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THE DIRICHLET PROBLEM FOR THE LAGRANGIAN MEAN CURVATURE EQUATION

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We solve the Dirichlet problem with continuous boundary data for the Lagrangian mean curvature equation on a uniformly convex, bounded domain in $\mathbb{R}^n$.

1. Introduction

We consider the Dirichlet problem for the Lagrangian mean curvature equation on a uniformly convex, bounded domain $\Omega \subset \mathbb{R}^n$ given by

$$\begin{cases} F(D^2u) = \sum_{i=1}^{n} \arctan \lambda_i = \psi(x) & \text{in } \Omega, \\ u = \phi & \text{on } \partial\Omega, \end{cases} \tag{1-1}$$

where the λ_i are the eigenvalues of the Hessian matrix D^2u, ψ is the potential for the mean curvature of the Lagrangian submanifold $\{(x, Du(x)) \mid x \in \Omega\} \subseteq \mathbb{R}^n \times \mathbb{R}^n$, and ϕ is a given continuous function on $\partial\Omega$.

Our main result is the following:

Theorem 1.1. *Suppose that $\phi \in C^0(\partial\Omega)$ and $\psi : \bar{\Omega} \to \big[(n-2)\frac{\pi}{2} + \delta, n\frac{\pi}{2}\big)$ is in $C^{1,1}(\bar{\Omega})$, where Ω is a uniformly convex, bounded domain in $\mathbb{R}^n$ and $\delta > 0$. Then there exists a unique solution $u \in C^{2,\alpha}(\Omega) \cap C^0(\partial\Omega)$ to the Dirichlet problem (1-1).*

We also provide a viscosity-based proof for the following well-known result established in [Harvey and Lawson 2009].

Theorem 1.2. *Suppose that $\phi \in C^0(\partial\Omega)$ and $\psi : \bar{\Omega} \to \big(-n\frac{\pi}{2}, n\frac{\pi}{2}\big)$ is a constant, where Ω is a uniformly convex, bounded domain in $\mathbb{R}^n$. Then there exists a unique solution $u \in C^0(\bar{\Omega})$ to the Dirichlet problem (1-1).*

When the phase ψ is constant, denoted by c, we have that u solves the special Lagrangian equation

$$\sum_{i=1}^{n} \arctan \lambda_i = c, \tag{1-2}$$

or equivalently,

$$\cos c \sum_{1 \le 2k+1 \le n} (-1)^k \sigma_{2k+1} - \sin c \sum_{0 \le 2k \le n} (-1)^k \sigma_{2k} = 0.$$

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Equation (1-2) originates in the special Lagrangian geometry of Harvey and Lawson [1982]. The Lagrangian graph $(x, Du(x)) \subset \mathbb{R}^n \times \mathbb{R}^n$ is called special when the argument of the complex number $(1+i\lambda_1)\cdots(1+i\lambda_n)$ or the phase ψ is constant, and it is special if and only if $(x, Du(x))$ is a (volume-minimizing) minimal surface in $(\mathbb{R}^n \times \mathbb{R}^n, dx^2+dy^2)$ [Harvey and Lawson 1982].

A dual form of (1-2) is the Monge–Ampère equation

$$\sum_{i=1}^{n} \ln \lambda_i = c.$$

This is the potential equation for special Lagrangian submanifolds in $(\mathbb{R}^n \times \mathbb{R}^n, dx\,dy)$ as interpreted in [Hitchin 1997]. The gradient graph $(x, Du(x))$ is volume-maximizing in this pseudo-Euclidean space as shown in [Warren 2010]. Mealy [1989] showed that an equivalent algebraic form of the above equation is the potential equation for his volume-maximizing special Lagrangian submanifolds in $(\mathbb{R}^n \times \mathbb{R}^n, dx^2 - dy^2)$.

A key prerequisite for the smooth solvability of the Dirichlet problem for fully nonlinear, elliptic equations is the concavity of the operator on the space of symmetric matrices. The arctangent operator or the logarithmic operator is concave if u is convex, or if the Hessian of u has a lower bound $\lambda \geq 0$. Certain concavity properties of the arctangent operator are still preserved for saddle u. The concavity of the arctangent operator in (1-1) depends on the range of the Lagrangian phase. The phase $(n-2)\frac{\pi}{2}$ is called critical because the level set $\{\lambda \in \mathbb{R}^n \mid \lambda \text{ satisfying (1-1)}\}$ is convex only when $|\psi| \geq (n-2)\frac{\pi}{2}$ [Yuan 2006, Lemma 2.2]. The concavity of the level set is evident for $|\psi| \geq (n-1)\frac{\pi}{2}$ since that implies $\lambda > 0$ and then F is concave. For a supercritical phase $|\psi| \geq (n-2)\frac{\pi}{2} + \delta$ the operator F can be extended to a concave operator [Chen and Warren 2019; Collins et al. 2017].

The Dirichlet problem for fully nonlinear, elliptic equations of the form $F(\lambda[D^2u]) = \psi(x)$ was studied by Caffarelli, Nirenberg, and Spruck in [Caffarelli et al. 1985], where they proved the existence of classical solutions under various hypotheses on the function F and the domain. Their results extended the work of Krylov [1983b], Ivochkina [1983], and their previous work [Caffarelli et al. 1984] on equations of Monge–Ampère-type. For the Monge–Ampère equation, continuous boundary data leads to only Lipschitz continuous solutions; Pogorelov [1978] constructed his famous counterexamples for the three dimensional Monge–Ampère equation $\sigma_3(D^2u) = \det(D^2u) = 1$, which also serve as counterexamples for cubic and higher-order symmetric σ_k equations. Trudinger [1995] proved a priori estimates and existence of smooth solutions to fully nonlinear equations of the type of Hessian equations. In [Ivochkina et al. 2004], Ivochkina, Trudinger, and Wang studied the Dirichlet problem for a class of fully nonlinear, degenerate elliptic equations which depend only on the eigenvalues of the Hessian matrix. Harvey and Lawson [2009] studied the Dirichlet problem for fully nonlinear, degenerate elliptic equations of the form $F(D^2u) = 0$ on a smoothly bounded domain in $\mathbb{R}^n$. Interior regularity for viscosity solutions of (1-2) with critical and supercritical constant phase $|\psi| \geq (n-2)\frac{\pi}{2}$ was shown in [Warren and Yuan 2010; 2014]. For a subcritical phase $|\psi| < (n-2)\frac{\pi}{2}$, singular solutions of (1-2) were constructed in [Nadirashvili and Vlăduţ 2010; Wang and Yuan 2013]. The existence and uniqueness of continuous viscosity solutions to the Dirichlet problem for (1-2) with continuous boundary data was shown in Yuan [2008]. Brendle and Warren [2010] studied a second boundary value problem for the special Lagrangian equation.

The Lagrangian mean curvature equation (1-1), which was introduced by Harvey and Lawson, is far from being completely understood. This gives rise to several challenging problems concerning the regularity of solutions and the well-posedness for general phase functions. Recently, regularity and effective Hessian estimates for viscosity solutions of equation (1-1) were studied in [Bhattacharya 2021; Bhattacharya and Shankar 2020; 2023] under certain assumptions on the regularity of the phase and convexity properties of the solution. In [Collins et al. 2017], Collins, Picard, and Wu solved the Dirichlet problem (1-1) on a compact domain with C^4 boundary value under the assumption of the existence of a subsolution and a supercritical phase restriction using techniques accumulated since the 1980s. In [Dinew et al. 2019], Dinew, Do, and Tô showed the existence and uniqueness of a C^0 solution to (1-1) on a bounded C^2 domain with C^0 boundary value under the assumption of the existence of a subsolution and a supercritical phase restriction.

