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DYNAMICAL TORSION FOR CONTACT ANOSOV FLOWS

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We introduce a new object, the dynamical torsion, which extends the potentially ill-defined value at 0 of the Ruelle zeta function of a contact Anosov flow, twisted by an acyclic representation of the fundamental group. We show important properties of the dynamical torsion: it is invariant under deformations among contact Anosov flows, it is holomorphic in the representation and it has the same logarithmic derivative as some refined combinatorial torsion of Turaev. This shows that the ratio between this torsion and the Turaev torsion is locally constant on the space of acyclic representations.

In particular, for contact Anosov flows path-connected to the geodesic flow of some hyperbolic manifold among contact Anosov flows, we relate the leading term of the Laurent expansion of ζ at the origin, the Reidemeister torsion and the torsions of the finite-dimensional complexes of the generalized resonant states of both flows for the resonance 0. This extends previous work of Dang, Guillarmou, Rivière and Shen (*Invent. Math.* **220**:2 (2020), 525–579) on the Fried conjecture near geodesic flows of hyperbolic 3-manifolds, to hyperbolic manifolds of any odd dimension.

1. Introduction

Let M be a closed odd-dimensional manifold and (E, ∇) be a flat vector bundle over M. The parallel transport of the connection ∇ induces a conjugacy class of representation $\rho \in \mathrm{Hom}(\pi_1(M), \mathrm{GL}(\mathbb{C}^d))$ (every representation of the fundamental group can be obtained in this way; see Section 11.1). Moreover, ∇ defines a differential on the complex $\Omega^\bullet(M, E)$ of E-valued differential forms on M and thus cohomology groups $H^\bullet(M, \nabla) = H^\bullet(M, \rho)$ (note that we use the notation ∇ also for the twisted differential induced by ∇, whereas it can be denoted by d^∇ in other references). We will say that ∇ (or ρ) is acyclic if those cohomology groups are trivial.

If ρ is unitary (or equivalently, if there exists a hermitian structure on E preserved by ∇) and acyclic, Reidemeister [1935] introduced a combinatorial invariant $\tau_R(\rho)$ of the pair (M, ρ), the so-called *Franz–Reidemeister torsion* (or R-torsion), which is a positive number. This allowed him to classify lens spaces in dimension 3; this result was then extended in higher dimensions by Franz [1935] and de Rham [1936].

On the analytic side, Ray and Singer [1971] introduced another invariant $\tau_{RS}(\rho)$, the *analytic torsion*, defined via the derivative at 0 of the spectral zeta function of the Laplacian given by the Hermitian metric on E and some Riemannian metric on M. They conjectured the equality of the analytic and Reidemeister torsions. This conjecture was proved independently by Cheeger [1979] and Müller [1978], assuming only that ρ is unitary (both R-torsion and analytic torsion have a natural extension if ρ is unitary and not

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acyclic). The Cheeger–Müller theorem was extended to unimodular flat vector bundles by Müller [1993] and to arbitrary flat vector bundles by Bismut and Zhang [1992].

In the context of hyperbolic dynamical systems, Fried [1987] was interested in the link between the R-torsion and the Ruelle zeta function of an Anosov flow X, which is defined by

$$\zeta_{X,\rho}(s) = \prod_{\gamma \in \mathcal{G}_X^{\#}} \det(1 - \varepsilon_\gamma \rho([\gamma]) e^{-s\ell(\gamma)}), \quad \operatorname{Re}(s) \gg 0,$$

where $\mathcal{G}_X^{\#}$ is the set of primitive closed orbits of X, $\ell(\gamma)$ is the period of γ and $\varepsilon_\gamma = 1$ if the stable bundle of γ is orientable and $\varepsilon_\gamma = -1$ otherwise. Using Selberg's trace formula, Fried could relate the behavior of $\zeta_{X,\rho}(s)$ near $s = 0$ with τ_{R}, as follows.

Theorem 1 [Fried 1986]. *Let $M = SZ$ be the unit tangent bundle of some closed oriented hyperbolic manifold Z, and denote by X its geodesic vector field on M. Assume that $\rho : \pi_1(M) \to O(d)$ is an acyclic and unitary representation. Then $\zeta_{X,\rho}$ extends meromorphically to $\mathbb{C}$. Moreover, it is holomorphic near $s = 0$ and*

$$|\zeta_{X,\rho}(0)|^{(-1)^q} = \tau_{\mathrm{R}}(\rho), \tag{1-1}$$

where $2q + 1 = \dim M$ and $\tau_{\mathrm{R}}(\rho)$ is the Reidemeister torsion of (M, ρ).

Fried [1987] conjectured that the same holds true for negatively curved locally symmetric spaces. This was proved by Moscovici and Stanton [1991] and Shen [2018].

For analytic Anosov flows, the meromorphic continuation of $\zeta_{X,\rho}$ was proved by Rugh [1996] in dimension 3 and by Fried [1995] in higher dimensions. Then Sánchez-Morgado [1993; 1996] proved in dimension 3 that if ρ is acyclic, unitary, and satisfies that $\rho([\gamma]) - \varepsilon_\gamma^j$ is invertible for $j \in \{0, 1\}$ for some closed orbit γ, then (1-1) is true.

For general smooth Anosov flows, the meromorphic continuation of $\zeta_{X,\rho}$ was proved by Giuletti, Liverani and Pollicott [Giulietti et al. 2013] and alternatively by Dyatlov and Zworski [2016]. The Axiom A case was treated by Dyatlov and Guillarmou [2018]. Quoting the commentary from Zworski [2018] on Smale's seminal paper [1967], equation (1-1) "would link dynamical, spectral and topological quantities. [...] In the case of smooth manifolds of variable negative curvature, equation (1-1) remains completely open". However in [Dyatlov and Zworski 2017], the authors were able to prove the following.

Theorem 2 (Dyatlov–Zworski). *Suppose (Σ, g) is a negatively curved orientable Riemannian surface. Let X denote the associated geodesic vector field on the unitary cotangent bundle $M = S^*\Sigma$. Then, for some $c \neq 0$, we have as $s \to 0$*

$$\zeta_{X,\mathbf{1}}(s) = c s^{|\chi(\Sigma)|}(1 + \mathcal{O}(s)), \tag{1-2}$$

where $\mathbf{1}$ is the trivial representation $\pi_1(S^*\Sigma) \to \mathbb{C}^*$ and $\chi(\Sigma)$ is the Euler characteristic of Σ. In particular, the length spectrum $\{\ell(\gamma) : \gamma \in \mathcal{G}_X^{\#}\}$ determines the genus.

This result was generalized in the recent preprint [Cekić and Paternain 2020] to volume-preserving Anosov flows in dimension 3.

In the same spirit and using similar microlocal methods, Guillarmou, Rivière, Shen and the second author [Dang et al. 2020] showed:

Theorem 3 (Dang–Rivière–Guillarmou–Shen). *Let ρ be an acyclic representation of $\pi_1(M)$. Then the map*

$$X \mapsto \zeta_{X,\rho}(0)$$

is locally constant on the open set of smooth vector fields which are Anosov and for which 0 *is not a Ruelle resonance, that is,* $0 \notin \operatorname{Res}(\mathcal{L}_X^{\nabla})$. *If X preserves a smooth volume form and* $\dim(M) = 3$, (1-1) *holds true if $b_1(M) \neq 0$ or under the same assumption used in* [Sánchez-Morgado 1996].

Let us comment on the notion of Ruelle resonance to explain the assumptions in the above theorem. All recent works on the analytic continuation of the Ruelle zeta function are important by-products of new functional methods to study hyperbolic flows. They rely on the construction of spaces of anisotropic distributions adapted to the dynamics, initiated by Kitaev [1999], Blank, Keller and Liverani [Blank et al. 2002], Baladi [2005; 2018], Baladi and Tsujii [2007], Gouëzel and Liverani [2006], Liverani [2005], Butterley and Liverani [2007; 2013], and many others, where we refer to the recent book [Baladi 2018] for precise references. These spaces allow one to define a suitable notion of spectrum for the operator $\mathcal{L}_X^{\nabla} = \nabla \iota_X + \iota_X \nabla$, where ι is the interior product, acting on $\Omega^\bullet(M, E)$. This spectrum is the set of so-called Pollicott–Ruelle resonances $\operatorname{Res}(\mathcal{L}_X^{\nabla})$, which forms a discrete subset of $\mathbb{C}$ and contains all zeros and poles of $\zeta_{X,\rho}$. Faure, Roy and Sjöstrand [Faure et al. 2008] and Faure and Sjöstrand [2011] initiated the use of microlocal methods to describe these anisotropic spaces of distributions giving a purely microlocal approach to study Ruelle resonances. This was further developed by Dyatlov and Zworski to study Ruelle zeta functions.

However, if $0 \in \operatorname{Res}(\mathcal{L}_X^{\nabla})$ then the results of [Dang et al. 2020] no longer apply since the zeta function $\zeta_{X,\rho}$ might have a pole or zero at $s = 0$ (recall zeros and poles of $\zeta_{X,\rho}$ are contained in $\operatorname{Res}(\mathcal{L}_X^{\nabla})$). One goal of this article is to remove the assumption that 0 is not a Ruelle resonance. In the spirit of Theorem 2 and the Fried conjecture, we can state a theorem which follows from more general results of the present paper (see Section 2).

Theorem 4. *Let (Z, g_0) be a compact hyperbolic manifold of dimension q and ρ be the lift to S^*Z of some acyclic unitary representation $\pi_1(Z) \to \mathrm{GL}(\mathbb{C}^d)$. Then, for every metric g which is path-connected to g_0 in the space of negatively curved metrics, there exists $m(g,\rho) \in \mathbb{Z}$ such that*

$$|\zeta_{X_g,\rho}(s)|^{(-1)^q} = |s|^{(-1)^q m(g,\rho)} \underbrace{\tau_{\mathrm{R}}(\rho)}_{\mathrm{R}\text{-}torsion} \left| \frac{\tau(C^\bullet(X_{g_0},\rho))}{\tau(C^\bullet(X_g,\rho))} \right| (1 + \mathcal{O}(s)), \tag{1-3}$$

where X_g denotes the geodesic vector field of g and $\tau(C^\bullet(X_g,\rho))$ is the refined torsion of the finite-dimensional space of resonant states for the resonance 0 *of* (X_g, ρ).

In the above statement, the vector field X_g generates a contact Anosov flow on the contact manifold $S_g^*Z = \{(x,\xi) \in T^*Z : |\xi|_g = 1\}$.[1] The finite-dimensional torsion $\tau(C^\bullet(X_g,\rho))$ will be described in Section 2 below.

[1]This means concretely that changing the metric g on Z affects both the contact form ϑ and Reeb field X on S^*Z.

2. Main results

There are two restrictions in Theorem 3 of [Dang et al. 2020]. The first restriction is that

$$|\zeta_{X,\rho}(0)|^{(-1)^r} = \tau_{\mathrm{R}}(\rho)$$

is an equality of positive real numbers and the representation ρ is unitary. For arbitrary acyclic representations $\rho : \pi_1(M) \to \mathrm{GL}(\mathbb{C}^d)$, one could wonder if the phase of the complex number $\zeta_{X,\rho}(0)$ contains topological information. For instance, if it can be compared with some complex-valued torsion defined for general acyclic representations $\rho : \pi_1(M) \to \mathrm{GL}(\mathbb{C}^d)$. The second restriction concerns the assumption that 0 is not a Ruelle resonance. Apart from the low-dimension cases studied in [Dang et al. 2020], this assumption is particularly hard to control and is difficult to check for explicit examples.

Our goal in the present work is to partially overcome these two obstacles. In the case where X induces a contact flow, which means that $X = X_\vartheta$ is the Reeb vector field of some contact form ϑ on M, we deal with these difficulties by introducing a *dynamical torsion* $\tau_\vartheta(\rho)$ which is a new object defined for any acyclic ρ and which coincides with $\zeta_{X,\rho}(0)^{\pm 1}$ if $0 \notin \mathrm{Res}(\mathcal{L}_X^\nabla)$. Before stating our main results, let us introduce the two main characters of our discussion in the next two subsections.

2.1. ***Refined versions of torsion.*** The Franz–Reidemeister torsion τ_{R} is given by the modulus of some alternate product of determinants and is therefore real-valued. One cannot get a canonical object by removing the modulus since one has to make some choices to define the combinatorial torsion, and the ambiguities in these choices affect the alternate product of determinants. To remove indeterminacies arising in the definition of the combinatorial torsion, Turaev [1986; 1989; 1997] introduced in the acyclic case a refined version of the combinatorial R-torsion, the *refined combinatorial torsion*. It is a complex number $\tau_{\mathfrak{e},\mathfrak{o}}(\rho)$ which depends on additional combinatorial data, namely an Euler structure $\mathfrak{e}$ and a homology orientation $\mathfrak{o}$ of M, and which satisfies $|\tau_{\mathfrak{e},\mathfrak{o}}(\rho)| = \tau_{\mathrm{R}}(\rho)$ if ρ is acyclic and unitary. We refer the reader to Section 9.2 for precise definitions. Later, Farber and Turaev [2000] extended this object to nonacyclic representations. In this case, $\tau_{\mathfrak{e},\mathfrak{o}}(\rho)$ is an element of the determinant line of cohomology $\det H^\bullet(M, \rho)$.

Motivated by the work of Turaev, but from the analytic side, Braverman and Kappeler [2007b; 2007c; 2008] introduced a refined version of the Ray–Singer analytic torsion called *refined analytic torsion* $\tau_{\mathrm{an}}(\rho)$. It is complex-valued in the acyclic case. Their construction heavily relies on the existence of a chirality operator Γ_g, that is,

$$\Gamma_g : \Omega^\bullet(M, E) \to \Omega^{n-\bullet}(M, E), \qquad \Gamma_g^2 = \mathrm{Id},$$

which is a renormalized version of the Hodge star operator associated with some metric g. They showed that the ratio

$$\rho \mapsto \frac{\tau_{\mathrm{an}}(\rho)}{\tau_{\mathfrak{e},\mathfrak{o}}(\rho)}$$

is a holomorphic function on the representation variety given by an explicit local expression, up to a local constant of modulus 1. This result is an extension of the Cheeger–Müller theorem. Simultaneously, Burghelea and Haller [2007] introduced a complex-valued analytic torsion, which is closely related to the refined analytic torsion [Braverman and Kappeler 2007a] when it is defined; see [Huang 2007] for comparison theorems.

2.2. *Dynamical torsion.* We now assume that $X = X_\vartheta$ is the Reeb vector field of some contact form ϑ on M. Let us briefly describe the construction of the dynamical torsion. In the spirit of [Braverman and Kappeler 2007c], we use a chirality operator associated with the contact form ϑ,

$$\Gamma_\vartheta : \Omega^\bullet(M, E) \to \Omega^{n-\bullet}(M, E), \quad \Gamma_\vartheta^2 = \mathrm{Id},$$

see Section 6, analogous to the usual Hodge star operator associated with a Riemannian metric. Let $C^\bullet \subset \mathcal{D}'^\bullet(M, E)$ be the finite-dimensional space of Pollicott–Ruelle generalized resonant states of $\mathcal{L}_X^\nabla$ for the resonance 0, that is,

$$C^\bullet = \{u \in \mathcal{D}'^\bullet(M, E) : \mathrm{WF}(u) \subset E_u^*, \text{ there exists } N \in \mathbb{N} \text{ such that } (\mathcal{L}_X^\nabla)^N u = 0\},$$

where WF is the Hörmander wavefront set, $E_u^* \subset T^*M$ is the unstable cobundle of X,[2] see Section 5, and $\mathcal{D}'(M, E)$ denotes the space of E-valued currents. Then ∇ induces a differential on $C^\bullet$ which makes it a finite-dimensional cochain complex. Then a result from [Dang and Rivière 2020b] implies that the complex $(C^\bullet, \nabla)$ is acyclic if we assume that ∇ is. Because Γ_ϑ commutes with $\mathcal{L}_X^\nabla$, it induces a chirality operator on $C^\bullet$. Therefore we can compute the torsion $\tau(C^\bullet, \Gamma_\vartheta)$ of the finite-dimensional complex $(C^\bullet, \nabla)$ with respect to Γ_ϑ, as described in [Braverman and Kappeler 2007c] (see Section 3). Then we define the *dynamical torsion* τ_ϑ as the product

$$\tau_\vartheta(\rho)^{(-1)^q} = \pm \underbrace{\tau(C^\bullet, \Gamma_\vartheta)^{(-1)^q}}_{\text{finite-dimensional torsion}} \times \underbrace{\lim_{s\to 0} s^{-m(X,\rho)} \zeta_{X,\rho}(s)}_{\text{renormalized Ruelle zeta function at } s=0} \in \mathbb{C} \setminus 0,$$

where the sign $\pm$ will be given later, $m(X, \rho)$ is the order of $\zeta_{X,\rho}(s)$ at $s = 0$ and $q = (\dim(M) - 1)/2$ is the dimension of the unstable bundle of X. Note that the order $m(X, \rho) \in \mathbb{Z}$ is a priori not stable under perturbations of (X, ρ), in fact both terms in the product may not be invariant under small changes of ϑ, whereas the dynamical torsion τ_ϑ has interesting invariance properties as we will see below.

2.3. *Statement of the results.* We denote by $\mathrm{Rep}_{\mathrm{ac}}(M, d)$ the set of acyclic representations $\pi_1(M) \to \mathrm{GL}(\mathbb{C}^d)$ and by $\mathcal{A} \subset \mathcal{C}^\infty(M, TM)$ the space of contact forms on M whose Reeb vector field induces an Anosov flow. This is an open subset of the space of contact forms. For any $\vartheta \in \mathcal{A}$, we denote by X_ϑ its Reeb vector field. Recall that we want to study the value at 0 without taking the modulus. As in Fried's case, $\zeta_{X,\rho}(0)$ might be ill–defined since $0 \in \mathrm{Res}(\mathcal{L}_X^\nabla)$ and this was the reason for introducing the more general object $\tau_\vartheta(\rho)$. Our goal is to compare this new complex number with the refined torsion. As a first step towards this, our first result shows $\tau_\vartheta(\rho)$ is invariant by small perturbations of the contact form $\vartheta \in \mathcal{A}$.

Theorem 5. *Let (M, ϑ) be a contact manifold such that the Reeb vector field of ϑ induces an Anosov flow. Let $(\vartheta_\tau)_{\tau\in(-\varepsilon,\varepsilon)}$ be a smooth family in $\mathcal{A}$. Then $\partial_\tau \log \tau_{\vartheta_\tau}(\rho) = 0$ for any $\rho \in \mathrm{Rep}_{\mathrm{ac}}(M, d)$.*

[2] The annihilator of $E_u \oplus \mathbb{R}X$ where $E_u \subset TM$ denotes the unstable bundle of the flow.

Remark 2.1. In the case where the representation ρ is not acyclic, we can still define $\tau_\vartheta(\rho)$ as an element of the determinant line $\det H^\bullet(M,\rho)$; see Remark 6.5. This element is invariant under perturbations of $\vartheta \in \mathcal{A}$; see Remark 7.1.

This result implies that the map $\vartheta \in \mathcal{A} \mapsto \tau_\vartheta(\rho)$ is locally constant for all $\rho \in \mathrm{Rep}_{\mathrm{ac}}(M,d)$. To apply Theorem 3 in the case of contact Anosov flows, we need to make small perturbations near a contact Anosov flow such that $0 \notin \mathrm{Res}(\mathcal{L}_X^\nabla)$: if we have a C^1 family of contact Anosov flows $(X_t)_{t\in[0,1]}$ such that 0 is not a resonance of $\mathcal{L}_{X_0}^\nabla$ and $\mathcal{L}_{X_1}^\nabla$ but is a resonance of $\mathcal{L}_{X_u}^\nabla$ for some $u \in]0,1[$, then we cannot claim that $\zeta_{X_0,\rho}(0) = \zeta_{X_1,\rho}(0)$ using Theorem 3; however, we can claim that $\tau_{\vartheta_0}(\rho) = \tau_{\vartheta_1}(\rho)$ with Theorem 5.

Our second result aims to compare τ_ϑ with Turaev's refined version of the Reidemeister torsion $\tau_{\mathfrak{e},\mathfrak{o}}$, which depends on some choice of Euler structure $\mathfrak{e}$ and a homology orientation $\mathfrak{o}$. An analog of the Fried conjecture would be to prove the equality $\tau_\vartheta(\rho) = \tau_{\mathfrak{e},\mathfrak{o}}(\rho)$ for some $(\mathfrak{e},\mathfrak{o})$ and for all $\rho \in \mathrm{Rep}_{\mathrm{ac}}(M,d)$ (this would imply $|\tau_R(\rho)| = |\zeta_{X,\rho}(0)|^{\pm 1}$ if ρ is acyclic and unitary and if $0 \notin \mathrm{Res}(\mathcal{L}_X^\nabla)$). We prove a weaker result, which shows that the derivatives in $\rho \in \mathrm{Rep}_{\mathrm{ac}}(M,d)$ of $\log \tau_\vartheta(\rho)$ and $\log \tau_{\mathfrak{e},\mathfrak{o}}(\rho)$ coincide.

Theorem 6. *Let (M,ϑ) be a contact manifold such that the Reeb vector field of ϑ induces an Anosov flow. Then $\rho \in \mathrm{Rep}_{\mathrm{ac}}(M,d) \mapsto \tau_\vartheta(\rho)$ is holomorphic*[3] *and there exists an Euler structure $\mathfrak{e}$ such that, for any homology orientation $\mathfrak{o}$ and any smooth family $(\rho_u)_{u\in(-\varepsilon,\varepsilon)}$ of $\mathrm{Rep}_{\mathrm{ac}}(M,d)$,*

$$\partial_u \log \tau_\vartheta(\rho_u) = \partial_u \log \tau_{\mathfrak{e},\mathfrak{o}}(\rho_u).$$

Moreover, if $\dim M = 3$ and $b_1(M) \neq 0$, the map $\rho \mapsto \tau_\vartheta(\rho)/\tau_{\mathfrak{e},\mathfrak{o}}(\rho)$ is of modulus 1 on the connected components of $\mathrm{Rep}_{\mathrm{ac}}(M,d)$ containing an acyclic and unitary representation.

In [Dang et al. 2020], for ρ acyclic, the authors proved that $0 \notin \mathrm{Res}(\mathcal{L}_X^\nabla)$ implies that $X \mapsto \zeta_{X,\rho}(0)$ is locally constant. Then the equality $|\zeta_{X,\rho}(0)| = \tau_{\mathrm{R}}(\rho)$ was proved indirectly by working near analytic Anosov flows in dimension 3 or near geodesic flows of hyperbolic 3-manifolds, where the equality is known by the works of Sanchez Morgado and Fried, relying on the fact that $\zeta_{X,\rho}(0)$ remains constant by small perturbations of the vector field X. Whereas in the above theorem, for any contact Anosov flow in any odd dimension, we directly compare the log derivatives of the dynamical and refined torsions as holomorphic functions on the representation variety. We do not need to work near some vector field X for which the equality $|\zeta_{X,\rho}(0)| = \tau_{\mathrm{R}}(\rho)$ is already known.

Finally, our third result aims to describe how $\partial_u \log \tau_\vartheta(\rho_u)$ depends on the choice of the contact Anosov vector field X_ϑ.

Theorem 7. *Let (M,ϑ) be a contact manifold such that the Reeb vector field of ϑ induces an Anosov flow. Let $(\rho_u)_{|u|\leqslant\varepsilon}$ be a smooth family in $\mathrm{Rep}_{\mathrm{ac}}(M,d)$. Then, for any $\eta \in \mathcal{A}$,*

$$\partial_u \log \tau_\eta(\rho_u) = \partial_u \log \tau_\vartheta(\rho_u) + \partial_u \log \underbrace{\det \rho_u(\mathrm{cs}(X_\vartheta, X_\eta))}_{\textit{topological}}$$

as differential 1-forms on $\mathrm{Rep}_{\mathrm{ac}}(M,d)$ and where $\mathrm{cs}(X_\vartheta,X_\eta) \in H_1(M,\mathbb{Z})$ is the Chern–Simons class of the pair of vector fields (X_ϑ,X_η).

[3] $\mathrm{Rep}_{\mathrm{ac}}(M,d)$ is a variety over $\mathbb{C}$; see Section 11.2 for the right notion of holomorphicity.

Here, by $\det \rho_u(\mathrm{cs}(X_\vartheta, X_\eta))$ we mean $\det \rho_u(c)$ where $c \in \pi_1(M)$ is any element such that its homology class $[c] \in H_1(M, \mathbb{Z})$ coincides with $\mathrm{cs}(X_\vartheta, X_\eta)$ (note that the value of the determinant does not depend on the choice of c). This underbraced term is indeed topological as the Chern–Simons class $\mathrm{cs}(X_\vartheta, X_\eta) \in H_1(M, \mathbb{Z})$ measures the obstruction to find a homotopy among nonsingular vector fields connecting X_ϑ and X_η. In particular, if ϑ and η are connected by some path in $\mathcal{A}$, then $\mathrm{cs}(X_\vartheta, X_\eta) = 0$, which yields $\det \rho(\mathrm{cs}(X_\vartheta, X_\eta)) = 1$; hence $\partial_u \log \tau_\eta(\rho_u) = \partial_u \log \tau_\vartheta(\rho_u)$ for any acyclic ρ. We refer the reader to Section 9.1 for the definition of Chern–Simons classes.

Because the dynamical torsion is constructed with the help of the dynamical zeta function $\zeta_{X,\rho}$, we deduce from the above theorem some information about the behavior of $\zeta_{X,\rho}(s)$ near $s = 0$, as follows.

Corollary 8. *Let M be a closed odd-dimensional manifold. Then, for all connected open subsets $\mathcal{U} \subset \mathrm{Rep}_{\mathrm{ac}}(M, d)$ and $\mathcal{V} \subset \mathcal{A}$, there exists a constant C such that, for every Anosov contact form $\vartheta \in \mathcal{V}$ and every representation $\rho \in \mathcal{U}$,*

$$\zeta_{X_\vartheta,\rho}(s)^{(-1)^q} = C s^{(-1)^q m(\rho, X_\vartheta)} \frac{\tau_{\mathfrak{e}_{X_\vartheta},\mathfrak{o}}(\rho)}{\tau(C^\bullet(\vartheta, \rho), \Gamma_\vartheta)} (1 + \mathcal{O}(s)), \tag{2-1}$$

where X_ϑ is the Reeb vector field of ϑ, (E_ρ, ∇_ρ) is the flat vector bundle over M induced by ρ, $C^\bullet(\vartheta, \rho) \subset \mathcal{D}'^\bullet(M, E_\rho)$ is the space of generalized resonant states for the resonance 0 of $\mathcal{L}^{\nabla_\rho}_{X_\vartheta}$ and $m(X_\vartheta, \rho)$ is the vanishing order of $\zeta_{X_\vartheta,\rho}(s)$ at $s = 0$.

2.4. ***Methods of proof.*** Let us briefly sketch the proof of Theorems 5 and 6, which relies essentially on two variational arguments: we compute the variation of $\tau_\vartheta(\nabla)$ when we perturb the contact form ϑ and the connection ∇. As we do so, the space $C^\bullet(\vartheta, \nabla)$ of Pollicott–Ruelle resonant states of $\mathcal{L}^\nabla_{X_\vartheta}$ for the resonance 0 may radically change. Therefore, it is convenient to consider the space $C^\bullet_{[0,\lambda]}(\vartheta, \nabla)$ instead, which consists of the generalized resonant states for $\mathcal{L}^\nabla_{X_\vartheta}$ for resonances s such that $|s| \leqslant \lambda$, where $\lambda \in (0, 1)$ is chosen so that $\{|s| = \lambda\} \cap \mathrm{Res}(\mathcal{L}^\nabla_{X_\vartheta}) = \varnothing$. Then using [Braverman and Kappeler 2007c, Proposition 5.6] and multiplicativity of torsion, one can show that

$$\tau_\vartheta(\nabla) = \pm \tau(C^\bullet_{[0,\lambda]}(\vartheta, \nabla), \Gamma_\vartheta) \zeta^{(\lambda,\infty)}_{X_\vartheta,\rho}(0)^{(-1)^q}, \tag{2-2}$$

where $\zeta^{(\lambda,\infty)}_{X_\vartheta,\rho}$ is a renormalized version of $\zeta_{X_\vartheta,\rho}$ (we remove all the poles and zeros of $\zeta_{X_\vartheta,\rho}$ within $\{s \in \mathbb{C} : |s| \leq \lambda\}$); see Section 6. Thus we can work with the space $C^\bullet_{[0,\lambda]}(\vartheta, \nabla)$, which behaves nicely under perturbations of X thanks to Bonthonneau's construction [Bonthonneau 2020] of uniform anisotropic Sobolev spaces for families of Anosov flows, and also under perturbations of ∇.

Now consider a smooth family of contact forms $(\vartheta_t)_t$ for $|t| < \varepsilon$ such that their Reeb vector fields $(X_t)_t$ induce Anosov flows. Then Theorem 9 says that for any acyclic ∇, the map $t \mapsto \tau_{\vartheta_t}(\nabla)$ is differentiable and its derivative vanishes. This follows from a result of [Braverman and Kappeler 2007c] which allows one to compute the variation of the torsion of a finite-dimensional complex when the chirality operator is perturbed, and on a variation formula of the map $t \mapsto \zeta_{X_t,\rho}(s)$ for $\mathrm{Re}(s)$ big enough obtained in [Dang et al. 2020].

Next, consider a smooth family of flat connections $z \mapsto \nabla(z)$, where z is a complex number varying in a small neighborhood of the origin and write $\nabla(z) = \nabla + z\alpha + o(z)$, where $\alpha \in \Omega^1(M, \mathrm{End}(E))$. Then we show in Section 8, in the same spirit as before, that $z \mapsto \tau_\vartheta(\nabla(z))$ is complex differentiable and its logarithmic derivative reads

$$\partial_z|_{z=0} \log \tau_\vartheta(\nabla(z)) = -\mathrm{tr}_\mathrm{s}^\flat \alpha K e^{-\varepsilon \mathcal{L}^\nabla_{X_\vartheta}},$$

where $\varepsilon > 0$ is small enough, $\mathrm{tr}_\mathrm{s}^\flat$ is the super flat trace, see Section 4.4, and $K : \Omega^\bullet(M, E) \to \mathcal{D}'^\bullet(M, E)$ is a cochain contraction, that is, it satisfies $\nabla K + K\nabla = \mathrm{Id}_{\Omega^\bullet(M,E)}$. On the other hand, we can compute, using the formalism of [Dang and Rivière 2020a],

$$\partial_z|_{z=0} \log \tau_{\mathfrak{e}_\vartheta,\mathfrak{o}}(\nabla(z)) = -\mathrm{tr}_\mathrm{s}^\flat \alpha \widetilde{K} e^{-\varepsilon \mathcal{L}^\nabla_{-\widetilde{X}}} - \int_e \mathrm{tr}\, \alpha,$$

where $\mathfrak{e}_\vartheta$ is an Euler structure canonically associated with ϑ, $\widetilde{K}$ is another cochain contraction, $\widetilde{X}$ is a Morse–Smale gradient vector field and $e \in C_1(M, \mathbb{Z})$ is a singular one-chain representing the Euler structure $\mathfrak{e}_\vartheta$; see Section 9. Now using the fact that K and $\widetilde{K}$ are cochain contractions, one can see that

$$\alpha(K e^{-\varepsilon \mathcal{L}^\nabla_{X_\vartheta}} - \widetilde{K} e^{-\varepsilon \mathcal{L}^\nabla_{\widetilde{X}}}) = \alpha R_\varepsilon + [\nabla, \alpha G_\varepsilon],$$

where R_ε is an operator of degree -1 whose kernel is, roughly speaking, the union of graphs of the maps $e^{-\varepsilon X_u}$, where $(X_u)_u$ is a nondegenerate family of vector fields interpolating X_ϑ and $\widetilde{X}$, see Section 9.3, and G_ε is some operator of degree -2. Therefore we obtain by cyclicity of the flat trace

$$\partial_z|_{z=0} \log \frac{\tau_\vartheta(\nabla(z))}{\tau_{\mathfrak{e}_\vartheta,\mathfrak{o}}(\nabla(z))} = \mathrm{tr}_\mathrm{s}^\flat \alpha R_\varepsilon - \int_e \mathrm{tr}\, \alpha = 0, \tag{2-3}$$

where the last equality comes from differential topology arguments. Using the analytical structure of the representation variety, we may deduce from (2-3) the claim of Theorem 6. Theorem 7 then follows from the invariance of the dynamical torsion under small perturbations of the flow, the fact that $\tau_{\mathfrak{e},\mathfrak{o}}(\rho) = \tau_{\mathfrak{e}',\mathfrak{o}}(\rho)\langle \det \rho, h\rangle$ for any other Euler structure $\mathfrak{e}'$, where $h \in H_1(M, \mathbb{Z})$ satisfies $\mathfrak{e} = \mathfrak{e}' + h$ (we have that $H_1(M, \mathbb{Z})$ acts freely and transitively on the set of Euler structures; see Section 9), and the fact that, in our notation, $\mathfrak{e}_\eta - \mathfrak{e}_\vartheta = \mathrm{cs}(X_\vartheta, X_\eta)$ for any other contact form η.

2.5. ***Related works.*** Some analogs of our dynamical torsion were introduced by Burghelea and Haller [2008b] for vector fields which admit a Lyapunov closed 1–form generalizing previous works by Hutchings [2002] and Hutchings and Lee [1999a; 1999b] dealing with Morse–Novikov flows. In that case, the dynamical torsion depends on a choice of Euler structure and is a partially defined function on $\mathrm{Rep}_{\mathrm{ac}}(M, d)$; if $d = 1$, it is shown in [Burghelea and Haller 2008a] that it extends to a rational map on the Zariski closure of $\mathrm{Rep}_{\mathrm{ac}}(M, 1)$, which coincides, up to sign, with Turaev's refined combinatorial torsion (for the same choice of Euler structure). This follows from previous works of Hutchings and Lee [1999a; 1999b] who introduced some topological invariant involving circle-valued Morse functions. In both works, the considered object has the form

$$\text{dynamical zeta function}(0) \times \text{correction term},$$

where the correction term is the torsion of some finite-dimensional complex whose chains are generated by the critical points of the vector field. The chosen Euler structure gives a distinguished basis of the complex and thus a well-defined torsion. This is one of the main differences with our work since in the Anosov case, there are no such choices of distinguished currents in $C^\bullet$. However, the chirality operator allows us to overcome this problem as described above.

