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UNIFORM SKODA INTEGRABILITY AND CALABI–YAU DEGENERATION

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We study polarised algebraic degenerations of Calabi–Yau manifolds. We prove a uniform Skoda-type estimate and a uniform L^∞-estimate for the Calabi–Yau Kähler potentials.

1. Introduction

Let (Y, ω) be a compact Kähler manifold, and let $d\mu$ be a measure on Y. We say $(Y, \omega, d\mu)$ satisfies the *Skoda-type inequality*, if, for any Kähler potential $u \in \mathrm{PSH}(Y, \omega)$ normalised to $\sup u = 0$,

$$\int_Y e^{-\alpha u}\, d\mu \le A, \tag{1}$$

where α and A are independent of u. A prototype theorem is:

Theorem 1.1 [Tian 1987]. *On a fixed compact Kähler manifold* (Y, ω), *the Skoda-type inequality holds for* $d\mu = \omega^n$.

Remark 1.2. Here the supremum of all such α is known as Tian's alpha invariant and is important for existence questions of Kähler–Einstein metrics.

We are interested in keeping track of the constants α and A as $(Y, \omega, d\mu)$ varies. The main theme of this paper is that oftentimes the Skoda constants can be chosen uniformly for quite flexible choices of probability measures $d\mu$, even when the complex structure degenerates severely. In the literature, α is much studied, see [Guedj and Zeriahi 2005; Tian 1987], and a very recent preprint [Di Nezza et al. 2023] made aware to the author after the completion of this work contains a uniform estimate for both α and A in the related context of Kähler–Einstein manifolds.

Our main application is to algebraic degenerations of Calabi–Yau manifolds. We work over $\mathbb{C}$. Let S be a smooth affine algebraic curve, with a point $0 \in S$. An *algebraic degeneration family* is given by a submersive projective morphism $\pi : X \to S \setminus \{0\}$ with smooth connected n-dimensional fibres X_t for $t \in S \setminus \{0\}$. A *polarisation* is given by an ample line bundle L over X; the sections of a sufficiently high power of L induces an embedding $X \to \mathbb{CP}^N$, and hence a Fubini–Study metric ω_X on (X, L). For $0 < |t| \ll 1$, a fixed choice of ω_X induces *rescaled* background metrics $\omega_t = \omega_X|_{X_t}/\big|\log|t|\big|$ on X_t in the class $c_1(L)/\big|\log|t|\big|$.

A *model* of X is a normal flat projective S-scheme $\mathcal{X}$ which agrees with $\pi : X \to S \setminus \{0\}$ over the punctured curve. It is called a *semistable snc model* if $\mathcal{X}$ is smooth, the central fibre over $0 \in S$ is *reduced*

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and is a simple normal crossing (snc) divisor in $\mathcal{X}$. By the semistable reduction theorem [Kempf et al. 1973, Chapter 2], after finite base change to another smooth algebraic curve S', we can always find some semistable snc model for the degeneration family $X \times_S (S' \setminus \{0\})$. Everything here is quasiprojective.

We say the degeneration family is *Calabi–Yau* if there is a trivialising section Ω of the canonical bundle K_X. Over a small disc $\mathbb{D}_t$ around $0 \in S$, this induces holomorphic volume forms Ω_t on X_t via $\Omega = dt \wedge \Omega_t$. The *normalised Calabi–Yau measure* on X_t is the probability measure

$$d\mu_t = \frac{\Omega_t \wedge \bar{\Omega}_t}{\int_{X_t} \Omega_t \wedge \bar{\Omega}_t}. \tag{2}$$

We are ready to state our main result.

Theorem 1.3 (uniform Skoda estimate). *Given a polarised algebraic Calabi–Yau degeneration family $\pi : X \to S \setminus \{0\}$ as above, there are uniform positive constants α and A independent of t for $0 < |t| \ll 1$ such that, for the normalised Calabi–Yau measures $d\mu_t$,*

$$\int_{X_t} e^{-\alpha u}\, d\mu_t \le A \quad \textit{for all } u \in \mathrm{PSH}(X_t, \omega_t) \textit{ with } \sup_{X_t} u = 0.$$

This is proved by reducing to the semistable snc model case and proving a general Skoda-type estimate there (see Theorem 2.9). A major consequence, readily reaped using Kołodziej's estimate (see Theorem 3.1), follows.

Theorem 1.4 (uniform L^∞-estimate). *Let ϕ_t be the Kähler potential of the Calabi–Yau metric in the class $(X_t, [\omega_t])$, namely*

$$\frac{(\omega_t + \sqrt{-1}\partial\bar{\partial}\phi)^n}{\int_{X_t} \omega_t^n} = d\mu_t, \qquad \sup_{X_t} \phi_t = 0.$$

Then $\|\phi_t\|_{L^\infty} \le C$ independently of t for $0 < |t| \ll 1$.

Remark 1.5. Applications of pluripotential theory to Calabi–Yau metrics when the Kähler class is degenerating can be found in [Eyssidieux et al. 2008], which is used further in [Tosatti 2010]. Our main results generalise certain aspects of [Li 2022] which focuses on degenerating projective hypersurfaces near the large complex structure limit.

2. Uniform Skoda inequality

We work in the context of semistable simple normal crossing (snc) models. Concretely, let $\pi : \mathcal{X} \to \mathbb{D}_t$ be a flat projective family of n-dimensional varieties over a small disc $\mathbb{D}_t$ such that the total space $\mathcal{X}$ is smooth, π is a submersion over the punctured disc with connected fibres, and the central fibre X_0 is *reduced* and is an snc divisor in $\mathcal{X}$. Denote the components of X_0 by E_i with $i \in I$. We equip $\mathcal{X}$ with a fixed background Kähler metric $\omega_{\mathcal{X}}$, inducing a distance function $d_{\omega_{\mathcal{X}}}$. This induces a family of *rescaled* Kähler metrics $\omega_t = \omega_{\mathcal{X}}|_{X_t} / \big|\log|t|\big|$. We shall derive a uniform Skoda-type estimate (1) for $(X_t, \omega_t, d\mu_t)$, where $d\mu_t$ belongs to a natural class of measures. The main result is Theorem 2.9.

Quantitative stratification and good test functions. There is a *quantitative stratification* on any smooth fibre X_t induced by the intersection pattern of E_i: for $J \subset I$ such that $E_J = \bigcap_{i\in J} E_i \neq \varnothing$, the corresponding stratum is

$$E_J^0 = \{x \in X_t \mid d_{\omega_{\mathcal{X}}}(x, E_J) \lesssim \epsilon\} \setminus \{x \in X_t \mid d_{\omega_{\mathcal{X}}}(x, E_{J'}) \lesssim \epsilon \text{ for some } J' \supsetneq J\},$$

namely a small "ϵ-tubular neighbourhood" of E_J minus the deeper strata. For $J = \{i\}$ we write $E_i^0 = E_{\{i\}}^0$. Here the disc $\mathbb{D}_t$ and the small parameter $\epsilon \ll 1$ can be shrunk for convenience; the essential thing is that all parameters should be independent of the coordinate t.

It is useful to introduce local coordinates $\{z_i\}_0^n$ around $E_J \subset \mathcal{X}$ such that $z_0, \dots, z_p$ with $p = |J| - 1$ are the local defining equations of E_j for $j \in J$, and locally the fibration map is $t = z_0 \cdots z_p$. Then up to uniform equivalence, locally

$$\omega_{\mathcal{X}} \sim \sum_0^n \sqrt{-1}\, dz_i \wedge d\bar{z}_i .$$

The rest of this section is devoted to the construction of good test functions. Given any of these divisors E_0, we can find a nonnegative function $h = h_{E_0}$ on $\mathcal{X}$ such that:

- In the local charts near E_0 with z_0 being the defining function for E_0,

$$h = |z_0|^2 \tilde{h}(z_0, \dots, z_n)$$

  for some positive smooth function $\tilde{h}$.
- Away from E_0, the function h is comparable to 1.

We observe that:

- The form $\partial\bar{\partial} \log h = \partial\bar{\partial} \log \tilde{h}$ extends smoothly.
- For $|t|^2 \ll h \lesssim \delta \ll 1$ inside X_t, so that $|z_0| \gg |t|$, by a local calculation near E_J with $0 \in J$,

$$\sqrt{-1}\partial \log h \wedge \bar{\partial} \log h \wedge \omega_{\mathcal{X}}|_{X_t}^{n-1} \geq \frac{\sqrt{-1}}{2|z_0|^2} dz_0 \wedge d\bar{z}_0 \wedge \omega_{\mathcal{X}}|_{X_t}^{n-1} \gtrsim \min\left\{ \frac{1}{|z_0|^2}, \max_{1\leq i\leq p} |z_i|^{-2} \right\} \omega_{\mathcal{X}}|_{X_t}^n$$
$$\gtrsim \min\left\{ \frac{1}{h}, h^{1/p}|t|^{-2/p} \right\} \omega_{\mathcal{X}}|_{X_t}^n \gtrsim \min\left\{ \frac{1}{h}, h^{1/n}|t|^{-2/n} \right\} \omega_{\mathcal{X}}|_{X_t}^n .$$

  Here in the first inequality we need to fix $\delta \ll 1$ so that the effect of $\partial \log h$ is dominated by $d \log z_0$. The second inequality uses that, for $1 \leq k \leq p$, the volume forms on X_t satisfy

$$\frac{1}{|z_0|^2} dz_0 \wedge d\bar{z}_0 \wedge \prod_{j\neq k, 1\leq j\leq n} \sqrt{-1}\, dz_j \wedge d\bar{z}_j \sim \frac{1}{|z_k|^2} \prod_{1\leq j\leq n} \sqrt{-1}\, dz_j \wedge d\bar{z}_j ,$$

  and the third inequality uses $|t| = |z_0 \cdots z_p| \sim h^{1/2}|z_1 \cdots z_p|$.
- On X_t, we have that $h \gtrsim |t|^2$. The region $\{|t|^2 \sim h\} \subset X_t$ can be identified as E_0^0, namely the vicinity of E_0 away from deeper strata. Here

$$\sqrt{-1}\partial \log h \wedge \bar{\partial} \log h \wedge \omega_{\mathcal{X}}|_{X_t}^{n-1} \geq 0 \quad \text{and} \quad \sqrt{-1}\partial\bar{\partial} \log h \wedge \omega_{\mathcal{X}}|_{X_t}^{n-1} \gtrsim -\omega_{\mathcal{X}}|_{X_t}^n .$$

Lemma 2.1 (good test functions). *Given the divisor E_0, we can choose a C^2 test function v on X_t such that the following hold uniformly for small $t \neq 0$:*

- *v is zero for $h \geq \delta$.*
- *Globally, $0 \leq v \leq -\log|t|$.*
- *For any divisor E_j intersecting E_0, there is a subset of E_j^0 with measure at least C_2 on which $\sqrt{-1}\partial\bar{\partial}v \wedge \omega_{\mathcal{X}}|_{X_t}^{n-1} \geq C_3\omega_{\mathcal{X}}|_{X_t}^n$.*
- *For $C_4|t|^2 \leq h \leq \delta$, the form $\sqrt{-1}\partial\bar{\partial}v \wedge \omega_{\mathcal{X}}|_{X_t}^{n-1}$ is greater than or equal to 0.*
- *For $h \leq C_4|t|^2$, the form $\sqrt{-1}\partial\bar{\partial}v \wedge \omega_{\mathcal{X}}|_{X_t}^{n-1}$ is greater than or equal to $-C_5\omega_{\mathcal{X}}|_{X_t}^n$.*

Proof. We seek the test function in the form $v = \Phi \circ \log h$ for some convex, nonincreasing, nonnegative C^2-function Φ. Compute

$$\partial\bar{\partial}v = \Phi''\partial\log h \wedge \bar{\partial}\log h + \Phi'(\partial\bar{\partial}\log\tilde{h}),$$

and using the properties of h above,

$$\sqrt{-1}\partial\bar{\partial}v \wedge \omega_{\mathcal{X}}|_{X_t}^{n-1} \geq \begin{cases} \left(\Phi''C_1'\min\{\frac{1}{h}, h^{1/n}|t|^{-2/n}\} + \Phi'C_2'\right)\omega_{\mathcal{X}}|_{X_t}^n, & |t|^2 \lesssim h \leq \delta, \\ C_3'\Phi'\omega_{\mathcal{X}}|_{X_t}^n, & h \lesssim |t|^2. \end{cases}$$

To satisfy our conditions on v, it is enough to have:

- $\Phi(x) = 0$ for $x \geq \log\delta$.
- $|\Phi'(x)| \lesssim 1$ for $2\log|t| \lesssim x \leq \log\delta$.
- For $h = e^x \leq \delta$,
$$-\frac{d}{dx}\log|\Phi'| = \frac{\Phi''}{|\Phi'|} \geq C_4'\max\{h, h^{-1/n}|t|^{2/n}\},$$
where $C_4' > C_2'/C_1'$. Moreover, for $x < \delta$, we need $\Phi' < 0$, so that $\sqrt{-1}\partial\bar{\partial}v \wedge \omega_{\mathcal{X}}|_{X_t}^{n-1}$ has some strict positivity for $\frac{1}{2}\delta < h < \delta$. Notice convexity of Φ is a consequence of these conditions.

To construct such a Φ, we can prescribe the behaviour near $x = \log\delta$ by

$$\Phi'(x) = -e^{1/(x-\log\delta)}$$

for $x < \log\delta$ and match this with a solution to

$$-\frac{d}{dx}\log|\Phi'| = C_4'\max\{e^x, e^{-x/n}|t|^{2/n}\}, \quad x < \log\delta,$$

for some large enough C_4' such that Φ' remains C^1 at the matching point. Integration shows that $|\Phi'|$ remains uniformly bounded at $h \sim |t|^2$, or equivalently $x \sim 2\log|t|$. □

Convexity. Consider $u \in \mathrm{PSH}(X_t, \omega_t)$ normalised to $\sup_{X_t} u = 0$. Equivalently, we can cover X_t by a bounded number of charts as before and use the local potentials of $\omega_{\mathcal{X}}$ to represent u as a collection of local plurisubharmonic (psh) functions $\{u_\beta\}$ with $|u_\beta - u| \leq C$.

Lemma 2.2 (convexity). *Let ϕ be any psh function on the open subset of*

$$\{1 < |z_i| < \Lambda,\ i = 1, \dots, p;\ |z_k| < 1,\ k = p+1, \dots, n\} \subset (\mathbb{C}^*)^p \times \mathbb{C}^{n-p}.$$

Then the function

$$\bar{\phi}(x_1, \dots, x_p) = \frac{1}{(2\pi)^n} \int_{\{|z_k| \le 1\ \forall k > p\}} \prod_{p+1}^{n} \sqrt{-1}\, dz_k \wedge d\bar{z}_k \int_{T^p} \phi(e^{x_1 + i\theta_1}, \dots, e^{x_p + i\theta_p})\, d\theta_1 \cdots d\theta_p$$

is convex.

Proof. For any choice of θ_i, the function $\phi(z_1 e^{i\theta_1}, \dots, z_p e^{i\theta_n}, z_{p+1}, \dots, z_n)$ is psh, since the T^p-action on $(\mathbb{C}^*)^p$ is holomorphic. Thus the average function $\bar{\phi}$ is also psh as a function of $z_1, \dots, z_p$. Any T^p-invariant psh function must be convex in the log coordinates because, for $x_i = \log |z_i|$,

$$\sqrt{-1}\partial\bar{\partial}\bar{\phi} = \frac{1}{4} \sum \frac{\partial^2 \bar{\phi}}{\partial x_i \partial x_j} \sqrt{-1}\, d \log z_i \wedge d\, \overline{\log z_j} \ge 0. \qquad \square$$

Harnack-type inequality.

Lemma 2.3 (almost maximum on top strata I). *For $u \in \mathrm{PSH}(X_t, \omega_t)$ normalised to $\sup_{X_t} u = 0$, there is some $i \in I$ such that*

$$\sup_{E_i^0} u \ge -C, \qquad \int_{E_i^0} u \omega_{\mathcal{X}}|_{X_t}^n \ge -C'.$$

Proof. Let the global maximum of u be achieved at $q_0 \in E_J^0$, and denote the local potential of u by u_β. Without loss of generality, $u_\beta \le 0$. We have $u_\beta(q_0) \ge -C$ since $|u - u_\beta| \le C$. Applying the mean value inequality around q_0, we find that the local average function $\bar{u}_\beta$ produced in Lemma 2.2 satisfies $\sup \bar{u}_\beta \ge -C$ for another uniform constant C. By the convexity of $\bar{u}_\beta$, its supremum is almost achieved at the boundary of the chart, which is contained in a union of less deep strata $E_{J'}^0$, with $J' \subsetneq J$. Thus we can find a point q' with $u(q') \ge -C$ that belongs to a less deep stratum; an induction shows that there is some $i \in I$ such that $\sup_{E_i^0} u \ge -C$.

For the L^1-bound we recall the following Harnack inequality argument. Suppose a coordinate ball $B(q, 3R)$ is contained in a local chart in a small neighbourhood of E_i^0. Applying the mean value inequality to the local psh function associated to u, we see, for $y \in B(q, R)$, that

$$u(y) \le C + \fint_{B(y,2R)} u \lesssim 1 + \fint_{B(q,R)} u.$$

Hence the Harnack inequality yields

$$\fint_{B(q,R)} |u| \lesssim 1 + \inf_{B(q,R)} (-u).$$

Applying this to a chain of balls connecting any two points in E_i^0 gives the L^1-bound $\int_{E_i^0} u \omega_{\mathcal{X}}|_{X_t}^n \ge -C'$; the bound is uniform because the number of balls involved in the chain can be controlled independently of t. $\square$

Proposition 2.4 (almost maximum on top strata II). *There is a uniform lower bound for all $|t| \ll 1$ and all $i \in I$*:

$$\sup_{E_i^0} u \geq -C, \quad \int_{E_i^0} u\omega_{\mathcal{X}}|_{X_t}^n \geq -C'. \tag{3}$$

Proof. The L^1-estimate follows from the supremum estimate as above, so the real problem is to transfer bounds between different E_i^0. This is nontrivial because the necks connecting E_i^0 with each other are highly degenerate.

Given one divisor E_0 such that $\int_{E_0^0} u\omega_{\mathcal{X}}|_{X_t}^n \geq -C$, we produce a good test function v by Lemma 2.1. Integrating by parts,

$$\int_{X_t} v\sqrt{-1}\partial\bar{\partial}u \wedge \omega_{\mathcal{X}}|_{X_t}^{n-1} = \int_{X_t} u\sqrt{-1}\partial\bar{\partial}v \wedge \omega_{\mathcal{X}}|_{X_t}^{n-1}.$$

The left-hand side is the difference between

$$\int_{X_t} v(\omega_t + \sqrt{-1}\partial\bar{\partial}u) \wedge \omega_{\mathcal{X}}|_{X_t}^{n-1} \quad \text{and} \quad \int_{X_t} v\omega_t \wedge \omega_{\mathcal{X}}|_{X_t}^{n-1},$$

and since $-\log|t| \gtrsim v \geq 0$, both terms are bounded between 0 and C. Thus

$$\left|\int_{X_t} u\sqrt{-1}\partial\bar{\partial}v \wedge \omega_{\mathcal{X}}|_{X_t}^{n-1}\right| \leq C.$$

Now the form $\sqrt{-1}\partial\bar{\partial}v \wedge \omega_{\mathcal{X}}|_{X_t}^{n-1}$ can only be negative on $\{h \sim |t|^2\} = E_0^0$ and is bounded below by $-C\omega_{\mathcal{X}}|_{X_t}^n$. Thus the positive part of the signed measure $u\sqrt{-1}\partial\bar{\partial}v \wedge \omega_{\mathcal{X}}|_{X_t}^{n-1}$ has total mass controlled by $\int_{E_0^0} |u|\omega_{\mathcal{X}}|_{X_t}^n \leq C$. Consequently, the negative part of the signed measure must also have total mass $\leq C$.

By construction, for any divisor E_j intersecting E_0, there is a nontrivial amount of $\sqrt{-1}\partial\bar{\partial}v \wedge \omega_{\mathcal{X}}|_{X_t}^{n-1}$-measure inside E_j^0. This forces $\sup_{E_j^0} u \geq -C$. To summarise, we have transferred the supremum bound from E_0^0 to any E_j^0 with $E_j \cap E_0 \neq \varnothing$. Since the central fibre X_0 is connected, in at most $|I|$ steps this supremum bound is transferred to all E_i^0 with $i \in I$. □

Remark 2.5. This proof is inspired by the intersection-theoretic argument of [Boucksom et al. 2016, Section 6.1], which can be viewed as a non-Archimedean analogue.

Local L^1 estimate. For a given local chart on E_J^0 with $\mathbb{C}^*$-coordinates $z_1, \dots, z_p$ and $\mathbb{C}$-coordinates $z_{p+1}, \dots, z_n$, and a point q therein, we shall refer to the subregion

$$\left\{\tfrac{1}{2}|z_i(q)| \lesssim |z_i| \lesssim 2|z_i(q)|,\ 1 \leq i \leq p\right\}$$

as a *log scale*.

Lemma 2.6 (local L^1-estimate). *Within every log scale there is a uniform bound on the L^1-average integral*:

$$⨍_{\text{loc}} |u| \prod_1^p \sqrt{-1}\, d\log z_i \wedge d\log \bar{z}_i \wedge \prod_{p+1}^n \sqrt{-1}\, dz_k \wedge d\bar{z}_k \leq C.$$

Proof. For $p=0$ this follows from Proposition 2.4. Given a depth p chart, we consider the local psh function u_β associated to u and produce the convex average function $\bar{u}_\beta$ as in Lemma 2.2. We claim $|\bar{u}_\beta| \le C$ in the chart. The upper bound holds as u_β is bounded above, and by convexity it suffices to achieve a lower bound at the barycentre of the simplex. However, when we blow up the depth p intersection E_J, there is a new vertex in the dual complex corresponding to the new divisor component in the central fibre. The dual complex is subdivided, and the new vertex is situated at the original barycentre. The same argument in Proposition 2.4 then achieves a lower bound on the local average of the Kähler potential near this new divisor, which amounts to the desired lower bound on $\bar{u}_\beta$ at the barycentre, whence the claim follows.

Within any log scale, by construction the local average $⨍_{\mathrm{loc}}(u_\beta - \bar{u}_\beta)$ equals zero. But $u_\beta \le C$ since $u \le 0$, and hence

$$⨍_{\mathrm{loc}} |u_\beta - \bar{u}_\beta| \lesssim ⨍_{\mathrm{loc}} (u_\beta - \bar{u}_\beta)_+ \le C.$$

Using $|u - u_\beta| \le C$, we conclude the local L^1-estimate on u. □

Local Skoda estimate. We recall a basic version of the *Skoda inequality*:

Proposition 2.7 [Zeriahi 2001, Theorem 3.1]. *If ϕ is psh on $B_2 \subset \mathbb{C}^n$, with $\int_{B_2} |\phi| \omega_E^n \le 1$ with respect to the standard Euclidean metric ω_E, then there are dimensional constants α and C such that*

$$\int_{B_1} e^{-\alpha\phi} \omega_E^n \le C.$$

Applying this along with Lemma 2.6, we get the following corollary.

Corollary 2.8 (local Skoda estimate). *Within every log scale, there are uniform positive constants α and C such that*

$$⨍_{\mathrm{loc}} e^{-\alpha u} \prod_1^p \sqrt{-1}\, d\log z_i \wedge d\log \bar{z}_i \wedge \prod_{p+1}^n \sqrt{-1}\, dz_k \wedge d\bar{z}_k \le C.$$

Uniform global Skoda estimate. We are interested in the following class of measures, motivated by Calabi–Yau measures (see Section 3). Let a_i be nonnegative real numbers assigned to $i \in I$, with $\min a_i = 0$. Let

$$m = \max\{|J| - 1 : E_J \neq \varnothing,\ a_i = 0 \text{ for } i \in J\}.$$

We say the measures $d\mu_t$ on X_t satisfy a *uniform upper bound of class* (a_i) if, on the local charts of each E_J^0,

$$d\mu_t \le \frac{C}{|\log|t||^m} |z_0|^{2a_0} \cdots |z_p|^{2a_p} \prod_1^p \sqrt{-1}\, d\log z_i \wedge d\log \bar{z}_i \wedge \prod_{p+1}^n \sqrt{-1}\, dz_k \wedge d\bar{z}_k. \tag{4}$$

The normalisation factor ensures that $\int_{X_t} d\mu_t \le C$ independently of t by a straightforward local calculation.

Theorem 2.9 (uniform Skoda estimate). *Suppose the measures $d\mu_t$ on X_t satisfy a uniform upper bound of class (a_i). Then there are uniform positive constants α and A such that*

$$\int_{X_t} e^{-\alpha u}\, d\mu_t \le A \quad \textit{for all } u \in \mathrm{PSH}(X_t, \omega_t) \textit{ with } \sup_{X_t} u = 0.$$

Proof. We choose the charts so that each point on X_t is covered by $\le C$ log scales. Summing over the local Skoda estimates from all log scales, $\int_{X_t} e^{-\alpha u}\, d\mu_t$ is bounded by

$$\frac{C}{\big|\log|t|\big|^m} \sum_{\text{log scales}} \int_{\text{loc}} |z_0|^{2a_0} \cdots |z_p|^{2a_p} \prod_1^p \sqrt{-1}\, d\log z_i \wedge d\log \bar{z}_i \wedge \prod_{p+1}^n \sqrt{-1}\, dz_k \wedge d\bar{z}_k \le C. \qquad \square$$

3. Application to Calabi–Yau degeneration

We work in the setting of polarised algebraic degeneration of Calabi–Yau manifolds, as in the Introduction.

Calabi–Yau measure. The Calabi–Yau measure (2) is studied thoroughly in [Boucksom and Jonsson 2017], but it is illustrative to recall it explicitly on a semistable snc model $\mathcal{X}$. The discussion is local on the base, and we will follow the notations of Section 2, e.g., the components of the central fibre are denoted by E_i for $i \in I$.

The canonical divisor $K_{\mathcal{X}} = \sum_i a_i E_i$ is supported on the central fibre, since K_X is trivialised. Multiplying Ω by a power of t, which does not change $d\mu_t$, we may assume $\min a_i = 0$. In the local coordinates around E_J away from the deeper strata,

$$\Omega = f_J \prod_0^p z_i^{a_i}\, dz_i \wedge \prod_{p+1}^n dz_j$$

for some nowhere-vanishing local holomorphic function f_J. Since $t = z_0 \cdots z_p$,

$$\Omega_t = f_J z_0^{a_0} \cdots z_p^{a_p} \prod_1^p d\log z_i \wedge \prod_{p+1}^n dz_j,$$

and hence

$$\sqrt{-1}^{n^2}\, \Omega_t \wedge \overline{\Omega}_t = |f_J|^2 |z_0|^{2a_0} \cdots |z_p|^{2a_p} \prod_1^p \sqrt{-1}\, d\log z_i \wedge d\,\overline{\log z_i} \wedge \prod_{p+1}^n \sqrt{-1}\, dz_j \wedge d\bar{z}_j.$$

The total measure $\int_{X_t} \sqrt{-1}^{n^2}\, \Omega_t \wedge \overline{\Omega}_t$ is of the order $O\big(\big|\log|t|\big|^m\big)$, where

$$m = \max\{|J| - 1 : E_J \ne \varnothing,\ a_i = 0 \text{ for } i \in J\}.$$

Thus $d\mu_t$ satisfies a uniform upper bound of class (a_i); see (4).

Uniform Skoda estimate. We now prove the main Theorem 1.3.

Proof. First we observe that the choice of the Fubini–Study metric ω_X is immaterial. Given any two choices, the relative Kähler potential between them is bounded by $O\big(\big|\log|t|\big|\big)$ for $0 < |t| \ll 1$ because the pole order of a section near $t = 0$ must be finite. Thus the relative Kähler potential between two choices of ω_t is bounded by $O(1)$ independent of t, which affects the Skoda constant A but not its uniform nature.

We now pass to a finite base change and find a semistable reduction. The Calabi–Yau measure $d\mu_t$ on X_t is independent of the parametrisation of the base and is preserved under finite base change. Thus it is enough to prove it assuming ω_X agrees with a smooth Kähler metric on a semistable snc model $\mathcal{X}$; this is a special case of Theorem 2.9. □

Uniform L^∞-estimate. We recall the following result proved using Kołodziej's pluripotential-theoretic methods (see [Li 2022, Section 2.2] for an exposition based on [Eyssidieux et al. 2009; 2008]):

Theorem 3.1. *Let (Y, ω) be a compact Kähler manifold, and let the Kähler potential ϕ solve the complex Monge–Ampère equation*

$$\frac{(\omega+\sqrt{-1}\partial\bar{\partial}\phi)^n}{\int_Y \omega^n} = d\mu, \quad \sup\phi = 0.$$

Assume there are positive constants α and A such that the Skoda-type estimate (1) holds for $(Y, \omega, d\mu)$:

$$\int_Y e^{-\alpha u}\, d\mu \leq A \quad \text{for all } u \in \mathrm{PSH}(Y, \omega) \text{ with } \sup_Y u = 0.$$

Then $\|\phi\|_{C^0} \leq C(n, \alpha, A)$.

The uniform L^∞-estimate for the Calabi–Yau potentials in Theorem 1.4 is an immediate consequence.

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UNIQUE CONTINUATION FOR THE HEAT OPERATOR WITH POTENTIALS IN WEAK SPACES

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We prove the strong unique continuation property for the differential inequality

$$|(\partial_t + \Delta)u(x,t)| \le V(x,t)|u(x,t)|,$$

with V contained in weak spaces. In particular, we establish the strong unique continuation property for $V \in L_t^\infty L_x^{d/2,\infty}$, which has been left open since the works of Escauriaza (2000) and Escauriaza and Vega (2001). Our results are consequences of the Carleman estimates for the heat operator in the Lorentz spaces.

1. Introduction

We consider the differential inequality

$$|(\partial_t + \Delta)u(x,t)| \le |V(x,t)u(x,t)|, \quad (x,t) \in \mathbb{R}^d \times (0,T). \tag{1-1}$$

For a differential operator P on a domain Ω, the strong unique continuation property (abbreviated *sucp* hereafter) for $|Pu| \le |Vu|$ means that a nontrivial solution u to $|Pu| \le |Vu|$ cannot vanish to infinite order (in a suitable sense) at any point. The *sucp* for second-order parabolic operators has been studied by many authors; see [Banerjee and Manna 2020; Chen 1998; Escauriaza 2000; Escauriaza and Fernández 2003; Escauriaza and Vega 2001; Fernández 2003; Koch and Tataru 2009; Lin 1990; Poon 1996; Sogge 1990]. In particular, the study of *sucp* for the heat operator with time-dependent potentials goes back to Poon [1996] and Chen [1998], who considered bounded potentials. Escauriaza [2000] and Escauriaza and Vega [2001] extended the results to unbounded potentials V under the parabolic vanishing condition: for a given $\delta \in (0,1)$ and any $k \in \mathbb{N}$, there is a constant C_k such that

$$|u(x,t)| \le C_k(|x| + \sqrt{t})^k e^{(1-\delta)|x|^2/(8t)}, \quad (x,t) \in \mathbb{R}^d \times (0,T). \tag{1-2}$$

The growth condition at infinity is necessary since there exists a nonzero solution u to $(\partial_t + \Delta)u = 0$ such that u vanishes to infinite order in the space-time variables at any point $(x,0)$, $x \in \mathbb{R}^d$; see, for example, [Escauriaza 2000; John 1971].

The *sucp* for the Laplacian $-\Delta$ is better understood. Since the pioneering work of [Carleman 1939], most subsequent results were obtained using the Carleman weighted inequality. In particular, [Jerison and Kenig 1985] proved the *sucp* for the Laplacian, with $V \in L^{d/2}_{\mathrm{loc}}$, $d \ge 3$. Their result was extended by

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Stein in the Appendix of [Jerison and Kenig 1985] to potentials $V \in L^{d/2,\infty}$ under the assumption that $\|V\|_{L^{d/2,\infty}}$ is small enough. Here, $\|\cdot\|_{p,r}$ denotes the Lorentz space norm

$$\|f\|_{L^{p,r}} = \left(\frac{r}{p}\int_0^\infty t^{r/p-1}(f^*(t))^r\,dt\right)^{1/r} < \infty$$

for $r \neq \infty$, and $\|f\|_{L^{p,\infty}} = \sup_{t>0} t^{1/p} f^*(t) < \infty$ for $r = \infty$, where f^* is the decreasing rearrangement of f on $\mathbb{R}^d$; for example, see [Stein and Weiss 1971]. Later, [Wolff 1992] showed that the smallness assumption is indispensable if $V \in L^{d/2,\infty}$. By the aforementioned results due to Escauriaza [2000] and Escauriaza and Vega [2001], the *sucp* for (1-1) is known when $t^{1-d/(2\mathfrak{r})-1/\mathfrak{s}}V(x,t) \in L^{\mathfrak{s}}_{t,\mathrm{loc}}L^{\mathfrak{r}}_x$ and $\mathfrak{r}$ and $\mathfrak{s}$ satisfy

$$\frac{d}{2\mathfrak{r}} + \frac{1}{\mathfrak{s}} \le 1, \quad 1 \le \mathfrak{r}, \mathfrak{s} \le \infty. \tag{1-3}$$

However, in view of those results concerning the (abovementioned) *sucp* for the Laplacian, it seems natural to expect that the class of potentials for which the *sucp* for (1-1) holds can be further expanded to certain weak spaces.

In this paper, we extend the results in [Escauriaza 2000; Escauriaza and Vega 2001] to a larger class of potentials, that is to say,

$$t^{1-d/(2\mathfrak{r})-1/\mathfrak{s}}V(x,t) \in L^{\mathfrak{s}}_{t,\mathrm{loc}}L^{\mathfrak{r},\infty}_x, \quad \frac{d}{2\mathfrak{r}} + \frac{1}{\mathfrak{s}} \le 1, \ \ \frac{d}{2} \le \mathfrak{r} \le \infty.$$

As in the Appendix of [Jerison and Kenig 1985], our result is a consequence of new Carleman estimates for the heat operator in the Lorentz spaces.

Carleman estimate. Write $L^s_t L^{q,b}_x = L^s_t(\mathbb{R}_+; L^{q,b}_x(\mathbb{R}^d))$. We consider the Carleman inequality for the heat operator of the form

$$\|t^{-\alpha}e^{-|x|^2/(8t)}g\|_{L^s_tL^{q,b}_x} \le C\|t^{-\alpha+1-(d/2)(1/p-1/q)-(1/r-1/s)}e^{-|x|^2/(8t)}(\Delta+\partial_t)g\|_{L^r_tL^{p,a}_x}, \tag{1-4}$$

with C independent of α, which holds for $g \in C^\infty_c(\mathbb{R}^{d+1}\setminus\{(0,0)\})$ under a suitable condition on the exponents α, p, q, r, s, a, and b. For $\alpha \in \mathbb{R}$, we set

$$\beta = \beta(\alpha, q, s) = 2\alpha - \frac{d}{q} - \frac{2}{s}.$$

Estimate (1-4) was formerly considered with $p = a$ and $q = b$. It was Escauriaza [2000] who first obtained (1-4) for some $p = a$, $q = b$, r, and s. More precisely, he showed that (1-4) holds with the Lebesgue spaces (i.e., $a = p$ and $b = q$) for p, q satisfying $q = p'$ and $0 \le 1/p - 1/q < 2/d$ when $d \ge 2$, and $0 \le 1/p - 1/q \le 1$ when $d = 1$, provided that

$$\mathrm{dist}(\beta, \mathbb{N}_0) \ge c$$

for some $c > 0$, where $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$. Subsequently, estimate (1-4) was extended by [Escauriaza and Vega 2001] to the exponents p, q which lie outside of the line of duality. They obtained (1-4) for

$$\frac{2d}{d+2} \le p \le 2 \le q \le \frac{2d}{d-2} \quad \text{when } d \ge 3,$$

and for $1 \le p \le 2 \le q \le \infty$ except $(p, q, d) \neq (1, \infty, 2)$ when $d = 1, 2$.

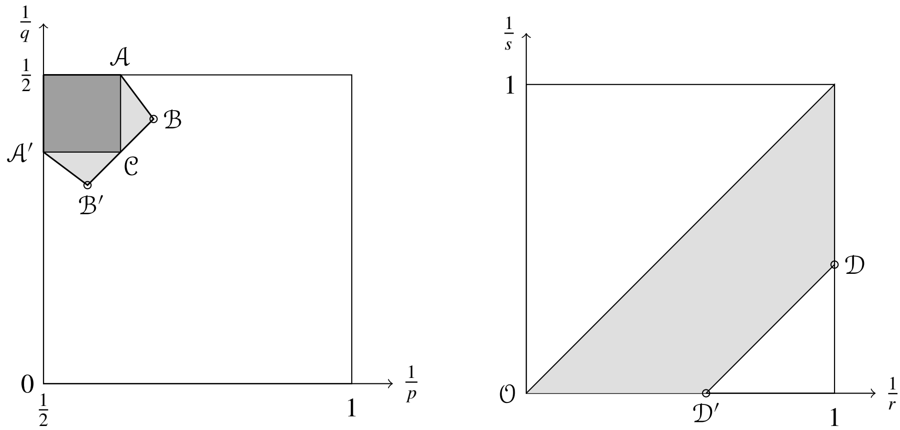

Figure 1. The regions of (p, q) and (r, s) for which (1-4) holds when $d \geq 3$: the dark gray square in the left figure represents the earlier result due to [Escauriaza and Vega 2001], and the light gray region represents the newly extended range. In the right figure, $\mathcal{O} = (0, 0)$ and $\mathcal{D} = \big(1, \frac{1}{2}d(1/p - 1/q)\big)$.

We extend the previously known results not only to Lorentz spaces but also on a wider range of exponents p and q. To present our result, for $d \geq 3$ we define $\mathcal{A} = \mathcal{A}(d),\ \mathcal{B} = \mathcal{B}(d),\ \mathcal{C} = \mathcal{C}(d) \in \big[\frac{1}{2}, 1\big] \times \big[0, \frac{1}{2}\big]$ by

$$\mathcal{A} = \left(\frac{d+2}{2d}, \frac{1}{2}\right), \qquad \mathcal{B} = \left(\frac{d^2+2d-4}{2d(d-1)}, \frac{d-2}{2(d-1)}\right), \qquad \mathcal{C} = \left(\frac{d+2}{2d}, \frac{d-2}{2d}\right).$$

By $\mathfrak{T}$ we denote the closed pentagon with vertices $\big(\frac{1}{2}, \frac{1}{2}\big)$, $\mathcal{A}$, $\mathcal{B}$, $\mathcal{B}'$, and $\mathcal{A}'$ from which the two vertices $\mathcal{B}$ and $\mathcal{B}'$ are removed. Here, $X' = (1-b, 1-a)$ (the dual point) if $X = (a, b)$. See Figure 1.

Theorem 1.1. *Let $d \geq 3$ and $(1/p, 1/q) \in \mathfrak{T}$. Let $1 \leq r \leq s \leq \infty$ satisfy*

$$\left(\frac{1}{r}, \frac{1}{s}\right) \neq \left(1, \frac{d}{2}\left(\frac{1}{p} - \frac{1}{q}\right)\right), \left(1 - \frac{d}{2}\left(\frac{1}{p} - \frac{1}{q}\right), 0\right)$$

and

$$0 \leq \frac{1}{r} - \frac{1}{s} \leq 1 - \frac{d}{2}\left(\frac{1}{p} - \frac{1}{q}\right). \tag{1-5}$$

Suppose $\beta \notin \mathbb{N}_0$. Then, if $1/p - 1/q < 2/d$, $p \neq 2$, and $q \neq 2$, estimate (1-4) holds for $1 \leq a = b \leq \infty$ with C depending only on p, q, a, b, r, s, and $\operatorname{dist}(\beta, \mathbb{N}_0)$*; if*

$$\frac{1}{p} - \frac{1}{q} = \frac{2}{d}, \tag{1-6}$$

the same estimate (1-4) holds for $a = b = 2$.

It is remarkable that Theorem 1.1 gives (1-4) for $(1/p, 1/q)$ contained in the open line segment $(\mathcal{B}, \mathcal{B}')$ (see Figure 1). The exponents p, q satisfying (1-6) constitute the critical case in that (1-4) is no longer true if $1/p - 1/q > 2/d$. (See the remark on page 2270 and the condition (1-5).) Consequently, it is more difficult to obtain estimate (1-4) for p, q with (1-6) than that for p, q with $1/p - 1/q < 2/d$. Only estimate (1-4) with $(1/p, 1/q) = \mathcal{C}$, $a = p$, and $b = q$ was previously shown by [Escauriaza and Vega 2001].

If $p,\ q$ satisfy (1-6) and $r=s$, then the estimate (1-4) implies the Carleman inequality for the Laplacian (see [Escauriaza and Vega 2001]):

$$\||x|^{-\sigma} f\|_{L^{q,b}} \le C\||x|^{-\sigma}\Delta f\|_{L^{p,a}}, \quad f\in C_c^\infty(\mathbb{R}^d\setminus\{0\}) \tag{1-7}$$

for $\sigma>0$, with $C>0$ depending on $d,\ p,\ q$, and $\operatorname{dist}(\sigma,\mathbb{N}+d/q)$ (if $\operatorname{dist}(\sigma,\mathbb{N}+d/q)>0$). By this implication the estimates in Theorem 1.1 with $p,\ q$ satisfying (1-6) give (1-7) for $(1/p,1/q)\in(\mathcal{B},\mathcal{B}')$. However, it does not extend the previously known range of $p,\ q$ for which (1-7) holds. When $d\ge 5$, the range of $p,\ q$ coincides with that in [Kwon and Lee 2018], which was obtained by making use of the sharp estimate for the spherical harmonic projection. The optimal range of $p,\ q$ for (1-7) remains open.

To obtain *sucp* for potentials in $L^s_{t,\mathrm{loc}}L^{r,\infty}_x$, we need to obtain (1-4) with $a=b$. To this end, we are basically relying on real interpolation to upgrade $L^r_tL^p_x$-$L^s_tL^q_x$ estimates to those of $L^r_tL^{p,a}_x$-$L^s_tL^{q,b}_x$ with $a=b$. However, such an extension of the inequality (1-4) to the Lorentz spaces is not so straightforward as in the Appendix of [Jerison and Kenig 1985], since real interpolation does not behave well in mixed-norm spaces; see [Cwikel 1974]. We are only able to obtain (1-4) with $a=b=2$ when p,q satisfy (1-6); also see Lemma 4.1.

Theorem 1.1 provides estimate (1-4) for exponents on an extended range, but the problem of determining the optimal range of $p,\ q$ for which (1-4) holds remains open. When $1/p-1/q<d/2$, by Theorem 1.1, estimate (1-4) holds for all $a,\ b$ satisfying $1\le a\le b\le\infty$, since $L^{p,r_1}\subset L^{p,r_2}$ if $r_1\le r_2$. Because of limitation of the real interpolation in the mixed-norm spaces, we have (1-4) only for $1\le a\le 2\le b\le\infty$ when $1/p-1/q=d/2$. However, we expect that the same continues to be true even for $p,\ q$ satisfying $1/p-1/q=d/2$.

Strong unique continuation property for the heat operator. Our extension of the Carleman estimate to the Lorentz spaces (Theorem 1.1) allows a larger class of potentials for the strong unique continuation property for the heat operator. In this regard we obtain Theorems 1.2 and 1.3, which improve the results in [Escauriaza and Vega 2001]. Once we have the Carleman estimate (1-4), those theorems can be shown by routine adaptation of the argument in that paper. We state them without providing the proofs.

Theorem 1.2. *Let $d\ge 3,\ 0<T<\infty$, and $\mathfrak{r},\ \mathfrak{s}$ satisfy (1-3). Let $(1/p,1/q)\in\mathfrak{T}$ satisfy $1/p-1/q=1/\mathfrak{r}$. Suppose that $u\in W^{1,a}((0,T);\ W^{2,p}(\mathbb{R}^d)),\ a\le\min\{2,\mathfrak{s}\}$, is a solution to the differential inequality (1-1), and suppose that, for any $k\in\mathbb{N}$, there is a constant C_k such that (1-2) holds for some $\delta>0$. Then u is identically zero on $\mathbb{R}^d\times(0,T)$ provided that $\|t^{1-d/(2\mathfrak{r})-1/\mathfrak{s}}V\|_{L^{\mathfrak{s}}((0,T);L^{\mathfrak{r},\infty}_x(\mathbb{R}^d))}$ is small enough.*

Most significantly, Theorem 1.2 gives the *sucp* with $V\in L^\infty((0,T);\ L^{d/2,\infty}_x(\mathbb{R}^d))$. This extends the result obtained by [Escauriaza and Vega 2001] under the assumption that $\|V\|_{L^\infty((0,T);L^{d/2}_x)}$ is small enough. Using Wolff's construction [1992], we can show that the smallness assumption is necessary in general for $V\in L^\infty((0,T);\ L^{d/2,\infty}(\mathbb{R}^d))$, or $V\in L^{d/2,\infty}(\mathbb{R}^d;\ L^\infty((0,T)))$. Indeed, Wolff showed that there is a bounded nonzero function w such that $|\Delta w|\le|V_*w|$ with $V_*\in L^{d/2,\infty}$ and which vanishes to infinite order at the origin. Since the function w in [Wolff 1992] is bounded, by considering the time independent function $u(x,t):=w(x)$, it is easy to see that $u(x,t)$ satisfies (1-2) and obviously the differential inequality $|\Delta u+\partial_t u|\le|V_*u|$.

We also have the following *sucp* result for a local solution.

Theorem 1.3. *Let $d \ge 3$ and $\mathfrak{r}$, $\mathfrak{s}$ satisfy (1-3). Suppose that u is a continuous solution to $|\Delta u + \partial_t u| \le |Vu|$ on $B(0,2) \times (0,2)$, and suppose that, for any $k \in \mathbb{N}$, there is a constant C_k such that*

$$\|e^{-|x|^2/(8t)} u\|_{L^2((0,\varepsilon);L^2_x(B(0,2)))} \le C_k \varepsilon^k, \quad 0 < \varepsilon < 2.$$

Then $u(x,0)$ vanishes on $B(0,2)$ if $\|t^{1-d/(2\mathfrak{r})-1/\mathfrak{s}} V\|_{L^{\mathfrak{s}}((0,2);L^{\mathfrak{r},\infty}_x(B(0,2)))}$ is small enough.

Uniform resolvent estimate for the Hermite operator. We now consider the resolvent estimate for the Hermite operator $H = -\Delta + |x|^2$ in $\mathbb{R}^d$:

$$\|(H-z)^{-1} f\|_q \le C\|f\|_p, \quad z \in \mathbb{C} \setminus (2\mathbb{N}_0 + d), \tag{1-8}$$

with a constant C independent of z. The estimate has independent interest while it plays an important role in proving Theorem 1.1; see Lemma 4.1. Since H has the discrete spectrum $2\mathbb{N}_0 + d$, the points $z \in 2\mathbb{N}_0 + d$ are excluded. In contrast with the operator with a continuous spectrum, it is impossible for (1-8) to hold with C independent of z, so we need to impose the assumption that

$$\operatorname{dist}(z, 2\mathbb{N}_0 + d) \ge c \tag{1-9}$$

for some $1 \gg c > 0$; see the remark on page 2265. Estimate (1-8) may be compared with the corresponding estimate for the resolvent of the Laplacian which is due to Kenig, Ruiz, and Sogge [Kenig et al. 1987]. It was shown in that paper that the estimate

$$\|(-\Delta - z)^{-1} f\|_q \le C\|f\|_p, \quad z \in \mathbb{C} \setminus (0,\infty),$$

holds with C independent of z if and only if

$$\frac{1}{p} - \frac{1}{q} = \frac{2}{d}, \quad \frac{2d}{d+3} < p < \frac{2d}{d+1}, \quad \text{and} \quad d \ge 3.$$

Also, see [Jeong et al. 2016] for the uniform estimates for more general second-order differential operators and [Kwon and Lee 2020] for the sharp bounds which depend on z. Under the assumption (1-9), the uniform resolvent estimate for H continues to hold with p, q away from the critical line $1/p - 1/q = 2/d$, whereas this cannot be true for $-\Delta$ because of scaling structure; see [Kenig et al. 1987; Kwon and Lee 2020].

The uniform estimate (1-8) was obtained by Escauriaza and Vega [2001] for

$$\frac{2d}{d+2} \le p \le 2 \le q \le \frac{2d}{d-2}, \quad d \ge 3.$$

However, (1-8) fails to hold if $1/p - 1/q > 2/d$ (see the remark on page 2270), and the proof of (1-8) is more involved when p, q satisfy (1-6). As for such (p,q) of the critical case, the estimate has been known only for $(p,q) = (2d/(d+2), 2d/(d-2))$. In what follows we establish (1-8) for $(1/p, 1/q) \in (\mathcal{B}, \mathcal{B}')$ under assumption (1-9). Those estimates in the expanded range are crucial for obtaining (1-4) with $a = b$ when p, q satisfy (1-6).

Theorem 1.4. *Let $d \ge 3$. Suppose $(1/p, 1/q) \in \mathfrak{T}$ and (1-9) holds. Then there is a constant $C > 0$ such that estimate (1-8) holds. Furthermore, if $(1/p, 1/q) = \mathcal{B}$ or $\mathcal{B}'$, we have the restricted weak-type (uniform) estimate for $(H-z)^{-1}$.*

The proof of (1-8) with $(p,q)=(2d/(d+2),2d/(d-2))$ in [Escauriaza and Vega 2001] heavily relies on the uniform bound on the spectral projection operator Π_k, which is the projection onto the k-th eigenspace of the Hermite operator H; see Section 2. In fact, they also used interpolation along an analytic family of operators which are motivated by Mehler's formula for the Hermite function. However, their argument is not enough to prove (1-8) for $(1/p,1/q)\in(\mathcal{B},\mathcal{B}')$. We develop a different approach which is more direct and significantly simpler. We make use of a representation formula (2-1) for Π_k which was observed in [Jeong et al. 2022a] and an estimate for the Hermite–Schrödinger propagator $e^{-itH}f$ (see Proposition 2.1) which is a consequence of the representation formula and the endpoint Strichartz estimate [Keel and Tao 1998].

Organization of the paper. The rest of this paper is organized as follows. In Section 2 we provide useful properties of the Hermite operator H and the Hermite spectral projection operator Π_k. We prove boundedness of more general multiplier operators for the Hermite operator in Section 3, which implies Theorem 1.4. Finally, the proof of the Carleman estimate for the heat operator is given in Section 4.

2. Properties of the Hermite operator

For any multi-index $\alpha\in\mathbb{N}_0^d$, the L^2-normalized Hermite function Φ_α, which is a tensor product of one dimensional Hermite functions, is an eigenfunction of H with eigenvalue $2|\alpha|+d$. Here $|\alpha|:=\alpha_1+\cdots+\alpha_d$. The set $\{\Phi_\alpha:\alpha\in\mathbb{N}_0^d\}$ forms an orthonormal basis of $L^2(\mathbb{R}^d)$. Thus, for any $f\in L^2(\mathbb{R}^d)$, we have the Hermite expansion $f=\sum_\alpha\langle f,\Phi_\alpha\rangle\Phi_\alpha$.

We consider the Hermite spectral projection operator Π_k which is defined by

$$\Pi_k f=\sum_{\alpha\in\mathbb{N}_0^d:|\alpha|=k}\langle f,\Phi_\alpha\rangle\Phi_\alpha,\qquad f\in\mathcal{S}(\mathbb{R}^d).$$

Then, the Hermite–Schrödinger propagator is given by

$$e^{-itH}f=\sum_{k\in\mathbb{N}_0}e^{-it(2k+d)}\Pi_k f,\qquad f\in\mathcal{S}(\mathbb{R}^d),$$

which is the solution to the Cauchy problem $(i\partial_t-H)u=0,\ u(x,0)=f(x)$. If $f\in\mathcal{S}(\mathbb{R}^d)$, it is easy to see that $\Pi_k f$ decays rapidly in k, thus $\sum_{k=0}^\infty e^{-it(2k+d)}\Pi_k f$ converges uniformly. Clearly,

$$\Pi_k f=\sum_{k'\in\mathbb{N}_0}\frac{1}{2\pi}\left(\int_{-\pi}^{\pi}e^{it(k-k')}\,dt\right)\Pi_{k'}f.$$

Therefore, we obtain

$$\Pi_k f=\frac{1}{2\pi}\int_{-\pi}^{\pi}e^{it(2k+d-H)/2}f\,dt \tag{2-1}$$

for $f\in\mathcal{S}(\mathbb{R}^d)$. Meanwhile, the operator e^{-itH} has the kernel formula

$$e^{-itH}f(x)=C_d(\sin(2t))^{-d/2}\int_{\mathbb{R}^d}e^{i((|x|^2+|y|^2)\cot(2t)/2-\langle x,y\rangle\csc(2t))}f(y)\,dy \tag{2-2}$$

for $f\in\mathcal{S}(\mathbb{R}^d)$, which is shown by using Mehler's formula [Sjögren and Torrea 2010; Thangavelu 1987]. Combining this with (2-1) gives an explicit expression of the kernel of Π_k.

In order to prove the uniform resolvent estimate (Theorem 1.4), we make use of the following mixed-norm estimate for e^{-itH}, which strengthens the uniform bound (2-4) in a different direction.

Proposition 2.1. *Let $d \geq 3$ and $(1/p, 1/q) = \mathcal{B}'$. Then we have*

$$\left\| \int_{-\pi}^{\pi} |e^{-itH/2} f| \, dt \right\|_{q,\infty} \leq C \|f\|_{p,1}. \tag{2-3}$$

Various authors [Jeong et al. 2022a; 2024; Karadzhov 1994; Koch and Tataru 2005; Thangavelu 1998] studied the problem of characterizing the sharp asymptotic bound on the operator norm $\|\Pi_k\|_{p\to q}$ of Π_k from L^p to L^q as $k \to \infty$. In particular, [Karadzhov 1994] showed

$$\|\Pi_k\|_{p\to q} \leq C \tag{2-4}$$

for a constant C when $p = 2$ and $q = 2d/(d-2)$. By duality and the TT^* argument, the bound (2-4) with $(p, q) = (2d/(d+2), 2)$ and $(p, q) = (2d/(d+2), 2d/(d-2))$ follows. Interpolating those estimates with the trivial bound $\|\Pi_k\|_{2\to 2} \leq 1$, we have (2-4) for p, q satisfying

$$\frac{2d}{d+2} \leq p \leq 2 \leq q \leq \frac{2d}{d-2}.$$

Recently, the authors showed in [Jeong et al. 2024, Theorem 1.6] that (2-4) holds on an extended range of p, q for $d \geq 3$; see [Jeong et al. 2022b] for a related result. By means of Proposition 2.1, we can provide a simple alternative proof of this result. Indeed, by (2-1) and Proposition 2.1, it follows that $\|\Pi_k f\|_{q,\infty} \leq C\|f\|_{p,1}$ if $(1/p, 1/q) = \mathcal{B}'$. By duality, the same estimate also holds for $(1/p, 1/q) = \mathcal{B}$. Interpolating these estimates with the above mentioned estimate (2-4) for

$$\frac{2d}{d+2} \leq p \leq 2 \leq q \leq \frac{2d}{d-2}$$

gives the following. See Figure 1.

Corollary 2.2 [Jeong et al. 2024, Theorem 1.6]. *Let $d \geq 3$. For p, q satisfying $(1/p, 1/q) \in \mathfrak{T}$, there is a constant $C > 0$, independent of k, such that (2-4) holds. Furthermore, the uniform restricted weak-type estimate for Π_k holds if $(1/p, 1/q) = \mathcal{B}$ or $\mathcal{B}'$.*

Proof of Proposition 2.1. We make use of the endpoint Strichartz estimate for e^{-itH}:

$$\|e^{-itH/2} f\|_{L^2_t([-\pi,\pi]; L^{p_0}_x(\mathbb{R}^d))} \leq C\|f\|_2, \tag{2-5}$$

where $p_0 = 2d/(d-2)$. By periodicity, one can easily show estimate (2-5) using the dispersive estimate $\|e^{-itH/2} f\|_\infty \lesssim |t|^{-d/2}\|f\|_1$, $t \in \left(0, \frac{\pi}{2}\right)$, which follows from (2-2) and the standard argument in [Keel and Tao 1998]; for example, see [Sjögren and Torrea 2010]. We choose a smooth partition of unity, so that

$$\psi^0 + \sum_{j\geq 4} \big(\psi(2^j t) + \psi(-2^j t) + \psi(2^j(t+\pi)) + \psi(2^j(\pi - t))\big) = 1$$

for $t \in (-\pi, \pi) \setminus \{0\}$. Here $\psi \in C_c^\infty\left(\left[\frac{1}{4}, 1\right]\right)$ satisfy $\sum_j \psi(2^j t) = 1$ for $t > 0$, and ψ^0 is a smooth function which is supported in the interval $[-\pi, \pi]$ and vanishes near 0, π, and $-\pi$.

Set $\psi_j^{\pm} = \psi(\pm 2^j\,\cdot)$ and $\psi_j^{\pm\pi} = \psi(2^j(\pi - \pm\,\cdot))$. Then, for $\sigma = \pm, \pm\pi$, we have

$$\int |\psi_j^\sigma e^{-itH/2} f|\,dt \lesssim 2^{(d-2)j/2}\|f\|_1$$

because $|\psi_j^\sigma e^{-itH/2} f| \lesssim 2^{dj/2}\|f\|_1$ by (2-2). Using (2-5) and Hölder's inequality followed by Minkowski's inequality, we also obtain the estimate

$$\left\| \int |\psi_j^\sigma e^{-itH/2} f|\,dt \right\|_{2d/(d-2)} \lesssim 2^{-j/2}\|f\|_2.$$

In other words, for the sublinear operators $T_j^\sigma f = \int |\psi_j^\sigma e^{-itH/2} f|\,dt$, $\sigma = \pm, \pm\pi$, two estimates

$$\|T_j^\sigma f\|_{q_\ell} \lesssim 2^{j(-1)^\ell \varepsilon_\ell}\|f\|_{p_\ell}, \quad \ell = 0, 1,$$

hold, where $p_0 = 1$, $q_0 = \infty$, $\varepsilon_0 = \frac{1}{2}d$, and $p_1 = 2$, $q_1 = 2d/(d-2)$, $\varepsilon_1 = \frac{1}{2}$. Note that the exponents of 2^j in the two estimates have different signs. Thus, applying Bourgain's summation trick (for example, see [Jeong et al. 2024, Lemma 2.4]), we obtain

$$\left\| \int \left| \sum_j \psi_j^\sigma e^{-itH/2} f \right| dt \right\|_{q,\infty} \le \left\| \sum_j T_j^\sigma \right\|_{q,\infty} \lesssim \|f\|_{p,1}, \quad \sigma = \pm, \pm\pi,$$

for $(1/p, 1/q) = \mathcal{B}'$. By a similar argument, it is easy to show $\left\| \int |\psi^0 e^{-itH/2} f|\,dt \right\|_q \lesssim \|f\|_p$ for $(1/p, 1/q) = \mathcal{B}'$. Hence, combining all of those estimates, we get (2-3). □

We now consider the L^p-L^q estimate for the operator H^{-s}, $s > 0$, which is defined by

$$H^{-s} f = \sum_{k=0}^{\infty} (2k+d)^{-s}\Pi_k f.$$

The operator can also be written as

$$H^{-s} f = \frac{1}{\Gamma(s)} \int_0^\infty t^{s-1} e^{-tH} f\,dt^{1}$$

by making use of the heat semigroup e^{-tH} associated to H. By means of the explicit kernel expression of e^{-tH} which is based on Mehler's formula (see [Thangavelu 1993]), Bongioanni and Torrea [2006] obtained L^p-L^q boundedness for H^{-s}. Sharpness of their result was later verified by [Nowak and Stempak 2013]. Thus, the results completely characterize L^p-L^q boundedness of H^{-s}.

Theorem 2.3 [Bongioanni and Torrea 2006, Theorem 8; Nowak and Stempak 2013, Theorem 3.1]. *Let $d \ge 1$, $1 < p, q < \infty$, and $0 < s < \frac{1}{2}d$. Then H^{-s} is bounded from L^p to L^q if and only if*

$$-\frac{2s}{d} < \frac{1}{p} - \frac{1}{q} \le \frac{2s}{d}.$$

There are weak/restricted weak-type estimates in the borderline cases which are not included in the above theorem, and we refer the readers to [Nowak and Stempak 2013] for more details regarding such endpoint estimates.

[1] For a bounded function $\mathfrak{m}$ on $\mathbb{R}_+$, the operator $\mathfrak{m}(H)$ is formally defined by $\mathfrak{m}(H) = \sum_{k\in\mathbb{N}_0} \mathfrak{m}(2k+d)\Pi_k$.

3. Proof of Theorem 1.4

We consider the more general operator $(H-z)^{-m}$, $m \in \mathbb{N}$, which is given by

$$(H-z)^{-m} f = \sum_{k=0}^{\infty} \frac{\Pi_k f}{(2k+d-z)^m} = (-2)^{-m} \sum_{k=0}^{\infty} \frac{\Pi_k f}{(i\tau+\beta-k)^m},$$

with $z = 2\beta + d + 2\tau i$, $\beta \notin \mathbb{N}_0$, and $\tau \in \mathbb{R}$. We prove the following.

Theorem 3.1. *Let $d \geq 3$, and let m be a positive integer. Suppose that (1-9) holds for some constant $c > 0$. If $(1/p, 1/q) \in (\mathcal{B}, \mathcal{B}')$, then there is a constant $C = C(m)$, independent of z, such that*

$$\|(H-z)^{-m} f\|_q \leq C(1+|\operatorname{Im} z|)^{1-m} \|f\|_p. \tag{3-1}$$

Furthermore, we have $\|(H-z)^{-m} f\|_{q,\infty} \leq C(1+|\operatorname{Im} z|)^{1-m} \|f\|_{p,1}$ if $(1/p, 1/q) = \mathcal{B}$ or $\mathcal{B}'$.

While the estimates for $m \geq 2$ are rather straightforward from (2-4), the proof of (3-1) for $m = 1$ is more involved. This case is handled in Proposition 3.2 below.

Remark. The gap condition (1-9) is necessary for the uniform estimate (3-1) to hold. In fact,

$$\|(H-z)^{-m}\|_{p \to q} \geq \frac{|2k+d-z|^{-m} \|f\|_q}{\|f\|_p}$$

if f is an eigenfunction with eigenvalue $2k+d$. Therefore, the operator norm cannot be bounded as $z \to 2k+d$ unless (1-9) holds.

For positive numbers $\mathcal{B}$ and t_0, let $\mathcal{C}(\mathcal{B}, t_0)$ denote the class of functions which are contained in $C^{[(d+2)/2]}(\mathbb{R} \setminus [-t_0, t_0])$ and satisfy the following:

$$|G(n)| \leq \mathcal{B}, \quad n \in \mathbb{Z}, \tag{3-2}$$

$$\sum_{k=1}^{\infty} |G(k) + G(-k)| \leq \mathcal{B}, \tag{3-3}$$

$$\sum_{k=1}^{\infty} |kG(k) - (k+1)G(k+1)| \leq \mathcal{B}, \tag{3-4}$$

$$\left|\left(\frac{d}{dt}\right)^l G(t)\right| \leq \mathcal{B}(1+|t|)^{-l-1}, \quad t_0 < |t|, \tag{3-5}$$

for $0 \leq l \leq \frac{1}{2}(d+2)$. Particular examples satisfying the conditions (3-2)–(3-5) are $G_{\mu,\tau}(t) = 1/(i\tau + t + \mu)$, where $(\mu, \tau) \in \left(-\frac{1}{2}, \frac{1}{2}\right) \times \mathbb{R}$ and $|(\mu, \tau)| \geq c$ for some small $c > 0$.

Proposition 3.2. *Let $d \geq 3$ and $(1/p, 1/q) \in (\mathcal{B}, \mathcal{B}')$. Suppose that G is in $\mathcal{C}(\mathcal{B}, t_0)$. Then, there is a constant C, depending only on $\mathcal{B}$ and t_0, such that*

$$\left\|G\left(\frac{2n+d-H}{2}\right) f\right\|_q \leq C\|f\|_p \tag{3-6}$$

holds for every $n \in \mathbb{N}_0$. Furthermore, if $(1/p, 1/q) = \mathcal{B}$ or $\mathcal{B}'$, the restricted weak-type (p, q) estimate holds for $G\left(\frac{1}{2}(2n+d-H)\right)$ with a uniform bound depending only on $\mathcal{B}$ and t_0.

Proof. Let p_* and q_* be given by $(1/p_*, 1/q_*) = \mathfrak{B}'$. In order to show Proposition 3.2, it is sufficient to show the restricted weak-type (p_*, q_*) estimate for $G\big(\frac{1}{2}(2n+d-H)\big)$. Note that the adjoint operator $G\big(\frac{1}{2}(2n+d-H)\big)^*$ is given by

$$G\Big(\frac{2n+d-H}{2}\Big)^* f = \sum_{k=0}^{\infty} \overline{G}(n-k)\Pi_k f.$$

Clearly $\overline{G} \in \mathcal{C}(\mathcal{B}, t_0)$. Hence, the same argument shows that the restricted weak-type (p_*, q_*) estimate holds for $G\big(\frac{1}{2}(2n+d-H)\big)^*$. This in turn gives the restricted weak-type estimate (q_*', p_*') for $G\big(\frac{1}{2}(2n+d-H)\big)$ by duality. Real interpolation between these two (restricted weak-type) estimates for $G\big(\frac{1}{2}(2n+d-H)\big)$ yields the desired estimates for $(1/p, 1/q) \in (\mathfrak{B}, \mathfrak{B}')$.

No differentiability assumption is made on G for $|t| \le t_0$. So, we handle the cases $n \ge n_0$ and $n < n_0$ separately, where n_0 is an integer satisfying $n_0 \ge 2t_0$. We first consider the case $n \ge n_0$. Recalling

$$G\Big(\frac{2n+d-H}{2}\Big) = \sum_{k=0}^{\infty} G(n-k)\Pi_k,$$

we write the decomposition

$$G\Big(\frac{2n+d-H}{2}\Big) =: \mathcal{J}_n + \mathcal{K}_n,$$

where

$$\mathcal{J}_n := \sum_{k=0}^{\infty} G(n-k)\phi\Big(\frac{n-k}{n}\Big)\Pi_k \quad \text{and} \quad \mathcal{K}_n := \sum_{k=0}^{\infty} G(n-k)\Big(1-\phi\Big(\frac{n-k}{n}\Big)\Big)\Pi_k.$$

Here, we choose a nonnegative smooth even function ϕ on $\mathbb{R}$ such that $\phi(t) = 1$ on $\big[-\frac{1}{2}, \frac{1}{2}\big]$, $\phi = 0$ if $1 \le |t|$, and ϕ is nonincreasing on the half-line $t > 0$. This monotonicity assumption plays an important role in obtaining a bound on a sum of trigonometric functions.

The sum $\mathcal{J}_n$ is the major contribution to the estimate (3-6) and is to be handled by the integral formula for Π_k and Proposition 2.1. The second sum $\mathcal{K}_n$ behaves like the operator H^{-1}, which is actually bounded from L^p-L^q on a larger range of p, q. We consider $\mathcal{J}_n$ first.

We set

$$\mathcal{I}_1 = \sum_{k=1}^{n} G(k)\phi\Big(\frac{k}{n}\Big)(\Pi_{n-k} - \Pi_{n+k}) \quad \text{and} \quad \mathcal{I}_2 = \sum_{k=1}^{n} (G(-k)+G(k))\phi\Big(\frac{k}{n}\Big)\Pi_{n+k}.$$

Since ϕ is an even function and supported in $[-1, 1]$, after reindexing by $(n-k) \to k$, we see

$$\mathcal{J}_n = \sum_{k=1}^{n} G(k)\phi\Big(\frac{k}{n}\Big)\Pi_{n-k} + G(0)\Pi_n + \sum_{k=1}^{n} G(-k)\phi\Big(\frac{k}{n}\Big)\Pi_{n+k}.$$

Thus,

$$\mathcal{J}_n = \mathcal{I}_1 + \mathcal{I}_2 + G(0)\Pi_n.$$

By (3-2), (3-3), and the uniform restricted weak-type (p_*, q_*) estimate for Π_λ in Corollary 2.2, it follows that

$$\|G(0)\Pi_n f\|_{q_*,\infty} \lesssim \mathcal{B}\|f\|_{p_*,1} \quad \text{and} \quad \|\mathcal{I}_2\|_{q_*,\infty} \lesssim \mathcal{B}\|f\|_{p_*,1}.$$

So, it suffices to deal with $\mathcal{I}_1$. Using the formula (2-1), we note

$$\Pi_{n-k} f - \Pi_{n+k} f = -\frac{i}{\pi}\int_{-\pi}^{\pi} \sin(tk) e^{it(2n+d-H)/2} f\, dt.$$

Thus, we have

$$\mathcal{I}_1 f = \int_{-\pi}^{\pi} \zeta_n(t) e^{-itH/2} f\, dt,$$

where

$$\zeta_n(t) = -\frac{i}{\pi} e^{it(2n+d)/2} \sum_{k=1}^{n} G(k)\sin(tk)\phi\Big(\frac{k}{n}\Big), \quad -\pi \le t \le \pi.$$

Using Proposition 2.1, it is sufficient to show

$$|\zeta_n(t)| \le C, \tag{3-7}$$

with C independent of n and G. By the property of ϕ, it is clear that

$$|\zeta_n(t)| \lesssim \bigg|\sum_{k=1}^{\lfloor n/2\rfloor} \sin(tk)G(k)\bigg| + \bigg|\sum_{k=\lfloor n/2\rfloor+1}^{n} \sin(tk)G(k)\phi\Big(\frac{k}{n}\Big)\bigg|.$$

Boundedness of the second term is easy to show. Indeed, since the condition (3-5) holds for $|t| > \frac{1}{2}n$ by our choice of n_0, we see

$$\bigg|\sum_{k=\lfloor n/2\rfloor+1}^{n} \sin(tk)G(k)\phi\Big(\frac{k}{n}\Big)\bigg| \lesssim \mathcal{B} \sum_{k=\lfloor n/2\rfloor+1}^{n} k^{-1}\phi\Big(\frac{k}{n}\Big) \lesssim \mathcal{B}.$$

So, for (3-7), we only have to show $\big|\sum_{k=1}^{n} \sin(tk)G(k)\big| \lesssim 1$ for any n. Setting $\sigma_k(t) = \sum_{j=1}^{k} j^{-1}\sin(jt)$, by summation by parts we write

$$\sum_{k=1}^{n} \sin(tk)G(k) = \sum_{k=1}^{n-1} \sigma_k(t)(kG(k) - (k+1)G(k+1)) + \sigma_n(t)nG(n).$$

Since $|\sigma_k(t)| \lesssim 1$ for any k and t as can be shown by an elementary argument,[2] by the conditions (3-4) and (3-5) it follows that $\big|\sum_{k=1}^{n} \sin(tk)G(k)\big| \lesssim 1$.

We now turn to the operator $\mathcal{K}_n$. Clearly, we may write $\mathcal{K}_n = H^{-1} \circ m_n(H)$, where m_n is given by

$$m_n(t) = tG\Big(\frac{2n+d-t}{2}\Big)\Big(1 - \phi\Big(\frac{2n+d-t}{2n}\Big)\Big),$$

which is in $C^\infty(\mathbb{R})$. Using (3-2), (3-5), and the support property of ϕ, a simple calculation shows

$$\Big|\frac{d^l}{dt^l} m_n(t)\Big| \lesssim (1+t)^{-l} \quad \text{for } l = 0, 1, 2, \dots, \tfrac{1}{2}(d+2)$$

whenever $t > 0$. Here the implicit constants are independent of n. Thus, the Marcinkiewicz multiplier theorem [Thangavelu 1993, Theorem 4.2.1] implies that $m_n(H)$ is bounded on L^p, $1 < p < \infty$, uniformly

[2] This can be seen by approximating Dirichlet's kernel, or again by summation by parts.

in n. By Theorem 2.3, H^{-1} is also bounded from L^p to L^q for $1 < p, q < \infty$ satisfying $1/p - 1/q = 2/d$. Hence, we have

$$\|\mathcal{K}_n\|_{p\to q} \le \|H^{-1}\|_{p\to q}\|m_n(H)\|_{p\to p} \lesssim 1,$$

with the implicit constant independent of n.

We now consider the case $n < n_0$, which is much simpler to show than the case $n \ge n_0$. To prove (3-6), we write the following decomposition for $G\big(\frac{1}{2}(2n+d-H)\big)$:

$$G\Big(\frac{2n+d-H}{2}\Big) = \widetilde{\mathcal{J}}_n + \widetilde{\mathcal{K}}_n,$$

where

$$\widetilde{\mathcal{J}}_n = \sum_{k=0}^{\infty} G(n-k)\phi\Big(\frac{k}{2n_0}\Big)\Pi_k \quad \text{and} \quad \widetilde{\mathcal{K}}_n = \sum_{k=0}^{\infty} G(n-k)\Big(1-\phi\Big(\frac{k}{2n_0}\Big)\Big)\Pi_k.$$

Clearly, the multiplier

$$G\Big(\frac{2n+d-\cdot}{2}\Big)\Big(1-\phi\Big(\frac{2n+d-\cdot}{2n_0}\Big)\Big)$$

of the operator $\widetilde{\mathcal{K}}_n$ satisfies the condition (3-5). So, in the same manner as in the above we obtain the bound $\|\widetilde{\mathcal{K}}_n\|_{p\to q} \lesssim 1$ if $1 < p, q < \infty$ and $1/p - 1/q = 2/d$. By condition (3-2) and Corollary 2.2 it follows that

$$\|\widetilde{\mathcal{J}}_n f\|_{q_*,\infty} \le \mathcal{B} \sum_{k=0}^{2n_0} \|\Pi_k f\|_{q_*,\infty} \lesssim \|f\|_{p_*,1}$$

uniformly in $n \le n_0$. This completes the proof of Proposition 3.2. □

We are ready to prove Theorem 3.1.

Proof of Theorem 3.1. Let p_* and q_* be given by $(1/p_*, 1/q_*) = \mathcal{B}'$. As in the proof of Proposition 3.2, it is enough to show the restricted weak-type (p_*, q_*) estimate for $(H-z)^{-m}$ with bound $C(1+|\operatorname{Im} z|)^{1-m}$, since the adjoint operator of $(H-z)^{-m}$ is given by $(H-\bar{z})^{-m}$. We can handle $(H-\bar{z})^{-m}$ in exactly the same way to obtain the restricted weak-type (p_*, q_*) estimate for $(H-z)^{-m}$ with bound $C(1+|\operatorname{Im} z|)^{1-m}$. By duality and interpolation, we get all the desired estimates.

By Corollary 2.2, we have the estimate $\|\Pi_k f\|_{q_*,\infty} \le C\|f\|_{p_*,1}$, with C independent of k. Using this estimate, for $m \ge 2$, we get

$$\|(H-z)^{-m} f\|_{q_*,\infty} \lesssim \sum_{k=0}^{\infty} |2k+d-z|^{-m}\|f\|_{p_*,1} \lesssim (1+|\operatorname{Im} z|)^{1-m}\|f\|_{p_*,1}$$

because

$$\sum_{k=0}^{\infty} |2k+d-z|^{-m} \le C_m(1+|\operatorname{Im} z|)^{1-m},$$

with C_m independent of z for $m \ge 2$ if (1-9) holds. Thus, we need only to show

$$\|(H-z)^{-1} f\|_{q_*,\infty} \le C\|f\|_{p_*,1}. \tag{3-8}$$

If $\operatorname{Re} z > d-1$, we have $z = 2(n+\mu)+d+2i\tau$ for some $n \in \mathbb{N}_0$, $\mu \in \left(-\frac{1}{2}, \frac{1}{2}\right)$, and $\tau \in \mathbb{R}$ satisfying $|(\mu, \tau)| \geq \frac{1}{2}c$ because of (1-9). We note that

$$(H-z)^{-1} = G_{\mu,\tau}\left(\frac{2n+d-H}{2}\right),$$

where $G_{\mu,\tau}(t) = 1/(i\tau+t+\mu)$. It is easy to see that $G_{\mu,\tau} \in \mathcal{C}(\mathcal{B}, 1)$ for some $\mathcal{B} > 0$ provided that $\mu \in \left(-\frac{1}{2}, \frac{1}{2}\right)$ and $\tau \in \mathbb{R}$ satisfy $|(\mu, \tau)| \geq \frac{1}{2}c$. Indeed, since $|k+\mu| \geq |\mu|$ for $k \in \mathbb{Z}$ and $\mu \in \left(-\frac{1}{2}, \frac{1}{2}\right)$, it follows that $|G_{\mu,\tau}(k)| \leq |\mu+i\tau|^{-1} \leq 2/c$ for all $k \in \mathbb{Z}$. Moreover, we have

$$\sum_{k=1}^{\infty} |G_{\mu,\tau}(k) + G_{\mu,\tau}(-k)| \lesssim \sum_{k=1}^{\infty} \frac{|i\tau+\mu|}{k^2+\tau^2} \lesssim 1,$$

$$\sum_{k=1}^{\infty} |kG_{\mu,\tau}(k) - (k+1)G_{\mu,\tau}(k+1)| \leq \sum_{k=1}^{\infty} \frac{|i\tau+\mu|}{|i\tau+k+\mu|^2} \lesssim 1,$$

and, for $0 \leq l \leq \frac{1}{2}(d+2)$,

$$\left|\left(\frac{d}{dt}\right)^l G_{\mu,\tau}(t)\right| \lesssim (1+|t|)^{-l-1}, \quad |t| \geq 1,$$

whenever $\mu \in \left(-\frac{1}{2}, \frac{1}{2}\right)$ and $\tau \in \mathbb{R}$ satisfy $|(\mu, \tau)| \geq \frac{1}{2}c$. Obviously, the implicit constants are independent of specific values of μ and τ. Hence, taking a sufficiently large constant $\mathcal{B} \geq 2/c$, we see $G_{\mu,\tau} \in \mathcal{C}(\mathcal{B}, 1)$.

Thus, by Proposition 3.2, the estimate (3-8) holds uniformly in z. For the remaining case, i.e., $\operatorname{Re} z < d-1$, we have that z clearly stays away from the eigenvalues of H, so $(H-z)^{-1}$ behaves like H^{-1}. More precisely, we obtain the uniform estimate (3-8) repeating the same argument as in the case $n < n_0$ of the proof of Proposition 3.2. This completes the proof. □

The uniform resolvent estimate in Theorem 1.4 is a special case of the following.

Corollary 3.3. *Let $d \geq 3$ and m be a positive integer, and let p, q be given as in Theorem 1.1. Then, there is a constant $C = C(m)$ such that*

$$\|(H-z)^{-m} f\|_q \leq C(1+|\operatorname{Im} z|)^{d(1/p-1/q)/2-m} \|f\|_p \tag{3-9}$$

provided (1-9) holds. Furthermore, if $(1/p, 1/q) = \mathfrak{B}$ or $\mathfrak{B}'$, we have the restricted weak-type estimate for $(H-z)^{-m}$ with bound $C(1+|\operatorname{Im} z|)^{d(1/p-1/q)/2-m}$.

Proof. By Theorem 3.1, we have estimate (3-9) for $(1/p, 1/q) \in (\mathfrak{B}, \mathfrak{B}')$. In view of interpolation, it is enough to show (3-9) with $(p, q) = (2, 2)$, $(2d/(d+2), 2)$, or $(2, 2d/(d-2))$. These estimates are easy to show by using orthogonality between the projection operators Π_k. In fact, we have

$$\|(H-z)^{-m} f\|_2 \leq \left(\sum_{k=0}^{\infty} |2k+d-z|^{-2m} \|\Pi_k f\|_2^2\right)^{1/2}.$$

So, taking the supremum over k of $|2k+d-z|^{-2m}$, we obtain (3-9) when $p = q = 2$. We note that $\sum_{k=0}^{\infty} |2k+d-z|^{-2m} \leq C(1+|\operatorname{Im} z|)^{-2m+1}$ with C independent of z as long as (1-9) holds. Applying the uniform $L^{2d/(d+2)}$-L^2 estimate in Corollary 2.2, we get (3-9) with $p = 2d/(d+2)$ and $q = 2$. Since the adjoint of $(H-z)^{-m}$ is $(H-\bar{z})^{-m}$, estimate (3-9) with $(p, q) = (2d/(d+2), 2)$ implies that with $(p, q) = (2, 2d/(d-2))$ by duality. □

4. Proof of Theorem 1.1

We now prove the estimate (1-4) by adapting the argument in [Escauriaza and Vega 2001] (also see [Escauriaza 2000]) which deduces the Carleman estimate for the heat operator from the uniform resolvent estimate for the Hermite operator. We are basically relying on real interpolation as in the Appendix of [Jerison and Kenig 1985]. However, there are some nontrivial issues which are related to a shortcoming of the real interpolation in mixed-norm spaces.

Lemma 4.1. *Let $1 < p \le 2 \le q < \infty$, $1 \le r, s \le \infty$, $1 \le a \le b \le \infty$, and $0 \le \gamma \le 1$, and let $\beta \notin \mathbb{N}_0$ be a real number. Suppose that the estimate*

$$\Big\| \sum_{k=0}^{\infty} \frac{\Pi_k f}{(\tau i + \beta - k)^m} \Big\|_{q,b} \le C_m (1 + |\tau|)^{\gamma - m} \|f\|_{p,a} \tag{4-1}$$

holds for positive integers m, with C_m independent of $\tau \in \mathbb{R}$ and β, provided $\operatorname{dist}(\beta, \mathbb{N}_0) \ge c$ for some $c > 0$. Then, if $\operatorname{dist}(\beta, \mathbb{N}_0) \ge c$ for some $c > 0$, estimate (1-4) holds uniformly in β whenever the following hold:

- *$\gamma < 1$, $0 \le 1/r - 1/s \le 1 - \gamma$, and $(1/r, 1/s) \ne (1, \gamma)$, $(1 - \gamma, 0)$,*
- *$\gamma = 1$, $a = b = 2$, and $1 < r = s < \infty$.*

Lemma 4.1 was implicit in [Escauriaza and Vega 2001] with the Lebesgue spaces instead of the Lorentz spaces. The extra condition $a = b = 2$ when $\gamma = 1$ is due to a limitation of the real interpolation in mixed-norm spaces. Once we have Lemma 4.1, the proof of Theorem 1.1 is rather simple.

Proof of Theorem 1.1. Let $(1/p, 1/q)$ be in $\mathfrak{T}$. By real interpolation between the estimates in Corollary 3.3 and inclusion relations between Lorentz spaces, we get (4-1) with $\gamma = \frac{1}{2}d(1/p - 1/q)$ for any $1 \le a \le b \le \infty$ if $p \ne 2$ and $q \ne 2$. Thus Lemma 4.1 gives estimate (1-4) in the Lorentz spaces if the exponents satisfy the condition in Theorem 1.1. □

Estimate (1-4) is equivalent to the Sobolev-type inequality

$$\|h\|_{L^s(\mathbb{R}; L^{q,b}_x)} \le C \|(\Delta - |x|^2 + \partial_t + 2\beta + d)h\|_{L^r(\mathbb{R}; L^{p,a}_x)}, \quad h \in C_c^\infty(\mathbb{R}^{d+1}). \tag{4-2}$$

One can easily see this by following the argument in [Escauriaza 2000]. In particular, if $r = s$, the inequality (4-2) implies $\|f\|_q \le C \|(\Delta - |x|^2 + 2\beta + d) f\|_p$ for $f \in C_c^\infty(\mathbb{R}^d)$, which is, in fact, a special case of (1-8), where $z = 2\beta + d \notin 2\mathbb{N}_0 + d$. Indeed, let f_1 be a compactly supported smooth function on $\mathbb{R}$ with $f_1(0) = 1$. Then the above estimate follows by applying (4-2) to the function $h(x, t) = f(x) f_1(t/R) R^{-1/r}$, $R > 1$, and letting $R \to \infty$.

Remark. When $r = s$, the implication from (4-2) to (1-8) with $z = 2\beta + d \notin 2\mathbb{N}_0 + d$ can be used to show that the Carleman estimate (1-4) holds only if

$$\frac{1}{p} - \frac{1}{q} \le \frac{2}{d}.$$

By the Marcinkiewicz multiplier theorem for the Hermite operator H [Thangavelu 1993, Theorem 4.2.1], $(H - z)^{-1} H$ with $z = 2\beta + d \notin 2\mathbb{N}_0 + d$ is bounded on L^p, $1 < p < \infty$. Thus, we see that estimate (1-4) implies $\|H^{-1} u\|_q \lesssim \|u\|_p$ for $u \in C_c^\infty(\mathbb{R}^d)$. By Theorem 2.3, the inequality holds only if $1/p - 1/q \le 2/d$.

Proof of Lemma 4.1. To prove Lemma 4.1, we basically rely on the argument in [Escauriaza 2000; Escauriaza and Vega 2001], so we shall be brief. By scaling, it is easy to see that (1-4) is equivalent to (4-2). See [Escauriaza 2000] for the details. Thus, we need to show (4-2) by replacing h with $(\Delta-|x|^2+\partial_t+2\beta+d)^{-1}g$. Applying the projection operator Π_λ in x-variables and taking the Fourier transform in t, we see the operator $S_\beta := (\Delta-|x|^2+\partial_t+2\beta+d)^{-1}$ is given by

$$S_\beta g(x,t)=\int_{\mathbb{R}} K_\beta(t-s)(g(\cdot,s))(x)\,ds,$$

where the operator-valued kernel K_β is given by

$$K_\beta(t)(f)=\frac{1}{2}\int_{\mathbb{R}} e^{2\pi i t\tau}\sum_{k=0}^{\infty}\frac{\Pi_k(f)}{\pi i\tau+\beta-k}\,d\tau,\qquad f\in C_c^\infty(\mathbb{R}^d).$$

To prove (1-4), it is enough to show

$$\|S_\beta g\|_{L^s(\mathbb{R};L_x^{q,b})}\lesssim \|g\|_{L^r(\mathbb{R};L_x^{p,a})},\qquad g\in C_c^\infty(\mathbb{R}^{d+1}),\tag{4-3}$$

with an implicit constant independent of β as long as $\operatorname{dist}(\beta,\mathbb{N}_0)\ge c$ for some $c>0$.

We regard S_β as a vector-valued convolution operator. Let us first consider the case $\gamma<1$ which is easier. Let $\phi\in C_c^\infty([-1,1])$ be such that $\phi(t)=1$ on $\left[-\frac{1}{2},\frac{1}{2}\right]$. Breaking the integral with functions $\phi(t\tau)$ and $1-\phi(t\tau)$ and using integration by parts and (4-1), it is easy to see that $\|K_\beta(t)\|_{L_x^{p,a}\to L_x^{q,b}}\lesssim\min\{|t|^{-\gamma},|t|^{-2}\}$. Since $\gamma<1$, for r and s satisfying $0\le 1/r-1/s\le 1-\gamma$ and $(1/r,1/s)\neq(1,\gamma),\ (1-\gamma,0)$, we obtain estimate (4-3) by Young's convolution inequality and the Hardy–Littlewood–Sobolev inequality.

We now turn to the case $\gamma=1$. We claim that the kernel K_β satisfies the Hörmander condition

$$\sup_{s\neq 0}\int_{|t|>2|s|}\|K_\beta(t-s)-K_\beta(t)\|_{L^{p,2}\to L^{q,2}}\,dt\le A<\infty,\tag{4-4}$$

where A depends only on the constant $c>0$ such that $\operatorname{dist}(\beta,\mathbb{N}_0)\ge c$. To show (4-4) it is sufficient to show $\|K'_\beta(t)\|_{L^{p,2}\to L^{q,2}}\lesssim|t|^{-2}$. In fact, if $\|K'_\beta(t)\|_{L_x^{p,2}\to L_x^{q,2}}\lesssim|t|^{-2}$, then

$$\|K_\beta(t-s)-K_\beta(t)\|_{L_x^{p,2}\to L_x^{q,2}}=\left\|\int_t^{t-s}K'_\beta(\sigma)\,d\sigma\right\|_{L_x^{p,2}\to L_x^{q,2}}\lesssim|s|\,|t|^{-2}.$$

This clearly yields (4-4). Integrating by parts, we have

$$(-2\pi i t)^2K'_\beta(t)=2^2(\pi i)^3\int_{-\infty}^{\infty}\tau e^{2\pi i\tau t}\sum_{k=0}^{\infty}\frac{1}{(\pi\tau i+\beta-k)^3}\Pi_k\,d\tau.$$

The assumption (4-1) (with $\gamma=1$) gives $\||t|^2K'_\beta(t)\|_{L^{p,2}\to L^{q,2}}\lesssim 1$ uniformly in t and β satisfying $\operatorname{dist}(\beta,\mathbb{N}_0)\ge c$, which proves the claim (4-4) (see [Escauriaza and Vega 2001] for details). Thanks to (4-4) and the usual vector-valued singular integral theory, in order to prove (4-3) for $1<r=s<\infty$, it suffices to obtain estimate (4-3) with $r=s=2$ and $a=b=2$.

For $\eta\in C_c^\infty(\mathbb{R})$, we define $\eta(D_t)$ by $\mathcal{F}_t(\eta(D_t)g)(x,\tau)=\eta(\tau)\mathcal{F}_tg(x,\tau)$, where $\mathcal{F}_t$ denotes the Fourier transform in t. We use the following Littlewood–Paley-type inequality in the Lorentz spaces.

Lemma 4.2. *Let $1 < p, r < \infty$. Suppose η is a smooth function supported in $[2^{-2}, 1]$ which satisfies $\sum_{j=-\infty}^{\infty} |\eta(2^{-j}t)|^2 \sim 1$ for all $t > 0$. Then we have*

$$\|g\|_{L_t^r(\mathbb{R}; L_x^{p,r})} \lesssim \Big\| \Big(\sum_{j\in\mathbb{Z}} |\eta(2^{-j}|D_t|)g|^2 \Big)^{1/2} \Big\|_{L_t^r(\mathbb{R}; L_x^{p,r})} \lesssim \|g\|_{L_t^r(\mathbb{R}; L_x^{p,r})}. \tag{4-5}$$

Proof. It is sufficient to show the second inequality in (4-5) because the first inequality follows from the second one via the standard polarization argument and duality. For every $1 < p, r < \infty$, we have

$$\Big\| \Big(\sum_{j\in\mathbb{Z}} |\eta(2^{-j}|D_t|)g|^2 \Big)^{1/2} \Big\|_{L^r(\mathbb{R}; L^p(\mathbb{R}^d))} \lesssim \|g\|_{L^r(\mathbb{R}; L^p(\mathbb{R}^d))}$$

by means of the usual Littlewood–Paley inequality and the vector-valued singular integral theorem; see [Escauriaza and Vega 2001, Lemma 2.1]. We interpolate these estimates using the real interpolation in the mixed-norm spaces, in particular,

$$(L^{p_0}(\mathbb{R}; L^{q_0}), L^{p_1}(\mathbb{R}; L^{q_1}))_{\theta,p} = L^p(\mathbb{R}; L^{q,p})$$

whenever $p_0, q_0, p_1, q_1 \in [1, \infty)$ and $(1/p, 1/q) = (1-\theta)(1/p_0, 1/q_0) = \theta(1/p_1, 1/q_1)$ with $\theta \in (0, 1)$; see [Cwikel 1974; Lions and Peetre 1964]. Therefore, we obtain the second inequality in (4-5). $\square$

We now note that

$$\psi(2^{-j}|D_t|)S_\beta g(x,t) = \int_{\mathbb{R}} K_{\beta,j}(t-s) g(\cdot, s)(x)\, dt,$$

where

$$K_{\beta,j}(t) f(x) := \frac{1}{2} \int_{\mathbb{R}} e^{2\pi i t\tau} \psi\Big(\frac{|\tau|}{2^j}\Big) \sum_{k=0}^{\infty} \frac{1}{\pi i\tau + \beta - k} \Pi_k f(x)\, d\tau.$$

Using (4-1) with $a = b = 2$ and integration by parts, we see that $\|K_{\beta,j}(t)\|_{L_x^{p,2} \to L_x^{q,2}} \le C 2^j (1 + 2^j|t|)^{-2}$, with C independent of j and β if $\operatorname{dist}(\beta, \mathbb{N}_0) \ge c > 0$. Thus, Young's convolution inequality gives

$$\|\psi(2^{-j}|D_t|)S_\beta g\|_{L^2(\mathbb{R}; L_x^{q,2})} \lesssim \|g\|_{L^2(\mathbb{R}; L_x^{p,2})}, \tag{4-6}$$

with the implicit constant independent of j and β. To get the desired (4-3) with $r = s = 2$, we combine this inequality and Lemma 4.2. Since $2 \le q < \infty$, the space $L^{(q/2),(2/2)}$ is normable. So,

$$\Big\| \Big(\sum_j |h_j|^2 \Big)^{1/2} \Big\|_{L_x^{q,2}} \lesssim \Big(\sum_j \|h_j\|_{L_x^{q,2}}^2 \Big)^{1/2}. \tag{4-7}$$

Since $S_\beta g = \sum_{j\in\mathbb{Z}} \psi(2^{-j}|D_t|)S_\beta g$, applying Lemma 4.2 and then (4-7), we have

$$\|S_\beta g\|_{L^2(\mathbb{R}; L_x^{q,2})} \lesssim \Big(\sum_{j\in\mathbb{Z}} \|\psi(2^{-j}|D_t|)S_\beta g\|_{L^2(\mathbb{R}; L_x^{q,2})}^2 \Big)^{1/2}.$$

Let $\tilde\psi \in \mathrm{C}_c([2^{-2}, 1])$ such that $\psi\tilde\psi = \psi$, so

$$\psi(2^{-j}|D_t|)S_\beta g = \psi(2^{-j}|D_t|)S_\beta \tilde\psi(2^{-j}|D_t|)g.$$

Using (4-6) followed by (4-5), we get

$$\|S_\beta g\|_{L^2(\mathbb{R};L_x^{q,2})} \lesssim \Big(\sum_{j\in\mathbb{Z}} \|\tilde{\psi}(2^{-j}|D_t|)g\|^2_{L^2(\mathbb{R};L_x^{p,2})}\Big)^{1/2}.$$

By duality, the inequality (4-7) is equivalent to $\big(\sum_j \|h_j\|^2_{L_x^{p,2}}\big)^{1/2} \lesssim \big\|\big(\sum_j |h_j|^2\big)^{1/2}\big\|_{L_x^{p,2}}$ for $1 < p \le 2$. Thus, using Lemma 4.2, we get

$$\|S_\beta g\|_{L^2(\mathbb{R};L_x^{q,2})} \lesssim \Big\|\Big(\sum_{j\in\mathbb{Z}} |\tilde{\psi}(2^{-j}|D_t|)g|^2\Big)^{1/2}\Big\|_{L^2(\mathbb{R};L_x^{p,2})} \lesssim \|g\|_{L^2(\mathbb{R};L_x^{p,2})}. \qquad \square$$

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NONNEGATIVE RICCI CURVATURE AND MINIMAL GRAPHS WITH LINEAR GROWTH

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We study minimal graphs with linear growth on complete manifolds M^m with $\mathrm{Ric} \geq 0$. Under the further assumption that the $(m-2)$-th Ricci curvature in radial direction is bounded below by $Cr(x)^{-2}$, we prove that any such graph, if nonconstant, forces tangent cones at infinity of M to split off a line. Note that M is not required to have Euclidean volume growth. We also show that M may not split off any line. Our result parallels that obtained by Cheeger, Colding and Minicozzi for harmonic functions. The core of the paper is a new refinement of Korevaar's gradient estimate for minimal graphs, together with heat equation techniques.

1. Introduction

The theory of entire minimal graphs in Euclidean space $\mathbb{R}^m$, that is, of functions $u : \mathbb{R}^m \to \mathbb{R}$ solving the minimal (hyper-)surface equation

$$\mathrm{div}\left(\frac{Du}{\sqrt{1+|Du|^2}}\right) = 0 \tag{MSE}$$

is built upon the following foundational results:

($\mathscr{B}1$) The *Bernstein theorem*: solutions to (MSE) are all affine if and only if $m \leq 7$.

($\mathscr{B}2$) For each $m \geq 2$, positive solutions to (MSE) are constant.

($\mathscr{B}3$) For each $m \geq 2$, solutions to (MSE) with at most linear growth on one side are affine (i.e., the Hessian $D^2u \equiv 0$).

Here, u is said to have at most linear growth on one side if, up to changing the sign of u,

$$u(x) \geq -C(1+r(x))$$

holds on $\mathbb{R}^m$ for some constant $C > 0$, where r is the distance from a fixed origin. The validity of ($\mathscr{B}1$) is due, as well-known, to the combined effort of S. Bernstein [1915] ($m = 2$, see also [Mickle 1950; Hopf 1950]), W. H. Fleming [1962] (still for $m = 2$), E. De Giorgi [1965] ($m = 3$), F. Almgren [1966] ($m = 4$), J. Simons [1968] ($m \leq 7$) and E. Bombieri, De Giorgi and E. Giusti [Bombieri et al. 1969a] (counterexamples if $m \geq 8$). On the other hand, ($\mathscr{B}2$) and ($\mathscr{B}3$) were both proved by Bombieri, De Giorgi and M. Miranda [Bombieri et al. 1969b] for $m \geq 3$; in particular, ($\mathscr{B}3$) refines J. Moser's theorem [1961], which states that u is affine provided that $|Du| \in L^\infty(M)$. Further properties of entire minimal graphs in Euclidean space were obtained by Bombieri and Giusti [1972]: among them, we mention the fact that u is

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affine whenever $m-1$ of its partial derivatives are bounded. The result was improved in recent years by A. Farina [2015; 2018], who showed that u is affine if $m-7$ partial derivatives of u are bounded on one side. Further enhancements of Moser's result, proving that $D^2u \equiv 0$ by only assuming that $|Du| = o(r)$ as $r(x) \to \infty$, were obtained in [Caffarelli et al. 1988; Ecker and Huisken 1990; Simon 1989]. We also mention the recent [Farina 2022], where the rigidity of a minimal graph is obtained by assuming that an upper level set contains, or is contained in, a half-space.

In a Riemannian setting, it is natural to ask the following:

Question 1. *For which classes of complete Riemannian manifolds M one could expect results like* $(\mathscr{B}1)$, $(\mathscr{B}2)$, $(\mathscr{B}3)$?

The problem motivated our previous works [Bianchini et al. 2021b; Colombo et al. 2022], as well as the present paper. Recall that a solution to (MSE) on a Riemannian manifold (M^m, σ) gives rise to a graph

$$F : M \to \mathbb{R} \times M, \qquad F(x) = (u(x), x),$$

which is minimal if the ambient space $\mathbb{R} \times M$ is endowed with the product metric $\mathrm{d}t^2 + \sigma$. Hereafter, we say that the graph is entire if u is defined on the whole of M.

If M is close to hyperbolic space $\mathbb{H}^m$, namely, M is a Cartan–Hadamard manifolds with suitably pinched negative curvature, $(\mathscr{B}1)$, $(\mathscr{B}2)$, $(\mathscr{B}3)$ drastically fail, since each continuous function on the boundary at infinity of M can be attained as the limit value of an entire minimal graph, which is therefore bounded. An exhaustive literature on the problem can be found in the survey [Heinonen 2021]; see also the introduction of [Bianchini et al. 2021a].

Denoting by $g = F^*(\mathrm{d}t^2 + \sigma)$ the graph metric and with Δ_g its Laplace–Beltrami operator, (MSE) can be written as $\Delta_g u = 0$, making contact with the theory of harmonic functions. In Euclidean space $M = \mathbb{R}^m$, $(\mathscr{B}2)$ and $(\mathscr{B}3)$ hold as well when considering harmonic functions instead of solutions to (MSE), while the analogy fails for $(\mathscr{B}1)$ since there is no rigidity for entire harmonic functions without imposing any growth condition. This suggests that, for $(\mathscr{B}2)$ and $(\mathscr{B}3)$, an answer to the above question may be guided by the global behavior of harmonic functions on Riemannian manifolds, according to which it is natural to consider the problem on manifolds satisfying either

$$\mathrm{Sec} \geq 0 \quad \text{or} \quad \mathrm{Ric} \geq 0, \tag{1}$$

where Sec, Ric are the sectional and Ricci curvatures of (M, σ). Indeed, if $\mathrm{Ric} \geq 0$, positive harmonic functions on M are constant, by S.Y. Cheng and S.T. Yau's gradient estimate [Yau 1975; Cheng and Yau 1975], while a harmonic function with linear growth forces any tangent cone at infinity of M to split, by work of J. Cheeger, T. Colding and W. Minicozzi [Cheeger et al. 1995]. Furthermore, M itself splits off a line if $\mathrm{Ric} \geq 0$ is strengthened to $\mathrm{Sec} \geq 0$ (see [Antonelli et al. 2022] for a complete proof), or if M is parabolic (see [Li and Tam 1989] and Remark 5 below).

In view of the convergence theory developed in the past 50 years for manifolds with $\mathrm{Sec} \geq 0$ or $\mathrm{Ric} \geq 0$, some of the tools used to prove the Bernstein theorem in $\mathbb{R}^m$ are available on manifolds satisfying (1), making these assumptions a natural setting also for the study of $(\mathscr{B}1)$. However, much has to be done and ($\mathscr{B}1$) seems very challenging to prove even on manifolds with $\mathrm{Sec} \geq 0$. In fact, we are aware of no results in this direction.

The situation is different for ($\mathscr{B}2$) and ($\mathscr{B}3$), for which, as we shall detail below, the main difficulty is to prove the results by only requiring $\mathrm{Ric} \geq 0$, arguably the sharp condition for their validity (in this case, however, ($\mathscr{B}3$) has to be suitably weakened, see later).

Regarding ($\mathscr{B}2$), after previous work by H. Rosenberg, F. Schulze and J. Spruck [Rosenberg et al. 2013], a complete answer was obtained by the first, third and fourth authors together with M. Magliaro [Colombo et al. 2022], and independently by Q. Ding [2021a] with different methods:

Theorem 2 [Colombo et al. 2022; Ding 2021a]. *Connected, complete manifolds M with* $\mathrm{Ric} \geq 0$ *satisfy* ($\mathscr{B}2$); *that is, entire positive minimal graphs over M are constant.*

In this paper, we address ($\mathscr{B}3$). In view of the result in [Cheeger et al. 1995], it is reasonable to formulate the following:

Conjecture 3. *Let M be a connected, complete manifold with* $\mathrm{Ric} \geq 0$ *and possessing a nonconstant entire minimal graph with at most linear growth on one side. Then, every tangent cone at infinity of M splits off a line.*

The problem seems to be considerably harder compared to the case of harmonic functions. We are aware of only two results in the direction of Conjecture 3. Ding, J. Jost and Y. Xin [Ding et al. 2016] proved that $\mathbb{R}^m$ is the only manifold satisfying the following assumptions:

$$\mathrm{Ric} \geq 0, \qquad \lim_{r\to\infty} \frac{|B_r|}{r^m} > 0, \tag{2.α}$$

$$\text{the curvature tensor decays quadratically} \tag{2.β}$$

and admits an entire, nonconstant minimal graph with at most linear growth on one side. Very recently, Ding [2021b] posted on arXiv a paper where he proved Conjecture 3 on manifolds satisfying the assumptions in (2.α). The bulk of his argument is to show the remarkable property that the isoperimetric inequality, satisfied by (M, σ) in view of (2.α), is inherited by the graph of u. This allowed Ding to adapt, in a nontrivial way, tools from [Bombieri et al. 1969b; Bombieri and Giusti 1972] and from Cheeger–Colding theory to reach the goal. We stress that his method heavily depends on the Euclidean volume growth condition in (2.α).

In our work, we address Conjecture 3 without requiring the Euclidean volume growth assumption, but rather a mild further curvature condition. To formulate our main result, we first recall the definition of the ℓ-th Ricci curvature:

Definition 4. Let (M, σ) be a manifold of dimension $m \geq 2$. For $\ell \in \{1, \ldots, m-1\}$, the ℓ-th (normalized) Ricci curvature is the function

$$v \in T_x M \longmapsto \mathrm{Ric}^{(\ell)}(v) \doteq \inf_{\substack{\mathcal{W} \leq v^\perp \\ \dim \mathcal{W} = \ell}} \left(\frac{1}{\ell} \sum_{j=1}^{\ell} \mathrm{Sec}(v \wedge e_j) \right),$$

where $\{e_j\}$ is an orthonormal basis of $\mathcal{W}$.

The function $\mathrm{Ric}^{(\ell)}$ interpolates between the sectional and Ricci curvatures, obtained respectively for $\ell = 1$ and, up to the normalization constant $m-1$ for $\ell = m-1$. In particular, with our chosen normalization the following implications are immediate:

$$\mathrm{Sec} \geq c \quad \Longrightarrow \quad \mathrm{Ric}^{(\ell-1)} \geq c \quad \Longrightarrow \quad \mathrm{Ric}^{(\ell)} \geq c \quad \Longrightarrow \quad \mathrm{Ric} \geq (m-1)c.$$

Hereafter, given $H \in C([0,\infty))$ and denoting by r the distance from a fixed origin $o \in M$, we use the short-hand notation $\mathrm{Ric}^{(\ell)}(\nabla r) \geq -H(r)$ on M to mean the inequality

$$\mathrm{Ric}^{(\ell)}(\nabla r(x)) \geq -H(r(x)) \quad \text{for all } x \in M\backslash(\{o\} \cup \mathrm{cut}(o)),$$

where $\mathrm{cut}(o)$ is the cut-locus of o.

A relevant class of manifolds for which rigidity holds without imposing any growth of u is that of parabolic ones. Recall that a manifold M is said to be parabolic if every positive superharmonic function on M is constant.

Remark 5. By work of N. Varopoulos [1981] and Li and Yau [1986], if $\mathrm{Ric} \geq 0$ the parabolicity of M is equivalent to

$$\int^{\infty} \frac{s\, \mathrm{d}s}{|B_s|} = \infty, \tag{3}$$

where B_s is a geodesic ball centered at a fixed origin o. Indeed, (3) is sufficient for the parabolicity of a complete manifold, independently of any curvature requirement; see [Grigoryan 1999].

Lastly, we recall that a tangent cone at infinity for a complete (noncompact) manifold M is any metric space obtained as a blow-down of M. More precisely, a pointed metric space $(X_\infty, \mathrm{d}_\infty, x_\infty)$, $x_\infty \in X_\infty$, is a tangent cone at infinity for (M, σ) if, for some base point $x \in M$ and some sequence $\{\lambda_n\}$ of positive real numbers such that $\lambda_n \to \infty$, one has

$$(M, \lambda_n^{-1} \mathrm{dist}_\sigma, x) \to (X_\infty, \mathrm{d}_\infty, x_\infty)$$

in the pointed Gromov–Hausdorff (pGH) sense. If (M, σ) has nonnegative Ricci curvature, then tangent cones at infinity exist based at any point $x \in M$, by Gromov's precompactness theorem [2007].

We are ready to state:

Theorem 6. *Let (M, σ) be a connected, complete Riemannian manifold of dimension $m \geq 2$ with*

$$\mathrm{Ric} \geq 0,$$

and let $u \in C^\infty(M)$ be a nonconstant entire solution to (MSE).

(i) *If M is parabolic, then it admits a splitting $M = N \times \mathbb{R}$ with the product metric $\sigma_N + \mathrm{d}s^2$ for some complete manifold N with $\mathrm{Ric}_N \geq 0$ such that in the variables $(y, s) \in N \times \mathbb{R}$ it holds $u(y, s) = as + b$ for some $a, b \in \mathbb{R}$.*

(ii) *If M is nonparabolic and*

- *u has at most linear growth on one side,*

- *there exists an origin $o \in M$ such that, denoting by r the distance from o,*

$$\mathrm{Ric}^{(m-2)}(\nabla r) \geq -\frac{\bar{\kappa}^2}{1+r^2} \quad \text{on } M, \tag{4}$$

  for some constant $\bar{\kappa} \geq 0$,

then every tangent cone at infinity of M splits off a line.

Remark 7. In case (i), the claimed splitting $M = N \times \mathbb{R}$ for which u is independent of N may not be the unique splitting of the manifold as a product of a line and a complete manifold, as the case of affine graphs on $M = \mathbb{R}^2$ shows. In case (ii), since M is nonparabolic then necessarily $m \geq 3$ (see below), so $\mathrm{Ric}^{(m-2)}$ is well-defined. In the statement of (ii), we also emphasize that tangent cones at infinity may be based at any point of M, not necessarily at o.

Case (i) in Theorem 6 is easy to obtain, and might be well-known among specialists. We included it for the sake of completeness. Regarding case (ii), the curvature condition in (4) is only used to infer that u has bounded gradient on M. In other words, as a consequence of our proof we obtain a generalization of Moser's result [1961] to the following:

Theorem 8. *Let M be a connected, complete manifold with $\mathrm{Ric} \geq 0$. If u is a nonconstant solution to (MSE) and $|Du| \in L^\infty(M)$, then every tangent cone at infinity of M splits off a line.*

It was already observed in [Cheeger et al. 1995] that a manifold M with $\mathrm{Ric} \geq 0$ may not split off any line despite each of its tangent cones at infinity does. A counterexample was constructed in [Kasue and Washio 1990], and building on it we get the following result:

Proposition 9. *For $m \geq 4$, there exists a connected, complete manifold M with*

$$\mathrm{Ric}^{(2)} \geq 0, \qquad \mathrm{Ric} > 0, \qquad |\mathrm{Sec}\,| \leq \bar{\kappa}^2$$

for some constant $\bar{\kappa} > 0$, which carries a nonconstant minimal graph $u : M \to \mathbb{R}$ with $|Du| \in L^\infty(M)$.

Note that $\mathrm{Ric}^{(m-2)} \geq 0$ and that, having positive Ricci curvature, M does not split off any line. Whence, in assumption (ii) of Theorem 6 the conclusion cannot be strengthened to a splitting of M itself, at least if $m \geq 4$.

When $\mathrm{Sec} \geq 0$, however, the above phenomenon does not happen. Leaving aside dimension $m = 2$, covered by case (i) in Theorem 6, we obtain

Corollary 10. *Let (M, σ) be a connected, complete Riemannian manifold of dimension $m \geq 3$ satisfying $\mathrm{Sec} \geq 0$. If there exists a nonconstant entire solution $u \in C^\infty(M)$ of (MSE) with at most linear growth on one side, then M admits a splitting $M = N \times \mathbb{R}$ with the product metric $\sigma_N + \mathrm{d}s^2$ for some complete manifold N with $\mathrm{Sec}_N \geq 0$ such that in the variables $(y, s) \in N \times \mathbb{R}$ it holds $u(y, s) = as + b$ for some $a, b \in \mathbb{R}$.*

The corresponding problem for harmonic functions was also studied by A. Kasue [1990]. Corollary 10 relates to the results obtained by P. Li and J. Wang [2004]. There, the authors study connected, complete, stable minimal hypersurfaces $\Sigma \to \overline{M}$ properly immersed into a complete manifold $\overline{M}$ whose sectional curvature is nonnegative, and prove that either Σ has only one end or Σ is a totally geodesic cylinder $P \times \mathbb{R}$, for some compact manifold P with nonnegative sectional curvature. Our setting falls into their framework, since a minimal graph in $\mathbb{R} \times M$ is stable, properly embedded and $\overline{M} = \mathbb{R} \times M$ has nonnegative sectional curvature. However, our conclusion is stronger, since it allows M to have only one end and it also implies a splitting of M itself.

To conclude, we prove the next result for graphs with slower than linear growth on one side, that should be compared to [Ding et al. 2016, Theorem 3.6; 2021b, Theorem 1.4].

Theorem 11. *Let (M, σ) be a connected, complete Riemannian manifold of dimension $m \geq 2$ with $\mathrm{Ric} \geq 0$, and let $u \in C^\infty(M)$ solve (MSE) on M and satisfy*

$$\lim_{r(x) \to \infty} \frac{u_-(x)}{r(x)} = 0, \tag{5}$$

where $u_-(x) = \max\{-u(x), 0\}$. Assume that either

- *M is parabolic, or*
- *M is nonparabolic and there exists an origin $o \in M$ such that, denoting by r the distance from o,*

$$\mathrm{Ric}^{(m-2)}(\nabla r) \geq -\frac{\bar{\kappa}^2}{1+r^2} \quad \text{on } M,$$

  for some constant $\bar{\kappa} \geq 0$.

Then, u is constant.

Remark 12 (more general curvature bounds). It is natural to wonder whether conditions $\mathrm{Ric} \geq 0$ or $\mathrm{Sec} \geq 0$ can be weakened still allowing for some rigidity of u and M. In this respect we quote [Bianchini et al. 2021a, Example 3.1], where the authors constructed a manifold M of dimension $m \geq 3$ with two ends, satisfying

$$\mathrm{Sec} \geq -\frac{\bar{\kappa}^2}{1+r^2}, \qquad \mathrm{vol}(B_r) \leq C r^m$$

for constants $\bar{\kappa}, C > 0$ and supporting a nonconstant, bounded entire minimal graph. On the other hand, the existence of such a solution to (MSE) is forbidden if M has asymptotically nonnegative sectional curvature and only one end; see [Casteras et al. 2020]. An interesting class for which one might try to obtain rigidity results is that of manifolds with quadratically decaying (or asymptotically nonnegative) Ricci curvature and linear volume growth; see [Sormani 2000].

Strategy of the proof. Case (i) in Theorem 6 is a direct consequence of the parabolicity of (M, σ), which in our setting can be transplanted to the graph of u. In particular, since every surface with $\mathrm{Ric} \geq 0$ is parabolic, the result holds if $m = 2$, so we focus on dimension $m \geq 3$. In [Bombieri et al. 1969b], the authors obtain ($\mathscr{B}3$) in $\mathbb{R}^m$ via the following steps:

(a) a sharp gradient estimate, implying that a solution $u \in C^\infty(\mathbb{R}^m)$ to (MSE) with at most linear growth on one side satisfies $|Du| \in L^\infty(\mathbb{R}^m)$;

(b) an argument of [Moser 1961]: since $|Du| \in L^\infty(\mathbb{R}^m)$, for each coordinate field ∂_j the partial derivative $\partial_j u$ is a bounded solution to a uniformly elliptic PDE. The global Harnack inequality implies that $\partial_j u$ is constant, which implies that u is affine.

Step (b) cannot be implemented on manifolds, which in general lack parallel fields. An alternative idea was proposed in [Cheeger et al. 1995] to study harmonic functions, a blowdown argument which exploits the convergence theory of manifolds with $\mathrm{Ric} \geq 0$. Our strategy closely follows the one in that work, and can be split into the following steps:

(a) We prove that a solution $u \in C^\infty(M)$ to (MSE) with at most linear growth on one side satisfies $|Du| \in L^\infty(M)$.

(b) For fixed $x_0 \in M$, we show that the functions $|Du|$ and $|D^2u|$ satisfy

$$\lim_{R\to\infty} \frac{1}{|B_R(x_0)|} \int_{B_R(x_0)} |Du|^2 \, \mathrm{d}x = \sup_M |Du|^2, \tag{6}$$

$$\lim_{R\to\infty} \frac{R^2}{|B_R(x_0)|} \int_{B_R(x_0)} |D^2u|^2 \, \mathrm{d}x = 0, \tag{7}$$

where $B_R(x_0)$ is the geodesic ball of radius R and center x_0 in (M, σ).

(c) We use the blowdown argument to guarantee the splitting of any tangent cone at infinity with base point x_0.

To be more precise, step (a) will be achieved by assuming

$$\mathrm{Ric} \geq 0 \quad \text{and} \quad \mathrm{Ric}^{(m-2)}(\nabla r) \geq -\frac{\bar{\kappa}^2}{1+r^2}, \tag{8}$$

while (b) will be shown by requiring

$$\mathrm{Ric} \geq 0 \quad \text{and} \quad |Du| \in L^\infty(M). \tag{9}$$

Though the strategy is the same as that in [Cheeger et al. 1995], we emphasize that the techniques in the current literature to prove (a) (respectively, (b)) do not apply under the sole assumptions in (8) (respectively, in (9)). We shall justify this claim in the next sections. Our strategy to obtain (a) is to refine a method due to N. Korevaar [1986], see Theorem 15 below, while to get (b) in our needed generality we exploit heat equation techniques, inspired by works of P. Li [1986] and L. Saloff-Coste [1992]. In this respect, we underline Theorem 27 below, yielding to (7), that in the stated generality seems to us new and of an independent interest.

2. Preliminaries

We briefly review some formulas for minimal graphs that will be used later on. In local coordinates (x^i) on M, the background metric σ and the graph metric $g = F^*(\mathrm{d}t^2 + \sigma)$ can be written as

$$\sigma = \sigma_{ij}\mathrm{d}x^i \otimes \mathrm{d}x^j, \quad g = g_{ij}\mathrm{d}x^i \otimes \mathrm{d}x^j, \quad \mathrm{d}u = u_i\mathrm{d}x^i,$$

and $g_{ij} = \sigma_{ij} + u_iu_j$. Letting σ^{ij} and g^{ij} be the components of the inverse matrices of (σ_{ij}), (g_{ij}), respectively, it holds

$$g^{ij} = \sigma^{ij} - \frac{u^iu^j}{W^2},$$

where $W = \sqrt{1+|Du|^2}$ and $u^i = \sigma^{ij} u_j$. In general, if $\phi \in C^1(M)$ then the symbols $D\phi$ and $\nabla\phi$ will denote the gradients of ϕ in the metrics σ and g, respectively, and in local notation we write

$$\mathrm{d}\phi = \phi_i \mathrm{d}x^i, \quad D\phi = \phi^i \partial_{x_i} \equiv \sigma^{ij}\phi_j \partial_{x_i}, \quad \nabla\phi = g^{ij}\phi_j \partial_{x_i}.$$

Differentiating the upward-pointing unit normal vector $\boldsymbol{n} = W^{-1}(\partial_t - u^i e_i)$, the second fundamental form II in the direction of $\boldsymbol{n}$ has components

$$\mathrm{II}_{ij} = \frac{u_{ij}}{W}, \tag{10}$$

where u_{ij} are the components of the Hessian D^2u in the metric σ. Let $H = g^{ij}h_{ij}$ be the mean curvature, which we assume to vanish. Using the relation

$$\Gamma^k_{ij} - \gamma^k_{ij} = \frac{u^k u_{ij}}{W^2}$$

between the Christoffel coefficients Γ^k_{ij} of g and γ^k_{ij} of σ, for every $\phi : M \to \mathbb{R}$ the Laplace–Beltrami operator Δ_g of g can be written as

$$\Delta_g \phi = g^{ij}\phi_{ij} - \phi_k u^k \frac{H}{W} = g^{ij}\phi_{ij},$$

where we used the minimality of Σ. Also, Δ_g has the local expression

$$\Delta_g \phi = \frac{1}{\sqrt{|g|}} \partial_{x_j}(\sqrt{|g|} g^{ij}\phi_i) = \frac{1}{W} \operatorname{div}(W g^{ij}\phi_i \partial_{x_j}), \tag{11}$$

where div is, as before, the divergence operator in (M, σ) and $|g|$ is the determinant of (g_{ij}). Next, for every Killing field $\overline{X}$ defined in $\mathbb{R} \times M$, the angle function $\Theta_{\overline{X}} \doteq \langle \boldsymbol{n}, \overline{X} \rangle$ solves the Jacobi equation

$$\Delta_g \Theta_{\overline{X}} + (\|\mathrm{II}\|^2 + \overline{\mathrm{Ric}}(\boldsymbol{n}, \boldsymbol{n}))\Theta_{\overline{X}} = 0, \tag{12}$$

with $\overline{\mathrm{Ric}}$ the Ricci curvature of $\mathbb{R} \times M$. This is the case, for instance, of the angle function

$$\Theta_{\partial_t} = \langle \boldsymbol{n}, \partial_t \rangle = W^{-1}$$

associated to the Killing field ∂_t. As a consequence, W satisfies

$$\mathscr{L}_W W = (\|\mathrm{II}\|^2 + \overline{\mathrm{Ric}}(\boldsymbol{n}, \boldsymbol{n}))W, \tag{13}$$

where we defined

$$\mathscr{L}_W \phi \doteq W^2 \operatorname{div}_g(W^{-2}\nabla\phi) = \Delta_g \phi - 2\langle \nabla \log W, \nabla\phi \rangle.$$

Observe that, in terms of the metric σ,

$$\mathscr{L}_W \phi = W \operatorname{div}(W^{-1} g^{ij}\phi_i \partial_{x_j}). \tag{14}$$

If X is a Killing field in (M, σ) then we can extend it by parallel transport on $\mathbb{R} \times M$ to a Killing field $\overline{X}$ satisfying $\langle \partial_t, \overline{X} \rangle = 0$, with corresponding angle function

$$\Theta_{\overline{X}} = \langle \boldsymbol{n}, \overline{X} \rangle = -W^{-1}\sigma(Du, X).$$

Since (12) holds both for Θ_{∂_t} and for $\Theta_{\overline{X}}$, it can be checked that the quotient

$$v \doteq -\frac{\Theta_{\overline{X}}}{\Theta_{\partial_y}} = \sigma(Du, X)$$

is a solution to

$$\mathscr{L}_W v = 0. \tag{15}$$

We next discuss the implications of ℓ-th Ricci curvature lower bounds. Hereafter, we set $\mathbb{R}^+ = (0, \infty)$ and $\mathbb{R}^+_0 = [0, \infty)$.

Proposition 13. *Let (M, σ) be a connected, complete manifold of dimension $m \geq 2$ satisfying*

$$\mathrm{Ric}^{(\ell)}(\nabla r) \geq -H(r) \quad \textit{for } \ell = \max\{1, m-2\},$$

where r is the distance from a fixed origin $o \in M$, and $0 \leq H \in C(\mathbb{R}^+_0)$. Let $h \in C^2(\mathbb{R}^+_0)$ solve

$$\begin{cases} h'' - Hh \geq 0 & \textit{on } \mathbb{R}^+, \\ \liminf\limits_{t\to 0}\left(\dfrac{h'}{h} - \dfrac{1}{t}\right) \geq 0. \end{cases} \tag{16}$$

Let $u : M \to \mathbb{R}$ solve (MSE). Then, denoting by Δ_g the Laplacian in the graph metric g,

$$\Delta_g r \leq m\frac{h'(r)}{h(r)}$$

pointwise on $M\backslash(\{o\} \cup \mathrm{cut}(o))$ and in the barrier sense on $M\backslash\{o\}$.

Proof. Outside of $\{o\} \cup \mathrm{cut}(o)$, denote by $\{\lambda_j(D^2 r)\}$ the eigenvalues of $D^2 r$ in increasing order. The comparison theorem in [Mari and Pessoa 2020, Proposition 7.4] guarantees that

$$\sum_{j=2}^m \lambda_j(D^2 r) \leq (m-1)\frac{h'(r)}{h(r)} \tag{17}$$

pointwise on $M\backslash(\{o\} \cup \mathrm{cut}(o))$ and in the barrier sense on $M\backslash\{o\}$. Note that the initial assumptions on h therein are $h(0) = 0,\ h'(0) \geq 1$, but the same proof works for the more general (16). In this respect, note that $h' > 0$ on $\mathbb{R}^+$ follows from $H \geq 0$.

To estimate $\Delta_g r = g^{ij} r_{ij}$, pick a point x where r is smooth. If $Du(x) = 0$, then $g^{ij} = \sigma^{ij}$. In our assumptions, the Ricci curvature satisfies

$$\mathrm{Ric}(\nabla r, \nabla r) \geq -(m-1)H(r),$$

so by the Laplacian comparison theorem and since $h' > 0$,

$$\Delta_g r = \mathrm{Tr}(D^2 r) \leq (m-1)\frac{h'(r)}{h(r)} \leq m\frac{h'(r)}{h(r)}.$$

Assume that $Du(x) \neq 0$, write $\nu = Du/|Du|$ in a neighborhood of x and complete it to a local σ-orthonormal basis $\{\nu, e_\alpha\}$ with $2 \leq \alpha \leq m$. Note that g^{ij} is diagonalized with eigenvalues $W^{-2} \leq 1$ in direction ν and 1 in directions $\{e_\alpha\}$. Expressing $\Delta_g r$ in the basis $\{e_\alpha, \nu\}$ we get

$$\begin{aligned}\Delta_g r &= \frac{1}{W^2} D^2 r(\nu,\nu) + \sum_{\alpha=2}^m D^2 r(e_\alpha, e_\alpha) \\ &= \frac{1}{W^2}\left[\mathrm{Tr}(D^2 r) - \sum_{\alpha=2}^m D^2 r(e_\alpha, e_\alpha)\right] + \sum_{\alpha=2}^m D^2 r(e_\alpha, e_\alpha) \\ &= \frac{1}{W^2}\,\mathrm{Tr}(D^2 r) + \left[\frac{W^2-1}{W^2}\right]\sum_{\alpha=2}^m D^2 r(e_\alpha, e_\alpha). \end{aligned} \tag{18}$$

By min-max and since the eigenvalues are ordered,

$$\mathrm{Tr}(D^2 r) \le \frac{m}{m-1}\sum_{\alpha=2}^m \lambda_\alpha(D^2 r), \qquad \sum_{\alpha=2}^m D^2 r(e_\alpha, e_\alpha) \le \sum_{\alpha=2}^m \lambda_\alpha(D^2 r).$$

Therefore,

$$\begin{aligned}\Delta_g r &\le \left[\frac{m}{(m-1)W^2} + \frac{W^2-1}{W^2}\right]\sum_{\alpha=2}^m \lambda_\alpha(D^2 r) \\ &\le \left[\frac{1}{(m-1)W^2} + 1\right](m-1)\frac{h'(r)}{h(r)} \le m\frac{h'(r)}{h(r)}, \end{aligned} \tag{19}$$

as claimed. The validity of (17) in the barrier sense on the entire $M\backslash\{o\}$ easily follows by Calabi's trick; see [Mari and Pessoa 2020, Proposition 7.4]. □

Remark 14. In particular, letting $\bar\kappa \in \mathbb{R}^+$, for $\ell = \max\{1, m-2\}$ it holds

$$\mathrm{Ric}^{(\ell)}(\nabla r) \ge -\bar\kappa^2 \quad \Longrightarrow \quad \Delta_g r \le m\bar\kappa \coth(\bar\kappa r),$$

$$\mathrm{Ric}^{(\ell)}(\nabla r) \ge -\frac{\bar\kappa^2}{1+r^2} \quad \Longrightarrow \quad \Delta_g r \le \frac{m(1+\sqrt{1+4\bar\kappa^2})}{2r}$$

pointwise outside of $\{o\}\cup \mathrm{cut}(o)$ and in the barrier sense on $M\backslash\{o\}$. Indeed, it is enough to consider for h, respectively, the functions

$$h(t) = \frac{\sinh(\bar\kappa t)}{\bar\kappa} \qquad \text{and} \qquad h(t) = t^{\bar\kappa'}, \quad \text{where } \bar\kappa' = \frac{1+\sqrt{1+4\bar\kappa^2}}{2}.$$

3. Proof of Theorem 6(i)

Since Ric ≥ 0 and M is parabolic, by Remark 5 the manifold (M,σ) satisfies

$$\int^\infty \frac{s\,\mathrm{d}s}{|B_s|} = \infty. \tag{20}$$

We apply an argument outlined in [Colding and Minicozzi 2011, p. 48] for minimal graphs in $\mathbb{R}^3$. First, a calibration method in [Li and Wang 2001] (see also [Colding and Minicozzi 2011; Trudinger 1972] for the case $M = \mathbb{R}^m$) shows that the volume of the graph $\Sigma = (M, g)$ inside an extrinsic ball $\mathbb{B}_r \subset \mathbb{R}\times M$

centered at a point $(u(o), o)$ satisfies

$$|\Sigma \cap \mathbb{B}_r| \le |B_r| + \tfrac{1}{2}|B_{3r} \backslash B_r| \le 2|B_{3r}|.$$

Hence, the volume of a geodesic ball B^g_s in Σ centered at o is bounded by

$$|B^g_r| \le |\Sigma \cap \mathbb{B}_r| \le 2|B_{3r}|, \tag{21}$$

which implies

$$\int^{\infty} \frac{s\,\mathrm{d}s}{|B^g_s|} = \infty.$$

Therefore, by Remark 5, the graph $\Sigma = (M, g)$ is parabolic. Because of the Jacobi equation

$$\Delta_g \frac{1}{W} = -(\|\mathrm{II}\|^2 + \overline{\mathrm{Ric}}(\boldsymbol{n}, \boldsymbol{n}))\frac{1}{W},$$

the bounded function $1/W$ is superharmonic on Σ, and hence constant by parabolicity. Since $\mathrm{Ric} \ge 0$ implies $\overline{\mathrm{Ric}} \ge 0$, again from the Jacobi equation we deduce $\mathrm{II} \equiv 0$ and Σ is totally geodesic in $M \times \mathbb{R}$. Equivalently, by (10), $D^2 u \equiv 0$ in M. As a consequence, since u is nonconstant, Du is a nonzero parallel vector field. The flow of Du therefore splits M isometrically as a product $N \times \mathbb{R}$, and u is an affine function of the $\mathbb{R}$-coordinate alone.

4. A local gradient estimate

Let $u : B_R \subset M \to \mathbb{R}$ solve (MSE) on a geodesic ball $B_R = B_R(x)$. The original argument in [Bombieri et al. 1969b] to prove the gradient estimate in Euclidean setting ($M = \mathbb{R}^m$)

$$|Du(x)| \le c_1 \exp\left\{ c_2 \frac{u(x) - \inf_{B_R} u}{R} \right\} \tag{22}$$

for some constants $c_j = c_j(m)$ makes use of the isoperimetric inequality, which does not hold for minimal graphs over manifolds (M, σ) with $\mathrm{Ric} \ge 0$ unless M has maximal volume growth compatible with the Bishop–Gromov inequality, in the sense that (2.α) is satisfied. Indeed, the isoperimetric inequality forces geodesic balls B^g_r in the graph $\Sigma = (M, g)$ to satisfy $|B^g_r| \ge Cr^m$, which coupled with (21) imply that M has Euclidean volume growth.

The exponential bound in (22) is sharp; see [Finn 1963]. On the other hand, in their seminal paper Bombieri and Giusti [1972] proved a different estimate for entire solutions: if $u : \mathbb{R}^m \to \mathbb{R}$ solves (MSE), then for any $x \in \mathbb{R}^m$ and $R > 0$

$$|Du(x)| \le c_1 \left\{ 1 + \frac{\sup_{B_R} |u|}{R} \right\}^m, \tag{23}$$

where $c_1 = c_1(m)$. Whence, for entire solutions, an exponentially growing bound in terms of $|u|$ is not sharp.

If M has $\mathrm{Ric} \ge 0$ and Euclidean volume growth, the validity of an isoperimetric inequality on entire minimal graphs was recently shown in [Ding 2021b]; see also [Brendle 2023] for the case $\mathrm{Sec} \ge 0$. As a consequence, in [Ding 2021b, Theorems 1.3 and 6.2] the author was able to extend (22) and (23) to such manifolds.

An alternative method to prove (22) in Euclidean setting was given by N. Trudinger [1972]. His strategy hinges on a mean value inequality on Σ which, remarkably, is obtained without needing the isoperimetric inequality and is therefore suited to apply to manifolds whose volume growth is not Euclidean. However, to adapt the proof to minimal graphs over M, it seems that an upper bound on the sectional curvature of M is necessary; see also the related [Ding et al. 2016]. Later, N. Korevaar [1986] gave new insight into the problem, finding a striking argument to get gradient estimates that only requires lower bounds on the curvatures of M. Exploiting Korevaar's method, in [Rosenberg et al. 2013] the authors obtained the slightly different estimate

$$|Du(x)| \leq c_1 \exp\left\{c_2[1+\bar{\kappa}R\coth(\bar{\kappa}R)]\frac{(u(x)-\inf_{B_R} u)^2}{R^2}\right\}, \tag{24}$$

provided that $\mathrm{Ric} \geq 0$ and $\mathrm{Sec} \geq -\bar{\kappa}^2$. Note that, unless $\bar{\kappa} = 0$, the estimate explodes as $R \to \infty$ if $u : M \to \mathbb{R}$ is of linear growth. Extensions to more general ambient spaces were later given in [Casteras et al. 2020; Dajczer and de Lira 2015; 2017], but they only consider graphs which are bounded on one side or have logarithmic growth.

Inspecting the proofs in [Rosenberg et al. 2013; Dajczer and de Lira 2015; 2017; Ding et al. 2016], to reach the inequality $|Du(x)| \leq C$ for solutions of linear growth, with C uniform with respect to x, the bounds on Sec are instrumental to guarantee that the distance r_x from x satisfies $\Delta_g r_x \leq C_1/r_x$ for some absolute constant C_1. In view of the arbitrariness of the point x, assumption $\mathrm{Sec} \geq 0$ in [Rosenberg et al. 2013] seems therefore difficult to replace by a weaker control on Sec from below. For instance, if one considers the inequality $\mathrm{Sec} \geq -\bar{\kappa}^2/(1+r_o^2)$ for some constant $\bar{\kappa} > 0$ and some origin o, comparison theory and standard estimates for ODE would yield to a constant C_1, hence C, that depends on the distance of x from o and explodes as $r_o(x) \to \infty$, making the estimate on $\Delta_g r_x$ insufficient to imply the desired uniform gradient bound.

From a different perspective, we mention that a *global* gradient estimate for *positive* entire solutions was obtained in [Colombo et al. 2022] under the sole curvature assumption

$$\mathrm{Ric} \geq -(m-1)\kappa^2, \qquad \kappa \in \mathbb{R}_0^+,$$

namely, a positive solution to (MSE) on the entire M shall satisfy

$$\sqrt{1+|Du(x)|^2} \leq e^{\kappa u(x)\sqrt{m-1}} \quad \text{for all } x \in M.$$

Note that ($\mathscr{B}2$) directly follows if $\kappa = 0$. However, modifying the argument in [Colombo et al. 2022] to allow for linearly growing solutions seems challenging.

Our first main result, Theorem 15, provides an improvement of Korevaar's method that apply to the more general assumption (8).

Theorem 15. *Let (M, σ) be a complete Riemannian manifold with dimension $m \geq 2$. Let $B_R = B_R(o) \subseteq M$ be a geodesic open ball of radius $R > 0$ centered at $o \in M$ and let $u \in C^3(\bar{B}_R)$ be a nonconstant solution to (MSE). Assume that*

$$\mathrm{Ric} \geq -(m-1)\kappa^2, \qquad \mathrm{Ric}^{(\ell)}(\nabla r) \geq -\frac{\bar{\kappa}^2}{1+r^2} \quad \text{on } B_R, \quad \ell = \max\{1, m-2\},$$

for some $\kappa, \bar\kappa \in \mathbb{R}_0^+$, *where* r *denotes the distance from* o. *Let* $0 < R_1 < R$. *Then,*

$$\sqrt{1+|Du(x)|^2} \le \max\left\{\sqrt{1+a_0^2(\gamma^*)^2}, \sqrt{\frac{a_3}{a_3-a_2}}\right\}\left(\frac{e^{LR(\sqrt{\varepsilon^2+1}-\varepsilon)}-1}{e^{LR(\sqrt{\varepsilon^2+1}-\sqrt{\varepsilon^2+r(x)^2/R^2}-q\gamma(x))}-1}\right)$$

for every $x \in B_{R_1}(o)$, *where*

$$\gamma(x) = \frac{u(x)-\inf_{B_R} u}{R}, \quad \gamma^* = \sup_{x\in B_{R_1}} \gamma(x) = \frac{\sup_{B_{R_1}} u - \inf_{B_R} u}{R},$$

$\varepsilon > 0$ *and* $\tau \in (0, 1)$ *are fixed arbitrarily*, $q, a_0 \in \mathbb{R}^+$ *satisfy*

$$\frac{\sqrt{1+\varepsilon^2}-\sqrt{(R_1/R)^2+\varepsilon^2}}{\gamma^*} > q > \frac{1}{\sqrt{\tau}a_0\gamma^*} > 0$$

and $L \in \mathbb{R}^+$ *satisfies*

$$(1-\tau)\left(q^2 - \frac{1}{\tau a_0^2(\gamma^*)^2}\right)L^2 - \frac{(m+1)\bar\kappa_0 L}{\varepsilon R} > (m-1)\kappa^2,$$

with $\bar\kappa_0 = \max\{1, \bar\kappa\}$. *Finally*, a_2, a_3 *are defined by*

$$a_2 = \frac{(m+1)\bar\kappa_0 L}{\varepsilon R} \quad a_3 = (1-\tau)\left(q^2 - \frac{1}{\tau a_0^2(\gamma^*)^2}\right)L^2 - (m-1)\kappa^2.$$

Remark 16. The assumption that u is nonconstant ensures that $\gamma_* > 0$ by the maximum principle.

Before proving the theorem, we give some applications, starting from the case where $\kappa = 0$.

Corollary 17. *Let* (M^m, σ) *be a complete Riemannian manifold with* $\mathrm{Ric} \ge 0$ *and*

$$\mathrm{Ric}^{(\ell)}(\nabla r) \ge -\frac{\bar\kappa^2}{1+r^2}, \quad \ell = \max\{1, m-2\},$$

for some $\bar\kappa \in \mathbb{R}_0^+$, *where* r *denotes the distance from* o. *Let* $u \in C^3(\bar B_R)$ *solve* (MSE). *Then, for every* $\delta \in (0, 1)$ *and for every* $R_1 \in (0, \delta R)$,

$$\sup_{B_{R_1}} \sqrt{1+|Du|^2} \le C_1 \exp\left(C_2 m \bar\kappa_0 \frac{[\sup_{B_{R_1}} u - \inf_{B_R} u]^2}{R^2}\right), \tag{25}$$

with $\bar\kappa_0 = \max\{1, \bar\kappa\}$ *and* $C_1, C_2 > 0$ *only depending on* δ.

Proof. The desired inequality is trivial if u is constant, so assume that u is nonconstant. It suffices to prove the claim for $\delta \in \left[\frac{1}{2}, 1\right)$. Let

$$\gamma^* = \frac{\sup_{B_{R_1}} u - \inf_{B_R} u}{R}.$$

Choose

$$\tau = \frac{1}{2}, \quad \varepsilon = \delta, \quad q = \frac{1-\delta}{2\sqrt{2}\gamma^*}, \quad a_0 = \frac{2}{q\gamma^*}, \quad L = \frac{8(m+1)\bar\kappa_0}{\delta R q^2}.$$

With this choice, we have

$$\frac{\sqrt{1+\varepsilon^2}-\sqrt{(R_1/R)^2+\varepsilon^2}}{\gamma^*} \ge \frac{\sqrt{1+\varepsilon^2}-\sqrt{\delta^2+\varepsilon^2}}{\gamma^*} = \frac{\sqrt{1+\delta^2}-\sqrt{2}\delta}{\gamma^*} \ge 2q, \tag{26}$$

where, from $\delta < 1$, we used

$$\sqrt{1+\delta^2}-\sqrt{2}\delta = \frac{1-\delta^2}{\sqrt{1+\delta^2}+\sqrt{2}\delta} \ge \frac{1-\delta^2}{\sqrt{2}+\sqrt{2}\delta} = \frac{1-\delta}{\sqrt{2}}.$$

We also have

$$q^2 - \frac{1}{a_0^2(\gamma^*)^2\tau} = q^2 - \frac{q^2}{2} = \frac{q^2}{2}$$

and then

$$a_3 = (1-\tau)\left(q^2 - \frac{1}{a_0^2(\gamma^*)^2\tau}\right)L^2 = \frac{L^2q^2}{4} = 2\frac{(m+1)\bar{\kappa}_0 L}{\varepsilon R} = 2a_2.$$

Hence, all assumptions of Theorem 15 are satisfied and for every $x \in B_{R_1}$ we have

$$\sqrt{1+|Du(x)|^2} \le \max\left\{\sqrt{1+a_0^2(\gamma^*)^2}, \sqrt{2}\right\} \cdot \frac{e^{LR(\sqrt{1+\varepsilon^2}-\varepsilon)}-1}{e^{LR(\sqrt{1+\varepsilon^2}-\sqrt{(r(x)/R)^2+\varepsilon^2}-q\gamma(x))}-1}.$$

Note that, for every $x \in B_{R_1}$ and taking into account (26),

$$\begin{aligned}\sqrt{1+\varepsilon^2}-\sqrt{(r(x)/R)^2+\varepsilon^2}-q\gamma(x) &\ge \sqrt{1+\varepsilon^2}-\sqrt{\delta^2+\varepsilon^2}-q\gamma^* \\ &\ge 2q\gamma^* - q\gamma^* = q\gamma^*,\end{aligned}$$

and also

$$\sqrt{1+\varepsilon^2}-\varepsilon = \frac{1}{\sqrt{1+\varepsilon^2}+\varepsilon} \le \frac{1}{\sqrt{1+\varepsilon^2}} = \frac{1}{\sqrt{1+\delta^2}} \le \frac{1}{\sqrt{2}\delta} = \frac{2}{\delta(1-\delta)}q\gamma^*.$$

Therefore, we can estimate

$$\frac{e^{LR(\sqrt{1+\varepsilon^2}-\varepsilon)}-1}{e^{LR(\sqrt{1+\varepsilon^2}-\sqrt{(r(x)/R)^2+\varepsilon^2}-q\gamma(x))}-1} \le \frac{e^{LR\frac{2}{\delta(1-\delta)}q\gamma^*}-1}{e^{LRq\gamma^*}-1} \le C(\delta)e^{LR(\frac{2}{\delta(1-\delta)}-1)q\gamma^*}.$$

Here, we have exploited the fact that for every $\alpha \in \mathbb{R}$ one has the validity of an inequality of the form

$$\frac{y^\alpha - 1}{y-1} \le C(\alpha)y^{\alpha-1} \quad \text{for all } y > 1$$

for a suitable constant $C(\alpha) > 0$. Recalling that

$$a_0^2(\gamma^*)^2 = \frac{32(\gamma^*)^2}{(1-\delta)^2}, \quad LRq\gamma^* = \frac{8(m+1)\bar{\kappa}_0\gamma^*}{\delta q} = \frac{16\sqrt{2}(m+1)\bar{\kappa}_0}{\delta(1-\delta)}(\gamma^*)^2,$$

we obtain

$$\sqrt{1+|Du(x)|^2} \le \max\left\{\sqrt{1+\frac{32(\gamma^*)^2}{(1-\delta)^2}}, \sqrt{2}\right\} \cdot C(\delta)\exp\left(\frac{16\sqrt{2}(m+1)\bar{\kappa}_0}{\delta(1-\delta)}\left(\frac{2}{\delta(1-\delta)}-1\right)(\gamma^*)^2\right)$$

and the conclusion follows. □

Assuming that u has at most linear growth, we get:

Corollary 18. *Let* (M^m, σ) *be a connected, complete Riemannian manifold with* $\mathrm{Ric} \geq 0$ *and*

$$\mathrm{Ric}^{(\ell)}(\nabla r) \geq -\frac{\bar{\kappa}^2}{1+r^2}, \qquad \ell = \max\{1, m-2\},$$

for some $\bar{\kappa} \in \mathbb{R}_0^+$. *If* $u \in C^\infty(M)$ *solves* (MSE) *and has at most linear growth on one side, then* $|Du| \in L^\infty(M)$.

Proof. Without loss of generality we can assume that the negative part of u has at most linear growth, so that there exists $a > 0$ such that $u(x) \geq -a(1 + r(x))$ for every $x \in M$. Let $R_1 > 0$ be fixed. Choosing $\delta = \frac{1}{2}$ and letting $R \to \infty$ in estimate (25) we get

$$\sup_{B_{R_1}} \sqrt{1 + |Du|^2} \leq C_1 \exp(C_2 m \bar{\kappa}_0 a^2),$$

where $C_1, C_2 > 0$ do not depend on R_1. Since $R_1 > 0$ was arbitrary, the conclusion follows. $\square$

To prove Theorem 15, we need the following:

Lemma 19. *Let* (M^m, σ) *be a complete Riemannian manifold with*

$$\mathrm{Ric}^{(\ell)}(\nabla r) \geq -\frac{\bar{\kappa}^2}{1+r^2}, \qquad \ell = \max\{1, m-2\},$$

for some $\bar{\kappa} \in \mathbb{R}_0^+$, *where* r *is the distance from a fixed origin* $o \in M$, *and let* $u \in C^\infty(B_R)$ *solve* (MSE) *on a geodesic ball* B_R *centered at* o.

For any given $a > 0$, *the function* $\psi = \sqrt{a^2 + r^2}$ *satisfies*

$$|D\psi| < 1, \qquad \Delta_g \psi \leq (m+1)\frac{\max\{1, \bar{\kappa}\}}{a}$$

in the barrier sense on B_R *and pointwise on* $B_R \setminus \mathrm{cut}(o)$.

Remark 20. By its very definition, a solution in the barrier sense is also a solution in the viscosity sense; see [Mantegazza et al. 2014] for comments.

Proof. Outside of $\{o\} \cup \mathrm{cut}(o)$, a direct computation yields $|D\psi| = (r/\psi)|Dr| < 1$ and

$$\Delta_g \psi = \frac{r \Delta_g r}{\sqrt{a^2 + r^2}} + \frac{a^2 \|\nabla r\|^2}{(a^2 + r^2)^{3/2}}.$$

From $\|\nabla r\|^2 = g^{ij} r_i r_j \leq |Dr|^2 = 1$ and Remark 14,

$$\Delta_g \psi \leq \frac{1}{\sqrt{a^2+r^2}} \left(\frac{m(1 + \sqrt{1 + 4\bar{\kappa}^2})}{2} + \frac{a^2}{a^2 + r^2} \right)$$

and the conclusion follows by observing that $1/\sqrt{a^2 + r^2} \leq 1/a$ and that

$$\frac{m(1 + \sqrt{1 + 4\bar{\kappa}^2})}{2} + \frac{a^2}{a^2 + r^2} \leq m(1 + \bar{\kappa}) + 1 \leq (m+1)\max\{1, \bar{\kappa}\}.$$

The validity of the inequality in the barrier sense can be proved by Calabi's trick; see for instance [Mari and Pessoa 2020, Proposition 7.4]. □

Proof of Theorem 15. Without loss of generality, we can assume $\inf_{B_R} u = 0$. Then

$$\gamma^* = \sup_{x\in B_{R_1}} \gamma(x) = \frac{\sup_{B_{R_1}} u}{R}.$$

As in the statement of the theorem, fix $\tau \in (0,1)$ and $\varepsilon > 0$, choose $q > 0$ and $a_0 > 0$ such that

$$\frac{\sqrt{\varepsilon^2+1}-\sqrt{(R_1/R)^2+\varepsilon^2}}{\gamma^*} > q > \frac{1}{\sqrt{\tau a_0 \gamma^*}} \tag{27}$$

and then $L > 0$, which satisfies

$$(1-\tau)\left(q^2 - \frac{1}{\tau a_0^2(\gamma^*)^2}\right)L^2 - \frac{(m+1)\bar{\kappa}_0 L}{\varepsilon R} > (m-1)\kappa^2, \tag{28}$$

where $\bar{\kappa}_0 = \max\{1, \bar{\kappa}\}$. Set

$$C = qL, \quad \delta = e^{-LR\sqrt{\varepsilon^2+1}},$$

define the function

$$\psi = \sqrt{\varepsilon^2 R^2 + r^2},$$

where $r(x) = \mathrm{dist}_\sigma(o, x)$, and let

$$\eta = e^{-Cu - L\psi} - \delta, \quad z = W\eta.$$

By writing

$$\eta = \delta\big(e^{LR(\sqrt{\varepsilon^2+1}-\sqrt{\varepsilon^2+(r/R)^2}-qu/R)} - 1\big),$$

we see that for every $x \in B_{R_1}$

$$\begin{aligned} \eta(x) &\geq \delta\big(e^{LR(\sqrt{\varepsilon^2+1}-\sqrt{\varepsilon^2+r(x)^2/R^2}-q\gamma(x))} - 1\big) \\ &\geq \delta\big(e^{LR(\sqrt{1+\varepsilon^2}-\sqrt{(R_1/R)^2+\varepsilon^2}-q\gamma^*)} - 1\big) > 0 \end{aligned}$$

as a consequence of (27). Noting that, on ∂B_R,

$$\eta = \delta(e^{-qLu} - 1) \leq 0,$$

the set

$$\Omega = \{x \in \bar{B}_R : z(x) > 0\} \equiv \{x \in \bar{B}_R : \eta(x) > 0\}$$

is nonempty and satisfies $B_{R_1} \subseteq \Omega \subseteq B_R$. Therefore, there exists $x_0 \in \Omega$ such that

$$0 < z(x_0) = \max_{\Omega} z.$$

The function z satisfies

$$\Delta_g z - 2\langle \nabla z, \nabla \log W\rangle \geq \left(-(m-1)\kappa^2 \|\nabla u\|^2 + \frac{\Delta_g \eta}{\eta}\right) z \quad \text{on } \Omega.$$

The above inequality has to be interpreted in the viscosity sense, in case x_0 is not a point where r (hence ψ) is smooth. By the maximum principle, necessarily

$$-(m-1)\kappa^2\|\nabla u\|^2 + \frac{\Delta_g \eta}{\eta} \leq 0 \quad \text{at } x_0 \tag{29}$$

in the viscosity sense. We compute

$$\Delta_g \eta = (\eta+\delta)(-C\Delta_g u - L\Delta_g \psi + \|C\nabla u + L\nabla\psi\|^2).$$

We recall that $W^{-2}(\sigma^{ij})_{i,j} \leq (g^{ij})_{i,j} \leq (\sigma^{ij})_{i,j}$ in the sense of quadratic forms; hence

$$\begin{aligned}\|C\nabla u + L\nabla\psi\|^2 &= g^{ij}(Cu_i + L\psi_i)(Cu_j + L\psi_j)\\ &\geq \frac{1}{W^2}\sigma^{ij}(Cu_i + L\psi_i)(Cu_j + L\psi_j)\\ &\geq \frac{1}{W^2}|CDu + LD\psi|^2.\end{aligned}$$

It follows that

$$\frac{\Delta_g \eta}{\eta+\delta} \geq -C\Delta_g u - L\Delta_g\psi + \frac{1}{W^2}|CDu + LD\psi|^2.$$

Using $\Delta_g u = 0$ and Young's inequality we obtain

$$\frac{\Delta_g \eta}{\eta+\delta} \geq -L\Delta_g\psi + (1-\tau)C^2\frac{|Du|^2}{W^2} - L^2\frac{1-\tau}{\tau}\frac{|D\psi|^2}{W^2}.$$

Taking into account Lemma 19 we infer

$$|D\psi| < 1, \qquad \Delta_g\psi \leq \frac{(m+1)\bar{\kappa}_0}{\varepsilon R}.$$

Substituting these estimates in the above inequality, we deduce

$$\frac{\Delta_g \eta}{\eta+\delta} \geq (1-\tau)C^2\frac{|Du|^2}{W^2} - L\left(\frac{(m+1)\bar{\kappa}_0}{\varepsilon R} + \frac{1-\tau}{\tau}\frac{L}{W^2}\right).$$

If $|Du(x_0)| \geq a_0\gamma^*$ then

$$\frac{|Du(x_0)|^2}{W^2 a_0^2(\gamma^*)^2} \geq \frac{1}{W^2}.$$

Thus, we can further estimate

$$\frac{\Delta_g \eta}{\eta+\delta} \geq (1-\tau)\left(q^2 - \frac{1}{\tau a_0^2(\gamma^*)^2}\right)L^2\frac{|Du|^2}{W^2} - \frac{(m+1)\bar{\kappa}_0 L}{\varepsilon R} \quad \text{at } x_0,$$

that is,

$$\frac{\Delta_g \eta}{\eta+\delta} \geq a_1\|\nabla u\|^2 - a_2, \tag{30}$$

with

$$a_1 = (1-\tau)\left(q^2 - \frac{1}{\tau a_0^2(\gamma^*)^2}\right)L^2 > 0, \quad a_2 = \frac{(m+1)\bar{\kappa}_0 L}{\varepsilon R} > 0.$$

Since

$$a_3 = a_1 - (m-1)\kappa^2,$$

we have $a_1 \geq a_3 > 0$ by condition (28). We claim that

$$\frac{|Du(x_0)|^2}{W^2(x_0)} = \|\nabla u(x_0)\|^2 \leq \frac{a_2}{a_3},$$

that is,

$$W(x_0) \leq \sqrt{\frac{a_3}{a_3 - a_2}}. \tag{31}$$

Indeed, assume by contradiction that

$$\|\nabla u(x_0)\|^2 > \frac{a_2}{a_3}. \tag{32}$$

Then, from (30) it follows

$$\frac{\Delta_g \eta}{\eta + \delta} \geq a_3 \|\nabla u\|^2 - a_2 > 0 \quad \text{at } x_0;$$

hence $\Delta_g \eta > 0$ and, by (29) and (30) again,

$$(m-1)\kappa^2 \|\nabla u\|^2 \geq \frac{\Delta_g \eta}{\eta} \geq \frac{\Delta_g \eta}{\eta + \delta} \geq a_1 \|\nabla u\|^2 - a_2 \quad \text{at } x_0,$$

leading to

$$a_2 \geq (a_1 - (m-1)\kappa^2)\|\nabla u\|^2 = a_3 \|\nabla u\|^2 \quad \text{at } x_0,$$

which contradicts (32) and proves our claim.

On the other hand, if $|Du(x_0)| \leq a_0 \gamma^*$, then

$$W(x_0) \leq \sqrt{1 + a_0^2 (\gamma^*)^2}. \tag{33}$$

Since x_0 is a global maximum point for z in Ω, we have $z(x) \leq z(x_0)$, that is,

$$W(x) \leq W(x_0) \frac{\eta(x_0)}{\eta(x)}$$

for every $x \in B_{R_1} \subseteq \Omega$. Note that

$$\frac{\eta(x_0)}{\eta(x)} = \frac{e^{LR(\sqrt{\varepsilon^2+1} - \sqrt{\varepsilon^2 + r(x_0)^2/R^2} - qu(x_0)/R)} - 1}{e^{LR(\sqrt{\varepsilon^2+1} - \sqrt{\varepsilon^2 + r(x)^2/R^2} - qu(x)/R)} - 1} \leq \frac{e^{LR(\sqrt{\varepsilon^2+1} - \varepsilon)} - 1}{e^{LR(\sqrt{\varepsilon^2+1} - \sqrt{\varepsilon^2 + r(x)^2/R^2} - q\gamma(x))} - 1};$$

hence

$$W(x) \leq W(x_0) \left(\frac{e^{LR(\sqrt{\varepsilon^2+1} - \varepsilon)} - 1}{e^{LR(\sqrt{\varepsilon^2+1} - \sqrt{\varepsilon^2 + r(x)^2/R^2} - q\gamma(x))} - 1} \right).$$

The latter, together with (31) and (33), implies the desired estimate. □

5. Uniformly elliptic operators on manifolds with Ric ≥ 0

Having shown that an entire minimal graph with at most linear growth on one side has globally bounded gradient, we need to show (6) and (7). We shall prove both of them under the only conditions

$$\mathrm{Ric} \geq 0, \quad |Du| \in L^\infty(M). \tag{34}$$

In such generality, it seems difficult to apply the "elliptic" approach in [Cheeger et al. 1995], adapted in [Ding et al. 2016; Ding 2021b]. To justify the statement, we observe that the method in [Cheeger et al. 1995] relies on the construction of a function ϱ satisfying

$$C^{-1}r \leq \varrho \leq Cr, \quad |D\varrho| \leq C, \quad \Delta_g \varrho \leq \frac{C}{\varrho} \tag{35}$$

for some absolute constant C. When considering harmonic functions, the third condition is replaced by $\Delta\varrho \leq C/\varrho$; thus by comparison theory the choice $\varrho = r$ is admissible. On the contrary, to our knowledge, for minimal graphs the existence of ϱ satisfying (35) is currently unknown under the sole assumptions (34). If M has Euclidean volume growth, we mention that in [Ding et al. 2016; Ding 2021b] the authors used as ϱ a reparametrization of the Green kernel of the Laplacian on M. Although the inequality $\Delta_g\varrho \leq C/\varrho$ may not hold pointwise, the integral estimates for $|D^2\varrho|$ provided in [Colding and Minicozzi 1997] suffice to estimate $\Delta_g\varrho$ and apply the method in [Cheeger et al. 1995], as done in [Ding 2021b, Lemma 7.1]. However, to our knowledge, estimates like those in [Colding and Minicozzi 1997] are not yet (if ever) available on manifolds with Ric ≥ 0 but whose volume growth is less than Euclidean.

For these reasons, inspired by [Li 1986; Saloff-Coste 1992] we choose a different approach via the heat equation. Throughout this section, let (M, σ) be a connected, complete Riemannian manifold of dimension $m \geq 2$ with Ric ≥ 0. Let L be the linear uniformly elliptic operator defined by

$$L\psi = \mathrm{div}(AD\psi), \tag{36}$$

where A is a measurable section of $T^{1,1}M$ satisfying

$$\alpha^{-1}|X|^2 \leq \langle AX, X\rangle \quad \text{and} \quad |AX| \leq \alpha|X| \quad \text{for all } X \in TM, \tag{37}$$

for some constant $\alpha > 0$. Hereafter, we shall assume that A is smooth, the general case being obtainable by approximation.

We denote by $H_L(x, y, t)$ the minimal heat kernel associated to the parabolic operator $\partial_t - L$, that is, the unique continuous function on $M \times M \times \mathbb{R}^+$ such that for every $\psi \in C_0^\infty(M)$ the function u defined by

$$u(t, x) = \int_M H_L(x, y, t)\psi(y)\,\mathrm{d}y \quad \text{for all } (t, x) \in \mathbb{R}^+ \times M$$

is a solution to

$$\partial_t u = Lu \tag{38}$$

on $\mathbb{R}^+ \times M$ satisfying

(i) $u(t, \cdot) \to \psi$ pointwise on M as $t \searrow 0$,

(ii) $u \le v$ on $(0, T) \times M$ for every $v \in C^2([0, T) \times M)$, $T > 0$, such that

$$\begin{cases} \partial_t v = Lv & \text{on } (0, T) \times M, \\ \psi \le v(0, \cdot) & \text{on } M. \end{cases}$$

If the endomorphism A is self-adjoint with respect to $\langle \cdot , \cdot \rangle$, the minimal heat kernel H_L is a symmetric function of the space variables, that is,

$$H_L(x, y, t) = H_L(y, x, t) \quad \text{for all } x, y \in M, \ \text{for all } t > 0. \tag{39}$$

By [Saloff-Coste 1992], see Corollary 6.2 and Theorem 6.3, there exist positive constants $C_i > 0$, $1 \le i \le 6$, depending only on m and α such that, for every $x, y \in M$ and $t > 0$,

$$C_1 \frac{\exp(-C_2 \operatorname{dist}(x, y)^2/t)}{\sqrt{|B_{\sqrt{t}}(x)||B_{\sqrt{t}}(y)|}} \le H_L(x, y, t) \le C_3 \frac{\exp(-C_4 \operatorname{dist}(x, y)^2/t)}{\sqrt{|B_{\sqrt{t}}(x)||B_{\sqrt{t}}(y)|}} \tag{40}$$

and

$$|\partial_t H_L(x, y, t)| \le \frac{C_5}{t} \frac{\exp(-C_6 \operatorname{dist}(x, y)^2/t)}{\sqrt{|B_{\sqrt{t}}(x)||B_{\sqrt{t}}(y)|}}. \tag{41}$$

Remarks on (40) will be given in the Appendix. We first need the following simple estimate on the volume of geodesic balls.

Lemma 21. *Let (M^m, σ) be a complete manifold with* Ric ≥ 0. *For every $x, y \in M$ and for every $R > 0$ it holds*

$$|B_R(x)|\left(1 + \frac{\operatorname{dist}(x, y)}{R}\right)^{-\frac{m}{2}} \le \sqrt{|B_R(x)||B_R(y)|} \le |B_R(x)|\left(1 + \frac{\operatorname{dist}(x, y)}{R}\right)^{\frac{m}{2}}.$$

Proof. By the Bishop–Gromov comparison theorem we have

$$\frac{|B_R(x)|}{|B_r(x)|} \le \left(\frac{R}{r}\right)^m, \qquad 0 < r \le R < \infty;$$

thus

$$|B_R(y)| \le |B_{R+\operatorname{dist}(x,y)}(x)| \le |B_R(x)|\left(1 + \frac{\operatorname{dist}(x, y)}{R}\right)^m,$$

and the thesis follows. □

Next, we recall that L generates a diffusion which is stochastically complete (see [Grigoryan 1999]), that is, the following holds:

Lemma 22. *Let M be a complete manifold with* Ric ≥ 0, *and let A, L be as in (36)–(37), with A self-adjoint and smooth. Then*

$$\int_M H_L(x, y, t)\, dy = 1 \quad \textit{for all } (t, x) \in \mathbb{R}^+ \times M.$$

The result is stated with no proof in the discussion following [Saloff-Coste 1992, Theorem 7.4]. We here provide an argument for the convenience of the reader.

Proof. Since L is uniformly elliptic and M has polynomial volume growth as a consequence of $\mathrm{Ric} \geq 0$, by Theorem 4.1 of [Alías et al. 2016] we have that for any $\lambda > 0$ the only entire bounded solution v of $Lv = \lambda v$ on M is $v \equiv 0$. Then the conclusion follows by [Pigola et al. 2005, Theorem 3.11]. □

With the above preparation, we are ready to state the following asymptotic mean value theorem. Our method is inspired by the one in [Li 1986], where the author considered the case $L = \Delta$, but with a difference to be stressed. Indeed, in that work the author uses the Li–Yau differential Harnack inequality to get rid of a boundary term at infinity. The inequality holds for solutions of the heat equation, but in general it may fail for solutions of $\partial_t u = Lu$, unless one has a uniform control on the gradient of A on the entire M; see for instance [Saloff-Coste 1992, p. 433]. As we will apply our results to $A = W\mathrm{Id} - W^{-1}\, \mathrm{d}u \otimes Du$, with $W = \sqrt{1 + |Du|^2}$, in our setting only an L^∞ control on A is available. One may therefore use De Giorgi-Nash-Moser theory to get Hölder estimates in space for u, see [Saloff-Coste 1992, Corollary 5.5], but these seem insufficient to treat the boundary term.

In view of the above, we shall modify the method in [Li 1986]. The main idea here is the use of upper level sets of H_L rather than geodesic balls. Note that we do not assume a Euclidean volume growth. We start with the following:

Lemma 23. *Let $(M^m, \langle \cdot\,, \cdot \rangle)$ be a connected, complete, noncompact manifold with* $\mathrm{Ric} \geq 0$ *and let A, L be as in (36)–(37), with A self-adjoint and smooth. If $f \in C^2(M) \cap L^\infty(M)$ satisfies $Lf \leq 0$ on M then the function $u : \mathbb{R}^+ \times M \to \mathbb{R}$ given by*

$$u(t, x) = \int_M f(y) H_L(x, y, t)\, \mathrm{d}y \quad \textit{for all } (t, x) \in \mathbb{R}^+ \times M \tag{42}$$

satisfies

$$\partial_t u \leq 0 \quad \textit{on } \mathbb{R}^+ \times M. \tag{43}$$

Proof. Note that the integral on the right-hand side of (42) converges for every $(t, x) \in \mathbb{R}^+ \times M$ since $f \in L^\infty(M)$ and because of (40) and Lemma 21. Also note that H_L is smooth as a consequence of the regularity assumptions on A. By Lemma 22, u only varies by an additive constant if so does f; hence without loss of generality we can assume $\inf_M f = 0$. Let $(t, x) \in \mathbb{R}^+ \times M$ be fixed. For notational convenience, for every $a > 0$ we define

$$\varphi_a(y) = H_L(x, y, t) - a \quad \text{for all } y \in M, \qquad \Omega_a = \{y \in M : \varphi_a(y) > 0\}. \tag{44}$$

Because of (40) it holds $H_L(x, y, t) \to 0$ as $y \to \infty$ in M; hence the collection $\{\Omega_a\}_{a>0}$ is an exhaustion of M by relatively compact open subsets, with $\Omega_a \subseteq \Omega_b$ when $a \geq b$. By (41) and boundedness of f, we can apply Lebesgue's dominated convergence theorem to get

$$\partial_t u(t, x) = \int_M f(y) \partial_t H_L(x, y, t)\, \mathrm{d}y = \lim_{a \to 0^+} \int_{\Omega_a} f(y) \partial_t H_L(x, y, t)\, \mathrm{d}y. \tag{45}$$

Therefore, since $Lf \leq 0$ and $\varphi_a > 0$ on Ω_a, (43) holds by monotone convergence if we prove the inequality

$$\int_{\Omega_a} f(y) \partial_t H_L(x, y, t)\, \mathrm{d}y \leq \int_{\Omega_a} \varphi_a(y) Lf(y)\, \mathrm{d}y. \tag{46}$$

Because of $\partial_t H_L(x, y, t) = L_y H_L(x, y, t) = L\varphi(y) = L\varphi_a(y)$, we have

$$\int_{\Omega_a} f(y)\partial_t H_L(x, y, t)\,\mathrm{d}y = \int_{\Omega_a} f(y) L_y H_L(x, y, t)\,\mathrm{d}y = \int_{\Omega_a} f(y) L\varphi_a(y)\,\mathrm{d}y.$$

Since $H_L \in C^\infty(\mathbb{R}^+ \times M)$, we have $\varphi \in C^\infty(M)$ and for almost every $a > 0$ the set Ω_a has smooth boundary. Let $a > 0$ be a regular value for φ. By Green's identity, since $\varphi_a = 0$ on $\partial\Omega_a$

$$\int_{\Omega_a} f(y) L\varphi_a(t, y)\,\mathrm{d}y = \int_{\Omega_a} \varphi_a(t, y) Lf(y)\,\mathrm{d}y + \int_{\partial\Omega_a} f(y)\langle AD\varphi_a(y), \nu\rangle\,\mathrm{d}\mathcal{H}^{m-1}(y),$$

where $\nu = -D\varphi_a/|D\varphi_a|$ is the outward-pointing normal on $\partial\Omega_a$. Noting that $f \geq 0$, that φ_a is nonincreasing in the direction of ν and that A is positive definite, we see that $f\langle AD\varphi_a, \nu\rangle \leq 0$ on $\partial\Omega_a$ and therefore the second integral is nonpositive, which implies the desired inequality (46). □

Proposition 24. *Let* $(M^m, \langle\,\cdot\,,\cdot\,\rangle)$ *be a connected, complete, noncompact manifold with* Ric ≥ 0 *and let* A, L *be as in* (36)–(37), *with* A *self-adjoint and smooth. If* $f \in C^2(M) \cap L^\infty(M)$ *satisfies* $Lf \leq 0$ *on* M, *then for any* $x \in M$

$$\lim_{R\to\infty} \frac{1}{|B_R(x)|}\int_{B_R(x)} f(y)\,\mathrm{d}y = \inf_M f. \tag{47}$$

Proof. Without loss of generality, we assume $\inf_M f = 0$. Let $u : \mathbb{R}^+ \times M \to \mathbb{R}$ be the function defined by (42). Note that u is the minimal solution to the parabolic equation $\partial_t u = Lu$ on $\mathbb{R}^+ \times M$ corresponding to the initial datum $u(0^+, \cdot\,) = f$. Hence, by the maximum principle and the monotonicity (43) we have

$$\inf_M f \leq u(t, x) \leq f(x) \quad \text{for all } (t, x) \in \mathbb{R}^+ \times M.$$

In particular, the limit

$$u_\infty(x) = \lim_{t\to\infty} u(t, x)$$

is well-defined for every $x \in M$. The convergence $u(t, \cdot\,) \to u_\infty$ is uniform on compact subsets, u_∞ is bounded and $Lu_\infty = 0$. Since M is complete and has nonnegative Ricci curvature, the operator L enjoys a Liouville property; see Theorem 7.4 of [Saloff-Coste 1992]. In particular, u_∞ must be constant. Since $\inf_M f \leq u_\infty \leq f$, it must be $u_\infty \equiv \inf_M f = 0$, that is,

$$\lim_{t\to\infty} u(t, x) = 0 \quad \text{for all } x \in M. \tag{48}$$

To conclude the proof of (47), we observe that

$$\begin{aligned} u(t, x) &= \int_M H_L(x, y, t) f(y)\,\mathrm{d}y \\ &\geq \frac{C_1}{|B_{\sqrt{t}}(x)|}\int_M \left(1 + \frac{\mathrm{dist}(x, y)}{\sqrt{t}}\right)^{-m/2} \exp\left(-C_2\frac{\mathrm{dist}(x, y)^2}{t}\right) f(y)\,\mathrm{d}y \\ &= \frac{C_1}{|B_{\sqrt{t}}(x)|}\int_0^\infty \left(1 + \frac{r}{\sqrt{t}}\right)^{-m/2} \exp\left(-C_2\frac{r^2}{t}\right) \int_{\partial B_r(x)} f(y)\,\mathrm{d}\mathcal{H}^{m-1}(y)\,\mathrm{d}r \end{aligned}$$

$$\geq \frac{C_1}{|B_{\sqrt{t}}(x)|}\int_0^{\sqrt{t}}\left(1+\frac{r}{\sqrt{t}}\right)^{-m/2}\exp\left(-C_2\frac{r^2}{t}\right)\int_{\partial B_r(x)} f(y)\,\mathrm{d}\mathcal{H}^{m-1}(y)\,\mathrm{d}r$$
$$\geq \frac{2^{-m/2}e^{-C_2}C_1}{|B_{\sqrt{t}}(x)|}\int_0^{\sqrt{t}}\int_{\partial B_r(x)} f(y)\,\mathrm{d}\mathcal{H}^{m-1}(y)\,\mathrm{d}r$$
$$= \frac{2^{-m/2}e^{-C_2}C_1}{|B_{\sqrt{t}}(x)|}\int_{B_{\sqrt{t}}(x)} f(y)\,\mathrm{d}y.$$

Since $f \geq 0$, by comparison we have

$$\lim_{t\to\infty}\frac{1}{|B_{\sqrt{t}}(x)|}\int_{B_{\sqrt{t}}(x)} f(y)\,\mathrm{d}y = 0 = \inf_M f$$

as desired. □

From the above result, we also obtain information on the spherical mean of u. This follows from the next variant of de L'Hôpital's theorem.

Lemma 25. *Let $h, g \in L^\infty_{\mathrm{loc}}(\mathbb{R}^+)$ satisfy $h \geq 0,\ g > 0$ a.e. and $g \notin L^1(\infty)$. Then,*

$$\operatorname{ess}\liminf_{r\to\infty}\frac{h(r)}{g(r)} \leq \liminf_{r\to\infty}\frac{\int_0^r h(t)\,\mathrm{d}t}{\int_0^r g(t)\,\mathrm{d}t}. \tag{49}$$

Proof. Denote by A and B, respectively, the left-hand side and right-hand side of (49). For $A' < A$, fix R_0 such that $h \geq A'g$ a.e. on (R_0, ∞). Then, for each $r > R_0$,

$$\frac{\int_0^r h(t)\,\mathrm{d}t}{\int_0^r g(t)\,\mathrm{d}t} = \frac{\int_0^{R_0} h(t)\,\mathrm{d}t + \int_{R_0}^r h(t)\,\mathrm{d}t}{\int_0^{R_0} g(t)\,\mathrm{d}t + \int_{R_0}^r g(t)\,\mathrm{d}t} \geq \frac{\int_0^{R_0} h(t)\,\mathrm{d}t + A'\int_{R_0}^r g(t)\,\mathrm{d}t}{\int_0^{R_0} g(t)\,\mathrm{d}t + \int_{R_0}^r g(t)\,\mathrm{d}t}.$$

Since $g \notin L^1(\infty)$, letting $r \to \infty$ along a sequence realizing B we get $B \geq A'$, and the thesis follows by letting $A' \uparrow A$. □

Corollary 26. *Let $(M^m, \langle\cdot,\cdot\rangle)$ be a complete (connected) Riemannian manifold with infinite volume. Let $0 \leq f \in L^1_{\mathrm{loc}}(M)$ and $x \in M$ and assume that*

$$\liminf_{R\to\infty}\frac{1}{|B_R(x)|}\int_{B_R(x)} f(y)\,\mathrm{d}y = \inf_M f.$$

Then

$$\operatorname{ess}\liminf_{R\to\infty}\frac{1}{|\partial B_R(x)|}\int_{\partial B_R(x)} f(y)\,\mathrm{d}\mathcal{H}^{m-1}(y)\,\mathrm{d}y = \inf_M f.$$

Proof. The functions h and g defined by

$$h(t) = \int_{\partial B_t(x)} f(y)\,\mathrm{d}\mathcal{H}^{m-1}(y) \qquad \text{and} \qquad g(t) = |\partial B_t(x)| \quad \text{for all } t > 0$$

satisfy the assumptions of the previous lemma (note that $1/g \in L^\infty_{\mathrm{loc}}(\mathbb{R}^+)$ by [Bianchini et al. 2013, Proposition 1.6], since M is noncompact). The thesis follows from the next chain of inequalities:

$$\inf_M f \le \operatorname{ess}\liminf_{R\to\infty} \frac{1}{|\partial B_R(x)|}\int_{\partial B_R(x)} f(y)\,\mathrm{d}\mathcal{H}^{m-1}(y)\,\mathrm{d}y \le \liminf_{R\to\infty} \frac{1}{|B_R(x)|}\int_{B_R(x)} f(y)\,\mathrm{d}y \le \inf_M f. \qquad \square$$

We are ready to state our second main result of the section, which will enable us to prove the Hessian estimate (7). The argument below seems to be new.

Theorem 27. *Let $(M^m, \langle\,\cdot\,,\cdot\,\rangle)$ be a connected, complete manifold with* Ric ≥ 0 *and let A, L be as in (36)–(37), with A self-adjoint and smooth. If $f \in L^\infty(M)$ satisfies $Lf \le 0$ on M, then for any $x \in M$*

$$\lim_{R\to\infty} \frac{R^2}{|B_R(x)|}\int_{B_R(x)} Lf(y)\,\mathrm{d}y = 0. \tag{50}$$

Proof. Without loss of generality, we assume $\inf_M f = 0$. Fix $x \in M$. We refer to the proof of Lemma 23 for notation, and in particular, for $t > 0$ and $a > 0$ we define $\varphi_a(y)$ and Ω_a as in (44). As already observed, $\{\Omega_a\}$ is an exhaustion of M, increasing as a decreases. Furthermore, for almost every $a > 0$ the boundary $\partial\Omega_a$ is smooth. From the proof of Lemma 23 we get

$$\int_{\Omega_a} f(y)\partial_t H_L(x,y,t)\,\mathrm{d}y \le \int_{\Omega_a} \varphi_a(y) Lf(y)\,\mathrm{d}y. \tag{51}$$

On the other hand, since $f \ge 0$, by (41) and Lemma 21 we can estimate

$$\int_{\Omega_a} f(y)\partial_t H_L(x,y,t)\,\mathrm{d}y \ge -\frac{C_5}{t}\frac{1}{|B_{\sqrt{t}}(x)|}\int_{\Omega_a} f(y)\left(1+\frac{\mathrm{dist}(x,y)}{\sqrt{t}}\right)^{\frac{m}{2}} \exp\left(-C_6\frac{\mathrm{dist}(x,y)^2}{t}\right)\mathrm{d}y.$$

By (40) and Lemma 21 we also have the bounds

$$\frac{C_1}{|B_{\sqrt{t}}(x)|}\left(1+\frac{\mathrm{dist}(x,y)}{\sqrt{t}}\right)^{-\frac{m}{2}}\exp\left(-C_2\frac{\mathrm{dist}(x,y)^2}{t}\right) \le H_L(x,y,t) \le \frac{C_3}{|B_{\sqrt{t}}(x)|}\left(1+\frac{\mathrm{dist}(x,y)}{\sqrt{t}}\right)^{\frac{m}{2}}\exp\left(-C_4\frac{\mathrm{dist}(x,y)^2}{t}\right).$$

Now, fix $k > 1$ large enough so that

$$C_3(1+s)^{\frac{m}{2}}e^{-C_4 s^2} \le \tfrac{1}{2}C_1 2^{-\frac{m}{2}}e^{-C_2} \quad \text{for all } s \ge k$$

and pick

$$a = \frac{C_1 2^{-\frac{m}{2}}e^{-C_2}}{2|B_{\sqrt{t}}(x)|}.$$

With this choice, we have

$$\begin{cases} \varphi \le a & \text{on } M \setminus B_{k\sqrt{t}}(x), \\ \varphi \ge 2a & \text{on } B_{\sqrt{t}}(x); \end{cases}$$

hence $B_{\sqrt{t}}(x) \subseteq \Omega_a \subseteq B_{k\sqrt{t}}(x)$ and $\varphi_a \geq a$ on $B_{\sqrt{t}}(x)$. Thus, using also (51) we can estimate

$$\begin{aligned} 0 &\geq \frac{C_1 2^{-\frac{m}{2}} e^{-C_2}}{2|B_{\sqrt{t}}(x)|} \int_{B_{\sqrt{t}}(x)} Lf(y)\,\mathrm{d}y = a \int_{B_{\sqrt{t}}(x)} Lf(y)\,\mathrm{d}y \\ &\geq \int_{B_{\sqrt{t}}(x)} \varphi_a(y) Lf(y)\,\mathrm{d}y \geq \int_{\Omega_a} \varphi_a(y) Lf(y)\,\mathrm{d}y \geq \int_{\Omega_a} f(y) \partial_t H_L(x,y,t)\,\mathrm{d}y \\ &\geq -\frac{C_5}{t|B_{\sqrt{t}}(x)|} \int_{\Omega_a} f(y) \Big(1 + \frac{\mathrm{dist}(x,y)}{\sqrt{t}}\Big)^{\frac{m}{2}} \exp\Big(-C_6 \frac{\mathrm{dist}(x,y)^2}{t}\Big)\,\mathrm{d}y \\ &\geq -\frac{C_7}{t|B_{\sqrt{t}}(x)|} \int_{B_{k\sqrt{t}}(x)} f(y)\,\mathrm{d}y, \end{aligned}$$

where

$$C_7 = C_5 \sup\{(1+s)^{\frac{m}{2}} e^{-C_6 s^2} : s > 0\} < \infty.$$

Summing up, there exists a constant $C > 0$, depending only on C_i, $1 \leq i \leq 7$, such that

$$0 \geq \frac{t}{|B_{\sqrt{t}}(x)|} \int_{B_{\sqrt{t}}(x)} Lf(y)\,\mathrm{d}y \geq -\frac{C}{|B_{\sqrt{t}}(x)|} \int_{B_{k\sqrt{t}}(x)} f(y)\,\mathrm{d}y.$$

Since $f \geq 0$, by the Bishop–Gromov theorem we also have

$$0 \geq \frac{t}{|B_{\sqrt{t}}(x)|} \int_{B_{\sqrt{t}}(x)} Lf(y)\,\mathrm{d}y \geq -\frac{Ck^m}{|B_{k\sqrt{t}}(x)|} \int_{B_{k\sqrt{t}}(x)} f(y)\,\mathrm{d}y.$$

By Proposition 24 we have that the right-hand side of this inequality converges to $\inf_M f = 0$ as $t \to \infty$, and the conclusion follows. □

6. Proof of Theorem 6(ii)

Combining Corollary 18, Proposition 24 and Theorem 27, we get:

Proposition 28. *Let (M^m, σ) be a connected, complete Riemannian manifold with* $\mathrm{Ric} \geq 0$ *and*

$$\mathrm{Ric}^{(\ell)}(\nabla r) \geq -\frac{\bar{\kappa}^2}{1+r^2}, \qquad \ell = \max\{1, m-2\},$$

for some $\bar{\kappa} \in \mathbb{R}_0^+$ and where r is the distance from a fixed origin. Let $u \in C^\infty(M)$ be a nonconstant solution to (MSE) *which grows at most linearly on one side. Then, for each $x \in M$,*

$$\lim_{R\to\infty} \frac{1}{|B_R(x)|} \int_{B_R(x)} |Du|^2\,\mathrm{d}x = \sup_M |Du|^2, \tag{52}$$

$$\lim_{R\to\infty} \frac{R^2}{|B_R(x)|} \int_{B_R(x)} |D^2u|^2\,\mathrm{d}x = 0. \tag{53}$$

Proof. Because of Corollary 18, in our assumptions $|Du| \in L^\infty(M)$; hence by (11) the operator

$$L\phi \doteq W\Delta_g \phi = \mathrm{div}(W g^{ij} \phi_i \partial_{x_j})$$

is uniformly elliptic on M. By the Jacobi equation, $f = 1/W$ is a nonnegative solution to $Lf \le -\|\mathrm{II}\|^2 \le 0$, and therefore $-W^2 \in L^\infty(M)$ satisfies $L(-W^2) \le 0$. Applying Proposition 24 to $-W^2$ and Theorem 27 to f we deduce

$$\lim_{R\to\infty} \frac{1}{|B_R(x)|} \int_{B_R(x)} W^2 \, \mathrm{d}x = \sup_M W^2, \tag{54}$$

$$\limsup_{R\to\infty} \frac{R^2}{|B_R(x)|} \int_{B_R(x)} \|\mathrm{II}\|^2 \, \mathrm{d}x \le -\lim_{R\to\infty} \frac{R^2}{|B_R(x)|} \int_{B_R(x)} Lf \, \mathrm{d}x = 0. \tag{55}$$

From (54) we readily deduce (52). On the other hand, note that

$$\|\mathrm{II}\|^2 = W^{-2} g^{ik} u_{kj} g^{jl} u_{li} = W^{-2} \left\{ |D^2 u|^2 - 2\left| D^2 u\left(\frac{Du}{W}, \cdot\right)\right|^2 + \left[D^2 u\left(\frac{Du}{W}, \frac{Du}{W}\right)\right]^2 \right\}.$$

If $\mathrm{d}u(x) = 0$, then $\|\mathrm{II}\|^2 \ge W^{-2}|D^2u|^2$. Otherwise, let $e_1 = Du/|Du|$ and choose a local orthonormal frame $\{e_\alpha\}$ for $e_1^\perp$ around x, where $2 \le \alpha \le m$. Then,

$$\begin{aligned} |D^2u|^2 - 2\left| D^2 u\left(\frac{Du}{W}, \cdot\right)\right|^2 + \left[D^2 u\left(\frac{Du}{W}, \frac{Du}{W}\right)\right]^2 \\ &= \sum_{\alpha,\beta} u_{\alpha\beta}^2 + 2\sum_\alpha u_{1\alpha}^2 + u_{11}^2 - 2\frac{W^2-1}{W^2}\sum_j u_{1j}^2 + \frac{(W^2-1)^2}{W^4} u_{11}^2 \\ &= \sum_{\alpha,\beta} u_{\alpha\beta}^2 + \frac{2}{W^2}\sum_\alpha u_{1\alpha}^2 + \frac{1}{W^4} u_{11}^2 \ge W^{-4}|D^2u|^2. \end{aligned}$$

Summarizing, we have $\|\mathrm{II}\|^2 \ge W^{-6}|D^2u|^2$; thus from the boundedness of W and from (55) we conclude (53). ☐

We now conclude the proof of Theorem 6 with a blow-down procedure, for which we use some basic convergence results in the theory of limit spaces and nonsmooth spaces with Ricci curvature bounded below. All the tools needed herein can be found in [Honda 2015; Ambrosio and Honda 2017; 2018].