The major difficulty in proving Theorem 1.1 is the unavailability of smooth boundary data: our boundary value is merely C^0. We use a standard continuity method and uniform approximation of the C^0 boundary value to overcome this. Another hurdle lies in estimating the double normal derivatives at the boundary: we use Trudinger's technique and a change of basis argument to construct a lower linear barrier function for u_n. Once we obtain uniform $C^{2,\alpha}$ estimates up to the boundary, we use the a priori interior Hessian estimates proved in [Bhattacharya 2021] to approximate the C^0 boundary value. Note that we assume $\psi \geq (n-2)\frac{\pi}{2}+\delta$ since, by symmetry, $\psi \leq -(n-2)\frac{\pi}{2}-\delta$ can be treated similarly.

In Theorem 1.2, we consider all values of the constant Lagrangian phase, which include subcritical values. The main difficulty here is the lack of uniform ellipticity and concavity. Harvey and Lawson [2009] established the existence and uniqueness of continuous solutions of fully nonlinear, degenerate elliptic equations of the form $F(D^2u)=0$ on a smoothly bounded domain in $\mathbb{R}^n$ under an explicit geometric F-convexity assumption on the boundary of the domain. The key ingredients of their proof were the use of subaffine functions and Dirichlet duality. As an application, the continuous solvability of the constant phase equation (1-2) is obtained. Here in Theorem 1.2, we focus only on the continuous solvability of the Dirichlet problem of equation (1-2) and provide a short proof that solely relies on a certain comparison principle. Note that our methods of proving Theorem 1.2 are much different in nature than the proof by Harvey and Lawson: our brief proof follows via Perron's method using an idea that was introduced in [Ishii 1989], and it requires comparison principles for strictly elliptic,[1] nonconcave, fully nonlinear equations [Yuan 2004].

Remark 1.3. For Theorem 1.1, an assumption weaker than C^1 on ψ will lead to counterexamples with continuous boundary data. For example, in two dimensions, we consider a boundary value problem of (1-1) on the unit ball $B_1(0)$, where the phase is in C^α with $\alpha \in (0,1)$:

$$\psi(x)=\frac{\pi}{2}-\arctan(\alpha^{-1}|x|^{1-\alpha}) \qquad \text{and} \qquad u(x)=\int_0^{|x|} t^\alpha\, dt \quad \text{on } \partial B_1.$$

This problem admits a non-C^2 viscosity solution u with gradient $Du=|x|^{\alpha-1}x$, thereby proving a contradiction. If the Lagrangian phase is subcritical, i.e., $|\psi|<(n-2)\frac{\pi}{2}$, then even for the constant

[1] $F(D^2u)=\psi$ is strictly elliptic in the sense that $(F_{u_{ij}}(D^2u))>0$

phase equation (1-2) with analytic boundary data, C^0 viscosity solutions may only be C^{1,ε_0} but no more, as shown in [Wang and Yuan 2013]. However, the existence of $C^{2,\alpha}$ solutions to (1-1) with critical and supercritical phase, i.e., $|\psi| \geq (n-2)\frac{\pi}{2}$, where $\psi \in C^{1,\varepsilon_0}$, or even $|\psi| \geq (n-2)\frac{\pi}{2}$, where $\psi \in C^{1,1}$, are still open questions. As of now, it is also unknown if C^0 viscosity solutions of (1-2) are Lipschitz for subcritical phases.

Remark 1.4. In Theorem 1.2, if we replace the constant phase with any continuous function lying in the subcritical or critical range, then the existence and uniqueness of C^0 viscosity solutions of (1-1) remain open questions. This is due to the lack of a suitable comparison principle for strictly elliptic, nonconcave, fully nonlinear equations with a variable right-hand side. Harvey and Lawson [2019] introduced a condition called "tameness" on the operator F, which is a little stronger than strict ellipticity and allows one to prove comparison. Harvey and Lawson [2021] further proved that, for the Lagrangian mean curvature equation, one can only show tamability in the supercritical phase interval. Cirant and Payne [2021] established comparison for this equation when the range of the phase is restricted to the intervals $\big((n-2k)\frac{\pi}{2}, (n-2(k-1))\frac{\pi}{2}\big)$, where $1 \leq k \leq n$. This in turn solves the Dirichlet problem on these intervals, as shown in [Harvey and Lawson 2021, Theorem 6.2(C)]. For σ_k equations with a variable right-hand side, results analogous to Theorem 1.2 exist. This is due to the fact that the linearized operator has a positive lower bound in determinant unlike the Lagrangian mean curvature equation (1-1).

This article is divided into the following sections: in Section 2, we state some well-known algebraic and trigonometric inequalities satisfied by solutions of (1-1). In Section 3, we prove $C^{2,\alpha}$ estimates up to the boundary assuming C^4 boundary data. In Section 4, we first solve the Dirichlet problem with C^4 boundary data using the method of continuity and then combine it with the Hessian estimates proved in [Bhattacharya 2021] to solve the Dirichlet problem with continuous boundary data. In Section 5, we prove Theorem 1.2. In the Appendix, we state a well-known linear algebra lemma that we use in estimating the Hessian of u on the boundary, and we provide the proof of a certain comparison principle that is essential for the proof of Theorem 1.2.

2. Preliminaries

The induced Riemannian metric on the Lagrangian submanifold $\{(x, Du(x)) \mid x \in \Omega\} \subset \mathbb{R}^n \times \mathbb{R}^n$ is given by

$$g = I_n + (D^2u)^2. \tag{2-1}$$

On taking the gradient of both sides of the Lagrangian mean curvature equation (1-1), we get

$$\sum_{a,b=1}^{n} g^{ab} u_{jab} = \psi_j, \tag{2-2}$$

where g^{ab} is the inverse of the induced Riemannian metric g. From [Harvey and Lawson 1982, (2.19)], we see that the mean curvature vector $\vec{H}$ of this Lagrangian submanifold $\{(x, Du(x)) \mid x \in \Omega\}$ is given by $\vec{H} = J\nabla_g \psi$, where ∇_g is the gradient operator for the metric g and J is the complex structure, or the $\frac{\pi}{2}$ rotation matrix in $\mathbb{R}^n \times \mathbb{R}^n$.

Lemma 2.1. *Suppose that the ordered real numbers $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$ satisfy (1-1) with $\psi \geq (n-2)\frac{\pi}{2}$. Then we have*

(1) $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_{n-1} > 0,\ \lambda_{n-1} \geq |\lambda_n|$,

(2) $\lambda_1 + (n-1)\lambda_n \geq 0$,

(3) $\sigma_k(\lambda_1, \dots, \lambda_n) \geq 0$ *for all* $1 \leq k < n$ *and* $n \geq 2$,

(4) *if* $\psi \geq (n-2)\frac{\pi}{2} + \delta$, *then* $D^2 u \geq -\cot(\delta I_n)$.