We also would like to mention the interesting related work [Rumin and Seshadri 2012], where the authors relate some dynamical zeta function involving the Reeb flow and some analytic contact torsion on 3-dimensional Seifert CR manifolds.

Finally, recently Spilioti [2020] and Müller [2020] were able to compare the Ruelle zeta function for odd-dimensional compact hyperbolic manifolds with some of the complex-valued torsions mentioned above.

2.6. ***Plan of the paper.*** The paper is organized as follows. In Section 3, we give some preliminaries about torsion of finite-dimensional complexes computed with respect to a chirality operator. In Section 4, we present our geometrical setting and conventions. In Section 5, we introduce Pollicott–Ruelle resonances. In Section 6, we compute the refined torsion of a space of generalized eigenvectors for nonzero resonances and we define the dynamical torsion. In Section 7, we prove that our torsion is insensitive to small perturbations of the dynamics. In Section 8, we compute the variation of our torsion with respect to the connection. In Section 9, we introduce Euler structures which are some topological tools used to fix ambiguities of the refined torsion. In Section 10, we introduce the refined combinatorial torsion of Turaev using Morse theory and we compute its variation with respect to the connection. We finally compare it to the dynamical torsion in Section 11.

3. Torsion of finite-dimensional complexes

We recall the definition of the refined torsion of a finite-dimensional acyclic complex computed with respect to a chirality operator, following [Braverman and Kappeler 2007c]. Then we compute the variation of the torsion of such a complex when the differential is perturbed.

3.1. ***The determinant line of a complex.*** For a nonzero complex vector space V, the determinant line of V is the line defined by $\det(V) = \bigwedge^{\dim V} V$. We declare the determinant line of the trivial vector space $\{0\}$ to be $\mathbb{C}$. If L is a 1-dimensional vector space, we will denote by L^{-1} its dual line. Any basis $(v_1, \dots, v_n)$ of V defines a nonzero element $v_1 \wedge \cdots \wedge v_n \in \det(V)$. Thus elements of the determinant line of $\det(V)$ should be thought of as equivalence classes of oriented basis of V.

Let

$$(C^\bullet, \partial) : 0 \xrightarrow{\partial} C^0 \xrightarrow{\partial} C^1 \xrightarrow{\partial} \cdots \xrightarrow{\partial} C^n \xrightarrow{\partial} 0$$

be a finite-dimensional complex, i.e., $\dim C^j < \infty$ for all $j = 0, \dots, n$. We define the *determinant line* of the complex $C^\bullet$ by

$$\det(C^\bullet) = \bigotimes_{j=0}^{n} \det(C^j)^{(-1)^j}.$$

Let $H^\bullet(\partial)$ be the cohomology of $(C^\bullet, \partial)$, that is,

$$H^\bullet(\partial) = \bigoplus_{j=0}^{n} H^j(\partial), \qquad H^j(\partial) = \frac{\ker(\partial : C^j \to C^{j+1})}{\operatorname{ran}(\partial : C^{j-1} \to C^j)}.$$

We will say that the complex $(C^\bullet, \partial)$ is acyclic if $H^\bullet(\partial) = 0$. In that case, $\det H^\bullet(\partial)$ is canonically isomorphic to $\mathbb{C}$.

It remains to define the fusion homomorphism that we will later need to define the torsion of a finite-dimensional based complex [Farber and Turaev 2000, §2.3]. For any finite-dimensional vector spaces $V_1, \dots, V_r$, we have a fusion isomorphism

$$\mu_{V_1,\dots,V_r} : \det(V_1) \otimes \cdots \otimes \det(V_r) \to \det(V_1 \oplus \cdots \oplus V_r)$$

defined by

$$\mu_{V_1,\dots,V_r}(v_1^1 \wedge \cdots \wedge v_1^{m_1} \otimes \cdots \otimes v_r^1 \wedge \cdots \wedge v_r^{m_r}) = v_1^1 \wedge \cdots \wedge v_1^{m_1} \wedge \cdots \wedge v_r^1 \wedge \cdots \wedge v_r^{m_r},$$

where $m_j = \dim V_j$ for $j \in \{1, \dots, r\}$.

3.2. ***Torsion of finite-dimensional acyclic complexes.*** In the present paper, we want to think of torsion of finite-dimensional acyclic complexes as a map $\varphi_{C^\bullet}$ from the determinant line of the complex to $\mathbb{C}$. We have a canonical isomorphism

$$\varphi_{C^\bullet} : \det(C^\bullet) \xrightarrow{\sim} \mathbb{C}, \tag{3-1}$$

defined as follows. Fix a decomposition

$$C^j = B^j \oplus A^j, \quad j = 0, \dots, n,$$

with $B^j = \ker(\partial) \cap C^j$ and $B^j = \partial(A^{j-1}) = \partial(C^{j-1})$ for every j. Then $\partial|_{A^j} : A^j \to B^{j+1}$ is an isomorphism for every j.

Fix nonzero elements $c_j \in \det C^j$ and $a_j \in \det A^j$ for any j. Let $\partial(a_j) \in \det B^{j+1}$ denote the image of a_j under the isomorphism $\det A^j \to \det B^{j+1}$ induced by the isomorphism $\partial|_{A^j} : A^j \to B^{j+1}$. Then, for each $j = 0, \dots, n$, there exists a unique $\lambda_j \in \mathbb{C}$ such that

$$c_j = \lambda_j \mu_{B^j, A^j}(\partial(a_{j-1}) \otimes a_j),$$

where μ_{B^j, A^j} is the fusion isomorphism defined in Section 3.1. Then define the isomorphism $\varphi_{C^\bullet}$ by

$$\varphi_{C^\bullet} : c_0 \otimes c_1^{-1} \otimes \cdots \otimes c_n^{(-1)^n} \mapsto (-1)^{N(C^\bullet)} \prod_{j=0}^{n} \lambda_j^{(-1)^j} \in \mathbb{C},$$

where

$$N(C^\bullet) = \frac{1}{2} \sum_{j=0}^{n} \dim A^j (\dim A^j + (-1)^{j+1}).$$

One easily shows that $\varphi_{C^\bullet}$ is independent of the choices of a_j [Turaev 2001, Lemma 1.3]. The number $\tau(C^\bullet, c) = \varphi_{C^\bullet}(c)$ is called the *refined torsion* of $(C^\bullet, \partial)$ with respect to the element c.

The torsion will depend on the choices of $c_j \in \det C^j$. Here the sign convention (that is, the choice of the prefactor $(-1)^{N(C^\bullet)}$ in the definition of $\varphi_{C^\bullet}$) follows [Braverman and Kappeler 2007c, §2] and is consistent with [Nicolaescu 2003, §1]. This prefactor was introduced by Turaev and differs from [Turaev 1986]. See [Nicolaescu 2003] for the motivation for the choice of sign.

Remark 3.1. If the complex $(C^\bullet, \partial)$ is not acyclic, we can still define a torsion $\tau(C^\bullet, c)$, which is this time an element of the determinant line $\det H^\bullet(\partial)$; see [Braverman and Kappeler 2007c, §2.4].

3.3. ***Torsion with respect to a chirality operator.*** We saw above that torsion depends on the choice of an element of the determinant line. A way to fix the value of the torsion without choosing an explicit basis is to use a chirality operator as in [Braverman and Kappeler 2007c]. Take $n = 2r + 1$ an odd integer and consider a complex $(C^\bullet, \partial)$ of length n. We will call a *chirality* operator an operator $\Gamma : C^\bullet \to C^\bullet$ such that $\Gamma^2 = \mathrm{Id}_{C^\bullet}$, and

$$\Gamma(C^j) = C^{n-j}, \quad j = 0, \dots, n.$$

Γ induces isomorphisms $\det(C^j) \to \det(C^{n-j})$ that we will still denote by Γ. If $\ell \in L$ is a nonzero element of a complex line, we will denote by $\ell^{-1} \in L^{-1}$ the unique element such that $\ell^{-1}(\ell) = 1$. Fix nonzero elements $c_j \in \det(C^j)$ for $j \in \{0, \dots, r\}$ and define

$$c_\Gamma = (-1)^{m(C^\bullet)} c_0 \otimes c_1^{-1} \otimes \cdots \otimes c_r^{(-1)^r} \otimes (\Gamma c_r)^{(-1)^{r+1}} \otimes (\Gamma c_{r-1})^{(-1)^r} \otimes \cdots \otimes (\Gamma c_0)^{-1},$$

where

$$m(C^\bullet) = \frac{1}{2} \sum_{j=0}^{r} \dim C^j (\dim C^j + (-1)^{r+j}).$$

Definition 3.2. The element c_Γ is independent of the choices of c_j for $j \in \{0, \dots, r\}$; the *refined torsion of* $(C^\bullet, \partial)$ *with respect to* Γ is the element

$$\tau(C^\bullet, \Gamma) = \tau(C^\bullet, c_\Gamma).$$

We also have the following result, which is [Braverman and Kappeler 2007c, Lemma 4.7] in the acyclic case about the multiplicativity of torsion.

Proposition 3.3. *Let* $(C^\bullet, \partial)$ *and* $(\widetilde{C}^\bullet, \tilde{\partial})$ *be two acyclic complexes of same length endowed with two chirality operators* Γ *and* $\widetilde{\Gamma}$. *Then*

$$\tau(C^\bullet \oplus \widetilde{C}^\bullet, \Gamma \oplus \widetilde{\Gamma}) = \tau(C^\bullet, \Gamma)\tau(\widetilde{C}^\bullet, \widetilde{\Gamma}).$$

3.4. ***Computation of the torsion with the contact signature operator.*** Let

$$B = \Gamma\partial + \partial\Gamma : C^\bullet \to C^\bullet.$$

B is called the *signature operator*. Let $B_+ = \Gamma\partial$ and $B_- = \partial\Gamma$. Define

$$C^j_\pm = C^j \cap \ker(B_\mp), \quad j = 0, \dots, n.$$

We have that $B_\pm$ preserves $C^\bullet_\pm$. Note that $B_+(C^j_+) \subset C^{n-j-1}_+$, so that $B_+(C^j_+ \oplus C^{n-j-1}_+) \subset C^j_+ \oplus C^{n-j-1}_+$. Note that if B is invertible on $C^\bullet$, B_+ is invertible on $C^\bullet_+$. If B is invertible, we can compute the refined torsion of $(C^\bullet, \partial)$ using the following:

Proposition 3.4 [Braverman and Kappeler 2007c, Proposition 5.6]. *Assume that B is invertible. Then* $(C^\bullet, \partial)$ *is acyclic so that* $\det(H^\bullet(\partial))$ *is canonically isomorphic to* $\mathbb{C}$. *Moreover,*

$$\tau(C^\bullet, \Gamma) = (-1)^{r \dim C^r_+} \det(\Gamma\partial|_{C^r_+})^{(-1)^r} \prod_{j=0}^{r-1} \det(\Gamma\partial|_{C^j_+ \oplus C^{n-j-1}_+})^{(-1)^j},$$

where we recall that $n = 2r + 1$.

3.5. ***Super traces and determinants.*** Let $V^\bullet = \bigoplus_{j=0}^p V^j$ be a graded finite-dimensional vector space and $A : V^\bullet \to V^\bullet$ be a degree-preserving linear map. We define the *super trace* and the *super determinant* of A by

$$\operatorname{tr}_{\mathrm{s},V^\bullet} A = \sum_{j=0}^p (-1)^j \operatorname{tr}_{V^j} A, \quad \det_{\mathrm{s},V^\bullet} A = \prod_{j=0}^p (\det_{V^j} A)^{(-1)^j}.$$

We also define the *graded trace* and the *graded determinant* of A by

$$\operatorname{tr}_{\mathrm{gr},V^\bullet} A = \sum_{j=0}^p (-1)^j j \operatorname{tr}_{V^j} A, \quad \det_{\mathrm{gr},V^\bullet} A = \prod_{j=0}^p (\det_{V^j} A)^{(-1)^j j}.$$

3.6. ***Analytic families of differentials.*** The goal of the present subsection is to give a variation formula for the torsion of a finite-dimensional complex when we vary the differential. This formula plays a crucial role in the variation formula of the dynamical torsion, when the representation is perturbed. Indeed, we split the dynamical torsion as the product of the torsion $\tau(C^\bullet(\vartheta, \rho), \Gamma_\vartheta)$ of some finite-dimensional space of Ruelle resonant states and a renormalized value at $s = 0$ of the dynamical zeta function $\zeta_{X,\rho}(s)$. Then the following formula allows us to deal with the variation of $\tau(C^\bullet(\vartheta, \rho), \Gamma_\vartheta)$.

Let $(C^\bullet, \partial)$ be an acyclic finite-dimensional complex of finite odd length n. If $S : C^\bullet : C^\bullet$ is a linear operator, we will say that it is of degree s if $S(C^k) \subset C^{k+s}$ for any k. If S and T are two operators on $C^\bullet$ of degrees s and t respectively then the supercommutator of S and T by

$$[S, T] = ST - (-1)^{st} TS.$$

Cyclicity of the usual trace gives $\operatorname{tr}_{\mathrm{s},C^\bullet}[S, T] = 0$ for any S, T.

Let U be a neighborhood of the origin in the complex plane and $\partial(z)$, $z \in U$, be a family of acyclic differentials on $C^\bullet$ which is real differentiable at $z = 0$, that is,

$$\partial(\sigma) = \partial + \operatorname{Re}(\sigma)\mu + \operatorname{Im}(\sigma)\nu + o(\sigma), \quad \sigma \to 0, \tag{3-2}$$

where $\mu, \nu : C^\bullet \to C^\bullet$ are degree-1 operators. Note that $\partial(\sigma) \circ \partial(\sigma) = 0$ implies that the supercommutator

$$[\partial, a(\sigma)] = \partial a(\sigma) + a(\sigma)\partial = 0, \quad \sigma \in \mathbb{C}, \tag{3-3}$$

where $a(\sigma) = \operatorname{Re}(\sigma)\mu + \operatorname{Im}(\sigma)\nu$. We will denote by $C^\bullet(z)$ the complex $(C^\bullet, \partial(z))$. Finally, let $k : C^\bullet \to C^\bullet$ be a cochain contraction, that is a linear map of degree 1 such that

$$\partial k + k\partial = \mathrm{Id}_{C^\bullet}. \tag{3-4}$$

The existence of such map is ensured by the acyclicity of $(C^\bullet, \partial)$.

Lemma 3.5. *In the above notation, for any chirality operator Γ on $C^\bullet$, the map $z \mapsto \tau(C^\bullet(z), \Gamma)$ is real differentiable at $z = 0$ and, for any $c \in \det C^\bullet$, one has*

$$\frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} \log \tau(C^\bullet(t\sigma), \Gamma) = \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} \log \tau(C^\bullet(t\sigma), c) = -\operatorname{tr}_{\mathrm{s},C^\bullet}(a(\sigma)k).$$

Note that this implies in particular that $\mathrm{tr}_{\mathrm{s},C^\bullet}(a(\sigma)k)$ does not depend on the chosen cochain contraction k. This is expected since if k' is another cochain contraction,

$$[\partial, a(\sigma)kk'] = \partial a(\sigma)kk' + a(\sigma)kk'\partial = a(\sigma)(k-k')$$

by (3-3), and the supertrace of a supercommutator vanishes.

Proof. First note that for nonzero elements $c, c' \in \det C^\bullet$, we have

$$\tau(C^\bullet(z), c) = [c : c'] \cdot \tau(C^\bullet(z), c'), \tag{3-5}$$

where $[c : c'] \in \mathbb{C}$ satisfies $c = [c : c'] \cdot c'$.

For every $j = 0, \dots, n$, fix a decomposition

$$C^j = A^j \oplus B^j,$$

where $B^j = \ker \partial \cap C^j$ and A^j is any complementary of B^j in C^j. Fix some basis $a_j^1, \dots, a_j^{\ell_j}$ of A^j; then $\partial a_j^1, \dots, \partial a_j^{\ell_j}$ is a basis of B^{j+1} by acyclicity of $(C^\bullet, \partial)$. Now let

$$c_j = \partial a_{j-1}^1 \wedge \dots \wedge \partial a_{j-1}^{\ell_{j-1}} \wedge a_j^1 \wedge \dots \wedge a_j^{\ell_j} \in \det C^j$$

and

$$c = c_0 \otimes (c_1)^{-1} \otimes c_2 \otimes \dots \otimes (c_n)^{(-1)^n} \in \det C^\bullet.$$

Now by definition of the refined torsion, we have for $|z|$ small enough

$$\tau(C^\bullet(t\sigma), c) = \pm \prod_{j=0}^{n} \det(A_j(t\sigma))^{(-1)^{j+1}}, \tag{3-6}$$

where the sign $\pm$ is independent of z and $A_j(z)$ is the matrix sending the basis

$$\partial a_{j-1}^1, \dots, \partial a_{j-1}^{\ell_{j-1}}, a_j^1, \dots, a_j^{\ell_j}$$

to the basis

$$\partial(t\sigma) a_{j-1}^1, \dots, \partial(t\sigma) a_{j-1}^{\ell_{j-1}}, a_j^1, \dots, a_j^{\ell_j}$$

(which is indeed a basis of C^j for $|z|$ small enough). Let $k : C^\bullet \to C^\bullet$ of degree -1 defined by

$$k\partial a_j^m = a_j^m, \quad k a_j^m = 0$$

for every j and $m \in \{0, \dots, \ell_j\}$. Then $k\partial + \partial k = \mathrm{Id}_{C^\bullet}$ and

$$\det A_j(t\sigma) = \det_{\partial B^{j-1} \oplus B^j}(\partial(t\sigma)k \oplus \mathrm{Id}).$$

Now (3-2) and (3-6) imply the desired result, because $\tau(C^\bullet(t\sigma), \Gamma) = [c_\Gamma : c] \cdot \tau(C^\bullet(t\sigma), c)$ by (3-5). □

4. Geometrical setting and notations

We introduce here our geometrical conventions and notation. In particular, we adopt the formalism of [Harvey and Polking 1979], which will be convenient to compute flat traces and relate the variation of the Ruelle zeta function with topological objects.

4.1. ***Twisted cohomology.*** We consider M an oriented closed connected manifold of odd dimension $n = 2r+1$. Let $E \to M$ be a flat vector bundle over M of rank $d \geqslant 1$. For $k \in \{0, \dots, n\}$, we will denote the bundle $\Lambda^k T^*M$ by Λ^k for simplicity. We will denote by $\Omega^k(M, E) = \mathcal{C}^\infty(M, \Lambda^k \otimes E)$ the space of E-valued k-forms. We set

$$\Omega^\bullet(M, E) = \bigoplus_{k=0}^{n} \Omega^k(M, E).$$

Let ∇ be a flat connection on E. We view the connection as a degree-1 operator (as an operator of the graded vector space $\Omega^\bullet(M, E)$)

$$\nabla : \Omega^k(M, E) \to \Omega^{k+1}(M, E), \quad k = 0, \dots, n.$$

The flatness of the connection reads $\nabla^2 = 0$ and thus we obtain a cochain complex $(\Omega^\bullet(M, E), \nabla)$. We will assume that the connection ∇ is acyclic, that is, the complex $(\Omega^\bullet(M, E), \nabla)$ is acyclic, or equivalently, the cohomology groups

$$H^k(M, \nabla) = \frac{\{u \in \Omega^k(M, E) : \nabla u = 0\}}{\{\nabla v : v \in \Omega^{k-1}(M, E)\}}, \quad k = 0, \dots, n,$$

are trivial.

4.2. ***Currents and Schwartz kernels.*** Let

$$\mathcal{D}'^\bullet(M, E) = \bigoplus_{k=0}^{n} \mathcal{D}'(M, \Lambda^k \otimes E)$$

be the space of E-valued currents. Let $E^\vee$ denote the dual bundle of E. We will identify $\mathcal{D}'^k(M, E)$ and the topological dual of $\Omega^{n-k}(M, E^\vee)$ via the nondegenerate bilinear pairing

$$\langle \alpha, \beta \rangle = \int_M \alpha \wedge \beta, \quad \alpha \in \Omega^k(M, E),\ \beta \in \Omega^{n-k}(M, E^\vee),$$

where $\wedge$ is the usual wedge product between E-valued forms and $E^\vee$-valued forms.

A continuous linear operator $G : \Omega^\bullet(M, E) \to \mathcal{D}'^\bullet(M, E)$ is called homogeneous if, for some $p \in \mathbb{Z}$, we have $G(\Omega^k(M, E)) \subset \mathcal{D}'^{k+p}(M, E)$ for every $k = 0, \dots, n$; the number p is called the degree of G and is denoted by $\deg G$. In that case, the Schwartz kernel theorem gives us a twisted current $\mathcal{G} \in \mathcal{D}'^{n+p}(M \times M, \pi_1^* E^\vee \otimes \pi_2^* E)$ satisfying

$$\langle Gu, v \rangle_M = \langle \mathcal{G}, \pi_1^* u \wedge \pi_2^* v \rangle_{M \times M}, \quad u \in \Omega^k(M, E),\ v \in \Omega^{n-k-p}(M, E^\vee),$$

where π_1 and π_2 are the projections of $M \times M$ onto its first and second factors respectively.

4.3. ***Integration currents.*** Let N be an oriented submanifold of M of dimension d, possibly with boundary. The associated integration current $[N] \in \mathcal{D}'^{n-d}(M)$ is given by

$$\langle [N], \omega \rangle = \int_N i_N^* \omega, \quad \omega \in \Omega^d(M),$$

where $i_N : N \to M$ is the inclusion. Note that Stokes' formula yields

$$\mathrm{d}[N] = (-1)^{n-d+1}[\partial N]. \tag{4-1}$$

For $f \in \mathrm{Diff}(M)$, we will set $\mathrm{Gr}(f) = \{(f(x), x) : x \in M\}$ to be the graph of f. Note that $\mathrm{Gr}(f)$ is an n-dimensional submanifold of $M \times M$ which is canonically oriented since M is. Therefore, we can consider the integration current over $\mathrm{Gr}(f)$. By definition, we have for any $\alpha, \beta \in \Omega^\bullet(M)$

$$\langle [\mathrm{Gr}(f)], \pi_1^*\alpha \wedge \pi_2^*\beta \rangle = \int_M f^*\alpha \wedge \beta.$$

In particular, $[\mathrm{Gr}(f)]$ is the Schwartz kernel of $f^* : \Omega^\bullet(M) \to \Omega^\bullet(M)$.

4.4. *Flat traces.* Let $G : \Omega^\bullet(M, E) \to \mathcal{D}'^\bullet(M, E)$ be an operator of degree 0. We denote its Schwartz kernel by $\mathcal{G}$ and we define

$$\mathrm{WF}'(\mathcal{G}) = \{(x, y, \xi, \eta) : (x, y, \xi, -\eta) \in \mathrm{WF}(\mathcal{G})\} \subset T^*(M \times M),$$

where WF denotes the classical Hörmander wavefront set; see [Hörmander 1990, §8]. We will also use the notation $\mathrm{WF}(G) = \mathrm{WF}(\mathcal{G})$ and $\mathrm{WF}'(G) = \mathrm{WF}'(\mathcal{G})$. Assume that

$$\mathrm{WF}'(\mathcal{G}) \cap \Delta(T^*M) = \varnothing, \quad \Delta(T^*M) = \{(x, x, \xi, \xi) : (x, \xi) \in T^*M\}. \tag{4-2}$$

Let $\iota : M \to M \times M$, $x \mapsto (x, x)$, be the diagonal inclusion. Then by [Hörmander 1990, Theorem 8.2.4] the pull back $\iota^*\mathcal{G} \in \mathcal{D}'^n(M, E^\vee \otimes E)$ is well-defined and we define the *super flat trace* of G by

$$\mathrm{tr}_\mathrm{s}^\flat G = \langle \mathrm{tr}\, \iota^*\mathcal{G}, 1 \rangle,$$

where tr denotes the trace on $E^\vee \otimes E$. We will also use the notation

$$\mathrm{tr}_\mathrm{gr}^\flat G = \mathrm{tr}_\mathrm{s}^\flat NG,$$

where $N : \Omega^\bullet(M, E) \to \Omega^\bullet(M, E)$ is the number operator, that is, $N\omega = k\omega$ for every $\omega \in \Omega^k(M, E)$.

The notation $\mathrm{tr}_\mathrm{s}^\flat$ is motivated by the following. Let $A : \mathcal{C}^\infty(M, F) \to \mathcal{D}'(M, F)$ be an operator acting on sections of a vector bundle F. If A satisfies (4-2), we can also define a flat trace $\mathrm{tr}^\flat A$ as in [Dyatlov and Zworski 2016, §2.4]. Now if $G : \Omega^\bullet(M, E) \to \mathcal{D}'^\bullet(M, E)$ is an operator of degree 0, it gives rise to an operator $G_k : \mathcal{C}^\infty(M, F_k) \to \mathcal{D}'(M, F_k)$ for each $k = 0, \dots, n$, where $F_k = \Lambda^k \otimes E$. Then the link between the two notions of flat trace mentioned above is given by

$$\mathrm{tr}_\mathrm{s}^\flat G = \sum_{k=0}^n (-1)^k \, \mathrm{tr}^\flat G_k.$$

If $\Gamma \subset T^*M$ is a closed conical subset, we let

$$\mathcal{D}'^\bullet_\Gamma(M, E) = \{u \in \mathcal{D}'^\bullet(M, E), \mathrm{WF}(u) \subset \Gamma\} \tag{4-3}$$

be the space of E-valued current whose wavefront set is contained in Γ, endowed with its usual topology; see [Hörmander 1990, §8]. If Γ is a closed conical subset of $T^*(M \times M)$ not intersecting the conormal to the diagonal

$$N^*\Delta(T^*M) = \{(x, x, \xi, -\xi) : (x, \xi) \in T^*M\},$$

then the flat trace is continuous as a map $\mathcal{D}'^{\bullet}_{\Gamma}(M \times M, \pi_1^* E^{\vee} \otimes \pi_2^* E) \to \mathbb{R}$.

4.5. ***Cyclicity of the flat trace.*** Let $G, H : \Omega^{\bullet}(M, E) \to \mathcal{D}'^{\bullet}(M, E)$ be two homogeneous operators. We denote by $\mathcal{G}, \mathcal{H}$ their respective kernels. If $\Gamma \subset T^*(M \times M)$ is a closed conical subset, we define

$$\Gamma^{(1)} = \{(y, \eta) : \text{there exists } x \in M \text{ such that } (x, y, 0, \eta) \in \Gamma\},$$
$$\Gamma^{(2)} = \{(y, \eta) : \text{there exists } x \in M \text{ such that } (x, y, -\eta, 0) \in \Gamma\}.$$

Then under the assumption

$$\mathrm{WF}(\mathcal{G})^{(2)} \cap \mathrm{WF}(\mathcal{H})^{(1)} = \varnothing,$$

the operator $F = G \circ H$ is well-defined by [Hörmander 1990, Theorem 8.2.14] and its Schwartz kernel $\mathcal{F}$ satisfies the wavefront set estimate:

$$\mathrm{WF}(\mathcal{F}) \subset \{(x, y, \xi, \eta) : \text{there exists } (z, \zeta) \text{ such that } (x, z, \xi, \zeta) \in \mathrm{WF}'(\mathcal{G}) \text{ and } (z, y, \zeta, \eta) \in \mathrm{WF}(\mathcal{H})\}.$$

If both compositions $G \circ H$ and $H \circ G$ are defined, we will denote by

$$[G, H] = G \circ H - (-1)^{\deg G \deg H} H \circ G$$

the graded commutator of G and H. We have the following:

Proposition 4.1. *Let G, H be two homogeneous operators with $\deg G + \deg H = 0$ and such that both compositions $G \circ H$ and $H \circ G$ are defined and satisfy the bound (4-2). Then we have*

$$\mathrm{tr}_{\mathrm{s}}^{\flat}[G, H] = 0.$$

The above result follows from the cyclicity of the L^2-trace, the approximation result [Dyatlov and Zworski 2016, Lemma 2.8], the relation

$$\mathrm{tr}_{\mathrm{s}}^{\flat}[G, H] = \mathrm{tr}^{\flat}[(-1)^N G, H],$$

where N is the number operator and $\mathrm{tr}^{\flat}$ is the flat trace with the convention from [Dyatlov and Zworski 2016, §2.4] (see Section 4.4), and the fact that the map $(G, H) \mapsto G \circ H$ is continuous

$$\mathcal{D}'^{\bullet}_{\Gamma}(M \times M, \pi_1^* E^{\vee} \otimes \pi_2^* E) \times \mathcal{D}'^{\bullet}_{\widetilde{\Gamma}}(M \times M, \pi_1^* E^{\vee} \otimes \pi_2^* E) \to \mathcal{D}'^{\bullet}_{\Upsilon}(M \times M, \pi_1^* E^{\vee} \otimes \pi_2^* E)$$

for any closed conical subsets $\Gamma, \widetilde{\Gamma} \subset T^*(M \times M)$ such that $\Gamma^{(2)} \cap \widetilde{\Gamma}^{(1)} = \varnothing$, and where Υ is a closed conical subset given in [Hörmander 1990, 8.2.14].

4.6. ***Perturbation of holonomy.*** Let $\gamma : [0, 1] \to M$ be a smooth curve and $\alpha \in \Omega^1(M, \mathrm{End}(E))$. Let P_{t} (resp. $\widetilde{P}_{\mathrm{t}}$) be the parallel transport $E_{\gamma(0)} \to E_{\gamma(t)}$ of ∇ (resp. $\tilde{\nabla} = \nabla + \alpha$) along $\gamma|_{[0,t]}$. Then

$$\widetilde{P}_{\mathrm{t}} = P_{\mathrm{t}} \exp\left(- \int_0^t P_{-\tau} \alpha(\dot{\gamma}(\tau)) P_{\tau} \, \mathrm{d}\tau \right). \tag{4-4}$$

The above formula will be useful in some occasion. For simplicity, we will denote for any $A \in \mathcal{C}^\infty(M, \mathrm{End}(E))$

$$\int_\gamma A = \int_0^t P_{-\tau} A(\gamma(\tau)) P_\tau \,\mathrm{d}\tau \in \mathrm{End}(E_{\gamma(0)}),$$

so that $\widetilde{P}_1 = P_1 \exp\big(-\int_\gamma \alpha(X)\big)$.

5. Pollicott–Ruelle resonances

5.1. ***Anosov dynamics.*** Let X be a smooth vector field on M and denote by φ^t its flow. We will assume that X generates an Anosov flow, that is, there exists a splitting of the tangent space T_xM at every $x \in M$

$$T_xM = \mathbb{R}X(x) \oplus E_s(x) \oplus E_u(x),$$

where $E_u(x)$, $E_s(x)$ are subspaces of T_xM depending continuously on x and invariant by the flow φ^t, such that for some constants $C, \nu > 0$ and some smooth metric $|\cdot|$ on TM one has

$$|(\mathrm{d}\varphi^t)_x v_s| \leqslant Ce^{-\nu t}|v_s|, \quad t \geq 0,\ v_s \in E_s(x),$$
$$|(\mathrm{d}\varphi^t)_x v_u| \leqslant Ce^{-\nu|t|}|v_u|, \quad t \leq 0,\ v_u \in E_u(x).$$

We will use the dual decomposition $T^*M = E_0^* \oplus E_u^* \oplus E_s^*$, where E_0^*, E_u^* and E_s^* are defined by

$$E_0^*(E_s \oplus E_u) = 0, \quad E_s^*(E_0 \oplus E_s) = 0, \quad E_u^*(E_0 \oplus E_u) = 0. \tag{5-1}$$

5.2. ***Pollicott–Ruelle resonances.*** Let ι_X denote the interior product with X and

$$\mathcal{L}_X^\nabla = \nabla\iota_X + \iota_X\nabla : \Omega^\bullet(M, E) \to \Omega^\bullet(M, E)$$

be the Lie derivative along X acting on E-valued forms. Locally, the action of $\mathcal{L}_X^\nabla$ is given by the following. Take U a domain of a chart and write $\nabla = \mathrm{d} + A$, where $A \in \Omega^1(M, \mathrm{End}(E))$. Take $w_1, \dots, w_\ell$ (resp. $e_1, \dots, e_d$) some local basis of Λ^k (resp. E) on U. Then, for any $1 \leqslant i \leqslant \ell$ and $1 \leqslant j \leqslant d$,

$$\mathcal{L}_X^\nabla(f w_i \otimes e_j) = (Xf) w_i \otimes e_j + f(\mathcal{L}_X w_i) \otimes e_j + f w_i \otimes A(X) e_j, \quad f \in \mathcal{C}^\infty(U),$$

where $\mathcal{L}_X$ is the standard Lie derivative acting on forms. In particular, $\mathcal{L}_X^\nabla$ is a differential operator of order 1 acting on sections of the bundle $\Lambda^\bullet T^*M \otimes E$, whose principal part is diagonal and given by X. This operator generates a transfer operator

$$e^{t\mathcal{L}_X^\nabla} : \Omega^\bullet(M, E) \to \Omega^\bullet(M, E),$$

which is defined by the relation

$$\frac{\mathrm{d}}{\mathrm{d}t}(e^{t\mathcal{L}_X^\nabla} u) = e^{t\mathcal{L}_X^\nabla}(\mathcal{L}_X^\nabla u).$$

For $\mathrm{Re}(s)$ big enough, the operator $\mathcal{L}_X^\nabla + s$ acting on $\Omega^\bullet(M, E)$ is invertible with inverse

$$(\mathcal{L}_X^\nabla + s)^{-1} = \int_0^\infty e^{-t\mathcal{L}_X^\nabla} e^{-st}\,\mathrm{d}t, \tag{5-2}$$

as it follows by an integration by parts. The results of [Faure and Sjöstrand 2011] generalize to the flat bundle case as in [Dang and Rivière 2020b, §3] and the resolvent $(\mathcal{L}_X^\nabla + s)^{-1}$, viewed as a family of operators $\Omega^\bullet(M, E) \to \mathcal{D}'^\bullet(M, E)$, admits a meromorphic continuation to $s \in \mathbb{C}$ with poles of finite multiplicities; we will still denote by $(\mathcal{L}_X^\nabla + s)^{-1}$ this extension. Those poles are the *Pollicott–Ruelle resonances* of $\mathcal{L}_X^\nabla$, and we will denote this set by $\mathrm{Res}(\mathcal{L}_X^\nabla)$.

