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MEASURE PROPAGATION ALONG A $\mathscr{C}^0$-VECTOR FIELD AND WAVE CONTROLLABILITY ON A ROUGH COMPACT MANIFOLD

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The celebrated Rauch–Taylor/Bardos–Lebeau–Rauch geometric control condition is central in the study of the observability of the wave equation linking this property to high-frequency propagation along geodesics that are the rays of geometric optics. This connection is best understood through the propagation properties of microlocal defect measures that appear as solutions to the wave equation concentrate. For a sufficiently smooth metric this propagation occurs along the bicharacteristic flow. If one considers a merely $\mathscr{C}^1$-metric, this bicharacteristic flow may however not exist. The Hamiltonian vector field is only continuous; bicharacteristics do exist (as integral curves of this continuous vector field) but uniqueness is lost. Here, on a compact manifold without boundary, we consider this low-regularity setting, revisit the geometric control condition, and address the question of support propagation for a measure solution to an ODE with continuous coefficients. This leads to a sufficient condition for the observability and equivalently the exact controllability of the wave equation. Moreover, we investigate the stability of the observability property and the sensitivity of the control process under a perturbation of the metric of regularity as low as Lipschitz.

1. Introduction

The observability property for the wave equation has been intensively studied during the last decades mainly because of its deep connection with the problem of exact controllability. Until the end of the 80s, most of the positive results of observability were established under a (global) geometric assumption, the so-called Γ-condition introduced by J.-L. Lions [1988], essentially based on and well-adapted to a multiplier method. Later, following [Rauch and Taylor 1974], Bardos, Lebeau and Rauch [Bardos et al. 1992] established boundary observability inequalities under a geometric control condition (GCC for short), linking the set on which the control acts and the generalized geodesic flow. Proofs of this result are based on microlocal tools, such as the propagation in phase space of wavefront sets in [Bardos et al. 1992] or

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the propagation of microlocal defect measures in more modern proofs [Burq and Gérard 1997]. For the latter approach, microlocal defect measures originate from the concentration phenomena for sequences of waves if one assumes that observability does not hold. Away from boundaries one obtains

$$ {}^{t}\mathrm{H}_p \mu = 0, \tag{1-1} $$

yielding the transport of the measure μ along the bicharacteristic flow in phase space. This flow is generated by the Hamiltonian vector field H_p associated with the symbol of the wave operator p. However, note that despite their high efficiency and robustness, these methods present the great disadvantage of requiring too much regularity in the coefficients of the wave operator and the geometry. To define the generalized bicharacteristic flow and prove the propagation properties mentioned above a minimal smoothness of the metric and the boundary domain is needed. To our knowledge, the best result, in the context of $\mathscr{C}^2$ metrics, was proven in [Burq 1997a], and barely misses the natural minimal smoothness required to define the geodesic flow ($W^{2,\infty}$) and thus the geometric control condition.

In this context, in the present article, we address the following natural question: how can one derive observability estimate for the wave equation from optimal observation regions in the case of a nonsmooth metric? This problem has already received some attention and answers by E. Zuazua and his collaborators, in [Castro and Zuazua 2002], and more recently in [Fanelli and Zuazua 2015] (see also the result of [Dehman and Ervedoza 2017]). More precisely, in [Castro and Zuazua 2002], the authors prove a lack of observability of waves in highly heterogeneous media, that is, if the density is of low regularity. In [Fanelli and Zuazua 2015], the authors establish observability with coefficients in the Zygmund class and also observability with loss when the coefficients are log-Zygmund or log-Lipschitz. Furthermore, this result is proven sharp since one observes an infinite loss of derivatives for a regularity lower than log-Lipschitz. Note that these analyses are carried out in one space dimension. This calls for the following comments. First, in this simplified framework, for smooth coefficients all the geodesics reach the observability region in uniform time: captive geodesics are not an issue. Second, proofs are based on a sidewise energy estimate, a technique that is specific to the one-dimensional setting; the underlying idea consists of exchanging the roles of the time and space variables and, finally in proving hyperbolic energy estimates for waves with rough coefficients. Unfortunately, such a method does not extend to higher space dimensions. Furthermore, for the low regularity considered in these articles, the geodesic flow is not well-defined. Proving propagation results for wavefront sets or microlocal defect measure appears quite out of reach in such cases.

The present work is the first in a series of three articles devoted to the question of observability (and equivalently exact controllability) of wave equations with nonsmooth coefficients. Here, we initiate this study on a compact Riemannian manifold with a rough metric, yet *without boundary*, while the two forthcoming articles will present the counterpart analysis on manifolds with boundary (or bounded domains of $\mathbb{R}^d$) [Burq et al. 2024a; 2024b]. The presence of a boundary yields a much more involved analysis and in [Burq et al. 2024a; 2024b] we develop Melrose–Sjöstrand generalized propagation theory in a low-regularity framework. In the present article, our main result is the observability of the wave equation with a $\mathscr{C}^1$-metric, completed with the stability of the observability property for small Lipschitz ($W^{1,\infty}$) perturbations of the metric. More precisely, we first show that if the geometric control condition in time T holds for geodesics associated with a $\mathscr{C}^1$-metric g, then the observability property holds for the wave equation, and equivalently exact controllability. For this low-regularity case one has to carefully consider the meaning of the geometric condition (or more generally the meaning of a geodesic) since the metric does not define a natural geodesic flow: geodesics are not uniquely defined. Only their existence is guaranteed. Second, we consider a reference $\mathscr{C}^1$-metric g^0 as above and we prove that observability also holds for any Lipschitz metric g chosen sufficiently close to g^0 (in the Lipschitz topology). It has to be noticed that Lipschitz metrics are too rough to permit the use of microlocal tools and a direct proof of the observability property. Even worse for such a metric, the geometric control condition itself does not seem make sense (as the generating vector field is only L^∞), and we have to use a perturbation argument near the (not so) smooth $\mathscr{C}^1$ reference metric.

Following the strategy of [Burq 1997a], we argue by contradiction and we prove a propagation result for microlocal defect measures in a low-regularity setting. We prove that these measures are solutions to the ODE (1-1) with here H_p having $\mathscr{C}^0$-coefficients. Then, we deduce some general properties about their support. Namely we show that their support is a union of integral curves of the vector field. This latter step also follows from Ambrosio and Crippa's superposition principle [2014]. Yet, we give a completely different proof which is of interest since it can be extended to the case of a domain with a boundary [Burq et al. 2024a; 2024b]. We have not been able to extend the approach of [Ambrosio and Crippa 2014] to that case. To derive the ODE fulfilled by the microlocal defect measure, we heavily rely on some harmonic analysis results due to R. Coifman and Y. Meyer [1978, Proposition IV.7] that express that the commutator of a pseudodifferential operator of order 1 and a Lipschitz function is a bounded operator on L^2.

Finally, going further in the analysis, we investigate another stability property with respect to perturbations of the metric. We prove that the HUM optimal control associated with a fixed initial data is *not* stable with respect to perturbations of the metric.

1A. ***Outline.*** The article is organized as follows. In Section 1B we set up the geometric framework we shall use and in Section 1C we precisely recall the equivalence of observability and exact controllability for the wave equation. In Section 1D we state the main results of the article.

In Section 2 we recall some geometric facts and the notions of pseudodifferential calculus and microlocal defect (density) measures on a manifold. In addition, using bicharacteristics we state the geometric control condition of [Bardos et al. 1992] in its classical form ($\mathscr{C}^2$-metric) and generalized form ($\mathscr{C}^1$-metric).

In Section 3 we recall what microlocal defect measures are and we show how, if associated with sequences of solutions of PDEs, their support can be estimated and how a transport ODE can be derived, in the particular context of low regularity of coefficients.

Section 4 is devoted to our proof of the support propagation for measures solutions of a ODE with $\mathscr{C}^0$-coefficients, Theorem 1.10.

In Section 5 we use the results of Section 3 and the propagation result of Theorem 1.10 to prove the observability and controllability results for the wave equation, Theorems 1.11 and 1.12.

Finally, in Section 6 we prove the results related to stability properties of the HUM control process.

1B. ***Setting and well-posedness.*** Throughout the article, we consider $\mathcal{M}$, a d-dimensional $\mathscr{C}^\infty$-compact manifold, that is, a manifold without boundary with a topology that makes it compact, equipped with a $\mathscr{C}^\infty$-atlas. We assume that the topology is also given by a Riemannian metric g, to be chosen either Lipschitz or of class $\mathscr{C}^k$ for some value of k to be made precise below.[1]

We denote by μ_g the canonical positive Riemannian density on $\mathcal{M}$, that is, the density measure associated with the density function $(\det g)^{1/2}$. We also consider a positive Lipschitz or of class $\mathscr{C}^k$-function κ and we define the density $\kappa\mu_g$.

The L^2-inner product and norm are considered with respect to this density $\kappa\mu_g$, that is,

$$(u,v)_{L^2(\mathcal{M})} = \int_{\mathcal{M}} u\bar{v}\,\kappa\mu_g, \quad \|u\|^2_{L^2(\mathcal{M})} = \int_{\mathcal{M}} |u|^2\,\kappa\mu_g. \tag{1-2}$$

We denote by $L^2V(\mathcal{M})$ the space of L^2-vector fields on $\mathcal{M}$, equipped with the norm

$$\|v\|^2_{L^2V(\mathcal{M})} = \int_{\mathcal{M}} g(v,\bar{v})\,\kappa\mu_g, \quad v \in L^2V(\mathcal{M}).$$

We recall that the Riemannian gradient and divergence are given by

$$g(\nabla_g f, v) = v(f) \quad \text{and} \quad \int_{\mathcal{M}} f \operatorname{div}_g v\mu_g = -\int_{\mathcal{M}} v(f)\,\mu_g$$

for f a function and v a vector field, yielding in local coordinates

$$(\nabla_g f)^i = \sum_{1\le j\le d} g^{ij}\partial_{x_j} f, \quad \operatorname{div}_g v = (\det g)^{-1/2} \sum_{1\le i\le d} \partial_{x_i}((\det g)^{1/2} v^i),$$

with $(g_x^{ij}) = (g_{x,ij})^{-1}$.

We introduce the elliptic operator $A = A_{\kappa,g} = \kappa^{-1}\operatorname{div}_g(\kappa\nabla_g)$, that is, in local coordinates

$$Af = \kappa^{-1}(\det g)^{-1/2} \sum_{1\le i,j\le d} \partial_{x_i}(\kappa(\det g)^{1/2} g^{ij}(x)\partial_{x_j} f).$$

Its principal symbol is simply $a(x,\xi) = -\sum_{1\le i,j\le d} g_x^{ij}\xi_i\xi_j$. Note that for $\kappa = 1$, one has $A = \Delta_g$, the Laplace–Beltrami operator associated with g on $\mathcal{M}$. Similarly to Δ_g, the operator A is unbounded on $L^2(\mathcal{M})$. With the domain $D(A) = H^2(\mathcal{M})$, one finds that A is self-adjoint, with respect to the L^2-inner product given in (1-2), and negative. Moreover, one has

$$(Au,v)_{L^2(\mathcal{M})} = -\int_{\mathcal{M}} g(\nabla_g u, \nabla_g \bar{v})\,\kappa\mu_g, \quad u \in H^2(\mathcal{M}),\ v \in H^1(\mathcal{M}).$$

Together with A we consider the wave operator $P_{\kappa,g} = \partial_t^2 - A_{\kappa,g} + m$, with $m > 0$ a constant and the equation

$$\begin{cases} P_{\kappa,g} y = f & \text{in } (0,+\infty)\times\mathcal{M}, \\ y_{|t=0} = y^0,\ \partial_t y_{|t=0} = y^1 & \text{in } \mathcal{M}. \end{cases} \tag{1-3}$$

It is well-posed in the energy space $H^1(\mathcal{M}) \oplus L^2(\mathcal{M})$.

[1]Note that despite considering $\mathscr{C}^k$ metrics with $k < \infty$, we still impose the condition that underlying manifold is smooth. This is due to our use of pseudodifferential techniques that are simple to introduce on a smooth manifold. See Section 2C.

Proposition 1.1. *Consider κ and g both of Lipschitz class. Let $(y^0, y^1) \in H^1(\mathcal{M}) \times L^2(\mathcal{M})$ and let $f \in L^2(0, T; L^2(\mathcal{M}))$ for any $T > 0$. There exists a unique*

$$y \in \mathscr{C}^0([0, +\infty); H^1(\mathcal{M})) \cap \mathscr{C}^1([0, +\infty); L^2(\mathcal{M}))$$

that is a weak solution of (1-3), *that is, $y_{|t=0} = y^0$ and $\partial_t y_{|t=0} = y^1$ and*

$$P_{\kappa,g} y = f \quad \text{in } \mathscr{D}'((0, +\infty) \times \mathcal{M}).$$

Remark 1.2. At this level of regularity of κ and g, the well-posedness of the wave equation is classical. For less regular coefficients we refer to [Colombini and Del Santo 2009; Colombini et al. 2013].

In what follows, for simplicity we shall consider the case $m = 1$, that is, for

$$P_{\kappa,g} = \partial_t^2 - A_{\kappa,g} + 1.$$

In this case, we denote by

$$\begin{aligned} \mathcal{E}_{\kappa,g}(y)(t) &= \tfrac{1}{2}(\|y(t)\|^2_{H^1(\mathcal{M})} + \|\partial_t y(t)\|^2_{L^2(\mathcal{M})}) \\ &= \tfrac{1}{2}(\|y(t)\|^2_{L^2(\mathcal{M})} + \|\nabla_g y(t)\|^2_{L^2 V(\mathcal{M})} + \|\partial_t y(t)\|^2_{L^2(\mathcal{M})}), \end{aligned}$$

the energy of this solution at time t. For a weak solution y of (1-3), if $f = 0$, this energy is independent of time t, that is,

$$\mathcal{E}_{\kappa,g}(y)(t) = \mathcal{E}_{\kappa,g}(y)(0) = \tfrac{1}{2}(\|y^0\|^2_{H^1(\mathcal{M})} + \|y^1\|^2_{L^2(\mathcal{M})}).$$

Remark 1.3. The equation we consider, with the constant $m > 0$, is often referred to the Klein–Gordon equation. Here, we keep the name wave equation. We choose this equation instead of the classical wave equation that corresponds to the case $m = 0$. In fact, on a compact manifold without boundary, constants are eigenfunctions of the elliptic operator $A_{\kappa,g}$ with 0 as an eigenvalue. Hence, constant functions are solutions to the wave equation and are so-called *invisible* solutions, as far as the observability property we are interested in is concerned. If one considers a manifold with boundary and say, homogeneous Dirichlet conditions, this issue becomes irrelevant. We could have dealt with the case $m = 0$ (the usual wave equation) at the cost of additional technical complications.

1C. ***Exact controllability and observability.*** Let ω be a nonempty open subset of $\mathcal{M}$ and $T > 0$. The notion of exact controllability for the wave equation from ω at time T is stated as follows.

Definition 1.4 (exact controllability in $H^1(\mathcal{M}) \oplus L^2(\mathcal{M})$). One says that the wave equation is exactly controllable from ω at time $T > 0$ if, for any $(y^0, y^1) \in H^1(\mathcal{M}) \times L^2(\mathcal{M})$, there exists $f \in L^2((0, T) \times \mathcal{M})$ such that the weak solution y to

$$P_{\kappa,g} y = \mathbf{1}_{(0,T)\times\omega} f, \quad (y_{|t=0}, \partial_t y_{|t=0}) = (y^0, y^1), \tag{1-4}$$

as given by Proposition 1.1, satisfies $(y, \partial_t y)_{|t=T} = (0, 0)$. The function f is called the control function or simply the control.

Observability of the wave equation from the open set ω in time T is the following notion.

Definition 1.5 (observability). One says that the wave equation is observable from ω at time T if there exists $C_{\text{obs}} > 0$ such that for any $(u^0, u^1) \in H^1(\mathcal{M}) \times L^2(\mathcal{M})$ one has

$$\mathcal{E}_{\kappa,g}(u)(0) \le C_{\text{obs}} \|\mathbf{1}_{(0,T)\times\omega}\, \partial_t u\|^2_{L^2(\mathcal{L})} \tag{1-5}$$

for $u \in \mathscr{C}^0([0,T]; H^1(\mathcal{M})) \cap \mathscr{C}^1([0,T]; L^2(\mathcal{M}))$ the weak solution of $P_{\kappa,g} u = 0$ with $u_{|t=0} = u^0$ and $\partial_t u_{|t=0} = u^1$ as given by Proposition 1.1; see [Lions 1988].

Proposition 1.6. *Let ω be an open subset of $\mathcal{M}$ and $T > 0$. The wave equation is exactly controllable from ω at time T if and only if it is observable from ω at time T.*

Remark 1.7. In the case $m = 0$, the energy function is given by

$$\mathcal{E}_{\kappa,g}(u)(t) = \tfrac{1}{2}(\|\partial_t u(t)\|^2_{L^2(\mathcal{M})} + \|\nabla_g u(t)\|^2_{L^2V(\mathcal{M})}).$$

It follows that a constant function u, a solution to the wave equation $(\partial_t^2 - A)u = 0$, has zero energy. Since $\|\mathbf{1}_{(0,T)\times\omega}\, \partial_t u\|^2_{L^2(\mathcal{L})}$ also vanishes, one sees that such solutions are invisible for an observability inequality of the form of (1-5). Possibilities to overcome this difficulty are to work in a quotient space or to change the wave operator into the Klein–Gordon operator. Here, we chose for simplicity the latter option.

1D. ***Main results.*** We introduce the following spaces for the coefficients (κ, g) to distinguish various levels of regularity:

$$\mathcal{X}^2(\mathcal{M}) = \{(\kappa, g) : \kappa \in \mathscr{C}^2(\mathcal{M}) \text{ and } g \text{ is a } \mathscr{C}^2\text{-metric on } \mathcal{M}\},$$
$$\mathcal{X}^1(\mathcal{M}) = \{(\kappa, g) : \kappa \in \mathscr{C}^1(\mathcal{M}) \text{ and } g \text{ is a } \mathscr{C}^1\text{-metric on } \mathcal{M}\},$$
$$\mathcal{Y}(\mathcal{M}) = \{(\kappa, g) : \kappa \in W^{1,\infty}(\mathcal{M}) \text{ and } g \text{ is a } W^{1,\infty}\text{-metric on } \mathcal{M}\}.$$

We start by recalling the controllability result known for regularity higher than or equal to $\mathscr{C}^2$, under the Rauch–Taylor geometric control condition.

Definition 1.8 (Rauch–Taylor, geometric control condition). Let g be a $\mathscr{C}^k$ metric, $k = 1$ or 2, and let ω be an open set of $\mathcal{M}$ and $T > 0$. One says that (ω, T) fulfills the geometric control condition if all maximal geodesics associated with g, traveled at speed 1, encounter ω for some time $t \in (0, T)$.

A second formulation of this geometric condition based on the dual notion of bicharacteristics is given in Section 2B below.

Theorem 1.9 (exact controllability: $\mathscr{C}^2$-regularity). *Consider $(\kappa, g) \in \mathcal{X}^2(\mathcal{M})$, ω an open subset of $\mathcal{M}$ and $T > 0$ such that (ω, T) fulfills the geometric control condition of Definition 1.8. Then, the wave equation is exactly controllable from ω at time T.*

This result was first proven by Rauch and Taylor [1974] for a smooth metric. The case $(\kappa, g) \in \mathcal{X}^2(\mathcal{M})$ was proven by the first author in [Burq 1997a]. On smooth open sets of $\mathbb{R}^d$, or equivalently on manifolds with boundary equipped with smooth (κ, g), for instance in the case of homogeneous Dirichlet boundary conditions, this result is given in the celebrated articles [Bardos et al. 1988; 1992].