Fix $o \in M$, and write $B_R = B_R(o)$. Because of Corollary 18 and Proposition 28,

$$\lim_{R\to\infty} \frac{1}{|B_R|} \int_{B_R} |Du|^2 \, \mathrm{d}x = \sup_M |Du|^2, \tag{56}$$

$$\lim_{R\to\infty} \frac{R^2}{|B_R|} \int_{B_R} |D^2u|^2 \, \mathrm{d}x = 0. \tag{57}$$

Consider a tangent cone at infinity M_∞ for M based at o. By statement (2.1) in [De Philippis and Gigli 2018], the limit space M_∞ also supports a Borel measure $\mathfrak{m}_\infty$ such that, up to a subsequence,

$$(M, \lambda_n^{-1}\,\mathrm{dist}_\sigma, \lambda_n^{-m}\mathrm{d}x, o) \xrightarrow{\text{pmGH}} (M_\infty, \mathrm{d}_\infty, \mathfrak{m}_\infty, o_\infty) \tag{58}$$

in the pointed-measured-Gromov–Hausdorff (pmGH) sense. For the precise definition of pGH and pmGH convergence we refer to [Gigli et al. 2015]. Here, $\{\lambda_n\} \subset \mathbb{R}^+$, $\lambda_n \to \infty$ as $n \to \infty$, and $\lambda_n^{-1}\,\mathrm{dist}_\sigma$ is the distance function induced by the rescaled metric $\sigma_n \doteq \lambda_n^{-2}\sigma$. Denote with D_n and $\mathrm{d}x_n$ the induced connection and volume measure, and B^n_R the metric balls centered at o in (M, σ_n). Therefore, $B^n_R = B_{\lambda_n R}$. Define $u_n = u/\lambda_n$. Then,

$$|D_n u_n|_{\sigma_n} = |Du|, \quad |D^2_n u_n|_{\sigma_n} = \lambda_n |D^2 u| \tag{59}$$

and therefore, by the Arzelà–Ascoli theorem, up to subsequences $u_n \to u_\infty \in \mathrm{Lip}(M_\infty)$ locally uniformly; hence $u_n \to u_\infty$ strongly in L^2 on $B^\infty_R = B^{\mathrm{d}_\infty}_R(x_\infty)$, that is,

$$\lim_{n\to\infty} \int_{B^n_R} |u_n|^2 \, \mathrm{d}x_n = \int_{B^\infty_R} |u_\infty|^2 \, \mathrm{d}\mathfrak{m}_\infty,$$
$$\lim_{n\to\infty} \int_{B^n_R} u_n \varphi \, \mathrm{d}x_n = \int_{B^\infty_R} u_\infty \varphi \, \mathrm{d}\mathfrak{m}_\infty$$

for each φ bounded and continuous on a metric space Z in which (B^n_R, d_n) and $(B^\infty_R, \mathrm{d}_\infty)$ are isometrically embedded and converge in Hausdorff sense, with $o_n \to o_\infty$ and o_n the center of B^n_R. From

$$W_n \doteq \sqrt{1 + |D_n u_n|^2_{\sigma_n}} = \sqrt{1 + |Du|^2} = W.$$

Scaling (56) and (57) we therefore get, for each fixed $R > 0$,

$$\lim_{n\to\infty} \frac{1}{|B^n_R|_{\sigma_n}} \int_{B^n_R} |D_n u_n|^2_{\sigma_n} \, \mathrm{d}x_n = \sup_M |Du|^2, \tag{60}$$

$$\lim_{n\to\infty} \frac{R^2}{|B^n_R|_{\sigma_n}} \int_{B^n_R} |D^2_n u_n|^2_{\sigma_n} \, \mathrm{d}x_n = 0. \tag{61}$$

In particular, from Newton's inequality $|\Delta_n u_n|^2 \le m |D^2_n u_n|^2_{\sigma_n}$ and the Bishop–Gromov theorem, $|B^n_R|_{\sigma_n} \le \omega_{m-1} R^m / m$ we deduce

$$\int_{B^n_R} |\Delta_n u_n|^2 \, \mathrm{d}x_n \le \frac{\omega_{m-1} R^m}{|B^n_R|_{\sigma_n}} \int_{B^n_R} |D^2_n u_n|^2_{\sigma_n} \, \mathrm{d}x_n \to 0 \tag{62}$$

as $n \to \infty$, and therefore

$$\begin{aligned} \int_{B^n_R} \varphi \Delta_n u_n \, \mathrm{d}x_n &\le \left(\int_{B^n_R} \varphi^2 \, \mathrm{d}x_n \right)^{\frac{1}{2}} \left(\int_{B^n_R} |\Delta_n u_n|^2 \, \mathrm{d}x_n \right)^{\frac{1}{2}} \\ &\le \max |\varphi| \left[\frac{\omega_{m-1} R^m}{m} \right]^{\frac{1}{2}} \left(\int_{B^n_R} |\Delta_n u_n|^2 \, \mathrm{d}x_n \right)^{\frac{1}{2}} \to 0. \end{aligned} \tag{63}$$

By (62) and (63), $\Delta_n u_n \to 0$ strongly in L^2. Combining $u_n \to u_\infty$ strongly in L^2 with

$$\sup_n \left(\int_{B^n_R} [|u_n|^2 + |D_n u_n|^2_{\sigma_n} + (\Delta_n u_n)^2] \, \mathrm{d}x_n \right) < \infty,$$

we infer by [Ambrosio and Honda 2018, Theorem 4.4] that

(i) $u_\infty \in \mathcal{D}(\Delta, B^\infty_R)$, the domain of the Laplacian on B^∞_R,

(ii) $\Delta_n u_n \to \Delta u_\infty$ on B^n_R weakly in L^2, so in particular $\Delta u_\infty = 0$,

(iii) $|D_n u_n|^2_{\sigma_n} \to |D_\infty u_\infty|^2_\infty$ in L^1-strongly in B^n_r for each $r < R$.

In particular, setting $P \doteq \sup_M |Du|^2$, from (59) and (60) we get

$$\lim_{n\to\infty} \int_{B_R^n} \big||D_n u_n|^2_{\sigma_n} - P\big| \,\mathrm{d}x_n \le \frac{\omega_{m-1} R^m}{|B_R^n|_{\sigma_n}} \int_{B_R^n} (P - |D_n u_n|^2_{\sigma_n}) \,\mathrm{d}x_n = 0.$$

Using (iii) and [Bruè et al. 2023, Proposition 1.27(i)] (see also [Ambrosio and Honda 2017]), we therefore deduce $|D_n u_n|^2_{\sigma_n} - P \to |D_\infty u_\infty|^2_\infty - P$ strongly in L^1 on B_r^∞ for each $r < R$, and thus

$$0 = \lim_{n\to\infty} \int_{B_r^n} \big||D_n u_n|^2_{\sigma_n} - P\big| \,\mathrm{d}x_n = \int_{B_r^\infty} \big||D_\infty u_\infty|^2_\infty - P\big| \,\mathrm{d}\mathfrak{m}_\infty.$$

Concluding, u_∞ solves

$$\Delta u_\infty = 0, \quad |D_\infty u_\infty|^2 = P \neq 0$$

on the RCD$(0, m)$ space $(M_\infty, \mathrm{d}_\infty, \mathfrak{m}_\infty, x_\infty)$. Bochner inequality (see [Honda 2015, Theorem 1.4]) guarantees that $|D^2 u_\infty| \equiv 0$ on M_∞. One concludes that $M_\infty = N \times \mathbb{R}$ by using [Antonelli et al. 2019, Lemma 1.21].

7. Proof of Theorem 11

If M is parabolic, clearly the result follows from Theorem 6. If M is nonparabolic, the argument goes as in [Ding et al. 2016, Theorem 3.6], so we only sketch the main steps. In our assumptions, by Corollary 18, $|Du| \in L^\infty(M)$; hence $L = W\Delta_g$ is uniformly elliptic. The Harnack inequality in [Saloff-Coste 1992] together with (5) imply that $|u(x)| = o(r(x))$ as x diverges. By a standard cutoff argument using $Lu = 0$, the next Caccioppoli inequality holds: for each $\varphi \in \mathrm{Lip}_c(M)$

$$\int_M \varphi^2 |Du|^2 \,\mathrm{d}x \le 4\alpha^2 \int_M u^2 |D\varphi|^2 \,\mathrm{d}x. \tag{64}$$

In particular, having fixed $\varepsilon > 0$, by condition $u = o(r)$ we can also fix $R_0 = R_0(\varepsilon) > 0$ such that for every $R \ge R_0$ we have $u^2 \le \varepsilon R^2$ on B_{2R}. Considering the Lipschitz cutoff function φ which is 1 on B_R, 0 outside of B_{2R} and satisfies $|D\varphi| \le 1/R$, we get

$$\int_{B_R} |Du|^2 \,\mathrm{d}x \le \frac{4\alpha^2}{R^2} \int_{B_{2R}\setminus B_R} u^2 \,\mathrm{d}x \le \varepsilon |B_{2R}| \le C\varepsilon |B_R|$$

for every $R \ge R_0$, where we used the doubling property on M coming from condition $\mathrm{Ric} \ge 0$. From (52) we finally infer

$$\sup_M |Du|^2 = \lim_{R\to\infty} \frac{1}{|B_R|} \int_{B_R} |Du|^2 \,\mathrm{d}x \le C\varepsilon,$$

and the thesis follows by letting $\varepsilon \to 0$.

8. Proof of Corollary 10

By Theorem 6, $|Du| \in L^\infty(M)$ and any tangent cone at infinity of M splits off a line. It is a general fact that, if $\mathrm{Sec} \ge 0$, a tangent cone splits if and only if M itself splits. A proof of this result can be found in

[Antonelli et al. 2022, Theorem 4.6]. Therefore, it remains to prove that u only depends on the coordinate of a split line. Write $M = N^{m-1} \times \mathbb{R}$ with coordinates (y_1, s_1), for some complete manifold N^{m-1} with $\mathrm{Sec} \geq 0$, and consider the function $v_1 = \sigma(Du, \partial_{s_1})$, which by (15) satisfies $Lv_1 = 0$ on M, where we set

$$L\phi \doteq W^{-1}\mathscr{L}_W\phi = \mathrm{div}(W^{-1}g^{ij}\phi_i\partial_{x_j}).$$

Our gradient estimate guarantees that v_1 is bounded and that L is uniformly elliptic on M, and therefore, by [Saloff-Coste 1992, Theorem 7.4] we deduce that v_1 is constant on M. Hence,

$$u(y_1, s_1) = a_1 s_1 + b_1 + u_2(y_1)\sqrt{1 + a_1^2}$$

for some smooth function $u_2 : N^{m-1} \to \mathbb{R}$ and some $a_1, b_1 \in \mathbb{R}$. One easily checks that u_2 solves (MSE) on N^{m-1}. Since u_2 has at most linear growth on one side, and N^{m-1} has nonnegative sectional curvature, by the first part of the proof we deduce that either u_2 is constant or that $N^{m-1} = N^{m-2} \times \mathbb{R}$ and $u_2(y_2, s_2) = a_2 s_2 + b_2 + u_3(y_2)\sqrt{1 + a_2^2}$. Iterating, we can write $M = N^{m-k} \times \mathbb{R}^k$ for some $k \in \{1, \dots, m-2\}$ and for some complete manifold N^{m-k} with $\mathrm{Sec} \geq 0$, and

$$u(z, (s_1, \dots, s_k)) = \sum_{j=1}^{k} a_j s_j + b + u_{k+1}(z)\sqrt{1 + a_k^2}$$

for some $a_i, b \in \mathbb{R}$ and $u_{k+1} : N^{m-k} \to \mathbb{R}$. Indeed, we can continue the iteration procedure up until either u_{k+1} is constant, or $k = m-2$ and u_{m-1} is nonconstant. In the latter case, observe that N^2 is a complete surface with $\mathrm{Sec} \geq 0$; hence N^2 is parabolic. Being u_{m-1} nonconstant, both N^2 and u_{m-1} split as indicated in Theorem 6(i). Summarizing, in each case we can conclude that $M = N^{m-k} \times \mathbb{R}^k$ for some $k \in \{1, \dots, m-1\}$, and that

$$u(z, (s_1, \dots, s_k)) = \sum_{j=1}^{k} a_j s_j + b \tag{65}$$

for some $b \in \mathbb{R}$, as required. It is therefore sufficient to consider the splitting $\mathbb{R}^k = \mathbb{R}^{k-1} \times \mathbb{R}$ along a line in direction $(a_1, \dots, a_k)$ to get the desired splitting $M = N \times \mathbb{R}$ of M in such a way that $u(y, s) = as + b$.

9. Proof of Proposition 9

The following example is essentially that in [Kasue and Washio 1990, p. 913]. Let $m \geq 4$. We consider a manifold (P^{m-2}, h) and smooth functions $f, \eta \in C^\infty(\mathbb{R}^+)$ to be chosen later, and define the following metric on $M \doteq \mathbb{R} \times \mathbb{R}^+ \times P$:

$$\sigma = f(r)^2\,\mathrm{d}t^2 + \mathrm{d}r^2 + \eta(r)^2 h.$$

To compute the curvatures of M, we use the index agreement $1 \leq a, b, c, l \leq m$, $3 \leq \alpha, \beta, \gamma, \delta \leq m$. Let $\{\theta^\alpha\}$ be a local orthonormal coframe on P, with associated connection forms ω^α_β obeying the structure equations

$$\begin{cases} \mathrm{d}\theta^\alpha = -\omega^\alpha_\beta \wedge \theta^\beta, \\ \omega^\alpha_\beta = -\omega^\beta_\alpha \end{cases}$$

and related curvature forms $\Theta^\alpha_\beta = \mathrm{d}\omega^\alpha_\beta + \omega^\alpha_\gamma \wedge \omega^\gamma_\beta$. Then, a local orthonormal coframe $\{\bar\theta^a\}$ on M is given by

$$\bar\theta^1 = f\,\mathrm{d}t, \quad \bar\theta^2 = \mathrm{d}r, \quad \bar\theta^\alpha = \eta\theta^\alpha,$$

where, as usual, pull-backs to M via the canonical projections onto $\mathbb{R}$, $\mathbb{R}^+$ and P are implicit. Differentiating, one checks that the forms

$$\bar\omega^\alpha_1 = 0, \quad \bar\omega^\alpha_2 = \frac{\eta'}{\eta}\bar\theta^\alpha, \quad \bar\omega^\alpha_\beta = \omega^\alpha_\beta, \quad \bar\omega^2_1 = -\frac{f'}{f}\bar\theta^1$$

satisfy the structure equations on M for the coframe $\{\bar\theta^a\}$; hence they are the connection forms of $\{\bar\theta^a\}$. The associated curvature forms $\bar\Theta^a_b = \mathrm{d}\bar\omega^a_b + \bar\omega^a_c \wedge \bar\omega^c_b$ are therefore

$$\begin{aligned} \bar\Theta^\alpha_1 &= -\frac{\eta' f'}{\eta f}\bar\theta^\alpha \wedge \bar\theta^1, & \bar\Theta^\alpha_2 &= \frac{\eta''}{\eta}\bar\theta^2 \wedge \bar\theta^\alpha, \\ \bar\Theta^\alpha_\beta &= \Theta^\alpha_\beta - \left(\frac{\eta'}{\eta}\right)^2 \bar\theta^\alpha \wedge \bar\theta^\beta, & \bar\Theta^2_1 &= -\frac{f''}{f}\bar\theta^2 \wedge \bar\theta^1. \end{aligned} \tag{66}$$

The components $R^\alpha_{\beta\gamma\delta}$ and $\bar R^a_{bcl}$ of the (3, 1) curvature tensors of, respectively, P and M, are given by the identities

$$\Theta^\alpha_\beta = \tfrac{1}{2} R^\alpha_{\beta\gamma\delta}\theta^\gamma \wedge \theta^\delta, \quad \bar\Theta^a_b = \tfrac{1}{2}\bar R^a_{bcl}\bar\theta^c \wedge \bar\theta^l,$$

and thus, from (66), we deduce

$$\begin{aligned} 0 = \bar R^2_{12\alpha} = \bar R^2_{1\alpha 1} &= \bar R^2_{1\alpha\beta} = \bar R^\alpha_{12\beta} = \bar R^\alpha_{1\gamma\delta} = \bar R^\alpha_{2\gamma\delta}, \\ \bar R^2_{121} = -\frac{f''}{f}, \quad \bar R^\alpha_{1\beta 1} &= -\frac{\eta' f'}{\eta f}\delta^\alpha_\beta, \quad \bar R^\alpha_{2\beta 2} = -\frac{\eta''}{\eta}\delta^\alpha_\beta, \\ \bar R^\alpha_{\beta\gamma\delta} &= \frac{1}{\eta^2}R^\alpha_{\beta\gamma\delta} - \left(\frac{\eta'}{\eta}\right)^2[\delta^\alpha_\gamma\delta_{\beta\delta} - \delta^\alpha_\delta\delta_{\beta\gamma}]. \end{aligned} \tag{67}$$

Assume that (P, h) is the round sphere with curvature 1, and let $\{e_\alpha\}$ and $\{\bar e_a\}$ be, respectively, the dual frames of $\{\theta^\alpha\}$ and $\{\bar\theta^a\}$. From (67) we deduce that the curvature operator is diagonalized by the simple planes $\{\bar e_a \wedge \bar e_b\}$, so for $m \geq 4$ we get

$$|\overline{\mathrm{Sec}}(\pi)| \leq \max\left\{\left|\frac{f''}{f}\right|, \left|\frac{\eta' f'}{\eta f}\right|, \left|\frac{1-(\eta')^2}{\eta^2}\right|, \left|\frac{\eta''}{\eta}\right|\right\}.$$

In [Kasue and Washio 1990], the authors chose the following functions f, η: given $\alpha, \beta \in (0, 1)$ such that $m - 1 - \beta > 2 + \alpha$, let $0 < \zeta_1, \zeta_2 \in C^\infty(\mathbb{R}^+)$ satisfy

$$\zeta_1(t) = \begin{cases} t & \text{if } t \in (0, 1], \\ t^{-1-\alpha} & \text{if } t \in [2, \infty), \end{cases} \qquad \zeta_2(t) = \int_t^\infty \zeta_1(s)\,\mathrm{d}s.$$

Then, for $b, c \in \mathbb{R}^+$ they defined

$$\eta(r) = \frac{1}{2}r + \frac{1}{2\zeta_2(0)}\int_0^r \zeta_2(s)\,\mathrm{d}s, \qquad f(r) = (b + r^2)^{\frac{\beta+3-m}{2}} + c.$$

Note that with such a choice the metric extends in a C^2 way at $r=0$, giving rise to a complete manifold. Since the curvature operator is diagonalized by $\{\bar{e}_a \wedge \bar{e}_b\}$,

$$\begin{aligned}\overline{\mathrm{Ric}}^{(2)} \ge \min\Big\{ &-\frac{f''}{f}+\frac{1-(\eta')^2}{\eta^2},\ -\frac{f''}{f}-\frac{\eta' f'}{\eta f},\ -\frac{f''}{f}-\frac{\eta''}{\eta},\\ &\frac{1-(\eta')^2}{\eta^2}-\frac{\eta' f'}{\eta f},\ \frac{1-(\eta')^2}{\eta^2}-\frac{\eta''}{\eta},\ -\frac{\eta''}{\eta}-\frac{\eta' f'}{\eta f},\ -2\frac{\eta' f'}{\eta f},\\ &2\frac{1-(\eta')^2}{\eta^2},\ -2\frac{\eta''}{\eta}\Big\}. \end{aligned} \tag{68}$$

By the expression of η, f, the four terms in the second line of (68) are positive, and it is easy to see that, when b, c are large enough, the three terms in the first line are positive as well. The two terms in the third line are positive except at $r=0$. Whence, $\overline{\mathrm{Ric}}^{(2)} \ge 0$, and moreover $|\overline{\mathrm{Sec}}| \le \bar{\kappa}^2$ holds for a suitable $\bar{\kappa} > 0$. Moreover, from the fact that $\overline{\mathrm{Ric}}$ is diagonal in the basis $\{\bar{e}_a\}$ with

$$\overline{\mathrm{Ric}}_{11} = -\frac{f''}{f}-(m-3)\frac{\eta' f'}{\eta f}, \quad \overline{\mathrm{Ric}}_{22} = -\frac{f''}{f}-(m-3)\frac{\eta''}{\eta}, \quad \overline{\mathrm{Ric}}_{\alpha\beta} = \left[-\frac{\eta' f'}{\eta f}-\frac{\eta''}{\eta}+(m-3)\frac{1-(\eta')^2}{\eta^2}\right]\delta_{\alpha\beta},$$

we deduce that $\mathrm{Ric} > 0$ if b, c are chosen large enough. To construct linearly growing minimal graphs, consider a function $u : M \to \mathbb{R}$ of the coordinate t alone. It follows that $\mathrm{d}u = u_a \bar{\theta}^a$ with $u_1 = (\partial_t u)/f$ and $u_a = 0$ for $a \ge 2$. The components of the Hessian $D^2 u$ obey the relation

$$u_{ab}\bar{\theta}^b = \mathrm{d}u_a - u_c \bar{\omega}_a^c,$$

and from the expression of $\bar{\omega}_a^c$ we get

$$u_{11} = \frac{\partial_t^2 u}{f^2}, \quad u_{21} = -\frac{f'}{f^2}\partial_t u, \quad u_{1\alpha} = u_{22} = u_{2\alpha} = u_{\alpha\beta} = 0.$$

In particular, setting $W = \sqrt{1+|Du|^2} = \sqrt{1+u_1^2}$,

$$\mathrm{div}\left(\frac{Du}{\sqrt{1+|Du|^2}}\right) = \frac{\Delta u}{W} - \frac{D^2 u(Du, Du)}{W^3} = \frac{\partial_t^2 u}{f^2 W} - \frac{(\partial_t^2 u)u_1^2}{f^2 W^3} = \frac{\partial_t^2 u}{f^2 W^3}.$$

It follows that any affine function $u(t) = at + b$ gives rise to a minimal graph. Furthermore, $|Du| = a/f$ is bounded on M since f is bounded below by a positive constant, thus u has at most linear growth.

Appendix

Let M be a connected, complete Riemannian manifold with nonnegative Ricci curvature, $\dim M = m$, and let A, L, H_L be as in Section 5. In this Appendix, we discuss the two-sided bound in (40) for H_L. While the upper bound is shown in [Saloff-Coste 1992], the argument for the lower bound is merely indicated with no proof. The approach relies on the following parabolic Harnack inequality in [Saloff-Coste 1992, Corollary 5.4]: given $p \in M$, $R > 0$, $T > 0$ and $\delta \in (0, 1)$, if u is a positive solution to $\partial_t u = Lu$ on

$B_R(p)\times(0,T)$, then

$$\log\left(\frac{u(t,y)}{u(s,x)}\right)\le C\left(\frac{\operatorname{dist}(x,y)^2}{s-t}+\left(\frac{1}{R^2}+\frac{1}{t}\right)(s-t)+1\right) \tag{69}$$

for every $x, y\in B_{\delta R}(p)$ and $0<t<s<T$, with $C=C(m,\delta,\alpha)>0$. A note of warning: in [Saloff-Coste 1992, Corollary 5.4], the final $+1$ in brackets in (69) is missing. However, necessity of this correction becomes apparent by direct inspection of Moser's original proof [1964, pages 110–112] in Euclidean setting (the analogue of (69) is [Moser 1964, Formula (1.5)]). For the reader's convenience, we give a proof that the lower bound in (40) follows from the upper one coupled with (69), along the lines of the argument developed by Aronson and Serrin [1967] in the Euclidean case. A few observations are in order.

First, in view of Lemma 21 the upper bound in (40) implies

$$H_L(x,y,t)\le\frac{C_3'}{|B_{\sqrt{t}}(x)|}\exp\left(-C_4'\frac{\operatorname{dist}(x,y)^2}{t}\right)\quad\text{for all } x,y\in M,\ t>0, \tag{70}$$

with $C_3', C_4'>0$ depending only on m and α (the ellipticity constant of A). Secondly, the differential Harnack inequality (69) applied to $u=H_L(x,\cdot,\cdot)$ yields

$$H_L(x,y_1,t_1)\le H_L(x,y_2,t_2)\exp\left(C\frac{\operatorname{dist}(y_1,y_2)^2}{t_2-t_1}+C\frac{t_2}{t_1}\right) \tag{71}$$

for every $y_1, y_2\in M$ and $0<t_1<t_2<\infty$, with $C=C(m,\alpha)>0$. Lastly, note that if we have the validity of a lower bound of the form

$$H_L(x,y,t)\ge\frac{C_1'}{|B_{\sqrt{t}}(x)|}\exp\left(-C_2'\frac{\operatorname{dist}(x,y)^2}{t}\right)\quad\text{for all } x,y\in M,\ t>0, \tag{72}$$

with $C_1', C_2'>0$ depending only on m and α, then, again by Lemma 21, a lower bound as that in (40) holds for suitable constants $C_1\in(0,C_1')$ and $C_2>C_2'$ depending only on C_1', C_2' and m. Hence, we limit ourselves to the proof that (72) follows from (70) and (71) under the assumption $\operatorname{Ric}\ge 0$.

Fix a constant $c_0>2$ such that

$$\gamma\doteq mC_3'\int_{\sqrt{c_0}}^{+\infty}s^{m-1}e^{-C_4's^2}\,ds<1. \tag{73}$$

Let $(x,y,t)\in M\times M\times\mathbb{R}^+$ be given. By (71) we have

$$H_L\left(x,x,\frac{t}{2}\right)\le H_L(x,y,t)\exp\left(2C\frac{\operatorname{dist}(x,y)^2}{t}+2C\right) \tag{74}$$

and also

$$H_L\left(x,z,\frac{t}{c_0}\right)\le H_L\left(x,x,\frac{t}{2}\right)\exp\left(c_0^*C\frac{\operatorname{dist}(x,z)^2}{t}+\frac{c_0}{2}C\right)$$

for every $z\in M$, with

$$c_0^*=\frac{2c_0}{c_0-2}=\left(\frac{1}{2}-\frac{1}{c_0}\right)^{-1}.$$

Integrating on $B_{\sqrt{t}}(x)$ we get

$$\int_{B_{\sqrt{t}}(x)} H_L\Big(x,z,\frac{t}{c_0}\Big)\,\mathrm{d}z \le e^{(c_0^*+\frac{1}{2}c_0)C}|B_{\sqrt{t}}(x)|H_L\Big(x,x,\frac{t}{2}\Big). \tag{75}$$

Putting together (74) and (75) we obtain

$$H_L(x,y,t) \ge \frac{e^{-(2+\frac{1}{2}c_0+c_0^*)C}}{|B_{\sqrt{t}}(x)|}\exp\Big(-2C\frac{\mathrm{dist}(x,y)^2}{t}\Big)\int_{B_{\sqrt{t}}(x)} H_L\Big(x,z,\frac{t}{c_0}\Big)\,\mathrm{d}z. \tag{76}$$

From the upper bound (70) and the coarea formula we have

$$\begin{aligned}\int_{M\setminus B_{\sqrt{t}}(x)} H_L\Big(x,z,\frac{t}{c_0}\Big)\,\mathrm{d}z &\le C_3'\int_{\sqrt{t}}^{\infty}\frac{|\partial B_r(x)|}{|B_{\sqrt{t/c_0}}(x)|}\exp\Big(-c_0C_4'\frac{r^2}{t}\Big)\,\mathrm{d}r\\ &= C_3'\int_{\sqrt{c_0}}^{\infty}\frac{\sqrt{t/c_0}\,|\partial B_{s\sqrt{t/c_0}}(x)|}{|B_{\sqrt{t/c_0}}(x)|}e^{-C_4's^2}\,\mathrm{d}s,\end{aligned}$$

where we have changed variable $s = r\sqrt{c_0/t}$. Since $\mathrm{Ric} \ge 0$ we have

$$\frac{\sqrt{t/c_0}\,|\partial B_{s\sqrt{t/c_0}}(x)|}{|B_{\sqrt{t/c_0}}(x)|} \le s^{m-1}\frac{\sqrt{t/c_0}\,|\partial B_{\sqrt{t/c_0}}(x)|}{|B_{\sqrt{t/c_0}}(x)|} \le ms^{m-1},$$

where the first inequality follows by the Bishop–Gromov theorem and the second from the inequality $R|\partial B_R(x)| \le m|B_R(x)|$, holding for every $R > 0$ and for any base point x on a Riemannian manifold with $\mathrm{Ric} \ge 0$; see for instance [Li 1986, Formula (19)]. Substituting in the above estimate and recalling (73) and Lemma 22 we get

$$\int_{B_{\sqrt{t}}(x)} H_L(x,z,t/c_0)\,\mathrm{d}z = 1-\int_{M\setminus B_{\sqrt{t}}(x)} H_L(x,z,t/c_0)\,\mathrm{d}z \ge 1-\gamma > 0$$

and from (76) we obtain

$$H_L(x,y,t) \ge \frac{C_1'}{|B_{\sqrt{t}}(x)|}\exp\Big(-2C\frac{\mathrm{dist}(x,y)^2}{t}\Big),$$

where $C_1' = (1-\gamma)e^{-(2+c_0/2+c_0^*)C} > 0$ only depends on m and α. This proves (72).

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NONLINEAR PERIODIC WAVES ON THE EINSTEIN CYLINDER

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Motivated by the study of small amplitude nonlinear waves in the anti-de Sitter spacetime and in particular the conjectured existence of periodic in time solutions to the Einstein equations, we construct families of arbitrary small time-periodic solutions to the conformal cubic wave equation and the spherically symmetric Yang–Mills equations on the Einstein cylinder $\mathbb{R}\times\mathbb{S}^3$. For the conformal cubic wave equation, we consider both spherically symmetric solutions and complex-valued aspherical solutions with an ansatz relying on the Hopf fibration of the 3-sphere. In all three cases, the equations reduce to 1+1 semilinear wave equations.

Our proof relies on a theorem of Bambusi–Paleari for which the main assumption is the existence of a seed solution, given by a nondegenerate zero of a nonlinear operator associated with the resonant system. For the problems that we consider, such seed solutions are simply given by the mode solutions of the linearized equations. Provided that the Fourier coefficients of the systems can be computed, the nondegeneracy conditions then amount to solving infinite dimensional linear systems. Since the eigenfunctions for all three cases studied are given by Jacobi polynomials, we derive the different Fourier and resonant systems using linearization and connection formulas as well as integral transformation of Jacobi polynomials.

In the Yang–Mills case, the original version of the theorem of Bambusi–Paleari is not applicable because the nonlinearity of smallest degree is nonresonant. The resonant terms are then provided by the next order nonlinear terms with an extra correction due to backreaction terms of the smallest degree of nonlinearity, and we prove an analogous theorem in this setting.

1. Introduction

1A. ***Stability/instability of the anti-de Sitter spacetime.*** The anti-de Sitter (AdS) spacetime is the maximally symmetric solution to the vacuum Einstein equations with a negative cosmological constant:

$$\mathrm{Ric}(g) = -\Lambda g, \qquad \Lambda < 0. \tag{1-1}$$

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Given $\Lambda < 0$, this is the simplest, or trivial, solution to (1-1), in the sense that the Minkowski or de Sitter spacetimes are the trivial solutions to the vacuum Einstein equation with $\Lambda = 0$ or $\Lambda > 0$. While the stability properties of the Minkowski or de Sitter spacetimes are now well understood [Christodoulou and Klainerman 1993; Friedrich 1986], the study of perturbations of AdS spacetime is still an active subject of research. One important aspect is that the AdS spacetime, or more generally spacetimes which are asymptotically AdS, are not globally hyperbolic. Hence, any evolution problem for these solutions is only well-posed after boundary conditions are imposed at the conformal boundary. Two naturally opposite classes of boundary conditions are the fully reflective and dissipative boundary conditions. In the reflective case, we expect — as originally conjectured by Dafermos and Holzegel [2006] and independently by Anderson [2006] — that the AdS spacetime is unstable. Strong evidence for this instability was first presented by Bizoń and Rostworowski [2011], who pioneered the study of the spherically symmetric Einstein–Klein–Gordon system using numerical and Fourier based analysis and proposed weak turbulence as the nonlinear source of instability. A proof of instability for the spherically symmetric Einstein–Vlasov[1] system was obtained in the work of Moschidis [2020; 2023] and is based on a physical space mechanism. In the dissipative case, one has strong decay of solutions for the linearized Einstein equations [Holzegel et al. 2020], and this should lead to stability even at the nonlinear level.

1B. ***The time-periodic solutions of Rostworowski–Maliborski.*** In the reflective case, parallel to the study of the instability conjecture, an interesting class of solutions was introduced by Rostworowski and Maliborski [2013], who constructed perturbatively and numerically small data, time-periodic solutions of the spherically symmetric Einstein-scalar field system. They furthermore suggested, based on their numerical analysis, that these solutions should enjoy stronger stability properties than the original AdS spacetime. The present paper is directly motivated by this work. We prove the existence of arbitrary small time-periodic solutions for various toy models, which mimic certain properties of nonlinear waves in the AdS spacetime.

1C. ***The conformal wave and the Yang–Mills equations.*** More precisely, we study the conformal wave and the Yang–Mills equations on the Einstein cylinder $\mathbb{R} \times \mathbb{S}^3$. The AdS spacetime is conformal to half of the Einstein cylinder, so that solutions to the conformal wave and the Yang–Mill equations on the AdS spacetime can be mapped to solutions on the entire Einstein cylinder with a certain reflection symmetry at the equator. The conformal cubic wave equation on the Einstein cylinder can be written as

$$-\partial_t^2 \phi(t,\omega) + \Delta_{\mathbb{S}^3}\phi(t,\omega) - \phi(t,\omega) = |\phi(t,\omega)|^2 \phi(t,\omega) \tag{1-2}$$

for a scalar field $\phi : \mathbb{R} \times \mathbb{S}^3 \to \mathbb{C}$ with $\phi = \phi(t,\omega)$. We will consider perturbations around the static solution $\phi_0 = 0$ and, for simplicity, with zero initial velocity.

In the spherically symmetric case, the initial value problem for (1-2) reduces to

$$\begin{cases} (\partial_t^2 + L)u = f(u), & (t,x) \in \mathbb{R} \times (0,\pi), \\ (u(0,x), \partial_t u(0,x)) = (u_0(x), 0), & x \in (0,\pi), \end{cases} \tag{1-3}$$

[1] Moschidis [2021] has further announced similar results for the spherically symmetric Einstein-scalar-field system.

for a scalar field $u : \mathbb{R} \times (0, \pi) \to \mathbb{R}$ with $u = u(t, x)$ and

$$Lu := -\Delta^{ss}_{\mathbb{S}^3} u + u, \quad -\Delta^{ss}_{\mathbb{S}^3} u = -\frac{1}{\sin^2(x)} \partial_x (\sin^2(x) \partial_x u), \quad f(u) = -u^3, \tag{1-4}$$

where $\Delta^{ss}_{\mathbb{S}^3}$ stands for the spherically symmetric Laplace–Beltrami operator on $\mathbb{S}^3$.

When the spherical symmetry assumption is removed [Ben Achour et al. 2016; Evnin 2021], we use an ansatz based on *Hopf coordinates*[2] $(\eta, \xi_1, \xi_2) \in \left[0, \frac{\pi}{2}\right] \times [0, 2\pi) \times [0, 2\pi)$ rather than the standard spherical coordinates. The Laplace–Beltrami operator on $\mathbb{S}^3$ in these coordinates reads as

$$\Delta^{\mathbb{S}^3}_{(\eta,\xi_1,\xi_2)} \chi = \partial^2_\eta \chi + \left(\frac{\cos \eta}{\sin \eta} - \frac{\sin \eta}{\cos \eta} \right) \partial_\eta \chi + \frac{1}{\sin^2 \eta} \partial^2_{\xi_1} \chi + \frac{1}{\cos^2 \eta} \partial^2_{\xi_2} \chi .$$

While in principle the Fourier expansion with respect to ξ_1 and ξ_2 of a solution $\chi(t, \eta, \xi_1, \xi_2)$ to (1-2) may include all possible admissible frequencies, we will pick a fixed pair of frequencies (μ_1, μ_2) and force the Fourier expansion to excite only this particular pair by implementing the ansatz

$$\chi(t, \eta, \xi_1, \xi_2) = u(t, \eta) e^{i\mu_1 \xi_1} e^{i\mu_2 \xi_2}. \tag{1-5}$$

This leads us to consider the initial value problem

$$\begin{cases} (\partial_t^2 + \mathsf{L}^{(\mu_1,\mu_2)}) u = \mathsf{f}(u), & (t, \eta) \in \mathbb{R} \times \left(0, \frac{\pi}{2}\right), \\ (u(0, \eta), \partial_t u(0, \eta)) = (u_0(\eta), 0), & \eta \in \left(0, \frac{\pi}{2}\right), \end{cases} \tag{1-6}$$

for a scalar field $u : \mathbb{R} \times \left(0, \frac{\pi}{2}\right) \to \mathbb{R}$ with $u = u(t, \eta)$ and

$$\mathsf{L}^{(\mu_1,\mu_2)} u = -\partial^2_\eta u - \left(\frac{\cos \eta}{\sin \eta} - \frac{\sin \eta}{\cos \eta} \right) \partial_\eta u + \left(\frac{\mu_1^2}{\sin^2 \eta} + \frac{\mu_2^2}{\cos^2 \eta} + 1 \right) u, \quad \mathsf{f}(u) = -u^3. \tag{1-7}$$

Finally, we consider the spherically symmetric (equivariant) Yang–Mills equation for the SU(2) connection A on the Einstein cylinder $\mathbb{R} \times \mathbb{S}^3$ endowed with the metric

$$g(t, \omega) = -dt^2 + dx^2 + \sin^2(x)\, d\omega^2, \tag{1-8}$$

where $d\omega^2$ stands for the standard round metric on the 2-sphere. The connection $A_\mu = A^\nu_\mu \tau_\nu$ is a 1-form that takes values in the Lie algebra su(2). Here, τ_α stand for the generators of su(2) that satisfy $[\tau_a, \tau_b] = i\epsilon_{abc}\tau_c$. Furthermore, the curvature F is a $(2, 0)$-tensor defined by $F_{\mu\nu} = \nabla_\mu A_\nu - \nabla_\nu A_\mu + [A_\mu, A_\nu]$. The Euler–Lagrange equations associated to the action

$$\int_{\mathbb{R}\times\mathbb{S}^3} \operatorname{tr}(F_{\mu\nu} F^{\mu\nu}) \sqrt{-\det(g)}$$

are provided by the Yang–Mills equation

$$\nabla_\mu F^{\mu\nu} + [A_\mu, F^{\mu\nu}] = 0. \tag{1-9}$$

Following [Bizoń 1993; 2014; Bizoń and Mach 2017], we assume the spherically symmetric purely magnetic ansatz

$$A = \phi(t, x)\eta + \tau_3 \cos(\vartheta)\, d\varphi, \quad \eta = \tau_1\, d\vartheta + \tau_2 \sin(\vartheta)\, d\varphi,$$

[2]We would like to thank Oleg Evnin who suggested the Hopf coordinate ansatz.

which yields

$$F = \partial_t \phi(t,x)\, dt \wedge \eta + \partial_x \phi(t,x)\, dx \wedge \eta - (1-\phi^2(t,x))\tau_3\, d\vartheta \wedge \sin(\vartheta)\, d\varphi.$$

In this case, a straightforward computation shows that (1-9) reduce to

$$-\partial_t^2 \phi(t,x) + \partial_x^2 \phi(t,x) + \frac{\phi(t,x)}{\sin^2(x)} = \frac{\phi^3(t,x)}{\sin^2(x)} \tag{1-10}$$

for a scalar field $\phi : \mathbb{R} \times (0,\pi) \to \mathbb{R}$ with $\phi = \phi(t,x)$. We will study perturbations of the static solution $\phi_0 = 1$ to the equation above [Bizoń 2014]. Writing $\phi(t,x) = 1 + \sin^2(x) u(t,x)$, we are led to the initial value problem

$$\begin{cases} (\partial_t^2 + \mathfrak{L})u = \mathfrak{f}(x,u), & (t,x) \in \mathbb{R} \times (0,\pi), \\ (u(0,x), \partial_t u(0,x)) = (u_0(x), 0), & x \in (0,\pi), \end{cases} \tag{1-11}$$

where

$$\mathfrak{L}u = -\frac{1}{\sin^4 x}\partial_x(\sin^4 x\, \partial_x u) + 4u, \quad \mathfrak{f}(x,u) = -3u^2 - \sin^2(x) u^3. \tag{1-12}$$

1D. ***Connection of the models to the fixed AdS spacetime.*** In the following lines, we discuss the connection of the models (1-3), (1-6) and (1-11), and we consider two dynamical problems related to the AdS spacetime. The Einstein static universe is the cylinder $\mathbb{R} \times \mathbb{S}^3$ endowed with the metric given by (1-8), and the AdS spacetime is conformal to only the part of the entire Einstein cylinder which is given by $\mathbb{R} \times \mathbb{S}^3_+$, where $\mathbb{S}^3_+$ denotes the upper hemisphere of $\mathbb{S}^3$. Since both the cubic conformal wave equation and the Yang Mills equation are conformally invariant, this implies that solutions to the cubic conformal wave and Yang–Mills equations can be mapped to solutions of the same equations on $\mathbb{R} \times \mathbb{S}^3_+$. Depending on the choice of boundary conditions at the conformal infinity of the AdS spacetime, these solutions can then be extended on the whole of the Einstein cylinder via a reflection symmetry; see for example [Bizoń et al. 2017, Remark 1].

The models we consider here also preserve several key features of the Einstein–Klein–Gordon system in spherical symmetry, and for which we have reliable numerical evidence for the existence of time-periodic solutions due to [Maliborski and Rostworowski 2013] (see Section 1B). Indeed, in all cases, the spectrum is completely resonant and the eigenfunctions to the linearized operators are weighted Jacobi polynomials. As a consequence, the derivation and analysis of the Fourier and resonant systems share many properties. In addition, although quasilinear, the Einstein–Klein–Gordon system in spherical symmetry has a cubic leading-order nonlinearity as in the models we considered here. More importantly, the existence of the time-periodic solutions we construct here depends on the so-called nondegeneracy condition (Section 7). This is a system of infinitely many nonlinear conditions for oscillatory integrals that quantify the mode couplings, relying on the analysis of the underlying Fourier system. In this paper, we develop a rigorous and delicate analysis for the Fourier coefficients (Section 5) by establishing closed formulas, as well as rigorous asymptotic analysis in the case where the closed formulas for these integrals are too complicated to handle. Besides their strong numerical evidence, Rostworowski and Maliborski [2013] also suggest that there should be an analogous nondegeneracy condition for the quasilinear Einstein–Klein–Gordon system in spherical symmetry. The computation and the analysis of the Fourier system for the Einstein–Klein–Gordon system is a challenge in itself, see for example [Chatzikaleas 2024; Craps et al. 2014; 2015a; 2015b; Evnin and Jai-akson 2016], and we believe that the type of analysis for the Fourier coefficients developed here should find applications there as well.

1E. ***Main results and general strategy.*** In the following, we use the abbreviations

- CW: conformal cubic wave equation in spherical symmetry, that is (1-3)–(1-4),
- CH: conformal cubic wave equation out of spherical symmetry in Hopf coordinates according to the ansatz (1-5), that is (1-6)–(1-7),
- YM: Yang–Mills equation in spherical symmetry, that is (1-11)–(1-12),

and study the evolution of the perturbations

$$u : \mathbb{R} \times I \to \mathbb{R}, \qquad u = u(t,x), \qquad I = \begin{cases} (0,\pi) & \text{for CW}, \\ \left(0, \frac{\pi}{2}\right) & \text{for CH}, \\ (0,\pi) & \text{for YM}, \end{cases}$$

driven by the partial differential equations

$$(\partial_t^2 + \boldsymbol{L})u = \boldsymbol{f}(x,u), \qquad (t,x) \in \mathbb{R} \times I, \tag{1-13}$$

subject to the initial data $u_0(x) = u(0,x)$ with zero initial velocity $u_1(x) = \partial_t u(0,x) = 0$ for all $x \in I$. Here, the linear operators and the nonlinearities are given, respectively, by

$$\boldsymbol{L}u = \begin{cases} -\dfrac{1}{\sin^2(x)}\partial_x(\sin^2(x)\partial_x u) + u & \text{for CW}, \\ -\partial_x^2 u - \left(\dfrac{\cos x}{\sin x} - \dfrac{\sin x}{\cos x}\right)\partial_x u + \left(\dfrac{\mu_1^2}{\sin^2 x} + \dfrac{\mu_2^2}{\cos^2 x} + 1\right)u & \text{for CH}, \\ -\dfrac{1}{\sin^4 x}\partial_x(\sin^4 x\,\partial_x u) + 4u & \text{for YM}, \end{cases} \tag{1-14}$$

$$\boldsymbol{f}(x,u) = \begin{cases} -u^3 & \text{for CW and CH}, \\ -3u^2 - \sin^2(x)u^3 & \text{for YM}. \end{cases} \tag{1-15}$$

Associated to the linear operators given by (1-14), one can introduce natural Hilbert spaces, and with suitable definitions for their domains (Section 3), the linear operators are then all self-adjoint operators with compact resolvent. In order to simplify the presentation below, we denote by $\{e_n(x) : n \geq 0\}$ the set of eigenfunctions of any of these operators[3] and by $\{\omega_n^2 : n \geq 0\}$ the set of corresponding eigenvalues. Recall that, in all three models considered, the sequences $\{\omega_n : n \geq 0\}$ are all strictly increasing with $\omega_n \sim n$ as $n \to +\infty$.

We also denote by $\Phi^t(\xi)$ the flow associated to any of the linearized equations with initial data $(u_{t=0}, \partial_t u_{t=0}) = (\xi, 0)$. If we use ξ_n to denote the Fourier coefficients of ξ associated to the eigenbasis $\{e_n(x) : n \geq 0\}$, then

$$\Phi^t(\xi) = \sum_{n=0}^{\infty} \cos(t\omega_n)\xi_n e_n(x). \tag{1-16}$$

[3] Of course, the eigenfunctions are different for the different operators, so this is just a generic name.

To state our result, we need to introduce a set of frequencies verifying a certain Diophantine condition [Bambusi and Paleari 2001]. Given $0 < \alpha < \frac{1}{3}$, define

$$\mathcal{W}_\alpha = \left\{ \omega \in \mathbb{R} : |\omega \cdot l - \omega_j| \geq \frac{\alpha}{l} \, \forall (l, j) \in \mathbb{N}^2, \, l \geq 1, \, \omega_j \neq l \right\}. \tag{1-17}$$

According to [Bambusi and Paleari 2001, Remark 2.4] and [Schmidt 1980, p. 23], the set $\mathcal{W}_\alpha$ contains infinitely many irrationals, is uncountable and accumulates at 1 from above and below. Consider any of the problems CW, CH or YM, and let e_γ be one of the eigenfunctions to the corresponding linear operator. In addition to α, the statements of our results depend on the constant $\gamma \in \mathbb{N} \cup \{0\}$, the index of the eigenfunction, and $s > 0$, which defines the Sobolev space[4] H^s where the solutions will belong. Our assumptions are slightly different depending on the problems addressed.

Assumptions 1.1. Specifically, we make the following assumptions:

- CW: We take $\gamma \in \{0, 1, 2, \dots\}$ and $s \in \mathbb{N}$ with $s \geq 2$.
- CH: We take $\gamma \in \{0, 1, 2, 3, 4, 5\}$ and $s \in \mathbb{N}$ with $s \geq 2$. Moreover, we assume that the parameters μ_1 and μ_2 appearing in (1-6) satisfy $\mu_1 = \mu_2 = \mu$, with μ either sufficiently large, or $\mu \in \{0, 1, 2, 3, 4, 5\}$.
- YM: We take $\gamma \in \{0, 1, 2, 3, 4, 5\}$ and $s \in \mathbb{N}$ with $s \geq 3$.

Remark 1.2 (range of γ). We note that our proof is based on closed formulas for the Fourier coefficients, integrals that quantify the mode couplings. Although we derive these formulas uniformly with respect to γ (see Section 5), we also need to check the validity of particular nonlinear conditions depending on the Fourier coefficients. On the one hand, for the CW model, the Fourier coefficients have a relatively simple closed formula. Hence, there is no need to restrict the range of γ and we establish the validity of the conditions needed uniformly with respect to γ. On the other hand, for the CH and YM models, the complexity of the Fourier coefficients requires us to restrict the range of γ to any finite set. Since the smaller the range the easier one can verify our computations, we fix $\gamma \in \{0, 1, 2, 3, 4, 5\}$ solely for the purpose of computing and verifying all computations in the manuscript by hand. However, we believe that our result stated below also holds true for larger values of γ. The interested reader can access our Mathematica notebooks as ancillary files posted with the present paper on arXiv at https://arxiv.org/abs/2201.05447 to both easily verify our computations for small γ as well as derive and verify the analogous closed formulas for larger values of γ.

Under Assumptions 1.1, we prove the following result.

Theorem 1.3 (main result 1: existence of time-periodic solutions to all three models bifurcating from various 1-modes). *Let $(\gamma, s) \in (\mathbb{N} \cup \{0\}) \times \mathbb{R}$ satisfy Assumptions 1.1, and let e_γ be the eigenfunction to the corresponding linear operator. Also, let $0 < \alpha < \frac{1}{3}$ and $\mathcal{W}_\alpha$ be the corresponding set of frequencies, defined in (1-17). Then, there exists a family $\{u_\epsilon : \epsilon \in \mathcal{E}_{\alpha,\gamma}\}$ of time-periodic solutions to either CW, CH or YM, where $\mathcal{E}_{\alpha,\gamma}$ is an uncountable set that has 0 as an accumulation point. In addition, each element u_ϵ has the following properties*:

[4]The definition of the H^s spaces is adapted to each problem; see Section 2.

(1) *u_ϵ has period $T_\epsilon = 2\pi/\omega_\epsilon$ with $\omega_\epsilon \in \mathcal{W}_\alpha$ being ϵ-close to* 1.

(2) $u_\epsilon \in H^1([0, T_\epsilon]; H^s)$.

(3) *u_ϵ stays close to the solution to the linearized equation with initial data $(u_{t=0}, \partial_t u_{t=0}) = (\epsilon\kappa_\gamma e_\gamma, 0)$ for all times*:

$$\sup_{t\in\mathbb{R}} \|u_\epsilon(t, \cdot) - \Phi^{t\omega_\epsilon}(\epsilon\kappa_\gamma e_\gamma)\|_{H^s} \lesssim \epsilon^2,$$

where κ_γ is a normalization constant.

Proof. The result follows by applying the original version of Bambusi–Paleari's theorem (Theorem 2.4 for CW and CH) and our modified version (Theorems 1.4 and 2.5 for YM) by verifying their main conditions; see Sections 6 and 7. □

For the CW and CH models, the results above are proven using a theorem of Bambusi and Paleari [2001], while for the YM model, the original version of their theorem (stated as Theorem 2.4 below) is not applicable and we will adapt their work. To explain this, we follow [Bambusi and Paleari 2001] and consider any of the models above in the Fourier space by projecting the equations on the eigenbasis $\{e_n : n \geq 0\}$, so that, schematically, the equations take the form

$$\ddot{u}^j(t) + (Au(t))^j = (f(u))^j \tag{1-18}$$

for all integers $j \geq 0$, where $u = \{u^j : j \geq 0\}$ denotes the array of the coefficients in the Fourier space, A is a multiplication operator defined by $(Au)^j = \omega_j^2 u^j$ and $(f(u))^j$ are the coefficients of the nonlinearity written in Fourier space, which takes the form of a polynomial in the u^j. In addition, we assume that

$$f(u) = f^{(0)}(u) + f^{(1)}(u),$$

where $f^{(0)}$ is a homogeneous polynomial of degree $r \geq 2$ and $f^{(1)}$ is a polynomial of degree at least $r+1$. Then, one looks for solutions $u(t)$ to (1-18), where $u(t)$ belongs to the Hilbert space

$$l_s^2 = \{u = \{u^j : j \geq 0\} : |u|_s^2 < \infty\}, \quad |u|_s^2 = \sum_{j=0}^{\infty} j^{2s}|u_j|^2.$$

Besides some regularity considerations, the main theorem in [Bambusi and Paleari 2001] asserts that, given any nondegenerate zero of the operator

$$\mathcal{M}\xi = A\xi + \langle f^{(0)}\rangle(\xi), \quad \xi = \{\xi^j : j \geq 0\} \in l_s^2,$$

where $\langle f^{(0)}\rangle(\xi)$ denotes the average in time of the nonlinearity $f^{(0)}$ along the linearized flow, one can construct a family of small data periodic in time solutions, with properties similar to those stated in Theorem 1.3. The operator $\mathcal{M}$ is in fact linked to the resonant system associated to the original equation. If $u(t)$ is periodic in time with frequency ω, let q be defined by $u(t) = q(\omega t)$, and let L_ω be the operator

$$L_\omega q = \omega^2 \frac{d^2}{dt^2} q + Aq.$$

The proof of [Bambusi and Paleari 2001] is based on a Lyapunov–Schmidt decomposition $q = q_\perp + v$, with $v \in \ker L_1$ and $q_\perp \in (\ker L_1)^\perp$, together with the projections of the equations onto the range and kernel of L_1, leading to the so-called P-equation and Q-equation, defined, respectively, as

$$L_{\varpi} q_{\perp} = Pf(v+q_{\perp}), \tag{1-19}$$

$$(1-\omega^2)Av = Qf(v+q_{\perp}). \tag{1-20}$$

The Diophantine condition (1-17) is then used to solve the P-equation, while the nondegeneracy assumption and an implicit function argument is used to solve the Q-equation.

For the CW and CH models, one easily verifies that the eigenfunctions $\kappa_n e_n$, where κ_n is an appropriate rescaling, are all zeroes of $\mathcal{M}$, so that the main difficulty is to establish the nondegeneracy condition, i.e., to prove that the kernel of $d\mathcal{M}(\kappa_n e_n)$ is trivial. In the YM case, however, the nonlinearity contains both quadratic and cubic terms, so that a priori, only the quadratic terms would contribute to the definition of the operator $\mathcal{M}$. On the other hand, it turns out that the average along the flow of the quadratic terms actually vanishes identically, leading to a degenerate, linear operator $\mathcal{M}$. Thus, we introduce a replacement for the operator $\mathcal{M}$ that takes into account also the cubic terms. However, the quadratic terms still play a role in this modified operator. Indeed, the solution to the P-equation roughly takes the form $q_{\perp}(v) = q_{\perp,\text{quadratic}}(v) + q_{\perp,\text{cubic}}(v) + \cdots$, where the term $q_{\perp,\text{quadratic}}(v)$ arises from the quadratic nonlinearity, and after substituting $q_{\perp}(v)$ into the Q-equation, these terms will generate new additional cubic terms. Thus, in some sense, the backreaction of the quadratic terms into the Q-equation must also be taken into account. This type of difficulty, where the contribution of the lowest degree part of the nonlinearity is nonresonant, has been treated in some situations; see for instance [Berti and Bolle 2003, Section 4.2] and [Berti and Bolle 2006, Section 1.2.3], where equations of the form $-\partial_{tt}u + \partial_{xx}u = u^{2p} + \mathcal{O}(u^{2p+1})$ were considered. Here, we prove a modified abstract version of the Bambusi–Paleari theorem which we then apply to the YM model.

Theorem 1.4 (main result 2: modification of the Bambusi–Paleari theorem for the YM model). *Consider the partial differential equation in the Fourier space*

$$\ddot{u}^j(t) + (\mathfrak{A}u(t))^j = (\mathfrak{f}(u(t)))^j, \quad j \geq 0, \tag{1-21}$$

where the dots stand for derivatives with respect to time and $\mathfrak{A}$ is a positive multiplication self-adjoint operator with pure point and resonant spectrum $\{\varpi_j^2 > 0 : j \geq 0\}$, with $\varpi_j \sim j$ as $j \to \infty$, defined by

$$\mathfrak{A} : \mathcal{D}(\mathfrak{A}) \simeq l^2_{s+2} \to l^2_s, \quad (\mathfrak{A}u)^j = \varpi_j^2 u_j,$$

with $\mathcal{D}(\mathfrak{A})$ being its maximal domain of definition.[5] *In addition, assume that the nonlinearity is given by*

$$\mathfrak{f}(u) = \mathfrak{f}^{(2)}(u) + \mathfrak{f}^{(3)}(u),$$

where each $\mathfrak{f}^{(k)}$ is a homogeneous polynomial of order k which is well defined and smooth in l^2_s, with $\mathfrak{f}^{(2)}$ being ***nonresonant****, that is*

$$\langle \mathfrak{f}^{(2)} \rangle (x) = \frac{1}{2\pi} \int_0^{2\pi} \Phi^t(\mathfrak{f}^{(2)}(\Phi^t(x)))\, dt = 0 \tag{1-22}$$

[5]Later, we will take l^2_{s+1} instead of l^2_s as our base Hilbert space, so that we will consider $\mathfrak{A}$ as an operator from $l^2_{s+3} \to l^2_{s+1}$. This allows for the construction of classical solutions, instead of solutions defined via the Duhamel formula or duality.

*for all initial data x, where $\Phi^t(x)$ denotes the solution to the linearized equation with initial data $(x, 0)$. Furthermore, define the **modified** operator*

$$\mathfrak{M}_\pm(\xi) = \pm\mathfrak{A}\xi + \langle\mathfrak{f}^{(3)}\rangle(\xi) + \mathfrak{F}_0(\xi),$$

where $\mathfrak{F}_0(\xi)$ is given by Lemma 2.13, and let $\xi_0 \in l^2_{s+3}$ be initial data such that

- *ξ_0 is a **zero** of $\mathfrak{M}_\pm$,*

$$\mathfrak{M}_\pm(\xi_0) = 0,$$

- *and ξ_0 satisfies the **nondegeneracy condition***

$$\ker(d\mathfrak{M}_\pm(\xi_0)) = \{0\}.$$

Also, for some $0 < \alpha < \frac{1}{3}$, define $\mathcal{W}_\alpha$ according to (1-17). Then, there exists a family $\{u_\epsilon(t, \cdot) : \epsilon \in \mathcal{E}_{\alpha,\gamma}\}$ of time-periodic solutions to (1-21), where $\mathcal{E}_{\alpha,\gamma}$ is an uncountable set that has 0 as an accumulation point. In addition, each element u_ϵ has the following properties:

(1) *u_ϵ has period $T_\epsilon = 2\pi/\omega_\epsilon$, with $\omega_\epsilon \in \mathcal{W}_\alpha$ being ϵ-close to 1.*

(2) *$u_\epsilon \in H^1([0, T_\epsilon]; l^2_s)$.*

(3) *u_ϵ stays close to the solution to the linearized equation with initial data $(u_{t=0}, \partial_t u_{t=0}) = (\epsilon\xi_0, 0)$ for all times:*

$$\sup_{t\in\mathbb{R}} |u_\epsilon(t, \cdot) - \Phi^{t\omega_\epsilon}(\epsilon\xi_0)|_s \lesssim \epsilon^2.$$

1F. *Remarks.* • Minimal periods: Theorems 1.3 and 1.4 give no information on the minimal periods T_ϵ of the time-periodic solutions $u_\epsilon(t, \cdot)$. However, one can relate T_ϵ to the minimal period T_0 of the solutions to the linearized system with 1-mode initial data; see [Bambusi and Paleari 2001, Theorem 5.3].

• Cantor-like set: We emphasize that the time-periodic solutions we construct exist only when the small perturbative parameter belongs to a Cantor-like set (of measure 0). This set together with the Diophantine condition introduced in Theorem 1.3 are closely related to the presence of small divisors in the perturbation series around equilibrium points, a classical topic in the context of Kolmogorov–Arnold–Moser (KAM) theory in infinite dimensions. Although this type of condition is essential in proving the existence of time-periodic solutions as in [Bambusi and Paleari 2001], it can be removed in some very special cases; see for example [Chatzikaleas 2020]. On the other hand, we note that the numerical constructions [Choptuik et al. 2018; Fodor et al. 2015; Maliborski and Rostworowski 2013] do not seem to see the small divisors obstructions.

• Proof: The proof of Theorem 1.4 follows the general strategy of [Bambusi and Paleari 2001], the main and essential difference being the backscattering contribution of the quadratic terms. An alternative approach to ours would be to find a normal form, in the spirit of [Shatah 1985], that allows us to eliminate the quadratic terms and then apply the original result of [Bambusi and Paleari 2001].

• The works of Berti and Bolle: In [Berti and Bolle 2003; 2004; 2006], a different strategy, based on variational methods, was introduced to solve the Q-equation (1-20) instead of the implicit function theorem as in [Bambusi and Paleari 2001]. This in particular leads to a strong improvement in [Berti and Bolle 2006] concerning the size of the frequency set, using an extra Nash–Moser iteration. We have not implemented this here for simplicity and leave a possible implementation of this improvement for future works. The works [Berti and Bolle 2004; 2006] also treat the case of nonresonant quadratic terms — as we have here in the Yang–Mills case — in the specific case of the wave equation on an interval with Dirichlet boundary conditions.

• Regularity of the solutions: The solutions constructed here are H^1 in time with values in H^s, with s arbitrarily large but fixed a priori. A posteriori, one can then use the equation to obtain additional regularity properties of the solutions. For instance, one easily has $\partial_t^2 u \in L_t^2 H_x^{s-1}$. Since some of the estimates depend a priori on the value of s, we cannot directly take $s = \infty$, but it is likely that a refinement of the methods presented would lead to such an improvement.

• Jacobi polynomials: One of the difficulties to proving Theorem 1.3 comes from the fact that the eigenfunctions $e_n(x)$ of the linearized operators are given by Jacobi polynomials instead of simpler explicit functions. This fact is not specific to our model problem and is a general feature of nonlinear wave equations on AdS-like background. In particular, in the CH and YM models,[6] the computation and the analysis of the Fourier coefficients associated to the resonant terms are nontrivial and constitute one of the contributions of this paper. To this end, we use linearization and connection formulas as well as particular Mellin transforms for the Jacobi polynomials. On the one hand, a linearization formula (also called addition formula) represents a product of two orthogonal polynomials with some parameters as a linear combination of orthogonal polynomials of the same kind with the same parameters. On the other hand, a connection formula represents a single orthogonal polynomial with some parameters as a linear combination of orthogonal polynomials of different kinds with new parameters. In both cases, these are computationally efficient only in the case where the coefficients in the expansions are known in closed formulas. These computations also motivate our choice of $\mu_1 = \mu_2 = \mu$ for the CH model, since in this case, the eigenfunctions are reduced to Gegenbauer polynomials, a special class of Jacobi polynomials with additional algebraic properties that lead to closed formulas for the linearization and connection coefficients described above. Moreover, we also use particular Mellin transforms of Gegenbauer polynomials. These are integral transforms that may be regarded as the multiplicative version of the Laplace transform.

• Mathematica files: For the CH and YM models, Theorem 1.3 ensures the existence of time-periodic solutions bifurcating only from finitely many 1-mode initial data. As stated in Remark 1.2, this is solely for the purpose of computing and verifying all computations in the manuscript by hand. Furthermore, one can use Mathematica to verify that our result still holds true also for larger values of γ. For the convenience of the reader, our Mathematica notebooks — available as ancillary files to the present paper on arXiv at https://arxiv.org/abs/2201.05447 — can help the reader to both easily verify our computations for small γ as well as derive and verify the analogous computations for larger values of γ.

[6] The eigenfunctions for the CW case are given by Chebyshev polynomials of the second kind. The derivation of the resonant system in this case had been previously addressed in [Bizoń et al. 2017].

1G. ***Previous works.*** The conformal wave equation (1-2) was introduced as a toy problem for the study of nonlinear waves in confined geometries in [Bizoń et al. 2017] and has been studied further in [Bizoń et al. 2019; 2020; Chatzikaleas 2020]. In particular, [Chatzikaleas 2020] proved that solutions emanating from the first mode e_0 stay proportional to e_0 for all times and are periodic in time. The fact these data do not excite further modes is, however, specific to the first mode and to this equation.

Concerning the well-posedness theories for the different models, since we do not focus here on low regularity solutions, we will simply recall that global well-posedness holds for the conformal cubic wave equation in the energy space, while the Yang–Mills equations in curved geometry have been shown to be globally well-posed in $H^2 \times H^1$ [Choquet-Bruhat et al. 1983; Chruściel and Shatah 1997] and on AdS with reflective boundary conditions [Choquet-Bruhat 1989]. We were motivated to study the Yang–Mills model by [Bizoń 2014].

Since the pioneering work [Maliborski and Rostworowski 2013], there have been many investigations of time-periodic solutions for nonlinear equations with completely resonant spectrum [Berti and Bolle 2003; 2004; 2006; Paleari et al. 2001]. For the conformal wave equations, there exist also several constructions of time-periodic weak solutions via the variational techniques first introduced by Rabinowitz [1978a; 1978b]; see [Chang and Hong 1985; Zhou 1986].

1H. ***Organization of the paper.*** We split the paper into the following sections:

- Section 2: We describe the methods we are about to use. For CW and CH, we will use the original version of Bambusi and Paleari's theorem [2001] (Theorem 2.4). However, for YM, as explained above, we need to revise the original version and establish an extension of their result (Theorem 2.5) as stated in Theorem 1.4. In particular, we define the operators $\mathcal{M}$ and $\mathfrak{M}_{\pm}$, which determine the "special" initial data leading to time-periodic solutions.

- Section 3: We study the linear eigenvalue problems where the linearized operators are given by (1-14). As it turns out, the associated eigenfunctions are given by Jacobi polynomials, which is a common feature with the Einstein–Klein–Gordon system in spherical symmetry [Maliborski and Rostworowski 2013].

- Section 4: We express the partial differential equations (1-13) in the Fourier space and obtain infinite dimensional systems of coupled harmonic oscillators.

- Section 5: We define and study the mode couplings given by the Fourier coefficients. Specifically, we derive explicit closed formulas for all the Fourier coefficients on resonant indices.

- Section 6: We study 1-mode initial data. In particular, we show that these modes satisfy the resonant systems (are zeros of the operators $\mathcal{M}$ and $\mathfrak{M}_{\pm}$ defined in Section 2). In addition, we derive their differentials at these modes.

- Section 7: Firstly, we derive the crucial nondegeneracy conditions for the 1-mode initial data. As it turns out, these are nonlinear conditions for the Fourier coefficients. Then, we use the analysis on the Fourier coefficients from Section 5 to rigorously establish these conditions and prove the existence of time-periodic solutions as stated in Theorem 1.3.

1I. *Notation.* We use different notation for each of the models we consider, which we summarize here:

model	CW	CH	YM
equation	(1-3)–(1-4)	(1-6)–(1-7)	(1-11)–(1-12)
font	standard	serif	fraktur
linearized operator	L	$\mathsf{L}^{(\mu_1,\mu_2)}$	$\mathfrak{L}$
eigenvalues	ω_n	$\omega_n^{(\mu_1,\mu_2)}$	ϖ_n
eigenfunctions	e_n	$\mathsf{e}_n^{(\mu_1,\mu_2)}$	$\mathfrak{e}_n$
inner product	$(\,\cdot\,\|\,\cdot\,)$	$\langle\,\cdot\,\|\,\cdot\,\rangle$	$[\,\cdot\,\|\,\cdot\,]$
linear flow	Φ^t	ϕ^t	$\varPhi^t$
fourier coefficients	C	$\mathsf{C}^{(\mu_1,\mu_2)}$	$\mathfrak{C}$, $\overline{\mathfrak{C}}$

2. The method of Bambusi–Paleari revisited

In order to establish Theorem 1.3 and construct our time-periodic solutions, we rely on the method of Bambusi and Paleari [2001]. This is an effective method to construct families of small amplitude time-periodic solutions close to a resonant equilibrium point for semilinear partial differential equations.

2A. *The original version of the theorem.* Let $s \geq 0$ be a real number, and define the Hilbert space l_s^2 to be the space of sequences such that

$$u = \{u^j : j \geq 0\}, \quad |u|_s^2 = \sum_{j=0}^{\infty} |j^s u^j|^2 < \infty.$$

We endow l_s^2 with the natural inner product associated with the norm $|\cdot|_s$ and consider the following differential equation in l_s^2:

$$\ddot{u}^j + (\mathfrak{A}u)^j = (\mathfrak{f}(u))^j, \quad (\mathfrak{A}u)^j = \varpi_j^2 u_j, \tag{2-1}$$

for all integers $j \geq 0$, where the dots denote derivatives with respect to time. Here, $\mathfrak{A} : \mathcal{D}(\mathfrak{A}) \to l_s^2$ is a positive multiplication self-adjoint operator with pure point and resonant spectrum $\{\varpi_j^2 : j \geq 0\}$, meaning $\{\varpi_j : j \geq 0\} \subset \mathbb{N}$, and $\mathcal{D}(\mathfrak{A})$ stands for its maximal domain of definition endowed with the norm

$$\|u\|_{\mathcal{D}(\mathfrak{A})}^2 = |u|_s^2 + |\mathfrak{A}u|_s^2 = \sum_{j=0}^{\infty} j^{2s} |\xi^j|^2 + \sum_{j=0}^{\infty} j^{2s} |\varpi_j^2 \xi^j|^2.$$

Moreover, we also assume that $\mathfrak{A}$ and f verify the following conditions:

(1) The injection of $(\mathcal{D}(\mathfrak{A}), \|\cdot\|_{\mathcal{D}(\mathfrak{A})})$ into l_s^2 is compact.

(2) The nonlinearity $\mathfrak{f}(u)$ can be decomposed into

$$\mathfrak{f}(u) = \mathfrak{f}^{(0)}(u) + \mathfrak{f}^{(1)}(u). \tag{2-2}$$

(3) The lowest-degree term $\mathfrak{f}^{(0)}(u)$ is a homogeneous polynomial of order $r \geq 2$ and is a bounded operator from $\mathcal{D}(\mathfrak{A})$ to $\mathcal{D}(\mathfrak{A})$ with the domain $\mathcal{D}(\mathfrak{A})$ being invariant under $\mathfrak{f}^{(0)}$.

(4) The highest-degree term $\mathfrak{f}^{(1)}(u)$ (treated perturbatively as an error term) has a zero of order $r+1$ at 0, is differentiable in l_s^2, and its differential is Lipschitz and satisfies the estimate

$$|d\mathfrak{f}^{(1)}(u_1) - d\mathfrak{f}^{(1)}(u_2)|_s \le C\epsilon^{r-1}|u_1 - u_2|_s$$

for all $u_1, u_2 \in l_s^2$ with $|u_1|_s \le \epsilon$ and $|u_2|_s \le \epsilon$.

Remark 2.1. In our case, the conditions above are obtained by starting from any of the equations (1-3), (1-6), (1-11) and projecting them on the eigenfunctions to the corresponding linear operator. Specifically, condition (1) follows automatically from the fact that $\varpi_j \sim j$ as $j \to \infty$, while conditions (3) and (4), which refer to the nonlinearities in the Fourier space, essentially follow from the facts that the original nonlinearities are smooth and that the Sobolev spaces of sufficiently high regularity form an algebra; see Section 2C.

Let $\mathfrak{e}_n$ be the eigenfunctions to the associated linearized operators. On the Fourier side, these can be identified with $\mathfrak{e}_n = \{\delta_n^i : i \ge 0\} \in l_s^2$. Then, for any initial data ξ, we denote by

$$\Phi^t(\xi) = \{\xi^n \cos(\varpi_n t) : n \ge 0\}, \quad \xi = \{\xi^n : n \ge 0\} = \sum_{n=0}^{\infty} \xi^n \mathfrak{e}_n$$

its linear flow, that is the solution to the initial value problem

$$\begin{cases} \ddot{u}^n(t) + \varpi_n^2 u^n(t) = 0, & t \in \mathbb{R}, \\ u^n(0) = \xi^n, & \dot{u}^n(0) = 0. \end{cases}$$

We note that $\Phi^t(\xi) = \Phi^{-t}(\xi)$. Moreover, we define the operator

$$\mathcal{M}(\xi) := \mathfrak{A}\xi + \langle \mathfrak{f}^{(0)} \rangle(\xi), \quad \langle \mathfrak{f}^{(0)} \rangle(\xi) = \frac{1}{2\pi} \int_0^{2\pi} \Phi^{-t}[\mathfrak{f}^{(0)}(\Phi^t(\xi))]\, dt, \tag{2-3}$$

where $\langle \mathfrak{f}^{(0)} \rangle(\xi)$ is the average of $\mathfrak{f}^{(0)}$ along the linear flow. Here we note that the highest-degree term $\mathfrak{f}^{(1)}$ in (2-2) does not contribute to the definition of the operator $\mathcal{M}$. Bambusi and Paleari [2001] used a Lyapunov–Schmidt decomposition together with averaging theory and established the existence of a family of small amplitude time-periodic solutions with frequencies that satisfy the strong Diophantine condition

$$\varpi \in \mathcal{W}_\alpha = \Big\{ \varpi \in \mathbb{R} : |\varpi \cdot l - \varpi_j| \ge \frac{\alpha}{l}\ \forall (l, j) \in \mathbb{N}^2,\ l \ge 1,\ \varpi_j \ne l \Big\}. \tag{2-4}$$

Remark 2.2 (accumulation to 1). For $0 < \alpha < \frac{1}{3}$, the set $\mathcal{W}_\alpha$ is an uncountable Cantor-like set that accumulates to 1 from above and below; see [Bambusi and Paleari 2001, Remark 2.4] and [Schmidt 1980, p. 23].

Remark 2.3 (connection to Hurwitz's theorem). According to Hurwitz's theorem, for every irrational number ϖ, there are infinitely many relatively prime integers ϖ_j and l such that $|\varpi \cdot l - \varpi_j| < 1/(\sqrt{5}l)$, and moreover the constant $\sqrt{5}$ is optimal. Consequently, $\mathcal{W}_\alpha = \varnothing$ for $\alpha \ge 1/\sqrt{5}$. In this note, we pick a suitable α with $0 < \alpha < \frac{1}{3}$.

The main result of [Bambusi and Paleari 2001] reads as follows.

Theorem 2.4 (original version of Bambusi and Paleari's theorem [2001]). *For $0<\alpha<\frac{1}{3}$, define $\mathcal{W}_\alpha$ according to (2-4) and consider the operator $\mathcal{M}$ defined in (2-3). Assume that conditions (1)–(4) are verified. Moreover, let ξ_0 be a nondegenerate zero of $\mathcal{M}$, that is*

$$\mathcal{M}(\xi_0)=0, \quad \ker(d\mathcal{M}(\xi_0))=\{0\}.$$

Then, there exists a family $\{u_\epsilon:\epsilon\in\mathcal{E}_{\alpha,\gamma}\}\subset H^1([0,T_\epsilon];l_s^2)$ of time-periodic solutions to (2-1)–(2-2), where $\mathcal{E}_{\alpha,\gamma}$ is an uncountable set that has 0 as an accumulation point. In addition, each element u_ϵ has the following properties:

(1) *u_ϵ has period $T_\epsilon=2\pi/\varpi_\epsilon$, and there exists $\varpi_\star>0$ such that the map*

$$\epsilon\in\mathcal{E}_{\alpha,\gamma}\mapsto\varpi_\epsilon\in\mathcal{W}_\alpha\cap[1,1+\varpi_\star)$$

is a monotone, one-to-one map that stays close to 1: $|1-\varpi_\epsilon|\lesssim\epsilon^{r-1}$,

(2) *u_ϵ stays close to the solution to the linearized equation with initial data $(u_{t=0},\partial_t u_{t=0})=(\epsilon\xi_0,0)$ for all times*:

$$\sup_{t\in\mathbb{R}}|u_\epsilon(t,\cdot)-\Phi^{t\varpi_\epsilon}(\epsilon\xi_0)|_s\lesssim\epsilon^2.$$

2B. ***A modified Bambusi–Paleari theorem.*** As we will see in Section 4C, in the case of the YM model, the nonlinearity is given by

$$\mathfrak{f}(u)=\mathfrak{f}^{(2)}(u)+\mathfrak{f}^{(3)}(u), \tag{2-5}$$

where

(1) the lowest-degree term $\mathfrak{f}^{(2)}$ is a homogeneous polynomial of order 2:

$$(\mathfrak{f}^{(2)}(\{u^j(t):j\geq 0\}))^m=-3\sum_{i,j=0}^{\infty}\overline{\mathfrak{C}}_{ijm}u^i(t)u^j(t), \tag{2-6}$$

(2) the highest-degree term $\mathfrak{f}^{(3)}$ is a homogeneous polynomial of order 3:

$$(\mathfrak{f}^{(3)}(\{u^j(t):j\geq 0\}))^m=-\sum_{i,j,k=0}^{\infty}\mathfrak{C}_{ijkm}u^i(t)u^j(t)u^k(t). \tag{2-7}$$

Thus, according to the original version of Bambusi–Paleari's theorem (Theorem 2.4), one may argue that $\mathfrak{f}^{(2)}$ is the main nonlinearity and $\mathfrak{f}^{(3)}$ can be treated perturbatively. However, in this setting, the original version of Bambusi–Paleari's theorem would not be applicable, because $\mathfrak{f}^{(2)}$ is *nonresonant* (Lemma 6.5), that is

$$\langle\mathfrak{f}^{(2)}\rangle(\xi)=\frac{1}{2\pi}\int_0^{2\pi}\Phi^t(\mathfrak{f}^{(2)}(\Phi^t(\xi)))\,dt=0 \tag{2-8}$$

for all initial data ξ, and therefore $\mathcal{M}(\xi)=\mathfrak{A}\xi$ leads to trivial zeros of the operator $\mathcal{M}$. Consequently, we need to revisit the theorem of Bambusi–Paleari in this context. Specifically, we consider (2-1)–(2-5)–(2-8), replace $\mathfrak{f}^{(0)}(u)$ by $\mathfrak{f}^{(2)}(u)+\mathfrak{f}^{(3)}(u)$ and establish the following theorem.

Theorem 2.5 (modification of Bambusi–Paleari's theorem for the YM model). *Let* $0 < \alpha < \frac{1}{3}$, *and define* $\mathcal{W}_\alpha$ *according to* (2-4). *Let* $\mathfrak{A}$ *be a positive multiplication self-adjoint operator with spectrum* $\{\varpi_j^2 > 0 : j \geq 0\}$ *such that* $\{\varpi_j : j \geq 0\} \subset \mathbb{N}$ *and* $\varpi_j \simeq j$ *as* $j \to \infty$, *defined by*

$$\mathfrak{A} : \mathcal{D}(\mathfrak{A}) \simeq l^2_{s+2} \to l^2_s, \quad (\mathfrak{A}u)^j = \varpi_j^2 u_j,$$

with $\mathcal{D}(\mathfrak{A})$ *being its maximal domain of definition. Assume that* $\mathfrak{f} = \mathfrak{f}^{(2)} + \mathfrak{f}^{(3)}$, *where* $\mathfrak{f}^{(2)}$ *and* $\mathfrak{f}^{(3)}$ *admit the representations* (2-6) *and* (2-7) *respectively.*[7] *Moreover, assume that both* $\mathfrak{f}^{(2)}$ *and* $\mathfrak{f}^{(3)}$ *are differentiable, with Lipschitz differentials, and define the* ***modified*** *operator*

$$\mathfrak{M}_\pm(\xi) = \pm\mathfrak{A}\xi + \langle \mathfrak{f}^{(3)}\rangle(\xi) + \mathfrak{F}_0(\xi),$$

where $\mathfrak{F}_0(\xi) = \{(\mathfrak{F}_0(\xi))^m : m \geq 0\}$ *is a bounded map on* l^2_s *that is given by*

$$(\mathfrak{F}_0(\xi))^m = \frac{9}{4}\sum_{\kappa,\nu\geq 0} \overline{\mathfrak{C}}_{\kappa\nu m} \sum_{\substack{i,j\geq 0\\ \varpi_i - \varpi_j \neq \pm\varpi_\nu}} \frac{\overline{\mathfrak{C}}_{ij\nu}}{\varpi_\nu^2 - (\varpi_i - \varpi_j)^2} \xi^i \xi^j \xi^\kappa \sum_{\pm} \mathbb{1}(\varpi_i - \varpi_j \pm \varpi_\kappa \pm \varpi_m = 0)$$
$$+ \frac{9}{4}\sum_{\kappa,\nu\geq 0} \overline{\mathfrak{C}}_{\kappa\nu m} \sum_{\substack{i,j\geq 0\\ \varpi_i + \varpi_j \neq \pm\varpi_\nu}} \frac{\overline{\mathfrak{C}}_{ij\nu}}{\varpi_\nu^2 - (\varpi_i + \varpi_j)^2} \xi^i \xi^j \xi^\kappa \sum_{\pm} \mathbb{1}(\varpi_i + \varpi_j \pm \varpi_\kappa \pm \varpi_m = 0).$$

Also, let $\xi_0 \in l^2_{s+3}$ *be a nondegenerate zero of* $\mathfrak{M}_\pm$, *that is*

$$\mathfrak{M}_\pm(\xi_0) = 0, \quad \ker(d\mathfrak{M}_\pm(\xi_0)) = \{0\}.$$

Then, there exists a family $\{u_\epsilon : \epsilon \in \mathcal{E}_{\alpha,\gamma}\} \subset H^1([0, T_\epsilon]; l^2_s])$ *of time-periodic solutions to* (2-1)–(2-5)–(2-8), *where* $\mathcal{E}_{\alpha,\gamma}$ *is an uncountable set that has* 0 *as an accumulation point. In addition, each element* u_ϵ *has the following properties*:

(1) u_ϵ *has period* $T_\epsilon = 2\pi/\varpi_\epsilon$ *where there exists* $\varpi_\star > 0$ *such that the maps* $\epsilon \mapsto \varpi_\epsilon \in \mathcal{W}_\alpha \cap [1, 1+\varpi_\star)$ *for* $\mathfrak{M}_+$ *and* $\epsilon \mapsto \varpi_\epsilon \in \mathcal{W}_\alpha \cap (1-\varpi_\star, 1]$ *for* $\mathfrak{M}_-$, *are monotone, one-to-one maps that stay close to* 1 *with* $|1 - \varpi_\epsilon| \lesssim \epsilon$,

(2) u_ϵ *stays close to the solution to the linearized equation with the same initial data as above and zero initial velocity*:

$$\sup_{t\in\mathbb{R}} |u_\epsilon(t,\cdot) - \Phi^{t\varpi_\epsilon}(\epsilon\xi_0)|_s \lesssim \epsilon^2.$$

The rest of this section is devoted to the proof of the theorem above.

2C. *Preliminaries.* The core of the proof follows that of [Bambusi and Paleari 2001]. Let $0 < \alpha < \frac{1}{3}$ and pick a frequency $\varpi \in \mathcal{W}_\alpha$. We are looking for a solution to (2-1) with frequency ϖ, that is

$$u(t) = q(\varpi t). \tag{2-9}$$

[7]For Theorem 2.5, we need very little information about the Fourier coefficients $\overline{\mathfrak{C}}_{ijm}$. However, for the application of this abstract theorem, see Section 7C and Proposition 7.6, we also need additional vanishing properties, see Lemma 5.8.

For any integer $k \geq 0$, we define the Banach space

$$\mathcal{H}_s^k = \left\{ q \in H^k([0, 2\pi]; l_s^2) : q(t) = \sum_{j=0}^{\infty} q^j(t) e_j = \sum_{j=0}^{\infty} \left(\sum_{l=0}^{\infty} q^{lj} \cos(lt) \right) e_j,\ \|q\|_{\mathcal{H}_s^k}^2 < \infty \right\}$$

endowed with the norm

$$\|q\|_{\mathcal{H}_s^k}^2 = \sum_{j=0}^{\infty} j^{2s} \left(2|q^{0j}|^2 + \sum_{l=1}^{\infty} |q^{lj}|^2 (1+l^2)^k \right).$$

In particular, we aim to construct q in the Hilbert space $\mathcal{H}_s^1$. To do so, we substitute (2-9) into (2-1) and obtain the nonlinear equation

$$L_{\varpi} q = f(q), \tag{2-10}$$

where

$$L_{\varpi} : \mathcal{D}(L_{\varpi}) \subset \mathcal{H}_s^1 \to \mathcal{H}_s^1, \quad L_{\varpi} q = \varpi^2 \frac{d^2}{dt^2} q + \mathfrak{A} q.$$

Now, we are looking for a solution with frequency close to 1. For this reason, we split $\mathcal{H}_s^1$ into

$$\mathcal{H}_s^1 = K \oplus R, \quad K = \ker(L_1), \quad R = K^{\perp},$$

and write

$$q \in \mathcal{H}_s^1, \quad q = v + q_{\perp}, \quad v \in K, \quad q_{\perp} \in R.$$

Taking into account the fact that K is generated by $\{\cos(\varpi_j t) : j \geq 0\}$, since

$$v \in K \quad \Longleftrightarrow \quad v(t) = \{ v^j(t) = c^j \cos(\varpi_j t) : j \geq 0 \}$$

for some constants c^j, the latter simply means that we split $q = \{q^j : j \geq 0\} \in \mathcal{H}_s^1$ into

$$q^j(t) = v^j(t) + q_{\perp}^j(t), \quad v^j(t) = c^j \cos(\varpi_j t), \quad q_{\perp}^j(t) = \sum_{l \neq \varpi_j} d^{jl} \cos(lt),$$

for some constants c^j and d^{jl}. In addition, we define the associated projections

$$P : \mathcal{H}_s^1 \to R, \quad P(q) = P(v + q_{\perp}) = q_{\perp}, \quad Q : \mathcal{H}_s^1 \to K, \quad Q(q) = Q(v + q_{\perp}) = v,$$

and project (2-10) onto R and K, respectively. We obtain the coupled nonlinear system

$$L_{\varpi} q_{\perp} = P f(v + q_{\perp}), \tag{2-11}$$

$$-2\beta \mathfrak{A} v = Q f(v + q_{\perp}), \tag{2-12}$$

where we also set

$$\varpi^2 = 1 + 2\beta. \tag{2-13}$$

As is usual in this setting, we refer to (2-11) and (2-12) as the P-equation and Q-equation, respectively.

2D. ***Solution to the P-equation.*** As we will now see, the Diophantine condition $\varpi \in \mathcal{W}_\alpha$ guarantees the existence of a solution to the P-equation.

Lemma 2.6 (solution to the P-equation [Bambusi and Paleari 2001, Lemma 4.6]). *Let* $0 < \alpha < \frac{1}{3}$, *and pick* $\varpi \in \mathcal{W}_\alpha$. *Then, the operator* L_ϖ *restricted to* R *admits a bounded inverse*

$$L_\varpi^{-1} : \mathcal{H}_s^1 \cap R \to \mathcal{H}_s^1 \cap R, \quad \|L_\varpi^{-1}\| \le c_0 \alpha^{-1},$$

for some positive constant c_0. *Moreover, there exists* $\rho = \rho(\alpha) > 0$ *and a* C^1*-function* $q_\perp : B_\rho \to R$ *with* $v \mapsto q_\perp(v)$ *that solves the* P*-equation, where* B_ρ *denotes the ball of radius* ρ *in* K *centered at* 0. *Furthermore, we have the estimates*

$$\|q_\perp(v)\|_{\mathcal{H}_s^1} \lesssim_\alpha \|v\|^2_{\mathcal{H}_s^1}, \quad \|q_\perp(v) - L_\varpi^{-1} P \mathfrak{f}^{(2)}(v)\|_{\mathcal{H}_s^1} \lesssim_\alpha \|v\|^3_{\mathcal{H}_s^1}.$$

Proof. Apart from the C^1 regularity of $q_\perp$ (which is stated only as Lipschitz in [Bambusi and Paleari 2001]), the proof coincides with the one of Lemma 4.6 in the aforementioned paper, where the $f^{(0)}$ there is replaced by $\mathfrak{f}^{(2)}$. Once a Lipschitz solution $q_\perp$ has been found, one can read off the C^1 regularity of $q_\perp$ based on the regularity of $\mathfrak{f}$. However, for the convenience of the reader, we give a proof below of the construction of $q_\perp$. Let $0 < \alpha < \frac{1}{3}$, and pick $\varpi \in \mathcal{W}_\alpha$. The eigenvalues of L_ϖ are given by

$$\lambda_{jl} = \varpi_j^2 - l^2 \varpi^2 = (\varpi_j - l\varpi)(\varpi_j + l\varpi). \tag{2-14}$$

Then, for all $(l, j) \in \mathbb{N}^2$ with $l \ge 1$ and $l \ne \varpi_j$, we have that $|\lambda_{jl}| \ge (\alpha/l)(\varpi_j + l\varpi) \ge \alpha\varpi \ge \frac{1}{2}\alpha$. Therefore, $L_\varpi|_R$ has a bounded inverse and there exists a positive constant c_0 such that $\|L_\varpi^{-1}\| \le c_0 \alpha^{-1}$. In addition, we let $\epsilon > 0$ be sufficiently small, let $\|v\|_{\mathcal{H}_s^1} \le \epsilon$, let $\delta > 0$ be sufficiently large, define the closed ball of radius $\delta \|v\|^3_{\mathcal{H}_s^1}$ centered at $L_\varpi^{-1} P \mathfrak{f}^{(2)}(v)$, that is

$$B = \{w \in \mathcal{H}_s^1 : \|w - L_\varpi^{-1} P \mathfrak{f}^{(2)}(v)\|_{\mathcal{H}_s^1} \le \delta \|v\|^3_{\mathcal{H}_s^1}\},$$

and rewrite the P-equation in the fixed-point formulation as

$$q_\perp = \mathcal{F}(q_\perp) = L_\varpi^{-1}[P\mathfrak{f}^{(2)}(v) + P(\mathfrak{f}^{(2)}(v + q_\perp) - \mathfrak{f}^{(2)}(v)) + P\mathfrak{f}^{(3)}(v + q_\perp)].$$

Next, we show that $\mathcal{F}$ maps the closed ball to itself. Indeed, for all $w \in B$, we have

$$\begin{aligned}\|w\|_{\mathcal{H}_s^1} &\le \|w - L_\varpi^{-1} P \mathfrak{f}^{(2)}(v)\|_{\mathcal{H}_s^1} + \|L_\varpi^{-1} P \mathfrak{f}^{(2)}(v)\|_{\mathcal{H}_s^1} \le \delta \|v\|^3_{\mathcal{H}_s^1} + \|L_\varpi^{-1}\| \|\mathfrak{f}^{(2)}(v)\|_{\mathcal{H}_s^1} \\ &\le \delta \|v\|^3_{\mathcal{H}_s^1} + c_0 \alpha^{-1} k_s \|v\|^2_{\mathcal{H}_s^1} \le c_1 \|v\|^2_{\mathcal{H}_s^1},\end{aligned}$$

and Lemma 4.5 implies

$$\begin{aligned}\|\mathfrak{f}^{(2)}(v + w) - \mathfrak{f}^{(2)}(v)\|_{\mathcal{H}_s^1} &\le k_s (\|v + w\|_{\mathcal{H}_s^1} + \|v\|_{\mathcal{H}_s^1}) \|w\|_{\mathcal{H}_s^1} \le k_s (\|w\|_{\mathcal{H}_s^1} + 2\|v\|_{\mathcal{H}_s^1}) \|w\|_{\mathcal{H}_s^1} \\ &\le c_1 k_s (c_1 \|v\|^2_{\mathcal{H}_s^1} + 2\|v\|_{\mathcal{H}_s^1}) \|v\|^2_{\mathcal{H}_s^1} \le c_2 \|v\|^3_{\mathcal{H}_s^1}, \\ \|\mathfrak{f}^{(3)}(v + w)\|_{\mathcal{H}_s^1} &\le k_s \|v + w\|^3_{\mathcal{H}_s^1} \lesssim k_s (\|v\|^3_{\mathcal{H}_s^1} + c_1^3 \|v\|^6_{\mathcal{H}_s^1}) \le c_3 \|v\|^3_{\mathcal{H}_s^1}.\end{aligned}$$

Hence, we infer

$$\begin{aligned}\|\mathcal{F}(w)-L_{\varpi}^{-1}P\mathfrak{f}^{(2)}(v)\|_{\mathcal{H}_s^1} &= \|L_{\varpi}^{-1}[P(\mathfrak{f}^{(2)}(v+w)-\mathfrak{f}^{(2)}(v))+P\mathfrak{f}^{(3)}(v+w)]\|_{\mathcal{H}_s^1}\\ &\le \|L_{\varpi}^{-1}\|\,[\|\mathfrak{f}^{(2)}(v+w)-\mathfrak{f}^{(2)}(v)\|_{\mathcal{H}_s^1}+\|\mathfrak{f}^{(3)}(v+w)\|_{\mathcal{H}_s^1}]\\ &\le c_0\alpha^{-1}[c_2\|v\|_{\mathcal{H}_s^1}^3+c_3\|v\|_{\mathcal{H}_s^1}^3]\le \delta\|v\|_{\mathcal{H}_s^1}^3\end{aligned}$$

by choosing δ sufficiently large. The contraction property follows similarly. For the C^1 regularity, we set $\mathfrak{F}^{(2)}=L_{\varpi}^{-1}P\mathfrak{f}^{(2)}$ and, for $v, v+h\in B_\rho$, we have

$$q_\perp(v)=\mathfrak{F}^{(2)}(v+q_\perp(v)),\quad q_\perp(v+h)=\mathfrak{F}^{(2)}(v+h+q_\perp(v+h)),$$

so that

$$\begin{aligned}q_\perp(v+h) &= \mathfrak{F}^{(2)}(v+q_\perp(v))+d\mathfrak{F}^{(2)}_{v+q_\perp(v)}(h+q_\perp(v+h)-q_\perp(v))+\mathcal{O}(h+q_\perp(v+h)-q_\perp(v))\\ &= q_\perp(v)+d\mathfrak{F}^{(2)}_{v+q_\perp(v)}(h+q_\perp(v+h)-q_\perp(v))+\mathcal{O}(h),\end{aligned}$$

where we used that $q_\perp$ is Lipschitz. Assuming that ρ is small enough, we can ensure that

$$\|d\mathfrak{F}^{(2)}_{v+q_\perp(v)}\|_{\mathcal{H}_s^1}\le c\|v\|_{\mathcal{H}_s^1}<\tfrac{1}{2}$$

uniformly in v, and hence

$$q_\perp(v+h)=q_\perp(v)+(\mathrm{Id}-d\mathfrak{F}^{(2)}_{v+q_\perp(v)})^{-1}d\mathfrak{F}^{(2)}_{v+q_\perp(v)}(h)+\mathcal{O}(h),$$

so that $q_\perp(v)$ is C^1 with differential $(\mathrm{Id}-d\mathfrak{F}^{(2)}_{v+q_\perp(v)})^{-1}d\mathfrak{F}^{(2)}_{v+q_\perp(v)}$. □

2E. *Solution to the Q-equation.* Next, we turn our attention to the existence of a solution to the Q-equation. Firstly, we define two Banach spaces of initial data

$$\mathcal{Q}=\left\{\xi=\sum_{j=0}^{\infty}\xi^j e_j:\|\xi\|_{\mathcal{Q}}^2<\infty\right\}\simeq l_{s+1}^2\subseteq l_s^2,\quad \mathcal{D}(\mathfrak{A})=\left\{\xi=\sum_{j=0}^{\infty}\xi^j e_j:\|\xi\|_{\mathcal{D}(\mathfrak{A})}^2<\infty\right\}\simeq l_{s+2}^2\subseteq l_s^2,$$

endowed with the norms

$$\begin{aligned}\|\xi\|_{\mathcal{Q}}^2 &= \sum_{j=0}^{\infty}j^{2s}|\xi^j|^2+\sum_{j=0}^{\infty}j^{2s}|\varpi_j\xi^j|^2\simeq\sum_{j=0}^{\infty}j^{2(s+1)}|\xi^j|^2=|\xi|_{s+1}^2,\\ \|\xi\|_{\mathcal{D}(\mathfrak{A})}^2 &= |\xi|_s^2+|\mathfrak{A}\xi|_s^2=\sum_{j=0}^{\infty}j^{2s}|\xi^j|^2+\sum_{j=0}^{\infty}j^{2s}|\varpi_j^2\xi^j|^2\simeq\sum_{j=0}^{\infty}j^{2(s+2)}|\xi^j|^2=|\xi|_{s+2}^2,\end{aligned}$$

since $\varpi_j\sim j$ as $j\to\infty$. We call the Hilbert space $(\mathcal{Q},\|\cdot\|_{\mathcal{Q}})$ the configuration space. In fact, $\mathcal{Q}$ is isomorphic to $K=\ker(L_1)$, and the isomorphism is given by the linear flow

$$I:\mathcal{Q}\to K,\quad (I(x))(t)=\Phi^t(x).$$

Also, recall the Banach space of spacetime functions

$$\mathcal{H}_s^k=\left\{q(t)=\sum_{j=0}^{\infty}q^j(t)e_j=\sum_{j=0}^{\infty}\left(\sum_{l=0}^{\infty}q^{lj}\cos(lt)\right)e_j:\|q\|_{\mathcal{H}_s^k}^2<\infty\right\}\subseteq H^k([0,2\pi];l_s^2)$$

endowed with the norm

$$\|q\|^2_{\mathcal{H}^k_s}=\sum_{j=0}^{\infty} j^{2s}\biggl(2|q^{0j}|^2+\sum_{l=1}^{\infty}|q^{lj}|^2(1+l^2)^k\biggr).$$

Notice that, since

$$I(\xi)(t)=\sum_{j=0}^{\infty}(I(\xi)(t))^j e_j=\sum_{j=0}^{\infty}(\Phi^t(\xi))^j e_j=\sum_{j=0}^{\infty}\xi^j\cos(\varpi_j t)e_j$$

and $\varpi_j\neq 0$ for all integers $j\geq 0$, we have

$$\begin{aligned}\|I(\xi)\|^2_{\mathcal{H}^0_s}&=\sum_{j=0}^{\infty} j^{2s}|\xi^j|^2=|\xi|^2_s,\\ \|I(\xi)\|^2_{\mathcal{H}^1_s}&=\sum_{j=0}^{\infty} j^{2s}|\xi^j|^2(1+\varpi_j^2)=\sum_{j=0}^{\infty} j^{2s}(|\xi^j|^2+|\varpi_j\xi^j|^2)\simeq|\xi|^2_{s+1}\simeq\|\xi\|^2_{\mathcal{Q}}.\end{aligned}\tag{2-15}$$

Secondly, we prove the following averaging identity that generalizes the one in Lemma 4.7 in [Bambusi and Paleari 2001] from vector fields $F:\mathcal{Q}\to\mathcal{Q}$ to $F:\mathcal{H}^k_s\to\mathcal{H}^k_s$.

Lemma 2.7 (averaging identity). *Let $F:\mathcal{H}^k_s\to\mathcal{H}^k_s$ be any vector field. Then, for all $x\in l^2_s$, we have*

$$\langle F\rangle(x)=\frac{1}{2\pi}\int_0^{2\pi}\Phi^t[F(\Phi^t(x))]\,dt=\frac{1}{2}I^{-1}Q[F(I(x))].$$

Proof. Let $F:\mathcal{H}^k_s\to\mathcal{H}^k_s$ be a vector field in $\mathcal{H}^k_s$ (not necessarily in $\mathcal{Q}$), pick any $x\in l^2_s$ and set $w=I(x)$. By the definition of the Banach space $\mathcal{H}^k_s$, we have

$$F(w)=\sum_{m=0}^{\infty}(F(w))^m e_m=\sum_{m=0}^{\infty}\biggl(\sum_{l=0}^{\infty}(F(w))^m_l\cos(lt)\biggr)e_m,\quad (F(w))^m_l=\frac{1}{\pi}\int_0^{2\pi}(F(w))^m\cos(lt)\,dt.$$

Then, the definition of the linear flow together with the definition of the projection Q yield

$$\begin{aligned}Q[F(w)]&=\sum_{m=0}^{\infty}(Q[F(w)])^m e_m=\sum_{m=0}^{\infty}(F(w))^m_{\omega_m}\cos(\varpi_m t)e_m,\\ \frac{1}{2}I^{-1}Q[F(w)]&=\frac{1}{2}\sum_{m=0}^{\infty}(F(w))^m_{\varpi_m}e_m=\frac{1}{2}\sum_{m=0}^{\infty}\biggl(\frac{1}{\pi}\int_0^{2\pi}(F(w))^m\cos(\varpi_m t)\,dt\biggr)e_m\\ &=\frac{1}{2\pi}\int_0^{2\pi}\sum_{m=0}^{\infty}((\Phi^t[F(w)])^m e_m)\,dt=\frac{1}{2\pi}\int_0^{2\pi}\Phi^t[F(w)]\,dt=\langle F\rangle(x).\end{aligned}$$ □

Then, we express the Q-equation in the configuration space introduced above.

Lemma 2.8 (the Q-equation in the configuration space). *Let $\rho>0$ and $q_\perp:B_\rho\subset K\to R$ be the solution map to the P-equation derived in Lemma 2.6. Also, let $x\in l^2_{s+2}$, and set $v=I(x)\in B_\rho$. Then, the Q-equation (2-12) for v is equivalent to*

$$\beta\mathfrak{A}x+\langle\mathfrak{f}\rangle(x)=-\tfrac{1}{2}I^{-1}Q[\mathfrak{f}(I(x)+q_\perp(I(x)))-\mathfrak{f}(I(x))].\tag{2-16}$$

Proof. Let $\rho > 0$ and $q_\perp : B_\rho \subset K \to R$ be the solution map to the P-equation derived in Lemma 2.6. Also, let $x \in \mathcal{Q}$, and set $v = I(x) \in B_\rho$. Then, we rewrite the Q-equation given in (2-12), that is $-2\beta A v = Q f(v + q_\perp)$, as

$$-2\beta \mathfrak{A} I(x) - Q\mathfrak{f}(I(x)) = Q\mathfrak{f}(I(x) + q_\perp(I(x))) - Q\mathfrak{f}(I(x)).$$

Since $\mathfrak{A} I(x) = I \mathfrak{A} x$, by applying $-\frac{1}{2} I^{-1}$ to both sides, we get

$$\beta \mathfrak{A} x + \tfrac{1}{2} I^{-1} Q\mathfrak{f}(I(x)) = -\tfrac{1}{2} I^{-1} Q[\mathfrak{f}(I(x) + q_\perp(I(x))) - \mathfrak{f}(I(x))].$$

Now, the claim follows by the averaging identity due to Lemma 2.7. □

It remains to show that there exists a solution to (2-16). To this end, we define

$$x = \epsilon\xi, \quad |\beta| = \epsilon^2, \tag{2-17}$$

and (2-16) becomes

$$\pm \epsilon^2 \mathfrak{A}(\epsilon\xi) + \langle \mathfrak{f} \rangle(\epsilon\xi) = -\tfrac{1}{2} I^{-1} Q[\mathfrak{f}(I(\epsilon\xi) + q_\perp(I(\epsilon\xi))) - \mathfrak{f}(I(\epsilon\xi))].$$

On the one hand, (2-5) and (2-8) yield

$$\pm \epsilon^2 \mathfrak{A}(\epsilon\xi) + \langle \mathfrak{f} \rangle(\epsilon\xi) = \pm \epsilon^3 \mathfrak{A}\xi + \epsilon^2 \langle \mathfrak{f}^{(2)} \rangle(\xi) + \epsilon^3 \langle \mathfrak{f}^{(3)} \rangle(\xi) = \epsilon^3 (\pm \mathfrak{A}\xi + \langle \mathfrak{f}^{(3)} \rangle(\xi)).$$

On the other hand, (2-5), (2-8) and the averaging identity from Lemma 2.7 yield

$$\begin{aligned} &\tfrac{1}{2} I^{-1} Q[\mathfrak{f}(I(\epsilon\xi) + q_\perp(I(\epsilon\xi))) - \mathfrak{f}(I(\epsilon\xi))] \\ &\qquad = \tfrac{1}{2} I^{-1} Q[\mathfrak{f}^{(2)}(I(\epsilon\xi) + q_\perp(I(\epsilon\xi)))] + \tfrac{1}{2} I^{-1} Q[\mathfrak{f}^{(3)}(I(\epsilon\xi) + q_\perp(I(\epsilon\xi))) - \mathfrak{f}^{(3)}(I(\epsilon\xi))] \\ &\qquad = \tfrac{1}{2} I^{-1} Q[\mathfrak{f}^{(2)}(I(\epsilon\xi) + L_\varpi^{-1} P\mathfrak{f}^{(2)}(I(\epsilon\xi)))] + \epsilon^3 \mathfrak{G}_\epsilon(\xi), \end{aligned}$$

where we set

$$\begin{aligned} \mathfrak{G}_\epsilon(\xi) = \epsilon^{-3} \big[&\tfrac{1}{2} I^{-1} Q[\mathfrak{f}^{(2)}(I(\epsilon\xi) + q_\perp(I(\epsilon\xi))) - \mathfrak{f}^{(2)}(I(\epsilon\xi) + L_\varpi^{-1} P\mathfrak{f}^{(2)}(I(\epsilon\xi)))] \\ &+ \tfrac{1}{2} I^{-1} Q[\mathfrak{f}^{(3)}(I(\epsilon\xi) + q_\perp(I(\epsilon\xi))) - \mathfrak{f}^{(3)}(I(\epsilon\xi))] \big]. \end{aligned} \tag{2-18}$$

We apply the averaging identity and the notation from Lemma 2.7 to the map $v \to \mathfrak{f}^{(2)}(v + L_1^{-1} P\mathfrak{f}^{(2)}(v))$, which is a vector field from $\mathcal{H}_s^1$ to $\mathcal{H}_s^1$, to obtain

$$\begin{aligned} &\tfrac{1}{2} I^{-1} Q[\mathfrak{f}(I(\epsilon\xi) + q_\perp(I(\epsilon\xi))) - \mathfrak{f}(I(\epsilon\xi))] \\ &\quad = \tfrac{1}{2} I^{-1} Q[\mathfrak{f}^{(2)}(I(\epsilon\xi) + L_\varpi^{-1} P\mathfrak{f}^{(2)}(I(\epsilon\xi)))] + \epsilon^3 \mathfrak{G}_\epsilon(\xi) \\ &\quad = \tfrac{1}{2} I^{-1} Q[\mathfrak{f}^{(2)}(I(\epsilon\xi) + L_1^{-1} P\mathfrak{f}^{(2)}(I(\epsilon\xi)))] + \epsilon^3 \mathfrak{G}_\epsilon(\xi) + \epsilon^3 \mathfrak{R}_\epsilon(\xi, \varpi) \\ &\quad = \frac{1}{2} \int_0^{2\pi} \Phi^{-t} \big(\mathfrak{f}^{(2)}(\Phi^t(\epsilon\xi) + L_1^{-1} P\mathfrak{f}^{(2)}(\Phi^t(\epsilon\xi))) \big)\, dt + \epsilon^3 \mathfrak{G}_\epsilon(\xi) + \epsilon^3 \mathfrak{R}_\epsilon(\xi, \varpi) \\ &\quad = \langle \mathfrak{f}^{(2)}((\cdot) + L_1^{-1} P\mathfrak{f}^{(2)}(\cdot)) \rangle(\epsilon\xi) + \epsilon^3 \mathfrak{G}_\epsilon(\xi) + \epsilon^3 \mathfrak{R}_\epsilon(\xi, \varpi) = \epsilon^3 (\mathfrak{F}_\epsilon(\xi) + \mathfrak{G}_\epsilon(\xi) + \mathfrak{R}_\epsilon(\xi, \varpi)), \end{aligned}$$

where we set

$$\begin{aligned} \mathfrak{R}_\epsilon(\xi, \varpi) &= \epsilon^{-3} \tfrac{1}{2} I^{-1} Q[\mathfrak{f}^{(2)}(I(\epsilon\xi) + L_\varpi^{-1} P\mathfrak{f}^{(2)}(I(\epsilon\xi))) - \mathfrak{f}^{(2)}(I(\epsilon\xi) + L_1^{-1} P\mathfrak{f}^{(2)}(I(\epsilon\xi)))], \\ \mathfrak{F}_\epsilon(\xi) &= \epsilon^{-3} \langle \mathfrak{f}^{(2)}((\cdot) + L_1^{-1} P\mathfrak{f}^{(2)}(\cdot)) \rangle(\epsilon\xi). \end{aligned}$$

In conclusion, the Q-equation (2-16) can be written equivalently, for $\epsilon > 0$ sufficiently small, as

$$\pm\mathfrak{A}\xi + \langle \mathfrak{f}^{(3)}\rangle(\xi) = -(\mathfrak{F}_\epsilon(\xi) + \mathfrak{G}_\epsilon(\xi) + \mathfrak{R}_\epsilon(\xi, \varpi)). \tag{2-19}$$

However, instead of (2-19), we focus on a modified version, namely

$$\pm\mathfrak{A}\xi + \langle \mathfrak{f}^{(3)}\rangle(\xi) = -\left(\mathfrak{F}_\epsilon(\xi) + \mathfrak{G}_\epsilon(\xi) \pm \frac{2\epsilon^2}{\varpi^2 - 1}\mathfrak{R}_\epsilon(\xi, \varpi)\right). \tag{2-20}$$

Notice that (2-19) coincides with (2-20) provided that $\varpi^2 - 1 = \pm 2\epsilon^2$.

Remark 2.9. Since $\mathfrak{f}^{(2)}$ is differentiable and quadratic, and $\langle \mathfrak{f}^{(2)}\rangle(\xi) = 0$ for all initial data $\xi \in l^2_{s+3}$, it follows that $\mathfrak{F}_\epsilon(\xi)$ is differentiable and $\|\mathfrak{F}_\epsilon(\xi)\|_{\mathcal{Q}} \lesssim 1$. Later, in Section 2F, we compute the exact expressions of $\mathfrak{F}_0(\xi)$ for general initial data (Lemma 2.13), $\mathfrak{F}_\epsilon(\xi)$ for small ϵ close to zero and 1-mode initial data (Lemma 2.14), as well as the differential $d\mathfrak{F}_0(\xi)$ at the 1-mode initial data (Lemma 2.15).

In the following, we estimate the error terms. To begin with, we estimate $\mathfrak{R}_\epsilon(\xi, \varpi)$.

Lemma 2.10 (estimate for $\mathfrak{R}_\epsilon(\xi, \varpi)$ and $d_\xi\mathfrak{R}_\epsilon(\xi, \varpi)$). *Let $0 < \alpha < \frac{1}{3}$, and pick any $\varpi \in \mathcal{W}_\alpha$. Also, let $\xi \in l^2_{s+3}$ be any initial data. Then, we have*

$$\|\mathfrak{R}_\epsilon(\xi, \varpi)\|_{\mathcal{Q}} \lesssim |\varpi^2 - 1|, \qquad \|d_\xi\mathfrak{R}_\epsilon(\xi, \varpi)[h]\|_{\mathcal{Q}} \lesssim |\varpi^2 - 1|\,|h|_{s+3}.$$

Proof. Let $0 < \alpha < \frac{1}{3}$, and pick any $\varpi \in \mathcal{W}_\alpha$. Also, let $\xi \in l^2_{s+3}$ be any initial data. Firstly, we pick any $\epsilon > 0$, set $v = I(\epsilon\xi)$ and compute

$$\begin{aligned}
\mathfrak{f}^{(2)}(v) &= \sum_{j=0}^{\infty} (\mathfrak{f}^{(2)}(v))^j e_j = \sum_{j=0}^{\infty}\left(\sum_{l=0}^{\infty} (\mathfrak{f}^{(2)}(v))^j_l \cos(lt)\right) e_j,\\
P\mathfrak{f}^{(2)}(v) &= \sum_{j=0}^{\infty}\left(\sum_{\substack{l=0\\ l\neq\varpi_j}}^{\infty} (\mathfrak{f}^{(2)}(v))^j_l \cos(lt)\right) e_j,\\
L_\varpi^{-1}P\mathfrak{f}^{(2)}(v) &= \sum_{j=0}^{\infty}\left(\sum_{\substack{l=0\\ l\neq\varpi_j}}^{\infty} \frac{1}{\varpi_j^2 - l^2\varpi^2}(\mathfrak{f}^{(2)}(v))^j_l \cos(lt)\right) e_j,\\
(L_\varpi^{-1} - L_1^{-1})P\mathfrak{f}^{(2)}(v) &= \sum_{j=0}^{\infty}\left(\sum_{\substack{l=0\\ l\neq\varpi_j}}^{\infty} \left(\frac{1}{\varpi_j^2 - l^2\varpi^2} - \frac{1}{\varpi_j^2 - l^2}\right)(\mathfrak{f}^{(2)}(v))^j_l \cos(lt)\right) e_j\\
&= \sum_{j=0}^{\infty}\left(\sum_{\substack{l=0\\ l\neq\varpi_j}}^{\infty} \frac{l^2(\varpi^2 - 1)}{(\varpi_j^2 - l^2\varpi^2)(\varpi_j^2 - l^2)}(\mathfrak{f}^{(2)}(v))^j_l \cos(lt)\right) e_j.
\end{aligned}$$

Secondly, we note that

$$\|L_\varpi^{-1}P\mathfrak{f}^{(2)}(I(\epsilon\xi))\|_{\mathcal{H}^1_s} \lesssim \epsilon^2, \quad \|L_1^{-1}P\mathfrak{f}^{(2)}(I(\epsilon\xi))\|_{\mathcal{H}^1_s} \lesssim \epsilon^2. \tag{2-21}$$

These can be easily proved using the Diophantine condition, the elementary inequality $|\varpi_j^2 - l^2| \geq 1$ (since $\varpi \in \mathcal{W}_\alpha$, both $\varpi_j^2 \geq 1$ and $l^2 \geq 0$ are integers with $\varpi_j \neq l$), the Lipschitz estimate $\|\mathfrak{f}^{(2)}(u)\|_{\mathcal{H}^k_s} \lesssim_s \|u\|^2_{\mathcal{H}^k_s}$

for all $u \in \mathcal{H}^k_s$ with $\|u\|_{\mathcal{H}^k_s} \leq \epsilon$ (which follows from Lemma 4.5), together with (2-15). Indeed, we infer

$$\begin{aligned}
\|L_{\varpi}^{-1} P\mathfrak{f}^{(2)}(I(\epsilon\xi))\|^2_{\mathcal{H}^1_s} &= \sum_{j=0}^{\infty} j^{2s} \sum_{l=0}^{\infty} \left| \frac{1}{\varpi_j^2 - l^2\varpi^2} (\mathfrak{f}^{(2)}(I(\epsilon\xi)))^j_l \right|^2 (1+l^2) \\
&\lesssim_\alpha \sum_{j=0}^{\infty} j^{2s} \sum_{l=0}^{\infty} |(\mathfrak{f}^{(2)}(I(\epsilon\xi)))^j_l|^2 (1+l^2) \\
&\leq \|\mathfrak{f}^{(2)}(I(\epsilon\xi))\|^2_{\mathcal{H}^1_s} \lesssim \|I(\epsilon\xi)\|^4_{\mathcal{H}^1_s} = \epsilon^4 \|I(\xi)\|^4_{\mathcal{H}^1_s} = \epsilon^4 \|\xi\|^4_{\mathcal{Q}} \leq \epsilon^4 |\xi|^4_{s+3}, \\
\|L_1^{-1} P\mathfrak{f}^{(2)}(I(\epsilon\xi))\|^2_{\mathcal{H}^1_s} &= \sum_{j=0}^{\infty} j^{2s} \sum_{l=0}^{\infty} \left| \frac{1}{\varpi_j^2 - l^2} (\mathfrak{f}^{(2)}(I(\epsilon\xi)))^j_l \right|^2 (1+l^2) \\
&\lesssim \sum_{j=0}^{\infty} j^{2s} \sum_{l=0}^{\infty} |(\mathfrak{f}^{(2)}(I(\epsilon\xi)))^j_l|^2 (1+l^2) \\
&\leq \|\mathfrak{f}^{(2)}(I(\epsilon\xi))\|^2_{\mathcal{H}^1_s} \lesssim \|I(\epsilon\xi)\|^4_{\mathcal{H}^1_s} = \epsilon^4 \|I(\xi)\|^4_{\mathcal{H}^1_s} = \epsilon^4 \|\xi\|^4_{\mathcal{Q}} \leq \epsilon^4 |\xi|^4_{s+3}.
\end{aligned}$$

Next, we use the above together with the Lipschitz estimate for $\mathfrak{f}^{(2)}$ (see Lemma 4.5) and the fact that $I^{-1} : \mathcal{H}^1_s \to \mathcal{Q}$ to obtain

$$\begin{aligned}
\epsilon^3 \|\mathfrak{R}_\epsilon(\xi,\varpi)\|_{\mathcal{Q}} &= \left\| \tfrac{1}{2} I^{-1} Q[\mathfrak{f}^{(2)}(I(\epsilon\xi) + L_{\varpi}^{-1} P\mathfrak{f}^{(2)}(I(\epsilon\xi))) - \mathfrak{f}^{(2)}(I(\epsilon\xi) + L_1^{-1} P\mathfrak{f}^{(2)}(I(\epsilon\xi)))] \right\|_{\mathcal{Q}} \\
&\lesssim \|\mathfrak{f}^{(2)}(I(\epsilon\xi) + L_{\varpi}^{-1} P\mathfrak{f}^{(2)}(I(\epsilon\xi))) - \mathfrak{f}^{(2)}(I(\epsilon\xi) + L_1^{-1} P\mathfrak{f}^{(2)}(I(\epsilon\xi)))\|_{\mathcal{H}^1_s} \\
&\lesssim (\|I(\epsilon\xi) + L_{\varpi}^{-1} P\mathfrak{f}^{(2)}(I(\epsilon\xi))\|_{\mathcal{H}^1_s} + \|I(\epsilon\xi) + L_1^{-1} P\mathfrak{f}^{(2)}(I(\epsilon\xi))\|_{\mathcal{H}^1_s}) \\
&\qquad \cdot \|L_{\varpi}^{-1} P\mathfrak{f}^{(2)}(I(\epsilon\xi)) - L_1^{-1} P\mathfrak{f}^{(2)}(I(\epsilon\xi))\|_{\mathcal{H}^1_s} \\
&\lesssim \epsilon \|(L_{\varpi}^{-1} - L_1^{-1}) P\mathfrak{f}^{(2)}(I(\epsilon\xi))\|_{\mathcal{H}^1_s}.
\end{aligned}$$

Once again, the Diophantine condition, the elementary inequality $|\varpi_j^2 - l^2| \geq 1$, the Lipschitz estimate $\|\mathfrak{f}^{(2)}(u)\|_{\mathcal{H}^k_s} \lesssim_s \|u\|^2_{\mathcal{H}^k_s}$ for all $u \in \mathcal{H}^k_s$ with $\|u\|_{\mathcal{H}^k_s} \leq \epsilon$ (which follows from Lemma 4.5), together with (2-15) imply that

$$\begin{aligned}
&\|(L_{\varpi}^{-1} - L_1^{-1}) P\mathfrak{f}^{(2)}(I(\epsilon\xi))\|^2_{\mathcal{H}^1_s} \\
&\quad = |\varpi^2 - 1|^2 \sum_{j=0}^{\infty} j^{2s} \sum_{l=1}^{\infty} \left| \frac{l^2}{(\varpi_j^2 - l^2\varpi^2)(\varpi_j^2 - l^2)} (\mathfrak{f}^{(2)}(I(\epsilon\xi)))^j_l \right|^2 (1+l^2) \\
&\quad \lesssim_\alpha |\varpi^2 - 1|^2 \sum_{j=0}^{\infty} j^{2s} \sum_{l=1}^{\infty} |l^2 (\mathfrak{f}^{(2)}(I(\epsilon\xi)))^j_l|^2 (1+l^2) \lesssim |\varpi^2 - 1|^2 \sum_{j=0}^{\infty} j^{2s} \sum_{l=1}^{\infty} |(\mathfrak{f}^{(2)}(I(\epsilon\xi)))^j_l|^2 (1+l^2)^3 \\
&\quad \leq |\varpi^2 - 1|^2 \|\mathfrak{f}^{(2)}(I(\epsilon\xi))\|^2_{\mathcal{H}^3_s} \lesssim |\varpi^2 - 1|^2 \|I(\epsilon\xi)\|^4_{\mathcal{H}^3_s} = |\varpi^2 - 1|^2 \epsilon^4 \|I(\xi)\|^4_{\mathcal{H}^3_s} \lesssim |\varpi^2 - 1|^2 \epsilon^4 |\xi|^4_{s+3}.
\end{aligned}$$

Finally, putting this all together yields $\|\mathfrak{R}_\epsilon(\xi,\varpi)\|_{\mathcal{Q}} \lesssim |\varpi^2 - 1|$, which completes the first part of the proof. The estimate for the differential follows similarly. □

Next, we estimate $\mathfrak{G}_\epsilon(\xi)$ and its differential.

Lemma 2.11 (estimate for $\mathfrak{G}_\epsilon(\xi)$ and $d_\xi\mathfrak{G}_\epsilon(\xi)$). *Let $\xi \in l^2_{s+3}$ be any initial data. Then, $\mathfrak{G}_\epsilon(\xi)$ is continuously differentiable with respect to ξ, and we have*

$$\|\mathfrak{G}_\epsilon(\xi)\|_{\mathcal{Q}} \lesssim \epsilon, \quad \|d_\xi\mathfrak{G}_\epsilon(\xi)[h]\|_{\mathcal{Q}} \lesssim \epsilon|h|_{s+3}.$$

Proof. Let $\xi \in l^2_{s+3}$ be any initial data, and recall the definition of $\mathfrak{G}_\epsilon(\xi)$ from (2-18). The claim follows by Lemmas 4.5 and 2.6 together with (2-15) and the fact that $I^{-1} : \mathcal{H}^1_s \to \mathcal{Q}$. Indeed, since

$$\|I(\epsilon\xi)\|_{\mathcal{H}^1_s} \lesssim \epsilon\|\xi\|_{\mathcal{Q}}, \quad \|q_\perp(I(\epsilon\xi))\|_{\mathcal{H}^1_s} \lesssim \|I(\epsilon\xi)\|^2_{\mathcal{H}^1_s} \lesssim \epsilon^2\|\xi\|^2_{\mathcal{Q}}, \tag{2-22}$$

we can estimate

$$\begin{aligned}\Big\|\tfrac{1}{2}I^{-1}Q[\mathfrak{f}^{(2)}(I(\epsilon\xi)+q_\perp(I(\epsilon\xi))) - \mathfrak{f}^{(2)}(I(\epsilon\xi)+L_\varpi^{-1}P\mathfrak{f}^{(2)}(I(\epsilon\xi)))]\Big\|_{\mathcal{Q}}&\\ \lesssim \|\mathfrak{f}^{(2)}(I(\epsilon\xi)+q_\perp(I(\epsilon\xi))) - \mathfrak{f}^{(2)}(I(\epsilon\xi)+L_\varpi^{-1}P\mathfrak{f}^{(2)}(I(\epsilon\xi)))\|_{\mathcal{H}^1_s}&\\ \lesssim \epsilon\|q_\perp(I(\epsilon\xi)) - L_\varpi^{-1}P\mathfrak{f}^{(2)}(I(\epsilon\xi))\|_{\mathcal{H}^1_s} \lesssim \epsilon\|I(\epsilon\xi)\|^3_{\mathcal{H}^1_s} \lesssim \epsilon^4&\end{aligned}$$

and

$$\begin{aligned}&\Big\|\tfrac{1}{2}I^{-1}Q[\mathfrak{f}^{(3)}(I(\epsilon\xi)+q_\perp(I(\epsilon\xi))) - \mathfrak{f}^{(3)}(I(\epsilon\xi))]\Big\|_{\mathcal{Q}}\\ &\lesssim \|\mathfrak{f}^{(3)}(I(\epsilon\xi)+q_\perp(I(\epsilon\xi))) - \mathfrak{f}^{(3)}(I(\epsilon\xi))\|_{\mathcal{H}^1_s} \lesssim [\|I(\epsilon\xi)+q_\perp(I(\epsilon\xi))\|^2_{\mathcal{H}^1_s} + \|I(\epsilon\xi)\|^2_{\mathcal{H}^1_s}]\|q_\perp(I(\epsilon\xi))\|_{\mathcal{H}^1_s}\\ &\lesssim [\|I(\epsilon\xi)\|^2_{\mathcal{H}^1_s} + \|q_\perp(I(\epsilon\xi))\|^2_{\mathcal{H}^1_s}]\|q_\perp(I(\epsilon\xi))\|_{\mathcal{H}^1_s} \lesssim \epsilon^4.\end{aligned}$$

The estimate for the differential follows similarly. □

It remains to show that there exists a solution to (2-20), that is

$$\pm\mathfrak{A}\xi + \langle\mathfrak{f}^{(3)}\rangle(\xi) = -\Big(\mathfrak{F}_\epsilon(\xi) + \mathfrak{G}_\epsilon(\xi) \pm \frac{2\epsilon^2}{\varpi^2-1}\mathfrak{R}_\epsilon(\xi,\varpi)\Big) \iff \mathfrak{M}_\pm(\xi) = \mathfrak{H}_\epsilon(\xi), \tag{2-23}$$

where we set

$$\mathfrak{M}_\pm(\xi) = \pm\mathfrak{A}\xi + \langle\mathfrak{f}^{(3)}\rangle(\xi) + \mathfrak{F}_0(\xi), \quad \mathfrak{H}_\epsilon(\xi) = \mathfrak{F}_0(\xi) - \mathfrak{F}_\epsilon(\xi) - \mathfrak{G}_\epsilon(\xi) \mp \frac{2\epsilon^2}{\varpi^2-1}\mathfrak{R}_\epsilon(\xi,\varpi).$$

We refer to $\mathfrak{M}_\pm$ as the *modified operator*. Note that Lemmas 2.10 and 2.11 and the smoothness[8] of $\mathfrak{F}_\epsilon(\xi)$ with respect to ϵ yield

$$\|\mathfrak{H}_\epsilon(\xi)\|_{\mathcal{Q}} \lesssim \epsilon, \quad \|d_\xi\mathfrak{H}_\epsilon(\xi)[h]\|_{\mathcal{Q}} \lesssim \epsilon|h|_{s+3}.$$

The following result constitutes the main modification of Bambusi–Paleari's theorem.

Lemma 2.12 (solution to the Q-equation). *Define the **modified operator***

$$\mathfrak{M}_\pm(x) = \pm\mathfrak{A}x + \langle\mathfrak{f}^{(3)}\rangle(x) + \mathfrak{F}_0(x),$$

and let $\xi_0 \in l^2_{s+3}$ be a nondegenerate zero of $\mathfrak{M}_\pm$, that is

$$\mathfrak{M}_\pm(\xi_0) = 0, \quad \ker(d\mathfrak{M}_\pm(\xi_0)) = \{0\}.$$

[8] In Lemma 2.13 we show that $\mathfrak{F}_\epsilon(\xi)$ is smooth with respect to ϵ. As one can see in the proof of Lemma 2.13, $\mathfrak{F}_\epsilon(\xi)$ is in fact linear with respect to ϵ. See (2-25).

Then, there exists a positive ϵ_0 *and a Lipschitz map* $\xi : [0,\epsilon_0) \to l^2_{s+3}$, $\epsilon \mapsto \xi(\epsilon)$ *that solves the* Q*-equation* (2-19) *with the plus sign. Furthermore, we have the estimate* $|\xi(\epsilon)-\xi_0|_{s+3} \lesssim \epsilon$.

Proof. The proof follows from the implicit function theorem and is similar to the one of Proposition 4.8 in [Bambusi and Paleari 2001]. Let $\xi_0 \in l^2_{s+3}$ be a nondegenerate zero of the modified operator $\mathfrak{M}_\pm$, and define the map

$$\mathcal{G} : \mathbb{R} \times l^2_{s+3} \to \mathcal{H}^1_s, \quad (\epsilon,\xi) \mapsto \mathcal{G}(\epsilon,\xi) = \mathfrak{M}_\pm(\xi) - \mathfrak{H}_\epsilon(\xi),$$

and note that it is Lipschitz, differentiable at $\epsilon = 0$ and it vanishes at $(\epsilon,\xi) = (0,\xi_0)$. It remains to show that its differential with respect to ξ at $(0,\xi_0)$, namely

$$d\mathcal{G}(0,\xi_0) : l^2_{s+3} \to \mathcal{H}^1_s, \quad X \mapsto d\mathcal{G}(0,\xi_0)(X) = d\mathfrak{M}_\pm(\xi_0)(X),$$

is an isomorphism. Equivalently, this means that, for all $Y \in \mathcal{H}^1_s$, there exists $X \in l^2_{s+3}$ that solves the equation $d\mathfrak{M}_\pm(\xi_0)(X) = Y$, with

$$d\mathfrak{M}_\pm(\xi_0) = \pm\mathfrak{A} + d\langle \mathfrak{f}^{(3)}\rangle(\xi_0) + d\mathfrak{F}_0(\xi_0).$$

Now, the operator $d\mathfrak{M}_\pm$ is a Fredholm operator since it is the sum of a Fredholm and a compact operator due to the facts that $\mathfrak{f}^{(3)}$ and $\mathfrak{F}_\epsilon(\xi)$ are bounded on l^2_{s+3} and that they are differentiable with bounded differential. Since the defect index of $d\mathfrak{M}_\pm(\xi_0)$ is 0 from the nondegeneracy condition, it follows that it is an isomorphism, and thus we can apply the implicit function theorem. Note finally, that the range of ϵ, defined by ϵ_0, does not depend on ϖ, since $\mathcal{G}$ depends continuously on ϖ and all the necessary bounds hold uniformly with respect to ϖ. □

Finally, we prove Theorem 2.5.

Proof of Theorem 2.5. Let $\varpi \in \mathcal{W}_\alpha$ be fixed. Then, according to Lemmas 2.6 and 2.12, there exists $\epsilon_0 > 0$ such that the map $[0,\epsilon_0) \ni \epsilon \mapsto \big(\epsilon I(\xi(\epsilon)), q_\perp(\epsilon I(\xi(\epsilon)))\big)$ solves both the P-equation (2-11) and the Q-equation (2-12). Furthermore, pick $\varpi_\star$ such that $\epsilon(\varpi_\star) = \epsilon_0$. Then, the function $\epsilon^2(\varpi) = \pm\frac{1}{2}(\varpi^2 - 1)$ solves (2-13), and the map $\varpi \mapsto \big(\epsilon I(\xi(\epsilon(\varpi))), q_\perp\big(\epsilon I(\xi(\epsilon(\varpi)))\big)\big)$ defines a family of solutions to (2-10) labeled by $\varpi \in \mathcal{W}_\alpha \cap [1, 1+\varpi_\star]$ or $\varpi \in \mathcal{W}_\alpha \cap [1-\varpi_\star, 1]$. Finally, the map $\epsilon \mapsto \epsilon(\varpi)$ is one-to-one, and hence this family can be also parametrized by

$$\epsilon \in \mathcal{E}_{\alpha,\gamma} = \epsilon(\mathcal{W}_\alpha \cap [1, 1+\varpi_\star]) \quad \text{or} \quad \epsilon \in \mathcal{E}_{\alpha,\gamma} = \epsilon(\mathcal{W}_\alpha \cap [1-\varpi_\star, 1]).$$ □

2F. ***The function*** $\mathfrak{F}_\epsilon(\xi)$**.** In Lemma 5.8, we prove that $\overline{\mathfrak{C}}_{ijm} = 0$ for all integers $i, j, m \geq 0$ with $i + j < m$. Moreover, by its definition (2-6), $\overline{\mathfrak{C}}_{ijm}$ is also invariant under any permutation of the indices i, j, m. For future reference, we now use these additional properties of the Fourier coefficient $\overline{\mathfrak{C}}$ to compute:

- $\mathfrak{F}_0(\xi)$ for general initial data (Lemma 2.13),
- $\mathfrak{F}_\epsilon(\xi)$ for small ϵ close to zero and 1-mode initial data (Lemma 2.14),
- $d\mathfrak{F}_0(\xi)$ at the 1-mode initial data (Lemma 2.15).

Lemma 2.13 (computation of $\mathfrak{F}_0(\xi)$ for general initial data). *Let $\xi = \{\xi^m : m \geq 0\} \in l^2_{s+3}$. Then, for all integers $m \geq 0$, we have*

$$(\mathfrak{F}_0(\xi))^m = \frac{9}{4} \sum_{\kappa,\nu \geq 0} \overline{\mathfrak{C}}_{\kappa\nu m} \sum_{\substack{i,j \geq 0 \\ \varpi_i - \varpi_j \neq \pm\varpi_\nu}} \frac{\overline{\mathfrak{C}}_{ij\nu}}{\varpi_\nu^2 - (\varpi_i - \varpi_j)^2} \xi^i \xi^j \xi^\kappa \sum_{\pm} \mathbb{1}(\varpi_i - \varpi_j \pm \varpi_\kappa \pm \varpi_m = 0)$$
$$+ \frac{9}{4} \sum_{\kappa,\nu \geq 0} \overline{\mathfrak{C}}_{\kappa\nu m} \sum_{\substack{i,j \geq 0 \\ \varpi_i + \varpi_j \neq \pm\varpi_\nu}} \frac{\overline{\mathfrak{C}}_{ij\nu}}{\varpi_\nu^2 - (\varpi_i + \varpi_j)^2} \xi^i \xi^j \xi^\kappa \sum_{\pm} \mathbb{1}(\varpi_i + \varpi_j \pm \varpi_\kappa \pm \varpi_m = 0).$$

In addition, the function $\mathfrak{F}_\epsilon$ is smooth with respect to ϵ.

Proof. Let $\xi = \{\xi^m : m \geq 0\} \in l^2_{s+3}$ be any initial data, let $\epsilon > 0$, set $x = \epsilon\xi$ and pick any integer $m \geq 0$. Then, we compute

$$(\mathfrak{f}^{(2)}(\{u^k : k \geq 0\}))^m = -3 \sum_{i,j \geq 0} \overline{\mathfrak{C}}_{ijm} u^i u^j,$$
$$(\mathfrak{f}^{(2)}(\Phi^t(x)))^m = -3 \sum_{i,j \geq 0} \overline{\mathfrak{C}}_{ijm} (\Phi^t(x))^i (\Phi^t(x))^j = -3 \sum_{i,j \geq 0} \overline{\mathfrak{C}}_{ijm} x^i x^j \cos(\varpi_i t) \cos(\varpi_j t)$$
$$= -\frac{3}{2} \sum_{i,j \geq 0} \overline{\mathfrak{C}}_{ijm} x^i x^j \cos((\varpi_i - \varpi_j)t) - \frac{3}{2} \sum_{i,j \geq 0} \overline{\mathfrak{C}}_{ijm} x^i x^j \cos((\varpi_i + \varpi_j)t).$$

Then, $(P\mathfrak{f}^{(2)}(\Phi^t(x)))^m$ is given by

$$-\frac{3}{2}\Bigg[\sum_{\substack{i,j \geq 0 \\ \varpi_i - \varpi_j \neq \pm\varpi_m}} \overline{\mathfrak{C}}_{ijm} x^i x^j \cos((\varpi_i - \varpi_j)t) + \sum_{\substack{i,j \geq 0 \\ \varpi_i + \varpi_j \neq \pm\varpi_m}} \overline{\mathfrak{C}}_{ijm} x^i x^j \cos((\varpi_i + \varpi_j)t) \Bigg],$$

and $(L_\varpi^{-1} P\mathfrak{f}^{(2)}(\Phi^t(x)))^m$ reads

$$-\frac{3}{2}\Bigg[\sum_{\substack{i,j \geq 0 \\ \varpi_i - \varpi_j \neq \pm\varpi_m}} \frac{\overline{\mathfrak{C}}_{ijm}}{\lambda_{m,\varpi_i - \varpi_j}} x^i x^j \cos((\varpi_i - \varpi_j)t) + \sum_{\substack{i,j \geq 0 \\ \varpi_i + \varpi_j \neq \pm\varpi_m}} \frac{\overline{\mathfrak{C}}_{ijm}}{\lambda_{m,\varpi_i + \varpi_j}} x^i x^j \cos((\varpi_i + \varpi_j)t) \Bigg],$$

where we used the fact that the eigenvalues of L_ϖ are given by $\lambda_{ml} = \varpi_m^2 - l^2\varpi^2$. Hence, using the above together with the symmetries of the Fourier coefficients $\overline{\mathfrak{C}}_{\kappa\nu m} = \overline{\mathfrak{C}}_{\nu\kappa m}$ for all integers $\kappa, \nu, m \geq 0$, we deduce that

$$\big(\mathfrak{f}^{(2)}(\Phi^t(x) + L_1^{-1} P\mathfrak{f}^{(2)}(\Phi^t(x)))\big)^m$$
$$= -3 \sum_{\kappa,\nu \geq 0} \overline{\mathfrak{C}}_{\kappa\nu m} (\Phi^t(x))^\kappa (\Phi^t(x))^\nu - 6 \sum_{\kappa,\nu \geq 0} \overline{\mathfrak{C}}_{\kappa\nu m} (\Phi^t(x))^\kappa (L_1^{-1} P\mathfrak{f}^{(2)}(\Phi^t(x)))^\nu$$
$$- 3 \sum_{\kappa,\nu \geq 0} \overline{\mathfrak{C}}_{\kappa\nu m} (L_1^{-1} P\mathfrak{f}^{(2)}(\Phi^t(x)))^\kappa (L_1^{-1} P\mathfrak{f}^{(2)}(\Phi^t(x)))^\nu$$

and, by setting $x = \epsilon\xi$, we infer that

$$\big(\mathfrak{f}^{(2)}(\Phi^t(\epsilon\xi) + L_1^{-1} P\mathfrak{f}^{(2)}(\Phi^t(\epsilon\xi)))\big)^m = \epsilon^2 (\mathfrak{f}^{(2)}(\Phi^t(\xi)))^m + \epsilon^3 (E(\xi))^m + \epsilon^4 (F(\xi))^m, \qquad (2\text{-}24)$$

where

$$(F(\xi))^m = -3 \sum_{\kappa,\nu\geq 0} \overline{\mathfrak{C}}_{\kappa\nu m} (L_1^{-1} P\mathfrak{f}^{(2)}(\Phi^t(\xi)))^\kappa (L_1^{-1} P\mathfrak{f}^{(2)}(\Phi^t(\xi)))^\nu,$$

$$\begin{aligned}(E(\xi))^m &= -6 \sum_{\kappa,\nu\geq 0} \overline{\mathfrak{C}}_{\kappa\nu m} (\Phi^t(\xi))^\kappa (L_1^{-1} P\mathfrak{f}^{(2)}(\Phi^t(\xi)))^\nu \\ &= -6 \sum_{\kappa,\nu\geq 0} \overline{\mathfrak{C}}_{\kappa\nu m} \xi^\kappa \cos(\varpi_\kappa t) (L_1^{-1} P\mathfrak{f}^{(2)}(\Phi^t(\xi)))^\nu.\end{aligned}$$

We set $(E(\xi))^m = (E(\xi))^m_- + (E(\xi))^m_+$, where

$$(E(\xi))^m_\pm = 9 \sum_{\kappa,\nu\geq 0} \overline{\mathfrak{C}}_{\kappa\nu m} \sum_{\substack{i,j\geq 0 \\ |\varpi_i \pm \varpi_j| \neq |\varpi_\nu|}} \frac{\overline{\mathfrak{C}}_{ij\nu}}{\varpi_\nu^2 - (\varpi_i \pm \varpi_j)^2} \xi^i \xi^j \xi^\kappa \cos((\varpi_i \pm \varpi_j)t) \cos(\varpi_\kappa t).$$

Next, we first apply the linear flow and then the average in time to obtain

$$\begin{aligned}(\mathfrak{F}_\epsilon(\xi))^m &= \epsilon^{-3} (\langle \mathfrak{f}^{(2)}((\cdot) + L_1^{-1} P\mathfrak{f}^{(2)}(\cdot))\rangle(\epsilon\xi))^m \\ &= \frac{\epsilon^{-3}}{2\pi} \int_0^{2\pi} \big(\Phi^t(\mathfrak{f}^{(2)}(\Phi^t(\epsilon\xi) + L_1^{-1} P\mathfrak{f}^{(2)} \Phi^t(\epsilon\xi)))\big)^m \, dt \\ &= \frac{\epsilon^{-1}}{2\pi} \int_0^{2\pi} \big(\Phi^t(\mathfrak{f}^{(2)}(\Phi^t(\xi)))\big)^m \, dt + \frac{1}{2\pi} \int_0^{2\pi} (\Phi^t(E(\xi)))^m \, dt + \frac{\epsilon}{2\pi} \int_0^{2\pi} (\Phi^t(F(\xi)))^m \, dt \\ &= \epsilon^{-1} \langle \mathfrak{f}^{(2)}\rangle(\xi) + \frac{1}{2\pi} \int_0^{2\pi} (\Phi^t(E(\xi)))^m \, dt + \frac{\epsilon}{2\pi} \int_0^{2\pi} (\Phi^t(F(\xi)))^m \, dt \\ &= \frac{1}{2\pi} \int_0^{2\pi} (\Phi^t(E(\xi)))^m \, dt + \frac{\epsilon}{2\pi} \int_0^{2\pi} (\Phi^t(F(\xi)))^m \, dt, \end{aligned} \tag{2-25}$$

where we used the condition that $\mathfrak{f}^{(2)}$ is nonresonant (2-8). Then, since $(E(\xi))^m = (E(\xi))^m_- + (E(\xi))^m_+$, the latter at $\epsilon = 0$ boils down to

$$\mathfrak{F}_0(\xi) = \frac{1}{2\pi} \int_0^{2\pi} (\Phi^t(E(\xi)))^m \, dt = \frac{1}{2\pi} \int_0^{2\pi} ((E(\xi))^m_- + (E(\xi))^m_+) \cos(\varpi_m t) \, dt.$$

Finally, we use the facts that

$$\int_0^{2\pi} \cos((\varpi_i - \varpi_j)t) \cos(\varpi_\kappa t) \cos(\varpi_m t) \, dt = \frac{\pi}{2} \sum_{\pm} \mathbb{1}(\varpi_i - \varpi_j \pm \varpi_\kappa \pm \varpi_m = 0)$$

to compute

$$\begin{aligned}&\frac{1}{2\pi} \int_0^{2\pi} (E(\xi))^m_- \cos(\varpi_m t) \, dt \\ &\qquad = \frac{9}{4} \sum_{\kappa,\nu\geq 0} \overline{\mathfrak{C}}_{\kappa\nu m} \sum_{\substack{i,j\geq 0 \\ |\varpi_i - \varpi_j| \neq |\varpi_\nu|}} \frac{\overline{\mathfrak{C}}_{ij\nu}}{\varpi_\nu^2 - (\varpi_i - \varpi_j)^2} \xi^i \xi^j \xi^\kappa \sum_{\pm} \mathbb{1}(\varpi_i - \varpi_j \pm \varpi_\kappa \pm \varpi_m = 0).\end{aligned}$$

The term with the $(E(\xi))^m_+$ follows similarly and completes the proof. □

Lemma 2.14 (computation of $\mathfrak{F}_\epsilon(\xi)$ for small ϵ close to zero and 1-mode initial data). *Assume that $\varpi_n = n+2$ for all integers $n \geq 0$. Let $\gamma \geq 0$ be an integer, $\mathfrak{K}_\gamma \in \mathbb{R}$ and ξ be the* 1*-mode initial data, that is,*

$$\xi^m = \mathfrak{K}_\gamma \mathbb{1}(m = \gamma), \quad m \geq 0.$$

Then, for all integers $m \geq 0$, we have

$$(\mathfrak{F}_\epsilon(\xi))^m = \mathfrak{q}_\gamma \mathfrak{K}_\gamma^3 \mathbb{1}(m = \gamma) + \mathcal{O}(\epsilon), \quad \mathfrak{q}_\gamma = \frac{9}{4}\sum_{\nu=0}^{2\gamma}(\overline{\mathfrak{C}}_{\gamma\gamma\nu})^2\left(\frac{2}{\varpi_\nu^2} + \frac{\mathbb{1}(\varpi_\nu^2 \neq (2\varpi_\gamma)^2)}{\varpi_\nu^2 - (2\varpi_\gamma)^2}\right).$$

Proof. Let $\gamma \geq 0$ be an integer, and define ξ as the 1-mode initial data, that is $\xi^m = \mathfrak{K}_\gamma \mathbb{1}(m = \gamma)$, for all integers $m \geq 0$. Then, for any integer $m \geq 0$, we use the definition (2-6) to compute

$$\begin{aligned}(\mathfrak{f}^{(2)}(\Phi^t(\xi)))^m &= -3\sum_{i,j\geq 0}\overline{\mathfrak{C}}_{ijm}(\Phi^t(\xi))^i(\Phi^t(\xi))^j \\ &= -3\sum_{i,j\geq 0}\overline{\mathfrak{C}}_{ijm}\xi^i\xi^j\cos(\varpi_i t)\cos(\varpi_j t) \\ &= -3\mathfrak{K}_\gamma^2\overline{\mathfrak{C}}_{\gamma\gamma m}\cos^2(\varpi_\gamma t) = -\tfrac{3}{2}\mathfrak{K}_\gamma^2\overline{\mathfrak{C}}_{\gamma\gamma m}[1+\cos(2\varpi_\gamma t)].\end{aligned}$$

Recall the definition of the eigenvalues $\varpi_i = i+2$ for all integers $i \geq 0$. Then, we have

$$\varpi_m \neq 0 \iff m \geq 0 \quad \text{and} \quad \varpi_m \neq 2\varpi_\gamma \iff m \neq 2\gamma+2.$$

Hence, we infer

$$\begin{aligned}(P\mathfrak{f}^{(2)}(\Phi^t(\xi)))^m &= -\tfrac{3}{2}\mathfrak{K}_\gamma^2\overline{\mathfrak{C}}_{\gamma\gamma m}[1+\mathbb{1}(m\neq 2\gamma+2)\cos(2\varpi_\gamma t)], \\ (L_\varpi^{-1}P\mathfrak{f}^{(2)}(\Phi^t(\xi)))^m &= -\frac{3}{2}\mathfrak{K}_\gamma^2\overline{\mathfrak{C}}_{\gamma\gamma m}\left[\frac{1}{\varpi_m^2} + \frac{\mathbb{1}(m\neq 2\gamma+2)}{\varpi_m^2-(2\varpi_\gamma)^2\varpi^2}\cos(2\varpi_\gamma t)\right],\end{aligned}$$

where we used the fact that the eigenvalues of L_ϖ are given by (2-14), i.e., $\lambda_{ml} = \varpi_m^2 - l^2\varpi^2$. In addition, we set $x = \epsilon\xi$, and $\left(\mathfrak{f}^{(2)}(\Phi^t(x) + L_1^{-1}P\mathfrak{f}^{(2)}(\Phi^t(x)))\right)^m$ is given by

$$\begin{aligned}&-3\sum_{\kappa,\nu\geq 0}\overline{\mathfrak{C}}_{\kappa\nu m}(\Phi^t(\epsilon\xi)+L_1^{-1}P\mathfrak{f}^{(2)}(\Phi^t(\epsilon\xi)))^\kappa(\Phi^t(\epsilon\xi)+L_1^{-1}P\mathfrak{f}^{(2)}(\Phi^t(\epsilon\xi)))^\nu \\ &\quad = -3\sum_{\kappa,\nu\geq 0}\overline{\mathfrak{C}}_{\kappa\nu m}[(\Phi^t(\epsilon\xi))^\kappa + (L_1^{-1}P\mathfrak{f}^{(2)}(\Phi^t(\epsilon\xi)))^\kappa][(\Phi^t(\epsilon\xi))^\nu + (L_1^{-1}P\mathfrak{f}^{(2)}(\Phi^t(\epsilon\xi)))^\nu] \\ &\quad = -3\sum_{\kappa,\nu\geq 0}\overline{\mathfrak{C}}_{\kappa\nu m}\big[(\Phi^t(\epsilon\xi))^\kappa(\Phi^t(\epsilon\xi))^\nu + (\Phi^t(\epsilon\xi))^\kappa(L_1^{-1}P\mathfrak{f}^{(2)}(\Phi^t(\epsilon\xi)))^\nu \\ &\qquad\qquad + (\Phi^t(\epsilon\xi))^\nu(L_1^{-1}P\mathfrak{f}^{(2)}(\Phi^t(\epsilon\xi)))^\kappa + (L_1^{-1}P\mathfrak{f}^{(2)}(\Phi^t(\epsilon\xi)))^\kappa(L_1^{-1}P\mathfrak{f}^{(2)}(\Phi^t(\epsilon\xi)))^\nu\big] \\ &\quad = -3\sum_{\kappa,\nu\geq 0}\overline{\mathfrak{C}}_{\kappa\nu m}(\Phi^t(\epsilon\xi))^\kappa(\Phi^t(\epsilon\xi))^\nu - 6\sum_{\kappa,\nu\geq 0}\overline{\mathfrak{C}}_{\kappa\nu m}(\Phi^t(\epsilon\xi))^\kappa(L_1^{-1}P\mathfrak{f}^{(2)}(\Phi^t(\epsilon\xi)))^\nu \\ &\qquad\qquad - 3\sum_{\kappa,\nu\geq 0}\overline{\mathfrak{C}}_{\kappa\nu m}(L_1^{-1}P\mathfrak{f}^{(2)}(\Phi^t(\epsilon\xi)))^\kappa(L_1^{-1}P\mathfrak{f}^{(2)}(\Phi^t(\epsilon\xi)))^\nu \\ &\quad = \epsilon^2(\mathfrak{f}^{(2)}(\Phi^t(\xi)))^m + \epsilon^3(E(\xi))^m + \epsilon^4(F(\xi))^m,\end{aligned}$$

where we set

$$
\begin{aligned}
(F(\xi))^m &= -3\sum_{\kappa,\nu\geq 0}\overline{\mathfrak{C}}_{\kappa\nu m}(L_1^{-1}P\mathfrak{f}^{(2)}(\Phi^t(\xi)))^\kappa(L_1^{-1}P\mathfrak{f}^{(2)}(\Phi^t(\xi)))^\nu,\\
(E(\xi))^m &= -6\sum_{\kappa,\nu\geq 0}\overline{\mathfrak{C}}_{\kappa\nu m}(\Phi^t(\xi))^\kappa(L_1^{-1}P\mathfrak{f}^{(2)}(\Phi^t(\xi)))^\nu\\
&= 9\mathfrak{K}_\gamma^2\sum_{\kappa,\nu\geq 0}\overline{\mathfrak{C}}_{\kappa\nu m}\xi^\kappa\cos(\varpi_\kappa t)\overline{\mathfrak{C}}_{\gamma\gamma\nu}\left[\frac{1}{\varpi_\nu^2}+\frac{\mathbb{1}(\nu\neq 2\gamma+2)}{\varpi_\nu^2-(2\varpi_\gamma)^2}\cos(2\varpi_\gamma t)\right]\\
&= 9\mathfrak{K}_\gamma^3\sum_{\nu\geq 0}\overline{\mathfrak{C}}_{\gamma\nu m}\overline{\mathfrak{C}}_{\gamma\gamma\nu}\left[\left(\frac{1}{\varpi_\nu^2}+\frac{1}{2}\frac{\mathbb{1}(\nu\neq 2\gamma+2)}{\varpi_\nu^2-(2\varpi_\gamma)^2}\right)\cos(\varpi_\gamma t)+\frac{1}{2}\frac{\mathbb{1}(\nu\neq 2\gamma+2)}{\varpi_\nu^2-(2\varpi_\gamma)^2}\cos(3\varpi_\gamma t)\right].
\end{aligned}
$$

Now, we first apply the linear flow and then the average in time to obtain

$$
\begin{aligned}
\mathfrak{F}_\epsilon(\xi) &= \epsilon^{-3}\langle\mathfrak{f}^{(2)}((\cdot)+L_1^{-1}P\mathfrak{f}^{(2)}(\cdot))\rangle(\epsilon\xi)\\
&= \frac{\epsilon^{-3}}{2\pi}\int_0^{2\pi}\big(\Phi^t\big(\mathfrak{f}^{(2)}(\Phi^t(\epsilon\xi)+L_1^{-1}P\mathfrak{f}^{(2)}(\Phi^t(\epsilon\xi)))\big)\big)^m\,dt\\
&= \frac{\epsilon^{-1}}{2\pi}\int_0^{2\pi}\big(\Phi^t(\mathfrak{f}^{(2)}(\Phi^t(\xi)))\big)^m\,dt+\frac{1}{2\pi}\int_0^{2\pi}(\Phi^t(E(\xi)))^m\,dt+\frac{\epsilon}{2\pi}\int_0^{2\pi}(\Phi^t(F(\xi)))^m\,dt.
\end{aligned}
$$

On the one hand, due to (2-8), we have

$$
\frac{1}{2\pi}\int_0^{2\pi}\big(\Phi^t(\mathfrak{f}^{(2)}(\Phi^t(\xi)))\big)^m\,dt=\langle\mathfrak{f}^{(2)}\rangle(\xi)=0.
$$

On the other hand, we use the orthogonality of the cosine function together with

$$
\varpi_m=\varpi_\gamma \iff m=\gamma \qquad\text{and}\qquad \varpi_m=3\varpi_\gamma \iff m=3\gamma+4
$$

to compute

$$
\begin{aligned}
&\frac{1}{2\pi}\int_0^{2\pi}(\Phi^t(E(\xi)))^m\,dt\\
&\quad=\frac{1}{2\pi}\int_0^{2\pi}(E(\xi))^m\cos(\varpi_m t)\,dt\\
&\quad=9\mathfrak{K}_\gamma^3\sum_{\nu\geq 0}\overline{\mathfrak{C}}_{\gamma\nu m}\overline{\mathfrak{C}}_{\gamma\gamma\nu}\Bigg[\left(\frac{1}{\varpi_\nu^2}+\frac{1}{2}\frac{\mathbb{1}(\nu\neq 2\gamma+2)}{\varpi_\nu^2-(2\varpi_\gamma)^2}\right)\frac{1}{2\pi}\int_0^{2\pi}\cos(\varpi_\gamma t)\cos(\varpi_m t)\,dt\\
&\qquad\qquad+\frac{1}{2}\frac{\mathbb{1}(\nu\neq 2\gamma+2)}{\varpi_\nu^2-(2\varpi_\gamma)^2}\frac{1}{2\pi}\int_0^{2\pi}\cos(3\varpi_\gamma t)\cos(\varpi_m t)\,dt\Bigg]\\
&\quad=9\mathfrak{K}_\gamma^3\sum_{\nu\geq 0}\overline{\mathfrak{C}}_{\gamma\nu m}\overline{\mathfrak{C}}_{\gamma\gamma\nu}\left[\left(\frac{1}{2\varpi_\nu^2}+\frac{1}{4}\frac{\mathbb{1}(\nu\neq 2\gamma+2)}{\varpi_\nu^2-(2\varpi_\gamma)^2}\right)\mathbb{1}(m=\gamma)+\frac{1}{4}\frac{\mathbb{1}(\nu\neq 2\gamma+2)}{\varpi_\nu^2-(2\varpi_\gamma)^2}\mathbb{1}(m=3\gamma+4)\right]\\
&\quad=\frac{9}{4}\mathfrak{K}_\gamma^3\sum_{\nu\geq 0}(\overline{\mathfrak{C}}_{\gamma\gamma\nu})^2\left(\frac{2}{\varpi_\nu^2}+\frac{\mathbb{1}(\nu\neq 2\gamma+2)}{\varpi_\nu^2-(2\varpi_\gamma)^2}\right)\mathbb{1}(m=\gamma)+\frac{9}{4}\mathfrak{K}_\gamma^3\sum_{\substack{\nu\geq 0\\ \nu\neq 2\gamma+2}}\frac{\overline{\mathfrak{C}}_{\gamma,\nu,3\gamma+4}\overline{\mathfrak{C}}_{\gamma\gamma\nu}}{\varpi_\nu^2-(2\varpi_\gamma)^2}\mathbb{1}(m=3\gamma+4).
\end{aligned}
$$

Finally, we note that $\overline{\mathfrak{C}}_{\gamma,\nu,3\gamma+4}\overline{\mathfrak{C}}_{\gamma\gamma\nu} = 0$ for all integers $\gamma \geq 0$ and $\nu \geq 0$. This follows immediately from the fact that $\overline{\mathfrak{C}}_{ijm} = 0$ for all integers $i, j, m \geq 0$ with $i + j < m$ due to Lemma 5.8 below. Specifically, we have $\overline{\mathfrak{C}}_{\gamma\gamma\nu} = 0$ since $\nu > 2\gamma$, and $\overline{\mathfrak{C}}_{\gamma,\nu,3\gamma+4} = 0$ since

$$0 \leq \nu \leq 2\gamma \quad \Longrightarrow \quad \gamma + \nu \leq 3\gamma < 3\gamma + 4.$$

Consequently, we conclude that

$$\frac{1}{2\pi}\int_0^{2\pi} (\Phi^t(E(\xi)))^m \, dt = \frac{9}{4}\mathfrak{K}_\gamma^3 \sum_{\nu=0}^{2\gamma} (\overline{\mathfrak{C}}_{\gamma\gamma\nu})^2 \left(\frac{2}{\varpi_\nu^2} + \frac{1}{\varpi_\nu^2 - (2\varpi_\gamma)^2}\right) \mathbb{1}(m = \gamma). \qquad \square$$

Finally, we compute the differential of $\mathfrak{F}_0(\xi)$ at the 1-mode initial data.

Lemma 2.15 (differential of $\mathfrak{F}_0(\xi)$ at the 1-mode initial data). *Let $\gamma \geq 0$ be an integer, $\mathfrak{K}_\gamma \in \mathbb{R}$ and ξ be the 1-mode initial data, that is*

$$\xi^m = \mathfrak{K}_\gamma \mathbb{1}(m = \gamma), \quad m \geq 0.$$

Then, for all $h \in l^2_{s+3}$ and integers $m \geq 0$, we have

$$(d\mathfrak{F}_0(\xi)[h])^m = \mathfrak{K}_\gamma^2 [\mathfrak{a}_{\gamma m} h^m + \mathbb{1}(0 \leq m \leq 2\gamma)\mathfrak{b}_{\gamma m} h^{2\gamma - m}],$$

where

$$\mathfrak{a}_{\gamma m} = \frac{9}{2}\sum_{\nu=0}^{m+\gamma} \frac{(\overline{\mathfrak{C}}_{\gamma\nu m})^2}{\varpi_\nu^2 - (\varpi_m + \varpi_\gamma)^2} + \frac{9}{2}\sum_{\substack{\nu=0 \\ \nu \neq \pm(m-\gamma)-2}}^{m+\gamma} \frac{(\overline{\mathfrak{C}}_{m\gamma\nu})^2}{\varpi_\nu^2 - (\varpi_m - \varpi_\gamma)^2} + \frac{9}{4}\sum_{\nu=0}^{2\gamma} \frac{\overline{\mathfrak{C}}_{m\nu m}\overline{\mathfrak{C}}_{\gamma\gamma\nu}}{\varpi_\nu^2}$$

and

$$\mathfrak{b}_{\gamma m} = \frac{9}{4}\sum_{\nu=0}^{2\gamma} \frac{\overline{\mathfrak{C}}_{2\gamma-m,\nu,m}\overline{\mathfrak{C}}_{\gamma\gamma\nu}}{\varpi_\nu^2 - (2\varpi_\gamma)^2} + \frac{9}{2}\sum_{\substack{\nu=0 \\ \nu \neq \pm(m-\gamma)-2}}^{m+\gamma} \frac{\overline{\mathfrak{C}}_{\gamma\nu m}\overline{\mathfrak{C}}_{2\gamma-m,\gamma,\nu}}{\varpi_\nu^2 - (\varpi_{2\gamma-m} - \varpi_\gamma)^2}.$$

Proof. Let $\gamma \geq 0$ be an integer, $\mathfrak{K}_\gamma \in \mathbb{R}$ and ξ to be the 1-mode initial data as above, and pick any $h \in l^2_{s+3}$, $\epsilon > 0$ and integer $m \geq 0$. Then, we set $\hat{\xi} = \xi + \epsilon h$, and according to Lemma 2.13, we have

$$(\mathfrak{F}_0(\hat{\xi}))^m = (\mathfrak{F}_{0-}(\hat{\xi}))^m + (\mathfrak{F}_{0+}(\hat{\xi}))^m,$$

where

$$(\mathfrak{F}_{0-}(\hat{\xi}))^m = \frac{9}{4}\sum_{\kappa,\nu \geq 0} \overline{\mathfrak{C}}_{\kappa\nu m} \sum_{\substack{i,j \geq 0 \\ \varpi_i - \varpi_j \neq \pm\varpi_\nu}} \frac{\overline{\mathfrak{C}}_{ij\nu}}{\varpi_\nu^2 - (\varpi_i - \varpi_j)^2} \hat{\xi}^i \hat{\xi}^j \hat{\xi}^\kappa \sum_{\pm} \mathbb{1}(\varpi_i - \varpi_j \pm \varpi_\kappa \pm \varpi_m = 0)$$

and

$$(\mathfrak{F}_{0+}(\hat{\xi}))^m = \frac{9}{4}\sum_{\kappa,\nu \geq 0} \overline{\mathfrak{C}}_{\kappa\nu m} \sum_{\substack{i,j \geq 0 \\ \varpi_i + \varpi_j \neq \pm\varpi_\nu}} \frac{\overline{\mathfrak{C}}_{ij\nu}}{\varpi_\nu^2 - (\varpi_i + \varpi_j)^2} \hat{\xi}^i \hat{\xi}^j \hat{\xi}^\kappa \sum_{\pm} \mathbb{1}(\varpi_i + \varpi_j \pm \varpi_\kappa \pm \varpi_m = 0).$$

We expand[9] $\mathfrak{F}_{0\pm}(\hat{\xi}) = \mathfrak{F}_{0\pm}(\xi) + \epsilon \cdot d\mathfrak{F}_{0\pm}(\xi)[h] + \mathcal{O}(\epsilon^2)$ and, using the definition of the 1-mode initial data, we obtain

$$
\begin{aligned}
(d\mathfrak{F}_{0-}(\xi)[h])^m &= \frac{9}{4}\sum_{\kappa,\nu\geq 0} \overline{\mathfrak{C}}_{\kappa\nu m} \sum_{\substack{i,j\geq 0\\ \varpi_i-\varpi_j\neq\pm\varpi_\nu}} \frac{\overline{\mathfrak{C}}_{ij\nu}}{\varpi_\nu^2-(\varpi_i-\varpi_j)^2} \\
&\qquad \cdot [h^i\xi^j\xi^\kappa + \xi^i h^j \xi^\kappa + \xi^i\xi^j h^\kappa] \sum_{\pm} \mathbb{1}(\varpi_i-\varpi_j\pm\varpi_\kappa\pm\varpi_m = 0) \\
&= \mathfrak{K}_\gamma^2 \Bigg\{ \frac{9}{4}\sum_{\nu\geq 0} \overline{\mathfrak{C}}_{\gamma\nu m} \sum_{\substack{i\geq 0\\ \varpi_i-\varpi_\gamma\neq\pm\varpi_\nu}} \frac{\overline{\mathfrak{C}}_{i\gamma\nu}}{\varpi_\nu^2-(\varpi_i-\varpi_\gamma)^2} h^i \sum_{\pm}\mathbb{1}(\varpi_i-\varpi_\gamma\pm\varpi_\gamma\pm\varpi_m=0) \\
&\qquad + \frac{9}{4}\sum_{\nu\geq 0} \overline{\mathfrak{C}}_{\gamma\nu m} \sum_{\substack{j\geq 0\\ \varpi_\gamma-\varpi_j\neq\pm\varpi_\nu}} \frac{\overline{\mathfrak{C}}_{\gamma j\nu}}{\varpi_\nu^2-(\varpi_\gamma-\varpi_j)^2} h^j \sum_{\pm}\mathbb{1}(\varpi_\gamma-\varpi_j\pm\varpi_\gamma\pm\varpi_m=0) \\
&\qquad + \frac{9}{4}\sum_{\kappa,\nu\geq 0} \overline{\mathfrak{C}}_{\kappa\nu m} \frac{\overline{\mathfrak{C}}_{\gamma\gamma\nu}}{\varpi_\nu^2-(\varpi_\gamma-\varpi_\gamma)^2} h^\kappa \sum_{\pm}\mathbb{1}(\pm\varpi_\kappa\pm\varpi_m=0) \Bigg\}.
\end{aligned}
$$

Recall the definition of the eigenvalues $\varpi_i = i+2$ for all integers $i\geq 0$ and also recall that $m, i, j, \kappa, \nu, \gamma \geq 0$. Then, we have

$$
\begin{cases} \varpi_i - \varpi_\gamma \pm \varpi_\gamma \pm \varpi_m = 0, \\ \varpi_i - \varpi_\gamma \neq \pm\varpi_\nu \end{cases} \iff \begin{cases} i = m \text{ and } \nu \neq \pm(m-\gamma)-2, \\ i = 2\gamma - m \text{ and } m \leq 2\gamma \text{ and } \nu \neq \pm(m-\gamma)-2, \\ i = 2\gamma+m+4 \text{ and } \nu \neq 2+m+\gamma, \end{cases}
$$

$$
\begin{cases} \varpi_\gamma - \varpi_j \pm \varpi_\gamma \pm \varpi_m = 0, \\ \varpi_\gamma - \varpi_j \neq \pm\varpi_\nu \end{cases} \iff \begin{cases} j = 2\gamma+m+4 \text{ and } \nu \neq 2+m+\gamma, \\ j = 2\gamma - m \text{ and } m \leq 2\gamma \text{ and } \nu \neq \pm(m-\gamma)-2, \\ j = m \text{ and } \nu \neq \pm(m-\gamma)-2, \end{cases}
$$

$$
\pm\varpi_\kappa \pm \varpi_m = 0 \iff \begin{cases} m = -\kappa - 4, \\ m = \kappa, \end{cases} \iff \kappa = m.
$$

Therefore, we infer

$$
\begin{aligned}
\mathfrak{K}_\gamma^{-2}(d\mathfrak{F}_{0-}(\xi)[h])^m &= h^m\Bigg[\frac{9}{2}\sum_{\substack{\nu=0\\ \nu\neq\pm(m-\gamma)-2}}^{\infty} \frac{(\overline{\mathfrak{C}}_{m\gamma\nu})^2}{\varpi_\nu^2-(\varpi_m-\varpi_\gamma)^2} + \frac{9}{4}\sum_{\nu=0}^{\infty}\frac{\overline{\mathfrak{C}}_{m\nu m}\overline{\mathfrak{C}}_{\gamma\gamma\nu}}{\varpi_\nu^2}\Bigg] \\
&\quad + h^{2\gamma-m}\mathbb{1}(0\leq m\leq 2\gamma)\Bigg[\frac{9}{2}\sum_{\substack{\nu=0\\ \nu\neq\pm(m-\gamma)-2}}^{\infty} \frac{\overline{\mathfrak{C}}_{\gamma\nu m}\overline{\mathfrak{C}}_{2\gamma-m,\gamma,\nu}}{\varpi_\nu^2-(\varpi_{2\gamma-m}-\varpi_\gamma)^2}\Bigg] \\
&\quad + h^{2\gamma+m+4}\Bigg[\frac{9}{2}\sum_{\substack{\nu=0\\ \nu\neq 2+m+\gamma}}^{\infty} \frac{\overline{\mathfrak{C}}_{\gamma\nu m}\overline{\mathfrak{C}}_{2\gamma+m+4,\gamma,\nu}}{\varpi_\nu^2-(\varpi_{2\gamma+m+4}-\varpi_\gamma)^2}\Bigg].
\end{aligned}
$$

[9] Here, the notation $\mathcal{O}(\epsilon^2)$ for a function of ξ or h refers to a function that is bounded by ϵ^2 in the $\mathcal{Q}$-norm using the l_{s+3}-norm of ξ or h.

Similarly, using the definition of the 1-mode initial data, we obtain

$$(d\mathfrak{F}_{0+}(\xi)[h])^m = \frac{9}{4}\sum_{\kappa,\nu\geq 0}\overline{\mathfrak{C}}_{\kappa\nu m}\sum_{\substack{i,j\geq 0\\ \varpi_i+\varpi_j\neq\pm\varpi_\nu}}\frac{\overline{\mathfrak{C}}_{ij\nu}}{\varpi_\nu^2-(\varpi_i+\varpi_j)^2} \cdot [h^i\xi^j\xi^\kappa+\xi^i h^j\xi^\kappa+\xi^i\xi^j h^\kappa]\sum_{\pm}\mathbb{1}(\varpi_i+\varpi_j\pm\varpi_\kappa\pm\varpi_m=0)$$

$$= \mathfrak{K}_\gamma^2\Bigg\{\frac{9}{4}\sum_{\nu\geq 0}\overline{\mathfrak{C}}_{\gamma\nu m}\sum_{\substack{i\geq 0\\ \varpi_i+\varpi_\gamma\neq\pm\varpi_\nu}}\frac{\overline{\mathfrak{C}}_{i\gamma\nu}}{\varpi_\nu^2-(\varpi_i+\varpi_\gamma)^2}h^i\sum_{\pm}\mathbb{1}(\varpi_i+\varpi_\gamma\pm\varpi_\gamma\pm\varpi_m=0)$$

$$+\frac{9}{4}\sum_{\nu\geq 0}\overline{\mathfrak{C}}_{\gamma\nu m}\sum_{\substack{j\geq 0\\ \varpi_\gamma+\varpi_j\neq\pm\varpi_\nu}}\frac{\overline{\mathfrak{C}}_{\gamma j\nu}}{\varpi_\nu^2-(\varpi_\gamma+\varpi_j)^2}h^j\sum_{\pm}\mathbb{1}(\varpi_\gamma+\varpi_j\pm\varpi_\gamma\pm\varpi_m=0)$$

$$+\frac{9}{4}\sum_{\kappa,\nu\geq 0}\overline{\mathfrak{C}}_{\kappa\nu m}\frac{\mathbb{1}(\varpi_\gamma+\varpi_\gamma\neq\pm\varpi_\nu)\overline{\mathfrak{C}}_{\gamma\gamma\nu}}{\varpi_\nu^2-(\varpi_\gamma+\varpi_\gamma)^2}h^\kappa\sum_{\pm}\mathbb{1}(\varpi_\gamma+\varpi_\gamma\pm\varpi_\kappa\pm\varpi_m=0)\Bigg\}.$$

As before, recall the definition of the eigenvalues $\varpi_i = i+2$ for all integers $i\geq 0$ and also recall that $m, i, j, \kappa, \nu, \gamma \geq 0$. Then, we have

$$\begin{cases}\varpi_i+\varpi_\gamma\pm\varpi_\gamma\pm\varpi_m=0,\\ \varpi_i+\varpi_\gamma\neq\pm\varpi_\nu\end{cases} \iff \begin{cases} i=-4+m-2\gamma \text{ and } m\geq 4+2\gamma \text{ and } \nu\neq\pm(m-\gamma)-2,\\ i=m \text{ and } \nu\neq 2+m+\gamma,\end{cases}$$

$$\begin{cases}\varpi_\gamma+\varpi_j\pm\varpi_\gamma\pm\varpi_m=0,\\ \varpi_\gamma+\varpi_j\neq\pm\varpi_\nu\end{cases} \iff \begin{cases} j=-4+m-2\gamma \text{ and } m\geq 4+2\gamma \text{ and } \nu\neq\pm(m-\gamma)-2,\\ j=m \text{ and } \nu\neq 2+m+\gamma,\end{cases}$$

$$\begin{cases}\varpi_\gamma+\varpi_\gamma\pm\varpi_\kappa\pm\varpi_m=0,\\ \varpi_\gamma+\varpi_\gamma\neq\pm\varpi_\nu\end{cases} \iff \begin{cases} \kappa=-4+m-2\gamma \text{ and } m\geq 4+2\gamma \text{ and } \nu\neq 2+2\gamma,\\ \kappa=4+m+2\gamma \text{ and } \nu\neq 2+2\gamma,\\ \kappa=2\gamma-m \text{ and } m\leq 2\gamma \text{ and } \nu\neq 2+2\gamma.\end{cases}$$

Therefore, we infer

$$\mathfrak{K}_\gamma^{-2}(d\mathfrak{F}_{0+}(\xi)[h])^m$$
$$= h^m\Bigg[\frac{9}{2}\sum_{\substack{\nu=0\\ \nu\neq 2+m+\gamma}}^{\infty}\frac{(\overline{\mathfrak{C}}_{\gamma\nu m})^2}{\varpi_\nu^2-(\varpi_m+\varpi_\gamma)^2}\Bigg]+h^{4+m+2\gamma}\Bigg[\frac{9}{4}\sum_{\substack{\nu=0\\ \nu\neq 2+2\gamma}}^{\infty}\frac{\overline{\mathfrak{C}}_{4+m+2\gamma,\nu,m}\overline{\mathfrak{C}}_{\gamma\gamma\nu}}{\varpi_\nu^2-(2\varpi_\gamma)^2}\Bigg]$$
$$+h^{-4+m-2\gamma}\mathbb{1}(m\geq 4+2\gamma)\Bigg[\frac{9}{2}\sum_{\substack{\nu=0\\ \nu\neq\pm(m-\gamma)-2}}^{\infty}\frac{\overline{\mathfrak{C}}_{\gamma\nu m}\overline{\mathfrak{C}}_{\gamma,-4+m-2\gamma,\nu}}{\varpi_\nu^2-(\varpi_\gamma+\varpi_{-4+m-2\gamma})^2}+\frac{9}{4}\sum_{\substack{\nu=0\\ \nu\neq 2+2\gamma}}^{\infty}\frac{\overline{\mathfrak{C}}_{-4+m-2\gamma,\nu,m}\overline{\mathfrak{C}}_{\gamma\gamma\nu}}{\varpi_\nu^2-(2\varpi_\gamma)^2}\Bigg]$$
$$+h^{2\gamma-m}\mathbb{1}(0\leq m\leq 2\gamma)\Bigg[\frac{9}{4}\sum_{\substack{\nu=0\\ \nu\neq 2+2\gamma}}^{\infty}\frac{\overline{\mathfrak{C}}_{2\gamma-m,\nu,m}\overline{\mathfrak{C}}_{\gamma\gamma\nu}}{\varpi_\nu^2-(2\varpi_\gamma)^2}\Bigg].$$

Putting this all together yields that

$$\mathfrak{K}_\gamma^{-2}(d\mathfrak{F}_0(\xi)[h])^m = \mathfrak{K}_\gamma^{-2}[(d\mathfrak{F}_{0-}(\xi)[h])^m+(d\mathfrak{F}_{0+}(\xi)[h])^m]$$

is equal to

$$h^m\Bigg[\frac{9}{2}\sum_{\substack{\nu=0\\ \nu\neq 2+m+\gamma}}^{\infty}\frac{(\overline{\mathfrak{C}}_{\gamma\nu m})^2}{\varpi_\nu^2-(\varpi_m+\varpi_\gamma)^2}+\frac{9}{2}\sum_{\substack{\nu=0\\ \nu\neq\pm(m-\gamma)-2}}^{\infty}\frac{(\overline{\mathfrak{C}}_{m\gamma\nu})^2}{\varpi_\nu^2-(\varpi_m-\varpi_\gamma)^2}+\frac{9}{4}\sum_{\nu=0}^{\infty}\frac{\overline{\mathfrak{C}}_{m\nu m}\overline{\mathfrak{C}}_{\gamma\gamma\nu}}{\varpi_\nu^2}\Bigg]$$
$$+h^{2\gamma-m}\mathbb{1}(0\leq m\leq 2\gamma)\Bigg[\frac{9}{4}\sum_{\substack{\nu=0\\ \nu\neq 2+2\gamma}}^{\infty}\frac{\overline{\mathfrak{C}}_{2\gamma-m,\nu,m}\overline{\mathfrak{C}}_{\gamma\gamma\nu}}{\varpi_\nu^2-(2\varpi_\gamma)^2}+\frac{9}{2}\sum_{\substack{\nu=0\\ \nu\neq\pm(m-\gamma)-2}}^{\infty}\frac{\overline{\mathfrak{C}}_{\gamma\nu m}\overline{\mathfrak{C}}_{2\gamma-m,\gamma,\nu}}{\varpi_\nu^2-(\varpi_{2\gamma-m}-\varpi_\gamma)^2}\Bigg]$$
$$+h^{4+m+2\gamma}\Bigg[\frac{9}{4}\sum_{\substack{\nu=0\\ \nu\neq 2+2\gamma}}^{\infty}\frac{\overline{\mathfrak{C}}_{4+m+2\gamma,\nu,m}\overline{\mathfrak{C}}_{\gamma\gamma\nu}}{\varpi_\nu^2-(2\varpi_\gamma)^2}+\frac{9}{2}\sum_{\substack{\nu=0\\ \nu\neq 2+m+\gamma}}^{\infty}\frac{\overline{\mathfrak{C}}_{\gamma\nu m}\overline{\mathfrak{C}}_{2\gamma+m+4,\gamma,\nu}}{\varpi_\nu^2-(\varpi_{2\gamma+m+4}-\varpi_\gamma)^2}\Bigg]$$
$$+h^{-4+m-2\gamma}\mathbb{1}(m\geq 4+2\gamma)\Bigg[\frac{9}{2}\sum_{\substack{\nu=0\\ \nu\neq\pm(m-\gamma)-2}}^{\infty}\frac{\overline{\mathfrak{C}}_{\gamma\nu m}\overline{\mathfrak{C}}_{\gamma,-4+m-2\gamma,\nu}}{\varpi_\nu^2-(\varpi_\gamma+\varpi_{-4+m-2\gamma})^2}+\frac{9}{4}\sum_{\substack{\nu=0\\ \nu\neq 2+2\gamma}}^{\infty}\frac{\overline{\mathfrak{C}}_{-4+m-2\gamma,\nu,m}\overline{\mathfrak{C}}_{\gamma\gamma\nu}}{\varpi_\nu^2-(2\varpi_\gamma)^2}\Bigg].$$

Finally, we simplify the formula above and show that

$$\overline{\mathfrak{C}}_{4+m+2\gamma,\nu,m}\overline{\mathfrak{C}}_{\gamma\gamma\nu}=\overline{\mathfrak{C}}_{\gamma\nu m}\overline{\mathfrak{C}}_{2\gamma+m+4,\gamma,\nu}=0, \tag{2-26}$$
$$\overline{\mathfrak{C}}_{\gamma\nu m}\overline{\mathfrak{C}}_{\gamma,-4+m-2\gamma,\nu}=\overline{\mathfrak{C}}_{-4+m-2\gamma,\nu,m}\overline{\mathfrak{C}}_{\gamma\gamma\nu}=0, \tag{2-27}$$

for all $\gamma,\nu,m\geq 0$ and all $\gamma,\nu\geq 0$ and $m\geq 2\gamma+4$, respectively. These follow immediately from the fact that $\overline{\mathfrak{C}}_{ijm}=0$ for all integers $i,j,m\geq 0$ with $i+j<m$ (Lemma 5.8). Specifically, we have $\overline{\mathfrak{C}}_{\gamma\gamma\nu}=0$ since $\nu>2\gamma$, $\overline{\mathfrak{C}}_{4+m+2\gamma,\nu,m}=0$ since

$$0\leq\nu\leq 2\gamma \iff m+\nu\leq m+2\gamma<4+m+2\gamma,$$

as well as $\overline{\mathfrak{C}}_{\gamma\nu m}=0$ since $\nu>\gamma+m$ and $\overline{\mathfrak{C}}_{2\gamma+m+4,\gamma,\nu}=0$ since

$$0\leq\nu\leq\gamma+m \iff \gamma+\nu\leq 2\gamma+m<2\gamma+m+4$$

which prove (2-26). Similarly, we have $\overline{\mathfrak{C}}_{\gamma\nu m}=0$ since $m>\gamma+\nu$, $\overline{\mathfrak{C}}_{\gamma,-4+m-2\gamma,\nu}=0$ since

$$0\leq m\leq\gamma+\nu \iff -4+m-2\gamma+\gamma=-4+m-\gamma\leq -4+\nu<\nu,$$

as well as $\overline{\mathfrak{C}}_{\gamma\gamma\nu}=0$ since $\nu>2\gamma$ and $\overline{\mathfrak{C}}_{-4+m-2\gamma,\nu,m}=0$ since

$$0\leq\nu\leq 2\gamma \iff -4+m-2\gamma+\nu\leq -4+m<m,$$

which prove (2-27). Using the same argument as above (Lemma 5.8), we also infer $\overline{\mathfrak{C}}_{\gamma\gamma\nu}=0$ since $\nu>2\gamma$ and $\overline{\mathfrak{C}}_{\gamma\nu m}=0$ since $\nu>\gamma+m$. Hence, the latter reduces to

$$h^m\Bigg[\frac{9}{2}\sum_{\nu=0}^{m+\gamma}\frac{(\overline{\mathfrak{C}}_{\gamma\nu m})^2}{\varpi_\nu^2-(\varpi_m+\varpi_\gamma)^2}+\frac{9}{2}\sum_{\substack{\nu=0\\ \nu\neq\pm(m-\gamma)-2}}^{m+\gamma}\frac{(\overline{\mathfrak{C}}_{m\gamma\nu})^2}{\varpi_\nu^2-(\varpi_m-\varpi_\gamma)^2}+\frac{9}{4}\sum_{\nu=0}^{2\gamma}\frac{\overline{\mathfrak{C}}_{m\nu m}\overline{\mathfrak{C}}_{\gamma\gamma\nu}}{\varpi_\nu^2}\Bigg]$$
$$+h^{2\gamma-m}\mathbb{1}(0\leq m\leq 2\gamma)\Bigg[\frac{9}{4}\sum_{\nu=0}^{2\gamma}\frac{\overline{\mathfrak{C}}_{2\gamma-m,\nu,m}\overline{\mathfrak{C}}_{\gamma\gamma\nu}}{\varpi_\nu^2-(2\varpi_\gamma)^2}+\frac{9}{2}\sum_{\substack{\nu=0\\ \nu\neq\pm(m-\gamma)-2}}^{m+\gamma}\frac{\overline{\mathfrak{C}}_{\gamma\nu m}\overline{\mathfrak{C}}_{2\gamma-m,\gamma,\nu}}{\varpi_\nu^2-(\varpi_{2\gamma-m}-\varpi_\gamma)^2}\Bigg]. \qquad \square$$

3. The linear eigenvalue problems

Next, we the study the linear eigenvalue problems where the linearized operators are given by (1-14). In all three models, the associated eigenfunctions are given by Jacobi polynomials, which is a common feature with the Einstein–Klein–Gordon system in spherical symmetry [Maliborski and Rostworowski 2013].