5.3. *Generalized resonant states.* Let $s_0 \in \mathrm{Res}(\mathcal{L}_X^\nabla)$. By [Dyatlov and Zworski 2016, Proposition 3.3] we have a Laurent expansion

$$(\mathcal{L}_X^\nabla + s)^{-1} = Y_{s_0}(s) + \sum_{j=1}^{J(s_0)} (-1)^{j-1} \frac{(\mathcal{L}_X^\nabla + s_0)^{j-1} \Pi_{s_0}}{(s - s_0)^j}, \tag{5-3}$$

where $Y_{s_0}(s)$ is holomorphic near $s = s_0$, and

$$\Pi_{s_0} = \frac{1}{2\pi i} \int_{C_\varepsilon(s_0)} (\mathcal{L}_X^\nabla + s)^{-1} \, \mathrm{d}s : \Omega^\bullet(M, E) \to \mathcal{D}'^\bullet(M, E) \tag{5-4}$$

is an operator of finite rank. Here $C_\varepsilon(s_0) = \{|z - s_0| = \varepsilon\}$ with $\varepsilon > 0$ small enough is a small circle around s_0 such that $\mathrm{Res}(\mathcal{L}_X^\nabla) \cap \{|z - s_0| \leqslant \varepsilon\} = \{s_0\}$. Moreover the operators $Y_{s_0}(s)$ and Π_{s_0} extend to continuous operators

$$Y_{s_0}(s), \Pi_{s_0} : \mathcal{D}'^\bullet_{E_u^*}(M, E) \to \mathcal{D}'^\bullet_{E_u^*}(M, E). \tag{5-5}$$

The space

$$C^\bullet(s_0) = \mathrm{ran}(\Pi_{s_0}) \subset \mathcal{D}'^\bullet_{E_u^*}(M, E)$$

is called the space of generalized resonant states of $\mathcal{L}_X^\nabla$ associated with the resonance s_0.

5.4. *The twisted Ruelle zeta function.* Fix a base point $x_\star \in M$ and identify $\pi_1(M)$ with $\pi_1(M, x_\star)$. Let $\mathrm{Per}(X)$ be the set of periodic orbits of X. For every $\gamma \in \mathrm{Per}(X)$ we fix some base point $x_\gamma \in \mathrm{Im}(\gamma)$ and an arbitrary path c_γ joining x_γ to $x_\star$. This path defines an isomorphism $\psi_\gamma : \pi_1(M, x_\gamma) \cong \pi_1(M)$ and we can thus define for every $\gamma \in \mathrm{Per}(X)$

$$\rho_\nabla([\gamma]) = \rho_\nabla(\psi_\gamma[\gamma]).$$

The *twisted Ruelle zeta function* associated with the pair (X, ∇) is defined by

$$\zeta_{X,\nabla}(s) = \prod_{\gamma \in \mathcal{G}_X} \det(\mathrm{Id} - \varepsilon_\gamma \rho_\nabla([\gamma]) e^{-s\ell(\gamma)}), \quad \mathrm{Re}(s) > C. \tag{5-6}$$

Here $\mathcal{G}_X$ is the set of all primitive closed orbits of X (that is, the closed orbits that generate their class in $\pi_1(M)$), $\ell(\gamma)$ is the length of the orbit γ and $C > 0$ is some big constant depending on ρ and X, which satisfies

$$\|\rho_\nabla([\gamma])\| \leqslant \exp(C\ell(\gamma)), \quad \gamma \in \mathcal{G}_X, \tag{5-7}$$

for some norm $\|\cdot\|$ on $\mathrm{End}(E_{x_\star})$. Finally $\varepsilon_\gamma = 1$ if $E_u|_\gamma$ is orientable, and -1 if not.

In what follows, we will denote by P_γ the linearized Poincaré return map of γ, that is,

$$P_\gamma = \mathrm{d}_x \varphi^{-\ell(\gamma)}|_{E_s(x) \oplus E_u(x)}$$

for some $x \in \mathrm{Im}(\gamma)$ (if we choose another point in $\mathrm{Im}(\gamma)$, the new map will be conjugated to the first one). Then one has

$$\varepsilon_\gamma = \operatorname{sgn}\det(P_\gamma|_{E_s}) = (-1)^q \frac{\det(\mathrm{Id}-P_\gamma)}{|\det(\mathrm{Id}-P_\gamma)|}, \quad \text{where } q = \dim E_s. \tag{5-8}$$

Giuletti, Pollicott and Liverani [Giulietti et al. 2013] and Dyatlov and Zworski [2016] showed that $\zeta_{X,\nabla}$ has a meromorphic continuation to $\mathbb{C}$ whose poles and zeros are contained in $\mathrm{Res}(\mathcal{L}_X^\nabla)$. In fact, a consequence of the Guillemin trace formula [1977], together with (5-8) and the identity

$$\det(\mathrm{Id}-P_\gamma) = \sum_{k=0}^{n} (-1)^{k+1} k \operatorname{tr} \Lambda^k \mathrm{d}_x \varphi^{-\ell(\gamma)},$$

is that whenever $\mathrm{Re}(s)$ is large enough, we have, for every small $\varepsilon > 0$,

$$\partial_s \log \zeta_{X,\nabla}(s) = (-1)^{q+1} \operatorname{tr}_{\mathrm{gr}}^\flat ((\mathcal{L}_X^\nabla + s)^{-1} e^{-\varepsilon(\mathcal{L}_X^\nabla + s)}), \tag{5-9}$$

where the flat trace makes sense, because the wavefront set of $(\mathcal{L}_X^\nabla + s)^{-1} e^{-\varepsilon(\mathcal{L}_X^\nabla + s)}$ does not encounter the conormal to the diagonal in $T^*(M \times M)$ (see Section 8.4). In particular, one can see that the order of $\zeta_{X,\nabla}$ near a resonance $s_0 \in \mathrm{Res}(\mathcal{L}_X^\nabla)$ is given by

$$m(s_0) = (-1)^{q+1} \sum_{k=0}^{n} (-1)^k k m_k(s_0), \tag{5-10}$$

where $m_k(s_0)$ is the rank of the spectral projector $\Pi_{s_0}|_{\Omega^k(M,E)}$.[4]

5.5. ***Topology of resonant states.*** Since ∇ commutes with $\mathcal{L}_X^\nabla$, it induces a differential on the complexes $C^\bullet(s_0)$ for any $s_0 \in \mathrm{Res}(\mathcal{L}_X^\nabla)$. It is shown in [Dang and Rivière 2020b] that the complexes $(C^\bullet(s_0), \nabla)$ are acyclic whenever $s_0 \neq 0$. Moreover, for $s_0 = 0$, the map

$$\Pi_{s_0=0} : \Omega^\bullet(M, \nabla) \longrightarrow C^\bullet(s_0 = 0) = C^\bullet$$

is a quasi-isomorphism, that is, it induces isomorphisms at the level of cohomology groups. Since we assumed ∇ to be acyclic, the complex $(C^\bullet, \nabla)$ is also acyclic.

6. The dynamical torsion of a contact Anosov flow

From now on, we will assume that the flow φ^t is contact, that is, there exists a smooth one form $\vartheta \in \Omega^1(M)$ such that $\vartheta \wedge (\mathrm{d}\vartheta)^r$ is a volume form on M, $\iota_X \vartheta = 1$ and $\iota_X \mathrm{d}\vartheta = 0$. The purpose of this section is to define the dynamical torsion of the pair (ϑ, ∇). We first introduce a chirality operator Γ_ϑ acting on $\Omega^\bullet(M, E)$, which is defined thanks to the contact structure. Then the dynamical torsion is a renormalized version of the twisted Ruelle zeta function corrected by the torsion of the finite-dimensional space of the generalized resonant states for resonance $s_0 = 0$ computed with respect to Γ_ϑ.

This construction was inspired by the work of Braverman and Kappeler [2007c] on the refined analytic torsion.

[4] In [Dyatlov and Zworski 2016], the authors study the action of $\mathcal{L}_X^\nabla$ on $(\Lambda^k T^*M \cap \ker \iota_X) \otimes E$ and they get $m(s_0) = (-1)^q \sum_{k=0}^{n-1} (-1)^k m_k^0(s_0)$, where $m_k^0(s_0)$ is the dimension of $\Pi_{s_0}(\Omega^k(M, E) \cap \ker \iota_X)$. Here we study the action of $\mathcal{L}_X^\nabla$ on the full bundle $\Lambda^k T^*M \otimes E$, which leads to (5-9) and (5-10).

6.1. ***The chirality operator associated with a contact structure.*** Let $V_X \to M$ denote the bundle $T^*M \cap \ker \iota_X$. Note that for $k \in \{0, \dots, n\}$, we have the decomposition

$$\Lambda^k T^*M = \Lambda^{k-1} V_X \wedge \vartheta \oplus \Lambda^k V_X. \tag{6-1}$$

Indeed, if $\alpha \in \Lambda^k T^*M$ we may write

$$\alpha = \underbrace{(-1)^{k+1}\iota_X\alpha \wedge \vartheta}_{\in \Lambda^{k-1}V_X \wedge \vartheta} + \underbrace{\alpha - (-1)^{k+1}\iota_X\alpha \wedge \vartheta.}_{\in \Lambda^k V_X}$$

Let us introduce the Lefschetz map

$$\mathscr{L} : \Lambda^\bullet V_X \to \Lambda^{\bullet+2} V_X, \quad u \mapsto u \wedge \mathrm{d}\vartheta.$$

Since $\mathrm{d}\vartheta$ is a symplectic form on V_X, the maps $\mathscr{L}^{r-k}$ induce bundle isomorphisms

$$\mathscr{L}^{r-k} : \Lambda^k V_X \xrightarrow{\sim} \Lambda^{2r-k} V_X, \quad k = 0, \dots, r; \tag{6-2}$$

see for example [Libermann and Marle 1987, Theorem 16.3]. Using the above Lefschetz isomorphisms, we are now ready to introduce our chirality operator.

Definition 6.1. The chirality operator associated with the contact form ϑ is the operator $\Gamma_\vartheta : \Lambda^\bullet T^*M \to \Lambda^{n-\bullet} T^*M$ defined by $\Gamma_\vartheta^2 = 1$ and

$$\Gamma_\vartheta(f \wedge \vartheta + g) = \mathscr{L}^{r-k} g \wedge \vartheta + \mathscr{L}^{r-k+1} f, \quad f \in \Lambda^{k-1} V_X,\ g \in \Lambda^k V_X,\ k \in \{0, \dots, r\}, \tag{6-3}$$

where we used the decomposition (6-1).

Note that in particular one has, for $k \in \{r+1, \dots, n\}$,

$$\Gamma_\vartheta(f \wedge \vartheta + g) = (\mathscr{L}^{k-r})^{-1} g \wedge \vartheta + (\mathscr{L}^{k-1-r})^{-1} f.$$

6.2. ***The refined torsion of a space of generalized eigenvectors.*** The operator Γ_ϑ acts also on $\Omega^\bullet(M, E)$ by acting trivially on E-coefficients. Since $\mathcal{L}_X \vartheta = 0$, Γ_ϑ and $\mathcal{L}_X^\nabla$ commute so that Γ_ϑ induces a chirality operator

$$\Gamma_\vartheta : C^\bullet(s_0) \to C^{n-\bullet}(s_0)$$

for every $s_0 \in \mathrm{Res}(\mathcal{L}_X^\nabla)$. Recall from Section 5.5 that the complexes $(C^\bullet(s_0), \nabla)$ are acyclic. The following formula motivates the upcoming definition of the dynamical torsion.

Proposition 6.2. *Let $s_0 \in \mathrm{Res}(\mathcal{L}_X^\nabla) \setminus \{0, 1\}$. We have*

$$\tau(C^\bullet(s_0), \Gamma_\vartheta)^{-1} = (-1)^{Q_{s_0}} \det{}_{\mathrm{gr}, C^\bullet(s_0)} \mathcal{L}_X^\nabla,$$

where

$$Q_{s_0} = \sum_{k=0}^{r} (-1)^k (r + 1 - k) \dim C^k(s_0),$$

and $\tau(C^\bullet(s_0), \Gamma_\vartheta) \in \mathbb{C} \setminus 0$ is the refined torsion of the acyclic complex $(C^\bullet(s_0), \nabla)$ with respect to the chirality Γ_ϑ; see Definition 3.2.

Let us first admit the above proposition; the proof will be given in Sections 6.5 and 6.6.

6.3. *Spectral cuts.* If $\mathcal{I} \subset [0, 1)$ is an interval, we set

$$\Pi_{\mathcal{I}} = \sum_{\substack{s_0 \in \mathrm{Res}(\mathcal{L}_X^\nabla) \\ |s_0| \in \mathcal{I}}} \Pi_{s_0}, \quad C^\bullet_{\mathcal{I}} = \bigoplus_{\substack{s_0 \in \mathrm{Res}(\mathcal{L}_X^\nabla) \\ |s_0| \in \mathcal{I}}} C^\bullet(s_0) \quad \text{and} \quad Q_{\mathcal{I}} = \sum_{\substack{s_0 \in \mathrm{Res}(\mathcal{L}_X^\nabla) \\ |s_0| \in \mathcal{I}}} Q_{s_0}. \tag{6-4}$$

Note that $\mathcal{L}_X^\nabla + s$ acts on $C^\bullet(s_0)$ for every $s_0 \in \mathrm{Res}(\mathcal{L}_X^\nabla)$ as $-s_0 \,\mathrm{Id} + J$, where J is nilpotent. We thus have for $s \notin \mathrm{Res}(\mathcal{L}_X^\nabla)$

$$\det\nolimits_{\mathrm{gr}, C^\bullet_{\mathcal{I}}} (\mathcal{L}_X^\nabla + s)^{(-1)^{q+1}} = \prod_{\substack{s_0 \in \mathrm{Res}(\mathcal{L}_X^\nabla) \\ |s_0| \in \mathcal{I}}} (s - s_0)^{m(s_0)}, \tag{6-5}$$

where $\det_{\mathrm{gr}}$ is the graded determinant; see Section 3.5.

Let $\lambda \in [0, 1)$ such that $\mathrm{Res}(\mathcal{L}_X^\nabla) \cap \{s \in \mathbb{C} : |s| = \lambda\} = \varnothing$. Now define the meromorphic function

$$\zeta_{X,\nabla}^{(\lambda,\infty)}(s) = \zeta_{X,\nabla}(s) \det\nolimits_{\mathrm{gr}, C^\bullet_{[0,\lambda]}} (\mathcal{L}_X^\nabla + s)^{(-1)^q}. \tag{6-6}$$

Then (5-10) and (6-5) show that $\zeta_{X,\nabla}^{(\lambda,\infty)}$ has neither pole nor zero in $\{|s| \leqslant \lambda\}$, so that the number $\zeta_{X,\nabla}^{(\lambda,\infty)}(0)$ is well-defined.

6.4. *Definition of the dynamical torsion.* Let $0 < \mu < \lambda < 1$ such that, for every $s_0 \in \mathrm{Res}(\mathcal{L}_X^\nabla)$, one has $|s_0| \neq \lambda, \mu$. Using Propositions 3.3 and 6.2 we obtain, with notation of Section 6.3,

$$\tau(C^\bullet_{[0,\lambda]}, \Gamma_\vartheta) = (-1)^{-Q_{(\mu,\lambda]}} (\det\nolimits_{\mathrm{gr}, C^\bullet_{(\mu,\lambda]}} \mathcal{L}_X^\nabla)^{-1} \tau(C^\bullet_{[0,\mu]}, \Gamma_\vartheta). \tag{6-7}$$

This allows us to give the following:

Proposition/Definition 6.3 (dynamical torsion). *The number*

$$\tau_\vartheta(\nabla) = (-1)^{Q_{[0,\lambda]}} \zeta_{X,\nabla}^{(\lambda,\infty)}(0)^{(-1)^q} \cdot \tau(C^\bullet_{[0,\lambda]}, \Gamma_\vartheta) \in \mathbb{C} \setminus 0 \tag{6-8}$$

*is independent of the spectral cut $\lambda \in (0, 1)$. We will call this number the **dynamical torsion** of the pair (ϑ, ∇).*

Proof. Let $0 < \mu < \lambda < 1$ be such that $|s_0| \neq \lambda, \mu$ for each $s_0 \in \mathrm{Res}(\mathcal{L}_X^\nabla)$. Denote by $\tau_\vartheta(\nabla, \lambda)$ the right-hand side of (6-8) and define $\tau_\vartheta(\nabla, \mu)$ identically. Then we have, by (6-7),

$$\begin{aligned}\tau_\vartheta(\nabla, \lambda) &= (-1)^{Q_{[0,\lambda]}} \zeta_{X,\nabla}^{(\lambda,\infty)}(0)^{(-1)^q} \cdot \tau(C^\bullet_{[0,\lambda]}, \Gamma_\vartheta) \\ &= (-1)^{Q_{[0,\lambda]}} \zeta_{X,\nabla}^{(\lambda,\infty)}(0)^{(-1)^q} (-1)^{-Q_{(\mu,\lambda]}} (\det\nolimits_{\mathrm{gr}, C^\bullet_{(\mu,\lambda]}} \mathcal{L}_X^\nabla)^{-1} \tau(C^\bullet_{[0,\mu]}, \Gamma_\vartheta).\end{aligned}$$

Now, we have $Q_{[0,\lambda]} - Q_{(\mu,\lambda]} = Q_{[0,\mu]}$ by (6-4); moreover

$$\zeta_{X,\nabla}^{(\lambda,\infty)}(0)^{(-1)^q} (\det\nolimits_{\mathrm{gr}, C^\bullet_{(\mu,\lambda]}} \mathcal{L}_X^\nabla)^{-1} = \zeta_{X,\nabla}^{(\mu,\infty)}(0)^{(-1)^q}$$

by (6-6). Thus $\tau_\vartheta(\nabla, \lambda) = \tau_\vartheta(\nabla, \mu)$, which concludes the proof. □

Remark 6.4. If $c_{X,\nabla} s^{m(0)}$ is the leading term of the Laurent expansion of $\zeta_{X,\nabla}(s)$ at $s = 0$, then taking λ small enough actually shows that

$$\tau_\vartheta(\nabla) = (-1)^{Q_0} c_{X,\nabla}^{(-1)^q} \cdot \tau(C^\bullet, \Gamma_\vartheta). \tag{6-9}$$

In particular, if $0 \notin \mathrm{Res}(\mathcal{L}_X^\nabla)$,

$$\tau_\vartheta(\nabla) = \zeta_{X,\nabla}(0)^{(-1)^q}. \tag{6-10}$$

Note that we could have taken (6-9) as a definition of the dynamical torsion; however, (6-8) is more convenient to study the regularity of the $\tau_\vartheta(\nabla)$ with respect to ϑ and ∇.

Remark 6.5. This definition actually makes sense even if ∇ is not acyclic. Indeed, in that case, formula (6-8) defines an element of the determinant line $\det H^\bullet(C^\bullet_{[0,\lambda]}\nabla)$; see Remark 3.1. Under the identification $H^\bullet(M,\nabla) = H^\bullet(C^\bullet_{[0,\lambda]}\nabla)$ given by the quasi-isomorphism $\Pi_{[0,\lambda]} : \Omega^\bullet(M, E) \to C^\bullet_{[0,\lambda]}$ (see Section 5.5), we thus get an element of $\det H^\bullet(M,\nabla)$.

The rest of this section is devoted to the proof of Proposition 6.2, which computes the value of the torsion $\tau(C^\bullet(s_0), \Gamma_\vartheta)$. The strategy goes at follows. First, we introduce the signature operator $B_\vartheta = \Gamma_\vartheta\nabla + \nabla\Gamma_\vartheta$, and show that it is invertible on $C^\bullet(s_0)$ for $s_0 \neq 0, 1$ (Proposition 6.6). This property will allow us to use Proposition 3.4 in order to compute $\tau(C^\bullet(s_0), \Gamma_\vartheta)$.

6.5. *Invertibility of the contact signature operator.* To prove Proposition 6.2 we shall use Section 3.4 and introduce the *contact signature operator*

$$B_\vartheta = \Gamma_\vartheta\nabla + \nabla\Gamma_\vartheta : \mathcal{D}'^\bullet(M, E) \to \mathcal{D}'^\bullet(M, E),$$

where Γ_ϑ acts trivially on E. We fix in what follows some $s_0 \in \mathrm{Res}(\mathcal{L}_X^\nabla) \setminus \{0, 1\}$ and we denote $C^\bullet(s_0)$ by $C^\bullet(s_0)$ for simplicity. We also set $C_0^\bullet(s_0) = C^\bullet(s_0) \cap \ker(\iota_X)$.

Proposition 6.6. *The operator B_ϑ is invertible $C^\bullet(s_0) \to C^\bullet(s_0)$.*

Note that, as $\nabla^2 = 0$ and $\Gamma_\vartheta^2 = \mathrm{Id}$, we have that B_ϑ is invertible on $C^\bullet(s_0)$ if and only if

$$\ker(\Gamma_\vartheta\nabla) \cap \ker(\nabla\Gamma_\vartheta) = \{0\} \tag{6-11}$$

on $C^\bullet(s_0)$. Indeed, assume that (6-11) holds and let $\beta \in \ker B_\vartheta$. Set $\mu = \Gamma_\vartheta\nabla\beta = -\nabla\Gamma_\vartheta\beta$; we have

$$\Gamma_\vartheta\nabla\mu = 0 = \nabla\Gamma_\vartheta\mu,$$

hence $\mu = 0$ by (6-11), and therefore $\beta = 0$, again by (6-11), yielding $\ker B_\vartheta = \{0\}$.

In order to prove (6-11) (and thus Proposition 6.6) and Proposition 3.4, we introduce several notations that will help us understand the action of the operator $\Gamma_\vartheta\nabla$ restricted to $\ker(\nabla\Gamma_\vartheta)$. First, because ∇ does not leave the decomposition (6-1) stable, we need to introduce an operator $\Psi : C_0^\bullet(s_0) \to C_0^{\bullet+1}(s_0)$ which mimics the action of ∇. More precisely, we define

$$\Psi\mu = \nabla\mu - (-1)^k \mathcal{L}_X^\nabla\mu \wedge \vartheta, \quad \mu \in C_0^k(s_0). \tag{6-12}$$

Because $\mathcal{L}_X\, d\vartheta = 0$, the map Ψ satisfies the simple relation

$$\Psi(\mu \wedge d\vartheta^j) = (\Psi\mu) \wedge d\vartheta^j, \quad \mu \in C_0^\bullet(s_0),\ j \in \mathbb{N}, \tag{6-13}$$

that is, Ψ commutes with $\mathscr{L}$. Also, observe that

$$\Psi^2\mu = -\mathcal{L}_X^\nabla\mu \wedge d\vartheta, \quad \mu \in C_0^\bullet(s_0). \tag{6-14}$$

Indeed, using the fact that $\mathcal{L}_X^\nabla$ and ∇ commute,

$$\begin{aligned}\Psi^2\mu &= \nabla\big(\nabla\mu - (-1)^k\mathcal{L}_X^\nabla\mu\wedge\vartheta\big) - (-1)^{k+1}\big(\mathcal{L}_X^\nabla(\nabla\mu - (-1)^k\mathcal{L}_X^\nabla\mu\wedge\vartheta)\big)\wedge\vartheta\\ &= \nabla^2\mu + (-1)^{k+1}\nabla(\mathcal{L}_X^\nabla\mu\wedge\vartheta) + (-1)^k\mathcal{L}_X^\nabla\nabla\mu\wedge\vartheta - \mathcal{L}_X^{\nabla 2}\mu\wedge\vartheta\wedge\vartheta\\ &= (-1)^{k+1}(-1)^k\mathcal{L}_X^\nabla\mu\wedge d\vartheta.\end{aligned}$$

For $k\in\{0,\dots,r\}$, we also define the operator $J_k : C^k(s_0)\to C^k(s_0)$ by the formula

$$J_k\beta = f\wedge\vartheta - (-1)^k\Psi f \tag{6-15}$$

for any $\beta = f\wedge\vartheta + g\in C^k(s_0)$, with $f\in C_0^{k-1}(s_0)$. We finally set, as in Section 3.4,

$$C_+^\bullet(s_0) = C^\bullet(s_0)\cap\ker(\nabla\Gamma_\vartheta)\quad\text{and}\quad C_-^\bullet(s_0) = C^\bullet(s_0)\cap\ker(\Gamma_\vartheta\nabla).$$

Lemma 6.7. *J_k is a projector and is valued in $C_+^k(s_0)$.*

Proof. Indeed, we have for any $f\in C_0^{k-1}(s_0)$ and $g\in C_0^k(s_0)$,

$$\begin{aligned}\nabla\Gamma_\vartheta(f\wedge\vartheta+g) &= \nabla(g\wedge d\vartheta^{r-k}\wedge\vartheta + f\wedge d\vartheta^{r-k+1})\\ &= \Psi g\wedge d\vartheta^{r-k}\wedge\vartheta + (-1)^k g\wedge d\vartheta^{r-k+1} + \Psi f\wedge d\vartheta^{r-k+1} + (-1)^{k+1}\mathcal{L}_X^\nabla f\wedge d\vartheta^{r-k+1}\wedge\vartheta,\end{aligned}$$

which implies that $\beta = f\wedge\vartheta + g$ lies in $C_+^k(s_0)$ if and only if

$$(\Psi g + (-1)^{k+1}\mathcal{L}_X^\nabla f\wedge d\vartheta)\wedge d\vartheta^{r-k} = 0\quad\text{and}\quad(\Psi f + (-1)^k g)\wedge d\vartheta^{r-k+1} = 0. \tag{6-16}$$

But now note that if $\beta = f\wedge\vartheta + g = J_k\beta' = f'\wedge\vartheta - (-1)^k\Psi f'$ for some $\beta' = f'\wedge\vartheta + g'$ then $f = f'$ and $g = -(-1)^k\Psi f$, and thus β satisfies the second part of (6-16). We also obtain

$$\Psi g = -(-1)^k\Psi^2 f = -(-1)^k\mathcal{L}_X^\nabla f\wedge d\vartheta$$

by (6-14), so the first part of (6-16) is also satisfied. Therefore $J_k : C^k(s_0)\to C_+^k(s_0)$; it is clear that J_k is a projector. □

We start by a lemma which tells us how $(\Gamma_\vartheta\nabla)^2$ acts on $C_+^k(s_0)$ with $k<r$.

Lemma 6.8. *Take $k\in\{0,\dots,r-1\}$. Then, for any $\beta\in C_+^k(s_0)$, one has*

$$(\Gamma_\vartheta\nabla)^2\beta = \mathcal{L}_X^\nabla(\mathcal{L}_X^\nabla - \mathrm{Id})\beta - (\mathcal{L}_X^\nabla - \mathrm{Id})J_k\beta.$$

Proof. Since $k<r$ we can write, thanks to (6-20),

$$\Gamma_\vartheta\nabla\beta = \nabla\beta\wedge\vartheta\wedge d\vartheta^{r-k-1} + (-1)^k\iota_X\nabla\beta\wedge d\vartheta^{r-k}.$$

Therefore

$$\begin{aligned}\nabla\Gamma_\vartheta\nabla\beta &= -(-1)^k\nabla\beta\wedge d\vartheta^{r-k} + (-1)^k\nabla\iota_X\nabla\beta\wedge d\vartheta^{r-k}\\ &= (-1)^k(\mathcal{L}_X^\nabla - \mathrm{Id})\nabla\beta\wedge d\vartheta^{r-k}\\ &= (\iota_X\nabla\iota_X\nabla\beta - \iota_X\nabla\beta)\wedge\vartheta\wedge d\vartheta^{r-k} + (-1)^k(\mathcal{L}_X^\nabla - \mathrm{Id})(\nabla\beta - (-1)^k\iota_X\nabla\beta\wedge\vartheta)\wedge d\vartheta^{r-k},\end{aligned}$$

where we used $\nabla\iota_X\nabla\beta=\mathcal{L}_X^{\nabla}\nabla\beta$ and $\iota_X\nabla\iota_X\nabla\beta=\mathcal{L}_X^{\nabla}\iota_X\nabla\beta$. Since $\beta\in C_+^k(s_0)$, one has with (6-20)

$$(\nabla\beta-(-1)^k\iota_X\nabla\beta\wedge\vartheta)\wedge \mathrm{d}\vartheta^{r-k}=(\iota_X\beta-\iota_X\nabla\iota_X\beta)\wedge \mathrm{d}\vartheta^{r-k+1}.$$

This leads to

$$\nabla\Gamma_\vartheta\nabla\beta=(\iota_X\nabla\iota_X\nabla\beta-\iota_X\nabla\beta)\wedge\vartheta\wedge \mathrm{d}\vartheta^{r-k}+(-1)^k(\mathcal{L}_X^{\nabla}-\mathrm{Id})(\iota_X\beta-\iota_X\nabla\iota_X\beta)\wedge \mathrm{d}\vartheta^{r-k+1}.$$

Since $\iota_X\nabla\iota_X\nabla\beta-\iota_X\nabla\beta=(\mathcal{L}_X^{\nabla}-\mathrm{Id})\iota_X\nabla\beta$ and $\iota_X\beta-\iota_X\nabla\iota_X\beta=(\mathrm{Id}-\mathcal{L}_X^{\nabla})\iota_X\beta$, we obtain

$$\nabla\Gamma_\vartheta\nabla\beta=(\mathcal{L}_X^{\nabla}-\mathrm{Id})\iota_X\nabla\beta\wedge\vartheta\wedge \mathrm{d}\vartheta^{r-k}+(-1)^k(\mathcal{L}_X^{\nabla}-\mathrm{Id})(\mathrm{Id}-\mathcal{L}_X^{\nabla})\iota_X\beta\wedge \mathrm{d}\vartheta^{r-k+1},$$

and thus by the definition of Γ_ϑ

$$\Gamma_\vartheta\nabla\Gamma_\vartheta\nabla\beta=-(-1)^k(\mathrm{Id}-\mathcal{L}_X^{\nabla})^2\iota_X\beta\wedge\vartheta+(\mathcal{L}_X^{\nabla}-\mathrm{Id})\iota_X\nabla\beta. \tag{6-17}$$

Now, writing $\beta=f\wedge\vartheta+g$, where $\iota_Xf=0$ and $\iota_Xg=0$, we have

$$\begin{aligned}\nabla\beta&=\nabla f\wedge\vartheta-(-1)^k f\wedge \mathrm{d}\vartheta+\nabla g,\\ \iota_X\nabla\beta&=\mathcal{L}_X^{\nabla}f\wedge\vartheta+(-1)^k\nabla f+\mathcal{L}_X^{\nabla}g,\\ \iota_X\beta\wedge\vartheta&=-(-1)^k f\wedge\vartheta.\end{aligned}\tag{6-18}$$

Injecting those relations in (6-17) we get

$$\Gamma_\vartheta\nabla\Gamma_\vartheta\nabla\beta=\mathcal{L}_X^{\nabla}(\mathcal{L}_X^{\nabla}-\mathrm{Id})(f\wedge\vartheta+g)-(\mathcal{L}_X^{\nabla}-\mathrm{Id})\big(f\wedge\vartheta-(-1)^k(\nabla f+(-1)^k\mathcal{L}_X^{\nabla}f\wedge\vartheta)\big),$$

which concludes in view of (6-12) and (6-15). □

We now deal with the case $k=r$.

