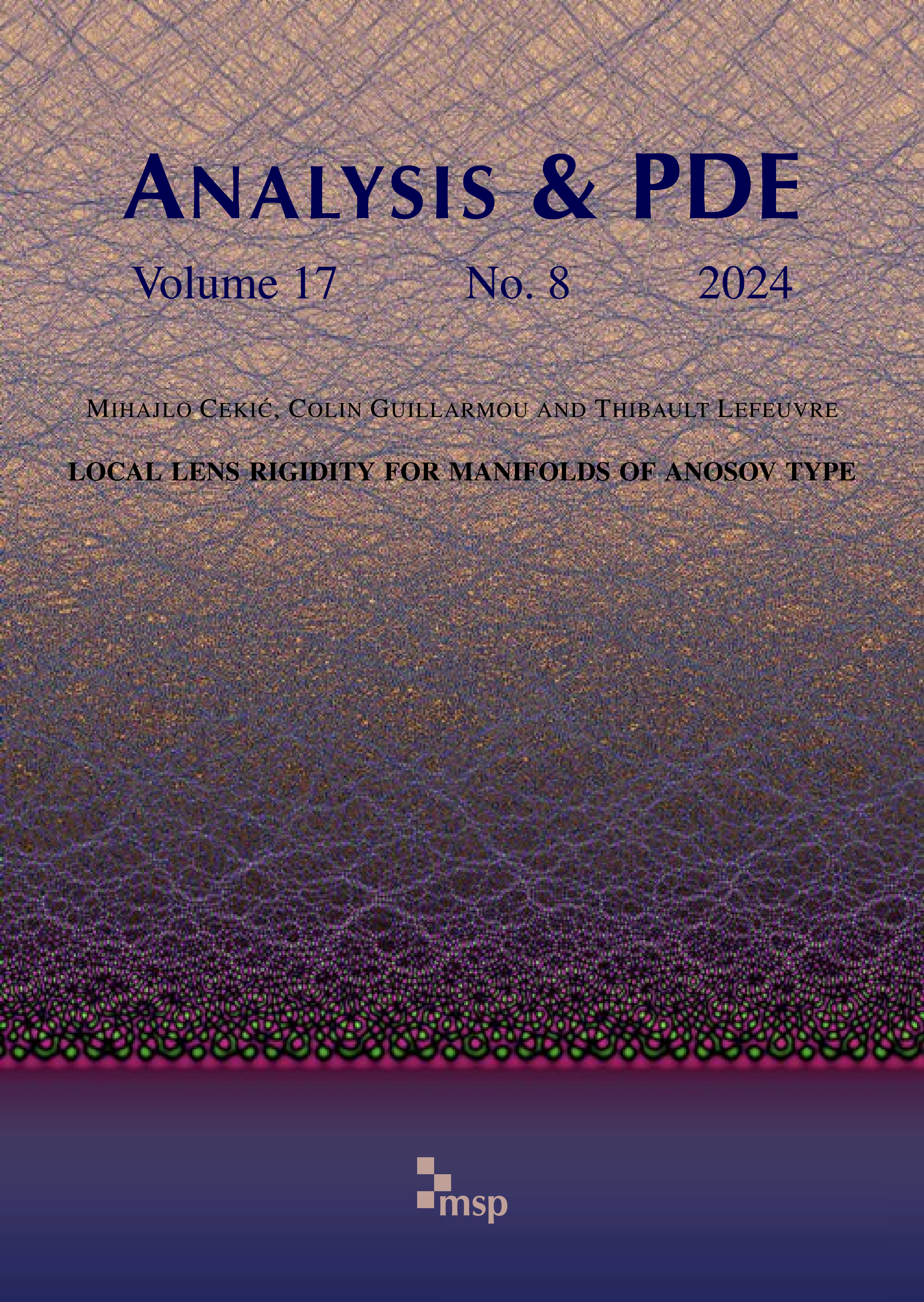

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LOCAL LENS RIGIDITY FOR MANIFOLDS OF ANOSOV TYPE

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The *lens data* of a Riemannian manifold with boundary is the collection of lengths of geodesics with endpoints on the boundary, together with their incoming and outgoing vectors. We show that negatively curved Riemannian manifolds with strictly convex boundary are *locally lens rigid* in the following sense: if g_0 is such a metric, then any metric g sufficiently close to g_0 and with the same lens data is isometric to g_0, up to a boundary-preserving diffeomorphism. More generally, we consider the same problem for a wider class of metrics with strictly convex boundary, called metrics of *Anosov type*. We prove that the same rigidity result holds within that class in dimension 2 and in any dimension, further assuming that the curvature is nonpositive.

1. Introduction

1A. ***The lens rigidity problem.*** Let (M, g) be a smooth compact connected Riemannian manifold with strictly convex boundary (i.e., the second fundamental form is positive on ∂M). Let $\mathcal{M} := SM$ be the unit tangent bundle of (M, g), and define the incoming $(-)$ and outgoing $(+)$ boundary of $\mathcal{M}$ as

$$\partial_\pm \mathcal{M} := \{(x, v) \in \mathcal{M} \mid x \in \partial M,\ \pm g_x(v, \nu(x)) > 0\},$$

where ν is the unit outward-pointing normal vector to the boundary. For any $(x, v) \in \partial_- \mathcal{M}$, the maximally extended geodesic $\gamma_{(x,v)}$, with initial condition $\gamma_{(x,v)}(0) = x$, $\dot{\gamma}_{(x,v)} = v$, is defined on a time interval $[0, \ell_g(x, v)]$, where $\ell_g(x, v) \in \mathbb{R}_+ \cup \{\infty\}$. When $\ell_g(x, v) < \infty$, we define

$$S_g(x, v) := (\gamma_{(x,v)}(\ell_g(x, v)), \dot{\gamma}_{(x,v)}(\ell_g(x, v)))$$

to be the outgoing tangent vector at $\partial_+ \mathcal{M}$; see Figure 1.

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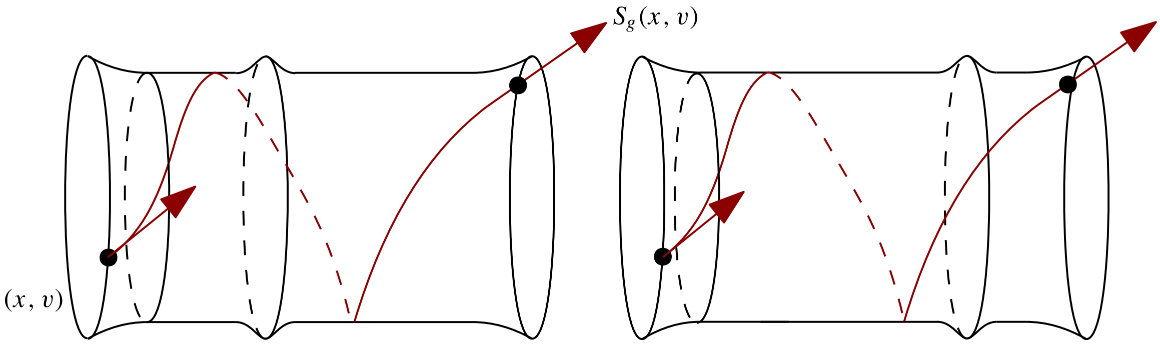

Figure 1. A surface with strictly convex boundary which is not lens rigid. Example taken from [Croke and Herreros 2016].

Definition 1.1 (lens data). The map $S_g : \partial_-\mathcal{M} \setminus \{\ell_g = \infty\} \to \partial_+\mathcal{M}$ is called the *scattering map* and the function $\ell_g : \partial_-\mathcal{M} \setminus \{\ell_g = \infty\} \to \mathbb{R}_+$ the *length map*. The pair (ℓ_g, S_g) is the *lens data* of the Riemannian manifold (M, g).

The lens data encodes the boundary data one can measure on the geodesic flow from "outside of the manifold". A natural inverse problem that arises from tomography consists in determining the geometry, namely, the Riemannian metric g inside M, from the measurement of the lens data (ℓ_g, S_g). In geophysics, this is related to recovering the speed of propagation of waves inside a domain such as the Earth, for instance; see [Paternain et al. 2014]. When two metrics g and g' agree on ∂M, it makes sense to say that they have the same lens data as there is a natural identification between the boundary of their respective unit tangent bundles via the unit disk bundle of the boundary; see Section 2A1 for further details. The *lens rigidity problem* is concerned with the following question:

Question 1.2. *Assume that (M, g) and (M', g') are two Riemannian metrics with strictly convex boundary such that there exists an isometry $I \in \mathrm{Diff}(\partial M, \partial M')$ with $I^*(g'|_{T\partial M'}) = g|_{T\partial M}$. Does the implication*

$$(\ell_g, S_g) = I^*(\ell_{g'}, S_{g'}) \quad \Longrightarrow \quad \textit{there exists } \psi \in \mathrm{Diffeo}(M, M') \textit{ such that } \psi|_{\partial M} = I \textit{ and } \psi^* g' = g$$

hold true?

We say that a manifold (M, g) is *lens rigid* if there is no other Riemannian manifold (up to isometry) having the same lens data as (ℓ_g, S_g). In the following, in order to simplify the notation, we will assume that $M = M'$ and $I = \mathrm{id}$.

There are simple counterexamples of manifolds for which lens rigidity does not hold: considering certain perturbations of the flat cylinder $\mathbb{S}^1 \times [0, 1]$ (see Figure 1 and [Croke and Herreros 2016], where this is further discussed), one can easily obtain nonisometric metrics with the same lens data. Such cases have *trapped geodesics*, that is some maximally extended geodesics with infinite length, or equivalently $\ell_g(x, v) = \infty$ for some $(x, v) \in \partial_-\mathcal{M}$. It turns out that all existing counterexamples to lens rigidity have trapped geodesics.

1B. ***Lens rigidity for nontrapping manifolds.*** Even among manifolds without a trapped set, the lens rigidity problem is still widely open. The closest result in this direction is the recent breakthrough of Stefanov, Uhlmann and Vasy [Stefanov et al. 2021], showing lens rigidity in dimensions $n \geq 3$ under the additional assumption that the manifold (M, g) is foliated by strictly convex hypersurfaces. This includes all simply connected nonpositively curved manifolds with strictly convex boundary. In the class of real analytic metrics such that from each $x \in \partial M$ there is a maximal geodesic free of conjugate points, the lens rigidity was proved by Vargo [2009]. A local lens rigidity result was also proved near analytic metrics by Stefanov and Uhlmann [2009] under certain assumptions on the conjugate points.

There is also a subclass of metrics that have attracted a lot of attention since the work of Michel [1981], namely the class of *simple manifolds*, which are manifolds with strictly convex boundary that have no trapped geodesics and no conjugate points. These manifolds are diffeomorphic to the unit ball in $\mathbb{R}^n$. In this case, knowing the lens data is equivalent to knowing the restriction $d_g|_{\partial M \times \partial M}$ of the Riemannian distance function $d_g \in C^0(M \times M)$ to the boundary, also called the *boundary distance*. The lens rigidity problem for this subclass of metrics is also called the *boundary rigidity problem*. In dimension $n = 2$, it was proved by Otal [1990b] (in negative curvature), Croke [1991] (in nonpositive curvature), and Pestov and Uhlmann [2005] (in general) that simple surfaces are boundary rigid and thus lens rigid. We also mention the results by Croke, Dairbekov and Sharafutdinov [Croke et al. 2000] and Stefanov and Uhlmann [2004] for local boundary rigidity results, the work by Gromov [1983] and Burago and Ivanov [2010] for rigidity results of flat and close to flat simple manifolds, and we finally refer more generally to the review article by Croke [2004] and the recent book of Paternain, Salo and Uhlmann [Paternain et al. 2023] for an overview of the boundary rigidity problem.

1C. ***Lens rigidity for manifolds with nonempty trapped set.*** Trapped geodesics appear in most situations since all Riemannian manifolds (M, g) with strictly convex boundary and nontrivial topology, i.e., nontrivial fundamental group, always have trapped geodesics (and they even have closed geodesics in the interior M°). As far as manifolds with trapped geodesics are concerned, very little is known on the lens rigidity problem. It is not even clear what would be the most general class of manifolds for which lens rigidity could hold, and the example above in Figure 1 shows that it seems hopeless to consider general manifolds with both trapped geodesics and conjugate points.

The only available result considering cases with both trapped geodesics and conjugate points seems to be the local rigidity result of [Stefanov and Uhlmann 2009]. In dimensions $n \geq 3$, under a certain topological assumption, it is proved that if (M, g_0) is real analytic,[1] with strictly convex boundary, and for each $(x, v) \in SM$ there is $w \in v^\perp$ such that the maximally extended geodesic tangent to w at x has finite length (it is not trapped) and is free of conjugate points, then the following holds: if g is another metric with $\|g - g_0\|_{C^N}$ small enough for some $N \gg 1$ and $(\ell_g, S_g) = (\ell_{g_0}, S_{g_0})$, then g and g_0 are isometric via a boundary-preserving diffeomorphism. On the other hand, it is not clear (geometrically speaking) what type of manifolds are contained in this class and there are many interesting geometric cases not contained in it. For example, there exist convex cocompact hyperbolic 3-manifolds $M := \Gamma \backslash \mathbb{H}^3$ (with constant

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[1] Or more generally if a certain localized X-ray transform is injective.

sectional curvature -1) whose convex core $\mathcal{C}$ has positive measure and totally geodesic boundary. Thus, cutting the ends of such examples at a finite positive distance of $\mathcal{C}$, one obtains a metric not satisfying the assumptions of [Stefanov and Uhlmann 2009] due to the totally geodesic surfaces bounding $\mathcal{C}$.

From our point of view, there is a very natural class of metrics with nontrivial trapped set where the lens rigidity problem seems well-posed and interesting from a geometrical point of view. We call elements of this class manifolds of *Anosov type*; it contains as a strict subclass the set of negatively curved metrics with strictly convex boundary.

Definition 1.3. A compact Riemannian manifold (M, g) with boundary is of *Anosov type* if:

(1) It has strictly convex boundary.

(2) It has no conjugate points.

(3) The trapped set for the geodesic flow $(\varphi_t^g)_{t\in\mathbb{R}}$ on $\mathcal{M} := SM$, defined by
$$K^g := \bigcap_{t\in\mathbb{R}} \varphi_t^g(\mathcal{M}^\circ) \subset \mathcal{M}^\circ,$$
is *hyperbolic* in the following sense. There exist a continuous flow-invariant splitting
$$\text{for all } y \in K^g, \quad T_y\mathcal{M} = \mathbb{R}X_g(y) \oplus E_-(y) \oplus E_+(y),$$
where X_g is the geodesic vector field, and constants $\nu, C > 0$ such that,
$$\text{for all } \pm t \geq 0, \ \text{ for all } y \in K^g, \ \text{ for all } v \in E_\mp(y), \quad \|d\varphi_t^g(y)v\| \leq Ce^{-\nu|t|}\|v\| \tag{1-1}$$
for an arbitrary choice of metric $\|\cdot\|$ on $\mathcal{M}$.

Example 1.4. The main two examples of manifolds of Anosov type are

(1) Riemannian manifolds with negative sectional curvature and strictly convex boundary (see [Klingenberg 1995, Theorem 3.2.17 and Section 3.9]),

(2) strictly convex subdomains of closed Riemannian manifolds with Anosov geodesic flows.

Manifolds of Anosov type have a trapped set with fractal structure and zero Lebesgue measure. It implies that almost-every point in $\mathcal{M}$ is reachable from geodesics with endpoints on $\partial\mathcal{M}$. This case can be interpreted as an intermediate rigidity problem between the *length spectrum rigidity* of manifolds with Anosov geodesic flows, where one asks if the lengths of closed geodesics determine the metric up to isometry, and the boundary rigidity problem of simple manifolds.

In the closed case, Vignéras [1980] exhibited counterexamples to the length spectrum rigidity: in constant negative curvature, there are nonisometric metrics on surfaces with the same length spectrum. The well-posed rigidity problem is rather that of the *marked length spectrum* problem, also known as the Burns–Katok conjecture [Burns and Katok 1985]: on a manifold (M, g) with Anosov geodesic flow, each free homotopy class of loops c on M contains a unique geodesic representative $\gamma_c(g)$ whose length is denoted by $L_g(c)$; if g_1 and g_2 are two such Anosov metrics on M with $L_{g_1}(c) = L_{g_2}(c)$ for all c, it is then conjectured that g_1 should be isometric to g_2. This conjecture was proved in dimension 2 by

Otal [1990a] and Croke [1990], and in all dimensions for pairs of metrics that are close enough in C^k norm for $k \gg 1$ large enough by the last two authors [Guillarmou and Lefeuvre 2019] (local rigidity). However, it is still open in general.

Similarly, for manifolds with boundary and nontrivial topology, the same problem of "marking" of geodesics is a serious difficulty. The first natural question one may consider is the following, known as the *marked lens rigidity* or *marked boundary rigidity* problem for Riemannian manifolds of Anosov type.

Definition 1.5 (marked lens data). Let g_1, g_2 be two metrics of Anosov type on M. We say that g_1 and g_2 have the same *marked lens data* if, for each $(x, v) \in \partial_-\mathcal{M} \setminus \{\ell_g = \infty\}$, one has $(\ell_{g_1}(x, v), S_{g_1}(x, v)) = (\ell_{g_2}(x, v), S_{g_2}(x, v))$ and the g_1- and g_2-geodesics with initial conditions (x, v) are homotopic via a homotopy fixing the endpoints.

Technically, having the same marked lens data is the same as having same boundary distance function on the universal cover $\widetilde{M}$ (which is now a noncompact space). The following conjecture is somehow similar to the Burns–Katok conjecture in the closed case and to the boundary rigidity problem of negatively curved simple metrics.

Conjecture 1.6 (marked lens rigidity of manifolds of Anosov type). *Let M be a smooth manifold with boundary, and assume that g_1, g_2 are two smooth metrics of Anosov type on M in the sense of Definition 1.3 such that $g_1|_{T(\partial M)} = g_2|_{T(\partial M)}$. If g_1 and g_2 have the same marked lens data, then there exists a smooth diffeomorphism ψ, homotopic to the identity and equal to the identity on the boundary ∂M, such that $\psi^* g_2 = g_1$.*

In dimension 2, Conjecture 1.6 was recently solved by the third author with Erchenko in [Erchenko and Lefeuvre 2024] (an earlier result had also been obtained by the second author together with Mazzuchelli in [Guillarmou and Mazzucchelli 2018] for negatively curved surfaces using the method of Otal [1990a]). In higher dimensions, the third author [Lefeuvre 2020] proved Conjecture 1.6 for pairs of negatively curved metrics g_1, g_2 that are close enough in C^k norm for $k \gg 1$ large enough (local marked lens rigidity). The fact that there is no smooth 1-parameter family $(g_s)_{s\in(-1,1)}$ of nonisometric negatively curved metrics with the same marked lens data[2] is called *infinitesimal rigidity* and was first proved by the second author [Guillarmou 2017b].

In this paper, we consider the more difficult problem of lens rigidity in the class of manifolds of Anosov type. Since, contrary to the closed case, there are still no counterexamples to lens rigidity, we make the following conjecture of lens rigidity in the class of metrics of Anosov type.

Conjecture 1.7 (lens rigidity of manifolds of Anosov type). *Let (M_1, g_1), (M_2, g_2) be two smooth Riemannian manifolds of Anosov type such that $(\partial M_1, g_1|_{\partial M_1}) = (\partial M_2, g_2|_{\partial M_1})$. If $(\ell_{g_1}, S_{g_1}) = (\ell_{g_2}, S_{g_2})$, then there exists a smooth diffeomorphism ψ, equal to the identity on the boundary, such that $\psi^* g_2 = g_1$.*

There are already partial answers to Conjecture 1.7:

(1) In dimension 2, Croke and Herreros [2016] proved that negatively curved cylinders with strictly convex boundary are lens rigid.

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[2] In this case, having the same marked lens data is equivalent to having the same lens data.

(2) In dimension 2, the second author shows in [Guillarmou 2017b] that the scattering map S_g determines (M, g) up to conformal diffeomorphism fixing the boundary. Recovering the conformal factor of the metric is still an open question.

(3) In dimensions $n \geq 3$, Stefanov, Uhlmann and Vasy [Stefanov et al. 2021] prove that, for general metrics with strictly convex boundary, the lens data determines the metric in a neighborhood of ∂M; applying this result in the setting of negatively curved manifolds, one can recover the metric outside the convex core of the manifold (which contains the projection of the trapped set).

(4) In [Guedes-Bonthonneau et al. 2024], Guedes-Bonthonneau, Jézéquel, and the second author proved Conjecture 1.7 under the extra assumption that (M_1, g_1), (M_2, g_2) are real analytic, but only using the equality $S_{g_1} = S_{g_2}$ of the scattering maps.

Our first result in this article is the following local rigidity result answering Conjecture 1.7 for metrics close to each other.

Theorem 1.8. *Let* (M, g_0) *be a Riemannian manifold of Anosov type. Assume that either* $\dim M = 2$ *or that the curvature of* g_0 *is nonpositive. Then there exist* $N \gg 1$, $\delta > 0$ *such that the following holds*: *for any smooth metric* g *on* M *such that* $\|g - g_0\|_{C^N} < \delta$, *if* $(\ell_g, S_g) = (\ell_{g_0}, S_{g_0})$, *then there exists a smooth diffeomorphism* $\psi : M \to M$ *such that* $\psi|_{\partial M} = \mathrm{id}$ *and* $\psi^* g = g_0$.

More generally, Theorem 1.8 holds under the general assumption that g_0 is of Anosov type and its *X-ray transform operator* $I_2^{g_0}$ on divergence-free symmetric 2-tensors is injective; see (1-2) for a definition of $I_2^{g_0}$ and Section 3A2 where this is further discussed. The fact that $I_2^{g_0}$ is injective on divergence-free tensors was proved in [Guillarmou 2017b] in nonpositive curvature and in general on Anosov surfaces by [Lefeuvre 2019a] (without any assumption on the curvature). It was also proved in [Guedes-Bonthonneau et al. 2024] that $I_2^{g_0}$ is injective for real-analytic metrics g_0 which implies that generic smooth metrics of Anosov type have an injective X-ray transform operator $I_2^{g_0}$; generic injectivity of $I_2^{g_0}$ follows from the work of the first and third authors [Cekić and Lefeuvre 2021] as well, admitting also Theorem 1.10 below. As a corollary of Theorem 1.8, we obtain:

Corollary 1.9. *Let* (M, g_0) *be a negatively curved Riemannian manifold with strictly convex boundary. Then*, *there exist* $N \gg 1$, $\delta > 0$ *such that the following holds*: *for any smooth metric* g *on* M *such that* $\|g - g_0\|_{C^N} < \delta$, *if* $(\ell_g, S_g) = (\ell_{g_0}, S_{g_0})$, *then there exists a smooth diffeomorphism* $\psi : M \to M$ *such that* $\psi|_{\partial M} = \mathrm{id}$ *and* $\psi^* g = g_0$.

We observe that Corollary 1.9 and Theorem 1.8 are not a consequence of [Stefanov and Uhlmann 2009] (nor of [Stefanov et al. 2021]) mentioned above since: (1) our result contains the case of surfaces (dimension $n = 2$) and (2) the assumption on the trapped set in [Stefanov and Uhlmann 2009] does not cover all hyperbolic trapped sets (typically, the example $M = \Gamma \backslash \mathbb{H}^3$ mentioned above is not covered when the boundary of the convex core $\mathcal{C}$ is totally geodesic), whereas we do not make any specific assumption on the topology, and neither do we assume that g_0 is analytic or that it has an injective localized X-ray transform. Theorem 1.8 is also clearly stronger than the marked local rigidity result of the third author [Lefeuvre 2020], since we are now able to remove the *marking* assumption on the lens data.

Let us finally mention that there are interesting and related results for Euclidean billiards: Noakes and Stoyanov [2015] show that the lens data for the billiard flow on $\mathbb{R}^n \setminus \mathcal{O}$ (where $\mathcal{O}$ is a collection of strictly convex domains) is rigid, and De Simoi, Kaloshin and Leguil [De Simoi et al. 2023] prove that the lengths of the marked periodic orbits generically determine the obstacles under a $\mathbb{Z}^2 \times \mathbb{Z}^2$ symmetry assumption.

1D. ***Removing the marking assumption, idea of proof.*** The removal of the marking assumption is not simply a technical artifact: it is rather a crucial aspect in our work. Indeed, without the marking assumption, one can no longer use the fact that the geodesic flows of g and g_0 are conjugate with a conjugacy preserving the Liouville measure. This conjugacy was a fundamental aspect of both proofs of [Guillarmou and Mazzucchelli 2018; Lefeuvre 2020]. In the proof of Theorem 1.8, one has to rely on a completely different argument, which is the linearization of the pair (ℓ_g, S_g). Nevertheless, since g has a big set of trapped geodesics (typically a fractal set), this creates many singularities for (ℓ_g, S_g) and its linearization. The analysis one has to perform is then quite involved. One needs to combine several different key tools, in particular,

(1) the proof of the C^2-regularity with respect to g of the operator $\mathcal{S}_g : C^\infty(\partial_+\mathcal{M}) \to \mathcal{D}'(\partial_-\mathcal{M})$ defined by $\mathcal{S}_g f := f \circ S_g$,

(2) the exponential decay in $t \to \infty$ of the volume of points $(x, v) \in \mathcal{M} = SM$ that remain trapped for time t.

The first item is obtained by reproving certain results of [Dyatlov and Guillarmou 2016] on the resolvent of an Axiom A vector field X, but now with an explicit control of the dependence with respect to the vector field X. In particular, as a byproduct of this analysis we show the following result that could prove useful for other applications such as Fried's conjecture for manifolds with boundary, in the spirit of [Dang et al. 2020].

Theorem 1.10. *Let $\mathcal{M}$ be a smooth manifold with boundary, and let X_0 be a smooth vector field so that $\partial\mathcal{M}$ is strictly convex for the flow of X_0. Assume that the trapped set*

$$K^{X_0} := \bigcap_{t\in\mathbb{R}} \varphi_t^{X_0}(\mathcal{M}^\circ)$$

of the flow $(\varphi_t^{X_0})_{t\in\mathbb{R}}$ of X_0 is hyperbolic. Then, there exist $\delta > 0$, $N \gg 1$, such that, for all $X \in C^\infty(\mathcal{M}, T\mathcal{M})$ with $\|X - X_0\|_{C^N} < \delta$, the following hold:

(1) *The resolvent $R^X(z) := (-X + z)^{-1} : L^2(\mathcal{M}) \to L^2(\mathcal{M})$, initially defined in the half-plane $\{z \in \mathbb{C} \mid \Re(z) \gg 1\}$, extends meromorphically to $\mathbb{C}$ as a bounded operator $R^X(z) : C_c^\infty(\mathcal{M}^\circ) \to \mathcal{D}'(\mathcal{M}^\circ)$.*

(2) *If $z_0 \in \mathbb{C}$ is not a pole of $R^{X_0}(z)$, then the map*

$$C^\infty(\mathcal{M}, T\mathcal{M}) \ni X \mapsto R^X(z_0) \in \mathcal{L}(C_c^\infty(\mathcal{M}^\circ), \mathcal{D}'(\mathcal{M}^\circ))$$

is C^2-regular[3] *with respect to X.*

Here, we denote by $\mathcal{L}(A, B)$ the space of continuous linear maps between functional spaces A and B. The space $\mathcal{L}(C_c^\infty(\mathcal{M}^\circ), \mathcal{D}'(\mathcal{M}^\circ))$ can be naturally identified with $\mathcal{D}'(\mathcal{M}^\circ \times \mathcal{M}^\circ)$ via the Schwartz kernel

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[3] Even though we only need C^2, our proof actually shows it is C^k for all $k \in \mathbb{N}$.

theorem; the space $\mathcal{D}'(\mathcal{M}^\circ \times \mathcal{M}^\circ)$ is equipped with the standard topology on distributions. In fact, we prove the result above in anisotropic Sobolev spaces, and refer to Theorem 5.14 for a more detailed statement. We show that the scattering operator $\mathcal{S}_g$ has a Schwartz kernel that can be written as a restriction of the Schwartz kernel of $R^{X_g}(0)$ on $\partial_-\mathcal{M} \times \partial_+\mathcal{M}$, implying that the map $g \mapsto \mathcal{S}_g$ is C^2-regular as operators acting on some appropriate Sobolev spaces.

The strategy of the proof then goes as follows. First of all, we put the metric g in solenoidal gauge (with respect to g_0), namely we find a first diffeomorphism $\psi \in \operatorname{Diff}(M)$ such that $\psi|_{\partial M} = \mathrm{id}$ and $g' = \psi^* g$ is divergence-free with respect to g_0, see Lemma 3.6. Secondly, letting

$$I_2^{g_0} : C^\infty(M, \otimes_S^2 T^*M) \to L^\infty_{\mathrm{loc}}(\partial_-\mathcal{M} \setminus \{\ell_{g_0} = \infty\})$$

be the X-ray transform on symmetric 2-tensors with respect to g_0, defined as

$$I_2^{g_0} h(x, v) := \int_0^{\ell_{g_0}(x,v)} h_{\gamma(t)}(\dot\gamma(t), \dot\gamma(t))\, \mathrm{d}t \quad \text{if } \varphi_t^{g_0}(x, v) = (\gamma(t), \dot\gamma(t)) \in \mathcal{M}, \tag{1-2}$$

we show in Section 4A the following key estimate: there are $C, \mu > 0$ such that, if $(\ell_{g_0}, \mathcal{S}_{g_0}) = (\ell_g, \mathcal{S}_g)$ and $\|g' - g_0\|_{C^N} < \delta$ for some small $\delta > 0$, then

$$\|I_2^{g_0}(g' - g_0)\|_{H^{-6}(\partial_-\mathcal{M})} \le C \|g' - g_0\|^{1+\mu}_{C^N(M, \otimes_S^2 T^*M)}. \tag{1-3}$$

The proof of this estimate is involved. It is based on some complex interpolation argument using the holomorphic map

$$\mathbb{C} \ni z \mapsto e^{-z\ell_{g_0}} I_2^{g_0}(g' - g_0)$$

and the C^2-smoothness of the scattering map $g \mapsto \mathcal{S}_g$ as a continuous map from $C^\infty(\partial_+\mathcal{M})$ to $H^{-6}(\partial_-\mathcal{M})$. This is established in Section 5. It also relies on some volume estimates on the set of geodesics trapped for time $t \to \infty$ that follow from [Guillarmou 2017b].

Finally, slightly extending (M, g_0) to some (M_e, g_{0e}), using the mapping properties of the adjoint $(I_2^{g_{0e}})^*$, interpolation arguments, and (1-3), one obtains, for $h := g' - g_0$,

$$\|h\|_{L^2} \le C \|\Pi_2^{g_{0e}} E_0 h\|_{H^1} \le C \|h\|^{1+\mu}_{C^N}, \tag{1-4}$$

where E_0 is the zero extension operator to M_e, $\Pi_2^{g_{0e}} = (I_2^{g_{0e}})^* I_2^{g_{0e}}$ is the normal operator, and the estimate on the left is an elliptic estimate proved in Proposition 3.8. It is left to interpolate C^N between L^2 and $C^{N'}$ in (1-4), where $N' \gg N$, to get, for some $0 < \mu' < \mu$,

$$\|h\|_{L^2} \le C \|h\|_{L^2} \|h\|^{\mu'}_{C^{N'}} \le C \|h\|_{L^2} \|g - g_0\|^{\mu'}_{C^{N'}}.$$

For $\|g - g_0\|_{C^{N'}}$ small enough, this readily implies that $g' = \phi^* g = g_0$, concluding the proof.

2. Geometric and dynamical preliminaries

Following [Guillarmou 2017b, Section 2], we describe the scattering and length maps in our geometric setting, and relate them to the resolvent of the geodesic flow.