3A. ***Conformal cubic wave equation in spherical symmetry.*** We consider $L^2((0,\pi);\sin^2(x)\,dx)$, a Hilbert space, and associate it with the inner product

$$(f \mid g) = \frac{2}{\pi}\int_0^{\pi} f(x)g(x)\sin^2(x)\,dx.$$

For the conformal wave equation in spherical symmetry, the operator that governs the solutions to the linearized equation is given by

$$Lu = -\frac{1}{\sin^2(x)}\partial_x(\sin^2(x)\partial_x u)+u, \qquad \mathcal{D}(L)=\{u\in L^2((0,\pi);\sin^2(x)\,dx) : Lu \in L^2((0,\pi);\sin^2(x)\,dx)\}.$$

The operator L is generated by the closed sesquilinear form a defined on $(H^1((0,\pi);\sin^2(x)\,dx))^2$ that is given by

$$a(u,v) = \int_0^{\pi}(\partial_x u\partial_x v + uv)\sin^2(x)\,dx$$

and $a(u,u) \simeq \|u\|^2_{H^1((0,\pi);\sin^2(x)\,dx)}$. In particular, L is self-adjoint on $\mathcal{D}(L)$. Now, the eigenvalue problem $Lu = \omega^2 u$ reads

$$\partial_x(\sin^2(x)\partial_x u) + (\omega^2-1)\sin^2(x)u = 0,$$

and, by setting $u(x) = v(y)$ and $y = \cos(x)$, it becomes

$$(1-y^2)v''(y) - 3yv'(y) + (\omega^2-1)v(y) = 0.$$

The latter has nontrivial solutions if and only if the solutions are given by the Chebyshev polynomials of the second kind [Szegő 1975], that is $v(y) = U_n(y)$. Hence, the solutions to the eigenvalue problem $Lu = \omega^2 u$ are given by

$$e_n(x) = U_n(\cos(x)), \qquad \omega_n^2 = (n+1)^2, \tag{3-1}$$

for all integers $n \geq 0$. In addition, the set $\{e_n : n \geq 0\}$ forms an orthonormal and complete basis for $L^2((0,\pi);\sin^2(x)\,dx)$. In fact, $(e_n \mid e_m) = \mathbb{1}(n=m)$ for any $n, m \geq 0$ due to the orthogonality of the Chebyshev polynomials of the second kind.

3B. ***Conformal cubic wave equation out of spherical symmetry.*** We consider $L^2\big((0,\frac{\pi}{2});\sin(2x)\,dx\big)$, a Hilbert space, and associate it with the inner product

$$\langle f \mid g\rangle = \int_0^{\pi/2} f(x)g(x)\sin(2x)\,dx.$$

For the conformal cubic wave equation out of spherical symmetry, the operator that governs the solutions to the linearized equation is given by

$$\mathsf{L}^{(\mu_1,\mu_2)}u = -\frac{1}{\sin(2x)}\partial_x(\sin(2x)\partial_x u) + \left(\frac{\mu_1^2}{\sin^2 x} + \frac{\mu_2^2}{\cos^2 x} + 1\right)u$$

endowed with the domain $\mathcal{D}(\mathsf{L}^{(\mu_1,\mu_2)})$ defined by

$$\mathcal{D}(\mathsf{L}^{(\mu_1,\mu_2)}) = \left\{u \in L^2\big((0,\tfrac{\pi}{2}); \sin(2x)\,dx\big) : \mathsf{L}^{(\mu_1,\mu_2)}u \in L^2\big((0,\tfrac{\pi}{2}); \sin(2x)\,dx\big)\right\}.$$

The operator $\mathsf{L}^{(\mu_1,\mu_2)}$ is generated by the closed sesquilinear form a defined on $\big(H^1\big((0,\tfrac{\pi}{2}); \sin(2x)\,dx\big)\big)^2$ that is given by

$$\mathsf{a}(u,v) = \int_0^{\pi}\left(\partial_x u \partial_x v + \left(\frac{\mu_1^2}{\sin^2 x} + \frac{\mu_2^2}{\cos^2 x} + 1\right)uv\right)\sin(2x)\,dx,$$

and Hardy's inequality yields $\mathsf{a}(u,u) \simeq \|u\|_{H^1((0,\pi/2);\sin(2x)\,dx)^2}$. In particular, $\mathsf{L}^{(\mu_1,\mu_2)}$ is self-adjoint on $\mathcal{D}(\mathsf{L}^{(\mu_1,\mu_2)})$. Now, the eigenvalue problem $\mathsf{L}^{(\mu_1,\mu_2)}u = \omega^2 u$ reads

$$\partial_x^2 u + \left(\frac{\cos x}{\sin x} - \frac{\sin x}{\cos x}\right)\partial_x u - \left(\frac{\mu_1^2}{\sin^2 x} + \frac{\mu_2^2}{\cos^2 x} + 1 - \omega^2\right)u = 0,$$

and by setting $u(x) = v(y),\ v(y) = (1-y)^{\mu_1/2}(1+y)^{\mu_2/2}w(y)$ and $y = \cos(2x)$, it becomes

$$(1-y^2)w''(y) + [(\mu_2-\mu_1) - (2+\mu_1+\mu_2)y]w'(y) + \tfrac{1}{4}[\omega^2 - (1+\mu_1+\mu_2)^2]w(y) = 0.$$

The latter has nontrivial solutions if and only if the solutions are given by the Jacobi polynomial with parameters (μ_1,μ_2) and degree $n \geq 0$, that is $w(y) = P_n^{(\mu_1,\mu_2)}(y)$. Hence, the solutions to the eigenvalue problem $\mathsf{L}^{(\mu_1,\mu_2)}u = \omega^2 u$ are given by

$$\begin{aligned} \mathsf{e}_n^{(\mu_1,\mu_2)}(x) &= \mathsf{N}_n^{(\mu_1,\mu_2)}(1-\cos(2x))^{\mu_1/2}(1+\cos(2x))^{\mu_2/2}P_n^{(\mu_1,\mu_2)}(\cos(2x)), \\ (\omega_n^{(\mu_1,\mu_2)})^2 &= (2n+1+\mu_1+\mu_2)^2, \end{aligned} \tag{3-2}$$

for all integers $n \geq 0$, where the normalization constant reads

$$\mathsf{N}_n^{(\mu_1,\mu_2)} = \sqrt{\frac{\omega_n^{(\mu_1,\mu_2)}}{2^{\mu_1+\mu_2}}\frac{\Gamma(n+1)\Gamma(n+\mu_1+\mu_2+1)}{\Gamma(n+\mu_1+1)\Gamma(n+\mu_2+1)}}. \tag{3-3}$$

In addition, the set $\{\mathsf{e}_n^{(\mu_1,\mu_2)} : n \geq 0\}$ forms an orthonormal and complete basis for $L^2\big((0,\tfrac{\pi}{2}); \sin(2x)\,dx\big)$. In fact,

$$\langle \mathsf{e}_n^{(\mu_1,\mu_2)} \mid \mathsf{e}_m^{(\mu_1,\mu_2)}\rangle = \mathbb{1}(n=m)$$

for any $n, m \geq 0$ due to the orthogonality of the Jacobi polynomials.

3C. ***Yang–Mills equation in spherical symmetry.*** We consider the Hilbert space $L^2((0,\pi); \sin^4(x)\,dx)$ associated with the inner product

$$[f|g] = \int_0^{\pi} f(x)g(x)\sin^4(x)\,dx.$$

For the Yang–Mills equation in spherical symmetry, the operator that governs the solutions to the linearized equation is given by

$$\mathfrak{L}u = -\frac{1}{\sin^4 x}\partial_x(\sin^4 x\,\partial_x u) + 4u, \quad \mathcal{D}(\mathfrak{L}) = \{u \in L^2((0,\pi); \sin^4 x\,dx) : \mathfrak{L}u \in L^2((0,\pi); \sin^4 x\,dx)\}.$$

The operator $\mathfrak{L}$ is generated by the closed sesquilinear form $\mathfrak{a}$ defined on $(H^1((0,\pi);\sin^4 x\,dx))^2$ that is given by

$$\mathfrak{a}(u,v)=\int_0^\pi(\partial_x u\partial_x v+4uv)\sin^4(x)\,dx$$

and $\mathfrak{a}(u,u)\simeq\|u\|^2_{H^1((0,\pi);\sin^4 x\,dx)}$. In particular, $\mathfrak{L}$ is self-adjoint on $\mathcal{D}(\mathfrak{L})$. Now, the eigenvalue problem $\mathfrak{L}u=\varpi^2 u$ reads

$$\partial_x^2 u+\frac{4}{\tan(x)}\partial_x u+(\omega^2-4)u=0,$$

and by setting $u(x)=w(y)$ and $y=\cos(x)$ it becomes

$$(1-y^2)w''(y)-5yw'(y)+(\omega^2-4)w(y)=0.$$

The latter has nontrivial solutions if and only if the solutions are given by the Jacobi polynomials with parameters $\left(\frac{3}{2},\frac{3}{2}\right)$ and degree n, that is $w(y)=P_n^{(3/2,3/2)}(y)$. Hence, the solutions to the eigenvalue problem $\mathfrak{L}u=\varpi^2 u$ are given by

$$\mathfrak{e}_n(x)=\mathfrak{N}_n P_n^{(3/2,3/2)}(\cos(x)),\quad \varpi_n^2=(n+2)^2,\tag{3-4}$$

for all integers $n\geq 0$, where the normalization constant reads

$$\mathfrak{N}_n=\frac{\sqrt{\varpi_n\Gamma(1+n)\Gamma(4+n)}}{2\sqrt{2}\Gamma\left(\frac{5}{2}+n\right)}.\tag{3-5}$$

In addition, the set $\{\mathfrak{e}_n:n\geq 0\}$ forms an orthonormal and complete basis for $L^2((0,\pi);\sin^4(x)\,dx)$. In fact, $[\mathfrak{e}_n\mid\mathfrak{e}_m]=\mathbb{1}(n=m)$ for any $n,m\geq 0$ due to the orthogonality of the Jacobi polynomials.

4. The PDEs in Fourier space

In this section, we express the partial differential equations (1-13),

$$(\partial_t^2+\boldsymbol{L})u=\boldsymbol{f}(x,u),\quad (t,x)\in\mathbb{R}\times\boldsymbol{I},$$

in the Fourier space to obtain infinite dimensional systems of coupled, nonlinear harmonic oscillators, and we provide basic estimates for the nonlinearities. Here, the nonlinearities are given by (1-15), namely

$$\boldsymbol{f}(x,u)=\begin{cases}-u^3 & \text{for CW and WH,}\\ -3u^2-\sin^2(x)u^3 & \text{for YM.}\end{cases}$$

Let $u(t,\cdot)$ be a solution to any of the three models

CW: (1-3)–(1-4), WH: (1-6)–(1-7)–(1-5), YM: (1-11)–(1-12),

and recall that the sets of the associated eigenfunctions

CW: $\{e_n:n\geq 0\}$ by (3-1), WH: $\{\mathrm{e}_n^{(\mu_1,\mu_2)}:n\geq 0\}$ by (3-2), YM: $\{\mathfrak{e}_n:n\geq 0\}$ by (3-4) form an orthonormal and complete basis of the Hilbert spaces

$$\text{CW: } L^2([0,\pi];\sin^2(x)\,dx),\qquad \text{WH: } L^2\big([0,\tfrac{\pi}{2}];\sin(2x)\,dx\big),\qquad \text{YM: } L^2([0,\pi];\sin^4(x)\,dx).$$

Then, we expand $u(t,\cdot)$ in terms of the eigenfunctions and substitute the expression into (1-13) to find infinite systems of nonlinear harmonic oscillators.

4A. ***Conformal cubic wave equation in spherical symmetry.*** For the conformal cubic wave equation in spherical symmetry, we expand

$$u(t,\cdot)=\sum_{n=0}^{\infty}u^n(t)e_n,\qquad e_ie_je_k=\sum_{m=0}^{\infty}C_{ijkm}e_m,\tag{4-1}$$

to find the infinite system of nonlinear harmonic oscillators

$$\ddot{u}^m(t)+(Au(t))^m=(f(\{u^j(t):j\geq 0\}))^m\tag{4-2}$$

for all integers $m\geq 0$, where the dots denote derivatives with respect to time and

$$(Au(t))^m=\omega_m^2u^m(t),\qquad (f(\{u^j(t):j\geq 0\}))^m=-\sum_{i,j,k=0}^{\infty}C_{ijkm}u^i(t)u^j(t)u^k(t).\tag{4-3}$$

4B. ***Conformal cubic wave equation out of spherical symmetry.*** For the conformal cubic wave equation out of spherical symmetry, we expand

$$u(t,\cdot)=\sum_{n=0}^{\infty}u^n(t)\mathsf{e}_n^{(\mu_1,\mu_2)},\qquad \mathsf{e}_i^{(\mu_1,\mu_2)}\mathsf{e}_j^{(\mu_1,\mu_2)}\mathsf{e}_k^{(\mu_1,\mu_2)}=\sum_{m=0}^{\infty}\mathsf{C}_{ijkm}^{(\mu_1,\mu_2)}\mathsf{e}_m^{(\mu_1,\mu_2)}\tag{4-4}$$

to find the infinite system of nonlinear harmonic oscillators

$$\ddot{u}^m(t)+(\mathsf{A}u(t))^m=(\mathsf{f}(\{u^j(t):j\geq 0\}))^m\tag{4-5}$$

for all integers $m\geq 0$, where the dots denote derivatives with respect to time and

$$(\mathsf{A}u(t))^m=(\omega_n^{(\mu_1,\mu_2)})^2u^m(t),\qquad (\mathsf{f}(\{u^j(t):j\geq 0\}))^m=-\sum_{i,j,k=0}^{\infty}\mathsf{C}_{ijkm}^{(\mu_1,\mu_2)}u^i(t)u^j(t)u^k(t).\tag{4-6}$$

4C. ***Yang–Mills equation in spherical symmetry.*** For the Yang–Mills equation in spherical symmetry, we expand

$$u(t,\cdot)=\sum_{n=0}^{\infty}u^n(t)\mathfrak{e}_n,\quad \mathfrak{e}_i(x)\mathfrak{e}_j(x)=\sum_{m=0}^{\infty}\overline{\mathfrak{C}}_{ijm}\mathfrak{e}_m(x),\quad \sin^2(x)\mathfrak{e}_i(x)\mathfrak{e}_j(x)\mathfrak{e}_k(x)=\sum_{m=0}^{\infty}\mathfrak{C}_{ijkm}\mathfrak{e}_m(x)\tag{4-7}$$

to find the infinite system of nonlinear harmonic oscillators

$$\ddot{u}^m(t)+(\mathfrak{A}u(t))^m=(\mathfrak{f}(\{u^j(t):j\geq 0\}))^m\tag{4-8}$$

for all integers $m\geq 0$, where the dots denote derivatives with respect to time and

$$(\mathfrak{A}u(t))^m=\varpi_m^2u^m(t),\qquad (\mathfrak{f}(\{u^j(t):j\geq 0\}))^m=(\mathfrak{f}^{(2)}(\{u^j(t):j\geq 0\}))^m+(\mathfrak{f}^{(3)}(\{u^j(t):j\geq 0\}))^m,$$

with

$$(\mathfrak{f}^{(2)}(\{u^j(t):j\geq 0\}))^m = -3\sum_{i,j=0}^{\infty}\overline{\mathfrak{C}}_{ijm}u^i(t)u^j(t), \tag{4-9}$$

$$(\mathfrak{f}^{(3)}(\{u^j(t):j\geq 0\}))^m = -\sum_{i,j,k=0}^{\infty}\mathfrak{C}_{ijkm}u^i(t)u^j(t)u^k(t). \tag{4-10}$$

4D. *Lipschitz bounds.* Recall Section 2 where we define the Banach space

$$\mathcal{H}^k_s = \left\{q\in H^k([0,2\pi];l^2_s): q(t)=\sum_{j=0}^{\infty}q^j(t)e_j=\sum_{j=0}^{\infty}\left(\sum_{l=0}^{\infty}q^{lj}\cos(lt)\right)e_j,\ \|q\|^2_{\mathcal{H}^k_s}<\infty\right\}$$

endowed with the norm

$$\|q\|^2_{\mathcal{H}^k_s}=\sum_{j=0}^{\infty}j^{2s}\left(2|q^{0j}|^2+\sum_{l=1}^{\infty}|q^{lj}|^2(1+l^2)^k\right)=\frac{1}{\pi}\int_0^{2\pi}\sum_{\lambda=0}^{k}|q^{(\lambda)}(t)|^2_s\,dt,$$

where $q^{(\lambda)}(t)$ denotes the λ-th derivative of $q(t)$ with respect to t. Next, we show that the nonlinear terms we consider satisfy the following Lipschitz bounds, and we begin by considering the conformal cubic wave equation in spherical symmetry.

Lemma 4.1 (Lipschitz bounds for the CW model). *Let f be given by (4-3). Then, for all integers $k\geq 0$ and $s\geq 2$, there exists a positive constant (depending only on k and s) such that*

$$\|f(u)-f(v)\|_{\mathcal{H}^k_s}\lesssim(\|u\|^2_{\mathcal{H}^k_s}+\|v\|^2_{\mathcal{H}^k_s})\|u-v\|_{\mathcal{H}^k_s},$$
$$\|df(u)[h]-df(v)[h]\|_{\mathcal{H}^k_s}\lesssim(\|u\|_{\mathcal{H}^k_s}+\|v\|_{\mathcal{H}^k_s})\|h\|_{\mathcal{H}^k_s}\|u-v\|_{\mathcal{H}^k_s},$$

for all $u,v,h\in\mathcal{H}^k_s$ with $\|u\|_{\mathcal{H}^k_s}\leq\epsilon$, $\|v\|_{\mathcal{H}^k_s}\leq\epsilon$ and $\|h\|_{\mathcal{H}^k_s}\leq\epsilon$.

Remark 4.2 (regularity of the initial data for the CW model). As stated above, for the CW model, we require $s\in\mathbb{N}$ with $s\geq 2$. This means that the space of initial data $\mathcal{Q}\simeq l^2_{s+1}$ is at least l^2_3 (Theorem 2.4).

Proof. Let $s\geq 2$ be an integer, and pick any $u=\{u^j:j\geq 0\}\in l^2_s$. We also denote by $u(x)=\sum_{j=0}^{\infty}u^je_j(x)$ the corresponding function in the physical space and recall the definition of the linear operator L given in Section 3A. On the one hand, for any integer $s\geq 1$, we define the Sobolev space H^s_{CW} for spherically symmetric functions and find

$$\|u\|^2_{H^s_{\mathrm{CW}}}=\int_0^{\pi}uL^su\sin^2x\,dx=\sum_{j=0}^{+\infty}\omega_j^{2s}|u_j|^2\simeq|u|^2_s$$

since $\omega_j\simeq j$. On the other hand, note that, with a slight abuse of notation (we denote by u the original variable as well as the spherically symmetric version of it), we have that the Sobolev space above is equivalent to the standard Sobolev on $\mathbb{S}^3$:

$$\|u\|^2_{H^s_{\mathrm{CW}}}\simeq\|u\|^2_{H^s(\mathbb{S}^3)}=\int_{\mathbb{S}^3}u(-\Delta^s_{\mathbb{S}^3}u)\,d\mathrm{vol}_{\mathbb{S}^3}+\|u\|^2_{L^2(\mathbb{S}^3)}.$$

Here, $\Delta_{\mathbb{S}^3}$ stands for the standard Laplacian on $\mathbb{S}^3$ for the round metric and the standard volume form $d\mathrm{vol}_{\mathbb{S}^3}$. This equivalence yields that H^s_{CW} is an algebra since $H^s(\mathbb{S}^3)$ is an algebra provided that $s > \frac{3}{2}$. Then, picking an integer $s \geq 2$ and $u, v \in l^2_s$ and using the algebra property and the triangular inequality together with Plancherel's theorem yield

$$
\begin{aligned}
|f(u)|_s &= \|u^3\|_{H^s_{\mathrm{CW}}} \lesssim \|u\|^3_{H^s_{\mathrm{CW}}} = |u|^3_s, \\
|f(u)-f(v)|_s &= \|u^3 - v^3\|_{H^s_{\mathrm{CW}}} = \|(u-v)(u^2+uv+v^2)\|_{H^s_{\mathrm{CW}}} \\
&\lesssim \|u-v\|_{H^s_{\mathrm{CW}}}(\|u\|^2_{H^s_{\mathrm{CW}}} + \|u\|_{H^s_{\mathrm{CW}}}\|v\|_{H^s_{\mathrm{CW}}} + \|v\|^2_{H^s_{\mathrm{CW}}}) \\
&\lesssim \|u-v\|_{H^s_{\mathrm{CW}}}(\|u\|^2_{H^s_{\mathrm{CW}}} + \|v\|^2_{H^s_{\mathrm{CW}}}) = |u-v|_s(|u|^2_s + |v|^2_s), \\
|df(u)[h]-df(v)[h]|_s &= \|df(u)[h]-df(v)[h]\|_{H^s_{\mathrm{CW}}} = \|d(u^3)[h]-d(v^3)[h]\|_{H^s_{\mathrm{CW}}} \\
&= \|3u^2h - 3v^2h\|_{H^s_{\mathrm{CW}}} \lesssim \|u^2-v^2\|_{H^s_{\mathrm{CW}}}\|h\|_{H^s_{\mathrm{CW}}} \\
&\lesssim \|u-v\|_{H^s_{\mathrm{CW}}}(\|u\|_{H^s_{\mathrm{CW}}} + \|v\|_{H^s_{\mathrm{CW}}})\|h\|_{H^s_{\mathrm{CW}}} \lesssim |u-v|_s(|u|_s+|v|_s)|h|_s.
\end{aligned}
$$

This proves the claim for $k=0$. Finally, we present the proof for $k=1$. In this case, Plancherel's theorem yields

$$
\|f(u)\|_{\mathcal{H}^1_s} = \|f(u)\|_{H^1_t l^2_s} = \|f(u)\|_{L^2_t l^2_s} + \|\partial_t f(u)\|_{L^2_t l^2_s} = \|f(u)\|_{L^2_t H^s_x} + \|\partial_t f(u)\|_{L^2_t H^s_x}.
$$

Furthermore, the algebra property and Holder's inequality together with the embedding $H^1 \hookrightarrow L^\infty$ yield

$$
\begin{aligned}
\|f(u)\|_{L^2_t H^s_x} &= \|u^3\|_{L^2_t H^s_x} = \big\| \|u^3\|_{H^s_x} \big\|_{L^2_t} \lesssim \big\| \|u\|^3_{H^s_x} \big\|_{L^2_t} \leq \big\| \|u\|^2_{H^s_x} \big\|_{L^\infty_t} \big\| \|u\|_{H^s_x} \big\|_{L^2_t} \\
&= \big\| \|u\|_{H^s_x} \big\|^2_{L^\infty_t} \big\| \|u\|_{H^s_x} \big\|_{L^2_t} \lesssim \big\| \|u\|_{H^s_x} \big\|^2_{H^1_t} \big\| \|u\|_{H^s_x} \big\|_{L^2_t} = \|u\|^2_{H^1_t H^s_x} \|u\|_{H^0_t H^s_x} \leq \|u\|^3_{H^1_t H^s_x}, \\
\|\partial_t f(u)\|_{L^2_t H^s_x} &= \|3u^2 \partial_t u\|_{L^2_t H^s_x} \simeq \big\| \|u^2\partial_t u\|_{H^s_x} \big\|_{L^2_t} \lesssim \big\| \|u\|^2_{H^s_x} \|\partial_t u\|_{H^s_x} \big\|_{L^2_t} \leq \big\| \|u\|^2_{H^s_x} \big\|_{L^\infty_t} \big\| \|\partial_t u\|_{H^s_x} \big\|_{L^2_t} \\
&= \big\| \|u\|_{H^s_x} \big\|^2_{L^\infty_t} \big\| \|\partial_t u\|_{H^s_x} \big\|_{L^2_t} \lesssim \big\| \|u\|_{H^s_x} \big\|^2_{H^1_t} \big\| \|u\|_{H^s_x} \big\|_{H^1_t} \leq \|u\|^3_{H^1_t H^s_x},
\end{aligned}
$$

and hence $\|f(u)\|_{\mathcal{H}^1_s} \lesssim \|u\|^3_{\mathcal{H}^1_s}$. All the other bounds follow similarly. □

Next, we consider the conformal cubic wave equation out of spherical symmetry.

Lemma 4.3 (Lipschitz bounds for the CH model). *Let* f *be given by* (4-6). *Then, for all integers* $k \geq 0$ *and* $s \geq 2$, *there exists a positive constant* (*depending only on* k *and* s) *such that*

$$
\begin{aligned}
\|\mathsf{f}(u)-\mathsf{f}(v)\|_{\mathcal{H}^k_s} &\lesssim (\|u\|^2_{\mathcal{H}^k_s} + \|v\|^2_{\mathcal{H}^k_s})\|u-v\|_{\mathcal{H}^k_s}, \\
\|d\mathsf{f}(u)[h]-d\mathsf{f}(v)[h]\|_{\mathcal{H}^k_s} &\lesssim (\|u\|_{\mathcal{H}^k_s} + \|v\|_{\mathcal{H}^k_s})\|h\|_{\mathcal{H}^k_s}\|u-v\|_{\mathcal{H}^k_s},
\end{aligned}
$$

for all $u, v, h \in \mathcal{H}^k_s$ *with* $\|u\|_{\mathcal{H}^k_s} \leq \epsilon$, $\|v\|_{\mathcal{H}^k_s} \leq \epsilon$ *and* $\|h\|_{\mathcal{H}^k_s} \leq \epsilon$.

Remark 4.4 (regularity of the initial data for the CH model). As stated above, for the CH model, we require $s \in \mathbb{N}$ with $s \geq 2$. This means that the space of initial data $\mathcal{Q} \simeq l^2_{s+1}$ is at least l^2_3 (Theorem 2.4).

Proof. The proof coincides with the one of Lemma 4.1. □

Finally, we consider the Yang–Mills equation in spherical symmetry.

Lemma 4.5 (Lipschitz bounds for the YM model). *Let $\mathfrak{f}^{(2)}$ and $\mathfrak{f}^{(3)}$ be given by* (4-9) *and* (4-10), *respectively. Then, for all integers $k \geq 0$ and $s \geq 3$, there exists a positive constant (depending only on k and s) such that*

$$\|\mathfrak{f}^{(2)}(u) - \mathfrak{f}^{(2)}(v)\|_{\mathcal{H}^k_s} \lesssim (\|u\|_{\mathcal{H}^k_s} + \|v\|_{\mathcal{H}^k_s})\|u - v\|_{\mathcal{H}^k_s},$$
$$\|d\mathfrak{f}^{(2)}(u)[h] - d\mathfrak{f}^{(2)}(v)[h]\|_{\mathcal{H}^k_s} \lesssim \|h\|_{\mathcal{H}^k_s}\|u - v\|_{\mathcal{H}^k_s},$$
$$\|\mathfrak{f}^{(3)}(u) - \mathfrak{f}^{(3)}(v)\|_{\mathcal{H}^k_s} \lesssim (\|u\|^2_{\mathcal{H}^k_s} + \|v\|^2_{\mathcal{H}^k_s})\|u - v\|_{\mathcal{H}^k_s},$$
$$\|d\mathfrak{f}^{(2)}(u)[h] - d\mathfrak{f}^{(2)}(v)[h]\|_{\mathcal{H}^k_s} \lesssim (\|u\|_{\mathcal{H}^k_s} + \|v\|_{\mathcal{H}^k_s})\|h\|_{\mathcal{H}^k_s}\|u - v\|_{\mathcal{H}^k_s},$$

for all $u, v, h \in \mathcal{H}^k_s$ with $\|u\|_{\mathcal{H}^k_s} \leq \epsilon$, $\|v\|_{\mathcal{H}^k_s} \leq \epsilon$ and $\|h\|_{\mathcal{H}^k_s} \leq \epsilon$.

Remark 4.6 (regularity of the initial data for the YM model). As stated above, for the YM model, we require $s \in \mathbb{N}$ with $s \geq 3$. This means that the space of initial data l^2_{s+3} is at least l^2_6 (Lemma 2.10, Theorem 2.5).

Proof. Let $s \geq 3$ be an integer and pick any $u = \{u^j : j \geq 0\} \in l^2_s$. We also denote by $u(x) = \sum_{j=0}^{\infty} u^j \mathfrak{e}_j(x)$ the corresponding function in the physical space and recall the definition of the linear operator $\mathfrak{L}$ given in Section 3C. In the following, we claim that the operator

$$\Delta_{\text{YM}} u = \frac{1}{\sin^4(x)} \partial_x (\sin^4(x) \partial_x u)$$

coincides with the Laplace–Beltrami operator $\Delta_{\mathbb{S}^5} u$ on the sphere $\mathbb{S}^5 \hookrightarrow \mathbb{R}^6$ restricted to a class of symmetric functions. Indeed, we endow $\mathbb{S}^5$ with the round metric and consider the standard Eulerian coordinates $(x_1 = x, x_2, x_3, x_4, x_5) \in (0, \pi)^4 \times (0, 2\pi)$, so that $y = (y^1, y^2, y^3, y^4, y^5, y^6) \in \mathbb{S}^5$ with $y^1 = \cos x_1$,

$$y^i = \cos x_i \prod_{j=1}^{i-1} \sin x_j$$

for all $i \in \{2, 3, 4, 5\}$ and $y^6 = \sin x_1 \sin x_2 \sin x_3 \sin x_4 \sin x_5$. The metric element in these coordinates is given by the standard round metric on $\mathbb{S}^5$. Then, for a function u defined on $\mathbb{S}^5$ that is invariant under all rotations around the y^6-axis, the operator $\Delta_{\text{YM}} u$ coincides with $\Delta_{\mathbb{S}^5} u$. We call such functions on $\mathbb{S}^5$ "spherically symmetric". Here, $H^s(\mathbb{S}^5)$ is an algebra provided that $s > \frac{5}{2}$. We pick an integer $s \geq 3$, and the rest of the proof coincides with the one of Lemma 4.1. □

5. The Fourier coefficients

Here we study the Fourier coefficients, as defined by (4-1), (4-4) and (4-7). Since the eigenfunctions are given by Jacobi polynomials and since the Fourier coefficients involve products of the eigenfunctions, these are a priori complicated integrals, depending on the indices of the eigenfunctions. Nonetheless, we will derive here explicit closed formulas for the various Fourier coefficients on resonant indices.

5A. ***Conformal cubic wave equation in spherical symmetry.*** In this case, the Fourier coefficients are given by (4-1). By taking the inner product $(\cdot \mid \cdot)$, defined in Section 3A, in both sides of (4-1), we deduce that

$$C_{ijkm} = \frac{2}{\pi} \int_{-1}^{1} U_i(y)U_j(y)U_k(y)U_m(y)\sqrt{1-y^2}\,dy,$$

where we also used the definition of the Chebyshev polynomials of the second kind:

$$e_n(x) = U_n(\cos(x)) = \frac{\sin(\omega_n x)}{\sin(x)}$$

for all $n \in \{i, j, k, m\}$. Next, we call a quadruple (i, j, k, m) of indices *resonant* if

$$\omega_i \pm \omega_j \pm \omega_k \pm \omega_m = 0 \tag{5-1}$$

and study the Fourier coefficients on resonant indices.

Vanishing Fourier coefficients. Firstly, we show the Fourier coefficients vanish on some resonant indices.

Lemma 5.1 (vanishing Fourier coefficients on resonant indices). *For any integers $i, j, k, m \geq 0$ such that (5-1) holds with only one minus sign, we have $C_{ijkm} = 0$.*

Proof. Let i, j, k, m be positive integers such that $\omega_i + \omega_j + \omega_k - \omega_m = 0$. Then, $m = 2+i+j+k$ and, according to the computation above, we have

$$C_{ijkm} = \int_{-1}^{1} R_N(y)U_m(y)\sqrt{1-y^2}\,dy, \quad R_N(y) = \frac{2}{\pi}U_i(y)U_j(y)U_k(y),$$

where $R_N(y)$ is a polynomial of degree $N = i+j+k < m$, and hence the Fourier coefficient vanishes since $U_m(y)$ forms an orthonormal and complete basis with respect to the weight $\sqrt{1-y^2}$. The other results now follow immediately using the symmetries of the Fourier coefficients with respect to i, j, k, m. □

Nonvanishing Fourier coefficients. Secondly, we study the nonvanishing Fourier coefficients on resonant indices. In order to deal with these constants, one needs a computationally efficient formula. In the spherically symmetric case, where the basis consists of the Chebyshev polynomials, there exists the addition formula [Szegő 1975]

$$U_p(y)U_q(y) = \sum_{\substack{r=|q-p| \\ \text{step } 2}}^{p+q} U_r(y) = \sum_{s=0}^{\min(p,q)} U_{|q-p|+2s}(y) \tag{5-2}$$

for all $p, q \geq 0$. Consequently, we implement the addition formula (5-2) together with the orthogonality property of the Chebyshev polynomials with respect to the weight $\sqrt{1-y^2}$ to obtain

$$C_{ijkm} = \frac{2}{\pi} \sum_{\substack{r=|j-i| \\ \text{step } 2}}^{i+j} \sum_{\substack{s=|m-k| \\ \text{step } 2}}^{m+k} \int_{-1}^{1} U_r(y)U_s(y)\sqrt{1-y^2}\,dy = \sum_{\substack{r=|j-i| \\ \text{step } 2}}^{j+i} \sum_{\substack{s=|m-k| \\ \text{step } 2}}^{m+k} \mathbb{1}(r=s). \tag{5-3}$$

Next, we use (5-3) to derive closed formulas for the nonvanishing Fourier coefficients.

Lemma 5.2 (nonvanishing Fourier coefficients on resonant indices: closed formulas). *For any integers $i, j, k, m \geq 0$ such that (5-1) holds with only two minus signs, we have $C_{ijkm} = \omega_{\min\{i,j,k,m\}}$.*

Proof. Let $i, j, k, m \geq 0$ be integers such that $\omega_i + \omega_j - \omega_k - \omega_m = 0$. Hence $m = i + j - k$, and assume for simplicity that $i \leq j$, $k \leq m$ and $i \leq k$. Then, setting $r = i + j - p$ and $s = i + j - q$ as well as $t = 2p$ and $\tau = 2q$, equation (5-3) yields

$$C_{ijkm} = \sum_{\substack{p=0 \\ \text{step } 2}}^{2i} \sum_{\substack{q=0 \\ \text{step } 2}}^{2k} \mathbb{1}(p = q) = \sum_{t=0}^{i} \sum_{\tau=0}^{k} \mathbb{1}(t = \tau) = \sum_{t=0}^{i} \sum_{\tau=0}^{i} \mathbb{1}(t = \tau) = i + 1 = \omega_i.$$

All the other results follow immediately by the symmetries of the Fourier coefficients with respect to i, j, k and m, □

5B. ***Conformal cubic wave equation out of spherical symmetry.*** In this case, the Fourier coefficients are given by (4-4). By taking the inner product $\langle \cdot | \cdot \rangle$, defined in Section 3B, in both sides of (4-4), we deduce that

$$\mathsf{C}^{(\mu_1,\mu_2)}_{ijkm} = \frac{1}{2} \prod_{\lambda_1 \in \{i,j,k,m\}} \mathsf{N}^{(\mu_1,\mu_2)}_{\lambda_1} \int_{-1}^{1} (1-x)^{2\mu_1} (1+x)^{2\mu_2} \prod_{\lambda_2 \in \{i,j,k,m\}} P^{(\mu_1,\mu_2)}_{\lambda_2}(x)\, dx,$$

where the normalization constant $\mathsf{N}^{(\mu_1,\mu_2)}_{\lambda_1}$ is given by (3-3). As before, we call a quadruple (i, j, k, m) of indices *resonant* if

$$\omega^{(\mu_1,\mu_2)}_i \pm \omega^{(\mu_1,\mu_2)}_j \pm \omega^{(\mu_1,\mu_2)}_k \pm \omega^{(\mu_1,\mu_2)}_m = 0 \tag{5-4}$$

and study the Fourier coefficients on resonant indices.

Vanishing Fourier coefficients. Firstly, we show the Fourier coefficients vanish on some resonant indices.

Lemma 5.3 (vanishing Fourier coefficients on resonant indices). *For any integers $i, j, k, m \geq 0$ such that (5-4) holds with only one minus sign, we have $\mathsf{C}^{(\mu_1,\mu_2)}_{ijkm} = 0$.*

Proof. The proof is similar to the one of Lemma 5.1. □

Nonvanishing Fourier coefficients. Next, we study the nonvanishing Fourier coefficients. In principle, in order to deal with these constants, one needs a computationally efficient formula as in the spherically symmetric case. However, out of spherical symmetry, where the basis consists of the Jacobi polynomials — although there exists the addition formula

$$P^{(\mu_1,\mu_2)}_p(x) P^{(\mu_1,\mu_2)}_q(x) = \sum_{r=|p-q|}^{p+q} L(p, q, r) P^{(\mu_1,\mu_2)}_r(x)$$

for all $p, q \geq 0$, similar to (5-2) — the linearization coefficients $L(p, q, r)$ remain unknown in closed form for generic values of μ_1 and μ_2, and hence a closed formula similar to (5-3) is not available in general. We note that Rahman[10] [1981, p. 919] was able to prove that the linearization coefficients of Jacobi polynomials can be represented as a well-posed hypergeometric function ${}_9F_8(1)$. On the other hand, in the special case where $\mu_1 = \mu_2$, the Jacobi polynomials reduce to Gegenbauer polynomials for which the

[10] According to [Cohl 2016], there was a minor typo in Rahman's published result; in the linearization coefficient in [Rahman 1981], the term $(-\alpha - \beta - 2m)$ should be replaced by the Pochhammer symbol $(-\alpha - \beta - 2m)_k$. The corrected linearization formula is given in [Cohl 2016].

linearization coefficients are given by well-posed and closed formulas [NIST 2010; Sánchez-Ruiz 2001; Szegő 1975]. Hence, we restrict ourselves to the case $\mu_1=\mu_2=\mu$ and also denote by γ the index referring to the fixed choice of the 1-mode initial data. Then, we derive closed formulas for the nonvanishing Fourier coefficients for a resonant pair of indices (i, j, k, m) by using the formula [Sánchez-Ruiz 2001, (20)] for the Gegenbauer polynomials:

$$(C_m^{(\mu+1/2)}(x))^2=\sum_{\lambda=0}^{m} L_{\mu m}(\lambda) C_{2\lambda}^{(2\mu+1/2)}(x), \tag{5-5}$$

where the coefficients are given by

$$L_{\mu m}(\lambda)=\frac{(2\mu+1)_m}{\Gamma(m+1)}\frac{\left(\frac{1}{2}\right)_\lambda\left(\frac{1}{2}\right)_{m-\lambda}\left(\mu+\frac{1}{2}\right)_\lambda(\lambda+2\mu+1)_m}{\Gamma(m-\lambda+1)(\mu+1)_\lambda\left(2\mu+\frac{1}{2}\right)_{2\lambda}\left(2\lambda+2\mu+\frac{3}{2}\right)_{m-\lambda}}, \tag{5-6}$$

valid for any real $x\in[-1,1]$ and integers $\mu, m, \lambda \geq 0$, and $(a)_n=\Gamma(a+n)/\Gamma(a)$ stands for the Pochhammer's symbol defined for any $a\in\mathbb{R}$ with $a\notin\{0,-1,-2,\dots\}$ and $n\in\mathbb{N}$. Notice that (5-5) is a combination of a linearization and a connection formula for Gegenbauer polynomials. Specifically, we establish the following result.

Lemma 5.4 (nonvanishing Fourier coefficients on resonant indices: closed formulas). *Let $\gamma\geq 0$ and $m\geq\gamma$ be any integers. Then, we have*

$$\mathsf{C}_{\gamma\gamma mm}^{(\mu,\mu)}=\frac{1}{2}\sum_{\lambda=0}^{\gamma}\mathsf{M}_\gamma^{(\mu)}(\lambda)\mathsf{M}_m^{(\mu)}(\lambda)\xi_\lambda(\mu),$$

where

$$\xi_\lambda(\mu)=\frac{\pi 2^{1-4\mu}\Gamma(2\lambda+4\mu+1)}{(4\lambda+4\mu+1)\Gamma(2\lambda+1)\left(\Gamma\left(2\mu+\frac{1}{2}\right)\right)^2},$$

$$\mathsf{M}_m^{(\mu)}(\lambda)=\frac{1}{2\pi^{3/2}}\frac{(4\lambda+4\mu+1)\Gamma\left(\lambda+\frac{1}{2}\right)\Gamma\left(2\mu+\frac{1}{2}\right)\Gamma\left(\lambda+\mu+\frac{1}{2}\right)}{\Gamma(\lambda+\mu+1)\Gamma(\lambda+2\mu+1)}\cdot\frac{(2\mu+2m+1)\Gamma\left(m-\lambda+\frac{1}{2}\right)\Gamma(m+\lambda+2\mu+1)}{\Gamma(m-\lambda+1)\Gamma\left(m+\lambda+2\mu+\frac{3}{2}\right)}.$$

Proof. Let $\gamma\geq 1$ be a fixed integer and pick any integer $m\geq\gamma$. Then, by the definition of the Fourier coefficient, the fact that the Jacobi polynomials with equal parameters can be written in terms of the Gegenbauer polynomials [NIST 2010, 18.7.1],

$$P_m^{(\mu,\mu)}(x)=w_m^{(\mu)}C_m^{(\mu+1/2)}(x), \qquad w_m^{(\mu)}=\frac{\Gamma(2\mu+1)\Gamma(m+\mu+1)}{\Gamma(\mu+1)\Gamma(m+2\mu+1)},$$

together with (5-5)–(5-6), we have

$$\begin{aligned}\mathsf{C}_{\gamma\gamma mm}^{(\mu,\mu)}&=\frac{1}{2}(w_\gamma^{(\mu)}\mathsf{N}_\gamma^{(\mu,\mu)})^2(w_m^{(\mu)}\mathsf{N}_m^{(\mu,\mu)})^2\int_{-1}^{1}(1-x^2)^{2\mu}(C_\gamma^{(\mu+1/2)}(x))^2(C_m^{(\mu+1/2)}(x))^2\,dx\\&=\frac{1}{2}(w_\gamma^{(\mu)}\mathsf{N}_\gamma^{(\mu,\mu)})^2(w_m^{(\mu)}\mathsf{N}_m^{(\mu,\mu)})^2\sum_{\nu=0}^{\gamma}L_{\mu\gamma}(\nu)\sum_{\lambda=0}^{m}L_{\mu m}(\lambda)\int_{-1}^{1}(1-x^2)^{2\mu}C_{2\nu}^{(2\mu+1/2)}(x)C_{2\lambda}^{(2\mu+1/2)}(x)\,dx.\end{aligned}$$

Now, the orthogonality of the Gegenbauer polynomials,

$$\int_{-1}^{1} C_{2\nu}^{(2\mu+1/2)}(x)C_{2\lambda}^{(2\mu+1/2)}(x)(1-x^2)^{2\mu}\,dx = \xi_\lambda(\mu)\mathbb{1}(\nu=\lambda),$$

where $\xi_\lambda(\mu)$ is defined above, together with the fact that $0 \le \gamma \le m$ yields

$$\begin{aligned}\mathsf{C}_{\gamma\gamma mm}^{(\mu,\mu)} &= \frac{1}{2}(w_\gamma^{(\mu)}\mathsf{N}_\gamma^{(\mu,\mu)})^2(w_m^{(\mu)}\mathsf{N}_m^{(\mu,\mu)})^2\sum_{\nu=0}^{\gamma}L_{\mu\gamma}(\nu)\sum_{\lambda=0}^{\gamma}L_{\mu m}(\lambda)\xi_\lambda(\mu)\mathbb{1}(\nu=\lambda)\\ &= \frac{1}{2}(w_\gamma^{(\mu)}\mathsf{N}_\gamma^{(\mu,\mu)})^2(w_m^{(\mu)}\mathsf{N}_m^{(\mu,\mu)})^2\sum_{\lambda=0}^{\gamma}L_{\mu\gamma}(\lambda)L_{\mu m}(\lambda)\xi_\lambda(\mu)\\ &= \frac{1}{2}\sum_{\lambda=0}^{\gamma}\mathsf{M}_\gamma^{(\mu)}(\lambda)\mathsf{M}_m^{(\mu)}(\lambda)\xi_\lambda(\mu).\end{aligned}$$

Finally, setting $\mathsf{M}_m^{(\mu)}(\lambda) = (w_m^{(\mu)}\mathsf{N}_m^{(\mu,\mu)})^2L_{\mu m}(\lambda)$, a direct computation using (3-3) and (5-6) yields the closed formula for $\mathsf{M}_m^{(\mu)}(\lambda)$ stated above and completes the proof. □

Next, we show that the closed formulas we derived above are in fact monotone with respect to m.

Lemma 5.5 (monotonicity of $\mathsf{M}_m^{(\mu)}(\lambda)$). *Let $\gamma \ge 0$, $m \ge \gamma$ and $0 \le \lambda \le \gamma$ be any integers. Then, the function $\mathsf{M}_m^{(\mu)}(\lambda)$ defined in Lemma 5.4 is decreasing with respect to m.*

Proof. Let $\gamma \ge 0$, $m \ge \gamma$ and $0 \le \lambda \le \gamma$ be any integers. The claim follows immediately by computing the difference $\mathsf{M}_{m+1}^{(\mu)}(\lambda) - \mathsf{M}_m^{(\mu)}(\lambda)$. Indeed, the identity for the ratio of two Gamma functions, $\Gamma(x+1) = x\Gamma(x)$, valid for all $x \in \mathbb{R}$, yields that $\mathsf{M}_{m+1}^{(\mu)}(\lambda) - \mathsf{M}_m^{(\mu)}(\lambda)$ equals

$$-\frac{(4\lambda+4\mu+1)\Gamma\big(\lambda+\frac{1}{2}\big)\Gamma\big(2\mu+\frac{1}{2}\big)\Gamma\big(\lambda+\mu+\frac{3}{2}\big)\Gamma\big(m-\lambda+\frac{1}{2}\big)\Gamma(m+\lambda+2\mu+1)}{\pi^{3/2}\Gamma(\lambda+\mu)\Gamma(\lambda+2\mu+1)\Gamma(m-\lambda+2)\Gamma\big(m+\lambda+2\mu+\frac{5}{2}\big)},$$

which is strictly negative for all m, λ and μ, and hence $\mathsf{M}_m^{(\mu)}(\lambda)$ is decreasing with respect to m for all λ and μ, which completes the proof. □

Remark 5.6 (closed formulas for $\mathsf{C}_{\gamma\gamma mm}^{(\mu,\mu)}$ for small values of γ). Finally, we note that one can use Lemma 5.4 to find closed formulas for the Fourier coefficients provided that γ is sufficiently small. For example, for $\gamma \in \{0, 1\}$, we find that

$$\begin{aligned}\mathsf{C}_{00mm}^{(\mu,\mu)} &= \frac{1}{2\pi}(2\mu+1)\left(\frac{\Gamma\big(\mu+\frac{1}{2}\big)}{\Gamma(\mu+1)}\right)^2\frac{(2\mu+2m+1)\Gamma\big(m+\frac{1}{2}\big)\Gamma(m+2\mu+1)}{\Gamma(m+1)\Gamma\big(m+2\mu+\frac{3}{2}\big)},\\ \mathsf{C}_{11mm}^{(\mu,\mu)} &= \frac{1}{8\pi}(\mu+1)(2\mu+1)(2\mu+3)\left(\frac{\Gamma\big(\mu+\frac{1}{2}\big)}{\Gamma(\mu+2)}\right)^2\\ &\qquad\cdot\frac{(2\mu+2m+1)(-\mu+2m(2\mu+m+1)-1)\Gamma\big(m-\frac{1}{2}\big)\Gamma(m+2\mu+1)}{\Gamma(m+1)\Gamma\big(m+2\mu+\frac{5}{2}\big)}.\end{aligned}$$

Figure 1 illustrates the Fourier coefficients $\mathsf{C}_{\gamma\gamma mm}^{(\mu,\mu)}$ for $\mu = 30$ and $\gamma \in \{0, 1, 2\}$, respectively, as m varies within $\{1, 2, \dots, 50\}$.

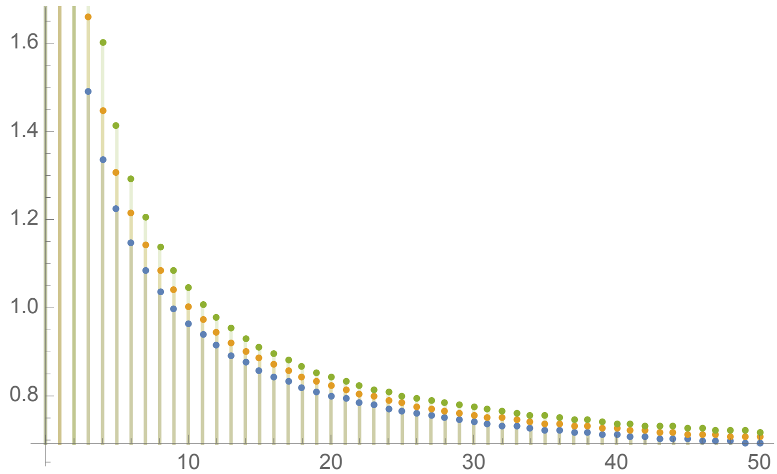

Figure 1. The Fourier coefficients $\mathsf{C}^{(30,30)}_{\gamma\gamma mm}$ for $\gamma=0$ (blue/bottom), $\gamma=1$ (orange/middle) and $\gamma=2$ (green/top) as m varies within the interval $[1,50]$. They are all decreasing for $m\geq 2\gamma+1$.

5C. ***Yang–Mills equation in spherical symmetry.*** In this case, the Fourier coefficients are given by (4-7). By taking the inner product $[\cdot\,|\,\cdot]$, defined in Section 3C, in both sides of (4-7), we deduce that

$$\overline{\mathfrak{C}}_{ijm}=\prod_{\lambda_1\in\{i,j,m\}}\mathfrak{N}_{\lambda_1}\int_{-1}^{1}(1-y^2)^{3/2}\prod_{\lambda_2\in\{i,j,m\}}P^{(3/2,3/2)}_{\lambda_2}(y)\,dy,$$

$$\mathfrak{C}_{ijkm}=\prod_{\lambda_1\in\{i,j,k,m\}}\mathfrak{N}_{\lambda_1}\int_{-1}^{1}(1-y^2)^{5/2}\prod_{\lambda_2\in\{i,j,k,m\}}P^{(3/2,3/2)}_{\lambda_2}(y)\,dy,$$

where the normalization constant $\mathfrak{N}_{\lambda_1}$ is given by (3-5). As before, we call a triple $(i,\,j,\,m)$ or a quadruple $(i,\,j,\,k,\,m)$ of indices *resonant* if

$$\varpi_i\pm\varpi_j\pm\varpi_m=0, \tag{5-7}$$

$$\varpi_i\pm\varpi_j\pm\varpi_k\pm\varpi_m=0, \tag{5-8}$$

respectively, and study the Fourier coefficients on resonant indices.

Vanishing Fourier coefficients. Firstly, we show the Fourier coefficients vanish on some resonant indices.

Lemma 5.7 (vanishing Fourier coefficients on resonant indices). *For any integers $i,\,j,\,m\geq 0$ such that (5-7) holds with only one minus sign and for any integers $i,\,j,\,k,\,m\geq 0$ such that (5-8) holds with only one minus sign, we have $\overline{\mathfrak{C}}_{ijm}=0$ and $\mathfrak{C}_{ijkm}=0$, respectively.*

Proof. The proof is similar to the one of Lemma 5.1. ☐

Nonvanishing Fourier coefficients. Next, we study the nonvanishing Fourier coefficients. In order to deal with these constants, one needs a computationally efficient formula as in the two previous cases. In the spherically symmetric case we consider here, the basis consists of the Jacobi polynomials with equal parameters, and these are weighted Gegenbauer polynomials [NIST 2010, 18.7.1]

$$C_n^{(2)}(x) = \frac{(4)_n}{\left(\frac{5}{2}\right)_n} P_n^{(3/2,3/2)}(x) = \frac{\sqrt{\pi}}{8} \frac{\Gamma(n+4)}{\Gamma\left(n+\frac{5}{2}\right)} P_n^{(3/2,3/2)}(x)$$

for $n \in \{\gamma, m\}$ and $x \in [-1, 1]$. The latter, together with the definition of the normalization constant $\mathfrak{N}_n$ from (3-5), expresses the normalized Jacobi polynomials in terms of the normalized Gegenbauer polynomials as

$$\mathfrak{N}_n P_n^{(3/2,3/2)}(x) = \mathfrak{N}_n \frac{8}{\sqrt{\pi}} \frac{\Gamma\left(n+\frac{5}{2}\right)}{\Gamma(n+4)} C_n^{(2)}(x) = \mathfrak{w}_n C_n^{(2)}(x), \quad \mathfrak{w}_n = \sqrt{\frac{8}{\pi}} \frac{1}{\sqrt{(n+1)(n+3)}},$$

for all $n \in \{\gamma, m\}$. Consequently, the Fourier coefficients can be written in terms of the Gegenbauer polynomials as follows:

$$\overline{\mathfrak{C}}_{ijm} = \mathfrak{w}_i \mathfrak{w}_j \mathfrak{w}_m \int_{-1}^{1} C_i^{(2)}(y) C_j^{(2)}(y) C_m^{(2)}(y) (1-y^2)^{3/2} \, dy,$$

$$\mathfrak{C}_{ijkm} = \mathfrak{w}_i \mathfrak{w}_j \mathfrak{w}_k \mathfrak{w}_m \int_{-1}^{1} C_i^{(2)}(y) C_j^{(2)}(y) C_k^{(2)}(y) C_m^{(2)}(y) (1-y^2)^{5/2} \, dy.$$

Then, we derive closed formulas for the nonvanishing Fourier coefficients for a resonant quadruple (i, j, k, m) as follows:

• Whenever a Gegenbauer polynomial has an order μ such that the weight $(1-y^2)^{\mu-1/2}$ (with respect to which it forms an orthonormal basis) does not coincide with the weights $(1-y^2)^p$ with $p \in \left\{\frac{3}{2}, \frac{5}{2}\right\}$ (that define the integrals above), we use the connection formula [NIST 2010, 18.18.16]

$$C_n^{(\mu)}(y) = \sum_{\ell=0}^{\lfloor n/2 \rfloor} \alpha_{n\mu\lambda}(\ell) C_{n-2\ell}^{(\lambda)}(y), \quad \alpha_{n\mu\lambda}(\ell) = \frac{\lambda+n-2\ell}{\lambda} \frac{(\mu)_{n-\ell}}{(\lambda+1)_{n-\ell}} \frac{(\mu-\lambda)_\ell}{\ell!}. \tag{5-9}$$

In particular, we are going to use this only for $\lambda = \mu + 1$. In this case, the latter is equivalent to the recurrence relation [NIST 2010, 18.9.7]

$$C_n^{(\mu)}(x) = \frac{\mu}{n+\mu} \left(C_n^{(\mu+1)}(x) - C_{n-2}^{(\mu+1)}(x) \right),$$

valid for all integers $\mu \geq 0$ and $n \geq 2$.

• Whenever a Gegenbauer polynomial is multiplied by itself, we use the addition formula [NIST 2010, 18.18.22; Sánchez-Ruiz 2001, (19)]

$$(C_n^{(\lambda)}(y))^2 = \sum_{\ell=0}^{n} \beta_{n\lambda}(\ell) C_{2\ell}^{(\lambda)}(y), \quad \beta_{n\lambda}(\ell) = \frac{1}{n!} \binom{n}{\ell} \frac{(2\ell)!\,(\lambda)_\ell (\lambda)_{n-\ell} (2\ell+2\lambda)_{n-\ell}}{\ell!\,(\ell+\lambda)_\ell (2\ell+\lambda+1)_{n-\ell}}. \tag{5-10}$$

• Whenever two Gegenbauer polynomials of different degrees but of the same order are multiplied, we use the addition formula [NIST 2010, 18.18.22]

$$C_m^{(\lambda)}(y) C_n^{(\lambda)}(y) = \sum_{\ell=0}^{\min(m,n)} \zeta_{mn\lambda}(\ell) C_{m+n-2\ell}^{(\lambda)}(y), \tag{5-11}$$

where the coefficients are given by

$$\zeta_{mn\lambda}(\ell)=\frac{(m+n+\lambda-2\ell)(m+n-2\ell)!}{(m+n+\lambda-\ell)\ell!\,(m-\ell)!\,(n-\ell)!}\frac{(\lambda)_\ell(\lambda)_{m-\ell}(\lambda)_{n-\ell}(2\lambda)_{m+n-\ell}}{(\lambda)_{m+n-\ell}(2\lambda)_{m+n-2\ell}}.$$

• Whenever a product of a monomial with a Gegenbauer polynomial is integrated, we use the formula [NIST 2010, 18.17.37]

$$\int_0^1 x^{z-1}C_n^{(\lambda)}(x)(1-x^2)^{\lambda-1/2}\,dx=\frac{\pi 2^{1-2\lambda-z}\Gamma(n+2\lambda)\Gamma(z)}{n!\,\Gamma(\lambda)\Gamma\left(\frac{1}{2}+\frac{1}{2}n+\lambda+\frac{1}{2}z\right)\Gamma\left(\frac{1}{2}+\frac{1}{2}z-\frac{1}{2}n\right)},\tag{5-12}$$

valid for all integers $\lambda\geq 0$ and real numbers $z>0$. Notice that this is the Mellin transform of the function $C_n^{(\lambda)}(x)(1-x^2)^{\lambda-1/2}$ restricted to [0, 1].

In particular, we will only need to study $\overline{\mathfrak{C}}_{\gamma,\gamma,2\tau}$, $\overline{\mathfrak{C}}_{\gamma,2\tau+m-\gamma,m}$, $\overline{\mathfrak{C}}_{m,2\tau,m}$ and $\mathfrak{C}_{\gamma\gamma mm}$ for all integers $\tau\in\{0,1,\dots,\gamma\}$ and $m\geq 2\gamma+1$. To begin with, we focus on $\overline{\mathfrak{C}}_{ijm}$ for any integers $i,j,m\geq 0$ and establish the follow result.

Lemma 5.8 (nonvanishing Fourier coefficients $\overline{\mathfrak{C}}_{ijm}$ on resonant indices: closed formula). *For any integers* $i,j,m\geq 0$, *we have*

$$\begin{aligned}\overline{\mathfrak{C}}_{ijm}=&\frac{(i+j-m+2)(i-j+m+2)(-i+j+m+2)(i+j+m+6)}{4\sqrt{2\pi(i+1)(i+3)(j+1)(j+3)(m+1)(m+3)}}\\&\cdot\mathbb{1}(|j-m|\leq i\leq j+m)\mathbb{1}(|i-m|\leq j\leq i+m)\mathbb{1}(|i-j|\leq m\leq i+j)\\&\cdot\mathbb{1}(j+m-i\in 2\mathbb{N}\cup\{0\})\mathbb{1}(i+m-j\in 2\mathbb{N}\cup\{0\})\mathbb{1}(i+j-m\in 2\mathbb{N}\cup\{0\}).\end{aligned}$$

Proof. The result follows immediately from (5-11) together with the orthogonality of the Gegenbauer polynomials,

$$\int_{-1}^1 C_n^{(2)}(y)C_m^{(2)}(y)(1-y^2)^{3/2}\,dy=\frac{\pi}{8}(m+1)(m+3)\mathbb{1}(m=n)$$

for all integers $m,n\geq 0$. Indeed, for any integers $i,j,m\geq 0$, we have

$$\begin{aligned}\overline{\mathfrak{C}}_{ijm}&=\mathfrak{w}_i\mathfrak{w}_j\mathfrak{w}_m\sum_{\ell=0}^{\min(i,j)}\zeta_{ij2}(\ell)\int_{-1}^1 C_{i+j-2\ell}^{(2)}(y)C_m^{(2)}(y)(1-y^2)^{3/2}\,dy\\&=\frac{\pi}{8}(m+1)(m+3)\mathfrak{w}_i\mathfrak{w}_j\mathfrak{w}_m\sum_{\ell=0}^{\min(i,j)}\zeta_{ij2}(\ell)\mathbb{1}(2\ell=i+j-m).\end{aligned}$$

On the one hand, for all integers i, j and m such that $i+j-m\notin 2\mathbb{N}\cup\{0\}$, the Fourier coefficient vanishes. Furthermore, we have

$$0\leq i+j-m\leq 2\min(i,j)\quad\Longleftrightarrow\quad |i-j|\leq m\leq i+j.$$

Consequently, for all integers i, j and m such that the condition $|i-j|\leq m\leq i+j$ is not fulfilled, the Fourier coefficient vanishes. On the other hand, for all i, j and m such that both $i+j-m\in 2\mathbb{N}\cup\{0\}$

and $|i-j| \leq m \leq i+j$ hold true, we compute

$$\begin{aligned}\overline{\mathfrak{C}}_{ijm} &= \frac{\pi}{8}(m+1)(m+3)\mathfrak{w}_i\mathfrak{w}_j\mathfrak{w}_m\zeta_{ij2}\left(\frac{i+j-m}{2}\right)\sum_{\ell=0}^{\min(i,j)} \mathbb{1}(2\ell=i+j-m)\\ &= \frac{(i+j-m+2)(i-j+m+2)(-i+j+m+2)(i+j+m+6)}{4\sqrt{2\pi(i+1)(i+3)(j+1)(j+3)(m+1)(m+3)}},\end{aligned}$$

where we used the fact that $\sum_{l=0}^{\min(i,j)} \mathbb{1}(2\ell=i+j-m)=1$. Finally, using the symmetries of the Fourier coefficient with respect to i, j and m completes the proof. □

Next, we apply the previous result to obtain closed formulas for the Fourier coefficient $\overline{\mathfrak{C}}_{ijm}$ on the particular resonant indices we are interested in. Specifically, we establish the following result.

Lemma 5.9 (nonvanishing Fourier coefficients $\overline{\mathfrak{C}}_{ijm}$ on particular resonant indices: closed formulas). *Let γ, τ, m be integers such that $\gamma \geq 0$, $\tau \in \{0, 1, \ldots, \gamma\}$ and $m \geq 2\gamma+1$. Then, we have*

$$\overline{\mathfrak{C}}_{\gamma,\gamma,2\tau} = 2\sqrt{\frac{2}{\pi}}\frac{(\tau+1)^2(\gamma-\tau+1)(\gamma+\tau+3)}{(\gamma+1)(\gamma+3)\sqrt{4\tau(\tau+2)+3}}, \quad \overline{\mathfrak{C}}_{m,2\tau,m} = 2\sqrt{\frac{2}{\pi}}\frac{(\tau+1)^2(m-\tau+1)(m+\tau+3)}{(m+1)(m+3)\sqrt{4\tau(\tau+2)+3}},$$

$$\overline{\mathfrak{C}}_{\gamma,2\tau+m-\gamma,m} = 2\sqrt{\frac{2}{\pi}}\frac{(\tau+1)(\gamma-\tau+1)(m+\tau+3)(-\gamma+m+\tau+1)}{\sqrt{(\gamma+1)(\gamma+3)(m+1)(m+3)(-\gamma+m+2\tau+1)(-\gamma+m+2\tau+3)}}.$$

Proof. Let γ, τ, m be integers such that $\gamma \geq 0$, $\tau \in \{0, 1, \ldots, \gamma\}$ and $m \geq 2\gamma+1$. Firstly, notice that all the indices of the Fourier coefficients above satisfy all the conditions in the Booleans in Lemma 5.8. Then, the result follows immediately from Lemma 5.8 by direct substitution. □

Now, we focus on $\mathfrak{C}_{\gamma\gamma mm}$ and derive the following result.

Lemma 5.10 (nonvanishing Fourier coefficients $\mathfrak{C}_{\gamma\gamma mm}$ on resonant indices: closed formulas). *Let $\gamma \geq 0$ be a fixed integer. Then, for all $m \geq 2\gamma+1$, we have*

$$\mathfrak{C}_{\gamma\gamma mm} = \mathfrak{w}_\gamma^2\mathfrak{w}_m^2\sum_{\ell_2=0}^{\gamma}\sum_{\nu_2=0}^{\ell_2}\delta_\gamma(\ell_2,\nu_2)J_m(\ell_2,\nu_2),$$

where

$$\delta_\gamma(\ell_2,\nu_2) = \frac{(\ell_2+1)^2(-1)^{\nu_2}(\gamma-\ell_2+1)(\gamma+\ell_2+3)2^{2(\ell_2-\nu_2)}\Gamma(2\ell_2-\nu_2+2)}{(4\ell_2(\ell_2+2)+3)\Gamma(\nu_2+1)\Gamma(2\ell_2-2\nu_2+1)},$$

$$\begin{aligned}J_m(\ell_2,\nu_2) &= \frac{3\sqrt{\pi}(5\ell_2(3m(m+4)+1)+m(m+4)(4-15\nu_2)-5(\nu_2-4))\Gamma\left(\ell_2-\nu_2+\frac{1}{2}\right)}{8\Gamma(\ell_2-\nu_2+5)}\\ &+\sum_{\ell_1=2}^{\ell_2-\nu_2+1}\frac{\pi\ell_1(\ell_1+1)^2(2\ell_1-1)4^{-\ell_2+\nu_2-2}(\ell_1-m-1)(\ell_1+m+3)\Gamma(2\ell_2-2\nu_2+1)}{\Gamma(-\ell_1+\ell_2-\nu_2+2)\Gamma(\ell_1+\ell_2-\nu_2+3)}\\ &-\sum_{\ell_1=2}^{\ell_2-\nu_2}\frac{\pi(\ell_1+1)^2(\ell_1+2)(2\ell_1+5)4^{-\ell_2+\nu_2-2}(\ell_1-m-1)(\ell_1+m+3)\Gamma(2\ell_2-2\nu_2+1)}{\Gamma(-\ell_1+\ell_2-\nu_2+1)\Gamma(\ell_1+\ell_2-\nu_2+4)}.\end{aligned}$$

Proof. Let $\gamma \geq 0$ be a fixed integer and pick any integer $m \geq 2\gamma+1$. Then, by the definition of the Fourier coefficients, we have

$$\mathfrak{C}_{\gamma\gamma mm} = \mathfrak{w}_\gamma^2 \mathfrak{w}_m^2 \int_{-1}^{1} (C_\gamma^{(2)}(y))^2 (C_m^{(2)}(y))^2 (1-y^2)^{5/2}\, dy.$$

On the one hand, we use the linearization formula (5-10) together with the special cases $C_0^{(2)}(y) = 1$ and $C_2^{(2)}(y) = 12y^2 - 2$ to obtain

$$\begin{aligned}(C_m^{(2)}(y))^2 &= \sum_{\ell_1=0}^{m} \beta_{m2}(\ell_1) C_{2\ell_1}^{(2)}(y) = \sum_{\ell_1=0}^{m} \frac{(\ell_1+1)^2(-\ell_1+m+1)(\ell_1+m+3)}{4\ell_1(\ell_1+2)+3} C_{2\ell_1}^{(2)}(y)\\ &= \frac{1}{3}(m+1)(m+3) + \frac{4}{15}m(m+4)(12y^2-2) + \sum_{\ell_1=2}^{m} \frac{(\ell_1+1)^2(-\ell_1+m+1)(\ell_1+m+3)}{4\ell_1(\ell_1+2)+3} C_{2\ell_1}^{(2)}(y).\end{aligned}$$

Furthermore, for all $\ell_1 \geq 2$, the connection formula (5-9) yields

$$\begin{aligned}C_{2\ell_1}^{(2)}(y) &= \sum_{\nu_1=0}^{\ell_1} \alpha_{2\ell_1,2,3}(\nu_1) C_{2(\ell_1-\nu_1)}^{(3)}(y) = \sum_{\nu_1=0}^{\ell_1} \frac{(2\ell_1-2\nu_1+3)(-1)_{\nu_1}\Gamma(2\ell_1-\nu_1+2)}{3\nu_1!\,(4)_{2\ell_1-\nu_1}} C_{2(\ell_1-\nu_1)}^{(3)}(y)\\ &= \frac{(2\ell_1+3)(2)_{2\ell_1}}{3(4)_{2\ell_1}} C_{2\ell_1}^{(3)}(y) - \frac{(2\ell_1+1)(2)_{2\ell_1-1}}{3(4)_{2\ell_1-1}} C_{2(\ell_1-1)}^{(3)}(y) = \frac{1}{\ell_1+1}\big(C_{2\ell_1}^{(3)}(y) - C_{2(\ell_1-1)}^{(3)}(y)\big)\end{aligned}$$

since $(-1)_{\nu_1} = 0$ for all $\nu_1 \geq 2$. Consequently, we have

$$\begin{aligned}(C_m^{(2)}(y))^2 &= \tfrac{1}{3}(m+1)(m+3) + \tfrac{4}{15}m(m+4)(12y^2-2)\\ &\quad + \sum_{\ell_1=2}^{m} \frac{(\ell_1+1)(\ell_1-m-1)(\ell_1+m+3)}{4\ell_1(\ell_1+2)+3} \big(C_{2(\ell_1-1)}^{(3)}(y) - C_{2\ell_1}^{(3)}(y)\big).\end{aligned}$$

On the other hand, the linearization formula (5-10) together with the definition of the Gegenbauer polynomials

$$C_{2\ell_2}^{(2)}(y) = \sum_{\nu_2=0}^{\ell_2} d_{\ell_2}(\nu_2) y^{2(\ell_2-\nu_2)}, \quad d_{\ell_2}(\nu_2) = \frac{(-1)^{\nu_2}(4^{\ell_2-\nu_2}\Gamma(2\ell_2-\nu_2+2))}{\Gamma(\nu_2+1)\Gamma(2\ell_2-2\nu_2+1)},$$

yield

$$\begin{aligned}(C_\gamma^{(2)}(y))^2 = \sum_{\ell_2=0}^{\gamma} \beta_{\gamma 2}(\ell_2) C_{2\ell_2}^{(2)}(y) &= \sum_{\ell_2=0}^{\gamma} \beta_{\gamma 2}(\ell_2) \sum_{\nu_2=0}^{\ell_2} d_{\ell_2}(\nu_2) y^{2(\ell_2-\nu_2)}\\ &= \sum_{\ell_2=0}^{\gamma} \sum_{\nu_2=0}^{\ell_2} \delta_\gamma(\ell_2,\nu_2) y^{2(\ell_2-\nu_2)},\end{aligned}$$

where we set

$$\delta_\gamma(\ell_2,\nu_2) = \beta_{\gamma 2}(\ell_2) d_{\ell_2}(\nu_2) = \frac{(\ell_2+1)^2(-1)^{\nu_2}(\gamma-\ell_2+1)(\gamma+\ell_2+3)2^{2(\ell_2-\nu_2)}\Gamma(2\ell_2-\nu_2+2)}{(4\ell_2(\ell_2+2)+3)\Gamma(\nu_2+1)\Gamma(2\ell_2-2\nu_2+1)}.$$

Now, putting this all together, we infer

$$\begin{aligned}\mathfrak{C}_{\gamma\gamma mm} &= \mathfrak{w}_\gamma^2 \mathfrak{w}_m^2 \sum_{\ell_2=0}^{\gamma} \sum_{\nu_2=0}^{\ell_2} \delta_\gamma(\ell_2, \nu_2) \\ &\quad \times \int_{-1}^{1} y^{2(\ell_2-\nu_2)} \Bigg[\tfrac{1}{3}(m+1)(m+3) + \tfrac{4}{15} m(m+4)(12y^2-2) \\ &\qquad + \sum_{\ell_1=2}^{m} \frac{(\ell_1+1)(\ell_1-m-1)(\ell_1+m+3)}{4\ell_1(\ell_1+2)+3} (C^{(3)}_{2(\ell_1-1)}(y) - C^{(3)}_{2\ell_1}(y)) \Bigg] (1-y^2)^{5/2}\, dy \\ &= \mathfrak{w}_\gamma^2 \mathfrak{w}_m^2 \sum_{\ell_2=0}^{\gamma} \sum_{\nu_2=0}^{\ell_2} \delta_\gamma(\ell_2, \nu_2) J_m(\ell_2, \nu_2),\end{aligned}$$

where

$$\begin{aligned} J_m(\ell_2, \nu_2) &= \frac{(m+1)(m+3)}{3} \int_{-1}^{1} y^{2(\ell_2-\nu_2)} (1-y^2)^{5/2}\, dy \\ &\quad + \frac{4m(m+4)}{15} \int_{-1}^{1} y^{2(\ell_2-\nu_2)} (12y^2-2)(1-y^2)^{5/2}\, dy \\ &\quad + \sum_{\ell_1=2}^{m} \frac{(\ell_1+1)(\ell_1-m-1)(\ell_1+m+3)}{4\ell_1(\ell_1+2)+3} \int_{-1}^{1} y^{2(\ell_2-\nu_2)} C^{(3)}_{2(\ell_1-1)}(y)(1-y^2)^{5/2}\, dy \\ &\quad - \sum_{\ell_1=2}^{m} \frac{(\ell_1+1)(\ell_1-m-1)(\ell_1+m+3)}{4\ell_1(\ell_1+2)+3} \int_{-1}^{1} y^{2(\ell_2-\nu_2)} C^{(3)}_{2\ell_1}(y)(1-y^2)^{5/2}\, dy. \end{aligned}$$

We compute

$$\int_{-1}^{1} y^{2(\ell_2-\nu_2)} (1-y^2)^{5/2}\, dy = \frac{15\sqrt{\pi}\,\Gamma\big(\ell_2-\nu_2+\frac{1}{2}\big)}{8\Gamma(\ell_2-\nu_2+4)},$$

$$\int_{-1}^{1} y^{2(\ell_2-\nu_2)} (12y^2-2)(1-y^2)^{5/2}\, dy = \frac{15\sqrt{\pi}(5\ell_2-5\nu_2-1)\Gamma\big(\ell_2-\nu_2+\frac{1}{2}\big)}{4\Gamma(\ell_2-\nu_2+5)}.$$

On the one hand, for all $\ell_1 \geq 2$ with $\ell_1 > \ell_2-\nu_2+1$, we have $2(\ell_2-\nu_2) < 2(\ell_1-1)$, and hence

$$\int_{-1}^{1} y^{2(\ell_2-\nu_2)} C^{(3)}_{2(\ell_1-1)}(y)(1-y^2)^{5/2}\, dy = 0$$

since the Gegenbauer polynomial in the integrand forms an orthonormal and complete basis with respect to the weight $(1-y^2)^{5/2}$. On the other hand, for all $2 \leq \ell_1 \leq \ell_2-\nu_2+1$, the identity $C^{(3)}_{2\lambda}(-y) = C^{(3)}_{2\lambda}(y)$, valid for all real $y \in [-1, 1]$ and integers $\lambda \geq 0$, yields

$$\begin{aligned}\int_{-1}^{1} y^{2(\ell_2-\nu_2)} C^{(3)}_{2(\ell_1-1)}(y)(1-y^2)^{5/2}\, dy &= 2\int_{0}^{1} y^{2(\ell_2-\nu_2)} C^{(3)}_{2(\ell_1-1)}(y)(1-y^2)^{5/2}\, dy \\ &= \frac{\pi 4^{-\ell_2+\nu_2-3}\Gamma(2\ell_1+4)\Gamma(2\ell_2-2\nu_2+1)}{\Gamma(2\ell_1-1)\Gamma(-\ell_1+\ell_2-\nu_2+2)\Gamma(\ell_1+\ell_2-\nu_2+3)},\end{aligned}$$

where we used (5-12) to compute the last integral. In other words, we have

$$\sum_{\ell_1=2}^{m} \frac{(\ell_1+1)(\ell_1-m-1)(\ell_1+m+3)}{4\ell_1(\ell_1+2)+3} \int_{-1}^{1} y^{2(\ell_2-\nu_2)} C^{(3)}_{2(\ell_1-1)}(y)(1-y^2)^{5/2}\,dy$$
$$= \sum_{\ell_1=2}^{\ell_2-\nu_2+1} \frac{(\ell_1+1)(\ell_1-m-1)(\ell_1+m+3)}{4\ell_1(\ell_1+2)+3} \int_{-1}^{1} y^{2(\ell_2-\nu_2)} C^{(3)}_{2(\ell_1-1)}(y)(1-y^2)^{5/2}\,dy$$
$$= \sum_{\ell_1=2}^{\ell_2-\nu_2+1} \frac{\pi\ell_1(\ell_1+1)^2(2\ell_1-1)4^{-\ell_2+\nu_2-2}(\ell_1-m-1)(\ell_1+m+3)\Gamma(2\ell_2-2\nu_2+1)}{\Gamma(-\ell_1+\ell_2-\nu_2+2)\Gamma(\ell_1+\ell_2-\nu_2+3)}.$$

Similarly, on the one hand, for all $\ell_1 \geq 2$ with $\ell_1 > \ell_2-\nu_2$, we have $2(\ell_2-\nu_2) < 2\ell_1$, and hence

$$\int_{-1}^{1} y^{2(\ell_2-\nu_2)} C^{(3)}_{2\ell_1}(y)(1-y^2)^{5/2}\,dy = 0$$

since the Gegenbauer polynomial in the integrand forms an orthonormal and complete basis with respect to the weight $(1-y^2)^{5/2}$. On the other hand, for all $2 \leq \ell_1 \leq \ell_2-\nu_2$, the identity $C^{(3)}_{2\lambda}(-y) = C^{(3)}_{2\lambda}(y)$, valid for all real $y \in [-1, 1]$ and integers $\lambda \geq 0$, yields

$$\int_{-1}^{1} y^{2(\ell_2-\nu_2)} C^{(3)}_{2\ell_1}(y)(1-y^2)^{5/2}\,dy = 2\int_{0}^{1} y^{2(\ell_2-\nu_2)} C^{(3)}_{2\ell_1}(y)(1-y^2)^{5/2}\,dy$$
$$= \frac{\pi 4^{-\ell_2+\nu_2-3}\Gamma(2\ell_1+6)\Gamma(2\ell_2-2\nu_2+1)}{\Gamma(2\ell_1+1)\Gamma(-\ell_1+\ell_2-\nu_2+1)\Gamma(\ell_1+\ell_2-\nu_2+4)},$$

where we used once again (5-12) to compute the last integral. In other words, we have

$$\sum_{\ell_1=2}^{m} \frac{(\ell_1+1)(\ell_1-m-1)(\ell_1+m+3)}{4\ell_1(\ell_1+2)+3} \int_{-1}^{1} y^{2(\ell_2-\nu_2)} C^{(3)}_{2\ell_1}(y)(1-y^2)^{5/2}\,dy$$
$$= \sum_{\ell_1=2}^{\ell_2-\nu_2} \frac{(\ell_1+1)(\ell_1-m-1)(\ell_1+m+3)}{4\ell_1(\ell_1+2)+3} \int_{-1}^{1} y^{2(\ell_2-\nu_2)} C^{(3)}_{2\ell_1}(y)(1-y^2)^{5/2}\,dy$$
$$= \sum_{\ell_1=2}^{\ell_2-\nu_2} \frac{\pi(\ell_1+1)^2(\ell_1+2)(2\ell_1+5)4^{-\ell_2+\nu_2-2}(\ell_1-m-1)(\ell_1+m+3)\Gamma(2\ell_2-2\nu_2+1)}{\Gamma(-\ell_1+\ell_2-\nu_2+1)\Gamma(\ell_1+\ell_2-\nu_2+4)}.$$

Putting this all together yields $J_m(\ell_2, \nu_2)$ as stated above and completes the proof. □

Remark 5.11 (closed formulas for $\mathfrak{C}_{\gamma\gamma mm}$ for small values of γ). Finally, we note that one can use Lemma 5.10 to find closed formulas for the Fourier coefficients provided that γ is sufficiently small. For example, for $\gamma \in \{0, 1, 2\}$, we find

$$\mathfrak{C}_{00mm} = \frac{4(m(m+4)+5)}{3\pi(m+1)(m+3)}, \quad \mathfrak{C}_{11mm} = \frac{2(m(m+4)+7)}{\pi(m+1)(m+3)}, \quad \mathfrak{C}_{22mm} = \frac{8(5m(m+4)+49)}{15\pi(m+1)(m+3)}$$

for all $m \geq 2\gamma+1$. Figure 2 illustrates the Fourier coefficients $\mathfrak{C}_{\gamma\gamma mm}$ for $\gamma \in \{0, 1, 2\}$, respectively, as m varies within $\{1, 2, \ldots, 50\}$.

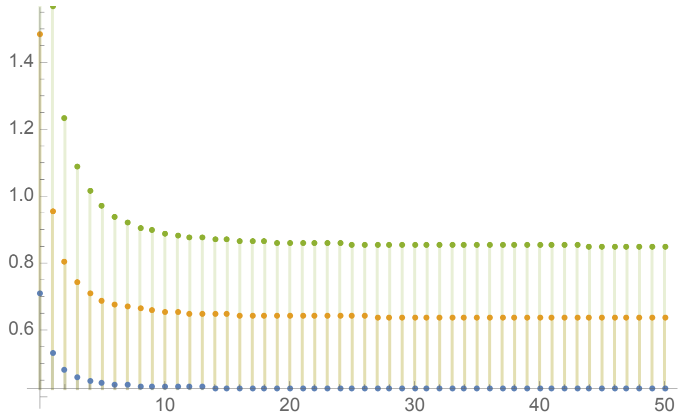

Figure 2. The Fourier coefficients $\mathfrak{C}_{\gamma\gamma mm}$ for $\gamma = 0$ (blue/bottom), $\gamma = 1$ (orange/middle) and $\gamma = 2$ (green/top) as m varies within $\{1, 2, \dots, 50\}$. They are all decreasing for $m \geq 2\gamma + 1$.

6. 1-mode initial data

In this section, we study the operators $\mathcal{M}$ and $\mathfrak{M}_{\pm}$ (Section 2) for 1-mode initial data. Specifically, we verify that all the 1-modes are zeros of the operators $\mathcal{M}$ (for CW and CH) and $\mathfrak{M}_{-}$ (for YM) and compute the differentials $d\mathcal{M}$ and $d\mathfrak{M}_{-}$ at the 1-mode initial data.

6A. ***Conformal cubic wave equation in spherical symmetry.*** Recall that the eigenfunctions $\{e_n : n \geq 0\}$ are given by (3-1) and the PDE in the Fourier space from (4-2) reads

$$\ddot{u}^m(t) + (Au(t))^m = (f(\{u^j(t) : j0\}))^m$$

for all integers $m \geq 0$, where the dots denote derivatives with respect to time and

$$(Au(t))^m = \omega_m^2 u^m(t), \quad (f(\{u^j(t) : j \geq 0\}))^m = -\sum_{i,j,k=0}^{\infty} C_{ijkm} u^i(t) u^j(t) u^k(t).$$

For any initial data $u(0, \cdot)$, we denote by

$$\Phi^t(\xi) = \{\xi^n \cos(\omega_n t) : n \geq 0\}, \quad u(0, \cdot) = \sum_{n=0}^{\infty} \xi^n e_n, \quad \xi = \{\xi^n : n \geq 0\},$$

the linear flow, that is the solution to the linear problem

$$\ddot{u}^n(t) + \omega_n^2 u^n(t) = 0, \quad (u^n(0), \dot{u}^n(0)) = (\xi^n, 0),$$

for all times $t \in \mathbb{R}$. For this model, we aim towards implementing the original version of Bambusi–Paleari's theorem (Theorem 2.4) and define

$$\mathcal{M}(\xi) := A\xi + \langle f \rangle(\xi), \quad \langle f \rangle(\xi) = \frac{1}{2\pi} \int_0^{2\pi} \Phi^t[f(\Phi^t(\xi))]\, dt.$$

To begin with, we show that the 1-modes are zeros of the operator $\mathcal{M}$.

Lemma 6.1 (zeros of the operator $\mathcal{M}$). *Let $\xi=\{\xi^m : m\geq 0\}$ be the rescaled 1-mode initial data,*

$$\xi^m = K_\gamma \mathbb{1}(m=\gamma), \qquad K_\gamma=\pm 2\omega_\gamma\sqrt{\frac{2}{3C_{\gamma\gamma\gamma\gamma}}}, \tag{6-1}$$

for all integers $m\geq 0$. Then, we have $\mathcal{M}(\xi)=0$.

Proof. Let $\xi=\{\xi^m : m\geq 0\}$ be given by (6-1), and pick any integer $m\geq 0$. Then, we compute

$$\begin{aligned}
(A\xi)^m &= \omega_m^2\xi^m = K_\gamma\omega_\gamma^2\mathbb{1}(m=\gamma),\\
(\Phi^t(\xi))^m &= \xi^m\cos(\omega_m t) = K_\gamma\cos(\omega_\gamma t)\mathbb{1}(m=\gamma),\\
(f(\Phi^t(\xi)))^m &= -\sum_{i,j,k} C_{ijkm}(\Phi^t(\xi))^i(\Phi^t(\xi))^j(\Phi^t(\xi))^k = -K_\gamma^3 C_{\gamma\gamma\gamma m}\cos^3(\omega_\gamma t),\\
(\Phi^t[f(\Phi^t(\xi))])^m &= (f(\Phi^t(\xi)))^m\cos(\omega_m t) = -K_\gamma^3 C_{\gamma\gamma\gamma m}\cos^3(\omega_\gamma t)\cos(\omega_m t),\\
(\langle f\rangle(\xi))^m &= -\frac{C_{\gamma\gamma\gamma m}}{2\pi}K_\gamma^3\int_0^{2\pi}\cos^3(\omega_\gamma t)\cos(\omega_m t)\,dt\\
&= -C_{\gamma\gamma\gamma m}K_\gamma^3\big(\tfrac{3}{8}\mathbb{1}(m=\gamma)+\tfrac{1}{8}\mathbb{1}(m=3\gamma+2)\big) = -\tfrac{3}{8}C_{\gamma\gamma\gamma\gamma}K_\gamma^3\mathbb{1}(m=\gamma),\\
(\mathcal{M}(\xi))^m &= (A\xi)^m+(\langle f\rangle(\xi))^m = K_\gamma\big(\omega_\gamma^2-\tfrac{3}{8}C_{\gamma\gamma\gamma\gamma}K_\gamma^2\big)\mathbb{1}(m=\gamma)=0,
\end{aligned}$$

where we used the facts that $\omega_m+\omega_\gamma\neq 0,\ \omega_m+3\omega_\gamma\neq 0,$

$$\omega_m-\omega_\gamma=0 \iff m=\gamma \quad\text{and}\quad \omega_m-3\omega_\gamma=0 \iff m=3\gamma+2,$$

as well as $C_{\gamma\gamma\gamma m}=0$ for $m=3\gamma+2$ according to Lemma 5.1. □

Next, we derive the differential of $\mathcal{M}$ at the rescaled 1-modes.

Lemma 6.2 (differential of $\mathcal{M}$ at the 1-modes). *Let $\xi=\{\xi^m : m\geq 0\}$ be given by (6-1). Then, for all $h=\{h^j : j\geq 0\}\in l^2_{s+3}$, we have that*

$$\begin{aligned}
C_{\gamma\gamma\gamma\gamma}(d\mathcal{M}(\xi)[h])^m &= [(\omega_m^2 C_{\gamma\gamma\gamma\gamma}-2\omega_\gamma^2 C_{\gamma\gamma mm})h^m-\omega_\gamma^2 C_{\gamma,2\gamma-m,\gamma,m}h^{2\gamma-m}]\mathbb{1}(0\leq m\leq\gamma-1)\\
&\quad+[-2\omega_\gamma^2 C_{\gamma\gamma\gamma\gamma}h^\gamma]\mathbb{1}(m=\gamma)\\
&\quad+[(\omega_m^2 C_{\gamma\gamma\gamma\gamma}-2\omega_\gamma^2 C_{\gamma\gamma mm})h^m-\omega_\gamma^2 C_{\gamma,2\gamma-m,\gamma,m}h^{2\gamma-m}]\mathbb{1}(\gamma+1\leq m\leq 2\gamma)\\
&\quad+[(\omega_m^2 C_{\gamma\gamma\gamma\gamma}-2\omega_\gamma^2 C_{\gamma\gamma mm})h^m]\mathbb{1}(m\geq 2\gamma+1),
\end{aligned}$$

where C_{ijkm} are given in closed formulas in Lemma 5.2.

Proof. Let $\xi=\{\xi^m : m\geq 0\}$ be given by (6-1), $\epsilon>0$, $h=\{h^j : j\geq 0\}\in l^2_{s+3}$, and pick any integer $m\geq 0$. Then, a similar computation with the one of Lemma 6.1 yields[11]

$$(\langle f\rangle(\xi+\epsilon h))^m = (\langle f\rangle(\xi))^m-\frac{3\epsilon K_\gamma^2}{2\pi}\sum_i C_{i\gamma\gamma m}h^i\int_0^{2\pi}\cos^2(\omega_\gamma t)\cos(\omega_i t)\cos(\omega_m t)\,dt+\mathcal{O}(\epsilon^2).$$

[11]Here, the notation $\mathcal{O}(\epsilon^2)$ for a function of ξ or h refers to a function that is bounded by ϵ^2 in the $\mathcal{Q}$-norm using the l_{s+3}-norm of ξ or h.

Therefore, we infer

$$\begin{aligned}(d\langle f\rangle(\xi)[h])^m &= -\frac{3K_\gamma^2}{2\pi}\sum_i C_{i\gamma\gamma m}h^i\int_0^{2\pi}\cos^2(\omega_\gamma t)\cos(\omega_i t)\cos(\omega_m t)\,dt\\ &= -\frac{3K_\gamma^2}{8}\sum_i C_{i\gamma\gamma m}h^i\sum_\pm \mathbb{1}(\omega_i\pm\omega_\gamma\pm\omega_\gamma\pm\omega_m=0)\\ &= -\frac{3K_\gamma^2}{8}\Big[\sum_i C_{i\gamma\gamma m}h^i\mathbb{1}(i=m)+\sum_i C_{i\gamma\gamma m}h^i\mathbb{1}(i=m)+\sum_i C_{i\gamma\gamma m}h^i\mathbb{1}(i=2\gamma-m\geq 0)\Big]\\ &= -\frac{3K_\gamma^2}{8}[2C_{m\gamma\gamma m}h^m+C_{2\gamma-m,\gamma,\gamma,m}h^{2\gamma-m}\mathbb{1}(0\leq m\leq 2\gamma)]\\ &= -\frac{\omega_\gamma^2}{C_{\gamma\gamma\gamma\gamma}}[2C_{\gamma\gamma mm}h^m+\mathbb{1}(0\leq m\leq 2\gamma)C_{\gamma,2\gamma-m,\gamma,m}h^{2\gamma-m}],\end{aligned}$$

where we also used the fact that $C_{ijkm}=0$ for $\omega_i\pm\omega_j\pm\omega_k\pm\omega_m=0$ with only 1 minus sign according to Lemma 5.1; so we are left with $\omega_i\pm\omega_j\pm\omega_k\pm\omega_m=0$ with only 2 minus signs, and there are three such terms in total, that is $i=m$, $i=m$ and $i=2\gamma-m$ with $i\geq 0$. Finally, we obtain

$$\begin{aligned}(d\mathcal{M}(\xi)[h])^m &= \omega_m^2 h^m + (d\langle f\rangle(\xi)[h])^m\\ &= \left[\omega_m^2-\frac{2\omega_\gamma^2 C_{\gamma\gamma mm}}{C_{\gamma\gamma\gamma\gamma}}\right]h^m - \mathbb{1}(0\leq m\leq 2\gamma)\frac{\omega_\gamma^2 C_{\gamma,2\gamma-m,\gamma,m}}{C_{\gamma\gamma\gamma\gamma}}h^{2\gamma-m}. \qquad \square\end{aligned}$$

6B. *Conformal cubic wave equation out of spherical symmetry.* We first recall that the eigenfunctions $\{\mathsf{e}_n^{(\mu_1,\mu_2)}:n\geq 0\}$ are given by (3-2) and the PDE in the Fourier space from (4-5) reads

$$\ddot{u}^m(t)+(\mathsf{A}u(t))^m=(\mathsf{f}(\{u^j(t):j\geq 0\}))^m$$

for all integers $m\geq 0$, where the dots denote derivatives with respect to time and

$$(\mathsf{A}u(t))^m=(\omega_n^{(\mu_1,\mu_2)})^2u^m(t),\qquad (\mathsf{f}(\{u^j(t):j\geq 0\}))^m=-\sum_{i,j,k=0}^{\infty}\mathsf{C}_{ijkm}^{(\mu_1,\mu_2)}u^i(t)u^j(t)u^k(t).$$

For any initial data $u(0,\cdot)$, we denote by

$$\phi^t(\xi)=\{\xi^n\cos(\omega_n^{(\mu_1,\mu_2)}t):n\geq 0\},\qquad u(0,\cdot)=\sum_{n=0}^{\infty}\xi^n\mathsf{e}_n^{(\mu_1,\mu_2)},\qquad \xi=\{\xi^n:n\geq 0\},$$

the linear flow, that is the solution to the linear problem

$$\ddot{u}^n(t)+(\omega_n^{(\mu_1,\mu_2)})^2u^n(t)=0,\qquad (u^n(0),\dot{u}(0))=(\xi^n,0),$$

for all times $t\in\mathbb{R}$. For this model, we aim towards implementing the original version of Bambusi–Paleari's theorem (Theorem 2.4) and define

$$\mathcal{M}(\xi):=\mathsf{A}\xi+\langle\mathsf{f}\rangle(\xi),\qquad \langle\mathsf{f}\rangle(\xi)=\frac{1}{2\pi}\int_0^{2\pi}\phi^t[f(\phi^t(\xi))]\,dt.$$

To begin with, we show that the 1-modes are zeros of the operator $\mathcal{M}$.

Lemma 6.3 (zeros of the operator $\mathcal{M}$). *Let $\xi=\{\xi^m : m\geq 0\}$ be the rescaled* 1*-mode initial data,*

$$\xi^m=\mathsf{K}_\gamma^{(\mu_1,\mu_2)}\mathbb{1}(m=\gamma),\quad \mathsf{K}_\gamma^{(\mu_1,\mu_2)}=\pm 2\omega_\gamma^{(\mu_1,\mu_2)}\sqrt{\frac{2}{3C_{\gamma\gamma\gamma\gamma}^{(\mu_1,\mu_2)}}},\tag{6-2}$$

for all integers $m\geq 0$. Then, we have $\mathcal{M}(\xi)=0$.

Proof. The proof is similar to the one of Lemma 6.1. □

Next, we derive the differential of $\mathcal{M}$ at the rescaled 1-modes.

Lemma 6.4 (differential of $\mathcal{M}$ at the 1-modes). *Let $\xi=\{\xi^m : m\geq 0\}$ be given by (6-2). Then, for all $h=\{h^j : j\geq 0\}\in l^2_{s+3}$, we have*

$$\begin{aligned}\mathsf{C}_{\gamma\gamma\gamma\gamma}^{(\mu_1,\mu_2)}(d\mathcal{M}(\xi)[h])^m &= [(\omega_m^2\mathsf{C}_{\gamma\gamma\gamma\gamma}^{(\mu_1,\mu_2)}-2\omega_\gamma^2\mathsf{C}_{\gamma\gamma mm}^{(\mu_1,\mu_2)})h^m-\omega_\gamma^2\mathsf{C}_{\gamma,2\gamma-m,\gamma,m}^{(\mu_1,\mu_2)}h^{2\gamma-m}]\mathbb{1}(0\leq m\leq\gamma-1)\\ &\quad+[-2\omega_\gamma^2\mathsf{C}_{\gamma\gamma\gamma\gamma}^{(\mu_1,\mu_2)}h^\gamma]\mathbb{1}(m=\gamma)\\ &\quad+[(\omega_m^2\mathsf{C}_{\gamma\gamma\gamma\gamma}^{(\mu_1,\mu_2)}-2\omega_\gamma^2\mathsf{C}_{\gamma\gamma mm}^{(\mu_1,\mu_2)})h^m-\omega_\gamma^2\mathsf{C}_{\gamma,2\gamma-m,\gamma,m}^{(\mu_1,\mu_2)}h^{2\gamma-m}]\mathbb{1}(\gamma+1\leq m\leq 2\gamma)\\ &\quad+[(\omega_m^2\mathsf{C}_{\gamma\gamma\gamma\gamma}^{(\mu_1,\mu_2)}-2\omega_\gamma^2\mathsf{C}_{\gamma\gamma mm}^{(\mu_1,\mu_2)})h^m]\mathbb{1}(m\geq 2\gamma+1),\end{aligned}$$

where $\mathsf{C}_{\gamma\gamma mm}^{(\mu_1,\mu_2)}$ are given by closed formulas in Lemma 5.4.

Proof. The proof is similar to the one of Lemma 6.2 due to the fact that $\mathsf{C}_{ijkm}^{(\mu_1,\mu_2)}=0$ for

$$\omega_i^{(\mu_1,\mu_2)}\pm\omega_j^{(\mu_1,\mu_2)}\pm\omega_k^{(\mu_1,\mu_2)}\pm\omega_m^{(\mu_1,\mu_2)}=0$$

with only 1 minus sign according to Lemma 5.3; so we are left with

$$\omega_i^{(\mu_1,\mu_2)}\pm\omega_j^{(\mu_1,\mu_2)}\pm\omega_k^{(\mu_1,\mu_2)}\pm\omega_m^{(\mu_1,\mu_2)}=0$$

with only 2 minus signs, and there are again the same three such terms in total, that is $i=m$, $i=m$ and $i=2\gamma-m$ with $i\geq 0$, which completes the proof. □

6C. *Yang–Mills equation in spherical symmetry.* Recall that the eigenfunctions $\{\mathfrak{e}_n : n\geq 0\}$ are given by (3-4) and the PDE in the Fourier space from (4-8) reads

$$\ddot{u}^m(t)+(\mathfrak{A}u(t))^m=(\mathfrak{f}(\{u^j(t) : j\geq 0\}))^m$$

for all integers $m\geq 0$, where the dots denote derivatives with respect to time and

$$(\mathfrak{A}u(t))^m=\varpi_n^2u^m(t),\quad (\mathfrak{f}(u))^m=(\mathfrak{f}^{(2)}(u))^m+(\mathfrak{f}^{(3)}(u))^m,$$

with

$$(\mathfrak{f}^{(2)}(\{u^j(t) : j\geq 0\}))^m=-3\sum_{i,j=0}^{\infty}\overline{\mathfrak{C}}_{ijm}u^i(t)u^j(t),$$

$$(\mathfrak{f}^{(3)}(\{u^j(t) : j\geq 0\}))^m=-\sum_{i,j,k=0}^{\infty}\mathfrak{C}_{ijkm}u^i(t)u^j(t)u^k(t).$$

For any initial data $u(0,\cdot)$, we denote by

$$\Phi^t(\xi)=\{\xi^n\cos(\varpi_nt) : n\geq 0\}.\quad u(0,\cdot)=\sum_{n=0}^{\infty}\xi^n\mathfrak{e}_n,\quad \xi=\{\xi^n : n\geq 0\},$$

the linear flow, that is the solution to the linear problem

$$\ddot{u}^n(t)+\varpi_n^2 u^n(t)=0, \quad (u^n(0),\dot{u}^n(0))=(\xi^n,0),$$

for all times $t\in\mathbb{R}$. As a starting point, we show that the original version of Bambusi–Paleari's theorem (Theorem 2.4) is not applicable.

Lemma 6.5 (nonresonant $\mathfrak{f}^{(2)}$). *For all initial data $\xi\in l_s^2$, we have*

$$\langle\mathfrak{f}^{(2)}\rangle(\xi)=\frac{1}{2\pi}\int_0^{2\pi}\Phi^t[\mathfrak{f}^{(2)}(\Phi^t(\xi))]\,dt=0.$$

Proof. Let $\xi=\{\xi^m:m\geq 0\}\in l_s^2$ be any initial data, and pick an integer $m\geq 0$. Then, similar computations with the ones in Lemma 6.1 yield

$$\langle\mathfrak{f}^{(2)}\rangle(\xi)=-\frac{3}{2\pi}\sum_{i,j=0}^{\infty}\overline{\mathfrak{C}}_{ijm}\xi^i\xi^j\int_0^{2\pi}\prod_{\lambda\in\{i,j,m\}}\cos(\varpi_\lambda t)\,dt=-\frac{3}{4}\sum_{i,j=0}^{\infty}\overline{\mathfrak{C}}_{ijm}\xi^i\xi^j\sum_{\pm}\mathbb{1}(\varpi_i\pm\varpi_j\pm\varpi_m=0).$$

Now, notice that all the possible conditions are those with only 1 minus sign, and according to Lemma 5.7 the corresponding Fourier coefficients vanish. □

Consequently, for this model, we aim towards implementing the modified version of Bambusi–Paleari's theorem (Theorem 2.5) and define

$$\mathfrak{M}_\pm(\xi)=\pm\mathfrak{A}\xi+\langle\mathfrak{f}^{(3)}\rangle(\xi)+\mathfrak{F}_0(\xi), \quad \langle\mathfrak{f}^{(3)}\rangle(\xi)=\frac{1}{2\pi}\int_0^{2\pi}\Phi^t[\mathfrak{f}^{(3)}(\Phi^t(\xi))]\,dt,$$

where $\mathfrak{F}_0(\xi)$ is given for any initial data by Lemma 2.13 and for the 1-mode initial data by Lemma 2.14. Also, recall the Diophantine condition $\varpi\in\mathcal{W}_\alpha$ for some $0<\alpha<\frac{1}{3}$ from Theorem 2.5.

To begin with, we show that the 1-modes are zeros of the operator $\mathfrak{M}_-$.

Lemma 6.6 (zeros of the operator $\mathfrak{M}_-$). *Let $\gamma\in\{0,1,\dots,5\}$ and*

$$\mathfrak{q}_\gamma=\frac{9}{4}\sum_{\nu=0}^{2\gamma}(\overline{\mathfrak{C}}_{\gamma\gamma\nu})^2\biggl(\frac{2}{\varpi_\nu^2}+\frac{1}{\varpi_\nu^2-(2\varpi_\gamma)^2}\biggr).$$

Then, we have that $8\mathfrak{q}_\gamma>3\mathfrak{C}_{\gamma\gamma\gamma\gamma}$. Moreover, let $\xi=\{\xi^m:m\geq 0\}$ be the rescaled 1-mode initial data,

$$\xi^m=\mathfrak{K}_\gamma\mathbb{1}(m=\gamma), \quad \mathfrak{K}_\gamma=\pm 2\omega_\gamma\sqrt{\frac{2}{8\mathfrak{q}_\gamma-3\mathfrak{C}_{\gamma\gamma\gamma\gamma}}}, \tag{6-3}$$

for all integers $m\geq 0$. Then, we have that $\mathfrak{M}_-(\xi)=0$.

Proof. Let $\gamma\in\{0,1,\dots,5\}$, define $\xi=\{\xi^m:m\geq 0\}$ to be the rescaled 1-mode initial data given by (6-3) and pick any integer $m\geq 0$. Firstly, we compute $-\mathfrak{A}\xi+\langle\mathfrak{f}^{(3)}\rangle(\xi)$, and a similar computation with the one in Lemma 6.1 yields

$$-(\mathfrak{A}\xi)^m+(\langle\mathfrak{f}^{(3)}\rangle(\xi))^m=-\mathfrak{K}_\gamma\bigl(\varpi_\gamma^2+\tfrac{3}{8}\mathfrak{C}_{\gamma\gamma\gamma\gamma}\mathfrak{K}_\gamma^2\bigr)\mathbb{1}(m=\gamma),$$

where we used the fact that $\varpi_m+\varpi_\gamma\neq 0$, $\varpi_m+3\varpi_\gamma\neq 0$,

$$\varpi_m-\varpi_\gamma=0 \iff m=\gamma \quad\text{and}\quad \varpi_m-3\varpi_\gamma=0 \iff m=3\gamma+4,$$

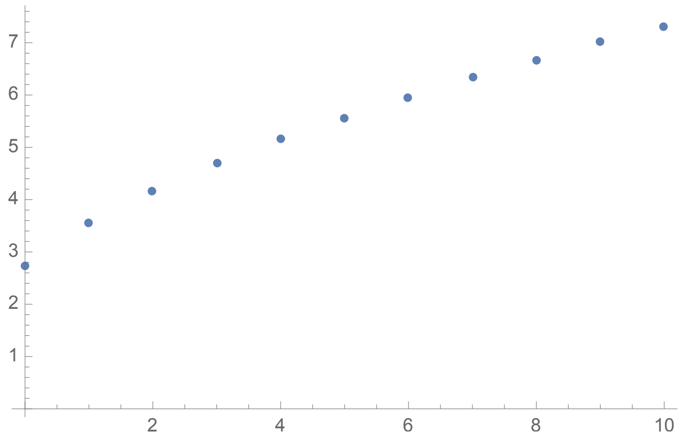

Figure 3. The constants $\mathfrak{K}_\gamma$ as γ varies within $\{0, 1, \dots, 10\}$. They are all real numbers.

as well as $\mathfrak{C}_{\gamma\gamma\gamma m} = 0$ for $m = 3\gamma + 4$ according to Lemma 5.7. Furthermore, we use the computation for $\mathfrak{F}_0(\xi)$ at the 1-mode initial data we derived in Lemma 2.14, that is

$$(\mathfrak{F}_0(\xi))^m = \mathfrak{q}_\gamma \mathfrak{K}_\gamma^3 \mathbb{1}(m=\gamma), \qquad \mathfrak{q}_\gamma = \frac{9}{4}\sum_{\nu=0}^{2\gamma} (\overline{\mathfrak{C}}_{\gamma\gamma\nu})^2 \left(\frac{2}{\varpi_\nu^2} + \frac{1}{\varpi_\nu^2 - (2\varpi_\gamma)^2} \right),$$

to conclude that

$$\begin{aligned}(\mathfrak{M}_-(\xi))^m &= -(\mathfrak{A}\xi)^m + (\langle \mathfrak{f}^{(3)} \rangle(\xi))^m + (\mathfrak{F}_0(\xi))^m \\ &= -\mathfrak{K}_\gamma \left(\varpi_\gamma^2 + \tfrac{3}{8}\mathfrak{C}_{\gamma\gamma\gamma\gamma}\mathfrak{K}_\gamma^2 - \mathfrak{q}_\gamma \mathfrak{K}_\gamma^2 \right) \mathbb{1}(m=\gamma) \\ &= -\tfrac{1}{8}\mathfrak{K}_\gamma (8\varpi_\gamma^2 + 3\mathfrak{C}_{\gamma\gamma\gamma\gamma}\mathfrak{K}_\gamma^2 - 8\mathfrak{q}_\gamma\mathfrak{K}_\gamma^2)\mathbb{1}(m=\gamma) = 0\end{aligned}$$

provided that $\mathfrak{K}_\gamma^2 = -8\varpi_\gamma^2/(3\mathfrak{C}_{\gamma\gamma\gamma\gamma} - 8\mathfrak{q}_\gamma)$. Finally, it remains to show that this choice is well defined, that is $\mathfrak{K}_\gamma \in \mathbb{R}$ for all $\gamma \in \{0, 1, \dots, 10\}$. To this end, we fix $\gamma \in \{0, 1, \dots, 10\}$ and use the definition of the Fourier coefficients to compute each $\mathfrak{K}_\gamma$ and verify that they are all real numbers. Figure 3 illustrates the constants $\mathfrak{K}_\gamma$ as γ varies within $\{0, 1, \dots, 10\}$. □

Next, we derive the differential of $\mathfrak{M}_-$ at the rescaled 1-modes.

Lemma 6.7 (differential of $\mathfrak{M}_-$ at the 1-modes). *Let $\gamma \in \{0, 1, \dots, 10\}$, and let $\xi = \{\xi^m : m \geq 0\}$ be given by (6-3). Then, for all $h = \{h^j : j \geq 0\} \in l^2_{s+3}$, we have*

$$\begin{aligned}-\mathfrak{K}_\gamma^{-2}(d\mathfrak{M}_-(\xi)[h])^m &= \mathbb{1}(0 \leq m \leq \gamma - 1)[h^m \mathfrak{u}_{\gamma m} + h^{2\gamma - m}\mathfrak{v}_{\gamma m}] + \mathbb{1}(m=\gamma)[h^\gamma(\mathfrak{u}_{\gamma\gamma} + \mathfrak{v}_{\gamma\gamma})] \\ &\quad + \mathbb{1}(\gamma+1 \leq m \leq 2\gamma)[h^m \mathfrak{u}_{\gamma m} + h^{2\gamma-m}\mathfrak{v}_{\gamma m}] + \mathbb{1}(m \geq 2\gamma+1)[h^m \mathfrak{u}_{\gamma m}],\end{aligned}$$

where

$$\mathfrak{u}_{\gamma m} = \left(\frac{\varpi_m}{\mathfrak{K}_\gamma} \right)^2 + \tfrac{3}{4}\mathfrak{C}_{\gamma\gamma mm} - \mathfrak{a}_{\gamma m}, \qquad \mathfrak{v}_{\gamma m} = \tfrac{3}{8}\mathfrak{C}_{\gamma, 2\gamma - m, \gamma, m} - \mathfrak{b}_{\gamma m},$$

and $\overline{\mathfrak{C}}_{ijm}$ and $\mathfrak{C}_{\gamma\gamma mm}$ are given by Lemmas 5.8 and 5.10, respectively, whereas $\mathfrak{a}_{\gamma m}$ and $\mathfrak{b}_{\gamma m}$ are given by Lemma 2.15.

Proof. Let $\gamma \in \{0, 1, \dots, 10\}$, $\xi = \{\xi^m : m \geq 0\}$ be given by (6-3), $h = \{h^j : j \geq 0\} \in l^2_{s+3}$, and pick any integer $m \geq 0$. Firstly, we use similar computations to the ones derived in Lemma 6.2 to obtain

$$\begin{aligned}(d\langle \mathfrak{f}^{(3)}\rangle(\xi)[h])^m &= -\frac{3}{8}\mathfrak{K}_\gamma^2 \sum_i \mathfrak{C}_{i\gamma\gamma m} h^i \sum_{\pm} \mathbb{1}(\varpi_i \pm \varpi_\gamma \pm \varpi_\gamma \pm \varpi_m = 0) \\ &= -\frac{3}{8}\mathfrak{K}_\gamma^2 \Big[\sum_i \mathfrak{C}_{i\gamma\gamma m} h^i \mathbb{1}(i=m) + \sum_i \mathfrak{C}_{i\gamma\gamma m} h^i \mathbb{1}(i=m) + \sum_i \mathfrak{C}_{i\gamma\gamma m} h^i \mathbb{1}(i = 2\gamma - m \geq 0)\Big] \\ &= -\tfrac{3}{8}\mathfrak{K}_\gamma^2 [2\mathfrak{C}_{m\gamma\gamma m} h^m + \mathfrak{C}_{2\gamma-m,\gamma,\gamma,m} h^{2\gamma-m}\mathbb{1}(0 \leq m \leq 2\gamma)],\end{aligned}$$

where we used the fact that $\mathfrak{C}_{ijkm} = 0$ for $\varpi_i \pm \varpi_j \pm \varpi_k \pm \varpi_m = 0$ with only 1 minus sign according to Lemma 5.7; so we are left with $\varpi_i \pm \varpi_j \pm \varpi_k \pm \varpi_m = 0$ with only 2 minus signs, and there are three such terms in total, that is $i = m$, $i = m$ and $i = 2\gamma - m$ with $i \geq 0$. Then, we infer

$$\begin{aligned}-(d\mathfrak{A}\xi[h])^m + (d\langle \mathfrak{f}^{(3)}\rangle(\xi)[h])^m &= -\varpi_m^2 h^m + (d\langle \mathfrak{f}^{(3)}\rangle(\xi)[h])^m \\ &= -\varpi_m^2 h^m - \tfrac{3}{8}\mathfrak{K}_\gamma^2[2\mathfrak{C}_{\gamma\gamma mm} h^m + \mathbb{1}(0 \leq m \leq 2\gamma)\mathfrak{C}_{\gamma,2\gamma-m,\gamma,m} h^{2\gamma-m}] \\ &= -\big[\varpi_m^2 + \tfrac{3}{4}\mathfrak{K}_\gamma^2 \mathfrak{C}_{\gamma\gamma mm}\big]h^m - \mathbb{1}(0 \leq m \leq 2\gamma)\tfrac{3}{8}\mathfrak{K}_\gamma^2 \mathfrak{C}_{\gamma,2\gamma-m,\gamma,m} h^{2\gamma-m}.\end{aligned}$$

Recall that the differential of $\mathfrak{F}_0$ at the 1-modes, $(d\mathfrak{F}_0(\xi)[h])^m$, is given by Lemma 2.15. Putting this all together yields that $(d\mathfrak{M}_-(\xi)[h])^m$ is given by

$$-h^m\big[\varpi_m^2 + \mathfrak{K}_\gamma^2(\omega)\big(\tfrac{3}{4}\mathfrak{C}_{\gamma\gamma mm} - \mathfrak{a}_{\gamma m}\big)\big] - \mathbb{1}(0 \leq m \leq 2\gamma) h^{2\gamma-m}\mathfrak{K}_\gamma^2(\omega)\big[\tfrac{3}{8}\mathfrak{C}_{\gamma,2\gamma-m,\gamma,m} - \mathfrak{b}_{\gamma m}\big].$$

Finally, one can rewrite the latter as stated above, which completes the proof. □

7. Nondegeneracy conditions for the 1-modes

In this section, we derive and establish the crucial nondegeneracy conditions for 1-mode initial data according to Theorem 2.4 (for CW and CH) and Theorem 2.5 (for YM).

7A. *Conformal cubic wave equation in spherical symmetry.* Firstly, we consider the conformal cubic wave equation in spherical symmetry and derive the nondegeneracy condition for the 1-modes.

Lemma 7.1 (CW model: derivation of the nondegeneracy condition for the 1-modes). *Let $\gamma \geq 0$ be any integer, and define ξ to be the rescaled 1-mode according to (6-1). Then, the nondegeneracy condition*

$$\ker(d\mathcal{M}(\xi)) = \{0\}$$

is equivalent to

$$\begin{cases}\omega_m^2 C_{\gamma\gamma\gamma\gamma} - 2\omega_\gamma^2 C_{\gamma\gamma mm} \neq 0 & \textit{for all } m \geq 2\gamma+1, \\ D_{\gamma n} \neq 0 & \textit{for all } n \in \{0, 1, \dots, \gamma-1\},\end{cases} \tag{7-1}$$

where

$$D_{\gamma n} = [\omega_n^2 C_{\gamma\gamma\gamma\gamma} - 2\omega_\gamma^2 C_{\gamma\gamma nn}][\omega_{2\gamma-n}^2 C_{\gamma\gamma\gamma\gamma} - 2\omega_\gamma^2 C_{\gamma,\gamma,2\gamma-n,2\gamma-n}] - [\omega_\gamma^2 C_{\gamma,2\gamma-n,\gamma,n}]^2.$$

Proof. Let $\gamma \geq 0$ be any integer and define ξ to be the rescaled 1-mode according to (6-1). Furthermore, pick any $h = \{h^j : j \geq 0\} \in l^2_{s+3}$ such that $d\mathcal{M}(\xi)[h] = 0$, and fix an integer $m \geq 0$. Then, according to Lemma 6.2, we have that $d\mathcal{M}(\xi)[h] = 0$ is equivalent to

$$(\omega_m^2 C_{\gamma\gamma\gamma\gamma} - 2\omega_\gamma^2 C_{\gamma\gamma mm})h^m - \omega_\gamma^2 C_{\gamma,2\gamma-m,\gamma,m} h^{2\gamma-m} = 0 \quad \text{for } 0 \leq m \leq \gamma - 1, \tag{7-2}$$

$$(\omega_m^2 C_{\gamma\gamma\gamma\gamma} - 2\omega_\gamma^2 C_{\gamma\gamma mm})h^m - \omega_\gamma^2 C_{\gamma,2\gamma-m,\gamma,m} h^{2\gamma-m} = 0 \quad \text{for } \gamma + 1 \leq m \leq 2\gamma, \tag{7-3}$$

$$(\omega_m^2 C_{\gamma\gamma\gamma\gamma} - 2\omega_\gamma^2 C_{\gamma\gamma mm})h^m = 0 \quad \text{for } m \geq 2\gamma + 1, \tag{7-4}$$

coupled to

$$\omega_\gamma^2 C_{\gamma\gamma\gamma\gamma} h^\gamma = 0. \tag{7-5}$$

We will show that $h = \{h^m : m \geq 0\} = 0$ is the unique solution to the linear system above if and only if (7-1) holds. Firstly, (7-5) yields $h^\gamma = 0$ due to the fact that $C_{\gamma\gamma\gamma\gamma} \neq 0$ and $\omega_\gamma \neq 0$ for all $\gamma \geq 0$, whereas, for (7-4), one has that $h^m = 0$ for all integers $m \geq 2\gamma + 1$ if and only if $\omega_m^2 C_{\gamma\gamma\gamma\gamma} - 2\omega_\gamma^2 C_{\gamma\gamma mm} \neq 0$ for all integers $m \geq 2\gamma + 1$. Next, we rearrange (7-2) and (7-3) by setting $m = n$ and $m = 2\gamma - n$, respectively, and obtain

$$\begin{bmatrix} \omega_n^2 C_{\gamma\gamma\gamma\gamma} - 2\omega_\gamma^2 C_{\gamma\gamma nn} & -\omega_\gamma^2 C_{\gamma,2\gamma-n,\gamma,n} \\ -\omega_\gamma^2 C_{\gamma,n,\gamma,2\gamma-n} & \omega_{2\gamma-n}^2 C_{\gamma\gamma\gamma\gamma} - 2\omega_\gamma^2 C_{\gamma,\gamma,2\gamma-n,2\gamma-n} \end{bmatrix} \begin{bmatrix} h^n \\ h^{2\gamma-n} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

for all $n \in \{0, 1, \ldots, \gamma - 1\}$. Observe that there are γ in total (2×2)-linear systems where the unknowns are h^m for $m \in \{0, 1, \ldots, 2\gamma\} \setminus \{\gamma\}$. Finally, these systems have only the trivial solution $h^m = 0$ for all $m \in \{0, 1, \ldots, 2\gamma\} \setminus \{\gamma\}$ if and only if the determinants $D_{\gamma n}$ are nonzero for all $n \in \{0, 1, \ldots, \gamma - 1\}$, which completes the proof. ☐

Next, we establish the nondegeneracy condition for this model.

Proposition 7.2 (nondegeneracy condition for the 1-modes and the CW model). *Let $\gamma \geq 0$ be any integer. Then, the nondegeneracy condition (7-1) holds.*

Proof. Let $\gamma \geq 0$ be any integer. Also, pick any integers $m \geq 2\gamma + 1$ and $n \in \{0, 1, \ldots, \gamma - 1\}$. Then, according to Lemma 5.2, we have that $C_{ijkm} = \omega_{\min\{i,j,k,m\}}$ provided that either $\omega_i + \omega_j - \omega_k - \omega_m = 0$, $\omega_i - \omega_j + \omega_k - \omega_m = 0$ or $\omega_i - \omega_j - \omega_k + \omega_m = 0$. One can easily show that all the indices (i, j, k, m) of the Fourier coefficients that appear in Lemma 7.1 satisfy at least one of these conditions, and hence we infer

$$C_{\gamma\gamma\gamma\gamma} = \omega_\gamma, \quad C_{\gamma\gamma nn} = \omega_n, \quad C_{\gamma\gamma mm} = \omega_\gamma, \quad C_{\gamma,\gamma,2\gamma-n,2\gamma-n} = \omega_\gamma, \quad C_{\gamma,2\gamma-n,\gamma,n} = \omega_n.$$

Putting this all together yields

$$\omega_m^2 C_{\gamma\gamma\gamma\gamma} - 2\omega_\gamma^2 C_{\gamma\gamma mm} = \omega_\gamma(\omega_m^2 - 2\omega_\gamma^2) \geq \omega_\gamma(\omega_{2\gamma+1}^2 - 2\omega_\gamma^2) \geq 2,$$

$$D_{\gamma n} = \omega_n \omega_\gamma^2 (n - 3 - 4\gamma)(n - \gamma)^2 \neq 0,$$

for all $m \geq 2\gamma + 1$ and $n \in \{0, 1, \ldots, \gamma - 1\}$, which completes the proof. ☐

7B. *Conformal cubic wave equation out of spherical symmetry.* Next, we consider the conformal cubic wave equation out of spherical symmetry and show that the nondegeneracy condition is a condition on the Fourier coefficients.

Lemma 7.3 (derivation of the nondegeneracy condition for the 1-modes and the CH model). *Let γ and μ_1, μ_2 be any integers and define ξ according to (6-2). Then, the nondegeneracy condition*

$$\ker(d\mathcal{M}(\xi)) = \{0\}$$

is equivalent to

$$\begin{cases} (\omega_m^{(\mu_1,\mu_2)})^2 \mathsf{C}_{\gamma\gamma\gamma\gamma}^{(\mu_1,\mu_2)} - 2(\omega_\gamma^{(\mu_1,\mu_2)})^2 \mathsf{C}_{\gamma\gamma mm}^{(\mu_1,\mu_2)} \neq 0 & \text{for all } m \geq 2\gamma+1, \\ \mathsf{D}_{\gamma n}^{(\mu_1,\mu_2)} \neq 0 & \text{for all } n \in \{0,1,\dots,\gamma-1\}, \end{cases} \tag{7-6}$$

where

$$\mathsf{D}_{\gamma n}^{(\mu_1,\mu_2)} = [(\omega_n^{(\mu_1,\mu_2)})^2 \mathsf{C}_{\gamma\gamma\gamma\gamma}^{(\mu_1,\mu_2)} - 2(\omega_\gamma^{(\mu_1,\mu_2)})^2 \mathsf{C}_{\gamma\gamma nn}^{(\mu_1,\mu_2)}][(\omega_{2\gamma-n}^{(\mu_1,\mu_2)})^2 \mathsf{C}_{\gamma\gamma\gamma\gamma}^{(\mu_1,\mu_2)} - 2(\omega_\gamma^{(\mu_1,\mu_2)})^2 \mathsf{C}_{\gamma,\gamma,2\gamma-n,2\gamma-n}^{(\mu_1,\mu_2)}] - [(\omega_\gamma^{(\mu_1,\mu_2)})^2 \mathsf{C}_{\gamma,2\gamma-n,\gamma,n}^{(\mu_1,\mu_2)}]^2.$$

Proof. The proof is similar to the one of Lemma 7.1. ☐

Next, we establish the nondegeneracy condition for this model.

Proposition 7.4 (nondegeneracy condition for the 1-modes and the CH model). *Let γ, μ_1, $\mu_2 \geq 0$ be any integers with $\gamma \in \{0,1,2,3,4,5\}$ and*

$$\mu_1 = \mu_2 =: \mu,$$

where μ is either sufficiently small with $\mu \in \{0,1,2,3,4,5\}$ or sufficiently large. Then, the nondegeneracy condition (7-6) holds true.

Proof. Let γ, μ_1, $\mu_2 \geq 0$ be any integers with $\gamma \in \{0,1,2,3,4,5\}$ and $\mu_1 = \mu_2 =: \mu$, where μ is either sufficiently small with $\mu \in \{0,1,2,3,4,5\}$ or sufficiently large. Also, pick any integer $m \geq 2\gamma+1$. Recall that, according to Lemma 5.4, we have that $\mathsf{C}_{\gamma\gamma\gamma\gamma}^{(\mu,\mu)}$ and $\mathsf{C}_{\gamma\gamma mm}^{(\mu,\mu)}$ are given in terms of the function $\mathsf{M}_m^{(\mu)}(\lambda)$ that is also given in a closed formula. Moreover, according to Lemma 5.5, $\mathsf{M}_m^{(\mu)}(\lambda)$ is decreasing with respect to m. In addition, recall that the eigenvalues are given by

$$(\omega_m^{(\mu,\mu)})^2 = (2m+1+2\mu)^2$$

and they are clearly increasing with respect to $m \geq 0$. In other words, the function

$$\mathsf{P}_m^{(\mu)}(\lambda) = \frac{\mathsf{M}_m^{(\mu)}(\lambda)}{(\omega_m^{(\mu,\mu)})^2}$$

is decreasing with respect to m for all $m \geq \gamma$ as a product of two positive and decreasing functions. In the following, we show that

$$(\omega_m^{(\mu,\mu)})^2 \mathsf{C}_{\gamma\gamma\gamma\gamma}^{(\mu,\mu)} - 2(\omega_\gamma^{(\mu,\mu)})^2 \mathsf{C}_{\gamma\gamma mm}^{(\mu,\mu)}$$

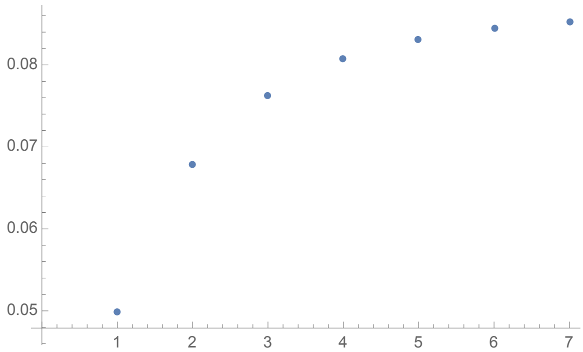

Figure 4. The constants σ_γ with $\gamma \in \{1, 2, 3, 4, 5\}$. They are all strictly positive.

stays away from zero for all $m \geq 2\gamma + 1$ and $\gamma \in \{0, 1, 2, 3, 4, 5\}$ provided that μ is either sufficiently small or sufficiently large. To this end, we use the monotonicity of $\mathsf{P}_m^{(\mu)}(\lambda)$ with respect to m to infer

$$\begin{aligned}
&(\omega_m^{(\mu,\mu)})^2 \mathsf{C}_{\gamma\gamma\gamma\gamma}^{(\mu,\mu)} - 2(\omega_\gamma^{(\mu,\mu)})^2 \mathsf{C}_{\gamma\gamma mm}^{(\mu,\mu)} \\
&\quad = (\omega_m^{(\mu,\mu)})^2 (\omega_\gamma^{(\mu,\mu)})^2 \left[\frac{\mathsf{C}_{\gamma\gamma\gamma\gamma}^{(\mu,\mu)}}{(\omega_\gamma^{(\mu,\mu)})^2} - 2\frac{\mathsf{C}_{\gamma\gamma mm}^{(\mu,\mu)}}{(\omega_m^{(\mu,\mu)})^2} \right] \\
&\quad = \frac{1}{2}(\omega_m^{(\mu,\mu)})^2 (\omega_\gamma^{(\mu,\mu)})^2 \left[\sum_{\lambda=0}^{\gamma} \mathsf{P}_\gamma^{(\mu)}(\lambda)\mathsf{M}_\gamma^{(\mu)}(\lambda)\xi_\lambda(\mu) - 2\sum_{\lambda=0}^{\gamma} \mathsf{M}_\gamma^{(\mu)}(\lambda)\mathsf{P}_m^{(\mu)}(\lambda)\xi_\lambda(\mu) \right] \\
&\quad = \frac{1}{2}(\omega_m^{(\mu,\mu)})^2 (\omega_\gamma^{(\mu,\mu)})^2 \sum_{\lambda=0}^{\gamma} \mathsf{M}_\gamma^{(\mu)}(\lambda)[\mathsf{P}_\gamma^{(\mu)}(\lambda) - 2\mathsf{P}_m^{(\mu)}(\lambda)]\xi_\lambda(\mu) \\
&\quad \geq \frac{1}{2}(\omega_m^{(\mu,\mu)})^2 (\omega_\gamma^{(\mu,\mu)})^2 \sum_{\lambda=0}^{\gamma} \mathsf{M}_\gamma^{(\mu)}(\lambda)[\mathsf{P}_\gamma^{(\mu)}(\lambda) - 2\mathsf{P}_{2\gamma+1}^{(\mu)}(\lambda)]\xi_\lambda(\mu) = (\omega_m^{(\mu,\mu)})^2 \mathsf{S}_\gamma^{(\mu)},
\end{aligned}$$

where we set

$$\mathsf{S}_\gamma^{(\mu)} = \frac{1}{2}(\omega_\gamma^{(\mu,\mu)})^2 \sum_{\lambda=0}^{\gamma} \mathsf{M}_\gamma^{(\mu)}(\lambda)[\mathsf{P}_\gamma^{(\mu)}(\lambda) - 2\mathsf{P}_{2\gamma+1}^{(\mu)}(\lambda)]\xi_\lambda(\mu).$$

On the one hand, for all $\gamma \in \{0, 1, 2, 3, 4, 5\}$ and $\mu \in \{0, 1, 2, 3, 4, 5\}$, we compute $\mathsf{S}_\gamma^{(\mu)}$ and verify that all $\mathsf{S}_\gamma^{(\mu)}$ are strictly positive. On the other hand, for all $\gamma \in \{0, 1, 2, 3, 4, 5\}$ and sufficiently large μ, we firstly compute $\mathsf{S}_\gamma^{(\mu)}$ in terms of μ and then derive its asymptotic expansion as $\mu \to \infty$. For $\gamma = 0$, we find

$$\mathsf{S}_0^{(\mu)} = \frac{4^\mu(2\mu+1)(10\mu+7)\Gamma\left(\mu+\frac{1}{2}\right)^2\Gamma\left(\mu+\frac{5}{2}\right)}{\pi(2\mu+3)^2\Gamma(\mu+1)\Gamma\left(2\mu+\frac{5}{2}\right)},$$

which is strictly positive for all $\mu \geq 0$, and for $\gamma \in \{1, 2, 3, 4, 5\}$, we expand

$$\mathsf{S}_\gamma^{(\mu)} = \sigma_\gamma \mu^{1/2} + \mathcal{O}(\mu^{-1/2})$$

as $\mu \to \infty$ for some strictly positive constants σ_γ. Figure 4 above illustrates the constants σ_γ with $\gamma \in \{1, 2, 3, 4, 5\}$.

Consequently, in both cases, we can ensure that $\mathsf{S}_{\gamma}^{(\mu)} > 0$, and hence we conclude that

$$(\omega_m^{(\mu,\mu)})^2 \mathsf{C}_{\gamma\gamma\gamma\gamma}^{(\mu,\mu)} - 2(\omega_\gamma^{(\mu,\mu)})^2 \mathsf{C}_{\gamma\gamma mm}^{(\mu,\mu)} \geq (\omega_m^{(\mu,\mu)})^2 \mathsf{S}_{\gamma}^{(\mu)} \geq (\omega_{2\gamma+1}^{(\mu,\mu)})^2 \mathsf{S}_{\gamma}^{(\mu)} > 0$$

for all $m \geq 2\gamma+1$. Finally, it remains to show that the determinants $\mathsf{D}_{\gamma n}^{(\mu,\mu)}$ are all nonzero for $n \in \{0, 1, \ldots, \gamma-1\}$. To this end, for all $\gamma \in \{1, 2, 3, 4, 5\}$ and $n \in \{0, 1, \ldots, \gamma-1\}$, we firstly compute each of the Fourier coefficients in the determinants above, find a closed formula for each of the determinants in terms of μ, and then either compute the determinants when μ is sufficiently small or compute their limits when $\mu \to \infty$. For example, for $\gamma = 1$, we have $n = 0$ and compute

$$\mathsf{D}_{10}^{(\mu,\mu)} = -\frac{3\cdot 16^{\mu-1}}{\pi^2}(\mu+1)(2\mu+3)^4(2\mu+5)(4\mu+7) \cdot (20\mu^4+328\mu^3+1029\mu^2+1155\mu+435)\frac{\Gamma\left(\mu+\frac{1}{2}\right)^2\Gamma\left(\mu+\frac{3}{2}\right)^3\Gamma\left(\mu+\frac{3}{2}\right)}{\Gamma(\mu+2)^2\Gamma\left(2\mu+\frac{9}{2}\right)^2},$$

which is clearly strictly negative for all integers $\mu \geq 0$. For all $\gamma \in \{1, 2, 3, 4, 5\}$ and $n \in \{0, 1, \ldots, \gamma-1\}$, we find either $\mathsf{D}_{\gamma n}^{(\mu,\mu)} < 0$ when μ is sufficiently small or $\mathsf{D}_{\gamma n}^{(\mu,\mu)} \to \pm\infty$ when $\mu \to \infty$, which completes the proof. □

7C. *Yang–Mills equation in spherical symmetry.* Finally, we consider the Yang–Mills equation in spherical symmetry and show that the nondegeneracy condition is a condition on the Fourier coefficients.

Lemma 7.5 (derivation of the nondegeneracy condition for the 1-modes and the YM model). *Let $\gamma \in \{0, 1, 2, 3, 4, 5\}$, and define ξ according to (6-3). Then, the nondegeneracy condition*

$$\ker(d\mathfrak{M}_{-}(\xi)) = \{0\}$$

is equivalent to

$$\begin{cases} \mathfrak{u}_{\gamma\gamma} + \mathfrak{v}_{\gamma\gamma} \neq 0, & \\ \mathfrak{u}_{\gamma m} \neq 0 & \text{for all } m \geq 2\gamma+1, \\ \mathfrak{D}_{\gamma n} \neq 0 & \text{for all } n \in \{0, 1, \ldots, \gamma-1\}, \end{cases} \tag{7-7}$$

where

$$\mathfrak{D}_{\gamma n} := \mathfrak{u}_{\gamma n}\mathfrak{u}_{\gamma,2\gamma-n} - \mathfrak{v}_{\gamma n}\mathfrak{v}_{\gamma,2\gamma-n}$$

and

$$\mathfrak{u}_{\gamma m} = \left(\frac{\varpi_m}{\mathfrak{K}_\gamma}\right)^2 + \frac{3}{4}\mathfrak{C}_{\gamma\gamma mm} - \frac{9}{2}\sum_{\nu=0}^{m+\gamma}\frac{(\overline{\mathfrak{C}}_{\gamma\nu m})^2}{\varpi_\nu^2 - (\varpi_m + \varpi_\gamma)^2} - \frac{9}{4}\sum_{\nu=0}^{2\gamma}\frac{\overline{\mathfrak{C}}_{m\nu m}\overline{\mathfrak{C}}_{\gamma\gamma\nu}}{\varpi_\nu^2} - \frac{9}{2}\sum_{\substack{\nu=0 \\ \nu\neq\pm(m-\gamma)-2}}^{m+\gamma}\frac{(\overline{\mathfrak{C}}_{m\gamma\nu})^2}{\varpi_\nu^2 - (\varpi_m - \varpi_\gamma)^2}, \tag{7-8}$$

$$\mathfrak{v}_{\gamma m} = \frac{3}{8}\mathfrak{C}_{\gamma,2\gamma-m,\gamma,m} - \frac{9}{4}\sum_{\nu=0}^{2\gamma}\frac{\overline{\mathfrak{C}}_{2\gamma-m,\nu,m}\overline{\mathfrak{C}}_{\gamma\gamma\nu}}{\varpi_\nu^2 - (2\varpi_\gamma)^2} - \frac{9}{2}\sum_{\substack{\nu=0 \\ \nu\neq\pm(m-\gamma)-2}}^{m+\gamma}\frac{\overline{\mathfrak{C}}_{\gamma\nu m}\overline{\mathfrak{C}}_{2\gamma-m,\gamma,\nu}}{\varpi_\nu^2 - (\varpi_{2\gamma-m} - \varpi_\gamma)^2}. \tag{7-9}$$

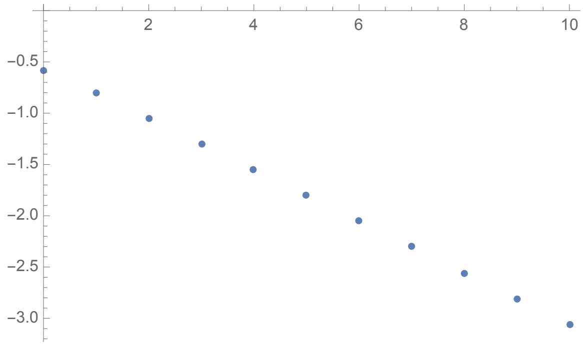

Figure 5. The constants $\mathfrak{u}_{\gamma\gamma}+\mathfrak{v}_{\gamma\gamma}$ for $\gamma\in\{0,1,2,3,4,5\}$. They decrease and stay away from zero.

Proof. Let $\gamma\in\{0,1,2,3,4,5\}$, and define ξ according to (6-3). Furthermore, assume $\ker(d\mathfrak{M}_-(\xi))=\{0\}$, and pick any $h=\{h^j : j\geq 0\}\in l^2_{s+3}$ such that $d\mathfrak{M}_-(\xi)[h]=0$. Also, fix any integer $m\geq 0$. Then, according to Lemma 6.7, we have that the system

$$\mathfrak{u}_{\gamma m}h^m+\mathfrak{v}_{\gamma m}h^{2\gamma-m}=0 \quad \text{for } 0\leq m\leq \gamma-1, \tag{7-10}$$

$$(\mathfrak{u}_{\gamma\gamma}+\mathfrak{v}_{\gamma\gamma})h^{\gamma}=0 \quad \text{for } m=\gamma, \tag{7-11}$$

$$\mathfrak{u}_{\gamma m}h^m+\mathfrak{v}_{\gamma m}h^{2\gamma-m}=0 \quad \text{for } \gamma+1\leq m\leq 2\gamma, \tag{7-12}$$

$$\mathfrak{u}_{\gamma m}h^m=0 \quad \text{for } m\geq 2\gamma+1, \tag{7-13}$$

has $h=\{h^i : i\geq 0\}=0$ as the unique solution, where $\mathfrak{u}_{\gamma m}$ and $\mathfrak{v}_{\gamma m}$ are given explicitly in terms of the auxiliary sequences $\mathfrak{a}_{\gamma m}$ and $\mathfrak{b}_{\gamma m}$ as Lemma 6.7 states. Furthermore, $\mathfrak{a}_{\gamma m}$ and $\mathfrak{b}_{\gamma m}$ are given by Lemma 2.15, and putting this all together yields the closed formulas (7-8) and (7-9) for $\mathfrak{u}_{\gamma m}$ and $\mathfrak{v}_{\gamma m}$, respectively, as stated above. Now, (7-11) and (7-13) yield $\mathfrak{u}_{\gamma\gamma}+\mathfrak{v}_{\gamma\gamma}\neq 0$ and $\mathfrak{u}_{\gamma m}\neq 0$ for all $m\geq 2\gamma+1$. Next, we rearrange (7-10) and (7-12) by setting $m=n$ and $m=2\gamma-n$, respectively, to obtain

$$\begin{bmatrix}\mathfrak{u}_{\gamma n} & \mathfrak{v}_{\gamma n}\\ \mathfrak{v}_{\gamma,2\gamma-n} & \mathfrak{u}_{\gamma,2\gamma-n}\end{bmatrix}\begin{bmatrix}h^n\\ h^{2\gamma-n}\end{bmatrix}=\begin{bmatrix}0\\0\end{bmatrix}$$

for all $n\in\{0,1,\dots,\gamma-1\}$. Observe that there are γ in total (2×2)-linear systems where the unknowns are h^m for $m\in\{0,1,\dots,2\gamma\}\setminus\{\gamma\}$. Finally, these systems have only the trivial solution $h^m=0$ for all $m\in\{0,1,\dots,2\gamma\}\setminus\{\gamma\}$ if and only if the determinants $\mathfrak{D}_{\gamma n}$ are nonzero for all $n\in\{0,1,\dots,\gamma-1\}$, which completes the proof. □

Next, we establish the nondegeneracy condition for this model.

Proposition 7.6 (nondegeneracy condition for the 1-modes and the YM model). *Let* $\gamma\in\{0,1,2,3,4,5\}$. *Then, the nondegeneracy condition* (7-7) *holds.*

Proof. Let $\gamma\in\{0,1,2,3,4,5\}$, $\alpha=1/\sqrt{6}$ and pick any frequency $\varpi\in\mathcal{W}_\alpha$ with $\varpi<1$. Also, pick an integer $m\geq 2\gamma+1$, and define $\mathfrak{u}_{\gamma m}$ and $\mathfrak{v}_{\gamma m}$ according to (7-8) and (7-9), respectively. Firstly, we show that $\mathfrak{u}_{\gamma\gamma}+\mathfrak{v}_{\gamma\gamma}\neq 0$. In this case, all the sums in the definitions of $\mathfrak{u}_{\gamma\gamma}$ and $\mathfrak{v}_{\gamma\gamma}$ are finite as the index variable ν varies within $\{0, 1, \dots, 2\gamma\}$. Hence, we compute $\mathfrak{u}_{\gamma\gamma} + \mathfrak{v}_{\gamma\gamma}$ for all $\gamma \in \{0, 1, 2, 3, 4, 5\}$ and verify that they are all nonzero. Figure 5 illustrates the constants $\mathfrak{u}_{\gamma\gamma} + \mathfrak{v}_{\gamma\gamma}$ for $\gamma \in \{0, 1, 2, 3, 4, 5\}$. Secondly, we show that $\mathfrak{u}_{\gamma m} \neq 0$ for $m \geq 2\gamma + 1$. To this end, we note that $\overline{\mathfrak{C}}_{ijm} = 0$ for all integers $i, j, m \geq 0$ with either $m > i + j$ or $|i - j| > m$ (Lemma 5.8). Specifically, for any integer $m \geq 2\gamma + 1$, we focus on

$$\sum_{\nu=0}^{m+\gamma} \frac{(\overline{\mathfrak{C}}_{\gamma\nu m})^2}{\varpi_\nu^2 - (\varpi_m + \varpi_\gamma)^2}, \qquad \sum_{\substack{\nu=0 \\ \nu \neq \pm(m-\gamma)-2}}^{m+\gamma} \frac{(\overline{\mathfrak{C}}_{m\gamma\nu})^2}{\varpi_\nu^2 - (\varpi_m - \varpi_\gamma)^2},$$

and note that we must have $m - \gamma \leq \nu \leq m + \gamma$. Indeed, $\overline{\mathfrak{C}}_{\gamma\nu m} = 0$ since $m - \gamma = |m - \gamma| > \nu$, and $\overline{\mathfrak{C}}_{\gamma\nu m} = 0$ since $\nu > m + \gamma$. In addition, for all such ν, the conditions $\nu \neq \pm(m - \gamma) - 2$ are satisfied since

$$\nu \geq m - \gamma \quad \Longrightarrow \quad (m - \gamma) - 2 < m - \gamma \leq \nu \quad \Longrightarrow \quad \nu \neq (m - \gamma) - 2$$

and

$$m \geq 2\gamma + 1 \quad \Longrightarrow \quad -(m - \gamma) - 2 < 0 \quad \Longrightarrow \quad \nu \neq -(m - \gamma) - 2.$$

Consequently, for all $m \geq 2\gamma + 1$, we have

$$\mathfrak{u}_{\gamma m} = \left(\frac{\varpi_m}{\mathfrak{K}_\gamma}\right)^2 + \frac{3}{4}\mathfrak{C}_{\gamma\gamma mm} - \frac{9}{2}\sum_{\nu=m-\gamma}^{m+\gamma} \frac{(\overline{\mathfrak{C}}_{\gamma\nu m})^2}{\varpi_\nu^2 - (\varpi_m + \varpi_\gamma)^2} - \frac{9}{4}\sum_{\nu=0}^{2\gamma} \frac{\overline{\mathfrak{C}}_{m\nu m}\overline{\mathfrak{C}}_{\gamma\gamma\nu}}{\varpi_\nu^2} - \frac{9}{2}\sum_{\nu=m-\gamma}^{m+\gamma} \frac{(\overline{\mathfrak{C}}_{m\gamma\nu})^2}{\varpi_\nu^2 - (\varpi_m - \varpi_\gamma)^2},$$

and by setting $\nu = \sigma + m - \gamma$ and $\nu = \sigma$, respectively, we can rewrite the latter as

$$\begin{aligned}\mathfrak{u}_{\gamma m} = \left(\frac{\varpi_m}{\mathfrak{K}_\gamma}\right)^2 + \frac{3}{4}\mathfrak{C}_{\gamma\gamma mm} - \frac{9}{2}\sum_{\sigma=0}^{2\gamma} \frac{(\overline{\mathfrak{C}}_{\gamma,\sigma+m-\gamma,m})^2}{\varpi_{\sigma+m-\gamma}^2 - (\varpi_m + \varpi_\gamma)^2} \\ - \frac{9}{4}\sum_{\sigma=0}^{2\gamma} \frac{\overline{\mathfrak{C}}_{m\sigma m}\overline{\mathfrak{C}}_{\gamma\gamma\sigma}}{\varpi_\sigma^2} - \frac{9}{2}\sum_{\sigma=0}^{2\gamma} \frac{(\overline{\mathfrak{C}}_{m,\gamma,\sigma+m-\gamma})^2}{\varpi_{\sigma+m-\gamma}^2 - (\varpi_m - \varpi_\gamma)^2}.\end{aligned}$$

Now, recall from Lemma 6.6 that

$$\begin{aligned}\mathfrak{K}_\gamma^{-2} = \frac{3\mathfrak{C}_{\gamma\gamma\gamma\gamma} - 8\mathfrak{q}_\gamma}{-8\varpi_\gamma^2} &= -\frac{3}{8\varpi_\gamma^2}\mathfrak{C}_{\gamma\gamma\gamma\gamma} + \frac{1}{\varpi_\gamma^2}\mathfrak{q}_\gamma \\ &= -\frac{3}{8\varpi_\gamma^2}\mathfrak{C}_{\gamma\gamma\gamma\gamma} + \frac{9}{4\varpi_\gamma^2}\sum_{\nu=0}^{2\gamma}(\overline{\mathfrak{C}}_{\gamma\gamma\nu})^2\left(\frac{2}{\varpi_\nu^2} + \frac{1}{\varpi_\nu^2 - (2\varpi_\gamma)^2}\right),\end{aligned}$$

which yields

$$\left(\frac{\varpi_m}{\mathfrak{K}_\gamma}\right)^2 = -\frac{3\varpi_m^2}{8\varpi_\gamma^2}\mathfrak{C}_{\gamma\gamma\gamma\gamma} + \frac{9\varpi_m^2}{4\varpi_\gamma^2}\sum_{\nu=0}^{2\gamma}(\overline{\mathfrak{C}}_{\gamma\gamma\nu})^2\left(\frac{2}{\varpi_\nu^2} + \frac{1}{\varpi_\nu^2 - (2\varpi_\gamma)^2}\right).$$

Putting this all together, we obtain

$$\begin{aligned}\mathfrak{u}_{\gamma m} = -\frac{3\varpi_m^2}{8\varpi_\gamma^2}\mathfrak{C}_{\gamma\gamma\gamma\gamma} + \frac{9\varpi_m^2}{4\varpi_\gamma^2}\sum_{\sigma=0}^{2\gamma}(\overline{\mathfrak{C}}_{\gamma\gamma\sigma})^2\left(\frac{2}{\varpi_\sigma^2} + \frac{1}{\varpi_\sigma^2 - (2\varpi_\gamma)^2}\right) + \frac{3}{4}\mathfrak{C}_{\gamma\gamma mm} \\ - \frac{9}{2}\sum_{\sigma=0}^{2\gamma} \frac{(\overline{\mathfrak{C}}_{\gamma,\sigma+m-\gamma,m})^2}{\varpi_{\sigma+m-\gamma}^2 - (\varpi_m + \varpi_\gamma)^2} - \frac{9}{4}\sum_{\sigma=0}^{2\gamma} \frac{\overline{\mathfrak{C}}_{m\sigma m}\overline{\mathfrak{C}}_{\gamma\gamma\sigma}}{\varpi_\sigma^2} - \frac{9}{2}\sum_{\sigma=0}^{2\gamma} \frac{(\overline{\mathfrak{C}}_{m,\gamma,\sigma+m-\gamma})^2}{\varpi_{\sigma+m-\gamma}^2 - (\varpi_m - \varpi_\gamma)^2}.\end{aligned}$$

In addition, we also note that $\overline{\mathfrak{C}}_{ijm}=0$ for all integers $i, j, m \geq 0$ with $i+j-m \notin 2\mathbb{N}\cup\{0\}$ (Lemma 5.8). Specifically, we must have $\sigma \in 2\mathbb{N}\cup\{0\}$. Indeed,

$$\overline{\mathfrak{C}}_{\gamma\gamma\sigma}=0 \quad \text{since } \sigma=\sigma+\gamma-\gamma \notin 2\mathbb{N}\cup\{0\},$$

and

$$\overline{\mathfrak{C}}_{\gamma,\sigma+m-\gamma,m}=\overline{\mathfrak{C}}_{m,\gamma,\sigma+m-\gamma}=0 \quad \text{since } \sigma=\gamma+\sigma+m-\gamma-m \notin 2\mathbb{N}\cup\{0\}.$$

Therefore, by setting $\sigma=2\tau$, we arrive at

$$\begin{aligned}\mathfrak{u}_{\gamma m}=&-\frac{3\varpi_m^2}{8\varpi_\gamma^2}\mathfrak{C}_{\gamma\gamma\gamma\gamma}+\frac{9\varpi_m^2}{2\varpi_\gamma^2}\sum_{\tau=0}^{\gamma}\frac{(\overline{\mathfrak{C}}_{\gamma,\gamma,2\tau})^2}{\varpi_{2\tau}^2}+\frac{9\varpi_m^2}{4\varpi_\gamma^2}\sum_{\tau=0}^{\gamma}\frac{(\overline{\mathfrak{C}}_{\gamma,\gamma,2\tau})^2}{\varpi_{2\tau}^2-(2\varpi_\gamma)^2}+\frac{3}{4}\mathfrak{C}_{\gamma\gamma mm}\\&-\frac{9}{2}\sum_{\tau=0}^{\gamma}\frac{(\overline{\mathfrak{C}}_{\gamma,2\tau+m-\gamma,m})^2}{\varpi_{2\tau+m-\gamma}^2-(\varpi_m+\varpi_\gamma)^2}-\frac{9}{4}\sum_{\tau=0}^{\gamma}\frac{\overline{\mathfrak{C}}_{m,2\tau,m}\overline{\mathfrak{C}}_{\gamma,\gamma,2\tau}}{\varpi_{2\tau}^2}-\frac{9}{2}\sum_{\tau=0}^{\gamma}\frac{(\overline{\mathfrak{C}}_{m,\gamma,2\tau+m-\gamma})^2}{\varpi_{2\tau+m-\gamma}^2-(\varpi_m-\varpi_\gamma)^2}.\end{aligned}$$

Now, all the Fourier coefficients above are nonzero and, according to Lemma 5.9, we have $\overline{\mathfrak{C}}_{\gamma,\gamma,2\tau}$, $\overline{\mathfrak{C}}_{m,2\tau,m}$ and $\overline{\mathfrak{C}}_{\gamma,2\tau+m-\gamma,m}$ in closed formulas. These allow us to compute

$$\sum_{\tau=0}^{\gamma}\frac{(\overline{\mathfrak{C}}_{\gamma,\gamma,2\tau})^2}{\varpi_{2\tau}^2}=\frac{(\gamma+2)(2\gamma+3)(2\gamma+5)}{15\pi(\gamma+1)(\gamma+3)},$$
$$\sum_{\tau=0}^{\gamma}\frac{\overline{\mathfrak{C}}_{m,2\tau,m}\overline{\mathfrak{C}}_{\gamma,\gamma,2\tau}}{\varpi_{2\tau}^2}=\frac{(\gamma+2)(-\gamma(\gamma+4)+5m(m+4)+15)}{15\pi(m+1)(m+3)},$$

for all integers $\gamma \geq 0$. Recall that $\mathfrak{C}_{\gamma\gamma mm}$ and $\mathfrak{C}_{\gamma\gamma\gamma\gamma}$ are also given by closed formulas (Remark 5.11). Consequently, we rescale $\mathfrak{u}_{\gamma m}$ and obtain

$$\frac{\mathfrak{u}_{\gamma m}}{\varpi_m^2}=\mathfrak{I}_{\gamma m}+\mathfrak{E}_{\gamma m}, \tag{7-14}$$

where $\mathfrak{I}_{\gamma m}$ stands for the part that can be explicitly computed:

$$\begin{aligned}\mathfrak{I}_{\gamma m}=&-\frac{3}{8\varpi_\gamma^2}\mathfrak{C}_{\gamma\gamma\gamma\gamma}+\frac{9}{2\varpi_\gamma^2}\sum_{\tau=0}^{\gamma}\frac{(\overline{\mathfrak{C}}_{\gamma,\gamma,2\tau})^2}{\varpi_{2\tau}^2}+\frac{9}{4\varpi_\gamma^2}\sum_{\tau=0}^{\gamma}\frac{(\overline{\mathfrak{C}}_{\gamma,\gamma,2\tau})^2}{\varpi_{2\tau}^2-(2\varpi_\gamma)^2}\\&+\frac{3}{4\varpi_m^2}\mathfrak{C}_{\gamma\gamma mm}-\frac{9}{4\varpi_m^2}\sum_{\tau=0}^{\gamma}\frac{\overline{\mathfrak{C}}_{m,2\tau,m}\overline{\mathfrak{C}}_{\gamma,\gamma,2\tau}}{\varpi_{2\tau}^2},\end{aligned}$$

and $\mathfrak{E}_{\gamma m}$ stands for the part that cannot be explicitly computed:

$$\mathfrak{E}_{\gamma m}=-\frac{9}{2\varpi_m^2}\sum_{\tau=0}^{\gamma}\frac{(\overline{\mathfrak{C}}_{\gamma,2\tau+m-\gamma,m})^2}{\varpi_{2\tau+m-\gamma}^2-(\varpi_m+\varpi_\gamma)^2}-\frac{9}{2\varpi_m^2}\sum_{\tau=0}^{\gamma}\frac{(\overline{\mathfrak{C}}_{m,\gamma,2\tau+m-\gamma})^2}{\varpi_{2\tau+m-\gamma}^2-(\varpi_m-\varpi_\gamma)^2}.$$

Now, using the elementary inequalities

$$|\varpi_{2\tau+m-\gamma}^2-(\varpi_m+\varpi_\gamma)^2|=|4(\gamma-\tau+1)(m+\tau+3)|\geq 4(m+3),$$
$$|\varpi_{2\tau+m-\gamma}^2-(\varpi_m-\varpi_\gamma)^2|=|4(\tau+1)(-\gamma+m+\tau+1)|\geq 2(m+3),$$

for all $0 \le \tau \le \gamma$ and $\gamma \ge 0$, we estimate

$$\begin{aligned}|\mathfrak{E}_{\gamma m}| &\le \frac{9}{2\varpi_m^2}\sum_{\tau=0}^{\gamma}\frac{(\overline{\mathfrak{C}}_{\gamma,2\tau+m-\gamma,m})^2}{|\varpi_{2\tau+m-\gamma}^2-(\varpi_m+\varpi_\gamma)^2|}+\frac{9}{2\varpi_m^2}\sum_{\tau=0}^{\gamma}\frac{(\overline{\mathfrak{C}}_{m,\gamma,2\tau+m-\gamma})^2}{|\varpi_{2\tau+m-\gamma}^2-(\varpi_m-\varpi_\gamma)^2|}\\
&\le \frac{9}{2\varpi_m^2}\frac{1}{4(m+3)}\sum_{\tau=0}^{\gamma}(\overline{\mathfrak{C}}_{\gamma,2\tau+m-\gamma,m})^2+\frac{9}{2\varpi_m^2}\frac{1}{2(m+3)}\sum_{\tau=0}^{\gamma}(\overline{\mathfrak{C}}_{\gamma,2\tau+m-\gamma,m})^2\\
&=\frac{9}{2\varpi_m^2}\left(\frac{1}{4(m+3)}+\frac{1}{2(m+3)}\right)\sum_{\tau=0}^{\gamma}(\overline{\mathfrak{C}}_{\gamma,2\tau+m-\gamma,m})^2\\
&=\frac{9(\gamma+2)}{70\pi(m+1)(m+2)^2(m+3)^2}\big[-3\gamma^4-24\gamma^3-40\gamma^2+32\gamma+7\gamma^2m^2+28\gamma m^2+35m^2\\
&\qquad\qquad +28\gamma^2m+112\gamma m+140m+105\big]=\mathfrak{P}_{\gamma m},\end{aligned}$$

where we used the closed formula for $\overline{\mathfrak{C}}_{\gamma,2\tau+m-\gamma,m}$ from above. Hence, for all $m \ge 2\gamma+1$, we obtain

$$\frac{\mathfrak{u}_{\gamma m}}{\varpi_m^2}=\mathfrak{I}_{\gamma m}+\mathfrak{E}_\gamma \ge \mathfrak{I}_{\gamma m}-\mathfrak{P}_{\gamma m}=\mathfrak{O}_{\gamma m}. \tag{7-15}$$

Finally, for each $\gamma \in \{0, 1, 2, 3, 4, 5\}$, we use the closed formulas for $\mathfrak{C}_{\gamma\gamma mm}$ and $\mathfrak{C}_{\gamma\gamma\gamma\gamma}$ (see Remark 5.11) to firstly explicitly compute $\mathfrak{I}_{\gamma m}$ in terms of m and then explicitly compute $\mathfrak{O}_{\gamma m}$ in terms of m. Once the closed formula is derived, we show that $\mathfrak{O}_{\gamma m} > 0$ for all $m \ge 2\gamma+1$. For example, for $\gamma = 0$, we find

$$\mathfrak{I}_{0m}=\frac{m(m+4)(5m(m+4)+29)+66}{12\pi(m+1)(m+2)^2(m+3)},$$

and hence

$$\mathfrak{O}_{0m}=\frac{5m^4+40m^3+109m^2+8m-42}{12\pi(m+1)(m+2)^2(m+3)},$$

which is greater than 10^{-3} provided that $m \ge 1$. Similarly, for $\gamma = 1$, we compute

$$\mathfrak{I}_{1m}=\frac{(m^2+2)(m(m+8)+18)}{4\pi(m+1)(m+2)^2(m+3)},$$

and hence

$$\mathfrak{O}_{1m}=\frac{m(m^4+11m^3+44m^2-32m-348)}{4\pi(m+1)(m+2)^2(m+3)^2},$$

which is greater than 10^{-3} provided that $m \ge 3$. For all the other cases with $\gamma \in \{2, 3, 4, 5\}$, we find

$$\mathfrak{O}_{2m}=\frac{109m^5+1199m^4+4523m^3-30347m^2-132936m+107244}{600\pi(m+1)(m+2)^2(m+3)^2},$$

$$\mathfrak{O}_{3m}=\frac{43m^5+473m^4+1646m^3-33554m^2-129372m+238248}{300\pi(m+1)(m+2)^2(m+3)^2},$$

$$\mathfrak{O}_{4m}=\frac{83m^5+913m^4+2851m^3-139159m^2-515982m+1611198}{700\pi(m+1)(m+2)^2(m+3)^2},$$

$$\mathfrak{O}_{5m}=\frac{17m^5+187m^4+505m^3-53329m^2-194760m+905292}{168\pi(m+1)(m+2)^2(m+3)^2}$$

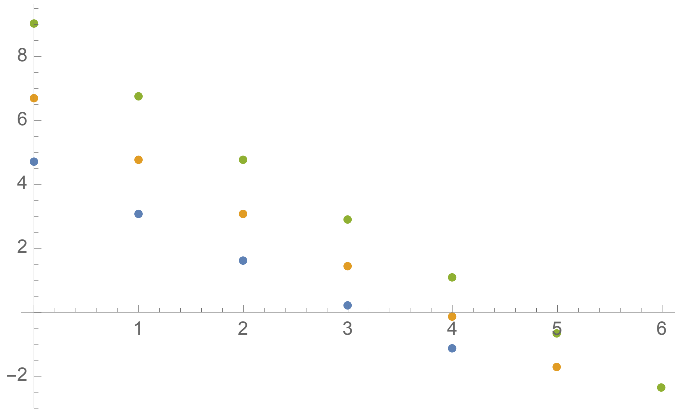

Figure 6. The determinants $\mathfrak{D}_{\gamma n}$ for $\gamma = 5$ (blue/bottom), $\gamma = 6$ (orange/middle) and $\gamma = 7$ (green/top) for all $n \in \{0, 1, \ldots, \gamma - 1\}$. They are all in fact nonzero.

and the claim follows similarly.[12] Finally, it remains to show that the determinants

$$\mathfrak{D}_{\gamma n} := \mathfrak{u}_{\gamma n}\mathfrak{u}_{\gamma,2\gamma-n} - \mathfrak{v}_{\gamma n}\mathfrak{v}_{\gamma,2\gamma-n}$$

are all nonzero for $n \in \{0, 1, \ldots, \gamma - 1\}$, which follows by a direct computation using the definition of the Fourier coefficients. Specifically, we compute $\mathfrak{D}_{\gamma n}$ for all $\gamma \in \{0, 1, 2, 3, 4, 5\}$ and $n \in \{0, 1, \ldots, \gamma - 1\}$ and verify that they are all strictly negative, which completes the proof. Figure 6 illustrates the determinants $\mathfrak{D}_{\gamma n}$ for $\gamma \in \{5, 6, 7\}$ and $n \in \{0, 1, \ldots, \gamma - 1\}$. □

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[12]For $\gamma \in \{2, 3\}$, the estimates $\mathfrak{D}_{\gamma m} \geq 10^{-3}$ hold true for all integers $m \geq 2\gamma + 1$, whereas, for $\gamma \in \{4, 5\}$, we have that $\mathfrak{D}_{\gamma m} \geq 10^{-3}$ provided that $m \geq 2\gamma + 3$ instead of $m \geq 2\gamma + 1$. In this case, we use the definition of the Fourier coefficients to explicitly compute $\mathfrak{u}_{\gamma m}\varpi_m^{-2}$ for $m \in \{2\gamma + 1, 2\gamma + 2\}$ and verify that it is still strictly positive.

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HOST–KRA FACTORS FOR $\bigoplus_{p\in P}\mathbb{Z}/p\mathbb{Z}$ ACTIONS AND FINITE-DIMENSIONAL NILPOTENT SYSTEMS

OR SHALOM

Let $\mathcal{P}$ be a countable multiset of primes and let $G=\bigoplus_{p\in P}\mathbb{Z}/p\mathbb{Z}$. We study the universal characteristic factors associated with the Gowers–Host–Kra seminorms for the group G. We show that the universal characteristic factor of order $<k+1$ is a factor of an inverse limit of *finite-dimensional k-step nilpotent homogeneous spaces*. The latter is a counterpart of a k-step nilsystem where the homogeneous group is not necessarily a Lie group. As an application of our structure theorem we derive an alternative proof for the L^2-convergence of multiple ergodic averages associated with k-term arithmetic progressions in G and derive a formula for the limit in the special case where the underlying space is a nilpotent homogeneous system. Our results provide a counterpart of the structure theorem of Host and Kra (2005) and Ziegler (2007) concerning $\mathbb{Z}$-actions and generalize the results of Bergelson, Tao and Ziegler (2011, 2015) concerning $\mathbb{F}_p^\omega$-actions. This is also the first instance of studying the Host–Kra factors of nonfinitely generated groups of unbounded torsion.

1. Introduction

The universal characteristic factors for multiple ergodic averages play an important role in ergodic Ramsey theory. For instance, in the case of $\mathbb{Z}$-actions they are related to the theorem of Szemerédi [1975] about the existence of arbitrary large arithmetic progressions in sets of positive upper Banach density in the integers.

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The universal characteristic factors associated with multiple ergodic averages in $\mathbb{Z}$-actions were studied by Host and Kra [2005] and independently by Ziegler [2007]. Later, Bergelson, Tao and Ziegler proved a counterpart for the nonfinitely generated group $G=\mathbb{F}_p^\omega$. The goal of this paper is to generalize these results further for the group $G=\bigoplus_{p\in P}\mathbb{Z}/p\mathbb{Z}$, where P is a multiset of primes. Moreover, in Section 2D we discuss the general case where G is any countable abelian group. In particular, we identify a result (Conjecture 2.15) which leads to a general structure theorem for the Gowers–Host–Kra seminorms.

Conventions 1. We use X to denote a probability space. For technical reasons we assume that any probability space $X=(X,\mathcal{B},\mu)$ is regular[1] and separable modulo null sets. We let $(U,\cdot)$ denote a compact abelian group and we assume that all topological groups in this paper are metrizable. Let $(G,+)$ be a countable abelian group, a G-system is a probability space $X=(X,\mathcal{B},\mu)$ together with an action of G on X by measure-preserving transformations $T_g:X\to X$. Throughout most of this paper we use G to denote the group $\bigoplus_{p\in P}\mathbb{Z}/p\mathbb{Z}$, where P is a given countable multiset of primes.

Host and Kra proved that the universal characteristic factors for $\mathbb{Z}$-actions are closely related to an infinite version of the Gowers norms. The following version of the Gowers–Host–Kra (GHK) seminorms in the special case where $G=\mathbb{Z}$ was essentially introduced by Host and Kra [2005] (see [Host and Kra 2018, Proposition 16, Chapter 8] for this version).

Definition 1.1 (Gowers–Host–Kra seminorms). Let (X,T_g) be a G-system, let $\phi\in L^\infty(X)$, and let $k\geq 1$ be an integer. The GHK seminorm $\|\phi\|_{U^k}$ of order k of ϕ is defined recursively by the formula

$$\|\phi\|_{U^1}:=\lim_{N\to\infty}\frac{1}{|\Phi_N^1|}\Big\|\sum_{g\in\Phi_N^1}\phi\circ T_g\Big\|_{L^2}$$

for $k=1$, and

$$\|\phi\|_{U^k}:=\lim_{N\to\infty}\bigg(\frac{1}{|\Phi_N^k|}\sum_{g\in\Phi_N^k}\|\Delta_g\phi\|_{U^{k-1}}^{2^{k-1}}\bigg)^{1/2^k}$$

for $k>1$, where $\phi_N^1,\ldots,\phi_N^k$ are arbitrary Følner sequences and $\Delta_g\phi(x)=\phi(T_gx)\cdot\overline{\phi(x)}$.

These seminorms were first introduced in the special case where $G=\mathbb{Z}/N\mathbb{Z}$ in [Gowers 2001], where he proved quantitative bounds for Szemerédi's theorem [1975]. As mentioned above, Host and Kra [2005] generalized these seminorms for the infinite group $\mathbb{Z}$ and proved that each seminorm corresponds to a unique factor of X. Later, Leibman [2006] proved that these factors coincide with the universal characteristic factors for multiple ergodic averages which were studied by Ziegler [2007].

Proposition 1.2 (existence and uniqueness of the universal characteristic factors). *Let G be a countable abelian group, let X be a G-system, and let $k\geq 1$. Then there exists a factor*

$$Z_{<k}(X)=(Z_{<k}(X),\mathcal{B}_{Z_{<k}(X)},\mu_{Z_{<k}(X)},\pi^X_{Z_{<k}(X)})$$

of X with the property that, for every $f\in L^\infty(X)$, $\|f\|_{U^k(X)}=0$ if and only if $E(f|Z_{<k}(X))=0$. This factor is unique up to isomorphism and is called the k-th universal characteristic factor of X. If $X=Z_{<k}(X)$, we say that X is of order $<k$.

[1] Meaning that X is a compact metric space, $\mathcal{B}$ is the completion of the σ-algebra of Borel sets, and μ is a Borel measure.

In Appendix A we summarize previous work. In particular, we survey the definitions and various results by Host and Kra [2005] and Bergelson, Tao and Ziegler [2010]. We also state a structure theorem for totally disconnected $\bigoplus_{p\in P}\mathbb{Z}/p\mathbb{Z}$-systems (Theorem A.21) from the author's previous work [Shalom 2023]. This theorem will be used as a black box in this paper.

Another related approach for the study of the universal characteristic factors and related problems like the inverse problem for the Gowers norms (see [Green and Tao 2008; Green et al. 2012; Tao and Ziegler 2010] for more details), which we do not pursue in this paper, is the study of nilspaces and nilspace systems. Antolín Camarena and Szegedy [2010] introduced a purely combinatorial counterpart of the Host–Kra factors called *nilspaces*. The idea was to give a more abstract and general notion which describes the "cubic structure" of an ergodic system (see [Host and Kra 2005, Section 2]). They proved that connected nilspaces are inverse limits of nilmanifolds in the category of nilspaces, which justifies the *nil* in nilspace. A nilspace system is a compact nilspace equipped with a continuous action of a group which preserves its cube structure. Candela, González-Sánchez and Szegedy [Candela et al. 2020] studied nilspace systems and proved that the theory of nilspaces passes through to nilspace systems when the group acting on the space is finitely generated (Gutman, Manners and Varjú [Gutman et al. 2020b] generalized this result to compactly generated groups). Candela and Szegedy [2023] used nilspaces to prove a structure theorem for characteristic factors for GHK seminorms associated with any nilpotent group. They proved that the characteristic factors for the GHK seminorms of a nilpotent group form a nilspace system and obtained from previous result an alternative proof of Host–Kra structure theorem for finitely generated nilpotent groups. In a series of papers Gutman, Manners, and Varjú [Gutman et al. 2020a; 2019; 2020b] studied further the structure of nilspaces. By imposing another measure-theoretical aspect to these nilspaces, Gutman and Lian [2023] gave yet another alternative proof of Host and Kra's theorem for arbitrary finitely generated abelian groups.

The structure of *nilspace systems* is currently only well understood when the acting group is finitely generated (or compactly generated in general). The only exception is a new result for actions of the group $G=\mathbb{F}_p^\omega$ which was recently obtained by Candela, González-Sánchez and Szegedy [Candela et al. 2023]. We believe that the tools developed in this paper may turn out to be useful also in the study of nilspace systems associated when the acting group is not finitely generated and of unbounded torsion. For instance, it is evident from this work that the Host–Kra factors of a nonfinitely generated group with unbounded torsion are not necessarily isomorphic to an inverse limit of nilsystems. Thus, contrary to the finitely generated case, it is impossible to describe arbitrary nilspace systems as an inverse limit of nilmanifolds. The main results in this paper suggests that the notion of *nilpotent systems* in Definition 2.9 (see also the double-coset construction from [Shalom 2022, Theorem 1.21]) may replace the notion of a nilmanifold when studying arbitrary nilspace systems.

2. Main results

Recall that a k-step nilsystem is a quadruple $(\mathcal{G}/\Gamma,\mathcal{B},\mu,R)$, where $\mathcal{G}$ is a k-step nilpotent Lie group, Γ is a discrete cocompact subgroup, $\mathcal{B}$ is the Borel σ-algebra, μ is the Haar measure and $R:\mathcal{G}/\Gamma\to\mathcal{G}/\Gamma$ is a left multiplication by some element $r\in\mathcal{G}$. Host and Kra [2005, Theorem 10.1] and independently

Ziegler [2007, Theorem 1.7] proved the following structure theorem for the universal characteristic factors concerning $\mathbb{Z}$-actions.

Theorem 2.1. *Let $(X, \mathcal{B}, \mu, T)$ be an ergodic invertible system. Then, for every $k \geq 1$, the factor $Z_{<k+1}(X)$ is isomorphic to an inverse limit of k-step nilsystems.*

Our goal is to generalize this result to $\bigoplus_{p\in P} \mathbb{Z}/p\mathbb{Z}$-actions, where P is a countable multiset of primes. As a first step we explain how to interpret the results of Bergelson, Tao and Ziegler [Bergelson et al. 2010] about $\mathbb{F}_p^\omega$-systems in this language. We need the following version of nilsystems.

Definition 2.2 (zero-dimensional nilpotent system). Let $k \geq 1$ and let G be a countable abelian group. A *zero-dimensional k-step nilpotent system* is a quadruple $X = (\mathcal{G}/\Gamma, \mathcal{B}, \mu, (R_g)_{g\in G})$, where $\mathcal{G}$ is a zero-dimensional[2] k-step nilpotent group, Γ is a closed (not necessarily discrete) cocompact subgroup, $\mathcal{B}$ is the Borel σ-algebra, μ is a left $\mathcal{G}$-invariant measure and there exists a homomorphism $\varphi : G \to \mathcal{G}$ such that for every $g \in G$ the transformation $R_g : \mathcal{G}/\Gamma \to \mathcal{G}/\Gamma$ is given by a left multiplication by $\varphi(g)$.

The following structure theorem for the universal characteristic factors concerning $\mathbb{F}_p^\omega$ can be derived from [Bergelson et al. 2010].[3]

Theorem 2.3. *Let $k \geq 1$, any ergodic $\mathbb{F}_p^\omega$-system of order $< k+1$ is a zero-dimensional k-step nilpotent system whenever $k < p$. In the low-characteristic case $k \geq p$, our argument shows that there exists some $m = O_k(1)$ and an m-extension* (*see Definition 2.5*) *isomorphic to a zero-dimensional k-step nilpotent $\mathbb{Z}/p^m\mathbb{Z}$-system.*

Remark 2.4. We cautiously note that it is possible that the group $\mathcal{G}$ in Definition 2.2 is not locally compact. For instance consider the ergodic $\mathbb{F}_p^\omega$-system

$$X = \prod_{i=1}^{\mathbb{N}} C_p \times_\sigma \prod_{i=1}^{\mathbb{N}} C_p$$

of order < 3, where σ is any phase polynomial of degree < 2. If $\mathcal{G}(X)$ is the Host–Kra group of X (see Definition 2.10), then since σ is a phase polynomial of degree < 2, one can show that $\mathcal{G}(X)$ is a semidirect product of $\prod_{i=1}^{\mathbb{N}} C_p$ with $\prod_{i=1}^{\mathbb{N}} C_p \oplus \operatorname{Hom}\big(\prod_{i=1}^{\mathbb{N}} C_p, \prod_{i=1}^{\mathbb{N}} C_p\big) \cong \prod_{i=1}^{\mathbb{N}} C_p \oplus (\mathbb{F}_p^\omega)^{\mathbb{N}}$. The group $\Gamma = (\mathbb{F}_p^\omega)^{\mathbb{N}}$ is a nonlocally compact totally disconnected subgroup and $\mathcal{G}(X)/\Gamma \cong X$.

Let P be a countable multiset of primes and $G = \bigoplus_{p\in P} \mathbb{Z}/p\mathbb{Z}$. In order to prove a counterpart of the theorem above for G-systems, we introduce a new notion of extensions (Definition 2.5). Our main theorem (Theorem 2.12) asserts, roughly speaking, that every ergodic G-system of order $< k+1$ is a (special) factor of an inverse limit of k-step *finite-dimensional nilpotent systems*.