Lemma 6.9. *One has, for* $\beta\in C_+^r(s_0)$,

$$\Gamma_\vartheta\nabla\beta=(-1)^r\big((\mathcal{L}_X^{\nabla}-\mathrm{Id})\beta+(\mathrm{Id}-J_r)\beta\big).$$

Proof. We have

$$\Gamma_\vartheta\nabla\beta=\mathscr{L}^{-1}\big(\nabla\beta-(-1)^r\iota_X\nabla\beta\wedge\vartheta\big)+(-1)^r\iota_X\nabla\beta.$$

Since $\beta\in C_+^r(s_0)$, we have with (6-20) that $\nabla\beta-(-1)^r\iota_X\nabla\beta\wedge\vartheta=(\iota_X\beta-\iota_X\nabla\iota_X\beta)\wedge \mathrm{d}\vartheta$. Therefore,

$$\Gamma_\vartheta\nabla\beta=(\iota_X\beta-\iota_X\nabla\iota_X\beta)\wedge\vartheta+(-1)^r\iota_X\nabla\beta.$$

We now conclude as in the previous lemma, using (6-18). □

Proof of Proposition 6.6. To prove that B_ϑ is invertible on $C^\bullet(s_0)$, recall that it suffices to show that (6-11) holds. Let $\beta\in C^\bullet(s_0)$ lying in the left-hand side of (6-11), and write

$$\beta=\sum_{k=0}^{2r+1}\beta_k,$$

where $\beta_k\in C^k(s_0)$. Then $\beta_k\in C_+^k(s_0)\cap C_-^k(s_0)$ for each k. Therefore, Lemma 6.8 yields, for $k<r$,

$$0=(\Gamma_\vartheta\nabla)^2\beta_k=\mathcal{L}_X^{\nabla}(\mathcal{L}_X^{\nabla}-\mathrm{Id})\beta_k-(\mathcal{L}_X^{\nabla}-\mathrm{Id})J_k\beta_k,$$

that is, $(\mathcal{L}_X^\nabla - \mathrm{Id})(\mathcal{L}_X^\nabla \beta_k - J_k\beta_k) = 0$, which gives

$$\mathcal{L}_X^\nabla \beta_k = J_k\beta_k$$

since $\mathcal{L}_X^\nabla - \mathrm{Id}$ is invertible on $C^\bullet(s_0)$. However, writing $\beta_k = f_{k-1}\wedge\vartheta + g_k$, with $f_{k-1}, g_k \in C_0^\bullet(s_0)$, we have by (6-15)

$$\mathcal{L}_X^\nabla f_{k-1}\wedge\vartheta + \mathcal{L}_X^\nabla g_k = f_{k-1}\wedge\vartheta - (-1)^k\Psi f_{k-1}.$$

Therefore $\mathcal{L}_X^\nabla f_{k-1} = f_{k-1}$ and $\mathcal{L}_X^\nabla g_k = -(-1)^k\Psi f_{k-1}$ and $f_{k-1} = 0$ by invertibility of $\mathcal{L}_X^\nabla - \mathrm{Id}$. Hence $g_k = 0$ by invertibility of $\mathcal{L}_X^\nabla$, and thus $\beta_k = 0$. For $k = r$, Lemma 6.9 yields

$$\mathcal{L}_X^\nabla \beta_r = J_r\beta_r,$$

which gives, as above, $\beta_r = 0$. Applying the above arguments to $\tilde\beta = \Gamma_\vartheta\beta$, which lies in the intersection (6-11), yields $\beta_{n-k} = 0$ for each $k \leqslant r$. Thus $\beta = 0$ and the equality (6-11) is proven. □

6.6. ***Proof of Proposition 6.2.*** We start from Proposition 3.4 which gives us, in view of Proposition 6.6,

$$\tau(C^\bullet(s_0), \Gamma_\vartheta) = (-1)^{r\dim C_+^r(s_0)} \det(\Gamma_\vartheta\nabla|_{C_+^r(s_0)})^{(-1)^r} \prod_{j=0}^{r-1} \det(\Gamma_\vartheta\nabla|_{C_+^j(s_0)\oplus C_+^{n-j-1}(s_0)})^{(-1)^j}. \tag{6-19}$$

We first note that for $k \in \{0, \dots, r\}$ and $\beta \in \Omega^k(M, E)$, one has

$$\begin{aligned}\nabla\Gamma_\vartheta\beta &= \mathscr{L}^{r-k}\big(\nabla\beta - (-1)^k\iota_X\nabla\beta\wedge\vartheta + \mathscr{L}(\iota_X\nabla\iota_X\beta - \iota_X\beta)\big)\wedge\vartheta \\ &\qquad + (-1)^k\mathscr{L}^{r-k+1}\big(\beta - \nabla\iota_X\beta + (-1)^k\iota_X(\beta - \nabla\iota_X\beta)\wedge\vartheta\big), \\ \Gamma_\vartheta\nabla\beta &= \mathscr{L}^{r-k-1}\big(\nabla\beta - (-1)^k\iota_X\nabla\beta\wedge\vartheta\big)\wedge\vartheta + (-1)^k\mathscr{L}^{r-k}(\iota_X\nabla\beta),\end{aligned} \tag{6-20}$$

where $\mathscr{L}^{j-r} = (\mathscr{L}^{r-j}|_{\Lambda^j V_X})^{-1}$ for $0 \leqslant j \leqslant r$. Indeed, using the decomposition (6-1),

$$\begin{aligned}\Gamma_\vartheta\beta &= (-1)^{k+1}\iota_X\beta\wedge d\vartheta^{r-k+1} + (\beta + (-1)^k\iota_X\beta\wedge\vartheta)\wedge d\vartheta^{r-k}\wedge\vartheta \\ &= (-1)^{k+1}\iota_X\beta\wedge d\vartheta^{r-k+1} + \beta\wedge d\vartheta^{r-k}\wedge\vartheta,\end{aligned}$$

which leads to

$$\begin{aligned}\nabla\Gamma_\vartheta\beta &= (-1)^{k+1}\nabla\iota_X\beta\wedge d\vartheta^{r-k+1} + \nabla\beta\wedge d\vartheta^{r-k}\wedge\vartheta + (-1)^k\beta\wedge d\vartheta^{r-k+1} \\ &= (-1)^{k+1}\big((-1)^{k+1}\iota_X\nabla\iota_X\beta\wedge\vartheta\wedge d\vartheta^{r-k+1}\big) \\ &\quad + (-1)^{k+1}\big(\nabla\iota_X\beta + (-1)^k\iota_X\nabla\iota_X\beta\wedge\vartheta\big)\wedge d\vartheta^{r-k+1} \\ &\quad + \big(\nabla\beta - (-1)^k\iota_X\nabla\beta\wedge\vartheta\big)\wedge d\vartheta^{r-k}\wedge\vartheta \\ &\quad + (-1)^k\big(\beta + (-1)^k\iota_X\beta\wedge\vartheta\big)\wedge d\vartheta^{r-k+1} \\ &\quad - \iota_X\beta\wedge d\vartheta^{r-k+1}\wedge\vartheta,\end{aligned}$$

which is exactly the first part of (6-20). The second part follows directly from the decomposition (6-1). We will set, for $0 \leqslant k \leqslant n$,

$$m_k = \dim C^k(s_0), \quad m_k^0 = \dim C_0^k(s_0), \quad m_k^\pm = \dim C_\pm^k(s_0).$$

First, take $k \in \{0, \dots, r-1\}$. Because B_ϑ is invertible on $C^\bullet(s_0)$, $\Gamma_\vartheta \nabla$ induces an isomorphism $C_+^k(s_0) \to C_+^{n-k-1}(s_0)$. Take any basis γ of $C_+^k(s_0)$. Then $\Gamma_\vartheta \nabla \gamma$ is a basis of C_+^{n-k-1} and the matrix of $\Gamma_\vartheta \nabla|_{C_+^k(s_0)\oplus C_+^{n-k+1}(s_0)}$ in the basis $\gamma \oplus \Gamma_\vartheta \nabla \gamma$ is

$$\begin{pmatrix} 0 & [(\Gamma_\vartheta \nabla)^2]_\gamma \\ \mathrm{Id} & 0 \end{pmatrix}, \tag{6-21}$$

where $[(\Gamma_\vartheta \nabla)^2]_\gamma$ is the matrix of $(\Gamma_\vartheta \nabla)^2|_{C_+^k(s_0)}$ in the basis γ. Define

$$\widetilde{J}_k = \mathrm{Id} - J_k : C_+^k(s_0) \to C_+^k(s_0).$$

Then $\widetilde{J}_k$ is a projector (since J_k is by Lemma 6.7) and J_k (and thus $\widetilde{J}_k$) commutes with $\mathcal{L}_X^\nabla$ (since Ψ commutes with $\mathcal{L}_X^\nabla$). Moreover one has

$$(\Gamma_\vartheta \nabla)^2|_{\ker \widetilde{J}_k} = (\mathcal{L}_X^\nabla - \mathrm{Id})^2, \quad (\Gamma_\vartheta \nabla)^2|_{\mathrm{ran}\, \widetilde{J}_k} = \mathcal{L}_X^\nabla(\mathcal{L}_X^\nabla - \mathrm{Id}).$$

As a consequence,

$$\det((\Gamma_\vartheta \nabla)^2|_{C_+^k(s_0)}) = [s_0(1+s_0)]^{m_k^+ - m_{k-1}^0}(1+s_0)^{2m_{k-1}^0} = s_0^{m_k^+ - m_{k-1}^0}(1+s_0)^{m_k^+ + m_{k-1}^0},$$

because on $C^\bullet(s_0)$ (and in particular on $C_+^k(s_0)$), one has $\mathcal{L}_X^\nabla = -s_0 \,\mathrm{Id} + \nu$, where ν is nilpotent, and one has $\dim\ker \widetilde{J}_k = \dim \mathrm{ran}\, J_k = m_{k-1}^0$. Indeed, by (6-15) we can view J_k as a map $C_0^{k-1}(s_0) \to C_+^k(s_0)$, which is of course injective. We finally obtain with (6-21)

$$\det(\Gamma_\vartheta \nabla|_{C_+^k(s_0)\oplus C_+^{n-k+1}(s_0)}) = (-1)^{m_k^+} s_0^{m_k^+ - m_{k-1}^0}(1+s_0)^{m_k^+ + m_{k-1}^0}. \tag{6-22}$$

We now deal with the case $k = r$. Lemma 6.9 gives

$$\Gamma_\vartheta \nabla|_{\ker \widetilde{J}_r} = (-1)^r(\mathcal{L}_X^\nabla - \mathrm{Id}), \quad \Gamma_\vartheta \nabla|_{\mathrm{ran}\, \widetilde{J}_r} = (-1)^r \mathcal{L}_X^\nabla.$$

As before, we obtain

$$\det(\Gamma_\vartheta \nabla|_{C_+^r(s_0)}) = (-1)^{rm_r^+}(-1)^{m_r^+} s_0^{m_r^+ - m_{r-1}^0}(1+s_0)^{m_{r-1}^0}. \tag{6-23}$$

Combining (6-19) with (6-22) and (6-23) we finally obtain

$$\tau(C^\bullet(s_0), \Gamma_\vartheta) = (-1)^J s_0^K (1+s_0)^L, \tag{6-24}$$

where

$$J = \sum_{k=0}^r (-1)^k m_k^+, \quad K = \sum_{k=0}^r (-1)^k (m_k^+ - m_{k-1}^0), \quad L = \sum_{k=0}^{r-1} (-1)^k (m_k^+ - m_k^0).$$

Note that for $0 \leqslant k \leqslant r-1$ one has by acyclicity and because Γ_ϑ induces isomorphisms $C_+^k(s_0) \simeq C_-^{n-k}(s_0)$ (since B_ϑ is invertible),

$$m_k^+ = m_{n-k}^- = \dim\ker(\nabla|_{C^{n-k}(s_0)}) = \dim \mathrm{ran}(\nabla|_{C^{n-k-1}(s_0)}) = m_{n-k-1} - m_{n-k-1}^-.$$

Since $m_{n-k-1} - m_{n-k-1}^- = m_{k+1} - m_{k+1}^+$, one obtains

$$m_k^+ + m_{k+1}^+ = m_{k+1}, \quad 0 \leqslant k \leqslant r-1, \tag{6-25}$$

which leads to $m_k^+ + m_{k+1}^+ = m_k^0 + m_{k+1}^0$. As a consequence, since $m_0^+ = m_0 = m_0^0$, we get

$$m_r^+ - m_r^0 = -(m_{r-1}^+ - m_{r-1}^0) = \cdots = (-1)^r (m_0^+ - m_0^0) = 0.$$

This implies

$$m_k^0 = m_k^+, \quad 0 \leqslant k \leqslant r, \tag{6-26}$$

which leads to $L = 0$. Moreover, since $m_k^0 = m_{2r-k}^0$, we get

$$K = \sum_{k=0}^{r} (-1)^k (m_k^0 - m_{k-1}^0) = \sum_{k=0}^{2r} (-1)^k m_k^0 = -\sum_{k=0}^{n} (-1)^k k m_k = (-1)^q m(s_0),$$

where we used (5-10) in the last equality. Finally, again because $m_k^0 = m_{2r-k}^0$,

$$2J = (-1)^r m_r^0 + \sum_{k=0}^{2r} (-1)^k m_k^0 = (-1)^r m_r^0 - \sum_{k=0}^{n} (-1)^k k m_k.$$

We have

$$(-1)^r m_r^0 = \sum_{k=0}^{r} (-1)^k m_k \quad \text{and} \quad \sum_{k=0}^{n} (-1)^k k m_k = \sum_{k=0}^{r} (-1)^k (2k - n) m_k,$$

where the first equality comes from (6-25) and (6-26) and the second from the fact that $m_k = m_{n-k}$. We thus obtained

$$J = \sum_{k=0}^{r} (-1)^k (r + 1 - k) m_k = Q_{s_0},$$

and finally by (6-24)

$$\tau(C^\bullet(s_0), \Gamma_\vartheta) = (-1)^{Q_{s_0}} (-s_0)^{(-1)^q m(s_0)}.$$

But now recall from (6-5) that $\det_{\mathrm{gr}, C^\bullet}(\mathcal{L}_X^\nabla)^{(-1)^{q+1}} = (-s_0)^{m(s_0)}$. This completes the proof.

7. Invariance of the dynamical torsion under small perturbations of the contact form

In this section, we are interested in the behavior of the dynamical torsion when we deform the contact form. Namely, we prove here:

Theorem 9. *Assume that $(\vartheta_\mathrm{t})_{\mathrm{t} \in (-\delta, \delta)}$ is a smooth family of contact forms such that their Reeb vector fields X_t generate a contact Anosov flow for each* t. *Let (E, ∇) be an acyclic flat vector bundle. Then the map* $\mathrm{t} \mapsto \tau_{\vartheta_\mathrm{t}}(\nabla)$ *is real differentiable and we have*

$$\frac{\mathrm{d}}{\mathrm{dt}} \tau_{\vartheta_\mathrm{t}}(\nabla) = 0.$$

Remark 7.1. In view of Remark 6.5, if ∇ is not assumed acyclic, then it is not hard to see that the proof (given below) of Theorem 9 is still valid and we have that $\partial_\mathrm{t} \tau_{\vartheta_\mathrm{t}}(\nabla) = 0$ in $\det H^\bullet(M, \nabla)$.

We will thus consider a family of contact forms and set $\vartheta = \vartheta_0$ and $X = X_0$. We also fix an acyclic flat vector bundle (E, ∇).

7.1. ***Anisotropic spaces for a family of vector fields.*** To study the dynamical torsion when the dynamics is perturbed, we construct with the help of [Bonthonneau 2020] some anisotropic Sobolev spaces on which each X_{t} has nice spectral properties. We refer to Appendix B where we briefly recall the construction of these spaces.

By Section B.4, the set
$$\{(\mathrm{t}, s) : s \notin \mathrm{Res}(\mathcal{L}^{\nabla}_{X_{\mathrm{t}}})\}$$
is open in $(-\delta, \delta) \times \mathbb{C}$. Fix $\lambda \in (0, 1)$ such that
$$\mathrm{Res}(\mathcal{L}^{\nabla}_{X}) \cap \{|s| \leqslant \lambda\} \subset \{0\}. \tag{7-1}$$
Then for t close enough to 0, we have $\mathrm{Res}(\mathcal{L}^{\nabla}_{X_{\mathrm{t}}}) \cap \{|s| = \lambda\} = \varnothing$ so that the spectral projectors
$$\Pi_{\mathrm{t}} = \frac{1}{2i\pi} \int_{|s|=\lambda} (\mathcal{L}^{\nabla}_{X_{\mathrm{t}}} + s)^{-1}\, \mathrm{d}s : \Omega^{\bullet}(M, E) \to \mathcal{D}'^{\bullet}(M, E) \tag{7-2}$$
are well-defined. The next proposition is a brief summary of the results from Appendix B. For any $C, \rho > 0$, we will let
$$\Omega(c, \rho) = \{\mathrm{Re}(s) > c\} \cup \{|s| \leqslant \rho\} \subset \mathbb{C}. \tag{7-3}$$

Proposition 7.2. *There is $c, \varepsilon_0 > 0$ such that for any $\rho > 0$ there exists anisotropic Sobolev spaces*
$$\Omega^{\bullet}(M, E) \subset \mathcal{H}^{\bullet}_1 \subset \mathcal{H}^{\bullet} \subset \mathcal{D}'^{\bullet}(M, E),$$
each inclusion being continuous with dense image, such that the following hold:

(1) *For each $\mathrm{t} \in [-\varepsilon_0, \varepsilon_0]$, the family $s \mapsto \mathcal{L}^{\nabla}_{X_{\mathrm{t}}} + s$ is a holomorphic family of (unbounded) Fredholm operators $\mathcal{H}^{\bullet}_1 \to \mathcal{H}^{\bullet}_1$ and $\mathcal{H}^{\bullet} \to \mathcal{H}^{\bullet}$ of index 0 in the region $\Omega(c, \rho)$. Moreover*
$$\mathcal{L}^{\nabla}_{X_{\mathrm{t}}} \in \mathcal{C}^1([-\varepsilon_0, \varepsilon_0]_{\mathrm{t}}, \mathcal{L}(\mathcal{H}^{\bullet}_1, \mathcal{H}^{\bullet})).$$

(2) *For every relatively compact open region $\mathcal{Z} \subset \mathrm{int}\, \Omega(c, \rho)$ such that $\mathrm{Res}(\mathcal{L}^{\nabla}_{X}) \cap \overline{\mathcal{Z}} = \varnothing$, there exists $\mathrm{t}_{\mathcal{Z}} > 0$ such that*
$$(\mathcal{L}^{\nabla}_{X_{\mathrm{t}}} + s)^{-1} \in \mathcal{C}^0\big([-t_{\mathcal{Z}}, t_{\mathcal{Z}}]_{\mathrm{t}}, \mathrm{Hol}(\mathcal{Z}_s, \mathcal{L}(\mathcal{H}^{\bullet}_1, \mathcal{H}^{\bullet}))\big).$$

(3) $\Pi_{\mathrm{t}} \in \mathcal{C}^1([-\varepsilon_0, \varepsilon_0]_{\mathrm{t}}, \mathcal{L}(\mathcal{H}^{\bullet}, \mathcal{H}^{\bullet}_1))$.

We will thus fix such Hilbert spaces for some $\rho > c + 1$. We let $C^{\bullet}_{\mathrm{t}} = \mathrm{ran}\, \Pi_{\mathrm{t}} \subset \mathcal{H}^{\bullet}$, $\Pi = \Pi_{\mathrm{t}=0}$ and $C^{\bullet} = \mathrm{ran}\, \Pi$.

7.2. ***Variation of the torsion part.*** Let $\Gamma_{\mathrm{t}} : C^{\bullet}_{\mathrm{t}} \to C^{n-\bullet}_{\mathrm{t}}$ be the chirality operator associated with X_{t}; see Section 6.1. The next lemma allows us to compute the variation of the finite-dimensional torsion part of the dynamical torsion.

Lemma 7.3. *We have that $\mathrm{t} \mapsto \tau(C^{\bullet}_{\mathrm{t}}, \Gamma_{\mathrm{t}})$ is real differentiable and*
$$\frac{\mathrm{d}}{\mathrm{dt}} \tau(C^{\bullet}_{\mathrm{t}}, \Gamma_{\mathrm{t}}) = -\mathrm{tr}_{\mathrm{s}, C^{\bullet}_{\mathrm{t}}}(\Pi_{\mathrm{t}} \vartheta_{\mathrm{t}} \iota_{\dot{X}_{\mathrm{t}}}) \tau(C^{\bullet}_{\mathrm{t}}, \Gamma_{\mathrm{t}}),$$
where $\dot{X}_{\mathrm{t}} = (\mathrm{d}/\mathrm{dt}) X_{\mathrm{t}}$.

Proof. By Proposition 7.2, the operator $\Pi_t|_{C^\bullet} : C^\bullet \to C_t^\bullet$ is invertible for t close enough to 0 and we will denote by Q_t its inverse. Then for t close enough to 0, one has

$$\tau(C_t^\bullet, \Gamma_t) = \tau(C^\bullet, \widetilde{\Gamma}_t),$$

where $\widetilde{\Gamma}_t$ is defined by $\widetilde{\Gamma}_t = \Pi Q_t \Gamma_t \Pi_t \Pi$, because ∇ and Π_t commute and the image of a $\widetilde{\Gamma}_t$-invariant basis of $C^\bullet$ by the projector Π_t is a Γ_t-invariant basis of $C_t^\bullet$.

Therefore [Braverman and Kappeler 2007c, Proposition 4.9] gives

$$\frac{\mathrm{d}}{\mathrm{dt}}\tau(C_t^\bullet, \Gamma_t) = \tfrac{1}{2}\mathrm{tr}_{s,C^\bullet}(\dot{\widetilde{\Gamma}}_t\widetilde{\Gamma}_t)\tau(C_t^\bullet, \Gamma_t),$$

where $\dot{\widetilde{\Gamma}}_t = (\mathrm{d}/\mathrm{dt})\widetilde{\Gamma}_t : C^\bullet \to C^\bullet$. Since Γ_t and Π_t commute, and by the two first points of Proposition 7.2, we can apply (A-2) to get

$$\widetilde{\Gamma}_t = \Pi\Gamma_t\Pi + \mathrm{t}\Pi\dot{\Gamma}\Pi + o_{C^\bullet\to C^\bullet}(\mathrm{t}).$$

This leads to

$$\dot{\widetilde{\Gamma}}\widetilde{\Gamma} = \Pi\dot{\Gamma}\Gamma|_{C^\bullet},$$

where we removed the subscripts t to signify that we take all the t-dependent objects at $\mathrm{t} = 0$. Therefore,

$$\tfrac{1}{2}\mathrm{tr}_{s,C^\bullet}(\dot{\widetilde{\Gamma}}\widetilde{\Gamma}) = \tfrac{1}{2}\mathrm{tr}_{s,C^\bullet}(\Pi\dot{\Gamma}\Gamma).$$

Now notice that $\Gamma_t^2 = 1$ implies $\Gamma\dot{\Gamma} + \dot{\Gamma}\Gamma = 0$. Therefore, for every $k \in \{0, \dots, r\}$,

$$\mathrm{tr}_{C^{n-k}}\,\Gamma\dot{\Gamma} = \mathrm{tr}_{C^k}\,\Gamma\Gamma\dot{\Gamma}\Gamma = \mathrm{tr}_{C^k}\,\dot{\Gamma}\Gamma = -\,\mathrm{tr}_{C^k}\,\Gamma\dot{\Gamma}.$$

Therefore we only need to compute $\mathrm{tr}_{C^k}(\Gamma\dot{\Gamma})$ for $k \in \{0, \dots, r\}$ to get the full super trace $\mathrm{tr}_{s,C^\bullet}(\dot{\Gamma}\Gamma)$. Since n is odd, we have

$$\tfrac{1}{2}\mathrm{tr}_{s,C^\bullet}(\dot{\widetilde{\Gamma}}\widetilde{\Gamma}) = \tfrac{1}{2}\,\mathrm{tr}_{C^\bullet}((-1)^{N+1}\Pi\Gamma\dot{\Gamma}) = \sum_{k=0}^{r}(-1)^{k+1}\,\mathrm{tr}_{C^k}(\Pi\Gamma\dot{\Gamma}).$$

Let $k \in \{0, \dots, r\}$ and $\alpha \in \Omega^k(M)$. Using the decomposition

$$\alpha = (-1)^{k-1}\iota_{X_t}\alpha \wedge \vartheta_t + (\alpha + (-1)^k\iota_{X_t}\alpha \wedge \vartheta_t),$$

we get by the definition of Γ_t

$$\Gamma_t\alpha = (-1)^{k-1}\iota_{X_t}\alpha \wedge (\mathrm{d}\vartheta_t)^{r-k+1} + (\alpha + (-1)^k\iota_{X_t}\alpha \wedge \vartheta_t) \wedge (\mathrm{d}\vartheta_t)^{r-k} \wedge \vartheta_t.$$

Therefore,

$$\begin{aligned}\dot{\Gamma}_t\alpha = {} & (-1)^{k-1}\iota_{\dot{X}_t}\alpha \wedge (\mathrm{d}\vartheta_t)^{r-k+1} \\ & + (r-k+1)(-1)^{k-1}\iota_{X_t}\alpha \wedge \mathrm{d}\dot{\vartheta}_t \wedge (\mathrm{d}\vartheta_t)^{r-k} \\ & + (-1)^k\big(\iota_{\dot{X}_t}\alpha \wedge \vartheta_t + \iota_{X_t}\alpha \wedge \dot{\vartheta}_t\big) \wedge (\mathrm{d}\vartheta_t)^{r-k} \wedge \vartheta_t \\ & + \big(\alpha + (-1)^k\iota_{X_t}\alpha \wedge \vartheta_t\big) \wedge (\mathrm{d}\vartheta_t)^{r-k} \wedge \dot{\vartheta}_t \\ & + (r-k)\big(\alpha + (-1)^k\iota_{X_t}\alpha \wedge \vartheta_t\big) \wedge \mathrm{d}\dot{\vartheta}_t \wedge (\mathrm{d}\vartheta_t)^{r-k-1} \wedge \vartheta_t.\end{aligned}$$

Now we use the decompositions

$$\begin{aligned}\mathrm{d}\dot{\vartheta}_{\mathrm{t}} &= -\iota_{X_{\mathrm{t}}}\mathrm{d}\dot{\vartheta}_{\mathrm{t}}\wedge\vartheta_{\mathrm{t}} + (\mathrm{d}\dot{\vartheta}_{\mathrm{t}} + \iota_{X_{\mathrm{t}}}\mathrm{d}\dot{\vartheta}_{\mathrm{t}}\wedge\vartheta_{\mathrm{t}}),\\ \dot{\vartheta}_{\mathrm{t}} &= \dot{\vartheta}_{\mathrm{t}}(X_{\mathrm{t}})\vartheta + (\dot{\vartheta}_{\mathrm{t}} - \dot{\vartheta}_{\mathrm{t}}(X_{\mathrm{t}})\vartheta),\\ \iota_{\dot{X}_{\mathrm{t}}}\alpha &= (-1)^k\iota_{X_{\mathrm{t}}}\iota_{\dot{X}_{\mathrm{t}}}\alpha\wedge\vartheta_{\mathrm{t}} + (\iota_{\dot{X}_{\mathrm{t}}}\alpha + (-1)^{k+1}\iota_{X_{\mathrm{t}}}\iota_{\dot{X}_{\mathrm{t}}}\alpha\wedge\vartheta_{\mathrm{t}})\end{aligned}$$

to get, again by definition,

$$\begin{aligned}\Gamma\dot{\Gamma}\alpha = {}&(-1)^{k-1}(\iota_{\dot{X}}\alpha + (-1)^{k+1}\iota_X\iota_{\dot{X}}\alpha\wedge\vartheta)\wedge\vartheta\\ &+(-1)^{k-1}(\mathscr{L}^{r-k})^{-1}\big((-1)^k\iota_X\iota_{\dot{X}}\alpha\wedge(\mathrm{d}\vartheta)^{r-k+1}\big)\\ &+(r-k+1)(\mathscr{L}^{r-k+1})^{-1}\big((-1)^{k-1}\iota_X\alpha\wedge(\mathrm{d}\dot{\vartheta}+\iota_X\,\mathrm{d}\dot{\vartheta}\wedge\vartheta)\wedge(\mathrm{d}\vartheta)^{r-k}\big)\wedge\vartheta\\ &-(r-k+1)((-1)^{k-1}\iota_X\alpha)\wedge\iota_X\,\mathrm{d}\dot{\vartheta}\\ &+(-1)^k\iota_X\alpha\wedge(\dot{\vartheta}-\dot{\vartheta}(X)\vartheta)\\ &+(\mathscr{L}^{r-k+1})^{-1}\big((\alpha+(-1)^k\iota_X\alpha\wedge\vartheta)\wedge(\mathrm{d}\vartheta)^{r-k}\wedge(\dot{\vartheta}-\dot{\vartheta}(X)\vartheta)\big)\wedge\vartheta\\ &+(\alpha+(-1)^k\iota_X\alpha\wedge\vartheta)\dot{\vartheta}(X)\\ &+(r-k)(\mathscr{L}^{r-k})^{-1}\big((\alpha+(-1)^k\iota_X\alpha\wedge\vartheta)\wedge(\mathrm{d}\dot{\vartheta}+\iota_X\,\mathrm{d}\dot{\vartheta}\wedge\vartheta)\wedge(\mathrm{d}\vartheta)^{r-k-1}\big),\end{aligned}\tag{7-4}$$

where again we removed the subscripts t to signify that we take everything at $\mathrm{t}=0$. Now let $A_k : C_0^k \to C_0^k$ (note that here C_0^k is $C^k\cap\ker\iota_X$, see Section 6.1, and not C_{t}^k at $\mathrm{t}=0$) defined by

$$A_k u = (r-k)(\mathscr{L}^{r-k})^{-1}\big(u\wedge(\mathrm{d}\dot{\vartheta}+\iota_X\,\mathrm{d}\dot{\vartheta})\wedge(\mathrm{d}\vartheta)^{r-k-1}\big).$$

Note that the maps defined by the second, the fourth, the fifth and the sixth terms of the right-hand side of (7-4) are antidiagonal, that is, they have the form $\left(\begin{smallmatrix}0 & \star\\ \star & 0\end{smallmatrix}\right)$ in the decomposition $C^\bullet = C_0^{\bullet-1}\wedge\vartheta\oplus C_0^\bullet$. Therefore, since $A_r=0$ (we also set $A_{-1}=0$),

$$\begin{aligned}\sum_{k=0}^r(-1)^{k+1}\operatorname{tr}_{C^k}(\Pi\Gamma\dot{\Gamma}) &= \sum_{k=0}^r(-1)^{k+1}(\operatorname{tr}_{C^k}\Pi\vartheta\iota_{\dot{X}}+\operatorname{tr}_{C_0^k}\Pi\dot{\vartheta}(X))+\sum_{k=0}^r(-1)^{k+1}(\operatorname{tr}_{C_0^{k-1}}\Pi A_{k-1}+\operatorname{tr}_{C_0^k}\Pi A_k)\\ &= \sum_{k=0}^r(-1)^{k+1}(\operatorname{tr}_{C^k}\Pi\vartheta\iota_{\dot{X}}+\operatorname{tr}_{C_0^k}\Pi\dot{\vartheta}(X)).\end{aligned}\tag{7-5}$$

Here, the first and seventh terms of (7-4) correspond to the first sum of the right-hand side of the first equality of (7-5), while the third and eighth correspond to the second one. If $\alpha = f\wedge\vartheta+g\in C_0^{k-1}\wedge\vartheta\oplus C_0^k$, then

$$\vartheta\wedge\iota_{\dot{X}}\alpha = \vartheta(\dot{X})(f\wedge\vartheta)+\vartheta\wedge\iota_{\dot{X}}g.$$

This shows that for every $k\in\{0,\dots,n\}$ one has

$$\operatorname{tr}_{C^k}\Pi\vartheta\iota_{\dot{X}} = \operatorname{tr}_{C_0^{k-1}}\Pi\vartheta(\dot{X}).\tag{7-6}$$

Injecting this relation in (7-5) we obtain, with $\vartheta(\dot{X}) = -\dot{\vartheta}(X)$ and the formula $\dot{\vartheta}(X)|_{C_0^{2r-k}}\mathscr{L}^{r-k} = \mathscr{L}^{r-k}\dot{\vartheta}(X)|_{C_0^k}$,

$$\sum_{k=0}^r(-1)^{k+1}\operatorname{tr}_{C^k}(\Pi\Gamma\dot{\Gamma}) = \sum_{k=0}^r(-1)^{k+1}(\operatorname{tr}_{C_0^{k-1}}\Pi\vartheta(\dot{X})-\operatorname{tr}_{C_0^k}\Pi\vartheta(\dot{X})) = \sum_{k=0}^{2r}(-1)^k\operatorname{tr}_{C_0^k}\Pi\vartheta(\dot{X}).$$

However by (7-6) we have

$$\sum_{k=0}^{2r}(-1)^k \operatorname{tr}_{C_0^k} \Pi\vartheta(\dot X) = \operatorname{tr}_{C^\bullet}((-1)^{N+1}\Pi\vartheta\iota_{\dot X}),$$

which completes the proof. □

7.3. *Variation of the rest.* Let us now interest ourselves in the variation of $\mathrm{t}\mapsto \zeta^{(\lambda,\infty)}_{X_\mathrm{t},\nabla}(0)$; see Section 6.3. For t close enough to 0, let $P_\mathrm{t}: TM\to TM$ be defined by

$$\begin{array}{rcccccc} P_\mathrm{t}: & \ker\vartheta & \oplus & \mathbb{R}X & \to & \ker\vartheta \oplus \mathbb{R}X_\mathrm{t}, \\ & v & + & \mu X & \mapsto & v + \mu X_\mathrm{t}. \end{array}$$

For simplicity, we will still denote $\Lambda^k({}^T P_\mathrm{t}) : \Lambda^k T^*M \to \Lambda^k T^*M$ by P_t.

Proposition 7.4 (variation of the dynamical zeta function with respect to the vector field). *For any relatively compact open set $\mathcal{Z}\subset\mathbb{C}$ such that $\overline{\mathcal{Z}}\cap\operatorname{Res}(\mathcal{L}_X^\nabla)=\varnothing$, there is $\mathrm{t}_\mathcal{Z}>0$ so that $\mathrm{t}\mapsto\zeta_{X_\mathrm{t},\nabla}(s)$ is $\mathcal{C}^1$ as a map*

$$[-\mathrm{t}_\mathcal{Z},\mathrm{t}_\mathcal{Z}]\to\operatorname{Hol}(\mathcal{Z},\mathbb{C}).$$

Moreover for each $s\notin\operatorname{Res}(\mathcal{L}_{X_\mathrm{t}})$ we have

$$\partial_\mathrm{t}\log\zeta_{X_\mathrm{t},\nabla}(s) = (-1)^q s \operatorname{tr}^\flat_\mathrm{s}(\vartheta_\mathrm{t}\iota_{\dot X_\mathrm{t}}(\mathcal{L}^\nabla_{X_\mathrm{t}}+s)^{-1}e^{-\varepsilon(\mathcal{L}^\nabla_{X_\mathrm{t}}+s)}). \tag{7-7}$$

Proof. Take a relatively compact open set $\mathcal{Z}\subset\mathbb{C}$ such that $\overline{\mathcal{Z}}\cap\operatorname{Res}(\mathcal{L}_X^\nabla)=\varnothing$. We denote by

$$\mathcal{Q}_\mathrm{t}(s)\in\mathcal{D}'^n(M\times M, E^\vee\boxtimes E)$$

the Schwartz kernel of the operator $(\mathcal{L}^\nabla_{X_\mathrm{t}}+s)^{-1}e^{-\varepsilon(\mathcal{L}^\nabla_{X_\mathrm{t}}+s)}$. Then it follows from [Dang et al. 2020, Proposition 6.3] that there is $t_\mathcal{Z}>0$ and a closed conical subset Γ not intersecting $N^*\Delta$ such that the map $(\mathrm{t},s)\mapsto\mathcal{Q}_\mathrm{t}(s)$ is bounded as a map

$$[-t_\mathcal{Z},t_\mathcal{Z}]\times\overline{\mathcal{Z}}\to\mathcal{D}'^n_\Gamma(M\times M, E^\vee\boxtimes E). \tag{7-8}$$

In fact it is actually $\mathcal{C}^2$ as a map $[-t_\mathcal{Z},t_\mathcal{Z}]\times\overline{\mathcal{Z}}\to\mathcal{D}'^n(M\times M,E^\vee\boxtimes E)$ and from this it is not hard to see that the map (7-8) is actually $\mathcal{C}^1$. Next, by (5-9) we have

$$\zeta_{X_\mathrm{t},\nabla}(s)=\exp\left(\operatorname{tr}^\flat_{\mathrm{gr}}\int_\infty^s(\mathcal{L}^\nabla_{X_\mathrm{t}}+\tau)^{-1}e^{-(\mathcal{L}^\nabla_{X_\mathrm{t}}+\tau)}\,\mathrm{d}\tau\right)^{(-1)^{q+1}}$$

for $s\in\mathcal{Z}$, where ∞ means $\operatorname{Re}\tau\to+\infty$. The first part of the proposition follows.