In the present article, we extend the result of Theorem 1.9 to cases of rougher coefficients. Our extension is twofold: we treat the case $(\kappa, g) \in \mathcal{X}^1(\mathcal{M})$ and, we treat small perturbations in $\mathcal{Y}(\mathcal{M})$ of some $(\kappa, g) \in \mathcal{X}^1(\mathcal{M})$. Most importantly, these two results rely on the understanding of the structure of the support of a nonnegative measure subject to a homogeneous transport equation with continuous coefficients.

1D1. *Transport equation and measure support.* Let $\mathcal{O}$ be an open set of a smooth manifold. We denote by ${}^1\mathscr{D}'(\mathcal{O})$ and ${}^1\mathscr{D}'^{,0}(\mathcal{O})$ the spaces of density distributions and density Radon measures on $\mathcal{O}$.

Consider a continuous vector field X on $\mathcal{O}$ and let μ be a nonnegative measure density on $\mathcal{O}$. Assume that μ is such that ${}^tX\mu = 0$ in the sense of distributions, that is,

$$\langle {}^tX\mu, a\rangle_{{}^1\mathscr{D}'(\mathcal{O}),\mathscr{C}_c^\infty(\mathcal{O})} = \langle \mu, Xa\rangle_{{}^1\mathscr{D}'^{,0}(\mathcal{O}),\mathscr{C}_c^0(\mathcal{O})} = 0, \quad a \in \mathscr{C}_c^\infty(\mathcal{O}). \tag{1-6}$$

If X is moreover Lipschitz, one concludes that μ is invariant along the flow that X generates. However, if X is not Lipschitz, there is no such flow in general. Yet, integral curves do exist by the Cauchy–Peano theorem. The following theorem provides a structure of the support of μ.

Theorem 1.10. *Let X be a continuous vector field on $\mathcal{O}$ and μ be a nonnegative density measure on $\mathcal{O}$ that is a solution to ${}^tX\mu = 0$ in the sense of distributions. Then, the support of μ is a union of maximally extended integral curves of the vector field X.*

In other words, if $m^0 \in \mathcal{O}$ is in $\operatorname{supp}(\mu)$, then there exist an interval I in $\mathbb{R}$ with $0 \in I$ and a $\mathscr{C}^1$ curve $\gamma : I \to \mathcal{O}$ that cannot be extended such that $\gamma(0) = m^0$ and

$$\frac{d}{ds}\gamma(s) = X(\gamma(s)), \quad s \in I,$$

and $\gamma(I) \subset \operatorname{supp}(\mu)$.

Theorem 1.10 can actually be obtained as a consequence of the superposition principle of L. Ambrosio and G. Crippa [2014, Theorem 3.4]. Here, we provide an alternative proof that is of interest as it allows one to extend this measure support structure result to the case of an open set or a manifold with boundary [Burq et al. 2024b] as needed for our application to observability and controllability. Ambrosio and Crippa's proof is based on a smoothing-by-convolution argument. Extending this approach does not seem to be straightforward in the context of a boundary.

Theorem 1.10 is proven in Section 4 and its proof is independent of the other sections of the article. A reader only interested in our proof of Theorem 1.10 may thus head to Section 4 directly.

1D2. *Exact controllability results.* If $(\kappa, g) \in \mathcal{X}^2(\mathcal{M})$, $x \in \mathcal{M}$ and $v \in T_x\mathcal{M}$ there is a unique geodesic originating from x in direction v. In the case $(\kappa, g) \in \mathcal{X}^1(\mathcal{M})$ uniqueness is *lost*. Existence holds however and maximal (here global, see below) geodesics can still be defined by the Cauchy–Peano theorem. In particular, the geometric control condition of Definition 1.8 still makes sense. As announced above, our first result is the following theorem.

Theorem 1.11 (exact controllability: $\mathscr{C}^1$-regularity). *Consider $(\kappa, g) \in \mathcal{X}^1(\mathcal{M})$, ω an open subset of $\mathcal{M}$ and $T > 0$ such that (ω, T) fulfills the geometric control condition of Definition 1.8. Then, the wave equation is exactly controllable from ω at time T.*

A second result is the following *perturbation* result.

Theorem 1.12 (exact controllability: Lipschitz perturbation). *Let $(\kappa^0, g^0) \in \mathcal{X}^1(\mathcal{M})$, ω be an open subset of $\mathcal{M}$ and $T > 0$ be such that (ω, T) fulfills the geometric control condition of Definition 1.8 with respect to the metric g^0. There exists $\varepsilon > 0$ such that for any $(\kappa, g) \in \mathcal{Y}(\mathcal{M})$ satisfying*

$$\|(\kappa, g) - (\kappa^0, g^0)\|_{\mathcal{Y}(\mathcal{M})} \leq \varepsilon,$$

the wave equation associated with (κ, g) is exactly controllable by ω in time T.

Observe that Theorem 1.11 is a direct consequence of Theorem 1.12. We shall thus concentrate on this second more general result. Its proof relies on the measure support structure result of Theorem 1.10.

The sequence of Theorems 1.9, 1.11, and 1.12 calls for the following important comment. Under the assumption of Theorem 1.9, that is, $(\kappa, g) \in \mathcal{X}^2(\mathcal{M})$, there is a *geodesic flow* and the geometric condition of Definition 1.8 is actually a condition on the flow. Under the assumption of Theorem 1.11, that is, $(\kappa, g) \in \mathcal{X}^1(\mathcal{M})$, as pointed out above there is no geodesic flow in general. Yet, maximal geodesics are still well-defined and, the geometric condition of Definition 1.8 makes sense because it does not refer to a flow. However, under the assumption of Theorem 1.12, that is, $(\kappa, g) \in \mathcal{Y}(\mathcal{M})$, geodesics cannot be defined in general. No geometric condition can be formulated. Yet, Theorem 1.12 is a perturbation result and a geometric condition is expressed for a reference pair $(\kappa^0, g^0) \in \mathcal{X}^1(\mathcal{M})$ around which a (small) neighborhood in $\mathcal{Y}(\mathcal{M})$ is considered.

The following remark further emphasizes that the perturbation is to be considered around a pair $(\kappa^0, g^0) \in \mathcal{X}^1(\mathcal{M})$ for which the geometric control condition holds and not around a pair $(\kappa^0, g^0) \in \mathcal{X}^1(\mathcal{M})$ for which exact controllability (or equivalently observability) holds.

Remark 1.13 (on the perturbation result). Having both our results, geometric control for $\mathscr{C}^1$ metrics and Lipschitz stability of exact controllability around a reference metric satisfying the geometric control condition, a natural question is whether the exact controllability property is itself stable by perturbation. On the one hand, it is classical that the exact controllability property is *stable* under lower-order perturbations of the elliptic operator $A_{\kappa,g}$, but on the other hand, it is possible to show that it is *not* stable under (smooth) perturbations of the geometry or the metric.

Let us illustrate this instability property with a quite simple example. Consider the wave equation on the sphere

$$\mathbb{S}^d = \Big\{x \in \mathbb{R}^{d+1} : \sum_i x_i^2 = 1\Big\},$$

endowed with its standard metric and with control domain the open hemisphere

$$\omega = \{x \in \mathbb{S}^d : x_1 > 0\}.$$

Even though ω does not fulfill the geometric control condition of Definition 1.8 exact controllability holds for this geometry by an unpublished result by G. Lebeau (see [Lebeau 1992, Section VI.B] and [Zhu 2018] for extensions). Consider now the sphere endowed with the above standard metric, with the smaller control domain

$$\omega_\varepsilon = \{x \in \mathbb{S}^d : x_1 > \varepsilon\}$$

for some $\varepsilon > 0$. This second geometry is ε-close to the Lebeau example in the $\mathscr{C}^\infty$-topology. Yet, for all $\varepsilon > 0$, exact controllability does *not* hold, because there exists a geodesic (the equator, $\{x \in \mathbb{S}^d : x_1 = 0\}$) that does not encounter $\bar{\omega}_\varepsilon$. This shows that in Theorem 1.12, the assumption that the reference geometry should satisfy the geometric control condition *cannot* be replaced by the weaker assumption that it should satisfy the exact controllability property. This also shows that our perturbation argument will have to be performed on the actual proof that geometric control implies exact controllability and *not* on the final property itself.

1D3. *Further results on the control operator.* We finish this section with results analyzing the influence of some metric perturbations on the control process.

We introduce further levels of regularity for the coefficients by setting, for $k \in \mathbb{N} \cup \{+\infty\}$,

$$\mathcal{X}^k(\mathcal{M}) = \{(\kappa, g) : \kappa \in \mathscr{C}^k(\mathcal{M}) \text{ and } g \text{ is a } \mathscr{C}^k\text{-metric on } \mathcal{M}\}.$$

First, we consider $k \geq 2$. We recall the notation $P_{\kappa,g} = \partial_t^2 - A_{\kappa,g} + 1$ with $A_{\kappa,g} = \kappa^{-1} \operatorname{div}_g(\kappa \nabla_g)$, and we assume that $(\kappa, g) \in \mathcal{X}^k(\mathcal{M})$, and that (ω, T) satisfies the geometric control condition of Definition 1.8 for geodesics given by the metric g. Then, by Theorem 1.9, given $(y^0, y^1) \in H^1(\mathcal{M}) \times L^2(\mathcal{M})$, there exists $f \in L^2((0, T) \times \omega)$ such that the solution to (1-4) satisfies $y(T) = 0$ and $\partial_t y(T) = 0$. One can prove that among all possible control functions there is one of minimal L^2-norm. We denote by $f_{\kappa,g}^{y^0,y^1}$ this control function usually named the HUM control function; see for instance [Lions 1988]. Moreover, the map

$$H_{\kappa,g} : H^1(\mathcal{M}) \oplus L^2(\mathcal{M}) \to L^2((0, T) \times \mathcal{M}), \quad (y^0, y^1) \mapsto f_{\kappa,g}^{y^0,y^1}, \tag{1-7}$$

is continuous. Note that $f_{\kappa,g}^{y^0,y^1}$ is actually a weak solution of the wave equation with initial data in $L^2(\mathcal{M}) \times H^{-1}(\mathcal{M})$, meaning that one moreover has $f_{\kappa,g}^{y^0,y^1} \in \mathscr{C}^0([0, T], L^2(\mathcal{M}))$.

Theorem 1.14 (lack of continuity of the HUM-operator: the case $k \geq 2$). *Let $k \geq 2$ and (κ, g) as above. For any neighborhood $\mathcal{U}$ of (κ, g) in $\mathcal{X}^k(\mathcal{M})$, there exist $(\tilde{\kappa}, \tilde{g}) \in \mathcal{U}$ and an initial data $(y^0, y^1) \in H^1(\mathcal{M}) \times L^2(\mathcal{M})$, with $\|y^0\|_{H^1}^2 + \|y^1\|_{L^2}^2 = 1$, such that the respective solutions y and $\tilde{y}$ of*

$$\begin{cases} P_{\kappa,g} y = \mathbf{1}_{(0,T)\times\omega} f_{\kappa,g}^{y^0,y^1} & \text{in } (0, T) \times \mathcal{M}, \\ (y, \partial_t y)_{|t=0} = (y^0, y^1) & \text{in } \mathcal{M}, \end{cases} \qquad \begin{cases} P_{\tilde{\kappa},\tilde{g}} \tilde{y} = \mathbf{1}_{(0,T)\times\omega} f_{\kappa,g}^{y^0,y^1} & \text{in } (0, T) \times \mathcal{M}, \\ (\tilde{y}, \partial_t \tilde{y})_{|t=0} = (y^0, y^1) & \text{in } \mathcal{M} \end{cases} \tag{1-8}$$

are such that

$$\mathcal{E}_{\kappa,g}(\tilde{y} - y)(T) = \mathcal{E}_{\kappa,g}(\tilde{y})(T) \geq \tfrac{1}{2}. \tag{1-9}$$

Moreover, there exists $C_T > 0$ such that

$$\left\|(H_{\kappa,g} - H_{\tilde{\kappa},\tilde{g}})(y^0, y^1)\right\|_{L^2((0,T)\times\omega)} = \|f_{\kappa,g}^{y^0,y^1} - f_{\tilde{\kappa},\tilde{g}}^{y^0,y^1}\|_{L^2((0,T)\times\omega)} \geq C_T \tag{1-10}$$

for (y^0, y^1) as given above.

Remark 1.15. The result of Theorem 1.14 states that starting from the same initial data and solving the two wave equations with the same control vector $f_{\kappa,g}$ associated with $P_{\kappa,g}$, a small perturbation of the metric can induce a large error for the final state $(y(T), \partial_t y(T))$. In other words, the two dynamics are no longer close. In particular, the map

$$\mathcal{X}^k(\mathcal{M}) \to \mathscr{L}\big(H^1(\mathcal{M}) \oplus L^2(\mathcal{M}), L^2((0, T) \times \mathcal{M})\big), \quad (\kappa, g) \mapsto H_{\kappa,g},$$

is not continuous.

Remark 1.16. The result of Theorem 1.14 can also be stated on open bounded smooth domains of $\mathbb{R}^n$ in the case of homogeneous Dirichlet condition. In fact, as can be checked in what follows, its proof only relies on basic properties of microlocal defect measures (support localization and propagation) that are known to be valid in this framework; see [Burq 1997a].

Remark 1.17. In the statement of Theorem 1.14 if the neighborhood $\mathcal{U}$ of (κ, g) in $\mathcal{X}^k$ is small enough, the pair (ω, T) also satisfies the geometric control condition of Definition 1.8 for $(\tilde{\kappa}, \tilde{g})$ and therefore $f_{\tilde{\kappa},\tilde{g}}^{y^0,y^1}$ is well-defined. In particular, this is clear as in the case $k \geq 2$ there is a well-defined and unique geodesic flow.

The case $k = 1$ is quite different as there is no geodesic flow, as already mentioned above. However, given $(\kappa, g) \in \mathcal{X}^1$ and (ω, T) if the Rauch–Taylor geometric control condition of Definition 1.8 holds for (ω, T) for the geodesics associated with g, given any neighborhood $\mathcal{U}$ of (κ, g) in $\mathcal{X}^1$ one can still find $(\tilde{\kappa}, \tilde{g}) \in \mathcal{U}$ such that

(1) the geometric control condition still holds for the geodesics associated with $\tilde{g}$,

(2) the result of Theorem 1.14 also holds.

Theorem 1.14′ (lack of continuity of the HUM-operator: the case $k = 1$). *Let $k = 1$ and $(\kappa, g) \in \mathcal{X}^1$ as above. For any neighborhood $\mathcal{U}$ of (κ, g) in $\mathcal{X}^1(\mathcal{M})$, there exist $(\tilde{\kappa}, \tilde{g}) \in \mathcal{U}$ and an initial data $(y^0, y^1) \in H^1(\mathcal{M}) \times L^2(\mathcal{M})$, with $\|y^0\|_{H^1}^2 + \|y^1\|_{L^2}^2 = 1$, such that the geometric control condition of Definition 1.8 for geodesics given by the metric $\tilde{g}$ holds and moreover the results listed in Theorem 1.14 hold.*

The proofs of Theorems 1.14 and 1.14′ are given in Section 6A.

We finish this section with some remarks and some questions.

Remark 1.18. In all results above we have used $1_{(0,T)\times\omega}$ as a control operator, that is, the characteristic function of an open set. We could have also considered a control operator given by $1_{(0,T)}(t)\chi(x)$, with χ a smooth function on $\mathcal{M}$. The controlled wave equation then has the form

$$P_{\kappa,g}\, y = \mathbf{1}_{(0,T)}\, \chi\, f, \quad (y_{|t=0}, \partial_t y_{|t=0}) = (y^0, y^1). \tag{1-11}$$

In such case, the open set to be used in the geometric control condition is $\omega = \{\chi \neq 0\}$. This is often done this way, in particular since the smoothness of the function χ allows one to use some microlocal techniques that require regularity in the operator coefficients. The results and proofs of the present article can be written mutatis mutandis for this type of control operator.

1D4. *Comparison with the smooth case and some open questions.* Following on the previous remark, with a smooth-in-space control operator, one can wonder above the smoothness of the HUM operator. This question is addressed in the joint work of the second author [Dehman and Lebeau 2009]. In fact, a gain of regularity in the initial data (y^0, y^1) yields an equivalent gain of regularity in the HUM control function $f_{\kappa,g}^{y^0,y^1}$. For instance, for $(y^0, y^1) \in H^2(\mathcal{M}) \times H^1(\mathcal{M})$ one finds $f_{\kappa,g}^{y^0,y^1} \in \mathscr{C}^0([0, T], H^1(\mathcal{M}))$. Note that the result of [Dehman and Lebeau 2009] is proven in the case of smooth coefficients, that is, $(\kappa, g) \in \mathcal{X}^\infty$. We thus consider this smooth case in the discussion that ends this introductory section. Open questions around the results of Theorems 1.14 and 1.14′ are then raised.

As we shall see in their proofs, the results of Theorems 1.14 and 1.14′ rely on the high-frequency behavior of the solutions to (1-8). In the case of smooth coefficients and a smooth control operator, if we assume smoother data (y^0, y^1) in the HUM control process, the result of Theorem 1.14 does not hold any more. The HUM control process becomes regular with respect to (κ, g) as expressed in the following proposition.

Proposition 1.19 (HUM control process for smooth data). *Consider* $(\kappa, g) \in \mathcal{X}^\infty(\mathcal{M})$ *and let* $\chi \in \mathscr{C}^\infty(\mathcal{M})$. *Set* $\omega = \{\chi \neq 0\}$ *and assume that* (ω, T) *fulfills the geometric control condition of Definition 1.8 for the geodesics associated with* (κ, g). *Let* $\alpha \in (0, 1]$. *There exists* $C_\alpha > 0$ *such that, for any* $(\tilde{\kappa}, \tilde{g}) \in \mathcal{X}^\infty(\mathcal{M})$ *and any* $(y^0, y^1) \in H^{1+\alpha}(\mathcal{M}) \times H^\alpha(\mathcal{M})$, *the respective solutions* y *and* $\tilde{y}$ *to*

$$\begin{cases} P_{\kappa,g} y = \mathbf{1}_{(0,T)}\, \chi\, f_{\kappa,g}^{y^0,y^1} & \text{in } (0,T)\times\mathcal{M}, \\ (y, \partial_t y)_{|t=0} = (y^0, y^1) & \text{in } \mathcal{M}, \end{cases} \qquad \begin{cases} P_{\tilde{\kappa},\tilde{g}} \tilde{y} = \mathbf{1}_{(0,T)}\, \chi\, f_{\kappa,g}^{y^0,y^1} & \text{in } (0,T)\times\mathcal{M}, \\ (\tilde{y}, \partial_t \tilde{y})_{|t=0} = (y^0, y^1) & \text{in } \mathcal{M} \end{cases}$$

satisfy

$$\mathcal{E}_{\kappa,g}(y-\tilde{y})(T)^{1/2} \le C_\alpha \|(\kappa, g) - (\tilde{\kappa}, \tilde{g})\|^\alpha_{\mathcal{X}^1(\mathcal{M})}\, \|(y^0, y^1)\|_{H^{1+\alpha}(\mathcal{M})\oplus H^\alpha(\mathcal{M})}.$$

The proof of Proposition 1.19 is given in Section 6B.

In the above proposition coefficients are chosen smooth, quite in contrast with the rest of this article. As explained above, and as the reader can check in the proof, this lies in the use of the regularity of the HUM operator with respect to the data (y^0, y^1), a result proven for smooth coefficients in [Dehman and Lebeau 2009]. The result of Proposition 1.19 raises the following natural questions:

(1) Does the HUM operator exhibit regularity with respect to the data (y^0, y^1) similar to what is proven in [Dehman and Lebeau 2009] in the case of not so smooth coefficients?

(2) If so, if one increases the smoothness of the data (y^0, y^1) as in Proposition 1.19, does the HUM control process also become regular with respect of the metric?