2A. ***k-extensions and finite-dimensional groups.*** Let (X, T_g) be a G-system and $\varphi : H \to G$ be a homomorphism from a countable abelian group H onto G. The homomorphism φ gives rise to an action of H on X by $S_h x = T_{\varphi(h)} x$. This observation allows us to define extensions outside of the category of G-systems.

[2] A zero-dimensional group is a topological group with a totally disconnected topology. That is, every point has a basis of clopen sets.

[3] A proof of this result is not explicitly given in [Bergelson et al. 2010]. One way to prove this theorem is by following the arguments in this paper in the simple case where $P = \{p, p, p, \dots\}$.

Definition 2.5 (k-extensions). Let $P=\{p_1,p_2,\dots\}$ be a multiset of primes and $G=\bigoplus_{p\in P}\mathbb{Z}/p\mathbb{Z}$. For a natural number $k\geq 1$, we define $G^{(k)}=\bigoplus_{p\in P}\mathbb{Z}/p^k\mathbb{Z}$ and let

$$\varphi_k : G^{(k)}\to G,\quad \varphi(g)_i=g_i \mod p_i.$$

We say that a $G^{(k)}$-system Y is a k-extension of X if it is an extension of $(X,G^{(k)})$.

Example 2.6. Let $G=\mathbb{Z}/2\mathbb{Z}$ and $X=\{-1,1\}$ and define an action of G on X by $T_g x=(-1)^g x$. Similarly, let $H=\mathbb{Z}/4\mathbb{Z}$ and $Y=\{-1,-i,i,1\}$. Then H acts on Y by $S_h y=i^h y$. The system (Y,H) defines a 2-extension of (X,G) with respect to the homomorphism

$$\varphi : H\to G,\quad \varphi(h)=h \mod 2,$$

and the factor map $\pi : Y\to X$, where $\pi(y)=y^2$.

There are multiple notions of dimension for topological spaces in the literature, which do not coincide for nonlocally compact groups (see [Arhangelskii and van Mill 2018] for a survey about the different notions of dimension and the history behind them). Throughout we say that a topological space X is totally disconnected if, for every $x,y\in X$, there exists a clopen subset $C\subseteq X$ such that $x\in C$ and $y\notin C$. Another possible definition for a totally disconnected space X is to require that all connected components are singletons. These definitions do not coincide in general, but in this paper the results remain the same if we interchange one definition with another. It is well known that all products and closed subsets of totally disconnected sets are totally disconnected.

Since there is no concrete notion of dimension for nonlocally compact groups, we will use the following natural definition instead.

Definition 2.7. A topological group H is said to be *zero-dimensional* if it is totally disconnected. A topological group H is said to be *finite-dimensional* if it is contained in the family of groups $\mathcal{FD}$, where $\mathcal{FD}$ is the minimal family satisfying:

(i) $\mathcal{FD}$ contains all Lie groups and all totally disconnected groups.

(ii) If $K\leq\mathcal{FD}$ then any closed subgroup of K is in $\mathcal{FD}$.

(iii) If $L\leq K$ is a closed subgroup and $K/L, L\in\mathcal{FD}$, then $K\in\mathcal{FD}$.

In the specific case of compact abelian groups, this is equivalent to the following definition.

Proposition 2.8 (finite-dimensional compact abelian groups [Hofmann and Morris 2013, Theorem 8.22]). *The following conditions are equivalent for a compact abelian group U and a natural number n:*

(1) *U is of dimension n.*

(2) *There exists a compact totally disconnected subgroup Δ of U and a short exact sequence*

$$1\to\Delta\to U\to (S^1)^n\to 1.$$

(3) *There exists a compact zero-dimensional subgroup Δ of U and a continuous surjective homomorphism $\varphi:\Delta\times\mathbb{R}^n\to U$ such that $\Gamma:=\ker\varphi$ is discrete. Hence, $U\cong(\Delta\times\mathbb{R}^n)/\Gamma$ as topological groups.*

We generalize Definition 2.2.

Definition 2.9 (nilpotent systems). Let $k \geq 1$ and let G be a countable abelian group. A quadruple $X = (\mathcal{G}/\Gamma, \mathcal{B}, \mu, (R_g)_{g\in G})$, where $\mathcal{G}$ is a k-step nilpotent Polish group, Γ is a closed cocompact zero-dimensional subgroup, and $\mathcal{B}$, μ and R_g as in Definition 2.2 is called a *k-step nilpotent system*. If in addition $\mathcal{G}$ is a finite-dimensional group, we say that X is a finite-dimensional k-step nilpotent system.

We note that any zero-dimensional subgroup of a Lie group is discrete. Therefore, if $\mathcal{G}$ is a Lie group, then the nilpotent system X is a nilsystem. Moreover, even though the notion of dimension in nonlocally compact groups may exhibit some pathologies, the quotient space $\mathcal{G}/\Gamma$ is compact and so it is of finite dimension with respect to any natural notion of dimension of compact topological spaces (e.g., Lebesgue covering dimension or the small or large inductive dimension).

2B. *The Host–Kra group.* The Host–Kra group plays an important role in this paper. We generalize the definition from [Host and Kra 2005, Section 5] to arbitrary countable abelian group G.

Definition 2.10. Let G be a countable abelian group and let (X, G) be a G-system. We denote by $\mathcal{G}(X)$ the group of all transformations $t : X \to X$ with the property that, for every $l > 0$, the measure $\mu^{[l]}$ is $t^{[l]}$-invariant and $t^{[l]}$ acts trivially on $\mathcal{I}^{[l]}(X)$.

The measure $\mu^{[l]}$, the transformation $t^{[l]} : X^{[l]} \to X^{[l]}$ and the σ-algebra $\mathcal{I}^{[l]}(X)$ are defined in Appendix A. We note that if X is a system of order $< k+1$ (i.e., $X = Z_{<k+1}(X)$), then $\mathcal{G}(X)$ is a k-step nilpotent locally compact Polish group [Host and Kra 2005, Corollary 5.9].

Host and Kra [2005] proved the following stronger version of Theorem 2.1.

Theorem 2.11. *Let $k \geq 0$. Let X be an ergodic $\mathbb{Z}$-system of order $< k+1$. Then, for every $n \in \mathbb{N}$ there exists a factor X_n of X such that*:

(1) *X_n is an increasing sequence (i.e., X_n is a factor of X_{n+1} for every n) and X is the inverse limit of X_n.*

(2) *For each n, X_n is isomorphic to the system $(\mathcal{G}(X_n)/\Gamma(X_n), \mathcal{B}_n, \mu_n, S_n)$ where the Host–Kra group $\mathcal{G}(X_n)$ is a k-step nilpotent locally compact Lie group, $\Gamma(X_n)$ is a discrete cocompact subgroup of $\mathcal{G}(X_n)$, $\mathcal{B}_n$ is the Borel σ-algebra and μ_n is the Haar measure. The action of S_n on $\mathcal{G}(X_n)/\Gamma(X_n)$ is given by left multiplication by an element in $\mathcal{G}(X_n)$.*

Our main result is the following counterpart of Theorem 2.11 for the group $G = \bigoplus_{p\in P} \mathbb{Z}/p\mathbb{Z}$.

Theorem 2.12 (structure theorem). *Let $k \geq 0$ and let X be an ergodic $\bigoplus_{p\in P} \mathbb{Z}/p\mathbb{Z}$-system of order $< k+1$. Then for some $m = O_k(1)$*[4] *there exists an m-extension Y of X that is an inverse limit of finite-dimensional k-step nilpotent systems. Moreover, for each $n \in \mathbb{N}$ there exists a factor Y_n of Y such that*:

(1) *Y_n is an increasing sequence and Y is the inverse limit of Y_n.*

(2) *For each n, Y_n is isomorphic to the system $(\mathcal{G}(Y_n)/\Gamma(Y_n), \mathcal{B}_n, \mu_n, (S_{n,g})_{g\in\bigoplus_{p\in P}\mathbb{Z}/p^m\mathbb{Z}})$, where the Host–Kra group $\mathcal{G}(Y_n)$ is a finite-dimensional k-step nilpotent group, $\Gamma(Y_n)$ is a zero-dimensional closed cocompact subgroup of $\mathcal{G}(Y_n)$, $\mathcal{B}_n$ is the Borel σ-algebra and μ_n is a left $\mathcal{G}(Y_n)$-invariant measure. For*

[4]We use $O_k(1)$ to denote a quantity which is bounded by a constant depending only on k.

every $g\in\bigoplus_{p\in P}\mathbb{Z}/p^m\mathbb{Z}$, the action of $S_{n,g}$ on $\mathcal{G}(Y_n)/\Gamma(Y_n)$ is given by left multiplication by an element in $\mathcal{G}(Y_n)$.

In particular, it follows that if X is a G-system where $G=\bigoplus_{p\in P}\mathbb{Z}/p\mathbb{Z}$, then for every $k\in\mathbb{N}$ there exists $m=O_k(1)$ and an m-extension $(Y,G^{(m)})$ such that the following diagram commutes:

$$\begin{array}{ccc} (X,G) & \xleftarrow{\quad\pi\quad} & (Y,G^{(m)}) \\ \Big\downarrow \pi_k^X & & \Big\downarrow \pi_k^Y \\ (Z_k(X),G) & \longleftarrow & (Z_k(Y),G^{(m)})\cong \varprojlim(\mathcal{G}_n/\Gamma_n,G^{(m)}) \end{array}$$

The case $k=2$ of Theorem 2.12 was established by the author in [Shalom 2023] without the use of extensions (see also Theorem 4.7). We do not know whether this result (without m-extensions) holds for higher values of k. In Section 2E we explain how m-extensions are used to overcome certain difficulties in the simple case where $k=3$.

2C. *Convergence of multiple ergodic averages and limit formula.* As an application of our structure theory, we derive an alternative proof for the convergence of some multiple ergodic averages and a limit formula in the special case where the underlying system is a nilpotent system and the homogeneous group is the Host–Kra group. More concretely, fix $k\geq 0$ and let (X,T_g) be an ergodic G-system where $G=\bigoplus_{p\in P}\mathbb{Z}/p\mathbb{Z}$ and $f_1,\dots,f_{k+1}\in L^\infty(X)$. We study the limit of the following multiple ergodic averages as N goes to infinity:

$$\mathbb{E}_{g\in\Phi_N}T_g f_1 T_{2g} f_2\cdot\dots\cdot T_{(k+1)g}f_{k+1}. \tag{2-1}$$

In the case of $\mathbb{Z}$-actions, Host and Kra [2005] and Ziegler [2007] proved the convergence of these averages by studying the universal characteristic factors. In the special case where X is a nilsystem, Ziegler [2005] proved a limit formula for average (2-1); see (2-2) below. A simpler proof of this result and some applications to multiple recurrence can be found in [Bergelson et al. 2005]. Bergelson, Tao and Ziegler [Bergelson et al. 2015] proved a variant of this formula for $\mathbb{F}_p^\omega$-systems from which they deduced a Khintchine-type recurrence for various configurations (see [Bergelson et al. 2015, Theorem 1.13] for more details). In the special case where $k=2$ this formula and the multiple recurrence results were generalized to other abelian groups in [Shalom 2022; Ackelsberg et al. 2021]. We note that the norm convergence of average (2-1) as $N\to\infty$ was proved by Walsh [2012] for any countable nilpotent group G. We give an alternative proof for this result in the special case where $G=\bigoplus_{p\in P}\mathbb{Z}/p\mathbb{Z}$.

Theorem 2.13 (convergence of the multiple ergodic averages). *Let (X,T_g) be an ergodic G-system. Let $k\geq 0$ be such that $k+1\leq\min_{p\in P}p$ and $f_1,\dots,f_{k+1}\in L^\infty(X)$. Then, the multiple ergodic average (2-1) converges in $L^2(X)$ as N goes to infinity.*

The properties of the Host–Kra group are needed in the proof of the following formula for the limit of average (2-1) in the special case where the underlying system X is a nilpotent system. In other words, it is important that the homogeneous groups in Theorem 2.12 are the Host–Kra groups of the system.

Let $\mathcal{G}$ be a k-step nilpotent group and $\mu_{\mathcal{G}/\Gamma}$ be a left $\mathcal{G}$-invariant measure on $\mathcal{G}/\Gamma$. Let $\mathcal{G}_1=\mathcal{G}$ and for every $2\le r\le k$ let $\mathcal{G}_r$ be the closed subgroup generated by all the r-commutators $[\dots[[x_1,x_2],x_3],\dots x_r]$, where $x_1,\dots x_r\in\mathcal{G}$. Let $\Gamma_r=\mathcal{G}_r\cap\Gamma$ and let m_r be a left $\mathcal{G}_r$-invariant measure on the quotient space $\mathcal{G}_r/\mathcal{G}_{r+1}\Gamma_r$. We have the following formula for the limit of average (2-1).

Theorem 2.14 (limit formula). *Let $m\ge 1$ and let $G=\bigoplus_{p\in P}\mathbb{Z}/p^m\mathbb{Z}$. Fix $k\ge 0$ with $k+1<\min_{p\in P}p$ and let $X=\mathcal{G}(X)/\Gamma$ be a k-step nilpotent G-system, where $\mathcal{G}(X)$ is the Host–Kra group of X. Then, for every $f_1,\dots,f_{k+1}\in L^\infty(X)$, every Følner sequence Φ_N of G and $\mu_{\mathcal{G}/\Gamma}$-almost every $x\in\mathcal{G}/\Gamma$ we have*

$$\lim_{N\to\infty}\mathbb{E}_{g\in\Phi_N}T_gf_1(x)T_{2g}f_2(x)\cdot\dots\cdot T_{(k+1)g}f_{k+1}(x)\\ =\int_{\mathcal{G}/\Gamma}\int_{\mathcal{G}_2/\Gamma_2}\cdots\int_{\mathcal{G}_k/\Gamma_k}\prod_{i=1}^{k+1}f_i(x\cdot y_1^i\cdot y_2^{\binom{i}{2}}\cdot\dots\cdot y_i^{\binom{i}{i}})\,d\prod_{i=1}^{k}m_i(y_i\Gamma_i),\tag{2-2}$$

with the abuse of notation $f(x)=f(x\Gamma)$.

2D. ***Discussion of the main steps in the proof of Theorem 2.12 and generalizations to other countable abelian groups.*** We summarize the main steps in the proof of Theorem 2.12. In each step we survey previous work concerning $\mathbb{Z}$-actions [Host and Kra 2005; Ziegler 2007] and $\mathbb{F}_p^\omega$-actions [Bergelson et al. 2010], describe the counterpart that we prove for $\bigoplus_{p\in P}\mathbb{Z}/p\mathbb{Z}$-actions and discuss the general case of G-actions, where G is any countable abelian group.

Let $m\ge 0$. We denote by $Z^1_{<m}(G,X,S^1)$ the group of cocycles of type $<m$ (Definition A.5) and by $PC_{<m}(G,X,S^1)=P_{<m}(G,X,S^1)\cap Z^1(G,X,S^1)$ the phase polynomial cocycles of degree $<m$. The first step in the proof of Theorem 2.12 is to study how *large* this subgroup is.

Step 1 (Theorem 3.1): Let X be an ergodic $\bigoplus_{p\in P}\mathbb{Z}/p\mathbb{Z}$-system. Then

$$PC_{<m}\Big(\bigoplus_{p\in P}\mathbb{Z}/p\mathbb{Z},X,S^1\Big)\cdot B^1\Big(\bigoplus_{p\in P}\mathbb{Z}/p\mathbb{Z},X,S^1\Big)$$

has at most countable index in $Z^1_{<m}\big(\bigoplus_{p\in P}\mathbb{Z}/p\mathbb{Z},X,S^1\big)$.

A well-known theorem of Moore and Schmidt states that, for any countable abelian group G and any ergodic G-system X,

$$Z^1_{<1}(G,X,S^1)=PC_{<1}(G,X,S^1)\cdot B^1(G,X,S^1).$$

In the special case where $G=\mathbb{F}_p^\omega$, this equality holds for higher values of m. Formally, Bergelson, Tao and Ziegler [Bergelson et al. 2010] proved that

$$Z^1_{<m}(\mathbb{F}_p^\omega,X,S^1)=PC_{<m}(\mathbb{F}_p^\omega,X,S^1)\cdot B^1(\mathbb{F}_p^\omega,X,S^1)$$

for every $m<p$ and an ergodic $\mathbb{F}_p^\omega$-system X.

This equality fails in general. For instance Furstenberg [1990] and Weiss proved that there exists an ergodic $\mathbb{Z}$-system X and a $\mathbb{Z}$-cocycle ρ of type <2 that is not cohomologous to a phase polynomial of any order. In previous work [Shalom 2023], we find an "if and only if" criterion for this equality to hold for $\bigoplus_{p\in P}\mathbb{Z}/p\mathbb{Z}$-systems and construct a counterpart of the Furstenberg and Weiss example in the case where

P is an unbounded multiset of primes. If (X, T) is a group rotation, Host and Kra [2005, Lemma 9.2] proved that $P_{<1}(\mathbb{Z}, X, S^1)\cdot B^1(\mathbb{Z}, X, S^1)$ is of countable index in $Z^1_{<2}(\mathbb{Z}, X, S^1)$ and mentioned that the counterpart for higher values of m fails. Instead, they proved the following result: let (X, T) be an ergodic $\mathbb{Z}$-system and let (Ω, P) be a probability space. Let $\omega\mapsto\rho_\omega$ be a measurable map into $Z^1_{<m}(\mathbb{Z}, X, S^1)$. Then there exists a set of positive measure $\mathcal{A}\subseteq\Omega$ such that $\rho_\omega/\rho_{\omega'}$ is cohomologous to a constant for every $\omega, \omega'\in A$.

We thus conjecture that the following general version holds.

Conjecture 2.15. *Let G be a countable abelian group and (X, G) be an ergodic G-system. Let $m\geq 1$ and (Ω, P) be a probability space. Then, for any measurable map $\omega\mapsto\rho_\omega$ from Ω to $Z^1_{<m}(G, X, S^1)$, there exists a set of positive measure $\mathcal{A}\subseteq\Omega$ such that $\rho_\omega/\rho_{\omega'}\in PC_{<m}(G, X, S^1)\cdot B^1(G, X, S^1)$.*

The next step in the proof of Theorem 2.12 is to reduce matters to the case where X is a *finite-dimensional system* (see Definition 4.1) using inverse limits. Recall (Proposition A.19) that every ergodic system X of order $<k$ is a tower of abelian extensions. Namely,

$$X = U_0\times_{\rho_1}U_1\times\cdots\times_{\rho_{k-1}}U_{k-1}.$$

We refer to the compact abelian groups $U_0, \ldots, U_{k-1}$ as the structure groups of the system X.

Step 2 (Theorem 4.3): Let $k\geq 1$. Then any ergodic $\bigoplus_{p\in P}\mathbb{Z}/p\mathbb{Z}$-system is an inverse limit of finite-dimensional systems (see Definition 4.1). Bergelson, Tao and Ziegler [Bergelson et al. 2010] proved that the structure groups of an ergodic $\mathbb{F}_p^\omega$-system of order $<k$ are totally disconnected (zero-dimensional). We refer to these systems as totally disconnected systems. The following definition is due to Host and Kra: a system X of order $<k$ is called *toral* if U_1 is a Lie group and any other structure group is a finite-dimensional torus. In the case of $\mathbb{Z}$-actions Host and Kra proved that X is an inverse limit of toral systems. In the generality of countable abelian groups, it is impossible to approximate every system with finite-dimensional systems. As a counterexample consider the action of the group $\mathbb{Z}^\omega$ on $(S^1)^\mathbb{N}$ by $R_n x = (\alpha^{n_1}x, \alpha^{n_2}x, \ldots)$, where $n = (n_1, n_2, \ldots)$. If α is irrational then the action is ergodic. Let $(e_1, e_2, \ldots)$ denote the natural basis for $\mathbb{Z}^\omega$ and $\rho:\mathbb{Z}^\omega\times(S^1)^\mathbb{N}\to S^1$ be the unique cocycle with $\rho(e_i, x) = x_i$. Then, the extension $(S^1)^\mathbb{N}\times_\rho S^1$ is an ergodic system of order <3 that is not an inverse limit of finite-dimensional systems.

Step 3: The last and most technically difficult step in the proof of Theorem 2.12 is solving the following lifting problem. Let $X = Z_{<k}(X)\times_\rho U$ be a finite-dimensional ergodic system of order $<k+1$. Using a proof by induction and passing to an extension, we may assume that $Z_{<k}(X) = \mathcal{G}/\Gamma$ is a nilpotent system. Let $\mathcal{G}_k\leq\mathcal{G}_{k-2}\leq\cdots\leq\mathcal{G}_2\leq\mathcal{G}_1 = \mathcal{G}$ be the lower central series for $\mathcal{G}$. We adapt an inductive argument of [Host and Kra 2005], where in step j we lift some elements from the group $\mathcal{G}_{k-j+1}$ to transformations on X which belongs to $\mathcal{G}(X)$. The following difficulties arise in the case where X is a finite-dimensional system that is not a toral system.

(1) The near-action defined by group generated by the connected component of $\mathcal{G}(X)$ and $\{T_g : g\in G\}$ may not be transitive on X.

(2) The cocycles in the process may take values in a compact abelian group U which is not necessarily a torus. In particular, it is difficult to apply the results from step (1).

To deal with these difficulties we use extensions and k-extensions (see Definition 2.5). For instance, in order to overcome the second difficulty, we prove the following result (Lemma 5.11): Let $G=\bigoplus_{p\in P}\mathbb{Z}/p\mathbb{Z}$, and $\rho:G\times X\to U$ be a cocycle into some finite-dimensional group U. If $\chi\circ\rho\in PC_{<k}(G,X,S^1)\cdot B^1(G,X,S^1)$ for every $\chi\in\widehat{U}$, then there exists some $m=O_k(1)$ and an m-extension $\pi:Y\to X$ with the property that $\rho\circ\pi\in PC_{<k}(G^{(m)},Y,U)\cdot B^1(G^{(m)},Y,U)$. To deal with the first difficulty we also need extensions, but of different type. Roughly speaking, we show that for every countable set of transformations of bounded torsion in $\mathcal{G}(X)$, there exists an extension Y where these transformations have a lift in $\mathcal{G}(Y)$. (See Lemma 7.6, and Section 7 for more details). We note that the passing to extensions of X leads to other difficulties which we discuss in detail in Section 6.

The reduction to the case where X is a finite-dimensional system is only necessary to ensure that U is finite-dimensional and Y is an m-extension. This is no longer necessary when working in the generality of all countable abelian groups. Therefore, by following the arguments in this paper and assuming that Conjecture 2.15 holds we can prove a structure theorem for all countable abelian groups.

Theorem 2.16 (structure theorem). *Let G be a countable abelian group and $k\geq 1$. Let X be an ergodic G-system of order $<k+1$ and assume that Conjecture 2.15 holds. Then X is a factor of a k-step nilpotent system (Y,H), where H is a countable abelian group extending G and the homogeneous group $\mathcal{G}(Y)$ is a k-step nilpotent Polish group.*

Since any countable abelian group is a quotient of $\mathbb{Z}^\omega$, one can always take $H=\mathbb{Z}^\omega$ in the theorem above. The case $k=2$ of this theorem was established in [Shalom 2022]. The proof of the theorem above follows by arguing as in Sections 6 and 7 together with the counterpart of Lemma 5.11 and Theorem 5.9 for abelian groups that is given in [Shalom 2022, Proposition 3.8 and Theorem 3.14].

2E. ***A simple case of Theorem 2.12 and k-extensions.*** We emphasize the part of the proof where we used k-extensions. For the sake of the example, let X be an arbitrary G-system of order <4. By Proposition A.19, there exist compact abelian groups Z, U and W and cocycles σ and ρ such that

$$X=Z\times_\sigma U\times_\rho W.$$

In Section 4 we reduce matters to the case where X is a finite-dimensional system (see Definition 4.1) using inverse limits. Let us therefore assume that Z and U are finite-dimensional groups and W is a Lie group. Since W is a Lie group, modifying the arguments of Host and Kra one can show that X is (a factor of) a 3-step nilpotent system with respect to its Host–Kra group if and only if $Y=Z_{<3}(X)$ is (a factor of) a 2-step nilpotent system with respect to its Host–Kra group. In that case the Host–Kra group takes the convenient form

$$\mathcal{G}(Y)=\{S_{s,F}:s\in Z,\ F\in\mathcal{M}(Z,U)\text{ such that there exists }c\in\mathrm{Hom}(G,U)\text{ with }\Delta_s\sigma(g,x)=c(g)\cdot\Delta_g F(x)\},$$

where $S_{s,F}:Y\to Y$ is the transformation $S_{s,F}(z,u)=(sz,F(z)u)$.

If U is a torus, then for every $s\in Z$ there exists a character $c_s:G\to U$ and a measurable map $F_s:Z\to U$ such that $\Delta_s\sigma=c_s\cdot\Delta F_s$ (see Lemma 3.2). Equivalently, this means that the transformation $s\in Z$ has a lift S_{s,F_s} in $\mathcal{G}(Y)$. If this holds for every $s\in Z$ then the action of $\mathcal{G}(Y)$ on Y is transitive and Y is a nilpotent system. Below we discuss the case where U is a finite-dimensional group that is not a Lie group. (If U is a Lie group then it is a direct product of a torus and a finite group. The case where U is finite is covered in [Shalom 2023].)

Observe that if $\chi:U\to S^1$ is a character, then by the torus case there exist $c_{s,\chi}\in\mathrm{Hom}(G,S^1)$ and $F_{s,\chi}\in\mathcal{M}(Z,S^1)$ such that

$$\Delta_s\chi\circ\sigma(g,x)=c_{s,\chi}(g)\cdot\Delta_g F_{s,\chi}(x). \tag{2-3}$$

By Pontryagin duality, s has a lift in $\mathcal{G}(Y)$ if and only if there is a choice of $F_{s,\chi}$ and $c_{s,\chi}$ for which (2-3) holds and $\chi\mapsto F_{s,\chi}$ is a homomorphism. Equivalently, for every $s\in Z$ we can consider the map $k_s:\widehat{U}\times\widehat{U}\to P_{<2}(Z,S^1)$, where

$$k_s(\chi,\chi')=\frac{F_{s,\chi\cdot\chi'}}{F_{s,\chi}\cdot F_{s,\chi'}}.$$

The map k_s defines an abelian multiplication on the Cartesian product $\widehat{U}\times P_{<2}(Z,S^1)$ by $(\chi,p)\cdot(\chi',p')=(\chi\chi',k_s(\chi,\chi')pp')$. We denote this group by $U\times_{k_s}P_{<2}(Z,S^1)$ and observe that the short exact sequence

$$1\to P_{<2}(Z,S^1)\to\widehat{U}\times_{k_s}P_{<2}(Z,S^1)\to\widehat{U}\to 1 \tag{2-4}$$

splits if and only if s has a lift in $\mathcal{G}(Y)$.

Since U is not a torus, $\widehat{U}$ is not a projective object in the category of discrete abelian groups. Moreover, it is not necessary that $P_{<2}(Z,S^1)$ is a divisible group (an injective object). In other words, other properties of k_s must be used in order to prove that this extension always splits. Instead, we chose a different method which involves k-extensions.

Let $n=\dim U$. A careful analysis of the finite-dimensional group U shows that we can find a multiset of primes P and vectors $v_p\in\mathbb{Z}^n$ for every $p\in P$ such that $\widehat{U}\cong\mathbb{Z}^n[(1/p)\cdot v_p]$. It is easy to find a cross-section from the subgroup $\mathbb{Z}^n$ to $\widehat{U}\times_{k_s}P_{<2}(X,S^1)$ and so it is left to find p-th roots for certain phase polynomials in $P_{<2}(X,S^1)$. We do not know whether or not such roots exist and therefore we use extensions. In Section 5 we prove, roughly speaking, that by extending Y to a 2-extension Y' we can find a p-th root for any phase polynomial of degree <2 on Z in $P_{<2}(Z_{<2}(Y'),S^1)$ (see Theorem 5.5 or Theorem 5.9). This means that by passing to an extension and replacing $P_{<2}(Z,S^1)$ with $P_{<2}(Z_{<2}(Y'),S^1)$, the short exact sequence (2-4) splits and we can lift s to $\mathcal{G}(Y')$. Then, since we passed from Y to Y', we need to make sure that we can also lift all of the new transformations which arise from the extension $Z_{<2}(Y')\to Z$. A formal proof is given in Sections 6 and 7.

3. Conze–Lesigne-type equation

Throughout, fix a multiset of primes P. Let $m\geq 0$, let G be a countable abelian group and denote by $Z^1_{<m}(G,X,S^1)$ the group of (G,X,S^1)-cocycles of type $<m$ and by $PC_{<m}(G,X,S^1)$ the phase

polynomial cocycles of degree $< m$. It follows by Lemma A.15 that

$$PC_{<m}(G, X, S^1) \cdot B^1(G, X, S^1) \leq Z^1_{<m}(G, X, S^1). \tag{3-1}$$

The following theorem is the main result in this section.

Theorem 3.1. *Let X be an ergodic $\bigoplus_{p\in P} \mathbb{Z}/p\mathbb{Z}$-system. Then, for every $m \geq 0$, the subgroup*

$$PC_{<m}\Big(\bigoplus_{p\in P} \mathbb{Z}/p\mathbb{Z}, X, S^1\Big) \cdot B^1\Big(\bigoplus_{p\in P} \mathbb{Z}/p\mathbb{Z}, X, S^1\Big)$$

is of at most countable index in $Z^1_{<m}\big(\bigoplus_{p\in P} \mathbb{Z}/p\mathbb{Z}, X, S^1\big)$.

We recall some relevant results from previous work. In the case where $m = 1$, we have the following theorem of [Moore and Schmidt 1980].

Lemma 3.2 (cocycles of type < 1 are cohomologous to constants). *Let G be a countable abelian group, and let X be an ergodic G-system. Suppose that $\rho : G \times X \to S^1$ is a cocycle of type < 1. Then, there exists a character $c : G \to S^1$ such that ρ is (G, X, S^1)-cohomologous to c. Equivalently, $Z^1_{<1}(G, X, S^1) = PC_{<1}(G, X, S^1) \cdot B^1(G, X, S^1)$.*

The following result [Host and Kra 2005, Corollary 7.9] allow us to reduce matters to the case where X is of order $< m$.

Proposition 3.3. *Let G be a countable abelian group and X an ergodic G-system. If $m \geq 0$ and $\rho : G \times X \to U$ is a cocycle of type $< m$ into some compact abelian group U, then ρ is (G, X, U)-cohomologous to a cocycle $\rho' : G \times X \to U$ that is measurable with respect to $Z_{<m+1}(X)$.*

From this and Theorem A.21 we conclude the following result.

Theorem 3.4. *Let $m \geq 0$ and P be a multiset of primes. If X is an ergodic totally disconnected $\bigoplus_{p\in P} \mathbb{Z}/p\mathbb{Z}$-system* (see Definition A.20) *then every cocycle $\rho : \bigoplus_{p\in P} \mathbb{Z}/p\mathbb{Z} \times X \to S^1$ of type $< m$ is $\big(\bigoplus_{p\in P} \mathbb{Z}/p\mathbb{Z}, X, S^1\big)$-cohomologous to a phase polynomial of degree $< d$ for some $d = O_m(1)$.*

If hypothetically we knew that the quantity d in the theorem equals to m, then this would imply Theorem 3.1 for a totally disconnected system X. In fact, in this case

$$PC_{<m}\Big(\bigoplus_{p\in P} \mathbb{Z}/p\mathbb{Z}, X, S^1\Big) \cdot B^1\Big(\bigoplus_{p\in P} \mathbb{Z}/p\mathbb{Z}, X, S^1\Big) = Z^1_{<m}\Big(\bigoplus_{p\in P} \mathbb{Z}/p\mathbb{Z}, X, S^1\Big).$$

In order to deal with the fact that d is potentially higher than m we need the following generalization of Lemma A.17.

Lemma 3.5. *Let X be an ergodic G-system. Let $d \geq m \geq 0$ and $p : G \times X \to S^1$ be a phase polynomial of degree $< d$ and type $< m$. Write $d^{[m]}p = \Delta Q$, and let $r = d - m$. If $\|Q - 1\|_{L^2(X^{[m]}, \mu^{[m]})} < \sqrt{2}/2^{r+d-1}$ then p is a phase polynomial of degree $< m$.*

Proof. If $m = 0$ then the claim follows by Lemma A.17. Assume that $m \geq 1$. We have:

Claim. *Let $P : X^{[m]} \to S^1$ be a phase polynomial of degree $< r$ and suppose that*

$$\|P-1\|_{L^2(X^{[m]},\mu^{[m]})} < \sqrt{2}/2^{r-1}. \tag{3-2}$$

Then P is invariant with respect to the diagonal action of G on $X^{[m]}$ if and only if, for every $(m-1)$-face α and $g \in G$, $\Delta_{g_\alpha^{[m]}} P$ is invariant with respect to that action.

Proof. Let $g \in G$ and α be an $(m-1)$-face. Since $g_\alpha^{[m]}$ is measure-preserving (Lemma A.3) and commutes with $h^{[m]}$, the first direction follows. We prove the other direction. By the ergodic decomposition theorem there exists a probability measure (Ω_m, P_m) such that

$$\mu^{[m]} = \int_{\Omega_m} \mu_\omega \, dP_m(\omega). \tag{3-3}$$

Let α be an $(m-1)$-dimensional face. By Lemma A.4, the transformation $g_\alpha^{[m]}$ defines an isomorphism of ergodic components and an action on (Ω_m, P_m). Moreover, by the same lemma, the action generated by these transformations for every $g \in G$ and $(m-1)$-dimensional face α is ergodic. Let

$$A = \{\omega \in \Omega_m : \Delta P = 1 \ \mu_\omega\text{-a.e.}\}.$$

Since $\Delta_{g_\alpha^{[m]}} P$ is invariant, we have that $\omega \in A$ if and only if $g_\alpha^{[m]} w \in A$. In other words, A is invariant. On the other hand, from (3-3) and (3-2) we conclude that the set

$$B := \{\omega \in \Omega_m : \|P-1\|_{L^2(\mu_\omega)} < \sqrt{2}/2^{r-1}\}$$

is of positive measure with respect to P_m. Since μ_ω is ergodic it follows by Lemma A.17 that $B \subseteq A$ and therefore by ergodicity (Lemma A.4) A is of measure 1. This proves the claim. □

We return to the proof of the lemma. For any $g_1, \ldots, g_d \in G$ and $(m-1)$-dimensional faces, $\alpha_1, \ldots, \alpha_d$ we have

$$d_\alpha^{[m]} \Delta_{g_1} \cdots \Delta_{g_d} p = \Delta \Delta_{g_{1\alpha_1}^{[m]}} \cdots \Delta_{g_{d\alpha_d}^{[m]}} Q,$$

where α is the intersection of $\alpha_1, \ldots, \alpha_d$. Since p is of degree $< d$ we conclude that $\Delta_{g_{1\alpha_1}^{[m]}} \cdots \Delta_{g_{d\alpha_d}^{[m]}} Q$ is invariant with respect to the diagonal action of G. The claim above imply that if d-derivatives of Q are invariant and these derivatives are sufficiently close to 1, then only $d-1$ derivatives of Q are invariant. Repeating this claim iteratively d times, we deduce that when Q sufficiently small (as in the lemma), it is invariant with respect to that action. Since $d^{[m]} p = \Delta Q = 1$, Lemma A.15 implies that p is of degree $< m$ as required. □

We have the following reduction of Theorem 3.1.

Lemma 3.6. *Let $G = \bigoplus_{p\in P} \mathbb{Z}/p\mathbb{Z}$. In order to prove Theorem 3.1 it is enough to show that for some $d > m$*

$$PC_{<d}(G, X, S^1) \cdot B^1(G, X, S^1) \cap Z^1_{<m}(G, X, S^1) \le Z^1_{<m}(G, X, S^1)$$

is of at most countable index.

Proof. Let $\mathcal{W} \subseteq L^2(X^{[m]})$ denote the space of phase polynomials of degree $< d-m+1$ in $X^{[m]}$. Since $L^2(X^{[m]}, \mu^{[m]})$ is separable, we can decompose $\mathcal{W}$ into a countable union of balls $\{B_i\}_{i\in\mathbb{N}}$ of diameter $< \sqrt{2}/2^{2d-m+1}$. For each ball, we choose (if it exists) a cocycle $\rho_i \in P_{<d}(G, X, S^1) \cdot B^1(G, X, S^1)$

such that $\rho_i = p_i \cdot \Delta F_i$ for a measurable map $F_i : X \to S^1$ and a phase polynomial cocycle $p_i \in P_{<d}(G, X, S^1)$ which satisfies that $d^{[m]} p_i = \Delta Q_i$, where $Q_i \in B_i$. We conclude that, for every $\rho \in PC_{<d}(G, X, S^1) \cdot B^1(G, X, S^1)$, there exists i such that $\rho/\rho_i \in PC_{<m}(G, X, S^1) \cdot B^1(G, X, S^1)$. Indeed, write $\rho = p \cdot \Delta F$ for some $p \in PC_{<d}(G, X, S^1)$. Since p is cohomologous to ρ, it is of type $< m$. Therefore, we can write $d^{[m]} p = \Delta Q$ for some phase polynomial $Q : X^{[m]} \to S^1$ of degree $< d - m + 1$ (by Lemma A.15). We conclude that $Q \in \mathcal{W}$ and there exists i such that $Q \in B_i$. Let ρ_i as above. We conclude that $\rho/\rho_i = p/p_i \cdot \Delta F/F_i$ and $d^{[m]} p/p_i = \Delta Q/Q_i$. Since

$$\|Q/Q_i - 1\|_{L^2(\mu^{[m]})} = \|Q - Q_i\|_{L^2(\mu^{[m]})} < \sqrt{2}/2^{2d-m+1},$$

Lemma 3.5 implies that p/p_i is of degree $< m$. It follows that $PC_{<m}(G, X, S^1) \cdot B^1(G, X, S^1)$ is of countable index in $PC_{<d}(G, X, S^1) \cdot B^1(G, X, S^1) \cap Z^1_{<m}(G, X, S^1)$. By the assumption, the latter is of at most countable index in $Z^1_{<m}(G, X, S^1)$. We conclude that $PC_{<m}(G, X, S^1) \cdot B^1(G, X, S^1)$ is of countable index in $Z^1_{<m}(G, X, S^1)$, as required. □

The main difficulty in the proof of Theorem 3.1 is therefore to reduce matters to the case where X is totally disconnected. Before we turn to the proof of Theorem 3.1, we prove some corollaries. Since we prove Theorem 3.1 by induction on the order of X, we will be able to use these corollaries for systems of smaller order.

3A. *Corollaries.* Throughout and unless specified otherwise $G = \bigoplus_{p \in P} \mathbb{Z}/p\mathbb{Z}$. We begin with the following counterpart of [Host and Kra 2005, Lemma 9.2].

Theorem 3.7. *Let $m \geq 0$ be a natural number and X be an ergodic G-system. Let (Ω, P) be a probability space and $\omega \mapsto \rho_\omega$ be a measurable map from Ω into $Z^1_{<m}(G, X, S^1)$. Then, there exists a set of positive measure $\mathcal{A} \subset \Omega$ such that $\rho_\omega/\rho_{\omega'} \in PC_{<m}(G, X, S^1) \cdot B^1(G, X, S^1)$ for every $\omega, \omega' \in \mathcal{A}$.*

We need some notation: An analytic subset of a measurable space X is the continuous image of a Polish space in X. The Lusin separation theorem [Kechris 1995, Theorem 14.7] implies that if A and $X \backslash A$ are analytic, then A is Borel measurable.

Proof. Observe that since the analytic set $PC_{<m}(G, X, S^1) \cdot B^1(G, X, S^1)$ is of at most countable index in $Z^1_{<m}(G, X, S^1)$. Then by the separation theorem it is Borel. Therefore the map

$$\omega \mapsto \rho_\omega \cdot PC_{<m}(G, X, S^1) \cdot B^1(G, X, S^1)$$

is a measurable map into a countable set. We conclude that there exists a measurable set $\mathcal{B} \subseteq \Omega$ of positive measure such that for every $\omega, \omega' \in \mathcal{B}$, ρ_ω and $\rho_{\omega'}$ belong to the same coset. □

As a corollary we have the following result.

Theorem 3.8. *Let $m \geq 0$, X be an ergodic G-system and U be a compact abelian group which acts on X by automorphisms. Let $\rho : G \times X \to S^1$ be a cocycle and suppose that, for every $u \in U$, $\Delta_u \rho$ is of type $< m$. Then there exists an open subgroup $U' \leq U$ such that $\Delta_u \rho \in PC_{<m}(G, X, S^1) \cdot B^1(G, X, S^1)$ for all $u \in U'$.*

Proof. From the cocycle identity it is easy to see that

$$U' = \{u \in U : \Delta_u \rho \in PC_{<m}(G, X, S^1) \cdot B^1(G, X, S^1)\}$$

is a subgroup of U. We use Theorem 3.7 with $\Omega = U$ and $\rho_u = \Delta_u \rho$. We see that there exists a set of positive measure $\mathcal{A} \subseteq U$ such that $\Delta_u \rho/\Delta_{u'} \rho \in PC_{<m}(G, X, S^1) \cdot B^1(G, X, S^1)$ for all $u, u' \in \mathcal{A}$. A direct computation shows that $\Delta_{uu'^{-1}} \rho = V_{u'^{-1}} \Delta_u \rho/\Delta_{u'} \rho$ for every $u, u' \in U$. Since U commutes with G, we conclude that $\mathcal{A}\cdot\mathcal{A}^{-1} \subseteq U'$. Therefore, by Lemma B.1, U' is an open subgroup and the proof is complete. □

Given a compact abelian group U and an integer m we define $U^m := \{u^m : u \in U\}$. Observe that the subgroup U' in the previous theorem depends on the cocycle ρ. In the next lemma we compute this group for a root of ρ.

Lemma 3.9. *Let X be an ergodic G-system. Let U be a compact abelian group which acts on X by automorphisms and let $n, m, d \in \mathbb{N}$. If $\rho : G \times X \to S^1$ is a cocycle of type $< m$ and $\Delta_u \rho^n \in PC_{<d}(G, X, S^1) \cdot B^1(G, X, S^1)$ for every $u \in U$, then $\Delta_u \rho \in PC_{<d}(G, X, S^1) \cdot B^1(G, X, S^1)$ for every $u \in U^{nm}$.*

Proof. The claim follows immediately by induction on m and from the identity

$$\Delta_{u^n} \tilde{\rho} = \Delta_u \tilde{\rho}^n \cdot \prod_{k=0}^{n-1} \Delta_u \Delta_{u^k} \tilde{\rho}$$

We have $\Delta_u \tilde{\rho}^n \in PC_{<d}(G, X, S^1) \cdot B^1(G, X, S^1)$ and, for every $1 \le k \le n$, $\Delta_u \Delta_{u^k} \tilde{\rho}$ is of smaller type. □

3B. ***The proof of Theorem 3.1.*** We briefly explain the method in the proof. We prove the claim by induction on m: The case $m = 0$ is trivial, and the case $m = 1$ follows from Lemma 3.2. Fix $m \ge 2$ and assume inductively that the theorem holds for smaller values of m. Let $\rho : G \times X \to S^1$ be a cocycle of type $< m$. By Proposition 3.3 we can assume without loss of generality that X is of order $< m + 1$. By Proposition A.19 we can find compact abelian groups $U_0, U_1, U_2, \ldots, U_m$ where U_0 is trivial and

$$X = U_0 \times_{\rho_1} U_1 \times \cdots \times_{\rho_m} U_m.$$

We construct a sequence of factors

$$X = X_m \to X_{m-1} \to X_{m-2} \to \cdots \to X_0,$$

where in each step we quotient out the connected component of the identity in the next structure group. The last factor, X_0 is a totally disconnected system.

Observe that the factor maps define a sequence of injections

$$Z^1_{<m}(G, X_0, S^1) \hookrightarrow Z^1_{<m}(G, X_1, S^1) \hookrightarrow \cdots \hookrightarrow Z^1_{<m}(G, X, S^1).$$

Adapting the arguments of Host and Kra, we show that the image of each of these embeddings is of at most countable index in the next group. Then we apply Theorem A.21 to the system X_0 and complete the proof.

One difficulty which arise in this process is that the connected component of the identity of the structure groups may not act on X by automorphisms. For this reason we study under which conditions we can lift an automorphism from $Z_{<k}(X)$ to $Z_{<k+1}(X)$ for every $1 \le k \le m$. We have:

Lemma 3.10 (going up). *Let X be an ergodic G-systems. Let U be a compact abelian group and $Y = X \times_\rho U$ be an extension of X by a cocycle $\rho : G \times X \to U$. Let A be a connected compact abelian group of automorphisms of X and suppose that for every $a \in A$, $\Delta_a \rho \in B^1(G, X, U)$. Then, there exists a compact connected abelian group of automorphisms $\tilde{A}$ of Y such that the induced action of $\tilde{A}$ on X coincides with the action of A.*

Proof. Let X, ρ, A as above. For every $a \in A$ and a measurable map $F : X \to U$ we define a measure-preserving transformation $S_{a,F}$ on $X \times U$ by $S_{a,F}(x, u) := (ax, F(x)u)$. Direct computation shows that the group

$$\mathcal{K} := \{S_{a,F} : \Delta_a \rho = \Delta F\}$$

acts on $X \times U$ by automorphisms. Indeed,

$$S_{a,F} T_g^Y(x, u) = (a \cdot T_g^X x, F(T_g x)\rho(g, x)u) = (T_g^X ax, F(x)\rho(g, ax)u) = T_g^Y S_{a,F}(x, u).$$

Equipped with the topology of convergence in probability $\mathcal{K}$ is a Polish group with respect to the multiplication $S_{a,F} \circ S_{a',F'} = S_{aa',F'V_{a'}F}$ (see, e.g., [Host and Kra 2005, Appendix A]). By the assumption the projection map $p : \mathcal{K} \to A$ is onto. Moreover, by ergodicity we see that the kernel of p is isomorphic to U. In particular, it follows from Corollary B.4 that $\mathcal{K}$ is a compact Polish group. Finally, since A is abelian, direct computation reveals that $\mathcal{K}$ is 2-step nilpotent. We let $\tilde{A}$ be the connected component of $\mathcal{K}$. By all of the above and Proposition B.15 we deduce that $\tilde{A}$ is a compact connected abelian group. Since p is open (Theorem B.3) it maps the connected group $\tilde{A}$ onto A. □

Given a system X and a group A acting freely on X. We define the quotient space X/A to be the space of all equivalent classes $[x] := \{ax : a \in A\}$ with the quotient σ-algebra. We let the measure $\mu_{X/A}$ be the push-forward of μ_X under the factor map $\pi : X \to X/A$, $\pi(x) = [x]$. Finally, if $gAg^{-1} \subseteq A$ for every $g \in G$ then the action of G on X/A by $T_g[x] = [T_g x]$ is well-defined.

Lemma 3.11 (going down). *Let $Y = X \times_\rho U$ be an ergodic abelian extension of a G-system X by a compact abelian group. Let A be a compact connected abelian group of automorphisms of X and suppose that A acts freely on X and $\Delta_a \rho \in B^1(G, X, S^1)$ for every $a \in A$. Then $\tilde{A}$ from the previous lemma acts freely on Y and the factor $Y' = Y/\tilde{A}$ is an extension of $X' := X/A$ by some quotient of U.*

Proof. Let Y' as in the lemma. The G-action on Y' is given by $g[y] = [gy]$, where $[y] = \{ay : a \in \tilde{A}\}$. This action is well-defined since the action of $\tilde{A}$ on Y commutes with the action of G. Let $\mathcal{K}$ be as in the proof of the lemma above, and $p : \mathcal{K} \to A$ the projection map. As before we can identify U with the kernel of $p : \mathcal{K} \to A$. We show that $Y' \cong X' \times_\sigma U/V$ for some cocycle σ, where $V = \tilde{A} \cap U$.

Let $\bar{\rho} : G \times X \to U/V$ be the composition of ρ with the projection map $U \to U/V$ and consider the extension $X \times_{\bar{\rho}} U/V$. For every $a \in A$ choose a measurable cross-section $a \mapsto F_a$ such that $S_{a,F_a} \in \tilde{A}$. Since

$$\Delta_a \rho = \Delta F_a, \tag{3-4}$$

the cocycle identity implies that $F_{aa'}/(F_a V_a F'_a)$ is a constant. Since $S_{1,F_{aa'}/(F_a V_a F'_a)} \in \tilde{A}$, we conclude that $F_{aa'}/(F_a V_a F'_a) \in V$. Now, let $\bar{F}_a$ be the projection of F_a to U/V. It follows that $\bar{F}_{aa'} = \bar{F}_a V_a \bar{F}_{a'}$. Since A acts freely on X, we can write $X = X_0 \times A$ measurably. Choose some generic point $a_0 \in A$

and set $F(x, aa_0) := \overline{F}_a(x, a_0)$. A direct computation reveals that $\Delta_a F = \overline{F}_a$. From (3-4) we conclude that $\bar{\rho}/\Delta F$ is invariant to A. Let $\sigma : G \times X' \to U/V$ be the push-forward of $\bar{\rho}/\Delta F$ and let $\widetilde{F}$ be any measurable lift of F to a map into U. Then, for every $x \in X$ we have that $\widetilde{F}(ax)/\widetilde{F}(x) \cdot F_a(x) \in V$. This implies that $\pi([x]_A, uV) = [(x, \widetilde{F}(x)^{-1}u)]_{\tilde{A}}$ is a well-defined isomorphism from $X' \times_\sigma U/V$ to Y'. Since A acts freely on X, we can write $X \cong X' \times A$ and therefore $Y \cong X' \times A \times U/V \times V$. By identifying $A \times V$ (measurably) with $\tilde{A}$ we see that $\tilde{A}$ acts freely on Y, as required. □

Next we modify the argument from [Host and Kra 2005, Proposition 8.9] for connected groups in the context of $\bigoplus_{p\in P}\mathbb{Z}/p\mathbb{Z}$-actions.

Lemma 3.12. *Let X be a G-system of order $< m+1$ for some $m \geq 0$ and let U be a connected compact abelian group which acts freely on X by automorphisms. We abuse notation and identify $Z^1_{<m}(G, X/U, S^1)$ with a subgroup of $Z^1_{<m}(G, X, S^1)$. Then, $Z^1_{<m}(G, X/U, S^1) \cdot B^1(G, X, S^1)$ is of at most countable index in $Z^1_{<m}(G, X, S^1)$.*

Proof. Equip $\mathcal{M}(X, S^1)$ with the L^2 topology. For every cocycle $\rho \in Z^1_{<m}(G, X, S^1)$ we consider the group

$$\mathcal{H}_\rho := \{S_{u,F} : u \in U,\ F \in \mathcal{M}(X, S^1),\ \text{there exists } p \in P_{<m}(G, X, S^1) \text{ such that } \Delta_u \rho = p_u \cdot \Delta F\}.$$

Equipped with the topology of convergence in probability $\mathcal{H}_\rho$ is a Polish group and we have a short exact sequence

$$1 \to P_{<m+1}(X, S^1) \to \mathcal{H}_\rho \to U \to 1.$$

By Corollary B.4, $\mathcal{H}_\rho$ is locally compact. Moreover, since U is connected, $\mathcal{H}_\rho$ is 2-step nilpotent. To see this observe that if $S_{u,F}, S_{u',F'} \in \mathcal{H}_\rho$, then $[S_{u,F}, S_{u',F'}] = S_{1,\Delta_{u'}F/\Delta_u F'}$. Since phase polynomials cocycles are invariant with respect to translations by connected groups (Proposition C.5), we conclude that

$$\Delta \frac{\Delta_u F'_u}{\Delta_{u'} F_u} = \frac{\Delta_u \Delta_{u'} \rho}{\Delta_{u'} \Delta_u \rho} = 1.$$

Therefore by ergodicity $\Delta_u F'_u/\Delta_{u'} F_u$ is a constant and $S_{1,\Delta_{u'}F/\Delta_u F'}$ is in the center of $\mathcal{H}_\rho$.

Let $\mathcal{M}(X, S^1)$ denote the space of all measurable maps on X with values in S^1, equipped with the topology of convergence in probability. Let $\mathcal{F}$ be the group of all continuous maps from U to $\mathcal{M}(X, S^1)/P_{<m+1}(X, S^1)$, where $\mathcal{M}(X, S^1)/P_{<m+1}(X, S^1)$ is equipped with the quotient metric (i.e., $d(f, g) = \inf_{p\in P_{<m+1}(X,S^1)} d_{\mathcal{M}(X,S^1)}(f - p, g)$). Equipped with the supremum metric, $\mathcal{F}$ is a Polish group. We define a map $\Phi_\rho \in \mathcal{F}$ by giving each $u \in U$ the equivalence class of F_u in $\mathcal{M}(X, S^1)/P_{<m+1}(X, S^1)$. If Φ_ρ is sufficiently small (say $\|\Phi_\rho\|_{\mathcal{F}} < 1/20^m$) we show that we can linearize the term $u \mapsto F_u$:

For such ρ we can define a subset $\mathcal{K} \leq \mathcal{H}_\rho$ by

$$\mathcal{K} := \left\{ S_{s,F} : \text{ there exists } c \in S^1 \text{ such that } |F - c| \leq \frac{1}{10^m} \right\}.$$

A direct computation shows that $\mathcal{K}$ is a closed subgroup of $\mathcal{H}_\rho$ (see [Host and Kra 2005, Proposition 8.9] for the details). Observe that since $\|\Phi_\rho\|_{\mathcal{F}} < 1/20^m$ the projection $p_{\mathcal{K}} : \mathcal{K} \to U$ is onto. Since U is connected and p is open (Theorem B.3), the same holds if we restrict ourselves to the connected component of the identity $\mathcal{K}_0$ of $\mathcal{K}$. Since $\mathcal{K}_0$ is 2-step nilpotent and connected, it is abelian (Proposition B.15) and so it splits as $\mathcal{K}_0 \cong S^1 \times U$. In other words, for every $u \in U$ we can find F_u such that $(u, F_u) \in \mathcal{K}_0$ and $F_{uv} = F_u V_u F_v$.

Since the group U acts freely on X we can write $X = Y \times U$. Fix any generic point $u_0 \in U$ and define $F(y, uu_0) := F_u(y, u_0)$ for all $y \in Y, u \in U$. It follows that

$$\Delta_u(\rho/\Delta F) = p'_u \tag{3-5}$$

for some phase polynomial $p'_u \in P_{<m}(G, X, S^1)$.

The phase polynomial term p'_u is in fact trivial. To see this notice that $u \mapsto p'_u$ is a cocycle. Since U is connected, by Proposition C.5 $u \mapsto p'_u$ is a homomorphism. Since there are only countably many polynomials up to constants and U is connected, we conclude that p'_u is a constant in x. Finally, since p'_u is a (G, X, S^1)-cocycle and a constant in x, it can be identified with an element in $\widehat{G}$. Therefore, $u \mapsto p'_u$ is a continuous homomorphism from U to $\widehat{G}$, and hence trivial. We conclude from (3-5) that $\Delta_u(\rho/\Delta F)$ is a coboundary for every $u \in U$. By Lemma A.24, ρ is (G, X, S^1)-cohomologous to a function that is invariant under U.

Now, since $\mathcal{F}$ is separable we can decompose $\mathcal{F}$ as a union of countably many balls $\{B_i\}_{i\in\mathbb{N}}$ of diameter $< 1/20^m$. For each ball B_i choose (if exists) a cocycle ρ_i of type $< m$ such that $\Phi_{\rho_i} \in B_i$. We conclude that if ρ is an arbitrary cocycle of type $< m$, then there exists ρ_i such that $\|\phi_{\rho/\rho_i}\|_{\mathcal{F}} < 1/20^m$. Therefore ρ/ρ_i is cohomologous to a cocycle which is invariant to the action of U. By Lemma A.9 the push-forward of ρ/ρ_i to X/U is a cocycle of type $< m$. □

It is left to prove Theorem 3.1.

Proof. We already considered the cases $m = 0$ and $m = 1$ above. Fix $m \geq 2$ and let X be as in the theorem. By Proposition 3.3 we can assume that the G-system X is of order $< m + 1$. Therefore, by Proposition A.19, X can be written as $X = U_0 \times_{\rho_1} U_1 \times \cdots \times_{\rho_m} U_m$ for some compact abelian groups $U_0, \dots, U_m$ and cocycles $\rho_1, \dots, \rho_m$. Let $l = l(X)$ denote the smallest number for which $U_{l+1}, \dots, U_m$ are totally disconnected. We prove the claim by induction on l. If $l = 0$ then X is a totally disconnected system. In this case the proof follows by Theorem A.21 and Lemma 3.6. Fix $l \geq 1$ and suppose that the claim holds for all smaller values of l. Let $U_{l,0}$ be the connected component of the identity of U_l and recall that $U_{l,0}$ acts freely on $Z_{<l+1}(X)$ by vertical rotations. In particular, if $l = m$ then $U_{l,0}$ acts by automorphisms on X. Otherwise suppose that $l < m$. In this case we lift $U_{l,0}$ to a group of automorphism of X using Lemma 3.10. We argue as follows: Let $\chi \in \widehat{U}_{l+1}$, using the induction hypothesis we know that the claim in Theorem 3.1 holds for $Z_{<l}(X)$ and so we can apply Theorem 3.8. We conclude that there exists a phase polynomial $p_u \in P_{<k}(G, X, S^1)$ (in fact we can take p_u of degree < 1) and a measurable map $F_u : X \to S^1$ such that

$$\Delta_u \chi \circ \rho_{l+1} = p_u \cdot \Delta F_u \tag{3-6}$$

for every $u \in U_{l,0}$. By assumption, U_{l+1} is totally disconnected and therefore there exists some $n \in \mathbb{N}$ such that $\chi^n = 1$ (Corollary B.8). Let $u \in U_{l,0}$, the cocycle identity gives that

$$\Delta_{u^n} \chi \circ \rho_{l+1} = (\Delta_u \chi \circ \rho_{l+1})^n \cdot \prod_{k=0}^{n-1} \Delta_{u^k} \Delta_u \chi \circ \rho.$$

Since $\chi^n = 1$, the first term in the right-hand side of the equation above vanishes. By Proposition C.5 and (3-6) the other term (the product) is a coboundary. Therefore, we see that p_{u^n} is a (G, X, S^1)-coboundary for every $u \in U_{l,0}$. Since connected groups are divisible this implies that p_u is a coboundary for every $u \in U_{l,0}$. From this and (3-6) we see that $\chi(\Delta_u \rho_{l+1})$ is a coboundary for every $u \in U_{l,0}$. As χ was arbitrary Theorem A.6 implies that $\Delta_u \rho_{l+1}$ is a $(G, Z_{<l+1}(X), U_{l+1})$-coboundary.

Therefore we are in a situation as in Lemma 3.10 and so we can lift $U_{l,0}$ to a group of automorphisms on $Z_{<l+2}(X)$. Repeating this argument iteratively we conclude that we can lift $U_{l,0}$ to a compact abelian connected group of automorphisms on $X = Z_{<m+1}(X)$. We denote this group by $\mathcal{H}_l$ and let $X' = X/\mathcal{H}_l$. Now, by applying Lemma 3.11 iteratively we see that $\mathcal{H}_l$ acts freely on X and $l(X') = l-1$. Therefore, by Lemma 3.12 we have that $Z^1_{<m}(G, X', S^1)\cdot B^1(G, X, S^1)$ is of at most countable index in $Z^1_{<m}(G, X, S^1)$. Moreover, by the induction hypothesis $PC_{<m}(G, X', S^1)\cdot B^1(G, X', S^1)$ is of at most countable index in $Z^1_{<m}(G, X', S^1)$. We conclude that $PC_{<m}(G, X', S^1)\cdot B^1(G, X, S^1)$ is of at most countable index in $Z^1_{<m}(G, X, S^1)$ (we identify $PC_{<m}(G, X', S^1)$ with a subgroup of $PC_{<m}(G, X, S^1)$ using the factor map). Finally, by Proposition C.5 phase polynomial cocycles are invariant with respect to the action of connected groups and therefore $PC_{<m}(G, X', S^1) = PC_{<m}(G, X, S^1)$ and the claim follows. □

4. Inverse limit of finite-dimensional systems

We begin with the following definition of a finite-dimensional system. The main result in this section (Theorem 4.3) asserts that every ergodic G-system of order $< k$ is an inverse limit of these systems.

Definition 4.1 (finite-dimensional system). Let $k \geq 1$. An ergodic G-system X of order $< k$ is called a finite-dimensional system if for every $1 \leq r \leq k-1$ the system $Z_{<r+1}(X)$ is an extension of $Z_{<r}(X)$ by a finite-dimensional compact abelian group.

Note that by Proposition A.19 this means that we can write $X = U_0 \times_{\rho_1} U_1 \times \cdots \times_{\rho_{k-1}} U_{k-1}$, where $U_0, U_1, \ldots, U_{k-1}$ are finite-dimensional compact abelian groups.

We are particularly interested in finite-dimensional systems which also have a finite *exponent*.

Definition 4.2 (exponent of a finite-dimensional system). Let $m \geq 0$.

- A totally disconnected group Δ is said to be of exponent m if for any prime p, the p-sylow subgroup of Δ is a p^m-torsion group. Equivalently, by Theorem B.9, $\Delta \cong \prod_{p\in I} C_{p^{d_p}}$ for some multiset of primes I and $d_p \leq m$.
- We say that a compact abelian finite-dimensional group U is of exponent m if there exists a closed totally disconnected subgroup Δ of exponent m such that U/Δ is a Lie group.
- A finite-dimensional system X is of exponent m if the structure groups are of exponent m.

We prove that every ergodic G-system of order $< k$ is an inverse limit of finite-dimensional systems of some bounded exponent.

Theorem 4.3 (systems of order $< k$ are inverse limits of finite-dimensional systems). *Let X be an ergodic G-system of order $< k$. There exists $m = O_k(1)$ and a sequence X_n of increasing factors of X such that, for each $n \in \mathbb{N}$, X_n is a finite-dimensional system of exponent m and X is the inverse limit of the sequence X_n.*

Let $X = Z_{<k-1}(X) \times_\rho U$ be an ergodic G-system of order $< k$. Since every compact abelian group is an inverse limit of compact abelian Lie groups (Theorem B.14), we can assume that U is a torus times a finite group (Theorem B.13). We note that in general replacing a structure group of X with one of its quotients will not necessarily be a factor of X and therefore this approximation is only possible for the last structure group.

In the next lemma we study cocycles with values in a Lie group. By taking coordinates it is enough to study cocycles into the torus and into a finite cyclic group.

Lemma 4.4. *Let X be an ergodic G-system of order $< k$. Suppose that U is a compact abelian group which acts freely on X by automorphisms. Let H be either S^1 or C_{p^n} for some prime p and a natural number n and let $\rho : G \times X \to H$ be a cocycle of type $< m$. Then there exists a subgroup $V \leq U$ such that U/V is a finite-dimensional group of exponent m and ρ is (G, X, H)-cohomologous to a cocycle that is invariant with respect to the action of V.*

Proof. If $m = 0$ then ρ is a coboundary and the claim follows. We assume that $m \geq 1$. By embedding H in S^1 (if necessary) and applying Theorem 3.8 we see that there exists an open subgroup $U' \leq U$ such that for every $u \in U$ we have

$$\Delta_u \rho = p_u \cdot \Delta F_u$$

for some phase polynomial $p_u \in P_{<m}(G, X, S^1)$ and a measurable map F_u.

Using Lemma A.26 and then Theorem B.14 we can find a closed subgroup $J \leq U'$ such that U'/J and U/J are Lie groups and $p_{jj'} = p_j V_j p_{j'} = p_j p_{j'} \cdot \Delta_j p_{j'}$ for every $j, j' \in J$. Since $\Delta_j p_{j'}$ is a phase polynomial of degree $< m-1$, we conclude that the map $j \mapsto p_j \cdot P_{<m-1}(G, X, S^1)$ from J to the quotient $P_{<m}(G, X, S^1)/P_{<m-1}(G, X, S^1)$ is a homomorphism. Write $G = \bigoplus_{p \in \mathcal{P}} \mathbb{Z}/p\mathbb{Z}$. For every prime $q \in \mathcal{P}$, we denote by G_q be the q-component of G (i.e., $G_q = \{g \in G : qg = 0\}$). By Lemma C.2 we know that for every $g \in G_q$ and for all $j \in J$ we have that $p_j(g, \cdot)^q = p_{j^q}(g, \cdot)$ is phase polynomial cocycle of degree $< m-1$. Inductively (see the proof of Corollary C.4), we have that $p_{j^{q^m}}(g, \cdot) = 1$. Let $J_q := J^{q^m}$ and $J' = \bigcap_{q \in \mathcal{P}} J_q$. The quotient J/J' is a totally disconnected group of exponent m and we have that $\Delta_j \rho \in B^1(G, X, S^1)$ for every $j \in J'$. Therefore, by Lemma A.24, ρ is (G, X, S^1)-cohomologous to a cocycle ρ' that is invariant with respect to the action of some open subgroup $J'' \leq J'$. To complete the proof we notice that since $J'' \leq J'$ is an open subgroup and J/J' is a totally disconnected group of exponent m then the groups U/J' and U/J'' are finite-dimensional of exponent m. Thus, if $H = S^1$ we can take $V = J''$ and the proof is complete. Otherwise suppose that $H = C_{p^n}$. By embedding H in S^1 and arguing as before we can find a measurable map $F : X \to S^1$ such that

$$\Delta_j \rho = \Delta \Delta_j F$$

for all $j \in J''$. Our goal is to replace F with a function which takes values in C_{p^n}. Since $\rho^{p^n} = 1$ the equation above implies that $\Delta_j F^{p^n}$ is a constant. Since $j \mapsto \Delta_j F^{p^n}$ is a cocycle we conclude that there exists a character $\chi : J'' \to S^1$ such that $\Delta_j F^{p^n} = \chi(j)$ for every $j \in J''$. Let $V := \ker(\chi)$, since J'' is totally disconnected Corollary B.8 implies that the image of χ is discrete. In particular, it follows that V is an open subgroup of J'' and therefore U/V is a finite-dimensional group of exponent m. Finally as F^{p^n} is invariant under V we can find a measurable map $\widetilde{F}: X\to S^1$ that is invariant under V with $\widetilde{F}^{p^n}=F^{p^n}$. Since $\widetilde{F}$ is invariant to V, we conclude that

$$\Delta_j\rho=\Delta_j\Delta F/\widetilde{F}$$

for every $j\in V$. Moreover, since $F/\widetilde{F}^{p^n}=1$, $\rho/\Delta(F/\widetilde{F})$ is (G,X,C_{p^n})-cohomologous to ρ and is invariant under V, as required. □

From the lemma above we conclude the following result.

Proposition 4.5. *Let X be an ergodic G-system and let H be a compact abelian group which acts on X by automorphisms. Let $m,l\geq 0$ be integers, let U be a finite-dimensional group exponent m and let $\rho: G\times X\to U$ be a cocycle of type $<l$. Then, there exists a subgroup $\widetilde{H}\leq H$ such that for every $h\in\widetilde{H}$, $\Delta_h\rho\in B^1(G,X,U)$ and $H/\widetilde{H}$ is finite-dimensional of exponent $<m\cdot l$.*

Proof. Let U be as in the proposition and find a closed totally disconnected subgroup $\Delta\leq U$ of exponent m such that U/Δ is a Lie group. By Theorem B.13 we can write $U/\Delta=(S^1)^n\times\prod_{i=1}^r C_{p_i^{a_i}}$ where $r\in\mathbb{N}$, $p_1,\dots,p_r$ are primes and $a_1,\dots,a_r\in\mathbb{N}$. Let $\chi_1,\dots,\chi_n,\tau_1,\dots,\tau_r$ be the coordinate maps and lift each of them to a character of U.

By Lemma 4.4, we can find a subgroup H' of H such that H/H' is of exponent l and $\Delta_h\chi\circ\rho$ is a coboundary for all $h\in H'$ and $\chi\in\langle\chi_1,\dots,\chi_n,\tau_1,\dots,\tau_r\rangle$.

The group Δ is of exponent m and so we can write it as $\prod_{i\in I}C_{p_i^{b_i}}$, where $b_i\leq m$ (see Definition 4.2). Let $\{\pi_i: i\in I\}$ denote the coordinates of Δ and lift each of them arbitrarily to a character of $\widehat{U}$. Then the countable set $\chi_1,\dots,\chi_n,\tau_1,\dots,\tau_r,\pi_1,\pi_2,\dots$ generates $\widehat{U}$.

Fix a coordinate π of Δ of order p^b. Then π^{p^b} is invariant under Δ and therefore is a character of U/Δ. In particular, $\Delta_h\pi\circ\rho^{p^b}$ is a coboundary for all $h\in H'$. We conclude by Lemma 3.9 that $\Delta_{h^{p^b\cdot l}}\rho\in B^1(G,X,S^1)$ for every $h\in H'$. For every $i\in I$, let $H_{p_i}=H^{p_i^{b_i}\cdot l}$ and $\widetilde{H}:=\bigcap_{i\in I}H_{p_i}$. It follows that the quotient $H/\widetilde{H}$ is a totally disconnected group of exponent $m\cdot l$. Since $\Delta_h\chi\circ\rho$ is a coboundary for every $h\in\widetilde{H}$ and $\chi\in\widehat{U}$, the claim follows by Theorem A.6. □

We also need the following technical group theoretical lemma.

Lemma 4.6. *Let H and K be compact abelian groups and suppose that K is finite-dimensional of exponent m for some $m\in\mathbb{N}$. We give* $\mathrm{Hom}(H,K)$ *the topology of uniform convergence. Then, for any continuous homomorphism $\varphi: H\to\mathrm{Hom}(H,K)$, we have that the group $H/\ker\varphi$ admits a totally disconnected open subgroup of exponent m. Moreover, if the group K is totally disconnected of exponent m, then $H/\ker\varphi$ is totally disconnected of exponent m.*

Proof. Let Δ be a totally disconnected subgroup of K of exponent m such that K/Δ is a Lie group. Let $\tilde{\varphi}: H\to\mathrm{Hom}(H,K/\Delta)$ be the composition of $\tilde{\varphi}$ with the projection $\mathrm{Hom}(H,K)\to\mathrm{Hom}(H,K/\Delta)$. Since K/Δ is embedded in a finite-dimensional torus, we conclude that $\mathrm{Hom}(H,K/\Delta)$ is discrete. It follows that $\ker\tilde{\varphi}$ is an open subgroup of H. We denote by $\widetilde{H}$ the kernel of $\tilde{\varphi}$ and by φ' the restriction of φ to $\widetilde{H}$. Then the map $\varphi':\widetilde{H}\to\mathrm{Hom}(H,K)$ takes values in $\mathrm{Hom}(H,\Delta)$. We prove that $\widetilde{H}/\ker\varphi'$ is totally disconnected of exponent m. First recall that Δ can written as $\Delta=\prod_{i\in I}C_{p_i^{b_i}}$ for some $b_i\leq m$

and a countable set of indices I. For each prime p let $\pi_p : \mathrm{Hom}(H, \Delta) \to \mathrm{Hom}(H, \Delta_p)$ be the projection map, where Δ_p is the p-sylow subgroup of Δ. Clearly, the group $H_p := \widetilde{H}^{p^m}$ is in the kernel of $\pi_p \circ \varphi$. Let H' be the intersection of all H_p over all primes. We see that $H' \leq \ker \varphi'$ and $\widetilde{H}/H'$ is a totally disconnected group of exponent m from which we conclude that $\widetilde{H}/\ker \varphi'$ is totally disconnected of exponent m. Since $\widetilde{H}$ is open in H, we have that $\widetilde{H}/\ker \varphi'$ is open in $H/\ker \varphi$. In the case where K is totally disconnected we get that $\widetilde{H} = H$ and so the claim follows from the same argument. □

We can now prove Theorem 4.3.

Proof. Let X be a G-system of order $< k$. and write $X = Z_{<k-1}(X) \times_\rho U$. Let U_n be a sequence of Lie groups such that $U = \varprojlim U_n$.

The idea is to construct a sequence of k factors of $Z_{<k-1}(X) \times_{\rho_n} U_n$ each time replacing one of the structure groups of $Z_{<k-1}(X)$ with a finite-dimensional group. More concretely, we construct systems $X_{l,n}$ recursively as follows: let $X_{k,n} := X_n$. Fix $l \leq k$ and suppose inductively that we have already constructed

$$X_{l,n} = U_0 \times_{\rho_1} U_1 \times \cdots \times_{\rho_{l-1}} U_{l-1} \times_{\rho_{l,n}} U_{l,n} \times \cdots \times_{\rho_{k-1,n}} U_{k-1,n},$$

where $U_{l,n}, \ldots, U_{k-1,n}$ are finite-dimensional quotients of $U_l, \ldots, U_{k-1}$ respectively of exponent $m = O_k(1)$ and that $X_{l,n}$ is a factor of $X_{l,n+1}$ for every $n \in \mathbb{N}$. To construct $X_{l-1,n}$, we use a similar argument as in Lemma 3.10 to lift a subgroup of the vertical rotations by $U_{l-1,n}$ to automorphisms of $X_{l,n}$. We argue as follows: the group U_{l-1} acts on $Z_{<l}(X)$ by automorphisms. Therefore, by Proposition 4.5 we can find a subgroup $\widetilde{U}_{l-1,n}$ such that $\Delta_u \rho_{l,n} \in B^1(G, Z_{<l}(X), U_{l,n})$ for every $u \in \widetilde{U}_{l-1,n}$. We consider the group

$$\mathcal{H}_{l-1,n} := \{S_{u,F} : u \in U_{l-1,n},\, F \in \mathcal{M}(Z_{<l}(X), U_{l,n}),\, \Delta_u \rho_{l,n} = \Delta F\}.$$

Since $\widetilde{U}_{l-1,n}$ is abelian, $\mathcal{H}_{l-1,n}$ is a 2-step nilpotent group. Let $p : \mathcal{H}_{l-1,n} \to U_{l-1,n}$ be the projection map. The kernel of p consists of transformations of the form $S_{1,c}$, where c is a constant in $U_{l,n}$. We can therefore identify $\ker p$ with the compact group $U_{l,n}$. By Theorem B.3 and Corollary B.4 we conclude that $\mathcal{H}_{l-1}$ is compact and $\mathcal{H}_{l-1,n}/U_{l,n} \cong \widetilde{U}_{l-1,n}$. This implies that the commutator map on $\mathcal{H}_{l-1,n}$ induces a bilinear map $b : \widetilde{U}_{l-1,n} \times \widetilde{U}_{l-1,n} \to U_{l,n}$. Using Pontryagin duality, we see that b can be identified with a continuous homomorphism $\widetilde{U}_{l-1,n} \to \mathrm{Hom}(\widetilde{U}_{l-1,n}, U_{l,n})$. Since $U_{l,n}$ is finite-dimensional of exponent $m = O_k(1)$ we conclude from the previous lemma that the kernel is a subgroup $U'_{l-1,n} \leq \widetilde{U}_{l-1,n}$ such that the quotient $\widetilde{U}_{l-1,n}/U'_{l-1,n}$ admits a totally disconnected group of exponent $m' = O_k(1)$ as an open subgroup. By shrinking $U'_{l-1,n}$ if necessary we can assume that $\widetilde{U}_{l-1,n}/U'_{l-1,n}$ is increasing in n. The preimage of $U'_{l-1,n}$ under the projection p is a compact abelian finite-dimensional group and it acts by automorphisms on $Z_{<l+1}(X_{l,n})$. We repeat this process inductively another $k - l$ steps each time lifting a subgroup of $U_{l-1,n}$ to a group of automorphisms of the next universal characteristic factor of $X_{l,n}$. At the end of the day we obtain a compact abelian finite-dimensional group of exponent $m'' = O_k(1)$, $\widetilde{H}_{l-1,n}$ which acts by automorphisms on $X_{l,n}$. Let $\tilde{p} : \widetilde{H}_{l-1,n} \to U_{l-1,n}$ be the projection map, we see that $U_{l-1,n}/\tilde{p}(\widetilde{H}_{l-1,n})$ is a finite-dimensional group of exponent $m'' = O_k(1)$. Now in the last step we can use the fact that $U_{k-1,n}$ is a Lie group. In that step we invoke Lemma 4.4 instead of Proposition 4.5. We conclude that $\rho_{k-1,n}$ is $(G, X, U_{l,n})$-cohomologous to a cocycle that is invariant under the action of $\widetilde{\mathcal{H}}_{l-1,n}$.

Let $X_{l-1,n}=X_{l,n}/\widetilde{\mathcal{H}}_{l-1,n}$ (as in Lemma 3.11). Clearly, $X_{l-1,n}$ is a factor of $X_{l,n}$ and the $(l-1)$-th structure group of $X_{l-1,n}$ is finite-dimensional of exponent $m''=O_k(1)$. Since in every step in the proof we extended the structure groups of $X_{l-1,n}$ to exceeds those of $X_{l-1,n-1}$ we have that $X_{l-1,n}$ is increasing in n. □

4A. *Proof of Theorem 2.12 for systems of order < 3.* The proof of Theorem 2.12 for systems of order <3 is significantly easier than the general case.

Theorem 4.7. *Let $G=\bigoplus_{p\in P}\mathbb{Z}/p\mathbb{Z}$, and X be an ergodic G-system of order <3. Then X is an inverse limit of finite-dimensional 2-step nilpotent systems.*

A version of this theorem without the finite-dimensional result was given in [Shalom 2023]. For the sake of completeness we repeat the proof here. Recall that a system of order <3 takes the form $X=Z\times_\rho U$, where U and Z are compact abelian groups and Z is the Kronecker factor. By Theorem 4.3 we may assume that Z is finite-dimensional and U is a Lie group.

Definition 4.8 (Host and Kra group for systems of order <3). Let X be a system of order <3. Let $s\in Z$ and $F:Z\to U$ be a measurable map. We denote by $S_{s,F}$ the measure-preserving transformation $(z,u)\mapsto(sz,F(z)u)$, and let $\mathcal{G}(X)$ denote the group of all such transformations with the property that there exists $c_s:G\to U$ such that $\Delta_s\rho=c_s\cdot\Delta F_s$.

Host and Kra [2005] proved that this definition of $\mathcal{G}(X)$ coincides with Definition 2.10 for systems of order <3. In particular, this means that $\mathcal{G}(X)$ is a 2-step nilpotent Polish group.

Observe that the kernel of the projection $p:\mathcal{G}(X)\to Z$ can be identified with $P_{<2}(Z,U)$. We claim that in order to prove Theorem 4.7 it is enough to show that p is onto. Indeed, in that case, p is an open map (Theorem B.3) and the group $\mathcal{G}(X)$ acts transitively on X. Moreover, if Z is finite-dimensional and U is a Lie group then $P_{<2}(Z,U)$ is finite-dimensional. Since the projection $p:\mathcal{G}(X)\to Z$ is onto, this implies that $\mathcal{G}(X)$ is also a finite-dimensional group. At this point we would want to use Theorem B.5, but unfortunately we only have a near-action of $\mathcal{G}(X)$ on X. Yet, the identification $X\cong\mathcal{G}(X)/\Gamma$ for $\Gamma=\{S_{1,\chi}:\chi\in\mathrm{Hom}(Z,U)\}$ was obtained in [Meiri 1990, Theorem 3.21]. This completes the proof of the claim. We note that for the higher order case we will use a different argument (see Section 8).

Proof of Theorem 4.7. The projection $p:\mathcal{G}(X)\to Z$ is onto if and only if for every $s\in Z$ there exist a measurable map $F_s:Z\to U$ and a homomorphism $c_s:G\to U$ such that $\Delta_s\rho=c_s\cdot\Delta F_s$. Since U is a Lie group, by studying each coordinate separately it is enough to show that the equation holds in the case where U is a torus or equals to C_{p^n} for some prime p and $n\in\mathbb{N}$. By Lemma A.12 the cocycle $\Delta_s\rho$ is of type <1; therefore if U is a torus the equation follows by Lemma 3.2. Otherwise, assume that $U=C_{p^n}$. By embedding C_{p^n} in S^1 and applying Lemma 3.2, we see that, for every $s\in Z$,

$$\Delta_s\rho=c_s\cdot\Delta F_s \tag{4-1}$$

for some constant $c_s:G\to S^1$ and $F_s:Z\to S^1$.

Our goal is to replace F_s and c_s with some F_s' and c_s' such that (4-1) holds and F_s', c_s' takes values in C_{p^n}.

As a first step, we show that ρ is (G, Z, S^1)-cohomologous to a phase polynomial of degree < 2. Observe that by the cocycle identity we have

$$\Delta_{s^{p^n}} \rho = \Delta_s \rho^{p^n} \cdot \prod_{k=0}^{p^n-1} \Delta_s \Delta_{s^k} \rho.$$

From (4-1) we see that $\prod_{k=0}^{p^n-1} \Delta_s \Delta_{s^k} \rho$ is a coboundary. Since ρ takes values in C_{p^n}, the term $\Delta_s \rho^{p^n}$ vanishes and we conclude that $\Delta_{s^{p^n}} \rho$ is a coboundary for every $s \in Z$. Let Z_0 be the connected component of the identity in Z. Since connected groups are divisible (Lemma B.11), we conclude that $\Delta_s \rho$ is a (G, Z, S^1)-coboundary for every $s \in Z_0$. By Lemma A.24, ρ is (G, Z, S^1)-cohomologous to a cocycle ρ' that is invariant with respect to the action of Z_0. Let $\pi_\star \rho'$ be the push-forward of ρ' to Z/Z_0. By Lemma A.9, $\pi_\star \rho'$ is of type < 2. Therefore, by Theorem A.21 it is cohomologous to a phase polynomial of degree < 2.[5] Lifting everything back to Z we conclude that ρ' and ρ are (G, Z, S^1)-cohomologous to a phase polynomial $Q : G \times Z \to S^1$ of degree < 2. Moreover, Q is invariant to translations by Z_0. We write

$$\rho = Q \cdot \Delta F \tag{4-2}$$

for some $F : Z \to S^1$.

Since ρ takes values in C_{p^n}, we have

$$1 = Q^{p^n} \cdot \Delta F^{p^n}. \tag{4-3}$$

By taking the derivative of both sides of the equation above by $s \in Z$, we conclude that $\Delta_s F^{p^n}$ is a phase polynomial of degree < 2. Our next goal is to replace F with a function F' such that F'/F is a phase polynomial of degree < 3 and at the same time $\Delta_s F'^{p^n}$ is a constant.

We study the phase polynomial Q. It is a fact that every phase polynomial of degree < 2 is a constant multiple of a homomorphism. Therefore, we can write $Q(g, x) = c(g) \cdot q(g, x)$, where $c : G \to S^1$ is a constant and $q : G \times Z \to S^1$ is a homomorphism in the Z-coordinate. Since Q is a cocycle,

$$c(g + g')q(g + g', x) = c(g)c(g')\Delta_{g'} q(g, x) \cdot q(g, x) \cdot q(g', x).$$

It follows that q is bilinear in g and x. Let

$$Z'_p = \ker(q^{p^n}) = \{s \in Z : q(g, s)^{p^n} = 1 \text{ for every } g \in G\}.$$

Then Z/Z'_p is isomorphic to a subgroup of $\widehat{G}^{p^n} = \prod_{p \neq q \in \mathcal{P}} C_q$. By taking the derivative of both sides of (4-3) by $s \in Z'_p$, we conclude by the ergodicity of the Kronecker factor that $\Delta_s F^{p^n}$ is a constant. Recall that F^{p^n} is a phase polynomial. Therefore, by Corollary C.4 and the above we see that there exists an open subgroup $Z' \leq Z$, which contains Z'_p, such that $\Delta_s F^{p^n}$ is a constant for every $s \in Z'$. By the cocycle identity we conclude that $\Delta_s F^{p^n} = \chi(s)$ for some character $\chi : Z' \to S^1$. Lift χ to a character of Z arbitrarily. We conclude that F^{p^n}/χ is a phase polynomial which is invariant under translations by Z'. Since Z' is open, the quotient Z/Z' is a finite group. Moreover, since Z' contains Z'_p, we conclude that the order of Z/Z' is coprime to p. Let $m = |Z/Z'|$. Since F^{p^n}/χ is of degree < 3, we conclude that

[5] This result requires a slightly stronger version of Theorem A.21. If $l < \min_{p \in \mathcal{P}} p$ then one can replace the quantity $O_{k,m,l}(1)$ in Theorem A.21 with l. A proof for this can be found in [Shalom 2023].

$\Delta F^{p^n}/\chi$ is a constant multiple of a homomorphism from Z to S^1. This homomorphism is invariant to Z' and therefore $\Delta(F^{p^n}/\chi)^m$ is a constant. This implies that $(F^{p^n}/\chi)^m$ is a polynomial of degree <2 and by the same argument we conclude that $(F^{p^n}/\chi)^{m^2}$ is a constant. Let $c\in S^1$ be an m-th root of that constant, we conclude that $F^{p^n}/c\cdot\chi$ take values in C_{m^2}. Since p and m are coprime, we can find an integer l such that $l\cdot p^n=1 \mod m^2$. We conclude that $R:=(F^{p^n}/c\cdot\chi)^l$ is a phase polynomial of degree <3 and that $R^{p^n}=F^{p^n}/c\cdot\chi$. Let $Q':=Q\cdot\Delta R$ and $F':=F/R$. Then, as in (4-2), we have

$$\rho=Q'\cdot\Delta F'$$

and $\Delta_s F'^{p^n}=c\cdot\chi(s)$.

Now, by taking the derivative by $s\in Z$ on both sides of the equation above, we conclude that

$$\Delta_s\rho=\Delta_s Q'\cdot\Delta\Delta_s F'.$$

Observe that $c'_s:=\Delta_s Q'$ is a character of G and

$$c_s'^{p^n}=\Delta_s Q^{p^n}\cdot\Delta\Delta_s F^{p^n}/c\cdot\chi=1,$$

where the last equality follows from (4-3) and the fact that $\Delta\Delta_s c\cdot\chi$ vanishes. It is left to change the term $\Delta_s F$. Set $F'_s:=\Delta_s F'/\phi(s)$, where $\phi(s)$ is a p^n-th root of $\chi(s)$ in S^1. Then, as before we have that

$$\Delta_s\rho=c'_s\cdot\Delta F'_s,$$

but this time $c_s'^{p^n}=F_s'^{p^n}=1$. This implies that $S_{s,F'_s}\in\mathcal{G}(X)$ and therefore p is onto. □

We note that this argument fails for systems of higher order. The main reason for this is that Theorem 4.3 only allows to approximate the last structure group by Lie groups. In particular, we do not know how to prove a counterpart of this result in the case where U is not a Lie group without the use of k-extensions.

5. Extension trick

Let X be an ergodic G-system. Under certain conditions we show that there exist an extension $\pi:Y\to X$ with the following property: for any prime p, a natural number $n\in\mathbb{N}$ and a phase polynomial $P:X\to S^1$, there exists a phase polynomial $Q:Y\to S^1$ such that $Q^{p^n}=P\circ\pi$.

We begin with an example which illustrates the idea.

Example 5.1. Let $X=S^1$ and $G=\bigoplus_{p\in P}\mathbb{Z}/p\mathbb{Z}$, where P is an infinite subset of odd prime numbers. We define a homomorphism $\varphi:G\to X$ by

$$\varphi((g_p)_{p\in P})=\prod_{p\in P} w_p^{g_p},$$

where ω_p is the first p-th root of unity. The action of G on X by $T_g x=\varphi(g)x$ is ergodic, and the identity map $\chi:X\to S^1$ is a phase polynomial of degree <2. Our goal is to construct an extension $\pi:Y\to X$ and a phase polynomial $Q:Y\to S^1$ of degree <2 such that $Q^2=\chi\circ\pi$.

We let $c(g):=\Delta_g\chi$ and observe that since $2\notin P$, there exists some $d\in\widehat{G}$ such that $d^2=c$. Fix any measurable map $F:X\to S^1$ with $F^2=\chi$ and let $\tau=d\cdot\Delta\overline{F}$. The cocycle $\tau:G\times X\to C_2$ defines an extension $Y := X \times_\tau C_2$. On Y we have that d is an eigenvalue of the eigenfunction $Q(x,u) := u \cdot F(x)$. Moreover, $Q^2 = \chi \circ \pi$.

Since X is ergodic and τ is minimal (see Lemma 5.4), we conclude that Y is ergodic. Moreover τ is of type < 1 and therefore Y is of order < 2. In fact, it is easy to see that $(x,u) \mapsto u \cdot \overline{F}(x)$ defines an isomorphism of Y and (S^1, G), where the action of G on S^1 is given by $T_g y = d(g)y$.

Let $G = \bigoplus_{p\in P} \mathbb{Z}/p\mathbb{Z}$ as usual. We state a simple technical proposition about G-cocycles.[6]

Proposition 5.2 (conditions for cocycle). *Let X be a G-system and $f : G \times X \to S^1$ be a function such that*

$$\Delta_h f(g,x) = \Delta_g f(h,x) \tag{5-1}$$

for all $h, g \in G$. Let $E = \{e_1, e_2, \dots\}$ denote the natural basis of G and suppose that, for every $g \in E$, we have

$$\prod_{k=0}^{\mathrm{order}(g)-1} T_g^k f(g,x) = 1. \tag{5-2}$$

Then the function $\tilde{f} : G \times X \to S^1$ below is a cocycle which agrees with f for every $g \in E$:

$$\tilde{f}(g,x) := \prod_{i=1}^{\infty} T_{g_1} \cdots T_{g_{i-1}} \prod_{k=0}^{g_i - 1} T_{ke_i} f(e_i, x), \tag{5-3}$$

where $g = (\bar{g}_1, \bar{g}_2, \dots)$, the constants $g_1, g_2, \dots$ are any representatives of $\bar{g}_1, \bar{g}_2, \dots$ in $\mathbb{N}$ respectively and the product $\prod_{k=0}^{-1}(\text{anything})$ is assumed to be 1.

Convention. We refer to an element $g \in E$ as a generator.

We need the following results about group extensions.

Definition 5.3 (image and minimal cocycles). Let X be a G-system and $\rho : G \times X \to U$ be a cocycle into a compact abelian group U. The subgroup $U_\rho \le U$ generated by $\{\rho(g,x) : g \in G, x \in X\}$ is called the *image* of ρ. We say that ρ is *minimal* if it is not (G, X, U)-cohomologous to a cocycle σ with $U_\sigma \lneqq U_\rho$.

Lemma 5.4 [Zimmer 1976, Corollary 3.8]. *Let X be an ergodic G-system. Let $\rho : G \times X \to U$ be a cocycle into a compact abelian group. Then*:

- *There exists a minimal cocycle $\sigma : G \times X \to U$ such that ρ is (G, X, U)-cohomologous to σ.*
- *$X \times_\rho U$ is ergodic if and only if X is ergodic and ρ is minimal with image $U_\rho = U$.*

We describe an obstacle. Suppose that $P \in P_{<2}(X, S^1)$. Then ΔP can be identified with an element in $\widehat{G}$. If for some $g \in G$ of order p we have that $\Delta_g P \ne 1$ then ΔP does not have a p-th root in $\widehat{G}$. In particular, it is impossible to find a p-th root for P in $P_{<2}(X, S^1)$, even if one passes to an extension of the original system. We deal with this problem later using k-extensions (Definition 2.5), but as for now we assume that there is no such obstacle.

[6]The proposition below is the cocycle-counterpart of the fact that any homomorphism is uniquely determined by the values it gives to a generating set.

Theorem 5.5 (roots for phase polynomials in an extension). *Let X be an ergodic G-system. Fix $d\geq 1$ and suppose that $P_1, P_2,\dots$ are at most countably many (X,S^1)-phase polynomials of degree $<d$. Let $p_1, p_2,\dots$ be (not necessarily distinct) prime numbers and assume that, for every $i\in\mathbb{N}$, $\Delta_g P_i=1$ for all $g\in G$ of order p_i. Then, there exist a totally disconnected group Δ and a cocycle $\tau: G\times X\to\Delta$ of type $<d-1$ such that the extension $Y=X\times_\tau\Delta$ is ergodic and for every $n\in\mathbb{N}$ and $i=1,2,\dots$ there exist (G,X,S^1)-phase polynomials $Q_{i,n}:Y\to S^1$ of degree $<d$ such that $Q_{i,n}^{p_i^n}=P_i\circ\pi$, where $\pi:Y\to X$ is the factor map.*

Proof. Let p be a prime number. Let $P:X\to S^1$ be a polynomial of degree $<d$ and assume that P is T_g-invariant for every $g\in G$ of order p. Let $c(g,x):=\Delta_g P(x)$ and observe that by Proposition C.1 the phase polynomial $c(g,x)$ takes values in C_m for some $m=\text{order}(g)^{d-1}$. Let $n\in\mathbb{N}$ be a natural number, fix $g\in G$ of order coprime to p and let $m=\text{order}(g)^{d-1}$. Since p^n and m are coprime, we can find a natural number $l_g(n)$ such that $p^n\cdot l_g(n)=1$ modulo m. It follows that the phase polynomial $d_n(g,x):=c(g,x)^{l_g(n)}$ is a p^n-th root of $c(g,x)$. We extend d_n to G by decomposing every $g\in G$ as $g=g_p+g'$, where g_p is of order p and g' of order coprime to p, and setting $d_n(g,x):=d_n(g',x)$. Our next goal is to replace d_n with a cocycle using Proposition 5.2. Observe first that since $d_n(g,\cdot)$ is a power of $c(g,\cdot)$, we have

$$\prod_{i=0}^{\text{order}(g)-1} d_n(g,T_{g^i}x)=1. \tag{5-4}$$

Now we claim that, for every $g,h\in G$,

$$\frac{\Delta_h d_n(g,x)}{\Delta_g d_n(h,x)}=1. \tag{5-5}$$

On one hand, $d_n^{p^n}=c$ and therefore this quotient is of order p^n. On the other hand, $d_n(g,x)$ and $d_n(h,x)$ are of order coprime to p; hence the quotient is trivial. Therefore by Proposition 5.2 there exists a cocycle $\tilde{d}_n: G\times X\to S^1$ which agrees with d_n on a generating set. Since ${d_n}^{p^n}=c$ and c is a cocycle, we conclude that $\tilde{d}_n^{p^n}=c$.

Now, we apply the argument above for each of the polynomials in the theorem. Set $c_i(g,x):=\Delta_g P_i$. We conclude that for every $i,n\in\mathbb{N}$ there exists a phase polynomial cocycle $\tilde{d}_{i,n}$ of degree $<d-1$ such that $\tilde{d}_{i,n}^{p_i^n}=c_i$. For each $i,n\in\mathbb{N}$ fix a measurable map[7] $F_{i,n}:X\to S^1$ such that $F_{i,n}^{p_i^n}=P_i$ and let $\tau:=(\tilde{d}_{i,n}\cdot\Delta F_{i,n})_{i,n\in\mathbb{N}}$ be a cocycle, $\tau:G\times X\to\prod_{i,n\in\mathbb{N}}C_{p_i^n}$.

The extension of X by τ may not be ergodic, so we choose a minimal cocycle τ' that is $(G,X,\prod_{i,n\in\mathbb{N}}C_{p_i^n})$-cohomologous to τ (Lemma 5.4) and write $\tau'=\tau\cdot\Delta F$ for some measurable map $F:X\to\prod_{i,n\in\mathbb{N}}C_{p_i^n}$. We denote the image of τ' by Δ and consider the extension $Y:=X\times_{\tau'}\Delta$. The closed subgroup $\Delta\leq\prod_{i,n\in\mathbb{N}}C_{p_i^n}$ is totally disconnected. Moreover, it follows from the construction that the system Y is ergodic. Finally, since τ is of type $<d-1$, we conclude that so is τ'. For $i,n\in\mathbb{N}$ let $\pi_{i,n}:\prod_{i,n\in\mathbb{N}}C_{p_i^n}\to C_{p_i^n}$ be the (i,n)-th coordinate map. We conclude that the function

$$Q_{i,n}(x,u):=\pi_{i,n}|_{\Delta'}(u)\cdot\overline{F}_{i,n}(x)\cdot\pi_{i,n}\circ\overline{F}(x)$$

is a phase polynomial in Y (whose derivative is $\tilde{d}_{i,n}$) and it satisfies that $Q_{i,n}^{p_i^n}=P_i\circ\pi$, as required. □

[7] One way to do so is by identifying S^1 with $\mathbb{R}/\mathbb{Z}$ and setting $F_{i,n}(x)=\{P_i(x)\}/n$, where $\{\cdot\}$ is the fractional part.

Remark 5.6. Following the same argument as above we have the following generalizations:

- If P is a phase polynomial and $\Delta_g P = 1$ for every $g \in G$ of order p and of order q then we can adapt the proof and find an $p^n q^m$ root for all $n, m \in \mathbb{N}$. The same goes for multiple primes.
- If instead of $G = \bigoplus_{p\in P} \mathbb{Z}/p\mathbb{Z}$ we take an extension $G^{(l)} = \bigoplus_{p\in P} \mathbb{Z}/p^l\mathbb{Z}$ then the same proof still holds.

Moreover, observe that if P is T_g-invariant for some $g \in G$, then by choosing d_n in the proof above with $d_n(g, \cdot) = 1$ for these g's, we can construct an n-th root of P which is also T_g-invariant.

Now, we want to remove the hypothesis that $\Delta_g P_i = 1$ for every $g \in G$ of some order p_i. To do this we use m-extensions (see Definition 2.5). We begin with the following definitions.

Definition 5.7 (multicocycles). Let $m \geq 1$. Let X be an ergodic G-system and U a compact abelian group. We say that a function $q : G^m \times X \to U$ is a multicocycle if it is a cocycle in each coordinate. Namely, for every $1 \leq i \leq m$, every $g_1, g_2, \ldots, g_m \in G$ and $g_i' \in G$ we have

$$q(g_1, \ldots, g_{i-1}, g_i \cdot g_i', g_{i+1}, \ldots, g_m, x) = q(g_1, \ldots, g_i, \ldots, g_m, x) \cdot q(g_1, \ldots, g_i', \ldots, g_m, T_{g_i} x).$$

We say that q is symmetric if it is invariant under permutations of coordinates of G^m and we denote by $\mathrm{SMC}_m(G, X, U)$ the group of symmetric multicocycles $q : G^m \times X \to U$.

If the multicocycle q is a constant in x then we say that q is multilinear and denote by $\mathrm{SML}_m(G, U)$ the group of symmetric multilinear maps $\lambda : G^m \to U$.

We say that a multicocycle $q : G^m \times X \to U$ is a phase polynomial of degree $< r$ if for every $g_1, \ldots, g_m \in G$ the map $x \mapsto q(g_1, \ldots, g_m, x)$ is a phase polynomial of degree $< r$. We have the following result.

Lemma 5.8. *Let X be an ergodic G-system and let $m, r \in \mathbb{N}$. Let $q \in \mathrm{SMC}_m(G, X, S^1)$ be a phase polynomial of degree $< r$. Then, there exists an $O_{m,r}(1)$-extension Y with factor map $\pi : Y \to X$ and a phase polynomial Q of degree $< r + m$ such that $q(g_1, \ldots, g_m, \pi(y)) = \Delta_{g_1} \cdots \Delta_{g_m} Q(y)$.*

Proof. We prove the lemma by induction on m. For $m = 1$, q is a cocycle. Therefore, by Lemma 5.4, q is cohomologous to a minimal cocycle σ. Let V be the image of σ and consider the extension $Y = X \times_\sigma V$. Arguing in Theorem 5.5, we see that q is a coboundary in Y and the claim follows. Now, let $m \geq 2$ and suppose that the claim holds for all smaller values of m. Let $q : G^m \times X \to S^1$ be a multicocycle. Then, for every $g_1, \ldots, g_{m-1}$, the map $g \mapsto p(g, g_1, \ldots, g_{m-1}, x)$ is a cocycle. Therefore, as in the case $m = 1$ we can find an extension $\tilde{\pi} : \widetilde{X} \to X$ and phase polynomials $Q_{g_1,\ldots,g_m} : \widetilde{X} \to S^1$ such that $q(g, g_1, \ldots, g_{m-1}, \tilde{\pi}(x)) = \Delta_g Q_{g_1,\ldots,g_{m-1}}(x)$ for every $g, g_1, \ldots, g_{m-1} \in G$. By choosing the same $Q_{g_1,\ldots,g_{m-1}}$ for any permutation of $g_1, \ldots, g_{m-1}$, we can assume that $(g_1, \ldots, g_{m-1}) \mapsto Q_{g_1,\ldots,g_{m-1}}$ is symmetric. It is left to show that we can choose $Q_{g_1,\ldots,g_{m-1}}$ to be a cocycle in every coordinate. The fact that q is symmetric implies

$$\Delta_h Q_{g_1,\ldots,g_{m-1}} = \Delta_{g_i} Q_{g_1,\ldots,g_{i-1},h,g_{i+1},\ldots,g_{m-1}} \tag{5-6}$$

for every $1 \leq i \leq m - 1$ and $h, g_1, \ldots, g_{m-1} \in G$. Observe that by Proposition C.1 the order of the left-hand side is some power of order(h) and the order of the right-hand side some power of order(g_i).

It follows that if one of the g_i is of order coprime to p, then $Q_{g_1,\dots,g_{m-1}}$ is invariant with respect to the action of the subgroup $G_p=\{g\in G : pg=0\}$. In particular, if g_i is coprime to g_j then $Q_{g_1,\dots,g_{m-1}}$ is a constant. By changing the choice of $Q_{g_1,\dots,g_{m-1}}$ we can assume that $Q_{g_1,\dots,g_{m-1}}=\prod_{p\in P}Q_{g_1^{(p)},\dots,g_{m-1}^{(p)}}$, where $g_i^{(p)}$ is the p-component of g_i (we note that the infinite product is well defined because all but finitely many p-components are trivial).

Suppose now that all $g_1,\dots,g_{m-1}$ are of order p. Then, since q is a multicocycle, for every $1\le i\le k$ we get that

$$\prod_{k=0}^{p-1}T_{g_i}^k Q_{g_1,\dots,g_{m-1}}=c_p(g_1,\dots,g_{m-1}), \tag{5-7}$$

where $c_p(g_1,\dots,g_{m-1})$ is symmetric and independent of i. Let $c'_p(g_1,\dots,g_{m-1})$ be a p-th root of $c_p(g_1,\dots,g_{m-1})$. By picking the same root for all permutations of $g_1,\dots,g_{m-1}$ we can assume that c'_p is symmetric. Let $Q'_{g_1,\dots,g_{m-1}}=Q_{g_1,\dots,g_{m-1}}/c'_p(g_1,\dots,g_{m-1})$ whenever $g_1,\dots,g_{m-1}$ are of order p. Then, for every $1\le i\le m-1$

$$\prod_{k=0}^{p-1}T_{g_i}^k Q'_{g_1,\dots,g_{m-1}}=1. \tag{5-8}$$

Now set

$$Q''_{g_1,\dots,g_{m-1}}=\prod_{p\in P}Q'_{g_1^{(p)},\dots,g_{m-1}^{(p)}}.$$

By Proposition 5.2 (applied for all coordinates), there exists a symmetric multicocycle $Q''_{g_1,\dots,g_{m-1}}$ that agrees with $Q'_{g_1,\dots,g_{m-1}}$ whenever $g_1,\dots,g_{m-1}$ are elements in a basis of G.

Since $Q''_{g_1,\dots,g_{m-1}}$ is symmetric, it is a cocycle in every coordinate. In other words $Q''\in \mathrm{SML}_{m-1}(G,X,S^1)$ is a phase polynomial of degree $<r+1$. Therefore, by the induction hypothesis we can find an $O_{m,r}(1)$-extension $\pi:Y\to\widetilde{X}$ and a phase polynomial $Q:Y\to S^1$ of degree $<r+m$ such that $Q''_{g_1,\dots,g_{m-1}}(\pi(y))=\Delta_{g_1}\cdots\Delta_{g_{m-1}}Q(y)$. Observe moreover that since q is a multicocycle, we have that $\Delta_g Q''_{g_1,\dots,g_{m-1}}=\Delta_g Q_{g_1,\dots,g_{m-1}}$ for every $g_1\in G$. Therefore, $Q''_{g_1,\dots,g_{m-1}}/Q_{g_1,\dots,g_{m-1}}$ is a constant and we conclude that for every $g\in G$ and every $g_1,g_2,\dots,g_{m-1}$ in a basis of G, we have that

$$q(g,g_1,\dots,g_{m-1},\tilde{\pi}(\pi(y)))=\Delta_g\Delta_{g_1}\cdots\Delta_{g_{m-1}}Q(y)$$

since q is a cocycle in each coordinate, the same holds if the generators $g_1,g_2,\dots,g_{m-1}$ are replaced with any elements of G. □

We can finally prove the desired result.

Theorem 5.9 (roots for phase polynomials in a k-extension). *Let X be an ergodic G-system. Let $P_1,P_2,\dots$ be at most countably many (G,X,S^1)-phase polynomials of degree $<m$ and let $p_1,p_2,\dots$ be prime numbers. Then, for every natural number l there exists an $O_{m,l}(1)$-extension Y with the following property: for every $n=p_1^{n_1}\cdot p_2^{n_2}\cdot\dots\cdot p_j^{n_j}$ where $j\in\mathbb{N}$ and $n_1,n_2,\dots,n_j\le l$, there exist $(G^{(O_{m,l}(1))},Y,S^1)$ phase polynomials $Q_{i,n}:Y\to S^1$ of degree $<m$ such that $Q_{i,n}^n=P_i\circ\pi$, where $\pi:Y\to X$ is the factor map.*

Proof. We prove the theorem by induction on m. If $m=1$, then by ergodicity $P_1,P_2,\dots$ are constants and the claim follows without extensions. Fix $m\ge 1$ and assume inductively that the claim holds for this m and let P be a phase polynomial of degree $< m+1$. For every $g_1, \dots, g_m \in G$ we have that

$$c(g_1, \dots, g_m) := \Delta_{g_1} \cdots \Delta_{g_m} P$$

is a symmetric multilinear map. Let $\widetilde{G} = G^{(l)}$ be the extension of G. We can lift c to an element in $\mathrm{SML}_m(\widetilde{G}, S^1)$. Once lifted, we can find $d \in \mathrm{SML}_m(\widetilde{G}, S^1)$ with $d^n = c$. By the previous lemma applied to d there exists an extension $\pi : Y \to X$ and a phase polynomial Q such that $\Delta_{g_1} \cdots \Delta_{g_m} Q^n = \Delta_{g_1} \cdots \Delta g_m P \circ \pi$ and so by ergodicity $P \circ \pi / Q^n$ is of degree $< m-1$ and the claim for this P follows by induction hypothesis. The same argument holds if applied to all $P_1, P_2, \dots$ simultaneously, as required. □

The results in Theorems 5.5 and 5.9 require to extend X by a zero-dimensional group multiple times. For this reason it will be convenient to use the following definition.

Definition 5.10 (zero-dimensional extension). Let X be an ergodic G-system. We say that an extension Y is a zero-dimensional extension of X if there exists finitely many zero-dimensional groups $\Delta_1, \dots, \Delta_n$ such that $Y = ((X \times_{\rho_1} \Delta_1) \times_{\rho_2} \Delta_2 \times \dots) \times_{\rho_n} \Delta_n$ for some cocycles $\rho_1, \dots, \rho_n$. We say that a zero-dimensional extension is of exponent l if $\Delta_1, \dots, \Delta_n$ are of exponent l.

We note that by the Mackey-Zimmer theory, Y can be written as a single extension of X by a zero-dimensional group, but we do not use this here.

Below we prove various corollaries of Theorem 5.5 and Theorem 5.9. We begin with the following important lemma which allows us to reduce any Conze–Lesigne-type equation to the torus.

Lemma 5.11. *Let $l, m \geq 1$, X be an ergodic G-system and U be a finite-dimensional compact abelian group of exponent l. Let $\rho : G \times X \to U$ a cocycle of type $< m$ and suppose that for every $\chi \in \widehat{U}$ the cocycle $\chi \circ \rho$ is (G, X, S^1)-cohomologous to a phase polynomial of degree $< m$. Then, there exists some $r = O_{m,l}(1)$ and a zero-dimensional r-extension $\pi : Y \to X$ such that $\rho \circ \pi$ is $(G^{(r)}, Y, U)$-cohomologous to a phase polynomial of degree $< m$. Moreover, the extension Y is independent of ρ.*

Proof. Let Δ be a zero-dimensional subgroup of exponent l such that $U/\Delta \cong (S^1)^\alpha \times \prod_{i=1}^{\beta} C_{p_i^\beta}$ is a Lie group (i.e., α, β are finite natural numbers). We denote by $\chi_1, \dots, \chi_n$ the lifts of the coordinate maps of U/Δ to $\widehat{U}$.

By assumption, for every $\chi \in \widehat{U}$ there exists a phase polynomial P_χ and a measurable map F_χ such that

$$\chi \circ \rho = P_\chi \cdot \Delta F_\chi. \tag{5-9}$$

Our goal is to find a choice of P_χ and F_χ such that $\chi \mapsto P_\chi$ and $\chi \mapsto F_\chi$ are homomorphisms. Recall that the dual group of U/Δ takes the form $\mathbb{Z}^\alpha \oplus \mathbb{Z}/p_1^{n_1}\mathbb{Z} \oplus \cdots \oplus \mathbb{Z}/p_\beta^{n_\beta}\mathbb{Z}$. As a first step we show that for every prime p and a natural number n, if $\chi : U/\Delta \to S^1$ is a character of order p^n then we can replace P_χ, F_χ such that (5-9) holds for the new replacements and at the same time P_χ and F_χ takes values in C_{p^n}.

From (5-9) and the fact that χ is of order p^n, it follows that $\Delta F_\chi^{p^n} = \overline{P}_\chi^{p^n}$. In particular, $F_\chi^{p^n}$ is a phase polynomial of degree $< m+1$. If n is sufficiently large with respect to m (Proposition C.1) then $P_\chi^{p^n}(g, \cdot)$ is trivial for every $g \in G$ of order p. For such n apply Theorem 5.5. Otherwise, $n = O_m(1)$ and we apply Theorem 5.9. We conclude that there exists a zero-dimensional $O_m(1)$-extension $\pi_1 : \widetilde{X} \to X$ and a phase polynomial $Q_\chi : \widetilde{X} \to S^1$ of degree $< m+1$ such that $Q_\chi^{p^n} = F_\chi^{p^n} \circ \pi_1$. Now, we replace F_χ with $F_\chi \circ \pi_1 / Q_\chi$ and P_χ with $P_\chi \circ \pi_1 \cdot \Delta Q_\chi$. We can therefore assume that

$$\chi \circ (\rho \circ \pi_1) = P_\chi \cdot \Delta F_\chi \tag{5-10}$$

and at the same time F_χ, P_χ takes values in C_{p^n}, as desired.

We conclude that there exist homomorphisms $\chi \mapsto P_\chi$ and $\chi \mapsto F_\chi$ from the dual of U/Δ to $P_{<m}(G, \widetilde{X}, S^1)$ and $\mathcal{M}(\widetilde{X}, S^1)$ respectively such that (5-10) holds.

Our next step is to extend these homomorphisms to $\widehat{U}$. Since Δ is of exponent l, by Theorem B.9 there exists a countable multiset of primes $I = \{p_1, p_2, \dots\}$ such that $\Delta = \prod_{i\in\mathbb{N}} C_{p_i^{n_i}}$, where $n_i \le l$ for every i. Let $\tau_1, \tau_2, \dots \in \hat{\Delta}$ be the coordinate maps. Using the Pontryagin duality, we lift each of the τ_i to $\widehat{U}$ arbitrarily. Abusing notation, we denote the lifts of these characters by $\tau_1, \tau_2, \dots$ as well. Observe that $\tau_1, \tau_2, \dots$ and $\chi_1, \dots, \chi_n$ form a generating set of $\widehat{U}$. Therefore, in order to extend the homomorphisms above to $\widehat{U}$ it is left to work out the relations of the form

$$\tau_i^{p_i^{n_i}} = \chi_1^{l_1} \cdots \chi_n^{l_n},$$

where $n_i \le l$ is as before and $l_1, \dots, l_n$ are natural numbers.

Fix some $\tau : U \to S^1$ as above and suppose that $\tau^{p^r} = \chi_1^{l_1} \cdot \dots \cdot \chi_n^{l_n}$ for some prime p and a natural number $r \le l$. Equation (5-10) implies

$$P_\tau^{p^r} \cdot \Delta F_\tau^{p^r} = \prod_{i=1}^{n} \chi_i^{l_i} P_{\chi_i}^{l_i} \cdot \Delta F_{\tau_i}^{l_i}$$

and we conclude that

$$\frac{F_\tau^{p^r}}{\prod_{i=1}^n F_{\chi_i}^{l_i}}$$

is a phase polynomial of degree $< m+1$.

Therefore by Theorem 5.9 there exists a zero-dimensional $O_l(1)$-extension $\pi_2 : Y \to \widetilde{X}$ of $\widetilde{X}$ (which is independent of τ) of exponent l and a phase polynomial $R_\tau : Y \to S^1$ such that

$$R_\tau^{p^r} = \left(\frac{F_\tau^{p^r}}{\prod_{i=1}^n F_{\chi_i}^{l_i}} \right) \circ \pi_2.$$

Now, we replace F_τ with $(F_\tau \circ \pi_2)/R_\tau$ and P_τ with $P_\tau \circ \pi_2 \cdot \Delta R_\tau$ and we lift the other polynomials to Y as well. After the replacement, we have that $P_\tau^{p^r} = \prod_{i=1}^n P_{\chi_i}^{l_i}$. This means that we can find homomorphisms $\chi \mapsto P_\chi$ and $\chi \mapsto F_\chi$ from $\widehat{U}$ to $P_{<m}(G, Y, S^1)$ and $\mathcal{M}(Y, S^1)$ respectively such that (5-9) holds. Using the Pontryagin duality, we see that there exists a phase polynomial $P \in P_{<m}(G, Y, U)$ and a measurable map $F \in \mathcal{M}(Y, U)$ such that $\rho \circ \pi_2 = P \cdot \Delta F$. □

In a similar manner we have the following result.

Lemma 5.12. *Let $l, m \ge 1$ be natural numbers, let X be an ergodic G-system and U be a compact abelian group. Let $\varphi : V \to U$ be a surjective homomorphism from a compact abelian group V onto U and suppose that the kernel of φ is a totally disconnected group of exponent l. Then, there exists an $O_{m,l}(1)$-extension $\pi : Y \to X$ with the property that for every phase polynomial $p : X \to U$ of degree $< m$ there exists a phase polynomial $\tilde{p} : Y \to V$ such that $\varphi \circ \tilde{p} = p \circ \pi$.*

Proof. Using the Pontryagin duality, we see that the surjective homomorphism φ gives rise to an injective homomorphism $\hat{\varphi} : \widehat{U} \to \widehat{V}$. Therefore, we can assume without loss of generality that $\widehat{U} \leq \widehat{V}$. Let $p : X \to U$ be as in the theorem. Then, for every $\chi \in \widehat{U}$, $\chi \circ p : X \to S^1$ is also a phase polynomial of degree $< m$ and $\chi \mapsto \chi \circ p$ is clearly a homomorphism. Arguing as in the previous lemma we see that by passing to an extension we can extend this homomorphism to $\widehat{V}$. Namely, there exists an $O_{m,l}(1)$-extension $\pi : Y \to X$ and a homomorphism $\chi \mapsto \tilde{p}_\chi$ from $\widehat{V}$ to $P_{<m}(X, S^1)$ such that $\tilde{p}_\chi = p_\chi \circ \pi$ for every $\chi \in \widehat{U}$. By the Pontryagin duality, there exists a phase polynomial $\tilde{p} : Y \to V$ such that $\chi \circ \tilde{p} = \tilde{p}_\chi$. □

We recall some definitions from [Bergelson et al. 2010].

Definition 5.13 (quasicocycles). Let X be a G-system and $k \geq 0$ be a natural number. We say that f is a quasicocycle of order $< k$ if $d^{[k]} f : G \times X^{[k]} \to S^1$ is a cocycle.

Note that by Lemma A.15 this means that for all $g, g' \in G$ there exists a phase polynomial $p_{g,g'}$ of degree $< k$ such that

$$\frac{f(g+g', x)}{f(g, x) \cdot f(g', T_g x)} = p_{g,g'}(x).$$

We also need the following definition.

Definition 5.14. We say that a function $f : G \times X \to S^1$ is a line-cocycle if for any $g \in G$ we have

$$\prod_{k=0}^{\operatorname{order}(g)-1} f(g, T_g^k x) = 1.$$

We weaken the assumptions in Theorem 3.7.

Theorem 5.15. *Theorem 3.7 holds with the weaker assumption that each ρ_ω is a quasicocycle of order $< k-1$ and a line-cocycle.*

Proof. The proof is the same as in Theorem 3.7 and therefore will not be repeated. The main observation is that in the proof of Theorem 3.1 we only used the fact that ρ is a cocycle for two purposes: First, so we can apply Lemma A.9 which we now can replace with Lemma A.10 and second, to eliminate the term p'_u in (3-5). This time, $p'_u : G \to S^1$ is a line-cocycle. In particular, $p'_u(g)$ is of finite order for every $g \in G$. Since $\mathcal{H}_l$ is connected, it is divisible and therefore the homomorphism $u \mapsto p'_u$ is trivial. □

Let X be an ergodic G-system. In the next lemma we see how the extension theorems are useful to construct line-cocycles from arbitrary functions of finite type.

Lemma 5.16. *Let $k, m \geq 1$ and X be an ergodic G-system of order $< k$. Then there exists $r = O_{k,m}(1)$ an ergodic zero-dimensional $O_{k,m}(1)$-extension $\pi : Y \to X$, with the following property: for every function $f : G \times X \to S^1$ of type $< m$ there exists a phase polynomial $p : G^{(r)} \times Y \to S^1$ of degree $< m$ such that $f \circ \pi / p$ is a line cocycle.*

Proof. Let $g \in G$ and denote by n the order of g. Let f be of type $< m$. Since $d^{[m]} f$ is a coboundary, it is also a cocycle. We conclude that

$$d^{[m]} \prod_{k=0}^{n-1} f(g, T_g^k x) = 1.$$

Lemma A.15 implies that $\prod_{k=0}^{n-1} f(g, T_g^k x)$ is a T_g-invariant phase polynomial of degree $< m$. We apply Theorem 5.9 for every $g \in G$ (simultaneously). We see that there exist an $O_{k,m}(1)$-extension Y and a phase polynomial $p : G \times Y \to S^1$ of degree $< m$ such that $p(g, \cdot)$ is T_g-invariant for every $g \in G$ (see Remark 5.6) and $p(g, \cdot)^n = \prod_{k=0}^{n-1} f(g, T_g^k x)$. It follows that $f \circ \pi / p$ is a line-cocycle, as required. □

The following theorem summarizes the main results in this section.

Theorem 5.17. *Let $k, m, l, \alpha \geq 1$ and let X be an ergodic system of order $< k$. Let U be a finite-dimensional group of exponent α and $\varphi : V \to U$ a surjective homomorphism such that $\ker\varphi$ is a zero-dimensional group of exponent l. Then, there exists a zero-dimensional $O_{k,m,l,\alpha}(1)$-extension Y of X with a factor map $\pi : Y \to X$ such that the following properties hold*:

- *Let $\rho : G \times X \to U$ be a cocycle. If $\chi \circ \rho \in P_{<m}(G, X, S^1) \cdot B^1(G, X, S^1)$ for every $\chi \in \widehat{U}$, then $\rho \circ \pi \in P_{<m}(G, Y, U) \cdot B^1(G, Y, U)$.*
- *For every phase polynomial $p : X \to U$ of degree $< m$, there exists a phase polynomial $\tilde{p} : Y \to V$ such that $\varphi \circ \tilde{p} = p \circ \pi$.*
- *For every function $f : G \times X \to S^1$ of type $< m$, there exists a phase polynomial of degree $< m$, $p : G \times X \to S^1$ such that $f \circ \pi / p$ is a line cocycle.*

6. Proof of Theorem 2.12, part I

Throughout the rest of this paper we let G denote a group of the form $\bigoplus_{p\in P}\mathbb{Z}/p^m\mathbb{Z}$, where P is a multiset of primes and $m \in \mathbb{N}$. Moreover, we will no longer deal with general nilpotent systems and so whenever we say a nilpotent system we implicitly assume that the homogeneous group is the Host–Kra group. Our goal is to prove the following result.

Theorem 6.1 (any finite-dimensional system is a factor of a nilpotent system). *Let $k, \alpha \in \mathbb{N}$ and let X be an ergodic finite-dimensional G-system of order $< k + 1$ and exponent α. Then, there exists a finite-dimensional ergodic $O_{k,\alpha}(1)$-extension Y of exponent $\beta = O_{k,\alpha,m}(1)$ such that $Y \cong \mathcal{G}(Y)/\Gamma$ for some totally disconnected group Γ.*

Let $\alpha \in \mathbb{N}$ and X be as in the theorem above. Then, $X = Z_{<k}(X) \times_\rho U$, where U is a finite-dimensional compact abelian group of exponent α. Proving the theorem above by induction, we can replace $Z_{<k}(X)$ with a finite-dimensional nilpotent system $\mathcal{G}/\Gamma$. It is natural to ask when an element $s \in \mathcal{G}$ has a lift in $\mathcal{G}(X)$.

Definition 6.2. Let $\mathcal{G}/\Gamma$ be a k-step nilpotent system, $\rho : G \times \mathcal{G}/\Gamma \to U$ be a cocycle into some compact abelian group U and $Y = \mathcal{G}/\Gamma \times_\rho U$. We say that an element $s \in \mathcal{G}$ has a lift in $\mathcal{G}(Y)$ if there exists a transformation $\bar{s} \in \mathcal{G}(Y)$ which induces the same action as s on $\mathcal{G}/\Gamma$. In this context we let

$$\mathcal{G}^\star = \{s \in \mathcal{G} : s \text{ has a lift in } \mathcal{G}(Y)\}.$$

Note that translations by U are automatically in $\mathcal{G}(Y)$. Therefore, if the action of $\mathcal{G}^\star$ on $\mathcal{G}/\Gamma$ is transitive then the action of $\mathcal{G}(Y)$ on Y is transitive.

We need the following easy lemma.

Lemma 6.3. *Let $k \geq 1$ and $X = \mathcal{G}/\Gamma$ be a k-step nilpotent G-system. Let $\mathcal{L}$ be an open subgroup of $\mathcal{G}$ which contains T_g for every $g \in G$. Then the action of $\mathcal{L}$ on X is transitive.*

Proof. Let $\Gamma_{\mathcal{L}} = \Gamma \cap \mathcal{L}$. The quotient $\mathcal{L}/\Gamma_{\mathcal{L}}$ can be identified with a G-invariant open and closed subset of $\mathcal{G}/\Gamma_{\mathcal{G}}$. Ergodicity implies that $\mathcal{L}/\Gamma_{\mathcal{L}} = \mathcal{G}/\Gamma_{\mathcal{G}}$ under the identification $l \cdot \Gamma_{\mathcal{L}} \mapsto l \cdot \Gamma_{\mathcal{G}}$. In particular, the action of $\mathcal{L}$ on X is transitive. □

We use the following criterion for lifting due to [Host and Kra 2005, Lemma 10.6].[8]

Lemma 6.4. *Let $k \geq 0$ and let X be an ergodic G-system of order $< k+1$. Write $X = Z_{<k}(X) \times_\rho U$ for some compact abelian group U and a cocycle $\rho : G \times Z_{<k}(X) \to U$ of type $< k$ with $d^{[k]}\rho = \Delta F$. Let $t \in \mathcal{G}(X)$. If there exists a map $\phi : Z_{<k}(X) \to U$ with the property that*

$$\Delta_{t^{[k]}} F = d^{[k]}\phi, \tag{6-1}$$

then the transformation

$$\bar{t}(x, u) := (tx, \phi(x)u) \tag{6-2}$$

is a lift of t in $\mathcal{G}(X)$.

Conversely, every element in $\mathcal{G}(X)$ is of the form (6-2).

Let $p : \mathcal{G}(X) \to \mathcal{G}(Z_{<k}(X))$ be the projection map (Lemma 9.5). By the lemma above, the kernel of p consists of transformations of the form $S_{1,F}$, where $F \in P_{<k+1}(Z_{<k}(X), U)$. As a consequence we have the following result.

Lemma 6.5. *If X is a finite-dimensional system, then $\mathcal{G}(X)$ is a finite-dimensional group* (see Definition 2.7).

Proof. Using a proof by induction it is enough to show that $P_{<k+1}(Z_{<k}(X), U)$ is finite-dimensional. Let $\Delta \leq U$ be a zero-dimensional group such that U/Δ is a finite-dimensional torus. The projection $U \to U/\Delta$ gives rise to a short exact sequence

$$1 \to A \to P_{<k+1}(Z_{<k}(X), U) \to B \to 1, \tag{6-3}$$

where $A \leq P_{<k+1}(Z_{<k}(X), \Delta)$ and $B \leq P_{<k+1}(Z_{<k}(X), U/\Delta)$ are closed subgroups. By Lemma A.17 we conclude that the subgroup of constants $P_{<1}(Z_{<k}(X), U/\Delta) \cong U/\Delta$ is an open subgroup of $P_{<k+1}(Z_{<k}, U/\Delta)$ and therefore, B is finite-dimensional. Moreover, since every profinite group is embedded in a direct product of finite groups we can assume that $\Delta \leq \prod_{i=1}^{\infty} F_i$, which implies $P_{<k+1}(Z_{<k}(X), \Delta) \leq \prod_{i=1}^{\infty} P_{<k+1}(Z_{<k}(X), F_i)$. For every i, $P_{<k+1}(Z_{<k}(X), F_i)$ is discrete, this implies that the product $\prod_{i=1}^{\infty} P_{<k+1}(Z_{<k}(X), F_i)$ is totally disconnected. In particular, the closed subgroup $P_{<k+1}(Z_{<k}(X), \Delta)$ and the closed subgroup A are zero-dimensional. The short exact sequence (6-3) implies that $P_{<k+1}(Z_{<k}(X), U)$ is finite-dimensional, as required. □

The proof of Theorem 6.1 is reduced to solving the following lifting problem.

[8] Lemma 10.6 in [Host and Kra 2005] is formulated only in the case where U is a torus. However, since this assumption does not play a role in their proof, the claim holds for every compact abelian group U.

Theorem 6.6. *Let $k, m \geq 1$. Let $X = \mathcal{G}/\Gamma$ be a finite-dimensional k-step nilpotent system of exponent m and suppose that $\mathcal{G}$ is an open subgroup of $\mathcal{G}(X)$ which contains T_g for every $g \in G$. Let $\rho : G \times X \to U$ be a cocycle of type $< k+1$ into some finite-dimensional compact abelian group U of exponent α. Then, there exists an ergodic zero-dimensional $O_{k,m}(1)$-extension $\pi : Y_0 \to X$ of order $< k+1$ and exponent $m' = O_{k,m}(1)$ which is independent on ρ such that the following properties hold*:

(1) *The subgroup*
$$\mathcal{L}_0 = \{s \in \mathcal{G} : \text{ there exists a lift of } s \text{ in } \mathcal{G}(Y_0)^\star\}$$
is open, where $\mathcal{G}(Y_0)^\star$ is defined with respect to the extension $Y_0 \times_{\rho\circ\pi} U$.

(2) *The subgroup of $\mathcal{G}(Y_0)^\star$ generated by all of the lifts of the elements in $\mathcal{L}_0$ acts transitively on Y_0. In particular, Y_0 is a finite-dimensional k-step nilpotent system.*

Given this result we prove Theorem 6.1 in Section 8.

The extension theorems from Section 5 play an important role in the proof of Theorem 6.6. However the use of these theorems require passing to an extension (or k-extension) of the original system. This leads to various difficulties which we will explain soon. First, we need to distinguish between two different types of extensions. These are, weakly mixing extensions and degenerate extensions. We begin with a definition for the former.

Definition 6.7 (weakly mixing abelian extensions). Let X be a system of order $< k$, U be a compact abelian group and $\sigma : G \times X \to U$ a cocycle of type $< k$. The extension $X \times_\sigma U$ is called weakly mixing if $Z_{<k}(X \times_\sigma U) = X$. In this case we also say that $\sigma : G \times X \to U$ is weakly mixing.

A classical result [Host and Kra 2005, Corollary 7.7] asserts that an extension of a system of order $< k$ by a cocycle of type $< k-1$ is also of order $< k$. We say that this kind of extensions are *degenerate* and note that all of the extensions from Section 5 are degenerate.

Moreover, a cocycle $\sigma : G \times X \to U$ is not weakly mixing if and only if there exists a character $1 \neq \chi \in \widehat{U}$ such that $\chi \circ \sigma$ is of type $< k-1$ [Host and Kra 2002, Proposition 4]. We deduce the following result.

Lemma 6.8. *Let X be a system of order $< k$, and $\sigma : G \times X \to U$ a weakly mixing cocycle on X. Let $\pi : Y \to X$ be an ergodic extension of order $< k$, then $\sigma \circ \pi$ is weakly mixing.*

Proof. Suppose by contradiction that, for some $1 \neq \chi \in \widehat{U}$, $\chi \circ \sigma \circ \pi$ is of type $< k-1$. Therefore, the extension $Y \times_{\chi\circ\sigma\circ\pi} \chi(U)$ is of order $< k$. Since $X \times_{\chi\circ\sigma} \chi(U)$ is a factor of $Y \times_{\chi\circ\sigma\circ\pi} \chi(U)$, it is also of order $< k$. But then we have a contradiction because $Z_{<k}(X \times_\sigma U)$ must be a nontrivial extension of X. □

The following result will allow us to strengthen Lemma 5.11 for weakly mixing cocycles.

Lemma 6.9. *Let $m, d \geq 1$. Fix a multiset of primes P and let $G = \bigoplus_{p\in P} \mathbb{Z}/p^m\mathbb{Z}$ and X be an ergodic G-system. Then for every prime $p \in P$ and a natural number $n \in \mathbb{N}$, if $\rho : G \times Z_{<d}(X) \to C_{p^n}$ is a weakly mixing cocycle and $(G, Z_{<d}(X), S^1)$-cohomologous to a phase polynomial of degree $< d$, then $n = O_{m,d}(1)$. Moreover, if $p \notin P$ then $n = 0$.*

Proof. Write $\rho = P \cdot \Delta F$, where $P : G \times Z_{<d}(X) \to S^1$ is a phase polynomial cocycle of degree $< d$ and $F : Z_{<d}(X) \to S^1$. By assumption $P^{p^n} \cdot \Delta F^{p^n} = 1$, we conclude that F^{p^n} is a phase polynomial of degree $< d+1$. If $n < d$ the claim follows; otherwise by Proposition C.1 we see that F^{p^n} is T_g-invariant for every $g \in G$ of order p (this part is trivial if $p \notin P$). By Theorem 5.5 we can pass to an extension of X and assume that F^{p^n} has a phase polynomial root Q of degree $< d$. Let $P' = P \cdot \Delta Q$ and $F = F/Q$ then $\rho = P' \cdot \Delta F'$. Moreover, P' takes values in C_{p^n} and therefore by Proposition C.1 in C_{p^l} for some $l = O_{m,d}(1)$ (or that P' is trivial if $p \notin P$). We conclude that ρ^{p^l} is a coboundary on an extension of $Z_{<d}(X)$. This contradicts Lemma 6.8, unless $n = l$ (or $n = 0$ if $p \notin P$). □

The role of Theorem 5.5 in the proof of Lemma 5.11 is to show that if χ takes values in some C_{p^n} then we can replace P_χ and F_χ in (5-9) with P'_χ and F'_χ which takes values in $C_{p^{O_{m,d}(1)}}$. Therefore, the argument in the proof of Lemma 5.11 requires passing to extensions by totally disconnected groups of potentially unbounded exponent. However now we know (by the lemma above) that if the cocycle is weakly mixing then the quantity n must be bounded. In other words we have the following stronger version of Lemma 5.11.

Corollary 6.10. *In the settings of Lemma 5.11. If X is of order $< k$ then there exists $\alpha = O_{k,m,l}(1)$ such that the extension Y is a (bounded) tower of extensions of X by totally disconnected groups of exponent α.*

It is classical (see [Host and Kra 2005, Proposition 7.6]) that every extension can be decomposed as a weakly mixing extension and a *degenerate* extension. More formally, let X be a system of order $< k$ and let $\sigma : G \times X \to U$ be a cocycle of type $< k+1$. If we let W be the annihilator of $\{\chi \in \widehat{U} : \chi \circ \sigma \text{ is of type } < k\}$, then $X \times_\sigma U = X \times_{\sigma \bmod W} U/W \times_\tau W$, where $X \times_{\sigma \bmod W} U/W$ is a system of order $< k$ and τ is a weakly mixing cocycle. In particular, we see that it is enough to prove Theorem 6.6 in the case where ρ is of type $< k$ (the degenerate case) and in the case where ρ is weakly mixing. In this section we prove the former, we begin with the following lemma about degenerate extensions.

Lemma 6.11. *Let $k \geq 1$ and let $X = Z_{<k}(X) \times_\sigma W$ be an ergodic G-system of order $< k+1$. Let $Y = Z_{<k}(Y) \times_\tau V$ be an ergodic extension of X of the same order. Then, there exists a surjective homomorphism $\varphi : V \to W$ such that $\varphi \circ \tau$ is $(G, Z_{<k}(Y), W)$-cohomologous to $\sigma \circ \pi_k$, where $\pi_k : Z_{<k}(Y) \to Z_{<k}(X)$ is the factor map.*

Proof. Let $\pi : Y \to X$ be the factor map. It is classical that π defines a factor $\pi_k : Z_{<k}(Y) \to Z_{<k}(X)$ and we have

$$\pi(y, v) = (\pi_k(y), p(y, v)),$$

where $p : Y \to W$.

Since π commutes with G, we conclude that $\Delta_g p(y, v) = \sigma(g, \pi_k(y))$. In particular, we see that Δp is invariant under the action of V. Since Y is ergodic, $\Delta_v p$ is a constant. We conclude that there exists a homomorphism $\varphi : V \to W$ and a measurable map $F : Z_{<k}(Y) \to W$ such that $p(y, v) = \varphi(v) \cdot F(y)$. Therefore, $\sigma(g, \pi_k(y)) = \Delta_g p(y, v) = \varphi \circ \tau(g, y) \cdot \Delta F$. Since $Z_{<k}(Y) \times_{\sigma \circ \pi_k} W$ is a factor of Y, it is ergodic. It follows by Lemma 5.4 that the image of $\varphi \circ \tau$ is W. We conclude that φ is surjective as required. □

Our next result is that Theorem 6.6 holds for degenerate extensions if the group U is totally disconnected of bounded exponent.

Theorem 6.12 (a degenerate version of Theorem 6.6). *Let $k, m \geq 1$ and let $X = \mathcal{G}(X)/\Gamma$ be a finite-dimensional k-step nilpotent system. Let $\rho : G \times X \to U$ be a cocycle of type $<k$ into some zero-dimensional compact abelian group U of exponent m. Then, there exists an ergodic $O_{k,m}(1)$-extension $\pi : Y_0 \to X$ with the same properties as in Theorem 6.6.*

Recall that by Lemma 6.11 if Y is an extension of X of the same order then we can find compact abelian groups V and U such that $Y = Z_{<k}(Y) \times_\tau V$ and $X = Z_{<k}(X) \times_\sigma U$. Moreover, there exists a surjective homomorphism $\varphi : V \to U$ such that $\varphi \circ \tau$ is cohomologous to $\sigma \circ \pi_k$, where $\pi_k : Z_{<k}(Y) \to Z_{<k}(X)$ is a factor map. It will be convenient to lift the elements of $\mathcal{G}(X)$ to an intermediate factor $\widetilde{X} = Z_{<k}(Y) \times_{\sigma\circ\pi_k} U$ and only then to lift them to Y. Each of these steps will require extending the k-th Host–Kra factor further. For the first lift we need the following corollary of Lemma 6.4.

Corollary 6.13. *Let $k \geq 1$. Let X be an ergodic G-system of order $<k$ and $\rho : G \times X \to U$ a weakly mixing cocycle. Let $\pi : Y \to X$ be an ergodic extension of X of order $<k$ and $s \in \mathcal{G}(X)$. If s has a lift in $\mathcal{G}(X \times_\rho U)$ and a lift in $\mathcal{G}(Y)$, then s can be lifted to $\mathcal{G}(Y \times_{\rho\circ\pi} U)$.*

Proof. Write $d^{[k]}\rho = \Delta F$ and let $s \in \mathcal{G}(X)$ be as in the claim. By Lemma 6.4 there exists a measurable map $\phi_s : X \to U$ such that $S_{s,\phi_s} \in \mathcal{G}(X \times_\rho U)$. Let $\tilde{s}$ be a lift of s in $\mathcal{G}(Y)$, since

$$\Delta_{\tilde{s}^{[k]}} F \circ \pi = (\Delta_{s^{[k]}} F) \circ \pi = d^{[k]}\phi_s \circ \pi,$$

we conclude by Lemmas 6.8 and 6.4 that the transformation $S_{\tilde{s},\phi_s\circ\pi}$ is a lift of $\tilde{s}$ in $\mathcal{G}(Y \times_{\rho\circ\pi} U)$, as required. □

For convenience, if X is a factor of a system Y, $\pi^Y_X : Y \to X$ is the factor map and F is a function on X we refer to $F \circ \pi^Y_X$ as the lift of F to Y. We turn to the proof of Theorem 6.12.

Proof. Let $X = \mathcal{G}/\Gamma$ and $\rho : G \times X \to U$ be as in Theorem 6.12 and assume that ρ is of type $<k$ and U is totally disconnected of exponent m. By Proposition A.19 there exists a compact abelian group W and a cocycle $\sigma : G \times Z_{<k}(X) \to W$ such that $X = Z_{<k}(X) \times_\sigma W$. Let $Y = X \times_\rho U$ and write $Y = Z_{<k}(Y) \times_\tau V$ for some compact abelian groups V and a cocycle τ. Let $\pi_k : Z_{<k}(Y) \to Z_{<k}(X)$ be the factor map. Then by Lemma 6.11 there exist a surjective homomorphism $\varphi : V \to W$ and a measurable map $F : Z_{<k}(Y) \to W$ such that $\sigma \circ \pi_k = \varphi \circ \tau \cdot \Delta F$:

$$\begin{array}{ccc}
Z_{<k}(Y) & \xleftarrow{\text{weakly mixing}} & Z_{<k}(Y) \times_\tau V \\
{\scriptstyle\text{degenerate}}\big\downarrow & & \big\downarrow{\scriptstyle\text{degenerate}} \\
Z_{<k}(X) & \xleftarrow{\text{weakly mixing}} & Z_{<k}(X) \times_\sigma W
\end{array}$$

Observe that $Z_{<k}(Y)$ is a degenerate extension of $Z_{<k}(X)$ by a zero-dimensional group and Y is a weakly mixing extension of $Z_{<k}(Y)$. Therefore we can apply the induction hypothesis in order to lift transformations from $Z_{<k}(X)$ to $Z_{<k}(Y)$ and then use the weakly mixing case to lift from $Z_{<k}(Y)$ to

$Z_{<k}(Y)\times_\tau V$. Each time we have to pass to an extension. We conclude that there exists an extension $Y_0=Z_{<k}(Y_0)\times_{\tau_0} V$, where $\tau_0=\tau\circ\pi^{Z_{<k}(Y_0)}_{Z_{<k}(Y)}$ and such that

$$\mathcal{H}=\{s\in\mathcal{G}(Z_{<k}(X)) : \text{ there exists a lift of } s \text{ in } \mathcal{G}(Y_0)\}$$

is an open subgroup of $\mathcal{G}(Z_{<k}(X))$. Indeed, $\mathcal{H}$ here equals to the group $\mathcal{L}$ defined in Theorem 6.6.

Now let $S_{s,\phi}$ be any transformation in $\mathcal{G}(X)$. If $s\in\mathcal{H}$ then there exists a lift $\bar{s}\in\mathcal{G}(Z_{<k}(Y_0))$ and a measurable map $\psi : Y_0\to V$ such that $S_{\bar{s},\psi}\in\mathcal{G}(Y_0)$.

Let $\tilde{\sigma} : G\times Z_{<k}(Y_0)\to W$ be the lift of σ to $Z_{<k}(Y_0)$, that is, $\tilde{\sigma}=\sigma\circ\pi^{Z_{<k}(Y_0)}_{Z_{<k}(X)}$. Similarly let $\tilde{\phi}$ and $\widetilde{F}$ be the lifts of F and ϕ to $Z_{<k}(Y_0)$, respectively. We consider the intermediate factor $\widetilde{X}=Z_{<k}(Y_0)\times_{\tilde{\sigma}} W$. By Lemma 6.4 and Corollary 6.13 we see that $S_{\bar{s},\tilde{\phi}}$ and $S_{\bar{s},\varphi\circ\psi\cdot\Delta_{\bar{s}}\widetilde{F}}$ belong to $\mathcal{G}(\widetilde{X})$. Therefore,

$$\frac{\tilde{\phi}}{\varphi\circ\psi\cdot\Delta_{\bar{s}}\widetilde{F}}\in P_{<k}(Z_{<k}(Y_0),W).$$

By Theorem 5.17 we can pass to an extension $Y_1=Z_{<k}(Y_1)\times_{\tau_1} V$ where we can find a phase polynomial $p : Z_{<k}(Y_1)\to V$ such that $\varphi\circ p=\tilde{\phi}/(\varphi\circ\psi\cdot\Delta_{\bar{s}}\widetilde{F})$. Arguing as before we can pass to another extension $Y_2=Z_{<k}(Y_2)\times_{\tau_2} V$ and find an open subgroup $\mathcal{H}'$ of $\mathcal{H}$ of transformations in $\mathcal{G}(Z_{<k}(X))$, which has a lift in $\mathcal{G}(Y_2)$. Let $\bar{p}$ and $\bar{\psi}$ be the lifts of p and ψ to $Z_{<k}(Y_2)$, respectively. We conclude that for every $s\in\mathcal{H}'$ the transformation $S_{\tilde{s},\bar{\psi}\cdot p}$ induces the action of $S_{s,\phi}$ on X. Since $\mathcal{H}'$ is open, the group of all elements in $\mathcal{H}$ which admits a lift in $\mathcal{G}(X)$ contains the open subgroup $\mathcal{H}'$ and is therefore also open. □

The case where the group U in Theorem 6.12 is of dimension greater than zero follows by a similar argument. Indeed, if V is an extension of W by a finite-dimensional group U of dimension n, then V is an extension of $(S^1)^n\times W$ by a zero-dimensional group. Since the extensions which arise from Theorem 5.9 are zero-dimensional, this case is not needed in the proof of Theorem 6.1 and we leave the details for the interested reader.

7. Proof of Theorem 2.12, part II

As mentioned in the previous section it is enough to prove Theorem 6.6 in the case where the extension is weakly mixing. In this case we have the following result [Host and Kra 2005, Lemma 10.8].

Lemma 7.1. *Let $k\geq 0$ and let X be an ergodic G-system of order $<k+1$ and $\sigma : G\times X\to U$ be a weakly mixing cocycle with $d^{[k+1]}\sigma=\Delta F$. Let j be an integer with $0\leq j<k+1$ and let $\mathcal{G}_j$ be the j-th group in the lower central series for $\mathcal{G}$. Then, for every $t\in\mathcal{G}_j$ and a measurable map $\phi : X\to U$ the following are equivalent*:

(1) *For every $(k-j+1)$-face β, $\Delta_{t_\beta^{[k+1]}}F=d_\beta^{[k+1]}\phi$.*

(2) *For every $(k-j)$-face α, $\Delta_{t_\alpha^{[k+1]}}F/d_\alpha^{[k+1]}\phi$ is an invariant function on $X^{[k+1]}$.*

7A. *The main objects in the proof.* We begin by describing the objects which will be used in the proof of Theorem 6.6. For $0\leq j\leq k+1$ we construct a tower of extensions $Y_0\to Y_1\to\cdots\to Y_{k+1}=X$, where each Y_j is an $O_{k,m,j}(1)$- extension of Y_{j+1} (and therefore of X) of order $<k$ with the following properties:

- The subgroup
$$\mathcal{L}_j := \{s \in \mathcal{G}(X) : s \text{ has a lift in } \mathcal{G}(Y_j)\}$$
is open in $\mathcal{G}(X)$.
- The subgroup
$$V_j := \{s \in \mathcal{G}_j(X) \cap \mathcal{L}_j : s \text{ has a lift in } \mathcal{G}(Y_j)^\star\}$$
is open in $\mathcal{G}_j(X)$.
- For technical reasons we will also prove that V_j contains the subgroup generated by the commutators $\{[s, g] : s \in \mathcal{G}_{j-1}(X) \cap \mathcal{L}_j, g \in G\}$.

We note that in each step the group G is replaced with some $O_{k,m,j}(1)$-extension, we abuse notation and denote all of them by G. Since we only construct $k+1$ extensions $(Y_k, Y_{k-1}, \dots, Y_0)$, the final extension Y_0 is an $O_{k,m}(1)$-extension of X.

We construct these objects by downward induction on j: For $j=k+1$ we can take $Y_{k+1}=X$, $\mathcal{L}_{k+1}=\mathcal{G}$ and $V_{k+1}=\{e\}$. For $j=k$ we have the following lemma.

Lemma 7.2. *In the setting of Theorem 6.6, there exists an ergodic zero-dimensional $O_{k,m}(1)$-extension Y_k of order $<k$ such that $\mathcal{L}_k$ is open and $V_k = \mathcal{G}_k \cap \mathcal{L}_k$.*

The proof is a modification of the arguments of [Host and Kra 2005, Lemma 10.9].

Proof. Let $F : X^{[k+1]} \to U$ be such that $d^{[k+1]}\rho = \Delta F$. Since $\mathcal{G}(Z_{<k}(X))$ is $(k-1)$-step nilpotent, we conclude that any $t \in \mathcal{G}_k$ is an automorphism which fixes $Z_{<k}(X)$. Therefore, by Lemma A.12, $\Delta_t\rho$ is of type <1. Let $\chi \in \widehat{U}$. Then by Lemma 3.2 $\Delta_t\chi \circ \rho$ is (G, X, S^1)-cohomologous to a constant. We conclude by Lemma 5.11 that there exists a zero-dimensional $O_{k,m}(1)$-extension $\tilde{\pi} : \widetilde{X} \to X$ by a cocycle of type <1 such that
$$(\Delta_t\rho) \circ \tilde{\pi} = c_t \cdot \Delta F_t \tag{7-1}$$
for some constant $c_t : G \to U$ and a measurable map $F_t : \widetilde{X} \to U$. It follows that, for every 1-dimensional face α,
$$\frac{(\Delta_{t_\alpha^{[k]}} F) \circ \tilde{\pi}}{d_\alpha^{[k]} F_t} \tag{7-2}$$
is invariant in $\widetilde{X}^{[k]}$. Notice that $\widetilde{X}$ is a degenerate extension of X, therefore by the induction hypothesis of Theorem 6.12 there exists an $O_{k,m}(1)$-extension $\pi : Y_k \to \widetilde{X}$ (and therefore of X). By the same theorem the group $\mathcal{L}_k \le \mathcal{G}$ with respect to this extension is open. Any $t \in \mathcal{L}_k$ has a lift in $\mathcal{G}(Y_k)$. If in addition $t \in \mathcal{G}_k$ then (7-2) and Lemma 7.1 imply that t has a lift in $\mathcal{G}(Y_k)^\star$, where $\mathcal{G}(Y_k)^\star$ is defined with respect to the extension $Y_k \times_{\rho\circ\pi_X^{Y_k}} U$, as required. □

We climb up along the central series inductively. Suppose by induction that we have already constructed Y_{j+1}, $\mathcal{L}_{j+1}$ and V_{j+1} as above. We prove:

Lemma 7.3. *There exists an ergodic $O_{k,m}(1)$-extension Y_j' of Y_{j+1} such that*
$$\mathcal{L}_j' := \{s \in \mathcal{L}_{j+1} : s \textit{ has a lift in } \mathcal{G}(Y_j')\}$$

is open in $\mathcal{L}_{j+1}$ *and at the same time*

$$V'_j := \{s \in \mathcal{L}'_j \cap \mathcal{G}_j : s \textit{ has a lift in } \mathcal{G}(Y'_j)^\star\}$$

is open in $\mathcal{L}'_j \cap \mathcal{G}_j$.

Remark 7.4. For technical reasons, we will postpone the proof of this lemma until after Lemma 7.5. In other words, we will assume for now that this lemma was already established, and proceed to prove Lemma 7.5 below. At the end of this section we will prove Lemma 7.3 without relying on Lemma 7.5. The advantage is that some ideas and notions used in the proof of Lemma 7.3 are more natural in the settings of Lemma 7.5 and so we prefer to define these in the proof of Lemma 7.5 first.

The following lemma is the final step in the proof. We describe a process which allow (by passing to an extension) to add arbitrary countable set of transformations of the form $[s, g]$ to V'_j, where $s \in \mathcal{L}'_j \cap \mathcal{G}_{j-1}$ and $g \in G$.

Lemma 7.5. *There exists an ergodic zero-dimensional* $O_{k,m,j}(1)$*-extension* Y_j *of* Y'_j *such that* $\mathcal{L}_j = \{s \in \mathcal{L}'_j : s \textit{ has a lift in } \mathcal{G}(Y_j)\}$ *is open and* $V_j \subseteq \mathcal{G}_j \cap \mathcal{L}_j$ *satisfy the properties in Section 7A.*

Proof. Let $s \in \mathcal{L}'_j$, and let $g \in G$ be a generator in the natural basis of G. By the structure of G, the order of g is $p^{O_{k,m,j}(1)}$ for some prime p (the exact power is not important). Since $\mathcal{G}$ is k-step nilpotent, the order of $[s, g]$ is also $p^{O_{k,m,j}(1)}$ for possibly higher but bounded power. Since V'_j is open in $\mathcal{L}'_j \cap \mathcal{G}_j$, it is of at most countable index in that group. Therefore, we can find a countable set $\{s_n\}_{n\in\mathbb{N}}$ of transformations in $\mathcal{L}'_j \cap \mathcal{G}_j$, where each s_n is of order $p_n^{O_{k,m,j}(1)}$ for some primes p_n where the power is bounded uniformly for all n and such that $\bigcup_{n\in\mathbb{N}} s_n V'_j$ contains $\{[s, g] : s \in \mathcal{G}_{j-1} \cap \mathcal{L}'_j, g \in G\}$. By adding inverses, we may assume that $\{s_n : n \in \mathbb{N}\}$ contains all of its inverses.

Let $C_0 := \{s_n : n \in \mathbb{N}\}$, and for every $n \geq 1$ let $C_n := \{[s, g] : s \in C_{n-1}, g \in G\}$ and $C = \bigcup_{n\in\mathbb{N}} C_n$.

Definition. For $s \in C$ we define the complexity of s by

$$\operatorname{comp}(s) := \max\{n : s \in C_n\}.$$

We need the following result.

Lemma 7.6. *For every* $n \in \mathbb{N}$*, there exists a zero-dimensional* $O_{k,m}(1)$*-extension* $\pi_n : Y_{j,n} \to Y'_j$ *such that for every* $s \in C_n$ *and for every* 0*-dimensional face* α*, there exists* $\psi_s : Y_{j,n} \to U$ *such that*

$$(\Delta_{s_\alpha^{[k+1]}} F) \circ \pi_n - d_\alpha^{[k+1]} \psi_s \tag{7-3}$$

is $T_g^{[k+1]}$*-invariant for every* $g \in G$.

Proof. We prove the claim by downward induction on n. If $n \geq j$, then C_j is trivial and the claim follows. Fix some $0 \leq n < j$ and assume that Lemma 7.6 holds for all values greater than n. Let $s \in C_n$. For every $g \in G$, $[s^{-1}, g^{-1}]$ has complexity greater than n. Therefore, by the induction hypothesis there exists an extension $\pi_{n+1} : Y_{j,n+1} \to Y'_j$ and a measurable map $\psi_{s,g} : Y_{j,n+1} \to U$ such that $(\Delta_{[s^{-1},g^{-1}]_\alpha^{[k+1]}} F) \circ \pi_{n+1} - d_\alpha^{[k+1]} \psi_{s,g}$ is invariant for any 0-face α. Let $\bar{s}$ be any measure-preserving transformation on $Y_{j,n+1}$ (not necessarily in $\mathcal{G}(Y_{j,n+1})$), which induces the same action of s on Y'_j.[9]

[9]Such lift always exists. Recall that Y'_j is a tower of group extensions of X. Therefore, there exists a compact group K such that $Y'_j \cong X \times K$ as measure spaces. In particular, $\bar{s}(x, k) = (sx, k)$ is a lift for s.

Consider the function

$$\theta_s(g,y) := \psi_{s,g}(g\bar{s}y)\cdot\Delta_s\rho(g,\pi_{n+1}(y)). \tag{7-4}$$

As in the previous lemma, $\theta_s : G\times Y_{j,n+1}\to U$ is a function of type $< k-j+1$. Note that if the order of g is coprime to p then s commutes with g and so we can take $\psi_{s,g}=1$.

We replace θ_s with a cocycle:

Claim. *By embedding U into $(S^1)^{\mathbb{N}}$ using the Pontryagin dual, we view θ_s as a map into $(S^1)^{\mathbb{N}}$. We claim that there exists a constant $c_s : G\to(S^1)^{\mathbb{N}}$ and a cocycle $\theta'_s : G\times Y_{j,n+1}\to(S^1)^{\mathbb{N}}$ such that, for any generator $g\in G$, $\theta'_s(g)=c_s(g)\cdot\theta_s(g)$. Moreover, $c_s(g)=1$ whenever the order of g is coprime to the order of s.*

Proof of claim. Fix any 0-face α and $h\in G$, by the induction hypothesis of Lemma 7.6 we know that $\Delta_h^{[k+1]}(\Delta_{[s^{-1},g^{-1}]_\alpha^{[k+1]}}F)\circ\pi_{n+1}=d_\alpha^{[k+1]}\Delta_h\psi_{s,g}$. The same calculation as in (7-14) gives

$$d_\alpha^{[k+1]}\Delta_h\theta_s(g)=\Delta_{h^{[k+1]}}\Delta_{g^{[k+1]}}(\Delta_{\bar{s}_\alpha^{[k+1]}}F\circ\pi_{n+1}).$$

Since g and h commute, we have $d_\alpha^{[k+1]}\Delta_h\theta_s(g)=d_\alpha^{[k+1]}\Delta_g\theta_s(h)$ for any 0-face α. We conclude that

$$\Delta_h\theta_s(g)=\Delta_g\theta_s(h). \tag{7-5}$$

Therefore, by ergodicity we have

$$\frac{\theta_s(g+g')}{\theta_s(g)T_g\theta_s(g')}=c_s(g,g') \tag{7-6}$$

for all $g,g'\in G$ and some constant $c_s(g,g')$.

From this we conclude that $\prod_{k=0}^{\mathrm{order}(g)-1}T_g^k\theta_s(g)$ is a constant in U. In order to take roots we embed U in the divisible group $(S^1)^{\mathbb{N}}$ and choose $c_{s,\chi}(g)\in S^1$ such that $c_{s,\chi}(g)^{\mathrm{order}(g)}=\prod_{k=0}^{p^d-1}T_g^k\chi\circ\theta_s$. Note that if g is of order coprime to p, then s and g commute. In this case $\psi_{s,g}=1$ and $\prod_{k=0}^{\mathrm{order}(g)-1}T_g^k\chi\circ\theta_s=1$. In other words, if g is of order coprime to p, we can take $c_{s,\chi}(g)=1$. Let $\tilde{\theta}_{s,\chi}(g):=\chi\circ\theta_s(g)/c_{s,\chi}(g)$. We see that $\prod_{k=0}^{\mathrm{order}(g)-1}T_g^k\tilde{\theta}_{s,\chi}(g)=1$ and $\Delta_h\tilde{\theta}_{s,\chi}(g)=\Delta_g\tilde{\theta}_{s,\chi}(h)$ for every $h,g\in G$. Let $\theta'_{s,\chi}$ be the cocycle as in Proposition 5.2. □

We return to the proof of Lemma 7.6. Consider the cocycle

$$\theta := (\theta'_{s,\chi})_{s\in C_n,\chi\in\widehat{U}} : G\times Y_{j,n+1}\to(S^1)^{\mathbb{N}}$$

and choose a minimal cocycle $\sigma : G\times Y_{j,n+1}\to(S^1)^{\mathbb{N}}$ which is cohomologous to θ. Let $Y_{j,n}=Y_{j,n+1}\times_\sigma W$, where W is the image of σ and assume for now that W is zero-dimensional (proof below). Since θ is cohomologous to σ it is a coboundary on $Y_{j,n}$. This implies that $\chi\circ\theta_s$ is cohomologous to a constant for every $\chi\in\widehat{U}$ and $s\in C_n$. Using Theorem 5.17 we can replace $Y_{j,n}$ with an extension such that θ_s is $(G,Y_{j,n},U)$-cohomologous to a constant on that extension. In that case we can find $\psi_s : Y_{j,n}\to U$ such that $\theta_s-d\psi_s$ is a constant and this ψ_s satisfies (7-3). This completes the proof of Lemma 7.6. □

We return to the proof of Lemma 7.5. Consider the extension $Y_{j,0}$ from the previous lemma. This is a degenerate extension of Y'_j. Therefore, by Theorem 6.12 we can find an extension Y_j such that the subgroup $\mathcal{L}_j=\{s\in\mathcal{L}'_j : s \text{ has a lift in } \mathcal{G}(Y_j)\}$ is open. Let V_j be as in the theorem. Since $V'_j\cap\mathcal{L}_j\subseteq V_j$, we have that V_j is open in $\mathcal{L}_j\cap\mathcal{G}_j$. Recall that $\{[s,g]:s\in\mathcal{G}_{j-1}\cap\mathcal{L}'_j,\, g\in G\}\subseteq C_0\cdot V'_j$. Let $s\in\mathcal{L}'_j\cap\mathcal{G}_{j-1}$ and $g\in G$. From (7-3) and Lemma 6.4 it follows that if s has a lift in $\mathcal{G}(Y_{j,0})$, then $[s,g]$ has a lift in $\mathcal{G}(Y_{j,0})^\star$. From this and Corollary 6.13, we conclude that if s has a lift in $\mathcal{G}(Y'_j)$, then $[s,g]$ it has a lift in $\mathcal{G}(Y'_j)^\star$. Since all elements in $\mathcal{L}_j$ has lifts in $\mathcal{G}(Y'_j)$ we conclude that if $s\in\mathcal{L}_j\cap\mathcal{G}_{j-1}$ then $[s,g]$ has a lift in $\mathcal{G}(Y'_j)^\star$. We conclude that any element of the form $[s,g]$ where $s\in\mathcal{L}_j\cap\mathcal{G}_{j-1}$ and $g\in G$ has a lift in $\mathcal{G}(Y'_j)^\star$. In other words $\{[s,g]:s\in\mathcal{L}_j\cap\mathcal{G}_{j-1},\, g\in G\}\subseteq V_j$, as required.

It is left to show that W is zero-dimensional. Fix $s\in C_n$ and $\chi\in\widehat{U}$, we prove that there exists N such that $(\theta'_{s,\chi})^N$ is a coboundary. We need the following lemma.

Lemma 7.7. *Let Y be an ergodic extension of a G-system X of order $<k$. We denote by $\mathcal{G}$ be the Host–Kra group of X. Let p be a prime number and write $G=G_p\oplus G_p^\perp$, where G_p is the p-component of G. Let $m\geq 0$ and suppose that $f:G\times Y\to S^1$ satisfies that $d^{[m]}f=\Delta F$, for some $F:Y^{[m]}\to S^1$ which is measurable with respect to $X^{[m]}$. Then, for every $s\in\mathcal{G}$ of order p^n for some $n\in\mathbb{N}$, there exists a function $\sigma_s:G\times X\to S^1$ and a natural number $N=O_{k,n,m}(1)$ such that with $\sigma_s(g,x)=1$ for all $g\in G_p^\perp$ and*

$$(\Delta_s f\cdot\sigma_s)^N$$

is a (G,X,S^1)-coboundary. Furthermore $(d^{[m]}\Delta_s f\cdot\sigma_s)^{N'}=\Delta\Delta_s^{[m]}F^{N'}$ for some natural number $N'=O_{k,n,m}(1)$.

We briefly explain the idea behind this result. Let n be a natural number and let s be a transformation of order n. Since $\Delta_{s^n}f=1$, the cocycle identity gives

$$\Delta_s f^n=\prod_{k=0}^{n-1}\Delta_s\Delta_{s^k}\bar{f}.$$

Assume hypothetically that s is an automorphism. Then, by Lemma A.12, the type of $\Delta_s f^n$ is smaller than the type of $\Delta_s f$. If we repeat this process iteratively, we will eventually get that some power of $\Delta_s f$ is a coboundary.

In the lemma we do not assume that s is an automorphism. However, (7-4) indicates that up to a multiplication by some function it still behaves like one. The formal proof is given below.

Proof of Lemma 7.7. We prove the lemma by induction on m. For $m=0$ we have that $f=\Delta F$. Therefore, since F is measurable with respect to X we have

$$\Delta_s f=\Delta_s\Delta F=\Delta\Delta_s F\cdot V_sT_g\Delta_{[s^{-1},g^{-1}]}F$$

and the claim follows by taking $\sigma_s(g,x):=V_sT_g\Delta_{[s^{-1},g^{-1}]}\overline{F}(x)$. Note that if $g\in G_p^\perp$ then s and g commutes, in this case $\sigma_s(g,x)=1$.

Claim. *Fix $1\leq j\leq k$, let $s\in\mathcal{G}_j$ of order p and β be an $(m-j)$-dimensional face (or a vertex if $j\geq m$). Then, there exists a natural number $M=O_{m,j,n}(1)$ and a function $\phi_s:Y\to S^1$ such that $\Delta_{s_\beta^{[m]}}F^M=d_\beta^{[m]}\phi_s^M$.*

We prove the claim by downward induction on j. If $j=k$, then $s=e$ and the claim is trivial. Fix $j<k$ and assume that the claim holds for all values greater than j. Let $s\in\mathcal{G}_j$ be as in the lemma. Then

by the induction hypothesis we see that for every $g\in G$ and every $(m-j-1)$-dimensional face β there exist a power M and $\phi_{s,g}:X\to S^1$ such that

$$\Delta_{[s^{-1},g^{-1}]^{[m]}_\beta}F^M=d^{[m]}_\beta\phi^M_{s,g}.$$

We use this to prove Lemma 7.7 for all $s\in\mathcal{G}_j$ for this specific j and then we use the lemma to prove the rest of the claim. Let $\sigma_s(g,x):=V_sT_g\phi_{s,g}(x)$ and observe that if $g\in G_p^\perp$ then s and g commute and so we can take $\phi_{s,g}=1$. Let $f'_s:=\Delta_sf\cdot\sigma_s$. As in (7-14) we have

$$\begin{aligned}\Delta(\Delta_{s^{[m]}_\beta}F^M)&=V^{[m]}_{s_\beta}T^{[m]}_g\Delta_{[s^{-1},g^{-1}]^{[m]}_\beta}F^M\cdot\Delta_{s^{[m]}_\beta}\Delta F^M\\&=d^{[m]}_\beta V_s\circ T_g\phi^M_{s,g}\cdot\Delta_{s^{[m]}_\beta}d^{[m]}\rho^M=d^{[m]}_\beta f'^M_s.\end{aligned}\tag{7-7}$$

Since this is true for every $(m-j)$-dimensional face β, we see by Lemma A.8 that f'^M_s is of type $<m-j$ (or a coboundary if $j\ge m$). We conclude that there exists a measurable map $F_s:Y^{[m-j]}\to S^1$, which is measurable with respect to $X^{[m-j]}$ such that $d^{[m-j]}f'^M_s=\Delta F^M_s$. Moreover, we note here that $d^{[j]}F^M_s=\Delta_{s^{[m]}}F^M$. Since f'^M_s is of smaller type, we can apply the induction hypothesis for the transformations $s,s^2,s^3,\dots,s^{p^n-1}$. We conclude that there exist N, $N'=O_{k,m,n}(1)$ and $\sigma'_{s,l}$ with

$$(d^{[m-j]}\Delta_{s^l}f'_s\cdot\sigma'_{s,l})^{N'}=\Delta\Delta_{s^{l[m-j]}}F^{N'}_s\tag{7-8}$$

and

$$\Delta_{s^l}f'^N_s\cdot{\sigma'_{s,l}}^N\in B^1(G,X,S^1).\tag{7-9}$$

By replacing N and N' with NM and $N'M$ we can assume without loss of generality that N and N' are multiples of M. Recall, that $f'_s=\Delta_sf\cdot\sigma_s$. Let p^n be the order of s then by the cocycle identity we have

$$1=\Delta_{s^{p^n}}f=\prod_{k=0}^{p^n-1}\Delta_sV_{s^k}f=(\Delta_sf)^{p^n}\cdot\prod_{k=1}^{p^n-1}\Delta_{s^k}\Delta_sf$$

and it follows that $(\Delta_sf)^{p^n}=\prod_{k=1}^{p^n-1}\Delta_{s^k}\Delta_s\bar f$.

From all of this we conclude that

$$(\Delta_s\bar f)^{N'Np^n}\prod_{k=1}^{p^n-1}\Delta_{s^k}\sigma_s^{N'N}\cdot\prod_{k=1}^{p^n-1}\sigma'^{N'N}_{s,k}=\prod_{k=1}^{p^n-1}(\Delta_{s^k}(\Delta_sf^{N'N}\sigma_s^{N'N})\cdot\sigma'^{N'N}_{s,k})=\prod_{k=1}^{p^n-1}(\Delta_{s^k}f'_s\cdot\sigma_{s,k})^{N'N},\tag{7-10}$$

which by (7-9) is a coboundary. Moreover by (7-8) we have that

$$d^{[m-j]}(\Delta_s\bar f)^{N'Np^n}\prod_{k=1}^{p^n-1}\Delta_{s^k}\sigma_s^{N'N}\cdot\prod_{k=1}^{p^n-1}\sigma'^{N'N}_{s,k}=\Delta\prod_{k=1}^{p^n-1}\Delta_{s^{k[m-j]}}\bar F^{NN'}_s.\tag{7-11}$$

Choose $\widetilde N=N'Np^n$ and $\tilde\sigma_s$ any measurable function which satisfies that

$$\sigma_s^{\widetilde N}=\Big(\prod_{k=1}^{p^n-1}\Delta_{s^k}\sigma_s^{N'N}\cdot\prod_{k=1}^{p^n-1}\sigma'^{N'N}_{s,k}\Big)^{-1}$$

and that $\tilde{\sigma}_s(g,x)=1$ whenever $g\in G_p^{\perp}$. We conclude that

$$d^{[m-j]}\Delta_s f^{\widetilde{N}}\cdot\tilde{\sigma}_s^{\widetilde{N}}=\Delta\prod_{k=1}^{p^n-1}\Delta_{s^{k[m-j]}}F_s^{N'N}. \tag{7-12}$$

From (7-11) and since $d^{[j]}F_s^M=\Delta_{s^{[m]}}F^M$, we get that, for every $(m-j)$-dimensional face β,

$$d_\beta^{[m]}\Delta_s f\tilde{\sigma}_s^{\widetilde{N}}=\Delta\Delta_{s_\beta^{[m]}}F^{\widetilde{N}}. \tag{7-13}$$

Since this is true for every β we conclude that $d^{[m]}\Delta_s f^{\widetilde{N}}\cdot\tilde{\sigma}_s^{\widetilde{N}}=\Delta\Delta_{s^{[m]}}F^{\widetilde{N}}$, which completes the proof of the lemma. It is left to prove the claim for this j.

Observe that

$$d_\beta^{[m]}\Delta_s f=\Delta_{s_\beta^{[m]}}\Delta F=\Delta\Delta_{s_\beta^{[m]}}F\cdot V_{s_\beta^{[k+1]}}T_{g^{[k+1]}}\Delta_{[s^{-1},g^{-1}]_\beta^{[m]}}F.$$

Plugging this above we get that

$$(\Delta_{[s^{-1},g^{-1}]_\beta^{[m]}}F^{\widetilde{N}}(y))(d_\beta^{[m]}T_g^{-1}V_s^{-1}\sigma_s(g,y)^{\widetilde{N}})^{-1}$$

is invariant with respect to the diagonal action of G on $Y^{[m]}$. Since β is an $(m-j)$-dimensional face, the claim follows by Lemma 7.1. □

We return to the proof of Lemma 7.5. By what we just proved, there exists a power N and a function $\sigma_{s,\chi}$ such that ${\theta'_{s,\chi}}^N\cdot\sigma_{s,\chi}$ is a coboundary. Since $\theta'_{s,\chi}$ is a cocycle of type $<k-j+1$, so is $\sigma_{s,\chi}$. As in the lemma above, $\sigma_{s,\chi}(g,\cdot)$ is trivial for any $g\in G_p^{\perp}$. Therefore, by the cocycle equation it is invariant under the action of $G_p^{\perp}$. Recall that the action of G on $Y_{j,n+1}$ is ergodic and let $Y'_{j,n+1}$ be the factor of $Y_{j,n+1}$ which corresponds to the σ-algebra of the $G_p^{\perp}$-invariant functions. The induced action of G_p on $Y'_{j,n+1}$ is therefore ergodic.

We consider this system as an $\mathbb{Z}/p^d\mathbb{Z}^\omega$-system for some fixed d. By the main theorem of [Bergelson et al. 2010], any $\mathbb{Z}/p^d\mathbb{Z}^\omega$-cocycle is cohomologous to a phase polynomial of some bounded degree. By Proposition C.1 there exists some power p^n such that $\sigma_{s,\chi}^{p^n}$ is a coboundary. Therefore ${\theta'_{s,\chi}}^{Np^n}$ is a coboundary. As $n=O_{k,m}(1)$, we conclude that the image of the minimal cocycle cohomologous to θ'_s takes values in a finite-dimensional group of exponent $O_{k,m}(1)$. The proof of Theorem 6.6 is now complete. □

It is left to prove Lemma 7.3.

Proof. Let $s\in\mathcal{G}(Y_{j+1})$ be any lift of an element $s'\in\mathcal{G}_j\cap\mathcal{L}_{j+1}$. By assumption, $[s'^{-1},g^{-1}]\in V_{j+1}$ for every $g\in G$. We conclude that there exists $\psi_{s,g}:Y_{j+1}\to U$ such that for every $(k-j+1)$-face β we have

$$\Delta_{[s^{-1},g^{-1}]_\beta^{[k+1]}}F\circ\pi=d_\beta^{[k+1]}\psi_{s,g},$$

where $\pi:Y_{j+1}^{[k+1]}\to X$ is the factor map. Consider the function $\theta_s(g,y):=\psi_{s,g}(g\bar{s}y)\cdot\Delta_s\rho(g,\pi(y))$, where $\pi:Y_{j+1}\to X$ is the factor map. The following computation is taken from [Host and Kra 2005, Proposition 10.10]:

$$\Delta_{g^{[k+1]}}(\Delta_{s_\beta^{[k+1]}}F(\pi(y))) = V_{s_\beta^{[k+1]}}\circ T_{g^{[k+1]}}(\Delta_{[s^{-1},g^{-1}]_\beta^{[k+1]}}F(\pi(y)))\cdot\Delta_{s_\beta^{[k+1]}}\Delta_{g^{[k+1]}}F(\pi(y))$$
$$= d_\beta^{[k+1]}V_s\circ T_g(\psi_{s,g}(y))\cdot\Delta_{s_\beta^{[k+1]}}d^{[k+1]}\rho(g,\pi(y)) = d_\beta^{[k+1]}\theta_s, \tag{7-14}$$

where the last equality follows from the fact that $\Delta_{s_\beta^{[k+1]}}d^{[k+1]}\rho(g,\pi(y)) = d_\beta^{[k+1]}\Delta_s\rho(g,\pi(y))$. It follows that $d_\beta^{[k+1]}\theta_s$ is a $(G, Y_{j+1}^{[k+1]}, S^1)$-coboundary for every $(k-j+1)$-face β and therefore θ_s is a function of type $<k-j+1$ (Lemma A.8). By Lemma 5.16 we can extend Y_j and assume that all θ_s are line-cocycles (the polynomial term p in Lemma 5.16 can be ignored by changing $\psi_{s,g}$ with $\psi_{s,g}/p$).

The map $s\mapsto\theta_s$ is a measurable map from $\mathcal{G}_j$ to functions of type $<k-j+1$. By Theorem 3.1 and Baire theorem we have for every χ a nonmeagre measurable set $\mathcal{A}_\chi\subseteq\mathcal{G}_j$ such that, for every $s,t\in\mathcal{A}_\chi$, $\chi(\theta_s/\theta_t)$ is (G, Y_{j+1}, S^1)-cohomologous to a phase polynomial of degree $<k-j+1$. Assume for now that we can choose the same set $\mathcal{A}$ for all $\chi\in\widehat{U}$ simultaneously. This assumption will be explained at the end of the proof.

By Lemma 5.11 we can find an $<O_{k,m}(1)$-extension $\widetilde{Y}_{j+1}$ of Y_{j+1} such that as a function on $\widetilde{Y}_{j+1}$, θ_s/θ_t is $(G,\widetilde{Y}_{j+1},U)$-cohomologous to a phase polynomial of degree $<k-j+1$. Therefore, for every $s,t\in\mathcal{G}_j\cap\mathcal{L}_j$ there exists a measurable function $\theta_{s,t}:\widetilde{Y}_{j+1}\to U$ with $d_\beta^{[k+1]}\theta_s/\theta_t = \Delta d_\beta^{[k+1]}\theta_{s,t}$. Since $\widetilde{Y}_{j+1}$ is a degenerate extension of Y_{j+1}, we can use Theorem 6.12. Thus, we can pass to an extension Y_j' of order $<k+1$ such that

$$\mathcal{L}_j' := \{s\in\mathcal{L}_{j+1} : s \text{ has a lift in } \mathcal{G}(Y_j')\}$$

is open. By lifting everything to Y_j' it follows from (7-14) that

$$\Delta(V_{\bar{s}_\beta^{[k+1]}}F\circ\pi_X^{Y_j'} - V_{\bar{t}_\beta^{[k+1]}}F\circ\pi_X^{Y_j'}) = \Delta d_\beta^{[k+1]}\theta_{s,t},$$

where F and $\theta_{s,t}$ are viewed as functions on Y_j' and $\bar{s},\bar{t}$ are any lifts of s and t. It follows that

$$\Delta_{\bar{s}_\beta^{[k+1]}}F\circ\pi_X^{Y_j'} - V_{\bar{t}_\beta^{[k+1]}}F\circ\pi_X^{Y_j'} - d_\beta^{[k+1]}\theta$$

is invariant in $(Y_j')^{[k+1]}$. Since $t\in\mathcal{G}_j$, it maps the σ-algebra $\mathcal{I}_{k+1}(X)$ to itself. Moreover, since $F\circ\pi_X^{Y_j'}$ is measurable with respect to X, we have that

$$\Delta_{\bar{s}\bar{t}_\beta^{-1[k+1]}}F\circ\pi_X^{Y_j'} - \Delta d_\beta^{[k+1]}V_{\bar{t}^{-1}}\theta_{s,t}$$

is invariant with respect to the diagonal action of G on $X^{[k+1]}$. Now, by Lemma 6.4, we conclude that for every lifts of $s,t\in\mathcal{A}$ the element corresponding to st^{-1} in $\mathcal{G}(Y_j')$ is in $\mathcal{G}(Y_j')^\star$, where $\mathcal{G}(Y_j')^\star$ is defined with respect to the extension

$$Y_j'\times_{\rho\circ\pi_X^{Y_j'}}U.$$

Thus, V_j' contains $\mathcal{A}\cdot\mathcal{A}^{-1}$ and so the proof is complete by Lemma B.2.

It is left to establish the assumption above about the existence of a measurable set $\mathcal{A}$ of positive measure, which satisfies that $\chi\circ\theta_s/\chi\circ\theta_t$ is cohomologous to a polynomial of degree $<k-j+1$ for every $s,t\in\mathcal{A}$

and every $\chi \in \widehat{U}$. Let $\Delta \leq U$ be a subgroup of bounded exponent such that U/Δ is a Lie group. Let $\chi_1, \ldots, \chi_n \in \widehat{U}$ be a lift of a basis of the dual of U/Δ and $\pi_1, \pi_2, \ldots$ a lift of the coordinate maps in the dual of Δ. Since $\{\chi_1, \ldots, \chi_n\}$ is a finite set of characters, we can apply Theorem 5.15 and find a set $\mathcal{A}$ of positive measure such that, for every $1 \leq i \leq n$, $\chi_i(\theta_s/\theta_t)$ is cohomologous to a phase polynomial of degree $< k - j + 1$, simultaneously.

We also notice, as in the proof of Lemma 7.3 that $\chi(\theta_s/\theta_t)$ is cohomologous to a phase polynomial of degree $< k - j + 1$ if and only if $\Delta_{(st^{-1})_\beta^{[m]}} \chi \circ F = d_\beta^{[m]} \phi$ for some $\phi : Y_{j+1} \to S^1$. Let $\mathcal{H}$ be the subgroup generated by $\mathcal{A} \cdot \mathcal{A}^{-1}$. Then, for every $1 \leq i \leq n$, $h \in H$ and an $(k-j+1)$-dimensional face β there exists $\phi_{h,i}$ such that

$$\Delta_{h_\beta^{[m]}} \chi_i \circ F = d_\beta^{[m]} \phi_{h,i}.$$

Now, let π be one of the maps $\pi_1, \pi_2, \ldots$. Then, by Theorem 5.15, we can find a set of positive measure $\mathcal{A}_\pi \subseteq \mathcal{A}$ such that $\pi(\theta_s/\theta_t)$ is cohomologous to a phase polynomial of degree $< k - j + 1$. As before, let $\mathcal{H}_\pi$ be the group generated by $\mathcal{A}_\pi \cdot \mathcal{A}_\pi^{-1}$. We conclude that for every $h \in \mathcal{H}_\pi$ we have

$$\Delta_{h_\beta^{[k+1]}} \pi \circ F = d_\beta^{[k+1]} \phi_{h,\pi}.$$

In particular, we see that for every $s \in \mathcal{H}_\pi$, $\pi \circ \theta_s$ is cohomologous to a phase polynomial of degree $< k - j + 1$. We want to extend $\mathcal{H}_\pi$ to $\mathcal{H}$. For every i, $\mathcal{H}_{\pi_i}$ is an open subgroup of $\mathcal{H}$. We conclude that the index $[\mathcal{H} : \mathcal{H}_{\pi_i}]$ is at most countable. Thus, for each $\pi_i \in \{\pi_i : i \in \mathbb{N}\}$ we find a set of countably many transformations $\{s_{n,\pi_i} : n \in \mathbb{N}\}$ such that $\bigcup_{n\in\mathbb{N}} s_{n,\pi_i} \mathcal{H}_{\pi_i} = \mathcal{H}$. It is left to show that $\Delta_{s_{n,\pi_i}}^{[k+1]} \pi_i \circ F = d_\beta^{[k+1]} \phi_{n,\pi_i}$ for some $\phi_{n,\pi_i} : Y_{j+1} \to S^1$. Indeed, in this case we have that $s \in \mathcal{H}$ and $i \in \mathbb{N}$, the cocycle $\pi_i(\theta_s)$ is cohomologous to a phase polynomial of degree $< k - j + 1$. In particular, we can take $\mathcal{A} = \mathcal{H}$ and the proof is complete.

