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DYNAMICAL TORSION FOR CONTACT ANOSOV FLOWS

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We introduce a new object, the dynamical torsion, which extends the potentially ill-defined value at 0 of the Ruelle zeta function of a contact Anosov flow, twisted by an acyclic representation of the fundamental group. We show important properties of the dynamical torsion: it is invariant under deformations among contact Anosov flows, it is holomorphic in the representation and it has the same logarithmic derivative as some refined combinatorial torsion of Turaev. This shows that the ratio between this torsion and the Turaev torsion is locally constant on the space of acyclic representations.

In particular, for contact Anosov flows path-connected to the geodesic flow of some hyperbolic manifold among contact Anosov flows, we relate the leading term of the Laurent expansion of ζ at the origin, the Reidemeister torsion and the torsions of the finite-dimensional complexes of the generalized resonant states of both flows for the resonance 0. This extends previous work of Dang, Guillarmou, Rivière and Shen (*Invent. Math.* **220**:2 (2020), 525–579) on the Fried conjecture near geodesic flows of hyperbolic 3-manifolds, to hyperbolic manifolds of any odd dimension.

1. Introduction

Let M be a closed odd-dimensional manifold and (E, ∇) be a flat vector bundle over M. The parallel transport of the connection ∇ induces a conjugacy class of representation $\rho \in \mathrm{Hom}(\pi_1(M), \mathrm{GL}(\mathbb{C}^d))$ (every representation of the fundamental group can be obtained in this way; see Section 11.1). Moreover, ∇ defines a differential on the complex $\Omega^\bullet(M, E)$ of E-valued differential forms on M and thus cohomology groups $H^\bullet(M, \nabla) = H^\bullet(M, \rho)$ (note that we use the notation ∇ also for the twisted differential induced by ∇, whereas it can be denoted by d^∇ in other references). We will say that ∇ (or ρ) is acyclic if those cohomology groups are trivial.

If ρ is unitary (or equivalently, if there exists a hermitian structure on E preserved by ∇) and acyclic, Reidemeister [1935] introduced a combinatorial invariant $\tau_{\mathrm{R}}(\rho)$ of the pair (M, ρ), the so-called *Franz–Reidemeister torsion* (or R-torsion), which is a positive number. This allowed him to classify lens spaces in dimension 3; this result was then extended in higher dimensions by Franz [1935] and de Rham [1936].

On the analytic side, Ray and Singer [1971] introduced another invariant $\tau_{\mathrm{RS}}(\rho)$, the *analytic torsion*, defined via the derivative at 0 of the spectral zeta function of the Laplacian given by the Hermitian metric on E and some Riemannian metric on M. They conjectured the equality of the analytic and Reidemeister torsions. This conjecture was proved independently by Cheeger [1979] and Müller [1978], assuming only that ρ is unitary (both R-torsion and analytic torsion have a natural extension if ρ is unitary and not

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acyclic). The Cheeger–Müller theorem was extended to unimodular flat vector bundles by Müller [1993] and to arbitrary flat vector bundles by Bismut and Zhang [1992].

In the context of hyperbolic dynamical systems, Fried [1987] was interested in the link between the R-torsion and the Ruelle zeta function of an Anosov flow X, which is defined by

$$\zeta_{X,\rho}(s) = \prod_{\gamma \in \mathcal{G}_X^\#} \det(1 - \varepsilon_\gamma \rho([\gamma]) e^{-s\ell(\gamma)}), \quad \operatorname{Re}(s) \gg 0,$$

where $\mathcal{G}_X^\#$ is the set of primitive closed orbits of X, $\ell(\gamma)$ is the period of γ and $\varepsilon_\gamma = 1$ if the stable bundle of γ is orientable and $\varepsilon_\gamma = -1$ otherwise. Using Selberg's trace formula, Fried could relate the behavior of $\zeta_{X,\rho}(s)$ near $s = 0$ with τ_{R}, as follows.

Theorem 1 [Fried 1986]. *Let $M = SZ$ be the unit tangent bundle of some closed oriented hyperbolic manifold Z, and denote by X its geodesic vector field on M. Assume that $\rho : \pi_1(M) \to O(d)$ is an acyclic and unitary representation. Then $\zeta_{X,\rho}$ extends meromorphically to $\mathbb{C}$. Moreover, it is holomorphic near $s = 0$ and*

$$|\zeta_{X,\rho}(0)|^{(-1)^q} = \tau_{\mathrm{R}}(\rho), \tag{1-1}$$

where $2q + 1 = \dim M$ and $\tau_{\mathrm{R}}(\rho)$ is the Reidemeister torsion of (M, ρ).

Fried [1987] conjectured that the same holds true for negatively curved locally symmetric spaces. This was proved by Moscovici and Stanton [1991] and Shen [2018].

For analytic Anosov flows, the meromorphic continuation of $\zeta_{X,\rho}$ was proved by Rugh [1996] in dimension 3 and by Fried [1995] in higher dimensions. Then Sánchez-Morgado [1993; 1996] proved in dimension 3 that if ρ is acyclic, unitary, and satisfies that $\rho([\gamma]) - \varepsilon_\gamma^j$ is invertible for $j \in \{0, 1\}$ for some closed orbit γ, then (1-1) is true.

For general smooth Anosov flows, the meromorphic continuation of $\zeta_{X,\rho}$ was proved by Giuletti, Liverani and Pollicott [Giulietti et al. 2013] and alternatively by Dyatlov and Zworski [2016]. The Axiom A case was treated by Dyatlov and Guillarmou [2018]. Quoting the commentary from Zworski [2018] on Smale's seminal paper [1967], equation (1-1) "would link dynamical, spectral and topological quantities. [...] In the case of smooth manifolds of variable negative curvature, equation (1-1) remains completely open". However in [Dyatlov and Zworski 2017], the authors were able to prove the following.

Theorem 2 (Dyatlov–Zworski). *Suppose (Σ, g) is a negatively curved orientable Riemannian surface. Let X denote the associated geodesic vector field on the unitary cotangent bundle $M = S^*\Sigma$. Then, for some $c \neq 0$, we have as $s \to 0$*

$$\zeta_{X,\mathbf{1}}(s) = c s^{|\chi(\Sigma)|}(1 + \mathcal{O}(s)), \tag{1-2}$$

where $\mathbf{1}$ is the trivial representation $\pi_1(S^*\Sigma) \to \mathbb{C}^*$ and $\chi(\Sigma)$ is the Euler characteristic of Σ. In particular, the length spectrum $\{\ell(\gamma) : \gamma \in \mathcal{G}_X^\#\}$ determines the genus.

This result was generalized in the recent preprint [Cekić and Paternain 2020] to volume-preserving Anosov flows in dimension 3.

In the same spirit and using similar microlocal methods, Guillarmou, Rivière, Shen and the second author [Dang et al. 2020] showed:

Theorem 3 (Dang–Rivière–Guillarmou–Shen). *Let ρ be an acyclic representation of $\pi_1(M)$. Then the map*

$$X \mapsto \zeta_{X,\rho}(0)$$

is locally constant on the open set of smooth vector fields which are Anosov and for which 0 *is not a Ruelle resonance, that is,* $0 \notin \mathrm{Res}(\mathcal{L}_X^{\nabla})$. *If X preserves a smooth volume form and* $\dim(M) = 3$, (1-1) *holds true if $b_1(M) \neq 0$ or under the same assumption used in* [Sánchez-Morgado 1996].

Let us comment on the notion of Ruelle resonance to explain the assumptions in the above theorem. All recent works on the analytic continuation of the Ruelle zeta function are important by-products of new functional methods to study hyperbolic flows. They rely on the construction of spaces of anisotropic distributions adapted to the dynamics, initiated by Kitaev [1999], Blank, Keller and Liverani [Blank et al. 2002], Baladi [2005; 2018], Baladi and Tsujii [2007], Gouëzel and Liverani [2006], Liverani [2005], Butterley and Liverani [2007; 2013], and many others, where we refer to the recent book [Baladi 2018] for precise references. These spaces allow one to define a suitable notion of spectrum for the operator $\mathcal{L}_X^{\nabla} = \nabla\iota_X + \iota_X\nabla$, where ι is the interior product, acting on $\Omega^\bullet(M, E)$. This spectrum is the set of so-called Pollicott–Ruelle resonances $\mathrm{Res}(\mathcal{L}_X^{\nabla})$, which forms a discrete subset of $\mathbb{C}$ and contains all zeros and poles of $\zeta_{X,\rho}$. Faure, Roy and Sjöstrand [Faure et al. 2008] and Faure and Sjöstrand [2011] initiated the use of microlocal methods to describe these anisotropic spaces of distributions giving a purely microlocal approach to study Ruelle resonances. This was further developed by Dyatlov and Zworski to study Ruelle zeta functions.

However, if $0 \in \mathrm{Res}(\mathcal{L}_X^{\nabla})$ then the results of [Dang et al. 2020] no longer apply since the zeta function $\zeta_{X,\rho}$ might have a pole or zero at $s = 0$ (recall zeros and poles of $\zeta_{X,\rho}$ are contained in $\mathrm{Res}(\mathcal{L}_X^{\nabla})$). One goal of this article is to remove the assumption that 0 is not a Ruelle resonance. In the spirit of Theorem 2 and the Fried conjecture, we can state a theorem which follows from more general results of the present paper (see Section 2).

Theorem 4. *Let (Z, g_0) be a compact hyperbolic manifold of dimension q and ρ be the lift to S^*Z of some acyclic unitary representation $\pi_1(Z) \to \mathrm{GL}(\mathbb{C}^d)$. Then, for every metric g which is path-connected to g_0 in the space of negatively curved metrics, there exists $m(g, \rho) \in \mathbb{Z}$ such that*

$$|\zeta_{X_g,\rho}(s)|^{(-1)^q} = |s|^{(-1)^q m(g,\rho)} \underbrace{\tau_{\mathrm{R}}(\rho)}_{\mathrm{R}\text{-}torsion} \left|\frac{\tau(C^\bullet(X_{g_0}, \rho))}{\tau(C^\bullet(X_g, \rho))}\right| (1 + \mathcal{O}(s)), \tag{1-3}$$

where X_g denotes the geodesic vector field of g and $\tau(C^\bullet(X_g, \rho))$ is the refined torsion of the finite-dimensional space of resonant states for the resonance 0 *of (X_g, ρ).*

In the above statement, the vector field X_g generates a contact Anosov flow on the contact manifold $S_g^*Z = \{(x, \xi) \in T^*Z : |\xi|_g = 1\}$.[1] The finite-dimensional torsion $\tau(C^\bullet(X_g, \rho))$ will be described in Section 2 below.

[1] This means concretely that changing the metric g on Z affects both the contact form ϑ and Reeb field X on S^*Z.

2. Main results

There are two restrictions in Theorem 3 of [Dang et al. 2020]. The first restriction is that

$$|\zeta_{X,\rho}(0)|^{(-1)^r} = \tau_{\mathrm{R}}(\rho)$$

is an equality of positive real numbers and the representation ρ is unitary. For arbitrary acyclic representations $\rho : \pi_1(M) \to \mathrm{GL}(\mathbb{C}^d)$, one could wonder if the phase of the complex number $\zeta_{X,\rho}(0)$ contains topological information. For instance, if it can be compared with some complex-valued torsion defined for general acyclic representations $\rho : \pi_1(M) \to \mathrm{GL}(\mathbb{C}^d)$. The second restriction concerns the assumption that 0 is not a Ruelle resonance. Apart from the low-dimension cases studied in [Dang et al. 2020], this assumption is particularly hard to control and is difficult to check for explicit examples.

Our goal in the present work is to partially overcome these two obstacles. In the case where X induces a contact flow, which means that $X = X_\vartheta$ is the Reeb vector field of some contact form ϑ on M, we deal with these difficulties by introducing a *dynamical torsion* $\tau_\vartheta(\rho)$ which is a new object defined for any acyclic ρ and which coincides with $\zeta_{X,\rho}(0)^{\pm 1}$ if $0 \notin \mathrm{Res}(\mathcal{L}_X^\nabla)$. Before stating our main results, let us introduce the two main characters of our discussion in the next two subsections.

2.1. ***Refined versions of torsion.*** The Franz–Reidemeister torsion τ_{R} is given by the modulus of some alternate product of determinants and is therefore real-valued. One cannot get a canonical object by removing the modulus since one has to make some choices to define the combinatorial torsion, and the ambiguities in these choices affect the alternate product of determinants. To remove indeterminacies arising in the definition of the combinatorial torsion, Turaev [1986; 1989; 1997] introduced in the acyclic case a refined version of the combinatorial R-torsion, the *refined combinatorial torsion*. It is a complex number $\tau_{\mathfrak{e},\mathfrak{o}}(\rho)$ which depends on additional combinatorial data, namely an Euler structure $\mathfrak{e}$ and a homology orientation $\mathfrak{o}$ of M, and which satisfies $|\tau_{\mathfrak{e},\mathfrak{o}}(\rho)| = \tau_{\mathrm{R}}(\rho)$ if ρ is acyclic and unitary. We refer the reader to Section 9.2 for precise definitions. Later, Farber and Turaev [2000] extended this object to nonacyclic representations. In this case, $\tau_{\mathfrak{e},\mathfrak{o}}(\rho)$ is an element of the determinant line of cohomology $\det H^\bullet(M, \rho)$.

Motivated by the work of Turaev, but from the analytic side, Braverman and Kappeler [2007b; 2007c; 2008] introduced a refined version of the Ray–Singer analytic torsion called *refined analytic torsion* $\tau_{\mathrm{an}}(\rho)$. It is complex-valued in the acyclic case. Their construction heavily relies on the existence of a chirality operator Γ_g, that is,

$$\Gamma_g : \Omega^\bullet(M, E) \to \Omega^{n-\bullet}(M, E), \qquad \Gamma_g^2 = \mathrm{Id},$$

which is a renormalized version of the Hodge star operator associated with some metric g. They showed that the ratio

$$\rho \mapsto \frac{\tau_{\mathrm{an}}(\rho)}{\tau_{\mathfrak{e},\mathfrak{o}}(\rho)}$$

is a holomorphic function on the representation variety given by an explicit local expression, up to a local constant of modulus 1. This result is an extension of the Cheeger–Müller theorem. Simultaneously, Burghelea and Haller [2007] introduced a complex-valued analytic torsion, which is closely related to the refined analytic torsion [Braverman and Kappeler 2007a] when it is defined; see [Huang 2007] for comparison theorems.

2.2. ***Dynamical torsion.*** We now assume that $X = X_\vartheta$ is the Reeb vector field of some contact form ϑ on M. Let us briefly describe the construction of the dynamical torsion. In the spirit of [Braverman and Kappeler 2007c], we use a chirality operator associated with the contact form ϑ,

$$\Gamma_\vartheta : \Omega^\bullet(M, E) \to \Omega^{n-\bullet}(M, E), \quad \Gamma_\vartheta^2 = \mathrm{Id},$$

see Section 6, analogous to the usual Hodge star operator associated with a Riemannian metric. Let $C^\bullet \subset \mathcal{D}'^\bullet(M, E)$ be the finite-dimensional space of Pollicott–Ruelle generalized resonant states of $\mathcal{L}_X^\nabla$ for the resonance 0, that is,

$$C^\bullet = \{u \in \mathcal{D}'^\bullet(M, E) : \mathrm{WF}(u) \subset E_u^*, \text{ there exists } N \in \mathbb{N} \text{ such that } (\mathcal{L}_X^\nabla)^N u = 0\},$$

where WF is the Hörmander wavefront set, $E_u^* \subset T^*M$ is the unstable cobundle of X,[2] see Section 5, and $\mathcal{D}'(M, E)$ denotes the space of E-valued currents. Then ∇ induces a differential on $C^\bullet$ which makes it a finite-dimensional cochain complex. Then a result from [Dang and Rivière 2020b] implies that the complex $(C^\bullet, \nabla)$ is acyclic if we assume that ∇ is. Because Γ_ϑ commutes with $\mathcal{L}_X^\nabla$, it induces a chirality operator on $C^\bullet$. Therefore we can compute the torsion $\tau(C^\bullet, \Gamma_\vartheta)$ of the finite-dimensional complex $(C^\bullet, \nabla)$ with respect to Γ_ϑ, as described in [Braverman and Kappeler 2007c] (see Section 3). Then we define the *dynamical torsion* τ_ϑ as the product

$$\tau_\vartheta(\rho)^{(-1)^q} = \pm \underbrace{\tau(C^\bullet, \Gamma_\vartheta)^{(-1)^q}}_{\text{finite-dimensional torsion}} \times \underbrace{\lim_{s\to 0} s^{-m(X,\rho)} \zeta_{X,\rho}(s)}_{\text{renormalized Ruelle zeta function at } s=0} \in \mathbb{C} \setminus 0,$$

where the sign $\pm$ will be given later, $m(X, \rho)$ is the order of $\zeta_{X,\rho}(s)$ at $s = 0$ and $q = (\dim(M) - 1)/2$ is the dimension of the unstable bundle of X. Note that the order $m(X, \rho) \in \mathbb{Z}$ is a priori not stable under perturbations of (X, ρ), in fact both terms in the product may not be invariant under small changes of ϑ, whereas the dynamical torsion τ_ϑ has interesting invariance properties as we will see below.

2.3. ***Statement of the results.*** We denote by $\mathrm{Rep}_{\mathrm{ac}}(M, d)$ the set of acyclic representations $\pi_1(M) \to \mathrm{GL}(\mathbb{C}^d)$ and by $\mathcal{A} \subset \mathcal{C}^\infty(M, TM)$ the space of contact forms on M whose Reeb vector field induces an Anosov flow. This is an open subset of the space of contact forms. For any $\vartheta \in \mathcal{A}$, we denote by X_ϑ its Reeb vector field. Recall that we want to study the value at 0 without taking the modulus. As in Fried's case, $\zeta_{X,\rho}(0)$ might be ill–defined since $0 \in \mathrm{Res}(\mathcal{L}_X^\nabla)$ and this was the reason for introducing the more general object $\tau_\vartheta(\rho)$. Our goal is to compare this new complex number with the refined torsion. As a first step towards this, our first result shows $\tau_\vartheta(\rho)$ is invariant by small perturbations of the contact form $\vartheta \in \mathcal{A}$.

Theorem 5. *Let (M, ϑ) be a contact manifold such that the Reeb vector field of ϑ induces an Anosov flow. Let $(\vartheta_\tau)_{\tau\in(-\varepsilon,\varepsilon)}$ be a smooth family in $\mathcal{A}$. Then $\partial_\tau \log \tau_{\vartheta_\tau}(\rho) = 0$ for any $\rho \in \mathrm{Rep}_{\mathrm{ac}}(M, d)$.*

[2] The annihilator of $E_u \oplus \mathbb{R}X$ where $E_u \subset TM$ denotes the unstable bundle of the flow.

Remark 2.1. In the case where the representation ρ is not acyclic, we can still define $\tau_\vartheta(\rho)$ as an element of the determinant line $\det H^\bullet(M,\rho)$; see Remark 6.5. This element is invariant under perturbations of $\vartheta \in \mathcal{A}$; see Remark 7.1.

This result implies that the map $\vartheta \in \mathcal{A} \mapsto \tau_\vartheta(\rho)$ is locally constant for all $\rho \in \mathrm{Rep}_{\mathrm{ac}}(M,d)$. To apply Theorem 3 in the case of contact Anosov flows, we need to make small perturbations near a contact Anosov flow such that $0 \notin \mathrm{Res}(\mathcal{L}_X^\nabla)$: if we have a C^1 family of contact Anosov flows $(X_t)_{t\in[0,1]}$ such that 0 is not a resonance of $\mathcal{L}_{X_0}^\nabla$ and $\mathcal{L}_{X_1}^\nabla$ but is a resonance of $\mathcal{L}_{X_u}^\nabla$ for some $u \in]0,1[$, then we cannot claim that $\zeta_{X_0,\rho}(0) = \zeta_{X_1,\rho}(0)$ using Theorem 3; however, we can claim that $\tau_{\vartheta_0}(\rho) = \tau_{\vartheta_1}(\rho)$ with Theorem 5.

Our second result aims to compare τ_ϑ with Turaev's refined version of the Reidemeister torsion $\tau_{\mathfrak{e},\mathfrak{o}}$, which depends on some choice of Euler structure $\mathfrak{e}$ and a homology orientation $\mathfrak{o}$. An analog of the Fried conjecture would be to prove the equality $\tau_\vartheta(\rho) = \tau_{\mathfrak{e},\mathfrak{o}}(\rho)$ for some $(\mathfrak{e},\mathfrak{o})$ and for all $\rho \in \mathrm{Rep}_{\mathrm{ac}}(M,d)$ (this would imply $|\tau_R(\rho)| = |\zeta_{X,\rho}(0)|^{\pm 1}$ if ρ is acyclic and unitary and if $0 \notin \mathrm{Res}(\mathcal{L}_X^\nabla)$). We prove a weaker result, which shows that the derivatives in $\rho \in \mathrm{Rep}_{\mathrm{ac}}(M,d)$ of $\log \tau_\vartheta(\rho)$ and $\log \tau_{\mathfrak{e},\mathfrak{o}}(\rho)$ coincide.

Theorem 6. *Let (M,ϑ) be a contact manifold such that the Reeb vector field of ϑ induces an Anosov flow. Then $\rho \in \mathrm{Rep}_{\mathrm{ac}}(M,d) \mapsto \tau_\vartheta(\rho)$ is holomorphic*[3] *and there exists an Euler structure $\mathfrak{e}$ such that, for any homology orientation $\mathfrak{o}$ and any smooth family $(\rho_u)_{u\in(-\varepsilon,\varepsilon)}$ of $\mathrm{Rep}_{\mathrm{ac}}(M,d)$,*

$$\partial_u \log \tau_\vartheta(\rho_u) = \partial_u \log \tau_{\mathfrak{e},\mathfrak{o}}(\rho_u).$$

Moreover, if $\dim M = 3$ and $b_1(M) \neq 0$, the map $\rho \mapsto \tau_\vartheta(\rho)/\tau_{\mathfrak{e},\mathfrak{o}}(\rho)$ is of modulus 1 on the connected components of $\mathrm{Rep}_{\mathrm{ac}}(M,d)$ containing an acyclic and unitary representation.

In [Dang et al. 2020], for ρ acyclic, the authors proved that $0 \notin \mathrm{Res}(\mathcal{L}_X^\nabla)$ implies that $X \mapsto \zeta_{X,\rho}(0)$ is locally constant. Then the equality $|\zeta_{X,\rho}(0)| = \tau_{\mathrm{R}}(\rho)$ was proved indirectly by working near analytic Anosov flows in dimension 3 or near geodesic flows of hyperbolic 3-manifolds, where the equality is known by the works of Sanchez Morgado and Fried, relying on the fact that $\zeta_{X,\rho}(0)$ remains constant by small perturbations of the vector field X. Whereas in the above theorem, for any contact Anosov flow in any odd dimension, we directly compare the log derivatives of the dynamical and refined torsions as holomorphic functions on the representation variety. We do not need to work near some vector field X for which the equality $|\zeta_{X,\rho}(0)| = \tau_{\mathrm{R}}(\rho)$ is already known.

Finally, our third result aims to describe how $\partial_u \log \tau_\vartheta(\rho_u)$ depends on the choice of the contact Anosov vector field X_ϑ.

Theorem 7. *Let (M,ϑ) be a contact manifold such that the Reeb vector field of ϑ induces an Anosov flow. Let $(\rho_u)_{|u|\leqslant\varepsilon}$ be a smooth family in $\mathrm{Rep}_{\mathrm{ac}}(M,d)$. Then, for any $\eta \in \mathcal{A}$,*

$$\partial_u \log \tau_\eta(\rho_u) = \partial_u \log \tau_\vartheta(\rho_u) + \partial_u \log \underbrace{\det \rho_u(\mathrm{cs}(X_\vartheta, X_\eta))}_{\textit{topological}}$$

as differential 1-forms on $\mathrm{Rep}_{\mathrm{ac}}(M,d)$ and where $\mathrm{cs}(X_\vartheta,X_\eta) \in H_1(M,\mathbb{Z})$ is the Chern–Simons class of the pair of vector fields (X_ϑ, X_η).

[3] $\mathrm{Rep}_{\mathrm{ac}}(M,d)$ is a variety over $\mathbb{C}$; see Section 11.2 for the right notion of holomorphicity.

Here, by $\det \rho_u(\mathrm{cs}(X_\vartheta, X_\eta))$ we mean $\det \rho_u(c)$ where $c \in \pi_1(M)$ is any element such that its homology class $[c] \in H_1(M, \mathbb{Z})$ coincides with $\mathrm{cs}(X_\vartheta, X_\eta)$ (note that the value of the determinant does not depend on the choice of c). This underbraced term is indeed topological as the Chern–Simons class $\mathrm{cs}(X_\vartheta, X_\eta) \in H_1(M, \mathbb{Z})$ measures the obstruction to find a homotopy among nonsingular vector fields connecting X_ϑ and X_η. In particular, if ϑ and η are connected by some path in $\mathcal{A}$, then $\mathrm{cs}(X_\vartheta, X_\eta) = 0$, which yields $\det \rho(\mathrm{cs}(X_\vartheta, X_\eta)) = 1$; hence $\partial_u \log \tau_\eta(\rho_u) = \partial_u \log \tau_\vartheta(\rho_u)$ for any acyclic ρ. We refer the reader to Section 9.1 for the definition of Chern–Simons classes.

Because the dynamical torsion is constructed with the help of the dynamical zeta function $\zeta_{X,\rho}$, we deduce from the above theorem some information about the behavior of $\zeta_{X,\rho}(s)$ near $s = 0$, as follows.

Corollary 8. *Let M be a closed odd-dimensional manifold. Then, for all connected open subsets $\mathcal{U} \subset \mathrm{Rep}_{\mathrm{ac}}(M, d)$ and $\mathcal{V} \subset \mathcal{A}$, there exists a constant C such that, for every Anosov contact form $\vartheta \in \mathcal{V}$ and every representation $\rho \in \mathcal{U}$,*

$$\zeta_{X_\vartheta,\rho}(s)^{(-1)^q} = C s^{(-1)^q m(\rho, X_\vartheta)} \frac{\tau_{\mathfrak{e}_{X_\vartheta},\mathfrak{o}}(\rho)}{\tau(C^\bullet(\vartheta, \rho), \Gamma_\vartheta)} (1 + \mathcal{O}(s)), \tag{2-1}$$

where X_ϑ is the Reeb vector field of ϑ, (E_ρ, ∇_ρ) is the flat vector bundle over M induced by ρ, $C^\bullet(\vartheta, \rho) \subset \mathcal{D}'^\bullet(M, E_\rho)$ is the space of generalized resonant states for the resonance 0 of $\mathcal{L}_{X_\vartheta}^{\nabla_\rho}$ and $m(X_\vartheta, \rho)$ is the vanishing order of $\zeta_{X_\vartheta,\rho}(s)$ at $s = 0$.

2.4. ***Methods of proof.*** Let us briefly sketch the proof of Theorems 5 and 6, which relies essentially on two variational arguments: we compute the variation of $\tau_\vartheta(\nabla)$ when we perturb the contact form ϑ and the connection ∇. As we do so, the space $C^\bullet(\vartheta, \nabla)$ of Pollicott–Ruelle resonant states of $\mathcal{L}_{X_\vartheta}^\nabla$ for the resonance 0 may radically change. Therefore, it is convenient to consider the space $C^\bullet_{[0,\lambda]}(\vartheta, \nabla)$ instead, which consists of the generalized resonant states for $\mathcal{L}_{X_\vartheta}^\nabla$ for resonances s such that $|s| \leqslant \lambda$, where $\lambda \in (0, 1)$ is chosen so that $\{|s| = \lambda\} \cap \mathrm{Res}(\mathcal{L}_{X_\vartheta}^\nabla) = \varnothing$. Then using [Braverman and Kappeler 2007c, Proposition 5.6] and multiplicativity of torsion, one can show that

$$\tau_\vartheta(\nabla) = \pm \tau(C^\bullet_{[0,\lambda]}(\vartheta, \nabla), \Gamma_\vartheta) \zeta_{X_\vartheta,\rho}^{(\lambda,\infty)}(0)^{(-1)^q}, \tag{2-2}$$

where $\zeta_{X_\vartheta,\rho}^{(\lambda,\infty)}$ is a renormalized version of $\zeta_{X_\vartheta,\rho}$ (we remove all the poles and zeros of $\zeta_{X_\vartheta,\rho}$ within $\{s \in \mathbb{C} : |s| \le \lambda\}$); see Section 6. Thus we can work with the space $C^\bullet_{[0,\lambda]}(\vartheta, \nabla)$, which behaves nicely under perturbations of X thanks to Bonthonneau's construction [Bonthonneau 2020] of uniform anisotropic Sobolev spaces for families of Anosov flows, and also under perturbations of ∇.

Now consider a smooth family of contact forms $(\vartheta_{\mathrm{t}})_{\mathrm{t}}$ for $|\mathrm{t}| < \varepsilon$ such that their Reeb vector fields $(X_{\mathrm{t}})_{\mathrm{t}}$ induce Anosov flows. Then Theorem 9 says that for any acyclic ∇, the map $\mathrm{t} \mapsto \tau_{\vartheta_{\mathrm{t}}}(\nabla)$ is differentiable and its derivative vanishes. This follows from a result of [Braverman and Kappeler 2007c] which allows one to compute the variation of the torsion of a finite-dimensional complex when the chirality operator is perturbed, and on a variation formula of the map $\mathrm{t} \mapsto \zeta_{X_{\mathrm{t}},\rho}(s)$ for $\mathrm{Re}(s)$ big enough obtained in [Dang et al. 2020].

Next, consider a smooth family of flat connections $z \mapsto \nabla(z)$, where z is a complex number varying in a small neighborhood of the origin and write $\nabla(z) = \nabla + z\alpha + o(z)$, where $\alpha \in \Omega^1(M, \mathrm{End}(E))$. Then we show in Section 8, in the same spirit as before, that $z \mapsto \tau_\vartheta(\nabla(z))$ is complex differentiable and its logarithmic derivative reads

$$\partial_z|_{z=0} \log \tau_\vartheta(\nabla(z)) = -\mathrm{tr}_\mathrm{s}^\flat \alpha K e^{-\varepsilon \mathcal{L}^\nabla_{X_\vartheta}},$$

where $\varepsilon > 0$ is small enough, $\mathrm{tr}_\mathrm{s}^\flat$ is the super flat trace, see Section 4.4, and $K : \Omega^\bullet(M, E) \to \mathcal{D}'^\bullet(M, E)$ is a cochain contraction, that is, it satisfies $\nabla K + K\nabla = \mathrm{Id}_{\Omega^\bullet(M,E)}$. On the other hand, we can compute, using the formalism of [Dang and Rivière 2020a],

$$\partial_z|_{z=0} \log \tau_{\mathfrak{e}_\vartheta,\mathfrak{o}}(\nabla(z)) = -\mathrm{tr}_\mathrm{s}^\flat \alpha \widetilde{K} e^{-\varepsilon \mathcal{L}^\nabla_{-\widetilde{X}}} - \int_e \mathrm{tr}\, \alpha,$$

where $\mathfrak{e}_\vartheta$ is an Euler structure canonically associated with ϑ, $\widetilde{K}$ is another cochain contraction, $\widetilde{X}$ is a Morse–Smale gradient vector field and $e \in C_1(M, \mathbb{Z})$ is a singular one-chain representing the Euler structure $\mathfrak{e}_\vartheta$; see Section 9. Now using the fact that K and $\widetilde{K}$ are cochain contractions, one can see that

$$\alpha(K e^{-\varepsilon \mathcal{L}^\nabla_{X_\vartheta}} - \widetilde{K} e^{-\varepsilon \mathcal{L}^\nabla_{\widetilde{X}}}) = \alpha R_\varepsilon + [\nabla, \alpha G_\varepsilon],$$

where R_ε is an operator of degree -1 whose kernel is, roughly speaking, the union of graphs of the maps $e^{-\varepsilon X_u}$, where $(X_u)_u$ is a nondegenerate family of vector fields interpolating X_ϑ and $\widetilde{X}$, see Section 9.3, and G_ε is some operator of degree -2. Therefore we obtain by cyclicity of the flat trace

$$\partial_z|_{z=0} \log \frac{\tau_\vartheta(\nabla(z))}{\tau_{\mathfrak{e}_\vartheta,\mathfrak{o}}(\nabla(z))} = \mathrm{tr}_\mathrm{s}^\flat \alpha R_\varepsilon - \int_e \mathrm{tr}\, \alpha = 0, \tag{2-3}$$

where the last equality comes from differential topology arguments. Using the analytical structure of the representation variety, we may deduce from (2-3) the claim of Theorem 6. Theorem 7 then follows from the invariance of the dynamical torsion under small perturbations of the flow, the fact that $\tau_{\mathfrak{e},\mathfrak{o}}(\rho) = \tau_{\mathfrak{e}',\mathfrak{o}}(\rho)\langle \det \rho, h\rangle$ for any other Euler structure $\mathfrak{e}'$, where $h \in H_1(M, \mathbb{Z})$ satisfies $\mathfrak{e} = \mathfrak{e}' + h$ (we have that $H_1(M, \mathbb{Z})$ acts freely and transitively on the set of Euler structures; see Section 9), and the fact that, in our notation, $\mathfrak{e}_\eta - \mathfrak{e}_\vartheta = \mathrm{cs}(X_\vartheta, X_\eta)$ for any other contact form η.

2.5. ***Related works.*** Some analogs of our dynamical torsion were introduced by Burghelea and Haller [2008b] for vector fields which admit a Lyapunov closed 1–form generalizing previous works by Hutchings [2002] and Hutchings and Lee [1999a; 1999b] dealing with Morse–Novikov flows. In that case, the dynamical torsion depends on a choice of Euler structure and is a partially defined function on $\mathrm{Rep}_\mathrm{ac}(M, d)$; if $d = 1$, it is shown in [Burghelea and Haller 2008a] that it extends to a rational map on the Zariski closure of $\mathrm{Rep}_\mathrm{ac}(M, 1)$, which coincides, up to sign, with Turaev's refined combinatorial torsion (for the same choice of Euler structure). This follows from previous works of Hutchings and Lee [1999a; 1999b] who introduced some topological invariant involving circle-valued Morse functions. In both works, the considered object has the form

$$\text{dynamical zeta function}(0) \times \text{correction term},$$

where the correction term is the torsion of some finite-dimensional complex whose chains are generated by the critical points of the vector field. The chosen Euler structure gives a distinguished basis of the complex and thus a well-defined torsion. This is one of the main differences with our work since in the Anosov case, there are no such choices of distinguished currents in $C^\bullet$. However, the chirality operator allows us to overcome this problem as described above.

We also would like to mention the interesting related work [Rumin and Seshadri 2012], where the authors relate some dynamical zeta function involving the Reeb flow and some analytic contact torsion on 3-dimensional Seifert CR manifolds.

Finally, recently Spilioti [2020] and Müller [2020] were able to compare the Ruelle zeta function for odd-dimensional compact hyperbolic manifolds with some of the complex-valued torsions mentioned above.

2.6. ***Plan of the paper.*** The paper is organized as follows. In Section 3, we give some preliminaries about torsion of finite-dimensional complexes computed with respect to a chirality operator. In Section 4, we present our geometrical setting and conventions. In Section 5, we introduce Pollicott–Ruelle resonances. In Section 6, we compute the refined torsion of a space of generalized eigenvectors for nonzero resonances and we define the dynamical torsion. In Section 7, we prove that our torsion is insensitive to small perturbations of the dynamics. In Section 8, we compute the variation of our torsion with respect to the connection. In Section 9, we introduce Euler structures which are some topological tools used to fix ambiguities of the refined torsion. In Section 10, we introduce the refined combinatorial torsion of Turaev using Morse theory and we compute its variation with respect to the connection. We finally compare it to the dynamical torsion in Section 11.

3. Torsion of finite-dimensional complexes

We recall the definition of the refined torsion of a finite-dimensional acyclic complex computed with respect to a chirality operator, following [Braverman and Kappeler 2007c]. Then we compute the variation of the torsion of such a complex when the differential is perturbed.

3.1. ***The determinant line of a complex.*** For a nonzero complex vector space V, the determinant line of V is the line defined by $\det(V) = \bigwedge^{\dim V} V$. We declare the determinant line of the trivial vector space $\{0\}$ to be $\mathbb{C}$. If L is a 1-dimensional vector space, we will denote by L^{-1} its dual line. Any basis $(v_1, \dots, v_n)$ of V defines a nonzero element $v_1 \wedge \cdots \wedge v_n \in \det(V)$. Thus elements of the determinant line of $\det(V)$ should be thought of as equivalence classes of oriented basis of V.

Let

$$(C^\bullet, \partial) : 0 \xrightarrow{\partial} C^0 \xrightarrow{\partial} C^1 \xrightarrow{\partial} \cdots \xrightarrow{\partial} C^n \xrightarrow{\partial} 0$$

be a finite-dimensional complex, i.e., $\dim C^j < \infty$ for all $j = 0, \dots, n$. We define the *determinant line* of the complex $C^\bullet$ by

$$\det(C^\bullet) = \bigotimes_{j=0}^{n} \det(C^j)^{(-1)^j}.$$

Let $H^\bullet(\partial)$ be the cohomology of $(C^\bullet, \partial)$, that is,

$$H^\bullet(\partial) = \bigoplus_{j=0}^{n} H^j(\partial), \qquad H^j(\partial) = \frac{\ker(\partial : C^j \to C^{j+1})}{\operatorname{ran}(\partial : C^{j-1} \to C^j)}.$$

We will say that the complex $(C^\bullet, \partial)$ is acyclic if $H^\bullet(\partial) = 0$. In that case, $\det H^\bullet(\partial)$ is canonically isomorphic to $\mathbb{C}$.

It remains to define the fusion homomorphism that we will later need to define the torsion of a finite-dimensional based complex [Farber and Turaev 2000, §2.3]. For any finite-dimensional vector spaces $V_1, \dots, V_r$, we have a fusion isomorphism

$$\mu_{V_1,\dots,V_r} : \det(V_1) \otimes \cdots \otimes \det(V_r) \to \det(V_1 \oplus \cdots \oplus V_r)$$

defined by

$$\mu_{V_1,\dots,V_r}(v_1^1 \wedge \cdots \wedge v_1^{m_1} \otimes \cdots \otimes v_r^1 \wedge \cdots \wedge v_r^{m_r}) = v_1^1 \wedge \cdots \wedge v_1^{m_1} \wedge \cdots \wedge v_r^1 \wedge \cdots \wedge v_r^{m_r},$$

where $m_j = \dim V_j$ for $j \in \{1, \dots, r\}$.

3.2. ***Torsion of finite-dimensional acyclic complexes.*** In the present paper, we want to think of torsion of finite-dimensional acyclic complexes as a map $\varphi_{C^\bullet}$ from the determinant line of the complex to $\mathbb{C}$. We have a canonical isomorphism

$$\varphi_{C^\bullet} : \det(C^\bullet) \xrightarrow{\sim} \mathbb{C}, \tag{3-1}$$

defined as follows. Fix a decomposition

$$C^j = B^j \oplus A^j, \quad j = 0, \dots, n,$$

with $B^j = \ker(\partial) \cap C^j$ and $B^j = \partial(A^{j-1}) = \partial(C^{j-1})$ for every j. Then $\partial|_{A^j} : A^j \to B^{j+1}$ is an isomorphism for every j.

Fix nonzero elements $c_j \in \det C^j$ and $a_j \in \det A^j$ for any j. Let $\partial(a_j) \in \det B^{j+1}$ denote the image of a_j under the isomorphism $\det A^j \to \det B^{j+1}$ induced by the isomorphism $\partial|_{A^j} : A^j \to B^{j+1}$. Then, for each $j = 0, \dots, n$, there exists a unique $\lambda_j \in \mathbb{C}$ such that

$$c_j = \lambda_j \mu_{B^j, A^j}(\partial(a_{j-1}) \otimes a_j),$$

where μ_{B^j, A^j} is the fusion isomorphism defined in Section 3.1. Then define the isomorphism $\varphi_{C^\bullet}$ by

$$\varphi_{C^\bullet} : c_0 \otimes c_1^{-1} \otimes \cdots \otimes c_n^{(-1)^n} \mapsto (-1)^{N(C^\bullet)} \prod_{j=0}^{n} \lambda_j^{(-1)^j} \in \mathbb{C},$$

where

$$N(C^\bullet) = \frac{1}{2} \sum_{j=0}^{n} \dim A^j (\dim A^j + (-1)^{j+1}).$$

One easily shows that $\varphi_{C^\bullet}$ is independent of the choices of a_j [Turaev 2001, Lemma 1.3]. The number $\tau(C^\bullet, c) = \varphi_{C^\bullet}(c)$ is called the *refined torsion* of $(C^\bullet, \partial)$ with respect to the element c.

The torsion will depend on the choices of $c_j \in \det C^j$. Here the sign convention (that is, the choice of the prefactor $(-1)^{N(C^\bullet)}$ in the definition of $\varphi_{C^\bullet}$) follows [Braverman and Kappeler 2007c, §2] and is consistent with [Nicolaescu 2003, §1]. This prefactor was introduced by Turaev and differs from [Turaev 1986]. See [Nicolaescu 2003] for the motivation for the choice of sign.

Remark 3.1. If the complex $(C^\bullet, \partial)$ is not acyclic, we can still define a torsion $\tau(C^\bullet, c)$, which is this time an element of the determinant line $\det H^\bullet(\partial)$; see [Braverman and Kappeler 2007c, §2.4].

3.3. ***Torsion with respect to a chirality operator.*** We saw above that torsion depends on the choice of an element of the determinant line. A way to fix the value of the torsion without choosing an explicit basis is to use a chirality operator as in [Braverman and Kappeler 2007c]. Take $n = 2r + 1$ an odd integer and consider a complex $(C^\bullet, \partial)$ of length n. We will call a *chirality* operator an operator $\Gamma : C^\bullet \to C^\bullet$ such that $\Gamma^2 = \mathrm{Id}_{C^\bullet}$, and

$$\Gamma(C^j) = C^{n-j}, \quad j = 0, \dots, n.$$

Γ induces isomorphisms $\det(C^j) \to \det(C^{n-j})$ that we will still denote by Γ. If $\ell \in L$ is a nonzero element of a complex line, we will denote by $\ell^{-1} \in L^{-1}$ the unique element such that $\ell^{-1}(\ell) = 1$. Fix nonzero elements $c_j \in \det(C^j)$ for $j \in \{0, \dots, r\}$ and define

$$c_\Gamma = (-1)^{m(C^\bullet)} c_0 \otimes c_1^{-1} \otimes \cdots \otimes c_r^{(-1)^r} \otimes (\Gamma c_r)^{(-1)^{r+1}} \otimes (\Gamma c_{r-1})^{(-1)^r} \otimes \cdots \otimes (\Gamma c_0)^{-1},$$

where

$$m(C^\bullet) = \frac{1}{2} \sum_{j=0}^{r} \dim C^j (\dim C^j + (-1)^{r+j}).$$

Definition 3.2. The element c_Γ is independent of the choices of c_j for $j \in \{0, \dots, r\}$; the *refined torsion of* $(C^\bullet, \partial)$ *with respect to* Γ is the element

$$\tau(C^\bullet, \Gamma) = \tau(C^\bullet, c_\Gamma).$$

We also have the following result, which is [Braverman and Kappeler 2007c, Lemma 4.7] in the acyclic case about the multiplicativity of torsion.

Proposition 3.3. *Let* $(C^\bullet, \partial)$ *and* $(\widetilde{C}^\bullet, \tilde{\partial})$ *be two acyclic complexes of same length endowed with two chirality operators* Γ *and* $\widetilde{\Gamma}$. *Then*

$$\tau(C^\bullet \oplus \widetilde{C}^\bullet, \Gamma \oplus \widetilde{\Gamma}) = \tau(C^\bullet, \Gamma)\tau(\widetilde{C}^\bullet, \widetilde{\Gamma}).$$

3.4. ***Computation of the torsion with the contact signature operator.*** Let

$$B = \Gamma\partial + \partial\Gamma : C^\bullet \to C^\bullet.$$

B is called the *signature operator*. Let $B_+ = \Gamma\partial$ and $B_- = \partial\Gamma$. Define

$$C^j_\pm = C^j \cap \ker(B_\mp), \quad j = 0, \dots, n.$$

We have that $B_\pm$ preserves $C^\bullet_\pm$. Note that $B_+(C^j_+) \subset C^{n-j-1}_+$, so that $B_+(C^j_+ \oplus C^{n-j-1}_+) \subset C^j_+ \oplus C^{n-j-1}_+$. Note that if B is invertible on $C^\bullet$, B_+ is invertible on $C^\bullet_+$. If B is invertible, we can compute the refined torsion of $(C^\bullet, \partial)$ using the following:

Proposition 3.4 [Braverman and Kappeler 2007c, Proposition 5.6]. *Assume that* B *is invertible. Then* $(C^\bullet, \partial)$ *is acyclic so that* $\det(H^\bullet(\partial))$ *is canonically isomorphic to* $\mathbb{C}$. *Moreover*,

$$\tau(C^\bullet, \Gamma) = (-1)^{r \dim C^r_+} \det(\Gamma\partial|_{C^r_+})^{(-1)^r} \prod_{j=0}^{r-1} \det(\Gamma\partial|_{C^j_+ \oplus C^{n-j-1}_+})^{(-1)^j},$$

where we recall that $n = 2r + 1$.

3.5. ***Super traces and determinants.*** Let $V^\bullet = \bigoplus_{j=0}^{p} V^j$ be a graded finite-dimensional vector space and $A : V^\bullet \to V^\bullet$ be a degree-preserving linear map. We define the *super trace* and the *super determinant* of A by

$$\operatorname{tr}_{s, V^\bullet} A = \sum_{j=0}^{p} (-1)^j \operatorname{tr}_{V^j} A, \quad \det_{s, V^\bullet} A = \prod_{j=0}^{p} (\det_{V^j} A)^{(-1)^j}.$$

We also define the *graded trace* and the *graded determinant* of A by

$$\operatorname{tr}_{\mathrm{gr}, V^\bullet} A = \sum_{j=0}^{p} (-1)^j j \operatorname{tr}_{V^j} A, \quad \det_{\mathrm{gr}, V^\bullet} A = \prod_{j=0}^{p} (\det_{V^j} A)^{(-1)^j j}.$$

3.6. ***Analytic families of differentials.*** The goal of the present subsection is to give a variation formula for the torsion of a finite-dimensional complex when we vary the differential. This formula plays a crucial role in the variation formula of the dynamical torsion, when the representation is perturbed. Indeed, we split the dynamical torsion as the product of the torsion $\tau(C^\bullet(\vartheta, \rho), \Gamma_\vartheta)$ of some finite-dimensional space of Ruelle resonant states and a renormalized value at $s = 0$ of the dynamical zeta function $\zeta_{X,\rho}(s)$. Then the following formula allows us to deal with the variation of $\tau(C^\bullet(\vartheta, \rho), \Gamma_\vartheta)$.

Let $(C^\bullet, \partial)$ be an acyclic finite-dimensional complex of finite odd length n. If $S : C^\bullet : C^\bullet$ is a linear operator, we will say that it is of degree s if $S(C^k) \subset C^{k+s}$ for any k. If S and T are two operators on $C^\bullet$ of degrees s and t respectively then the supercommutator of S and T by

$$[S, T] = ST - (-1)^{st} TS.$$

Cyclicity of the usual trace gives $\operatorname{tr}_{s, C^\bullet}[S, T] = 0$ for any S, T.

Let U be a neighborhood of the origin in the complex plane and $\partial(z)$, $z \in U$, be a family of acyclic differentials on $C^\bullet$ which is real differentiable at $z = 0$, that is,

$$\partial(\sigma) = \partial + \operatorname{Re}(\sigma)\mu + \operatorname{Im}(\sigma)\nu + o(\sigma), \quad \sigma \to 0, \tag{3-2}$$

where $\mu, \nu : C^\bullet \to C^\bullet$ are degree-1 operators. Note that $\partial(\sigma) \circ \partial(\sigma) = 0$ implies that the supercommutator

$$[\partial, a(\sigma)] = \partial a(\sigma) + a(\sigma)\partial = 0, \quad \sigma \in \mathbb{C}, \tag{3-3}$$

where $a(\sigma) = \operatorname{Re}(\sigma)\mu + \operatorname{Im}(\sigma)\nu$. We will denote by $C^\bullet(z)$ the complex $(C^\bullet, \partial(z))$. Finally, let $k : C^\bullet \to C^\bullet$ be a cochain contraction, that is a linear map of degree 1 such that

$$\partial k + k\partial = \mathrm{Id}_{C^\bullet}. \tag{3-4}$$

The existence of such map is ensured by the acyclicity of $(C^\bullet, \partial)$.

Lemma 3.5. *In the above notation, for any chirality operator* Γ *on* $C^\bullet$, *the map* $z \mapsto \tau(C^\bullet(z), \Gamma)$ *is real differentiable at* $z = 0$ *and, for any* $c \in \det C^\bullet$, *one has*

$$\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \log \tau(C^\bullet(t\sigma), \Gamma) = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \log \tau(C^\bullet(t\sigma), c) = -\operatorname{tr}_{s, C^\bullet}(a(\sigma)k).$$

Note that this implies in particular that $\mathrm{tr}_{s,C^\bullet}(a(\sigma)k)$ does not depend on the chosen cochain contraction k. This is expected since if k' is another cochain contraction,

$$[\partial, a(\sigma)kk'] = \partial a(\sigma)kk' + a(\sigma)kk'\partial = a(\sigma)(k-k')$$

by (3-3), and the supertrace of a supercommutator vanishes.

Proof. First note that for nonzero elements $c, c' \in \det C^\bullet$, we have

$$\tau(C^\bullet(z), c) = [c : c'] \cdot \tau(C^\bullet(z), c'), \tag{3-5}$$

where $[c : c'] \in \mathbb{C}$ satisfies $c = [c : c'] \cdot c'$.

For every $j = 0, \dots, n$, fix a decomposition

$$C^j = A^j \oplus B^j,$$

where $B^j = \ker \partial \cap C^j$ and A^j is any complementary of B^j in C^j. Fix some basis $a_j^1, \dots, a_j^{\ell_j}$ of A^j; then $\partial a_j^1, \dots, \partial a_j^{\ell_j}$ is a basis of B^{j+1} by acyclicity of $(C^\bullet, \partial)$. Now let

$$c_j = \partial a_{j-1}^1 \wedge \dots \wedge \partial a_{j-1}^{\ell_{j-1}} \wedge a_j^1 \wedge \dots \wedge a_j^{\ell_j} \in \det C^j$$

and

$$c = c_0 \otimes (c_1)^{-1} \otimes c_2 \otimes \dots \otimes (c_n)^{(-1)^n} \in \det C^\bullet.$$

Now by definition of the refined torsion, we have for $|z|$ small enough

$$\tau(C^\bullet(t\sigma), c) = \pm \prod_{j=0}^{n} \det(A_j(t\sigma))^{(-1)^{j+1}}, \tag{3-6}$$

where the sign $\pm$ is independent of z and $A_j(z)$ is the matrix sending the basis

$$\partial a_{j-1}^1, \dots, \partial a_{j-1}^{\ell_{j-1}}, a_j^1, \dots, a_j^{\ell_j}$$

to the basis

$$\partial(t\sigma) a_{j-1}^1, \dots, \partial(t\sigma) a_{j-1}^{\ell_{j-1}}, a_j^1, \dots, a_j^{\ell_j}$$

(which is indeed a basis of C^j for $|z|$ small enough). Let $k : C^\bullet \to C^\bullet$ of degree -1 defined by

$$k\partial a_j^m = a_j^m, \quad k a_j^m = 0$$

for every j and $m \in \{0, \dots, \ell_j\}$. Then $k\partial + \partial k = \mathrm{Id}_{C^\bullet}$ and

$$\det A_j(t\sigma) = \det_{\partial B^{j-1} \oplus B^j}(\partial(t\sigma)k \oplus \mathrm{Id}).$$

Now (3-2) and (3-6) imply the desired result, because $\tau(C^\bullet(t\sigma), \Gamma) = [c_\Gamma : c] \cdot \tau(C^\bullet(t\sigma), c)$ by (3-5). □

4. Geometrical setting and notations

We introduce here our geometrical conventions and notation. In particular, we adopt the formalism of [Harvey and Polking 1979], which will be convenient to compute flat traces and relate the variation of the Ruelle zeta function with topological objects.

4.1. ***Twisted cohomology.*** We consider M an oriented closed connected manifold of odd dimension $n = 2r + 1$. Let $E \to M$ be a flat vector bundle over M of rank $d \geqslant 1$. For $k \in \{0, \dots, n\}$, we will denote the bundle $\Lambda^k T^*M$ by Λ^k for simplicity. We will denote by $\Omega^k(M, E) = \mathcal{C}^\infty(M, \Lambda^k \otimes E)$ the space of E-valued k-forms. We set

$$\Omega^\bullet(M, E) = \bigoplus_{k=0}^{n} \Omega^k(M, E).$$

Let ∇ be a flat connection on E. We view the connection as a degree-1 operator (as an operator of the graded vector space $\Omega^\bullet(M, E)$)

$$\nabla : \Omega^k(M, E) \to \Omega^{k+1}(M, E), \quad k = 0, \dots, n.$$

The flatness of the connection reads $\nabla^2 = 0$ and thus we obtain a cochain complex $(\Omega^\bullet(M, E), \nabla)$. We will assume that the connection ∇ is acyclic, that is, the complex $(\Omega^\bullet(M, E), \nabla)$ is acyclic, or equivalently, the cohomology groups

$$H^k(M, \nabla) = \frac{\{u \in \Omega^k(M, E) : \nabla u = 0\}}{\{\nabla v : v \in \Omega^{k-1}(M, E)\}}, \quad k = 0, \dots, n,$$

are trivial.

4.2. ***Currents and Schwartz kernels.*** Let

$$\mathcal{D}'^\bullet(M, E) = \bigoplus_{k=0}^{n} \mathcal{D}'(M, \Lambda^k \otimes E)$$

be the space of E-valued currents. Let $E^\vee$ denote the dual bundle of E. We will identify $\mathcal{D}'^k(M, E)$ and the topological dual of $\Omega^{n-k}(M, E^\vee)$ via the nondegenerate bilinear pairing

$$\langle \alpha, \beta \rangle = \int_M \alpha \wedge \beta, \quad \alpha \in \Omega^k(M, E),\ \beta \in \Omega^{n-k}(M, E^\vee),$$

where $\wedge$ is the usual wedge product between E-valued forms and $E^\vee$-valued forms.

A continuous linear operator $G : \Omega^\bullet(M, E) \to \mathcal{D}'^\bullet(M, E)$ is called homogeneous if, for some $p \in \mathbb{Z}$, we have $G(\Omega^k(M, E)) \subset \mathcal{D}'^{k+p}(M, E)$ for every $k = 0, \dots, n$; the number p is called the degree of G and is denoted by $\deg G$. In that case, the Schwartz kernel theorem gives us a twisted current $\mathcal{G} \in \mathcal{D}'^{n+p}(M \times M, \pi_1^* E^\vee \otimes \pi_2^* E)$ satisfying

$$\langle Gu, v \rangle_M = \langle \mathcal{G}, \pi_1^* u \wedge \pi_2^* v \rangle_{M \times M}, \quad u \in \Omega^k(M, E),\ v \in \Omega^{n-k-p}(M, E^\vee),$$

where π_1 and π_2 are the projections of $M \times M$ onto its first and second factors respectively.

4.3. ***Integration currents.*** Let N be an oriented submanifold of M of dimension d, possibly with boundary. The associated integration current $[N] \in \mathcal{D}'^{n-d}(M)$ is given by

$$\langle [N], \omega \rangle = \int_N i_N^* \omega, \quad \omega \in \Omega^d(M),$$

where $i_N : N \to M$ is the inclusion. Note that Stokes' formula yields

$$\mathrm{d}[N] = (-1)^{n-d+1}[\partial N]. \tag{4-1}$$

For $f \in \mathrm{Diff}(M)$, we will set $\mathrm{Gr}(f) = \{(f(x), x) : x \in M\}$ to be the graph of f. Note that $\mathrm{Gr}(f)$ is an n-dimensional submanifold of $M \times M$ which is canonically oriented since M is. Therefore, we can consider the integration current over $\mathrm{Gr}(f)$. By definition, we have for any $\alpha, \beta \in \Omega^\bullet(M)$

$$\langle [\mathrm{Gr}(f)], \pi_1^*\alpha \wedge \pi_2^*\beta \rangle = \int_M f^*\alpha \wedge \beta.$$

In particular, $[\mathrm{Gr}(f)]$ is the Schwartz kernel of $f^* : \Omega^\bullet(M) \to \Omega^\bullet(M)$.

4.4. ***Flat traces.*** Let $G : \Omega^\bullet(M, E) \to \mathcal{D}'^\bullet(M, E)$ be an operator of degree 0. We denote its Schwartz kernel by $\mathcal{G}$ and we define

$$\mathrm{WF}'(\mathcal{G}) = \{(x, y, \xi, \eta) : (x, y, \xi, -\eta) \in \mathrm{WF}(\mathcal{G})\} \subset T^*(M \times M),$$

where WF denotes the classical Hörmander wavefront set; see [Hörmander 1990, §8]. We will also use the notation $\mathrm{WF}(G) = \mathrm{WF}(\mathcal{G})$ and $\mathrm{WF}'(G) = \mathrm{WF}'(\mathcal{G})$. Assume that

$$\mathrm{WF}'(\mathcal{G}) \cap \Delta(T^*M) = \varnothing, \quad \Delta(T^*M) = \{(x, x, \xi, \xi) : (x, \xi) \in T^*M\}. \tag{4-2}$$

Let $\iota : M \to M \times M$, $x \mapsto (x, x)$, be the diagonal inclusion. Then by [Hörmander 1990, Theorem 8.2.4] the pull back $\iota^*\mathcal{G} \in \mathcal{D}'^n(M, E^\vee \otimes E)$ is well-defined and we define the *super flat trace* of G by

$$\mathrm{tr}_\mathrm{s}^\flat G = \langle \mathrm{tr}\, \iota^*\mathcal{G}, 1 \rangle,$$

where tr denotes the trace on $E^\vee \otimes E$. We will also use the notation

$$\mathrm{tr}_\mathrm{gr}^\flat G = \mathrm{tr}_\mathrm{s}^\flat N G,$$

where $N : \Omega^\bullet(M, E) \to \Omega^\bullet(M, E)$ is the number operator, that is, $N\omega = k\omega$ for every $\omega \in \Omega^k(M, E)$.

The notation $\mathrm{tr}_\mathrm{s}^\flat$ is motivated by the following. Let $A : \mathcal{C}^\infty(M, F) \to \mathcal{D}'(M, F)$ be an operator acting on sections of a vector bundle F. If A satisfies (4-2), we can also define a flat trace $\mathrm{tr}^\flat A$ as in [Dyatlov and Zworski 2016, §2.4]. Now if $G : \Omega^\bullet(M, E) \to \mathcal{D}'^\bullet(M, E)$ is an operator of degree 0, it gives rise to an operator $G_k : \mathcal{C}^\infty(M, F_k) \to \mathcal{D}'(M, F_k)$ for each $k = 0, \dots, n$, where $F_k = \Lambda^k \otimes E$. Then the link between the two notions of flat trace mentioned above is given by

$$\mathrm{tr}_\mathrm{s}^\flat G = \sum_{k=0}^n (-1)^k \,\mathrm{tr}^\flat G_k.$$

If $\Gamma \subset T^*M$ is a closed conical subset, we let

$$\mathcal{D}'^\bullet_\Gamma(M, E) = \{u \in \mathcal{D}'^\bullet(M, E), \mathrm{WF}(u) \subset \Gamma\} \tag{4-3}$$

be the space of E-valued current whose wavefront set is contained in Γ, endowed with its usual topology; see [Hörmander 1990, §8]. If Γ is a closed conical subset of $T^*(M \times M)$ not intersecting the conormal to the diagonal

$$N^*\Delta(T^*M) = \{(x, x, \xi, -\xi) : (x, \xi) \in T^*M\},$$

then the flat trace is continuous as a map $\mathcal{D}'^{\bullet}_{\Gamma}(M \times M, \pi_1^* E^\vee \otimes \pi_2^* E) \to \mathbb{R}$.

4.5. ***Cyclicity of the flat trace.*** Let $G, H : \Omega^\bullet(M, E) \to \mathcal{D}'^\bullet(M, E)$ be two homogeneous operators. We denote by $\mathcal{G}, \mathcal{H}$ their respective kernels. If $\Gamma \subset T^*(M \times M)$ is a closed conical subset, we define

$$\Gamma^{(1)} = \{(y, \eta) : \text{there exists } x \in M \text{ such that } (x, y, 0, \eta) \in \Gamma\},$$
$$\Gamma^{(2)} = \{(y, \eta) : \text{there exists } x \in M \text{ such that } (x, y, -\eta, 0) \in \Gamma\}.$$

Then under the assumption

$$\mathrm{WF}(\mathcal{G})^{(2)} \cap \mathrm{WF}(\mathcal{H})^{(1)} = \varnothing,$$

the operator $F = G \circ H$ is well-defined by [Hörmander 1990, Theorem 8.2.14] and its Schwartz kernel $\mathcal{F}$ satisfies the wavefront set estimate:

$$\mathrm{WF}(\mathcal{F}) \subset \{(x, y, \xi, \eta) : \text{there exists } (z, \zeta) \text{ such that } (x, z, \xi, \zeta) \in \mathrm{WF}'(\mathcal{G}) \text{ and } (z, y, \zeta, \eta) \in \mathrm{WF}(\mathcal{H})\}.$$

If both compositions $G \circ H$ and $H \circ G$ are defined, we will denote by

$$[G, H] = G \circ H - (-1)^{\deg G \deg H} H \circ G$$

the graded commutator of G and H. We have the following:

Proposition 4.1. *Let G, H be two homogeneous operators with $\deg G + \deg H = 0$ and such that both compositions $G \circ H$ and $H \circ G$ are defined and satisfy the bound (4-2). Then we have*

$$\mathrm{tr}_{\mathrm{s}}^\flat [G, H] = 0.$$

The above result follows from the cyclicity of the L^2-trace, the approximation result [Dyatlov and Zworski 2016, Lemma 2.8], the relation

$$\mathrm{tr}_{\mathrm{s}}^\flat [G, H] = \mathrm{tr}^\flat [(-1)^N G, H],$$

where N is the number operator and $\mathrm{tr}^\flat$ is the flat trace with the convention from [Dyatlov and Zworski 2016, §2.4] (see Section 4.4), and the fact that the map $(G, H) \mapsto G \circ H$ is continuous

$$\mathcal{D}'^{\bullet}_{\Gamma}(M \times M, \pi_1^* E^\vee \otimes \pi_2^* E) \times \mathcal{D}'^{\bullet}_{\widetilde{\Gamma}}(M \times M, \pi_1^* E^\vee \otimes \pi_2^* E) \to \mathcal{D}'^{\bullet}_{\Upsilon}(M \times M, \pi_1^* E^\vee \otimes \pi_2^* E)$$

for any closed conical subsets $\Gamma, \widetilde{\Gamma} \subset T^*(M \times M)$ such that $\Gamma^{(2)} \cap \widetilde{\Gamma}^{(1)} = \varnothing$, and where Υ is a closed conical subset given in [Hörmander 1990, 8.2.14].

4.6. ***Perturbation of holonomy.*** Let $\gamma : [0, 1] \to M$ be a smooth curve and $\alpha \in \Omega^1(M, \mathrm{End}(E))$. Let P_{t} (resp. $\widetilde{P}_{\mathrm{t}}$) be the parallel transport $E_{\gamma(0)} \to E_{\gamma(t)}$ of ∇ (resp. $\tilde{\nabla} = \nabla + \alpha$) along $\gamma|_{[0,t]}$. Then

$$\widetilde{P}_{\mathrm{t}} = P_{\mathrm{t}} \exp\left(- \int_0^t P_{-\tau} \alpha(\dot{\gamma}(\tau)) P_\tau \, \mathrm{d}\tau \right). \tag{4-4}$$

The above formula will be useful in some occasion. For simplicity, we will denote for any $A \in \mathcal{C}^\infty(M, \mathrm{End}(E))$

$$\int_\gamma A = \int_0^t P_{-\tau} A(\gamma(\tau)) P_\tau \, \mathrm{d}\tau \in \mathrm{End}(E_{\gamma(0)}),$$

so that $\widetilde{P}_1 = P_1 \exp\bigl(-\int_\gamma \alpha(X)\bigr)$.

5. Pollicott–Ruelle resonances

5.1. ***Anosov dynamics.*** Let X be a smooth vector field on M and denote by φ^t its flow. We will assume that X generates an Anosov flow, that is, there exists a splitting of the tangent space $T_x M$ at every $x \in M$

$$T_x M = \mathbb{R}X(x) \oplus E_s(x) \oplus E_u(x),$$

where $E_u(x)$, $E_s(x)$ are subspaces of $T_x M$ depending continuously on x and invariant by the flow φ^t, such that for some constants $C, \nu > 0$ and some smooth metric $|\cdot|$ on TM one has

$$|(\mathrm{d}\varphi^t)_x v_s| \leqslant Ce^{-\nu t}|v_s|, \quad t \geq 0, \; v_s \in E_s(x),$$
$$|(\mathrm{d}\varphi^t)_x v_u| \leqslant Ce^{-\nu |t|}|v_u|, \quad t \leq 0, \; v_u \in E_u(x).$$

We will use the dual decomposition $T^*M = E_0^* \oplus E_u^* \oplus E_s^*$, where E_0^*, E_u^* and E_s^* are defined by

$$E_0^*(E_s \oplus E_u) = 0, \quad E_s^*(E_0 \oplus E_s) = 0, \quad E_u^*(E_0 \oplus E_u) = 0. \tag{5-1}$$

5.2. ***Pollicott–Ruelle resonances.*** Let ι_X denote the interior product with X and

$$\mathcal{L}_X^\nabla = \nabla \iota_X + \iota_X \nabla : \Omega^\bullet(M, E) \to \Omega^\bullet(M, E)$$

be the Lie derivative along X acting on E-valued forms. Locally, the action of $\mathcal{L}_X^\nabla$ is given by the following. Take U a domain of a chart and write $\nabla = \mathrm{d} + A$, where $A \in \Omega^1(M, \mathrm{End}(E))$. Take $w_1, \ldots, w_\ell$ (resp. $e_1, \ldots, e_d$) some local basis of Λ^k (resp. E) on U. Then, for any $1 \leqslant i \leqslant \ell$ and $1 \leqslant j \leqslant d$,

$$\mathcal{L}_X^\nabla(f w_i \otimes e_j) = (Xf) w_i \otimes e_j + f(\mathcal{L}_X w_i) \otimes e_j + f w_i \otimes A(X) e_j, \quad f \in \mathcal{C}^\infty(U),$$

where $\mathcal{L}_X$ is the standard Lie derivative acting on forms. In particular, $\mathcal{L}_X^\nabla$ is a differential operator of order 1 acting on sections of the bundle $\Lambda^\bullet T^*M \otimes E$, whose principal part is diagonal and given by X. This operator generates a transfer operator

$$e^{t\mathcal{L}_X^\nabla} : \Omega^\bullet(M, E) \to \Omega^\bullet(M, E),$$

which is defined by the relation

$$\frac{\mathrm{d}}{\mathrm{d}t}(e^{t\mathcal{L}_X^\nabla} u) = e^{t\mathcal{L}_X^\nabla}(\mathcal{L}_X^\nabla u).$$

For $\mathrm{Re}(s)$ big enough, the operator $\mathcal{L}_X^\nabla + s$ acting on $\Omega^\bullet(M, E)$ is invertible with inverse

$$(\mathcal{L}_X^\nabla + s)^{-1} = \int_0^\infty e^{-t\mathcal{L}_X^\nabla} e^{-st} \, \mathrm{d}t, \tag{5-2}$$

as it follows by an integration by parts. The results of [Faure and Sjöstrand 2011] generalize to the flat bundle case as in [Dang and Rivière 2020b, §3] and the resolvent $(\mathcal{L}_X^\nabla + s)^{-1}$, viewed as a family of operators $\Omega^\bullet(M, E) \to \mathcal{D}'^\bullet(M, E)$, admits a meromorphic continuation to $s \in \mathbb{C}$ with poles of finite multiplicities; we will still denote by $(\mathcal{L}_X^\nabla + s)^{-1}$ this extension. Those poles are the *Pollicott–Ruelle resonances* of $\mathcal{L}_X^\nabla$, and we will denote this set by $\mathrm{Res}(\mathcal{L}_X^\nabla)$.

5.3. ***Generalized resonant states.*** Let $s_0 \in \mathrm{Res}(\mathcal{L}_X^\nabla)$. By [Dyatlov and Zworski 2016, Proposition 3.3] we have a Laurent expansion

$$(\mathcal{L}_X^\nabla + s)^{-1} = Y_{s_0}(s) + \sum_{j=1}^{J(s_0)} (-1)^{j-1} \frac{(\mathcal{L}_X^\nabla + s_0)^{j-1}\Pi_{s_0}}{(s-s_0)^j}, \tag{5-3}$$

where $Y_{s_0}(s)$ is holomorphic near $s = s_0$, and

$$\Pi_{s_0} = \frac{1}{2\pi i}\int_{C_\varepsilon(s_0)} (\mathcal{L}_X^\nabla + s)^{-1}\,\mathrm{d}s : \Omega^\bullet(M, E) \to \mathcal{D}'^\bullet(M, E) \tag{5-4}$$

is an operator of finite rank. Here $C_\varepsilon(s_0) = \{|z - s_0| = \varepsilon\}$ with $\varepsilon > 0$ small enough is a small circle around s_0 such that $\mathrm{Res}(\mathcal{L}_X^\nabla) \cap \{|z - s_0| \leqslant \varepsilon\} = \{s_0\}$. Moreover the operators $Y_{s_0}(s)$ and Π_{s_0} extend to continuous operators

$$Y_{s_0}(s), \Pi_{s_0} : \mathcal{D}'^\bullet_{E_u^*}(M, E) \to \mathcal{D}'^\bullet_{E_u^*}(M, E). \tag{5-5}$$

The space

$$C^\bullet(s_0) = \mathrm{ran}(\Pi_{s_0}) \subset \mathcal{D}'^\bullet_{E_u^*}(M, E)$$

is called the space of generalized resonant states of $\mathcal{L}_X^\nabla$ associated with the resonance s_0.

5.4. ***The twisted Ruelle zeta function.*** Fix a base point $x_\star \in M$ and identify $\pi_1(M)$ with $\pi_1(M, x_\star)$. Let $\mathrm{Per}(X)$ be the set of periodic orbits of X. For every $\gamma \in \mathrm{Per}(X)$ we fix some base point $x_\gamma \in \mathrm{Im}(\gamma)$ and an arbitrary path c_γ joining x_γ to $x_\star$. This path defines an isomorphism $\psi_\gamma : \pi_1(M, x_\gamma) \cong \pi_1(M)$ and we can thus define for every $\gamma \in \mathrm{Per}(X)$

$$\rho_\nabla([\gamma]) = \rho_\nabla(\psi_\gamma[\gamma]).$$

The *twisted Ruelle zeta function* associated with the pair (X, ∇) is defined by

$$\zeta_{X,\nabla}(s) = \prod_{\gamma \in \mathcal{G}_X} \det(\mathrm{Id} - \varepsilon_\gamma \rho_\nabla([\gamma]) e^{-s\ell(\gamma)}), \quad \mathrm{Re}(s) > C. \tag{5-6}$$

Here $\mathcal{G}_X$ is the set of all primitive closed orbits of X (that is, the closed orbits that generate their class in $\pi_1(M)$), $\ell(\gamma)$ is the length of the orbit γ and $C > 0$ is some big constant depending on ρ and X, which satisfies

$$\|\rho_\nabla([\gamma])\| \leqslant \exp(C\ell(\gamma)), \quad \gamma \in \mathcal{G}_X, \tag{5-7}$$

for some norm $\|\cdot\|$ on $\mathrm{End}(E_{x_\star})$. Finally $\varepsilon_\gamma = 1$ if $E_u|_\gamma$ is orientable, and -1 if not.

In what follows, we will denote by P_γ the linearized Poincaré return map of γ, that is,

$$P_\gamma = \mathrm{d}_x\varphi^{-\ell(\gamma)}|_{E_s(x)\oplus E_u(x)}$$

for some $x \in \mathrm{Im}(\gamma)$ (if we choose another point in $\mathrm{Im}(\gamma)$, the new map will be conjugated to the first one). Then one has

$$\varepsilon_\gamma = \operatorname{sgn}\det(P_\gamma|_{E_s}) = (-1)^q \frac{\det(\mathrm{Id}-P_\gamma)}{|\det(\mathrm{Id}-P_\gamma)|}, \quad \text{where } q = \dim E_s. \tag{5-8}$$

Giuletti, Pollicott and Liverani [Giulietti et al. 2013] and Dyatlov and Zworski [2016] showed that $\zeta_{X,\nabla}$ has a meromorphic continuation to $\mathbb{C}$ whose poles and zeros are contained in $\mathrm{Res}(\mathcal{L}_X^\nabla)$. In fact, a consequence of the Guillemin trace formula [1977], together with (5-8) and the identity

$$\det(\mathrm{Id}-P_\gamma) = \sum_{k=0}^{n} (-1)^{k+1} k \operatorname{tr} \Lambda^k \mathrm{d}_x \varphi^{-\ell(\gamma)},$$

is that whenever $\mathrm{Re}(s)$ is large enough, we have, for every small $\varepsilon > 0$,

$$\partial_s \log \zeta_{X,\nabla}(s) = (-1)^{q+1} \operatorname{tr}^\flat_{\mathrm{gr}}((\mathcal{L}_X^\nabla + s)^{-1} e^{-\varepsilon(\mathcal{L}_X^\nabla + s)}), \tag{5-9}$$

where the flat trace makes sense, because the wavefront set of $(\mathcal{L}_X^\nabla + s)^{-1} e^{-\varepsilon(\mathcal{L}_X^\nabla + s)}$ does not encounter the conormal to the diagonal in $T^*(M \times M)$ (see Section 8.4). In particular, one can see that the order of $\zeta_{X,\nabla}$ near a resonance $s_0 \in \mathrm{Res}(\mathcal{L}_X^\nabla)$ is given by

$$m(s_0) = (-1)^{q+1} \sum_{k=0}^{n} (-1)^k k m_k(s_0), \tag{5-10}$$

where $m_k(s_0)$ is the rank of the spectral projector $\Pi_{s_0}|_{\Omega^k(M,E)}$.[4]

5.5. *Topology of resonant states.* Since ∇ commutes with $\mathcal{L}_X^\nabla$, it induces a differential on the complexes $C^\bullet(s_0)$ for any $s_0 \in \mathrm{Res}(\mathcal{L}_X^\nabla)$. It is shown in [Dang and Rivière 2020b] that the complexes $(C^\bullet(s_0), \nabla)$ are acyclic whenever $s_0 \neq 0$. Moreover, for $s_0 = 0$, the map

$$\Pi_{s_0=0} : \Omega^\bullet(M, \nabla) \longrightarrow C^\bullet(s_0 = 0) = C^\bullet$$

is a quasi-isomorphism, that is, it induces isomorphisms at the level of cohomology groups. Since we assumed ∇ to be acyclic, the complex $(C^\bullet, \nabla)$ is also acyclic.

6. The dynamical torsion of a contact Anosov flow

From now on, we will assume that the flow φ^t is contact, that is, there exists a smooth one form $\vartheta \in \Omega^1(M)$ such that $\vartheta \wedge (\mathrm{d}\vartheta)^r$ is a volume form on M, $\iota_X \vartheta = 1$ and $\iota_X \mathrm{d}\vartheta = 0$. The purpose of this section is to define the dynamical torsion of the pair (ϑ, ∇). We first introduce a chirality operator Γ_ϑ acting on $\Omega^\bullet(M, E)$, which is defined thanks to the contact structure. Then the dynamical torsion is a renormalized version of the twisted Ruelle zeta function corrected by the torsion of the finite-dimensional space of the generalized resonant states for resonance $s_0 = 0$ computed with respect to Γ_ϑ.

This construction was inspired by the work of Braverman and Kappeler [2007c] on the refined analytic torsion.

[4] In [Dyatlov and Zworski 2016], the authors study the action of $\mathcal{L}_X^\nabla$ on $(\Lambda^k T^*M \cap \ker \iota_X) \otimes E$ and they get $m(s_0) = (-1)^q \sum_{k=0}^{n-1} (-1)^k m_k^0(s_0)$, where $m_k^0(s_0)$ is the dimension of $\Pi_{s_0}(\Omega^k(M, E) \cap \ker \iota_X)$. Here we study the action of $\mathcal{L}_X^\nabla$ on the full bundle $\Lambda^k T^*M \otimes E$, which leads to (5-9) and (5-10).

6.1. ***The chirality operator associated with a contact structure.*** Let $V_X \to M$ denote the bundle $T^*M \cap \ker \iota_X$. Note that for $k \in \{0, \dots, n\}$, we have the decomposition

$$\Lambda^k T^*M = \Lambda^{k-1} V_X \wedge \vartheta \oplus \Lambda^k V_X. \tag{6-1}$$

Indeed, if $\alpha \in \Lambda^k T^*M$ we may write

$$\alpha = \underbrace{(-1)^{k+1}\iota_X\alpha \wedge \vartheta}_{\in \Lambda^{k-1}V_X \wedge \vartheta} + \underbrace{\alpha - (-1)^{k+1}\iota_X\alpha \wedge \vartheta.}_{\in \Lambda^k V_X}$$

Let us introduce the Lefschetz map

$$\mathscr{L} : \Lambda^\bullet V_X \to \Lambda^{\bullet+2} V_X, \quad u \mapsto u \wedge \mathrm{d}\vartheta.$$

Since $\mathrm{d}\vartheta$ is a symplectic form on V_X, the maps $\mathscr{L}^{r-k}$ induce bundle isomorphisms

$$\mathscr{L}^{r-k} : \Lambda^k V_X \xrightarrow{\sim} \Lambda^{2r-k} V_X, \quad k = 0, \dots, r; \tag{6-2}$$

see for example [Libermann and Marle 1987, Theorem 16.3]. Using the above Lefschetz isomorphisms, we are now ready to introduce our chirality operator.

Definition 6.1. The chirality operator associated with the contact form ϑ is the operator $\Gamma_\vartheta : \Lambda^\bullet T^*M \to \Lambda^{n-\bullet} T^*M$ defined by $\Gamma_\vartheta^2 = 1$ and

$$\Gamma_\vartheta(f \wedge \vartheta + g) = \mathscr{L}^{r-k} g \wedge \vartheta + \mathscr{L}^{r-k+1} f, \quad f \in \Lambda^{k-1} V_X,\ g \in \Lambda^k V_X,\ k \in \{0, \dots, r\}, \tag{6-3}$$

where we used the decomposition (6-1).

Note that in particular one has, for $k \in \{r+1, \dots, n\}$,

$$\Gamma_\vartheta(f \wedge \vartheta + g) = (\mathscr{L}^{k-r})^{-1} g \wedge \vartheta + (\mathscr{L}^{k-1-r})^{-1} f.$$

6.2. ***The refined torsion of a space of generalized eigenvectors.*** The operator Γ_ϑ acts also on $\Omega^\bullet(M, E)$ by acting trivially on E-coefficients. Since $\mathcal{L}_X \vartheta = 0$, Γ_ϑ and $\mathcal{L}_X^\nabla$ commute so that Γ_ϑ induces a chirality operator

$$\Gamma_\vartheta : C^\bullet(s_0) \to C^{n-\bullet}(s_0)$$

for every $s_0 \in \mathrm{Res}(\mathcal{L}_X^\nabla)$. Recall from Section 5.5 that the complexes $(C^\bullet(s_0), \nabla)$ are acyclic. The following formula motivates the upcoming definition of the dynamical torsion.

Proposition 6.2. *Let $s_0 \in \mathrm{Res}(\mathcal{L}_X^\nabla) \setminus \{0, 1\}$. We have*

$$\tau(C^\bullet(s_0), \Gamma_\vartheta)^{-1} = (-1)^{Q_{s_0}} \det\nolimits_{\mathrm{gr}, C^\bullet(s_0)} \mathcal{L}_X^\nabla,$$

where

$$Q_{s_0} = \sum_{k=0}^{r} (-1)^k (r+1-k) \dim C^k(s_0),$$

and $\tau(C^\bullet(s_0), \Gamma_\vartheta) \in \mathbb{C} \setminus 0$ is the refined torsion of the acyclic complex $(C^\bullet(s_0), \nabla)$ with respect to the chirality Γ_ϑ; see Definition 3.2.

Let us first admit the above proposition; the proof will be given in Sections 6.5 and 6.6.

6.3. ***Spectral cuts.*** If $\mathcal{I} \subset [0, 1)$ is an interval, we set

$$\Pi_{\mathcal{I}} = \sum_{\substack{s_0 \in \mathrm{Res}(\mathcal{L}_X^\nabla) \\ |s_0| \in \mathcal{I}}} \Pi_{s_0}, \quad C_{\mathcal{I}}^\bullet = \bigoplus_{\substack{s_0 \in \mathrm{Res}(\mathcal{L}_X^\nabla) \\ |s_0| \in \mathcal{I}}} C^\bullet(s_0) \quad \text{and} \quad Q_{\mathcal{I}} = \sum_{\substack{s_0 \in \mathrm{Res}(\mathcal{L}_X^\nabla) \\ |s_0| \in \mathcal{I}}} Q_{s_0}. \tag{6-4}$$

Note that $\mathcal{L}_X^\nabla + s$ acts on $C^\bullet(s_0)$ for every $s_0 \in \mathrm{Res}(\mathcal{L}_X^\nabla)$ as $-s_0 \,\mathrm{Id} + J$, where J is nilpotent. We thus have for $s \notin \mathrm{Res}(\mathcal{L}_X^\nabla)$

$$\det\nolimits_{\mathrm{gr}, C_{\mathcal{I}}^\bullet} (\mathcal{L}_X^\nabla + s)^{(-1)^{q+1}} = \prod_{\substack{s_0 \in \mathrm{Res}(\mathcal{L}_X^\nabla) \\ |s_0| \in \mathcal{I}}} (s - s_0)^{m(s_0)}, \tag{6-5}$$

where $\det_{\mathrm{gr}}$ is the graded determinant; see Section 3.5.

Let $\lambda \in [0, 1)$ such that $\mathrm{Res}(\mathcal{L}_X^\nabla) \cap \{s \in \mathbb{C} : |s| = \lambda\} = \varnothing$. Now define the meromorphic function

$$\zeta_{X,\nabla}^{(\lambda,\infty)}(s) = \zeta_{X,\nabla}(s) \det\nolimits_{\mathrm{gr}, C_{[0,\lambda]}^\bullet} (\mathcal{L}_X^\nabla + s)^{(-1)^q}. \tag{6-6}$$

Then (5-10) and (6-5) show that $\zeta_{X,\nabla}^{(\lambda,\infty)}$ has neither pole nor zero in $\{|s| \leqslant \lambda\}$, so that the number $\zeta_{X,\nabla}^{(\lambda,\infty)}(0)$ is well-defined.

6.4. ***Definition of the dynamical torsion.*** Let $0 < \mu < \lambda < 1$ such that, for every $s_0 \in \mathrm{Res}(\mathcal{L}_X^\nabla)$, one has $|s_0| \neq \lambda, \mu$. Using Propositions 3.3 and 6.2 we obtain, with notation of Section 6.3,

$$\tau(C_{[0,\lambda]}^\bullet, \Gamma_\vartheta) = (-1)^{-Q_{(\mu,\lambda]}} (\det\nolimits_{\mathrm{gr}, C_{(\mu,\lambda]}^\bullet} \mathcal{L}_X^\nabla)^{-1} \tau(C_{[0,\mu]}^\bullet, \Gamma_\vartheta). \tag{6-7}$$

This allows us to give the following:

Proposition/Definition 6.3 (dynamical torsion). *The number*

$$\tau_\vartheta(\nabla) = (-1)^{Q_{[0,\lambda]}} \zeta_{X,\nabla}^{(\lambda,\infty)}(0)^{(-1)^q} \cdot \tau(C_{[0,\lambda]}^\bullet, \Gamma_\vartheta) \in \mathbb{C} \setminus 0 \tag{6-8}$$

*is independent of the spectral cut $\lambda \in (0, 1)$. We will call this number the **dynamical torsion** of the pair (ϑ, ∇).*

Proof. Let $0 < \mu < \lambda < 1$ be such that $|s_0| \neq \lambda, \mu$ for each $s_0 \in \mathrm{Res}(\mathcal{L}_X^\nabla)$. Denote by $\tau_\vartheta(\nabla, \lambda)$ the right-hand side of (6-8) and define $\tau_\vartheta(\nabla, \mu)$ identically. Then we have, by (6-7),

$$\begin{aligned} \tau_\vartheta(\nabla, \lambda) &= (-1)^{Q_{[0,\lambda]}} \zeta_{X,\nabla}^{(\lambda,\infty)}(0)^{(-1)^q} \cdot \tau(C_{[0,\lambda]}^\bullet, \Gamma_\vartheta) \\ &= (-1)^{Q_{[0,\lambda]}} \zeta_{X,\nabla}^{(\lambda,\infty)}(0)^{(-1)^q} (-1)^{-Q_{(\mu,\lambda]}} (\det\nolimits_{\mathrm{gr}, C_{(\mu,\lambda]}^\bullet} \mathcal{L}_X^\nabla)^{-1} \tau(C_{[0,\mu]}^\bullet, \Gamma_\vartheta). \end{aligned}$$

Now, we have $Q_{[0,\lambda]} - Q_{(\mu,\lambda]} = Q_{[0,\mu]}$ by (6-4); moreover

$$\zeta_{X,\nabla}^{(\lambda,\infty)}(0)^{(-1)^q} (\det\nolimits_{\mathrm{gr}, C_{(\mu,\lambda]}^\bullet} \mathcal{L}_X^\nabla)^{-1} = \zeta_{X,\nabla}^{(\mu,\infty)}(0)^{(-1)^q}$$

by (6-6). Thus $\tau_\vartheta(\nabla, \lambda) = \tau_\vartheta(\nabla, \mu)$, which concludes the proof. □

Remark 6.4. If $c_{X,\nabla} s^{m(0)}$ is the leading term of the Laurent expansion of $\zeta_{X,\nabla}(s)$ at $s = 0$, then taking λ small enough actually shows that

$$\tau_\vartheta(\nabla) = (-1)^{Q_0} c_{X,\nabla}^{(-1)^q} \cdot \tau(C^\bullet, \Gamma_\vartheta). \tag{6-9}$$

In particular, if $0 \notin \mathrm{Res}(\mathcal{L}_X^\nabla)$,

$$\tau_\vartheta(\nabla) = \zeta_{X,\nabla}(0)^{(-1)^q}. \tag{6-10}$$

Note that we could have taken (6-9) as a definition of the dynamical torsion; however, (6-8) is more convenient to study the regularity of the $\tau_\vartheta(\nabla)$ with respect to ϑ and ∇.

Remark 6.5. This definition actually makes sense even if ∇ is not acyclic. Indeed, in that case, formula (6-8) defines an element of the determinant line $\det H^\bullet(C^\bullet_{[0,\lambda]}\nabla)$; see Remark 3.1. Under the identification $H^\bullet(M,\nabla) = H^\bullet(C^\bullet_{[0,\lambda]}\nabla)$ given by the quasi-isomorphism $\Pi_{[0,\lambda]} : \Omega^\bullet(M,E) \to C^\bullet_{[0,\lambda]}$ (see Section 5.5), we thus get an element of $\det H^\bullet(M,\nabla)$.

The rest of this section is devoted to the proof of Proposition 6.2, which computes the value of the torsion $\tau(C^\bullet(s_0),\Gamma_\vartheta)$. The strategy goes at follows. First, we introduce the signature operator $B_\vartheta = \Gamma_\vartheta\nabla + \nabla\Gamma_\vartheta$, and show that it is invertible on $C^\bullet(s_0)$ for $s_0 \neq 0, 1$ (Proposition 6.6). This property will allow us to use Proposition 3.4 in order to compute $\tau(C^\bullet(s_0),\Gamma_\vartheta)$.

6.5. *Invertibility of the contact signature operator.* To prove Proposition 6.2 we shall use Section 3.4 and introduce the *contact signature operator*

$$B_\vartheta = \Gamma_\vartheta\nabla + \nabla\Gamma_\vartheta : \mathcal{D}'^\bullet(M,E) \to \mathcal{D}'^\bullet(M,E),$$

where Γ_ϑ acts trivially on E. We fix in what follows some $s_0 \in \mathrm{Res}(\mathcal{L}_X^\nabla)\setminus\{0,1\}$ and we denote $C^\bullet(s_0)$ by $C^\bullet(s_0)$ for simplicity. We also set $C_0^\bullet(s_0) = C^\bullet(s_0)\cap\ker(\iota_X)$.

Proposition 6.6. *The operator B_ϑ is invertible $C^\bullet(s_0) \to C^\bullet(s_0)$.*

Note that, as $\nabla^2 = 0$ and $\Gamma_\vartheta^2 = \mathrm{Id}$, we have that B_ϑ is invertible on $C^\bullet(s_0)$ if and only if

$$\ker(\Gamma_\vartheta\nabla)\cap\ker(\nabla\Gamma_\vartheta) = \{0\} \tag{6-11}$$

on $C^\bullet(s_0)$. Indeed, assume that (6-11) holds and let $\beta \in \ker B_\vartheta$. Set $\mu = \Gamma_\vartheta\nabla\beta = -\nabla\Gamma_\vartheta\beta$; we have

$$\Gamma_\vartheta\nabla\mu = 0 = \nabla\Gamma_\vartheta\mu,$$

hence $\mu = 0$ by (6-11), and therefore $\beta = 0$, again by (6-11), yielding $\ker B_\vartheta = \{0\}$.

In order to prove (6-11) (and thus Proposition 6.6) and Proposition 3.4, we introduce several notations that will help us understand the action of the operator $\Gamma_\vartheta\nabla$ restricted to $\ker(\nabla\Gamma_\vartheta)$. First, because ∇ does not leave the decomposition (6-1) stable, we need to introduce an operator $\Psi : C_0^\bullet(s_0) \to C_0^{\bullet+1}(s_0)$ which mimics the action of ∇. More precisely, we define

$$\Psi\mu = \nabla\mu - (-1)^k\mathcal{L}_X^\nabla\mu\wedge\vartheta, \quad \mu \in C_0^k(s_0). \tag{6-12}$$

Because $\mathcal{L}_X\,d\vartheta = 0$, the map Ψ satisfies the simple relation

$$\Psi(\mu\wedge d\vartheta^j) = (\Psi\mu)\wedge d\vartheta^j, \quad \mu \in C_0^\bullet(s_0),\ j \in \mathbb{N}, \tag{6-13}$$

that is, Ψ commutes with $\mathscr{L}$. Also, observe that

$$\Psi^2\mu = -\mathcal{L}_X^\nabla\mu\wedge d\vartheta, \quad \mu \in C_0^\bullet(s_0). \tag{6-14}$$

Indeed, using the fact that $\mathcal{L}_X^\nabla$ and ∇ commute,

$$\begin{aligned}\Psi^2\mu &= \nabla\big(\nabla\mu - (-1)^k\mathcal{L}_X^\nabla\mu\wedge\vartheta\big) - (-1)^{k+1}\big(\mathcal{L}_X^\nabla(\nabla\mu - (-1)^k\mathcal{L}_X^\nabla\mu\wedge\vartheta)\big)\wedge\vartheta \\ &= \nabla^2\mu + (-1)^{k+1}\nabla(\mathcal{L}_X^\nabla\mu\wedge\vartheta) + (-1)^k\mathcal{L}_X^\nabla\nabla\mu\wedge\vartheta - \mathcal{L}_X^{\nabla 2}\mu\wedge\vartheta\wedge\vartheta \\ &= (-1)^{k+1}(-1)^k\mathcal{L}_X^\nabla\mu\wedge d\vartheta.\end{aligned}$$

For $k\in\{0,\dots,r\}$, we also define the operator $J_k : C^k(s_0)\to C^k(s_0)$ by the formula

$$J_k\beta = f\wedge\vartheta - (-1)^k\Psi f \tag{6-15}$$

for any $\beta = f\wedge\vartheta + g \in C^k(s_0)$, with $f\in C_0^{k-1}(s_0)$. We finally set, as in Section 3.4,

$$C_+^\bullet(s_0) = C^\bullet(s_0)\cap\ker(\nabla\Gamma_\vartheta)\quad\text{and}\quad C_-^\bullet(s_0) = C^\bullet(s_0)\cap\ker(\Gamma_\vartheta\nabla).$$

Lemma 6.7. *J_k is a projector and is valued in $C_+^k(s_0)$.*

Proof. Indeed, we have for any $f\in C_0^{k-1}(s_0)$ and $g\in C_0^k(s_0)$,

$$\begin{aligned}\nabla\Gamma_\vartheta(f\wedge\vartheta+g) &= \nabla(g\wedge d\vartheta^{r-k}\wedge\vartheta + f\wedge d\vartheta^{r-k+1}) \\ &= \Psi g\wedge d\vartheta^{r-k}\wedge\vartheta + (-1)^k g\wedge d\vartheta^{r-k+1} + \Psi f\wedge d\vartheta^{r-k+1} + (-1)^{k+1}\mathcal{L}_X^\nabla f\wedge d\vartheta^{r-k+1}\wedge\vartheta,\end{aligned}$$

which implies that $\beta = f\wedge\vartheta + g$ lies in $C_+^k(s_0)$ if and only if

$$(\Psi g + (-1)^{k+1}\mathcal{L}_X^\nabla f\wedge d\vartheta)\wedge d\vartheta^{r-k} = 0\quad\text{and}\quad(\Psi f + (-1)^k g)\wedge d\vartheta^{r-k+1} = 0. \tag{6-16}$$

But now note that if $\beta = f\wedge\vartheta + g = J_k\beta' = f'\wedge\vartheta - (-1)^k\Psi f'$ for some $\beta' = f'\wedge\vartheta + g'$ then $f = f'$ and $g = -(-1)^k\Psi f$, and thus β satisfies the second part of (6-16). We also obtain

$$\Psi g = -(-1)^k\Psi^2 f = -(-1)^k\mathcal{L}_X^\nabla f\wedge d\vartheta$$

by (6-14), so the first part of (6-16) is also satisfied. Therefore $J_k : C^k(s_0)\to C_+^k(s_0)$; it is clear that J_k is a projector. □

We start by a lemma which tells us how $(\Gamma_\vartheta\nabla)^2$ acts on $C_+^k(s_0)$ with $k<r$.

Lemma 6.8. *Take $k\in\{0,\dots,r-1\}$. Then, for any $\beta\in C_+^k(s_0)$, one has*

$$(\Gamma_\vartheta\nabla)^2\beta = \mathcal{L}_X^\nabla(\mathcal{L}_X^\nabla - \mathrm{Id})\beta - (\mathcal{L}_X^\nabla - \mathrm{Id})J_k\beta.$$

Proof. Since $k<r$ we can write, thanks to (6-20),

$$\Gamma_\vartheta\nabla\beta = \nabla\beta\wedge\vartheta\wedge d\vartheta^{r-k-1} + (-1)^k\iota_X\nabla\beta\wedge d\vartheta^{r-k}.$$

Therefore

$$\begin{aligned}\nabla\Gamma_\vartheta\nabla\beta &= -(-1)^k\nabla\beta\wedge d\vartheta^{r-k} + (-1)^k\nabla\iota_X\nabla\beta\wedge d\vartheta^{r-k} \\ &= (-1)^k(\mathcal{L}_X^\nabla - \mathrm{Id})\nabla\beta\wedge d\vartheta^{r-k} \\ &= (\iota_X\nabla\iota_X\nabla\beta - \iota_X\nabla\beta)\wedge\vartheta\wedge d\vartheta^{r-k} + (-1)^k(\mathcal{L}_X^\nabla - \mathrm{Id})(\nabla\beta - (-1)^k\iota_X\nabla\beta\wedge\vartheta)\wedge d\vartheta^{r-k},\end{aligned}$$

where we used $\nabla\iota_X\nabla\beta = \mathcal{L}_X^\nabla\nabla\beta$ and $\iota_X\nabla\iota_X\nabla\beta = \mathcal{L}_X^\nabla\iota_X\nabla\beta$. Since $\beta \in C_+^k(s_0)$, one has with (6-20)

$$(\nabla\beta - (-1)^k\iota_X\nabla\beta\wedge\vartheta)\wedge d\vartheta^{r-k} = (\iota_X\beta - \iota_X\nabla\iota_X\beta)\wedge d\vartheta^{r-k+1}.$$

This leads to

$$\nabla\Gamma_\vartheta\nabla\beta = (\iota_X\nabla\iota_X\nabla\beta - \iota_X\nabla\beta)\wedge\vartheta\wedge d\vartheta^{r-k} + (-1)^k(\mathcal{L}_X^\nabla - \mathrm{Id})(\iota_X\beta - \iota_X\nabla\iota_X\beta)\wedge d\vartheta^{r-k+1}.$$

Since $\iota_X\nabla\iota_X\nabla\beta - \iota_X\nabla\beta = (\mathcal{L}_X^\nabla - \mathrm{Id})\iota_X\nabla\beta$ and $\iota_X\beta - \iota_X\nabla\iota_X\beta = (\mathrm{Id} - \mathcal{L}_X^\nabla)\iota_X\beta$, we obtain

$$\nabla\Gamma_\vartheta\nabla\beta = (\mathcal{L}_X^\nabla - \mathrm{Id})\iota_X\nabla\beta\wedge\vartheta\wedge d\vartheta^{r-k} + (-1)^k(\mathcal{L}_X^\nabla - \mathrm{Id})(\mathrm{Id} - \mathcal{L}_X^\nabla)\iota_X\beta\wedge d\vartheta^{r-k+1},$$

and thus by the definition of Γ_ϑ

$$\Gamma_\vartheta\nabla\Gamma_\vartheta\nabla\beta = -(-1)^k(\mathrm{Id} - \mathcal{L}_X^\nabla)^2\iota_X\beta\wedge\vartheta + (\mathcal{L}_X^\nabla - \mathrm{Id})\iota_X\nabla\beta. \tag{6-17}$$

Now, writing $\beta = f\wedge\vartheta + g$, where $\iota_X f = 0$ and $\iota_X g = 0$, we have

$$\begin{aligned} \nabla\beta &= \nabla f\wedge\vartheta - (-1)^k f\wedge d\vartheta + \nabla g, \\ \iota_X\nabla\beta &= \mathcal{L}_X^\nabla f\wedge\vartheta + (-1)^k\nabla f + \mathcal{L}_X^\nabla g, \\ \iota_X\beta\wedge\vartheta &= -(-1)^k f\wedge\vartheta. \end{aligned} \tag{6-18}$$

Injecting those relations in (6-17) we get

$$\Gamma_\vartheta\nabla\Gamma_\vartheta\nabla\beta = \mathcal{L}_X^\nabla(\mathcal{L}_X^\nabla - \mathrm{Id})(f\wedge\vartheta + g) - (\mathcal{L}_X^\nabla - \mathrm{Id})\big(f\wedge\vartheta - (-1)^k(\nabla f + (-1)^k\mathcal{L}_X^\nabla f\wedge\vartheta)\big),$$

which concludes in view of (6-12) and (6-15). □

We now deal with the case $k = r$.