2. Geometric aspects and operators

We define the smooth manifold $\mathcal{L} = \mathbb{R} \times \mathcal{M}$ and $T^*\mathcal{L}$ its cotangent bundle. We denote by $\pi : T^*\mathcal{L} \to \mathcal{L}$ the natural projection. Elements in $T^*\mathcal{L}$ are denoted by (t, x, τ, ξ). One has $\pi(t, x, \tau, \xi) = (t, x)$.

Setting $|\xi|_x^2 = g_x(\xi, \xi)$ the Riemannian norm in the cotangent space of $\mathcal{M}$ at x, we define

$$S^*\mathcal{L} = \{(t, x, \tau, \xi) \in T^*\mathcal{L} : \tau^2 + |\xi|_x^2 = 1\},$$

the cosphere bundle of $\mathcal{L}$. We shall also use the associated cosphere bundle in the spatial variables only,

$$S^*\mathcal{M} = \left\{(x, \xi) \in T^*\mathcal{M} : |\xi|_x^2 = \tfrac{1}{2}\right\}.$$

For a $\mathscr{C}^k$-metric both $S^*\mathcal{M}$ and $S^*\mathcal{L}$ are $\mathscr{C}^k$-manifolds.

Consider a $\mathscr{C}^\infty$-atlas $\mathcal{A}^\mathcal{M} = (\mathcal{C}_j^\mathcal{M})_{j\in J}$ of $\mathcal{M}$, $\#J < \infty$, with $\mathcal{C}_j^\mathcal{M} = (O_j, \theta_j)$, where O_j is an open set of $\mathcal{M}$ and $\theta_j : O_j \to \widetilde{O}_j$ is a bijection for $\widetilde{O}_j$ an open set of $\mathbb{R}^d$. For $j \in J$, we define $\mathcal{C}_j = (\mathcal{O}_j, \vartheta_j)$ with $\mathcal{O}_j = \mathbb{R} \times O_j$ and

$$\vartheta_j : \mathcal{O}_j \to \widetilde{\mathcal{O}}_j, \quad (t, x) \mapsto (t, \theta_j(x)),$$

with $\widetilde{\mathcal{O}}_j = \mathbb{R} \times \widetilde{O}_j$. Then $\mathcal{A} = (\mathcal{C}_j)_{j\in J}$ is a $\mathscr{C}^\infty$-atlas for $\mathcal{L}$.

In what follows for simplicity we shall use the same notation for an element of $T^*\mathcal{L}$ and its local representative if no confusion arises.

2A. ***Hamiltonian vector field and bicharacteristics.*** Let $(\kappa, g) \in \mathcal{X}^k$, $k = 1$ or 2. The principal symbol of the wave operator $P_{\kappa,g}$ is given by

$$p(t, x, \tau, \xi) = p_{\kappa,g}(t, x, \tau, \xi) = -\tau^2 + |\xi|_x^2, \quad (t, x, \tau, \xi) \in T^*\mathcal{L}. \tag{2-1}$$

In local charts, one has

$$p(t, x, \tau, \xi) = -\tau^2 + \sum_{1\le i,j\le d} g^{ij}(x)\xi_i\xi_j.$$

Note that $(g^{ij}(x))_{i,j}$ is the inverse of $(g_{ij}(x))_{i,j}$, the latter being the local representative of the metric.

We denote by H_p the Hamiltonian vector field associated with p, that is, the unique vector field such that $\{p, f\} = \mathrm{H}_p f$ for any smooth function f. Here, $\{\cdot, \cdot\}$ denotes the Poisson bracket, that is, in local chart

$$\{p, f\} = \partial_\tau p\, \partial_t f - \partial_t p\, \partial_\tau f + \sum_{1\le j\le d} (\partial_{\xi_j} p\, \partial_{x_j} f - \partial_{x_j} p\, \partial_{\xi_j} f),$$

yielding

$$\mathrm{H}_p = -2\tau\partial_t + \nabla_\xi p \cdot \nabla_x - \nabla_x p \cdot \nabla_\xi,$$

as p is in fact independent of the time variable t. The Hamiltonian vector field H_p is of class $\mathscr{C}^{k-1}$. Observe that, for a function f of the variables (t, x, τ, ξ), one has

$${}^t\mathrm{H}_p f = 2\tau\partial_t f - \operatorname{div}_x(f\nabla_\xi p) + \operatorname{div}_\xi(f\nabla_x p),$$

with which one deduces

$${}^t\mathrm{H}_p = -\mathrm{H}_p, \tag{2-2}$$

even in the case $(\kappa, g) \in \mathcal{X}^1$.

First, consider the case $k = 2$. Thus, H_p is a $\mathscr{C}^1$-vector field. For $\varrho \in T^*\mathcal{L}$ one denotes by $s \mapsto \phi_s(\varrho)$ the unique maximal solution to

$$\frac{d}{ds}\phi_s(\varrho) = \mathrm{H}_p\, \phi_s(\varrho), \quad s \in \mathbb{R}, \quad \text{and} \quad \phi_{s=0}(\varrho) = \varrho, \tag{2-3}$$

as given by the Cauchy–Lipschitz theorem. One calls $(s, \varrho) \mapsto \phi_s(\varrho)$ the Hamiltonian flow map. Let $s \mapsto \gamma(s)$ be an integral curve of H_p, that is, $\gamma(s) = \phi_s(\varrho)$ for some $\varrho \in T^*\mathcal{L}$. For any smooth function f on $T^*\mathcal{L}$ one has

$$\frac{d}{ds} f \circ \gamma(s) = \mathrm{H}_p\, f(\gamma(s)).$$

Note that $\mathrm{H}_p\, \tau = 0$, meaning that the variable τ is constant along γ. Note also that the value of p remains constant along γ since $\mathrm{H}_p\, p = \{p, p\} = 0$. Hence, $|\xi|_x^2 = g_x(\xi, \xi)$ is also constant. Thus, if $\gamma(0) \in S^*\mathcal{L}$ then $\gamma(s)$ remains in $S^*\mathcal{L}$, and, for $\varrho \in S^*\mathcal{L}$, the vector field H_p at ϱ is tangent to $S^*\mathcal{L}$. Consequently, we may consider H_p as a tangent vector field on the $\mathscr{C}^2$-manifold $S^*\mathcal{L}$. In particular $\mathrm{H}_p\, a$ makes sense if $a \in \mathscr{C}_c^1(S^*\mathcal{L})$. If moreover $a \in \mathscr{C}_c^{2+\ell}(S^*\mathcal{L})$, $\ell \ge 0$, one has $\mathrm{H}_p\, a \in \mathscr{C}_c^1(S^*\mathcal{L})$.

Since $\mathrm{H}_p\, p = 0$, the flow ϕ_s preserves $\mathrm{Char}(p) = p^{-1}(\{0\})$, the characteristic set of p. As is done classically, we call bicharacteristic an integral curve for which $p = 0$. Observe then that (2-3) defines a flow on the $\mathscr{C}^2$-manifold

$$\mathrm{Char}(p) \cap S^*\mathcal{L} = \left\{(t, x, \tau, \xi) : \tau^2 = \tfrac{1}{2} \text{ and } |\xi|_x^2 = \tfrac{1}{2}\right\}.$$

Second, consider the case $k = 1$. Then H_p is only a continuous vector field. Thus, for any $\varrho \in \mathrm{Char}(p)$ there exists a maximal bicharacteristic $s \mapsto \gamma(s)$ defined on $\mathbb{R}$ such that $\gamma(0) = \varrho$, that is,

$$\frac{d}{ds}\gamma(s) = \mathrm{H}_p(\gamma(s)), \quad s \in \mathbb{R},$$

by the Cauchy–Peano theorem. Uniqueness is however not guaranteed and the notion of flow cannot be used in the case $k = 1$. Since the value of $|\xi|_x$ remains constant and the manifold $\mathcal{M}$ is compact, maximal bicharacteristics are actually defined *globally*.

As above, if $\gamma(0) \in S^*\mathcal{L}$ (resp. $\mathrm{Char}(p) \cap S^*\mathcal{L}$) one has $\gamma(s) \in S^*\mathcal{L}$ (resp. $\mathrm{Char}(p) \cap S^*\mathcal{L}$) for all $s \in \mathbb{R}$. The Hamiltonian vector field H_p can be viewed as a $\mathscr{C}^0$-vector field on the $\mathscr{C}^1$-manifold $S^*\mathcal{L}$ (resp. on the $\mathscr{C}^1$-manifold $\mathrm{Char}(p) \cap S^*\mathcal{L}$). For $a \in \mathscr{C}_c^{1+\ell}(S^*\mathcal{L})$, $\ell \geq 0$, one finds $\mathrm{H}_p\, a \in \mathscr{C}_c^0(S^*\mathcal{L})$.

Finally, connection between bicharacteristic and geodesics can be made. For this we recall that if $\xi \in T_x^*\mathcal{M}$ for some $x \in \mathcal{M}$, one can define $v \in T_x\mathcal{M}$ by $v = \xi^\sharp$, which reads in local coordinates $v^i = \sum_j g^{ij}(x)\xi_j$. In particular $|v|_x^2 = g_x(v, v) = |\xi|_x$. If now $\varrho^0 = (t^0, x^0, \tau^0, \xi^0) \in \mathrm{Char}(p) \cap S^*\mathcal{L}$ and letting $s \mapsto \varrho(s) = (t(s), x(s), \tau, \xi(s))$ be a bicharacteristic such that $\varrho(0) = \varrho^0$, one has $\tau = \tau^0$ and $t(s) = t^0 - 2\tau^0 s$. The map

$$X : t \mapsto x\left(\frac{t^0 - t}{2\tau^0}\right)$$

can be proven to be the geodesic originating from x^0 in the direction given by $v^0 = (\xi^0)^\sharp$ and parametrized by t.

We now compute the speed at which the geodesic is traveled. We have

$$\frac{dX}{dt}(t) = -\frac{1}{2\tau^0}\frac{dx(s)}{ds},$$

which yields

$$\frac{dX}{dt}(t) = -\frac{1}{2\tau^0}\nabla_\xi\, p(x(s), \xi(s)) = -\frac{\xi(s)^\sharp}{\tau^0}.$$

It follows that

$$\left|\frac{dX}{dt}(t)\right|_x = \frac{|\xi(s)^\sharp|_x}{|\tau^0|} = \frac{|\xi(s)|_x}{|\tau^0|} = \frac{|\xi^0|_x}{|\tau^0|} = 1,$$

since $\varrho^0 \in \mathrm{Char}(p)$. Hence, the projection of the bicharacteristic $s \mapsto \gamma(s)$ yields a geodesic traveled at speed 1.

2B. ***Geometric control condition.*** As the projections of bicharacteristics onto $\mathcal{L}$ yield geodesics, in the case $k \geq 2$, we can state the Rauch–Taylor geometric control condition [1974] formulated in Definition 1.8 with the notion of Hamiltonian flow introduced above.

Definition 1.8′ (geometric control condition, $k \geq 2$). Let g be a $\mathscr{C}^2$ metric and let ω be an open set of $\mathcal{M}$ and $T > 0$. One says that (ω, T) fulfills the geometric control condition if for all $\varrho \in \mathrm{Char}(p)$ one has $\pi(\phi_s(\varrho)) \in (0, T) \times \omega$ for some $s \in \mathbb{R}$.

In the case $k = 1$, since g is only $\mathscr{C}^1$ there is no flow in general, one rather writes the geometric control condition by means of maximal bicharacteristics.

Definition 1.8″ (generalized geometric control condition, $k = 1$). Let g be a $\mathscr{C}^1$ metric and let ω be an open set of $\mathcal{M}$ and $T > 0$. One says that (ω, T) fulfills the geometric control condition if for *any* maximal bicharacteristic $s \mapsto \gamma(s)$ in $\mathrm{Char}(p)$ one has $\pi(\gamma(s)) \in (0, T) \times \omega$ for some $s \in \mathbb{R}$.

In other words, for all $\varrho \in \mathrm{Char}(p)$, all bicharacteristics that go through ϱ meet the cotangent bundle above $(0, T) \times \omega$.

Naturally, Definitions 1.8′ and 1.8″ coincide in the case $k = 2$ because of the uniqueness of a bicharacteristic going through a point of $\mathrm{Char}(p)$.

2C. *Symbols and pseudodifferential operators.* Here, we follow [Burq 1997b, Section 1.1] for the notation. We denote by $H^k(X)$ or $H^k_{\mathrm{loc}}(X)$, with $X = \mathcal{M}$ or $\mathcal{L}$, the usual Sobolev space for complex valued functions, endowed with its natural inner product and norm. In particular, the $L^2(X)$-inner product is denoted by $(\,\cdot\,,\,\cdot\,)_{L^2(X)}$.

Classical polyhomogeneous symbol classes on $T^*\mathbb{R}^n \simeq \mathbb{R}^n \times \mathbb{R}^n$ are denoted by $S^m_{\mathrm{ph}}(\mathbb{R}^n \times \mathbb{R}^n)$ and the classes of associated operators by $\Psi^m_{\mathrm{ph}}(\mathbb{R}^n)$. We recall that symbols in the class $S^m_{\mathrm{ph}}(\mathbb{R}^n \times \mathbb{R}^n)$ behave well with respect to changes of variables, up to symbols in $S^{m-1}_{\mathrm{ph}}(\mathbb{R}^n \times \mathbb{R}^n)$; see [Hörmander 1985, Theorem 18.1.17 and Lemma 18.1.18].

We define $S^m_{c,\mathrm{ph}}(T^*\mathcal{L})$ as the set of polyhomogeneous symbols of order m on $T^*\mathcal{L}$ with compact support in the variables $(t, x) \in \mathcal{L}$ (note that compactness with respect to $x \in \mathcal{M}$ is obvious). Having the manifold $\mathcal{M}$ smooth is important for symbols and following pseudodifferential operators to be simply defined.

For any m, the restriction to the sphere

$$S^m_{c,\mathrm{ph}}(T^*\mathcal{L}) \to \mathscr{C}^\infty_c(S^*\mathcal{L}), \quad a \to a_{|S^*\mathcal{L}}, \tag{2-4}$$

is onto. This allows one to identify a homogeneous symbol with a smooth function on $S^*\mathcal{L}$ with compact support.

We denote by $\Psi^m_{c,\mathrm{ph}}(\mathcal{L})$ the space of polyhomogeneous pseudodifferential operators of order m on $\mathcal{L}$: one says that $Q \in \Psi^m_{c,\mathrm{ph}}(\mathcal{L})$ if Q maps $\mathscr{C}^\infty_c(\mathcal{L})$ into $\mathscr{D}'(\mathcal{L})$ and

(1) its kernel $K(x, y) \in \mathscr{D}'(\mathcal{L} \times \mathcal{L})$ is such that $\mathrm{supp}(K)$ is compact in $\mathcal{L} \times \mathcal{L}$;

(2) $K(x, y)$ is smooth away from the diagonal $\Delta_{\mathcal{L}} = \{(t, x; t, x) : (t, x) \in \mathcal{L}\}$;

(3) for any local chart $\mathcal{C}_j = (\mathcal{O}_j, \vartheta_j)$ and all $\phi_0, \phi_1 \in \mathscr{C}^\infty_c(\widetilde{\mathcal{O}}_j)$ one has

$$\phi_1 \circ (\vartheta_j^{-1})^* \circ Q \circ \vartheta_j^* \circ \phi_0 \in \mathrm{Op}(S^m_{c,\mathrm{ph}}(\mathbb{R}^{d+1} \times \mathbb{R}^{d+1})).$$

For $Q \in \Psi^m_{c,\mathrm{ph}}(\mathcal{L})$, we denote by $\sigma_m(Q) \in S^m_{c,\mathrm{ph}}(T^*\mathcal{L})$ the principal symbol of Q; see [Hörmander 1985, Chapter 18.1]. Note that the principal symbol is uniquely defined in $S^m_{c,\mathrm{ph}}(T^*\mathcal{L})$ because of the polyhomogeneous structure; see the remark following Definition 18.1.20 in [Hörmander 1985]. The application σ_m enjoys the following properties:

(1) The map $\sigma_m : \Psi^m_{c,\text{ph}}(\mathcal{L}) \to S^m_{c,\text{ph}}(T^*\mathcal{L})$ is onto.

(2) For all $Q \in \Psi^m_{c,\text{ph}}(\mathcal{L})$, $\sigma_m(Q) = 0$ if and only if $Q \in \Psi^{m-1}_{c,\text{ph}}(\mathcal{L})$.

(3) For all $Q \in \Psi^m_{c,\text{ph}}(\mathcal{L})$, $\sigma_m(Q^*) = \overline{\sigma_m(Q)}$.

(4) For all $Q_1 \in \Psi^{m_1}_{c,\text{ph}}(\mathcal{L})$ and $Q_2 \in \Psi^{m_2}_{c,\text{ph}}(\mathcal{L})$, one has $Q_1Q_2 \in \Psi^{m_1+m_2}_{c,\text{ph}}(\mathcal{L})$ with
$$\sigma_{m_1+m_2}(Q_1Q_2) = \sigma_{m_1}(Q_1)\sigma_{m_2}(Q_2).$$

(5) For all $Q_1 \in \Psi^{m_1}_{c,\text{ph}}(\mathcal{L})$ and $Q_2 \in \Psi^{m_2}_{c,\text{ph}}(\mathcal{L})$, one has $[Q_1, Q_2] = Q_1Q_2 - Q_2Q_1 \in \Psi^{m_1+m_2-1}_{c,\text{ph}}(\mathcal{L})$, with
$$\sigma_{m_1+m_2-1}([Q_1, Q_2]) = \frac{1}{i}\{\sigma_{m_1}(Q_1), \sigma_{m_2}(Q_2)\}.$$

(6) If $Q \in \Psi^m_{c,\text{ph}}(\mathcal{L})$, then Q maps continuously $H^k_{\text{loc}}(\mathcal{L})$ into $H^{k-m}_{\text{comp}}(\mathcal{L})$. In particular, for $m < 0$, Q is compact on $L^2_{\text{loc}}(\mathcal{L})$.

Given an operator $Q \in \Psi^m_{c,\text{ph}}(\mathcal{L})$, one sets
$$\text{Char}(Q) = \text{Char}(\sigma_m(Q)) = \{\varrho \in T^*\mathcal{L} : \sigma_m(Q)(\varrho) = 0\}.$$

3. Microlocal defect measure and propagation properties

A defect measure is used to characterize locally the failure of a sequence to strongly converge, meaning some concentration phenomenon. This characterization can be made finer by further considering microlocal concentration phenomena.

3A. ***Microlocal defect density measures.*** We define $\mathcal{M}_+(S^*\mathcal{L})$ as the set of positive density measures on $S^*\mathcal{L}$. For $\mu \in \mathcal{M}_+(S^*\mathcal{L})$ and $a \in \mathscr{C}^0_c(S^*\mathcal{L})$, we shall write
$$\langle \mu, a\rangle_{S^*\mathcal{L}} = \int_{S^*\mathcal{L}} a(\varrho)\mu(d\varrho)$$
for the duality bracket. This notation will also be used for $a \in S^0_{c,\text{ph}}(T^*\mathcal{L})$ according to the identification map (2-4).

Consider a sequence $(u^k)_{k\in\mathbb{N}} \subset L^2_{\text{loc}}(\mathcal{L})$ that converges weakly to 0. Here, to define the L^2-norm and inner product on $\mathcal{L}$ we use a fixed (κ^0, g^0) chosen in $\mathcal{X}^1(\mathcal{M})$; see (1-2).

As a consequence of [Gérard 1991, Theorem 1], there exists a subsequence of $(u^k)_{k\in\mathbb{N}}$ (still denoted by $(u^k)_{k\in\mathbb{N}}$ in what follows) and a density measure $\mu \in \mathcal{M}_+(S^*\mathcal{L})$ such that
$$\lim_{k\to\infty} \langle Qu^k, \overline{u^k}\rangle_{L^2_{\text{comp}}(\mathcal{L}), L^2_{\text{loc}}(\mathcal{L})} = \langle \mu, \sigma_0(Q)\rangle_{S^*\mathcal{L}} \tag{3-1}$$
for any $Q \in \Psi^0_{c,\text{ph}}(\mathcal{L})$. Recall that symbols in $S^0_{c,\text{ph}}(T^*\mathcal{L})$ are compactly supported in time t here. We also refer to [Tartar 1990; Burq 1997b]. One calls μ a microlocal defect (density) measure associated with $(u^k)_{k\in\mathbb{N}}$.