Next we prove (7-7) for $\mathrm{t}=0$, the proof being the same for arbitrary t. Note that we have

$$\partial_\mathrm{t}(\mathcal{L}^\nabla_{X_\mathrm{t}}+\tau)^{-1} = -(\mathcal{L}^\nabla_{X_\mathrm{t}}+\tau)^{-1}\mathcal{L}_{\dot X_\mathrm{t}}(\mathcal{L}^\nabla_{X_\mathrm{t}}+\tau)^{-1},$$

which leads to

$$\partial_\mathrm{t}\log\zeta_{X_\mathrm{t},\nabla}(s) = (-1)^q\int_\infty^s \operatorname{tr}^\flat_{\mathrm{gr}}(\mathcal{L}^\nabla_{X_\mathrm{t}}+\tau)^{-1}\mathcal{L}_{\dot X_\mathrm{t}}(\mathcal{L}^\nabla_{X_\mathrm{t}}+\tau)^{-1}e^{-\varepsilon(\mathcal{L}^\nabla_{X_\mathrm{t}}+\tau)}\,\mathrm{d}\tau \\ + (-1)^{q+1}\int_\infty^s \operatorname{tr}^\flat_{\mathrm{gr}}(\mathcal{L}^\nabla_{X_\mathrm{t}}+\tau)^{-1}\partial_\mathrm{t}\, e^{-\varepsilon(\mathcal{L}^\nabla_{X_\mathrm{t}}+\tau)}\,\mathrm{d}\tau. \tag{7-9}$$

By cyclicity of the trace, and using $(\mathcal{L}^\nabla_{X_t}+\tau)^{-2} = -\partial_\tau(\mathcal{L}^\nabla_{X_t}+\tau)^{-1}$, one gets

$$\begin{aligned}\operatorname{tr}^\flat_{\mathrm{gr}}(\mathcal{L}^\nabla_{X_t}+\tau)^{-1}\mathcal{L}_{\dot X_t}(\mathcal{L}^\nabla_{X_t}+\tau)^{-1}e^{-\varepsilon(\mathcal{L}^\nabla_{X_t}+\tau)}\\ = -\partial_\tau \operatorname{tr}^\flat_{\mathrm{gr}}\mathcal{L}^\nabla_{\dot X_t}(\mathcal{L}^\nabla_{X_t}+\tau)^{-1}e^{-\varepsilon(\mathcal{L}^\nabla_{X_t}+\tau)} + \operatorname{tr}^\flat_{\mathrm{gr}}\mathcal{L}^\nabla_{\dot X_t}(\mathcal{L}^\nabla_{X_t}+\tau)^{-1}\partial_\tau e^{-\varepsilon(\mathcal{L}^\nabla_{X_t}+\tau)}.\end{aligned}$$

Next, one has $\partial_\tau e^{-\varepsilon(\mathcal{L}^\nabla_{X_t}+\tau)} = -\varepsilon\, e^{-\varepsilon(\mathcal{L}^\nabla_{X_t}+\tau)}$ and moreover

$$\partial_t e^{-\varepsilon(\mathcal{L}^\nabla_{X_t}+\tau)} = -e^{-\varepsilon(\mathcal{L}^\nabla_{X_t}+\tau)}\int_0^\varepsilon e^{u(\mathcal{L}^\nabla_{X_t}+\tau)}\mathcal{L}^\nabla_{\dot X_t}e^{-u(\mathcal{L}^\nabla_{X_t}+\tau)}\,\mathrm{d}u$$

by Duhamel's principle, and notice that the integral

$$\int_\infty^s \operatorname{tr}^\flat_{\mathrm{gr}}(\mathcal{L}^\nabla_{X_t}+\tau)^{-1}e^{-\varepsilon(\mathcal{L}_{\dot X_t}+\tau)}\left[\varepsilon\,\mathcal{L}^\nabla_{\dot X_t} - \int_0^\varepsilon e^{u(\mathcal{L}^\nabla_{X_t}+\tau)}\mathcal{L}^\nabla_{\dot X_t}e^{-u(\mathcal{L}^\nabla_{X_t}+\tau)}\,\mathrm{d}u\right]\mathrm{d}\tau$$

vanishes by cyclicity of the trace. Thus by (7-9) one gets

$$\partial_t \log\zeta_{X_t,\nabla}(s) = (-1)^{q+1}\operatorname{tr}^\flat_{\mathrm{gr}}\mathcal{L}^\nabla_{\dot X_t}(\mathcal{L}^\nabla_{X_t}+s)^{-1}e^{-\varepsilon(\mathcal{L}_{\dot X_t}+s)}. \tag{7-10}$$

Setting $A_t = P_t^{-1}\dot P_t$, one can verify that

$$\iota_{X_t} = P_t^{-1}\iota_X P_t,$$

which yields

$$\mathcal{L}^\nabla_{\dot X_t} = -\nabla A_t\iota_{X_t} + \nabla\iota_{X_t}A_t - A_t\iota_{X_t}\nabla + \iota_{X_t}A_t\nabla. \tag{7-11}$$

Notice that if N is the number operator, we have

$$(-1)^N N\nabla = \nabla(-1)^{N+1}(N+1) \quad\text{and}\quad (-1)^N N\iota_{X_t} = \iota_{X_t}(-1)^{N-1}(N-1). \tag{7-12}$$

Combining (7-11), (7-12) and the fact that ι_{X_t} and ∇ commute with $\mathcal{L}^\nabla_{X_t}$, one can show that

$$(-1)^N N\mathcal{L}^\nabla_{\dot X_t}(\mathcal{L}^\nabla_{X_t}+s)^{-1}e^{-\varepsilon(\mathcal{L}_{\dot X_t}+s)} = (-1)^N A_t\mathcal{L}^\nabla_{X_t}(\mathcal{L}^\nabla_{X_t}+s)^{-1}e^{-\varepsilon(\mathcal{L}_{\dot X_t}+s)} + B, \tag{7-13}$$

where B is a commutator. Note that $A_{t=0} = \dot P_{t=0}$ since $P_{t=0} = \mathrm{Id}$; therefore

$$\dot P_{t=0} = \vartheta\wedge\iota_{\dot X}.$$

Moreover we have $\mathcal{L}^\nabla_{X_t}(\mathcal{L}^\nabla_{X_t}+s)^{-1} = \mathrm{Id} - s\,(\mathcal{L}^\nabla_{X_t}+s)^{-1}$ and injecting those two last identities in (7-13) one obtains, by (7-10),

$$\partial_t|_{t=0}\log\zeta_{X_t,\nabla}(s) = (-1)^q s\operatorname{tr}^\flat_s(\vartheta\wedge\iota_{\dot X}(\mathcal{L}^\nabla_X+s)^{-1}e^{-\varepsilon(\mathcal{L}_{\dot X}+s)}),$$

where we used that the flat trace of $A_t e^{-\varepsilon(\mathcal{L}^\nabla_{X_t}+s)} = 0$ vanishes. □

Now we compute the variation of the $[0,\lambda]$-part of $\zeta^{(\lambda,\infty)}(s)$.

Lemma 7.5. *We have*

$$\frac{\mathrm{d}}{\mathrm{d}t}\log\det_{\mathrm{gr},C_t^\bullet}(\mathcal{L}^\nabla_{X_t}+s) = \operatorname{tr}_{s,C_t^\bullet}(\Pi_t\vartheta_t\iota_{\dot X_t}) - s\operatorname{tr}_{s,C_t^\bullet}(\Pi_t\vartheta_t\iota_{\dot X_t}(\mathcal{L}^\nabla_{X_t}+s)^{-1}).$$

Proof. Again it suffices to prove the lemma for $\mathrm{t}=0$. We are in a position to apply Lemma A.2, which gives

$$\frac{\mathrm{d}}{\mathrm{dt}} \log \det_{\mathrm{gr},C_{\mathrm{t}}^\bullet}(\mathcal{L}_{X_{\mathrm{t}}}^{\nabla}+s)^{(-1)^{q+1}} = (-1)^{q+1}\mathrm{tr}_{\mathrm{gr},C_{\mathrm{t}}^\bullet}(\Pi_{\mathrm{t}}\mathcal{L}_{\dot{X}_{\mathrm{t}}}^{\nabla}(\mathcal{L}_{X_{\mathrm{t}}}^{\nabla}+s)^{-1}).$$

Now we may conclude as in the proof of Proposition 7.4, using that

$$(-1)^N N\Pi_{\mathrm{t}}\mathcal{L}_{\dot{X}_{\mathrm{t}}}^{\nabla}(\mathcal{L}_{X_{\mathrm{t}}}^{\nabla}+s)^{-1} = (-1)^N\Pi_{\mathrm{t}}A_{\mathrm{t}}\mathcal{L}_{X_{\mathrm{t}}}^{\nabla}(\mathcal{L}_{X_{\mathrm{t}}}^{\nabla}+s)^{-1}+C,$$

where C is a commutator. □

7.4. *Proof of Theorem 9.* Combining Proposition 7.4 and Lemma 7.5, we obtain, for $s \notin \mathrm{Res}(\mathcal{L}_{X_{\mathrm{t}}}^{\nabla})$,

$$\begin{aligned}\partial_{\mathrm{t}} \log \zeta_{X_{\mathrm{t}},\nabla}^{(\lambda,\infty)}(s) = (-1)^q\, &\mathrm{tr}_{\mathrm{s},C_{\mathrm{t}}^\bullet}(\Pi_{\mathrm{t}}\vartheta_{\mathrm{t}}\iota_{\dot{X}_{\mathrm{t}}}) \\ &+(-1)^q s\, \mathrm{tr}_{\mathrm{s}}^\flat\big(\vartheta_{\mathrm{t}}\wedge\iota_{\dot{X}_{\mathrm{t}}}(\mathcal{L}_{X_{\mathrm{t}}}^{\nabla}+s)^{-1}e^{-\varepsilon(\mathcal{L}_{\dot{X}}+s)}(1-\Pi_{\mathrm{t}})\big) \\ &\quad+(-1)^q s\, \mathrm{tr}_{\mathrm{s},C_{\mathrm{t}}^\bullet}\big(\Pi_{\mathrm{t}}\vartheta_{\mathrm{t}}\iota_{\dot{X}_{\mathrm{t}}}(\mathcal{L}_{X_{\mathrm{t}}}^{\nabla}+s)^{-1}(e^{-\varepsilon(\mathcal{L}_{\dot{X}}+s)}-\mathrm{Id})\big).\end{aligned}$$

Now it is a simple observation that the last two terms in the right-hand side of the above equality vanish at $s=0$; hence we get

$$\partial_{\mathrm{t}} \log \zeta_{X_{\mathrm{t}},\nabla}^{(\lambda,\infty)}(0) = (-1)^q\, \mathrm{tr}_{\mathrm{s},C_{\mathrm{t}}^\bullet}(\Pi_{\mathrm{t}}\vartheta_{\mathrm{t}}\iota_{\dot{X}_{\mathrm{t}}}).$$

Comparing this with Lemma 7.3, we obtain Theorem 9 by the definition of the dynamical torsion; see Section 6.4.

8. Variation of the connection

In this section we compute the variation of the dynamical torsion when the connection is perturbed. This formula will be crucial to compare the dynamical torsion and Turaev's refined combinatorial torsion.

8.1. *Real-differentiable families of flat connections.* Let $U \subset \mathbb{C}$ be some open set and consider $\nabla(z)$, $z \in U$, a family of flat connections on E. We will assume that the map $z \mapsto \nabla(z)$ is $\mathcal{C}^1$,[5] that is, there exists continuous maps $z \mapsto \mu_z, \nu_z \in \Omega^1(M,\mathrm{End}(E))$ such that for any $z_0 \in U$ one has

$$\nabla(z) = \nabla(z_0) + \mathrm{Re}(z-z_0)\mu_{z_0} + \mathrm{Im}(z-z_0)\nu_{z_0} + o(z-z_0), \tag{8-1}$$

where $o(z-z_0)$ is understood in the Fréchet topology of $\Omega^1(M,\mathrm{End}(E))$. We will denote for any $\sigma \in \mathbb{C}$

$$\alpha_{z_0}(\sigma) = \mathrm{Re}(\sigma)\mu_{z_0} + \mathrm{Im}(\sigma)\nu_{z_0} \in \Omega^1(M,\mathrm{End}(E)). \tag{8-2}$$

Note that since the connections $\nabla(z)$ are assumed to be flat, we have

$$[\nabla(z),\alpha_z(\sigma)] = \nabla(z)\alpha_z(\sigma) + \alpha_z(\sigma)\nabla(z) = 0. \tag{8-3}$$

[5]Note that, even if in the following we will consider *holomorphic* families of representations $\rho(z)$, $|z|<\delta$, it is not clear that we may find a holomorphic family of connections $\nabla(z)$, $|z|<\delta$, such that $\rho_{\nabla(z)} = \rho(z)$, but only such *real-differentiable* families; see Section 11.3. Therefore we need to consider the class of real-differentiable families of connections.

8.2. ***A cochain contraction induced by the Anosov flow.*** For $z \in U$ let

$$(\mathcal{L}_X^{\nabla(z)}+s)^{-1}=\sum_{j=1}^{J(0)}\frac{(-\mathcal{L}_X^{\nabla(z)})^{j-1}\Pi_0(z)}{s^j}+Y(z)+\mathcal{O}(s) \tag{8-4}$$

be the development (5-3) for the resonance $s_0=0$. Let $C^\bullet(0;z)=\operatorname{ran}\Pi_0(z)$. Recall from Section 5.5 that since $\nabla(z)$ is acyclic, the complex $(C^\bullet(0;z),\nabla(z))$ is acyclic. Therefore there exists a cochain contraction $k(z):C^\bullet(0;z)\to C^\bullet(0;z)$, i.e., a map of degree -1 such that

$$\nabla(z)k(z)+k(z)\nabla(z)=\mathrm{Id}_{C^\bullet(0;z)}\,. \tag{8-5}$$

We now define

$$K(z)=\iota_X Y(z)(\mathrm{Id}-\Pi_0(z))+k(z)\Pi_0(z):\Omega^\bullet(M,E)\to\mathcal{D}'^\bullet(M,E). \tag{8-6}$$

A crucial property of the operator K is that it satisfies the chain homotopy equation

$$\nabla(z)K(z)+K(z)\nabla(z)=\mathrm{Id}_{\Omega^\bullet(M,E)}, \tag{8-7}$$

as follows from the development (8-4).

8.3. ***The variation formula.*** For simplicity, we will set for every $z\in U$

$$\tau(z)=\tau_\vartheta(\nabla(z)).$$

The operators $K(z)$ defined above are involved in the variation formula of the dynamical torsion, as follows.

Proposition 8.1. *The map $z\mapsto\tau(z)$ is real differentiable; we have for every $z\in U$ and $\varepsilon>0$ small enough*

$$\mathrm{d}(\log\tau)_z\sigma=-\mathrm{tr}_{\mathrm{s}}^\flat(\alpha_z(\sigma)K(z)e^{-\varepsilon\mathcal{L}_X^{\nabla(z)}}),\quad \sigma\in\mathbb{C}. \tag{8-8}$$

The proof of the previous proposition is similar of that of the last subsection, i.e., we compute the variation of each part of the dynamical torsion. The rest of this section is devoted to the proof of Proposition 8.1.

8.4. ***Anisotropic Sobolev spaces for a family of connections.*** Fix some $z_0\in U$. Recall from Section 7.1 that we chose some anisotropic Sobolev spaces $\mathcal{H}_1^\bullet\subset\mathcal{H}^\bullet$. Notice that

$$\mathcal{L}_X^{\nabla(z)}=\mathcal{L}_X^{\nabla(z_0)}+\beta(z)(X), \tag{8-9}$$

where $\beta(z)\in\Omega^1(M,\mathrm{End}(E))$ is defined by

$$\nabla(z)=\nabla(z_0)+\beta(z).$$

Therefore (8-1) implies that $z\mapsto\mathcal{L}_X^{\nabla(z)}-\mathcal{L}_X^{\nabla(z_0)}$ is a $\mathcal{C}^1$ family of multiplication operators and thus forms a $\mathcal{C}^1$ family of bounded operators $\mathcal{H}^\bullet\to\mathcal{H}^\bullet$ and $\mathcal{H}_1^\bullet\to\mathcal{H}_1^\bullet$ by construction of the anisotropic spaces and standard rules of pseudodifferential calculus (see for example [Faure and Sjöstrand 2011]). As a consequence and thanks to Proposition 7.2, we are in position to apply [Kato 1976, Theorem 3.11]; thus if δ is small enough we have that

$$R_\rho = \{(z, s) \in \mathbb{C}^2 : |z - z_0| < \delta,\ s \in \Omega(c, \rho),\ s \notin \sigma_{\mathcal{H}^\bullet}(\mathcal{L}_X^{\nabla(z)})\} \text{ is open,} \tag{8-10}$$

where $\sigma_{\mathcal{H}^\bullet}(\mathcal{L}_X^{\nabla(z)})$ denotes the resolvent set of $\mathcal{L}_X^{\nabla(z)}$ on $\mathcal{H}^\bullet$, and $\Omega(c, \rho)$ is defined in (7-3). Moreover (8-1) and (8-9) imply that for any open set $\mathcal{Z} \subset \Omega(c, \rho)$ such that $\mathrm{Res}(\mathcal{L}_X^{\nabla(z_0)}) \cap \overline{\mathcal{Z}} = \varnothing$, there exists $\delta_{\mathcal{Z}} > 0$ such that for any $j \in \{0, 1\}$,

$$(\mathcal{L}_X^{\nabla(z)} + s)^{-1} \in \mathcal{C}^1\big(\{|z - z_0| < \delta_{\mathcal{Z}}\}, \mathrm{Hol}(\mathcal{Z}_s, \mathcal{L}(\mathcal{H}_j^\bullet, \mathcal{H}_j^\bullet))\big). \tag{8-11}$$

For all z, the map $s \mapsto (\mathcal{L}_X^{\nabla(z)} + s)^{-1}$ is meromorphic in the region $\Omega(c, \rho)$ with poles (of finite multiplicity) which coincide with the resonances of $\mathcal{L}_X^{\nabla(z)}$ in this region.

Moreover, the arguments from the proof of [Dyatlov and Zworski 2016, Proposition 3.4] can be made uniformly for the family $z \mapsto (\mathcal{L}_X^{\nabla(z)} + s)^{-1}$ to obtain that for some closed conical set $\Gamma \subset T^*(M \times M)$ not intersecting the conormal to the diagonal and any $\varepsilon > 0$ small enough, the map

$$\mathcal{Z} \times \{|z - z_0| < \delta_{\mathcal{Z}}\} \to \mathcal{D}'_\Gamma(M \times M, \pi_1^* E^\vee \otimes \pi_2^* E), \quad (s, z) \mapsto \mathcal{K}(s, z),$$

is bounded, where $\mathcal{K}(s, z)$ is the Schwartz kernel of the shifted resolvent $(\mathcal{L}_X^{\nabla(z)} + s)^{-1} e^{-\varepsilon \mathcal{L}_X^{\nabla(z)}}$.

8.5. ***A family of spectral projectors.*** Fix $\lambda \in (0, 1)$ such that

$$\{s \in \mathbb{C} : |s| \leqslant \lambda\} \cap \mathrm{Res}(\mathcal{L}_X^{\nabla(z_0)}) \subset \{0\}. \tag{8-12}$$

Thanks to (8-10), if z is close enough to z_0,

$$\{s \in \mathbb{C} : |s| = \lambda\} \cap \mathrm{Res}(\mathcal{L}_X^{\nabla(z)}) = \varnothing, \tag{8-13}$$

by compactness of the circle. For $z \in U$ we will denote by

$$\Pi(z) = \frac{1}{2i\pi} \int_{|s|=\lambda} (\mathcal{L}_X^{\nabla(z)} + s)^{-1}\, \mathrm{d}s \tag{8-14}$$

the spectral projector of $\mathcal{L}_X^{\nabla(z)}$ on generalized eigenvectors for resonances in $\{s \in \mathbb{C} : |s| \leqslant \lambda\}$, and $C^\bullet(z) = \mathrm{ran}\, \Pi(z)$. It follows from (8-11), (8-13) and (8-14) that the map

$$\Pi : z \mapsto \Pi(z) \in \mathcal{L}(\mathcal{H}_j^\bullet, \mathcal{H}_j^\bullet)$$

is $\mathcal{C}^1$ for $j = 0, 1$. We can therefore apply A.3 to get, for δ small enough,

$$\Pi \in \mathcal{C}^1(\{|z - z_0| < \delta\}_z : \mathcal{L}(\mathcal{H}^\bullet, \mathcal{H}_1^\bullet)). \tag{8-15}$$

8.6. ***Variation of the finite-dimensional part.*** Because $(C^\bullet(z_0), \nabla(z_0))$ is acyclic, there exists a cochain contraction $k(z_0) : C^\bullet(z_0) \to C^{\bullet-1}(z_0)$; see Section 3.6. The next lemma computes the variation of the finite-dimensional part of the dynamical torsion.

Lemma 8.2. *The map $z \mapsto c(z) = \tau(C^\bullet(z), \Gamma)$ is real differentiable at $z = z_0$ and*

$$\mathrm{d}(\log c)_{z_0}\sigma = -\mathrm{tr}_{\mathrm{s},C^\bullet}\Pi(z_0)\alpha_{z_0}(\sigma)k(z_0), \quad \sigma \in \mathbb{C}.$$

Note that here, $a_{z_0}(\sigma)$ is identified with the map $\omega \mapsto a_{z_0}(\sigma) \wedge \omega$.

Proof. By continuity of the family $z \mapsto \Pi(z)$, we have that $\Pi(z)|_{C^\bullet(z_0)} : C^\bullet(z_0) \to C^\bullet(z)$ is an isomorphism, for $|z - z_0|$ small enough, of inverse denoted by $Q(z)$. For those z we denote by $\widehat{C}^\bullet(z)$ the graded vector space $C^\bullet(z_0)$ endowed with the differential

$$\hat{\nabla}(z) = Q(z)\nabla(z)\Pi(z) : C^\bullet(z_0) \to C^\bullet(z_0).$$

Then because Γ commutes with every $\Pi(z)$ one has

$$\tau(\widehat{C}^\bullet(z), \Gamma) = \tau(C^\bullet(z), \Gamma). \tag{8-16}$$

By (8-15) we can apply (A-2) in the proof of Lemma A.2 which gives, as $\sigma \to 0$,

$$\hat{\nabla}(z_0 + \sigma)\Pi(z_0) = \Pi(z_0)\nabla(z_0)\Pi(z_0) + \Pi(z_0)\alpha_{z_0}(\sigma)\Pi(z_0) + o_{C^\bullet(z_0)\to C^\bullet(z_0)}(\sigma).$$

Therefore Lemma 3.5 implies the desired result. □

8.7. *Variation of the zeta part.* We give a first proposition which computes the variation of the Ruelle zeta function in its convergence region.

Proposition 8.3 (variation of the dynamical zeta function with respect to the connection). *For any relatively compact open set $\mathcal{Z} \subset \mathbb{C}$ such that $\overline{\mathcal{Z}} \cap \mathrm{Res}(\mathcal{L}_X^{\nabla(z_0)}) = \varnothing$, there is $\delta_{\mathcal{Z}} > 0$ so that $(z, s) \mapsto \zeta_{X,\nabla(z)}(s)$ is $\mathcal{C}^1$ as a map*

$$\{|z - z_0| < \delta\} \times \mathcal{Z} \to \mathbb{C}$$

and for every $\varepsilon > 0$ small enough it holds

$$\mathrm{d}_z(\zeta_{X,\nabla(z)}(s))|_{z=z_0}\sigma = (-1)^{q+1}e^{-\varepsilon s}\mathrm{tr}_\mathrm{s}^\flat\big(\alpha_{z_0}(\sigma)\iota_X(\mathcal{L}_X^{\nabla(z_0)} + s)^{-1}e^{-\varepsilon\mathcal{L}_X^{\nabla(z_0)}}\big).$$

Proof. The proof is very similar to that of Proposition 7.4, using the identities

$$\frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0}(\mathcal{L}_X^{\nabla(z+t\sigma)} + \tau)^{-1} = -(\mathcal{L}_X^{\nabla(z)} + \tau)^{-1}a_{z_0}(\sigma)(X)(\mathcal{L}_X^{\nabla(z)} + \tau)^{-1}$$

and $\alpha_{z_0}(\sigma)(X) = [\alpha_{z_0}(\sigma), \iota_X] = \alpha_{z_0}(\sigma) \circ \iota_X + \iota_X \circ \alpha_{z_0}(\sigma)$, and we shall omit the details. □

The following lemma is a direct consequence of Lemma A.2 and the fact that $\Pi_0(z_0) = \Pi(z_0)$ by (8-12).

Lemma 8.4. *For $s \notin \mathrm{Res}(\mathcal{L}_X^\nabla(z_0))$, the map $z \mapsto h_s(z) = \det_{\mathrm{gr},C^\bullet(z)}(\mathcal{L}_X^{\nabla(z)} + s)^{(-1)^{q+1}}$ is $\mathcal{C}^1$ near $z = z_0$ and*

$$\mathrm{d}(\log h_s)_{z_0}\sigma = (-1)^{q+1}\mathrm{tr}_{\mathrm{s},C^\bullet(z_0)}\big(\Pi_0(z_0)\alpha_{z_0}(\sigma)\iota_X(\mathcal{L}_X^{\nabla(z_0)} + s)^{-1}\big).$$

8.8. ***Proof of Proposition 8.1.*** Combining the two lemmas of the preceding subsection we obtain that for $s \notin \mathrm{Res}(\mathcal{L}_X^{\nabla(z_0)})$, the map $z \mapsto \zeta_{X,\nabla(z)}^{(\lambda,\infty)}(s) = g_s(z)/h_s(z)$ is real differentiable at $z = z_0$ (and therefore on U since we may vary z_0). Moreover for every $\varepsilon > 0$ small enough

$$\mathrm{d}\left(\log \frac{g_s}{h_s}\right)_z \sigma = (-1)^{q+1}\left(e^{-\varepsilon s}\mathrm{tr}_\mathrm{s}^\flat \alpha_z(\sigma)\iota_X(\mathcal{L}_X^{\nabla(z)}+s)^{-1}e^{-\varepsilon\mathcal{L}_X^{\nabla(z)}} - \mathrm{tr}_{\mathrm{s},C^\bullet(z)}\Pi_0(z)\alpha_z(\sigma)\iota_X(\mathcal{L}_X^{\nabla(z)}+s)^{-1}\right). \tag{8-17}$$

Letting $s \to 0$, this yields

$$(-1)^{q+1}\,\mathrm{d}(\log b)_z\sigma = \mathrm{tr}_\mathrm{s}^\flat\left(\alpha_z(\sigma)\iota_X Y(z)(\mathrm{Id} - \Pi_0(z))e^{-\varepsilon\mathcal{L}_X^{\nabla(z)}}\right) + \mathrm{tr}_{\mathrm{s},C^\bullet(z)}(\Pi_0(z)\alpha_z(\sigma)\iota_X Q_z(\varepsilon)),$$

where we set $b(z) = \zeta_{X,\nabla(z)}^{(\lambda,\infty)}(0)$ and

$$Q_z(\varepsilon) = \sum_{n \geqslant 1} \frac{(-\varepsilon)^n}{n!}(\mathcal{L}_X^{\nabla(z)})^{n-1} : C^\bullet(z) \to C^\bullet(z).$$

Recall that if $c(z) = \tau(C^\bullet(z), \Gamma)$ one has $\tau(z) = c(z)b(z)^{(-1)^q}$. Therefore Lemma 8.2 gives, with what precedes, and with $K(z)$ given by (8-6),

$$\mathrm{d}(\log\tau)_z\sigma = -\mathrm{tr}_\mathrm{s}^\flat(\alpha_z(\sigma)K(z)e^{-\varepsilon\mathcal{L}_X^{\nabla(z)}}) - \mathrm{tr}_{\mathrm{s},C^\bullet(z)}\left(\Pi_0(z)\alpha_z(\sigma)\left(k(z)(\mathrm{Id} - e^{-\varepsilon\mathcal{L}_X^{\nabla(z)}}) + \iota_X Q_z(\varepsilon)\right)\right). \tag{8-18}$$

Moreover, by using (8-3) and (8-7), we see that

$$\begin{aligned}\alpha_z(\sigma)K(z)\mathcal{L}_X^{\nabla(z)}e^{-\varepsilon\mathcal{L}_X^{\nabla(z)}} &= \alpha_z(\sigma)K(z)[\nabla(z), \iota_X]e^{-\varepsilon\mathcal{L}_X^{\nabla(z)}} \\ &= \alpha_z(\sigma)\iota_X e^{-\varepsilon\mathcal{L}_X^{\nabla(z)}} + [\alpha_z(\sigma)K(z)\iota_X e^{-\varepsilon\mathcal{L}_X^{\nabla}}, \nabla(z)],\end{aligned}$$

and hence, by cyclicity of the trace, $(\mathrm{d}/\mathrm{d}\varepsilon)\mathrm{tr}_\mathrm{s}^\flat(\alpha_z(\sigma)K(z)e^{-\varepsilon\mathcal{L}_X^{\nabla(z)}}) = 0$. In particular, the last term in the right-hand side of (8-18) does not depend on ε; since it goes to zero as $\varepsilon \to 0$, it vanishes, and Proposition 8.1 follows.

9. Euler structures, Chern–Simons classes

The Turaev torsion is defined using *Euler structures*, introduced by Turaev [1989], whose purpose is to fix sign ambiguities of combinatorial torsions. We shall use however the representation in terms of vector fields used by Burghelea and Haller [2006]. The goal of the present section is to introduce these Euler structures, in view of the definition of the Turaev torsion.

9.1. ***The Chern–Simons class of a pair of vector fields.*** If $X \in \mathcal{C}^\infty(M, TM)$ is a vector field with isolated nondegenerate zeros, we define the singular 0-chain

$$\mathrm{div}(X) = -\sum_{x \in \mathrm{Crit}(X)} \mathrm{ind}_X(x)[x] \in C_0(M, \mathbb{Z}),$$

where $\mathrm{Crit}(X)$ is the set of critical points of X and $\mathrm{ind}_X(x)$ denotes the Poincaré–Hopf index of x as a critical point of X.[6] Note also that $\mathrm{div}(-X) = -\mathrm{div}(X)$ since M is odd-dimensional.

[6] $\mathrm{ind}_X(x) = (-1)^{\dim E_s(x)}$ if x is hyperbolic and $E_s(x) \subset T_xM$ is the stable subspace of x.

Let X_0, X_1 be two vector fields with isolated nondegenerate zeros. Let $p : M \times [0, 1] \to M$ be the projection over the first factor and choose a smooth section H of the bundle $p^*TM \to M \times [0, 1]$, transversal to the zero section, such that H restricts to X_i on $\{i\} \times M$ for $i = 0, 1$. Then the set $H^{-1}(0) \subset M \times [0, 1]$ is an oriented smooth submanifold of dimension 1 with boundary (it is oriented because M and $[0, 1]$ are), and we denote by $[H^{-1}(0)]$ its fundamental class.

Definition 9.1. The class

$$p_*[H^{-1}(0)] \in C_1(M, \mathbb{Z})/\partial C_2(M, \mathbb{Z}),$$

where p_* is the pushforward by p, does not depend on the choice of the homotopy H relating X_0 and X_1; see [Burghelea and Haller 2006, §2.2]. This is the *Chern–Simons class* of the pair (X_0, X_1), denoted by $\mathrm{cs}(X_0, X_1)$.

We have the fundamental formulae

$$\partial\, \mathrm{cs}(X_0, X_1) = \mathrm{div}(X_1) - \mathrm{div}(X_0),$$
$$\mathrm{cs}(X_0, X_1) + \mathrm{cs}(X_1, X_2) = \mathrm{cs}(X_0, X_2) \tag{9-1}$$

for any other vector field with nondegenerate zeros X_2. Notice also that if X_0 and X_1 are nonsingular vector fields, then $\mathrm{cs}(X_0, X_1)$ defines a homology class in $H_1(M, \mathbb{Z})$.

9.2. ***Euler structures.*** Let X be a smooth vector field on M with nondegenerate zeros. An *Euler chain* for X is a singular one-chain $e \in C_1(M, \mathbb{Z})$ such that $\partial e = \mathrm{div}(X)$. Euler chains for X always exist because M is odd-dimensional and thus $\chi(M) = 0$.

Two pairs (X_0, e_0) and (X_1, e_1), with X_i a vector field with nondegenerate zeros and e_i an Euler chain for X_i, $i = 0, 1$, will be said to be equivalent if

$$[e_1] = [e_0] + \mathrm{cs}(X_0, X_1) \in C_1(M, \mathbb{C})/\partial C_2(M, \mathbb{Z}), \tag{9-2}$$

where $[e_i]$ is the class of e_i in $C_1(M, \mathbb{C})/\partial C_2(M, \mathbb{Z})$ for $i = 1, 2$.

Definition 9.2. An *Euler structure* is an equivalence class $[X, e]$ for the relation (9-2). We will denote by $\mathrm{Eul}(M)$ the set of Euler structures.

There is a free and transitive action of $H_1(M, \mathbb{Z})$ on $\mathrm{Eul}(M)$ given by

$$[X, e] + h = [X, e + h], \quad h \in H_1(M, \mathbb{Z}).$$

9.3. ***Homotopy formula relating flows.*** Let X_0, X_1 be two vector fields with nondegenerate zeros. Let H be a smooth homotopy between X_0 and X_1 as in Section 9.1 and set $X_{\mathrm{t}} = H(\mathrm{t}, \cdot) \in C^\infty(M, TM)$. For $\varepsilon > 0$ we define $\Phi_\varepsilon : M \times [0, 1] \to M \times M \times [0, 1]$ via

$$\Phi_\varepsilon(x, \mathrm{t}) = (e^{-\varepsilon X_{\mathrm{t}}}(x), x, \mathrm{t}), \quad x \in M, \ \mathrm{t} \in [0, 1].$$

We also set

$$H_\varepsilon = \{\Phi_\varepsilon(x, \mathrm{t}) : (x, \mathrm{t}) \in M \times [0, 1]\} \subset M \times M \times \mathbb{R}.$$

Then H_ε is a submanifold with boundary of $M \times M \times \mathbb{R}$ which is oriented (since M and $\mathbb{R}$ are). Define

$$[H_\varepsilon] = (\Phi_\varepsilon)_*([M] \times [\![0, 1]\!]) \in \mathcal{D}'^n(M \times M \times \mathbb{R})$$

to be the associated integration current; see Section 4.3. Let g be any metric on M and let $\rho > 0$ be smaller than its injectivity radius. Then for any $x, y \in M$ with $\mathrm{dist}(x, y) \leqslant \rho$, we denote by $P(x, y) \in \mathrm{Hom}(E_x, E_y)$ the parallel transport by ∇ along the minimizing geodesic joining x to y. Then P is a smooth section of $\pi_1^* E^\vee \otimes \pi_2^* E$ defined in some neighborhood of the diagonal in $M \times M$. Take ε small enough so that

$$\mathrm{dist}(x, e^{-sX_t}(x)) \leqslant \rho, \quad s \in [0, \varepsilon],\ t \in [0, 1],\ x \in M, \tag{9-3}$$

so that $\mathrm{supp}\, \pi_*[H_\varepsilon] \subset \{(x, y) : \mathrm{dist}(x, y) \leqslant \rho\}$. Here, $\pi : M \times M \times \mathbb{R} \to M \times M$ is the projection over the two first factors and $\pi_* : \mathcal{D}'^n(M \times M \times [0, 1]) \to \mathcal{D}'^{n-1}(M \times M)$ is the push-forward operator which is simply defined by

$$\int_{M \times M} \pi_* u \wedge v = \int_{M \times M \times [0,1]} u \wedge \pi^* v, \quad u \in \mathcal{D}'^n(M \times M \times [0, 1]),\ v \in \Omega^{n+1}(M \times M).$$

Then we define

$$\mathcal{R}_\varepsilon = -\pi_*[H_\varepsilon] \cdot P \in \mathcal{D}'^{n-1}(M \times M, \pi_1^* E^\vee \otimes \pi_2^* E).$$

Finally, we denote by $R_\varepsilon : \Omega^\bullet(M, E) \to \mathcal{D}'^{\bullet-1}(M, E)$ the operator of degree -1 whose Schwartz kernel is $\mathcal{R}_\varepsilon$.