Lemma 6.9. *One has, for* $\beta \in C_+^r(s_0)$,

$$\Gamma_\vartheta\nabla\beta = (-1)^r\big((\mathcal{L}_X^\nabla - \mathrm{Id})\beta + (\mathrm{Id} - J_r)\beta\big).$$

Proof. We have

$$\Gamma_\vartheta\nabla\beta = \mathscr{L}^{-1}\big(\nabla\beta - (-1)^r\iota_X\nabla\beta\wedge\vartheta\big) + (-1)^r\iota_X\nabla\beta.$$

Since $\beta \in C_+^r(s_0)$, we have with (6-20) that $\nabla\beta - (-1)^r\iota_X\nabla\beta\wedge\vartheta = (\iota_X\beta - \iota_X\nabla\iota_X\beta)\wedge d\vartheta$. Therefore,

$$\Gamma_\vartheta\nabla\beta = (\iota_X\beta - \iota_X\nabla\iota_X\beta)\wedge\vartheta + (-1)^r\iota_X\nabla\beta.$$

We now conclude as in the previous lemma, using (6-18). □

Proof of Proposition 6.6. To prove that B_ϑ is invertible on $C^\bullet(s_0)$, recall that it suffices to show that (6-11) holds. Let $\beta \in C^\bullet(s_0)$ lying in the left-hand side of (6-11), and write

$$\beta = \sum_{k=0}^{2r+1}\beta_k,$$

where $\beta_k \in C^k(s_0)$. Then $\beta_k \in C_+^k(s_0)\cap C_-^k(s_0)$ for each k. Therefore, Lemma 6.8 yields, for $k < r$,

$$0 = (\Gamma_\vartheta\nabla)^2\beta_k = \mathcal{L}_X^\nabla(\mathcal{L}_X^\nabla - \mathrm{Id})\beta_k - (\mathcal{L}_X^\nabla - \mathrm{Id})J_k\beta_k,$$

that is, $(\mathcal{L}_X^\nabla - \mathrm{Id})(\mathcal{L}_X^\nabla \beta_k - J_k\beta_k) = 0$, which gives

$$\mathcal{L}_X^\nabla \beta_k = J_k\beta_k$$

since $\mathcal{L}_X^\nabla - \mathrm{Id}$ is invertible on $C^\bullet(s_0)$. However, writing $\beta_k = f_{k-1}\wedge\vartheta + g_k$, with $f_{k-1}, g_k \in C_0^\bullet(s_0)$, we have by (6-15)

$$\mathcal{L}_X^\nabla f_{k-1}\wedge\vartheta + \mathcal{L}_X^\nabla g_k = f_{k-1}\wedge\vartheta - (-1)^k\Psi f_{k-1}.$$

Therefore $\mathcal{L}_X^\nabla f_{k-1} = f_{k-1}$ and $\mathcal{L}_X^\nabla g_k = -(-1)^k\Psi f_{k-1}$ and $f_{k-1} = 0$ by invertibility of $\mathcal{L}_X^\nabla - \mathrm{Id}$. Hence $g_k = 0$ by invertibility of $\mathcal{L}_X^\nabla$, and thus $\beta_k = 0$. For $k = r$, Lemma 6.9 yields

$$\mathcal{L}_X^\nabla \beta_r = J_r\beta_r,$$

which gives, as above, $\beta_r = 0$. Applying the above arguments to $\tilde\beta = \Gamma_\vartheta\beta$, which lies in the intersection (6-11), yields $\beta_{n-k} = 0$ for each $k \leqslant r$. Thus $\beta = 0$ and the equality (6-11) is proven. □

6.6. ***Proof of Proposition 6.2.*** We start from Proposition 3.4 which gives us, in view of Proposition 6.6,

$$\tau(C^\bullet(s_0), \Gamma_\vartheta) = (-1)^{r\dim C_+^r(s_0)} \det(\Gamma_\vartheta\nabla|_{C_+^r(s_0)})^{(-1)^r} \prod_{j=0}^{r-1} \det(\Gamma_\vartheta\nabla|_{C_+^j(s_0)\oplus C_+^{n-j-1}(s_0)})^{(-1)^j}. \tag{6-19}$$

We first note that for $k \in \{0, \dots, r\}$ and $\beta \in \Omega^k(M, E)$, one has

$$\begin{aligned}\nabla\Gamma_\vartheta\beta = \mathscr{L}^{r-k}\big(\nabla\beta - (-1)^k\iota_X\nabla\beta\wedge\vartheta + \mathscr{L}(\iota_X\nabla\iota_X\beta - \iota_X\beta)\big)\wedge\vartheta \\ + (-1)^k\mathscr{L}^{r-k+1}\big(\beta - \nabla\iota_X\beta + (-1)^k\iota_X(\beta - \nabla\iota_X\beta)\wedge\vartheta\big), \\ \Gamma_\vartheta\nabla\beta = \mathscr{L}^{r-k-1}\big(\nabla\beta - (-1)^k\iota_X\nabla\beta\wedge\vartheta\big)\wedge\vartheta + (-1)^k\mathscr{L}^{r-k}(\iota_X\nabla\beta),\end{aligned} \tag{6-20}$$

where $\mathscr{L}^{j-r} = (\mathscr{L}^{r-j}|_{\Lambda^j V_X})^{-1}$ for $0 \leqslant j \leqslant r$. Indeed, using the decomposition (6-1),

$$\begin{aligned}\Gamma_\vartheta\beta &= (-1)^{k+1}\iota_X\beta\wedge d\vartheta^{r-k+1} + (\beta + (-1)^k\iota_X\beta\wedge\vartheta)\wedge d\vartheta^{r-k}\wedge\vartheta \\ &= (-1)^{k+1}\iota_X\beta\wedge d\vartheta^{r-k+1} + \beta\wedge d\vartheta^{r-k}\wedge\vartheta,\end{aligned}$$

which leads to

$$\begin{aligned}\nabla\Gamma_\vartheta\beta &= (-1)^{k+1}\nabla\iota_X\beta\wedge d\vartheta^{r-k+1} + \nabla\beta\wedge d\vartheta^{r-k}\wedge\vartheta + (-1)^k\beta\wedge d\vartheta^{r-k+1} \\ &= (-1)^{k+1}\big((-1)^{k+1}\iota_X\nabla\iota_X\beta\wedge\vartheta\wedge d\vartheta^{r-k+1}\big) \\ &\quad + (-1)^{k+1}\big(\nabla\iota_X\beta + (-1)^k\iota_X\nabla\iota_X\beta\wedge\vartheta\big)\wedge d\vartheta^{r-k+1} \\ &\quad + \big(\nabla\beta - (-1)^k\iota_X\nabla\beta\wedge\vartheta\big)\wedge d\vartheta^{r-k}\wedge\vartheta \\ &\quad + (-1)^k\big(\beta + (-1)^k\iota_X\beta\wedge\vartheta\big)\wedge d\vartheta^{r-k+1} \\ &\quad - \iota_X\beta\wedge d\vartheta^{r-k+1}\wedge\vartheta,\end{aligned}$$

which is exactly the first part of (6-20). The second part follows directly from the decomposition (6-1). We will set, for $0 \leqslant k \leqslant n$,

$$m_k = \dim C^k(s_0), \quad m_k^0 = \dim C_0^k(s_0), \quad m_k^\pm = \dim C_\pm^k(s_0).$$

First, take $k \in \{0, \dots, r-1\}$. Because B_ϑ is invertible on $C^\bullet(s_0)$, $\Gamma_\vartheta \nabla$ induces an isomorphism $C_+^k(s_0) \to C_+^{n-k-1}(s_0)$. Take any basis γ of $C_+^k(s_0)$. Then $\Gamma_\vartheta \nabla \gamma$ is a basis of C_+^{n-k-1} and the matrix of $\Gamma_\vartheta \nabla|_{C_+^k(s_0) \oplus C_+^{n-k+1}(s_0)}$ in the basis $\gamma \oplus \Gamma_\vartheta \nabla \gamma$ is

$$\begin{pmatrix} 0 & [(\Gamma_\vartheta \nabla)^2]_\gamma \\ \mathrm{Id} & 0 \end{pmatrix}, \tag{6-21}$$

where $[(\Gamma_\vartheta \nabla)^2]_\gamma$ is the matrix of $(\Gamma_\vartheta \nabla)^2|_{C_+^k(s_0)}$ in the basis γ. Define

$$\widetilde{J}_k = \mathrm{Id} - J_k : C_+^k(s_0) \to C_+^k(s_0).$$

Then $\widetilde{J}_k$ is a projector (since J_k is by Lemma 6.7) and J_k (and thus $\widetilde{J}_k$) commutes with $\mathcal{L}_X^\nabla$ (since Ψ commutes with $\mathcal{L}_X^\nabla$). Moreover one has

$$(\Gamma_\vartheta \nabla)^2|_{\ker \widetilde{J}_k} = (\mathcal{L}_X^\nabla - \mathrm{Id})^2, \quad (\Gamma_\vartheta \nabla)^2|_{\mathrm{ran}\, \widetilde{J}_k} = \mathcal{L}_X^\nabla(\mathcal{L}_X^\nabla - \mathrm{Id}).$$

As a consequence,

$$\det((\Gamma_\vartheta \nabla)^2|_{C_+^k(s_0)}) = [s_0(1+s_0)]^{m_k^+ - m_{k-1}^0}(1+s_0)^{2m_{k-1}^0} = s_0^{m_k^+ - m_{k-1}^0}(1+s_0)^{m_k^+ + m_{k-1}^0},$$

because on $C^\bullet(s_0)$ (and in particular on $C_+^k(s_0)$), one has $\mathcal{L}_X^\nabla = -s_0 \,\mathrm{Id} + \nu$, where ν is nilpotent, and one has $\dim \ker \widetilde{J}_k = \dim \mathrm{ran}\, J_k = m_{k-1}^0$. Indeed, by (6-15) we can view J_k as a map $C_0^{k-1}(s_0) \to C_+^k(s_0)$, which is of course injective. We finally obtain with (6-21)

$$\det(\Gamma_\vartheta \nabla|_{C_+^k(s_0) \oplus C_+^{n-k+1}(s_0)}) = (-1)^{m_k^+} s_0^{m_k^+ - m_{k-1}^0}(1+s_0)^{m_k^+ + m_{k-1}^0}. \tag{6-22}$$

We now deal with the case $k = r$. Lemma 6.9 gives

$$\Gamma_\vartheta \nabla|_{\ker \widetilde{J}_r} = (-1)^r(\mathcal{L}_X^\nabla - \mathrm{Id}), \quad \Gamma_\vartheta \nabla|_{\mathrm{ran}\, \widetilde{J}_r} = (-1)^r \mathcal{L}_X^\nabla.$$

As before, we obtain

$$\det(\Gamma_\vartheta \nabla|_{C_+^r(s_0)}) = (-1)^{r m_r^+}(-1)^{m_r^+} s_0^{m_r^+ - m_{r-1}^0}(1+s_0)^{m_{r-1}^0}. \tag{6-23}$$

Combining (6-19) with (6-22) and (6-23) we finally obtain

$$\tau(C^\bullet(s_0), \Gamma_\vartheta) = (-1)^J s_0^K (1+s_0)^L, \tag{6-24}$$

where

$$J = \sum_{k=0}^{r}(-1)^k m_k^+, \quad K = \sum_{k=0}^{r}(-1)^k(m_k^+ - m_{k-1}^0), \quad L = \sum_{k=0}^{r-1}(-1)^k(m_k^+ - m_k^0).$$

Note that for $0 \leqslant k \leqslant r-1$ one has by acyclicity and because Γ_ϑ induces isomorphisms $C_+^k(s_0) \simeq C_-^{n-k}(s_0)$ (since B_ϑ is invertible),

$$m_k^+ = m_{n-k}^- = \dim \ker(\nabla|_{C^{n-k}(s_0)}) = \dim \mathrm{ran}(\nabla|_{C^{n-k-1}(s_0)}) = m_{n-k-1} - m_{n-k-1}^-.$$

Since $m_{n-k-1} - m_{n-k-1}^- = m_{k+1} - m_{k+1}^+$, one obtains

$$m_k^+ + m_{k+1}^+ = m_{k+1}, \quad 0 \leqslant k \leqslant r-1, \tag{6-25}$$

which leads to $m_k^+ + m_{k+1}^+ = m_k^0 + m_{k+1}^0$. As a consequence, since $m_0^+ = m_0 = m_0^0$, we get

$$m_r^+ - m_r^0 = -(m_{r-1}^+ - m_{r-1}^0) = \cdots = (-1)^r (m_0^+ - m_0^0) = 0.$$

This implies

$$m_k^0 = m_k^+, \quad 0 \leqslant k \leqslant r, \tag{6-26}$$

which leads to $L = 0$. Moreover, since $m_k^0 = m_{2r-k}^0$, we get

$$K = \sum_{k=0}^{r} (-1)^k (m_k^0 - m_{k-1}^0) = \sum_{k=0}^{2r} (-1)^k m_k^0 = -\sum_{k=0}^{n} (-1)^k k m_k = (-1)^q m(s_0),$$

where we used (5-10) in the last equality. Finally, again because $m_k^0 = m_{2r-k}^0$,

$$2J = (-1)^r m_r^0 + \sum_{k=0}^{2r} (-1)^k m_k^0 = (-1)^r m_r^0 - \sum_{k=0}^{n} (-1)^k k m_k.$$

We have

$$(-1)^r m_r^0 = \sum_{k=0}^{r} (-1)^k m_k \quad \text{and} \quad \sum_{k=0}^{n} (-1)^k k m_k = \sum_{k=0}^{r} (-1)^k (2k - n) m_k,$$

where the first equality comes from (6-25) and (6-26) and the second from the fact that $m_k = m_{n-k}$. We thus obtained

$$J = \sum_{k=0}^{r} (-1)^k (r + 1 - k) m_k = Q_{s_0},$$

and finally by (6-24)

$$\tau(C^\bullet(s_0), \Gamma_\vartheta) = (-1)^{Q_{s_0}} (-s_0)^{(-1)^q m(s_0)}.$$

But now recall from (6-5) that $\det_{\mathrm{gr}, C^\bullet}(\mathcal{L}_X^\nabla)^{(-1)^{q+1}} = (-s_0)^{m(s_0)}$. This completes the proof.

7. Invariance of the dynamical torsion under small perturbations of the contact form

In this section, we are interested in the behavior of the dynamical torsion when we deform the contact form. Namely, we prove here:

Theorem 9. *Assume that $(\vartheta_{\mathrm{t}})_{\mathrm{t} \in (-\delta, \delta)}$ is a smooth family of contact forms such that their Reeb vector fields X_{t} generate a contact Anosov flow for each* t. *Let (E, ∇) be an acyclic flat vector bundle. Then the map* $\mathrm{t} \mapsto \tau_{\vartheta_{\mathrm{t}}}(\nabla)$ *is real differentiable and we have*

$$\frac{\mathrm{d}}{\mathrm{dt}} \tau_{\vartheta_{\mathrm{t}}}(\nabla) = 0.$$

Remark 7.1. In view of Remark 6.5, if ∇ is not assumed acyclic, then it is not hard to see that the proof (given below) of Theorem 9 is still valid and we have that $\partial_{\mathrm{t}} \tau_{\vartheta_{\mathrm{t}}}(\nabla) = 0$ in $\det H^\bullet(M, \nabla)$.

We will thus consider a family of contact forms and set $\vartheta = \vartheta_0$ and $X = X_0$. We also fix an acyclic flat vector bundle (E, ∇).

7.1. ***Anisotropic spaces for a family of vector fields.*** To study the dynamical torsion when the dynamics is perturbed, we construct with the help of [Bonthonneau 2020] some anisotropic Sobolev spaces on which each X_{t} has nice spectral properties. We refer to Appendix B where we briefly recall the construction of these spaces.

By Section B.4, the set

$$\{(\mathrm{t}, s) : s \notin \mathrm{Res}(\mathcal{L}^{\nabla}_{X_{\mathrm{t}}})\}$$

is open in $(-\delta, \delta) \times \mathbb{C}$. Fix $\lambda \in (0, 1)$ such that

$$\mathrm{Res}(\mathcal{L}^{\nabla}_{X}) \cap \{|s| \leqslant \lambda\} \subset \{0\}. \tag{7-1}$$

Then for t close enough to 0, we have $\mathrm{Res}(\mathcal{L}^{\nabla}_{X_{\mathrm{t}}}) \cap \{|s| = \lambda\} = \varnothing$ so that the spectral projectors

$$\Pi_{\mathrm{t}} = \frac{1}{2i\pi} \int_{|s|=\lambda} (\mathcal{L}^{\nabla}_{X_{\mathrm{t}}} + s)^{-1}\, \mathrm{d}s : \Omega^{\bullet}(M, E) \to \mathcal{D}'^{\bullet}(M, E) \tag{7-2}$$

are well-defined. The next proposition is a brief summary of the results from Appendix B. For any $C, \rho > 0$, we will let

$$\Omega(c, \rho) = \{\mathrm{Re}(s) > c\} \cup \{|s| \leqslant \rho\} \subset \mathbb{C}. \tag{7-3}$$

Proposition 7.2. *There is $c, \varepsilon_0 > 0$ such that for any $\rho > 0$ there exists anisotropic Sobolev spaces*

$$\Omega^{\bullet}(M, E) \subset \mathcal{H}^{\bullet}_1 \subset \mathcal{H}^{\bullet} \subset \mathcal{D}'^{\bullet}(M, E),$$

each inclusion being continuous with dense image, such that the following hold:

(1) *For each $\mathrm{t} \in [-\varepsilon_0, \varepsilon_0]$, the family $s \mapsto \mathcal{L}^{\nabla}_{X_{\mathrm{t}}} + s$ is a holomorphic family of (unbounded) Fredholm operators $\mathcal{H}^{\bullet}_1 \to \mathcal{H}^{\bullet}_1$ and $\mathcal{H}^{\bullet} \to \mathcal{H}^{\bullet}$ of index 0 in the region $\Omega(c, \rho)$. Moreover*

$$\mathcal{L}^{\nabla}_{X_{\mathrm{t}}} \in \mathcal{C}^1([-\varepsilon_0, \varepsilon_0]_{\mathrm{t}}, \mathcal{L}(\mathcal{H}^{\bullet}_1, \mathcal{H}^{\bullet})).$$

(2) *For every relatively compact open region $\mathcal{Z} \subset \mathrm{int}\,\Omega(c, \rho)$ such that $\mathrm{Res}(\mathcal{L}^{\nabla}_{X}) \cap \overline{\mathcal{Z}} = \varnothing$, there exists $\mathrm{t}_{\mathcal{Z}} > 0$ such that*

$$(\mathcal{L}^{\nabla}_{X_{\mathrm{t}}} + s)^{-1} \in \mathcal{C}^0\big([-t_{\mathcal{Z}}, t_{\mathcal{Z}}]_{\mathrm{t}}, \mathrm{Hol}(\mathcal{Z}_s, \mathcal{L}(\mathcal{H}^{\bullet}_1, \mathcal{H}^{\bullet}))\big).$$

(3) $\Pi_{\mathrm{t}} \in \mathcal{C}^1([-\varepsilon_0, \varepsilon_0]_{\mathrm{t}}, \mathcal{L}(\mathcal{H}^{\bullet}, \mathcal{H}^{\bullet}_1))$.

We will thus fix such Hilbert spaces for some $\rho > c + 1$. We let $C^{\bullet}_{\mathrm{t}} = \mathrm{ran}\,\Pi_{\mathrm{t}} \subset \mathcal{H}^{\bullet}$, $\Pi = \Pi_{\mathrm{t}=0}$ and $C^{\bullet} = \mathrm{ran}\,\Pi$.

7.2. ***Variation of the torsion part.*** Let $\Gamma_{\mathrm{t}} : C^{\bullet}_{\mathrm{t}} \to C^{n-\bullet}_{\mathrm{t}}$ be the chirality operator associated with X_{t}; see Section 6.1. The next lemma allows us to compute the variation of the finite-dimensional torsion part of the dynamical torsion.

Lemma 7.3. *We have that $\mathrm{t} \mapsto \tau(C^{\bullet}_{\mathrm{t}}, \Gamma_{\mathrm{t}})$ is real differentiable and*

$$\frac{\mathrm{d}}{\mathrm{dt}} \tau(C^{\bullet}_{\mathrm{t}}, \Gamma_{\mathrm{t}}) = -\mathrm{tr}_{\mathrm{s}, C^{\bullet}_{\mathrm{t}}}(\Pi_{\mathrm{t}} \vartheta_{\mathrm{t}} \iota_{\dot{X}_{\mathrm{t}}}) \tau(C^{\bullet}_{\mathrm{t}}, \Gamma_{\mathrm{t}}),$$

where $\dot{X}_{\mathrm{t}} = (\mathrm{d}/\mathrm{dt}) X_{\mathrm{t}}$.

Proof. By Proposition 7.2, the operator $\Pi_t|_{C^\bullet} : C^\bullet \to C_t^\bullet$ is invertible for t close enough to 0 and we will denote by Q_t its inverse. Then for t close enough to 0, one has

$$\tau(C_t^\bullet, \Gamma_t) = \tau(C^\bullet, \widetilde{\Gamma}_t),$$

where $\widetilde{\Gamma}_t$ is defined by $\widetilde{\Gamma}_t = \Pi Q_t \Gamma_t \Pi_t \Pi$, because ∇ and Π_t commute and the image of a $\widetilde{\Gamma}_t$-invariant basis of $C^\bullet$ by the projector Π_t is a Γ_t-invariant basis of $C_t^\bullet$.

Therefore [Braverman and Kappeler 2007c, Proposition 4.9] gives

$$\frac{\mathrm{d}}{\mathrm{dt}}\tau(C_t^\bullet, \Gamma_t) = \tfrac{1}{2}\mathrm{tr}_{s,C^\bullet}(\dot{\widetilde{\Gamma}}_t\widetilde{\Gamma}_t)\tau(C_t^\bullet, \Gamma_t),$$

where $\dot{\widetilde{\Gamma}}_t = (\mathrm{d}/\mathrm{dt})\widetilde{\Gamma}_t : C^\bullet \to C^\bullet$. Since Γ_t and Π_t commute, and by the two first points of Proposition 7.2, we can apply (A-2) to get

$$\widetilde{\Gamma}_t = \Pi\Gamma_t\Pi + \mathrm{t}\Pi\dot{\Gamma}\Pi + o_{C^\bullet \to C^\bullet}(\mathrm{t}).$$

This leads to

$$\dot{\widetilde{\Gamma}}\widetilde{\Gamma} = \Pi\dot{\Gamma}\Gamma|_{C^\bullet},$$

where we removed the subscripts t to signify that we take all the t-dependent objects at t = 0. Therefore,

$$\tfrac{1}{2}\mathrm{tr}_{s,C^\bullet}(\dot{\widetilde{\Gamma}}\widetilde{\Gamma}) = \tfrac{1}{2}\mathrm{tr}_{s,C^\bullet}(\Pi\dot{\Gamma}\Gamma).$$

Now notice that $\Gamma_t^2 = 1$ implies $\Gamma\dot{\Gamma} + \dot{\Gamma}\Gamma = 0$. Therefore, for every $k \in \{0, \dots, r\}$,

$$\mathrm{tr}_{C^{n-k}}\,\Gamma\dot{\Gamma} = \mathrm{tr}_{C^k}\,\Gamma\Gamma\dot{\Gamma}\Gamma = \mathrm{tr}_{C^k}\,\dot{\Gamma}\Gamma = -\,\mathrm{tr}_{C^k}\,\Gamma\dot{\Gamma}.$$

Therefore we only need to compute $\mathrm{tr}_{C^k}(\Gamma\dot{\Gamma})$ for $k \in \{0, \dots, r\}$ to get the full super trace $\mathrm{tr}_{s,C^\bullet}(\dot{\Gamma}\Gamma)$. Since n is odd, we have

$$\tfrac{1}{2}\mathrm{tr}_{s,C^\bullet}(\dot{\widetilde{\Gamma}}\widetilde{\Gamma}) = \tfrac{1}{2}\,\mathrm{tr}_{C^\bullet}((-1)^{N+1}\Pi\Gamma\dot{\Gamma}) = \sum_{k=0}^{r}(-1)^{k+1}\,\mathrm{tr}_{C^k}(\Pi\Gamma\dot{\Gamma}).$$

Let $k \in \{0, \dots, r\}$ and $\alpha \in \Omega^k(M)$. Using the decomposition

$$\alpha = (-1)^{k-1}\iota_{X_t}\alpha \wedge \vartheta_t + (\alpha + (-1)^k\iota_{X_t}\alpha \wedge \vartheta_t),$$

we get by the definition of Γ_t

$$\Gamma_t\alpha = (-1)^{k-1}\iota_{X_t}\alpha \wedge (\mathrm{d}\vartheta_t)^{r-k+1} + (\alpha + (-1)^k\iota_{X_t}\alpha \wedge \vartheta_t) \wedge (\mathrm{d}\vartheta_t)^{r-k} \wedge \vartheta_t.$$

Therefore,

$$\begin{aligned}\dot{\Gamma}_t\alpha = {} & (-1)^{k-1}\iota_{\dot{X}_t}\alpha \wedge (\mathrm{d}\vartheta_t)^{r-k+1} \\ & + (r-k+1)(-1)^{k-1}\iota_{X_t}\alpha \wedge \mathrm{d}\dot{\vartheta}_t \wedge (\mathrm{d}\vartheta_t)^{r-k} \\ & + (-1)^k\big(\iota_{\dot{X}_t}\alpha \wedge \vartheta_t + \iota_{X_t}\alpha \wedge \dot{\vartheta}_t\big) \wedge (\mathrm{d}\vartheta_t)^{r-k} \wedge \vartheta_t \\ & + \big(\alpha + (-1)^k\iota_{X_t}\alpha \wedge \vartheta_t\big) \wedge (\mathrm{d}\vartheta_t)^{r-k} \wedge \dot{\vartheta}_t \\ & + (r-k)\big(\alpha + (-1)^k\iota_{X_t}\alpha \wedge \vartheta_t\big) \wedge \mathrm{d}\dot{\vartheta}_t \wedge (\mathrm{d}\vartheta_t)^{r-k-1} \wedge \vartheta_t.\end{aligned}$$

Now we use the decompositions

$$\begin{aligned}\mathrm{d}\dot{\vartheta}_{\mathrm{t}} &= -\iota_{X_{\mathrm{t}}}\mathrm{d}\dot{\vartheta}_{\mathrm{t}} \wedge \vartheta_{\mathrm{t}} + (\mathrm{d}\dot{\vartheta}_{\mathrm{t}} + \iota_{X_{\mathrm{t}}}\mathrm{d}\dot{\vartheta}_{\mathrm{t}} \wedge \vartheta_{\mathrm{t}}),\\ \dot{\vartheta}_{\mathrm{t}} &= \dot{\vartheta}_{\mathrm{t}}(X_{\mathrm{t}})\vartheta + (\dot{\vartheta}_{\mathrm{t}} - \dot{\vartheta}_{\mathrm{t}}(X_{\mathrm{t}})\vartheta),\\ \iota_{\dot{X}_{\mathrm{t}}}\alpha &= (-1)^k \iota_{X_{\mathrm{t}}}\iota_{\dot{X}_{\mathrm{t}}}\alpha \wedge \vartheta_{\mathrm{t}} + (\iota_{\dot{X}_{\mathrm{t}}}\alpha + (-1)^{k+1}\iota_{X_{\mathrm{t}}}\iota_{\dot{X}_{\mathrm{t}}}\alpha \wedge \vartheta_{\mathrm{t}})\end{aligned}$$

to get, again by definition,

$$\begin{aligned}\Gamma\dot{\Gamma}\alpha = {}&(-1)^{k-1}(\iota_{\dot{X}}\alpha + (-1)^{k+1}\iota_X\iota_{\dot{X}}\alpha \wedge \vartheta)\wedge\vartheta\\ &+(-1)^{k-1}(\mathscr{L}^{r-k})^{-1}\big((-1)^k\iota_X\iota_{\dot{X}}\alpha\wedge(\mathrm{d}\vartheta)^{r-k+1}\big)\\ &+(r-k+1)(\mathscr{L}^{r-k+1})^{-1}\big((-1)^{k-1}\iota_X\alpha\wedge(\mathrm{d}\dot{\vartheta}+\iota_X\,\mathrm{d}\dot{\vartheta}\wedge\vartheta)\wedge(\mathrm{d}\vartheta)^{r-k}\big)\wedge\vartheta\\ &-(r-k+1)((-1)^{k-1}\iota_X\alpha)\wedge\iota_X\,\mathrm{d}\dot{\vartheta}\\ &+(-1)^k\iota_X\alpha\wedge(\dot{\vartheta}-\dot{\vartheta}(X)\vartheta)\\ &+(\mathscr{L}^{r-k+1})^{-1}\big((\alpha+(-1)^k\iota_X\alpha\wedge\vartheta)\wedge(\mathrm{d}\vartheta)^{r-k}\wedge(\dot{\vartheta}-\dot{\vartheta}(X)\vartheta)\big)\wedge\vartheta\\ &+(\alpha+(-1)^k\iota_X\alpha\wedge\vartheta)\dot{\vartheta}(X)\\ &+(r-k)(\mathscr{L}^{r-k})^{-1}\big((\alpha+(-1)^k\iota_X\alpha\wedge\vartheta)\wedge(\mathrm{d}\dot{\vartheta}+\iota_X\,\mathrm{d}\dot{\vartheta}\wedge\vartheta)\wedge(\mathrm{d}\vartheta)^{r-k-1}\big),\end{aligned}\tag{7-4}$$

where again we removed the subscripts t to signify that we take everything at $\mathrm{t}=0$. Now let $A_k : C_0^k \to C_0^k$ (note that here C_0^k is $C^k \cap \ker\iota_X$, see Section 6.1, and not C_{t}^k at $\mathrm{t}=0$) defined by

$$A_k u = (r-k)(\mathscr{L}^{r-k})^{-1}\big(u\wedge(\mathrm{d}\dot{\vartheta}+\iota_X\,\mathrm{d}\dot{\vartheta})\wedge(\mathrm{d}\vartheta)^{r-k-1}\big).$$

Note that the maps defined by the second, the fourth, the fifth and the sixth terms of the right-hand side of (7-4) are antidiagonal, that is, they have the form $\left(\begin{smallmatrix}0 & \star\\ \star & 0\end{smallmatrix}\right)$ in the decomposition $C^\bullet = C_0^{\bullet-1}\wedge\vartheta\oplus C_0^\bullet$. Therefore, since $A_r = 0$ (we also set $A_{-1}=0$),

$$\begin{aligned}\sum_{k=0}^r(-1)^{k+1}\operatorname{tr}_{C^k}(\Pi\Gamma\dot{\Gamma}) &= \sum_{k=0}^r(-1)^{k+1}(\operatorname{tr}_{C^k}\Pi\vartheta\iota_{\dot{X}}+\operatorname{tr}_{C_0^k}\Pi\dot{\vartheta}(X))+\sum_{k=0}^r(-1)^{k+1}(\operatorname{tr}_{C_0^{k-1}}\Pi A_{k-1}+\operatorname{tr}_{C_0^k}\Pi A_k)\\ &= \sum_{k=0}^r(-1)^{k+1}(\operatorname{tr}_{C^k}\Pi\vartheta\iota_{\dot{X}}+\operatorname{tr}_{C_0^k}\Pi\dot{\vartheta}(X)).\end{aligned}\tag{7-5}$$

Here, the first and seventh terms of (7-4) correspond to the first sum of the right-hand side of the first equality of (7-5), while the third and eighth correspond to the second one. If $\alpha = f\wedge\vartheta+g\in C_0^{k-1}\wedge\vartheta\oplus C_0^k$, then

$$\vartheta\wedge\iota_{\dot{X}}\alpha = \vartheta(\dot{X})(f\wedge\vartheta)+\vartheta\wedge\iota_{\dot{X}}g.$$

This shows that for every $k\in\{0,\dots,n\}$ one has

$$\operatorname{tr}_{C^k}\Pi\vartheta\iota_{\dot{X}} = \operatorname{tr}_{C_0^{k-1}}\Pi\vartheta(\dot{X}).\tag{7-6}$$

Injecting this relation in (7-5) we obtain, with $\vartheta(\dot{X}) = -\dot{\vartheta}(X)$ and the formula $\dot{\vartheta}(X)|_{C_0^{2r-k}}\mathscr{L}^{r-k} = \mathscr{L}^{r-k}\dot{\vartheta}(X)|_{C_0^k}$,

$$\sum_{k=0}^r(-1)^{k+1}\operatorname{tr}_{C^k}(\Pi\Gamma\dot{\Gamma}) = \sum_{k=0}^r(-1)^{k+1}(\operatorname{tr}_{C_0^{k-1}}\Pi\vartheta(\dot{X})-\operatorname{tr}_{C_0^k}\Pi\vartheta(\dot{X})) = \sum_{k=0}^{2r}(-1)^k\operatorname{tr}_{C_0^k}\Pi\vartheta(\dot{X}).$$

However by (7-6) we have

$$\sum_{k=0}^{2r}(-1)^k \operatorname{tr}_{C_0^k} \Pi\vartheta(\dot X) = \operatorname{tr}_{C^\bullet}((-1)^{N+1}\Pi\vartheta\iota_{\dot X}),$$

which completes the proof. □

7.3. ***Variation of the rest.*** Let us now interest ourselves in the variation of $\mathrm{t}\mapsto \zeta^{(\lambda,\infty)}_{X_\mathrm{t},\nabla}(0)$; see Section 6.3. For t close enough to 0, let $P_\mathrm{t} : TM \to TM$ be defined by

$$\begin{array}{rcccccl} P_\mathrm{t}: \ker\vartheta & \oplus & \mathbb{R}X & \to & \ker\vartheta & \oplus & \mathbb{R}X_\mathrm{t}, \\ v & + & \mu X & \mapsto & v & + & \mu X_\mathrm{t}. \end{array}$$

For simplicity, we will still denote $\Lambda^k({}^T P_\mathrm{t}) : \Lambda^k T^*M \to \Lambda^k T^*M$ by P_t.

Proposition 7.4 (variation of the dynamical zeta function with respect to the vector field). *For any relatively compact open set $\mathcal{Z}\subset\mathbb{C}$ such that $\overline{\mathcal{Z}}\cap\operatorname{Res}(\mathcal{L}^\nabla_X)=\varnothing$, there is $\mathrm{t}_\mathcal{Z}>0$ so that $\mathrm{t}\mapsto\zeta_{X_\mathrm{t},\nabla}(s)$ is $\mathcal{C}^1$ as a map*

$$[-\mathrm{t}_\mathcal{Z},\mathrm{t}_\mathcal{Z}]\to\operatorname{Hol}(\mathcal{Z},\mathbb{C}).$$

Moreover for each $s\notin\operatorname{Res}(\mathcal{L}_{X_\mathrm{t}})$ we have

$$\partial_\mathrm{t}\log\zeta_{X_\mathrm{t},\nabla}(s) = (-1)^q s\operatorname{tr}^\flat_\mathrm{s}(\vartheta_\mathrm{t}\iota_{\dot X_\mathrm{t}}(\mathcal{L}^\nabla_{X_\mathrm{t}}+s)^{-1}e^{-\varepsilon(\mathcal{L}^\nabla_{X_\mathrm{t}}+s)}). \tag{7-7}$$

Proof. Take a relatively compact open set $\mathcal{Z}\subset\mathbb{C}$ such that $\overline{\mathcal{Z}}\cap\operatorname{Res}(\mathcal{L}^\nabla_X)=\varnothing$. We denote by

$$\mathcal{Q}_\mathrm{t}(s)\in\mathcal{D}'^n(M\times M, E^\vee\boxtimes E)$$

the Schwartz kernel of the operator $(\mathcal{L}^\nabla_{X_\mathrm{t}}+s)^{-1}e^{-\varepsilon(\mathcal{L}^\nabla_{X_\mathrm{t}}+s)}$. Then it follows from [Dang et al. 2020, Proposition 6.3] that there is $t_\mathcal{Z}>0$ and a closed conical subset Γ not intersecting $N^*\Delta$ such that the map $(\mathrm{t},s)\mapsto\mathcal{Q}_\mathrm{t}(s)$ is bounded as a map

$$[-t_\mathcal{Z},t_\mathcal{Z}]\times\overline{\mathcal{Z}}\to\mathcal{D}'^n_\Gamma(M\times M,E^\vee\boxtimes E). \tag{7-8}$$

In fact it is actually $\mathcal{C}^2$ as a map $[-t_\mathcal{Z},t_\mathcal{Z}]\times\overline{\mathcal{Z}}\to\mathcal{D}'^n(M\times M,E^\vee\boxtimes E)$ and from this it is not hard to see that the map (7-8) is actually $\mathcal{C}^1$. Next, by (5-9) we have

$$\zeta_{X_\mathrm{t},\nabla}(s)=\exp\left(\operatorname{tr}^\flat_\mathrm{gr}\int_\infty^s(\mathcal{L}^\nabla_{X_\mathrm{t}}+\tau)^{-1}e^{-(\mathcal{L}^\nabla_{X_\mathrm{t}}+\tau)}\,\mathrm{d}\tau\right)^{(-1)^{q+1}}$$

for $s\in\mathcal{Z}$, where ∞ means $\operatorname{Re}\tau\to+\infty$. The first part of the proposition follows.

Next we prove (7-7) for $\mathrm{t}=0$, the proof being the same for arbitrary t. Note that we have

$$\partial_\mathrm{t}(\mathcal{L}^\nabla_{X_\mathrm{t}}+\tau)^{-1}=-(\mathcal{L}^\nabla_{X_\mathrm{t}}+\tau)^{-1}\mathcal{L}_{\dot X_\mathrm{t}}(\mathcal{L}^\nabla_{X_\mathrm{t}}+\tau)^{-1},$$

which leads to

$$\begin{aligned}\partial_\mathrm{t}\log\zeta_{X_\mathrm{t},\nabla}(s) = (-1)^q\int_\infty^s \operatorname{tr}^\flat_\mathrm{gr}(\mathcal{L}^\nabla_{X_\mathrm{t}}+\tau)^{-1}\mathcal{L}_{\dot X_\mathrm{t}}(\mathcal{L}^\nabla_{X_\mathrm{t}}+\tau)^{-1}e^{-\varepsilon(\mathcal{L}^\nabla_{X_\mathrm{t}}+\tau)}\,\mathrm{d}\tau \\ +(-1)^{q+1}\int_\infty^s\operatorname{tr}^\flat_\mathrm{gr}(\mathcal{L}^\nabla_{X_\mathrm{t}}+\tau)^{-1}\partial_\mathrm{t}\,e^{-\varepsilon(\mathcal{L}^\nabla_{X_\mathrm{t}}+\tau)}\,\mathrm{d}\tau.\end{aligned} \tag{7-9}$$

By cyclicity of the trace, and using $(\mathcal{L}_{X_t}^\nabla+\tau)^{-2}=-\partial_\tau(\mathcal{L}_{X_t}^\nabla+\tau)^{-1}$, one gets

$$\operatorname{tr}_{\mathrm{gr}}^\flat(\mathcal{L}_{X_t}^\nabla+\tau)^{-1}\mathcal{L}_{\dot X_t}(\mathcal{L}_{X_t}^\nabla+\tau)^{-1}e^{-\varepsilon(\mathcal{L}_{X_t}^\nabla+\tau)} \\ =-\partial_\tau \operatorname{tr}_{\mathrm{gr}}^\flat \mathcal{L}_{\dot X_t}^\nabla(\mathcal{L}_{X_t}^\nabla+\tau)^{-1}e^{-\varepsilon(\mathcal{L}_{X_t}^\nabla+\tau)}+\operatorname{tr}_{\mathrm{gr}}^\flat \mathcal{L}_{\dot X_t}^\nabla(\mathcal{L}_{X_t}^\nabla+\tau)^{-1}\partial_\tau e^{-\varepsilon(\mathcal{L}_{X_t}^\nabla+\tau)}.$$

Next, one has $\partial_\tau e^{-\varepsilon(\mathcal{L}_{X_t}^\nabla+\tau)}=-\varepsilon\, e^{-\varepsilon(\mathcal{L}_{X_t}^\nabla+\tau)}$ and moreover

$$\partial_t e^{-\varepsilon(\mathcal{L}_{X_t}^\nabla+\tau)}=-e^{-\varepsilon(\mathcal{L}_{X_t}^\nabla+\tau)}\int_0^\varepsilon e^{u(\mathcal{L}_{X_t}^\nabla+\tau)}\mathcal{L}_{\dot X_t}^\nabla e^{-u(\mathcal{L}_{X_t}^\nabla+\tau)}\,\mathrm{d}u$$

by Duhamel's principle, and notice that the integral

$$\int_\infty^s \operatorname{tr}_{\mathrm{gr}}^\flat(\mathcal{L}_{X_t}^\nabla+\tau)^{-1}e^{-\varepsilon(\mathcal{L}_{\dot X_t}+\tau)}\left[\varepsilon\,\mathcal{L}_{\dot X_t}^\nabla-\int_0^\varepsilon e^{u(\mathcal{L}_{X_t}^\nabla+\tau)}\mathcal{L}_{\dot X_t}^\nabla e^{-u(\mathcal{L}_{X_t}^\nabla+\tau)}\,\mathrm{d}u\right]\mathrm{d}\tau$$

vanishes by cyclicity of the trace. Thus by (7-9) one gets

$$\partial_t \log\zeta_{X_t,\nabla}(s)=(-1)^{q+1}\operatorname{tr}_{\mathrm{gr}}^\flat \mathcal{L}_{\dot X_t}^\nabla(\mathcal{L}_{X_t}^\nabla+s)^{-1}e^{-\varepsilon(\mathcal{L}_{\dot X_t}+s)}. \tag{7-10}$$

Setting $A_t=P_t^{-1}\dot P_t$, one can verify that

$$\iota_{X_t}=P_t^{-1}\iota_X P_t,$$

which yields

$$\mathcal{L}_{\dot X_t}^\nabla=-\nabla A_t\iota_{X_t}+\nabla\iota_{X_t}A_t-A_t\iota_{X_t}\nabla+\iota_{X_t}A_t\nabla. \tag{7-11}$$

Notice that if N is the number operator, we have

$$(-1)^N N\nabla=\nabla(-1)^{N+1}(N+1)\quad\text{and}\quad(-1)^N N\iota_{X_t}=\iota_{X_t}(-1)^{N-1}(N-1). \tag{7-12}$$

Combining (7-11), (7-12) and the fact that ι_{X_t} and ∇ commute with $\mathcal{L}_{X_t}^\nabla$, one can show that

$$(-1)^N N\mathcal{L}_{\dot X_t}^\nabla(\mathcal{L}_{X_t}^\nabla+s)^{-1}e^{-\varepsilon(\mathcal{L}_{\dot X_t}+s)}=(-1)^N A_t\mathcal{L}_{X_t}^\nabla(\mathcal{L}_{X_t}^\nabla+s)^{-1}e^{-\varepsilon(\mathcal{L}_{\dot X_t}+s)}+B, \tag{7-13}$$

where B is a commutator. Note that $A_{t=0}=\dot P_{t=0}$ since $P_{t=0}=\mathrm{Id}$; therefore

$$\dot P_{t=0}=\vartheta\wedge\iota_{\dot X}.$$

Moreover we have $\mathcal{L}_{X_t}^\nabla(\mathcal{L}_{X_t}^\nabla+s)^{-1}=\mathrm{Id}-s\,(\mathcal{L}_{X_t}^\nabla+s)^{-1}$ and injecting those two last identities in (7-13) one obtains, by (7-10),

$$\partial_t|_{t=0}\log\zeta_{X_t,\nabla}(s)=(-1)^q s\operatorname{tr}_s^\flat(\vartheta\wedge\iota_{\dot X}(\mathcal{L}_X^\nabla+s)^{-1}e^{-\varepsilon(\mathcal{L}_{\dot X}+s)}),$$

where we used that the flat trace of $A_t e^{-\varepsilon(\mathcal{L}_{X_t}^\nabla+s)}=0$ vanishes. □

Now we compute the variation of the $[0,\lambda]$-part of $\zeta^{(\lambda,\infty)}(s)$.

Lemma 7.5. *We have*

$$\frac{\mathrm{d}}{\mathrm{dt}}\log\det_{\mathrm{gr},C_t^\bullet}(\mathcal{L}_{X_t}^\nabla+s)=\operatorname{tr}_{s,C_t^\bullet}(\Pi_t\vartheta_t\iota_{\dot X_t})-s\operatorname{tr}_{s,C_t^\bullet}(\Pi_t\vartheta_t\iota_{\dot X_t}(\mathcal{L}_{X_t}^\nabla+s)^{-1}).$$

Proof. Again it suffices to prove the lemma for $\mathrm{t}=0$. We are in a position to apply Lemma A.2, which gives

$$\frac{\mathrm{d}}{\mathrm{dt}} \log \det_{\mathrm{gr}, C_{\mathrm{t}}^{\bullet}} (\mathcal{L}_{X_{\mathrm{t}}}^{\nabla} + s)^{(-1)^{q+1}} = (-1)^{q+1} \mathrm{tr}_{\mathrm{gr}, C_{\mathrm{t}}^{\bullet}} (\Pi_{\mathrm{t}} \mathcal{L}_{\dot{X}_{\mathrm{t}}}^{\nabla} (\mathcal{L}_{X_{\mathrm{t}}}^{\nabla} + s)^{-1}).$$

Now we may conclude as in the proof of Proposition 7.4, using that

$$(-1)^N N \Pi_{\mathrm{t}} \mathcal{L}_{\dot{X}_{\mathrm{t}}}^{\nabla} (\mathcal{L}_{X_{\mathrm{t}}}^{\nabla} + s)^{-1} = (-1)^N \Pi_{\mathrm{t}} A_{\mathrm{t}} \mathcal{L}_{X_{\mathrm{t}}}^{\nabla} (\mathcal{L}_{X_{\mathrm{t}}}^{\nabla} + s)^{-1} + C,$$

where C is a commutator. □

7.4. ***Proof of Theorem 9.*** Combining Proposition 7.4 and Lemma 7.5, we obtain, for $s \notin \mathrm{Res}(\mathcal{L}_{X_{\mathrm{t}}}^{\nabla})$,

$$\begin{aligned}\partial_{\mathrm{t}} \log \zeta_{X_{\mathrm{t}}, \nabla}^{(\lambda, \infty)}(s) = (-1)^q \, &\mathrm{tr}_{\mathrm{s}, C_{\mathrm{t}}^{\bullet}} (\Pi_{\mathrm{t}} \vartheta_{\mathrm{t}} \iota_{\dot{X}_{\mathrm{t}}}) \\ &+ (-1)^q s \, \mathrm{tr}_{\mathrm{s}}^{\flat} \big(\vartheta_{\mathrm{t}} \wedge \iota_{\dot{X}_{\mathrm{t}}} (\mathcal{L}_{X_{\mathrm{t}}}^{\nabla} + s)^{-1} e^{-\varepsilon(\mathcal{L}_{\dot{X}} + s)} (1 - \Pi_{\mathrm{t}}) \big) \\ &\quad + (-1)^q s \, \mathrm{tr}_{\mathrm{s}, C_{\mathrm{t}}^{\bullet}} \big(\Pi_{\mathrm{t}} \vartheta_{\mathrm{t}} \iota_{\dot{X}_{\mathrm{t}}} (\mathcal{L}_{X_{\mathrm{t}}}^{\nabla} + s)^{-1} (e^{-\varepsilon(\mathcal{L}_{\dot{X}} + s)} - \mathrm{Id}) \big).\end{aligned}$$

Now it is a simple observation that the last two terms in the right-hand side of the above equality vanish at $s = 0$; hence we get

$$\partial_{\mathrm{t}} \log \zeta_{X_{\mathrm{t}}, \nabla}^{(\lambda, \infty)}(0) = (-1)^q \, \mathrm{tr}_{\mathrm{s}, C_{\mathrm{t}}^{\bullet}} (\Pi_{\mathrm{t}} \vartheta_{\mathrm{t}} \iota_{\dot{X}_{\mathrm{t}}}).$$

Comparing this with Lemma 7.3, we obtain Theorem 9 by the definition of the dynamical torsion; see Section 6.4.

8. Variation of the connection

In this section we compute the variation of the dynamical torsion when the connection is perturbed. This formula will be crucial to compare the dynamical torsion and Turaev's refined combinatorial torsion.

8.1. ***Real-differentiable families of flat connections.*** Let $U \subset \mathbb{C}$ be some open set and consider $\nabla(z)$, $z \in U$, a family of flat connections on E. We will assume that the map $z \mapsto \nabla(z)$ is $\mathcal{C}^1$,[5] that is, there exists continuous maps $z \mapsto \mu_z, \nu_z \in \Omega^1(M, \mathrm{End}(E))$ such that for any $z_0 \in U$ one has

$$\nabla(z) = \nabla(z_0) + \mathrm{Re}(z - z_0) \mu_{z_0} + \mathrm{Im}(z - z_0) \nu_{z_0} + o(z - z_0), \tag{8-1}$$

where $o(z - z_0)$ is understood in the Fréchet topology of $\Omega^1(M, \mathrm{End}(E))$. We will denote for any $\sigma \in \mathbb{C}$

$$\alpha_{z_0}(\sigma) = \mathrm{Re}(\sigma) \mu_{z_0} + \mathrm{Im}(\sigma) \nu_{z_0} \in \Omega^1(M, \mathrm{End}(E)). \tag{8-2}$$

Note that since the connections $\nabla(z)$ are assumed to be flat, we have

$$[\nabla(z), \alpha_z(\sigma)] = \nabla(z) \alpha_z(\sigma) + \alpha_z(\sigma) \nabla(z) = 0. \tag{8-3}$$

[5]Note that, even if in the following we will consider *holomorphic* families of representations $\rho(z)$, $|z| < \delta$, it is not clear that we may find a holomorphic family of connections $\nabla(z)$, $|z| < \delta$, such that $\rho_{\nabla(z)} = \rho(z)$, but only such *real-differentiable* families; see Section 11.3. Therefore we need to consider the class of real-differentiable families of connections.

8.2. ***A cochain contraction induced by the Anosov flow.*** For $z \in U$ let

$$(\mathcal{L}_X^{\nabla(z)} + s)^{-1} = \sum_{j=1}^{J(0)} \frac{(-\mathcal{L}_X^{\nabla(z)})^{j-1}\Pi_0(z)}{s^j} + Y(z) + \mathcal{O}(s) \tag{8-4}$$

be the development (5-3) for the resonance $s_0 = 0$. Let $C^\bullet(0; z) = \operatorname{ran} \Pi_0(z)$. Recall from Section 5.5 that since $\nabla(z)$ is acyclic, the complex $(C^\bullet(0; z), \nabla(z))$ is acyclic. Therefore there exists a cochain contraction $k(z) : C^\bullet(0; z) \to C^\bullet(0; z)$, i.e., a map of degree -1 such that

$$\nabla(z)k(z) + k(z)\nabla(z) = \mathrm{Id}_{C^\bullet(0;z)} . \tag{8-5}$$

We now define

$$K(z) = \iota_X Y(z)(\mathrm{Id} - \Pi_0(z)) + k(z)\Pi_0(z) : \Omega^\bullet(M, E) \to \mathcal{D}'^\bullet(M, E). \tag{8-6}$$

A crucial property of the operator K is that it satisfies the chain homotopy equation

$$\nabla(z)K(z) + K(z)\nabla(z) = \mathrm{Id}_{\Omega^\bullet(M,E)}, \tag{8-7}$$

as follows from the development (8-4).

8.3. ***The variation formula.*** For simplicity, we will set for every $z \in U$

$$\tau(z) = \tau_\vartheta(\nabla(z)).$$

The operators $K(z)$ defined above are involved in the variation formula of the dynamical torsion, as follows.

Proposition 8.1. *The map $z \mapsto \tau(z)$ is real differentiable; we have for every $z \in U$ and $\varepsilon > 0$ small enough*

$$\mathrm{d}(\log \tau)_z \sigma = -\mathrm{tr}_{\mathrm{s}}^\flat(\alpha_z(\sigma)K(z)e^{-\varepsilon\mathcal{L}_X^{\nabla(z)}}), \quad \sigma \in \mathbb{C}. \tag{8-8}$$

The proof of the previous proposition is similar of that of the last subsection, i.e., we compute the variation of each part of the dynamical torsion. The rest of this section is devoted to the proof of Proposition 8.1.

8.4. ***Anisotropic Sobolev spaces for a family of connections.*** Fix some $z_0 \in U$. Recall from Section 7.1 that we chose some anisotropic Sobolev spaces $\mathcal{H}_1^\bullet \subset \mathcal{H}^\bullet$. Notice that

$$\mathcal{L}_X^{\nabla(z)} = \mathcal{L}_X^{\nabla(z_0)} + \beta(z)(X), \tag{8-9}$$

where $\beta(z) \in \Omega^1(M, \mathrm{End}(E))$ is defined by

$$\nabla(z) = \nabla(z_0) + \beta(z).$$

Therefore (8-1) implies that $z \mapsto \mathcal{L}_X^{\nabla(z)} - \mathcal{L}_X^{\nabla(z_0)}$ is a $\mathcal{C}^1$ family of multiplication operators and thus forms a $\mathcal{C}^1$ family of bounded operators $\mathcal{H}^\bullet \to \mathcal{H}^\bullet$ and $\mathcal{H}_1^\bullet \to \mathcal{H}_1^\bullet$ by construction of the anisotropic spaces and standard rules of pseudodifferential calculus (see for example [Faure and Sjöstrand 2011]). As a consequence and thanks to Proposition 7.2, we are in position to apply [Kato 1976, Theorem 3.11]; thus if δ is small enough we have that

$$R_\rho = \{(z, s) \in \mathbb{C}^2 : |z - z_0| < \delta,\ s \in \Omega(c, \rho),\ s \notin \sigma_{\mathcal{H}^\bullet}(\mathcal{L}_X^{\nabla(z)})\} \text{ is open,} \tag{8-10}$$

where $\sigma_{\mathcal{H}^\bullet}(\mathcal{L}_X^{\nabla(z)})$ denotes the resolvent set of $\mathcal{L}_X^{\nabla(z)}$ on $\mathcal{H}^\bullet$, and $\Omega(c, \rho)$ is defined in (7-3). Moreover (8-1) and (8-9) imply that for any open set $\mathcal{Z} \subset \Omega(c, \rho)$ such that $\operatorname{Res}(\mathcal{L}_X^{\nabla(z_0)}) \cap \overline{\mathcal{Z}} = \varnothing$, there exists $\delta_{\mathcal{Z}} > 0$ such that for any $j \in \{0, 1\}$,

$$(\mathcal{L}_X^{\nabla(z)} + s)^{-1} \in \mathcal{C}^1\big(\{|z - z_0| < \delta_{\mathcal{Z}}\}, \operatorname{Hol}(\mathcal{Z}_s, \mathcal{L}(\mathcal{H}_j^\bullet, \mathcal{H}_j^\bullet))\big). \tag{8-11}$$

For all z, the map $s \mapsto (\mathcal{L}_X^{\nabla(z)} + s)^{-1}$ is meromorphic in the region $\Omega(c, \rho)$ with poles (of finite multiplicity) which coincide with the resonances of $\mathcal{L}_X^{\nabla(z)}$ in this region.

Moreover, the arguments from the proof of [Dyatlov and Zworski 2016, Proposition 3.4] can be made uniformly for the family $z \mapsto (\mathcal{L}_X^{\nabla(z)} + s)^{-1}$ to obtain that for some closed conical set $\Gamma \subset T^*(M \times M)$ not intersecting the conormal to the diagonal and any $\varepsilon > 0$ small enough, the map

$$\mathcal{Z} \times \{|z - z_0| < \delta_{\mathcal{Z}}\} \to \mathcal{D}'_\Gamma(M \times M, \pi_1^* E^\vee \otimes \pi_2^* E), \quad (s, z) \mapsto \mathcal{K}(s, z),$$

is bounded, where $\mathcal{K}(s, z)$ is the Schwartz kernel of the shifted resolvent $(\mathcal{L}_X^{\nabla(z)} + s)^{-1} e^{-\varepsilon \mathcal{L}_X^{\nabla(z)}}$.

8.5. ***A family of spectral projectors.*** Fix $\lambda \in (0, 1)$ such that

$$\{s \in \mathbb{C} : |s| \leqslant \lambda\} \cap \operatorname{Res}(\mathcal{L}_X^{\nabla(z_0)}) \subset \{0\}. \tag{8-12}$$

Thanks to (8-10), if z is close enough to z_0,

$$\{s \in \mathbb{C} : |s| = \lambda\} \cap \operatorname{Res}(\mathcal{L}_X^{\nabla(z)}) = \varnothing, \tag{8-13}$$

by compactness of the circle. For $z \in U$ we will denote by

$$\Pi(z) = \frac{1}{2i\pi} \int_{|s|=\lambda} (\mathcal{L}_X^{\nabla(z)} + s)^{-1}\, \mathrm{d}s \tag{8-14}$$

the spectral projector of $\mathcal{L}_X^{\nabla(z)}$ on generalized eigenvectors for resonances in $\{s \in \mathbb{C} : |s| \leqslant \lambda\}$, and $C^\bullet(z) = \operatorname{ran} \Pi(z)$. It follows from (8-11), (8-13) and (8-14) that the map

$$\Pi : z \mapsto \Pi(z) \in \mathcal{L}(\mathcal{H}_j^\bullet, \mathcal{H}_j^\bullet)$$

is $\mathcal{C}^1$ for $j = 0, 1$. We can therefore apply A.3 to get, for δ small enough,

$$\Pi \in \mathcal{C}^1(\{|z - z_0| < \delta\}_z : \mathcal{L}(\mathcal{H}^\bullet, \mathcal{H}_1^\bullet)). \tag{8-15}$$

8.6. ***Variation of the finite-dimensional part.*** Because $(C^\bullet(z_0), \nabla(z_0))$ is acyclic, there exists a cochain contraction $k(z_0) : C^\bullet(z_0) \to C^{\bullet-1}(z_0)$; see Section 3.6. The next lemma computes the variation of the finite-dimensional part of the dynamical torsion.

Lemma 8.2. *The map $z \mapsto c(z) = \tau(C^\bullet(z), \Gamma)$ is real differentiable at $z = z_0$ and*

$$\mathrm{d}(\log c)_{z_0}\sigma = -\mathrm{tr}_{\mathrm{s},C^\bullet}\Pi(z_0)\alpha_{z_0}(\sigma)k(z_0), \quad \sigma \in \mathbb{C}.$$

Note that here, $a_{z_0}(\sigma)$ is identified with the map $\omega \mapsto a_{z_0}(\sigma) \wedge \omega$.

Proof. By continuity of the family $z \mapsto \Pi(z)$, we have that $\Pi(z)|_{C^\bullet(z_0)} : C^\bullet(z_0) \to C^\bullet(z)$ is an isomorphism, for $|z - z_0|$ small enough, of inverse denoted by $Q(z)$. For those z we denote by $\widehat{C}^\bullet(z)$ the graded vector space $C^\bullet(z_0)$ endowed with the differential

$$\hat{\nabla}(z) = Q(z)\nabla(z)\Pi(z) : C^\bullet(z_0) \to C^\bullet(z_0).$$

Then because Γ commutes with every $\Pi(z)$ one has

$$\tau(\widehat{C}^\bullet(z), \Gamma) = \tau(C^\bullet(z), \Gamma). \tag{8-16}$$

By (8-15) we can apply (A-2) in the proof of Lemma A.2 which gives, as $\sigma \to 0$,

$$\hat{\nabla}(z_0 + \sigma)\Pi(z_0) = \Pi(z_0)\nabla(z_0)\Pi(z_0) + \Pi(z_0)\alpha_{z_0}(\sigma)\Pi(z_0) + o_{C^\bullet(z_0)\to C^\bullet(z_0)}(\sigma).$$

Therefore Lemma 3.5 implies the desired result. □

8.7. ***Variation of the zeta part.*** We give a first proposition which computes the variation of the Ruelle zeta function in its convergence region.

Proposition 8.3 (variation of the dynamical zeta function with respect to the connection). *For any relatively compact open set $\mathcal{Z} \subset \mathbb{C}$ such that $\overline{\mathcal{Z}} \cap \mathrm{Res}(\mathcal{L}_X^{\nabla(z_0)}) = \varnothing$, there is $\delta_{\mathcal{Z}} > 0$ so that $(z, s) \mapsto \zeta_{X,\nabla(z)}(s)$ is $\mathcal{C}^1$ as a map*

$$\{|z - z_0| < \delta\} \times \mathcal{Z} \to \mathbb{C}$$

and for every $\varepsilon > 0$ small enough it holds

$$\mathrm{d}_z(\zeta_{X,\nabla(z)}(s))|_{z=z_0}\sigma = (-1)^{q+1}e^{-\varepsilon s}\mathrm{tr}_{\mathrm{s}}^\flat\big(\alpha_{z_0}(\sigma)\iota_X(\mathcal{L}_X^{\nabla(z_0)} + s)^{-1}e^{-\varepsilon\mathcal{L}_X^{\nabla(z_0)}}\big).$$

Proof. The proof is very similar to that of Proposition 7.4, using the identities

$$\frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0}(\mathcal{L}_X^{\nabla(z+t\sigma)} + \tau)^{-1} = -(\mathcal{L}_X^{\nabla(z)} + \tau)^{-1}a_{z_0}(\sigma)(X)(\mathcal{L}_X^{\nabla(z)} + \tau)^{-1}$$

and $\alpha_{z_0}(\sigma)(X) = [\alpha_{z_0}(\sigma), \iota_X] = \alpha_{z_0}(\sigma) \circ \iota_X + \iota_X \circ \alpha_{z_0}(\sigma)$, and we shall omit the details. □

The following lemma is a direct consequence of Lemma A.2 and the fact that $\Pi_0(z_0) = \Pi(z_0)$ by (8-12).

Lemma 8.4. *For $s \notin \mathrm{Res}(\mathcal{L}_X^\nabla(z_0))$, the map $z \mapsto h_s(z) = \det_{\mathrm{gr},C^\bullet(z)}(\mathcal{L}_X^{\nabla(z)} + s)^{(-1)^{q+1}}$ is $\mathcal{C}^1$ near $z = z_0$ and*

$$\mathrm{d}(\log h_s)_{z_0}\sigma = (-1)^{q+1}\mathrm{tr}_{\mathrm{s},C^\bullet(z_0)}\big(\Pi_0(z_0)\alpha_{z_0}(\sigma)\iota_X(\mathcal{L}_X^{\nabla(z_0)} + s)^{-1}\big).$$

8.8. ***Proof of Proposition 8.1.*** Combining the two lemmas of the preceding subsection we obtain that for $s \notin \mathrm{Res}(\mathcal{L}_X^{\nabla(z_0)})$, the map $z \mapsto \zeta_{X,\nabla(z)}^{(\lambda,\infty)}(s) = g_s(z)/h_s(z)$ is real differentiable at $z = z_0$ (and therefore on U since we may vary z_0). Moreover for every $\varepsilon > 0$ small enough

$$\mathrm{d}\left(\log \frac{g_s}{h_s}\right)_z \sigma = (-1)^{q+1}\left(e^{-\varepsilon s}\mathrm{tr}_\mathrm{s}^\flat \alpha_z(\sigma)\iota_X(\mathcal{L}_X^{\nabla(z)}+s)^{-1}e^{-\varepsilon\mathcal{L}_X^{\nabla(z)}} - \mathrm{tr}_{\mathrm{s},C^\bullet(z)}\Pi_0(z)\alpha_z(\sigma)\iota_X(\mathcal{L}_X^{\nabla(z)}+s)^{-1}\right). \quad (8\text{-}17)$$

Letting $s \to 0$, this yields

$$(-1)^{q+1}\,\mathrm{d}(\log b)_z\sigma = \mathrm{tr}_\mathrm{s}^\flat\left(\alpha_z(\sigma)\iota_X Y(z)(\mathrm{Id} - \Pi_0(z))e^{-\varepsilon\mathcal{L}_X^{\nabla(z)}}\right) + \mathrm{tr}_{\mathrm{s},C^\bullet(z)}(\Pi_0(z)\alpha_z(\sigma)\iota_X Q_z(\varepsilon)),$$

where we set $b(z) = \zeta_{X,\nabla(z)}^{(\lambda,\infty)}(0)$ and

$$Q_z(\varepsilon) = \sum_{n\geqslant 1}\frac{(-\varepsilon)^n}{n!}(\mathcal{L}_X^{\nabla(z)})^{n-1} : C^\bullet(z) \to C^\bullet(z).$$

Recall that if $c(z) = \tau(C^\bullet(z), \Gamma)$ one has $\tau(z) = c(z)b(z)^{(-1)^q}$. Therefore Lemma 8.2 gives, with what precedes, and with $K(z)$ given by (8-6),

$$\mathrm{d}(\log\tau)_z\sigma = -\mathrm{tr}_\mathrm{s}^\flat(\alpha_z(\sigma)K(z)e^{-\varepsilon\mathcal{L}_X^{\nabla(z)}}) - \mathrm{tr}_{\mathrm{s},C^\bullet(z)}\left(\Pi_0(z)\alpha_z(\sigma)\left(k(z)(\mathrm{Id} - e^{-\varepsilon\mathcal{L}_X^{\nabla(z)}}) + \iota_X Q_z(\varepsilon)\right)\right). \quad (8\text{-}18)$$

Moreover, by using (8-3) and (8-7), we see that

$$\begin{aligned}\alpha_z(\sigma)K(z)\mathcal{L}_X^{\nabla(z)}e^{-\varepsilon\mathcal{L}_X^{\nabla(z)}} &= \alpha_z(\sigma)K(z)[\nabla(z), \iota_X]e^{-\varepsilon\mathcal{L}_X^{\nabla(z)}}\\ &= \alpha_z(\sigma)\iota_X e^{-\varepsilon\mathcal{L}_X^{\nabla(z)}} + [\alpha_z(\sigma)K(z)\iota_X e^{-\varepsilon\mathcal{L}_X^{\nabla}}, \nabla(z)],\end{aligned}$$

and hence, by cyclicity of the trace, $(\mathrm{d}/\mathrm{d}\varepsilon)\mathrm{tr}_\mathrm{s}^\flat(\alpha_z(\sigma)K(z)e^{-\varepsilon\mathcal{L}_X^{\nabla(z)}}) = 0$. In particular, the last term in the right-hand side of (8-18) does not depend on ε; since it goes to zero as $\varepsilon \to 0$, it vanishes, and Proposition 8.1 follows.

9. Euler structures, Chern–Simons classes

The Turaev torsion is defined using *Euler structures*, introduced by Turaev [1989], whose purpose is to fix sign ambiguities of combinatorial torsions. We shall use however the representation in terms of vector fields used by Burghelea and Haller [2006]. The goal of the present section is to introduce these Euler structures, in view of the definition of the Turaev torsion.

9.1. ***The Chern–Simons class of a pair of vector fields.*** If $X \in \mathcal{C}^\infty(M, TM)$ is a vector field with isolated nondegenerate zeros, we define the singular 0-chain

$$\mathrm{div}(X) = -\sum_{x\in\mathrm{Crit}(X)}\mathrm{ind}_X(x)[x] \in C_0(M, \mathbb{Z}),$$

where $\mathrm{Crit}(X)$ is the set of critical points of X and $\mathrm{ind}_X(x)$ denotes the Poincaré–Hopf index of x as a critical point of X.[6] Note also that $\mathrm{div}(-X) = -\mathrm{div}(X)$ since M is odd-dimensional.

[6] $\mathrm{ind}_X(x) = (-1)^{\dim E_s(x)}$ if x is hyperbolic and $E_s(x) \subset T_xM$ is the stable subspace of x.

Let X_0, X_1 be two vector fields with isolated nondegenerate zeros. Let $p : M \times [0, 1] \to M$ be the projection over the first factor and choose a smooth section H of the bundle $p^*TM \to M \times [0, 1]$, transversal to the zero section, such that H restricts to X_i on $\{i\} \times M$ for $i = 0, 1$. Then the set $H^{-1}(0) \subset M \times [0, 1]$ is an oriented smooth submanifold of dimension 1 with boundary (it is oriented because M and $[0, 1]$ are), and we denote by $[H^{-1}(0)]$ its fundamental class.

Definition 9.1. The class

$$p_*[H^{-1}(0)] \in C_1(M, \mathbb{Z})/\partial C_2(M, \mathbb{Z}),$$

where p_* is the pushforward by p, does not depend on the choice of the homotopy H relating X_0 and X_1; see [Burghelea and Haller 2006, §2.2]. This is the *Chern–Simons class* of the pair (X_0, X_1), denoted by $\mathrm{cs}(X_0, X_1)$.

We have the fundamental formulae

$$\begin{aligned} \partial\,\mathrm{cs}(X_0, X_1) &= \mathrm{div}(X_1) - \mathrm{div}(X_0), \\ \mathrm{cs}(X_0, X_1) + \mathrm{cs}(X_1, X_2) &= \mathrm{cs}(X_0, X_2) \end{aligned} \tag{9-1}$$

for any other vector field with nondegenerate zeros X_2. Notice also that if X_0 and X_1 are nonsingular vector fields, then $\mathrm{cs}(X_0, X_1)$ defines a homology class in $H_1(M, \mathbb{Z})$.

9.2. ***Euler structures.*** Let X be a smooth vector field on M with nondegenerate zeros. An *Euler chain* for X is a singular one-chain $e \in C_1(M, \mathbb{Z})$ such that $\partial e = \mathrm{div}(X)$. Euler chains for X always exist because M is odd-dimensional and thus $\chi(M) = 0$.

Two pairs (X_0, e_0) and (X_1, e_1), with X_i a vector field with nondegenerate zeros and e_i an Euler chain for X_i, $i = 0, 1$, will be said to be equivalent if

$$[e_1] = [e_0] + \mathrm{cs}(X_0, X_1) \in C_1(M, \mathbb{C})/\partial C_2(M, \mathbb{Z}), \tag{9-2}$$

where $[e_i]$ is the class of e_i in $C_1(M, \mathbb{C})/\partial C_2(M, \mathbb{Z})$ for $i = 1, 2$.

Definition 9.2. An *Euler structure* is an equivalence class $[X, e]$ for the relation (9-2). We will denote by $\mathrm{Eul}(M)$ the set of Euler structures.

There is a free and transitive action of $H_1(M, \mathbb{Z})$ on $\mathrm{Eul}(M)$ given by

$$[X, e] + h = [X, e + h], \quad h \in H_1(M, \mathbb{Z}).$$

9.3. ***Homotopy formula relating flows.*** Let X_0, X_1 be two vector fields with nondegenerate zeros. Let H be a smooth homotopy between X_0 and X_1 as in Section 9.1 and set $X_\mathrm{t} = H(\mathrm{t}, \cdot) \in C^\infty(M, TM)$. For $\varepsilon > 0$ we define $\Phi_\varepsilon : M \times [0, 1] \to M \times M \times [0, 1]$ via

$$\Phi_\varepsilon(x, \mathrm{t}) = (e^{-\varepsilon X_\mathrm{t}}(x), x, \mathrm{t}), \quad x \in M,\ \mathrm{t} \in [0, 1].$$

We also set

$$H_\varepsilon = \{\Phi_\varepsilon(x, \mathrm{t}) : (x, \mathrm{t}) \in M \times [0, 1]\} \subset M \times M \times \mathbb{R}.$$

Then H_ε is a submanifold with boundary of $M \times M \times \mathbb{R}$ which is oriented (since M and $\mathbb{R}$ are). Define

$$[H_\varepsilon] = (\Phi_\varepsilon)_*([M] \times [\![0, 1]\!]) \in \mathcal{D}'^n(M \times M \times \mathbb{R})$$

to be the associated integration current; see Section 4.3. Let g be any metric on M and let $\rho > 0$ be smaller than its injectivity radius. Then for any $x, y \in M$ with $\operatorname{dist}(x, y) \leqslant \rho$, we denote by $P(x, y) \in \operatorname{Hom}(E_x, E_y)$ the parallel transport by ∇ along the minimizing geodesic joining x to y. Then P is a smooth section of $\pi_1^* E^\vee \otimes \pi_2^* E$ defined in some neighborhood of the diagonal in $M \times M$. Take ε small enough so that

$$\operatorname{dist}(x, e^{-sX_t}(x)) \leqslant \rho, \quad s \in [0, \varepsilon],\ t \in [0, 1],\ x \in M, \tag{9-3}$$

so that $\operatorname{supp} \pi_*[H_\varepsilon] \subset \{(x, y) : \operatorname{dist}(x, y) \leqslant \rho\}$. Here, $\pi : M \times M \times \mathbb{R} \to M \times M$ is the projection over the two first factors and $\pi_* : \mathcal{D}'^n(M \times M \times [0, 1]) \to \mathcal{D}'^{n-1}(M \times M)$ is the push-forward operator which is simply defined by

$$\int_{M \times M} \pi_* u \wedge v = \int_{M \times M \times [0,1]} u \wedge \pi^* v, \quad u \in \mathcal{D}'^n(M \times M \times [0, 1]),\ v \in \Omega^{n+1}(M \times M).$$

Then we define

$$\mathcal{R}_\varepsilon = -\pi_*[H_\varepsilon] \cdot P \in \mathcal{D}'^{n-1}(M \times M, \pi_1^* E^\vee \otimes \pi_2^* E).$$

Finally, we denote by $R_\varepsilon : \Omega^\bullet(M, E) \to \mathcal{D}'^{\bullet-1}(M, E)$ the operator of degree -1 whose Schwartz kernel is $\mathcal{R}_\varepsilon$.

Lemma 9.3. *We have the homotopy formula*

$$[\nabla, R_\varepsilon] = \nabla R_\varepsilon + R_\varepsilon \nabla = e^{-\varepsilon \mathcal{L}_{X_1}^\nabla} - e^{-\varepsilon \mathcal{L}_{X_0}^\nabla}. \tag{9-4}$$

Proof. First note that because M is odd-dimensional, the boundary (computed with orientations) of the manifold H_ε is calculated using the Leibniz rule [Krantz and Parks 2008, (7.15), p. 190] as

$$\begin{aligned} \partial H_\varepsilon &= \partial\big((\Phi_\varepsilon)_*([M] \times [\![0, 1]\!])\big) = (-1)^{\dim(M)} (\Phi_\varepsilon)_*([M] \times (\partial [\![0, 1]\!])) \\ &= (-1)^{\dim(M)} (\Phi_\varepsilon)_*([M] \times (\{1\} - \{0\})) = \operatorname{Gr}(e^{-\varepsilon X_0}) \times \{0\} - \operatorname{Gr}(e^{-\varepsilon X_1}) \times \{1\}. \end{aligned}$$

Therefore we have, see (4-1),

$$(-1)^n \, \mathrm{d}^{M \times M} \pi_*[H_\varepsilon] = \pi_*[\partial H_\varepsilon] = [\operatorname{Gr}(e^{-\varepsilon X_0})] - [\operatorname{Gr}(e^{-\varepsilon X_1})],$$

where $[\operatorname{Gr}(e^{-\varepsilon X_i})]$ denotes the integration current on the manifold $\operatorname{Gr}(e^{-\varepsilon X_i})$ for $i = 0, 1$. Now note that we have by construction $\nabla^{E^\vee \boxtimes E} P = 0$. Therefore

$$\nabla^{E^\vee \boxtimes E} \mathcal{R}_\varepsilon = (-1)^n \big([\operatorname{Gr}(e^{-\varepsilon X_1})] - [\operatorname{Gr}(e^{-\varepsilon X_0})]\big) \otimes P.$$

Note that by definition of $e^{-\mathcal{L}_{X_i}^\nabla}$ (see Section 5.2), the bound (9-3) and the flatness of ∇ imply that the Schwartz kernel of $e^{-\varepsilon \mathcal{L}_{X_i}^\nabla}$ is $[\operatorname{Gr}(e^{-\varepsilon X_i})] \otimes P$. This concludes because the Schwartz kernel of $[\nabla, R_\varepsilon]$ is $(-1)^n \nabla^{E^\vee \boxtimes E} \mathcal{R}_\varepsilon$; see [Harvey and Lawson 2001, Lemma 2.2]. ☐

The next formula follows from the definition of the flat trace and the Chern–Simons classes. It will be crucial for the topological interpretation of the variation formula obtained in Section 8.

Lemma 9.4. *We have for any $\alpha \in \Omega^\bullet(M, \mathrm{End}(E))$ such that $\operatorname{tr}\alpha$ is closed and $\varepsilon > 0$ small enough*

$$\operatorname{tr}_{\mathrm{s}}^{\flat} \alpha R_\varepsilon = \langle \operatorname{tr}\alpha, \operatorname{cs}(X_0, X_1)\rangle. \tag{9-5}$$

Here α is identified with the operator $u \mapsto \alpha \wedge u$. Note that because H is transverse to the zero section, we have

$$\mathrm{WF}(\mathcal{R}_\varepsilon) \cap N^*\Delta = \varnothing, \tag{9-6}$$

where $N^*\Delta$ denotes the conormal to the diagonal Δ in $M \times M$, so that the above flat trace is well-defined.

Proof. We denote by $i : M \hookrightarrow M \times M$ the diagonal inclusion. Note that the Schwartz kernel of αR_ε is $(-1)^n \pi_2^*\alpha \wedge \mathcal{R}_\varepsilon = -\pi_2^*\alpha \wedge \mathcal{R}_\varepsilon$ since n is odd. From the definition of the super flat trace $\operatorname{tr}_{\mathrm{s}}^{\flat}$, we find that

$$\operatorname{tr}_{\mathrm{s}}^{\flat} \alpha R_\varepsilon = \langle \operatorname{tr} i^*(\pi_2^*\alpha \wedge \pi_*[H_\varepsilon]\cdot P), 1\rangle, \tag{9-7}$$

where $\pi_2 : M \times M \to M$ is the projection over the second factor. Of course we have $i^*P = \mathrm{Id}_E \in \mathcal{C}^\infty(M, \mathrm{End}(E))$. We therefore have

$$\operatorname{tr} i^*(\pi_2^*\alpha \wedge \pi_*[H_\varepsilon]\cdot P) = \operatorname{tr}\alpha \wedge i^*\pi_*[H_\varepsilon] = \operatorname{tr}\alpha \wedge p_* j^*[H_\varepsilon],$$

where $j : M \times [0, 1] \hookrightarrow M \times M \times [0, 1]$, $(x, t) \mapsto (x, x, t)$. Now, it holds $j^*[H_\varepsilon] = [H^{-1}(0)]$ and thus $p_* j^*[H_\varepsilon] = \operatorname{cs}(X_0, X_1)$. This finally leads to

$$\operatorname{tr}_{\mathrm{s}}^{\flat} \alpha R_\varepsilon = \langle \operatorname{tr}\alpha \wedge \operatorname{cs}(X_0, X_1), 1\rangle = \langle \operatorname{tr}\alpha, \operatorname{cs}(X_0, X_1)\rangle. \qquad \square$$

10. Morse theory and variation of Turaev torsion

We introduce here the Turaev torsion which is defined in terms of CW decompositions. In the spirit of the seminal work [Bismut and Zhang 1992] based on geometric constructions of [Laudenbach 1992], we use a CW decomposition which comes from the unstable cells of a Morse–Smale gradient flow induced by a Morse function. This allows us to interpret the variation of the Turaev torsion as a supertrace on the space of generalized resonant states for the Morse–Smale flow. This interpretation will be convenient for the comparison of the Turaev torsion with the dynamical torsion.

10.1. ***Morse theory and CW-decompositions.*** Let f be a Morse function on M and $\widetilde{X} = -\operatorname{grad}_g f$ be its associated gradient vector field with respect to some Riemannian metric g (the tilde notation is used to make the difference with the Anosov flows we studied until now). For any $a \in \mathrm{Crit}(f)$, we denote by

$$W^s(a) = \Big\{ y \in M : \lim_{t\to\infty} e^{t\widetilde{X}} y = a \Big\}, \quad W^u(a) = \Big\{ y \in M : \lim_{t\to\infty} e^{-t\widetilde{X}} y = a \Big\},$$

the stable and unstable manifolds of a. Then it is well known that $W^s(a)$ (resp. $W^u(x)$) is a smooth embedded open disk of dimension $n - \mathrm{ind}_f(a)$ (resp. $\mathrm{ind}_f(a)$), where $\mathrm{ind}_f(a)$ is the index of a as a critical point of f, that is, in a Morse chart $(z_1, \ldots, z_n)$ near a,

$$f(z_1, \ldots, z_n) = f(a) - z_1^2 - \cdots - z_{\mathrm{ind}_f(a)}^2 + z_{\mathrm{ind}_f(a)+1}^2 + \cdots + z_n^2.$$

For simplicity, we will let

$$|a| = \mathrm{ind}_f(a) = \dim W^u(a),$$

and we fix an orientation of every $W^u(a)$.

We assume that $\widetilde{X}$ satisfies the Morse–Smale condition, that is, for any $a, b \in \mathrm{Crit}(f)$, the manifolds $W^s(a)$ and $W^u(b)$ are transverse. Also, we assume that, for every $a \in \mathrm{Crit}(f)$, the metric g is flat near a and reads $\sum_{i=1}^n (\mathrm{d}x^i)^2$ in the Morse charts. This assumption on the metric is crucial to ensure one can compactify the unstable and stable manifolds as smooth manifolds with corners. The existence of such a compactification and of the CW structure is unknown without the flatness assumption. Let us summarize some results from [Qin 2010, Theorems 3.2, 3.8 and 3.9] which apply to f. We would like to mention that such results can be found in a slightly different form in [Laudenbach 1992] and are used in [Bismut and Zhang 1992]. A difference is that Laudenbach only needs to compactify the unstable cells as C^1-manifolds with conical singularities (as opposed to C^∞) to show that the unstable manifolds have finite mass near the boundary — he is also able to obtain the CW-complex structure. On the other hand, Qin obtains a smooth compactification as manifolds with corners which is stronger than the result of Laudenbach[7] and hence his results recover all those of [Laudenbach 1992]. In the work [Dang and Rivière 2020b], no assumption is made on the flatness of the metric g and only the fact that $\widetilde{X}$ is C^1 linearizable near critical points is needed. In this context, the unstable currents are resonant states for the Lie derivative $\mathcal{L}_{\widetilde{X}}$ and belong to some anisotropic Sobolev spaces. This allows to bound the wavefront set of the unstable currents. Yet this method does not allow to show the finiteness of the mass as in the work of Laudenbach. This nevertheless gives a spectral interpretation of the Morse complex, but this approach does not show that the unstable manifolds form a CW-complex, and the latter is crucial in the topological approach of the torsion. Making such strong assumptions on the pair (f, g) in the present paper allows us to benefit from the best of both worlds — we can use the results from [Dang and Rivière 2020b] together with those from [Qin 2010].

First, $W^u(a)$ admits a compactification to a smooth $|a|$-dimensional manifold with corner $\overline{W}^u(a)$, endowed with a smooth map $e_a : \overline{W}^u(a) \to M$ that extends the inclusion $W^u(a) \subset M$. Then the collection $W = \{\overline{W}^u(a)\}_{a \in \mathrm{Crit}(f)}$ and the applications e_a induce a CW-decomposition on M. Moreover, the boundary operator of the cellular chain complex is given by

$$\partial \overline{W}^u(a) = \sum_{|b|=|a|-1} \#\mathcal{L}(a, b) \overline{W}^u(b),$$

where $\mathcal{L}(a, b)$ is the set of gradient lines joining a to b and $\#\mathcal{L}(a, b)$ is the sum of the orientations induced by the orientations of the unstable manifolds of (a, b); see [Qin 2010, Theorem 3.9].

10.2. ***The Thom–Smale complex.*** We set $C_\bullet(W, E^\vee) = \bigoplus_{k=0}^n C_k(W, E^\vee)$, where

$$C_k(W, E^\vee) = \bigoplus_{\substack{a \in \mathrm{Crit}(f) \\ |a|=k}} E_a^\vee, \quad k = 0, \dots, n.$$

[7] As discussed in detail in https://mathoverflow.net/questions/346822/unstable-manifolds-of-a-morse-function-give-a-cw-complex.

We endow the complex $C_\bullet(W, E^\vee)$ with the boundary operator $\partial^{\nabla^\vee}$ defined by

$$\partial^{\nabla^\vee} u = \sum_{|b|=|a|-1} \sum_{\gamma \in \mathcal{L}(a,b)} \varepsilon_\gamma P_\gamma(u), \quad a \in \mathrm{Crit}(f),\ u \in E_a^\vee,$$

where for $\gamma \in \mathcal{L}(a, b)$, $P_\gamma \in \mathrm{End}(E_a^\vee, E_b^\vee)$ is the parallel transport of $\nabla^\vee$ along the curve γ and $\varepsilon_\gamma = \pm 1$ is the orientation number of $\gamma \in \mathcal{L}(a, b)$.

Then by [Laudenbach 1992] (see also [Dang and Rivière 2020b] for a different approach), there is a canonical isomorphism

$$H_\bullet(M, \nabla^\vee) \simeq H_\bullet(W, \nabla^\vee),$$

where $H_\bullet(M, \nabla^\vee)$ is the singular homology of flat sections of $(E^\vee, \nabla^\vee)$ and $H_\bullet(W, \nabla^\vee)$ denotes the homology of the complex $C_\bullet(W, E^\vee)$ endowed with the boundary map $\partial^{\nabla^\vee}$. Therefore this complex is acyclic since ∇ (and thus $\nabla^\vee$) is.

10.3. ***The Turaev torsion.*** Fix some base point $x_\star \in M$ and, for every $a \in \mathrm{Crit}(f)$, let γ_a be some path in M joining $x_\star$ to a. Define

$$e = \sum_{a \in \mathrm{Crit}(f)} (-1)^{|a|} \gamma_a \in C_1(M, \mathbb{Z}). \tag{10-1}$$

Note that the Poincaré–Hopf index of $\widetilde{X}$ near $a \in \mathrm{Crit}(f)$ is $-(-1)^{|a|}$ so that

$$\partial e = \mathrm{div}(\widetilde{X}) \tag{10-2}$$

because $\sum_{a \in \mathrm{Crit}(f)} (-1)^{|a|} = \chi(M) = 0$ by the Poincaré–Hopf index theorem. Therefore e is an Euler chain for $\widetilde{X}$ and

$$\mathfrak{e} = [\widetilde{X}, e]$$

defines an Euler structure.

Next, choose some basis $u_1, \dots, u_d$ of $E_{x_\star}^\vee$. For each $a \in \mathrm{Crit}(f)$, we propagate this basis via the parallel transport of ∇ along γ_a to obtain a basis $u_{1,a}, \dots, u_{d,a}$ of E_a. We choose an ordering of the cells $\{\overline{W}^u(a)\}$; this gives us a homology orientation $\mathfrak{o}$, that is, an orientation on the line $\det H_\bullet(W, \mathbb{R})$ (see [Farber and Turaev 2000, §6.3]). Moreover, this ordering and the chosen basis of $E_a^\vee$ give us (using the wedge product) an element $c_k \in \det C_k(W, E^\vee)$ for each k, and thus an element $c \in \det C_\bullet(W, E^\vee)$.

The *Turaev torsion* of ∇ with respect to the choices $\mathfrak{e}, \mathfrak{o}$ is then defined by [Farber and Turaev 2000, §9.2, p. 218]

$$\tau_{\mathfrak{e},\mathfrak{o}}(\nabla)^{-1} = \varphi_{C_\bullet(W,\nabla^\vee)}(c) \in \mathbb{C} \setminus 0, \tag{10-3}$$

where $\varphi_{C_\bullet(W,\nabla^\vee)} : \det C_\bullet(W, \nabla^\vee) \simeq \mathbb{C} \setminus 0$ is the canonical isomorphism from [Farber and Turaev 2000, §2.2] — the homology version of the isomorphism (3-1). Note that $\nabla^\vee$ (and not ∇) is involved in the definition of $\tau_{\mathfrak{e},\mathfrak{o}}(\nabla)$; indeed, we use here the cohomological version of Turaev's torsion, which is more convenient for our purposes, and which is consistent with [Braverman and Kappeler 2007b; 2008, p. 252].

10.4. ***Resonant states of the Morse–Smale flow.*** In [Dang and Rivière 2020b], it was shown that we can define Ruelle resonances for the Morse–Smale gradient flow $\mathcal{L}_{\widetilde{X}}^{\nabla}$ as described in Section 5 in the context of Anosov flows. More precisely, we have that the resolvent

$$(\mathcal{L}^{\nabla}_{\widetilde{X}} + s)^{-1} : \Omega^\bullet(M, E) \to \mathcal{D}'^\bullet(M, E)$$

is well-defined for $\mathrm{Re}(s) \gg 0$ and has a meromorphic continuation to all $s \in \mathbb{C}$. The poles of this continuation are the Ruelle resonances of $\mathcal{L}^{\nabla}_{\widetilde{X}}$ and the set of those will be denoted by $\mathrm{Res}(\mathcal{L}^{\nabla}_{\widetilde{X}})$. In fact, the set $\mathrm{Res}(\mathcal{L}^{\nabla}_{\widetilde{X}})$ does not depend on the flat vector bundle (E, ∇). It only depends on the Lyapunov exponents of the Morse–Smale vector field at critical points. In fact $\mathrm{Res}(\mathcal{L}^{\nabla}_{\widetilde{X}}) \subset \mathbb{Z}_{\geqslant 0}$ in the present case since the Lyapunov exponents are only ± 1 and the Ruelle spectrum was proved to be equal to integer combinations of absolute values of Lyapunov exponents [Dang and Rivière 2020a, Theorem 6.3, p. 571]. Let $\lambda > 0$ be such that $\mathrm{Res}(\mathcal{L}^{\nabla}_{\widetilde{X}}) \cap \{|s| \leqslant \lambda\} \subset \{0\}$; let

$$\widetilde{\Pi} = \frac{1}{2\pi i} \int_{|s|=\lambda} (\mathcal{L}^{\nabla}_{\widetilde{X}} + s)^{-1}\, \mathrm{d}s \tag{10-4}$$

be the spectral projector associated with the resonance 0, and denote by

$$\widetilde{C}^\bullet = \mathrm{ran}\, \widetilde{\Pi} \subset \mathcal{D}'^\bullet(M, E)$$

the associated space of generalized eigenvectors for $\mathcal{L}^{\nabla}_{\widetilde{X}}$. Since ∇ and $\mathcal{L}^{\nabla}_{\widetilde{X}}$ commute, ∇ induces a differential on the complex $\widetilde{C}^\bullet$. Moreover, $\widetilde{\Pi}$ maps $\mathcal{D}'^\bullet_\Gamma(M, E)$ to itself continuously, where

$$\Gamma = \bigcup_{a \in \mathrm{Crit}(f)} \overline{N^* W^u(a)} \subset T^*M.$$

10.5. ***A variation formula for the Turaev torsion.*** Assume that we are given a $\mathcal{C}^1$ family of acyclic connections $\nabla(z)$ on E as in Section 8. We denote by $\widetilde{\Pi}_-(z)$ the spectral projector (10-4) associated with $\nabla(z)$ and $-\widetilde{X}$, and set $\widetilde{C}^\bullet_-(z) = \mathrm{ran}\, \widetilde{\Pi}_-(z)$. By [Dang and Rivière 2020b] we have that all the complexes $(\widetilde{C}^\bullet(z), \nabla(z))$ are acyclic and there exists cochain contractions $\tilde{k}_-(z) : \widetilde{C}^\bullet_-(z) \to \widetilde{C}^{\bullet-1}_-(z)$. As in Section 8.3 we have a variation formula for the Turaev torsion.

Proposition 10.1. *The map $z \mapsto \tilde{\tau}(z) = \tau_{\mathfrak{e},\mathfrak{o}}(\nabla(z))$ is real differentiable on U and for any $z \in U$*

$$\mathrm{d}(\log \tilde{\tau})_z \sigma = -\mathrm{tr}_{\mathrm{s}, \widetilde{C}^\bullet(z)}(\widetilde{\Pi}_-(z) \alpha_z(\sigma) \tilde{k}_-(z)) - \int_e \mathrm{tr}\, \alpha_z(\sigma), \qquad \sigma \in \mathbb{C},$$

where $\alpha_z(\sigma)$ is given by (8-2) and e is given by (10-1).

The rest of this section is devoted to the proof of Proposition 10.1. For convenience, we will first study the variation of $z \mapsto \tau_{\mathfrak{e},\mathfrak{o}}(\nabla(z)^\vee)$, in order to make computations on E instead of $E^\vee$ (indeed, $\tau_{\mathfrak{e},\mathfrak{o}}(\nabla(z))$ is defined with the dual connection $\nabla(z)^\vee$; see (10-3)). Then a simple duality relation will allow us to obtain the variation formula for $z \mapsto \tau_{\mathfrak{e},\mathfrak{o}}(\nabla(z))$.

10.6. ***A preferred basis.*** Let $a \in \mathrm{Crit}(f)$ and $k = |a|$. We denote by $[W^u(a)] \in \mathcal{D}'^{n-k}_\Gamma(M)$ the integration current over the unstable manifold $W^u(a)$ of $\widetilde{X}$; it is a well-defined current far from $\partial W^u(a)$. We also pick a cut-off function $\chi_a \in \mathcal{C}^\infty(M)$ valued in $[0, 1]$ with $\chi_a \equiv 1$ near a and χ_a is supported in a small neighborhood Ω_a of a, with $\overline{\Omega}_a \cap \partial W^u(a) = \varnothing$. Recall from Section 10.3 that we have a basis $u_{1,a}, \dots, u_{d,a}$ of E_a. Using the parallel transport of ∇, we obtain flat sections of E over $W^u(a)$ that we will still denote by $u_{1,a}, \dots, u_{d,a}$. Define

$$\tilde{u}_{j,a} = \widetilde{\Pi}(\chi_a[W^u(a)] \otimes u_{j,a}) \in \widetilde{C}^{n-k}, \quad j = 1, \dots, d. \tag{10-5}$$

By [Dang and Rivière 2020a] we have that $\{\tilde{u}_{j,a} : a \in \mathrm{Crit}(f),\ 1 \leqslant j \leqslant d\}$ is a basis of $\widetilde{C}^\bullet$. Adapting the proof of [Dang and Rivière 2021, Theorem 2.6] to the bundle case, we obtain the following proposition which will allow us to compute the Turaev torsion with the help of the complex $\widetilde{C}^\bullet$.

Proposition 10.2. *The map $\Phi : C_\bullet(W, \nabla) \to \widetilde{C}^{n-\bullet}$ defined by*

$$\Phi(u_{j,a}) = \tilde{u}_{j,a}, \quad a \in \mathrm{Crit}(f),\ j = 1, \dots, d,$$

is an isomorphism and satisfies[8]

$$\Phi \circ \partial^\nabla = (-1)^{\bullet+1} \nabla \circ \Phi.$$

An immediate corollary of the above proposition and (10-3) is that (using the notation of Section 3.2)

$$\tau_{\mathfrak{e},\mathfrak{o}}(\nabla^\vee) = \varphi_{C_\bullet(W,\nabla)}(u)^{-1} = \tau(\widetilde{C}^\bullet, \tilde{u}), \tag{10-6}$$

where $u \in \det C_\bullet(W, \nabla)$ (resp. $\tilde{u} \in \det \widetilde{C}^\bullet$) is the element given by the basis $\{u_{j,a}\}$ (resp. $\{\tilde{u}_{j,a}\}$) and the ordering of the cells $W^u(a)$.

10.7. ***Proof of Proposition 10.1.*** For any $a \in \mathrm{Crit}(f)$ we denote by $P_{\gamma_a}(z) \in \mathrm{Hom}(E_{x_\star}, E_a)$ the parallel transport of $\nabla(z)$ along γ_a. We set

$$u_{j,a}(z) = P_{\gamma_a}(z) P_{\gamma_a}(z_0)^{-1} u_{j,a}$$

and

$$\tilde{u}_{j,a}(z) = \widetilde{\Pi}(z)(\chi_a[W^u(a)] \otimes u_{j,a}(z)),$$

where again we consider $u_{j,a}(z)$ as a $\nabla(z)$-flat section of E over $W^u(a)$ using the parallel transport of $\nabla(z)$. The construction of Ruelle resonances for Morse–Smale gradient flow follows from the construction of anisotropic Sobolev spaces

$$\Omega^\bullet(M, E) \subset \widetilde{\mathcal{H}}_1^\bullet \subset \widetilde{\mathcal{H}}^\bullet \subset \mathcal{D}'^\bullet(M, E),$$

see [Dang and Rivière 2019], on which $\mathcal{L}_X^\nabla + s$ is a holomorphic family of Fredholm operators of index 0 in the region $\{\mathrm{Re}(s) > -2\}$, and such that $\nabla(z)$ is bounded $\widetilde{\mathcal{H}}_1^\bullet \to \widetilde{\mathcal{H}}^\bullet$. Every argument made in Section 8.4 also stands here and $z \mapsto \widetilde{\Pi}(z)$ is a $\mathcal{C}^1$ family of bounded operators $\widetilde{\mathcal{H}}^\bullet \to \widetilde{\mathcal{H}}_1^\bullet$.

Note that by continuity, $\widetilde{\Pi}(z)$ induces an isomorphism $\widetilde{C}^\bullet(z_0) \to \widetilde{C}^\bullet(z)$ for z close enough to zero. In fact, this isomorphism holds true for all z since we have an explicit description of the range of $\widetilde{\Pi}(z)$ for all z using the basis of resonant states of $\mathcal{L}_X^\nabla$. Let $\tilde{u}(z) \in \det \widetilde{C}^\bullet(z)$ be the element given by the basis $\{\tilde{u}_{j,a}(z)\}$ and the ordering of the cells $W^u(a)$. Then by (10-6) and (3-5) we have

$$\tau_{\mathfrak{e},\mathfrak{o}}(\nabla(z)^\vee) = \tau(\widetilde{C}^\bullet(z), \tilde{u}(z)) = [\tilde{u}(z) : \widetilde{\Pi}(z)\tilde{u}(z_0)]\tau(\widetilde{C}^\bullet(z), \widetilde{\Pi}(z)\tilde{u}(z_0)), \tag{10-7}$$

[8] $(-1)^\bullet$ comes from $\partial = (-1)^{\deg +1}\,\mathrm{d}$ comparing the boundary ∂ and De Rham differential d.

where $\widetilde{\Pi}(z)\tilde{u}(z_0) \in \det \widetilde{C}^\bullet(z)$ is the image of $\tilde{u}$ by the isomorphism $\det \widetilde{C}^\bullet(z_0) \to \det \widetilde{C}^\bullet(z)$ induced by $\widetilde{\Pi}(z)$, and $\tilde{u}(z) = [\tilde{u}(z) : \widetilde{\Pi}(z)\tilde{u}(z_0)]\widetilde{\Pi}(z)\tilde{u}(z_0)$. Doing exactly as in Section 8.6, we obtain that $z \mapsto \hat{\tau}(z) = \tau(\widetilde{C}^\bullet(z), \widetilde{\Pi}(z)\tilde{u})$ is $\mathcal{C}^1$ and

$$\mathrm{d}(\log \hat{\tau})_{z_0}\sigma = -\mathrm{tr}_{\mathrm{s},\widetilde{C}^\bullet}\widetilde{\Pi}(z_0)\alpha_{z_0}(\sigma)\tilde{k}(z_0). \tag{10-8}$$

Therefore it remains to compute the variation of $[\tilde{u}(z) : \widetilde{\Pi}(z)\tilde{u}(z_0)]$. This is the purpose of the next formula.

Lemma 10.3. *We have*

$$[\tilde{u}(z) : \widetilde{\Pi}(z)\tilde{u}(z_0)] = \prod_{a\in \mathrm{Crit}(f)} \det(P_{\gamma_a}(z)P_{\gamma_a}(z_0)^{-1})^{(-1)^{n-|a|}}.$$

Proof. By the definition of the basis $\{u_{a,j}\}$ in Section 10.3 it suffices to show that for z small enough

$$\widetilde{\Pi}(z)\tilde{u}_{a,i} = \sum_{j=1}^{d} A^j_{a,i}(z)\tilde{u}_{a,j}(z), \quad a \in \mathrm{Crit}(f),\ 1 \leqslant i, j \leqslant d, \tag{10-9}$$

where the coefficients $A^j_{a,i}(z)$ are defined by $u_{a,i}(z_0)(a) = \sum_{j=1}^{d} A^j_{a,i}(z)u_{a,j}(z)(a)$.

Everything relies on the fact that one has a decomposition of the projector

$$\widetilde{\Pi}(z) = \sum_{a,i}\langle \tilde{s}_{a,i}(z), \cdot\rangle \tilde{u}_{a,i}(z)$$

which originates from [Harvey and Lawson 2001] and was also used in [Dang and Rivière 2019, Theorem 2.4, p. 1409].

Consider the dual operator $\mathcal{L}^{\nabla(z)^\vee}_{-\widetilde{X}} : \Omega^\bullet(M, E^\vee) \to \Omega^\bullet(M, E^\vee)$. The above constructions, starting from a dual basis $s_1, \dots, s_d \in E^\vee_{x_\star}$ of $u_1, \dots, u_d$, give a basis $\{s_{a,i}(z)\}$ of each $\Gamma(W^s(a), \nabla(z)^\vee)$ (the space of flat section of $\nabla(z)^\vee$ over $W^s(a)$), since the unstable manifolds of $-\widetilde{X}$ are the stable ones of $\widetilde{X}$. Let $\widetilde{C}^\bullet_\vee(z)$ be the range of the spectral projector $\widetilde{\Pi}^\vee(z)$ from (10-4) associated with the vector field $-\widetilde{X}$ and the connection $\nabla(z)^\vee$. We have a basis $\{\tilde{s}_{a,i}(z)\}$ of $\widetilde{C}^\bullet_\vee(z)$ given by

$$\tilde{s}_{a,i}(z) = \widetilde{\Pi}^\vee(z)(\chi_a[W^s(a)] \otimes s_{a,i}(z)).$$

We will prove that for any $a, b \in \mathrm{Crit}(f)$ with same Morse index we have, for any $1 \leqslant i, j \leqslant d$,

$$\langle \tilde{s}_{a,j}(z), \tilde{u}_{a,i}(z_0)\rangle = \begin{cases} \langle s_{a,j}(z)(a), u_{a,i}(z_0)(a)\rangle_{E^\vee_a, E_a} & \text{if } a = b, \\ 0 & \text{if } a \neq b. \end{cases} \tag{10-10}$$

First assume that $a \neq b$. Then $W^u(a) \cap W^s(b) = \varnothing$ by the transversality condition, since a and b have same Morse index. Therefore for any $t_1, t_2 \geqslant 0$, we have

$$\left\langle e^{-t_1\mathcal{L}^{\nabla(z)^\vee}_{-\widetilde{X}}}(\chi_b[W^s(b)] \otimes s_{b,j}(z)), e^{-t_2\mathcal{L}^{\nabla(z_0)}_{\widetilde{X}}}(\chi_a[W^u(a)] \otimes u_{a,i}(z))\right\rangle = 0, \tag{10-11}$$

since the currents in the pairing have disjoint support because they are respectively contained in $W^s(b)$ and $W^u(a)$. Now notice that for $\mathrm{Re}(s)$ big enough, one has

$$(\mathcal{L}^{\nabla(z)^\vee}_{-\widetilde{X}} + s)^{-1} = \int_0^\infty e^{-t\mathcal{L}^{\nabla(z)^\vee}_{-\widetilde{X}}} e^{-ts}\,\mathrm{d}t \quad \text{and} \quad (\mathcal{L}^{\nabla(z_0)}_{\widetilde{X}} + s)^{-1} = \int_0^\infty e^{-t\mathcal{L}^{\nabla(z_0)}_{\widetilde{X}}} e^{-ts}\,\mathrm{d}t.$$

Therefore the representation (10-4) of the spectral projectors and the analytic continuation of the above resolvents imply with (10-11) that $\langle \tilde{s}_{b,j}(z), \tilde{u}_{a,i}\rangle = 0$.

Next assume that $a = b$. Then $W^u(a) \cap W^s(a) = \{a\}$. Since the support of $\tilde{s}_{a,i}(z)$ (resp. $\tilde{u}_{a,i}(z_0)$) is contained in the closure of $W^s(a)$ (resp. $W^u(a)$), we can compute

$$\begin{aligned}\big\langle \widetilde{\Pi}^\vee(z)(\chi_a[W^s(a)]\otimes s_{a,j}(z)), \widetilde{\Pi}(\chi_a[W^u(a)]\otimes u_{a,i}(z_0))\big\rangle &= \big\langle \chi_a[W^s(a)]\otimes s_{a,j}(z), \chi_a[W^u(a)]\otimes u_{a,i}(z_0)\big\rangle \\ &= \big\langle [a], \langle s_{a,j}(z), u_{a,i}(z_0)\rangle_{E^\vee,E}\big\rangle,\end{aligned}$$

where the first equality stands because $\tilde{s}_a(z) = [W^s(a)] \otimes s_{a,j}(z)$ near a by [Dang and Rivière 2020a, Proposition 7.1]. This gives (10-10).

This identity immediately yields (10-9) with $A^j_{a,i}(z) = \langle s_{a,j}(z)(a), u_{a,i}(z_0)(a)\rangle_{E_a^\vee,E_a}$ since we have

$$\widetilde{\Pi}(z) = \sum_{a,i} \langle \tilde{s}_{a,j}(z), \cdot\,\rangle \tilde{u}_{a,j}(z), \tag{10-12}$$

completing the proof. □

Using the lemma, we obtain, if $\mu(z) = [\tilde{u}(z) : \widetilde{\Pi}(z)\tilde{u}(z_0)]$,

$$\mathrm{d}(\log \mu)_{z_0}\sigma = \sum_{a\in \mathrm{Crit}(f)} (-1)^{n-|a|} \operatorname{tr}(A_{\gamma_a}(z_0,\sigma) P_{\gamma_a}(z_0)^{-1}),$$

where $A_{\gamma_a}(z_0,\sigma) = \mathrm{d}(P_{\gamma_a})_{z_0}\sigma$. Since n is odd, we obtain by definition of e and (4-4)

$$\mathrm{d}(\log \mu)_{z_0}\sigma = \sum_{a\in \mathrm{Crit}(f)} (-1)^{|a|} \int_{\gamma_a} \operatorname{tr} \alpha_{z_0}(\sigma) = \int_e \operatorname{tr} \alpha_{z_0}(\sigma).$$

This equation combined with (10-7) and (10-8) yields, if $\tilde{\tau}^\vee(z) = \tau_{\mathfrak{e},\mathfrak{o}}(\nabla(z)^\vee)$

$$\mathrm{d}(\log \tilde{\tau}^\vee)_{z_0}\sigma = -\operatorname{tr}_{\mathrm{s},\widetilde{C}^\bullet} \widetilde{\Pi}(z_0)\alpha_{z_0}(\sigma)\tilde{k}(z_0) + \int_e \operatorname{tr}\alpha_{z_0}(\sigma).$$

The proof is almost finished. We first studied the variation of $z \mapsto \tau(\nabla(z)^\vee)$; we now recover the variation of $z \mapsto \tau(\nabla(z))$, which was the goal of Proposition 10.1. Let us introduce some notation. Recall that the operator $\widetilde{\Pi}$ is the spectral projector on the kernel of $\mathcal{L}^{\nabla}_{\widetilde{X}}$; now, we need to work with the spectral projector on $\ker(\mathcal{L}^{\nabla(z_0)^\vee}_{\widetilde{X}})$ (resp. $\mathcal{L}^{\nabla(z_0)}_{-\widetilde{X}}$), which we denote by $\widetilde{\Pi}^\vee_+(z_0)$ (resp. $\widetilde{\Pi}_-(z_0)$) — the sign $+$ (resp. $-$) emphasize the fact that we deal with $+\widetilde{X}$ (resp. $-\widetilde{X}$). Next, we have

$$\nabla(z)^\vee = \nabla(z_0)^\vee - {}^T(\alpha_{z_0}(z-z_0)) + o(z-z_0).$$

Therefore, applying what precedes to $\tilde{\tau}(z)$ we get

$$\mathrm{d}(\log \tilde{\tau})_{z_0}\sigma = -\operatorname{tr}_{\mathrm{s},\widetilde{C}^\bullet_{\vee,+}} \big(\widetilde{\Pi}^\vee_+(z_0)(-{}^T\alpha_{z_0}(\sigma))\tilde{k}^\vee_+(z_0)\big) + \int_e \operatorname{tr}(-{}^T\alpha_{z_0}(\sigma)), \tag{10-13}$$

where $\widetilde{\Pi}^\vee_+(z_0)$ is the spectral projector (10-4) associated with $\nabla(z_0)^\vee$ and $+\widetilde{X}$, $\widetilde{C}^\bullet_{\vee,+} = \operatorname{ran} \widetilde{\Pi}^\vee_+(z_0)$, and $\tilde{k}^\vee_+(z_0)$ is any cochain contraction on the complex $(\widetilde{C}^\bullet_{\vee,+}, \nabla(z_0)^\vee)$. Now, we have the identification

$$(\widetilde{C}^k_{\vee,+})^\vee \simeq \widetilde{C}^{n-k}_-,$$

where $\widetilde{C}_-^\bullet$ is the range of $\widetilde{\Pi}_-(z_0)$, the spectral projector (10-4) associated with $\nabla(z_0)$ and $-\widetilde{X}$. This identification can be thought of as a chain level version of Poincaré duality, the coresonant states of the resonant states for the operator $\mathcal{L}^{\nabla}_{\widetilde{X}}$ acting on the sections of the flat bundle (E,∇) are nothing but the resonant states of $\mathcal{L}^{\nabla^\vee}_{-\widetilde{X}}$ acting on the sections of the dual flat bundle $(E^\vee,\nabla^\vee)$. Moreover, one can show that under this identification, the operators $(\widetilde{\Pi}_+^\vee({}^T\alpha_{z_0}(\sigma))\tilde{k}(z_0))^\vee$ and $\widetilde{\Pi}_-(z_0)\alpha_{z_0}(\sigma)k_-(z_0)$ coincide modulo a supercommutator. More precisely, it holds

$$(\widetilde{\Pi}_+^\vee({}^T\alpha_{z_0}(\sigma))\tilde{k}(z_0))^\vee=\widetilde{\Pi}_-(z_0)\alpha_{z_0}(\sigma)k_-(z_0)+[\widetilde{\Pi}_-(z_0)\alpha_{z_0}(\sigma),k_-(z_0)],$$

where for any $j\in\{0,\dots,n\}$ we set

$$k_-(z_0)|_{\widetilde{C}_-^{n-j}}=(-1)^{j+1}(\tilde{k}_+^\vee(z_0)|_{\widetilde{C}^{j+1}})^\vee:\widetilde{C}_-^{n-j}\to\widetilde{C}_-^{n-j-1}.$$

The operator $k_-(z_0)$ is a cochain contraction on the complex $(\widetilde{C}_-^\bullet,\nabla(z_0))$. As a consequence, since n is odd,

$$\operatorname{tr}_{s,\widetilde{C}^\bullet_{\vee,+}}(\widetilde{\Pi}_+^\vee(z_0)(-{}^T\alpha_{z_0}(\sigma))\tilde{k}_+^\vee(z_0))=\operatorname{tr}_{s,\widetilde{C}_-^\bullet}\widetilde{\Pi}_-(z_0)\alpha_{z_0}(\sigma)k_-(z_0).$$

This concludes the proof of Proposition 10.1 by (10-13) since $\operatorname{tr}(-{}^T\beta)=-\operatorname{tr}\beta$ for any $\beta\in\Omega^1(M,\operatorname{End}(E))$.