2A. *Unit tangent bundle and extensions.*

2A1. *Geometry of the unit tangent bundle.* Let (M, g) be a smooth compact oriented Riemannian manifold with strictly convex boundary (in the sense that the second fundamental form is positive), and let $S^gM = \{(x, v) \in TM \mid |v|_{g_x} = 1\}$ be the unit tangent bundle with projection on the base denoted by $\pi_0 : S^gM \to M$. For a point $y = (x, v) \in S^gM$, we shall write $-y := (x, -v)$. Denote by $\varphi_t^g : S^gM \to S^gM$ the geodesic flow at time $t \in \mathbb{R}$, and by X_g its generating vector field. Let α be the canonical Liouville 1-form on S^gM, defined by $\alpha(x, v)(\xi) := g_x(d\pi_0(x, v)\xi, v)$ for any $\xi \in T_{(x,v)}S^gM$, and define $\mu := \alpha \wedge d\alpha^{n-1}$, the associated Liouville volume form, which we will freely identify with the Liouville measure. It satisfies $\mathcal{L}_{X_g}\mu = 0$, where $\mathcal{L}_{X_g}$ denotes the Lie derivative along X_g.

Recall that we introduced the incoming $(-)$ and outgoing $(+)$ boundaries as

$$\partial_\pm S^gM = \{(x, v) \in \partial S^gM \mid \pm g_x(v, \nu) > 0\},$$

where ν is the outward-pointing unit normal to ∂M. Using the orthogonal decomposition

$$T_{\partial M}M = T(\partial M) \oplus^\perp \mathbb{R}\nu, \tag{2-1}$$

the boundary $\partial_\pm S^gM$ can be naturally identified with the boundary ball

$$B(\partial M) := \{(x, v) \in TM \mid x \in \partial M,\ v \in T_x(\partial M),\ |v|_g \leq 1\}$$

by means of the orthogonal projection onto the first factor in (2-1). As a consequence, if g' is any other smooth metric on M such that $g|_{T\partial M} = g'|_{T\partial M}$, the boundaries $\partial_\pm S^gM$ and $\partial_\pm S^{g'}M$ can be naturally identified and it makes sense to say that $(\ell_g, S_g) = (\ell_{g'}, S_{g'})$. When this equality holds, we say that the manifolds (M, g) and (M', g') have the same *lens data*.

When we consider a set of metrics g, the unit tangent bundles S^gM depend on g. For convenience, we will thus fix the manifold

$$\mathcal{M} := S^{g_0}M,$$

associated to an arbitrary metric of reference g_0. We can always rescale the flow φ_t^g so that it becomes defined on $\mathcal{M}$. Indeed, define $\Phi_{g_0\to g} : S^{g_0}M \to S^gM$ by

$$\Phi_{g_0\to g}(x, v) := (x, v/|v|_g).$$

Then $\Phi_{g_0\to g}^{-1} \circ \varphi_t^g \circ \Phi_{g_0\to g}$ is a flow on $\mathcal{M}$ which we shall still denote by φ_t^g, and its vector field will also be denoted by X_g for simplicity.

We shall always work with metrics g such that $g|_{T\partial M} = g_0|_{T\partial M}$. The boundary of $\mathcal{M}$ splits into a disjoint union

$$\partial\mathcal{M} = \partial_-\mathcal{M} \cup \partial_+\mathcal{M} \cup \partial_0\mathcal{M}, \tag{2-2}$$

where $\partial_\pm\mathcal{M} := \{(x, v) \in \partial\mathcal{M} \mid \pm g_x(v, \nu) > 0\}$ and $\partial_0\mathcal{M} := \{(x, v) \in \partial\mathcal{M} \mid g_x(v, \nu) = 0\}$. Note that the normal ν depends on g, and that the splitting (2-2) does not depend on the choice of $g = g_0$ on $T\partial M$. This will be important to compare for $g \neq g'$ the length functions ℓ_g with $\ell_{g'}$ and the scattering maps S_g with $S_{g'}$ (see Definition 2.2 below).

There is a symplectic form on $\partial_\pm\mathcal{M}$ obtained by restricting $\iota_\partial^* d\alpha$ to $\partial_\pm\mathcal{M}$, where $\iota_\partial : \partial\mathcal{M} \to \mathcal{M}$ is the inclusion map. We denote by

$$\mu_\partial := |\iota_\partial^*(i_{X_g}\mu)| = |\iota_\partial^*(d\alpha)^{n-1}|$$

the induced measure on $\partial\mathcal{M}$, where i_{X_g} denotes the contraction with X_g. In what follows we will write $L^p(\partial_\pm\mathcal{M})$ for the usual L^p space with respect to any smooth Riemannian measure $\mathrm{d}v_h$ on $\partial\mathcal{M}$ (for some metric h on $\partial\mathcal{M}$), while we will write $L^p(\partial_\pm\mathcal{M}, \mu_\partial)$ when we use the measure μ_∂. We note that $\mu_\partial = \omega\,\mathrm{d}v_h$, where $\omega \in C^\infty(\partial\mathcal{M})$ is positive outside $\partial_0\mathcal{M}$ and vanishes to order 1 at $\partial_0\mathcal{M}$, thus $L^p(\partial_\pm\mathcal{M}) \hookrightarrow L^p(\partial_\pm\mathcal{M}, \mu_\partial)$ continuously.

2A2. *Extension of the manifold.* It will be convenient to consider an embedding of $\mathcal{M}$ into a smooth closed manifold $\mathcal{N}$. This can be done by considering an embedding $M \hookrightarrow N$, where N is a smooth closed manifold (this is always possible by doubling the manifold M across its boundary for instance, i.e., gluing $M \sqcup M$ along ∂M by means of the identity map), then extending smoothly the metric g_0 to N (denoted by g_{0N}) and taking $\mathcal{N} := S^{g_{0N}}N$. If g_0 is of Anosov type (see Definition 1.3), it will be also convenient to have a slightly larger manifold with boundary M_e at our disposal such that $M \hookrightarrow M_e \hookrightarrow N$ and the extension of the metric g_0 to M_e, which we denote by g_{0e}, is of Anosov type; see [Guillarmou 2017b, Section 2] where this is further discussed. Set $\mathcal{M}_e := S^{g_{0e}}M_e$. We have the successive embeddings $\mathcal{M} \hookrightarrow \mathcal{M}_e \hookrightarrow \mathcal{N}$. For a metric g close to g_0 in C^N norm and such that $g = g_0$ on $T\partial M$, we consider an extension g_e of Anosov type on M_e. The map $g \mapsto g_e$ can be chosen to be smooth and so that

$$\|g_e - g_{0e}\|_{C^N(M_e, \otimes_S^2 T^*M_e)} \le C_N \|g - g_0\|_{C^N(M, \otimes_S^2 T^*M)}$$

for all $N \ge 0$ and some constants $C_N > 0$, where $\otimes_S^2 T^*M$ is the bundle of symmetric 2-tensors.

Definition 2.1. Let $c \in \mathbb{R}$. We say that a level set $\{\rho = c\}$ of a function $\rho \in C^\infty(\mathcal{N})$ is *strictly convex* with respect to a vector field $Y \in C^\infty(\mathcal{N}, T\mathcal{N})$ if, for all $y \in \{\rho = c\}$, one has

$$Y\rho(y) = 0 \quad \Longrightarrow \quad Y^2\rho(y) < 0.$$

We say that a smooth submanifold $\mathcal{H} \subset \mathcal{N}$ is strictly convex with respect to Y if $\mathcal{H}$ is in a neighborhood of $\mathcal{H}$ given by a level set $\{\rho = 0\}$ of some function ρ, and this level set is strictly convex with respect to Y. This is independent of the choice of ρ.

It can be easily checked that (M, g_0) has strictly convex boundary in the Riemannian sense if and only if $\partial\mathcal{M}$ is strictly convex with respect to the geodesic vector field X_{g_0}.

We now consider an arbitrary smooth extension $\widetilde{X}_{g_0}$ of $X_{g_{0e}}|_{\mathcal{M}_e}$ to $\mathcal{N}$. Let $\rho \in C^\infty(\mathcal{N})$ be a global boundary-defining function for $\mathcal{M}$, i.e., such that $\rho > 0$ on the interior of $\mathcal{M}$, $\partial\mathcal{M} = \{\rho = 0\}$ and $\rho < 0$ on $\mathcal{N} \setminus \mathcal{M}$. Since X_{g_0} does not vanish on $\mathcal{M} = \{\rho \ge 0\}$, we can consider $\rho_0 > 0$ small enough that $\widetilde{X}_{g_0}$ does not vanish in $\{\rho > -2\rho_0\}$. A continuity argument shows that, for all $\rho_0 > 0$ small enough, the level set $\{\rho = -\rho_0\}$ is strictly convex with respect to $\widetilde{X}_{g_0}$. We can assume that

$$\mathcal{M}_e = \left\{x \in \mathcal{N} \;\middle|\; \rho(x) \ge -\tfrac{1}{2}\rho_0\right\}.$$

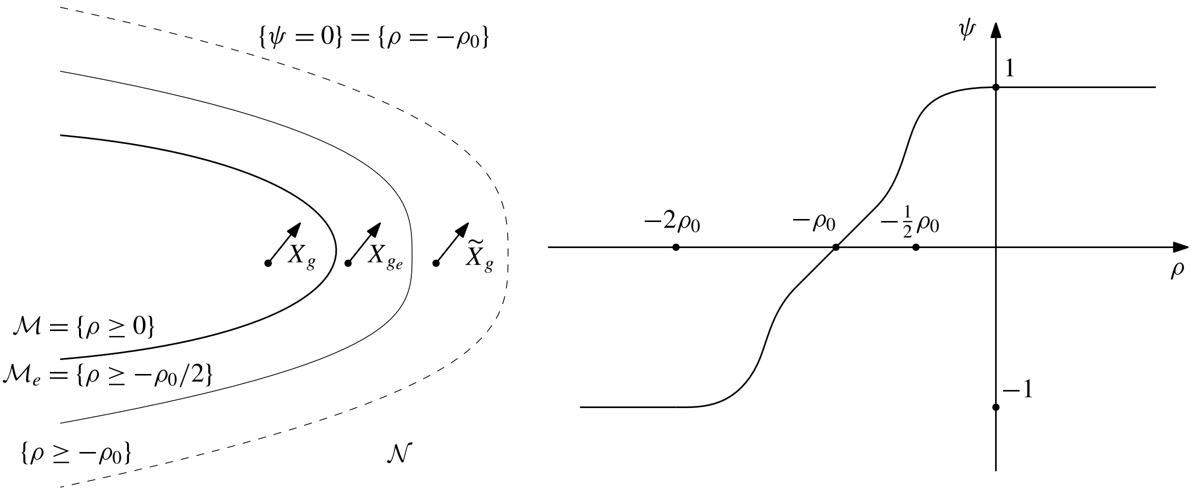

Figure 2. One the left: the extension of the vector field X_g from $\mathcal{M}$ to X_{g_e} on $\mathcal{M}_e$, and further to $\widetilde{X}_g$ on $\mathcal{N}$. The vector field $X = \psi \widetilde{X}_g$ is *complete* on the set $\{\rho \geq -\rho_0\}$ and vanishes on $\{\rho = -\rho_0\}$. On the right: the auxiliary function ψ as a function of ρ.

In the following, we will consider smooth perturbations X of the vector field X_{g_0} in $\mathcal{M}$ (small in the C^N-topology, for $N \gg 1$ large enough). They will mostly be induced by a metric g close to g_0, but it might be better to have in mind a more general picture than just geodesic flows. It will be convenient to extend the vector fields X_g to vector fields $\widetilde{X}_g$ on $\mathcal{N}$ such that $\widetilde{X}_g = \widetilde{X}_{g_0}$ on the set $\left\{\rho \leq -\frac{2}{3}\rho_0\right\}$ and $\widetilde{X}_g = X_{g_e}$ on $\mathcal{M}_e$. Moreover, it is possible to construct such an extension with, for any $N \in \mathbb{N}$,

$$\|\widetilde{X}_g - \widetilde{X}_{g_0}\|_{C^N(\mathcal{N}, T\mathcal{N})} \leq C \|X_g - X_{g_0}\|_{C^N(\mathcal{M}, T\mathcal{M})}$$

for some constant $C > 0$ (depending only on $\mathcal{M}$, $\mathcal{N}$, and N). Also observe that strict convexity of the boundary is stable by a C^2-perturbation of the vector field.

We introduce the smooth function $\psi \in C^\infty(\mathcal{N})$ with values in $[-1, 1]$ such that

- $\psi = \rho + \rho_0$ on the set $\left\{-\rho_0 - \frac{1}{10}\rho_0 \leq \rho \leq -\rho_0 + \frac{1}{10}\rho_0\right\}$,
- $\psi = 1$ on $\mathcal{M} = \{\rho \geq 0\}$, and $\psi > 0$ on $\{\rho > -\rho_0\}$,
- $\psi = -1$ on $\{\rho \leq -2\rho_0\}$, and $\psi < 0$ on $\{\rho < -\rho_0\}$.

With some abuse of notation, we then denote by X and X_0 the vector fields on $\mathcal{N}$ defined by $X := \psi \widetilde{X}_g$ and $X_0 := \psi \widetilde{X}_{g_0}$, respectively. This construction ensures that the restriction of X to $\mathcal{M}$ is the original vector field initially defined on $\mathcal{M}$ and that $\{\rho \geq -\rho_0\}$ is preserved by all the flows $(\varphi_t^X)_{t \in \mathbb{R}}$ for all $t \in \mathbb{R}$, and finally that each trajectory leaving $\mathcal{M}$ never comes back to $\mathcal{M}$, with the same property for $\mathcal{M}_e$. See Figure 2 for a visual summary of this construction.

2B. ***Scattering and length maps.*** For $(x, v) \in \mathcal{M}$, the escape time $\tau_g(x, v)$ is defined to be the maximal time of existence of the integral curve $(\varphi_t^g(x, v))_{t \geq 0}$ in $\mathcal{M}$:

$$\tau_g : \mathcal{M} \to [0, \infty], \quad \tau_g(x, v) := \sup\{t \geq 0 \mid \varphi_t^g(x, v) \in \mathcal{M}\}.$$

The forward ($-$) and backward ($+$) trapped sets $\Gamma^g_\pm$ are defined by

$$\Gamma^g_\pm := \{(x,v)\in\mathcal{M} \mid \tau_g(x,\mp v)=\infty\};$$

they are closed sets in $\mathcal{M}$, and the trapped set is the closed invariant set

$$K^g := \Gamma^g_+\cap\Gamma^g_- = \bigcap_{t\in\mathbb{R}}\varphi^g_t(\mathcal{M}).$$

Since ∂M is strictly convex, it is straightforward to check that $\Gamma^g_\mp\cap\partial_\pm\mathcal{M}=\varnothing$ and $K^g\cap\partial\mathcal{M}=\varnothing$. We now recall the definition (see Definition 1.1) of the *lens data.*

Definition 2.2 (lens data). The length map $\ell_g:\partial_-\mathcal{M}\setminus\Gamma^g_-\to\mathbb{R}_+$ and the scattering map $S_g:\partial_-\mathcal{M}\setminus\Gamma^g_-\to\partial_+\mathcal{M}\setminus\Gamma^g_+$ are defined by

$$\ell_g(x,v) := \tau_g(x,v) \quad\text{and}\quad S_g(x,v) := \varphi^g_{\tau_g(x,v)}(x,v).$$

The pair (ℓ_g, S_g) is called the lens data of (M,g).

When unnecessary, we will drop the index g in the notation. It will be convenient to view the scattering map as acting on functions on $\partial_+\mathcal{M}$ by pull-back. We define the *scattering operator* as

$$\mathcal{S}_g : C^\infty_{\mathrm{c}}(\partial_+\mathcal{M}\setminus\Gamma^g_+)\to C^\infty_{\mathrm{c}}(\partial_-\mathcal{M}\setminus\Gamma^g_-),\quad \mathcal{S}_g\omega := \omega\circ S_g.$$

Under the assumption that $\mu_\partial((\Gamma^g_-\cup\Gamma^g_+)\cap\partial\mathcal{M})=0$, it is not difficult to show (see [Guillarmou 2017b, Lemma 3.4]) that, for all $f\in C^\infty_{\mathrm{c}}(\partial_+\mathcal{M}\setminus\Gamma_+)$, one has

$$\|\mathcal{S}_g f\|_{L^2(\partial_-\mathcal{M},\mu_\partial)} = \|f\|_{L^2(\partial_+\mathcal{M},\mu_\partial)},$$

and thus $\mathcal{S}_g$ extends continuously to an isometry $L^2(\partial_+\mathcal{M},\mu_\partial)\to L^2(\partial_-\mathcal{M},\mu_\partial)$. The scattering operator $\mathcal{S}_g$ determines S_g, and conversely.

By the implicit function theorem (since ∂M is strictly convex), we also have that

$$\tau_g \in C^\infty(\mathcal{M}\setminus(\Gamma^g_-\cup\partial_0\mathcal{M})) \quad\text{and}\quad \ell_g\in C^\infty(\overline{\partial_-\mathcal{M}}\setminus\Gamma^g_-)$$

(here $\overline{\partial_-\mathcal{M}}=\partial_0\mathcal{M}\cup\partial_-\mathcal{M}$); see [Sharafutdinov 1994, Lemmas 4.1.1 and 4.1.2] for further details. Since we shall need the dependence of ℓ_g with respect to g, we first prove a result outside the trapped sets.

Lemma 2.3. *Let (M,g_0) be a smooth compact Riemannian manifold with strictly convex boundary, and let $p\in\mathbb{N}$. There exists $\varepsilon>0$ small enough that the following holds: for all metrics $g\in U_{g_0}$, where*

$$U_{g_0} := \{g\in C^{p+2}(M,\otimes^2_S T^*M) \mid \|g-g_0\|_{C^{p+2}}<\varepsilon,\ g|_{T\partial M}=g_0|_{T\partial M}\}, \tag{2-3}$$

the following map is C^p-regular:

$$\ell : V\to\mathbb{R}_+,\quad (g,y)\mapsto \ell_g(y),$$

where $V:=\{(g,y)\in U_{g_0}\times\partial_-\mathcal{M}\mid y\notin\Gamma^g_-\}$. Moreover, for all $\chi\in C^\infty_{\mathrm{c}}(\partial_-\mathcal{M})$, there exists a constant $C>0$ (depending only on g_0, p and χ) such that, for all $j\le p$ and $h\in C^\infty(M,\otimes^2_S T^*M)$,

$$\text{for all } (g,y)\in V,\qquad |\chi d^j_y\ell_g(y)|\le Ce^{C\ell_g(y)} \quad\text{and}\quad |\chi\partial^j_g\ell_g(y)(\otimes^j h)|\le Ce^{C\ell_g(y)}\|h\|^j_{C^{j+1}}.$$

Proof. We shall use the implicit function theorem. Let ρ be the boundary-defining function of $\mathcal{M}$ defined in Section 2A2. As explained in this paragraph, for g close to g_0, we can consider a vector field X on $\mathcal{N}$ such that X vanishes (to first order) on $\{\rho = -\rho_0\}$. For the sake of simplicity, we still denote by $(\varphi_t^g)_{t\in\mathbb{R}}$ the extended flow on $\mathcal{N}$, and by $X_g := X$ its generator.

We consider the C^p-regular map

$$F : U_{g_0} \times \mathbb{R}_+ \to \mathbb{R}, \quad (g, y, t) \mapsto \rho(\varphi_t^g(y)).$$

The function $\ell_g(y)$ satisfies the implicit equation $F(g, y, \ell_g(y)) = 0$. Let us take a point $(g_0, y_0) \in V$ and differentiate, for (g, y) near (g_0, y_0),

$$\partial_t F(g, y, t) = (X_g\rho)(\varphi_t^g(y)).$$

Notice that this is nonzero if $y \in \partial_-\mathcal{M}$, and $\varphi_t(y) \in \partial_+\mathcal{M}$ by strict convexity of $\partial\mathcal{M}$. Thus the implicit function theorem guarantees that there are neighborhoods $U'_{g_0} \subset U_{g_0}$ of g_0 and $B_{y_0}(\varepsilon') \subset \partial_-\mathcal{M}$ of y_0 such that $(g, y) \mapsto \ell_g(y)$ is a well-defined $C^p(U'_{g_0} \times B_{y_0}(\varepsilon'))$ function and

$$d_y\ell_g(y) = -\frac{d\rho(\varphi^g_{\ell_g(y)}(y)) \circ (d\varphi^g_{\ell_g(y)})(y)}{(X_g\rho)(\varphi^g_{\ell_g(y)}(y))}.$$

Notice in particular that this implies that V is an open set. By the Grönwall lemma, there is a constant $C > 0$ uniform in $g \in U_{g_0}$ such that, for each $(g, y) \in V$ and all $t > 0$, where $\|\cdot\|$ denotes an arbitrary fixed metric on $\mathcal{N}$,

$$\|d_y\varphi_t^g(y)\| \le Ce^{Ct}. \tag{2-4}$$

The constant $C > 0$ provided by the Grönwall lemma is uniform in the metric g as long as it is C^3-close to g_0. More generally, (2-4) holds for the j-th derivative $d_y^j\varphi_t^g$ with a constant $C > 0$ uniform for g which is C^{j+2}-close to g_0. On the other hand, we know that $X_g\rho \neq 0$ on $\partial\mathcal{M} \setminus \partial_0\mathcal{M}$. So we obtain a constant $C > 0$ such that,

$$\text{for all } (g, y) \in V, \quad |\chi(y)d_y\ell_g(y)| \le Ce^{C\ell_g(y)}.$$

Next, we compute the derivative with respect to g for some $h \in C^\infty(M, \otimes_S^2 T^*M)$:

$$(\partial_g\ell_g.h)(y) = -\frac{d\rho(\varphi^g_{\ell_g(y)}(y)) \circ (\partial_g\varphi^g_{\ell_g(y)}.h)(y)}{(X_g\rho)(\varphi^g_{\ell_g(y)}(y))}.$$

Again, by the Grönwall lemma, we obtain a constant $C > 0$ such that, for all $t > 0$, $(g, y) \in V$,

$$\|(\partial_g\varphi_t^g.h)(y)\| \le Ce^{Ct}\|h\|_{C^2}, \tag{2-5}$$

which provides the desired estimate for the C^2-norm. (The C^2-norm of h appears as the vector field X_g involves the 1-derivative of g, so that X_{g+sh} is C^1 for all $s \in \mathbb{R}$ small). The constant $C > 0$ is uniform for g that is C^3-close to g_0. More generally, the bound $|\partial_g^j\varphi_t^g(\otimes^j h)(y)| \le Ce^{Ct}\|h\|^j_{C^{j+1}}$ holds with a constant $C > 0$ depending on the C^{j+2}-norm of g. The case of higher-order derivatives works exactly the same way by differentiating as many times as needed the implicit equation defining $\ell_g(y)$ with respect to (g, y), and using that the derivatives of the flow satisfy the bounds $\|D^j\varphi_t^g(y)\| \le Ce^{Ct}$ (where $D^j = \partial_g^j$ or d_y^j) for some uniform $C > 0$ with respect to $t > 0$, y and $g \in U_{g_0}$. $\square$

2C. *Hyperbolic trapped set.*

2C1. *Axiom A property.* We say that the trapped set is *hyperbolic* if there is a continuous flow-invariant splitting of $T(SM)$ restricted to K^g into three subbundles:

$$\text{for all } y \in K^g, \quad T_y\mathcal{M} = \mathbb{R}X_g(y) \oplus E_s^g(y) \oplus E_u^g(y),$$

and $C, \nu > 0$ such that, for all $y \in K^g$ and $t \ge 0$,

$$\begin{aligned} v \in E_s^g(y) \quad &\Longrightarrow \quad \|d\varphi_t^g(y)v\| \le Ce^{-\nu t}\|v\|, \\ v \in E_u^g(y) \quad &\Longrightarrow \quad \|d\varphi_{-t}^g(y)v\| \le Ce^{-\nu t}\|v\|. \end{aligned} \tag{2-6}$$

There is a continuous extension of the bundles E_s^g and E_u^g to the bundles E_-^g and E_+^g over the sets Γ_-^g and Γ_+^g, respectively, on which (2-6) is still satisfied; see [Dyatlov and Guillarmou 2016, Lemma 2.10]. For $y \in K^g$, these bundles coincide with E_s^g and E_u^g, namely $E_s^g(y) = E_-^g(y)$ and $E_u^g(y) = E_+^g(y)$. We define $C_{\mathrm{hyp}}^k(M, \otimes_S^2 T^*M_+)$ to be the set of C^k Riemannian metrics on M with strictly convex boundary and hyperbolic trapped set. For such metrics, the geodesic flow is a typical example of what is known as an *Axiom A flow*. Since these metrics could have conjugate points, this set is larger than the set of metrics of Anosov type.

If g_0 is some fixed metric on M and M_e denotes the extension defined in Section 2A2 with ρ a boundary-defining function of $\mathcal{M}$, we can always choose $\rho_0 > 0$ small enough that, for all $|t| \le \rho_0$, the level set $\{\rho = t\}$ is strictly convex with respect to the extension g_{0e} of g_0 to M_e. This also holds for any metric g close to g_0 in the C^2-topology. Recall that we denote by g_e the extension of g from M to M_e.

Observe that if $y \in \partial_\pm\mathcal{M}$ then $\bigcup_{\pm t>0}\varphi_t^{g_e}(y) \subset \mathcal{N} \setminus \mathcal{M}$. The trapped sets of (M, g) and (M_e, g_e) then coincide and $\Gamma_\pm^g = \Gamma_\pm^{g_e} \cap \mathcal{M}$. Moreover, if (M, g) has no conjugate points, then by taking $\rho_0 > 0$ small enough (M_e, g_e) does not have conjugate points either; see [Guillarmou 2017b, Lemma 2.3].

Define the set of points that are trapped for time less than $t \ge 0$ as

$$\mathcal{T}^g(t) := \{y \in \mathcal{M} \mid \forall s \in (0, t),\ \varphi_s^g(y) \in \mathcal{M}^\circ\} = \tau_g^{-1}(t, \infty).$$

It is proved in [Guillarmou 2017b, Proposition 2.4] that there exist $C_g, Q_g > 0$ (depending on the metric g) such that, for all $t \ge 0$,

$$\mu(\mathcal{T}^g(t)) \le C_g e^{-Q_g t}. \tag{2-7}$$

(Here μ is the Liouville measure for the fixed g_0.) In particular, $\mu(\Gamma_\pm^g) = 0$. The quantity Q_g is called the *escape rate* and is given by $-Q_g = P_g(-J_u^g) < 0$: the topological pressure of negative the unstable Jacobian $J_u^g(y) := \partial_t(\det d\varphi_t^g(y)|_{E_u(y)})|_{t=0}$ of the flow $(\varphi_t^g)_{t\in\mathbb{R}}$. Recall that the topological pressure of a Hölder potential $V \in C^\beta(S^gM)$ (for some $\beta > 0$) with respect to g can be defined as follows:

$$P_g(V) := \lim_{T\to\infty}\frac{1}{T}\log\sum_{\gamma\in\mathcal{P},T_\gamma\in[T,T+1]}\exp\biggl(\int_\gamma V\biggr),$$

where $\mathcal{P}$ is the set of periodic orbits of the geodesic flow $(\varphi_t^g)_{t\in\mathbb{R}}$, and T_γ is the period of $\gamma \in \mathcal{P}$.

The following formula for $f \in L^1(\mathcal{M})$ is known as *Santaló's formula* (see [Guillarmou 2017b, Section 2.5]):

$$\int_{\mathcal{M}} f(y)\,\mathrm{d}\mu(y) = \int_{\partial_-\mathcal{M}} \int_0^{+\infty} f(\varphi_t^g(y))\,\mathrm{d}t\,\mathrm{d}\mu_\partial(y). \tag{2-8}$$

It implies, together with (2-7), that there is $C_g > 0$ such that, for all $t > 0$,

$$\mu_\partial(\ell_g^{-1}(t,\infty)) \le C_g e^{-Q_g t}. \tag{2-9}$$

Using Cavalieri's principle, estimates (2-7) and (2-9), it is straightforward to derive the following bounds:

$$\begin{aligned} &\text{for all } p \in [1,\infty), && \tau_g \in L^p(\mathcal{M}), && \ell_g \in L^p(\partial_-\mathcal{M}),\\ &\text{for all } \lambda \in (0,Q_g), && e^{\lambda\tau_g} \in L^1(\mathcal{M}), && e^{\lambda\ell_g} \in L^1(\partial_-\mathcal{M}). \end{aligned} \tag{2-10}$$

Here note that ℓ_g is bounded near $\partial_0\mathcal{M}$, so that this region is trivial to deal with.

2C2. *Robinson structural stability.* In this paragraph, we recall some results about the stability of flows with hyperbolic trapped set, due to [Robinson 1980, Theorem C]. First, the stable and unstable manifolds of a point $y \in K^g$ are defined by

$$W_s(y) := \{y' \in \mathcal{M} \mid \lim_{t\to+\infty} d(\varphi_t^g(y'),\varphi_t^g(y)) \to 0\},$$
$$W_u(y) := \{y' \in \mathcal{M} \mid \lim_{t\to-\infty} d(\varphi_t^g(y'),\varphi_t^g(y)) \to 0\}.$$

They are smooth injectively immersed submanifolds. We also set

$$W_u(K^g) := \bigcup_{y\in K^g} W_u(y) \quad\text{and}\quad W_s(K^g) := \bigcup_{y\in K^g} W_s(y).$$

It is proved in [Guillarmou 2017b, Lemma 2.2] that

$$W_s(K^g) = \Gamma_-^g \quad\text{and}\quad W_u(K^g) = \Gamma_+^g. \tag{2-11}$$

The tangent spaces to $W_s(y)$ and $W_u(y)$ are $E_s(y)$ and $E_u(y)$, respectively. The flow satisfies the following *transversality property* for the stable and unstable manifolds $W_s(y)$ and $W_u(y)$: for each $y, y' \in K^g$ and $z \in W_s(y) \cap W_u(y') \subset K^g$, we have

$$T_z(\mathcal{M}) = T_z(W_s(y)) \oplus T_z(W_u(y')) \oplus \mathbb{R}X_g(z).$$

Indeed, such z must belong to K^g, and the identity of the tangent space can be rewritten as

$$E_s(z) \oplus E_u(z) \oplus \mathbb{R}X_g(z) = T_z(\mathcal{M}),$$

which holds since K^g is assumed hyperbolic. For a Riemannian manifold with strictly convex boundary and hyperbolic trapped set, the geodesic flow $(\varphi_t^g)_{t\in\mathbb{R}}$ on $\mathcal{M}$ satisfies the following:

- The nonwandering set $\Omega \subset K^g$ is hyperbolic.
- The stable and unstable manifolds have the transversality property.
- The boundary is strictly convex with respect to the vector field X_g.

Proposition 2.4 [Robinson 1980]. *Let* (M, g_0) *be a smooth Riemannian manifold with strictly convex boundary and hyperbolic trapped set* $K^{g_0} \subset \mathcal{M} := SM$. *Then, there exists* $\varepsilon_0 > 0$ *such that, for each smooth vector field* X *on* $\mathcal{M}$ *with* $\|X - X_{g_0}\|_{C^2(\mathcal{M})} \le \varepsilon_0$, *there is a homeomorphism* $h : \mathcal{M} \to \mathcal{M}$ *and* $a \in C^0(U)$, *where* $U = \{(y, t) \in \mathcal{M} \times \mathbb{R} \mid t \in [-\tau_{g_0}(-h(y)), \tau_{g_0}(h(y))]\}$, *such that the following holds: for all* $y \in \mathcal{M}$, *we have that* $t \mapsto a(y, t)$ *is strictly increasing in* t *and satisfies*

$$\varphi_t^{X_{g_0}}(h(y)) = h(\varphi^X_{a(y,t)}(y))$$

for all $(y, t) \in \mathcal{M} \times \mathbb{R}$ *such that* $\varphi^X_{a(y,t)}(y) \in \mathcal{M}$. *Moreover, for each* $\delta > 0$ *there exists* $\varepsilon > 0$ *small enough that if* $\|X - X_{g_0}\|_{C^2(\mathcal{M})} \le \varepsilon$, *then* $d(h(y), y) \le \delta$ *for* $y \in \mathcal{M}$, *where* d *denotes a Riemannian distance on* $\mathcal{M}$, *that is*, $\|h - \mathrm{id}_{\mathcal{M}}\|_{C^0} \le \delta$.