Similarly, one can use the notion of H^1-microlocal defect density measure. Consider $(u^k)_{k\in\mathbb{N}} \subset H^1_{\mathrm{loc}}(\mathcal{L})$ that converges weakly to 0. Then, there exists a subsequence of $(u^k)_{k\in\mathbb{N}}$ (still denoted by $(u^k)_{k\in\mathbb{N}}$) and a density measure $\mu \in \mathcal{M}_+(S^*\mathcal{L})$ such that for any $Q \in \Psi^2_{c,\mathrm{ph}}(\mathcal{L})$

$$\lim_{k\to\infty} \langle Qu^k, \overline{u^k}\rangle_{H^{-1}_{\mathrm{comp}}(\mathcal{L}),H^1_{\mathrm{loc}}(\mathcal{L})} = \langle \mu, \sigma_2(Q)\rangle_{S^*\mathcal{L}}. \tag{3-2}$$

Naturally, in either cases, the density measure μ depends on the choice made of $(\kappa^0, g^0) \in \mathcal{X}^1(\mathcal{M})$. In what follows we shall make clear what choice is made.

3B. *Local representatives.* Consider a finite atlas $\mathcal{A} = (\mathcal{C}_j)_{j\in J}$ on $\mathcal{L}$, as introduced in Section 2, with $\mathcal{C}_j = (\mathcal{O}_j, \vartheta_j)$. Consider a smooth partition of unity $(\chi_j)_{j\in J}$ subordinated to the covering by the open sets $(\mathcal{O}_j)_j$. We consider also $\tilde{\chi}_j, \hat{\chi}_j \in \mathscr{C}^\infty(\mathcal{L})$ supported in $\mathcal{O}_j$ such that $\tilde{\chi}_j \equiv 1$ on a neighborhood of $\mathrm{supp}(\chi_j)$ and $\hat{\chi}_j \equiv 1$ on a neighborhood of $\mathrm{supp}(\tilde{\chi}_j)$. Set also $\chi_j^{\mathcal{C}_j} = (\vartheta_j^{-1})^*\chi_j$, $\tilde{\chi}_j^{\mathcal{C}_j} = (\vartheta_j^{-1})^*\tilde{\chi}_j$, and $\hat{\chi}_j^{\mathcal{C}_j} = (\vartheta_j^{-1})^*\hat{\chi}_j$. One has $\chi_j^{\mathcal{C}_j}, \tilde{\chi}_j^{\mathcal{C}_j}, \hat{\chi}_j^{\mathcal{C}_j} \in \mathscr{C}^\infty_c(\widetilde{\mathcal{O}}_j)$, with $\widetilde{\mathcal{O}}_j = \vartheta_j(\mathcal{O}_j)$.

Let $(u^k)_k \subset H^1_{\mathrm{loc}}(\mathcal{L})$ that converges weakly to 0, $Q \in \Psi^2_{c,\mathrm{ph}}(\mathcal{L})$, and $j \in J$. One can write

$$\chi_j Q = \chi_j Q \tilde{\chi}_j + \chi_j Q(1 - \tilde{\chi}_j).$$

Since $\chi_j Q(1-\tilde{\chi}_j)$ is a regularizing operator one finds

$$\begin{aligned}\langle \mu, \chi_j\sigma_2(Q)\rangle_{S^*\mathcal{L}} &\sim \langle \chi_j Qu^k, \overline{u^k}\rangle_{H^{-1}_{\mathrm{comp}}(\mathcal{L}),H^1_{\mathrm{loc}}(\mathcal{L})}\\ &\sim \langle \chi_j\tilde{\chi}_j Q\tilde{\chi}_j v_j^k, \overline{v_j^k}\rangle_{H^{-1}_{\mathrm{comp}}(\mathcal{L}),H^1_{\mathrm{loc}}(\mathcal{L})} \quad \text{as } k \to +\infty,\end{aligned}$$

for $v_j^k = \hat{\chi}_j u^k$.

The operator $Q_j = (\vartheta_j^{-1})^*\tilde{\chi}_j Q\tilde{\chi}_j(\vartheta_j)^*$ is a pseudodifferential operator of order 2 on $\mathbb{R}^{d+1}$ with principal symbol $q_j = \tilde{\chi}_j^2 q^{\mathcal{C}_j}$, where $q^{\mathcal{C}_j}$ is the local representative of $\sigma_2(Q)$. Set also $v_j^{k,\mathcal{C}_j} = (\vartheta_j^{-1})^* v_j^k$. It converges weakly to 0 in $H^1(\mathbb{R}^{d+1})$. Associated with this sequence is a microlocal defect measure μ_j. If one writes

$$\langle \chi_j\tilde{\chi}_j Q\tilde{\chi}_j v_j^k, \overline{v_j^k}\rangle_{H^{-1}_{\mathrm{comp}}(\mathcal{L}),H^1_{\mathrm{loc}}(\mathcal{L})} = \langle \chi_j^{\mathcal{C}_j} Q_j v_j^{k,\mathcal{C}_j}, \overline{v_j^{k,\mathcal{C}_j}}\rangle_{H^{-1}_{\mathrm{comp}}(\mathbb{R}^{d+1}),H^1_{\mathrm{loc}}(\mathbb{R}^{d+1})},$$

one obtains

$$\langle \mu, \chi_j\sigma_2(Q)\rangle_{S^*\mathcal{L}} = \langle \mu_j, \chi_j^{\mathcal{C}_j} q_j\rangle_{S^*\widetilde{\mathcal{O}}^j} = \langle \mu_j, \chi_j^{\mathcal{C}_j} q^{\mathcal{C}_j}\rangle_{S^*\widetilde{\mathcal{O}}^j}.$$

Note that here, the L^2 and H^s-norms on $\mathbb{R}^{d+1}$ are based on the local representative of the density measure $\kappa^0\mu_{g^0}\,dt$. One thus sees that the local representative of $\chi_j\mu$ is precisely $\chi_j^{\mathcal{C}_j}\mu_j$, that is, $\chi_j\mu = \vartheta_j^*(\chi_j^{\mathcal{C}_j}\mu_j) = \chi_j\vartheta_j^*\mu_j$. Summing up, we thus have

$$\mu = \sum_{j\in J}\chi_j\mu = \sum_{j\in J}\chi_j\vartheta_j^*\mu_j$$

and

$$\langle \mu, \sigma_2(Q)\rangle_{S^*\mathcal{L}} = \sum_{j\in J}\langle \mu, \chi_j\sigma_2(Q)\rangle_{S^*\mathcal{L}} = \sum_{j\in J}\langle \mu_j, \chi_j^{\mathcal{C}_j} q^{\mathcal{C}_j}\rangle_{S^*\widetilde{\mathcal{O}}^j}.$$

Remark 3.1. Local properties of microlocal defect measures like μ can be deduced from the properties of $\chi_j^{\mathcal{C}_j}\mu_j$. In what follows most results are of local nature. In such cases we shall work in local charts and use Sections 2C and 3B to bring the analysis to open domains of $\mathbb{R}^{d+1}$.

3C. ***Operators with a low regularity.*** An important tool we use to handle low-regularity terms in what follows is a result due to R. Coifman and Y. Meyer [1978, Proposition IV.7] and some of its consequences that we list below.

Theorem 3.2 (Coifman–Meyer). *Let $Q \in \Psi^1_{\mathrm{ph}}(\mathbb{R}^n \times \mathbb{R}^n)$. If $m \in W^{1,\infty}(\mathbb{R}^n)$ the commutator $[Q, m]$ maps $L^2(\mathbb{R}^n)$ into itself continuously. Moreover there exists $C > 0$ such that*

$$\big\|[Q, m]\big\|_{L^2\to L^2} \le C\|m\|_{W^{1,\infty}}, \quad m \in W^{1,\infty}(\mathbb{R}^n).$$

We deduce the following corollary.

Corollary 3.3. *Let $Q \in \Psi^1_{\mathrm{ph}}(\mathbb{R}^n \times \mathbb{R}^n)$ be such that its kernel has compact support in $\mathbb{R}^n \times \mathbb{R}^n$. With $q \in S^1_{\mathrm{ph}}(\mathbb{R}^n \times \mathbb{R}^n)$ its principal symbol.*

Let $m \in \mathscr{C}^1(\mathbb{R}^n)$. There exist K_1 and K_2, compact operators on $L^2(\mathbb{R}^n)$, with compactly supported kernels, such that

$$[Q, m] = \frac{1}{i}\nabla_x m \cdot \mathrm{Op}(\nabla_\xi q) + K_1 = \frac{1}{i}\,\mathrm{Op}(\nabla_\xi q)\cdot \nabla_x m + K_2. \tag{3-3}$$

Proof. Consider a sequence $(m^k)_{k\in\mathbb{N}} \subset \mathscr{C}^\infty(\mathbb{R}^n)$ such that

$$\sum_{|\alpha|\le 1} \|\partial_x^\alpha (m^k - m)\|_{L^\infty} \to 0 \quad \text{as } k \to +\infty.$$

Classical symbolic calculus gives

$$[Q, m^k] = \frac{1}{i}\nabla_x m^k \cdot \mathrm{Op}(\nabla_\xi q) + K_1^k, \tag{3-4}$$

with $K_1^k = \mathrm{Op}(r_1^k)$ for some $r_1^k \in S^{-1}_{\mathrm{ph}}$, $j = 1, 2$. Thus, K_1^k is bounded from $L^2(\mathbb{R}^n)$ into $H^1(\mathbb{R}^n)$. In addition, since K_1^k has a kernel with compact supports in $\mathbb{R}^n \times \mathbb{R}^n$, it is compact on $L^2(\mathbb{R}^n)$. Note that the support of the kernel of K_1^k lies in a compact $\mathcal{K}$ of $\mathbb{R}^n \times \mathbb{R}^n$ that is uniform with respect to k.

On the other hand, observe that

$$\nabla_x m^k \cdot \mathrm{Op}(\nabla_\xi q) \to \nabla_x m \cdot \mathrm{Op}(\nabla_\xi q) \quad \text{in } \mathscr{L}(L^2(\mathbb{R}^n)).$$

Moreover, from Theorem 3.2 applied to $m^k - m$, one also has

$$[Q, m^k] \to [Q, m] \quad \text{in } \mathscr{L}(L^2(\mathbb{R}^n)).$$

Using then (3-4) we deduce that $(K_1^k)_{n\in\mathbb{N}}$ converges to some K^1 in $\mathscr{L}(L^2(\mathbb{R}^n))$, and from the closedness of the set of compact operators in $\mathscr{L}(L^2(\mathbb{R}^n))$ we find that K^1 is compact. Moreover, K^1 has a kernel supported in $\mathcal{K}$. The limits above give the first equality in (3-3). The second equality follows similarly. □

Let Ω be a bounded open set of $\mathbb{R}^n$ and $(\kappa^0, g^0) \in \mathcal{X}^1(\Omega)$, with definition adapted from that of $\mathcal{X}^1(\mathcal{M})$. The L^2-inner product and norm are given by the density $\kappa^0\mu_{g^0}$. The following result is also a consequence of Theorem 3.2.

Proposition 3.4. *Let $(u^k)_{k\in\mathbb{N}} \subset H^1_{\mathrm{loc}}(\Omega)$ be a sequence that converges weakly to 0 and let μ be an H^1-microlocal defect density measure on $S^*\Omega$ associated with the sequence $(u^k)_k$.*

Let $b_1 \in W^{1,\infty}(\mathbb{R}^n)$ and $b_2 \in \mathscr{C}^0(\mathbb{R}^n)$. Consider also $Q_1, Q_2 \in \Psi^1_{\mathrm{ph}}(\mathbb{R}^n)$, both with kernels compactly supported in $\Omega \times \Omega$, with $q_1, q_2 \in S^1_{\mathrm{ph}}(\mathbb{R}^n \times \mathbb{R}^n)$ for respective principal symbol. Then, one has

$$\langle b_1 Q_1\, b_2\, Q_2 u^k, \overline{u^k}\rangle_{H^{-1}_{\mathrm{comp}}(\Omega), H^1_{\mathrm{loc}}(\Omega)} \xrightarrow[k\to+\infty]{} \langle \mu, b_1 b_2 q_1 q_2\rangle_{S^*\Omega}. \tag{3-5}$$

More generally, assume that $(b_1^k)_{k\in\mathbb{N}} \subset W^{1,+\infty}(\mathbb{R}^n)$ and $(b_2^k)_{k\in\mathbb{N}} \subset L^\infty(\mathbb{R}^n)$, and $(\kappa_k, g_k)_{k\in\mathbb{N}} \subset \mathcal{Y}(\Omega)$ with

$$\|b_1^k - b_1\|_{W^{1,+\infty}(\mathbb{R}^n)} + \|b_2^k - b_2\|_{L^\infty(\mathbb{R}^n)} + \|(\kappa_k, g_k) - (\kappa^0, g^0)\|_{\mathcal{Y}(\Omega)} \to 0 \quad \text{as } k \to +\infty.$$

Then

$$\langle b_1^k Q_1\, b_2^k\, Q_2 u^k, \overline{u^k}\rangle_{H^{-1}_{\mathrm{comp}}(\Omega,\kappa_k\mu_{g_k}), H^1_{\mathrm{loc}}(\Omega,\kappa_k\mu_{g_k})} \xrightarrow[k\to+\infty]{} \langle \mu, b_1 b_2 q_1 q_2\rangle_{S^*\Omega}. \tag{3-6}$$

Remark 3.5. Note that b_1 is chosen in $W^{1,\infty}(\mathbb{R}^n)$ because one cannot multiply an element in H^{-1} by a bounded function. One derivative is needed.

Proof of Proposition 3.4. With Lemma 3.6 below we may replace the density $\kappa_k \mu_{g_k}$ in the L^2-inner product by $\kappa^0 \mu_{g^0}$ and thus in the H^{-1}_{comp}-H^1_{loc} duality.

We write

$$b_1^k Q_1\, b_2^k\, Q_2 = b_1 Q_1\, b_2\, Q_2 + R^k, \quad R^k = b_1 Q_1\,(b_2^k - b_2)\,Q_2 + (b_1^k - b_1) Q_1\, b_2^k\, Q_2.$$

Note that R^k maps $H^1_{\mathrm{loc}}(\Omega)$ into $H^{-1}_{\mathrm{comp}}(\Omega)$ continuously. Moreover because of the convergences of b_1^k and b_2^k, and the boundedness of $(u^k)_{k\in\mathbb{N}}$ in $H^1_{\mathrm{loc}}(\Omega)$, one finds that $R^k u^k \to 0$ strongly in $H^{-1}_{\mathrm{comp}}(\Omega)$. Thus we can write

$$\langle b_1^k Q_1\, b_2^k\, Q_2 u^k, \overline{u^k}\rangle_{H^{-1}_{\mathrm{comp}}(\Omega), H^1_{\mathrm{loc}}(\Omega)} = \langle b_1 Q_1\, b_2\, Q_2 u^k, \overline{u^k}\rangle_{H^{-1}_{\mathrm{comp}}(\Omega), H^1_{\mathrm{loc}}(\Omega)} + o(1)_{k\to+\infty}$$

and (3-6) follows if we prove (3-5).

According to Theorem 3.2 the commutator $[b_1, Q_1]$ is bounded on $L^2(\Omega)$ implying $[b_1, Q_1]\, b_2\, Q_2 u^k$ is bounded in $L^2(\Omega)$ yielding

$$\langle [b_1, Q_1]\, b_2\, Q_2 u^k, \overline{u^k}\rangle_{H^{-1}_{\mathrm{comp}}(\Omega), H^1_{\mathrm{loc}}(\Omega)} = ([b_1, Q_1]\, b_2\, Q_2 u^k, u^k)_{L^2(\Omega)} \xrightarrow[k\to+\infty]{} 0,$$

since $u^k \to 0$ strongly in $L^2(\Omega)$. We may thus assume that $b_1 = 1$ without any loss of generality.

Let $\varepsilon > 0$ and let $b_2^\varepsilon \in \mathscr{C}^\infty(\Omega)$ be such that $\|b_2 - b_2^\varepsilon\|_{L^\infty} \le \varepsilon$. Write

$$Q_1\, b_2\, Q_2 = Q_1\, b_2^\varepsilon\, Q_2 + R^\varepsilon, \quad R^\varepsilon = Q_1\,(b_2 - b_2^\varepsilon)\,Q_2.$$

One has $|\langle R^\varepsilon u^k, \overline{u^k}\rangle_{H^{-1}_{\mathrm{comp}}(\Omega), H^1_{\mathrm{loc}}(\Omega)}| \le C\varepsilon$, and this leads to

$$\langle Q_1\, b_2\, Q_2 u^k, \overline{u^k}\rangle_{H^{-1}_{\mathrm{comp}}(\Omega), H^1_{\mathrm{loc}}(\Omega)} = \langle Q_1\, b_2^\varepsilon\, Q_2 u^k, \overline{u^k}\rangle_{H^{-1}_{\mathrm{comp}}(\Omega), H^1_{\mathrm{loc}}(\Omega)} + o(1)_{\varepsilon\to 0} + o(1)_{k\to+\infty}. \tag{3-7}$$

Since b_2^ε is smooth, by symbolic calculus one has

$$\langle Q_1\, b_2^\varepsilon\, Q_2 u^k, \overline{u^k}\rangle_{H^{-1}_{\mathrm{comp}}(\Omega), H^1_{\mathrm{loc}}(\Omega)} \xrightarrow[k\to+\infty]{} \langle \mu, b_2^\varepsilon q_1 q_2\rangle_{S^*\Omega}. \tag{3-8}$$

Finally, since $\langle \mu, b_2^\varepsilon q_1 q_2\rangle_{S^*\Omega} \to \langle \mu, b_2 q_1 q_2\rangle_{S^*\Omega}$ as $\varepsilon \to 0$, with (3-7) and (3-8) one concludes that (3-5) holds. □

Lemma 3.6. *Assume that* $\|(\kappa_k, g_k) - (\kappa^0, g^0)\|_{\mathcal{Y}(\Omega)} \to 0$ *and consider a sequence* $(f_k, h_k)_{k\in\mathbb{N}}$ *bounded in* $L^2_{\mathrm{comp}}(\Omega) \oplus L^2_{\mathrm{loc}}(\Omega)$. *Then*

$$(f_k, h_k)_{L^2(\Omega, \kappa_k \mu_{g_k})} = (f_k, h_k)_{L^2(\Omega)} + o(1)_{k\to+\infty}.$$

If $(f_k, h_k)_{k\in\mathbb{N}}$ *is bounded in* $H^{-1}_{\mathrm{comp}}(\Omega) \oplus H^1_{\mathrm{loc}}(\Omega)$ *then*

$$\langle f_k, \overline{h^k}\rangle_{H^{-1}_{\mathrm{comp}}(\Omega, \kappa_k\mu_{g_k}), H^1_{\mathrm{loc}}(\Omega, \kappa_k\mu_{g_k})} = \langle f_k, \overline{h^k}\rangle_{H^{-1}_{\mathrm{comp}}(\Omega), H^1_{\mathrm{loc}}(\Omega)} + o(1)_{k\to+\infty}.$$

Here, Lemma 3.6 is written in the case of a bounded open set of the Euclidean space but the same result holds in the case of a compact manifold.