11. Comparison of the dynamical torsion with the Turaev torsion

In this section we see the dynamical torsion and the Turaev torsion as functions on the space of acyclic representations. This is an open subset of a complex affine algebraic variety. Therefore we can compute the derivative of $\tau_\vartheta/\tau_{\mathfrak{e},\mathfrak{o}}$ along holomorphic curves, using the variation formulae obtained in Sections 8 and 10. From this computation we will deduce Theorem 6.

11.1. ***The algebraic structure of the representation variety.*** We describe here the analytic structure of the space

$$\operatorname{Rep}(M,d)=\operatorname{Hom}(\pi_1(M),\operatorname{GL}(\mathbb{C}^d))$$

of complex representations of degree d of the fundamental group. Since M is compact, $\pi_1(M)$ is generated by a finite number of elements $c_1,\dots,c_L\in\pi_1(M)$ which satisfy finitely many relations. A representation $\rho\in\operatorname{Rep}(M,d)$ is thus given by $2L$ invertible $d\times d$ matrices $\rho(c_1),\dots,\rho(c_L),\rho(c_1^{-1}),\dots\rho(c_L^{-1})$ with complex coefficients satisfying finitely many polynomial equations. Therefore the set $\operatorname{Rep}(M,d)$ has a natural structure of a complex affine algebraic set. We will denote the set of its singular points by $\Sigma(M,d)$. In what follows, we will only consider the classical topology of $\operatorname{Rep}(M,d)$, and not the Zariski one.

For any $\rho\in\operatorname{Rep}(M,d)$, we define

$$E_\rho=\widetilde{M}\times\mathbb{C}^d/\sim_\rho,$$

where $\widetilde{M}$ is the universal cover of M and $\sim_\rho$ is the equivalence relation given by

$$(\tilde{x},v)\sim_\rho(\gamma\cdot\tilde{x},\rho(\gamma)\cdot v),\quad \tilde{x}\in M,\ \gamma\in\pi_1(M).$$

Then E_ρ is vector bundle over M which we endow with the flat connection ∇_ρ induced by the trivial connection on $\widetilde{M}\times\mathbb{C}^d$.

We will say that a representation $\rho \in \operatorname{Rep}(M, d)$ is acyclic if ∇_ρ is acyclic. We denote by $\operatorname{Rep}_{\mathrm{ac}}(M, d) \subset \operatorname{Rep}(M, d)$ the space of acyclic representations. This is an open set (in the Zariski topology, thus in the classical one) in $\operatorname{Rep}(M, d)$; see [Burghelea and Haller 2006, §4.1]. For any $\rho \in \operatorname{Rep}_{\mathrm{ac}}(M, d)$ we set

$$\tau_\vartheta(\rho) = \tau_\vartheta(\nabla_\rho), \qquad \tau_{\mathfrak{e},\mathfrak{o}}(\rho) = \tau_{\mathfrak{e},\mathfrak{o}}(\nabla_\rho)$$

for any Euler structure $\mathfrak{e}$ and any homology orientation $\mathfrak{o}$.

11.2. ***Holomorphic families of acyclic representations.*** Let $\rho_0 \in \operatorname{Rep}_{\mathrm{ac}}(M, d) \setminus \Sigma(M, d)$ be a regular point. Take $\delta > 0$ and $\rho(z)$, $|z| < \delta$, a holomorphic curve in $\operatorname{Rep}_{\mathrm{ac}}(M, d) \setminus \Sigma(M, d)$ such that $\rho(0) = \rho_0$. Theorems 6 and 7 will be a consequence of the following

Proposition 11.1. *Let X be a contact Anosov vector field on M. Let $\mathfrak{e} = [\widetilde{X}, e]$ be the Euler structure defined in Section 10.3. Note that $-\operatorname{cs}(-\widetilde{X}, X) + e$ is a cycle and defines a homology class $h \in H_1(M, \mathbb{Z})$. Then $z \mapsto \tau_\vartheta(\rho(z))/\tau_{\mathfrak{e},\mathfrak{o}}(\rho(z))$ is complex differentiable and*

$$\frac{\mathrm{d}}{\mathrm{d}z}\left(\frac{\tau_\vartheta(\rho(z))}{\tau_{\mathfrak{e},\mathfrak{o}}(\rho(z))}\langle \det \rho(z), h\rangle\right) = 0$$

for any homology orientation $\mathfrak{o}$.

Proposition 11.1 relies on the variation formulae given by Propositions 8.1 and 10.1, and Lemma 9.4, which gives a topological interpretation of those.

11.3. ***An adapted family of connections.*** By [Braverman and Vertman 2017, Lemma 4.3], there exists a flat vector bundle E over M and a $\mathcal{C}^1$ family of connections $\nabla(z)$, $|z| < \delta$, in the sense of Section 8.1, such that

$$\rho_{\nabla(z)} = \rho(z) \tag{11-1}$$

for every z; we can moreover ask the family $\nabla(z)$ to be complex differentiable at $z = 0$, that is,

$$\nabla(z) = \nabla + z\alpha + o(z), \tag{11-2}$$

where $\nabla = \nabla(0)$ and $\alpha \in \Omega^1(M, \operatorname{End}(E))$. Note that flatness of $\nabla(z)$ implies

$$[\nabla, \alpha] = \nabla\alpha + \alpha\nabla = 0. \tag{11-3}$$

11.4. ***A cochain contraction induced by the Morse–Smale gradient flow.*** Let

$$(\mathcal{L}^{\nabla}_{-\widetilde{X}} + s)^{-1} = \frac{\widetilde{\Pi}_-}{s} + \widetilde{Y} + \mathcal{O}(s)$$

be the Laurent expansion of $(\mathcal{L}^{\nabla}_{-\widetilde{X}} + s)^{-1}$ near $s = 0$. The fact that $s = 0$ is a simple pole comes from [Dang and Rivière 2019, Proposition 6.1, p. 1431], where it is proved that there are no Jordan blocks for the resonance $s = 0$. As in Section 8.2, we consider the operator

$$\widetilde{K} = \iota_{-\widetilde{X}}\widetilde{Y}(\mathrm{Id} - \widetilde{\Pi}_-) + \tilde{k}_-\widetilde{\Pi}_- : \Omega^\bullet(M, E) \to \mathcal{D}'^\bullet(M, E),$$

where $\tilde{k}_-$ is any cochain contraction on $\widetilde{C}^\bullet_- = \operatorname{ran}\widetilde{\Pi}_-$. Note that we have the identity

$$[\nabla, \widetilde{K}] = \nabla\widetilde{K} + \widetilde{K}\nabla = \mathrm{Id}. \tag{11-4}$$

The next proposition allows us to interpret the term $\operatorname{tr}_{s,\widetilde{C}^\bullet}\widetilde{\Pi}_-(z)\alpha_z(\sigma)\tilde{k}_-(z)$ appearing in Proposition 10.1 as a flat trace similar to the one appearing in Proposition 8.1. This will be crucial for the comparison between τ_ϑ and $\tau_{\mathfrak{e},\mathfrak{o}}$.

Proposition 11.2. *For $\varepsilon > 0$ small enough, the wavefront set of the Schwartz kernel of the operator $\iota_{-\widetilde{X}}\widetilde{Y}(\mathrm{Id}-\widetilde{\Pi}_-)e^{-\varepsilon\mathcal{L}^\nabla_{-\widetilde{X}}}$ does not meet the conormal to the diagonal in $M \times M$ and we have for any $\alpha \in \Omega^1(M, \mathrm{End}(E))$*

$$\operatorname{tr}^\flat_s(\alpha\iota_{-\widetilde{X}}\widetilde{Y}(\mathrm{Id}-\widetilde{\Pi}_-)e^{-\varepsilon\mathcal{L}^\nabla_{-\widetilde{X}}}) = 0.$$

Proof of Proposition 11.2. Fix $\varepsilon > 0$. We start from the Atiyah–Bott–Lefschetz trace formula [Atiyah and Bott 1967], which gives

$$\operatorname{tr}^\flat_s\alpha\iota_{-\widetilde{X}}e^{(t+\varepsilon)\widetilde{X}} = 0$$

for all $t \geqslant 0$ since the flat trace $\operatorname{tr}^\flat_s$ localizes at the critical points of $\widetilde{X}$ and the contribution from the term $\alpha\iota_{-\widetilde{X}}$ vanishes at the critical points. Now we would like to integrate this equality in time t on $[0, +\infty)$ and then connect with the resolvent $(\mathcal{L}_{-\widetilde{X}} + s)^{-1}$; we have to argue rigorously why we can interchange the flat trace and the integral over time t. This relies in an essential way on some explicit bound of the wavefront set of the resolvent that can be deduced from Lemma C.1 in Appendix C, where we bound the wavefront of the propagator near the conormal of the diagonal. Assuming that the inversion is justified, we obtain, for large $\mathrm{Re}(s)$,

$$\begin{aligned}0 &= \int_0^\infty e^{-ts}\operatorname{tr}^\flat_s(\alpha\iota_{-\widetilde{X}}e^{(t+\varepsilon)\widetilde{X}})\,\mathrm{d}t = \int_0^\infty e^{-ts}\operatorname{tr}^\flat_s(\iota_{-\widetilde{X}}e^{(t+\varepsilon)\widetilde{X}}(\mathrm{Id}-\widetilde{\Pi}_-))\,\mathrm{d}t\\ &= \operatorname{tr}^\flat_s\big(\alpha\iota_{-\widetilde{X}}(\mathcal{L}_{-\widetilde{X}}+s)^{-1}(\mathrm{Id}-\widetilde{\Pi}_-)e^{\varepsilon\mathcal{L}_{\widetilde{X}}}\big),\end{aligned}$$

where we used the fact that $\iota_{-\widetilde{X}}\widetilde{\Pi}_- = 0$, which follows from the proof of [Dang and Rivière 2019, Proposition 7.7, p. 1448]. Actually, both resonant and coresonant states of $-\widetilde{X}$ are killed by the contraction operator $\iota_{-\widetilde{X}}$. Our wavefront bound implies that the above identity still makes sense for s near the origin; we then conclude by noting that

$$\underbrace{\operatorname{tr}^\flat_s\big(\alpha\iota_{-\widetilde{X}}(\mathcal{L}_{-\widetilde{X}}+s)^{-1}(\mathrm{Id}-\widetilde{\Pi}_-)e^{\varepsilon\mathcal{L}_{\widetilde{X}}}\big)}_{0} = \operatorname{tr}^\flat_s(\alpha\iota_{-\widetilde{X}}\widetilde{Y}e^{\varepsilon\mathcal{L}_{\widetilde{X}}}) + \mathcal{O}(s)$$

since $\widetilde{Y}(\mathrm{Id}-\widetilde{\Pi}_-) = \widetilde{Y}$. Thus letting $s \to 0$ concludes the proof of the proposition, provided that we can justify the interchange of the flat trace and the integration over t.

For $a \in \mathrm{Crit}(f)$, take c_a, Γ_a, χ_a as in Lemma C.1 proved in Appendix C. The proof of Lemma C.1 actually shows that for $\mathrm{Re}(s) > -c_a$, the integral

$$G_{\chi_a,\varepsilon,s} = \int_0^\infty e^{-ts}\chi_a e^{(t+\varepsilon)\widetilde{X}}(\mathrm{Id}-\widetilde{\Pi}_-)\chi_a\,\mathrm{d}t$$

converges as an operator $\Omega^\bullet(M) \to \mathcal{D}'^\bullet(M)$. Moreover, its Schwartz kernel $\mathcal{G}_{\chi_a,\varepsilon,s}$ is locally bounded in $\mathcal{D}'^n_{\Gamma_a}(M \times M)$ in the region $\{\mathrm{Re}(s) > -c_a\}$. We will need the following lemma, which is also proved in Appendix C.

Lemma 11.3. *For any $\mu > 0$, there is $\nu > 0$ with the following property. For every $x \in M$ such that* $\mathrm{dist}(x, \mathrm{Crit}(f)) \geqslant \mu$, *it holds*

$$\mathrm{dist}(x, e^{-(t+\varepsilon)\widetilde{X}}(x)) \geqslant \nu, \quad t \geqslant 0.$$

By (10-12) we have $\operatorname{supp} \mathcal{K}_{\widetilde{\Pi}_-} \cap \Delta = \mathrm{Crit}(f)$, where $\mathcal{K}_{\widetilde{\Pi}_-}$ is the Schwartz kernel of $\widetilde{\Pi}_-$ and Δ is the diagonal in $M \times M$; the same holds for $e^{(t+\varepsilon)\widetilde{X}}\widetilde{\Pi}_- = \widetilde{\Pi}_-$ (see [Dang and Rivière 2021]). Moreover, Lemma 11.3 implies that if $\chi \in C^\infty(M, [0, 1])$ satisfies $\chi \equiv 1$ near Δ and has support close enough to Δ, we have

$$\chi e^{(t+\varepsilon)\widetilde{X}}\chi = \sum_a \chi_a e^{(t+\varepsilon)\widetilde{X}}\chi_a.$$

Let $c = \min_{a\in\mathrm{Crit}(f)} c_a$. For $\mathrm{Re}(s) > -c$,

$$G_{\chi,\varepsilon,s} = \int_0^\infty e^{-ts}\chi e^{(t+\varepsilon)\widetilde{X}}(\mathrm{Id} - \widetilde{\Pi}_-)\chi \, \mathrm{d}t$$

defines an operator $\Omega^\bullet(M) \to \mathcal{D}'^\bullet(M)$, whose Schwartz kernel $\mathcal{G}_{\chi,\varepsilon,s}$ is locally bounded in $\mathcal{D}'^n_\Gamma(M \times M)$ in the region $\{\mathrm{Re}(s) > -c\}$, where $\Gamma = \bigcup_{a\in\mathrm{Crit}(f)} \Gamma_a$.

Now for $\mathrm{Re}(s) \gg 0$, we have as a consequence of the Hille–Yosida theorem applied to $\mathcal{L}_{-\widetilde{X}}$ acting on suitable anisotropic spaces [Dang and Rivière 2021, 3.2.3]:

$$(\mathcal{L}_{-\widetilde{X}} + s)^{-1} = \int_0^\infty e^{-ts}e^{t\widetilde{X}} \, \mathrm{d}t : \Omega^\bullet(M) \to \mathcal{D}'^\bullet(M).$$

Therefore for $\mathrm{Re}(s) \gg 0$, it holds

$$G_{\chi,\varepsilon,s} = \chi(\mathcal{L}_{-\widetilde{X}} + s)^{-1}(\mathrm{Id} - \widetilde{\Pi}_-)e^{\varepsilon\widetilde{X}}\chi.$$

Since both members are holomorphic in the region $\{\mathrm{Re}(s) > -c\}$ and coincide for $\mathrm{Re}(s) \gg 0$, they coincide in the region $\mathrm{Re}(s) > -c$. We may compute, for $\mathrm{Re}(s) \gg 0$,

$$\mathrm{tr}^\flat_s\big(\alpha\iota_{-\widetilde{X}}(\mathcal{L}_{-\widetilde{X}} + s)^{-1}(\mathrm{Id} - \widetilde{\Pi}_-)e^{\varepsilon\mathcal{L}_{\widetilde{X}}}\big) = \mathrm{tr}^\flat_s \alpha\iota_{-\widetilde{X}} G_{\chi,\varepsilon,s} = \int_0^\infty e^{-ts}\mathrm{tr}^\flat_s\big(\alpha\iota_{-\widetilde{X}}e^{(t+\varepsilon)\widetilde{X}}(\mathrm{Id} - \widetilde{\Pi}_-)\big) \, \mathrm{d}t.$$

By holomorphy this holds true for any s such that $\mathrm{Re}(s) > -c$, which concludes the proof. ☐

As a consequence, we have the formula

$$\mathrm{tr}_{s,\widetilde{C}^\bullet_-}\widetilde{\Pi}_-\alpha\tilde{k}_- = \mathrm{tr}^\flat_s \alpha\widetilde{K}e^{-\varepsilon\mathcal{L}^\nabla_{-\widetilde{X}}}. \tag{11-5}$$

Indeed, since $\mathcal{L}^\nabla_{-\widetilde{X}}\widetilde{\Pi}_- = 0$, we have $\widetilde{\Pi}_- e^{-\varepsilon\mathcal{L}^\nabla_{-\widetilde{X}}} = \widetilde{\Pi}_-$. Moreover, since the trace of finite-rank operators coincides with the flat trace, we have $\mathrm{tr}_{s,\widetilde{C}^\bullet_-}\widetilde{\Pi}_-\alpha\tilde{k}_- = \mathrm{tr}_{s,\widetilde{C}^\bullet_-}\widetilde{\Pi}_-\alpha\tilde{k}_- e^{-\varepsilon\mathcal{L}^\nabla_{-\widetilde{X}}} = \mathrm{tr}^\flat_s\alpha\tilde{k}_-\widetilde{\Pi}_- e^{-\varepsilon\mathcal{L}^\nabla_{-\widetilde{X}}}$. Therefore we obtain with Proposition 11.2

$$\mathrm{tr}_{s,\widetilde{C}^\bullet}\widetilde{\Pi}_-\alpha\tilde{k}_- = \mathrm{tr}^\flat_s\alpha\iota_{-\widetilde{X}}\widetilde{Y}(\mathrm{Id} - \widetilde{\Pi}_-)e^{-\varepsilon\mathcal{L}^\nabla_{-\widetilde{X}}} + \mathrm{tr}^\flat_s\alpha\tilde{k}_-\widetilde{\Pi}_- e^{-\varepsilon\mathcal{L}^\nabla_{-\widetilde{X}}},$$

which gives (11-5).

11.5. ***Proof of Proposition 11.1.*** Note that we have by (11-1)

$$\tau_\vartheta(\rho(z)) = \tau_\vartheta(\nabla(z)), \qquad \tau_{\mathfrak{e},\mathfrak{o}}(\rho(z)) = \tau_{\mathfrak{e},\mathfrak{o}}(\nabla(z)).$$

We will set $f(z) = \tau_\vartheta(\nabla(z))/\tau_{\mathfrak{e},\mathfrak{o}}(\nabla(z))$ for simplicity. Now we apply Propositions 8.1 and 10.1 to obtain that $z \mapsto f(z)$ is real differentiable (since $z \mapsto \nabla(z)$ is); moreover it is complex differentiable at $z = 0$ by (11-2) and for $\varepsilon > 0$ small enough we have

$$\frac{\mathrm{d}}{\mathrm{d}z}\bigg|_{z=0} \log f(z) = -\mathrm{tr}_\mathrm{s}^\flat \alpha K e^{-\varepsilon \mathcal{L}_X^\nabla} + \mathrm{tr}_\mathrm{s}^\flat \alpha \widetilde{K} e^{-\varepsilon \mathcal{L}_{-\widetilde{X}}^\nabla} + \langle \mathrm{tr}\,\alpha, e\rangle, \tag{11-6}$$

where we used (11-5).

Lemma 11.4. *It holds* $\mathrm{tr}_\mathrm{s}^\flat[\alpha(K e^{-\varepsilon \mathcal{L}_X^\nabla} - \widetilde{K} e^{-\varepsilon \mathcal{L}_{-\widetilde{X}}^\nabla})] = \mathrm{tr}_\mathrm{s}^\flat \alpha R_\varepsilon$, *where* R_ε *is the interpolator at time* ε *defined in Section 9.3 for the pair of vector fields* $(-\widetilde{X}, X)$.

Let us admit the lemma for now (we shall prove it later). The identity $[\nabla, \alpha] = 0$ also implies that $\mathrm{d}\,\mathrm{tr}\,\alpha = \mathrm{tr}\,\nabla^{E\otimes E^\vee}\alpha = \mathrm{tr}[\nabla, \alpha] = 0$. As a consequence we can apply (9-5) to obtain

$$\mathrm{tr}_\mathrm{s}^\flat \alpha R_\varepsilon = \langle \mathrm{tr}\,\alpha, \mathrm{cs}(-\widetilde{X}, X)\rangle.$$

Now note that $\partial(-\mathrm{cs}(-\widetilde{X}, X) + e) = -(\mathrm{div}(X) - \mathrm{div}(-\widetilde{X})) + \mathrm{div}(\widetilde{X}) = 0$ by (9-1) and (10-2) since X is nonsingular. Therefore we obtain

$$\frac{\mathrm{d}}{\mathrm{d}z}\bigg|_{z=0} \log f(z) = \langle \mathrm{tr}\,\alpha, h\rangle,$$

where $h = [-\mathrm{cs}(-\widetilde{X}, X) + e] \in H_1(M, \mathbb{Z})$. Finally, let us note that by (4-4),

$$\frac{\mathrm{d}}{\mathrm{d}z}\bigg|_{z=0} \log \det \rho(z)(h) = -\langle \mathrm{tr}\,\alpha, h\rangle,$$

since $\rho(z) = \rho_{\nabla(z)}$. Therefore the proposition is proved for $z = 0$. However the same argument holds for every z close enough to 0, which gives the conclusion of Proposition 11.1. It remains to prove Lemma 11.4.

Proof of Lemma 11.4. Using the identities (8-6), (9-4), (11-3) and (11-4) one can see that

$$[\nabla, \alpha(K e^{-\varepsilon \mathcal{L}_X^\nabla} - \widetilde{K} e^{-\varepsilon \mathcal{L}_{-\widetilde{X}}^\nabla} + R_\varepsilon)] = 0. \tag{11-7}$$

Next, it is a general fact that, for a finite-dimensional acyclic cochain complex $(C^\bullet, \partial)$ and an operator $b : C^\bullet \to C^\bullet$ of order zero such that $[\partial, b] = 0$, it holds $\mathrm{tr}_{\mathrm{s},C^\bullet}\, b = 0$. Indeed, if $k : C^\bullet \to C^\bullet$ satisfies $k\partial + \partial k = \mathrm{Id}_{C^\bullet}$, we have $[\partial, kb] = [\partial, k]b = b$ since $[\partial, b] = b\partial - \partial b = 0$. Thus b is a supercommutator and its supertrace vanishes. Here (11-7) shows that we are in the same situation, with an infinite-dimensional complex; we will use Hodge theory to obtain a cochain contraction J (that takes the role of k in the above argument), and such that the composition $J B_\varepsilon$, where

$$B_\varepsilon = \alpha(K e^{-\varepsilon \mathcal{L}_X^\nabla} - \widetilde{K} e^{-\varepsilon \mathcal{L}_{-\widetilde{X}}^\nabla} - R_\varepsilon),$$

is well-defined. Let

$$\Delta = \nabla\nabla^\star + \nabla^\star\nabla : \Omega^\bullet(M, E) \to \Omega^\bullet(M, E)$$

be the Hodge–Laplace operator induced by any metric on M and any Hermitian product on E. Because ∇ is acyclic, Δ is invertible and Hodge theory gives that its inverse Δ^{-1} is a pseudodifferential operator of order -2. Define

$$J = \nabla^\star \Delta^{-1} : \mathcal{D}'^\bullet(M, E) \to \mathcal{D}'^{\bullet-1}(M, E).$$

We have of course

$$[\nabla, J] = \nabla J + J\nabla = \mathrm{Id}_{\mathcal{D}'^\bullet(M,E)}. \tag{11-8}$$

As above, this gives $B_\varepsilon = [\nabla, J B_\varepsilon]$. Moreover, it follows from wavefront composition [Hörmander 1990, Theorem 8.2.14] that $\mathrm{WF}(G_\varepsilon) \cap N^*\Delta = \varnothing$. Therefore, the operators ∇, G_ε satisfy the assumptions of Proposition 4.1 which gives $\mathrm{tr}_s^\flat B_\varepsilon = \mathrm{tr}_s^\flat[\nabla, G_\varepsilon] = 0$, which concludes the proof of Lemma 11.4. □

11.6. ***Proof of Theorems 6 and 7.*** By Hartogs' theorem and Proposition 11.1, we have that the map

$$\rho \mapsto \frac{\tau_\vartheta(\rho)}{\tau_{\mathfrak{e},\mathfrak{o}}(\rho)} \langle \det \rho, h\rangle \tag{11-9}$$

is locally constant on $\mathrm{Rep}_{\mathrm{ac}}(M, d) \setminus \Sigma(M, d)$.

Moreover, we can reproduce all the arguments we made in the continuous category to obtain that $\rho \mapsto \tau_\vartheta(\rho)/\tau_{\mathfrak{e},\mathfrak{o}}(\rho)$ is actually continuous on $\mathrm{Rep}_{\mathrm{ac}}(M, d)$. Because $\mathrm{Rep}_{\mathrm{ac}}(M, d) \setminus \Sigma(M, d)$ is open and dense in $\mathrm{Rep}_{\mathrm{ac}}(M, d)$, we get that the map (11-9) is locally constant on $\mathrm{Rep}_{\mathrm{ac}}(M, d)$.

By [Farber and Turaev 2000, p. 211] we have, if $\mathfrak{e}'$ is another Euler structure, then $\tau_{\mathfrak{e}',\mathfrak{o}}(\rho) = \langle \det \rho, \mathfrak{e}' - \mathfrak{e}\rangle \tau_{\mathfrak{e},\mathfrak{o}}(\rho)$. As a consequence, if we set $\mathfrak{e}_\vartheta = [-X, 0]$, which defines an Euler structure since X is nonsingular (see Section 9.2), we have $\mathfrak{e} - \mathfrak{e}_\vartheta = h$ and we obtain that $\rho \mapsto \tau_\vartheta(\rho)/\tau_{\mathfrak{e}_\vartheta,\mathfrak{o}}(\rho)$ is locally constant on $\mathrm{Rep}_{\mathrm{ac}}(M, d)$.

Now let η be another contact form inducing an Anosov Reeb flow and denote by X_η its Reeb flow. Then if $\mathfrak{e}_\eta = [-X_\eta, 0]$, we have

$$\mathfrak{e}_\eta - \mathfrak{e}_\vartheta = \mathrm{cs}(X, X_\eta)$$

by definition. Therefore $\tau_{\mathfrak{e}_\vartheta,\mathfrak{o}}(\rho) = \tau_{\mathfrak{e}_\eta,\mathfrak{o}}(\rho)\langle \det \rho, \mathfrak{e}_\vartheta - \mathfrak{e}_\eta\rangle = \tau_{\mathfrak{e}_\eta,\mathfrak{o}}(\rho)\langle \det \rho, \mathrm{cs}(X_\eta, X)\rangle$ and we obtain that

$$\rho \mapsto \frac{\tau_\vartheta(\rho)}{\tau_\eta(\rho)} \langle \det \rho, \mathrm{cs}(X, X_\eta)\rangle$$

is locally constant on $\mathrm{Rep}_{\mathrm{ac}}(M, d)$. By Theorem 9 we thus obtain Theorem 7.

Finally assume that $\dim M = 3$ and $b_1(M) \neq 0$. Take $\mathcal{R}$ a connected component of $\mathrm{Rep}_{\mathrm{ac}}(M, d)$ and assume that it contains an acyclic and unitary representation ρ_0. We invoke [Dang et al. 2020, Theorem 1] and the Cheeger–Müller theorem [Cheeger 1979; Müller 1978] to obtain that $0 \notin \mathrm{Res}(\mathcal{L}_X^{\nabla^{\rho_0}})$ and

$$|\tau_\vartheta(\rho_0)| = |\zeta_{X,\nabla_{\rho_0}}(0)|^{-1} = \tau_{\mathrm{RS}}(\rho_0),$$

where the first equality comes from (6-10) (we have $q = 1$ since $\dim M = 3$) and $\tau_{\mathrm{RS}}(\rho_0)$ is the Ray–Singer torsion of (M, ρ_0) [1971]. On the other hand, we have by [Farber and Turaev 2000, Theorem 10.2] that $\tau_{\mathrm{RS}}(\rho_0) = |\tau_{\mathfrak{e},\mathfrak{o}}(\rho_0)|$ since ρ_0 is unitary. Therefore the map $\rho \mapsto \tau_\vartheta(\rho)/\tau_{\mathfrak{e}_\vartheta,\mathfrak{o}}(\rho)$ is of modulus 1 on $\mathcal{R}$. This concludes the proof of Theorem 6.

Appendix A: Projectors of finite rank

A.1. ***Traces on variable finite-dimensional spaces.*** In what follows, we consider two Hilbert spaces $\mathcal{G} \subset \mathcal{H}$, the inclusion being dense and continuous. We will denote by $\mathcal{L}(\mathcal{H}, \mathcal{G})$ the space of bounded linear operators $\mathcal{H} \to \mathcal{G}$ endowed with the operator norm. Let $\delta > 0$ and Π_t, $|t| \leqslant \delta$, be a family of finite-rank projectors on $\mathcal{H}$ such that $\operatorname{ran} \Pi_t \subset \mathcal{G}$. Assume that $t \mapsto \Pi_t$ is differentiable at $t = 0$ as a family of bounded operators $\mathcal{H} \to \mathcal{G}$, that is,

$$\Pi_t = \Pi + tP + o_{\mathcal{H}\to\mathcal{G}}(t) \tag{A-1}$$

for some $P \in \mathcal{L}(\mathcal{H}, \mathcal{G})$, where $\Pi = \Pi_0$. Let $C_t = \operatorname{ran} \Pi_t$ and $C = \operatorname{ran} \Pi$. Note that by continuity, $\Pi_t|_C : C \to C_t$ is invertible for $|t|$ small enough; we denote by $Q_t : C_t \to C$ its inverse.

Lemma A.1. *We have*

(i) $P = \Pi P + P\Pi$,

(ii) $Q_t \Pi_t = \Pi \Pi_t + o_{\mathcal{H}\to\mathcal{G}}(t)$.

Proof. Using (A-1) and $\Pi_t^2 = \Pi_t$ we obtain (i). This implies

$$\begin{aligned}\Pi_t \circ \Pi \circ \Pi_t &= (\Pi + tP + o(t))\Pi(\Pi + tP + o(t)) \\ &= \Pi + t(P\Pi + \Pi P) + o(t) = \Pi + tP + o(t) = \Pi_t + o(t),\end{aligned}$$

where all the $o(t)$ are taken in $\mathcal{L}(\mathcal{H}, \mathcal{G})$. Therefore $Q_t \circ \Pi_t \circ \Pi \circ \Pi_t = Q_t \Pi_t + o(t)$. Since $Q_t \circ \Pi_t \circ \Pi = \Pi$ by definition, one obtains

$$Q_t \circ \Pi_t = \Pi \circ \Pi_t + o(t),$$

which concludes the proof of the lemma. □

Lemma A.2. *Let* A_t, $|t| \leqslant \delta$, *be a* $\mathcal{C}^1$ *family of bounded operators* $\mathcal{G} \to \mathcal{H}$ *such that* A_t *commutes with* Π_t *for every* t. *Let* $A = A_0$. *Then* $t \mapsto \operatorname{tr}_{C_t}(A_t)$ *is real differentiable at* $t = 0$ *and*

$$\left.\frac{d}{dt}\right|_{t=0} \operatorname{tr}_{C_t}(A_t) = \operatorname{tr}_C(\Pi \dot{A}),$$

where $\dot{A}_t = (d/dt)A_t$. *If moreover* A *is invertible on* C, *we have*

$$\left.\frac{d}{dt}\right|_{t=0} \log \det_{C_t}(A_t) = \operatorname{tr}_C(\Pi \dot{A}(A|_C)^{-1}).$$

Proof. We start from

$$\operatorname{tr}_{C_t}(A_t) = \operatorname{tr}_C(Q_t A_t \Pi_t).$$

Now since A_t commutes with Π_t we have by the second part of Lemma A.1

$$\begin{aligned}Q_t A_t \Pi_t \Pi &= \Pi \Pi_t A_t \Pi + o_{C\to C}(t) \\ &= \Pi A \Pi + t\Pi(\dot{A} + PA\Pi + \Pi AP)\Pi + o_{C\to C}(t).\end{aligned}$$

But now the first part of Lemma A.1 gives $\Pi P \Pi = 0$. We therefore obtain, because A and Π commute,

$$Q_t A_t \Pi_t \Pi = \Pi A \Pi + t\Pi \dot{A} \Pi + o_{C\to C}(t), \tag{A-2}$$

which concludes the proof. □

A.2. ***Gain of regularity.*** Assume that we are given four Hilbert spaces $\mathcal{E} \subset \mathcal{F} \subset \mathcal{G} \subset \mathcal{H}$ with continuous and dense inclusions. Let Π_{t}, $|\mathrm{t}| < \delta$, be a family of finite-rank projectors on $\mathcal{H}$ which is differentiable at $\mathrm{t} = 0$ as family of bounded operators $\mathcal{G} \to \mathcal{H}$ (note that this differs from the last subsection where we had $\mathcal{H} \to \mathcal{G}$ instead), that is,

$$\Pi_{\mathrm{t}} = \Pi + \mathrm{t}P + o_{\mathcal{G}\to\mathcal{H}}(\mathrm{t}) \tag{A-3}$$

for some $P \in \mathcal{L}(\mathcal{G}, \mathcal{H})$. We will write $C_{\mathrm{t}} = \mathrm{ran}(\Pi_{\mathrm{t}}) \subset \mathcal{H}$ and $C = \mathrm{ran}(\Pi)$.

Lemma A.3. *Under the above assumptions, assume that Π_{t} is bounded $\mathcal{E} \to \mathcal{F}$ and that Π_{t} is differentiable at $\mathrm{t} = 0$ as a family of $\mathcal{L}(\mathcal{E}, \mathcal{F})$. Assume also that $\operatorname{rank} \Pi_{\mathrm{t}}$ does not depend on t. Then P is actually bounded $\mathcal{G} \to \mathcal{F}$ and*

$$\Pi_{\mathrm{t}} = \Pi + \mathrm{t}P + o_{\mathcal{G}\to\mathcal{F}}(\mathrm{t}).$$

Proof. Because $\mathcal{E}$ is dense in $\mathcal{H}$ we know that $C \subset \mathcal{F}$. There exists $\varphi^1, \dots, \varphi^m \in \mathcal{E}$ such that $\varphi^1_{\mathrm{t}}, \dots, \varphi^m_{\mathrm{t}}$ is a basis of C_{t} for t small enough, where we set $\varphi^j_{\mathrm{t}} = \Pi_{\mathrm{t}}(\varphi^j) \in \mathcal{F}$. Let $\tilde{\varphi}^j_{\mathrm{t}} = \Pi(\varphi^j_{\mathrm{t}}) \in C$. The family $\mathrm{t} \mapsto \tilde{\varphi}^j_{\mathrm{t}} \in C$ is differentiable at $\mathrm{t} = 0$. Let $\nu^1_{\mathrm{t}}, \dots, \nu^m_{\mathrm{t}} \in C^*$ be the dual basis of $\tilde{\varphi}^1_{\mathrm{t}}, \dots, \tilde{\varphi}^m_{\mathrm{t}}$. Because C is finite-dimensional, Π is actually bounded $\mathcal{H} \to \mathcal{F}$. As a consequence the map

$$\mathrm{t} \mapsto \ell^j_{\mathrm{t}} = \nu^j_{\mathrm{t}} \circ \Pi \circ \Pi_{\mathrm{t}} \in \mathcal{G}'$$

is differentiable at $\mathrm{t} = 0$. Noting that

$$\Pi_{\mathrm{t}} = \sum_{j=1}^{m} \varphi^j_{\mathrm{t}} \otimes \ell^j_{\mathrm{t}} : \mathcal{G} \to \mathcal{F},$$

we finally obtain that $\mathrm{t} \mapsto \Pi_{\mathrm{t}} \in \mathcal{L}(\mathcal{G}, \mathcal{F})$ is differentiable at $\mathrm{t} = 0$. □

Appendix B: Continuity of the Pollicott–Ruelle spectrum

In this appendix, we describe the spaces used in Sections 7 and 8; everything in this appendix is more or less folklore, but we chose to provide a short summary of the results that we use in the main body of the article, because we did not find any satisfying presentation in the literature. In what follows, M is a compact manifold, (E, ∇) is a flat vector bundle on M and X_0 is a vector field on M generating an Anosov flow; see Section 5.1. We denote by $T^*M = E^*_{u,0} \oplus E^*_{s,0} \oplus E^*_{0,0}$ its Anosov decomposition of T^*M.

B.1. ***Bonthonneau's uniform weight function.*** We state here [Bonthonneau 2020, Lemma 3]. This gives us an escape function having uniform good properties for a family of vector fields. A consequence is that one can define some uniform anisotropic Sobolev spaces on which each vector field of the family has good spectral properties. In what follows, $|\cdot|$ is a smooth norm on T^*M.

Lemma B.1. *There exist conical neighborhoods N_u and N_s of $E^*_{u,0}$ and $E^*_{s,0}$, some constants $C, \beta, T, \eta > 0$, and a weight function $m \in C^\infty(T^*M, [0, 1])$ such that the following hold. Let X be any vector field satisfying $\|X - X_0\|_{C^1} < \eta$, and denote by Φ^t its induced flow on T^*M and by E^*_u and E^*_s its (dual) unstable and stable bundles. Then:*

(1) *For* $E_\bullet^* \subset N_\bullet$, *for* $\bullet = s, u$ *and for any* $t > 0$, $\xi_u \in E_u^*$ *and* $\xi_s \in E_s^*$, *one has*

$$|\Phi^t(\xi_u)| \geqslant \frac{1}{C} e^{\beta t} |\xi_u|, \quad |\Phi^{-t}(\xi_s)| \geqslant \frac{1}{C} e^{\beta t} |\xi_s|.$$

(2) *For every* $t \geqslant T$ *it holds*

$$\Phi^t(\complement N_s \cap X^\perp) \subset N_u, \quad \Phi^{-t}(\complement N_u \cap X^\perp) \subset N_s,$$

where $X^\perp = \{\xi \in T^*M : \xi \cdot X = 0\}$.

(3) *If* $\boldsymbol{X}$ *is the Lie derivative induced by* Φ^t, *then*

$$m \equiv 1 \ \text{near} \ N_s, \quad m \equiv -1 \ \text{near} \ N_u, \quad \boldsymbol{X}.m \geqslant 0.$$

B.2. *Anisotropic Sobolev spaces.* Take the weight function m of Lemma B.1. Define the escape function g by

$$g(x,\xi) = m(x,\xi)\log(1+|\xi|), \quad (x,\xi) \in T^*M.$$

We set $G = \mathrm{Op}(g) \in \Psi^{0+}(M)$ for any quantization procedure Op. Then by [Zworski 2012, §8.3, 9.3, 14.2] we have $\exp(\pm\mu G) \in \Psi^{\mu+}(M)$ for any $\mu > 0$. For any $\mu > 0$ and $j \in \mathbb{Z}$ we define the spaces

$$\mathcal{H}^\bullet_{\mu G, j} = \exp(-\mu G) H^j(M, \Lambda^\bullet \otimes E) \subset \mathcal{D}'^\bullet(M, E),$$

where $H^j(M, \Lambda^\bullet \otimes E)$ is the usual Sobolev space of order j on M with values in the bundle $\Lambda^\bullet \otimes E$. Note that any pseudodifferential operator of order m is bounded $\mathcal{H}^\bullet_{\mu G, j} \to \mathcal{H}^\bullet_{\mu G, j-m}$ for any μ, m, j.

B.3. *Uniform parametrices.* Let us consider a smooth family of vector fields X_{t}, $|\mathrm{t}| < \varepsilon$, perturbing X_0. For any $c, \rho > 0$ we will set

$$\Omega(c,\rho) = \{\mathrm{Re}(s) > c\} \cup \{|s| \leqslant \rho\} \subset \mathbb{C}.$$

The spaces defined in the last subsection yield a uniform version of [Dyatlov and Zworski 2016, Proposition 3.4], as follows.

Proposition B.2 [Bonthonneau 2020, Lemma 9]. *Let* Q *be a pseudodifferential operator microlocally supported near the zero section in* T^*M *and elliptic there. There exists* $c, \varepsilon_0 > 0$ *such that, for any* $\rho > 0$ *and* $J \in \mathbb{N}$, *there is* $\mu_0, h_0 > 0$ *such that the following holds. For each* $\mu \geqslant \mu_0$, $0 < h < h_0$, $j \in \mathbb{Z}$ *such that* $|j| \leqslant J$ *and* $s \in \Omega(c,\rho)$ *the operator*

$$\mathcal{L}^\nabla_{X_{\mathrm{t}}} - h^{-1}Q + s : \mathcal{H}^\bullet_{\mu G, j+1} \to \mathcal{H}^\bullet_{\mu G, j}$$

is invertible for $|\mathrm{t}| \leqslant \varepsilon_0$ *and the inverse is bounded* $\mathcal{H}^\bullet_{\mu G, j} \to \mathcal{H}^\bullet_{\mu G, j}$ *independently of* t.

B.4. *Continuity of the Pollicott–Ruelle spectrum.* We fix ρ, $J \geqslant 4$ and μ_0, μ, h_0, h, j as in Proposition B.2. We first observe that

$$(\mathcal{L}^\nabla_{X_{\mathrm{t}}} + s)(\mathcal{L}^\nabla_{X_{\mathrm{t}}} - h^{-1}Q + s)^{-1} = \mathrm{Id} + h^{-1}Q(\mathcal{L}^\nabla_{X_{\mathrm{t}}} - h^{-1}Q + s)^{-1}. \tag{B-1}$$

Since Q is supported near 0 in T^*M, it is smoothing and thus trace class on any $\mathcal{H}^\bullet_{\mu G,j}$. By analytic Fredholm theory, the family $s \mapsto K(\mathrm{t}, s) = h^{-1}Q(\mathcal{L}^\nabla_{X_\mathrm{t}} - h^{-1}Q + s)^{-1}$ is a holomorphic family of trace class operators on $\mathcal{H}^\bullet_{\mu G,j}$ in the region $\Omega(c, \rho)$. We can therefore consider the Fredholm determinant

$$D(\mathrm{t}, s) = \det_{\mathcal{H}^\bullet_{\mu G,j}}(\mathrm{Id} + K(\mathrm{t}, s)).$$

It follows from [Simon 2005, Corollary 2.5] that for each t, $s \mapsto D(\mathrm{t}, s)$ is holomorphic on $\Omega(c, \rho)$. Moreover (B-1) shows that its zeros coincide, on $\Omega(c, \rho)$, with the Pollicott–Ruelle resonances of $\mathcal{L}^\nabla_{X_\mathrm{t}}$. In addition, we have, for any $s \in \Omega(c, \rho)$,

$$(\mathcal{L}^\nabla_{X_\mathrm{t}} - h^{-1}Q + s)^{-1} - (\mathcal{L}^\nabla_{X_{\mathrm{t}'}} - h^{-1}Q + s)^{-1} = -(\mathcal{L}^\nabla_{X_\mathrm{t}} - h^{-1}Q + s)^{-1}(\mathcal{L}^\nabla_{X_\mathrm{t}} - \mathcal{L}^\nabla_{X_{\mathrm{t}'}})(\mathcal{L}^\nabla_{X_{\mathrm{t}'}} - h^{-1}Q + s)^{-1}. \quad \text{(B-2)}$$

We have

$$\frac{\mathcal{L}^\nabla_{X_\mathrm{t}} - \mathcal{L}^\nabla_{X_{\mathrm{t}'}}}{\mathrm{t} - \mathrm{t}'} \xrightarrow[\mathrm{t} \to \mathrm{t}']{} \mathcal{L}^\nabla_{\dot X_\mathrm{t}} \quad \text{in } \mathcal{L}(\mathcal{H}^\bullet_{\mu G,j+1}, \mathcal{H}^\bullet_{\mu G,j}), \tag{B-3}$$

where $\dot X_\mathrm{t} = (\mathrm{d}/\mathrm{dt})X_\mathrm{t}$ and $\mathcal{L}(\mathcal{H}^\bullet_{\mu G,j+1}, \mathcal{H}^\bullet_{\mu G,j})$ is the space of bounded linear operators $\mathcal{H}^\bullet_{\mu G,j+1} \to \mathcal{H}^\bullet_{\mu G,j}$ endowed with the operator norm. We therefore obtain by Proposition B.2 and because Q is smoothing (and thus trace class $\mathcal{H}^\bullet_{\mu G,j} \to \mathcal{H}^\bullet_{\mu G,j'}$ for any μ, j, j') that $K(\mathrm{t}', s) \to K(\mathrm{t}, s)$ as $\mathrm{t}' \to \mathrm{t}$ in $\mathcal{L}^1(\mathcal{H}^\bullet_{\mu G,0})$ locally uniformly in s, where $\mathcal{L}^1(\mathcal{H}^\bullet_{\mu G,0})$ is the space of trace class operators on $\mathcal{H}^\bullet_{\mu G,0}$ endowed with its usual norm. As a consequence, we obtain with [Simon 2005, Corollary 2.5]

$$D(\mathrm{t}, s) \in \mathcal{C}^0\big([-\varepsilon_0, \varepsilon_0]_\mathrm{t}, \mathrm{Hol}(\Omega(c, \rho)_s)\big). \tag{B-4}$$

B.5. *Regularity of the resolvent.* Let $\mathcal{Z}$ be an open set of $\mathbb{C}$ whose closure is contained in the interior of $\Omega(c, \rho)$. We assume that $\bar{\mathcal{Z}} \cap \mathrm{Res}(\mathcal{L}^\nabla_{X_0}) = \varnothing$. Up to taking ε_0 smaller, Rouché's theorem and (B-4) imply that there exists $\delta > 0$ such that $\mathrm{dist}(\mathcal{Z}, \mathrm{Res}(\mathcal{L}^\nabla_{X_\mathrm{t}})) > \delta$ for any $|\mathrm{t}| \leqslant \varepsilon_0$. As a consequence, we obtain that, for every $|j| \leqslant J$, the map $(\mathcal{L}^\nabla_{X_\mathrm{t}} + s)^{-1} : \mathcal{H}^\bullet_{\mu G,j} \to \mathcal{H}^\bullet_{\mu G,j}$ is bounded independently of $(\mathrm{t}, s) \in [-\varepsilon_0, \varepsilon_0] \times \mathcal{Z}$. Noting that

$$\frac{(\mathcal{L}^\nabla_{X_\mathrm{t}} + s)^{-1} - (\mathcal{L}^\nabla_{X_{\mathrm{t}'}} + s)^{-1}}{\mathrm{t} - \mathrm{t}'} = -(\mathcal{L}^\nabla_{X_\mathrm{t}} + s)^{-1} \frac{\mathcal{L}^\nabla_{X_\mathrm{t}} - \mathcal{L}^\nabla_{X_{\mathrm{t}'}}}{\mathrm{t} - \mathrm{t}'} (\mathcal{L}^\nabla_{X_{\mathrm{t}'}} + s)^{-1}, \tag{B-5}$$

we obtain by (B-3) that $\mathrm{t}' \mapsto (\mathcal{L}^\nabla_{X_{\mathrm{t}'}} + s)^{-1}$ is continuous in $\mathcal{L}(\mathcal{H}^\bullet_{\mu G,j+1}, \mathcal{H}^\bullet_{\mu G,j})$. Therefore, applying (B-5) again, we get that

$$(\mathcal{L}^\nabla_{X_\mathrm{t}} + s)^{-1} \in \mathcal{C}^1\big([-\varepsilon_0, \varepsilon_0]_\mathrm{t}, \mathrm{Hol}(\mathcal{Z}_s, \mathcal{L}(\mathcal{H}^\bullet_{\mu G,j+1}, \mathcal{H}^\bullet_{\mu G,j-2}))\big). \tag{B-6}$$

Note that here we need $|j-2|, |j+1| \leqslant J$.

B.6. *Regularity of the spectral projectors.* Let $0 < \lambda < 1$ such that $\{|s| = \lambda\} \cap \mathrm{Res}(\mathcal{L}^\nabla_{X_0}) = \varnothing$. Applying the last subsection with $\mathcal{Z} = \{|s| = \lambda\}$, we get $\{|s| = \lambda\} \cap \mathrm{Res}(\mathcal{L}^\nabla_{X_\mathrm{t}}) = \varnothing$ for any $|\mathrm{t}| \leqslant \varepsilon_0$. We can therefore define for those t

$$\Pi_\mathrm{t} = \frac{1}{2\pi i} \int_{|s|=\lambda} (\mathcal{L}^\nabla_{X_\mathrm{t}} + s)^{-1} \, \mathrm{d}s : \mathcal{H}^\bullet_{\mu G,j} \to \mathcal{H}^\bullet_{\mu G,j}.$$

Then (B-6) gives that $\Pi_t \in \mathcal{C}^1([-\varepsilon_0, \varepsilon_0]_t, \mathcal{Z}_s, \mathcal{L}(\mathcal{H}^\bullet_{\mu G, j+1}, \mathcal{H}^\bullet_{\mu G, j-2}))$. This is true for $j = 3$ and $j = -1$ because $J \geqslant 4$. Moreover by Rouché's theorem, the number m of zeros of $s \mapsto D(t, s)$ does not depend on t. Noting that

$$\partial_s K(t, s)(1 + K(t, s))^{-1} = -K(t, s)(\mathcal{L}^\nabla_{X_t} - h^{-1}Q + s)^{-1}(1 + K(t, s))^{-1},$$

we obtain by [Dyatlov and Zworski 2019, Theorem C.11] and the cyclicity of the trace that m is equal to

$$\begin{aligned}\frac{1}{2\pi i}\operatorname{tr}\int_{|s|=\lambda} \partial_s K(t, s)(1 + K(t, s))^{-1}\,ds &= -\frac{1}{2\pi i}\operatorname{tr}\int_{|s|=\lambda} (\mathcal{L}^\nabla_{X_t} - h^{-1}Q + s)^{-1}(1 + K(t, s))^{-1}K(t, s)\,ds \\ &= \frac{1}{2\pi i}\operatorname{tr}\int_{|s|=\lambda} (\mathcal{L}^\nabla_{X_t} - h^{-1}Q + s)^{-1}(1 + K(t, s))^{-1},\end{aligned}$$

where we used that $s \mapsto (\mathcal{L}^\nabla_{X_t} - h^{-1}Q + s)^{-1}$ is holomorphic on $\{|s| \leqslant \lambda\}$. The last integral is equal to $\operatorname{tr}\Pi_t = \operatorname{rank}\Pi_t$ by (B-1). As a consequence we can apply Lemma A.3 to obtain that

$$\Pi_t \in \mathcal{C}^1\big([-\varepsilon_0, \varepsilon_0]_t, \mathcal{L}(\mathcal{H}^\bullet_{\mu G,0}, \mathcal{H}^\bullet_{\mu G,1})\big). \tag{B-7}$$

Appendix C: The wavefront set of the Morse–Smale resolvent

The purpose of this section is to prove the wavefront bound needed to conclude the proof of Proposition 11.2. For simplicity we prove it for $\widetilde{X}$ instead of $-\widetilde{X}$. We will denote by $\widehat{\Pi}$ the spectral projector (10-4) for the trivial bundle $(\mathbb{C}, d)$. Recall that $\mathcal{D}'_\Gamma(M \times M)$ denotes distributions whose wavefront set is contained in the closed conic set $\Gamma \subset T^\bullet(M \times M)$. A family $(f_t)_{t\geqslant 0}$ of distributions will be $\mathcal{O}_{\mathcal{D}'_\Gamma}(1)$ if it is bounded in $\mathcal{D}'_\Gamma$ in the sense of [Dang 2013, p. 31]. We will need the following:

Lemma C.1. *Let $\varepsilon > 0$ and $a \in \operatorname{Crit}(f)$. There exists $c > 0$, a closed conic set $\Gamma \subset T^*(M \times M)$ with $\Gamma \cap N^*\Delta(T^*M) = \varnothing$ and $\chi \in \mathcal{C}^\infty(M, [0, 1])$ such that $\chi \equiv 1$ near a such that*

$$\mathcal{K}_{\chi,t+\varepsilon} = \mathcal{O}_{\mathcal{D}'^n_\Gamma(M\times M)}(e^{-tc}),$$

where, for $t \geqslant 0$, $\mathcal{K}_{\chi,t}$ is the Schwartz kernel of the operator $\chi e^{-t\mathcal{L}_{\widetilde{X}}}(\mathrm{Id} - \widehat{\Pi})\chi$.

Proof. Because $\widetilde{X}$ is $\mathcal{C}^\infty$-linearizable, we can take $U \subset \mathbb{R}^n$ to be a coordinate patch centered in a so that, in those coordinates, $e^{-t\widetilde{X}}(x) = e^{-tA}(x)$, where A is a matrix whose eigenvalues have nonvanishing real parts. Denoting $(x^1, \dots, x^n)$ the coordinates of the patch, $\widetilde{X}$ reads

$$\widetilde{X} = \sum_{1\leqslant i,j\leqslant n} A^j_i x^i \partial_j.$$

We have a decomposition $\mathbb{R}^n = W^u \oplus W^s$ stable by A such that $A|_{W^u}$ (resp. $A|_{W^s}$) have eigenvalues with positive (resp. negative) real parts, $d_{u/s} = \dim W^{u/s}$, this induces a decomposition of the coordinates $x = (x_s, x_u)$. We will denote by $A_u = A|_{W^u} \oplus 0_{W^s}$, $A_s = 0_{W^u} \oplus A|_{W^s}$ and $c > 0$ such that

$$c < \inf_{\lambda\in \operatorname{sp}(A)} |\operatorname{Re}(\lambda)|,$$

where $\operatorname{sp}(A)$ is the spectrum of A.

Let $\chi_1, \chi_2 \in \Omega^\bullet(M)$ such that $\operatorname{supp} \chi_i \subset \operatorname{supp} \chi$ for $i = 1, 2$. For simplicity, we identify e^{-tA} and its action on differential forms and currents given by the pull-back, $\delta^d(x)$ denotes the Dirac δ distribution at $0 \in \mathbb{R}^d$, π_1, π_2 are the projections $M \times M \mapsto M$ on the first and second factors, respectively.

$$\begin{aligned}\langle \mathcal{K}_{\chi,t}, \pi_1^*\chi_1 \wedge \pi_2^*\chi_2\rangle &= \langle \chi_2, e^{-tA}(\mathrm{Id} - \widehat{\Pi})\chi_1\rangle \\ &= \left\langle \chi_2, e^{-tA}\left(\chi_1 - \delta^{d_u}(x_u)\,\mathrm{d}x_u \int_{W^s} \pi_{s,0}^*\chi_1\right)\right\rangle \\ &= \langle e^{tA_s}\chi_2, e^{-tA_u}\chi_1\rangle - \left(\int_{W^u} \pi_{u,0}^*\chi_2\right)\left(\int_{W^s} \pi_{s,0}^*\chi_1\right) \\ &= \int_0^1 \int_U \partial_\tau(e^{tA_s}\pi_{u,\tau}^*\chi_2 \wedge e^{-tA_u}\pi_{s,\tau}^*\chi_1)\,\mathrm{d}\tau,\end{aligned}$$

where $\pi_{u,\tau}, \pi_{s,\tau} : U \to U$ are defined by $\pi_{u,\tau}(x_u, x_s) = (x_u, \tau x_s)$ and $\pi_{s,\tau}(x_u, x_s) = (\tau x_u, x_s)$. Now write $\chi_2 = \sum_{|I|=k} \beta_I \,\mathrm{d}x_s^{I_s} \wedge \mathrm{d}x_u^{I_u}$. We have

$$\begin{aligned}\partial_\tau \pi_{u,\tau}^*\chi_2(x_u, x_s) &= \partial_\tau \sum_I \tau^{|I_s|}\beta_I(x_u, \tau x_s)\,\mathrm{d}x_u^{I_u} \wedge \mathrm{d}x_s^{I_s} \\ &= \sum_I |I_s|\tau^{|I_s|-1}\beta_I(x_u, \tau x_s)\,\mathrm{d}x_u^{I_u} \wedge \mathrm{d}x_s^{I_s} + \sum_I \tau^{|I_s|}(\partial_{x_s}\beta_I)_{(x_u,\tau x_s)}(x_s)\,\mathrm{d}x_u^{I_u} \wedge \mathrm{d}x_s^{I_s}.\end{aligned}$$

Therefore

$$\partial_\tau e^{tA_s}\pi_{u,\tau}^*\chi_2 = \sum_I \big(|I_s|\tau^{|I_s|-1}\beta_I(x_u, \tau e^{tA_s}x_s) + \tau^{|I_s|}(\partial_{x_s}\beta_I)_{(x_u,\tau x_s)}(e^{tA_s}x_s)\big)e^{tA_s}\,\mathrm{d}x^I.$$

Because $|e^{tA_s}x_s| = \mathcal{O}(e^{-tc})$ and $e^{tA_s}\,\mathrm{d}x^I = \mathcal{O}(e^{-ct|I_s|})$, $I = (I_s, I_u)$ is a multi-index and repeating the same argument for $\partial_\tau e^{-tA_u}\pi_{s,\tau}^*\chi_1$, we obtain the bound

$$\partial_\tau(e^{tA_s}\pi_{u,\tau}^*\chi_2 \wedge e^{-tA_u}\pi_{s,\tau}^*\chi_1) = \mathcal{O}_{\chi_1,\chi_2}(e^{-tc}). \tag{C-1}$$

Replacing χ_1 and χ_2 by $\chi_1 e^{i\langle\xi,\cdot\rangle}$ and $\chi_2 e^{i\langle\eta,\cdot\rangle}$ with $\xi, \eta \in \mathbb{R}^n$, one gets

$$\begin{aligned}&\langle \mathcal{K}_{\chi,t}, \pi_1^*(\chi_1 e^{i\langle\xi,\cdot\rangle}) \wedge \pi_2^*(\chi_2 e^{i\langle\eta,\cdot\rangle})\rangle \\ &\quad= \int_0^1\int_U \partial_\tau(e^{tA_s}\pi_{u,\tau}^*\chi_2 \wedge e^{-tA_u}\pi_{s,\tau}^*\chi_1)e^{i\langle e^{tA_s}(x_u,\tau x_s),\eta\rangle}e^{i\langle e^{-tA_u}(\tau x_u,x_s),\xi\rangle}\,\mathrm{d}\tau \\ &\qquad+ \int_0^1\int_U e^{tA_s}\pi_{u,\tau}^*\chi_2 \wedge e^{-tA_u}\pi_{s,\tau}^*\chi_1\partial_\tau(e^{i\langle e^{tA_s}(x_u,\tau x_s),\eta\rangle}e^{i\langle e^{-tA_u}(\tau x_u,x_s),\xi\rangle})\,\mathrm{d}\tau.\end{aligned}$$

Setting $g(\tau, x_u, x_s) = e^{i\langle e^{tA_s}(x_u,\tau x_s),\eta\rangle}e^{i\langle e^{-tA_u}(\tau x_u,x_s),\xi\rangle}$, we have

$$\partial_\tau g(\tau, x_u, x_s) = i(\langle e^{tA_s}x_s, \eta_s\rangle + \langle e^{-tA_u}x_u, \xi_u\rangle)g(\tau, x_u, x_s) = \mathcal{O}_{C^\infty(M)}(e^{-tc}),$$

because $|e^{tA_s}x_s|, |e^{-tA_u}x_u| = \mathcal{O}(e^{-tc})$. Repeating the process that led to (C-1) but for derivatives of χ_1, χ_2 as test forms with successive integration by parts, we therefore obtain for any $N \in \mathbb{N}$

$$\begin{aligned}&|\langle \mathcal{K}_{\chi,t}, \pi_1^*(\chi_1 e^{i\langle\xi_1,\cdot\rangle}) \wedge \pi_2^*(\chi_2 e^{i\langle\xi_2,\cdot\rangle})\rangle| \\ &\qquad\leqslant C_{N,\chi_1,\chi_2}e^{-tc}(1 + |e^{tA_s}\eta_s| + |e^{-tA_u}\xi_u|)\int_0^1 (1 + |\tau e^{tA_s}\eta_s + \xi_s| + |\tau e^{-tA_u}\xi_u + \eta_u|)^{-N}\,\mathrm{d}\tau,\end{aligned}$$

where $\xi = (\xi_u, \xi_s)$ and $\eta = (\eta_u, \eta_s)$. Now assume (ξ, η) is close to $N^*\Delta(T^*M)$, say

$$\left| \frac{\xi}{|\xi|} + \frac{\eta}{|\eta|} \right| < \nu \quad \text{and} \quad 1 - \nu < \frac{|\xi|}{|\eta|} < 1 + \nu$$

for some $\nu > 0$. Then we have for any $\tau \in [0, 1]$

$$|\tau e^{tA_s} \eta_s + \xi_s| + |\tau e^{-tA_u} \xi_u + \eta_u| \geqslant (1 - e^{-tc}(1 + \nu))(|\xi_s| + |\eta_u|).$$

As a consequence, if $\nu > 0$ is small enough so that $(1 + \nu)e^{-(t+\varepsilon)c} < 1$, for every $t \geqslant 0$, we obtain

$$|\langle \mathcal{K}_{\chi, t+\varepsilon}, \pi_1^*(\chi_1 e^{i\langle \xi, \cdot \rangle}) \wedge \pi_2^*(\chi_2 e^{i\langle \eta, \cdot \rangle}) \rangle| \leqslant C'_{N, \chi_1, \chi_2}(1 + |\xi| + |\eta|)^{-N},$$

which concludes. □

To conclude the proof of Proposition 11.2, we also need to prove Lemma 11.3:

Proof of Lemma 11.3. We proceed by contradiction. Suppose that there is $\mu > 0$ and sequences $x_m \in M$ and $t_m \geqslant \varepsilon$ such that $\mathrm{dist}(x_m, e^{-t_m \widetilde{X}}(x_m)) \to 0$ as $m \to \infty$ and $\mathrm{dist}(x_m, \mathrm{Crit}(f)) \geqslant \mu$. Extracting a subsequence we may assume that $x_m \to x$, $t_m \to \infty$ (indeed if $t_m \to t_\infty < \infty$ then x is a periodic point for $\widetilde{X}$, which does not exist) and, for any m,

$$e^{-t\widetilde{X}}(x_m) \to a \quad \text{and} \quad e^{t\widetilde{X}}(x_m) \to b \quad \text{as } t \to \infty,$$

for some $a, b \in \mathrm{Crit}(f)$. Since the space of broken curves $\bar{\mathcal{L}}(a, b)$ is compact (see [Audin and Damian 2014]), we may assume that the sequence of curves $\gamma_m = \{e^{t\widetilde{X}}(x_m) : t \in \mathbb{R}\}$ converges to a broken curve $\ell = (\ell^1, \dots, \ell^q) \in \bar{\mathcal{L}}(a, b)$, with $\ell^j \in \mathcal{L}(c_{j-1}, c_j)$ for some $c_0, \dots, c_q \in \mathrm{Crit}(f)$, with $c_0 = a$ and $c_q = b$. Because $x_m \to x$, the proof of [Audin and Damian 2014, Theorem 3.2.2] implies $x \in \ell^j$ for some j so that $e^{-t\widetilde{X}} x \to c_{j-1}$ as $t \to \infty$. Therefore replacing x by $e^{-t\widetilde{X}}(x)$ for t big enough, we may assume that x is contained in a Morse chart $\Omega(c_{j-1})$ near c_{j-1}. Then $c_{j-1} \neq a$. Indeed if it was not the case then we would have $e^{-t_m \widetilde{X}} x_m \to a$ as $m \to \infty$ (since x_m would be contained in $\Omega(a) \cap W^u(a)$ for big enough m and $t_m \to \infty$), which is not the case since $\mathrm{dist}(x, \mathrm{Crit}(f)) \geqslant \mu \implies x \neq a$ and $\mathrm{dist}(x_m, e^{-t_m \widetilde{X}}(x_m)) \to 0$ as $m \to \infty$. Therefore the flow line of x_m exists $\Omega(c_{j-1})$ in the past. We therefore obtain, since $e^{-t_m \widetilde{X}} x_m \to x$, that there is $i < j - 1$ so that $c_i = c_{j-1}$. This is absurd since the sequence $(\mathrm{ind}_f(c_i))_{i=0,\dots,q}$ is strictly decreasing. □

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MEASURE PROPAGATION ALONG A $\mathscr{C}^0$-VECTOR FIELD AND WAVE CONTROLLABILITY ON A ROUGH COMPACT MANIFOLD

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The celebrated Rauch–Taylor/Bardos–Lebeau–Rauch geometric control condition is central in the study of the observability of the wave equation linking this property to high-frequency propagation along geodesics that are the rays of geometric optics. This connection is best understood through the propagation properties of microlocal defect measures that appear as solutions to the wave equation concentrate. For a sufficiently smooth metric this propagation occurs along the bicharacteristic flow. If one considers a merely $\mathscr{C}^1$-metric, this bicharacteristic flow may however not exist. The Hamiltonian vector field is only continuous; bicharacteristics do exist (as integral curves of this continuous vector field) but uniqueness is lost. Here, on a compact manifold without boundary, we consider this low-regularity setting, revisit the geometric control condition, and address the question of support propagation for a measure solution to an ODE with continuous coefficients. This leads to a sufficient condition for the observability and equivalently the exact controllability of the wave equation. Moreover, we investigate the stability of the observability property and the sensitivity of the control process under a perturbation of the metric of regularity as low as Lipschitz.

1. Introduction

The observability property for the wave equation has been intensively studied during the last decades mainly because of its deep connection with the problem of exact controllability. Until the end of the 80s, most of the positive results of observability were established under a (global) geometric assumption, the so-called Γ-condition introduced by J.-L. Lions [1988], essentially based on and well-adapted to a multiplier method. Later, following [Rauch and Taylor 1974], Bardos, Lebeau and Rauch [Bardos et al. 1992] established boundary observability inequalities under a geometric control condition (GCC for short), linking the set on which the control acts and the generalized geodesic flow. Proofs of this result are based on microlocal tools, such as the propagation in phase space of wavefront sets in [Bardos et al. 1992] or

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the propagation of microlocal defect measures in more modern proofs [Burq and Gérard 1997]. For the latter approach, microlocal defect measures originate from the concentration phenomena for sequences of waves if one assumes that observability does not hold. Away from boundaries one obtains

$$ {}^t\mathrm{H}_p\mu = 0, \tag{1-1} $$

yielding the transport of the measure μ along the bicharacteristic flow in phase space. This flow is generated by the Hamiltonian vector field H_p associated with the symbol of the wave operator p. However, note that despite their high efficiency and robustness, these methods present the great disadvantage of requiring too much regularity in the coefficients of the wave operator and the geometry. To define the generalized bicharacteristic flow and prove the propagation properties mentioned above a minimal smoothness of the metric and the boundary domain is needed. To our knowledge, the best result, in the context of $\mathscr{C}^2$ metrics, was proven in [Burq 1997a], and barely misses the natural minimal smoothness required to define the geodesic flow ($W^{2,\infty}$) and thus the geometric control condition.

In this context, in the present article, we address the following natural question: how can one derive observability estimate for the wave equation from optimal observation regions in the case of a nonsmooth metric? This problem has already received some attention and answers by E. Zuazua and his collaborators, in [Castro and Zuazua 2002], and more recently in [Fanelli and Zuazua 2015] (see also the result of [Dehman and Ervedoza 2017]). More precisely, in [Castro and Zuazua 2002], the authors prove a lack of observability of waves in highly heterogeneous media, that is, if the density is of low regularity. In [Fanelli and Zuazua 2015], the authors establish observability with coefficients in the Zygmund class and also observability with loss when the coefficients are log-Zygmund or log-Lipschitz. Furthermore, this result is proven sharp since one observes an infinite loss of derivatives for a regularity lower than log-Lipschitz. Note that these analyses are carried out in one space dimension. This calls for the following comments. First, in this simplified framework, for smooth coefficients all the geodesics reach the observability region in uniform time: captive geodesics are not an issue. Second, proofs are based on a sidewise energy estimate, a technique that is specific to the one-dimensional setting; the underlying idea consists of exchanging the roles of the time and space variables and, finally in proving hyperbolic energy estimates for waves with rough coefficients. Unfortunately, such a method does not extend to higher space dimensions. Furthermore, for the low regularity considered in these articles, the geodesic flow is not well-defined. Proving propagation results for wavefront sets or microlocal defect measure appears quite out of reach in such cases.

The present work is the first in a series of three articles devoted to the question of observability (and equivalently exact controllability) of wave equations with nonsmooth coefficients. Here, we initiate this study on a compact Riemannian manifold with a rough metric, yet *without boundary*, while the two forthcoming articles will present the counterpart analysis on manifolds with boundary (or bounded domains of $\mathbb{R}^d$) [Burq et al. 2024a; 2024b]. The presence of a boundary yields a much more involved analysis and in [Burq et al. 2024a; 2024b] we develop Melrose–Sjöstrand generalized propagation theory in a low-regularity framework. In the present article, our main result is the observability of the wave equation with a $\mathscr{C}^1$-metric, completed with the stability of the observability property for small Lipschitz ($W^{1,\infty}$) perturbations of the metric. More precisely, we first show that if the geometric control condition in time T holds for geodesics associated with a $\mathscr{C}^1$-metric g, then the observability property holds for the wave equation, and equivalently exact controllability. For this low-regularity case one has to carefully consider the meaning of the geometric condition (or more generally the meaning of a geodesic) since the metric does not define a natural geodesic flow: geodesics are not uniquely defined. Only their existence is guaranteed. Second, we consider a reference $\mathscr{C}^1$-metric g^0 as above and we prove that observability also holds for any Lipschitz metric g chosen sufficiently close to g^0 (in the Lipschitz topology). It has to be noticed that Lipschitz metrics are too rough to permit the use of microlocal tools and a direct proof of the observability property. Even worse for such a metric, the geometric control condition itself does not seem make sense (as the generating vector field is only L^∞), and we have to use a perturbation argument near the (not so) smooth $\mathscr{C}^1$ reference metric.

Following the strategy of [Burq 1997a], we argue by contradiction and we prove a propagation result for microlocal defect measures in a low-regularity setting. We prove that these measures are solutions to the ODE (1-1) with here H_p having $\mathscr{C}^0$-coefficients. Then, we deduce some general properties about their support. Namely we show that their support is a union of integral curves of the vector field. This latter step also follows from Ambrosio and Crippa's superposition principle [2014]. Yet, we give a completely different proof which is of interest since it can be extended to the case of a domain with a boundary [Burq et al. 2024a; 2024b]. We have not been able to extend the approach of [Ambrosio and Crippa 2014] to that case. To derive the ODE fulfilled by the microlocal defect measure, we heavily rely on some harmonic analysis results due to R. Coifman and Y. Meyer [1978, Proposition IV.7] that express that the commutator of a pseudodifferential operator of order 1 and a Lipschitz function is a bounded operator on L^2.

Finally, going further in the analysis, we investigate another stability property with respect to perturbations of the metric. We prove that the HUM optimal control associated with a fixed initial data is *not* stable with respect to perturbations of the metric.

1A. ***Outline.*** The article is organized as follows. In Section 1B we set up the geometric framework we shall use and in Section 1C we precisely recall the equivalence of observability and exact controllability for the wave equation. In Section 1D we state the main results of the article.

In Section 2 we recall some geometric facts and the notions of pseudodifferential calculus and microlocal defect (density) measures on a manifold. In addition, using bicharacteristics we state the geometric control condition of [Bardos et al. 1992] in its classical form ($\mathscr{C}^2$-metric) and generalized form ($\mathscr{C}^1$-metric).

In Section 3 we recall what microlocal defect measures are and we show how, if associated with sequences of solutions of PDEs, their support can be estimated and how a transport ODE can be derived, in the particular context of low regularity of coefficients.

Section 4 is devoted to our proof of the support propagation for measures solutions of a ODE with $\mathscr{C}^0$-coefficients, Theorem 1.10.

In Section 5 we use the results of Section 3 and the propagation result of Theorem 1.10 to prove the observability and controllability results for the wave equation, Theorems 1.11 and 1.12.

Finally, in Section 6 we prove the results related to stability properties of the HUM control process.

1B. ***Setting and well-posedness.*** Throughout the article, we consider $\mathcal{M}$, a d-dimensional $\mathscr{C}^\infty$-compact manifold, that is, a manifold without boundary with a topology that makes it compact, equipped with a $\mathscr{C}^\infty$-atlas. We assume that the topology is also given by a Riemannian metric g, to be chosen either Lipschitz or of class $\mathscr{C}^k$ for some value of k to be made precise below.[1]

We denote by μ_g the canonical positive Riemannian density on $\mathcal{M}$, that is, the density measure associated with the density function $(\det g)^{1/2}$. We also consider a positive Lipschitz or of class $\mathscr{C}^k$-function κ and we define the density $\kappa\mu_g$.

The L^2-inner product and norm are considered with respect to this density $\kappa\mu_g$, that is,

$$(u, v)_{L^2(\mathcal{M})} = \int_{\mathcal{M}} u\bar{v}\,\kappa\mu_g, \quad \|u\|^2_{L^2(\mathcal{M})} = \int_{\mathcal{M}} |u|^2\,\kappa\mu_g. \tag{1-2}$$

We denote by $L^2V(\mathcal{M})$ the space of L^2-vector fields on $\mathcal{M}$, equipped with the norm

$$\|v\|^2_{L^2V(\mathcal{M})} = \int_{\mathcal{M}} g(v, \bar{v})\,\kappa\mu_g, \quad v \in L^2V(\mathcal{M}).$$

We recall that the Riemannian gradient and divergence are given by

$$g(\nabla_g f, v) = v(f) \quad \text{and} \quad \int_{\mathcal{M}} f \operatorname{div}_g v\mu_g = -\int_{\mathcal{M}} v(f)\,\mu_g$$

for f a function and v a vector field, yielding in local coordinates

$$(\nabla_g f)^i = \sum_{1\le j\le d} g^{ij}\partial_{x_j} f, \quad \operatorname{div}_g v = (\det g)^{-1/2} \sum_{1\le i\le d} \partial_{x_i}((\det g)^{1/2} v^i),$$

with $(g_x^{ij}) = (g_{x,ij})^{-1}$.

We introduce the elliptic operator $A = A_{\kappa,g} = \kappa^{-1}\operatorname{div}_g(\kappa\nabla_g)$, that is, in local coordinates

$$Af = \kappa^{-1}(\det g)^{-1/2} \sum_{1\le i,j\le d} \partial_{x_i}(\kappa(\det g)^{1/2} g^{ij}(x)\partial_{x_j} f).$$

Its principal symbol is simply $a(x,\xi) = -\sum_{1\le i,j\le d} g_x^{ij}\xi_i\xi_j$. Note that for $\kappa = 1$, one has $A = \Delta_g$, the Laplace–Beltrami operator associated with g on $\mathcal{M}$. Similarly to Δ_g, the operator A is unbounded on $L^2(\mathcal{M})$. With the domain $D(A) = H^2(\mathcal{M})$, one finds that A is self-adjoint, with respect to the L^2-inner product given in (1-2), and negative. Moreover, one has

$$(Au, v)_{L^2(\mathcal{M})} = -\int_{\mathcal{M}} g(\nabla_g u, \nabla_g \bar{v})\,\kappa\mu_g, \quad u \in H^2(\mathcal{M}),\ v \in H^1(\mathcal{M}).$$

Together with A we consider the wave operator $P_{\kappa,g} = \partial_t^2 - A_{\kappa,g} + m$, with $m > 0$ a constant and the equation

$$\begin{cases} P_{\kappa,g} y = f & \text{in } (0,+\infty)\times\mathcal{M}, \\ y_{|t=0} = y^0,\ \partial_t y_{|t=0} = y^1 & \text{in } \mathcal{M}. \end{cases} \tag{1-3}$$

It is well-posed in the energy space $H^1(\mathcal{M}) \oplus L^2(\mathcal{M})$.

[1] Note that despite considering $\mathscr{C}^k$ metrics with $k < \infty$, we still impose the condition that underlying manifold is smooth. This is due to our use of pseudodifferential techniques that are simple to introduce on a smooth manifold. See Section 2C.

Proposition 1.1. *Consider κ and g both of Lipschitz class. Let $(y^0, y^1) \in H^1(\mathcal{M}) \times L^2(\mathcal{M})$ and let $f \in L^2(0, T; L^2(\mathcal{M}))$ for any $T > 0$. There exists a unique*

$$y \in \mathscr{C}^0([0, +\infty); H^1(\mathcal{M})) \cap \mathscr{C}^1([0, +\infty); L^2(\mathcal{M}))$$

that is a weak solution of (1-3), that is, $y_{|t=0} = y^0$ and $\partial_t y_{|t=0} = y^1$ and

$$P_{\kappa,g} y = f \quad \text{in } \mathscr{D}'((0, +\infty) \times \mathcal{M}).$$

Remark 1.2. At this level of regularity of κ and g, the well-posedness of the wave equation is classical. For less regular coefficients we refer to [Colombini and Del Santo 2009; Colombini et al. 2013].

In what follows, for simplicity we shall consider the case $m = 1$, that is, for

$$P_{\kappa,g} = \partial_t^2 - A_{\kappa,g} + 1.$$

In this case, we denote by

$$\begin{aligned} \mathcal{E}_{\kappa,g}(y)(t) &= \tfrac{1}{2}(\|y(t)\|^2_{H^1(\mathcal{M})} + \|\partial_t y(t)\|^2_{L^2(\mathcal{M})}) \\ &= \tfrac{1}{2}(\|y(t)\|^2_{L^2(\mathcal{M})} + \|\nabla_g y(t)\|^2_{L^2 V(\mathcal{M})} + \|\partial_t y(t)\|^2_{L^2(\mathcal{M})}), \end{aligned}$$

the energy of this solution at time t. For a weak solution y of (1-3), if $f = 0$, this energy is independent of time t, that is,

$$\mathcal{E}_{\kappa,g}(y)(t) = \mathcal{E}_{\kappa,g}(y)(0) = \tfrac{1}{2}(\|y^0\|^2_{H^1(\mathcal{M})} + \|y^1\|^2_{L^2(\mathcal{M})}).$$

Remark 1.3. The equation we consider, with the constant $m > 0$, is often referred to the Klein–Gordon equation. Here, we keep the name wave equation. We choose this equation instead of the classical wave equation that corresponds to the case $m = 0$. In fact, on a compact manifold without boundary, constants are eigenfunctions of the elliptic operator $A_{\kappa,g}$ with 0 as an eigenvalue. Hence, constant functions are solutions to the wave equation and are so-called *invisible* solutions, as far as the observability property we are interested in is concerned. If one considers a manifold with boundary and say, homogeneous Dirichlet conditions, this issue becomes irrelevant. We could have dealt with the case $m = 0$ (the usual wave equation) at the cost of additional technical complications.

1C. *Exact controllability and observability.* Let ω be a nonempty open subset of $\mathcal{M}$ and $T > 0$. The notion of exact controllability for the wave equation from ω at time T is stated as follows.

Definition 1.4 (exact controllability in $H^1(\mathcal{M}) \oplus L^2(\mathcal{M})$). One says that the wave equation is exactly controllable from ω at time $T > 0$ if, for any $(y^0, y^1) \in H^1(\mathcal{M}) \times L^2(\mathcal{M})$, there exists $f \in L^2((0, T) \times \mathcal{M})$ such that the weak solution y to

$$P_{\kappa,g} y = \mathbf{1}_{(0,T)\times\omega} f, \quad (y_{|t=0}, \partial_t y_{|t=0}) = (y^0, y^1), \tag{1-4}$$

as given by Proposition 1.1, satisfies $(y, \partial_t y)_{|t=T} = (0, 0)$. The function f is called the control function or simply the control.

Observability of the wave equation from the open set ω in time T is the following notion.

Definition 1.5 (observability). One says that the wave equation is observable from ω at time T if there exists $C_{\mathrm{obs}} > 0$ such that for any $(u^0, u^1) \in H^1(\mathcal{M}) \times L^2(\mathcal{M})$ one has

$$\mathcal{E}_{\kappa,g}(u)(0) \leq C_{\mathrm{obs}} \|\mathbf{1}_{(0,T)\times\omega}\, \partial_t u\|^2_{L^2(\mathcal{L})} \tag{1-5}$$

for $u \in \mathscr{C}^0([0, T]; H^1(\mathcal{M})) \cap \mathscr{C}^1([0, T]; L^2(\mathcal{M}))$ the weak solution of $P_{\kappa,g} u = 0$ with $u_{|t=0} = u^0$ and $\partial_t u_{|t=0} = u^1$ as given by Proposition 1.1; see [Lions 1988].

Proposition 1.6. *Let ω be an open subset of $\mathcal{M}$ and $T > 0$. The wave equation is exactly controllable from ω at time T if and only if it is observable from ω at time T.*

Remark 1.7. In the case $m = 0$, the energy function is given by

$$\mathcal{E}_{\kappa,g}(u)(t) = \tfrac{1}{2}(\|\partial_t u(t)\|^2_{L^2(\mathcal{M})} + \|\nabla_g u(t)\|^2_{L^2 V(\mathcal{M})}).$$

It follows that a constant function u, a solution to the wave equation $(\partial_t^2 - A)u = 0$, has zero energy. Since $\|\mathbf{1}_{(0,T)\times\omega}\, \partial_t u\|^2_{L^2(\mathcal{L})}$ also vanishes, one sees that such solutions are invisible for an observability inequality of the form of (1-5). Possibilities to overcome this difficulty are to work in a quotient space or to change the wave operator into the Klein–Gordon operator. Here, we chose for simplicity the latter option.

1D. *Main results.* We introduce the following spaces for the coefficients (κ, g) to distinguish various levels of regularity:

$$\mathcal{X}^2(\mathcal{M}) = \{(\kappa, g) : \kappa \in \mathscr{C}^2(\mathcal{M}) \text{ and } g \text{ is a } \mathscr{C}^2\text{-metric on } \mathcal{M}\},$$
$$\mathcal{X}^1(\mathcal{M}) = \{(\kappa, g) : \kappa \in \mathscr{C}^1(\mathcal{M}) \text{ and } g \text{ is a } \mathscr{C}^1\text{-metric on } \mathcal{M}\},$$
$$\mathcal{Y}(\mathcal{M}) = \{(\kappa, g) : \kappa \in W^{1,\infty}(\mathcal{M}) \text{ and } g \text{ is a } W^{1,\infty}\text{-metric on } \mathcal{M}\}.$$

We start by recalling the controllability result known for regularity higher than or equal to $\mathscr{C}^2$, under the Rauch–Taylor geometric control condition.

Definition 1.8 (Rauch–Taylor, geometric control condition). Let g be a $\mathscr{C}^k$ metric, $k = 1$ or 2, and let ω be an open set of $\mathcal{M}$ and $T > 0$. One says that (ω, T) fulfills the geometric control condition if all maximal geodesics associated with g, traveled at speed 1, encounter ω for some time $t \in (0, T)$.

A second formulation of this geometric condition based on the dual notion of bicharacteristics is given in Section 2B below.

Theorem 1.9 (exact controllability: $\mathscr{C}^2$-regularity). *Consider $(\kappa, g) \in \mathcal{X}^2(\mathcal{M})$, ω an open subset of $\mathcal{M}$ and $T > 0$ such that (ω, T) fulfills the geometric control condition of Definition 1.8. Then, the wave equation is exactly controllable from ω at time T.*

This result was first proven by Rauch and Taylor [1974] for a smooth metric. The case $(\kappa, g) \in \mathcal{X}^2(\mathcal{M})$ was proven by the first author in [Burq 1997a]. On smooth open sets of $\mathbb{R}^d$, or equivalently on manifolds with boundary equipped with smooth (κ, g), for instance in the case of homogeneous Dirichlet boundary conditions, this result is given in the celebrated articles [Bardos et al. 1988; 1992].

In the present article, we extend the result of Theorem 1.9 to cases of rougher coefficients. Our extension is twofold: we treat the case $(\kappa, g) \in \mathcal{X}^1(\mathcal{M})$ and, we treat small perturbations in $\mathcal{Y}(\mathcal{M})$ of some $(\kappa, g) \in \mathcal{X}^1(\mathcal{M})$. Most importantly, these two results rely on the understanding of the structure of the support of a nonnegative measure subject to a homogeneous transport equation with continuous coefficients.

1D1. *Transport equation and measure support.* Let $\mathcal{O}$ be an open set of a smooth manifold. We denote by ${}^1\mathscr{D}'(\mathcal{O})$ and ${}^1\mathscr{D}'^{,0}(\mathcal{O})$ the spaces of density distributions and density Radon measures on $\mathcal{O}$.

Consider a continuous vector field X on $\mathcal{O}$ and let μ be a nonnegative measure density on $\mathcal{O}$. Assume that μ is such that ${}^tX\mu = 0$ in the sense of distributions, that is,

$$\langle {}^tX\mu, a\rangle_{{}^1\mathscr{D}'(\mathcal{O}),\mathscr{C}_c^\infty(\mathcal{O})} = \langle \mu, Xa\rangle_{{}^1\mathscr{D}'^{,0}(\mathcal{O}),\mathscr{C}_c^0(\mathcal{O})} = 0, \quad a \in \mathscr{C}_c^\infty(\mathcal{O}). \tag{1-6}$$

If X is moreover Lipschitz, one concludes that μ is invariant along the flow that X generates. However, if X is not Lipschitz, there is no such flow in general. Yet, integral curves do exist by the Cauchy–Peano theorem. The following theorem provides a structure of the support of μ.

Theorem 1.10. *Let X be a continuous vector field on $\mathcal{O}$ and μ be a nonnegative density measure on $\mathcal{O}$ that is a solution to ${}^tX\mu = 0$ in the sense of distributions. Then, the support of μ is a union of maximally extended integral curves of the vector field X.*

In other words, if $m^0 \in \mathcal{O}$ is in $\operatorname{supp}(\mu)$, then there exist an interval I in $\mathbb{R}$ with $0 \in I$ and a $\mathscr{C}^1$ curve $\gamma : I \to \mathcal{O}$ that cannot be extended such that $\gamma(0) = m^0$ and

$$\frac{d}{ds}\gamma(s) = X(\gamma(s)), \quad s \in I,$$

and $\gamma(I) \subset \operatorname{supp}(\mu)$.

Theorem 1.10 can actually be obtained as a consequence of the superposition principle of L. Ambrosio and G. Crippa [2014, Theorem 3.4]. Here, we provide an alternative proof that is of interest as it allows one to extend this measure support structure result to the case of an open set or a manifold with boundary [Burq et al. 2024b] as needed for our application to observability and controllability. Ambrosio and Crippa's proof is based on a smoothing-by-convolution argument. Extending this approach does not seem to be straightforward in the context of a boundary.

Theorem 1.10 is proven in Section 4 and its proof is independent of the other sections of the article. A reader only interested in our proof of Theorem 1.10 may thus head to Section 4 directly.

1D2. *Exact controllability results.* If $(\kappa, g) \in \mathcal{X}^2(\mathcal{M})$, $x \in \mathcal{M}$ and $v \in T_x\mathcal{M}$ there is a unique geodesic originating from x in direction v. In the case $(\kappa, g) \in \mathcal{X}^1(\mathcal{M})$ uniqueness is *lost*. Existence holds however and maximal (here global, see below) geodesics can still be defined by the Cauchy–Peano theorem. In particular, the geometric control condition of Definition 1.8 still makes sense. As announced above, our first result is the following theorem.

Theorem 1.11 (exact controllability: $\mathscr{C}^1$-regularity). *Consider $(\kappa, g) \in \mathcal{X}^1(\mathcal{M})$, ω an open subset of $\mathcal{M}$ and $T > 0$ such that (ω, T) fulfills the geometric control condition of Definition 1.8. Then, the wave equation is exactly controllable from ω at time T.*

A second result is the following *perturbation* result.

Theorem 1.12 (exact controllability: Lipschitz perturbation). *Let $(\kappa^0, g^0) \in \mathcal{X}^1(\mathcal{M})$, ω be an open subset of $\mathcal{M}$ and $T > 0$ be such that (ω, T) fulfills the geometric control condition of Definition 1.8 with respect to the metric g^0. There exists $\varepsilon > 0$ such that for any $(\kappa, g) \in \mathcal{Y}(\mathcal{M})$ satisfying*

$$\|(\kappa, g) - (\kappa^0, g^0)\|_{\mathcal{Y}(\mathcal{M})} \leq \varepsilon,$$

the wave equation associated with (κ, g) is exactly controllable by ω in time T.

Observe that Theorem 1.11 is a direct consequence of Theorem 1.12. We shall thus concentrate on this second more general result. Its proof relies on the measure support structure result of Theorem 1.10.

The sequence of Theorems 1.9, 1.11, and 1.12 calls for the following important comment. Under the assumption of Theorem 1.9, that is, $(\kappa, g) \in \mathcal{X}^2(\mathcal{M})$, there is a *geodesic flow* and the geometric condition of Definition 1.8 is actually a condition on the flow. Under the assumption of Theorem 1.11, that is, $(\kappa, g) \in \mathcal{X}^1(\mathcal{M})$, as pointed out above there is no geodesic flow in general. Yet, maximal geodesics are still well-defined and, the geometric condition of Definition 1.8 makes sense because it does not refer to a flow. However, under the assumption of Theorem 1.12, that is, $(\kappa, g) \in \mathcal{Y}(\mathcal{M})$, geodesics cannot be defined in general. No geometric condition can be formulated. Yet, Theorem 1.12 is a perturbation result and a geometric condition is expressed for a reference pair $(\kappa^0, g^0) \in \mathcal{X}^1(\mathcal{M})$ around which a (small) neighborhood in $\mathcal{Y}(\mathcal{M})$ is considered.

The following remark further emphasizes that the perturbation is to be considered around a pair $(\kappa^0, g^0) \in \mathcal{X}^1(\mathcal{M})$ for which the geometric control condition holds and not around a pair $(\kappa^0, g^0) \in \mathcal{X}^1(\mathcal{M})$ for which exact controllability (or equivalently observability) holds.

Remark 1.13 (on the perturbation result). Having both our results, geometric control for $\mathscr{C}^1$ metrics and Lipschitz stability of exact controllability around a reference metric satisfying the geometric control condition, a natural question is whether the exact controllability property is itself stable by perturbation. On the one hand, it is classical that the exact controllability property is *stable* under lower-order perturbations of the elliptic operator $A_{\kappa,g}$, but on the other hand, it is possible to show that it is *not* stable under (smooth) perturbations of the geometry or the metric.

Let us illustrate this instability property with a quite simple example. Consider the wave equation on the sphere

$$\mathbb{S}^d = \Big\{x \in \mathbb{R}^{d+1} : \sum_i x_i^2 = 1\Big\},$$

endowed with its standard metric and with control domain the open hemisphere

$$\omega = \{x \in \mathbb{S}^d : x_1 > 0\}.$$

Even though ω does not fulfill the geometric control condition of Definition 1.8 exact controllability holds for this geometry by an unpublished result by G. Lebeau (see [Lebeau 1992, Section VI.B] and [Zhu 2018] for extensions). Consider now the sphere endowed with the above standard metric, with the smaller control domain

$$\omega_\varepsilon = \{x \in \mathbb{S}^d : x_1 > \varepsilon\}$$

for some $\varepsilon > 0$. This second geometry is ε-close to the Lebeau example in the $\mathscr{C}^\infty$-topology. Yet, for all $\varepsilon > 0$, exact controllability does *not* hold, because there exists a geodesic (the equator, $\{x \in \mathbb{S}^d : x_1 = 0\}$) that does not encounter $\bar{\omega}_\varepsilon$. This shows that in Theorem 1.12, the assumption that the reference geometry should satisfy the geometric control condition *cannot* be replaced by the weaker assumption that it should satisfy the exact controllability property. This also shows that our perturbation argument will have to be performed on the actual proof that geometric control implies exact controllability and *not* on the final property itself.

1D3. *Further results on the control operator.* We finish this section with results analyzing the influence of some metric perturbations on the control process.

We introduce further levels of regularity for the coefficients by setting, for $k \in \mathbb{N} \cup \{+\infty\}$,

$$\mathcal{X}^k(\mathcal{M}) = \{(\kappa, g) : \kappa \in \mathscr{C}^k(\mathcal{M}) \text{ and } g \text{ is a } \mathscr{C}^k\text{-metric on } \mathcal{M}\}.$$

First, we consider $k \geq 2$. We recall the notation $P_{\kappa,g} = \partial_t^2 - A_{\kappa,g} + 1$ with $A_{\kappa,g} = \kappa^{-1} \operatorname{div}_g(\kappa \nabla_g)$, and we assume that $(\kappa, g) \in \mathcal{X}^k(\mathcal{M})$, and that (ω, T) satisfies the geometric control condition of Definition 1.8 for geodesics given by the metric g. Then, by Theorem 1.9, given $(y^0, y^1) \in H^1(\mathcal{M}) \times L^2(\mathcal{M})$, there exists $f \in L^2((0, T) \times \omega)$ such that the solution to (1-4) satisfies $y(T) = 0$ and $\partial_t y(T) = 0$. One can prove that among all possible control functions there is one of minimal L^2-norm. We denote by $f_{\kappa,g}^{y^0,y^1}$ this control function usually named the HUM control function; see for instance [Lions 1988]. Moreover, the map

$$H_{\kappa,g} : H^1(\mathcal{M}) \oplus L^2(\mathcal{M}) \to L^2((0, T) \times \mathcal{M}), \quad (y^0, y^1) \mapsto f_{\kappa,g}^{y^0,y^1}, \tag{1-7}$$

is continuous. Note that $f_{\kappa,g}^{y^0,y^1}$ is actually a weak solution of the wave equation with initial data in $L^2(\mathcal{M}) \times H^{-1}(\mathcal{M})$, meaning that one moreover has $f_{\kappa,g}^{y^0,y^1} \in \mathscr{C}^0([0, T], L^2(\mathcal{M}))$.

Theorem 1.14 (lack of continuity of the HUM-operator: the case $k \geq 2$). *Let $k \geq 2$ and (κ, g) as above. For any neighborhood $\mathcal{U}$ of (κ, g) in $\mathcal{X}^k(\mathcal{M})$, there exist $(\tilde{\kappa}, \tilde{g}) \in \mathcal{U}$ and an initial data $(y^0, y^1) \in H^1(\mathcal{M}) \times L^2(\mathcal{M})$, with $\|y^0\|_{H^1}^2 + \|y^1\|_{L^2}^2 = 1$, such that the respective solutions y and $\tilde{y}$ of*

$$\begin{cases} P_{\kappa,g} y = \mathbf{1}_{(0,T)\times\omega} f_{\kappa,g}^{y^0,y^1} & \text{in } (0, T) \times \mathcal{M}, \\ (y, \partial_t y)_{|t=0} = (y^0, y^1) & \text{in } \mathcal{M}, \end{cases} \qquad \begin{cases} P_{\tilde{\kappa},\tilde{g}} \tilde{y} = \mathbf{1}_{(0,T)\times\omega} f_{\kappa,g}^{y^0,y^1} & \text{in } (0, T) \times \mathcal{M}, \\ (\tilde{y}, \partial_t \tilde{y})_{|t=0} = (y^0, y^1) & \text{in } \mathcal{M} \end{cases} \tag{1-8}$$

are such that

$$\mathcal{E}_{\kappa,g}(\tilde{y} - y)(T) = \mathcal{E}_{\kappa,g}(\tilde{y})(T) \geq \tfrac{1}{2}. \tag{1-9}$$

Moreover, there exists $C_T > 0$ such that

$$\left\|(H_{\kappa,g} - H_{\tilde{\kappa},\tilde{g}})(y^0, y^1)\right\|_{L^2((0,T)\times\omega)} = \|f_{\kappa,g}^{y^0,y^1} - f_{\tilde{\kappa},\tilde{g}}^{y^0,y^1}\|_{L^2((0,T)\times\omega)} \geq C_T \tag{1-10}$$

for (y^0, y^1) as given above.

Remark 1.15. The result of Theorem 1.14 states that starting from the same initial data and solving the two wave equations with the same control vector $f_{\kappa,g}$ associated with $P_{\kappa,g}$, a small perturbation of the metric can induce a large error for the final state $(y(T), \partial_t y(T))$. In other words, the two dynamics are no longer close. In particular, the map

$$\mathcal{X}^k(\mathcal{M}) \to \mathscr{L}\big(H^1(\mathcal{M}) \oplus L^2(\mathcal{M}), L^2((0, T) \times \mathcal{M})\big), \quad (\kappa, g) \mapsto H_{\kappa,g},$$

is not continuous.

Remark 1.16. The result of Theorem 1.14 can also be stated on open bounded smooth domains of $\mathbb{R}^n$ in the case of homogeneous Dirichlet condition. In fact, as can be checked in what follows, its proof only relies on basic properties of microlocal defect measures (support localization and propagation) that are known to be valid in this framework; see [Burq 1997a].

Remark 1.17. In the statement of Theorem 1.14 if the neighborhood $\mathcal{U}$ of (κ, g) in $\mathcal{X}^k$ is small enough, the pair (ω, T) also satisfies the geometric control condition of Definition 1.8 for $(\tilde{\kappa}, \tilde{g})$ and therefore $f_{\tilde{\kappa},\tilde{g}}^{y^0,y^1}$ is well-defined. In particular, this is clear as in the case $k \geq 2$ there is a well-defined and unique geodesic flow.

The case $k = 1$ is quite different as there is no geodesic flow, as already mentioned above. However, given $(\kappa, g) \in \mathcal{X}^1$ and (ω, T) if the Rauch–Taylor geometric control condition of Definition 1.8 holds for (ω, T) for the geodesics associated with g, given any neighborhood $\mathcal{U}$ of (κ, g) in $\mathcal{X}^1$ one can still find $(\tilde{\kappa}, \tilde{g}) \in \mathcal{U}$ such that

(1) the geometric control condition still holds for the geodesics associated with $\tilde{g}$,

(2) the result of Theorem 1.14 also holds.

Theorem 1.14′ (lack of continuity of the HUM-operator: the case $k = 1$). *Let $k = 1$ and $(\kappa, g) \in \mathcal{X}^1$ as above. For any neighborhood $\mathcal{U}$ of (κ, g) in $\mathcal{X}^1(\mathcal{M})$, there exist $(\tilde{\kappa}, \tilde{g}) \in \mathcal{U}$ and an initial data $(y^0, y^1) \in H^1(\mathcal{M}) \times L^2(\mathcal{M})$, with $\|y^0\|_{H^1}^2 + \|y^1\|_{L^2}^2 = 1$, such that the geometric control condition of Definition 1.8 for geodesics given by the metric $\tilde{g}$ holds and moreover the results listed in Theorem 1.14 hold.*

The proofs of Theorems 1.14 and 1.14′ are given in Section 6A.

We finish this section with some remarks and some questions.

Remark 1.18. In all results above we have used $1_{(0,T)\times\omega}$ as a control operator, that is, the characteristic function of an open set. We could have also considered a control operator given by $1_{(0,T)}(t)\chi(x)$, with χ a smooth function on $\mathcal{M}$. The controlled wave equation then has the form

$$P_{\kappa,g} y = \mathbf{1}_{(0,T)}\, \chi\, f, \qquad (y_{|t=0}, \partial_t y_{|t=0}) = (y^0, y^1). \tag{1-11}$$

In such case, the open set to be used in the geometric control condition is $\omega = \{\chi \neq 0\}$. This is often done this way, in particular since the smoothness of the function χ allows one to use some microlocal techniques that require regularity in the operator coefficients. The results and proofs of the present article can be written mutatis mutandis for this type of control operator.

1D4. *Comparison with the smooth case and some open questions.* Following on the previous remark, with a smooth-in-space control operator, one can wonder above the smoothness of the HUM operator. This question is addressed in the joint work of the second author [Dehman and Lebeau 2009]. In fact, a gain of regularity in the initial data (y^0, y^1) yields an equivalent gain of regularity in the HUM control function $f_{\kappa,g}^{y^0,y^1}$. For instance, for $(y^0, y^1) \in H^2(\mathcal{M}) \times H^1(\mathcal{M})$ one finds $f_{\kappa,g}^{y^0,y^1} \in \mathscr{C}^0([0, T], H^1(\mathcal{M}))$. Note that the result of [Dehman and Lebeau 2009] is proven in the case of smooth coefficients, that is, $(\kappa, g) \in \mathcal{X}^\infty$. We thus consider this smooth case in the discussion that ends this introductory section. Open questions around the results of Theorems 1.14 and 1.14′ are then raised.

As we shall see in their proofs, the results of Theorems 1.14 and 1.14′ rely on the high-frequency behavior of the solutions to (1-8). In the case of smooth coefficients and a smooth control operator, if we assume smoother data (y^0, y^1) in the HUM control process, the result of Theorem 1.14 does not hold any more. The HUM control process becomes regular with respect to (κ, g) as expressed in the following proposition.

Proposition 1.19 (HUM control process for smooth data). *Consider $(\kappa, g) \in \mathcal{X}^\infty(\mathcal{M})$ and let $\chi \in \mathscr{C}^\infty(\mathcal{M})$. Set $\omega = \{\chi \neq 0\}$ and assume that (ω, T) fulfills the geometric control condition of Definition 1.8 for the geodesics associated with (κ, g). Let $\alpha \in (0, 1]$. There exists $C_\alpha > 0$ such that, for any $(\tilde\kappa, \tilde g) \in \mathcal{X}^\infty(\mathcal{M})$ and any $(y^0, y^1) \in H^{1+\alpha}(\mathcal{M}) \times H^\alpha(\mathcal{M})$, the respective solutions y and $\tilde y$ to*

$$\begin{cases} P_{\kappa,g}\, y = \mathbf{1}_{(0,T)}\, \chi\, f_{\kappa,g}^{y^0,y^1} & \text{in } (0,T)\times\mathcal{M},\\ (y, \partial_t y)_{|t=0} = (y^0, y^1) & \text{in } \mathcal{M}, \end{cases} \qquad \begin{cases} P_{\tilde\kappa,\tilde g}\, \tilde y = \mathbf{1}_{(0,T)}\, \chi\, f_{\kappa,g}^{y^0,y^1} & \text{in } (0,T)\times\mathcal{M},\\ (\tilde y, \partial_t \tilde y)_{|t=0} = (y^0, y^1) & \text{in } \mathcal{M} \end{cases}$$

satisfy

$$\mathcal{E}_{\kappa,g}(y - \tilde y)(T)^{1/2} \le C_\alpha \|(\kappa, g) - (\tilde\kappa, \tilde g)\|^\alpha_{\mathcal{X}^1(\mathcal{M})}\, \|(y^0, y^1)\|_{H^{1+\alpha}(\mathcal{M}) \oplus H^\alpha(\mathcal{M})}.$$

The proof of Proposition 1.19 is given in Section 6B.