Proof. This is a direct consequence of [Robinson 1980, Theorems A and C]. We note that Robinson's "quadratic external boundary conditions" are equivalent to our strict convexity of the boundary, and that the *chain-recurrent set* (see [Robinson 1980] for the definition) is contained in the trapped set, which by assumption has a hyperbolic structure with transversal stable and unstable manifolds. Finally, the last statement about the continuity of h is stated in [Robinson 1980, Theorem A]. ☐

As a consequence, we see that, for g close enough to g_0 in C^3 norm, applying Proposition 2.4 with $X = X_g$, we get

$$K^g = h^{-1}(K^{g_0}) \quad \text{and} \quad h^{-1}(\Gamma_\pm^{g_0}) = \Gamma_\pm^g,$$

and the trapped set varies continuously with respect to the metric.

2C3. *Symplectic lift to the cotangent bundle.* Recall that we introduced the vector field X on $\mathcal{N}$ in Section 2A2. In Section 5, it will be convenient to work on the cotangent bundle $T^*\mathcal{N}$ of the extended manifold $\mathcal{N}$. Denote by $\boldsymbol{X}$ the symplectic lift of the vector field X to $T^*\mathcal{N}$. It generates the flow

$$\varphi_t^{\boldsymbol{X}}(y, \xi) = (\varphi_t^X(y), (d\varphi_t^X(y))^{-\top}\xi), \tag{2-12}$$

where ${}^{-\top}$ stands for the inverse transpose. Note that this flow is linear in the second variable and thus induces a flow on the spherical bundle $S^*\mathcal{N} := (T^*\mathcal{N} \setminus \{0\})/\mathbb{R}_+$. Let $\pi : S^*\mathcal{N} \to \mathcal{N}$ and $\kappa : T^*\mathcal{N} \to S^*\mathcal{N}$ be the natural projections, and still write π for the projection $T^*\mathcal{N} \to \mathcal{N}$. The dual subbundles $(E^X_{\pm,0})^* \subset T^*\mathcal{N}$ are defined as the following symplectic orthogonals:

$$(E_0^X)^*(E_+^X \oplus E_-^X) = (E_+^X)^*(E_+^X \oplus E_0^X) = (E_-^X)^*(E_-^X \oplus E_0^X) = \{0\}.$$

With some abuse of notation, the spaces $(E^X_{\pm,0})^*$ will be identified with the projections $\kappa((E^X_{\pm,0})^*) \subset S^*\mathcal{N}$. Eventually, we record the following definition to be found useful later:

$$\Sigma_\pm := \bigcup_{\|X - X_0\|_{C^2} \le \delta, \pm t \ge 0} \varphi_t^{\boldsymbol{X}}(\mathcal{M}), \tag{2-13}$$

where $\delta > 0$ is small enough. Finally, we note that the tails $\Gamma_\pm^X$ and the bundles $(E^X_{\pm,0})^*$ admit an extension to the set $\{\rho > -\rho_0\}$.

2D. ***Resolvent and X-ray transform.*** Since we will work with Sobolev spaces on the manifolds $\mathcal{M}$ and $\partial_\pm\mathcal{M}$, let us clarify what this means as these are manifolds with boundary or open manifolds. First, since $\mathcal{M}$ is a smooth manifold with boundary, the spaces $H^s(\mathcal{M})$ are defined intrinsically for $s \geq 0$ (as the restriction of H^s-functions defined on $\mathcal{N}$ for instance). Set $H^s_0(\mathcal{M}) = \overline{C^\infty_{\mathrm{c}}(\mathcal{M}^\circ)}$, where the closure is for the H^s norm, and write $H^{-s}(\mathcal{M}) := (H^s_0(\mathcal{M}))^*$ for $s > 0$, where the upper star denotes the continuous dual. For $\partial_\pm\mathcal{M}$, write $H^s(\partial\mathcal{M}) := H^s(\overline{\partial_\pm\mathcal{M}})$, where $\overline{\partial_\pm\mathcal{M}} := \partial_\pm\mathcal{M} \cup \partial_0\mathcal{M}$ is a smooth manifold with boundary, and $H^{-s}(\partial_\pm\mathcal{M}) = (H^s_0(\partial_\pm\mathcal{M}))^*$.

Define the resolvent of X_g to be the family of operators, for $\Re(z) \geq 0$,

$$R_g(z) : C^\infty_{\mathrm{c}}(\mathcal{M}^\circ \setminus \Gamma^g_-) \to C^\infty(\mathcal{M}), \quad R_g(z) f(y) := -\int_0^{\tau_g(y)} e^{-zt} f(\varphi^g_t(y))\, \mathrm{d}t. \tag{2-14}$$

For $z = 0$, simply write $R_g := R_g(0)$. It solves $X_g R_g = \mathbb{1}$ on $C^\infty_{\mathrm{c}}(\mathcal{M}^\circ \setminus \Gamma^g_-)$ with boundary condition $(R_g f)|_{\partial_+\mathcal{M}} = 0$.

Assuming that (M, g) has strictly convex boundary and hyperbolic trapped set, we have by [Guillarmou 2017b, Propositions 4.2 and 4.4] the following boundedness properties:

$$\text{for all } p \in [1, \infty), \qquad R_g : L^\infty(\mathcal{M}) \to L^p(\mathcal{M}), \tag{2-15}$$

$$\text{for all } \alpha \in (0, 1), \text{ there exists } s > 0 \text{ such that} \qquad R_g : C^\alpha_{\mathrm{c}}(\mathcal{M}^\circ) \to H^s(\mathcal{M}), \tag{2-16}$$

$$\text{for all } s > 0, \qquad R_g : H^s(\mathcal{M}) \to H^{-s}(\mathcal{M}), \tag{2-17}$$

where $C^\alpha(\mathcal{M})$ is the Hölder space of order α. Note that if $\varepsilon > 0$ is chosen small enough,

$$U := \bigcup_{t \in (-\varepsilon, \varepsilon)} \varphi^g_t(\partial_-\mathcal{M})$$

is a neighborhood of $\partial_-\mathcal{M}$ in $\mathcal{M}_e$ which is diffeomorphic to $(-\varepsilon, \varepsilon) \times \partial_-\mathcal{M}$ by $(t, y) \mapsto \varphi^g_t(y)$, and $\partial_t(\tau_g \circ \varphi^g_t) = -1$ in U. Using (2-15), Santaló's formula (2-8), and the fact that ℓ_g is smooth near $\partial_0\mathcal{M}$ in $\partial_-\mathcal{M} \cup \partial_0\mathcal{M}$ (see [Sharafutdinov 1994, Lemma 4.1.1]), we consequently obtain

$$\ell_g = -(R_g \mathbf{1}_{\mathcal{M}})|_{\partial_-\mathcal{M}} \in L^p(\partial_-\mathcal{M}, \mu_\partial) \tag{2-18}$$

for all $1 \leq p < \infty$. The *X-ray transform* is defined as the operator

$$I^g : C^\infty_{\mathrm{c}}(\mathcal{M} \setminus \Gamma^g_-) \to C^\infty_{\mathrm{c}}(\partial_-\mathcal{M} \setminus \Gamma^g_-), \quad I^g f := -(R_g f)|_{\partial_-\mathcal{M}},$$

and, by [Guillarmou 2017b, Lemma 5.1], it extends as a bounded map for all $p > 2$:

$$I^g : L^p(\mathcal{M}) \to L^2(\partial_-\mathcal{M}, \mu_\partial). \tag{2-19}$$

We now show the following boundedness property.

Lemma 2.5. *Let (M, g) be a compact Riemannian manifold with strictly convex boundary and hyperbolic trapped set. Then, there exists $s > 0$ such that the operator I^g is bounded as a map*:

$$I^g : C^2(\mathcal{M}) \to H^s(\partial_-\mathcal{M}).$$

Proof. First of all, if $\chi \in C^\infty(\overline{\partial_-\mathcal{M}})$ is supported close to $\partial_0\mathcal{M}$, one can check that $\chi I^g f \in C^2(\overline{\partial_-\mathcal{M}})$ for $f \in C^2(\mathcal{M})$; see [Sharafutdinov 1994, Lemma 4.1.1]. It thus remains to analyze $\chi I^g f$ when $\chi \in C^\infty_c(\partial_-\mathcal{M})$. Let $\gamma > 0$ be a large enough constant (it will be determined later), $\varepsilon \in (0, Q_g/(2\gamma))$, and let Δ_h be the Riemannian Laplacian associated to an arbitrarily chosen smooth Riemannian metric h on $\overline{\partial_-\mathcal{M}}$, with Dirichlet condition at $\partial_0\mathcal{M}$. It is self-adjoint on $H^1_0(\overline{\partial_-\mathcal{M}}) \cap H^2(\overline{\partial_-\mathcal{M}})$ with respect to the Riemannian volume measure $\mathrm{d}v_h$. Note that $\mathrm{d}v_h$ is smoothly equivalent to μ_∂ on each compact set of $\partial_-\mathcal{M}$ as μ_∂ vanishes to first order on the boundary $\partial_0\mathcal{M}$.

For $f \in C^2(\mathcal{M})$, consider the holomorphic map

$$\{-\varepsilon \le \Re(z) \le 1-\varepsilon\} \ni z \mapsto u(z) := (1+\Delta_h)^{z+\varepsilon}(e^{-z\gamma\ell_g}\chi I^g f) \in \mathcal{D}'(\partial_-\mathcal{M}).$$

We are going to apply the Hadamard three-line theorem (see [Rudin 1987, Theorem 12.8]) to the holomorphic family of distributions $u(z)$. From (2-19), we have $I^g f \in L^2(\partial_-\mathcal{M}, \mu_\partial)$, but we can also write the pointwise bound,

$$\text{for all } y \in \partial_-\mathcal{M} \setminus \Gamma_-, \quad |I^g f(y)| \le \|f\|_{L^\infty}\ell_g(y). \tag{2-20}$$

From (2-10), we get, using that $\varepsilon < Q_g/(2\gamma)$,

$$\chi e^{\varepsilon\gamma\ell_g} I^g f \in L^2(\partial_-\mathcal{M}, \mathrm{d}v_h).$$

Therefore on the line $\{\Re(z) = -\varepsilon\}$ with $0 < \varepsilon < Q_g/(2\gamma)$, there exists a constant $C > 0$ independent of z and f (but depending on χ) such that

$$\|u(z)\|_{L^2} \le \|(1+\Delta_h)^{i\Im(z)}\|_{L^2\to L^2}\|\chi e^{\varepsilon\gamma\ell_g} I^g(f)\|_{L^2} \le C\|f\|_{L^\infty}, \tag{2-21}$$

where $L^2 = L^2(\partial_-\mathcal{M}, \mathrm{d}v_h)$. Note that we used the spectral theorem for Δ_h in order to bound

$$\|(1+\Delta_h)^{i\Im(z)}\|_{L^2\to L^2} \le 1.$$

Now, using that $I^g f(y) = \int_0^{\ell_g(y)} f(\varphi^g_t(y))\,\mathrm{d}t$, we obtain, using Lemma 2.3, (2-4), and (2-20), the pointwise bound on $\partial_-\mathcal{M} \setminus \Gamma^g_-$:

$$|\Delta_h(e^{-z\gamma\ell_g} I^g f)(y)| \le C(1+|z|^2)\|f\|_{C^2(\mathcal{M})} e^{(C_0-\gamma\Re(z))\ell_g(y)}$$

for some uniform constants $C, C_0 > 0$ (depending only on the metric g). We therefore see that, for $\Re(z) = 1-\varepsilon$, the function $\Delta_h(e^{-\gamma z\ell_g}\chi I^g(f))$ can be extended from $\partial_-\mathcal{M} \setminus \Gamma_-$ continuously to $\partial_-\mathcal{M}$ by setting it to be 0 on Γ_- as long as $\gamma(1-\varepsilon) > C_0$. Here, we see that, in order to achieve this, we can choose $\gamma > 2022C_0$ at the very beginning (the constant C_0 only depends on the metric g).

Claim 2.6. *The continuous extension by* 0 *of* $\Delta_h(e^{-z\gamma\ell_g}\chi I^g f)$ *on* Γ^g_- *matches with the distributional derivative* $\Delta_h(e^{-z\gamma\ell_g}\chi I^g f) \in \mathcal{D}'(\partial_-\mathcal{M})$.

The proof of this claim is postponed until below. Then $\Delta_h(e^{-z\gamma\ell_g}\chi I^g f) \in L^2(\partial_-\mathcal{M})$, and on the line $\{\Re(z) = 1-\varepsilon\}$ we have

$$\|u(z)\|_{L^2} \le \|(1+\Delta_h)^{i\Im(z)}\|_{L^2\to L^2}\|(1+\Delta_h)(e^{-z\gamma\ell_g}\chi I^g f)\|_{L^2} \le C(1+|z|^2)\|f\|_{C^2}. \tag{2-22}$$

We can then use the Hadamard three-line interpolation theorem applied to the holomorphic function

$$\{-\varepsilon \leq \Re(z) \leq 1-\varepsilon\} \ni z \mapsto v(z) := \int_{\partial_-\mathcal{M}} (1+z)^{-2} u(z)\psi \, \mathrm{d}v_h \in \mathbb{C},$$

where $\psi \in C_{\mathrm{c}}^{\infty}(\partial_-\mathcal{M})$ is arbitrary. Note that this is well defined and holomorphic in the strip $\Re(z) \in [-\varepsilon, 1-\varepsilon]$ since we have the bound

$$|v(z)| \leq \frac{1}{(1-\varepsilon)^2} \|\psi\|_{H^{2(\Re(z)+\varepsilon)}} \|e^{\varepsilon\gamma\ell_g} \chi I_g f\|_{L^2} \leq C\|\psi\|_{H^2}\|f\|_{C^2}.$$

From (2-21) and (2-22), we deduce the existence of a constant $C>0$, independent of ψ, such that, for all z with $\Re(z) \in [-\varepsilon, 1-\varepsilon]$, one has

$$|v(z)| \leq C\|\psi\|_{L^2}.$$

This shows that $u(z) \in L^2(\partial_-\mathcal{M})$ for all such z with the bound $|u(z)| \leq C$. In particular, taking $z=0$, we obtain that $(1+\Delta_h)^{\varepsilon}(\chi I^g f) \in L^2$, thus showing the claimed result.

It thus remains to prove Claim 2.6 above. Denote by F the continuous extension of $\Delta_h(e^{-z\gamma\ell_g}\chi I^g(f))$ by 0 on Γ_-^g. We need to show that, for each $\psi \in C_{\mathrm{c}}^{\infty}(\partial_-\mathcal{M})$,

$$\int_{\partial_-\mathcal{M}} \chi e^{-z\gamma\ell_g} I^g(f)\Delta_h\psi \, \mathrm{d}v_h = \int_{\partial_-\mathcal{M}} F\psi \, \mathrm{d}v_h. \tag{2-23}$$

Take $\theta \in C_{\mathrm{c}}^{\infty}([0,2))$ equal to 1 in $[0,1]$. We write the left-hand side as

$$\begin{aligned}\lim_{T\to\infty} \int_{\partial_-\mathcal{M}} \theta(\ell_g/T)\chi e^{-z\gamma\ell_g} I^g(f)\Delta_h\psi \, \mathrm{d}v_h &= \lim_{T\to\infty} \int_{\partial_-\mathcal{M}} \Delta_h(\theta(\ell_g/T)\chi e^{-z\gamma\ell_g} I^g(f))\psi \, \mathrm{d}v_h \\ &= \lim_{T\to\infty} A_1(T) + A_2(T),\end{aligned}$$

where

$$\begin{aligned}A_1(T) &:= \int_{\partial_-\mathcal{M}} \Delta_h(\theta(\ell_g/T))\chi e^{-z\gamma\ell_g} I^g(f)\psi \, \mathrm{d}v_h + 2\int_{\partial_-\mathcal{M}} \nabla(\theta(\ell_g/T)) \cdot \nabla(\chi e^{-z\gamma\ell_g} I^g(f))\psi \, \mathrm{d}v_h, \\ A_2(T) &:= \int_{\partial_-\mathcal{M}} \theta(\ell_g/T)\Delta_h(\chi e^{-z\gamma\ell_g} I^g(f))\psi \, \mathrm{d}v_h = \int_{\partial_-\mathcal{M}} \theta(\ell_g/T)F\psi \, \mathrm{d}v_h.\end{aligned}$$

In order to show (2-23), it thus suffices to show that $A_1(T) \to 0$ as $T \to \infty$. The derivatives $d_y^j(\theta(\ell_g/T))$ of order $j=1,2$ are supported in $\{\ell_g \in [T,2T]\}$, where we can use the pointwise bound of Lemma 2.3:

$$|d_y^j(\theta(\ell_g(y)/T))| \leq Ce^{C_0\ell_g(y)} \leq Ce^{2C_0T}$$

for some uniform $C, C_0 > 0$. Since all terms in the integrand of A_1 are multiplied by the weight $|e^{-\gamma z\ell_g(y)}| \leq e^{-\gamma(1-\varepsilon)T}$, we easily see, using Lemma 2.3 once again, that

$$A_1(T) = \mathcal{O}((1+|z|)e^{(3C_0-\gamma(1-\varepsilon))T}).$$

Taking $\gamma > 6C_0$ at the beginning and $\varepsilon < \frac{1}{2}$, one obtains that $A_1(T) \to 0$, and this proves our claim. $\square$

Note that, as a corollary of Lemma 2.5, we obtain that there is $s > 0$ such that

$$\ell_g = I^g(\mathbf{1}_{\mathcal{M}}) \in H^s(\partial_-\mathcal{M}). \tag{2-24}$$

2E. ***Scattering operator.*** Working with the scattering operator $\mathcal{S}_g$ has several advantages over working directly with S_g. The main reason is that its Schwartz kernel can be expressed in terms of restriction of the Schwartz kernel of the resolvent R_g of the geodesic vector field X_g. This is the content of Lemma 2.7 below. This will be important so that we can work in a good functional setting in order to apply the Taylor expansion of the lens data with respect to g. We denote by R_{g_e} the resolvent on $\mathcal{M}_e$ for the extension g_e (for the definition of g_e recall Section 2A2), which has all the properties of R_g.

Lemma 2.7. *Let (M, g) be a compact Riemannian manifold with strictly convex boundary and hyperbolic trapped set. Let $\iota_{\partial_\pm} : \partial_\pm\mathcal{M} \to \mathcal{M}$ be the inclusion map. The restriction $(\iota_{\partial_-} \times \iota_{\partial_+})^* R_{g_e}$ of the Schwartz kernel of the resolvent on $\partial_-\mathcal{M} \times \partial_+\mathcal{M}$ makes sense as a distribution, and the Schwartz kernel of $\mathcal{S}_g$ is given by*

$$\mathcal{S}_g(y, y') = -(\iota_{\partial_-} \times \iota_{\partial_+})^* R_{g_e}(y, y'), \quad (y, y') \in \partial_-\mathcal{M} \times \partial_+\mathcal{M}.$$

Proof. First, we define the operator $\mathcal{E}_g : C^\infty_{\mathrm{c}}(\partial_+\mathcal{M}) \to H^s(\mathcal{M})$ for $s > 0$ as follows: for $\delta > 0$ small, let $\Omega = \{(x, v) \in \partial\mathcal{M} \mid |g_x(\nu, v)| \le \delta\}$; define $\Omega_e = \mathcal{M}_e \cap \bigcup_{t\in\mathbb{R}} \varphi_t^{g_e}(\Omega)$ to be the flowout of Ω by $\varphi_t^{g_e}$; and let $\psi \in C^\infty(\mathcal{M}_e, \mathbb{R}_+)$ such that $\psi|_{\Omega_e\cup\partial_-\mathcal{M}} = 0$, ψ is supported in a small neighborhood of $\partial_+\mathcal{M} \setminus \Omega$ and $X_{g_e}\psi = 0$ in $\mathcal{M}_e \setminus \mathcal{M}$ and near $\partial_+\mathcal{M}$. Then set, for $\omega \in C^\infty_{\mathrm{c}}(\partial_+\mathcal{M})$,

$$\mathcal{E}_g\omega := \tilde{\omega}\psi - R_{g_e}X_{g_e}(\tilde{\omega}\psi) \in H^s(\mathcal{M}_e) \cap L^p(\mathcal{M}_e) \cap C^\infty(\mathcal{M}_e \setminus (\Gamma_- \cup \Gamma_+))$$

for some $s > 0$ and all $p < \infty$ using (2-15) and (2-16), where $\tilde{\omega}$ is defined on $\operatorname{supp}(\psi)$ by extending ω from $\partial_+\mathcal{M}$ to be constant on the flow lines of X_{g_e}. This can be done by using the diffeomorphism

$$\Psi_+ : \left\{(t, y) \in \left(-\tfrac{1}{2}\delta, \infty\right) \times (\partial_+\mathcal{M} \setminus \Omega) \;\middle|\; t \le \tau_{g_e}(y)\right\} \ni (t, y) \mapsto \varphi_t^{g_e}(y) \in \mathcal{M}_e$$

and using that the flow $\varphi_t^{g_e}$ is the translation in t in these coordinates. One clearly has that $\mathcal{E}_g\omega$ is smooth near $\partial_+\mathcal{M}$ and

$$X_{g_e}\mathcal{E}_g\omega = 0, \quad (\mathcal{E}_g\omega)|_{\partial_+\mathcal{M}} = \psi|_{\partial_+\mathcal{M}}\omega.$$

In particular, we see that, outside Γ_-, we have

$$(\mathcal{E}_g\omega)|_{\partial_-\mathcal{M}\setminus\Gamma_-} = (\mathcal{S}_g(\omega\psi|_{\partial_+\mathcal{M}}))|_{\partial_-\mathcal{M}\setminus\Gamma_-}. \tag{2-25}$$

On the other hand, using the diffeomorphism

$$\Psi_- : \left\{(t, y) \in \left(-\infty, \tfrac{1}{2}\delta\right) \times (\partial_-\mathcal{M} \setminus \Omega) \;\middle|\; t \ge -\tau_{g_e}(-y)\right\} \ni (t, y) \mapsto \varphi_t^{g_e}(y) \in \mathcal{M}_e$$

mapping to a neighborhood of $\partial_-\mathcal{M} \setminus \Omega$, we see that $\Psi_-^*\mathcal{E}_g\omega$ is independent of t and can be viewed as a function in $H^s(\partial_-\mathcal{M}) \cap L^p(\partial_-\mathcal{M})$, i.e., the restriction $(\mathcal{E}_g\omega)|_{\partial_-\mathcal{M}}$ makes sense as an $H^s(\partial_-\mathcal{M}) \cap L^p(\partial_-\mathcal{M})$ function. (This fact can also be proved using the Hörmander pull-back theorem for distributions using wave-front analysis with the fact that X is transverse to $\partial_-\mathcal{M}$.) Since $\mu_\partial(\Gamma_-^g \cap \partial_-\mathcal{M}) = 0$, this implies with (2-25) that $(\mathcal{E}_g\omega)|_{\partial_-\mathcal{M}} = \mathcal{S}_g(\omega\psi|_{\partial_+\mathcal{M}})$. But this is also given by $(\mathcal{E}_g\omega)|_{\partial_-\mathcal{M}} = -(R_{g_e}X_{g_e}(\tilde{\omega}\psi))|_{\partial_-\mathcal{M}}$.

Since $X_{g_e}R_{g_e}=R_{g_e}X_{g_e}=\mathrm{Id}$ in $C^\infty_{\mathrm{c}}(\mathcal{M}_e^\circ)$ (this follows for instance by analytic extension of the identity $R_{g_e}(z)(X_{g_e}-z)=(X_{g_e}-z)R_{g_e}(z)=\mathrm{Id}$ on $C_c^\infty(M_e^\circ)$ for $\Re(z)\gg 1$), one has $(X_{g_e}R_{g_e})(y,y')=0$ and $(X'_{g_e}R_{g_e})(y,y')$ in the distribution sense for y close to $\partial_-\mathcal{M}\setminus\Omega$ and y' close to $\partial_+\mathcal{M}\setminus\Omega$, where X_{g_e} and X'_{g_e} denotes the action of X_{g_e} on the left and right variable of $\mathcal{M}_e\times\mathcal{M}_e$, respectively. This implies as above that the restriction $(\iota_{\partial_-}\times\iota_{\partial_+})^*R_{g_e}$ makes sense and we can apply Green's formula in the right variable: if $\omega'\in C^\infty_{\mathrm{c}}(\partial_-\mathcal{M})$,

$$\begin{aligned}-\langle\iota_{\partial_-}^*(R_{g_e}X_{g_e}(\tilde{\omega}\psi)),\omega'\rangle&=-\int_{\partial_-\mathcal{M}}\int_{\mathcal{M}}R_{g_e}(y,y')X_{g_e}(\tilde{\omega}\psi)(y')\omega'(y)\,\mathrm{d}\mu(y')\,\mathrm{d}\mu_\partial(y)\\&=-\int_{\partial_-\mathcal{M}}\int_{\partial_+\mathcal{M}}R_{g_e}(y,y')(\psi\omega)(y')\omega'(y)i_{X_{g_e}}\,\mathrm{d}\mu(y')\,\mathrm{d}\mu_\partial(y),\end{aligned}$$

where we used $X_{g_e}(\tilde{\omega}\psi)=0$ on $\mathcal{M}_e\setminus\mathcal{M}$ and that $(X_{g_e}R_{g_e})(y,y')=0$ for the interior term from Green's formula. This means, using $i_{X_{g_e}}d\mu=d\mu_\partial$ at $\partial_+\mathcal{M}$, that

$$-\langle\iota_{\partial_-}^*(R_{g_e}X_{g_e}(\tilde{\omega}\psi)),\omega'\rangle=-\langle(\iota_{\partial_-}\times\iota_{\partial_+})^*R_{g_e},\omega'\otimes\psi|_{\partial_+\mathcal{M}}\omega\rangle.$$

This shows that $\mathcal{S}_g(y,y')\psi(y')=-(\iota_{\partial_-}\times\iota_{\partial_+})^*R_g(y,y')\psi(y')$ as a distribution of $(y,y')\in\partial_-\mathcal{M}\times\partial_+\mathcal{M}$. Since Ω can be chosen with $\delta>0$ arbitrarily small, we obtain the result by choosing $\psi=1$ outside a $\frac{1}{4}\delta$ neighborhood of $\Omega\cap\partial_+\mathcal{M}$ in $\partial_+\mathcal{M}$. ☐

We will also need the following regularity bound.

Lemma 2.8. *Let $g\in C^\infty(M,\otimes_S^2T^*M_+)$ be a metric with strictly convex boundary and hyperbolic trapped set, $\chi\in C^\infty_{\mathrm{c}}(\partial_-\mathcal{M})$, $f\in C^\infty(\partial_+\mathcal{M})$ and $p\in\mathbb{N}$. Then:*

1. *There exists $\beta\gg 0$ large enough that, for all $z\in i\mathbb{R}+\beta$, we have that $\chi e^{-z\ell_g}\mathcal{S}_g f$ extends by 0 on Γ_-^g with an extension belonging to $W^{p+1,\infty}(\partial_-\mathcal{M})$, and also that the weak distributional derivative $(1+\Delta_h)^{(p+1)/2}(\chi e^{-z\ell_g}\mathcal{S}_g f)\in\mathcal{D}'(\partial_-\mathcal{M})$ coincides with the derivative of the $W^{p+1,\infty}(\partial_-\mathcal{M})$-extension.*
2. *The map*
$$C^{p+1}(\partial_+\mathcal{M})\ni f\mapsto e^{-z\ell_g}\mathcal{S}_g f\in W^{p+1,\infty}(\partial_-\mathcal{M})$$
is bounded, and there exists a uniform constant $C>0$ (independent of z) such that
$$\|(1+z)^{-(p+1)}\chi e^{-z\ell_g}\mathcal{S}_g f\|_{W^{p+1,\infty}(\partial_-\mathcal{M})}\le C\|f\|_{C^{p+1}(\partial_+\mathcal{M})}.\tag{2-26}$$
3. *In particular, by the Sobolev embedding $W^{p+1,\infty}(\partial_-\mathcal{M})\hookrightarrow C^p(\partial_-\mathcal{M})$, the function $\chi e^{-z\ell_g}\mathcal{S}_g f$ extends to a C^p-function with C^p-norm bounded by (2-26).*

Proof. The proof is rather similar to that of Lemma 2.5 so we will be more succinct. First, if $\Re(z)>0$ and $f\in C^{p+1}(\partial_+\mathcal{M})$, the function $F_z(y):=e^{-z\ell_g(y)}(\mathcal{S}_g f)(y)$ is C^{p+1} outside Γ_-^g and can be extended by continuity by 0 on Γ_-^g. We compute its derivative on $\partial_-\mathcal{M}\setminus\Gamma_-^g$: if Y is a smooth vector field on $\partial_-\mathcal{M}$, then

$$YF_z(y)=F_z(y)(-zd_y\ell_g(y)Y+df_{S_g(y)}(d\varphi^g_{\ell_g(y)}(y)Y+d_y\ell_g(y)(Y)X_g(S_g(y)))).$$

We can use Lemma 2.3 and the fact that $\|d_y\varphi_t^g\| \le Ce^{C_0|t|}$ for some uniform $C, C_0 > 0$ with respect to t: this gives on $\mathrm{supp}(\chi)$ that

$$|YF_z(y)| \le C(1+|z|)\|Y\|_{C^0}\|f\|_{C^1}e^{(C_0-\beta)\ell_g(y)}$$

for some $C, C_0 > 0$ uniform in y. In particular, if $\beta > C_0$ we obtain that $|Y(\chi F_z)(y)| \le C(1+|z|)\|Y\|_{C^0}$ almost everywhere. Now, we claim that this function is also equal to the weak distributional derivative $Y(\chi F_z) \in H^{-1}(\partial_- SM)$. As in the proof of Lemma 2.5, we need to show that, for each $\psi \in C_c^\infty(\partial_-\mathcal{M})$,

$$\int_{\partial_-\mathcal{M}} \chi e^{-z\gamma\ell_g}\mathcal{S}_g(f)Y(\psi)\,dv_h = \lim_{T\to\infty}\int_{\partial_-\mathcal{M}} \theta(\ell_g/T)Y(e^{-z\gamma\ell_g}\chi\mathcal{S}_g(f))\psi\,dv_h,$$

where $\theta \in C_c^\infty([0,2))$ is equal to 1 in $[0,1]$ and h is a smooth metric on $\partial_-\mathcal{M}$ as in the proof of Lemma 2.5. Since the proof of the equality is exactly the same as in the proof of Lemma 2.5, we do not repeat the argument. This shows that $\chi F_z \in W^{1,\infty}(\partial_-\mathcal{M})$ with bound

$$\|\chi F_z\|_{W^{1,\infty}(\partial_-\mathcal{M})} \le C(1+|z|)\|f\|_{C^1}$$

for some C uniform with respect to z. The bound $\|\chi F_z\|_{C^0(\partial_-\mathcal{M})} \le C(1+|z|)\|f\|_{C^1}$ also follows immediately by Sobolev embedding.

For higher-order derivatives, it suffices to repeat this argument, noting by Lemma 2.3 that there are $C > 0$ and $C_0 > 0$ such that, for $j \le p+1$, we have

$$\|d_y^j\ell_g(y)\| \le Ce^{C_0|t|} \quad \text{and} \quad \|d_y^j\varphi_t^g\| \le Ce^{C_0 t}$$

on $(\partial_-\mathcal{M}\cap\mathrm{supp}(\chi))\setminus\Gamma_-^g$. This means that, taking $\beta > 0$ large enough depending on C_0, the argument explained above works the same way. This proves the claimed result. ☐

Given $\chi \in C_c^\infty(\partial_-\mathcal{M})$, define the following function on $\partial_-\mathcal{M}$:

$$\mathcal{L}_g(z) := \chi e^{-z\ell_g} = (z(R_g(z)\mathbf{1}_{\mathcal{M}})|_{\partial_-\mathcal{M}} + 1)\chi.$$

We will need the following regularity property.