Since the set $\{s_{n,\pi_i} : n \in \mathbb{N}\}$ is countable, we can use the same argument as in Lemma 7.6 with one minor modification. This time the elements s_{n,π_i} are not of finite order (but the commutators are). Therefore in the last step, we cannot use Lemma 7.7 in order to deduce that W is zero-dimensional. Instead, recall that for each π_i there exists a constant m_i such that $\pi_i^{m_i} \in \langle \chi_1, \ldots, \chi_n \rangle$. This means, in particular, that some power $d = O_{k,m}(1)$ of $\pi_i(\theta_s)$ is (G, Y_{j+1}, S^1)-cohomologous to a phase polynomial $p_{s,i}$ of degree $< k - j + 1$. As in the claim in Lemma 7.6 we can find a constant $c_{s,i}$ and a cocycle $\theta'_{s,i} = \pi_i(\theta_s) \cdot c_{s,i}$. It follows that $\theta'^d_{s,i}$ is a phase polynomial of degree $< k - j + 1$. By Theorem 5.9 we can also find a phase polynomial cocycle $q_{s,i}$ of degree $< k - j + 1$ such that $q_{s,i}^d = p_{s,i}$ (by passing to an extension). We conclude that $\theta'_{s,i}/q_{s,i}$ is a cocycle, and the d-th power of this cocycle is a coboundary. As in Lemma 7.6, by extending with a minimal cocycle which is cohomologous to $\theta'_{s,i}/q_{i,s}$ we can assume that the latter is a coboundary. Therefore, $\pi_i \circ \theta_s$ is cohomologous to a phase polynomial of degree $< k - j + 1$ for all $s \in \mathcal{H}$, which completes the proof. □

7B. *Concluding everything.* To finish the proof of Theorem 2.12 we need to following variant of a theorem of [Furstenberg and Weiss 1996].

Lemma 7.8. *Let X and Y be ergodic G-systems and $\pi : Y \to X$ be the factor map. Let $\rho : G \times X \to U$ be a cocycle and suppose that $X' = X \times_\rho U$ is ergodic. If σ is the minimal cocycle cohomologous to $\rho \circ \pi : G \times Y \to U$ and V is the image of σ, then X' is a factor of $Y \times_\sigma V$.*

Proof. Consider the (possibly nonergodic) system $Y\times_{\rho\circ\pi}U$. It follows by the theory of Mackey (see [Furstenberg and Weiss 1996, Proposition 7.1]) that every ergodic component of $Y\times_{\rho\circ\pi}U$ is isomorphic to $Y\times_\sigma V$ for some $V\leq U$. Choose any ergodic invariant measure $\mu_{Y'}$ on $Y\times_{\rho\circ\pi}U$. It is easy to see that the push-forward of $\mu_{Y'}$ to Y is μ_Y. Moreover, since X is a factor of Y we conclude that the push-forward of $\mu_{Y'}$ to X is μ_X. Let $\mu_{X'}$ be the push-forward of $\mu_{Y'}$ to X'. Since X' is ergodic $\mu_{X'}$ must be the product measure $\mu_X\times m_U$, where m_U is the Haar measure on U (see [Host and Kra 2018, Section 2.2, Lemma 4]). In other words, X' is a factor of $Y\times_\sigma V$ as required. □

Given an ergodic G-system X, by Theorem 4.3 it is an inverse limit of finite-dimensional systems $X=\varprojlim X_n$. By Theorem 6.1, we can find a constant $l=O_k(1)$ and for each $(X_n,G^{(l)})$ we can find an extension $(Y_n,G^{(l)})$ which is an finite-dimensional nilsystem. By increasing Y_n we may assume that Y_{n-1} is a factor of Y_n. More concretely, in the proof of Theorem 6.6 we build Y_n as a sequence of extensions of X_n (by zero-dimensional groups). In each step, instead of extending by the groups associated to X_n we can also extend by the groups associated to the previous systems $X_{n-1},\dots,X_1$ (and replace with minimal cocycles, as in Lemma 7.8). In this case $Y:=\varprojlim Y_n$ is an inverse limit of nilsystems. It is standard that $(X,G^{(l)})$ is a factor of $(Y,G^{(l)})$ or equivalently that Y is an l-extension of X as required.[10]

8. Proving the identification $\mathcal{G}(X)/\Gamma\cong X$

The goal of this section is to deduce Theorem 6.1 from Theorem 6.6 and thus complete the proof of Theorem 2.12. Given a finite-dimensional system X, we have already established the existence of a finite-dimensional extension Y which is an inverse limit of systems Y_n, where the action of $\mathcal{G}(Y_n)$ on Y_n is transitive. It is therefore enough to derive the identification $Y_n\cong\mathcal{G}(Y_n)/\Lambda_n$ for some totally disconnected closed subgroup Λ_n of $\mathcal{G}(Y_n)$. In other words it suffices to prove the following theorem.

Theorem 8.1. *Let X be an ergodic G-system of order k and suppose that $\mathcal{G}(X)$ acts transitively on X (as a near-action), then there exists a totally disconnected subgroup $\Lambda\leq\mathcal{G}(X)$ so that X and $\mathcal{G}(X)/\Lambda$ are isomorphic as $\mathcal{G}(X)$-systems.*

In order to prove this theorem we construct a topological model for X. That is a compact Hausdorff space $\widehat{X}$ with a continuous action $\widehat{T}:\mathcal{G}(X)\times\widehat{X}\to\widehat{X}$ such that X and $\widehat{X}$ are isomorphic as measure spaces. Write $X=Z_k(X)\times_\rho U$, by induction hypothesis we may write $Z_{<k}(X)=\mathcal{L}/\Lambda_{\mathcal{L}}$ and we have a projection map $p:\mathcal{G}(X)\to\mathcal{L}$ which is onto.

Definition 8.2. Let H be a Polish group with a near-action on X. A function $f\in L^\infty(X)$ is said to be H-continuous if $f(hx)\to f(x)$ in $L^\infty(X)$ as $h\to 1_H$.

Proposition 8.3. *Let $\mathcal{A}\subseteq L^\infty(X)$ denote the algebra of $\mathcal{G}(X)$-continuous functions. We claim that the unit ball of this algebra is dense in the unit ball of $L^\infty(X)$ with respect to the L^2-topology.*

The following result [Gleason 1950, Theorem 3.3] will be used to lift the $\mathcal{L}$-continuity to a $\mathcal{G}(X)$-continuity.

[10] Another approach would be to use a version of Lemma A.4 from [Frantzikinakis and Host 2018] and replace Y_n with an ergodic joining of $Y_n,Y_{n-1},\dots,Y_1$.

Theorem 8.4. *Let U be a compact Lie group acting freely and continuously on a completely regular topological space X. Let $q : X \to X/U$ be the quotient map where $x \sim_U y$ if there exists $u \in U$ so that $ux = y$ and X/U is equipped with the quotient topology. Then every point $x \in X/U$ has an open neighborhood $x \in V \subseteq U/X$ such that there is a local continuous section $s : V \to X$ so that $p \circ s = \mathrm{Id}_V$.*

Proof of Proposition 8.3. Let f be a continuous function on X with $\|f\|_\infty \le 1$ and let $\varepsilon > 0$ be arbitrary. Recall that $X = \mathcal{L}/\Lambda_{\mathcal{L}} \times U$. Since f is uniformly continuous, we can find an open subset $U' \le U$ so that $\|f(ux) - f(x)\|_\infty < \varepsilon/2$ for all $u \in U'$. By Gleason–Yamabe, we can find a subgroup $J \le U$ so that $J \subseteq U'$ and U/J is a Lie group. Let $\tilde{f}(x) = \int_J f(jx)\, dj$, where dj is the Haar measure on J. We see that $\tilde{f}$ is J-invariant and

$$\|\tilde{f}(x) - f(x)\| < \varepsilon/2. \tag{8-1}$$

Recall that we have a surjective homomorphism $p : \mathcal{G}(X) \to \mathcal{L}$. By Lemma 6.4 we can identify the kernel of p with $P_{<k+1}(Y, U)$, where $Y := Z_{k-1}(X) = \mathcal{L}/\Lambda_{\mathcal{L}}$. Quotienting by $P_{<k+1}(Y, J)$ we get the short exact sequence

$$1 \to P_{<k+1}(Y, U)/P_{<k+1}(Y, J) \to \mathcal{G}(X)/P_{<k+1}(Y, J) \to \mathcal{L} \to 1. \tag{8-2}$$

Since U/J is a Lie group, we deduce by Lemma A.17 that so is $K := P_{<k+1}(Y, U)/P_{<k+1}(Y, J)$. In particular, K is locally compact and admits a Haar measure dK. Observe that $\tilde{f}$ is invariant to translations by $P_{<k+1}(Y, J)$; hence for every continuous function ϕ on K the convolution

$$\tilde{f} * \phi(x) := \int_K \tilde{f}(kx)\phi(k)\, dk$$

is well-defined on $X/J := Y \times U/J$ and K-continuous. Letting ϕ be a suitable approximation to the identity (nonnegative, supported on a small neighborhood of the identity, and of total mass one) we deduce that

$$\|\tilde{f} - \tilde{f} * \phi\|_{L^2(X/J)} < \varepsilon/2. \tag{8-3}$$

Moreover, $\tilde{f} * \phi$ is a K-continuous function and an $\mathcal{L}$-continuous function.

We use Gleason theorem and (8-2) to show that $\tilde{f}*\phi$ is $\mathcal{G}(X)/P_{<k+1}(Y, J)$-continuous. By Lemma A.17, $K = U/J \times D$, where D is a countable discrete group. Fix a complete metric d on U which induces its topology and normalize so that $d(u, v) \le 1$ for all $u, v \in U$. For every two functions f, f' taking values in U we define $\|f - f'\|_\infty = \sup d(f(x), f'(x))$. We make an observation that D remains discrete even when $\mathcal{G}(X)$ is equipped with the following finer metric[11] $d(S_{l,f}, S_{l',f'}) = d_{\mathcal{L}}(l, l') + \|f - f'\|_\infty$. Moreover, the action of K on $\mathcal{G}(X)$ is still continuous.

Let $B \subseteq \mathcal{G}(X)$ be an open neighborhood of the identity so that $B \cap D = \{1\}$. Identify U/J with the subgroup $U/J \times \{1\} \subseteq K$ and replacing B with $U/J \cdot B$ we can assume that $K \cap B = U/J$. The Lie group U/J then acts continuously on B. Moreover, any element in the quotient $B/(U/J)$ can be identified with a unique element of $\mathcal{L}$. By Gleason theorem, we can find an open neighborhood V of the identity in $\mathcal{L}$ and a local continuous section $l \mapsto S_{l,\phi_l} \in B \subseteq \mathcal{G}(X)$, where the latter is equipped with the finer metric

[11] We note that $\mathcal{G}(X)$ is no longer a topological group with respect to that metric.

introduced above. Since $\tilde{f}$ is continuous on X/J, so is $\tilde{f}*\phi$. Without loss of generality we can take $\phi_1=1$ by replacing ϕ_l with ϕ_l/ϕ_1 for all $l\in V$.

We now prove that $\tilde{f}*\phi$ is $\mathcal{G}(X)/P_{<k+1}(Y,J)$-continuous. Since $\tilde{f}*\phi$ is continuous on X, it suffices to show that $g\mapsto \tilde{f}*\phi(gx)$ is continuous at $g=1$. Let $\varepsilon'>0$, by uniform continuity, there exists $\delta>0$ sufficiently small so that

$$\|\tilde{f}*\phi(x)-\tilde{f}*\phi(y)\|_{L^\infty}<\varepsilon' \tag{8-4}$$

for all $|x-y|<\delta$. By shrinking V, we can assume that

$$\|\phi_l-1\|_\infty<\delta/2 \tag{8-5}$$

for all $l\in V$. We need to find $\delta'>0$ so that if $S_{l,\phi}\in\mathcal{G}(X)/P_{<k+1}(Y,J)$ satisfies $d(S_{l,\phi},1)<\delta'$, then $\|S_{l,\phi_l}\tilde{f}*\phi-\tilde{f}*\phi\|_{L^\infty}<\varepsilon''$, where here $\mathcal{G}(X)/P_{<k+1}(Y,J)$ is equipped with the usual quotient metric.

By taking $\delta'>0$ sufficiently small, we can guarantee that $l\in V$. In that case we must have that $\phi=\phi_l\cdot p_l$ for some phase polynomial $p_l:Y\to U/J$ of degree $k+1$. By (8-5) and the triangle inequality we deduce that $\|p_l-1\|_2\le\delta/2+\delta'$. Choosing $\delta'<\delta/6$ and then choosing δ sufficiently small we can guarantee by Lemma A.17 that p_l is a constant and $|p_l-1|<\frac{2}{3}\delta$. By the triangle inequality

$$\|\phi-1\|_\infty=\|\phi_l\cdot p_l-1\|_\infty\le\|\phi_l-1\|+|p_l-1|<\tfrac{2}{3}\delta+\delta'<\delta.$$

Equation (8-4) now gives $\|S_{l,\phi}\tilde{f}*\phi-\tilde{f}*\phi\|_{L^\infty}<\varepsilon'$, as required. We deduce that $\tilde{f}*\phi$ is $\mathcal{G}(X)/P_{<k+1}(Y,J)$-continuous. We now combine everything to deduce that f can be approximated by a $\mathcal{G}(X)$ continuous function. Let f' denote the lift of $\tilde{f}*\phi$ to X under the canonical projection map $X\to X/J$. It follows that f' is $\mathcal{G}(X)$-continuous. Moreover, by (8-1), (8-3) and the triangle inequality it follows that $\|f-\tilde{f}*\psi\|_{L^2(X)}<\varepsilon$. Since $\varepsilon>0$ is arbitrary, we deduce that the $\mathcal{G}(X)$-continuous functions are dense in the unit ball of $C(X)$, and therefore also in the unit ball of $L^\infty(X)$. □

We can now construct $\widehat{X}$. Applying the Gelfand–Riesz theorem and letting $\widehat{X}$ be the spectrum of $\mathcal{A}$, we see that $\widehat{X}$ is a compact Hausdorff topological space satisfying that $C(\widehat{X})$ is isomorphic as a $C^\star$-algebra to $\mathcal{A}$. The following properties of $\widehat{X}$ were established in [Jamneshan et al. 2021].

Lemma 8.5. *Let $\widehat{X}$ as above. Then*:

(i) *There exists a Radon measure on $\widehat{X}$. In particular, every open subset in $\widehat{X}$ has positive measure.*

(ii) *The natural action $\widehat{T}:\mathcal{G}(X)\times\widehat{X}\to\widehat{X}$ is jointly continuous in $\mathcal{G}(X)$ and $\widehat{X}$.*

(iii) *Every G-continuous function $f\in\mathcal{A}$ has a unique continuous representative $\hat{f}$ on $\widehat{X}$.*

From property (ii) and Theorem B.5 we see that in order to show that $X\cong\mathcal{G}(X)/\Lambda$ is suffices to show that $\mathcal{G}(X)$ acts transitively on $\widehat{X}$ and the stabilizer is totally disconnected. We prove these in two separate lemmas.

Lemma 8.6. *The action of $\mathcal{G}(X)$ on $\widehat{X}$ is transitive.*

Proof. Observe that any continuous function $f\in C(\mathcal{L}/\Lambda_{\mathcal{L}})$ gives rise to a $\mathcal{G}(X)$-continuous function on $X=\mathcal{L}/\Lambda_{\mathcal{L}}\times U$ by $(x,u)\mapsto f(x)$. This gives a $C^\star$-algebra homomorphism from the continuous functions on $\mathcal{L}/\Lambda_{\mathcal{L}}$ to $\mathcal{A}$ which by Gelfand–Riesz gives rise to a continuous factor map $\pi : \widehat{X} \to \mathcal{L}/\Lambda_{\mathcal{L}}$ of $\mathcal{G}(X)$-systems, where the group $\mathcal{G}(X)$ acts on $\mathcal{L}/\Lambda_{\mathcal{L}}$ through the projection $p : \mathcal{G}(X) \to \mathcal{L}$. Since p is surjective, the action of $\mathcal{G}(X)$ on $\mathcal{L}/\Lambda_{\mathcal{L}}$ is transitive. Thus it suffices to show that for every $x_1, x_2 \in \widehat{X}$ with $\pi(x_1) = \pi(x_2)$ we can find $g \in \mathcal{G}(X)$ with $x_1 = gx_2$. We follow the argument from [Host and Kra 2018, Section 19.3.3 Lemma 10]: if not, then by continuity we can find an open neighborhood $V \subseteq \widehat{X}$ of x_1 which contains no element in the orbit of x_2 with respect to the action of $\mathcal{G}(X)$. By Urysohn's lemma we can find a nonnegative continuous function $f : \widehat{X} \to \mathbb{R}$ which is supported on V with $f(x_1) > 0$. Recall that all translations by $u \in U$ belong to $\mathcal{G}(X)$ and let $\tilde{f}(x) = \int_U \tilde{f}(ux)\, du$ where du is the Haar measure on U. Then $\tilde{f}$ is a U-invariant function on $\widehat{X}$ satisfying $\tilde{f}(x_1) > 0$ and $\tilde{f}(x_2) = 0$. By property (iii) of the Lemma above we can identify $\tilde{f}$ with a $\mathcal{G}(X)$-continuous function which is also U-invariant, which is therefore identified with a continuous function f' on $\mathcal{L}/\Lambda_{\mathcal{L}}$ and $\tilde{f} = f' \circ \pi$. This gives a contradiction as x_1, x_2 lie in the same fiber of π. □

Lemma 8.7. *The stabilizer of some point in $\widehat{X}$ under the action of $\mathcal{G}(X)$ is totally disconnected.*

Proof. Recall that for every $u \in U$, the translations V_u are automorphisms which belong in $\mathcal{G}(X)$. It is easy to see that the action of U on $\widehat{X}$ is free. Indeed, if not then there exists $x \in \widehat{X}$ so that $ux = x$. But since u commutes with any $g \in \mathcal{G}(X)$ we see that u stabilizes the orbit of x, which by the previous lemma is everything. Now let $\pi : \widehat{X} \to \mathcal{L}/\Lambda_{\mathcal{L}}$ as before and let x_0 be any element in the preimage of the coset $1 \cdot \Lambda_{\mathcal{L}}$. Let Λ be the stabilizer of x_0. By construction we have $p(\Lambda) = \Lambda_{\mathcal{L}}$ and the kernel is isomorphic to a subgroup of $P_k(Z_k(X), U)$. Since the action of $U \le P_k(Z_k(X), U)$ is free, we have that $U \cap \Lambda = \{e\}$. Thus, we can identify the kernel of p with a closed subgroup of $P_k(Z_k(X), U)/P_1(Z_k(X), U)$ which is totally disconnected (by Pontryagin duality it is embedded as a closed subgroup in the direct product of countably many copies of $P_k(Z_k(X), S^1)/P_1(Z_k(X), S^1)$ which by Lemma A.17 is a product of discrete groups). □

By Theorem B.5 we deduce that $\widehat{X} \cong \mathcal{G}(X)/\Lambda$, where Λ is the stabilizer of some $x_0 \in \widehat{X}$. Since X is isomorphic to $\widehat{X}$ as $\mathcal{G}(X)$-systems and $T_g \in \mathcal{G}(X)$ for every $g \in G$, they are also isomorphic as G-systems.

9. Limit formula and convergence result

In this section we deduce the convergence result (Theorem 2.13) and the limit formula (Theorem 2.14). We begin with the following proposition of [Bergelson et al. 2015, Theorem 3.2] generalized for $\bigoplus_{p\in P} \mathbb{Z}/p\mathbb{Z}$-systems.

Lemma 9.1 (characteristic factors). *Let X be an ergodic $\bigoplus_{p\in P} \mathbb{Z}/p\mathbb{Z}$ system and $f_1, f_2, \ldots, f_{k+1} \in L^\infty(X)$. If, for some i, we have that $E(f_i | Z_{<k+1}(X)) = 0$ then*

$$\limsup_{N\to\infty} \|\mathbb{E}_{g\in\Phi_N} T_g f_1 T_{2g} f_2 \cdot \cdots \cdot T_{(k+1)g} f_{k+1}\|_{L^2} = 0.$$

The proof is the same as in [Bergelson et al. 2015] and therefore is omitted. We deduce that in order to prove Theorem 2.13, it is enough to prove Theorem 2.14.

Proposition 9.2. *Theorem 2.13 follows from Theorem 2.14.*

Proof. We denote by $\tilde{f}$ the projection of f to $L^2(Z_{<k-1}(X))$. By Lemma 9.1, the limit of the average (2-1) exists if and only if it exists for $\tilde{f}_1,\dots,\tilde{f}_{k+1}$. Now let $(Y, G^{(m)})$ be as in Theorem 2.12 and let $h_1,\dots,h_{k+1}$ be the lifts of $\tilde{f}_1,\dots,\tilde{f}_{k+1}$ to Y respectively. Note that, for any Følner sequence Φ_N of G, there exists a Følner sequence $\widetilde{\Phi}_N$ for $G^{(m)}$ such that, for every $y\in Y$,

$$\mathbb{E}_{g\in\widetilde{\Phi}_N} T_g h_1(y)\cdot\dots\cdot T_{(k+1)g}h_{k+1}(y) = \mathbb{E}_{g\in\Phi_N} T_g\tilde{f}_1(\pi(y))\cdot\dots\cdot T_{(k+1)g}\tilde{f}_{k+1}(\pi(y)). \tag{9-1}$$

Therefore, since Y is an inverse limit of k-step nilpotent systems we can approximate $h_1,\dots,h_{k+1}$ in L^2 by bounded functions $h_{1,n},\dots,h_{k+1,n}$ such that, for every $1\le i\le k+1$ and $n\in\mathbb{N}$, $h_{i,n}$ is measurable with respect to the k-step nilpotent system Y_n. The dominated convergence theorem implies that the pointwise convergence in Theorem 2.14 is also an L^2 convergence. Since $L^2(X)$ is a complete metric space, we conclude that the average associated with $h_1,\dots,h_{k+1}$ converges. By (9-1), we conclude that the average associated with $\tilde{f}_1,\dots,\tilde{f}_{k+1}$ also exists, as required. □

We prove Theorem 2.14 and the following theorem simultaneously by induction on k.

Theorem 9.3. *Let $X=\mathcal{G}/\Gamma$ be as in Theorem 2.14. Then, for every $1\le r\le k+1$, $Z_{<r}(X)\cong\mathcal{G}/\mathcal{G}_r\Gamma$.*

Let $k=0$. Then $Z_{<1}(X)$ is trivial and the claim in Theorem 9.3 follows. As for Theorem 2.14, the case $k=0$ follows by the pointwise mean ergodic theorem and Lemma 6.3. Fix $k\ge 1$. Throughout the rest of this section we assume that Theorems 9.3 and 2.14 hold for all smaller values of k. We prove Theorem 9.3 for this value of k and then, we deduce Theorem 2.14 from this result.

Claim 9.4 (the induction hypothesis). *Let $k\ge 1$ be such that Theorems 2.14 and 9.3 hold for all smaller values of k. Let X be as in Theorem 2.14. Then, for every $1\le r\le k$ and every $f_1,\dots,f_r\in L^\infty(X)$, the following r-term formula holds*:

$$\lim_{N\to\infty}\mathbb{E}_{g\in\Phi_N} T_g f_1(x)T_{2g}f_2(x)\cdot\dots\cdot T_{rg}f_r(x) = \int_{\mathcal{G}/\Gamma}\int_{\mathcal{G}_2/\Gamma_2}\cdots\int_{\mathcal{G}_r/\Gamma_r}\prod_{i=1}^r f_i\big(x\cdot y_1^i\cdot y_2^{\binom{i}{2}}\cdot\dots\cdot y_i^{\binom{i}{i}}\big)\,d\prod_{i=1}^r m_i(y_i\Gamma_i), \tag{9-2}$$

with the abuse of notation that $f(x)=f(x\Gamma)$.

Proof. Let $\tilde{f}_1,\dots,\tilde{f}_r$ be the projections of $f_1,\dots,f_r$ respectively into $L^2(Z_{<r}(X))$. By the induction hypothesis of Theorem 2.14 and Theorem 9.3, we have that

$$\lim_{N\to\infty}\mathbb{E}_{g\in\Phi_N} T_g \tilde{f}_1(x)T_{2g}\tilde{f}_2(x)\cdot\dots\cdot T_{rg}\tilde{f}_r(x) = \int_{\mathcal{G}/\mathcal{G}_r\Gamma}\int_{\mathcal{G}_2/\mathcal{G}_r\Gamma_2}\cdots\int_{\mathcal{G}_{r-1}/\mathcal{G}_r}\prod_{i=1}^r \tilde{f}_i\big(x\cdot y_1^i\cdot y_2^{\binom{i}{2}}\cdot\dots\cdot y_i^{\binom{i}{i}}\big)\,d\prod_{i=1}^r m_i(y_i\Gamma_i). \tag{9-3}$$

We lift each $\tilde{f}_i$ to X. Then, (9-3) remains unchanged and by Lemma 9.1 and the fact that each lift is invariant to $\mathcal{G}_r$, we get that (9-2) holds. As required. □

9A. *Proof of Theorem 9.3 and corollaries.* We recall that the Host–Kra group induces an action on each of the universal characteristic factors.

Lemma 9.5 ($\mathcal{G}$ induces an action on the universal characteristic factors). *Let X be an ergodic G-system of order $< k$ and $\mathcal{G}(X)$ be the Host–Kra group. Then for every $1 \leq l < k$ there exists a projection $p_l : \mathcal{G}(X) \to \mathcal{G}(Z_{<l}(X))$, where $\mathcal{G}(Z_{<l}(X))$ is the Host–Kra group of the factor $Z_{<l}(X)$.*

The proof of Theorem 9.3 is a modification of the argument from [Ziegler 2007, Lemma 4.5].

Proof of Theorem 9.3. Let $X = Z_{<k}(X) = \mathcal{G}/\Gamma$ be as in Theorem 2.14. Let $1 \leq r \leq k-1$ and consider the factor map $\pi : \mathcal{G}/\Gamma \to Z_{<r}(X)$. By Lemma 9.5 the action of $\mathcal{G}_r$ on $Z_{<r}(X)$ is trivial and so π induces a factor $\pi_r : \mathcal{G}/\mathcal{G}_r\Gamma \to Z_{<r}(X)$. We prove that π_r an isomorphism. Let $f : \mathcal{G}/\mathcal{G}_r\Gamma \to S^1$. Then, since the coset $g\mathcal{G}_r$ is uniquely determined by the cosets associated with $gy_1, gy_1^2y_2, \dots, gy_1^r y_2^{\binom{r}{2}} \cdots y_{r-1}^{\binom{r}{r-1}}$, we conclude that there is a measurable map $F : (\mathcal{G}/\mathcal{G}_r\Gamma)^r \to S^1$ with $F(gy_1, gy_1^2y_2, \dots, gy_1^r y_2^{\binom{r}{2}} \cdots y_{r-1}^{\binom{r}{r-1}}) = f(g\Gamma)$ for almost all $g \in \mathcal{G}$, $y_1 \in \mathcal{G}_1$, $y_2 \in \mathcal{G}_2, \dots, y_{r-1} \in \mathcal{G}_{r-1}$. We conclude that

$$f(g\Gamma) = \int_{\mathcal{G}/\Gamma} \int_{\mathcal{G}_2/\Gamma_2} \cdots \int_{\mathcal{G}_k/\Gamma_k} F(gy_1, gy_1^2y_2, \dots, gy_1^r y_2^{\binom{r}{2}} \cdots y_{r-1}^{\binom{r}{r-1}})\, d\prod_{i=1}^{r} m_i(y_i\Gamma_i).$$

By approximating F with functions of the form $(x_1, \dots, x_r) \mapsto f_1(x_1) \cdot f_2(x_2) \cdot \dots \cdot f_r(x_r)$, it follows from Claim 9.4 that F is spanned by limits of the form

$$\lim_{N\to\infty} \mathbb{E}_{g\in\Phi_N} T_g f_1 \cdot \dots \cdot T_{rg} f_r.$$

By Lemma 9.1, these terms are measurable with respect to $Z_{<r}(X)$. □

Let X be an ergodic G-system. It is well known [Host and Kra 2005, Proposition 4.11] that every factor of X of order $< k$ factors through $Z_{<k}(X)$. We refer to this fact as the maximal property of the k-th universal characteristic factor. We have the following result.

Lemma 9.6. *Let X be a $(k+1)$-step nilpotent system and let $\mathcal{G}$ be an open subgroup of the Host–Kra group of X and Γ be the stabilizer of $(1, 1, \dots, 1) \in U_0 \times U_1 \times \dots \times U_{k-1}$, where $U_0, U_1, \dots, U_{k-1}$ are the structure groups of X. Then, for every $0 \leq i \leq k-1$, we have that $U_i \cong \mathcal{G}_i/\mathcal{G}_{i+1}\Gamma_i$ as topological groups and measure spaces, where $\mathcal{G}_{i+1}\Gamma_i$ is the closed subgroup generated by all the products of elements in $\mathcal{G}_{i+1}$ and Γ_i.*

Proof. By Lemma 6.3, we have that $X = \mathcal{G}/\Gamma$ and by Theorem 9.3 that $Z_{<r}(X) = \mathcal{G}/\mathcal{G}_r\Gamma$ for every $1 \leq r \leq k$. Let $A_r := \mathcal{G}_r/\mathcal{G}_{r+1}\Gamma_r$. Then, by Lemma 9.5, A_r acts on $Z_{<r+1}(X)$, fixes $Z_{<r}(X)$ and commutes with $\mathcal{G}(Z_{<r+1}(X))$. It follows that the action of any $t \in A_r$ equals to a translations by an element in U_r. In particular, this means that we can identify A_r with a closed subgroup of U_r (as topological groups and measure spaces). Therefore, $Z_{<r+1}(X)/A_r$ is an extension of $Z_{<r}(X)$ by U_r/A_r. On the other hand $Z_{<r+1}(X)/A_r = \mathcal{G}/\mathcal{G}_r\Gamma \cong Z_{<r}(X)$ is a system of order $< r$. By the maximal property of $Z_{<r}(X)$ it follows that U_r/A_r is trivial, as required. □

As a corollary we have the following result.

Corollary 9.7. *Let X be as in Lemma 9.6, and let $\mathcal{G}$ be an open subgroup of $\mathcal{G}(X)$ which contains T_g for every $g \in G$. Then, for every $1 \leq i \leq k$, $\mathcal{G}_i/\mathcal{G}_{i+1}\Gamma_{\mathcal{G}_i} \cong \mathcal{G}(X)_i/\mathcal{G}(X)_{i+1}\Gamma(X)_i$.*

Let $(H,\cdot)$ be any group and $n\in\mathbb{N}$. We say that H is n-divisible if for every $h\in H$ there exists $x\in H$ with $x^{n!}=h$, where $n!=n\cdot(n-1)\cdot\cdots\cdot 1$. Our goal is to show that $\mathcal{G}_r/\mathcal{G}_{r+1}\Gamma_r$ is k-divisible for every $k<\min_{p\in P} p$ and every $1\le r\le k$. We use a type argument by Host and Kra which requires analysis of the ergodic components of $(X^{[1]},\mu^{[1]})$. We recall the following result by Host and Kra [2005, Lemmas 9.1 and 9.3].

Lemma 9.8. *Let X be an ergodic G-system, U a compact abelian group, $\rho: G\times X\to U$ a cocycle and $k\ge 0$ an integer. Let $(Z_{<2}(X),\nu)$ be the Kronecker factor and $\mu^{[1]}=\int_{Z_{<2}(X)}\mu_s\,d\nu(s)$ be the ergodic decomposition of $\mu^{[1]}$ with respect to the diagonal action of G. The set*

$$A=\{s\in Z_{<2}(X): d^{[1]}\rho \text{ is a cocycle of type } <k \text{ of } X_s\}$$

is measurable. Furthermore, the cocycle ρ is of type $<k+1$ if and only if $\nu(A)=1$. Moreover, if X is of order $<k$, then for ν-almost every s, the ergodic component $(X^{[1]},\mu_s)$ is a system of order $<k$.

We deduce the following result.

Lemma 9.9. *Let $1\le m\le l$ and $k<\min_{p\in P} p$. Let X be an ergodic G-system of order $<m$ and $\rho: G\times X\to S^1$ a cocycle of type $<l$ such that $\rho^{k!}$ of type $<m-1$. Then ρ is of type $<m-1$.*

Assume this lemma for now, we prove the following result.

Theorem 9.10. *Let $r\ge 1$, $k<\min_{p\in P} P$ and X be an ergodic r-step nilpotent G-system. Then, for each $1\le i\le r$ we have that $\mathcal{G}_i/\mathcal{G}_{i+1}\Gamma_i$ is k-divisible.*

Proof. Write $Z_{<r+1}(X)=Z_{<r}(X)\times_\rho U$. Since $U\cong\mathcal{G}_r/\mathcal{G}_{r+1}\Gamma_r$, it is enough to prove that U is k-divisible. Assume by contradiction that this is not the case and let $\chi: U\to C_{k!}$ be a lift of a nontrivial character of the quotient $U/U^{k!}$. By Lemma 9.9, we see that $\chi\circ\rho$ is a cocycle of type $<r-1$. This means that the extension $Z_{<r}(X)\times_{\chi\circ\rho}\chi(U)$ is degenerate. In particular, the maximal property of $Z_{<r}(X)$ provides a contradiction. □

Proof of Lemma 9.9. We prove the lemma by induction on m. If $m=1$, then X is trivial. By assumption $\rho^{k!}$ is a coboundary; hence $\rho^{k!}\equiv 1$. We conclude that $\rho: G\to C_{k!}$ is a homomorphism. Since $k<\min_{p\in P} p$, ρ is trivial and the claim follows. Fix $2\le m$ and assume inductively that the claim holds for smaller values of m. Let X be as in the lemma and write $X=Z_{<m-1}(X)\times_\sigma U$. By the induction hypothesis we also know that Theorem 9.10 holds and so we can assume that $U^{k!}=U$. Our goal is to show that ρ is cohomologous to a cocycle that is measurable with respect to $Z_{<m-1}(X)$. The first step is to reduce matters to the case where U is finite. By Theorem 3.8, there exists an open subgroup $U'\le U$ such that, for every $u\in U'$, there exists a phase polynomial $p_u\in P_{<l-1}(X,S^1)$ and a measurable map $F: X\to S^1$ such that

$$\Delta_u\rho=p_u\cdot\Delta F_u. \tag{9-4}$$

We claim that p_u is trivial. The cocycle identity implies that

$$\Delta_{u^{k!}}\rho=\Delta_u\rho^{k!}\cdot\prod_{i=0}^{k!-1}\Delta_{u^i}\Delta_u\rho.$$

Since $\rho^{k!}$ is of type $< m-1$, we conclude by Lemma A.12 that $\Delta_u \rho^{k!}$ is a coboundary. Moreover, by (9-4) and Lemma C.3, we see that $\prod_{l=0}^{k!-1} \Delta_{u^l} \Delta_u \rho$ is cohomologous to a phase polynomial of degree $< l-2$. It follows that $\Delta_{u^{k!}} \rho$ is cohomologous to a phase polynomial of degree $< l-2$. Since $U^{k!} = U$, we conclude that $\Delta_u \rho$ is cohomologous to a phase polynomial of degree $< l-2$. Repeating this argument (by induction on the degree of p_u), we conclude that $\Delta_u \rho$ is a coboundary for every $u \in U'$. Therefore by Lemma A.24, ρ is cohomologous to a cocycle ρ' which is invariant with respect to some open subgroup $U'' \le U$.

Let $X' = X \times_{\sigma'} U/U''$ where σ' is the composition of σ with the quotient map $U \mapsto U/U''$. We view ρ' as a cocycle on X'. By Lemma A.9, ρ' is of type $< l$ and $\rho'^{k!}$ of type $< m-1$.

Now we deal with the finite case. Let $n = |U/U''|$ and let $u \in U/U''$. By Lemma A.12, the cocycle $\Delta_u \rho'$ is of type $< l-m+1$ and $(\Delta_u \rho')^{k!}$ is a coboundary. We prove that $\Delta_u \rho'$ is also a coboundary. By the cocycle identity

$$1 = \Delta_{u^n} \rho' = (\Delta_u \rho')^n \cdot \prod_{l=0}^{n-1} \Delta_{u^l} \Delta_u \rho'.$$

By Lemma A.12, $\prod_{l=0}^{n-1} \Delta_{u^l} \Delta_u \rho'$ is of type $< l-2m+2$ and it follows that so is $(\Delta_u \rho')^m$. Since $U^{k!} = U$, we conclude that n is coprime to $k!$. In particular, there exists a natural number d such that $nd = 1 \mod k!$. We conclude that $\Delta_u \rho'$ is cohomologous to $\Delta_u \rho'^{nd}$ which is of type $< l-2m+2$; hence $\Delta_u \rho'$ is of type $< l-2m+2$. Since $m \ge 2$, we can continue this argument by induction until the type of $\Delta_u \rho'$ is < 0. Therefore, for every $u \in U/U''$ we can find a measurable map $F_u : Z_{<m-1}(X) \to S^1$ such that

$$\Delta_u \rho' = \Delta F_u. \tag{9-5}$$

By ergodicity and the cocycle identity, we conclude that for every $u, v \in U/U''$ there exists a constant $c(u, v)$ such that

$$\frac{F_{uv}}{F_u V_u F_v} = c(u, v). \tag{9-6}$$

Let $b(u, v) = (\Delta_u F_v)/(\Delta_v F_u)$. Since $\Delta_u \Delta_v \rho' = \Delta_v \Delta_u \rho'$, (9-5) implies that $b(u, v)$ is a constant in x. Direct computation using (9-6) shows that b is a bilinear map. For instance we have

$$b(uu', v) = \frac{\Delta_{uu'} F_v}{\Delta_v F_{uu'}} = \frac{\Delta_u F_v V_u \Delta_{u'} F_v}{\Delta_v F_u V_u F'_u} = \frac{\Delta_u F_v}{\Delta_v F_u} V_u \left(\frac{\Delta'_u F_v}{\Delta_v F'_u} \right) = b(u, v) \cdot b(u', v).$$

In particular, we see that $b^n(u, v) = 1$ for every $u, v \in U/U''$. Since n is coprime to $k!$, it is enough to show that ρ'^n is of type $< m-1$. Therefore, we can assume without loss of generality that $b = 1$. In this case it follows that the group

$$H = \{S_{u,F} : u \in U/U'', F \in \mathcal{M}(Z_{<m-1}(X), S^1), \Delta_u \rho' = \Delta F\}$$

is abelian. By (9-5), the projection $p : H \to U/U''$ is onto. Moreover, the kernel is isomorphic to S^1 and so H is a compact group (Corollary B.4). Since the torus is injective in the category of compact abelian groups we conclude that there exists a cross-section $u \mapsto S_{u,F_u}$ such that $\Delta_u \rho' = \Delta F_u$. In particular, $F_{uv} = F_u V_u F_v$ for every $u, v \in U/U''$. Let $F(x, u) = F_u(x, 1_{U/U''})$, direct computation shows that $\Delta_v F(x, u) = F_v(x, u)$

for almost every $x \in Z_{<m-1}(X)$ and every $u, v \in U/U''$. We conclude that $\rho'/\Delta F$ is invariant to U/U'. In other words, ρ' is cohomologous to a cocycle ρ'' which is measurable with respect to $Z_{<m-1}(X)$.

We view ρ'' as a cocycle of $Z_{<m-1}(X)$. By Lemma A.9, ρ'' is of type $< l$ and $\rho''^{k!}$ of type $< m-1$.

Now we use an inductive type argument. By Lemma 9.8, $d^{[1]}\rho''$ is of type $< l-1$ and $d^{[1]}\rho''^{k!}$ is of type $< m-2$ on every ergodic component of $Z_{<m-1}(X)^{[1]}$. Since $Z_{<m-1}(X)$ is a system of order $< m-1$ so is every ergodic component of $Z_{<m-1}(X)^{[1]}$. We conclude, by the induction hypothesis that $d^{[1]}\rho''$ is of type $< m-2$ on every ergodic component. Therefore, by Lemma 9.8, ρ'' is of type $< m-1$ on $Z_{<m-1}(X)$. Lifting everything up using the factor map $X \to Z_{<m-1}(X)$, we conclude that ρ is of type $< m-1$, as required. □

9B. *The group of arithmetic progressions.* The proof of Theorem 2.14 follows the methods of [Bergelson et al. 2005]. Let $X = \mathcal{G}/\Gamma$ be as in Theorem 2.14, we define a function

$$\iota : \mathcal{G} \times \mathcal{G}_1 \times \mathcal{G}_2 \times \cdots \times \mathcal{G}_k \to \mathcal{G}^{k+1}$$

by

$$\iota(g, g_1, g_2, \ldots, g_k) = (g, gg_1, gg_1^2 g_2, \ldots, gg_1^k g_2^{\binom{k}{2}} \cdot \cdots \cdot g_k^{\binom{k}{k}})$$

and let $\widetilde{\mathcal{G}}$ to be the image of ι in $\mathcal{G}^{k+1}$.

Theorem 9.11 [Leibman 1998]. *$\widetilde{\mathcal{G}}$ is a group.*

The group $\widetilde{\Gamma} := \Gamma^{k+1} \cap \widetilde{\mathcal{G}}$ is a closed zero-dimensional cocompact subgroup of $\widetilde{\mathcal{G}}$. Let

$$T_g^{\star} = \mathrm{Id} \times T_g \times T_g^2 \times \cdots \times T_g^k \text{ and } T_g^{\triangle} = T_g \times T_g \times \cdots \times T_g.$$

It is easy to see that $T_g^{\star}$ and $T_g^{\triangle}$ belongs to $\widetilde{\mathcal{G}}$ and therefore acts on $\widetilde{G}/\widetilde{\Gamma}$. Our next goal is to prove that the action of $G \times G$ on $\widetilde{G}/\widetilde{\Gamma}$ by $T_g^{\triangle} \circ T_h^{\star}$ is uniquely ergodic.

9C. *Green's theorem.* Green's theorem [Auslander et al. 1963] states that in a nilsystem $(\mathcal{G}/\Gamma, R_a)$ where $\mathcal{G}$ is a connected simply connected Lie group the action of R_a on $\mathcal{G}/\Gamma$ is ergodic if and only if the induced action of R_a on the factor $\mathcal{G}/\mathcal{G}_2\Gamma$ is ergodic. Parry [1970] gave an alternative simpler proof which was then used by Leibman [2005, Theorem 2.17] to generalize this result to arbitrary nilsystems. Parry's proof relies on the fact that on a connected nilsystem N/Γ the eigenfunctions are invariant with respect to N_2. In Theorem 9.13 below we generalize this result for polynomials of higher order and some special nilpotent systems that may not be connected. First we need the following technical lemma.

Lemma 9.12. *Let $f : \widetilde{\mathcal{G}}/\widetilde{\Gamma} \to S^1$ be a measurable function. Let $V \le \widetilde{\mathcal{G}}$ be an open subgroup which contains the elements $T_g^{\star}$ and $T_g^{\triangle}$ for every $g \in G$ and $2 \le r \le k+1$. Then, if f is invariant with respect to left multiplication by the r-commutator subgroup V_r of V, then f is invariant to the action of $\widetilde{\mathcal{G}} \cap \mathcal{G}_r^{k+1}$.*

Proof. Since V is open and ι is a continuous map, we have that $\iota^{-1}(V)$ contains a subgroup of the form $\mathcal{L} \times \mathcal{L} \times \{e\} \times \cdots \times \{e\}$, where $\mathcal{L} \le \mathcal{G}$ is open. Moreover, since V contains $T_g^{\star}$ and $T_g^{\triangle}$ we can also assume that $\mathcal{L}$ contains T_g. For each $2 \le r \le k+1$ let,

$$\mathcal{H}_r := \widetilde{\mathcal{G}} \cap \mathcal{G}_r^{k+1}.$$

Now, let f be as in the lemma. We prove by downward induction on $2 \le r \le k+1$ that f is also invariant with respect to the action of $\mathcal{H}_r$. If $r = k+1$, then $\mathcal{H}_{k+1}$ is trivial and the claim follows. Let $2 \le r < k+1$ and assume inductively that f is already invariant to left multiplication by elements in $\mathcal{H}_{r+1}$.

For convenient, we write the elements of $\widetilde{\mathcal{G}}$ as sequences $x(n)$ where $x : \{0, 1, \ldots, k\} \to \mathcal{G}$. Since $\widetilde{\mathcal{G}}$ is a group, a general form of an element in $\mathcal{H}_r$ is

$$x(n) = g_0 g_1^n g_2^{\binom{n}{2}} \cdot \cdots \cdot g_k^{\binom{n}{k}},$$

where $g_0, g_1, \ldots, g_r \in \mathcal{G}_t$ and $g_{r+1} \in \mathcal{G}_{r+1}, \ldots, g_k \in \mathcal{G}_k$ with the convention that $\binom{n}{m} = n!/((n-m)!\,m!)$ when $m \le n$ and zero otherwise.

Fix $0 \le m \le k$ and let $x_m(n) = g_m^{\binom{n}{m}} \in \widetilde{G}$, it is enough to show that f is invariant to left multiplication by x_m. If $r < m < k+1$, then $x_m \in \widetilde{H}_{r+1}$ and the claim follows by induction hypothesis. Otherwise, we can assume that $m \le r$. In that case $g_m \in \mathcal{G}_r$.

Step 1: We replace g_m with an $m!$-root. By Theorem 9.10 we can find an element $h \in \mathcal{G}_r$ such that $h^{m!} \cdot g' = g_m$, where $g' \in \mathcal{G}_{r+1} \cdot \Gamma_r$. Hence, $g'^{\binom{n}{m}} = g''^{\binom{n}{m}} \cdot \gamma^{\binom{m}{n}} \cdot y(n)$, where $g'' \in \mathcal{G}_{r+1}$, $\gamma \in \Gamma_r$ and y takes values in $\mathcal{H}_{r+1}$. Since $\widetilde{G}$ is a group we see that $y(n)$ and $g''^{\binom{n}{m}}$ are in $\mathcal{H}_{r+1}$. As for $\gamma^{\binom{m}{n}}$, we have

$$f(\gamma^{\binom{m}{n}} x\Gamma) = f([(\gamma^{-1})^{\binom{m}{n}}, x]x\Gamma)$$

and $[(\gamma^{-1})^{\binom{m}{n}}, x] \in \mathcal{H}_{r+1}$. We conclude that f is invariant to left multiplication by $g_m^{\binom{n}{m}}$ if and only if it is invariant to left multiplication by $h^{p_m(n)}$, where $p_m(n) = n!/(n-m)!$ is a polynomial of degree $< n-m$ with natural coefficients.

Step 2: We replace h with an element in $\mathcal{L}_r$. By Corollary 9.7 we can write $h = l \cdot h' \cdot \delta$, where $l \in \mathcal{L}_r$, $h' \in \mathcal{G}_{r+1}$ and $\delta \in \Gamma_r$. Then, we have

$$h^{p_m(n)} = l^{p_m(n)} \cdot \delta^{p_m(n)} \cdot y'(n),$$

where $y'(n)$ takes values in $\mathcal{G}_{r+1}$. Since $\widetilde{G}$ is a group we conclude that $y' \in \mathcal{H}_{r+1}$. As in the previous step we also know that f is invariant to left multiplication by $\delta^{p_m(n)}$. Therefore, f is $h^{p_m(n)}$-invariant if and only if it is invariant to left multiplication by $l^{p_m(n)}$.

Step 3: We show that f is invariant to $l^{p_m(n)}$ and complete the proof. Since $\mathcal{L}_r$ is generated by commutators of r elements and f is invariant with respect to the action of $\mathcal{H}_{r+1}$, we can assume that l is an r-commutator. Write $l = [s_1, s_2, s_3, \ldots, s_r]$ for some $s_1, s_2, \ldots, s_r \in \mathcal{L}$. We consider two sequences in $\widetilde{\mathcal{G}}$ for each s_i. The first is the constant sequence which we denote by $c_i(n) = s_i$. The second is the arithmetic progression with no constant term, namely $d_i(n) = s_i^n$. Observe that for each $1 \le j \le r$ we have $[d_1, d_2, \ldots, d_j, c_{j+1}, \ldots, c_r] = l^{n^j} \cdot z_j(n)$, where $z_j(n)$ takes values in $\mathcal{G}_{r+1}$, and so is in $\mathcal{H}_{r+1}$. We conclude that f is l^{n^j}-invariant for all $1 \le j \le r$ and so it is also invariant to left multiplication by $l^{p_m(n)}$. □

We note that since $\mathcal{G}_r \trianglelefteq \mathcal{G}$, it follows that $\mathcal{H}_r$ is a normal subgroup of $\widetilde{G}$.

Conventions 2. For the sake of the proof of the ergodicity of $\widetilde{G}/\widetilde{\Gamma}$ with respect to $T_g^{\star}$ and $T_g^{\triangle}$ we say that a homogeneous space N/Γ with an action of $\varphi : G \to N$ is *special* if the induced action of φ on $N/N_2\Gamma$

is ergodic and for every open subgroup $V \le N$ which contains $\varphi(G)$ we have that for every $2 \le r \le k$ any function $f : N/\Gamma \to S^1$ is invariant with respect to the action of V_r if and only if it is invariant with respect to N_r.

We note that by the previous lemma, $\widetilde{G}/\widetilde{\mathcal{G}}_l\widetilde{\Gamma}$ is an l-step special homogeneous space for every $1 \le l \le k$. We generalize Green's theorem.

Theorem 9.13 (Green's theorem for special homogeneous spaces). *Let N/Γ be a k-step special homogeneous space. Then, for every $1 \le d \le k$ and $1 \le r < d$, we have the following results*:

(1) *f is invariant with respect to the action of N_d.*

(2) *For every $n \in N_r$, $\Delta_n f$ is a phase polynomial of degree $< d - r$.*

(3) *For every $n \in N$, $V_n f$ is a phase polynomial of degree $< d$.*

Note that from the case $d = 1$ in the theorem above we can deduce that every special homogeneous space is ergodic.

Corollary 9.14. *Let $X = N/\Gamma$ be a special k-step homogeneous space. Then X is ergodic. In particular, $\widetilde{\mathcal{G}}/\widetilde{\Gamma}$ is ergodic with respect to the action generated by $T_g^\star$ and $T_g^\triangle$.*

A nilpotent system is ergodic if and only if it is uniquely ergodic [Parry 1969b, Section 2, Lemma 1] (see also [Parry 1969a, Theorem 5]).

Theorem 9.15. *The action generated by $T_g^\triangle$ and $T_g^\star$ on $\widetilde{\mathcal{G}}/\widetilde{\Gamma}$ is uniquely ergodic.*

Proof of Theorem 9.13. We prove the claims by induction on k and then by induction on d. If $k = 1$ then the claims follow because the system is ergodic and every $n \in N$ is an automorphism. Fix $k \ge 2$ and assume that the claims hold for all smaller values of k.

Induction basis: The case $d = 1$ follows by adapting the argument of Parry. Let $f : N/\Gamma \to \mathbb{C}$ be an invariant function. The compact abelian group N_k/Γ_k defines a unitary action on $L^2(N/\Gamma)$ by translations. In particular, there is a decomposition of f to eigenfunctions with respect to this action. Namely $f = \sum_\lambda f_\lambda$, where $\Delta_n f_\lambda = \lambda(n)$ for every $n \in N_k/\Gamma_k$, where $\lambda : N_k/\Gamma_k \to S^1$ is a character. Since the action of G commutes with the action of N_k we can also assume that the f_λ are eigenfunctions with respect to the G-action. Thus, $|f_\lambda|$ is G-invariant and invariant with respect to N_k and so by induction hypothesis and Corollary 9.14 we can write $f = \sum_\lambda a_\lambda f_\lambda$ where $a_\lambda \in \mathbb{C}$ and f_λ take values in S^1. Now, we claim by downward induction on $1 \le r \le k$ that:

Claim 9.16. *For every $n \in N_r$ and λ, $\Delta_n f_\lambda$ is a phase polynomial of degree $< k - r + 1$.*

Proof of claim. If $r = k$ then $\Delta_n f_\lambda = \lambda(n)$ is a constant. Fix $r < k$ and assume inductively that the claim holds for larger values of r and let $n \in N_r$. Observe that for every $g \in G$ we have

$$\Delta_g \Delta_n f_\lambda = \Delta_n \Delta_g f_\lambda \cdot V_n T_g \Delta_{[n^{-1},g^{-1}]} f_\lambda.$$

Since $\Delta_g f_\lambda$ is a constant, the term $\Delta_n \Delta_g f_\lambda$ vanishes. Moreover, by the induction hypothesis $\Delta_{[n^{-1},g^{-1}]} f_\lambda$ is a phase polynomial of degree $< k - r$. Observe that $\Delta_{[n^{-1},g^{-1}]} f_\lambda$ is invariant with respect to the action of N_k and therefore by the induction hypothesis on k, we conclude that $V_n T_g \Delta_{[n^{-1},g^{-1}]} f_\lambda$ is also of degree $< k-r$. It follows that $\Delta_g\Delta_n f_\lambda$ is of degree $< k-r$ for every $g \in G$ and therefore $\Delta_n f_\lambda$ of degree $< k-r+1$, as required. $\square$

Now, we apply the claim with $r=1$. We deduce that for every $n \in N$, $\Delta_n f_\lambda$ is a phase polynomial of degree $< k$. Since $\Delta_n f_\lambda$ and is invariant with respect to the action of N_k and $N/N_k\Gamma$ is ergodic, Lemma A.17 implies that the group

$$V_\lambda := \{n \in N : \Delta_n f_\lambda \text{ is a constant}\}$$

is open. The map $v \mapsto \Delta_v f_\lambda$ is a homomorphism from V_λ to the abelian group S^1 and so is trivial on $(V_\lambda)_2 = N_2$. We conclude that $f = \sum_\lambda a_\lambda f_\lambda$ is also invariant with respect to N_2. Since $N/N_2\Gamma$ is ergodic, f is a constant and the rest of the claims follow.

Induction step: Fix $d > 1$ and assume inductively that the claims hold for smaller values of d.

Observe that by the case $d=1$ we can assume that N/Γ is ergodic. Let $f : N/\Gamma \to \mathbb{C}$ be a phase polynomial of degree $< d$. By setting $g_1 = \cdots = g_d = 0$ we conclude that $|f|^{2^d} = 1$ and therefore f takes values in S^1. We show that f is invariant with respect to N_d by adapting the argument from the induction basis. As before, we prove by induction on r that $\Delta_n f$ is of degree $< k-r+1$. If $r=k$, then since $\Delta_g f$ is invariant to N_{d-1} we see that $\Delta_g\Delta_n f = \Delta_n\Delta_g f = 1$, as required. Fix $r < k$ and let $n \in N_r$. Then

$$\Delta_g\Delta_n f = \Delta_n\Delta_g f \cdot V_n T_g \Delta_{[n^{-1},g^{-1}]} f. \tag{9-7}$$

By induction hypothesis $\Delta_{[n^{-1},g^{-1}]} f$ is of degree $< k-r$, since this function is invariant with respect to N_k, the same argument as in the induction basis gives that $V_n T_g \Delta_{[n^{-1},g^{-1}]} f$ is also a phase polynomial degree $< k-r$. If $r \geq d-1$ then $\Delta_n\Delta_g f$ vanishes and therefore $\Delta_n f$ is of degree $< k-r+1$, as required. If $r < d-1$ then, by the induction hypothesis on d, we conclude that $\Delta_n\Delta_g f$ is of degree $< d-r$. Since $d < k$, (9-7) implies that $\Delta_n f$ is of degree $< k-r+1$.

In particular, by the case $r=1$, we conclude that for every $n \in N$, $\Delta_n f$ is a phase polynomial of degree $< k$. This time, consider the subgroup

$$V = \{n \in N : \Delta_n f \text{ is a phase polynomial of degree } < d-1\}.$$

As in the induction basis this is an open subgroup which contains the image of $\varphi : G \to N$.

Write $\Delta_v f = p_v$ and observe that since p_v is invariant to N_k, we have by induction hypothesis that, for every $v' \in V$, $\Delta_{v'} p_v$ is of degree $< d-1$. Moreover, by the cocycle identity we have that $p_{vv'} = p_v \cdot p_{v'} \cdot \Delta_{v'} p_v$. It follows that $v \mapsto p_v \cdot P_{<d-2}(X, S^1)$ is a homomorphism and so trivial with respect to V_2. In other words, for every $v \in V_2$, p_v is a phase polynomial of degree $< d-2$. Continue this way by induction, we see that $p_v = 1$ for every $v \in V_d$. Since N is special, $V_d = N_d$ and the first claim follows. Viewing f as a polynomial of degree $< d$ on $N/N_d\Gamma$ the rest of the claims follow by the induction hypothesis on k. $\square$

9D. ***The proof of Theorem 2.14.*** We construct new systems.

Let $X = \mathcal{G}/\Gamma$ and $\widetilde{X} = \widetilde{\mathcal{G}}/\widetilde{\Gamma}$ as in the previous sections. For every $x \in \mathcal{G}/\Gamma$ the set

$$\widetilde{X}_x := \{(x_1, x_2, \ldots, x_k) \in X^k : (x, x_1, x_2, \ldots, x_k) \in \widetilde{X}\}$$

is a compact subset of X^k. As in [Bergelson et al. 2005], the group $\widetilde{\mathcal{G}}^\star$ acts on $\widetilde{X}_x$ transitively. Moreover, if $\widetilde{\Gamma}_x \le \widetilde{\mathcal{G}}^\star$ is the stabilizer of $(x, x, x, \dots, x)$, then $\widetilde{X}_x \cong \widetilde{\mathcal{G}}^\star/\widetilde{\Gamma}_x$. Let μ be the Haar measure on X, $\tilde{\mu}$ the Haar measure on $\widetilde{X}$ and $\tilde{\mu}_x$ on $\widetilde{X}_x$. Using the fact that $\widetilde{X}$ is uniquely ergodic Bergelson, Host and Kra proved,

$$\tilde{\mu} = \int_X \delta_x \otimes \tilde{\mu}_x \, d\mu(x). \tag{9-8}$$

We prove the limit formula in Theorem 2.14 following the argument in [Bergelson et al. 2005, Theorem 5.4].

Proof. We first prove the claim in the case where the functions are continuous. Since $(x, x, x, \dots, x) \in \widetilde{X}_x$, we can apply the pointwise mean ergodic theorem for the space $\widetilde{X}_x$ with respect to the action of $(T_g, T_g^2, T_g^3, \dots, T_g^k) \in \widetilde{\mathcal{G}}^\star$. The limit is some function ϕ on X. Let f be any continuous function on X. We have

$$\int f(x)\phi(x)\, d\mu(x) = \lim_{N\to\infty} \mathbb{E}_{g\in\Phi_N} \int_X f(x) \prod_{j=1}^k f_j(T_g^j x)\, d\mu(x). \tag{9-9}$$

We translate the functions in (9-9) by T_h and then take an average over $h \in G$. Since μ is T_h invariant for every $h \in G$, the limit above is equal to

$$\lim_{N\to\infty} \mathbb{E}_{g\in\Phi_N}\mathbb{E}_{h\in\Phi_N} \int_X f(T_h x) \prod_{j=1}^k f_j(T_g^j T_h x)\, d\mu(x).$$

The action generated by $T_h^\triangle$ and $T_g^\star$ on $\widetilde{X}$ is uniquely ergodic. Therefore, by the mean ergodic theorem, the limit above converges everywhere to

$$\int_{\widetilde{X}} f(x_0) f_1(x_1)\cdot\dots\cdot f_k(x_k)\, d\tilde{\mu}(x_0, x_1, \dots, x_k) = \int_X f(x) \int_{\widetilde{X}_x} f_1(x_1)\cdot\dots\cdot f_k(x_k)\, d\tilde{\mu}(x_1, \dots, x_k)\, d\mu(x_0).$$

Since this holds for all continuous functions f, we conclude that

$$\phi(x) = \int_{\widetilde{X}_x} f_1(x_1)\cdot\dots\cdot f_k(x_k)\, d\tilde{\mu}(x_1, \dots, x_k)\, d\mu(x_0),$$

whenever $f_1, \dots, f_k$ are continuous.

The map $(g_1, g_2, \dots, g_k) \mapsto (gg_1, gg_1^2 g_2, \dots, gg_1^k g_2^{\binom{k}{2}} \cdot\dots\cdot g_k^{\binom{k}{k}})$ from $\mathcal{G}/\Gamma \times \mathcal{G}_2/\Gamma_2 \times \dots \times \mathcal{G}_k$ to $\widetilde{X}_x$ is an isomorphism and so $\phi(x)$ is equal to the function in (2-2). This completes the proof for continuous functions and by approximation argument the convergence holds for all bounded functions as well. □

Appendix A: Survey of some notation and previous results

The goal of this section is to survey some definitions and known results from previous work. Most of these theorems and definitions appear in [Bergelson et al. 2010] or in [Host and Kra 2005].

A1. *Notation.*

Definition A.1 (abelian cohomology). Let G be a countable discrete abelian group. Let $X = (X, \mathcal{B}, \mu, G)$ be a G-system and let $U = (U, \cdot)$ be a compact abelian group.

- We denote by $\mathcal{M}(X, U)$ or $\mathcal{M}(G, X, U)$ the group of all measurable functions $\phi : X \to U$ or $\phi : G \times X \to U$, respectively. We say that two functions $f_1, f_2 \in \mathcal{M}(X, U)$ are equal if $f_1(x) = f_2(x)$ for μ-almost every x. Similarly, if $f_1, f_2 \in \mathcal{M}(G, X, U)$ then they are equal if $f_1(g, x) = f_2(g, x)$ for μ-almost $x \in X$ and every $g \in G$.

- A (G, X, U)-cocycle is a measurable function $\rho : G \times X \to U$ which satisfies that $\rho(g + g', x) = \rho(g, x)\rho(g', T_g x)$ for all $g, g' \in G$ and μ-almost every $x \in X$. We let $Z^1(G, X, U)$ denote the subgroup of all cocycles.

- Given a cocycle $\rho : G \times X \to U$, we define the abelian extension $X \times_\rho U$ of X by ρ to be the product space $(X \times U, \mathcal{B}_X \otimes \mathcal{B}_U, \mu_X \otimes \mu_U)$, where $\mathcal{B}_U$ is the Borel σ-algebra on U and μ_U the Haar measure. We define the action of G on this product space by $(x, u) \mapsto (T_g x, \rho(g, x)u)$ for every $g \in G$. In this situation, we define by $V_u(x, t) = (x, ut)$ the vertical rotation of some $u \in U$ on $X \times_\rho U$.

- If $F \in \mathcal{M}(X, U)$, we define the derivative $\Delta F \in \mathcal{M}(G, X, U)$ of F to be the function $\Delta F(g, x) := \Delta_g F(x)$. We write $B^1(G, X, U)$ for the image of $\mathcal{M}(X, U)$ under the derivative operation. We refer to the elements in $B^1(G, X, U)$ as (G, X, U)-coboundaries.

- We say that $\rho, \rho' \in \mathcal{M}(G, X, U)$ are (G, X, U)-cohomologous if $\rho/\rho' \in B^1(G, X, U)$. In that case it is easy to see that the abelian extensions defined by ρ and ρ' are isomorphic.

A2. ***Cubic measure spaces and type of functions.*** We begin by introducing the cubic spaces from [Host and Kra 2005, Section 3] (generalized for arbitrary countable abelian group).

Definition A.2 (cubic measure spaces). Let $X = (X, \mathcal{B}, \mu, G)$ be a G-system for some countable abelian group G. For each $k \geq 0$ we define $X^{[k]} = (X^{[k]}, \mathcal{B}^{[k]}, \mu^{[k]}, G)$, where $X^{[k]}$ is the Cartesian product of 2^k copies of X, endowed with the product σ-algebra $\mathcal{B}^{[k]} = \mathcal{B}^{2^k}$, and G acting on $X^{[k]}$ diagonally (i.e., $T_g((x_\omega)_{\omega\in\{0,1\}^k}) := (T_g x_\omega)_{\omega\in\{0,1\}^k}$.). We define the cubic measures $\mu^{[k]}$ and σ-algebras $\mathcal{I}_k \subseteq \mathcal{B}^{[k]}$ inductively. $\mathcal{I}_0$ is defined to be the σ-algebra of invariant sets in X and $\mu^{[0]} := \mu$. Let $k \geq 0$ and suppose that $\mu^{[k]}$ and $\mathcal{I}_k$ are already defined. We identify $X^{[k+1]}$ with $X^{[k]} \times X^{[k]}$ and define $\mu^{[k+1]}$ by the formula

$$\int f_1(x) f_2(y)\, d\mu^{[k+1]}(x, y) = \int E(f_1|\mathcal{I}_k)(x) E(f_2|\mathcal{I}_k)(x)\, d\mu^{[k]}(x).$$

For f_1, f_2 functions on $X^{[k]}$ and $E(\cdot\,|\mathcal{I}_k)$ the conditional expectation and let $\mathcal{I}_{k+1}$ be the σ-algebra of invariant sets in $X^{[k+1]}$.

We adapt the notion of face from [Host and Kra 2005, Section 2]. Let $V_k := \{-1, 1\}^{2^k}$ and for every $0 \leq l \leq k$, if $J \in 2^k$ (equivalently $J \subseteq \{1, \ldots, k\}$) is a set of size $k - l$ and $\eta \in \{-1, 1\}^J$ then the subset

$$\alpha := \{\varepsilon \in V_k : \varepsilon_j = \eta_j \text{ for all } j \in J\}$$

is called an l-dimensional face. For any transformation $u : X \to X$ and a face α we define a transformation $u_\alpha^{[k]}$ on $X^{[k]}$ by

$$(u_\alpha^{[k]})_{\varepsilon\in V_k} = \begin{cases} u & \varepsilon \in \alpha, \\ \mathrm{Id} & \text{otherwise.} \end{cases}$$

We survey some results from [Host and Kra 2005]. We begin with the following result about the relation between measure-preserving transformations, faces and the measure $\mu^{[k]}$.

Lemma A.3 [Host and Kra 2005, Lemma 5.3]. *Let G be a countable abelian group and X be an ergodic G-system. Let $0 \le l \le k$ be integers. For a measure-preserving transformation $t : X \to X$ the following are equivalent*:

(1) *For any l-dimensional face α of V_k, the transformation $t_\alpha^{[k]}$ leaves the measure $\mu^{[k]}$ invariant and maps the σ-algebra $\mathcal{I}^{[k]}$ to itself.*

(2) *For any $(l+1)$-dimensional face β of V_{k+1}, the transformation $t_\beta^{[k+1]}$ leaves $\mu^{[k+1]}$-invariant.*

(3) *For any $(l+1)$-dimensional face γ of V_k, the transformation $t_\gamma^{[k]}$ leaves the measure $\mu^{[k]}$-invariant and acts trivially on the σ-algebra $\mathcal{I}^{[k]}$.*

We also need the following result related to the ergodic decomposition of $\mu^{[k]}$.

Lemma A.4 [Host and Kra 2005, Corollary 3.5]. *Let X be an ergodic G-system and $k \ge 1$ then the following hold*:

- *There exists a measure space (Ω_k, P_k) and an ergodic decomposition*

$$\mu^{[k]} := \int_{\Omega_k} \mu_\omega \, dP(\omega).$$

- *For every $(k-1)$-face α and every $g \in G$ the transformation $g_\alpha^{[k]}$ sends an ergodic component to an ergodic component. In other words, $g_\alpha^{[k]}$ acts on (Ω_k, P_k).*
- *The action of the group generated by $g_\alpha^{[k]}$ for all $g \in G$ and all $(k-1)$-faces α on (Ω_k, P_k) is ergodic.*

The definition of cubic measure spaces (Definition A.2) leads to the following definition of type for measurable functions.

Definition A.5 (functions of type $< k$ [Bergelson et al. 2010, Definition 4.1]). Let G be a countable abelian group, and let $X = (X, \mathcal{B}, \mu, G)$ be a G-system. Let $k \ge 0$ and let $X^{[k]}$ be the cubic system associated with X.

- For each measurable $f : X \to U$, we define $d^{[k]} f : X^{[k]} \to U$ by

$$d^{[k]} f((x_w)_{w\in\{-1,1\}^k}) := \prod_{w\in\{-1,1\}^k} f(x_w)^{\operatorname{sgn}(w)},$$

where $\operatorname{sgn}(w) = w_1 \cdot w_2 \cdot \dots \cdot w_k$.

- Similarly for each measurable $\rho : G \times X \to U$ we define $d^{[k]}\rho : G \times X^{[k]} \to U$ by

$$d^{[k]}\rho(g, (x_w)_{w\in\{-1,1\}^k}) := \prod_{w\in\{-1,1\}^k} \rho(g, x_w)^{\operatorname{sgn}(w)}.$$

- A function $\rho : G \times X \to U$ is said to be a function of type $< k$ if $d^{[k]}\rho$ is a $(G, X^{[k]}, U)$-coboundary. We let $\mathcal{M}_{<k}(G, X^{[k]}, U)$ denote the subspace of functions $\rho : G \times X \to U$ of type $< k$.

Using the Pontryagin dual, Moore and Schmidt [1980, Theorem 4.3] proved the following result.

Theorem A.6. *Let X be a G-system and U a compact abelian group. Let $k \geq 0$ be an integer and $f : G \times X \to U$ a measurable map. Then, f is of type $< k$ if and only if $\chi \circ f : G \times X \to S^1$ is of type $< k$ for every $\chi \in \widehat{U}$.*

We summarize previous results about type of functions. We begin with the following definition.

Definition A.7. Let X be a G-system. Let $U = (U, \cdot)$ be a group and $f : G \times X \to U$ a function. For every $k \in \mathbb{N}$ and every face $\alpha \in V_k$ we define a function $d_\alpha^{[k]} f : G \times X^{[k]} \to U$ by $d_\alpha^{[k]} f(g, (x_\omega)_{\omega \in 2^k}) = \prod_{\omega \in \alpha} f(g, x_\omega)^{\mathrm{sgn}(\omega)}$.

We have the following result [Host and Kra 2005, Lemma C.7].

Lemma A.8. *Let X be an ergodic G-system and U be a compact abelian group. Let $f : X \to U$ be a function and α be an m-dimensional face of V_k for some $1 \leq m < k$. If $d_\alpha^{[k]} f$ is a $(G, X^{[k]}, U)$-coboundary, then f is of type $< m$.*

The following lemma studies the interactions between type and factors.

Lemma A.9 (decent lemma [Bergelson et al. 2010, Proposition 8.11]). *Let X be an ergodic G-system of order $< k$. Let Y be a factor of X, with factor map $\pi : X \to Y$. Let $\rho : G \times Y \to S^1$ be a cocycle. If $\rho \circ \pi$ is of type $< k$, then ρ is of type $< k$.*

We also have the following more general version for quasicocycles [Bergelson et al. 2010, Proposition 8.11].

Lemma A.10. *Let X be an ergodic G-system of order $< k$ and $\pi : X \to Y$ be a factor of X. If $f : G \times Y \to S^1$ is a quasicocycle of order $< k - 1$, such that $f \circ \pi$ is of type $< k$ then f is of type $< k$.*

We are particularly interested in certain measure-preserving transformations $t : X \to X$ on a G-system X.

Definition A.11 (automorphism). Let X be a G-system. A measure-preserving transformation $u : X \to X$ is called an automorphism if the action it induces on $L^2(X)$ by $f \mapsto f \circ u$ commutes with the action of G. In particular we set $\Delta_u f = f \circ u \cdot \bar{f}$.

Automorphisms arise naturally from Host–Kra's theory. For instance, given an abelian extension $Y \times_\rho U$, the group U acts on this extensions by automorphisms defined by $V_u(y, t) = (y, tu)$.

Given a function $f : G \times X \to U$ of type $< k$, the derivative of f by an automorphism t decreases the type of $\Delta_t f$.

Lemma A.12 (differentiation lemma [Bergelson et al. 2010, Lemma 5.3]). *Let $k \geq 1$, and let X be a G-system of order $< k$. Let $f : G \times X \to S^1$ be a function of type $< m$ for some $m \geq 1$. Then for every automorphism $t : X \to X$ which preserves $Z_{<k}(X)$ the function $\Delta_t f(g, x) := f(g, tx) \cdot \bar{f}(g, x)$ is of type $< m - \min(m, k)$.*

A3. ***Phase polynomials.*** Phase polynomials play an important role throughout this paper. We begin with the following definition.

Definition A.13 (phase polynomials). Let G be a countable abelian discrete group, X be a G-system, let $\phi \in L^\infty(X)$, and let $k \geq 0$ be an integer. A function $\phi : X \to \mathbb{C}$ is said to be a phase polynomial of degree $< k$ if we have $\Delta_{h_1}\cdots\Delta_{h_k}\phi = 1$ μ_X-almost everywhere for all $h_1, \ldots, h_k \in G$. (In particular, setting $h_1 = \cdots = h_k = 0$, we see that ϕ must take values in S^1, μ_X-a.e.). We write $P_{<k}(X) = P_{<k}(X, S^1)$ for the set of all phase polynomials of degree $< k$. Similarly, a function $\rho : G \times X \to \mathbb{C}$ is said to be a phase polynomial of degree $< k$ if $\rho(g, \cdot) \in P_{<k}(X, S^1)$ for every $g \in G$. We let $P_{<k}(G, X, S^1)$ denote the set of all phase polynomials $\rho : G \times X \to \mathbb{C}$ of degree $< k$.

Remark A.14. The notion of phase polynomials can be generalized for an arbitrary abelian group $(U, \cdot)$. Let $\phi : X \to U$ be a measurable function and $g \in G$, we can define the derivative $\Delta_g\phi(x)$ by the formula $\phi(T_g x)\cdot\phi(x)^{-1}$. A function $\phi : X \to U$ is said to be a phase polynomial of degree $< k$ if $\Delta_{h_1}\cdots\Delta_{h_k}\phi = 1$ μ_X-a.e. for every $h_1, \ldots, h_k \in G$. We let $P_{<k}(X, U)$ denote the phase polynomials of degree $< k$ which take values in U.

We have the following characterization of phase polynomials [Bergelson et al. 2010, Lemma 5.3].

Lemma A.15. *Let $k \geq 0$ be an integer. Let G be a countable abelian group and X be an ergodic G-system. $f : X \to U$ be a measurable map into compact abelian group $U = (U, \cdot)$. Then, f is a phase polynomial of degree $< k$ if and only if $d^{[k]} f = 1$, $\mu^{[k]}$-almost everywhere.*

We need the following a counterpart of Lemma A.12 for phase polynomials [Bergelson et al. 2010, Lemma 8.8].

Lemma A.16. *In the settings of Lemma A.12 if f is a phase polynomial of degree $< m$ then $\Delta_t f(x)$ is of degree $< m - \min(m, k)$.*

The following lemma implies, in particular, that there are at most countably many (X, S^1)-phase polynomials in any ergodic G-system X.

Lemma A.17 (separation lemma [Bergelson et al. 2010, Lemma C.1]). *Let X be an ergodic G-system, let $k \geq 1$, and $\phi, \psi \in P_{<k}(X, S^1)$ be such that ϕ/ψ is nonconstant. Then $\|\phi - \psi\|_{L^2(X)} \geq \sqrt{2}/2^{k-2}$.*

The famous theorem of [Bergelson et al. 2010] states the following result.

Theorem A.18 (structure theorem for $Z_{<k}(X)$ for ergodic $\mathbb{Z}/p\mathbb{Z}^\omega$-systems). *There exists a constant $C(k)$ such that, for any ergodic $\mathbb{F}_p^\omega$-system X, $L^2(Z_{<k}(X))$ is generated by phase polynomials of degree $< C(k)$. Moreover, if p is sufficiently large with respect to k then $C(k) = k$.*

In [Shalom 2023] we generalized this result for totally disconnected systems (see Definition A.20 and Theorem A.21 below).

A4. ***The structure of systems of order $< k$.*** Let X be an ergodic G-system of order $< k$. Then X can be written as a tower of abelian extensions [Host and Kra 2005, Proposition 6.3].

Proposition A.19 (order $< k+1$ systems are abelian extensions of order $< k$ systems). *Let G be a discrete countable abelian group, let $k \geq 1$ and X be an ergodic G-system of order $< k+1$. Then X is an abelian extension $X = Z_{<k}(X) \times_\rho U$ for some compact abelian metrizable group U and a cocycle $\rho : G \times Z_{<k}(X) \to U$ of type $< k$.*

In particular, it follows that every ergodic G-system of order $< k+1$ is isomorphic to a tower of abelian extensions $U_0 \times_{\rho_1} U_1 \times \cdots \times_{\rho_k} U_k$ where $\rho_i : G \times Z_{<i-1}(X) \to U_i$ is a cocycle of type $< i$. This leads to the following definitions.

Definition A.20 (totally disconnected and Weil systems). Let X be an ergodic G-system of order $< k$ and write $X = U_0 \times_{\rho_1} U_1 \times_{\rho_2} \times \cdots \times_{\rho_{k-1}} U_{k-1}$. We say that X is a totally disconnected system if $U_0, U_1, \ldots, U_{k-1}$ are totally disconnected groups.

In [Shalom 2023] we proved a generalization of Theorem A.18 for totally disconnected $\bigoplus_{p\in P} \mathbb{Z}/p\mathbb{Z}$-systems.

Theorem A.21 (functions of finite type on totally disconnected systems are cohomologous to phase polynomials). *Let $k, m, l \geq 0$ be integers and P be a multiset of primes. If X is an ergodic totally disconnected $\bigoplus_{p\in P} \mathbb{Z}/p^m\mathbb{Z}$-system of order $< k$, then every function $f : \bigoplus_{p\in P} \mathbb{Z}/p^m\mathbb{Z} \times X \to S^1$ of type $< l$ is $\big(\bigoplus_{p\in P} \mathbb{Z}/p^m\mathbb{Z}, X, S^1\big)$-cohomologous to a phase polynomial of degree $< O_{k,m,l}(1)$.*[12]

Note that the proof in [Shalom 2023] is only given in the case $m = 1$, but the general case follows similarly.

A5. ***Conze–Lesigne equations.*** Conze and Lesigne [1984; 1988a; 1988b] studied the structure of ergodic $\mathbb{Z}$-systems of order < 3. They identified a particular functional equation involving the cocycle ρ defining the extension $Z_{<3}(X) = Z_{<2}(X) \times_\rho U$. We refer to this equation ((A-1) below) as a Conze–Lesigne-type equation.

Definition A.22. Let X be an ergodic G-system, $\rho : G \times X \to S^1$ be a cocycle and U a compact abelian group which acts on X. Let $m \geq 0$ we say that ρ is a *Conze–Lesigne cocycle of degree $< m$ with respect to U* if for every $u \in U$ we have

$$\Delta_u \rho(g, x) = p_u(g, x) \cdot \Delta_g F_u(x) \tag{A-1}$$

for some (G, X, S^1)-phase polynomial p_u of degree $< m$ and a measurable map $F_u : X \to S^1$ is a measurable map.

Below are some results regarding Conze–Lesigne cocycles. The first lemma implies that we can choose the terms p_u and F_u measurable in u [Bergelson et al. 2010, Lemma C.4].

[12] We denote by $O_{k,m,l}(1)$ a quantity which is bound by a constant depending only on k, m and l.

Lemma A.23 (measure selection lemma). *Let X be an ergodic G-system, and let $k \geq 1$. Let U be a compact abelian group. If $u \mapsto h_u$ is Borel measurable map from U to $\mathcal{P}_{<k}(G, X, S^1)\cdot\mathcal{B}^1(G, X, S^1) \subseteq \mathcal{M}(G, X, S^1)$, where $\mathcal{M}(G, X, S^1)$ is the group of measurable maps of the form $G \times X \to S^1$ endowed with the topology of convergence in measure, then there is a Borel measurable choice of f_u, ψ_u (as functions from U to $\mathcal{M}(X, S^1)$ and U to $P_{<k}(G, X, S^1)$ respectively) obeying that $h_u = \psi_u \cdot \Delta f_u$.*

The following lemma studies Conze–Lesigne cocycles of degree < 0 [Host and Kra 2005, Lemma C.9].

Lemma A.24 (straightening nearly translation-invariant cocycles). *Let X be an ergodic G-system, let K be a compact abelian group acting freely on X and commuting with the G-action, and let $\rho : G \times X \to S^1$ be such that $\Delta_k \rho$ is a (G, X, S^1)-coboundary for every $k \in K$. Then ρ is (G, X, S^1)-cohomologous to a function which is invariant under the action of some open subgroup of K.*

Remark A.25. Note that if K is connected then it has no nontrivial open subgroups (see Lemma B.10). In this case we have that ρ is (G, X, S^1)-cohomologous to a function which is invariant under K. Moreover it is important to note that such result does not work for cocycles which take values in an arbitrary compact abelian group.

The next lemma asserts that we can *locally linearize* the term p_u in the Conze–Lesigne equation.

Lemma A.26 (linearization of the p_u-term). *Let X be an ergodic G-system, let U be a compact abelian group acting freely on X and commuting with the action of G. Let $\rho : G \times X \to S^1$ be a cocycle and suppose that there exists $m \in \mathbb{N}$ such that for every $u \in U$ there exist phase polynomials $p_u \in P_{<m}(G, X, S^1)$ and a measurable map $F_u : X \to S^1$ such that $\Delta_u \rho = p_u \cdot \Delta F_u$. Then there exists a measurable choice $u \mapsto p'_u$ and $u \mapsto F'_u$ such that $\Delta_u \rho = p'_u \cdot \Delta F'_u$ for phase polynomials $p'_u \in P_{<m}(G, X, S^1)$ which satisfies that $p'_{uv} = p'_u \cdot V_u p'_v$ whenever $u, v, uv \in U'$, where U' is some neighborhood of U.*

The proof of this lemma is given in [Bergelson et al. 2010] as part of the proof of Proposition 6.1 (see in particular (6.5) in that proof).

Appendix B: Topological groups and measurable homomorphisms

In this section we survey some results about topological groups.

Lemma B.1 (A. Weil [Rosendal 2009, Lemma 2.3]). *Let G be a locally compact Polish group and let $A \subseteq G$ be a measurable subset of positive measure. Then $A \cdot A^{-1}$ contains an open neighborhood of the identity.*

At some point we have to work with nonlocally compact groups. The following variant of the theorem above will therefore be useful for us (see [Kechris 1995, Theorem 9.9]).

Lemma B.2 (Pettis lemma). *Let $\mathcal{G}$ be a Polish group and $A \subseteq \mathcal{G}$ be a Baire-measurable, nonmeagre subset of G. Then 1_G lies in the interior of $A \cdot A^{-1}$.*

The following is a variant of the open mapping theorem for Polish groups

Theorem B.3 [Becker and Kechris 1996, Chapter 1]. *Let G and H be Polish groups and let $p : G \to H$ be a group homomorphism that is continuous and onto. Then p is an open map. Moreover, p admits a Borel cross section, that is, a Borel map $s : H \to G$ with $p \circ s = \mathrm{Id}$.*

From this it is easy to conclude the following result.

Corollary B.4. *Let H be a closed normal subgroup of the Polish group G. If H and G/H are (locally) compact, then G is (locally) compact.*

Another corollary of Theorem B.3 is the following result about quotient spaces due to [Effros 1965].

Theorem B.5. *If $\mathcal{G}$ is a Polish group which acts transitively on a compact metric space X, then for any $x \in X$ the stabilizer $\Gamma = \{g \in \mathcal{G} : gx = x\}$ is a closed subgroup of $\mathcal{G}$ and X is homeomorphic to $\mathcal{G}/\Gamma$.*

B1. ***Totally disconnected groups.***

Definition B.6 [Hofmann and Morris 2013, Exercise E8.6]. Let X be a compact Hausdorff space. Then the following are equivalent:

- Every connected component in X is a singleton.
- X has a basis consisting of open closed sets.

We say that X is totally disconnected if one of the above is satisfied.

In this section we will be interested in compact (Hausdorff) totally disconnected groups. These groups are also called profinite groups.

Proposition B.7. *Let G be a compact Hausdorff totally disconnected group. Let $1 \in U \subseteq G$ be an open neighborhood of the identity. Then U contains an open subgroup of G.*

The proof of this proposition can be found in [Neukirch et al. 2000, Proposition 1.1.3]. As a corollary we have the following result.

Corollary B.8 (the dual of totally disconnected group is a torsion group). *Let G be a compact abelian totally disconnected group. Then the image of any continuous character $\chi : G \to S^1$ is finite.*

Proof. Choose an open neighborhood of the identity U in S^1 that contains no nontrivial subgroups. Then, $\chi^{-1}(U)$ is an open neighborhood of G. By the previous proposition there exists an open subgroup H such that $H \subseteq \chi^{-1}(U)$. It follows that $\chi(H)$ is trivial and so χ factors through the finite group G/H. This implies that the image is finite. □

We need the following classical structure theorem [Morris 1977, Chapter 5, Theorem 18].

Theorem B.9 (structure theorem for abelian groups of bounded torsion). *Let G be a compact abelian group and suppose that there exists some $n \in \mathbb{N}$ such that $g^n = 1_G$ for every $g \in G$. Then, G is topologically and algebraically isomorphic to $\prod_{i=1}^{\infty} C_{m_i}$ where for every i, m_i is an integer which divides n.*

One way to generate totally disconnected groups is to begin with an arbitrary compact abelian group and quotient it out by its connected component.

Lemma B.10. *Let G be a compact abelian group and G_0 be the connected component of the identity in G. Since the multiplication and the inversion maps are continuous, one has that G_0 is a subgroup of G and:*

- *G_0 has no nontrivial open subgroups.*
- *Every open subgroup of G contains G_0.*
- *G/G_0 equipped with the quotient topology is totally disconnected compact group.*

We also need the following important fact that connected groups are divisible [Hofmann and Morris 2013, Corollary 8.5].

Lemma B.11. *Let G be a compact abelian connected group. Then for every $g \in G$ and $n \in \mathbb{N}$ there exists $h \in G$ such that $h^n = g$.*

B2. ***Lie groups.***

Definition B.12. A topological group G is said to be a Lie group if it has the structure of a finite-dimensional differentiable manifold over $\mathbb{R}$ such that the multiplication and inversion maps are smooth.

A compact abelian group is a Lie group if and only if its Pontryagin dual is finitely generated. The structure theorem for finitely generated abelian group gives:

Theorem B.13 (structure theorem for compact abelian Lie groups [Sepanski 2007, Theorem 5.2]). *A compact abelian group G is a Lie group if and only if there exists $n \in \mathbb{N}$ such that $G \cong (S^1)^n \times C_k$, where C_k is some finite group with discrete topology.*

The famous Gleason–Yamabe theorem implies that every compact abelian group can be approximated by compact abelian Lie groups using inverse limits.

Theorem B.14 [Hofmann and Morris 2013, Corollary 8.18]. *Let G be a compact abelian group and let U be a neighborhood of the identity in G. Then U contains a subgroup N such that G/N is a Lie group.*

It follows from the above (see also [Iwasawa 1949, Lemma 2.2]) that any compact connected nilpotent group is abelian.

Proposition B.15. *If G is a compact connected k-step nilpotent group, then G is abelian.*

Appendix C: Some results about phase polynomials

In this appendix we survey some results about phase polynomials from [Shalom 2023; Bergelson et al. 2010]. Let $G = \bigoplus_{p\in P}\mathbb{Z}/p\mathbb{Z}$ for some multiset of primes P. For a natural number $m \in \mathbb{N}$ we let C_m denote the group of m-roots of unity in the unit circle.

Proposition C.1 (values of phase polynomial cocycles). *Let X be an ergodic G-system. Let $d \geq 0$, and let $q : G \times X \to S^1$ be a phase polynomial of degree $< d$ that is also a cocycle. Then for $g \in G$, $q(g, \cdot)$ takes values in C_m where m is the order of g to the power of d.*

Proof. We prove the proposition by induction on d. If $d = 0$ then $q \equiv 1$ and the claim is trivial. Fix $d \geq 1$ and assume inductively that the claim holds for smaller values of d. Let $q : G \times X \to S^1$ be a phase polynomial of degree $< d$ and fix $g \in G$ of order n. The cocycle identity implies that

$$1 = q(ng, x) = \prod_{k=0}^{n-1} q(g, T_{kg}x).$$

Since $q(g, T_{kg}x) = q(g, x) \cdot \Delta_{kg} q(g, x)$ we have that $q(g, x)^n \cdot \prod_{k=0}^{n-1} \Delta_{kg} q(g, x) = 1$. By the induction hypothesis, $\prod_{k=0}^{n-1} \Delta_{kg} q(g, x)$ is in $C_{n^{d-1}}$ and it follows that $q(g, x) \in C_{n^d}$, as required. □

We need the following version of [Bergelson et al. 2010, Lemma D.3(i)].

Lemma C.2. *Let X be an ergodic G-system. Let p^n be a power of a prime number p and let $Q : X \to C_{p^n}$ be a phase polynomial of degree $< d$ for some $d > 1$. Then Q^p is a phase polynomial of degree $< d - 1$.*

Proof. Let $G_p = \{g \in G : pg = 0\}$ and let G' be the complement so that $G = G_q \oplus G'$. By the proposition above and the assumption, P is invariant with respect to the action of G'. Let X_p be the factor of X generated by the G'-invariant functions. The induced action of G_p on X_p is ergodic and so X_p admits an ergodic action of $\mathbb{F}_p^\omega$ (note that if G_p is finite, one can still define an action of $\mathbb{F}_p^\omega$ by letting some of the coordinates act trivially). Therefore, the claim in the lemma follows by [Bergelson et al. 2010, Lemma D.3(i)]. □

The following lemma is a simple but useful case of Lemma A.16.

Lemma C.3 (vertical derivatives of phase polynomials are phase polynomials of smaller degree). *Let X be an ergodic G-system. Let U be a compact abelian group acting freely on X by automorphisms and $P : X \to S^1$ be a phase polynomial of degree $< d$ for some integer $d \geq 1$. Then $\Delta_u P$ is a phase polynomial of degree $< d - 1$ for every $u \in U$.*

Proposition C.1 and Lemma C.3 implies the following result.

Corollary C.4. *Let X be an ergodic G-system and U be a compact abelian group acting freely on X by automorphisms. Suppose that there exists a measurable map $u \mapsto f_u$ from U to $P_{<d}(X, S^1)$ which satisfies the cocycle identity (i.e., $f_{uv} = f_u V_u f_v$) for all $u, v \in U$. Then there exists an open subgroup V of U such that $f_v \in P_{<1}(X, S^1)$ for every $v \in V$.*

Proof. We prove the claim by induction on d; for $d = 1$ we can take $V = U$. Let $d > 1$ and assume by induction that the claim holds for all smaller values of d. Let $u \mapsto f_u$ be a map from U to $P_{<d}(X, S^1)$. The cocycle identity implies that $f_{uv} = f_u f_v \cdot \Delta_u f_v$ for every $u, v \in U$. By Lemma C.3, $\Delta_u f_v \in P_{<d-1}(X, S^1)$; therefore $u \mapsto f_u \cdot P_{<d-1}(X, S^1)$ is a homomorphism. Since $d > 1$, Lemma A.17 (separation lemma) implies that $P_{<d-1}(X, S^1)$ has at most countable index in $P_{<d}(X, S^1)$. Let U' be the kernel of $u \mapsto f_u \cdot P_{<d-1}(X, S^1)$. Since U' has positive Haar measure it is open.

Since $f_{u'} \in P_{<d-1}(X, S^1)$ for all $u' \in U'$, the induction hypothesis implies that there exists an open subgroup V of U' such that $f_v \in P_{<1}(X, S^1)$ for all $v \in V$. As V is open in U' and U' is open in U we have that V is open in U, as required. □

Proposition C.5 (phase polynomial are invariant under connected components; see [Bergelson et al. 2015, Lemma 2.1]). *Let X be a G-system of order $<k$; let U be a compact abelian connected group acting freely on X (not necessarily commuting with the G-action). Let $P : G\times X\to S^1$ be a phase polynomial of degree $<d$ such that for every $g\in G$ there exists $M_g\in\mathbb{N}$ such that $P(g,\cdot)$ takes at most M_g values.* (e.g., P is a phase polynomial cocycle — Proposition C.1). *Then P is invariant under the action of U.*

Proof. Fix $g\in G$ and consider the map $u\mapsto\Delta_u P(g,\cdot)$. Since $P(g,\cdot)$ is a measurable map $X\to S^1$, we have that $\Delta_u P$ converges in measure to the constant 1 as u converges to the identity in U. Since convergence in measure implies convergence in L^2, we can use Lemma A.17 to conclude that $\Delta_u P(g,\cdot)$ must be almost everywhere constant for u close to the identity. From the cocycle identity, we have that the subset $U'_g=\{u\in U:\Delta_u P(g,\cdot)\text{ is a constant}\}$ is an open subgroup of U. As U is connected, we conclude that $U'_g=U$ for every $g\in G$. We conclude that, for every $g\in G$, there exists a character $\chi_g:U\to S^1$ such that $\Delta_u P(g,\cdot)=\chi_g(u)$ for every $u\in U$. Since U is connected and χ_g is continuous we have that the image of χ_g is either trivial or is S^1. But, the latter contradicts the assumption that $P(g,\cdot)$ takes finitely many values. It follows that $\Delta_u P(g,\cdot)=1$ for every $u\in U$ and $g\in G$. In other words, P is invariant under the action of U, as required. □

Remark C.6. In some cases the group S^1 in the proposition can be replaced by any compact abelian group using Pontryagin duality. For instance, if $P:G\times X\to V$ is a phase polynomial cocycle for some compact abelian group V, then for every $\chi\in\widehat{V}$ we have that $\chi\circ P:G\times X\to S^1$ is a phase polynomial cocycle of the same degree. By Propositions C.1 and C.5 we have that $\chi(\Delta_u P)=1$ for every $u\in U$. As the characters separate points, this would imply that $\Delta_u P=1$; hence P is invariant with respect to the action of U.

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A FAST POINT CHARGE INTERACTING WITH THE SCREENED VLASOV–POISSON SYSTEM

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We consider the long-time behavior of a fast, charged particle interacting with an initially spatially homogeneous background plasma. The background is modeled by the screened Vlasov–Poisson equations, whereas the interaction potential of the point charge is assumed to be smooth. We rigorously prove the validity of the *stopping power theory* in physics, which predicts a decrease of the velocity $V(t)$ of the point charge given by $\dot{V} \sim -|V|^{-3}V$, a formula that goes back to Bohr (1915). Our result holds for all initial velocities larger than a threshold value that is larger than the velocity of all background particles and remains valid until the particle slows down to the threshold velocity or the time is exponentially long compared to the velocity of the point charge.

The long-time behavior of this coupled system is related to the question of Landau damping, which has remained open in this setting so far. Contrary to other results in nonlinear Landau damping, the long-time behavior of the system is driven by the nontrivial electric field of the plasma, and the damping only occurs in regions that the point charge has already passed.

1. Introduction

We consider the screened Vlasov–Poisson equation coupled to the motion of a point charge. Let $F(t, x, v)$ be a phase space density of the plasma on $\mathbb{R}^3 \times \mathbb{R}^3$ and $X(t), V(t) \in \mathbb{R}^3$ be the position and velocity of

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the point charge. We are interested in the coupled system

$$\begin{aligned}
\partial_t F + v\cdot\nabla_x F + E\cdot\nabla_v F &= -e_0 \nabla\Phi(x - X(t))\cdot\nabla_v F,\\
F(0,x,v) &= \mu(v),\\
\rho[F] = \int_{\mathbb{R}^3} F(x,v)\,\mathrm{d}v, &\qquad E(t,x) = -\nabla\phi *_x \rho[F],\\
\dot{X}(t) = V(t), &\qquad X(0) = 0,\\
\dot{V}(t) = -\alpha e_0 E(t, X(t)), &\qquad V(0) = V_0 e_1.
\end{aligned} \tag{1-1}$$

Here $\mu(v)$ is a probability density, determining the spatially homogeneous initial datum of the density F. Moreover, the initial velocity of the point charge is $V_0 > 0$, and oriented in direction of the first coordinate vector e_1, without loss of generality. The parameter $\alpha > 0$ is related to the coupling strength, and $e_0 = \pm 1$ distinguishes whether the interaction of the point charge with the background is attractive or repulsive.

We consider the screened Vlasov–Poisson equation, i.e., $\phi(x)$ is the screened Coulomb potential. Moreover, Φ is a smooth decaying potential. We refer to Assumption 1.1 for details. The screened potential ϕ takes into account the shielding of interactions beyond the Debye length. We refer to [Bardos et al. 2018; Bouchut 1991; Boyd and Sanderson 2003] for details on this mechanism. The assumptions on Φ are made for technical reasons. Note that by considering the screened Coulomb potential, we have $\nabla\phi \in L^1(\mathbb{R}^3)$ such that E is well-defined for homogeneous ρ and there is no need to subtract a constant as for the unscreened potential

In this paper we rigorously prove that the large-time behavior of the system (1-1) is governed by a deceleration of the point charge. More precisely, after some initial layer where the self-consistent field approaches a traveling wave solution, we show that for $|V(t)|$ sufficiently large, the friction force experienced by the point charge is given by

$$-e_0 E(t, X(t)) \sim -\frac{V(t)}{|V(t)|^3}. \tag{1-2}$$

This means that for large initial velocity of the point charge, i.e., $V_0 \gg 1$, the particle decelerates on a slow time scale $V_0^3 \tau = t$.

The friction force of order $|V(t)|^{-2}$ can be heuristically understood as follows: the swiftly moving point charge induces a perturbation in the spatial density $\rho[F]$ of the plasma. The perturbation will be asymmetric with respect to the direction of motion, since the particle has affected the region behind it for longer than the region ahead of it. For $e_0 = 1$, i.e., if the charge attracts plasma particles, $\rho[F]$ will be larger behind the moving charge than in front of it, so that $-e_0 E(t, X(t))$ is a friction force. For $e_0 = -1$, the argument is analogous.

The typical size of the perturbation is proportional to the time spent in a region of order 1, i.e., of order $|V(t)|^{-1}$. On the other hand, the force (1-2) acting on the point charge is of order $|V(t)|^{-2}$ and therefore much smaller. This is due to the fact that $E(t,x)$ can be expressed through $\nabla_{e_1}\rho[F(t)]$. As a result of the swift motion of the charged particle, the characteristic length scale along the direction of motion is stretched by $|V(t)|$; hence $|\nabla_{e_1}\rho[F(t)]| \sim |V(t)|^{-2}$. Consequently, very detailed estimates in the vicinity of the point charge are required in order to make (1-2) rigorous.

For a more precise description of (1-2), we proceed as follows: For $t_* > 1$ and $V_* := V(t_*) \gg 1$, we show that $F(t_*, \cdot)$ is close to a travelling wave solution. More precisely, we write $F = \mu + f$ and show that for $|x| \ll V(t_*)$, $\rho[f](t_*, X(t_*) + x) \approx \lim_{t\to\infty} \rho[h_{V_*}](t, x)$, where h_{V_*} is the solution to the linearized equation in the inertial frame of the point charge, namely

$$\partial_s h_{V_*} + (v - V_*) \cdot \nabla_x h_{V_*} - \nabla(\phi *_x \rho[h_{V_*}]) \cdot \nabla_v \mu = -e_0 \nabla\Phi(x) \cdot \nabla_v \mu, \quad h_{V_*}(0, \cdot) = 0. \tag{1-3}$$

This traveling wave solution h_{V_*} is explicitly computable in Fourier variables and satisfies the friction relation (1-2).

The linearization (1-3) is only valid over time intervals where $V(t)$ can be approximated by a fixed value V_*. Hence (1-3) is only valid as a short-time linearization on a timescale much shorter than the timescale on which we observe deceleration of the point charge. This allows us to get precise information on the response of the plasma to the presence of the point charge.

In order to obtain estimates for the equation on the long timescale, we first perform a long-time linearization. Here we cannot approximate the velocity of the point charge by a constant and pass to an inertial frame. This is then only done in the short-time linearization that yields (1-3).

We show that the perturbation on the background induced by the point charge is (roughly) of order $|V(t)|^{-1}$ near the point charge and decays algebraically in the distance to the point charge in regions that have not (yet) been penetrated by it. In order to bootstrap this argument, we show that in regions the point charge has already passed, Landau damping occurs as a result of dispersion. A precise description of Landau damping is necessary already for the long-time well-posedness of (1-1), which is a byproduct of our result.

1A. *Previous results.* The model (1-1) and the resulting friction force (1-2) are widely studied in plasma physics to describe the stopping of a fast ion passing through plasma, see for instance [Boine-Frankenheim 1996; Grabowski et al. 2013; Peter and Meyer-ter-Vehn 1991]. The formula (1-2) (with additional logarithmic corrections accounting for Coulomb interactions) goes back to [Bohr 1915].

The Vlasov(–Poisson) equation and its large-time behavior (Landau damping) is the subject of numerous important mathematical works over the last decades. The celebrated paper [Mouhot and Villani 2011] gave a first proof for Landau damping on the torus, while the analysis on the full space goes back to [Bardos and Degond 1985; Glassey and Schaeffer 1994; 1995]. The analysis has since been significantly extended and refined. For small (absolutely continuous) perturbations of the spatially homogeneous plasma described by the screened Vlasov–Poisson equation, this was first achieved in [Bedrossian et al. 2018; Han-Kwan et al. 2021a]. Recently, sharp estimates for this problem have been proved in [Huang et al. 2022; 2024]. Moreover, in [Ionescu et al. 2022], the results in [Bedrossian et al. 2018; Han-Kwan et al. 2021a] have been extended to the Coulomb case for slowly decaying velocity profile μ.

The presence of a point charge gives rise to additional problems for the qualitative and quantitative behavior. In particular, the coupled system enjoys much weaker dispersive properties, since the point charge does not disperse at all. Due to these difficulties and its physical relevance, Vlasov-point charge models have been extensively studied in recent years. Most results concern the coupled system (1-1) with $\phi = \Phi$ given by the Coulomb potential. For existence and growth bounds for plasmas with density decaying for $|x| \to \infty$, we refer to [Caprino et al. 2015; Caprino and Marchioro 2010; Chen et al. 2015a]. Let us point out that the result in [Caprino et al. 2015] does not require the spatial density to be integrable. For the case of a plasma with finite mass, existence and growth-bounds for solutions can be found in [Crippa et al. 2018; Desvillettes et al. 2015; Marchioro et al. 2011]. Global existence of weak solutions has been shown in [Chen et al. 2015b] for a finite plasma and attractive Coulomb interaction.

The existing results assume some decay of the initial data $f_0(x, v)$ for $|x| \to \infty$ in order to handle the problem explained above. To our knowledge, the long-time existence of (1-1) for homogeneous plasmas remained an open problem so far.

Even less is known on the asymptotic behavior of solutions. The publications [Arroyo-Rabasa and Winter 2021; Pausader and Widmayer 2021] investigate the properties of radially symmetric Vlasov–Poisson systems in interaction with a point charge at rest. For the spatially homogeneous plasma with infinite mass and energy, existence and Debye screening for stationary solutions is shown in [Arroyo-Rabasa and Winter 2021]. For small initial data with finite mass and finite energy of the plasma density, the result in [Pausader and Widmayer 2021] gives a precise characterization of the asymptotic scattering. A common feature of the asymptotic results in [Bedrossian et al. 2018; Han-Kwan et al. 2021a; Pausader and Widmayer 2021] is the decay of the plasma's electric field for $t \to \infty$.

The key novelty and difficulty of the present paper is the analysis of the nontrivial long-time behavior of the self-consistent electric field. This poses major difficulties, both for the long-time well-posedness and the long-time behavior of the system (1-1). The system (1-1) combines the difficulties of lack of dispersion of the point charge, and a plasma of infinite mass and energy. This results in the persistence of the electric field

$$\|E_f(t, \cdot)\|_{L^\infty(\mathbb{R}^3)} = O(1) \quad \text{for } t \gg 1, \tag{1-4}$$

and a linear growth of the mass of the perturbation

$$\|\rho[f(t)](\cdot)\|_{L^1(\mathbb{R}^3)} = O(t) \quad \text{for } t \gg 1. \tag{1-5}$$

Due to (1-4) and (1-5), the characteristics of the system do not return to free transport or an explicitly computable ODE for $t \gg 1$. Instead, we derive stronger pointwise estimates (see (4-1)) for the perturbation, which are strongly related to the scattering-geometry of plasma particles by the point charge (see Definition 2.5). This allows us to separate characteristics which are close to free transport from those which are nonexplicit; see Corollary 5.2.

1B. *Statement of the main result.*

Assumption 1.1 (potentials). In the following, let ϕ be the screened Coulomb potential. More precisely, with the convention (1-14) for Fourier transforms,

$$\hat{\phi}(\xi) = \frac{1}{1 + |\xi|^2}.$$

We assume Φ satisfies $\widehat{\Phi} > 0$ and, for some constants $C_\Phi, c_\Phi > 0$,

$$(|\Phi| + |\nabla\Phi| + |\nabla^2\Phi|)(x) \le C_\Phi e^{-c_\Phi |x|}. \tag{1-6}$$

Assumption 1.2 (radial symmetry and regularity of μ). Let $\mu \in C^\infty(\mathbb{R}^3)$ be a radially symmetric probability density which satisfies

$$|\nabla^k \mu(v)| \le C_k e^{-c_k|v|} \tag{1-7}$$

for some $c_k > 0,\ C_k > 0$.

We also assume that the initial distribution μ is monotone.

Assumption 1.3 (monotonicity of μ). We assume that $\mu(v)$ satisfies the monotonicity assumption

$$\nabla_v \mu(v) = -v\psi(v) \tag{1-8}$$

for some nonnegative function $\psi \in C^\infty(\mathbb{R}^3)$.

Assumption 1.4 (Penrose stability). We assume μ satisfies the Penrose stability criterion. More precisely, let $a(z)$ for $z \in \mathbb{C},\ \Im(z) \le 0$ be defined by

$$a(z) = -\int_0^\infty e^{-ipz} p\hat{\mu}(pe_1)\,\mathrm{d}p. \tag{1-9}$$

We then assume that μ is Penrose stable in the sense that there exists a constant $\kappa > 0$ such that

$$\inf_{\Im(z)\le 0, \xi\in\mathbb{R}^3} |1 - \hat{\phi}(\xi)a(z)| \ge \kappa. \tag{1-10}$$

Sufficient conditions for Penrose stability for screened Coulomb interactions can be found in [Bedrossian et al. 2018]. Since we consider compactly supported densities $\mu(v)$ in this paper, we include a sufficient criterion for this case, which is an adaptation of Proposition 2.7 in [Bedrossian et al. 2018]. The proof is postponed to Appendix A.

Proposition 1.5 (Penrose criterion, compactly supported functions). *Let μ satisfy Assumption 1.2. Then there exists a constant $\bar{C} > 0$, depending only on the constants C_k, c_k, such that*

$$\mu(v) > 0 \quad \textit{for all } |v| \le \bar{C},$$

implies the Penrose stability criterion (1-10) for some $\kappa > 0$.

We will work with strong solutions F to (1-1) in the following function space.

Definition 1.6. For $k > 0$, let $C_k^1(\mathbb{R}^3 \times \mathbb{R}^3)$ be the space given by the norm

$$\|F\|_{C_k^1(\mathbb{R}^3\times\mathbb{R}^3)} := \|\langle v\rangle^k F\|_{L^\infty} + \|\langle v\rangle^k \nabla_{x,v} F\|_{L^\infty}. \tag{1-11}$$

Our main result is the following theorem.

Theorem 1.7. *Let ϕ, Φ, μ satisfy Assumptions 1.1–1.4. Then, there exist n, $A_{\min}$, $A_{\max} > 0$, $\bar{V}$ depending only on the constants in Assumptions 1.1–1.4 such that for all $V_0 > \bar{V}$ and all $\alpha > 0$, the following holds true:*

There exists $T > 0$ and a function $F \in C([0, T); C_k^1(\mathbb{R}^3 \times \mathbb{R}^3)) \cap C^1([0, T); C(\mathbb{R}^3 \times \mathbb{R}^3))$ for all $k > 3$ and $X, V \in C^1([0, T))$ uniquely solving the system (1-1) on $(0, T)$. Moreover, if μ has compact support, then for all $8V_0^{-3/5} < t < T$

$$-\frac{\alpha A_{\max}}{|V(t)|} \le \dot{V}(t)\cdot V(t) \le -\frac{\alpha A_{\min}}{|V(t)|}, \tag{1-12}$$

and on $(0, T)$

$$(V_0^3 - 1 - 3\alpha A_{\max} t)^{1/3} \le |V(t)| \le (V_0^3 + 1 - 3\alpha A_{\min} t)^{1/3}. \tag{1-13}$$

Furthermore, at time T *at least one of the following conditions holds*:

(1) $V(T) = \overline{V}$,

(2) $V(T) = \log^n V_0$,

(3) $\operatorname{supp} \mu \cap B^c_{V(T)/5} \neq \varnothing$.

A few comments are in order on the conditions at time T.

(1) The threshold velocity $\overline{V}$ is related to the critical velocity of the point charge which is necessary to study the system perturbatively.

(2) We are only able to bootstrap the estimates as long as the velocity of the point charge still satisfies a lower bound in terms of its initial velocity. This leads to the second condition, $V(T) = \log^n V_0$. The constant n arising from our proof could be made explicit, but we do not pursue to optimize this constant.

(3) The third condition, $\operatorname{supp} \mu \cap B^c_{V(T)/5} \neq \varnothing$, means that the theorem only makes a statement about the deceleration of the point charge as long as the point charge remains faster than all the background particles at time zero. We remark that the ball $B_{V(T)/5}$ could be replaced by $B_{\theta V(T)}$ for any fixed $\theta < 1$ and we only choose $\theta = \frac{1}{5}$ for definiteness.

We also remark that the condition $t > 8V_0^{-3/5}$ for the validity of (1-12) could be improved but we do not pursue this question either. In the initial layer the velocity of the point charge does not significantly change anyway. In (1-13), this is expressed by the term $\pm 1 \ll V_0^3$ which could be further improved without difficulty.

We believe Theorem 1.7 remains valid under more general assumptions. First of all, in Assumption 1.2, it suffices to assume, for some $n \in \mathbb{N}$ sufficiently large, $\mu \in C^n(\mathbb{R}^3)$ and the bound (1-7) for all $k \le n$. All proofs directly apply in this case.

Weakening the assumption of compact support of μ should be possible with the methods of this paper, at least to super-exponential decay of μ. The assumption ensures that the collision time between the point particle and background particles is bounded above. For μ with unbounded support, analogous estimates on the characteristics as in Section 4 would grow exponentially in time for particles with similar velocity to the point charge. Hence our argument can only work if the fraction of such particles is super-exponentially small.

Due to the corresponding Grönwall estimates, it seems difficult to apply the current method for profiles μ with only exponential or slower decay.

We assume Φ to be nonsingular at the origin. An appealing, and likely very challenging problem would be the extension of Theorem 1.7 to the case where $\phi = \Phi$ are both given by the (screened) Coulomb potential. The main difficulty for treating the Coulomb singularity for Φ is the lack of an a priori bound for the exchanged momentum between plasma particles and the point charge in a collision. In particular, the deviation of background particles by the point charge cannot be bounded by $|V|^{-1}$. Moreover, the presence of collisions with very small impact parameter formally leads to a logarithmic correction of the timescale of deceleration of the point charge, known as *Coulomb logarithm* (see [Peter and Meyer-ter-Vehn 1991]). An additional difficulty in treating the full Coulomb potential is due to its slow decay (for both ϕ and Φ), and the slow, logarithmic damping of perturbations. We refer to [Bedrossian et al. 2022; Han-Kwan et al. 2021b] for results on the linearized problem, and to [Ionescu et al. 2022] for proof of nonlinear Landau damping around equilibria with very slow decay in velocities.

For the Vlasov–Poisson equation without a point charge, this has recently been treated in [Ionescu et al. 2022]. However, this work crucially assumes slow polynomial decay for μ, which is in conflict with our assumptions on μ.

In the case of a radially symmetric plasma with finite mass and energy and a point charge at rest, a stability analysis has been achieved in [Pausader and Widmayer 2021] through a delicate geometric argument.

The fact that we are only able to treat velocities $V(T) \geq \log^2(V_0)$ is related to errors that grow logarithmically in time. This problem is also present in [Han-Kwan et al. 2021a] which precludes the treatment of the 2-dimensional case there. The removal of these logarithmic errors by using suitable Hölder-type norms has been achieved in [Huang et al. 2024] which appeared after the first version of the present paper. In [Huang et al. 2022], the authors also deal with the 2-dimensional case, and these papers thus open a perspective on removing the constraint $V(T) \geq \log^2(V_0)$ in our result.

More precisely, we make use of the fact that the perturbation induced by the moving point charge disperses in the two-dimensional orthogonal complement to its direction of motion. However, the current techniques fail to show global-in-time well-posedness of the screened Vlasov–Poisson equation in two dimensions due to a logarithmic divergence (see [Han-Kwan et al. 2021a]).

Another challenge consists in the behavior of system (1-1) when $V(t)$ becomes of order 1. This seems a very hard problem because of the lack of any small parameter that allows for a linearization. For a large range of physically relevant problems, it seems that there is a small parameter in front of the right-hand side in the first line of (1-1), which corresponds to the ratio of the so-called effective charge of the ion to the Debye number. If this parameter is small, a linearization is again formally possible (see, e.g., [Boine-Frankenheim 1996; Peter and Meyer-ter-Vehn 1991]), but we are currently not able to treat this case rigorously.

1C. ***Outline of the rest of the paper.*** As indicated above, the main challenge of the analysis of the coupled system (1-1) is to rigorously prove nonlinear Landau damping in this setting. Our basic strategy is inspired by [Han-Kwan et al. 2021a] where Landau damping is shown using a Lagrangian approach for the screened Vlasov–Poisson system in the whole space. The argument in [loc. cit.] roughly proceeds as follows: First, the screened Vlasov–Poisson equation is reformulated as a linear system with a solution-dependent source term. In a second step, estimates for the linear system are shown via Fourier analysis. Finally, the solution-dependent source term is estimated by means of a bootstrap argument and a representation of the solution through characteristics. This last step is accomplished by a careful analysis of the characteristics. More precisely, it is shown that the characteristics can be well-approximated by rectilinear trajectories ("straightening") under the bootstrap assumption.

Such a Lagrangian approach seems particularly suitable for the system (1-1) in order to quantify dispersion, which only occurs after the point charge has passed a region and only acts in the directions orthogonal to the trajectory of the point charge. However, our analysis is much more delicate than the one in [Han-Kwan et al. 2021a] in several ways. For instance, the point charge induces a perturbation which is large in the L^1- and L^∞-norms considered in the bootstrap argument of [loc. cit.] (see (1-5)). Instead we need to consider a solution-dependent weighted norm adapted to the expected dispersive effects.

Moreover, it is not possible to globally straighten the characteristics as in [loc. cit.]: two background particles with the same initial position but different initial velocities might attain the same position at later time due to the influence of the point charge. The straightening argument therefore only applies to background particles that are not scattered too much by the point charge.

The rest of the paper is organized as follows.

In Section 2, we collect some key ingredients for the proof of Theorem 1.7. The proof itself is given in Section 2D.

In Section 3, we provide additional pointwise estimates for the linear equation already studied in [loc. cit.].

Section 4 is devoted to estimates for the characteristics of the nonlinear equation, which leads to their straightening in suitable regions in Section 5.

We gather the results of the preceding sections to estimate the source term in the linear formulation (in Section 6), as well as the difference of the forces on the point charge corresponding to the system (1-1) and its linearization (1-3) (in Section 7).

Finally, in Section 8, we show that the force corresponding to the linearized equation, (1-3) satisfies (1-2).

In Appendix A, we prove Proposition 1.5, a Penrose stability criterion for compactly supported velocity distributions. Appendix B gathers two standard auxiliary lemmas.

1D. ***Some notation.*** To lighten the notation, we will set the constants from Assumptions 1.1 and 1.2 to 1, as well as the coupling strength α in (1-1), i.e.,

$$\alpha = C_k = c_k = C_\Phi = c_\Phi = 1.$$

The value of these constants does not affect any of the proofs.

Throughout the paper we will use the Japanese brackets defined for any $a \in \mathbb{R}^d$, $d > 0$ by

$$\langle a \rangle := \sqrt{1 + |a|^2}.$$

For $x \in \mathbb{R}^3$, we introduce the orthogonal part $x^\perp \in \mathbb{R}^2$ such that

$$x = (x_1, x^\perp).$$

We use the following conventions for the Fourier transform in space and space-time respectively

$$\hat{g}(\xi) = \int_{\mathbb{R}^3} e^{-ix\cdot\xi} g(x)\,\mathrm{d}x, \quad \tilde{h}(\tau,\xi) = \int_{\mathbb{R}} \int_{\mathbb{R}^3} e^{-i\tau t} e^{-ix\cdot\xi} h(t,x)\,\mathrm{d}x\,\mathrm{d}t. \tag{1-14}$$

For radial functions we will use the convention

$$g(k) = g(|k|),$$

whenever there is no risk of confusion.

We use C for a constant that may change from line to line and use $A \lesssim B$ for $A \leq CB$.

2. Outline of the proof of the main result

This section contains the proof of Theorem 1.7 and sets the structure of the remainder of the paper. We start by giving estimates on the linearized friction force in Section 2A. We then reformulate (1-1) in terms of the Green's function of the linearized problem in Section 2B. In Section 2C, we introduce scattering variables for the interaction of the plasma with the moving charged particle, as well as associated norms. At this point we also state the estimates which are used for the bootstrap argument and are proved in the remaining sections. Finally, in Section 2D we give the proof of Theorem 1.7.

2A. ***The force on the point charge for the linearized equation.*** As outlined in the Introduction, the proof of the main result is based on a rigorous linearization of the equation.

The solution h_{V_*} to the linearized equation (1-3), has an integral representation through the space-time Fourier transformation, which gives access to the force on the point charge corresponding to h_{V_*}. More precisely, we prove the following proposition.

Proposition 2.1. *Recall the function h_{V_*} defined in (1-3). For any $0 \neq V_* \in \mathbb{R}^3$, the following limit exists and is negative*:

$$\lim_{s \to \infty} e_0 \nabla \phi *_x \rho[h_{V_*}(s, \cdot)](0) \cdot V_* < 0. \tag{2-1}$$

More precisely, there exists a constant $A > 0$ and $c > 0$ such that

$$\lim_{s \to \infty} \left| A + |V_*| e_0 \nabla \phi *_x \rho[h_{V_*}(s, \cdot)](0) \cdot V_* \right| \lesssim e^{-c|V_*|}. \tag{2-2}$$

The proof of Proposition 2.1 will be given in Section 8.

Although the short-time linearized equation (1-3) in the inertial frame of the point charge is very practical for computing this force, we rewrite it in the original coordinate frame to compare with the nonlinear equation (1-1). It is then convenient to introduce the parameter $R > 0$ that will play the role of a large time and consider the short-time linearized equation where the charged particle starts at position $X_* - RV_*$ at time zero and moves with constant velocity V_*. More precisely, we define for $R > 0$, $X_* \in \mathbb{R}^3$

$$g_{R,X_*,V_*}(t, x, v) = h_{V_*}(t, x - X_* + (R - t)V_*, v).$$

Observe that g_{R,X_*,V_*} solves

$$\begin{aligned} \partial_s g_{R,X_*,V_*} + v \cdot \nabla_x g_{R,X_*,V_*} - \nabla(\phi *_x \rho[g_{R,X_*,V_*}]) \cdot \nabla_v \mu &= -e_0 \nabla \Phi((x - X_* + (R - s)V_*) \cdot \nabla_v \mu, \\ g_{R,X_*,V_*}(0, \cdot) &= 0. \end{aligned} \tag{2-3}$$

Then, on the one hand, we have the following relation of the forces:

$$\lim_{s\to\infty} \nabla\phi *_x \rho[h_{V_*}(s,\cdot)](0) = \lim_{R\to\infty} \nabla\phi *_x \rho[g_{R,X_*,V_*}(R,\cdot)](X_*).$$

On the other hand, for $t_* \gg V_{\min}^{-1}$ and $t \approx t_*$ such that $V(t) \approx V_*$, we expect $g_{t_*,X(t_*),V(t_*)}(t,\cdot)$ to be close to the solution $f = F - \mu$ of (1-1).

2B. *Representation of the solution through a Green's function.* Let F be a solution to (1-1). We decompose F as

$$F(t,x,v) = \mu(v) + f(t,x,v).$$

Then f solves the equation

$$\begin{gathered} \partial_t f + v\cdot\nabla_x f + \nabla_x(e_0\Phi(\cdot - X(t)) - (\phi *_x \rho[f]))\cdot\nabla_v f = \nabla_x((\phi *_x \rho[f]) - e_0\Phi(\cdot - X(t)))\cdot\nabla_v\mu, \\ f(0,\cdot) = f_0, \\ \dot X(t) = V(t), \quad \dot V(t) = e_0\nabla(\phi *_x \rho[f])(X(t)), \quad X(0)=0,\ V(0)=V_0, \end{gathered} \tag{2-4}$$

with $f_0 = 0$.

Since $\mu(v)$ is spatially homogeneous, the self-consistent force field E in (1-1) can be expressed as

$$E(t,x) = -(\nabla\phi *_x \rho[f(t,\cdot)])(x). \tag{2-5}$$

We introduce the total force $\bar E$, defined by

$$\bar E(t,x) = E(t,x) + e_0\nabla\Phi(x - X(t)).$$

Let $X_{s,t}$, $V_{s,t}$ be the characteristics associated to $\bar E$. More precisely, for $x, v \in \mathbb{R}^3$, $0 \le s \le t$,

$$\frac{d}{ds} X_{s,t}(x,v) = V_{s,t}(x,v), \qquad X_{t,t}(x,v) = x, \tag{2-6}$$

$$\frac{d}{ds} V_{s,t}(x,v) = \bar E(s, X_{s,t}(x,v)), \quad V_{t,t}(x,v) = v. \tag{2-7}$$

Then, if f is sufficiently regular

$$f(t,x,v) = -\int_0^t E(s, X_{s,t}(x,v))\cdot\nabla_v\mu(V_{s,t}(x,v)) - \int_0^t e_0\nabla\Phi(X_{s,t}(x,v) - X(s))\cdot\nabla_v\mu(V_{s,t}(x,v)).$$

This we can rewrite as

$$\begin{aligned} f(t,x,v) = &\int_0^t (\nabla\phi * \rho[f])(s, x-(t-s)v)\cdot\nabla\mu(v)\mathrm{d}s \\ &+ \int_0^t E(s, x-(t-s)v)\cdot\nabla\mu(v)\mathrm{d}s - \int_0^t E(s, X_{s,t}(x,v))\cdot\nabla_v\mu(V_{s,t}(x,v)) \\ &- \int_0^t e_0\nabla\Phi(X_{s,t}(x,v) - X(s))\cdot\nabla_v\mu(V_{s,t}(x,v)), \end{aligned}$$

and therefore the density $\rho[f]$ solves the following integral equation for $t \ge 0$:

$$\rho[f](t,x) = \int_0^t\int_{\mathbb{R}^3} (\nabla\phi * \rho[f])(s, x-(t-s)v)\cdot\nabla\mu(v)\,\mathrm{d}v\,\mathrm{d}s + \mathcal{S}(t,x), \tag{2-8}$$

where $\mathcal{S}$ for $t \geq 0$ is given by

$$\mathcal{S}(t,x) = \mathcal{R}(t,x) + S_{\mathrm{P}}(t,x), \tag{2-9}$$

$$\mathcal{R}(t,x) = \int_0^t \int_{\mathbb{R}^3} \big(E(s, x-(t-s)v)\cdot\nabla_v\mu(v) - E(s, X_{s,t}(x,v))\cdot\nabla_v\mu(V_{s,t}(x,v))\big)\,\mathrm{d}v\,\mathrm{d}s, \tag{2-10}$$

$$S_{\mathrm{P}} = -\int_0^t \int_{\mathbb{R}^3} e_0 \nabla\Phi(X_{s,t}(x,v) - X(s))\cdot\nabla_v\mu(V_{s,t}(x,v))\,\mathrm{d}v\,\mathrm{d}s. \tag{2-11}$$

We will call $\mathcal{S}$ the source term, $\mathcal{R}$ the reaction term, and S_{P} the contribution of the point charge.

We can write the solution to the short-time linearization of the equation in (2-3) analogously. Then, for $t \geq 0$, the density $\rho[g_{R,X_*,V_*}]$ satisfies the equation

$$\rho[g_{R,X_*,V_*}](t,x) = \int_0^t \int_{\mathbb{R}^3} (\nabla\phi * \rho[g_{R,X_*,V_*}])(s, x-(t-s)v)\cdot\nabla\mu(v)\,\mathrm{d}v\,\mathrm{d}s + S_{R,X_*,V_*}(t,x), \tag{2-12}$$

$$S_{R,X_*,V_*}(t,x) = -\int_0^t \int_{\mathbb{R}^3} \nabla\Phi(x-(t-s)v-(X_*-(R-s)V_*))\cdot\nabla_v\mu(v)\,\mathrm{d}v\,\mathrm{d}s. \tag{2-13}$$

We extend both $\mathcal{S}$ and S_{R,X_*,V_*} by 0 for negative times.

Following [Han-Kwan et al. 2021a], we obtain a representation of $\rho[f]$ and $\rho[g_{R,X_*,V_*}]$ through a Green's function G of the form

$$\rho(t,x) = G *_{t,x} S + S, \tag{2-14}$$

with $\rho = \rho[f]$ and $S = \mathcal{S}$, respectively, $\rho = \rho[g_{R,X_*,V_*}]$ and $S = S_{R,X_*,V_*}$. More precisely, corresponding to [loc. cit., Theorem 2.1], we have the following proposition. In addition to [loc. cit., Theorem 2.1], we show also pointwise estimates for G that will be needed later on.

Proposition 2.2. *Let μ satisfy Assumptions 1.2 and 1.4 and let ϕ be given as in Assumption 1.1. Then, for all $S \in L^1_{\mathrm{loc}}(\mathbb{R}; L^2(\mathbb{R}^3)\cap L^\infty(\mathbb{R}^3))$, there exists a unique solution $\rho \in L^1_{\mathrm{loc}}(\mathbb{R}; L^2(\mathbb{R}^3))$ to (2-8) that can be expressed through (2-14) with a kernel $G : \mathbb{R}\times\mathbb{R}^3 \to \mathbb{R}$ that satisfies $G(t,\cdot) = 0$ for $t < 0$ and for $t \geq 0$*

$$\|G(t,\cdot)\|_{L^1} \leq \frac{C}{1+t}. \tag{2-15}$$

Moreover, for all $t \geq 0$ and $x \in \mathbb{R}^3$, G satisfies the pointwise estimates

$$|G(t,x)| \lesssim \frac{1}{t^4+|x|^4}, \tag{2-16}$$

$$|\nabla G(t,x)| \lesssim \frac{1}{t^5+|x|^5}. \tag{2-17}$$

This proposition is proved in Section 3A.

2C. ***Bootstrap estimates.*** The proof of long-time well-posedness of the solution to (2-4) relies on local well-posedness and a bootstrap argument. We start by stating the local well-posedness result, the proof of which is a standard fixed-point argument and will be omitted for the sake of conciseness.

Theorem 2.3 (local well-posedness). *Let μ, ϕ and Φ satisfy the Assumptions 1.1, 1.4 and (1-6) respectively and let $V_0 > 0$. Further let $k > 3$, and $f_0 \in C_k^1(\mathbb{R}^3 \times \mathbb{R}^3)$ (see (1-11)).*

Then there exists a time $T_* > 0$, a function $f \in C([0, T_*); C_k^1(\mathbb{R}^3 \times \mathbb{R}^3)) \cap C^1([0, T_*); C(\mathbb{R}^3 \times \mathbb{R}^3))$ and $X, V \in C^1([0, T_*))$ uniquely solving the system (2-4). Moreover, $T_* = \infty$ or

$$\limsup_{t \to T_*} \|\rho(t, \cdot)\|_{W^{1,\infty}} = \infty. \tag{2-18}$$

Furthermore, for all $0 \le t < T_*$

$$X(t) \in \operatorname{span}\{e_1\}. \tag{2-19}$$

The relation (2-19) follows immediately from symmetry considerations.

The bootstrap argument consists in estimating ρ by S and vice versa in a weighted $W^{1,\infty}$-norm which is adapted to the scattering of the plasma by a fast moving charged particle. More precisely, we will assign weights that reflect that we expect the following decay of both S and ρ:

- For regions with a large component orthogonal to the particle trajectory (i.e., $x^\perp \gg 1$): decay in $x^\perp$ because the charged particle never reaches these regions.
- For regions in front of the charged particle (i.e., $x_1 > X_1(t)$): decay in terms of the distance $x_1 - X_1(t)$; the charged particle has not yet significantly disturbed these regions.
- For regions behind the charged particle (i.e., $x_1 < X_1(t)$): decay in terms of the time passed since $x_1 = X_1(s)$ due to dispersion.

In order to formalize this decay, we introduce several parameters that depend on the trajectory of the charged particle. Because this trajectory is a priori only defined for short times, we first introduce the following linear extension. First, for $T < T_*$, we define the minimum of the first component of the velocity of the charged particle

$$V_{\min}(T) := \min_{t \in [0,T]} V_1(t). \tag{2-20}$$

Definition 2.4. Let $0 < T < T_*$, where T_* is the maximal existence time from Theorem 2.3 and assume that $V_{\min}(T) > 0$. Then, we define

$$X^T(t) := \begin{cases} X(t) & \text{in } [0, T], \\ X(T) + (t - T)V(T) & \text{in } [T, \infty), \\ tV_0 & \text{in } (-\infty, 0]. \end{cases}$$

Definition 2.5. Given $T > 0$, and X^T as in Definition 2.4 we introduce the following:

(i) Let $x \in \mathbb{R}^3$. Then, there exists a unique $\tau_x := \tau_{x_1} \in (-\infty, \infty)$ such that

$$X_1^T(\tau_{x_1}) = x_1,$$

which we call the *passage time* at x_1. We also define

$$\check{\tau}_{t,x} = \check{\tau}_{t,x_1} := [t - \tau_{x_1}]_+.$$

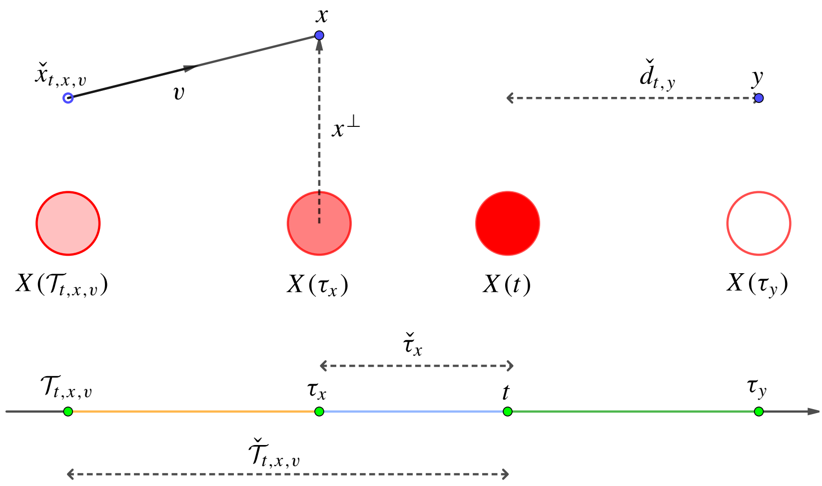

Figure 1. The quantities τ_x, $\check{\tau}_{t,x}$, $\mathcal{T}_{t,x,v}$, $\check{\mathcal{T}}_{t,x,v}$, $\check{x}_{t,x,v}$ and $\check{d}_{t,x}$.

(ii) For $x \in \mathbb{R}^3$, $s \in \mathbb{R}$ we denote the distance to the approaching charged particle with respect to the first component by

$$\check{d}_{s,x} = \check{d}_{s,x_1} := [x_1 - X_1^T(s)]_+.$$

(iii) For $\Psi \in L^1_{\text{loc}}((0,T); W^{1,1}_{\text{loc}}(\mathbb{R}^3))$ we define the weighted norm

$$\|\Psi\|_{Y_T} = \sup_{s\in[0,T],x\in\mathbb{R}^3} |\Psi(s,x)|\langle \check{\tau}^2_{s,x} + \check{d}^2_{s,x} + |x^\perp|^2\rangle + |\nabla\Psi(s,x)|\langle \check{\tau}^3_{s,x} + \check{d}^3_{s,x} + |x^\perp|^3\rangle. \tag{2-21}$$

The quantities τ_x, $\check{\tau}_{t,x}$ and $\check{d}_{s,x}$ are visualized in Figure 1 (together with further quantities that will be defined later on): Point x lies *behind* and point y *in front* of the point charge at time t which is located at $X(t)$. Times τ_x and τ_y are then the times when the point charge has passed by x and will pass by y, respectively.

On the one hand, the distance $\check{d}_{t,y}$ is the distance between the point charge at time t and the first coordinate of the position y. Note that by definition $\check{d}_{t,x} = 0$ as x lies *behind* the point charge at time t. On the other hand $\check{\tau}_{t,x} = t - \tau_x$ is the time difference between the present time t and the passage time τ_x, i.e., the time the point charge needs to travel from the $X(\tau_x)$ to $X(t)$. Note that by definition $\check{\tau}_{t,y} = 0$ as y lies *in front* of the point charge at time t.

We point out that in Figure 1 both time and the first space coordinate is visualized on the horizontal line. However, since the point charge is very fast, the relative times are much smaller than the relative distances, i.e., $\check{\tau}_{t,x} \ll X_1(t) - X_1(\tau_x)$.

We mark quantities with $\check{\cdot}$ to indicate that they are differences of quantities inherent to the point charge and external quantities. Consequently, those quantities appear in the norm $\|\cdot\|_{Y_T}$ which anticipates the expected decay of the perturbation of the background density. For consistency, one could also replace $x^\perp$ by $\check{x}^\perp = x^\perp - X^\perp(\tau_x)$ in that norm, but we prefer not to further complicate the notation in this way.

Note that τ_x, $\check{\tau}_{t,x}$ and $\check{d}_{s,x}$ all implicitly depend on T. Since the time T will always be fixed when dealing with these quantities, no confusion will arise from this implicit dependence.

We will for simplicity mostly write τ_x, $\check{\tau}_{t,x}$ and $\check{d}_{s,x}$ and only use τ_{x_1}, $\check{\tau}_{t,x_1}$ and $\check{d}_{s,x_1}$ when we want to emphasize that these quantities only depend on x_1.

The quadratic and cubic weight in the definition of Ψ are dictated by the expected dispersion. Indeed, since the fast charged particle effectively creates a disturbance on the whole line, the dispersion only takes place with respect to the orthogonal direction. Since the orthogonal space is 2-dimensional, this gives rise to $\check{\tau}_{t,x}^2$ and $\check{\tau}_{t,x}^3$ in (2-21). The pointwise decay of the Green's function dictates the powers in $\check{d}_{s,x}$ and $x^\perp$ to be the same.

A consequence of Proposition 2.2 are the following estimates for the linear equation (2-14).

Corollary 2.6. *There exists a constant $C > 0$ with the following property. Let $T > 0$, and X^T as in Definition 2.4 and assume in addition that $V_{\min}(T) \geq 1$. Then, for all $S \in L^1_{\mathrm{loc}}(\mathbb{R}; L^2(\mathbb{R}^3) \cap L^\infty(\mathbb{R}^3))$ the unique solution $\rho \in L^1_{\mathrm{loc}}(\mathbb{R}; L^2(\mathbb{R}^3)$ to (2-8) satisfies*

$$\|\rho\|_{Y_T} \leq C \log^2(2+T) \|S\|_{Y_T}.$$

This corollary is proved in Section 3B.

To close a bootstrap argument, we need to estimate $\mathcal{S}$ (see (2-9)) in terms of ρ. This is the content of the following proposition which contains two estimates: (2-22) gives control of the source term $\mathcal{S}$ on the long time scale $V_0^3 \tau = t$, and the estimate implicitly takes into account the trajectory of the charged particle $X(t)$. On the other hand, (2-23) allows us to approximate the force $E(T, X(T))$ on the charged particle with the short-time linearization of the force. Note that, compared to (2-22), this requires a much finer estimate of the error in the vicinity of $X(T)$.

Proposition 2.7. *There exists $0 < \delta_0 < 1$, $V_{\mathrm{nl}} > 0$, $n_{\mathrm{nl}} > 0$, $C > 0$ such that the following holds true. Let T and X^T be as in Definition 2.4. Then, if $V_{\min}(T) \geq V_{\mathrm{nl}}$, $\operatorname{supp} \mu \subset B_{V_{\min}(T)/5}(0)$, $V_{\min}(T)^{-1} < \delta < \delta_0$, $\|\rho[f]\|_{Y_T} < \delta$ and $\log^{n_{\mathrm{nl}}}(2+T) < \delta^{-1}$, then?*

(i) *The source $\mathcal{S}$ can be estimated by*

$$\|\mathcal{S}\|_{Y_T} \leq C V_{\min}(T)^{-1} + C\delta^{3/2}. \tag{2-22}$$

(ii) *If in addition $T \geq 8 V_{\min}(T)^{-3/5}$, then*

$$\lim_{s\to\infty} |(\nabla\phi * \rho[h_{V(T)}])(s, 0) + E(T, X(T))| \leq C\delta^{13/6}. \tag{2-23}$$

The proof of Proposition 2.7 is the main technical part of the paper. Part (i) will be shown in Section 6D. The proof of Proposition 2.7(ii) can be found in Section 7C.

The powers of δ in (2-22) and (2-23) are probably not optimal, neither the restriction $T \geq 8 V_{\min}(T)^{-3/5}$. Indeed, the two terms on the right-hand side of (2-22) correspond to the reaction term $\mathcal{R}$ and the contribution of the point charge S_{P} respectively (see (2-9)–(2-11)). The estimate of the contribution of S_{P} is given in Section 6C. It makes rigorous the heuristics that the point charge only induces a perturbation of order $V_{\min}$ since this is the order of the time that it can interact with any given point x where it passes by. On the other hand, one could expect $\mathcal{R}$ to be of order δ^2. Indeed, the self-consistent force field E is bounded by $\|\rho[f]\|_{Y_T} < \delta$ (see Lemma 2.8 below). Additionally, $\mathcal{R}$ is given by the difference obtained by evaluating the same function on the true characteristics $(X_{s,t}(x,v), V_{s,t}(x,v))$ and on their rectilinear counterparts $x-(t-s)v$, v. Since the error in the rectilinear approximation is due to the total force field $\bar{E} = E + e_0\nabla\Phi(\cdot - X)$, this yields additional smallness of order $\delta + V_{\min}^{-1} \lesssim \delta$.

Estimates on the characteristics will be given in Section 4. These allow us to straighten the characteristics (see Section 5), except on a very small set.

Note that the forces whose difference is estimated in (2-23) can be rewritten as

$$\begin{aligned} E(T, X(T)) &= -(\nabla\phi * (G * \mathcal{S} + \mathcal{S}))(t, X(t)), \\ \lim_{s\to\infty} \nabla\phi * \rho[h_{V(T)}](s, 0) &= \lim_{R\to\infty} (\nabla\phi * (G * S_{R,X(T),V(T)} + S_{R,X(T),V(T)})(R, X(T)). \end{aligned} \tag{2-24}$$

Therefore, it is enough to analyze the source term $\mathcal{S}$ and its counterpart S_{R,X_*,V_*} for the second part of the above proposition, too. The estimate (2-23) cannot be valid for very small times, since the force $E(T, X(T))$ vanishes at $T = 0$. Instead, we can at best expect the estimate to hold for times that are large compared to the timescale of the point charge, i.e., $T \gg V(0)^{-1}$.

Once we have established estimates for $\nabla\rho$, we immediately obtain estimates for the force field E and its derivative. This convolution estimate is stated in the following lemma, the proof is elementary and will be skipped.

Lemma 2.8. *The force field E given by (2-5) and ∇E can be estimated by*

$$|E(t,x)| + |\nabla E(t,x)| \lesssim \frac{1}{1+\check{\tau}_{t,x}^3 + \check{d}_{t,x}^3 + |x^\perp|^3}\Big(\sup_{x\in\mathbb{R}^3} |\nabla\rho(t,x)|(1+\check{\tau}_{t,x}^3 + \check{d}_{t,x}^3 + |x^\perp|^3)\Big). \tag{2-25}$$

2D. *Proof of Theorem 1.7.*

Proof of Theorem 1.7. Let f, (X, V) be the solution of (2-4) with maximal time of existence $T_* > 0$. Recall the maximal existence time T_* from Theorem 2.3 and the constants n_{nl}, δ_0, V_{nl} from Proposition 2.7. Define $n = \max\{n_{\text{nl}}, 8\}$, $V_\mu := 5\sup_{v\in\operatorname{supp}\mu} |v|$ and $\delta(t) := C_0 V_{\min}^{(-n+2)/n}(t)$. Then, for all $C_0 > 0$ there exists V_{C_0} such that $V_{\min}(t)^{-1} < \delta(t) < \delta_0$ provided $V_{\min}(t) \geq V_{C_0}$. Let $\overline{V} \geq \max\{V_{\text{nl}}, V_{C_0}, 1\}$. The constants C_0, $\overline{V}$ will be chosen later.

Consider the time $T > 0$ given by

$$T := \sup\{t \in [0, T_*) : \|\rho\|_{Y_t} \leq \delta(t),\ V_{\min}(t) \geq \max\{\overline{V}, V_\mu\}\}.$$

Then, by Corollary 2.6 and Proposition 2.7, for $0 \leq t < \min\{T, e^{\delta^{-1/n}(t)} - 2\}$

$$\begin{aligned} \|\rho\|_{Y_t} &\leq C\log^2(2+t)\|\mathcal{S}\|_{Y_t} \leq C\log^2(2+t)V_{\min}(t)^{-1} + C\log^2(2+t)\|\rho\|_{Y_t}^{3/2} \\ &\leq C\delta^{-2/n}(t)V_{\min}^{-1}(t) + C\delta^{-2/n}(t)\delta^{3/2}(t) \\ &\leq \frac{C}{C_0}\delta(t) + C\delta^{5/4}(t). \end{aligned}$$

Now pick $C_0 = 4C$ and choose $\overline{V}$ large enough such that the above estimate implies

$$\|\rho\|_{Y_t} \leq \tfrac{1}{2}\delta(t) \quad \text{for } t < \min\{T, e^{\delta^{-(1/n)}(t)} - 2\}.$$

By continuity of $\|\rho\|_{Y_t}$ and the blow-up criterion (2-18), we infer that one of the following statements holds (after possibly further increasing $\overline{V}$):

- $T \geq e^{\delta^{-1/n}(T)} - 2 \geq e^{V_{\min}^{1/(2n)}(T)}$.
- $V_{\min}(T) = \overline{V}$.
- $V_{\min}(T) = V_\mu$.

It remains to show (1-12)–(1-13), and that $T \geq e^{V_{\min}^{1/(2n)}(T)}$ implies $V(T) \leq \log^{3n} V_0$. We first show the latter assuming that (1-13) holds. Indeed, by (1-13), if $T \geq e^{V_{\min}^{1/(2n)}(T)}$, then

$$0 \leq V_{\min}(T) = V(T) \leq (2V_0^3 - A_{\min} e^{V_{\min}^{1/(2n)}(T)})^{1/3},$$

and thus (after possibly increasing $\overline{V}$)

$$V(T) \leq (C + 3\log V_0)^{2n} \leq \log^{3n} V_0.$$

It remains to show (1-12)–(1-13). We will first show the validity of (1-13) for $t \in [0, \min\{8V_0^{-3/5}, T\}]$. Then, if $T < 8V_0^{-3/5}$, the proof is complete. Otherwise, we show the validity of (1-12) on $[8V_0^{-3/5}, T]$ and how that implies (1-13) on in $[8V_0^{-3/5}, T]$ to conclude.

From (2-25) we get for all $0 \leq t < T$ (after possibly further increasing $\overline{V}$)

$$|\dot{V}(t)| \leq C\delta(t) \leq 1.$$

In particular, we can estimate the velocity of the charged particle by

$$V_0 - t \leq V_1(t) \leq V_0 + t. \tag{2-26}$$

This implies the validity of (1-13) for $t \in [0, \min\{8V_0^{-3/5}, T\}]$. Moreover, we infer (after possibly further increasing $\overline{V}$ and thus V_0)

$$t \geq 4V_{\min}^{-3/5}(t) \quad \text{for all } t \in [8V_0^{-3/5}, T],$$

where we first observe that the estimate follows immediately from the definition of T and $\overline{V} \geq 4$ if $t \geq 1$, and from the first inequality in (2-26) for $t \in [8V_0^{-3/5}, 1]$. Thus, combining Propositions 2.7 and 2.1 yields (1-12). Moreover, (2-26) and (1-12) yield that (1-13) holds on $[0, T]$. □

3. Estimates for the Green's function

In this section, we give the proofs of Proposition 2.2 and Corollary 2.6.

3A. *Proof of Proposition 2.2.*

We start with a simple estimate for the function a defined in (1-9).

Lemma 3.1. *The function $a(r)$ defined in (1-9) satisfies $a \in C^\infty(\mathbb{R})$ and for all $j \in \mathbb{N}$*

$$\left| a^{(j)}(r) - \frac{d^j}{d^j r}\left(\frac{1}{r^2}\right) \right| \leq \frac{C_j}{|r|^{3+j}}.$$

The proof is simply integration by parts and making use of $\hat{\mu}(0) = 1$ since μ is a probability density by assumption.

Proof of Proposition 2.2. The first part of the assertion, including the L^1-estimate (2-15), is taken from Theorem 2.1 in [Han-Kwan et al. 2021a].

It remains to prove (2-16) and (2-17). We only present the proof of (2-16). The proof of (2-17) is similar and will be skipped for the sake of conciseness.

Step 1: formulation in space-time Fourier transform. We start with the Fourier representation taken from (see [Han-Kwan et al. 2021a, equation (2.9)])

$$\widetilde{K}(\tau,\xi) := \hat{\phi}(\xi)\int_0^\infty e^{-i\tau t} i\xi\cdot\widehat{\nabla\mu}(t\xi)\,\mathrm{d}t = \hat{\phi}(\xi)a(\tau/|\xi|),$$
$$\widetilde{G}(\tau,\xi) = \frac{\widetilde{K}(\tau,\xi)}{1-\widetilde{K}(\tau,\xi)},$$

where we used the rotational symmetry of μ. Now we define $\Psi_\xi(r)$ by

$$\Psi_\xi(r) = \frac{a(r)}{1-\hat{\phi}(\xi)a(r)}, \quad r\in\mathbb{R}, \tag{3-1}$$

and take $\hat{\psi}_\xi(p) := (\mathcal{F}_r^{-1}\Psi_\xi(r))(p)$ to be the Fourier transform in r.

Relying on Assumption 1.4 and the smoothness of a, we observe that ψ as defined in (3-1) satisfies

$$\left|\nabla_\xi^j \frac{d^\ell}{d^\ell p}\hat{\psi}_\xi(p)\right| \lesssim_{j,l,M} \begin{cases} \dfrac{1}{1+|\xi|^{2+j}}\dfrac{1}{1+|p|^M} & \text{for any } M\in\mathbb{N},\ j\geq 1, \\ \dfrac{1}{1+|p|^M} & \text{for any } M\in\mathbb{N},\ j=0. \end{cases} \tag{3-2}$$

This allows us to rewrite

$$\widehat{G}(t,\xi) = \hat{\phi}(\xi)|\xi|\hat{\psi}_\xi(t|\xi|).$$

Step 2: the case $t\leq 1$. We take $\eta(\xi)$ to be a nonnegative bump function which takes value 1 on $B_{1/2}$ and vanishes outside of B_1. We decompose $\widehat{G}$ into

$$\begin{aligned}\widehat{G}(t,\xi) &= \eta(|\xi|^2)\hat{\phi}(\xi)|\xi|(\hat{\psi}_\xi(t|\xi|)-\hat{\psi}_\xi(0)) + \eta(|\xi|^2)\hat{\phi}(\xi)|\xi|\hat{\psi}_\xi(0) + (1-\eta(|\xi|^2))\hat{\phi}(\xi)|\xi|\hat{\psi}_\xi(t|\xi|) \\ &= R^{(a)}(\xi,t|\xi|) + R^{(b)}(\xi) + R^{(c)}(\xi,t|\xi|).\end{aligned}$$

We have

$$R^{(a)}(\xi,t|\xi|) = \eta(|\xi|^2)\hat{\phi}(\xi)t|\xi|^2\frac{\hat{\psi}_\xi(t|\xi|)-\hat{\psi}_\xi(0)}{t|\xi|} = \eta(|\xi|^2)\hat{\phi}(\xi)t|\xi|^2 h_\xi(t|\xi|),$$

where h is a smooth function with bounded derivatives to any order. Hence, $\nabla_\xi^4(R^{(a)}(\xi,t|\xi|))\in L^1(\mathbb{R}^3)$, uniformly for $t\leq 1$, and thus we can bound

$$|\mathcal{F}_\xi^{-1}[R^{(a)}(\xi,t|\xi|)](x)| \lesssim \frac{1}{1+|x|^4} \quad \text{for } t\leq 1.$$

Next,

$$R^{(b)}(\xi) = \eta(|\xi|^2)|\xi|e^{-|\xi|}\hat{\psi}_0(0) + \eta(|\xi|^2)\big(\hat{\phi}(\xi)|\xi|\hat{\psi}_\xi(0) - |\xi|e^{-|\xi|}\hat{\psi}_0(0)\big). \tag{3-3}$$

The Fourier transform of the first term can be explicitly estimated using

$$|\mathcal{F}_\xi^{-1}(|\xi|e^{-|\xi|})(x)| \lesssim \left|\Delta_x\left(\frac{1}{1+|x|^2}\right)\right| \lesssim \frac{1}{1+|x|^4}.$$

We then use (3-2) and proceed much as we did for $R^{(a)}$ to estimate the second term on the right-hand side of (3-3) and obtain a total bound

$$|\mathcal{F}_\xi^{-1}[R^{(b)}(\xi)](x)| \lesssim \frac{1}{1+|x|^4}.$$

The contribution of $R^{(c)}(\xi, t|\xi|)$ can be estimated by

$$|\mathcal{F}_\xi^{-1}[R^{(c)}(\xi,t|\xi|)](x)| \lesssim \frac{1}{|x|^4}\left\|\nabla_\xi^4\big((1-\eta(|\xi|^2))\hat{\phi}(\xi)|\xi|\hat{\psi}_\xi(t|\xi|)\big)\right\|_{L^1_\xi} \lesssim \frac{1}{|x|^4},$$

due to (3-2). Similarly, we obtain the bound

$$|\mathcal{F}_\xi^{-1}[R^{(c)}(\xi,t|\xi|)](x)| \lesssim \left\|\big((1-\eta(|\xi|^2))\hat{\phi}(\xi)|\xi|\hat{\psi}_\xi(t|\xi|)\big)\right\|_{L^1_\xi} \lesssim \frac{1}{t^4},$$

and we obtain the claim, (2-16), for $t \le 1$.

Step 3: the case $t \ge 1$. We rewrite $\widehat{G}$ as

$$\widehat{G}(t,\xi) = \frac{\hat{\phi}(\xi)}{t}\zeta_\xi(t|\xi|), \qquad \zeta_\xi(p) = |p|\hat{\psi}_\xi(|p|).$$

The Fourier transformation of ζ in ξ can be rewritten as

$$|\mathcal{F}_\xi^{-1}[\zeta_\xi(t|\xi|)](x)| = \frac{1}{t^3}|\mathcal{F}_\xi[\zeta_{\xi/t}(|\xi|)](x/t)|.$$

Much as above, we now decompose the function ζ further into

$$\begin{aligned}\zeta_{\xi/t}(|\xi|) &= |\xi|\big(\hat{\psi}_{\xi/t}(|\xi|)-\hat{\psi}_{\xi/t}(0)\big)\eta(|\xi|^2) + |\xi|\hat{\psi}_{\xi/t}(0)\eta(|\xi|^2) + |\xi|\hat{\psi}_{\xi/t}(|\xi|)(1-\eta(|\xi|^2)) \\ &= R^{(a)}_{\xi/t}(|\xi|) + R^{(b)}_{\xi/t}(|\xi|) + R^{(c)}_{\xi/t}(|\xi|).\end{aligned}$$

Arguing as in Step 2, we obtain

$$|\mathcal{F}_\xi^{-1}(\zeta_{\xi/t}(|\xi|))(x)| \lesssim \frac{1}{1+|x|^4}.$$

We conclude

$$|G(t,x)| \lesssim \frac{1}{t}\left|\frac{e^{-|x|}}{|x|} *_x \mathcal{F}_\xi[\zeta_\xi(t|\xi|)](x)\right| \lesssim \frac{1}{t^4+|x|^4} \quad t \ge 1,\, x \in \mathbb{R}^3,$$

as claimed. □

3B. *Proof of Corollary 2.6.* We start with a simple observation on the passage time τ_x that we will use frequently.

Lemma 3.2. *Let $T < T_*$ from Theorem 2.3. Recall the passage time τ_x introduced in Definition 2.5. Then, we have for all $x \in \mathbb{R}^3$*

$$|\nabla_x \tau_x| \le \frac{1}{V_{\min}(T)}. \tag{3-4}$$

The proof follows immediately from the definition of $V_{\min}(T)$ and τ_x.

Proof of Corollary 2.6. By the definition of ρ (see (2-14)) we have

$$\rho(t,x) = S(t,x) + \int_0^t \int_{\mathbb{R}^3} G(t-s, x-y) S(s,y)\, \mathrm{d}y\, \mathrm{d}s.$$

To prove the assertion, it suffices to estimate the convolution. We first consider the integral over the region

$$B_{t,x} := \left\{(s,y) \in (0,t)\times\mathbb{R}^3 : |t-s| + |x-y| \le \tfrac{1}{2}\max\{1, \check{\tau}_{t,x}, \check{d}_{t,x}, |x^\perp|\}\right\}.$$

We observe that for $s \le t$, and all $x, y \in \mathbb{R}^3$

$$|x^\perp| - |y^\perp| \le |t-s| + |x-y|, \quad \check{\tau}_{t,x} - \check{\tau}_{s,y} \le |t-s| + |x-y|, \quad \check{d}_{t,x} - \check{d}_{s,y} \le |t-s| + |x-y|.$$

Indeed, the first inequality follows immediately from the reverse triangle inequality. For the second inequality, we use in addition (3-4) and that $V_{\min} \ge 1$ by assumption. For the third inequality, we use in addition that $\check{d}_{t,x} \le \check{d}_{s,x}$ for $s \le t$ since $\dot{X}_1 > 0$ by assumption. Thus,

$$1 + \check{\tau}_{s,y} + \check{d}_{s,y} + |y^\perp| \ge \tfrac{1}{2}\max\{\check{\tau}_{t,x}, \check{d}_{t,x}, |x^\perp|\} \quad \text{in } B_{t,x}. \tag{3-5}$$

Combining this with (2-15) and the definition of $\|\cdot\|_{Y_t}$ (see (2-21)) yields

$$\left| \int_{B_{t,x}} G(t-s, x-y) S(s,y)\, \mathrm{d}s\, \mathrm{d}y \right| \lesssim \frac{\log(2+t)\|S\|_{Y_t}}{1 + \check{\tau}_{t,x}^2 + \check{d}_{t,x}^2 + |x^\perp|^2}.$$

It remains to estimate the excluded regions of the integral. By (2-16) we can estimate

$$\begin{aligned}
&\left| \int_{B_{t,x}^c} G(t-s, x-y) S(s,y)\, \mathrm{d}s\, \mathrm{d}y \right| \\
&\quad \lesssim \frac{\|S\|_{Y_t}}{1 + \check{\tau}_{t,x}^2 + d_{t,x}^2 + |x^\perp|^2} \int_0^t \int_{\mathbb{R}^3} \frac{\mathbb{1}_{|y^\perp| \le |x^\perp|/2} + \mathbb{1}_{|x^\perp|/2 \le |y^\perp| \le 2|x^\perp|} + \mathbb{1}_{|y^\perp| \ge 2|x^\perp|}}{(1 + (t-s)^2 + |x-y|^2)(1 + |y^\perp|^2)}\, \mathrm{d}y\, \mathrm{d}s \\
&\quad \lesssim \frac{\|S\|_{Y_t}}{1 + \check{\tau}_{t,x}^2 + d_{t,x}^2 + |x^\perp|^2} \left(\int_0^t \frac{2\log(2 + |x^\perp|)}{1 + (t-s) + |x^\perp|}\, \mathrm{d}s + \int_0^t \int_{\mathbb{R}^2} \frac{1}{(1 + s + |y^\perp|)(1 + |y^\perp|^2)}\, \mathrm{d}y^\perp\, \mathrm{d}s \right) \\
&\quad \lesssim \frac{\log^2(2+t)\|S\|_{Y_t}}{1 + \check{\tau}_{t,x}^2 + \check{d}_{t,x}^2 + |x^\perp|^2}.
\end{aligned}$$

The last inequality follows by separating the cases $|x^\perp| \le t$, $|x^\perp| \ge t$ for the first term and the regions $|y^\perp| \le s$, $|y^\perp| \ge s$ for the second term. Hence we have

$$|\rho(t,x)| \lesssim \frac{\log^2(2+t)\|S\|_{Y_t}}{1 + \check{\tau}_{t,x}^2 + \check{d}_{t,x}^2 + |x^\perp|^2}.$$

For the gradient $\nabla_x \rho$ we observe again that by (2-15)

$$\left| \int_{B_{t,x}} G(t-s, x-y) \nabla S(s,y)\, \mathrm{d}s\, \mathrm{d}y \right| \lesssim \frac{\log(2+t)\|S\|_{Y_t}}{1 + \check{\tau}_{t,x}^3 + \check{d}_{t,x}^3 + |x^\perp|^3}.$$

Again, it remains to estimate the convolution on the excluded region. Integrating by parts yields

$$\left|\int_{B^c_{t,x}} G(t-s,x-y)\nabla S(s,y)\,\mathrm{d}s\,\mathrm{d}y\right|$$
$$\leq \left|\int_{B^c_{t,x}} \nabla G(t-s,x-y)S(s,y)\,\mathrm{d}s\,\mathrm{d}y\right| + \left|\int_0^t \int_{|t-s|+|x-y|=\max\{1,\check{\tau}_{t,x},\check{d}_{t,x},|x^\perp|\}/2} G(t-s,x-y)S(s,y)\,\mathrm{d}y\,\mathrm{d}s\right|.$$

The first term on the right-hand side is estimated exactly as above. Moreover, by (2-16) and (3-5)

$$\left|\int_0^t \int_{|t-s|+|x-y|=\max\{1,\check{\tau}_{t,x},\check{d}_{t,x},|x^\perp|\}/2} G(t-s,x-y)S(s,y)\,\mathrm{d}s\,\mathrm{d}y\right|$$
$$\lesssim \int_0^t \int_{|t-s|+|x-y|=\max\{1,\check{\tau}_{t,x},\check{d}_{t,x},|x^\perp|\}/2} \frac{\|S\|_{Y_t}}{1+\check{\tau}^6_{t,x}+\check{d}^6_{t,x}+|x^\perp|^6}\,\mathrm{d}y\,\mathrm{d}s \lesssim \frac{\|S\|_{Y_t}}{1+\check{\tau}^3_{t,x}+\check{d}^3_{t,x}+|x^\perp|^3}. \qquad \square$$

4. Estimates on the characteristics

In this and the following sections we will always work under the following bootstrap assumptions: We consider $T>0$, $\delta>0$ as in Proposition 2.7. More precisely, recalling the maximal existence time T_* from Theorem 2.3, and the notation $V_{\min}$ and $\|\cdot\|_{Y_T}$ from (2-20) and Definition 2.5, respectively, we assume

$$T < T_*, \tag{B1}$$
$$V^{-1}_{\min}(T) < \delta < \min\{\delta_0, \log^{-n}(2+T)\}, \tag{B2}$$
$$\|\rho[f]\|_{Y_T} < \delta, \tag{B3}$$
$$\operatorname{supp}\mu \subset B_{V_{\min}(T)/5}(0) \tag{B4}$$

for some constants $\delta_0, n>0$ to be chosen later. We will refer to (B1)–(B4) as the bootstrap assumptions. In the following, we will often write $V_{\min}$ instead of $V_{\min}(T)$.

We recall from Lemma 2.8 that $\|\rho[f]\|_{Y_T}<\delta$ implies for all $0\le t\le T$ and all $x\in\mathbb{R}^3$

$$|E(t,x)|+|\nabla E(t,x)| \le \frac{\delta}{1+\check{\tau}^3_{t,x}+\check{d}^3_{t,x}+|x^\perp|^3}. \tag{4-1}$$

The objective of this section is to derive estimates for the characteristics defined in (2-6)–(2-7) which in integrated form read

$$X_{s,t}(x,v) = x-(t-s)v+\int_s^t (\sigma-s)\bar{E}(\sigma,X_{\sigma,t}(x,v))\,\mathrm{d}\sigma,$$
$$V_{s,t}(x,v) = v-\int_s^t \bar{E}(\sigma,X_{\sigma,t}(x,v)).$$

Definition 4.1. We define $\widetilde{W}_{s,t}$, $\widetilde{Y}_{s,t}$ as the functions given by

$$V_{s,t}(x,v) = v+\widetilde{W}_{s,t}(x,v),$$
$$X_{s,t}(x,v) = x-(t-s)v+\widetilde{Y}_{s,t}(x,v). \tag{4-2}$$

We are interested in the backwards characteristics, i.e., $0 \le s \le t < T$. We distinguish estimates for initial positions x "in front" and "behind" the point charge which are characterized by $\check{d}_{t,x} > 0$ and $t \ge \tau_x$, respectively.

We start by giving estimates for the characteristics for background particles which are in front of the point charge at time t. This is the easiest case, since those particles stay in front of the point charge along the backwards characteristics.

4A. *Estimates on the characteristics for particles in front of the point charge.*

Proposition 4.2. *For all $\delta_0, n > 0$ sufficiently small and large, respectively, we have under the bootstrap assumptions (B1)–(B4) for all $0 \le s \le t \le T$ and all $x, v \in \mathbb{R}^3$ with $|v| \le V_{\min}/2$ and $x_1 > X_1(t)$*

$$|\widetilde{Y}_{s,t}(x,v)| + |\nabla_x \widetilde{Y}_{s,t}(x,v)| \lesssim \frac{\delta(t-s)}{1+\check{d}_{t,x}^2+|x^\perp|^2}, \qquad |\nabla_v \widetilde{Y}_{s,t}(x,v)| \lesssim \frac{\delta(t-s)}{1+\check{d}_{t,x}+|x^\perp|},$$

$$|\widetilde{W}_{s,t}(x,v)| + |\nabla_x \widetilde{W}_{s,t}(x,v)| \lesssim \frac{\delta}{1+\check{d}_{t,x}^2+|x^\perp|^2}, \qquad |\nabla_v \widetilde{W}_{s,t}(x,v)| \lesssim \frac{\delta}{1+\check{d}_{t,x}+|x^\perp|}.$$

Proof. Since all the estimates are analogous, we only give the proof of for the estimate of $\widetilde{Y}$. By a continuity argument, we have $|\widetilde{Y}| \le \frac{1}{2}$ for δ sufficiently small, i.e., for δ_0 in (B2) sufficiently small. Therefore, using $|v| \le V_{\min}/2$, for all $\sigma \le t$

$$\begin{aligned}
\langle \check{d}_{\sigma,X_{\sigma,t}(x,v)} + |X^\perp_{\sigma,t}(x,v)|\rangle &= \langle |x_1-(t-\sigma)v_1-X_1(\sigma)+(\widetilde{Y}_{\sigma,t})_1(x,v)| + |x^\perp-(t-\sigma)v^\perp+Y^\perp_{\sigma,t}(x,v)|\rangle \\
&\gtrsim \big\langle |x_1-X_1(t)| + \tfrac{1}{4}(t-\sigma)V_{\min} + \tfrac{1}{2}|x^\perp| - \tfrac{3}{4}\big\rangle \\
&\gtrsim \langle \check{d}_{t,x} + (t-\sigma)V_{\min} + |x^\perp|\rangle.
\end{aligned}$$

Starting from the definition (4-2) of $\widetilde{Y}$ and using (4-1), we estimate

$$\begin{aligned}
|\widetilde{Y}_{s,t}(x,v)| &= \left| \int_s^t (\sigma-s)\bar{E}(\sigma, X_{\sigma,t}(x,v))\,\mathrm{d}\sigma \right| \\
&\lesssim \int_s^t (\sigma-s)\left(\frac{\delta}{\langle \check{d}_{t,x}+(t-\sigma)V_{\min}+x^\perp\rangle^3} + e^{-c\langle \check{d}_{t,x}+(t-\sigma)V_{\min}+|x^\perp|\rangle} \right) \mathrm{d}\sigma \\
&\lesssim \frac{\delta(t-s)}{1+\check{d}_{t,x}^2+|x^\perp|^2},
\end{aligned}$$

where we used $V_{\min}^{-1} < \delta$ by (B2). □

4B. *Estimate on the characteristics for particles behind the point charge.*

For estimating the characteristics behind the point charge we introduce further notation.

Definition 4.3. Let T be as in (B1), $t \in [0,T]$, $x \in \mathbb{R}^3$ and $v \in B_{V_{\min}/2}(0)$.

(1) Recalling the definition X^T from Definition 2.4, we define the *collision time* $\mathcal{T}_{t,x,v} := \mathcal{T}_{t,x_1,v_1}$ to be the unique solution to

$$X_1^T(\tau) = x_1 - (t-\tau)v_1.$$

We also define

$$\breve{\mathcal{T}}_{t,x,v} := \breve{\mathcal{T}}_{t,x_1,v_1} := [t - \mathcal{T}_{t,x_1,v_1}]_+.$$

(2) Finally, we introduce

$$\breve{x}_{t,x,v} := x - \breve{\mathcal{T}}_{t,x,v} v, \tag{4-3}$$

and we will call $\breve{x}^{\perp}_{t,x,v}$ the *impact parameter* of the collision, following the convention in collisional kinetic theory.

The quantities $\mathcal{T}_{t,x,v}$, $\breve{\mathcal{T}}_{t,x,v}$ and $\breve{x}_{t,x,v}$ correspond to τ_x, $\check{\tau}_{t,x}$ and x (see Definition 2.5): Instead of considering relations between the point charge and a fixed point in space x, these new quantities are the corresponding relations to the straight characteristic $x-(t-s)v$ for a given velocity v. The quantities $\mathcal{T}_{t,x,v}$, $\breve{\mathcal{T}}_{t,x,v}$ and $\breve{x}_{t,x,v}$ are also visualized in Figure 1: The collision time $\mathcal{T}_{t,x,v}$ is the time where the characteristic "collides" with the point charge with respect to the first coordinate. This is the time where the distance between the point charge and the characteristic is minimized. The difference $\breve{\mathcal{T}}_{t,x,v}$ is the time passed since this collision. We emphasize that the collision time $\mathcal{T}_{t,x,v}$ can lie before or after the passage time of τ_x depending on the sign of v_1. Moreover, if $v_1 \ll V_{\min}$, then, contrary to the visualization in Figure 1, the passage time and collision time are close in relation to the difference to the present time t, i.e., $|\tau_x - \mathcal{T}_{t,x,v}| \ll |t-\tau_x|$. In particular, $\check{\tau}_x$ and $\mathcal{T}_{t,x,v}$ are of the same order (see (4-7) below).

Finally, $\breve{x}_{t,x,v}$ is the position of the characteristic at the collision time. In particular $(\breve{x}_{t,x,v})_1 = X_1(\mathcal{T}_{t,x,v})$ and therefore we will be mostly interested in the impact parameter $\breve{x}^{\perp}_{t,x,v}$.

Note that the collision time and impact parameters are defined with respect to the straight characteristics. These will turn out to be sufficiently good approximations for the collision time and impact parameters for the true characteristics for our purposes.

In order to estimate the error of the backwards characteristics to the straight characteristics for particles in front of the point charge, it is suitable to consider the following error functions W, Y. Their definition is inspired by the intuition that the error can be best expressed in terms of the particle positions at the "collision".

Definition 4.4. For $0 \le s \le t < T$, with T as in (B1), and $(x,v) \in \mathbb{R}^3 \times \mathbb{R}^3$ with $\tau_x \le t$, we define the error functions Y and W by

$$\begin{aligned} W_{s,t}(x - \breve{\mathcal{T}}_{t,x_1,v_1} v, v) &= V_{s,t}(x,v) - v, \\ Y_{s,t}(x - \breve{\mathcal{T}}_{t,x_1,v_1} v, v) &= X_{s,t}(x,v) - (x-(t-s)v). \end{aligned}$$

Using (4-6), we infer the representation

$$Y_{s,t}(x,v) = X_{s,t}(x + \check{\tau}_{t,x} v, v) - (x + (s-\tau_x)v),$$

and hence

$$Y_{s,t}(x,v) = \int_s^t (\sigma - s)\bar{E}(\sigma, x + (\sigma-\tau_x)v + Y_{\sigma,t}(x,v))\,\mathrm{d}\sigma. \tag{4-4}$$

Before we proof estimates for Y and W, we give some basic facts regarding the passage time, the impact parameter and the collision time.

Lemma 4.5. *Recall the quantities τ_x, $\mathcal{T}_{t,x,v}$ introduced in Definitions 2.5 and 4.3. Then, we have the following identities for all $0 \le s \le t \le T$ with T as in (B1) and all $x, v \in \mathbb{R}^3$ with $|v| \le V_{\min}/2$ provided $V_{\min}(T) \ge 4$:*

$$\tau_x = \mathcal{T}_{t,x,0}, \qquad \check{\tau}_{t,x} = \check{\mathcal{T}}_{t,x,0}, \tag{4-5}$$

$$\mathcal{T}_{t,x,v} = \tau_{x-\check{\mathcal{T}}_{t,x,v}v} \quad \text{provided } \check{\mathcal{T}}_{t,x,v} > 0. \tag{4-6}$$

Moreover, we can estimate

$$\tfrac{1}{2}\check{\tau}_{t,x} \le \check{\mathcal{T}}_{t,x,v} \le 2\check{\tau}_{t,x}, \tag{4-7}$$

and if $\check{\mathcal{T}}_{t,x,v} > 0$

$$|\nabla_x \check{\mathcal{T}}_{t,x,v}| \le \frac{2}{V_{\min}}, \tag{4-8}$$

$$|\nabla_v \check{\mathcal{T}}_{t,x,v}| \lesssim \frac{\check{\mathcal{T}}_{t,x,v}}{V_{\min}}. \tag{4-9}$$

Furthermore, we have the lower bound

$$\langle \check{\tau}_{s,x-(t-s)v} \rangle \gtrsim \langle s - \mathcal{T}_{t,x,v} \rangle \quad \text{for all } \check{\mathcal{T}}_{t,x,v} > 0 \text{ and } s \ge \mathcal{T}_{t,x,v} - 5. \tag{4-10}$$

Finally, we have for $s \le t$

$$\langle (\tau_x - s)V_{\min} + |x^\perp| \rangle \lesssim \langle \check{d}_{s,x-(\tau_x-s)v} + |x^\perp - (\tau_x - s)v^\perp| \rangle \quad \text{for } s \le \tau_x, \tag{4-11}$$

$$\langle (\mathcal{T}_{t,x,v} - s)V_{\min} + |\check{x}^\perp_{t,x,v}| \rangle \lesssim \langle \check{d}_{s,x-(t-s)v} + |x^\perp - (t-s)v^\perp| \rangle \quad \text{for } s \le \mathcal{T}_{t,x,v} \le t, \tag{4-12}$$

$$\langle \check{d}_{t,x} + (t-s)V_{\min} + |x^\perp| \rangle \lesssim \langle \check{d}_{s,x-(t-s)v} + |x^\perp - (t-s)v^\perp| \rangle \quad \text{for } \check{d}_{t,x} > 0. \tag{4-13}$$

Proof. The identities (4-5) and (4-6) follow immediately from the definition of these quantities, and (3-4) and (4-8) are a consequence of $\dot{X}_1 \ge V_{\min} \ge 2|v_1|$. Estimate (4-9) follows from the identity (4-6), estimate (3-4) and the chain rule.

For (4-7), we first observe that $\check{\tau}_{t,x} = 0$ if and only if $\check{\mathcal{T}}_{t,x,v} = 0$. Otherwise, (4-7) follows from (4-6) and (3-4). Regarding (4-10), we observe that the estimate trivially holds for $\mathcal{T}_{t,x,v} - 5 \le s \le \mathcal{T}_{t,x,v}$. For $s \ge \mathcal{T}_{t,x,v}$ use once again (4-6) and (3-4) to find

$$[s - \tau_{x-(t-s)v}]_+ \ge [s - \mathcal{T}_{t,x,v}]_+ - \frac{|v|}{V_{\min}} |s - \mathcal{T}_{t,x_1,v_1}| \ge \frac{1}{2}(s - \mathcal{T}_{t,x,v}).$$

Finally, we turn to (4-11)–(4-13). Observe that (4-11) follows from (4-12) by choosing $t = \tau_x$. For the proof of (4-12), we insert the definition of $\check{x}_{t,x,v}$ (see (4-3)) to rewrite

$$x - (t-s)v = \check{x}_{t,x,v} - (\mathcal{T}_{t,x,v} - s)v.$$

Therefore, using the definition of $\mathcal{T}_{t,x,v}$ and $V_{\min}$,

$$(x - (t-s)v - X(s))_1 = (s - \mathcal{T}_{t,x,v})v_1 + X_1(\mathcal{T}_{t,x,v}) - X_1(s) \ge (V_{\min} - v_1)(\mathcal{T}_{t,x,v} - s).$$

Since $|v_1| + |v^\perp| \le \sqrt{2}|v| \le V_{\min}/\sqrt{2}$, this implies

$$\check{d}_{s,x-(t-s)v} + |x^\perp - (t-s)v^\perp| \ge (\mathcal{T}_{t,x,v} - s)V_{\min} + |\check{x}^\perp_{t,x,v}| - \frac{1}{\sqrt{2}}(\mathcal{T}_{t,x,v} - s)V_{\min},$$

which proves (4-12). The estimate (4-13) is shown analogously. □

Proposition 4.6. *For all δ_0, $n > 0$ sufficiently small and large, respectively, the following estimates hold under the bootstrap assumptions (B1)–(B4) for all $0 \le s \le t \le T$, $x, v \in \mathbb{R}^3$ such that $|v| \le V_{\min}/2$ and $-\infty < \tau_x \le t$:*

$$(|Y_{s,t}|+|\nabla_x Y_{s,t}|)(x,v) \lesssim \delta \min\left\{\frac{1}{\langle \check{\tau}_{s,x}\rangle+\langle x^\perp\rangle/\langle |v^\perp|\rangle}, \frac{t-s}{\langle \check{\tau}_{s,x}\rangle^2+\langle x^\perp\rangle^2/\langle |v^\perp|^2\rangle}\right\} \quad \text{for } s \ge \tau_x - 5, \tag{4-14}$$

$$(|Y_{s,t}|+|\nabla_x Y_{s,t}|)(x,v) \lesssim \frac{\delta}{1+\langle x^\perp\rangle/\langle v^\perp\rangle}\left(\frac{\tau_x - s}{1+\langle x^\perp\rangle/\langle v^\perp\rangle}+\min\left\{1, \frac{\check{\tau}_{t,x}}{1+\langle x^\perp\rangle/\langle v^\perp\rangle}\right\}\right) \quad \text{for } s \le \tau_x, \tag{4-15}$$

$$|\nabla_v Y_{s,t}(x,v)| \lesssim \log(2+t)\delta \min\left\{1, \frac{t-s}{\langle \check{\tau}_{s,x}\rangle+\langle x^\perp\rangle/\langle |v^\perp|\rangle}\right\} \quad \text{for } s \ge \tau_x - 5, \tag{4-16}$$

$$|\nabla_v Y_{s,t}(x,v)| \lesssim \log(2+t)\delta\left(\frac{\tau_x - s}{1+\langle x^\perp\rangle/\langle v^\perp\rangle}+\min\left\{1, \frac{\check{\tau}_{t,x}}{1+\langle x^\perp\rangle/\langle v^\perp\rangle}\right\}\right) \quad \text{for } s \le \tau_x. \tag{4-17}$$

Moreover,

$$|W_{s,t}(x,v)| + |\nabla_x W_{s,t}(x,v)| \lesssim \frac{\delta}{\langle \check{\tau}_{s,x}\rangle^2 + \langle x^\perp\rangle^2/\langle v^\perp\rangle^2} \quad \text{for } s \ge 0,$$

$$|\nabla_v W_{s,t}(x,v)| \lesssim \frac{\delta}{\langle \check{\tau}_{s,x}\rangle + \langle x^\perp\rangle/\langle v^\perp\rangle} \quad \text{for } s \ge 0.$$

Proof. We observe that for $s \in [\tau_x - 5, \tau_x]$, (4-14) and (4-16) follow from (4-15) and (4-17), respectively.

We prove (4-14) for $s \ge \tau_x$. For $\delta > 0$ sufficiently small, the right-hand side of (4-14) is bounded by 1. By a standard continuity argument we can therefore use $|Y_{s,t}| + |\nabla_x Y_{s,t}| \le 1$ for $\tau_x \le s \le t$. We use $|v| \le V_{\min}/2$ and (3-4) to find for all $\sigma \in [s, t]$

$$1 + \check{\tau}_{\sigma, x+\check{\tau}_{\sigma,x} v + Y_{\sigma,t}(x,v)} \gtrsim 1 + \check{\tau}_{\sigma,x}. \tag{4-18}$$

Moreover, $|v| \le V_{\min}/2$ and $|X_1(\sigma) - x_1| \ge V_{\min}\check{\tau}_{\sigma,x}$ implies

$$\begin{aligned} |x + \check{\tau}_{\sigma,x} v - X(\sigma)| &\ge \tfrac{1}{4}|x^\perp + \check{\tau}_{\sigma,x} v^\perp| + \tfrac{1}{2}|x_1 + \check{\tau}_{\sigma,x} v_1 - X_1(\sigma)| \\ &\ge \tfrac{1}{4}|x^\perp| + \tfrac{1}{8}\check{\tau}_{\sigma,x} V_{\min}. \end{aligned} \tag{4-19}$$

Resorting to (4-4) and using estimates (4-1), (1-6), (4-18) and (4-19), we deduce

$$\begin{aligned} |Y_{s,t}(x,v)| &= \left|\int_s^t (\sigma - s)\bar{E}(\sigma, x + \check{\tau}_{\sigma,x} v + Y_{\sigma,t}(x,v))\,\mathrm{d}\sigma\right| \\ &\lesssim \left|\int_s^t (\sigma - s)\left(\frac{\delta}{1+\check{\tau}_{\sigma,x}^3 + |x^\perp + \check{\tau}_{\sigma,x} v^\perp|^3} + e^{-(|x^\perp|+\check{\tau}_{\sigma,x} V_{\min})/8}\right)\mathrm{d}\sigma\right|. \end{aligned}$$

Observing that for $\tau_x \le \sigma \le t$ (by distinguishing the cases $|x^\perp| \ge 2\check{\tau}_{\sigma,x}|v^\perp|$ and $|x^\perp| \le 2\check{\tau}_{\sigma,x}|v^\perp|$)

$$1 + \check{\tau}_{\sigma,x} + |x^\perp + \check{\tau}_{\sigma,x} v^\perp| \gtrsim 1 + \check{\tau}_{\sigma,x} + \frac{\langle x^\perp\rangle}{\langle v^\perp\rangle}, \tag{4-20}$$

we obtain

$$|Y_{s,t}(x,v)| \lesssim \int_s^t (\sigma - s)\left(\frac{\delta}{1+\check{\tau}_{\sigma,x}^3 + \langle x^\perp\rangle^3/\langle v^\perp\rangle^3} + e^{-(|x^\perp|+V_{\min}\check{\tau}_{\sigma,x})/8}\right)\mathrm{d}\sigma. \tag{4-21}$$

To conclude the estimate (4-14) for $|Y|$, we use

$$\int_s^t (\sigma - s) e^{-(|x^\perp| + V_{\min} \check{\tau}_{s,x})/8} \, d\sigma \lesssim e^{-(|x^\perp| + V_{\min} \check{\tau}_{s,x})/8} \int_s^t (\sigma - s) e^{-(\sigma - s) V_{\min}/8} \, d\sigma \lesssim \min\left\{ \frac{1}{V_{\min}^2}, \frac{t-s}{V_{\min}} \right\} e^{-(|x^\perp| + V_{\min} \check{\tau}_{s,x})/8},$$

similar considerations for the first term in (4-21) and we recall that $V_{\min}^{-1} \leq \delta$ by (B2). The estimate of $|\nabla_x Y|$ is analogous.