In the above proposition coefficients are chosen smooth, quite in contrast with the rest of this article. As explained above, and as the reader can check in the proof, this lies in the use of the regularity of the HUM operator with respect to the data (y^0, y^1), a result proven for smooth coefficients in [Dehman and Lebeau 2009]. The result of Proposition 1.19 raises the following natural questions:

(1) Does the HUM operator exhibit regularity with respect to the data (y^0, y^1) similar to what is proven in [Dehman and Lebeau 2009] in the case of not so smooth coefficients?

(2) If so, if one increases the smoothness of the data (y^0, y^1) as in Proposition 1.19, does the HUM control process also become regular with respect of the metric?

2. Geometric aspects and operators

We define the smooth manifold $\mathcal{L} = \mathbb{R} \times \mathcal{M}$ and $T^*\mathcal{L}$ its cotangent bundle. We denote by $\pi : T^*\mathcal{L} \to \mathcal{L}$ the natural projection. Elements in $T^*\mathcal{L}$ are denoted by (t, x, τ, ξ). One has $\pi(t, x, \tau, \xi) = (t, x)$.

Setting $|\xi|_x^2 = g_x(\xi, \xi)$ the Riemannian norm in the cotangent space of $\mathcal{M}$ at x, we define

$$S^*\mathcal{L} = \{(t, x, \tau, \xi) \in T^*\mathcal{L} : \tau^2 + |\xi|_x^2 = 1\},$$

the cosphere bundle of $\mathcal{L}$. We shall also use the associated cosphere bundle in the spatial variables only,

$$S^*\mathcal{M} = \left\{(x, \xi) \in T^*\mathcal{M} : |\xi|_x^2 = \tfrac{1}{2}\right\}.$$

For a $\mathscr{C}^k$-metric both $S^*\mathcal{M}$ and $S^*\mathcal{L}$ are $\mathscr{C}^k$-manifolds.

Consider a $\mathscr{C}^\infty$-atlas $\mathcal{A}^\mathcal{M} = (\mathcal{C}_j^\mathcal{M})_{j\in J}$ of $\mathcal{M}$, $\#J < \infty$, with $\mathcal{C}_j^\mathcal{M} = (O_j, \theta_j)$, where O_j is an open set of $\mathcal{M}$ and $\theta_j : O_j \to \widetilde{O}_j$ is a bijection for $\widetilde{O}_j$ an open set of $\mathbb{R}^d$. For $j \in J$, we define $\mathcal{C}_j = (\mathcal{O}_j, \vartheta_j)$ with $\mathcal{O}_j = \mathbb{R} \times O_j$ and

$$\vartheta_j : \mathcal{O}_j \to \widetilde{\mathcal{O}}_j, \quad (t, x) \mapsto (t, \theta_j(x)),$$

with $\widetilde{\mathcal{O}}_j = \mathbb{R} \times \widetilde{O}_j$. Then $\mathcal{A} = (\mathcal{C}_j)_{j\in J}$ is a $\mathscr{C}^\infty$-atlas for $\mathcal{L}$.

In what follows for simplicity we shall use the same notation for an element of $T^*\mathcal{L}$ and its local representative if no confusion arises.

2A. ***Hamiltonian vector field and bicharacteristics.*** Let $(\kappa, g) \in \mathcal{X}^k$, $k = 1$ or 2. The principal symbol of the wave operator $P_{\kappa,g}$ is given by

$$p(t, x, \tau, \xi) = p_{\kappa,g}(t, x, \tau, \xi) = -\tau^2 + |\xi|_x^2, \quad (t, x, \tau, \xi) \in T^*\mathcal{L}. \tag{2-1}$$

In local charts, one has

$$p(t, x, \tau, \xi) = -\tau^2 + \sum_{1\le i,j\le d} g^{ij}(x)\xi_i\xi_j.$$

Note that $(g^{ij}(x))_{i,j}$ is the inverse of $(g_{ij}(x))_{i,j}$, the latter being the local representative of the metric.

We denote by H_p the Hamiltonian vector field associated with p, that is, the unique vector field such that $\{p, f\} = \mathrm{H}_p f$ for any smooth function f. Here, $\{\cdot, \cdot\}$ denotes the Poisson bracket, that is, in local chart

$$\{p, f\} = \partial_\tau p\, \partial_t f - \partial_t p\, \partial_\tau f + \sum_{1\le j\le d} (\partial_{\xi_j} p\, \partial_{x_j} f - \partial_{x_j} p\, \partial_{\xi_j} f),$$

yielding

$$\mathrm{H}_p = -2\tau\partial_t + \nabla_\xi p \cdot \nabla_x - \nabla_x p \cdot \nabla_\xi,$$

as p is in fact independent of the time variable t. The Hamiltonian vector field H_p is of class $\mathscr{C}^{k-1}$. Observe that, for a function f of the variables (t, x, τ, ξ), one has

$${}^t\mathrm{H}_p f = 2\tau\partial_t f - \mathrm{div}_x(f\nabla_\xi p) + \mathrm{div}_\xi(f\nabla_x p),$$

with which one deduces

$${}^t\mathrm{H}_p = -\mathrm{H}_p, \tag{2-2}$$

even in the case $(\kappa, g) \in \mathcal{X}^1$.

First, consider the case $k = 2$. Thus, H_p is a $\mathscr{C}^1$-vector field. For $\varrho \in T^*\mathcal{L}$ one denotes by $s \mapsto \phi_s(\varrho)$ the unique maximal solution to

$$\frac{d}{ds}\phi_s(\varrho) = \mathrm{H}_p\, \phi_s(\varrho), \quad s \in \mathbb{R}, \quad \text{and} \quad \phi_{s=0}(\varrho) = \varrho, \tag{2-3}$$

as given by the Cauchy–Lipschitz theorem. One calls $(s, \varrho) \mapsto \phi_s(\varrho)$ the Hamiltonian flow map. Let $s \mapsto \gamma(s)$ be an integral curve of H_p, that is, $\gamma(s) = \phi_s(\varrho)$ for some $\varrho \in T^*\mathcal{L}$. For any smooth function f on $T^*\mathcal{L}$ one has

$$\frac{d}{ds} f \circ \gamma(s) = \mathrm{H}_p\, f(\gamma(s)).$$

Note that $\mathrm{H}_p\, \tau = 0$, meaning that the variable τ is constant along γ. Note also that the value of p remains constant along γ since $\mathrm{H}_p\, p = \{p, p\} = 0$. Hence, $|\xi|_x^2 = g_x(\xi, \xi)$ is also constant. Thus, if $\gamma(0) \in S^*\mathcal{L}$ then $\gamma(s)$ remains in $S^*\mathcal{L}$, and, for $\varrho \in S^*\mathcal{L}$, the vector field H_p at ϱ is tangent to $S^*\mathcal{L}$. Consequently, we may consider H_p as a tangent vector field on the $\mathscr{C}^2$-manifold $S^*\mathcal{L}$. In particular $\mathrm{H}_p\, a$ makes sense if $a \in \mathscr{C}_c^1(S^*\mathcal{L})$. If moreover $a \in \mathscr{C}_c^{2+\ell}(S^*\mathcal{L})$, $\ell \ge 0$, one has $\mathrm{H}_p\, a \in \mathscr{C}_c^1(S^*\mathcal{L})$.

Since $\mathrm{H}_p\, p = 0$, the flow ϕ_s preserves $\mathrm{Char}(p) = p^{-1}(\{0\})$, the characteristic set of p. As is done classically, we call bicharacteristic an integral curve for which $p = 0$. Observe then that (2-3) defines a flow on the $\mathscr{C}^2$-manifold

$$\mathrm{Char}(p) \cap S^*\mathcal{L} = \left\{(t, x, \tau, \xi) : \tau^2 = \tfrac{1}{2} \text{ and } |\xi|_x^2 = \tfrac{1}{2}\right\}.$$

Second, consider the case $k = 1$. Then H_p is only a continuous vector field. Thus, for any $\varrho \in \mathrm{Char}(p)$ there exists a maximal bicharacteristic $s \mapsto \gamma(s)$ defined on $\mathbb{R}$ such that $\gamma(0) = \varrho$, that is,

$$\frac{d}{ds}\gamma(s) = \mathrm{H}_p(\gamma(s)), \quad s \in \mathbb{R},$$

by the Cauchy–Peano theorem. Uniqueness is however not guaranteed and the notion of flow cannot be used in the case $k = 1$. Since the value of $|\xi|_x$ remains constant and the manifold $\mathcal{M}$ is compact, maximal bicharacteristics are actually defined *globally*.

As above, if $\gamma(0) \in S^*\mathcal{L}$ (resp. $\mathrm{Char}(p) \cap S^*\mathcal{L}$) one has $\gamma(s) \in S^*\mathcal{L}$ (resp. $\mathrm{Char}(p) \cap S^*\mathcal{L}$) for all $s \in \mathbb{R}$. The Hamiltonian vector field H_p can be viewed as a $\mathscr{C}^0$-vector field on the $\mathscr{C}^1$-manifold $S^*\mathcal{L}$ (resp. on the $\mathscr{C}^1$-manifold $\mathrm{Char}(p) \cap S^*\mathcal{L}$). For $a \in \mathscr{C}_c^{1+\ell}(S^*\mathcal{L})$, $\ell \geq 0$, one finds $\mathrm{H}_p\, a \in \mathscr{C}_c^0(S^*\mathcal{L})$.

Finally, connection between bicharacteristic and geodesics can be made. For this we recall that if $\xi \in T_x^*\mathcal{M}$ for some $x \in \mathcal{M}$, one can define $v \in T_x\mathcal{M}$ by $v = \xi^\sharp$, which reads in local coordinates $v^i = \sum_j g^{ij}(x)\xi_j$. In particular $|v|_x^2 = g_x(v, v) = |\xi|_x$. If now $\varrho^0 = (t^0, x^0, \tau^0, \xi^0) \in \mathrm{Char}(p) \cap S^*\mathcal{L}$ and letting $s \mapsto \varrho(s) = (t(s), x(s), \tau, \xi(s))$ be a bicharacteristic such that $\varrho(0) = \varrho^0$, one has $\tau = \tau^0$ and $t(s) = t^0 - 2\tau^0 s$. The map

$$X : t \mapsto x\left(\frac{t^0 - t}{2\tau^0}\right)$$

can be proven to be the geodesic originating from x^0 in the direction given by $v^0 = (\xi^0)^\sharp$ and parametrized by t.

We now compute the speed at which the geodesic is traveled. We have

$$\frac{dX}{dt}(t) = -\frac{1}{2\tau^0}\frac{dx(s)}{ds},$$

which yields

$$\frac{dX}{dt}(t) = -\frac{1}{2\tau^0}\nabla_\xi p(x(s), \xi(s)) = -\frac{\xi(s)^\sharp}{\tau^0}.$$

It follows that

$$\left|\frac{dX}{dt}(t)\right|_x = \frac{|\xi(s)^\sharp|_x}{|\tau^0|} = \frac{|\xi(s)|_x}{|\tau^0|} = \frac{|\xi^0|_x}{|\tau^0|} = 1,$$

since $\varrho^0 \in \mathrm{Char}(p)$. Hence, the projection of the bicharacteristic $s \mapsto \gamma(s)$ yields a geodesic traveled at speed 1.

2B. ***Geometric control condition.*** As the projections of bicharacteristics onto $\mathcal{L}$ yield geodesics, in the case $k \geq 2$, we can state the Rauch–Taylor geometric control condition [1974] formulated in Definition 1.8 with the notion of Hamiltonian flow introduced above.

Definition 1.8′ (geometric control condition, $k \geq 2$). Let g be a $\mathscr{C}^2$ metric and let ω be an open set of $\mathcal{M}$ and $T > 0$. One says that (ω, T) fulfills the geometric control condition if for all $\varrho \in \mathrm{Char}(p)$ one has $\pi(\phi_s(\varrho)) \in (0, T) \times \omega$ for some $s \in \mathbb{R}$.

In the case $k = 1$, since g is only $\mathscr{C}^1$ there is no flow in general, one rather writes the geometric control condition by means of maximal bicharacteristics.

Definition 1.8″ (generalized geometric control condition, $k = 1$). Let g be a $\mathscr{C}^1$ metric and let ω be an open set of $\mathcal{M}$ and $T > 0$. One says that (ω, T) fulfills the geometric control condition if for *any* maximal bicharacteristic $s \mapsto \gamma(s)$ in $\mathrm{Char}(p)$ one has $\pi(\gamma(s)) \in (0, T) \times \omega$ for some $s \in \mathbb{R}$.

In other words, for all $\varrho \in \mathrm{Char}(p)$, all bicharacteristics that go through ϱ meet the cotangent bundle above $(0, T) \times \omega$.

Naturally, Definitions 1.8′ and 1.8″ coincide in the case $k = 2$ because of the uniqueness of a bicharacteristic going through a point of $\mathrm{Char}(p)$.

2C. *Symbols and pseudodifferential operators.* Here, we follow [Burq 1997b, Section 1.1] for the notation. We denote by $H^k(X)$ or $H^k_{\mathrm{loc}}(X)$, with $X = \mathcal{M}$ or $\mathcal{L}$, the usual Sobolev space for complex valued functions, endowed with its natural inner product and norm. In particular, the $L^2(X)$-inner product is denoted by $(\,\cdot\,,\,\cdot\,)_{L^2(X)}$.

Classical polyhomogeneous symbol classes on $T^*\mathbb{R}^n \simeq \mathbb{R}^n \times \mathbb{R}^n$ are denoted by $S^m_{\mathrm{ph}}(\mathbb{R}^n \times \mathbb{R}^n)$ and the classes of associated operators by $\Psi^m_{\mathrm{ph}}(\mathbb{R}^n)$. We recall that symbols in the class $S^m_{\mathrm{ph}}(\mathbb{R}^n \times \mathbb{R}^n)$ behave well with respect to changes of variables, up to symbols in $S^{m-1}_{\mathrm{ph}}(\mathbb{R}^n \times \mathbb{R}^n)$; see [Hörmander 1985, Theorem 18.1.17 and Lemma 18.1.18].

We define $S^m_{c,\mathrm{ph}}(T^*\mathcal{L})$ as the set of polyhomogeneous symbols of order m on $T^*\mathcal{L}$ with compact support in the variables $(t, x) \in \mathcal{L}$ (note that compactness with respect to $x \in \mathcal{M}$ is obvious). Having the manifold $\mathcal{M}$ smooth is important for symbols and following pseudodifferential operators to be simply defined.

For any m, the restriction to the sphere

$$S^m_{c,\mathrm{ph}}(T^*\mathcal{L}) \to \mathscr{C}^\infty_c(S^*\mathcal{L}), \quad a \to a_{|S^*\mathcal{L}}, \tag{2-4}$$

is onto. This allows one to identify a homogeneous symbol with a smooth function on $S^*\mathcal{L}$ with compact support.

We denote by $\Psi^m_{c,\mathrm{ph}}(\mathcal{L})$ the space of polyhomogeneous pseudodifferential operators of order m on $\mathcal{L}$: one says that $Q \in \Psi^m_{c,\mathrm{ph}}(\mathcal{L})$ if Q maps $\mathscr{C}^\infty_c(\mathcal{L})$ into $\mathscr{D}'(\mathcal{L})$ and

(1) its kernel $K(x, y) \in \mathscr{D}'(\mathcal{L} \times \mathcal{L})$ is such that $\mathrm{supp}(K)$ is compact in $\mathcal{L} \times \mathcal{L}$;

(2) $K(x, y)$ is smooth away from the diagonal $\Delta_{\mathcal{L}} = \{(t, x; t, x) : (t, x) \in \mathcal{L}\}$;

(3) for any local chart $\mathcal{C}_j = (\mathcal{O}_j, \vartheta_j)$ and all $\phi_0, \phi_1 \in \mathscr{C}^\infty_c(\widetilde{\mathcal{O}}_j)$ one has

$$\phi_1 \circ (\vartheta_j^{-1})^* \circ Q \circ \vartheta_j^* \circ \phi_0 \in \mathrm{Op}(S^m_{c,\mathrm{ph}}(\mathbb{R}^{d+1} \times \mathbb{R}^{d+1})).$$

For $Q \in \Psi^m_{c,\mathrm{ph}}(\mathcal{L})$, we denote by $\sigma_m(Q) \in S^m_{c,\mathrm{ph}}(T^*\mathcal{L})$ the principal symbol of Q; see [Hörmander 1985, Chapter 18.1]. Note that the principal symbol is uniquely defined in $S^m_{c,\mathrm{ph}}(T^*\mathcal{L})$ because of the polyhomogeneous structure; see the remark following Definition 18.1.20 in [Hörmander 1985]. The application σ_m enjoys the following properties:

(1) The map $\sigma_m : \Psi^m_{c,\text{ph}}(\mathcal{L}) \to S^m_{c,\text{ph}}(T^*\mathcal{L})$ is onto.

(2) For all $Q \in \Psi^m_{c,\text{ph}}(\mathcal{L})$, $\sigma_m(Q) = 0$ if and only if $Q \in \Psi^{m-1}_{c,\text{ph}}(\mathcal{L})$.

(3) For all $Q \in \Psi^m_{c,\text{ph}}(\mathcal{L})$, $\sigma_m(Q^*) = \overline{\sigma_m(Q)}$.

(4) For all $Q_1 \in \Psi^{m_1}_{c,\text{ph}}(\mathcal{L})$ and $Q_2 \in \Psi^{m_2}_{c,\text{ph}}(\mathcal{L})$, one has $Q_1 Q_2 \in \Psi^{m_1+m_2}_{c,\text{ph}}(\mathcal{L})$ with

$$\sigma_{m_1+m_2}(Q_1 Q_2) = \sigma_{m_1}(Q_1)\sigma_{m_2}(Q_2).$$

(5) For all $Q_1 \in \Psi^{m_1}_{c,\text{ph}}(\mathcal{L})$ and $Q_2 \in \Psi^{m_2}_{c,\text{ph}}(\mathcal{L})$, one has $[Q_1, Q_2] = Q_1 Q_2 - Q_2 Q_1 \in \Psi^{m_1+m_2-1}_{c,\text{ph}}(\mathcal{L})$, with

$$\sigma_{m_1+m_2-1}([Q_1, Q_2]) = \frac{1}{i}\{\sigma_{m_1}(Q_1), \sigma_{m_2}(Q_2)\}.$$

(6) If $Q \in \Psi^m_{c,\text{ph}}(\mathcal{L})$, then Q maps continuously $H^k_{\text{loc}}(\mathcal{L})$ into $H^{k-m}_{\text{comp}}(\mathcal{L})$. In particular, for $m < 0$, Q is compact on $L^2_{\text{loc}}(\mathcal{L})$.

Given an operator $Q \in \Psi^m_{c,\text{ph}}(\mathcal{L})$, one sets

$$\text{Char}(Q) = \text{Char}(\sigma_m(Q)) = \{\varrho \in T^*\mathcal{L} : \sigma_m(Q)(\varrho) = 0\}.$$

3. Microlocal defect measure and propagation properties

A defect measure is used to characterize locally the failure of a sequence to strongly converge, meaning some concentration phenomenon. This characterization can be made finer by further considering microlocal concentration phenomena.

3A. *Microlocal defect density measures.* We define $\mathcal{M}_+(S^*\mathcal{L})$ as the set of positive density measures on $S^*\mathcal{L}$. For $\mu \in \mathcal{M}_+(S^*\mathcal{L})$ and $a \in \mathscr{C}^0_c(S^*\mathcal{L})$, we shall write

$$\langle \mu, a\rangle_{S^*\mathcal{L}} = \int_{S^*\mathcal{L}} a(\varrho)\mu(d\varrho)$$

for the duality bracket. This notation will also be used for $a \in S^0_{c,\text{ph}}(T^*\mathcal{L})$ according to the identification map (2-4).

Consider a sequence $(u^k)_{k\in\mathbb{N}} \subset L^2_{\text{loc}}(\mathcal{L})$ that converges weakly to 0. Here, to define the L^2-norm and inner product on $\mathcal{L}$ we use a fixed (κ^0, g^0) chosen in $\mathcal{X}^1(\mathcal{M})$; see (1-2).

As a consequence of [Gérard 1991, Theorem 1], there exists a subsequence of $(u^k)_{k\in\mathbb{N}}$ (still denoted by $(u^k)_{k\in\mathbb{N}}$ in what follows) and a density measure $\mu \in \mathcal{M}_+(S^*\mathcal{L})$ such that

$$\lim_{k\to\infty} \langle Qu^k, \overline{u^k}\rangle_{L^2_{\text{comp}}(\mathcal{L}), L^2_{\text{loc}}(\mathcal{L})} = \langle \mu, \sigma_0(Q)\rangle_{S^*\mathcal{L}} \tag{3-1}$$

for any $Q \in \Psi^0_{c,\text{ph}}(\mathcal{L})$. Recall that symbols in $S^0_{c,\text{ph}}(T^*\mathcal{L})$ are compactly supported in time t here. We also refer to [Tartar 1990; Burq 1997b]. One calls μ a microlocal defect (density) measure associated with $(u^k)_{k\in\mathbb{N}}$.

Similarly, one can use the notion of H^1-microlocal defect density measure. Consider $(u^k)_{k\in\mathbb{N}} \subset H^1_{\mathrm{loc}}(\mathcal{L})$ that converges weakly to 0. Then, there exists a subsequence of $(u^k)_{k\in\mathbb{N}}$ (still denoted by $(u^k)_{k\in\mathbb{N}}$) and a density measure $\mu \in \mathcal{M}_+(S^*\mathcal{L})$ such that for any $Q \in \Psi^2_{c,\mathrm{ph}}(\mathcal{L})$

$$\lim_{k\to\infty} \langle Qu^k, \overline{u^k}\rangle_{H^{-1}_{\mathrm{comp}}(\mathcal{L}),H^1_{\mathrm{loc}}(\mathcal{L})} = \langle \mu, \sigma_2(Q)\rangle_{S^*\mathcal{L}}. \tag{3-2}$$

Naturally, in either cases, the density measure μ depends on the choice made of $(\kappa^0, g^0) \in \mathcal{X}^1(\mathcal{M})$. In what follows we shall make clear what choice is made.

3B. *Local representatives.* Consider a finite atlas $\mathcal{A} = (\mathcal{C}_j)_{j\in J}$ on $\mathcal{L}$, as introduced in Section 2, with $\mathcal{C}_j = (\mathcal{O}_j, \vartheta_j)$. Consider a smooth partition of unity $(\chi_j)_{j\in J}$ subordinated to the covering by the open sets $(\mathcal{O}_j)_j$. We consider also $\tilde{\chi}_j, \hat{\chi}_j \in \mathscr{C}^\infty(\mathcal{L})$ supported in $\mathcal{O}_j$ such that $\tilde{\chi}_j \equiv 1$ on a neighborhood of $\operatorname{supp}(\chi_j)$ and $\hat{\chi}_j \equiv 1$ on a neighborhood of $\operatorname{supp}(\tilde{\chi}_j)$. Set also $\chi_j^{\mathcal{C}_j} = (\vartheta_j^{-1})^*\chi_j$, $\tilde{\chi}_j^{\mathcal{C}_j} = (\vartheta_j^{-1})^*\tilde{\chi}_j$, and $\hat{\chi}_j^{\mathcal{C}_j} = (\vartheta_j^{-1})^*\hat{\chi}_j$. One has $\chi_j^{\mathcal{C}_j}, \tilde{\chi}_j^{\mathcal{C}_j}, \hat{\chi}_j^{\mathcal{C}_j} \in \mathscr{C}_c^\infty(\widetilde{\mathcal{O}}_j)$, with $\widetilde{\mathcal{O}}_j = \vartheta_j(\mathcal{O}_j)$.

Let $(u^k)_k \subset H^1_{\mathrm{loc}}(\mathcal{L})$ that converges weakly to 0, $Q \in \Psi^2_{c,\mathrm{ph}}(\mathcal{L})$, and $j \in J$. One can write

$$\chi_j Q = \chi_j Q \tilde{\chi}_j + \chi_j Q(1 - \tilde{\chi}_j).$$

Since $\chi_j Q(1 - \tilde{\chi}_j)$ is a regularizing operator one finds

$$\begin{aligned}\langle \mu, \chi_j\sigma_2(Q)\rangle_{S^*\mathcal{L}} &\sim \langle \chi_j Qu^k, \overline{u^k}\rangle_{H^{-1}_{\mathrm{comp}}(\mathcal{L}),H^1_{\mathrm{loc}}(\mathcal{L})}\\ &\sim \langle \chi_j\tilde{\chi}_j Q\tilde{\chi}_j v_j^k, \overline{v_j^k}\rangle_{H^{-1}_{\mathrm{comp}}(\mathcal{L}),H^1_{\mathrm{loc}}(\mathcal{L})} \quad \text{as } k \to +\infty,\end{aligned}$$

for $v_j^k = \hat{\chi}_j u^k$.

The operator $Q_j = (\vartheta_j^{-1})^*\tilde{\chi}_j Q \tilde{\chi}_j (\vartheta_j)^*$ is a pseudodifferential operator of order 2 on $\mathbb{R}^{d+1}$ with principal symbol $q_j = \tilde{\chi}_j^2 q^{\mathcal{C}_j}$, where $q^{\mathcal{C}_j}$ is the local representative of $\sigma_2(Q)$. Set also $v_j^{k,\mathcal{C}_j} = (\vartheta_j^{-1})^* v_j^k$. It converges weakly to 0 in $H^1(\mathbb{R}^{d+1})$. Associated with this sequence is a microlocal defect measure μ_j. If one writes

$$\langle \chi_j\tilde{\chi}_j Q\tilde{\chi}_j v_j^k, \overline{v_j^k}\rangle_{H^{-1}_{\mathrm{comp}}(\mathcal{L}),H^1_{\mathrm{loc}}(\mathcal{L})} = \langle \chi_j^{\mathcal{C}_j} Q_j v_j^{k,\mathcal{C}_j}, \overline{v_j^{k,\mathcal{C}_j}}\rangle_{H^{-1}_{\mathrm{comp}}(\mathbb{R}^{d+1}),H^1_{\mathrm{loc}}(\mathbb{R}^{d+1})},$$

one obtains

$$\langle \mu, \chi_j\sigma_2(Q)\rangle_{S^*\mathcal{L}} = \langle \mu_j, \chi_j^{\mathcal{C}_j} q_j\rangle_{S^*\widetilde{\mathcal{O}}^j} = \langle \mu_j, \chi_j^{\mathcal{C}_j} q^{\mathcal{C}_j}\rangle_{S^*\widetilde{\mathcal{O}}^j}.$$

Note that here, the L^2 and H^s-norms on $\mathbb{R}^{d+1}$ are based on the local representative of the density measure $\kappa^0\mu_{g^0}\,dt$. One thus sees that the local representative of $\chi_j\mu$ is precisely $\chi_j^{\mathcal{C}_j}\mu_j$, that is, $\chi_j\mu = \vartheta_j^*(\chi_j^{\mathcal{C}_j}\mu_j) = \chi_j\vartheta_j^*\mu_j$. Summing up, we thus have

$$\mu = \sum_{j\in J}\chi_j\mu = \sum_{j\in J}\chi_j\vartheta_j^*\mu_j$$

and

$$\langle \mu, \sigma_2(Q)\rangle_{S^*\mathcal{L}} = \sum_{j\in J}\langle \mu, \chi_j\sigma_2(Q)\rangle_{S^*\mathcal{L}} = \sum_{j\in J}\langle \mu_j, \chi_j^{\mathcal{C}_j} q^{\mathcal{C}_j}\rangle_{S^*\widetilde{\mathcal{O}}^j}.$$

Remark 3.1. Local properties of microlocal defect measures like μ can be deduced from the properties of $\chi_j^{\mathcal{C}_j}\mu_j$. In what follows most results are of local nature. In such cases we shall work in local charts and use Sections 2C and 3B to bring the analysis to open domains of $\mathbb{R}^{d+1}$.

3C. ***Operators with a low regularity.*** An important tool we use to handle low-regularity terms in what follows is a result due to R. Coifman and Y. Meyer [1978, Proposition IV.7] and some of its consequences that we list below.

Theorem 3.2 (Coifman–Meyer). *Let $Q \in \Psi^1_{\mathrm{ph}}(\mathbb{R}^n \times \mathbb{R}^n)$. If $m \in W^{1,\infty}(\mathbb{R}^n)$ the commutator $[Q, m]$ maps $L^2(\mathbb{R}^n)$ into itself continuously. Moreover there exists $C > 0$ such that*

$$\big\| [Q, m] \big\|_{L^2 \to L^2} \le C \|m\|_{W^{1,\infty}}, \quad m \in W^{1,\infty}(\mathbb{R}^n).$$

We deduce the following corollary.

Corollary 3.3. *Let $Q \in \Psi^1_{\mathrm{ph}}(\mathbb{R}^n \times \mathbb{R}^n)$ be such that its kernel has compact support in $\mathbb{R}^n \times \mathbb{R}^n$. With $q \in S^1_{\mathrm{ph}}(\mathbb{R}^n \times \mathbb{R}^n)$ its principal symbol.*

Let $m \in \mathscr{C}^1(\mathbb{R}^n)$. There exist K_1 and K_2, compact operators on $L^2(\mathbb{R}^n)$, with compactly supported kernels, such that

$$[Q, m] = \frac{1}{i} \nabla_x m \cdot \mathrm{Op}(\nabla_\xi q) + K_1 = \frac{1}{i} \mathrm{Op}(\nabla_\xi q) \cdot \nabla_x m + K_2. \tag{3-3}$$

Proof. Consider a sequence $(m^k)_{k\in\mathbb{N}} \subset \mathscr{C}^\infty(\mathbb{R}^n)$ such that

$$\sum_{|\alpha|\le 1} \|\partial_x^\alpha (m^k - m)\|_{L^\infty} \to 0 \quad \text{as } k \to +\infty.$$

Classical symbolic calculus gives

$$[Q, m^k] = \frac{1}{i} \nabla_x m^k \cdot \mathrm{Op}(\nabla_\xi q) + K^k_1, \tag{3-4}$$

with $K^k_1 = \mathrm{Op}(r^k_1)$ for some $r^k_1 \in S^{-1}_{\mathrm{ph}}$, $j = 1, 2$. Thus, K^k_1 is bounded from $L^2(\mathbb{R}^n)$ into $H^1(\mathbb{R}^n)$. In addition, since K^k_1 has a kernel with compact supports in $\mathbb{R}^n \times \mathbb{R}^n$, it is compact on $L^2(\mathbb{R}^n)$. Note that the support of the kernel of K^k_1 lies in a compact $\mathcal{K}$ of $\mathbb{R}^n \times \mathbb{R}^n$ that is uniform with respect to k.

On the other hand, observe that

$$\nabla_x m^k \cdot \mathrm{Op}(\nabla_\xi q) \to \nabla_x m \cdot \mathrm{Op}(\nabla_\xi q) \quad \text{in } \mathscr{L}(L^2(\mathbb{R}^n)).$$

Moreover, from Theorem 3.2 applied to $m^k - m$, one also has

$$[Q, m^k] \to [Q, m] \quad \text{in } \mathscr{L}(L^2(\mathbb{R}^n)).$$

Using then (3-4) we deduce that $(K^k_1)_{n\in\mathbb{N}}$ converges to some K^1 in $\mathscr{L}(L^2(\mathbb{R}^n))$, and from the closedness of the set of compact operators in $\mathscr{L}(L^2(\mathbb{R}^n))$ we find that K^1 is compact. Moreover, K^1 has a kernel supported in $\mathcal{K}$. The limits above give the first equality in (3-3). The second equality follows similarly. ☐

Let Ω be a bounded open set of $\mathbb{R}^n$ and $(\kappa^0, g^0) \in \mathcal{X}^1(\Omega)$, with definition adapted from that of $\mathcal{X}^1(\mathcal{M})$. The L^2-inner product and norm are given by the density $\kappa^0 \mu_{g^0}$. The following result is also a consequence of Theorem 3.2.

Proposition 3.4. *Let $(u^k)_{k\in\mathbb{N}} \subset H^1_{\mathrm{loc}}(\Omega)$ be a sequence that converges weakly to 0 and let μ be an H^1-microlocal defect density measure on $S^*\Omega$ associated with the sequence $(u^k)_k$.*

Let $b_1 \in W^{1,\infty}(\mathbb{R}^n)$ and $b_2 \in \mathscr{C}^0(\mathbb{R}^n)$. Consider also $Q_1, Q_2 \in \Psi^1_{\mathrm{ph}}(\mathbb{R}^n)$, both with kernels compactly supported in $\Omega \times \Omega$, with $q_1, q_2 \in S^1_{\mathrm{ph}}(\mathbb{R}^n \times \mathbb{R}^n)$ for respective principal symbol. Then, one has

$$\langle b_1 Q_1\, b_2\, Q_2 u^k, \overline{u^k}\rangle_{H^{-1}_{\mathrm{comp}}(\Omega), H^1_{\mathrm{loc}}(\Omega)} \xrightarrow[k\to+\infty]{} \langle \mu, b_1 b_2 q_1 q_2\rangle_{S^*\Omega}. \tag{3-5}$$

More generally, assume that $(b_1^k)_{k\in\mathbb{N}} \subset W^{1,+\infty}(\mathbb{R}^n)$ and $(b_2^k)_{k\in\mathbb{N}} \subset L^\infty(\mathbb{R}^n)$, and $(\kappa_k, g_k)_{k\in\mathbb{N}} \subset \mathcal{Y}(\Omega)$ with

$$\|b_1^k - b_1\|_{W^{1,+\infty}(\mathbb{R}^n)} + \|b_2^k - b_2\|_{L^\infty(\mathbb{R}^n)} + \|(\kappa_k, g_k) - (\kappa^0, g^0)\|_{\mathcal{Y}(\Omega)} \to 0 \quad \text{as } k \to +\infty.$$

Then

$$\langle b_1^k Q_1\, b_2^k\, Q_2 u^k, \overline{u^k}\rangle_{H^{-1}_{\mathrm{comp}}(\Omega,\kappa_k\mu_{g_k}), H^1_{\mathrm{loc}}(\Omega,\kappa_k\mu_{g_k})} \xrightarrow[k\to+\infty]{} \langle \mu, b_1 b_2 q_1 q_2\rangle_{S^*\Omega}. \tag{3-6}$$

Remark 3.5. Note that b_1 is chosen in $W^{1,\infty}(\mathbb{R}^n)$ because one cannot multiply an element in H^{-1} by a bounded function. One derivative is needed.

Proof of Proposition 3.4. With Lemma 3.6 below we may replace the density $\kappa_k \mu_{g_k}$ in the L^2-inner product by $\kappa^0 \mu_{g^0}$ and thus in the H^{-1}_{comp}-H^1_{loc} duality.

We write

$$b_1^k Q_1\, b_2^k\, Q_2 = b_1 Q_1\, b_2\, Q_2 + R^k, \qquad R^k = b_1 Q_1\, (b_2^k - b_2)\, Q_2 + (b_1^k - b_1) Q_1\, b_2^k\, Q_2.$$

Note that R^k maps $H^1_{\mathrm{loc}}(\Omega)$ into $H^{-1}_{\mathrm{comp}}(\Omega)$ continuously. Moreover because of the convergences of b_1^k and b_2^k, and the boundedness of $(u^k)_{k\in\mathbb{N}}$ in $H^1_{\mathrm{loc}}(\Omega)$, one finds that $R^k u^k \to 0$ strongly in $H^{-1}_{\mathrm{comp}}(\Omega)$. Thus we can write

$$\langle b_1^k Q_1\, b_2^k\, Q_2 u^k, \overline{u^k}\rangle_{H^{-1}_{\mathrm{comp}}(\Omega), H^1_{\mathrm{loc}}(\Omega)} = \langle b_1 Q_1\, b_2\, Q_2 u^k, \overline{u^k}\rangle_{H^{-1}_{\mathrm{comp}}(\Omega), H^1_{\mathrm{loc}}(\Omega)} + o(1)_{k\to+\infty}$$

and (3-6) follows if we prove (3-5).

According to Theorem 3.2 the commutator $[b_1, Q_1]$ is bounded on $L^2(\Omega)$ implying $[b_1, Q_1]\, b_2\, Q_2 u^k$ is bounded in $L^2(\Omega)$ yielding

$$\langle [b_1, Q_1]\, b_2\, Q_2 u^k, \overline{u^k}\rangle_{H^{-1}_{\mathrm{comp}}(\Omega), H^1_{\mathrm{loc}}(\Omega)} = ([b_1, Q_1]\, b_2\, Q_2 u^k, u^k)_{L^2(\Omega)} \xrightarrow[k\to+\infty]{} 0,$$

since $u^k \to 0$ strongly in $L^2(\Omega)$. We may thus assume that $b_1 = 1$ without any loss of generality.

Let $\varepsilon > 0$ and let $b_2^\varepsilon \in \mathscr{C}^\infty(\Omega)$ be such that $\|b_2 - b_2^\varepsilon\|_{L^\infty} \le \varepsilon$. Write

$$Q_1\, b_2\, Q_2 = Q_1\, b_2^\varepsilon\, Q_2 + R^\varepsilon, \qquad R^\varepsilon = Q_1\, (b_2 - b_2^\varepsilon)\, Q_2.$$

One has $|\langle R^\varepsilon u^k, \overline{u^k}\rangle_{H^{-1}_{\mathrm{comp}}(\Omega), H^1_{\mathrm{loc}}(\Omega)}| \le C\varepsilon$, and this leads to

$$\langle Q_1\, b_2\, Q_2 u^k, \overline{u^k}\rangle_{H^{-1}_{\mathrm{comp}}(\Omega), H^1_{\mathrm{loc}}(\Omega)} = \langle Q_1\, b_2^\varepsilon\, Q_2 u^k, \overline{u^k}\rangle_{H^{-1}_{\mathrm{comp}}(\Omega), H^1_{\mathrm{loc}}(\Omega)} + o(1)_{\varepsilon\to 0} + o(1)_{k\to+\infty}. \tag{3-7}$$

Since b_2^ε is smooth, by symbolic calculus one has

$$\langle Q_1\, b_2^\varepsilon\, Q_2 u^k, \overline{u^k}\rangle_{H^{-1}_{\mathrm{comp}}(\Omega), H^1_{\mathrm{loc}}(\Omega)} \xrightarrow[k\to+\infty]{} \langle \mu, b_2^\varepsilon q_1 q_2\rangle_{S^*\Omega}. \tag{3-8}$$

Finally, since $\langle \mu, b_2^\varepsilon q_1 q_2\rangle_{S^*\Omega} \to \langle \mu, b_2 q_1 q_2\rangle_{S^*\Omega}$ as $\varepsilon \to 0$, with (3-7) and (3-8) one concludes that (3-5) holds. ☐

Lemma 3.6. *Assume that* $\|(\kappa_k, g_k) - (\kappa^0, g^0)\|_{\mathcal{Y}(\Omega)} \to 0$ *and consider a sequence* $(f_k, h_k)_{k\in\mathbb{N}}$ *bounded in* $L^2_{\mathrm{comp}}(\Omega) \oplus L^2_{\mathrm{loc}}(\Omega)$. *Then*

$$(f_k, h_k)_{L^2(\Omega,\kappa_k\mu_{g_k})} = (f_k, h_k)_{L^2(\Omega)} + o(1)_{k\to+\infty}.$$

If $(f_k, h_k)_{k\in\mathbb{N}}$ *is bounded in* $H^{-1}_{\mathrm{comp}}(\Omega) \oplus H^1_{\mathrm{loc}}(\Omega)$ *then*

$$\langle f_k, \overline{h^k}\rangle_{H^{-1}_{\mathrm{comp}}(\Omega,\kappa_k\mu_{g_k}), H^1_{\mathrm{loc}}(\Omega,\kappa_k\mu_{g_k})} = \langle f_k, \overline{h^k}\rangle_{H^{-1}_{\mathrm{comp}}(\Omega), H^1_{\mathrm{loc}}(\Omega)} + o(1)_{k\to+\infty}.$$

Here, Lemma 3.6 is written in the case of a bounded open set of the Euclidean space but the same result holds in the case of a compact manifold.

Proof. One has $\mu_{g^0} = (\det g^0)^{1/2}\,dx$ and $\mu_{g_k} = (\det g_k)^{1/2}\,dx$. Therefore $\kappa_k\mu_{g_k} = \alpha_k\kappa^0\mu_{g^0}$, with

$$\alpha_k = \frac{\kappa_k}{\kappa^0}\left(\frac{\det g_k}{\det g^0}\right)^{1/2}$$

and $\alpha_k \to 1$ in the Lipschitz norm. □

3D. *Measures and partial differential equations.* Microlocal defect measures associated with sequences of solutions of partial differential equations with smooth coefficients can have properties such as support localization in the characteristic set and invariance along the Hamiltonian flow. With the material developed above, we extend these results to the case of $\mathscr{C}^1$-coefficients. We focus on the case of wave operators.

Proposition 3.7. *Let* $(\kappa^0, g^0) \in \mathcal{X}^1(\mathcal{M})$ *and set* $P^0 = P_{\kappa^0,g^0}$. *Denote by* $p^0(x,\tau,\xi) = -\tau^2 + g^0_x(\xi,\xi)$ *its principal symbol. Let* $(\kappa_k, g_k)_{k\in\mathbb{N}} \subset \mathcal{Y}(\mathcal{M})$ *be such that* $\|(\kappa_k, g_k) - (\kappa^0, g^0)\|_{\mathcal{Y}(\mathcal{M})} \to 0$ *as* $k \to +\infty$ *and set* $P_k = P_{\kappa_k,g_k}$.

Consider a sequence $(u^k)_{k\in\mathbb{N}} \subset H^1_{\mathrm{loc}}(\mathcal{L})$ *that converges to* 0 *weakly and* μ *an* H^1*-microlocal defect density measure associated with* $(u^k)_{k\in\mathbb{N}}$.

Let $T_1 < T_2$. *The following properties hold*:

(1) *If* $P_k u^k \to 0$ *strongly in* $H^{-1}_{\mathrm{loc}}((T_1,T_2)\times\mathcal{M})$ *then*

$$\operatorname{supp}(\mu) \cap S^*((T_1,T_2)\times\mathcal{M}) \subset \operatorname{Char}(p^0). \tag{3-9}$$

(2) *If moreover* $P_k u^k \to 0$ *strongly in* $L^2_{\mathrm{loc}}((T_1,T_2)\times\mathcal{M})$ *then one has*

$$ {}^t\mathrm{H}_{p^0}\mu = 0 \text{ in the sense of distributions on } S^*((T_1,T_2)\times\mathcal{M}), \tag{3-10}$$

that is, $\langle \mu, \mathrm{H}_{p^0} q\rangle_{S^*\mathcal{L}} = 0$ *for all* $q \in \mathscr{C}^\infty_c(S^*((T_1,T_2)\times\mathcal{M}))$.

Since H_{p^0} is a tangent vector field on $S^*\mathcal{L}$ where μ lives (see Section 2A) note that ${}^t\mathrm{H}_{p^0}\mu$ makes sense in the second item of the proposition. Moreover note that H_{p^0} is a tangent vector field on $S^*\mathcal{L}\cap\operatorname{Char}(p^0)$ and one has $\operatorname{supp}(\mu)\cap S^*((T_1,T_2)\times\mathcal{M}) \subset \operatorname{Char}(p^0)$ by the first item of the proposition. Finally, notice that for a Hamiltonian vector field, $\mathrm{H}_{p^0} = -{}^t\mathrm{H}_{p^0}$ as recalled in Section 2A even in the case $(\kappa^0, g^0) \in \mathcal{X}^1(\mathcal{M})$.

Naturally, Proposition 3.7 and its proof can be adapted to the other energy levels. We shall also need the following result.

Proposition 3.7′. *With the notation of Proposition 3.7, consider a sequence* $(u^k)_{k\in\mathbb{N}} \subset L^2_{\mathrm{loc}}(\mathcal{L})$ *that converges to* 0 *weakly and* μ *an* L^2*-microlocal defect density measure associated with* $(u^k)_{k\in\mathbb{N}}$.

Let $T_1 < T_2$. The following properties hold.

(1) *If $P_k u^k \to 0$ strongly in $H^{-2}_{\mathrm{loc}}((T_1, T_2) \times \mathcal{M})$ then*

$$\operatorname{supp}(\mu) \cap S^*((T_1, T_2) \times \mathcal{M}) \subset \operatorname{Char}(p^0).$$

(2) *If moreover $P_k u^k \to 0$ strongly in $H^{-1}_{\mathrm{loc}}((T_1, T_2) \times \mathcal{M})$ then one has*

$$ {}^t\mathrm{H}_{p^0}\mu = 0 \textit{ in the sense of distributions on } S^*((T_1, T_2) \times \mathcal{M}).$$

Proof of Proposition 3.7. Consider $B \in \Psi^0_{c,\mathrm{ph}}(\mathcal{L})$ with kernel supported in $((T_1, T_2) \times \mathcal{M})^2$ and $b \in S^0_{c,\mathrm{ph}}(\mathcal{L})$ its principal symbol. For the definition of the L^2-inner product we use (κ^0, g^0). We also use the partition of unity $1 = \sum_{j \in J} \chi_j$, with $\chi_j \in \mathscr{C}^\infty_c(\mathcal{O}_j)$ associated with the atlas $\mathcal{A}$ and the additional cutoff functions $\tilde{\chi}_j, \hat{\chi}_j \in \mathscr{C}^\infty_c(\mathcal{O}_j)$ that are introduced in Section 3B and, as obtained in that section, we write

$$\begin{aligned}\langle B P_k u^k, \overline{u^k}\rangle_{H^{-1}_{\mathrm{comp}}(\mathcal{L}), H^1_{\mathrm{loc}}(\mathcal{L})} &= \sum_{j \in J} \langle \chi_j B P_k u^k, \overline{u^k}\rangle_{H^{-1}_{\mathrm{comp}}(\mathcal{L}), H^1_{\mathrm{loc}}(\mathcal{L})} \\ &= \sum_{j \in J} \langle \chi_j \tilde{\chi}_j B P_k \tilde{\chi}_j v^k_j, \overline{v^k_j}\rangle_{H^{-1}_{\mathrm{comp}}(\mathcal{L}), H^1_{\mathrm{loc}}(\mathcal{L})} + o(1)_{k \to +\infty},\end{aligned} \tag{3-11}$$

with $v^k_j = \hat{\chi}_j u^k$. Associated with $(\vartheta_j^{-1})^* v^k_j$, the local representative of v^k_j, is a microlocal defect measure μ_j in $\vartheta_j(\mathcal{O}_j) = \widetilde{\mathcal{O}}_j = \mathbb{R} \times \widetilde{O}_j$ and $\chi_j^{C_j} \mu_j$ is the local representative of $\chi_j \mu$ in this chart. See Section 3B.

Note that we use local representatives of the operators, functions, and measures without introducing any new symbols. Yet to keep clear that the analysis is carried out in a local chart we use the notation $L^2(\widetilde{\mathcal{O}}_j)$, $H^s(\widetilde{\mathcal{O}}_j)$ and not $L^2(\mathcal{L})$, $H^s(\mathcal{L})$. To further lighten notation we set $\tilde{\kappa}_k = (\det g_k)^{1/2} \kappa_k$. One has

$$P_k = \partial_t^2 - (\tilde{\kappa}_k)^{-1} \sum_{p,q} \partial_p \tilde{\kappa}_k g_k^{pq} \partial_q + 1 = \widetilde{P}_k - \sum_{p,q} R_k^{p,q},$$

with $\widetilde{P}_k = \partial_t^2 - \sum_{p,q} \partial_p g_k^{pq} \partial_q + 1$ and $R_k^{p,q} = (\tilde{\kappa}_k)^{-1} [\partial_p, \tilde{\kappa}_k] g_k^{pq} \partial_q$. Note that $\tilde{\chi}_j B R_k^{p,q} \tilde{\chi}_j$ defines a sequence of bounded operators from $H^1(\mathcal{L})$ into $L^2(\mathcal{L})$, uniformly with respect to k. Consequently, one has

$$\langle \chi_j \tilde{\chi}_j B R_k^{p,q} \tilde{\chi}_j v^k_j, \overline{v^k_j}\rangle_{H^{-1}_{\mathrm{comp}}(\widetilde{\mathcal{O}}_j), H^1_{\mathrm{loc}}(\widetilde{\mathcal{O}}_j)} = (\chi_j \tilde{\chi}_j B R_k^{p,q} \tilde{\chi}_j v^k_j, v^k_j)_{L^2(\widetilde{\mathcal{O}}_j)} \xrightarrow[k \to +\infty]{} 0$$

since v^k_j converges strongly to 0 in $L^2(\widetilde{\mathcal{O}}_j)$. This leads to

$$\begin{aligned}\langle \chi_j \tilde{\chi}_j B P_k \tilde{\chi}_j v^k_j, \overline{v^k_j}\rangle_{H^{-1}_{\mathrm{comp}}(\widetilde{\mathcal{O}}_j), H^1_{\mathrm{loc}}(\widetilde{\mathcal{O}}_j)} &= \langle \chi_j \tilde{\chi}_j B \widetilde{P}_k \tilde{\chi}_j v^k_j, \overline{v^k_j}\rangle_{H^{-1}_{\mathrm{comp}}(\widetilde{\mathcal{O}}_j), H^1_{\mathrm{loc}}(\widetilde{\mathcal{O}}_j)} + o(1)_{k \to +\infty} \\ &= \langle \mu_j, \chi_j b p^0\rangle_{S^*(\widetilde{\mathcal{O}}_j)} + o(1)_{k \to +\infty},\end{aligned}$$

by Proposition 3.4. Since $\chi_j \mu_j = \chi_j \mu$ locally, lifting back the analysis to the manifold level, with (3-11), one finds

$$\langle B P_k u^k, \overline{u^k}\rangle_{H^{-1}_{\mathrm{comp}}(\mathcal{L}), H^1_{\mathrm{loc}}(\mathcal{L})} = \sum_{j \in J} \langle \mu, \chi_j b p^0\rangle_{S^*(\mathcal{L})} = \langle \mu, b p^0\rangle_{S^*(\mathcal{L})} + o(1)_{k \to +\infty}.$$

Now, one has

$$\langle B P_k u^k, \overline{u^k}\rangle_{H^{-1}_{\mathrm{comp}}(\mathcal{L}), H^1_{\mathrm{loc}}(\mathcal{L})} = \langle P_k u^k, \overline{{}^t B u^k}\rangle_{H^{-1}_{\mathrm{loc}}(\mathcal{L}), H^1_{\mathrm{comp}}(\mathcal{L})} + o(1)_{k \to +\infty},$$

with the transpose operator ${}^tB$ bounded from $H^1_{\mathrm{loc}}(\mathcal{L})$ into $H^1_{\mathrm{comp}}(\mathcal{L})$ since B is itself bounded from $H^{-1}_{\mathrm{loc}}(\mathcal{L})$ into $H^{-1}_{\mathrm{comp}}(\mathcal{L})$. If one assumes that $P_k u^k \to 0$ strongly in $H^{-1}_{\mathrm{loc}}((T_1, T_2) \times \mathcal{M})$, one obtains

$$\langle BP_k u^k, \overline{u^k}\rangle_{H^{-1}_{\mathrm{comp}}(\mathcal{L}), H^1_{\mathrm{loc}}(\mathcal{L})} \xrightarrow[k\to+\infty]{} 0,$$

and thus

$$\langle \mu, bp^0\rangle_{S^*(\mathcal{L})} = 0 \quad \text{for all } b \in S^0_{c,\mathrm{ph}}(\mathcal{L}) \text{ with } \operatorname{supp}(b) \subset T^*((T_1, T_2) \times \mathcal{M}),$$

and one obtains the support estimation (3-9).

We now prove the second item of the proposition. We assume that $P_k u^k$ lies in $L^2_{\mathrm{loc}}((T_1, T_2)\times)$ and converges strongly to 0 in this space. Consider $B \in \Psi^1_{c,\mathrm{ph}}(\mathcal{L})$ with kernel supported in $((T_1, T_2) \times \mathcal{M})^2$ and $b \in S^1_{c,\mathrm{ph}}(\mathcal{L})$ its principal symbol. We are interested in the limit of $\langle [P_k, B]u^k, \overline{u^k}\rangle_{H^{-1}_{\mathrm{comp}}(\mathcal{L}), H^1_{\mathrm{loc}}(\mathcal{L})}$, which makes sense since $[P_k, B]$ is of order 2. We have $[P_k, B]u^k = P_k B u^k - BP_k u^k \in H^{-1}((T_1, T_2) \times \mathcal{M})$. Since $P_k u^k$ lies in $L^2((T_1, T_2) \times \mathcal{M})$ by assumption, $BP_k u^k$ lies in $H^{-1}((T_1, T_2) \times \mathcal{M})$ and the same holds for $P_k B u^k$. We may thus write

$$\langle [P_k, B]u^k, \overline{u^k}\rangle_{H^{-1}_{\mathrm{comp}}(\mathcal{L}), H^1_{\mathrm{loc}}(\mathcal{L})} = \langle P_k B u^k, \overline{u^k}\rangle_{H^{-1}_{\mathrm{comp}}(\mathcal{L}), H^1_{\mathrm{loc}}(\mathcal{L})} - \langle P_k u^k, \overline{B^* u^k}\rangle_{L^2_{\mathrm{loc}}(\mathcal{L}), L^2_{\mathrm{comp}}(\mathcal{L})},$$

where the adjoint is computed with respect to the L^2-inner product associated with (k^0, g^0) here. As B maps continuously $L^2_{\mathrm{loc}}((T_1, T_2) \times \mathcal{M})$ into $H^{-1}_{\mathrm{comp}}((T_1, T_2) \times \mathcal{M})$, we have B^* maps continuously $H^1_{\mathrm{loc}}(\mathcal{L})$ into $L^2_{\mathrm{comp}}(\mathcal{L})$. Thus, one has

$$(P_k u^k, B^* u^k)_{L^2(\mathcal{L})} \xrightarrow[k\to+\infty]{} 0.$$

By Lemma 3.6 it is asymptotically equivalent to use (κ^0, g^0) or (κ_k, g_k) for the definition of the L^2-inner product and H^{-1}_{comp}-H^1_{loc} duality, that is,

$$\langle P_k B u^k, \overline{u^k}\rangle_{H^{-1}_{\mathrm{comp}}(\mathcal{L}), H^1_{\mathrm{loc}}(\mathcal{L})} = \langle P_k B u^k, \overline{u^k}\rangle_{H^{-1}_{\mathrm{comp}}(\mathcal{L}, \kappa_k \mu_{g_k} dt), H^1_{\mathrm{loc}}(\mathcal{L}, \kappa_k \mu_{g_k} dt)} + o(1)_{k\to+\infty}.$$

Since P_k is selfadjoint for this latter L^2-inner product, one obtains

$$\begin{aligned}\langle P_k B u^k, \overline{u^k}\rangle_{H^{-1}_{\mathrm{comp}}(\mathcal{L}), H^1_{\mathrm{loc}}(\mathcal{L})} &= \langle B u^k, \overline{P_k u^k}\rangle_{L^2_{\mathrm{comp}}(\mathcal{L}, \kappa_k \mu_{g_k} dt), L^2_{\mathrm{loc}}(\mathcal{L}, \kappa_k \mu_{g_k} dt)} + o(1)_{k\to+\infty} \\ &= \langle B u^k, \overline{P_k u^k}\rangle_{L^2_{\mathrm{comp}}(\mathcal{L}), L^2_{\mathrm{loc}}(\mathcal{L})} + o(1)_{k\to+\infty}.\end{aligned}$$

Using again that $P_k u^k \to 0$ strongly to 0 in $L^2_{\mathrm{loc}}((T_1, T_2) \times \mathcal{M})$, we obtain

$$\langle P_k B u^k, \overline{u^k}\rangle_{H^{-1}_{\mathrm{comp}}(\mathcal{L}), H^1_{\mathrm{loc}}(\mathcal{L})} \xrightarrow[k\to+\infty]{} 0,$$

and finally

$$\langle [P_k, B]u^k, \overline{u^k}\rangle_{H^{-1}_{\mathrm{comp}}(\mathcal{L}), H^1_{\mathrm{loc}}(\mathcal{L})} \xrightarrow[k\to+\infty]{} 0. \tag{3-12}$$

As above, with the partition of unity $1 = \sum_{j\in J} \chi_j$ we write

$$\langle [P_k, B]u^k, \overline{u^k}\rangle_{H^{-1}_{\mathrm{comp}}(\mathcal{L}), H^1_{\mathrm{loc}}(\mathcal{L})} = \sum_{j\in J} \langle \chi_j [P_k, B]u^k, \overline{u^k}\rangle_{H^{-1}_{\mathrm{comp}}(\mathcal{L}), H^1_{\mathrm{loc}}(\mathcal{L})}. \tag{3-13}$$

For each term in the sum one has

$$\langle \chi_j [P_k, B]u^k, \overline{u^k}\rangle_{H^{-1}_{\mathrm{comp}}(\mathcal{L}), H^1_{\mathrm{loc}}(\mathcal{L})} = \langle \chi_j [P_k, \widetilde{B}_j] v^k_j, \overline{v^k_j}\rangle_{H^{-1}_{\mathrm{comp}}(\mathcal{L}), H^1_{\mathrm{loc}}(\mathcal{L})} + o(1)_{k\to+\infty},$$

with $\widetilde{B}_j = \tilde{\chi}_j B \tilde{\chi}_j$. This allows one to work in a local chart and write

$$\langle [P_k, B] u^k, \overline{u^k} \rangle_{H^{-1}_{\mathrm{comp}}(\mathcal{L}), H^1_{\mathrm{loc}}(\mathcal{L})} = \sum_{j \in J} \langle \chi_j [P_k, \widetilde{B}_j] v_j^k, \overline{v_j^k} \rangle_{H^{-1}_{\mathrm{comp}}(\widetilde{\mathcal{O}}_j), H^1_{\mathrm{loc}}(\widetilde{\mathcal{O}}_j)}, \tag{3-14}$$

with the (manifold-local chart) identifications described above. With $A_k = A_{\kappa_k, g_k}$, in the local chart $\mathcal{C}_j$ one writes

$$\chi_j [P_k, \widetilde{B}_j] = \chi_j [\partial_t^2, \widetilde{B}_j] - \chi_j [A_k, \widetilde{B}_j] = \chi_j [\partial_t^2, \widetilde{B}_j] - \sum_{1 \le p,q \le d} (Q_1^{pq} + Q_2^{pq} + Q_3^{pq} + Q_4^{pq}),$$

with

$$Q_1^{pq} = \chi_j \tilde{\kappa}_k^{-1} \partial_{x_p} \tilde{\kappa}_k g_k^{pq} [\partial_{x_q}, \widetilde{B}_j], \quad Q_2^{pq} = \chi_j \tilde{\kappa}_k^{-1} \partial_{x_p} [\tilde{\kappa}_k g_k^{pq}, \widetilde{B}_j] \partial_{x_q},$$
$$Q_3^{pq} = \chi_j \tilde{\kappa}_k^{-1} [\partial_{x_p}, \widetilde{B}_j] \tilde{\kappa}_k g_k^{pq} \partial_{x_q}, \quad Q_4^{pq} = \chi_j [\tilde{\kappa}_k^{-1}, \widetilde{B}_j] \partial_{x_p} \tilde{\kappa}_k g_k^{pq} \partial_{x_q}.$$

We now compute the limit of each term associated with this decomposition of $[P_k, \widetilde{B}_j]$ on the right-hand side of (3-14). The principal symbol of $\chi_j [\partial_t^2, \widetilde{B}_j]$ is $i \chi_j \{\tau^2, b\}$ and thus

$$\langle \chi_j [\partial_t^2, \widetilde{B}_j] v_j^k, \overline{v_j^k} \rangle_{H^{-1}_{\mathrm{comp}}(\widetilde{\mathcal{O}}_j), H^1_{\mathrm{loc}}(\widetilde{\mathcal{O}}_j)} = \langle \mu_j, i \chi_j \{\tau^2, b\} \rangle_{S^*(\widetilde{\mathcal{O}}_j)} + o(1)_{k \to +\infty}.$$

Proposition 3.4 applies and yields

$$\langle Q_1^{pq} v_j^k, \overline{v_j^k} \rangle_{H^{-1}_{\mathrm{comp}}(\widetilde{\mathcal{O}}_j), H^1_{\mathrm{loc}}(\widetilde{\mathcal{O}}_j)} = \langle \mu_j, i \chi_j g^{0,pq} \xi_p \partial_{x_q} b \rangle_{S^*(\widetilde{\mathcal{O}}_j)} + o(1)_{k \to +\infty},$$
$$\langle Q_3^{pq} v_j^k, \overline{v_j^k} \rangle_{H^{-1}_{\mathrm{comp}}(\widetilde{\mathcal{O}}_j), H^1_{\mathrm{loc}}(\widetilde{\mathcal{O}}_j)} = \langle \mu_j, i \chi_j g^{0,pq} (\partial_{x_p} b) \xi_q \rangle_{S^*(\widetilde{\mathcal{O}}_j)} + o(1)_{k \to +\infty}.$$

With Theorem 3.2 one has $[\tilde{\kappa}_k g_k^{pq}, \widetilde{B}_j] \to [\tilde{\kappa}^0 g^{0,pq}, \widetilde{B}_j]$ in $\mathscr{L}(L^2(\widetilde{\mathcal{O}}_j))$ as $k \to +\infty$. It follows that

$$\langle Q_2^{pq} v_j^k, \overline{v_j^k} \rangle_{H^{-1}_{\mathrm{comp}}(\widetilde{\mathcal{O}}_j), H^1_{\mathrm{loc}}(\widetilde{\mathcal{O}}_j)} = \langle Q_{2,a}^{pq} v_j^k, \overline{v_j^k} \rangle_{H^{-1}_{\mathrm{comp}}(\widetilde{\mathcal{O}}_j), H^1_{\mathrm{loc}}(\widetilde{\mathcal{O}}_j)} + o(1)_{k \to +\infty},$$

with

$$Q_{2,a}^{pq} = \chi_j \tilde{\kappa}_k^{-1} \partial_{x_p} [\tilde{\kappa}^0 g^{0,pq}, \widetilde{B}_j] \partial_{x_q}.$$

With Corollary 3.3 one writes

$$[\tilde{\kappa}^0 g^{0,pq}, \widetilde{B}_j] = -\frac{1}{i} \nabla_x (\tilde{\kappa}^0 g^{0,pq}) \cdot \mathrm{Op}(\nabla_\xi (\tilde{\chi}_j^2 b)) + K_1,$$

with K_1 a compact operator on $L^2(\mathbb{R}^{d+1})$, with compactly supported kernel. One thus obtains

$$\langle Q_2^{pq} v_j^k, \overline{v_j^k} \rangle_{H^{-1}_{\mathrm{comp}}(\widetilde{\mathcal{O}}_j), H^1_{\mathrm{loc}}(\widetilde{\mathcal{O}}_j)} = \langle Q_{2,b}^{pq} v_j^k, \overline{v_j^k} \rangle_{H^{-1}_{\mathrm{comp}}(\widetilde{\mathcal{O}}_j), H^1_{\mathrm{loc}}(\widetilde{\mathcal{O}}_j)} + o(1)_{k \to +\infty},$$

with

$$Q_{2,b}^{pq} = -\frac{1}{i} \chi_j \tilde{\kappa}_k^{-1} \partial_{x_p} \nabla_x (\tilde{\kappa}^0 g^{0,pq}) \cdot \mathrm{Op}(\nabla_\xi (\tilde{\chi}_j^2 b)) \partial_{x_q}.$$

Proposition 3.4 applies and yields

$$\langle Q_2^{pq} v_j^k, \overline{v_j^k} \rangle_{H^{-1}_{\mathrm{comp}}(\widetilde{\mathcal{O}}_j), H^1_{\mathrm{loc}}(\widetilde{\mathcal{O}}_j)} = \langle \mu_j, -i \chi_j \xi_p \xi_q (\tilde{\kappa}^0)^{-1} \nabla_x (\tilde{\kappa}^0 g^{0,pq}) \cdot \nabla_\xi b \rangle_{S^*(\widetilde{\mathcal{O}}_j)} + o(1)_{k \to +\infty}.$$

We now treat the term associated with Q_4^{pq}. Note that one has $\sum_{p,q} Q_4^{pq} = \chi_j [\tilde{\kappa}_k^{-1}, \widetilde{B}_j] \tilde{\kappa}_k A_k$. We write, lifting temporarily the analysis back to the manifold,

$$\begin{aligned} \sum_{p,q} \langle Q_4^{pq}, v \rangle_j^k \overline{v_j^k}_{H^{-1}_{\mathrm{comp}}(\widetilde{\mathcal{O}}_j), H^1_{\mathrm{loc}}(\widetilde{\mathcal{O}}_j)} &= \langle \chi_j [\tilde{\kappa}_k^{-1}, B] \tilde{\kappa}_k A_k v_j^k, \overline{v_j^k} \rangle_{H^{-1}_{\mathrm{comp}}(\mathcal{L}), H^1_{\mathrm{loc}}(\mathcal{L})} \\ &= \langle \chi_j [\tilde{\kappa}_k^{-1}, B] \tilde{\kappa}_k A_k u^k, \overline{u^k} \rangle_{H^{-1}_{\mathrm{comp}}(\mathcal{L}), H^1_{\mathrm{loc}}(\mathcal{L})} + o(1)_{k \to +\infty}. \end{aligned}$$

Setting $f^k = (\partial_t^2 - A_k)u^k$ with $f^k \to 0$ strongly in $L^2_{\mathrm{loc}}((T_1, T_2) \times \mathcal{M})$, we thus find

$$\begin{aligned}\sum_{p,q} \langle Q_4^{pq} v_j^k, \overline{v_j^k} \rangle_{H^{-1}_{\mathrm{comp}}(\widetilde{\mathcal{O}}_j), H^1_{\mathrm{loc}}(\widetilde{\mathcal{O}}_j)} &= \langle \chi_j [\tilde{\kappa}_k^{-1}, B] \tilde{\kappa}_k \partial_t^2 u^k, \overline{u^k} \rangle_{H^{-1}_{\mathrm{comp}}(\mathcal{L}), H^1_{\mathrm{loc}}(\mathcal{L})} - \langle \chi_j [\tilde{\kappa}_k^{-1}, B] \tilde{\kappa}_k f^k, \overline{u^k} \rangle_{H^{-1}_{\mathrm{comp}}(\mathcal{L}), H^1_{\mathrm{loc}}(\mathcal{L})} + o(1)_{k\to+\infty} \\ &= \langle \chi_j [\tilde{\kappa}_k^{-1}, \widetilde{B}_j] \tilde{\kappa}_k \partial_t^2 v_j^k, \overline{v_j^k} \rangle_{H^{-1}_{\mathrm{comp}}(\widetilde{\mathcal{O}}_j), H^1_{\mathrm{loc}}(\widetilde{\mathcal{O}}_j)} + o(1)_{k\to+\infty},\end{aligned}$$

bringing again the analysis at the level of the local chart.

Using that $\tilde{\kappa}_k$ is independent of t, we may write

$$\chi_j [\tilde{\kappa}_k^{-1}, \widetilde{B}_j] \tilde{\kappa}_k \partial_t = \chi_j \partial_t [\tilde{\kappa}_k^{-1}, \widetilde{B}_j] \tilde{\kappa}_k + \chi_j [\tilde{\kappa}_k^{-1}, E_j] \tilde{\kappa}_k,$$

where $E_j = [\partial_t, \widetilde{B}_j] \in \Psi^1_{c,\mathrm{ph}}(\widetilde{\mathcal{O}}_j)$, with $\partial_t b \in S^1_{c,\mathrm{ph}}(\widetilde{\mathcal{O}}_j)$ for principal symbol. With Theorem 3.2 we see that $[\tilde{\kappa}_k^{-1}, E_j]$ maps $L^2(\widetilde{\mathcal{O}}_j)$ into itself continuously and moreover $[\tilde{\kappa}_k^{-1}, E_j] \to [(\tilde{\kappa}^0)^{-1}, E_j]$ in $\mathscr{L}(L^2(\widetilde{\mathcal{O}}_j))$. Thus we obtain

$$\begin{aligned}\langle \chi_j [\tilde{\kappa}_k^{-1}, E_j] \tilde{\kappa}_k \partial_t v_j^k, \overline{v_j^k} \rangle_{H^{-1}_{\mathrm{comp}}(\widetilde{\mathcal{O}}_j), H^1_{\mathrm{loc}}(\widetilde{\mathcal{O}}_j)} &= \langle \chi_j [(\tilde{\kappa}^0)^{-1}, E_j] \tilde{\kappa}_k \partial_t v_j^k, \overline{v_j^k} \rangle_{H^{-1}_{\mathrm{comp}}(\widetilde{\mathcal{O}}_j), H^1_{\mathrm{loc}}(\widetilde{\mathcal{O}}_j)} + o(1)_{k\to+\infty} \xrightarrow[k\to+\infty]{} 0,\end{aligned}$$

arguing as above. Similarly we write

$$\langle \chi_j \partial_t [\tilde{\kappa}_k^{-1}, \widetilde{B}_j] \tilde{\kappa}_k \partial_t v_j^k, \overline{v_j^k} \rangle_{H^{-1}_{\mathrm{comp}}(\widetilde{\mathcal{O}}_j), H^1_{\mathrm{loc}}(\widetilde{\mathcal{O}}_j)} \sim_{k\to+\infty} \langle \chi_j \partial_t [(\tilde{\kappa}^0)^{-1}, \widetilde{B}_j] \tilde{\kappa}_k^0 \partial_t v_j^k, \overline{v_j^k} \rangle_{H^{-1}_{\mathrm{comp}}(\widetilde{\mathcal{O}}_j), H^1_{\mathrm{loc}}(\widetilde{\mathcal{O}}_j)}.$$

Arguing as we did for the term associated with $Q_2^{p,q}$ we thus find

$$\langle \chi_j \partial_t [\tilde{\kappa}_k^{-1}, \widetilde{B}_j] \tilde{\kappa}_k \partial_t v_j^k, \overline{v_j^k} \rangle_{H^{-1}_{\mathrm{comp}}(\widetilde{\mathcal{O}}_j), H^1_{\mathrm{loc}}(\widetilde{\mathcal{O}}_j)} = \langle \mu_j, -i \chi_j \tau^2 \tilde{\kappa}^0 (\nabla_x (\tilde{\kappa}^0)^{-1}) \cdot \nabla_\xi b \rangle_{S^*(\widetilde{\mathcal{O}}_j)} + o(1)_{k\to+\infty}.$$

Collecting the various estimates we found we obtain

$$\langle \chi_j [P_k, \widetilde{B}_j] v_j^k, \overline{v_j^k} \rangle_{H^{-1}_{\mathrm{comp}}(\widetilde{\mathcal{O}}_j), H^1_{\mathrm{loc}}(\widetilde{\mathcal{O}}_j)} = \langle \mu_j, \chi_j \sigma \rangle_{S^*(\widetilde{\mathcal{O}}_j)} + o(1)_{k\to+\infty}, \tag{3-15}$$

with

$$\sigma = i\{\tau^2, b\} - i \sum_{p,q} \big(g^{0,pq} \xi_p \partial_{x_q} b + g^{0,pq} (\partial_{x_p} b) \xi_q - \xi_p \xi_q (\tilde{\kappa}^0)^{-1} \nabla_x (\tilde{\kappa}^0 g^{0,pq}) \cdot \nabla_\xi b \big) + i \tau^2 \tilde{\kappa}^0 (\nabla_x (\tilde{\kappa}^0)^{-1}) \cdot \nabla_\xi b.$$

Recalling that $p^0 = -\tau^2 + \sum_{p,q} g^{0,pq} \xi_p \xi_q$, one finds

$$\sigma = -i\{p^0, b\} + i p^0 (\tilde{\kappa}^0)^{-1} \nabla_x (\tilde{\kappa}^0) \cdot \nabla_\xi b.$$

Since μ, and thus μ_j, is supported in $\mathrm{Char}(p^0)$ by the first part of the proposition, one concludes that

$$\langle \chi_j [P_k, \widetilde{B}_j] v_j^k, \overline{v_j^k} \rangle_{H^{-1}_{\mathrm{comp}}(\widetilde{\mathcal{O}}_j), H^1_{\mathrm{loc}}(\widetilde{\mathcal{O}}_j)} = -i \langle \mu_j, \chi_j \{p^0, b\} \rangle_{S^*(\widetilde{\mathcal{O}}_j)} + o(1)_{k\to+\infty}.$$

Since $\chi_j \mu = \chi_j \mu_j$ (see Section 3B), with (3-13)–(3-14) one obtains

$$\langle [P_k, B] u^k, \overline{u^k} \rangle_{H^{-1}_{\mathrm{comp}}(\mathcal{L}), H^1_{\mathrm{loc}}(\mathcal{L})} = -i \langle \mu, \{p^0, b\} \rangle_{S^*(\mathcal{L})} + o(1)_{k\to+\infty}.$$

With (3-12), this concludes the proof of the second part of the proposition since $\{p^0, b\} = \mathrm{H}_{p^0} b$. □

4. Measure support propagation: proof of Theorem 1.10

Theorem 1.10 is stated on an open subset of a smooth manifold. Yet, its result is of a local nature. Using a local chart we may assume that we consider an open of set Ω of $\mathbb{R}^d$ instead without any loss of generality.

The strategy we follow is very much inspired by the approach of Melrose and Sjöstrand [1978] to the propagation of singularities and relies on careful choices of test functions allowing one to construct sequences of points in the support of the measure relying on nonnegativity.[2] Then, a limiting procedure leads to the conclusion, in the spirit of the classical proof of the Cauchy–Peano theorem.

The proof of Theorem 1.10 is made of two steps that are stated in the following propositions.

Proposition 4.1. *Let X be a $\mathscr{C}^0$-vector field on Ω an open set of $\mathbb{R}^d$. For a closed set F of Ω, the following two properties are equivalent*:

(1) *The set F is a union of maximally extended integral curves of the vector field X.*

(2) *For any compact $K \subset \Omega$ where the vector field X does not vanish*,

$$\forall \varepsilon > 0,\ \exists \delta_0 > 0,\ \forall x \in K \cap F,\ \forall \delta \in [-\delta_0, \delta_0], \quad B(x + \delta X(x), \delta\varepsilon) \cap F \neq \varnothing.$$

Proposition 4.2. *Let X be a $\mathscr{C}^0$-vector field on Ω an open set of $\mathbb{R}^d$. Consider a nonnegative measure μ on Ω that is a solution to ${}^tX\mu = 0$ in the sense of distributions, that is*,

$$\langle {}^tX\mu, a\rangle_{\mathscr{D}'(\Omega),\mathscr{C}_c^\infty(\Omega)} = \langle \mu, Xa\rangle_{\mathscr{D}'^{,0}(\Omega),\mathscr{C}_c^0(\Omega)} = 0, \quad a \in \mathscr{C}_c^\infty(\Omega). \tag{4-1}$$

Then, the closed set $F = \operatorname{supp}(\mu)$ satisfies the second property in Proposition 4.1.

Proof of Proposition 4.1. First, we prove that property (1) implies property (2) and consider a compact set K of $\mathbb{R}^d$ such that $K \subset \Omega$ and $K \cap F \neq \varnothing$.

There exists $\eta > 0$ such that $K \subset K_\eta \subset \Omega$ with $K_\eta = \{x \in \Omega : \operatorname{dist}(x, K) \le \eta\}$. One has $\|X\| \le C_0$ on K_η for some $C_0 > 0$. Let $x \in K$ and let $\gamma(s)$ be a maximal integral curve defined on an interval $]a, b[$, $a, b \in \overline{\mathbb{R}}$ and such that $0 \in]a, b[$ and $\gamma(0) = x$. If $b < \infty$ then there exists $s^1 \in]0, b[$ such that $\gamma(s^1) \notin K_\eta$. Since $\gamma(s) \in K_\eta$ if $s < \eta/C_0$, one finds that $b \ge \eta/C_0$. Similarly, one has $|a| \ge \eta/C_0$. Consequently, there exists $S > 0$ such that any maximal integral curve $\gamma(s)$ of the vector field X with $\gamma(0) \in K$ is defined for $s \in I = (-S, S)$.

Let us pick $x \in K$. According to property (1), there exists

$$\gamma : I \to F \text{ such that } \dot\gamma(s) = X(\gamma(s)) \text{ and } \gamma(0) = x.$$

By uniform continuity of the vector field X in a compact neighborhood of K we have

$$\gamma(s) = \gamma(0) + \int_0^s \dot\gamma(s)\,ds = \gamma(0) + \int_0^s X(\gamma(s))\,ds = x + sX(x) + r(s), \quad s \in (-S, S),$$

where $\lim_{s\to 0} \|r(s)\|/s = 0$, *uniformly* with respect to x. We deduce that for any $\varepsilon > 0$ there exists $0 < \delta_0 < S$ such that $\|r(s)\| < s\varepsilon$ for any $s \in (-\delta_0, \delta_0)$, which implies

$$F \ni \gamma(s) \in B(x + sX(x), s\varepsilon).$$

[2] Of the measure in our case and of some operators for Melrose and Sjöstrand, via the Gårding inequality.

Second, we prove that property (2) implies property (1). It suffices to prove that for any $x \in F$ there exist an interval $I \ni 0$ and an integral curve

$$\gamma : I \to F \text{ such that } \dot\gamma(s) = X(\gamma(s)) \text{ and } \gamma(0) = x.$$

Then, the standard continuation argument shows that this local integral curve included in F can be extended to a maximal integral curve also included in F.

If $X(x) = 0$, then the trivial integral curve $\gamma(s) = x$, $s \in \mathbb{R}$, is included in F. As a consequence, we assume $X(x) \neq 0$ and we pick a compact neighborhood K of x containing $B(x, \eta)$ with $\eta > 0$ and where, for some $0 < c_K < C_K$,

$$c_K \le \|X(y)\| \le C_K, \quad y \in K.$$

Let $n \in \mathbb{N}^*$. Set $x_{n,0} = x$ and $\varepsilon = 1/n$ and apply property (2). One deduces that there exist $0 < \delta_n \le 1/n$ and a point

$$x_{n,1} \in F \cap B(x_{n,0} + \delta_n X(x_{n,0}), \delta_n/n).$$

If $x_{n,1} \in K$, one can perform this construction again, starting from $x_{n,1}$ instead of $x_{n,0}$. If a sequence of points $x_{n,0}, x_{n,1}, \dots, x_{n,L^+}$ is obtained in this manner, one has

$$x_{n,\ell+1} \in F \cap B(x_{n,\ell} + \delta_n X(x_{n,\ell}), \delta_n/n), \quad \ell = 0, \dots, L^+ - 1. \tag{4-2}$$

One can carry on the construction as long as $x_{n,L^+} \in K$. We can perform the same construction for $\ell \le 0$, with the property

$$x_{n,\ell-1} \in F \cap B(x_{n,\ell} - \delta_n X(x_{n,\ell}), \delta_n/n), \quad |\ell| = 0, \dots, L^- - 1. \tag{4-3}$$

Having $\|X\| \le C_K$ on K and $B(x, \eta) \subset K$ ensures that we can construct the sequence at least for

$$L^+ = L^- = L_n = \left\lfloor \frac{\eta}{\delta_n(C_K + 1)} \right\rfloor + 1 \le \left\lfloor \frac{\eta}{\delta_n(C_K + 1/n)} \right\rfloor + 1,$$

where $\lfloor \cdot \rfloor$ denotes the floor function. With the points $x_{n,\ell}$, $|\ell| \le L_n$, we have constructed we define the following continuous curve $\gamma_n(s)$ for $|s| \le L_n\delta_n$:

$$\gamma_n(s) = x_{n,\ell} + (s - \ell\delta_n)\frac{x_{n,\ell+1} - x_{n,\ell}}{\delta_n} \quad \text{for } s \in [\ell\delta_n, (\ell+1)\delta_n) \text{ and } |\ell| \le L_n - 1.$$

This curve and its construction is illustrated in Figure 1. Note that $\gamma_n(s)$ remains in a compact set, uniformly with respect to n. In this compact set X is uniformly continuous.

We set $S = \eta/(C_K + 1)$. Since $S \le L_n\delta_n$, we shall in fact only consider the function $\gamma_n(s)$ for $|s| \le S$ in what follows. Note that since $x_{n,\ell} \in F$ for $|\ell| \le L_n$, one has

$$\operatorname{dist}(\gamma_n(s), F) \le \delta_n(C_K + 1/n), \quad |s| \le S. \tag{4-4}$$

From (4-2), for $\ell \ge 0$ and $s \in (\ell\delta_n, (\ell+1)\delta_n)$, we have

$$\dot\gamma_n(s) = \frac{x_{n,\ell+1} - x_{n,\ell}}{\delta_n} = X(x_{n,\ell}) + \mathcal{O}(1/n).$$

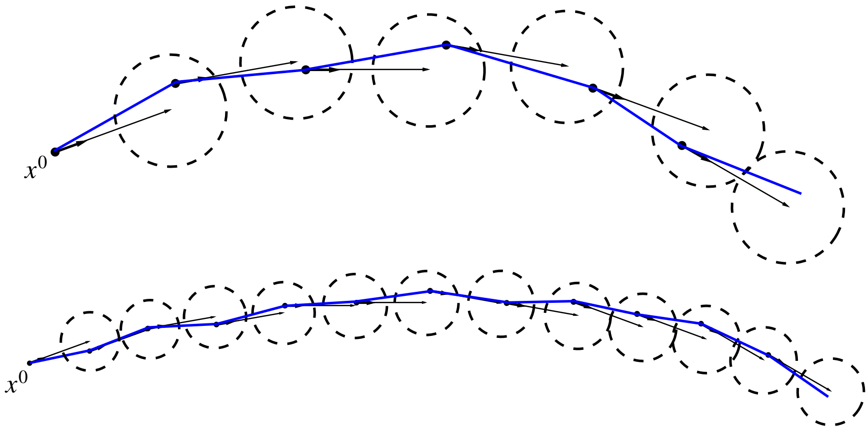

Figure 1. Top: iterative construction of the curve γ_n. Bottom: convergence of γ_n as n increases.

Similarly, from (4-3), for $\ell \le 0$ and $s \in ((\ell-1)\delta_n, \ell\delta_n)$, we have

$$\dot{\gamma}_n(s) = \frac{x_{n,\ell} - x_{n,\ell-1}}{\delta_n} = X(x_{n,\ell}) + \mathcal{O}(1/n).$$

In any case, using the uniform continuity of the vector field X, we find

$$\dot{\gamma}_n(s) = X(\gamma_n(s)) + e_n(s),$$

where the error $|e_n|$ goes to zero *uniformly* with respect to $|s| \le S$ as $n \to +\infty$.

Since the curve γ_n is absolutely continuous (and differentiable except at isolated points), we find

$$\gamma_n(s) = \gamma_n(0) + \int_0^s \dot{\gamma}_n(\sigma)\, d\sigma = x + \int_0^s X(\gamma_n(\sigma))\, d\sigma + \int_0^s e_n(\sigma)\, d\sigma. \tag{4-5}$$

We now let n grow to infinity. With (4-5), the family of curves $(s \mapsto \gamma_n(s), |s| \le S)_{n\in\mathbb{N}^*}$ is equicontinuous and pointwise bounded; by the Arzelà-Ascoli theorem we can extract a subsequence $(s \mapsto \gamma_{n_p})_{p\in\mathbb{N}}$ that converges uniformly to a curve $\gamma(s)$, $|s| \le S$. Convergence is illustrated in Figure 1. Passing to the limit $n_p \to +\infty$ in (4-5) we find that $\gamma(s)$ is solution to

$$\gamma(s) = x + \int_0^s X(\gamma(\sigma))\, d\sigma.$$

From estimation (4-4), for any $|s| \le S$, there exists $(y_p)_p \subset F$ such that $\lim_{p\to+\infty} y_p = \gamma(s)$. Since F is closed we conclude that $\gamma(s) \in F$. □

Positivity argument and proof of Proposition 4.2. We consider a compact set K where the vector field X does not vanish. By continuity of the vector field there exist $0 < c_K \le C_K$ such that $0 < c_K \le \|X(x)\| \le C_K$ for all $x \in K$.

Let us consider $x^0 \in K \cap \operatorname{supp}(\mu)$. By performing a rotation and a dilation by a factor $\|X(x^0)\|$, we can assume that $X(x^0) = (1, 0, \dots, 0) \in \mathbb{R}^d$. We shall write $x = (x_1, x')$ with $x' \in \mathbb{R}^{d-1}$.

Let $\chi \in \mathscr{C}^\infty(\mathbb{R})$ be given by

$$\chi(s) = \mathbf{1}_{s<1} \exp(1/(s-1)), \tag{4-6}$$

and $\beta \in \mathscr{C}^\infty(\mathbb{R})$ be such that

$$\beta \equiv 0 \quad \text{on }]-\infty, -1], \qquad \beta' > 0 \quad \text{on } \left]-1, -\tfrac{1}{2}\right[, \qquad \beta \equiv 1 \quad \text{on } \left[-\tfrac{1}{2}, +\infty\right[. \tag{4-7}$$

We then set

$$q_{\varepsilon,\delta,x^0} = (\chi \circ v)(\beta \circ w), \quad g_{\varepsilon,\delta,x^0} = (\chi' \circ v)(\beta \circ w) X v, \quad h_{\varepsilon,\delta,x^0} = (\chi \circ v)(\beta' \circ w) X w, \tag{4-8}$$

with

$$v(x) = \tfrac{1}{2} - \delta^{-1}(x_1 - x_1^0) + 8(\varepsilon\delta)^{-2}\|x' - x^{0\prime}\|^2 \quad \text{and} \quad w(x) = 2\varepsilon^{-1}(1 - \delta^{-1}(x_1 - x_1^0))$$

for $\varepsilon > 0$ and $\delta > 0$ both meant to be chosen small in what follows. We have $X q_{\varepsilon,\delta,x^0} = g_{\varepsilon,\delta,x^0} + h_{\varepsilon,\delta,x^0}$.

The function $q_{\varepsilon,\delta,x^0}$ is compactly supported. Indeed, in the support of $\beta \circ w$, one has $w \geq -1$, implying

$$x_1 - x_1^0 \leq \delta\left(1 + \tfrac{1}{2}\varepsilon\right),$$

while on the support of $\chi \circ v$ one has $v \leq 1$, which gives

$$-\tfrac{1}{2} + 8(\varepsilon\delta)^{-2}\|x' - x^{0\prime}\|^2 \leq \delta^{-1}(x_1 - x_1^0).$$

On the supports of $q_{\varepsilon,\delta,x^0}$ and $(\chi' \circ v)(\beta \circ w)$, one thus finds

$$-\tfrac{1}{2}\delta \leq x_1 - x_1^0 \leq \delta\left(1 + \tfrac{1}{2}\varepsilon\right) \quad \text{and} \quad 8(\varepsilon\delta)^{-2}\|x' - x^{0\prime}\|^2 \leq \tfrac{3}{2} + \tfrac{1}{2}\varepsilon. \tag{4-9}$$

Similarly, on the support of $\beta' \circ w$ one has $-1 \leq w \leq -\frac{1}{2}$

$$\delta\left(1 + \tfrac{1}{4}\varepsilon\right) \leq x_1 - x_1^0 \leq \delta\left(1 + \tfrac{1}{2}\varepsilon\right),$$

which implies that on the support of $h_{\varepsilon,\delta,x^0}$ one has

$$\delta\left(1 + \tfrac{1}{4}\varepsilon\right) \leq x_1 - x_1^0 \leq \delta\left(1 + \tfrac{1}{2}\varepsilon\right) \quad \text{and} \quad 8(\varepsilon\delta)^{-2}\|x' - x^{0\prime}\|^2 \leq \tfrac{3}{2} + \tfrac{1}{2}\varepsilon. \tag{4-10}$$

In particular, in the case $\varepsilon \leq 1$, one finds

$$\operatorname{supp}(h_{\varepsilon,\delta,x^0}) \subset B(x^0 + \delta X(x^0), \varepsilon\delta). \tag{4-11}$$

These estimations of the supports of $q_{\varepsilon,\delta,x^0}$ and $h_{\varepsilon,\delta,x^0}$ are illustrated in Figure 2.

Lemma 4.3. *For any $0 < \varepsilon \leq 1$ there exists $\delta_0 > 0$ such that, for any $x^0 \in K$ and $0 < \delta \leq \delta_0$, the function $g_{\varepsilon,\delta,x^0}$ is nonnegative. Moreover, $g_{\varepsilon,\delta,x^0}$ is positive in a neighborhood of x^0.*

Proof. Let $0 < \varepsilon \leq 1$. We have $g_{\varepsilon,\delta,x^0} = (\chi' \circ v)(\beta \circ w) X v$. Since $\beta \geq 0$ and $\chi' < 0$, it suffices to prove that $X v(x) \leq 0$ for x in the support of $(\chi' \circ v)(\beta \circ w)$ for $\delta > 0$ chosen sufficiently small, uniformly with respect to $x^0 \in K$.

We write

$$X(x) - X(x^0) = \alpha^1(x, x^0)\partial_{x_1} + \alpha'(x, x^0) \cdot \nabla_{x'},$$

with $\alpha^1(x, x^0) \in \mathbb{R}$ and $\alpha'(x, x^0) \in \mathbb{R}^{d-1}$. By (4-9), for $x \in \operatorname{supp}(\chi' \circ v)(\beta \circ w)$ we have $\|x - x^0\| \lesssim \delta$. From the uniform continuity of X in any compact set we conclude that

$$|\alpha^1(x, x^0)| + \|\alpha'(x, x^0)\| = o(1) \quad \text{as } \delta \to 0^+, \tag{4-12}$$

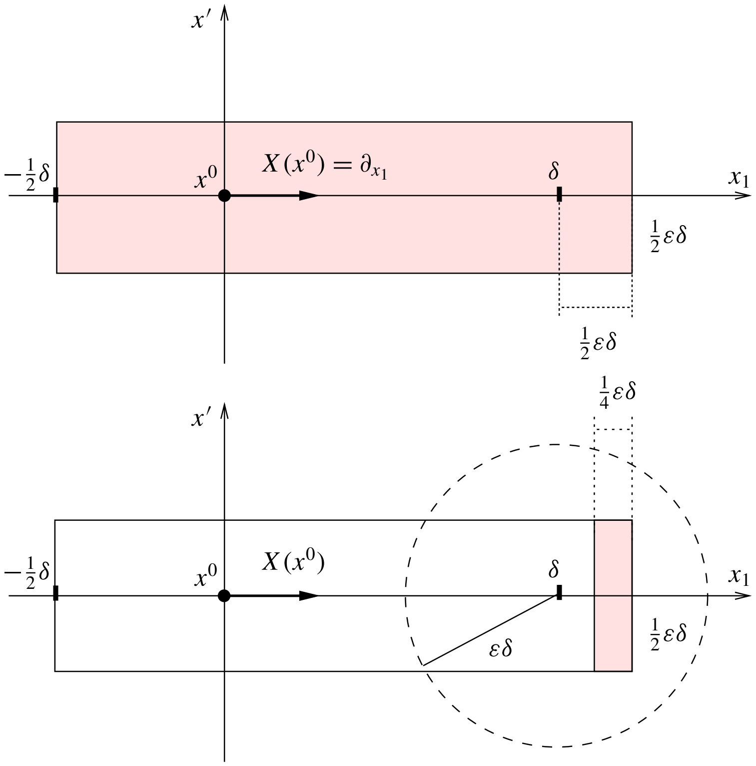

Figure 2. Estimation of the test function supports in the case $\varepsilon \le 1$. Top: support of $h_{\varepsilon,\delta,x^0}$. Bottom: support of $q_{\varepsilon,\delta,x^0}$.

uniformly[3] with respect to $x^0 \in K$ and $x \in \operatorname{supp}(\chi' \circ v)(\beta \circ w)$. Using that $X(x^0) = \partial_{x_1}$ and the form of v given above, we write

$$\begin{aligned} Xv(x) &= (X(x)v)(x) = \big(\partial_{x_1} v + (X(x) - X(x^0))v\big)(x) \\ &= -\delta^{-1}\big(1 + \alpha^1(x, x^0) - 16\varepsilon^{-1}(\varepsilon\delta)^{-1}\alpha'(x, x^0) \cdot (x' - x^{0\prime})\big). \end{aligned} \tag{4-13}$$

Using again (4-9), we thus find for $x \in \operatorname{supp}(\chi' \circ v)(\beta \circ w)$

$$|\alpha^1(x, x^0) - 16\varepsilon^{-1}(\varepsilon\delta)^{-1}\alpha'(x, x^0) \cdot (x' - x^{0\prime})| \lesssim |\alpha^1(x, x^0)| + \varepsilon^{-1}\|\alpha'(x, x^0)\|.$$

With ε fixed above and with (4-12), we find that $Xv(x) \sim -\delta^{-1}$ as $\delta \to 0^+$ uniformly with respect to $x^0 \in K$ and $x \in \operatorname{supp}(\chi' \circ v)(\beta \circ w)$.

Finally, we have $g_{\varepsilon,\delta,x^0}(x^0) = -\delta^{-1}\chi'\big(\frac{1}{2}\big)\beta(2\varepsilon^{-1}) > 0$ and thus $g_{\varepsilon,\delta,x^0}$ is positive in a neighborhood of x^0. □

We are now in a position to conclude the proof of Proposition 4.2. Note that it suffices to prove the result for $0 < \varepsilon \le 1$. We choose $\delta_0 > 0$ as given by Lemma 4.3. Let then $x^0 \in K \cap \operatorname{supp}(\mu)$. We apply (4-1)

[3] Observe that the change of variables made above for $X(x^0) = (1, 0, \dots, 0)$ does not affect uniformity since the dilation is made by a factor in $[c_K, C_K]$.

to the family $q_{\varepsilon,\delta,x^0}$ of test functions with $0 < \delta \le \delta_0$:

$$0 = \langle \mu, X(q_{\varepsilon,\delta,x^0})\rangle = \langle \mu, g_{\varepsilon,\delta,x^0}\rangle + \langle \mu, h_{\varepsilon,\delta,x^0}\rangle. \tag{4-14}$$

By Lemma 4.3, $g_{\varepsilon,\delta,x^0} \ge 0$ and $g_{\varepsilon,\delta,x^0}$ is positive in a neighborhood of x^0. As $x^0 \in \operatorname{supp}(\mu)$ we find $\langle \mu, g_{\varepsilon,\delta,x^0}\rangle > 0$. Consequently, $\langle \mu, h_{\varepsilon,\delta,x^0}\rangle \neq 0$. By the support estimate for $h_{\varepsilon,\delta,x^0}$ given in (4-11) the conclusion follows: $\operatorname{supp}(\mu) \cap B(x^0 + \delta X(x^0), \varepsilon\delta) \neq \varnothing$. □

5. Exact controllability: proof of Theorem 1.12

Let $(\kappa^0, g^0) \in \mathcal{X}^1(\mathcal{M})$ and assume that (ω, T) fulfills the geometric control condition of Definition 1.8″.

Let also $(\kappa, g) \in \mathcal{Y}(\mathcal{M})$. With Proposition 1.6, the result of Theorem 1.12 follows if we prove that there exists $\varepsilon > 0$ and $C_{\text{obs}} > 0$ such that

$$\mathcal{E}_{\kappa,g}(u)(0) \le C_{\text{obs}} \|\mathbf{1}_{(0,T)\times\omega}\, \partial_t u\|^2_{L^2(\mathcal{L},\kappa\mu_g dt)}$$

for any weak solution u of the wave equation associated with (κ, g) chosen such that

$$\|(\kappa, g) - (\kappa^0, g^0)\|_{\mathcal{Y}(\mathcal{M})} \le \varepsilon.$$

The L^2-norm on the right-hand side is associated with (κ, g), that is,

$$\|\mathbf{1}_{(0,T)\times\omega}\, \partial_t u\|^2_{L^2(\mathcal{L},\kappa\mu_g dt)} = \int_0^T \int_\omega |\partial_t u|^2\, \kappa \mu_g\, dt.$$

Yet, for $\varepsilon > 0$ chosen sufficiently small one has $\|\cdot\|_{L^2(\mathcal{L},\kappa^0\mu_{g^0})} \eqsim \|\cdot\|_{L^2(\mathcal{L},\kappa\mu_g)}$, where $A \eqsim B$ means $c_1 \le A/B \le c_2$ for some $c_1, c_2 > 0$. In other words, we have equivalence with constants uniform with respect to (κ, g). In what follows, L^2- and more generally H^s-norms on $\mathcal{M}$ are chosen with respect to $\kappa^0\mu_{g^0}$ unless explicitly written. Our goal is thus to prove the observability inequality

$$\mathcal{E}_{\kappa^0,g^0}(u)(0) \le C_{\text{obs}} \|\mathbf{1}_{(0,T)\times\omega}\, \partial_t u\|^2_{L^2(\mathcal{L})}. \tag{5-1}$$

The Bardos–Lebeau–Rauch uniqueness compactness argument reduces the proof of (5-1) to the proof of the weaker estimate

$$\mathcal{E}_{\kappa^0,g^0}(u)(0) \le C \|\mathbf{1}_{(0,T)\times\omega}\, \partial_t u\|^2_{L^2(\mathcal{L})} + C' \big\|(u(0), \partial_t u(0))\big\|^2_{L^2(\mathcal{M})\oplus H^{-1}(\mathcal{M})}, \tag{5-2}$$

which exhibits an additional compact term, and expresses observability for high frequencies. Low frequencies are dealt with by means of a unique continuation argument.

To prove (5-2) we argue by contradiction and we assume that there exists a sequence $(\kappa_k, g_k)_{k\in\mathbb{N}} \subset \mathcal{Y}(\mathcal{M})$ such that

$$\lim_{k\to+\infty} \|(\kappa_k, g_k) - (\kappa^0, g^0)\|_{\mathcal{Y}(\mathcal{M})} = 0, \tag{5-3}$$

and yet for each $k \in \mathbb{N}$ the associated observability inequality does not hold. Thus, for each $k \in \mathbb{N}$, there exists a sequence of initial data $(v^{k,p,0}, v^{k,p,1})_{p\in\mathbb{N}} \subset H^1(\mathcal{M}) \times L^2(\mathcal{M})$ with associated solution $(v^{k,p})_{p\in\mathbb{N}}$, that is,

$$\begin{cases} P_k v^{k,p} = 0 & \text{in } (0,+\infty)\times\mathcal{M}, \\ v^{k,p}{}_{|t=0} = v^{k,p,0}, \;\; \partial_t v^{k,p}{}_{|t=0} = v^{k,p,1} & \text{in } \mathcal{M}, \end{cases}$$

with $P_k = P_{\kappa_k, g_k}$, that moreover has the properties

$$\mathcal{E}_{\kappa^0,g^0}(v^{k,p})(0) = 1 \quad \text{and} \quad \|\mathbf{1}_{(0,T)\times\omega}\, \partial_t v^{k,p}\|_{L^2(\mathcal{L})} + \|(v^{k,p,0}, v^{k,p,1})\|_{L^2(\mathcal{M})\oplus H^{-1}(\mathcal{M})} \le \frac{1}{p+1}.$$

We take $p = k$ and we set$(u^{k,0}, u^{k,1}) = (v^{k,k,0}, v^{k,k,1})$ and $u^k = v^{k,k}$; one obtains $P_k u^k = 0$ in $\mathcal{L}$ and

$$\mathcal{E}_{\kappa^0,g^0}(u^k)(0) = 1 \quad \text{and} \quad \|\mathbf{1}_{(0,T)\times\omega}\, \partial_t u^k\|_{L^2(\mathcal{L})} + \|(u^{k,0}, u^{k,1})\|_{L^2(\mathcal{M})\oplus H^{-1}(\mathcal{M})} \le \frac{1}{k+1}. \tag{5-4}$$

From (5-4) one has $u^k \rightharpoonup 0$ weakly in $H^1_{\mathrm{loc}}(\mathcal{L})$. With (3-1)–(3-2), we can associate with (a subsequence of) $(u^k)_k$ an H^1-microlocal defect measure μ on $S^*(\mathcal{L})$. Here, the measure is understood with respect to $L^2(\mathcal{L}, \kappa^0\mu_{g^0}\, dt)$.

From the second part of (5-4) one has

$$\mu = 0 \quad \text{in } S^*((0,T)\times\omega). \tag{5-5}$$

In fact, for any $\psi \in \mathscr{C}^\infty((0,T)\times\omega)$ one has $\|\psi\partial_t u^k\|_{L^2(\mathcal{L})} \sim 0$ and thus $\langle \mu, \tau^2\psi^2\rangle = 0$. Hence, $\operatorname{supp}(\mu) \cap S^*((0,T)\times\omega) \subset \{\tau = 0\}$. Since $\{\tau = 0\} \cap \operatorname{Char}(p^0) \cap S^*(\mathcal{L}) = \varnothing$ with (3-9) one obtains (5-5).

With the first part of (5-4) one has the following lemma.

Lemma 5.1. *The measure μ does not vanish on $S^*(\mathcal{L})$.*

A proof is given below.

We now use Proposition 3.7 to obtain a precise description of the measure μ. First, one has $\operatorname{supp}(\mu) \cap S^*((0,T)\times\mathcal{M}) \subset \operatorname{Char}(p^0)$. Furthermore, one has ${}^t\mathrm{H}_{p^0}\mu = 0$ in the sense of distributions on $S^*((0,T)\times\mathcal{M})$. Since H_{p^0} is a $\mathscr{C}^0$-vector field on the manifold $S^*\mathcal{L}$, Theorem 1.10 implies that $\operatorname{supp}(\mu)$ is a union of maximally extended bicharacteristics in $S^*((0,T)\times\mathcal{M})$.

Under the geometric control condition of Definition 1.8″, any maximal bicharacteristic meets $S^*((0,T)\times\omega)$ where μ vanishes by (5-5). Thus $\operatorname{supp}(\mu) = \varnothing$, yielding a contradiction with the result of Lemma 5.1. We thus obtain that (5-1) holds. This concludes the proof of Theorem 1.12. ☐

Proof of Lemma 5.1. Let $T_1 < T_2$ and $\phi \in \mathscr{C}^\infty_c(\mathbb{R})$ nonnegative and equal to 1 on a neighborhood of $[T_1, T_2]$. On $\mathcal{L}$, consider the *elliptic* operator $Q = -\partial_t^2 - A_{\kappa^0,g^0} + 1$ with symbol $q = \tau^2 + \sum_{p,q} g^{0\,p,q}(x)\xi_p\xi_q$. Taking (3-2) and Lemma 3.6 into account one can write

$$\langle \phi^2 Q u^k, \overline{u^k}\rangle_{H^{-1}_{\mathrm{comp}}(\mathcal{L}), H^1_{\mathrm{loc}}(\mathcal{L})} \underset{k\to+\infty}{\sim} \langle \mu, \phi^2 q\rangle_{S^*\mathcal{L}}. \tag{5-6}$$

Integrating by parts one obtains

$$\begin{aligned}\langle \phi^2 Q u^k, \overline{u^k}\rangle_{H^{-1}_{\mathrm{comp}}(\mathcal{L}), H^1_{\mathrm{loc}}(\mathcal{L})} &= \int_{\mathcal{L}} \phi(t)^2\big(|\partial_t u^k|^2 + g^0(\nabla_{g^0} u^k, \nabla_{g^0}\bar{u}^k) + |u^k|^2\big)\kappa^0\mu_{g^0}\, dt + 2(\phi'\phi\,\partial_t u^k, u^k)_{L^2(\mathcal{L})}\\ &= \int_{\mathbb{R}} \phi(t)^2 \mathcal{E}_{\kappa^0,g^0}(u^k)(t)\, dt + 2(\phi'\phi\,\partial_t u^k, u^k)_{L^2(\mathcal{L})}.\end{aligned}$$

Since the energy built on κ^p, g^p is preserved by the evolution given by P_p, we have by (5-4)

$$\mathcal{E}_{\kappa^0,g^0}(u^k)(t) = \mathcal{E}_{\kappa^k,g^k}(u^k)(t) + o(1) = \mathcal{E}_{\kappa^k,g^k}(u^k)(0) + o(1) = \mathcal{E}_{\kappa^0,g^0}(u^k)(0) + o(1) = 1 + o(1) \tag{5-7}$$

and since $(\phi'\phi\, \partial_t u^k, u^k)_{L^2(\mathcal{L})} \to 0$ as $u^k \to 0$ strongly in $L^2_{\mathrm{loc}}(\mathcal{L})$, one obtains

$$\langle \phi^2 Q u^k, \overline{u^k}\rangle_{H^{-1}_{\mathrm{comp}}(\mathcal{L}), H^1_{\mathrm{loc}}(\mathcal{L})} \underset{k\to+\infty}{\sim} \|\phi\|^2_{L^2(\mathbb{R})}.$$

With (5-6) this proves that $\mu \neq 0$. □

6. Lack of continuity of the control operator with respect to coefficients

6A. ***Proof of Theorems 1.14 and 1.14′.*** We prove the result of both theorems, that is, in the case $k \geq 1$. In the case $k = 1$ we are simply required to prove additionally that the geometric control condition of Definition 1.8 is fulfilled for geodesics given by the chosen metric $\tilde{g}$; see Remark 1.17.