Proof. One has $\mu_{g^0} = (\det g^0)^{1/2}\, dx$ and $\mu_{g_k} = (\det g_k)^{1/2}\, dx$. Therefore $\kappa_k \mu_{g_k} = \alpha_k \kappa^0 \mu_{g^0}$, with

$$\alpha_k = \frac{\kappa_k}{\kappa^0}\left(\frac{\det g_k}{\det g^0}\right)^{1/2}$$

and $\alpha_k \to 1$ in the Lipschitz norm. □

3D. *Measures and partial differential equations.* Microlocal defect measures associated with sequences of solutions of partial differential equations with smooth coefficients can have properties such as support localization in the characteristic set and invariance along the Hamiltonian flow. With the material developed above, we extend these results to the case of $\mathscr{C}^1$-coefficients. We focus on the case of wave operators.

Proposition 3.7. *Let* $(\kappa^0, g^0) \in \mathcal{X}^1(\mathcal{M})$ *and set* $P^0 = P_{\kappa^0, g^0}$. *Denote by* $p^0(x, \tau, \xi) = -\tau^2 + g^0_x(\xi, \xi)$ *its principal symbol. Let* $(\kappa_k, g_k)_{k\in\mathbb{N}} \subset \mathcal{Y}(\mathcal{M})$ *be such that* $\|(\kappa_k, g_k) - (\kappa^0, g^0)\|_{\mathcal{Y}(\mathcal{M})} \to 0$ *as* $k \to +\infty$ *and set* $P_k = P_{\kappa_k, g_k}$.

Consider a sequence $(u^k)_{k\in\mathbb{N}} \subset H^1_{\mathrm{loc}}(\mathcal{L})$ *that converges to* 0 *weakly and* μ *an* H^1*-microlocal defect density measure associated with* $(u^k)_{k\in\mathbb{N}}$.

Let $T_1 < T_2$. *The following properties hold*:

(1) *If* $P_k u^k \to 0$ *strongly in* $H^{-1}_{\mathrm{loc}}((T_1, T_2) \times \mathcal{M})$ *then*

$$\operatorname{supp}(\mu) \cap S^*((T_1, T_2) \times \mathcal{M}) \subset \operatorname{Char}(p^0). \tag{3-9}$$

(2) *If moreover* $P_k u^k \to 0$ *strongly in* $L^2_{\mathrm{loc}}((T_1, T_2) \times \mathcal{M})$ *then one has*

$${}^t\mathrm{H}_{p^0}\mu = 0 \textit{ in the sense of distributions on } S^*((T_1, T_2) \times \mathcal{M}), \tag{3-10}$$

that is, $\langle \mu, \mathrm{H}_{p^0} q\rangle_{S^*\mathcal{L}} = 0$ *for all* $q \in \mathscr{C}^\infty_c(S^*((T_1, T_2) \times \mathcal{M}))$.

Since H_{p^0} is a tangent vector field on $S^*\mathcal{L}$ where μ lives (see Section 2A) note that ${}^t\mathrm{H}_{p^0}\mu$ makes sense in the second item of the proposition. Moreover note that H_{p^0} is a tangent vector field on $S^*\mathcal{L} \cap \operatorname{Char}(p^0)$ and one has $\operatorname{supp}(\mu) \cap S^*((T_1, T_2) \times \mathcal{M}) \subset \operatorname{Char}(p^0)$ by the first item of the proposition. Finally, notice that for a Hamiltonian vector field, $\mathrm{H}_{p^0} = -{}^t\mathrm{H}_{p^0}$ as recalled in Section 2A even in the case $(\kappa^0, g^0) \in \mathcal{X}^1(\mathcal{M})$.

Naturally, Proposition 3.7 and its proof can be adapted to the other energy levels. We shall also need the following result.

Proposition 3.7′. *With the notation of Proposition 3.7, consider a sequence* $(u^k)_{k\in\mathbb{N}} \subset L^2_{\mathrm{loc}}(\mathcal{L})$ *that converges to* 0 *weakly and* μ *an* L^2*-microlocal defect density measure associated with* $(u^k)_{k\in\mathbb{N}}$.

Let $T_1 < T_2$. The following properties hold.

(1) *If $P_k u^k \to 0$ strongly in $H^{-2}_{\mathrm{loc}}((T_1, T_2) \times \mathcal{M})$ then*

$$\operatorname{supp}(\mu) \cap S^*((T_1, T_2) \times \mathcal{M}) \subset \operatorname{Char}(p^0).$$

(2) *If moreover $P_k u^k \to 0$ strongly in $H^{-1}_{\mathrm{loc}}((T_1, T_2) \times \mathcal{M})$ then one has*

$$ {}^t\mathrm{H}_{p^0}\mu = 0 \text{ in the sense of distributions on } S^*((T_1, T_2) \times \mathcal{M}).$$

Proof of Proposition 3.7. Consider $B \in \Psi^0_{c,\mathrm{ph}}(\mathcal{L})$ with kernel supported in $((T_1, T_2) \times \mathcal{M})^2$ and $b \in S^0_{c,\mathrm{ph}}(\mathcal{L})$ its principal symbol. For the definition of the L^2-inner product we use (κ^0, g^0). We also use the partition of unity $1 = \sum_{j \in J} \chi_j$, with $\chi_j \in \mathscr{C}^\infty_c(\mathcal{O}_j)$ associated with the atlas $\mathcal{A}$ and the additional cutoff functions $\tilde{\chi}_j, \hat{\chi}_j \in \mathscr{C}^\infty_c(\mathcal{O}_j)$ that are introduced in Section 3B and, as obtained in that section, we write

$$\begin{aligned}\langle B P_k u^k, \overline{u^k}\rangle_{H^{-1}_{\mathrm{comp}}(\mathcal{L}), H^1_{\mathrm{loc}}(\mathcal{L})} &= \sum_{j \in J} \langle \chi_j B P_k u^k, \overline{u^k}\rangle_{H^{-1}_{\mathrm{comp}}(\mathcal{L}), H^1_{\mathrm{loc}}(\mathcal{L})} \\ &= \sum_{j \in J} \langle \chi_j \tilde{\chi}_j B P_k \tilde{\chi}_j v^k_j, \overline{v^k_j}\rangle_{H^{-1}_{\mathrm{comp}}(\mathcal{L}), H^1_{\mathrm{loc}}(\mathcal{L})} + o(1)_{k \to +\infty},\end{aligned} \tag{3-11}$$

with $v^k_j = \hat{\chi}_j u^k$. Associated with $(\vartheta_j^{-1})^* v^k_j$, the local representative of v^k_j, is a microlocal defect measure μ_j in $\vartheta_j(\mathcal{O}_j) = \widetilde{\mathcal{O}}_j = \mathbb{R} \times \widetilde{O}_j$ and $\chi_j^{\mathcal{C}_j} \mu_j$ is the local representative of $\chi_j \mu$ in this chart. See Section 3B.

Note that we use local representatives of the operators, functions, and measures without introducing any new symbols. Yet to keep clear that the analysis is carried out in a local chart we use the notation $L^2(\widetilde{\mathcal{O}}_j)$, $H^s(\widetilde{\mathcal{O}}_j)$ and not $L^2(\mathcal{L})$, $H^s(\mathcal{L})$. To further lighten notation we set $\tilde{\kappa}_k = (\det g_k)^{1/2}\kappa_k$. One has

$$P_k = \partial_t^2 - (\tilde{\kappa}_k)^{-1} \sum_{p,q} \partial_p \tilde{\kappa}_k g_k^{pq} \partial_q + 1 = \widetilde{P}_k - \sum_{p,q} R_k^{p,q},$$

with $\widetilde{P}_k = \partial_t^2 - \sum_{p,q} \partial_p g_k^{pq} \partial_q + 1$ and $R_k^{p,q} = (\tilde{\kappa}_k)^{-1}[\partial_p, \tilde{\kappa}_k] g_k^{pq} \partial_q$. Note that $\tilde{\chi}_j B R_k^{p,q} \tilde{\chi}_j$ defines a sequence of bounded operators from $H^1(\mathcal{L})$ into $L^2(\mathcal{L})$, uniformly with respect to k. Consequently, one has

$$\langle \chi_j \tilde{\chi}_j B R_k^{p,q} \tilde{\chi}_j v^k_j, \overline{v^k_j}\rangle_{H^{-1}_{\mathrm{comp}}(\widetilde{\mathcal{O}}_j), H^1_{\mathrm{loc}}(\widetilde{\mathcal{O}}_j)} = (\chi_j \tilde{\chi}_j B R_k^{p,q} \tilde{\chi}_j v^k_j, v^k_j)_{L^2(\widetilde{\mathcal{O}}_j)} \xrightarrow[k \to +\infty]{} 0$$

since v^k_j converges strongly to 0 in $L^2(\widetilde{\mathcal{O}}_j)$. This leads to

$$\begin{aligned}\langle \chi_j \tilde{\chi}_j B P_k \tilde{\chi}_j v^k_j, \overline{v^k_j}\rangle_{H^{-1}_{\mathrm{comp}}(\widetilde{\mathcal{O}}_j), H^1_{\mathrm{loc}}(\widetilde{\mathcal{O}}_j)} &= \langle \chi_j \tilde{\chi}_j B \widetilde{P}_k \tilde{\chi}_j v^k_j, \overline{v^k_j}\rangle_{H^{-1}_{\mathrm{comp}}(\widetilde{\mathcal{O}}_j), H^1_{\mathrm{loc}}(\widetilde{\mathcal{O}}_j)} + o(1)_{k \to +\infty} \\ &= \langle \mu_j, \chi_j b p^0\rangle_{S^*(\widetilde{\mathcal{O}}_j)} + o(1)_{k \to +\infty},\end{aligned}$$

by Proposition 3.4. Since $\chi_j \mu_j = \chi_j \mu$ locally, lifting back the analysis to the manifold level, with (3-11), one finds

$$\langle B P_k u^k, \overline{u^k}\rangle_{H^{-1}_{\mathrm{comp}}(\mathcal{L}), H^1_{\mathrm{loc}}(\mathcal{L})} = \sum_{j \in J} \langle \mu, \chi_j b p^0\rangle_{S^*(\mathcal{L})} = \langle \mu, b p^0\rangle_{S^*(\mathcal{L})} + o(1)_{k \to +\infty}.$$

Now, one has

$$\langle B P_k u^k, \overline{u^k}\rangle_{H^{-1}_{\mathrm{comp}}(\mathcal{L}), H^1_{\mathrm{loc}}(\mathcal{L})} = \langle P_k u^k, {}^t B \overline{u^k}\rangle_{H^{-1}_{\mathrm{loc}}(\mathcal{L}), H^1_{\mathrm{comp}}(\mathcal{L})} + o(1)_{k \to +\infty},$$

with the transpose operator ${}^tB$ bounded from $H^1_{\mathrm{loc}}(\mathcal{L})$ into $H^1_{\mathrm{comp}}(\mathcal{L})$ since B is itself bounded from $H^{-1}_{\mathrm{loc}}(\mathcal{L})$ into $H^{-1}_{\mathrm{comp}}(\mathcal{L})$. If one assumes that $P_k u^k \to 0$ strongly in $H^{-1}_{\mathrm{loc}}((T_1, T_2) \times \mathcal{M})$, one obtains

$$\langle B P_k u^k, \overline{u^k} \rangle_{H^{-1}_{\mathrm{comp}}(\mathcal{L}), H^1_{\mathrm{loc}}(\mathcal{L})} \xrightarrow[k\to+\infty]{} 0,$$

and thus

$$\langle \mu, bp^0 \rangle_{S^*(\mathcal{L})} = 0 \quad \text{for all } b \in S^0_{c,\mathrm{ph}}(\mathcal{L}) \text{ with } \operatorname{supp}(b) \subset T^*((T_1, T_2) \times \mathcal{M}),$$

and one obtains the support estimation (3-9).

We now prove the second item of the proposition. We assume that $P_k u^k$ lies in $L^2_{\mathrm{loc}}((T_1, T_2)\times)$ and converges strongly to 0 in this space. Consider $B \in \Psi^1_{c,\mathrm{ph}}(\mathcal{L})$ with kernel supported in $((T_1, T_2) \times \mathcal{M})^2$ and $b \in S^1_{c,\mathrm{ph}}(\mathcal{L})$ its principal symbol. We are interested in the limit of $\langle [P_k, B] u^k, \overline{u^k} \rangle_{H^{-1}_{\mathrm{comp}}(\mathcal{L}), H^1_{\mathrm{loc}}(\mathcal{L})}$, which makes sense since $[P_k, B]$ is of order 2. We have $[P_k, B]u^k = P_k B u^k - B P_k u^k \in H^{-1}((T_1, T_2) \times \mathcal{M})$. Since $P_k u^k$ lies in $L^2((T_1, T_2) \times \mathcal{M})$ by assumption, $B P_k u^k$ lies in $H^{-1}((T_1, T_2) \times \mathcal{M})$ and the same holds for $P_k B u^k$. We may thus write

$$\langle [P_k, B] u^k, \overline{u^k} \rangle_{H^{-1}_{\mathrm{comp}}(\mathcal{L}), H^1_{\mathrm{loc}}(\mathcal{L})} = \langle P_k B u^k, \overline{u^k} \rangle_{H^{-1}_{\mathrm{comp}}(\mathcal{L}), H^1_{\mathrm{loc}}(\mathcal{L})} - \langle P_k u^k, \overline{B^* u^k} \rangle_{L^2_{\mathrm{loc}}(\mathcal{L}), L^2_{\mathrm{comp}}(\mathcal{L})},$$

where the adjoint is computed with respect to the L^2-inner product associated with (k^0, g^0) here. As B maps continuously $L^2_{\mathrm{loc}}((T_1, T_2) \times \mathcal{M})$ into $H^{-1}_{\mathrm{comp}}((T_1, T_2) \times \mathcal{M})$, we have B^* maps continuously $H^1_{\mathrm{loc}}(\mathcal{L})$ into $L^2_{\mathrm{comp}}(\mathcal{L})$. Thus, one has

$$(P_k u^k, B^* u^k)_{L^2(\mathcal{L})} \xrightarrow[k\to+\infty]{} 0.$$

By Lemma 3.6 it is asymptotically equivalent to use (κ^0, g^0) or (κ_k, g_k) for the definition of the L^2-inner product and H^{-1}_{comp}-H^1_{loc} duality, that is,

$$\langle P_k B u^k, \overline{u^k} \rangle_{H^{-1}_{\mathrm{comp}}(\mathcal{L}), H^1_{\mathrm{loc}}(\mathcal{L})} = \langle P_k B u^k, \overline{u^k} \rangle_{H^{-1}_{\mathrm{comp}}(\mathcal{L}, \kappa_k \mu_{g_k} dt), H^1_{\mathrm{loc}}(\mathcal{L}, \kappa_k \mu_{g_k} dt)} + o(1)_{k\to+\infty}.$$

Since P_k is selfadjoint for this latter L^2-inner product, one obtains

$$\begin{aligned} \langle P_k B u^k, \overline{u^k} \rangle_{H^{-1}_{\mathrm{comp}}(\mathcal{L}), H^1_{\mathrm{loc}}(\mathcal{L})} &= \langle B u^k, \overline{P_k u^k} \rangle_{L^2_{\mathrm{comp}}(\mathcal{L}, \kappa_k \mu_{g_k} dt), L^2_{\mathrm{loc}}(\mathcal{L}, \kappa_k \mu_{g_k} dt)} + o(1)_{k\to+\infty} \\ &= \langle B u^k, \overline{P_k u^k} \rangle_{L^2_{\mathrm{comp}}(\mathcal{L}), L^2_{\mathrm{loc}}(\mathcal{L})} + o(1)_{k\to+\infty}. \end{aligned}$$

Using again that $P_k u^k \to 0$ strongly to 0 in $L^2_{\mathrm{loc}}((T_1, T_2) \times \mathcal{M})$, we obtain

$$\langle P_k B u^k, \overline{u^k} \rangle_{H^{-1}_{\mathrm{comp}}(\mathcal{L}), H^1_{\mathrm{loc}}(\mathcal{L})} \xrightarrow[k\to+\infty]{} 0,$$

and finally

$$\langle [P_k, B] u^k, \overline{u^k} \rangle_{H^{-1}_{\mathrm{comp}}(\mathcal{L}), H^1_{\mathrm{loc}}(\mathcal{L})} \xrightarrow[k\to+\infty]{} 0. \tag{3-12}$$

As above, with the partition of unity $1 = \sum_{j\in J} \chi_j$ we write

$$\langle [P_k, B] u^k, \overline{u^k} \rangle_{H^{-1}_{\mathrm{comp}}(\mathcal{L}), H^1_{\mathrm{loc}}(\mathcal{L})} = \sum_{j\in J} \langle \chi_j [P_k, B] u^k, \overline{u^k} \rangle_{H^{-1}_{\mathrm{comp}}(\mathcal{L}), H^1_{\mathrm{loc}}(\mathcal{L})}. \tag{3-13}$$

For each term in the sum one has

$$\langle \chi_j [P_k, B] u^k, \overline{u^k} \rangle_{H^{-1}_{\mathrm{comp}}(\mathcal{L}), H^1_{\mathrm{loc}}(\mathcal{L})} = \langle \chi_j [P_k, \widetilde{B}_j] v^k_j, \overline{v^k_j} \rangle_{H^{-1}_{\mathrm{comp}}(\mathcal{L}), H^1_{\mathrm{loc}}(\mathcal{L})} + o(1)_{k\to+\infty},$$

with $\widetilde{B}_j = \tilde{\chi}_j B \tilde{\chi}_j$. This allows one to work in a local chart and write

$$\langle [P_k, B]u^k, \overline{u^k}\rangle_{H^{-1}_{\mathrm{comp}}(\mathcal{L}), H^1_{\mathrm{loc}}(\mathcal{L})} = \sum_{j\in J} \langle \chi_j [P_k, \widetilde{B}_j] v^k_j, \overline{v^k_j}\rangle_{H^{-1}_{\mathrm{comp}}(\widetilde{\mathcal{O}}_j), H^1_{\mathrm{loc}}(\widetilde{\mathcal{O}}_j)}, \tag{3-14}$$

with the (manifold-local chart) identifications described above. With $A_k = A_{\kappa_k, g_k}$, in the local chart $\mathcal{C}_j$ one writes

$$\chi_j[P_k, \widetilde{B}_j] = \chi_j[\partial_t^2, \widetilde{B}_j] - \chi_j[A_k, \widetilde{B}_j] = \chi_j[\partial_t^2, \widetilde{B}_j] - \sum_{1\le p,q\le d} (Q_1^{pq} + Q_2^{pq} + Q_3^{pq} + Q_4^{pq}),$$

with

$$Q_1^{pq} = \chi_j \tilde{\kappa}_k^{-1} \partial_{x_p} \tilde{\kappa}_k g_k^{pq} [\partial_{x_q}, \widetilde{B}_j], \quad Q_2^{pq} = \chi_j \tilde{\kappa}_k^{-1} \partial_{x_p} [\tilde{\kappa}_k g_k^{pq}, \widetilde{B}_j] \partial_{x_q},$$
$$Q_3^{pq} = \chi_j \tilde{\kappa}_k^{-1} [\partial_{x_p}, \widetilde{B}_j] \tilde{\kappa}_k g_k^{pq} \partial_{x_q}, \quad Q_4^{pq} = \chi_j [\tilde{\kappa}_k^{-1}, \widetilde{B}_j] \partial_{x_p} \tilde{\kappa}_k g_k^{pq} \partial_{x_q}.$$

We now compute the limit of each term associated with this decomposition of $[P_k, \widetilde{B}_j]$ on the right-hand side of (3-14). The principal symbol of $\chi_j[\partial_t^2, \widetilde{B}_j]$ is $i\chi_j\{\tau^2, b\}$ and thus

$$\langle \chi_j[\partial_t^2, \widetilde{B}_j] v_j^k, \overline{v_j^k}\rangle_{H^{-1}_{\mathrm{comp}}(\widetilde{\mathcal{O}}_j), H^1_{\mathrm{loc}}(\widetilde{\mathcal{O}}_j)} = \langle \mu_j, i\chi_j\{\tau^2, b\}\rangle_{S^*(\widetilde{\mathcal{O}}_j)} + o(1)_{k\to+\infty}.$$