Lemma 2.9. *Let (M, g_0) be a smooth compact Riemannian manifold with hyperbolic trapped set, and let $p \in 2\mathbb{N}$. There exists $\varepsilon > 0$ small enough and $\beta \gg 0$ large enough that the following holds: setting*

$$U_{g_0} := \{g \in C^{p+2}(M, \otimes_S^2 T^*M) \mid \|g-g_0\|_{C^{p+2}} < \varepsilon,\ g|_{T\partial M} = g_0|_{T\partial M}\} \tag{2-27}$$

as in Lemma 2.3, we have that, for $\Re(z) = \beta$, the map

$$\mathcal{L} : U_{g_0}\times\{\Re(z)=\beta\} \ni (g,z) \mapsto \mathcal{L}_g(z) = e^{-z\ell_g}\chi \in L^\infty(\partial_-\mathcal{M}) \subset L^2(\partial_-\mathcal{M})$$

is C^{p-1}-regular. Moreover, there exists a uniform constant $C > 0$ such that, for all $j \le p-1$,

$$\text{for all } h \in C^{p+2}(M, \otimes_S^2 T^*M), \quad \|\partial_g^j\mathcal{L}_g(z)(\otimes^j h)\|_{L^2} \le C(1+|z|)^j\|h\|_{C^{p+2}}^j.$$

Proof. First of all, note by [Guillarmou and Mazzucchelli 2018, Proposition 2.1] that all metrics in a C^2-neighborhood of g_0 have hyperbolic trapped set and strictly convex boundary. Hence $\varepsilon > 0$ is chosen so that this holds. Pick an arbitrary $g_0' \in U_{g_0}$, and let $h \in C^{p+2}(M, \otimes_S^2 T^*M)$ such that $g_t := g_0' + th \in U_{g_0}$ for $t \in (-\delta, 1+\delta)$ for some $\delta > 0$ small. Consider the map

$$F : (-\delta, 1+\delta) \times \partial_-\mathcal{M} \times \{\Re(z) = \beta\} \ni (t, y, z) \mapsto \mathcal{L}_{g_t}(z)(y) = e^{-z\ell_{g_t}(y)}\chi(y),$$

where by convention $e^{-z\ell_{g_t}(y)} := 0$ when $\ell_{g_t}(y) = \infty$. Lemma 2.3 implies that F is C^p in the open set

$$\mathcal{O} := \{(t, y, z) \in (-\delta, 1+\delta) \times \partial_-\mathcal{M} \times \{\Re(z) = \beta\} \mid y \notin \Gamma_-^{g_t}\},$$

and one can write $\partial_t^{j_1}\partial_y^{j_2}\partial_z^{j_3}\mathcal{L}_{g_t}(z)(y) = H(t, y, z, h)(\otimes^{j_1}h)$, where $H(t, y, z, h)$ is a continuous function on $(-\delta, 1+\delta) \times \partial_-\mathcal{M} \times C^{p+2}(M, \otimes_S^2 T^*M)$ with values in j_1-multilinear functions on $C^{p+2}(M, \otimes_S^2 T^*M)$ and satisfying the following: there is $C > 0$ such that, for all $j_1 + j_2 + j_3 \le p$ and all $(t, y, z) \in \mathcal{O}$,

$$|\partial_t^{j_1}\partial_y^{j_2}\partial_z^{j_3}\mathcal{L}_{g_t}(z)(y)| \le C(1+|z|)^{j_1+j_2} e^{(C-\beta)\ell_{g_t}(y)} \|h\|_{C^{p+2}}^{j_1}. \tag{2-28}$$

First, we observe that F is continuous on $(-\delta, 1+\delta) \times \partial_-\mathcal{M} \times \{\Re(z) = \beta\}$. Indeed, if $(t_n, y_n) \to (t, y)$ is a sequence such that $\ell_{g_{t_n}}(y_n) \le T$ for some $T < \infty$, by Proposition 2.4 we deduce that the trajectories $\mathcal{M} \cap \bigcup_{s\ge 0} \varphi_s^{g_{t_n}}(y_n)$ converge to the trajectory $\mathcal{M} \cap \bigcup_{s\ge 0} \varphi_s^{g_t}(y)$ as $n \to \infty$, and therefore $\ell_{g_t}(y) < \infty$, and so the limit point belongs to $\mathcal{O}$. On the other hand, if there is no such T, this also implies that $\ell_{g_{t_n}}(y_n) \to \infty$, and in turn $F(t_n, y_n, z) \to 0$ as $n \to \infty$, and (t, y, z) belongs to the set

$$S := \bigcup_{t \in (-\delta, 1+\delta)} (\{t\} \times \Gamma_-^{g_t} \times \{\Re(z) = \beta\}).$$

Since $\ell_{g_{t_n}}(y_n) \to \infty$ if (t_n, y_n) converge to a point in S as $n \to \infty$, we see from (2-28) that if $\beta \gg 1$ is large enough, the derivative $H(t, y, z, h)$ of F on $\mathcal{O}$ converges to 0 when approaching S, and can thus be extended from $\mathcal{O}$ by 0 as a continuous function on $(-\delta, 1+\delta) \times \partial_-\mathcal{M} \times \{\Re(z) = \beta\} \times C^{p+2}(M, \otimes_S^2 T^*M)$. Next, we are going to show that F is a C^{p-1} map, with $\partial_t^{j_1}\partial_y^{j_2}\partial_z^{j_3}F(t, y, z) = H(t, y, z, h)(\otimes^{j_1}h)$ and with H the continuous extension by 0 on S just discussed, and that there exists $C > 0$ independent of h, t, y, z such that, for all $(t, y, z) \in (-\delta, 1+\delta) \times \partial_-\mathcal{M} \times \{\Re(z) = \beta\}$ and all $j_1 + j_2 + j_3 \le p - 1$,

$$|\partial_t^{j_1}\partial_y^{j_2}\partial_z^{j_3}F(t, y, z)| \le C(1+|z|)^{j_1+j_2} \|h\|_{C^{j_1+1}}^{j_1}. \tag{2-29}$$

This would prove that the Gateaux derivatives of order $p - 1$ are continuous and thus the function $\mathcal{L}$ is C^{p-1} and with the desired bounds on the derivatives.

We proceed in a way similar to the proof of Claim 2.6. We will show that, for each fixed h, the distributional derivatives of F of order $j \le p$ are bounded and coincide with the continuous extension of $H(t, y, z, h)(\otimes^{j_1}h)$ from $\mathcal{O}$ to $W := (-\delta, 1+\delta) \times \partial_-\mathcal{M} \times \{\Re(z) = \beta\}$. First we let Δ be a Laplacian associated to a fixed smooth product metric $\hat{g} := dt^2 + g_- + ds^2$ on $(-\delta, 1+\delta) \times \partial_-\mathcal{M} \times \{\beta + is \mid s \in \mathbb{R}\}$. Let $\psi \in C_c^\infty((-\delta, 1+\delta) \times \partial_-\mathcal{M} \times (\beta + i\mathbb{R}))$. We want to show that, for $2j \le p$,

$$\int_W \chi e^{-z\ell_{g_t}} \Delta^j \psi \, dv_{\hat{g}} = \int_{\mathcal{O}} (\Delta^j F)\psi \, dv_{\hat{g}}.$$

Take $\theta \in C_c^\infty([0,2);[0,1])$ equal to 1 in $[0,1]$, and write the left-hand side as

$$\lim_{T\to\infty} \int_W \theta\left(\frac{\ell_{g_t}}{T}\right) \chi e^{-z\ell_g} \Delta^j \psi \, dv_{\hat{g}} = \lim_{T\to\infty} A_1(T) + A_2(T), \tag{2-30}$$

where

$$A_1(T) := \sum_{k=1}^{2j} \int_W P_k\left(\theta\left(\frac{\ell_{g_t}}{T}\right)\right) Q_{2j-k}(\chi e^{-z\ell_{g_t}}) \psi \, dv_{\hat{g}},$$

with P_k and Q_k some differential operators of order $k \geq 1$ in the variable (t, y, z) and such that $P_k(1) = Q_k(1) = 0$ and

$$A_2(T) := \int_W \theta\left(\frac{\ell_{g_t}}{T}\right) (\Delta^j F) \psi \, dv_{\hat{g}}.$$

In order to show (2-30), it suffices to show that $A_1(T) \to 0$ as $T \to \infty$. The derivatives $D^k_{t,y,z}(\theta(\ell_{g_t}/T))$ of order $k \in [1, 2j]$ are supported in $\{\ell_{g_t} \in [T, 2T]\}$, where we can use the pointwise bound of Lemma 2.3: there exists $C > 0$ such that, for all (t, y, z) with $\ell_{g_t}(y) \in [T, 2T]$,

$$|D^k_{t,y,z}(\theta(\ell_{g_t}(y)/T))| \leq C e^{C\ell_{g_t}(y)} \leq C e^{2CT}.$$

Since all terms in the integrand of A_1 are multiplied by the weight $|e^{-\beta\ell_{g_t}(y)}| \leq e^{-\beta T}$, we see using Lemma 2.3 that

$$A_1(T) = \mathcal{O}(e^{(4C-\beta)T}).$$

Thus if β is chosen large enough we obtain that $A_1(T) \to 0$ as $T \to \infty$. We thus deduce that $F \in W^{p,\infty}_{\mathrm{loc}}(W)$ and by Sobolev embedding that $F \in C^{p-1,\alpha}(W)$ for all $\alpha < 1$. Finally, the bound (2-29) follows from (2-28) by continuity. ☐

3. Symmetric tensors and the normal operator

3A. ***Symmetric tensors.*** In this paragraph, we recall standard facts on symmetric tensors on Riemannian manifolds. We refer to [Gouëzel and Lefeuvre 2021; Guillarmou 2017a; Heil et al. 2016] for further details.

3A1. *Definitions.* Let (M, g) be a smooth connected Riemannian manifold with boundary. Let $m \in \mathbb{Z}_{\geq 0}$. Let $\otimes^m_S T^*M \to M$ be the vector bundle of symmetric tensors over M (for $m = 0$ we just take the trivial line bundle $\mathbb{R} \times M \to M$). We will also write $\otimes^2_S T^*M_+ \subset \otimes^2_S T^*M$ for the open convex subset consisting of positive definite tensors (Riemannian metrics). Since $\otimes^m_S T^*M$ is a subbundle of the vector bundle $\otimes^m T^*M \to M$ of m-tensors over M, it inherits the natural metric $g^{\otimes m}$. Define the pullback operator

$$\pi_m^* : L^2(M, \otimes^m_S T^*M) \to L^2(\mathcal{M}), \quad \pi_m^* f(x, v) := f_x(v^{\otimes m}),$$

where M is equipped with the Riemannian volume, $\otimes^m_S T^*M$ with the metric $g^{\otimes m}$ and $\mathcal{M}$ with the Liouville measure μ. We denote by π_{m*} the adjoint of π_m^* with respect to these scalar products and volume forms.

The symmetric covariant derivative

$$D_g : C^\infty(M, \otimes_S^m T^*M) \to C^\infty(M, \otimes_S^{m+1} T^*M)$$

is defined as $D_g := \sigma \circ \nabla^g$, where ∇^g is the Levi-Civita connection induced by g and $\sigma : \otimes^m T^*M \to \otimes_S^m T^*M$ is the symmetrization operator defined as:

$$\sigma(\eta_1 \otimes \cdots \otimes \eta_m) := \frac{1}{m!} \sum_{\pi \in \mathfrak{S}_m} \eta_{\pi(1)} \otimes \cdots \otimes \eta_{\pi(m)},$$

where $\eta_1, \dots, \eta_m \in T^*M$. The operator D_g is of *gradient type*, namely it has injective principal symbol. Moreover, it is injective when m is odd and has kernel given by $\mathbb{R}g^{\otimes m/2}$ for even m. It satisfies the relation

$$X_g \pi_m^* = \pi_{m+1}^* D_g, \tag{3-1}$$

where we recall that X_g is the geodesic vector field of g. We let

$$D_g^* : C^\infty(M, \otimes_S^{m+1} T^*M) \to C^\infty(M, \otimes_S^m T^*M)$$

be the formal adjoint of D_g, which is nothing more than the divergence $D_g^* u = -\mathrm{Tr}(\nabla^g u)$, where $\mathrm{Tr}(\,\cdot\,)$ is the trace operator.

For $m \geq 1$, $k \geq 0$ and $\alpha \in (0, 1)$, there exists a unique decomposition

$$C^{k,\alpha}(M, \otimes_S^m T^*M) = D_g(C_0^{k+1,\alpha}(M, \otimes_S^{m-1} T^*M)) \oplus^\perp \ker D_g^*|_{C^{k,\alpha}(M, \otimes_S^m T^*M)}, \tag{3-2}$$

where $C_0^{k+1,\alpha}(M, \otimes_S^{m-1} T^*M)$ denotes the space of tensors of Hölder–Zygmund regularity $k+1+\alpha$, vanishing on the boundary, and the sum is orthogonal with respect to the L^2-scalar product. The decomposition (3-2) also holds in the scale of Sobolev spaces $H^s(M, \otimes_S^m T^*M)$ for $s \geq 0$. We call *potential tensors* the tensors in $\operatorname{ran} D_g$ and *solenoidal tensors* (or divergence free tensors) those in $\ker D_g^*$.

Lemma 3.1. *For $m \geq 1$, there exist bounded projections*

$$\pi_{\ker D_g^*} : L^2(M, \otimes_S^m T^*M) \to L^2(M, \otimes_S^m T^*M) \cap \ker D_g^*,$$
$$\pi_{\operatorname{ran} D_g} : L^2(M, \otimes_S^m T^*M) \to L^2(M, \otimes_S^m T^*M) \cap \operatorname{ran} D_g|_{H_0^1},$$

which are pseudodifferential operators of order 0 *on M°. Moreover, for all $f \in L^2(M, \otimes_S^m T^*M)$, there is a unique $h \in H_0^1(M, \otimes_S^{m-1} T^*M)$ and $f_s \in \ker D_g^* \cap L^2$ such that $f = D_g h + f_s$, and it is given by $\pi_{\ker D_g^*} f = f_s$ and $\pi_{\operatorname{ran} D_g} f = D_g h$.*

Proof. The Dirichlet Laplacian $D_g^* D_g : H^2(M, \otimes_S^m T^*M) \cap H_0^1(M, \otimes_S^m T^*M) \to L^2(M)$ is an elliptic self-adjoint operator which is invertible since there are no symmetric Killing tensors vanishing at ∂M by [Dairbekov and Sharafutdinov 2010]. Its inverse $(D_g^* D_g)^{-1} : H^{-1}(M, \otimes_S^m T^*M) \to H_0^1(M, \otimes_S^m T^*M)$, when restricted to $C_c^\infty(M^\circ)$, is a pseudodifferential operator of order -2 on M° by standard elliptic microlocal analysis. We then set

$$\pi_{\operatorname{ran} D_g} := D_g (D_g^* D_g)^{-1} D_g^*, \quad \pi_{\ker D_g^*} =: \mathrm{Id} - \pi_{\operatorname{ran} D_g}.$$

By construction, they satisfy the desired properties. □

3A2. *X-ray transform of tensors.* We now further assume that the metric g is of Anosov type in the sense of Definition 1.3. We introduce the X-ray transform of symmetric m-tensors.

Definition 3.2. The X-ray transform on the space of symmetric m-tensors is defined by $I_m^g := I^g \circ \pi_m^*$, where I_m^g is a map from $C^\infty(M, \otimes_S^m T^*M)$ to $L^2(\partial_-\mathcal{M})$.

It is clear from (3-1) that the following inclusion holds:

$$D_g(C_0^{k+1,\alpha}(M, \otimes_S^{m-1} T^*M)) \subset \ker I_m^g. \tag{3-3}$$

Definition 3.3. The X-ray transform I_m^g is said to be *solenoidal injective* on $C^{k,\alpha}(M, \otimes_S^m T^*M)$ if (3-3) is an equality.

In other words, I_m^g is solenoidal injective if it is injective in restriction to solenoidal tensors, i.e., on the second factor of the decomposition (3-2). When (M, g) is of Anosov type, solenoidal injectivity of the X-ray transform has been proved so far in the following cases:

(1) In dimensions $n \geq 2$, when g is of Anosov type with nonpositive sectional curvature, see [Guillarmou 2017b].

(2) On all surfaces of Anosov type; see [Lefeuvre 2019a].

(3) In dimensions $n \geq 2$, on all real analytic manifolds of Anosov type, injectivity of I_2^g is proved in [Guedes-Bonthonneau et al. 2024].

We conjecture that the following holds.

Conjecture 3.4 (solenoidal injectivity of the X-ray transform on manifolds of Anosov type). *Let (M, g) be a smooth Riemannian manifold of Anosov type in the sense of Definition 1.3. Then I_m^g is solenoidal injective.*

Eventually, we conclude this paragraph by the following variational formula which relates the length map and the X-ray transform on 2-tensors.

Lemma 3.5. *Let (M, g_0) be a compact Riemannian manifold with strictly convex boundary and hyperbolic trapped set. Let $(x, v) \in \partial_-\mathcal{M} \setminus \Gamma_-^{g_0}$. Let $(g_t)_{t\in(-1,1)}$ be a smooth family of metrics on M with $g_t|_{t=0} = g_0$, and write $h := \partial_t g_t|_{t=0}$. Then $t \mapsto \ell_{g_t}(x, v)$ is C^2-regular for small t, and*

$$\partial_t \ell_{g_t}(x, v)|_{t=0} = \tfrac{1}{2} I_2^{g_0} h(x, v) + \alpha_{S_{g_0}(x,v)}(\partial_t S_{g_t}(x, v)|_{t=0}),$$

where we recall that α is the Liouville 1-form.

Proof. First, we use the fact that, for t small enough, g_t must have hyperbolic trapped set by [Guillarmou and Mazzucchelli 2018, Proposition 2.1]. Let $c_0(s)$ be a geodesic for g_0 parametrize by arc-length, and $s \mapsto c_t(s)$ for $t \in (-1, 1)$ be a C^1 family of curves for $s \in [0, \ell_{g_0}(c_0)]$. Let $Y(s) := \partial_t c_t(s)|_{t=0}$ be the vector field along $c_0(s)$ determined by the family $(c_t)_{t\in(-1,1)}$. Define $\dot{g} := \partial_t g_t|_{t=0}$, and denote by ∇ the Levi-Civita derivative defined by g_0.

By definition, $\ell_{g_t}(c_t)=\int_0^{\ell_{g_0}(c_0)}\sqrt{g_t(\partial_s c_t(s),\partial_s c_t(s))}\,\mathrm{d}s$, so differentiating we obtain

$$\begin{aligned}\partial_t(\ell_{g_t}(c_t))|_{t=0}&=\frac{1}{2}\int_0^{\ell_{g_0}(c_0)}\frac{2g_0(\nabla_{\partial_t}\partial_s c_t(s)|_{t=0},\partial_s c_0(s))+\dot g(\partial_s c_0(s),\partial_s c_0(s))}{|\partial_s c_0(s)|_{g_0}}\,\mathrm{d}s\\&=\frac{1}{2}\int_0^{\ell_{g_0}(c_0)}\dot g(\partial_s c_0(s),\partial_s c_0(s))\,\mathrm{d}s\\&\qquad+\int_0^{\ell_{g_0}(c_0)}\Big(\partial_s(g_0(\partial_t c_t(s),\partial_s c_t(s)))|_{t=0}-g_0(\partial_t c_t(s)|_{t=0},\underbrace{\nabla_{\partial_s}\partial_s c_0(s)}_{=0})\Big)\,\mathrm{d}s\\&=\frac{1}{2}\int_0^{\ell_{g_0}(c_0)}\dot g(\partial_s c(s),\partial_s c(s))\,\mathrm{d}s+g_0(Y(s),\partial_s c_0(s))|_0^{\ell_{g_0}(c_0)}.\end{aligned}\tag{3-4}$$

Here we used that $|\partial_s c_0(s)|_g=1$ since the parametrization of c_0 is by arc-length, and that $\nabla_{\partial_t}\partial_s=\nabla_{\partial_s}\partial_t$ (this is seen on the pullback bundle c^*TM of the tangent bundle by the family c since the connection is torsion-free and $[\partial_t,\partial_s]=0$). In the third line, we used the compatibility of g_0 with ∇, and the last term is zero since $\nabla_{\partial_s}\partial_s c_0(s)=0$ is the geodesic equation.

If $(x,v)\in\partial_-\mathcal{M}\setminus\Gamma_-^{g_0}$, then, for t small enough, $(x,v)\notin\Gamma_-^{g_t}$ by Proposition 2.4 and $\ell_{g_t}(x,v)$ is C^2 near $t=0$ by Lemma 2.3. Then we get, from (3-4),

$$\begin{aligned}\partial_t\ell_{g_t}(x,v)|_{t=0}&=\tfrac{1}{2}I_2^{g_0}(\dot g)(x,v)+g_0\Bigg(\underbrace{\partial_t\bigg(\pi\circ S_{g_t}\bigg(x,\frac{v}{|v|_{g_t}}\bigg)\bigg)\bigg|_{t=0}}_{=d\pi\circ\partial_t S_{g_t}(x,v)|_{t=0}},S_{g_0}(x,v)\Bigg)\\&=\tfrac{1}{2}I_2^{g_0}(\dot g)(x,v)+\alpha_{S_{g_0}(x,v)}(\partial_t S_{g_t}(x,v)|_{t=0}).\end{aligned}$$ ☐

3A3. *Solenoidal gauge.* The following lemma asserts that any metric in a neighborhood of a fixed metric g_0 can be put in a *solenoidal gauge*.

Lemma 3.6. *Let (M,g_0) be a smooth Riemannian manifold with metric of Anosov type, and let $k\geq 2$ and $\alpha\in(0,1)$. There exists $C,\delta>0$ such that the following holds: for all metrics g such that $\|g-g_0\|_{C^{k,\alpha}}<\delta$, there exists a $C^{k+1,\alpha}$-diffeomorphism ψ, with $\psi|_{\partial M}=\mathrm{Id}$, such that ψ^*g is divergence-free with respect to g_0, namely $D_{g_0}^*(\psi^*g-g_0)=0$, and $\|\psi^*g-g_0\|_{C^{k,\alpha}}\leq C\|g-g_0\|_{C^{k,\alpha}}$.*

Proof. The proof is contained in [Croke et al. 2000, Lemma 2.2]. ☐

3B. *Normal operator.*

Let (M,g) be a smooth Riemannian manifold with metric g of Anosov type. The normal operator on m-symmetric tensors is defined by

$$\Pi_m^g:=(I_m^g)^*I_m^g.$$

It enjoys strong analytic properties, as proved in [Guillarmou 2017b]:

Proposition 3.7. *The operator $\Pi_m^g\in\Psi^{-1}(M^\circ,\otimes_S^m T^*M^\circ)$ is a pseudodifferential operator of order -1 on M°. It is elliptic on solenoidal tensors, in the sense that there exists pseudodifferential operators Q, K_L, K_R on M° of orders 1, $-\infty$, $-\infty$, respectively, such that*

$$Q\Pi_m^g=\pi_{\ker D_g^*}+K_L,\qquad \Pi_m^g Q=\pi_{\ker D_g^*}+K_R,$$

and the equalities hold when applied to all distributions $f \in \mathcal{E}'(M^\circ, \otimes_S^m T^*M^\circ)$ *with compact support in* M°*. The operator* Q *can be taken to be properly supported in* M°*. Moreover,* Π_m^g *is solenoidal injective, i.e., injective in restriction to* $\ker D_g^*$*, if and only if the X-ray transform* I_m^g *is solenoidal injective.*

We now prove an elliptic estimate for the operator Π_m^g. Recall from Section 2C that $(M_e, g_e) \supset (M, g)$ is a Riemannian extension of the manifold (M, g), which is also of Anosov type in the sense of Definition 1.3. We will denote by

$$E_0 : L^2(M, \otimes_S^m T^*M) \to L^2(M_e, \otimes_S^m T^*M_e)$$

the operator of extension by 0.

Proposition 3.8. *Let* (M, g) *be a manifold of Anosov type, and further assume that* I_2^g *is solenoidal injective. Let* (M_e, g_e) *be an extension of Anosov type of* (M, g)*. Then, there exists* $C > 0$ *such that, for all* $f \in L^2(M, \otimes_S^2 T^*M) \cap \ker D_g^*$,

$$\|f\|_{L^2(M)} \le C \|\Pi_2^{g_e} E_0 f\|_{H^1(M_e)}.$$

Proof. It will be convenient in the proof to consider a second extension of Anosov type $(M_{ee}, g_{ee}) \supset (M_e, g_e)$ and to work on it. The argument follows [Stefanov and Uhlmann 2004]. The operator $\Pi_2^{g_{ee}}$ is a (not properly supported) pseudodifferential operator of order -1 on $M_{ee}{}^\circ$ which is elliptic on solenoidal tensors. By Proposition 3.7, we can construct a properly supported pseudodifferential operator $Q \in \Psi^1(M_{ee}^\circ, \otimes_S^2 T^*M_{ee}^\circ)$ such that

$$Q\Pi_2^{g_{ee}} = \pi_{\ker D_{g_{ee}}^*} + K,$$

where $K \in \Psi^{-\infty}(M_{ee}^\circ)$ is smoothing. We let $\iota : M_e \hookrightarrow M_{ee}$ be the embedding. Observe that, taking a cutoff function $\chi \in C_c^\infty(M_e^\circ)$ with value 1 in an open neighborhood of M, we get

$$\begin{aligned}\iota^* Q\Pi_2^{g_{ee}} E_0 &= \iota^*\pi_{\ker D_{g_{ee}}^*} E_0 + \iota^* K E_0 \\ &= \pi_{\ker D_{g_e}^*} E_0 + \chi(\iota^*\pi_{\ker D_{g_{ee}}^*} - \pi_{\ker D_{g_e}^*})\chi E_0 + \iota^* K E_0 + (1-\chi)(\iota^*\pi_{\ker D_{g_{ee}}^*} - \pi_{\ker D_{g_e}^*})E_0.\end{aligned}$$

By the pseudolocality of pseudodifferential operators (they preserve the singular support of distributions), the term $(\iota^*\pi_{\ker D_{g_{ee}}^*} - \pi_{\ker D_{g_e}^*})E_0$ maps continuously L^2 sections to sections that are smooth outside M, and thus

$$(1-\chi)(\iota^*\pi_{\ker D_{g_{ee}}^*} - \pi_{\ker D_{g_e}^*})E_0 : L^2(M, \otimes_S^2 T^*M) \to L^2(M_e, \otimes_S^2 T^*M_e)$$

is a compact operator. As for the term $\chi(\iota^*\pi_{\ker D_{g_{ee}}^*} - \pi_{\ker D_{g_e}^*})\chi$, we observe that it has Schwartz kernel supported in the interior of $M_{ee} \times M_{ee}$. It is a priori a pseudodifferential operator of order 0, but its principal symbol vanishes (see Lemma 3.1) and thus it is a pseudodifferential operator of order -1, i.e., it is compact as a map $L^2(M_e) \to L^2(M_e)$. (We now drop the notation of the vector bundle in the functional spaces in order to avoid repetition.) As a consequence, we see that, up to changing the compact remainder,

$$\iota^* Q\Pi_2^{g_{ee}} E_0 = \pi_{\ker D_{g_e}^*} E_0 + K, \tag{3-5}$$

where K is compact as a map $L^2(M) \to L^2(M_e)$.

Given $f \in L^2(M) \cap \ker D_g^*$, by Lemma 3.1 we may write $E_0 f = D_g p + h$, where $p \in H^1(M_e, T^*M_e)$ and $p|_{\partial M_e} = 0$, $h = \pi_{\ker D_{g_e}^*} E_0 f$. Now, using (3-5), there is $C > 0$ independent of f such that

$$\begin{aligned} \|f\|_{L^2(M)} = \|E_0 f\|_{L^2(M_e)} &\leq \|\pi_{\ker D_{g_e}^*} E_0 f\|_{L^2(M_e)} + \|(\mathrm{Id} - \pi_{\ker D_{g_e}^*}) E_0 f\|_{L^2(M_e)} \\ &\leq \|\iota^* Q \Pi_2^{g_{ee}} E_0 f\|_{L^2(M_e)} + \|Kf\|_{L^2(M_e)} + \|D_{g_e} p\|_{L^2(M_e)} \\ &\leq C(\|\Pi_2^{g_{ee}} E_0 f\|_{H^1(M_{ee})} + \|Kf\|_{L^2(M_e)} + \|D_{g_e} p\|_{L^2(M_e)}). \end{aligned} \tag{3-6}$$

It remains now to bound the potential term $D_{g_e} p$. We have

$$\|D_{g_e} p\|_{L^2(M_e)} \leq \|D_{g_e} p\|_{L^2(\Omega)} + \|D_g p\|_{L^2(M)}, \tag{3-7}$$

where $\Omega := M_e \setminus M^\circ$. We observe that, on Ω, $D_g p = -h = -\pi_{\ker D_{g_e}^*} E_0 f$. Hence, using (3-5), we get

$$\|D_{g_e} p\|_{L^2(\Omega)} \leq \|\iota^* Q \Pi_2^{g_{ee}} E_0 f\|_{L^2(\Omega)} + \|Kf\|_{L^2(\Omega)}. \tag{3-8}$$

The boundary $\partial\Omega = \partial M_e \sqcup \partial M$ splits into two components. We define ν to be the outward-pointing unit normal vector to $\partial\Omega$ and $j := p|_{\partial M}$. In M, we have $D_g^* f = 0 = D_g^* h + D_g^* D_g p = \Delta_D p$, where $\Delta_D := D_g^* D_g$ is the (symmetric) Laplacian on 1-forms. Hence, in M, p satisfies the elliptic system $\Delta_D p = 0$, $p|_{\partial M} = j \in H^{1/2}(\partial M, \otimes_S^2 T^*M)$ (by the trace theorem), so by standard elliptic estimates [Taylor 2011, Chapter 5, Proposition 1.7], we get $\|p\|_{H^1(M)} \lesssim \|j\|_{H^{1/2}(\partial M)}$. Observe that the H^1-norm in M can be defined by $\|p\|_{H^1(M)} := \|p\|_{L^2(M)} + \|D_g p\|_{L^2(M)}$. As a consequence, using the boundedness of the trace map $H^1(\Omega) \to H^{1/2}(\partial\Omega)$, we get (for some C uniform that can change from line to line)

$$\begin{aligned} \|D_g p\|_{L^2(M)} \leq C\|p\|_{H^1(M)} \leq C\|j\|_{H^{1/2}(\partial M)} \leq C\|p\|_{H^1(\Omega)} &\leq C(\|p\|_{L^2(\Omega)} + \|D_{g_e} p\|_{L^2(\Omega)}) \\ &\leq C(\|p\|_{L^2(\Omega)} + \|\iota^* Q \Pi_2^{g_{ee}} E_0 f\|_{L^2(\Omega)} + \|Kf\|_{L^2(\Omega)}) \end{aligned} \tag{3-9}$$

by (3-8). It remains to bound $\|p\|_{L^2(\Omega)}$. Recall that $D_g p = \pi_{\operatorname{ran} D_g} E_0 f$, and by pseudolocality of the pseudodifferential operator $\pi_{\operatorname{ran} D_g}$ (see Lemma 3.1) we get that $p|_\Omega$ belongs to $C^\infty(\Omega, T^*\Omega)$. For any point $(x, v) \in S\Omega$, there is a uniformly bounded time $\tau(x, v)$ (possibly negative) such that $\pi(\varphi_{\tau(x,v)}(x, v)) \in \partial M_e$, and using that p vanishes on ∂M_e, we can thus write, using (3-1),

$$|\pi_1^* p(x, v)| = \left| \int_0^{\tau(x,v)} (X_{g_e} \pi_1^* p)(\varphi_t^{g_e}(x, v))\, \mathrm{d}t \right| = \left| \int_0^{\tau(x,v)} (\pi_2^* D_{g_e} p)(\varphi_t^{g_e}(x, v))\, \mathrm{d}t \right|.$$

This equality implies that $\|p\|_{L^2(\Omega)} \leq C\|D_{g_e} p\|_{L^2(\Omega)}$. Hence, combining (3-6) with (3-7)–(3-9), we get that, for all $f \in L^2(M, \otimes_S^2 T^*M) \cap \ker D_g^*$, the following inequality holds for some uniform $C > 0$:

$$\|f\|_{L^2(M)} \leq C(\|\Pi_2^{g_{ee}} E_0 f\|_{H^1(M_{ee})} + \|Kf\|_{L^2(M_e)}),$$

where $K : L^2(M, \otimes_S^2 T^*M) \to L^2(M_e, \otimes_S^2 T^*M_e)$ is compact. The solenoidal injectivity of Π_2^g on M implies that $\Pi_2^{g_{ee}} E_0$ is also solenoidal injective (see [Lefeuvre 2019a, Proof of Lemma 2.3] for instance) and thus by standard arguments, one can remove the compact remainder K from the previous inequality. Hence there is uniform $C > 0$ such that

$$\|f\|_{L^2(M)} \leq C\|\Pi_2^{g_{ee}} E_0 f\|_{H^1(M_{ee})}.$$

The claimed estimate is proved by observing that in the above proof one can replace (M_{ee}, g_{ee}) by (M_e, g_e), and (M_e, g_e) by a slightly smaller manifold (M'_e, g'_e) of Anosov type containing (M, g). ☐

4. Local lens rigidity, proof of the main result

In this section, we prove the main Theorem 1.8.