Let $\varepsilon > 0$. We set $\tilde{g} = (1+\varepsilon)g$. Given any neighborhood $\mathcal{U}$ of (κ, g) in $\mathcal{X}^k(\mathcal{M})$, for $\varepsilon > 0$ chosen sufficiently small one has $(\kappa, \tilde{g}) \in \mathcal{U}$.

Moreover, observe that, for $\varepsilon > 0$ chosen sufficiently small, geodesics associated with $\tilde{g}$ can be made arbitrarily close to those associated with g uniformly in $t \in [0, T]$. Hence, for such $\varepsilon > 0$ the geometric control condition is fulfilled for geodesics associated with $\tilde{g}$.

Observe that one has

$$\mathrm{Char}(p_{\kappa,g}) \cap \mathrm{Char}(p_{\kappa,\tilde{g}}) \cap S^*\mathcal{L} = \varnothing. \tag{6-1}$$

We consider a sequence $(y^{k,0}, y^{k,1}) \rightharpoonup (0,0)$ weakly in $H^1(\mathcal{M}) \oplus L^2(\mathcal{M})$ such that

$$\tfrac{1}{2}(\|y^{k,0}\|^2_{H^1(\mathcal{M})} + \|y^{k,1}\|^2_{L^2(\mathcal{M})}) = 1.$$

L^2- and H^1-norms are based on the $\kappa\mu_g\, dt$ measure on $\mathcal{L}$.

Setting $f^k_{\kappa,g} = H_{\kappa,g}(y^{k,0}, y^{k,1}) \in L^2((0,T)\times\mathcal{M})$ with $H_{\kappa,g}$ defined in (1-7), one obtains a sequence of control functions. According to the HUM method [Lions 1988], $f^k_{\kappa,g}$ is itself a (weak) solution to the following free wave equation

$$P_{\kappa,g} f^k_{\kappa,g} = 0, \tag{6-2}$$

in the energy space $L^2(\mathcal{M}) \oplus H^{-1}(\mathcal{M})$, that is, $(f^k_{\kappa,g}(0), \partial_t f^k_{\kappa,g}(0)) \in L^2(\mathcal{M}) \times H^{-1}(\mathcal{M})$. Moreover, $(f^k_{\kappa,g}(0), \partial_t f^k_{\kappa,g}(0))$ depend continuously on $(y^{k,0}, y^{k,1})$. The function $f^k_{\kappa,g}$ is thus bounded in $\mathscr{C}^0((T_1,T_2), L^2(\mathcal{M}))$ uniformly with respect to k for any $T_1 < T_2$. Since the map $H_{\kappa,g}$ is continuous, $f^k_{\kappa,g} \rightharpoonup 0$ weakly in $L^2_{\mathrm{loc}}(\mathcal{L})$. Up to extraction of a subsequence, it is associated with an L^2-microlocal defect measure μ_f. With Proposition 3.7′ one has

$$\mathrm{supp}(\mu_f) \subset \mathrm{Char}(p_{\kappa,g}). \tag{6-3}$$

We consider the sequences of solutions $(y^k)_k$ and $(\tilde{y}^k)_k$ to

$$\begin{cases} P_{\kappa,g} y^k = \mathbf{1}_{(0,T)\times\omega} f^k_{\kappa,g} & \text{in } \mathcal{L},\\ (y^k, \partial_t y^k)_{|t=0} = (y^{k,0}, y^{k,1}) & \text{in } \mathcal{M},\end{cases} \qquad \begin{cases} P_{\kappa,\tilde{g}} \tilde{y}^k = \mathbf{1}_{(0,T)\times\omega} f^k_{\kappa,g} & \text{in } \mathcal{L},\\ (\tilde{y}^k, \partial_t \tilde{y}^k)_{|t=0} = (y^{k,0}, y^{k,1}) & \text{in } \mathcal{M}.\end{cases}$$

Both are bounded and weakly converge to 0 in $H^1_{\mathrm{loc}}(\mathcal{L})$. Up to extraction of subsequences, both are associated with H^1-microlocal defect density measures μ and $\tilde{\mu}$ respectively. Since $\mathbf{1}_{(0,T)\times\omega} f^k_{\kappa,g} \rightharpoonup 0$ weakly in $L^2_{\mathrm{loc}}(\mathcal{L})$ we have $\mathbf{1}_{(0,T)\times\omega} f^k_{\kappa,g} \to 0$ strongly in $H^{-1}_{\mathrm{loc}}(\mathcal{L})$ and, with Proposition 3.7, one finds

$\text{supp}(\tilde{\mu}) \subset \text{Char}(p_{\kappa,\tilde{g}})$. Thus one has

$$\text{supp}(\tilde{\mu}) \cap \text{supp}(\mu_f) = \varnothing. \tag{6-4}$$

The sequence $(\partial_t \tilde{y}^k)$ converges to 0 weakly in $L^2_{\text{loc}}(\mathcal{L})$ and can be associated with an L^2-microlocal defect density measure whose support is given by $\text{supp}(\tilde{\mu})$.

Lemma 6.1. *One has* $(\mathbf{1}_{(0,T)\times\omega}\, f^k_{\kappa,g}, \partial_t \tilde{y}^k)_{L^2(\mathcal{L},\kappa\mu_{\tilde{g}}dt)} \to 0$ *as* $k \to +\infty$.

A proof is given below.

Using the density of strong solutions of the wave equation, with integration by parts, one finds the classical energy estimate

$$\mathcal{E}_{\kappa,\tilde{g}}(\tilde{y}^k)(T) - \mathcal{E}_{\kappa,\tilde{g}}(\tilde{y}^k)(0) = (\mathbf{1}_{(0,T)\times\omega}\, f^k_{\kappa,g}, \partial_t \tilde{y}^k)_{L^2(\kappa\mu_{\tilde{g}}dt)}.$$

With Lemma 6.1 one obtains

$$\mathcal{E}_{\kappa,\tilde{g}}(\tilde{y}^k)(T) \underset{k\to+\infty}{\sim} \mathcal{E}_{\kappa,\tilde{g}}(\tilde{y}^k)(0).$$

With the form of $\tilde{g}$ chosen above one has

$$\mathcal{E}_{\kappa,g}(\tilde{y}^k)(t) = (1 + \mathcal{O}(\varepsilon))\mathcal{E}_{\kappa,\tilde{g}}(\tilde{y}^k)(t),$$

uniformly with respect to $t \in [0, T]$. Choosing $\varepsilon > 0$ sufficiently small and k sufficiently large, the first part of Theorem 1.14 follows since $\mathcal{E}_{\kappa,g}(\tilde{y}^k)(0) = 1$.

We use the values of ε and k chosen above. To prove (1-10), we write $\tilde{y}^k$ in the form $\tilde{y}^k = v_1 + v_2$, where v_1 and v_2 are solutions to

$$\begin{cases} P_{\kappa,\tilde{g}} v_1 = \mathbf{1}_{(0,T)\times\omega}\, f^k_{\kappa,\tilde{g}} & \text{in } \mathcal{L}, \\ (v_1, \partial_t v_1)_{|t=0} = (y^{k,0}, y^{k,1}) & \text{in } \mathcal{M}, \end{cases} \qquad \begin{cases} P_{\kappa,\tilde{g}} v_2 = \mathbf{1}_{(0,T)\times\omega}(f^k_{\kappa,g} - f^k_{\kappa,\tilde{g}}) & \text{in } \mathcal{L}, \\ (v_2, \partial_t v_2)_{|t=0} = (0, 0) & \text{in } \mathcal{M}, \end{cases} \tag{6-5}$$

with $f^k_{\kappa,\tilde{g}} = H_{\kappa,\tilde{g}}(y^{k,0}, y^{k,1})$. A hyperbolic energy estimation for the solution v_2 to the second equation in (6-5) gives

$$\mathcal{E}_{\kappa,\tilde{g}}(v_2)(T) \le C_T \|\mathbf{1}_{(0,T)\times\omega}(f^k_{\kappa,g} - f^k_{\kappa,\tilde{g}})\|^2_{L^2(\mathcal{L})}.$$

Since one has $(v_1(T), \partial_t v_1(T)) = (0, 0)$, because of the definition of $f^k_{\kappa,\tilde{g}}$ one finds

$$\mathcal{E}_{\kappa,\tilde{g}}(v_2)(T) = \mathcal{E}_{\kappa,\tilde{g}}(\tilde{y}^k)(T) \ge \tfrac{1}{2},$$

which gives the second result of Theorem 1.14. □

Proof of Lemma 6.1. The key point in the proof is the following lemma.

Lemma 6.2 [Gérard 1991, Proposition 3.1]. *Assume that u_k and v_k are two sequences bounded in L^2_{loc} that converge weakly to zero and are associated with defect measures μ and ν respectively. Assume that $\mu \perp \nu$, that is, μ and ν are supported on disjoint sets. Then, for any $\psi \in \mathscr{C}^0_c$,*

$$\lim_{k\to+\infty} (\psi u_k, v_k)_{L^2} = 0.$$

To apply this result, we just need to exchange the rough cutoff $\mathbf{1}_{(0,T)\times\omega}$ for a smooth cutoff $\psi(t,x)$. First, note that one has

$$(\mathbf{1}_{(0,T)\times\omega}\, f^k_{\kappa,g}, \partial_t \tilde{y}^k)_{L^2(\mathcal{L},\kappa\mu_{\tilde{g}}dt)} \underset{k\to+\infty}{\sim} (\mathbf{1}_{(0,T)\times\omega}\, f^k_{\kappa,g}, \partial_t \tilde{y}^k)_{L^2(\mathcal{L},\kappa\mu_{g}dt)}.$$

We may thus simply consider the L^2-norm and inner product associated with $\kappa\mu_g\,dt$.

Second, let $\delta > 0$. Since $(f^k_{\kappa,g})_k$ and $(\tilde{y}^k)_k$ are both bounded in $\mathscr{C}^0((0,T), L^2(\mathcal{M}))$ uniformly with respect to k, there exists $0 < T_1 < T_2 < T$ and $\mathcal{O} \Subset \omega$ such that

$$\iint_K |f^k_{\kappa,g}||\partial_t \tilde{y}_k|\kappa\mu_g\,dt \le \delta,$$

with $K = ((0,T)\times\omega)\setminus((T_1,T_2)\times\mathcal{O})$. Let $\psi \in \mathscr{C}^\infty_c((0,T)\times\omega)$ such that $0 \le \psi \le 1$ and equal to 1 in a neighborhood of $[T_1,T_2]\times\mathcal{O}$. One thus has

$$\begin{aligned}|(\mathbf{1}_{(0,T)\times\omega}\, f^k_{\kappa,g}, \partial_t \tilde{y}^k)_{L^2(\mathcal{L})}| &\le |(\psi f^k_{\kappa,g}, \partial_t \tilde{y}^k)_{L^2(\mathcal{L})}| + |((\mathbf{1}_{(0,T)\times\omega} - \psi) f^k_{\kappa,g}, \partial_t \tilde{y}^k)_{L^2(\mathcal{L})}| \\ &\le |(\psi f^k_{\kappa,g}, \partial_t \tilde{y}^k)_{L^2(\mathcal{L})}| + \delta.\end{aligned}$$

With (6-4) and Lemma 6.2, one finds

$$(\psi f^k_{\kappa,g}, \partial_t \tilde{y}^k)_{L^2(\mathcal{L})} \xrightarrow[k\to+\infty]{} 0, \tag{6-6}$$

and the conclusion of the lemma follows. □

6B. *Proof of Proposition 1.19.* We consider first the case $\alpha = 1$. As proven in [Dehman and Lebeau 2009] one has $f^{y^0,y^1}_{\kappa,g} \in \mathscr{C}^0([0,T], H^1(\mathcal{M}))$ and the estimate

$$\|f^{y^0,y^1}_{\kappa,g}\|_{L^\infty(0,T;H^1(\mathcal{M}))} \lesssim \|(y^0,y^1)\|_{H^2(\mathcal{M})\oplus H^1(\mathcal{M})}.$$

With this regularity of the source term in the right-hand-side of the wave equations in (1-8), one finds $y, \tilde{y} \in \mathscr{C}^0([0,T], H^2(\mathcal{M}))$. Computing the difference in (1-8) one writes

$$P_{\kappa,g}(y-\tilde{y}) = (A_{\kappa,g} - A_{\tilde{\kappa},\tilde{g}})\tilde{y}. \tag{6-7}$$

A hyperbolic energy estimate yields

$$\begin{aligned}\mathcal{E}_{\kappa,g}(y-\tilde{y})(T)^{1/2} &\lesssim \|(A_{\kappa,g} - A_{\tilde{\kappa},\tilde{g}})\tilde{y}\|_{L^\infty(0,T;L^2(\mathcal{M}))} \lesssim \|(\kappa,g)-(\tilde{\kappa},\tilde{g})\|_{\mathcal{X}^1}\|\tilde{y}\|_{L^\infty(0,T;H^2(\mathcal{M}))} \\ &\lesssim \|(\kappa,g)-(\tilde{\kappa},\tilde{g})\|_{\mathcal{X}^1}\|f^{y^0,y^1}_{\kappa,g}\|_{L^\infty(0,T;H^1(\mathcal{M}))} \\ &\lesssim \|(\kappa,g)-(\tilde{\kappa},\tilde{g})\|_{\mathcal{X}^1}\|(y^0,y^1)\|_{H^2(\mathcal{M})\oplus H^1(\mathcal{M})}.\end{aligned}$$

In the case $\alpha = 0$, one writes

$$\begin{aligned}\mathcal{E}_{\kappa,g}(y-\tilde{y})(T)^{1/2} &\lesssim \mathcal{E}_{\kappa,g}(y)(T)^{1/2} + \mathcal{E}_{\kappa,g}(\tilde{y})(T)^{1/2} \lesssim \mathcal{E}_{\kappa,g}(y)(T)^{1/2} + \mathcal{E}_{\tilde{\kappa},\tilde{g}}(\tilde{y})(T)^{1/2} \\ &\lesssim \|(y^0,y^1)\|_{H^1(\mathcal{M})\oplus L^2(\mathcal{M})}.\end{aligned}$$

Finally, the result follows from interpolation between the two cases $\alpha = 0$ and $\alpha = 1$. □

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THE DIRICHLET PROBLEM FOR THE LAGRANGIAN MEAN CURVATURE EQUATION

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We solve the Dirichlet problem with continuous boundary data for the Lagrangian mean curvature equation on a uniformly convex, bounded domain in $\mathbb{R}^n$.

1. Introduction

We consider the Dirichlet problem for the Lagrangian mean curvature equation on a uniformly convex, bounded domain $\Omega \subset \mathbb{R}^n$ given by

$$\begin{cases} F(D^2u) = \sum_{i=1}^n \arctan \lambda_i = \psi(x) & \text{in } \Omega, \\ u = \phi & \text{on } \partial\Omega, \end{cases} \tag{1-1}$$

where the λ_i are the eigenvalues of the Hessian matrix D^2u, ψ is the potential for the mean curvature of the Lagrangian submanifold $\{(x, Du(x)) \mid x \in \Omega\} \subseteq \mathbb{R}^n \times \mathbb{R}^n$, and ϕ is a given continuous function on $\partial\Omega$.

Our main result is the following:

Theorem 1.1. *Suppose that $\phi \in C^0(\partial\Omega)$ and $\psi : \bar{\Omega} \to \left[(n-2)\frac{\pi}{2} + \delta, n\frac{\pi}{2}\right)$ is in $C^{1,1}(\bar{\Omega})$, where Ω is a uniformly convex, bounded domain in $\mathbb{R}^n$ and $\delta > 0$. Then there exists a unique solution $u \in C^{2,\alpha}(\Omega) \cap C^0(\partial\Omega)$ to the Dirichlet problem (1-1).*

We also provide a viscosity-based proof for the following well-known result established in [Harvey and Lawson 2009].

Theorem 1.2. *Suppose that $\phi \in C^0(\partial\Omega)$ and $\psi : \bar{\Omega} \to \left(-n\frac{\pi}{2}, n\frac{\pi}{2}\right)$ is a constant, where Ω is a uniformly convex, bounded domain in $\mathbb{R}^n$. Then there exists a unique solution $u \in C^0(\bar{\Omega})$ to the Dirichlet problem (1-1).*

When the phase ψ is constant, denoted by c, we have that u solves the special Lagrangian equation

$$\sum_{i=1}^n \arctan \lambda_i = c, \tag{1-2}$$

or equivalently,

$$\cos c \sum_{1 \le 2k+1 \le n} (-1)^k \sigma_{2k+1} - \sin c \sum_{0 \le 2k \le n} (-1)^k \sigma_{2k} = 0.$$

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Equation (1-2) originates in the special Lagrangian geometry of Harvey and Lawson [1982]. The Lagrangian graph $(x, Du(x)) \subset \mathbb{R}^n \times \mathbb{R}^n$ is called special when the argument of the complex number $(1+i\lambda_1)\cdots(1+i\lambda_n)$ or the phase ψ is constant, and it is special if and only if $(x, Du(x))$ is a (volume-minimizing) minimal surface in $(\mathbb{R}^n \times \mathbb{R}^n, dx^2+dy^2)$ [Harvey and Lawson 1982].

A dual form of (1-2) is the Monge–Ampère equation

$$\sum_{i=1}^{n} \ln \lambda_i = c.$$

This is the potential equation for special Lagrangian submanifolds in $(\mathbb{R}^n \times \mathbb{R}^n, dx\, dy)$ as interpreted in [Hitchin 1997]. The gradient graph $(x, Du(x))$ is volume-maximizing in this pseudo-Euclidean space as shown in [Warren 2010]. Mealy [1989] showed that an equivalent algebraic form of the above equation is the potential equation for his volume-maximizing special Lagrangian submanifolds in $(\mathbb{R}^n \times \mathbb{R}^n, dx^2-dy^2)$.

A key prerequisite for the smooth solvability of the Dirichlet problem for fully nonlinear, elliptic equations is the concavity of the operator on the space of symmetric matrices. The arctangent operator or the logarithmic operator is concave if u is convex, or if the Hessian of u has a lower bound $\lambda \geq 0$. Certain concavity properties of the arctangent operator are still preserved for saddle u. The concavity of the arctangent operator in (1-1) depends on the range of the Lagrangian phase. The phase $(n-2)\frac{\pi}{2}$ is called critical because the level set $\{\lambda \in \mathbb{R}^n \mid \lambda \text{ satisfying (1-1)}\}$ is convex only when $|\psi| \geq (n-2)\frac{\pi}{2}$ [Yuan 2006, Lemma 2.2]. The concavity of the level set is evident for $|\psi| \geq (n-1)\frac{\pi}{2}$ since that implies $\lambda > 0$ and then F is concave. For a supercritical phase $|\psi| \geq (n-2)\frac{\pi}{2}+\delta$ the operator F can be extended to a concave operator [Chen and Warren 2019; Collins et al. 2017].

The Dirichlet problem for fully nonlinear, elliptic equations of the form $F(\lambda[D^2u]) = \psi(x)$ was studied by Caffarelli, Nirenberg, and Spruck in [Caffarelli et al. 1985], where they proved the existence of classical solutions under various hypotheses on the function F and the domain. Their results extended the work of Krylov [1983b], Ivochkina [1983], and their previous work [Caffarelli et al. 1984] on equations of Monge–Ampère-type. For the Monge–Ampère equation, continuous boundary data leads to only Lipschitz continuous solutions; Pogorelov [1978] constructed his famous counterexamples for the three dimensional Monge–Ampère equation $\sigma_3(D^2u) = \det(D^2u) = 1$, which also serve as counterexamples for cubic and higher-order symmetric σ_k equations. Trudinger [1995] proved a priori estimates and existence of smooth solutions to fully nonlinear equations of the type of Hessian equations. In [Ivochkina et al. 2004], Ivochkina, Trudinger, and Wang studied the Dirichlet problem for a class of fully nonlinear, degenerate elliptic equations which depend only on the eigenvalues of the Hessian matrix. Harvey and Lawson [2009] studied the Dirichlet problem for fully nonlinear, degenerate elliptic equations of the form $F(D^2u) = 0$ on a smoothly bounded domain in $\mathbb{R}^n$. Interior regularity for viscosity solutions of (1-2) with critical and supercritical constant phase $|\psi| \geq (n-2)\frac{\pi}{2}$ was shown in [Warren and Yuan 2010; 2014]. For a subcritical phase $|\psi| < (n-2)\frac{\pi}{2}$, singular solutions of (1-2) were constructed in [Nadirashvili and Vlăduţ 2010; Wang and Yuan 2013]. The existence and uniqueness of continuous viscosity solutions to the Dirichlet problem for (1-2) with continuous boundary data was shown in Yuan [2008]. Brendle and Warren [2010] studied a second boundary value problem for the special Lagrangian equation.

The Lagrangian mean curvature equation (1-1), which was introduced by Harvey and Lawson, is far from being completely understood. This gives rise to several challenging problems concerning the regularity of solutions and the well-posedness for general phase functions. Recently, regularity and effective Hessian estimates for viscosity solutions of equation (1-1) were studied in [Bhattacharya 2021; Bhattacharya and Shankar 2020; 2023] under certain assumptions on the regularity of the phase and convexity properties of the solution. In [Collins et al. 2017], Collins, Picard, and Wu solved the Dirichlet problem (1-1) on a compact domain with C^4 boundary value under the assumption of the existence of a subsolution and a supercritical phase restriction using techniques accumulated since the 1980s. In [Dinew et al. 2019], Dinew, Do, and Tô showed the existence and uniqueness of a C^0 solution to (1-1) on a bounded C^2 domain with C^0 boundary value under the assumption of the existence of a subsolution and a supercritical phase restriction.

The major difficulty in proving Theorem 1.1 is the unavailability of smooth boundary data: our boundary value is merely C^0. We use a standard continuity method and uniform approximation of the C^0 boundary value to overcome this. Another hurdle lies in estimating the double normal derivatives at the boundary: we use Trudinger's technique and a change of basis argument to construct a lower linear barrier function for u_n. Once we obtain uniform $C^{2,\alpha}$ estimates up to the boundary, we use the a priori interior Hessian estimates proved in [Bhattacharya 2021] to approximate the C^0 boundary value. Note that we assume $\psi \geq (n-2)\frac{\pi}{2}+\delta$ since, by symmetry, $\psi \leq -(n-2)\frac{\pi}{2}-\delta$ can be treated similarly.

In Theorem 1.2, we consider all values of the constant Lagrangian phase, which include subcritical values. The main difficulty here is the lack of uniform ellipticity and concavity. Harvey and Lawson [2009] established the existence and uniqueness of continuous solutions of fully nonlinear, degenerate elliptic equations of the form $F(D^2u)=0$ on a smoothly bounded domain in $\mathbb{R}^n$ under an explicit geometric F-convexity assumption on the boundary of the domain. The key ingredients of their proof were the use of subaffine functions and Dirichlet duality. As an application, the continuous solvability of the constant phase equation (1-2) is obtained. Here in Theorem 1.2, we focus only on the continuous solvability of the Dirichlet problem of equation (1-2) and provide a short proof that solely relies on a certain comparison principle. Note that our methods of proving Theorem 1.2 are much different in nature than the proof by Harvey and Lawson: our brief proof follows via Perron's method using an idea that was introduced in [Ishii 1989], and it requires comparison principles for strictly elliptic,[1] nonconcave, fully nonlinear equations [Yuan 2004].

Remark 1.3. For Theorem 1.1, an assumption weaker than C^1 on ψ will lead to counterexamples with continuous boundary data. For example, in two dimensions, we consider a boundary value problem of (1-1) on the unit ball $B_1(0)$, where the phase is in C^α with $\alpha \in (0,1)$:

$$\psi(x)=\frac{\pi}{2}-\arctan(\alpha^{-1}|x|^{1-\alpha}) \qquad \text{and} \qquad u(x)=\int_0^{|x|} t^\alpha\, dt \quad \text{on } \partial B_1.$$

This problem admits a non-C^2 viscosity solution u with gradient $Du=|x|^{\alpha-1}x$, thereby proving a contradiction. If the Lagrangian phase is subcritical, i.e., $|\psi|<(n-2)\frac{\pi}{2}$, then even for the constant

[1] $F(D^2u)=\psi$ is strictly elliptic in the sense that $(F_{u_{ij}}(D^2u))>0$

phase equation (1-2) with analytic boundary data, C^0 viscosity solutions may only be C^{1,ε_0} but no more, as shown in [Wang and Yuan 2013]. However, the existence of $C^{2,\alpha}$ solutions to (1-1) with critical and supercritical phase, i.e., $|\psi| \geq (n-2)\frac{\pi}{2}$, where $\psi \in C^{1,\varepsilon_0}$, or even $|\psi| \geq (n-2)\frac{\pi}{2}$, where $\psi \in C^{1,1}$, are still open questions. As of now, it is also unknown if C^0 viscosity solutions of (1-2) are Lipschitz for subcritical phases.

Remark 1.4. In Theorem 1.2, if we replace the constant phase with any continuous function lying in the subcritical or critical range, then the existence and uniqueness of C^0 viscosity solutions of (1-1) remain open questions. This is due to the lack of a suitable comparison principle for strictly elliptic, nonconcave, fully nonlinear equations with a variable right-hand side. Harvey and Lawson [2019] introduced a condition called "tameness" on the operator F, which is a little stronger than strict ellipticity and allows one to prove comparison. Harvey and Lawson [2021] further proved that, for the Lagrangian mean curvature equation, one can only show tamability in the supercritical phase interval. Cirant and Payne [2021] established comparison for this equation when the range of the phase is restricted to the intervals $\big((n-2k)\frac{\pi}{2}, (n-2(k-1))\frac{\pi}{2}\big)$, where $1 \leq k \leq n$. This in turn solves the Dirichlet problem on these intervals, as shown in [Harvey and Lawson 2021, Theorem 6.2(C)]. For σ_k equations with a variable right-hand side, results analogous to Theorem 1.2 exist. This is due to the fact that the linearized operator has a positive lower bound in determinant unlike the Lagrangian mean curvature equation (1-1).

This article is divided into the following sections: in Section 2, we state some well-known algebraic and trigonometric inequalities satisfied by solutions of (1-1). In Section 3, we prove $C^{2,\alpha}$ estimates up to the boundary assuming C^4 boundary data. In Section 4, we first solve the Dirichlet problem with C^4 boundary data using the method of continuity and then combine it with the Hessian estimates proved in [Bhattacharya 2021] to solve the Dirichlet problem with continuous boundary data. In Section 5, we prove Theorem 1.2. In the Appendix, we state a well-known linear algebra lemma that we use in estimating the Hessian of u on the boundary, and we provide the proof of a certain comparison principle that is essential for the proof of Theorem 1.2.

2. Preliminaries

The induced Riemannian metric on the Lagrangian submanifold $\{(x, Du(x)) \mid x \in \Omega\} \subset \mathbb{R}^n \times \mathbb{R}^n$ is given by

$$g = I_n + (D^2u)^2. \tag{2-1}$$

On taking the gradient of both sides of the Lagrangian mean curvature equation (1-1), we get

$$\sum_{a,b=1}^{n} g^{ab} u_{jab} = \psi_j, \tag{2-2}$$

where g^{ab} is the inverse of the induced Riemannian metric g. From [Harvey and Lawson 1982, (2.19)], we see that the mean curvature vector $\vec{H}$ of this Lagrangian submanifold $\{(x, Du(x)) \mid x \in \Omega\}$ is given by $\vec{H} = J\nabla_g \psi$, where ∇_g is the gradient operator for the metric g and J is the complex structure, or the $\frac{\pi}{2}$ rotation matrix in $\mathbb{R}^n \times \mathbb{R}^n$.

Lemma 2.1. *Suppose that the ordered real numbers* $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$ *satisfy* (1-1) *with* $\psi \geq (n-2)\frac{\pi}{2}$. *Then we have*

(1) $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_{n-1} > 0,\ \lambda_{n-1} \geq |\lambda_n|$,

(2) $\lambda_1 + (n-1)\lambda_n \geq 0$,

(3) $\sigma_k(\lambda_1, \dots, \lambda_n) \geq 0$ *for all* $1 \leq k < n$ *and* $n \geq 2$,

(4) *if* $\psi \geq (n-2)\frac{\pi}{2} + \delta$, *then* $D^2u \geq -\cot(\delta I_n)$.

Proof. Properties (1), (2), and (3) follow from [Wang and Yuan 2014, Lemma 2.1]. Property (4) follows from [Yuan 2006, p. 1356]. □

3. $C^{2,\alpha}$ estimate up to the boundary

We first prove the following $C^{2,\alpha}$ estimate up to the boundary of Ω.

Theorem 3.1. *Let* $\phi \in C^4(\bar{\Omega})$ *and* $\psi : \bar{\Omega} \to \big[(n-2)\frac{\pi}{2} + \delta, n\frac{\pi}{2}\big)$ *be in* $C^{2,\alpha}(\bar{\Omega})$, *where* Ω *is a uniformly convex domain in* $\mathbb{R}^n$ *with* $\partial\Omega \in C^2$. *Then there exists a universal constant* $\alpha \in (0,1)$ *such that if* $u \in C^{4,\alpha}(\bar{\Omega})$ *is a solution of* (1-1), *then*

$$\|u\|_{C^{2,\alpha}(\bar{\Omega})} \leq C(\|\psi\|_{C^{1,1}(\bar{\Omega})}, \|\phi\|_{C^4(\bar{\Omega})}, n, \delta, \partial\Omega). \tag{3-1}$$

Proof. We first make the following observation, which will be used for Steps 1, 2, 3.2, and 3.3 below. We pick an arbitrary boundary point $x_0 \in \partial\Omega$. By a rotation and translation, we choose a coordinate system such that the chosen boundary point is the origin and Ω lies above the hyperplane $\{x_n = 0\}$, with e_n as the inner unit normal at 0. For such a domain, we can write

$$\partial\Omega = \big\{(x', x_n) \mid x_n = h(x') = \tfrac{1}{2}(k_1 x_1^2 + \cdots + k_{n-1}x_{n-1}^2) + o(|x'|^2)\big\}, \tag{3-2}$$

where the $\{k_i\}_{1 \leq i \leq n}$ denote the principal curvatures of $\partial\Omega$ at 0. At $0 \in \partial\Omega$ the boundary value satisfies

$$\begin{aligned}
\phi(x', x_n) &= \phi(x', h(x')) \\
&= \phi(0) + \phi_{x'}(0)\cdot x' + \phi_{x_n}(0)h(x') \\
&\qquad + \tfrac{1}{2}(x')^T\phi_{x'x'}(0)x' + \tfrac{1}{2}\phi_{x'x_n}(0)\cdot x' h(x') + \tfrac{1}{2}\phi_{x_n x_n}(0)h(x')h(x') + o(|x'|^2 + h^2(x')) \\
&= Q(x) + o(1)|x'|^2.
\end{aligned}$$

Without loss of generality, one may subtract the linear part in x' of the above Taylor expansion to get $C_0 = C_0(\|\phi\|_{C^2(\partial\Omega)}, n, k)$ such that

$$L^- = -C_0 x_n \leq \phi \leq C_0 x_n = L^+ \quad \text{on } \partial\Omega. \tag{3-3}$$

We now prove estimate (3-1) in the following four steps. We will estimate all the boundary derivatives of u at the origin.

<u>Step 1</u>: Bound for $\|u\|_{L^\infty(\bar{\Omega})}$.

Claim 1. *We show the following*:

$$\|u\|_{L^\infty(\bar{\Omega})} \leq C(\|\phi\|_{C^2(\bar{\Omega})}, n, |\partial\Omega|_{C^2}). \tag{3-4}$$

Proof. The function $\psi : \overline{\Omega} \to \big[(n-2)\frac{\pi}{2}+\delta, n\frac{\pi}{2}\big)$ is in $C^{1,1}(\overline{\Omega})$, so there exists $\varepsilon > 0$ such that $\psi < n\frac{\pi}{2} - \varepsilon$. Fixing this ε, we define $\underline{\psi} = (n-2)\frac{\pi}{2}+\delta$ and $\overline{\psi} = n\frac{\pi}{2} - \varepsilon$. Recalling (3-3) we find constants c_0 and C_0' depending on C_0 above such that, on $\partial\Omega$, we have

$$-c_0|x|^2 + \frac{1}{2}|x|^2 \tan\frac{\overline{\psi}}{n} = -C_0'|x|^2 \le -C_0 x_n \le \phi \le C_0'|x|^2 + \frac{1}{2}|x|^2 \tan\frac{\underline{\psi}}{n}. \tag{3-5}$$

Using relation (3-2), we define

$$-Cx_n + \frac{1}{2}|x|^2 \tan\frac{\overline{\psi}}{n} = B^-, \tag{3-6}$$

$$Cx_n + \frac{1}{2}|x|^2 \tan\frac{\underline{\psi}}{n} = B^+, \tag{3-7}$$

where $C = C(\|\phi\|_{C^2(\partial\Omega)}, n, k_i)$. We observe that

$$\begin{aligned} F(D^2B^-) \ge F(D^2u) \ge F(D^2B^+) &\quad \text{in } \Omega, \\ B^- \le u \le B^+ &\quad \text{on } \partial\Omega, \quad \text{with equality holding at } 0. \end{aligned} \tag{3-8}$$

Using comparison principles we see that (3-4) holds. □

Step 2: Bound for $\|Du\|_{L^\infty(\overline{\Omega})}$.

Claim 2. *We show the following*:

$$\|Du\|_{L^\infty(\overline{\Omega})} \le C(\|\psi\|_{C^1(\overline{\Omega})}, \|\phi\|_{C^2(\overline{\Omega})}, n, \delta, |\partial\Omega|_{C^2}). \tag{3-9}$$

Proof. From Lemma 2.1, we see that u is semiconvex: $D^2u \ge -\cot(\delta I_n)$. We modify u to the convex function $u(x)+\cot(\delta|x|^2/2)$. Since the gradient of this convex function, given by $Du(x)+x\cot\delta$, attains its supremum on the boundary of Ω, we get

$$\sup_{\overline{\Omega}} |Du(x)| \le \sup_{\partial\Omega} |Du(x)| + \cot\delta. \tag{3-10}$$

For $1 \le i < n$, we have $u_i = \phi_i$, so we only need to estimate $u_n(0)$. Recalling (3-8), we again use comparison principles, and on taking the normal derivative at 0, we get

$$|u_n(0)| \le C(\|\psi\|_{C^1(\overline{\Omega})}, \|\phi\|_{C^2(\overline{\Omega})}, n, |\partial\Omega|_{C^2}).$$

Combining (3-10) with the above we get (3-9). □

Step 3: Bound for $\|D^2u\|_{L^\infty(\overline{\Omega})}$.

Claim 3. *We prove the following*:

$$\|D^2u\|_{L^\infty(\overline{\Omega})} \le C(\|\psi\|_{C^{1,1}(\overline{\Omega})}, \|\phi\|_{C^4(\overline{\Omega})}, n, \delta, |\partial\Omega|_{C^4}). \tag{3-11}$$

The proof of the above claim is achieved by from the following steps.

Step 3.1: We first prove that the Hessian attains its supremum on the boundary of Ω. We show that

$$\|D^2u\|_{L^\infty(\overline{\Omega})} \le C(\|\psi\|_{C^{1,1}(\overline{\Omega})}, \|D^2u\|_{L^\infty(\partial\Omega)}, \delta). \tag{3-12}$$

Since the phase is supercritical, we can modify the operator F to a concave operator as shown in [Collins et al. 2017, Lemma 2.2] or [Chen and Warren 2019, p. 347]. (For a detailed proof of this fact, see [Collins et al. 2017, Lemma 2.2].) Following the notation used in [Chen and Warren 2019, p. 347], we denote the modified concave operator by $\widetilde{F} = -\exp(-A(\delta)F)$ and the modified phase by $\tilde{\psi}(\lambda) = -\exp(-A(\delta)\psi(\lambda))$, where $A(\delta)$ is large enough. On differentiating (1-1) twice, we get

$$\widetilde{F}^{ij}\partial_{ij}u_{ee} + \widetilde{F}^{ij,kl}\partial_{ij}u_e\partial_{kl}u_e = \tilde{\psi}_{ee},$$

$$\widetilde{F}^{ij}\partial_{ij}\Delta u = \Delta\tilde{\psi} - \sum_e \widetilde{F}^{ij,kl}\partial_{ij}u_e\partial_{kl}u_e \geq \Delta\tilde{\psi},$$

where the last inequality follows from the concavity of the operator. Let p_0 be an interior point of Ω. By an orthogonal transformation, we assume D^2u to be diagonalized at p_0. We observe that

$$g^{ij}\partial_{ij}\big(\Delta u + \tfrac{1}{2}C_1|x|^2\big)(p_0) \geq -C(\|\psi\|_{C^{1,1}(\Omega)}) + C_1\sum_{i=1}^{n}\frac{1}{1+\lambda_i^2} > 0.$$

The last two inequalities follow from using the structure of the metric g (defined in (2-1)) and then choosing a large enough constant C_1 by exploiting the semiconvexity of u. The maximal principle implies that $|D^2u|$ attains its supremum on the boundary. Next, we estimate the Hessian on the boundary in the following steps: we first estimate the double tangential derivatives $u_{TT}(0)$, followed by the mixed tangent normal derivatives $u_{TN}(0)$, followed by the double normal derivative $u_{NN}(0)$.

Step 3.2: The double tangential estimate. Denoting the second fundamental form by II, we observe that

$$D^2(u-\phi)|_T(0) = -(u-\phi)_n(0)II|_{\partial\Omega}(0),$$

where

$$(D^2u)|_T = \{u_{T_iT_j} \mid 1 \leq i,\, j < n\}$$

is the Riemannian Hessian. By estimate (3-9) derived in Step 2, for $1 \leq i, j < n$, we get the estimate:

$$|u_{ij}(0)| \leq C(\|\psi\|_{C^1(\bar{\Omega})}, \|\phi\|_{C^2(\bar{\Omega})}, n, \delta, \Omega).$$

Step 3.3: The mixed tangent normal estimate. Observe that (1-1) is dependent only on the eigenvalues of the Hessian and hence is invariant under rotation of coordinates. In light of [Caffarelli et al. 1985, p. 281], we observe that, since $x_i\partial_j - x_j\partial_i$ for $i \neq j$ is the infinitesimal generator of a rotation, we get

$$g^{ij}\partial_{ij}(x_i\partial_j - x_j\partial_i)u = (x_i\partial_j - x_j\partial_i)\psi.$$

For $i < n$, we define the annular vector field

$$\tau(x) = \partial_i + \sum_{j=1}^{n-1} h_{ij}(0)(x_j\partial_n - x_n\partial_j),$$

where h is as defined in (3-2); $\tau(0) = e_i$ for $i < n$. This is an approximated tangent vector up to the second-order on the boundary. Indeed, at a point $(x', h(x'))$ on $\partial\Omega$, near the origin, we can write

$$\tau(x) = \partial_i + \partial_i h(x')\partial_n + O(|x'|^2)\partial_n - \sum_{j=1}^{n-1} h_{ij}(0)h(x')\partial_j.$$

Denoting the rotational derivative of u along the boundary by u_τ, we get $g^{ij}\partial_{ij}u_\tau = \psi_\tau$ in Ω and $u_\tau = \phi_\tau$ on $\partial\Omega$.

Replacing ϕ with ϕ_τ and repeating the argument in (3-3), we get the following on $\partial\Omega$:

$$-Cx_n \le \phi_\tau \le Cx_n, \tag{3-13}$$

where $C = C(\|\phi_\tau\|_{C^2(\bar{\Omega})}, n, k)$. Repeating the argument in (3-5) and choosing $c_1 > 0$ suitably, we get

$$-c_1|x|^2 + \frac{1}{2}|x|^2 \tan\frac{\overline{\psi}}{n} = -C|x|^2 \le \phi_\tau \le C|x|^2 + \frac{1}{2}|x|^2 \tan\frac{\psi}{n} \quad \text{on } \partial\Omega.$$

We define u_0 to be the subsolution

$$u_0 = -Cx_n + \frac{1}{2}|x|^2 \tan\frac{\overline{\psi}}{n},$$

where $C = C(\|\phi\|_{C^3(\bar{\Omega})}, \|\psi\|_{C^1(\bar{\Omega})}, n, |\partial\Omega|_{C^2})$. Let $w = u - u_0$. Since the phase lies in the supercritical range, as before we extend the operator F to the concave operator $\widetilde{F}$ and denote the corresponding linearization by $\tilde{g}^{ij}$. Using concavity, for some $\varepsilon_0 > 0$, we get the following on a small ball of radius r around the origin:

$$\begin{aligned} \tilde{g}^{ij} w_{ij} &\le -\varepsilon_0 \quad \text{inside } \Omega \cap B_r(0), \\ w &\ge 0 \qquad \text{on } \partial(\Omega \cap B_r(0)), \\ w(0) &= 0. \end{aligned} \tag{3-14}$$

We now choose α and β large enough that

$$\begin{aligned} \tilde{g}^{ij}\partial_{ij}(\alpha w + \beta|x|^2 \pm u_\tau) &\le 0 \quad \text{in } \Omega \cap B_r(0), \\ \alpha w + \beta|x|^2 \pm u_\tau &\ge 0 \quad \text{on } \partial(\Omega \cap B_r(0)). \end{aligned} \tag{3-15}$$

Since $w \ge 0$ on $\partial(\Omega \cap B_r(0))$, we only need to choose β large enough that

$$\beta|x|^2 \pm u_\tau \ge 0 \quad \text{on } \partial(\Omega \cap B_r(0)).$$

We observe that, on $\Omega \cap \partial B_r(0)$, we have $\beta \ge C/r^2$, where $C = C(\|\psi\|_{C^1(\bar{\Omega})}, \|\phi\|_{C^2(\bar{\Omega})}, \delta, n, |\partial\Omega|_{C^2})$ is obtained by using the gradient estimate in (3-9). Using (3-13) we get the required value of β on $\partial\Omega \cap B_r(0)$. Fixing the larger of the two values to be the constant β we now choose α such that (3-15) holds. We have

$$\tilde{g}^{ij}\partial_{ij}(\alpha w + \beta|x|^2 \pm u_\tau) \le -\alpha\varepsilon_0 + C,$$

where $C = C(\beta, |\psi|_{C^1(\bar{\Omega})})$. We now choose α large enough that $-\alpha\varepsilon_0 + C \le 0$ and observe that $\alpha w + \beta|x|^2 \pm u_\tau(0) = 0$ at 0. Using Hopf's lemma we see that

$$\partial_n(\alpha w + \beta|x|^2 \pm u_\tau)(0) \ge 0 \implies \pm u_{\tau n}(0) \ge \mp\partial_n(\alpha w + \beta|x|^2 \pm u_\tau)(0) \implies |u_{\tau n}(0)| \le |\alpha w_n(0)| \le C.$$

Therefore, for $1 \le i < n$, we have

$$|u_{in}(0)| \le C(\|\psi\|_{C^{1,1}(\bar{\Omega})}, \|\phi\|_{C^3(\bar{\Omega})}, n, \delta, |\partial\Omega|_{C^2}).$$

<u>Step 3.4</u>: The double normal estimate. By Lemma 2.1, D^2u is bounded from below, so we only need to prove an upper bound, which we find using an idea of Trudinger [1995].

Let the unit normal direction vector be denoted by e_γ. Denoting the eigenvalues of the $(n-1)\times(n-1)$ matrix u_{TT} by λ', we write the Hessian as

$$D^2u=\begin{bmatrix}u_{TT} & u_{T\gamma}\\ u_{\gamma T} & u_{\gamma\gamma}\end{bmatrix}=\begin{bmatrix}\lambda' & u_{T\gamma}\\ u_{\gamma T} & u_{\gamma\gamma}\end{bmatrix}.$$

Let x_0' be the minimal point of $\widetilde{\Theta}(\lambda')|_{\partial\Omega}$, where

$$\widetilde{\Theta}(\lambda')=\sum_{i=1}^{n-1}\arctan\lambda_i'-\psi,$$

and we write $\lambda_0'=\lambda'(x_0')$.

Our goal is to find a lower linear barrier function for u_γ at x_0'. Then, with the help of a change of basis technique, we find a lower linear barrier function for u_n at x_0'. This leads us to find an upper bound of $u_{nn}(x_0')$ followed by an upper bound of $u_{nn}(x)$ for all $x\in\partial\Omega$. Now we estimate the lower bound of

$$\operatorname{tr}(D^2u)|_T=\sum_{i=1}^{n-1}\lambda_i'.$$

Observe that $\widetilde{\Theta}(\lambda')\geq\widetilde{\Theta}(\lambda_0')>\psi-\frac{\pi}{2}>(n-3)\frac{\pi}{2}$. So the level set $\{\lambda'\in\mathbb{R}^{n-1}\mid\widetilde{\Theta}(\lambda')=\widetilde{\Theta}(\lambda_0')\}$ should be convex. Heuristically, this property means the following:

$$\langle D\widetilde{\Theta}(\lambda_0'),\lambda'\rangle\geq\langle D\widetilde{\Theta}(\lambda_0'),\lambda_0'\rangle=K_0,\quad\text{with equality holding at } x_0',$$

where K_0 is a constant depending on $|\psi|_{C^1(\Omega)}$, $|\phi|_{C^2(\partial\Omega)}$, and δ. Writing

$$\left[\frac{\partial\widetilde{\Theta}(D^2u(x_0))|_T}{\partial D^2u|_T}\right]=A_{ij}(\lambda_0'),$$

where $1\leq i,\ j<n$, we see that

$$\operatorname{tr}(A_{ij}(\lambda_0'))(D^2u(x)|_T)\geq K_0,\quad\text{with equality holding at } x_0'.$$

Again denoting the second fundamental form by II, we observe that

$$D^2(u-\phi)|_T=(u-\phi)_\gamma II|_{\partial\Omega}\quad\text{and}$$
$$\operatorname{tr}[A_{ij}(\lambda_0')(D^2\phi|_T-\phi_\gamma II|_{\partial\Omega}+u_\gamma II|_{\partial\Omega})]\geq K_0,\quad\text{with equality holding at } x_0'.$$

Writing $\widetilde{\Theta}_i(\lambda')=(\partial/\partial\lambda_i')\widetilde{\Theta}(\lambda')$, we get

$$u_\gamma\geq\frac{1}{\sum_{i=1}^{n-1}\widetilde{\Theta}_i(\lambda_0')\kappa_i(x')}[K_0-\operatorname{tr}(A_{ij}(\lambda_0')(D^2\phi|_T-\phi_\gamma II|_{\partial\Omega}))],\quad\text{with equality holding at } x_0',\tag{3-16}$$
$$\implies\quad u_\gamma\geq C(|\phi|_{C^4(\overline{\Omega})},|\partial\Omega|_{C^4},|\psi|_{C^1(\Omega)},\delta),\quad\text{with equality holding at } x_0',$$

where the last inequality follows from the observation that, for all the terms in the right-hand side of (3-16), one can find a lower linear barrier function whose Lipschitz norm depends on the $C^{3,1}$ norm of ϕ and the C^1 norm of ψ. Next, we consider a unit local basis at x_0' denoted by $\mathcal{B}=\{e_n,e_{T_\alpha}|1\leq\alpha<n\}$, where e_n is

used to denote the outward unit normal and e_{T_α} denotes vectors in the tangential direction at x_0'. By a change of basis, we write $e_\gamma = ae_n + be_{T_\alpha}$. A simple computation shows that

$$e_\gamma = \frac{\langle e_\gamma, e_n\rangle}{1-\langle e_n, e_{T_\alpha}\rangle^2} e_n - \frac{\langle e_\gamma, e_n\rangle\langle e_n, e_{T_\alpha}\rangle}{1-\langle e_n, e_{T_\alpha}\rangle^2} e_{T_\alpha},$$

from which one can easily find a lower linear barrier for u_n at x_0'. So far we have the following:

$$u_n \geq L_1^-(x', x_n) \quad \text{on } \partial\Omega, \quad \text{with equality holding at } x_0', \tag{3-17}$$

where

$$L_1^-(x', x_n) = -C(|\phi|_{C^4}, |\partial\Omega|_{C^4}, |\psi|_{C^1(\Omega)}, \delta)x_n \geq -C|x|^2.$$

Now we choose coordinates such that x_0' is the origin and the $(n-1)\times(n-1)$ matrix $u_{TT}(0)$ is diagonalized.

Claim 4. *We show that*

$$u_{nn}(0) \leq C,$$

where $C = C(\|\psi\|_{C^{1,1}(\bar\Omega)}, \|\phi\|_{C^4(\bar\Omega)}, n, \delta, |\partial\Omega|_{C^4})$.

To be clear, the notation e_n now denotes the outward unit normal unlike earlier in the proof where it was used to denote the inner unit normal (see page 2723).

Proof. We repeat the process in Step 3.3. First observe that, on taking the gradient of both sides of (1-1) in the direction e_n, we get

$$|g^{ij}\partial_{ij}u_n| \leq C(\|\psi\|_{C^1(\Omega)}). \tag{3-18}$$

We define $w = u - B^-$, where B^- is the subsolution defined in (3-6), and we see that w satisfies condition (3-14). We choose α and β large enough that

$$\begin{aligned} g^{ij}\partial_{ij}(\alpha w + \beta|x|^2 + u_n) &\leq 0 \quad \text{in } \Omega\cap B_r(0), \\ \alpha w + \beta|x|^2 + u_n &\geq 0 \quad \text{on } \partial(\Omega\cap B_r(0)). \end{aligned} \tag{3-19}$$

As $w \geq 0$ on $\partial(B_r(0)\cap\Omega))$, we first choose β. On $\partial B_r(0)\cap\Omega$, we have $\beta \geq -C/r^2$, where $C = C(\|\psi\|_{C^1(\bar\Omega)}, \delta, \|\phi\|_{C^2(\bar\Omega)}, n, |\partial\Omega|_{C^2})$ is the constant from the estimates in (3-9) and (3-4). On $\partial\Omega\cap B_r(0)$, we find β using (3-17). Choosing the larger of the two values we get the required value of β. Fixing this β, we choose α such that (3-19) holds. Using the constant C from (3-18), we choose α large enough that $-\alpha\varepsilon_0 + C < 0$, where $C = C(\beta, \|\psi\|_{C^1(\bar\Omega)})$. Now since $(\alpha w + \beta|x|^2 + u_n)(0) = 0$, using Hopf's lemma, we get

$$\frac{\partial}{\partial_n}(\alpha w + \beta|x|^2 + u_n)(0) \leq 0 \quad \Longrightarrow \quad u_{nn}(0) \leq C(\|\psi\|_{C^{1,1}(\bar\Omega)}, \|\phi\|_{C^4(\bar\Omega)}, n, \delta, |\partial\Omega|_{C^4}). \qquad \square$$

Claim 5. *If* $u_{nn}(0)$ *is bounded from above, then* $u_{nn}(x)$ *will be bounded from above for all* $x \in \partial\Omega$.

Proof. Suppose that $u_{nn}(x_p) \geq K$ for some $x_p \in \partial\Omega$, where K is a large constant to be chosen shortly. From Claim 4, we see that, at 0,

$$\begin{aligned} &F(D^2u + Ne_n\times e_n) - F(D^2u) = \delta_0(\|\phi\|_{C^4(\partial\Omega)}, \|\psi\|_{C^{1,1}(\bar\Omega)}) > 0 \\ &\qquad \Longrightarrow \quad \lim_{a\to\infty} F(D^2u + ae_n\times e_n) \geq F(D^2u + Ne_n\times e_n) \geq F(D^2u) + \delta_0 = \psi + \delta_0. \end{aligned}$$

From Lemma A.1, we see that

$$\sum_{i=1}^{n-1} \arctan \lambda_i'(x_p) \geq \psi + \delta_0 - \frac{\pi}{2}$$

and

$$\psi = F(D^2u) = \sum_{i=1}^{n-1} \arctan \lambda_i' + o(1) + \arctan(u_{nn} + O(1)) \geq \psi + \delta_0 - \frac{\pi}{2} - \frac{\delta_0}{2} + \arctan(u_{nn} + O(1)).$$

Now if we choose K large enough that

$$u_{nn}(x_p) > \tan\Big(\frac{\pi}{2} - \frac{\delta_0}{2}\Big) - O(1),$$

we arrive at a contradiction. Therefore, choosing

$$K \leq \tan\Big(\frac{\pi}{2} - \frac{\delta_0}{2}\Big) - O(1) = C(\|\psi\|_{C^{1,1}(\bar{\Omega})}, \|\phi\|_{C^4(\bar{\Omega})}, n, \delta, |\partial\Omega|_{C^4}),$$

we see that $u_{nn}(x) \leq K$ for all $x \in \partial\Omega$. Combining all the estimates in Step 3 above we obtain (3-11). □

Step 4: Bound for $\|D^2u\|_{C^\alpha(\bar{\Omega})}$. This follows from the interior $C^{2,\alpha}$ estimates in [Evans 1982; Krylov 1983a] and the boundary $C^{2,\alpha}$ estimates in [Krylov 1983a, Theorem 4.1]. Therefore, combining all the four steps above we obtain estimate (3-1). □

4. Proof of Theorem 1.1

In this section we use the $C^{2,\alpha}$ estimate up to the boundary to solve the following Dirichlet problem using the method of continuity.

Theorem 4.1. *Suppose that $\phi \in C^4(\bar{\Omega})$ and $\psi : \bar{\Omega} \to \big[(n-2)\frac{\pi}{2} + \delta, n\frac{\pi}{2}\big)$ is in $C^{1,1}(\bar{\Omega})$, where Ω is a uniformly convex, bounded domain in $\mathbb{R}^n$ and $\delta > 0$. Then there exists a unique solution $u \in C^{2,\alpha}(\bar{\Omega})$ to the Dirichlet problem (1-1).*

Proof. For each $t \in [0, 1]$, consider the family of equations

$$\begin{cases} F(D^2u) = t\psi + (1-t)c_0 & \text{in } \Omega, \\ u = \phi & \text{on } \partial\Omega, \end{cases} \tag{4-1}$$

where $c_0 = (n-2)\frac{\pi}{2} + \delta$ and $\psi \in C^{2,\alpha}(\bar{\Omega})$. Let $I = \{t \in [0, 1] \mid \exists u_t \in C^{4,\alpha}(\bar{\Omega}) \text{ solving (4-1)}\}$. As a consequence of the interior Hessian estimates proved by Wang and Yuan [2014, p. 482, second paragraph], we have that $0 \in I$. The fact that I is open is a consequence of the implicit function theorem and invertibility of the linearized operator (2-2). The closedness of I follows from the a priori estimates. Hence, $1 \in I$. Now using a smooth approximation[2] we solve (1-1) for $\psi \in C^{1,1}$. Uniqueness follows from the maximum principle for fully nonlinear equations. □

Remark 4.2. There exists a unique smooth solution to the Dirichlet problem (1-1) if all data is smooth and if the phase lies in the supercritical range.

[2] When ψ is in $C^{1,1}(\bar{\Omega})$, we can take a sequence of smooth functions ψ_k approximating ψ and a sequence of solutions u_k solving (1-1) with ψ_k as the right-hand side. Applying the uniform $C^{2,\alpha}$ estimate and taking a limit solves the equation.

Proof of Theorem 1.1. We approximate $\phi \in C^0(\partial\Omega)$ uniformly on $\partial\Omega$ by a sequence $\{\phi_k\}_{k\geq 1}$ of C^4 functions and solve

$$\begin{cases} F(D^2u_k) = \psi & \text{in } \Omega, \\ u_k = \phi_k & \text{on } \partial\Omega \end{cases}$$

using Theorem 4.1. Applying the interior Hessian estimates proved in [Bhattacharya 2021, Theorem 1.1] and the compactness in C^2 of bounded sets in $C^{2,\alpha}$ along with maximum principles, we get convergence of $\{u_k\}$ to the desired solution $u \in C^{2,\alpha}$ on the interior and convergence of $\{\phi_k\}$ to the desired boundary function $\phi \in C^0$ on the boundary. □

Remark 4.3. The above existence proof can be extended to prove the existence of a unique C^0 viscosity solution to (1-1), where ψ is in $C^0(\bar{\Omega})$ and lies in the supercritical range. The existence part is based on smooth solution approximations, with smooth approximations of the phase and the boundary data in the C^0 continuous norm: the C^0 limit of smooth approximating solutions is a viscosity solution. The uniqueness part follows from [Trudinger 1990, p. 155]: Trudinger's condition is satisfied since the minimum eigenvalue is bounded for a uniform, supercritical phase. Note that this existence proof is different from the one shown in [Dinew et al. 2019, Theorem 40].

5. Proof of Theorem 1.2

Proof. We denote upper/lower semicontinuous functions by usc/lsc. We define

$$\begin{aligned} A &= \{u \in \operatorname{usc}(\bar{\Omega}) \mid F(D^2u) \geq \psi \text{ in } \Omega,\ u \leq \phi \text{ on } \partial\Omega\}, \\ w(x) &= \sup\{u(x) \mid u \in A\}. \end{aligned}$$

Claim 6. *The above function w is the unique continuous viscosity solution of (1-1), where ψ is a constant.*

Remark 5.1. The proof follows from the following four steps. It is noteworthy that the first three steps of the proof hold for any continuous function ψ. The fourth step requires a certain comparison principle (see Theorem A.2 of the Appendix), which is only available for a constant right-hand side. As of now, it is unknown if such a comparison principle holds for a continuous right-hand side. In order to highlight this distinction, we present the first three steps of the proof assuming ψ is any continuous function. In the final step, we assume ψ to be a constant, thereby proving Theorem 1.2.

Step 1: We define the functions

$$\underline{z}(x) = \overline{\lim_{y\to x}}\, w(y),$$

$$\bar{z}(x) = \underline{\lim}_{y\to x}\, w(y).$$

We first show that A is nonempty and w, $\underline{z}$, $\bar{z}$ are well defined. Since $\psi \in C(\bar{\Omega})$, there exists $\varepsilon' > 0$ such that $-n\frac{\pi}{2} + \varepsilon' < \psi(x) < n\frac{\pi}{2} - \varepsilon'$ for all $x \in \bar{\Omega}$. Fixing this ε' we define the functions

$$\psi_* = -n\tfrac{\pi}{2} + \varepsilon' < \psi < n\tfrac{\pi}{2} - \varepsilon' = \psi^*.$$

Recalling (3-6) and (3-7), we define

$$\begin{aligned} \underline{w}(x) &= -Cx_n + \frac{1}{2}|x|^2 \tan\frac{\psi^*}{n}, \\ \overline{w}(x) &= Cx_n + \frac{1}{2}|x|^2 \tan\frac{\psi_*}{n}, \end{aligned} \tag{5-1}$$

where $C = C(\|\phi\|_{C^2(\partial\Omega)}, n, |\partial\Omega|_{C^2})$. By definition $\underline{w} \in A$, which shows that A is nonempty. Next, $\max\{u, \underline{w}\}$ is upper semicontinuous and still a subsolution of (1-1), so we replace $u \in A$ by $\max\{u, \underline{w}\}$. This shows $u \geq \underline{w}$ and, therefore, w is well defined. Next, we observe that since $\underline{w}$ and $\overline{w}$ are sub- and supersolutions of (1-1), respectively, we have

$$\underline{w} \leq u \leq \overline{w},$$

which shows $\underline{z}$ and $\bar{z}$ are well defined.

Step 2: We show that $\underline{z}$ is a subsolution of (1-1). Suppose not. Then we can find a quadratic polynomial P such that $P(x) \geq \underline{z}(x)$ in $B_\rho(0)$, with equality holding at 0, such that $F(D^2P) < \psi_*$ in $B_\rho(0)$. Now we choose $\varepsilon > 0$ such that

$$F(D^2P + 4\varepsilon I) < \psi_*. \tag{5-2}$$

From the definition of w and $\underline{z}$, we can find sequences $\{u_k\} \subset A$ and $\{x_k\} \subset \Omega$, with $x_k \to 0$, such that

$$\underline{z}(0) = \overline{\lim_{y\to 0}}\, w(y) = \lim_{x_k \to 0} u_k(x_k).$$

For k large enough, we see that

$$\left|u_k(x_k) - P(x_k) - 2\varepsilon|x_k|^2\right| = \left|u_k(x_k) - P(0) + P(0) - P(x_k) - 2\varepsilon|x_k|^2\right| = o(1) < \varepsilon\rho^2.$$

On $\partial B_\rho(0)$, we see

$$u_k(x) \leq w(x) \leq \underline{z}(x) \leq P(x) + 2\varepsilon|x|^2 - \varepsilon\rho^2.$$

Using the definition of w and $\underline{z}$, we see that, for any k, the following holds in $B_\rho(0)$:

$$Q(x) = P(x) + 2\varepsilon|x|^2 \geq u_k(x).$$

Fixing a k large enough, we observe the following. The functions $u_k(x_k)$ and $Q(x_k)$ are less than $\varepsilon\rho^2$ apart, but u_k is at a distance of more than $\varepsilon\rho^2$ below Q on $\partial B_\rho(0)$. So we drop Q at most $\varepsilon\rho^2$ so that it touches u_k at a point inside $B_\rho(0)$ while still remaining above u_k on $\partial B_\rho(0)$. So there exists $\gamma \leq \varepsilon\rho^2$ such that, in $B_\rho(0)$,

$$u_k(x) \leq P(x) + 2\varepsilon|x|^2 - \gamma,$$

with equality holding at an interior point of B_ρ. Now since u_k is a subsolution, we have

$$\psi \leq F(D^2P + 4\varepsilon I).$$

This contradicts (5-2). Noting that $\underline{z}$ is upper semicontinuous, we see that it is a subsolution of (1-1).

Step 3: We show that $\bar{z}$ is a supersolution of (1-1). Suppose not. Then we can find a quadratic polynomial P such that $P(x) \le \bar{z}(x)$ in $B_\rho(0)$, with equality holding at 0, such that $F(D^2P) > \psi^*$ in $B_\rho(0)$. We choose $\varepsilon > 0$ small enough that

$$F(D^2P - 2\varepsilon I) > \psi^*. \tag{5-3}$$

We have $\bar{z} \ge P - \varepsilon|x|^2$. We define a new quadratic $Q(x) = P(x) - \varepsilon|x|^2 + \varepsilon\rho^2$. Observe that, since $\bar{z}(0) = \underline{\lim}_{x_k \to 0} w(x_k)$, for k large enough, we have

$$\begin{aligned} w(x_k) = \bar{z}(0) + o(1) &= P(0) - P(x_k) + P(x_k) + o(1) \\ &= P(x_k) + o(1) = Q(x_k) - \varepsilon\rho^2 + o(1) < Q(x_k). \end{aligned}$$

This contradicts the supremum definition of w since Q is a subsolution of (1-1) by (5-3). Noting that $\bar{z}$ is lower semicontinuous, we see that it is a supersolution of (1-1).

Step 4: We take care of the boundary value in this final step. This is where we assume (for the first time) that ψ is a constant. Note that now we may assume the boundary value ϕ is in $C^2(\partial\Omega)$ since we can always approximate ϕ by a sequence of smooth functions ϕ_δ that solve

$$\begin{cases} F(D^2u_\delta) = \psi & \text{in } \Omega, \\ u_\delta = \phi_\delta & \text{on } \partial\Omega \end{cases}$$

and apply the comparison principle[3] to get

$$\max_{\Omega} |u_{\delta_1} - u_{\delta_2}| \le \max_{x \to \partial\Omega} |(\phi_{\delta_1} - \phi_{\delta_2})(x)| \to 0$$

as $\delta_1, \delta_2 \to 0$. We have $u_\delta \to u$ in C^0 as $\delta \to 0$. Next, we pick an arbitrary point $x_0 \in \partial\Omega$ and recall the construction of $\underline{w}$ and $\overline{w}$ from (5-1). Defining similar functions at x_0 and on using the comparison principle, we get $\underline{w} \le u \le \overline{w}$, with equality holding at x_0 for all $u \in A$. Again, since $\max(u, \underline{w}) \in A$ for all $u \in A$, we can replace

$$w(x) = \sup_{u \in A} \max(u, \underline{w}).$$

We get $\underline{w} \le u \le \overline{w}$, with equality holding at x_0, which shows

$$\bar{z}(x_0) = \phi(x_0) = \underline{z}(x_0).$$

Since $x_0 \in \partial\Omega$ is arbitrary, we have $\bar{z} = \underline{z} = \phi$ on $\partial\Omega$. Combining the above steps and on using the comparison principle, we see

$$\bar{z} = \underline{z} = w \in C^0(\overline{\Omega})$$

is the desired solution. This proves the existence part of Claim 6. Uniqueness again follows from the comparison principle. □

[3] See the Appendix.

Appendix

We state the following linear algebra lemma that was used in proving the double normal estimate in Step 3.4 of Section 3.

Lemma A.1 [Caffarelli et al. 1985, Lemma 1.2]. *Consider the $n \times n$ symmetric matrix*

$$M = \begin{bmatrix} \lambda_1' & & & a_1 \\ & \ddots & & \vdots \\ & & \lambda_{n-1}' & a_{n-1} \\ a_1 & \cdots & a_{n-1} & a \end{bmatrix},$$

where $\lambda_1', \lambda_2', \ldots, \lambda_{n-1}'$ are fixed, $|a_i| < C$ for $1 \le i < n$, and $|a| \to +\infty$. Then the eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$ of M behave like

$$\lambda_1' + o(1), \quad \lambda_2' + o(1), \quad \ldots, \quad \lambda_n' + o(1), \quad a + O(1),$$

where $o(1)$ and $O(1)$ are uniform as $a \to \infty$.

For the sake of completeness we state and prove the following comparison principle for strictly elliptic equations, which is well known to experts.[4]

Theorem A.2. *Suppose that u is a* usc *subsolution and v is an* lsc *supersolution of the strictly elliptic equation (1-2) in $\Omega \subset \mathbb{R}^n$. If $u \le v$ on $\partial\Omega$, then $u \le v$ in Ω.*

Proof. Without loss of generality, we assume $\Omega = B_1(0)$ and $u \le v - 2\delta$ on ∂B_1 for some small $\delta > 0$. We rewrite (1-2) as

$$F(D^2u) = \sum_{i=1}^{n} \arctan \lambda_i - c = 0.$$

Let u^ε be an upper parabolic envelope[5] satisfying

$$F(D^2u^\varepsilon) \ge 0, \quad D^2u^\varepsilon \ge -C/\varepsilon, \quad \|u^\varepsilon\|_{C^{0,1}} \le C/\varepsilon$$

outside a measure-zero subset, where u^ε is punctually second-order differentiable and C is chosen such that

$$u^\varepsilon - v_\varepsilon \le C - \varepsilon|x - x_0|^2 \quad \text{on } \partial B_1,$$

with equality holding at $x_0 \in B_1$. We see that

$$0 \le u^\varepsilon(x) - u(x) \le u(x^*) - u(x) + \varepsilon,$$

[4] We learned this proof from [Yuan 2004]. Indeed, the arguments presented in [Caffarelli and Cabré 1995, p. 43–46] toward the comparison principle for fully nonlinear, uniformly elliptic equations work for strictly elliptic equations as well.

[5] For $\varepsilon > 0$, we define the upper ε-envelope of u to be

$$u^\varepsilon(x_0) = \sup_{x \in \overline{H}} \{u(x) + \varepsilon - |x - x_0|^2/\varepsilon\} \quad \text{for } x_0 \in H,$$

where H is an open set such that $\overline{H} \subset B_1$.

where $x^* \to x$ as $\varepsilon \to 0$. By symmetry, the lower parabolic envelope v_ε satisfies

$$F(D^2 v_\varepsilon) \le 0, \quad D^2 v_\varepsilon \le C/\varepsilon, \quad \|v_\varepsilon\|_{C^{0,1}} \le C/\varepsilon$$

and

$$0 \ge v_\varepsilon(x) - v(x) \ge v(x_*) - v(x) - \varepsilon,$$

where $x_* \to x$ as $\varepsilon \to 0$. Note that $v_\varepsilon - u^\varepsilon \le L + (C/\varepsilon)|x - x_0|^2$ for $x_0 \in B_1$, where L is a linear function. The convex envelope $\Gamma(v_\varepsilon - u^\varepsilon)$ is in $C^{1,1}$. From the Alexandroff estimate, we have

$$\sup_{B_1}(v_\varepsilon - u^\varepsilon)^- \le C(n)\left[\int_\Sigma \det D^2\Gamma\right]^{1/n},$$

where

$$\Sigma = \{x \in B_1 \mid \Gamma(x) = v_\varepsilon(x) - u^\varepsilon(x)\}.$$

Now in Σ, we have

$$0 \le D^2\Gamma \le D^2(v_\varepsilon - u^\varepsilon) \quad \text{or} \quad L(x) \le v_\varepsilon(x) - u^\varepsilon(x)$$

near $x_0 \in \Sigma$. For K large, since $u^\varepsilon + (K/\varepsilon)|x|^2$ is convex and $v_\varepsilon - (K/\varepsilon)|x|^2$ is concave, we have the following for a.e. $x_0 \in B_1$:

$$v_\varepsilon = \Gamma + \frac{K}{\varepsilon}|x|^2 + O(|x - x_0|^2),$$
$$u^\varepsilon = \Gamma + \frac{K}{\varepsilon}|x|^2 + O(|x - x_0|^2).$$

Again, since v_ε is a supersolution and u^ε is a subsolution, for a.e. $x_0 \in B_1$, we have

$$F(D^2 v_\varepsilon(x_0)) \le 0, \quad F(D^2 u^\varepsilon(x_0)) \ge 0, \quad F(D^2 v_\varepsilon(x_0)) - F(D^2 u^\varepsilon(x_0)) \le 0.$$

Also, a.e. $x_0 \in \Gamma$, we have $D^2 v_\varepsilon(x_0) - D^2 u^\varepsilon(x_0) \ge 0$. However, F is strictly elliptic, so we must have $F(D^2 v_\varepsilon) - F(D^2 u^\varepsilon) \ge 0$, which shows

$$F(D^2 v_\varepsilon(x_0)) = F(D^2 u^\varepsilon(x_0)) \quad \text{a.e } x_0 \in \Sigma.$$

Again, given that F is strictly elliptic, the line with the positive direction $D^2 v_\varepsilon(x_0) - D^2 u^\varepsilon(x_0)$ intersects the level set $\{F = C\}$ only once, which implies $D^2 v_\varepsilon(x_0) = D^2 u^\varepsilon(x_0)$. This shows $\sup_{B_1}(v_\varepsilon - u^\varepsilon)^- \le 0$, which proves that

$$v \ge v_\varepsilon \ge u^\varepsilon \ge u \quad \text{in } B_1. \qquad \square$$

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LOCAL LENS RIGIDITY FOR MANIFOLDS OF ANOSOV TYPE

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The *lens data* of a Riemannian manifold with boundary is the collection of lengths of geodesics with endpoints on the boundary, together with their incoming and outgoing vectors. We show that negatively curved Riemannian manifolds with strictly convex boundary are *locally lens rigid* in the following sense: if g_0 is such a metric, then any metric g sufficiently close to g_0 and with the same lens data is isometric to g_0, up to a boundary-preserving diffeomorphism. More generally, we consider the same problem for a wider class of metrics with strictly convex boundary, called metrics of *Anosov type*. We prove that the same rigidity result holds within that class in dimension 2 and in any dimension, further assuming that the curvature is nonpositive.

1. Introduction

1A. ***The lens rigidity problem.*** Let (M, g) be a smooth compact connected Riemannian manifold with strictly convex boundary (i.e., the second fundamental form is positive on ∂M). Let $\mathcal{M} := SM$ be the unit tangent bundle of (M, g), and define the incoming $(-)$ and outgoing $(+)$ boundary of $\mathcal{M}$ as

$$\partial_\pm \mathcal{M} := \{(x, v) \in \mathcal{M} \mid x \in \partial M,\ \pm g_x(v, \nu(x)) > 0\},$$

where ν is the unit outward-pointing normal vector to the boundary. For any $(x, v) \in \partial_- \mathcal{M}$, the maximally extended geodesic $\gamma_{(x,v)}$, with initial condition $\gamma_{(x,v)}(0) = x$, $\dot{\gamma}_{(x,v)} = v$, is defined on a time interval $[0, \ell_g(x, v)]$, where $\ell_g(x, v) \in \mathbb{R}_+ \cup \{\infty\}$. When $\ell_g(x, v) < \infty$, we define

$$S_g(x, v) := (\gamma_{(x,v)}(\ell_g(x, v)), \dot{\gamma}_{(x,v)}(\ell_g(x, v)))$$

to be the outgoing tangent vector at $\partial_+ \mathcal{M}$; see Figure 1.

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Keywords: lens data, lens rigidity, microlocal analysis, hyperbolic dynamics, anisotropic spaces, inverse problems, resolvent, X-ray transform.

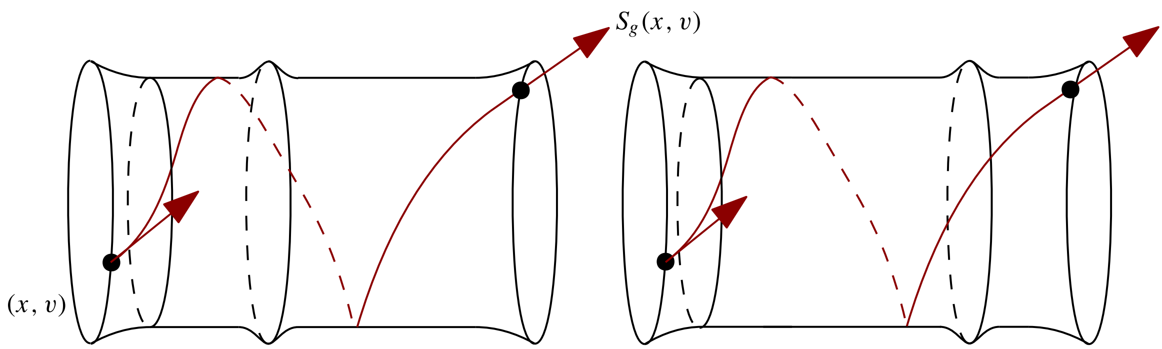

Figure 1. A surface with strictly convex boundary which is not lens rigid. Example taken from [Croke and Herreros 2016].

Definition 1.1 (lens data). The map $S_g : \partial_-\mathcal{M} \setminus \{\ell_g = \infty\} \to \partial_+\mathcal{M}$ is called the *scattering map* and the function $\ell_g : \partial_-\mathcal{M} \setminus \{\ell_g = \infty\} \to \mathbb{R}_+$ the *length map*. The pair (ℓ_g, S_g) is the *lens data* of the Riemannian manifold (M, g).

The lens data encodes the boundary data one can measure on the geodesic flow from "outside of the manifold". A natural inverse problem that arises from tomography consists in determining the geometry, namely, the Riemannian metric g inside M, from the measurement of the lens data (ℓ_g, S_g). In geophysics, this is related to recovering the speed of propagation of waves inside a domain such as the Earth, for instance; see [Paternain et al. 2014]. When two metrics g and g' agree on ∂M, it makes sense to say that they have the same lens data as there is a natural identification between the boundary of their respective unit tangent bundles via the unit disk bundle of the boundary; see Section 2A1 for further details. The *lens rigidity problem* is concerned with the following question:

Question 1.2. *Assume that* (M, g) *and* (M', g') *are two Riemannian metrics with strictly convex boundary such that there exists an isometry* $I \in \mathrm{Diff}(\partial M, \partial M')$ *with* $I^*(g'|_{T\partial M'}) = g|_{T\partial M}$. *Does the implication*

$$(\ell_g, S_g) = I^*(\ell_{g'}, S_{g'}) \quad \Longrightarrow \quad \textit{there exists } \psi \in \mathrm{Diffeo}(M, M') \textit{ such that } \psi|_{\partial M} = I \textit{ and } \psi^* g' = g$$

hold true?

We say that a manifold (M, g) is *lens rigid* if there is no other Riemannian manifold (up to isometry) having the same lens data as (ℓ_g, S_g). In the following, in order to simplify the notation, we will assume that $M = M'$ and $I = \mathrm{id}$.

There are simple counterexamples of manifolds for which lens rigidity does not hold: considering certain perturbations of the flat cylinder $\mathbb{S}^1 \times [0, 1]$ (see Figure 1 and [Croke and Herreros 2016], where this is further discussed), one can easily obtain nonisometric metrics with the same lens data. Such cases have *trapped geodesics*, that is some maximally extended geodesics with infinite length, or equivalently $\ell_g(x, v) = \infty$ for some $(x, v) \in \partial_-\mathcal{M}$. It turns out that all existing counterexamples to lens rigidity have trapped geodesics.

1B. ***Lens rigidity for nontrapping manifolds.*** Even among manifolds without a trapped set, the lens rigidity problem is still widely open. The closest result in this direction is the recent breakthrough of Stefanov, Uhlmann and Vasy [Stefanov et al. 2021], showing lens rigidity in dimensions $n \geq 3$ under the additional assumption that the manifold (M, g) is foliated by strictly convex hypersurfaces. This includes all simply connected nonpositively curved manifolds with strictly convex boundary. In the class of real analytic metrics such that from each $x \in \partial M$ there is a maximal geodesic free of conjugate points, the lens rigidity was proved by Vargo [2009]. A local lens rigidity result was also proved near analytic metrics by Stefanov and Uhlmann [2009] under certain assumptions on the conjugate points.

There is also a subclass of metrics that have attracted a lot of attention since the work of Michel [1981], namely the class of *simple manifolds*, which are manifolds with strictly convex boundary that have no trapped geodesics and no conjugate points. These manifolds are diffeomorphic to the unit ball in $\mathbb{R}^n$. In this case, knowing the lens data is equivalent to knowing the restriction $d_g|_{\partial M \times \partial M}$ of the Riemannian distance function $d_g \in C^0(M \times M)$ to the boundary, also called the *boundary distance*. The lens rigidity problem for this subclass of metrics is also called the *boundary rigidity problem*. In dimension $n = 2$, it was proved by Otal [1990b] (in negative curvature), Croke [1991] (in nonpositive curvature), and Pestov and Uhlmann [2005] (in general) that simple surfaces are boundary rigid and thus lens rigid. We also mention the results by Croke, Dairbekov and Sharafutdinov [Croke et al. 2000] and Stefanov and Uhlmann [2004] for local boundary rigidity results, the work by Gromov [1983] and Burago and Ivanov [2010] for rigidity results of flat and close to flat simple manifolds, and we finally refer more generally to the review article by Croke [2004] and the recent book of Paternain, Salo and Uhlmann [Paternain et al. 2023] for an overview of the boundary rigidity problem.

1C. ***Lens rigidity for manifolds with nonempty trapped set.*** Trapped geodesics appear in most situations since all Riemannian manifolds (M, g) with strictly convex boundary and nontrivial topology, i.e., nontrivial fundamental group, always have trapped geodesics (and they even have closed geodesics in the interior M°). As far as manifolds with trapped geodesics are concerned, very little is known on the lens rigidity problem. It is not even clear what would be the most general class of manifolds for which lens rigidity could hold, and the example above in Figure 1 shows that it seems hopeless to consider general manifolds with both trapped geodesics and conjugate points.

The only available result considering cases with both trapped geodesics and conjugate points seems to be the local rigidity result of [Stefanov and Uhlmann 2009]. In dimensions $n \geq 3$, under a certain topological assumption, it is proved that if (M, g_0) is real analytic,[1] with strictly convex boundary, and for each $(x, v) \in SM$ there is $w \in v^\perp$ such that the maximally extended geodesic tangent to w at x has finite length (it is not trapped) and is free of conjugate points, then the following holds: if g is another metric with $\|g - g_0\|_{C^N}$ small enough for some $N \gg 1$ and $(\ell_g, S_g) = (\ell_{g_0}, S_{g_0})$, then g and g_0 are isometric via a boundary-preserving diffeomorphism. On the other hand, it is not clear (geometrically speaking) what type of manifolds are contained in this class and there are many interesting geometric cases not contained in it. For example, there exist convex cocompact hyperbolic 3-manifolds $M := \Gamma \backslash \mathbb{H}^3$ (with constant

[1]Or more generally if a certain localized X-ray transform is injective.

sectional curvature -1) whose convex core $\mathcal{C}$ has positive measure and totally geodesic boundary. Thus, cutting the ends of such examples at a finite positive distance of $\mathcal{C}$, one obtains a metric not satisfying the assumptions of [Stefanov and Uhlmann 2009] due to the totally geodesic surfaces bounding $\mathcal{C}$.

From our point of view, there is a very natural class of metrics with nontrivial trapped set where the lens rigidity problem seems well-posed and interesting from a geometrical point of view. We call elements of this class manifolds of *Anosov type*; it contains as a strict subclass the set of negatively curved metrics with strictly convex boundary.

Definition 1.3. A compact Riemannian manifold (M, g) with boundary is of *Anosov type* if:

(1) It has strictly convex boundary.

(2) It has no conjugate points.

(3) The trapped set for the geodesic flow $(\varphi_t^g)_{t\in\mathbb{R}}$ on $\mathcal{M} := SM$, defined by

$$K^g := \bigcap_{t\in\mathbb{R}} \varphi_t^g(\mathcal{M}^\circ) \subset \mathcal{M}^\circ,$$

is *hyperbolic* in the following sense. There exist a continuous flow-invariant splitting

$$\text{for all } y \in K^g, \quad T_y\mathcal{M} = \mathbb{R}X_g(y) \oplus E_-(y) \oplus E_+(y),$$

where X_g is the geodesic vector field, and constants $\nu, C > 0$ such that,

$$\text{for all } \pm t \geq 0, \text{ for all } y \in K^g, \text{ for all } v \in E_\mp(y), \quad \|d\varphi_t^g(y)v\| \leq Ce^{-\nu|t|}\|v\| \tag{1-1}$$

for an arbitrary choice of metric $\|\cdot\|$ on $\mathcal{M}$.

Example 1.4. The main two examples of manifolds of Anosov type are

(1) Riemannian manifolds with negative sectional curvature and strictly convex boundary (see [Klingenberg 1995, Theorem 3.2.17 and Section 3.9]),

(2) strictly convex subdomains of closed Riemannian manifolds with Anosov geodesic flows.

Manifolds of Anosov type have a trapped set with fractal structure and zero Lebesgue measure. It implies that almost-every point in $\mathcal{M}$ is reachable from geodesics with endpoints on $\partial\mathcal{M}$. This case can be interpreted as an intermediate rigidity problem between the *length spectrum rigidity* of manifolds with Anosov geodesic flows, where one asks if the lengths of closed geodesics determine the metric up to isometry, and the boundary rigidity problem of simple manifolds.

In the closed case, Vignéras [1980] exhibited counterexamples to the length spectrum rigidity: in constant negative curvature, there are nonisometric metrics on surfaces with the same length spectrum. The well-posed rigidity problem is rather that of the *marked length spectrum* problem, also known as the Burns–Katok conjecture [Burns and Katok 1985]: on a manifold (M, g) with Anosov geodesic flow, each free homotopy class of loops c on M contains a unique geodesic representative $\gamma_c(g)$ whose length is denoted by $L_g(c)$; if g_1 and g_2 are two such Anosov metrics on M with $L_{g_1}(c) = L_{g_2}(c)$ for all c, it is then conjectured that g_1 should be isometric to g_2. This conjecture was proved in dimension 2 by

Otal [1990a] and Croke [1990], and in all dimensions for pairs of metrics that are close enough in C^k norm for $k \gg 1$ large enough by the last two authors [Guillarmou and Lefeuvre 2019] (local rigidity). However, it is still open in general.

Similarly, for manifolds with boundary and nontrivial topology, the same problem of "marking" of geodesics is a serious difficulty. The first natural question one may consider is the following, known as the *marked lens rigidity* or *marked boundary rigidity* problem for Riemannian manifolds of Anosov type.

Definition 1.5 (marked lens data). Let g_1, g_2 be two metrics of Anosov type on M. We say that g_1 and g_2 have the same *marked lens data* if, for each $(x, v) \in \partial_-\mathcal{M} \setminus \{\ell_g = \infty\}$, one has $(\ell_{g_1}(x, v), S_{g_1}(x, v)) = (\ell_{g_2}(x, v), S_{g_2}(x, v))$ and the g_1- and g_2-geodesics with initial conditions (x, v) are homotopic via a homotopy fixing the endpoints.

Technically, having the same marked lens data is the same as having same boundary distance function on the universal cover $\widetilde{M}$ (which is now a noncompact space). The following conjecture is somehow similar to the Burns–Katok conjecture in the closed case and to the boundary rigidity problem of negatively curved simple metrics.

Conjecture 1.6 (marked lens rigidity of manifolds of Anosov type). *Let M be a smooth manifold with boundary, and assume that g_1, g_2 are two smooth metrics of Anosov type on M in the sense of Definition 1.3 such that $g_1|_{T(\partial M)} = g_2|_{T(\partial M)}$. If g_1 and g_2 have the same marked lens data, then there exists a smooth diffeomorphism ψ, homotopic to the identity and equal to the identity on the boundary ∂M, such that $\psi^* g_2 = g_1$.*

In dimension 2, Conjecture 1.6 was recently solved by the third author with Erchenko in [Erchenko and Lefeuvre 2024] (an earlier result had also been obtained by the second author together with Mazzuchelli in [Guillarmou and Mazzucchelli 2018] for negatively curved surfaces using the method of Otal [1990a]). In higher dimensions, the third author [Lefeuvre 2020] proved Conjecture 1.6 for pairs of negatively curved metrics g_1, g_2 that are close enough in C^k norm for $k \gg 1$ large enough (local marked lens rigidity). The fact that there is no smooth 1-parameter family $(g_s)_{s\in(-1,1)}$ of nonisometric negatively curved metrics with the same marked lens data[2] is called *infinitesimal rigidity* and was first proved by the second author [Guillarmou 2017b].

In this paper, we consider the more difficult problem of lens rigidity in the class of manifolds of Anosov type. Since, contrary to the closed case, there are still no counterexamples to lens rigidity, we make the following conjecture of lens rigidity in the class of metrics of Anosov type.

Conjecture 1.7 (lens rigidity of manifolds of Anosov type). *Let (M_1, g_1), (M_2, g_2) be two smooth Riemannian manifolds of Anosov type such that $(\partial M_1, g_1|_{\partial M_1}) = (\partial M_2, g_2|_{\partial M_1})$. If $(\ell_{g_1}, S_{g_1}) = (\ell_{g_2}, S_{g_2})$, then there exists a smooth diffeomorphism ψ, equal to the identity on the boundary, such that $\psi^* g_2 = g_1$.*

There are already partial answers to Conjecture 1.7:

(1) In dimension 2, Croke and Herreros [2016] proved that negatively curved cylinders with strictly convex boundary are lens rigid.

[2] In this case, having the same marked lens data is equivalent to having the same lens data.

(2) In dimension 2, the second author shows in [Guillarmou 2017b] that the scattering map S_g determines (M, g) up to conformal diffeomorphism fixing the boundary. Recovering the conformal factor of the metric is still an open question.

(3) In dimensions $n \geq 3$, Stefanov, Uhlmann and Vasy [Stefanov et al. 2021] prove that, for general metrics with strictly convex boundary, the lens data determines the metric in a neighborhood of ∂M; applying this result in the setting of negatively curved manifolds, one can recover the metric outside the convex core of the manifold (which contains the projection of the trapped set).

(4) In [Guedes-Bonthonneau et al. 2024], Guedes-Bonthonneau, Jézéquel, and the second author proved Conjecture 1.7 under the extra assumption that (M_1, g_1), (M_2, g_2) are real analytic, but only using the equality $S_{g_1} = S_{g_2}$ of the scattering maps.

Our first result in this article is the following local rigidity result answering Conjecture 1.7 for metrics close to each other.

Theorem 1.8. *Let (M, g_0) be a Riemannian manifold of Anosov type. Assume that either* $\dim M = 2$ *or that the curvature of g_0 is nonpositive. Then there exist $N \gg 1$, $\delta > 0$ such that the following holds: for any smooth metric g on M such that $\|g - g_0\|_{C^N} < \delta$, if $(\ell_g, S_g) = (\ell_{g_0}, S_{g_0})$, then there exists a smooth diffeomorphism $\psi : M \to M$ such that $\psi|_{\partial M} = \mathrm{id}$ and $\psi^* g = g_0$.*

More generally, Theorem 1.8 holds under the general assumption that g_0 is of Anosov type and its *X-ray transform operator* $I_2^{g_0}$ on divergence-free symmetric 2-tensors is injective; see (1-2) for a definition of $I_2^{g_0}$ and Section 3A2 where this is further discussed. The fact that $I_2^{g_0}$ is injective on divergence-free tensors was proved in [Guillarmou 2017b] in nonpositive curvature and in general on Anosov surfaces by [Lefeuvre 2019a] (without any assumption on the curvature). It was also proved in [Guedes-Bonthonneau et al. 2024] that $I_2^{g_0}$ is injective for real-analytic metrics g_0 which implies that generic smooth metrics of Anosov type have an injective X-ray transform operator $I_2^{g_0}$; generic injectivity of $I_2^{g_0}$ follows from the work of the first and third authors [Cekić and Lefeuvre 2021] as well, admitting also Theorem 1.10 below. As a corollary of Theorem 1.8, we obtain:

Corollary 1.9. *Let (M, g_0) be a negatively curved Riemannian manifold with strictly convex boundary. Then, there exist $N \gg 1$, $\delta > 0$ such that the following holds: for any smooth metric g on M such that $\|g - g_0\|_{C^N} < \delta$, if $(\ell_g, S_g) = (\ell_{g_0}, S_{g_0})$, then there exists a smooth diffeomorphism $\psi : M \to M$ such that $\psi|_{\partial M} = \mathrm{id}$ and $\psi^* g = g_0$.*

We observe that Corollary 1.9 and Theorem 1.8 are not a consequence of [Stefanov and Uhlmann 2009] (nor of [Stefanov et al. 2021]) mentioned above since: (1) our result contains the case of surfaces (dimension $n = 2$) and (2) the assumption on the trapped set in [Stefanov and Uhlmann 2009] does not cover all hyperbolic trapped sets (typically, the example $M = \Gamma \backslash \mathbb{H}^3$ mentioned above is not covered when the boundary of the convex core $\mathcal{C}$ is totally geodesic), whereas we do not make any specific assumption on the topology, and neither do we assume that g_0 is analytic or that it has an injective localized X-ray transform. Theorem 1.8 is also clearly stronger than the marked local rigidity result of the third author [Lefeuvre 2020], since we are now able to remove the *marking* assumption on the lens data.

Let us finally mention that there are interesting and related results for Euclidean billiards: Noakes and Stoyanov [2015] show that the lens data for the billiard flow on $\mathbb{R}^n \setminus \mathcal{O}$ (where $\mathcal{O}$ is a collection of strictly convex domains) is rigid, and De Simoi, Kaloshin and Leguil [De Simoi et al. 2023] prove that the lengths of the marked periodic orbits generically determine the obstacles under a $\mathbb{Z}^2 \times \mathbb{Z}^2$ symmetry assumption.

1D. *Removing the marking assumption, idea of proof.* The removal of the marking assumption is not simply a technical artifact: it is rather a crucial aspect in our work. Indeed, without the marking assumption, one can no longer use the fact that the geodesic flows of g and g_0 are conjugate with a conjugacy preserving the Liouville measure. This conjugacy was a fundamental aspect of both proofs of [Guillarmou and Mazzucchelli 2018; Lefeuvre 2020]. In the proof of Theorem 1.8, one has to rely on a completely different argument, which is the linearization of the pair (ℓ_g, S_g). Nevertheless, since g has a big set of trapped geodesics (typically a fractal set), this creates many singularities for (ℓ_g, S_g) and its linearization. The analysis one has to perform is then quite involved. One needs to combine several different key tools, in particular,

(1) the proof of the C^2-regularity with respect to g of the operator $\mathcal{S}_g : C^\infty(\partial_+\mathcal{M}) \to \mathcal{D}'(\partial_-\mathcal{M})$ defined by $\mathcal{S}_g f := f \circ S_g$,

(2) the exponential decay in $t \to \infty$ of the volume of points $(x, v) \in \mathcal{M} = SM$ that remain trapped for time t.

The first item is obtained by reproving certain results of [Dyatlov and Guillarmou 2016] on the resolvent of an Axiom A vector field X, but now with an explicit control of the dependence with respect to the vector field X. In particular, as a byproduct of this analysis we show the following result that could prove useful for other applications such as Fried's conjecture for manifolds with boundary, in the spirit of [Dang et al. 2020].

Theorem 1.10. *Let $\mathcal{M}$ be a smooth manifold with boundary, and let X_0 be a smooth vector field so that $\partial\mathcal{M}$ is strictly convex for the flow of X_0. Assume that the trapped set*

$$K^{X_0} := \bigcap_{t\in\mathbb{R}} \varphi_t^{X_0}(\mathcal{M}^\circ)$$

of the flow $(\varphi_t^{X_0})_{t\in\mathbb{R}}$ of X_0 is hyperbolic. Then, there exist $\delta > 0$, $N \gg 1$, such that, for all $X \in C^\infty(\mathcal{M}, T\mathcal{M})$ with $\|X - X_0\|_{C^N} < \delta$, the following hold:

(1) *The resolvent $R^X(z) := (-X + z)^{-1} : L^2(\mathcal{M}) \to L^2(\mathcal{M})$, initially defined in the half-plane $\{z \in \mathbb{C} \mid \Re(z) \gg 1\}$, extends meromorphically to $\mathbb{C}$ as a bounded operator $R^X(z) : C_c^\infty(\mathcal{M}^\circ) \to \mathcal{D}'(\mathcal{M}^\circ)$.*

(2) *If $z_0 \in \mathbb{C}$ is not a pole of $R^{X_0}(z)$, then the map*

$$C^\infty(\mathcal{M}, T\mathcal{M}) \ni X \mapsto R^X(z_0) \in \mathcal{L}(C_c^\infty(\mathcal{M}^\circ), \mathcal{D}'(\mathcal{M}^\circ))$$

is C^2-regular[3] *with respect to X.*

Here, we denote by $\mathcal{L}(A, B)$ the space of continuous linear maps between functional spaces A and B. The space $\mathcal{L}(C_c^\infty(\mathcal{M}^\circ), \mathcal{D}'(\mathcal{M}^\circ))$ can be naturally identified with $\mathcal{D}'(\mathcal{M}^\circ \times \mathcal{M}^\circ)$ via the Schwartz kernel

[3] Even though we only need C^2, our proof actually shows it is C^k for all $k \in \mathbb{N}$.

theorem; the space $\mathcal{D}'(\mathcal{M}^\circ \times \mathcal{M}^\circ)$ is equipped with the standard topology on distributions. In fact, we prove the result above in anisotropic Sobolev spaces, and refer to Theorem 5.14 for a more detailed statement. We show that the scattering operator $\mathcal{S}_g$ has a Schwartz kernel that can be written as a restriction of the Schwartz kernel of $R^{X_g}(0)$ on $\partial_-\mathcal{M} \times \partial_+\mathcal{M}$, implying that the map $g \mapsto \mathcal{S}_g$ is C^2-regular as operators acting on some appropriate Sobolev spaces.

The strategy of the proof then goes as follows. First of all, we put the metric g in solenoidal gauge (with respect to g_0), namely we find a first diffeomorphism $\psi \in \mathrm{Diff}(M)$ such that $\psi|_{\partial M} = \mathrm{id}$ and $g' = \psi^* g$ is divergence-free with respect to g_0, see Lemma 3.6. Secondly, letting

$$I_2^{g_0} : C^\infty(M, \otimes_S^2 T^*M) \to L^\infty_{\mathrm{loc}}(\partial_-\mathcal{M} \setminus \{\ell_{g_0} = \infty\})$$

be the X-ray transform on symmetric 2-tensors with respect to g_0, defined as

$$I_2^{g_0} h(x, v) := \int_0^{\ell_{g_0}(x,v)} h_{\gamma(t)}(\dot\gamma(t), \dot\gamma(t))\, \mathrm{d}t \quad \text{if } \varphi_t^{g_0}(x, v) = (\gamma(t), \dot\gamma(t)) \in \mathcal{M}, \tag{1-2}$$

we show in Section 4A the following key estimate: there are $C, \mu > 0$ such that, if $(\ell_{g_0}, S_{g_0}) = (\ell_g, S_g)$ and $\|g' - g_0\|_{C^N} < \delta$ for some small $\delta > 0$, then

$$\|I_2^{g_0}(g' - g_0)\|_{H^{-6}(\partial_-\mathcal{M})} \le C \|g' - g_0\|^{1+\mu}_{C^N(M, \otimes_S^2 T^*M)}. \tag{1-3}$$

The proof of this estimate is involved. It is based on some complex interpolation argument using the holomorphic map

$$\mathbb{C} \ni z \mapsto e^{-z\ell_{g_0}} I_2^{g_0}(g' - g_0)$$

and the C^2-smoothness of the scattering map $g \mapsto \mathcal{S}_g$ as a continuous map from $C^\infty(\partial_+\mathcal{M})$ to $H^{-6}(\partial_-\mathcal{M})$. This is established in Section 5. It also relies on some volume estimates on the set of geodesics trapped for time $t \to \infty$ that follow from [Guillarmou 2017b].

Finally, slightly extending (M, g_0) to some (M_e, g_{0e}), using the mapping properties of the adjoint $(I_2^{g_{0e}})^*$, interpolation arguments, and (1-3), one obtains, for $h := g' - g_0$,

$$\|h\|_{L^2} \le C \|\Pi_2^{g_{0e}} E_0 h\|_{H^1} \le C \|h\|^{1+\mu}_{C^N}, \tag{1-4}$$

where E_0 is the zero extension operator to M_e, $\Pi_2^{g_{0e}} = (I_2^{g_{0e}})^* I_2^{g_{0e}}$ is the normal operator, and the estimate on the left is an elliptic estimate proved in Proposition 3.8. It is left to interpolate C^N between L^2 and $C^{N'}$ in (1-4), where $N' \gg N$, to get, for some $0 < \mu' < \mu$,

$$\|h\|_{L^2} \le C \|h\|_{L^2} \|h\|^{\mu'}_{C^{N'}} \le C \|h\|_{L^2} \|g - g_0\|^{\mu'}_{C^{N'}}.$$

For $\|g - g_0\|_{C^{N'}}$ small enough, this readily implies that $g' = \phi^* g = g_0$, concluding the proof.

2. Geometric and dynamical preliminaries

Following [Guillarmou 2017b, Section 2], we describe the scattering and length maps in our geometric setting, and relate them to the resolvent of the geodesic flow.

2A. *Unit tangent bundle and extensions.*

2A1. *Geometry of the unit tangent bundle.* Let (M, g) be a smooth compact oriented Riemannian manifold with strictly convex boundary (in the sense that the second fundamental form is positive), and let $S^g M = \{(x, v) \in TM \mid |v|_{g_x} = 1\}$ be the unit tangent bundle with projection on the base denoted by $\pi_0 : S^g M \to M$. For a point $y = (x, v) \in S^g M$, we shall write $-y := (x, -v)$. Denote by $\varphi_t^g : S^g M \to S^g M$ the geodesic flow at time $t \in \mathbb{R}$, and by X_g its generating vector field. Let α be the canonical Liouville 1-form on $S^g M$, defined by $\alpha(x, v)(\xi) := g_x(d\pi_0(x, v)\xi, v)$ for any $\xi \in T_{(x,v)} S^g M$, and define $\mu := \alpha \wedge d\alpha^{n-1}$, the associated Liouville volume form, which we will freely identify with the Liouville measure. It satisfies $\mathcal{L}_{X_g} \mu = 0$, where $\mathcal{L}_{X_g}$ denotes the Lie derivative along X_g.

Recall that we introduced the incoming $(-)$ and outgoing $(+)$ boundaries as

$$\partial_\pm S^g M = \{(x, v) \in \partial S^g M \mid \pm g_x(v, \nu) > 0\},$$

where ν is the outward-pointing unit normal to ∂M. Using the orthogonal decomposition

$$T_{\partial M} M = T(\partial M) \oplus^\perp \mathbb{R}\nu, \tag{2-1}$$

the boundary $\partial_\pm S^g M$ can be naturally identified with the boundary ball

$$B(\partial M) := \{(x, v) \in TM \mid x \in \partial M,\ v \in T_x(\partial M),\ |v|_g \le 1\}$$

by means of the orthogonal projection onto the first factor in (2-1). As a consequence, if g' is any other smooth metric on M such that $g|_{T\partial M} = g'|_{T\partial M}$, the boundaries $\partial_\pm S^g M$ and $\partial_\pm S^{g'} M$ can be naturally identified and it makes sense to say that $(\ell_g, S_g) = (\ell_{g'}, S_{g'})$. When this equality holds, we say that the manifolds (M, g) and (M', g') have the same *lens data*.

When we consider a set of metrics g, the unit tangent bundles $S^g M$ depend on g. For convenience, we will thus fix the manifold

$$\mathcal{M} := S^{g_0} M,$$

associated to an arbitrary metric of reference g_0. We can always rescale the flow φ_t^g so that it becomes defined on $\mathcal{M}$. Indeed, define $\Phi_{g_0 \to g} : S^{g_0} M \to S^g M$ by

$$\Phi_{g_0 \to g}(x, v) := (x, v/|v|_g).$$

Then $\Phi_{g_0 \to g}^{-1} \circ \varphi_t^g \circ \Phi_{g_0 \to g}$ is a flow on $\mathcal{M}$ which we shall still denote by φ_t^g, and its vector field will also be denoted by X_g for simplicity.

We shall always work with metrics g such that $g|_{T\partial M} = g_0|_{T\partial M}$. The boundary of $\mathcal{M}$ splits into a disjoint union

$$\partial \mathcal{M} = \partial_- \mathcal{M} \cup \partial_+ \mathcal{M} \cup \partial_0 \mathcal{M}, \tag{2-2}$$

where $\partial_\pm \mathcal{M} := \{(x, v) \in \partial \mathcal{M} \mid \pm g_x(v, \nu) > 0\}$ and $\partial_0 \mathcal{M} := \{(x, v) \in \partial \mathcal{M} \mid g_x(v, \nu) = 0\}$. Note that the normal ν depends on g, and that the splitting (2-2) does not depend on the choice of $g = g_0$ on $T\partial M$. This will be important to compare for $g \ne g'$ the length functions ℓ_g with $\ell_{g'}$ and the scattering maps S_g with $S_{g'}$ (see Definition 2.2 below).

There is a symplectic form on $\partial_\pm\mathcal{M}$ obtained by restricting $\iota_\partial^* d\alpha$ to $\partial_\pm\mathcal{M}$, where $\iota_\partial : \partial\mathcal{M} \to \mathcal{M}$ is the inclusion map. We denote by

$$\mu_\partial := |\iota_\partial^*(i_{X_g}\mu)| = |\iota_\partial^*(d\alpha)^{n-1}|$$

the induced measure on $\partial\mathcal{M}$, where i_{X_g} denotes the contraction with X_g. In what follows we will write $L^p(\partial_\pm\mathcal{M})$ for the usual L^p space with respect to any smooth Riemannian measure $\mathrm{d}v_h$ on $\partial\mathcal{M}$ (for some metric h on $\partial\mathcal{M}$), while we will write $L^p(\partial_\pm\mathcal{M}, \mu_\partial)$ when we use the measure μ_∂. We note that $\mu_\partial = \omega\,\mathrm{d}v_h$, where $\omega \in C^\infty(\partial\mathcal{M})$ is positive outside $\partial_0\mathcal{M}$ and vanishes to order 1 at $\partial_0\mathcal{M}$, thus $L^p(\partial_\pm\mathcal{M}) \hookrightarrow L^p(\partial_\pm\mathcal{M}, \mu_\partial)$ continuously.