For the proof of (4-15), the continuity argument shows $|Y_{s,t}| + |\nabla_x Y_{s,t}| \leq 1 + (\tau_x - s)$. We then split the integral

$$|Y_{s,t}(x, v)| \leq \left| \int_s^{\tau_x} (\sigma - s) \bar{E}(\sigma, x + (\sigma - \tau_x) v + Y_{\sigma,t}(x, v)) \, d\sigma \right| + \left| \int_{\tau_x}^t (\sigma - s) \bar{E}(\sigma, x + \check{\tau}_{\sigma,x} v + Y_{\sigma,t}(x, v)) \, d\sigma \right|. \tag{4-22}$$

Arguing much as we did for (4-11), we have for $\sigma \leq \tau_x$

$$\langle \check{d}_{\sigma, x + (\sigma - \tau_x) v + Y_{\sigma,t}(x,v)} + |x^\perp + (\sigma - \tau_x) v^\perp| \rangle \gtrsim \langle (\tau_x - \sigma) V_{\min} + |x^\perp| \rangle.$$

Using this, we can bound the first term in (4-22) as

$$\left| \int_s^{\tau_x} (\sigma - s) \bar{E}(\sigma, x + (\sigma - \tau_x) v + Y_{\sigma,t}(x, v)) \, d\sigma \right| \lesssim (\tau_x - s) \int_s^{\tau_x} \frac{\delta}{\langle x^\perp \rangle^3 + ((\tau_x - \sigma) V_{\min})^3} + e^{-(|x^\perp| + (\tau_x - \sigma) V_{\min})/8} \, d\sigma \lesssim \delta \frac{\tau_x - s}{\langle x^\perp \rangle^2}.$$

Therefore and using again (4-18) and (4-19), we can bound $|Y|$ by

$$|Y_{s,t}(x, v)| \lesssim \left| \int_{\tau_x}^t \delta \frac{(\tau_x - s)}{1 + \check{\tau}_{\sigma,x}^3 + |x^\perp + \check{\tau}_{\sigma,x} v^\perp|^3} \, d\sigma \right| + \left| \int_{\tau_x}^t \delta \frac{(\sigma - \tau_x)}{1 + \check{\tau}_{\sigma,x}^3 + |x^\perp + \check{\tau}_{\sigma,x} v^\perp|^3} \, d\sigma \right| + \int_{\tau_x}^t (\sigma - s) e^{-(|x^\perp| + \check{\tau}_{\sigma,x} V_{\min})/8} \, d\sigma + \delta \frac{\tau_x - s}{\langle x^\perp \rangle^2}.$$

Using the inequality (4-20) as above to bound the remaining integrals yields the desired estimate.

The remaining inequalities are proved analogously. □

In the following it will sometimes be convenient to use the estimates above for the functions $\widetilde{Y}$, $\widetilde{W}$ instead. They satisfy the relations

$$\widetilde{W}_{s,t}(x, v) = W_{s,t}(x - \check{\mathcal{T}}_{t,x,v} v, v),$$
$$\widetilde{Y}_{s,t}(x, v) = Y_{s,t}(x - \check{\mathcal{T}}_{t,x,v} v, v).$$

Using Lemma 4.5, we then obtain the following corollary.

Corollary 4.7. *Recall the notation* $\check{x}_{t,x,v}$ *from* (4-3). *For all* $\delta_0, n > 0$ *sufficiently small and large, respectively, the following estimates hold under the bootstrap assumptions* (B1)–(B4) *for all* $0 \le s \le t \le T$, $x, v \in \mathbb{R}^3$ *such that* $|v| \le V_{\min}/2$ *and* $-\infty < \tau_x \le t$:

- *If* $s \ge \mathcal{T}_{t,x,v} - 5$,

$$|\widetilde{Y}_{s,t}(x, v)| + |\nabla_x \widetilde{Y}_{s,t}(x, v)| \lesssim \delta \min\left\{\frac{1}{\langle s - \mathcal{T}_{t,x,v}\rangle + \langle \check{x}^\perp_{t,x,v}\rangle/\langle v^\perp\rangle}, \frac{t-s}{\langle s - \mathcal{T}_{t,x,v}\rangle^2 + \langle \check{x}^\perp_{t,x,v}\rangle^2/\langle v^\perp\rangle^2}\right\},$$
$$|\nabla_v \widetilde{Y}_{s,t}(x, v)| \lesssim \delta \log(2+t) \min\left\{1, \frac{t-s}{\langle s - \mathcal{T}_{t,x,v}\rangle + \langle \check{x}^\perp_{t,x,v}\rangle/\langle v^\perp\rangle}\right\} + \check{\mathcal{T}}_{t,x,v} |\nabla_x \widetilde{Y}_{s,t}(x, v)|.$$

- *If* $s \le \mathcal{T}_{t,x,v}$,

$$|\widetilde{Y}_{s,t}(x, v)| + |\nabla_x \widetilde{Y}_{s,t}(x, v)| \lesssim \frac{\delta}{1 + \langle \check{x}^\perp_{t,x,v}\rangle/\langle v^\perp\rangle}\left(\frac{\mathcal{T}_{t,x,v} - s}{1 + \langle \check{x}^\perp_{t,x,v}\rangle/\langle v^\perp\rangle} + \min\left\{1, \frac{\check{\mathcal{T}}_{t,x,v}}{1 + \langle \check{x}^\perp_{t,x,v}\rangle/\langle v^\perp\rangle}\right\}\right)$$
$$|\nabla_v \widetilde{Y}_{s,t}(x, v)| \lesssim \delta \log(2+t)\left(\frac{\mathcal{T}_{t,x,v} - s}{1 + \langle \check{x}^\perp_{t,x,v}\rangle/\langle v^\perp\rangle} + \min\left\{1, \frac{\check{\mathcal{T}}_{t,x,v}}{1 + \langle \check{x}^\perp_{t,x,v}\rangle/\langle v^\perp\rangle}\right\}\right) + \check{\mathcal{T}}_{t,x,v} |\nabla_x \widetilde{Y}_{s,t}(x, v)|.$$

- *For the function* $\widetilde{W}$ *we obtain for all* $0 \le s \le t$

$$(|\widetilde{W}_{s,t}| + |\nabla_x \widetilde{W}_{s,t}|)(x, v) \lesssim \frac{\delta}{\langle [s - \mathcal{T}_{t,x,v}]_+\rangle^2 + \langle \check{x}^\perp_{t,x,v}\rangle^2/\langle v^\perp\rangle^2},$$
$$|\nabla_v \widetilde{W}_{s,t}(x, v)| \lesssim \frac{\delta}{\langle [s - \mathcal{T}_{t,x,v}]_+\rangle + \langle \check{x}^\perp_{t,x,v}\rangle/\langle v^\perp\rangle}.$$

4C. *Some direct consequences of the error estimates of the characteristics.* As a first consequence of the estimates above, we deduce the following inequalities.

Corollary 4.8. *For all* $\delta_0, n > 0$ *sufficiently small respectively large, under the bootstrap assumptions* (B1)–(B4) *the following holds true for all* $x \in \mathbb{R}^3$, $0 \le s \le t \le T$ *and all* $v \in B_{V_{\min}/2}(0)$

$$\begin{gathered} |V_{s,t}(x, v) - v| \le 1, \quad \langle V_{s,t}(x, v)\rangle \ge \tfrac{1}{2}\langle v\rangle, \\ \langle X_{s,t}(x, v) - X(s)\rangle \ge \tfrac{1}{2}\langle x - (t-s)v - X(s)\rangle, \\ \langle \check{d}_{s,X_{s,t}(x,v)}\rangle \gtrsim \langle \check{d}_{s,x-(t-s)v}\rangle, \\ \langle \check{\tau}_{s,X_{s,t}(x,v)}\rangle \gtrsim \langle \check{\tau}_{s,x-(t-s)v}\rangle, \\ ||X^\perp_{s,t}(x, v)| - |x^\perp - (t-s)v^\perp|| \lesssim \delta \langle [t \wedge \mathcal{T}_{t,x,v} - s]_+\rangle. \end{gathered} \tag{4-23}$$

As a second consequence, we deduce that the support of $f(t, x, \cdot)$ remains contained in $B_{V_{\min}/2}$ under the bootstrap assumptions (B1)–(B4).

Corollary 4.9. *For all* $\delta_0, n > 0$ *sufficiently small respectively large, under the bootstrap assumptions* (B1)–(B4) *we have for all* $0 \le s \le t \le T$, $x, v \in \mathbb{R}^3$

$$|V_{s,t}(x, v)| \ge \tfrac{1}{5} V_{\min} \quad \textit{for all } |v| \ge \tfrac{1}{4} V_{\min}.$$

In particular, $\operatorname{supp} f(t, x, \cdot) \subset B_{V_{\min}/4}(0)$.

Proof. Assume the contrary, i.e., $|v| \geq \frac{1}{4} V_{\min}$ and $|V_{s,t}(x, v)| \leq \frac{1}{5} V_{\min}$. By continuity, there exist $s' \in [s, t]$ and $v' \in \partial B_{V_{\min}/4}(0)$ such that

$$V_{s',t}(x, v) = v', \quad X_{s',t}(x, v) = x', \quad V_{s,t}(x, v) = V_{s,s'}(x', v').$$

In particular we know that

$$|V_{s,s'}(x', v') - v'| \geq ||V_{s,s'}(x', v')| - |v'|| \geq \tfrac{1}{20} V_{\min}.$$

However by the corollary above, we have

$$|V_{s,s'}(x', v') - v'| \leq 1,$$

which is a contradiction for $V_{\min}^{-1} < \delta_0$ sufficiently small. □

To future reference, we summarize the decay of the background field in the following lemma.

Lemma 4.10. *Under the bootstrap assumptions (B1)–(B4) with $\delta_0, n > 0$ sufficiently small we have for all $0 \leq s \leq t \leq T$ and all $x \in \mathbb{R}^3$, $v \in B_{V_{\min}/2}$*

$$|\nabla_v \mu(V_{s,t}(x, v))| + |\nabla_v^2 \mu(V_{s,t}(x, v))| \lesssim e^{-|v|}. \tag{4-24}$$

Moreover,

(i) *For $\check{d}_{t,x} > 0$*

$$|E(s, x-(t-s)v)| + |\nabla E(s, x-(t-s)v)| + |E(s, X_{s,t}(x, v))| + |\nabla E(s, X_{s,t}(x, v))| \lesssim \frac{\delta}{\langle \check{d}_{t,x} + (t-s)V_{\min} + |x^\perp| \rangle^3}.$$

(ii) *For $\check{d}_{t,x} = 0$ and $s \geq \mathcal{T}_{t,x,v} - 5$*

$$|E(s, x-(t-s)v)| + |\nabla E(s, x-(t-s)v)| + |E(s, X_{s,t}(x, v))| + |\nabla E(s, X_{s,t}(x, v))| \lesssim \frac{\delta}{\langle s - \mathcal{T}_{t,x,v} \rangle^3 + \langle x^\perp - (t-s)v^\perp \rangle^3}.$$

(iii) *For $\check{d}_{t,x} = 0$ and $s \leq \mathcal{T}_{t,x,v}$*

$$|E(s, x-(t-s)v)| + |\nabla E(s, x-(t-s)v)| + |E(s, X_{s,t}(x, v))| + |\nabla E(s, X_{s,t}(x, v))| \lesssim \frac{\delta}{\langle V_{\min}(\mathcal{T}_{t,x,v} - s) \rangle^3 + \langle \check{x}_{t,x,v}^\perp \rangle^3}.$$

Proof. The estimate (4-24) follows immediately from the decay of μ from Assumption 1.2 together with the estimate $|\widetilde{W}_{s,t}(x, v)| \lesssim 1$ due to Proposition 4.2 and Corollary 4.7.

The estimates in items (i)–(iii) for the background field E along the straight characteristics $x - (t-s)v$ follow immediately from the decay of E, (4-1), together with the estimates (4-13), (4-10) and (4-12), respectively. Finally, along the true characteristics, estimates analogous to (4-13), (4-10) and (4-12) also hold which can be seen by combining them with the estimates from Corollary 4.8. □

5. Straightening the characteristics

As explained in Section 2C, a key ingredient for the proof of Proposition 2.7 consists of straightening the characteristics except for a small region which is the purpose of this section. As we will see, roughly speaking, this straightening is possible in regions where both $|\widetilde{Y}_{s,t}|$, $|\nabla_v \widetilde{Y}_{s,t}|$ are small compared to $t-s$. We distinguish four regions in which we can make use of this. In the following, we first give the characterization and necessary estimates in these regions, then prove an abstract result about the possibility of straightening the characteristics and finally apply the abstract result to those four regions.

(1) Due to Proposition 4.2, this is guaranteed in the region $\check{d}_{t,x} > 0$. More precisely, under the assumptions of that proposition, we have

$$|\widetilde{Y}_{s,t}(x,v)| \lesssim \frac{\delta(t-s)}{\langle \check{d}_{t,x}\rangle^2 + \langle x^\perp\rangle^2}, \quad |\nabla_v \widetilde{Y}_{s,t}(x,v)| \lesssim \frac{\delta(t-s)}{\langle \check{d}_{t,x}\rangle + \langle x^\perp\rangle}. \tag{5-1}$$

(2) Moreover, Corollary 4.7 provides sufficient estimates for straightening the characteristics on the set given by $\check{\tau}_{t,x} > 0$ and $s > \mathcal{T}_{t,x,v} - 5$.

More precisely, we have

$$|\widetilde{Y}_{s,t}(x,v)| \lesssim \delta \min\left\{ \frac{t-s}{\langle s-\mathcal{T}_{t,x,v}\rangle^2 + \langle \check{x}^\perp_{t,x,v}\rangle^2/\langle v^\perp\rangle^2}, \frac{t-s}{\check{\tau}_{t,x}} \right\}, \tag{5-2}$$

$$|\nabla_v \widetilde{Y}_{s,t}(x,v)| \lesssim \frac{\log(2+t)\delta(t-s)}{\langle s-\mathcal{T}_{t,x,v}\rangle + \langle \check{x}^\perp_{t,x,v}\rangle/\langle v^\perp\rangle}. \tag{5-3}$$

Here, we have used (4-7) and distinguished the cases $t-s \geq \check{\mathcal{T}}_{t,x,v}/2$ and $t-s \leq \check{\mathcal{T}}_{t,x,v}/2$ to obtain the second term in the minimum of the right-hand side in (5-2). Moreover, regarding the estimate (5-3), we have split the factor $\check{\mathcal{T}}_{t,x,v} = (t-s) + (s-\mathcal{T}_{t,x,v})$ to estimate the term $\check{\mathcal{T}}_{t,x,v}|\nabla_x \widetilde{Y}_{s,t}(x,v)|$.

(3) The straightening is more subtle in the regions where $\check{\tau}_{t,x} > 0$ and $s < \mathcal{T}_{t,x,v} - 1$. Indeed, since the error $\nabla_v \widetilde{Y}$ (due to the term $\check{\mathcal{T}}_{t,x,v}|\nabla_x \widetilde{Y}|$) grows linearly in $(\mathcal{T}_{t,x,v} - s)\check{\mathcal{T}}_{t,x,v}$, the straightening only works well if this factor is balanced by a sufficiently large impact parameter $\check{x}^\perp_{t,x,v}$.

This is the case if the time $\check{\mathcal{T}}_{t,x,v}$ (or equivalently $\check{\tau}_{t,x}$) is small compared to $|x^\perp|$ such that the impact parameter $\check{x}^\perp_{t,x,v}$ is still comparable to $x^\perp$. We therefore introduce

$$K_{s,t,x} := \left\{ v : s < \mathcal{T}_{t,x,v} - 1,\ |v| < \delta^{-\beta},\ \check{\tau}_{t,x}\langle v^\perp\rangle < \tfrac{1}{4}\langle x^\perp\rangle \right\} \tag{5-4}$$

for some $\beta > 0$.

Then, using (4-7), we observe that on $K_{s,t,x}$

$$\langle \check{x}^\perp_{t,x,v}\rangle \geq \tfrac{1}{2}\langle x^\perp\rangle. \tag{5-5}$$

Hence, Corollary 4.7 implies (recalling $V_{\min}^{-1} \leq \delta$) on $K_{s,t,x}$

$$\begin{aligned} |\widetilde{Y}_{s,t}(x,v)| &\lesssim \frac{\delta(t-s)}{1 + \langle x^\perp\rangle^2/\langle v^\perp\rangle^2}, \\ |\nabla_v \widetilde{Y}_{s,t}(x,v)| &\lesssim \log(2+t)\frac{\delta(t-s)}{1 + \langle x^\perp\rangle/\langle v^\perp\rangle}. \end{aligned} \tag{5-6}$$

(4) If the time $\check{\mathcal{T}}_{t,x,v}$ is not small compared to $|x^\perp|$, there are still a lot of trajectories with large $\check{x}^\perp_{t,x,v}$. To characterize these, we introduce

$$v_*^\perp(t,x,v_1) = \frac{x^\perp}{\check{\mathcal{T}}_{t,x_1,v_1}}. \tag{5-7}$$

Then, if v is far from $v^*(t,x,v_1)$, $\check{x}^\perp_{t,x,v}$ will be large. More precisely, by (4-7)

$$|\check{x}^\perp| = |x^\perp - \check{\mathcal{T}}_{t,x_1,v_1} v^\perp| = \check{\mathcal{T}}_{t,x_1,v_1}|v_*^\perp(t,x,v_1) - v^\perp| \geq \tfrac{1}{2}\check{\tau}_{t,x}|v_*^\perp(t,x,v_1) - v^\perp|. \tag{5-8}$$

Therefore, we define for $s < \mathcal{T}_{t,x_1,v_1} - 1$

$$F_{s,t,x} := \Bigl\{v \in \mathbb{R}^3 : s < \mathcal{T}_{t,x_1,v_1} - 1,\ |v| < \delta^{-\beta},\ |v_*^\perp(t,x,v_1) - v^\perp|\check{\tau}_{t,x} > \sqrt{\mathcal{T}_{t,x_1,v_1} - s}\Bigr\}. \tag{5-9}$$

Let us assume that n from the bootstrap assumption (B2) satisfies $n \geq 1/\beta$, and thus $\log(2+t) \leq \delta^{-\beta}$. Then, under the assumptions of Corollary 4.7, we find on $F_{s,t,x}$

$$\begin{aligned}
|\widetilde{Y}_{s,t}(x,v)| &\lesssim \frac{\delta^{1-3\beta}}{\langle \check{x}^\perp_{t,x,v}\rangle}\left(1 + \frac{\mathcal{T}_{t,x_1,v_1} - s}{\langle \check{x}^\perp_{t,x,v}\rangle}\right)\\
&\lesssim \frac{\delta^{1-3\beta}}{\langle \check{\tau}_{t,x}|v_*^\perp(t,x,v_1) - v^\perp|\rangle}\left(1 + \frac{\mathcal{T}_{t,x_1,v_1} - s}{\langle \check{\tau}_{t,x}|v_*^\perp(t,x,v_1) - v^\perp|\rangle}\right)\\
&\lesssim \frac{\delta^{1-3\beta}}{\langle \sqrt{\mathcal{T}_{t,x_1,v_1} - s}\rangle}\left(1 + \frac{\mathcal{T}_{t,x_1,v_1} - s}{\langle \sqrt{\mathcal{T}_{t,x_1,v_1} - s}\rangle}\right) \leq \delta^{1-3\beta},\\
|\nabla_v \widetilde{Y}_{s,t}(x,v)| &\lesssim \delta^{1-3\beta}\left(1 + \frac{\check{\mathcal{T}}_{t,x_1,v_1}}{\langle \check{\tau}_{t,x}|v_*^\perp(t,x,v_1) - v^\perp|\rangle}\right)\left(1 + \frac{\mathcal{T}_{t,x_1,v_1} - s}{\langle \check{\tau}_{t,x}|v_*^\perp(t,x,v_1) - v^\perp|\rangle}\right)\\
&\lesssim \delta^{1-3\beta}\left(1 + \frac{\check{\mathcal{T}}_{t,x_1,v_1}}{\langle \sqrt{\mathcal{T}_{t,x_1,v_1} - s}\rangle}\right)\frac{\mathcal{T}_{t,x_1,v_1} - s}{\langle \sqrt{\mathcal{T}_{t,x_1,v_1} - s}\rangle}.
\end{aligned} \tag{5-10}$$

We emphasize the right-hand sides above are bounded by $\delta^{1-3\beta}(t-s)$ since $t - s \geq 1$.

Notice that the derivative of the average velocity deviation $\nabla_v \widetilde{Y}_{s,t}/(t-s)$ satisfies stronger estimates than the derivative of the deviation $\nabla_v \widetilde{W}_{s,t}$. Intuitively, a deviation only significantly affects the average velocity if $\mathcal{T}_{t,x,v} - s$ becomes large. This gain of decay will be crucial to our argument.

Lemma 5.1. *Let $x \in \mathbb{R}^3$ be arbitrary and $0 \leq s \leq t$. Suppose there are open sets $\Omega'' \subset \Omega' \subset \Omega \subset \mathbb{R}^3$ such that*

(i) *for some $0 < \eta < \frac{1}{2}$, we have the estimate*

$$\sup_{v\in\Omega} \frac{|\widetilde{Y}_{s,t}(x,v)|}{t-s} < \eta, \quad \sup_{v\in\Omega} \frac{|\nabla_v \widetilde{Y}_{s,t}(x,v)|}{t-s} \leq \frac{1}{2}, \tag{5-11}$$

(ii) *and the following inclusions hold*

$$\{v \in \mathbb{R}^3 : \operatorname{dist}(v,\Omega'') \leq \eta\} \subset \Omega', \quad \{v \in \mathbb{R}^3 : \operatorname{dist}(v,\Omega') \leq \eta\} \subset \Omega. \tag{5-12}$$

Then there exists an open set $\Omega'' \subset \Omega^* \subset \Omega$, *and a diffeomorphism* $\Psi_{s,t}(x,\cdot):\Omega^* \to \Omega'$ *such that for all* $v \in \Omega^*$

$$X_{s,t}(x, \Psi_{s,t}(x,v)) = x-(t-s)v.$$

Moreover, Ψ *satisfies the estimates*

$$\begin{aligned} |\Psi_{s,t}(x,v)-v| &\le 2\frac{|\widetilde{Y}_{s,t}(x,v)|}{t-s}, \\ |\nabla_v \Psi_{s,t}(x,v)-\mathrm{Id}\,| &\le \sup_{w\in\Omega':|w-v|\le 2|\widetilde{Y}_{s,t}(x,v)|/(t-s)} \frac{|\nabla_v \widetilde{Y}_{s,t}(x,w)|}{t-s}. \end{aligned} \tag{5-13}$$

Proof. Let $\zeta_{s,t,x}(v)$ be the mapping defined by

$$\zeta_{s,t,x}(v) := v - \frac{\widetilde{Y}_{s,t}(x,v)}{t-s}.$$

With this definition, $X_{s,t}(x,v)$ can be rewritten as

$$X_{s,t}(x,v) = x-(t-s)\zeta_{s,t,x}(v).$$

Due to the second inequality in (5-11), $\zeta_{s,t,x}(v)$ is injective on Ω'. Therefore, the function has an inverse $\psi_{s,t}(x,\cdot)$ on the set $\Omega^* = \zeta(\Omega')$ which satisfies $\Omega^* \subset \Omega$ due to the first inequality in (5-11). Moreover, for any $w \in \Omega''$ the mapping

$$\Gamma : \bar{B}_\eta(w) \to \bar{B}_\eta(w), \quad v \mapsto w + \frac{\widetilde{Y}_{s,t}(x,v)}{t-s},$$

is a contraction and thus there exists $v \in \bar{B}_\eta(v_*) \subset \Omega'$ such that $\zeta_{s,t,x}(v) = w$. Therefore $\Omega'' \subset \Omega^* \subset \Omega$.

By (5-11), the inverse mapping $\Psi_{s,t}$ satisfies the estimate

$$|\Psi_{s,t}(x,v)-v| = |\Psi_{s,t}(x,v)-\zeta(\Psi_{s,t}(x,v))| \le \frac{|\widetilde{Y}_{s,t}(x,\Psi_{s,t}(x,v))|}{t-s} \le \frac{|\widetilde{Y}_{s,t}(x,v)|}{t-s} + \frac{1}{2}|\Psi_{s,t}(x,v)-v|,$$

which yields (5-13). Similarly, we can estimate its derivative in v by

$$|\nabla_v \Psi_{s,t}(x,v)-\mathrm{Id}\,| \lesssim \frac{|\nabla_v \widetilde{Y}_{s,t}(x,\Psi_{s,t}(x,v))|}{t-s} \le \sup_{w\in\Omega':|w-v|\le 2|\widetilde{Y}_{s,t}(x,v)|/(t-s)} \frac{|\nabla_v \widetilde{Y}_{s,t}(x,w)|}{t-s},$$

which finishes the proof. □

Corollary 5.2. *For all* $\beta > 0$, *under the bootstrap assumptions* (B1)–(B4) *with* $\delta_0, n > 0$ *sufficiently small, respectively large, the following hold true for all* $0 \le s \le t$ *and* $x \in \mathbb{R}^3$:

(1) ***Suppose*** $\check{d}_{t,x} > 0$**.** *Then there exists an open set* $B_{V_{\min}/4} \subset \mathcal{G}_{s,t,x} \subset B_{V_{\min}/2}$ *and a diffeomorphism* $\Psi_{s,t}(x,\cdot):\mathcal{G}_{s,t,x} \to B_{V_{\min}/3}$ *such that*

$$X_{s,t}(x, \Psi_{s,t}(x,v)) = x-(t-s)v. \tag{5-14}$$

Moreover, Ψ *satisfies the estimates*

$$|\Psi_{s,t}(x,v)-v| + |\nabla_v \Psi_{s,t}(x,v)-\mathrm{Id}| \lesssim \frac{\delta}{(1+\check{d}_{t,x}+|x^\perp|)}.$$

(2) ***Suppose*** $\check{\boldsymbol{d}}_{t,x} = \mathbf{0}$ *and*:

(a) $s > \mathcal{T}_{t,x_1,v_1} - 5$,

$$A_{s,t,x} := \{v \in B_{V_{\min}/2} : s > \mathcal{T}_{t,x,v} - 5\},$$
$$A'_{s,t,x} := \{v \in B_{V_{\min}/3} : s > \mathcal{T}_{t,x,v} - 4\},$$
$$A''_{s,t,x} := \{v \in B_{V_{\min}/4} : s > \mathcal{T}_{t,x,v} - 3\}.$$

Then, if $\tau_x \le t$, *there exists an open set* $A''_{s,t,x} \subset \mathcal{A}_{s,t,x} \subset A_{s,t,x}$ *and a diffeomorphism* $\Psi_{s,t}(x,\cdot) : \mathcal{A}_{s,t,x} \to A'_{s,t,x}$ *such that* (5-14) *holds. Moreover*, Ψ *satisfies the estimate*

$$|\Psi_{s,t}(x,v) - v| + |\nabla_v \Psi_{s,t}(x,v) - \mathrm{Id}| \lesssim \frac{\delta^{1-\beta}}{\langle s - \mathcal{T}_{t,x,v}\rangle + \langle \check{x}^{\perp}_{t,x,v}\rangle / \langle v^{\perp}\rangle}.$$

(b) $s < \mathcal{T}_{t,x_1,v_1} - 2$.

(i) *Next, let* $K_{s,t,x}$ *be as in* (5-4),

$$K'_{s,t,x} := \left\{v : s < \mathcal{T}_{t,x_1,v_1} - 2,\ |v| < \tfrac{1}{2}\delta^{-\beta},\ \check{\tau}_{t,x}\langle v^{\perp}\rangle < \tfrac{1}{5}\langle x^{\perp}\rangle\right\},$$
$$K''_{s,t,x} := \left\{v : s < \mathcal{T}_{t,x_1,v_1} - 3,\ |v| < \tfrac{1}{3}\delta^{-\beta},\ \check{\tau}_{t,x}\langle v^{\perp}\rangle < \tfrac{1}{6}\langle x^{\perp}\rangle\right\}.$$

Then, if $\tau_x \le t$ *and* $\langle x^{\perp}\rangle\delta^{\beta} \ge \check{\tau}_{t,x}$, *there exist an open set* $K''_{s,t,x} \subset \mathcal{K}_{s,t,x} \subset K_{s,t,x}$ *and a diffeomorphism* $\Psi_{s,t}(x,\cdot) : \mathcal{K}_{s,t,x} \to K'_{s,t,x}$ *such that* (5-14) *holds. Moreover*, Ψ *satisfies the estimate*

$$|\Psi_{s,t}(x,v) - v| + |\nabla_v \Psi_{s,t}(x,v) - \mathrm{Id}| \lesssim \frac{\delta^{1-2\beta}}{\langle x^{\perp}\rangle}.$$

(ii) *Similarly, let* $F_{s,t,x}$ *be as defined in* (5-9) *and recall the definition of* $v_*^{\perp} = v_*^{\perp}(t,x,v_1)$ (*see* (5-7)) *and define*

$$F'_{s,t,x} := \left\{v \in \mathbb{R}^3 : s < \mathcal{T}_{t,x_1,v_1} - 2,\ |v| < \tfrac{1}{2}\delta^{-\beta},\ |v^{\perp} - v_*^{\perp}|\check{\tau}_{t,x} > 2\sqrt{\mathcal{T}_{t,x_1,v_1} - s}\right\},$$
$$F''_{s,t,x} := \left\{v \in \mathbb{R}^3 : s < \mathcal{T}_{t,x_1,v_1} - 3,\ |v| < \tfrac{1}{3}\delta^{-\beta},\ |v^{\perp} - v_*^{\perp}|\check{\tau}_{t,x} > 3\sqrt{\mathcal{T}_{t,x_1,v_1} - s}\right\}.$$

Then, if $\langle x^{\perp}\rangle \le \check{\tau}_{t,x}\delta^{-\beta}$ *there exists an open set* $F''_{s,t,x} \subset \mathcal{F}_{s,t,x} \subset F_{s,t,x}$ *and a diffeomorphism* $\Psi_{s,t}(x,\cdot) : \mathcal{F}_{s,t,x} \to F'_{s,t,x}$ *which satisfies* (5-14) *and*

$$|\Psi_{s,t}(x,v) - v| + |\nabla_v \Psi_{s,t} - \mathrm{Id}| \lesssim \frac{\delta^{1-3\beta}}{t-s}\left(1 + \frac{t-s}{\langle \check{\tau}_{t,x}|v^{\perp} - v_*^{\perp}|\rangle} + \frac{\check{\tau}_{t,x}(\mathcal{T}_{t,x_1,v_1} - s)}{\langle \check{\tau}_{t,x}|v^{\perp} - v_*^{\perp}|\rangle^2}\right).$$

Proof. We choose $n \ge \beta^{-1}$ and assume also that δ_0 and n are chosen sufficiently small respectively large such that we can apply Proposition 4.2 and Corollary 4.8 and such that in particular the estimates (5-1)–(5-10) hold. We then apply Lemma 5.1 as follows.

Proof of (1). We apply Lemma 5.1 first in the case $\check{d}_{t,x} > 0$ to $\Omega = B_{V_{\min}/2}(0)$, $\Omega' = B_{V_{\min}/3}(0)$, $\Omega'' = B_{V_{\min}/4}(0)$. By (5-1), there is $C > 0$ such that we may choose $\eta = C\delta$ to satisfy the assumptions of Lemma 5.1 and the first assertion follows.

Proof of (a). Next, we apply Lemma 5.1 to $\Omega = A_{s,t,x}$, $\Omega' = A'_{s,t,x}$, $\Omega'' = A''_{s,t,x}$. By (5-2), we may choose $\eta = C\delta/\langle\check{\tau}_{t,x}\rangle$ to satisfy (5-11).

Using (4-9) and (4-7), we have for $v, v' \in \mathbb{R}^3$ with $|v' - v| \le \eta$

$$|\check{\mathcal{T}}_{t,x,v} - \check{\mathcal{T}}_{t,x,v'}| \lesssim \frac{\eta\check{\tau}_{t,x}}{V_{\min}} \lesssim \frac{\delta}{V_{\min}}, \tag{5-15}$$

which guarantees that (5-12) is satisfied. Combining Lemma 5.1 with (5-2)–(5-3) yields the second assertion.

Proof of (i). We apply Lemma 5.1 to $\Omega = K_{s,t,x}$, $\Omega' = K'_{s,t,x}$, $\Omega'' = K''_{s,t,x}$, with $\eta = C\delta^{1-\beta}/\langle x^\perp\rangle$ for some $C > 0$ sufficiently large. Using (5-6) we verify (5-11). Since $\langle x^\perp\rangle\delta^\beta \ge \check{\tau}_{t,x}$ by assumption, (5-15), and therefore (5-12), is satisfied. The estimate then follows from (5-6).

Proof of (ii). Finally, we choose $\Omega = F_{s,t,x}$, $\Omega' = F'_{s,t,x}$, $\Omega'' = F''_{s,t,x}$, and set $\eta = C\delta^{1-3\beta}/(t-s) \le C\delta^{1-3\beta}/\langle\check{\tau}_{t,x}\rangle$. Using (5-10) we verify that (5-11) is satisfied.

For the inclusions (5-12) we observe that for $v, v' \in B_{V_{\min}/2}$ with $|v' - v| \le \eta$, (5-7), (4-9) and (4-7) yield

$$\begin{aligned}\check{\tau}_{t,x}|v_*^\perp(t,x,v') - v'^\perp| &\ge \check{\tau}_{t,x}|v_*^\perp(t,x,v) - v^\perp| - \eta\check{\tau}_{t,x} - \frac{|x^\perp|\eta}{V_{\min}} \\ &\ge \check{\tau}_{t,x}|v_*^\perp(t,x,v) - v^\perp| - C\delta^{1-3\beta} - \frac{C\delta^{1-4\beta}}{V_{\min}},\end{aligned}$$

where we used the assumption $\langle x^\perp\rangle \le \check{\tau}_{t,x}\delta^{-\beta}$ in the last inequality.

Note that by assumption we have $\sqrt{\mathcal{T}_{t,x_1,v_1} - s} \ge 1$ in $F_{s,t,x}$. Hence, for δ sufficiently small, the last two terms on the right-hand side are smaller than 1 and (5-12) follows. Combining the assertion of Lemma 5.1 with (5-10) yields the desired estimates. □

6. Estimate of the direct contribution of the reaction term and the point charge

In the subsections below, we estimate the reaction term $\mathcal{R}$ (see (2-10)), which we rewrite as

$$\mathcal{R}(t,x) = R_{\mathrm{L}}(t,x) - R_{\mathrm{NL}}(t,x),$$

where

$$\begin{aligned}R_{\mathrm{L}}(t,x) &= \int_0^t \int_{\mathbb{R}^3} E(s, x-(t-s)v)\cdot\nabla_v\mu(v),\\ R_{\mathrm{NL}}(t,x) &= \int_0^t \int_{\mathbb{R}^3} E(s, X_{s,t}(x,v))\cdot\nabla_v\mu(V_{s,t}(x,v)).\end{aligned}$$

We need to estimate both $\mathcal{R}$ and $\nabla\mathcal{R}$. The general strategy is as follows. In regions where the change of variables $v \mapsto \Psi_{s,t}(x,v)$ is well-defined and $\Psi_{s,t}(x,v) \approx v$, we can use cancellations between the linear and nonlinear reaction terms. Here we rely on the analysis of $\Psi_{s,t}$ in the previous section. Otherwise, we do not control well the deviations of the straightened characteristics from the linear characteristics, and we cannot exploit cancellations between the linear and nonlinear reaction terms, R_{L} and R_{NL}. In that case, the desired estimates will follow from "smallness" of these regions and from the decay of μ.

6A. *Estimates for the reaction term $\mathcal{R}$.*

Proposition 6.1. *For all $\gamma \in (0, 1)$ there exists $C > 0$ such that the following estimate holds under the bootstrap assumptions (B1)–(B4) for all $\delta_0, n > 0$ sufficiently small*

$$|\mathcal{R}(t, x)| \le C \frac{\delta^{1+\gamma}}{1 + \check{\tau}_{t,x}^2 + \check{d}_{t,x}^2 + |x^\perp|^2}.$$

Proof. Step 1: structure of the proof. It suffices to show that there exists $M > 0$ such that for all $\beta > 0$ (sufficiently small)

$$|\mathcal{R}(t, x)| \lesssim \frac{\delta^{2-M\beta}}{1 + \check{\tau}_{t,x}^2 + \check{d}_{t,x}^2 + |x^\perp|^2}.$$

We use this peculiar reformulation for the sake of analogy of the parameter β with the one from Corollary 5.2. We emphasize that throughout the proof (implicit) constants may depend on β. By choosing $n \ge \beta^{-1}$, we can always absorb logarithmic errors in time due to (B2) by

$$\log(2 + t) \le \delta^{-\beta}. \tag{6-1}$$

We split the proof into three different cases, depending on which of the terms in $\check{\tau}_{t,x}^2$, $\check{d}_{t,x}$, $|x^\perp|$ is dominant, and whether the point charge has already passed x, i.e., whether $x_1 \ge X_1(t)$.

In each of these cases, we will make use of the estimate

$$\begin{aligned}|\mathcal{R}(t, x)| \le \int_{G_k} &|E(s, x - (t - s)v)| |\nabla_v \mu(v) - \nabla_v \mu(V_{s,t}(x, \Psi_{s,t}(x, v))) \det(\nabla_v \Psi_{s,t}(x, v))| \,dv\, ds \\ &+ \int_{B_k} |E(s, x - (t - s)v) \nabla_v \mu(v)| + |E(s, X_{s,t}(x, v)) \nabla_v \mu(V_{s,t}(x, v))| \,dv\, ds \\ =: &\int_{G_k} r_d(s, x, v) \,dv\, ds + \int_{B_k} r_s(s, x, v) \,dv\, ds,\end{aligned}$$

where the choice of $G_k, B_k \subset [0, t] \times \mathbb{R}^3$ depends on the case k under consideration, $k = 1, 2, 3$, such that the change of variables $\Psi_{t,s}(x, \cdot)$ from Corollary 5.2 is well-defined on $G_k^s := \{v : (s, v) \in G_k\}$ and

$$B_k^s \cup (G_k^s \cap \Psi(G_k^s)) \supset B_{V_{\min}/4}(0), \tag{6-2}$$

where we also set $B_k^s := \{v : (v, s) \in B_k\}$. Note that Corollary 4.9 together with the bootstrap equation (B4) implies $\mu(v) = \mu(V_{s,t}(x, v)) = 0$ for all $v \in B^c_{V_{\min}/4}(0)$.

In the following we will only rely on the estimates in Lemma 4.10 with squares in the denominator of all the estimates instead of cubes. This will prove useful for drawing analogies to the estimate of $\nabla\mathcal{R}$ later on.

Step 2: the case $\check{d}_{t,x} > 0$. In this case, we choose $G_1^s = \mathcal{G}$ from Corollary 5.2 and $B_1^s = \varnothing$. By Corollary 5.2, we have (6-2). Combining (4-24) with the estimates from Corollary 5.2 and Proposition 4.2, we infer on G_1

$$\begin{aligned}&|\nabla_v \mu(v) - \nabla_v \mu(V_{s,t}(x, \Psi_{s,t}(x, v))) \det(\nabla_v \Psi_{s,t}(x, v))| \\ &\qquad \lesssim (|\Psi_{s,t}(x, v) - v| + |\widetilde{W}_{s,t}(x, \Psi_{s,t}(x, v))| + |1 - \det(\nabla_v \Psi_{s,t}(x, v))|) e^{-|v|} \lesssim \delta \frac{e^{-|v|}}{1 + \check{d}_{t,x} + |x^\perp|}.\end{aligned}$$

Using now Lemma 4.10(i) yields

$$\int_{G_1} r_d(s,x,v)\,ds\,dv \lesssim \frac{\delta^2}{\langle \check{d}_{t,x}+|x^\perp|\rangle}\int_{\mathbb{R}^3}\int_0^t \frac{1}{\langle \check{d}_{t,x}+(t-s)V_{\min}+|x^\perp|\rangle^2} e^{-|v|}\,ds\,dv \lesssim \frac{\delta^3}{\langle \check{d}_{t,x}+|x^\perp|\rangle^2},$$

where we used that $V_{\min}^{-1}\le\delta$ by the bootstrap assumption (B2). For future reference, we point out that, had we used that E actually decays with the third power of $\check{d}_{t,x}+|x^\perp|$, we would have gained one power more in the denominator.

Step 3: the case $\check{d}_{t,x}=0$ and $\langle x^\perp\rangle\delta^\beta\le\check{\tau}_{t,x}$. Note that the assumption $\langle x^\perp\rangle\delta^\beta\le\check{\tau}_{t,x}$ implies that it suffices to show that

$$\int_{G_3} r_d(s,x,v)\,dv\,ds+\int_{B_3} r_s(s,x,v)\,dv\,ds \lesssim \frac{\delta^{2-M\beta}}{\check{\tau}_{t,x}^2}$$

for some M independent of β (note that there is no Japanese bracket in the denominator). We write $G_3=G_{3,1}\cup G_{3,2}$ with

$$\begin{aligned} G_{3,1} &:= \{(s,v)\in[0,t]\times\mathbb{R}^3 : v\in\mathcal{F}_{s,t,x}\},\\ G_{3,2} &:= \{(s,v)\in[0,t]\times\mathbb{R}^3 : v\in\mathcal{A}_{s,t,x}\},\\ B_3 &:= \{(s,v)\in[0,t]\times\mathbb{R}^3 : v\in B_{V_{\min}/4}\setminus F''_{s,t,x},\ s<\mathcal{T}_{t,x_1,v_1}-3\}, \end{aligned}$$

with the sets $\mathcal{F}_{s,t,x}$, $F''_{s,t,x}$, $\mathcal{A}_{s,t,x}$ as defined in Corollary 5.2. By Corollary 5.2, we have

$$\mathcal{A}_{s,t,x}\cap\Psi_{s,t}(x,\mathcal{A}_{t,s,x})\supset\{v\in B_{V_{\min}/4} : s>\mathcal{T}_{t,x_1,v_1}-3\}$$

and $\mathcal{F}_{s,t,x}\cap\Psi_{s,t}(x,\mathcal{F}_{t,s,x})\supset F''_{s,t,x}$. In particular, we verify the condition (6-2).

We first deal with the estimate on the set $G_{3,2}$. By Corollaries 5.2 and 4.7 we have on $G_{3,2}$

$$|\nabla_v\mu(v)-\nabla_v\mu(V_{s,t}(x,\Psi_{s,t}(x,v)))\det(\nabla_v\Psi_{s,t}(x,v))| \lesssim \frac{\delta^{1-\beta}}{\langle s-\mathcal{T}_{t,x_1,v_1}\rangle}e^{-|v|}.$$

Combining with Lemma 4.10(ii) yields

$$\int_{G_{3,2}} r_d(s,x,v) \lesssim \int_{B_{V_{\min}/2}}\int_{[\mathcal{T}_{t,x_1,v_1}-3]_+}^t \frac{\delta^{2-\beta}}{\langle s-\mathcal{T}_{t,x_1,v_1}\rangle}\frac{e^{-|v|}}{\langle s-\mathcal{T}_{t,x_1,v_1}\rangle^2+\langle x^\perp-(t-s)v^\perp\rangle^2}\,ds\,dv. \tag{6-3}$$

For $\check{\tau}_{t,x}\le 1$, the desired estimate follows immediately. If $\check{\tau}_{t,x}\ge 1$, we split the time integral: We observe that by (4-7), we have for $|v|\le V_{\min}/2$

$$s-\mathcal{T}_{t,x_1,v_1}=\check{\mathcal{T}}_{t,x_1,v_1}-(t-s)\ge\tfrac{1}{2}\check{\mathcal{T}}_{t,x_1,v_1}\ge\tfrac{1}{4}\check{\tau}_{t,x}\quad\text{for all } s\ge t-\tfrac{1}{2}\check{\mathcal{T}}_{t,x_1,v_1}=:s^*_{t,x,v_1}.$$

Thus, changing variables $w=x^\perp-(t-s)v^\perp$ for $s<s^*_{t,x,v_1}$, and using once more (4-7), we have

$$\begin{aligned}\int_{G_{3,2}} r_d(s,x,v) &\lesssim \frac{1}{\check{\tau}_{t,x}^2}\int_{-V_{\min}/2}^{V_{\min}/2}\int_{(\mathcal{T}_{t,x_1,v_1}-3)\vee 0}^{s^*_{t,x,v_1}\vee 0}\int_{B_{V_{\min}/2(t-s)}(x^\perp)}\frac{\delta^{2-\beta}}{\langle s-\mathcal{T}_{t,x_1,v_1}\rangle}\frac{e^{-|w-x^\perp|/(4|t-s|)}e^{-|v_1|/4}}{\langle w\rangle^2}\,dw\,ds\,dv_1\\ &\quad+\frac{1}{\check{\tau}_{t,x}^2}\int_{B_{V_{\min}/2}}\int_{s^*_{t,x,v_1}\vee 0}^t\frac{\delta^{2-\beta}}{\langle s-\mathcal{T}_{t,x_1,v_1}\rangle}e^{-|v|}\,dv\,ds \lesssim \frac{\delta^{2-2\beta}}{\check{\tau}_{t,x}^2}.\end{aligned}$$

The last inequality follows by separating the region $|w-x^\perp|\ge|t-s|$ and its complement.

We now turn to the estimate on the set $G_{3,1}$. By definition of $\mathcal{F}_{s,t,x} \subset F_{s,t,x}$ we have the inclusion $G_{3,1} \subset \{(s,v) : |v| \le V_{\min}/2,\ 0 < s < \mathcal{T}_{t,x_1,v_1} - 1\}$. In particular, if $\check{\tau}_{t,x} \ge 2t$, we have by (4-7) $\mathcal{T}_{t,x_1,v_1} = t - \check{\mathcal{T}}_{t,x_1,v_1} \le 0$ and thus $G_{3,1} = 0$. Therefore, it suffices to consider the case $\check{\tau}_{t,x} \le 2t$.

In this case, Corollary 5.2 implies that

$$G_{3,1} \subset \{(s,v) \in [0,t] \times \mathbb{R}^3 : v \in F_{s,t,x}\}.$$

We introduce

$$w = \delta^{\beta} \langle \check{\tau}_{t,x} \rangle (v^{\perp} - v_*^{\perp}(t,x,v_1)),$$

where $v_*^{\perp} = v_*^{\perp}(t,x,v_1)$ is defined as in (5-7). Since $\langle x^{\perp} \rangle \delta^{\beta} \le \check{\tau}_{t,x}$ by the assumption in this step, we can estimate $\delta^{\beta} |v^{\perp} - v_*^{\perp}| \le \delta^{\beta}(|v^{\perp}| + |v_*^{\perp}|) \lesssim 1$ on $G_{3,1}$. Therefore, and by (5-8),

$$|w| \lesssim \langle \check{\tau}_{t,x} \rangle, \qquad \langle w \rangle \lesssim \langle \check{\tau}_{t,x}(v^{\perp} - v_*^{\perp}) \rangle, \qquad \frac{\langle \check{x}^{\perp}_{t,x,v} \rangle}{\langle v^{\perp} \rangle} \ge |w|.$$

Therefore Corollaries 5.2 and 4.7 imply on $G_{3,1}$

$$|\nabla_v \mu(v) - \nabla_v \mu(V_{s,t}(x, \Psi_{s,t}(x,v))) \det(\nabla_v \Psi_{s,t}(x,v))| \lesssim \frac{\delta^{1-3\beta}}{t-s} \left(1 + \frac{t-s}{\langle w \rangle} + \frac{\check{\tau}_{t,x}(\mathcal{T}_{t,x_1,v_1} - s)}{\langle w \rangle^2}\right) e^{-|v|}.$$

We note also that by Lemma 4.10(iii), on $G_{3,1}$

$$|E(s, x-(t-s)v)| \le \frac{\delta}{\langle |w| + (\mathcal{T}_{t,x_1,v_1} - s) V_{\min} \rangle^2}.$$

Combining the two preceding estimates, we obtain, using also (4-7) and $\mathcal{T}_{t,x,v} \le t$ due to the assumption $\check{d}_{t,x} = 0$ in this step,

$$\begin{aligned}
&\int_{G_{3,1}} r_d(s,x,v)\, ds\, dv \\
&\quad\lesssim \delta^{2-3\beta} \int_{\mathbb{R}^3} \int_0^{\mathcal{T}_{t,x_1,v_1}-1} \frac{\mathbb{1}_{|w| \le C\langle \check{\tau}_{t,x} \rangle}}{t-s} \left(1 + \frac{t-s}{\langle w \rangle} + \frac{\check{\tau}_{t,x}(\mathcal{T}_{t,x_1,v_1} - s)}{\langle w \rangle^2}\right) \frac{e^{-|v|}}{\langle |w| + (\mathcal{T}_{t,x_1,v_1} - s) V_{\min} \rangle^2}\, ds\, dv \\
&\quad\lesssim \delta^{2-3\beta} \int_{\mathbb{R}^3} \int_1^t \frac{\mathbb{1}_{|w| \le C\langle \check{\tau}_{t,x} \rangle}}{\check{\mathcal{T}}_{t,x_1,v_1} + \sigma} \left(1 + \frac{\check{\mathcal{T}}_{t,x_1,v_1} + \sigma}{\langle w \rangle} + \frac{\check{\tau}_{t,x}\sigma}{\langle w \rangle^2}\right) \frac{e^{-|v|}}{\langle |w| + \sigma V_{\min} \rangle^2}\, d\sigma\, dv \\
&\quad\lesssim \delta^{2-4\beta} \int_{\mathbb{R}^2} \left(\frac{1}{V_{\min} \langle w \rangle \langle \check{\tau}_{t,x} \rangle} + \frac{1}{\langle w \rangle^2}\right) \mathbb{1}_{|w| \le C\langle \check{\tau}_{t,x} \rangle}\, dv_{\perp} \\
&\quad\lesssim \frac{\delta^{2-6\beta}}{\langle \check{\tau}_{t,x} \rangle^2} \int_{\mathbb{R}^2} \left(\frac{1}{V_{\min} \langle w \rangle \langle \check{\tau}_{t,x} \rangle} + \frac{1}{\langle w \rangle^2}\right) \mathbb{1}_{|w| \le C\langle \check{\tau}_{t,x} \rangle}\, dw \lesssim \frac{\delta^{2-5\beta}}{\langle \check{\tau}_{t,x} \rangle^2},
\end{aligned}$$

where we used in the last inequality that $\log(2 + \check{\tau}_{t,x}) \lesssim \log(2+t) \lesssim \delta^{-\beta}$ since we can assume $\check{\tau}_{t,x} \le 2t$ as argued above.

Finally, we turn to B_3. We split $B_3 = B_{3,1} + B_{3,2}$, where

$$\begin{aligned}
B_{3,1} &= \{(s,v) \in [0,t] \times \mathbb{R}^3 : v \in B_{V_{\min}/2} \setminus B_{\delta^{-\beta}},\ s < \mathcal{T}_{t,x_1,v_1} - 3\}, \\
B_{3,2} &= \{(s,v) \in [0,t] \times \mathbb{R}^3 : v \in B_{\delta^{-\beta}} \setminus F''_{s,t,x},\ s < \mathcal{T}_{t,x_1,v_1} - 3\}.
\end{aligned}$$

As above, we observe that (4-7) implies that $B_{3,1} = B_{3,2} = \varnothing$ if $\tau_{x_1} < -t$. If $\tau_{x_1} \ge -t$, and thus $\check{\tau}_{t,x_1} \le 2t$, the desired estimate on $B_{3,1}$ is trivial, since the exponential decay of μ together with the choice of n, δ_0 in (B2) gives $e^{-\delta^{-\beta}} \le C\delta^2/\langle \check{\tau} \rangle^2$ for n sufficiently large and δ_0 sufficiently small.

On $B_{3,2}$ we estimate using the definition of $F''_{s,t,x}$, Lemma 4.10(iii) and $V_{\min}^{1-} \le \delta$

$$\int_{B_{3,2}} r_s(s,x,v)\,\mathrm{d}v\,\mathrm{d}s \lesssim \delta \int_{\mathbb{R}} \int_0^{[\mathcal{T}_{t,x_1,v_1}-3]_+} \int_{|v^\perp - v_*^\perp||\check{\tau}_{t,x}| \le \sqrt{\mathcal{T}_{t,x_1,v_1}-s}} \mathrm{d}v^\perp \frac{1}{\langle V_{\min}(\mathcal{T}_{t,x_1,v_1}-s)\rangle^2}\,\mathrm{d}s e^{-|v_1|}\,\mathrm{d}v_1$$
$$\lesssim \frac{\delta^{2-\beta}}{|\check{\tau}_{t,x}|^2}.$$

Step 4: the case $\check{d}_{t,x} = 0$ and $\langle x^\perp\rangle\delta^\beta \ge \check{\tau}_{t,x}$. Arguing as above, we write again $G_4 = G_{4,1} \cup G_{4,2}$ and

$$G_{4,1} := \{(s,v) \in [0,t] \times \mathbb{R}^3 : v \in \mathcal{K}_{s,t,x}\},$$
$$G_{4,2} := \{(s,v) \in [0,t] \times \mathbb{R}^3 : v \in \mathcal{A}_{s,t,x}\},$$
$$B_4 := \{(s,v) \in [0,t] \times \mathbb{R}^3 : v \in B_{V_{\min}} \setminus K''_{s,t,x},\, s < \mathcal{T}_{t,x_1,v_1} - 3\},$$

with the sets $\mathcal{K}_{s,t,x}$, $K''_{s,t,x}$, $\mathcal{A}_{s,t,x}$ as defined in Corollary 5.2.

We first turn to $G_{4,2}$. Note that $G_{4,2} = G_{3,2}$ so in particular (6-3) holds in $G_{4,2}$. However, this time we want to gain a factor $\langle x^\perp\rangle^2$ instead of the factor $\langle \check{\tau}_{t,x}\rangle^2$ from the previous step. As above, the case $|x^\perp| \le 1$ is straightforward and we therefore only consider $|x^\perp| \ge 1$ in the following. We split $G_{4,2}$ further and first consider

$$G_{4,2}^1 := G_{4,2} \cap \left\{(s,v) \in [0,t] \times \mathbb{R}^3 : |v| \ge \frac{|x^\perp|}{2(\check{\mathcal{T}}_{t,x_1,v_1}+5)}\right\}.$$

On this set, by (4-7), we have

$$e^{-|v|} \lesssim \frac{\langle \check{\tau}_{t,x}^2\rangle}{|x^\perp|^2} e^{-|v|/2}.$$

Inserting this estimate within (6-3) yields the desired estimate on $G_{4,2}^1$.

On the other hand, on $G_{4,2}^2 := G_{4,2}^1 \setminus G_{4,2}$ we have

$$|x^\perp - (t-s)v^\perp| \ge \tfrac{1}{2}|x^\perp| \quad \text{for all } s \in [\mathcal{T}_{t,x_1,v_1} - 5, t].$$

Resorting to (6-3) leads again to the desired estimate.

We next turn to $G_{4,1}$. By Corollary 5.2, we have

$$G_{4,1} \subset \{(s,v) \in [0,t] \times \mathbb{R}^3 : v \in K_{s,t,x}\}.$$

Combining the estimates from Corollary 5.2 and Corollary 4.7 with (5-5) yields

$$|\nabla_v \mu(v) - \nabla_v \mu(V_{s,t}(x, \Psi_{s,t}(x,v))) \det(\nabla_v \Psi_{s,t}(x,v))| \lesssim \frac{\delta^{1-2\beta}}{\langle x^\perp\rangle} e^{-|v|}.$$

By Lemma 4.10(iii) and (5-5) we have

$$|E(s, x-(s-t)v)| \lesssim \frac{\delta}{\langle x^\perp\rangle^2 + \langle(\mathcal{T}_{t,x_1,v_1}-s)|V_{\min}|\rangle^2}.$$

Therefore,

$$\int_{G_{4,1}} r_d(s,x,v)\,\mathrm{d}s\,\mathrm{d}v \lesssim \frac{\delta^{2-2\beta}}{\langle x^\perp\rangle} \int_{\mathbb{R}^3} \int_0^{[\mathcal{T}_{t,x_1,v_1}-1]_+} \frac{e^{-|v|}}{\langle x^\perp\rangle^2 + \frac{1}{2}\langle(\mathcal{T}_{t,x_1,v_1}-s)|V_{\min}|\rangle^2}\,\mathrm{d}s\,\mathrm{d}v \lesssim \frac{\delta^{2-2\beta}}{V_{\min}\langle x^\perp\rangle^2}.$$

Regarding B_4, we first argue

$$B_4 \subset \{(s, v) \in [0, t] \times \mathbb{R}^3 : v \in B_{V_{\min}/4} \setminus B_{\delta^{-\beta}/6}, s < \mathcal{T}_{t,x_1,v_1} - 3\}.$$

Indeed, for $(s, v) \in B_4$ with $s < \mathcal{T}_{t,x_1,v_1} - 1$ and $|v| \le \delta^{-\beta}/6$, we find, due to the assumption $\langle x^\perp \rangle \delta^\beta \ge \check{\tau}_{t,x}$ that we made in this step, that

$$\check{\tau}_{t,x} \langle v^\perp \rangle \le \check{\tau}_{t,x} + \check{\tau}_{t,x} |v| \le \langle x^\perp \rangle \delta^\beta \big(1 + \tfrac{1}{6} \delta^{-\beta}\big) \le \tfrac{1}{6} \langle x^\perp \rangle,$$

and thus $v \in K''_{s,t,x}$. Moreover, as above, either $B_4 = \varnothing$ or $\check{\tau}_{t,x} \le 2t$; the latter we assume in the following. We split again, much as we did for $G_{4,2}$,

$$B_{4,1} := B_4 \cap \left\{ (s, v) \in [0, t] \times \mathbb{R}^3 : |v| \ge \frac{\langle x^\perp \rangle}{2 \langle \check{\mathcal{T}}_{t,x_1,v_1} \rangle} \right\}.$$

On this set, by (4-7), we have for $n \ge 3\beta^{-1}$

$$e^{-|v|} \lesssim e^{-\delta^{-\beta}/18} \frac{\langle \check{\tau}_{t,x} \rangle^3}{\langle x^\perp \rangle^3} e^{-|v|/3} \lesssim e^{-\delta^{-\beta}/18} \frac{\langle t \rangle^3}{\langle x^\perp \rangle^3} e^{-|v|/3} \lesssim \frac{1}{\langle x^\perp \rangle^3} e^{-|v|/3}. \tag{6-4}$$

Combining this estimate with Lemma 4.10(iii) and using $V_{\min}^{-1} \le \delta$ yields

$$\int_{B_{4,1}} r_s(s, x, v) \, \mathrm{d}v \, \mathrm{d}s \lesssim \frac{\delta}{\langle x^\perp \rangle^3} \int_{\mathbb{R}^3} \int_0^{[\mathcal{T}_{t,x_1,v_1} - 3]_+} \frac{e^{-|v|/3}}{\langle V_{\min} (\mathcal{T}_{t,x_1,v_1} - s) \rangle^2} \, \mathrm{d}v \, \mathrm{d}s \lesssim \frac{\delta^2}{\langle x^\perp \rangle^3}.$$

Finally, on $B_{4,2} := B_4 \setminus B_{4,1}$, we estimate

$$\int_{B_{4,2}} r_s(s, x, v) \, \mathrm{d}v \, \mathrm{d}s \lesssim \delta \int_{B_{V_{\min}/2} \setminus B_{\delta^{-\beta}/6}} \int_0^{[\mathcal{T}_{t,x_1,v_1} - 3]_+} \frac{e^{-|v|}}{\langle x^\perp \rangle^2 + \langle V_{\min} (\mathcal{T}_{t,x_1,v_1} - s) \rangle^2} \, \mathrm{d}v \, \mathrm{d}s \lesssim \frac{\delta^2}{\langle x^\perp \rangle^2},$$

where we used $e^{-\delta^{-\beta}} \lesssim \delta / \langle t \rangle^2$ for n sufficiently large. □

6B. *Estimates for* $\nabla \mathcal{R}$.

Proposition 6.2. *For all* $0 < \gamma < 1$ *there exists* $C > 0$ *such that under the bootstrap assumptions* (B1)–(B4) *with* $\delta_0, n > 0$ *sufficiently small we have for all* $t \le T$

$$|\nabla \mathcal{R}(t, x)| \le C \frac{\delta^{1+\gamma}}{1 + \check{\tau}_{t,x}^3 + \check{d}_{t,x}^3 + |x^\perp|^3}. \tag{6-5}$$

Proof. The proof is in large parts analogous to the proof of Proposition 6.1. We therefore just highlight the main differences. The main difficulty consists in extracting the third power in the denominator of (6-5) in comparison with the second power obtained in Proposition 6.1. To this end we must exploit once more the dispersion.

We will again distinguish the same three different cases as in the proof of Proposition 6.1. In the case $\check{d}_{t,x} > 0$, the estimates are easiest, since the backwards characteristics do not come close to the point charge. Therefore, the error estimates along the backwards characteristics are sufficient in this case.

For $\check{d}_{t,x} = 0$, the estimates are more delicate. Let us briefly explain why we expect better decay for $\nabla_x \mathcal{R}$ than for $\mathcal{R}$ itself also in this case. Basically, for characteristics close to free transport, we can make use of $\nabla_x \sim (1/t)\nabla_v$. More precisely, by integration by parts we find

$$\int \nabla_x f_0(x - tv) g(v) \, dv = \frac{1}{t} \int f_0(x - tv) \nabla g(v) \, dv.$$

This can also obtained through the following change of variables

$$\begin{aligned} \nabla_x \int f_0(x - tv) g(v) \, dv &= \frac{1}{t} \nabla_x \frac{1}{t^3} \int f_0(w) g\left(\frac{x - w}{t}\right) dw \\ &= \frac{1}{t^4} \int f_0(w) \nabla g\left(\frac{x - w}{t}\right) dw = \frac{1}{t} \int f_0(x - tv) \nabla g(v) \, dv. \end{aligned}$$

This argument still works well in our setting when we are close to free transport. We will thus manipulate $\mathcal{R}(t, x)$ through a change of variables before taking the gradient. Roughly speaking, the change of variables consists in replacing v by a point along the straightened backwards characteristics which corresponds to a time after the (potential) "approximate collision" along this characteristics. By taking the gradient after this change of variables we will gain the desired power. Indeed, we know that the time after an approximate collision is $\check{\tau}_{t,x}$.

Moreover, if $|x^\perp|$ is dominant over $\tau_{t,x}$ (and $|v|$ is of order 1) the decay of E in $|x^\perp|$ allows us to choose $\langle x^\perp \rangle$ as this corresponding time. Indeed, in view of the estimates in Corollary 4.7, the error for the backwards characteristics until times $s \geq t - |x^\perp|$ can still be controlled by $\delta \log(2 + t)$, whereas for larger times, this error grows linearly in s, just as if there was a collision at time $t - |x^\perp|$.

Step 1: the case $\check{d}_{t,x} > 0$. We have

$$\begin{aligned} \nabla \mathcal{R}(t, x) = & \int_0^t \int_{\mathbb{R}^3} \nabla E(s, x - (t - s)v) \cdot \nabla_v \mu(v) - \nabla E(s, X_{s,t}(x, v)) \cdot \nabla_v \mu(V_{s,t}(x, v)) \, dv \, ds \\ & - \int_0^t \int_{\mathbb{R}^3} \nabla_x \widetilde{Y}_{s,t}(x, v) \cdot \nabla E(s, X_{s,t}(x, v)) \nabla_v \mu(V_{s,t}(x, v)) \, dv \, ds \\ & - \int_0^t \int_{\mathbb{R}^3} E(s, X_{s,t}(x, v)) \nabla_x \widetilde{W}_{s,t}(x, v) \cdot \nabla_v \mu(V_{s,t}(x, v)) \, dv \, ds \\ =: & \, \mathcal{R}_a + \mathcal{R}_c + \mathcal{R}_d. \end{aligned}$$

The estimate of $\mathcal{R}_a$ works exactly as before. Indeed, as we pointed out above in Step 2 of the proof of Proposition 6.1, we could have already gained three powers of $\check{d}_{t,x} + |x^\perp|$ for $\mathcal{R}(t, x)$ in this case.

The terms $\mathcal{R}_c$ and $\mathcal{R}_d$ are estimated analogously, since the estimates of $\nabla \widetilde{W}$ and $\nabla \widetilde{Y}$ bring an additional power of $\check{d}_{t,x} + |x^\perp|$. More precisely, combining Proposition 4.2 and Lemma 4.10(i) yields

$$|\nabla E(s, X_{s,t}(x, v))| \, |\nabla_x \widetilde{Y}_{s,t}(x, v)| \lesssim \frac{\delta \log^2(2 + t)}{V_{\min}} \frac{1}{\langle \check{d}_{t,x} + |x|^\perp \rangle^2} \frac{t - s}{\langle \check{d}_{t,x} + (t - s) V_{\min} + |x|^\perp \rangle^3},$$

and the same bound holds for $|E(s, X_{s,t}(x, v))| \, |\nabla_x \widetilde{W}_{s,t}(x, v)|$. Integrating this bound in s and using the exponential decay of μ for the integration in v immediately yields the desired estimate.

Step 2: the case $\check{d}_{t,x} = 0$ and $\langle x^\perp \rangle \delta^\beta \le \check{\tau}_{t,x}$. Much as in the proof of Proposition 6.1, in this case it suffices to show

$$|\nabla \mathcal{R}(t,x)| \lesssim \frac{\delta^{2-M\beta}}{\check{\tau}_{t,x}^3}$$

for some M independent of β.

The key idea is to use the change of variables

$$\omega = \check{x}_{t,x,v} = x - \check{\mathcal{T}}_{t,x_1,v_1} v \quad \iff \quad v = \frac{x-\omega}{t-\tau_\omega},$$

since by (4-6) $\tau_\omega = \mathcal{T}_{t,x,v}$. Performing this change of variables (and recalling the definition of the error functions Y, W from 4.4) yields

$$\begin{aligned}\mathcal{R}_{\mathrm{NL}}(x) &= \int_0^t \int_{\mathbb{R}^3} E(s, x-(t-s)v + Y_{s,t}(x-\check{\mathcal{T}}_{t,x_1,v_1} v, v)) \cdot \nabla_v \mu(v + W_{s,t}(x-\check{\mathcal{T}}_{t,x_1,v_1} v, v))\, \mathrm{d}v\, \mathrm{d}s \\ &= \int_0^t \int_{\mathbb{R}^3} \frac{1}{\check{\tau}_{t,\omega}^3} E\left(s, \omega-(\tau_\omega - s)\frac{x-\omega}{t-\tau_\omega} + Y_{s,t}\left(\omega, \frac{x-\omega}{t-\tau_\omega}\right)\right) \cdot \nabla_v \mu\left(\frac{x-\omega}{t-\tau_\omega} + W_{s,t}\left(\omega, \frac{x-\omega}{t-\tau_\omega}\right)\right) \mathrm{d}\omega\, \mathrm{d}s.\end{aligned}$$

Taking the gradient in x yields

$$\begin{aligned}&\nabla_x \mathcal{R}_{\mathrm{NL}}(x) \\ &= \int_0^t \int_{\mathbb{R}^3} \frac{s-\tau_\omega}{\check{\tau}_{t,\omega}^4} \nabla_x E\left(s, \omega-(\tau_\omega - s)\frac{x-\omega}{t-\tau_\omega} + Y_{s,t}\left(\omega, \frac{x-\omega}{t-\tau_\omega}\right)\right) \cdot \nabla_v \mu\left(\frac{x-\omega}{t-\tau_\omega} + W_{s,t}\left(\omega, \frac{x-\omega}{t-\tau_\omega}\right)\right) \mathrm{d}\omega\, \mathrm{d}s \\ &\quad + \int_0^t \int_{\mathbb{R}^3} \frac{1}{\check{\tau}_{t,\omega}^4} E\left(s, \omega-(\tau_\omega - s)\frac{x-\omega}{t-\tau_\omega} + Y_{s,t}\left(\omega, \frac{x-\omega}{t-\tau_\omega}\right)\right) \cdot \nabla_v^2 \mu\left(\frac{x-\omega}{t-\tau_\omega} + W_{s,t}\left(\omega, \frac{x-\omega}{t-\tau_\omega}\right)\right) \mathrm{d}\omega\, \mathrm{d}s \\ &\quad + \int_0^t \int_{\mathbb{R}^3} \frac{\nabla_v Y_{s,t}(\omega, (x-\omega)/(t-\tau_\omega))}{\check{\tau}_{t,\omega}^4} \\ &\qquad \cdot \nabla_x E\left(s, \omega-(\tau_\omega - s)\frac{x-\omega}{t-\tau_\omega} + Y_{s,t}\left(\omega, \frac{x-\omega}{t-\tau_\omega}\right)\right) \cdot \nabla_v \mu\left(\frac{x-\omega}{t-\tau_\omega} + W_{s,t}\left(\omega, \frac{x-\omega}{t-\tau_\omega}\right)\right) \mathrm{d}\omega\, \mathrm{d}s \\ &\quad + \int_0^t \int_{\mathbb{R}^3} \frac{\nabla_v W_{s,t}(\omega, (x-\omega)/(t-\tau_\omega))}{\check{\tau}_{t,\omega}^4} \\ &\qquad \cdot E\left(s, \omega-(\tau_\omega - s)\frac{x-\omega}{t-\tau_\omega} + Y_{s,t}\left(\omega, \frac{x-\omega}{t-\tau_\omega}\right)\right) \cdot \nabla_v^2 \mu\left(\frac{x-\omega}{t-\tau_\omega} + W_{s,t}\left(\omega, \frac{x-\omega}{t-\tau_\omega}\right)\right) \mathrm{d}\omega\, \mathrm{d}s,\end{aligned}$$

and changing back to the original set of variables we obtain

$$\begin{aligned}\nabla_x \mathcal{R}_{\mathrm{NL}}(x) = &\int_0^t \int_{\mathbb{R}^3} \frac{s-\mathcal{T}_{t,x_1,v_1}}{\check{\mathcal{T}}_{t,x_1,v_1}} \nabla_x E(s, X_{s,t}(x,v)) \cdot \nabla_v \mu(V_{s,t}(x,v))\, \mathrm{d}\omega\, \mathrm{d}s \\ &+ \int_0^t \int_{\mathbb{R}^3} \frac{1}{\check{\mathcal{T}}_{t,x_1,v_1}} E(s, X_{s,t}(x,v)) \cdot \nabla_v^2 \mu(V_{s,t}(x,v))\, \mathrm{d}v\, \mathrm{d}s \\ &+ \int_0^t \int_{\mathbb{R}^3} \frac{\nabla_v Y_{s,t}(x-\check{\mathcal{T}}_{t,x_1,v_1} v, v)}{\check{\mathcal{T}}_{t,x_1,v_1}} \cdot \nabla_x E(s, X_{s,t}(x,v)) \cdot \nabla_v \mu(V_{s,t}(x,v))\, \mathrm{d}v\, \mathrm{d}s \\ &+ \int_0^t \int_{\mathbb{R}^3} \frac{\nabla_v W_{s,t}(x-\check{\mathcal{T}}_{t,x_1,v_1} v, v)}{\check{\mathcal{T}}_{t,x_1,v_1}} \cdot E(s, X_{s,t}(x,v)) \cdot \nabla_v^2 \mu(V_{s,t}(x,v))\, \mathrm{d}v\, \mathrm{d}s.\end{aligned}$$

Performing the same manipulations on the linear term leads to

$$\begin{aligned}\nabla\mathcal{R}(t,x) &= \int_0^t\int_{\mathbb{R}^3} \frac{s-\mathcal{T}_{t,x_1,v_1}}{\check{\mathcal{T}}_{t,x_1,v_1}}(\nabla_x E(s,x-(t-s)v)\cdot\nabla_v\mu(v) - \nabla_x E(s,X_{s,t}(x,v))\cdot\nabla_v\mu(V_{s,t}))\,\mathrm{d}v\,\mathrm{d}s \\ &\quad + \int_0^t\int_{\mathbb{R}^3} \frac{1}{\check{\mathcal{T}}_{t,x_1,v_1}}(E(s,x-(t-s)v)\cdot\nabla_v^2\mu(v) - E(s,X_{s,t}(x,v))\cdot\nabla_v^2\mu(V_{s,t}(x,v)))\,\mathrm{d}v\,\mathrm{d}s \\ &\quad - \int_0^t\int_{\mathbb{R}^3} \frac{\nabla_v Y_{s,t}(x-\check{\mathcal{T}}_{t,x_1,v_1}v,v)}{\check{\mathcal{T}}_{t,x_1,v_1}}\nabla_x E(s,X_{s,t}(x,v))\cdot\nabla_v\mu(V_{s,t}(x,v))\,\mathrm{d}v\,\mathrm{d}s \\ &\quad - \int_0^t\int_{\mathbb{R}^3} \frac{\nabla_v W_{s,t}(x-\check{\mathcal{T}}_{t,x_1,v_1}v,v)}{\check{\mathcal{T}}_{t,x_1,v_1}}E(s,X_{s,t}(x,v))\cdot\nabla_v^2\mu(V_{s,t}(x,v))\,\mathrm{d}v\,\mathrm{d}s \\ &=: \mathcal{R}_a+\mathcal{R}_b+\mathcal{R}_c+\mathcal{R}_d.\end{aligned}$$

The estimates of $\mathcal{R}_a$ and $\mathcal{R}_b$ are analogous to those in Proposition 6.1. The additional factor $\mathcal{T}_{t,x_1,v_1}-s$ in $\mathcal{R}_a$ does not pose a problem if one uses the third power of the decay of E instead of the second power as in the proof of Proposition 6.1. More precisely, by Lemma 4.10(ii) and (iii) we have

$$|\mathcal{T}_{t,x_1,v_1}-s|(|\nabla_x E(s,x-(t-s)v|+|\nabla_x E(s,X_{s,t}(x,v))|) \lesssim \frac{\delta}{\langle s-\mathcal{T}_{t,x_1,v_1}\rangle^2+\langle x^\perp-(t-s)v^\perp\rangle^2}$$

and

$$|\mathcal{T}_{t,x_1,v_1}-s|(|\nabla_x E(s,x-(t-s)v|+|\nabla_x E(s,X_{s,t}(x,v))|) \lesssim \frac{\delta}{\langle V_{\min}(\mathcal{T}_{t,x_1,v_1}-s)\rangle^2+\langle \check{x}^\perp_{t,x,v}\rangle^2}$$

for $s\ge\mathcal{T}_{t,x_1,v_1}-5$ and $s\le\mathcal{T}_{t,x_1,v_1}$ respectively, which are precisely the estimates we used for E in the proof of Proposition 6.1.

It remains to estimate $\mathcal{R}_c$ and $\mathcal{R}_d$. We use the estimates from Proposition 4.6 for $\nabla_v Y$, $\nabla_v W$. Since the estimates for $\nabla_v Y$ are weaker than those for $\nabla_v W$, it suffices to show the desired estimates for $\mathcal{R}_c$. We write

$$\mathcal{R}_c =: \int_0^t\int_{\mathbb{R}^3} \mathfrak{r}_c(s,x,v)\,\mathrm{d}v\,\mathrm{d}s.$$

Much as before, we split the integral in $\widetilde{G}_{3,1}:=\{(s,v):|v|\le V_{\min}/4,\ 0\le s\le\mathcal{T}_{t,x_1,v_1}\}$ and $\widetilde{G}_{3,2}:=\{(s,v):|v|\le V_{\min}/4,\ t\ge s\ge[\mathcal{T}_{t,x_1,v_1}]_+\}$.

We note that the identity (4-6) implies that we can use (4-16) in the set $\widetilde{G}_{3,2}$ to estimate the term $\nabla_v Y_{s,t}(x-\check{\mathcal{T}}_{t,x_1,v_1}v,v)$, and thus

$$|\nabla_v Y_{s,t}(x-\check{\mathcal{T}}_{t,x_1,v_1}v,v)| \lesssim \delta\log(2+t)\le\delta^{1-\beta},$$

where we used again (6-1).

Combining this estimate with Lemma 4.10(iii) and (4-7) yields on $\widetilde{G}_{3,2}$

$$|\mathfrak{r}_c(s,x,v)| \le \frac{1}{\check{\tau}_{t,x}}\frac{\delta^{2-\beta}}{\langle(\mathcal{T}_{t,x_1,v_1}-s)^3+|x^\perp-(t-s)v^\perp|^3\rangle}e^{-|v|}.$$

Then $\check{\tau}_{t,x}\int_{\widetilde{G}_{3,2}}|\mathfrak{r}_c(s,x,v)|$ is bounded by the right-hand side in (6-3). We thus the desired estimate by the estimates after (6-3).

Similarly, on $\widetilde{G}_{3,1}$, Lemma 4.10 (ii) and (4-17) imply

$$|\mathfrak{r}_c(s,x,v)| \le \frac{1}{\check{\tau}_{t,x}}\left(\frac{(\mathcal{T}_{t,x_1,v_1}-s)\langle v^\perp\rangle}{\langle \check{x}^\perp_{t,x,v}\rangle}+1\right)\frac{\delta^2\log(2+t)}{\langle (V_{\min}(\mathcal{T}_{t,x_1,v_1}-s))^3+|\check{x}^\perp_{t,x,v}|^3\rangle}e^{-|v|}$$
$$\lesssim \frac{1}{\check{\tau}_{t,x}}\frac{\delta^{2-\beta}}{\langle (V_{\min}(\mathcal{T}_{t,x_1,v_1}-s))^2+|\check{x}^\perp_{t,x,v}|^2\rangle}\frac{1}{\langle \check{x}^\perp_{t,x,v}\rangle}e^{-|v|}.$$

We now proceed much as we did in the estimate on $G_{3,1}$ in the proof of Proposition 6.1. We recall that either $\widetilde{G}_{3,1}=\varnothing$ or $\tau_x\ge -t$. Thus, using the change of variables

$$\omega^\perp=\check{x}^\perp=x^\perp-\check{\mathcal{T}}_{t,x_1,v_1}v^\perp,$$

and (4-7), we obtain the estimate

$$\int_{\mathbb{R}^3}\int_0^{[\mathcal{T}_{t,x,v}]_+}\mathfrak{r}_c(s,x,v)\,\mathrm{d}v\,\mathrm{d}s \lesssim \frac{\delta^{2-\beta}}{\check{\tau}^3_{t,x}}\int_{-\infty}^{\infty}\int_{\mathbb{R}^2}\int_0^{\mathcal{T}_{t,x_1,v_1}}\frac{1}{\langle\omega^\perp\rangle}\frac{e^{-|x^\perp-\omega^\perp|/(4\check{\mathcal{T}}_{t,x,v})}}{(\mathcal{T}_{t,x_1,v_1}-s)^2+\langle\omega^\perp\rangle^2}\,\mathrm{d}s\,\mathrm{d}\omega^\perp e^{-|v_1|/4}\,\mathrm{d}v_1$$
$$\lesssim \frac{\delta^{2-\beta}}{\check{\tau}^3_{t,x}}\int_{\mathbb{R}^2}\frac{1}{\langle\omega^\perp\rangle^2}e^{-|x^\perp-\omega^\perp|/\check{\tau}_{t,x}}\,\mathrm{d}\omega^\perp \lesssim \frac{\delta^{2-2\beta}}{\check{\tau}^3_{t,x}}.$$

Step 3: the case $\check{d}_{t,x}=0$ and $\langle x^\perp\rangle\delta^\beta\ge\check{\tau}_{t,x}$. In this case, we use a different change of variables before taking the gradient. More precisely, for $R>0$ (which we will later choose as $R=\langle x^\perp\rangle$), we write $v=(x-\omega)/R$ to find

$$\mathcal{R}_{\mathrm{NL}}(t,x)=\frac{1}{R^3}\int_0^t\int_{\mathbb{R}^3}E\left(s,\omega-(t-R-s)\frac{x-\omega}{R}+\widetilde{Y}\left(x,\frac{x-\omega}{R}\right)\right)\nabla_v\mu\left(\frac{x-\omega}{R}+\widetilde{W}\left(x,\frac{x-\omega}{R}\right)\right)\mathrm{d}\omega\,\mathrm{d}s.$$

Taking the gradient on this term as well as the corresponding linear term $\mathcal{R}_{\mathrm{L}}$, then reverting the change of variables and finally setting $R=\langle x^\perp\rangle$ yields

$$\begin{aligned}\nabla\mathcal{R}(t,x)&=\int_0^t\int_{\mathbb{R}^3}\frac{t-s-\langle x^\perp\rangle}{\langle x^\perp\rangle}(\nabla_xE(s,x-(t-s)v)\cdot\nabla_v\mu(v)-\nabla_xE(s,X_{s,t}(x,v))\cdot\nabla_v\mu(V_{s,t}))\,\mathrm{d}v\,\mathrm{d}s\\&\quad+\int_0^t\int_{\mathbb{R}^3}\frac{1}{\langle x^\perp\rangle}(E(s,x-(t-s)v)\cdot\nabla_v^2\mu(v)-E(s,X_{s,t}(x,v))\cdot\nabla_v^2\mu(V_{s,t}(x,v)))\,\mathrm{d}v\,\mathrm{d}s\\&\quad-\int_0^t\int_{\mathbb{R}^3}\left(\nabla_x\widetilde{Y}_{s,t}(x,v)+\frac{\nabla_v\widetilde{Y}_{s,t}(x,v)}{\langle x^\perp\rangle}\right)\cdot\nabla_xE(s,X_{s,t}(x,v))\cdot\nabla_v\mu(V_{s,t}(x,v))\,\mathrm{d}v\,\mathrm{d}s\\&\quad-\int_0^t\int_{\mathbb{R}^3}\left(\nabla_x\widetilde{W}+\frac{\nabla_v\widetilde{W}_{s,t}(x,v)}{\langle x^\perp\rangle}\right)\cdot E(s,X_{s,t}(x,v))\cdot\nabla_v^2\mu(V_{s,t}(x,v))\,\mathrm{d}v\,\mathrm{d}s\\&=:\mathcal{R}_a+\mathcal{R}_b+\mathcal{R}_c+\mathcal{R}_d.\end{aligned}$$

We argue that $\mathcal{R}_a+\mathcal{R}_b$ can be estimated as in the proof of Proposition 6.1. For $\mathcal{R}_b$ this is obvious. For $\mathcal{R}_a$, we consider the set

$$\widetilde{G}_4:=\left\{(s,v)\in[0,t]\times B_{V_{\min}/4}(0):|v|\le\frac{\langle x^\perp\rangle}{10\langle\check{\mathcal{T}}_{t,x_1,v_1}\rangle}\right\}.$$

Then, on $\widetilde{G}_4$, we have, recalling that we are in the case $\langle x^\perp\rangle\delta^\beta \geq \check{\tau}_{t,x}$ and using (4-7),

$$\begin{aligned} |\check{x}^\perp_{t,x,v}| &\gtrsim |x^\perp|, \\ |x^\perp - (t-s)v| &\gtrsim |x^\perp| \quad \text{for all } \mathcal{T}_{t,s,x} - 5 \leq s \leq t, \\ |(t-s) - \langle x^\perp\rangle| &\leq \langle x^\perp\rangle + [\mathcal{T}_{t,x_1,v_1} - s]_+. \end{aligned} \tag{6-6}$$

Thus, by Lemma 4.10(ii) and (iii),

$$|(t-s) - \langle x^\perp\rangle|(|\nabla_x E(s, x-(t-s)v| + |\nabla_x E(s, X_{s,t}(x,v))|) \lesssim \frac{\delta}{\langle s - \mathcal{T}_{t,x_1,v_1}\rangle^2 + \langle x^\perp\rangle^2}$$

and

$$|(t-s) - \langle x^\perp\rangle|(|\nabla_x E(s, x-(t-s)v| + |\nabla_x E(s, X_{s,t}(x,v))|) \lesssim \frac{\delta^{1-\beta}\log(2+t)}{\langle V_{\min}(\mathcal{T}_{t,x_1,v_1} - s)\rangle^2 + \langle x^\perp\rangle^2}$$

for $s \geq \mathcal{T}_{t,x_1,v_1} - 5$ and $s \leq \mathcal{T}_{t,x_1,v_1}$ respectively. With these bounds at hand, the desired estimate follows precisely as in the proof of Proposition 6.1.

We continue with the estimate on the set

$$\widetilde{B}_4 := \left\{ (s,v) \in [0,t] \times B_{V_{\min}/4}(0) : |v| \geq \frac{\langle x^\perp\rangle}{10\langle \check{\mathcal{T}}_{t,x_1,v_1}\rangle} \right\}.$$

Notice that on this set we have $|v| \geq \frac{1}{20}\delta^{-\beta}$ due to the assumption $\langle x^\perp\rangle\delta^\beta \geq \check{\tau}_{t,x}$ in the case under consideration. This allows us to argue analogous to the estimate on $B_{4,1}$ in the proof of Proposition 6.1: Similar to how we obtained (6-4), we find on $\widetilde{B}_4$

$$e^{-|v|} \lesssim \frac{\delta e^{-|v|/3}}{\langle t\rangle^2 \langle x^\perp\rangle^3},$$

which allows us to deduce the desired estimate by just using the estimate $|\nabla E| \lesssim \delta$.

Regarding, $\mathcal{R}_c$ and $\mathcal{R}_d$, we use the estimates from Corollary 4.7. Again, since the estimates on $\widetilde{W}$ are better than those on $\widetilde{Y}$, it suffices to estimate $\mathcal{R}_c$. Moreover, since $\nabla_v \widetilde{Y}_{s,t}(x,v) \lesssim \delta t^2$, we can argue as above on the set $\widetilde{B}_4$ and it therefore suffices to consider the set $\widetilde{G}_4$. Since by the assumption $\langle x^\perp\rangle\delta^\beta \geq \check{\tau}_{t,x}$ and (4-7), we have $\check{\mathcal{T}}_{t,x_1,v_1} \leq \langle x^\perp\rangle$, Corollary 4.7 and (6-6) yield on $\widetilde{G}_4$

$$|\nabla_x \widetilde{Y}_{s,t}(x,v)| + \left|\frac{\nabla_v \widetilde{Y}_{s,t}(x,v)}{\langle x^\perp\rangle}\right| \lesssim \begin{cases} \dfrac{\delta\log(2+t)\langle v^\perp\rangle}{\langle x^\perp\rangle}\left(\dfrac{(\mathcal{T}_{t,x_1,v_1} - s)\langle v^\perp\rangle}{\langle x^\perp\rangle} + 1\right) & \text{for } 0 < s < \mathcal{T}_{t,x_1,v_1}, \\ \dfrac{\delta\log(2+t)\langle v^\perp\rangle}{\langle x^\perp\rangle} & \text{for } \mathcal{T}_{t,x_1,v_1} < s < t. \end{cases}$$

Combining again with the estimates from Lemma 4.10 and using $|v^\perp|^2 e^{-|v|} \lesssim e^{-|v|/2}$ yields on both $\widetilde{G}_{3,1}$ and $\widetilde{G}_{3,2}$

$$|\mathfrak{r}_c(s,x,v)| \lesssim \frac{\delta^{2-\beta}\log(2+t)^2}{\langle(\mathcal{T}_{t,x,v} - s)^2 + |x^\perp|^2\rangle} \frac{e^{-|v|/2}}{\langle x^\perp\rangle^2}.$$

Integrating over $\widetilde{G}_{3,1}$ and $\widetilde{G}_{3,2}$ yields the desired estimate. ∎

6C. *Contribution of the point charge.* In this section we derive estimates for the function S_{P} defined in (2-11). For future reference, we also introduce the function $\bar{S}_{\mathrm{P}}(t,x)$ defined by

$$\bar{S}_{\mathrm{P}}(t,x) = -\int_{-\infty}^{t}\int_{\mathbb{R}^3} \nabla\Phi(x-(t-s)v-(X(t)-(t-s)V(t)))\cdot\nabla_v\mu(v)\,\mathrm{d}v\,\mathrm{d}s. \tag{6-7}$$

Compared to S_{P}, this corresponds to a linearization of both the characteristics and the trajectory of the point charge and in addition to a extension to all negative times. In particular, $\bar{S}_{\mathrm{P}}$ resembles the function S_{R,X_*,V_*} from (2-13).

Proposition 6.3. *Under the bootstrap assumptions* (B1)–(B4) *with* $\delta_0, n > 0$ *sufficiently small*, *we have for all* $t \le T$

$$|S_{\mathrm{P}}(t,x)| + |\bar{S}_{\mathrm{P}}(t,x)| \le \frac{C}{V_{\min}(1+|x^{\perp}|^2+\check{d}_{t,x_1}^2+\check{\tau}_{t,x_1}^2)}, \tag{6-8}$$

$$|\nabla S_{\mathrm{P}}(t,x)| \le \frac{C}{V_{\min}(1+|x^{\perp}|^3+\check{d}_{t,x_1}^3+\check{\tau}_{t,x_1}^3)}. \tag{6-9}$$

Proof. Step 1: Proof of (6-8). Recall the definition of S_{P}

$$S_{\mathrm{P}}(t,x) = -\int_0^t\int_{\mathbb{R}^3} \nabla\Phi(X_{s,t}-X(s))\cdot\nabla_v\mu(V_{s,t})\,\mathrm{d}v\,\mathrm{d}s.$$

We observe that by (4-23) and the definitions of $\mathcal{T}_{t,x,v}$ and $v_* = v_*(t,x,v_1)$ from Definition 4.3 and (5-7), for $|v| \le V_{\min}/2$

$$\langle X_{s,t}(x,v) - X(s)\rangle \gtrsim |s-\mathcal{T}_{t,x_1,v_1}|V_{\min} + |v^{\perp}-v_*^{\perp}|\check{\mathcal{T}}_{t,x_1,v_1} \quad \text{if } \check{d}_{t,x_1} = 0, \tag{6-10}$$

$$\langle X_{s,t}(x,v) - X(s)\rangle \gtrsim \check{d}_{t,x_1} + |x^{\perp}| + V_{\min}(t-s) \quad \text{if } \check{d}_{t,x_1} > 0. \tag{6-11}$$

Consider first the case $\check{d}_{t,x} = 0$, i.e., $\tau_x \le t$. which by (4-7) is equivalent to $\check{\mathcal{T}}_{t,x_1,v_1} \ge \check{\tau}_{t,x}/2$ and thus $\check{\mathcal{T}}_{t,x_1,v_1} > 0$ is equivalent to $\check{\tau}_{t,x} > 0$. Then, by (6-10) and the decay of Φ (see (1-6)),

$$|S_{\mathrm{P}}(t,x)| \lesssim \int_0^t\int_{\mathbb{R}^3} e^{-|s-\mathcal{T}_{t,x_1,v_1}|V_{\min}-|v^{\perp}-v_*^{\perp}|\check{\mathcal{T}}_{t,x_1,v_1}} e^{-(|v_1|+|v^{\perp}|)/4}\,\mathrm{d}v\,\mathrm{d}s \lesssim \frac{1}{V_{\min}(1+\check{\tau}_{t,x_1}^2)}.$$

For the desired decay in $|x^{\perp}|$, we consider again the sets

$$G := \left\{v \in B_{V_{\min}/2}(0) : |v| \le \frac{\langle x^{\perp}\rangle}{2\check{\mathcal{T}}_{t,x_1,v_1}}\right\},$$

$$B := \left\{v \in B_{V_{\min}/2}(0) : |v| \ge \frac{\langle x^{\perp}\rangle}{2\check{\mathcal{T}}_{t,x_1,v_1}}\right\}.$$

For $v \in G$, we have $\langle |v^{\perp}-v_*^{\perp}|\check{\mathcal{T}}_{t,x_1,v_1}\rangle = \langle x^{\perp} - \check{\mathcal{T}}_{t,x_1,v_1}v^{\perp}\rangle \gtrsim \langle x^{\perp}\rangle$ and thus

$$\int_0^t\int_G e^{-|s-\mathcal{T}_{t,x_1,v_1}|V_{\min}-|v^{\perp}-v_*^{\perp}|\check{\mathcal{T}}_{t,x_1,v_1}} e^{-(|v_1|+|v^{\perp}|)/4}\,\mathrm{d}v\,\mathrm{d}s \lesssim \frac{1}{V_{\min}(1+|x^{\perp}|^2)}.$$

Moreover, in B we use that $e^{-|v|} \lesssim e^{-|v|/2}\check{\tau}_{t,x}^2/\langle x^\perp\rangle^2$ to deduce

$$\int_0^t\int_B e^{-|s-\mathcal{T}_{t,x_1,v_1}|V_{\min}-|v^\perp-v_*^\perp|\check{\mathcal{T}}_{t,x_1,v_1}} e^{-(|v_1|+|v^\perp|)/4}\,\mathrm{d}v\,\mathrm{d}s \lesssim \frac{\check{\tau}_{t,x}^2}{V_{\min}(1+\check{\tau}_{t,x_1}^2)\langle x^\perp\rangle^2} \lesssim \frac{1}{V_{\min}(1+|x^\perp|^2)}.$$

Finally, if $\check{d}_{t,x} > 0$, we use (6-11) to deduce the following estimate for S_{P}:

$$|S_{\mathrm{P}}(t,x)| \lesssim \frac{1}{V_{\min}(1+\check{d}_{t,x_1}^2+|x^\perp|^2}.$$

Collecting the above estimates, we obtain (6-8) for S_{P}. The estimate for $\bar{S}_{\mathrm{P}}$ is analogous.

Step 2: Proof of (6-9). We observe that Proposition 4.2 and Corollary 4.7 gives

$$|\nabla_x X_{s,t}| + |\nabla_x V_{s,t}| \lesssim 1 + |t \wedge \mathcal{T}_{t,x_1,v_1} - s|.$$

Since this term can be always absorbed by the exponential decay coming from $\nabla^2\Phi$, the desired estimates are analogous to the above in the case $\check{\tau}_{t,x_1} \le 1$ (so in particular for $\check{d}_{t,x} > 0$). On the other hand, if $\check{\tau}_{t,x_1} \ge 1$, then we rewrite S_{P} similarly to Step 1 of the proof of Proposition 6.2 as

$$\begin{aligned}
&-S_{\mathrm{P}}(t,x)\\
&\quad= \int_0^t\int_{\mathbb{R}^3} \nabla\Phi(x-(t-s)v+Y_{s,t}(x-\check{\mathcal{T}}_{t,x,v}v,v)-X(s))\nabla_v\mu(v+W_{s,t}(x-\check{\mathcal{T}}_{t,x,v}v,v))\,\mathrm{d}v\\
&\quad= \frac{1}{(t-\tau_\omega)^3}\int_0^t\int_{\mathbb{R}^3}\nabla\Phi\left(\omega-(\tau_\omega-s)\frac{x-\omega}{t-\tau_\omega}+Y_{s,t}\left(\omega,\frac{x-\omega}{t-\tau_\omega}\right)-X(s)\right)\nabla_v\mu\left(\frac{x-\omega}{t-\tau_\omega}+W_{s,t}\left(\omega,\frac{x-\omega}{t-\tau_\omega}\right)\right)\mathrm{d}\omega.
\end{aligned}$$

Taking the gradient in x we obtain (omitting the arguments of $Y_{s,t}$ and $W_{s,t}$)

$$\begin{aligned}
|\nabla S_{\mathrm{P}}(t,x)| \lesssim &\int_0^t\int_{\mathbb{R}^3}\frac{(\tau_\omega-s)}{(t-\tau_\omega)^4}\left|\nabla^2\Phi\left(\omega-(\tau_\omega-s)\frac{x-\omega}{t-\tau_\omega}+Y_{s,t}-X(s)\right)\right|\left|\nabla_v\mu\left(\frac{x-\omega}{t-\tau_\omega}+W_{s,t}\right)\right|\mathrm{d}\omega\\
&+\int_0^t\int_{\mathbb{R}^3}\frac{1}{(t-\tau_\omega)^4}\left|\nabla^2\Phi\left(\omega-(\tau_\omega-s)\frac{x-\omega}{t-\tau_\omega}+Y_{s,t}-X(s)\right)\right||\nabla_v Y_{s,t}|\left|\nabla_v\mu\left(\frac{x-\omega}{t-\tau_\omega}+W_{s,t}\right)\right|\mathrm{d}\omega\\
&+\int_0^t\int_{\mathbb{R}^3}\frac{1}{(t-\tau_\omega)^4}\left|\nabla\Phi\left(\omega-(\tau_\omega-s)\frac{x-\omega}{t-\tau_\omega}+Y_{s,t}-X(s)\right)\right|\left|\nabla_v^2\mu\left(\frac{x-\omega}{t-\tau_\omega}+W_{s,t}\right)\right|\mathrm{d}\omega\\
&+\int_0^t\int_{\mathbb{R}^3}\frac{1}{(t-\tau_\omega)^4}\left|\nabla\Phi\left(\omega-(\tau_\omega-s)\frac{x-\omega}{t-\tau_\omega}+Y_{s,t}-X(s)\right)\right|\left|\nabla_v^2\mu\left(\frac{x-\omega}{t-\tau_\omega}+W_{s,t}\right)\right||\nabla_v W_{s,t}|\,\mathrm{d}\omega.
\end{aligned}$$

We change variables back to v and find

$$\begin{aligned}
|\nabla S_{\mathrm{P}}(t,x)| \lesssim &\int_0^t\int_{\mathbb{R}^3}\frac{|s-\mathcal{T}_{t,x_1,v_1}|}{\check{\tau}_{t,x}}|\nabla^2\Phi(X_{s,t}-X(s))||\nabla_v\mu(V_{s,t})|\,\mathrm{d}v\\
&+\int_0^t\int_{\mathbb{R}^3}\frac{1}{\check{\tau}_{t,x}}|\nabla^2\Phi(X_{s,t}-X(s))||\nabla_v Y_{s,t}(x-\check{\mathcal{T}}_{t,x_1,v_1}v,v)||\nabla_v\mu(V_{s,t})|\,\mathrm{d}v\\
&+\int_0^t\int_{\mathbb{R}^3}\frac{1}{\check{\tau}_{t,x}}|\nabla\Phi(X_{s,t}-X(s))||\nabla_v^2\mu(V_{s,t})|\,\mathrm{d}v\\
&+\int_0^t\int_{\mathbb{R}^3}\frac{1}{\check{\tau}_{t,x}}|\nabla\Phi(X_{s,t}-X(s))||\nabla_v^2\mu(V_{s,t})||\nabla_v W_{s,t}(x-\check{\mathcal{T}}_{t,x_1,v_1}v,v)|\,\mathrm{d}v.
\end{aligned}$$

By Proposition 4.6 and (4-6) we have, using $\log(2+t) \le \delta$,

$$|\nabla_v Y_{s,t}(x - \check{\mathcal{T}}_{t,x_1,v_1} v, v)| + |\nabla_v W_{s,t}(x - \check{\mathcal{T}}_{t,x_1,v_1} v, v)| \lesssim \langle s - \mathcal{T}_{t,x_1,v_1} \rangle,$$

which can be absorbed again in the exponential decay coming from Φ. The claim (6-9) then follows by repeating the argument of Step 1. □

6D. *Proof of Proposition 2.7(i).* Resorting to the definition of $\mathcal{S}$ in (2-9) and the definition of the norm $\|\cdot\|_{Y_T}$ in Definition 2.5, the proof of Proposition 2.7(i) just consists of combining the estimates from Propositions 6.1, 6.2 with $\gamma = \frac{1}{2}$ with Proposition 6.3.

7. Error estimates for the friction force

In this section, we prove Proposition 2.7(ii) which asserts that the force acting on the point charge is given, to the leading order, by the linearization of the system. To this end, we recall (2-9)–(2-14) and rewrite for $R > 0$

$$\begin{aligned} E(t, X(t)) &= -\nabla(\phi * \rho(t, \cdot))(X(t)) \\ &= -\nabla(\phi * (G *_{s,x} \mathcal{S} + \mathcal{S}))(X(t)) \\ &= \mathcal{F}^R(t) + \mathcal{E}_1^R(t) + \mathcal{E}_2(t) + \mathcal{E}_3(t), \end{aligned}$$

where the linearized friction force $\mathcal{F}(t)$ and the error terms $\mathcal{E}_1^R$, $\mathcal{E}_2$ and $\mathcal{E}_3$ are given by

$$\mathcal{F}^R(t) = -(\nabla\phi * \rho[g_{R,X(t),V(t)}])(R, X(t)),$$

$$\mathcal{E}_1^R(t) = -(\nabla\phi * (S_{\mathrm{P}} + G * S_{\mathrm{P}}))(t, X(t)) + (\nabla\phi * (S_{R,X(t),V(t)} + G * S_{R,X(t),V(t)}))(R, X(t)), \tag{7-1}$$

$$\mathcal{E}_2(t) = -(\nabla\phi * \mathcal{R})(t, X(t)), \tag{7-2}$$

$$\mathcal{E}_3(t) = -(\nabla\phi * G * \mathcal{R}))(t, X(t)). \tag{7-3}$$

7A. *Contribution of the self-consistent field.*

Lemma 7.1. *Under the bootstrap assumptions (B1)–(B4) with $\delta_0, n > 0$ sufficiently small respectively large, the error term $\mathcal{E}_2$ (see (7-2)) can be estimated for all $t \in [0, T]$ by*

$$|\mathcal{E}_2(t)| \lesssim \log(2+t)\delta^2 V_{\min}^{-1/2}.$$

Proof. Step 1: We start by rewriting $\mathcal{R}$ (see (2-10)) as

$$\begin{aligned} \mathcal{R}(t, x) &= \int_0^t \int_{\mathbb{R}^3} (E(s, x - (t-s)v) \cdot \nabla_v \mu(v) - E(s, X_{s,t}) \cdot \nabla_v \mu(V_{s,t}))\, \mathrm{d}v\, \mathrm{d}s \\ &= \int_0^t \int_{\mathbb{R}^3} \operatorname{div} E(s, x - (t-s)v)(t-s)\mu(v) - \operatorname{div} E(s, X_{s,t})(t-s)\mu(V_{s,t})\, \mathrm{d}v\, \mathrm{d}s \\ &\quad + \int_0^t \int_{\mathbb{R}^3} E(s, X_{s,t}) \cdot \nabla_v \widetilde{W}_{s,t}(x, v) \cdot \nabla_v \mu(V_{s,t})\, \mathrm{d}v\, \mathrm{d}s \\ &\quad - \int_0^t \int_{\mathbb{R}^3} \nabla_x E(s, X_{s,t}) \cdot \nabla_v \widetilde{Y}_{s,t}(x, v) \mu(V_{s,t})\, \mathrm{d}v\, \mathrm{d}s. \end{aligned} \tag{7-4}$$

Step 2: We show that for $|x - X(t)| \le V_{\min}^{1/2}$

$$|\mathcal{R}(t,x)| \lesssim \log(2+t)\delta^2 V_{\min}^{-1/2}. \tag{7-5}$$

We observe that $|x - X(t)| \le V_{\min}^{1/2}$ implies for $|v| < V_{\min}/2$ by (4-7) and (3-4)

$$|\check{T}_{t,x,v}| \le 2|\check{\tau}_{t,x}| \le 2V_{\min}^{-1/2},$$

and thus by Lemma 4.10

$$|E(s, X_{s,t})| + |\nabla E(s, X_{s,t})| \lesssim \begin{cases} \dfrac{\delta}{\langle V_{\min}(t-s)\rangle^3} & \text{for } s \le t - 4V_{\min}^{-1/2}, \\ \delta & \text{for } s \ge t - 4V_{\min}^{-1/2}, \end{cases} \tag{7-6}$$

and by Proposition 4.2 and Corollary 4.7

$$|\widetilde{Y}_{s,t}(x,v)| + |\widetilde{W}_{s,t}(x,v)| + |\nabla_v \widetilde{Y}_{s,t}(x,v)| + |\nabla_v \widetilde{W}_{s,t}(x,v)| \lesssim \log(2+t)\delta\langle t-s\rangle. \tag{7-7}$$

The last two terms in (7-4) can thus be estimated by

$$\begin{aligned} \int_0^t \int_{\mathbb{R}^3} |E(s,X_{s,t}) \cdot \nabla_v \widetilde{W}_{s,t}(x,v) \cdot \nabla_v \mu(V_{s,t})| + |\nabla_x E(s,X_{s,t}) \cdot \nabla_v \widetilde{Y}_{s,t}(x,v)\mu(V_{s,t})| \, dv \, ds \\ \lesssim \log(2+t)\delta^2 \biggl(\int_0^t \int_{\mathbb{R}^3} \frac{\langle t-s\rangle e^{-|v|}}{\langle V_{\min}(t-s)\rangle^3} \, dv \, ds + \int_{t-4V_{\min}^{-1/2}}^t \int_{\mathbb{R}^3} \langle t-s\rangle e^{-|v|} \, dv \, ds \biggr) \\ \lesssim \log(2+t)\delta^2 V_{\min}^{-1/2}. \end{aligned} \tag{7-8}$$

For the first term on the right-hand side of (7-4), we furthermore use that since ϕ is the fundamental solution to $-\Delta + 1$, we have

$$\operatorname{div} E = -(\rho * \phi - \rho),$$

and in particular $\nabla \operatorname{div} E = E + \nabla\rho$ Using the assumption (B3) together with Lemma 2.8, the same arguments that lead to the estimates in Lemma 4.10 and thus to (7-6) also show uniformly for all $\lambda \in [0,1]$

$$|\nabla \operatorname{div} E(s, \lambda(x-(t-s)v) + (1-\lambda)X_{s,t})| \lesssim \begin{cases} \dfrac{\delta}{\langle V_{\min}(t-s)\rangle^3} & \text{for } s > t - 4V_{\min}^{-1/2}, \\ \delta & \text{for } s < t - 4V_{\min}^{-1/2}. \end{cases}$$

Combining these inequalities with (7-6) and (7-7) and splitting the integral as in (7-8) we obtain

$$\left| \int_0^t \int_{\mathbb{R}^3} \operatorname{div} E(s, x-(t-s)v)(t-s)\mu(v) - \operatorname{div} E(s, X_{s,t})(t-s)\mu(V_{s,t}) \, dv \, ds \right| \lesssim \log(2+t)\delta^2 V_{\min}^{-1/2},$$

which finishes the proof of (7-5).

Step 3: conclusion of the proof. We insert the estimate (7-5) into the definition of $\mathcal{E}_2(t)$ (see (7-2)) and use the exponential decay of ϕ to find

$$\begin{aligned} |\mathcal{E}_2(t)| &= \left| \int_{\mathbb{R}^3} \nabla\phi(y) * \mathcal{R}(X(t)-y) \, dy \right| \\ &\le \left| \int_{\{|y| \le V_{\min}^{1/2}\}} \nabla\phi(y) * \mathcal{R}(X(t)-y) \, dy \right| + \left| \int_{\{|y| \ge V_{\min}^{1/2}\}} \nabla\phi(y) * \mathcal{R}(X(t)-y) \, dy \right| \lesssim \log(2+t)\delta^2 V_{\min}^{-1/2}, \end{aligned}$$

where we used for the estimate of the second term that $|\mathcal{R}(X(t)-y)| \le 1$ by Proposition 6.1 and that $V_{\min}^{-1} \le \delta$. □

Lemma 7.2. *Under the bootstrap assumptions (B1)–(B4) with $\delta_0, n > 0$ sufficiently small respectively large, the error term $\mathcal{E}_3$ (see (7-3)) can be estimated for all $t \in [0, T]$ by*

$$|\mathcal{E}_3(t)| = |(\phi * (\nabla\mathcal{R} *_{t,x} G))(X(t)| \lesssim \delta^{7/4} V_{\min}^{-1/2}.$$

Proof. Let $|z| \le V_{\min}^{1/2}/8$, and consider

$$(\nabla\mathcal{R} *_{t,x} G)(X(t)+z) = \int_0^t \int_{\mathbb{R}^3} \nabla\mathcal{R}(t-s, X(t)+z-y) G(s,y)\,\mathrm{d}y\,\mathrm{d}s.$$

We split the integral in the regions

$$A_1 = \left\{(s,y) \in [0,t]\times\mathbb{R}^3 : |t-s| \ge V_{\min}^{-1/2},\ |y| \le \tfrac{1}{4} V_{\min}^{1/2}\right\}, \qquad A_2 = ([0,t]\times\mathbb{R}^3)\setminus A_1.$$

In the region A_1 we have

$$\check{d}_{t-s,X(t)+z-y} \ge \tfrac{1}{2} V_{\min}^{1/2}.$$

Using the a priori estimate on $\mathcal{R}$ from Proposition 6.2 together with (2-15) we therefore have for any $\beta \in (0,1)$

$$\begin{aligned}\left|\int_{A_1} \nabla\mathcal{R}(t-s, X(t)+z-y) G(s,y)\,\mathrm{d}y\,\mathrm{d}s\right| &\lesssim \delta^{2-\beta} V_{\min}^{-3/2} \int_0^t \int_{B_{V_{\min}^{1/2}}} |G(s,y)|\,\mathrm{d}y \\ &\lesssim \log(2+t)\delta^{2-\beta} V_{\min}^{-3/2} \lesssim \delta^{2-2\beta} V_{\min}^{-3/2}, \end{aligned} \tag{7-9}$$

by using (B2) with $n \ge \beta^{-1}$.

On the complement of A_1, we can use that $|t-s| \le V_{\min}^{-1/2}$ or $|y| \ge V_{\min}^{1/2}/4$. Therefore using (2-15) and (2-16) together with Proposition 6.2

$$\begin{aligned}&\left|\int_{A_1^c} \nabla\mathcal{R}(t-s, X(t)+z-y) G(s,y)\,\mathrm{d}y\,\mathrm{d}s\right| \\ &\lesssim \delta^{2-\beta}\left(V_{\min}^{-1/2} \sup_{0\le s\le t} \int_{\mathbb{R}^3} |G(s,y)|\,\mathrm{d}y + \int_0^t \int_{B^c_{V_{\min}^{1/2}/4}} \frac{1}{1+|z^\perp - y^\perp|^3} |G(s,y)|\,\mathrm{d}y\,\mathrm{d}s\right) \lesssim \delta^{2-\beta} V_{\min}^{-1/2}. \end{aligned} \tag{7-10}$$

Choosing $\beta = \frac{1}{8}$ and combining the estimates (7-9)–(7-10) yields for all $|z| \le V_{\min}^{1/2}$

$$|(G * \nabla\mathcal{R})(t, X(t)+z)| \lesssim \delta^{7/4} V_{\min}^{-1/2}.$$

Moreover, combining Propositions 2.2 and 6.2 yields $|(G * \nabla\mathcal{R})(t, X(t)+z)| \lesssim 1$ for all $z \in \mathbb{R}^3$.

Combining these estimates with the decay of ϕ as in Step 3 of the previous proof yields the assertion. □

7B. ***Estimate of $\mathcal{E}_1^R$.*** In order to estimate $\mathcal{E}_1^R$, we will first provide separate estimates for $S_R - \bar{S}_{\mathrm{P}}$ and for $\bar{S}_{\mathrm{P}} - S_{\mathrm{P}}$, where $\bar{S}_{\mathrm{P}}$ is defined in (6-7) and where we denote for shortness $S_R = S_{R,X(T),V(T)}$.

Lemma 7.3. *Under the bootstrap assumptions (B1)–(B4) with $\delta_0, n > 0$ sufficiently small respectively large, the function $S_R = S_{R,X(T),V(T)}$ defined in (2-13) can be estimated for all $x \in \mathbb{R}^3$ and all $0 \le t \le T \le R$ by*

$$|S_R(t,x)| \lesssim \frac{1}{V_{\min}\langle x^\perp\rangle^2}.$$

Moreover, for all $x \in \mathbb{R}^3$ with $|x - X(T)| \le V_{\min}^{2/5}$ and all $0 \le t \le T \le R$, we can estimate

(1) *For $t \ge 4V_{\min}^{-3/5}$*

$$|\bar{S}_P(T-t,x)| + |S_R(R-t,x)| \lesssim e^{-ctV_{\min}}. \tag{7-11}$$

(2) *For $t \le 4V_{\min}^{-3/5}$*

$$|\bar{S}_P(T-t,x) - S_R(R-t,x)| \lesssim \delta V_{\min}^{-6/5}. \tag{7-12}$$

Proof. We rewrite $\bar{S}_P$ and S_R as

$$\bar{S}_P(T-t,x) = -\int_{-\infty}^{R-t}\int_{\mathbb{R}^3} \nabla\Phi(x - X(T-t) - (R-t-s)(v - V(T-t)))\cdot\nabla\mu(v)\,\mathrm{d}v\,\mathrm{d}s,$$
$$S_R(R-t,x) = -\int_{0}^{R-t}\int_{\mathbb{R}^3} \nabla\Phi(x - X(T) - (R-t-s)v + (R-s)V(T))\cdot\nabla\mu(v)\,\mathrm{d}v\,\mathrm{d}s.$$

For $|v| \le V_{\min}/2$ and $s \le R-t$, we have

$$\begin{aligned}|x - X(T-t) - (R-t-s)(v - V(T-t))| &\ge \left|(R-t-s)V(T-t) + \int_{T-t}^{T} V(\tau)\,\mathrm{d}\tau\right| - |x - X(T)| - |(R-t-s)v| \\ &\ge \tfrac{1}{2}(R-s)V_{\min} - |x - X(T)|\end{aligned}$$

and

$$|x - X(T) - (R-t-s)v + (R-s)V(T)| \ge \tfrac{1}{2}(R-s)V_{\min} - |x - X(T)|.$$

Since $|x - X(T)| \le V_{\min}^{2/5}$ and $\operatorname{supp}\mu \subset B_{V_{\min}/5}$, the integrands of both integrals above thus satisfy the bound

$$|\nabla\Phi(x - X(T-t) - (R-t-s)(v - V(T-t)))| \lesssim e^{-c(R-s)V_{\min}} \quad \text{for } R-s \ge 4V_{\min}^{-3/5},$$
$$|\nabla\Phi(x - X(T) - (R-t-s)v + (R-s)V(T))| \lesssim e^{-c(R-s)V_{\min}} \quad \text{for } R-s \ge 4V_{\min}^{-3/5}.$$

In particular, for $t \ge 4V_{\min}^{-3/5}$ we immediately (7-11). On the other hand, for $s \le R$

$$\begin{aligned}|x - X(T-t) - (R-t-s)(v - V(T-t)) - (x - X(T) - (R-t-s)v + (R-s)V(T))| &\le \int_{T-t}^{T} |V(\sigma) - V(T-t)|\,\mathrm{d}\sigma + (R-s)|V(T) - V(t-T)| \\ &\le t(t+R-s)\sup|\dot{V}| \le C\delta t(t+R-s),\end{aligned}$$

where we used $|\dot{V}| \le \|E\|_\infty \le \delta$ by (4-1). Therefore, if $t \le 4V_{\min}^{-3/5}$, the difference is bounded by

$$|\bar{S}_P(T-t,x) - S_R(R-t,x)| \lesssim e^{-cV_{\min}^{2/5}} + \int_{R-4V_{\min}^{-3/5}}^{R-t}\int_{\mathbb{R}^3} I_\Phi|\nabla\mu(v)|\,\mathrm{d}v\,\mathrm{d}s,$$

where by Taylor expansion of $\nabla\Phi$, I_Φ is given by

$$I_\Phi := \|\nabla^2\Phi\|_{L^\infty}\delta(4V_{\min}^{-3/5})^2 \sup \lesssim \delta V_{\min}^{-6/5},$$

and the claim (7-12) follows. Finally, the proof of 7.3 follows analogous to (6-8). □

The following lemma shows that $\bar{S}_{\mathrm{P}}$ is a good approximation for S_{P}.

Lemma 7.4. *Under the bootstrap assumptions (B1)–(B4) with δ_0, $n>0$ sufficiently small, if $T \geq 4V_{\min}^{-3/5}$, we have for all $x\in\mathbb{R}^3$ with $|x-X(T)|\leq V_{\min}^{2/5}$ and all $0\leq t\leq T$:*

(1) *If $t\geq 4V_{\min}^{-3/5}$, we have the estimate*

$$|\bar{S}_{\mathrm{P}}(T-t,x)|+|S_{\mathrm{P}}(T-t,x)| \lesssim e^{-ctV_{\min}}. \tag{7-13}$$

(2) *If $t\leq 4V_{\min}^{-3/5}$, we have the estimate*

$$|\bar{S}_{\mathrm{P}}(T-t,x)-S_{\mathrm{P}}(T-t,x)| \lesssim \log(2+T)\delta V_{\min}^{-6/5}. \tag{7-14}$$

Proof. The proof is largely analogous to the previous lemma and we only detail the differences. We first observe that for $|x-X(T)|\leq V_{\min}^{2/5}$ and $T-s\geq 4V_{\min}^{-3/5}$, we have

$$|\nabla\Phi(x-X(T-t)-(T-t-s)(v-V(T-t)))|+|\nabla\Phi(X_{s,T-t}(x,v)-X(s))| \lesssim e^{-c(T-s)V_{\min}}, \tag{7-15}$$

and (7-13) follows as above.

It remains to show (7-14). Let $t\leq 4V_{\min}^{-3/5}$. With the notation $\lambda=T-s$ and omitting arguments of $X_{s,\lambda}$ and $V_{s,\lambda}$, we split the error into

$$\begin{aligned}
&|(\bar{S}_{\mathrm{P}}-S_{\mathrm{P}})(T-t,x)|\\
&\leq \left|\int_{-\infty}^{0}\int_{\mathbb{R}^3}\nabla\Phi\big(x-X(\lambda)-(\lambda-s)(v-V(\lambda))\big)\cdot\nabla_v\mu(v)\,\mathrm{d}v\,\mathrm{d}s\right|\\
&\quad+\left|\int_{0}^{T-4V_{\min}^{-3/5}}\int_{\mathbb{R}^3}\nabla\Phi\big(x-X(\lambda)-(\lambda-s)(v-V(\lambda))\big)\cdot\nabla_v\mu(v)-\nabla\Phi(X_{s,\lambda}-X(s))\cdot\nabla_v\mu(V_{s,\lambda})\,\mathrm{d}v\,\mathrm{d}s\right|\\
&\quad\;+\left|\int_{T-4V_{\min}^{-3/5}}^{\lambda}\int_{\mathbb{R}^3}\nabla\Phi\big(x-X(\lambda)-(\lambda-s)(v-V(\lambda))\big)\cdot\nabla_v\mu(v)-\nabla\Phi(X_{s,\lambda}-X(s))\cdot\nabla_v\mu(V_{s,\lambda})\,\mathrm{d}v\,\mathrm{d}s\right|.
\end{aligned} \tag{7-16}$$

Relying on (7-15), the first two lines can be estimated as before, by

$$\begin{aligned}
&\left|\int_{0}^{T-4V_{\min}^{-3/5}}\int_{\mathbb{R}^3}[\nabla\Phi(x-X(\lambda)-(\lambda-s)(v-V(\lambda)))\cdot\nabla_v\mu(v)-\nabla\Phi(X_{s,\lambda}-X(s))\cdot\nabla_v\mu(V_{s,\lambda})]\,\mathrm{d}v\,\mathrm{d}s\right|\\
&\qquad+\left|\int_{-\infty}^{0}\int_{\mathbb{R}^3}\nabla\Phi(x-X(\lambda)-(\lambda-s)(v-V(\lambda)))\cdot\nabla_v\mu(v)\,\mathrm{d}v\,\mathrm{d}s\right| \lesssim e^{-cV_{\min}^{2/5}}.
\end{aligned}$$

For the last term in (7-16), we first take a closer look at the velocity integral. We integrate by parts

$$\begin{aligned}\int_{\mathbb{R}^3} \nabla\Phi(x - X(\lambda) - (\lambda - s)(v - V(\lambda))) \cdot \nabla_v \mu(v) - \nabla\Phi(X_{s,\lambda} - X(s)) \cdot \nabla_v \mu(V_{s,\lambda})\, \mathrm{d}v \\ = -\int_{\mathbb{R}^3} (\lambda - s)[\Delta\Phi(x - X(\lambda) - (\lambda - s)(v - V(\lambda)))\mu(v) - \Delta\Phi(X_{s,\lambda} - X(s))\mu(V_{s,\lambda})]\, \mathrm{d}v \\ + \int_{\mathbb{R}^3} \nabla\Phi(X_{s,\lambda} - X(s)) \cdot \nabla_v \widetilde{W}_{s,\lambda} \mu(V_{s,\lambda})\, \mathrm{d}v - \int_{\mathbb{R}^3} \nabla_v \widetilde{Y}_{s,\lambda} \nabla^2\Phi(X_{s,\lambda} - X(s)) \cdot \nabla_v \mu(V_{s,\lambda})\, \mathrm{d}v \\ =: I_1 + I_2 + I_3,\end{aligned}$$

and estimate I_1, I_2, I_3 separately. For I_1 we use again $|\dot V(s)| \lesssim \delta$ as well as the estimates from Proposition 4.2 and Corollary 4.7 to deduce that, for $|\lambda - s| \le 4V_{\min}^{-3/5}$, we have

$$|x - X(\lambda) - (\lambda - s)(v - V(\lambda)) - (X_{s,\lambda} - X(s))| + |v - V_{s,\lambda}| \lesssim \delta.$$

This yields the bound

$$|I_1| \lesssim \|\nabla^3\Phi\|_{L^\infty(\mathbb{R}^3)} V_{\min}^{-3/5}\delta.$$

For I_2, I_3 we observe that $|x - X(T)| \le V_{\min}^{2/5}$ and $t \le 4V_{\min}^{-3/5}$ implies $|\check{\mathcal{T}}_{\lambda,x,v}| \lesssim V_{\min}^{-3/5}$ due to (4-8). Combining this with Corollary 4.7 and Proposition 4.2 we obtain for $\lambda - s \le V_{\min}^{-3/5}$

$$|I_2| + |I_3| \lesssim \log(2+T)\delta V_{\min}^{-3/5}.$$

Using these estimates in the last term in (7-16) finishes the proof. □

Inserting the estimates from Lemmas 7.3 and 7.4 into the definition of the error term $\mathcal{E}_1$ (see (7-1)) yields the following estimate.

Corollary 7.5. *Under the bootstrap assumptions (B1)–(B4) with $\delta_0, n > 0$ sufficiently small, and if $T \ge 4V_{\min}^{-3/5}$, we have for all $R \ge T$*

$$|\mathcal{E}_1^R(T)| \lesssim \delta \log(2+T) V_{\min}^{-6/5}.$$

Proof. We split $\mathcal{E}_1^R$ into

$$\begin{aligned}\mathcal{E}_1^R &= \mathcal{E}_1^1 + \mathcal{E}_1^2, \\ \mathcal{E}_1^1(T) &:= (\nabla\phi * S_{\mathrm{P}})(T, X(T)) - (\nabla\phi * (S_R))(R, X(T)), \\ \mathcal{E}_1^2(T) &:= (\nabla\phi * (G * S_{\mathrm{P}}))(t, X(T)) - (\nabla\phi * (G * S_R))(R, X(T)).\end{aligned}$$

Then the desired estimate for $\mathcal{E}_1^1$ follows directly from the decay of ϕ and Lemmas 7.3 and 7.4 applied with $t = 0$.

To estimate $\mathcal{E}_1^2$, we write $S = S_{\mathrm{P}} - \bar{S}_{\mathrm{P}}$ and first observe that we can split the convolution as

$$\begin{aligned}(\nabla\phi * (G * S))(T, X(T)) = \int_0^{4V_{\min}^{-3/5}} ((\nabla\phi) * G(t,\cdot) * S(T-t,\cdot))(X(T))\, \mathrm{d}t \\ + \int_{4V_{\min}^{-3/5}}^{\infty} (\phi * (\nabla G(t,\cdot)) * S(T-t,\cdot))(X(T))\, \mathrm{d}t. \quad (7\text{-}17)\end{aligned}$$

Defining $B = B_{V_{\min}^{2/5}/2}(0)$, using Proposition 6.3 and Lemma 7.4, as well as Proposition 2.2, we estimate for $|x| \le V_{\min}^{2/5}/4$

$$\begin{aligned}\int_0^{4V_{\min}^{-3/5}} &|(G(t)*S(T-t))(X(T)-x)|\,\mathrm{d}t \\ &\le \int_0^{4V_{\min}^{-3/5}}\int_B |G(t,y)|\,|S|(T-t,X(T)-x-y)\,\mathrm{d}y\,\mathrm{d}t + \int_0^{4V_{\min}^{-3/5}}\int_{B^c} |G(t,y)|\,|S|(T-t,X(T)-x-y)\,\mathrm{d}y\,\mathrm{d}t \\ &\lesssim \int_0^{4V_{\min}^{-3/5}} \delta\log(2+T)V_{\min}^{-6/5}\,\mathrm{d}t + \int_0^{4V_{\min}^{-3/5}}\int_{B^c} \frac{1}{|y|^4}\frac{1}{V_{\min}|y^{\perp}|^2}\,\mathrm{d}y\,\mathrm{d}t \\ &\lesssim \log(2+T)\delta V_{\min}^{-7/5}. \end{aligned} \tag{7-18}$$

Moreover, relying on the pointwise estimates for ∇G from Proposition 2.2, we find

$$\begin{aligned}\int_{4V_{\min}^{-3/5}}^{\infty} &|(\nabla G(t,\cdot)*S(T-t,\cdot))(X(T)-x)|\,\mathrm{d}t \\ &\le \int_{4V_{\min}^{-3/5}}^{T}\int_B |\nabla G(t,y)|\,|S|(T-t,X(T)-x-y)\,\mathrm{d}y\,\mathrm{d}t \\ &\qquad\qquad + \int_{4V_{\min}^{-3/5}}^{\infty}\int_{B^c} |\nabla G(t,y)|\,|S|(T-t,X(T)-x-y)\,\mathrm{d}y\,\mathrm{d}t \\ &\lesssim \int_{4V_{\min}^{-3/5}}^{\infty}\int_{\mathbb{R}^3} \frac{1}{|y|^5+t^5}e^{-ctV_{\min}}\,\mathrm{d}y\,\mathrm{d}t + \int_0^{\infty}\int_{B^c} \frac{1}{|y|^5+t^5}\frac{1}{V_{\min}\langle y_\perp\rangle^2}\,\mathrm{d}y\,\mathrm{d}t \lesssim V_{\min}^{-11/5}. \end{aligned} \tag{7-19}$$

For $|x| \ge V_{\min}^{2/5}/4$, Propositions 6.3 and 2.2 imply

$$\int_0^{4V_{\min}^{-3/5}} |(G(t,\cdot)*S(T-t,\cdot))(X(T)-x)|\,\mathrm{d}t \lesssim 1, \tag{7-20}$$

$$\int_{4V_{\min}^{-3/5}}^{\infty} |(\nabla G(t,\cdot)*S(T-t,\cdot))(X(T)-x)|\,\mathrm{d}t \lesssim 1. \tag{7-21}$$

Inserting (7-18)–(7-21) into (7-17) and using the exponential decay of ϕ yields

$$|(\nabla\phi*(G*(S_{\mathrm{P}}-\bar{S}_{\mathrm{P}})))(T,X(T))| \lesssim \delta\log(2+T)V_{\min}^{-6/5}. \tag{7-22}$$

Similarly, relying on Lemma 7.3 yields

$$|(\nabla\phi*G*\bar{S}_{\mathrm{P}})(T,X(T)) - (\nabla\phi*G*S_R)(R,X(T))| \lesssim \delta V_{\min}^{-6/5}. \tag{7-23}$$

Combining (7-22)–(7-23) yields the desired bound for $\mathcal{E}_1^2(T)$ which concludes the proof. □

7C. *Proof of Proposition 2.7*(ii). We recall the identities (2-12)–(2-14) and (2-24) to rewrite

$$\lim_{s\to\infty} |(\nabla\phi*\rho[h_{V(T)}])(s,0) + E(T,X(T))| \le \sup_{R\ge T} |\mathcal{E}_1^R(T)| + |\mathcal{E}_2(T)| + |\mathcal{E}_3(T)|.$$

Now it remains to apply Corollary 7.5 for $\mathcal{E}_1^R$, Lemma 7.1 for $\mathcal{E}_2$ and Lemma 7.2 for $\mathcal{E}_3$. Since we assume $T \geq 4V_{\min}(T)^{-3/5}$ and we have $V_{\min}^{-1} \leq \delta$, we obtain

$$\sup_{R\geq T} |\mathcal{E}_1^R(T)|+|\mathcal{E}_2(T)|+|\mathcal{E}_3(T)| \lesssim \delta \log(2+T) V_{\min}^{-6/5} + \delta^2 \log(2+T) V_{\min}^{-1/2} + \delta^{7/2} V_{\min}^{-1/2} \lesssim \delta^{11/5} \log(2+T).$$

Hence, by a suitable choice of δ_0 and n, from (B2) we deduce

$$\sup_{R\geq T} |\mathcal{E}_1^R(T)| + |\mathcal{E}_2(T)| + |\mathcal{E}_3(T)| \leq C\delta^{13/6},$$

which proves the claim.

8. The linearized friction force

Proof of Proposition 2.1. Fix $V_* \in \mathbb{R}^3$, and recall the defining equation for $h = h_{V_*}$ from (1-3):

$$\partial_s h + (v - V_*) \cdot \nabla_x h - \nabla(\phi *_x \rho[h]) \cdot \nabla_v \mu = -e_0 \nabla \Phi(x) \cdot \nabla_v \mu, \quad h(0, \cdot) = 0.$$

We extend h by zero for negative times. The equation for h can be explicitly solved in space-time Fourier variables. Let $\tilde{h}(z, k, v)$ be given according to (1-14). Then

$$(\tau + k \cdot (v - V_*))\tilde{h} - \hat{\phi}(k)\rho[\tilde{h}](\tau, k)k \cdot \nabla_v \mu = \frac{-e_0 \widehat{\Phi}(k) k \cdot \nabla_v \mu}{i\tau}$$

for negative imaginary part, $\Im(\tau) < 0$. This yields the explicit representation

$$\rho[\tilde{h}](\tau, k) = \frac{-e_0 \widehat{\Phi}(k)}{i\tau\varepsilon(\tau, |k|, \hat{k} \cdot V_*)} \int_{\mathbb{R}^3} \frac{k \cdot \nabla_v \mu(v)}{\tau + k \cdot (v - V_*)} \, \mathrm{d}v = \frac{-e_0 \widehat{\Phi}(k)}{i\tau\varepsilon(\tau, |k|, \hat{k} \cdot V_*)} \frac{1 - \varepsilon(\tau, |k|, \hat{k} \cdot V_*)}{\hat{\phi}(|k|)},$$

where $\hat{k} = k/|k|$, $k \neq 0$, and the dielectric function $\varepsilon(\tau, |k|, \hat{k} \cdot V_*)$ is given by

$$\varepsilon(\tau, r, \hat{k} \cdot V_*) = 1 - \hat{\phi}(r) \int_{\mathbb{R}^3} \frac{\hat{k} \cdot \nabla_v \mu(v)}{\tau/r + \hat{k} \cdot (v - V_*)} \, \mathrm{d}v.$$

Notice that the integral indeed only depends on V_* and $\hat{k}$ through $\hat{k} \cdot V_*$ since by (1-8)

$$\int_{\mathbb{R}^3} \frac{\hat{k} \cdot \nabla_v \mu(v)}{\tau/r + \hat{k} \cdot (v - V_*)} \, \mathrm{d}v = \int_{\mathbb{R}^3} \frac{-(\hat{k} \cdot v)\psi(v)}{\tau/r + \hat{k} \cdot v - \hat{k} \cdot V_*} \, \mathrm{d}v,$$

and ψ is radially symmetric by Assumption 1.2. We remark, that by elementary computation ε and a (see (1-9)) are related by

$$\varepsilon(\tau, |k|, \hat{k} \cdot V_*) = 1 - \hat{\phi}(k) a(\tau/|k| - \hat{k} \cdot V_*).$$

The Penrose condition (1-10), and Assumption 1.2 then ensure a uniform bound for $|\varepsilon|$

$$0 < \kappa \leq |\varepsilon| \leq C.$$

We now compute the limit $s\to\infty$ of the associated force. Using Lemma B.1 yields

$$\lim_{s\to\infty}(\rho[h(R,\cdot)]*\nabla\phi)(0)=\lim_{s\to\infty}\frac{1}{(2\pi)^3}\int_{\mathbb{R}^3} ik\hat{\rho}[h(s,\cdot)]\hat{\phi}\,\mathrm{d}k=\lim_{i\tau\to0^+}\frac{i\tau}{(2\pi)^3}\int_{\mathbb{R}^3} ik\rho[\tilde{h}](\tau,k)\hat{\phi}\,\mathrm{d}k$$
$$=\lim_{i\tau\to0^+}\frac{e_0}{(2\pi)^3}\int_{\mathbb{R}^3} ik\widehat{\Phi}(k)\,\mathrm{d}k-\lim_{i\tau\to0^+}\frac{e_0}{(2\pi)^3}\int_{\mathbb{R}^3}\frac{ik\widehat{\Phi}(k)}{\varepsilon(\tau,|k|,\hat{k}\cdot V_*)}\,\mathrm{d}k.$$

The first term vanishes since $\widehat{\Phi}(k)=\widehat{\Phi}(-k)$, and we can simplify

$$\lim_{s\to\infty}(\rho[h(s,\cdot)]*\nabla\phi)(0)=\lim_{i\tau\to0^+}\frac{-e_0}{(2\pi)^3}\int_{\mathbb{R}^3}\frac{ik\widehat{\Phi}(k)}{\varepsilon(\tau,|k|,\hat{k}\cdot V_*)}\,\mathrm{d}k$$
$$=\lim_{i\tau\to0^+}\frac{-e_0}{(2\pi)^3}\int_{\mathbb{R}^3}\frac{ik\widehat{\Phi}(k)\varepsilon^*(\tau,k)}{|\varepsilon(\tau,|k|,\hat{k}\cdot V_*)|^2}\,\mathrm{d}k. \tag{8-1}$$

By rotational symmetry of the potential ϕ, $\hat{\phi}$ is real. Thus, by Plemelj's formula, Lemma B.2, for $k\neq0$,

$$\lim_{i\tau\to0^+}\Im\varepsilon^*(\tau,|k|,\hat{k}\cdot V_*)=\hat{\phi}(k)\lim_{i\tau\to0^+}\Im\int_{\mathbb{R}^3}\frac{k\cdot\nabla_v\mu(v)}{k\cdot(v-V_*)+\tau}\,\mathrm{d}v$$
$$=\hat{\phi}(k)\lim_{i\tau\to0^+}\Im\int_{\{w\cdot k=0\}}\int_{\mathbb{R}}\frac{k\cdot\nabla_v\mu(V_*+\lambda\hat{k}+w)}{\lambda|k|+\tau}\,\mathrm{d}\lambda\,\mathrm{d}w$$
$$=\pi\hat{\phi}(k)\int_{\{w\cdot k=0\}}\hat{k}\cdot\nabla_v\mu(V_*+w)\,\mathrm{d}w.$$

By the radial symmetry of the potential Φ, $\widehat{\Phi}$ is real. Since the left hand side of (8-1) is real, we can simplify the above to

$$\lim_{s\to\infty}(\rho[h(s,\cdot)]*\nabla\phi)(0)=\lim_{i\tau\to0^+}\frac{-e_0}{(2\pi)^3}\int_{\mathbb{R}^3}\frac{ik\widehat{\Phi}(k)\varepsilon^*(\tau,k)}{|\varepsilon(\tau,|k|,\hat{k}\cdot V_*)|^2}\,\mathrm{d}k$$
$$=\lim_{i\tau\to0^+}\frac{e_0}{8\pi^2}\int_{\mathbb{R}^3}\frac{k\widehat{\Phi}(k)\hat{\phi}(k)\int_{\{k\cdot v=k\cdot V_*\}}\hat{k}\cdot\nabla_v\mu(v)}{|\varepsilon(\tau,|k|,\hat{k}\cdot V_*)|^2}\,\mathrm{d}k. \tag{8-2}$$

Recall Assumption 1.3, i.e., we have

$$\nabla_v\mu(v)=-v\psi(v),$$

for some nonnegative, continuous, exponentially decaying, positive function ψ. This finally yields

$$\lim_{s\to\infty}e_0(\rho[h(s,\cdot)]*\nabla\phi)(0)\cdot V_*=-\frac{e_0^2}{8\pi^2}\int_{\mathbb{R}^3}\frac{\widehat{\Phi}(k)|k|\hat{\phi}(k)(\hat{k}V_*)^2}{|\varepsilon(-i0^+,|k|,\hat{k}\cdot V_*)|^2}\varphi(\hat{k}\cdot V_*)\,\mathrm{d}k,$$

where $\varphi(u)$ is a nonnegative, continuous, exponentially decaying function given by

$$\varphi(u)=\int_{\{e_1\cdot v=u\}}\psi(v)\,\mathrm{d}v.$$

Since ψ is radial, nonnegative and not everywhere vanishing, we also have $\varphi(0)>0$. In particular, since $\hat{\phi}$ and $\widehat{\Phi}$ are both positive (see Assumption 1.1) (2-1) holds, i.e.,

$$\lim_{s\to\infty}e_0(\rho[h(s,\cdot)]*\nabla\phi)(0)\cdot V_*<0.$$

It remains to determine the asymptotics of the integral for $|V_*| \to \infty$. We rewrite the integral in terms of the variable $u = \hat{k} \cdot \widehat{V}_*$. Multiplying with $|V_*|$ we obtain

$$\begin{aligned}\lim_{s\to\infty} e_0|V_*|(\rho[h(s,\cdot)] * \nabla\phi)(0)\cdot V_* &= -\frac{e_0^2|V_*|}{4\pi}\int_0^\infty\int_{-1}^1 \frac{\widehat{\Phi}(r) r^3 \hat{\phi}(r)(u|V_*|)^2}{|\varepsilon(-i0^+, r, u|V_*|)|^2}\varphi(u|V_*|)\,\mathrm{d}r\,\mathrm{d}u \\ &= -\frac{1}{4\pi}\int_0^\infty\int_{-|V_*|}^{|V_*|}\frac{\widehat{\Phi}(r) r^3\hat{\phi}(r)U^2}{|\varepsilon(-i0^+, r, U)|^2}\varphi(U)\,\mathrm{d}r\,\mathrm{d}U.\end{aligned}$$

The integral converges exponentially fast to a positive limit for $|V_*| \to \infty$. This establishes (2-2). ☐

Remark 8.1. The friction force is related to the Balescu–Lenard correction of the Landau equation. More precisely, consider the case $\phi = \Phi$ in (8-2). We obtain

$$\begin{aligned}\lim_{s\to\infty} e_0(\rho[h(s,\cdot)] * \nabla\phi)(0) &= -\frac{1}{8\pi^2}\int_{\mathbb{R}^3}\frac{k|\hat{\phi}(k)|^2\int_{\{k\cdot v = k\cdot V_*\}}\hat{k}\cdot\nabla_v\mu(v)}{|\varepsilon(-i0^+, |k|, \hat{k}\cdot V_*)|^2}\,\mathrm{d}k \\ &= -\frac{1}{8\pi^2}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\frac{\delta(k\cdot(v-v_*))|\hat{\phi}(k)|^2(k\otimes k)\cdot\nabla_v\mu(v)}{|\varepsilon(-i0^+, |k|, \hat{k}\cdot V_*)|^2}\,\mathrm{d}k\,\mathrm{d}v,\end{aligned}$$

which gives the friction coefficient of the Balescu–Lenard equation

$$\begin{aligned}\partial_t G &= \mathrm{LB}(G), \\ \mathrm{LB}(G)(v) &= \nabla_v\cdot\left(\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} B(v, v-v_*; \nabla G)(\nabla G G_* - G\nabla_* G_*)\,\mathrm{d}v_*\right), \\ B(v, v-v_*; \nabla G) &= \int_{\mathbb{R}^3}\frac{\delta(k\cdot(v-v_*))|\hat{\phi}(k)|^2(k\otimes k)}{|\varepsilon(-i0^+, |k|, \hat{k}\cdot v_*; \nabla G)|^2}\,\mathrm{d}k, \\ \varepsilon(\tau, r, \hat{k}\cdot V_*; \nabla G) &= 1 - \hat{\phi}(r)\int_{\mathbb{R}^3}\frac{\hat{k}\cdot\nabla_v G(v)}{\tau/r + \hat{k}\cdot(v - V_*)}\,\mathrm{d}v.\end{aligned}$$

The equation was formally derived in [Balescu 1960; Lenard 1960]; for a recent well-posedness result see [Duerinckx and Winter 2023]. Notice that we recover the Landau equation from the Balescu–Lenard equation when we neglect collective effects, i.e., replace $\varepsilon \equiv 1$.

Appendix A: Proof of Proposition 1.5

Proof of Proposition 1.5. By Assumptions 1.1 and 1.2, the function $a(z)$ defined in (1-9) decays for $|z| \to \infty$, $\Im(z) \le 0$. Therefore the infimum in (1-10) can be replaced by a minimum. This allows us to argue by contradiction. For $\overline{C} > 0$ given, assume there exist $\xi^* \in \mathbb{R}^3$, $\Im(z^*) \le 0$ such that

$$a(z^*) = (\hat{\phi}(k))^{-1} > 1. \tag{A-1}$$

As in the proof of Proposition 2.7 in [Bedrossian et al. 2018], we use Penrose's argument principle [1960]: the function $z \mapsto a(z)$ is a holomorphic function on the lower half plane, vanishing for $|z| \to \infty$. The boundary behavior of the function is given by the curve $\gamma : \mathbb{R} \to \mathbb{C}$ given by

$$\vec{\gamma}(x) = a(x - i0) := \lim_{\varepsilon\to 0} a(x - i\varepsilon).$$

By the argument principle, (A-1) can only hold if the curve $\vec{\gamma}$ intersects the half-line $\{y \in \mathbb{R} : y > 1\}$.

Writing $\mu(v) = \mu(|v|)$ by slight abuse of notation, we have the representation (see [Bedrossian et al. 2018, Appendix] and [Mouhot and Villani 2011, Section 3])

$$\vec{\gamma}(x) = \mathrm{PV} \int_{\mathbb{R}} \frac{-2\pi u \mu(|u|)}{u-x} \, \mathrm{d}u - i 2\pi^2 u \mu(|u|).$$

By Assumption 1.2, there exists $\bar{C} > 0$ such that

$$\left| \mathrm{PV} \int_{\mathbb{R}} \frac{-2\pi u \mu(|u|)}{u-x} \, \mathrm{d}u \right| < \frac{1}{2}, \qquad |x| \geq \bar{C}.$$

Now it suffices to observe that the imaginary part does not vanish if $\mu(v) > 0$ for $|v| \leq \bar{C}$. This contradicts the assumption for $\bar{C}$ large enough and finishes the proof. □

Appendix B: Two standard auxiliary lemmas

In this section, we recall two standard results which we use to compute the linearized force in Section 8.

Lemma B.1. *Assume $f \in C_b^1(\mathbb{R})$, $f = 0$ in $(-\infty, 0]$ and let $\tilde{f}$ be it's Fourier transform. Then,*

$$\lim_{t \to \infty} f(t) = \lim_{z \downarrow 0} z \tilde{f}(-iz),$$

whenever the limit on the right-hand side exists.

Proof. Provided the right-hand side above exists, we have

$$\begin{aligned} \lim_{z \downarrow 0} z \tilde{f}(-iz) &= \lim_{z \downarrow 0} \int_0^\infty f(t) z e^{-zt} \, \mathrm{d}t = \lim_{z \downarrow 0} \int_0^\infty f'(t) e^{-zt} \, \mathrm{d}t - [f e^{-zt}]_0^\infty \\ &= \lim_{z \downarrow 0} \int_0^\infty f'(t) e^{-zt} \, \mathrm{d}t - [f e^{-zt}]_0^\infty = \int_0^\infty f'(t) \, \mathrm{d}t - f(0) = \lim_{t \to \infty} f(t). \end{aligned}$$

as claimed. □

Lemma B.2 (Plemelj's formula, e.g., [Muskhelishvili 1958]). *For $f \in L^2(\mathbb{R}) \cap C^1(\mathbb{R})$ we have the identity*

$$\lim_{\delta \to 0^+} \int_{\mathbb{R}} \frac{f(y)}{(x-y) \pm i\delta} \, \mathrm{d}y = \mp i\pi f(x) + \lim_{\delta \to 0^+} \int_{\{|x-y| \geq \delta\}} \frac{f(y)}{x-y} \, \mathrm{d}y.$$

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HAAGERUP'S PHASE TRANSITION AT POLYDISC SLICING

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We establish a sharp comparison inequality between the negative moments and the second moment of the magnitude of sums of independent random vectors uniform on three-dimensional Euclidean spheres. This provides a probabilistic extension of the Oleszkiewicz–Pełczyński polydisc slicing result. The Haagerup-type phase transition occurs exactly when the p-norm recovers volume, in contrast to the real case. We also obtain partial results in higher dimensions.

1. Introduction

Khinchin-type inequalities concern estimates on L_p norms of (weighted) sums of independent random variables, typically involving a norm which is easily understood (or explicit in given parameters) such as the L_2 norm. They can be traced back to Khinchin's work [25] on the law of the iterated logarithm, where he established such bounds for Rademacher random variables (random signs). Beyond their original use, most notably, such inequalities have played an important role in Banach space theory (in connection with topics such as unconditional convergence or type and cotype); see [13; 22; 33; 49]. Considerable work has been devoted to the pursuit of sharp constants in Khinichin-type inequalities, see for instance [3; 6; 16; 17; 19; 21; 30; 31; 32; 36; 37; 38; 39; 40; 41; 43; 45; 48; 50], in particular for sums of random vectors uniform on Euclidean spheres [4; 9; 11; 26; 28] (as a natural generalisation of Rademacher and Steinhaus random variables, intimately related to uniform convergence in real and complex Banach spaces, respectively). This paper continues that line of research.

Throughout, $|\cdot|$ denotes the standard Euclidean norm on $\mathbb{R}^d$, inherited from the standard inner product $\langle\cdot,\cdot\rangle$. For a random vector X in $\mathbb{R}^d$ and a real parameter p, we write $\|X\|_p = (\mathbb{E}|X|^p)^{1/p}$ for the L_p norm (p-th moment) of the magnitude of X (whenever the expectation exists, with $p=0$ understood as usual as $\|X\|_0 = e^{\mathbb{E}\log|X|}$, arising from taking the limit as $p \to 0$).

Let $\xi_1, \xi_2, \ldots$ be independent random vectors, each uniform on the unit Euclidean sphere S^{d-1} in $\mathbb{R}^d$. In particular, when $d=1$, these are Rademacher random variables, that is symmetric random signs in $\mathbb{R}$, whereas when $d=2$, they are often referred to as Steinhaus random variables (especially when $\mathbb{R}^2$ is treated as $\mathbb{C}$). For $q > -(d-1)$, let $c_d(q)$ be the best positive constant such that the following

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Khinchin-type inequality holds: for every $n \geqslant 1$ and real scalars $a_1, \dots, a_n$, we have

$$\Big\|\sum_{k=1}^{n} a_k\xi_k\Big\|_q \geqslant c_d(q)\Big\|\sum_{k=1}^{n} a_k\xi_k\Big\|_2. \tag{1}$$

In other words, thanks to homogeneity, $c(q)$ is the infimal value of $\big\|\sum_{k=1}^n a_k\xi_k\big\|_q$ over all $n \geqslant 1$ and $a_1, \dots, a_n \in \mathbb{R}$ with $\sum a_k^2 = 1$. We stress that, when $d \geqslant 1$ and $q > -(d-1)$, this L_q norm exists regardless of the coefficients, e.g., seen by noting that then $\mathbb{E}|\xi_1 + x|^q = \mathbb{E}(|x|^2 + 2\langle x, \xi_1\rangle + 1)^{q/2}$ is finite for every $x \in \mathbb{R}^d$, using that $\langle x, \xi_1\rangle$ has density proportional to $(1-(u/|x|)^2)^{(d-3)/2}$ on $-|x| \leqslant u \leqslant |x|$ (of course, for a given sequence of coefficients a_j, the range of q may be larger, for instance when $d = 1$, it is all $q \in \mathbb{R}$ as long as $\sum_{j=1}^n \pm a_j$ never vanishes).

Plainly, $c_d(q) = 1$ for $q \geqslant 2$ (by the monotonicity of $p \mapsto \|\cdot\|_p$). When $q \geqslant 2$, the reverse inequality to (1) is nontrivial and interesting, but we do not discuss it here at all, referring instead to, for instance, [4; 20; 37] for a comprehensive account of known as well as recent results.

From now on we consider $-(d-1) < q < 2$. We define two constants arising from two particular choices of weights in (1): $a_1 = a_2 = 1/\sqrt{2}$ with $n = 2$ and $a_1 = \dots = a_n = 1/\sqrt{n}$ with $n \to \infty$,

$$c_{d,2}(q) = \Big\|\frac{\xi_1+\xi_2}{\sqrt{2}}\Big\|_q = \frac{1}{\sqrt{2}}\left(\frac{\Gamma\big(\frac{d}{2}\big)\Gamma(d+q-1)}{\Gamma\big(\frac{d+q}{2}\big)\Gamma\big(d+\frac{q}{2}-1\big)}\right)^{1/q}, \tag{2}$$

$$c_{d,\infty}(q) = \lim_{n\to\infty}\Big\|\frac{\xi_1+\dots+\xi_n}{\sqrt{n}}\Big\|_q = \Big\|\frac{Z}{\sqrt{d}}\Big\|_q = \sqrt{\frac{2}{d}}\left(\frac{\Gamma\big(\frac{d+q}{2}\big)}{\Gamma\big(\frac{d}{2}\big)}\right)^{1/q}, \tag{3}$$

where Z is a standard Gaussian random vector in $\mathbb{R}^d$ (emerging by the central limit theorem). The expression for $c_{d,2}(q)$ will be justified later (see Corollary 14), whereas the expression for $c_{d,\infty}(q)$ follows by a simple integration in polar coordinates. Note that

$$c_d(q) \leqslant \min\{c_{d,2}(q), c_{d,\infty}(q)\}. \tag{4}$$

It can be checked that, in fact,

$$\min\{c_{d,2}(q), c_{d,\infty}(q)\} = \begin{cases} c_{d,2}(q), & -(d-1) < q \leqslant q_d^*, \\ c_{d,\infty}(q), & q_d^* \leqslant q \leqslant 2, \end{cases} \tag{5}$$

where q_d^* is the unique solution of the equation $c_{d,2}(q) = c_{d,\infty}(q)$ in $(-(d-1), 2)$. We have included a sketch of the proof of this fact in the Appendix. In Table 1 we list some numerical values of q_d^*. We are grateful to Hermann König for sharing his notes on these topics (personal communication, 2021).

1.1. *Known results.* The pursuit of the value of $c_d(q)$ has a rich history which can be summarised in one simple statement that in all known cases the trivial bound (4) is tight. Of course, the history begins with the one-dimensional case of Rademacher random variables. In his study [34] on bilinear forms, Littlewood conjectured that $c_1(1) = c_{1,2}(1) = 1/\sqrt{2}$, which was confirmed by Szarek in [45] (see also [31] and [46]). Haagerup's pivotal work [19] addressed the entire range $0 < q < 2$, showing the following

d	q_d^*	range where $c(q)$ is known	phase transition	left open
1	$1.82\dots$	$0 < q < 2$ [19]	[19]	—
2	$0.47\dots$	$0 < q < 2$ [4; 26; 28]	[26]	$-1 < q < 0$
3	$-0.79\dots$	$-1 < q < 2$ [11; 9; 32]	[9]	$-2 < q < -1$
4	-2	$-3 < q < 2$ Theorem 2	Theorem 2	—
5	$-3.16\dots$	$-1 < q < 2$ [4; 28], Theorem 1	?	$-4 < q < -1$
$\vdots$				
d	$-(d-1)+o(1)$	$-(d-4) < q < 2$ [4; 28], Theorem 1	?	$-(d-1) < q < -(d-4)$

Table 1. Numerical values of q_d^* (see (38) for its asymptotics), known results and open questions about the best constant in Khinchin inequality (1).

phase transition in the behaviour of $c_1(q)$:

$$c_1(q) = \begin{cases} c_{1,2}(q), & 0 < q \leqslant q_1^*, \\ c_{1,\infty}(q), & q_1^* \leqslant q < 2, \end{cases}$$

where $q_1^* = 1.84\dots$ is the unique solution of the equation $c_{1,2}(q) = c_{1,\infty}(q)$ in $(0, 2)$; in particular, when $d = 1$, we have equality in (4). We also refer to Nazarov and Podkorytov's paper [38] which offered great simplifications. Haagerup devised a very efficient argument, crucially relying on Fourier-analytic formulae for L_p norms, which together with [38] paved the path for many further results.

That a similar behaviour occurs in the case $d = 2$ (Steinhaus variables) was conjectured by Haagerup and later confirmed by König in [26]: when $d = 2$, $0 \leqslant q < 2$, we have equality in (4) and the phase transition occurs now at $q_2^* = 0.47\dots$. The range $1 \leqslant q < 2$ was in fact earlier dealt with by König and Kwapień in [28] (with $q = 1$ handled even earlier by Sawa in [44]), whereas $-1 < q < 0$ (to the best of our knowledge) appears to be left open, with a natural conjecture that $c(q) = c_{2,2}(q)$.

For the case $d = 3$, Latała and Oleszkiewicz showed in [32] that $c_3(q) = c_{3,\infty}(q)$ for $1 \leqslant q < 2$, which was extended to $0 < q < 1$ in our joint work [11] with Gurushankar (see Proposition 3 below for a connection to uniform distribution on intervals). The phase transition occurs in the range $-1 < q < 0$ at $q_3^* = -0.79\dots$, as established in our joint work [9] with König, so when $d = 3$ and $-1 < q < 2$, (4) holds with equality. Again, $-2 < q < -1$ appears to be open with a natural conjecture that $c(q) = c_{3,2}(q)$.

In higher dimensions $d \geqslant 4$, there are precise Schur-convexity results available for positive moments due to Baernstein II and Culverhouse from [4] and, independently, König and Kwapień from [28]: when $0 \leqslant q < 2$, it follows in particular that $c_d(q) = c_{d,\infty}(q)$. However, nothing seems to be known about the value of $c_d(q)$ for negative q, except it being (nontrivially) finite, as shown by Gorin and Favorov in [18] (in a much more general setting). This paper partially fills out this gap.

1.2. ***Our contribution.*** Our first result concerns the best constant $c_d(q)$ in the inequality (1) when $q > -(d-4)$. It turns out that this is a consequence of a Schur-concavity type statement that follows directly from the main result of [4] (see Theorem 6 below).

Theorem 1. *For every $d \geqslant 5$ and $-(d-4) \leqslant q < 0$, we have $c_d(q) = c_{d,\infty}(q)$.*

Note that the restriction $-(d-4) \leqslant q < 0$ already makes the statement of Theorem 1 meaningful only for dimensions $d \geqslant 5$. Our second result covers the entire range $-3 < q < 0$ for dimension $d = 4$, which exhibits Haagerup's phase transition at exactly $q_4^* = -2$ (see also Table 1 for other values of q_d^* and a summary of known results and open questions).

Theorem 2. *For $-3 < q < 0$, we have*

$$c_4(q) = \begin{cases} c_{4,2}(q), & -3 < q \leqslant -2, \\ c_{4,\infty}(q), & -2 \leqslant q < 0. \end{cases}$$

1.3. ***Relation to volume.*** It can perhaps be traced back to Kalton and Koldobsky's paper [24] that the volume of hyperplane sections of convex bodies can be expressed in terms of negative moments (of linear forms in vectors uniform on the body). Brzezinski's work [8] makes the same connection for sections of products of Euclidean balls by block subspaces, and our recent work with Nayar [10] explores this further. In particular, as [9] extends Ball's cube slicing result from [5] (in the form of sharp Khinchin inequality (1) when $d = 3$), Theorem 2 can be viewed as a probabilistic extension of Oleszkiewicz and Pełczyński's polydisc slicing from [42]. In fact, this connection was the main motivation of this work. It is very intriguing that the phase transition occurs exactly at $q = -2$ which is when (1) recovers the result for volume from [42].

More specifically, let $\mathbb{D} = \{z \in \mathbb{C}, |z| < 1\}$ be the unit disc in the complex plane. Oleszkiewicz and Pełczyński in [42] proved the following sharp inequality about extremal-volume (complex) hyperplane sections of the polydisc $\mathbb{D}^n$ in $\mathbb{C}^n$: for every (complex) codimension 1 subspace H in $\mathbb{C}^n$, we have

$$\mathrm{vol}_{2n-2}(\mathbb{D}^n \cap H) \leqslant \mathrm{vol}_{2n-2}(\mathbb{D}^n \cap (1, 1, 0, \dots, 0)^\perp), \tag{6}$$

$$\mathrm{vol}_{2n-2}(\mathbb{D}^n \cap H) \geqslant \mathrm{vol}_{2n-2}(\mathbb{D}^n \cap (1, 0, \dots, 0)^\perp). \tag{7}$$

Here $a^\perp = \{z \in \mathbb{C}^n, \langle a, z\rangle = 0\}$ is the (codimension 1) hyperplane orthogonal to a vector a in $\mathbb{C}^n$ and $\langle \cdot, \cdot \rangle$ is the standard inner product in $\mathbb{C}^n$. If we let $U_1, \dots, U_n$ be independent random vectors, each uniform on $\mathbb{D}$, and let $a = (a_1, \dots, a_n)$ be a unit vector in $\mathbb{C}^n$, then

$$\mathrm{vol}_{2n-2}(\mathbb{D}^n \cap a^\perp) = \frac{\pi^{n-1}}{2} \lim_{p \to 2-} (2-p)\mathbb{E}\left|\sum_{k=1}^n a_k U_k\right|^{-p}$$

(such formulae hold for arbitrary origin-symmetric convex sets, and this one follows immediately from Corollary 11 in [10]). Moreover, the moments of sums of vectors uniform on balls are proportional to sums of vectors uniform on spheres (in a slightly higher dimension).

Proposition 3 [4; 28]. *Let $d \geqslant 3$, let $\xi_1, \xi_2, \dots$ be independent random vectors uniform on the unit Euclidean sphere S^{d-1} in $\mathbb{R}^d$ and let $U_1, U_2, \dots$ be independent random vectors uniform on the unit Euclidean ball $\mathbb{B}^{d-2}$ in $\mathbb{R}^{d-2}$. For every $q > -(d-2)$, $n \geqslant 1$ and scalars $a_1, \dots, a_n$, we have*

$$\mathbb{E}\left|\sum_{k=1}^n a_k U_k\right|^q = \frac{d-2}{d-2+q}\mathbb{E}\left|\sum_{k=1}^n a_k \xi_k\right|^q.$$

This identity can be seen in a number of ways, but essentially it follows from the folklore result that if a random vector $\xi = (\xi_1, \dots, \xi_d)$ is uniform on S^{d-1}, then its projection $(\xi_1, \dots, \xi_{d-2})$ onto $\mathbb{R}^{d-2}$ is uniform on $\mathbb{B}^{d-2}$. Specialised to $d=4$ and combined with the previous formula, it yields

$$\mathrm{vol}_{2n-2}(\mathbb{D}^n \cap a^\perp) = \pi^{n-1}\mathbb{E}\left|\sum_{k=1}^n a_k\xi_k\right|^{-2}$$

(see also [27] and [29] for generalisations to noncentral sections). Thus, the upper bound (6) is Theorem 2 at $q=-2$, that is $c_4(-2)=c_{4,2}(-2)$. Incidentally, the lower bound (7) follows immediately from Jensen's inequality (see, e.g., [8], or [27], as well as [10] for a stability result).

The sequel is devoted to proofs. First we provide some background and give a brief summary. Then we move to the proof of Theorem 1 (which is very short), and the rest is occupied with the proof of Theorem 2.

2. Proofs of the main results

2.1. ***Some background and outline.*** Theorem 1 will follow easily from the main result of [4]. As for positive moments, the point is that the range $-(d-4)<q<0$ still warrants *enough* convexity of the underlying moment functional, specifically the function $|x|^q$ (in fact, its C^∞ regularisation/approximation) is bisubharmonic.

When $d=4$, as in Theorem 2, this range is empty, Schur convexity/concavity does not hold, and more subtle arguments are needed. We will employ a Fourier-analytic approach (pioneered by Haagerup for random signs in [19]). On its own however, this does not dispense of all cases. We extend an inductive argument of Nazarov and Podkorytov from [38] to our multidimensional setting and *all* negative moments (building on [9] with new ideas needed to go beyond the range $q\in(-1,0)$). The Fourier-analytic approach relies on the following integral representation of Gorin and Favorov for negative moments.

Lemma 4 [18, Lemma 3]. *For a random vector X in $\mathbb{R}^d$ and $0<p<d$, we have*

$$\mathbb{E}|X|^{-p} = K_{p,d}\int_{\mathbb{R}^d}(\mathbb{E}e^{i\langle t,X\rangle})|t|^{p-d}\,\mathrm{d}t, \tag{8}$$

provided that the right-hand side integral exists, where

$$K_{p,d} = 2^{-p}\pi^{-d/2}\frac{\Gamma\left(\frac{d-p}{2}\right)}{\Gamma\left(\frac{p}{2}\right)}.$$

Of course, the Fourier transform (the characteristic function) goes hand in hand with independence. The trade-off is that when applied to sums of independent random vectors uniform on spheres, highly oscillating integrands appear, more precisely, the Bessel functions. To recall, for integral $k\geqslant 0$ and real x, we use the notation

$$(x)_k = \frac{\Gamma(x+k)}{\Gamma(x)} = x(x+1)\cdots(x+k-1)$$

for the rising factorial (Pochhammer symbol). Throughout,

$$J_\nu(t) = \sum_{k=0}^\infty \frac{(-1)^k}{k!\,\Gamma(k+\nu+1)}\left(\frac{t}{2}\right)^{2k+\nu}$$

is the Bessel function of the first kind with parameter $\nu > 0$. We also introduce the function

$$\mathrm{j}_\nu(t) = 2^\nu \Gamma(\nu+1) t^{-\nu} J_\nu(t) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!(\nu+1)_k}\left(\frac{t}{2}\right)^{2k}. \tag{9}$$

Its importance stems from the fact that for a random vector ξ uniform on the unit Euclidean sphere S^{d-1} in $\mathbb{R}^d$ and a vector v in $\mathbb{R}^d$, we have

$$\mathbb{E} e^{i\langle v,\xi\rangle} = \mathrm{j}_{d/2-1}(|v|) \tag{10}$$

(see, e.g., the proof of Proposition 10 in [28]). This combined with Lemma 4 gives the following corollary.

Corollary 5. *For independent, rotationally invariant random vectors $X_1, \dots, X_n$ in $\mathbb{R}^d$ and $0 < p < d$, we have*

$$\mathbb{E}\left|\sum_{k=1}^n X_k\right|^{-p} = \kappa_{p,d} \int_0^\infty \prod_{k=1}^n (\mathbb{E}\mathrm{j}_{d/2-1}(t|X_k|)) t^{p-1}\,\mathrm{d}t, \tag{11}$$

provided the integral on the right-hand side exists, where

$$\kappa_{p,d} = 2^{1-p}\frac{\Gamma\left(\frac{d-p}{2}\right)}{\Gamma\left(\frac{d}{2}\right)\Gamma\left(\frac{p}{2}\right)}.$$

Proof. Let $\xi_1, \dots, \xi_n$ be independent random vectors, each uniform on the unit Euclidean sphere S^{d-1}, chosen independently of the X_k. Then X_k has the same distribution as $|X_k|\xi_k$, and (8) together with (10) and integration in polar coordinates give

$$\mathbb{E}\left|\sum_{k=1}^n X_k\right|^{-p} = K_{p,d}\int_{\mathbb{R}^d}\left(\prod_{k=1}^n \mathbb{E} e^{i\langle t,|X_k|\xi_k\rangle}\right)|t|^{p-d}\,\mathrm{d}t = K_{p,d}\int_{\mathbb{R}^d}\left(\prod_{k=1}^n \mathbb{E}\mathrm{j}_{d/2-1}(|t||X_k|)\right)|t|^{p-d}\,\mathrm{d}t$$
$$= K_{p,d}|S^{d-1}|\int_0^\infty\left(\prod_{k=1}^n \mathbb{E}\mathrm{j}_{d/2-1}(t|X_k|)\right)t^{p-1}\,\mathrm{d}t,$$

where $|S^{d-1}| = 2\pi^{d/2}/(\Gamma(d/2))$ is the $(d-1)$-dimensional volume of the unit sphere in $\mathbb{R}^d$. □

2.2. *Proof of Theorem 1*. Theorem 1 is a straightforward corollary of the following stronger Schur-concavity result. For background on Schur-majorisation, we refer for example to [7].

Theorem 6. *Let $d \geqslant 5$, and let $\xi_1, \xi_2, \dots$ be independent random vectors uniform on the unit Euclidean sphere S^{d-1} in $\mathbb{R}^d$. For every $n \geqslant 1$ and $0 < p \leqslant d-4$, the function*

$$(x_1, \dots, x_n) \mapsto \mathbb{E}\left|\sum_{k=1}^n \sqrt{x_k}\xi_k\right|^{-p}$$

is Schur-concave on $\mathbb{R}_+^n$.

Proof. Thanks to Lebesgue's monotone convergence theorem, it suffices to show that, for every $\delta > 0$, the theorem holds with $|\cdot|^{-p}$ replaced by the function $\Psi_\delta(x) = (|x|^2+\delta)^{-p/2}$. The gain is that Ψ_δ is C^∞

on $\mathbb{R}^d$. In view of the result of Baernstein II and Culverhouse from [4], it suffices to show that Ψ_δ is bisubharmonic, that is $\Delta\Delta\Psi_\delta \geqslant 0$ on $\mathbb{R}^d$. We approach this directly. Recall that

$$\Delta f(|x|) = \frac{d-1}{|x|} f'(|x|) + f''(|x|)$$

for a rotation-invariant function $f(|x|)$ on $\mathbb{R}^d$, $f \in C^2(\mathbb{R}_+)$. We have

$$\Delta\Delta\Psi_\delta(x) = p(p+2)(|x|^2+\delta)^{-p/2-4}(A|x|^4 + B|x|^2 + C),$$

where $A = (p-d+2)(p-d+4)$, $B = 2\delta(d+2)(-p+d-4)$ and $C = \delta^2 d(d+2)$. For $p < d-4$, plainly $A > 0$ and $B^2 - 4AC = 8\delta^2(d+2)(p+4)(p-d+4) < 0$. This shows that Ψ_δ is bisubharmonic on $\mathbb{R}^d$ for every $\delta > 0$. □

Remark 7. The crux of Baernstein II and Culverhouse's work is the observation that the bisubharmonicity of a continuous function Ψ on $\mathbb{R}^d$ on one hand is sufficient for the Schur-convexity of the corresponding moment functional from Theorem 6, $\mathbb{E}\Psi\big(\sum_{k=1}^n \sqrt{x_k}\xi_k\big)$ (and necessary when Ψ is radial), and on the other hand, it is equivalent to the convexity of the function

$$t \mapsto \mathbb{E}\Psi(v + \sqrt{t}\xi)$$

on $\mathbb{R}_+$ for every $v \in \mathbb{R}^d$. In the sequel, we will need to examine the behaviour of this function on (0, 1) for unit vectors v when $\Psi(x) = |x|^{-p}$ (see Section 3.1 below).

2.3. ***Outline of the proof of Theorem 2.*** Recall that here $d = 4$ and $\xi_1, \xi_2, \dots$ are independent random vectors uniform on the unit Euclidean sphere S^3 in $\mathbb{R}^4$. For notational convenience, we put $q = -p$, $0 < p < 3$ and set

$$C_2(p) = c_{4,2}(q)^q = \mathbb{E}\left|\frac{\xi_1+\xi_2}{\sqrt{2}}\right|^{-p} = 2^{p/2}\frac{\Gamma(3-p)}{\Gamma\big(2-\frac{p}{2}\big)\Gamma\big(3-\frac{p}{2}\big)}, \tag{12}$$

$$C_\infty(p) = c_{4,\infty}(q)^q = \mathbb{E}\left|\frac{Z}{2}\right|^{-p} = 2^{p/2}\Gamma\Big(2-\frac{p}{2}\Big), \tag{13}$$

where Z is a standard Gaussian random vector in $\mathbb{R}^4$ (consult (2) and (3) to justify the explicit expressions on the right-hand sides). Moreover, let $C(p)$ be the best constant such that the equivalent form of (1),

$$\mathbb{E}\left|\sum_{k=1}^n a_k\xi_k\right|^{-p} \leqslant C(p)\left(\sum_{k=1}^n a_k^2\right)^{-p/2}, \tag{14}$$

holds for every $n \geqslant 1$ and all real scalars $a_1, \dots, a_n$.

Theorem 2 is a consequence of the next two results, where we break it up into two regimes.

Theorem 8. *For* $0 < p \leqslant 2$, *we have* $C(p) = C_\infty(p)$.

Theorem 9. *For* $2 < p < 3$, *we have* $C(p) = C_2(p)$.

As optimality is clear, for the proofs of these theorems, we need to show that (14) holds with the specified values of $C(p)$.

2.3.1. *Outline of the proof of Theorem 8.* Thanks to homogeneity, we can assume that the a_k are positive with $\sum a_k^2 = 1$. Using the Fourier-analytic formula for negative moments (11) and Hölder's inequality, we obtain

$$\begin{aligned}\mathbb{E}\left|\sum_{k=1}^n a_k\xi_k\right|^{-p} &= \kappa_{p,4}\int_0^\infty\left(\prod_{k=1}^n \mathrm{j}_1(a_kt)\right)t^{p-1}\,\mathrm{d}t \\ &\leqslant \kappa_{p,4}\prod_{k=1}^n\left(\int_0^\infty |\mathrm{j}_1(a_kt)|^{a_k^{-2}}t^{p-1}\,\mathrm{d}t\right)^{a_k^2} = \kappa_{p,4}\prod_{k=1}^n (a_k^{-p}F(p,a_k^{-2}))^{a_k^2},\end{aligned} \tag{15}$$

where the following function has emerged (after a change of variables in the last line):

$$F(p,s) = \int_0^\infty |\mathrm{j}_1(t)|^s t^{p-1}\,\mathrm{d}t, \qquad p, s > 0. \tag{16}$$

This integral is finite as long as $p < \frac{3}{2}s$ because $\mathrm{j}_1(t) = O(t^{-3/2})$ (see (22) below).

The next step is to maximise, individually, the terms in the product on the right-hand side of the second line of (15), that is to look into $\sup_{s\geqslant 1} s^{p/2}F(p,s)$. Heuristically, if we aim at proving that the worst case is Gaussian, that is when $a_1 = \cdots = a_n = 1/\sqrt{n}$ with $n\to\infty$, a natural candidate for this supremum is then given by $s\to\infty$, which would correspond to the inequality

$$s^{p/2}F(p,s) \leqslant \lim_{s\to\infty} s^{p/2}\int_0^\infty |\mathrm{j}_1(t)|^s t^{p-1}\,\mathrm{d}t = \lim_{s\to\infty}\int_0^\infty \left|\mathrm{j}_1\left(\frac{t}{\sqrt{s}}\right)\right|^s t^{p-1}\,\mathrm{d}t = \int_0^\infty e^{-t^2/8}t^{p-1}\,\mathrm{d}t \tag{17}$$

(the last line can be justified using $\mathrm{j}_1(t) = 1 - t^2/8 + o(t^2) = e^{-t^2/8} + o(t^2)$, recall the power series definition (9) of j_1). Were it true for all values of p and s, we would get

$$\mathbb{E}\left|\sum_{k=1}^n a_k\xi_k\right|^{-p} \leqslant \kappa_{p,4}\int_0^\infty e^{-t^2/8}t^{p-1}\,\mathrm{d}t = C_\infty(p),$$

finishing the proof. Unfortunately, the integral inequality (17) fails in certain ranges of p and s, where additional arguments and ideas are needed. This is how we will proceed.

Step 1: Inequality (17) holds for all $0 < p \leqslant 2$ and $s \geqslant 2$.

As above, this gives the following partial case of the theorem when all coefficients a_k are small.

Corollary 10. *When $0 < p \leqslant 2$, inequality (14) holds with $C(p) = C_\infty(p)$ for every $n \geqslant 1$ and all real numbers $a_1, \dots, a_n$ with $\max_{k\leqslant n}|a_k| \leqslant \frac{1}{\sqrt{2}}\left(\sum_{k=1}^n a_k^2\right)^{1/2}$.*

Step 2: For $\frac{1}{4} \leqslant p \leqslant 2$, we employ induction on n to cover the case $\max_{k\leqslant n}|a_k| > \frac{1}{\sqrt{2}}\left(\sum_{k=1}^n a_k^2\right)^{1/2}$.

This will give the theorem when $p \geqslant \frac{1}{4}$. For the induction to work, (14) is strengthened, but the base of the induction fails for small p (roughly $p < 0.2$), hence the next two steps. Fortunately, when p is *small*, the integral inequality holds for a wider range of s.

Step 3: Inequality (17) holds for all $0 < p \leqslant \frac{1}{4}$ and $s \geqslant 1.3$.

Corollary 11. *When $0 < p \leqslant \frac{1}{4}$, inequality (14) holds with $C(p) = C_\infty(p)$ for every $n \geqslant 1$ and all real numbers $a_1, \dots, a_n$ such that $\max_{k\leqslant n}|a_k| \leqslant \sqrt{\frac{10}{13}}\left(\sum_{k=1}^n a_k^2\right)^{1/2}$.*

Finally, when one of the coefficients a_k is *large*, the inequality holds for a different reason (we will use a sort of projection-type argument).

Step 4: When $0 < p \leqslant \frac{1}{4}$, inequality (14) holds with $C(p) = C_\infty(p)$ for every $n \geqslant 1$ and all real numbers $a_1, \dots, a_n$ with $\max_{k \leqslant n} |a_k| > \sqrt{\frac{10}{13}}\big(\sum_{k=1}^n a_k^2\big)^{1/2}$.

2.3.2. *Outline of the proof of Theorem 9.* If we want to prove that the worst case is now $n = 2$ with $a_1 = a_2 = 1/\sqrt{2}$, it is only natural to expect that $\sup_{s \geqslant 1} s^{p/2} F(p,s)$ is attained at $s = 2$, corresponding to the integral inequality

$$s^{p/2} F(p,s) \leqslant 2^{p/2} F(p,2). \tag{18}$$

We will proceed similarly, with only the first two steps sufficing, as the inductive base now holds in the entire range.

Step 1: Inequality (18) holds for all $2 < p < 3$ and $s \geqslant 2$.

Taking this statement for granted for now, we derive the following corollary.

Corollary 12. *When $2 < p < 3$, inequality (14) holds with $C(p) = C_2(p)$ for every $n \geqslant 1$ and all real numbers $a_1, \dots, a_n$ with $\max_{k \leqslant n} |a_k| \leqslant \frac{1}{\sqrt{2}}\big(\sum_{k=1}^n a_k^2\big)^{1/2}$.*

Proof. Assuming $\sum a_k^2 = 1$ and applying (18) to the right-hand side of (15) yields

$$\mathbb{E}\left|\sum_{k=1}^n a_k \xi_k\right|^{-p} \leqslant \kappa_{p,4} \cdot 2^{p/2} F(p,2) = 2^{p/2} \kappa_{p,4} \int_0^\infty \mathrm{j}_1(t)^2 t^{p-1}\,\mathrm{d}t$$
$$= 2^{p/2} \mathbb{E}|\xi_1 + \xi_2|^{-p} = C_2(p)$$

(for the penultimate step, recall again (15)). □

Step 2: For $2 < p < 3$, we employ induction on n to cover the case $\max_{k \leqslant n} |a_k| > \frac{1}{\sqrt{2}}\big(\sum_{k=1}^n a_k^2\big)^{1/2}$.

To carry out these steps, we first establish a variety of indispensable technical estimates. After this has been done in the next section, we will conclude the proof in Sections 4 and 5.

3. Ancillary results

3.1. ***Two-coefficient function.*** By rotational invariance,

$$\mathbb{E}|a_1\xi_1 + a_2\sqrt{t}\xi_2|^{-p} = \mathbb{E}|a_1 e_1 + a_2\sqrt{t}\xi_2|^{-p}.$$

We begin with some properties of the function $t \mapsto \mathbb{E}|a_1 e_1 + a_2\sqrt{t}\xi_2|^{-p}$, particularly important in the inductive part of our proof. Recall the definition of the (Gaussian) hypergeometric function which shows up very naturally, as explained in the next lemma. For real parameters a, b, c, it is defined for $|z| < 1$ by the power series,

$${}_2F_1(a,b;c;z) = \sum_{k=0}^\infty \frac{(a)_k (b)_k}{(c)_k} \frac{z^k}{k!}.$$

Lemma 13. *Let $d \geqslant 1$, and let ξ be a random vector uniform on the unit Euclidean sphere S^{d-1} in $\mathbb{R}^d$. Let $p < d-1$. Then*

$$\mathbb{E}|e_1 + \sqrt{t}\xi|^{-p} = {}_2F_1\left(\frac{p}{2}, \frac{p-d+2}{2}; \frac{d}{2}; t\right) = \sum_{k=0}^{\infty} \frac{\left(\frac{p}{2}\right)_k \left(\frac{p-d+2}{2}\right)_k}{\left(\frac{d}{2}\right)_k} \frac{t^k}{k!}, \qquad 0 < t < 1.$$

Proof. Fix $0 < t < 1$. Let $\theta = \langle e_1, \xi\rangle$ be the first coordinate of ξ. Thus

$$\mathbb{E}|e_1 + \sqrt{t}\xi|^{-p} = \mathbb{E}(1 + 2\sqrt{t}\theta + t)^{-p/2} = (1+t)^{-p/2}\mathbb{E}\left(1 + \frac{2\sqrt{t}}{1+t}\theta\right)^{-p/2}$$
$$= (1+t)^{-p/2} \sum_{k=0}^{\infty} \binom{-p/2}{2k} (\mathbb{E}\theta^{2k}) \left(\frac{2\sqrt{t}}{1+t}\right)^{2k}.$$

From (10),

$$\mathbb{E}\theta^{2k} = \frac{(2k)!}{2^{2k} \cdot k! \left(\frac{d}{2}\right)_k},$$

hence

$$\mathbb{E}|e_1 + \sqrt{t}\xi|^{-p} = (1+t)^{-p/2} \sum_{k=0}^{\infty} \frac{\left(\frac{p}{2}\right)_{2k}}{2^{2k}\left(\frac{d}{2}\right)_k} \frac{1}{k!} \left(\frac{4t}{(1+t)^2}\right)^k.$$

Since

$$\left(\frac{p}{2}\right)_{2k} 2^{-2k} = \left(\frac{p}{4}\right)_k \left(\frac{p+2}{4}\right)_k,$$

we get

$$\mathbb{E}|e_1 + \sqrt{t}\xi|^{-p} = (1+t)^{-p/2} {}_2F_1\left(\frac{p}{4}, \frac{p+2}{4}; \frac{d}{2}; \frac{4t}{(1+t)^2}\right) = {}_2F_1\left(\frac{p}{2}, \frac{p-d+2}{2}; \frac{d}{2}; t\right),$$

where the last identity follows from Kummer's quadratic transformations for the hypergeometric function ${}_2F_1$ (see, e.g., 15.3.26 in [1]). The desired power series expansion now follows from the definition of ${}_2F_1$. □

This in particular yields the explicit expression for $c_{d,2}(q)$ from (2).