Proposition 3.4 applies and yields

$$\langle Q_1^{pq} v_j^k, \overline{v_j^k}\rangle_{H^{-1}_{\mathrm{comp}}(\widetilde{\mathcal{O}}_j), H^1_{\mathrm{loc}}(\widetilde{\mathcal{O}}_j)} = \langle \mu_j, i\chi_j g^{0,pq} \xi_p \partial_{x_q} b\rangle_{S^*(\widetilde{\mathcal{O}}_j)} + o(1)_{k\to+\infty},$$
$$\langle Q_3^{pq} v_j^k, \overline{v_j^k}\rangle_{H^{-1}_{\mathrm{comp}}(\widetilde{\mathcal{O}}_j), H^1_{\mathrm{loc}}(\widetilde{\mathcal{O}}_j)} = \langle \mu_j, i\chi_j g^{0,pq} (\partial_{x_p} b) \xi_q\rangle_{S^*(\widetilde{\mathcal{O}}_j)} + o(1)_{k\to+\infty}.$$

With Theorem 3.2 one has $[\tilde{\kappa}_k g_k^{pq}, \widetilde{B}_j] \to [\tilde{\kappa}^0 g^{0,pq}, \widetilde{B}_j]$ in $\mathscr{L}(L^2(\widetilde{\mathcal{O}}_j))$ as $k\to+\infty$. It follows that

$$\langle Q_2^{pq} v_j^k, \overline{v_j^k}\rangle_{H^{-1}_{\mathrm{comp}}(\widetilde{\mathcal{O}}_j), H^1_{\mathrm{loc}}(\widetilde{\mathcal{O}}_j)} = \langle Q_{2,a}^{pq} v_j^k, \overline{v_j^k}\rangle_{H^{-1}_{\mathrm{comp}}(\widetilde{\mathcal{O}}_j), H^1_{\mathrm{loc}}(\widetilde{\mathcal{O}}_j)} + o(1)_{k\to+\infty},$$

with

$$Q_{2,a}^{pq} = \chi_j \tilde{\kappa}_k^{-1} \partial_{x_p} [\tilde{\kappa}^0 g^{0,pq}, \widetilde{B}_j] \partial_{x_q}.$$

With Corollary 3.3 one writes

$$[\tilde{\kappa}^0 g^{0,pq}, \widetilde{B}_j] = -\frac{1}{i} \nabla_x(\tilde{\kappa}^0 g^{0,pq}) \cdot \mathrm{Op}(\nabla_\xi(\tilde{\chi}_j^2 b)) + K_1,$$

with K_1 a compact operator on $L^2(\mathbb{R}^{d+1})$, with compactly supported kernel. One thus obtains

$$\langle Q_2^{pq} v_j^k, \overline{v_j^k}\rangle_{H^{-1}_{\mathrm{comp}}(\widetilde{\mathcal{O}}_j), H^1_{\mathrm{loc}}(\widetilde{\mathcal{O}}_j)} = \langle Q_{2,b}^{pq} v_j^k, \overline{v_j^k}\rangle_{H^{-1}_{\mathrm{comp}}(\widetilde{\mathcal{O}}_j), H^1_{\mathrm{loc}}(\widetilde{\mathcal{O}}_j)} + o(1)_{k\to+\infty},$$

with

$$Q_{2,b}^{pq} = -\frac{1}{i} \chi_j \tilde{\kappa}_k^{-1} \partial_{x_p} \nabla_x(\tilde{\kappa}^0 g^{0,pq}) \cdot \mathrm{Op}(\nabla_\xi(\tilde{\chi}_j^2 b)) \partial_{x_q}.$$

Proposition 3.4 applies and yields

$$\langle Q_2^{pq} v_j^k, \overline{v_j^k}\rangle_{H^{-1}_{\mathrm{comp}}(\widetilde{\mathcal{O}}_j), H^1_{\mathrm{loc}}(\widetilde{\mathcal{O}}_j)} = \langle \mu_j, -i\chi_j \xi_p \xi_q (\tilde{\kappa}^0)^{-1} \nabla_x(\tilde{\kappa}^0 g^{0,pq}) \cdot \nabla_\xi b\rangle_{S^*(\widetilde{\mathcal{O}}_j)} + o(1)_{k\to+\infty}.$$

We now treat the term associated with Q_4^{pq}. Note that one has $\sum_{p,q} Q_4^{pq} = \chi_j [\tilde{\kappa}_k^{-1}, \widetilde{B}_j] \tilde{\kappa}_k A_k$. We write, lifting temporarily the analysis back to the manifold,

$$\begin{aligned}\sum_{p,q} \langle Q_4^{pq}, v\rangle_j^k \overline{v_j^k}_{H^{-1}_{\mathrm{comp}}(\widetilde{\mathcal{O}}_j), H^1_{\mathrm{loc}}(\widetilde{\mathcal{O}}_j)} &= \langle \chi_j [\tilde{\kappa}_k^{-1}, B] \tilde{\kappa}_k A_k v_j^k, \overline{v_j^k}\rangle_{H^{-1}_{\mathrm{comp}}(\mathcal{L}), H^1_{\mathrm{loc}}(\mathcal{L})} \\ &= \langle \chi_j [\tilde{\kappa}_k^{-1}, B] \tilde{\kappa}_k A_k u^k, \overline{u^k}\rangle_{H^{-1}_{\mathrm{comp}}(\mathcal{L}), H^1_{\mathrm{loc}}(\mathcal{L})} + o(1)_{k\to+\infty}.\end{aligned}$$

Setting $f^k = (\partial_t^2 - A_k)u^k$ with $f^k \to 0$ strongly in $L^2_{\mathrm{loc}}((T_1, T_2) \times \mathcal{M})$, we thus find

$$\sum_{p,q} \langle Q_4^{pq} v_j^k, \overline{v_j^k} \rangle_{H^{-1}_{\mathrm{comp}}(\widetilde{\mathcal{O}}_j), H^1_{\mathrm{loc}}(\widetilde{\mathcal{O}}_j)} = \langle \chi_j [\tilde{\kappa}_k^{-1}, B] \tilde{\kappa}_k \partial_t^2 u^k, \overline{u^k} \rangle_{H^{-1}_{\mathrm{comp}}(\mathcal{L}), H^1_{\mathrm{loc}}(\mathcal{L})} - \langle \chi_j [\tilde{\kappa}_k^{-1}, B] \tilde{\kappa}_k f^k, \overline{u^k} \rangle_{H^{-1}_{\mathrm{comp}}(\mathcal{L}), H^1_{\mathrm{loc}}(\mathcal{L})} + o(1)_{k\to+\infty}$$
$$= \langle \chi_j [\tilde{\kappa}_k^{-1}, \widetilde{B}_j] \tilde{\kappa}_k \partial_t^2 v_j^k, \overline{v_j^k} \rangle_{H^{-1}_{\mathrm{comp}}(\widetilde{\mathcal{O}}_j), H^1_{\mathrm{loc}}(\widetilde{\mathcal{O}}_j)} + o(1)_{k\to+\infty},$$

bringing again the analysis at the level of the local chart.

Using that $\tilde{\kappa}_k$ is independent of t, we may write

$$\chi_j [\tilde{\kappa}_k^{-1}, \widetilde{B}_j] \tilde{\kappa}_k \partial_t = \chi_j \partial_t [\tilde{\kappa}_k^{-1}, \widetilde{B}_j] \tilde{\kappa}_k + \chi_j [\tilde{\kappa}_k^{-1}, E_j] \tilde{\kappa}_k,$$

where $E_j = [\partial_t, \widetilde{B}_j] \in \Psi^1_{c,\mathrm{ph}}(\widetilde{\mathcal{O}}_j)$, with $\partial_t b \in S^1_{c,\mathrm{ph}}(\widetilde{\mathcal{O}}_j)$ for principal symbol. With Theorem 3.2 we see that $[\tilde{\kappa}_k^{-1}, E_j]$ maps $L^2(\widetilde{\mathcal{O}}_j)$ into itself continuously and moreover $[\tilde{\kappa}_k^{-1}, E_j] \to [(\tilde{\kappa}^0)^{-1}, E_j]$ in $\mathscr{L}(L^2(\widetilde{\mathcal{O}}_j))$. Thus we obtain

$$\langle \chi_j [\tilde{\kappa}_k^{-1}, E_j] \tilde{\kappa}_k \partial_t v_j^k, \overline{v_j^k} \rangle_{H^{-1}_{\mathrm{comp}}(\widetilde{\mathcal{O}}_j), H^1_{\mathrm{loc}}(\widetilde{\mathcal{O}}_j)} = \langle \chi_j [(\tilde{\kappa}^0)^{-1}, E_j] \tilde{\kappa}_k \partial_t v_j^k, \overline{v_j^k} \rangle_{H^{-1}_{\mathrm{comp}}(\widetilde{\mathcal{O}}_j), H^1_{\mathrm{loc}}(\widetilde{\mathcal{O}}_j)} + o(1)_{k\to+\infty} \xrightarrow[k\to+\infty]{} 0,$$

arguing as above. Similarly we write

$$\langle \chi_j \partial_t [\tilde{\kappa}_k^{-1}, \widetilde{B}_j] \tilde{\kappa}_k \partial_t v_j^k, \overline{v_j^k} \rangle_{H^{-1}_{\mathrm{comp}}(\widetilde{\mathcal{O}}_j), H^1_{\mathrm{loc}}(\widetilde{\mathcal{O}}_j)} \sim_{k\to+\infty} \langle \chi_j \partial_t [(\tilde{\kappa}^0)^{-1}, \widetilde{B}_j] \tilde{\kappa}_k^0 \partial_t v_j^k, \overline{v_j^k} \rangle_{H^{-1}_{\mathrm{comp}}(\widetilde{\mathcal{O}}_j), H^1_{\mathrm{loc}}(\widetilde{\mathcal{O}}_j)}.$$

Arguing as we did for the term associated with $Q_2^{p,q}$ we thus find

$$\langle \chi_j \partial_t [\tilde{\kappa}_k^{-1}, \widetilde{B}_j] \tilde{\kappa}_k \partial_t v_j^k, \overline{v_j^k} \rangle_{H^{-1}_{\mathrm{comp}}(\widetilde{\mathcal{O}}_j), H^1_{\mathrm{loc}}(\widetilde{\mathcal{O}}_j)} = \langle \mu_j, -i \chi_j \tau^2 \tilde{\kappa}^0 (\nabla_x (\tilde{\kappa}^0)^{-1}) \cdot \nabla_\xi b \rangle_{S^*(\widetilde{\mathcal{O}}_j)} + o(1)_{k\to+\infty}.$$

Collecting the various estimates we found we obtain

$$\langle \chi_j [P_k, \widetilde{B}_j] v_j^k, \overline{v_j^k} \rangle_{H^{-1}_{\mathrm{comp}}(\widetilde{\mathcal{O}}_j), H^1_{\mathrm{loc}}(\widetilde{\mathcal{O}}_j)} = \langle \mu_j, \chi_j \sigma \rangle_{S^*(\widetilde{\mathcal{O}}_j)} + o(1)_{k\to+\infty}, \tag{3-15}$$

with

$$\sigma = i\{\tau^2, b\} - i \sum_{p,q} \big(g^{0,pq} \xi_p \partial_{x_q} b + g^{0,pq} (\partial_{x_p} b) \xi_q - \xi_p \xi_q (\tilde{\kappa}^0)^{-1} \nabla_x (\tilde{\kappa}^0 g^{0,pq}) \cdot \nabla_\xi b \big) + i \tau^2 \tilde{\kappa}^0 (\nabla_x (\tilde{\kappa}^0)^{-1}) \cdot \nabla_\xi b.$$

Recalling that $p^0 = -\tau^2 + \sum_{p,q} g^{0,pq} \xi_p \xi_q$, one finds

$$\sigma = -i\{p^0, b\} + i p^0 (\tilde{\kappa}^0)^{-1} \nabla_x (\tilde{\kappa}^0) \cdot \nabla_\xi b.$$

Since μ, and thus μ_j, is supported in $\mathrm{Char}(p^0)$ by the first part of the proposition, one concludes that

$$\langle \chi_j [P_k, \widetilde{B}_j] v_j^k, \overline{v_j^k} \rangle_{H^{-1}_{\mathrm{comp}}(\widetilde{\mathcal{O}}_j), H^1_{\mathrm{loc}}(\widetilde{\mathcal{O}}_j)} = -i \langle \mu_j, \chi_j \{p^0, b\} \rangle_{S^*(\widetilde{\mathcal{O}}_j)} + o(1)_{k\to+\infty}.$$

Since $\chi_j \mu = \chi_j \mu_j$ (see Section 3B), with (3-13)–(3-14) one obtains

$$\langle [P_k, B] u^k, \overline{u^k} \rangle_{H^{-1}_{\mathrm{comp}}(\mathcal{L}), H^1_{\mathrm{loc}}(\mathcal{L})} = -i \langle \mu, \{p^0, b\} \rangle_{S^*(\mathcal{L})} + o(1)_{k\to+\infty}.$$

With (3-12), this concludes the proof of the second part of the proposition since $\{p^0, b\} = \mathrm{H}_{p^0} b$. $\square$

4. Measure support propagation: proof of Theorem 1.10

Theorem 1.10 is stated on an open subset of a smooth manifold. Yet, its result is of a local nature. Using a local chart we may assume that we consider an open of set Ω of $\mathbb{R}^d$ instead without any loss of generality.

The strategy we follow is very much inspired by the approach of Melrose and Sjöstrand [1978] to the propagation of singularities and relies on careful choices of test functions allowing one to construct sequences of points in the support of the measure relying on nonnegativity.[2] Then, a limiting procedure leads to the conclusion, in the spirit of the classical proof of the Cauchy–Peano theorem.

The proof of Theorem 1.10 is made of two steps that are stated in the following propositions.

Proposition 4.1. *Let X be a $\mathscr{C}^0$-vector field on Ω an open set of $\mathbb{R}^d$. For a closed set F of Ω, the following two properties are equivalent*:

(1) *The set F is a union of maximally extended integral curves of the vector field X.*

(2) *For any compact $K \subset \Omega$ where the vector field X does not vanish*,

$$\forall \varepsilon > 0,\ \exists \delta_0 > 0,\ \forall x \in K \cap F,\ \forall \delta \in [-\delta_0, \delta_0], \quad B(x + \delta X(x), \delta\varepsilon) \cap F \neq \varnothing.$$

Proposition 4.2. *Let X be a $\mathscr{C}^0$-vector field on Ω an open set of $\mathbb{R}^d$. Consider a nonnegative measure μ on Ω that is a solution to ${}^tX\mu = 0$ in the sense of distributions, that is*,

$$\langle {}^tX\mu, a\rangle_{\mathscr{D}'(\Omega),\mathscr{C}_c^\infty(\Omega)} = \langle \mu, Xa\rangle_{\mathscr{D}'^{,0}(\Omega),\mathscr{C}_c^0(\Omega)} = 0, \quad a \in \mathscr{C}_c^\infty(\Omega). \tag{4-1}$$

Then, the closed set $F = \operatorname{supp}(\mu)$ satisfies the second property in Proposition 4.1.

Proof of Proposition 4.1. First, we prove that property (1) implies property (2) and consider a compact set K of $\mathbb{R}^d$ such that $K \subset \Omega$ and $K \cap F \neq \varnothing$.

There exists $\eta > 0$ such that $K \subset K_\eta \subset \Omega$ with $K_\eta = \{x \in \Omega : \operatorname{dist}(x, K) \le \eta\}$. One has $\|X\| \le C_0$ on K_η for some $C_0 > 0$. Let $x \in K$ and let $\gamma(s)$ be a maximal integral curve defined on an interval $]a, b[$, $a, b \in \overline{\mathbb{R}}$ and such that $0 \in]a, b[$ and $\gamma(0) = x$. If $b < \infty$ then there exists $s^1 \in]0, b[$ such that $\gamma(s^1) \notin K_\eta$. Since $\gamma(s) \in K_\eta$ if $s < \eta/C_0$, one finds that $b \ge \eta/C_0$. Similarly, one has $|a| \ge \eta/C_0$. Consequently, there exists $S > 0$ such that any maximal integral curve $\gamma(s)$ of the vector field X with $\gamma(0) \in K$ is defined for $s \in I = (-S, S)$.

Let us pick $x \in K$. According to property (1), there exists

$$\gamma : I \to F \text{ such that } \dot\gamma(s) = X(\gamma(s)) \text{ and } \gamma(0) = x.$$

By uniform continuity of the vector field X in a compact neighborhood of K we have

$$\gamma(s) = \gamma(0) + \int_0^s \dot\gamma(s)\, ds = \gamma(0) + \int_0^s X(\gamma(s))\, ds = x + sX(x) + r(s), \quad s \in (-S, S),$$

where $\lim_{s\to 0} \|r(s)\|/s = 0$, *uniformly* with respect to x. We deduce that for any $\varepsilon > 0$ there exists $0 < \delta_0 < S$ such that $\|r(s)\| < s\varepsilon$ for any $s \in (-\delta_0, \delta_0)$, which implies

$$F \ni \gamma(s) \in B(x + sX(x), s\varepsilon).$$

[2] Of the measure in our case and of some operators for Melrose and Sjöstrand, via the Gårding inequality.

Second, we prove that property (2) implies property (1). It suffices to prove that for any $x \in F$ there exist an interval $I \ni 0$ and an integral curve

$$\gamma : I \to F \text{ such that } \dot{\gamma}(s) = X(\gamma(s)) \text{ and } \gamma(0) = x.$$

Then, the standard continuation argument shows that this local integral curve included in F can be extended to a maximal integral curve also included in F.

If $X(x) = 0$, then the trivial integral curve $\gamma(s) = x,\ s \in \mathbb{R}$, is included in F. As a consequence, we assume $X(x) \neq 0$ and we pick a compact neighborhood K of x containing $B(x, \eta)$ with $\eta > 0$ and where, for some $0 < c_K < C_K$,

$$c_K \le \|X(y)\| \le C_K, \quad y \in K.$$

Let $n \in \mathbb{N}^*$. Set $x_{n,0} = x$ and $\varepsilon = 1/n$ and apply property (2). One deduces that there exist $0 < \delta_n \le 1/n$ and a point

$$x_{n,1} \in F \cap B(x_{n,0} + \delta_n X(x_{n,0}), \delta_n/n).$$

If $x_{n,1} \in K$, one can perform this construction again, starting from $x_{n,1}$ instead of $x_{n,0}$. If a sequence of points $x_{n,0}, x_{n,1}, \ldots, x_{n,L^+}$ is obtained in this manner, one has

$$x_{n,\ell+1} \in F \cap B(x_{n,\ell} + \delta_n X(x_{n,\ell}), \delta_n/n), \quad \ell = 0, \ldots, L^+ - 1. \tag{4-2}$$

One can carry on the construction as long as $x_{n,L^+} \in K$. We can perform the same construction for $\ell \le 0$, with the property

$$x_{n,\ell-1} \in F \cap B(x_{n,\ell} - \delta_n X(x_{n,\ell}), \delta_n/n), \quad |\ell| = 0, \ldots, L^- - 1. \tag{4-3}$$

Having $\|X\| \le C_K$ on K and $B(x, \eta) \subset K$ ensures that we can construct the sequence at least for

$$L^+ = L^- = L_n = \left\lfloor \frac{\eta}{\delta_n(C_K + 1)} \right\rfloor + 1 \le \left\lfloor \frac{\eta}{\delta_n(C_K + 1/n)} \right\rfloor + 1,$$

where $\lfloor \cdot \rfloor$ denotes the floor function. With the points $x_{n,\ell},\ |\ell| \le L_n$, we have constructed we define the following continuous curve $\gamma_n(s)$ for $|s| \le L_n \delta_n$:

$$\gamma_n(s) = x_{n,\ell} + (s - \ell\delta_n)\frac{x_{n,\ell+1} - x_{n,\ell}}{\delta_n} \quad \text{for } s \in [\ell\delta_n, (\ell+1)\delta_n) \text{ and } |\ell| \le L_n - 1.$$

This curve and its construction is illustrated in Figure 1. Note that $\gamma_n(s)$ remains in a compact set, uniformly with respect to n. In this compact set X is uniformly continuous.