Lemma 9.3. *We have the homotopy formula*

$$[\nabla, R_\varepsilon] = \nabla R_\varepsilon + R_\varepsilon \nabla = e^{-\varepsilon \mathcal{L}^\nabla_{X_1}} - e^{-\varepsilon \mathcal{L}^\nabla_{X_0}}. \tag{9-4}$$

Proof. First note that because M is odd-dimensional, the boundary (computed with orientations) of the manifold H_ε is calculated using the Leibniz rule [Krantz and Parks 2008, (7.15), p. 190] as

$$\begin{aligned} \partial H_\varepsilon &= \partial\big((\Phi_\varepsilon)_*([M] \times [\![0, 1]\!])\big) = (-1)^{\dim(M)} (\Phi_\varepsilon)_*([M] \times (\partial [\![0, 1]\!])) \\ &= (-1)^{\dim(M)} (\Phi_\varepsilon)_*([M] \times (\{1\} - \{0\})) = \mathrm{Gr}(e^{-\varepsilon X_0}) \times \{0\} - \mathrm{Gr}(e^{-\varepsilon X_1}) \times \{1\}. \end{aligned}$$

Therefore we have, see (4-1),

$$(-1)^n\, \mathrm{d}^{M \times M} \pi_*[H_\varepsilon] = \pi_*[\partial H_\varepsilon] = [\mathrm{Gr}(e^{-\varepsilon X_0})] - [\mathrm{Gr}(e^{-\varepsilon X_1})],$$

where $[\mathrm{Gr}(e^{-\varepsilon X_i})]$ denotes the integration current on the manifold $\mathrm{Gr}(e^{-\varepsilon X_i})$ for $i = 0, 1$. Now note that we have by construction $\nabla^{E^\vee \boxtimes E} P = 0$. Therefore

$$\nabla^{E^\vee \boxtimes E} \mathcal{R}_\varepsilon = (-1)^n \big([\mathrm{Gr}(e^{-\varepsilon X_1})] - [\mathrm{Gr}(e^{-\varepsilon X_0})]\big) \otimes P.$$

Note that by definition of $e^{-\mathcal{L}^\nabla_{X_i}}$ (see Section 5.2), the bound (9-3) and the flatness of ∇ imply that the Schwartz kernel of $e^{-\varepsilon \mathcal{L}^\nabla_{X_i}}$ is $[\mathrm{Gr}(e^{-\varepsilon X_i})] \otimes P$. This concludes because the Schwartz kernel of $[\nabla, R_\varepsilon]$ is $(-1)^n \nabla^{E^\vee \boxtimes E} \mathcal{R}_\varepsilon$; see [Harvey and Lawson 2001, Lemma 2.2]. □

The next formula follows from the definition of the flat trace and the Chern–Simons classes. It will be crucial for the topological interpretation of the variation formula obtained in Section 8.

Lemma 9.4. *We have for any* $\alpha \in \Omega^\bullet(M, \mathrm{End}(E))$ *such that* $\operatorname{tr}\alpha$ *is closed and* $\varepsilon > 0$ *small enough*

$$\operatorname{tr}_{\mathrm{s}}^{\flat} \alpha R_\varepsilon = \langle \operatorname{tr}\alpha, \operatorname{cs}(X_0, X_1)\rangle. \tag{9-5}$$

Here α is identified with the operator $u \mapsto \alpha \wedge u$. Note that because H is transverse to the zero section, we have

$$\mathrm{WF}(\mathcal{R}_\varepsilon) \cap N^*\Delta = \varnothing, \tag{9-6}$$

where $N^*\Delta$ denotes the conormal to the diagonal Δ in $M \times M$, so that the above flat trace is well-defined.

Proof. We denote by $i : M \hookrightarrow M \times M$ the diagonal inclusion. Note that the Schwartz kernel of αR_ε is $(-1)^n \pi_2^*\alpha \wedge \mathcal{R}_\varepsilon = -\pi_2^*\alpha \wedge \mathcal{R}_\varepsilon$ since n is odd. From the definition of the super flat trace $\operatorname{tr}_{\mathrm{s}}^{\flat}$, we find that

$$\operatorname{tr}_{\mathrm{s}}^{\flat} \alpha R_\varepsilon = \langle \operatorname{tr} i^*(\pi_2^*\alpha \wedge \pi_*[H_\varepsilon]\cdot P), 1\rangle, \tag{9-7}$$

where $\pi_2 : M \times M \to M$ is the projection over the second factor. Of course we have $i^*P = \mathrm{Id}_E \in \mathcal{C}^\infty(M, \mathrm{End}(E))$. We therefore have

$$\operatorname{tr} i^*(\pi_2^*\alpha \wedge \pi_*[H_\varepsilon]\cdot P) = \operatorname{tr}\alpha \wedge i^*\pi_*[H_\varepsilon] = \operatorname{tr}\alpha \wedge p_* j^*[H_\varepsilon],$$

where $j : M \times [0, 1] \hookrightarrow M \times M \times [0, 1]$, $(x, t) \mapsto (x, x, t)$. Now, it holds $j^*[H_\varepsilon] = [H^{-1}(0)]$ and thus $p_* j^*[H_\varepsilon] = \operatorname{cs}(X_0, X_1)$. This finally leads to

$$\operatorname{tr}_{\mathrm{s}}^{\flat} \alpha R_\varepsilon = \langle \operatorname{tr}\alpha \wedge \operatorname{cs}(X_0, X_1), 1\rangle = \langle \operatorname{tr}\alpha, \operatorname{cs}(X_0, X_1)\rangle. \qquad \square$$

10. Morse theory and variation of Turaev torsion

We introduce here the Turaev torsion which is defined in terms of CW decompositions. In the spirit of the seminal work [Bismut and Zhang 1992] based on geometric constructions of [Laudenbach 1992], we use a CW decomposition which comes from the unstable cells of a Morse–Smale gradient flow induced by a Morse function. This allows us to interpret the variation of the Turaev torsion as a supertrace on the space of generalized resonant states for the Morse–Smale flow. This interpretation will be convenient for the comparison of the Turaev torsion with the dynamical torsion.

10.1. ***Morse theory and CW-decompositions.*** Let f be a Morse function on M and $\widetilde{X} = -\operatorname{grad}_g f$ be its associated gradient vector field with respect to some Riemannian metric g (the tilde notation is used to make the difference with the Anosov flows we studied until now). For any $a \in \mathrm{Crit}(f)$, we denote by

$$W^s(a) = \left\{ y \in M : \lim_{t\to\infty} e^{t\widetilde{X}} y = a \right\}, \quad W^u(a) = \left\{ y \in M : \lim_{t\to\infty} e^{-t\widetilde{X}} y = a \right\},$$

the stable and unstable manifolds of a. Then it is well known that $W^s(a)$ (resp. $W^u(x)$) is a smooth embedded open disk of dimension $n - \mathrm{ind}_f(a)$ (resp. $\mathrm{ind}_f(a)$), where $\mathrm{ind}_f(a)$ is the index of a as a critical point of f, that is, in a Morse chart $(z_1, \ldots, z_n)$ near a,

$$f(z_1, \ldots, z_n) = f(a) - z_1^2 - \cdots - z_{\mathrm{ind}_f(a)}^2 + z_{\mathrm{ind}_f(a)+1}^2 + \cdots + z_n^2.$$

For simplicity, we will let

$$|a| = \mathrm{ind}_f(a) = \dim W^u(a),$$

and we fix an orientation of every $W^u(a)$.

We assume that $\widetilde{X}$ satisfies the Morse–Smale condition, that is, for any $a, b \in \mathrm{Crit}(f)$, the manifolds $W^s(a)$ and $W^u(b)$ are transverse. Also, we assume that, for every $a \in \mathrm{Crit}(f)$, the metric g is flat near a and reads $\sum_{i=1}^n (\mathrm{d}x^i)^2$ in the Morse charts. This assumption on the metric is crucial to ensure one can compactify the unstable and stable manifolds as smooth manifolds with corners. The existence of such a compactification and of the CW structure is unknown without the flatness assumption. Let us summarize some results from [Qin 2010, Theorems 3.2, 3.8 and 3.9] which apply to f. We would like to mention that such results can be found in a slightly different form in [Laudenbach 1992] and are used in [Bismut and Zhang 1992]. A difference is that Laudenbach only needs to compactify the unstable cells as C^1-manifolds with conical singularities (as opposed to C^∞) to show that the unstable manifolds have finite mass near the boundary — he is also able to obtain the CW-complex structure. On the other hand, Qin obtains a smooth compactification as manifolds with corners which is stronger than the result of Laudenbach[7] and hence his results recover all those of [Laudenbach 1992]. In the work [Dang and Rivière 2020b], no assumption is made on the flatness of the metric g and only the fact that $\widetilde{X}$ is C^1 linearizable near critical points is needed. In this context, the unstable currents are resonant states for the Lie derivative $\mathcal{L}_{\widetilde{X}}$ and belong to some anisotropic Sobolev spaces. This allows to bound the wavefront set of the unstable currents. Yet this method does not allow to show the finiteness of the mass as in the work of Laudenbach. This nevertheless gives a spectral interpretation of the Morse complex, but this approach does not show that the unstable manifolds form a CW-complex, and the latter is crucial in the topological approach of the torsion. Making such strong assumptions on the pair (f, g) in the present paper allows us to benefit from the best of both worlds — we can use the results from [Dang and Rivière 2020b] together with those from [Qin 2010].

First, $W^u(a)$ admits a compactification to a smooth $|a|$-dimensional manifold with corner $\overline{W}^u(a)$, endowed with a smooth map $e_a : \overline{W}^u(a) \to M$ that extends the inclusion $W^u(a) \subset M$. Then the collection $W = \{\overline{W}^u(a)\}_{a \in \mathrm{Crit}(f)}$ and the applications e_a induce a CW-decomposition on M. Moreover, the boundary operator of the cellular chain complex is given by

$$\partial \overline{W}^u(a) = \sum_{|b|=|a|-1} \#\mathcal{L}(a, b) \overline{W}^u(b),$$

where $\mathcal{L}(a, b)$ is the set of gradient lines joining a to b and $\#\mathcal{L}(a, b)$ is the sum of the orientations induced by the orientations of the unstable manifolds of (a, b); see [Qin 2010, Theorem 3.9].

10.2. ***The Thom–Smale complex.*** We set $C_\bullet(W, E^\vee) = \bigoplus_{k=0}^n C_k(W, E^\vee)$, where

$$C_k(W, E^\vee) = \bigoplus_{\substack{a \in \mathrm{Crit}(f) \\ |a|=k}} E_a^\vee, \quad k = 0, \ldots, n.$$

[7] As discussed in detail in https://mathoverflow.net/questions/346822/unstable-manifolds-of-a-morse-function-give-a-cw-complex.

We endow the complex $C_\bullet(W, E^\vee)$ with the boundary operator $\partial^{\nabla^\vee}$ defined by

$$\partial^{\nabla^\vee} u = \sum_{|b|=|a|-1} \sum_{\gamma \in \mathcal{L}(a,b)} \varepsilon_\gamma P_\gamma(u), \quad a \in \mathrm{Crit}(f),\ u \in E_a^\vee,$$

where for $\gamma \in \mathcal{L}(a, b)$, $P_\gamma \in \mathrm{End}(E_a^\vee, E_b^\vee)$ is the parallel transport of $\nabla^\vee$ along the curve γ and $\varepsilon_\gamma = \pm 1$ is the orientation number of $\gamma \in \mathcal{L}(a, b)$.

Then by [Laudenbach 1992] (see also [Dang and Rivière 2020b] for a different approach), there is a canonical isomorphism

$$H_\bullet(M, \nabla^\vee) \simeq H_\bullet(W, \nabla^\vee),$$

where $H_\bullet(M, \nabla^\vee)$ is the singular homology of flat sections of $(E^\vee, \nabla^\vee)$ and $H_\bullet(W, \nabla^\vee)$ denotes the homology of the complex $C_\bullet(W, E^\vee)$ endowed with the boundary map $\partial^{\nabla^\vee}$. Therefore this complex is acyclic since ∇ (and thus $\nabla^\vee$) is.

10.3. ***The Turaev torsion.*** Fix some base point $x_\star \in M$ and, for every $a \in \mathrm{Crit}(f)$, let γ_a be some path in M joining $x_\star$ to a. Define

$$e = \sum_{a \in \mathrm{Crit}(f)} (-1)^{|a|} \gamma_a \in C_1(M, \mathbb{Z}). \tag{10-1}$$

Note that the Poincaré–Hopf index of $\widetilde{X}$ near $a \in \mathrm{Crit}(f)$ is $-(-1)^{|a|}$ so that

$$\partial e = \mathrm{div}(\widetilde{X}) \tag{10-2}$$

because $\sum_{a \in \mathrm{Crit}(f)} (-1)^{|a|} = \chi(M) = 0$ by the Poincaré–Hopf index theorem. Therefore e is an Euler chain for $\widetilde{X}$ and

$$\mathfrak{e} = [\widetilde{X}, e]$$

defines an Euler structure.

Next, choose some basis $u_1, \dots, u_d$ of $E_{x_\star}^\vee$. For each $a \in \mathrm{Crit}(f)$, we propagate this basis via the parallel transport of ∇ along γ_a to obtain a basis $u_{1,a}, \dots, u_{d,a}$ of E_a. We choose an ordering of the cells $\{\overline{W}^u(a)\}$; this gives us a homology orientation $\mathfrak{o}$, that is, an orientation on the line $\det H_\bullet(W, \mathbb{R})$ (see [Farber and Turaev 2000, §6.3]). Moreover, this ordering and the chosen basis of $E_a^\vee$ give us (using the wedge product) an element $c_k \in \det C_k(W, E^\vee)$ for each k, and thus an element $c \in \det C_\bullet(W, E^\vee)$.

The *Turaev torsion* of ∇ with respect to the choices $\mathfrak{e}, \mathfrak{o}$ is then defined by [Farber and Turaev 2000, §9.2, p. 218]

$$\tau_{\mathfrak{e},\mathfrak{o}}(\nabla)^{-1} = \varphi_{C_\bullet(W,\nabla^\vee)}(c) \in \mathbb{C} \setminus 0, \tag{10-3}$$

where $\varphi_{C_\bullet(W,\nabla^\vee)} : \det C_\bullet(W, \nabla^\vee) \simeq \mathbb{C} \setminus 0$ is the canonical isomorphism from [Farber and Turaev 2000, §2.2] — the homology version of the isomorphism (3-1). Note that $\nabla^\vee$ (and not ∇) is involved in the definition of $\tau_{\mathfrak{e},\mathfrak{o}}(\nabla)$; indeed, we use here the cohomological version of Turaev's torsion, which is more convenient for our purposes, and which is consistent with [Braverman and Kappeler 2007b; 2008, p. 252].

10.4. ***Resonant states of the Morse–Smale flow.*** In [Dang and Rivière 2020b], it was shown that we can define Ruelle resonances for the Morse–Smale gradient flow $\mathcal{L}_{\widetilde{X}}^{\nabla}$ as described in Section 5 in the context of Anosov flows. More precisely, we have that the resolvent

$$(\mathcal{L}_{\widetilde{X}}^{\nabla}+s)^{-1}:\Omega^{\bullet}(M,E)\to\mathcal{D}'^{\bullet}(M,E)$$

is well-defined for $\mathrm{Re}(s)\gg 0$ and has a meromorphic continuation to all $s\in\mathbb{C}$. The poles of this continuation are the Ruelle resonances of $\mathcal{L}_{\widetilde{X}}^{\nabla}$ and the set of those will be denoted by $\mathrm{Res}(\mathcal{L}_{\widetilde{X}}^{\nabla})$. In fact, the set $\mathrm{Res}(\mathcal{L}_{\widetilde{X}}^{\nabla})$ does not depend on the flat vector bundle (E,∇). It only depends on the Lyapunov exponents of the Morse–Smale vector field at critical points. In fact $\mathrm{Res}(\mathcal{L}_{\widetilde{X}}^{\nabla})\subset\mathbb{Z}_{\geqslant 0}$ in the present case since the Lyapunov exponents are only ± 1 and the Ruelle spectrum was proved to be equal to integer combinations of absolute values of Lyapunov exponents [Dang and Rivière 2020a, Theorem 6.3, p. 571]. Let $\lambda>0$ be such that $\mathrm{Res}(\mathcal{L}_{\widetilde{X}}^{\nabla})\cap\{|s|\leqslant\lambda\}\subset\{0\}$; let

$$\widetilde{\Pi}=\frac{1}{2\pi i}\int_{|s|=\lambda}(\mathcal{L}_{\widetilde{X}}^{\nabla}+s)^{-1}\,\mathrm{d}s \tag{10-4}$$

be the spectral projector associated with the resonance 0, and denote by

$$\widetilde{C}^{\bullet}=\operatorname{ran}\widetilde{\Pi}\subset\mathcal{D}'^{\bullet}(M,E)$$

the associated space of generalized eigenvectors for $\mathcal{L}_{\widetilde{X}}^{\nabla}$. Since ∇ and $\mathcal{L}_{\widetilde{X}}^{\nabla}$ commute, ∇ induces a differential on the complex $\widetilde{C}^{\bullet}$. Moreover, $\widetilde{\Pi}$ maps $\mathcal{D}'^{\bullet}_{\Gamma}(M,E)$ to itself continuously, where

$$\Gamma=\bigcup_{a\in\mathrm{Crit}(f)}\overline{N^{*}W^{u}(a)}\subset T^{*}M.$$

10.5. ***A variation formula for the Turaev torsion.*** Assume that we are given a $\mathcal{C}^1$ family of acyclic connections $\nabla(z)$ on E as in Section 8. We denote by $\widetilde{\Pi}_{-}(z)$ the spectral projector (10-4) associated with $\nabla(z)$ and $-\widetilde{X}$, and set $\widetilde{C}^{\bullet}_{-}(z)=\operatorname{ran}\widetilde{\Pi}_{-}(z)$. By [Dang and Rivière 2020b] we have that all the complexes $(\widetilde{C}^{\bullet}(z),\nabla(z))$ are acyclic and there exists cochain contractions $\tilde{k}_{-}(z):\widetilde{C}^{\bullet}_{-}(z)\to\widetilde{C}^{\bullet-1}_{-}(z)$. As in Section 8.3 we have a variation formula for the Turaev torsion.

Proposition 10.1. *The map $z\mapsto\tilde{\tau}(z)=\tau_{\mathfrak{e},\mathfrak{o}}(\nabla(z))$ is real differentiable on U and for any $z\in U$*

$$\mathrm{d}(\log\tilde{\tau})_z\sigma=-\mathrm{tr}_{\mathrm{s},\widetilde{C}^{\bullet}(z)}(\widetilde{\Pi}_{-}(z)\alpha_z(\sigma)\tilde{k}_{-}(z))-\int_e\mathrm{tr}\,\alpha_z(\sigma),\qquad\sigma\in\mathbb{C},$$

where $\alpha_z(\sigma)$ is given by (8-2) and e is given by (10-1).

The rest of this section is devoted to the proof of Proposition 10.1. For convenience, we will first study the variation of $z\mapsto\tau_{\mathfrak{e},\mathfrak{o}}(\nabla(z)^{\vee})$, in order to make computations on E instead of $E^{\vee}$ (indeed, $\tau_{\mathfrak{e},\mathfrak{o}}(\nabla(z))$ is defined with the dual connection $\nabla(z)^{\vee}$; see (10-3)). Then a simple duality relation will allow us to obtain the variation formula for $z\mapsto\tau_{\mathfrak{e},\mathfrak{o}}(\nabla(z))$.

10.6. ***A preferred basis.*** Let $a\in\mathrm{Crit}(f)$ and $k=|a|$. We denote by $[W^u(a)]\in\mathcal{D}'^{n-k}_{\Gamma}(M)$ the integration current over the unstable manifold $W^u(a)$ of $\widetilde{X}$; it is a well-defined current far from $\partial W^u(a)$. We also pick a cut-off function $\chi_a\in\mathcal{C}^{\infty}(M)$ valued in $[0,1]$ with $\chi_a\equiv 1$ near a and χ_a is supported in a small neighborhood Ω_a of a, with $\overline{\Omega}_a\cap\partial W^u(a)=\varnothing$. Recall from Section 10.3 that we have a basis $u_{1,a}, \dots, u_{d,a}$ of E_a. Using the parallel transport of ∇, we obtain flat sections of E over $W^u(a)$ that we will still denote by $u_{1,a}, \dots, u_{d,a}$. Define

$$\tilde{u}_{j,a} = \widetilde{\Pi}(\chi_a[W^u(a)] \otimes u_{j,a}) \in \widetilde{C}^{n-k}, \quad j = 1, \dots, d. \tag{10-5}$$

By [Dang and Rivière 2020a] we have that $\{\tilde{u}_{j,a} : a \in \mathrm{Crit}(f),\ 1 \leqslant j \leqslant d\}$ is a basis of $\widetilde{C}^\bullet$. Adapting the proof of [Dang and Rivière 2021, Theorem 2.6] to the bundle case, we obtain the following proposition which will allow us to compute the Turaev torsion with the help of the complex $\widetilde{C}^\bullet$.

Proposition 10.2. *The map $\Phi : C_\bullet(W, \nabla) \to \widetilde{C}^{n-\bullet}$ defined by*

$$\Phi(u_{j,a}) = \tilde{u}_{j,a}, \quad a \in \mathrm{Crit}(f),\ j = 1, \dots, d,$$

is an isomorphism and satisfies[8]

$$\Phi \circ \partial^\nabla = (-1)^{\bullet+1} \nabla \circ \Phi.$$

An immediate corollary of the above proposition and (10-3) is that (using the notation of Section 3.2)

$$\tau_{\mathfrak{e},\mathfrak{o}}(\nabla^\vee) = \varphi_{C_\bullet(W,\nabla)}(u)^{-1} = \tau(\widetilde{C}^\bullet, \tilde{u}), \tag{10-6}$$

where $u \in \det C_\bullet(W, \nabla)$ (resp. $\tilde{u} \in \det \widetilde{C}^\bullet$) is the element given by the basis $\{u_{j,a}\}$ (resp. $\{\tilde{u}_{j,a}\}$) and the ordering of the cells $W^u(a)$.

10.7. ***Proof of Proposition 10.1.*** For any $a \in \mathrm{Crit}(f)$ we denote by $P_{\gamma_a}(z) \in \mathrm{Hom}(E_{x_\star}, E_a)$ the parallel transport of $\nabla(z)$ along γ_a. We set

$$u_{j,a}(z) = P_{\gamma_a}(z) P_{\gamma_a}(z_0)^{-1} u_{j,a}$$

and

$$\tilde{u}_{j,a}(z) = \widetilde{\Pi}(z)(\chi_a[W^u(a)] \otimes u_{j,a}(z)),$$

where again we consider $u_{j,a}(z)$ as a $\nabla(z)$-flat section of E over $W^u(a)$ using the parallel transport of $\nabla(z)$. The construction of Ruelle resonances for Morse–Smale gradient flow follows from the construction of anisotropic Sobolev spaces

$$\Omega^\bullet(M, E) \subset \widetilde{\mathcal{H}}_1^\bullet \subset \widetilde{\mathcal{H}}^\bullet \subset \mathcal{D}'^\bullet(M, E),$$

see [Dang and Rivière 2019], on which $\mathcal{L}_X^\nabla + s$ is a holomorphic family of Fredholm operators of index 0 in the region $\{\mathrm{Re}(s) > -2\}$, and such that $\nabla(z)$ is bounded $\widetilde{\mathcal{H}}_1^\bullet \to \widetilde{\mathcal{H}}^\bullet$. Every argument made in Section 8.4 also stands here and $z \mapsto \widetilde{\Pi}(z)$ is a $\mathcal{C}^1$ family of bounded operators $\widetilde{\mathcal{H}}^\bullet \to \widetilde{\mathcal{H}}_1^\bullet$.

Note that by continuity, $\widetilde{\Pi}(z)$ induces an isomorphism $\widetilde{C}^\bullet(z_0) \to \widetilde{C}^\bullet(z)$ for z close enough to zero. In fact, this isomorphism holds true for all z since we have an explicit description of the range of $\widetilde{\Pi}(z)$ for all z using the basis of resonant states of $\mathcal{L}_X^\nabla$. Let $\tilde{u}(z) \in \det \widetilde{C}^\bullet(z)$ be the element given by the basis $\{\tilde{u}_{j,a}(z)\}$ and the ordering of the cells $W^u(a)$. Then by (10-6) and (3-5) we have

$$\tau_{\mathfrak{e},\mathfrak{o}}(\nabla(z)^\vee) = \tau(\widetilde{C}^\bullet(z), \tilde{u}(z)) = [\tilde{u}(z) : \widetilde{\Pi}(z)\tilde{u}(z_0)]\tau(\widetilde{C}^\bullet(z), \widetilde{\Pi}(z)\tilde{u}(z_0)), \tag{10-7}$$

[8] $(-1)^\bullet$ comes from $\partial = (-1)^{\deg+1}\,\mathrm{d}$ comparing the boundary ∂ and De Rham differential d.

where $\widetilde{\Pi}(z)\tilde{u}(z_0) \in \det \widetilde{C}^\bullet(z)$ is the image of $\tilde{u}$ by the isomorphism $\det \widetilde{C}^\bullet(z_0) \to \det \widetilde{C}^\bullet(z)$ induced by $\widetilde{\Pi}(z)$, and $\tilde{u}(z) = [\tilde{u}(z) : \widetilde{\Pi}(z)\tilde{u}(z_0)]\widetilde{\Pi}(z)\tilde{u}(z_0)$. Doing exactly as in Section 8.6, we obtain that $z \mapsto \hat{\tau}(z) = \tau(\widetilde{C}^\bullet(z), \widetilde{\Pi}(z)\tilde{u})$ is $\mathcal{C}^1$ and

$$\mathrm{d}(\log \hat{\tau})_{z_0}\sigma = -\mathrm{tr}_{\mathrm{s},\widetilde{C}^\bullet} \widetilde{\Pi}(z_0)\alpha_{z_0}(\sigma)\tilde{k}(z_0). \tag{10-8}$$

Therefore it remains to compute the variation of $[\tilde{u}(z) : \widetilde{\Pi}(z)\tilde{u}(z_0)]$. This is the purpose of the next formula.

Lemma 10.3. *We have*

$$[\tilde{u}(z) : \widetilde{\Pi}(z)\tilde{u}(z_0)] = \prod_{a \in \mathrm{Crit}(f)} \det(P_{\gamma_a}(z)P_{\gamma_a}(z_0)^{-1})^{(-1)^{n-|a|}}.$$

Proof. By the definition of the basis $\{u_{a,j}\}$ in Section 10.3 it suffices to show that for z small enough

$$\widetilde{\Pi}(z)\tilde{u}_{a,i} = \sum_{j=1}^{d} A_{a,i}^{j}(z)\tilde{u}_{a,j}(z), \quad a \in \mathrm{Crit}(f),\ 1 \leqslant i, j \leqslant d, \tag{10-9}$$

where the coefficients $A_{a,i}^j(z)$ are defined by $u_{a,i}(z_0)(a) = \sum_{j=1}^d A_{a,i}^j(z)u_{a,j}(z)(a)$.

Everything relies on the fact that one has a decomposition of the projector

$$\widetilde{\Pi}(z) = \sum_{a,i} \langle \tilde{s}_{a,i}(z), \cdot \rangle \tilde{u}_{a,i}(z)$$

which originates from [Harvey and Lawson 2001] and was also used in [Dang and Rivière 2019, Theorem 2.4, p. 1409].

Consider the dual operator $\mathcal{L}_{-\widetilde{X}}^{\nabla(z)^\vee} : \Omega^\bullet(M, E^\vee) \to \Omega^\bullet(M, E^\vee)$. The above constructions, starting from a dual basis $s_1, \dots, s_d \in E_{x_\star}^\vee$ of $u_1, \dots, u_d$, give a basis $\{s_{a,i}(z)\}$ of each $\Gamma(W^s(a), \nabla(z)^\vee)$ (the space of flat section of $\nabla(z)^\vee$ over $W^s(a)$), since the unstable manifolds of $-\widetilde{X}$ are the stable ones of $\widetilde{X}$. Let $\widetilde{C}_\vee^\bullet(z)$ be the range of the spectral projector $\widetilde{\Pi}^\vee(z)$ from (10-4) associated with the vector field $-\widetilde{X}$ and the connection $\nabla(z)^\vee$. We have a basis $\{\tilde{s}_{a,i}(z)\}$ of $\widetilde{C}_\vee^\bullet(z)$ given by

$$\tilde{s}_{a,i}(z) = \widetilde{\Pi}^\vee(z)(\chi_a[W^s(a)] \otimes s_{a,i}(z)).$$

We will prove that for any $a, b \in \mathrm{Crit}(f)$ with same Morse index we have, for any $1 \leqslant i, j \leqslant d$,

$$\langle \tilde{s}_{a,j}(z), \tilde{u}_{a,i}(z_0) \rangle = \begin{cases} \langle s_{a,j}(z)(a), u_{a,i}(z_0)(a) \rangle_{E_a^\vee, E_a} & \text{if } a = b, \\ 0 & \text{if } a \neq b. \end{cases} \tag{10-10}$$

First assume that $a \neq b$. Then $W^u(a) \cap W^s(b) = \varnothing$ by the transversality condition, since a and b have same Morse index. Therefore for any $t_1, t_2 \geqslant 0$, we have

$$\left\langle e^{-t_1 \mathcal{L}_{-\widetilde{X}}^{\nabla(z)^\vee}} (\chi_b[W^s(b)] \otimes s_{b,j}(z)), e^{-t_2 \mathcal{L}_{\widetilde{X}}^{\nabla(z_0)}} (\chi_a[W^u(a)] \otimes u_{a,i}(z)) \right\rangle = 0, \tag{10-11}$$

since the currents in the pairing have disjoint support because they are respectively contained in $W^s(b)$ and $W^u(a)$. Now notice that for $\mathrm{Re}(s)$ big enough, one has

$$(\mathcal{L}_{-\widetilde{X}}^{\nabla(z)^\vee} + s)^{-1} = \int_0^\infty e^{-t\mathcal{L}_{-\widetilde{X}}^{\nabla(z)^\vee}} e^{-ts}\,\mathrm{d}t \quad \text{and} \quad (\mathcal{L}_{\widetilde{X}}^{\nabla(z_0)} + s)^{-1} = \int_0^\infty e^{-t\mathcal{L}_{\widetilde{X}}^{\nabla(z_0)}} e^{-ts}\,\mathrm{d}t.$$

Therefore the representation (10-4) of the spectral projectors and the analytic continuation of the above resolvents imply with (10-11) that $\langle \tilde{s}_{b,j}(z), \tilde{u}_{a,i}\rangle = 0$.

Next assume that $a = b$. Then $W^u(a) \cap W^s(a) = \{a\}$. Since the support of $\tilde{s}_{a,i}(z)$ (resp. $\tilde{u}_{a,i}(z_0)$) is contained in the closure of $W^s(a)$ (resp. $W^u(a)$), we can compute

$$\begin{aligned}\big\langle \widetilde{\Pi}^\vee(z)(\chi_a[W^s(a)]\otimes s_{a,j}(z)), \widetilde{\Pi}(\chi_a[W^u(a)]\otimes u_{a,i}(z_0))\big\rangle &= \big\langle \chi_a[W^s(a)]\otimes s_{a,j}(z), \chi_a[W^u(a)]\otimes u_{a,i}(z_0)\big\rangle \\ &= \big\langle [a], \langle s_{a,j}(z), u_{a,i}(z_0)\rangle_{E^\vee,E}\big\rangle,\end{aligned}$$

where the first equality stands because $\tilde{s}_a(z) = [W^s(a)] \otimes s_{a,j}(z)$ near a by [Dang and Rivière 2020a, Proposition 7.1]. This gives (10-10).