Corollary 14. *For $d \geqslant 1$ and $p < d-1$, we have*

$$\mathbb{E}|\xi_1 + \xi_2|^{-p} = {}_2F_1\left(\frac{p}{2}, \frac{p-d+2}{2}; \frac{d}{2}; 1\right) = \frac{\Gamma\left(\frac{d}{2}\right)\Gamma(d-p-1)}{\Gamma\left(\frac{d-p}{2}\right)\Gamma\left(d - \frac{p}{2} - 1\right)}.$$

Proof. The expression on the right-hand side follows from Gauss' summation identity (see, e.g., 15.1.20 in [1]). □

Remark 15. In addition to the proof of Lemma 13 presented above we would like to sketch a different argument, in the spirit of Lemma 1 from [4], which bypasses the explicit use of the hypergeometric function. Let $\Psi(x) = |x|^{-p}$. Since on the unit sphere $\xi \in S^{d-1}$ is the outer-normal, by the divergence theorem (for the usual Lebesgue *non*normalised surface integral),

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\int_{S^{d-1}}|e_1+\sqrt{t}\xi|^{-p}\,\mathrm{d}\xi &= \frac{1}{2\sqrt{t}}\int_{S^{d-1}}\langle(\nabla\Psi)(e_1+\sqrt{t}\xi),\xi\rangle\,\mathrm{d}\xi\\ &= \frac{1}{2\sqrt{t}}\int_{\mathbb{B}^d}\mathrm{div}_x((\nabla\Psi)(e_1+\sqrt{t}x))\,\mathrm{d}x\\ &= \frac{1}{2}\int_{\mathbb{B}^d}(\Delta\Psi)(e_1+\sqrt{t}x)\,\mathrm{d}x\end{aligned}$$

for every $0<t<1$ (note that $e_1+\sqrt{t}x$ on B_2^d is away from the origin where Ψ is singular). Computing the Laplacian yields the identity

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{S^{d-1}}|e_1+\sqrt{t}\xi|^{-p}\,\mathrm{d}\xi = \frac{p(p-d+2)}{2}\int_{B_2^d}|e_1+\sqrt{t}x|^{-p-2}\,\mathrm{d}x.$$

Writing the last integral using polar coordinates allows us to compute the higher derivatives by simply iterating this identity. Thus

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\mathbb{E}|e_1+\sqrt{t}\xi|^{-p} &= \frac{p(p-d+2)}{2}\frac{1}{|S^{d-1}|}\int_{B_2^d}|e_1+\sqrt{t}x|^{-p-2}\,\mathrm{d}x\\ &= \frac{p(p-d+2)}{2}\frac{1}{|S^{d-1}|}\int_0^1\int_{S^{d-1}}r^{d-1}|e_1+\sqrt{tr^2}\xi|^{-p-2}\,\mathrm{d}\xi \qquad (19)\end{aligned}$$

and

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2}\mathbb{E}|e_1+\sqrt{t}\xi|^{-p} = \frac{p(p-d+2)}{2}\frac{(p+2)(p-d+4)}{2}\frac{1}{|S^{d-1}|}\int_0^1 r^{d+1}\int_{\mathbb{B}^d}|e_1+\sqrt{t}rx|^{-p-4}\,\mathrm{d}x\,\mathrm{d}r,\ \text{etc.}$$

It then remains to evaluate these derivatives at $t=0$ to get the power series expansion coefficients.

Corollary 16. *Let ξ be a random vector uniform on the unit Euclidean sphere S^3 in $\mathbb{R}^4$. Let $0<p\leqslant 2$. Then*

$$\mathbb{E}|e_1+\sqrt{t}\xi|^{-p}\leqslant 1-\frac{p(2-p)}{8}t-\frac{p^2(4-p^2)}{192}t^2,\qquad 0<t<1.$$

Proof. When $d=4$ and $0<p<2$, all the terms in the power series from Lemma 13 but the first one (which equals 1) are negative. Dropping all but the first three thus gives the desired bound. □

Corollary 17. *Let $d\geqslant 1$. Let ξ be a random vector uniform on the unit Euclidean sphere S^{d-1} in $\mathbb{R}^d$. Let $0<p\leqslant d-2$. Then, for every vector v in $\mathbb{R}^d$ and $a>0$, we have*

$$\mathbb{E}|v+a\xi|^{-p}\leqslant\min\{|v|^{-p},a^{-p}\}.$$

Proof. By homogeneity and rotational invariance, we can assume without loss of generality that $v=e_1$ and $0<a<1$ (note in particular that rotational invariance implies $\mathbb{E}|e_1+a\xi|^{-p}=\mathbb{E}|\xi_1+a\xi_2|^{-p}=\mathbb{E}|ae_1+\xi|^{-p}$, so the case $a>1$ reduces to the case $0<a<1$ by multiplying both sides by a^p). From the first line of (19) we see that the function $a\mapsto\mathbb{E}|e_1+a\xi|^{-p}$ is nonincreasing, in particular $\mathbb{E}|e_1+a\xi|^{-p}\leqslant 1$. □

3.2. ***Bounds for the inductive base.*** We remark that in several places we need to use numerical values of some special functions such as j_1, Γ, $\psi = (\log \Gamma)'$ and will implicitly do so (to the required precision).

Based on tables left by Gauss, Deming and Colcord in [12] found the value of $\min_{x>0} \Gamma(x)$ correct up to the 19th decimal which we record here (although we will not require such precision).

Lemma 18 [12]. *We have*

$$\min_{x>0} \Gamma(x) = 0.8856031944108886887\ldots$$

uniquely occurring at $x_0 = 1.4616321449683622 6\ldots.$

To check the base of the induction from Step 2 in Section 2.3.1, we will need the following two-point inequality.

Lemma 19. *For every* $\frac{1}{8} \leqslant q \leqslant 1$ *and* $0 \leqslant t \leqslant 1$, *we have*

$$1 - \frac{q(1-q)}{2}t - \frac{q^2(1-q^2)}{12}t^2 \leqslant \Gamma(2-q)\Big(2 - \Big(\frac{3-t}{2}\Big)^{-q}\Big).$$

Proof. We let $Q_q(t)$ and $R_q(t)$ denote the left-hand side and the right-hand side, respectively, and set $h_q(t) = R_q(t) - Q_q(t)$. We examine its second derivative

$$h_q''(t) = -2^q \Gamma(2-q) q(q+1)(3-t)^{-q-2} + \frac{q^2(1-q^2)}{6},$$

which is clearly decreasing in t. Therefore, for all $0 \leqslant t \leqslant 1$, we have $h_q''(t) \leqslant h_q''(0)$, and, for $0 < q < 1$, with the aid of Lemma 18,

$$-\frac{3^{q+2}}{2^q \cdot q(1+q)} h_q''(0) = \Gamma(2-q) - \Big(\frac{3}{2}\Big)^{q+1} q(1-q) > 0.88 - \Big(\frac{3}{2}\Big)^2 \cdot \frac{1}{4} = 0.3175.$$

As a result, $h_q(t)$ is concave on [0, 1]. To show that $h_q(t) \geqslant 0$ on [0, 1], it thus suffices to verify that

$$\text{(A) } h_q(0) \geqslant 0 \quad \text{and} \quad \text{(B) } h_q(1) \geqslant 0 \quad \text{for all } \tfrac{1}{8} \leqslant q \leqslant 1.$$

(A): $h_q(0) \geqslant 0$ is equivalent to $\Gamma(2-q)\big(2 - \big(\frac{2}{3}\big)^q\big) \geqslant 1$, or after taking logarithms, $g(q) \geqslant f(q)$ with

$$g(q) = \log \Gamma(2-q) \quad \text{and} \quad f(q) = -\log 2 - \log\big(1 - \tfrac{1}{2}\big(\tfrac{2}{3}\big)^q\big).$$

Both f and g are clearly convex (note $f(q) = -\log 2 + \sum_{k=1}^{\infty} \big[\frac{1}{2}\big(\frac{2}{3}\big)^q\big]^k / k$). For $\frac{1}{8} \leqslant q \leqslant 0.35$, we bound g from below by its supporting tangent at $q = \frac{1}{8}$:

$$g(q) \geqslant \ell(q) = g\big(\tfrac{1}{8}\big) + g'\big(\tfrac{1}{8}\big)\big(q - \tfrac{1}{8}\big).$$

Since $\ell\big(\frac{1}{8}\big) - f\big(\frac{1}{8}\big) > 0.0005$ and $\ell(0.35) - f(0.35) > 0.0003$, thanks to the convexity of f, we conclude that indeed $g(q) > f(q)$ for $\frac{1}{8} \leqslant q \leqslant 0.35$. For the remaining range $0.35 \leqslant q \leqslant 1$, we crudely have, using the monotonicity of f and Lemma 18,

$$f(q) \leqslant f(0.35) < -0.124 < \log(0.885) < \log \Gamma(2-q) = g(q).$$

(B): $h_q(1) \geqslant 0$ is equivalent to $\Gamma(2-q) \geqslant 1 - \frac{1}{2}q(1-q) - \frac{1}{12}q^2(1-q^2)$. Taking the logarithms and using $\log(1-x) \leqslant -x,\ x < 1$, it suffices to show that

$$f(q) = \log\Gamma(2-q) + \frac{q(1-q)}{2} + \frac{q^2(1-q^2)}{12}$$

is nonnegative. This in fact holds for all $0 \leqslant q \leqslant 1$. Indeed, $f(0) = f(1) = 0$ and, for $0 \leqslant q \leqslant 1$,

$$f''(q) = \sum_{k=0}^{\infty} \frac{1}{(2-q+k)^2} - q^2 - \frac{5}{6}.$$

It suffices to show that this is negative for $0 \leqslant q \leqslant 1$, so that the concavity of f will finish the argument. To this end, we bound the convex function

$$h(q) = \sum_{k=0}^{\infty} \frac{1}{(2-q+k)^2}$$

from above by linear chords. For $0 \leqslant q \leqslant \frac{1}{2}$, we have

$$h(q) \leqslant h_1(q) = \frac{\frac{1}{2}-q}{\frac{1}{2}} h(0) + \frac{q}{\frac{1}{2}} h\big(\tfrac{1}{2}\big),$$

and, since $h(0) = \frac{1}{6}\pi^2 - 1$ and $h\big(\frac{1}{2}\big) = \frac{1}{2}\pi^2 - 4$, we get

$$h_1(q) = \tfrac{2}{3}(\pi^2 - 9)q + \tfrac{1}{6}\pi^2 - 1.$$

We check that $h_1(q) - q^2 - \frac{5}{6}$ is maximised at $q = \frac{1}{3}(\pi^2 - 9)$ with the value less than -0.1. For $\frac{1}{2} \leqslant q \leqslant 1$, we have

$$h(q) \leqslant h_2(q) = \frac{1-q}{\frac{1}{2}} h\big(\tfrac{1}{2}\big) + \frac{q-\frac{1}{2}}{\frac{1}{2}} h(1),$$

and, since $h(1) = \frac{1}{6}\pi^2$, we get

$$h_2(q) = 2\big(\tfrac{1}{3}12 - \pi^2\big)q + \tfrac{5}{6}\pi^2 - 8.$$

Finally, we check that $h_2(q) - q^2 - \frac{5}{6}$ is maximised at $q = \frac{1}{3}(12 - \pi^2)$ with the value also less than -0.1. □

We emphasise that in part (B) of this proof, we have shown that, when $t = 1$, the inequality in Lemma 19 holds for all $0 \leqslant q \leqslant 1$. This combined with Corollary 16 leads to the following result, important in the sequel in the proof of integral inequality (17).

Corollary 20. *Let ξ be a random vector uniform on the unit Euclidean sphere S^3 in $\mathbb{R}^4$. Let $0 < p \leqslant 2$. Then*

$$\mathbb{E}|e_1 + \xi|^{-p} \leqslant \Gamma\Big(2 - \frac{p}{2}\Big),$$

equivalently

$$\int_0^\infty |\mathfrak{j}_1(t)|^2 t^{p-1}\,\mathrm{d}t \leqslant 2^{p-1}\Gamma\Big(\frac{p}{2}\Big). \tag{20}$$

Proof. To explain the equivalent form involving $\mathfrak{j}_1$, note that $\mathbb{E}|e_1+\xi|^{-p} = \mathbb{E}|\xi+\xi'|^{-p}$ for an independent copy ξ' of ξ, thanks to rotational invariance. It remains to use (11) which gives

$$\mathbb{E}|\xi_1+\xi_2|^{-p} = \kappa_{p,4}\int_0^\infty |\mathfrak{j}_1(t)|^2 t^{p-1}\,\mathrm{d}t$$

and plug in the value of $\kappa_{p,4}$. □

3.3. ***The integral inequality*: $0 < p \leqslant 2$.** We record for future use the bounds

$$|\mathfrak{j}_1(t)| \leqslant \exp\Big(-\frac{t^2}{8} - \frac{t^4}{3\cdot 2^7}\Big), \quad 0 \leqslant t \leqslant 4, \tag{21}$$

$$|\mathfrak{j}_1(t)| \leqslant \Big(\frac{8}{\pi}\Big)^{1/2} t^{-1}(t^2-1)^{-1/4}, \quad t \geqslant 1, \tag{22}$$

where the first bound appears as Lemma 3.1 in [42] (see also [8, Lemma 3.6] for the proof of a more general statement) and the second bound can be found in Watson's treatise (see [47, p. 447] as well as [14, Lemma 4.4]), which in particular gives

$$|\mathfrak{j}_1(t)| \leqslant \Big(\frac{8}{\pi}\Big)^{1/2}\Big(\frac{t_0^2}{t_0^2-1}\Big)^{1/4} t^{-3/2}, \quad t \geqslant t_0 \geqslant 1. \tag{23}$$

We define

$$H(p,s) = \int_0^\infty (e^{-st^2/8} - |\mathfrak{j}_1(t)|^s)t^{p-1}\,\mathrm{d}t, \quad 0 < p < 2,\ s > 1 \tag{24}$$

and immediately observe that after a change of variables one integral can be expressed in terms of the gamma function,

$$G(p,s) = \int_0^\infty e^{-st^2/8} t^{p-1}\,\mathrm{d}t = s^{-p/2}2^{3p/2-1}\Gamma\Big(\frac{p}{2}\Big). \tag{25}$$

Recall (16), $F(p,s) = \int_0^\infty |\mathfrak{j}_1(t)|^s t^{p-1}\,\mathrm{d}t$, so

$$H(p,s) = G(p,s) - F(p,s). \tag{26}$$

Then the crucial integral inequality (17) is equivalent to $H(p,s) \geqslant 0$.

Our main goal and result here is that the integral inequality $H(p,s) > 0$ holds in rather wide ranges of parameters (p,s) (however, it does not hold for all $0 < p < 2$ and $s > 1$ which, as already noted, would have been enough to deduce Theorem 8).

Lemma 21. *The inequality $H(p,s) > 0$ holds in the following cases*:

(a) $0 < p \leqslant 2$ *and* $s \geqslant 2$,

(b) $0 < p \leqslant \frac{1}{4}$ *and* $s \geqslant 1.3$.

For the proof, we will need several rather intricate estimates on various integrals. The general idea we employ here follows [42] and is to first use the explicit bounds on $\mathfrak{j}_1$ from (21) and (23) to get $H > 0$ in certain but *not all* cases and then extend them by interpolating in s (exploiting the simple dependence of G on s). This is in contrast to several works, e.g., [8; 9; 11; 14; 26; 36] which heavily rely on the approach developed by Nazarov and Podkorytov in [38] to integral inequalities with oscillatory integrands. We also refer to [2] as well as [35] for connections between such integral inequalities and majorisation.

We begin by setting

$$U(p,s) = \frac{4^p(2\pi\cdot 15^{1/2})^{-s/2}}{\frac{3s}{2}-p} + 2^{3p/2-1}s^{-p/2}\left(\Gamma\left(\frac{p}{2}\right) - \frac{\Gamma\left(\frac{p}{2}+2\right)}{6s} + \frac{\Gamma\left(\frac{p}{2}+4\right)}{72s^2}\right), \tag{27}$$

which emerges in the next lemma (following [42, Lemma 3.2]).

Lemma 22. *For $p < \frac{3}{2}s$, we have*

$$F(p,s) < U(p,s).$$

Proof. Using (21) and (23) with $t_0 = 4$, we get

$$F(p,s) = \int_0^\infty |j_1(t)|^s t^{p-1}\,dt < \int_0^\infty \exp\left(-s\frac{t^2}{8} - s\frac{t^4}{3\cdot 2^7}\right)t^{p-1}\,dt + \left(\frac{8}{15^{1/4}(2\pi)^{1/2}}\right)^s \frac{4^{p-3s/2}}{\frac{3s}{2}-p},$$

valid for $p < \frac{3}{2}s$. After the change of variables $u = \frac{1}{8}st^2$, the first integral becomes

$$2^{3p/2-1}s^{-p/2}\int_0^\infty e^{-u^2/(6s)}e^{-u}u^{p/2-1}\,du.$$

We estimate the first exponential using $e^{-x} \leqslant 1 - x + \frac{1}{2}x^2,\ x \geqslant 0$, which gives the bound

$$\int_0^\infty \left(1 - \frac{u^2}{6s} + \frac{u^4}{72s^2}\right)e^{-u}u^{p/2-1}\,du = \Gamma\left(\frac{p}{2}\right) - \frac{\Gamma\left(\frac{p}{2}+2\right)}{6s} + \frac{\Gamma\left(\frac{p}{2}+4\right)}{72s^2}$$

on the integral appearing in the above expression. □

Lemma 23. *The inequality*

$$U(p,s) < G(p,s)$$

holds in the following cases:

(i) $0 < p \leqslant \frac{1}{4}$ *and* $s \geqslant \frac{17}{10}$,

(ii) $0 < p \leqslant \frac{4}{5}$ *and* $s \geqslant 2$,

(iii) $0 \leqslant p \leqslant 2$ *and* $s \geqslant \frac{8}{3}$.

Proof. Note that $U < G$ is equivalent to the following inequality (after cancelling the terms containing $\Gamma\left(\frac{1}{2}p\right)$ on both sides, factoring out $\Gamma\left(\frac{1}{2}p+2\right)$ and moving terms across using that $\frac{3}{2}s - p > 0$):

$$(2\pi\cdot 15^{1/2})^{s/2}2^{-p/2}\left(\frac{3s}{2}-p\right)\frac{12s - \left(\frac{p}{2}+2\right)\left(\frac{p}{2}+3\right)}{144} > \frac{s^{p/2+2}}{\Gamma\left(\frac{p}{2}+2\right)}. \tag{28}$$

To shorten the notation, let $a = (2\pi)^{1/2}\cdot 15^{1/4}$ and

$$A(p,s) = 2^{-p/2}\left(\frac{3s}{2}-p\right)\frac{12s - \left(\frac{p}{2}+2\right)\left(\frac{p}{2}+3\right)}{144},$$

which is decreasing in p and increasing in s. In each of the cases we will simply replace A with its smallest possible value given the range of p and s, so we let $p_1 = \frac{1}{4},\ s_1 = \frac{17}{10},\ p_2 = \frac{4}{5},\ s_2 = 2$ and $p_3 = 2$,

$s_3 = \frac{8}{3}$ and have $A(p,s) \geqslant A_k$, where $A_k = A(p_k, s_k)$ for $k = 1, 2, 3$ in cases (i), (ii), (iii), respectively. Then it suffices to prove that

$$A_k a^s > \frac{s^{p/2+2}}{\Gamma\left(\frac{p}{2}+2\right)}.$$

We take the logarithm and consider

$$f(p,s) = s\log a + \log A_k - \left(\frac{p}{2}+2\right)\log s + \log\Gamma\left(\frac{p}{2}+2\right).$$

Our goal is to show that $f(p,s) > 0$. We observe that

$$\frac{\partial}{\partial p} f(p,s) = -\frac{1}{2}\log s + \frac{1}{2}\psi\left(\frac{p}{2}+2\right) \leqslant -\frac{1}{2}\log s_k + \frac{1}{2}\psi\left(\frac{p_k}{2}+2\right)$$

in each of the three cases and the resulting numerical values on the right-hand side are bounded above by -0.015, -0.02 and -0.029 for $k = 1, 2, 3$, respectively. Similarly,

$$\frac{\partial}{\partial s} f(p,s) = \log a - \frac{\frac{p}{2}+2}{s} \geqslant \log a - \frac{\frac{p_k}{2}+2}{s_k},$$

with the right-hand side bounded this time below by 0.34, 0.39 and 0.47 for $k = 1, 2, 3$, respectively. Thus $f(p,s)$ is decreasing in p and increasing in s, so

$$f(p,s) \geqslant f(p_k, s_k),$$

and after plugging in the explicit numerical values, the right-hand side is bounded below by 0.041, 0.049 and 0.032 for $k = 1, 2, 3$, thus proving (i), (ii) and (iii), respectively. □

The next two lemmas are vital for the interpolation argument.

Lemma 24. *For $\frac{4}{5} \leqslant p \leqslant 2$, we have*

$$F\left(p, \tfrac{8}{3}\right) < e^{-p/6} G(p, 2).$$

Proof. Using (23) with $t_0 = 5$, we get

$$\int_5^\infty |\mathrm{j}_1(t)|^{8/3} t^{p-1}\,\mathrm{d}t \leqslant \left(\frac{8}{\pi}\right)^{4/3}\left(\frac{25}{24}\right)^{2/3} \frac{5^{p-4}}{4-p}, \tag{29}$$

which for $p \leqslant 2$ gives

$$\int_5^\infty |\mathrm{j}_1(t)|^{8/3} t^{p-1}\,\mathrm{d}t \leqslant \left(\frac{8}{\pi}\right)^{4/3}\left(\frac{25}{24}\right)^{2/3} \frac{5^{p-4}}{2} = \frac{2}{3^{2/3}\cdot 5^{8/3}\pi^{4/3}} 5^p.$$

We divide the interval $[0, 5]$ into consecutive subintervals of the form $[k/m, (k+1)/m]$, $k=0, 1, \dots, 5m-1$ with $m = 100$, and crudely bound

$$\begin{aligned}\int_0^5 &|\mathrm{j}_1(t)|^{8/3} t^{p-1}\,\mathrm{d}t \\ &< \int_0^{1/m} t^{p-1}\,\mathrm{d}t + \frac{1}{m}\sum_{k=1}^{5m-1} \max\left\{\left|\mathrm{j}_1\left(\frac{k}{m}\right)\right|^{8/3}, \left|\mathrm{j}_1\left(\frac{k+1}{m}\right)\right|^{8/3}\right\} \max\left\{\left(\frac{k}{m}\right)^{p-1}, \left(\frac{k+1}{m}\right)^{p-1}\right\}\end{aligned} \tag{30}$$

i	0	1	2	3	4	5	6	7	8	9	10	11
$10^3 \cdot (\ell_i(u_i) - L(u_i))$	1	5	8	9	10	12	13	14	14	15	15	15
$10^3 \cdot (\ell_i(u_{i+1}) - L(u_{i+1}))$	4	8	9	10	11	13	14	14	15	15	15	14

Table 2. Proof of Lemma 24: lower bounds on the differences at the endpoints of the linear approximations ℓ_i to $R(p)$.

(we have used that $|\mathrm{j}_1| < 1$ and that j_1 is monotone on [0, 5], the former justified by (10) and the latter, e.g., in [42, p. 290] in the proof of Proposition 1.1). Now,

$$\int_0^{1/m} t^{p-1}\,\mathrm{d}t = \frac{1}{pm^p} < \frac{1}{0.8m^p}.$$

A resulting bound on $e^{p/6}2^{1-p}\int_0^\infty |\mathrm{j}_1(t)|^{8/3}t^{p-1}\,\mathrm{d}t$ is of the form

$$h(p) = \sum_k \lambda_k a_k^p$$

with explicit positive numbers λ_k, a_k. We check that $L(p) = \log h(p) < \log\Gamma\left(\frac{1}{2}p\right) = R(p)$ for $0.8 \leqslant p \leqslant 2$ relying on the fact that both sides are clearly convex (recall that summation preserves log-convexity). Specifically, we divide the interval [0.8, 2] into 12 consecutive subintervals $[u_i, u_{i+1}]$, $u_i = 0.8 + 0.1i$, $i = 0, 1, \dots, 11$ and on each interval we bound $R(p)$ from below by its tangent put at the middle $v_i = \frac{1}{2}(u_i + u_{i+1})$, $\ell_i(p) = R'(v_i)(p - v_i) + R(v_i)$ and then check that $\ell_i(p) > L(p)$ by checking the values at the endpoints $p = u_i, u_{i+1}$, which are gathered in Table 2. □

Lemma 25. *For* $0 < p \leqslant \frac{1}{4}$, *we have*

$$F(p, 1.3) < e^{2p/17}G(p, 1.7).$$

Proof. Fix $0 < p \leqslant \frac{1}{4}$. We break the integral on the left-hand side into the sum of four integrals $A_1 + \cdots + A_4$ over $(0, 1)$, $(1, 5)$, $(5, 10)$ and $(10, \infty)$. For the first one, we use (21):

$$|\mathrm{j}_1(t)|^{1.3} < \exp\left\{-\frac{13}{10}\left(\frac{t^2}{8} + \frac{t^4}{3\cdot 2^7}\right)\right\} < 1 - \frac{13}{80}t^2 + \frac{377}{38400}t^4, \qquad 0 < t < 1$$

(the last inequality obtained by taking the first terms in the power series expansion of the penultimate expression, which gives an upper bound as can be checked directly by differentiation). Integrating against t^{p-1} yields

$$A_1 \leqslant \frac{1}{p} - \frac{13}{80(p+2)} + \frac{377}{38400(p+4)} < \frac{1}{p} - \frac{13}{80(p+2)} + \frac{377}{38400\cdot 4}.$$

For the last one, we use (23) with $t_0 = 10$:

$$\begin{aligned} A_4 \leqslant \int_{10}^\infty \left(\left(\frac{8}{\pi}\right)^{1/2}\left(\frac{100}{99}\right)^{1/4} t^{-3/2}\right)^{1.3} t^{p-1}\,\mathrm{d}t &= \frac{2^{53/20}}{11^{13/40}\cdot 5^{3/10}(3\pi)^{13/20}}\frac{10^p}{39-20p} \\ &\leqslant \frac{2^{53/20}}{11^{13/40}\cdot 5^{3/10}(3\pi)^{13/20}}\frac{10^p}{34}. \end{aligned}$$

For A_2 and A_3, we use Riemann sums. First, without any error term thanks to the monotonicity of j_1 on $(1, 5)$,

$$A_2 \leqslant \sum_{k=0}^{4m-1} \max\left\{\left|\mathrm{j}_1\left(1+\frac{k}{m}\right)\right|^{1.3}, \left|\mathrm{j}_1\left(1+\frac{k+1}{m}\right)\right|^{1.3}\right\} \int_{1+k/m}^{1+(k+1)/m} t^{p-1}\,\mathrm{d}t$$
$$< \sum_{k=0}^{4m-1} \max\left\{\left|\mathrm{j}_1\left(1+\frac{k}{m}\right)\right|^{1.3}, \left|\mathrm{j}_1\left(1+\frac{k+1}{m}\right)\right|^{1.3}\right\} \frac{\left(1+\frac{k}{m}\right)^{p-1}}{m}.$$

Second, on $(5, 10)$, we choose the midpoints and bound the error simply using the supremum of the derivative via the crude (numerical) bound

$$\sup_{t\in[5,10]}\left|\frac{\mathrm{d}}{\mathrm{d}t}|\mathrm{j}_1(t)|^{1.3}\right| < 0.06$$

(since

$$\left|\frac{\mathrm{d}}{\mathrm{d}t}|\mathrm{j}_1(t)|^{1.3}\right| = 1.3|\mathrm{j}_1(t)|^{0.3}|\mathrm{j}_1'(t)| \quad \text{and} \quad j_1'(t) = -2\frac{J_2(t)}{t} = 2\frac{J_0(t)}{t} - 4\frac{J_1(t)}{t^2},$$

the function under the supremum can be expressed in terms of J_0 and J_1 and the supremum can be estimated by employing the precise polynomial-type approximations to J_0 and J_1 from [1]: 9.4.3 and 9.4.6, pp. 369–370). This leads to

$$A_3 \leqslant \sum_{k=0}^{5m-1}\left|\mathrm{j}_1\left(5+\frac{k+\frac{1}{2}}{m}\right)\right|^{1.3} \int_{5+k/m}^{5+(k+1)/m} t^{p-1}\,\mathrm{d}t + 0.06\frac{1}{2m}\int_5^{10} t^{p-1}\,\mathrm{d}t$$
$$< \sum_{k=0}^{5m-1}\left|\mathrm{j}_1\left(5+\frac{k+\frac{1}{2}}{m}\right)\right|^{1.3} \frac{\left(5+\frac{k}{m}\right)^{p-1}}{m} + \frac{3\cdot 5^p}{100m}.$$

With hindsight, we choose $m = 200$. Adding these four estimates together (call the rightmost sides of these bounds $B_1, \ldots, B_4$) and multiplying through p, it suffices to show that $L(p) < R(p)$ for $0 < p \leqslant \frac{1}{4}$, where

$$L(p) = p\cdot(B_1+\cdots+B_4) \quad \text{and} \quad R(p) = (e^{2/17}2^{3/2}1.7^{-1/2})^p\,\Gamma\left(\frac{p}{2}+1\right).$$

Plainly, $R(p)$ is convex (as being log-convex), whilst

$$L(p) = \frac{67}{80} + \frac{13}{40(p+2)} + \frac{377}{153600}p + c_1\cdot p10^p + c_2\cdot p5^p + \sum_i \lambda_i p a_i^p$$

with positive constants c_1, c_2, λ_i (specified above) and $a_i \geqslant 1$ (of the form $(1+k/m)$, $k \geqslant 0$). Thus, $L(p)$ is also convex and now we proceed similarly to what we did in the proof of Lemma 24. Note that $L(0) = R(0) = 1$. For $0 < p \leqslant 0.02$, we have $R(p) \geqslant \ell_0(p) = 1 + R'(0)p$ and check that $\ell_0(0.02) - L(0.02) > 10^{-5} > 0$ to conclude $R(p) \geqslant L(p)$, $0 \leqslant p \leqslant 0.02$. We divide the remaining interval $(0.02, 0.25)$ into six intervals: $(0.02, 0.05)$, $(0.05, 0.1)$, $(0.1, 0.15)$, $(0.15, 0.2)$, $(0.2, 0.23)$, $(0.23, 0.25)$ denoted by (u_i, u_{i+1}), $i = 1, \ldots, 6$, choose their midpoints $v_i = \frac{1}{2}(u_i + u_{i+1})$ and bound $R(p)$ from below by its tangent $\ell_i(p) = R'(v_i)(p - v_i) + R(v_i)$, and check that $\ell_i(p) > L(p)$ at $p = u_i, u_{i+1}$ (see Table 3) to conclude that $R(p) > L(p)$ for all $u_i \leqslant p \leqslant u_{i+1}$, $i = 1, \ldots, 6$, by convexity. □

i	1	2	3	4	5	6
$10^4 \cdot (\ell_i(u_i) - L(u_i))$	.7	1	3	4	5	3
$10^4 \cdot (\ell_i(u_{i+1}) - L(u_{i+1}))$	2	3	4	3	3	2

Table 3. Proof of Lemma 25: lower bounds on the differences at the endpoints of the linear approximations ℓ_i to $R(p)$.

We are ready to prove the main inequalities of this section.

Proof of Lemma 21. First we show (a). Lemma 22 combined with Lemma 23 part (ii) as well as (iii) gives (a) for all $0 < p \leqslant \frac{4}{5}$, $s \geqslant 2$, as well as all $0 < p \leqslant 2$, $s \geqslant \frac{8}{3}$, respectively. It remains to handle the case $\frac{4}{5} < p < 2$, $2 \leqslant s \leqslant \frac{8}{3}$. We apply Hölder's inequality, Lemma 24 and (20), equivalently $F(p,2) \leqslant G(p,2)$, to get

$$\begin{aligned} F(p,s) \leqslant F(p,2)^{(8-3s)/2} F\big(p, \tfrac{8}{3}\big)^{(3s-6)/2} &\leqslant (G(p,2))^{(8-3s)/2} (e^{-p/6} G(p,2))^{(3s-6)/2} \\ &= e^{-p(s-2)/4} 2^{p-1} \Gamma\Big(\frac{p}{2}\Big). \end{aligned}$$

By concavity, $\log s \leqslant \frac{1}{2}(s-2) + \log 2$, $s \geqslant 2$, thus

$$e^{-p(s-2)/4} \leqslant s^{-p/2} 2^{p/2}, \quad s \geqslant 2,\ p > 0, \tag{31}$$

which gives (a).

To show (b), we proceed similarly. Lemma 22 combined with Lemma 23(i) gives (b) for all $0 < p \leqslant \frac{1}{4}$ and $s \geqslant 1.7$. In the remaining case $1.3 \leqslant s \leqslant 1.7$, from Hölder's inequality, Lemma 23(i) and Lemma 25, we obtain

$$\begin{aligned} F(p,s) \leqslant F(p,1.7)^{(10s-13)/4} F(p,1.3)^{(17-10s)/4} &\leqslant (G(p,1.7))^{(10s-13)/4} (e^{2p/17} G(p,1.7))^{(17-10s)/4} \\ &= e^{(p/2)(17-10s)/17} 1.7^{-p/2} 2^{3p/2-1} \Gamma\Big(\frac{p}{2}\Big). \end{aligned}$$

Thanks to concavity, $\log s \leqslant \frac{10}{17}s - 1 + \log 1.7$, $s \leqslant 1.7$, which gives

$$e^{(p/2)(17-10s)/17} 1.7^{-p/2} \leqslant s^{-p/2},$$

whence (b). □

3.4. ***The integral inequality*: $2 < p < 3$.** We follow the general approach from the previous case $p < 2$. Recall (16), $F(p,s) = \int_0^\infty |\mathrm{j}_1(t)|^s t^{p-1}\,\mathrm{d}t$, and that the crucial integral inequality (18) reads $s^{p/2} F(p,s) \leqslant 2^{p/2} F(p,2)$. Thus here we let

$$\widetilde{H}(p,s) = s^{-p/2} 2^{p/2} F(p,2) - F(p,s), \quad 2 < p < 3,\ s > 1. \tag{32}$$

Note that we can express $F(p,2)$ explicitly: using Corollary 5 and (12), we obtain

$$F(p,2) = \int_0^\infty \mathrm{j}_1(t)^2 t^{p-1}\,\mathrm{d}t = \kappa_{p,4}^{-1} \mathbb{E}|\xi_1 + \xi_2|^{-p} = \kappa_{p,4}^{-1} 2^{-p/2} C_2(p) = 2^{p-1} \frac{\Gamma\big(\frac{p}{2}\big)\Gamma(3-p)}{\big[\Gamma\big(2-\frac{p}{2}\big)\big]^2 \Gamma\big(3-\frac{p}{2}\big)}.$$

In view of (32), we therefore set

$$\widetilde{G}(p,s) = s^{-p/2} 2^{3p/2-1} \Gamma\Big(\frac{p}{2}\Big) D(p) \tag{33}$$

with

$$D(p) = \frac{\Gamma(3-p)}{\big[\Gamma\big(2-\frac{p}{2}\big)\big]^2 \Gamma\big(3-\frac{p}{2}\big)}, \tag{34}$$

so that

$$\widetilde{H}(p,s) = \widetilde{G}(p,s) - F(p,s).$$

The main result of this section is that integral inequality (18) also holds for all $s \geqslant 2$. We emphasise that $\widetilde{H}(p,2) = 0$.

Lemma 26. *The inequality* $\widetilde{H}(p,s) > 0$ *holds for all* $2 < p < 3$ *and* $s \geqslant 2$.

This will be established in a very similar way to the previous section: crude pointwise bounds on j_1 will suffice to handle the case $s \geqslant \frac{8}{3}$ which will then be extended to $s \geqslant 2$ by interpolation.

Lemma 27. *With* $D(p)$ *defined in* (34), *the function* $p \mapsto \log D(p)$ *is increasing, convex and positive on* $(2,3)$.

Proof. Let $x = \frac{1}{2}(3-p)$, $0 < x < \frac{1}{2}$. By the Legendre duplication formula (see, e.g., 6.1.18 in [1]),

$$D(p) = \frac{\Gamma(2x)}{\Gamma\big(x+\frac{1}{2}\big)^2 \Gamma\big(x+\frac{3}{2}\big)} = \frac{2^{2x-1}\Gamma(x)}{\sqrt{\pi}\,\Gamma\big(x+\frac{1}{2}\big)\Gamma\big(x+\frac{3}{2}\big)}.$$

Thus the convexity of $\log D(p)$ on $(2,3)$ is equivalent to the convexity of

$$f(x) = \log\Gamma(x) - \log\Gamma\big(x+\tfrac{1}{2}\big) - \log\Gamma\big(x+\tfrac{3}{2}\big)$$

on $\big(0,\frac{1}{2}\big)$. Using the series representation of $(\log\Gamma(z))'' = \sum_{n=0}^{\infty}(z+n)^{-2}$ (see, e.g., 6.4.10 in [1]), we get

$$\begin{aligned} f''(x) &= \sum_{n=0}^{\infty}\frac{1}{(x+n)^2} - \sum_{n=0}^{\infty}\frac{1}{\big(x+n+\frac{1}{2}\big)^2} - \sum_{n=0}^{\infty}\frac{1}{\big(x+n+\frac{3}{2}\big)^2} \\ &= \frac{1}{x^2} - \frac{1}{\big(x+\frac{1}{2}\big)^2} + \sum_{n=1}^{\infty}\frac{1}{(x+n)^2} - 2\sum_{n=1}^{\infty}\frac{1}{\big(x+n+\frac{1}{2}\big)^2}. \end{aligned}$$

For $0 < x < \frac{1}{2}$,

$$\sum_{n=1}^{\infty}\frac{1}{(x+n)^2} - 2\sum_{n=1}^{\infty}\frac{1}{\big(x+n+\frac{1}{2}\big)^2} > \sum_{n=1}^{\infty}\frac{1}{\big(\frac{1}{2}+n\big)^2} - 2\sum_{n=1}^{\infty}\frac{1}{\big(n+\frac{1}{2}\big)^2} = -\frac{\pi^2}{2} + 4,$$

thus

$$f''(x) > \frac{1}{x^2} - \frac{1}{\big(x+\frac{1}{2}\big)^2} - \frac{\pi^2}{2} + 4.$$

The right-hand side is clearly decreasing (e.g., by looking at the derivative), so for $0 < x < \frac{1}{2}$, it is at least $4 - 1 - \frac{1}{2}\pi^2 + 4 = 7 - \frac{1}{2}\pi^2$ which is positive.

Moreover,

$$\frac{\mathrm{d}}{\mathrm{d}p} \log D(p)\Big|_{p=2} = \frac{1-\gamma}{2} > 0$$

($\gamma = 0.57\ldots$ is Euler's constant), so $D(p)$ is strictly increasing on $(2, 3)$ with $D(2) = 1$. □

Lemma 28. *For all* $2 < p < 3$ *and* $s \geqslant \frac{8}{3}$, *we have*

$$U(p, s) < \widetilde{G}(p, s).$$

Proof. We let $a = (2\pi)^{1/2} \cdot 15^{1/4}$, and inserting the definitions of U from (27) and $\widetilde{G}$ from (33), the desired inequality becomes

$$\frac{4^p a^{-s}}{\frac{3s}{2} - p} + s^{-p/2} 2^{3p/2-1} \left(\Gamma\left(\frac{p}{2}\right) - \frac{\Gamma\left(\frac{p}{2}+2\right)}{6s} + \frac{\Gamma\left(\frac{p}{2}+4\right)}{72s^2} \right) < s^{-p/2} 2^{3p/2-1} \Gamma\left(\frac{p}{2}\right) D(p),$$

equivalently,

$$\frac{2^{p/2+1} a^{-s}}{\frac{3s}{2} - p} s^{p/2+2} < s^2 \Gamma\left(\frac{p}{2}\right)(D(p)-1) + \Gamma\left(\frac{p}{2}+2\right) \frac{12s - \left(\frac{p}{2}+2\right)\left(\frac{p}{2}+3\right)}{72}.$$

The right-hand side is clearly increasing with s ($D(p) > 1$ by Lemma 27), whereas the left-hand side is decreasing with s (for every fixed $2 < p < 3$), as can be checked by examining the derivative of $\log(a^{-s} s^{p/2+2})$. Therefore, it suffices to prove this inequality for $s = \frac{8}{3}$. Moreover, after replacing $\Gamma\left(\frac{1}{2}p\right)$ on the right-hand side with 0.88 (see Lemma 18) and $\Gamma\left(\frac{1}{2}p+2\right)$ with $\Gamma(3) = 2$, it suffices to prove that the function

$$f(p) = 0.88\left(\frac{8}{3}\right)^2 (D(p)-1) + \frac{32 - \left(\frac{p}{2}+2\right)\left(\frac{p}{2}+3\right)}{36} - b\frac{\left(\frac{16}{3}\right)^{p/2}}{4-p},$$

where $b = 2a^{-8/3}\left(\frac{8}{3}\right)^2$, is positive for $2 < p < 3$. We put

$$L(p) = b\frac{\left(\frac{16}{3}\right)^{p/2}}{4-p} + \frac{1}{36}\left(\frac{p}{2}+2\right)\left(\frac{p}{2}+3\right)$$

and

$$R(p) = 0.88\left(\tfrac{8}{3}\right)^2 (D(p)-1) + \tfrac{8}{9},$$

which are both convex ($D(p)$ is even log-convex, see Lemma 27). For $2 < p < \frac{5}{2}$, we use the tangent $\ell_1(p) = R(2) + R'(2)(p-2)$ as a lower bound, $R(p) > \ell_1(p)$, and check that at $p = 2$ and $p = \frac{5}{2}$ the linear function ℓ_1 dominates L (the difference is $0.017\ldots$ and $0.076\ldots$, respectively), which then gives $R > \ell_1 > L$ on $\left(2, \frac{5}{2}\right)$. Similarly, for $\frac{5}{2} < p < 3$, we have $R(p) > \ell_2(p) = R\left(\frac{5}{2}\right) + R'\left(\frac{5}{2}\right)\left(p - \frac{5}{2}\right)$, and $\ell_2 - L$ at $p = \frac{5}{2}$ and $p = 3$ is $1.19\ldots$ and $3.77\ldots$, respectively. This finishes the proof. □

Lemma 29. *For all* $2 < p < 3$, *we have*

$$F\left(p, \tfrac{8}{3}\right) < e^{-p/6} \widetilde{G}(p, 2).$$

Proof. Consider

$$L(p) = \log F\big(p, \tfrac{8}{3}\big) \quad\text{and}\quad R(p) = \log(e^{-p/6}\widetilde{G}(p,2)),$$

which are both convex (recall Lemma 27). Using that, we crudely bound $R(p)$ from below by tangents: $r_1(p) = R(2) + R'(2)(p-2)$ on $(2, 2.5)$ and $r_2(p) = R(2.5) + R'(2.5)(p-2.5)$ on $(2.5, 3)$, and then compare their values at the endpoints with upper bounds on L to conclude that $r_1 > L$ on $(2, 2.5)$ and $r_2 > L$ on $(2.5, 3)$. Estimates (29) and (30) added together (applied with $m = 100$ as in Lemma 24) yield

$$L(2) < 0.35, \quad L(2.5) < 0.56, \quad L(3) < 0.96,$$

whereas we check directly that

$$r_1(2) > 0.359, \quad r_1(2.5) > 0.58, \quad r_2(3) > 1.48.$$

Comparing these values finish the argument. □

Proof of Lemma 26. Lemma 22 combined with Lemma 28 show that $\widetilde{H}(p,s) > 0$ for all $2 < p < 3$ and $s \geqslant \frac{8}{3}$. To cover the regime $2 \leqslant s < \frac{8}{3}$, we first apply Hölder's inequality in the exact same way as in the proof of Lemma 21(a):

$$F(p,s) \leqslant F(p,2)^{(8-3s)/2} F\big(p, \tfrac{8}{3}\big)^{(3s-6)/2},$$

and now, with $F(p,2) = \widetilde{G}(p,2)$ and Lemma 29, we get that

$$F(p,s) \leqslant e^{-p(s-2)/4}\widetilde{G}(p,2).$$

Finally, using (31), the right-hand side gets bounded from above by the desired $\widetilde{G}(p,s)$. □

3.5. *Miscellaneous facts.* Our first result here is a straightforward extension of Lemma 8 from [28] to negative moments (see also [9, Lemma 3]).

Lemma 30. *Let $0 < p < 1$. Let $n, d \geqslant 1$, and let $X_1, \dots, X_n$ be independent rotationally invariant random vectors in $\mathbb{R}^d$. Then*

$$\mathbb{E}\left|\sum_{k=1}^n v_k |X_k|\right|^{-p} = \beta_{p,d}\,\mathbb{E}\left|\sum_{k=1}^n \langle v_k, X_k\rangle\right|^{-p}$$

for arbitrary vectors $v_1, \dots, v_n$ in $\mathbb{R}^d$, where

$$\beta_{p,d} = \frac{\sqrt{\pi}\,\Gamma\big(\frac{d-p}{2}\big)}{\Gamma\big(\frac{1-p}{2}\big)\Gamma\big(\frac{d}{2}\big)}.$$

Proof. Thanks to homogeneity, we can assume that the v_k are *unit*. Thanks to rotational invariance and independence, we can assume without loss of generality that $v_1 = \dots = v_n = e_1$, but then it suffices to consider the case $n = 1$ (because sums of independent rotationally invariant random vectors are rotationally invariant). The latter can be easily justified in a number of ways.

For instance, it follows from a Fourier-analytic argument: we invoke (11), rewrite $\mathbb{E}\,\mathrm{j}_{d/2-1}(t|X_k|)$ as $\mathbb{E}e^{it\langle v_k, X_k\rangle}$ and apply (8) with $d = 1$ to $\sum \langle v_k, X_k\rangle$ which gives $\beta_{p,d} = \kappa_{p,d}/(2K_{p,1})$.

Alternatively, we can apply a standard embedding-type argument: if we take a random vector ξ uniform on the unit Euclidean sphere S^{d-1}, independent of the X_k, we have, for every vector x in $\mathbb{R}^d$,

$$\mathbb{E}|\langle x, \xi\rangle|^{-p} = \beta_{p,d}^{-1}|x|^{-p},$$

with

$$\beta_{p,d}^{-1} = \mathbb{E}|\langle e_1, \xi\rangle|^{-p} = \frac{\int_{-1}^{1} |t|^{-p}(1-t^2)^{(d-3)/2}\,\mathrm{d}t}{\int_{-1}^{1}(1-t^2)^{(d-3)/2}\,\mathrm{d}t} = \frac{\Gamma\left(\frac{1-p}{2}\right)\Gamma\left(\frac{d}{2}\right)}{\sqrt{\pi}\,\Gamma\left(\frac{d-p}{2}\right)}.$$

Applying this to $x = X_1$, taking the expectation over X_1 and noting that $\langle X_1, \xi\rangle$ has the same distribution as $\langle X_1, e_1\rangle$ finishes the argument. □

Lemma 31. *For every* $0 < q < 2$, *we have*

$$\left(\tfrac{13}{20}\right)^q < \Gamma(2-q).$$

Proof. The function $f(q) = \log\Gamma(2-q) - q\log\frac{13}{20}$ is convex on $(0, 2)$ with $f'(0) = \gamma - 1 - \log\frac{13}{20} > 0.007$. Thus f is strictly increasing and the lemma follows since $f(0) = 0$. □

4. End of the proof of Theorem 8

To finish the proof of Theorem 8, we only need to justify Steps 1–4 from Section 2.3.1.

4.1. ***Step 1 and 3: Integral inequality.*** Lemma 21(a) and (b) gives Steps 1 and 3, respectively.

4.2. ***Step 2: Induction.*** First note that, by homogeneity, (14) with $C(p) = C_\infty(p)$ is equivalent to

$$\mathbb{E}\left|\xi_1 + \sum_{k=2}^{n} a_k\xi_k\right|^{-p} \leqslant C_\infty(p)\left(1 + \sum_{k=2}^{n} a_k^2\right)^{-p/2}.$$

For $p > 0$ and $x \geqslant 0$, we define

$$\phi_p(x) = (1+x)^{-p/2}$$

and

$$\Phi_p(x) = \begin{cases} \phi_p(x), & x \geqslant 1,\\ 2\phi_p(1) - \phi_p(2-x), & 0 \leqslant x \leqslant 1.\end{cases}$$

Geometrically, on $[0, 1]$, the graph of $\Phi_p(x)$ is obtained from the graph of $\phi_p(x)$ on $[1, 2]$ by reflecting it about $(1, \phi_p(1))$. Crucially, $\Phi_p(x) \leqslant \phi_p(x)$ for all $x \geqslant 0$, since $2\phi_p(1) \leqslant \phi_p(x) + \phi_p(2-x)$, by the convexity of ϕ_p. By induction on n, we will show a strengthened version of the above with ϕ_p on the right-hand side replaced by Φ_p.

Theorem 32. *Let* $\frac{1}{4} \leqslant p \leqslant 2$. *Let* $\xi_1, \xi_2, \ldots$ *be independent random vectors uniform on the unit Euclidean sphere* S^3 *in* $\mathbb{R}^4$. *For every* $n \geqslant 2$ *and nonnegative numbers* $a_2, \ldots, a_n$, *we have*

$$\mathbb{E}\left|\xi_1 + \sum_{k=2}^{n} a_k\xi_k\right|^{-p} \leqslant C_\infty(p)\Phi_p\left(\sum_{k=2}^{n} a_k^2\right). \tag{35}$$

Proof. For the inductive base, when $n = 2$, (35) becomes

$$\mathbb{E}|\xi_1 + \sqrt{t}\xi_2|^{-p} \leqslant 2^{p/2}\Gamma\Big(2 - \frac{p}{2}\Big)\Phi_p(t), \quad t \geqslant 0,$$

where we have put $t = a_2^2$. By homogeneity and the fact that $\Phi_p \leqslant \phi_p$, the case $t \geqslant 1$ reduces to the case $0 \leqslant t \leqslant 1$. Indeed, if $t \geqslant 1$, we have $\Phi_p(t) = \phi_p(t) = (1+t)^{-p/2}$, so dividing both sides by $t^{-p/2}$, the inequality is equivalent to the one with $1/t$ instead of t and $\phi_p(1/t)$ on the right-hand side. The case $0 \leqslant t \leqslant 1$ follows by combining Corollary 16 and Lemma 19 (applied to $q = \frac{1}{2}p$, noting as usual that by rotational invariance, $\mathbb{E}|e_1 + \sqrt{t}\xi_2|^{-p} = \mathbb{E}|\xi_1 + \sqrt{t}\xi_2|^{-p}$).

For the inductive step, let $n \geqslant 2$ and suppose (35) holds for all $n-1$ nonnegative numbers $a_2, \dots, a_n$. To prove it for n nonnegative arbitrary numbers, say $a_2, \dots, a_n, a_{n+1}$, we let

$$x = a_2^2 + \dots + a_n^2 + a_{n+1}^2$$

and consider 3 cases.

Case 1: $a_k > 1$ for some $2 \leqslant k \leqslant n+1$. Then $x > 1$, so $\Phi_p(x) = \phi_p(x)$ and our goal is to show

$$\mathbb{E}\Big|\sum_{k=1}^{n+1} a_k\xi_k\Big|^{-p} \leqslant C_\infty(p)\Big(\sum_{k=1}^{n+1} a_k^2\Big)^{-p/2}, \tag{36}$$

where we put $a_1 = 1$. Let $a_1^*, \dots, a_{n+1}^*$ be a nonincreasing rearrangement of the sequence $a_1, \dots, a_{n+1}$, and set $a_k' = a_k^*/a_1^*$, $k = 1, \dots, n+1$. Thanks to homogeneity, to prove (36), it is enough to prove

$$\mathbb{E}\Big|\sum_{k=1}^{n+1} a_k'\xi_k\Big|^{-p} \leqslant C_\infty(p)\Phi_p\Big(\sum_{k=2}^{n+1} a_k'^2\Big),$$

which is handled by either of the next two cases because here $a_1' = 1$ and $a_k' \leqslant 1$ for all $k \geqslant 2$.

Case 2.1: $a_k \leqslant 1$ for all $2 \leqslant k \leqslant n+1$ and $x \geqslant 1$. Since $x \geqslant 1$, our goal is again (36) with $a_1 = 1$. We have

$$\max_{k \leqslant n+1} a_k = 1 \leqslant \frac{1}{\sqrt{2}}\sqrt{1+x} = \frac{1}{\sqrt{2}}\Big(\sum_{k=1}^{n+1} a_k^2\Big)^{1/2},$$

so Corollary 10 finishes the inductive argument in this case.

Case 2.2: $a_k \leqslant 1$ for all $2 \leqslant k \leqslant n+1$ and $x < 1$. Fix vectors $v_2, \dots, v_{n+1}$ in $\mathbb{R}^4$ with $|v_k| = a_k$, for each $k = 2, \dots, n+1$. Then, plainly,

$$\mathbb{E}\Big|\xi_1 + \sum_{k=2}^{n+1} a_k\xi_k\Big|^{-p} = \mathbb{E}\Big||e_1|\xi_1 + \sum_{k=2}^{n+1} |v_k|\xi_k\Big|^{-p},$$

and thanks to Lemma 30, when $0 < p < 1$, the right-hand side can be written as

$$\mathbb{E}\Big||e_1|\xi_1 + \sum_{k=2}^{n+1} |v_k|\xi_k\Big|^{-p} = \beta_{p,4}\mathbb{E}\Big|\langle e_1, \xi_1\rangle + \sum_{k=2}^{n+1}\langle v_k, \xi_k\rangle\Big|^{-p}.$$

If we let Q be a random orthogonal matrix, independent of the ξ_k, and note that (ξ_n, ξ_{n+1}) has the same distribution as $(\xi_n, Q\xi_n)$, we obtain

$$\mathbb{E}\left|\langle e_1, \xi_1\rangle + \sum_{k=2}^{n+1}\langle v_k, \xi_k\rangle\right|^{-p} = \mathbb{E}_Q\mathbb{E}_\xi\left|\langle e_1, \xi_1\rangle + \sum_{k=2}^{n-1}\langle v_k, \xi_k\rangle + \langle v_n + Q^\top v_{n+1}, \xi_n\rangle\right|^{-p}.$$

Going back to the vector sum again via Lemma 30, we arrive at the identity

$$\mathbb{E}\left|\xi_1 + \sum_{k=2}^{n+1} a_k\xi_k\right|^{-p} = \mathbb{E}_Q\mathbb{E}_\xi\left|\xi_1 + \sum_{k=2}^{n-1} |v_k|\xi_k + |v_n + Q^\top v_{n+1}|\xi_n\right|^{-p}.$$

The same identity continues to hold for all $0 < p < 3$: we know it holds for all $0 < p < 1$, and both sides are clearly analytic in p wherever the expectations exists, that is in $\{p \in \mathbb{C}, \operatorname{Re}(p) < 3\}$ because $\left|\mathbb{E}|\cdot|^z\right| \leqslant \mathbb{E}|\cdot|^{\operatorname{Re}(z)}$ for $z \in \mathbb{C}$ (the analyticity follows, e.g., from Morera's theorem by a standard argument). Conditioned on the value of Q, the inductive hypothesis applied to the $n-1$ nonnegative numbers $|v_2|, \dots, |v_{n-1}|, |v_n + Q^\top v_{n+1}|$ yields

$$\mathbb{E}\left|\xi_1 + \sum_{k=2}^{n+1} a_k\xi_k\right|^{-p} \leqslant \mathbb{E}_Q C_\infty(p)\Phi_p(|v_2|^2 + \cdots + |v_{n-1}|^2 + |v_n + Q^\top v_{n+1}|^2).$$

Note that

$$|v_2|^2 + \cdots + |v_{n-1}|^2 + |v_n \pm Q^\top v_{n+1}|^2 = x \pm 2\langle v_n, Q^\top v_{n+1}\rangle,$$

so thanks to the symmetry of the distribution of Q, we can rewrite the right-hand side as

$$C_\infty(p)\mathbb{E}_Q\frac{\Phi_p(x + 2\langle v_n, Q^\top v_{n+1}\rangle) + \Phi_p(x - 2\langle v_n, Q^\top v_{n+1}\rangle)}{2}.$$

The proof of the inductive step now follows from the following *extended* concavity property of Φ_p applied to $a_\pm = x \pm 2\langle v_n, Q^\top v_{n+1}\rangle$. □

Lemma 33. *Let $p > 0$. For every $a_-, a_+ \geqslant 0$ with $\frac{1}{2}(a_- + a_+) \leqslant 1$, we have*

$$\frac{\Phi_p(a_-) + \Phi_p(a_+)}{2} \leqslant \Phi_p\left(\frac{a_- + a_+}{2}\right).$$

Proof. This is Lemma 20 in [9] (stated there for *no* reason only for $0 < p < 1$, as the proof works for every $p > 0$ because it only uses the convexity of ϕ_p). □

4.3. *Step 4*: *Projection.* Let us say that $a_1 = \max_{k \leqslant n} |a_k|$, so that $a_1 > \sqrt{\frac{10}{13}}$. *Projecting* onto this coefficient — that is applying Corollary 17 to $a = a_1$ and $v = \sum_{k=2}^n a_k\xi_k$ (conditioning on its value) — we get

$$\mathbb{E}\left|\sum_{k=1}^{n} a_k\xi_k\right|^{-p} \leqslant a_1^{-p} \leqslant \left(\frac{13}{10}\right)^{p/2} \leqslant 2^{p/2}\Gamma\left(2 - \frac{p}{2}\right) = C_\infty(p),$$

where the last inequality results from Lemma 31 (applied to $q = \frac{1}{2}p$). We have now finished the proof of Theorem 8. □

5. End of the proof of Theorem 9

To finish the proof of Theorem 9, we only need to show here Steps 1 and 2 from Section 2.3.2.

5.1. ***Step 1*: *Integral inequality.*** Lemma 26 gives the desired claim.

5.2. ***Step 2*: *Induction.*** We repeat the entire inductive argument from Section 4.2 verbatim, replacing $\frac{1}{4} \leqslant p \leqslant 2$ with $2 < p < 3$ and $C_\infty(p)$ with $C_2(p)$. The only modification required is to check the inductive base which now amounts to verifying that

$$\mathbb{E}|\xi_1 + \sqrt{t}\xi_2|^{-p} \leqslant C_2(p)\Phi_p(t) = C_2(p)(2^{1-p/2} - (3-t)^{-p/2}), \quad 0 \leqslant t \leqslant 1.$$

By Lemma 13, the left-hand side is clearly increasing in t (when $2 < p < 3$ and $d = 4$, all the coefficients in the power series expansion therein are positive), whereas the right-hand side is clearly decreasing in t. By the definition of $C_2(p)$, there is equality at $t = 1$. This finishes the proof of Theorem 9. ☐

Appendix: Behaviour of the constants

We sketch an argument of the following proposition which justifies (5).

Proposition 34. *For every $d \geqslant 1$, the equation $c_{d,2}(q) = c_{d,\infty}(q)$ has a unique solution $q = q_d^*$ in $(-(d-1), 2)$. Moreover, $c_{d,2}(q) < c_{d,\infty}(q)$ for $-(d-1) < q < q_d^*$, and $c_{d,2}(q) > c_{d,\infty}(q)$ for $q_d^* < q < 2$. For $d \geqslant 5$, we have $q_d^* \in (-(d-1), -(d-2))$.*

Proof. Since the cases $1 \leqslant d \leqslant 4$ have been explicitly dealt with (see the discussion in the introduction), it is enough to analyse the case $d \geqslant 5$. Moreover, by the Schur-concavity result of [4] and [28], $c_{d,\infty}(q) < c_{d,2}(q)$ for every $0 < q < 2$, so we can further assume that $-(d-1) < q < 0$. We look into the sign of

$$h_d(q) = \log(c_{d,2}(q)^q) - \log(c_{d,\infty}(q)^q).$$

Note that for $q < 0$ the sign of $h_d(q)$ is opposite to the sign of $c_{d,2}(q) - c_{d,\infty}(q)$. Now, $h_d(q)$ can be equivalently recast as

$$\begin{aligned} h_d(q) &= \log\left(2^{-q/2}\frac{\Gamma\left(\frac{d}{2}\right)\Gamma(d+q-1)}{\Gamma\left(\frac{d+q}{2}\right)\Gamma\left(d+\frac{q}{2}-1\right)}\right) - \log\left(\left(\frac{2}{d}\right)^{q/2}\frac{\Gamma\left(\frac{d+q}{2}\right)}{\Gamma\left(\frac{d}{2}\right)}\right) \\ &= -q\log 2 + \frac{q}{2}\log d + \log\left(\frac{\Gamma\left(\frac{d}{2}\right)^2\Gamma(d+q-1)}{\Gamma\left(\frac{d+q}{2}\right)^2\Gamma\left(d+\frac{q}{2}-1\right)}\right). \end{aligned}$$

Writing $x = \frac{1}{2}(q+d-1) \in \left(0, \frac{1}{2}(d-1)\right)$ and $\tilde{h}_d(x) = h_d(2x+1-d)$, we get — using the Legendre duplication formula $\Gamma(2x)\sqrt{\pi} = 2^{2x-1}\Gamma(x)\Gamma\left(x+\frac{1}{2}\right)$ — that

$$\tilde{h}_d(x) = x\log d + \log\left(\frac{\Gamma(x)}{\Gamma\left(x+\frac{1}{2}\right)\Gamma\left(x+\frac{d-1}{2}\right)}\right) + \log\left(\frac{2^{d-2}\Gamma\left(\frac{d}{2}\right)^2}{\sqrt{\pi}d^{(d-1)/2}}\right).$$

We now make the following claims.

Claim 1. For all $0 < x < 1$, we have $\tilde{h}''_d(x) > 0.06$.

Claim 2. For every $d \geqslant 5$, we have $\inf_{1<x<(d-1)/2} \tilde{h}'_d(x) > 0$.

Claim 3. $\tilde{h}_d\left(\frac{1}{2}\right) < 0$.

The strict convexity from Claim 1, the simple observation that $\tilde{h}_d(0+) = +\infty$ and Claim 3 give that $\tilde{h}_d$ has a unique zero, say x_0, in $\left(0, \frac{1}{2}\right)$, is positive on $(0, x_0)$ and negative on $\left(x_0, \frac{1}{2}\right)$. Claim 2 and the simple observation that $\tilde{h}_d\left(\frac{1}{2}(d-1)\right) = 0$ gives that $\tilde{h}_d$ is negative on $\left[1, \frac{1}{2}(d-1)\right)$. Convexity also gives that $\tilde{h}_d$ is negative on $\left(\frac{1}{2}, 1\right)$ since $h_d\left(\frac{1}{2}\right)$ and $h_d(1)$ are negative. These give the desired behaviour of $c_{d,2}(q) - c_{d,\infty}(q)$ for $-(d-1) < q < 0$. Finally, it also follows from Claim 2 that $h'_d(0) > 0$, which gives $c_{d,2}(0) - c_{d,\infty}(0) > 0$. It remains to prove the claims. □

Proof of Claim 1. Differentiating twice yields

$$\tilde{h}''_d(x) = \sum_{n=0}^{\infty}\left(\frac{1}{(x+n)^2} - \frac{1}{\left(x+n+\frac{1}{2}\right)^2} - \frac{1}{\left(x+n+\frac{d-1}{2}\right)^2}\right).$$

Note that the first two terms make up a decreasing function, thus, for $0 < x < 1$ and $d \geqslant 5$, the right-hand side is greater than

$$\sum_{n=0}^{\infty}\left(\frac{1}{(1+n)^2} - \frac{1}{\left(1+n+\frac{1}{2}\right)^2} - \frac{1}{(n+2)^2}\right) = 5 - \frac{\pi^2}{2} > 0. \qquad \square$$

Proof of Claim 2. Differentiating once yields

$$\tilde{h}'_d(x) = \log d + \left(\psi(x) - \psi\left(x+\tfrac{1}{2}\right)\right) - \psi\left(x + \tfrac{1}{2}(d-1)\right),$$

where $\psi = (\log \Gamma)'$ as usual denotes the digamma function. By the well-known inequality

$$\psi(u) \leqslant \log u - \frac{1}{2u}, \quad u > 0$$

(see, e.g., 6.3.21 in [1]), we obtain

$$\tilde{h}'_d(x) \geqslant \log d - \log\left(x + \frac{d-1}{2}\right) + \frac{1}{2x+d-1} - \left(\psi\left(x+\frac{1}{2}\right) - \psi(x)\right).$$

Put $y = \frac{1}{2}(d-1)$ and call the right-hand side $F(x, y)$. Note that, for every fixed $x > 1$,

$$\frac{\partial F}{\partial y}(x, y) = \frac{1}{y+\frac{1}{2}} - \frac{1}{x+y} - \frac{1}{2}\frac{1}{(x+y)^2} > \frac{1}{y+\frac{1}{2}} - \frac{1}{1+y} - \frac{1}{2}\frac{1}{(1+y)^2},$$

which is clearly positive for all $y > 0$. Therefore, for all $1 < x < y$,

$$\tilde{h}'_d(x) \geqslant F(x, y) > F(x, x).$$

It remains to prove that $f(x) = F(x, x) > 0$ for every $x > 1$. We have

$$f(x) = \left(\log\left(1 + \frac{1}{2x}\right) + \frac{1}{4x}\right) - \left(\psi\left(x + \frac{1}{2}\right) - \psi(x)\right).$$

Note that each bracket is a decreasing function in x (for the second one, e.g., by taking the derivative). Thus, crudely, for $1 < x < 1.07$,

$$f(x) > \Big(\log\Big(1+\frac{1}{2\cdot 1.07}\Big)+\frac{1}{4\cdot 1.07}\Big)-\Big(\psi\Big(1+\frac{1}{2}\Big)-\psi(1)\Big) > 0.003.$$

For $x \geqslant 1.07$, using again

$$\psi\Big(x+\frac{1}{2}\Big) \leqslant \log\Big(x+\frac{1}{2}\Big)-\frac{1}{2x+1}$$

as well as

$$\psi(x) \geqslant \log\Big(x+\frac{1}{2}\Big)-\frac{1}{x}$$

(see [15]), we get

$$f(x) \geqslant \log\Big(1+\frac{1}{2x}\Big)+\frac{1}{4x}-\Big(\frac{1}{x}-\frac{1}{2x+1}\Big).$$

It is elementary to verify that the right-hand side is positive for $x \geqslant 1.07$ (it is in fact unimodal, e.g., by analysing its derivative). □

Proof of Claim 3. We have

$$\tilde{h}_d\Big(\frac{1}{2}\Big) = \log\Big(2^{d-2}d^{1-d/2}\Gamma\Big(\frac{d}{2}\Big)\Big).$$

Letting $u = \frac{1}{2}d \geqslant \frac{5}{2}$ and using

$$\Gamma(u) \leqslant \sqrt{2\pi}u^{u-1/2}e^{-u+1/(12u)}, \quad u > 0, \tag{37}$$

(see [23]), we get

$$\tilde{h}_d\Big(\frac{1}{2}\Big) < \log(\sqrt{2\pi}2^{u-1}e^{-u+1/(12u)}u^{1/2}) \leqslant \log\big(\sqrt{2\pi}2^{u-1}e^{-u+1/30}u^{1/2}\big).$$

Denoting the rightmost side by $f(u)$, we see that f is strictly concave. Since $f'\big(\frac{5}{2}\big) < -0.1$, we see that f is decreasing for $u \geqslant \frac{5}{2}$. Thus $f\big(\frac{5}{2}\big) < -0.04$ finishes the argument. □

Remark 35. We have

$$q_d^* = -(d-1)+O(d)\exp\Big(-\frac{1-\log 2}{2}d\Big), \quad d \to \infty. \tag{38}$$

As before, by Claim 1, to show $q_d^* < -(d-1)+2\alpha_d$ for some $\alpha_d > 0$, it suffices to check that $\tilde{h}_d(\alpha_d) < 0$. We have

$$\begin{aligned}\tilde{h}_d(\alpha_d) &= \alpha_d\log d+\log\frac{\Gamma(\alpha_d)}{\Gamma\big(\alpha_d+\frac{1}{2}\big)\Gamma\big(\alpha_d+\frac{d-1}{2}\big)}+\log\frac{2^{d-2}\Gamma\big(\frac{d}{2}\big)^2}{\sqrt{\pi}d^{(d-1)/2}}\\ &= \alpha_d\log d+\log\frac{\Gamma(\alpha_d)\Gamma\big(\frac{d}{2}\big)^2}{\Gamma\big(\alpha_d+\frac{1}{2}\big)\Gamma\big(\alpha_d+\frac{d-1}{2}\big)d^{(d-1)/2}}+d\log 2-\log(4\sqrt{\pi}).\end{aligned}$$

Note that $\Gamma(x) < 1/x$ for $0 < x < 1$ (since $\Gamma(1+x) < 1$). We consider $\alpha_d = Cde^{-cd}$ for positive constants c and C chosen soon. For d large enough, $\alpha_d < \frac{1}{2}$, so $\Gamma\big(\alpha_d+\frac{1}{2}\big) > 1$. Moreover, $\Gamma(\alpha_d) < 1/\alpha_d$,

$$\Gamma\Big(\alpha_d+\frac{d-1}{2}\Big) \geqslant \Gamma\Big(\frac{d-1}{2}\Big) = \frac{2}{d-1}\Gamma\Big(\frac{d+1}{2}\Big) > \frac{2}{d}\Gamma\Big(\frac{d}{2}\Big),$$

as well as $\alpha_d \log d = o(1)$, thus

$$\tilde{h}_d(\alpha_d) \leqslant O(1) + \log \frac{d \cdot \Gamma\left(\frac{d}{2}\right)}{\alpha_d d^{(d-1)/2}} + d \log 2 = O(1) + \log \frac{\Gamma\left(\frac{d}{2}\right)}{Ce^{-cd} d^{(d-1)/2}} + d \log 2.$$

Applying (37) to $\Gamma\left(\frac{1}{2}d\right)$, we obtain

$$\tilde{h}_d(\alpha_d) \leqslant O(1) - \log C + d\left(c + \tfrac{1}{2}\log 2 - \tfrac{1}{2}\right).$$

Choosing $c = \frac{1}{2}(1 - \log 2)$ and C large enough to offset $O(1)$, the right-hand side becomes negative.

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A SUBSTITUTE FOR KAZHDAN'S PROPERTY (T) FOR UNIVERSAL NONLATTICES

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The well-known theorem of Shalom–Vaserstein and Ershov–Jaikin-Zapirain states that the group $\mathrm{EL}_n(\mathcal{R})$, generated by elementary matrices over a finitely generated commutative ring $\mathcal{R}$, has Kazhdan's property (T) as soon as $n \geq 3$. This is no longer true if the ring $\mathcal{R}$ is replaced by a commutative rng (a ring but without the identity) due to nilpotent quotients $\mathrm{EL}_n(\mathcal{R}/\mathcal{R}^k)$. We prove that even in such a case the group $\mathrm{EL}_n(\mathcal{R})$ satisfies a certain property that can substitute property (T), provided that n is large enough.

1. Introduction

We continue and extend the scope of the study of [Kaluba et al. 2019; 2021; Netzer and Thom 2015; Nitsche 2020; Ozawa 2016], which develops the way of proving Kazhdan's property (T) via sum of squares methods. See [Bekka et al. 2008] for a comprehensive treatment of property (T). Let $\Gamma = \langle S \rangle$ be a group together with a finite symmetric generating subset S. We denote by $\mathbb{R}[\Gamma]$ the real group algebra with the involution $*$ that extends the inverse $* : x \mapsto x^{-1}$ on Γ. The positive elements in $\mathbb{R}[\Gamma]$ are sums of (hermitian) squares,

$$\Sigma^2\mathbb{R}[\Gamma] := \left\{ \sum_i \xi_i^* \xi_i : \xi_i \in \mathbb{R}[\Gamma] \right\}$$

and the combinatorial Laplacian is

$$\Delta := \frac{1}{2} \sum_{s \in S} (1-s)^*(1-s) = |S| - \sum_{s \in S} s \in \Sigma^2\mathbb{R}[\Gamma].$$

It is proved in [Ozawa 2016] that the group Γ has property (T) if and only if there is $\varepsilon > 0$ that satisfies

$$\Delta^2 - \varepsilon\Delta \in \Sigma^2\mathbb{R}[\Gamma].$$

Property (T) for the so-called *universal lattice* $\mathrm{EL}_n(\mathbb{Z}[t_1, \dots, t_d])$, $n \geq 3$, is proved in [Shalom 2006; Vaserstein 2006; Ershov and Jaikin-Zapirain 2010]. See also [Mimura 2015] for a simpler proof and [Kassabov and Nikolov 2006; Kaluba et al. 2019] for partial results. All the proofs (save for [Kaluba et al. 2019]) rely on relative property (T) of certain semidirect products. Our interest in this paper is in the infinite index subgroup $\mathrm{EL}_n(\mathbb{Z}\langle t_1, \dots, t_d\rangle)$ of $\mathrm{EL}_n(\mathbb{Z}[t_1, \dots, t_d])$. Here $\mathcal{R} := \mathbb{Z}\langle t_1, \dots, t_d\rangle$ is the commutative *rng* (i.e., a ring, but without assuming the existence of the identity; $\mathcal{R}$ is an ideal in the unitization $\mathcal{R}^1$) of polynomials in $t_1, \dots, t_d$ with zero constant terms and $\mathrm{EL}_n(\mathcal{R}) \subset \mathrm{SL}_n(\mathcal{R}^1)$ denotes

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the group generated by the elementary matrices over the rng $\mathcal{R}$. The elementary matrices are those $e_{i,j}(r) \in \mathrm{SL}_n(\mathcal{R}^1)$ with 1's on the diagonal, $r \in \mathcal{R}$ in the (i, j)-th entry, and zeros everywhere else. The group $\mathrm{EL}_n(\mathcal{R})$ does not have property (T), because it has infinite nilpotent quotients $\mathrm{EL}_n(\mathcal{R}/\mathcal{R}^k)$. The group does not seem to admit a good analogue of relative property (T) phenomenon, either. Still, we prove via sum of squares methods that $\mathrm{EL}_n(\mathcal{R})$ satisfies a property that can substitute property (T).

Main Theorem. *Let $d \in \mathbb{N}$ and consider the commutative rng $\mathcal{R} := \mathbb{Z}\langle t_1, \dots, t_d\rangle$. Then there are $n_0 \in \mathbb{N}$ and $\varepsilon > 0$ such that, for every $n \geq n_0$, the combinatorial Laplacians*

$$\Delta := \sum_{i\neq j} \sum_{r=1}^{d} (1 - e_{i,j}(t_r))^*(1 - e_{i,j}(t_r))$$

for $\mathrm{EL}_n(\mathcal{R})$ and

$$\Delta^{(2)} := \sum_{i\neq j} \sum_{r,s=1}^{d} (1 - e_{i,j}(t_r t_s))^*(1 - e_{i,j}(t_r t_s))$$

for $\mathrm{EL}_n(\mathcal{R}^2)$ satisfy

$$\Delta^2 - n\varepsilon\Delta^{(2)} \in \overline{\Sigma^2\mathbb{R}[\mathrm{EL}_n(\mathcal{R})]}.$$

Here $\overline{\Sigma^2\mathbb{R}[\Gamma]}$ denotes the archimedean closure of $\Sigma^2\mathbb{R}[\Gamma]$ (see Section 2). An upper bound for n_0 in the Main Theorem is in principle explicitly calculable, but we do not attempt to do that (nor attempt to optimize the proof for a better estimate). We conjecture[1] that the Main Theorem holds true with $n_0 = 3$ (in particular n_0 should not depend on d). Our proof is inspired by the work of Kaluba, Kielak and Nowak [Kaluba et al. 2021] that proves property (T) for $\mathrm{Aut}(F_d)$ for $d \geq 5$ via computer calculations and an ingenious idea on stability. Our proof does not rely on computers, but instead on analysis by Boca and Zaharescu [2005] on the almost Mathieu operators in the rotation C*-algebras. In fact, there is no known method of rigorously proving a result like the Main Theorem by computers. This is because the conclusion is *analytic* in nature—the archimedean closure is indispensable. See discussions in Section 6.

The above theorem has a couple of corollaries. The first one is reminiscent of one of the standard definitions of property (T) (see Definition 1.1.3 in [Bekka et al. 2008]).

Corollary A. *For every d, if n is large enough, then for every $\kappa > 0$ there is $\delta > 0$ satisfying the following property. For every orthogonal representation π of $\mathrm{EL}_n(\mathbb{Z}\langle t_1, \dots, t_d\rangle)$ on a Hilbert space $\mathcal{H}$ and every unit vector $v \in \mathcal{H}$ with $\max_{i,j,r} \|v - \pi(e_{i,j}(t_r))v\| \leq \delta$, there is a vector $w \in \mathcal{H}$ such that $\|v - w\| \leq \kappa$ and*

$$\lim_{l\to\infty} \max_{i,j,r} \|w - \pi(e_{i,j}(t_r^l))w\| = 0.$$

We remark that a certain strengthening of the above corollary does not hold. Namely, there is an orthogonal representation π of $\mathrm{EL}_n(\mathbb{Z}\langle t_1, \dots, t_d\rangle)$ that simultaneously admits asymptotically invariant vectors v_k and a sequence $x_l \in \mathrm{EL}_n(\mathbb{Z}\langle t_1^l, \dots, t_d^l\rangle)$ with $\pi(x_l) \to 0$ in the weak operator topology.

Corollary B. *For every d, if n is large enough, then the group $\mathrm{EL}_n(\mathbb{Z}\langle t_1, \dots, t_d\rangle)$ has property (τ) with respect to the finite quotients of the form $\mathrm{EL}_n(\mathcal{S})$, where $\mathcal{S}$ is a finite unital quotients of $\mathbb{Z}\langle t_1, \dots, t_d\rangle$.*

[1] NB: As the author is lame at the computer, no computer experiments have been carried out.

Property (τ) is a generalization of property (T) for finite quotients. See Section 7 for the definition and the proofs of the above corollaries. Corollary B says $\{\mathrm{EL}_n(\mathcal{S}) : \mathcal{S}\}$ forms an expander family with respect to elementary generating subsets of fixed size. The novel point compared to the previously known case of the universal lattice [Kassabov and Nikolov 2006] is that the generating subsets of the finite commutative rings $\mathcal{S}$ need not contain the unit although the $\mathcal{S}$ are assumed unital. For example, for n large enough, the Cayley graphs of $\mathrm{SL}_n(\mathbb{Z}/q\mathbb{Z})$ with respect to the generating subsets $\{e_{i,j}(p) : i \neq j\}$ form an expander family as relatively prime pairs (p, q) vary. The study of the expander property for $\mathrm{SL}_n(\mathbb{Z}/q\mathbb{Z})$ and alike is a very active area. See [Breuillard and Lubotzky 2022; Helfgott 2019; Kowalski 2019] for recent surveys on this.

2. Preliminaries

Let $\Gamma = \langle S \rangle$ be a group together with a finite symmetric generating subset S. We denote by $\mathbb{R}[\Gamma]$ the real group algebra with the involution $*$ which is the linear extension of $x^* := x^{-1}$ on Γ. The identity element of Γ as well as $\mathbb{R}[\Gamma]$ is simply denoted by 1. Recall the positive cone of *sums of* (*hermitian*) *squares* is given by

$$\Sigma^2\mathbb{R}[\Gamma] := \left\{ \sum_i \xi_i^* \xi_i : \xi_i \in \mathbb{R}[\Gamma] \right\} \subset \mathbb{R}[\Gamma]^{\mathrm{her}} := \{\xi \in \mathbb{R}[\Gamma] : \xi = \xi^*\}.$$

The elements in $\Sigma^2\mathbb{R}[\Gamma]$ are considered positive. For $\xi, \eta \in \mathbb{R}[\Gamma]^{\mathrm{her}}$, we write $\xi \preceq \eta$ if $\eta - \xi \in \Sigma^2\mathbb{R}[\Gamma]$. It is obvious that $\xi \succeq 0$ implies $\xi \geq 0$ in the full group C*-algebra $\mathrm{C}^*[\Gamma]$, that is to say, $\pi(\xi)$ is positive selfadjoint for every orthogonal (or unitary) representation π of Γ on a real (or complex) Hilbert space $\mathcal{H}$. The converse is true up to the *archimedean closure*:

$$\overline{\Sigma^2\mathbb{R}[\Gamma]} := \{\xi \in \mathbb{R}[\Gamma] : \text{for all } \varepsilon > 0 \ \xi + \varepsilon \cdot 1 \succeq 0\} = \{\xi \in \mathbb{R}[\Gamma] : \xi \geq 0 \text{ in } \mathrm{C}^*[\Gamma]\}.$$

See, e.g., [Cimprič 2009; Ozawa 2013; Schmüdgen 2009] for this. On this occasion, we recall the basic fact that $0 \preceq \xi \preceq \eta$ (or $0 \leq \xi \leq \eta$) need not imply $0 \leq \xi^2 \leq \eta^2$. Note that since any orthogonal representation of Γ dilates to an orthogonal representation of any supergroup $\Gamma_1 \geq \Gamma$ by induction (i.e., $\mathrm{C}^*[\Gamma] \subset \mathrm{C}^*[\Gamma_1]$ in short), whether $\xi \geq 0$ or not does not depend on the ambient group. The same holds true for $\xi \succeq 0$, by the coset decomposition. The *combinatorial Laplacian*, with respect to the (symmetric) generating subset S,

$$\Delta := \frac{1}{2} \sum_{s \in S} (1 - s)^*(1 - s) = |S| - \sum_{s \in S} s$$

satisfies, for every orthogonal representation $(\pi, \mathcal{H})$ and a vector $v \in \mathcal{H}$,

$$\langle \pi(\Delta)v, v \rangle = \frac{1}{2} \sum_{s \in S} \|v - \pi(s)v\|^2.$$

3. Proof of the Main Theorem, prelude

For any rng $\mathcal{R}$, we denote by $\mathrm{EL}_n(\mathcal{R}) \subset \mathrm{SL}_n(\mathcal{R}^1)$ the group generated by the elementary matrices over the rng $\mathcal{R}$. The elementary matrices are those $e_{i,j}(r) \in \mathrm{SL}_n(\mathcal{R}^1)$ with 1's on the diagonal, $r \in \mathcal{R}$ in the

(i, j)-th entry $(i \neq j)$, and zeros everywhere else. They satisfy the Steinberg relations:

- $e_{i,j}(r)e_{i,j}(s) = e_{i,j}(r+s)$.
- $[e_{i,j}(r), e_{j,k}(s)] = e_{i,k}(rs)$ if $i \neq k$.
- $[e_{i,j}(r), e_{k,l}(s)] = 1$ if $i \neq l$ and $j \neq k$.

We note that every rng homomorphism $\mathcal{R} \to \mathcal{S}$ induces by entrywise operation a group homomorphism $\mathrm{EL}_n(\mathcal{R}) \to \mathrm{EL}_n(\mathcal{S})$ and that $\mathrm{EL}_n(\mathcal{R}/\mathcal{R}^k)$ is nilpotent for every k, where $\mathcal{R}^k := \mathrm{span}\{r_1 \cdots r_k : r_i \in \mathcal{R}\}$. To ease notation, we will write

$$E_{i,j}(r) := (1 - e_{i,j}(r))^*(1 - e_{i,j}(r)) = 2 - e_{i,j}(r) - e_{i,j}(r)^* \in \mathbb{R}[\mathrm{EL}_n(\mathcal{R})].$$

We now consider the case $\mathcal{R} = \mathbb{Z}\langle t_1, \ldots, t_d\rangle$ and start proving the Main Theorem. Recall that the combinatorial Laplacians with respect to the generating subset $\{e_{i,j}(\pm t_r)\}$ are given by

$$\Delta_n := \sum_{i \neq j} \sum_{r=1}^{d} E_{i,j}(t_r) \quad \text{and} \quad \Delta_n^{(2)} := \sum_{i \neq j} \sum_{r,s=1}^{d} E_{i,j}(t_r t_s).$$

We follow the idea of [Kaluba et al. 2021] about the stability with respect to n of the relation like $\Delta_n^{(2)} \ll \Delta_n^2$. Here $\xi \ll \eta$ means that $\xi \leq R\eta$ for some $R > 0$ in the full group C^*-algebra. For each n, put $\mathrm{E}_n := \{\{i, j\} : 1 \leq i, j \leq n,\ i \neq j\}$ and, for $\mathrm{e}, \mathrm{f} \in \mathrm{E}_n$, write $\mathrm{e} \sim \mathrm{f}$ if $|\mathrm{e} \cap \mathrm{f}| = 1$ and $\mathrm{e} \perp \mathrm{f}$ if $\mathrm{e} \cap \mathrm{f} = \varnothing$. One has

$$\Delta_n = \sum_{\mathrm{e} \in \mathrm{E}_n} \Delta_{\mathrm{e}},$$

where $\Delta_{\{i,j\}} := \sum_{r=1}^{d} E_{i,j}(t_r) + E_{j,i}(t_r)$. Thus

$$\Delta_n^2 = \sum_{\mathrm{e}} \Delta_{\mathrm{e}}^2 + \sum_{\mathrm{e} \sim \mathrm{f}} \Delta_{\mathrm{e}} \Delta_{\mathrm{f}} + \sum_{\mathrm{e} \perp \mathrm{f}} \Delta_{\mathrm{e}} \Delta_{\mathrm{f}} =: \mathrm{Sq}_n + \mathrm{Adj}_n + \mathrm{Op}_n\,.$$

The elements Sq_n and Op_n are positive, while Adj_n is not and this causes trouble.

For $m < n$, we view $\mathrm{EL}_m(\mathcal{R})$ as a subgroup of $\mathrm{EL}_n(\mathcal{R})$ sitting at the left upper corner. The symmetric group $\mathrm{Sym}(n)$ acts on $\mathrm{EL}_n(\mathcal{R})$ by permutation of the indices. We note that

$$\begin{aligned} |\mathrm{E}_m| &= \tfrac{1}{2} m(m-1), \\ |\{(\mathrm{e}, \mathrm{f}) \in \mathrm{E}_m^2 : \mathrm{e} \sim \mathrm{f}\}| &= m(m-1)(m-2), \\ |\{(\mathrm{e}, \mathrm{f}) \in \mathrm{E}_m^2 : \mathrm{e} \perp \mathrm{f}\}| &= \tfrac{1}{4} m(m-1)(m-2)(m-3). \end{aligned}$$

Hence, as it is proved in [Kaluba et al. 2021], one has

$$\begin{aligned} \sum_{\sigma \in \mathrm{Sym}(n)} \sigma(\Delta_m^{(2)}) &= m(m-1) \cdot (n-2)! \cdot \Delta_n^{(2)}, \\ \sum_{\sigma \in \mathrm{Sym}(n)} \sigma(\mathrm{Adj}_m) &= m(m-1)(m-2) \cdot (n-3)! \cdot \mathrm{Adj}_n, \\ \sum_{\sigma \in \mathrm{Sym}(n)} \sigma(\mathrm{Op}_m) &= m(m-1)(m-2)(m-3) \cdot (n-4)! \cdot \mathrm{Op}_n\,. \end{aligned}$$

Thus if we know there are $m \in \mathbb{N}$, $R > 0$, and $\varepsilon > 0$ such that

$$\mathrm{Adj}_m + R\,\mathrm{Op}_m \geq \varepsilon \Delta_m^{(2)} \tag{$\heartsuit$}$$

holds true in $\mathrm{C}^*[\mathrm{EL}_m(\mathcal{R})]$, then it follows

$$\frac{n-2}{m-2}\varepsilon \Delta_n^{(2)} \leq \mathrm{Adj}_n + \frac{m-3}{n-3} R\,\mathrm{Op}_n \leq \Delta_n^2$$

for all n such that $R(m-3)/(n-3) \leq 1$ and the Main Theorem is proved. This is Proposition 4.1 in [Kaluba et al. 2021]. To apply this machinery, we further expand Adj_m:

$$\begin{aligned}\mathrm{Adj}_m &= \sum_{r,s} \sum_{i,j,k \text{ distinct}} (E_{i,j}(t_r) + E_{j,i}(t_r))(E_{j,k}(t_s) + E_{k,j}(t_s)) \\ &= \sum_{r,s} \sum_{i,j,k \text{ distinct}} (E_{i,j}(t_r)E_{j,k}(t_s) + E_{j,k}(t_s)E_{i,j}(t_r) + E_{i,j}(t_r)E_{i,k}(t_s) + E_{j,k}(t_s)E_{i,k}(t_r)).\end{aligned}$$

Therefore, if there are $m \in \mathbb{N}$, $R > 0$, $\varepsilon > 0$, and distinct indices i, j, k, l such that

$$E_{i,j}(t_r)E_{j,k}(t_s) + E_{j,k}(t_s)E_{i,j}(t_r) + E_{i,j}(t_r)E_{i,l}(t_s) + E_{j,k}(t_s)E_{l,k}(t_r) + R\,\mathrm{Op}_m \geq \varepsilon E_{i,k}(t_r t_s) \tag{$\diamondsuit$}$$

holds true, then we obtain ($\heartsuit$) (for different $R > 0$ and $\varepsilon > 0$) by summing up this over the $\mathrm{Sym}(m)$-orbit and over r, s. This is what we will prove in the next section.

4. The Heisenberg group and the rotation C*-algebras

In this section, we will work entirely in the C^*-algebra setting. Let's consider the *integral Heisenberg group*

$$\boldsymbol{H} := \left\{ \begin{bmatrix} 1 & a & c \\ & 1 & b \\ & & 1 \end{bmatrix} : a, b, c \in \mathbb{Z} \right\} \cong \langle x, y : z := [x, y] \text{ is central} \rangle,$$

where

$$x = \begin{bmatrix} 1 & 1 & \\ & 1 & \\ & & 1 \end{bmatrix}, \quad y = \begin{bmatrix} 1 & & \\ & 1 & 1 \\ & & 1 \end{bmatrix}, \quad z = \begin{bmatrix} 1 & & 1 \\ & 1 & \\ & & 1 \end{bmatrix}.$$

Note that every irreducible unitary representation of $\boldsymbol{H}$ sends the central element z to a scalar (multiplication operator) of modulus 1. For $\theta \in [0, 1)$, we consider the irreducible unitary representation π_θ of $\boldsymbol{H}$ on $\ell_2(\mathbb{Z})$ or $\ell_2(\mathbb{Z}/q\mathbb{Z})$, depending on whether θ irrational or $\theta = p/q$ is rational with $\gcd(p, q) = 1$, given by

$$\pi_\theta(x)\delta_j = \exp(2j\pi\imath\theta)\delta_j, \quad \pi_\theta(y)\delta_j = \delta_{j+1}, \quad \pi_\theta(z) = \exp(2\pi\imath\theta).$$

By convention, if $\theta = p/q$ is rational, then $\gcd(p, q) = 1$ is assumed, and if θ is irrational, we consider $q = \infty$, and $\mathbb{Z}/q\mathbb{Z}$ means $\mathbb{Z}$. Thus in either case π_θ is a representation on $\ell_2(\mathbb{Z}/q\mathbb{Z})$. The C^*-algebra $\mathcal{A}_\theta := \pi_\theta(\mathrm{C}^*[\boldsymbol{H}])$ is called the *rotation* C^**-algebra*.

We fix the notation used throughout this section. We define

$$X := (1-x)^*(1-x) = 2 - x - x^* \in \mathrm{C}^*[\boldsymbol{H}]_+, \quad X_\theta := \pi_\theta(X) \in \mathcal{A}_\theta,$$

and the same for y and z. Note that $X+Y$ is the combinatorial Laplacian of $\boldsymbol{H}$ with respect to the generating subset $\{x^{\pm}, y^{\pm}\}$, that $0 \le X \le 4$, and that the triplets $(X_\theta, Y_\theta, Z_\theta)$, $(Y_\theta, X_\theta, Z_\theta)$, and $(X_{1-\theta}, Y_{1-\theta}, Z_{1-\theta})$ are unitarily equivalent. For a parameter $\lambda > 0$, the *almost Mathieu operator* on $\ell_2(\mathbb{Z}/q\mathbb{Z})$ is given by

$$H_{\theta,\lambda} := \pi_\theta\Big(\frac{\lambda}{2}(x+x^*)+y+y^*\Big) = (\lambda+2) - \Big(\frac{\lambda}{2}X_\theta + Y_\theta\Big).$$

We also write $s = \sin\pi\theta$, $s_m = \sin 2m\pi\theta$, and $c_m = \cos 2m\pi\theta$. In particular,

$$Z_\theta = 2(1-\cos 2\pi\theta) = 4s^2.$$

See [Boca 2001] for more information about the almost Mathieu operators and [Nitsche 2020] for some discussion in connection with the semidefinite programming.

Eventually, we will prove a certain inequality (Theorem 9) about X, Y, and Z (in the full group C*-algebra of a higher-dimensional Heisenberg group) that leads to ($\diamond$) in the previous section. To prove inequalities about X, Y, and Z, it suffices to work with X_θ, Y_θ, and Z_θ for each $\theta \in [0, \frac{1}{2}]$ separately, thanks to the following well-known fact (Lemma 1). The critical estimate is the one for small $\theta > 0$ (Corollary 4 and Lemma 6). The rest will work out anyway.

Lemma 1. *For any dense subset $I \subset [0, 1)$, the representation $\bigoplus_{\theta\in I}\pi_\theta$ is faithful on the full group* C*-*algebra* $\mathrm{C}^*[\boldsymbol{H}]$.

Proof. For the readers' convenience, we sketch the proof. Let τ_θ denote the tracial state on $\mathrm{C}^*[\boldsymbol{H}]$ associated with π_θ. That is to say, if θ is irrational, then τ_θ arises from the canonical tracial state on the irrational rotation C*-algebra $\mathcal{A}_\theta$ and it is given by $\tau_\theta(x^i y^j) = 0$ for all $(i, j) \ne (0, 0)$. If $\theta = p/q$ is rational, then τ_θ is given by $\mathrm{tr}_q \circ \pi_\theta$, where tr_q is the tracial state on $\mathbb{M}_q(\mathbb{C})$, and it satisfies $\tau_\theta(x^i y^j) = 0$ for all $(i, j) \ne (0, 0)$ in $(\mathbb{Z}/q\mathbb{Z})^2$. It follows that $\theta \mapsto \tau_\theta$ is continuous at irrational points and the assumption of the lemma implies that $\tau := \int_0^1 \tau_\theta\, d\theta$ is a continuous state on $\bigoplus_{\theta\in I}\pi_\theta$. It is not hard to see that τ coincides with the tracial state associated with the left regular representation of $\boldsymbol{H}$, that is to say, $\tau(x^i y^j z^k) = 0$ for all $(i, j, k) \ne (0, 0, 0)$. Since $\boldsymbol{H}$ is amenable, the tracial state τ is faithful on the full group C*-algebra $\mathrm{C}^*[\boldsymbol{H}]$. □

Theorem 2 [Boca and Zaharescu 2005]. *Let $\theta \in [0, \frac{1}{2})$. One has*

$$\|H_{\theta,\lambda}\| \le \lambda + 2 - \frac{2\lambda}{\lambda+2}\sin\pi\theta.$$

More precisely, for any real unit vector ξ in $\ell_2(\mathbb{Z}/q\mathbb{Z})$,

$$\|H_{\lambda,\theta}\xi\|^2 = \lambda^2 + 4 + 2(1-\tan\pi\theta)\Big\langle\frac{\lambda}{2}\pi_\theta(x+x^*)\xi, \pi_\theta(y+y^*)\xi\Big\rangle - \sum_m |\xi_{m-1}-\xi_{m+1}-\lambda s_m\xi_m|^2.$$

Proof. Because the statements are formulated in a different way in [Boca and Zaharescu 2005], we replicate here the proof from that work:

$$\begin{aligned}\|H_{\lambda,\theta}\xi\|^2 &= \sum_m |\lambda c_m\xi_m+\xi_{m-1}+\xi_{m+1}|^2\\ &= \lambda^2+4+\sum_m\big(-\lambda^2 s_m^2\xi_m^2-|\xi_{m-1}-\xi_{m+1}|^2+2\lambda c_m\xi_m(\xi_{m-1}+\xi_{m+1})\big)\\ &= \lambda^2+4-\sum_m|\xi_{m-1}-\xi_{m+1}-\lambda s_m\xi_m|^2-2\lambda\sum_m s_m(\xi_{m-1}-\xi_{m+1})\xi_m+2\lambda\sum_m c_m\xi_m(\xi_{m-1}+\xi_{m+1}).\end{aligned}$$

We continue with the computation,

$$\sum_m c_m \xi_m(\xi_{m-1}+\xi_{m+1}) = \sum_m (c_{m-1}+c_m)\xi_{m-1}\xi_m = 2\cos\pi\theta \sum_m \xi_{m-1}\xi_m \cos(2m-1)\pi\theta$$

and similarly

$$\begin{aligned} -\sum_m s_m(\xi_{m-1}-\xi_{m+1})\xi_m &= \sum_m (s_{m-1}-s_m)\xi_{m-1}\xi_m \\ &= -2\sin\pi\theta \sum_m \xi_{m-1}\xi_m \cos(2m-1)\pi\theta \\ &= -\tan\theta \sum_m c_m\xi_m(\xi_{m-1}+\xi_{m+1}). \end{aligned}$$

Thus one obtains the purported formula for $\|H_{\lambda,\theta}\xi\|^2$. We also observe that

$$\begin{aligned} \|H_{\lambda,\theta}\xi\|^2 &\leq \lambda^2+4+4\lambda(\cos\pi\theta-\sin\pi\theta)\sum_m \xi_{m-1}\xi_m\cos(2m-1)\pi\theta \\ &\leq \lambda^2+4+4\lambda(1-\sin\pi\theta). \end{aligned}$$

This yields the purported estimate for $\|H_{\theta,\lambda}\|$. □

Corollary 3. *In the full group* C*-*algebra* $\mathrm{C}^*[\boldsymbol{H}]$, *one has*

$$X+Y \geq \tfrac{1}{2}\sqrt{Z}.$$

Proof. By Lemma 1, it suffices to show the assertion in $\mathcal{A}_\theta$ for each $\theta \in \left[0, \frac{1}{2}\right]$. It follows from Theorem 2 with $\lambda = 2$ that $X_\theta + Y_\theta = 4 - H_{\theta,2} \geq \frac{1}{2}\sqrt{Z_\theta}$. □

Since Z is central, $X+Y \geq \frac{1}{2}\sqrt{Z}$ is equivalent to $4(X+Y)^2 \geq Z$ in $\mathrm{C}^*[\boldsymbol{H}]$. However, there is no $R>0$ such that $R(X+Y)^2 \succeq Z$ in $\mathbb{R}[\boldsymbol{H}]$. We will elaborate this in Section 6.

Corollary 4. *Let* $R \geq 1,\ 0<\kappa<1$, *and*

$$\theta_0 := \min\left\{\frac{1}{4}, \frac{1}{\pi}\arcsin\left(\kappa\sqrt{\frac{1-\kappa}{R}}\right)\right\}.$$

Then, *for any* $\theta \in [0, \theta_0]$, *one has*

$$RX_\theta + Y_\theta \geq \frac{\sqrt{(1-\kappa)R}}{2}\sqrt{Z_\theta}.$$

Proof. We write

$$s_0 := \sin\pi\theta_0, \quad c := \mathrm{diag}_m\, c_m = \pi_\theta\left(\frac{x+x^*}{2}\right) = 1 - \frac{1}{2}X_\theta, \quad C = \sqrt{(1-\kappa)R}.$$

Let $\theta \in [0,\theta_0]$ and a real unit vector $\xi \in \ell_2(\mathbb{Z}/q\mathbb{Z})$ be given. We need to prove $\langle (RX_\theta+Y_\theta)\xi,\xi\rangle \geq Cs$. For this, we may assume that $\langle \pi_\theta(x+x^*)\xi, \pi_\theta(y+y^*)\xi\rangle > 0$ because otherwise

$$\langle (X_\theta+Y_\theta)\xi,\xi\rangle \geq 4 - \|\pi_\theta(x+x^*+y+y^*)\xi\| \geq 4-2\sqrt{2}.$$

Put $\varepsilon := 1 - \|c\xi\|$. If $\varepsilon \geq Cs/(2R)$, then $\langle RX_\theta\xi, \xi\rangle \geq 2R\varepsilon \geq Cs$ and we are done. From now on, we assume that $\varepsilon < Cs/(2R)$. By Theorem 2 for $\lambda := 2R/C$, one has

$$\begin{aligned}\|H_{\lambda,\theta}\xi\|^2 &\leq \lambda^2 + 4 + 2\lambda(1-s)\langle c\xi, (H_{\theta,\lambda} - \lambda c)\xi\rangle \\ &\leq \lambda^2 + 4 + 2\lambda(1-s)(1-\varepsilon)\|H_{\theta,\lambda}\xi\| - 2\lambda^2(1-s)(1-\varepsilon)^2\end{aligned}$$

and hence

$$\begin{aligned}\big(\|H_{\lambda,\theta}\xi\| - \lambda(1-s)(1-\varepsilon)\big)^2 &\leq 4 + \lambda^2\big(1 - 2(1-s)(1-\varepsilon)^2 + (1-s)^2(1-\varepsilon)^2\big) \\ &= 4 + \lambda^2(1 - (1-s^2)(1-\varepsilon)^2) \\ &\leq 4 + \lambda^2(s^2 + 2\varepsilon).\end{aligned}$$

Thus

$$\|H_{\lambda,\theta}\xi\| \leq 2 + \lambda s\Big(\frac{1}{4}\lambda s_0 + \frac{1}{2}\lambda\frac{\varepsilon}{s}\Big) + \lambda(1-s).$$

By our choices,

$$\lambda s_0 = \frac{2R}{C}\cdot\frac{\kappa\sqrt{(1-\kappa)}}{\sqrt{R}} = 2\kappa$$

and $\lambda\varepsilon/s \leq 1$. Therefore,

$$\|H_{\lambda,\theta}\xi\| \leq \lambda + 2 - \Big(1 - \frac{1}{4}\cdot 2\kappa - \frac{1}{2}\Big)\cdot 2\sqrt{\frac{R}{1-\kappa}}s = \lambda + 2 - Cs.$$

Since $\lambda + 2 - H_{\lambda,\theta} = (\lambda/2)X_\theta + Y_\theta \leq RX_\theta + Y_\theta$, we are done. □

Proposition 5. *In the full group* C^**-algebra* $\mathrm{C}^*[\boldsymbol{H}]$, *one has*

$$(X+Y)\sqrt{Z} + \tfrac{1}{2}(XY + YX) \geq 0.$$

Proof. By Lemma 1, it suffices to show the same for the X_θ. We write $b_m := 1 - c_m = 1 - \cos 2m\pi\theta = 2\sin^2 m\pi\theta$. We observe that

$$X_\theta = \begin{bmatrix}\ddots &&& \\ & 2b_{m-1} && \\ && 2b_m & \\ &&& \ddots\end{bmatrix}, \quad Y_\theta = \begin{bmatrix}\ddots &&& \\ & 2 & -1 & \\ & -1 & 2 & \\ &&& \ddots\end{bmatrix},$$

$$\tfrac{1}{2}(X_\theta Y_\theta + Y_\theta X_\theta) = \begin{bmatrix}\ddots &&& \\ & 4b_{m-1} & -(b_{m-1}+b_m) & \\ & -(b_{m-1}+b_m) & 4b_m & \\ &&& \ddots\end{bmatrix}.$$

These are the sums of the following 2-by-2 matrices sitting at the $(m-1)$-to-m-th corners:

$$X_{\theta,m} = \begin{bmatrix}b_{m-1} & \\ & b_m\end{bmatrix}, \quad Y_{\theta,m} = \begin{bmatrix}1 & -1 \\ -1 & 1\end{bmatrix},$$

$$\tfrac{1}{2}(XY+YX)_{\theta,m} := \begin{bmatrix}2b_{m-1} & -(b_{m-1}+b_m) \\ -(b_{m-1}+b_m) & 2b_m\end{bmatrix}.$$

Thus, it suffices to show

$$\begin{aligned} T_{\theta,m} &:= 2s(X_{\theta,m}+Y_{\theta,m}) + \tfrac{1}{2}(XY+YX)_{\theta,m} \\ &= \begin{bmatrix} 2(s+1)b_{m-1}+2s & -(2s+b_{m-1}+b_m) \\ -(2s+b_{m-1}+b_m) & 2(s+1)b_m+2s \end{bmatrix} \end{aligned}$$

is positive in $\mathbb{M}_2(\mathbb{C})$ for every m. We only need to calculate the determinant:

$$\begin{aligned} \det(T_{\theta,m}) &\geq 4b_{m-1}b_m + 4s(s+1)(b_{m-1}+b_m) + 4s^2 - (2s+b_{m-1}+b_m)^2 \\ &= 4s^2(b_{m-1}+b_m) - (b_{m-1}-b_m)^2 \\ &= 8s^2(\sin^2(m-1)\pi\theta + \sin^2 m\pi\theta) - 4s^2\sin^2(2m-1)\pi\theta \\ &\geq 0. \end{aligned}$$

Here, we have used the formulas

$$\begin{aligned} b_m &= 2\sin^2 m\pi\theta, \\ b_{m-1}-b_m &= -2s\theta\sin(2m-1)\pi\theta, \\ |\sin(2m-1)\pi\theta| &\leq |\sin(m-1)\pi\theta| + |\sin m\pi\theta|. \end{aligned}$$

□

A similar calculation shows $Z+\frac{1}{2}(XY+YX) \geq 0$ in $\mathrm{C}^*[\boldsymbol{H}]$. In fact, it is a sum of squares:

$$Z + \frac{1}{2}(XY+YX) = \frac{1}{4}(X+Y)Z + \frac{1}{8}\sum(1-b)^{\delta}(1-a)^{\varepsilon}(1-a)^{\bar{\varepsilon}}(1-b)^{\bar{\delta}},$$

where $\sum$ is over the eight terms $(a,b) \in \{(x,y),(y,x)\}$ and $(\varepsilon,\bar{\varepsilon}), (\delta,\bar{\delta}) \in \{(*,\ \cdot\ ),(\ \cdot\ ,*)\}$.

Now, we consider the C*-algebra $\mathcal{A}_\theta \otimes \mathcal{A}_\theta$ on $\ell_2(\mathbb{Z}/q\mathbb{Z}) \otimes \ell_2(\mathbb{Z}/q\mathbb{Z})$. We continue to view Z_θ as a scalar in $\mathcal{A}_\theta \otimes \mathcal{A}_\theta$. We want to find an inequality that leads to (◇). The following does the job for small $\theta > 0$. We note that it fails at $\theta_0 = \frac{1}{2}$.

Lemma 6. *There are $\theta_0 > 0$, $R > 1$, and $\varepsilon > 0$ such that, for every $\theta \in [0, \theta_0]$, one has*

$$R(X_\theta \otimes Y_\theta + Y_\theta \otimes X_\theta) + X_\theta \otimes X_\theta + Y_\theta \otimes Y_\theta + (X_\theta Y_\theta + Y_\theta X_\theta) \otimes 1 \geq \varepsilon Z_\theta.$$

Proof. By Corollary 4, there are $\theta_0 > 0$ and $R > 1$ such that $1 \otimes (RX_\theta + Y_\theta) \geq 8s$ for every $\theta \in [0, \theta_0]$. By Proposition 5 and Corollary 3, it follows that the left-hand side dominates

$$(X_\theta + Y_\theta)\cdot 8s + X_\theta Y_\theta + Y_\theta X_\theta \geq (X_\theta + Y_\theta)\cdot 4s \geq Z_\theta,$$

where we omitted writing $\otimes 1$. □

To deal with the case $\theta \geq \theta_0$, we need a few more auxiliary lemmas on $\mathcal{A}_\theta$.

Lemma 7. *For every $\theta \in \left[0, \frac{1}{2}\right]$, one has*

$$\|\pi_\theta((1-x)(1-y))\| \leq 4\cos(\pi\theta/2).$$

Proof. The expansion of $(1-y)^*(1-x)^*(1-x)(1-y)$ has 16 terms (counting multiplicity) and among them are $-(1+z)x$, $-(1+z)^*x^*$ and $x(zy^*+y)+x^*(z^*y^*+y)$. One has $|1+z|=2\cos\pi\theta$ and

$$\|x(zy^*+y)+x^*(z^*y^*+y)\| \le \|[xy^* \;\; x^*y^*]\| \left\| \begin{bmatrix} z+y^2 \\ z^*+y^2 \end{bmatrix} \right\|$$
$$\le \sqrt{2}\|(z+y^2)^*(z+y^2)+(z^*+y^2)^*(z^*+y^2)\|^{1/2} = 4\cos\pi\theta.$$

Hence $\|\pi_\theta((1-y)^*(1-x)^*(1-x)(1-y))\| \le 8+8\cos\pi\theta = 16\cos^2(\pi\theta/2)$. □

For a positive operator A, we denote by $\mathbb{P}_{A\le\delta}$ (resp. $\mathbb{P}_{A>\delta}=1-\mathbb{P}_{A\le\delta}$) the spectral projection of A corresponding to the spectrum $[0,\delta]$ (resp. (δ,∞)). We also write $\mathbb{P}_{A\le\delta\,\wedge\,B\le\delta}$ etc. for the orthogonal projection onto $\operatorname{ran}\mathbb{P}_{A\le\delta}\cap\operatorname{ran}\mathbb{P}_{B\le\delta}$ etc. Note that if A and B commute, then so do their spectral projections and $\mathbb{P}_{A\le\delta\,\wedge\,B\le\delta}=\mathbb{P}_{A\le\delta}\mathbb{P}_{B\le\delta}$.

Lemma 8. *For every* $\theta\in\left(0,\frac{1}{2}\right]$ *and* $0<\delta<2(1-\cos\pi\theta)$, *one has*

$$\mathbb{P}_{X_\theta\le\delta}Y_\theta\mathbb{P}_{X_\theta\le\delta}=2\mathbb{P}_{X_\theta\le\delta},$$

the same with X_θ *and* Y_θ *interchanged*, *and*

$$\|\mathbb{P}_{Y_\theta\le\delta}\mathbb{P}_{X_\theta\le\delta}\| \le \sqrt{\frac{2}{4-\delta}}.$$

In particular, $\ell_2(\mathbb{Z}/q\mathbb{Z})$ *is decomposed into a direct sum*

$$\ell_2(\mathbb{Z}/q\mathbb{Z}) = \operatorname{ran}\mathbb{P}_{X_\theta\le\delta}+\operatorname{ran}\mathbb{P}_{Y_\theta\le\delta}+\operatorname{ran}\mathbb{P}_{X_\theta>\delta\,\wedge\,Y_\theta>\delta}$$

and the corresponding (*not necessarily orthogonal*) *projections have norm at most* $\sqrt{(4-\delta)/(2-\delta)}$.

Proof. We observe that $\mathbb{P}_{X_\theta\le\delta}$ is the projection onto $\ell_2(E)$ with

$$E := \{m : 2(1-\cos 2m\pi\theta)\le\delta\} \subset \{m : m\theta\in(-\theta/2,\theta/2)+\mathbb{Z}\}.$$

The set E does not contain consecutive numbers and the first assertion follows. The second follows from the unitary equivalence of the pairs (X_θ, Y_θ) and (Y_θ, X_θ). Since $Y_\theta\le\delta\mathbb{P}_{Y_\theta\le\delta}+4(1-\mathbb{P}_{Y_\theta\le\delta})=4-(4-\delta)\mathbb{P}_{Y_\theta\le\delta}$, one has

$$2\mathbb{P}_{X_\theta\le\delta}\le 4\mathbb{P}_{X_\theta\le\delta}-(4-\delta)\mathbb{P}_{X_\theta\le\delta}\mathbb{P}_{Y_\theta\le\delta}\mathbb{P}_{X_\theta\le\delta}$$

and $\|\mathbb{P}_{Y_\theta\le\delta}\mathbb{P}_{X_\theta\le\delta}\|^2=\|\mathbb{P}_{X_\theta\le\delta}\mathbb{P}_{Y_\theta\le\delta}\mathbb{P}_{X_\theta\le\delta}\|\le 2/(4-\delta)$. This gives the desired estimate for $\|\mathbb{P}_{Y_\theta\le\delta}\mathbb{P}_{X_\theta\le\delta}\|$. We remark that this estimate can be improved to $\approx 1/\sqrt{3}$ if θ is away from $\frac{1}{2}$ and $\delta>0$ is small enough. Indeed, the gaps of E will have length at least 2 and hence any unit vectors $\xi\in\operatorname{ran}\mathbb{P}_{X_\theta\le\delta}$ and $\eta\in\mathbb{P}_{Y_\theta\le\delta}$ satisfy

$$|\langle\xi,\eta\rangle| \approx \left|\left\langle\xi, \tfrac{1}{3}\pi_\theta(1+y+y^*)\eta\right\rangle\right| = \left|\left\langle\tfrac{1}{3}\pi_\theta(1+y+y^*)\xi, \eta\right\rangle\right| \le \tfrac{1}{\sqrt{3}}.$$

The projection onto the third subspace is orthogonal. On the other hand, any $\xi+\eta\in\operatorname{ran}\mathbb{P}_{X_\theta\le\delta}+\operatorname{ran}\mathbb{P}_{Y_\theta\le\delta}$ satisfies

$$\|\xi+\eta\|^2 \ge \|\xi\|^2+\|\eta\|^2-2\|\mathbb{P}_{Y_\theta\le\delta}\mathbb{P}_{X_\theta\le\delta}\|\,\|\xi\|\,\|\eta\| \ge (1-\|\mathbb{P}_{Y_\theta\le\delta}\mathbb{P}_{X_\theta\le\delta}\|^2)\|\xi\|^2.$$

This gives the desired norm estimate. □

Now, we consider this time the cubic tensor product $\mathcal{A}_\theta \otimes \mathcal{A}_\theta \otimes \mathcal{A}_\theta$. This arises as an irreducible representation of the higher dimensional Heisenberg group

$$\boldsymbol{H}_3 := \left\{ \begin{bmatrix} 1 & * & * & * & * \\ & 1 & 0 & 0 & * \\ & & 1 & 0 & * \\ & & & 1 & * \\ & & & & 1 \end{bmatrix} \right\} \subset \mathrm{SL}(5,\mathbb{Z}).$$

We put $x_i := e_{1,i+1}(1)$, $y_i := e_{i+1,5}(1)$, and $z := e_{1,5}(1)$ in $\boldsymbol{H}_3$, where we recall that $e_{i,j}(1)$ is the elementary matrix defined in the beginning of the previous section. Note that $[x_i, y_i] = z$ and $[x_i, y_j] = 1$ for $i \neq j$. Hence $\boldsymbol{H}_3$ is isomorphic to the quotient of $\boldsymbol{H} \times \boldsymbol{H} \times \boldsymbol{H}$ modulo z are identified. As before, we write $X_i := (1-x_i)^*(1-x_i)$, etc. This should not be confused with X_θ in $\mathcal{A}_\theta$.

Theorem 9. *There are* $R > 0$ *and* $\varepsilon > 0$ *such that*

$$R(X_1Y_2 + Y_1X_2 + X_1Y_3 + Y_1X_3) + X_1X_2 + Y_1Y_2 + X_1Y_1 + Y_1X_1 \geq \varepsilon Z$$

holds in $\mathrm{C}^*[\boldsymbol{H}_3]$.

Proof. By Lemma 1 (adapted to this case), it suffices to prove the assertion in $\mathcal{A}_\theta \otimes \mathcal{A}_\theta \otimes \mathcal{A}_\theta$ for each $\theta \in \left[0, \frac{1}{2}\right]$. We write $X_{i,\theta}$ for X_θ in the i-th tensor component. For a unit vector

$$\zeta \in \ell_2(\mathbb{Z}/q\mathbb{Z}) \otimes \ell_2(\mathbb{Z}/q\mathbb{Z}) \otimes \ell_2(\mathbb{Z}/q\mathbb{Z}),$$

we need to prove

$$\big\langle \big(R(X_{1,\theta}Y_{2,\theta} + Y_{1,\theta}X_{2,\theta} + X_{1,\theta}Y_{3,\theta} + Y_{1,\theta}X_{3,\theta}) + X_{1,\theta}X_{2,\theta} + Y_{1,\theta}Y_{2,\theta} + X_{1,\theta}Y_{1,\theta} + Y_{1,\theta}X_{1,\theta}\big)\zeta, \zeta \big\rangle \geq \varepsilon Z_\theta.$$

By Lemma 6, we are already done for $\theta \in [0, \theta_0]$. To apply Lemma 8, fix $0 < \delta < 2(1 - \cos \pi\theta_0)$ small enough and consider $\theta \in \left[\theta_0, \frac{1}{2}\right]$. Since we may choose $R > 1$ arbitrarily large with respect to the fixed δ, we may assume

$$\max\{\|\mathbb{P}_{X_{1,\theta}Y_{2,\theta} > \delta^2}\zeta\|, \ \|\mathbb{P}_{Y_{1,\theta}X_{2,\theta} > \delta^2}\zeta\|, \ \|\mathbb{P}_{X_{1,\theta}Y_{3,\theta} > \delta^2}\zeta\|, \ \|\mathbb{P}_{Y_{1,\theta}X_{3,\theta} > \delta^2}\zeta\|\} < \delta.$$

As described in Lemma 8, we consider the decomposition

$$\zeta = \xi + \eta + \gamma \in \operatorname{ran} \mathbb{P}_{X_{1,\theta} \leq \delta} + \operatorname{ran} \mathbb{P}_{Y_{1,\theta} \leq \delta} + \operatorname{ran} \mathbb{P}_{X_{1,\theta} > \delta \,\wedge\, Y_{1,\theta} > \delta}.$$

Note that $\max\{\|\xi\|, \|\eta\|, \|\gamma\|\} \leq 2$. By writing $\approx_\delta$, we will mean that the difference is at most δ. Since $\zeta \approx_\delta \mathbb{P}_{X_{1,\theta}Y_{2,\theta} \leq \delta^2}\zeta$ and $\mathbb{P}_{Y_{2,\theta} > \delta \,\wedge\, X_{1,\theta}Y_{2,\theta} \leq \delta^2} \leq \mathbb{P}_{X_{1,\theta} \leq \delta \,\wedge\, Y_{2,\theta} > \delta}$, one has

$$\mathbb{P}_{Y_{2,\theta} > \delta}\zeta \approx_\delta \mathbb{P}_{X_{1,\theta} \leq \delta \,\wedge\, Y_{2,\theta} > \delta}\zeta.$$

It follows that

$$\mathbb{P}_{Y_{2,\theta} > \delta}\eta + \mathbb{P}_{Y_{2,\theta} > \delta}\gamma \approx_\delta \mathbb{P}_{X_{1,\theta} \leq \delta \,\wedge\, Y_{2,\theta} > \delta}(\xi + \eta + \gamma) - \mathbb{P}_{Y_{2,\theta} > \delta}\xi = \mathbb{P}_{X_{1,\theta} \leq \delta \,\wedge\, Y_{2,\theta} > \delta}\eta.$$

Since $\mathbb{P}_{Y_{2,\theta} > \delta}$ leaves $\operatorname{ran} \mathbb{P}_{X_{1,\theta} \leq \delta}$ and $\operatorname{ran} \mathbb{P}_{Y_{1,\theta} \leq \delta}$ invariant, this implies

$$\mathbb{P}_{Y_{2,\theta} > \delta}\eta \approx_\delta \mathbb{P}_{X_{1,\theta} \leq \delta \,\wedge\, Y_{2,\theta} > \delta}\eta \quad \text{and} \quad \mathbb{P}_{Y_{2,\theta} > \delta}\gamma \approx_\delta 0.$$

Hence, in combination with Lemma 8 that $\mathbb{P}_{Y_{1,\theta}\le\delta}\mathbb{P}_{X_{1,\theta}>\delta}\mathbb{P}_{Y_{1,\theta}\le\delta}\ge\frac{1}{4}\mathbb{P}_{Y_{1,\theta}\le\delta}$, one obtains

$$\delta^2\ge\|\mathbb{P}_{X_{1,\theta}>\delta}\mathbb{P}_{Y_{2,\theta}>\delta}\eta\|^2\ge\tfrac{1}{4}\|\mathbb{P}_{Y_{2,\theta}>\delta}\eta\|^2,$$

that is,

$$\eta\approx_{2\delta}\mathbb{P}_{Y_{2,\theta}\le\delta}\eta.$$

The same consideration on $Y_{1,\theta}X_{2,\theta}$ yields

$$\mathbb{P}_{X_{2,\theta}>\delta}\gamma\approx_\delta 0\quad\text{and}\quad\xi\approx_{2\delta}\mathbb{P}_{X_{2,\theta}\le\delta}\xi.$$

Thus $\mathbb{P}_{Y_{2,\theta}>\delta}\mathbb{P}_{X_{2,\theta}\le\delta}\gamma\approx_\delta\mathbb{P}_{Y_{2,\theta}>\delta}\gamma\approx_\delta 0$ and, by Lemma 8 again,

$$\|\gamma\|^2\approx_{\delta^2}\|\mathbb{P}_{X_{2,\theta}\le\delta}\gamma\|^2\le 4\|\mathbb{P}_{Y_{2,\theta}>\delta}\mathbb{P}_{X_{2,\theta}\le\delta}\gamma\|^2\le 16\delta^2.$$

Further, the same for $X_{1,\theta}Y_{3,\theta}$ and $Y_{1,\theta}X_{3,\theta}$ yields

$$\xi\approx_{2\delta}\mathbb{P}_{X_{3,\theta}\le\delta}\xi\quad\text{and}\quad\eta\approx_{2\delta}\mathbb{P}_{Y_{3,\theta}\le\delta}\eta.$$

Now a routine but tedious calculation with Lemma 8 yields

$$\langle X_{1,\theta}X_{2,\theta}\zeta,\zeta\rangle\approx_{C\delta}\langle X_{1,\theta}X_{2,\theta}\mathbb{P}_{Y_{1,\theta}\le\delta\,\wedge\,Y_{2,\theta}\le\delta}\eta,\mathbb{P}_{Y_{1,\theta}\le\delta\,\wedge\,Y_{2,\theta}\le\delta}\eta\rangle\approx_{16\delta}4\|\eta\|^2$$

for some absolute constant C (e.g., $C=1000$ should be enough), and likewise

$$\langle Y_{1,\theta}Y_{2,\theta}\zeta,\zeta\rangle\approx_{C\delta}4\|\xi\|^2.$$

On the other hand, by Lemmas 7 and 8,

$$\begin{aligned}|\langle(X_{1,\theta}Y_{1,\theta}+Y_{1,\theta}X_{1,\theta})\zeta,\zeta\rangle|&\approx_{C\delta}2|\langle X_{1,\theta}Y_{1,\theta}\mathbb{P}_{X_{1,\theta}\le\delta\,\wedge\,X_{2,\theta}\le\delta\,\wedge\,X_{3,\theta}\le\delta}\xi,\mathbb{P}_{Y_{1,\theta}\le\delta\,\wedge\,Y_{2,\theta}\le\delta\,\wedge\,Y_{3,\theta}\le\delta}\eta\rangle|\\&\le 2\|\mathbb{P}_{Y_{1,\theta}\le\delta}\pi_\theta(1-x_1^*)\|\,\|\pi_\theta((1-x_1)(1-y_1))\|\,\|\pi_\theta(1-y_1^*)\mathbb{P}_{X_{1,\theta}\le\delta}\|\\&\qquad\times\|\mathbb{P}_{X_{2,\theta}\le\delta}\mathbb{P}_{Y_{2,\theta}\le\delta}\|\,\|\mathbb{P}_{X_{3,\theta}\le\delta}\mathbb{P}_{Y_{3,\theta}\le\delta}\|\,\|\xi\|\,\|\eta\|\\&\le 16\Big(\cos\frac{\pi\theta}{2}\Big)\cdot\frac{2}{4-\delta}\|\xi\|\,\|\eta\|.\end{aligned}$$

If we have chosen $\delta>0$ small enough, then

$$\varepsilon:=8-16\Big(\cos\frac{\pi\theta_0}{2}\Big)\cdot\frac{2}{4-\delta}>4C\delta.$$

Observe that $\delta>0$ and $\varepsilon>0$ depends on the absolute constants $\theta_0>0$ and $C>0$, but not on $\theta\in\big[\theta_0,\frac{1}{2}\big]$. In the end,

$$\begin{aligned}|\langle(X_{1,\theta}Y_{1,\theta}+X_{1,\theta}Y_{1,\theta})\zeta,\zeta\rangle|&\le(8-\varepsilon)\|\xi\|\eta\|+C\delta\\&\le 4(1-\varepsilon/2)(\|\xi\|^2+\|\eta\|^2)+C\delta\\&\le\langle(X_{1,\theta}X_{2,\theta}+Y_{1,\theta}Y_{2,\theta})\zeta,\zeta\rangle-\varepsilon+3C\delta.\end{aligned}$$

This completes the proof. We remark that the above proof for $\theta\in\big[\theta_0,\frac{1}{2}\big]$ is not as tight as it appears (and $\varepsilon>0$ can be "visible"), because if θ is around $\frac{1}{2}$, then $\cos\frac{1}{2}\pi\theta\approx\frac{1}{\sqrt{2}}$, and if θ is away from $\frac{1}{2}$, then $\|\mathbb{P}_{X_\theta\le\delta}\mathbb{P}_{Y_\theta\le\delta}\|$ is bounded by $\approx\frac{1}{\sqrt{3}}$. □

5. Proof of the Main Theorem, postlude

Since $\mathcal{R} := \mathbb{Z}\langle t_1, \dots, t_d\rangle$ is *commutative*, we may apply Theorem 9 to $x_1 = e_{1,2}(t_r)$, $x_2 = e_{1,3}(t_s)$, $x_3 = e_{1,4}(t_r)$, $y_1 = e_{2,5}(t_s)$, $y_2 = e_{3,5}(t_r)$, $y_3 = e_{4,5}(t_s)$, and $z = e_{1,5}(t_r t_s)$ in $\mathrm{EL}_5(\mathcal{R})$. This yields ($\diamondsuit$) in Section 3 and the proof of the Main Theorem is complete. □

The terms $X_1 Y_2 = E_{1,2}(t_r)E_{3,5}(t_r)$ and $Y_1 X_2 = E_{2,5}(t_s)E_{1,3}(t_s)$ are diagonal with respect to $\{t_r, t_s\}$. This causes an annoying dependence of R on d in the formula ($\heartsuit$), which results in dependence of n_0 on d in the Main Theorem.

6. Real group algebras and property $\mathrm{H_T}$

In this section, we continue the study of [Netzer and Thom 2013; 2015; Nitsche 2020; Ozawa 2013; 2016] about positivity in real group algebras. In addition to the notation from Section 2, we denote by

$$I[\Gamma] := \mathrm{span}\{1 - x : x \in \Gamma\} \subset \mathbb{R}[\Gamma]$$

the *augmentation ideal*. We observe that $\Sigma^2 I[\Gamma] = I[\Gamma] \cap \Sigma^2\mathbb{R}[\Gamma]$ and hence there is no ambiguity about the order $\preceq$ on $I[\Gamma]$. In [Ozawa 2016], it was observed that the combinatorial Laplacian $\Delta \in \Sigma^2 I[\Gamma]$ is an *order unit* for $I[\Gamma]$ (more precisely for $I[\Gamma]^{\mathrm{her}}$, but this abuse of terminology should not cause any problem). That is to say, for every $\xi \in I[\Gamma]^{\mathrm{her}}$, there is $R > 0$ such that $\xi \preceq R\Delta$. We will indicate this by $\xi \lll \Delta$.

We review the relation between positive linear functionals on $I[\Gamma]$ and 1-*cocycles* (with unitary coefficients). A linear functional φ on $I[\Gamma]$ is said to be *positive* if it is selfadjoint and $\varphi(\Sigma^2 I[\Gamma]) \subset \mathbb{R}_{\geq 0}$. One has $\varphi(\Delta) = 0$ if and only if $\varphi = 0$. Every positive linear functional φ gives rise to a semi-inner product $\langle \xi, \eta\rangle := \varphi(\xi^*\eta)$ and the corresponding seminorm $\|\xi\| := \varphi(\xi^*\xi)^{1/2}$ on $I[\Gamma]$, with respect to which the left multiplication by an element of Γ is orthogonal. This is the Gelfand–Naimark construction. The map $b : \Gamma \to I[\Gamma]$, $t \mapsto 1 - t$, is a 1-cocycle, i.e., it satisfies $b(st) = b(s) + sb(t)$ for every $s, t \in \Gamma$. We note that $\varphi(1 - t) = \frac{1}{2}\varphi((1 - t)^*(1 - t)) = \frac{1}{2}\|b(t)\|^2$ and $\varphi(\Delta) = \frac{1}{2}\sum_{s\in S}\|b(s)\|^2$. In fact, every 1-cocycle arises in this way. See, e.g., Appendix C in [Bekka et al. 2008] and Appendix D in [Brown and Ozawa 2008] for a comprehensive treatment.

It is proved in [Ozawa 2016] that $\overline{\Sigma^2 I[\Gamma]} = I[\Gamma]^{\mathrm{her}} \cap \overline{\Sigma^2\mathbb{R}[\Gamma]}$. That is to say,

$$\begin{aligned}\overline{\Sigma^2 I[\Gamma]} &:= \{\xi \in I[\Gamma]^{\mathrm{her}} : \text{for all } \varepsilon > 0,\ \xi + \varepsilon\Delta \succeq 0\}\\ &= \{\xi \in I[\Gamma]^{\mathrm{her}} : \varphi(\xi) \geq 0 \text{ for every positive linear functional } \varphi \text{ on } I[\Gamma]\}\\ &= \{\xi \in I[\Gamma]^{\mathrm{her}} : \xi \geq 0 \text{ in } \mathrm{C}^*[\Gamma]\}.\end{aligned}$$

We also record an easy consequence of the Hahn–Banach separation theorem (a.k.a. the Eidelheit–Kakutani separation theorem in this context). For $\xi, \eta \in I[\Gamma]^{\mathrm{her}}$ (or in any real ordered vector space with an order unit Δ), the following are equivalent:

(1) $\varphi(\xi) = 0$ implies $\varphi(\eta) \leq 0$ for every positive linear functional φ on $I[\Gamma]$.

(2) $-\eta \in \overline{\Sigma^2 I[\Gamma] - \mathbb{R}\xi}$.

(3) For all $\varepsilon > 0$, there exists $R \in \mathbb{R}$ such that $R\xi - \eta + \varepsilon\Delta \succeq 0$.

We observe that since

$$\varphi(\Delta^2) = \langle \Delta, \Delta \rangle = \Big\| \sum_{s \in S} b(s) \Big\|^2,$$

one has $\varphi(\Delta^2) = 0$ if and only if the corresponding 1-cocycle b is *harmonic* in the sense $\sum_{s \in S} b(s) = 0$. This observation recovers Shalom's theorem [2000] that every finitely generated group without property (T) has a nonzero harmonic 1-cocycle. An essentially same proof was given in [Nitsche 2020].

We record the following well-known fact:

- If a 1-cocycle b vanishes on a normal subgroup $N \lhd \Gamma$, then N acts trivially on $\operatorname{span} b(\Gamma)$ and hence b factors through the quotient Γ/N.
- If b is a harmonic 1-cocycle on Γ, then the center $\mathcal{Z}(\Gamma)$ acts trivially on $\operatorname{span} b(\Gamma)$ and Γ acts trivially on $\operatorname{span} b(\mathcal{Z}(\Gamma))$.
- Every harmonic 1-cocycle on an abelian group is an additive homomorphism.

The first assertion is not difficult to show. The second follows from the identity $(1-x)b(z) = (1-z)b(x)$ for $x \in \Gamma$ and $z \in \mathcal{Z}(\Gamma)$. If b is harmonic, then $\big(|S| - \sum_{s \in S} s\big) b(z) = 0$ and, by strict convexity of a Hilbert space, $b(z) = sb(z)$ for $s \in S$ and hence for all $s \in \Gamma$.

An additive character $\chi : \Gamma \to \mathbb{R}$ can be viewed as a harmonic 1-cocycle. The corresponding positive linear functional $\varphi_\chi : I[\Gamma] \to \mathbb{R}$ is given by $\varphi_\chi(1-t) = \frac{1}{2}\chi(t)^2$. This should not be confused with the linear extension $\chi : I[\Gamma] \to \mathbb{R}$ which is not even selfadjoint. The positive linear functional φ_χ factors through the abelianization $I[\Gamma^{\mathrm{ab}}]$.

We denote the *augmentation power* by

$$I^k[\Gamma] := \operatorname{span}(I[\Gamma]^k) \subset \mathbb{R}[\Gamma].$$

It is well-known and easy to see from the formula

$$1 - xy = (1-x) + (1-y) - (1-x)(1-y) \in (1-x) + (1-y) + I^2[\Gamma]$$

that $I[\Gamma]$ is generated as a rng by $\{1 - s : s \in S\}$ and that $\Gamma \ni x \mapsto 1 - x \in I[\Gamma]/I^2[\Gamma]$ is an additive homomorphism. On the other hand, every additive homomorphism χ vanishes on $I^2[\Gamma]$, because $\chi((1-x)(1-y)) = \chi(1 - x - y + xy) = 0$. Hence $I^2[\Gamma] = \bigcap_\chi \ker \chi$, where the intersection is taken over the additive characters χ on Γ. We will see that $\Delta^2 \in \Sigma^2 I^2[\Gamma]$ need not be an order unit for $I^4[\Gamma]$, but the element

$$\square := \frac{1}{4} \sum_{s,t \in S} (1-s)^*(1-t)^*(1-t)(1-s) \in \Sigma^2 I^2[\Gamma]$$

is. Since $\square = \Delta^2$ in $I[\Gamma^{\mathrm{ab}}]$, one has $\varphi_\chi(\square) = \varphi_\chi(\Delta^2) = 0$ for every additive character χ. We will prove later that the converse is also true.

Theorem 10. *The element $\square$ is an order unit for $I^4[\Gamma]$. Namely*

$$I^4[\Gamma]^{\mathrm{her}} = \{\xi \in \mathbb{R}[\Gamma]^{\mathrm{her}} : \pm\xi \preccurlyeq \square\} = \operatorname{span} \Sigma^2 I^2[\Gamma]$$

and moreover $I^4[\Gamma] \cap \Sigma^2 \mathbb{R}[\Gamma] = \Sigma^2 I^2[\Gamma]$.

Proof. We first prove that the left is contained the middle. The proof is similar to that for Lemma 2 in [Ozawa 2016]. Since $\xi^*\eta+\eta^*\xi \preceq \xi^*\xi+\eta^*\eta$ for every ξ,η, it suffices to show that

$$(1-x)^*(1-y)^*(1-y)(1-x) \lll \square \quad \text{for all } x, y \in \Gamma.$$

By using the inequality

$$\begin{aligned}(1-x_1x_2)^*(1-y)^*(1-y)(1-x_1x_2) &= ((1-x_1)+x_1(1-x_2))^*(1-y)^*(\text{——})\\ &\preceq 2(1-x_1)^*(1-y)^*(\text{——})+2(1-x_2)^*(1-x_1^{-1}yx_1)^*(\text{——}),\end{aligned}$$

one can reduce this to the case $x\in S$, and similarly to the case $y\in S$, where the assertion is obvious. We next show that $\pm\xi \lll \square$ implies $\xi\in\operatorname{span}\Sigma^2 I^2[\Gamma]$. There is $R>0$ such that $0\preceq R\square-\xi\preceq 2R\square$. Thus it remains to show $\sum_i \eta_i^*\eta_i \lll \square$ implies $\eta_i\in I^2[\Gamma]$. Since $\varphi_\chi(\square)=0$ for every additive character χ on Γ, one has

$$0=\varphi_\chi\Big(\sum_i \eta_i^*\eta_i\Big) = -\frac{1}{2}\sum_{i,x,y}\eta_i(x)\eta_i(y)\chi(x^{-1}y)^2 = \sum_i\Big(\sum_x \eta_i(x)\chi(x)\Big)^2,$$

or equivalently $\eta_i\in\bigcap_\chi \ker\chi = I^2[\Gamma]$ for all i. □

Corollary 11. *A positive linear functional φ on $I[\Gamma]$ satisfies $\varphi(\square)=0$ if and only if the associated 1-cocycle is an additive homomorphism.*

Proof. We have already noted that $\varphi_\chi(\square)=0$ for all additive character χ. Conversely, suppose $\varphi(\square)=0$. Since this implies $\varphi(\Delta^2)=0$, the 1-cocycle b associated with φ is harmonic. Moreover, since

$$1-[x,y]=(xy-yx)x^{-1}y^{-1}=((1-x)(1-y)-(1-y)(1-x))x^{-1}y^{-1}\in I^2[\Gamma],$$

Theorem 10 implies that $b=0$ on the commutator subgroup $[\Gamma,\Gamma]$. Thus b factors through Γ^{ab} and is an additive homomorphism. □

We recall that a finitely generated group Γ is said to have *Shalom's property* $\mathrm{H_T}$ if every harmonic 1-cocycle on Γ is an additive homomorphism. Property $\mathrm{H_T}$ coincides with Kazhdan's property (T) for groups with finite abelianization. It is observed in [Shalom 2004] that finitely generated nilpotent groups have property $\mathrm{H_T}$. We conjecture that the group $\mathrm{EL}_n(\mathbb{Z}\langle t_1,\dots,t_d\rangle)$ has property $\mathrm{H_T}$. By the Hahn–Banach separation theorem, one obtains the following characterization of property $\mathrm{H_T}$, which does not seem useful though.

Corollary 12. *The finitely generated group Γ has finite abelianization if and only if $\Delta \lll \square$. The finitely generated group Γ has property $\mathrm{H_T}$ if and only if for every $\varepsilon>0$ there is $R>0$ such that $\square\preceq R\Delta^2+\varepsilon\Delta$.*

Property $\mathrm{H_T}$ for nilpotent groups also follows from Corollary 3 that if a commutator $z=[x,y]$ is central, then $(1-z)^*(1-z)\lll\Delta^2$ in $\mathrm{C}^*[\Gamma]$. It is tempting to conjecture that every finitely generated nilpotent group Γ satisfies $\square\lll\Delta^2$. Had it been true that $\square\lll\Delta^2$ for a given group Γ, it would have been able to rigorously prove this by computer calculations because $\square$ is an order unit for $I^4[\Gamma]$ (modulo a quantitative estimate, see [Netzer and Thom 2015]). However, we will observe here that $\square \not\lll \Delta^2$

in $\mathbb{R}[\boldsymbol{H}]$. Hence, unlike property (T), property $\mathrm{H_T}$ is probably not characterized by a "simple"[2] inequality in the real group algebra. This spoils the current methods of proving something like the Main Theorem by computer calculations. (Note that $\mathrm{EL}_n(\mathbb{Z}\langle t\rangle)$ has the Heisenberg group $\boldsymbol{H}_{n-2}$ as a quotient and the analogous statement to the following proposition holds true for this group.)

Proposition 13. *Let $\boldsymbol{H}$ be the integral Heisenberg group and $z := [x, y]$ be as described in the beginning of Section 4. Then $(1-z)^*(1-z) \not\preccurlyeq \Delta^2$ in $\mathbb{R}[\boldsymbol{H}]$. Moreover,*

$$\overline{\Sigma^2 I^2[\boldsymbol{H}]} \neq I^4[\boldsymbol{H}]^{\mathrm{her}} \cap \overline{\Sigma^2 \mathbb{R}[\boldsymbol{H}]}.$$

The proof of $\overline{\Sigma^2 I[\Gamma]} = I[\Gamma]^{\mathrm{her}} \cap \overline{\Sigma^2 \mathbb{R}[\Gamma]}$ given in [Ozawa 2016] is based on Schoenberg's theorem that any positive linear functional on $I[\Gamma]$ is approximable by those that extend on $\mathbb{R}[\Gamma]$. The above proposition says there is no good enough analogue of Schoenberg's theorem for augmentation powers. For the proof of the proposition, we need a description of the graded vector space $\cdots \supset I^4[\boldsymbol{H}] \supset I^5[\boldsymbol{H}] \supset \cdots$. To ease notation, we write $\bar{x} := 1 - x$ etc. and observe that $\bar{z} \in \mathcal{Z}(\mathbb{R}[\boldsymbol{H}]) \cap I^2[\boldsymbol{H}]$ and

$$\bar{y}\bar{x} = \bar{x}\bar{y} + \bar{z} - \bar{z}\bar{x} - \bar{z}\bar{y} + \bar{z}\bar{y}\bar{x} \in \bar{x}\bar{y} + \bar{z} + I^3[\boldsymbol{H}].$$

Lemma 14. *For every $n \in \mathbb{N}$, the set $\{\bar{x}^i \bar{y}^j \bar{z}^k + I^n[\boldsymbol{H}] : i, j, k \geq 0,\ i + j + 2k < n\}$ forms a basis for $\mathbb{R}[\boldsymbol{H}]/I^n[\boldsymbol{H}]$. In particular*

$$\dim I^n[\boldsymbol{H}]/I^{n+1}[\boldsymbol{H}] = (\lfloor n/2 \rfloor + 1)(n - \lfloor n/2 \rfloor + 1).$$

Proof. We first observe that the asserted set spans $\mathbb{R}[\boldsymbol{H}]/I^n[\boldsymbol{H}]$. Indeed, this follows from the above equation for $\bar{y}\bar{x}$ and the general facts that

$$1 - uv = (1-u) + (1-v) - (1-u)(1-v),$$
$$1 - u^{-1} = -(1-u) + (1-u^{-1})(1-u)$$

for every $u, v \in \boldsymbol{H}$. It is left to show that the asserted set is also linearly independent. Suppose that

$$\xi := \sum_{i+j+2k<n} \alpha_{i,j,k} \bar{x}^i \bar{y}^j \bar{z}^k \in I^n[\boldsymbol{H}].$$

By considering the abelianization $\pi^{\mathrm{ab}} : \mathrm{C}^*[\boldsymbol{H}] \to \mathrm{C}^*[\mathbb{Z}^2]$, one sees $\alpha_{i,j,k} = 0$ whenever $k = 0$. It follows that $\xi \in I^n[\boldsymbol{H}] \cap \bar{z}\mathbb{R}[\boldsymbol{H}]$. We claim that

$$I^n[\boldsymbol{H}] \cap \bar{z}\mathbb{R}[\boldsymbol{H}] = \bar{z} I^{n-2}[\boldsymbol{H}] \quad \text{for } n \geq 2.$$

Since $\bar{z}$ is not a zero divisor in $\mathbb{R}[\boldsymbol{H}]$ (e.g., because $\pi_\theta(\bar{z})$ are invertible for $\theta \in (0, 1)$), the lemma would follow from this claim by induction.

The homomorphisms $\mathbb{R}[\langle x\rangle] \hookrightarrow \mathbb{R}[\boldsymbol{H}]$ and $\mathbb{R}[\langle y\rangle] \hookrightarrow \mathbb{R}[\boldsymbol{H}]$ extend to a linear injection

$$\sigma : \mathbb{R}[\langle x\rangle] \otimes \mathbb{R}[\langle y\rangle] \hookrightarrow \mathbb{R}[\boldsymbol{H}], \quad \xi \otimes \eta \mapsto \xi\eta,$$

with the left inverse

$$\pi^{\mathrm{ab}} : \mathbb{R}[\boldsymbol{H}] \to \mathbb{R}[\mathbb{Z}^2] \cong \mathbb{R}[\langle x\rangle] \otimes \mathbb{R}[\langle y\rangle].$$

[2] The quantifier elimination techniques, which the author is not familiar with, may be relevant.

Since $\bar{y}\bar{x} \in \bar{x}\bar{y} + \bar{z}\mathbb{R}[\boldsymbol{H}]$ and likewise for $\bar{x}^*$ and $\bar{y}^*$ (thanks to suitable symmetries $x \leftrightarrow x^{-1}$ and $y \leftrightarrow y^{-1}$ on $\boldsymbol{H}$), one has

$$I^n[\boldsymbol{H}] \cap \bar{z}\mathbb{R}[\boldsymbol{H}] \subset (\operatorname{ran}\sigma + \bar{z}I^{n-2}[\boldsymbol{H}]) \cap \ker \pi^{\mathrm{ab}} = \bar{z}I^{n-2}[\boldsymbol{H}].$$

This proves the claim. □

Proof of Proposition 13. We observe that in $I^4[\boldsymbol{H}]/I^5[\boldsymbol{H}]$

$$(\bar{x}\bar{x}\bar{y}\bar{y})^* = \bar{y}\bar{y}\bar{x}\bar{x} = \bar{y}\bar{x}\bar{y}\bar{x} + \bar{y}\bar{x}\bar{z} = \bar{x}\bar{y}\bar{x}\bar{y} + 3\bar{x}\bar{y}\bar{z} + 2\bar{z}\bar{z} = \bar{x}\bar{x}\bar{y}\bar{y} + 4\bar{x}\bar{y}\bar{z} + 2\bar{z}\bar{z}.$$

We define a linear functional φ on $I^4[\boldsymbol{H}]/I^5[\boldsymbol{H}]$ by

$$\varphi(\bar{x}^4) = \varphi(\bar{y}^4) = 1, \quad \varphi(\bar{z}^2) = -2, \quad \varphi(\bar{x}^2\bar{y}^2) = -1, \quad \varphi(\bar{x}\bar{y}\bar{z}) = 1,$$

and zero on all the other basis elements. Then, the linear functional φ is selfadjoint. Moreover, with respect to the basis $\{\bar{x}\bar{x}, \bar{x}\bar{y}, \bar{y}\bar{x}, \bar{y}\bar{y}\}$ for $I^2[\boldsymbol{H}]/I^3[\boldsymbol{H}]$, the bilinear form $(\xi, \eta) \mapsto \varphi(\xi^*\eta)$ is represented by the matrix

$$\begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -1 & 0 & 0 & 1 \end{bmatrix}.$$

Since this matrix is positive semidefinite, the linear functional is positive on $I^4[\boldsymbol{H}]$, by Theorem 10. One sees that $\varphi(\bar{z}^*\bar{z}) = -\varphi(\bar{z}\bar{z}) = 2 > 0$, $\varphi(\Box) = 4$, and

$$\varphi(\Delta^2) = \varphi((\bar{x}\bar{x} + \bar{y}\bar{y})(\bar{x}\bar{x} + \bar{y}\bar{y})) = 0.$$

Therefore there cannot be $R > 0$ such that $\bar{z}^*\bar{z} \preceq R\Delta^2 + \frac{1}{4}\Box$. It follows that $4\Delta^2 - \bar{z}^*\bar{z} \notin \overline{\Sigma^2 I^2[\boldsymbol{H}]}$, while $4\Delta^2 - \bar{z}^*\bar{z} \in I^4[\boldsymbol{H}]^{\mathrm{her}} \cap \overline{\Sigma^2\mathbb{R}[\boldsymbol{H}]}$ by Corollary 3. □

7. Property (τ)

We say a finitely generated group $\Gamma = \langle S \rangle$ has *property* (τ) with respect to a family $\{\Gamma_i\}$ of finite quotients $\Gamma \twoheadrightarrow \Gamma_i$ if there is $\delta > 0$ such that any unitary representation π of Γ that factors through some $\Gamma \twoheadrightarrow \Gamma_i$ either admits a nonzero $\pi(\Gamma)$-invariant vector or admits no unit vector v such that $\max_{s \in S} \|v - \pi(s)v\| \leq \delta$. This is equivalent to that the Cayley graphs of $\{\Gamma_i\}$ with respect to the generating subset S form an expander family. In case the family $\{\Gamma_i\}_i$ is the set of all finite quotients of Γ, it is simply said Γ has property (τ). See [Kowalski 2019] for a comprehensive treatment of expander graphs. By the Main Theorem, $\mathrm{EL}_n(\mathcal{S})$ has property (T) if $\mathcal{S}$ is a finitely generated *irng* (i.e., a rng which is idempotent, $\mathcal{S} = \mathcal{S}^2$, see [Monod et al. 2012]) and n is large enough. Corollaries A and B say this happens uniformly for *finite* commutative irngs with a fixed number of generators.

Proof of Corollary A. Let n_0 be as in the Main Theorem for $\mathbb{Z}\langle T_1, \ldots, T_d, S_1, \ldots, S_d \rangle$ and $n \geq n_0$. By the Main Theorem applied to $T_r \mapsto t_r^k$ and $S_r \mapsto t_r^{k+1}$, there is $\varepsilon > 0$ such that

$$\Delta_k := \sum_{i \neq j} \sum_{r=1}^{d} (1 - e_{i,j}(t_r^k))^*(1 - e_{i,j}(t_r^k)) \in \mathbb{R}[\mathrm{EL}_n(\mathbb{Z}\langle t_1, \ldots, t_d \rangle)]$$

(so $\Delta_1 = \Delta$) satisfy

$$(\Delta_k + \Delta_{k+1})^2 \geq \varepsilon(\Delta_{2k} + \Delta_{2k+1} + \Delta_{2k+2})$$

for all k. We may also assume that $\varepsilon > 0$ satisfies $\Delta_1^2 \geq \varepsilon \Delta_2$.

Let π, $\mathcal{H}$ and v be given for $\mathrm{EL}_n(\mathbb{Z}\langle t_1, \dots, t_d\rangle)$ (but we will omit writing π to ease notation) and put

$$\delta := \left(\sum_{i,j,r} \|v - e_{i,j}(t_r)v\|^2\right)^{1/2} = \langle \Delta v, v\rangle^{1/2}.$$

We assume $\delta < \left(\frac{1}{2}\right)^{10}$ and put $\rho := \delta^{1/10}$. Recall that $\mathbb{P}_{\Delta \leq (\delta/\rho)^2}$ stands for the spectral projection of Δ for the interval $[0, (\delta/\rho)^2]$. For $v_0 := \mathbb{P}_{\Delta \leq (\delta/\rho)^2} v$, one has $\|v - v_0\| \leq \rho$ and

$$\langle (\Delta_1 + \Delta_2)v_0, v_0\rangle \leq \delta^2 + \varepsilon^{-1}(\delta/\rho)^4 =: \delta_0^2.$$

Now, $v_1 := \mathbb{P}_{\Delta_1 + \Delta_2 \leq (\delta_0/\rho^2)^2} v_0$ satisfies $\|v_0 - v_1\| \leq \rho^2$ and

$$\langle (\Delta_2 + \Delta_3)v_1, v_1\rangle \leq \varepsilon^{-1}(\delta_0/\rho^2)^4 =: \delta_1^2.$$

We continue this and obtain $v_2 := \mathbb{P}_{\Delta_2 + \Delta_3 \leq (\delta_1/\rho^3)^2} v_1, \dots$ such that $\|v_k - v_{k+1}\| \leq \rho^{k+2}$ and

$$\langle (\Delta_{2^k} + \Delta_{2^k+1})v_k, v_k\rangle \leq \varepsilon^{-1}(\delta_{k-1}/\rho^{k+1})^4 =: \delta_k^2.$$

Then the vector $w := \lim_k v_k$ satisfies $\|v_k - w\| \leq \rho^{k+1}$ $\left(\text{as } \rho < \frac{1}{2}\right)$. Moreover,

$$\begin{aligned} 2^{-k}|\log \delta_k| &= 2^{-(k-1)}|\log \delta_{k-1}| - 2^{-(k-1)}(k+1)|\log \rho| + 2^{-(k+1)} \log \varepsilon \\ &= |\log \delta_0| - \left(\sum_{m=1}^{k} 2^{-(m-1)}(m+1)\right)|\log \rho| + \tfrac{1}{2}(1 - 2^{-k}) \log \varepsilon \\ &> \tfrac{1}{10}|\log \delta| \end{aligned}$$

if $\delta > 0$ is small enough compared to $\varepsilon > 0$. Hence $\delta_k \to 0$ at a double exponential rate.

We need to show $\lim_l \max_{i,j,r} \|w - e_{i,j}(t_r^l)w\| = 0$. We first observe that

$$\|w - e_{i,j}(t_r^{2^k})w\| \leq 2\|v_k - w\| + \delta_k \leq \rho^k + \delta_k.$$

Let l be given. Take $k = k(l)$ such that $l \in [2^k, 2^{k+1})$ and write $l = 2^k + \sum_{m=0}^{k-1} a(m)2^m$ with $a(m) \in \{0, 1\}$. Then for $b := \sum_{m=0}^{\lfloor k/2\rfloor - 1} a(m)2^m$, one has

$$\|e_{i,j}(t_r^l)w - e_{i,j}(t_r^{2^k+b})w\| \leq \sum_{m=\lfloor k/2\rfloor}^{k-1} a(m)(\rho^m + \delta_m),$$

which tends to 0 as $l \to \infty$. Observe that the recurrence relation

$$p_0 := 2^{k - \lfloor k/2\rfloor}, \quad p_{m+1} := 2p_m + a(\lfloor k/2\rfloor - 1 - m)$$

gives $p_{\lfloor k/2\rfloor} = 2^k + b$. Now by arguing as in the previous paragraph, but starting at $v_{k-\lfloor k/2\rfloor}$ and using $(\Delta_{p_m} + \Delta_{p_m+1})^2 \geq \varepsilon(\Delta_{p_{m+1}} + \Delta_{p_{m+1}+1})$, one obtains

$$\|v_{k-\lfloor k/2\rfloor} - e_{i,j}(t_r^{2^k+b})v_{k-\lfloor k/2\rfloor}\| \leq \rho^{k-\lfloor k/2\rfloor} + \delta_k \to 0.$$

Since $\|v_{k-\lfloor k/2\rfloor} - w\| \to 0$ as $l \to \infty$, this completes the proof. □

We give a proof of the remark that was made after Corollary A. Let $\mathcal{R} := \mathbb{Z}\langle t_1, \dots, t_d\rangle$. Since $\mathrm{EL}_n(\mathcal{R}/\mathcal{R}^l)$ is nilpotent, there is a *proper* 1-cocycle b_l (see Section 2.7 in [Bekka et al. 2008] or Section 12 in [Brown and Ozawa 2008]). We view b_l as 1-cocycles on $\mathrm{EL}_n(\mathcal{R})$ and consider $b := \sum_l^{\oplus} b_l$, which we may assume convergent pointwise on $\mathrm{EL}_n(\mathcal{R})$. We denote by π_k the Gelfand–Naimark representation associated with the positive definite function $\varphi_k(x) := \exp\bigl(-\frac{1}{k}\|b(x)\|^2\bigr)$. Then, the representation $\pi := \bigoplus \pi_k$ simultaneously admits asymptotically invariant vectors and a weak operator topology null sequence $x_l \in \mathrm{EL}_n(\mathcal{R}^l)$.

Proof of Corollary B. Let $\mathcal{R}^1 := \mathbb{Z}[t_1, \dots, t_d]$ denote the unitization of $\mathcal{R} := \mathbb{Z}\langle t_1, \dots, t_d\rangle$. Any quotient map $\mathcal{R} \twoheadrightarrow \mathcal{S}$ with $\mathcal{S}$ unital gives rise to a group homomorphism $\mathrm{EL}_n(\mathcal{R}^1) \twoheadrightarrow \mathrm{EL}_n(\mathcal{S})$ that extends $\mathrm{EL}_n(\mathcal{R}) \twoheadrightarrow \mathrm{EL}_n(\mathcal{S})$. We need to show that an orthogonal representation of $\mathrm{EL}_n(\mathcal{R}^1)$ which factors through $\mathrm{EL}_n(\mathcal{S})$ has a nonzero invariant vector, provided that it has almost $\mathrm{EL}_n(\mathcal{R})$ invariant vector. Since we know $\mathrm{EL}_n(\mathcal{R}^1)$ has property (T), it suffices to show that every almost $\mathrm{EL}_n(\mathcal{R})$ invariant vector is also almost $\mathrm{EL}_n(\mathbb{Z}1)$ invariant. The latter is true when $\mathcal{S}$ is finite. Indeed, the vector w in Corollary A is invariant under those $e_{i,j}(t_r^{l_0})$ such that $t_r^{l_0}$ is an idempotent in the quotient $\mathcal{S}$. Since a finite commutative ring is a direct sum of local rings (see, e.g., [Kassabov and Nikolov 2006]), the rng generated by such idempotents contains the identity of $\mathcal{S}$ and hence w is invariant under $\mathrm{EL}_n(\mathbb{Z}1)$. □

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TRIGONOMETRIC CHAOS AND X_p INEQUALITIES, I: BALANCED FOURIER TRUNCATIONS OVER DISCRETE GROUPS

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We investigate L_p-estimates for balanced averages of Fourier truncations in group algebras, in terms of "differential operators" acting on them. Our results extend a fundamental inequality of Naor for the hypercube (with profound consequences in metric geometry) to discrete groups. Different inequalities are established in terms of "directional derivatives" which are constructed via affine representations determined by the Fourier truncations. Our proofs rely on the Banach X_p nature of noncommutative L_p-spaces and dimension-free estimates for noncommutative Riesz transforms. In the particular case of free groups we use an alternative approach based on free Hilbert transforms.

Introduction

This paper is motivated by a recent inequality due to Assaf Naor, which we now introduce. Let $\Omega = \{\pm 1\}$ be the cyclic group of two elements with multiplicative terminology (that we use for all groups unless otherwise stated) and more generally $\Omega^n = \Omega \times \Omega \times \cdots \times \Omega$ be the hypercube. In both cases, we view them as equipped with their normalized counting measure. Ω^n is its own Pontryagin dual when equipped with its natural discrete measure. If $[n] := \{1, 2, \ldots, n\}$, every function $f : \Omega^n \to \mathbb{C}$ admits a Fourier expansion in terms of Walsh characters W_{A}, which are defined by A-products of coordinate functions $\varepsilon \mapsto \varepsilon_j$ for any $\mathsf{A} \subset [n]$. Given a mean-zero f Naor proved in [19] the following inequality for each $p \geq 2$ and $k \in [n]$:

$$\frac{1}{\binom{n}{k}} \sum_{\substack{\mathsf{S} \subset [n] \\ |\mathsf{S}| = k}} \Big\| \sum_{\mathsf{A} \subset \mathsf{S}} \hat{f}(\mathsf{A}) W_{\mathsf{A}} \Big\|_{L_p(\Omega^n)}^p \lesssim_p \frac{k}{n} \sum_{j=1}^n \|\partial_j f\|_{L_p(\Omega^n)}^p + \Big(\frac{k}{n}\Big)^{\frac{p}{2}} \|f\|_{L_p(\Omega^n)}^p. \qquad (\mathrm{N}_p)$$

The above S-truncations of the Walsh expansion of f are conditional expectations denoted by $\mathsf{E}_{[n]\setminus \mathsf{S}} f$, while $\partial_j f$ stands for the j-th directional (discrete) derivative of f, given by $\varepsilon \mapsto f(\varepsilon) - f(\varepsilon_1, \ldots, -\varepsilon_j, \ldots, \varepsilon_n)$, so that $\partial_j W_{\mathsf{A}} = 1_{\mathsf{A}}(j) 2 W_{\mathsf{A}}$. This inequality has groundbreaking applications in metric geometry. More precisely, it implies the quantitatively optimal form of the so-called X_p inequality, introduced by Naor and Schechtman in [20]. In turn, this gives a purely metric criterion to estimate the L_p-distortion of a metric space X from below. Its metric nature is very useful in solving nonlinear problems around the nonembedability of L_q into L_p for $2 < q < p$. This includes, beyond the scope of linear L_p-embedding theory, the optimal L_p-distortion of (nonlinear) grids in ℓ_q^n or the critical L_p snowflake exponent of L_q. In conclusion, Naor's inequality (N_p) and subsequent X_p inequalities with sharp scaling parameter are a key

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contribution to the Ribe program, an effort to identify which properties from the local theory of Banach spaces ultimately rely on purely metric considerations and not on the whole strength of the linear structure.

Naor's inequality (N_p) for functions with a linear Walsh expansion becomes a form of Rosenthal inequality for symmetrically exchangeable random variables [6; 21]. More precisely, let Π_k be the space of sets $\mathsf{S} \subset [n]$ with $|\mathsf{S}| = k$ equipped with its normalized counting measure and define $\Sigma_{n,k} = \Omega^n \otimes \Pi_k$. Then, if $\hat{f}(\mathsf{A}) = 0$ when $|\mathsf{A}| \neq 1$, the left-hand side of (N_p) becomes

$$\Big\| \sum_{j=1}^n \hat{f}(\{j\}) \sigma_j \Big\|_{L_p(\Sigma_{n,k})}^p, \qquad \text{with } \sigma_j(\varepsilon, \mathsf{S}) = \varepsilon_j \otimes 1_{\mathsf{S}}.$$

Then, the linear model for Naor's inequality, which is the one pertaining functions of the form $f(\varepsilon) = \sum_j \hat{f}(\{j\}) \varepsilon_j$, follows from [6]

$$\Big\| \sum_{j=1}^n \hat{f}(\{j\}) \sigma_j \Big\|_{L_p(\Sigma_{n,k})} \asymp_p \Big(\frac{k}{n} \sum_{j=1}^n |\hat{f}(\{j\})|^p \Big)^{\frac{1}{p}} + \Big(\frac{k}{n} \sum_{j=1}^n |\hat{f}(\{j\})|^2 \Big)^{\frac{1}{2}}.$$

Its general form (N_p) can be regarded as an extension for Rademacher chaos. Our primary goal in this paper is to produce inequalities similar to (N_p). This amounts to understanding the Walsh expansion of f as a Fourier series with frequencies in the discrete predual group Ω^n, with W_{A} being regarded as a Fourier character. In the general (nonabelian) case, this forces us to use the language of group von Neumann algebras generated by the left regular representation. Indeed, Fourier series with frequencies on a general discrete group G must be written in terms of its left regular representation $\lambda : \mathrm{G} \to \mathcal{B}(\ell_2(\mathrm{G}))$. The unitaries $\lambda(g)$ replace Walsh characters and we work with operators of the form

$$f = \sum_{g \in \mathrm{G}} \hat{f}(g) \lambda(g).$$

The "quantum" probability space where we place these operators is the group von Neumann algebra $\mathcal{L}(\mathrm{G})$. The noncommutative L_p space $L_p(\mathcal{L}(\mathrm{G}))$ associated with $\mathcal{L}(\mathrm{G})$ is isometrically isomorphic to the classical L_p-space over the Pontryagin dual $\widehat{\mathrm{G}}$ of G whenever the group G is abelian. We shall make no distinction between a function in $L_p(\widehat{\mathrm{G}})$ and the corresponding operator in $L_p(\mathcal{L}(\mathrm{G}))$ throughout the paper. Precise definitions of all the relevant objects are given below. Understanding how to replace Rademacher chaos by some sort of "trigonometric chaos" has to do with identifying elementary generating families. Our construction is somehow delicate and we start with a model case which originally motivated us.

Let $\mathbb{F}_n = \mathbb{Z} * \mathbb{Z} * \cdots * \mathbb{Z}$ be the free group with n generators $e_1, e_2, \ldots, e_n$. The unitaries $\lambda(e_j)$ are an archetype of Voiculescu's free random variables, which play the role of coordinate functions ε_j above. The tensor products $\zeta_j(\mathsf{S}) = \lambda(e_j) \otimes 1_{\mathsf{S}}(j)$ in $\Sigma'_{n,k} = \mathcal{L}(\mathbb{F}_n) \otimes \Pi_k$ satisfy the inequality

$$\Big\| \sum_{j=1}^n \hat{f}(e_j) \zeta_j \Big\|_{L_p(\Sigma'_{n,k})} \asymp_p \Big(\frac{k}{n} \sum_{j=1}^n |\hat{f}(e_j)|^p \Big)^{\frac{1}{p}} + \Big(\frac{k}{n} \sum_{j=1}^n |\hat{f}(e_j)|^2 \Big)^{\frac{1}{2}}. \tag{FR_p}$$

The desired free form of Naor's inequality looks as follows:

$$\frac{1}{\binom{n}{k}} \sum_{\substack{\mathsf{S} \subset [n] \\ |\mathsf{S}| = k}} \Big\| \sum_{w \in \mathbb{F}_{\mathsf{S}}} \hat{f}(w) \lambda(w) \Big\|_p^p \lesssim_p \frac{k}{n} \sum_{j=1}^n \|\partial_j f\|_p^p + \Big(\frac{k}{n} \Big)^{\frac{p}{2}} \|f\|_p^p. \tag{FN_p}$$

Here $\mathbb{F}_S$ denotes the free subgroup with generators in S and

$$\partial_j f = \sum_{w \geq e_j} \hat{f}(w)\lambda(w) + \sum_{w \geq e_j^{-1}} \hat{f}(w)\lambda(w),$$

where $w \geq e_j$ is used to pick those words starting with the subchain e_j^k for some positive integer k when written in reduced form. Let us briefly comment on the two inequalities above. The inequality (FR_p) follows from the noncommutative Burkholder/Rosenthal inequality [8; 9], while (FN_p) reduces to (FR_p) when f lives in the linear span of the $\lambda(e_j)$ as a consequence of the free Khintchine inequality [4]. It is therefore an extension of the linear model for free chaos. A look at Naor's original inequality shows that both group elements and collections of generators (respectively denoted by A and S there) become subsets of $[n]$. This curious coincidence in the hypercube must be decoupled for other discrete groups and our inner sum in the left-hand side of (FN_p) is taken over those words w with letters living in free coordinates located in S. On the other hand, our choice for $\partial_j f$ comes from [13] and will be properly justified in due time. It is worth mentioning that some nonlinear extensions of the free Rosenthal inequality were investigated in [10] for free chaos, but none of them include a free form of Naor's inequality along the lines suggested above.

The above reasoning settles a free model for Naor's inequality and illustrates how trigonometric chaos fits in for free groups. What happens if we take products of more general discrete groups? What about discrete groups lacking a product structure? Answering these questions amounts to considering Fourier truncations and somehow related differential operators over discrete groups. Other than lattices of Lie groups, discrete groups fail to admit canonical differential structures. This difficulty was successfully solved in [11; 13] with affine representations. More precisely, an orthogonal cocycle of G is a pair (α, β) given by an orthogonal action $\alpha : \mathrm{G} \curvearrowright \mathcal{H}$ into some $\mathbb{R}$-Hilbert space together with a map $\beta : \mathrm{G} \to \mathcal{H}$ satisfying the cocycle law

$$\alpha_g(\beta(h)) = \beta(gh) - \beta(g).$$

The latter ensures that $g \mapsto \alpha_g(\,\cdot\,) + \beta(g)$ is an affine representation of G, so that the cocycle map β establishes a good Hilbert space lift of G and one can expect to import the differential structure of $\mathcal{H}$. Naively, we "identify" the unitary $\lambda(g)$ with the Euclidean character $\exp(2\pi i \langle \beta(g), \cdot \rangle)$ and define "$\mathcal{H}$-directional derivatives" on $\mathcal{L}(\mathrm{G})$ as follows for any $u \in \mathcal{H}$:

$$\partial_u(\lambda(g)) = \langle \beta(g), u \rangle \lambda(g) \quad \text{and} \quad \Delta(\lambda(g)) = \|\beta(g)\|^2 \lambda(g).$$

We remark that we use the word "derivative" — in quotes, that we suppress after the Introduction — in a loose way here. They are linear operators that do not satisfy Leibniz rules, so in general they are not derivations. In the same vein, the correspondence between $\lambda(g)$ and Euclidean characters that we take as inspiration only holds for $\mathrm{G} = \mathbb{Z}^n$ with a particular choice of cocycle, that we shall detail below, and should not be considered in a literal way. This strategy of construction of differential structures has been extremely useful to establish L_p-boundedness criteria for Fourier multipliers on group von Neumann algebras. We now introduce the right setup for the problem. Given a discrete group G equipped with an orthogonal cocycle (α, β) and a positive integer n, we say that

$$\mathcal{A} = \{\mathrm{B}_S \subset \mathrm{G} : S \subset [n]\}$$

is an *admissible family of Fourier truncations* when we have:

- $\Big\| \sum_{g \in \mathrm{B_S}} \hat{f}(g)\lambda(g) \Big\|_p \leq_{\mathrm{cb}} C_p \Big\| \sum_{g \in \mathrm{G}} \hat{f}(g)\lambda(g) \Big\|_p$ for $p \geq 2$.
- Pairwise β-orthogonality:

$$\mathcal{H} = \bigoplus_{j=1}^n \mathcal{H}_j, \quad \text{with } \beta(\mathrm{B_S}), \beta(\mathrm{B_S^{-1}}) \subset \bigoplus_{j \in \mathrm{S}} \mathcal{H}_j = \mathcal{H}_\mathrm{S}.$$

Given an orthonormal basis $(u_{j\ell})_\ell$ of $\mathcal{H}_j$, define the j-th "gradients" as the first column vectors

$$\mathrm{D}_j f = \sum_{\ell \geq 1} \partial_{u_{j\ell}} f \otimes e_{\ell 1} \quad \text{so that} \quad |\mathrm{D}_j f| = \Big(\sum_{\ell \geq 1} |\partial_{u_{j\ell}} f|^2 \Big)^{\frac{1}{2}}.$$

Theorem A. *Let* G *be a discrete group equipped with an orthogonal cocycle* (α, β) *whose associated Laplacian* Δ *has a positive spectral gap* $\sigma > 0$. *Let us consider an admissible family of Fourier truncations* $\mathcal{A} = \{\mathrm{B_S} : \mathrm{S} \subset [n]\}$. *Then, given* $p \geq 2$ *and* $k \in [n]$, *the following inequality holds for any mean-zero* f:

$$\frac{1}{\binom{n}{k}} \sum_{|\mathrm{S}|=k} \Big\| \sum_{g \in \mathrm{B_S}} \hat{f}(g)\lambda(g) \Big\|_{L_p(\mathcal{L}(\mathrm{G}))}^p \lesssim_{p,\sigma} \frac{k}{n} \sum_{j=1}^n [\| |\mathrm{D}_j(f)| \|_{L_p(\mathcal{L}(\mathrm{G}))}^p + \| |\mathrm{D}_j(f^*)| \|_{L_p(\mathcal{L}(\mathrm{G}))}^p] + \Big(\frac{k}{n}\Big)^{\frac{p}{2}} \|f\|_{L_p(\mathcal{L}(\mathrm{G}))}^p.$$

Naor's inequality follows as a particular case of Theorem A by taking $\mathrm{G} = \Omega^n$ equipped with an adequate cocycle that we detail below in Remark 2.5. Said cocycle sends Ω^n into the n-dimensional space $\mathcal{H} = \mathbb{R} \times \mathbb{R} \times \cdots \times \mathbb{R}$, and we use the truncations $\mathrm{B_S} = \{\mathrm{A} \subset \mathrm{S}\} = \beta^{-1}(\mathcal{H}_\mathrm{S})$. The length ψ in this example is the *word length* (the geodesic distance in the Cayley graph) as is the case in many of the examples below. Recall that $\mathrm{D}_j = \partial_j$ here since $\dim \mathcal{H}_j = 1$. Moreover $|\partial_j(f)| = |\partial_j(f^*)|$ in the abelian framework of the hypercube. Two generalizations of Naor's inequality for large classes of discrete groups follow from Theorem A:

(i) Direct products. If $\mathrm{G} = \mathrm{G}_1 \times \mathrm{G}_2 \times \cdots \times \mathrm{G}_n$ is a direct product of discrete groups equipped with orthogonal cocycles (α_j, β_j), consider the product cocycle (α, β) and let $\mathrm{B_S}$ be the subgroup of G generated by group elements whose nontrivial entries lie in S. Then, the Fourier truncations become (completely contractive) conditional expectations and we get an admissible family of Fourier truncations. The gradients D_j correspond to the different factors and cocycles in the direct product above. One case that we shall explore is $\mathrm{G} = \mathbb{Z}^n$, which probably yields the most natural continuous generalization of (N_p) in the torus $\mathbb{T}^n$. Our result in inequality (2-1) can be obtained using only commutative ingredients and following the original argument. However, we shall deduce it from Theorem A and later improve it below using our stronger result — in this case.

(ii) Equivariant decompositions. If G is a discrete group equipped with an orthogonal cocycle (α, β), any direct sum decomposition of the Hilbert space $\mathcal{H}$ into α-equivariant subspaces gives rise to an admissible family of Fourier truncations. More precisely, assume

$$\mathcal{H} = \bigoplus_{j=1}^n \mathcal{H}_j \qquad \text{and} \qquad \alpha_g(\mathcal{H}_j) \subset \mathcal{H}_j \quad \text{for every } (g, j) \in \mathrm{G} \times [n].$$

Then, the family of sets

$$\mathrm{B_S} = \beta^{-1}\Big(\bigoplus_{j\in \mathsf{S}} \mathcal{H}_j\Big)$$

are subgroups of G. In particular, the associated Fourier truncations are conditional expectations (henceforth L_p-contractions) and the $\mathrm{B_S}$ satisfy pairwise β-orthogonality. This more general construction does not impose a direct product structure on the discrete group G.

Let $\mathcal{A}$ be an admissible family of Fourier truncations on G as defined above. Let us say that a group element $g \in \mathrm{G}$ is an $\mathcal{A}$-generator when $\beta(g) \in \mathcal{H}_j$ for some $1 \le j \le n$. Theorem A may be regarded as a nonlinear form of an inequality for linear combinations of $\mathcal{A}$-generators

$$f = \sum_{j=1}^{n} \sum_{\beta(g)\in\mathcal{H}_j} \hat{f}(g)\lambda(g) = \sum_{j=1}^{n} A_j(f).$$

This inequality controls balanced averages of S-truncations $\sum_{j\in\mathsf{S}} A_j(f)$ in terms of f and the j-th gradients of $A_j(f)$. This linear model seems to be new for general discrete groups/cocycles and Theorem A gives a nonlinear generalization in terms of trigonometric chaos over $\mathcal{A}$-generators.

Theorem A does not recover the conjectured free form of Naor's inequality (FN_p). Indeed, the free inequality relies on the standard cocycle of $\mathbb{F}_n$ associated with the word length, which yields $\mathcal{H} \simeq \ell_2(\mathbb{F}_n \backslash \{e\})$ and infinitely many "free derivatives" of the form

$$\partial_u f = \sum_{w\ge u} \hat{f}(w)\lambda(w) \quad \text{for any } u \in \mathbb{F}_n \setminus \{e\},$$

so the ∂_u can be regarded as Fourier multipliers whose symbols take values on $\{0, 1\}$. However, we only need to use n free directional derivatives which are defined as

$$\partial_j = \partial_{e_j} + \partial_{e_j^{-1}}, \quad \text{with } 1 \le j \le n,$$

and these are not coupled into a family of gradients, as we do in Theorem A. The key point to achieve this is the fact that free derivatives associated to free generators include all free derivatives in the sense that

$$u \neq e \quad \Longrightarrow \quad u \ge e_j \text{ or } u \ge e_j^{-1} \text{ for some } 1 \le j \le n \quad \Longrightarrow \quad \partial_u \circ \partial_j = \partial_j \circ \partial_u = \partial_u.$$

In general, assume that $\mathcal{A} = \{\mathrm{B_S} : \mathsf{S} \subset [n]\}$ is an admissible family of Fourier truncations in G with respect to (α, β). We will say that $\mathcal{J} = \{\partial_j : 1 \le j \le n\}$ is a *distinguished family of "derivatives"* when $\partial_u \circ \partial_j = \partial_u$ for any $u \in \mathcal{H}_j$ with $1 \le j \le n$. Throughout the paper, we shall consistently use u for vectors in $\mathcal{H}$ and $j \in [n]$, so that no confusion should arise when using ∂_u and ∂_j. The following result refines Theorem A when we can find such a family.

Theorem B. *Let* G *be a discrete group equipped with an orthogonal cocycle* (α, β) *and an admissible family of Fourier truncations* $\mathcal{A} = \{\mathrm{B_S} : \mathsf{S} \subset [n]\}$. *Assume that* $\mathcal{J} = \{\partial_j : 1 \le j \le n\}$ *is a distinguished* *family of derivatives. Then, given $p \geq 2$ and $k \in [n]$, the following inequality holds for any mean-zero f:*

$$\frac{1}{\binom{n}{k}} \sum_{|\mathrm{S}|=k} \Big\| \sum_{g \in \mathrm{B_S}} \hat{f}(g)\lambda(g) \Big\|_p^p \lesssim_p \frac{k}{n} \sum_{j=1}^n \|\partial_j(f)\|_p^p + \|\partial_j(f^*)\|_p^p + \Big(\frac{k}{n}\Big)^{\frac{p}{2}} \|f\|_p^p.$$

When the distinguished family of derivatives ∂_j is a proper subset of the cocycle derivatives ∂_u, it turns out that Theorem B gives a stronger inequality (compared to that of Theorem A) at the cost of additional assumptions, which fortunately hold in several important cases considered below. Note as well that the spectral gap assumption is unnecessary under the presence of distinguished derivatives. Here are our main applications of Theorem B:

(i) Free chaos. Our discussion on free derivatives illustrates how to apply Theorem B to obtain an inequality which gets very close to (FN$_p$). The extra term $\partial_j(f^*)$ is anyway removable due to a special property of free groups, for which word-length derivatives become free forms of directional Hilbert transforms [17]. This "good pathology" leads us to an even stronger inequality than the free analog of Naor's inequality (FN$_p$), as can be seen comparing the statements of Theorem 3.1 and Corollary 3.2. This could be useful in other directions of free harmonic analysis. We shall also explore the free products $\mathbb{Z}_{2m} * \mathbb{Z}_{2m} * \cdots * \mathbb{Z}_{2m}$.

(ii) Continuous and discrete tori. We also analyze $\mathbb{T}^n = \widehat{\mathbb{Z}}^n$ and $\mathbb{Z}_m^n = \widehat{\mathbb{Z}}_m^n$ equipped with different geometries. Theorem B is applicable for the Cayley graph metric and the resulting inequality improves the one coming from the Euclidean metric. These forms of Naor's inequality can be regarded as refinements of the classical Poincaré inequality.

(iii) Infinite Coxeter groups. Any group presented by

$$\mathrm{G} = \langle g_1, g_2, \ldots, g_n \mid (g_j g_k)^{s_{jk}} = e \rangle,$$

with $s_{jj} = 1$ and $s_{jk} \geq 2$ for $j \neq k$, is called a Coxeter group. Bożejko proved in [1] that the word length is conditionally negative for any infinite Coxeter group. The Cayley graph of these groups is more involved and we will not construct here a natural orthonormal basis for the cocycle; we invite the reader to do it and to derive inequalities along the lines of those in Theorems A and B.

Our proof of Theorems A and B streamlines Naor's original argument. The key point in this general setting is to identify the right notions, such as admissible families of Fourier truncations or distinguished families of derivatives. Once this is done, the proof heavily relies on dimension-free estimates for noncommutative Riesz transforms [13] in the same way Naor's inequality did in terms of Lust-Piquard results [16]. Another crucial point in our argument is the Banach X$_p$ nature of noncommutative L_p-spaces. Generalizing previous work of Naor and Schechtman [20, Theorem 7.1], we shall establish it with a much simpler argument based on Junge and Xu's noncommutative Burkholder/Rosenthal inequalities [8; 9]. Of course, one could expect that Theorems A and B may lead to noncommutative X$_p$-type inequalities, very much like in [19]. We have obtained some inequalities in this direction [2]. Our hope was to deduce nontrivial bounds for L_p-distortions of Schatten q-classes or other noncommutative L_q-spaces. Unfortunately, our efforts so far have not been fruitful in this direction.

1. Trigonometric chaos

1A. ***Harmonic analysis on discrete groups.*** Let G be a discrete group. The left regular representation of G on $\ell_2(\mathrm{G})$ is the unitary representation determined by

$$[\lambda(g)\varphi](h) = \varphi(g^{-1}h), \quad g, h \in \mathrm{G},\ \varphi \in \ell_2(\mathrm{G}).$$

The group von Neumann algebra of G is denoted by $\mathcal{L}(\mathrm{G})$. It is the weak operator closure of the linear span of $\{\lambda(g)\}_{g\in\mathrm{G}}$ in $\mathcal{B}(\ell_2(\mathrm{G}))$. Its canonical trace τ is linearly determined by $\tau(\lambda(g)) = \langle \lambda(g)1_{\{e\}}, 1_{\{e\}}\rangle_{\ell_2(\mathrm{G})} = \delta_{g=e}$. Every element $f \in \mathcal{L}(\mathrm{G})$ admits a Fourier series

$$f = \sum_{g\in\mathrm{G}} \hat{f}(g)\lambda(g), \quad \text{where } \hat{f}(g) = \tau(\lambda(g)^* f).$$

This shows that $\tau(f) = \hat{f}(e)$. For $1 \leq p < \infty$, we denote by $L_p(\mathcal{L}(\mathrm{G}))$ the associated noncommutative L_p space. We emphasize here that in the case G is abelian, its Pontryagin dual $\widehat{\mathrm{G}}$ is a compact abelian group and we have

$$L_p(\mathcal{L}(\mathrm{G})) \simeq L_p(\widehat{\mathrm{G}}),$$

isometrically. Therefore, in that case $L_p(\mathcal{L}(\mathrm{G}))$ is a classical (commutative) L_p space. In all instances below, we will consider all of our L_p spaces as noncommutative ones so that we can give a unified treatment to all the examples.