Proof. Properties (1), (2), and (3) follow from [Wang and Yuan 2014, Lemma 2.1]. Property (4) follows from [Yuan 2006, p. 1356]. □

3. $C^{2,\alpha}$ estimate up to the boundary

We first prove the following $C^{2,\alpha}$ estimate up to the boundary of Ω.

Theorem 3.1. *Let $\phi \in C^4(\bar{\Omega})$ and $\psi : \bar{\Omega} \to \left[(n-2)\frac{\pi}{2} + \delta, n\frac{\pi}{2}\right)$ be in $C^{2,\alpha}(\bar{\Omega})$, where Ω is a uniformly convex domain in $\mathbb{R}^n$ with $\partial\Omega \in C^2$. Then there exists a universal constant $\alpha \in (0,1)$ such that if $u \in C^{4,\alpha}(\bar{\Omega})$ is a solution of (1-1), then*

$$\|u\|_{C^{2,\alpha}(\bar{\Omega})} \leq C(\|\psi\|_{C^{1,1}(\bar{\Omega})}, \|\phi\|_{C^4(\bar{\Omega})}, n, \delta, \partial\Omega). \tag{3-1}$$

Proof. We first make the following observation, which will be used for Steps 1, 2, 3.2, and 3.3 below. We pick an arbitrary boundary point $x_0 \in \partial\Omega$. By a rotation and translation, we choose a coordinate system such that the chosen boundary point is the origin and Ω lies above the hyperplane $\{x_n = 0\}$, with e_n as the inner unit normal at 0. For such a domain, we can write

$$\partial\Omega = \left\{(x', x_n) \;\middle|\; x_n = h(x') = \tfrac{1}{2}(k_1 x_1^2 + \cdots + k_{n-1} x_{n-1}^2) + o(|x'|^2)\right\}, \tag{3-2}$$

where the $\{k_i\}_{1 \leq i \leq n}$ denote the principal curvatures of $\partial\Omega$ at 0. At $0 \in \partial\Omega$ the boundary value satisfies

$$\begin{aligned}
\phi(x', x_n) &= \phi(x', h(x')) \\
&= \phi(0) + \phi_{x'}(0) \cdot x' + \phi_{x_n}(0) h(x') \\
&\qquad + \tfrac{1}{2}(x')^T \phi_{x'x'}(0) x' + \tfrac{1}{2}\phi_{x'x_n}(0) \cdot x' h(x') + \tfrac{1}{2}\phi_{x_n x_n}(0) h(x') h(x') + o(|x'|^2 + h^2(x')) \\
&= Q(x) + o(1)|x'|^2.
\end{aligned}$$

Without loss of generality, one may subtract the linear part in x' of the above Taylor expansion to get $C_0 = C_0(\|\phi\|_{C^2(\partial\Omega)}, n, k)$ such that

$$L^- = -C_0 x_n \leq \phi \leq C_0 x_n = L^+ \quad \text{on } \partial\Omega. \tag{3-3}$$

We now prove estimate (3-1) in the following four steps. We will estimate all the boundary derivatives of u at the origin.

Step 1: Bound for $\|u\|_{L^\infty(\bar{\Omega})}$.

Claim 1. *We show the following*:

$$\|u\|_{L^\infty(\bar{\Omega})} \leq C(\|\phi\|_{C^2(\bar{\Omega})}, n, |\partial\Omega|_{C^2}). \tag{3-4}$$

Proof. The function $\psi:\overline{\Omega}\to\big[(n-2)\frac{\pi}{2}+\delta, n\frac{\pi}{2}\big)$ is in $C^{1,1}(\overline{\Omega})$, so there exists $\varepsilon>0$ such that $\psi<n\frac{\pi}{2}-\varepsilon$. Fixing this ε, we define $\underline{\psi}=(n-2)\frac{\pi}{2}+\delta$ and $\overline{\psi}=n\frac{\pi}{2}-\varepsilon$. Recalling (3-3) we find constants c_0 and C_0' depending on C_0 above such that, on $\partial\Omega$, we have

$$-c_0|x|^2+\frac{1}{2}|x|^2\tan\frac{\overline{\psi}}{n}=-C_0'|x|^2\le -C_0x_n\le\phi\le C_0'|x|^2+\frac{1}{2}|x|^2\tan\frac{\underline{\psi}}{n}. \tag{3-5}$$

Using relation (3-2), we define

$$-Cx_n+\frac{1}{2}|x|^2\tan\frac{\overline{\psi}}{n}=B^-, \tag{3-6}$$

$$Cx_n+\frac{1}{2}|x|^2\tan\frac{\underline{\psi}}{n}=B^+, \tag{3-7}$$

where $C=C(\|\phi\|_{C^2(\partial\Omega)},n,k_i)$. We observe that

$$\begin{aligned} F(D^2B^-)\ge F(D^2u)\ge F(D^2B^+) \quad & \text{in } \Omega,\\ B^-\le u\le B^+ \quad & \text{on } \partial\Omega, \quad \text{with equality holding at 0.}\end{aligned} \tag{3-8}$$

Using comparison principles we see that (3-4) holds. □

Step 2: Bound for $\|Du\|_{L^\infty(\overline{\Omega})}$.

Claim 2. *We show the following*:

$$\|Du\|_{L^\infty(\overline{\Omega})}\le C(\|\psi\|_{C^1(\overline{\Omega})},\|\phi\|_{C^2(\overline{\Omega})},n,\delta,|\partial\Omega|_{C^2}). \tag{3-9}$$

Proof. From Lemma 2.1, we see that u is semiconvex: $D^2u\ge-\cot(\delta I_n)$. We modify u to the convex function $u(x)+\cot(\delta|x|^2/2)$. Since the gradient of this convex function, given by $Du(x)+x\cot\delta$, attains its supremum on the boundary of Ω, we get

$$\sup_{\overline{\Omega}}|Du(x)|\le\sup_{\partial\Omega}|Du(x)|+\cot\delta. \tag{3-10}$$

For $1\le i<n$, we have $u_i=\phi_i$, so we only need to estimate $u_n(0)$. Recalling (3-8), we again use comparison principles, and on taking the normal derivative at 0, we get

$$|u_n(0)|\le C(\|\psi\|_{C^1(\overline{\Omega})},\|\phi\|_{C^2(\overline{\Omega})},n,|\partial\Omega|_{C^2}).$$

Combining (3-10) with the above we get (3-9). □

Step 3: Bound for $\|D^2u\|_{L^\infty(\overline{\Omega})}$.

Claim 3. *We prove the following*:

$$\|D^2u\|_{L^\infty(\overline{\Omega})}\le C(\|\psi\|_{C^{1,1}(\overline{\Omega})},\|\phi\|_{C^4(\overline{\Omega})},n,\delta,|\partial\Omega|_{C^4}). \tag{3-11}$$

The proof of the above claim is achieved by from the following steps.