2A2. *Extension of the manifold.* It will be convenient to consider an embedding of $\mathcal{M}$ into a smooth closed manifold $\mathcal{N}$. This can be done by considering an embedding $M \hookrightarrow N$, where N is a smooth closed manifold (this is always possible by doubling the manifold M across its boundary for instance, i.e., gluing $M \sqcup M$ along ∂M by means of the identity map), then extending smoothly the metric g_0 to N (denoted by g_{0N}) and taking $\mathcal{N} := S^{g_{0N}}N$. If g_0 is of Anosov type (see Definition 1.3), it will be also convenient to have a slightly larger manifold with boundary M_e at our disposal such that $M \hookrightarrow M_e \hookrightarrow N$ and the extension of the metric g_0 to M_e, which we denote by g_{0e}, is of Anosov type; see [Guillarmou 2017b, Section 2] where this is further discussed. Set $\mathcal{M}_e := S^{g_{0e}}M_e$. We have the successive embeddings $\mathcal{M} \hookrightarrow \mathcal{M}_e \hookrightarrow \mathcal{N}$. For a metric g close to g_0 in C^N norm and such that $g = g_0$ on $T\partial M$, we consider an extension g_e of Anosov type on M_e. The map $g \mapsto g_e$ can be chosen to be smooth and so that

$$\|g_e - g_{0e}\|_{C^N(M_e, \otimes_S^2 T^*M_e)} \le C_N \|g - g_0\|_{C^N(M, \otimes_S^2 T^*M)}$$

for all $N \ge 0$ and some constants $C_N > 0$, where $\otimes_S^2 T^*M$ is the bundle of symmetric 2-tensors.

Definition 2.1. Let $c \in \mathbb{R}$. We say that a level set $\{\rho = c\}$ of a function $\rho \in C^\infty(\mathcal{N})$ is *strictly convex* with respect to a vector field $Y \in C^\infty(\mathcal{N}, T\mathcal{N})$ if, for all $y \in \{\rho = c\}$, one has

$$Y\rho(y) = 0 \quad \Longrightarrow \quad Y^2\rho(y) < 0.$$

We say that a smooth submanifold $\mathcal{H} \subset \mathcal{N}$ is strictly convex with respect to Y if $\mathcal{H}$ is in a neighborhood of $\mathcal{H}$ given by a level set $\{\rho = 0\}$ of some function ρ, and this level set is strictly convex with respect to Y. This is independent of the choice of ρ.

It can be easily checked that (M, g_0) has strictly convex boundary in the Riemannian sense if and only if $\partial\mathcal{M}$ is strictly convex with respect to the geodesic vector field X_{g_0}.

We now consider an arbitrary smooth extension $\widetilde{X}_{g_0}$ of $X_{g_{0e}}|_{\mathcal{M}_e}$ to $\mathcal{N}$. Let $\rho \in C^\infty(\mathcal{N})$ be a global boundary-defining function for $\mathcal{M}$, i.e., such that $\rho > 0$ on the interior of $\mathcal{M}$, $\partial\mathcal{M} = \{\rho = 0\}$ and $\rho < 0$ on $\mathcal{N} \setminus \mathcal{M}$. Since X_{g_0} does not vanish on $\mathcal{M} = \{\rho \ge 0\}$, we can consider $\rho_0 > 0$ small enough that $\widetilde{X}_{g_0}$ does not vanish in $\{\rho > -2\rho_0\}$. A continuity argument shows that, for all $\rho_0 > 0$ small enough, the level set $\{\rho = -\rho_0\}$ is strictly convex with respect to $\widetilde{X}_{g_0}$. We can assume that

$$\mathcal{M}_e = \left\{x \in \mathcal{N} \;\middle|\; \rho(x) \ge -\tfrac{1}{2}\rho_0\right\}.$$

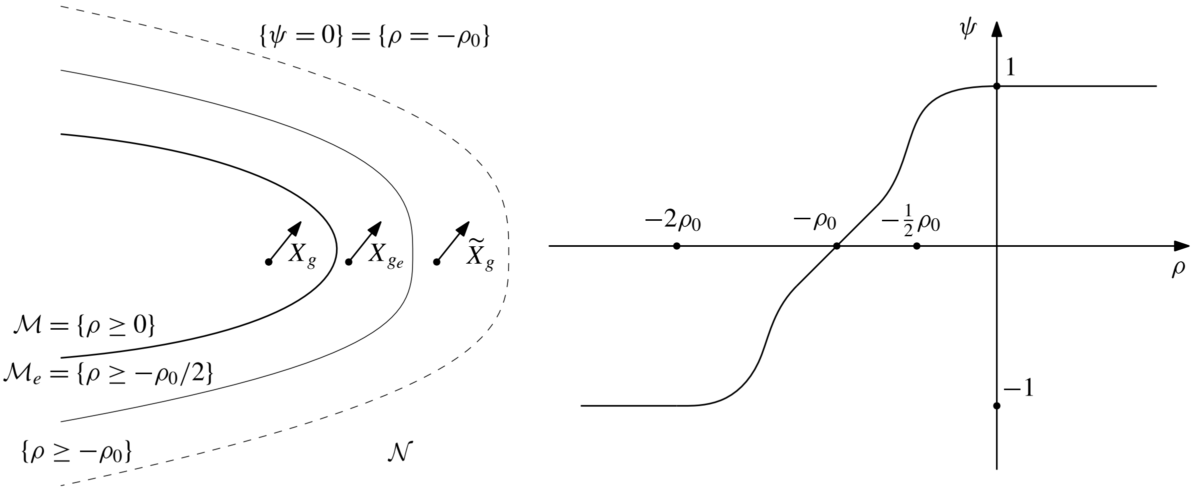

Figure 2. One the left: the extension of the vector field X_g from $\mathcal{M}$ to X_{g_e} on $\mathcal{M}_e$, and further to $\widetilde{X}_g$ on $\mathcal{N}$. The vector field $X = \psi \widetilde{X}_g$ is *complete* on the set $\{\rho \geq -\rho_0\}$ and vanishes on $\{\rho = -\rho_0\}$. On the right: the auxiliary function ψ as a function of ρ.

In the following, we will consider smooth perturbations X of the vector field X_{g_0} in $\mathcal{M}$ (small in the C^N-topology, for $N \gg 1$ large enough). They will mostly be induced by a metric g close to g_0, but it might be better to have in mind a more general picture than just geodesic flows. It will be convenient to extend the vector fields X_g to vector fields $\widetilde{X}_g$ on $\mathcal{N}$ such that $\widetilde{X}_g = \widetilde{X}_{g_0}$ on the set $\left\{\rho \leq -\frac{2}{3}\rho_0\right\}$ and $\widetilde{X}_g = X_{g_e}$ on $\mathcal{M}_e$. Moreover, it is possible to construct such an extension with, for any $N \in \mathbb{N}$,

$$\|\widetilde{X}_g - \widetilde{X}_{g_0}\|_{C^N(\mathcal{N}, T\mathcal{N})} \leq C \|X_g - X_{g_0}\|_{C^N(\mathcal{M}, T\mathcal{M})}$$

for some constant $C > 0$ (depending only on $\mathcal{M}$, $\mathcal{N}$, and N). Also observe that strict convexity of the boundary is stable by a C^2-perturbation of the vector field.

We introduce the smooth function $\psi \in C^\infty(\mathcal{N})$ with values in $[-1, 1]$ such that

- $\psi = \rho + \rho_0$ on the set $\left\{-\rho_0 - \frac{1}{10}\rho_0 \leq \rho \leq -\rho_0 + \frac{1}{10}\rho_0\right\}$,
- $\psi = 1$ on $\mathcal{M} = \{\rho \geq 0\}$, and $\psi > 0$ on $\{\rho > -\rho_0\}$,
- $\psi = -1$ on $\{\rho \leq -2\rho_0\}$, and $\psi < 0$ on $\{\rho < -\rho_0\}$.

With some abuse of notation, we then denote by X and X_0 the vector fields on $\mathcal{N}$ defined by $X := \psi \widetilde{X}_g$ and $X_0 := \psi \widetilde{X}_{g_0}$, respectively. This construction ensures that the restriction of X to $\mathcal{M}$ is the original vector field initially defined on $\mathcal{M}$ and that $\{\rho \geq -\rho_0\}$ is preserved by all the flows $(\varphi_t^X)_{t\in\mathbb{R}}$ for all $t \in \mathbb{R}$, and finally that each trajectory leaving $\mathcal{M}$ never comes back to $\mathcal{M}$, with the same property for $\mathcal{M}_e$. See Figure 2 for a visual summary of this construction.

2B. ***Scattering and length maps.*** For $(x, v) \in \mathcal{M}$, the escape time $\tau_g(x, v)$ is defined to be the maximal time of existence of the integral curve $(\varphi_t^g(x, v))_{t\geq 0}$ in $\mathcal{M}$:

$$\tau_g : \mathcal{M} \to [0, \infty], \quad \tau_g(x, v) := \sup\{t \geq 0 \mid \varphi_t^g(x, v) \in \mathcal{M}\}.$$

The forward (−) and backward (+) trapped sets $\Gamma^g_\pm$ are defined by

$$\Gamma^g_\pm := \{(x,v)\in\mathcal{M} \mid \tau_g(x,\mp v)=\infty\};$$

they are closed sets in $\mathcal{M}$, and the trapped set is the closed invariant set

$$K^g := \Gamma^g_+ \cap \Gamma^g_- = \bigcap_{t\in\mathbb{R}} \varphi^g_t(\mathcal{M}).$$

Since ∂M is strictly convex, it is straightforward to check that $\Gamma^g_\mp \cap \partial_\pm\mathcal{M} = \varnothing$ and $K^g \cap \partial\mathcal{M} = \varnothing$. We now recall the definition (see Definition 1.1) of the *lens data*.

Definition 2.2 (lens data). The length map $\ell_g : \partial_-\mathcal{M}\setminus\Gamma^g_- \to \mathbb{R}_+$ and the scattering map $S_g : \partial_-\mathcal{M}\setminus\Gamma^g_- \to \partial_+\mathcal{M}\setminus\Gamma^g_+$ are defined by

$$\ell_g(x,v) := \tau_g(x,v) \quad\text{and}\quad S_g(x,v) := \varphi^g_{\tau_g(x,v)}(x,v).$$

The pair (ℓ_g, S_g) is called the lens data of (M,g).

When unnecessary, we will drop the index g in the notation. It will be convenient to view the scattering map as acting on functions on $\partial_+\mathcal{M}$ by pull-back. We define the *scattering operator* as

$$\mathcal{S}_g : C^\infty_{\mathrm{c}}(\partial_+\mathcal{M}\setminus\Gamma^g_+) \to C^\infty_{\mathrm{c}}(\partial_-\mathcal{M}\setminus\Gamma^g_-), \quad \mathcal{S}_g\omega := \omega\circ S_g.$$

Under the assumption that $\mu_\partial((\Gamma^g_-\cup\Gamma^g_+)\cap\partial\mathcal{M})=0$, it is not difficult to show (see [Guillarmou 2017b, Lemma 3.4]) that, for all $f\in C^\infty_{\mathrm{c}}(\partial_+\mathcal{M}\setminus\Gamma_+)$, one has

$$\|\mathcal{S}_g f\|_{L^2(\partial_-\mathcal{M},\mu_\partial)} = \|f\|_{L^2(\partial_+\mathcal{M},\mu_\partial)},$$

and thus $\mathcal{S}_g$ extends continuously to an isometry $L^2(\partial_+\mathcal{M},\mu_\partial)\to L^2(\partial_-\mathcal{M},\mu_\partial)$. The scattering operator $\mathcal{S}_g$ determines S_g, and conversely.

By the implicit function theorem (since ∂M is strictly convex), we also have that

$$\tau_g \in C^\infty(\mathcal{M}\setminus(\Gamma^g_-\cup\partial_0\mathcal{M})) \quad\text{and}\quad \ell_g\in C^\infty(\overline{\partial_-\mathcal{M}}\setminus\Gamma^g_-)$$

(here $\overline{\partial_-\mathcal{M}} = \partial_0\mathcal{M}\cup\partial_-\mathcal{M}$); see [Sharafutdinov 1994, Lemmas 4.1.1 and 4.1.2] for further details. Since we shall need the dependence of ℓ_g with respect to g, we first prove a result outside the trapped sets.

Lemma 2.3. *Let (M,g_0) be a smooth compact Riemannian manifold with strictly convex boundary, and let $p\in\mathbb{N}$. There exists $\varepsilon>0$ small enough that the following holds: for all metrics $g\in U_{g_0}$, where*

$$U_{g_0} := \{g\in C^{p+2}(M,\otimes^2_S T^*M) \mid \|g-g_0\|_{C^{p+2}}<\varepsilon,\ g|_{T\partial M} = g_0|_{T\partial M}\}, \tag{2-3}$$

the following map is C^p-regular:

$$\ell : V\to\mathbb{R}_+, \quad (g,y)\mapsto \ell_g(y),$$

where $V := \{(g,y)\in U_{g_0}\times\partial_-\mathcal{M} \mid y\notin\Gamma^g_-\}$. Moreover, for all $\chi\in C^\infty_{\mathrm{c}}(\partial_-\mathcal{M})$, there exists a constant $C>0$ (depending only on g_0, p and χ) such that, for all $j\le p$ and $h\in C^\infty(M,\otimes^2_S T^*M)$,

$$\text{for all } (g,y)\in V, \qquad |\chi d^j_y \ell_g(y)| \le Ce^{C\ell_g(y)} \quad\text{and}\quad |\chi\partial^j_g\ell_g(y)(\otimes^j h)| \le Ce^{C\ell_g(y)}\|h\|^j_{C^{j+1}}.$$

Proof. We shall use the implicit function theorem. Let ρ be the boundary-defining function of $\mathcal{M}$ defined in Section 2A2. As explained in this paragraph, for g close to g_0, we can consider a vector field X on $\mathcal{N}$ such that X vanishes (to first order) on $\{\rho = -\rho_0\}$. For the sake of simplicity, we still denote by $(\varphi_t^g)_{t\in\mathbb{R}}$ the extended flow on $\mathcal{N}$, and by $X_g := X$ its generator.

We consider the C^p-regular map

$$F : U_{g_0} \times \mathbb{R}_+ \to \mathbb{R}, \quad (g, y, t) \mapsto \rho(\varphi_t^g(y)).$$

The function $\ell_g(y)$ satisfies the implicit equation $F(g, y, \ell_g(y)) = 0$. Let us take a point $(g_0, y_0) \in V$ and differentiate, for (g, y) near (g_0, y_0),

$$\partial_t F(g, y, t) = (X_g\rho)(\varphi_t^g(y)).$$

Notice that this is nonzero if $y \in \partial_-\mathcal{M}$, and $\varphi_t(y) \in \partial_+\mathcal{M}$ by strict convexity of $\partial\mathcal{M}$. Thus the implicit function theorem guarantees that there are neighborhoods $U'_{g_0} \subset U_{g_0}$ of g_0 and $B_{y_0}(\varepsilon') \subset \partial_-\mathcal{M}$ of y_0 such that $(g, y) \mapsto \ell_g(y)$ is a well-defined $C^p(U'_{g_0} \times B_{y_0}(\varepsilon'))$ function and

$$d_y\ell_g(y) = -\frac{d\rho(\varphi^g_{\ell_g(y)}(y)) \circ (d\varphi^g_{\ell_g(y)})(y)}{(X_g\rho)(\varphi^g_{\ell_g(y)}(y))}.$$

Notice in particular that this implies that V is an open set. By the Grönwall lemma, there is a constant $C > 0$ uniform in $g \in U_{g_0}$ such that, for each $(g, y) \in V$ and all $t > 0$, where $\|\cdot\|$ denotes an arbitrary fixed metric on $\mathcal{N}$,

$$\|d_y\varphi_t^g(y)\| \le Ce^{Ct}. \tag{2-4}$$

The constant $C > 0$ provided by the Grönwall lemma is uniform in the metric g as long as it is C^3-close to g_0. More generally, (2-4) holds for the j-th derivative $d_y^j\varphi_t^g$ with a constant $C > 0$ uniform for g which is C^{j+2}-close to g_0. On the other hand, we know that $X_g\rho \neq 0$ on $\partial\mathcal{M} \setminus \partial_0\mathcal{M}$. So we obtain a constant $C > 0$ such that,

$$\text{for all } (g, y) \in V, \quad |\chi(y) d_y\ell_g(y)| \le Ce^{C\ell_g(y)}.$$

Next, we compute the derivative with respect to g for some $h \in C^\infty(M, \otimes_S^2 T^*M)$:

$$(\partial_g\ell_g.h)(y) = -\frac{d\rho(\varphi^g_{\ell_g(y)}(y)) \circ (\partial_g\varphi^g_{\ell_g(y)}.h)(y)}{(X_g\rho)(\varphi^g_{\ell_g(y)}(y))}.$$

Again, by the Grönwall lemma, we obtain a constant $C > 0$ such that, for all $t > 0$, $(g, y) \in V$,

$$\|(\partial_g\varphi_t^g.h)(y)\| \le Ce^{Ct}\|h\|_{C^2}, \tag{2-5}$$

which provides the desired estimate for the C^2-norm. (The C^2-norm of h appears as the vector field X_g involves the 1-derivative of g, so that X_{g+sh} is C^1 for all $s \in \mathbb{R}$ small). The constant $C > 0$ is uniform for g that is C^3-close to g_0. More generally, the bound $|\partial_g^j\varphi_t^g(\otimes^j h)(y)| \le Ce^{Ct}\|h\|^j_{C^{j+1}}$ holds with a constant $C > 0$ depending on the C^{j+2}-norm of g. The case of higher-order derivatives works exactly the same way by differentiating as many times as needed the implicit equation defining $\ell_g(y)$ with respect to (g, y), and using that the derivatives of the flow satisfy the bounds $\|D^j\varphi_t^g(y)\| \le Ce^{Ct}$ (where $D^j = \partial_g^j$ or d_y^j) for some uniform $C > 0$ with respect to $t > 0$, y and $g \in U_{g_0}$. □

2C. *Hyperbolic trapped set.*

2C1. *Axiom A property.* We say that the trapped set is *hyperbolic* if there is a continuous flow-invariant splitting of $T(SM)$ restricted to K^g into three subbundles:

$$\text{for all } y \in K^g, \quad T_y\mathcal{M} = \mathbb{R}X_g(y) \oplus E_s^g(y) \oplus E_u^g(y),$$

and $C, \nu > 0$ such that, for all $y \in K^g$ and $t \ge 0$,

$$\begin{aligned} v \in E_s^g(y) \quad &\Longrightarrow \quad \|d\varphi_t^g(y)v\| \le Ce^{-\nu t}\|v\|, \\ v \in E_u^g(y) \quad &\Longrightarrow \quad \|d\varphi_{-t}^g(y)v\| \le Ce^{-\nu t}\|v\|. \end{aligned} \tag{2-6}$$

There is a continuous extension of the bundles E_s^g and E_u^g to the bundles E_-^g and E_+^g over the sets Γ_-^g and Γ_+^g, respectively, on which (2-6) is still satisfied; see [Dyatlov and Guillarmou 2016, Lemma 2.10]. For $y \in K^g$, these bundles coincide with E_s^g and E_u^g, namely $E_s^g(y) = E_-^g(y)$ and $E_u^g(y) = E_+^g(y)$. We define $C^k_{\text{hyp}}(M, \otimes_S^2 T^*M_+)$ to be the set of C^k Riemannian metrics on M with strictly convex boundary and hyperbolic trapped set. For such metrics, the geodesic flow is a typical example of what is known as an *Axiom A flow*. Since these metrics could have conjugate points, this set is larger than the set of metrics of Anosov type.

If g_0 is some fixed metric on M and M_e denotes the extension defined in Section 2A2 with ρ a boundary-defining function of $\mathcal{M}$, we can always choose $\rho_0 > 0$ small enough that, for all $|t| \le \rho_0$, the level set $\{\rho = t\}$ is strictly convex with respect to the extension g_{0e} of g_0 to M_e. This also holds for any metric g close to g_0 in the C^2-topology. Recall that we denote by g_e the extension of g from M to M_e.

Observe that if $y \in \partial_\pm\mathcal{M}$ then $\bigcup_{\pm t>0}\varphi_t^{g_e}(y) \subset \mathcal{N}\setminus\mathcal{M}$. The trapped sets of (M, g) and (M_e, g_e) then coincide and $\Gamma_\pm^g = \Gamma_\pm^{g_e} \cap \mathcal{M}$. Moreover, if (M, g) has no conjugate points, then by taking $\rho_0 > 0$ small enough (M_e, g_e) does not have conjugate points either; see [Guillarmou 2017b, Lemma 2.3].

Define the set of points that are trapped for time less than $t \ge 0$ as

$$\mathcal{T}^g(t) := \{y \in \mathcal{M} \mid \forall s \in (0, t),\ \varphi_s^g(y) \in \mathcal{M}^\circ\} = \tau_g^{-1}(t, \infty).$$

It is proved in [Guillarmou 2017b, Proposition 2.4] that there exist $C_g, Q_g > 0$ (depending on the metric g) such that, for all $t \ge 0$,

$$\mu(\mathcal{T}^g(t)) \le C_g e^{-Q_g t}. \tag{2-7}$$

(Here μ is the Liouville measure for the fixed g_0.) In particular, $\mu(\Gamma_\pm^g) = 0$. The quantity Q_g is called the *escape rate* and is given by $-Q_g = P_g(-J_u^g) < 0$: the topological pressure of negative the unstable Jacobian $J_u^g(y) := \partial_t(\det d\varphi_t^g(y)|_{E_u(y)})|_{t=0}$ of the flow $(\varphi_t^g)_{t\in\mathbb{R}}$. Recall that the topological pressure of a Hölder potential $V \in C^\beta(S^gM)$ (for some $\beta > 0$) with respect to g can be defined as follows:

$$P_g(V) := \lim_{T\to\infty}\frac{1}{T}\log\sum_{\gamma\in\mathcal{P},\, T_\gamma\in[T,T+1]}\exp\Big(\int_\gamma V\Big),$$

where $\mathcal{P}$ is the set of periodic orbits of the geodesic flow $(\varphi_t^g)_{t\in\mathbb{R}}$, and T_γ is the period of $\gamma \in \mathcal{P}$.

The following formula for $f \in L^1(\mathcal{M})$ is known as *Santaló's formula* (see [Guillarmou 2017b, Section 2.5]):

$$\int_{\mathcal{M}} f(y)\,\mathrm{d}\mu(y) = \int_{\partial_-\mathcal{M}} \int_0^{+\infty} f(\varphi_t^g(y))\,\mathrm{d}t\,\mathrm{d}\mu_\partial(y). \tag{2-8}$$

It implies, together with (2-7), that there is $C_g > 0$ such that, for all $t > 0$,

$$\mu_\partial(\ell_g^{-1}(t,\infty)) \le C_g e^{-Q_g t}. \tag{2-9}$$

Using Cavalieri's principle, estimates (2-7) and (2-9), it is straightforward to derive the following bounds:

$$\begin{aligned} &\text{for all } p \in [1,\infty), && \tau_g \in L^p(\mathcal{M}), && \ell_g \in L^p(\partial_-\mathcal{M}),\\ &\text{for all } \lambda \in (0, Q_g), && e^{\lambda\tau_g} \in L^1(\mathcal{M}), && e^{\lambda\ell_g} \in L^1(\partial_-\mathcal{M}). \end{aligned} \tag{2-10}$$

Here note that ℓ_g is bounded near $\partial_0\mathcal{M}$, so that this region is trivial to deal with.

2C2. *Robinson structural stability.* In this paragraph, we recall some results about the stability of flows with hyperbolic trapped set, due to [Robinson 1980, Theorem C]. First, the stable and unstable manifolds of a point $y \in K^g$ are defined by

$$W_s(y) := \{y' \in \mathcal{M} \mid \lim_{t\to+\infty} d(\varphi_t^g(y'), \varphi_t^g(y)) \to 0\},$$
$$W_u(y) := \{y' \in \mathcal{M} \mid \lim_{t\to-\infty} d(\varphi_t^g(y'), \varphi_t^g(y)) \to 0\}.$$

They are smooth injectively immersed submanifolds. We also set

$$W_u(K^g) := \bigcup_{y\in K^g} W_u(y) \quad \text{and} \quad W_s(K^g) := \bigcup_{y\in K^g} W_s(y).$$

It is proved in [Guillarmou 2017b, Lemma 2.2] that

$$W_s(K^g) = \Gamma_-^g \quad \text{and} \quad W_u(K^g) = \Gamma_+^g. \tag{2-11}$$

The tangent spaces to $W_s(y)$ and $W_u(y)$ are $E_s(y)$ and $E_u(y)$, respectively. The flow satisfies the following *transversality property* for the stable and unstable manifolds $W_s(y)$ and $W_u(y)$: for each $y, y' \in K^g$ and $z \in W_s(y) \cap W_u(y') \subset K^g$, we have

$$T_z(\mathcal{M}) = T_z(W_s(y)) \oplus T_z(W_u(y')) \oplus \mathbb{R}X_g(z).$$

Indeed, such z must belong to K^g, and the identity of the tangent space can be rewritten as

$$E_s(z) \oplus E_u(z) \oplus \mathbb{R}X_g(z) = T_z(\mathcal{M}),$$

which holds since K^g is assumed hyperbolic. For a Riemannian manifold with strictly convex boundary and hyperbolic trapped set, the geodesic flow $(\varphi_t^g)_{t\in\mathbb{R}}$ on $\mathcal{M}$ satisfies the following:

- The nonwandering set $\Omega \subset K^g$ is hyperbolic.
- The stable and unstable manifolds have the transversality property.
- The boundary is strictly convex with respect to the vector field X_g.

Proposition 2.4 [Robinson 1980]. *Let (M, g_0) be a smooth Riemannian manifold with strictly convex boundary and hyperbolic trapped set $K^{g_0} \subset \mathcal{M} := SM$. Then, there exists $\varepsilon_0 > 0$ such that, for each smooth vector field X on $\mathcal{M}$ with $\|X - X_{g_0}\|_{C^2(\mathcal{M})} \le \varepsilon_0$, there is a homeomorphism $h : \mathcal{M} \to \mathcal{M}$ and $a \in C^0(U)$, where $U = \{(y,t) \in \mathcal{M} \times \mathbb{R} \mid t \in [-\tau_{g_0}(-h(y)), \tau_{g_0}(h(y))]\}$, such that the following holds: for all $y \in \mathcal{M}$, we have that $t \mapsto a(y,t)$ is strictly increasing in t and satisfies*

$$\varphi_t^{X_{g_0}}(h(y)) = h(\varphi^X_{a(y,t)}(y))$$

for all $(y,t) \in \mathcal{M} \times \mathbb{R}$ such that $\varphi^X_{a(y,t)}(y) \in \mathcal{M}$. Moreover, for each $\delta > 0$ there exists $\varepsilon > 0$ small enough that if $\|X - X_{g_0}\|_{C^2(\mathcal{M})} \le \varepsilon$, then $d(h(y), y) \le \delta$ for $y \in \mathcal{M}$, where d denotes a Riemannian distance on $\mathcal{M}$, that is, $\|h - \mathrm{id}_{\mathcal{M}}\|_{C^0} \le \delta$.

Proof. This is a direct consequence of [Robinson 1980, Theorems A and C]. We note that Robinson's "quadratic external boundary conditions" are equivalent to our strict convexity of the boundary, and that the *chain-recurrent set* (see [Robinson 1980] for the definition) is contained in the trapped set, which by assumption has a hyperbolic structure with transversal stable and unstable manifolds. Finally, the last statement about the continuity of h is stated in [Robinson 1980, Theorem A]. ☐

As a consequence, we see that, for g close enough to g_0 in C^3 norm, applying Proposition 2.4 with $X = X_g$, we get

$$K^g = h^{-1}(K^{g_0}) \quad \text{and} \quad h^{-1}(\Gamma_\pm^{g_0}) = \Gamma_\pm^g,$$

and the trapped set varies continuously with respect to the metric.

2C3. *Symplectic lift to the cotangent bundle.* Recall that we introduced the vector field X on $\mathcal{N}$ in Section 2A2. In Section 5, it will be convenient to work on the cotangent bundle $T^*\mathcal{N}$ of the extended manifold $\mathcal{N}$. Denote by $\boldsymbol{X}$ the symplectic lift of the vector field X to $T^*\mathcal{N}$. It generates the flow

$$\varphi_t^{\boldsymbol{X}}(y,\xi) = (\varphi_t^X(y), (d\varphi_t^X(y))^{-\top}\xi), \tag{2-12}$$

where ${}^{-\top}$ stands for the inverse transpose. Note that this flow is linear in the second variable and thus induces a flow on the spherical bundle $S^*\mathcal{N} := (T^*\mathcal{N} \setminus \{0\})/\mathbb{R}_+$. Let $\pi : S^*\mathcal{N} \to \mathcal{N}$ and $\kappa : T^*\mathcal{N} \to S^*\mathcal{N}$ be the natural projections, and still write π for the projection $T^*\mathcal{N} \to \mathcal{N}$. The dual subbundles $(E^X_{\pm,0})^* \subset T^*\mathcal{N}$ are defined as the following symplectic orthogonals:

$$(E_0^X)^*(E_+^X \oplus E_-^X) = (E_+^X)^*(E_+^X \oplus E_0^X) = (E_-^X)^*(E_-^X \oplus E_0^X) = \{0\}.$$

With some abuse of notation, the spaces $(E^X_{\pm,0})^*$ will be identified with the projections $\kappa((E^X_{\pm,0})^*) \subset S^*\mathcal{N}$. Eventually, we record the following definition to be found useful later:

$$\Sigma_\pm := \bigcup_{\|X - X_0\|_{C^2} \le \delta, \pm t \ge 0} \varphi_t^{\boldsymbol{X}}(\mathcal{M}), \tag{2-13}$$

where $\delta > 0$ is small enough. Finally, we note that the tails $\Gamma_\pm^X$ and the bundles $(E^X_{\pm,0})^*$ admit an extension to the set $\{\rho > -\rho_0\}$.

2D. ***Resolvent and X-ray transform.*** Since we will work with Sobolev spaces on the manifolds $\mathcal{M}$ and $\partial_\pm\mathcal{M}$, let us clarify what this means as these are manifolds with boundary or open manifolds. First, since $\mathcal{M}$ is a smooth manifold with boundary, the spaces $H^s(\mathcal{M})$ are defined intrinsically for $s \geq 0$ (as the restriction of H^s-functions defined on $\mathcal{N}$ for instance). Set $H^s_0(\mathcal{M}) = \overline{C^\infty_{\mathrm{c}}(\mathcal{M}^\circ)}$, where the closure is for the H^s norm, and write $H^{-s}(\mathcal{M}) := (H^s_0(\mathcal{M}))^*$ for $s > 0$, where the upper star denotes the continuous dual. For $\partial_\pm\mathcal{M}$, write $H^s(\partial\mathcal{M}) := H^s(\overline{\partial_\pm\mathcal{M}})$, where $\overline{\partial_\pm\mathcal{M}} := \partial_\pm\mathcal{M} \cup \partial_0\mathcal{M}$ is a smooth manifold with boundary, and $H^{-s}(\partial_\pm\mathcal{M}) = (H^s_0(\partial_\pm\mathcal{M}))^*$.

Define the resolvent of X_g to be the family of operators, for $\Re(z) \geq 0$,

$$R_g(z) : C^\infty_{\mathrm{c}}(\mathcal{M}^\circ \setminus \Gamma^g_-) \to C^\infty(\mathcal{M}), \quad R_g(z) f(y) := -\int_0^{\tau_g(y)} e^{-zt} f(\varphi^g_t(y))\, \mathrm{d}t. \tag{2-14}$$

For $z = 0$, simply write $R_g := R_g(0)$. It solves $X_g R_g = \mathbb{1}$ on $C^\infty_{\mathrm{c}}(\mathcal{M}^\circ \setminus \Gamma^g_-)$ with boundary condition $(R_g f)|_{\partial_+\mathcal{M}} = 0$.

Assuming that (M, g) has strictly convex boundary and hyperbolic trapped set, we have by [Guillarmou 2017b, Propositions 4.2 and 4.4] the following boundedness properties:

$$\text{for all } p \in [1, \infty), \qquad R_g : L^\infty(\mathcal{M}) \to L^p(\mathcal{M}), \tag{2-15}$$

$$\text{for all } \alpha \in (0, 1),\ \text{there exists } s > 0 \text{ such that} \quad R_g : C^\alpha_{\mathrm{c}}(\mathcal{M}^\circ) \to H^s(\mathcal{M}), \tag{2-16}$$

$$\text{for all } s > 0, \qquad R_g : H^s(\mathcal{M}) \to H^{-s}(\mathcal{M}), \tag{2-17}$$

where $C^\alpha(\mathcal{M})$ is the Hölder space of order α. Note that if $\varepsilon > 0$ is chosen small enough,

$$U := \bigcup_{t \in (-\varepsilon, \varepsilon)} \varphi^g_t(\partial_-\mathcal{M})$$

is a neighborhood of $\partial_-\mathcal{M}$ in $\mathcal{M}_e$ which is diffeomorphic to $(-\varepsilon, \varepsilon) \times \partial_-\mathcal{M}$ by $(t, y) \mapsto \varphi^g_t(y)$, and $\partial_t(\tau_g \circ \varphi^g_t) = -1$ in U. Using (2-15), Santaló's formula (2-8), and the fact that ℓ_g is smooth near $\partial_0\mathcal{M}$ in $\partial_-\mathcal{M} \cup \partial_0\mathcal{M}$ (see [Sharafutdinov 1994, Lemma 4.1.1]), we consequently obtain

$$\ell_g = -(R_g \mathbf{1}_\mathcal{M})|_{\partial_-\mathcal{M}} \in L^p(\partial_-\mathcal{M}, \mu_\partial) \tag{2-18}$$

for all $1 \leq p < \infty$. The *X-ray transform* is defined as the operator

$$I^g : C^\infty_{\mathrm{c}}(\mathcal{M} \setminus \Gamma^g_-) \to C^\infty_{\mathrm{c}}(\partial_-\mathcal{M} \setminus \Gamma^g_-), \quad I^g f := -(R_g f)|_{\partial_-\mathcal{M}},$$

and, by [Guillarmou 2017b, Lemma 5.1], it extends as a bounded map for all $p > 2$:

$$I^g : L^p(\mathcal{M}) \to L^2(\partial_-\mathcal{M}, \mu_\partial). \tag{2-19}$$

We now show the following boundedness property.

Lemma 2.5. *Let (M, g) be a compact Riemannian manifold with strictly convex boundary and hyperbolic trapped set. Then, there exists $s > 0$ such that the operator I^g is bounded as a map*:

$$I^g : C^2(\mathcal{M}) \to H^s(\partial_-\mathcal{M}).$$

Proof. First of all, if $\chi \in C^\infty(\overline{\partial_-\mathcal{M}})$ is supported close to $\partial_0\mathcal{M}$, one can check that $\chi I^g f \in C^2(\overline{\partial_-\mathcal{M}})$ for $f \in C^2(\mathcal{M})$; see [Sharafutdinov 1994, Lemma 4.1.1]. It thus remains to analyze $\chi I^g f$ when $\chi \in C_c^\infty(\partial_-\mathcal{M})$. Let $\gamma > 0$ be a large enough constant (it will be determined later), $\varepsilon \in (0, Q_g/(2\gamma))$, and let Δ_h be the Riemannian Laplacian associated to an arbitrarily chosen smooth Riemannian metric h on $\overline{\partial_-\mathcal{M}}$, with Dirichlet condition at $\partial_0\mathcal{M}$. It is self-adjoint on $H_0^1(\overline{\partial_-\mathcal{M}}) \cap H^2(\overline{\partial_-\mathcal{M}})$ with respect to the Riemannian volume measure $\mathrm{d}v_h$. Note that $\mathrm{d}v_h$ is smoothly equivalent to μ_∂ on each compact set of $\partial_-\mathcal{M}$ as μ_∂ vanishes to first order on the boundary $\partial_0\mathcal{M}$.

For $f \in C^2(\mathcal{M})$, consider the holomorphic map

$$\{-\varepsilon \le \Re(z) \le 1-\varepsilon\} \ni z \mapsto u(z) := (1+\Delta_h)^{z+\varepsilon}(e^{-z\gamma\ell_g}\chi I^g f) \in \mathcal{D}'(\partial_-\mathcal{M}).$$

We are going to apply the Hadamard three-line theorem (see [Rudin 1987, Theorem 12.8]) to the holomorphic family of distributions $u(z)$. From (2-19), we have $I^g f \in L^2(\partial_-\mathcal{M}, \mu_\partial)$, but we can also write the pointwise bound,

$$\text{for all } y \in \partial_-\mathcal{M} \setminus \Gamma_-, \quad |I^g f(y)| \le \|f\|_{L^\infty} \ell_g(y). \tag{2-20}$$

From (2-10), we get, using that $\varepsilon < Q_g/(2\gamma)$,

$$\chi e^{\varepsilon\gamma\ell_g} I^g f \in L^2(\partial_-\mathcal{M}, \mathrm{d}v_h).$$

Therefore on the line $\{\Re(z) = -\varepsilon\}$ with $0 < \varepsilon < Q_g/(2\gamma)$, there exists a constant $C > 0$ independent of z and f (but depending on χ) such that

$$\|u(z)\|_{L^2} \le \|(1+\Delta_h)^{i\Im(z)}\|_{L^2\to L^2} \|\chi e^{\varepsilon\gamma\ell_g} I^g(f)\|_{L^2} \le C\|f\|_{L^\infty}, \tag{2-21}$$

where $L^2 = L^2(\partial_-\mathcal{M}, \mathrm{d}v_h)$. Note that we used the spectral theorem for Δ_h in order to bound

$$\|(1+\Delta_h)^{i\Im(z)}\|_{L^2\to L^2} \le 1.$$

Now, using that $I^g f(y) = \int_0^{\ell_g(y)} f(\varphi_t^g(y))\,\mathrm{d}t$, we obtain, using Lemma 2.3, (2-4), and (2-20), the pointwise bound on $\partial_-\mathcal{M} \setminus \Gamma_-^g$:

$$|\Delta_h(e^{-z\gamma\ell_g} I^g f)(y)| \le C(1+|z|^2)\|f\|_{C^2(\mathcal{M})} e^{(C_0-\gamma\Re(z))\ell_g(y)}$$

for some uniform constants $C, C_0 > 0$ (depending only on the metric g). We therefore see that, for $\Re(z) = 1-\varepsilon$, the function $\Delta_h(e^{-\gamma z\ell_g}\chi I^g(f))$ can be extended from $\partial_-\mathcal{M} \setminus \Gamma_-$ continuously to $\partial_-\mathcal{M}$ by setting it to be 0 on Γ_- as long as $\gamma(1-\varepsilon) > C_0$. Here, we see that, in order to achieve this, we can choose $\gamma > 2022C_0$ at the very beginning (the constant C_0 only depends on the metric g).

Claim 2.6. *The continuous extension by* 0 *of* $\Delta_h(e^{-z\gamma\ell_g}\chi I^g f)$ *on* Γ_-^g *matches with the distributional derivative* $\Delta_h(e^{-z\gamma\ell_g}\chi I^g f) \in \mathcal{D}'(\partial_-\mathcal{M})$.

The proof of this claim is postponed until below. Then $\Delta_h(e^{-z\gamma\ell_g}\chi I^g f) \in L^2(\partial_-\mathcal{M})$, and on the line $\{\Re(z) = 1-\varepsilon\}$ we have

$$\|u(z)\|_{L^2} \le \|(1+\Delta_h)^{i\Im(z)}\|_{L^2\to L^2} \|(1+\Delta_h)(e^{-z\gamma\ell_g}\chi I^g f)\|_{L^2} \le C(1+|z|^2)\|f\|_{C^2}. \tag{2-22}$$

We can then use the Hadamard three-line interpolation theorem applied to the holomorphic function

$$\{-\varepsilon \le \Re(z) \le 1-\varepsilon\} \ni z \mapsto v(z) := \int_{\partial_-\mathcal{M}} (1+z)^{-2} u(z)\psi \, dv_h \in \mathbb{C},$$

where $\psi \in C_c^\infty(\partial_-\mathcal{M})$ is arbitrary. Note that this is well defined and holomorphic in the strip $\Re(z) \in [-\varepsilon, 1-\varepsilon]$ since we have the bound

$$|v(z)| \le \frac{1}{(1-\varepsilon)^2} \|\psi\|_{H^{2(\Re(z)+\varepsilon)}} \|e^{\varepsilon\gamma\ell_g} \chi I_g f\|_{L^2} \le C\|\psi\|_{H^2} \|f\|_{C^2}.$$

From (2-21) and (2-22), we deduce the existence of a constant $C > 0$, independent of ψ, such that, for all z with $\Re(z) \in [-\varepsilon, 1-\varepsilon]$, one has

$$|v(z)| \le C\|\psi\|_{L^2}.$$

This shows that $u(z) \in L^2(\partial_-\mathcal{M})$ for all such z with the bound $|u(z)| \le C$. In particular, taking $z = 0$, we obtain that $(1+\Delta_h)^\varepsilon(\chi I^g f) \in L^2$, thus showing the claimed result.

It thus remains to prove Claim 2.6 above. Denote by F the continuous extension of $\Delta_h(e^{-z\gamma\ell_g}\chi I^g(f))$ by 0 on Γ_-^g. We need to show that, for each $\psi \in C_c^\infty(\partial_-\mathcal{M})$,

$$\int_{\partial_-\mathcal{M}} \chi e^{-z\gamma\ell_g} I^g(f)\Delta_h\psi \, dv_h = \int_{\partial_-\mathcal{M}} F\psi \, dv_h. \tag{2-23}$$

Take $\theta \in C_c^\infty([0,2))$ equal to 1 in $[0,1]$. We write the left-hand side as

$$\begin{aligned}\lim_{T\to\infty} \int_{\partial_-\mathcal{M}} \theta(\ell_g/T)\chi e^{-z\gamma\ell_g} I^g(f)\Delta_h\psi \, dv_h &= \lim_{T\to\infty} \int_{\partial_-\mathcal{M}} \Delta_h(\theta(\ell_g/T)\chi e^{-z\gamma\ell_g} I^g(f))\psi \, dv_h \\ &= \lim_{T\to\infty} A_1(T) + A_2(T),\end{aligned}$$

where

$$\begin{aligned}A_1(T) &:= \int_{\partial_-\mathcal{M}} \Delta_h(\theta(\ell_g/T))\chi e^{-z\gamma\ell_g} I^g(f)\psi \, dv_h + 2\int_{\partial_-\mathcal{M}} \nabla(\theta(\ell_g/T)) \cdot \nabla(\chi e^{-z\gamma\ell_g} I^g(f))\psi \, dv_h, \\ A_2(T) &:= \int_{\partial_-\mathcal{M}} \theta(\ell_g/T)\Delta_h(\chi e^{-z\gamma\ell_g} I^g(f))\psi \, dv_h = \int_{\partial_-\mathcal{M}} \theta(\ell_g/T)F\psi \, dv_h.\end{aligned}$$

In order to show (2-23), it thus suffices to show that $A_1(T) \to 0$ as $T \to \infty$. The derivatives $d_y^j(\theta(\ell_g/T))$ of order $j = 1, 2$ are supported in $\{\ell_g \in [T, 2T]\}$, where we can use the pointwise bound of Lemma 2.3:

$$|d_y^j(\theta(\ell_g(y)/T))| \le Ce^{C_0\ell_g(y)} \le Ce^{2C_0T}$$

for some uniform $C, C_0 > 0$. Since all terms in the integrand of A_1 are multiplied by the weight $|e^{-\gamma z\ell_g(y)}| \le e^{-\gamma(1-\varepsilon)T}$, we easily see, using Lemma 2.3 once again, that

$$A_1(T) = \mathcal{O}((1+|z|)e^{(3C_0-\gamma(1-\varepsilon))T}).$$

Taking $\gamma > 6C_0$ at the beginning and $\varepsilon < \frac{1}{2}$, one obtains that $A_1(T) \to 0$, and this proves our claim. ☐

Note that, as a corollary of Lemma 2.5, we obtain that there is $s > 0$ such that

$$\ell_g = I^g(\mathbf{1}_{\mathcal{M}}) \in H^s(\partial_-\mathcal{M}). \tag{2-24}$$

2E. *Scattering operator.* Working with the scattering operator $\mathcal{S}_g$ has several advantages over working directly with S_g. The main reason is that its Schwartz kernel can be expressed in terms of restriction of the Schwartz kernel of the resolvent R_g of the geodesic vector field X_g. This is the content of Lemma 2.7 below. This will be important so that we can work in a good functional setting in order to apply the Taylor expansion of the lens data with respect to g. We denote by R_{g_e} the resolvent on $\mathcal{M}_e$ for the extension g_e (for the definition of g_e recall Section 2A2), which has all the properties of R_g.

Lemma 2.7. *Let (M, g) be a compact Riemannian manifold with strictly convex boundary and hyperbolic trapped set. Let $\iota_{\partial_\pm} : \partial_\pm\mathcal{M} \to \mathcal{M}$ be the inclusion map. The restriction $(\iota_{\partial_-} \times \iota_{\partial_+})^* R_{g_e}$ of the Schwartz kernel of the resolvent on $\partial_-\mathcal{M} \times \partial_+\mathcal{M}$ makes sense as a distribution, and the Schwartz kernel of $\mathcal{S}_g$ is given by*

$$\mathcal{S}_g(y, y') = -(\iota_{\partial_-} \times \iota_{\partial_+})^* R_{g_e}(y, y'), \quad (y, y') \in \partial_-\mathcal{M} \times \partial_+\mathcal{M}.$$

Proof. First, we define the operator $\mathcal{E}_g : C^\infty_{\mathrm{c}}(\partial_+\mathcal{M}) \to H^s(\mathcal{M})$ for $s > 0$ as follows: for $\delta > 0$ small, let $\Omega = \{(x, v) \in \partial\mathcal{M} \mid |g_x(\nu, v)| \le \delta\}$; define $\Omega_e = \mathcal{M}_e \cap \bigcup_{t\in\mathbb{R}} \varphi_t^{g_e}(\Omega)$ to be the flowout of Ω by $\varphi_t^{g_e}$; and let $\psi \in C^\infty(\mathcal{M}_e, \mathbb{R}_+)$ such that $\psi|_{\Omega_e\cup\partial_-\mathcal{M}} = 0$, ψ is supported in a small neighborhood of $\partial_+\mathcal{M} \setminus \Omega$ and $X_{g_e}\psi = 0$ in $\mathcal{M}_e \setminus \mathcal{M}$ and near $\partial_+\mathcal{M}$. Then set, for $\omega \in C^\infty_{\mathrm{c}}(\partial_+\mathcal{M})$,

$$\mathcal{E}_g\omega := \tilde{\omega}\psi - R_{g_e}X_{g_e}(\tilde{\omega}\psi) \in H^s(\mathcal{M}_e) \cap L^p(\mathcal{M}_e) \cap C^\infty(\mathcal{M}_e \setminus (\Gamma_- \cup \Gamma_+))$$

for some $s > 0$ and all $p < \infty$ using (2-15) and (2-16), where $\tilde{\omega}$ is defined on $\mathrm{supp}(\psi)$ by extending ω from $\partial_+\mathcal{M}$ to be constant on the flow lines of X_{g_e}. This can be done by using the diffeomorphism

$$\Psi_+ : \left\{(t, y) \in \left(-\tfrac{1}{2}\delta, \infty\right) \times (\partial_+\mathcal{M} \setminus \Omega) \;\middle|\; t \le \tau_{g_e}(y)\right\} \ni (t, y) \mapsto \varphi_t^{g_e}(y) \in \mathcal{M}_e$$

and using that the flow $\varphi_t^{g_e}$ is the translation in t in these coordinates. One clearly has that $\mathcal{E}_g\omega$ is smooth near $\partial_+\mathcal{M}$ and

$$X_{g_e}\mathcal{E}_g\omega = 0, \quad (\mathcal{E}_g\omega)|_{\partial_+\mathcal{M}} = \psi|_{\partial_+\mathcal{M}}\omega.$$

In particular, we see that, outside Γ_-, we have

$$(\mathcal{E}_g\omega)|_{\partial_-\mathcal{M}\setminus\Gamma_-} = (\mathcal{S}_g(\omega\psi|_{\partial_+\mathcal{M}}))|_{\partial_-\mathcal{M}\setminus\Gamma_-}. \tag{2-25}$$

On the other hand, using the diffeomorphism

$$\Psi_- : \left\{(t, y) \in \left(-\infty, \tfrac{1}{2}\delta\right) \times (\partial_-\mathcal{M} \setminus \Omega) \;\middle|\; t \ge -\tau_{g_e}(-y)\right\} \ni (t, y) \mapsto \varphi_t^{g_e}(y) \in \mathcal{M}_e$$

mapping to a neighborhood of $\partial_-\mathcal{M} \setminus \Omega$, we see that $\Psi_-^*\mathcal{E}_g\omega$ is independent of t and can be viewed as a function in $H^s(\partial_-\mathcal{M}) \cap L^p(\partial_-\mathcal{M})$, i.e., the restriction $(\mathcal{E}_g\omega)|_{\partial_-\mathcal{M}}$ makes sense as an $H^s(\partial_-\mathcal{M}) \cap L^p(\partial_-\mathcal{M})$ function. (This fact can also be proved using the Hörmander pull-back theorem for distributions using wave-front analysis with the fact that X is transverse to $\partial_-\mathcal{M}$.) Since $\mu_\partial(\Gamma_-^g \cap \partial_-\mathcal{M}) = 0$, this implies with (2-25) that $(\mathcal{E}_g\omega)|_{\partial_-\mathcal{M}} = \mathcal{S}_g(\omega\psi|_{\partial_+\mathcal{M}})$. But this is also given by $(\mathcal{E}_g\omega)|_{\partial_-\mathcal{M}} = -(R_{g_e}X_{g_e}(\tilde{\omega}\psi))|_{\partial_-\mathcal{M}}$.

Since $X_{g_e}R_{g_e} = R_{g_e}X_{g_e} = \mathrm{Id}$ in $C_c^\infty(\mathcal{M}_e^\circ)$ (this follows for instance by analytic extension of the identity $R_{g_e}(z)(X_{g_e}-z) = (X_{g_e}-z)R_{g_e}(z) = \mathrm{Id}$ on $C_c^\infty(M_e^\circ)$ for $\Re(z) \gg 1$), one has $(X_{g_e}R_{g_e})(y, y') = 0$ and $(X'_{g_e}R_{g_e})(y, y')$ in the distribution sense for y close to $\partial_-\mathcal{M} \setminus \Omega$ and y' close to $\partial_+\mathcal{M} \setminus \Omega$, where X_{g_e} and X'_{g_e} denotes the action of X_{g_e} on the left and right variable of $\mathcal{M}_e \times \mathcal{M}_e$, respectively. This implies as above that the restriction $(\iota_{\partial_-} \times \iota_{\partial_+})^* R_{g_e}$ makes sense and we can apply Green's formula in the right variable: if $\omega' \in C_c^\infty(\partial_-\mathcal{M})$,

$$
\begin{aligned}
-\langle \iota_{\partial_-}^*(R_{g_e}X_{g_e}(\tilde{\omega}\psi)), \omega'\rangle &= -\int_{\partial_-\mathcal{M}}\int_{\mathcal{M}} R_{g_e}(y, y')X_{g_e}(\tilde{\omega}\psi)(y')\omega'(y)\,\mathrm{d}\mu(y')\,\mathrm{d}\mu_\partial(y) \\
&= -\int_{\partial_-\mathcal{M}}\int_{\partial_+\mathcal{M}} R_{g_e}(y, y')(\psi\omega)(y')\omega'(y) i_{X_{g_e}}\,\mathrm{d}\mu(y')\,\mathrm{d}\mu_\partial(y),
\end{aligned}
$$

where we used $X_{g_e}(\tilde{\omega}\psi) = 0$ on $\mathcal{M}_e \setminus \mathcal{M}$ and that $(X_{g_e}R_{g_e})(y, y') = 0$ for the interior term from Green's formula. This means, using $i_{X_{g_e}} d\mu = d\mu_\partial$ at $\partial_+\mathcal{M}$, that

$$
-\langle \iota_{\partial_-}^*(R_{g_e}X_{g_e}(\tilde{\omega}\psi)), \omega'\rangle = -\langle (\iota_{\partial_-} \times \iota_{\partial_+})^* R_{g_e}, \omega' \otimes \psi|_{\partial_+\mathcal{M}}\omega\rangle.
$$

This shows that $\mathcal{S}_g(y, y')\psi(y') = -(\iota_{\partial_-} \times \iota_{\partial_+})^* R_g(y, y')\psi(y')$ as a distribution of $(y, y') \in \partial_-\mathcal{M} \times \partial_+\mathcal{M}$. Since Ω can be chosen with $\delta > 0$ arbitrarily small, we obtain the result by choosing $\psi = 1$ outside a $\frac{1}{4}\delta$ neighborhood of $\Omega \cap \partial_+\mathcal{M}$ in $\partial_+\mathcal{M}$. ☐

We will also need the following regularity bound.

Lemma 2.8. *Let $g \in C^\infty(M, \otimes_S^2 T^*M_+)$ be a metric with strictly convex boundary and hyperbolic trapped set, $\chi \in C_c^\infty(\partial_-\mathcal{M})$, $f \in C^\infty(\partial_+\mathcal{M})$ and $p \in \mathbb{N}$. Then*:

1. *There exists $\beta \gg 0$ large enough that, for all $z \in i\mathbb{R} + \beta$, we have that $\chi e^{-z\ell_g}\mathcal{S}_g f$ extends by 0 on Γ_-^g with an extension belonging to $W^{p+1,\infty}(\partial_-\mathcal{M})$, and also that the weak distributional derivative $(1+\Delta_h)^{(p+1)/2}(\chi e^{-z\ell_g}\mathcal{S}_g f) \in \mathcal{D}'(\partial_-\mathcal{M})$ coincides with the derivative of the $W^{p+1,\infty}(\partial_-\mathcal{M})$-extension.*
2. *The map*
$$
C^{p+1}(\partial_+\mathcal{M}) \ni f \mapsto e^{-z\ell_g}\mathcal{S}_g f \in W^{p+1,\infty}(\partial_-\mathcal{M})
$$
is bounded, and there exists a uniform constant $C > 0$ (independent of z) such that
$$
\|(1+z)^{-(p+1)}\chi e^{-z\ell_g}\mathcal{S}_g f\|_{W^{p+1,\infty}(\partial_-\mathcal{M})} \le C\|f\|_{C^{p+1}(\partial_+\mathcal{M})}. \tag{2-26}
$$
3. *In particular, by the Sobolev embedding $W^{p+1,\infty}(\partial_-\mathcal{M}) \hookrightarrow C^p(\partial_-\mathcal{M})$, the function $\chi e^{-z\ell_g}\mathcal{S}_g f$ extends to a C^p-function with C^p-norm bounded by (2-26).*

Proof. The proof is rather similar to that of Lemma 2.5 so we will be more succinct. First, if $\Re(z) > 0$ and $f \in C^{p+1}(\partial_+\mathcal{M})$, the function $F_z(y) := e^{-z\ell_g(y)}(\mathcal{S}_g f)(y)$ is C^{p+1} outside Γ_-^g and can be extended by continuity by 0 on Γ_-^g. We compute its derivative on $\partial_-\mathcal{M} \setminus \Gamma_-^g$: if Y is a smooth vector field on $\partial_-\mathcal{M}$, then

$$
YF_z(y) = F_z(y)(-z d_y\ell_g(y)Y + df_{S_g(y)}(d\varphi^g_{\ell_g(y)}(y)Y + d_y\ell_g(y)(Y)X_g(S_g(y)))).
$$

We can use Lemma 2.3 and the fact that $\|d_y\varphi_t^g\| \le Ce^{C_0|t|}$ for some uniform $C, C_0 > 0$ with respect to t: this gives on $\mathrm{supp}(\chi)$ that

$$|YF_z(y)| \le C(1+|z|)\|Y\|_{C^0}\|f\|_{C^1}e^{(C_0-\beta)\ell_g(y)}$$

for some $C, C_0 > 0$ uniform in y. In particular, if $\beta > C_0$ we obtain that $|Y(\chi F_z)(y)| \le C(1+|z|)\|Y\|_{C^0}$ almost everywhere. Now, we claim that this function is also equal to the weak distributional derivative $Y(\chi F_z) \in H^{-1}(\partial_- SM)$. As in the proof of Lemma 2.5, we need to show that, for each $\psi \in C_c^\infty(\partial_-\mathcal{M})$,

$$\int_{\partial_-\mathcal{M}} \chi e^{-z\gamma\ell_g}\mathcal{S}_g(f)Y(\psi)\,\mathrm{d}v_h = \lim_{T\to\infty}\int_{\partial_-\mathcal{M}} \theta(\ell_g/T)Y(e^{-z\gamma\ell_g}\chi\mathcal{S}_g(f))\psi\,\mathrm{d}v_h,$$

where $\theta \in C_c^\infty([0,2))$ is equal to 1 in $[0,1]$ and h is a smooth metric on $\partial_-\mathcal{M}$ as in the proof of Lemma 2.5. Since the proof of the equality is exactly the same as in the proof of Lemma 2.5, we do not repeat the argument. This shows that $\chi F_z \in W^{1,\infty}(\partial_-\mathcal{M})$ with bound

$$\|\chi F_z\|_{W^{1,\infty}(\partial_-\mathcal{M})} \le C(1+|z|)\|f\|_{C^1}$$

for some C uniform with respect to z. The bound $\|\chi F_z\|_{C^0(\partial_-\mathcal{M})} \le C(1+|z|)\|f\|_{C^1}$ also follows immediately by Sobolev embedding.

For higher-order derivatives, it suffices to repeat this argument, noting by Lemma 2.3 that there are $C > 0$ and $C_0 > 0$ such that, for $j \le p+1$, we have

$$\|d_y^j\ell_g(y)\| \le Ce^{C_0|t|} \quad \text{and} \quad \|d_y^j\varphi_t^g\| \le Ce^{C_0 t}$$

on $(\partial_-\mathcal{M}\cap\mathrm{supp}(\chi))\setminus\Gamma_-^g$. This means that, taking $\beta > 0$ large enough depending on C_0, the argument explained above works the same way. This proves the claimed result. □

Given $\chi \in C_c^\infty(\partial_-\mathcal{M})$, define the following function on $\partial_-\mathcal{M}$:

$$\mathcal{L}_g(z) := \chi e^{-z\ell_g} = (z(R_g(z)\mathbf{1}_{\mathcal{M}})|_{\partial_-\mathcal{M}} + 1)\chi.$$

We will need the following regularity property.

Lemma 2.9. *Let (M, g_0) be a smooth compact Riemannian manifold with hyperbolic trapped set, and let $p \in 2\mathbb{N}$. There exists $\varepsilon > 0$ small enough and $\beta \gg 0$ large enough that the following holds: setting*

$$U_{g_0} := \{g \in C^{p+2}(M, \otimes_S^2 T^*M) \mid \|g-g_0\|_{C^{p+2}} < \varepsilon,\ g|_{T\partial M} = g_0|_{T\partial M}\} \tag{2-27}$$

as in Lemma 2.3, we have that, for $\Re(z) = \beta$, the map

$$\mathcal{L} : U_{g_0} \times \{\Re(z)=\beta\} \ni (g,z) \mapsto \mathcal{L}_g(z) = e^{-z\ell_g}\chi \in L^\infty(\partial_-\mathcal{M}) \subset L^2(\partial_-\mathcal{M})$$

is C^{p-1}-regular. Moreover, there exists a uniform constant $C > 0$ such that, for all $j \le p-1$,

$$\text{for all } h \in C^{p+2}(M, \otimes_S^2 T^*M), \quad \|\partial_g^j\mathcal{L}_g(z)(\otimes^j h)\|_{L^2} \le C(1+|z|)^j\|h\|_{C^{p+2}}^j.$$

Proof. First of all, note by [Guillarmou and Mazzucchelli 2018, Proposition 2.1] that all metrics in a C^2-neighborhood of g_0 have hyperbolic trapped set and strictly convex boundary. Hence $\varepsilon > 0$ is chosen so that this holds. Pick an arbitrary $g_0' \in U_{g_0}$, and let $h \in C^{p+2}(M, \otimes_S^2 T^*M)$ such that $g_t := g_0' + th \in U_{g_0}$ for $t \in (-\delta, 1+\delta)$ for some $\delta > 0$ small. Consider the map

$$F : (-\delta, 1+\delta) \times \partial_-\mathcal{M} \times \{\Re(z) = \beta\} \ni (t, y, z) \mapsto \mathcal{L}_{g_t}(z)(y) = e^{-z\ell_{g_t}(y)}\chi(y),$$

where by convention $e^{-z\ell_{g_t}(y)} := 0$ when $\ell_{g_t}(y) = \infty$. Lemma 2.3 implies that F is C^p in the open set

$$\mathcal{O} := \{(t, y, z) \in (-\delta, 1+\delta) \times \partial_-\mathcal{M} \times \{\Re(z) = \beta\} \mid y \notin \Gamma_-^{g_t}\},$$

and one can write $\partial_t^{j_1}\partial_y^{j_2}\partial_z^{j_3}\mathcal{L}_{g_t}(z)(y) = H(t, y, z, h)(\otimes^{j_1} h)$, where $H(t, y, z, h)$ is a continuous function on $(-\delta, 1+\delta) \times \partial_-\mathcal{M} \times C^{p+2}(M, \otimes_S^2 T^*M)$ with values in j_1-multilinear functions on $C^{p+2}(M, \otimes_S^2 T^*M)$ and satisfying the following: there is $C > 0$ such that, for all $j_1 + j_2 + j_3 \le p$ and all $(t, y, z) \in \mathcal{O}$,

$$|\partial_t^{j_1}\partial_y^{j_2}\partial_z^{j_3}\mathcal{L}_{g_t}(z)(y)| \le C(1+|z|)^{j_1+j_2} e^{(C-\beta)\ell_{g_t}(y)} \|h\|_{C^{p+2}}^{j_1}. \tag{2-28}$$

First, we observe that F is continuous on $(-\delta, 1+\delta) \times \partial_-\mathcal{M} \times \{\Re(z) = \beta\}$. Indeed, if $(t_n, y_n) \to (t, y)$ is a sequence such that $\ell_{g_{t_n}}(y_n) \le T$ for some $T < \infty$, by Proposition 2.4 we deduce that the trajectories $\mathcal{M} \cap \bigcup_{s \ge 0} \varphi_s^{g_{t_n}}(y_n)$ converge to the trajectory $\mathcal{M} \cap \bigcup_{s\ge 0} \varphi_s^{g_t}(y)$ as $n \to \infty$, and therefore $\ell_{g_t}(y) < \infty$, and so the limit point belongs to $\mathcal{O}$. On the other hand, if there is no such T, this also implies that $\ell_{g_{t_n}}(y_n) \to \infty$, and in turn $F(t_n, y_n, z) \to 0$ as $n \to \infty$, and (t, y, z) belongs to the set

$$S := \bigcup_{t \in (-\delta, 1+\delta)} (\{t\} \times \Gamma_-^{g_t} \times \{\Re(z) = \beta\}).$$

Since $\ell_{g_{t_n}}(y_n) \to \infty$ if (t_n, y_n) converge to a point in S as $n \to \infty$, we see from (2-28) that if $\beta \gg 1$ is large enough, the derivative $H(t, y, z, h)$ of F on $\mathcal{O}$ converges to 0 when approaching S, and can thus be extended from $\mathcal{O}$ by 0 as a continuous function on $(-\delta, 1+\delta) \times \partial_-\mathcal{M} \times \{\Re(z) = \beta\} \times C^{p+2}(M, \otimes_S^2 T^*M)$. Next, we are going to show that F is a C^{p-1} map, with $\partial_t^{j_1}\partial_y^{j_2}\partial_z^{j_3} F(t, y, z) = H(t, y, z, h)(\otimes^{j_1} h)$ and with H the continuous extension by 0 on S just discussed, and that there exists $C > 0$ independent of h, t, y, z such that, for all $(t, y, z) \in (-\delta, 1+\delta) \times \partial_-\mathcal{M} \times \{\Re(z) = \beta\}$ and all $j_1 + j_2 + j_3 \le p - 1$,

$$|\partial_t^{j_1}\partial_y^{j_2}\partial_z^{j_3} F(t, y, z)| \le C(1+|z|)^{j_1+j_2} \|h\|_{C^{j_1+1}}^{j_1}. \tag{2-29}$$

This would prove that the Gateaux derivatives of order $p-1$ are continuous and thus the function $\mathcal{L}$ is C^{p-1} and with the desired bounds on the derivatives.

We proceed in a way similar to the proof of Claim 2.6. We will show that, for each fixed h, the distributional derivatives of F of order $j \le p$ are bounded and coincide with the continuous extension of $H(t, y, z, h)(\otimes^{j_1} h)$ from $\mathcal{O}$ to $W := (-\delta, 1+\delta) \times \partial_-\mathcal{M} \times \{\Re(z) = \beta\}$. First we let Δ be a Laplacian associated to a fixed smooth product metric $\hat{g} := dt^2 + g_- + ds^2$ on $(-\delta, 1+\delta) \times \partial_-\mathcal{M} \times \{\beta + is \mid s \in \mathbb{R}\}$. Let $\psi \in C_c^\infty((-\delta, 1+\delta) \times \partial_-\mathcal{M} \times (\beta + i\mathbb{R}))$. We want to show that, for $2j \le p$,

$$\int_W \chi e^{-z\ell_{g_t}} \Delta^j \psi \, dv_{\hat{g}} = \int_{\mathcal{O}} (\Delta^j F)\psi \, dv_{\hat{g}}.$$

Take $\theta \in C_c^\infty([0,2);[0,1])$ equal to 1 in $[0,1]$, and write the left-hand side as

$$\lim_{T\to\infty}\int_W \theta\left(\frac{\ell_{g_t}}{T}\right)\chi e^{-z\ell_g}\Delta^j\psi\, dv_{\hat g} = \lim_{T\to\infty} A_1(T)+A_2(T), \tag{2-30}$$

where

$$A_1(T) := \sum_{k=1}^{2j}\int_W P_k\left(\theta\left(\frac{\ell_{g_t}}{T}\right)\right)Q_{2j-k}(\chi e^{-z\ell_{g_t}})\psi\, dv_{\hat g},$$

with P_k and Q_k some differential operators of order $k \geq 1$ in the variable (t,y,z) and such that $P_k(1) = Q_k(1) = 0$ and

$$A_2(T) := \int_W \theta\left(\frac{\ell_{g_t}}{T}\right)(\Delta^j F)\psi\, dv_{\hat g}.$$

In order to show (2-30), it suffices to show that $A_1(T)\to 0$ as $T\to\infty$. The derivatives $D^k_{t,y,z}(\theta(\ell_{g_t}/T))$ of order $k\in[1,2j]$ are supported in $\{\ell_{g_t}\in[T,2T]\}$, where we can use the pointwise bound of Lemma 2.3: there exists $C>0$ such that, for all (t,y,z) with $\ell_{g_t}(y)\in[T,2T]$,

$$|D^k_{t,y,z}(\theta(\ell_{g_t}(y)/T))| \leq Ce^{C\ell_{g_t}(y)} \leq Ce^{2CT}.$$

Since all terms in the integrand of A_1 are multiplied by the weight $|e^{-\beta\ell_{g_t}(y)}| \leq e^{-\beta T}$, we see using Lemma 2.3 that

$$A_1(T) = \mathcal{O}(e^{(4C-\beta)T}).$$

Thus if β is chosen large enough we obtain that $A_1(T)\to 0$ as $T\to\infty$. We thus deduce that $F\in W^{p,\infty}_{\mathrm{loc}}(W)$ and by Sobolev embedding that $F\in C^{p-1,\alpha}(W)$ for all $\alpha<1$. Finally, the bound (2-29) follows from (2-28) by continuity. □

3. Symmetric tensors and the normal operator

3A. *Symmetric tensors.* In this paragraph, we recall standard facts on symmetric tensors on Riemannian manifolds. We refer to [Gouëzel and Lefeuvre 2021; Guillarmou 2017a; Heil et al. 2016] for further details.

3A1. *Definitions.* Let (M,g) be a smooth connected Riemannian manifold with boundary. Let $m\in\mathbb{Z}_{\geq 0}$. Let $\otimes^m_S T^*M \to M$ be the vector bundle of symmetric tensors over M (for $m=0$ we just take the trivial line bundle $\mathbb{R}\times M\to M$). We will also write $\otimes^2_S T^*M_+ \subset \otimes^2_S T^*M$ for the open convex subset consisting of positive definite tensors (Riemannian metrics). Since $\otimes^m_S T^*M$ is a subbundle of the vector bundle $\otimes^m T^*M\to M$ of m-tensors over M, it inherits the natural metric $g^{\otimes m}$. Define the pullback operator

$$\pi_m^*: L^2(M,\otimes^m_S T^*M)\to L^2(\mathcal{M}),\qquad \pi_m^* f(x,v) := f_x(v^{\otimes m}),$$

where M is equipped with the Riemannian volume, $\otimes^m_S T^*M$ with the metric $g^{\otimes m}$ and $\mathcal{M}$ with the Liouville measure μ. We denote by π_{m*} the adjoint of π_m^* with respect to these scalar products and volume forms.

The symmetric covariant derivative

$$D_g : C^\infty(M, \otimes_S^m T^*M) \to C^\infty(M, \otimes_S^{m+1} T^*M)$$

is defined as $D_g := \sigma \circ \nabla^g$, where ∇^g is the Levi-Civita connection induced by g and $\sigma : \otimes^m T^*M \to \otimes_S^m T^*M$ is the symmetrization operator defined as:

$$\sigma(\eta_1 \otimes \cdots \otimes \eta_m) := \frac{1}{m!} \sum_{\pi \in \mathfrak{S}_m} \eta_{\pi(1)} \otimes \cdots \otimes \eta_{\pi(m)},$$

where $\eta_1, \ldots, \eta_m \in T^*M$. The operator D_g is of *gradient type*, namely it has injective principal symbol. Moreover, it is injective when m is odd and has kernel given by $\mathbb{R} g^{\otimes m/2}$ for even m. It satisfies the relation

$$X_g \pi_m^* = \pi_{m+1}^* D_g, \tag{3-1}$$

where we recall that X_g is the geodesic vector field of g. We let

$$D_g^* : C^\infty(M, \otimes_S^{m+1} T^*M) \to C^\infty(M, \otimes_S^m T^*M)$$

be the formal adjoint of D_g, which is nothing more than the divergence $D_g^* u = -\mathrm{Tr}(\nabla^g u)$, where $\mathrm{Tr}(\,\cdot\,)$ is the trace operator.

For $m \geq 1$, $k \geq 0$ and $\alpha \in (0,1)$, there exists a unique decomposition

$$C^{k,\alpha}(M, \otimes_S^m T^*M) = D_g(C_0^{k+1,\alpha}(M, \otimes_S^{m-1} T^*M)) \oplus^\perp \ker D_g^*|_{C^{k,\alpha}(M, \otimes_S^m T^*M)}, \tag{3-2}$$

where $C_0^{k+1,\alpha}(M, \otimes_S^{m-1} T^*M)$ denotes the space of tensors of Hölder–Zygmund regularity $k+1+\alpha$, vanishing on the boundary, and the sum is orthogonal with respect to the L^2-scalar product. The decomposition (3-2) also holds in the scale of Sobolev spaces $H^s(M, \otimes_S^m T^*M)$ for $s \geq 0$. We call *potential tensors* the tensors in $\mathrm{ran}\, D_g$ and *solenoidal tensors* (or divergence free tensors) those in $\ker D_g^*$.

Lemma 3.1. *For $m \geq 1$, there exist bounded projections*

$$\pi_{\ker D_g^*} : L^2(M, \otimes_S^m T^*M) \to L^2(M, \otimes_S^m T^*M) \cap \ker D_g^*,$$
$$\pi_{\mathrm{ran}\, D_g} : L^2(M, \otimes_S^m T^*M) \to L^2(M, \otimes_S^m T^*M) \cap \mathrm{ran}\, D_g|_{H_0^1},$$

which are pseudodifferential operators of order 0 *on M°. Moreover, for all $f \in L^2(M, \otimes_S^m T^*M)$, there is a unique $h \in H_0^1(M, \otimes_S^{m-1} T^*M)$ and $f_s \in \ker D_g^* \cap L^2$ such that $f = D_g h + f_s$, and it is given by $\pi_{\ker D_g^*} f = f_s$ and $\pi_{\mathrm{ran}\, D_g} f = D_g h$.*

Proof. The Dirichlet Laplacian $D_g^* D_g : H^2(M, \otimes_S^m T^*M) \cap H_0^1(M, \otimes_S^m T^*M) \to L^2(M)$ is an elliptic self-adjoint operator which is invertible since there are no symmetric Killing tensors vanishing at ∂M by [Dairbekov and Sharafutdinov 2010]. Its inverse $(D_g^* D_g)^{-1} : H^{-1}(M, \otimes_S^m T^*M) \to H_0^1(M, \otimes_S^m T^*M)$, when restricted to $C_c^\infty(M^\circ)$, is a pseudodifferential operator of order -2 on M° by standard elliptic microlocal analysis. We then set

$$\pi_{\mathrm{ran}\, D_g} := D_g (D_g^* D_g)^{-1} D_g^*, \quad \pi_{\ker D_g^*} =: \mathrm{Id} - \pi_{\mathrm{ran}\, D_g}.$$

By construction, they satisfy the desired properties. □

3A2. *X-ray transform of tensors.* We now further assume that the metric g is of Anosov type in the sense of Definition 1.3. We introduce the X-ray transform of symmetric m-tensors.

Definition 3.2. The X-ray transform on the space of symmetric m-tensors is defined by $I^g_m := I^g \circ \pi^*_m$, where I^g_m is a map from $C^\infty(M, \otimes^m_S T^*M)$ to $L^2(\partial_-\mathcal{M})$.

It is clear from (3-1) that the following inclusion holds:

$$D_g(C^{k+1,\alpha}_0(M, \otimes^{m-1}_S T^*M)) \subset \ker I^g_m. \tag{3-3}$$

Definition 3.3. The X-ray transform I^g_m is said to be *solenoidal injective* on $C^{k,\alpha}(M, \otimes^m_S T^*M)$ if (3-3) is an equality.

In other words, I^g_m is solenoidal injective if it is injective in restriction to solenoidal tensors, i.e., on the second factor of the decomposition (3-2). When (M, g) is of Anosov type, solenoidal injectivity of the X-ray transform has been proved so far in the following cases:

(1) In dimensions $n \geq 2$, when g is of Anosov type with nonpositive sectional curvature, see [Guillarmou 2017b].

(2) On all surfaces of Anosov type; see [Lefeuvre 2019a].

(3) In dimensions $n \geq 2$, on all real analytic manifolds of Anosov type, injectivity of I^g_2 is proved in [Guedes-Bonthonneau et al. 2024].

We conjecture that the following holds.

Conjecture 3.4 (solenoidal injectivity of the X-ray transform on manifolds of Anosov type). *Let (M, g) be a smooth Riemannian manifold of Anosov type in the sense of Definition 1.3. Then I^g_m is solenoidal injective.*

Eventually, we conclude this paragraph by the following variational formula which relates the length map and the X-ray transform on 2-tensors.

Lemma 3.5. *Let (M, g_0) be a compact Riemannian manifold with strictly convex boundary and hyperbolic trapped set. Let $(x, v) \in \partial_-\mathcal{M} \setminus \Gamma^{g_0}_-$. Let $(g_t)_{t\in(-1,1)}$ be a smooth family of metrics on M with $g_t|_{t=0} = g_0$, and write $h := \partial_t g_t|_{t=0}$. Then $t \mapsto \ell_{g_t}(x, v)$ is C^2-regular for small t, and*

$$\partial_t \ell_{g_t}(x, v)|_{t=0} = \tfrac{1}{2} I^{g_0}_2 h(x, v) + \alpha_{S_{g_0}(x,v)}(\partial_t S_{g_t}(x, v)|_{t=0}),$$

where we recall that α is the Liouville 1-form.

Proof. First, we use the fact that, for t small enough, g_t must have hyperbolic trapped set by [Guillarmou and Mazzucchelli 2018, Proposition 2.1]. Let $c_0(s)$ be a geodesic for g_0 parametrize by arc-length, and $s \mapsto c_t(s)$ for $t \in (-1, 1)$ be a C^1 family of curves for $s \in [0, \ell_{g_0}(c_0)]$. Let $Y(s) := \partial_t c_t(s)|_{t=0}$ be the vector field along $c_0(s)$ determined by the family $(c_t)_{t\in(-1,1)}$. Define $\dot{g} := \partial_t g_t|_{t=0}$, and denote by ∇ the Levi-Civita derivative defined by g_0.

By definition, $\ell_{g_t}(c_t) = \int_0^{\ell_{g_0}(c_0)} \sqrt{g_t(\partial_s c_t(s), \partial_s c_t(s))}\, \mathrm{d}s$, so differentiating we obtain

$$
\begin{aligned}
\partial_t(\ell_{g_t}(c_t))|_{t=0} &= \frac{1}{2}\int_0^{\ell_{g_0}(c_0)} \frac{2g_0(\nabla_{\partial_t}\partial_s c_t(s)|_{t=0}, \partial_s c_0(s)) + \dot{g}(\partial_s c_0(s), \partial_s c_0(s))}{|\partial_s c_0(s)|_{g_0}}\, \mathrm{d}s \\
&= \frac{1}{2}\int_0^{\ell_{g_0}(c_0)} \dot{g}(\partial_s c_0(s), \partial_s c_0(s))\, \mathrm{d}s \\
&\qquad + \int_0^{\ell_{g_0}(c_0)} \Big(\partial_s(g_0(\partial_t c_t(s), \partial_s c_t(s)))|_{t=0} - g_0(\partial_t c_t(s)|_{t=0}, \underbrace{\nabla_{\partial_s}\partial_s c_0(s)}_{=0})\Big)\, \mathrm{d}s \\
&= \frac{1}{2}\int_0^{\ell_{g_0}(c_0)} \dot{g}(\partial_s c(s), \partial_s c(s))\, \mathrm{d}s + g_0(Y(s), \partial_s c_0(s))\big|_0^{\ell_{g_0}(c_0)}.
\end{aligned} \tag{3-4}
$$

Here we used that $|\partial_s c_0(s)|_g = 1$ since the parametrization of c_0 is by arc-length, and that $\nabla_{\partial_t}\partial_s = \nabla_{\partial_s}\partial_t$ (this is seen on the pullback bundle c^*TM of the tangent bundle by the family c since the connection is torsion-free and $[\partial_t, \partial_s] = 0$). In the third line, we used the compatibility of g_0 with ∇, and the last term is zero since $\nabla_{\partial_s}\partial_s c_0(s) = 0$ is the geodesic equation.

If $(x, v) \in \partial_-\mathcal{M} \setminus \Gamma_-^{g_0}$, then, for t small enough, $(x, v) \notin \Gamma_-^{g_t}$ by Proposition 2.4 and $\ell_{g_t}(x, v)$ is C^2 near $t = 0$ by Lemma 2.3. Then we get, from (3-4),

$$
\begin{aligned}
\partial_t \ell_{g_t}(x, v)|_{t=0} &= \tfrac{1}{2} I_2^{g_0}(\dot{g})(x, v) + g_0\Bigg(\underbrace{\partial_t\bigg(\pi \circ S_{g_t}\Big(x, \frac{v}{|v|_{g_t}}\Big)\bigg)\bigg|_{t=0}}_{=d\pi\circ\partial_t S_{g_t}(x,v)|_{t=0}}, S_{g_0}(x, v)\Bigg) \\
&= \tfrac{1}{2} I_2^{g_0}(\dot{g})(x, v) + \alpha_{S_{g_0}(x,v)}(\partial_t S_{g_t}(x, v)|_{t=0}). \qquad \square
\end{aligned}
$$

3A3. *Solenoidal gauge.* The following lemma asserts that any metric in a neighborhood of a fixed metric g_0 can be put in a *solenoidal gauge*.

Lemma 3.6. *Let (M, g_0) be a smooth Riemannian manifold with metric of Anosov type, and let $k \geq 2$ and $\alpha \in (0, 1)$. There exists $C, \delta > 0$ such that the following holds: for all metrics g such that $\|g - g_0\|_{C^{k,\alpha}} < \delta$, there exists a $C^{k+1,\alpha}$-diffeomorphism ψ, with $\psi|_{\partial M} = \mathrm{Id}$, such that $\psi^* g$ is divergence-free with respect to g_0, namely $D_{g_0}^*(\psi^* g - g_0) = 0$, and $\|\psi^* g - g_0\|_{C^{k,\alpha}} \leq C\|g - g_0\|_{C^{k,\alpha}}$.*

Proof. The proof is contained in [Croke et al. 2000, Lemma 2.2]. □

3B. ***Normal operator.*** Let (M, g) be a smooth Riemannian manifold with metric g of Anosov type. The normal operator on m-symmetric tensors is defined by

$$\Pi_m^g := (I_m^g)^* I_m^g.$$

It enjoys strong analytic properties, as proved in [Guillarmou 2017b]:

Proposition 3.7. *The operator $\Pi_m^g \in \Psi^{-1}(M^\circ, \otimes_S^m T^*M^\circ)$ is a pseudodifferential operator of order -1 on M°. It is elliptic on solenoidal tensors, in the sense that there exists pseudodifferential operators Q, K_L, K_R on M° of orders $1, -\infty, -\infty$, respectively, such that*

$$Q\Pi_m^g = \pi_{\ker D_g^*} + K_L, \qquad \Pi_m^g Q = \pi_{\ker D_g^*} + K_R,$$

and the equalities hold when applied to all distributions $f \in \mathcal{E}'(M^\circ, \otimes_S^m T^*M^\circ)$ *with compact support in* M°. *The operator* Q *can be taken to be properly supported in* M°. *Moreover*, Π_m^g *is solenoidal injective*, *i.e.*, *injective in restriction to* $\ker D_g^*$, *if and only if the X-ray transform* I_m^g *is solenoidal injective.*

We now prove an elliptic estimate for the operator Π_m^g. Recall from Section 2C that $(M_e, g_e) \supset (M, g)$ is a Riemannian extension of the manifold (M, g), which is also of Anosov type in the sense of Definition 1.3. We will denote by

$$E_0 : L^2(M, \otimes_S^m T^*M) \to L^2(M_e, \otimes_S^m T^*M_e)$$

the operator of extension by 0.

Proposition 3.8. *Let* (M, g) *be a manifold of Anosov type, and further assume that* I_2^g *is solenoidal injective. Let* (M_e, g_e) *be an extension of Anosov type of* (M, g). *Then, there exists* $C > 0$ *such that, for all* $f \in L^2(M, \otimes_S^2 T^*M) \cap \ker D_g^*$,

$$\|f\|_{L^2(M)} \le C \|\Pi_2^{g_e} E_0 f\|_{H^1(M_e)}.$$

Proof. It will be convenient in the proof to consider a second extension of Anosov type $(M_{ee}, g_{ee}) \supset (M_e, g_e)$ and to work on it. The argument follows [Stefanov and Uhlmann 2004]. The operator $\Pi_2^{g_{ee}}$ is a (not properly supported) pseudodifferential operator of order -1 on ${M_{ee}}^\circ$ which is elliptic on solenoidal tensors. By Proposition 3.7, we can construct a properly supported pseudodifferential operator $Q \in \Psi^1(M_{ee}^\circ, \otimes_S^2 T^*M_{ee}^\circ)$ such that

$$Q\Pi_2^{g_{ee}} = \pi_{\ker D_{g_{ee}}^*} + K,$$

where $K \in \Psi^{-\infty}(M_{ee}^\circ)$ is smoothing. We let $\iota : M_e \hookrightarrow M_{ee}$ be the embedding. Observe that, taking a cutoff function $\chi \in C_c^\infty(M_e^\circ)$ with value 1 in an open neighborhood of M, we get

$$\begin{aligned}\iota^* Q\Pi_2^{g_{ee}} E_0 &= \iota^*\pi_{\ker D_{g_{ee}}^*} E_0 + \iota^* K E_0\\ &= \pi_{\ker D_{g_e}^*} E_0 + \chi(\iota^*\pi_{\ker D_{g_{ee}}^*} - \pi_{\ker D_{g_e}^*})\chi E_0 + \iota^* K E_0 + (1-\chi)(\iota^*\pi_{\ker D_{g_{ee}}^*} - \pi_{\ker D_{g_e}^*})E_0.\end{aligned}$$

By the pseudolocality of pseudodifferential operators (they preserve the singular support of distributions), the term $(\iota^*\pi_{\ker D_{g_{ee}}^*} - \pi_{\ker D_{g_e}^*})E_0$ maps continuously L^2 sections to sections that are smooth outside M, and thus

$$(1-\chi)(\iota^*\pi_{\ker D_{g_{ee}}^*} - \pi_{\ker D_{g_e}^*})E_0 : L^2(M, \otimes_S^2 T^*M) \to L^2(M_e, \otimes_S^2 T^*M_e)$$

is a compact operator. As for the term $\chi(\iota^*\pi_{\ker D_{g_{ee}}^*} - \pi_{\ker D_{g_e}^*})\chi$, we observe that it has Schwartz kernel supported in the interior of $M_{ee} \times M_{ee}$. It is a priori a pseudodifferential operator of order 0, but its principal symbol vanishes (see Lemma 3.1) and thus it is a pseudodifferential operator of order -1, i.e., it is compact as a map $L^2(M_e) \to L^2(M_e)$. (We now drop the notation of the vector bundle in the functional spaces in order to avoid repetition.) As a consequence, we see that, up to changing the compact remainder,

$$\iota^* Q\Pi_2^{g_{ee}} E_0 = \pi_{\ker D_{g_e}^*} E_0 + K, \tag{3-5}$$

where K is compact as a map $L^2(M) \to L^2(M_e)$.

Given $f \in L^2(M) \cap \ker D_g^*$, by Lemma 3.1 we may write $E_0 f = D_g p + h$, where $p \in H^1(M_e, T^*M_e)$ and $p|_{\partial M_e} = 0$, $h = \pi_{\ker D_{g_e}^*} E_0 f$. Now, using (3-5), there is $C > 0$ independent of f such that

$$\begin{aligned} \|f\|_{L^2(M)} = \|E_0 f\|_{L^2(M_e)} &\le \|\pi_{\ker D_{g_e}^*} E_0 f\|_{L^2(M_e)} + \|(\mathrm{Id} - \pi_{\ker D_{g_e}^*}) E_0 f\|_{L^2(M_e)} \\ &\le \|\iota^* Q \Pi_2^{g_{ee}} E_0 f\|_{L^2(M_e)} + \|K f\|_{L^2(M_e)} + \|D_{g_e} p\|_{L^2(M_e)} \\ &\le C(\|\Pi_2^{g_{ee}} E_0 f\|_{H^1(M_{ee})} + \|K f\|_{L^2(M_e)} + \|D_{g_e} p\|_{L^2(M_e)}). \end{aligned} \tag{3-6}$$

It remains now to bound the potential term $D_{g_e} p$. We have

$$\|D_{g_e} p\|_{L^2(M_e)} \le \|D_{g_e} p\|_{L^2(\Omega)} + \|D_g p\|_{L^2(M)}, \tag{3-7}$$

where $\Omega := M_e \setminus M^\circ$. We observe that, on Ω, $D_g p = -h = -\pi_{\ker D_{g_e}^*} E_0 f$. Hence, using (3-5), we get

$$\|D_{g_e} p\|_{L^2(\Omega)} \le \|\iota^* Q \Pi_2^{g_{ee}} E_0 f\|_{L^2(\Omega)} + \|K f\|_{L^2(\Omega)}. \tag{3-8}$$

The boundary $\partial\Omega = \partial M_e \sqcup \partial M$ splits into two components. We define ν to be the outward-pointing unit normal vector to $\partial\Omega$ and $j := p|_{\partial M}$. In M, we have $D_g^* f = 0 = D_g^* h + D_g^* D_g p = \Delta_D p$, where $\Delta_D := D_g^* D_g$ is the (symmetric) Laplacian on 1-forms. Hence, in M, p satisfies the elliptic system $\Delta_D p = 0$, $p|_{\partial M} = j \in H^{1/2}(\partial M, \otimes_S^2 T^*M)$ (by the trace theorem), so by standard elliptic estimates [Taylor 2011, Chapter 5, Proposition 1.7], we get $\|p\|_{H^1(M)} \lesssim \|j\|_{H^{1/2}(\partial M)}$. Observe that the H^1-norm in M can be defined by $\|p\|_{H^1(M)} := \|p\|_{L^2(M)} + \|D_g p\|_{L^2(M)}$. As a consequence, using the boundedness of the trace map $H^1(\Omega) \to H^{1/2}(\partial\Omega)$, we get (for some C uniform that can change from line to line)

$$\begin{aligned} \|D_g p\|_{L^2(M)} \le C\|p\|_{H^1(M)} \le C\|j\|_{H^{1/2}(\partial M)} \le C\|p\|_{H^1(\Omega)} &\le C(\|p\|_{L^2(\Omega)} + \|D_{g_e} p\|_{L^2(\Omega)}) \\ &\le C(\|p\|_{L^2(\Omega)} + \|\iota^* Q \Pi_2^{g_{ee}} E_0 f\|_{L^2(\Omega)} + \|K f\|_{L^2(\Omega)}) \end{aligned} \tag{3-9}$$

by (3-8). It remains to bound $\|p\|_{L^2(\Omega)}$. Recall that $D_g p = \pi_{\mathrm{ran}\, D_g} E_0 f$, and by pseudolocality of the pseudodifferential operator $\pi_{\mathrm{ran}\, D_g}$ (see Lemma 3.1) we get that $p|_\Omega$ belongs to $C^\infty(\Omega, T^*\Omega)$. For any point $(x, v) \in S\Omega$, there is a uniformly bounded time $\tau(x, v)$ (possibly negative) such that $\pi(\varphi_{\tau(x,v)}(x, v)) \in \partial M_e$, and using that p vanishes on ∂M_e, we can thus write, using (3-1),

$$|\pi_1^* p(x, v)| = \left| \int_0^{\tau(x,v)} (X_{g_e} \pi_1^* p)(\varphi_t^{g_e}(x, v))\, \mathrm{d}t \right| = \left| \int_0^{\tau(x,v)} (\pi_2^* D_{g_e} p)(\varphi_t^{g_e}(x, v))\, \mathrm{d}t \right|.$$

This equality implies that $\|p\|_{L^2(\Omega)} \le C\|D_{g_e} p\|_{L^2(\Omega)}$. Hence, combining (3-6) with (3-7)–(3-9), we get that, for all $f \in L^2(M, \otimes_S^2 T^*M) \cap \ker D_g^*$, the following inequality holds for some uniform $C > 0$:

$$\|f\|_{L^2(M)} \le C(\|\Pi_2^{g_{ee}} E_0 f\|_{H^1(M_{ee})} + \|K f\|_{L^2(M_e)}),$$

where $K : L^2(M, \otimes_S^2 T^*M) \to L^2(M_e, \otimes_S^2 T^*M_e)$ is compact. The solenoidal injectivity of Π_2^g on M implies that $\Pi_2^{g_{ee}} E_0$ is also solenoidal injective (see [Lefeuvre 2019a, Proof of Lemma 2.3] for instance) and thus by standard arguments, one can remove the compact remainder K from the previous inequality. Hence there is uniform $C > 0$ such that

$$\|f\|_{L^2(M)} \le C\|\Pi_2^{g_{ee}} E_0 f\|_{H^1(M_{ee})}.$$

The claimed estimate is proved by observing that in the above proof one can replace (M_{ee}, g_{ee}) by (M_e, g_e), and (M_e, g_e) by a slightly smaller manifold (M_e', g_e') of Anosov type containing (M, g). ☐

4. Local lens rigidity, proof of the main result

In this section, we prove the main Theorem 1.8.

4A. ***Key estimate.*** The goal of this paragraph is to show the following key estimate.

Proposition 4.1. *Let g_0 be of Anosov type. There exist $C, \varepsilon, \mu, N > 0$ such that, for all smooth metrics g such that $g|_{\partial M} = g_0|_{\partial M}$, $\|g - g_0\|_{C^N} < \varepsilon$, and $(\ell_g, S_g) = (\ell_{g_0}, S_{g_0})$, we have*

$$\|I_2^{g_0}(g - g_0)\|_{L^2} \leq C\|g - g_0\|_{C^N}^{1+\mu}.$$

In order to prove Proposition 4.1, we are still missing one ingredient, namely, the following C^2-regularity of the scattering operator.

Proposition 4.2. *Let (M, g_0) be a smooth compact Riemannian manifold with strictly convex boundary and hyperbolic trapped set. Let $\chi \in C_c^\infty(\partial_-\mathcal{M}, [0, 1])$ be a smooth cutoff function. Then, for each $\omega \in C^\infty(\partial_+\mathcal{M})$, the map*

$$C^\infty(M, \otimes_S^2 T^*M) \ni g \mapsto \chi \mathcal{S}_g(\omega) \in H^{-6}(\partial_-\mathcal{M})$$

is C^2-regular near g_0. As a consequence, there exists $C, N > 0$ large enough and $\delta > 0$ such that, for all $g \in C^\infty(M, \otimes_S^2 T^*M)$ with $\|g - g_0\|_{C^N} \leq \delta$, the following holds:

$$\|\chi \mathcal{S}_g(\omega) - \chi \mathcal{S}_{g_0}(\omega) + \chi\, \partial_g \mathcal{S}_g(\omega)|_{g=g_0} \cdot (g - g_0)\|_{H^{-6}(\partial_-\mathcal{M})} \leq C\|g - g_0\|^2_{C^N(M, \otimes_S^2 T^*M)}. \tag{4-1}$$

Since this result is quite technical, its proof is postponed to Section 5. In the following, we will write $h := g - g_0$. Using a complex interpolation argument, Proposition 4.1 is actually a direct consequence of the following technical lemma, which gives weighted estimates on the X-ray transform of $g - g_0$.

Lemma 4.3. *There exist $C, \varepsilon, \delta, \beta, N > 0$ such that, for all smooth metrics g such that $g|_{\partial M} = g_0|_{\partial M}$, $\|g - g_0\|_{C^N} < \varepsilon$, and $(\ell_g, S_g) = (\ell_{g_0}, S_{g_0})$, we have, for $h = g - g_0$,*

$$\|(1+z)^{-7} e^{-z\ell_{g_0}} I_2^{g_0} h\|_{H^{-6}(\partial_-\mathcal{M})} \leq \begin{cases} C\|h\|_{C^N} & \text{for all } z \in i\mathbb{R} - \delta, \\ C\|h\|^2_{C^N} & \text{for all } z \in i\mathbb{R} + \beta. \end{cases} \tag{4-2}$$

We now show that Lemma 4.3 implies Proposition 4.1. The rest of Section 4A is devoted to the proof of Lemma 4.3.

Proof of Proposition 4.1. By the Hadamard three-line theorem applied to the function

$$z \mapsto (1+z)^{-7} e^{-z\ell_{g_0}} I_2^{g_0}(h)$$

(which is bounded in $\Re(z) \in [-\delta, \beta]$ with values in $L^2(\partial_-\mathcal{M}) \subset H^{-6}(\partial_-\mathcal{M})$), Lemma 4.3 implies that

$$\|I_2^{g_0} h\|_{H^{-6}(\partial_-\mathcal{M})} \leq C\|h\|^{1+\mu}_{C^N(M)}$$

for some constants $C, \mu > 0$ independent of h (note that μ depends on δ and β). By Lemma 2.5, there is $C > 0$ and $s > 0$ depending on g_0 such that (for $N \geq 2$)

$$\|I_2^{g_0} h\|_{H^s(\partial_-\mathcal{M})} \leq C\|h\|_{C^N(M)}.$$

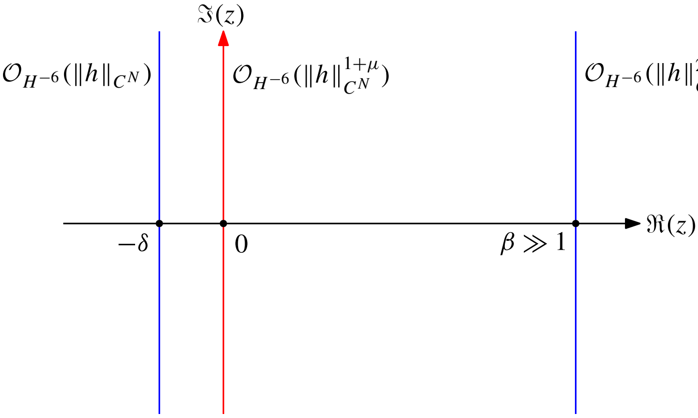

Figure 3. Estimates on $f(z) = e^{-z\ell_{g_0}} I_2^{g_0} h$ in (4-2). For z on the left blue line we have a "volume estimate" of $f(z)$, while for z on the right blue line we have a "microlocal estimate" of $f(z)$. For z on the middle red line, we have the interpolation estimate obtained in Proposition 4.1.

Interpolating $L^2(\partial_-\mathcal{M})$ between $H^{-6}(\partial_-\mathcal{M})$ and $H^s(\partial_-\mathcal{M})$, we deduce that there exists $\mu' > 0$ and $C > 0$ such that

$$\|I_2^{g_0} h\|_{L^2(\partial_-\mathcal{M})} \le C\|h\|_{C^N(M)}^{1+\mu'}. \qquad \square$$

We now start with the proof of Lemma 4.3. See Figure 3: on $\{\Re(z) = -\delta\}$ the bound will follow from an estimate on the volume of long trajectories, while the estimate on the line $\{\Re(z) = \beta\}$ may be thought of as a "microlocal estimate" since it crucially relies on the Taylor expansion of $g \mapsto \mathcal{S}_g$ obtained in Proposition 4.2.

The first bound in (4-2) for $z \in i\mathbb{R} - \delta$ follows directly from the following stronger bound.

Lemma 4.4. *There exists $\delta > 0$ small enough and $C > 0$ (depending on δ) such that, for all $h \in C^0(M, \otimes_S^2 T^*M)$,*

$$\|e^{\delta\ell_{g_0}} I_2^{g_0} h\|_{L^2(\partial_-\mathcal{M})} \le C\|h\|_{C^0(M)}.$$

Proof. For $y \notin \Gamma_-^{g_0}$, we have $|I_2^{g_0} h(y)| \le \|h\|_{C^0} |\ell_{g_0}(y)|$. Thus

$$\|e^{\delta\ell_{g_0}} I_2^{g_0} h\|_{L^2(\partial_-\mathcal{M})} \le \|e^{\delta\ell_{g_0}} \ell_{g_0}\|_{L^2(\partial_-\mathcal{M})} \|h\|_{C^0},$$

which gives the result by (2-10) if $\delta < \frac{1}{2} Q_{g_0}$. $\square$

We now study the second bound in (4-2). Let $\chi \in C_c^\infty(\partial_-\mathcal{M}, [0, 1])$ be a smooth cutoff function. First of all, near the boundary, we have the following:

Lemma 4.5. *There exist $C, \varepsilon > 0$ and $\chi \in C_c^\infty(\partial_-\mathcal{M}, [0, 1])$ such that $1 - \chi^2$ is supported near the boundary of $\partial_-\mathcal{M}$, such that if $\|g - g_0\|_{C^N} < \varepsilon$ and $(\ell_g, S_g) = (\ell_{g_0}, S_{g_0})$, then*

$$\|(1 - \chi^2) I_2^{g_0} h\|_{L^\infty(\partial_-\mathcal{M})} \le C\|h\|_{C^1}^2.$$

Proof. This follows from [Stefanov and Uhlmann 2004, Section 9] as we have the following Taylor expansion for $x, x' \in \partial M$ close enough:

$$d_g(x,x') = d_{g_0}(x,x') + \tfrac{1}{2} I_2^{g_0} h(x,x') + T_g(x,x'),$$

with the bound $|T_g(x,x')| \le C\|h\|_{C^1}^2 d_{g_0}(x,x')$, where $C > 0$ is a uniform constant depending only on g_0. Since the metrics have the same lens data, they also have the same boundary distance function for $x, x' \in \partial M$ close enough, that is, $d_g(x,x') = d_{g_0}(x,x')$, which easily implies the claimed estimate when $1-\chi^2$ is taken to have support near the boundary of $\partial_-\mathcal{M}$ (i.e., close to short geodesics). □

Using the continuous embeddings $L^\infty(\partial_-\mathcal{M}) \hookrightarrow L^2(\partial_-\mathcal{M}) \hookrightarrow H^{-6}(\partial_-\mathcal{M})$, from Lemma 4.5 we deduce that

$$\|(1+z)^{-7} e^{-z\ell_{g_0}} (1-\chi^2) I_2^{g_0} h\|_{H^{-6}(\partial_-\mathcal{M})} \le C\|h\|_{C^N}^2 \tag{4-3}$$

for all $z \in i\mathbb{R} + \beta$. It thus remains to prove the following estimate to deduce the second bound of (4-2).

Lemma 4.6. *There exist $C, \varepsilon, \beta, N > 0$ such that if $\|g - g_0\|_{C^N} < \varepsilon$ and $(\ell_g, S_g) = (\ell_{g_0}, S_{g_0})$, then, for $h := g - g_0$ and for all $z \in i\mathbb{R} + \beta$,*

$$\|(1+z)^{-7} e^{-z\ell_{g_0}} \chi^2 I_2^{g_0} h\|_{H^{-6}(\partial_-\mathcal{M})} \le C\|h\|_{C^N}^2.$$

Proof. We let $\iota_{\partial_-} : \partial_-\mathcal{M} \to \mathcal{M}$ be the inclusion map. For $\beta > 0$, we consider the space

$$E_\beta := C_b^0(\beta + i\mathbb{R}, L^2(\partial_-\mathcal{M})), \tag{4-4}$$

where C_b^0 denotes the vector space of bounded continuous functions, equipped with the L^∞ norm. It is a Banach space when equipped with the norm

$$\|F\|_{E_\beta} := \sup_{z \in \beta + i\mathbb{R}} \|F(z)\|_{L^2(\partial_-\mathcal{M})}.$$

Then, for $z \in \mathbb{C}$ with $\Re(z) = \beta$ large (it will be adjusted later), we define for U_{g_0} the neighborhood of g_0 introduced in (2-3) (with $p = N - 2$):

$$\mathcal{F} : U_{g_0} \ni g \mapsto \mathcal{F}(g)(z) := (1+z)^{-7} \chi^2 \frac{(1 - e^{-z\ell_g})}{z} \in E_\beta, \tag{4-5}$$

where the value at $z = 0$ is set to be $\chi^2 \ell_g$.

First, the function $\mathcal{F}$ is C^2 by Lemma 2.9 by taking $N \ge 5$. We compute its Taylor expansion in the space E_β: for some N large enough, g close enough to g_0, and $h := g - g_0$,

$$\begin{aligned} \mathcal{F}(g)(z) - \mathcal{F}(g_0)(z) &= \frac{\chi^2 e^{-z\ell_{g_0}}}{(1+z)^7} (\partial_g \ell_g)|_{g=g_0}.h + \mathcal{O}_{L^2(\partial_-\mathcal{M})}(\|h\|_{C^N}^2) \\ &= \frac{\chi^2 e^{-z\ell_{g_0}}}{(1+z)^7} \Big(\frac{1}{2} I_2^{g_0}(h) + \alpha_{S_{g_0}(\cdot)}(\partial_g S_g(\cdot)|_{g=g_0}.h) \Big) + \mathcal{O}_{L^2(\partial_-\mathcal{M})}(\|h\|_{C^N}^2), \end{aligned} \tag{4-6}$$

and the remainder is bounded uniformly in z (by Lemma 2.9 again), where we use Lemma 3.5 in the second line (recall α is the Liouville 1-form). If $\ell_g = \ell_{g_0}$, we obtain in particular $\mathcal{F}(g)(z) - \mathcal{F}(g_0)(z) = 0$, thus

$$\sup_{z\in\beta+i\mathbb{R}}\left\|\frac{\chi^2e^{-z\ell_{g_0}}}{(1+z)^7}\Big(\frac{1}{2}I_2^{g_0}(h)+\alpha_{S_{g_0}(\cdot)}(\partial_g S_g(\cdot)|_{g=g_0}.h)\Big)\right\|_{L^2(\partial_-\mathcal{M})}\le C\|h\|_{C^N}^2. \tag{4-7}$$

Note that, for $\Re(z)=\beta>0$, as a consequence of (2-19), we have $\chi^2e^{-z\ell_{g_0}}I_2^{g_0}(h)\in L^2(\partial_-\mathcal{M})$, thus, since by Lemma 2.9 we know that $\partial_g\mathcal{F}(g)(z)|_{g=g_0}.h\in L^2(\partial_-\mathcal{M})$ if β is large enough, we obtain that

$$\chi^2\alpha_{S_{g_0}(\cdot)}(\partial_g S_g(\cdot)|_{g=g_0}.h)e^{-z\ell_{g_0}}\in L^2(\partial_-\mathcal{M}).$$

We now claim the following lemma, the proof of which is deferred to the following paragraph.