This identity immediately yields (10-9) with $A^j_{a,i}(z) = \langle s_{a,j}(z)(a), u_{a,i}(z_0)(a)\rangle_{E_a^\vee,E_a}$ since we have

$$\widetilde{\Pi}(z) = \sum_{a,i} \langle \tilde{s}_{a,j}(z), \cdot\rangle \tilde{u}_{a,j}(z), \tag{10-12}$$

completing the proof. □

Using the lemma, we obtain, if $\mu(z) = [\tilde{u}(z) : \widetilde{\Pi}(z)\tilde{u}(z_0)]$,

$$\mathrm{d}(\log \mu)_{z_0}\sigma = \sum_{a\in \mathrm{Crit}(f)} (-1)^{n-|a|} \operatorname{tr}(A_{\gamma_a}(z_0,\sigma) P_{\gamma_a}(z_0)^{-1}),$$

where $A_{\gamma_a}(z_0,\sigma) = \mathrm{d}(P_{\gamma_a})_{z_0}\sigma$. Since n is odd, we obtain by definition of e and (4-4)

$$\mathrm{d}(\log \mu)_{z_0}\sigma = \sum_{a\in \mathrm{Crit}(f)} (-1)^{|a|} \int_{\gamma_a} \operatorname{tr} \alpha_{z_0}(\sigma) = \int_e \operatorname{tr} \alpha_{z_0}(\sigma).$$

This equation combined with (10-7) and (10-8) yields, if $\tilde{\tau}^\vee(z) = \tau_{\mathfrak{e},\mathfrak{o}}(\nabla(z)^\vee)$

$$\mathrm{d}(\log \tilde{\tau}^\vee)_{z_0}\sigma = -\operatorname{tr}_{\mathrm{s},\widetilde{C}^\bullet} \widetilde{\Pi}(z_0)\alpha_{z_0}(\sigma)\tilde{k}(z_0) + \int_e \operatorname{tr}\alpha_{z_0}(\sigma).$$

The proof is almost finished. We first studied the variation of $z \mapsto \tau(\nabla(z)^\vee)$; we now recover the variation of $z \mapsto \tau(\nabla(z))$, which was the goal of Proposition 10.1. Let us introduce some notation. Recall that the operator $\widetilde{\Pi}$ is the spectral projector on the kernel of $\mathcal{L}^{\nabla}_{\widetilde{X}}$; now, we need to work with the spectral projector on $\ker(\mathcal{L}^{\nabla(z_0)^\vee}_{\widetilde{X}})$ (resp. $\mathcal{L}^{\nabla(z_0)}_{-\widetilde{X}}$), which we denote by $\widetilde{\Pi}^\vee_+(z_0)$ (resp. $\widetilde{\Pi}_-(z_0)$) — the sign $+$ (resp. $-$) emphasize the fact that we deal with $+\widetilde{X}$ (resp. $-\widetilde{X}$). Next, we have

$$\nabla(z)^\vee = \nabla(z_0)^\vee - {}^T(\alpha_{z_0}(z-z_0)) + o(z-z_0).$$

Therefore, applying what precedes to $\tilde{\tau}(z)$ we get

$$\mathrm{d}(\log \tilde{\tau})_{z_0}\sigma = -\operatorname{tr}_{\mathrm{s},\widetilde{C}^\bullet_{\vee,+}} \big(\widetilde{\Pi}^\vee_+(z_0)(-{}^T\alpha_{z_0}(\sigma))\tilde{k}^\vee_+(z_0)\big) + \int_e \operatorname{tr}(-{}^T\alpha_{z_0}(\sigma)), \tag{10-13}$$

where $\widetilde{\Pi}^\vee_+(z_0)$ is the spectral projector (10-4) associated with $\nabla(z_0)^\vee$ and $+\widetilde{X}$, $\widetilde{C}^\bullet_{\vee,+} = \operatorname{ran}\widetilde{\Pi}^\vee_+(z_0)$, and $\tilde{k}^\vee_+(z_0)$ is any cochain contraction on the complex $(\widetilde{C}^\bullet_{\vee,+}, \nabla(z_0)^\vee)$. Now, we have the identification

$$(\widetilde{C}^k_{\vee,+})^\vee \simeq \widetilde{C}^{n-k}_-,$$

where $\widetilde{C}^\bullet_-$ is the range of $\widetilde{\Pi}_-(z_0)$, the spectral projector (10-4) associated with $\nabla(z_0)$ and $-\widetilde{X}$. This identification can be thought of as a chain level version of Poincaré duality, the coresonant states of the resonant states for the operator $\mathcal{L}^{\nabla}_{\widetilde{X}}$ acting on the sections of the flat bundle (E,∇) are nothing but the resonant states of $\mathcal{L}^{\nabla^\vee}_{-\widetilde{X}}$ acting on the sections of the dual flat bundle $(E^\vee,\nabla^\vee)$. Moreover, one can show that under this identification, the operators $(\widetilde{\Pi}^\vee_+({}^T\alpha_{z_0}(\sigma))\tilde{k}(z_0))^\vee$ and $\widetilde{\Pi}_-(z_0)\alpha_{z_0}(\sigma)k_-(z_0)$ coincide modulo a supercommutator. More precisely, it holds

$$(\widetilde{\Pi}^\vee_+({}^T\alpha_{z_0}(\sigma))\tilde{k}(z_0))^\vee = \widetilde{\Pi}_-(z_0)\alpha_{z_0}(\sigma)k_-(z_0) + [\widetilde{\Pi}_-(z_0)\alpha_{z_0}(\sigma), k_-(z_0)],$$

where for any $j \in \{0,\dots,n\}$ we set

$$k_-(z_0)|_{\widetilde{C}^{n-j}_-} = (-1)^{j+1}(\tilde{k}^\vee_+(z_0)|_{\widetilde{C}^{j+1}})^\vee : \widetilde{C}^{n-j}_- \to \widetilde{C}^{n-j-1}_-.$$

The operator $k_-(z_0)$ is a cochain contraction on the complex $(\widetilde{C}^\bullet_-,\nabla(z_0))$. As a consequence, since n is odd,

$$\operatorname{tr}_{s,\widetilde{C}^\bullet_{\vee,+}}(\widetilde{\Pi}^\vee_+(z_0)(-{}^T\alpha_{z_0}(\sigma))\tilde{k}^\vee_+(z_0)) = \operatorname{tr}_{s,\widetilde{C}^\bullet_-}\widetilde{\Pi}_-(z_0)\alpha_{z_0}(\sigma)k_-(z_0).$$

This concludes the proof of Proposition 10.1 by (10-13) since $\operatorname{tr}(-{}^T\beta) = -\operatorname{tr}\beta$ for any $\beta \in \Omega^1(M,\operatorname{End}(E))$.

11. Comparison of the dynamical torsion with the Turaev torsion

In this section we see the dynamical torsion and the Turaev torsion as functions on the space of acyclic representations. This is an open subset of a complex affine algebraic variety. Therefore we can compute the derivative of $\tau_\vartheta/\tau_{\mathfrak{e},\mathfrak{o}}$ along holomorphic curves, using the variation formulae obtained in Sections 8 and 10. From this computation we will deduce Theorem 6.

11.1. ***The algebraic structure of the representation variety.*** We describe here the analytic structure of the space

$$\operatorname{Rep}(M,d) = \operatorname{Hom}(\pi_1(M),\operatorname{GL}(\mathbb{C}^d))$$

of complex representations of degree d of the fundamental group. Since M is compact, $\pi_1(M)$ is generated by a finite number of elements $c_1,\dots,c_L \in \pi_1(M)$ which satisfy finitely many relations. A representation $\rho \in \operatorname{Rep}(M,d)$ is thus given by $2L$ invertible $d\times d$ matrices $\rho(c_1),\dots,\rho(c_L)$, $\rho(c_1^{-1}),\dots\rho(c_L^{-1})$ with complex coefficients satisfying finitely many polynomial equations. Therefore the set $\operatorname{Rep}(M,d)$ has a natural structure of a complex affine algebraic set. We will denote the set of its singular points by $\Sigma(M,d)$. In what follows, we will only consider the classical topology of $\operatorname{Rep}(M,d)$, and not the Zariski one.

For any $\rho \in \operatorname{Rep}(M,d)$, we define

$$E_\rho = \widetilde{M}\times\mathbb{C}^d/\sim_\rho,$$

where $\widetilde{M}$ is the universal cover of M and $\sim_\rho$ is the equivalence relation given by

$$(\tilde{x},v) \sim_\rho (\gamma\cdot\tilde{x},\rho(\gamma)\cdot v), \quad \tilde{x}\in M,\ \gamma\in\pi_1(M).$$

Then E_ρ is vector bundle over M which we endow with the flat connection ∇_ρ induced by the trivial connection on $\widetilde{M}\times\mathbb{C}^d$.

We will say that a representation $\rho \in \mathrm{Rep}(M,d)$ is acyclic if ∇_ρ is acyclic. We denote by $\mathrm{Rep}_{\mathrm{ac}}(M,d) \subset \mathrm{Rep}(M,d)$ the space of acyclic representations. This is an open set (in the Zariski topology, thus in the classical one) in $\mathrm{Rep}(M,d)$; see [Burghelea and Haller 2006, §4.1]. For any $\rho \in \mathrm{Rep}_{\mathrm{ac}}(M,d)$ we set

$$\tau_\vartheta(\rho) = \tau_\vartheta(\nabla_\rho), \qquad \tau_{\mathfrak{e},\mathfrak{o}}(\rho) = \tau_{\mathfrak{e},\mathfrak{o}}(\nabla_\rho)$$

for any Euler structure $\mathfrak{e}$ and any homology orientation $\mathfrak{o}$.

11.2. ***Holomorphic families of acyclic representations.*** Let $\rho_0 \in \mathrm{Rep}_{\mathrm{ac}}(M,d) \setminus \Sigma(M,d)$ be a regular point. Take $\delta > 0$ and $\rho(z)$, $|z| < \delta$, a holomorphic curve in $\mathrm{Rep}_{\mathrm{ac}}(M,d) \setminus \Sigma(M,d)$ such that $\rho(0) = \rho_0$. Theorems 6 and 7 will be a consequence of the following

Proposition 11.1. *Let X be a contact Anosov vector field on M. Let $\mathfrak{e} = [\widetilde{X}, e]$ be the Euler structure defined in Section 10.3. Note that $-\mathrm{cs}(-\widetilde{X}, X) + e$ is a cycle and defines a homology class $h \in H_1(M,\mathbb{Z})$. Then $z \mapsto \tau_\vartheta(\rho(z))/\tau_{\mathfrak{e},\mathfrak{o}}(\rho(z))$ is complex differentiable and*

$$\frac{\mathrm{d}}{\mathrm{d}z}\left(\frac{\tau_\vartheta(\rho(z))}{\tau_{\mathfrak{e},\mathfrak{o}}(\rho(z))}\langle \det \rho(z), h\rangle\right) = 0$$

for any homology orientation $\mathfrak{o}$.

Proposition 11.1 relies on the variation formulae given by Propositions 8.1 and 10.1, and Lemma 9.4, which gives a topological interpretation of those.

11.3. ***An adapted family of connections.*** By [Braverman and Vertman 2017, Lemma 4.3], there exists a flat vector bundle E over M and a $\mathcal{C}^1$ family of connections $\nabla(z)$, $|z| < \delta$, in the sense of Section 8.1, such that

$$\rho_{\nabla(z)} = \rho(z) \tag{11-1}$$

for every z; we can moreover ask the family $\nabla(z)$ to be complex differentiable at $z = 0$, that is,

$$\nabla(z) = \nabla + z\alpha + o(z), \tag{11-2}$$

where $\nabla = \nabla(0)$ and $\alpha \in \Omega^1(M, \mathrm{End}(E))$. Note that flatness of $\nabla(z)$ implies

$$[\nabla, \alpha] = \nabla\alpha + \alpha\nabla = 0. \tag{11-3}$$

11.4. ***A cochain contraction induced by the Morse–Smale gradient flow.*** Let

$$(\mathcal{L}^{\nabla}_{-\widetilde{X}} + s)^{-1} = \frac{\widetilde{\Pi}_-}{s} + \widetilde{Y} + \mathcal{O}(s)$$

be the Laurent expansion of $(\mathcal{L}^{\nabla}_{-\widetilde{X}} + s)^{-1}$ near $s = 0$. The fact that $s = 0$ is a simple pole comes from [Dang and Rivière 2019, Proposition 6.1, p. 1431], where it is proved that there are no Jordan blocks for the resonance $s = 0$. As in Section 8.2, we consider the operator

$$\widetilde{K} = \iota_{-\widetilde{X}}\widetilde{Y}(\mathrm{Id} - \widetilde{\Pi}_-) + \tilde{k}_-\widetilde{\Pi}_- : \Omega^\bullet(M,E) \to \mathcal{D}'^\bullet(M,E),$$

where $\tilde{k}_-$ is any cochain contraction on $\widetilde{C}^\bullet_- = \operatorname{ran} \widetilde{\Pi}_-$. Note that we have the identity

$$[\nabla, \widetilde{K}] = \nabla \widetilde{K} + \widetilde{K}\nabla = \mathrm{Id}\,. \tag{11-4}$$

The next proposition allows us to interpret the term $\operatorname{tr}_{s,\widetilde{C}^\bullet} \widetilde{\Pi}_-(z)\alpha_z(\sigma)\tilde{k}_-(z)$ appearing in Proposition 10.1 as a flat trace similar to the one appearing in Proposition 8.1. This will be crucial for the comparison between τ_ϑ and $\tau_{\mathfrak{e},\mathfrak{o}}$.

Proposition 11.2. *For $\varepsilon > 0$ small enough, the wavefront set of the Schwartz kernel of the operator $\iota_{-\widetilde{X}}\widetilde{Y}(\mathrm{Id} - \widetilde{\Pi}_-)e^{-\varepsilon\mathcal{L}^\nabla_{-\widetilde{X}}}$ does not meet the conormal to the diagonal in $M \times M$ and we have for any $\alpha \in \Omega^1(M, \mathrm{End}(E))$*

$$\operatorname{tr}^\flat_s(\alpha\iota_{-\widetilde{X}}\widetilde{Y}(\mathrm{Id} - \widetilde{\Pi}_-)e^{-\varepsilon\mathcal{L}^\nabla_{-\widetilde{X}}}) = 0.$$

Proof of Proposition 11.2. Fix $\varepsilon > 0$. We start from the Atiyah–Bott–Lefschetz trace formula [Atiyah and Bott 1967], which gives

$$\operatorname{tr}^\flat_s \alpha\iota_{-\widetilde{X}} e^{(t+\varepsilon)\widetilde{X}} = 0$$

for all $t \geqslant 0$ since the flat trace $\operatorname{tr}^\flat_s$ localizes at the critical points of $\widetilde{X}$ and the contribution from the term $\alpha\iota_{-\widetilde{X}}$ vanishes at the critical points. Now we would like to integrate this equality in time t on $[0, +\infty)$ and then connect with the resolvent $(\mathcal{L}_{-\widetilde{X}} + s)^{-1}$; we have to argue rigorously why we can interchange the flat trace and the integral over time t. This relies in an essential way on some explicit bound of the wavefront set of the resolvent that can be deduced from Lemma C.1 in Appendix C, where we bound the wavefront of the propagator near the conormal of the diagonal. Assuming that the inversion is justified, we obtain, for large $\mathrm{Re}(s)$,

$$\begin{aligned} 0 &= \int_0^\infty e^{-ts}\operatorname{tr}^\flat_s(\alpha\iota_{-\widetilde{X}}e^{(t+\varepsilon)\widetilde{X}})\,\mathrm{d}t = \int_0^\infty e^{-ts}\operatorname{tr}^\flat_s(\iota_{-\widetilde{X}}e^{(t+\varepsilon)\widetilde{X}}(\mathrm{Id} - \widetilde{\Pi}_-))\,\mathrm{d}t \\ &= \operatorname{tr}^\flat_s\big(\alpha\iota_{-\widetilde{X}}(\mathcal{L}_{-\widetilde{X}} + s)^{-1}(\mathrm{Id} - \widetilde{\Pi}_-)e^{\varepsilon\mathcal{L}_{\widetilde{X}}}\big), \end{aligned}$$

where we used the fact that $\iota_{-\widetilde{X}}\widetilde{\Pi}_- = 0$, which follows from the proof of [Dang and Rivière 2019, Proposition 7.7, p. 1448]. Actually, both resonant and coresonant states of $-\widetilde{X}$ are killed by the contraction operator $\iota_{-\widetilde{X}}$. Our wavefront bound implies that the above identity still makes sense for s near the origin; we then conclude by noting that

$$\underbrace{\operatorname{tr}^\flat_s\big(\alpha\iota_{-\widetilde{X}}(\mathcal{L}_{-\widetilde{X}} + s)^{-1}(\mathrm{Id} - \widetilde{\Pi}_-)e^{\varepsilon\mathcal{L}_{\widetilde{X}}}\big)}_{0} = \operatorname{tr}^\flat_s(\alpha\iota_{-\widetilde{X}}\widetilde{Y}e^{\varepsilon\mathcal{L}_{\widetilde{X}}}) + \mathcal{O}(s)$$

since $\widetilde{Y}(\mathrm{Id} - \widetilde{\Pi}_-) = \widetilde{Y}$. Thus letting $s \to 0$ concludes the proof of the proposition, provided that we can justify the interchange of the flat trace and the integration over t.

For $a \in \mathrm{Crit}(f)$, take c_a, Γ_a, χ_a as in Lemma C.1 proved in Appendix C. The proof of Lemma C.1 actually shows that for $\mathrm{Re}(s) > -c_a$, the integral

$$G_{\chi_a,\varepsilon,s} = \int_0^\infty e^{-ts}\chi_a e^{(t+\varepsilon)\widetilde{X}}(\mathrm{Id} - \widetilde{\Pi}_-)\chi_a\,\mathrm{d}t$$

converges as an operator $\Omega^\bullet(M) \to \mathcal{D}'^\bullet(M)$. Moreover, its Schwartz kernel $\mathcal{G}_{\chi_a,\varepsilon,s}$ is locally bounded in $\mathcal{D}'^n_{\Gamma_a}(M \times M)$ in the region $\{\mathrm{Re}(s) > -c_a\}$. We will need the following lemma, which is also proved in Appendix C.

Lemma 11.3. *For any $\mu > 0$, there is $\nu > 0$ with the following property. For every $x \in M$ such that* $\mathrm{dist}(x, \mathrm{Crit}(f)) \geqslant \mu$, *it holds*

$$\mathrm{dist}(x, e^{-(t+\varepsilon)\widetilde{X}}(x)) \geqslant \nu, \quad t \geqslant 0.$$

By (10-12) we have $\mathrm{supp}\,\mathcal{K}_{\widetilde{\Pi}_-} \cap \Delta = \mathrm{Crit}(f)$, where $\mathcal{K}_{\widetilde{\Pi}_-}$ is the Schwartz kernel of $\widetilde{\Pi}_-$ and Δ is the diagonal in $M \times M$; the same holds for $e^{(t+\varepsilon)\widetilde{X}}\widetilde{\Pi}_- = \widetilde{\Pi}_-$ (see [Dang and Rivière 2021]). Moreover, Lemma 11.3 implies that if $\chi \in C^\infty(M, [0,1])$ satisfies $\chi \equiv 1$ near Δ and has support close enough to Δ, we have

$$\chi e^{(t+\varepsilon)\widetilde{X}} \chi = \sum_a \chi_a e^{(t+\varepsilon)\widetilde{X}} \chi_a.$$

Let $c = \min_{a \in \mathrm{Crit}(f)} c_a$. For $\mathrm{Re}(s) > -c$,

$$G_{\chi,\varepsilon,s} = \int_0^\infty e^{-ts} \chi e^{(t+\varepsilon)\widetilde{X}} (\mathrm{Id} - \widetilde{\Pi}_-) \chi \, \mathrm{d}t$$

defines an operator $\Omega^\bullet(M) \to \mathcal{D}'^\bullet(M)$, whose Schwartz kernel $\mathcal{G}_{\chi,\varepsilon,s}$ is locally bounded in $\mathcal{D}'^n_\Gamma(M \times M)$ in the region $\{\mathrm{Re}(s) > -c\}$, where $\Gamma = \bigcup_{a \in \mathrm{Crit}(f)} \Gamma_a$.

Now for $\mathrm{Re}(s) \gg 0$, we have as a consequence of the Hille–Yosida theorem applied to $\mathcal{L}_{-\widetilde{X}}$ acting on suitable anisotropic spaces [Dang and Rivière 2021, 3.2.3]:

$$(\mathcal{L}_{-\widetilde{X}} + s)^{-1} = \int_0^\infty e^{-ts} e^{t\widetilde{X}} \, \mathrm{d}t : \Omega^\bullet(M) \to \mathcal{D}'^\bullet(M).$$

Therefore for $\mathrm{Re}(s) \gg 0$, it holds

$$G_{\chi,\varepsilon,s} = \chi (\mathcal{L}_{-\widetilde{X}} + s)^{-1} (\mathrm{Id} - \widetilde{\Pi}_-) e^{\varepsilon \widetilde{X}} \chi.$$

Since both members are holomorphic in the region $\{\mathrm{Re}(s) > -c\}$ and coincide for $\mathrm{Re}(s) \gg 0$, they coincide in the region $\mathrm{Re}(s) > -c$. We may compute, for $\mathrm{Re}(s) \gg 0$,

$$\mathrm{tr}_\mathrm{s}^\flat\big(\alpha \iota_{-\widetilde{X}} (\mathcal{L}_{-\widetilde{X}} + s)^{-1} (\mathrm{Id} - \widetilde{\Pi}_-) e^{\varepsilon \mathcal{L}_{\widetilde{X}}}\big) = \mathrm{tr}_\mathrm{s}^\flat \alpha \iota_{-\widetilde{X}} G_{\chi,\varepsilon,s} = \int_0^\infty e^{-ts} \mathrm{tr}_\mathrm{s}^\flat\big(\alpha \iota_{-\widetilde{X}} e^{(t+\varepsilon)\widetilde{X}} (\mathrm{Id} - \widetilde{\Pi}_-)\big) \, \mathrm{d}t.$$

By holomorphy this holds true for any s such that $\mathrm{Re}(s) > -c$, which concludes the proof. □

As a consequence, we have the formula

$$\mathrm{tr}_{\mathrm{s}, \widetilde{C}^\bullet_-} \widetilde{\Pi}_- \alpha \tilde{k}_- = \mathrm{tr}_\mathrm{s}^\flat \alpha \widetilde{K} e^{-\varepsilon \mathcal{L}^\nabla_{-\widetilde{X}}}. \tag{11-5}$$

Indeed, since $\mathcal{L}^\nabla_{-\widetilde{X}} \widetilde{\Pi}_- = 0$, we have $\widetilde{\Pi}_- e^{-\varepsilon \mathcal{L}^\nabla_{-\widetilde{X}}} = \widetilde{\Pi}_-$. Moreover, since the trace of finite-rank operators coincides with the flat trace, we have $\mathrm{tr}_{\mathrm{s}, \widetilde{C}^\bullet_-} \widetilde{\Pi}_- \alpha \tilde{k}_- = \mathrm{tr}_{\mathrm{s}, \widetilde{C}^\bullet_-} \widetilde{\Pi}_- \alpha \tilde{k}_- e^{-\varepsilon \mathcal{L}^\nabla_{-\widetilde{X}}} = \mathrm{tr}_\mathrm{s}^\flat \alpha \tilde{k}_- \widetilde{\Pi}_- e^{-\varepsilon \mathcal{L}^\nabla_{-\widetilde{X}}}$. Therefore we obtain with Proposition 11.2

$$\mathrm{tr}_{\mathrm{s}, \widetilde{C}^\bullet} \widetilde{\Pi}_- \alpha \tilde{k}_- = \mathrm{tr}_\mathrm{s}^\flat \alpha \iota_{-\widetilde{X}} \widetilde{Y} (\mathrm{Id} - \widetilde{\Pi}_-) e^{-\varepsilon \mathcal{L}^\nabla_{-\widetilde{X}}} + \mathrm{tr}_\mathrm{s}^\flat \alpha \tilde{k}_- \widetilde{\Pi}_- e^{-\varepsilon \mathcal{L}^\nabla_{-\widetilde{X}}},$$

which gives (11-5).

11.5. ***Proof of Proposition 11.1.*** Note that we have by (11-1)

$$\tau_\vartheta(\rho(z)) = \tau_\vartheta(\nabla(z)), \qquad \tau_{\mathfrak{e},\mathfrak{o}}(\rho(z)) = \tau_{\mathfrak{e},\mathfrak{o}}(\nabla(z)).$$

We will set $f(z) = \tau_\vartheta(\nabla(z))/\tau_{\mathfrak{e},\mathfrak{o}}(\nabla(z))$ for simplicity. Now we apply Propositions 8.1 and 10.1 to obtain that $z \mapsto f(z)$ is real differentiable (since $z \mapsto \nabla(z)$ is); moreover it is complex differentiable at $z = 0$ by (11-2) and for $\varepsilon > 0$ small enough we have

$$\frac{\mathrm{d}}{\mathrm{d}z}\bigg|_{z=0} \log f(z) = -\mathrm{tr}_\mathrm{s}^\flat \alpha K e^{-\varepsilon \mathcal{L}_X^\nabla} + \mathrm{tr}_\mathrm{s}^\flat \alpha \widetilde{K} e^{-\varepsilon \mathcal{L}_{-\widetilde{X}}^\nabla} + \langle \mathrm{tr}\, \alpha, e\rangle, \tag{11-6}$$

where we used (11-5).

Lemma 11.4. *It holds* $\mathrm{tr}_\mathrm{s}^\flat[\alpha(K e^{-\varepsilon \mathcal{L}_X^\nabla} - \widetilde{K} e^{-\varepsilon \mathcal{L}_{-\widetilde{X}}^\nabla})] = \mathrm{tr}_\mathrm{s}^\flat \alpha R_\varepsilon$, *where* R_ε *is the interpolator at time* ε *defined in Section 9.3 for the pair of vector fields* $(-\widetilde{X}, X)$.

Let us admit the lemma for now (we shall prove it later). The identity $[\nabla, \alpha] = 0$ also implies that $\mathrm{d}\,\mathrm{tr}\,\alpha = \mathrm{tr}\, \nabla^{E \otimes E^\vee} \alpha = \mathrm{tr}[\nabla, \alpha] = 0$. As a consequence we can apply (9-5) to obtain

$$\mathrm{tr}_\mathrm{s}^\flat \alpha R_\varepsilon = \langle \mathrm{tr}\,\alpha, \mathrm{cs}(-\widetilde{X}, X)\rangle.$$

Now note that $\partial(-\mathrm{cs}(-\widetilde{X}, X) + e) = -(\mathrm{div}(X) - \mathrm{div}(-\widetilde{X})) + \mathrm{div}(\widetilde{X}) = 0$ by (9-1) and (10-2) since X is nonsingular. Therefore we obtain

$$\frac{\mathrm{d}}{\mathrm{d}z}\bigg|_{z=0} \log f(z) = \langle \mathrm{tr}\,\alpha, h\rangle,$$

where $h = [-\mathrm{cs}(-\widetilde{X}, X) + e] \in H_1(M, \mathbb{Z})$. Finally, let us note that by (4-4),

$$\frac{\mathrm{d}}{\mathrm{d}z}\bigg|_{z=0} \log \det \rho(z)(h) = -\langle \mathrm{tr}\,\alpha, h\rangle,$$

since $\rho(z) = \rho_{\nabla(z)}$. Therefore the proposition is proved for $z = 0$. However the same argument holds for every z close enough to 0, which gives the conclusion of Proposition 11.1. It remains to prove Lemma 11.4.

Proof of Lemma 11.4. Using the identities (8-6), (9-4), (11-3) and (11-4) one can see that

$$[\nabla, \alpha(K e^{-\varepsilon \mathcal{L}_X^\nabla} - \widetilde{K} e^{-\varepsilon \mathcal{L}_{-\widetilde{X}}^\nabla} + R_\varepsilon)] = 0. \tag{11-7}$$

Next, it is a general fact that, for a finite-dimensional acyclic cochain complex $(C^\bullet, \partial)$ and an operator $b : C^\bullet \to C^\bullet$ of order zero such that $[\partial, b] = 0$, it holds $\mathrm{tr}_{\mathrm{s}, C^\bullet}\, b = 0$. Indeed, if $k : C^\bullet \to C^\bullet$ satisfies $k\partial + \partial k = \mathrm{Id}_{C^\bullet}$, we have $[\partial, kb] = [\partial, k]b = b$ since $[\partial, b] = b\partial - \partial b = 0$. Thus b is a supercommutator and its supertrace vanishes. Here (11-7) shows that we are in the same situation, with an infinite-dimensional complex; we will use Hodge theory to obtain a cochain contraction J (that takes the role of k in the above argument), and such that the composition $J B_\varepsilon$, where

$$B_\varepsilon = \alpha(K e^{-\varepsilon \mathcal{L}_X^\nabla} - \widetilde{K} e^{-\varepsilon \mathcal{L}_{-\widetilde{X}}^\nabla} - R_\varepsilon),$$

is well-defined. Let

$$\Delta = \nabla\nabla^\star + \nabla^\star\nabla : \Omega^\bullet(M, E) \to \Omega^\bullet(M, E)$$

be the Hodge–Laplace operator induced by any metric on M and any Hermitian product on E. Because ∇ is acyclic, Δ is invertible and Hodge theory gives that its inverse Δ^{-1} is a pseudodifferential operator of order -2. Define

$$J = \nabla^\star \Delta^{-1} : \mathcal{D}'^\bullet(M, E) \to \mathcal{D}'^{\bullet-1}(M, E).$$

We have of course

$$[\nabla, J] = \nabla J + J\nabla = \mathrm{Id}_{\mathcal{D}'^\bullet(M,E)}\,. \tag{11-8}$$

As above, this gives $B_\varepsilon = [\nabla, JB_\varepsilon]$. Moreover, it follows from wavefront composition [Hörmander 1990, Theorem 8.2.14] that $\mathrm{WF}(G_\varepsilon) \cap N^*\Delta = \varnothing$. Therefore, the operators ∇, G_ε satisfy the assumptions of Proposition 4.1 which gives $\mathrm{tr}_s^\flat B_\varepsilon = \mathrm{tr}_s^\flat[\nabla, G_\varepsilon] = 0$, which concludes the proof of Lemma 11.4. □

11.6. *Proof of Theorems 6 and 7.* By Hartogs' theorem and Proposition 11.1, we have that the map

$$\rho \mapsto \frac{\tau_\vartheta(\rho)}{\tau_{\mathfrak{e},\mathfrak{o}}(\rho)} \langle \det \rho, h\rangle \tag{11-9}$$

is locally constant on $\mathrm{Rep}_{\mathrm{ac}}(M, d) \setminus \Sigma(M, d)$.

Moreover, we can reproduce all the arguments we made in the continuous category to obtain that $\rho \mapsto \tau_\vartheta(\rho)/\tau_{\mathfrak{e},\mathfrak{o}}(\rho)$ is actually continuous on $\mathrm{Rep}_{\mathrm{ac}}(M, d)$. Because $\mathrm{Rep}_{\mathrm{ac}}(M, d) \setminus \Sigma(M, d)$ is open and dense in $\mathrm{Rep}_{\mathrm{ac}}(M, d)$, we get that the map (11-9) is locally constant on $\mathrm{Rep}_{\mathrm{ac}}(M, d)$.

By [Farber and Turaev 2000, p. 211] we have, if $\mathfrak{e}'$ is another Euler structure, then $\tau_{\mathfrak{e}',\mathfrak{o}}(\rho) = \langle \det \rho, \mathfrak{e}' - \mathfrak{e}\rangle \tau_{\mathfrak{e},\mathfrak{o}}(\rho)$. As a consequence, if we set $\mathfrak{e}_\vartheta = [-X, 0]$, which defines an Euler structure since X is nonsingular (see Section 9.2), we have $\mathfrak{e} - \mathfrak{e}_\vartheta = h$ and we obtain that $\rho \mapsto \tau_\vartheta(\rho)/\tau_{\mathfrak{e}_\vartheta,\mathfrak{o}}(\rho)$ is locally constant on $\mathrm{Rep}_{\mathrm{ac}}(M, d)$.

Now let η be another contact form inducing an Anosov Reeb flow and denote by X_η its Reeb flow. Then if $\mathfrak{e}_\eta = [-X_\eta, 0]$, we have

$$\mathfrak{e}_\eta - \mathfrak{e}_\vartheta = \mathrm{cs}(X, X_\eta)$$

by definition. Therefore $\tau_{\mathfrak{e}_\vartheta,\mathfrak{o}}(\rho) = \tau_{\mathfrak{e}_\eta,\mathfrak{o}}(\rho)\langle \det \rho, \mathfrak{e}_\vartheta - \mathfrak{e}_\eta\rangle = \tau_{\mathfrak{e}_\eta,\mathfrak{o}}(\rho)\langle \det \rho, \mathrm{cs}(X_\eta, X)\rangle$ and we obtain that

$$\rho \mapsto \frac{\tau_\vartheta(\rho)}{\tau_\eta(\rho)} \langle \det \rho, \mathrm{cs}(X, X_\eta)\rangle$$

is locally constant on $\mathrm{Rep}_{\mathrm{ac}}(M, d)$. By Theorem 9 we thus obtain Theorem 7.

Finally assume that $\dim M = 3$ and $b_1(M) \neq 0$. Take $\mathcal{R}$ a connected component of $\mathrm{Rep}_{\mathrm{ac}}(M, d)$ and assume that it contains an acyclic and unitary representation ρ_0. We invoke [Dang et al. 2020, Theorem 1] and the Cheeger–Müller theorem [Cheeger 1979; Müller 1978] to obtain that $0 \notin \mathrm{Res}(\mathcal{L}_X^{\nabla_{\rho_0}})$ and

$$|\tau_\vartheta(\rho_0)| = |\zeta_{X,\nabla_{\rho_0}}(0)|^{-1} = \tau_{\mathrm{RS}}(\rho_0),$$

where the first equality comes from (6-10) (we have $q = 1$ since $\dim M = 3$) and $\tau_{\mathrm{RS}}(\rho_0)$ is the Ray–Singer torsion of (M, ρ_0) [1971]. On the other hand, we have by [Farber and Turaev 2000, Theorem 10.2] that $\tau_{\mathrm{RS}}(\rho_0) = |\tau_{\mathfrak{e},\mathfrak{o}}(\rho_0)|$ since ρ_0 is unitary. Therefore the map $\rho \mapsto \tau_\vartheta(\rho)/\tau_{\mathfrak{e}_\vartheta,\mathfrak{o}}(\rho)$ is of modulus 1 on $\mathcal{R}$. This concludes the proof of Theorem 6.

Appendix A: Projectors of finite rank

A.1. ***Traces on variable finite-dimensional spaces.*** In what follows, we consider two Hilbert spaces $\mathcal{G} \subset \mathcal{H}$, the inclusion being dense and continuous. We will denote by $\mathcal{L}(\mathcal{H}, \mathcal{G})$ the space of bounded linear operators $\mathcal{H} \to \mathcal{G}$ endowed with the operator norm. Let $\delta > 0$ and Π_t, $|t| \leqslant \delta$, be a family of finite-rank projectors on $\mathcal{H}$ such that $\operatorname{ran} \Pi_t \subset \mathcal{G}$. Assume that $t \mapsto \Pi_t$ is differentiable at $t = 0$ as a family of bounded operators $\mathcal{H} \to \mathcal{G}$, that is,

$$\Pi_t = \Pi + tP + o_{\mathcal{H}\to\mathcal{G}}(t) \tag{A-1}$$

for some $P \in \mathcal{L}(\mathcal{H}, \mathcal{G})$, where $\Pi = \Pi_0$. Let $C_t = \operatorname{ran} \Pi_t$ and $C = \operatorname{ran} \Pi$. Note that by continuity, $\Pi_t|_C : C \to C_t$ is invertible for $|t|$ small enough; we denote by $Q_t : C_t \to C$ its inverse.

Lemma A.1. *We have*

(i) $P = \Pi P + P\Pi$,

(ii) $Q_t \Pi_t = \Pi \Pi_t + o_{\mathcal{H}\to\mathcal{G}}(t)$.