4A. *Key estimate.*

The goal of this paragraph is to show the following key estimate.

Proposition 4.1. *Let g_0 be of Anosov type. There exist $C, \varepsilon, \mu, N > 0$ such that, for all smooth metrics g such that $g|_{\partial M} = g_0|_{\partial M}$, $\|g - g_0\|_{C^N} < \varepsilon$, and $(\ell_g, S_g) = (\ell_{g_0}, S_{g_0})$, we have*

$$\|I_2^{g_0}(g - g_0)\|_{L^2} \le C\|g - g_0\|_{C^N}^{1+\mu}.$$

In order to prove Proposition 4.1, we are still missing one ingredient, namely, the following C^2-regularity of the scattering operator.

Proposition 4.2. *Let (M, g_0) be a smooth compact Riemannian manifold with strictly convex boundary and hyperbolic trapped set. Let $\chi \in C_c^\infty(\partial_-\mathcal{M}, [0, 1])$ be a smooth cutoff function. Then, for each $\omega \in C^\infty(\partial_+\mathcal{M})$, the map*

$$C^\infty(M, \otimes_S^2 T^*M) \ni g \mapsto \chi\mathcal{S}_g(\omega) \in H^{-6}(\partial_-\mathcal{M})$$

is C^2-regular near g_0. As a consequence, there exists $C, N > 0$ large enough and $\delta > 0$ such that, for all $g \in C^\infty(M, \otimes_S^2 T^*M)$ with $\|g - g_0\|_{C^N} \le \delta$, the following holds:

$$\|\chi\mathcal{S}_g(\omega) - \chi\mathcal{S}_{g_0}(\omega) + \chi\partial_g\mathcal{S}_g(\omega)|_{g=g_0} \cdot (g - g_0)\|_{H^{-6}(\partial_-\mathcal{M})} \le C\|g - g_0\|^2_{C^N(M, \otimes_S^2 T^*M)}. \tag{4-1}$$

Since this result is quite technical, its proof is postponed to Section 5. In the following, we will write $h := g - g_0$. Using a complex interpolation argument, Proposition 4.1 is actually a direct consequence of the following technical lemma, which gives weighted estimates on the X-ray transform of $g - g_0$.

Lemma 4.3. *There exist $C, \varepsilon, \delta, \beta, N > 0$ such that, for all smooth metrics g such that $g|_{\partial M} = g_0|_{\partial M}$, $\|g - g_0\|_{C^N} < \varepsilon$, and $(\ell_g, S_g) = (\ell_{g_0}, S_{g_0})$, we have, for $h = g - g_0$,*

$$\|(1 + z)^{-7} e^{-z\ell_{g_0}} I_2^{g_0} h\|_{H^{-6}(\partial_-\mathcal{M})} \le \begin{cases} C\|h\|_{C^N} & \text{for all } z \in i\mathbb{R} - \delta, \\ C\|h\|^2_{C^N} & \text{for all } z \in i\mathbb{R} + \beta. \end{cases} \tag{4-2}$$

We now show that Lemma 4.3 implies Proposition 4.1. The rest of Section 4A is devoted to the proof of Lemma 4.3.

Proof of Proposition 4.1. By the Hadamard three-line theorem applied to the function

$$z \mapsto (1 + z)^{-7} e^{-z\ell_{g_0}} I_2^{g_0}(h)$$

(which is bounded in $\Re(z) \in [-\delta, \beta]$ with values in $L^2(\partial_-\mathcal{M}) \subset H^{-6}(\partial_-\mathcal{M})$), Lemma 4.3 implies that

$$\|I_2^{g_0} h\|_{H^{-6}(\partial_-\mathcal{M})} \le C\|h\|^{1+\mu}_{C^N(M)}$$

for some constants $C, \mu > 0$ independent of h (note that μ depends on δ and β). By Lemma 2.5, there is $C > 0$ and $s > 0$ depending on g_0 such that (for $N \ge 2$)

$$\|I_2^{g_0} h\|_{H^s(\partial_-\mathcal{M})} \le C\|h\|_{C^N(M)}.$$

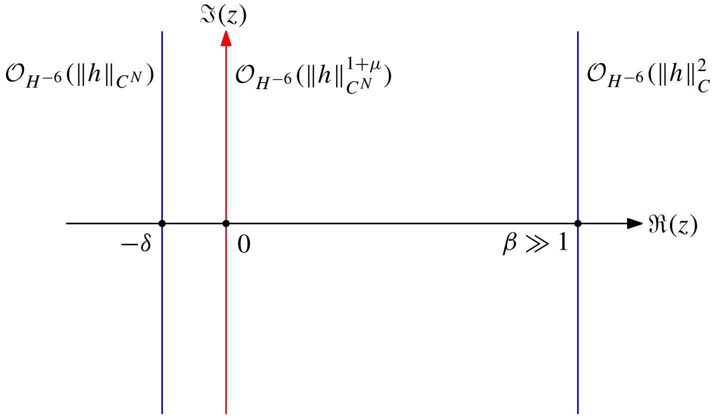

Figure 3. Estimates on $f(z) = e^{-z\ell_{g_0}} I_2^{g_0} h$ in (4-2). For z on the left blue line we have a "volume estimate" of $f(z)$, while for z on the right blue line we have a "microlocal estimate" of $f(z)$. For z on the middle red line, we have the interpolation estimate obtained in Proposition 4.1.

Interpolating $L^2(\partial_-\mathcal{M})$ between $H^{-6}(\partial_-\mathcal{M})$ and $H^s(\partial_-\mathcal{M})$, we deduce that there exists $\mu' > 0$ and $C > 0$ such that

$$\|I_2^{g_0} h\|_{L^2(\partial_-\mathcal{M})} \le C \|h\|_{C^N(M)}^{1+\mu'}. \qquad \square$$

We now start with the proof of Lemma 4.3. See Figure 3: on $\{\Re(z) = -\delta\}$ the bound will follow from an estimate on the volume of long trajectories, while the estimate on the line $\{\Re(z) = \beta\}$ may be thought of as a "microlocal estimate" since it crucially relies on the Taylor expansion of $g \mapsto \mathcal{S}_g$ obtained in Proposition 4.2.

The first bound in (4-2) for $z \in i\mathbb{R} - \delta$ follows directly from the following stronger bound.

Lemma 4.4. *There exists $\delta > 0$ small enough and $C > 0$ (depending on δ) such that, for all $h \in C^0(M, \otimes_S^2 T^*M)$,*

$$\|e^{\delta\ell_{g_0}} I_2^{g_0} h\|_{L^2(\partial_-\mathcal{M})} \le C \|h\|_{C^0(M)}.$$

Proof. For $y \notin \Gamma_-^{g_0}$, we have $|I_2^{g_0} h(y)| \le \|h\|_{C^0} |\ell_{g_0}(y)|$. Thus

$$\|e^{\delta\ell_{g_0}} I_2^{g_0} h\|_{L^2(\partial_-\mathcal{M})} \le \|e^{\delta\ell_{g_0}} \ell_{g_0}\|_{L^2(\partial_-\mathcal{M})} \|h\|_{C^0},$$

which gives the result by (2-10) if $\delta < \frac{1}{2} Q_{g_0}$. $\square$

We now study the second bound in (4-2). Let $\chi \in C_c^\infty(\partial_-\mathcal{M}, [0, 1])$ be a smooth cutoff function. First of all, near the boundary, we have the following:

Lemma 4.5. *There exist $C, \varepsilon > 0$ and $\chi \in C_c^\infty(\partial_-\mathcal{M}, [0, 1])$ such that $1 - \chi^2$ is supported near the boundary of $\partial_-\mathcal{M}$, such that if $\|g - g_0\|_{C^N} < \varepsilon$ and $(\ell_g, S_g) = (\ell_{g_0}, S_{g_0})$, then*

$$\|(1 - \chi^2) I_2^{g_0} h\|_{L^\infty(\partial_-\mathcal{M})} \le C \|h\|_{C^1}^2.$$

Proof. This follows from [Stefanov and Uhlmann 2004, Section 9] as we have the following Taylor expansion for $x, x' \in \partial M$ close enough:

$$d_g(x,x') = d_{g_0}(x,x') + \tfrac{1}{2} I_2^{g_0} h(x,x') + T_g(x,x'),$$

with the bound $|T_g(x,x')| \le C\|h\|_{C^1}^2 d_{g_0}(x,x')$, where $C > 0$ is a uniform constant depending only on g_0. Since the metrics have the same lens data, they also have the same boundary distance function for $x, x' \in \partial M$ close enough, that is, $d_g(x,x') = d_{g_0}(x,x')$, which easily implies the claimed estimate when $1-\chi^2$ is taken to have support near the boundary of $\partial_-\mathcal{M}$ (i.e., close to short geodesics). $\square$

Using the continuous embeddings $L^\infty(\partial_-\mathcal{M}) \hookrightarrow L^2(\partial_-\mathcal{M}) \hookrightarrow H^{-6}(\partial_-\mathcal{M})$, from Lemma 4.5 we deduce that

$$\|(1+z)^{-7} e^{-z\ell_{g_0}} (1-\chi^2) I_2^{g_0} h\|_{H^{-6}(\partial_-\mathcal{M})} \le C\|h\|_{C^N}^2 \tag{4-3}$$

for all $z \in i\mathbb{R} + \beta$. It thus remains to prove the following estimate to deduce the second bound of (4-2).

Lemma 4.6. *There exist $C, \varepsilon, \beta, N > 0$ such that if $\|g-g_0\|_{C^N} < \varepsilon$ and $(\ell_g, S_g) = (\ell_{g_0}, S_{g_0})$, then, for $h := g - g_0$ and for all $z \in i\mathbb{R} + \beta$,*

$$\|(1+z)^{-7} e^{-z\ell_{g_0}} \chi^2 I_2^{g_0} h\|_{H^{-6}(\partial_-\mathcal{M})} \le C\|h\|_{C^N}^2.$$

Proof. We let $\iota_{\partial_-} : \partial_-\mathcal{M} \to \mathcal{M}$ be the inclusion map. For $\beta > 0$, we consider the space

$$E_\beta := C_b^0(\beta + i\mathbb{R}, L^2(\partial_-\mathcal{M})), \tag{4-4}$$

where C_b^0 denotes the vector space of bounded continuous functions, equipped with the L^∞ norm. It is a Banach space when equipped with the norm

$$\|F\|_{E_\beta} := \sup_{z \in \beta + i\mathbb{R}} \|F(z)\|_{L^2(\partial_-\mathcal{M})}.$$

Then, for $z \in \mathbb{C}$ with $\Re(z) = \beta$ large (it will be adjusted later), we define for U_{g_0} the neighborhood of g_0 introduced in (2-3) (with $p = N-2$):

$$\mathcal{F} : U_{g_0} \ni g \mapsto \mathcal{F}(g)(z) := (1+z)^{-7} \chi^2 \frac{(1-e^{-z\ell_g})}{z} \in E_\beta, \tag{4-5}$$

where the value at $z = 0$ is set to be $\chi^2 \ell_g$.

First, the function $\mathcal{F}$ is C^2 by Lemma 2.9 by taking $N \ge 5$. We compute its Taylor expansion in the space E_β: for some N large enough, g close enough to g_0, and $h := g - g_0$,

$$\begin{aligned} \mathcal{F}(g)(z) - \mathcal{F}(g_0)(z) &= \frac{\chi^2 e^{-z\ell_{g_0}}}{(1+z)^7} (\partial_g \ell_g)|_{g=g_0}.h + \mathcal{O}_{L^2(\partial_-\mathcal{M})}(\|h\|_{C^N}^2) \\ &= \frac{\chi^2 e^{-z\ell_{g_0}}}{(1+z)^7} \Big(\frac{1}{2} I_2^{g_0}(h) + \alpha_{S_{g_0}(\cdot)}(\partial_g S_g(\cdot)|_{g=g_0}.h) \Big) + \mathcal{O}_{L^2(\partial_-\mathcal{M})}(\|h\|_{C^N}^2), \end{aligned} \tag{4-6}$$

and the remainder is bounded uniformly in z (by Lemma 2.9 again), where we use Lemma 3.5 in the second line (recall α is the Liouville 1-form). If $\ell_g = \ell_{g_0}$, we obtain in particular $\mathcal{F}(g)(z) - \mathcal{F}(g_0)(z) = 0$, thus

$$\sup_{z\in\beta+i\mathbb{R}}\left\| \frac{\chi^2 e^{-z\ell_{g_0}}}{(1+z)^7}\Big(\frac{1}{2}I_2^{g_0}(h)+\alpha_{S_{g_0}(\,\cdot\,)}(\partial_g S_g(\,\cdot\,)|_{g=g_0}.h)\Big)\right\|_{L^2(\partial_-\mathcal{M})} \le C\|h\|_{C^N}^2. \tag{4-7}$$

Note that, for $\Re(z)=\beta>0$, as a consequence of (2-19), we have $\chi^2 e^{-z\ell_{g_0}} I_2^{g_0}(h)\in L^2(\partial_-\mathcal{M})$, thus, since by Lemma 2.9 we know that $\partial_g\mathcal{F}(g)(z)|_{g=g_0}.h\in L^2(\partial_-\mathcal{M})$ if β is large enough, we obtain that

$$\chi^2\alpha_{S_{g_0}(\,\cdot\,)}(\partial_g S_g(\,\cdot\,)|_{g=g_0}.h)e^{-z\ell_{g_0}}\in L^2(\partial_-\mathcal{M}).$$

We now claim the following lemma, the proof of which is deferred to the following paragraph.

Lemma 4.7. *There exist $C,\varepsilon,\beta,N>0$ such that if $\|g-g_0\|_{C^N}<\varepsilon$ and $(\ell_g,S_g)=(\ell_{g_0},S_{g_0})$, then, for all $z\in i\mathbb{R}+\beta$ and $h=g-g_0$,*

$$\|(1+z)^{-7}\chi^2\alpha_{S_{g_0}(\,\cdot\,)}(\partial_g S_{g_0}(\,\cdot\,)|_{g=g_0}.h)e^{-z\ell_{g_0}}\|_{H^{-6}(\partial_-\mathcal{M})}\le C\|h\|_{C^N}^2.$$

Using (4-7) and Lemma 4.7, we deduce that, for all $\Re(z)=\beta$ with $\beta,N>0$ large enough,

$$\begin{aligned}\sup_{z\in\beta+i\mathbb{R}} & |1+z|^{-7}\|\chi^2 I_2^{g_0}(h)e^{-z\ell_{g_0}}\|_{H^{-6}(\partial_-\mathcal{M})}\\ &\le \sup_{z\in\beta+i\mathbb{R}}|1+z|^{-7}\|\chi^2\alpha_{S_{g_0}(\,\cdot\,)}(\partial_g S_{g_0}(\,\cdot\,)|_{g=g_0}.h)e^{-z\ell_{g_0}}\|_{H^{-6}(\partial_-\mathcal{M})}+C\|h\|_{C^N}^2\le C\|h\|_{C^N}^2,\end{aligned}$$

where the constant $C>0$ changes from line to line. This concludes the proof of Lemma 4.6. ☐

Proof of Lemma 4.7. Taking a finite cover of $\mathcal{M}=\bigcup_i U_i$, a partition of unity $\sum_i\chi_i=\mathbf{1}$ subordinate to that cover, we may write

$$\alpha=\sum_{i,j}\alpha_i^{(j)}dy_i^{(j)}, \tag{4-8}$$

where $\alpha_i^{(j)}, y_i^{(j)}\in C^\infty(\mathcal{M})$ are smooth functions compactly supported inside U_i, and thus, for $y\notin\Gamma_-^{g_0}$, we have

$$\begin{aligned}\chi^2\alpha_{S_{g_0}(y)}(\partial_g S_{g_0}(y)|_{g=g_0}.h)e^{-z\ell_{g_0}(y)} &= \chi^2\sum_{i,j}\alpha_i^{(j)}(S_{g_0}(y))\langle dy_i^{(j)},\partial_g S_g(y)|_{g=g_0}.h\rangle e^{-z\ell_{g_0}(y)}\\ &=\sum_{i,j}\chi\mathcal{S}_{g_0}(\alpha_i^{(j)})(y)e^{-z\ell_{g_0}(y)}\cdot\chi\partial_g\mathcal{S}_g(y_i^{(j)})(y)|_{g=g_0}.h. \end{aligned}\tag{4-9}$$

First, taking $\beta>0$ large enough, we can ensure by Lemma 2.8 the existence of a constant $C>0$ such that, for all $z\in i\mathbb{R}+\beta$ and for all i,j, one has $\chi\mathcal{S}_{g_0}^*\alpha_i^{(j)}e^{-z\ell_{g_0}}\in C^6(\partial_-\mathcal{M})$ with the uniform bound

$$\|(1+z)^{-7}\chi\mathcal{S}_{g_0}(\alpha_i^{(j)})e^{-z\ell_{g_0}}\|_{C^6(\partial_-\mathcal{M})}\le C. \tag{4-10}$$

We now let $f\in C^\infty(\mathcal{M})$ be one of the functions $y_i^{(j)}$ in (4-8). By Proposition 4.2, we have

$$\chi\mathcal{S}_g f=\chi\mathcal{S}_{g_0}f+\chi\partial_g\mathcal{S}_g f|_{g=g_0}.h+\mathcal{O}_{H^{-6}(\partial_-\mathcal{M})}(\|h\|_{C^N}^2).$$

(The constant in the $\mathcal{O}$ notation depends on the function f, but there are only finitely many functions $y_i^{(j)}$ considered in the end so the constant will be uniform.) Now, using that the scattering relations are the same, i.e., $S_g = S_{g_0}$, we have $\chi \mathcal{S}_g^* f = \chi \mathcal{S}_{g_0}^* f$, where the equality holds in $L^\infty(\partial_-\mathcal{M})$ and hence in $L^2(\partial_-\mathcal{M}) \subset H^{-6}(\partial_-\mathcal{M})$. As a consequence, we deduce that

$$\|\chi \partial_g \mathcal{S}_g^* y_i^{(j)}|_{g=g_0} . h\|_{H^{-6}(\partial_-\mathcal{M})} \le C\|h\|_{C^N}^2. \tag{4-11}$$

Using both (4-10) and (4-11) in (4-9) and the continuity of the multiplication

$$C^6(\partial_-\mathcal{M}) \times H^{-6}(\partial_-\mathcal{M}) \ni (u, v) \mapsto uv \in H^{-6}(\partial_-\mathcal{M}),$$

we deduce that, for some $C > 0$,

$$\|(1+z)^{-7} \alpha_{S_{g_0}(\,\cdot\,)} (\partial_g S_{g_0}(\,\cdot\,)|_{g=g_0} . h) e^{-z\ell_{g_0}}\|_{H^{-6}(\partial_-\mathcal{M})} \le C\|h\|_{C^N}^2.$$

This concludes the proof of Lemma 4.7. ☐

4B. *End of the proof.*

We can now complete the proof of Theorem 1.8.

Proof of Theorem 1.8. Assume that $(\ell_g, S_g) = (\ell_{g_0}, S_{g_0})$ and g is close enough to g_0 in the C^N-topology. By Lemma 3.6, we can find a diffeomorphism ψ such that $\psi|_{\partial M} = \mathrm{Id}_{\partial M}$ and $g' := \psi^* g$ is solenoidal with respect to g_0. Moreover, $(\ell_{g'}, S_{g'}) = (\ell_g, S_g) = (\ell_{g_0}, S_{g_0})$. Also note that $\|g' - g_0\|_{C^N} \le C\|g - g_0\|_{C^N}$ for some uniform $C > 0$ (depending on g_0).

Writing $h := g' - g_0$, Proposition 4.1 implies that

$$\|I_2^{g_0} h\|_{L^2} \le C\|h\|_{C^N}^{1+\mu}. \tag{4-12}$$

Now recall that, for any $\varepsilon > 0$, the adjoint $(I_2^{g_{0e}})^* : L^2 \to L^{p(\varepsilon)} \subset H^{-\varepsilon}$ is bounded (here $p(\varepsilon) < 2$ and $p(\varepsilon) \to 2$ as $\varepsilon \to 0$); see [Guillarmou 2017b, Lemma 5.1 and Equation (5.3)].

By (4-12), and since $\Pi_2^{g_{0e}}$ is of order -1 (by Proposition 3.7), and $E_0 h$ has regularity $H^{1/2-\varepsilon}$ for any $\varepsilon > 0$, we conclude that, for any $\varepsilon > 0$, where $C > 0$ changes from line to line,

$$\begin{aligned} \|\Pi_2^{g_{0e}} E_0 h\|_{H^{-\varepsilon}} &= \|(I_2^{g_{0e}})^* I_2^{g_{0e}} E_0 h\|_{H^{-\varepsilon}} \le C\|I_2^{g_{0e}} E_0 h\|_{L^2} \le C\|I_2^{g_0} h\|_{L^2} \le C\|h\|_{C^N}^{1+\mu}, \\ \|\Pi_2^{g_{0e}} E_0 h\|_{H^{3/2-\varepsilon}} &\le C\|E_0 h\|_{H^{1/2-\varepsilon}} \le C\|h\|_{C^N}. \end{aligned}$$

By interpolation in Sobolev spaces, we obtain from these two estimates that, for some (different) $C, \mu > 0$,

$$\|\Pi_2^{g_{0e}} E_0 h\|_{H^1} \le C\|h\|_{C^N}^{1+\mu}.$$

Applying the elliptic stability estimate for solenoidal tensors of Proposition 3.8 (using that our assumption implies that $I_2^{g_0}$ is solenoidal injective), we get

$$\|h\|_{L^2} \le C\|\Pi_2^{g_{0e}} E_0 h\|_{H^1} \le C\|h\|_{C^N}^{1+\mu}.$$

By interpolation, we then obtain, for some (much larger) other integer $N \in \mathbb{N}$,

$$\|h\|_{L^2} \le C\|h\|_{L^2}\|h\|_{C^N}^{\mu} \le C\|h\|_{L^2}\|g - g_0\|_{C^N}^{\mu}.$$

If $\|g - g_0\|_{C^N} < (1/C)^{1/\mu}$, this implies that $h = 0$, namely $g' = \psi^* g = g_0$. ☐

5. Smoothness of the scattering operator with respect to the metric

The goal of this section is to prove Theorem 1.10 and to derive Proposition 4.2 as a corollary. Theorem 1.10 will follow directly from Theorem 5.14 and Lemma 5.21 below. The scattering operator $\mathcal{S}_g$ can be expressed purely in terms of the resolvent R_{g_e} of X_{g_e} thanks to Lemma 2.7. Thus, in order to analyze the map $g \mapsto \mathcal{S}_g$, we shall study the regularity of the map $g \mapsto R_{g_e}$ in adequate functional spaces. Since working with g_e or g is equivalent (they share exactly the same properties), we shall consider R_g for simplicity of notation. The construction of R_g is done using microlocal methods as in [Dyatlov and Guillarmou 2016], but we need to understand the g-dependence in the construction. We fix a metric of Anosov type g_0 on M and we denote by X_0 its associated geodesic vector field on $\mathcal{M}$. We will consider the resolvent of X if X is any smooth vector field that is close enough to X_0 in $C^2(\mathcal{M}, T\mathcal{M})$. We refer to Section 2C3, where the notation for the cotangent bundle is introduced.

5A. ***Construction of the uniform escape function.*** In this paragraph, we construct a *uniform escape function*, i.e., an escape function[4] for X_0 which is also an escape function for all vector fields X that are sufficiently close to X_0. We will use an idea of [Bonthonneau 2020] in order to obtain an escape function adapted to all flows X close to X_0. Denote by $S^*\mathcal{M} := (T^*\mathcal{M} \setminus \{0\})/\mathbb{R}^+$ (and similarly $S^*\mathcal{N}$) the spherical bundle, by $\kappa : T^*\mathcal{M} \to S^*\mathcal{M}$ the quotient projection, by $\pi : S^*\mathcal{N} \to \mathcal{N}$ the footpoint map, and recall that $\boldsymbol{X}$ is the generator of the symplectic lift of φ_t defined in (2-12). Finally, recall that $\rho_0 > 0$ is the constant of Section 2A2 used to define the extension $\mathcal{M}_e$, and that $\widetilde{X}_0$ is some initial extension of the vector field from $\mathcal{M}$ to $\mathcal{N}$ (which does not need to vanish at $\{\rho = -\rho_0\}$).

Proposition 5.1. *There exist a smooth function $m \in C^\infty(S^*\mathcal{N}, [-1, 1])$, invariant by the antipodal map $(x, \xi) \mapsto (x, -\xi)$, and $\delta > 0$ such that, for all vector fields $X \in C^\infty(\mathcal{M}, T\mathcal{M})$ such that*

$$\|X - X_0\|_{C^2(\mathcal{M},T\mathcal{M})} \le \delta,$$

the following hold:

(1) *$m = 1$ in a neighborhood of $(E_-^X)^* \cap \pi^{-1}(\mathcal{M})$.*

(2) *$m = -1$ in a neighborhood of $(E_+^X)^* \cap \pi^{-1}(\mathcal{M})$.*

(3) *$\operatorname{supp}(m) \cap \pi^{-1}(\mathcal{M})$ is contained in a small conic neighborhood of $(E_-^X)^*$ and $(E_+^X)^*$.*

(4) *$\operatorname{supp}(m) \subset \{\rho > -2\rho_0\}$.*

(5) *$\operatorname{supp}(m) \cap \{\rho = -\rho_0\} \cap \{\widetilde{X}_0\rho = 0\} = \varnothing$.*

(6) *$\boldsymbol{X}m \le 0$.*

The fact that X and X_0 are C^2-close will ensure that the structural stability Proposition 2.4 applies. The function m will be constructed as

$$m = m_- - m_+ + \eta^{-1}(\pi^*\chi_- - \pi^*\chi_+), \tag{5-1}$$

---

[4] A function decreasing along the bicharacteristics of the symplectic lift of X to the cotangent bundle.

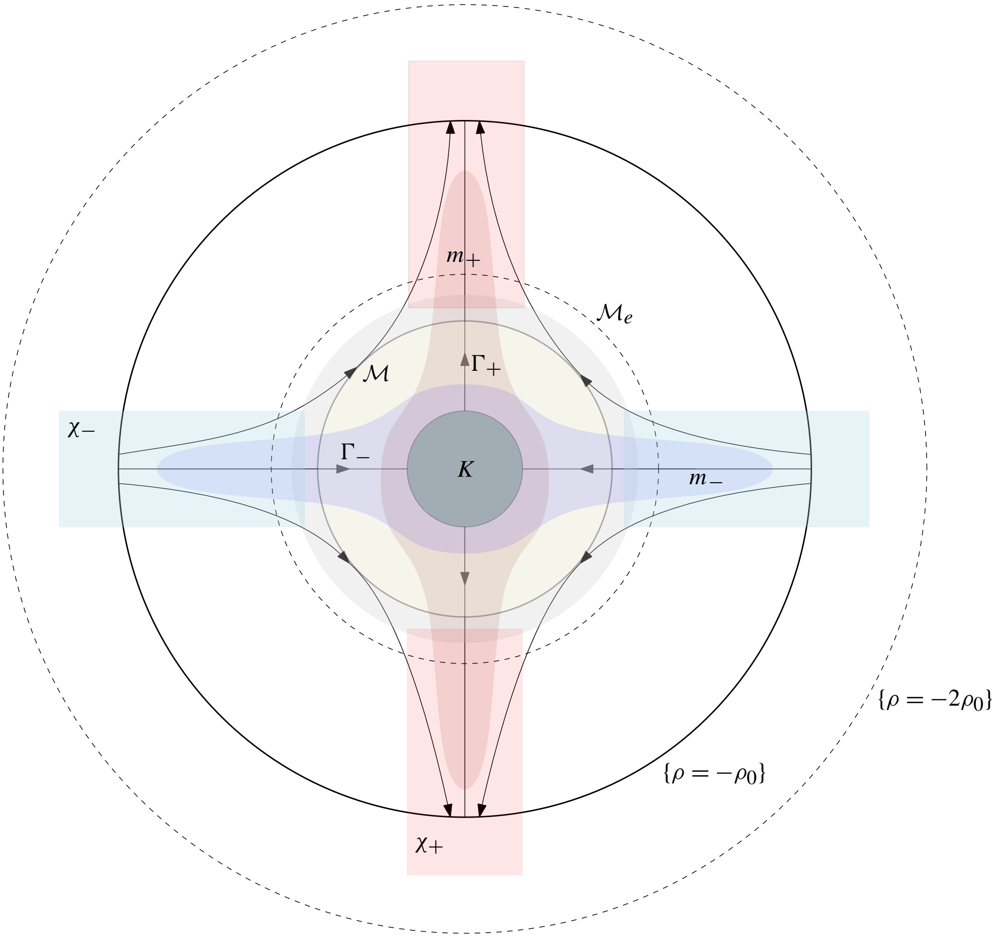

Figure 4. A schematic representation of the various sets and functions appearing in Lemmas 5.10 and 5.11. The disks represent (respectively, from the center to the outer disk): the trapped set K of X_0, the manifold $\mathcal{M}$, the set $\{q = 0\}$ (in light gray) defined in Section 5B, the extended manifold $\mathcal{M}_e$, the set $\{\rho \geq -\rho_0\}$, the set $\{\rho \geq -2\rho_0\}$. The support of the functions m_+, χ_+, m_-, χ_- are depicted, respectively, in: dark red, light red, dark blue, light blue. The flowlines of X_0 are represented in black with arrows indicating the flow direction.

where $m_\pm$ are smooth functions with support near $(E_\pm^X)^*$ and taking value 1 on $(E_\pm^X)^*$, $\chi_\pm$ are smooth functions with compact support in a slightly larger neighborhood of $\Sigma_\pm$ (defined in (2-13)), and $\eta > 0$ will be a small parameter chosen small enough in the end. We refer to Section 2C3 where all the previous notation are defined. The proof being rather technical, we advise the reader to have in mind Figure 4, where the various sets and functions of the construction are depicted.

Remark 5.2. More generally, one could construct a function m taking any positive (resp. negative) constant value near $(E_-^X)^*$ (resp. $(E_+^X)^*$) but this will not be needed.

5A1. *Uniform cone contraction.* We start with some technical lemmas on the contraction of cones in $T^*\mathcal{M}$. In order to abbreviate notation, we will sometimes write $X \sim X_0$ if $\|X - X_0\|_{C^2} \leq \delta$, where $\delta > 0$ is some small constant which will be chosen later. In what follows, we will use the notion of conic neighborhoods of conic sets in $T^*\mathcal{N} \setminus 0$, which may be identified with neighborhoods on the spherical bundle $S^*\mathcal{N}$. First of all, we have:

Lemma 5.3. *Let $\mathcal{U}$ be an open neighborhood of the trapped set K^{X_0}. Then, there exists $\delta > 0$ and $T \geq 0$ such that, for all $t \geq T$ and all smooth vector fields X such that $\|X - X_0\|_{C^2(\mathcal{M}, T\mathcal{M})} < \delta$,*

$$y,\ \varphi^X_{-t}(y),\ \varphi^X_t(y) \in \mathcal{M}_e \quad \Longrightarrow \quad y \in \mathcal{U}.$$

Taking $X \sim X_0$ close enough in the C^2-topology, we can ensure that $\mathcal{U}$ is also an open neighborhood of $\bigcup_{X \sim X_0} K^X$ by the structural stability Proposition 2.4.

Proof. We argue by contradiction. Assume that we can find sequences $(T_j)_{j \geq 1}$ such that $T_j \to +\infty$, $(X_j)_{j \geq 1}$ such that $X_j \to X_0$ in $C^2(\mathcal{M}, T\mathcal{M})$, and $(y_j)_{j \geq 1}$ such that $y_j \in \mathcal{M}_e$, $\varphi^{X_j}_{-T_j}(y_j) \in \mathcal{M}_e$ and $\varphi^{X_j}_{T_j}(y_j) \in \mathcal{M}_e$, but $y_j \notin \mathcal{U}$. By compactness of $\mathcal{M}_e$, we can always assume, up to extraction, that $y_j \to y_\infty \in \mathcal{M}_e$. But then $y_\infty \in K^{X_0}$, which contradicts $y_\infty \notin \mathcal{U}$. ☐

We now show the existence of small conic subsets in $T^*\mathcal{M}$, independent of the vector field X, on which the differential of the flow $(\varphi^X_t)_{t \in \mathbb{R}}$ is exponentially expanding/contracting. This may be compared to [Dyatlov and Guillarmou 2016, Lemma 2.11].