Step 3.1: We first prove that the Hessian attains its supremum on the boundary of Ω. We show that

$$\|D^2u\|_{L^\infty(\overline{\Omega})}\le C(\|\psi\|_{C^{1,1}(\overline{\Omega})},\|D^2u\|_{L^\infty(\partial\Omega)},\delta). \tag{3-12}$$

Since the phase is supercritical, we can modify the operator F to a concave operator as shown in [Collins et al. 2017, Lemma 2.2] or [Chen and Warren 2019, p. 347]. (For a detailed proof of this fact, see [Collins et al. 2017, Lemma 2.2].) Following the notation used in [Chen and Warren 2019, p. 347], we denote the modified concave operator by $\widetilde{F} = -\exp(-A(\delta)F)$ and the modified phase by $\tilde{\psi}(\lambda) = -\exp(-A(\delta)\psi(\lambda))$, where $A(\delta)$ is large enough. On differentiating (1-1) twice, we get

$$\widetilde{F}^{ij}\partial_{ij}u_{ee} + \widetilde{F}^{ij,kl}\partial_{ij}u_e\partial_{kl}u_e = \tilde{\psi}_{ee},$$

$$\widetilde{F}^{ij}\partial_{ij}\Delta u = \Delta\tilde{\psi} - \sum_e \widetilde{F}^{ij,kl}\partial_{ij}u_e\partial_{kl}u_e \geq \Delta\tilde{\psi},$$

where the last inequality follows from the concavity of the operator. Let p_0 be an interior point of Ω. By an orthogonal transformation, we assume D^2u to be diagonalized at p_0. We observe that

$$g^{ij}\partial_{ij}\big(\Delta u + \tfrac{1}{2}C_1|x|^2\big)(p_0) \geq -C(\|\psi\|_{C^{1,1}(\Omega)}) + C_1\sum_{i=1}^{n}\frac{1}{1+\lambda_i^2} > 0.$$

The last two inequalities follow from using the structure of the metric g (defined in (2-1)) and then choosing a large enough constant C_1 by exploiting the semiconvexity of u. The maximal principle implies that $|D^2u|$ attains its supremum on the boundary. Next, we estimate the Hessian on the boundary in the following steps: we first estimate the double tangential derivatives $u_{TT}(0)$, followed by the mixed tangent normal derivatives $u_{TN}(0)$, followed by the double normal derivative $u_{NN}(0)$.

Step 3.2: The double tangential estimate. Denoting the second fundamental form by II, we observe that

$$D^2(u-\phi)|_T(0) = -(u-\phi)_n(0)II|_{\partial\Omega}(0),$$

where

$$(D^2u)|_T = \{u_{T_iT_j} \mid 1 \leq i, j < n\}$$

is the Riemannian Hessian. By estimate (3-9) derived in Step 2, for $1 \leq i, j < n$, we get the estimate:

$$|u_{ij}(0)| \leq C(\|\psi\|_{C^1(\bar{\Omega})}, \|\phi\|_{C^2(\bar{\Omega})}, n, \delta, \Omega).$$

Step 3.3: The mixed tangent normal estimate. Observe that (1-1) is dependent only on the eigenvalues of the Hessian and hence is invariant under rotation of coordinates. In light of [Caffarelli et al. 1985, p. 281], we observe that, since $x_i\partial_j - x_j\partial_i$ for $i \neq j$ is the infinitesimal generator of a rotation, we get

$$g^{ij}\partial_{ij}(x_i\partial_j - x_j\partial_i)u = (x_i\partial_j - x_j\partial_i)\psi.$$

For $i < n$, we define the annular vector field

$$\tau(x) = \partial_i + \sum_{j=1}^{n-1} h_{ij}(0)(x_j\partial_n - x_n\partial_j),$$

where h is as defined in (3-2); $\tau(0) = e_i$ for $i < n$. This is an approximated tangent vector up to the second-order on the boundary. Indeed, at a point $(x', h(x'))$ on $\partial\Omega$, near the origin, we can write

$$\tau(x) = \partial_i + \partial_i h(x')\partial_n + O(|x'|^2)\partial_n - \sum_{j=1}^{n-1} h_{ij}(0)h(x')\partial_j.$$

Denoting the rotational derivative of u along the boundary by u_τ, we get $g^{ij}\partial_{ij}u_\tau = \psi_\tau$ in Ω and $u_\tau = \phi_\tau$ on $\partial\Omega$.

Replacing ϕ with ϕ_τ and repeating the argument in (3-3), we get the following on $\partial\Omega$:

$$-Cx_n \le \phi_\tau \le Cx_n, \tag{3-13}$$

where $C = C(\|\phi_\tau\|_{C^2(\bar{\Omega})}, n, k)$. Repeating the argument in (3-5) and choosing $c_1 > 0$ suitably, we get

$$-c_1|x|^2 + \frac{1}{2}|x|^2 \tan\frac{\overline{\psi}}{n} = -C|x|^2 \le \phi_\tau \le C|x|^2 + \frac{1}{2}|x|^2 \tan\frac{\underline{\psi}}{n} \quad \text{on } \partial\Omega.$$

We define u_0 to be the subsolution

$$u_0 = -Cx_n + \frac{1}{2}|x|^2 \tan\frac{\overline{\psi}}{n},$$

where $C = C(\|\phi\|_{C^3(\bar{\Omega})}, \|\psi\|_{C^1(\bar{\Omega})}, n, |\partial\Omega|_{C^2})$. Let $w = u - u_0$. Since the phase lies in the supercritical range, as before we extend the operator F to the concave operator $\widetilde{F}$ and denote the corresponding linearization by $\tilde{g}^{ij}$. Using concavity, for some $\varepsilon_0 > 0$, we get the following on a small ball of radius r around the origin:

$$\begin{aligned} \tilde{g}^{ij} w_{ij} &\le -\varepsilon_0 && \text{inside } \Omega \cap B_r(0), \\ w &\ge 0 && \text{on } \partial(\Omega \cap B_r(0)), \\ w(0) &= 0. \end{aligned} \tag{3-14}$$

We now choose α and β large enough that

$$\begin{aligned} \tilde{g}^{ij}\partial_{ij}(\alpha w + \beta|x|^2 \pm u_\tau) &\le 0 && \text{in } \Omega \cap B_r(0), \\ \alpha w + \beta|x|^2 \pm u_\tau &\ge 0 && \text{on } \partial(\Omega \cap B_r(0)). \end{aligned} \tag{3-15}$$

Since $w \ge 0$ on $\partial(\Omega \cap B_r(0))$, we only need to choose β large enough that

$$\beta|x|^2 \pm u_\tau \ge 0 \quad \text{on } \partial(\Omega \cap B_r(0)).$$

We observe that, on $\Omega \cap \partial B_r(0)$, we have $\beta \ge C/r^2$, where $C = C(\|\psi\|_{C^1(\bar{\Omega})}, \|\phi\|_{C^2(\bar{\Omega})}, \delta, n, |\partial\Omega|_{C^2})$ is obtained by using the gradient estimate in (3-9). Using (3-13) we get the required value of β on $\partial\Omega \cap B_r(0)$. Fixing the larger of the two values to be the constant β we now choose α such that (3-15) holds. We have

$$\tilde{g}^{ij}\partial_{ij}(\alpha w + \beta|x|^2 \pm u_\tau) \le -\alpha\varepsilon_0 + C,$$

where $C = C(\beta, |\psi|_{C^1(\bar{\Omega})})$. We now choose α large enough that $-\alpha\varepsilon_0 + C \le 0$ and observe that $\alpha w + \beta|x|^2 \pm u_\tau(0) = 0$ at 0. Using Hopf's lemma we see that

$$\partial_n(\alpha w + \beta|x|^2 \pm u_\tau)(0) \ge 0 \implies \pm u_{\tau n}(0) \ge \mp\partial_n(\alpha w + \beta|x|^2 \pm u_\tau)(0) \implies |u_{\tau n}(0)| \le |\alpha w_n(0)| \le C.$$

Therefore, for $1 \le i < n$, we have

$$|u_{in}(0)| \le C(\|\psi\|_{C^{1,1}(\bar{\Omega})}, \|\phi\|_{C^3(\bar{\Omega})}, n, \delta, |\partial\Omega|_{C^2}).$$

<u>Step 3.4</u>: The double normal estimate. By Lemma 2.1, D^2u is bounded from below, so we only need to prove an upper bound, which we find using an idea of Trudinger [1995].