Lemma 4.7. *There exist $C,\varepsilon,\beta,N>0$ such that if $\|g-g_0\|_{C^N}<\varepsilon$ and $(\ell_g,S_g)=(\ell_{g_0},S_{g_0})$, then, for all $z\in i\mathbb{R}+\beta$ and $h=g-g_0$,*

$$\|(1+z)^{-7}\chi^2\alpha_{S_{g_0}(\cdot)}(\partial_g S_{g_0}(\cdot)|_{g=g_0}.h)e^{-z\ell_{g_0}}\|_{H^{-6}(\partial_-\mathcal{M})}\le C\|h\|_{C^N}^2.$$

Using (4-7) and Lemma 4.7, we deduce that, for all $\Re(z)=\beta$ with $\beta,N>0$ large enough,

$$\begin{aligned}\sup_{z\in\beta+i\mathbb{R}}&|1+z|^{-7}\|\chi^2I_2^{g_0}(h)e^{-z\ell_{g_0}}\|_{H^{-6}(\partial_-\mathcal{M})}\\&\le\sup_{z\in\beta+i\mathbb{R}}|1+z|^{-7}\|\chi^2\alpha_{S_{g_0}(\cdot)}(\partial_g S_{g_0}(\cdot)|_{g=g_0}.h)e^{-z\ell_{g_0}}\|_{H^{-6}(\partial_-\mathcal{M})}+C\|h\|_{C^N}^2\le C\|h\|_{C^N}^2,\end{aligned}$$

where the constant $C>0$ changes from line to line. This concludes the proof of Lemma 4.6. ☐

Proof of Lemma 4.7. Taking a finite cover of $\mathcal{M}=\bigcup_i U_i$, a partition of unity $\sum_i\chi_i=\mathbf{1}$ subordinate to that cover, we may write

$$\alpha=\sum_{i,j}\alpha_i^{(j)}dy_i^{(j)}, \tag{4-8}$$

where $\alpha_i^{(j)},y_i^{(j)}\in C^\infty(\mathcal{M})$ are smooth functions compactly supported inside U_i, and thus, for $y\notin\Gamma_-^{g_0}$, we have

$$\begin{aligned}\chi^2\alpha_{S_{g_0}(y)}(\partial_g S_{g_0}(y)|_{g=g_0}.h)e^{-z\ell_{g_0}(y)}&=\chi^2\sum_{i,j}\alpha_i^{(j)}(S_{g_0}(y))\langle dy_i^{(j)},\partial_g S_g(y)|_{g=g_0}.h\rangle e^{-z\ell_{g_0}(y)}\\&=\sum_{i,j}\chi\mathcal{S}_{g_0}(\alpha_i^{(j)})(y)e^{-z\ell_{g_0}(y)}\cdot\chi\partial_g\mathcal{S}_g(y_i^{(j)})(y)|_{g=g_0}.h.\end{aligned} \tag{4-9}$$

First, taking $\beta>0$ large enough, we can ensure by Lemma 2.8 the existence of a constant $C>0$ such that, for all $z\in i\mathbb{R}+\beta$ and for all i,j, one has $\chi\mathcal{S}_{g_0}^*\alpha_i^{(j)}e^{-z\ell_{g_0}}\in C^6(\partial_-\mathcal{M})$ with the uniform bound

$$\|(1+z)^{-7}\chi\mathcal{S}_{g_0}(\alpha_i^{(j)})e^{-z\ell_{g_0}}\|_{C^6(\partial_-\mathcal{M})}\le C. \tag{4-10}$$

We now let $f\in C^\infty(\mathcal{M})$ be one of the functions $y_i^{(j)}$ in (4-8). By Proposition 4.2, we have

$$\chi\mathcal{S}_gf=\chi\mathcal{S}_{g_0}f+\chi\partial_g\mathcal{S}_gf|_{g=g_0}.h+\mathcal{O}_{H^{-6}(\partial_-\mathcal{M})}(\|h\|_{C^N}^2).$$

(The constant in the $\mathcal{O}$ notation depends on the function f, but there are only finitely many functions $y_i^{(j)}$ considered in the end so the constant will be uniform.) Now, using that the scattering relations are the same, i.e., $S_g = S_{g_0}$, we have $\chi \mathcal{S}_g^* f = \chi \mathcal{S}_{g_0}^* f$, where the equality holds in $L^\infty(\partial_-\mathcal{M})$ and hence in $L^2(\partial_-\mathcal{M}) \subset H^{-6}(\partial_-\mathcal{M})$. As a consequence, we deduce that

$$\|\chi \partial_g \mathcal{S}_g^* y_i^{(j)}|_{g=g_0} .h\|_{H^{-6}(\partial_-\mathcal{M})} \le C\|h\|_{C^N}^2. \tag{4-11}$$

Using both (4-10) and (4-11) in (4-9) and the continuity of the multiplication

$$C^6(\partial_-\mathcal{M}) \times H^{-6}(\partial_-\mathcal{M}) \ni (u, v) \mapsto uv \in H^{-6}(\partial_-\mathcal{M}),$$

we deduce that, for some $C > 0$,

$$\|(1+z)^{-7}\alpha_{S_{g_0}(\cdot)}(\partial_g S_{g_0}(\cdot)|_{g=g_0} .h)e^{-z\ell_{g_0}}\|_{H^{-6}(\partial_-\mathcal{M})} \le C\|h\|_{C^N}^2.$$

This concludes the proof of Lemma 4.7. □

4B. *End of the proof.* We can now complete the proof of Theorem 1.8.

Proof of Theorem 1.8. Assume that $(\ell_g, S_g) = (\ell_{g_0}, S_{g_0})$ and g is close enough to g_0 in the C^N-topology. By Lemma 3.6, we can find a diffeomorphism ψ such that $\psi|_{\partial M} = \mathrm{Id}_{\partial M}$ and $g' := \psi^* g$ is solenoidal with respect to g_0. Moreover, $(\ell_{g'}, S_{g'}) = (\ell_g, S_g) = (\ell_{g_0}, S_{g_0})$. Also note that $\|g' - g_0\|_{C^N} \le C\|g - g_0\|_{C^N}$ for some uniform $C > 0$ (depending on g_0).

Writing $h := g' - g_0$, Proposition 4.1 implies that

$$\|I_2^{g_0} h\|_{L^2} \le C\|h\|_{C^N}^{1+\mu}. \tag{4-12}$$

Now recall that, for any $\varepsilon > 0$, the adjoint $(I_2^{g_{0e}})^* : L^2 \to L^{p(\varepsilon)} \subset H^{-\varepsilon}$ is bounded (here $p(\varepsilon) < 2$ and $p(\varepsilon) \to 2$ as $\varepsilon \to 0$); see [Guillarmou 2017b, Lemma 5.1 and Equation (5.3)].

By (4-12), and since $\Pi_2^{g_{0e}}$ is of order -1 (by Proposition 3.7), and $E_0 h$ has regularity $H^{1/2-\varepsilon}$ for any $\varepsilon > 0$, we conclude that, for any $\varepsilon > 0$, where $C > 0$ changes from line to line,

$$\|\Pi_2^{g_{0e}} E_0 h\|_{H^{-\varepsilon}} = \|(I_2^{g_{0e}})^* I_2^{g_{0e}} E_0 h\|_{H^{-\varepsilon}} \le C\|I_2^{g_{0e}} E_0 h\|_{L^2} \le C\|I_2^{g_0} h\|_{L^2} \le C\|h\|_{C^N}^{1+\mu},$$
$$\|\Pi_2^{g_{0e}} E_0 h\|_{H^{3/2-\varepsilon}} \le C\|E_0 h\|_{H^{1/2-\varepsilon}} \le C\|h\|_{C^N}.$$

By interpolation in Sobolev spaces, we obtain from these two estimates that, for some (different) $C, \mu > 0$,

$$\|\Pi_2^{g_{0e}} E_0 h\|_{H^1} \le C\|h\|_{C^N}^{1+\mu}.$$

Applying the elliptic stability estimate for solenoidal tensors of Proposition 3.8 (using that our assumption implies that $I_2^{g_0}$ is solenoidal injective), we get

$$\|h\|_{L^2} \le C\|\Pi_2^{g_{0e}} E_0 h\|_{H^1} \le C\|h\|_{C^N}^{1+\mu}.$$

By interpolation, we then obtain, for some (much larger) other integer $N \in \mathbb{N}$,

$$\|h\|_{L^2} \le C\|h\|_{L^2}\|h\|_{C^N}^{\mu} \le C\|h\|_{L^2}\|g - g_0\|_{C^N}^{\mu}.$$

If $\|g - g_0\|_{C^N} < (1/C)^{1/\mu}$, this implies that $h = 0$, namely $g' = \psi^* g = g_0$. □

5. Smoothness of the scattering operator with respect to the metric

The goal of this section is to prove Theorem 1.10 and to derive Proposition 4.2 as a corollary. Theorem 1.10 will follow directly from Theorem 5.14 and Lemma 5.21 below. The scattering operator $\mathcal{S}_g$ can be expressed purely in terms of the resolvent R_{g_e} of X_{g_e} thanks to Lemma 2.7. Thus, in order to analyze the map $g \mapsto \mathcal{S}_g$, we shall study the regularity of the map $g \mapsto R_{g_e}$ in adequate functional spaces. Since working with g_e or g is equivalent (they share exactly the same properties), we shall consider R_g for simplicity of notation. The construction of R_g is done using microlocal methods as in [Dyatlov and Guillarmou 2016], but we need to understand the g-dependence in the construction. We fix a metric of Anosov type g_0 on M and we denote by X_0 its associated geodesic vector field on $\mathcal{M}$. We will consider the resolvent of X if X is any smooth vector field that is close enough to X_0 in $C^2(\mathcal{M}, T\mathcal{M})$. We refer to Section 2C3, where the notation for the cotangent bundle is introduced.

5A. ***Construction of the uniform escape function.*** In this paragraph, we construct a *uniform escape function*, i.e., an escape function[4] for X_0 which is also an escape function for all vector fields X that are sufficiently close to X_0. We will use an idea of [Bonthonneau 2020] in order to obtain an escape function adapted to all flows X close to X_0. Denote by $S^*\mathcal{M} := (T^*\mathcal{M} \setminus \{0\})/\mathbb{R}^+$ (and similarly $S^*\mathcal{N}$) the spherical bundle, by $\kappa : T^*\mathcal{M} \to S^*\mathcal{M}$ the quotient projection, by $\pi : S^*\mathcal{N} \to \mathcal{N}$ the footpoint map, and recall that $\boldsymbol{X}$ is the generator of the symplectic lift of φ_t defined in (2-12). Finally, recall that $\rho_0 > 0$ is the constant of Section 2A2 used to define the extension $\mathcal{M}_e$, and that $\widetilde{X}_0$ is some initial extension of the vector field from $\mathcal{M}$ to $\mathcal{N}$ (which does not need to vanish at $\{\rho = -\rho_0\}$).

Proposition 5.1. *There exist a smooth function $m \in C^\infty(S^*\mathcal{N}, [-1, 1])$, invariant by the antipodal map $(x, \xi) \mapsto (x, -\xi)$, and $\delta > 0$ such that, for all vector fields $X \in C^\infty(\mathcal{M}, T\mathcal{M})$ such that*

$$\|X - X_0\|_{C^2(\mathcal{M},T\mathcal{M})} \leq \delta,$$

the following hold:

(1) $m = 1$ *in a neighborhood of* $(E_-^X)^* \cap \pi^{-1}(\mathcal{M})$.

(2) $m = -1$ *in a neighborhood of* $(E_+^X)^* \cap \pi^{-1}(\mathcal{M})$.

(3) $\mathrm{supp}(m) \cap \pi^{-1}(\mathcal{M})$ *is contained in a small conic neighborhood of* $(E_-^X)^*$ *and* $(E_+^X)^*$.

(4) $\mathrm{supp}(m) \subset \{\rho > -2\rho_0\}$.

(5) $\mathrm{supp}(m) \cap \{\rho = -\rho_0\} \cap \{\widetilde{X}_0 \rho = 0\} = \varnothing$.

(6) $\boldsymbol{X} m \leq 0$.

The fact that X and X_0 are C^2-close will ensure that the structural stability Proposition 2.4 applies. The function m will be constructed as

$$m = m_- - m_+ + \eta^{-1}(\pi^*\chi_- - \pi^*\chi_+), \tag{5-1}$$

[4] A function decreasing along the bicharacteristics of the symplectic lift of X to the cotangent bundle.

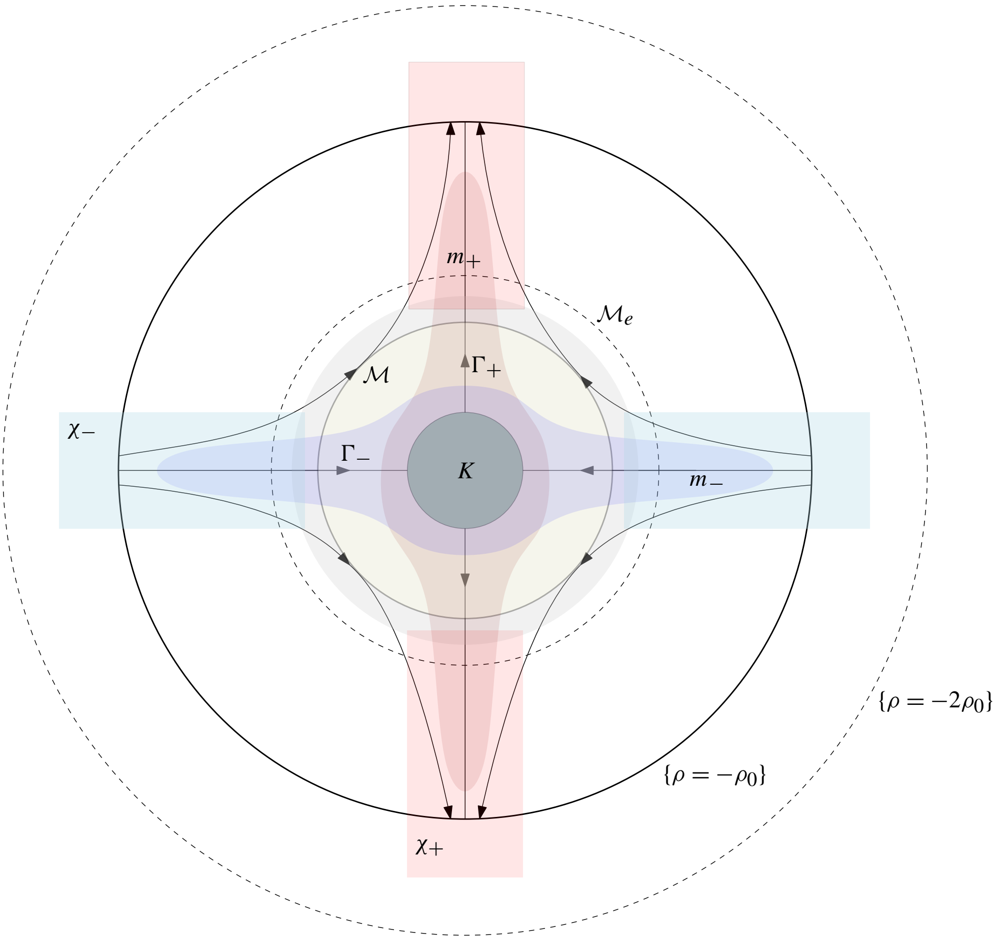

Figure 4. A schematic representation of the various sets and functions appearing in Lemmas 5.10 and 5.11. The disks represent (respectively, from the center to the outer disk): the trapped set K of X_0, the manifold $\mathcal{M}$, the set $\{q = 0\}$ (in light gray) defined in Section 5B, the extended manifold $\mathcal{M}_e$, the set $\{\rho \geq -\rho_0\}$, the set $\{\rho \geq -2\rho_0\}$. The support of the functions m_+, χ_+, m_-, χ_- are depicted, respectively, in: dark red, light red, dark blue, light blue. The flowlines of X_0 are represented in black with arrows indicating the flow direction.

where $m_\pm$ are smooth functions with support near $(E_\pm^X)^*$ and taking value 1 on $(E_\pm^X)^*$, $\chi_\pm$ are smooth functions with compact support in a slightly larger neighborhood of $\Sigma_\pm$ (defined in (2-13)), and $\eta > 0$ will be a small parameter chosen small enough in the end. We refer to Section 2C3 where all the previous notation are defined. The proof being rather technical, we advise the reader to have in mind Figure 4, where the various sets and functions of the construction are depicted.

Remark 5.2. More generally, one could construct a function m taking any positive (resp. negative) constant value near $(E_-^X)^*$ (resp. $(E_+^X)^*$) but this will not be needed.

5A1. *Uniform cone contraction.* We start with some technical lemmas on the contraction of cones in $T^*\mathcal{M}$. In order to abbreviate notation, we will sometimes write $X \sim X_0$ if $\|X - X_0\|_{C^2} \le \delta$, where $\delta > 0$ is some small constant which will be chosen later. In what follows, we will use the notion of conic neighborhoods of conic sets in $T^*\mathcal{N} \setminus 0$, which may be identified with neighborhoods on the spherical bundle $S^*\mathcal{N}$. First of all, we have:

Lemma 5.3. *Let $\mathcal{U}$ be an open neighborhood of the trapped set K^{X_0}. Then, there exists $\delta > 0$ and $T \ge 0$ such that, for all $t \ge T$ and all smooth vector fields X such that $\|X - X_0\|_{C^2(\mathcal{M}, T\mathcal{M})} < \delta$,*

$$y,\ \varphi^X_{-t}(y),\ \varphi^X_t(y) \in \mathcal{M}_e \quad \Longrightarrow \quad y \in \mathcal{U}.$$

Taking $X \sim X_0$ close enough in the C^2-topology, we can ensure that $\mathcal{U}$ is also an open neighborhood of $\bigcup_{X \sim X_0} K^X$ by the structural stability Proposition 2.4.

Proof. We argue by contradiction. Assume that we can find sequences $(T_j)_{j \ge 1}$ such that $T_j \to +\infty$, $(X_j)_{j \ge 1}$ such that $X_j \to X_0$ in $C^2(\mathcal{M}, T\mathcal{M})$, and $(y_j)_{j \ge 1}$ such that $y_j \in \mathcal{M}_e$, $\varphi^{X_j}_{-T_j}(y_j) \in \mathcal{M}_e$ and $\varphi^{X_j}_{T_j}(y_j) \in \mathcal{M}_e$, but $y_j \notin \mathcal{U}$. By compactness of $\mathcal{M}_e$, we can always assume, up to extraction, that $y_j \to y_\infty \in \mathcal{M}_e$. But then $y_\infty \in K^{X_0}$, which contradicts $y_\infty \notin \mathcal{U}$. □

We now show the existence of small conic subsets in $T^*\mathcal{M}$, independent of the vector field X, on which the differential of the flow $(\varphi^X_t)_{t \in \mathbb{R}}$ is exponentially expanding/contracting. This may be compared to [Dyatlov and Guillarmou 2016, Lemma 2.11].

Lemma 5.4. *There exist $\delta > 0$ small enough, constants $C, T, \lambda > 0$ and small open conic neighborhoods $U_\pm$ of $\bigcup_{X \sim X_0} (E^X_\pm)^*$, such that, for all X with $\|X - X_0\|_{C^2} \le \delta$, the following holds: for all $(y, \xi) \in U_\pm$, for all $t \ge T$ such that $y, \varphi^X_{\pm t}(y) \in \mathcal{M}_e$,*

$$\textit{for all } s \in [0, t - T], \quad e^{\pm sX}(y, \xi) \in U_\pm \qquad \textit{and}, \qquad \textit{for all } s \in [0, t], \quad |e^{\pm sX}(y, \xi)| \ge C e^{\lambda s} |\xi|.$$

Proof. We prove the lemma for the outgoing (+) direction, the proof being similar for the incoming (−) direction. Fix arbitrary small conic neighborhoods $\widetilde{U}^{(2)}_+ \Subset \widetilde{U}^{(1)}_+$ of $(E^{X_0}_+)^*$. By hyperbolicity, there is a $T_0 > 0$ large enough such that the following holds: for all $(y, \xi) \in T^*_{\Gamma^{X_0}_+} \mathcal{M}_e \cap \widetilde{U}^{(1)}_+$ such that $y, \varphi^{X_0}_{T_0}(y) \in \mathcal{M}_e$, one has

$$e^{T_0 X_0}(y, \xi) \in T^*_{\Gamma^{X_0}_+} \mathcal{M}_e \cap \widetilde{U}^{(2)}_+, \quad |e^{T_0 X_0}(y, \xi)| \ge 10 |\xi|.$$

By continuity, there exist small neighborhoods $U^{(j)}_+$ of $\widetilde{U}^{(j)}_+$ such that the following hold:

(1) The neighborhoods are chosen so that $\pi(U^{(1)}_+) \Subset \pi(U^{(2)}_+)$.

(2) Letting $W := \pi(U^{(1)}_+)$, one has $U^{(2)}_+ \cap W \Subset^{\text{fiber}} U^{(1)}_+ \cap W$, in the sense that, for all $y \in W$, we have $U^{(2)}_+ \cap T^*_y \mathcal{M}_e \Subset U^{(1)}_+ \cap T^*_y \mathcal{M}_e$.

(3) For all $(y, \xi) \in U^{(1)}_+$ such that $y, \varphi^{X_0}_{T_0}(y) \in \mathcal{M}_e$,

$$e^{T_0 X}(y, \xi) \in U^{(2)}_+, \quad |e^{T_0 X}(y, \xi)| \ge 5 |\xi|.$$

(4) There is a time $T_1 > T_0$ such that, if $y \in \pi(U^{(2)}_+) \setminus \pi(U^{(1)}_+)$, then $\varphi^{X_0}_t(y) \notin \mathcal{M}_e$ for all $t \ge T_1$.

By continuity, this can be achieved so that points (1-4) also hold for all smooth vector fields X such that $\|X - X_0\|_{C^1} \le \delta$, where $\delta > 0$ is chosen small enough. We will actually choose $\|X - X_0\|_{C^2} \le \delta$, where $\delta > 0$ is chosen small enough: by the structural stability Proposition 2.4, we can then ensure that the neighborhoods $U_+^{(j)}$ also contain $(E_+^X)^*$ for $X \sim X_0$ in the C^2-topology.

We set $U_+ := U_+^{(1)}$ and $T := 3T_1$, and we claim that these satisfy the required properties. Take $(y, \xi) \in U_+$ such that $y \in \mathcal{M}_e$, $\varphi_t(y) \in \mathcal{M}_e$ and $t \ge T$. Write $t = k_1T_1 + r_1$, with $k_1 \in \mathbb{Z}_{\ge 1}$, $r_1 \in [0, T_1)$, and $(k_1 - 1)T_1 = k_0T_0 + r_0$, with $k_0 \in \mathbb{Z}_{\ge 0}$, $r_0 \in [0, T_0)$, that is,

$$t = k_0T_0 + T_1 + r_1 + r_0.$$

Note that $T_1 + r_1 + r_0 < 3T_1 = T$.

For all $s \in [0, k_0T_0]$, one has $\varphi_s^X(y, \xi) \in \pi(U_+^{(1)})$ and $(y, \xi) \in U_+^{(1)}$. Indeed, otherwise, we would get, for some $s_\star \in [0, k_0T_0]$, that $\varphi_{s_\star}^X(y, \xi) \in \pi(U_+^{(2)}) \setminus \pi(U_+^{(1)})$, but then $\varphi_{s_\star+T_1}^X(y) \notin \mathcal{M}_e$, which contradicts the fact that $\varphi_t^X(y) \in \mathcal{M}_e$ since

$$s_\star + T_1 \le (k_1 - 1)T_1 + T_1 = kT_1 \le t.$$

Then, using the uniform lower bound $|e^{(T_1+r_0+r_1)\boldsymbol{X}}(y, \xi)| \ge C_0|\xi|$, we obtain

$$|e^{t\boldsymbol{X}}(y, \xi)| = |e^{(T_1+r_0+r_1)\boldsymbol{X}}(e^{T_0\boldsymbol{X}})^{k_0}(y, \xi)| \ge C_0 5^{k_0}|\xi| \ge Ce^{\lambda t}|\xi|$$

for some constant $C > 0$ and $\lambda = \log(5)/T_0$. □

We now let V_+ be a small conic neighborhood of $\bigcup_{X\sim X_0}(E_+^X)^*$ contained inside U_+, i.e., $V_+ \Subset U_+$. It will be convenient to use the following operation on the category of fibered conic subsets: if $V \subset T^*\mathcal{N}$ is an open conic subset, define the *fiberwise complement* of V as

$$V^{\complement_{\text{fiber}}} := \{(y, \xi) \in T^*\mathcal{N} \mid y \in \pi(V),\ \xi \in \overline{V}^{\complement} \cap T_y^*\mathcal{N}\},$$

where the superscript $\complement$ denotes the set theoretic complement.

Lemma 5.5. *There exists $\delta > 0$ and $T > 0$, and $V_- := (W_-)^{\complement_{\text{fiber}}}$, where W_- is a small conic neighborhood of $\bigcup_{X\sim X_0}(E_-^X)^* \oplus (E_0^X)^*$, such that, for all X with $\|X - X_0\|_{C^2(\mathcal{M},T\mathcal{M})} \le \delta$, one has $e^{T\boldsymbol{X}}V_- \Subset V_+$.*

The same lemma can be proved by reversing the direction of X, i.e., by swapping the roles of E_-^* and E_+^*.

Proof. We fix an arbitrary open conic set $\widetilde{V}_-$ near $\pi^{-1}(K^{X_0})$ such that $\widetilde{V}_- \cap ((E_-^{X_0})^* \oplus (E_0^{X_0})^*) = \varnothing$. In restriction to $\pi^{-1}(K^{X_0})$, hyperbolicity ensures the existence of a time $T > 0$ such that

$$e^{T\boldsymbol{X}_0}(\widetilde{V}_- \cap \pi^{-1}(K^{X_0})) \Subset V_+ \cap \pi^{-1}(K^{X_0}).$$

By continuity, this also holds for an open conic neighborhood V_- by taking $\pi(V_-)$ to be contained inside a small neighborhood of K^{X_0} (whose size depends on T), and it also holds uniformly for all vector fields X such that $\|X - X_0\|_{C^2} \le \delta$ if $\delta > 0$ is taken small enough (depending on T) by using the stability result of Proposition 2.4 and choosing $\delta > 0$ small enough that $\bigcup_{X\sim X_0} K^X \subset \pi(V_-)$. □

In order to simplify notation, we will write $\zeta = (y, \xi)$ for a point in $T^*\mathcal{N}$ and $p_X(x, \xi) := \xi(X)$ for the principal symbol of $-iX$. From Lemmas 5.4 and 5.5, we deduce:

Lemma 5.6. *Let Ω be a small conic neighborhood of $\bigcup_{X \sim X_0}\{p_X = 0\}$ in $T^*\mathcal{M}_e$. There exist $\delta > 0$ and $T > 0$ such that, for all X with $\|X - X_0\|_{C^2(\mathcal{M},T\mathcal{M})} \leq \delta$ and $t \geq T$, if $\zeta, e^{tX}(\zeta) \in \Omega \cap T^*\mathcal{M}_e \setminus \{0\}$, then*

$$\int_0^t \mathbf{1}_{U_+ \sqcup U_-}(e^{sX}(\zeta))\, \mathrm{d}s \geq t - T.$$

In other words, the flowline of ζ spends at least a time $t - T$ in $U_+ \sqcup U_-$, where there is some uniform contraction/expansion.

Proof. We use the sets $U_\pm$ and $V_\pm$ defined in Lemmas 5.4 and 5.5. Note that $\pi(V_\pm) \subset \pi(U_\pm)$ by construction, and we set $\mathcal{U} := \pi(U_+) \cap \pi(U_-)$. We introduce the following constants:

(1) Let $T_0 > 0$ be the time provided by Lemma 5.3 applied with the open neighborhood $\mathcal{U}$ of K^{X_0} and such that, for all X with $\|X - X_0\|_{C^2} \leq \delta$, for all $t \geq T_0$ and $y \in \mathcal{M}_e$ such that $\varphi_t^X(y) \in \mathcal{M}_e$, one has

$$\{\varphi_s^X(y) \mid s \in [T_0, t - T_0]\} \subset \mathcal{U}.$$

(2) Let $T_1 > 0$ be the time provided by Lemma 5.4.

(3) Let $T_2 > 0$ be the time provided by Lemma 5.5 such that $e^{T_2 X} V_- \Subset V_+$.

Take a point $\zeta \in \Omega \cap T^*\mathcal{M}_e \setminus \{0\}$ such that $e^{tX}(\zeta) \in T^*\mathcal{M}_e$ for some $t \geq 2T_0$, that is, $\varphi_s^X(\pi(\zeta)) \in \mathcal{U}$ for all $s \in [T_0, t - T_0]$. We treat different cases:

Case 1: Assume that $e^{T_0 X}(\zeta) \in U_-$. If $e^{sX}(\zeta) \in U_-$ for all $s \in [T_0, t - T_0]$, then the claim holds for ζ and $T = 2T_0$. If not, there is a time $s_\star \in [T_0, t - T_0]$ such that $e^{s_\star X}(\zeta) \in V_-$ and $e^{sX}(\zeta) \in U_-$ if $s \in [T_0, s_\star]$. By Lemma 5.5, we then deduce that $\zeta' := e^{(s_\star + T_2)X}(\zeta) \in V_+ \Subset U_+$. Observe that $\zeta' \in U_+$ and $e^{(t-(s_\star+T_2))X}(\zeta') \in T^*\mathcal{M}_e$. If $t - (s_\star + T_2) \geq T_1$, from Lemma 5.4 we deduce that, for all $s \in [T_0, s_\star] \cup [s_\star + T_2, t - T_1]$, we have $e^{sX}(\zeta) \in U_- \cup U_+$, that is, the flowline of ζ spends at least $t - (T_0 + T_1 + T_2)$ time in $U_- \cup U_+$. Thus, the claim holds with $T := T_0 + T_1 + T_2$. If $t - (s_\star + T_2) \leq T_1$, then the flowline of ζ has spent a time at least $s_\star - T_0 \geq t - (T_0 + T_1 + T_2)$ in U_-, and the claim holds with the same time T defined previously.

Case 2: Eventually, if $e^{T_0 X}(\zeta) \notin U_-$, then $e^{T_0 X}(\zeta) \in V_-$, and the claim is also straightforward, following the previous arguments. □

Eventually, we will need the following lemma.

Lemma 5.7. *Let $W_- = W'_- \cap (W''_-)^{\complement_{\text{fiber}}}$, where W'_- and W''_- are conic neighborhoods of $\pi^{-1}(K^{X_0})$ and $(E_+^{X_0})^*$, respectively. Let W_+ be a small conic neighborhood of $(E_+^{X_0})^*$. Then, there exists $T > 0$ such that, for all $t \geq T$, we have $e^{-tX_0} W_- \cap W_+ = \varnothing$.*

By *small* for W_+, it is understood that $W_+ \cap ((E_0^{X_0})^* \oplus (E_-^{X_0})^*) = \varnothing$.

Proof. This follows from the fact that there is a uniform time $T > 0$ such that, for each $(y, \xi) \in W_-$, either $\rho(\varphi_{-t}^{X_0}(y)) < 0$ for all $t > T$, or $e^{-tX_0}(y, \xi)$ belongs to a small conic neighborhood of $(E_0^{X_0})^* \oplus (E_-^{X_0})^*$ for all $t > T$, by the same argument as in Lemma 5.5. □

5A2. *Construction of* $m_\pm$. In this paragraph, we construct the functions $m_\pm$ involved in the expression (5-1) of the escape function m. We introduce a smooth function $m_0 \in C^\infty(S^*\mathcal{N}, [0,1])$, invariant by the antipodal map $(x,\xi) \mapsto (x,-\xi)$, such that $m_0 = 1$ in a small neighborhood of $\kappa((E_u^{X_0})^*)$ over K^{X_0} and $m_0 = 0$ on the complement of a slightly larger neighborhood of $\kappa((E_u^{X_0})^*)$. We will need the following.

Lemma 5.8. *For all $T > 0$ large enough, the following holds*:

$$\begin{cases} \zeta, e^{TX_0}(\zeta) \in S^*\mathcal{M}_e \\ m_0(\zeta) < 1 \end{cases} \implies \text{for all } t \in [T, 3T], \quad m_0(e^{-tX_0}(\zeta)) = 0.$$

Proof. We argue by contradiction. Assume that there exists

- an increasing sequence of values $(T_j)_{j\in\mathbb{Z}_{\geq 0}}$ such that $T_j \to +\infty$,
- a sequence of points $(\zeta_j)_{j\in\mathbb{Z}_{\geq 0}}$ such that $\zeta_j, e^{T_j X_0}(\zeta_j) \in S^*\mathcal{M}_e$ and $m_0(\zeta_j) < 1$, and
- a sequence of values $(S_j)_{j\in\mathbb{Z}_{\geq 0}}$ such that $S_j \geq T_j$ and $m_0(e^{-S_j X_0}(\zeta_j)) > 0$.

By compactness of $S^*\mathcal{M}_e$, up to extraction, we can always assume $\zeta_j \to \zeta_\infty$. Observe that $\zeta_\infty \in \pi^{-1}(K^{X_0})$ as $T_j \to +\infty$: indeed, since $T_j \to \infty$, we have that $\zeta_\infty \in \pi^{-1}(\Gamma_-^{X_0})$; if $\zeta_\infty \in \pi^{-1}(\Gamma_-^{X_0} \setminus K^{X_0})$, the exit time from $\mathcal{M}$ in the past of ζ_∞ is finite and since $S_j \to +\infty$, $m_0(e^{-S_j X_0}\zeta_j) > 0$ and m_0 vanishes outside of $\mathcal{M}$, we would get a contradiction for $j \geq 0$ large enough.

Since $m_0(\zeta_j) < 1$ and $m_0 = 1$ near $\kappa((E_u^{X_0})^*)$, we can find V_-, a small neighborhood of $\pi^{-1}(K^{X_0})$ whose closure is not intersecting $(E_-^{X_0})^*$ and such that $\zeta_\infty \in V_-$. Let V_+ be a small neighborhood of $\operatorname{supp}(m_0)$. By Lemma 5.7, there is $T > 0$ such that, for all $t \geq T$, $e^{-tX_0}V_- \cap V_+ = \varnothing$. In particular, for $j \geq 0$ large enough, $\zeta_j \in V_-$, and thus $e^{-S_j X_0}(\zeta_j) \notin V_+$, that is, $m_0(e^{-S_j X_0}(\zeta_j)) = 0$. But this contradicts $m_0(e^{-S_j X_0}(\zeta_j)) > 0$. □

We then set, for $T > 0$ large enough satisfying Lemma 5.8,

$$m_1(\zeta) := \frac{1}{2T} \int_T^{3T} m_0(e^{-tX_0}(\zeta))\, dt. \tag{5-2}$$

Lemma 5.9. *The function $m_1 \in C^\infty(S^*\mathcal{N}, [0,1])$ satisfies the following properties*:

(1) $m_1 = 1$ *near* $(E_+^{X_0})^* \cap \pi^{-1}(\mathcal{M}_e)$.

(2) $\operatorname{supp}(m_1) \subset \pi^{-1}(\Sigma_+)$ *and* $\operatorname{supp}(m_1)$ *is contained in a small neighborhood of* $(E_+^{X_0})^*$.

(3) $\boldsymbol{X}_0 m_1 \geq 0$ *on* $\pi^{-1}(\mathcal{M}_e)$.

(4) *There exist* $\varepsilon_0, \delta_0 > 0$ *such that, if* $\zeta \in \pi^{-1}(\mathcal{M}_e)$ *and* $\left|m_1(\zeta) - \frac{1}{2}\right| \leq \varepsilon_0$, *then* $\boldsymbol{X}_0 m_1(\zeta) \geq \delta_0$.

Proof. We prove each point separately.

(1) and (2) Taking $T > 0$ large enough in (5-2), the first two items are immediate to check.

(3) For $\zeta \in T^*\mathcal{M}_e$, we have

$$\boldsymbol{X}_0 m_1(\zeta) = \frac{1}{2T}\big(m_0(e^{-TX_0}(\zeta)) - m_0(e^{-3TX_0}(\zeta))\big),$$

and we want to show that $\boldsymbol{X}_0 m_1 \geq 0$ on $\pi^{-1}(\mathcal{M}_e)$. Observe that if $m_0(e^{-T\boldsymbol{X}_0}(\zeta)) = 1$, then the claim $\boldsymbol{X}_0 m_1(\zeta) \geq 0$ is immediate. We can thus assume that $m_0(e^{-T\boldsymbol{X}_0}(\zeta)) < 1$. If $e^{-T\boldsymbol{X}_0}(\zeta) \notin \pi^{-1}(\mathcal{M}_e)$, then $m_0(e^{-T\boldsymbol{X}_0}(\zeta)) = 0$ and, by convexity, $m_0(e^{-3T\boldsymbol{X}_0}(\zeta)) = 0$ and $\boldsymbol{X}_0 m_1(\zeta) = 0$. If $e^{-T\boldsymbol{X}_0}(\zeta) \in \pi^{-1}(\mathcal{M}_e)$, we can apply Lemma 5.8 which implies that $m_0(e^{-3T\boldsymbol{X}_0}(\zeta)) = 0$, and thus we also obtain $\boldsymbol{X}_0 m_1(\zeta) \geq 0$.

(4) In order to show the last item, it suffices to show that, on the compact set

$$\{\boldsymbol{X}_0 m_1 = 0\} \cap \pi^{-1}(\mathcal{M}_e),$$

one has $\left|m_1 - \tfrac{1}{2}\right| \geq \varepsilon_1$ for some positive $\varepsilon_1 > 0$, that is, the continuous function $\left|m_1 - \tfrac{1}{2}\right|$ does not vanish on this set. Let $\zeta \in \pi^{-1}(\mathcal{M}_e)$ be such that $\boldsymbol{X}_0 m_1(\zeta) = 0$. Then $m_0(e^{-T\boldsymbol{X}_0}\zeta) = m_0(e^{-3T\boldsymbol{X}_0}\zeta)$.

Assume that $m_0(e^{-T\boldsymbol{X}_0}\zeta) < 1$. If $e^{-T\boldsymbol{X}_0}\zeta \notin \pi^{-1}(\mathcal{M}_e)$, then, by convexity of $\mathcal{M}_e$, $e^{-t\boldsymbol{X}_0}(\zeta) \notin \pi^{-1}(\mathcal{M}_e)$ for all $t \geq T$, and thus $m_1(\zeta) = 0$, that is, $\left|m_1 - \tfrac{1}{2}\right| = \tfrac{1}{2} \neq 0$. We can thus assume that $e^{-T\boldsymbol{X}_0}(\zeta) \in \pi^{-1}(\mathcal{M}_e)$. By Lemma 5.8, we get that $m_0(e^{-3T\boldsymbol{X}_0}(\zeta)) = 0 = m_0(e^{-T\boldsymbol{X}_0}(\zeta))$. Lemma 5.8 also gives us that $m_0(e^{-t\boldsymbol{X}}(\zeta)) = 0$ for all $t \in [2T, 3T]$. As a consequence,

$$m_1(\zeta) = \frac{1}{2T}\int_T^{3T} m_0(e^{-t\boldsymbol{X}_0}\zeta)\,\mathrm{d}t = \frac{1}{2T}\int_T^{2T} m_0(e^{-t\boldsymbol{X}_0}\zeta)\,\mathrm{d}t < \tfrac{1}{2},$$

so $\left|m_1(\zeta) - \tfrac{1}{2}\right| \neq 0$.

We now assume that

$$m_0(e^{-T\boldsymbol{X}_0}(\zeta)) = 1 = m_0(e^{-3T\boldsymbol{X}_0}(\zeta)).$$

We claim that $m_0(e^{-t\boldsymbol{X}_0}\zeta) = 1$ for all $t \in [T, 2T]$. Indeed, assume that there exists some $t_0 \in [T, 2T]$ such that $\zeta_0 := e^{-t_0\boldsymbol{X}_0}(\zeta)$ satisfies $m_0(\zeta_0) < 1$. By Lemma 5.8, since $\zeta_0, e^{T\boldsymbol{X}_0}(\zeta_0) \in S^*\mathcal{M}_e$, we obtain that $m_0(e^{-t\boldsymbol{X}_0}(\zeta_0)) = 0$ for all $t \geq T$. Taking $t_1 := 3T - t_0 \geq T$, we deduce that

$$m_0(e^{-t_1\boldsymbol{X}_0}(\zeta_0)) = 0 = m_0(e^{-(3T-t_0)\boldsymbol{X}_0}e^{-t_0\boldsymbol{X}_0}(\zeta)) = m_0(e^{-3T\boldsymbol{X}_0}(\zeta)),$$

which is a contradiction. We then deduce that

$$m_1(\zeta) > \frac{1}{2T}\int_T^{2T} m_0(e^{-t\boldsymbol{X}_0}(\zeta))\,\mathrm{d}t = \tfrac{1}{2},$$

that is, $\left|m_1(\zeta) - \tfrac{1}{2}\right| \neq 0$. This eventually proves the fourth item. □

We now introduce

$$m_+ := \chi(m_1) \in C^\infty(S^*\mathcal{N}, [0, 1]), \tag{5-3}$$

where $\chi \in C^\infty(\mathbb{R})$ is a smooth cutoff function such that: $\chi' \geq 0$, $\chi = 0$ on $\left(-\infty, -\tfrac{1}{2} - \varepsilon_0\right]$, and $\chi = 1$ on $\left[\tfrac{1}{2} + \varepsilon_0, +\infty\right)$, where $\varepsilon_0 > 0$ is the constant provided by Lemma 5.9. By construction, this function takes value 1 near $(E_+^{\boldsymbol{X}_0})^*$. By the same process, one can also construct a function $m_- \in C^\infty(S^*\mathcal{N}, [0, 1])$ such that $m_- = 1$ near $(E_-^{\boldsymbol{X}_0})^*$.

Lemma 5.10. *There exists $\delta > 0$ small enough that, for all smooth vector fields X with*

$$\|X - X_0\|_{C^2(\mathcal{M}, T\mathcal{M})} < \delta,$$

the functions $m_\pm \in C^\infty(S^*\mathcal{N}, [0, 1])$ satisfy the following properties:

(1) $m_\pm = 1$ *near* $(E_\pm^X)^* \cap \pi^{-1}(\mathcal{M}_e)$.

(2) $\mathrm{supp}(m_\pm) \subset \pi^{-1}(\Sigma_\pm)$ *and* $\mathrm{supp}(m_\pm)$ *is contained in a small neighborhood of* $(E_\pm^X)^*$.

(3) *There exists* $\delta_1 > 0$ *small such that*

$$\mathrm{supp}(m_\pm) \subset \pi^{-1}(\{\rho > -(1-\delta_1)\rho_0\}), \tag{5-4}$$

$$\mathrm{supp}(m_\pm) \cap \pi^{-1}(\mathcal{M}^\complement) \subset \{\pm \widetilde{X}_0 \rho < -\delta_1\}. \tag{5-5}$$

(4) $\pm \boldsymbol{X} m_\pm \geq 0$ *on* $\pi^{-1}(\mathcal{M}_e)$.

We will argue on m_+, as the proof is similar for m_-.

Proof. We prove each item individually.

(1), (2) and (3) These are straightforward to check with $\delta_1 > 0$ small enough. The fact that X and X_0 are C^2-close implies by the structural stability Proposition 2.4 that $\bigcup_{X \sim X_0} (E_\pm^X)^*$ are contained in a small neighborhood of $(E_\pm^{X_0})^*$ where $m_\pm = 1$.

(4) Observe that

$$\boldsymbol{X} m_+ = \boldsymbol{X} m_1 \chi'(m_1) = ((\boldsymbol{X} - \boldsymbol{X}_0) m_1 + \boldsymbol{X}_0 m_1) \chi'(m_1).$$

The nonnegative function $\chi'(m_1) \geq 0$ vanishes everywhere, except on the set $\{|m_1 - \frac{1}{2}| \leq \varepsilon_0\}$. Observe that, on $\{|m_1 - \frac{1}{2}| \leq \varepsilon_0\}$, we have by Lemma 5.9 that

$$(\boldsymbol{X} - \boldsymbol{X}_0) m_1 + \boldsymbol{X}_0 m_1 \geq \delta_0 - \|X - X_0\|_{C^0} \|m_1\|_{C^1} \geq \tfrac{1}{2}\delta_0,$$

provided $\delta \leq \delta_0 / (2\|m_1\|_{C^1})$. As a consequence, we deduce that $\boldsymbol{X} m_+ \geq 0$ on $\pi^{-1}(\mathcal{M}_e)$. ☐

5A3. *Construction of the bump functions* $\chi_\pm$. In this paragraph, we construct the bump functions $\chi_\pm$ involved in the expression (5-1) of the escape function m.

Lemma 5.11. *There exist* $\delta_1, \delta > 0$ *small enough and cutoff functions* $\chi_\pm \in C^\infty(\mathcal{N}, [0, 1])$ *such that, for all smooth vector fields* X *such that* $\|X - X_0\|_{C^1(\mathcal{M}, T\mathcal{M})} < \delta$, *the following hold*:

(1) $\mathrm{supp}(\chi_\pm) \subset \{-2\rho_0 < \rho < -\delta_1\} \cap \{\pm \widetilde{X}_0 \rho < -\delta_1\}$.

(2) $X \chi_\pm \geq 0$.

(3) $X \chi_\pm > \frac{1}{2} \delta_1^3 \rho_0$ *on* $(\{-(1-\delta_1)\rho_0 < \rho < 0\} \cap \{\pm \widetilde{X}_0 \rho < -\delta_1\}) \setminus \mathcal{M}_e$.

Proof. We only deal with χ_+, the proof being similar for χ_-. First of all, for $j = 1, 2$, we define functions $\chi_j \in C^\infty(\mathbb{R})$ depending on some parameter $\delta_1 > 0$, which will be chosen small enough in the end. The function $\chi_1 \in C_c^\infty(\mathbb{R})$ is defined such that (see Figure 5)

- $\mathrm{supp}(\chi_1) \subset \{-2\rho_0 < \rho < -\delta_1\}$,
- $\chi_1 \geq 0$, $\chi_1(-\rho_0) = 1$, $\chi_1'(-\rho_0) = 0$,
- $\chi_1' \geq 0$ on $\{-2\rho_0 < \rho < -\rho_0\}$, $\chi_1' \leq 0$ on $\{-\rho_0 < \rho < -\delta_1\}$,
- $\chi_1' \leq -\delta_1$ on $\{-\rho_0(1-\delta_1) \leq \rho \leq -2\delta_1\}$.

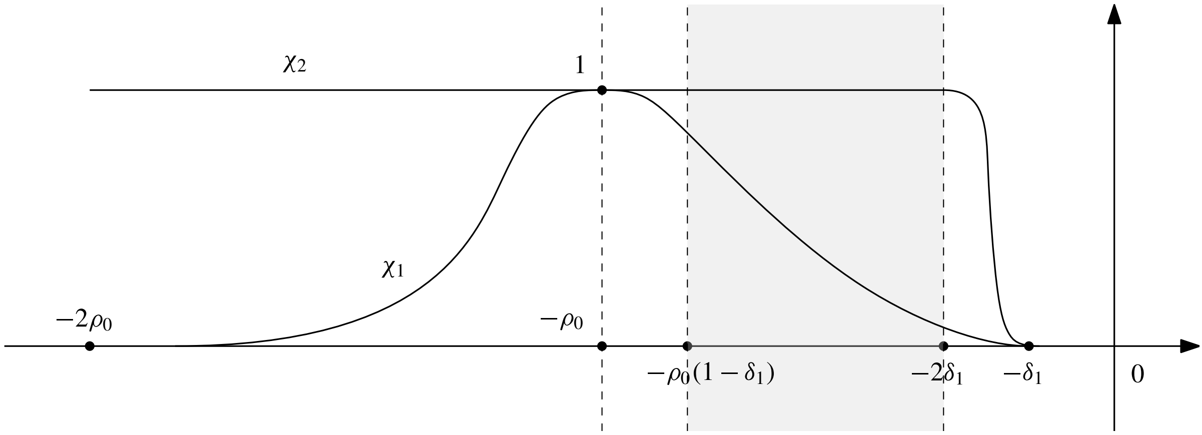

Figure 5. The cutoff functions χ_1 and χ_2.

The function $\chi_2 \in C^\infty(\mathbb{R})$ is defined such that

- $\operatorname{supp}(\chi_2) \subset (-\infty, -\delta_1]$,
- $\chi_2 \geq 0$,
- $\chi_2 = 1$ on $(-\infty, -2\delta_1]$.

We then set

$$\chi_+ := \chi_1(\rho)\chi_2(\widetilde{X}_0\rho), \tag{5-6}$$

and we claim that it satisfies the required properties. Recall from Section 2C3 that $X = \psi\widetilde{X}$, where $\widetilde{X}$ is some smooth extension of the vector field X, initially defined on $\mathcal{M}$ to the closed manifold $\mathcal{N}$.

We now study separately the three terms of

$$\begin{aligned} X\chi_+ &= X\rho\chi_1'(\rho)\chi_2(\widetilde{X}_0\rho) + (X\widetilde{X}_0\rho)\chi_1(\rho)\chi_2'(\widetilde{X}_0\rho) \\ &= \psi\cdot(\widetilde{X}\rho)\chi_1'(\rho)\chi_2(\widetilde{X}_0\rho) + \psi\cdot(\widetilde{X}_0^2\rho)\chi_1(\rho)\chi_2'(\widetilde{X}_0\rho) + \psi\cdot((\widetilde{X}-\widetilde{X}_0)\widetilde{X}_0\rho)\chi_1(\rho)\chi_2'(\widetilde{X}_0\rho). \end{aligned} \tag{5-7}$$

We study the first term in the last line of (5-7). On $\operatorname{supp}(\chi_2(\widetilde{X}_0\rho))$, one has $\widetilde{X}_0\rho \leq -\delta_1$. Thus, assuming $\|X - X_0\|_{C^0(\mathcal{M},T\mathcal{M})} < \delta$ is small enough (depending on δ_1), we obtain that $\widetilde{X}\rho \leq -\frac{1}{2}\delta_1$ on $\operatorname{supp}(\chi_2(\widetilde{X}_0\rho))$. As a consequence, we obtain (note that $\psi\chi_1' \leq 0$)

$$\psi\cdot(\widetilde{X}\rho)\chi_1'(\rho)\chi_2(\widetilde{X}_0\rho) \geq -\frac{\delta_1\psi}{2}\chi_1'(\rho)\chi_2(\widetilde{X}_0\rho) \geq 0.$$

Moreover, on the set $\{-(1-\delta_1)\rho_0 < \rho < -2\delta_1\} \cap \{\widetilde{X}_0\rho < -\delta_1\}$, using that $\psi = \rho + \rho_0$ near $\{\rho = -\rho_0\}$ (so $\psi \geq \delta_1\rho_0$ on the former set) and that $\chi_1'(\rho) \leq -\delta_1$, we obtain that this can be bounded from below by:

$$\psi\cdot(\widetilde{X}\rho)\chi_1'(\rho)\chi_2(\widetilde{X}_0\rho) \geq \frac{\delta_1^2\psi}{2} \geq \frac{\delta_1^3\rho_0}{2} > 0. \tag{5-8}$$

We now deal with the second and third term. The strict convexity property of the level sets $\{\rho = c\}$ (for $c \in [-2\rho_0, 0]$) with respect to $\widetilde{X}_0$ reads: $\widetilde{X}_0\rho = 0 \Rightarrow \widetilde{X}_0^2\rho < 0$. Since $\{\widetilde{X}_0\rho = 0\} \cap \{-2\rho_0 \leq \rho \leq 0\}$ is compact, we deduce that there exists $\delta_1 > 0$ small enough such that, on the set $\{|\widetilde{X}_0\rho| \le 2\delta_1\}$, one has $\widetilde{X}_0^2\rho \le -c < 0$ for some constant $c = c(\delta_1) > 0$. Using that $\mathrm{supp}(\chi_2'(\widetilde{X}_0\rho))$ has support in $\{|\widetilde{X}_0\rho| \le 2\delta_1\}$ and assuming $\|X - X_0\|_{C^0(\mathcal{M},T\mathcal{M})} \le \delta$, we obtain the existence of some constant $C > 0$ (depending on δ_1 but independent of $\delta > 0$) such that

$$\psi \cdot (\widetilde{X}_0^2\rho)\chi_1(\rho)\chi_2'(\widetilde{X}_0\rho) + \psi \cdot ((\widetilde{X} - \widetilde{X}_0)\widetilde{X}_0\rho)\chi_1(\rho)\chi_2'(\widetilde{X}_0\rho) \ge (C\delta - c)\psi\chi_1(\rho)\chi_2'(\widetilde{X}_0\rho).$$

Taking $\delta \le c/(2C)$ small enough (depending on $\delta_1 > 0$), we obtain that this last term is nonnegative.

Overall, we have thus proved (1) and (2), and (3) directly follows from (2) together with (5-8), since we can take $\delta_1 > 0$ small enough that $\{\rho \ge -2\delta_1\} \subset \mathcal{M}_e$. □

5A4. *Piecing together the functions.* The various sets appearing in the previous constructions and the functions $m_\pm$, $\chi_\pm$ can be seen in Figure 4. We now piece together the previous constructions and prove Proposition 5.1.

Proof of Proposition 5.1. Define m by (5-1), where $m_\pm$ and $\chi_\pm$ are provided by Lemmas 5.10 and 5.11, and the constant $\delta_1 > 0$ is chosen small enough that both Lemmas 5.10 and 5.11 hold.

Since $\chi_\pm$ have support outside of $\mathcal{M}$, $m_\pm = 1$ near $(E_\pm^X)^* \cap \pi^{-1}(\mathcal{M})$, and $m = m_- - m_+$ on $\pi^{-1}(\mathcal{M})$, we get that points (1), (2) and (3) are verified. The fact that $\mathrm{supp}(m) \subset \{\rho > -2\rho_0\}$ is also straightforward by Lemmas 5.10 and 5.11, which proves (4). Eventually, (5) is also immediate to verify.

We now show that (6) holds if we take $\eta > 0$ small enough. By Lemmas 5.10 (4) and 5.11 (2), the condition $\boldsymbol{X}m \le 0$ holds on $\pi^{-1}(\mathcal{M}_e)$. On the set $\{\rho \le -\rho_0(1-\delta_1)\}$, we have $m_\pm = 0$, and thus, by Lemma 5.11, the inequality $\boldsymbol{X}m \le 0$ also holds. It remains to check the inequality on $\{\rho \ge -\rho_0(1-\delta_1)\} \cap (\mathcal{M}_e)^\complement$. But there, we have, by Lemma 5.11 (3),

$$\boldsymbol{X}m = \boldsymbol{X}m_- - \boldsymbol{X}m_+ + \eta^{-1}(\pi^*X\chi_- - \pi^*X\chi_+) \le \|m_-\|_{C^1} + \|m_+\|_{C^1} - \eta^{-1}\frac{\delta_1^3\rho_0}{2} \le 0$$

if $\eta > 0$ is chosen small enough. □

5B. *Meromorphic extension of the resolvent.* We now study the meromorphic extension of the resolvent on anisotropic Sobolev spaces and its dependence with respect to the vector field X. This is the main difference with [Dyatlov and Guillarmou 2016]. We will be particularly interested by the resolvent at $z = 0$, namely R_g, for our application.

5B1. *Global resolvent on uniform anisotropic Sobolev spaces.* In the following, we assume that an arbitrary metric h was chosen on $T\mathcal{N} \to \mathcal{N}$. This induces a metric $h^\sharp$ on $T^*\mathcal{N} \to \mathcal{N}$ and, for $(y,\xi) \in T^*\mathcal{N}$, we will write $\langle\xi\rangle := (1 + h_y^\sharp(\xi,\xi))^{1/2}$ (the y is dropped from the Japanese bracket notation in order to avoid repetition). For $\varrho \in \left(\frac{1}{2}, 1\right]$, we denote by $S_\varrho^k(T^*\mathcal{N})$ the Fréchet space of symbols of order k, i.e., $a \in S^k(T^*\mathcal{N})$, if, in local coordinates,

$$\text{for all } \alpha, \beta, \quad \text{there exists } C > 0 \text{ such that} \quad |\partial_\xi^\alpha \partial_x^\beta a(y,\xi)| \le C\langle\xi\rangle^{k-\varrho|\alpha|+(1-\varrho)|\beta|},$$

and we denote by $\Psi_\varrho^k(\mathcal{N})$ the space of pseudodifferential operators of order k obtained by quantization of symbols in $S_\varrho^k(T^*\mathcal{N})$. We shall remove the ϱ index from the notation when $\varrho = 1$. Note that k can be a real number but also a variable *order function*; see [Faure et al. 2008, Appendix A] for further details.

The function $m \in C^\infty(S^*\mathcal{N}, [-1, 1])$ constructed in Section 5A yields a smooth, 0-homogeneous function $m \in C^\infty(T^*\mathcal{N} \setminus \{0\}, [-1, 1])$ — still denoted by m — which decreases along all flow lines of $\boldsymbol{X}$, the Hamiltonian vector field induced by X (and X is close to X_0). We can always modify m in a small neighborhood of the 0-section in $T^*\mathcal{N}$ to obtain a new function — still denoted by the same letter m to avoid unnecessary notation — such that $m \in C^\infty(T^*\mathcal{N}, [-1, 1])$ and $\boldsymbol{X}m(y, \xi) \le 0$ for all $(y, \xi) \in T^*\mathcal{N}$ such that $\langle \xi \rangle > 1$.

Define a *regularity pair* as a pair of indices $\boldsymbol{r} := (r_\perp, r_0)$, where $r_\perp > r_0 \ge 0$. Given such a regularity pair $\boldsymbol{r}$, we introduce (for all $\varepsilon > 0$ small enough)

$$A_{\boldsymbol{r}} := \mathrm{Op}(\langle \xi \rangle^{(r_\perp m(y,\xi) - r_0)/2})^* \, \mathrm{Op}(\langle \xi \rangle^{(r_\perp m(y,\xi) - r_0)/2}) \in \Psi^{r_\perp m - r_0}_{1-\varepsilon}(\mathcal{N}). \tag{5-9}$$

This is an elliptic and formally selfadjoint pseudodifferential operator belonging to an *anisotropic class*; see [Faure et al. 2008, Appendix A] for further details. As a consequence, up to a modification by a finite-rank formally selfadjoint smoothing operator, we can assume that $A_{\boldsymbol{r}}$ is invertible.

Definition 5.12. We define the *scale of anisotropic Sobolev spaces* with regularity $\boldsymbol{r} := (r_\perp, r_0)$, where $r_\perp > r_0 \ge 0$, as

$$\mathcal{H}^{\boldsymbol{r}}_\pm(\mathcal{N}) := A_{\boldsymbol{r}}^{\mp 1}(L^2(\mathcal{N})), \quad \|f\|_{\mathcal{H}^{\boldsymbol{r}}_\pm(\mathcal{N})} := \|A_{\boldsymbol{r}}^{\pm 1} f\|_{L^2(\mathcal{N})}.$$

Remark 5.13. (1) The spaces $\mathcal{H}^{\boldsymbol{r}}_\pm(\mathcal{N})$ are Hilbert spaces, equipped with the scalar product

$$\langle \cdot\,, \cdot \rangle_{\mathcal{H}^{\boldsymbol{r}}_\pm(\mathcal{N})} := \langle A_{\boldsymbol{r}}^{\pm 1} \cdot\,, A_{\boldsymbol{r}}^{\pm 1} \cdot \rangle_{L^2(\mathcal{N})}.$$

(2) This scale of spaces is *independent* of the vector field X, as long as it is close enough to X_0 in the C^2-topology, since the escape function m is independent of the vector field. This will be important when studying the regularity of the meromorphic extension of the resolvent $z \mapsto R^X_\pm(z)$ (given by (5-10)) with respect to the vector field X.

(3) Distributions in $\mathcal{H}^{\boldsymbol{r}}_+(\mathcal{N})$ are microlocally in $H^{r_\perp - r_0}(\mathcal{N})$ near $(E^X_-)^*$, $H^{-r_0}(\mathcal{N})$ near $(E^X_0)^*$, and $H^{-r_\perp - r_0}(\mathcal{N})$ near $(E^X_+)^*$ (in the sense that, after application of an $A \in \Psi^0(\mathcal{N})$ with wavefront set in the discussed region, they have the announced regularity). The choice of regularity is arbitrary here, and we did not try to optimize it. The only crucial point is that distributions in $\mathcal{H}^{\boldsymbol{r}}_+(\mathcal{N})$ have positive Sobolev regularity near $(E^X_-)^*$, while they have negative Sobolev regularity near $(E^X_+)^*$.

We let $q \in C^\infty(\mathcal{N}, [0, 1])$ be a smooth cutoff function such that

- $\mathrm{supp}(q)$ is contained in the complement of a small open neighborhood of $\mathcal{M}$,
- $q = 1$ on the complement of some slightly larger open neighborhood of $\mathcal{M}$,
- the closure of the set $\{q < 1\}$ is strictly convex with respect to all the vector fields X for $\|X - X_0\|_{C^2} \le \delta$ small enough.

Given a regularity pair $\boldsymbol{r} := (r_\perp, r_0)$ and a constant $\omega > 0$, we define, for X close enough to X_0 and $\Re(z) \gg 0$ large enough,

$$R^X_\mp(z) := -\int_0^{+\infty} e^{-tz} e^{-\omega \int_0^t (\varphi^X_{\mp s})^* q \, \mathrm{d}s} e^{\mp tX} \, \mathrm{d}t, \tag{5-10}$$

Although we do not indicate it in the notation, $R^X_\mp(z)$ *does* depend on a choice of ω. This satisfies the identity on $C^\infty(\mathcal{N})$:

$$(\mp X - z - \omega q)R^X_\mp(z) = \mathbf{1}_{\mathcal{N}}.$$

The constant $\omega > 0$ will be fixed later.

The aim of this section is to study the meromorphic extension of the resolvent $z \mapsto R^X_+(z)$ for X close to X_0 in the anisotropic Sobolev spaces of Definition 5.12, and the dependence with respect to the vector field X.

Theorem 5.14. *There exists $C_\star, \delta_\star, \Lambda > 0$ such that the following holds. For all $\delta \le \delta_\star$, for all regularity pairs $\boldsymbol{r} = (r_\perp, r_0)$, there exists a choice of constant $\omega := \omega(\boldsymbol{r}) > 0$ large enough that, for all smooth vector fields X on $\mathcal{M}$ such that $\|X - X_0\|_{C^2(\mathcal{M},T\mathcal{M})} \le \delta$, the family*

$$z \mapsto R^X_-(z) = (-X - z - \omega(\boldsymbol{r})q)^{-1} \in \mathcal{L}(\mathcal{H}^{\boldsymbol{r}}_+),$$

initially defined for $\Re(z) \gg 1$ by (5-10) and holomorphic for $\Re(z) \gg 1$ large enough, extends to a meromorphic family of operators on the half-space $\{\Re(z) > -\Lambda(r_\perp - r_0) + C_\star\delta\}$. The same holds for $R^X_+(z)$ on the space $\mathcal{H}^{\boldsymbol{r}}_-$.

Moreover, if $z_0 \in \{\Re(z) > -\Lambda(r_\perp - (r_0 + 2)) + C_\star\delta\}$ is not a pole of $z \mapsto R^{X_0}(z)$, then there exists $\varepsilon_0 > 0$ such that the map

$$C^\infty(\mathcal{N}, T\mathcal{N}) \times D(z_0, \varepsilon_0) \ni (X, z) \mapsto R^X(z) \in \mathcal{L}(\mathcal{H}^{(r_\perp,r_0)}_+, \mathcal{H}^{(r_\perp,r_0+2)}_+)$$

is C^2-regular[5] *with respect to X and holomorphic in z, where $D(z_0, \varepsilon_0) \subset \mathbb{C}$ is the disk centered at z_0 of radius ε_0.*

As usual, the poles do not depend on the choices made in the construction of the spaces. The rest of Section 5B is devoted to the proof of Theorem 5.14. We note that Theorem 5.14 obviously implies Theorem 1.10 stated in the introduction, since the resolvent on $\mathcal{M}$ can be expressed in terms of the resolvent on $\mathcal{N}$ and the restriction to $\mathcal{M}$ (as in Lemma 5.21 below in the analogous case of geodesic vector fields).

5B2. *Parametrix construction.* Denote by μ a smooth measure on $\mathcal{N}$ which restricts to the Liouville measure on $\mathcal{M}$. Note that X_0 is volume-preserving on $\mathcal{M}$ and, up to minor modifications, we can also assume that the extension of X_0 to $\mathcal{N}$ is volume-preserving on $\mathcal{M}_e$ (but not on $\mathcal{N}$, since X_0 vanishes on $\{\rho = -\rho_0\}$). In order to shorten notation, we will write $L^2(\mathcal{N}) := L^2(\mathcal{N}, \mu)$.

For $T > 0$, consider a smooth cutoff function $\chi_T \in C^\infty_c(\mathbb{R}_+)$, depending smoothly on T, such that $\chi_T = 1$ on $[0, T]$, $-2 \le \chi'_T \le 0$, and $\chi_T = 0$ on $[T + 1, \infty)$. For $\Re(z) \gg 1$ and $\omega \ge 1$, the following identity holds on $C^\infty(\mathcal{N})$:

$$-\int_0^{+\infty} \chi_T(t)e^{-tz}e^{-\int_0^t (\varphi^X_{-s})^*(\omega q)\,\mathrm{d}s}e^{-tX}\,\mathrm{d}t(-X - z - \omega q) = \mathbb{1} + \int_0^{+\infty} \chi'_T(t)e^{-tz}e^{-\int_0^t (\varphi^X_{-s})^*(\omega q)\,\mathrm{d}s}e^{-tX}\,\mathrm{d}t. \quad (5\text{-}11)$$

[5] Even though we only need C^2, our proof actually shows it is C^k for all $k \in \mathbb{N}$.

We now *fix* once and for all a regularity pair $\boldsymbol{r} := (r_\perp, r_0)$ and set $r := r_0 + r_\perp$. The constant $\omega \geq 1$ will be chosen to depend on $\boldsymbol{r}$ later. We conjugate the equality (5-11) by $A_{\boldsymbol{r}}$. We obtain

$$-A_{\boldsymbol{r}} \int_0^{+\infty} \chi_T(t) e^{-tz} e^{-\int_0^t (\varphi_{-s}^X)^*(\omega q)\,\mathrm{d}s} e^{-tX} A_{\boldsymbol{r}}^{-1}\,\mathrm{d}t\, A_{\boldsymbol{r}}(-X - z - \omega q) A_{\boldsymbol{r}}^{-1}$$
$$= \mathbb{1} + \int_0^{+\infty} \chi_T'(t) e^{-tz} e^{-tX} \underbrace{e^{tX} A_{\boldsymbol{r}} e^{-\int_0^t (\varphi_{-s}^X)^*(\omega q)\,\mathrm{d}s} A_{\boldsymbol{r}}^{-1} e^{-tX}}_{:= B_1^X(t)} \underbrace{e^{tX} A_{\boldsymbol{r}} e^{-tX} A_{\boldsymbol{r}}^{-1}}_{:= B_2^X(t)}\,\mathrm{d}t. \tag{5-12}$$

Since the second term on the right-hand side of (5-12) is defined as an integral over time in the flow direction e^{-tX}, it is smoothing outside $\{p_X = 0\}$. We let $\Omega' \Subset \Omega$ be two open nested conic neighborhoods of $\{p_{X_0} = 0\}$ in $T^*\mathcal{N} \cap \{\rho > -\rho_0\}$. Note that, by continuity, these are also conic neighborhoods of $\{p_X = 0\}$ for all $X \sim X_0$. We let $e \in S^0(T^*\mathcal{N})$ be a symbol of order 0 such that $e = 0$ outside Ω and $e = 1$ on Ω', and we set $E := \mathrm{Op}(e)$. We then decompose the second term on the right-hand side of (5-12) as

$$\int_0^{+\infty} \chi_T'(t) e^{-tz} e^{-tX} B_1^X(t) B_2^X(t)\,\mathrm{d}t = \int_0^{+\infty} \chi_T'(t) e^{-tz} e^{-tX} E B_1^X(t) B_2^X(t)\,\mathrm{d}t + K_1^X(T, z), \tag{5-13}$$

where

$$K_1^X(T, z) := \int_0^{+\infty} \chi_T'(t) e^{-tz} e^{-tX} (\mathbb{1} - E) B_1^X(t) B_2^X(t)\,\mathrm{d}t$$

and $K_1^X(T, z) \in \Psi^{-\infty}(\mathcal{N})$. In order to prove that $K_1^X(T, z)$ is smoothing, we remark that $K_1^X(T, z) = E' K_1^X(T, z)$ for some $E' \in \Psi^0(\mathcal{N})$ with microsupport that does not intersect a conic neighborhood of $\{p_X = 0\}$, and then show that $X^k K_1^X(T, z) \in \mathcal{L}(L^2)$ for all $k \in \mathbb{N}$, using that $X^k e^{-tX} = (-\partial_t)^k e^{-tX}$ and integrating by parts in t, and finally use that $E'(C - X^2)^{-1} \in \Psi^{-2}(\mathcal{N})$ for some $C \gg 1$ since $C - X^2$ is elliptic on the microsupport of E'. The dependence of $K_1^X(T, z)$ on its parameters is holomorphic in $z \in \mathbb{C}$ and smooth in the variables $T \in \mathbb{R}$ and $X \in C^\infty(\mathcal{M}, T\mathcal{M})$.

Below, we use the notation $\mathcal{L}(\mathcal{H})$ to denote continuous linear operators on a Hilbert space $\mathcal{H}$, and $\mathcal{K}(\mathcal{H})$ for compact operators.

Proposition 5.15. *There exist $C_\star, \delta_\star, \Lambda > 0$ such that the following holds. For all regularity pairs $\boldsymbol{r}$, there exist $C(\boldsymbol{r}), \omega(\boldsymbol{r}) > 0$ such that, for all smooth vector fields $\|X - X_0\|_{C^2} \leq \delta$ with $\delta \leq \delta_\star$, for all $t \geq 0$, there exist (Fourier integral) operators $M^X(t) \in \mathcal{L}(L^2(\mathcal{N}))$ and $S^X(t) \in \mathcal{K}(L^2(\mathcal{N}))$ such that*

$$e^{-tX} E B_1^X(t) B_2^X(t) = M^X(t) + S^X(t)$$

and

$$\|M^X(t)\|_{L^2(\mathcal{N})} \leq C(\boldsymbol{r}) e^{(-\Lambda(r_\perp - r_0) + C_\star \delta)t}.$$

Moreover, the map

$$\mathbb{R} \times C^\infty(\mathcal{M}, T\mathcal{M}) \ni (t, X) \mapsto (M^X(t), S^X(t)) \in \mathcal{L}(L^2(\mathcal{N})) \times \mathcal{K}(L^2(\mathcal{N}))$$

is smooth.

The rest of this paragraph is devoted to the proof of Proposition 5.15. It is split into several sublemmas. Given a regularity pair $\boldsymbol{r} = (r_\perp, r_0)$, in order to simplify notation we introduce

$$m_{\boldsymbol{r}} := r_\perp m - r_0. \tag{5-14}$$

Lemma 5.16. *For all $t \in \mathbb{R}$ and $\frac{1}{2} < \varrho < 1$, we have $B_1^X(t), B_2^X(t) \in \Psi_\varrho^0(\mathcal{N})$ with principal symbols*

$$\sigma_{B_1^X(t)}(y,\xi) = e^{-\omega \int_0^t (\varphi_s^X)^*(q)(y)\,\mathrm{d}s}, \quad \sigma_{B_2^X(t)}(y,\xi) = \frac{\langle e^{t\boldsymbol{X}}(y,\xi)\rangle^{m_{\boldsymbol{r}}(e^{t\boldsymbol{X}}(y,\xi))}}{\langle \xi\rangle^{m_{\boldsymbol{r}}(y,\xi)}}.$$

Proof. This follows directly from Egorov's lemma; see [Lefeuvre 2019b, Section 2.4.1]. □

In particular, Lemma 5.16 shows that the integrand $e^{-tX} B_1^X(t) B_2^X(t)$ on the right-hand side of (5-12) is a Fourier integral operator (FIO). We let

$$a^X(t)(y) := |\det d\varphi_{-t}^X(\varphi_t^X(y))|^{-1/2}, \tag{5-15}$$

where the Jacobian is defined with respect to the measure $\mathrm{d}\mu$ on $\mathcal{N}$.

Lemma 5.17. *For all $t \in \mathbb{R}$, we have $\|e^{-tX}(a^X(t))^{-1}\|_{\mathcal{L}(L^2(\mathcal{N}))} = 1$. Moreover, for all $y \in \mathcal{N}$ and $t \in \mathbb{R}$,*

$$a^X(t)(y) \le \exp\left(\int_0^t |\mathrm{div}_\mu X|(\varphi_s^X(y))\,\mathrm{d}s\right).$$

Proof. We have

$$\int_{\mathcal{N}} |e^{-tX}((a^X(t))^{-1} f)|^2 \,\mathrm{d}\mu = \int_{\mathcal{N}} (a^X(t))^{-2} |f|^2 |\det d\varphi_t^X| \,\mathrm{d}\mu = \|f\|_{L^2}^2.$$

The estimate on $a^X(t)(y)$ follows directly from the fact that $\mathrm{div}_\mu X \circ \varphi_t = \partial_t(\log|\det d\varphi_t^X|)$. □

By Lemma 5.16, the operator $a^X(t) E B_1^X(t) B_2^X(t)$ is a pseudodifferential operator of order 0. By the Calderón–Vaillancourt theorem [Grigis and Sjöstrand 1994, Theorem 4.5], up to a compact remainder in $\mathcal{K}(L^2(\mathcal{N}))$, its norm on $L^2(\mathcal{N})$ is given by the lim sup of its principal symbol as $|\xi| \to \infty$. We now bound the lim sup of its principal symbol.

Lemma 5.18. *There exists $\delta_\star, C_\star, \Lambda > 0$ such that the following holds. For all regularity pairs $\boldsymbol{r} := (r_\perp, r_0)$, there exists $C(\boldsymbol{r}), \omega(\boldsymbol{r}) > 0$ such that, for all smooth vector fields X with $\|X - X_0\|_{C^2(\mathcal{M}, T\mathcal{M})} \le \delta$, where $\delta \le \delta_\star$, for all $t \ge 0$,*

$$\limsup_{(y,\xi)\in T^*\mathcal{N}, |\xi|\to\infty} \sigma_{a^X(t) E B_1^X(t) B_2^X(t)}(y,\xi) \le C(\boldsymbol{r}) e^{(-\Lambda(r_\perp - r_0) + C_\star \delta)t}.$$

Proof. For $(y,\xi) \in T^*\mathcal{N}$, we have, by Lemma 5.16,

$$\sigma_{a^X(t) E B_1^X(t) B_2^X(t)}(y,\xi) = e(y,\xi) \exp\left(\int_0^t \left(\frac{1}{2}\,\mathrm{div}_\mu X - \omega q\right)(e^{sX}(y))\,\mathrm{d}s\right) \frac{\langle e^{t\boldsymbol{X}}(y,\xi)\rangle^{m_{\boldsymbol{r}}(e^{t\boldsymbol{X}}(y,\xi))}}{\langle \xi\rangle^{m_{\boldsymbol{r}}(y,\xi)}}. \tag{5-16}$$

Modulo the term $e(y,\xi) \le 1$, which we can neglect, this is a *cocycle* over the flow of $\boldsymbol{X}$, as it satisfies the relation

$$\sigma_{B_1^X(t')B_2^X(t')}(e^{t\boldsymbol{X}}(y,\xi))\sigma_{B_1^X(t)B_2^X(t)}(y,\xi) = \sigma_{B_1^X(t'+t)B_2^X(t'+t)}(y,\xi) \tag{5-17}$$

for all $t, t' \in \mathbb{R}$.

First, we need the following lemma.

Lemma 5.19. *For all regularity pairs $\boldsymbol{r} = (r_\perp, r_0)$, there exist constants $C(\boldsymbol{r}), \omega(\boldsymbol{r}) > 0$ such that, for all $(y, \xi) \in T^*\mathcal{N}$, $\omega > \omega(\boldsymbol{r})$ and for all $t \geq 0$,*

$$\{e^{sX}(y, \xi) \mid s \in [0, t]\} \subset \pi^{-1}(\{q = 1\}) \quad \Longrightarrow \quad \limsup_{(y,\xi)\in T^*\mathcal{N}, |\xi|\to\infty} \sigma_{a^X(t)EB_1^X(t)B_2^X(t)}(y, \xi) \leq C(\boldsymbol{r})e^{-rt},$$

where $r := r_\perp + r_0$.

Proof. Define $\nu := \sup_{\|X-X_0\|_{C^2}\leq\delta} \| \operatorname{div}_\mu X\|_{L^\infty(\mathcal{N})}$. We have, if $q(\varphi_s(x)) = 1$ for $s \in [0, t]$,

$$\begin{aligned}\sigma_{a^X(t)EB_1(t)B_2(t)}(y, \xi) &\leq e^{\nu t} e^{-\omega t} \frac{\langle e^{tX}(y, \xi)\rangle^{m_{\boldsymbol{r}}(e^{tX}(y,\xi))}}{\langle \xi\rangle^{m_{\boldsymbol{r}}(y,\xi)}} \\ &= e^{(\nu-\omega)t} \langle e^{tX}(y, \xi)\rangle^{m_{\boldsymbol{r}}(e^{tX}(y,\xi))-m_{\boldsymbol{r}}(y,\xi)} \left(\frac{\langle e^{tX}(y, \xi)\rangle}{\langle\xi\rangle}\right)^{m_{\boldsymbol{r}}(y,\xi)}.\end{aligned}$$

By construction, $m_{\boldsymbol{r}}$ is nonincreasing along the flow lines of $\boldsymbol{X}$ outside a neighborhood of the 0-section in $T^*\mathcal{N}$; see Proposition 5.1 (6). This implies that

$$\limsup_{(y,\xi)\in T^*\mathcal{N}, |\xi|\to\infty} \langle e^{tX}(y, \xi)\rangle^{m_{\boldsymbol{r}}(e^{tX}(y,\xi))-m_{\boldsymbol{r}}(y,\xi)} \leq 1.$$

Moreover, there exist a uniform exponent $\lambda > 0$ and $C > 0$ (depending only on X_0) such that, for all $X \sim X_0$, for all $t \geq 0$ and $(y, \xi) \in T^*\mathcal{N}$, one has

$$\langle e^{tX}(y, \xi)\rangle \leq Ce^{\lambda t}\langle\xi\rangle. \tag{5-18}$$

Using (5-18) and taking the lim sup as $|\xi| \to \infty$, we then obtain

$$\limsup_{(y,\xi)\in T^*\mathcal{N}, |\xi|\to\infty} \sigma_{a^X(t)EB_1^X(t)B_2^X(t)}(y, \xi) \leq C(\boldsymbol{r})e^{(\nu-\omega+r\lambda)t}.$$

Taking $\omega(\boldsymbol{r}) := \nu + r + r\lambda$, we obtain the announced result. □

From now on, given a regularity pair $\boldsymbol{r}$, the constant ω in (5-12) will always be taken to be fixed, equal to $\omega := \omega(\boldsymbol{r}) > 0$ provided by Lemma 5.19. Next we need the following lemma.

Lemma 5.20. *There exists $C_\star, \Lambda_1 > 0$ such that the following holds. For all regularity pairs $\boldsymbol{r}$, there exists a constant $C(\boldsymbol{r}) > 0$ such that, for all X with $\|X - X_0\|_{C^2} \leq \delta$ and $(y, \xi) \in T^*\mathcal{N}$, for all $t \geq 0$,*

$$(y, \xi), e^{tX}(y, \xi) \in T^*\mathcal{M}_e \quad \Longrightarrow \quad \limsup_{|\xi|\to\infty} \sigma_{a^X(t)EB_1^X(t)B_2^X(t)}(y, \xi) \leq C(\boldsymbol{r})e^{(-\Lambda_1(r_\perp-r_0)+C_\star\delta)t}.$$

Proof. We start with a preliminary observation: there exists a constant $C_\star > 0$ such that, if $y, \varphi_t^X(y) \in \mathcal{M}_e$ and $\|X - X_0\|_{C^2(\mathcal{M},T\mathcal{M})} \leq \delta$, then

$$a^X(t)(y) \leq e^{C_\star\delta t}. \tag{5-19}$$

This simply follows from the fact that X_0 is volume-preserving on $\mathcal{M}_e$ (that is, $a^{X_0}(t) = 1$).

We now consider the sets $U_\pm$ given by Lemma 5.4. These sets can always be constructed so that $U_\pm \subset \{m = \pm 1\}$. We also consider the sets $V_\pm$ given by Lemma 5.5. Denote by $T > 0$ the time provided by Lemma 5.6. If $t \leq T$, namely if the time is uniformly bounded, then the claim is immediate as $a^X(t)EB_1^X(t)B_2^X(t)$ is of order 0 by Lemma 5.16 and depends continuously on time. If $t \geq T$ and

(y,ξ), $e^{tX}(y,\xi)\in T^*\mathcal{M}_e\cap \mathrm{WF}(E)$, then the flow line $\{e^{sX}(y,\xi)\mid s\in[0,t]\}$ passes at least a time $t-T$ in $U_+\sqcup U_-$. We can thus introduce $0\le s_0<s_1\le t$ such that, for all $s\in[0,s_0]$, we have $e^{sX}(y,\xi)\in U_-$, for all $s\in[s_1,t]$, we have $e^{sX}(y,\xi)\in U_+$, and we have $s_0+(t-s_1)\ge t-T$. Hence, using the cocycle relation (5-17) and $\sigma_E\in[0,1]$,

$$\begin{aligned}\sigma_{a^X(t)EB_1^X(t)B_2^X(t)}(y,\xi)&\le \sigma_{a^X(t-s_1)B_1^X(t-s_1)B_2^X(t-s_1)}(e^{s_1X}(y,\xi))\\&\quad\cdot\sigma_{a^X(s_1-s_0)B_1^X(s_1-s_0)B_2^X(s_1-s_0)}(e^{s_0X}(y,\xi))\cdot\sigma_{a^X(s_0)B_1^X(s_0)B_2^X(s_0)}(y,\xi).\end{aligned}\tag{5-20}$$

Note that it suffices to bound the terms on the right-hand side of (5-20) on $\mathrm{WF}(E)$, that is, on a conic neighborhood of $\bigcup_{X\sim X_0}\{p_X=0\}$, since otherwise $\sigma_E=0$ and the symbol on the left-hand side vanishes.

Since $s_1-s_0\le T$ (independent of t) and $\sigma_{B_1^X(t)B_2^X(t)}\in\Psi^0_\varrho(\mathcal{N})$ for all $t\ge 0$ by Lemma 5.16, we get that the middle term in (5-20) is bounded uniformly by some constant, that is,

$$\sigma_{a^X(s_1-s_0)B_1(s_1-s_0)B_2(s_1-s_0)}(e^{s_0X}(y,\xi))\le C(\boldsymbol{r})\tag{5-21}$$

for some $C(\boldsymbol{r})>0$ which is independent of the point $(y,\xi)\in T^*\mathcal{N}$ and of the time t. As to the third factor in (5-20), we have, using that $m_{\boldsymbol{r}}=r_\perp-r_0$ on U_-, that q vanishes in $\mathcal{M}$, and (5-19),

$$\begin{aligned}\sigma_{a^X(s_0)B_1(s_0)B_2(s_0)}(y,\xi)&\le e^{C_\star\delta s_0}e^{-\int_0^{s_0}\omega(\boldsymbol{r})q(e^{sX}(y))\,ds}\frac{\langle e^{s_0X}(y,\xi)\rangle^{m_{\boldsymbol{r}}(e^{s_0X}(y,\xi))}}{\langle\xi\rangle^{m_{\boldsymbol{r}}(y,\xi)}}\\&\le C(\boldsymbol{r})e^{C_\star\delta s_0}\left(\frac{\langle e^{s_0X}(y,\xi)\rangle}{\langle\xi\rangle}\right)^{r_\perp-r_0}.\end{aligned}\tag{5-22}$$

Using the uniform contraction rate on U_- of Lemma 5.4, we get that $|e^{s_0X}(y,\xi)|\le Ce^{-\lambda s_0}|\xi|$ for some uniform constants $C,\lambda>0$ depending only on X_0. Taking the lim sup as $|\xi|\to\infty$ in (5-22), we thus obtain

$$\limsup_{|\xi|\to\infty}\sigma_{a^X(s_0)B_1(s_0)B_2(s_0)}(y,\xi)\le C(\boldsymbol{r})e^{C_\star\delta s_0}e^{-\lambda s_0(r_\perp-r_0)}.\tag{5-23}$$

Similarly, using the expansion rate on U_+ of Lemma 5.4 and that $m_{\boldsymbol{r}}=-r_\perp-r_0$ on U_+, the first term in (5-20) can be bounded by

$$\limsup_{|\xi|\to\infty}\sigma_{a^X(t-s_1)B_1(t-s_1)B_2(t-s_1)}(e^{s_1X}(y,\xi))\le C(\boldsymbol{r})e^{C_\star\delta(t-s_1)}e^{-\lambda(t-s_1)(r_\perp+r_0)}.\tag{5-24}$$

Taking $\Lambda_1:=\lambda$ and combining (5-21), (5-23), (5-24) in (5-20) completes the proof. □

We can now end the proof of Lemma 5.18. Given $(y,\xi)\in T^*\mathcal{N}$, the flowline of (y,ξ) under e^{tX} can be schematically described by one of the six following possibilities:

$$\{q=1\},\tag{5-25}$$

$$\mathcal{M}_e,\tag{5-26}$$

$$\{q=1\}\to\{0<q<1\}\to\{q=1\},\tag{5-27}$$

$$\{q=1\}\to\{0<q<1\}\to\mathcal{M}_e,\tag{5-28}$$

$$\mathcal{M}_e\to\{0<q<1\}\to\{q=1\},\tag{5-29}$$

$$\{q=1\}\to\{0<q<1\}\to\mathcal{M}_e\to\{0<q<1\}\to\{q=1\}.\tag{5-30}$$

Note that, for any flow line, there is a maximum time, bounded by some uniform constant $T_\star > 0$, spent in the region $\{0 < q < 1\}$. As a consequence, if the flowline of (y, ξ) falls into one of the cases (5-25) or (5-27), we get, using the cocycle relation (5-17) and Lemma 5.19,

$$\limsup_{|\xi|\to\infty} \sigma_{a^X(t)EB_1(t)B_2(t)}(y, \xi) \le C(\boldsymbol{r})e^{-rt}.$$

As to (5-26), (5-28), (5-29), the bound is obtained similarly to the bound for (5-30), which we now study.

So we assume that the flowline γ of (y, ξ) under e^{tX} passes successively through the six sets of (5-30). Define the times $s_0, s_1 \ge 0$ such that,

$$\begin{aligned} &\text{for all } s \in [0, s_0], && \varphi_s^X(y) \in \{q = 1\},\\ &\text{for all } s \in [s_0, s_1], && \varphi_s^X(y) \in \{q < 1\} \cup \mathcal{M}_e,\\ &\text{for all } s \in [s_1, t], && \varphi_s^X(y) \in \{q = 1\}. \end{aligned}$$

Combining the cocycle relation (5-17) and Lemmas 5.19 and 5.20, we get, on $\mathrm{WF}(E)$,

$$\begin{aligned} &\limsup_{|\xi|\to\infty} \sigma_{a^X(t)EB_1(t)B_2(t)}(y, \xi)\\ &\quad\le \limsup_{|\xi|\to\infty} \sigma_{a^X(t-s_1)B_1(t-s_1)B_2(t-s_1)}(e^{s_1X}(y, \xi))\\ &\qquad\qquad\cdot \limsup_{|\xi|\to\infty} \sigma_{a^X(s_1-s_0)B_1(s_1-s_0)B_2(s_1-s_0)}(e^{s_0X}(y, \xi)) \cdot \limsup_{|\xi|\to\infty} \sigma_{a^X(s_0)B_1(s_0)B_2(s_0)}(y, \xi)\\ &\quad\le C_r e^{-r(t-s_1)} \cdot C_r e^{(-(r_\perp-r_0)\Lambda_1+C_\star\delta)(s_1-s_0)} \cdot C_r e^{-rs_0} \le C_r e^{(-(r_\perp-r_0)\Lambda+C_\star\delta)t} \end{aligned}$$

by taking $\Lambda := \min(1, \Lambda_1)$. This concludes the proof. ☐

We now complete the proof of Proposition 5.15.

Proof of Proposition 5.15. Write

$$e^{-tX}EB_1(t)B_2(t) = e^{-tX}(a^X(t))^{-1}a^X(t)EB_1(t)B_2(t).$$

By Lemma 5.17, $e^{-tX}(a^X(t))^{-1} \in \mathcal{L}(L^2(\mathcal{N}))$ is unitary. By Lemma 5.18, $a^X(t)EB_1(t)B_2(t)$ is a pseudodifferential operator of order 0 such that

$$\limsup_{(y,\xi)\in T^*\mathcal{N}, |\xi|\to\infty} \sigma_{a^X(t)EB_1(t)B_2(t)}(y, \xi) \le C(\boldsymbol{r})e^{(-(r_\perp-r_0)\Lambda+C_\star\delta)t}.$$

By the Calderón–Vaillancourt theorem [Grigis and Sjöstrand 1994, Theorem 4.5] for pseudodifferential operators, we can thus write

$$a^X(t)EB_1(t)B_2(t) = M_0^X(t) + S_0^X(t),$$

where $M_0^X(t)$ is a pseudodifferential operator of order 0 and $S_0^X(t)$ is smoothing and

$$\|M_0^X(t)\|_{\mathcal{L}(L^2(\mathcal{N}))} \le 2C(\boldsymbol{r})e^{(-(r_\perp-r_0)\Lambda+C_\star\delta)t}.$$

Moreover, it is straightforward to check that these operators can be constructed so that they depend smoothly on the parameters $t \in \mathbb{R}$ and $X \in C^\infty(\mathcal{M}, T\mathcal{M})$ as $a^X(t)$, $B_1(t)$, $B_2(t)$ depend in an explicit (and smooth) fashion on X, and the decomposition in the Calderón–Vaillancourt Theorem depends smoothly on the operator. As a consequence, setting

$$M^X(t) := e^{-tX}(a^X(t))^{-1}M_0^X(t) \quad \text{and} \quad S^X(t) := e^{-tX}(a^X(t))^{-1}S_0^X(t),$$

we have

$$e^{-tX}EB_1(t)B_2(t) = M^X(t) + S^X(t),$$

and this concludes the proof. □

5B3. *Meromorphic extension on the closed manifold.* We now prove Theorem 5.14.

Proof of Theorem 5.14. Step 1: meromorphic extension. Fix $\boldsymbol{r} = (r_\perp, r_0)$ with $r_\perp > r_0$, and consider $z \in \mathbb{C}$ such that $\Re(z) > -\Lambda(r_\perp - r_0) + C_\star\delta$. By Proposition 5.15, we can consider a time $T > 0$ large enough, depending on $\boldsymbol{r}$, so that,

$$\text{for all } t \geq T, \quad e^{-\Re(z)t}\|M^X(t)\|_{\mathcal{L}(L^2(\mathcal{N}))} < \tfrac{1}{6}. \tag{5-31}$$

Using (5-12) and (5-13), we thus obtain

$$\int_0^{+\infty} \chi_T'(t)e^{-tz}e^{-tX}B_1^X(t)B_2^X(t)\,\mathrm{d}t = B^X(z) + K^X(z),$$

where

$$B^X(z) := \int_0^{+\infty} \chi_T'(t)e^{-tz}M^X(t)\,\mathrm{d}t$$

and $K^X(z) \in \Psi^{-\infty}(\mathcal{N})$ is the remainder. It is immediate to check that both $B^X(z)$ and $K^X(z)$ depend holomorphically on z and smoothly on $X \in C^\infty(\mathcal{M}, T\mathcal{M})$ as operators in $\mathcal{L}(L^2(\mathcal{N}))$.

Using that $\|\chi_T'\|_{L^\infty} \leq 2$, we get

$$\|B^X(z)\|_{\mathcal{L}(L^2(\mathcal{N}))} \leq 2\int_T^{T+1} e^{-\Re(z)t}\|M^X(t)\|_{\mathcal{L}(L^2(\mathcal{N}))}\,\mathrm{d}t \leq \tfrac{1}{3} < 1. \tag{5-32}$$

The equality (5-12) then reads

$$-A_{\boldsymbol{r}}\int_0^{+\infty} \chi_T(t)e^{-tz}e^{-\int_0^t(\varphi_{-s}^X)^*(\omega q)\,\mathrm{d}s}e^{-tX}A_{\boldsymbol{r}}^{-1}\,\mathrm{d}t\, \underbrace{A_{\boldsymbol{r}}(-X-z-\omega q)A_{\boldsymbol{r}}^{-1}}_{=:-P^X-z} = \mathbb{1} + B^X(z) + K^X(z), \tag{5-33}$$

and $\mathbb{1} + B^X(z)$ is invertible while $K^X(z)$ is compact. Moreover, for $\Re(z) \gg 1$, $\mathbb{1} + B^X(z) + K^X(z)$ is invertible on $\mathcal{L}(L^2(\mathcal{N}))$ since the L^2-norm of $B^X(z) + K^X(z)$ is exponentially decaying as $\Re(z) \to +\infty$. By the Fredholm analytic theorem [Zworski 2012, Theorem D.4], we deduce that

$$z \mapsto (\mathbb{1} + B^X(z) + K^X(z))^{-1} \in \mathcal{L}(L^2(\mathcal{N}))$$

is a meromorphic family of operators on $\{\Re(z) > -\Lambda(r_\perp - r_0) + C_\star\delta\}$. Equivalently,

$$z \mapsto -X - z - \omega(\boldsymbol{r})q,$$

is a holomorphic family of Fredholm operators[6] of index 0 on the anisotropic space $\mathcal{H}^r_+(\mathcal{N})$ that is invertible for $\Re(z) \gg 1$. Thus

$$z \mapsto R^X_-(z) = (-X - z - \omega(\boldsymbol{r})q)^{-1} \in \mathcal{L}(\mathcal{H}^{\boldsymbol{r}}_+)$$

is a meromorphic family of operators on $\{\Re(z) > -\Lambda(r_\perp - r_0) + C_\star \delta\}$. This proves the first part of the theorem; we next study the dependence in X and z.

Step 2: continuity of resonances. Assume z_0 is not a pole of $z \mapsto R^{X_0}(z)$ and furthermore that it does not have any poles in the closed disk $D(z_0, \varepsilon_0) \subset \mathbb{C}$ (since the resolvent is meromorphic, such $\varepsilon_0 > 0$ exists). We first show that, for X sufficiently close to X_0 in C^N for some N large enough, the map $z \mapsto R^X(z)$ does not have any poles in $D(z_0, \varepsilon_0)$. Let $z \in D(z_0, \varepsilon_0)$; we will use the identity (5-33). We first claim that we may pick the cutoff function χ suitably and T sufficiently large such that

$$\ker(\mathbb{1} + B^X(z) + K^X(z))|_{L^2} = 0.$$

Note that, as we will see below, this kernel could be nonzero even if z is not a resonance of $-X - q\omega$; we will show that generically this does not happen. We will argue by assuming that there is nonzero $u \in L^2(\mathcal{N})$ such that $(\mathbb{1} + B^X(z) + K^X(z))u = 0$. Since $K^X(z) \in \Psi^{-\infty}(\mathcal{N})$, we get

$$(\mathbb{1} + B^X(z))u \in C^\infty(\mathcal{N}) \subset \mathcal{D}(L^2) = \{f \in L^2(\mathcal{N}) \mid Xf \in L^2(\mathcal{N})\},$$

and since $\mathbb{1} + B^X(z)$ is invertible on $\mathcal{D}(L^2)$ (and on $L^2(\mathcal{N})$, by construction), we conclude that $u \in \mathcal{D}(L^2)$. Since $P^X + z$ commutes with $\mathbb{1} + B^X(z) + K^X(z)$, we have that $P^X + z$ acts on $\ker(\mathbb{1} + B^X(z) + K^X(z))|_{L^2}$, which is a finite-dimensional space by the Fredholm property shown above. Therefore, we can pick u such that $(P^X + z + \lambda)u = 0$ for some $\lambda \in \mathbb{C}$; by assumption, we have $\lambda \neq 0$. Write $u = A_{\boldsymbol{r}} v$ for some $v \in \mathcal{H}^{\boldsymbol{r}}_+$. This implies

$$e^{-tX} v = e^{(z+\lambda)t} e^{\int_0^t (\varphi^X_{-s})^*(q\omega)\, ds} v \quad \text{for all } t \in \mathbb{R},$$

and hence

$$\begin{aligned} 0 = (\mathbb{1} + B^X(z) + Q^X(z))u &= -A_{\boldsymbol{r}} \left(\mathbb{1} + \int_0^{+\infty} \chi'_T(t) e^{-tz} e^{-\int_0^t (\varphi^X_{-s})^*(q\omega)\, \mathrm{d}s} e^{-tX} \, \mathrm{d}t \right) v \\ &= -\Bigg(1 + \underbrace{\int_T^{T+1} \chi'_T(t) e^{\lambda t} \, \mathrm{d}t}_{F(\chi_T, \lambda) :=} \Bigg) u. \end{aligned}$$

If $\Re(\lambda) \le 0$, the integral in the last equality can be bounded by $\|\chi'_T\|_{C^0} e^{T\Re(\lambda)}$; then

$$\|\chi'_T\|_{C^0} e^{T\Re(\lambda)} < 1 \quad \Longleftrightarrow \quad \Re(\lambda) < -\frac{1}{T} \log(\|\chi'_T\|_{C^0}). \tag{5-34}$$

Moreover, integrating by parts once, we have

$$\int_T^{T+1} \chi'_T(t) e^{\lambda t} \, \mathrm{d}t = -\frac{1}{\lambda} \int_T^{T+1} \chi''_T(t) e^{\lambda t} \, \mathrm{d}t,$$

[6]Note that this is an unbounded family of operators. Since Fredholm operators are continuous by definition, one has to consider the operators on their domain $\mathcal{D}(\mathcal{H}^{\boldsymbol{r}}_+) := \{f \in \mathcal{H}^{\boldsymbol{r}}_+ \mid Xf \in \mathcal{H}^{\boldsymbol{r}}_+\}$.

which is in absolute value bounded by $(1/|\lambda|)\|\chi_T''\|_{C^0}e^{(T+1)|\Re(\lambda)|}$. Then

$$\frac{1}{|\lambda|}\|\chi_T''\|_{C^0}e^{(T+1)|\Re(\lambda)|} < 1 \iff |\Re(\lambda)| < \frac{\log|\lambda| - \log\|\chi_T''\|_{C^0}}{T+1}. \tag{5-35}$$

Using (5-34) and taking T large enough (changing χ_T in such a way that $\chi_T|_{[T,T+1]}$ is the same as before after a translation), we conclude $1+F(\chi_T,\lambda)$ has no zeroes (in λ) in $\{\Re(\lambda) > -\kappa\}$, where $\kappa = \kappa(T) > 0$ can be chosen arbitrarily small; we conclude that $z+\lambda$ is a resonance of $-X-q\omega$. Using additionally (5-35), we conclude that $z+\lambda$ belongs to a finite set of resonances $\mathcal{S}\subset\mathbb{C}$ of $-X-q\omega$ (in the regions defined by (5-34) and (5-35); note that there are no resonances with sufficiently large real part). Observe that the set $\mathcal{S}$ depends only on T, $\|\chi_T'\|_{C^0}$ and $\|\chi_T''\|_{C^0}$. Enumerate elements of the set $\mathcal{S}-z$ by $\lambda_1,\dots,\lambda_k$ for some $k\geq 0$.

We now perturb χ_T by considering $\chi_T + s\eta_T$, where $\eta_T \in C_c^\infty((T, T+1))$ is a smooth cutoff function and $s\in\mathbb{R}$ is small in absolute value. Assume $F(\chi_T,\lambda) = -1$ and $\Re(e^{i\Im(\lambda)t})$ to be positive on an interval $(T_1,T_2)\subset(T,T+1)$ (we argue similarly if it is negative), where $\lambda\in\mathcal{S}-z$. Taking $\eta\neq 0$ to be nonnegative and supported on (T_1,T_2), there is an $s>0$ small enough that

$$1+F(\chi_T+s\eta,\lambda) = -\lambda s\int_T^{T+1}\eta(t)e^{\lambda t}\,\mathrm{d}t \neq 0.$$

Arguing inductively, we ensure that $F(\tilde{\chi}_T,\lambda_i)\neq -1$ for $i=1,\dots,k$ for some new cutoff function $\tilde{\chi}_T$ (satisfying all the previously set out conditions of χ_T). We conclude that

$$\ker(\mathbb{1}+B^X(z)+K^X(z))|_{L^2} = \{0\}$$

with these new choices of T and χ_T, proving the claim.

As previously explained, since $B^X(z')$ and $K^X(z')$ depend continuously on X and z' in $\mathcal{L}(L^2)$, there is an $\varepsilon(z)>0$ small enough such that, for $\|X-X_0\|_{C^N}<\varepsilon(z)$ and $|z-z'|<\varepsilon(z)$, we have $\mathbb{1}+B^X(z)+K^X(z)$ invertible on L^2 (since it has empty kernel and is Fredholm of index 0). This implies that there are no resonances in $D(z,\varepsilon(z))$ for $z\in D(z_0,\varepsilon_0)$. By compactness of $D(z_0,\varepsilon_0)$, we conclude that there is an $\varepsilon>0$ such that there are no resonances in $D(z_0,\varepsilon_0)$ for $\|X-X_0\|_{C^N}<\varepsilon$, proving the desired claim.[7]

Step 3: smoothness of the resolvent. Now, using the following resolvent identity valid for $z\in D(z_0,\varepsilon_0)$ and X close to X_0 in C^N,

$$R_-^X(z) - R_-^{X'}(z) = R_-^{X'}(z)(X-X')R_-^X(z),$$

we obtain that $X\mapsto R_-^X(z)$ is twice differentiable in X, uniformly in $z\in D(z_0,\varepsilon_0)$, with

$$\partial_X(R_-^X(z)).Y = R_-^X(z)YR_-^X(z), \tag{5-36}$$

$$\partial_X^2(R_-^X(z)).(Y,Y') = R_-^X(z)Y'R_-^X(z)YR_-^X(z) + R_-^X(z)YR_-^X(z)Y'R_-^X(z), \tag{5-37}$$

where $Y, Y'\in C^\infty(\mathcal{N}, T\mathcal{N})$.

[7] A different proof of this step can be found in [Bonthonneau 2020].

Using the first part of Theorem 5.14, namely the boundedness of $R_-^X(z)$ on the spaces $\mathcal{H}_+^{\mathbf{r}}$ for X close to X_0 in C^2-norm, we deduce that the first derivative (5-36) is bounded as a map

$$\mathcal{H}_+^{(r_\perp,r_0)} \xrightarrow{R_-^X(z)} \mathcal{H}_-^{(r_\perp,r_0)} \xrightarrow{Y} \mathcal{H}_-^{(r_\perp,r_0+1)} \xrightarrow{R_-^X(z)} \mathcal{H}_-^{(r_\perp,r_0+1)},$$

and similarly the second derivative (5-37) is bounded as a map $\mathcal{H}_-^{(r_\perp,r_0)} \to \mathcal{H}_-^{(r_\perp,r_0+2)}$, and this holds for all X close enough to X_0 in the C^N-topology, with $N \gg 1$ large enough, and for all $z \in D(z_0, \varepsilon_0)$. Moreover, the dependence on z in (5-36) and (5-37) is holomorphic. This completes the proof of Theorem 5.14. □

5C. *Smoothness of the scattering map with respect to the metric.* The goal of this paragraph is to prove Proposition 4.2. We start with the following lemma.