Lemma 5.4. *There exist $\delta > 0$ small enough, constants $C, T, \lambda > 0$ and small open conic neighborhoods $U_\pm$ of $\bigcup_{X \sim X_0} (E^X_\pm)^*$, such that, for all X with $\|X - X_0\|_{C^2} \leq \delta$, the following holds: for all $(y, \xi) \in U_\pm$, for all $t \geq T$ such that $y, \varphi^X_{\pm t}(y) \in \mathcal{M}_e$,*

$$\text{for all } s \in [0, t - T], \quad e^{\pm sX}(y, \xi) \in U_\pm \qquad \text{and,} \qquad \text{for all } s \in [0, t], \quad |e^{\pm sX}(y, \xi)| \geq C e^{\lambda s} |\xi|.$$

Proof. We prove the lemma for the outgoing $(+)$ direction, the proof being similar for the incoming $(-)$ direction. Fix arbitrary small conic neighborhoods $\widetilde{U}^{(2)}_+ \Subset \widetilde{U}^{(1)}_+$ of $(E^{X_0}_+)^*$. By hyperbolicity, there is a $T_0 > 0$ large enough such that the following holds: for all $(y, \xi) \in T^*_{\Gamma^{X_0}_+}\mathcal{M}_e \cap \widetilde{U}^{(1)}_+$ such that $y, \varphi^{X_0}_{T_0}(y) \in \mathcal{M}_e$, one has

$$e^{T_0 X_0}(y, \xi) \in T^*_{\Gamma^{X_0}_+}\mathcal{M}_e \cap \widetilde{U}^{(2)}_+, \qquad |e^{T_0 X_0}(y, \xi)| \geq 10|\xi|.$$

By continuity, there exist small neighborhoods $U^{(j)}_+$ of $\widetilde{U}^{(j)}_+$ such that the following hold:

1. The neighborhoods are chosen so that $\pi(U^{(1)}_+) \Subset \pi(U^{(2)}_+)$.
2. Letting $W := \pi(U^{(1)}_+)$, one has $U^{(2)}_+ \cap W \Subset^{\text{fiber}} U^{(1)}_+ \cap W$, in the sense that, for all $y \in W$, we have $U^{(2)}_+ \cap T^*_y\mathcal{M}_e \Subset U^{(1)}_+ \cap T^*_y\mathcal{M}_e$.
3. For all $(y, \xi) \in U^{(1)}_+$ such that $y, \varphi^{X_0}_{T_0}(y) \in \mathcal{M}_e$,
$$e^{T_0 X}(y, \xi) \in U^{(2)}_+, \qquad |e^{T_0 X}(y, \xi)| \geq 5|\xi|.$$
4. There is a time $T_1 > T_0$ such that, if $y \in \pi(U^{(2)}_+) \setminus \pi(U^{(1)}_+)$, then $\varphi^{X_0}_t(y) \notin \mathcal{M}_e$ for all $t \geq T_1$.

By continuity, this can be achieved so that points (1-4) also hold for all smooth vector fields X such that $\|X - X_0\|_{C^1} \le \delta$, where $\delta > 0$ is chosen small enough. We will actually choose $\|X - X_0\|_{C^2} \le \delta$, where $\delta > 0$ is chosen small enough: by the structural stability Proposition 2.4, we can then ensure that the neighborhoods $U_+^{(j)}$ also contain $(E_+^X)^*$ for $X \sim X_0$ in the C^2-topology.

We set $U_+ := U_+^{(1)}$ and $T := 3T_1$, and we claim that these satisfy the required properties. Take $(y, \xi) \in U_+$ such that $y \in \mathcal{M}_e$, $\varphi_t(y) \in \mathcal{M}_e$ and $t \ge T$. Write $t = k_1T_1 + r_1$, with $k_1 \in \mathbb{Z}_{\ge 1}$, $r_1 \in [0, T_1)$, and $(k_1 - 1)T_1 = k_0T_0 + r_0$, with $k_0 \in \mathbb{Z}_{\ge 0}$, $r_0 \in [0, T_0)$, that is,

$$t = k_0T_0 + T_1 + r_1 + r_0.$$

Note that $T_1 + r_1 + r_0 < 3T_1 = T$.

For all $s \in [0, k_0T_0]$, one has $\varphi_s^X(y, \xi) \in \pi(U_+^{(1)})$ and $(y, \xi) \in U_+^{(1)}$. Indeed, otherwise, we would get, for some $s_\star \in [0, k_0T_0]$, that $\varphi_{s_\star}^X(y, \xi) \in \pi(U_+^{(2)}) \setminus \pi(U_+^{(1)})$, but then $\varphi_{s_\star+T_1}^X(y) \notin \mathcal{M}_e$, which contradicts the fact that $\varphi_t^X(y) \in \mathcal{M}_e$ since

$$s_\star + T_1 \le (k_1 - 1)T_1 + T_1 = kT_1 \le t.$$

Then, using the uniform lower bound $|e^{(T_1+r_0+r_1)X}(y, \xi)| \ge C_0|\xi|$, we obtain

$$|e^{tX}(y, \xi)| = |e^{(T_1+r_0+r_1)X}(e^{T_0X})^{k_0}(y, \xi)| \ge C_0 5^{k_0}|\xi| \ge Ce^{\lambda t}|\xi|$$

for some constant $C > 0$ and $\lambda = \log(5)/T_0$. ☐

We now let V_+ be a small conic neighborhood of $\bigcup_{X\sim X_0}(E_+^X)^*$ contained inside U_+, i.e., $V_+ \Subset U_+$. It will be convenient to use the following operation on the category of fibered conic subsets: if $V \subset T^*\mathcal{N}$ is an open conic subset, define the *fiberwise complement* of V as

$$V^{\complement_{\text{fiber}}} := \{(y, \xi) \in T^*\mathcal{N} \mid y \in \pi(V),\ \xi \in \overline{V}^{\complement} \cap T_y^*\mathcal{N}\},$$

where the superscript $\complement$ denotes the set theoretic complement.

Lemma 5.5. *There exists $\delta > 0$ and $T > 0$, and $V_- := (W_-)^{\complement_{\text{fiber}}}$, where W_- is a small conic neighborhood of $\bigcup_{X\sim X_0}(E_-^X)^* \oplus (E_0^X)^*$, such that, for all X with $\|X - X_0\|_{C^2(\mathcal{M},T\mathcal{M})} \le \delta$, one has $e^{TX}V_- \Subset V_+$.*

The same lemma can be proved by reversing the direction of X, i.e., by swapping the roles of E_-^* and E_+^*.

Proof. We fix an arbitrary open conic set $\widetilde{V}_-$ near $\pi^{-1}(K^{X_0})$ such that $\widetilde{V}_- \cap ((E_-^{X_0})^* \oplus (E_0^{X_0})^*) = \varnothing$. In restriction to $\pi^{-1}(K^{X_0})$, hyperbolicity ensures the existence of a time $T > 0$ such that

$$e^{TX_0}(\widetilde{V}_- \cap \pi^{-1}(K^{X_0})) \Subset V_+ \cap \pi^{-1}(K^{X_0}).$$

By continuity, this also holds for an open conic neighborhood V_- by taking $\pi(V_-)$ to be contained inside a small neighborhood of K^{X_0} (whose size depends on T), and it also holds uniformly for all vector fields X such that $\|X - X_0\|_{C^2} \le \delta$ if $\delta > 0$ is taken small enough (depending on T) by using the stability result of Proposition 2.4 and choosing $\delta > 0$ small enough that $\bigcup_{X\sim X_0} K^X \subset \pi(V_-)$. ☐

In order to simplify notation, we will write $\zeta = (y, \xi)$ for a point in $T^*\mathcal{N}$ and $p_X(x, \xi) := \xi(X)$ for the principal symbol of $-iX$. From Lemmas 5.4 and 5.5, we deduce:

Lemma 5.6. *Let Ω be a small conic neighborhood of $\bigcup_{X\sim X_0}\{p_X = 0\}$ in $T^*\mathcal{M}_e$. There exist $\delta > 0$ and $T > 0$ such that, for all X with $\|X - X_0\|_{C^2(\mathcal{M},T\mathcal{M})} \le \delta$ and $t \ge T$, if $\zeta, e^{tX}(\zeta) \in \Omega \cap T^*\mathcal{M}_e \setminus \{0\}$, then*

$$\int_0^t \mathbf{1}_{U_+\sqcup U_-}(e^{sX}(\zeta))\, \mathrm{d}s \ge t - T.$$

In other words, the flowline of ζ spends at least a time $t - T$ in $U_+ \sqcup U_-$, where there is some uniform contraction/expansion.

Proof. We use the sets $U_\pm$ and $V_\pm$ defined in Lemmas 5.4 and 5.5. Note that $\pi(V_\pm) \subset \pi(U_\pm)$ by construction, and we set $\mathcal{U} := \pi(U_+) \cap \pi(U_-)$. We introduce the following constants:

(1) Let $T_0 > 0$ be the time provided by Lemma 5.3 applied with the open neighborhood $\mathcal{U}$ of K^{X_0} and such that, for all X with $\|X - X_0\|_{C^2} \le \delta$, for all $t \ge T_0$ and $y \in \mathcal{M}_e$ such that $\varphi_t^X(y) \in \mathcal{M}_e$, one has

$$\{\varphi_s^X(y) \mid s \in [T_0, t - T_0]\} \subset \mathcal{U}.$$

(2) Let $T_1 > 0$ be the time provided by Lemma 5.4.

(3) Let $T_2 > 0$ be the time provided by Lemma 5.5 such that $e^{T_2X}V_- \Subset V_+$.

Take a point $\zeta \in \Omega \cap T^*\mathcal{M}_e \setminus \{0\}$ such that $e^{tX}(\zeta) \in T^*\mathcal{M}_e$ for some $t \ge 2T_0$, that is, $\varphi_s^X(\pi(\zeta)) \in \mathcal{U}$ for all $s \in [T_0, t - T_0]$. We treat different cases:

Case 1: Assume that $e^{T_0X}(\zeta) \in U_-$. If $e^{sX}(\zeta) \in U_-$ for all $s \in [T_0, t - T_0]$, then the claim holds for ζ and $T = 2T_0$. If not, there is a time $s_\star \in [T_0, t - T_0]$ such that $e^{s_\star X}(\zeta) \in V_-$ and $e^{sX}(\zeta) \in U_-$ if $s \in [T_0, s_\star]$. By Lemma 5.5, we then deduce that $\zeta' := e^{(s_\star+T_2)X}(\zeta) \in V_+ \Subset U_+$. Observe that $\zeta' \in U_+$ and $e^{(t-(s_\star+T_2))X}(\zeta') \in T^*\mathcal{M}_e$. If $t - (s_\star + T_2) \ge T_1$, from Lemma 5.4 we deduce that, for all $s \in [T_0, s_\star] \cup [s_\star + T_2, t - T_1]$, we have $e^{sX}(\zeta) \in U_- \cup U_+$, that is, the flowline of ζ spends at least $t - (T_0 + T_1 + T_2)$ time in $U_- \cup U_+$. Thus, the claim holds with $T := T_0 + T_1 + T_2$. If $t - (s_\star + T_2) \le T_1$, then the flowline of ζ has spent a time at least $s_\star - T_0 \ge t - (T_0 + T_1 + T_2)$ in U_-, and the claim holds with the same time T defined previously.

Case 2: Eventually, if $e^{T_0X}(\zeta) \notin U_-$, then $e^{T_0X}(\zeta) \in V_-$, and the claim is also straightforward, following the previous arguments. □

Eventually, we will need the following lemma.

Lemma 5.7. *Let $W_- = W'_- \cap (W''_-)^{\complement_{\text{fiber}}}$, where W'_- and W''_- are conic neighborhoods of $\pi^{-1}(K^{X_0})$ and $(E_+^{X_0})^*$, respectively. Let W_+ be a small conic neighborhood of $(E_+^{X_0})^*$. Then, there exists $T > 0$ such that, for all $t \ge T$, we have $e^{-tX_0}W_- \cap W_+ = \varnothing$.*

By *small* for W_+, it is understood that $W_+ \cap ((E_0^{X_0})^* \oplus (E_-^{X_0})^*) = \varnothing$.

Proof. This follows from the fact that there is a uniform time $T > 0$ such that, for each $(y, \xi) \in W_-$, either $\rho(\varphi_{-t}^{X_0}(y)) < 0$ for all $t > T$, or $e^{-tX_0}(y, \xi)$ belongs to a small conic neighborhood of $(E_0^{X_0})^* \oplus (E_-^{X_0})^*$ for all $t > T$, by the same argument as in Lemma 5.5. □

5A2. *Construction of $m_\pm$.* In this paragraph, we construct the functions $m_\pm$ involved in the expression (5-1) of the escape function m. We introduce a smooth function $m_0 \in C^\infty(S^*\mathcal{N}, [0,1])$, invariant by the antipodal map $(x,\xi) \mapsto (x,-\xi)$, such that $m_0 = 1$ in a small neighborhood of $\kappa((E_u^{X_0})^*)$ over K^{X_0} and $m_0 = 0$ on the complement of a slightly larger neighborhood of $\kappa((E_u^{X_0})^*)$. We will need the following.

Lemma 5.8. *For all $T > 0$ large enough, the following holds:*

$$\begin{cases} \zeta, e^{TX_0}(\zeta) \in S^*\mathcal{M}_e \\ m_0(\zeta) < 1 \end{cases} \implies \text{for all } t \in [T, 3T], \quad m_0(e^{-tX_0}(\zeta)) = 0.$$

Proof. We argue by contradiction. Assume that there exists

- an increasing sequence of values $(T_j)_{j \in \mathbb{Z}_{\geq 0}}$ such that $T_j \to +\infty$,
- a sequence of points $(\zeta_j)_{j \in \mathbb{Z}_{\geq 0}}$ such that $\zeta_j, e^{T_j X_0}(\zeta_j) \in S^*\mathcal{M}_e$ and $m_0(\zeta_j) < 1$, and
- a sequence of values $(S_j)_{j \in \mathbb{Z}_{\geq 0}}$ such that $S_j \geq T_j$ and $m_0(e^{-S_j X_0}(\zeta_j)) > 0$.

By compactness of $S^*\mathcal{M}_e$, up to extraction, we can always assume $\zeta_j \to \zeta_\infty$. Observe that $\zeta_\infty \in \pi^{-1}(K^{X_0})$ as $T_j \to +\infty$: indeed, since $T_j \to \infty$, we have that $\zeta_\infty \in \pi^{-1}(\Gamma_-^{X_0})$; if $\zeta_\infty \in \pi^{-1}(\Gamma_-^{X_0} \setminus K^{X_0})$, the exit time from $\mathcal{M}$ in the past of ζ_∞ is finite and since $S_j \to +\infty$, $m_0(e^{-S_j X_0}\zeta_j) > 0$ and m_0 vanishes outside of $\mathcal{M}$, we would get a contradiction for $j \geq 0$ large enough.

Since $m_0(\zeta_j) < 1$ and $m_0 = 1$ near $\kappa((E_u^{X_0})^*)$, we can find V_-, a small neighborhood of $\pi^{-1}(K^{X_0})$ whose closure is not intersecting $(E_-^{X_0})^*$ and such that $\zeta_\infty \in V_-$. Let V_+ be a small neighborhood of $\operatorname{supp}(m_0)$. By Lemma 5.7, there is $T > 0$ such that, for all $t \geq T$, $e^{-tX_0}V_- \cap V_+ = \varnothing$. In particular, for $j \geq 0$ large enough, $\zeta_j \in V_-$, and thus $e^{-S_j X_0}(\zeta_j) \notin V_+$, that is, $m_0(e^{-S_j X_0}(\zeta_j)) = 0$. But this contradicts $m_0(e^{-S_j X_0}(\zeta_j)) > 0$. ☐

We then set, for $T > 0$ large enough satisfying Lemma 5.8,

$$m_1(\zeta) := \frac{1}{2T} \int_T^{3T} m_0(e^{-tX_0}(\zeta))\, dt. \tag{5-2}$$

Lemma 5.9. *The function $m_1 \in C^\infty(S^*\mathcal{N}, [0,1])$ satisfies the following properties:*

1. *$m_1 = 1$ near $(E_+^{X_0})^* \cap \pi^{-1}(\mathcal{M}_e)$.*
2. *$\operatorname{supp}(m_1) \subset \pi^{-1}(\Sigma_+)$ and $\operatorname{supp}(m_1)$ is contained in a small neighborhood of $(E_+^{X_0})^*$.*
3. *$\mathbf{X}_0 m_1 \geq 0$ on $\pi^{-1}(\mathcal{M}_e)$.*
4. *There exist $\varepsilon_0, \delta_0 > 0$ such that, if $\zeta \in \pi^{-1}(\mathcal{M}_e)$ and $\left|m_1(\zeta) - \frac{1}{2}\right| \leq \varepsilon_0$, then $\mathbf{X}_0 m_1(\zeta) \geq \delta_0$.*

Proof. We prove each point separately.

(1) and (2) Taking $T > 0$ large enough in (5-2), the first two items are immediate to check.

(3) For $\zeta \in T^*\mathcal{M}_e$, we have

$$\mathbf{X}_0 m_1(\zeta) = \frac{1}{2T}\big(m_0(e^{-TX_0}(\zeta)) - m_0(e^{-3TX_0}(\zeta))\big),$$

and we want to show that $X_0 m_1 \geq 0$ on $\pi^{-1}(\mathcal{M}_e)$. Observe that if $m_0(e^{-TX_0}(\zeta)) = 1$, then the claim $X_0 m_1(\zeta) \geq 0$ is immediate. We can thus assume that $m_0(e^{-TX_0}(\zeta)) < 1$. If $e^{-TX_0}(\zeta) \notin \pi^{-1}(\mathcal{M}_e)$, then $m_0(e^{-TX_0}(\zeta)) = 0$ and, by convexity, $m_0(e^{-3TX_0}(\zeta)) = 0$ and $X_0 m_1(\zeta) = 0$. If $e^{-TX_0}(\zeta) \in \pi^{-1}(\mathcal{M}_e)$, we can apply Lemma 5.8 which implies that $m_0(e^{-3TX_0}(\zeta)) = 0$, and thus we also obtain $X_0 m_1(\zeta) \geq 0$.

(4) In order to show the last item, it suffices to show that, on the compact set

$$\{X_0 m_1 = 0\} \cap \pi^{-1}(\mathcal{M}_e),$$

one has $\left|m_1 - \tfrac{1}{2}\right| \geq \varepsilon_1$ for some positive $\varepsilon_1 > 0$, that is, the continuous function $\left|m_1 - \tfrac{1}{2}\right|$ does not vanish on this set. Let $\zeta \in \pi^{-1}(\mathcal{M}_e)$ be such that $X_0 m_1(\zeta) = 0$. Then $m_0(e^{-TX_0}\zeta) = m_0(e^{-3TX_0}\zeta)$.

Assume that $m_0(e^{-TX_0}\zeta) < 1$. If $e^{-TX_0}\zeta \notin \pi^{-1}(\mathcal{M}_e)$, then, by convexity of $\mathcal{M}_e$, $e^{-tX_0}(\zeta) \notin \pi^{-1}(\mathcal{M}_e)$ for all $t \geq T$, and thus $m_1(\zeta) = 0$, that is, $\left|m_1 - \tfrac{1}{2}\right| = \tfrac{1}{2} \neq 0$. We can thus assume that $e^{-TX_0}(\zeta) \in \pi^{-1}(\mathcal{M}_e)$. By Lemma 5.8, we get that $m_0(e^{-3TX_0}(\zeta)) = 0 = m_0(e^{-TX_0}(\zeta))$. Lemma 5.8 also gives us that $m_0(e^{-tX}(\zeta)) = 0$ for all $t \in [2T, 3T]$. As a consequence,

$$m_1(\zeta) = \frac{1}{2T}\int_T^{3T} m_0(e^{-tX_0}\zeta)\,\mathrm{d}t = \frac{1}{2T}\int_T^{2T} m_0(e^{-tX_0}\zeta)\,\mathrm{d}t < \tfrac{1}{2},$$

so $\left|m_1(\zeta) - \tfrac{1}{2}\right| \neq 0$.

We now assume that

$$m_0(e^{-TX_0}(\zeta)) = 1 = m_0(e^{-3TX_0}(\zeta)).$$

We claim that $m_0(e^{-tX_0}\zeta) = 1$ for all $t \in [T, 2T]$. Indeed, assume that there exists some $t_0 \in [T, 2T]$ such that $\zeta_0 := e^{-t_0 X_0}(\zeta)$ satisfies $m_0(\zeta_0) < 1$. By Lemma 5.8, since $\zeta_0, e^{TX_0}(\zeta_0) \in S^*\mathcal{M}_e$, we obtain that $m_0(e^{-tX_0}(\zeta_0)) = 0$ for all $t \geq T$. Taking $t_1 := 3T - t_0 \geq T$, we deduce that

$$m_0(e^{-t_1 X_0}(\zeta_0)) = 0 = m_0(e^{-(3T-t_0)X_0}e^{-t_0 X_0}(\zeta)) = m_0(e^{-3TX_0}(\zeta)),$$

which is a contradiction. We then deduce that

$$m_1(\zeta) > \frac{1}{2T}\int_T^{2T} m_0(e^{-tX_0}(\zeta))\,\mathrm{d}t = \tfrac{1}{2},$$

that is, $\left|m_1(\zeta) - \tfrac{1}{2}\right| \neq 0$. This eventually proves the fourth item. ☐

We now introduce

$$m_+ := \chi(m_1) \in C^\infty(S^*\mathcal{N}, [0, 1]), \tag{5-3}$$

where $\chi \in C^\infty(\mathbb{R})$ is a smooth cutoff function such that: $\chi' \geq 0$, $\chi = 0$ on $\left(-\infty, -\tfrac{1}{2} - \varepsilon_0\right]$, and $\chi = 1$ on $\left[\tfrac{1}{2} + \varepsilon_0, +\infty\right)$, where $\varepsilon_0 > 0$ is the constant provided by Lemma 5.9. By construction, this function takes value 1 near $(E_+^{X_0})^*$. By the same process, one can also construct a function $m_- \in C^\infty(S^*\mathcal{N}, [0, 1])$ such that $m_- = 1$ near $(E_-^{X_0})^*$.

Lemma 5.10. *There exists $\delta > 0$ small enough that, for all smooth vector fields X with*

$$\|X - X_0\|_{C^2(\mathcal{M}, T\mathcal{M})} < \delta,$$

the functions $m_\pm \in C^\infty(S^*\mathcal{N}, [0, 1])$ satisfy the following properties:

(1) $m_\pm = 1$ *near* $(E_\pm^X)^* \cap \pi^{-1}(\mathcal{M}_e)$.

(2) $\operatorname{supp}(m_\pm) \subset \pi^{-1}(\Sigma_\pm)$ *and* $\operatorname{supp}(m_\pm)$ *is contained in a small neighborhood of* $(E_\pm^X)^*$.

(3) *There exists* $\delta_1 > 0$ *small such that*

$$\operatorname{supp}(m_\pm) \subset \pi^{-1}(\{\rho > -(1-\delta_1)\rho_0\}), \tag{5-4}$$

$$\operatorname{supp}(m_\pm) \cap \pi^{-1}(\mathcal{M}^{\complement}) \subset \{\pm\widetilde{X}_0\rho < -\delta_1\}. \tag{5-5}$$

(4) $\pm\boldsymbol{X}m_\pm \geq 0$ *on* $\pi^{-1}(\mathcal{M}_e)$.

We will argue on m_+, as the proof is similar for m_-.

Proof. We prove each item individually.

(1), (2) and (3) These are straightforward to check with $\delta_1 > 0$ small enough. The fact that X and X_0 are C^2-close implies by the structural stability Proposition 2.4 that $\bigcup_{X\sim X_0}(E_\pm^X)^*$ are contained in a small neighborhood of $(E_\pm^{X_0})^*$ where $m_\pm = 1$.

(4) Observe that

$$\boldsymbol{X}m_+ = \boldsymbol{X}m_1\chi'(m_1) = ((\boldsymbol{X}-\boldsymbol{X}_0)m_1 + \boldsymbol{X}_0m_1)\chi'(m_1).$$

The nonnegative function $\chi'(m_1) \geq 0$ vanishes everywhere, except on the set $\{|m_1 - \frac{1}{2}| \leq \varepsilon_0\}$. Observe that, on $\{|m_1 - \frac{1}{2}| \leq \varepsilon_0\}$, we have by Lemma 5.9 that

$$(\boldsymbol{X}-\boldsymbol{X}_0)m_1 + \boldsymbol{X}_0m_1 \geq \delta_0 - \|X - X_0\|_{C^0}\|m_1\|_{C^1} \geq \tfrac{1}{2}\delta_0,$$

provided $\delta \leq \delta_0/(2\|m_1\|_{C^1})$. As a consequence, we deduce that $\boldsymbol{X}m_+ \geq 0$ on $\pi^{-1}(\mathcal{M}_e)$. ☐

5A3. *Construction of the bump functions* $\chi_\pm$. In this paragraph, we construct the bump functions $\chi_\pm$ involved in the expression (5-1) of the escape function m.

Lemma 5.11. *There exist* $\delta_1, \delta > 0$ *small enough and cutoff functions* $\chi_\pm \in C^\infty(\mathcal{N}, [0,1])$ *such that, for all smooth vector fields* X *such that* $\|X - X_0\|_{C^1(\mathcal{M},T\mathcal{M})} < \delta$, *the following hold*:

(1) $\operatorname{supp}(\chi_\pm) \subset \{-2\rho_0 < \rho < -\delta_1\} \cap \{\pm\widetilde{X}_0\rho < -\delta_1\}$.

(2) $X\chi_\pm \geq 0$.

(3) $X\chi_\pm > \frac{1}{2}\delta_1^3\rho_0$ *on* $(\{-(1-\delta_1)\rho_0 < \rho < 0\} \cap \{\pm\widetilde{X}_0\rho < -\delta_1\}) \setminus \mathcal{M}_e$.

Proof. We only deal with χ_+, the proof being similar for χ_-. First of all, for $j = 1, 2$, we define functions $\chi_j \in C^\infty(\mathbb{R})$ depending on some parameter $\delta_1 > 0$, which will be chosen small enough in the end. The function $\chi_1 \in C_c^\infty(\mathbb{R})$ is defined such that (see Figure 5)

- $\operatorname{supp}(\chi_1) \subset \{-2\rho_0 < \rho < -\delta_1\}$,
- $\chi_1 \geq 0,\ \chi_1(-\rho_0) = 1,\ \chi_1'(-\rho_0) = 0$,
- $\chi_1' \geq 0$ on $\{-2\rho_0 < \rho < -\rho_0\}$, $\chi_1' \leq 0$ on $\{-\rho_0 < \rho < -\delta_1\}$,
- $\chi_1' \leq -\delta_1$ on $\{-\rho_0(1-\delta_1) \leq \rho \leq -2\delta_1\}$.

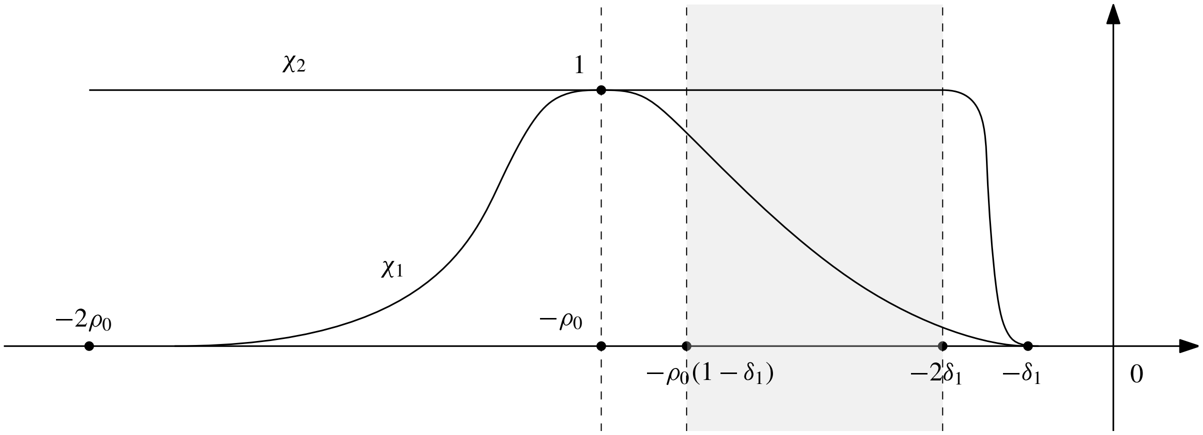

Figure 5. The cutoff functions χ_1 and χ_2.

The function $\chi_2 \in C^\infty(\mathbb{R})$ is defined such that

- $\operatorname{supp}(\chi_2) \subset (-\infty, -\delta_1]$,
- $\chi_2 \geq 0$,
- $\chi_2 = 1$ on $(-\infty, -2\delta_1]$.

We then set

$$\chi_+ := \chi_1(\rho)\chi_2(\widetilde{X}_0\rho), \tag{5-6}$$

and we claim that it satisfies the required properties. Recall from Section 2C3 that $X = \psi\widetilde{X}$, where $\widetilde{X}$ is some smooth extension of the vector field X, initially defined on $\mathcal{M}$ to the closed manifold $\mathcal{N}$.

We now study separately the three terms of

$$\begin{aligned} X\chi_+ &= X\rho\chi_1'(\rho)\chi_2(\widetilde{X}_0\rho) + (X\widetilde{X}_0\rho)\chi_1(\rho)\chi_2'(\widetilde{X}_0\rho) \\ &= \psi\cdot(\widetilde{X}\rho)\chi_1'(\rho)\chi_2(\widetilde{X}_0\rho) + \psi\cdot(\widetilde{X}_0^2\rho)\chi_1(\rho)\chi_2'(\widetilde{X}_0\rho) + \psi\cdot((\widetilde{X}-\widetilde{X}_0)\widetilde{X}_0\rho)\chi_1(\rho)\chi_2'(\widetilde{X}_0\rho). \end{aligned} \tag{5-7}$$

We study the first term in the last line of (5-7). On $\operatorname{supp}(\chi_2(\widetilde{X}_0\rho))$, one has $\widetilde{X}_0\rho \leq -\delta_1$. Thus, assuming $\|X - X_0\|_{C^0(\mathcal{M},T\mathcal{M})} < \delta$ is small enough (depending on δ_1), we obtain that $\widetilde{X}\rho \leq -\frac{1}{2}\delta_1$ on $\operatorname{supp}(\chi_2(\widetilde{X}_0\rho))$. As a consequence, we obtain (note that $\psi\chi_1' \leq 0$)

$$\psi\cdot(\widetilde{X}\rho)\chi_1'(\rho)\chi_2(\widetilde{X}_0\rho) \geq -\frac{\delta_1\psi}{2}\chi_1'(\rho)\chi_2(\widetilde{X}_0\rho) \geq 0.$$

Moreover, on the set $\{-(1-\delta_1)\rho_0 < \rho < -2\delta_1\} \cap \{\widetilde{X}_0\rho < -\delta_1\}$, using that $\psi = \rho + \rho_0$ near $\{\rho = -\rho_0\}$ (so $\psi \geq \delta_1\rho_0$ on the former set) and that $\chi_1'(\rho) \leq -\delta_1$, we obtain that this can be bounded from below by:

$$\psi\cdot(\widetilde{X}\rho)\chi_1'(\rho)\chi_2(\widetilde{X}_0\rho) \geq \frac{\delta_1^2\psi}{2} \geq \frac{\delta_1^3\rho_0}{2} > 0. \tag{5-8}$$

We now deal with the second and third term. The strict convexity property of the level sets $\{\rho = c\}$ (for $c \in [-2\rho_0, 0]$) with respect to $\widetilde{X}_0$ reads: $\widetilde{X}_0\rho = 0 \Rightarrow \widetilde{X}_0^2\rho < 0$. Since $\{\widetilde{X}_0\rho = 0\} \cap \{-2\rho_0 \leq \rho \leq 0\}$ is compact, we deduce that there exists $\delta_1 > 0$ small enough such that, on the set $\{|\widetilde{X}_0\rho| \le 2\delta_1\}$, one has $\widetilde{X}_0^2\rho \le -c < 0$ for some constant $c = c(\delta_1) > 0$. Using that $\operatorname{supp}(\chi_2'(\widetilde{X}_0\rho))$ has support in $\{|\widetilde{X}_0\rho| \le 2\delta_1\}$ and assuming $\|X - X_0\|_{C^0(\mathcal{M},T\mathcal{M})} \le \delta$, we obtain the existence of some constant $C > 0$ (depending on δ_1 but independent of $\delta > 0$) such that

$$\psi \cdot (\widetilde{X}_0^2\rho)\chi_1(\rho)\chi_2'(\widetilde{X}_0\rho) + \psi \cdot ((\widetilde{X} - \widetilde{X}_0)\widetilde{X}_0\rho)\chi_1(\rho)\chi_2'(\widetilde{X}_0\rho) \ge (C\delta - c)\psi\chi_1(\rho)\chi_2'(\widetilde{X}_0\rho).$$

Taking $\delta \le c/(2C)$ small enough (depending on $\delta_1 > 0$), we obtain that this last term is nonnegative.