Let the unit normal direction vector be denoted by e_γ. Denoting the eigenvalues of the $(n-1)\times(n-1)$ matrix u_{TT} by λ', we write the Hessian as

$$D^2u=\begin{bmatrix}u_{TT} & u_{T\gamma}\\ u_{\gamma T} & u_{\gamma\gamma}\end{bmatrix}=\begin{bmatrix}\lambda' & u_{T\gamma}\\ u_{\gamma T} & u_{\gamma\gamma}\end{bmatrix}.$$

Let x_0' be the minimal point of $\widetilde{\Theta}(\lambda')|_{\partial\Omega}$, where

$$\widetilde{\Theta}(\lambda')=\sum_{i=1}^{n-1}\arctan\lambda_i'-\psi,$$

and we write $\lambda_0'=\lambda'(x_0')$.

Our goal is to find a lower linear barrier function for u_γ at x_0'. Then, with the help of a change of basis technique, we find a lower linear barrier function for u_n at x_0'. This leads us to find an upper bound of $u_{nn}(x_0')$ followed by an upper bound of $u_{nn}(x)$ for all $x\in\partial\Omega$. Now we estimate the lower bound of

$$\operatorname{tr}(D^2u)|_T=\sum_{i=1}^{n-1}\lambda_i'.$$

Observe that $\widetilde{\Theta}(\lambda')\ge\widetilde{\Theta}(\lambda_0')>\psi-\frac{\pi}{2}>(n-3)\frac{\pi}{2}$. So the level set $\{\lambda'\in\mathbb{R}^{n-1}\mid\widetilde{\Theta}(\lambda')=\widetilde{\Theta}(\lambda_0')\}$ should be convex. Heuristically, this property means the following:

$$\langle D\widetilde{\Theta}(\lambda_0'),\lambda'\rangle\ge\langle D\widetilde{\Theta}(\lambda_0'),\lambda_0'\rangle=K_0,\quad\text{with equality holding at } x_0',$$

where K_0 is a constant depending on $|\psi|_{C^1(\Omega)}$, $|\phi|_{C^2(\partial\Omega)}$, and δ. Writing

$$\left[\frac{\partial\widetilde{\Theta}(D^2u(x_0))|_T}{\partial D^2u|_T}\right]=A_{ij}(\lambda_0'),$$

where $1\le i,\ j<n$, we see that

$$\operatorname{tr}(A_{ij}(\lambda_0'))(D^2u(x)|_T)\ge K_0,\quad\text{with equality holding at } x_0'.$$

Again denoting the second fundamental form by II, we observe that

$$D^2(u-\phi)|_T=(u-\phi)_\gamma II|_{\partial\Omega}\quad\text{and}$$
$$\operatorname{tr}[A_{ij}(\lambda_0')(D^2\phi|_T-\phi_\gamma II|_{\partial\Omega}+u_\gamma II|_{\partial\Omega})]\ge K_0,\quad\text{with equality holding at } x_0'.$$

Writing $\widetilde{\Theta}_i(\lambda')=(\partial/\partial\lambda_i')\widetilde{\Theta}(\lambda')$, we get

$$u_\gamma\ge\frac{1}{\sum_{i=1}^{n-1}\widetilde{\Theta}_i(\lambda_0')\kappa_i(x')}[K_0-\operatorname{tr}(A_{ij}(\lambda_0')(D^2\phi|_T-\phi_\gamma II|_{\partial\Omega}))],\quad\text{with equality holding at } x_0',\tag{3-16}$$
$$\implies\quad u_\gamma\ge C(|\phi|_{C^4(\bar{\Omega})},|\partial\Omega|_{C^4},|\psi|_{C^1(\Omega)},\delta),\quad\text{with equality holding at } x_0',$$

where the last inequality follows from the observation that, for all the terms in the right-hand side of (3-16), one can find a lower linear barrier function whose Lipschitz norm depends on the $C^{3,1}$ norm of ϕ and the C^1 norm of ψ. Next, we consider a unit local basis at x_0' denoted by $\mathcal{B}=\{e_n,e_{T_\alpha}|1\le\alpha<n\}$, where e_n is

used to denote the outward unit normal and e_{T_α} denotes vectors in the tangential direction at x_0'. By a change of basis, we write $e_\gamma = ae_n + be_{T_\alpha}$. A simple computation shows that

$$e_\gamma = \frac{\langle e_\gamma, e_n\rangle}{1-\langle e_n, e_{T_\alpha}\rangle^2} e_n - \frac{\langle e_\gamma, e_n\rangle\langle e_n, e_{T_\alpha}\rangle}{1-\langle e_n, e_{T_\alpha}\rangle^2} e_{T_\alpha},$$

from which one can easily find a lower linear barrier for u_n at x_0'. So far we have the following:

$$u_n \geq L_1^-(x', x_n) \quad \text{on } \partial\Omega, \quad \text{with equality holding at } x_0', \tag{3-17}$$

where

$$L_1^-(x', x_n) = -C(|\phi|_{C^4}, |\partial\Omega|_{C^4}, |\psi|_{C^1(\Omega)}, \delta)x_n \geq -C|x|^2.$$

Now we choose coordinates such that x_0' is the origin and the $(n-1)\times(n-1)$ matrix $u_{TT}(0)$ is diagonalized.

Claim 4. *We show that*

$$u_{nn}(0) \leq C,$$

where $C = C(\|\psi\|_{C^{1,1}(\bar\Omega)}, \|\phi\|_{C^4(\bar\Omega)}, n, \delta, |\partial\Omega|_{C^4})$.

To be clear, the notation e_n now denotes the outward unit normal unlike earlier in the proof where it was used to denote the inner unit normal (see page 2723).