An orthogonal cocycle for G is a triple $(\mathcal{H}, \alpha, \beta)$ given by a real Hilbert space $\mathcal{H}$, an orthogonal action $\alpha : \mathrm{G} \to \mathcal{O}(\mathcal{H})$, and a map $\beta : \mathrm{G} \to \mathcal{H}$ satisfying the cocycle law

$$\alpha_g(\beta(h)) = \beta(gh) - \beta(g).$$

Orthogonal cocycles are in one-to-one correspondence with length functions. We say that a map $\psi : \mathrm{G} \to \mathbb{R}_+$ is a length function if it vanishes at the identity e, it is symmetric $\psi(g) = \psi(g^{-1})$, and it is conditionally negative

$$\sum_{g\in\mathrm{G}} a_g = 0 \quad \Longrightarrow \quad \sum_{g,h\in\mathrm{G}} \bar{a}_g a_h \psi(g^{-1}h) \leq 0$$

for any finitely supported family $\{a_g\}_{g\in\mathrm{G}}$. Given a cocycle $(\mathcal{H}, \alpha, \beta)$, the function $\psi(g) = \|\beta(g)\|_{\mathcal{H}}^2$ is a length function. On the other hand, any length function ψ determines a Gromov form $\langle\,\cdot\,,\,\cdot\,\rangle_\psi$, a semidefinite positive form on

$$\mathbb{D}[\mathrm{G}] := \mathbb{R}\text{-span}\langle 1_{\{g\}} : g \in \mathrm{G}\rangle,$$

given by

$$\langle 1_{\{g\}}, 1_{\{h\}}\rangle_\psi = \frac{\psi(g) + \psi(h) - \psi(g^{-1}h)}{2}.$$

Then, the Hilbert completion $\mathcal{H}$ of $\mathbb{D}[\mathrm{G}]/\mathrm{Ker}(\langle\,\cdot\,,\,\cdot\,\rangle_\psi)$, equipped with $\langle\,\cdot\,,\,\cdot\,\rangle_\psi$, together with the map $\beta : g \mapsto 1_{\{g\}} + \mathrm{Ker}(\langle\,\cdot\,,\,\cdot\,\rangle_\psi)$, and the orthogonal action $\alpha_g(1_{\{h\}}) = 1_{\{gh\}} - 1_{\{g\}} + \mathrm{Ker}(\langle\,\cdot\,,\,\cdot\,\rangle_\psi)$ form a cocycle. Moreover, Schoenberg's theorem [22] claims that $\psi : \mathrm{G} \to \mathbb{R}_+$ is a length function if and only if the maps $S_t : \lambda(g) \mapsto e^{-t\psi(g)}\lambda(g)$ form a Markov semigroup $(S_t)_{t\geq 0}$ on $\mathcal{L}(\mathrm{G})$; see [11; 13]. In this case $(S_t)_{t\geq 0}$ admits an infinitesimal generator

$$-\Delta := \lim_{t\to 0^+} \frac{S_t - \mathrm{id}_{\mathcal{L}(\mathrm{G})}}{t} \quad \text{so that} \quad S_t = \exp(-t\Delta).$$

As is standard, we shall call the generator Δ the ψ-Laplacian on G. Since we have $\Delta(\lambda(g)) = \psi(g)\lambda(g)$ for $g \in \mathrm{G}$, it turns out that Δ is an unbounded Fourier multiplier whose fractional powers can be defined by

$$\Delta^\gamma f := \sum_{g\in\mathrm{G}} \psi(g)^\gamma f(g)\lambda(g).$$

Let $(\mathcal{H}, \alpha, \beta)$ be the orthogonal cocycle naturally associated to the length function $\psi : \mathrm{G} \to \mathbb{R}_+$ as explained above. Given an orthonormal basis $\{u_\ell\}_{\ell\geq 1}$ of $\mathcal{H}$, we consider the corresponding directional derivatives as follows:

$$\partial_{u_\ell}\lambda(g) := \langle \beta(g), u_\ell\rangle_\psi \lambda(g) \quad \text{so that} \quad \Delta = \sum_{\ell\geq 1} \partial_{u_\ell}^2.$$

The corresponding Riesz transforms associated to ψ are then defined as

$$R_\ell f = R_{u_\ell} f := \partial_{u_\ell}\Delta^{-\frac{1}{2}} f = \sum_{g\in\mathrm{G}} \frac{\langle \beta(g), u_\ell\rangle_\psi}{\sqrt{\psi(g)}} \hat{f}(g)\lambda(g).$$

Riesz transforms act on elements of $L_p(\mathcal{L}(\mathrm{G}))$ with null Fourier coefficients on the kernel of β. More precisely, maps R_ℓ are well-defined over the mean-zero subspaces

$$L_p^\circ(\mathcal{L}(\mathrm{G})) = \{f \in L_p(\mathcal{L}(\mathrm{G})) : \hat{f}(g) = 0 \text{ if } \psi(g) = 0\}.$$

Dimension-free estimates for noncommutative Riesz transforms were studied in [13].

Theorem 1.1 [13, Theorem A1]. *If* $2 \leq p < \infty$ *and* $f \in L_p^\circ(\mathcal{L}(\mathrm{G}))$

$$\|f\|_p \asymp_p \max\left\{ \left\| \Big(\sum_{\ell\geq 1} |R_\ell(f)|^2\Big)^{\frac{1}{2}} \right\|_p, \left\| \Big(\sum_{\ell\geq 1} |R_\ell(f^*)|^2\Big)^{\frac{1}{2}} \right\|_p \right\}.$$

Finally, our Fourier truncations will be written in the form

$$\mathsf{E}_{[n]\setminus \mathsf{S}} f = \sum_{g\in \mathrm{B}_\mathsf{S}} \hat{f}(g)\lambda(g), \quad \text{with } \mathsf{S} \subset [n].$$

When B_S is a subgroup of G, $\mathsf{E}_{[n]\setminus\mathsf{S}}$ is a (L_p-contractive) conditional expectation onto $\mathcal{L}(\mathrm{B}_\mathsf{S})$.

1B. ***Noncommutative L_p-spaces are Banach X_p spaces.*** Linear forms of X_p inequalities are vector-valued extensions of Rosenthal inequality for symmetrically exchangeable random variables [6]. More precisely, a Banach space $\mathbb{X}$ is said to satisfy a Banach X_p inequality if the inequality of Theorem 1.2 below is satisfied for vectors $\{x_j\}_{j\in[n]} \subset \mathbb{X}$ (and with norms taken in $\mathbb{X}$). In [20, Theorem 7.1] Naor and Schechtman proved such inequalities for Schatten p-classes. A noncommutative Burkholder martingale inequality for the conditioned square function [8] led Junge and Xu to obtain noncommutative Rosenthal inequalities for symmetric variables in [9]. The precise result that we use below is the following (see

[9, Corollary 6.6]): let $\mathcal{N}$ and $\mathcal{M}$ be von Neumann algebras, with $\mathcal{N}$ abelian, and $p \geq 2$. If $\{x_j\}_{j\in[n]} \subset L_p(\mathcal{M})$ satisfy that

$$\Big\| \sum_{j=1}^n s_j a_{\pi(j)} \otimes x_j \Big\|_{L_p(\mathcal{N}\bar{\otimes}\mathcal{M})} \lesssim \Big\| \sum_{j=1}^n a_j \otimes x_j \Big\|_{L_p(\mathcal{N}\bar{\otimes}\mathcal{M})}$$

holds for all random signs $s = (s_1, s_2, \dots, s_n) \in \Omega^n$, all permutations π on $[n]$ and coefficients $\{a_j\}_{j\in[n]} \subset L_p(\mathcal{N})$ — that is, the variables are symmetrically exchangeable — then

$$\Big\| \sum_{j=1}^n a_j \otimes x_j \Big\|_p \sim \frac{1}{n^{\frac{1}{p}}} \sum_{j,j'=1}^n \|a_j\|_p \|x_{j'}\|_p + \frac{1}{n^{\frac{1}{2}}} \Big\| \Big(\sum_{j=1}^n x_j^* x_j + x_j x_j^* \Big)^{\frac{1}{2}} \Big\|_p \Big\| \Big(\sum_{j=1}^n a_j^2 \Big)^{\frac{1}{2}} \Big\|_p . \tag{1-1}$$

We use this result below to establish the Banach X_p nature of arbitrary noncommutative L_p-spaces. Naor/Schechtman's argument can be extended to work as well for other noncommutative L_p-spaces, but our argument below is much shorter.

Theorem 1.2. *Let $(\mathcal{M}, \tau)$ be a von Neumann algebra equipped with a normal semifinite faithful trace. Then, if $\mathbb{E}$ denotes the expectation over independently equidistributed random signs $\varepsilon = (\varepsilon_1, \varepsilon_2, \dots, \varepsilon_n)$ and $x_j \in L_p(\mathcal{M})$, the following inequality holds for any $p \geq 2$ and $k \in [n]$:*

$$\frac{1}{\binom{n}{k}} \sum_{\substack{\mathsf{S}\subseteq[n] \\ |\mathsf{S}|=k}} \mathbb{E} \Big\| \sum_{j\in\mathsf{S}} \varepsilon_j x_j \Big\|_{L_p(\mathcal{M})}^p \lesssim_p \frac{k}{n} \sum_{j=1}^n \|x_j\|_{L_p(\mathcal{M})}^p + \Big(\frac{k}{n}\Big)^{\frac{p}{2}} \mathbb{E} \Big\| \sum_{j=1}^n \varepsilon_j x_j \Big\|_{L_p(\mathcal{M})}^p .$$

Proof. Define random variables $\sigma_j \in \Sigma_{n,k} = \Omega^n \otimes \Pi_k$ as defined in the Introduction by $\sigma_j(\varepsilon, \mathsf{S}) = \varepsilon_j \otimes 1_{\mathsf{S}}(j)$ for $1 \leq j \leq n$ and $\mathsf{S} \subset [n]$. We claim that the variables σ_j are symmetrically exchangeable in $L_p(\Sigma_{n,k} \bar{\otimes} \mathcal{M}) = L_p(\Pi_k; L_p(\Omega^n; L_p(\mathcal{M})))$, i.e., for any choice of signs $s_j = \pm 1$ and any permutation π of $[n]$, there holds

$$\mathrm{A} := \Big\| \sum_{j=1}^n s_j \sigma_{\pi(j)} \otimes x_j \Big\|_{L_p(\Sigma_{n,k}\bar{\otimes}\mathcal{M})} \lesssim \Big\| \sum_{j=1}^n \sigma_j \otimes x_j \Big\|_{L_p(\Sigma_{n,k}\bar{\otimes}\mathcal{M})} =: \mathrm{B}.$$

Indeed, applying the noncommutative Khintchine inequality [15] in $L_p(\mathcal{M})$ twice

$$\begin{aligned}
\mathrm{A}^p &= \frac{1}{\binom{n}{k}} \sum_{\substack{\mathsf{S}\subseteq[n] \\ |\mathsf{S}|=k}} \Big\| \sum_{\pi(j)\in\mathsf{S}} \varepsilon_{\pi(j)} \otimes s_j x_j \Big\|_{L_p(\Omega^n; L_p(\mathcal{M}))}^p \\
&\asymp_p \frac{1}{\binom{n}{k}} \sum_{\substack{\mathsf{S}\subseteq[n] \\ |\mathsf{S}|=k}} \max\Big\{ \Big\| \Big(\sum_{\pi(j)\in\mathsf{S}} x_j^* x_j \Big)^{\frac{1}{2}} \Big\|_{L_p(\mathcal{M})}, \Big\| \Big(\sum_{\pi(j)\in\mathsf{S}} x_j x_j^* \Big)^{\frac{1}{2}} \Big\|_{L_p(\mathcal{M})} \Big\} \\
&= \frac{1}{\binom{n}{k}} \sum_{\substack{\mathsf{S}\subseteq[n] \\ |\mathsf{S}|=k}} \max\Big\{ \Big\| \Big(\sum_{j\in\mathsf{S}} x_j^* x_j \Big)^{\frac{1}{2}} \Big\|_{L_p(\mathcal{M})}, \Big\| \Big(\sum_{j\in\mathsf{S}} x_j x_j^* \Big)^{\frac{1}{2}} \Big\|_{L_p(\mathcal{M})} \Big\} \asymp_p \mathrm{B}^p.
\end{aligned}$$

Hence, we can apply (1-1) (with the choice $\mathcal{N} = L_\infty(\Sigma_{n,k})$) to get

$$\mathrm{B}^p \lesssim_p \frac{1}{n} \sum_{j,j'=1}^n \|\sigma_j\|_p^p \|x_{j'}\|_p^p + \Big(\frac{1}{n}\Big)^{\frac{p}{2}} \Big\| \Big(\sum_{j=1}^n x_j^* x_j + x_j x_j^* \Big)^{\frac{1}{2}} \Big\|_p^p \Big\| \Big(\sum_{j=1}^n \sigma_j^2 \Big)^{\frac{1}{2}} \Big\|_p^p .$$

Now, we have

$$\|\sigma_j\|_{L_p(\Sigma_{n,k})}^p = \frac{1}{\binom{n}{k}} \sum_{\substack{S\subseteq[n]\\|S|=k}} 1_S(j) = \frac{k}{n},$$

$$\Big\|\Big(\sum_{j=1}^n \sigma_j^2\Big)^{\frac12}\Big\|_{L_p(\Sigma_{n,k})}^p = \frac{1}{\binom{n}{k}} \sum_{\substack{S\subseteq[n]\\|S|=k}} \Big(\sum_{j=1}^n 1_S(j)\Big)^{\frac{p}{2}} = k^{\frac{p}{2}}.$$

Therefore, we get

$$\begin{aligned} B^p &\lesssim_p \frac{k}{n}\sum_{j=1}^n \|x_j\|_{L_p(\mathcal{M})}^p + \Big(\frac{k}{n}\Big)^{\frac{p}{2}} \Big\|\Big(\sum_{j=1}^n x_j^*x_j + x_jx_j^*\Big)^{\frac12}\Big\|_{L_p(\mathcal{M})}^p \\ &\asymp_p \frac{k}{n}\sum_{j=1}^n \|x_j\|_{L_p(\mathcal{M})}^p + \Big(\frac{k}{n}\Big)^{\frac{p}{2}} \mathbb{E}\Big\|\sum_{j=1}^n \varepsilon_j x_j\Big\|_{L_p(\mathcal{M})}^p, \end{aligned}$$

applying once again the noncommutative Khintchine inequality. This proves the result since the random variables σ_j are chosen so that B equals the left-hand side in the inequality of the statement. □

Remark 1.3. Theorem 1.2 says that $L_p(\mathcal{M})$ is a Banach X_p space. The conclusion also holds in the completely bounded setting since the constants that appear in the inequality of the statement do not depend on the von Neumann algebra $\mathcal{M}$.

1C. *Proof of Theorem A.* According to Theorem 1.1

$$\begin{aligned} &\frac{1}{\binom{n}{k}} \sum_{\substack{S\subseteq[n]\\|S|=k}} \Big\|\sum_{g\in B_S} \hat{f}(g)\lambda(g)\Big\|_p^p \\ &\quad = \frac{1}{\binom{n}{k}} \sum_{\substack{S\subseteq[n]\\|S|=k}} \|\mathsf{E}_{[n]\setminus S} f\|_p^p \\ &\quad \asymp_p \frac{1}{\binom{n}{k}} \sum_{\substack{S\subseteq[n]\\|S|=k}} \Big\|\Big(\sum_{\substack{j\in[n]\\ \ell\ge 1}} |R_{j\ell}(\mathsf{E}_{[n]\setminus S} f)|^2\Big)^{\frac12}\Big\|_p^p + \frac{1}{\binom{n}{k}} \sum_{\substack{S\subseteq[n]\\|S|=k}} \Big\|\Big(\sum_{\substack{j\in[n]\\ \ell\ge 1}} |R_{j\ell}((\mathsf{E}_{[n]\setminus S} f)^*)|^2\Big)^{\frac12}\Big\|_p^p =: A + B, \end{aligned}$$

where $R_{j\ell} := R_{u_{j\ell}}$ and $\{u_{j\ell} : j\in[n],\ \ell\ge 1\}$ is the orthonormal basis of $\mathcal{H}$ considered before the statement of Theorem A. Since $\beta(B_S)\subset\mathcal{H}_S$, we observe that $\langle\beta(g), u_{j\ell}\rangle_\psi = 0$ whenever $g\in B_S$ and $j\notin S$. Moreover, Fourier truncations commute with Riesz transforms — as both are Fourier multipliers — and we deduce

$$R_{j\ell}\circ \mathsf{E}_{[n]\setminus S} = 1_S(j)\ \mathsf{E}_{[n]\setminus S}\circ R_{j\ell}.$$

Using the complete L_p-boundedness of our Fourier truncations, we get

$$A \lesssim_p \frac{1}{\binom{n}{k}} \sum_{\substack{S\subseteq[n]\\|S|=k}} \Big\|\Big(\sum_{j\in S}\Big[\sum_{\ell\ge1} |R_{j\ell} f|^2\Big]\Big)^{\frac12}\Big\|_p^p \lesssim_p \frac{1}{\binom{n}{k}} \sum_{\substack{S\subseteq[n]\\|S|=k}} \mathbb{E}\Big\|\sum_{j\in S}\varepsilon_j\Big[\sum_{\ell\ge1} |R_{j\ell} f|^2\Big]^{\frac12}\Big\|_p^p =: A'.$$

The last inequality follows from either the scalar (if G is abelian) or the noncommutative Khintchine inequality [15] otherwise, applied to independent equidistributed signs $\varepsilon_j = \pm 1$. Next, we use the

Banach $\mathrm{X_p}$ nature of either commutative or noncommutative L_p-spaces. More precisely, applying Theorem 1.2 we get

$$\mathrm{A}' \lesssim_p \frac{k}{n}\sum_{j=1}^n \bigg\| \Big(\sum_{\ell\geq 1} |R_{j\ell} f|^2 \Big)^{\frac12} \bigg\|_p^p + \Big(\frac{k}{n}\Big)^{\frac p2} \mathbb{E} \bigg\| \sum_{j=1}^n \varepsilon_j \Big[\sum_{\ell\geq 1} |R_{j\ell} f|^2 \Big]^{\frac12} \bigg\|_p^p = \mathrm{A}'_1 + \mathrm{A}'_2.$$

Since $R_{j\ell} = \partial_{u_{j\ell}} \Delta^{-1/2} = \Delta^{-1/2} \partial_{u_{j\ell}}$, [7, Proposition 1.1.5] yields

$$\mathrm{A}'_1 = \frac{k}{n}\sum_{j=1}^n \bigg\| \sum_{\ell\geq 1} R_{j\ell} f \otimes e_{\ell,1} \bigg\|_{S_p[L_p(\mathcal{L}(\mathrm{G}))]}^p \lesssim_{p,\sigma} \frac{k}{n}\sum_{j=1}^n \bigg\| \sum_{\ell\geq 1} \partial_{u_{j\ell}} f \otimes e_{\ell,1} \bigg\|_{S_p[L_p(\mathcal{L}(\mathrm{G}))]}^p = \frac{k}{n}\sum_{j=1}^n \|\mathrm{D}_j(f)\|_p^p.$$

Moreover, the Khintchine inequality and Theorem 1.1 give

$$\mathrm{A}'_2 \lesssim_p \Big(\frac{k}{n}\Big)^{\frac p2} \bigg\| \Big(\sum_{j=1}^n \sum_{\ell\geq 1} |R_{j\ell} f|^2 \Big)^{\frac12} \bigg\|_p^p \lesssim_p \Big(\frac{k}{n}\Big)^{\frac p2} \|f\|_p^p.$$

Therefore, the term A satisfies the expected estimate and it remains to justify the assertion for B. We now analyze the behavior of our Fourier truncations under adjoints. Observe that

$$(\mathsf{E}_{[n]\setminus \mathsf{S}} f)^* = \sum_{g\in \mathrm{B}_\mathsf{S}} \overline{\hat{f}(g)} \lambda(g^{-1}) =: \mathsf{E}'_{[n]\setminus \mathsf{S}}(f^*).$$

In particular, since $\mathsf{E}'_{[n]\setminus \mathsf{S}}$ commutes with $R_{j\ell}$

$$R_{j\ell}((\mathsf{E}_{[n]\setminus \mathsf{S}} f)^*) = \mathsf{E}'_{[n]\setminus \mathsf{S}}(R_{j\ell}(f^*)) = \mathsf{E}_{[n]\setminus \mathsf{S}}(R_{j\ell}(f^*)^*)^*.$$

This is the point where we need the condition $\beta(\mathrm{B}_\mathsf{S}^{-1}) \subset \mathcal{H}_\mathsf{S}$, to make sure that the above terms vanish when $j \notin \mathsf{S}$ since we find the inner products $\langle \beta(g^{-1}), u_{j\ell}\rangle_\psi$ for $g \in \mathrm{B}_\mathsf{S}$. Thus, we obtain

$$\begin{aligned} \bigg\| \Big(\sum_{\substack{j\in[n]\\ \ell\geq 1}} |R_{j\ell}((\mathsf{E}_{[n]\setminus \mathsf{S}} f)^*)|^2 \Big)^{\frac12} \bigg\|_p &= \bigg\| \sum_{j\in[n]} \sum_{\ell\geq 1} \mathsf{E}_{[n]\setminus \mathsf{S}}(R_{j\ell}(f^*)^*) \otimes e_{1,(j,\ell)} \bigg\|_p \\ &\lesssim_p \bigg\| \sum_{j\in \mathsf{S}} \sum_{\ell\geq 1} R_{j\ell}(f^*) \otimes e_{(j,\ell),1} \bigg\|_p = \bigg\| \Big(\sum_{j\in \mathsf{S}} \sum_{\ell\geq 1} |R_{j\ell}(f^*)|^2 \Big)^{\frac12} \bigg\|_p. \end{aligned}$$

Therefore, we may follow the above argument for A just replacing f by f^*. □

Remark 1.4. A careful reading of the proof of Theorems A and B shows that we may use different Hilbert space decompositions $\mathcal{H} = \oplus_j \mathcal{H}_j = \oplus_j \mathcal{K}_j$ for B_S and its inverse — with $\beta(\mathrm{B}_\mathsf{S}) \subset \mathcal{H}_\mathsf{S}$ and $\beta(\mathrm{B}_\mathsf{S}^{-1}) \subset \mathcal{K}_\mathsf{S}$ — as long as we can find an orthonormal basis $\{u_\ell : \ell \geq 1\}$ of $\mathcal{H}$ satisfying that

$$\text{for all } \ell \geq 1 \text{ there exists } j_1, j_2 \in [n] \text{ such that } u_\ell \in \mathcal{H}_{j_1} \cap \mathcal{K}_{j_2}. \tag{1-2}$$

More precisely, under this more flexible assumption we get

$$\frac{1}{\binom{n}{k}} \sum_{|\mathsf{S}|=k} \bigg\| \sum_{g\in \mathrm{B}_\mathsf{S}} \hat{f}(g)\lambda(g) \bigg\|_p^p \lesssim_{p,\sigma} \frac{k}{n}\sum_{j=1}^n [\|\mathrm{D}_j(f)\|_p^p + \|\mathrm{D}_j^\dagger(f^*)\|_p^p] + \Big(\frac{k}{n}\Big)^{\frac p2} \|f\|_p^p,$$

where $\mathrm{D}_j^\dagger := \sum_{u_\ell\in\mathcal{K}_j} \partial_{u_\ell}(\,\cdot\,) \otimes e_{\ell 1}$ is the gradient over the basis vectors living in $\mathcal{K}_j$.

Remark 1.5. The constant depending on σ in Theorem A grows as $\sigma^{-p/2}$. One can also track the dependence on p of the constant. Using free generators in place of random signs — Theorem 1.2 holds as well — we keep constants uniformly bounded replacing noncommutative by free Khintchine inequalities [4]. The constants in Theorem 1.1 are bounded by $p^{3/2}$, but it is still open whether this is optimal.

1D. ***Proof of Theorem B.*** Again Theorem 1.1 gives

$$\frac{1}{\binom{n}{k}} \sum_{\substack{\mathsf{S}\subseteq[n]\\ |\mathsf{S}|=k}} \|\mathsf{E}_{[n]\setminus\mathsf{S}} f\|_p^p \asymp_p \mathrm{A}+\mathrm{B}$$

as in the proof of Theorem A. Following our argument there, we use our estimate $\mathrm{A} \lesssim_p \mathrm{A}_1' + \mathrm{A}_2'$ and we bound A_2' in the same way. To estimate A_1' we use our distinguished family of derivatives and Theorem 1.1 to deduce

$$\mathrm{A}_1' = \frac{k}{n}\sum_{j=1}^n \bigg\| \Big(\sum_{\ell\geq 1} |R_{u_{j\ell}} \partial_j f|^2 \Big)^{\frac{1}{2}} \bigg\|_p^p \lesssim_p \frac{k}{n}\sum_{j=1}^n \|\partial_j f\|_p^p.$$

The estimate for B then follows by the same considerations as in Theorem A. □

2. Applications to abelian groups

We now focus our attention on concrete realizations of Theorems A and B for certain commutative group algebras. In all the cases in this section, we choose $\mathsf{E}_{[n]\setminus\mathsf{S}}$ of the form

$$\mathsf{E}_{[n]\setminus\mathsf{S}} f = \sum_{g\in \mathrm{B}_\mathsf{S}} \hat{f}(g)\lambda(g) \quad \text{for some subgroup } \mathrm{B}_\mathsf{S} \text{ of } \mathrm{G}.$$

Due to that fact, we know that they are conditional expectations, and therefore completely contractive maps. This allows us to safely apply Theorems A and B without checking that hypothesis. We will give the details for the cases $\mathrm{G}=\mathbb{Z}^n$ and $\mathrm{G}=\mathbb{Z}_{2m}^n$, yielding inequalities in $L_p(\mathbb{T}^n)$ and $L_p(\mathbb{Z}_{2m}^n)$, respectively. In this section, we must change to additive notation in our groups given their natural operations (and we reserve the product notation for the usual product of integer/real numbers). The necessary adjustments for the hypercube are discussed at the end of the section.

2A. ***Classical tori.*** Define

$$\psi_1(g) = |g_1| + \cdots + |g_n|,$$
$$\psi_2(g) = g_1^2 + g_2^2 + \cdots + g_n^2,$$

with $g=(g_1,g_2,\ldots,g_n)\in\mathbb{Z}^n$. Both functions are symmetric and vanish at 0. Moreover, conditional negativity follows easily. In the case of ψ_1, it suffices to check it for each summand $|g_j|$ which is conditionally negative from subordination with respect to g_j^2. These functions are denoted as the word and the Euclidean length respectively. We analyze balanced Fourier truncations using both geometries.

(A) The Euclidean length. The length ψ_2 induces the standard cocycle $(\mathcal{H},\alpha,\beta)$, where $\mathcal{H}=\mathbb{R}^n$ with the usual Euclidean inner product, the trivial action and the canonical inclusion $\beta=\mathrm{id}$. We use the standard decomposition $\mathcal{H}=\bigoplus_j \mathcal{H}_j$, with $\mathcal{H}_j=\mathbb{R}e_j$ the subspace generated by the j-th element of the canonical basis. Therefore, given $\mathsf{S} \subset [n]$, denote by $\mathbb{Z}^{\mathsf{S}}$ the subgroup of elements with vanishing entries outside S and consider the truncations

$$\mathsf{E}_{[n]\setminus \mathsf{S}} f(x) = \sum_{g \in \mathbb{Z}^{\mathsf{S}}} \hat{f}(g) e^{2\pi i \langle x, g\rangle} \quad \text{for any } f \in L_p(\mathbb{T}^n) \simeq L_p(\mathcal{L}(\mathbb{Z}^n)),$$

where $e^{2\pi i \langle \cdot, g\rangle} \mapsto \lambda(g)$ defines a trace-preserving $*$-homomorphism. The cocycle derivatives correspond in this case lup to a multiplicative constant — to the classical derivatives $(\partial / \partial x_j)$, and the infinitesimal generator Δ is the usual Laplacian (up to a universal constant) with spectral gap 1. Then, Theorem A yields

$$\frac{1}{\binom{n}{k}} \sum_{\substack{\mathsf{S} \subseteq [n] \\ |\mathsf{S}| = k}} \Big\| \sum_{g \in \mathbb{Z}^{\mathsf{S}}} \hat{f}(g) e^{2\pi i \langle \cdot, g\rangle} \Big\|_{L_p(\mathbb{T}^n)}^p \lesssim_p \frac{k}{n} \sum_{j=1}^n \Big\| \frac{\partial}{\partial x_j} f \Big\|_{L_p(\mathbb{T}^n)}^p + \Big(\frac{k}{n}\Big)^{\frac{p}{2}} \|f\|_{L_p(\mathbb{T}^n)}^p \tag{2-1}$$

for any mean-zero $f \in L_p(\mathbb{T}^n)$. This seems to be the most natural generalization of Naor's inequality for classical tori, but it is not the most efficient. Indeed, using the same Hilbert space decomposition as above, one can consider the alternative absorbent derivatives $\partial_j \lambda(g) = \delta_{g_j \neq 0} \lambda(g)$. In particular Theorem B yields

$$\frac{1}{\binom{n}{k}} \sum_{\substack{\mathsf{S} \subseteq [n] \\ |\mathsf{S}| = k}} \Big\| \sum_{g \in \mathbb{Z}^{\mathsf{S}}} \hat{f}(g) e^{2\pi i \langle \cdot, g\rangle} \Big\|_{L_p(\mathbb{T}^n)}^p \lesssim_p \frac{k}{n} \sum_{j=1}^n \|\partial_j f\|_{L_p(\mathbb{T}^n)}^p + \Big(\frac{k}{n}\Big)^{\frac{p}{2}} \|f\|_{L_p(\mathbb{T}^n)}^p, \tag{2-2}$$

where, abusing notation, we define $\partial_j e^{2\pi i \langle \cdot, g\rangle} = \delta_{g_j \neq 0} e^{2\pi i \langle \cdot, g\rangle}$. This is a stronger inequality since

$$\|\partial_j f\|_{L_p(\mathbb{T}^n)} = \frac{1}{2\pi} \Big\| \sum_{g_j \neq 0} \frac{1}{g_j} \Big(\frac{\partial}{\partial x_j} f \Big)^{\wedge}(g) e^{2\pi i \langle \cdot, g\rangle} \Big\|_{L_p(\mathbb{T}^n)} \leq C_p \Big\| \frac{\partial}{\partial x_j} f \Big\|_{L_p(\mathbb{T}^n)}.$$

Indeed, the symbol $m(g) = 1/g_j$ defines an L_p-bounded multiplier as a consequence of the K. de Leeuw restriction theorem and the Hörmander–Mikhlin multiplier theorem [5; 14; 18]. As we shall see (2-2) naturally appears using the word length.

Remark 2.1. Consider $f : \mathbb{T}^n \to \mathbb{C}$ with

$$f(x) = \sum_{g \in \mathbb{Z}^n} \hat{f}(g) e^{2\pi i \langle x, g\rangle} \quad \text{and} \quad \hat{f}(0) = 0.$$

Given $\mathsf{S} \subset [n]$, the classical Poincaré inequality gives

$$\begin{aligned}
\frac{1}{\binom{n}{k}} \sum_{\substack{\mathsf{S} \subset [n] \\ |\mathsf{S}| = k}} \Big\| \underbrace{\sum_{g \in \mathbb{Z}^{\mathsf{S}} \setminus \{0\}} \hat{f}(g) e^{2\pi i \langle \cdot, g\rangle}}_{f_{\mathsf{S}}} \Big\|_p^p &\leq \frac{1}{\binom{n}{k}} \sum_{\substack{\mathsf{S} \subset [n] \\ |\mathsf{S}| = k}} \| |\nabla f_{\mathsf{S}}| \|_p^p \asymp \frac{1}{\binom{n}{k}} \sum_{\substack{\mathsf{S} \subset [n] \\ |\mathsf{S}| = k}} \Big\| \sum_{j \in \mathsf{S}} \varepsilon_j \frac{\partial}{\partial x_j} f_{\mathsf{S}} \Big\|_p^p \\
&= \frac{1}{\binom{n}{k}} \sum_{\substack{\mathsf{S} \subset [n] \\ |\mathsf{S}| = k}} \Big\| \Big[\sum_{j \in \mathsf{S}} \varepsilon_j \frac{\partial}{\partial x_j} f \Big]_{\mathsf{S}} \Big\|_p^p \\
&\leq \frac{1}{\binom{n}{k}} \sum_{\substack{\mathsf{S} \subset [n] \\ |\mathsf{S}| = k}} \Big\| \sum_{j \in \mathsf{S}} \varepsilon_j \frac{\partial}{\partial x_j} f \Big\|_p^p = \Big\| \sum_{j=1}^n \sigma_j \frac{\partial}{\partial x_j} f \Big\|_p^p
\end{aligned}$$

for $\sigma_j(\varepsilon, \mathsf{S}) = \varepsilon_j \otimes 1_{\mathsf{S}}(j)$ as in the Introduction. Applying [6] gives

$$\frac{1}{\binom{n}{k}} \sum_{\substack{\mathsf{S}\subset[n]\\|\mathsf{S}|=k}} \Big\| \sum_{g\in\mathbb{Z}^{\mathsf{S}}} \hat{f}(g) e^{2\pi i\langle\,\cdot\,,g\rangle} \Big\|_p^p \lesssim_p \frac{k}{n} \sum_{j=1}^n \Big\| \frac{\partial}{\partial x_j} f \Big\|_p^p + \Big(\frac{k}{n}\Big)^{\frac{p}{2}} \||\nabla f|\|_p^p.$$

Inequalities (2-1) and (2-2) improve the above inequality replacing $|\nabla f|$ by f.

(B) The word length. Let us now study which inequality do we get with the word length. The cocycle associated to it is infinite-dimensional, with an orthonormal basis which can be described as oriented edges in the coordinate axes of the Cayley graph of $\mathbb{Z}^n$. More precisely, the associated Gromov form on $\mathbb{D}[\mathbb{Z}^n]$ is

$$\langle 1_{\{g\}}, 1_{\{h\}}\rangle_{\psi_1} = \tfrac{1}{2}(\psi_1(g) + \psi_1(h) - \psi_1(h-g)) = \sum_{j=1}^n \min\{|h_j|, |g_j|\}\delta_{g_j\cdot h_j>0}.$$

Given $g \in \mathbb{Z}^n$ and $j \in [n]$, define

$$g_{[j]}^- = g - \operatorname{sgn}(g_j)e_j, \quad \text{with } \operatorname{sgn}(0) = 0.$$

Then, we may construct the following elements in $\mathbb{D}[\mathbb{Z}^n]$:

$$w_{g,j} = 1_{\{g\}} - 1_{\{g_{[j]}^-\}} \quad \text{and} \quad u_j(\ell) = w_{\ell e_j, j}.$$

Below, it is convenient to keep in mind that $u_j(\ell) = 1_{\{\ell e_j\}} - 1_{\{(\ell-\operatorname{sgn}(\ell))e_j\}}$. If $\mathcal{H}_{\psi_1} = \mathbb{D}[\mathbb{Z}^n]/\operatorname{Ker}\langle\,\cdot\,,\cdot\,\rangle_{\psi_1}$, the following properties define an orthonormal basis:

- $\langle u_j(\ell), u_j(\ell)\rangle_{\psi_1} = 1$ for all $(j,\ell) \in [n]\times\mathbb{Z}\backslash\{0\}$.
- $\langle u_j(\ell), u_{j'}(\ell')\rangle_{\psi_1} = 0$ whenever $j \neq j'$ or $\ell \neq \ell'$.
- $w_{g,j} = u_j(\ell)$ if $g_j = \ell e_j$, since the difference belongs to $\operatorname{Ker}\langle\,\cdot\,,\cdot\,\rangle_{\psi_1}$.

Altogether, this implies that the image in $\mathcal{H}_{\psi_1}$ of the set

$$\{u_j(\ell) : (j,\ell) \in [n]\times\mathbb{Z}\setminus\{0\}\}$$

is an orthonormal basis for $\mathcal{H}_{\psi_1}$. We shall identify $1_{\{g\}}$ and $u_j(\ell)$ with their image in the quotient. The cocycle map is given by $\beta(g) = 1_{\{g\}}$ and the orthogonal action α satisfies $\alpha_g(1_{\{h\}}) = 1_{\{g+h\}} - 1_{\{g\}}$. This means that for any $g \in \mathbb{Z}^n$ we have

$$\alpha_g(u_j(\ell)) = 1_{\{g+\ell e_j\}} - 1_{\{g+\ell e_j-(\operatorname{sgn}(\ell))e_j\}}.$$

Therefore, the subspaces $\mathcal{H}_{\psi_1,j} = \operatorname{span}\{u_j(\ell) : \ell \in \mathbb{Z}\setminus\{0\}\}$ are α-invariant for $j \in [n]$. This proves that the same conditional expectations $\mathsf{E}_{[n]\setminus\mathsf{S}}$ considered before still define an admissible family of Fourier truncations. The cocycle derivative associated to $u_j(\ell)$ acts as follows:

$$\begin{aligned}\partial_{u_j(\ell)}\lambda(g) &= \langle u_j(\ell), 1_{\{g\}}\rangle_{\psi_1}\lambda(g)\\ &= (\min\{|g_j|, |\ell|\}\delta_{g_j\cdot\ell>0} - \min\{|g_j|, |\ell|-1\}\delta_{g_j\cdot(\ell-\operatorname{sgn}(\ell))>0})\lambda(g)\\ &= \delta_{\{g_j\cdot\ell>0,\,|g_j|\geq|\ell|\}}\lambda(g).\end{aligned}$$

The Laplacian is

$$\Delta_{\psi_1} f = \sum_{g \in \mathbb{Z}^n} \psi_1(g) \hat{f}(g) \lambda(g),$$

whose spectral gap is still $\sigma = \min_j \psi_1(e_j) = 1$. Theorem A yields

$$\frac{1}{\binom{n}{k}} \sum_{\substack{\mathsf{S} \subseteq [n] \\ |\mathsf{S}|=k}} \Big\| \sum_{g \in \mathbb{Z}^{\mathsf{S}}} \hat{f}(g) e^{2\pi i \langle \cdot, g \rangle} \Big\|_{L_p(\mathbb{T}^n)}^p \lesssim_p \frac{k}{n} \sum_{j=1}^n \| |\mathrm{D}_j f| \|_{L_p(\mathbb{T}^n)}^p + \Big(\frac{k}{n}\Big)^{\frac{p}{2}} \|f\|_{L_p(\mathbb{T}^n)}^p, \tag{2-3}$$

with

$$\| |\mathrm{D}_j(f)| \|_{L_p(\mathbb{T}^n)} = \| |\mathrm{D}_j(f^*)| \|_{L_p(\mathbb{T}^n)} = \Big\| \Big(\sum_{\ell \in \mathbb{Z} \setminus \{0\}} |\partial_{u_j(\ell)} f|^2 \Big)^{\frac{1}{2}} \Big\|_{L_p(\mathbb{T}^n)}.$$

Remark 2.2. Note that $|\partial_{u_j(\ell)}(f)| \neq |\partial_{u_j(\ell)}(f^*)|$. Thus, nontrivial cocycle actions lead to noncommutative phenomena even when working with abelian groups, as pointed out in [13]. In spite of that, observe that $\langle 1_{\{-g\}}, u_j(\ell) \rangle_\psi = \langle 1_{\{g\}}, u_j(-\ell) \rangle_\psi$, which implies $\| |\mathrm{D}_j(f)| \|_p = \| |\mathrm{D}_j(f^*)| \|_p$ as claimed above.

On the other hand, taking

$$\partial_j \lambda(g) := \partial_{u_j(1)} \lambda(g) + \partial_{u_j(-1)} \lambda(g) = \delta_{g_j \neq 0} \lambda(g),$$

we get $\partial_{u_j(\ell)} \circ \partial_j = \partial_{u_j(\ell)}$ for any $(j, \ell) \in [n] \times \mathbb{Z} \setminus \{0\}$. Thus, Theorem B gives

$$\frac{1}{\binom{n}{k}} \sum_{\substack{\mathsf{S} \subseteq [n] \\ |\mathsf{S}|=k}} \Big\| \sum_{g \in \mathbb{Z}^{\mathsf{S}}} \hat{f}(g) e^{2\pi i \langle \cdot, g \rangle} \Big\|_{L_p(\mathbb{T}^n)}^p \lesssim_p \frac{k}{n} \sum_{j=1}^n \| \partial_j f \|_{L_p(\mathbb{T}^n)}^p + \Big(\frac{k}{n}\Big)^{\frac{p}{2}} \|f\|_{L_p(\mathbb{T}^n)}^p$$

for any mean-zero $f \in L_p(\mathbb{T}^n)$. Here, ∂_j is the same as in (A), and so this recovers inequality (2-2) and improves (2-3). Note that above, both $\partial_{u_j(\ell)}$ and ∂_j are $\{0, 1\}$-valued multipliers.

2B. *Discrete tori.* Consider the word length $|g| = \min\{g, 2m - g\}$ in $\mathbb{Z}_{2m} = \widehat{\mathbb{Z}}_{2m}$. Therefore, for us $\mathbb{Z}_{2m} = \{0, 1, \dots, 2m-1\}$. As shown in [12], $|\cdot|$ defines a conditionally negative symmetric length. In particular the same holds for the corresponding length in the product $\mathbb{Z}_{2m}^n$

$$\psi(g) = |g_1| + |g_2| + \cdots + |g_n| \quad \text{for } g = (g_1, \dots, g_n) \in \mathbb{Z}_{2m}^n.$$

This word length has many similarities with the previous one

$$\langle 1_{\{g\}}, 1_{\{h\}} \rangle_\psi = \frac{1}{2}(\psi(g) + \psi(h) - \psi(h - g)) = \frac{1}{2} \sum_{j=1}^n |g_j| + |h_j| - |h_j - g_j|.$$

Given $g \in \mathbb{Z}_{2m}^n$ and $j \in [n]$, define

$$w_{g,j} = 1_{\{g\}} - 1_{\{g - e_j\}} \quad \text{and} \quad u_j(\ell) = w_{\ell e_j, j} \quad \text{for } 1 \leq \ell \leq 2m.$$

If $\mathcal{H}_\psi = \mathbb{D}[\mathbb{Z}_{2m}^n] / \operatorname{Ker}\langle \cdot, \cdot \rangle_\psi$, we find that:

- $\langle u_j(\ell), u_j(\ell) \rangle_\psi = 1$ for all $(j, \ell) \in [n] \times [m]$.
- $\langle u_j(\ell), u_{j'}(\ell') \rangle_\psi = 0$ whenever $j \neq j'$ or $\ell \neq \ell', \ell' + m$.

- $w_{g,j} = u_j(\ell)$ if $g_j = \ell e_j$ since the difference belongs to $\mathrm{Ker}\langle \cdot, \cdot \rangle_\psi$.
- $u_j(\ell) = -u_j(\ell + m)$ since the difference belongs to $\mathrm{Ker}\langle \cdot, \cdot \rangle_\psi$.
- $\langle 1_{\{\ell e_j\}}, 1_{\{\ell' e_j\}} \rangle_\psi = \min\{\ell, 2m - \ell', \max\{0, m - \ell' + \ell\}\}$ for $1 \leq \ell \leq \ell' \leq 2m$.

Altogether, this implies that the set

$$\{u_j(\ell) : (j, \ell) \in [n] \times [m]\}$$

is an orthonormal basis for $\mathcal{H}_\psi$. The cocycle map is given by $\beta(g) = 1_{\{g\}}$, by which we mean again that $\beta(g)$ is the image of $1_{\{g\}}$ in the quotient, and the orthogonal action α satisfies $\alpha_g(1_{\{h\}}) = 1_{\{g+h\}} - 1_{\{g\}}$. This means that for any $g \in \mathbb{Z}_{2m}^n$ we have

$$\alpha_g(u_j(\ell)) = 1_{\{g+\ell e_j\}} - 1_{\{g+(\ell-1)e_j\}}.$$

Therefore, the subspaces $\mathcal{H}_{\psi,j} = \mathrm{span}\{u_j(\ell) : \ell \in [m]\}$ give again an α-invariant splitting of $\mathcal{H}_\psi$ with j running over $[n]$. In particular, the conditional expectations $\mathsf{E}_{[n]\setminus \mathsf{S}}$ over the subgroups $\mathbb{Z}_{2m}^{\mathsf{S}}$ define an admissible family of Fourier truncations and the cocycle derivatives are given by

$$\partial_{u_j(\ell)} \lambda(g) = \delta_{\{\ell \leq g_j < \ell + m\}} \lambda(g).$$

The associated Laplacian has spectral gap equal to 1. As before, Theorem A yields a statement that we omit because it can readily be improved. If we set as before $\partial_j \lambda(g) := \delta_{g_j \neq 0} \lambda(g)$ for $j \in [n]$, we immediately see that $\partial_{u_j(\ell)} \circ \partial_j = \partial_{u_j(\ell)}$. Moreover, we can rewrite it as

$$\partial_j \lambda(g) = \partial_{u_j(1)} \lambda(g) + \partial_{u_j(m)} \lambda(g) - \delta_{g_j = m} \lambda(g).$$

Next, note that we may write the last term as

$$\delta_{g_j=m} \lambda(g) = \mathsf{E}_{\{0,m\},j}(\partial_{u_j(1)} \lambda(g)) = \tfrac{1}{2} \mathsf{E}_{\{0,m\},j}(\partial_{u_j(1)} \lambda(g) + \partial_{u_j(m)} \lambda(g)),$$

where $\mathsf{E}_{\{0,m\},j}$ is the conditional expectation onto $\mathbb{Z}_{2m}^{j-1} \times \{0, m\} \times \mathbb{Z}_{2m}^{n-j}$. Then, after applying Theorem B one gets the following for mean-zero $f : \mathbb{Z}_{2m}^n \to \mathbb{C}$:

$$\begin{aligned} \frac{1}{\binom{n}{k}} \sum_{\substack{\mathsf{S} \subseteq [n] \\ |\mathsf{S}| = k}} \Big\| \sum_{g \in \mathbb{Z}_{2m}^{\mathsf{S}}} \hat{f}(g) e^{\frac{\pi i}{m} \langle \cdot, g \rangle} \Big\|_{L_p(\mathbb{Z}_{2m}^n)}^p &\lesssim_p \frac{k}{n} \sum_{j=1}^n \|\partial_j f\|_{L_p(\mathbb{Z}_{2m}^n)}^p + \Big(\frac{k}{n}\Big)^{\frac{p}{2}} \|f\|_{L_p(\mathbb{Z}_{2m}^n)}^p \\ &\lesssim \frac{k}{n} \sum_{j=1}^n \|(\partial_{u_j(1)} + \partial_{u_j(m)}) f\|_{L_p(\mathbb{Z}_{2m}^n)}^p + \Big(\frac{k}{n}\Big)^{\frac{p}{2}} \|f\|_{L_p(\mathbb{Z}_{2m}^n)}^p. \end{aligned} \tag{2-4}$$

As before, both derivatives $\partial_{u_j(\ell)}$ and ∂_j turn out to be $\{0, 1\}$-valued multipliers.

Remark 2.3. It is natural to ask if the situation changes much when the cyclic groups under consideration have odd cardinal. The function $\psi(g) = \sum_{j=1}^n |g_j|$, with $|g_j| = \min\{g_j, 2m + 1 - g_j\}$, is a conditionally negative length on $\mathbb{Z}_{2m+1}^n$, and so there exists an associated cocycle induced by the Gromov form

$$\begin{aligned}\langle 1_{\{g\}}, 1_{\{h\}}\rangle &= \tfrac{1}{2}(\psi(g)+\psi(h)-\psi(g-h))\\ &= \sum_{j=1}^{n} \min\bigl\{g_j, 2m+1-h_j, \max\bigl\{0, m-h_j+g_j+\tfrac{1}{2}\bigr\}\bigr\}.\end{aligned}$$

It defines a cocycle Hilbert space $\mathcal{H}_\psi$ with dimension $2mn$. Theorems A and B apply but calculating an explicit expression for the orthonormal basis of $\mathcal{H}_\psi$ is more tedious and we shall leave it to the interested reader.

We end this subsection with a few comments on the case $m=1$ of the above construction.

Remark 2.4. Inequality (2-4) for $G=\mathbb{Z}_{2m}^n$ with $m=1$ recovers Naor's inequality (N_p) for the hypercube. Indeed, specializing the above computations in this case means that we take $\mathcal{H}=\mathbb{D}[\mathbb{Z}_2^n]/\operatorname{Ker}\langle\cdot,\cdot\rangle_\psi$ and consider the trivial cocycle β_0 given by $\{0,1\}^n \ni g \mapsto 1_{\{g\}}$, with the action

$$\alpha_{0,g}(\xi) = ((-1)^{g_1}\xi_1, (-1)^{g_2}\xi_2, \dots, (-1)^{g_n}\xi_n).$$

With this construction, given $L_p(\mathbb{Z}_2^n) \ni f = \sum_{g\in\mathbb{Z}_2^n} \hat{f}(g)\exp(\pi i\langle\cdot, g\rangle)$, $\partial_j^1 = 2\partial_j^2$, where ∂_j^1 is the discrete derivative used by Naor and ∂_j^2 is our choice of ∂_j in (2-4) for $m=1$.

Remark 2.5. One can also recover (N_p) from Theorem A using multiplicative notation directly. This however requires us to employ a nontrivial cocycle that we next describe. Set $G=\Omega^n$, $e=(1,\dots,1)$, and define the cocycle $\beta_1 : G \to \mathbb{R}^n$ by $G \ni h \mapsto e-h$ (the sum is the usual one in $\mathbb{R}^n$). This satisfies the cocycle law with respect to the — nontrivial — action

$$\alpha_{1,h}(\xi) = (h_1\xi_1, \dots, h_n\xi_n), \quad \xi \in \mathbb{R}^n.$$

One can see that if $g \in \mathbb{Z}_2^n$ is identified in the natural way with

$$h(g) := (\exp(\pi i g_1), \dots, \exp(\pi i g_1)) \in \Omega^n,$$

then

$$\beta_1(h(g)) = 2\beta_0(g).$$

Therefore, the cocycle derivatives are the same, up to a constant and modulo identification of characters, and the application of Theorem A yields the same inequality in both cases.

Remark 2.6. We can consider weighted forms of Naor's inequality by considering different measures on the same group Ω^n to get different cocycle representations. One could hope to get an improvement over the result in [19] in this way, but we next show that this is not the case. We borrow the aforementioned cocycle representations from the construction in Example B in [13, Section 1.4], that we use as follows: Let $G=\{-1,1\}^n$ and equip $\Gamma=\widehat{G}=\{-1,1\}^n$ with the measure

$$\mu = \sum_{j=1}^{n} \alpha_j 1_{\{w_j\}},$$

with $\alpha_j \geq 0$ and $w_j = (1, \dots, 1, -1, 1, \dots, 1)$ (change the sign in the j-th coordinate only) for $j \in [n]$. Viewing Γ as the power set of $[n]$, we identify w_j with $\{j\}$. We consider the conditionally negative length function

$$\psi(\mathsf{A}) := \|1 - W_\mathsf{A}\|_{L_2(\Gamma,\mu)}^2.$$

Then ψ may be represented by the cocycle $(\mathcal{H}_\psi, \alpha, \beta)$, with

$$\mathcal{H}_\psi = L_2(\Gamma, \mu), \quad \alpha_\mathsf{A}(u) = W_\mathsf{A} \cdot u, \quad \beta(\mathsf{A}) = 1 - W_\mathsf{A}.$$

The map β is indeed a cocycle. Then $\{u_j = \alpha_j^{-1/2} 1_{\{j\}} : j \in [n]\}$ is an orthonormal basis, and the cocycle derivatives are given by

$$\partial_{u_j} W_\mathsf{A} = \frac{1}{\sqrt{\alpha_j}} \langle \beta(\mathsf{A}), 1_{\{j\}} \rangle_\psi W_\mathsf{A} = 2\sqrt{\alpha_j} 1_\mathsf{A}(j) W_\mathsf{A} = \sqrt{\alpha_j} \partial_j W_\mathsf{A},$$

where ∂_j denotes the j-th discrete derivative. Riesz transforms take the form

$$R_{u_j} f = \sum_{\mathsf{A} \subset [n]} \frac{\langle \beta(\mathsf{A}), 1_{\{j\}} \rangle_\psi}{\sqrt{\alpha_j \psi(\mathsf{A})}} \hat{f}(\mathsf{A}) W_\mathsf{A} = \sum_{\substack{\mathsf{A} \subset [n] \\ j \in \mathsf{A}}} \frac{\sqrt{\alpha_j}}{\sqrt{\sum_{\ell \in A} \alpha_\ell}} \hat{f}(\mathsf{A}) W_\mathsf{A}.$$

Consider the decomposition $\mathcal{H}_{\psi,j} = \mathbb{R} 1_{\{j\}}$. Note $\alpha_\mathsf{A}(1_{\{j\}}) = W_\mathsf{A} 1_{\{j\}} = (-1)^{1_\mathsf{A}(j)} 1_{\{j\}}$, so $\alpha_\mathsf{A}(\mathcal{H}_{\psi,j}) \subset \mathcal{H}_{\psi,j}$ and the decomposition is equivariant. Therefore, the associated conditional expectation can be chosen to be

$$\mathsf{E}_{[n] \setminus \mathsf{S}} f = \sum_{\beta(\mathsf{A}) \in \mathcal{H}_\mathsf{S}} \hat{f}(\mathsf{A}) W_\mathsf{A} = \sum_{\mathsf{A} \subset \mathsf{S}} \hat{f}(\mathsf{A}) W_\mathsf{A}.$$

Then, Theorem A yields

$$\begin{aligned} \frac{1}{\binom{n}{k}} \sum_{\substack{\mathsf{S} \subseteq [n] \\ |\mathsf{S}| = k}} \Big\| \sum_{\mathsf{A} \subset \mathsf{S}} \hat{f}(\mathsf{A}) W_\mathsf{A} \Big\|_{L_p(\Omega^n)}^p &\lesssim \frac{1}{\sigma^{p/2}} \frac{k}{n} \sum_{j=1}^n \alpha_j^{\frac{p}{2}} \|\partial_j f\|_{L_p(\Omega^n)}^p + \Big(\frac{k}{n}\Big)^{\frac{p}{2}} \|f\|_{L_p(\Omega^n)}^p \\ &= \frac{k}{n} \sum_{j=1}^n \Big(\frac{\alpha_j}{\min_{k \in [n]} \alpha_k} \Big)^{\frac{p}{2}} \|\partial_j f\|_{L_p(\Omega^n)}^p + \Big(\frac{k}{n}\Big)^{\frac{p}{2}} \|f\|_{L_p(\Omega^n)}^p, \end{aligned}$$

since $\sigma = \min_{k \in [n]} \psi(\{k\}) = 4 \min_{k \in [n]} \alpha_k$. Thus taking $\alpha_j = 1$ for all j, which corresponds to (N_p), is the optimal choice.

3. Applications to free products

We now explore applications of Theorem B after replacing the direct products in the previous section by free products. Given a free product $\mathrm{G} = \mathrm{G}_1 * \mathrm{G}_2 * \cdots * \mathrm{G}_n$ a general element $g \in \mathrm{G}$ can always be written in reduced form $g = g_{i_1} g_{i_2} \cdots g_{i_s}$ where $g_{i_k} \in \mathrm{G}_{i_k}$ and $i_1 \neq i_2 \neq \cdots \neq i_s$. We shall be working with the free group $\mathbb{F}_n = \mathbb{Z} * \mathbb{Z} * \cdots * \mathbb{Z}$ and with the free product $\mathbb{Z}_{2m}^{*n}$ of n copies of $\mathbb{Z}_{2m}$. In both cases we shall write $e_1, e_2, \dots, e_n$ for the canonical generators and a generic element will be a word of the form

$$w = e_{i_1}^{\ell_1} e_{i_2}^{\ell_2} \cdots e_{i_s}^{\ell_s},$$

with $i_1 \neq i_2 \neq \cdots \neq i_s$ and ℓ_k in $\mathbb{Z}$ or $\mathbb{Z}_{2m}$ accordingly.

3A. ***The free group.*** Define

$$|w| = \sum_{j=1}^{r} |\ell_j| \quad \text{for } w = e_{i_1}^{\ell_1} \cdots e_{i_r}^{\ell_r}.$$

Haagerup proved in [3] that it is conditionally negative. The cocycle structure naturally induced by the word length $|\cdot|$ can be described through the Hilbert space orthonormaly generated by outgoing oriented edges in its Cayley graph. To be more precise, let us consider the following partial order on $\mathbb{F}_n$. Given $w_1 = e_{i_1}^{\ell_1} \cdots e_{i_r}^{\ell_r}$ and $w_2 = e_{j_1}^{t_1} \cdots e_{j_s}^{t_s}$, with $\ell_j, t_j \in \mathbb{Z} \setminus \{0\}$, we say that $w_1 \leq w_2$ when

- $r \leq s$,
- $e_{i_k}^{\ell_k} = e_{j_k}^{t_k}$ for $1 \leq k \leq r-1$,
- $e_{i_r} = e_{j_r}$, $\ell_r t_r > 0$ and $|\ell_r| \leq |t_r|$.

Any $w_1 \leq w_2$ is called an initial subchain of w_2. As we did with elements of cyclic groups equipped with their natural order structure, we can now define predecessors. If $w = e_{i_1}^{\ell_1} \cdots e_{i_r}^{\ell_r} \neq e$, we define

$$w^- = e_{i_1}^{\ell_1} \cdots e_{i_r}^{\ell_r - \mathrm{sgn}(\ell_r)}.$$

The Gromov form takes the following form in this case:

$$\langle 1_{\{w_1\}}, 1_{\{w_2\}} \rangle_{|\cdot|} = \tfrac{1}{2}(|w_1| + |w_2| - |w_1^{-1} w_2|) = |\min\{w_1, w_2\}|,$$

where $\min\{w_1, w_2\}$ denotes the longest word which is an initial chain of both w_1 and w_2. Given $w \neq e$ in $\mathbb{F}_n$, we define $u_w = 1_{\{w\}} - 1_{\{w^-\}} \in \mathbb{D}[\mathbb{F}_n]$. Very much like in the previous section, we find the following properties:

- $\mathrm{Ker}(\langle \cdot, \cdot \rangle_{|\cdot|}) = \mathbb{R} 1_{\{e\}}$.
- $\langle u_w, u_w \rangle_{|\cdot|} = 1$ for $w \in \mathbb{F}_n \setminus \{e\}$.
- $\langle u_{w_1}, u_{w_2} \rangle_{|\cdot|} = 0$ for $w_1 \neq w_2$ in $\mathbb{F}_n$.

This proves that

$$\{u_w : w \in \mathbb{F}_n \setminus \{e\}\}$$

is an orthonormal basis of $\mathcal{H}_{|\cdot|} = \mathbb{D}[\mathbb{F}_n]/\mathbb{R}1_{\{e\}}$. The cocycle map and the cocycle action are determined as usual by $\beta(w) = 1_{\{w\}}$ and $\alpha_w(1_{\{w'\}}) = 1_{\{ww'\}} - 1_{\{w\}}$.The cocycle derivative in the direction of u_w is

$$\partial_{u_w} \lambda(w') = \langle \beta(w'), u_w \rangle \lambda(w') = \delta_{w \leq w'} \lambda(w') \quad \Longrightarrow \quad \partial_{u_w} f = \sum_{w \leq w'} \hat{f}(w') \lambda(w').$$

Next, we decompose $\mathcal{H}_{|\cdot|}$ as

$$\mathcal{H}_{|\cdot|} = \bigoplus_{j=1}^{n} \mathcal{H}_{|\cdot|, j}, \quad \text{with } \mathcal{H}_{|\cdot|, j} = \mathrm{span}\{u_w : e_j \leq w \text{ or } e_j^{-1} \leq w\}.$$

This leads to consider the Fourier truncations

$$\mathsf{E}_{[n] \setminus \mathsf{S}} f := \sum_{w \in \mathbb{F}_{\mathsf{S}}} \hat{f}(w) \lambda(w).$$

Being conditional expectations, these Fourier truncations are completely contractive and pairwise β-orthogonality holds since we trivially have $\beta(\mathbb{F}_{\mathsf{S}}) = \beta(\mathbb{F}_{\mathsf{S}}^{-1}) \subset \mathcal{H}_{|\cdot|,\mathsf{S}}$. Define $\mathbb{A}_{\{j\}} \subset \mathbb{F}_n$ to be the set of reduced words that start with e_j^ℓ for some $\ell \in \mathbb{Z} \setminus \{0\}$, so that $\mathcal{H}_{|\cdot|,j} = \operatorname{span}\{u_w : w \in \mathbb{A}_{\{j\}}\}$. Taking the derivatives

$$\partial_j := \partial_{u_{e_j}} + \partial_{u_{e_j^{-1}}} \quad \text{for } j \in [n],$$

we can readily check that $\partial_{u_w} \circ \partial_j = \partial_{u_w}$ whenever $u_w \in \mathcal{H}_{|\cdot|,j}$, so that ∂_j is the projection onto words in $\mathbb{A}_{\{j\}}$. In conclusion, we have checked all the hypotheses to apply Theorem B for our family of Fourier truncations. In this case we get

$$\frac{1}{\binom{n}{k}} \sum_{\substack{\mathsf{S}\subset[n]\\ |\mathsf{S}|=k}} \Big\| \sum_{w\in\mathbb{F}_{\mathsf{S}}} \hat{f}(w)\lambda(w) \Big\|_p^p \lesssim_p \frac{k}{n} \sum_{j=1}^n [\|\partial_j(f)\|_p^p + \|\partial_j(f^*)\|_p^p] + \Big(\frac{k}{n}\Big)^{\frac{p}{2}} \|f\|_p^p. \tag{3-1}$$

Inequality (3-1) is very close to the conjectured free form of Naor's inequality (FN_p) in the Introduction, with an extra adjoint term which we shall eliminate at the end of the paper by proving an even stronger inequality.

3B. ***The free product*** $\mathbb{Z}_{2m}^{*n}$**.** A similar analysis applies as well in this case. Given two reduced words $w_1 = e_{i_1}^{\ell_1} \cdots e_{i_r}^{\ell_r}$ and $w_2 = e_{j_1}^{t_1} \cdots e_{j_s}^{t_s}$, with $\ell_j, t_j \in [2m-1]$, we say that $w_1 \le w_2$ if and only if

- $r \le s$,
- $i_k = j_k$ for any $k \in [r]$ and $\ell_k = t_k$ for any $k \in [r-1]$,
- either $\ell_r, t_r \in [m]$ and $i_r \le j_r$, or $i_r, j_r \in [2m-1] \setminus [m-1]$ and $i_r \ge j_r$.

The map $\psi : \mathbb{Z}_{2m}^{*n} \to \mathbb{R}_+$ given by

$$w = e_{i_1}^{\ell_1} \cdots e_{i_r}^{\ell_r} \mapsto \psi(w) = \sum_{k=1}^r |e_{i_j}^{\ell_j}| = \sum_{k=1}^r \min\{\ell_k, 2m - \ell_k\}$$

is a conditionally negative length function [3], with associated Gromov form

$$\begin{aligned} \langle 1_{\{w_1\}}, 1_{\{w_2\}} \rangle_\psi &= \tfrac{1}{2}(\psi(w_1) + \psi(w_2) - \psi(w_1^{-1}w_2)) \\ &= \psi(\min\{w_1, w_2\}) + \tfrac{1}{2}(\psi(\eta_1) + \psi(\eta_2) - \psi(\eta_1^{-1}\eta_2)), \end{aligned} \tag{3-2}$$

where $\min\{w_1, w_2\}$ is again the longest common subchain and $w_j = \min\{w_1, w_2\}\eta_j$ for $j = 1, 2$. The second term above is always 0 in the free group $\mathbb{F}_n$, but not necessarily in this case. Given $w = e_{i_1}^{\ell_1} \cdots e_{i_r}^{\ell_r} \neq e$ we define $w^- = e_{i_1}^{\ell_1} \cdots e_{i_r}^{\ell_r - 1}$ and construct $u_w = 1_{\{w\}} - 1_{\{w^-\}}$ as usual. Then, we find that:

- $\langle u_w, u_w \rangle_\psi = 1$ for every $w \in \mathbb{Z}_{2m}^{*n} \setminus \{e\}$.
- $\langle u_{w_1}, u_{w_2} \rangle_\psi = 0$ when $e \neq w_1^{-1}w_2 \neq e_j^m$ for $j \in [n]$.
- $\langle u_{w_1}, u_{w_2} \rangle_\psi = 0$ when $w_1^{-1}w_2 = e_j^m$ and both w_1, w_2 end with $e_j^{\pm 1}$.
- If $w = e_{i_1}^{\ell_1} \cdots e_{i_r}^{\ell_r}$, then $u_w = -u_{we_{i_r}^m}$ in $\mathcal{H}_\psi = \mathbb{D}[\mathbb{Z}_{2m}^{*n}]/\operatorname{Ker}\langle \cdot, \cdot \rangle_\psi$.

This proves that

$$\{u_w : w = e_{i_1}^{\ell_1} \cdots e_{i_r}^{\ell_r} \in \mathbb{Z}_{2m}^{*n} \setminus \{e\} \text{ with } \ell_r \in [m]\}$$

is an orthonormal basis of $\mathcal{H}_\psi = \mathbb{D}[\mathbb{Z}_{2m}^{*n}]/\operatorname{Ker}\langle\,\cdot\,,\cdot\,\rangle_\psi$. We set as usual $\beta(w) = 1_{\{w\}}$ and $\alpha_w(1_{\{w'\}}) = 1_{\{ww'\}} - 1_{\{w\}}$. Among the above properties it is perhaps convenient to justify the last one. Note that $\langle u_w + u_{we_{i_r}^m}, u_w + u_{we_{i_r}^m}\rangle_\psi = 0$ if and only if $\langle u_w, u_{we_{i_r}^m}\rangle_\psi = -1$ but we have

$$\begin{aligned}\langle u_w, u_{we_{i_r}^m}\rangle_\psi &= \tfrac{1}{2}(-\psi(e_{i_r}^m) + \psi((w^-)^{-1}we_{i_r}^m) + \psi(e_{i_r}^{m-1}) - \psi((w^-)^{-1}we_{i_r}^{m-1}))\\ &= \tfrac{1}{2}(-\psi(e_{i_r}^m) + \psi(e_{i_r}^{m+1}) + \psi(e_{i_r}^{m-1}) - \psi(e_{i_r}^m))\\ &= \tfrac{1}{2}(-m + m - 1 + m - 1 - m) = -1.\end{aligned}$$

If $w = e_{i_1}^{\ell_1} \cdots e_{i_r}^{\ell_r}$ with $\ell_r \in [m]$, derivatives are given by

$$\partial_{u_w}\lambda(w') = \langle \beta(w'), u_w\rangle_\psi \lambda(w') = \delta_{w'\in W(w)}\lambda(w'), \tag{3-3}$$

where $W(w)$ is the set of those words $w' = e_{j_1}^{t_1} \cdots e_{j_s}^{t_s}$ satisfying

$$r \le s, \qquad i_k = j_k \quad \text{for } k \le r, \qquad \ell_k = t_k \quad \text{for } k \le r-1 \qquad \text{and} \qquad \ell_r \le t_r \le \ell_r + m - 1. \tag{3-4}$$

Indeed, just write $\beta(w') = 1_{\{w'\}} = u_{w'} + 1_{\{w'^-\}} = u_{w'} + u_{w'^-} + 1_{\{w'^{--}\}}$ and so on. The inner product with u_w will be 0 unless we find u_w in our telescopic sum above just once, in which case we get the value 1. Note that it could appear twice due to the identity $u_w = -u_{we_{i_r}^m}$ recalled above. In that case, they get mutually canceled and we get 0. This happens when $t_r - \ell_r \in [2m-1] \setminus [m-1]$.

It remains to consider Fourier truncations. As for the free group, our choice is the conditional expectation into the subgroup $\mathbb{Z}_{2m}^{*\mathsf{S}} = \langle e_j : j \in \mathsf{S}\rangle$, which is the free group generated by e_j for $j \in \mathsf{S}$. Then we consider the decomposition

$$\mathcal{H}_\psi = \bigoplus_{j=1}^n \mathcal{H}_{\psi,j}, \qquad \text{with } \mathcal{H}_{\psi,j} = \operatorname{span}\{u_w : e_j \le w \text{ or } e_j^{-1} \le w\}.$$

Our Fourier truncations form an admissible family. Define

$$\partial_j\lambda(w) = \partial_{u_{e_j}}\lambda(w) + \partial_{u_{e_j^m}}\lambda(w) - \delta_{e_{i_1}^{\ell_1} = e_j^m}\lambda(w) \quad \text{for any } w = e_{i_1}^{\ell_1} \cdots e_{i_r}^{\ell_r}.$$

In other words, $\partial_j\lambda(w) = \delta_{i_1=j}\lambda(w)$ for $w \neq e$ and $\partial_{u_w} \circ \partial_j = \partial_{u_w}$ for $u_w \in \mathcal{H}_{\psi,j}$. The construction above yields the form of Theorem B on the von Neumann algebra of the free product $\mathbb{Z}_{2m}^{*n}$

$$\frac{1}{\binom{n}{k}} \sum_{\substack{\mathsf{S}\subseteq[n]\\|\mathsf{S}|=k}} \Big\| \sum_{w\in\mathbb{Z}_{2m}^{*\mathsf{S}}} \hat{f}(w)\lambda(w) \Big\|_p^p \lesssim_p \frac{k}{n} \sum_{j=1}^n \|\partial_j(f)\|_p^p + \|\partial_j(f^*)\|_p^p + \Big(\frac{k}{n}\Big)^{\frac{p}{2}} \|f\|_p^p.$$

3C. *Free Hilbert transforms.* Compared to (FN_p), the form of Theorem B for free groups gives the additional terms $\partial_j(f^*)$. These terms seem to be necessary in the general context of Theorem B, but they are removable for free groups — in fact, we shall prove an even stronger inequality — due to a singular behavior of word-length derivatives for free groups. Said behavior means in particular that word-length derivatives can be regarded as free forms of directional Hilbert transforms, which were recently investigated by Mei and Ricard in [17]. The free Hilbert transforms for mean-zero f are defined as

$$H_\varepsilon(f) = \sum_{j=1}^{n} \varepsilon_j \partial_j(f) \quad \text{for } \varepsilon_j = \pm 1.$$

Mei and Ricard proved in [17] the crucial inequality

$$\|H_\varepsilon f\|_{L_p(\mathcal{L}(\mathbb{F}_n))} \asymp_p \|f\|_{L_p(\mathcal{L}(\mathbb{F}_n))} \quad \text{for any } 1 < p < \infty. \tag{3-5}$$

Define

$$\mathbb{A}_{\mathsf{S}} = \bigcup_{j \in \mathsf{S}} \mathbb{A}_{\{j\}}.$$

Theorem 3.1. *If $p \geq 2$ and $k \in [n]$, every mean-zero $f \in L_p(\mathcal{L}(\mathbb{F}_n))$ satisfies*

$$\frac{1}{\binom{n}{k}} \sum_{\substack{\mathsf{S} \subseteq [n] \\ |\mathsf{S}| = k}} \Big\| \sum_{w \in \mathbb{A}_{\mathsf{S}}} \hat{f}(w)\lambda(w) \Big\|^p_{L_p(\mathcal{L}(\mathbb{F}_n))} \lesssim_p \frac{k}{n} \sum_{j=1}^{n} \|\partial_j(f)\|^p_{L_p(\mathcal{L}(\mathbb{F}_n))} + \Big(\frac{k}{n}\Big)^{\frac{p}{2}} \|f\|^p_{L_p(\mathcal{L}(\mathbb{F}_n))}.$$

Proof. Define

$$h = \sum_{w \in \mathbb{A}_{\mathsf{S}}} \hat{f}(w)\lambda(w) = \sum_{j \in \mathsf{S}} \sum_{w \in \mathbb{A}_{\{j\}}} \hat{f}(w)\lambda(w) = \sum_{j \in \mathsf{S}} \partial_j(f).$$

Applying inequality (3-5) we obtain

$$\|h\|_p \asymp_p \mathbb{E}\|H_\varepsilon(h)\|_p = \mathbb{E}\Big\| \sum_{j \in \mathsf{S}} \varepsilon_j \partial_j(f) \Big\|_p.$$

The result follows from Theorem 1.2 and another application of (3-5) for f. □

Corollary 3.2. *Inequality (FN$_p$) holds for $p \geq 2$ and any mean-zero $f \in L_p(\mathcal{L}(\mathbb{F}_n))$.*

Proof. It follows from Theorem 3.1 and the boundedness of the conditional expectation from $\mathcal{L}(\mathbb{F}_n)$ to $\mathcal{L}(\mathbb{F}_{\mathsf{S}})$

$$\Big\| \sum_{w \in \mathbb{F}_{\mathsf{S}}} \hat{f}(w)\lambda(w) \Big\|_p = \Big\| \sum_{w \in \mathbb{F}_{\mathsf{S}}} \hat{h}(w)\lambda(w) \Big\|_p \leq \|h\|_p = \Big\| \sum_{w \in \mathbb{A}_{\mathsf{S}}} \hat{f}(w)\lambda(w) \Big\|_p,$$

where h is defined as in the proof of Theorem 3.1, since we note that $\mathbb{F}_{\mathsf{S}} \subset \mathbb{A}_{\mathsf{S}}$. □

Remark 3.3. It is conceivable that Theorem 3.1 or at least Corollary 3.2 could have been proved as well from a generalized form of Theorem B in the line of Remark 1.4, but we have not found an argument using such an approach.

Remark 3.4. Hilbert transforms can also be constructed on $\mathcal{L}(\mathbb{Z}_{2m}^{*n})$. They are L_p-bounded maps as well there, as shown in [17, Theorem 3.5]. Therefore, Theorem 3.1 can also be proved with this technique replacing $\mathbb{F}_n$ by $\mathbb{Z}_{2m}^{*n}$ in the statement.

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BEURLING–CARLESON SETS, INNER FUNCTIONS AND A SEMILINEAR EQUATION

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Beurling–Carleson sets have appeared in a number of areas of complex analysis such as boundary zero sets of analytic functions, inner functions with derivative in the Nevanlinna class, cyclicity in weighted Bergman spaces, Fuchsian groups of Widom-type and the corona problem in quotient Banach algebras. After surveying these developments, we give a general definition of Beurling–Carleson sets and discuss some of their basic properties. We show that the Roberts decomposition characterizes measures that do not charge Beurling–Carleson sets.

For a positive singular measure μ on the unit circle, let S_μ denote the singular inner function with singular measure μ. In the second part of the paper, we use a corona-type decomposition to relate a number of properties of singular measures on the unit circle, such as membership of S'_μ in the Nevanlinna class $\mathcal{N}$, area conditions on level sets of S_μ and wepability. It was known that each of these properties holds for measures concentrated on Beurling–Carleson sets. We show that each of these properties implies that μ lives on a countable union of Beurling–Carleson sets. We also describe partial relations involving the membership of S'_μ in the Hardy space H^p, membership of S_μ in the Besov space B^p and $(1-p)$-Beurling–Carleson sets and give a number of examples which show that our results are optimal.

Finally, we show that measures that live on countable unions of α-Beurling–Carleson sets are almost in bijection with nearly maximal solutions of $\Delta u = u^p \cdot \chi_{u>0}$ when $p > 3$ and $\alpha = (p-3)/(p-1)$.

1. Introduction

A *Beurling–Carleson set* E is a closed subset of the unit circle $\partial\mathbb{D}$ of zero length whose complementary arcs $\{J\}$ satisfy

$$\|E\|_{\mathcal{BC}} = \sum_J |J| \log \frac{1}{|J|} < \infty. \tag{1-1}$$

Beurling–Carleson sets were introduced by A. Beurling [1940], who showed that they constitute boundary zero sets of holomorphic functions on the unit disk that are Hölder continuous up to the boundary. Several years later, L. Carleson [1952] constructed outer functions that vanished to arbitrary order on E. This construction was later improved to infinite order by Taylor and Williams [1970]. Since then, Beurling–Carleson sets appeared in a number of areas of complex analysis such as inner functions, weighted Bergman spaces, Fuchsian groups and the corona problem.

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In this paper, we will also consider Beurling–Carleson sets with respect to other gauge functions, although we will be mainly interested in usual Beurling–Carleson sets and α-Beurling–Carleson sets with $0 < \alpha < 1$. These are defined by the condition

$$\|E\|_{\mathcal{BC}_\alpha} = \sum_J |J|^\alpha < \infty \tag{1-2}$$

in place of (1-1).

1A. ***Derivative in Nevanlinna class.*** An inner function is a bounded analytic function on the unit disk $\mathbb{D}$ which has unimodular radial limits almost everywhere on $\partial\mathbb{D}$. Beurling–Carleson sets play an important role in understanding inner functions with derivative in the Nevanlinna class $\mathcal{N}$, which consists of analytic functions $f(z)$ on the unit disk for which

$$\lim_{r\to 1} \int_{|z|=r} \log^+ |f(z)| < \infty.$$

Suppose μ is a positive singular measure on the unit circle and

$$S_\mu(z) = \exp\biggl(-\int_{\partial\mathbb{D}} \frac{\zeta + z}{\zeta - z}\, d\mu(\zeta)\biggr), \quad |z| < 1,$$

is the associated singular inner function. On the unit circle, the radial boundary values of $|S'_\mu|$ are given by

$$|S'_\mu(z)| = 2\int_{\partial\mathbb{D}} \frac{|d\zeta|}{|\zeta - z|^2}, \quad |z| = 1,$$

which could be infinite. M. Cullen [1971] observed that if μ is concentrated on a Beurling–Carleson set, then $S'_\mu \in \mathcal{N}$. The converse does not hold in general: there are singular inner functions S_μ with $S'_\mu \in \mathcal{N}$ for which the support of μ is not contained in a single Beurling–Carleson set. One consequence of [Ivrii 2019] is that the condition $S'_\mu \in \mathcal{N}$ implies that μ lives on a countable union of Beurling–Carleson sets. The original proof used the classification of nearly maximal solutions of the Gauss curvature equation $\Delta u = e^{2u}$. In Section 4, we will give an elementary proof of this fact using a corona-type decomposition.

Theorem 1.1. *Let $\mu \geq 0$ be a singular measure on $\partial\mathbb{D}$. Consider the following conditions*:

(0) *The measure μ is supported on a Beurling–Carleson set.*

(1) $S'_\mu \in \mathcal{N}$.

(2) S_μ *satisfies the area condition*: *for every* $0 < c < 1$,

$$\int_{\{z\in\mathbb{D}:|S_\mu(z)|<c\}} \frac{dA(z)}{1-|z|} < \infty. \tag{1-3}$$

(3) *The measure μ is concentrated on a countable union of Beurling–Carleson sets.*

We have (0) $\Rightarrow$ (1) $\Rightarrow$ (2) $\Rightarrow$ (3).

1B. ***Quotient Banach algebras.*** Another important perspective on Beurling–Carleson sets stems from P. Gorkin, R. Mortini and N. Nikolskii [Gorkin et al. 2008] who studied the corona problem in the quotient space H^∞/IH^∞, where I is an inner function. They noticed that point evaluations at the zeros of I are dense in the maximal ideal space $\mathfrak{M}$ of H^∞/IH^∞ if and only if there exists a $0 < c < 1$ for which the sublevel set

$$\Omega_c = \{z \in \mathbb{D} : |I(z)| < c\}$$

is contained within a bounded hyperbolic distance of the zero set of I. In this case, one says that I has the *weak embedding property* (WEP). A. Borichev [2013] introduced the class of *wepable* inner functions, i.e., inner functions that could be made WEP if multiplied by a suitable Blaschke product. Consider the condition

(1′) *S_μ is wepable.*

In [Borichev et al. 2017], the authors proved that (0) ⇒ (1′) ⇒ (2). Together with the implication (2) ⇒ (3) from Theorem 1.1, this shows that up to countable unions, the collection of measures μ for which S_μ is wepable also coincides with measures that are concentrated on Beurling–Carleson sets.

Remark. Taking countable unions is necessary since there exist atomic measures μ for which S_μ is not wepable. See the proof of [Borichev et al. 2017, Theorem 3].

1C. ***Derivative in H^p.*** Next, we use a corona-type decomposition to study singular inner functions with derivative in the Hardy space H^p. We stick to the range of exponents $0 < p < \frac{1}{2}$ since derivatives of singular inner functions are never in $H^{1/2}$.

Theorem 1.2. *Suppose $0 < p < \frac{1}{2}$ and $\mu \geq 0$ is a singular measure on $\partial\mathbb{D}$. Consider the following conditions*:

(1) $S'_\mu \in H^p$.

(2) *S_μ satisfies the $(1+p)$-area condition*: *for every $0 < c < 1$*,

$$\int_{\{z\in\mathbb{D}:|S_\mu(z)|<c\}} \frac{dA(z)}{(1-|z|)^{1+p}} < \infty. \tag{1-4}$$

(3) *The measure μ is concentrated on a countable union of $(1-p)$-Beurling–Carleson sets.*

We have (1) ⇒ (2) ⇒ (3).

Unfortunately, it is no longer true that if μ is supported on a $(1-p)$-Beurling–Carleson set, then $S'_\mu \in H^p$.

We say that a finite measure $\mu \geq 0$ satisfies a property *up to countable sums* if it can be written as a countable sum of finite measures $\mu_k \geq 0$ satisfying the property. In Section 5, we will see that conditions (1) and (3) are different even after allowing countable sums. Nevertheless, in Section 6, we will show that conditions (1) and (2) agree after passing to countable sums.

We mention an additional condition on the measure μ, equivalent to (2), due to P. Ahern [1979] and A. Reijonen and T. Sugawa [Reijonen and Sugawa 2019]:

(2′) *We have*

$$\int_{\mathbb{D}} |S_\mu'(z)|^q (1-|z|^2)^{-p+(q-1)}\, dA(z) < \infty$$

for some (*and hence all*) $1 \le q \le 2$.

When $q = 1$, the above condition says that S_μ' belongs to the Besov space B^p. The implication (1) ⇒ (2′) can also be found in Ahern's paper.

1D. *Differential equations.* It was observed in [Ivrii 2019] that characterizing inner functions with derivative in Nevanlinna class amounts to understanding nearly maximal solutions of the Gauss curvature equation $\Delta u = e^{2u}$. These turn out to be in one-to-one correspondence with measures that live on countable unions of Beurling–Carleson sets. We refer the reader to Section 8 for the relevant definitions and background on semilinear equations.

In Section 9, we show the following theorem which partially characterizes the nearly maximal solutions of $\Delta u = u^p \cdot \chi_{u>0}$:

Theorem 1.3. (i) *When* $p > 3$, *deficiency measures of nearly maximal solutions are concentrated on countable unions of* α*-Beurling–Carleson sets, where* $\alpha = (p-3)/(p-1)$. *Conversely, any finite positive measure on the unit circle concentrated on a countable union of* β*-Beurling–Carleson sets for some* $\beta < \alpha$ *arises as the deficiency measure of some nearly maximal solution.*

(ii) *When* $1 < p \le 3$, *the only nearly maximal solution is the maximal one.*

It is natural to wonder if there is a precise correspondence between nearly maximal solutions of $\Delta u = u^p \cdot \chi_{u>0}$ and measures that live on countable unions of α-Beurling–Carleson sets. Unfortunately, with our current techniques, we are unable to either prove or disprove this tantalizing hypothesis.

2. Notes and references

2A. *Weighted Bergman spaces.* Beurling–Carleson sets also arise naturally in the study of cyclic functions in the weighted Bergman spaces A^p_α, which consists of holomorphic functions on the unit disk satisfying

$$\|f\|^p_{A^p_\alpha} = \int_{\mathbb{D}} |f(z)|^p (1-|z|)^\alpha\, dA(z) < \infty, \quad \alpha > -1, \quad 1 < p < \infty.$$

A function $f \in A^p_\alpha$ is *cyclic* if the closure of the set $\{pf : p \text{ polynomial}\}$ is dense in A^p_α. One question that puzzled mathematicians in the late 1960s was: *when is the singular inner function* S_μ *cyclic*? It was not difficult to show that if μ is concentrated on a Beurling–Carleson set, then the singular inner function S_μ could not be cyclic. In the other direction, it was known that if μ had modulus of continuity bounded by $Ct\log(1/t)$, then S_μ was cyclic. The gap between Beurling–Carleson sets and the $t \log 1/t$ condition stood for a number of years until it was resolved independently by B. Korenblum [1981] and J. Roberts [1985]. Roberts' approach used an elegant structure theorem for measures that do charge Beurling–Carleson sets. In Section 3, we will prove a converse of Roberts' result, thereby giving a description of positive singular measures that do not charge Beurling–Carleson sets.

2B. ***Model spaces.*** Let A^∞ denote the space of holomorphic functions on the open unit disk which extend to smooth functions on the closed unit disk. To an inner function $F(z)$, one can associate the *model space* $K_F = H^2 \ominus FH^2$. K. Dyakonov and D. Khavinson [Dyakonov and Khavinson 2006] were curious as to whether K_F contained smooth functions. They showed that $K_F \cap A^\infty = \{0\}$ if and only if $F = S_\mu$, where μ does not charge Beurling–Carleson sets.

In a recent work, A. Limani and B. Malman [Limani and Malman 2023a] asked the opposite question: when is $K_F \cap A^\infty$ dense in K_F? They showed that this occurs if and only if $F = BS_\mu$, where B is an arbitrary Blaschke product and μ is concentrated on a countable union of Beurling–Carleson sets.

2C. ***Character-automorphic functions.*** Widom [1971] and Pommerenke [1976a; 1976b] studied functions which were character-automorphic under Fuchsian groups of convergence type. A *character* v of a Fuchsian group $\Gamma \subset \mathrm{Aut}(\mathbb{D})$ is a homomorphism of Γ to the unit circle. A function f on the unit disk is called *character automorphic* if

$$f(\gamma(z)) = v(\gamma) \cdot f(z), \quad \gamma \in \Gamma.$$

One natural character automorphic function is the Blaschke product $g(z)$ whose zeros constitute an orbit of Γ (it is related to the Green's function of $\mathbb{D}/\Gamma$). If $g(z)$ has zeros at the points $\{\gamma(0) : \gamma \in \Gamma\}$, i.e.,

$$g(z) = \prod_{\gamma\in\Gamma} -\frac{\overline{\gamma(0)}}{|\gamma(0)|} \cdot \frac{z-\gamma(0)}{1-\overline{\gamma(0)}z},$$

then

$$|g'(z)| = \sum_{\gamma\in\Gamma} |\gamma'(z)|, \quad |z| = 1.$$

For a character v, let $H^\infty(\Gamma, v)$ denote the space of bounded holomorphic v-automorphic functions. Building on the work of Widom, Pommerenke [1976b] showed that

$$g' \in \mathcal{N} \quad \Longleftrightarrow \quad H^\infty(\Gamma, v) \neq \{\text{const}\} \text{ for every } v$$

and observed that the above condition is satisfied if the limit set $\Lambda(\Gamma)$ is a Beurling–Carleson set.

Pommerenke [1976a, Theorem 2] also showed that Λ is a Beurling–Carleson set if and only if there is a Γ-invariant holomorphic vector field $h(z)(\partial/\partial z)$ on the unit disk with $h'(z) \in H^\infty$.

2D. ***Fat Beurling–Carleson sets.*** A related class of sets was introduced by S. Khruschev, which is natural to call *fat Beurling–Carleson sets*. These are closed subsets of the unit circle which satisfy the entropy condition (1-1) but have positive Lebesgue measure. Amongst other things, Khruschev showed that if K is a closed subset of the unit circle which does not contain any fat Beurling–Carleson sets, then there is a sequence of polynomials $p_n(z)$ which tend to 1 in the Bergman space $A^2(\mathbb{D})$ but to 0 in $C(K)$. Conversely, if such a sequence of polynomials exists, then K cannot contain any fat Beurling–Carleson sets.

The proof presented in [Havin and Jöricke 1994, Chapter II.3] uses a structure theorem due to N. G. Makarov [1989]. Given a closed subset K of the circle which does not contain fat Beurling–Carleson sets and an arc $I \subset \partial\mathbb{D}$, there exists a measure $\mu = \mu_I$ supported on $I \setminus K$ which satisfies

(i) $\mu(I) \geq |I| \log \frac{1}{|I|}$,

(ii) $\mu(J) \leq 3|J| \log \frac{1}{|J|}$ for any arc $J \subseteq I$.

The first condition implies that μ has substantial mass, while the second condition says that μ is spread out.

For more applications of fat Beurling–Carleson sets, we refer the reader to [Limani and Malman 2023b; 2024; Malman 2023].

3. Beurling–Carleson sets

In this section, we give a general definition of Beurling–Carleson sets and discuss some of their basic properties. We say that $\phi : [0, 1] \to [0, \infty)$ is a *regular gauge function* if:

(G1) One can write

$$\phi(t) = t \cdot \phi_1(t) = t \int_t^1 \frac{ds}{\lambda(s)},$$

where $\lambda(t)$ is a nonnegative function such that $\int_0^1 (\lambda(s))^{-1}\, ds = \infty$.

(G2) The function $\lambda(t)$ satisfies the doubling condition

$$\lambda(\theta \cdot t) \asymp \lambda(t), \qquad \theta \in [1, 2]. \tag{3-1}$$

(G3) There exists a constant $C > 0$ such that

$$\sum_{k=0}^{\infty} \phi(2^{-k} t) \leq C\phi(t), \qquad t \in [0, 1].$$

A closed subset E of the unit circle of zero length is called a *ϕ-Beurling–Carleson set* if

$$\|E\|_{\mathcal{BC}_\phi} = \sum_k \phi(|J_k|) < \infty, \tag{3-2}$$

where the sum is over the complementary arcs $\{J_k\}$ of E.

For each $n \geq 0$, we can partition the unit circle into 2^n *dyadic arcs* of generation n:

$$\{z \in \partial\mathbb{D} : k \cdot 2^{-n} \cdot 2\pi < \arg z < (k+1) \cdot 2^{-n} \cdot 2\pi\}, \qquad k = 0, 1, \dots, 2^n - 1.$$

We denote the collection of dyadic arcs of generation n by $\mathcal{D}_n$. The *dyadic grid* $\mathcal{D} = \bigcup_{n=0}^{\infty} \mathcal{D}_n$ is the collection of all dyadic arcs.

Given a closed set E, the *Privalov star* K_E is defined as the union of the Stolz angles of opening $\frac{\pi}{2}$ emanating from points of E.

The following lemma provides several other characterizations of Beurling–Carleson sets:

Lemma 3.1. *Let E be a closed subset of the unit circle of zero length. Denote the complementary arcs by $\{J_k\}$, i.e., $\partial\mathbb{D} \setminus E = \bigcup J_k$. If ϕ is a regular gauge function, then the following quantities are comparable*:

(a) *Arc sum*: $$\sum_k \phi(|J_k|).$$

(b) *Distance integral*:
$$\int_{\partial\mathbb{D}\setminus E} \phi_1(\operatorname{dist}(x, E))\, dx.$$

(c) *Dyadic arc sum*:
$$\sum_{\substack{I \text{ dyadic} \\ I\cap E\neq\varnothing}} \frac{|I|^2}{\lambda(|I|)}.$$

(d) *Privalov star integral*:
$$\int_{K_E} \frac{dA(z)}{\lambda(1-|z|)}.$$

Remark. In (d), instead of integrating over the Privalov star K_E, one can also integrate over the region
$$\Omega_E = \mathbb{D} \setminus \bigcup_k Q_{J_k},$$
where
$$Q_J = \left\{ z \in \mathbb{D} : \frac{z}{|z|} \in J,\ 0 < 1 - |z| < |J| \right\}$$
is the Carleson box with base $J \subset \partial\mathbb{D}$. Alternatively, one can integrate over the domain
$$\Omega_E^{\text{dyadic}} = \bigcup_{\substack{I \text{ dyadic} \\ I\cap E\neq\varnothing}} T_I,$$
where
$$T_I = \left\{ z \in \mathbb{D} : \frac{z}{|z|} \in I,\ \frac{1}{2}|I| < 1 - |z| < |I| \right\}$$
denotes the top half of the Carleson box which rests on I.

Examples.

(i) If $\phi(t) = t \log t^{-1}$, then $\lambda(t) = t$ and we recover the usual Beurling–Carleson condition:
$$\sum_k |J_k| \log \frac{1}{|J_k|} \asymp \int_{[0,1]\setminus E} \log \frac{1}{\operatorname{dist}(x, E)} dx \asymp \sum_{\substack{I \text{ dyadic} \\ I\cap E\neq\varnothing}} |I| \asymp \int_{K_E} \frac{dA(z)}{1-|z|}.$$

(ii) If $\phi(t) = t^\alpha$ with $0 < \alpha < 1$, then $\lambda(t) \sim t^{2-\alpha}/(1-\alpha)$ as $t \to 0^+$ and we get the α-Beurling–Carleson condition:
$$\sum_k |J_k|^\alpha \asymp \int_{[0,1]\setminus E} \operatorname{dist}(x, E)^{\alpha-1} dx \asymp \sum_{\substack{I \text{ dyadic} \\ I\cap E\neq\varnothing}} |I|^\alpha \asymp \int_{K_E} \frac{dA(z)}{(1-|z|)^{2-\alpha}}.$$

Proof of Lemma 3.1. The comparability of the “arc sum” and the “distance integral” follows after subdividing each complementary interval J_k into Whitney arcs and applying the estimate (G3), while the comparability of the “distance integral” and the “Privalov star integral” follows from integrating in polar coordinates.

It remains to relate the “Privalov star integral” and the “dyadic arc sum”. By the doubling property (G2) of λ, we have
$$\int_{T_I} \frac{dA(z)}{\lambda(1-|z|)} \asymp \frac{|I|^2}{\lambda(|I|)}.$$

Summing over the dyadic arcs I which meet E gives

$$\int_{\Omega_E^{\text{dyadic}}} \frac{dA(z)}{\lambda(1-|z|)} \asymp \sum_{\substack{I \text{ dyadic} \\ I \cap E \neq \varnothing}} \frac{|I|^2}{\lambda(|I|)}.$$

Inspection shows that

$$\int_{\Omega_E^{\text{dyadic}}} \frac{dA(z)}{\lambda(1-|z|)} \asymp \int_{\Omega_E} \frac{dA(z)}{\lambda(1-|z|)} \asymp \int_{K_E} \frac{dA(z)}{\lambda(1-|z|)}. \tag{3-3}$$

The proof is complete. □

3A. *Dyadic grid with respect to a gauge function.* A ϕ-*dyadic grid* is a collection of dyadic arcs $\mathcal{D}_\phi = \bigcup_j \mathcal{D}_{n_j}$, where the sequence $\{n_j\}$ satisfies

$$\int_{2^{-n_{j+1}}}^{2^{-n_j}} \frac{dt}{\lambda(t)} \asymp \int_{2^{-n_j}}^{1} \frac{dt}{\lambda(t)} \asymp \phi_1(2^{-n_j}), \qquad j = 1, 2, \dots. \tag{3-4}$$

In particular, the above condition implies that $\phi_1(|I|) \asymp \phi_1(|J|)$ whenever $I \in \mathcal{D}_{n_{j+1}}$ and $J \in \mathcal{D}_{n_j}$.

Examples.

(i) If $\phi(t) = t \log t^{-1}$, one can take $n_j = 2^j$ and obtain the super-dyadic scales $2^{-n_j} = 2^{-2^j}$. In this case, $\lambda(t) = t$.

(ii) When $\phi(t) = t^\alpha$, $\alpha > 0$, one can take $n_j = j$ and get the standard dyadic scales 2^{-j}. In this case, $\lambda(t) \asymp t^{2-\alpha}/(1-\alpha)$ as $t \to 0$.

Dyadic shells and boxes. We can decompose the unit disk $\mathbb{D}$ into ϕ-dyadic shells:

$$\mathcal{A}_{\phi,0} = \{z \in \mathbb{D} : |z| < 1 - 2^{-n_1}\}$$

and

$$\mathcal{A}_{\phi,j} = \{z \in \mathbb{D} : 1 - 2^{-n_j} < |z| < 1 - 2^{-n_{j+1}}\}, \qquad j = 1, 2, \dots.$$

Each shell can be further subdivided into ϕ-dyadic boxes:

$$T_I^\phi = \mathcal{A}_{\phi,j} \cap Q(I) = \{re^{i\theta} \in \mathbb{D} : \theta \in I,\ 1 - 2^{-n_j} < r < 1 - 2^{-n_{j+1}}\},$$

where I ranges over $\mathcal{D}_{n_j}$. For further reference, we note that

$$\int_{T_I^\phi} \frac{dA(z)}{\lambda(1-|z|)} \asymp |I| \cdot \phi_1(|I|) = \phi(|I|). \tag{3-5}$$

3B. *Roberts decomposition.* In a remarkable work, Roberts [1985] came up with an elegant structure theorem for measures that do not charge Beurling–Carleson sets. This is done by *grating* a measure with respect to finer and finer partitions associated to a ϕ-dyadic grid.

Theorem 3.2. *Let $\phi : [0, 1] \to [0, \infty)$ be a regular gauge function and $\mathcal{D}_\phi = \bigcup \mathcal{D}_{n_k}$ be a ϕ-dyadic grid. Let μ be a finite positive measure on $\partial\mathbb{D}$. Then, for any integer $j_0 \geq 0$ and $C > 0$, one can decompose $\mu = \sum_{j=1}^\infty \mu_j + \mu_\infty$ such that $\mu_j(I) \leq C\phi(|I|)$ for any $I \in \mathcal{D}_{n_{j+j_0}}$ and μ_∞ is concentrated on a ϕ-Beurling–Carleson set.*

Proof. For each $j = 1, 2, \dots$, we can define a partition P_j of the unit circle into $2^{n_{j+j_0}}$ arcs of equal length (we consider half-open arcs which contain only one of the endpoints, for example, the left endpoint). Since $2^{n_{j+j_0}}$ divides $2^{n_{j+j_0+1}}$, each next partition can be chosen to be a refinement of the previous one.

To define μ_1, consider the arcs in the partition P_1. Call an arc $I \in P_1$ *light* if $\mu(I) \le C\phi(|I|)$ and *heavy* otherwise. On a light arc, take $\mu_1 = \mu$, while on a heavy arc, let μ_1 be a multiple of μ, so that the mass $\mu_1(I)$ equals $C\phi(|I|)$. The measure μ_1 will be called the *grated measure* of μ with respect to the partition P_1. Clearly, $\mu_1 \le \mu$. Consider the difference $\mu - \mu_1$ and grate it with respect to the partition P_2 to form the measure μ_2, then consider $\mu - \mu_1 - \mu_2$ and grate it with respect to P_3 to form μ_3, and so on. Continuing in this way, we obtain a sequence of measures $\mu - \mu_1,\ \mu - \mu_1 - \mu_2, \dots$, where each next measure is supported on the heavy arcs of the previous generation.

By construction, the bound $\mu_j(I) \le C\phi(|I|)$, $I \in \mathcal{D}_{n_{j+j_0}}$ holds for all j, while the residual measure μ_∞ is supported on the set of points which always lie in heavy arcs. A fortiori, the residual measure is supported on the complement of the light arcs and we need to show that $\sum_{I \text{ light}} \phi(|I|) < \infty$. The scaling condition (3-4) tells us that

$$\sum_{\substack{I \subset J \\ I \in \mathcal{D}_{n_{j+1}}}} \phi(|I|) = |J| \cdot \phi_1(|I|) \le C\phi(|J|), \qquad J \in \mathcal{D}_{n_j}.$$

Since a light arc of generation $j \ge 2$ is contained in a heavy one,

$$\begin{aligned}\sum_{\text{light}} \phi(|I|) \lesssim 2^{n_{j_0}} \phi(2^{-n_{j_0}}) + \sum_{\text{heavy}} \phi(|J|) &= 2^{n_{j_0}} \phi(2^{-n_{j_0}}) + \frac{1}{C} \sum_j \sum_{\substack{J \in \mathcal{D}_{n_{j+j_0}} \\ J \text{ heavy}}} \mu_j(J) \\ &\le 2^{n_{j_0}} \phi(2^{-n_{j_0}}) + \frac{1}{C} \cdot \mu(\partial \mathbb{D}).\end{aligned}$$

□

Corollary 3.3. *If μ does not charge ϕ-Beurling–Carleson sets, then, for any $j_0 \ge 0$ and $C > 0$, one can write $\mu = \sum \mu_j$, where $\mu_j(I) \le C\phi(|I|)$ for any $I \in \mathcal{D}_{n_{j+j_0}}$.*

We now show the converse of Corollary 3.3:

Corollary 3.4. *Suppose that there exists a constant $C > 0$ such that, for any offset $j_0 \ge 0$, one can decompose the measure μ into a countable sum $\mu = \sum \mu_j$ such that $\mu_j(I) \le C\phi(|I|)$ for any $I \in \mathcal{D}_{n_{j+j_0}}$. Then μ does not charge ϕ-Beurling–Carleson sets.*

Proof. Let E be a ϕ-Beurling–Carleson set. By Lemma 3.1, for any $\varepsilon > 0$, we can choose the offset $j_0 \ge 0$ large enough that

$$\sum_{j=1}^{\infty} \int_{K_E \cap \mathcal{A}_{\phi, j+j_0}} \frac{dA(z)}{\lambda(1-|z|)} < \varepsilon.$$

In view of (3-5), we have

$$\mu_j(E) = \sum_{\substack{I \in \mathcal{D}_{n_{j+j_0}} \\ I \cap E \ne \varnothing}} \mu_j(I) \le C \sum_{\substack{I \in \mathcal{D}_{n_{j+j_0}} \\ I \cap E \ne \varnothing}} \phi(|I|) \le C' \int_{K_E \cap \mathcal{A}_{\phi, j+j_0}} \frac{dA(z)}{\lambda(1-|z|)}.$$

Summing over $j = 1, 2, \dots$ yields $\mu(E) \le C'\varepsilon$. Since $\varepsilon > 0$ was arbitrary, $\mu(E) = 0$ as desired. □

3C. ***Local behavior.*** The following theorem roughly says that measures on the unit circle which are sufficiently spread out cannot charge Beurling–Carleson sets:

Theorem 3.5. *Suppose $w(\varepsilon)/\varepsilon$ is strictly decreasing on $(0, 1]$. Then $\mu(E) = 0$ for every ϕ-Beurling–Carleson set E and positive measure μ on the unit circle satisfying the modulus of continuity condition*

$$\mu(I) \le c \cdot w(|I|), \quad I \subset \partial\mathbb{D},$$

if and only if

$$\int_0^1 \frac{\varepsilon}{\lambda(\varepsilon) w(\varepsilon)}\, d\varepsilon = \infty. \tag{3-6}$$

In full generality, Theorem 3.5 was proved by R. D. Berman, L. Brown and W. S. Cohn [Berman et al. 1987, Corollary 4.1]. For usual Beurling–Carleson sets, Theorem 3.5 goes back to Ahern [1979] and J. H. Shapiro [1980].

Examples.

(i) If $\phi(t) = t \log t^{-1}$, the above condition reads $\int_0^1 w(\varepsilon)^{-1}\, d\varepsilon = \infty$.

(ii) For $\phi(t) = t^\alpha$, $\alpha > 0$, the condition becomes $\int_0^1 \varepsilon^{\alpha-1} w(\varepsilon)^{-1}\, d\varepsilon = \infty$.

Theorem 3.6. *Suppose μ is a measure on the unit circle supported on a countable union of ϕ-Beurling–Carleson sets. Let $\mu(x, \varepsilon) = \mu(I(x, \varepsilon))$, where $I(x, \varepsilon)$ is the arc on the unit circle centered at x of length 2ε. For almost every point x on the unit circle with respect to μ,*

$$\int_0^1 \frac{\varepsilon}{\lambda(\varepsilon)\mu(x, \varepsilon)}\, d\varepsilon < \infty.$$

Proof. It suffices to consider the case when μ is supported on a single ϕ-Beurling–Carleson set E. Since μ is a singular measure, for μ-a.e. $x \in \partial\mathbb{D}$, we have $\lim_{\varepsilon\to 0} \mu(x, \varepsilon)/\varepsilon = \infty$. To prove the lemma, we will show that

$$\int_E \int_0^1 \frac{\varepsilon}{\lambda(\varepsilon)\mu(x, \varepsilon)}\, d\varepsilon\, d\mu(x) \lesssim \|E\|_{\mathcal{BC}_\phi}.$$

For a point $x \in \partial\mathbb{D}$, we write $S(x)$ for the Stolz angle of opening $\frac{\pi}{2}$ with vertex at x. Recall that K_E denotes the union of the Stolz angles emanating from points $x \in E$. According to Lemma 3.1,

$$\|E\|_{\mathcal{BC}_\phi} \asymp \int_{K_E} \frac{dA(z)}{\lambda(1-|z|)}.$$

We subdivide the above integral over individual Stolz angles:

$$\int_{K_E} \frac{dA(z)}{\lambda(1-|z|)} = \int_E \int_{S(\zeta)} \eta(z) \cdot \frac{dA(z)}{\lambda(1-|z|)}\, d\mu(\zeta),$$

where the function $\eta(z) = \mu(I_z)^{-1}$ measures how many Stolz angles contain z. Here, I_z is the arc of the unit circle that consists of points ζ for which $z \in S(\zeta)$. From

$$\int_{S(\zeta)\cap\{1-|z|=\varepsilon\}} \eta(z) \cdot \frac{|dz|}{\lambda(1-|z|)} \ge \frac{\varepsilon}{\lambda(\varepsilon)} \cdot \min_{z\in S(\zeta)\cap\{1-|z|=\varepsilon\}} \mu(I_z)^{-1} \ge \frac{\varepsilon \cdot \mu(\zeta, 3\varepsilon)^{-1}}{\lambda(\varepsilon)},$$

we deduce that
$$\int_E \int_0^1 \frac{\varepsilon \cdot \mu(\zeta, 3\varepsilon)^{-1}}{\lambda(\varepsilon)}\, d\varepsilon\, d\mu(\zeta) \lesssim \|E\|_{\mathcal{BC}_\phi}. \qquad \square$$

Corollary 3.7. *Suppose μ is a measure on the unit circle supported on a countable union of ϕ-Beurling–Carleson sets. For any $c > 0$, the region*
$$\Omega_c = \{z \in \mathbb{D} : P_\mu(z) > c\}$$
is "thick" at almost every point x on the unit circle with respect to μ, in the sense that
$$\int_0^1 \frac{\eta(x, \varepsilon)}{\varepsilon \cdot \lambda(\varepsilon)}\, d\varepsilon < \infty, \tag{3-7}$$
where $\eta(x, \varepsilon) = \pi\varepsilon - |\partial B(x, \varepsilon) \cap \Omega_c|$.

To see the corollary, notice that if $\mu(x, \varepsilon) \geq \varepsilon$, then $\mu(x, \varepsilon)\eta(x, \varepsilon) \lesssim \varepsilon^2$.

Remark. For usual Beurling–Carleson sets, one has ε^2 in the denominator of (3-7). This is essentially the Rodin–Warschawski condition on the existence of a nonzero angular derivative of a Riemann map $\psi_c : \Omega_c \to \mathbb{D}$ at $x \in \partial\Omega_c \cap \partial\mathbb{D}$; see Theorem 7.1. (If Ω_c is disconnected, then we consider the Riemann map from an appropriate connected component of Ω_c.) For an application to critical values of inner functions, see [Ivrii and Kreitner 2024]. For α-Beurling–Carleson sets, the denominator of (3-7) is $\varepsilon^{3-\alpha}$.

4. A corona construction

In this section, we explore a number of conditions which guarantee that a singular measure is supported on a countable union of Beurling–Carleson sets and prove Theorems 1.1 and 1.2. Our main tool is a corona-type decomposition for singular measures.

4A. ***Decomposition of singular measures.*** Suppose μ is a singular measure on the unit circle. Fix a large constant $M > 0$ and consider the following corona-type decomposition. Let $\{I_j^{(1)}\}$ be the maximal (closed) dyadic arcs such that
$$\frac{\mu(I_j^{(1)})}{|I_j^{(1)}|} \geq M.$$
In each $I_j^{(1)}$, we consider the maximal dyadic subarcs $J_k^{(1)} \subset I_j^{(1)}$ for which
$$\frac{\mu(J_k^{(1)})}{|J_k^{(1)}|} \leq \frac{M}{100}.$$
In each $J_k^{(1)}$, we consider the maximal dyadic subarcs $I_j^{(2)} \subset J_k^{(1)}$ with
$$\frac{\mu(I_j^{(2)})}{|I_j^{(2)}|} \geq M.$$
Continuing in this way, we inductively define $I_j^{(m)}$ and $J_k^{(m)}$ for $m \geq 1$. We call the arcs $I_j^{(m)}$ *heavy* and the arcs $J_k^{(m)}$ *light*, $j, k, m \geq 1$.

Since μ is a singular measure, almost every point on the unit circle with respect to the Lebesgue measure is eventually contained in a light arc, so that

$$\sum_{J_k^{(m)} \subset I_j^{(m)}} |J_k^{(m)}| = |I_j^{(m)}|, \qquad j, m \geq 1.$$

From the definitions of light and heavy arcs, we have

$$\sum_{I_j^{(m+1)} \subset J_k^{(m)}} |I_j^{(m+1)}| \leq \frac{1}{M} \cdot \mu(J_k^{(m)}) \leq \frac{|J_k^{(m)}|}{100}, \qquad k, m \geq 1.$$

It follows that μ is concentrated on

$$\bigcup_{I_j^{(m)} \text{ heavy}} \Bigg(I_j^{(m)} \setminus \bigcup_{\text{light } J_k^{(m)} \subset I_j^{(m)}} \operatorname{Int} J_k^{(m)} \Bigg).$$

4B. *Proofs of Theorems 1.1 and 1.2.* For the convenience of the reader, we break the proofs of Theorems 1.1 and 1.2 into two lemmas:

Lemma 4.1. (i) *Let $\mu \geq 0$ be a finite singular measure on $\partial\mathbb{D}$ which satisfies*

$$\int_{\{z \in \mathbb{D}: P_\mu(z) > c\}} \frac{dA(z)}{1 - |z|} < \infty \tag{4-1}$$

for some $c \in \mathbb{R}$. Then μ is concentrated on a countable union of Beurling–Carleson sets.

(ii) *Let $\mu \geq 0$ be a finite singular measure on $\partial\mathbb{D}$ which satisfies*

$$\int_{\{z \in \mathbb{D}: P_\mu(z) > c\}} \frac{dA(z)}{(1 - |z|)^{1+p}} < \infty \tag{4-2}$$

for some $c \in \mathbb{R}$. Then μ is concentrated on a countable union of $(1-p)$-Beurling–Carleson sets.

Proof. We only prove (i) as (ii) is similar. We use the decomposition from Section 4A. To prove the theorem, it suffices to show that, for each heavy interval $I_j^{(m)}$,

$$E = I_j^{(m)} \setminus \bigcup_{\text{light } J_k^{(m)} \subset I_j^{(m)}} \operatorname{Int} J_k^{(m)}$$

is a Beurling–Carleson set. By Lemma 3.1, we may check that

$$\sum_{\substack{I \text{ dyadic} \\ I \cap E \neq \varnothing}} |I| < \infty.$$

By construction, if I is a dyadic interval in $I_j^{(m)}$ which meets E, then $\mu(I)/|I| > \frac{1}{100} M$ and $P_\mu(z) \gtrsim M$ for $z \in T_I$. Hence,

$$\sum_{\substack{I \text{ dyadic} \\ I \cap E \neq \varnothing}} |I| \lesssim \int_{\{z: P_\mu(z) \gtrsim M\}} \frac{dA(z)}{1 - |z|} < \infty$$

as desired. The proof is complete. □

Ahern and Clark gave an elegant formula for the angular derivative of a singular inner function on the unit circle:

$$|S'_\mu(z)| = 2\int_{\partial\mathbb{D}} \frac{d\mu(\zeta)}{|\zeta - z|^2}, \quad |z| = 1,$$

where at a given point $z \in \partial\mathbb{D}$, either both quantities are finite and equal or infinite. For a proof, see [Mashreghi 2013, Chapter 4.1].

Lemma 4.2. (i) *If $S'_\mu \in \mathcal{N}$, then the area condition (1-3) holds.*

(ii) *If $S'_\mu \in H^p$, then the $(1+p)$-area condition (1-4) holds.*

Proof. Observe that

$$\Omega_c = \{z \in \mathbb{D} : P_\mu(z) > c\} = \{z \in \mathbb{D} : |S_\mu(z)| < e^{-c}\}.$$

Let $e^{i\theta} \in \partial\mathbb{D}$ be a point at which S_μ has a finite angular derivative. According to a well-known result of Ahern and Clark [Mashreghi 2013, Theorem 4.15],

$$|S'_\mu(re^{i\theta})| \le 4|S'_\mu(e^{i\theta})|, \quad 0 < r < 1.$$

Let $[0, e^{i\theta}]$ denote the radial line segment from the origin to $e^{i\theta}$. As $1 - |S_\mu(re^{i\theta})| \le 4|S'_\mu(e^{i\theta})|(1-r)$,

$$\Omega_c \cap [0, e^{i\theta}] \subset \left[0, \left(1 - \frac{\varepsilon}{|S'_\mu(e^{i\theta})|}\right) \cdot e^{i\theta}\right],$$

where $\varepsilon > 0$ is a constant that depends on c. From this bound on Ω_c, (i) and (ii) follow quite easily. □

5. Derivative in Hardy spaces I

In this section, we explore conditions on a singular measure μ involving Beurling–Carleson sets that guarantee the membership of S'_μ in H^p. We show:

Theorem 5.1. *Fix $0 < p < \frac{1}{2}$. Let μ be a positive measure supported on a closed set $E \subset \partial\mathbb{D}$ of zero length whose complementary arcs $\{J\}$ satisfy*

$$\sum |J|^{1-q} < \infty \tag{5-1}$$

for some $q > p/(1-p)$. Then, $S'_\mu \in H^p$.

We will give two examples that show that the exponent $p/(1-p)$ in the theorem above is sharp. Theorem 5.1 improves a result of Cullen [1971], who showed that $S'_\mu \in H^p$ under the stronger hypothesis $q = 2p$.

5A. ***When is $S'_\mu \in H^p$?*** We begin by giving a simple criterion for a singular inner function to have derivative in H^p. As is standard, for an arc J on the unit circle with $|J| \le 1$, we write $z_J = \left(1 - \frac{1}{2}|J|\right) \cdot e^{i\theta_J}$, where $e^{i\theta_J}$ is the midpoint of J. For $0 < \beta < 1/|J|$, we write βJ for the arc of length $|\beta J|$ with the same midpoint as J.

Lemma 5.2. *Fix $0 < p < \frac{1}{2}$. Suppose $E \subset \partial\mathbb{D}$ is a closed set of zero length and $\{J\}$ is its complementary arcs. For a positive measure μ supported on E, we have $S'_\mu \in H^p$ if and only if*

$$\sum u(z_J)^p |J|^{1-p} < \infty, \tag{5-2}$$

where u is the Poisson integral of μ.

Proof. Differentiation shows that $S'_\mu(z) = h(z) S_\mu(z)$, where

$$h(z) = \int_E \frac{-2\zeta}{(\zeta - z)^2}\, d\mu(\zeta) = -\int_E \frac{2\zeta}{|\zeta - z|^2} \left(\frac{\bar{\zeta} - \bar{z}}{\zeta - z} \right) d\mu(\zeta).$$

Notice that if $z/|z| \in \frac{1}{2}J$, $|z| \geq 1 - \frac{1}{4}|J|$ and $\zeta \in E$, then the quantity

$$\zeta \cdot \frac{\bar{\zeta} - \bar{z}}{\zeta - z} = \frac{1 - \bar{z}\zeta}{\zeta - z}$$

is constrained in a sector of aperture strictly less than π. This tells us that

$$|h(z)| \asymp \int_E \frac{d\mu(\zeta)}{|\zeta - z|^2} \asymp \int_E \frac{d\mu(\zeta)}{|\zeta - z_J|^2} \asymp \frac{u(z_J)}{|J|}.$$

We see that

$$\int_{J/2} |S'_\mu(z)|^p\, |dz| \asymp u(z_J)^p |J|^{1-p},$$

so the condition (5-2) is necessary for $S'_\mu \in H^p$.

To prove the converse implication, we split $J = \bigcup_{k \in \mathbb{Z}} J_k$ into countably many Whitney arcs such that

$$|J_k| \asymp \operatorname{dist}(J_k, \partial\mathbb{D} \setminus J) \asymp 2^{-|k|}|J|.$$

For $z \in J_k$, we have

$$|S'_\mu(z)| = 2 \int_E \frac{d\mu(\zeta)}{|\zeta - z|^2} \asymp \frac{u(z_{J_k})}{|J_k|}.$$

By Harnack's inequality,

$$\frac{|J_k|}{|J|} \lesssim \frac{u(z_{J_k})}{u(z_J)} \lesssim \frac{|J|}{|J_k|}.$$

Therefore,

$$\int_J |S'_\mu(z)|^p\, |dz| \lesssim \sum_k |J_k| \cdot \frac{u(z_{J_k})^p}{|J_k|^p} \lesssim u(z_J)^p |J|^p \sum_k |J_k|^{1-2p} \asymp u(z_J)^p |J|^{1-p}.$$

Summing over J shows that $S'_\mu \in H^p$. □

With help of Lemma 5.2, the proof of Theorem 5.1 runs as follows:

Proof of Theorem 5.1. Let u be the Poisson integral of μ. Since u is a positive harmonic function, its nontangential maximal function is in L^δ for any $\delta < 1$. In particular, for any $\delta < 1$, we have

$$\sum_J u(z_J)^\delta |J| < \infty.$$

Applying Hölder's inequality with exponents $\delta/p > 1$ and $\delta/(\delta - p) > 1$, we obtain

$$\sum_J u(z_J)^p |J|^{1-p} = \sum_J u(z_J)^p |J|^{p/\delta} \cdot |J|^{(\delta-p)/\delta - p} \le \Big(\sum_J u(z_J)^\delta |J|\Big)^{p/\delta} \Big(\sum_J |J|^{1-\delta p/(\delta-p)}\Big)^{(\delta-p)/\delta}.$$

Choosing $\delta \in (p, 1)$ such that $\delta p/(\delta - p) = q$ gives

$$\sum_J u(z_J)^p |J|^{1-p} < \infty,$$

which implies that $S'_\mu \in H^p$ by Lemma 5.2. Note that as δ varies over $(p, 1)$, we have that $q = \delta p/(\delta - p) = (1/p - 1/\delta)^{-1}$ varies over $(p/(1-p), \infty)$. □

Next, we extend Theorem 5.1 to inner functions:

Corollary 5.3. *Fix $0 < p < \frac{1}{2}$. Let $E \subset \partial\mathbb{D}$ be a closed set of zero length whose complementary arcs $\{J\}$ satisfy*

$$\sum |J|^{1-q} < \infty$$

for some $q > p/(1-p)$. Let F be an inner function whose singular part is supported on E and whose zeros are contained in K_E. Then $F' \in H^p$.

Proof. By an approximation argument, we can assume that F is a finite Blaschke product with zeros $\{z_n\} \subset K_E$. For each zero z_n of F, pick a point z_n^* in E that is closest to z_n. Then,

$$|F'(e^{i\theta})| = \sum \frac{1-|z_n|^2}{|e^{i\theta} - z_n|^2} \lesssim \sum \frac{1-|z_n|^2}{|e^{i\theta} - z_n^*|^2} = \tfrac{1}{2}|S'_\sigma(e^{i\theta})|, \quad e^{i\theta} \in \partial\mathbb{D} \setminus E,$$

where $\sigma = \sum (1-|z_n|^2)\delta_{z_n^*}$. From Theorem 5.1, we know that $S'_\sigma \in H^p$, and by the above equation, $F' \in H^p$ as well. □

5B. *Sharpness.*

We now give two examples showing that the exponent in Theorem 5.1 is sharp:

Lemma 5.4. *There exists a measure μ supported on a closed set E of zero length whose complementary arcs $\{J\}$ satisfy $\sum |J|^{1-p/(1-p)} < \infty$ yet $S'_\mu \notin H^p$.*

Proof. Step 1: In our example, E will be a certain pruned Cantor set, and

$$\mu = \sum |J|^{(1-2p)/(1-p)} (\delta_{a(J)} + \delta_{b(J)}),$$

where $a(J)$ and $b(J)$ are the two endpoints of the complementary arc J. In order for the measure μ to be finite, we need to arrange that

$$\sum |J|^{(1-2p)/(1-p)} < \infty. \tag{5-3}$$

In addition, we will arrange that

$$\sum_J \mu(\beta J)^p |J|^{1-2p} = \infty \tag{5-4}$$

for some constant $\beta > 1$ to be chosen. As $P_\mu(z_J) \gtrsim \mu(\beta J)/|J|$,

$$\sum_J P_\mu(z_J)^p |J|^{1-p} = \infty$$

and $S'_\mu \notin H^p$ by Lemma 5.2.

Step 2. Let $N_j = \#\{J : |J| \asymp 2^{-j}\}$. To achieve (5-3), we request that $N_j \asymp j^{-\alpha} \cdot 2^{(1-2p)/(1-p)\cdot j}$ for some $\alpha > 1$ to be chosen. In this case, the total measure supported on the endpoints of arcs of length $\le 2^{-j}$ is

$$M_j = \sum_{|J| \le 2^{-j}} \mu(\bar{J}) \asymp \sum_{k=j}^{\infty} 2^{-(1-2p)/(1-p)\cdot k} N_k \asymp \sum_{k=j}^{\infty} \frac{1}{k^\alpha} \asymp \frac{1}{j^{\alpha-1}}.$$

Therefore, if we construct the arcs $\{J\}$ such that

$$\mu(\beta J) \asymp \frac{M_j}{N_j} \quad \text{for } |J| \asymp 2^{-j}, \tag{5-5}$$

then we would have

$$\sum_J \mu(\beta J)^p |J|^{1-2p} \asymp \sum_{j=1}^{\infty} N_j 2^{-j(1-2p)} \left(\frac{M_j}{N_j}\right)^p \asymp \sum_{j=1}^{\infty} \frac{1}{j^{\alpha-p}}.$$

In order to obtain (5-4), we may choose α to be any number in $(1, 1+p)$.

Step 3. Fix a real number $A > 2$. Consider the standard Cantor set E, which at generation n is formed from 2^n arcs of length A^{-n}. Inspection shows that $N_j \asymp 2^{j/\log_2 A}$. We choose A appropriately such that

$$\frac{1}{\log_2 A} = \frac{1-2p}{1-p} \in (0, 1).$$

In order to make N_j smaller, we slightly modify the construction of the standard Cantor set by removing a number of arcs. We call a generation *bad* if $N_j > j^{-\alpha} \cdot 2^{(1-2p)/(1-p)\cdot j}$ is too large. In a bad generation, we allow each arc to only have one descendant instead of two, say the left one. In the pruned Cantor set, we have $N_j \asymp j^{-\alpha} \cdot 2^{(1-2p)/(1-p)\cdot j}$ as desired.

We select $\beta > (1-2A)^{-1}$, so that if J is a complementary arc of some generation, then βJ covers the interval defining the Cantor set of the previous generation. Since the mass of μ is evenly spread out, μ satisfies (5-5). □

In our second example of the sharpness of the exponent in Theorem 5.1, we have a slightly stronger assumption and a slightly stronger conclusion:

Lemma 5.5. *Given $q < p/(1-p)$, there exists a $(1-q)$-Beurling–Carleson set E and a measure μ supported on E such that $S'_\nu \notin H^p$ for any $0 < \nu \le \mu$.*

Proof. Fix a real number $A > 2$. Consider the standard Cantor set E, which at generation n is formed from 2^n arcs of length A^{-n}. Let μ be the standard Cantor measure on E, that is, μ is the probability measure supported on E which gives equal mass to arcs of generation n.

Step 1: *When is E a Beurling–Carleson set*? In generation n, there are 2^{n-1} complementary arcs of length $A^{-n+1}(1-2A^{-1})$. If $\partial\mathbb{D}\setminus E=\bigcup I_k$, then

$$\sum |I_k|^{1-q} \asymp \sum_n 2^n A^{-(1-q)n},$$

which converges if $\log A > (\log 2)/(1-q)$. In other words, E is a q-Beurling–Carleson set when $\log A > (\log 2)/(1-q)$.

Step 2: *When is the measure μ invisible*? Fix a measure $0<\nu\le\mu$. Let $\mathcal{A}(n)$ be the collection of arcs I of generation n in the construction of the Cantor set E such that $\nu(I)\ge 2^{-n-1}\nu(\partial\mathbb{D})$. Since $\#\mathcal{A}(n)\le 2^n$, we have

$$\nu(\partial\mathbb{D}) \le \sum_{I\in\mathcal{A}(n)} \nu(I) + \sum_{I\notin\mathcal{A}(n)} \nu(I) \le \sum_{I\in\mathcal{A}(n)} \nu(I) + \tfrac{1}{2}\nu(\partial\mathbb{D}),$$

which simplifies to

$$\sum_{I\in\mathcal{A}(n)} \nu(I) \ge \tfrac{1}{2}\nu(\partial\mathbb{D}).$$

However, as $\nu(I)\le 2^{-n}$ for any $I\in\mathcal{A}(n)$,

$$\#\mathcal{A}(n) \ge 2^n\cdot\tfrac{1}{2}\nu(\partial\mathbb{D}).$$

Hence,

$$\sum_{I\in\mathcal{A}(n)} |I|^{1-p}P_\nu(z_I)^p \gtrsim \sum_{I\in\mathcal{A}(n)} |I|^{1-2p}\nu(I)^p \gtrsim 2^n\nu(\partial\mathbb{D})A^{-n(1-2p)}2^{-np} = \left(\frac{2^{1-p}}{A^{1-2p}}\right)^n \nu(\partial\mathbb{D}).$$

Since the lengths and locations of the arcs defining E of generation n are comparable to the complementary arcs of generation n, we may use Lemma 5.2 to conclude that $S'_\nu\notin H^p$ if $2^{1-p}>A^{1-2p}$.

Step 3: *Conclusion.* To prove the lemma, we need to find an $A>2$ satisfying

$$\frac{1}{1-q}\cdot\log 2 < \log A < \frac{1-p}{1-2p}\cdot\log 2,$$

which is possible if $1-q>(1-2p)/(1-p)$, that is, $q<p/(1-p)$. □

Remark. There may also be an example in the extreme case when $q=p/(1-p)$.

6. Derivative in Hardy spaces II

Suppose $0<p<\frac{1}{2}$ and $\mu\ge 0$ is a singular measure on $\partial\mathbb{D}$. Recall that, by Theorem 1.2, if $S'_\mu\in H^p$ then S_μ satisfies the $(1+p)$-area condition (1-4). We now show that if (1-4) holds, then $\mu=\sum\mu_i$ can be written as a countable sum of measures with $S'_{\mu_i}\in H^p$. In view of the implication (2) ⇒ (3) of Theorem 1.2, it is enough to prove the following lemma:

Lemma 6.1. *Fix $0<p<\frac{1}{2}$. Suppose μ is a measure supported on a $(1-p)$-Beurling–Carleson set. If S_μ satisfies the $(1+p)$-area condition (1-4), then $S'_\mu\in H^p$.*

Proof. Let $E = \operatorname{supp}\mu$, and write $\partial\mathbb{D} \setminus E = \bigcup J_k$. By Lemma 5.2, we need to show that

$$\sum_k P_\mu(z_{J_k})^p |J_k|^{1-p} < \infty.$$

Since $\sum |J_k|^{1-p} < \infty$, we only need to show that

$$\sum_{k:P_\mu(z_{J_k})\geq 1} P_\mu(z_{J_k})^p |J_k|^{1-p} < \infty.$$

Let $J \subset \partial\mathbb{D}$ be any arc with $J \cap E = \varnothing$. It is easy to see that

$$\frac{P_\mu(z_I)}{|I|} \gtrsim \frac{P_\mu(z_J)}{|J|}$$

for any arc $I \subset J$. Therefore, if $P_\mu(z_{J_k}) \geq 1$, then

$$\sum_{\substack{I\subset J_k \text{ dyadic}\\ P_\mu(z_I)\geq 1}} |I|^{1-p} \gtrsim \sum_{\substack{I\subset J_k \text{ dyadic}\\ |I|\gtrsim |J_k|/P_\mu(z_{J_k})}} |I|^{1-p} \asymp \sum_{n=0}^{\log_2 P_\mu(z_{J_k})} 2^n \cdot (2^{-n}|J_k|)^{1-p} \asymp |J_k|^{1-p} P_\mu(z_{J_k})^p.$$

By Harnack's inequality, one can find a constant $0 < c < 1$ such that

$$\sum_{k:P_\mu(z_{J_k})\geq 1} P_\mu(z_{J_k})^p |J_k|^{1-p} \lesssim \sum_{k:P_\mu(z_{J_k})\geq 1} \sum_{\substack{I\subset J_k \text{ dyadic}\\ P_\mu(z_I)\geq 1}} |I|^{1-p} \lesssim \int_{\{z\in\mathbb{D}:|S_\mu(z)|\leq c\}} \frac{dA(z)}{(1-|z|)^{1+p}},$$

which is finite by assumption. The proof is complete. □

We now give an example of a singular inner function S_μ which satisfies the $(1+p)$-area condition (1-4) yet $S'_\mu \notin H^p$.

Lemma 6.2. *For $0 < p < \frac{1}{2}$, there exists a singular inner function S_μ with $S'_\mu \notin H^p$ such that*

$$\int_{\{z\in\mathbb{D}:|S_\mu(z)|\leq c\}} \frac{dA(z)}{(1-|z|)^{1+p}} < \infty$$

for any $0 < c < 1$.

Sketch of proof. To get a feeling of why the lemma is true, we examine the situation for the measure μ which consists of n equally spaced point masses on the circle: $\mu = (1/n^{2-\varepsilon}) \sum_{k=0}^{n-1} \delta_{\xi_k}$, where $\xi_k = e^{2\pi i k/n}$, $k = 0, 1, 2, \dots, n-1$, and $\varepsilon > 0$ is a constant to be chosen. Since

$$|S'_\mu(e^{i\theta})| = \int_0^{2\pi} \frac{2\,d\mu(t)}{|e^{i\theta}-e^{it}|^2} = \frac{2}{n^{2-\varepsilon}} \sum_{k=0}^{n-1} \frac{1}{|e^{i\theta}-\xi_k|^2} \asymp \frac{1}{n^{2-\varepsilon}\cdot \operatorname{dist}(e^{i\theta}, \{\xi_k\})^2},$$

the integral

$$\int_0^{2\pi} |S'_\mu(e^{i\theta})|^p\, d\theta \asymp n \int_{-\pi/n}^{\pi/n} \left(\frac{1}{n^{2-\varepsilon}\theta^2}\right)^p d\theta \asymp n^{\varepsilon p}$$

tends to infinity as $n \to \infty$.

Let H_k be the horoball which rests at ξ_k of diameter $\alpha/n^{2-\varepsilon}$. It is not difficult to see that, for any $0 < c < 1$, there exists an $\alpha = \alpha(c) > 0$ such that

$$\{z \in \mathbb{D} : |S_\mu(z)| \le c\} \subseteq \bigcup_{k=0}^{n-1} H_k.$$

As the integral over a single horoball is

$$\int_{H_0} \frac{dA(z)}{(1-|z|^2)^{1+p}} \asymp \frac{1}{n^{(2-\varepsilon)(1-p)}},$$

the integral over their union is

$$\int_{\bigcup H_k} \frac{dA(z)}{(1-|z|^2)^{1+p}} \asymp n^{1-(2-\varepsilon)(1-p)}.$$

Since $0 < p < \frac{1}{2}$, we can choose $\varepsilon > 0$ small enough to make the exponent $1-(2-\varepsilon)(1-p)$ negative, so that the integrals

$$\int_{\{z\in\mathbb{D}:|S_\mu(z)|<c\}} \frac{dA(z)}{(1-|z|)^{1+p}}$$

tend to 0 as $n \to \infty$.

Independent copies of this construction provide an example of a singular inner function S with $S' \notin H^p$ for which

$$\int_{\{z\in\mathbb{D}:|S(z)|<c\}} \frac{dA(z)}{(1-|z|)^{1+p}} < \infty.$$

We leave the details to the reader. □

7. Background on angular derivatives

For $0 < \theta < \pi$ and $0 < \delta < 1$, let $S_{\theta,\delta}(p) = S_\theta(p) \cap B(p,\delta)$ denote the truncated Stolz angle of opening θ with vertex at $p \in \partial\mathbb{D}$.

Suppose $\Omega \subset \mathbb{D}$ is a domain in the unit disk bounded by a Jordan curve. We say that Ω has an *inner tangent* at a point $p \in \partial\Omega \cap \partial\mathbb{D}$ if, for any $0 < \theta < \pi$, Ω contains a truncated Stolz angle of opening θ with vertex at p.

Let $\varphi : \mathbb{D} \to \Omega$ be a conformal map. We say that φ has a (nonzero) *angular derivative* at $q = \varphi^{-1}(p)$ if the nontangential limit

$$\lim_{z\to q} |\varphi'(z)| = A$$

for some real number $A > 0$. While the number A depends on the choice of Riemann map φ, the existence of the angular derivative does not. In other words, possessing an angular derivative is an intrinsic property of (Ω, p), which we record by saying that Ω is *thick* at p. In the language of potential theory, one would say that the complement $\mathbb{D} \setminus \Omega$ is minimally thin at p, see [Burdzy 1986, Theorem 5.2], which means that Brownian motion conditioned to exit the unit disk at p is eventually contained in Ω.

To avoid dealing with the point q, we will simply say that the inverse conformal map $\psi : \Omega \to \mathbb{D}$ has an angular derivative at p and write $|\psi'(p)| = A^{-1}$. It is easy to see that if Ω is thick at p, then Ω possesses an inner tangent at p.

Rodin and Warschawski gave an if and only if condition for ψ to possess an angular derivative at p in terms of moduli of curve families, e.g., see [Garnett and Marshall 2005, Theorem V.5.7] or [Betsakos and Karamanlis 2022]. When Ω is a starlike domain with regular boundary, their condition takes a simpler form [Ivrii and Kreitner 2024]:

Theorem 7.1. *Suppose $\Omega = \{r\zeta : \zeta \in \partial\mathbb{D}, 0 \le r < 1 - h(\zeta)\}$, where $h : \partial\mathbb{D} \to \left[0, \frac{1}{2}\right]$ is a continuous function. Assume that h satisfies the doubling condition*

$$h(\zeta_1) \ge c \cdot h(\zeta_2) \quad \text{whenever } |\zeta_2 - \zeta_1| < c \cdot h(\zeta_1)$$

for some $c > 0$. Then, ψ has an angular derivative at $p \in \partial\Omega \cap \partial\mathbb{D}$ if and only if

$$\int_{\partial\mathbb{D}} \frac{h(\zeta)}{|\zeta - p|^2} |d\zeta| < \infty.$$

We will use the following elementary lemma about angular derivatives:

Lemma 7.2. *Let $\{\Omega_n\}_{n=1}^{\infty}$ be an increasing sequence of Jordan domains whose union is the unit disk. Suppose the conformal maps $\psi_n : \Omega_n \to \mathbb{D}$ converge uniformly on compact subsets to the identity. If ψ_1 has an angular derivative at $p \in \partial\Omega \cap \partial\mathbb{D}$, then the angular derivatives $|\psi_n'(p)|$ tend to 1.*

We will also need the following theorem from [Ivrii and Kreitner 2024] which describes how composition operators act on measures on the unit circle:

Theorem 7.3. *Suppose $\Omega \subset \mathbb{D}$ is a Jordan domain, $\varphi : \mathbb{D} \to \Omega$ is a conformal map and $\psi : \Omega \to \mathbb{D}$ is its inverse. Let $\mu \ge 0$ be a positive measure on the unit circle. Since $P_\mu(\varphi(z))$ is a positive harmonic function, it can be represented as the Poisson extension of some finite measure $\nu \ge 0$. If we use the normalization $0 \in \Omega$ and $\varphi(0) = 0$, then*

$$\varphi_* \nu = P_\mu \, d\omega_{\Omega,0} + |\psi'| \, d\mu, \tag{7-1}$$

provided that we interpret $|\psi'(p)| = 0$ if $p \notin \partial\Omega$ or Ω is not thick at p.

8. Background in PDE

In this section, we make some general observations about semilinear elliptic equations of the form

$$\Delta u = g(u), \tag{8-1}$$

which will be used in Section 9. We assume that the "nonlinearity" g is a nonnegative increasing convex function which satisfies the Keller–Osserman condition [Keller 1957; Osserman 1957]

$$\int_1^\infty \frac{ds}{\sqrt{G(s)}} < \infty, \tag{8-2}$$

where $G' = g$. Examples of g satisfying the above conditions include $g(t) = e^{2t}$ and $g(t) = t^p \cdot \chi_{t>0}$ with $p > 1$.

8A. *Basic properties.*

Traces. Given a function ϕ on the unit disk, we define its *boundary trace* as the weak limit of the measures $\phi(re^{i\theta})\,d\theta$ as $r \to 1$, provided that the limit exists. Otherwise, we say that ϕ does not possess a boundary trace.

Sub- and supersolutions. One says that a function $v : \mathbb{D} \to \mathbb{R}$ is a *subsolution* of (8-1) if $\Delta v \geq g(v)$ in the sense of distributions. Similarly, we say that v is a *supersolution* if $\Delta v \leq g(v)$ in the sense of distributions.

Theorem 8.1 (principle of sub- and supersolutions). *Suppose u_- is a subsolution and u_+ is a supersolution of (8-1) with $u_-(z) \leq u_+(z)$ for any $z \in \mathbb{D}$. Then, there exists at least one solution $u(z)$ with*

$$u_-(z) \leq u(z) \leq u_+(z), \quad z \in \mathbb{D}.$$

A proof using the Schauder fixed point theorem can be found in [Ponce 2016, Chapter 20].

Existence of solutions and the comparison principle.

Theorem 8.2. *Given a function $h \in L^\infty(\partial\mathbb{D})$, the boundary value problem*

$$\begin{cases} \Delta u = g(u) & \text{in } \mathbb{D}, \\ u = h & \text{on } \partial\mathbb{D} \end{cases} \tag{8-3}$$

admits a unique solution, where the boundary values are interpreted in the sense of weak limits of measures. If u_1 and u_2 are two solutions with boundary values $h_1 \leq h_2$, then $u_1 \leq u_2$ on $\mathbb{D}$.

Proof of Theorem 8.2. Step 1: *Uniqueness and monotonicity.* By Kato's inequality [Ponce 2016, Proposition 6.9],

$$\Delta(u_1 - u_2)^+ \geq \Delta(u_1 - u_2) \cdot \chi_{\{u_1 > u_2\}} = (g(u_1) - g(u_2)) \cdot \chi_{\{u_1 > u_2\}} \geq 0$$

is a subharmonic function. As $h_1 \leq h_2$, the function $(u_1 - u_2)^+$ has zero boundary values. The maximum principle shows that $(u_1 - u_2)^+ \leq 0$ or $u_1 \leq u_2$. The same argument also proves uniqueness.

Step 2: *Existence.* Let P_h denote the harmonic extension of h to the unit disk. Clearly, $u_+ = P_h$ is a supersolution of (8-1) with boundary data h. Similarly, if $G(z, w) = \log|(1 - \overline{w}z)/(w - z)|$ is the Green's function of the unit disk, then

$$u_-(z) = P_h(z) - \frac{1}{2\pi} \int_{\mathbb{D}} g(\|h\|_\infty) G(z, \zeta)\, dA(\zeta)$$

is a subsolution of (8-1) as

$$\Delta u_-(z) = g(\|h\|_\infty) \geq g(u_-(z)).$$

Since u_- also has boundary trace h, by the principle of sub- and supersolutions, there exists a solution with boundary trace h. □

The maximal solution.

Lemma 8.3. *The PDE (8-1) has a unique maximal solution $u_{\max}$ on the unit disk, which dominates all other solutions pointwise.*

Sketch of proof. We will simultaneously show that (8-1) has a maximal solution on every disk $\mathbb{D}_R = \{z : |z| < R\}$ with $R > 0$.

Keller [1957] and Osserman [1957] showed that, under the assumption (8-2), for any $R > 0$, there is a unique radially invariant solution $u_R(z)$ on $\mathbb{D}_R$ which tends to infinity as $|z| \to R$, and furthermore, the solutions $u_R(z)$ depend continuously on R.

Suppose $u : \mathbb{D}_R \to \mathbb{R}$ is any solution of (8-1). By the comparison principle, for any $S < R$, we have that $u(z) < u_S(z)$ on $\mathbb{D}_S$. Taking $S \to R$ yields $u(z) \le u_R(z)$. □

The above argument shows that if u is a solution of (8-1) on the unit disk which tends to infinity as $|z| \to 1$, then $u = u_{\max}$. As a consequence, the solutions u_n of (8-1) with constant boundary values n increase to $u_{\max}$ as $n \to \infty$.

Remark. For the existence and uniqueness of large solutions of semilinear equations on other domains, we refer the reader to [Bandle and Marcus 1992; García-Melián 2009]. Information about the asymptotic behavior of these solutions near the boundary can be found in [Bandle and Marcus 1998; 2004; del Pino and Letelier 2002; Lazer and McKenna 1994].

Minimal dominating solution. Let v be a subsolution of (8-1). For $0 < r < 1$, we write $\Lambda_r[v]$ for the unique solution of (8-1) on the disk $\mathbb{D}_r = \{z : |z| < r\}$ which agrees with v on $\partial\mathbb{D}_r$. An inspection of Step 1 of the proof of Theorem 8.2 shows that $\Lambda_r[v]$ is the pointwise-minimal solution which lies above v on $\mathbb{D}_r$. In particular, the solutions $\Lambda_r[v]$ are increasing in r. The limit $\Lambda[v] := \lim_{r\to 1} \Lambda_r[v]$ is finite on the unit disk because it is bounded above by the maximal solution.

For any test function $\phi \in C_c^\infty(\mathbb{D})$, we have

$$\int_{\mathbb{D}_r} u_r \Delta\phi \, dA = \int_{\mathbb{D}_r} g(u_r)\phi \, dA, \qquad u_r = \Lambda_r[v],$$

provided that $\mathbb{D}_r$ contains $\operatorname{supp}\phi$ in its interior. After taking $r \to 1$ and using the dominated convergence theorem, it follows that $\Lambda[v]$ is a solution of (8-1). From the construction, it is clear that $\Lambda[v]$ is the pointwise-minimal solution which satisfies $\Lambda[v] \ge v$.

Remark. This construction generalizes the notion of the *minimal harmonic majorant* for subharmonic functions on the unit disk. One small but important difference is that the minimal harmonic majorant does not always exist (i.e., may be identically $+\infty$).

8B. *Nearly maximal solutions.* A solution of (8-1) is called *nearly maximal* if

$$\limsup_{r\to 1} \int_{|z|=r} (u_{\max} - u)\, d\theta < \infty. \tag{8-4}$$

For each $0 < r < 1$, we may view $(u_{\max} - u)\, d\theta$ as a positive measure on the circle of radius r. Subharmonicity guarantees the existence of a weak limit as $r \to 1$, so we obtain a measure $\mu[u]$ on the unit circle associated to u. We refer to μ as the *deficiency measure* of u.

Notice that if $\mu \geq 0$ is a measure on the unit circle and P_μ is its Poisson extension to the unit disk, then $\Lambda[u_{\max} - P_\mu]$ is a nearly maximal solution. Clearly, the deficiency measure ν of $\Lambda[u_{\max} - P_\mu]$ is at most μ.

Lemma 8.4 (fundamental lemma). *If u is a nearly maximal solution of (8-1) with deficiency measure μ, then $u = \Lambda[u_{\max} - P_\mu]$.*

Proof. Step 1. Observe that $u_{\max} - P_\mu$ is a subsolution since

$$\Delta(u_{\max} - P_\mu) = g(u_{\max}) \geq g(u_{\max} - P_\mu).$$

We claim that $u \geq u_{\max} - P_\mu$ and thus $u \geq \Lambda[u_{\max} - P_\mu]$. To this end, we consider the function

$$\phi := u_{\max} - u - P_\mu.$$

Since ϕ is a subharmonic function with zero boundary trace, by the maximum principle, $\phi \leq 0$ in the unit disk.

Step 2. As $v := \Lambda[u_{\max} - P_\mu]$ is a nearly maximal solution, it possesses a deficiency measure ν. From Step 1, we know that

$$u \geq v = \Lambda[u_{\max} - P_\mu] \geq u_{\max} - P_\mu.$$

After rearranging, we get

$$u_{\max} - u \leq u_{\max} - v \leq P_\mu.$$

Taking the weak limit as $r \to 1$, we see that $\nu = \mu$.

Step 3. Finally, since $u - v$ is a nonnegative subharmonic function with zero boundary trace, $u = v$. □

In particular, Lemma 8.4 shows that the deficiency measure μ uniquely determines the nearly maximal solution u. Below, we will write u_μ for the nearly maximal solution associated to the measure μ, if it exists. Another simple consequence of Lemma 8.4 is the *monotonicity principle* for nearly maximal solutions:

Corollary 8.5 (monotonicity principle). *If $\nu < \mu$ then $u_\nu > u_\mu$.*

8C. *Constructible and invisible measures.* We say that a measure μ on the unit circle is *invisible* if, for any measure $0 < \nu \leq \mu$, there does not exist a nearly maximal solution u_ν with deficiency measure ν. In this section, we show that any positive measure on the unit circle can be uniquely decomposed into a deficiency measure and an invisible measure.

Theorem 8.6. *Suppose μ is a positive measure on the unit circle. If $u_\nu = \Lambda[u_{\max} - P_\mu]$, then ν is a deficiency measure and $\mu - \nu$ is an invisible measure.*

In particular, a measure μ is invisible if and only if $\Lambda[u_{\max} - P_\mu] = u_{\max}$. We will break the proof of Theorem 8.6 into a series of lemmas.

Lemma 8.7. *If μ is a deficiency measure, then any measure $0 \leq \mu_1 \leq \mu$ is also a deficiency measure.*

Proof. To show that μ_1 is a deficiency measure, we check that $\mu_1 = \nu_1$, where $u_{\nu_1} = \Lambda[u_{\max} - P_{\mu_1}]$. Since the inequality $\nu_1 \le \mu_1$ is always true, we only need to prove the opposite inequality $\mu_1 \le \nu_1$.

Let $\mu_2 = \mu - \mu_1$. Using the same argument as in the proof of Lemma 8.4, it is not difficult to show that

$$\Lambda[u_{\max} - P_{\mu_1+\mu_2}] \ge \Lambda[u_{\max} - P_{\mu_1}] - P_{\mu_2}$$

or

$$u_{\max} - \Lambda[u_{\max} - P_{\mu_1+\mu_2}] \le u_{\max} - \Lambda[u_{\max} - P_{\mu_1}] + P_{\mu_2}.$$

Taking traces, we see that $\mu_1 + \mu_2 \le \nu_1 + \mu_2$ or $\mu_1 \le \nu_1$ as desired. □

Lemma 8.8. (i) *The sum of two deficiency measures is a deficiency measure.*

(ii) *Suppose μ_i, $i = 1, 2, 3, \ldots$, are deficiency measures such that their sum $\mu = \sum \mu_i$ is a finite measure. Then, μ is also a deficiency measure.*

In the proof below, we will use the following elementary observation: if g is a convex function and $x_1 < x_2 < x_3 < x_4$ are four real numbers satisfying $x_1 + x_4 = x_2 + x_3$, then

$$g(x_2) + g(x_3) \le g(x_1) + g(x_4). \tag{8-5}$$

Moreover, if g is an *increasing* convex function, then (8-5) holds under the weaker assumption $x_1 + x_4 \ge x_2 + x_3$. This is a one-dimensional analogue of the fact that the composition $\phi \circ u$ of an increasing convex function ϕ and a subharmonic function u is subharmonic.

Proof of Lemma 8.8. (i) Suppose $\mu = \mu_1 + \mu_2$ is a measure on the unit circle. Set

$$u_\nu = \Lambda[u_{\max} - P_\mu].$$

In view of the discussion preceding Lemma 8.4, to prove (i), it is enough to show that

$$\mu_1 + \mu_2 \le \nu. \tag{8-6}$$

To verify (8-6), we check that

$$\Lambda[u_{\max} - P_{\mu_1}] + \Lambda[u_{\max} - P_{\mu_2}] \ge \Lambda[u_{\max}] + \Lambda[u_{\max} - P_\mu],$$

which we abbreviate as $B + C \ge A + D$. Clearly, $A \ge B \ge D$ and $A \ge C \ge D$. Consider the function

$$\phi = (A + D - B - C)^+.$$

Since g is an increasing convex function, at a point $z \in \mathbb{D}$ where $A + D > B + C$, we have

$$\Delta\phi(z) = g(A(z)) + g(D(z)) - g(B(z)) - g(C(z)) \ge 0.$$

In view of Kato's inequality, ϕ is subharmonic and nonnegative on the unit disk. If we knew that ϕ had zero trace, then we could immediately say that ϕ is identically 0.

Due to difficulties examining the trace of ϕ on $\partial\mathbb{D}$ directly, we use an approximation argument. For each $0 < r < 1$, we consider the function

$$\phi_r = (\Lambda_r[u_{\max} - P_{\mu_1}] + \Lambda_r[u_{\max} - P_{\mu_2}] - \Lambda_r[u_{\max}] - \Lambda_r[u_{\max} - P_\mu])^+,$$

defined on $\mathbb{D}_r$. The above argument shows that ϕ_r is a nonnegative subharmonic function on $\mathbb{D}_r$. As ϕ_r has zero boundary values on $\partial\mathbb{D}_r$, it is identically 0. Taking $r \to 1$, we see that ϕ is identically 0 as desired.

(ii) Set $\tilde{\mu}_j = \mu_1 + \mu_2 + \cdots + \mu_j$. By part (i), we have

$$\Lambda[u_{\max} - P_\mu] \le \Lambda[u_{\max} - P_{\tilde{\mu}_j}] = u_{\tilde{\mu}_j}.$$

The above equation shows that if

$$u_\nu = \Lambda[u_{\max} - P_\mu],$$

then $\nu \ge \tilde{\mu}_j$ for any j, which implies $\nu \ge \mu$. As the reverse inequality is always true, $\nu = \mu$ as desired. □

Lemma 8.9. *If $\mu \ge 0$ is a measure on the unit circle and $u_\nu = \Lambda[u_{\max} - P_\mu]$, then the difference $\mu - \nu$ is invisible.*

Proof. We need to show that any measure $0 < \omega \le \mu - \nu$ does not arise as a deficiency measure of some nearly maximal solution. The existence of u_ω would imply the existence of $u_{\nu+\omega}$ by Lemma 8.8, which would in turn lead to the estimate

$$u_{\max} - P_\mu \le u_{\max} - P_{\nu+\omega} \le u_{\nu+\omega} \le u_\nu$$

by the monotonicity principle and the fundamental lemma (Lemmas 8.5 and 8.4 respectively). This contradicts the definition of u_ν as the *least* solution that lies above $u_{\max} - P_\mu$. □

8D. ***A lemma on iterated majorants.*** For future reference, we record the following lemma:

Lemma 8.10. (i) *For two positive measures μ_1 and μ_2 on the unit circle,*

$$\Lambda[\Lambda[u_{\max} - P_{\mu_2}] - P_{\mu_1}] = \Lambda[u_{\max} - P_{\mu_1+\mu_2}].$$

(ii) *More generally,*

$$\Lambda[\cdots \Lambda[\Lambda[u_{\max} - P_{\mu_j}] - P_{\mu_{j-1}}] \cdots - P_{\mu_1}] = \Lambda[u_{\max} - P_{\mu_1+\mu_2+\cdots+\mu_j}].$$

(iii) *If $\mu = \sum_{j=1}^\infty \mu_j$ is a finite measure, then*

$$\lim_{j\to\infty} \Lambda[\cdots \Lambda[\Lambda[u_{\max} - P_{\mu_j}] - P_{\mu_{j-1}}] \cdots - P_{\mu_1}] = \Lambda[u_{\max} - P_\mu]$$

pointwise on the unit disk.

Proof. (i) The $\ge$ direction follows from the monotonicity of Λ. For the $\le$ direction, it suffices to show

$$\Lambda[u_{\max} - P_{\mu_2}] - P_{\mu_1} \le \Lambda[u_{\max} - P_{\mu_1+\mu_2}]$$

or

$$\Lambda_r[u_{\max} - P_{\mu_2}] - P_{\mu_1} \le \Lambda_r[u_{\max} - P_{\mu_1+\mu_2}]$$

for any $0 < r < 1$. To this end, we form the function

$$u_r = (\Lambda_r[u_{\max} - P_{\mu_2}] - P_{\mu_1}) - \Lambda_r[u_{\max} - P_{\mu_1+\mu_2}],$$

defined on $\mathbb{D}_r = \{z : |z| < r\}$. Since u_r is subharmonic and vanishes on $\partial\mathbb{D}_r$, it must be identically 0. This proves the $\leq$ direction.

(ii) This follows after applying (i) $j-1$ times.

(iii) Let $\tilde{\mu}_j = \mu_1 + \mu_2 + \cdots + \mu_j$. By part (i), we have

$$\Lambda[u_{\max} - P_{\tilde{\mu}_j}] - P_{\mu - \tilde{\mu}_j} \leq \Lambda[u_{\max} - P_\mu] \leq \Lambda[u_{\max} - P_{\tilde{\mu}_j}].$$

Since $P_{\mu - \tilde{\mu}_j} \to 0$ pointwise in the unit disk, the minimal dominating solutions $\Lambda[u_{\max} - P_{\tilde{\mu}_j}]$ decrease to $\Lambda[u_{\max} - P_\mu]$. □

9. Nearly maximal solutions

In this section, we prove Theorem 1.3 which partially characterizes the nearly maximal solutions of

$$\Delta u = u^p \cdot \chi_{u>0} \quad \text{on } \mathbb{D}, \tag{9-1}$$

with $p > 1$. From Section 8A, we know that (9-1) has a radially invariant solution $u_{\max}$ which dominates all the other solutions pointwise. By solving an ODE, one can write down an explicit formula for $u_{\max}$. Here, we will only need the asymptotic formula

$$u_{\max}(z) \sim C_\alpha (1 - |z|)^{\alpha - 1}, \quad |z| \to 1,$$

where $\alpha = (p-3)/(p-1)$. We will be especially interested in the case when $p > 3$, in which case $\alpha \in (0, 1)$.

The proof of Theorem 1.3 consists of two parts:

(1) First, we show that if μ does not charge α-Beurling–Carleson sets, then it is not the deficiency measure of any nearly maximal solution. As the proof is similar to the one in [Ivrii 2019] for $\Delta u = e^{2u}$, we only give a sketch of the argument in Section 9B.

(2) Secondly, we show that if μ is concentrated on an β-Beurling–Carleson set for some $\beta < \alpha$, then there is a nearly maximal solution u_μ with deficiency measure μ. The argument in [Ivrii 2019] relied on the Liouville correspondence between solutions of $\Delta u = e^{2u}$ and holomorphic self-mappings of the disk, which is unavailable in the present setting. We present a new approach to existence which involves special Privalov stars with round corners. The special Privalov stars will be constructed in Section 9C, and the existence will be explained in Section 9D.

9A. *Restoring property.* We focus on the case when $p > 3$. The following lemmas will be used in conjunction with Roberts decompositions to show that certain measures on the unit circle are invisible:

Lemma 9.1. *Let $n_i = 2^i$. For any $0 < a < 1$, there exists $a < b < 1$ such that*

$$\Lambda_{1-1/n_{i+1}}[a \cdot u_{\max}] > b \cdot u_{\max} \quad \text{on } \left\{z : |z| = 1 - \frac{1}{n_i}\right\}. \tag{9-2}$$

Proof. We prefer to work on the upper half-plane $\mathbb{H}$ since the expression for the maximal solution is simpler there: $u_{\max}(z) = C_\alpha y^{\alpha-1}$, where $y = \operatorname{Im} z$. We need to show that

$$\Lambda_{y_0}[a \cdot u_{\max}] > b \cdot u_{\max} \quad \text{on } \{\operatorname{Im} z = 2y_0\}.$$

When extending constant boundary values from a horizontal line, we get the maximal solution shifted vertically by an appropriate amount:

$$u = \Lambda_{y_0}[a \cdot u_{\max}] = C_\alpha (y + c)^{\alpha-1},$$

where c is determined by the equation

$$a \cdot C_\alpha y_0^{\alpha-1} = C_\alpha (y_0 + c)^{\alpha-1} \quad \Longrightarrow \quad c = a^{1/(\alpha-1)} \cdot y_0 - y_0.$$

In particular,

$$u(2y_0) = C_\alpha (1 + a^{1/(\alpha-1)})^{\alpha-1} \cdot y_0^{\alpha-1}.$$

This suggests that we should take

$$b = \frac{u(2y_0)}{u_{\max}(2y_0)} = \frac{(1 + a^{1/(\alpha-1)})^{\alpha-1}}{2^{\alpha-1}} > a. \qquad \square$$

A similar argument shows:

Lemma 9.2. *For any* $0 < a, \varepsilon, \rho < 1$, *there exists an* $0 < r < 1$ *such that*

$$\Lambda_r[a \cdot u_{\max}] > (1 - \varepsilon) \cdot u_{\max} \quad \text{on } \mathbb{D}_\rho. \tag{9-3}$$

9B. *Invisible measures.* Suppose μ is a measure on the unit circle that does not charge α-Beurling–Carleson sets. In order to show that μ is invisible, it is enough to check that $\Lambda[u_{\max} - P_\mu] = u_{\max}$, where Λ denotes the minimal dominating solution on the unit disk.

According to Corollary 3.3, for any parameters c and j_0, we can express μ as an infinite series

$$\mu = \mu_1 + \mu_2 + \cdots,$$

where μ_j satisfies the modulus of continuity estimate

$$\mu_j(I) \le c|I|^\alpha, \quad I \in \mathcal{D}_{j+j_0}. \tag{9-4}$$

One may express condition (9-4) in terms of the Poisson extension P_{μ_j} to the unit disk:

$$P_{\mu_j}(z) \le c_2(1 - |z|)^{\alpha-1} \le c_3 \cdot u_{\max}(z), \quad |z| = 1 - 2^{-(j+j_0)}.$$

We choose the parameter $c > 0$ in the Roberts decomposition small enough that the above equation holds with $c_3 = b - a$, where $0 < a < 1$ is arbitrary and $b = b(a)$ is given by Lemma 9.1.

By Lemma 8.10 and monotonicity properties of Λ, we have

$$\begin{aligned}\Lambda[u_{\max} - P_\mu] &= \lim_{j\to\infty} \Lambda[u_{\max} - P_{\mu_1+\mu_2+\cdots+\mu_j}] = \lim_{j\to\infty} \Lambda[\cdots \Lambda[u_{\max} - P_{\mu_j}] \cdots - P_{\mu_1}] \\ &\ge \lim_{j\to\infty} \Lambda_{1-1/n_1}[\cdots \Lambda_{1-1/n_j}[u_{\max} - P_{\mu_j}] \cdots - P_{\mu_1}].\end{aligned}$$

Since each time we apply Λ_{1-1/n_i} we shrink the domain of the definition, the above inequality is valid on $\mathbb{D}_{1-1/n_1}$. Using the restoring property j times, we get

$$\Lambda[u_{\max} - P_\mu] \geq a \cdot u_{\max} \quad \text{on } \mathbb{D}_{1-1/n_1}.$$

Applying the restoring property one more time shows that, for any given $0 < \rho < 1$ and $\varepsilon > 0$, one could choose the offset $j_0 \geq 0$ large enough to guarantee that

$$\Lambda[u_{\max} - P_\mu] \geq (1-\varepsilon) u_{\max} \quad \text{on } \mathbb{D}_\rho.$$

In other words, $\Lambda[u_{\max} - P_\mu] = u_{\max}$ as desired.

What happens when $1 < p \leq 3$? If $1 < p \leq 3$, then by Harnack's inequality,

$$P_\mu(z) \leq 2(1-|z|)^{-1} \mu(\partial\mathbb{D}) \lesssim u_{\max}(z), \quad |z| < 1,$$

is true for any measure on the unit circle. By multiplying μ by a small constant $\varepsilon > 0$, one can arrange that $P_{\varepsilon\mu} \leq \left(\frac{1}{2}\right) u_{\max}$ or $u_{\max} - P_{\varepsilon\mu} \geq \left(\frac{1}{2}\right) u_{\max}$. The argument above shows that $\Lambda[u_{\max} - P_\mu] = u_{\max}$, which means that the measure $\varepsilon\mu$ is invisible. In turn, this implies that μ itself is invisible.

9C. *Special Privalov Stars.* Suppose $E \subset \partial\mathbb{D}$ is a β-Beurling–Carleson set with $\beta < \alpha$ and μ is a measure supported on E. Given $\varepsilon > 0$, we will construct a special sawtooth domain $\widetilde{K}_E = \widetilde{K}_E(\varepsilon, \mu) \subset \mathbb{D}$ containing the origin which satisfies the following properties:

(1) Let ω_z denote the harmonic measure on $\partial\widetilde{K}_E$ as viewed from $z \in \widetilde{K}_E$. We require that

$$\int_{\partial\widetilde{K}_E} u_{\max}(z)\, d\omega_0(z) \asymp \int_{\partial\widetilde{K}_E} (1-|z|)^{\alpha-1}\, d\omega_0(z) < \infty.$$

(2a) Secondly, we want the Riemann map $\varphi : (\mathbb{D}, 0) \to (\partial\widetilde{K}_E, 0)$ to have a finite angular derivative at $\varphi^{-1}(\zeta)$ for μ a.e. $\zeta \in E = \partial\widetilde{K}_E \cap \partial\mathbb{D}$.

(2b) In view of the Schwarz lemma, for any $\zeta \in E$, the angular derivative satisfies $1 < |\varphi'(\varphi^{-1}(\zeta))| < \infty$, or alternatively, $0 < |(\varphi^{-1})'(\zeta)| < 1$. We will construct $\partial\widetilde{K}_E$ such that the set $E' \subset E$, where $1-\varepsilon < |(\varphi^{-1})'(\zeta)| < 1$, has measure $\mu(E') \geq (1-\varepsilon)\mu(E)$.

Fix a constant $1 < \gamma < 1/(1-\alpha)$. We fix a C^1 function $\phi : [0,1] \to [0,1]$ which satisfies

- $0 < \phi(t) \leq 1 - 2\left|t - \frac{1}{2}\right|$ for $0 < t < 1$,
- $\phi(0) = 0$ and $\phi(t) \sim t^\gamma$ as $t \to 0$,
- $\phi\left(\frac{1}{2}\right) = 1$,
- $\phi(1) = 0$ and $\phi(t) \sim (1-t)^\gamma$ as $t \to 1$,

and define the *tent* over $[0,1]$ with height h by

$$T^h_{[0,1]} = \{(x,y) \in \mathbb{R}^2 : 0 \leq x \leq 1,\ 0 \leq y \leq h \cdot \phi(x)\}.$$

Let $\{h(I)\} \subset (0, 1]$ be a collection of heights. Over each complementary arc $I = (e^{i\theta_1}, e^{i\theta_2}) \subset \partial\mathbb{D} \setminus E$, we build the tent

$$T_I = \left\{ re^{i\theta} : \theta_1 \le \theta \le \theta_2,\ 1 - \psi \cdot h(I) \cdot \phi\left(\frac{\theta - \theta_1}{\theta_2 - \theta_1}\right) \le r \le 1 \right\},$$

where $0 < \psi \le 1$ is an auxiliary parameter to be chosen. The special Privalov star $\widetilde{K}_E$ is then obtained by removing these tents from the unit disk. To achieve the above objectives, we use the heights

$$h(I) = \min\left(|I|, \frac{|I|^\alpha}{u(z_I)}\right), \quad u = P_\mu. \tag{9-5}$$

Condition (1). For an arc $J \subset \partial\mathbb{D}$, we denote by $\tau(J)$ the part of $\partial\widetilde{K}_E$ that is located above J in $\partial\widetilde{K}_E$, i.e., $\tau(J) = \{z \in \partial\widetilde{K}_E : z/|z| \in J\}$.

Lemma 9.3. *The harmonic measure on $\partial\widetilde{K}_E$ as viewed from the origin is bounded above by a multiple of arclength.*

Proof. To prove the lemma, we show that $\omega_{\widetilde{K}_E,0}(\tau(J)) \lesssim |J|$ for any arc J of the unit circle with $|J| \le \frac{1}{4}$. Let $B = B(J)$ be a ball centered at the midpoint of J of radius $3|J|$. Since $\tau(J) \subset B(J)$, by the monotonicity properties of harmonic measure, we have

$$\omega_{\widetilde{K}_E,0}(\tau(J)) \le \omega_{\mathbb{D}\setminus B,0}(\partial B \cap \mathbb{D}).$$

The latter quantity is easily seen to be $\lesssim |J|$. □

Corollary 9.4. *For a complementary arc $I \subset \partial\mathbb{D} \setminus E$, we have*

$$\int_{\tau(I)} u_{\max}(z)\, d\omega_0(z) \lesssim h(I)^{\alpha-1} \cdot |I|.$$

Proof. We split $I = \bigcup_{n\in\mathbb{Z}} I_n$ into countably many Whitney arcs, so that $|I_n| = \left(\frac{1}{2}\right)^{|n|} \cdot |I_0|$, and I_m and I_n have a common endpoint if $|m - n| = 1$. In view of the above lemma,

$$\int_{\tau(I_n)} u_{\max}(z)\, d\omega_0(z) \lesssim \frac{|I|}{2^{|n|}} \cdot \left\{\frac{h(I)}{2^{\gamma|n|}}\right\}^{\alpha-1}.$$

By the choice of γ, the corollary follows after summing a convergent geometric series. □

We now verify Condition (1). With the choice of heights (9-5),

$$\int_{\partial\widetilde{K}_E} u_{\max}(z)\, d\omega_0(z) \lesssim \sum |I|^{\alpha^2-\alpha+1} u(z_I)^{1-\alpha}.$$

Applying Hölder's inequality with exponents $1/\lambda$ and $1/(1-\lambda)$, we get

$$\begin{aligned}\sum |I|^{\alpha^2-\alpha+1} u(z_I)^{1-\alpha} &= \sum |I|^{\alpha(\alpha-1)+\lambda} \cdot |I|^{1-\lambda} u(z_I)^{1-\alpha} \\ &\le \left(\sum |I|^{\alpha(\alpha-1)+\lambda/(\lambda)}\right)^{\lambda} \left(\sum |I| u(z_I)^{(1-\alpha)/(1-\lambda)}\right)^{1-\lambda}.\end{aligned} \tag{9-6}$$

With the choice

$$\lambda = \alpha \cdot \frac{1-\alpha}{1-\beta} < \alpha,$$

we have

$$\beta = \frac{\alpha(\alpha-1)+\lambda}{\lambda} \quad \text{and} \quad \delta = \frac{1-\alpha}{1-\lambda} < 1.$$

The first sum in (9-6) is finite as E is a β-Beurling–Carleson set, while the second sum is finite since the nontangential maximal function of u lies in L^{δ}.

Conditions (2*a*) *and* (2*b*). In order to verify that the special sawtooth domain $\widetilde{K}_E$ satisfies Condition (2a), we need to check the Rodin–Warschawski condition for the existence of an angular derivative; see Theorem 7.1. This will be done in Lemmas 9.5 and 9.6 below.

For a point $\zeta \in \partial\mathbb{D}$, we write $H(\zeta)$ for the length of the radius $[0, \zeta]$ that lies outside of $\widetilde{K}_E$.

Lemma 9.5. *For a point $x \in E$ and a complementary arc $I \subset \partial\mathbb{D} \setminus E$, we have*

$$\int_{xe^{i\eta}\in I} \frac{H(xe^{i\eta})}{\eta^2}\, d\eta \lesssim \frac{h(I)\cdot|I|}{\operatorname{dist}\bigl(x, \tfrac{1}{2}I\bigr)^2}.$$

Proof. We decompose $I = \bigcup_{n\in\mathbb{Z}} I_n$ into a union of countably many Whitney arcs such that $|I_n| = \bigl(\tfrac{1}{2}\bigr)^{|n|}\cdot|I_0|$, and I_m and I_n have a common endpoint if $|m-n| = 1$. Since $\operatorname{dist}(x, I_n) \geq 2^{-|n|}\operatorname{dist}\bigl(x, \tfrac{1}{2}I\bigr)$,

$$\int_{xe^{i\eta}\in I_n} \frac{H(xe^{i\eta})}{\eta^2}\, d\eta \lesssim \frac{\{\max_{\zeta\in I_n} H(\zeta)\}\cdot|I_n|}{\operatorname{dist}(x, I_n)^2} \lesssim \frac{2^{-\gamma|n|}h(I)\cdot 2^{-|n|}|I|}{\bigl\{2^{-|n|}\operatorname{dist}\bigl(x, \tfrac{1}{2}I\bigr)\bigr\}^2} = 2^{-(\gamma-1)|n|}\cdot\frac{h(I)\cdot|I|}{\operatorname{dist}\bigl(x, \tfrac{1}{2}I\bigr)^2}.$$

The lemma follows after summing a convergent geometric series. □

Lemma 9.6. *For μ a.e. $x \in \partial\mathbb{D}$, we have the following when summing over complementary arcs*:

$$\sum \frac{h(I)\cdot|I|}{\operatorname{dist}\bigl(x, \tfrac{1}{2}I\bigr)^2} < \infty.$$

Proof. It is enough to check that

$$\int_{\partial\mathbb{D}}\biggl\{\sum_I \frac{h(I)\cdot|I|}{\operatorname{dist}\bigl(x, \tfrac{1}{2}I\bigr)^2}\biggr\}\, d\mu(x) \leq \int_{\partial\mathbb{D}}\biggl\{\sum_I \frac{|I|^{\alpha+1}}{u(z_I)\cdot\operatorname{dist}\bigl(x, \tfrac{1}{2}I\bigr)^2}\biggr\}\, d\mu(x)$$
$$= \sum_I |I|^{\alpha}\cdot\biggl\{\frac{1}{u(z_I)}\int_{\partial\mathbb{D}} \frac{|I|}{\operatorname{dist}\bigl(x, \tfrac{1}{2}I\bigr)^2}\, d\mu(x)\biggr\}$$

is finite. To see this, notice that the expression in the parentheses is $O(1)$ and use that E is a β-Beurling–Carleson set (and hence, an α-Beurling–Carleson set). □

In view of Lemma 7.2, to achieve Condition (2b), we only need to select a sufficiently small auxiliary parameter $0 < \psi \leq 1$.

9D. *Existence.* To prove Theorem 1.3, it remains to construct a nearly maximal solution with deficiency measure μ supported on a β-Beurling–Carleson set E.

For $n \in \mathbb{R}$, let u_n be the solution of $\Delta u = u^p\cdot\chi_{u>0}$ which is equal to n on the unit circle. Since $u_n - P_\mu$ is a subsolution and $n - P_\mu$ is a supersolution of $\Delta u = u^p\cdot\chi_{u>0}$ with the same boundary data, by the principle of sub- and supersolutions, there exists a unique solution $u_{\mu,n}$ such that

$$u_n - P_\mu \leq u_{\mu,n} \leq n - P_\mu. \tag{9-7}$$

As the solutions $u_{\mu,n}$ are increasing in n and bounded above by $u_{\max}$, the limit $u := \lim_{n\to\infty} u_{\mu,n}$ exists. Taking $n \to \infty$ in (9-7), we get

$$u_{\max} - P_\mu \le u,$$

which tells us that u is a nearly maximal solution whose deficiency measure is at most μ.

To show that the mass of the deficiency measure of u is at least $\mu(\partial\mathbb{D})$, we use the special sawtooth domain $\widetilde{K}_E$ constructed in Section 9C. For $0 < r < 1$, we form the truncated region $K_r = \widetilde{K}_E \cap \mathbb{D}_r$. Its boundary consists of two parts: a *sawtooth* part $\partial_{\text{saw}} K_r = \partial K_r \setminus \partial \mathbb{D}_r$ and a *round* part $\partial_{\text{round}} K_r = \partial K_r \cap \partial \mathbb{D}_r$. We estimate $u_{\mu,n}$ on ∂K_r by

$$u_{\mu,n} \le f := \begin{cases} u_{\max} & \text{on } \partial_{\text{saw}} K_r, \\ n - P_\mu & \text{on } \partial_{\text{round}} K_r. \end{cases} \tag{9-8}$$

By the maximum principle, u is bounded above on K_r by the harmonic extension of these boundary values. Taking $r \to 1$ while keeping n fixed, we get

$$u(z) \le \int_{\partial \widetilde{K}_E} u_{\max}(w)\, d\omega_z(w) - \lim_{r\to 1} \int_{\partial_{\text{round}} K_r} P_\mu(w)\, d\omega_{K_r,z}(w) = A(z) - B(z) \tag{9-9}$$

for $z \in \widetilde{K}_E$. In the equation above, ω_z and $\omega_{K_r,z}$ denote harmonic measures from the point z in the domains $\widetilde{K}_E$ and K_r, respectively. Condition (1) guarantees that $A(z)$ is finite. Below, we will see that Conditions (2a) and (2b) ensure that $B(z)$ is large enough to be responsible for the deficiency of u.

A lemma featuring Privalov stars. For a closed subset $F \subset \partial\mathbb{D}$, we write $K_{F,\theta}$ for the standard Privalov star, which is defined as the union of Stolz angles emanating from F with aperture $0 < \theta < \pi$. We will use the following elementary lemma:

Lemma 9.7. *Let μ be a positive measure on the unit circle and $F \subset \partial\mathbb{D}$ be a closed set. For any aperture $0 < \theta < \pi$,*

$$\limsup_{\rho\to 1} \int_{K_{F,\theta}\cap\partial\mathbb{D}_\rho} P_\mu(w)\, |dw| \le \mu(F).$$

Conversely, for any $\varepsilon > 0$, there exists an aperture $0 < \theta < \pi$ so that

$$\liminf_{\rho\to 1} \int_{K_{F,\theta}\cap\partial\mathbb{D}_\rho} P_\mu(w)\, |dw| \ge (1-\varepsilon)\mu(F).$$

Pruning the set E further. Recall that E' was defined as the subset of E where the angular derivative satisfies $1-\varepsilon < |(\varphi^{-1})'(\zeta)| < 1$, and we had arranged that $\mu(E') \ge (1-\varepsilon)\mu(E)$. By sacrificing a little bit more mass, we can obtain uniformity of nontangential limits and truncated Stolz angles. More precisely, for any $\varepsilon > 0$ and $\theta > 0$, one can find a closed subset $E'' \subset E'$ and $0 < \rho_0 < 1$ such that

$$\mu(E'') \ge (1-2\varepsilon)\mu(E), \tag{9-10}$$

$$1 - 2\varepsilon < |(\varphi^{-1})'(z)| < 1 + \varepsilon \quad \text{for } z \in K_{E'',\theta} \cap \{\rho_0 < |w| < 1\}, \tag{9-11}$$

$$K_{E'',\theta} \cap \{\rho_0 < |w| < 1\} \subset \widetilde{K}_E. \tag{9-12}$$

Strategy. To prove the existence part of Theorem 1.3, we show:

Lemma 9.8. *For any $\varepsilon > 0$, we can choose the aperture $0 < \theta < \pi$ close enough to π that*

$$\int_{K_{E'',\theta}\cap\partial\mathbb{D}_\rho} A(z)\,|dz| \le \varepsilon\cdot\mu(E'') \quad \text{and} \quad \int_{K_{E'',\theta}\cap\partial\mathbb{D}_\rho} B(z)\,|dz| \ge (1-\varepsilon)\cdot\mu(E'') \tag{9-13}$$

for all $\rho_0 < \rho < 1$ sufficiently close to 1.

Proof of existence in Theorem 1.3 assuming Lemma 9.8. Decompose $u = u_+ - u_-$ into positive and negative parts. For $\rho_0 < \rho < 1$, we have

$$\int_{|z|=\rho} (u_{\max}(z) - u(z))\,|dz| \ge \int_{|z|=\rho} u_-(z)\,|dz| \ge \int_{K_{E'',\theta}\cap\partial\mathbb{D}_\rho} (B(z)-A(z))\,|dz| \ge (1-2\varepsilon)\mu(E'') \ge (1-2\varepsilon)^2\mu(E).$$

Since $\varepsilon > 0$ was arbitrary, the mass of the deficiency measure of u is at least $\mu(E)$. □

The remainder of the paper is devoted to proving Lemma 9.8.

Estimating $A(z)$. Notice that $A(z)$ is a positive harmonic function on $\widetilde{K}_E$ which extends absolutely continuous boundary values $u_{\max} \in L^1(\partial\widetilde{K}_E, \omega_0)$. Therefore, if φ is a conformal map from $(\mathbb{D}, 0)$ to $(\widetilde{K}_E, 0)$, then $A\circ\varphi$ is a positive harmonic function on the unit disk with absolutely continuous boundary values on the unit circle. Since $\varphi^{-1}(E'')$ has Lebesgue measure zero by Loewner's lemma,

$$\lim_{\rho\to 1} \int_{K_{\varphi^{-1}(E''),\theta}\cap\partial\mathbb{D}_\rho} (A\circ\varphi)(w)\,|dw| = 0$$

by Lemma 9.7. From here, the first inequality in (9-13) follows after an application of Harnack's inequality.

Estimating $B(z)$. Since $\partial K_r = \partial_{\text{round}} K_r \cup \partial_{\text{saw}} K_r$,

$$\int_{\partial_{\text{round}} K_r} P_\mu(w)\,d\omega_{K_r,z}(w) = P_\mu(z) - \int_{\partial_{\text{saw}} K_r} P_\mu(w)\,d\omega_{K_r,z}(w), \quad z \in K_r.$$

By the monotonicity properties of harmonic measure, the integrals over $\partial_{\text{saw}} K_r$ are increasing in r. Taking $r \to 1$, we get

$$B(z) = P_\mu(z) - \int_{\partial\widetilde{K}_E\cap\mathbb{D}} P_\mu(w)\,d\omega_z(w), \quad z \in \widetilde{K}_E. \tag{9-14}$$

Since B is a positive harmonic function on $\widetilde{K}_E$, the composition $B\circ\varphi$ is a positive harmonic function on the unit disk. Inspection shows that $B\circ\varphi = P_\nu$ for a positive measure ν supported on $\varphi^{-1}(E)$. In fact, Theorem 7.3 tells us that

$$\nu = \varphi^*(|\psi'(\zeta)|\,d\mu(\zeta)).$$

Since $1-\varepsilon < |\psi'(\zeta)| < 1$ on $E' \supseteq E''$ by Condition (2b),

$$\nu(\varphi^{-1}(E'')) \ge (1-\varepsilon)\mu(E'').$$

Now, by Lemma 9.7, if the aperture θ is sufficiently close to π, then

$$\liminf_{\rho\to 1} \int_{K_{\varphi^{-1}(E''),\theta}\cap\partial\mathbb{D}_\rho} (B\circ\varphi)(w)\,|dw| \ge (1-\varepsilon)\nu(\varphi^{-1}(E'')) \ge (1-\varepsilon)^2\mu(E'').$$

The second estimate in (9-13) follows from Harnack's inequality as in *Estimating $A(z)$* above.

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