Overall, we have thus proved (1) and (2), and (3) directly follows from (2) together with (5-8), since we can take $\delta_1 > 0$ small enough that $\{\rho \ge -2\delta_1\} \subset \mathcal{M}_e$. ☐

5A4. *Piecing together the functions.* The various sets appearing in the previous constructions and the functions $m_\pm$, $\chi_\pm$ can be seen in Figure 4. We now piece together the previous constructions and prove Proposition 5.1.

Proof of Proposition 5.1. Define m by (5-1), where $m_\pm$ and $\chi_\pm$ are provided by Lemmas 5.10 and 5.11, and the constant $\delta_1 > 0$ is chosen small enough that both Lemmas 5.10 and 5.11 hold.

Since $\chi_\pm$ have support outside of $\mathcal{M}$, $m_\pm = 1$ near $(E_\pm^X)^* \cap \pi^{-1}(\mathcal{M})$, and $m = m_- - m_+$ on $\pi^{-1}(\mathcal{M})$, we get that points (1), (2) and (3) are verified. The fact that $\operatorname{supp}(m) \subset \{\rho > -2\rho_0\}$ is also straightforward by Lemmas 5.10 and 5.11, which proves (4). Eventually, (5) is also immediate to verify.

We now show that (6) holds if we take $\eta > 0$ small enough. By Lemmas 5.10 (4) and 5.11 (2), the condition $\boldsymbol{X}m \le 0$ holds on $\pi^{-1}(\mathcal{M}_e)$. On the set $\{\rho \le -\rho_0(1-\delta_1)\}$, we have $m_\pm = 0$, and thus, by Lemma 5.11, the inequality $\boldsymbol{X}m \le 0$ also holds. It remains to check the inequality on $\{\rho \ge -\rho_0(1-\delta_1)\} \cap (\mathcal{M}_e)^\complement$. But there, we have, by Lemma 5.11 (3),

$$\boldsymbol{X}m = \boldsymbol{X}m_- - \boldsymbol{X}m_+ + \eta^{-1}(\pi^*X\chi_- - \pi^*X\chi_+) \le \|m_-\|_{C^1} + \|m_+\|_{C^1} - \eta^{-1}\frac{\delta_1^3\rho_0}{2} \le 0$$

if $\eta > 0$ is chosen small enough. ☐

5B. *Meromorphic extension of the resolvent.* We now study the meromorphic extension of the resolvent on anisotropic Sobolev spaces and its dependence with respect to the vector field X. This is the main difference with [Dyatlov and Guillarmou 2016]. We will be particularly interested by the resolvent at $z = 0$, namely R_g, for our application.

5B1. *Global resolvent on uniform anisotropic Sobolev spaces.* In the following, we assume that an arbitrary metric h was chosen on $T\mathcal{N} \to \mathcal{N}$. This induces a metric $h^\sharp$ on $T^*\mathcal{N} \to \mathcal{N}$ and, for $(y,\xi) \in T^*\mathcal{N}$, we will write $\langle\xi\rangle := (1 + h_y^\sharp(\xi,\xi))^{1/2}$ (the y is dropped from the Japanese bracket notation in order to avoid repetition). For $\varrho \in \left(\frac{1}{2}, 1\right]$, we denote by $S_\varrho^k(T^*\mathcal{N})$ the Fréchet space of symbols of order k, i.e., $a \in S^k(T^*\mathcal{N})$, if, in local coordinates,

$$\text{for all } \alpha, \beta, \quad \text{there exists } C > 0 \text{ such that} \quad |\partial_\xi^\alpha \partial_x^\beta a(y,\xi)| \le C\langle\xi\rangle^{k-\varrho|\alpha|+(1-\varrho)|\beta|},$$

and we denote by $\Psi_\varrho^k(\mathcal{N})$ the space of pseudodifferential operators of order k obtained by quantization of symbols in $S_\varrho^k(T^*\mathcal{N})$. We shall remove the ϱ index from the notation when $\varrho = 1$. Note that k can be a real number but also a variable *order function*; see [Faure et al. 2008, Appendix A] for further details.

The function $m \in C^\infty(S^*\mathcal{N}, [-1,1])$ constructed in Section 5A yields a smooth, 0-homogeneous function $m \in C^\infty(T^*\mathcal{N} \setminus \{0\}, [-1,1])$ — still denoted by m — which decreases along all flow lines of $\boldsymbol{X}$, the Hamiltonian vector field induced by X (and X is close to X_0). We can always modify m in a small neighborhood of the 0-section in $T^*\mathcal{N}$ to obtain a new function — still denoted by the same letter m to avoid unnecessary notation — such that $m \in C^\infty(T^*\mathcal{N}, [-1,1])$ and $\boldsymbol{X}m(y,\xi) \le 0$ for all $(y,\xi) \in T^*\mathcal{N}$ such that $\langle\xi\rangle > 1$.

Define a *regularity pair* as a pair of indices $\boldsymbol{r} := (r_\perp, r_0)$, where $r_\perp > r_0 \ge 0$. Given such a regularity pair $\boldsymbol{r}$, we introduce (for all $\varepsilon > 0$ small enough)

$$A_{\boldsymbol{r}} := \mathrm{Op}(\langle\xi\rangle^{(r_\perp m(y,\xi)-r_0)/2})^* \,\mathrm{Op}(\langle\xi\rangle^{(r_\perp m(y,\xi)-r_0)/2}) \in \Psi^{r_\perp m - r_0}_{1-\varepsilon}(\mathcal{N}). \tag{5-9}$$

This is an elliptic and formally selfadjoint pseudodifferential operator belonging to an *anisotropic class*; see [Faure et al. 2008, Appendix A] for further details. As a consequence, up to a modification by a finite-rank formally selfadjoint smoothing operator, we can assume that $A_{\boldsymbol{r}}$ is invertible.

Definition 5.12. We define the *scale of anisotropic Sobolev spaces* with regularity $\boldsymbol{r} := (r_\perp, r_0)$, where $r_\perp > r_0 \ge 0$, as

$$\mathcal{H}^{\boldsymbol{r}}_\pm(\mathcal{N}) := A^{\mp 1}_{\boldsymbol{r}}(L^2(\mathcal{N})), \quad \|f\|_{\mathcal{H}^{\boldsymbol{r}}_\pm(\mathcal{N})} := \|A^{\pm 1}_{\boldsymbol{r}} f\|_{L^2(\mathcal{N})}.$$

Remark 5.13. (1) The spaces $\mathcal{H}^{\boldsymbol{r}}_\pm(\mathcal{N})$ are Hilbert spaces, equipped with the scalar product

$$\langle\,\cdot\,,\,\cdot\,\rangle_{\mathcal{H}^{\boldsymbol{r}}_\pm(\mathcal{N})} := \langle A^{\pm1}_{\boldsymbol{r}}\,\cdot\,, A^{\pm1}_{\boldsymbol{r}}\,\cdot\,\rangle_{L^2(\mathcal{N})}.$$

(2) This scale of spaces is *independent* of the vector field X, as long as it is close enough to X_0 in the C^2-topology, since the escape function m is independent of the vector field. This will be important when studying the regularity of the meromorphic extension of the resolvent $z \mapsto R^X_\pm(z)$ (given by (5-10)) with respect to the vector field X.

(3) Distributions in $\mathcal{H}^{\boldsymbol{r}}_+(\mathcal{N})$ are microlocally in $H^{r_\perp - r_0}(\mathcal{N})$ near $(E^X_-)^*$, $H^{-r_0}(\mathcal{N})$ near $(E^X_0)^*$, and $H^{-r_\perp - r_0}(\mathcal{N})$ near $(E^X_+)^*$ (in the sense that, after application of an $A \in \Psi^0(\mathcal{N})$ with wavefront set in the discussed region, they have the announced regularity). The choice of regularity is arbitrary here, and we did not try to optimize it. The only crucial point is that distributions in $\mathcal{H}^{\boldsymbol{r}}_+(\mathcal{N})$ have positive Sobolev regularity near $(E^X_-)^*$, while they have negative Sobolev regularity near $(E^X_+)^*$.

We let $q \in C^\infty(\mathcal{N}, [0,1])$ be a smooth cutoff function such that

- $\mathrm{supp}(q)$ is contained in the complement of a small open neighborhood of $\mathcal{M}$,
- $q = 1$ on the complement of some slightly larger open neighborhood of $\mathcal{M}$,
- the closure of the set $\{q < 1\}$ is strictly convex with respect to all the vector fields X for $\|X - X_0\|_{C^2} \le \delta$ small enough.

Given a regularity pair $\boldsymbol{r} := (r_\perp, r_0)$ and a constant $\omega > 0$, we define, for X close enough to X_0 and $\Re(z) \gg 0$ large enough,

$$R^X_\mp(z) := -\int_0^{+\infty} e^{-tz} e^{-\omega \int_0^t (\varphi^X_{\mp s})^* q \,\mathrm{d}s} e^{\mp tX}\,\mathrm{d}t, \tag{5-10}$$

Although we do not indicate it in the notation, $R^X_\mp(z)$ *does* depend on a choice of ω. This satisfies the identity on $C^\infty(\mathcal{N})$:

$$(\mp X - z - \omega q)R^X_\mp(z) = \mathbf{1}_{\mathcal{N}}.$$

The constant $\omega > 0$ will be fixed later.

The aim of this section is to study the meromorphic extension of the resolvent $z \mapsto R^X_\mp(z)$ for X close to X_0 in the anisotropic Sobolev spaces of Definition 5.12, and the dependence with respect to the vector field X.

Theorem 5.14. *There exists $C_\star, \delta_\star, \Lambda > 0$ such that the following holds. For all $\delta \le \delta_\star$, for all regularity pairs $\boldsymbol{r} = (r_\perp, r_0)$, there exists a choice of constant $\omega := \omega(\boldsymbol{r}) > 0$ large enough that, for all smooth vector fields X on $\mathcal{M}$ such that $\|X - X_0\|_{C^2(\mathcal{M},T\mathcal{M})} \le \delta$, the family*

$$z \mapsto R^X_-(z) = (-X - z - \omega(\boldsymbol{r})q)^{-1} \in \mathcal{L}(\mathcal{H}^{\boldsymbol{r}}_+),$$

initially defined for $\Re(z) \gg 1$ by (5-10) and holomorphic for $\Re(z) \gg 1$ large enough, extends to a meromorphic family of operators on the half-space $\{\Re(z) > -\Lambda(r_\perp - r_0) + C_\star\delta\}$. The same holds for $R^X_+(z)$ on the space $\mathcal{H}^{\boldsymbol{r}}_-$.

Moreover, if $z_0 \in \{\Re(z) > -\Lambda(r_\perp - (r_0 + 2)) + C_\star\delta\}$ is not a pole of $z \mapsto R^{X_0}(z)$, then there exists $\varepsilon_0 > 0$ such that the map

$$C^\infty(\mathcal{N}, T\mathcal{N}) \times D(z_0, \varepsilon_0) \ni (X, z) \mapsto R^X(z) \in \mathcal{L}(\mathcal{H}^{(r_\perp, r_0)}_+, \mathcal{H}^{(r_\perp, r_0+2)}_+)$$

is C^2-regular[5] *with respect to X and holomorphic in z, where $D(z_0, \varepsilon_0) \subset \mathbb{C}$ is the disk centered at z_0 of radius ε_0.*

As usual, the poles do not depend on the choices made in the construction of the spaces. The rest of Section 5B is devoted to the proof of Theorem 5.14. We note that Theorem 5.14 obviously implies Theorem 1.10 stated in the introduction, since the resolvent on $\mathcal{M}$ can be expressed in terms of the resolvent on $\mathcal{N}$ and the restriction to $\mathcal{M}$ (as in Lemma 5.21 below in the analogous case of geodesic vector fields).

5B2. *Parametrix construction.* Denote by μ a smooth measure on $\mathcal{N}$ which restricts to the Liouville measure on $\mathcal{M}$. Note that X_0 is volume-preserving on $\mathcal{M}$ and, up to minor modifications, we can also assume that the extension of X_0 to $\mathcal{N}$ is volume-preserving on $\mathcal{M}_e$ (but not on $\mathcal{N}$, since X_0 vanishes on $\{\rho = -\rho_0\}$). In order to shorten notation, we will write $L^2(\mathcal{N}) := L^2(\mathcal{N}, \mu)$.

For $T > 0$, consider a smooth cutoff function $\chi_T \in C^\infty_{\mathrm{c}}(\mathbb{R}_+)$, depending smoothly on T, such that $\chi_T = 1$ on $[0, T]$, $-2 \le \chi'_T \le 0$, and $\chi_T = 0$ on $[T + 1, \infty)$. For $\Re(z) \gg 1$ and $\omega \ge 1$, the following identity holds on $C^\infty(\mathcal{N})$:

$$-\int_0^{+\infty} \chi_T(t) e^{-tz} e^{-\int_0^t (\varphi^X_{-s})^*(\omega q)\,\mathrm{d}s} e^{-tX}\,\mathrm{d}t\,(-X - z - \omega q) = \mathbb{1} + \int_0^{+\infty} \chi'_T(t) e^{-tz} e^{-\int_0^t (\varphi^X_{-s})^*(\omega q)\,\mathrm{d}s} e^{-tX}\,\mathrm{d}t. \tag{5-11}$$

---

[5] Even though we only need C^2, our proof actually shows it is C^k for all $k \in \mathbb{N}$.

We now *fix* once and for all a regularity pair $\boldsymbol{r} := (r_\perp, r_0)$ and set $r := r_0 + r_\perp$. The constant $\omega \geq 1$ will be chosen to depend on $\boldsymbol{r}$ later. We conjugate the equality (5-11) by $A_{\boldsymbol{r}}$. We obtain

$$-A_{\boldsymbol{r}} \int_0^{+\infty} \chi_T(t) e^{-tz} e^{-\int_0^t (\varphi_{-s}^X)^*(\omega q)\,\mathrm{d}s} e^{-tX} A_{\boldsymbol{r}}^{-1}\,\mathrm{d}t\, A_{\boldsymbol{r}}(-X - z - \omega q) A_{\boldsymbol{r}}^{-1}$$

$$= \mathbb{1} + \int_0^{+\infty} \chi_T'(t) e^{-tz} e^{-tX} \underbrace{e^{tX} A_{\boldsymbol{r}} e^{-\int_0^t (\varphi_{-s}^X)^*(\omega q)\,\mathrm{d}s} A_{\boldsymbol{r}}^{-1} e^{-tX}}_{:=B_1^X(t)} \underbrace{e^{tX} A_{\boldsymbol{r}} e^{-tX} A_{\boldsymbol{r}}^{-1}}_{:=B_2^X(t)}\,\mathrm{d}t. \quad (5\text{-}12)$$

Since the second term on the right-hand side of (5-12) is defined as an integral over time in the flow direction e^{-tX}, it is smoothing outside $\{p_X = 0\}$. We let $\Omega' \Subset \Omega$ be two open nested conic neighborhoods of $\{p_{X_0} = 0\}$ in $T^*\mathcal{N} \cap \{\rho > -\rho_0\}$. Note that, by continuity, these are also conic neighborhoods of $\{p_X = 0\}$ for all $X \sim X_0$. We let $e \in S^0(T^*\mathcal{N})$ be a symbol of order 0 such that $e = 0$ outside Ω and $e = 1$ on Ω', and we set $E := \mathrm{Op}(e)$. We then decompose the second term on the right-hand side of (5-12) as

$$\int_0^{+\infty} \chi_T'(t) e^{-tz} e^{-tX} B_1^X(t) B_2^X(t)\,\mathrm{d}t = \int_0^{+\infty} \chi_T'(t) e^{-tz} e^{-tX} E B_1^X(t) B_2^X(t)\,\mathrm{d}t + K_1^X(T, z), \quad (5\text{-}13)$$

where

$$K_1^X(T, z) := \int_0^{+\infty} \chi_T'(t) e^{-tz} e^{-tX} (\mathbb{1} - E) B_1^X(t) B_2^X(t)\,\mathrm{d}t$$

and $K_1^X(T, z) \in \Psi^{-\infty}(\mathcal{N})$. In order to prove that $K_1^X(T, z)$ is smoothing, we remark that $K_1^X(T, z) = E' K_1^X(T, z)$ for some $E' \in \Psi^0(\mathcal{N})$ with microsupport that does not intersect a conic neighborhood of $\{p_X = 0\}$, and then show that $X^k K_1^X(T, z) \in \mathcal{L}(L^2)$ for all $k \in \mathbb{N}$, using that $X^k e^{-tX} = (-\partial_t)^k e^{-tX}$ and integrating by parts in t, and finally use that $E'(C - X^2)^{-1} \in \Psi^{-2}(\mathcal{N})$ for some $C \gg 1$ since $C - X^2$ is elliptic on the microsupport of E'. The dependence of $K_1^X(T, z)$ on its parameters is holomorphic in $z \in \mathbb{C}$ and smooth in the variables $T \in \mathbb{R}$ and $X \in C^\infty(\mathcal{M}, T\mathcal{M})$.

Below, we use the notation $\mathcal{L}(\mathcal{H})$ to denote continuous linear operators on a Hilbert space $\mathcal{H}$, and $\mathcal{K}(\mathcal{H})$ for compact operators.

Proposition 5.15. *There exist $C_\star, \delta_\star, \Lambda > 0$ such that the following holds. For all regularity pairs $\boldsymbol{r}$, there exist $C(\boldsymbol{r}), \omega(\boldsymbol{r}) > 0$ such that, for all smooth vector fields $\|X - X_0\|_{C^2} \leq \delta$ with $\delta \leq \delta_\star$, for all $t \geq 0$, there exist (Fourier integral) operators $M^X(t) \in \mathcal{L}(L^2(\mathcal{N}))$ and $S^X(t) \in \mathcal{K}(L^2(\mathcal{N}))$ such that*

$$e^{-tX} E B_1^X(t) B_2^X(t) = M^X(t) + S^X(t)$$

and

$$\|M^X(t)\|_{L^2(\mathcal{N})} \leq C(\boldsymbol{r}) e^{(-\Lambda(r_\perp - r_0) + C_\star \delta)t}.$$

Moreover, the map

$$\mathbb{R} \times C^\infty(\mathcal{M}, T\mathcal{M}) \ni (t, X) \mapsto (M^X(t), S^X(t)) \in \mathcal{L}(L^2(\mathcal{N})) \times \mathcal{K}(L^2(\mathcal{N}))$$

is smooth.

The rest of this paragraph is devoted to the proof of Proposition 5.15. It is split into several sublemmas. Given a regularity pair $\boldsymbol{r} = (r_\perp, r_0)$, in order to simplify notation we introduce

$$m_{\boldsymbol{r}} := r_\perp m - r_0. \tag{5-14}$$

Lemma 5.16. *For all $t \in \mathbb{R}$ and $\frac{1}{2} < \varrho < 1$, we have $B_1^X(t), B_2^X(t) \in \Psi_\varrho^0(\mathcal{N})$ with principal symbols*

$$\sigma_{B_1^X(t)}(y,\xi) = e^{-\omega \int_0^t (\varphi_s^X)^*(q)(y)\,\mathrm{d}s}, \qquad \sigma_{B_2^X(t)}(y,\xi) = \frac{\langle e^{t\boldsymbol{X}}(y,\xi)\rangle^{m_{\boldsymbol{r}}(e^{t\boldsymbol{X}}(y,\xi))}}{\langle \xi \rangle^{m_{\boldsymbol{r}}(y,\xi)}}.$$

Proof. This follows directly from Egorov's lemma; see [Lefeuvre 2019b, Section 2.4.1]. ☐

In particular, Lemma 5.16 shows that the integrand $e^{-tX} B_1^X(t) B_2^X(t)$ on the right-hand side of (5-12) is a Fourier integral operator (FIO). We let

$$a^X(t)(y) := |\det d\varphi_{-t}^X(\varphi_t^X(y))|^{-1/2}, \tag{5-15}$$

where the Jacobian is defined with respect to the measure $\mathrm{d}\mu$ on $\mathcal{N}$.

Lemma 5.17. *For all $t \in \mathbb{R}$, we have $\|e^{-tX}(a^X(t))^{-1}\|_{\mathcal{L}(L^2(\mathcal{N}))} = 1$. Moreover, for all $y \in \mathcal{N}$ and $t \in \mathbb{R}$,*

$$a^X(t)(y) \le \exp\biggl(\int_0^t |\mathrm{div}_\mu X|(\varphi_s^X(y))\,\mathrm{d}s\biggr).$$

Proof. We have

$$\int_{\mathcal{N}} |e^{-tX}((a^X(t))^{-1} f)|^2\,\mathrm{d}\mu = \int_{\mathcal{N}} (a^X(t))^{-2} |f|^2 |\det d\varphi_t^X|\,\mathrm{d}\mu = \|f\|_{L^2}^2.$$

The estimate on $a^X(t)(y)$ follows directly from the fact that $\mathrm{div}_\mu X \circ \varphi_t = \partial_t(\log|\det d\varphi_t^X|)$. ☐

By Lemma 5.16, the operator $a^X(t) E B_1^X(t) B_2^X(t)$ is a pseudodifferential operator of order 0. By the Calderón–Vaillancourt theorem [Grigis and Sjöstrand 1994, Theorem 4.5], up to a compact remainder in $\mathcal{K}(L^2(\mathcal{N}))$, its norm on $L^2(\mathcal{N})$ is given by the lim sup of its principal symbol as $|\xi| \to \infty$. We now bound the lim sup of its principal symbol.

Lemma 5.18. *There exists $\delta_\star, C_\star, \Lambda > 0$ such that the following holds. For all regularity pairs $\boldsymbol{r} := (r_\perp, r_0)$, there exists $C(\boldsymbol{r}), \omega(\boldsymbol{r}) > 0$ such that, for all smooth vector fields X with $\|X - X_0\|_{C^2(\mathcal{M}, T\mathcal{M})} \le \delta$, where $\delta \le \delta_\star$, for all $t \ge 0$,*

$$\limsup_{(y,\xi)\in T^*\mathcal{N}, |\xi|\to\infty} \sigma_{a^X(t) E B_1^X(t) B_2^X(t)}(y,\xi) \le C(\boldsymbol{r}) e^{(-\Lambda(r_\perp - r_0) + C_\star \delta)t}.$$

Proof. For $(y,\xi) \in T^*\mathcal{N}$, we have, by Lemma 5.16,

$$\sigma_{a^X(t) E B_1^X(t) B_2^X(t)}(y,\xi) = e(y,\xi) \exp\biggl(\int_0^t \Bigl(\frac{1}{2}\,\mathrm{div}_\mu X - \omega q\Bigr)(e^{sX}(y))\,\mathrm{d}s\biggr) \frac{\langle e^{t\boldsymbol{X}}(y,\xi)\rangle^{m_{\boldsymbol{r}}(e^{t\boldsymbol{X}}(y,\xi))}}{\langle \xi \rangle^{m_{\boldsymbol{r}}(y,\xi)}}. \tag{5-16}$$

Modulo the term $e(y,\xi) \le 1$, which we can neglect, this is a *cocycle* over the flow of $\boldsymbol{X}$, as it satisfies the relation

$$\sigma_{B_1^X(t') B_2^X(t')}(e^{t\boldsymbol{X}}(y,\xi))\,\sigma_{B_1^X(t) B_2^X(t)}(y,\xi) = \sigma_{B_1^X(t'+t) B_2^X(t'+t)}(y,\xi) \tag{5-17}$$

for all $t, t' \in \mathbb{R}$.

First, we need the following lemma.

Lemma 5.19. *For all regularity pairs* $\boldsymbol{r} = (r_\perp, r_0)$, *there exist constants* $C(\boldsymbol{r}), \omega(\boldsymbol{r}) > 0$ *such that, for all* $(y, \xi) \in T^*\mathcal{N}$, $\omega > \omega(\boldsymbol{r})$ *and for all* $t \geq 0$,

$$\{e^{sX}(y,\xi) \mid s \in [0,t]\} \subset \pi^{-1}(\{q=1\}) \quad \Longrightarrow \quad \limsup_{(y,\xi)\in T^*\mathcal{N}, |\xi|\to\infty} \sigma_{a^X(t)EB_1^X(t)B_2^X(t)}(y,\xi) \leq C(\boldsymbol{r})e^{-rt},$$

where $r := r_\perp + r_0$.

Proof. Define $\nu := \sup_{\|X-X_0\|_{C^2}\leq\delta} \|\operatorname{div}_\mu X\|_{L^\infty(\mathcal{N})}$. We have, if $q(\varphi_s(x)) = 1$ for $s \in [0,t]$,

$$\begin{aligned}\sigma_{a^X(t)EB_1(t)B_2(t)}(y,\xi) &\leq e^{\nu t}e^{-\omega t}\frac{\langle e^{tX}(y,\xi)\rangle^{m_r(e^{tX}(y,\xi))}}{\langle\xi\rangle^{m_r(y,\xi)}} \\ &= e^{(\nu-\omega)t}\langle e^{tX}(y,\xi)\rangle^{m_r(e^{tX}(y,\xi))-m_r(y,\xi)}\left(\frac{\langle e^{tX}(y,\xi)\rangle}{\langle\xi\rangle}\right)^{m_r(y,\xi)}.\end{aligned}$$

By construction, $m_{\boldsymbol{r}}$ is nonincreasing along the flow lines of $\boldsymbol{X}$ outside a neighborhood of the 0-section in $T^*\mathcal{N}$; see Proposition 5.1 (6). This implies that

$$\limsup_{(y,\xi)\in T^*\mathcal{N}, |\xi|\to\infty} \langle e^{tX}(y,\xi)\rangle^{m_r(e^{tX}(y,\xi))-m_r(y,\xi)} \leq 1.$$

Moreover, there exist a uniform exponent $\lambda > 0$ and $C > 0$ (depending only on X_0) such that, for all $X \sim X_0$, for all $t \geq 0$ and $(y,\xi) \in T^*\mathcal{N}$, one has

$$\langle e^{tX}(y,\xi)\rangle \leq Ce^{\lambda t}\langle\xi\rangle. \tag{5-18}$$

Using (5-18) and taking the lim sup as $|\xi| \to \infty$, we then obtain

$$\limsup_{(y,\xi)\in T^*\mathcal{N}, |\xi|\to\infty} \sigma_{a^X(t)EB_1^X(t)B_2^X(t)}(y,\xi) \leq C(\boldsymbol{r})e^{(\nu-\omega+r\lambda)t}.$$

Taking $\omega(\boldsymbol{r}) := \nu + r + r\lambda$, we obtain the announced result. ☐

From now on, given a regularity pair $\boldsymbol{r}$, the constant ω in (5-12) will always be taken to be fixed, equal to $\omega := \omega(\boldsymbol{r}) > 0$ provided by Lemma 5.19. Next we need the following lemma.

Lemma 5.20. *There exists* $C_\star, \Lambda_1 > 0$ *such that the following holds. For all regularity pairs* $\boldsymbol{r}$, *there exists a constant* $C(\boldsymbol{r}) > 0$ *such that, for all* X *with* $\|X - X_0\|_{C^2} \leq \delta$ *and* $(y,\xi) \in T^*\mathcal{N}$, *for all* $t \geq 0$,

$$(y,\xi), e^{tX}(y,\xi) \in T^*\mathcal{M}_e \quad \Longrightarrow \quad \limsup_{|\xi|\to\infty} \sigma_{a^X(t)EB_1^X(t)B_2^X(t)}(y,\xi) \leq C(\boldsymbol{r})e^{(-\Lambda_1(r_\perp-r_0)+C_\star\delta)t}.$$

Proof. We start with a preliminary observation: there exists a constant $C_\star > 0$ such that, if $y, \varphi_t^X(y) \in \mathcal{M}_e$ and $\|X - X_0\|_{C^2(\mathcal{M},T\mathcal{M})} \leq \delta$, then

$$a^X(t)(y) \leq e^{C_\star\delta t}. \tag{5-19}$$

This simply follows from the fact that X_0 is volume-preserving on $\mathcal{M}_e$ (that is, $a^{X_0}(t) = 1$).

We now consider the sets $U_\pm$ given by Lemma 5.4. These sets can always be constructed so that $U_\pm \subset \{m = \pm 1\}$. We also consider the sets $V_\pm$ given by Lemma 5.5. Denote by $T > 0$ the time provided by Lemma 5.6. If $t \leq T$, namely if the time is uniformly bounded, then the claim is immediate as $a^X(t)EB_1^X(t)B_2^X(t)$ is of order 0 by Lemma 5.16 and depends continuously on time. If $t \geq T$ and

$(y,\xi), e^{t\boldsymbol{X}}(y,\xi) \in T^*\mathcal{M}_e \cap \mathrm{WF}(E)$, then the flow line $\{e^{s\boldsymbol{X}}(y,\xi) \mid s \in [0,t]\}$ passes at least a time $t-T$ in $U_+ \sqcup U_-$. We can thus introduce $0 \le s_0 < s_1 \le t$ such that, for all $s \in [0,s_0]$, we have $e^{s\boldsymbol{X}}(y,\xi) \in U_-$, for all $s \in [s_1,t]$, we have $e^{s\boldsymbol{X}}(y,\xi) \in U_+$, and we have $s_0 + (t-s_1) \ge t-T$. Hence, using the cocycle relation (5-17) and $\sigma_E \in [0,1]$,

$$\begin{aligned}\sigma_{a^X(t)EB_1^X(t)B_2^X(t)}(y,\xi) &\le \sigma_{a^X(t-s_1)B_1^X(t-s_1)B_2^X(t-s_1)}(e^{s_1\boldsymbol{X}}(y,\xi)) \\ &\qquad \cdot \sigma_{a^X(s_1-s_0)B_1^X(s_1-s_0)B_2^X(s_1-s_0)}(e^{s_0\boldsymbol{X}}(y,\xi)) \cdot \sigma_{a^X(s_0)B_1^X(s_0)B_2^X(s_0)}(y,\xi). \end{aligned} \tag{5-20}$$

Note that it suffices to bound the terms on the right-hand side of (5-20) on $\mathrm{WF}(E)$, that is, on a conic neighborhood of $\bigcup_{X\sim X_0}\{p_X = 0\}$, since otherwise $\sigma_E = 0$ and the symbol on the left-hand side vanishes.