Proof. We repeat the process in Step 3.3. First observe that, on taking the gradient of both sides of (1-1) in the direction e_n, we get

$$|g^{ij}\partial_{ij}u_n| \leq C(\|\psi\|_{C^1(\Omega)}). \tag{3-18}$$

We define $w = u - B^-$, where B^- is the subsolution defined in (3-6), and we see that w satisfies condition (3-14). We choose α and β large enough that

$$\begin{aligned} g^{ij}\partial_{ij}(\alpha w + \beta|x|^2 + u_n) &\leq 0 \quad \text{in } \Omega\cap B_r(0), \\ \alpha w + \beta|x|^2 + u_n &\geq 0 \quad \text{on } \partial(\Omega\cap B_r(0)). \end{aligned} \tag{3-19}$$

As $w \geq 0$ on $\partial(B_r(0)\cap\Omega))$, we first choose β. On $\partial B_r(0)\cap\Omega$, we have $\beta \geq -C/r^2$, where $C = C(\|\psi\|_{C^1(\bar\Omega)}, \delta, \|\phi\|_{C^2(\bar\Omega)}, n, |\partial\Omega|_{C^2})$ is the constant from the estimates in (3-9) and (3-4). On $\partial\Omega\cap B_r(0)$, we find β using (3-17). Choosing the larger of the two values we get the required value of β. Fixing this β, we choose α such that (3-19) holds. Using the constant C from (3-18), we choose α large enough that $-\alpha\varepsilon_0 + C < 0$, where $C = C(\beta, \|\psi\|_{C^1(\bar\Omega)})$. Now since $(\alpha w + \beta|x|^2 + u_n)(0) = 0$, using Hopf's lemma, we get

$$\frac{\partial}{\partial_n}(\alpha w + \beta|x|^2 + u_n)(0) \leq 0 \quad\Longrightarrow\quad u_{nn}(0) \leq C(\|\psi\|_{C^{1,1}(\bar\Omega)}, \|\phi\|_{C^4(\bar\Omega)}, n, \delta, |\partial\Omega|_{C^4}). \qquad \square$$

Claim 5. *If $u_{nn}(0)$ is bounded from above, then $u_{nn}(x)$ will be bounded from above for all $x\in\partial\Omega$.*

Proof. Suppose that $u_{nn}(x_p) \geq K$ for some $x_p \in \partial\Omega$, where K is a large constant to be chosen shortly. From Claim 4, we see that, at 0,

$$\begin{aligned} &F(D^2u + Ne_n\times e_n) - F(D^2u) = \delta_0(\|\phi\|_{C^4(\partial\Omega)}, \|\psi\|_{C^{1,1}(\bar\Omega)}) > 0 \\ &\qquad\Longrightarrow\quad \lim_{a\to\infty} F(D^2u + ae_n\times e_n) \geq F(D^2u + Ne_n\times e_n) \geq F(D^2u) + \delta_0 = \psi + \delta_0. \end{aligned}$$

From Lemma A.1, we see that

$$\sum_{i=1}^{n-1} \arctan \lambda_i'(x_p) \geq \psi + \delta_0 - \frac{\pi}{2}$$

and

$$\psi = F(D^2u) = \sum_{i=1}^{n-1} \arctan \lambda_i' + o(1) + \arctan(u_{nn} + O(1)) \geq \psi + \delta_0 - \frac{\pi}{2} - \frac{\delta_0}{2} + \arctan(u_{nn} + O(1)).$$

Now if we choose K large enough that

$$u_{nn}(x_p) > \tan\left(\frac{\pi}{2} - \frac{\delta_0}{2}\right) - O(1),$$

we arrive at a contradiction. Therefore, choosing

$$K \leq \tan\left(\frac{\pi}{2} - \frac{\delta_0}{2}\right) - O(1) = C(\|\psi\|_{C^{1,1}(\bar{\Omega})}, \|\phi\|_{C^4(\bar{\Omega})}, n, \delta, |\partial\Omega|_{C^4}),$$

we see that $u_{nn}(x) \leq K$ for all $x \in \partial\Omega$. Combining all the estimates in Step 3 above we obtain (3-11). □

Step 4: Bound for $\|D^2u\|_{C^\alpha(\bar{\Omega})}$. This follows from the interior $C^{2,\alpha}$ estimates in [Evans 1982; Krylov 1983a] and the boundary $C^{2,\alpha}$ estimates in [Krylov 1983a, Theorem 4.1]. Therefore, combining all the four steps above we obtain estimate (3-1). □

4. Proof of Theorem 1.1

In this section we use the $C^{2,\alpha}$ estimate up to the boundary to solve the following Dirichlet problem using the method of continuity.

Theorem 4.1. *Suppose that $\phi \in C^4(\bar{\Omega})$ and $\psi : \bar{\Omega} \to \left[(n-2)\frac{\pi}{2} + \delta, n\frac{\pi}{2}\right)$ is in $C^{1,1}(\bar{\Omega})$, where Ω is a uniformly convex, bounded domain in $\mathbb{R}^n$ and $\delta > 0$. Then there exists a unique solution $u \in C^{2,\alpha}(\bar{\Omega})$ to the Dirichlet problem (1-1).*

Proof. For each $t \in [0, 1]$, consider the family of equations

$$\begin{cases} F(D^2u) = t\psi + (1-t)c_0 & \text{in } \Omega, \\ u = \phi & \text{on } \partial\Omega, \end{cases} \tag{4-1}$$

where $c_0 = (n-2)\frac{\pi}{2} + \delta$ and $\psi \in C^{2,\alpha}(\bar{\Omega})$. Let $I = \{t \in [0, 1] \mid \exists u_t \in C^{4,\alpha}(\bar{\Omega}) \text{ solving (4-1)}\}$. As a consequence of the interior Hessian estimates proved by Wang and Yuan [2014, p. 482, second paragraph], we have that $0 \in I$. The fact that I is open is a consequence of the implicit function theorem and invertibility of the linearized operator (2-2). The closedness of I follows from the a priori estimates. Hence, $1 \in I$. Now using a smooth approximation[2] we solve (1-1) for $\psi \in C^{1,1}$. Uniqueness follows from the maximum principle for fully nonlinear equations. □

Remark 4.2. There exists a unique smooth solution to the Dirichlet problem (1-1) if all data is smooth and if the phase lies in the supercritical range.

[2]When ψ is in $C^{1,1}(\bar{\Omega})$, we can take a sequence of smooth functions ψ_k approximating ψ and a sequence of solutions u_k solving (1-1) with ψ_k as the right-hand side. Applying the uniform $C^{2,\alpha}$ estimate and taking a limit solves the equation.

Proof of Theorem 1.1. We approximate $\phi \in C^0(\partial\Omega)$ uniformly on $\partial\Omega$ by a sequence $\{\phi_k\}_{k\geq 1}$ of C^4 functions and solve

$$\begin{cases} F(D^2u_k) = \psi & \text{in } \Omega, \\ u_k = \phi_k & \text{on } \partial\Omega \end{cases}$$

using Theorem 4.1. Applying the interior Hessian estimates proved in [Bhattacharya 2021, Theorem 1.1] and the compactness in C^2 of bounded sets in $C^{2,\alpha}$ along with maximum principles, we get convergence of $\{u_k\}$ to the desired solution $u \in C^{2,\alpha}$ on the interior and convergence of $\{\phi_k\}$ to the desired boundary function $\phi \in C^0$ on the boundary. □

Remark 4.3. The above existence proof can be extended to prove the existence of a unique C^0 viscosity solution to (1-1), where ψ is in $C^0(\overline{\Omega})$ and lies in the supercritical range. The existence part is based on smooth solution approximations, with smooth approximations of the phase and the boundary data in the C^0 continuous norm: the C^0 limit of smooth approximating solutions is a viscosity solution. The uniqueness part follows from [Trudinger 1990, p. 155]: Trudinger's condition is satisfied since the minimum eigenvalue is bounded for a uniform, supercritical phase. Note that this existence proof is different from the one shown in [Dinew et al. 2019, Theorem 40].