Proof. Using (A-1) and $\Pi_t^2 = \Pi_t$ we obtain (i). This implies

$$\begin{aligned}\Pi_t \circ \Pi \circ \Pi_t &= (\Pi + tP + o(t))\Pi(\Pi + tP + o(t)) \\ &= \Pi + t(P\Pi + \Pi P) + o(t) = \Pi + tP + o(t) = \Pi_t + o(t),\end{aligned}$$

where all the $o(t)$ are taken in $\mathcal{L}(\mathcal{H}, \mathcal{G})$. Therefore $Q_t \circ \Pi_t \circ \Pi \circ \Pi_t = Q_t \Pi_t + o(t)$. Since $Q_t \circ \Pi_t \circ \Pi = \Pi$ by definition, one obtains

$$Q_t \circ \Pi_t = \Pi \circ \Pi_t + o(t),$$

which concludes the proof of the lemma. □

Lemma A.2. *Let* A_t, $|t| \leqslant \delta$, *be a* $\mathcal{C}^1$ *family of bounded operators* $\mathcal{G} \to \mathcal{H}$ *such that* A_t *commutes with* Π_t *for every* t. *Let* $A = A_0$. *Then* $t \mapsto \operatorname{tr}_{C_t}(A_t)$ *is real differentiable at* $t = 0$ *and*

$$\left.\frac{d}{dt}\right|_{t=0} \operatorname{tr}_{C_t}(A_t) = \operatorname{tr}_C(\Pi \dot{A}),$$

where $\dot{A}_t = (d/dt)A_t$. *If moreover* A *is invertible on* C, *we have*

$$\left.\frac{d}{dt}\right|_{t=0} \log \det_{C_t}(A_t) = \operatorname{tr}_C(\Pi \dot{A}(A|_C)^{-1}).$$

Proof. We start from

$$\operatorname{tr}_{C_t}(A_t) = \operatorname{tr}_C(Q_t A_t \Pi_t).$$

Now since A_t commutes with Π_t we have by the second part of Lemma A.1

$$\begin{aligned}Q_t A_t \Pi_t \Pi &= \Pi \Pi_t A_t \Pi + o_{C\to C}(t) \\ &= \Pi A \Pi + t\Pi(\dot{A} + PA\Pi + \Pi A P)\Pi + o_{C\to C}(t).\end{aligned}$$

But now the first part of Lemma A.1 gives $\Pi P \Pi = 0$. We therefore obtain, because A and Π commute,

$$Q_t A_t \Pi_t \Pi = \Pi A \Pi + t\Pi \dot{A} \Pi + o_{C\to C}(t), \tag{A-2}$$

which concludes the proof. □

A.2. ***Gain of regularity.*** Assume that we are given four Hilbert spaces $\mathcal{E} \subset \mathcal{F} \subset \mathcal{G} \subset \mathcal{H}$ with continuous and dense inclusions. Let Π_{t}, $|\mathrm{t}| < \delta$, be a family of finite-rank projectors on $\mathcal{H}$ which is differentiable at $\mathrm{t} = 0$ as family of bounded operators $\mathcal{G} \to \mathcal{H}$ (note that this differs from the last subsection where we had $\mathcal{H} \to \mathcal{G}$ instead), that is,

$$\Pi_{\mathrm{t}} = \Pi + \mathrm{t}P + o_{\mathcal{G}\to\mathcal{H}}(\mathrm{t}) \tag{A-3}$$

for some $P \in \mathcal{L}(\mathcal{G}, \mathcal{H})$. We will write $C_{\mathrm{t}} = \mathrm{ran}(\Pi_{\mathrm{t}}) \subset \mathcal{H}$ and $C = \mathrm{ran}(\Pi)$.

Lemma A.3. *Under the above assumptions, assume that Π_{t} is bounded $\mathcal{E} \to \mathcal{F}$ and that Π_{t} is differentiable at $\mathrm{t} = 0$ as a family of $\mathcal{L}(\mathcal{E}, \mathcal{F})$. Assume also that* $\mathrm{rank}\, \Pi_{\mathrm{t}}$ *does not depend on* t*. Then P is actually bounded $\mathcal{G} \to \mathcal{F}$ and*

$$\Pi_{\mathrm{t}} = \Pi + \mathrm{t}P + o_{\mathcal{G}\to\mathcal{F}}(\mathrm{t}).$$

Proof. Because $\mathcal{E}$ is dense in $\mathcal{H}$ we know that $C \subset \mathcal{F}$. There exists $\varphi^1, \dots, \varphi^m \in \mathcal{E}$ such that $\varphi^1_{\mathrm{t}}, \dots, \varphi^m_{\mathrm{t}}$ is a basis of C_{t} for t small enough, where we set $\varphi^j_{\mathrm{t}} = \Pi_{\mathrm{t}}(\varphi^j) \in \mathcal{F}$. Let $\tilde{\varphi}^j_{\mathrm{t}} = \Pi(\varphi^j_{\mathrm{t}}) \in C$. The family $\mathrm{t} \mapsto \tilde{\varphi}^j_{\mathrm{t}} \in C$ is differentiable at $\mathrm{t} = 0$. Let $\nu^1_{\mathrm{t}}, \dots, \nu^m_{\mathrm{t}} \in C^*$ be the dual basis of $\tilde{\varphi}^1_{\mathrm{t}}, \dots, \tilde{\varphi}^m_{\mathrm{t}}$. Because C is finite-dimensional, Π is actually bounded $\mathcal{H} \to \mathcal{F}$. As a consequence the map

$$\mathrm{t} \mapsto \ell^j_{\mathrm{t}} = \nu^j_{\mathrm{t}} \circ \Pi \circ \Pi_{\mathrm{t}} \in \mathcal{G}'$$

is differentiable at $\mathrm{t} = 0$. Noting that

$$\Pi_{\mathrm{t}} = \sum_{j=1}^{m} \varphi^j_{\mathrm{t}} \otimes \ell^j_{\mathrm{t}} : \mathcal{G} \to \mathcal{F},$$

we finally obtain that $\mathrm{t} \mapsto \Pi_{\mathrm{t}} \in \mathcal{L}(\mathcal{G}, \mathcal{F})$ is differentiable at $\mathrm{t} = 0$. □

Appendix B: Continuity of the Pollicott–Ruelle spectrum

In this appendix, we describe the spaces used in Sections 7 and 8; everything in this appendix is more or less folklore, but we chose to provide a short summary of the results that we use in the main body of the article, because we did not find any satisfying presentation in the literature. In what follows, M is a compact manifold, (E, ∇) is a flat vector bundle on M and X_0 is a vector field on M generating an Anosov flow; see Section 5.1. We denote by $T^*M = E^*_{u,0} \oplus E^*_{s,0} \oplus E^*_{0,0}$ its Anosov decomposition of T^*M.

B.1. ***Bonthonneau's uniform weight function.*** We state here [Bonthonneau 2020, Lemma 3]. This gives us an escape function having uniform good properties for a family of vector fields. A consequence is that one can define some uniform anisotropic Sobolev spaces on which each vector field of the family has good spectral properties. In what follows, $|\cdot|$ is a smooth norm on T^*M.

Lemma B.1. *There exist conical neighborhoods N_u and N_s of $E^*_{u,0}$ and $E^*_{s,0}$, some constants $C, \beta, T, \eta > 0$, and a weight function $m \in C^\infty(T^*M, [0, 1])$ such that the following hold. Let X be any vector field satisfying $\|X - X_0\|_{C^1} < \eta$, and denote by Φ^t its induced flow on T^*M and by E^*_u and E^*_s its (dual) unstable and stable bundles. Then:*

(1) *For $E_\bullet^* \subset N_\bullet$, for $\bullet = s, u$ and for any $t > 0$, $\xi_u \in E_u^*$ and $\xi_s \in E_s^*$, one has*

$$|\Phi^t(\xi_u)| \geqslant \frac{1}{C} e^{\beta t} |\xi_u|, \quad |\Phi^{-t}(\xi_s)| \geqslant \frac{1}{C} e^{\beta t} |\xi_s|.$$

(2) *For every $t \geqslant T$ it holds*

$$\Phi^t(\complement N_s \cap X^\perp) \subset N_u, \quad \Phi^{-t}(\complement N_u \cap X^\perp) \subset N_s,$$

where $X^\perp = \{\xi \in T^*M : \xi \cdot X = 0\}$.

(3) *If $\boldsymbol{X}$ is the Lie derivative induced by Φ^t, then*

$$m \equiv 1 \ \text{near} \ N_s, \quad m \equiv -1 \ \text{near} \ N_u, \quad \boldsymbol{X}.m \geqslant 0.$$

B.2. ***Anisotropic Sobolev spaces.*** Take the weight function m of Lemma B.1. Define the escape function g by

$$g(x, \xi) = m(x, \xi) \log(1 + |\xi|), \quad (x, \xi) \in T^*M.$$

We set $G = \mathrm{Op}(g) \in \Psi^{0+}(M)$ for any quantization procedure Op. Then by [Zworski 2012, §8.3, 9.3, 14.2] we have $\exp(\pm\mu G) \in \Psi^{\mu+}(M)$ for any $\mu > 0$. For any $\mu > 0$ and $j \in \mathbb{Z}$ we define the spaces

$$\mathcal{H}^\bullet_{\mu G, j} = \exp(-\mu G) H^j(M, \Lambda^\bullet \otimes E) \subset \mathcal{D}'^\bullet(M, E),$$

where $H^j(M, \Lambda^\bullet \otimes E)$ is the usual Sobolev space of order j on M with values in the bundle $\Lambda^\bullet \otimes E$. Note that any pseudodifferential operator of order m is bounded $\mathcal{H}^\bullet_{\mu G, j} \to \mathcal{H}^\bullet_{\mu G, j-m}$ for any μ, m, j.

B.3. ***Uniform parametrices.*** Let us consider a smooth family of vector fields X_t, $|t| < \varepsilon$, perturbing X_0. For any $c, \rho > 0$ we will set

$$\Omega(c, \rho) = \{\mathrm{Re}(s) > c\} \cup \{|s| \leqslant \rho\} \subset \mathbb{C}.$$

The spaces defined in the last subsection yield a uniform version of [Dyatlov and Zworski 2016, Proposition 3.4], as follows.

Proposition B.2 [Bonthonneau 2020, Lemma 9]. *Let Q be a pseudodifferential operator microlocally supported near the zero section in T^*M and elliptic there. There exists $c, \varepsilon_0 > 0$ such that, for any $\rho > 0$ and $J \in \mathbb{N}$, there is $\mu_0, h_0 > 0$ such that the following holds. For each $\mu \geqslant \mu_0$, $0 < h < h_0$, $j \in \mathbb{Z}$ such that $|j| \leqslant J$ and $s \in \Omega(c, \rho)$ the operator*

$$\mathcal{L}^\nabla_{X_t} - h^{-1} Q + s : \mathcal{H}^\bullet_{\mu G, j+1} \to \mathcal{H}^\bullet_{\mu G, j}$$

is invertible for $|t| \leqslant \varepsilon_0$ and the inverse is bounded $\mathcal{H}^\bullet_{\mu G, j} \to \mathcal{H}^\bullet_{\mu G, j}$ independently of t.

B.4. ***Continuity of the Pollicott–Ruelle spectrum.*** We fix ρ, $J \geqslant 4$ and μ_0, μ, h_0, h, j as in Proposition B.2. We first observe that

$$(\mathcal{L}^\nabla_{X_t} + s)(\mathcal{L}^\nabla_{X_t} - h^{-1} Q + s)^{-1} = \mathrm{Id} + h^{-1} Q (\mathcal{L}^\nabla_{X_t} - h^{-1} Q + s)^{-1}. \tag{B-1}$$

Since Q is supported near 0 in T^*M, it is smoothing and thus trace class on any $\mathcal{H}^\bullet_{\mu G,j}$. By analytic Fredholm theory, the family $s \mapsto K(\mathrm{t}, s) = h^{-1}Q(\mathcal{L}^\nabla_{X_\mathrm{t}} - h^{-1}Q + s)^{-1}$ is a holomorphic family of trace class operators on $\mathcal{H}^\bullet_{\mu G,j}$ in the region $\Omega(c, \rho)$. We can therefore consider the Fredholm determinant

$$D(\mathrm{t}, s) = \det_{\mathcal{H}^\bullet_{\mu G,j}}(\mathrm{Id} + K(\mathrm{t}, s)).$$

It follows from [Simon 2005, Corollary 2.5] that for each t, $s \mapsto D(\mathrm{t}, s)$ is holomorphic on $\Omega(c, \rho)$. Moreover (B-1) shows that its zeros coincide, on $\Omega(c, \rho)$, with the Pollicott–Ruelle resonances of $\mathcal{L}^\nabla_{X_\mathrm{t}}$. In addition, we have, for any $s \in \Omega(c, \rho)$,

$$(\mathcal{L}^\nabla_{X_\mathrm{t}} - h^{-1}Q + s)^{-1} - (\mathcal{L}^\nabla_{X_{\mathrm{t}'}} - h^{-1}Q + s)^{-1} = -(\mathcal{L}^\nabla_{X_\mathrm{t}} - h^{-1}Q + s)^{-1}(\mathcal{L}^\nabla_{X_\mathrm{t}} - \mathcal{L}^\nabla_{X_{\mathrm{t}'}})(\mathcal{L}^\nabla_{X_{\mathrm{t}'}} - h^{-1}Q + s)^{-1}. \quad \text{(B-2)}$$

We have

$$\frac{\mathcal{L}^\nabla_{X_\mathrm{t}} - \mathcal{L}^\nabla_{X_{\mathrm{t}'}}}{\mathrm{t} - \mathrm{t}'} \xrightarrow[\mathrm{t} \to \mathrm{t}']{} \mathcal{L}^\nabla_{\dot{X}_\mathrm{t}} \quad \text{in } \mathcal{L}(\mathcal{H}^\bullet_{\mu G,j+1}, \mathcal{H}^\bullet_{\mu G,j}), \quad \text{(B-3)}$$

where $\dot{X}_\mathrm{t} = (\mathrm{d}/\mathrm{dt})X_\mathrm{t}$ and $\mathcal{L}(\mathcal{H}^\bullet_{\mu G,j+1}, \mathcal{H}^\bullet_{\mu G,j})$ is the space of bounded linear operators $\mathcal{H}^\bullet_{\mu G,j+1} \to \mathcal{H}^\bullet_{\mu G,j}$ endowed with the operator norm. We therefore obtain by Proposition B.2 and because Q is smoothing (and thus trace class $\mathcal{H}^\bullet_{\mu G,j} \to \mathcal{H}^\bullet_{\mu G,j'}$ for any μ, j, j') that $K(\mathrm{t}', s) \to K(\mathrm{t}, s)$ as $\mathrm{t}' \to \mathrm{t}$ in $\mathcal{L}^1(\mathcal{H}^\bullet_{\mu G,0})$ locally uniformly in s, where $\mathcal{L}^1(\mathcal{H}^\bullet_{\mu G,0})$ is the space of trace class operators on $\mathcal{H}^\bullet_{\mu G,0}$ endowed with its usual norm. As a consequence, we obtain with [Simon 2005, Corollary 2.5]

$$D(\mathrm{t}, s) \in \mathcal{C}^0\big([-\varepsilon_0, \varepsilon_0]_\mathrm{t}, \mathrm{Hol}(\Omega(c, \rho)_s)\big). \quad \text{(B-4)}$$

B.5. ***Regularity of the resolvent.*** Let $\mathcal{Z}$ be an open set of $\mathbb{C}$ whose closure is contained in the interior of $\Omega(c, \rho)$. We assume that $\overline{\mathcal{Z}} \cap \mathrm{Res}(\mathcal{L}^\nabla_{X_0}) = \varnothing$. Up to taking ε_0 smaller, Rouché's theorem and (B-4) imply that there exists $\delta > 0$ such that $\mathrm{dist}(\mathcal{Z}, \mathrm{Res}(\mathcal{L}^\nabla_{X_\mathrm{t}})) > \delta$ for any $|\mathrm{t}| \leqslant \varepsilon_0$. As a consequence, we obtain that, for every $|j| \leqslant J$, the map $(\mathcal{L}^\nabla_{X_\mathrm{t}} + s)^{-1} : \mathcal{H}^\bullet_{\mu G,j} \to \mathcal{H}^\bullet_{\mu G,j}$ is bounded independently of $(\mathrm{t}, s) \in [-\varepsilon_0, \varepsilon_0] \times \mathcal{Z}$. Noting that

$$\frac{(\mathcal{L}^\nabla_{X_\mathrm{t}} + s)^{-1} - (\mathcal{L}^\nabla_{X_{\mathrm{t}'}} + s)^{-1}}{\mathrm{t} - \mathrm{t}'} = -(\mathcal{L}^\nabla_{X_\mathrm{t}} + s)^{-1} \frac{\mathcal{L}^\nabla_{X_\mathrm{t}} - \mathcal{L}^\nabla_{X_{\mathrm{t}'}}}{\mathrm{t} - \mathrm{t}'} (\mathcal{L}^\nabla_{X_{\mathrm{t}'}} + s)^{-1}, \quad \text{(B-5)}$$

we obtain by (B-3) that $\mathrm{t}' \mapsto (\mathcal{L}^\nabla_{X_{\mathrm{t}'}} + s)^{-1}$ is continuous in $\mathcal{L}(\mathcal{H}^\bullet_{\mu G,j+1}, \mathcal{H}^\bullet_{\mu G,j})$. Therefore, applying (B-5) again, we get that

$$(\mathcal{L}^\nabla_{X_\mathrm{t}} + s)^{-1} \in \mathcal{C}^1\big([-\varepsilon_0, \varepsilon_0]_\mathrm{t}, \mathrm{Hol}(\mathcal{Z}_s, \mathcal{L}(\mathcal{H}^\bullet_{\mu G,j+1}, \mathcal{H}^\bullet_{\mu G,j-2}))\big). \quad \text{(B-6)}$$

Note that here we need $|j-2|, |j+1| \leqslant J$.

B.6. ***Regularity of the spectral projectors.*** Let $0 < \lambda < 1$ such that $\{|s| = \lambda\} \cap \mathrm{Res}(\mathcal{L}^\nabla_{X_0}) = \varnothing$. Applying the last subsection with $\mathcal{Z} = \{|s| = \lambda\}$, we get $\{|s| = \lambda\} \cap \mathrm{Res}(\mathcal{L}^\nabla_{X_\mathrm{t}}) = \varnothing$ for any $|\mathrm{t}| \leqslant \varepsilon_0$. We can therefore define for those t

$$\Pi_\mathrm{t} = \frac{1}{2\pi i} \int_{|s|=\lambda} (\mathcal{L}^\nabla_{X_\mathrm{t}} + s)^{-1} \, \mathrm{d}s : \mathcal{H}^\bullet_{\mu G,j} \to \mathcal{H}^\bullet_{\mu G,j}.$$

Then (B-6) gives that $\Pi_{\mathrm{t}} \in \mathcal{C}^1([-\varepsilon_0, \varepsilon_0]_{\mathrm{t}}, \mathcal{Z}_s, \mathcal{L}(\mathcal{H}^\bullet_{\mu G, j+1}, \mathcal{H}^\bullet_{\mu G, j-2}))$. This is true for $j = 3$ and $j = -1$ because $J \geqslant 4$. Moreover by Rouché's theorem, the number m of zeros of $s \mapsto D(\mathrm{t}, s)$ does not depend on t. Noting that

$$\partial_s K(\mathrm{t}, s)(1 + K(\mathrm{t}, s))^{-1} = -K(\mathrm{t}, s)(\mathcal{L}^\nabla_{X_{\mathrm{t}}} - h^{-1}Q + s)^{-1}(1 + K(\mathrm{t}, s))^{-1},$$

we obtain by [Dyatlov and Zworski 2019, Theorem C.11] and the cyclicity of the trace that m is equal to

$$\begin{aligned}\frac{1}{2\pi i}\operatorname{tr}\int_{|s|=\lambda} \partial_s K(\mathrm{t}, s)(1 + K(\mathrm{t}, s))^{-1}\,\mathrm{d}s &= -\frac{1}{2\pi i}\operatorname{tr}\int_{|s|=\lambda} (\mathcal{L}^\nabla_{X_{\mathrm{t}}} - h^{-1}Q + s)^{-1}(1 + K(\mathrm{t}, s))^{-1}K(\mathrm{t}, s)\,\mathrm{d}s \\ &= \frac{1}{2\pi i}\operatorname{tr}\int_{|s|=\lambda} (\mathcal{L}^\nabla_{X_{\mathrm{t}}} - h^{-1}Q + s)^{-1}(1 + K(\mathrm{t}, s))^{-1},\end{aligned}$$

where we used that $s \mapsto (\mathcal{L}^\nabla_{X_{\mathrm{t}}} - h^{-1}Q + s)^{-1}$ is holomorphic on $\{|s| \leqslant \lambda\}$. The last integral is equal to $\operatorname{tr}\Pi_{\mathrm{t}} = \operatorname{rank}\Pi_{\mathrm{t}}$ by (B-1). As a consequence we can apply Lemma A.3 to obtain that

$$\Pi_{\mathrm{t}} \in \mathcal{C}^1\big([-\varepsilon_0, \varepsilon_0]_{\mathrm{t}}, \mathcal{L}(\mathcal{H}^\bullet_{\mu G,0}, \mathcal{H}^\bullet_{\mu G,1})\big). \tag{B-7}$$

Appendix C: The wavefront set of the Morse–Smale resolvent

The purpose of this section is to prove the wavefront bound needed to conclude the proof of Proposition 11.2. For simplicity we prove it for $\widetilde{X}$ instead of $-\widetilde{X}$. We will denote by $\widehat{\Pi}$ the spectral projector (10-4) for the trivial bundle $(\mathbb{C}, \mathrm{d})$. Recall that $\mathcal{D}'_\Gamma(M \times M)$ denotes distributions whose wavefront set is contained in the closed conic set $\Gamma \subset T^\bullet(M \times M)$. A family $(f_t)_{t \geqslant 0}$ of distributions will be $\mathcal{O}_{\mathcal{D}'_\Gamma}(1)$ if it is bounded in $\mathcal{D}'_\Gamma$ in the sense of [Dang 2013, p. 31]. We will need the following:

Lemma C.1. *Let $\varepsilon > 0$ and $a \in \mathrm{Crit}(f)$. There exists $c > 0$, a closed conic set $\Gamma \subset T^*(M \times M)$ with $\Gamma \cap N^*\Delta(T^*M) = \varnothing$ and $\chi \in \mathcal{C}^\infty(M, [0, 1])$ such that $\chi \equiv 1$ near a such that*

$$\mathcal{K}_{\chi, t+\varepsilon} = \mathcal{O}_{\mathcal{D}'^n_\Gamma(M \times M)}(e^{-tc}),$$

where, for $t \geqslant 0$, $\mathcal{K}_{\chi,t}$ is the Schwartz kernel of the operator $\chi e^{-t\mathcal{L}_{\widetilde{X}}}(\mathrm{Id} - \widehat{\Pi})\chi$.

Proof. Because $\widetilde{X}$ is $\mathcal{C}^\infty$-linearizable, we can take $U \subset \mathbb{R}^n$ to be a coordinate patch centered in a so that, in those coordinates, $e^{-t\widetilde{X}}(x) = e^{-tA}(x)$, where A is a matrix whose eigenvalues have nonvanishing real parts. Denoting $(x^1, \ldots, x^n)$ the coordinates of the patch, $\widetilde{X}$ reads

$$\widetilde{X} = \sum_{1 \leqslant i, j \leqslant n} A^j_i x^i \partial_j.$$

We have a decomposition $\mathbb{R}^n = W^u \oplus W^s$ stable by A such that $A|_{W^u}$ (resp. $A|_{W^s}$) have eigenvalues with positive (resp. negative) real parts, $d_{u/s} = \dim W^{u/s}$, this induces a decomposition of the coordinates $x = (x_s, x_u)$. We will denote by $A_u = A|_{W^u} \oplus 0_{W^s}$, $A_s = 0_{W^u} \oplus A|_{W^s}$ and $c > 0$ such that

$$c < \inf_{\lambda \in \mathrm{sp}(A)} |\mathrm{Re}(\lambda)|,$$

where $\mathrm{sp}(A)$ is the spectrum of A.

Let $\chi_1, \chi_2 \in \Omega^\bullet(M)$ such that $\operatorname{supp} \chi_i \subset \operatorname{supp} \chi$ for $i = 1, 2$. For simplicity, we identify e^{-tA} and its action on differential forms and currents given by the pull-back, $\delta^d(x)$ denotes the Dirac δ distribution at $0 \in \mathbb{R}^d$, π_1, π_2 are the projections $M \times M \mapsto M$ on the first and second factors, respectively.

$$\begin{aligned}
\langle \mathcal{K}_{\chi,t}, \pi_1^*\chi_1 \wedge \pi_2^*\chi_2 \rangle &= \langle \chi_2, e^{-tA}(\mathrm{Id} - \widehat{\Pi})\chi_1 \rangle \\
&= \left\langle \chi_2, e^{-tA}\left(\chi_1 - \delta^{d_u}(x_u)\,\mathrm{d}x_u \int_{W^s} \pi_{s,0}^*\chi_1\right)\right\rangle \\
&= \langle e^{tA_s}\chi_2, e^{-tA_u}\chi_1 \rangle - \left(\int_{W^u} \pi_{u,0}^*\chi_2\right)\left(\int_{W^s} \pi_{s,0}^*\chi_1\right) \\
&= \int_0^1 \int_U \partial_\tau (e^{tA_s}\pi_{u,\tau}^*\chi_2 \wedge e^{-tA_u}\pi_{s,\tau}^*\chi_1)\,\mathrm{d}\tau,
\end{aligned}$$

where $\pi_{u,\tau}, \pi_{s,\tau} : U \to U$ are defined by $\pi_{u,\tau}(x_u, x_s) = (x_u, \tau x_s)$ and $\pi_{s,\tau}(x_u, x_s) = (\tau x_u, x_s)$. Now write $\chi_2 = \sum_{|I|=k} \beta_I\, \mathrm{d}x_s^{I_s} \wedge \mathrm{d}x_u^{I_u}$. We have

$$\begin{aligned}
\partial_\tau \pi_{u,\tau}^*\chi_2(x_u, x_s) &= \partial_\tau \sum_I \tau^{|I_s|}\beta_I(x_u, \tau x_s)\,\mathrm{d}x_u^{I_u} \wedge \mathrm{d}x_s^{I_s} \\
&= \sum_I |I_s|\tau^{|I_s|-1}\beta_I(x_u, \tau x_s)\,\mathrm{d}x_u^{I_u} \wedge \mathrm{d}x_s^{I_s} + \sum_I \tau^{|I_s|}(\partial_{x_s}\beta_I)_{(x_u,\tau x_s)}(x_s)\,\mathrm{d}x_u^{I_u} \wedge \mathrm{d}x_s^{I_s}.
\end{aligned}$$

Therefore

$$\partial_\tau e^{tA_s}\pi_{u,\tau}^*\chi_2 = \sum_I \big(|I_s|\tau^{|I_s|-1}\beta_I(x_u, \tau e^{tA_s}x_s) + \tau^{|I_s|}(\partial_{x_s}\beta_I)_{(x_u,\tau x_s)}(e^{tA_s}x_s)\big)e^{tA_s}\,\mathrm{d}x^I.$$

Because $|e^{tA_s}x_s| = \mathcal{O}(e^{-tc})$ and $e^{tA_s}\,\mathrm{d}x^I = \mathcal{O}(e^{-ct|I_s|})$, $I = (I_s, I_u)$ is a multi-index and repeating the same argument for $\partial_\tau e^{-tA_u}\pi_{s,\tau}^*\chi_1$, we obtain the bound

$$\partial_\tau (e^{tA_s}\pi_{u,\tau}^*\chi_2 \wedge e^{-tA_u}\pi_{s,\tau}^*\chi_1) = \mathcal{O}_{\chi_1,\chi_2}(e^{-tc}). \tag{C-1}$$

Replacing χ_1 and χ_2 by $\chi_1 e^{i\langle \xi,\cdot\rangle}$ and $\chi_2 e^{i\langle \eta,\cdot\rangle}$ with $\xi, \eta \in \mathbb{R}^n$, one gets

$$\begin{aligned}
&\langle \mathcal{K}_{\chi,t}, \pi_1^*(\chi_1 e^{i\langle \xi,\cdot\rangle}) \wedge \pi_2^*(\chi_2 e^{i\langle \eta,\cdot\rangle})\rangle \\
&\quad = \int_0^1 \int_U \partial_\tau (e^{tA_s}\pi_{u,\tau}^*\chi_2 \wedge e^{-tA_u}\pi_{s,\tau}^*\chi_1) e^{i\langle e^{tA_s}(x_u,\tau x_s),\eta\rangle} e^{i\langle e^{-tA_u}(\tau x_u,x_s),\xi\rangle}\,\mathrm{d}\tau \\
&\qquad + \int_0^1 \int_U e^{tA_s}\pi_{u,\tau}^*\chi_2 \wedge e^{-tA_u}\pi_{s,\tau}^*\chi_1 \partial_\tau (e^{i\langle e^{tA_s}(x_u,\tau x_s),\eta\rangle} e^{i\langle e^{-tA_u}(\tau x_u,x_s),\xi\rangle})\,\mathrm{d}\tau.
\end{aligned}$$

Setting $g(\tau, x_u, x_s) = e^{i\langle e^{tA_s}(x_u,\tau x_s),\eta\rangle} e^{i\langle e^{-tA_u}(\tau x_u,x_s),\xi\rangle}$, we have

$$\partial_\tau g(\tau, x_u, x_s) = i(\langle e^{tA_s}x_s, \eta_s\rangle + \langle e^{-tA_u}x_u, \xi_u\rangle) g(\tau, x_u, x_s) = \mathcal{O}_{C^\infty(M)}(e^{-tc}),$$

because $|e^{tA_s}x_s|, |e^{-tA_u}x_u| = \mathcal{O}(e^{-tc})$. Repeating the process that led to (C-1) but for derivatives of χ_1, χ_2 as test forms with successive integration by parts, we therefore obtain for any $N \in \mathbb{N}$

$$\begin{aligned}
&|\langle \mathcal{K}_{\chi,t}, \pi_1^*(\chi_1 e^{i\langle \xi_1,\cdot\rangle}) \wedge \pi_2^*(\chi_2 e^{i\langle \xi_2,\cdot\rangle})\rangle| \\
&\qquad \leqslant C_{N,\chi_1,\chi_2} e^{-tc}(1 + |e^{tA_s}\eta_s| + |e^{-tA_u}\xi_u|) \int_0^1 (1 + |\tau e^{tA_s}\eta_s + \xi_s| + |\tau e^{-tA_u}\xi_u + \eta_u|)^{-N}\,\mathrm{d}\tau,
\end{aligned}$$

where $\xi = (\xi_u, \xi_s)$ and $\eta = (\eta_u, \eta_s)$. Now assume (ξ, η) is close to $N^*\Delta(T^*M)$, say

$$\left|\frac{\xi}{|\xi|} + \frac{\eta}{|\eta|}\right| < \nu \quad \text{and} \quad 1-\nu < \frac{|\xi|}{|\eta|} < 1+\nu$$

for some $\nu > 0$. Then we have for any $\tau \in [0, 1]$

$$|\tau e^{tA_s}\eta_s + \xi_s| + |\tau e^{-tA_u}\xi_u + \eta_u| \geqslant (1 - e^{-tc}(1+\nu))(|\xi_s| + |\eta_u|).$$

As a consequence, if $\nu > 0$ is small enough so that $(1+\nu)e^{-(t+\varepsilon)c} < 1$, for every $t \geqslant 0$, we obtain

$$|\langle \mathcal{K}_{\chi,t+\varepsilon}, \pi_1^*(\chi_1 e^{i\langle \xi,\cdot\rangle}) \wedge \pi_2^*(\chi_2 e^{i\langle \eta,\cdot\rangle})\rangle| \leqslant C'_{N,\chi_1,\chi_2}(1+|\xi|+|\eta|)^{-N},$$

which concludes. □

To conclude the proof of Proposition 11.2, we also need to prove Lemma 11.3:

Proof of Lemma 11.3. We proceed by contradiction. Suppose that there is $\mu > 0$ and sequences $x_m \in M$ and $t_m \geqslant \varepsilon$ such that $\operatorname{dist}(x_m, e^{-t_m\widetilde{X}}(x_m)) \to 0$ as $m \to \infty$ and $\operatorname{dist}(x_m, \operatorname{Crit}(f)) \geqslant \mu$. Extracting a subsequence we may assume that $x_m \to x$, $t_m \to \infty$ (indeed if $t_m \to t_\infty < \infty$ then x is a periodic point for $\widetilde{X}$, which does not exist) and, for any m,

$$e^{-t\widetilde{X}}(x_m) \to a \quad \text{and} \quad e^{t\widetilde{X}}(x_m) \to b \quad \text{as } t \to \infty,$$

for some $a, b \in \operatorname{Crit}(f)$. Since the space of broken curves $\bar{\mathcal{L}}(a, b)$ is compact (see [Audin and Damian 2014]), we may assume that the sequence of curves $\gamma_m = \{e^{t\widetilde{X}}(x_m) : t \in \mathbb{R}\}$ converges to a broken curve $\ell = (\ell^1, \dots, \ell^q) \in \bar{\mathcal{L}}(a, b)$, with $\ell^j \in \mathcal{L}(c_{j-1}, c_j)$ for some $c_0, \dots, c_q \in \operatorname{Crit}(f)$, with $c_0 = a$ and $c_q = b$. Because $x_m \to x$, the proof of [Audin and Damian 2014, Theorem 3.2.2] implies $x \in \ell^j$ for some j so that $e^{-t\widetilde{X}}x \to c_{j-1}$ as $t \to \infty$. Therefore replacing x by $e^{-t\widetilde{X}}(x)$ for t big enough, we may assume that x is contained in a Morse chart $\Omega(c_{j-1})$ near c_{j-1}. Then $c_{j-1} \neq a$. Indeed if it was not the case then we would have $e^{-t_m\widetilde{X}}x_m \to a$ as $m \to \infty$ (since x_m would be contained in $\Omega(a) \cap W^u(a)$ for big enough m and $t_m \to \infty$), which is not the case since $\operatorname{dist}(x, \operatorname{Crit}(f)) \geqslant \mu \implies x \neq a$ and $\operatorname{dist}(x_m, e^{-t_m\widetilde{X}}(x_m)) \to 0$ as $m \to \infty$. Therefore the flow line of x_m exists $\Omega(c_{j-1})$ in the past. We therefore obtain, since $e^{-t_m\widetilde{X}}x_m \to x$, that there is $i < j-1$ so that $c_i = c_{j-1}$. This is absurd since the sequence $(\operatorname{ind}_f(c_i))_{i=0,\dots,q}$ is strictly decreasing. □

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