Since $s_1 - s_0 \le T$ (independent of t) and $\sigma_{B_1^X(t)B_2^X(t)} \in \Psi^0_\varrho(\mathcal{N})$ for all $t \ge 0$ by Lemma 5.16, we get that the middle term in (5-20) is bounded uniformly by some constant, that is,

$$\sigma_{a^X(s_1-s_0)B_1(s_1-s_0)B_2(s_1-s_0)}(e^{s_0\boldsymbol{X}}(y,\xi)) \le C(\boldsymbol{r}) \tag{5-21}$$

for some $C(\boldsymbol{r}) > 0$ which is independent of the point $(y,\xi) \in T^*\mathcal{N}$ and of the time t. As to the third factor in (5-20), we have, using that $m_{\boldsymbol{r}} = r_\perp - r_0$ on U_-, that q vanishes in $\mathcal{M}$, and (5-19),

$$\begin{aligned}\sigma_{a^X(s_0)B_1(s_0)B_2(s_0)}(y,\xi) &\le e^{C_\star\delta s_0} e^{-\int_0^{s_0}\omega(\boldsymbol{r})q(e^{sX}(y))\,\mathrm{d}s} \frac{\langle e^{s_0\boldsymbol{X}}(y,\xi)\rangle^{m_{\boldsymbol{r}}(e^{s_0\boldsymbol{X}}(y,\xi))}}{\langle\xi\rangle^{m_{\boldsymbol{r}}(y,\xi)}} \\ &\le C(\boldsymbol{r})e^{C_\star\delta s_0}\left(\frac{\langle e^{s_0\boldsymbol{X}}(y,\xi)\rangle}{\langle\xi\rangle}\right)^{r_\perp - r_0}. \end{aligned} \tag{5-22}$$

Using the uniform contraction rate on U_- of Lemma 5.4, we get that $|e^{s_0\boldsymbol{X}}(y,\xi)| \le Ce^{-\lambda s_0}|\xi|$ for some uniform constants $C, \lambda > 0$ depending only on X_0. Taking the lim sup as $|\xi| \to \infty$ in (5-22), we thus obtain

$$\limsup_{|\xi|\to\infty} \sigma_{a^X(s_0)B_1(s_0)B_2(s_0)}(y,\xi) \le C(\boldsymbol{r})e^{C_\star\delta s_0}e^{-\lambda s_0(r_\perp - r_0)}. \tag{5-23}$$

Similarly, using the expansion rate on U_+ of Lemma 5.4 and that $m_{\boldsymbol{r}} = -r_\perp - r_0$ on U_+, the first term in (5-20) can be bounded by

$$\limsup_{|\xi|\to\infty} \sigma_{a^X(t-s_1)B_1(t-s_1)B_2(t-s_1)}(e^{s_1\boldsymbol{X}}(y,\xi)) \le C(\boldsymbol{r})e^{C_\star\delta(t-s_1)}e^{-\lambda(t-s_1)(r_\perp + r_0)}. \tag{5-24}$$

Taking $\Lambda_1 := \lambda$ and combining (5-21), (5-23), (5-24) in (5-20) completes the proof. ☐

We can now end the proof of Lemma 5.18. Given $(y,\xi) \in T^*\mathcal{N}$, the flowline of (y,ξ) under $e^{t\boldsymbol{X}}$ can be schematically described by one of the six following possibilities:

$$\{q = 1\}, \tag{5-25}$$

$$\mathcal{M}_e, \tag{5-26}$$

$$\{q=1\} \to \{0 < q < 1\} \to \{q = 1\}, \tag{5-27}$$

$$\{q=1\} \to \{0 < q < 1\} \to \mathcal{M}_e, \tag{5-28}$$

$$\mathcal{M}_e \to \{0 < q < 1\} \to \{q = 1\}, \tag{5-29}$$

$$\{q=1\} \to \{0 < q < 1\} \to \mathcal{M}_e \to \{0 < q < 1\} \to \{q = 1\}. \tag{5-30}$$

Note that, for any flow line, there is a maximum time, bounded by some uniform constant $T_\star > 0$, spent in the region $\{0 < q < 1\}$. As a consequence, if the flowline of (y, ξ) falls into one of the cases (5-25) or (5-27), we get, using the cocycle relation (5-17) and Lemma 5.19,

$$\limsup_{|\xi|\to\infty} \sigma_{a^X(t)EB_1(t)B_2(t)}(y,\xi) \le C(\boldsymbol{r})e^{-rt}.$$

As to (5-26), (5-28), (5-29), the bound is obtained similarly to the bound for (5-30), which we now study.

So we assume that the flowline γ of (y, ξ) under e^{tX} passes successively through the six sets of (5-30). Define the times $s_0, s_1 \ge 0$ such that,

$$\begin{aligned} &\text{for all } s \in [0, s_0], && \varphi_s^X(y) \in \{q = 1\},\\ &\text{for all } s \in [s_0, s_1], && \varphi_s^X(y) \in \{q < 1\} \cup \mathcal{M}_e,\\ &\text{for all } s \in [s_1, t], && \varphi_s^X(y) \in \{q = 1\}. \end{aligned}$$

Combining the cocycle relation (5-17) and Lemmas 5.19 and 5.20, we get, on $\mathrm{WF}(E)$,

$$\begin{aligned} &\limsup_{|\xi|\to\infty} \sigma_{a^X(t)EB_1(t)B_2(t)}(y,\xi)\\ &\quad\le \limsup_{|\xi|\to\infty} \sigma_{a^X(t-s_1)B_1(t-s_1)B_2(t-s_1)}(e^{s_1\boldsymbol{X}}(y,\xi))\\ &\qquad\qquad \cdot \limsup_{|\xi|\to\infty} \sigma_{a^X(s_1-s_0)B_1(s_1-s_0)B_2(s_1-s_0)}(e^{s_0\boldsymbol{X}}(y,\xi)) \cdot \limsup_{|\xi|\to\infty} \sigma_{a^X(s_0)B_1(s_0)B_2(s_0)}(y,\xi)\\ &\quad\le C_r e^{-r(t-s_1)} \cdot C_r e^{(-(r_\perp-r_0)\Lambda_1+C_\star\delta)(s_1-s_0)} \cdot C_r e^{-rs_0} \le C_r e^{(-(r_\perp-r_0)\Lambda+C_\star\delta)t} \end{aligned}$$

by taking $\Lambda := \min(1, \Lambda_1)$. This concludes the proof. ☐

We now complete the proof of Proposition 5.15.

Proof of Proposition 5.15. Write

$$e^{-tX}EB_1(t)B_2(t) = e^{-tX}(a^X(t))^{-1}a^X(t)EB_1(t)B_2(t).$$

By Lemma 5.17, $e^{-tX}(a^X(t))^{-1} \in \mathcal{L}(L^2(\mathcal{N}))$ is unitary. By Lemma 5.18, $a^X(t)EB_1(t)B_2(t)$ is a pseudodifferential operator of order 0 such that

$$\limsup_{(y,\xi)\in T^*\mathcal{N},|\xi|\to\infty} \sigma_{a^X(t)EB_1(t)B_2(t)}(y,\xi) \le C(\boldsymbol{r})e^{(-(r_\perp-r_0)\Lambda+C_\star\delta)t}.$$

By the Calderón–Vaillancourt theorem [Grigis and Sjöstrand 1994, Theorem 4.5] for pseudodifferential operators, we can thus write

$$a^X(t)EB_1(t)B_2(t) = M_0^X(t) + S_0^X(t),$$

where $M_0^X(t)$ is a pseudodifferential operator of order 0 and $S_0^X(t)$ is smoothing and

$$\|M_0^X(t)\|_{\mathcal{L}(L^2(\mathcal{N}))} \le 2C(\boldsymbol{r})e^{(-(r_\perp-r_0)\Lambda+C_\star\delta)t}.$$

Moreover, it is straightforward to check that these operators can be constructed so that they depend smoothly on the parameters $t \in \mathbb{R}$ and $X \in C^\infty(\mathcal{M}, T\mathcal{M})$ as $a^X(t)$, $B_1(t)$, $B_2(t)$ depend in an explicit (and smooth) fashion on X, and the decomposition in the Calderón–Vaillancourt Theorem depends smoothly on the operator. As a consequence, setting

$$M^X(t) := e^{-tX}(a^X(t))^{-1}M_0^X(t) \quad \text{and} \quad S^X(t) := e^{-tX}(a^X(t))^{-1}S_0^X(t),$$

we have

$$e^{-tX}EB_1(t)B_2(t) = M^X(t) + S^X(t),$$

and this concludes the proof. ☐

5B3. *Meromorphic extension on the closed manifold.* We now prove Theorem 5.14.

Proof of Theorem 5.14. Step 1: meromorphic extension. Fix $\boldsymbol{r} = (r_\perp, r_0)$ with $r_\perp > r_0$, and consider $z \in \mathbb{C}$ such that $\Re(z) > -\Lambda(r_\perp - r_0) + C_\star\delta$. By Proposition 5.15, we can consider a time $T > 0$ large enough, depending on $\boldsymbol{r}$, so that,

$$\text{for all } t \geq T, \quad e^{-\Re(z)t}\|M^X(t)\|_{\mathcal{L}(L^2(\mathcal{N}))} < \tfrac{1}{6}. \tag{5-31}$$

Using (5-12) and (5-13), we thus obtain

$$\int_0^{+\infty} \chi_T'(t)e^{-tz}e^{-tX}B_1^X(t)B_2^X(t)\,\mathrm{d}t = B^X(z) + K^X(z),$$

where

$$B^X(z) := \int_0^{+\infty} \chi_T'(t)e^{-tz}M^X(t)\,\mathrm{d}t$$

and $K^X(z) \in \Psi^{-\infty}(\mathcal{N})$ is the remainder. It is immediate to check that both $B^X(z)$ and $K^X(z)$ depend holomorphically on z and smoothly on $X \in C^\infty(\mathcal{M}, T\mathcal{M})$ as operators in $\mathcal{L}(L^2(\mathcal{N}))$.

Using that $\|\chi_T'\|_{L^\infty} \leq 2$, we get

$$\|B^X(z)\|_{\mathcal{L}(L^2(\mathcal{N}))} \leq 2\int_T^{T+1} e^{-\Re(z)t}\|M^X(t)\|_{\mathcal{L}(L^2(\mathcal{N}))}\,\mathrm{d}t \leq \tfrac{1}{3} < 1. \tag{5-32}$$

The equality (5-12) then reads

$$-A_{\boldsymbol{r}}\int_0^{+\infty} \chi_T(t)e^{-tz}e^{-\int_0^t(\varphi_{-s}^X)^*(\omega q)\,\mathrm{d}s}e^{-tX}A_{\boldsymbol{r}}^{-1}\,\mathrm{d}t\, \underbrace{A_{\boldsymbol{r}}(-X - z - \omega q)A_{\boldsymbol{r}}^{-1}}_{=:-P^X-z} = \mathbb{1} + B^X(z) + K^X(z), \tag{5-33}$$

and $\mathbb{1} + B^X(z)$ is invertible while $K^X(z)$ is compact. Moreover, for $\Re(z) \gg 1$, $\mathbb{1} + B^X(z) + K^X(z)$ is invertible on $\mathcal{L}(L^2(\mathcal{N}))$ since the L^2-norm of $B^X(z) + K^X(z)$ is exponentially decaying as $\Re(z) \to +\infty$. By the Fredholm analytic theorem [Zworski 2012, Theorem D.4], we deduce that

$$z \mapsto (\mathbb{1} + B^X(z) + K^X(z))^{-1} \in \mathcal{L}(L^2(\mathcal{N}))$$

is a meromorphic family of operators on $\{\Re(z) > -\Lambda(r_\perp - r_0) + C_\star\delta\}$. Equivalently,

$$z \mapsto -X - z - \omega(\boldsymbol{r})q,$$

is a holomorphic family of Fredholm operators[6] of index 0 on the anisotropic space $\mathcal{H}^r_+(\mathcal{N})$ that is invertible for $\Re(z) \gg 1$. Thus

$$z \mapsto R^X_-(z) = (-X - z - \omega(\boldsymbol{r})q)^{-1} \in \mathcal{L}(\mathcal{H}^{\boldsymbol{r}}_+)$$

is a meromorphic family of operators on $\{\Re(z) > -\Lambda(r_\perp - r_0) + C_\star\delta\}$. This proves the first part of the theorem; we next study the dependence in X and z.

Step 2: continuity of resonances. Assume z_0 is not a pole of $z \mapsto R^{X_0}(z)$ and furthermore that it does not have any poles in the closed disk $D(z_0, \varepsilon_0) \subset \mathbb{C}$ (since the resolvent is meromorphic, such $\varepsilon_0 > 0$ exists). We first show that, for X sufficiently close to X_0 in C^N for some N large enough, the map $z \mapsto R^X(z)$ does not have any poles in $D(z_0, \varepsilon_0)$. Let $z \in D(z_0, \varepsilon_0)$; we will use the identity (5-33). We first claim that we may pick the cutoff function χ suitably and T sufficiently large such that

$$\ker(\mathbb{1} + B^X(z) + K^X(z))|_{L^2} = 0.$$

Note that, as we will see below, this kernel could be nonzero even if z is not a resonance of $-X - q\omega$; we will show that generically this does not happen. We will argue by assuming that there is nonzero $u \in L^2(\mathcal{N})$ such that $(\mathbb{1} + B^X(z) + K^X(z))u = 0$. Since $K^X(z) \in \Psi^{-\infty}(\mathcal{N})$, we get

$$(\mathbb{1} + B^X(z))u \in C^\infty(\mathcal{N}) \subset \mathcal{D}(L^2) = \{f \in L^2(\mathcal{N}) \mid Xf \in L^2(\mathcal{N})\},$$

and since $\mathbb{1} + B^X(z)$ is invertible on $\mathcal{D}(L^2)$ (and on $L^2(\mathcal{N})$, by construction), we conclude that $u \in \mathcal{D}(L^2)$. Since $P^X + z$ commutes with $\mathbb{1} + B^X(z) + K^X(z)$, we have that $P^X + z$ acts on $\ker(\mathbb{1} + B^X(z) + K^X(z))|_{L^2}$, which is a finite-dimensional space by the Fredholm property shown above. Therefore, we can pick u such that $(P^X + z + \lambda)u = 0$ for some $\lambda \in \mathbb{C}$; by assumption, we have $\lambda \neq 0$. Write $u = A_{\boldsymbol{r}} v$ for some $v \in \mathcal{H}^{\boldsymbol{r}}_+$. This implies

$$e^{-tX} v = e^{(z+\lambda)t} e^{\int_0^t (\varphi^X_{-s})^*(q\omega)\, ds} v \quad \text{for all } t \in \mathbb{R},$$

and hence

$$\begin{aligned} 0 = (\mathbb{1} + B^X(z) + Q^X(z))u &= -A_{\boldsymbol{r}}\left(\mathbb{1} + \int_0^{+\infty} \chi'_T(t) e^{-tz} e^{-\int_0^t (\varphi^X_{-s})^*(q\omega)\, \mathrm{d}s} e^{-tX}\, \mathrm{d}t\right) v \\ &= -\underbrace{\left(1 + \int_T^{T+1} \chi'_T(t) e^{\lambda t}\, \mathrm{d}t\right)}_{F(\chi_T, \lambda) :=} u. \end{aligned}$$

If $\Re(\lambda) \leq 0$, the integral in the last equality can be bounded by $\|\chi'_T\|_{C^0} e^{T\Re(\lambda)}$; then

$$\|\chi'_T\|_{C^0} e^{T\Re(\lambda)} < 1 \quad \iff \quad \Re(\lambda) < -\frac{1}{T}\log(\|\chi'_T\|_{C^0}). \tag{5-34}$$

Moreover, integrating by parts once, we have

$$\int_T^{T+1} \chi'_T(t) e^{\lambda t}\, \mathrm{d}t = -\frac{1}{\lambda}\int_T^{T+1} \chi''_T(t) e^{\lambda t}\, \mathrm{d}t,$$

---

[6] Note that this is an unbounded family of operators. Since Fredholm operators are continuous by definition, one has to consider the operators on their domain $\mathcal{D}(\mathcal{H}^{\boldsymbol{r}}_+) := \{f \in \mathcal{H}^{\boldsymbol{r}}_+ \mid Xf \in \mathcal{H}^{\boldsymbol{r}}_+\}$.

which is in absolute value bounded by $(1/|\lambda|)\|\chi_T''\|_{C^0}e^{(T+1)|\Re(\lambda)|}$. Then

$$\frac{1}{|\lambda|}\|\chi_T''\|_{C^0}e^{(T+1)|\Re(\lambda)|}<1 \quad\Longleftrightarrow\quad |\Re(\lambda)|<\frac{\log|\lambda|-\log\|\chi_T''\|_{C^0}}{T+1}. \tag{5-35}$$

Using (5-34) and taking T large enough (changing χ_T in such a way that $\chi_T|_{[T,T+1]}$ is the same as before after a translation), we conclude $1+F(\chi_T,\lambda)$ has no zeroes (in λ) in $\{\Re(\lambda)>-\kappa\}$, where $\kappa=\kappa(T)>0$ can be chosen arbitrarily small; we conclude that $z+\lambda$ is a resonance of $-X-q\omega$. Using additionally (5-35), we conclude that $z+\lambda$ belongs to a finite set of resonances $\mathcal{S}\subset\mathbb{C}$ of $-X-q\omega$ (in the regions defined by (5-34) and (5-35); note that there are no resonances with sufficiently large real part). Observe that the set $\mathcal{S}$ depends only on T, $\|\chi_T'\|_{C^0}$ and $\|\chi_T''\|_{C^0}$. Enumerate elements of the set $\mathcal{S}-z$ by $\lambda_1,\dots,\lambda_k$ for some $k\geq 0$.

We now perturb χ_T by considering $\chi_T+s\eta_T$, where $\eta_T\in C_c^\infty((T,T+1))$ is a smooth cutoff function and $s\in\mathbb{R}$ is small in absolute value. Assume $F(\chi_T,\lambda)=-1$ and $\Re(e^{i\Im(\lambda)t})$ to be positive on an interval $(T_1,T_2)\subset(T,T+1)$ (we argue similarly if it is negative), where $\lambda\in\mathcal{S}-z$. Taking $\eta\neq 0$ to be nonnegative and supported on (T_1,T_2), there is an $s>0$ small enough that

$$1+F(\chi_T+s\eta,\lambda)=-\lambda s\int_T^{T+1}\eta(t)e^{\lambda t}\,\mathrm{d}t\neq 0.$$

Arguing inductively, we ensure that $F(\tilde\chi_T,\lambda_i)\neq -1$ for $i=1,\dots,k$ for some new cutoff function $\tilde\chi_T$ (satisfying all the previously set out conditions of χ_T). We conclude that

$$\ker(\mathbb{1}+B^X(z)+K^X(z))|_{L^2}=\{0\}$$

with these new choices of T and χ_T, proving the claim.

As previously explained, since $B^X(z')$ and $K^X(z')$ depend continuously on X and z' in $\mathcal{L}(L^2)$, there is an $\varepsilon(z)>0$ small enough such that, for $\|X-X_0\|_{C^N}<\varepsilon(z)$ and $|z-z'|<\varepsilon(z)$, we have $\mathbb{1}+B^X(z)+K^X(z)$ invertible on L^2 (since it has empty kernel and is Fredholm of index 0). This implies that there are no resonances in $D(z,\varepsilon(z))$ for $z\in D(z_0,\varepsilon_0)$. By compactness of $D(z_0,\varepsilon_0)$, we conclude that there is an $\varepsilon>0$ such that there are no resonances in $D(z_0,\varepsilon_0)$ for $\|X-X_0\|_{C^N}<\varepsilon$, proving the desired claim.[7]

Step 3: smoothness of the resolvent. Now, using the following resolvent identity valid for $z\in D(z_0,\varepsilon_0)$ and X close to X_0 in C^N,

$$R_-^X(z)-R_-^{X'}(z)=R_-^{X'}(z)(X-X')R_-^X(z),$$

we obtain that $X\mapsto R_-^X(z)$ is twice differentiable in X, uniformly in $z\in D(z_0,\varepsilon_0)$, with

$$\partial_X(R_-^X(z)).Y=R_-^X(z)YR_-^X(z), \tag{5-36}$$

$$\partial_X^2(R_-^X(z)).(Y,Y')=R_-^X(z)Y'R_-^X(z)YR_-^X(z)+R_-^X(z)YR_-^X(z)Y'R_-^X(z), \tag{5-37}$$

where $Y,Y'\in C^\infty(\mathcal{N},T\mathcal{N})$.

---

[7] A different proof of this step can be found in [Bonthonneau 2020].

Using the first part of Theorem 5.14, namely the boundedness of $R_-^X(z)$ on the spaces $\mathcal{H}_+^{\boldsymbol{r}}$ for X close to X_0 in C^2-norm, we deduce that the first derivative (5-36) is bounded as a map

$$\mathcal{H}_+^{(r_\perp,r_0)} \xrightarrow{R_-^X(z)} \mathcal{H}_-^{(r_\perp,r_0)} \xrightarrow{Y} \mathcal{H}_-^{(r_\perp,r_0+1)} \xrightarrow{R_-^X(z)} \mathcal{H}_-^{(r_\perp,r_0+1)},$$

and similarly the second derivative (5-37) is bounded as a map $\mathcal{H}_-^{(r_\perp,r_0)} \to \mathcal{H}_-^{(r_\perp,r_0+2)}$, and this holds for all X close enough to X_0 in the C^N-topology, with $N \gg 1$ large enough, and for all $z \in D(z_0,\varepsilon_0)$. Moreover, the dependence on z in (5-36) and (5-37) is holomorphic. This completes the proof of Theorem 5.14. ☐

5C. *Smoothness of the scattering map with respect to the metric.* The goal of this paragraph is to prove Proposition 4.2. We start with the following lemma.

Lemma 5.21. *If R_g and R_{g_e} are the resolvents defined in (2-14) for (M,g) and (M_e,g_e), we have, for $X=\psi\widetilde{X}_g$ defined in Section 2A2, that, for all $z\in\mathbb{C}$,*

$$R_g(z) = \mathbf{1}_{\mathcal{M}} R_+^X(z)\mathbf{1}_{\mathcal{M}} \quad \textit{and} \quad R_{g_e}(z) = \mathbf{1}_{\mathcal{M}_e} R_+^X(z)\mathbf{1}_{\mathcal{M}_e},$$

when acting on $C_{\mathrm{c}}^\infty(\mathcal{M}^\circ)$ and $C_{\mathrm{c}}^\infty(\mathcal{M}_e^\circ)$, respectively.

Proof. This is an obvious consequence of the following fact: for $f\in C_{\mathrm{c}}^\infty(\mathcal{M}^\circ)$, writing $u_z = (R_g(z)f)|_{\mathcal{M}}$, if $\Re(z)\gg 1$, we have

$$u_z(y) = -\int_0^{\tau_g(y)} e^{-zt} f(\varphi_t^g(y))\,\mathrm{d}t,$$

and similarly for $R_{g_e}(z)$. Indeed, if $y\in\mathcal{M}$, the flow line $\gamma := \bigcup_{t\ge 0}\varphi_t^g(y)$ is contained in $\{\rho > -\rho_0\}$, and the convexity of $\mathcal{M}$ implies that $\gamma\cap\mathcal{M} = \bigcup_{t\in[0,\tau_g(y))}\varphi_t^g(y)$. ☐

We can now complete the proof of Proposition 4.2.

Proof of Proposition 4.2. Let $\omega\in C^\infty(\partial_+\mathcal{M})$. Observe that, by Lemmas 2.7 and 5.21,

$$\chi\mathcal{S}_g(\omega) = \chi[R_{g_e}(\tilde{\chi}\omega\delta_{\partial_+\mathcal{M}})]|_{\partial_-\mathcal{M}},$$

where $\tilde{\chi}$ is some smooth cutoff function equal to 1 everywhere except in a neighborhood of $\partial_0\mathcal{M}$, and where $\tilde{\chi}\omega\delta_{\partial_+\mathcal{M}}\in\mathcal{D}'(\mathcal{N})$ denotes the distribution defined by

$$\langle\tilde{\chi}\omega\delta_{\partial_+\mathcal{M}},\varphi\rangle := \int_{\partial_+\mathcal{M}} \tilde{\chi}\omega\varphi\,\mathrm{d}\mu_\partial.$$

Let $u := \tilde{\chi}\omega\delta_{\partial_+\mathcal{M}}$. Since $\partial_+\mathcal{M}$ is of codimension 1, we have that $u\in H^{-1/2-\varepsilon}(\mathcal{N})$ for all $\varepsilon>0$. Let $N^*\partial_+\mathcal{M}\subset T^*_{\partial_+\mathcal{M}}\mathcal{N}$ be the conormal of $\partial_+\mathcal{M}$ in $\mathcal{N}$ (i.e., $N^*\partial_+\mathcal{M}(T\partial_+\mathcal{M})=0$). By a standard argument of distribution theory, the wavefront set of u satisfies $\mathrm{WF}(u)\subset N^*\partial_+\mathcal{M}$.

The escape function m provided by Proposition 5.1 can be constructed so that, over $\mathcal{M}$, it has only support in a small conic neighborhood of $(E_-^{X_0})^*$ and $(E_+^{X_0})^*$. In particular, this construction can be achieved so that

$$N^*\partial_+\mathcal{M}\cap\operatorname{supp}(m) = \varnothing. \tag{5-38}$$

Indeed, a covector $V^* \in T^*_{\partial_+\mathcal{M}}\mathcal{N}$ such that $V^* \in (E_+^{X_0})^*$ must satisfy $V^*(X_0) = 0$ and $V^*(W)$ for all $W \in T\partial_+\mathcal{M}$, but since X_0 is transverse to $\partial_-\mathcal{M}$, one gets $V^* = 0$. We now take a regularity pair $\boldsymbol{r} := (r_\perp, r_0)$ with $\frac{1}{2} < r_0 < 1$, $r_0 + 2 < r_\perp < 3$ and a small $\delta > 0$ such that $-\Lambda(r_\perp - (r_0+2)) + C_\star\delta < 0$. By the previous discussion, $u \in \mathcal{H}^{\boldsymbol{r}}_-(\mathcal{N})$, i.e., since $\mathcal{H}^{\boldsymbol{r}}_-(\mathcal{N})$ is microlocally equivalent to H^{-r_0} near $N^*\partial_+\mathcal{M}$. Denote by $\theta \in C^\infty_{\mathrm{c}}(\mathcal{M}_e^\circ)$ a cutoff function equal to 1 near $\mathcal{M}$. We claim that the map

$$C^\infty(M, \otimes_S^2 T^*M) \ni g \mapsto \theta R_{g_e}\theta \in \mathcal{L}(\mathcal{H}_-^{(r_\perp, r_0)}(\mathcal{N}), \mathcal{H}_-^{(r_\perp, r_0+2)}(\mathcal{N}))$$

is C^2 for g close to g_0. Indeed, similar to the proof of Theorem 5.14 (alternatively we could simply use Theorem 1.10 along with the fact that $g \mapsto X_g$ is smooth; we give a direct argument instead), we can use the resolvent identity (recall $X = \psi\widetilde{X}_g$ and $X_0 = \psi\widetilde{X}_{g_0}$)

$$\theta R_{g_e}\theta - \theta R_{g_{0e}}\theta = \theta R_+^X(0)(X_0 - X)R_+^{X_0}(0)\theta$$

to deduce that $g \mapsto \theta R_{g_e}\theta$ is differentiable twice, with

$$\partial_g\theta R_{g_e}\theta = -\theta R_+^X(0)(\partial_g X)R_+^X(0)\theta, \tag{5-39}$$

$$\partial_g^2\theta R_{g_e}\theta = 2\theta R_+^X(0)(\partial_g X)R_+^X(0)(\partial_g X)R_+^X(0)\theta - \theta R_+^X(0)(\partial_g^2 X)R_+^X(0)\theta. \tag{5-40}$$

The first derivative (5-39) is bounded as a map

$$\mathcal{H}_-^{(r_\perp, r_0)} \xrightarrow{R_+^X(0)} \mathcal{H}_-^{(r_\perp, r_0)} \xrightarrow{\partial_g X} \mathcal{H}_-^{(r_\perp, r_0+1)} \xrightarrow{R_+^X(0)} \mathcal{H}_-^{(r_\perp, r_0+1)},$$

and similarly the second derivative (5-40) is bounded as a map $\mathcal{H}_-^{(r_\perp, r_0)} \to \mathcal{H}_-^{(r_\perp, r_0+2)}$, and this holds for all g close enough to g_0 in the C^N-topology, with $N \gg 1$ large enough.

As a consequence,

$$C^\infty(M, \otimes_S^2 T^*M) \ni g \mapsto \theta R_{g_e}\theta u = \theta R_{g_e}u \in \mathcal{H}_-^{(r_\perp, r_0+2)}(\mathcal{N})$$

is C^2-regular for g close to g_0. Note that, as $r_\perp + r_0 + 2 < 6$,

$$\mathcal{H}_-^{(r_\perp, r_0+2)}(\mathcal{N}) \hookrightarrow H^{-6}(\mathcal{N}).$$

Moreover, it satisfies $X_{g_e}\theta R_{g_e}u = 0$ near $\partial_-\mathcal{M}$, so that $\mathrm{WF}(\theta R_{g_e}u) \subset \{p_{X_{g_e}} = 0\}$. Therefore, the restriction $\chi[\theta R_{g_e}u]|_{\partial_-\mathcal{M}} = \chi[R_{g_e}u]|_{\partial_-\mathcal{M}} \in H^{-6}(\partial_-\mathcal{M})$ is well defined and depends in a C^2-fashion on the metric $g \in C^N(M, \otimes_S^2 T^*M)$, proving the first part of Proposition 4.2.

Using (5-39) and (5-40), and writing $g = g_0 + h$ with $\|h\|_{C^N} \le \delta$ for $\delta > 0$ small and N chosen large, we have as above, by Taylor expansion, for $u = \tilde{\chi}\omega\delta_{\partial_+\mathcal{M}}$,

$$\theta R_{g_e}u = \theta R_{g_{0e}}u - \theta R_+^{X_0}(0)((\partial_g X)|_{g=g_0}.h)R_+^{X_0}(0)u + \int_0^1 (1-t)\partial_g^2(\theta R_{g_{0e}+th}u).(h,h)\,dt. \tag{5-41}$$

Let $Y_g(h) := \partial_g X(h) \in C^\infty(\mathcal{N}, T\mathcal{N})$ for any smooth metric g close to g_0 in $C^N(M, \otimes_S^2 T^*M)$. For all $k \ge 1$, one has $\|Y_g(h)\|_{C^k(\mathcal{N}, T\mathcal{N})} \le C_k\|h\|_{C^{k+1}}$ for some $C_k > 0$ depending uniformly on $\|g\|_{C^{k+1}}$. Let $Z_g(h,h) = \partial_g^2 X(h,h) \in C^\infty(\mathcal{N}, T\mathcal{N})$. One has $\|Z_g(h,h)\|_{C^k(\mathcal{N}, T\mathcal{N})} \le C_k\|h\|^2_{C^{k+2}}$ for some $C_k > 0$

depending uniformly on $\|g\|_{C^{k+2}}$. Then the remainder term in (5-41) satisfies, for $g_e(t)$ the extension of $g(t) = g_0 + th$ (with $t \in [0, 1]$) and $X(t) = \psi \widetilde{X}_{g(t)}$,

$$\partial_g^2(\theta R_{g_e(t)}u)(h, h) = 2\theta R_+^{X(t)}(0)Y_{g(t)}(h)R_+^{X(t)}(0)Y_{g(t)}(h)R_+^{X(t)}(0)u - \theta R_+^{X(t)}(0)Z_{g(t)}(h, h)R_+^{X(t)}(0)u.$$

By the analysis above, for $\delta > 0$ small and $N > 0$ large enough, there exists a constant $C > 0$ such that, for $h = g(1) - g_0$ such that $\|h\|_{C^N} \le \delta$,

$$\begin{aligned} \sup_{t\in[0,1]} \|R_+^{X(t)}u\|_{\mathcal{H}_-^{(r_\perp, r_0+j)}(\mathcal{N})} &\le C && \text{for all } j \in \{0, 1, 2\},\\ \sup_{t\in[0,1]} \|Y_{g(t)}(h)\|_{\mathcal{H}_-^{(r_\perp, r_0+j)} \to \mathcal{H}_-^{(r_\perp, r_0+1+j)}} &\le C\|h\|_{C^N} && \text{for all } j \in \{0, 1\},\\ \sup_{t\in[0,1]} \|Z_{g(t)}(h, h)\|_{\mathcal{H}_-^{(r_\perp, r_0)} \to \mathcal{H}_-^{(r_\perp, r_0+2)}} &\le C\|h\|_{C^N}^2. \end{aligned}$$

Combining the last inequalities with (5-41), this shows (4-1) by applying the restriction to $\partial_-\mathcal{M}$ on the left of (5-41). Note that, in turn, this gives an expression of $\partial_g \mathcal{S}_g|_{g=g_0}$ in terms of $R_+^{X_0}(0)$ and $\partial_g X|_{g=g_0}$. This concludes the proof. ☐

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