We set $S = \eta/(C_K + 1)$. Since $S \le L_n\delta_n$, we shall in fact only consider the function $\gamma_n(s)$ for $|s| \le S$ in what follows. Note that since $x_{n,\ell} \in F$ for $|\ell| \le L_n$, one has

$$\operatorname{dist}(\gamma_n(s), F) \le \delta_n(C_K + 1/n), \quad |s| \le S. \tag{4-4}$$

From (4-2), for $\ell \ge 0$ and $s \in (\ell\delta_n, (\ell+1)\delta_n)$, we have

$$\dot{\gamma}_n(s) = \frac{x_{n,\ell+1} - x_{n,\ell}}{\delta_n} = X(x_{n,\ell}) + \mathcal{O}(1/n).$$

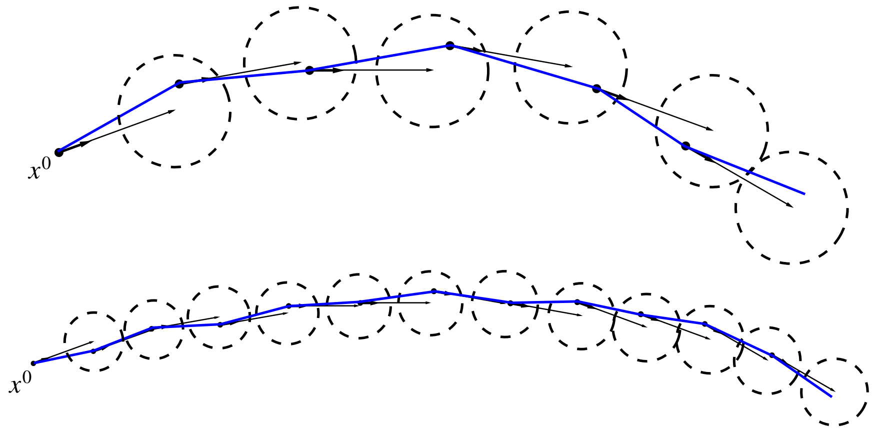

Figure 1. Top: iterative construction of the curve γ_n. Bottom: convergence of γ_n as n increases.

Similarly, from (4-3), for $\ell \le 0$ and $s \in ((\ell-1)\delta_n, \ell\delta_n)$, we have

$$\dot{\gamma}_n(s) = \frac{x_{n,\ell} - x_{n,\ell-1}}{\delta_n} = X(x_{n,\ell}) + \mathcal{O}(1/n).$$

In any case, using the uniform continuity of the vector field X, we find

$$\dot{\gamma}_n(s) = X(\gamma_n(s)) + e_n(s),$$

where the error $|e_n|$ goes to zero *uniformly* with respect to $|s| \le S$ as $n \to +\infty$.

Since the curve γ_n is absolutely continuous (and differentiable except at isolated points), we find

$$\gamma_n(s) = \gamma_n(0) + \int_0^s \dot{\gamma}_n(\sigma)\,d\sigma = x + \int_0^s X(\gamma_n(\sigma))\,d\sigma + \int_0^s e_n(\sigma)\,d\sigma. \tag{4-5}$$

We now let n grow to infinity. With (4-5), the family of curves $(s \mapsto \gamma_n(s), |s| \le S)_{n\in\mathbb{N}^*}$ is equicontinuous and pointwise bounded; by the Arzelà-Ascoli theorem we can extract a subsequence $(s \mapsto \gamma_{n_p})_{p\in\mathbb{N}}$ that converges uniformly to a curve $\gamma(s)$, $|s| \le S$. Convergence is illustrated in Figure 1. Passing to the limit $n_p \to +\infty$ in (4-5) we find that $\gamma(s)$ is solution to

$$\gamma(s) = x + \int_0^s X(\gamma(\sigma))\,d\sigma.$$

From estimation (4-4), for any $|s| \le S$, there exists $(y_p)_p \subset F$ such that $\lim_{p\to+\infty} y_p = \gamma(s)$. Since F is closed we conclude that $\gamma(s) \in F$. ☐

Positivity argument and proof of Proposition 4.2. We consider a compact set K where the vector field X does not vanish. By continuity of the vector field there exist $0 < c_K \le C_K$ such that $0 < c_K \le \|X(x)\| \le C_K$ for all $x \in K$.

Let us consider $x^0 \in K \cap \operatorname{supp}(\mu)$. By performing a rotation and a dilation by a factor $\|X(x^0)\|$, we can assume that $X(x^0) = (1, 0, \dots, 0) \in \mathbb{R}^d$. We shall write $x = (x_1, x')$ with $x' \in \mathbb{R}^{d-1}$.

Let $\chi \in \mathscr{C}^\infty(\mathbb{R})$ be given by

$$\chi(s) = \mathbf{1}_{s<1}\ \exp(1/(s-1)), \tag{4-6}$$

and $\beta \in \mathscr{C}^\infty(\mathbb{R})$ be such that

$$\beta \equiv 0 \quad \text{on }]-\infty, -1], \qquad \beta' > 0 \quad \text{on } \big]-1, -\tfrac{1}{2}\big[, \qquad \beta \equiv 1 \quad \text{on } \big[-\tfrac{1}{2}, +\infty\big[. \tag{4-7}$$

We then set

$$q_{\varepsilon,\delta,x^0} = (\chi \circ v)(\beta \circ w), \quad g_{\varepsilon,\delta,x^0} = (\chi' \circ v)(\beta \circ w)Xv, \quad h_{\varepsilon,\delta,x^0} = (\chi \circ v)(\beta' \circ w)Xw, \tag{4-8}$$

with

$$v(x) = \tfrac{1}{2} - \delta^{-1}(x_1 - x_1^0) + 8(\varepsilon\delta)^{-2}\|x' - x^{0\prime}\|^2 \quad \text{and} \quad w(x) = 2\varepsilon^{-1}(1 - \delta^{-1}(x_1 - x_1^0))$$

for $\varepsilon > 0$ and $\delta > 0$ both meant to be chosen small in what follows. We have $Xq_{\varepsilon,\delta,x^0} = g_{\varepsilon,\delta,x^0} + h_{\varepsilon,\delta,x^0}$.

The function $q_{\varepsilon,\delta,x^0}$ is compactly supported. Indeed, in the support of $\beta \circ w$, one has $w \geq -1$, implying

$$x_1 - x_1^0 \leq \delta\big(1 + \tfrac{1}{2}\varepsilon\big),$$

while on the support of $\chi \circ v$ one has $v \leq 1$, which gives

$$-\tfrac{1}{2} + 8(\varepsilon\delta)^{-2}\|x' - x^{0\prime}\|^2 \leq \delta^{-1}(x_1 - x_1^0).$$

On the supports of $q_{\varepsilon,\delta,x^0}$ and $(\chi' \circ v)(\beta \circ w)$, one thus finds

$$-\tfrac{1}{2}\delta \leq x_1 - x_1^0 \leq \delta\big(1 + \tfrac{1}{2}\varepsilon\big) \quad \text{and} \quad 8(\varepsilon\delta)^{-2}\|x' - x^{0\prime}\|^2 \leq \tfrac{3}{2} + \tfrac{1}{2}\varepsilon. \tag{4-9}$$

Similarly, on the support of $\beta' \circ w$ one has $-1 \leq w \leq -\frac{1}{2}$

$$\delta\big(1 + \tfrac{1}{4}\varepsilon\big) \leq x_1 - x_1^0 \leq \delta\big(1 + \tfrac{1}{2}\varepsilon\big),$$

which implies that on the support of $h_{\varepsilon,\delta,x^0}$ one has

$$\delta\big(1 + \tfrac{1}{4}\varepsilon\big) \leq x_1 - x_1^0 \leq \delta\big(1 + \tfrac{1}{2}\varepsilon\big) \quad \text{and} \quad 8(\varepsilon\delta)^{-2}\|x' - x^{0\prime}\|^2 \leq \tfrac{3}{2} + \tfrac{1}{2}\varepsilon. \tag{4-10}$$

In particular, in the case $\varepsilon \leq 1$, one finds

$$\operatorname{supp}(h_{\varepsilon,\delta,x^0}) \subset B(x^0 + \delta X(x^0), \varepsilon\delta). \tag{4-11}$$

These estimations of the supports of $q_{\varepsilon,\delta,x^0}$ and $h_{\varepsilon,\delta,x^0}$ are illustrated in Figure 2.

Lemma 4.3. *For any* $0 < \varepsilon \leq 1$ *there exists* $\delta_0 > 0$ *such that, for any* $x^0 \in K$ *and* $0 < \delta \leq \delta_0$, *the function* $g_{\varepsilon,\delta,x^0}$ *is nonnegative. Moreover,* $g_{\varepsilon,\delta,x^0}$ *is positive in a neighborhood of* x^0.

Proof. Let $0 < \varepsilon \leq 1$. We have $g_{\varepsilon,\delta,x^0} = (\chi' \circ v)(\beta \circ w)Xv$. Since $\beta \geq 0$ and $\chi' < 0$, it suffices to prove that $Xv(x) \leq 0$ for x in the support of $(\chi' \circ v)(\beta \circ w)$ for $\delta > 0$ chosen sufficiently small, uniformly with respect to $x^0 \in K$.

We write

$$X(x) - X(x^0) = \alpha^1(x, x^0)\partial_{x_1} + \alpha'(x, x^0) \cdot \nabla_{x'},$$

with $\alpha^1(x, x^0) \in \mathbb{R}$ and $\alpha'(x, x^0) \in \mathbb{R}^{d-1}$. By (4-9), for $x \in \operatorname{supp}(\chi' \circ v)(\beta \circ w)$ we have $\|x - x^0\| \lesssim \delta$. From the uniform continuity of X in any compact set we conclude that

$$|\alpha^1(x, x^0)| + \|\alpha'(x, x^0)\| = o(1) \quad \text{as } \delta \to 0^+, \tag{4-12}$$

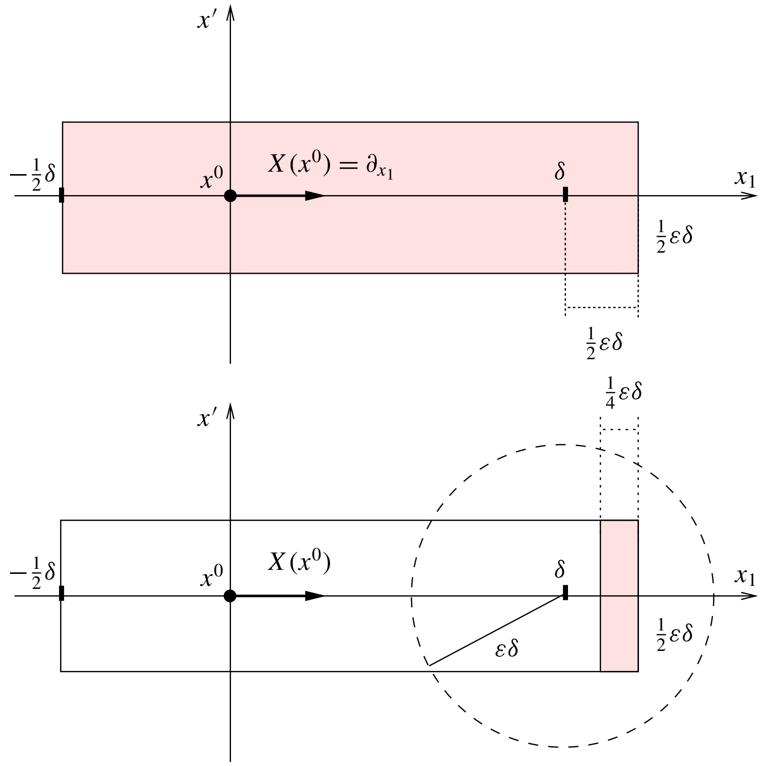

Figure 2. Estimation of the test function supports in the case $\varepsilon \le 1$. Top: support of $h_{\varepsilon,\delta,x^0}$. Bottom: support of $q_{\varepsilon,\delta,x^0}$.

uniformly[3] with respect to $x^0 \in K$ and $x \in \text{supp}(\chi' \circ v)(\beta \circ w)$. Using that $X(x^0) = \partial_{x_1}$ and the form of v given above, we write

$$\begin{aligned} Xv(x) &= (X(x)v)(x) = \big(\partial_{x_1} v + (X(x) - X(x^0))v\big)(x) \\ &= -\delta^{-1}\big(1 + \alpha^1(x, x^0) - 16\varepsilon^{-1}(\varepsilon\delta)^{-1}\alpha'(x, x^0) \cdot (x' - x^{0\prime})\big). \end{aligned} \tag{4-13}$$

Using again (4-9), we thus find for $x \in \text{supp}(\chi' \circ v)(\beta \circ w)$

$$|\alpha^1(x, x^0) - 16\varepsilon^{-1}(\varepsilon\delta)^{-1}\alpha'(x, x^0) \cdot (x' - x^{0\prime})| \lesssim |\alpha^1(x, x^0)| + \varepsilon^{-1}\|\alpha'(x, x^0)\|.$$

With ε fixed above and with (4-12), we find that $Xv(x) \sim -\delta^{-1}$ as $\delta \to 0^+$ uniformly with respect to $x^0 \in K$ and $x \in \text{supp}(\chi' \circ v)(\beta \circ w)$.

Finally, we have $g_{\varepsilon,\delta,x^0}(x^0) = -\delta^{-1}\chi'\big(\tfrac{1}{2}\big)\beta(2\varepsilon^{-1}) > 0$ and thus $g_{\varepsilon,\delta,x^0}$ is positive in a neighborhood of x^0. □

We are now in a position to conclude the proof of Proposition 4.2. Note that it suffices to prove the result for $0 < \varepsilon \le 1$. We choose $\delta_0 > 0$ as given by Lemma 4.3. Let then $x^0 \in K \cap \text{supp}(\mu)$. We apply (4-1)

[3]Observe that the change of variables made above for $X(x^0) = (1, 0, \dots, 0)$ does not affect uniformity since the dilation is made by a factor in $[c_K, C_K]$.

to the family $q_{\varepsilon,\delta,x^0}$ of test functions with $0 < \delta \le \delta_0$:

$$0 = \langle \mu, X(q_{\varepsilon,\delta,x^0})\rangle = \langle \mu, g_{\varepsilon,\delta,x^0}\rangle + \langle \mu, h_{\varepsilon,\delta,x^0}\rangle. \tag{4-14}$$

By Lemma 4.3, $g_{\varepsilon,\delta,x^0} \ge 0$ and $g_{\varepsilon,\delta,x^0}$ is positive in a neighborhood of x^0. As $x^0 \in \operatorname{supp}(\mu)$ we find $\langle \mu, g_{\varepsilon,\delta,x^0}\rangle > 0$. Consequently, $\langle \mu, h_{\varepsilon,\delta,x^0}\rangle \ne 0$. By the support estimate for $h_{\varepsilon,\delta,x^0}$ given in (4-11) the conclusion follows: $\operatorname{supp}(\mu) \cap B(x^0 + \delta X(x^0), \varepsilon\delta) \ne \varnothing$. □

5. Exact controllability: proof of Theorem 1.12

Let $(\kappa^0, g^0) \in \mathcal{X}^1(\mathcal{M})$ and assume that (ω, T) fulfills the geometric control condition of Definition 1.8″.

Let also $(\kappa, g) \in \mathcal{Y}(\mathcal{M})$. With Proposition 1.6, the result of Theorem 1.12 follows if we prove that there exists $\varepsilon > 0$ and $C_{\text{obs}} > 0$ such that

$$\mathcal{E}_{\kappa,g}(u)(0) \le C_{\text{obs}} \|\mathbf{1}_{(0,T)\times\omega}\, \partial_t u\|^2_{L^2(\mathcal{L},\kappa\mu_g dt)}$$

for any weak solution u of the wave equation associated with (κ, g) chosen such that

$$\|(\kappa, g) - (\kappa^0, g^0)\|_{\mathcal{Y}(\mathcal{M})} \le \varepsilon.$$

The L^2-norm on the right-hand side is associated with (κ, g), that is,

$$\|\mathbf{1}_{(0,T)\times\omega}\, \partial_t u\|^2_{L^2(\mathcal{L},\kappa\mu_g dt)} = \int_0^T\!\!\int_\omega |\partial_t u|^2\, \kappa\mu_g\, dt.$$

Yet, for $\varepsilon > 0$ chosen sufficiently small one has $\|\cdot\|_{L^2(\mathcal{L},\kappa^0\mu_{g^0})} \eqsim \|\cdot\|_{L^2(\mathcal{L},\kappa\mu_g)}$, where $A \eqsim B$ means $c_1 \le A/B \le c_2$ for some $c_1, c_2 > 0$. In other words, we have equivalence with constants uniform with respect to (κ, g). In what follows, L^2- and more generally H^s-norms on $\mathcal{M}$ are chosen with respect to $\kappa^0\mu_{g^0}$ unless explicitly written. Our goal is thus to prove the observability inequality

$$\mathcal{E}_{\kappa^0,g^0}(u)(0) \le C_{\text{obs}} \|\mathbf{1}_{(0,T)\times\omega}\, \partial_t u\|^2_{L^2(\mathcal{L})}. \tag{5-1}$$

The Bardos–Lebeau–Rauch uniqueness compactness argument reduces the proof of (5-1) to the proof of the weaker estimate

$$\mathcal{E}_{\kappa^0,g^0}(u)(0) \le C \|\mathbf{1}_{(0,T)\times\omega}\, \partial_t u\|^2_{L^2(\mathcal{L})} + C' \big\|(u(0), \partial_t u(0))\big\|^2_{L^2(\mathcal{M})\oplus H^{-1}(\mathcal{M})}, \tag{5-2}$$

which exhibits an additional compact term, and expresses observability for high frequencies. Low frequencies are dealt with by means of a unique continuation argument.

To prove (5-2) we argue by contradiction and we assume that there exists a sequence $(\kappa_k, g_k)_{k\in\mathbb{N}} \subset \mathcal{Y}(\mathcal{M})$ such that

$$\lim_{k\to+\infty} \|(\kappa_k, g_k) - (\kappa^0, g^0)\|_{\mathcal{Y}(\mathcal{M})} = 0, \tag{5-3}$$

and yet for each $k \in \mathbb{N}$ the associated observability inequality does not hold. Thus, for each $k \in \mathbb{N}$, there exists a sequence of initial data $(v^{k,p,0}, v^{k,p,1})_{p\in\mathbb{N}} \subset H^1(\mathcal{M}) \times L^2(\mathcal{M})$ with associated solution $(v^{k,p})_{p\in\mathbb{N}}$, that is,

$$\begin{cases} P_k v^{k,p} = 0 & \text{in } (0,+\infty)\times\mathcal{M}, \\ v^{k,p}{}_{|t=0} = v^{k,p,0}, \ \ \partial_t v^{k,p}{}_{|t=0} = v^{k,p,1} & \text{in } \mathcal{M}, \end{cases}$$

with $P_k = P_{\kappa_k, g_k}$, that moreover has the properties

$$\mathcal{E}_{\kappa^0,g^0}(v^{k,p})(0) = 1 \quad \text{and} \quad \|\mathbf{1}_{(0,T)\times\omega}\, \partial_t v^{k,p}\|_{L^2(\mathcal{L})} + \big\|(v^{k,p,0}, v^{k,p,1})\big\|_{L^2(\mathcal{M})\oplus H^{-1}(\mathcal{M})} \le \frac{1}{p+1}.$$

We take $p = k$ and we set$(u^{k,0}, u^{k,1}) = (v^{k,k,0}, v^{k,k,1})$ and $u^k = v^{k,k}$; one obtains $P_k u^k = 0$ in $\mathcal{L}$ and

$$\mathcal{E}_{\kappa^0,g^0}(u^k)(0) = 1 \quad \text{and} \quad \|\mathbf{1}_{(0,T)\times\omega}\, \partial_t u^k\|_{L^2(\mathcal{L})} + \big\|(u^{k,0}, u^{k,1})\big\|_{L^2(\mathcal{M})\oplus H^{-1}(\mathcal{M})} \le \frac{1}{k+1}. \tag{5-4}$$

From (5-4) one has $u^k \rightharpoonup 0$ weakly in $H^1_{\text{loc}}(\mathcal{L})$. With (3-1)–(3-2), we can associate with (a subsequence of) $(u^k)_k$ an H^1-microlocal defect measure μ on $S^*(\mathcal{L})$. Here, the measure is understood with respect to $L^2(\mathcal{L}, \kappa^0 \mu_{g^0}\, dt)$.