5. Proof of Theorem 1.2

Proof. We denote upper/lower semicontinuous functions by usc/lsc. We define

$$\begin{aligned} A &= \{u \in \mathrm{usc}(\overline{\Omega}) \mid F(D^2u) \geq \psi \text{ in } \Omega,\ u \leq \phi \text{ on } \partial\Omega\}, \\ w(x) &= \sup\{u(x) \mid u \in A\}. \end{aligned}$$

Claim 6. *The above function w is the unique continuous viscosity solution of (1-1), where ψ is a constant.*

Remark 5.1. The proof follows from the following four steps. It is noteworthy that the first three steps of the proof hold for any continuous function ψ. The fourth step requires a certain comparison principle (see Theorem A.2 of the Appendix), which is only available for a constant right-hand side. As of now, it is unknown if such a comparison principle holds for a continuous right-hand side. In order to highlight this distinction, we present the first three steps of the proof assuming ψ is any continuous function. In the final step, we assume ψ to be a constant, thereby proving Theorem 1.2.

Step 1: We define the functions

$$\underline{z}(x) = \overline{\lim_{y\to x}}\, w(y),$$

$$\bar{z}(x) = \underline{\lim}_{y\to x}\, w(y).$$

We first show that A is nonempty and w, $\underline{z}$, $\bar{z}$ are well defined. Since $\psi \in C(\overline{\Omega})$, there exists $\varepsilon' > 0$ such that $-n\frac{\pi}{2} + \varepsilon' < \psi(x) < n\frac{\pi}{2} - \varepsilon'$ for all $x \in \overline{\Omega}$. Fixing this ε' we define the functions

$$\psi_* = -n\tfrac{\pi}{2} + \varepsilon' < \psi < n\tfrac{\pi}{2} - \varepsilon' = \psi^*.$$

Recalling (3-6) and (3-7), we define

$$\begin{aligned}\underline{w}(x) &= -Cx_n + \frac{1}{2}|x|^2 \tan\frac{\psi^*}{n},\\ \overline{w}(x) &= Cx_n + \frac{1}{2}|x|^2 \tan\frac{\psi_*}{n},\end{aligned} \tag{5-1}$$

where $C = C(\|\phi\|_{C^2(\partial\Omega)}, n, |\partial\Omega|_{C^2})$. By definition $\underline{w} \in A$, which shows that A is nonempty. Next, $\max\{u, \underline{w}\}$ is upper semicontinuous and still a subsolution of (1-1), so we replace $u \in A$ by $\max\{u, \underline{w}\}$. This shows $u \geq \underline{w}$ and, therefore, w is well defined. Next, we observe that since $\underline{w}$ and $\overline{w}$ are sub- and supersolutions of (1-1), respectively, we have

$$\underline{w} \leq u \leq \overline{w},$$

which shows $\underline{z}$ and $\bar{z}$ are well defined.

Step 2: We show that $\underline{z}$ is a subsolution of (1-1). Suppose not. Then we can find a quadratic polynomial P such that $P(x) \geq \underline{z}(x)$ in $B_\rho(0)$, with equality holding at 0, such that $F(D^2P) < \psi_*$ in $B_\rho(0)$. Now we choose $\varepsilon > 0$ such that

$$F(D^2P + 4\varepsilon I) < \psi_*. \tag{5-2}$$

From the definition of w and $\underline{z}$, we can find sequences $\{u_k\} \subset A$ and $\{x_k\} \subset \Omega$, with $x_k \to 0$, such that

$$\underline{z}(0) = \overline{\lim_{y\to 0}}\, w(y) = \lim_{x_k\to 0} u_k(x_k).$$

For k large enough, we see that

$$\left|u_k(x_k) - P(x_k) - 2\varepsilon|x_k|^2\right| = \left|u_k(x_k) - P(0) + P(0) - P(x_k) - 2\varepsilon|x_k|^2\right| = o(1) < \varepsilon\rho^2.$$

On $\partial B_\rho(0)$, we see

$$u_k(x) \leq w(x) \leq \underline{z}(x) \leq P(x) + 2\varepsilon|x|^2 - \varepsilon\rho^2.$$

Using the definition of w and $\underline{z}$, we see that, for any k, the following holds in $B_\rho(0)$:

$$Q(x) = P(x) + 2\varepsilon|x|^2 \geq u_k(x).$$

Fixing a k large enough, we observe the following. The functions $u_k(x_k)$ and $Q(x_k)$ are less than $\varepsilon\rho^2$ apart, but u_k is at a distance of more than $\varepsilon\rho^2$ below Q on $\partial B_\rho(0)$. So we drop Q at most $\varepsilon\rho^2$ so that it touches u_k at a point inside $B_\rho(0)$ while still remaining above u_k on $\partial B_\rho(0)$. So there exists $\gamma \leq \varepsilon\rho^2$ such that, in $B_\rho(0)$,

$$u_k(x) \leq P(x) + 2\varepsilon|x|^2 - \gamma,$$

with equality holding at an interior point of B_ρ. Now since u_k is a subsolution, we have

$$\psi \leq F(D^2P + 4\varepsilon I).$$

This contradicts (5-2). Noting that $\underline{z}$ is upper semicontinuous, we see that it is a subsolution of (1-1).

Step 3: We show that $\bar{z}$ is a supersolution of (1-1). Suppose not. Then we can find a quadratic polynomial P such that $P(x) \le \bar{z}(x)$ in $B_\rho(0)$, with equality holding at 0, such that $F(D^2P) > \psi^*$ in $B_\rho(0)$. We choose $\varepsilon > 0$ small enough that

$$F(D^2P - 2\varepsilon I) > \psi^*. \tag{5-3}$$

We have $\bar{z} \ge P - \varepsilon|x|^2$. We define a new quadratic $Q(x) = P(x) - \varepsilon|x|^2 + \varepsilon\rho^2$. Observe that, since $\bar{z}(0) = \underline{\lim}_{x_k \to 0} w(x_k)$, for k large enough, we have

$$\begin{aligned} w(x_k) = \bar{z}(0) + o(1) &= P(0) - P(x_k) + P(x_k) + o(1) \\ &= P(x_k) + o(1) = Q(x_k) - \varepsilon\rho^2 + o(1) < Q(x_k). \end{aligned}$$

This contradicts the supremum definition of w since Q is a subsolution of (1-1) by (5-3). Noting that $\bar{z}$ is lower semicontinuous, we see that it is a supersolution of (1-1).

Step 4: We take care of the boundary value in this final step. This is where we assume (for the first time) that ψ is a constant. Note that now we may assume the boundary value ϕ is in $C^2(\partial\Omega)$ since we can always approximate ϕ by a sequence of smooth functions ϕ_δ that solve

$$\begin{cases} F(D^2u_\delta) = \psi & \text{in } \Omega, \\ u_\delta = \phi_\delta & \text{on } \partial\Omega \end{cases}$$

and apply the comparison principle[3] to get

$$\max_{\Omega} |u_{\delta_1} - u_{\delta_2}| \le \max_{x \to \partial\Omega} |(\phi_{\delta_1} - \phi_{\delta_2})(x)| \to 0$$

as $\delta_1, \delta_2 \to 0$. We have $u_\delta \to u$ in C^0 as $\delta \to 0$. Next, we pick an arbitrary point $x_0 \in \partial\Omega$ and recall the construction of $\underline{w}$ and $\overline{w}$ from (5-1). Defining similar functions at x_0 and on using the comparison principle, we get $\underline{w} \le u \le \overline{w}$, with equality holding at x_0 for all $u \in A$. Again, since $\max(u, \underline{w}) \in A$ for all $u \in A$, we can replace

$$w(x) = \sup_{u \in A} \max(u, \underline{w}).$$

We get $\underline{w} \le u \le \overline{w}$, with equality holding at x_0, which shows

$$\bar{z}(x_0) = \phi(x_0) = \underline{z}(x_0).$$

Since $x_0 \in \partial\Omega$ is arbitrary, we have $\bar{z} = \underline{z} = \phi$ on $\partial\Omega$. Combining the above steps and on using the comparison principle, we see

$$\bar{z} = \underline{z} = w \in C^0(\overline{\Omega})$$

is the desired solution. This proves the existence part of Claim 6. Uniqueness again follows from the comparison principle. □

[3] See the Appendix.

Appendix

We state the following linear algebra lemma that was used in proving the double normal estimate in Step 3.4 of Section 3.

Lemma A.1 [Caffarelli et al. 1985, Lemma 1.2]. *Consider the $n \times n$ symmetric matrix*

$$M = \begin{bmatrix} \lambda'_1 & & & a_1 \\ & \ddots & & \vdots \\ & & \lambda'_{n-1} & a_{n-1} \\ a_1 & \cdots & a_{n-1} & a \end{bmatrix},$$

where $\lambda'_1, \lambda'_2, \dots, \lambda'_{n-1}$ are fixed, $|a_i| < C$ for $1 \le i < n$, and $|a| \to +\infty$. Then the eigenvalues $\lambda_1, \lambda_2, \dots, \lambda_n$ of M behave like

$$\lambda'_1 + o(1), \quad \lambda'_2 + o(1), \quad \dots, \quad \lambda'_n + o(1), \quad a + O(1),$$

where $o(1)$ and $O(1)$ are uniform as $a \to \infty$.

For the sake of completeness we state and prove the following comparison principle for strictly elliptic equations, which is well known to experts.[4]

Theorem A.2. *Suppose that u is a* usc *subsolution and v is an* lsc *supersolution of the strictly elliptic equation (1-2) in $\Omega \subset \mathbb{R}^n$. If $u \le v$ on $\partial\Omega$, then $u \le v$ in Ω.*

Proof. Without loss of generality, we assume $\Omega = B_1(0)$ and $u \le v - 2\delta$ on ∂B_1 for some small $\delta > 0$. We rewrite (1-2) as

$$F(D^2u) = \sum_{i=1}^{n} \arctan \lambda_i - c = 0.$$

Let u^ε be an upper parabolic envelope[5] satisfying

$$F(D^2u^\varepsilon) \ge 0, \quad D^2u^\varepsilon \ge -C/\varepsilon, \quad \|u^\varepsilon\|_{C^{0,1}} \le C/\varepsilon$$

outside a measure-zero subset, where u^ε is punctually second-order differentiable and C is chosen such that

$$u^\varepsilon - v_\varepsilon \le C - \varepsilon|x - x_0|^2 \quad \text{on } \partial B_1,$$

with equality holding at $x_0 \in B_1$. We see that

$$0 \le u^\varepsilon(x) - u(x) \le u(x^*) - u(x) + \varepsilon,$$

[4] We learned this proof from [Yuan 2004]. Indeed, the arguments presented in [Caffarelli and Cabré 1995, p. 43–46] toward the comparison principle for fully nonlinear, uniformly elliptic equations work for strictly elliptic equations as well.

[5] For $\varepsilon > 0$, we define the upper ε-envelope of u to be

$$u^\varepsilon(x_0) = \sup_{x \in \overline{H}}\{u(x) + \varepsilon - |x - x_0|^2/\varepsilon\} \quad \text{for } x_0 \in H,$$

where H is an open set such that $\overline{H} \subset B_1$.

where $x^* \to x$ as $\varepsilon \to 0$. By symmetry, the lower parabolic envelope v_ε satisfies

$$F(D^2 v_\varepsilon) \le 0, \quad D^2 v_\varepsilon \le C/\varepsilon, \quad \|v_\varepsilon\|_{C^{0,1}} \le C/\varepsilon$$

and

$$0 \ge v_\varepsilon(x) - v(x) \ge v(x_*) - v(x) - \varepsilon,$$

where $x_* \to x$ as $\varepsilon \to 0$. Note that $v_\varepsilon - u^\varepsilon \le L + (C/\varepsilon)|x - x_0|^2$ for $x_0 \in B_1$, where L is a linear function. The convex envelope $\Gamma(v_\varepsilon - u^\varepsilon)$ is in $C^{1,1}$. From the Alexandroff estimate, we have

$$\sup_{B_1}(v_\varepsilon - u^\varepsilon)^- \le C(n)\left[\int_\Sigma \det D^2\Gamma\right]^{1/n},$$

where

$$\Sigma = \{x \in B_1 \mid \Gamma(x) = v_\varepsilon(x) - u^\varepsilon(x)\}.$$

Now in Σ, we have

$$0 \le D^2\Gamma \le D^2(v_\varepsilon - u^\varepsilon) \quad \text{or} \quad L(x) \le v_\varepsilon(x) - u^\varepsilon(x)$$

near $x_0 \in \Sigma$. For K large, since $u^\varepsilon + (K/\varepsilon)|x|^2$ is convex and $v_\varepsilon - (K/\varepsilon)|x|^2$ is concave, we have the following for a.e. $x_0 \in B_1$:

$$v_\varepsilon = \Gamma + \frac{K}{\varepsilon}|x|^2 + O(|x - x_0|^2),$$
$$u^\varepsilon = \Gamma + \frac{K}{\varepsilon}|x|^2 + O(|x - x_0|^2).$$

Again, since v_ε is a supersolution and u^ε is a subsolution, for a.e. $x_0 \in B_1$, we have

$$F(D^2 v_\varepsilon(x_0)) \le 0, \quad F(D^2 u^\varepsilon(x_0)) \ge 0, \quad F(D^2 v_\varepsilon(x_0)) - F(D^2 u^\varepsilon(x_0)) \le 0.$$

Also, a.e. $x_0 \in \Gamma$, we have $D^2 v_\varepsilon(x_0) - D^2 u^\varepsilon(x_0) \ge 0$. However, F is strictly elliptic, so we must have $F(D^2 v_\varepsilon) - F(D^2 u^\varepsilon) \ge 0$, which shows

$$F(D^2 v_\varepsilon(x_0)) = F(D^2 u^\varepsilon(x_0)) \quad \text{a.e } x_0 \in \Sigma.$$

Again, given that F is strictly elliptic, the line with the positive direction $D^2 v_\varepsilon(x_0) - D^2 u^\varepsilon(x_0)$ intersects the level set $\{F = C\}$ only once, which implies $D^2 v_\varepsilon(x_0) = D^2 u^\varepsilon(x_0)$. This shows $\sup_{B_1}(v_\varepsilon - u^\varepsilon)^- \le 0$, which proves that

$$v \ge v_\varepsilon \ge u^\varepsilon \ge u \quad \text{in } B_1. \qquad \square$$

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