From the second part of (5-4) one has

$$\mu = 0 \quad \text{in } S^*((0,T)\times\omega). \tag{5-5}$$

In fact, for any $\psi \in \mathscr{C}^\infty((0,T)\times\omega)$ one has $\|\psi\,\partial_t u^k\|_{L^2(\mathcal{L})} \sim 0$ and thus $\langle \mu, \tau^2\psi^2\rangle = 0$. Hence, $\operatorname{supp}(\mu) \cap S^*((0,T)\times\omega) \subset \{\tau = 0\}$. Since $\{\tau = 0\} \cap \operatorname{Char}(p^0) \cap S^*(\mathcal{L}) = \varnothing$ with (3-9) one obtains (5-5).

With the first part of (5-4) one has the following lemma.

Lemma 5.1. *The measure μ does not vanish on $S^*(\mathcal{L})$.*

A proof is given below.

We now use Proposition 3.7 to obtain a precise description of the measure μ. First, one has $\operatorname{supp}(\mu) \cap S^*((0,T)\times\mathcal{M}) \subset \operatorname{Char}(p^0)$. Furthermore, one has ${}^t\mathrm{H}_{p^0}\mu = 0$ in the sense of distributions on $S^*((0,T)\times\mathcal{M})$. Since H_{p^0} is a $\mathscr{C}^0$-vector field on the manifold $S^*\mathcal{L}$, Theorem 1.10 implies that $\operatorname{supp}(\mu)$ is a union of maximally extended bicharacteristics in $S^*((0,T)\times\mathcal{M})$.

Under the geometric control condition of Definition 1.8″, any maximal bicharacteristic meets $S^*((0,T)\times\omega)$ where μ vanishes by (5-5). Thus $\operatorname{supp}(\mu) = \varnothing$, yielding a contradiction with the result of Lemma 5.1. We thus obtain that (5-1) holds. This concludes the proof of Theorem 1.12. □

Proof of Lemma 5.1. Let $T_1 < T_2$ and $\phi \in \mathscr{C}^\infty_c(\mathbb{R})$ nonnegative and equal to 1 on a neighborhood of $[T_1, T_2]$. On $\mathcal{L}$, consider the *elliptic* operator $Q = -\partial_t^2 - A_{\kappa^0,g^0} + 1$ with symbol $q = \tau^2 + \sum_{p,q} g^{0\,p,q}(x)\xi_p\xi_q$. Taking (3-2) and Lemma 3.6 into account one can write

$$\langle \phi^2 Q u^k, \overline{u^k}\rangle_{H^{-1}_{\text{comp}}(\mathcal{L}), H^1_{\text{loc}}(\mathcal{L})} \underset{k\to+\infty}{\sim} \langle \mu, \phi^2 q\rangle_{S^*\mathcal{L}}. \tag{5-6}$$

Integrating by parts one obtains

$$\begin{aligned}\langle \phi^2 Q u^k, \overline{u^k}\rangle_{H^{-1}_{\text{comp}}(\mathcal{L}), H^1_{\text{loc}}(\mathcal{L})} &= \int_{\mathcal{L}} \phi(t)^2\big(|\partial_t u^k|^2 + g^0(\nabla_{g^0} u^k, \nabla_{g^0}\bar{u}^k) + |u^k|^2\big)\kappa^0\mu_{g^0}\, dt + 2(\phi'\phi\,\partial_t u^k, u^k)_{L^2(\mathcal{L})} \\ &= \int_{\mathbb{R}} \phi(t)^2 \mathcal{E}_{\kappa^0,g^0}(u^k)(t)\, dt + 2(\phi'\phi\,\partial_t u^k, u^k)_{L^2(\mathcal{L})}.\end{aligned}$$

Since the energy built on κ^p, g^p is preserved by the evolution given by P_p, we have by (5-4)

$$\mathcal{E}_{\kappa^0,g^0}(u^k)(t) = \mathcal{E}_{\kappa^k,g^k}(u^k)(t) + o(1) = \mathcal{E}_{\kappa^k,g^k}(u^k)(0) + o(1) = \mathcal{E}_{\kappa^0,g^0}(u^k)(0) + o(1) = 1 + o(1) \tag{5-7}$$

and since $(\phi'\phi\, \partial_t u^k, u^k)_{L^2(\mathcal{L})} \to 0$ as $u^k \to 0$ strongly in $L^2_{\text{loc}}(\mathcal{L})$, one obtains

$$\langle \phi^2 Q u^k, \overline{u^k}\rangle_{H^{-1}_{\text{comp}}(\mathcal{L}), H^1_{\text{loc}}(\mathcal{L})} \underset{k\to+\infty}{\sim} \|\phi\|^2_{L^2(\mathbb{R})}.$$

With (5-6) this proves that $\mu \neq 0$. ☐

6. Lack of continuity of the control operator with respect to coefficients

6A. *Proof of Theorems 1.14 and 1.14′.* We prove the result of both theorems, that is, in the case $k \geq 1$. In the case $k = 1$ we are simply required to prove additionally that the geometric control condition of Definition 1.8 is fulfilled for geodesics given by the chosen metric $\tilde{g}$; see Remark 1.17.

Let $\varepsilon > 0$. We set $\tilde{g} = (1+\varepsilon)g$. Given any neighborhood $\mathcal{U}$ of (κ, g) in $\mathcal{X}^k(\mathcal{M})$, for $\varepsilon > 0$ chosen sufficiently small one has $(\kappa, \tilde{g}) \in \mathcal{U}$.

Moreover, observe that, for $\varepsilon > 0$ chosen sufficiently small, geodesics associated with $\tilde{g}$ can be made arbitrarily close to those associated with g uniformly in $t \in [0, T]$. Hence, for such $\varepsilon > 0$ the geometric control condition is fulfilled for geodesics associated with $\tilde{g}$.

Observe that one has

$$\text{Char}(p_{\kappa,g}) \cap \text{Char}(p_{\kappa,\tilde{g}}) \cap S^*\mathcal{L} = \varnothing. \tag{6-1}$$

We consider a sequence $(y^{k,0}, y^{k,1}) \rightharpoonup (0,0)$ weakly in $H^1(\mathcal{M}) \oplus L^2(\mathcal{M})$ such that

$$\tfrac{1}{2}(\|y^{k,0}\|^2_{H^1(\mathcal{M})} + \|y^{k,1}\|^2_{L^2(\mathcal{M})}) = 1.$$

L^2- and H^1-norms are based on the $\kappa\mu_g\, dt$ measure on $\mathcal{L}$.

Setting $f^k_{\kappa,g} = H_{\kappa,g}(y^{k,0}, y^{k,1}) \in L^2((0,T)\times\mathcal{M})$ with $H_{\kappa,g}$ defined in (1-7), one obtains a sequence of control functions. According to the HUM method [Lions 1988], $f^k_{\kappa,g}$ is itself a (weak) solution to the following free wave equation

$$P_{\kappa,g} f^k_{\kappa,g} = 0, \tag{6-2}$$

in the energy space $L^2(\mathcal{M}) \oplus H^{-1}(\mathcal{M})$, that is, $(f^k_{\kappa,g}(0), \partial_t f^k_{\kappa,g}(0)) \in L^2(\mathcal{M}) \times H^{-1}(\mathcal{M})$. Moreover, $(f^k_{\kappa,g}(0), \partial_t f^k_{\kappa,g}(0))$ depend continuously on $(y^{k,0}, y^{k,1})$. The function $f^k_{\kappa,g}$ is thus bounded in $\mathscr{C}^0((T_1,T_2), L^2(\mathcal{M}))$ uniformly with respect to k for any $T_1 < T_2$. Since the map $H_{\kappa,g}$ is continuous, $f^k_{\kappa,g} \rightharpoonup 0$ weakly in $L^2_{\text{loc}}(\mathcal{L})$. Up to extraction of a subsequence, it is associated with an L^2-microlocal defect measure μ_f. With Proposition 3.7′ one has

$$\text{supp}(\mu_f) \subset \text{Char}(p_{\kappa,g}). \tag{6-3}$$

We consider the sequences of solutions $(y^k)_k$ and $(\tilde{y}^k)_k$ to

$$\begin{cases} P_{\kappa,g} y^k = \mathbf{1}_{(0,T)\times\omega}\, f^k_{\kappa,g} & \text{in } \mathcal{L}, \\ (y^k, \partial_t y^k)_{|t=0} = (y^{k,0}, y^{k,1}) & \text{in } \mathcal{M}, \end{cases} \qquad \begin{cases} P_{\kappa,\tilde{g}} \tilde{y}^k = \mathbf{1}_{(0,T)\times\omega}\, f^k_{\kappa,g} & \text{in } \mathcal{L}, \\ (\tilde{y}^k, \partial_t \tilde{y}^k)_{|t=0} = (y^{k,0}, y^{k,1}) & \text{in } \mathcal{M}. \end{cases}$$

Both are bounded and weakly converge to 0 in $H^1_{\text{loc}}(\mathcal{L})$. Up to extraction of subsequences, both are associated with H^1-microlocal defect density measures μ and $\tilde{\mu}$ respectively. Since $\mathbf{1}_{(0,T)\times\omega} f^k_{\kappa,g} \rightharpoonup 0$ weakly in $L^2_{\text{loc}}(\mathcal{L})$ we have $\mathbf{1}_{(0,T)\times\omega} f^k_{\kappa,g} \to 0$ strongly in $H^{-1}_{\text{loc}}(\mathcal{L})$ and, with Proposition 3.7, one finds

$\operatorname{supp}(\tilde{\mu}) \subset \operatorname{Char}(p_{\kappa,\tilde{g}})$. Thus one has

$$\operatorname{supp}(\tilde{\mu}) \cap \operatorname{supp}(\mu_f) = \varnothing. \tag{6-4}$$

The sequence $(\partial_t \tilde{y}^k)$ converges to 0 weakly in $L^2_{\mathrm{loc}}(\mathcal{L})$ and can be associated with an L^2-microlocal defect density measure whose support is given by $\operatorname{supp}(\tilde{\mu})$.

Lemma 6.1. *One has* $(\mathbf{1}_{(0,T)\times\omega} f^k_{\kappa,g}, \partial_t \tilde{y}^k)_{L^2(\mathcal{L},\kappa\mu_{\tilde{g}}dt)} \to 0$ *as* $k \to +\infty$.

A proof is given below.

Using the density of strong solutions of the wave equation, with integration by parts, one finds the classical energy estimate

$$\mathcal{E}_{\kappa,\tilde{g}}(\tilde{y}^k)(T) - \mathcal{E}_{\kappa,\tilde{g}}(\tilde{y}^k)(0) = (\mathbf{1}_{(0,T)\times\omega} f^k_{\kappa,g}, \partial_t \tilde{y}^k)_{L^2(\kappa\mu_{\tilde{g}}dt)}.$$

With Lemma 6.1 one obtains

$$\mathcal{E}_{\kappa,\tilde{g}}(\tilde{y}^k)(T) \underset{k\to+\infty}{\sim} \mathcal{E}_{\kappa,\tilde{g}}(\tilde{y}^k)(0).$$

With the form of $\tilde{g}$ chosen above one has

$$\mathcal{E}_{\kappa,g}(\tilde{y}^k)(t) = (1+\mathcal{O}(\varepsilon))\mathcal{E}_{\kappa,\tilde{g}}(\tilde{y}^k)(t),$$

uniformly with respect to $t \in [0, T]$. Choosing $\varepsilon > 0$ sufficiently small and k sufficiently large, the first part of Theorem 1.14 follows since $\mathcal{E}_{\kappa,g}(\tilde{y}^k)(0) = 1$.

We use the values of ε and k chosen above. To prove (1-10), we write $\tilde{y}^k$ in the form $\tilde{y}^k = v_1 + v_2$, where v_1 and v_2 are solutions to

$$\begin{cases} P_{\kappa,\tilde{g}} v_1 = \mathbf{1}_{(0,T)\times\omega} f^k_{\kappa,\tilde{g}} & \text{in } \mathcal{L}, \\ (v_1, \partial_t v_1)_{|t=0} = (y^{k,0}, y^{k,1}) & \text{in } \mathcal{M}, \end{cases} \qquad \begin{cases} P_{\kappa,\tilde{g}} v_2 = \mathbf{1}_{(0,T)\times\omega}(f^k_{\kappa,g} - f^k_{\kappa,\tilde{g}}) & \text{in } \mathcal{L}, \\ (v_2, \partial_t v_2)_{|t=0} = (0, 0) & \text{in } \mathcal{M}, \end{cases} \tag{6-5}$$

with $f^k_{\kappa,\tilde{g}} = H_{\kappa,\tilde{g}}(y^{k,0}, y^{k,1})$. A hyperbolic energy estimation for the solution v_2 to the second equation in (6-5) gives

$$\mathcal{E}_{\kappa,\tilde{g}}(v_2)(T) \le C_T \|\mathbf{1}_{(0,T)\times\omega}(f^k_{\kappa,g} - f^k_{\kappa,\tilde{g}})\|^2_{L^2(\mathcal{L})}.$$

Since one has $(v_1(T), \partial_t v_1(T)) = (0, 0)$, because of the definition of $f^k_{\kappa,\tilde{g}}$ one finds

$$\mathcal{E}_{\kappa,\tilde{g}}(v_2)(T) = \mathcal{E}_{\kappa,\tilde{g}}(\tilde{y}^k)(T) \ge \tfrac{1}{2},$$

which gives the second result of Theorem 1.14. □

Proof of Lemma 6.1. The key point in the proof is the following lemma.

Lemma 6.2 [Gérard 1991, Proposition 3.1]. *Assume that u_k and v_k are two sequences bounded in L^2_{loc} that converge weakly to zero and are associated with defect measures μ and ν respectively. Assume that $\mu \perp \nu$, that is, μ and ν are supported on disjoint sets. Then, for any $\psi \in \mathscr{C}^0_c$,*

$$\lim_{k\to+\infty} (\psi u_k, v_k)_{L^2} = 0.$$

To apply this result, we just need to exchange the rough cutoff $\mathbf{1}_{(0,T)\times\omega}$ for a smooth cutoff $\psi(t,x)$. First, note that one has

$$(\mathbf{1}_{(0,T)\times\omega} f^k_{\kappa,g}, \partial_t \tilde{y}^k)_{L^2(\mathcal{L},\kappa\mu_{\tilde{g}}dt)} \underset{k\to+\infty}{\sim} (\mathbf{1}_{(0,T)\times\omega} f^k_{\kappa,g}, \partial_t \tilde{y}^k)_{L^2(\mathcal{L},\kappa\mu_g dt)}.$$

We may thus simply consider the L^2-norm and inner product associated with $\kappa\mu_g\, dt$.

Second, let $\delta>0$. Since $(f^k_{\kappa,g})_k$ and $(\tilde{y}^k)_k$ are both bounded in $\mathscr{C}^0((0,T), L^2(\mathcal{M}))$ uniformly with respect to k, there exists $0<T_1<T_2<T$ and $\mathcal{O}\Subset\omega$ such that

$$\iint_K |f^k_{\kappa,g}||\partial_t \tilde{y}_k|\kappa\mu_g\, dt \le \delta,$$

with $K=((0,T)\times\omega)\setminus((T_1,T_2)\times\mathcal{O})$. Let $\psi\in\mathscr{C}^\infty_c((0,T)\times\omega)$ such that $0\le\psi\le 1$ and equal to 1 in a neighborhood of $[T_1,T_2]\times\mathcal{O}$. One thus has

$$\begin{aligned}|(\mathbf{1}_{(0,T)\times\omega} f^k_{\kappa,g}, \partial_t \tilde{y}^k)_{L^2(\mathcal{L})}| &\le |(\psi f^k_{\kappa,g}, \partial_t \tilde{y}^k)_{L^2(\mathcal{L})}| + |((\mathbf{1}_{(0,T)\times\omega}-\psi) f^k_{\kappa,g}, \partial_t \tilde{y}^k)_{L^2(\mathcal{L})}| \\ &\le |(\psi f^k_{\kappa,g}, \partial_t \tilde{y}^k)_{L^2(\mathcal{L})}| + \delta.\end{aligned}$$

With (6-4) and Lemma 6.2, one finds

$$(\psi f^k_{\kappa,g}, \partial_t \tilde{y}^k)_{L^2(\mathcal{L})} \xrightarrow[k\to+\infty]{} 0, \tag{6-6}$$

and the conclusion of the lemma follows. ☐

6B. *Proof of Proposition 1.19.* We consider first the case $\alpha=1$. As proven in [Dehman and Lebeau 2009] one has $f^{y^0,y^1}_{\kappa,g}\in\mathscr{C}^0([0,T], H^1(\mathcal{M}))$ and the estimate

$$\|f^{y^0,y^1}_{\kappa,g}\|_{L^\infty(0,T;H^1(\mathcal{M}))} \lesssim \|(y^0,y^1)\|_{H^2(\mathcal{M})\oplus H^1(\mathcal{M})}.$$

With this regularity of the source term in the right-hand-side of the wave equations in (1-8), one finds $y,\tilde{y}\in\mathscr{C}^0([0,T], H^2(\mathcal{M}))$. Computing the difference in (1-8) one writes

$$P_{\kappa,g}(y-\tilde{y}) = (A_{\kappa,g}-A_{\tilde{\kappa},\tilde{g}})\tilde{y}. \tag{6-7}$$

A hyperbolic energy estimate yields

$$\begin{aligned}\mathcal{E}_{\kappa,g}(y-\tilde{y})(T)^{1/2} &\lesssim \|(A_{\kappa,g}-A_{\tilde{\kappa},\tilde{g}})\tilde{y}\|_{L^\infty(0,T;L^2(\mathcal{M}))} \lesssim \|(\kappa,g)-(\tilde{\kappa},\tilde{g})\|_{\mathcal{X}^1}\|\tilde{y}\|_{L^\infty(0,T;H^2(\mathcal{M}))} \\ &\lesssim \|(\kappa,g)-(\tilde{\kappa},\tilde{g})\|_{\mathcal{X}^1}\|f^{y^0,y^1}_{\kappa,g}\|_{L^\infty(0,T;H^1(\mathcal{M}))} \\ &\lesssim \|(\kappa,g)-(\tilde{\kappa},\tilde{g})\|_{\mathcal{X}^1}\|(y^0,y^1)\|_{H^2(\mathcal{M})\oplus H^1(\mathcal{M})}.\end{aligned}$$

In the case $\alpha=0$, one writes

$$\begin{aligned}\mathcal{E}_{\kappa,g}(y-\tilde{y})(T)^{1/2} &\lesssim \mathcal{E}_{\kappa,g}(y)(T)^{1/2} + \mathcal{E}_{\kappa,g}(\tilde{y})(T)^{1/2} \lesssim \mathcal{E}_{\kappa,g}(y)(T)^{1/2} + \mathcal{E}_{\tilde{\kappa},\tilde{g}}(\tilde{y})(T)^{1/2} \\ &\lesssim \|(y^0,y^1)\|_{H^1(\mathcal{M})\oplus L^2(\mathcal{M})}.\end{aligned}$$

Finally, the result follows from interpolation between the two cases $\alpha=0$ and $\alpha=1$. ☐

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