Lemma 5.21. *If R_g and R_{g_e} are the resolvents defined in (2-14) for (M, g) and (M_e, g_e), we have, for $X = \psi \widetilde{X}_g$ defined in Section 2A2, that, for all $z \in \mathbb{C}$,*

$$R_g(z) = \mathbf{1}_{\mathcal{M}} R_+^X(z) \mathbf{1}_{\mathcal{M}} \quad \textit{and} \quad R_{g_e}(z) = \mathbf{1}_{\mathcal{M}_e} R_+^X(z) \mathbf{1}_{\mathcal{M}_e},$$

when acting on $C_c^\infty(\mathcal{M}^\circ)$ and $C_c^\infty(\mathcal{M}_e^\circ)$, respectively.

Proof. This is an obvious consequence of the following fact: for $f \in C_c^\infty(\mathcal{M}^\circ)$, writing $u_z = (R_g(z) f)|_{\mathcal{M}}$, if $\Re(z) \gg 1$, we have

$$u_z(y) = -\int_0^{\tau_g(y)} e^{-zt} f(\varphi_t^g(y))\, \mathrm{d}t,$$

and similarly for $R_{g_e}(z)$. Indeed, if $y \in \mathcal{M}$, the flow line $\gamma := \bigcup_{t \geq 0} \varphi_t^g(y)$ is contained in $\{\rho > -\rho_0\}$, and the convexity of $\mathcal{M}$ implies that $\gamma \cap \mathcal{M} = \bigcup_{t \in [0, \tau_g(y))} \varphi_t^g(y)$. □

We can now complete the proof of Proposition 4.2.

Proof of Proposition 4.2. Let $\omega \in C^\infty(\partial_+\mathcal{M})$. Observe that, by Lemmas 2.7 and 5.21,

$$\chi \mathcal{S}_g(\omega) = \chi [R_{g_e}(\tilde{\chi} \omega \delta_{\partial_+\mathcal{M}})]|_{\partial_-\mathcal{M}},$$

where $\tilde{\chi}$ is some smooth cutoff function equal to 1 everywhere except in a neighborhood of $\partial_0\mathcal{M}$, and where $\tilde{\chi}\omega\delta_{\partial_+\mathcal{M}} \in \mathcal{D}'(\mathcal{N})$ denotes the distribution defined by

$$\langle \tilde{\chi} \omega \delta_{\partial_+\mathcal{M}}, \varphi \rangle := \int_{\partial_+\mathcal{M}} \tilde{\chi} \omega \varphi \, \mathrm{d}\mu_\partial.$$

Let $u := \tilde{\chi}\omega\delta_{\partial_+\mathcal{M}}$. Since $\partial_+\mathcal{M}$ is of codimension 1, we have that $u \in H^{-1/2-\varepsilon}(\mathcal{N})$ for all $\varepsilon > 0$. Let $N^*\partial_+\mathcal{M} \subset T^*_{\partial_+\mathcal{M}}\mathcal{N}$ be the conormal of $\partial_+\mathcal{M}$ in $\mathcal{N}$ (i.e., $N^*\partial_+\mathcal{M}(T\partial_+\mathcal{M}) = 0$). By a standard argument of distribution theory, the wavefront set of u satisfies $\mathrm{WF}(u) \subset N^*\partial_+\mathcal{M}$.

The escape function m provided by Proposition 5.1 can be constructed so that, over $\mathcal{M}$, it has only support in a small conic neighborhood of $(E_-^{X_0})^*$ and $(E_+^{X_0})^*$. In particular, this construction can be achieved so that

$$N^*\partial_+\mathcal{M} \cap \mathrm{supp}(m) = \varnothing. \tag{5-38}$$

Indeed, a covector $V^* \in T^*_{\partial_+\mathcal{M}}\mathcal{N}$ such that $V^* \in (E_+^{X_0})^*$ must satisfy $V^*(X_0) = 0$ and $V^*(W)$ for all $W \in T\partial_+\mathcal{M}$, but since X_0 is transverse to $\partial_-\mathcal{M}$, one gets $V^* = 0$. We now take a regularity pair $\boldsymbol{r} := (r_\perp, r_0)$ with $\frac{1}{2} < r_0 < 1$, $r_0 + 2 < r_\perp < 3$ and a small $\delta > 0$ such that $-\Lambda(r_\perp - (r_0+2)) + C_\star\delta < 0$. By the previous discussion, $u \in \mathcal{H}^{\boldsymbol{r}}_-(\mathcal{N})$, i.e., since $\mathcal{H}^{\boldsymbol{r}}_-(\mathcal{N})$ is microlocally equivalent to H^{-r_0} near $N^*\partial_+\mathcal{M}$. Denote by $\theta \in C_c^\infty(\mathcal{M}_e^\circ)$ a cutoff function equal to 1 near $\mathcal{M}$. We claim that the map

$$C^\infty(M, \otimes_S^2 T^*M) \ni g \mapsto \theta R_{g_e}\theta \in \mathcal{L}(\mathcal{H}_-^{(r_\perp, r_0)}(\mathcal{N}), \mathcal{H}_-^{(r_\perp, r_0+2)}(\mathcal{N}))$$

is C^2 for g close to g_0. Indeed, similar to the proof of Theorem 5.14 (alternatively we could simply use Theorem 1.10 along with the fact that $g \mapsto X_g$ is smooth; we give a direct argument instead), we can use the resolvent identity (recall $X = \psi\widetilde{X}_g$ and $X_0 = \psi\widetilde{X}_{g_0}$)

$$\theta R_{g_e}\theta - \theta R_{g_{0e}}\theta = \theta R_+^X(0)(X_0 - X)R_+^{X_0}(0)\theta$$

to deduce that $g \mapsto \theta R_{g_e}\theta$ is differentiable twice, with

$$\partial_g \theta R_{g_e}\theta = -\theta R_+^X(0)(\partial_g X)R_+^X(0)\theta, \tag{5-39}$$

$$\partial_g^2 \theta R_{g_e}\theta = 2\theta R_+^X(0)(\partial_g X)R_+^X(0)(\partial_g X)R_+^X(0)\theta - \theta R_+^X(0)(\partial_g^2 X)R_+^X(0)\theta. \tag{5-40}$$

The first derivative (5-39) is bounded as a map

$$\mathcal{H}_-^{(r_\perp, r_0)} \xrightarrow{R_+^X(0)} \mathcal{H}_-^{(r_\perp, r_0)} \xrightarrow{\partial_g X} \mathcal{H}_-^{(r_\perp, r_0+1)} \xrightarrow{R_+^X(0)} \mathcal{H}_-^{(r_\perp, r_0+1)},$$

and similarly the second derivative (5-40) is bounded as a map $\mathcal{H}_-^{(r_\perp, r_0)} \to \mathcal{H}_-^{(r_\perp, r_0+2)}$, and this holds for all g close enough to g_0 in the C^N-topology, with $N \gg 1$ large enough.

As a consequence,

$$C^\infty(M, \otimes_S^2 T^*M) \ni g \mapsto \theta R_{g_e}\theta u = \theta R_{g_e} u \in \mathcal{H}_-^{(r_\perp, r_0+2)}(\mathcal{N})$$

is C^2-regular for g close to g_0. Note that, as $r_\perp + r_0 + 2 < 6$,

$$\mathcal{H}_-^{(r_\perp, r_0+2)}(\mathcal{N}) \hookrightarrow H^{-6}(\mathcal{N}).$$

Moreover, it satisfies $X_{g_e}\theta R_{g_e} u = 0$ near $\partial_-\mathcal{M}$, so that $\mathrm{WF}(\theta R_{g_e} u) \subset \{p_{X_{g_e}} = 0\}$. Therefore, the restriction $\chi[\theta R_{g_e} u]|_{\partial_-\mathcal{M}} = \chi[R_{g_e} u]|_{\partial_-\mathcal{M}} \in H^{-6}(\partial_-\mathcal{M})$ is well defined and depends in a C^2-fashion on the metric $g \in C^N(M, \otimes_S^2 T^*M)$, proving the first part of Proposition 4.2.

Using (5-39) and (5-40), and writing $g = g_0 + h$ with $\|h\|_{C^N} \le \delta$ for $\delta > 0$ small and N chosen large, we have as above, by Taylor expansion, for $u = \tilde{\chi}\omega\delta_{\partial_+\mathcal{M}}$,

$$\theta R_{g_e} u = \theta R_{g_{0e}} u - \theta R_+^{X_0}(0)((\partial_g X)|_{g=g_0}.h)R_+^{X_0}(0)u + \int_0^1 (1-t)\partial_g^2(\theta R_{g_{0e}+th} u).(h,h)\,dt. \tag{5-41}$$

Let $Y_g(h) := \partial_g X(h) \in C^\infty(\mathcal{N}, T\mathcal{N})$ for any smooth metric g close to g_0 in $C^N(M, \otimes_S^2 T^*M)$. For all $k \ge 1$, one has $\|Y_g(h)\|_{C^k(\mathcal{N}, T\mathcal{N})} \le C_k \|h\|_{C^{k+1}}$ for some $C_k > 0$ depending uniformly on $\|g\|_{C^{k+1}}$. Let $Z_g(h,h) = \partial_g^2 X(h,h) \in C^\infty(\mathcal{N}, T\mathcal{N})$. One has $\|Z_g(h,h)\|_{C^k(\mathcal{N}, T\mathcal{N})} \le C_k \|h\|^2_{C^{k+2}}$ for some $C_k > 0$

depending uniformly on $\|g\|_{C^{k+2}}$. Then the remainder term in (5-41) satisfies, for $g_e(t)$ the extension of $g(t) = g_0 + th$ (with $t \in [0, 1]$) and $X(t) = \psi \widetilde{X}_{g(t)}$,

$$\partial_g^2(\theta R_{g_e(t)} u)(h, h) = 2\theta R_+^{X(t)}(0) Y_{g(t)}(h) R_+^{X(t)}(0) Y_{g(t)}(h) R_+^{X(t)}(0) u - \theta R_+^{X(t)}(0) Z_{g(t)}(h, h) R_+^{X(t)}(0) u.$$

By the analysis above, for $\delta > 0$ small and $N > 0$ large enough, there exists a constant $C > 0$ such that, for $h = g(1) - g_0$ such that $\|h\|_{C^N} \leq \delta$,

$$\begin{aligned} \sup_{t\in[0,1]} \|R_+^{X(t)} u\|_{\mathcal{H}_-^{(r_\perp, r_0+j)}(\mathcal{N})} &\leq C && \text{for all } j \in \{0, 1, 2\}, \\ \sup_{t\in[0,1]} \|Y_{g(t)}(h)\|_{\mathcal{H}_-^{(r_\perp, r_0+j)} \to \mathcal{H}_-^{(r_\perp, r_0+1+j)}} &\leq C\|h\|_{C^N} && \text{for all } j \in \{0, 1\}, \\ \sup_{t\in[0,1]} \|Z_{g(t)}(h, h)\|_{\mathcal{H}_-^{(r_\perp, r_0)} \to \mathcal{H}_-^{(r_\perp, r_0+2)}} &\leq C\|h\|_{C^N}^2. \end{aligned}$$

Combining the last inequalities with (5-41), this shows (4-1) by applying the restriction to $\partial_-\mathcal{M}$ on the left of (5-41). Note that, in turn, this gives an expression of $\partial_g \mathcal{S}_g|_{g=g_0}$ in terms of $R_+^{X_0}(0)$ and $\partial_g X|_{g=g_0}$. This concludes the proof. □

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THE RANK-ONE THEOREM ON RCD SPACES

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We extend Alberti's rank-one theorem to $\mathrm{RCD}(K, N)$ metric measure spaces.

1. Introduction

1A. ***The rank-one theorem in the Euclidean setting.*** Let Ω be an open subset of $\mathbb{R}^n$ and $u \in \mathrm{BV}(\Omega; \mathbb{R}^k)$, i.e., $u = (u_1, \dots, u_k) \in (\mathrm{BV}(\Omega))^k$. By using the Lebesgue–Radon–Nikodým theorem, one can write the distributional derivative of u as

$$\mathrm{D}u = \mathrm{D}^a u + \mathrm{D}^s u,$$

where $\mathrm{D}^a u$ is the absolutely continuous part of $\mathrm{D}u$ with respect to the Lebesgue measure $\mathcal{L}^n$ and $\mathrm{D}^s u$ is the singular part of $\mathrm{D}u$. We denote by $\mathrm{D}u/|\mathrm{D}u|$ the matrix-valued Lebesgue–Radon–Nikodým density of $\mathrm{D}u$ with respect to the total variation $|\mathrm{D}u|$. Notice that the total variation of the singular part $|\mathrm{D}^s u|$ is equal to the singular part of the total variation $|\mathrm{D}u|^s$.

De Giorgi and Ambrosio [1988] — motivated by the study of some functionals coming from mathematical physics — conjectured the following:

<u>Rank-one property</u>: For every $u \in \mathrm{BV}(\Omega; \mathbb{R}^k)$, the matrix $\mathrm{D}u/|\mathrm{D}u|$ has rank-1 $|\mathrm{D}u|^s$-almost everywhere.

Alberti [1993] solved in the affirmative the previous conjecture; see also the account in [De Lellis 2008].

It is worth observing that the ideas used in [Alberti 1993] proved to be very robust for further developments of geometric measure theory and the rectifiability theory in Euclidean spaces and even beyond in the metric setting. As a main step of the proof, Alberti [1993] proved that, given an arbitrary Radon measure μ on a k-dimensional plane V in $\mathbb{R}^n$ that is singular with respect to $\mathcal{H}^k \llcorner V$, one can associate to μ a bundle $E(\mu, \cdot)$ whose fibers have dimension at most 1. The fiber $E(\mu, x)$ of this bundle is made by all the vectors $v \in \mathbb{R}^k$ such that $v\mu$ is *tangent* in x, in a precise sense, to the derivative of a BV function on V. What happens, moreover, is that the restriction of μ to the set where $E(\mu, \cdot)$ is one-dimensional can be written as $\int_I \mu_t \,\mathrm{d}t$, where $\mu_t = \mathcal{H}^{k-1} \llcorner S_t$ and S_t is $(k-1)$-rectifiable in V.

In the language of [Alberti et al. 2010], which collects several other fine results for the theory of rectifiability in $\mathbb{R}^n$, the previous result means that, on the set where the fiber is one-dimensional, μ is $(k-1)$-*representable*: namely, it can be written as the integral of measures that are $(k-1)$-rectifiable. At the basis of this possibility of representing a measure as the integral of rectifiable measures is the idea of the Alberti representations.

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Another interesting contribution that originated from this circle of ideas is a result by Alberti and Marchese [2016], where they associate to every Radon measure μ on $\mathbb{R}^n$ the minimal (unique μ-almost everywhere) bundle $V(\mu,\cdot)$ such that every real-valued Lipschitz function on $\mathbb{R}^n$ is differentiable along $V(\mu,x)$ for μ-almost every $x\in\mathbb{R}^n$. Alberti representations were also used by Bate [2015] and Bate and Li [2017] in the study of rectifiability in the general metric setting. For further readings, one can consult the survey by Mattila [2023], in particular, Chapters 8 and 13.

Besides its theoretical interest, the rank-one theorem soon gave important consequences in the calculus of variations. Ambrosio and Dal Maso [1992] exploited it to derive the expression of the relaxation (in BV) of a functional defined on C^1 functions as the integral of a quasiconvex function of linear growth of the gradient. See also [Kristensen and Rindler 2010] for a generalization. Moreover, Fonseca and Müller [1993] generalized the result in [Ambrosio and Dal Maso 1992] for integrands that might not depend solely on the gradient, but also on the space variable and the function itself. For further details we refer the reader to [Ambrosio et al. 2000, Chapter 5].

As an added value to the theoretical interest of the rank-one theorem, De Philippis and Rindler [2016] showed a general structure theorem for $\mathcal{A}$-free vector-valued Radon measures on Euclidean spaces, where $\mathcal{A}$ is a linear constant-coefficient differential operator, from which the rank-one theorem can be derived as a consequence. We also remark that Massaccesi and Vittone [2019] recently gave a very short proof of the rank-one theorem based on the theory of sets of finite perimeter, and with Don they used this simplified strategy to prove the analogue of the rank-one theorem in some Carnot groups [Don et al. 2019].

1B. ***Main result.*** Nowadays a well-established notion of a BV function is available in the metric measure setting. Such a notion was proposed by Miranda [2003] then studied by Ambrosio [2001; 2002] and more recently by Ambrosio and Di Marino [2014].

According to this approach, given a metric measure space $(\mathsf{X},\mathsf{d},\mathsf{m})$, the total variation of the derivative of $f\in L^1_{\mathrm{loc}}(\mathsf{X},\mathsf{m})$ is the relaxation in $L^1_{\mathrm{loc}}(\mathsf{X},\mathsf{m})$ of the energy given by the integral of the local Lipschitz constant. Such a definition can be readily extended to define the total variation of a vector-valued function whose components are in $\mathrm{BV}_{\mathrm{loc}}(\mathsf{X},\mathsf{d},\mathsf{m})$; see Definition 2.14 for the precise definition.

In this way one is giving a meaning to the total variation $|\mathrm{D}F|$ of an arbitrary $F\in\mathrm{BV}_{\mathrm{loc}}(\mathsf{X},\mathsf{d},\mathsf{m})^k$, while it is in general missing a good notion for the Lebesgue–Radon–Nikodým derivative $\mathrm{D}F/|\mathrm{D}F|$.

In the setting of RCD metric measure spaces, the study of calculus has been blossoming very fast in the last decade. In particular, in [Debin et al. 2021] the authors propose and study the notion of a $L^0(\mathrm{Cap})$-normed $L^0(\mathrm{Cap})$-module, and the notion of a capacitary tangent module $L^0_{\mathrm{Cap}}(T\mathsf{X})$, where Cap denotes the usual Capacity (2-3). We refer to Section 2B for the definitions and further details.

A fundamental contribution of [Bruè et al. 2023b], building on [Debin et al. 2021], is the fact that, in the setting of $\mathrm{RCD}(K,N)$ spaces, for an arbitrary set of finite perimeter E with finite mass, one can give a meaning to the unit normal $\nu_E=\mathrm{D}\chi_E/|\mathrm{D}\chi_E|$ as an element of the capacitary tangent module $L^0_{\mathrm{Cap}}(T\mathsf{X})$ such that the Gauss–Green formula holds; see [Bruè et al. 2023b, Theorem 2.4]. The Gauss–Green formula has been successfully employed, together with the former work by Ambrosio, Bruè and Semola [Ambrosio et al. 2019], to obtain the $(n-1)$-rectifiability of the essential boundary of any set of locally finite perimeter in an RCD space of essential dimension n; see [Bruè et al. 2023a; 2023b].

The Gauss–Green formula [Bruè et al. 2023b, Theorem 2.4] has been generalized by the second author together with Gigli for vector-valued BV functions [Brena and Gigli 2024]. We give below the statement of the Gauss–Green formula presented there, where the density $\nu_F = \mathrm{D}F/|\mathrm{D}F|$ is implicitly defined.

Theorem 1.1 [Brena and Gigli 2024, Theorem 3.13]. *Let $k \geq 1$ be a natural number, let $K \in \mathbb{R}$, and let $N \geq 1$. Let $(\mathsf{X}, \mathsf{d}, \mathsf{m})$ be an $\mathrm{RCD}(K, N)$ space, and let $F \in \mathrm{BV}(\mathsf{X}, \mathsf{d}, \mathsf{m})^k$. Then there exists a unique $\nu_F \in L^0_{\mathrm{Cap}}(T\mathsf{X})^k$, up to $|\mathrm{D}F|$-almost everywhere equality, such that $|\nu_F| = 1$ $|\mathrm{D}F|$-almost everywhere and*

$$\sum_{j=1}^{k} \int_{\mathsf{X}} F_j \,\mathrm{div}(v_j)\,\mathrm{dm} = -\int_{\mathsf{X}} \pi_{|\mathrm{D}F|}(v) \cdot \nu_F \,\mathrm{d}|\mathrm{D}F| \quad \text{for every } v = (v_1, \dots, v_k) \in \mathrm{TestV}(\mathsf{X})^k.$$

For the notion of divergence of a vector field, the notion of test vector fields $\mathrm{TestV}(\mathsf{X})$, the notion of the projection $\pi_{|\mathrm{D}F|}$ and of the norm $|\cdot|$ in $L^0_{\mathrm{Cap}}(T\mathsf{X})^k$, we refer the reader to Section 2B.

Theorem 1.1 tells us that in the setting of $\mathrm{RCD}(K, N)$ spaces we can give a precise meaning to $\mathrm{D}F/|\mathrm{D}F|$ for an arbitrary vector-valued BV function F. Hence it is meaningful to ask if $\mathrm{D}F/|\mathrm{D}F|$ is a rank-1 matrix $|\mathrm{D}F|^s$-almost everywhere, where $|\mathrm{D}F|^s$ is the singular part of the total variation $|\mathrm{D}F|$. Before giving the main result of this paper we clarify this last sentence by means of a definition. For the definition of the space $L^0(\mathrm{Cap})$, see Section 2B.

Definition 1.2. Let $k \geq 1$ be a natural number, let $K \in \mathbb{R}$, and let $N \geq 1$. Let $(\mathsf{X}, \mathsf{d}, \mathsf{m})$ be an $\mathrm{RCD}(K, N)$ space, let $\nu \in L^0_{\mathrm{Cap}}(T\mathsf{X})^k$, and let $\mu \ll \mathrm{Cap}$ be a Radon measure, where Cap is the usual Capacity (2-3). We say that

$$\mathrm{Rk}(\nu) = 1 \quad \mu\text{-almost everywhere}$$

if there exist $\omega \in L^0_{\mathrm{Cap}}(T\mathsf{X})$ and $\lambda_1, \dots, \lambda_k \in L^0(\mathrm{Cap})$ such that, for every $i = 1, \dots, k$,

$$\nu_i = \lambda_i \omega \quad \mu\text{-almost everywhere.}$$

We remark that this is one of the possible definitions we could have given of having rank 1. For example, one can give an alternative and equivalent definition exploiting the existence of a local basis (with respect to a decomposition of the space in Borel sets) of $L^0_{\mathrm{Cap}}(T\mathsf{X})$ to recover the language of rank of a matrix. It is however clear that in Euclidean spaces, the definition given above coincides with the usual one.

We are now ready to state the main theorem of this paper, which is the generalization of the rank-one theorem in the setting of $\mathrm{RCD}(K, N)$ metric measure spaces $(\mathsf{X}, \mathsf{d}, \mathsf{m})$.

Theorem 1.3 (rank-one theorem for $\mathrm{RCD}(K, N)$ spaces). *Let $k \geq 1$ be a natural number, let $K \in \mathbb{R}$, and let $N \geq 1$. Let $(\mathsf{X}, \mathsf{d}, \mathsf{m})$ be an $\mathrm{RCD}(K, N)$ space, and let $F \in \mathrm{BV}(\mathsf{X}, \mathsf{d}, \mathsf{m})^k$. Then*

$$\mathrm{Rk}(\nu_F) = 1 \quad |\mathrm{D}F|^s\text{-almost everywhere}$$

in the sense of Definition 1.2, where ν_F is defined in Theorem 1.1 and $|\mathrm{D}F|^s$ is the singular part of the total variation $|\mathrm{D}F|$.

As far as we know, apart from the result of Don, Massaccesi and Vittone [Don et al. 2019] that holds for a special class of Carnot groups, Theorem 1.3 is one of the first instances of the validity of the rank-one theorem in a large class of metric measure spaces.

We stress that, even if the proof of [Don et al. 2019] covers a large class of Carnot groups, some distinguished examples are still not covered. For example, as of today it is not known if the rank-one theorem holds for vector-valued BV functions in the first Heisenberg group $\mathbb{H}^1$. We stress that our strategy for the proof of Theorem 1.3 seems not to apply to the rank-one theorem in $\mathbb{H}^1$. Indeed, we are fundamentally exploiting the fact that we have good bi-Lipschitz charts on the space valued in the tangents. But, even if on $\mathbb{H}^1$ the boundary of a set of locally finite perimeter is intrinsic C^1-rectifiable, see [Franchi et al. 2001], it is nowadays not known whether intrinsic C^1 surfaces can be almost everywhere covered by (bi)-Lipschitz images of their tangents; see [Di Donato et al. 2022] for partial results in this direction.

We stress that our strategy cannot be easily adapted to prove rank-one-type results for BV functions in $\mathrm{RCD}(K,\infty)$ spaces. In fact, our proof works mainly by blow-up. Since $\mathrm{RCD}(K,\infty)$ spaces might not be locally doubling, we do not have a good notion of the Gromov–Hausdorff tangent at their points. In particular, it would even be challenging to understand whether the results in [Ambrosio 2001; 2002; Ambrosio et al. 2019; Bruè et al. 2023a; 2023b], which are the starting point of our analysis, can be adapted to the $\mathrm{RCD}(K,\infty)$ setting.

Moreover, we point out that very recently Lahti proposed an alternative formulation of Alberti's rank-one theorem that could make sense in arbitrary metric measure spaces [Lahti 2022, Section 6].

1C. ***Outline of the proof.*** Our proof is inspired by the one in [Massaccesi and Vittone 2019]. First, given $F \in \mathrm{BV}(\mathsf{X},\mathsf{d},\mathsf{m})^k$, the singular part of the total variation $|\mathrm{D}F|^s$ can be written as the sum of the jump part $|\mathrm{D}F|^j$, which is concentrated on the set where the approximate lower and upper limits of the components of F do not coincide, and the Cantor part $|\mathrm{D}F|^c$; see Definition 2.12. As a consequence of a result by the second author and Gigli, see the forthcoming Lemma 3.13, it is enough to show the rank-one theorem only on the Cantor part.

We stress that, in the proofs of the main results in Section 3, we shall always restrict to sets where the Cantor part of the components of F is concentrated and where we have good density and blow-up properties: we collect all the necessary properties in the technical Proposition 3.7.

The core and the most technically demanding part of the proof is Lemma 3.11, in which we adapt to our setting the main lemma of the short proof of the rank-one theorem in [Massaccesi and Vittone 2019]. In fact, those authors prove that, given two C^1-hypersurfaces Σ_1, Σ_2 in $\mathbb{R}^n\times\mathbb{R}$, the set T of points $p\in\Sigma_1$ such that there exists $q\in\Sigma_2$ for which p and q have the same first n coordinates, $\nu_{\Sigma_1}(p)_{n+1}=\nu_{\Sigma_2}(q)_{n+1}=0$, and $\nu_{\Sigma_1}(p)\neq\pm\nu_{\Sigma_2}(q)$, is $\mathcal{H}^n$-negligible. Clearly the latter statement makes no sense in our nonsmooth setting, but what one really needs for the proof of the rank-one theorem is Lemma 3.11.

Following the strategy in the proof of the lemma of [Massaccesi and Vittone 2019], one writes T as the projection of a set $\widetilde{T}\subset\mathbb{R}^n\times\mathbb{R}\times\mathbb{R}$ adding one fake coordinate, and proves that $T=\pi(\widetilde{T})$ is $\mathcal{H}^n$-negligible by means of the area formula. In Lemma 3.11 we adapt the same strategy; compare with the definition of the set (3-28). We prove the analogue of the Massaccesi–Vittone lemma substituting the hypersurfaces Σ_i with the (essential) boundaries of sets of the form $\mathcal{G}_f:=\{(x,t):t<f(x)\}$, where $f\in\mathrm{BV}(\mathsf{X},\mathsf{d},\mathsf{m})$. This

is enough to implement their strategy in our setting. However, to adapt the proof in [Don et al. 2019; Massaccesi and Vittone 2019] to our framework, one faces nontrivial technical difficulties. Indeed, the key ingredient used by Massaccesi–Vittone was a well-known transversality lemma: given two hypersurfaces in $\mathbb{R}^{n+2}$, their intersection is locally an n-dimensional manifold provided that at every intersection point the given hypersurfaces meet transversally, i.e., have different tangent spaces. This result then extends to the case of the intersection of two $(n+1)$-rectifiable subsets of $\mathbb{R}^{n+2}$: their intersection is σ-finite with respect to $\mathcal{H}^n$ provided that the transversality condition is satisfied and that one discards a set that turns out to be negligible when proving the rank-one property.

It is clear that one needs also information on codimension-2 objects (namely, the intersection of two transverse hypersurfaces) and this kind of information is unavailable on RCD spaces. Therefore, adopting directly this approach is not possible in our framework. Our strategy is then to translate part of the problem from the RCD setting to the Euclidean setting (which allows us to use transversality results as above) via the use of suitable δ-splitting maps that play the role of charts, relying heavily on the results of [Bruè et al. 2023a; 2023b]. The fact that the domains of these charts are not open sets is a source of difficulty and is morally the burden of the proof of Lemma 3.11. In other words, we could not work directly arguing with infinitesimal considerations in the RCD case (i.e., using directly difference of blow-ups) but we had to argue locally and then infinitesimally in a Euclidean space.

As an important part of the proof, to manipulate the vector that is normal to the boundary of the set $\mathcal{G}_f$, we need to introduce a family of charts in which we write those normals in coordinates; see Definition 3.6. We construct these charts in Definition 2.29, and we call them a *good collection of splitting maps*. The latter definition is based on the following fact, which is proved in Lemma 2.28. Given an RCD space of essential dimension n, we prove that, for every $\eta > 0$ small enough, we can find a sequence of n-tuples of harmonic maps $\{u_{k,\eta}\}_{k\in\mathbb{N}}$ defined on balls and a disjointed family of Borel sets $\{D_{k,\eta}\}_{k\in\mathbb{N}}$ such that, for every $x \in D_{k,\eta}$, we have that $u_{k,\eta}$ is an η-splitting map on $B_{r_k}(x)$ and the total variation of every $\mathrm{BV}_{\mathrm{loc}}$ function is concentrated on $\bigsqcup_{k\in\mathbb{N}} D_{k,\eta}$.

The other two ingredients to adapt in our setting the strategy of [Massaccesi and Vittone 2019] are Lemma 3.9 and Theorem 3.8. In the first we prove that, given $f \in \mathrm{BV}$, restricting to the good set on the Cantor part as in Proposition 3.7, we have that (in coordinates) the density $\nu_f(x)$ is equal to the first coordinates of the normal $\nu_{\mathcal{G}_f}(x, f(x))$, where $\mathcal{G}_f := \{(x,t) : t < f(x)\}$. In the second we prove that, restricting to the good set on the Cantor part as in Proposition 3.7, the $(n+1)$-th coordinate of the normal $\nu_{\mathcal{G}_f}$ is almost everywhere zero. This is essentially due to the fact that we are on the singular part of $\mathrm{D}f$.

Again, not having at our disposal a linear structure is a source of difficulty, as the distributional derivative has no more a direction-wise meaning, in the sense that it is not possible to define the distributional derivative of a function of bounded variation with respect to a given direction without giving up the differential meaning of this object. To overcome this difficulty, we employ blow-up arguments and density arguments.

Finally, putting together Lemmas 3.11 and 3.9 and Theorem 3.8, we conclude that, given two BV functions f, g, we have $\nu_f = \pm\nu_g$ $|\mathrm{D}f| \wedge |\mathrm{D}g|$-almost everywhere on the intersection of the good sets $C_f \cap C_g$ defined in Proposition 3.7; see Lemma 3.12. Here $\wedge$ stands for the minimum between the two measures, i.e., the biggest measure ϕ such that $\phi \le \mu$ and $\phi \le \nu$. This, together with the same property on the jump part, see Lemma 3.13, concludes the proof.

We stress that along the way in Section 3A, building on [Deng 2020] (compare with [Colding and Naber 2012] for the Hölder continuity property of tangents along geodesics in the Ricci-limit case), we improve a previous result of [Bruè et al. 2023a] by showing that every BV function on an RCD space of essential dimension n has total variation concentrated on the set $\mathcal{R}_n^*$ of n-regular points with positive and finite n-density, see Theorem 3.3. We exploit the latter result to answer in the affirmative a conjecture proposed in [Semola 2020] about the representation of the perimeter measure, see Theorem 3.4.

In the Appendix we exploit the previously described result proved in Theorem 3.3, together with the recently proved metric variant of the Marstrand–Mattila rectifiability criterion [Bate 2022], to give an alternative and shorter proof of the $(n-1)$-rectifiability of the essential boundaries of sets of locally finite perimeter in RCD spaces with essential dimension n. We believe that this result is of independent interest but we point out that it originated as a side remark due to the fact that we were interested in proving the rank-one property in general $\mathrm{RCD}(K, N)$ spaces without restricting ourselves to *noncollapsed* RCD spaces. Indeed, the information that the perimeter measure and the $\mathcal{H}^{n-1}$ measure restricted to the reduced boundary are mutually absolutely continuous (already known in the *noncollapsed* case) is crucial in the proof of Lemma 3.11. Anyway, we point out that even if the proof presented in the Appendix is much shorter than the original one, it is heavily based on the ideas and techniques exploited in [Bruè et al. 2023a; 2023b], i.e., looking at what happens at the space locally and infinitesimally by using well-behaved charts.

Structure of the paper. In Section 2 we discuss the basic tools and notation that we shall use throughout the paper.

In particular, in Section 2A we discuss the basic toolkit for metric measure spaces. We recall the definition of PI space, the notion of pointed measured Gromov–Hausdorff convergence and tangents, and the basic Sobolev and BV calculus in arbitrary metric measure spaces.

In Section 2B we recall basic structure results of RCD spaces and the main important notions of Sobolev and BV calculus on RCD spaces. We further recall the notion of *good coordinates* introduced in [Bruè et al. 2023a] and the notion of splitting maps, and finally we prove Lemma 2.28 that leads to the notion of *good collection of splitting maps*, see Definition 2.29.

In Section 3 we prove the main results of this paper, and in particular we give the proof of the rank-one theorem in Theorem 1.3.

In particular in Section 3A, building on [Deng 2020], we prove Theorem 3.3 described above.

In Section 3B we prove some auxiliary results toward the proof of the rank-one theorem, namely Proposition 3.7, Lemma 3.9, and Theorem 3.8.

Finally, in Section 3C we exploit the previous results together with the main result in Lemma 3.11, which is the adaptation to our setting of the lemma of [Massaccesi and Vittone 2019], to show the rank-one property on the Cantor part, see Lemma 3.12. This is enough to conclude the proof of the rank-one theorem by exploiting also Lemma 3.13, which is the rank-one property on the jump part.

In the Appendix we give the alternative proof of the rectifiability of the essential boundaries of sets of locally finite perimeter in RCD spaces that we described above.

2. Preliminaries

We often need to bound quantities in terms of constants that depend only on geometric parameters but whose precise value is not important. For this reason, we denote with $C_{a,b,\dots}$ a constant depending only on the parameters $a, b, \dots$, whose value might change from line to line or even within the same line.

Given $n \in \mathbb{N}$ and nonempty sets $\mathsf{X}_1, \dots, \mathsf{X}_n$, for any $i = 1, \dots, n$, we will always tacitly denote by π^i the projection of the Cartesian product $\mathsf{X}_1 \times \cdots \times \mathsf{X}_n$ onto its i-th factor:

$$\pi^i : \mathsf{X}_1 \times \cdots \times \mathsf{X}_n \to \mathsf{X}_i, \quad (x_1, \dots, x_n) \mapsto x_i.$$

Moreover, we denote by $\pi^{i,j}$ the projection of the Cartesian product $\mathsf{X}_1 \times \cdots \times \mathsf{X}_n$ onto its (i, j) factor, namely

$$\pi^{i,j} : \mathsf{X}_1 \times \cdots \times \mathsf{X}_n \to \mathsf{X}_i \times \mathsf{X}_j, \quad (x_1, \dots, x_n) \mapsto (x_i, x_j).$$

Finally, we denote by τ the inversion map on the last two factors on a product of three factors, namely

$$\tau : \mathsf{X}_1 \times \mathsf{X}_2 \times \mathsf{X}_3 \to \mathsf{X}_1 \times \mathsf{X}_3 \times \mathsf{X}_2, \quad (x_1, x_2, x_3) \mapsto (x_1, x_3, x_2). \tag{2-1}$$

2A. *Metric measure spaces.* For the purposes of this paper, a *metric measure space* is a triple $(\mathsf{X}, \mathsf{d}, \mathsf{m})$, where (X, d) is a complete and separable metric space while $\mathsf{m} \geq 0$ is a boundedly finite Borel measure on X. By a *pointed metric measure space* $(\mathsf{X}, \mathsf{d}, \mathsf{m}, p)$ we mean a metric measure space $(\mathsf{X}, \mathsf{d}, \mathsf{m})$ together with a distinguished point $p \in \operatorname{spt}(\mathsf{m})$, where

$$\operatorname{spt}(\mathsf{m}) := \{x \in \mathsf{X} \mid \mathsf{m}(B_r(x)) > 0 \text{ for every } r > 0\}$$

stands for the *support* of m. Given an open set $\Omega \subseteq \mathsf{X}$, we denote by $\mathrm{LIP}_{\mathrm{loc}}(\Omega)$ and $\mathrm{LIP}(\Omega)$ the spaces of all locally Lipschitz and Lipschitz functions on Ω, respectively, while we set

$$\mathrm{LIP}_{\mathrm{bs}}(\Omega) := \{f \in \mathrm{LIP}(\Omega) \mid \operatorname{spt}(f) \text{ is bounded and } \mathsf{d}(\partial\Omega, \operatorname{spt}(f)) > 0\}.$$

Given any $f \in \mathrm{LIP}_{\mathrm{loc}}(\Omega)$, its *local Lipschitz constant* $\operatorname{lip} f := \Omega \to [0, +\infty)$ is defined as

$$\operatorname{lip} f(x) := \begin{cases} \overline{\lim}_{y \to x} |f(x) - f(y)|/\mathsf{d}(x, y) & \text{if } x \in \Omega \text{ is an accumulation point,} \\ 0 & \text{if } x \in \Omega \text{ is an isolated point.} \end{cases}$$

For any $k \in [0, +\infty)$ and $\delta > 0$, we will denote by $\mathcal{H}^k_\delta$ and $\mathcal{H}^k$ the *k-dimensional Hausdorff* δ*-premeasure* and the *k-dimensional Hausdorff measure* on (X, d), respectively. Namely,

$$\mathcal{H}^k_\delta(E) := \inf\left\{\sum_{i=1}^{\infty} \omega_k \left(\frac{\operatorname{diam}(E_i)}{2}\right)^k \;\middle|\; E \subseteq \bigcup_{i\in\mathbb{N}} E_i \subseteq \mathsf{X},\ \sup_{i\in\mathbb{N}} \operatorname{diam}(E_i) < \delta\right\},$$

$$\mathcal{H}^k(E) := \lim_{\delta \searrow 0} \mathcal{H}^k_\delta(E) = \sup_{\delta > 0} \mathcal{H}^k_\delta(E)$$

for every set $E \subseteq \mathsf{X}$, where

$$\omega_k := \frac{\pi^{k/2}}{\Gamma\left(1 + \frac{k}{2}\right)}$$

and Γ stands for Euler's gamma function. For every $n \in \mathbb{N}$, notice that ω_n is the Euclidean volume of the unit ball in $\mathbb{R}^n$.

PI *spaces.* Throughout the whole paper, we will work in the setting of PI spaces. We say that a metric measure space $(\mathsf{X}, \mathsf{d}, \mathsf{m})$ is *uniformly locally doubling* provided that, for every radius $R > 0$, there exists a constant $C_D > 0$ such that

$$\mathsf{m}(B_{2r}(x)) \le C_D \mathsf{m}(B_r(x)) \quad \text{for every } x \in \mathsf{X} \text{ and } r \in (0, R).$$

Moreover, we say that $(\mathsf{X}, \mathsf{d}, \mathsf{m})$ supports a *weak local* $(1, 1)$-*Poincaré inequality* provided there exists a constant $\lambda \ge 1$ for which the following property holds: given any $R > 0$, there exists a constant $C_P > 0$ such that, for any function $f \in \mathrm{LIP}_{\mathrm{loc}}(\mathsf{X})$,

$$\fint_{B_r(x)} \left| f - \fint_{B_r(x)} f \, \mathrm{d}\mathsf{m} \right| \mathrm{d}\mathsf{m} \le C_P r \fint_{B_{\lambda r}(x)} \mathrm{lip}\, f \, \mathrm{d}\mathsf{m} \quad \text{for every } x \in \mathsf{X} \text{ and } r \in (0, R).$$

Definition 2.1 (PI space). We say that a metric measure space is a *PI space* provided it is uniformly locally doubling and it supports a weak local $(1, 1)$-Poincaré inequality.

In the context of PI spaces, we will consider the *codimension*-1 *Hausdorff* δ-*premeasure* $\mathcal{H}^h_\delta$ (for any $\delta > 0$) and the *codimension*-1 *Hausdorff measure* $\mathcal{H}^h$, which are given by

$$\begin{aligned} \mathcal{H}^h_\delta(E) &:= \inf\left\{ \sum_{i=1}^{\infty} \frac{\mathsf{m}(B_{r_i}(x_i))}{\mathrm{diam}(B_{r_i}(x_i))} \;\middle|\; E \subseteq \bigcup_{i \in \mathbb{N}} B_{r_i}(x_i),\ \sup_{i \in \mathbb{N}} \mathrm{diam}(B_{r_i}(x_i)) < \delta \right\}, \\ \mathcal{H}^h(E) &:= \lim_{\delta \searrow 0} \mathcal{H}^h_\delta(E) = \sup_{\delta > 0} \mathcal{H}^h_\delta(E), \end{aligned}$$

respectively, for every set $E \subseteq \mathsf{X}$.

Measured Gromov–Hausdorff convergence and tangents. Let us recall the notion of *pointed measured Gromov–Hausdorff* convergence (see, e.g., [Gigli et al. 2015]). We say that a pointed metric measure space $(\mathsf{X}, \mathsf{d}, \mathsf{m}, p)$ is *normalized* provided $C^1_p(\mathsf{m}) = 1$, where we set

$$C^r_p = C^r_p(\mathsf{m}) := \int_{B_r(p)} \left(1 - \frac{\mathsf{d}(\cdot, p)}{r}\right) \mathrm{d}\mathsf{m} \quad \text{for every } r > 0.$$

If $(\mathsf{X}, \mathsf{d}, \mathsf{m}, p)$ is any pointed metric measure space, then $(\mathsf{X}, \mathsf{d}, \mathsf{m}^1_p, p)$ is normalized, where

$$\mathsf{m}^r_p := C^r_p(\mathsf{m})^{-1} \mathsf{m} \quad \text{for every } r > 0.$$

Let $C \colon (0, +\infty) \to (0, +\infty)$ be a given nondecreasing function. Then we denote by $\mathbb{X}_{C(\cdot)}$ the family of all the equivalence classes of normalized pointed metric measure spaces that are $C(\cdot)$-doubling, in the sense that

$$\mathsf{m}(B_{2r}(x)) \le C(R) \mathsf{m}(B_r(x)) \quad \text{for every } x \in \mathsf{X} \text{ and } 0 < r \le R.$$

The equivalence classes are intended with respect to the following equivalence relation: we identify two pointed metric measure spaces $(\mathsf{X}_1, \mathsf{d}_1, \mathsf{m}_1, p_1)$ and $(\mathsf{X}_2, \mathsf{d}_2, \mathsf{m}_2, p_2)$ provided there exists a bijective isometry $\varphi \colon \mathrm{spt}(\mathsf{m}_1) \to \mathrm{spt}(\mathsf{m}_2)$ such that $\varphi(p_1) = p_2$ and $\varphi_* \mathsf{m}_1 = \mathsf{m}_2$.

Definition 2.2 (pointed measured Gromov–Hausdorff). Let $C\colon (0,+\infty)\to(0,+\infty)$ be nondecreasing. Let $(\mathsf{X},\mathsf{d},\mathsf{m},p)$, $(\mathsf{X}_i,\mathsf{d}_i,\mathsf{m}_i,p_i)\in\mathbb{X}_{C(\cdot)}$ for $i\in\mathbb{N}$ be given. Then we say that $(\mathsf{X}_i,\mathsf{d}_i,\mathsf{m}_i,p_i)\to(\mathsf{X},\mathsf{d},\mathsf{m},p)$ in the *pointed measured Gromov–Hausdorff sense* (briefly, in the *pmGH sense*) provided there exist a proper metric space $(\mathsf{Z},\mathsf{d}_\mathsf{Z})$ and isometric embeddings $\iota\colon\mathsf{X}\to\mathsf{Z}$ and $\iota_i\colon\mathsf{X}_i\to\mathsf{Z}$ for $i\in\mathbb{N}$ such that $\iota_i(p_i)\to\iota(p)$ and $(\iota_i)_*\mathsf{m}_i\rightharpoonup\iota_*\mathsf{m}$ in duality with $C_{\mathrm{bs}}(\mathsf{Z})$, meaning that $\int f\circ\iota_i\,\mathrm{d}\mathsf{m}_i\to\int f\circ\iota\,\mathrm{d}\mathsf{m}$ for every $f\in C_{\mathrm{bs}}(\mathsf{Z})$. The space Z is called a *realization* of the pmGH convergence $(\mathsf{X}_i,\mathsf{d}_i,\mathsf{m}_i,p_i)\to(\mathsf{X},\mathsf{d},\mathsf{m},p)$.

For brevity, we will identify $(\iota_i)_*\mathsf{m}_i$ with m_i itself. It is possible to construct a distance $\mathsf{d}_{\mathrm{pmGH}}$ on $\mathbb{X}_{C(\cdot)}$ whose converging sequences are exactly those converging in the pointed measured Gromov–Hausdorff sense. Moreover, the metric space $(\mathbb{X}_{C(\cdot)},\mathsf{d}_{\mathrm{pmGH}})$ is compact.

Definition 2.3 (pmGH tangent). Let $C\colon(0,+\infty)\to(0,+\infty)$ be nondecreasing. Then

$$\mathrm{Tan}_p(\mathsf{X},\mathsf{d},\mathsf{m}):=\Big\{(\mathsf{Y},\mathsf{d}_\mathsf{Y},\mathsf{m}_\mathsf{Y},q)\in\mathbb{X}_{C(\cdot)}\;\Big|\;\exists r_i\searrow 0:(\mathsf{X},r_i^{-1}\mathsf{d},\mathsf{m}_p^{r_i},p)\xrightarrow{\mathrm{pmGH}}(\mathsf{Y},\mathsf{d}_\mathsf{Y},\mathsf{m}_\mathsf{Y},q)\Big\}.$$

Notice that $(\mathsf{X},r^{-1}\mathsf{d},\mathsf{m}_p^r,p)\in\mathbb{X}_{C(\cdot)}$ holds for every $(\mathsf{X},\mathsf{d},\mathsf{m},p)\in\mathbb{X}_{C(\cdot)}$ and $r\in(0,1)$, and thus accordingly the family $\mathrm{Tan}_p(\mathsf{X},\mathsf{d},\mathsf{m})$ is (well defined and) nonempty.

Definition 2.4 (regular set). Let $n\in\mathbb{N}$ be given. Let $C\colon(0,+\infty)\to(0,+\infty)$ be any nondecreasing function such that $(\mathbb{R}^n,\mathsf{d}_e,\underline{\mathcal{L}}^n,0)\in\mathbb{X}_{C(\cdot)}$, where d_e stands for the Euclidean distance $\mathsf{d}_e(x,y):=|x-y|$ on $\mathbb{R}^n$ while $\underline{\mathcal{L}}^n$ is the normalized measure $(\mathcal{L}^n)_0^1=((n+1)/\omega_n)\mathcal{L}^n$. Then the set of *n-regular points* of a given element $(\mathsf{X},\mathsf{d},\mathsf{m},p)\in\mathbb{X}_{C(\cdot)}$ is defined as

$$\mathcal{R}_n=\mathcal{R}_n(\mathsf{X}):=\{x\in\mathsf{X}\mid\mathrm{Tan}_x(\mathsf{X},\mathsf{d},\mathsf{m})=\{(\mathbb{R}^n,\mathsf{d}_e,\underline{\mathcal{L}}^n,0)\}\}.$$

Remark 2.5. We point out that the set $\mathcal{R}_n(\mathsf{X})$ of n-regular points is Borel measurable. To check it, define $\phi\colon\mathsf{X}\to[0,+\infty)$ as

$$\phi(x):=\varlimsup_{r\searrow0}\mathsf{d}_{\mathrm{pmGH}}((\mathsf{X},r^{-1}\mathsf{d},\mathsf{m}_x^r,x),(\mathbb{R}^n,\mathsf{d}_e,\underline{\mathcal{L}}^n,0)).$$

One can readily verify that $(0,1)\ni r\mapsto(\mathsf{X},r^{-1}\mathsf{d},\mathsf{m}_x^r,x)\in\mathbb{X}_{C(\cdot)}$ is $\mathsf{d}_{\mathrm{pmGH}}$-continuous for any given $x\in\mathsf{X}$, whence

$$\phi(x)=\inf_{k\in\mathbb{N}}\sup_{q\in\mathbb{Q}\cap(0,1/k)}\mathsf{d}_{\mathrm{pmGH}}((\mathsf{X},q^{-1}\mathsf{d},\mathsf{m}_x^q,x),(\mathbb{R}^n,\mathsf{d}_e,\underline{\mathcal{L}}^n,0))\quad\text{for every }x\in\mathsf{X}.\tag{2-2}$$

Since $\mathsf{X}\ni x\mapsto(\mathsf{X},r^{-1}\mathsf{d},\mathsf{m}_x^r,x)\in\mathbb{X}_{C(\cdot)}$ is $\mathsf{d}_{\mathrm{pmGH}}$-continuous for any given $r\in(0,1)$, we deduce that $\mathsf{X}\ni x\mapsto\mathsf{d}_{\mathrm{pmGH}}((\mathsf{X},q^{-1}\mathsf{d},\mathsf{m}_x^q,x),(\mathbb{R}^n,\mathsf{d}_e,\underline{\mathcal{L}}^n,0))$ is a continuous function for any $q\in\mathbb{Q}\cap(0,1)$. Consequently, (2-2) ensures that $\mathcal{R}_n(\mathsf{X})=\{x\in\mathsf{X}:\phi(x)=0\}$ is a Borel set (in fact, a countable intersection of F_σ sets), as we claimed.

Definition 2.6 (convergences along pmGH converging sequences). Let $(\mathsf{X}_i,\mathsf{d}_i,\mathsf{m}_i,p_i)\in\mathbb{X}_{C(\cdot)}$ for $i\in\mathbb{N}$ and $(\mathsf{X},\mathsf{d},\mathsf{m},p)\in\mathbb{X}_{C(\cdot)}$ satisfy $(\mathsf{X}_i,\mathsf{d}_i,\mathsf{m}_i,p_i)\to(\mathsf{X},\mathsf{d},\mathsf{m},p)$ in the pmGH sense, with realization Z. Then we give the following definitions:

(i) Let $f_i:\mathsf{X}_i\to\mathbb{R}$ for $i\in\mathbb{N}$ and $f:\mathsf{X}\to\mathbb{R}$ be given functions. Then we say that f_i *uniformly converges* to f provided, for any $\varepsilon>0$, there exists $\delta>0$ such that $|f_i(x_i)-f(x)|\le\varepsilon$ for every $i\ge\delta^{-1}$ and $x_i\in\mathsf{X}_i$, $x\in\mathsf{X}$ with $\mathsf{d}_\mathsf{Z}(x_i,x)\le\delta$.

(ii) Let $f_i : \mathsf{X}_i \to \mathbb{R}$ for $i \in \mathbb{N}$ and $f : \mathsf{X} \to \mathbb{R}$ be given functions. Then we say that f_i *locally uniformly converges* to f provided, for any $R > 0$, we have that $f_{i|B_R(p_i)}$ uniformly converges to $f_{|B_R(p)}$.

(iii) Let $E_i \subseteq \mathsf{X}_i$ for $i \in \mathbb{N}$ and $E \subseteq \mathsf{X}$ be given Borel sets. Suppose that $\mathsf{m}_i(E_i) < +\infty$ for every $i \in \mathbb{N}$ and $\mathsf{m}(E) < +\infty$. Then we say that $E_i \to E$ *(strongly) in* L^1 provided $\mathsf{m}_i(E_i) \to \mathsf{m}(E)$ and $\mathsf{m}_i \llcorner E_i \rightharpoonup \mathsf{m} \llcorner E$ in duality with $C_{\mathrm{bs}}(\mathsf{Z})$.

(iv) Let $E_i \subseteq \mathsf{X}_i$ for $i \in \mathbb{N}$ and $E \subseteq \mathsf{X}$ be given Borel sets. Then we say that $E_i \to E$ *(strongly) in* L^1_{loc} provided $E_i \cap B_R(p_i) \to E \cap B_R(p)$ in L^1 for every $R > 0$.

Sobolev calculus. Given a metric measure space $(\mathsf{X}, \mathsf{d}, \mathsf{m})$, we define the *Sobolev space* $W^{1,2}(\mathsf{X})$ as the set of all functions $f \in L^2(\mathsf{m})$ for which there exists $(f_n)_{n\in\mathbb{N}} \subseteq \mathrm{LIP}_{\mathrm{bs}}(\mathsf{X})$ such that $f_n \to f$ in $L^2(\mathsf{m})$ and $(\mathrm{lip}\, f_n)_{n\in\mathbb{N}}$ is a bounded sequence in $L^2(\mathsf{m})$. Then $W^{1,2}(\mathsf{X})$ becomes a Banach space if endowed with the norm

$$\|f\|_{W^{1,2}(\mathsf{X})} := \left(\int |f|^2 \,\mathrm{d}\mathsf{m} + \inf_{(f_n)_n} \varliminf_{n\to\infty} \int \mathrm{lip}^2 f_n \,\mathrm{d}\mathsf{m} \right)^{1/2} \quad \text{for every } f \in W^{1,2}(\mathsf{X}),$$

where the infimum is taken among all those sequences $(f_n)_{n\in\mathbb{N}} \subseteq \mathrm{LIP}_{\mathrm{bs}}(\mathsf{X})$ such that $f_n \to f$ in $L^2(\mathsf{m})$ and $(\mathrm{lip}\, f_n)_{n\in\mathbb{N}}$ is bounded in $L^2(\mathsf{m})$. Given any function $f \in W^{1,2}(\mathsf{X})$, there exists a unique element $|\mathrm{D}f| \in L^2(\mathsf{m})$, called the *minimal relaxed slope* of f, such that the Sobolev norm of f can be expressed as $\|f\|^2_{W^{1,2}(\mathsf{X})} = \|f\|^2_{L^2(\mathsf{m})} + \||\mathrm{D}f|\|^2_{L^2(\mathsf{m})}$. Moreover, there exists a sequence $(f_n)_{n\in\mathbb{N}} \subseteq \mathrm{LIP}_{\mathrm{bs}}(\mathsf{X})$ such that $f_n \to f$ and $\mathrm{lip}\, f_n \to |\mathrm{D}f|$ in $L^2(\mathsf{m})$. This notion of Sobolev space, proposed in [Ambrosio et al. 2013], is an equivalent reformulation of the one introduced in [Cheeger 1999]. See [Ambrosio et al. 2013] for the equivalence between these two and other approaches.

The *Sobolev capacity* is the set-function on X defined as

$$\mathrm{Cap}(E) := \inf_f \|f\|^2_{W^{1,2}(\mathsf{X})} \quad \text{for every set } E \subseteq \mathsf{X}, \tag{2-3}$$

where the infimum is taken among all $f \in W^{1,2}(\mathsf{X})$ such that $f \geq 1$ holds m-a.e. on some open neighborhood of E. Here we adopt the convention that $\mathrm{Cap}(E) := +\infty$ whenever no such f exists. It holds that Cap is a submodular outer measure on X, which is finite on bounded sets and satisfies $\mathsf{m}(E) \leq \mathrm{Cap}(E)$ for every $E \subseteq \mathsf{X}$ Borel.

We shall also work with *local Sobolev spaces*, whose definition we are going to recall. Fix an open set $\Omega \subseteq \mathsf{X}$. Then we define $W^{1,2}_{\mathrm{loc}}(\Omega)$ as the space of all functions $f \in L^2_{\mathrm{loc}}(\Omega, \mathsf{m})$ such that $\eta f \in W^{1,2}(\mathsf{X})$ for every $\eta \in \mathrm{LIP}_{\mathrm{bs}}(\Omega)$. Since the minimal relaxed slope is a local object, meaning that, for any choice of $f_1, f_2 \in W^{1,2}(\mathsf{X})$,

$$|\mathrm{D}f_1| = |\mathrm{D}f_2| \quad \text{holds } \mathsf{m}\text{-a.e. on } \{f_1 = f_2\},$$

it makes sense to associate to any $f \in W^{1,2}_{\mathrm{loc}}(\Omega)$ the function $|\mathrm{D}f| \in L^2_{\mathrm{loc}}(\Omega, \mathsf{m})$ given by

$$|\mathrm{D}f| := |\mathrm{D}(\eta f)| \quad \mathsf{m}\text{-a.e. on } \{\eta = 1\}$$

for every $\eta \in \mathrm{LIP}_{\mathrm{bs}}(\Omega)$. The local Sobolev space $W^{1,2}(\Omega)$ is defined as

$$W^{1,2}(\Omega) := \{f \in W^{1,2}_{\mathrm{loc}}(\Omega) \mid f, |\mathrm{D}f| \in L^2(\mathsf{m})\}.$$

Finally, we define $W^{1,2}_0(\Omega)$ as the closure of $\mathrm{LIP}_{\mathrm{bs}}(\Omega)$ in $W^{1,2}(\Omega)$.

Following the terminology introduced in [Gigli 2015], we say that a given metric measure space $(\mathsf{X}, \mathsf{d}, \mathsf{m})$ is *infinitesimally Hilbertian* provided $W^{1,2}(\mathsf{X})$ (and thus also $W^{1,2}(\Omega)$ for any $\Omega \subseteq \mathsf{X}$ open) is a Hilbert space. Under this assumption, the mapping

$$W^{1,2}(\Omega) \times W^{1,2}(\Omega) \ni (f, g) \mapsto \nabla f \cdot \nabla g := \frac{|\mathrm{D}(f+g)|^2 - |\mathrm{D}f|^2 - |\mathrm{D}g|^2}{2} \in L^1(\Omega, \mathsf{m})$$

is bilinear and continuous. We say that a given function $f \in W^{1,2}(\Omega)$ has a *Laplacian*, briefly $f \in D(\Delta, \Omega)$, provided there exists a function $\Delta f \in L^2(\Omega, \mathsf{m})$ such that

$$\int_\Omega \nabla f \cdot \nabla g \,\mathrm{d}\mathsf{m} = -\int_\Omega g \Delta f \,\mathrm{d}\mathsf{m} \quad \text{for every } g \in W_0^{1,2}(\Omega). \tag{2-4}$$

No ambiguity may arise, since Δf is uniquely determined by (2-4). The set $D(\Delta, \Omega)$ is a linear subspace of $W^{1,2}(\Omega)$, and the resulting operator $\Delta \colon D(\Delta, \Omega) \to L^2(\Omega, \mathsf{m})$ is linear. For the sake of brevity, we shorten $D(\Delta, \mathsf{X})$ to $D(\Delta)$. By a *harmonic* function on Ω we mean an element $f \in D(\Delta, \Omega)$ such that $\Delta f = 0$.

BV *calculus.* We begin by recalling the notions of a function of bounded variation and of a set of finite perimeter in the context of metric measure spaces following [Miranda 2003].

Definition 2.7 (function of bounded variation). Let $(\mathsf{X}, \mathsf{d}, \mathsf{m})$ be a metric measure space. Let a function $f \in L^1_{\mathrm{loc}}(\mathsf{X}, \mathsf{m})$ be given. Then we define

$$|Df|(\Omega) := \inf\left\{ \varliminf_{i\to\infty} \int_\Omega \mathrm{lip}\, f_i \,\mathrm{d}\mathsf{m} \;\middle|\; (f_i)_{i\in\mathbb{N}} \subseteq \mathrm{LIP}_{\mathrm{loc}}(\Omega),\ f_i \to f \text{ in } L^1_{\mathrm{loc}}(\Omega, \mathsf{m}) \right\}$$

for any open set $\Omega \subseteq \mathsf{X}$. We declare that a function $f \in L^1_{\mathrm{loc}}(\mathsf{X}, \mathsf{m})$ is of *local bounded variation*, briefly $f \in \mathrm{BV}_{\mathrm{loc}}(\mathsf{X})$, if $|Df|(\Omega) < +\infty$ for every $\Omega \subseteq \mathsf{X}$ open and bounded. In this case, it is well known that $|\mathrm{D}f|$ extends to a locally finite measure on X. Moreover, a function $f \in L^1(\mathsf{X}, \mathsf{m})$ is said to belong to the space of *functions of bounded variation* $\mathrm{BV}(\mathsf{X}) = \mathrm{BV}(\mathsf{X}, \mathsf{d}, \mathsf{m})$ if $|Df|(\mathsf{X}) < +\infty$.

Definition 2.8 (set of finite perimeter). Let $(\mathsf{X}, \mathsf{d}, \mathsf{m})$ be a metric measure space. Let $E \subseteq \mathsf{X}$ be a Borel set and $\Omega \subseteq \mathsf{X}$ an open set. Then we define the *perimeter* of E in Ω as

$$P(E, \Omega) := \inf\left\{ \varliminf_{i\to\infty} \int_\Omega \mathrm{lip}\, f_i \,\mathrm{d}\mathsf{m} \;\middle|\; (f_i)_{i\in\mathbb{N}} \subseteq \mathrm{LIP}_{\mathrm{loc}}(\Omega),\ f_i \to \chi_E \text{ in } L^1_{\mathrm{loc}}(\Omega, \mathsf{m}) \right\},$$

in other words $P(E, \Omega) := |D\chi_E|(\Omega)$. We say that E has *locally finite perimeter* if $P(E, \Omega) < +\infty$ for every $\Omega \subseteq \mathsf{X}$ open and bounded. Moreover, we say that E has *finite perimeter* if $P(E, \mathsf{X}) < +\infty$, and we write $P(E) := P(E, \mathsf{X})$.

Given a uniformly locally doubling space $(\mathsf{X}, \mathsf{d}, \mathsf{m})$ and a Borel set $E \subseteq \mathsf{X}$, we define the *essential boundary* of E as

$$\partial^* E := \left\{ x \in \mathsf{X} \;\middle|\; \varlimsup_{r\searrow 0} \frac{\mathsf{m}(E \cap B_r(x))}{\mathsf{m}(B_r(x))} > 0,\ \varlimsup_{r\searrow 0} \frac{\mathsf{m}(E^c \cap B_r(x))}{\mathsf{m}(B_r(x))} > 0 \right\}.$$

Then $\partial^* E$ is a Borel subset of the topological boundary ∂E of E. Moreover, if $(\mathsf{X}, \mathsf{d}, \mathsf{m})$ is a PI space, then $P(E, \cdot)$ is concentrated on $\partial^* E$; see [Ambrosio 2002, Theorem 5.3].

Definition 2.9 (precise representative). Let $(\mathsf{X}, \mathsf{d}, \mathsf{m})$ be a metric measure space, and let $f : \mathsf{X} \to \mathbb{R}$ be a Borel function. Then we define the *approximate lower* and *upper limits* as

$$f^\wedge(x) := \operatorname{ap}\underline{\lim}_{y\to x} f(y) := \sup\left\{t \in \overline{\mathbb{R}} : \lim_{r\searrow 0} \frac{\mathsf{m}(B_r(x)\cap\{f<t\})}{\mathsf{m}(B_r(x))} = 0\right\},$$

$$f^\vee(x) := \operatorname{ap}\overline{\lim}_{y\to x} f(y) := \inf\left\{t \in \overline{\mathbb{R}} : \lim_{r\searrow 0} \frac{\mathsf{m}(B_r(x)\cap\{f>t\})}{\mathsf{m}(B_r(x))} = 0\right\}$$

for every $x \in \mathsf{X}$. Here we adopt the convention that

$$\inf\varnothing = +\infty \quad \text{and} \quad \sup\varnothing = -\infty.$$

Moreover, we define the *precise representative* $\bar{f} : \mathsf{X} \to \overline{\mathbb{R}}$ of f as

$$\bar{f}(x) := \tfrac{1}{2}(f^\wedge(x) + f^\vee(x)) \quad \text{for every } x \in \mathsf{X},$$

where we adopt the convention that $+\infty - \infty = 0$.

We define the *jump set* $J_f \subseteq \mathsf{X}$ of the function f as the Borel set

$$J_f := \{x \in \mathsf{X} : f^\wedge(x) < f^\vee(x)\}.$$

It is well known that if $(\mathsf{X}, \mathsf{d}, \mathsf{m})$ is a PI space and $f \in BV(\mathsf{X})$, then J_f is a countable union of essential boundaries of sets of finite perimeter, so that in particular $\mathsf{m}(J_f) = 0$. See [Ambrosio et al. 2004, Proposition 5.2]. Moreover, as proved in [Kinnunen et al. 2014, Lemma 3.2], we have that

$$|\mathrm{D}f|(\mathsf{X}\setminus\mathsf{X}_f) = 0, \quad \text{where } \mathsf{X}_f := \{x \in \mathsf{X} \mid -\infty < f^\wedge(x) \le f^\vee(x) < +\infty\}, \tag{2-5}$$

and thus in particular $-\infty < \bar{f}(x) < +\infty$ holds for $|\mathrm{D}f|$-a.e. $x \in \mathsf{X}$.

Definition 2.10. Let $(\mathsf{X}, \mathsf{d}, \mathsf{m})$ be a metric measure space, and let $f : \mathsf{X} \to \mathbb{R}$ be Borel. Then we define the *subgraph* of f, denoted by $\mathcal{G}_f \subseteq \mathsf{X}\times\mathbb{R}$, as the Borel set

$$\mathcal{G}_f := \{(x, t) \in \mathsf{X}\times\mathbb{R} : t < f(x)\}.$$

Lemma 2.11. *Let $(\mathsf{X}, \mathsf{d}, \mathsf{m})$ be a locally uniformly doubling metric measure space, and let $f : \mathsf{X} \to \mathbb{R}$ be a Borel function. Then*

$$(x,t) \in \partial^*\mathcal{G}_f \implies t \in [f^\wedge(x), f^\vee(x)] \qquad \textit{and} \qquad t \in (f^\wedge(x), f^\vee(x)) \implies (x,t) \in \partial^*\mathcal{G}_f.$$

In particular, if $x \in \mathsf{X}_f \setminus J_f$, then $\partial^*\mathcal{G}_f \cap (\{x\}\times\mathbb{R}) \subseteq \{(x, \bar{f}(x))\}$.

Proof. In the proof, the constant C_D may change from line to line and it only depends on the doubling constant at scale $R = 1$. We can compute, for $r \in (0, \varepsilon)$, using Fubini's theorem,

$$\begin{aligned}\frac{(\mathsf{m}\otimes\mathcal{L}^1)(B_r(x,t)\cap\mathcal{G}_f)}{(\mathsf{m}\otimes\mathcal{L}^1)(B_r(x,t))} &\le \frac{(\mathsf{m}\otimes\mathcal{L}^1)((B_r(x)\times B_r(t))\cap\mathcal{G}_f)}{(\mathsf{m}\otimes\mathcal{L}^1)(B_{r/2}(x)\times B_{r/2}(t))}\\ &\le C_D\frac{(\mathsf{m}\otimes\mathcal{L}^1)(\{(y,t)\in B_r(x)\times B_r(t) : t < f(y)\})}{r\,\mathsf{m}(B_r(x))}\\ &\le C_D\frac{⨍_{t-r}^{t+r}\mathsf{m}(\{y\in B_r(x) : s < f(y)\})\,\mathrm{d}s}{\mathsf{m}(B_r(x))} \le C_D\frac{\mathsf{m}(B_r(x)\cap\{f > t-\varepsilon\})}{\mathsf{m}(B_r(x))}.\end{aligned}$$

Therefore, if $(x,t)\in\partial^*\mathcal{G}_f$, then $t\le f^\vee(x)$. Similarly, we can show that if $r\in(0,\varepsilon)$,

$$\frac{(\mathsf{m}\otimes\mathcal{L}^1)(B_r(x,t)\setminus\mathcal{G}_f)}{(\mathsf{m}\otimes\mathcal{L}^1)(B_r(x,t))}\le C_D\frac{\mathsf{m}(B_r(x)\cap\{f<t+\varepsilon\})}{\mathsf{m}(B_r(x))},$$

which in turn shows that if $(x,t)\in\partial^*\mathcal{G}_f$, then $t\ge f^\wedge(x)$. Conversely, arguing as above, we can show that if $r\in(0,\varepsilon)$,

$$\frac{(\mathsf{m}\otimes\mathcal{L}^1)(B_{2r}(x,t)\cap\mathcal{G}_f)}{(\mathsf{m}\otimes\mathcal{L}^1)(B_{2r}(x,t))}\ge C_D\frac{\mathsf{m}(B_r(x)\cap\{f>t+\varepsilon\})}{\mathsf{m}(B_r(x))}$$

and

$$\frac{(\mathsf{m}\otimes\mathcal{L}^1)(B_r(x,t)\setminus\mathcal{G}_f)}{(\mathsf{m}\otimes\mathcal{L}^1)(B_r(x,t))}\ge C_D\frac{\mathsf{m}(B_r(x)\cap\{f<t-\varepsilon\})}{\mathsf{m}(B_r(x))},$$

which yield the second claim. □

Definition 2.12 (decomposition of the total variation measure). Let $(\mathsf{X},\mathsf{d},\mathsf{m})$ be a PI space and $f\in\mathrm{BV}(\mathsf{X})$. Then we write $|\mathrm{D}f|$ as $|\mathrm{D}f|^a+|\mathrm{D}f|^s$, where $|\mathrm{D}f|^a\ll\mathsf{m}$ and $|\mathrm{D}f|^s\perp\mathsf{m}$. We can decompose the singular part $|\mathrm{D}f|^s$ as $|\mathrm{D}f|^j+|\mathrm{D}f|^c$, where the *jump part* is given by $|\mathrm{D}f|^j:=|\mathrm{D}f|\llcorner J_f$ while the *Cantor part* is given by $|\mathrm{D}f|^c:=|\mathrm{D}f|^s\llcorner(\mathsf{X}\setminus J_f)$.

By [Ambrosio et al. 2015, Theorem 5.1] and its proof, taking into account the elementary inequality

$$a\le\sqrt{1+a^2}\le 1+a\quad\text{for every } a>0,$$

(or see [Ambrosio et al. 2004, Proposition 4.2]) we obtain the following proposition.

Proposition 2.13. *Let $(\mathsf{X},\mathsf{d},\mathsf{m})$ be a PI space and $f\in\mathrm{BV}(\mathsf{X})$. Then $\mathcal{G}_f$ is a set of locally finite perimeter in $\mathsf{X}\times\mathbb{R}$ and, denoting with π the projection map $\mathsf{X}\times\mathbb{R}\to\mathsf{X}$,*

$$|\mathrm{D}f|\le\pi_*|\mathrm{D}\chi_{\mathcal{G}_f}|\le|\mathrm{D}f|+\mathsf{m}.$$

In particular, if $C\subseteq\mathsf{X}$ is a Borel set satisfying $|\mathrm{D}f|^c=|\mathrm{D}f|\llcorner C$, then

$$\pi_*(|\mathrm{D}\chi_{\mathcal{G}_f}|\llcorner C\times\mathbb{R})=|\mathrm{D}f|\llcorner C.$$

Definition 2.14. Let $(\mathsf{X},\mathsf{d},\mathsf{m})$ be a metric measure space and $F\in\mathrm{BV}_{\mathrm{loc}}(\mathsf{X})^k$. We define

$$|\mathrm{D}F|(\Omega):=\inf\left\{\varliminf_{i\to\infty}\int_\Omega\left(\sum_{j=1}^k(\operatorname{lip}F_i^j)^2\right)^{1/2}\mathrm{d}\mathsf{m}\;\middle|\;(F_i)_i\subseteq\mathrm{LIP}_{\mathrm{loc}}(\Omega)^k,\ F_i\to F\text{ in }L^1_{\mathrm{loc}}(\Omega)^k\right\}$$

for any open set $\Omega\subseteq\mathsf{X}$. Then we extend this definition to Borel subsets of X, as done in the scalar case; see [Brena and Gigli 2024, Section 2.3]. We also define

$$J_F:=\bigcup_{i=1}^k J_{F_i}$$

It is clear that Definition 2.12 extends immediately to the vector-valued case.

2B. RCD *spaces.* We assume the reader is familiar with the language of $\mathrm{RCD}(K,N)$ spaces. Recall that an $\mathrm{RCD}(K,N)$ space is an infinitesimally Hilbertian metric measure space verifying the curvature-dimension condition $\mathrm{CD}(K,N)$, in the sense of Lott–Villani–Sturm, for some $K \in \mathbb{R}$ and $N \in [1,\infty)$. Here we only consider finite-dimensional $\mathrm{RCD}(K,N)$ spaces, namely we assume $N < \infty$. Finite-dimensional RCD spaces are PI. As proven in [Bruè and Semola 2020; De Philippis et al. 2017; Gigli and Pasqualetto 2021; Kell and Mondino 2018; Mondino and Naber 2019], the following structure theorem holds.

Theorem 2.15. *Let* $(\mathsf{X},\mathsf{d},\mathsf{m})$ *be an* $\mathrm{RCD}(K,N)$ *space. Then there exists a number* $n \in \mathbb{N}$ *with* $1 \le n \le N$, *called the* ***essential dimension*** *of* $(\mathsf{X},\mathsf{d},\mathsf{m})$, *such that* $\mathsf{m}(\mathsf{X} \setminus \mathcal{R}_n) = 0$. *Moreover, the regular set* $\mathcal{R}_n$ *is* (m, n)*-rectifiable and* $\mathsf{m} \ll \mathcal{H}^n \llcorner \mathcal{R}_n$.

Recall that $\mathcal{R}_n$ is said to be (m, n)*-rectifiable* provided there exist Borel subsets $(A_i)_{i\in\mathbb{N}}$ of $\mathcal{R}_n$ such that each A_i is bi-Lipschitz equivalent to a subset of $\mathbb{R}^n$ and $\mathsf{m}\big(\mathcal{R}_n \setminus \bigcup_i A_i\big) = 0$.

Sobolev calculus on RCD *spaces.* We assume the reader is familiar with the language of $L^p(\mathsf{m})$*-normed* $L^\infty(\mathsf{m})$*-modules* [Gigli 2018b] and $L^0(\mathrm{Cap})$*-normed* $L^0(\mathrm{Cap})$*-modules* [Debin et al. 2021]. Let $(\mathsf{X},\mathsf{d},\mathsf{m})$ be a given $\mathrm{RCD}(K,N)$ space. We denote by $L^2(T^*\mathsf{X})$ and $L^2(T\mathsf{X})$ the *cotangent module* and the *tangent module* of $(\mathsf{X},\mathsf{d},\mathsf{m})$, respectively. Moreover, $L^0(T\mathsf{X})$ stands for the $L^0(\mathsf{m})$-completion of $L^2(T\mathsf{X})$, in the sense of [Gigli 2018a, Theorem/Definition 2.7]. A fundamental class of Sobolev functions on X is the algebra of *test functions* [Savaré 2014; Gigli 2018b]:

$$\mathrm{Test}^\infty(\mathsf{X}) := \{ f \in D(\Delta) \cap L^\infty(\mathsf{m}) \mid |\mathrm{D}f| \in L^\infty(\mathsf{m}),\ \Delta f \in W^{1,2}(\mathsf{X}) \cap L^\infty(\mathsf{m}) \}.$$

Since RCD spaces enjoy the *Sobolev-to-Lipschitz property*, each function in $\mathrm{Test}^\infty(\mathsf{X})$ has a Lipschitz representative. Moreover, $\mathrm{Test}^\infty(\mathsf{X})$ is dense in $W^{1,2}(\mathsf{X})$ and $\nabla f \cdot \nabla g \in W^{1,2}(\mathsf{X})$ for every $f, g \in \mathrm{Test}^\infty(\mathsf{X})$. The class of *test vector fields* is then defined as

$$\mathrm{TestV}(\mathsf{X}) := \left\{ \sum_{i=1}^k f_i \nabla g_i \;\middle|\; k \in \mathbb{N},\ (f_i)_{i=1}^k,\ (g_i)_{i=1}^k \subseteq \mathrm{Test}^\infty(\mathsf{X}) \right\} \subseteq L^2(T\mathsf{X}).$$

We denote by $L^0_{\mathrm{Cap}}(T\mathsf{X})$ the *capacitary tangent module* on $(\mathsf{X},\mathsf{d},\mathsf{m})$ introduced in [Debin et al. 2021, Theorem 3.6] and by $\overline{\nabla}\colon \mathrm{Test}^\infty(\mathsf{X}) \to L^0_{\mathrm{Cap}}(T\mathsf{X})$ the capacitary gradient operator. Given any Borel measure μ on X such that $\mu \ll \mathrm{Cap}$ (meaning that $\mu(N) = 0$ for every $N \subseteq \mathsf{X}$ Borel with $\mathrm{Cap}(N) = 0$), we denote by $\pi_\mu\colon L^0(\mathrm{Cap}) \to L^0(\mu)$ the canonical projection.

Letting $L^0_\mu(T\mathsf{X})$ be the quotient of $L^0_{\mathrm{Cap}}(T\mathsf{X})$ up to μ-a.e. equality (where we identify two elements $v, w \in L^0_{\mathrm{Cap}}(T\mathsf{X})$ if $\pi_\mu(|v - w|) = 0$ holds μ-a.e.), we have a natural projection map $\pi_\mu\colon L^0_{\mathrm{Cap}}(T\mathsf{X}) \to L^0_\mu(T\mathsf{X})$, which satisfies $|\pi_\mu(v)| = \pi_\mu(|v|)$ μ-a.e. for all $v \in L^0_{\mathrm{Cap}}(T\mathsf{X})$. The space $L^0_\mu(T\mathsf{X})$ is an $L^0(\mu)$-normed $L^0(\mu)$-module. As pointed out in [Debin et al. 2021, Proposition 3.9], the quotient $L^0_{\mathsf{m}}(T\mathsf{X})$ can be identified with the tangent module $L^0(T\mathsf{X})$ and the projection $\pi_{\mathsf{m}}\colon L^0_{\mathrm{Cap}}(T\mathsf{X}) \to L^0(T\mathsf{X})$ satisfies $\nabla f = \pi_\mu(\overline{\nabla} f)$ for every $f \in \mathrm{Test}^\infty(\mathsf{X})$. Due to this consistency, to ease the notation we will indicate the capacitary gradient of a test function f with ∇f instead of $\overline{\nabla} f$.

The *Hessian* of $f \in \mathrm{Test}^\infty(\mathsf{X})$ is the unique tensor $\mathrm{Hess}(f) \in L^2(T^*\mathsf{X}) \otimes L^2(T^*\mathsf{X})$ with

$$2\int h\,\mathrm{Hess}(f)(\nabla g_1 \otimes \nabla g_2)\,\mathrm{d}\mathsf{m} = -\int \nabla f \cdot \nabla g_1 \,\mathrm{div}(h\nabla g_2) + \nabla f \cdot \nabla g_2 \,\mathrm{div}(h \nabla g_1) + h \nabla f \cdot \nabla(\nabla g_1 \cdot \nabla g_2)\,\mathrm{d}\mathsf{m}$$

for every $g_1, g_2, h \in \mathrm{Test}^\infty(\mathsf{X})$. Recall that a vector field $v \in L^2(T\mathsf{X})$ is said to have a *divergence*, briefly $v \in D(\mathrm{div})$, provided there exists a function $\mathrm{div}(v) \in L^2(\mathsf{m})$ such that

$$\int \nabla f \cdot v \, \mathrm{d}\mathsf{m} = -\int f \,\mathrm{div}(v) \,\mathrm{d}\mathsf{m} \quad \text{for every } f \in W^{1,2}(\mathsf{X}); \tag{2-6}$$

note that $\mathrm{div}(v)$ is uniquely determined by (2-6). The Hessian above is a local object:

$$\chi_{\{f_1=f_2\}} \cdot \mathrm{Hess}(f_1) = \chi_{\{f_1=f_2\}} \cdot \mathrm{Hess}(f_2) \quad \text{for every } f_1, f_2 \in \mathrm{Test}^\infty(\mathsf{X}). \tag{2-7}$$

The validity of this property allows us to define the Hessian of a harmonic function f defined on an open set $\Omega \subseteq \mathsf{X}$, as we are going to discuss. As proven in [Jiang 2014], the harmonic function $f : \Omega \to \mathbb{R}$ is locally Lipschitz. In particular, $\eta f \in \mathrm{Test}^\infty(\mathsf{X})$ for every cut-off function $\eta \in \mathrm{Test}^\infty(\mathsf{X})$ such that $\mathrm{spt}(\eta) \Subset \Omega$. As shown in [Ambrosio et al. 2016; Mondino and Naber 2019], there are plenty of cut-off test functions: given any $x \in \mathsf{X}$ and $0 < r < R$, there exists $\eta \in \mathrm{Test}^\infty(\mathsf{X})$ with $0 \le \eta \le 1$ such that $\eta = 1$ on $B_r(x)$ and $\mathrm{spt}(\eta) \Subset B_R(x)$. Thanks to this fact and to (2-7), it makes sense to m-a.e. define the measurable function $|\mathrm{Hess}(f)| : \Omega \to [0, +\infty)$ as

$$|\mathrm{Hess}(f)| := |\mathrm{Hess}(\eta f)| \quad \mathsf{m}\text{-a.e. on } \{\eta = 1\}$$

for every $\eta \in \mathrm{Test}^\infty(\mathsf{X})$ such that $\mathrm{spt}(\eta) \Subset \Omega$.

BV *calculus on* RCD *spaces.* Now we focus on BV functions and sets of finite perimeter on $\mathrm{RCD}(K, N)$ spaces. The following notion was introduced in [Ambrosio et al. 2019, Definition 4.1].

Definition 2.16 (tangents to a set of finite perimeter). Let $(\mathsf{X}, \mathsf{d}, \mathsf{m}, p)$ be a pointed $\mathrm{RCD}(K, N)$ space and $E \subseteq \mathsf{X}$ a set of locally finite perimeter. Then we define $\mathrm{Tan}_p(\mathsf{X}, \mathsf{d}, \mathsf{m}, E)$ as the family of all quintuplets $(\mathsf{Y}, \mathsf{d}_\mathsf{Y}, \mathsf{m}_\mathsf{Y}, q, F)$ that verify the following two conditions:

(1) $(\mathsf{Y}, \mathsf{d}_\mathsf{Y}, \mathsf{m}_\mathsf{Y}, q) \in \mathrm{Tan}_p(\mathsf{X}, \mathsf{d}, \mathsf{m})$.

(2) $F \subseteq \mathsf{Y}$ is a set of locally finite perimeter with $\mathsf{m}_\mathsf{Y}(F) > 0$ for which the following property holds: along a sequence $r_i \searrow 0$ such that $(\mathsf{X}, r_i^{-1}\mathsf{d}, \mathsf{m}_p^{r_i}, p) \to (\mathsf{Y}, \mathsf{d}_\mathsf{Y}, \mathsf{m}_\mathsf{Y}, q)$ in the pmGH sense, with realization Z, it holds that $\chi_E^i \to \chi_F$ in L^1_{loc}, where by χ_E^i we mean the characteristic function of E intended in the rescaled space $(\mathsf{X}, r_i^{-1}\mathsf{d})$. If this is the case, we write

$$(\mathsf{X}, r_i^{-1}\mathsf{d}, \mathsf{m}_p^{r_i}, p, E) \to (\mathsf{Y}, \mathsf{d}_\mathsf{Y}, \mathsf{m}_\mathsf{Y}, q, F).$$

The following theorem is extracted from [Brena and Gigli 2024, Theorem 3.13], see also [Bruè et al. 2023b, Theorem 2.4].

Theorem 2.17. *Let* $(\mathsf{X}, \mathsf{d}, \mathsf{m})$ *be an* $\mathrm{RCD}(K, N)$ *space and let* $F \in \mathrm{BV}(\mathsf{X})^k$. *Then there exists a unique, up to* $|\mathrm{D}F|$*-a.e. equality,* $\nu_F \in L^0_{\mathrm{Cap}}(T\mathsf{X})^k$ *such that* $|\nu_F| = 1$ $|\mathrm{D}F|$*-a.e. and*

$$\sum_{j=1}^k \int_\mathsf{X} F_j \,\mathrm{div}(v_j) \,\mathrm{d}\mathsf{m} = -\int_\mathsf{X} \pi_{|\mathrm{D}F|}(v) \cdot \nu_F \,\mathrm{d}|\mathrm{D}F| \quad \textit{for every } v = (v_1, \dots, v_k) \in \mathrm{TestV}(\mathsf{X})^k.$$

Notice that if $F \in \mathrm{BV}(\mathsf{X})^k$, we consider ν_F as an element of $L^0_{\mathrm{Cap}}(T\mathsf{X})^k$ that is defined $|\mathrm{D}F|$-a.e.. This allows us, via a standard localization procedure, to define ν_F even if F is a vector-valued function of locally bounded variation, or, in other words, if F is a k-tuple of functions of locally bounded variation. In particular, if E is a set of locally finite perimeter, we naturally have a unique, up to $|\mathrm{D}\chi_E|$-a.e. equality, $\nu_E \in L^0_{\mathrm{Cap}}(T\mathsf{X})$, where we understand $\nu_E = \nu_{\chi_E}$.

Next we recall that, as proven in [Bruè et al. 2023b], each set of locally finite perimeter E in an $\mathrm{RCD}(K, N)$ space $(\mathsf{X}, \mathsf{d}, \mathsf{m})$ satisfies $|\mathrm{D}\chi_E| \ll \mathrm{Cap}$. Notice however that the same result holds in every metric measure space; see [Brena and Gigli 2024, Theorem 2.5]. By the coarea formula, this absolute continuity extends immediately to total variations, so that

$$|\mathrm{D}F| \ll \mathrm{Cap} \quad \text{for every } F \in \mathrm{BV}_{\mathrm{loc}}(\mathsf{X})^n.$$

The following proposition summarizes results about sets of finite perimeter that are now well known in the context of PI spaces and are proved in [Ambrosio 2002; Eriksson-Bique et al. 2021]; see also [Ambrosio 2001].

Proposition 2.18. *Let $(\mathsf{X}, \mathsf{d}, \mathsf{m})$ be a PI space and let $E \subseteq \mathsf{X}$ be a set of locally finite perimeter. Then, for $|\mathrm{D}\chi_E|$-a.e. $x \in \mathsf{X}$ the following hold:*

(i) *E is **asymptotically minimal** at x, in the sense that there exist $r_x > 0$ and a function $\omega_x : (0, r_x) \to (0, \infty)$ with $\lim_{r\searrow 0} \omega_x(r) = 0$ satisfying*

$$|\mathrm{D}\chi_E|(B_r(x)) \le (1+\omega_x(r))|\mathrm{D}\chi_{E'}|(B_r(x)) \quad \textit{if } r \in (0, r_x) \textit{ and } E' \Delta E \Subset B_r(x).$$

(ii) *$|\mathrm{D}\chi_E|$ is **asymptotically doubling** at x:*

$$\varlimsup_{r\searrow 0} \frac{|\mathrm{D}\chi_E|(B_{2r}(x))}{|\mathrm{D}\chi_E|(B_r(x))} < \infty.$$

(iii) *We have the estimates*

$$0 < \varliminf_{r\searrow 0} \frac{r|\mathrm{D}\chi_E|(B_r(x))}{\mathsf{m}(B_r(x))} \le \varlimsup_{r\searrow 0} \frac{r|\mathrm{D}\chi_E|(B_r(x))}{\mathsf{m}(B_r(x))} < \infty.$$

(iv) *The following density estimate holds:*

$$\varliminf_{r\searrow 0} \min\left\{\frac{\mathsf{m}(B_r(x)\cap E)}{\mathsf{m}(B_r(x))}, \frac{\mathsf{m}(B_r(x)\setminus E)}{\mathsf{m}(B_r(x))}\right\} > 0.$$

Remark 2.19. It is well known (see [Heinonen et al. 2015, Theorem 3.4.3 and p. 77]) that for an asymptotically doubling measure the Lebesgue differentiation theorem holds. In particular, if E is a set of locally finite perimeter in a PI space and $f \in L^1(|\mathrm{D}\chi_E|)$, then, for $|\mathrm{D}\chi_E|$-a.e. x,

$$\lim_{r\searrow 0} \fint_{B_r(x)} |f(y) - f(x)| \,\mathrm{d}|\mathrm{D}\chi_E|(y) = 0.$$

Let us now introduce the notion of reduced boundary of a set of locally finite perimeter. First, we introduce the set $\mathcal{R}_n^*$. Following [Ambrosio and Tilli 2004], given a metric measure space $(\mathsf{X}, \mathsf{d}, \mu)$ and a real number $k \geq 0$, we define the *upper* and *lower k-dimensional densities* of μ as

$$\overline{\Theta}_k(\mu, x) := \overline{\lim_{r \searrow 0}} \frac{\mu(B_r(x))}{\omega_k r^k} \quad \text{and} \quad \underline{\Theta}_k(\mu, x) := \underline{\lim}_{r \searrow 0} \frac{\mu(B_r(x))}{\omega_k r^k} \qquad \text{for every } x \in \mathsf{X},$$

respectively. In the case where $\overline{\Theta}_k(\mu, x)$ and $\underline{\Theta}_k(\mu, x)$ coincide, we denote their common value by $\Theta_k(\mu, x) \in [0, +\infty]$, and we call it the *k-dimensional density* of μ at x.

Definition 2.20. Let $(\mathsf{X}, \mathsf{d}, \mathsf{m})$ be an $\mathrm{RCD}(K, N)$ space having essential dimension n. Then we define the set $\mathcal{R}_n^* = \mathcal{R}_n^*(\mathsf{X}) \subseteq \mathcal{R}_n$ as

$$\mathcal{R}_n^* := \{x \in \mathcal{R}_n \mid \exists \Theta_n(\mathsf{m}, x) \in (0, +\infty)\}.$$

In the case in which $\mathsf{m} = \mathcal{H}^N$, by the Bishop–Gromov comparison, one has that $\Theta_N(\mathcal{H}^N, x)$ exists and is positive for every $x \in \mathsf{X}$. Moreover, the volume convergence results in [De Philippis and Gigli 2018] and the lower semicontinuity of the density imply that $\Theta_N(\mathcal{H}^N, x) \leq 1$ for every $x \in \mathsf{X}$. Notice that the set $\mathcal{R}_n^*$ is Borel, see Remark 2.5. As shown in [Ambrosio et al. 2018, Theorem 4.1], $\mathsf{m}(\mathsf{X} \setminus \mathcal{R}_n^*) = 0$.

Definition 2.21 (reduced boundary). Let $(\mathsf{X}, \mathsf{d}, \mathsf{m})$ be an $\mathrm{RCD}(K, N)$ space. Let $E \subseteq \mathsf{X}$ be a set of locally finite perimeter. Then we define the *reduced boundary* $\mathcal{F}E \subseteq \partial^* E$ of E as the set of all points $x \in \mathcal{R}_n^*$ satisfying all four conclusions of Proposition 2.18 and such that

$$\mathrm{Tan}_x(\mathsf{X}, \mathsf{d}, \mathsf{m}, E) = \{(\mathbb{R}^n, \mathsf{d}_e, \underline{\mathcal{L}}^n, 0, \{x_n > 0\})\}, \tag{2-8}$$

where $n \in \mathbb{N}$, $n \leq N$ stands for the essential dimension of $(\mathsf{X}, \mathsf{d}, \mathsf{m})$. We recall that the set of points $x \in X$ that satisfy (2-8) is denoted by $\mathcal{F}_n E$.

As proven in [Bruè et al. 2023a] after [Ambrosio et al. 2019; Bruè et al. 2023b], taking into account the forthcoming Theorem 3.3, the perimeter measure $|\mathrm{D}\chi_E|$ is concentrated on the reduced boundary $\mathcal{F}E$.

Remark 2.22. By the proof of [Ambrosio et al. 2019, Corollary 4.10], by [Ambrosio et al. 2019, Corollary 3.4], and by the membership to $\mathcal{R}_n^*$, we see that the following hold for any $x \in \mathcal{F}E$:

(i) If $r_i \searrow 0$ is such that

$$(\mathsf{X}, r_i^{-1}\mathsf{d}, \mathsf{m}_x^{r_i}, x) \to (\mathbb{R}^n, \mathsf{d}_e, \underline{\mathcal{L}}^n, 0) \tag{2-9}$$

in a realization $(\mathsf{Z}, \mathsf{d}_\mathsf{Z})$, then, up to not relabeled subsequences and a change of coordinates in $\mathbb{R}^n$,

$$(\mathsf{X}, r_i^{-1}\mathsf{d}, \mathsf{m}_x^{r_i}, x, E) \to (\mathbb{R}^n, \mathsf{d}_e, \underline{\mathcal{L}}^n, 0, \{x_n > 0\})$$

in the same realization $(\mathsf{Z}, \mathsf{d}_\mathsf{Z})$. Notice that, given a sequence $r_i \searrow 0$, it is always possible to find a subsequence satisfying (2-9).

(ii) If $r_i \searrow 0$ is such that

$$(\mathsf{X}, r_i^{-1}\mathsf{d}, \mathsf{m}_x^{r_i}, x, E) \to (\mathbb{R}^n, \mathsf{d}_e, \underline{\mathcal{L}}^n, 0, \{x_n > 0\})$$

in a realization $(\mathsf{Z}, \mathsf{d}_\mathsf{Z})$, then $|\mathrm{D}\chi_E|$ weakly converges to $|\mathrm{D}\chi_{\{x_n>0\}}|$ in duality with $C_{\mathrm{bs}}(Z)$.

(iii) We have

$$\begin{aligned}\lim_{r\searrow 0}\frac{\mathsf{m}(B_r(x))}{r^n}&=\omega_n\Theta_n(\mathsf{m},x)\in(0,+\infty),\\ \lim_{r\searrow 0}\frac{C_x^r}{r^n}&=\frac{\omega_n}{n+1}\Theta_n(\mathsf{m},x),\\ \lim_{r\searrow 0}\frac{|\mathrm{D}\chi_E|(B_r(x))}{r^{n-1}}&=\omega_{n-1}\Theta_n(\mathsf{m},x).\end{aligned}\tag{2-10}$$

Definition 2.23 (good coordinates). Let $(\mathsf{X},\mathsf{d},\mathsf{m})$ be an $\mathrm{RCD}(K,N)$ space of essential dimension n. Let $E\subseteq\mathsf{X}$ be a set of locally finite perimeter and $x\in\mathcal{F}E$ be given. Then we say that an n-tuple $u=(u^1,\dots,u^n)$ of harmonic functions $u^\ell\colon B_{r_x}(x)\to\mathbb{R}$ is a *system of good coordinates* for E at x provided the following properties are satisfied:

(1) For any $\ell,j=1,\dots,n$,

$$\lim_{r\searrow 0}\fint_{B_r(x)}|\nabla u^\ell\cdot\nabla u^j-\delta_{\ell j}|\,\mathrm{d}\mathsf{m}=\lim_{r\searrow 0}\fint_{B_r(x)}|\nabla u^\ell\cdot\nabla u^j-\delta_{\ell j}|\,\mathrm{d}|\mathrm{D}\chi_E|=0.$$

(2) For any $\ell=1,\dots,n$, there exists $\nu_\ell(x)$ defined as follows:

$$\nu_\ell(x):=\lim_{r\searrow 0}\fint_{B_r(x)}\nu_E\cdot\nabla u^\ell\,\mathrm{d}|\mathrm{D}\chi_E|,\quad \lim_{r\searrow 0}\fint_{B_r(x)}|\nu_\ell(x)-\nu_E\cdot\nabla u^\ell|\,\mathrm{d}|\mathrm{D}\chi_E|=0.\tag{2-11}$$

(3) The resulting vector $\nu(x):=(\nu_1(x),\dots,\nu_n(x))\in\mathbb{R}^n$ satisfies $|\nu(x)|=1$.

The following theorem is proved in [Bruè et al. 2023a, Theorem 3.6].

Theorem 2.24. *Let $(\mathsf{X},\mathsf{d},\mathsf{m})$ be an $\mathrm{RCD}(K,N)$ space of essential dimension n. Let $E\subseteq\mathsf{X}$ be a set of locally finite perimeter and $x\in\mathcal{F}E$ be given. Then, good coordinates exist at $|\mathrm{D}\chi_E|$-a.e. point $x\in\mathcal{F}E$.*

Remark 2.25. Let $(\mathsf{X},\mathsf{d},\mathsf{m})$ be an $\mathrm{RCD}(K,N)$ space of essential dimension n, let $x\in\mathsf{X}$ and let $u=(u^1,\dots,u^n)$ be an n-tuple of harmonic functions satisfying

$$\lim_{r\searrow 0}\fint_{B_r(x)}|\nabla u^\ell\cdot\nabla u^j-\delta_{\ell j}|\,\mathrm{d}\mathsf{m}=0.$$

Given a sequence of radii $r_i\searrow 0$ such that

$$(\mathsf{X},r_i^{-1}\mathsf{d},\mathsf{m}_x^{r_i},x)\to(\mathbb{R}^n,\mathsf{d}_e,\underline{\mathcal{L}}^n,0)$$

and a fixed realization of such convergence, it follows from the results recalled in [Bruè et al. 2023b, Section 1.2.3] (see also [Bruè et al. 2023b, (1.22)], a consequence of the improved Bochner inequality in [Han 2018]) that, up to extracting a not relabeled subsequence, the functions in

$$\{r_i^{-1}u^j\}_i\quad\text{for } j=1,\dots,n$$

converge locally uniformly to orthogonal coordinate functions of $\mathbb{R}^n$.

The ensuing result is taken from [Bruè et al. 2023a, Proposition 4.8].

Proposition 2.26. *Let* $(\mathsf{X}, \mathsf{d}, \mathsf{m})$ *be an* $\mathrm{RCD}(K, N)$ *space of essential dimension* n. *Let* $E \subseteq \mathsf{X}$ *be a set of locally finite perimeter. Then, for* $|\mathrm{D}\chi_E|$*-a.e.* $x \in \mathsf{X}$, *the following property holds. Suppose that* $u = (u^1, \dots, u^n)\colon B_r(x) \to \mathbb{R}^n$ *is a system of good coordinates for* E *at* x. *Let* $\nu(x) \in \mathbb{R}^n$ *be as in Definition 2.23. If the coordinates* (x_ℓ) *on the (Euclidean) tangent space to* X *at* x *are chosen so that the maps* (u^ℓ) *converge to* $(x_\ell)\colon \mathbb{R}^n \to \mathbb{R}^n$ *when properly rescaled, then the blow-up* H *of* E *at* x *(in the sense of finite perimeter sets) is*

$$H = \{y \in \mathbb{R}^n \mid y \cdot \nu(x) \geq 0\}.$$

Splitting maps. Let us now present the notion of a δ-splitting map. We follow closely the presentation in [Bruè et al. 2023b], compare with [Bruè et al. 2023b, Definition 3.4].

Definition 2.27 (splitting map). Let $(\mathsf{X}, \mathsf{d}, \mathsf{m})$ be an $\mathrm{RCD}(K, N)$ space. Let $x \in \mathsf{X}$, $k \in \mathbb{N}$, and $r, \delta > 0$ be given. Then a map $u = (u_1, \dots, u_k)\colon B_r(x) \to \mathbb{R}^k$ is said to be a *δ-splitting map* provided the following properties hold:

(i) u_ℓ is harmonic, meaning that, for every $\ell = 1, \dots, k$, we have $u_\ell \in D(\Delta, B_r(x))$ and $\Delta u_\ell = 0$, and u_ℓ is C_N-Lipschitz for every $\ell = 1, \dots, k$.

(ii) $r^2 \fint_{B_r(x)} |\mathrm{Hess}(u_\ell)|^2 \,\mathrm{dm} \leq \delta$ for every $\ell = 1, \dots, k$.

(iii) $\fint_{B_r(x)} |\nabla u_\ell \cdot \nabla u_j - \delta_{\ell j}| \,\mathrm{dm} \leq \delta$ for every $\ell, j = 1, \dots, k$.

As already noticed in [Bruè et al. 2023b, Remark 3.6], in the classical definition of δ-splitting maps in the smooth setting, in item (i) above the stronger condition $|\nabla u| \leq 1 + \delta$ is required. Anyway we stress that when $(\mathsf{X}, \mathsf{d}, \mathsf{m})$ is an $\mathrm{RCD}(-\delta, N)$ space and u is a δ-splitting map as above, we have that $\sup_{y \in B_{r/2}(x)} |\nabla u|(y) \leq 1 + C_N \delta^{1/2}$, see [Bruè et al. 2022, Remark 3.3], and compare with [Cheeger and Naber 2015, Equations (3.42)–(3.46)]. This means that, for δ small enough, if u is a δ-splitting map on $B_r(x)$ on an $\mathrm{RCD}(-\delta, N)$ space as above, then it is a $C_N \delta^{1/2}$-splitting map on $B_{r/2}(x)$ in the classical smooth sense.

In the following lemma we slightly improve previous results obtained in [Bruè et al. 2023a; 2023b], and we show that we can find good coordinates with respect to *every* $\mathrm{BV}_{\mathrm{loc}}$ function.

Lemma 2.28. *Let* $(\mathsf{X}, \mathsf{d}, \mathsf{m})$ *be an* $\mathrm{RCD}(K, N)$ *space of essential dimension* n *and* $\eta \in (0, 1)$. *Then there exists a sequence of* n*-tuples of harmonic* $C_{K,N}$*-Lipschitz maps* $\{u_k\}_k$,

$$u_k = (u_k^1, \dots, u_k^n) : B_{2r_k}(x_k) \to \mathbb{R}^n,$$

and a sequence of pairwise disjoint Borel sets $\{D_k\}_k$ *with* $D_k \subseteq B_{r_k}(x_k)$ *such that*

(i) *for every* $f \in \mathrm{BV}_{\mathrm{loc}}(\mathsf{X})$,

$$|\mathrm{D}f|\Big(\mathsf{X} \setminus \bigcup_k D_k\Big) = 0,$$

(ii) *for every* $x \in D_k$, u_k *is an* η*-splitting map on* $B_r(x)$ *for any* $r \in (0, r_k)$,

(iii) *there exists a Borel matrix-valued map $M=(M_{\ell,j}):D_k\to\mathbb{R}^{n\times n}$ satisfying*

$$\lim_{r\searrow 0}\fint_{B_r(x)}|\nabla u_k^\ell\cdot\nabla u_k^j-M(x)_{\ell,j}|\,\mathsf{d}\mathsf{m}=0. \tag{2-12}$$

To any such collection of η-splitting maps, we can therefore associate a natural map

$$\bigcup_k D_k\to\mathbb{N},\qquad x\mapsto k(x).$$

Proof. The proof follows the arguments given in the proof of [Bruè et al. 2023b, Theorem 3.2]. However, as we need a slightly stronger statement, we include the details of the proof.

Fix a countable dense set $S\subseteq\mathcal{R}_n$. Let $y\in S$ be given. If $\varepsilon>0$ is small enough and $r\in(0,\sqrt{\varepsilon/|K|})\cap\mathbb{Q}$ is such that

$$\mathsf{d}_{\mathrm{pmGH}}((\mathsf{X},r^{-1}\mathsf{d},\mathsf{m}_y^r,y),(\mathbb{R}^n,\mathsf{d}_e,\underline{\mathcal{L}}^n,0))<\varepsilon,$$

then, by [Bruè et al. 2023b, Corollary 3.10], we obtain a δ-splitting map $u_{y,r}:B_{5r}(y)\to\mathbb{R}^n$ for some δ (which can be made arbitrarily small, taking ε small enough). Let

$$D_{y,r}:=\left\{x\in B_{(5/4)r}(y)\;\middle|\;u_{y,r}\text{ is an }\eta\text{-splitting map on }B_s(x)\text{ for every }s\in\left(0,\tfrac{5}{4}r\right)\right\}.$$

The claim of the lemma will be proved with the sequence of sets $\{D_{y,r}\}_{y,r}$ and maps $\{u_{y,r}\}_{y,r}$ after making the sets disjoint and restricting the maps.

Assume now, by contradiction, that the claim is false. Then, using a locality argument and the coarea formula, we find a set of finite perimeter $E\subseteq\mathsf{X}$ such that

$$|\mathrm{D}\chi_E|\Big(\mathsf{X}\setminus\bigcup_{y,r}D_{y,r}\Big)>0. \tag{2-13}$$

Fix $\varepsilon>0$ to be determined later. If $x\in\mathcal{F}E$, then there exists $r=r(x)\in\mathbb{Q}\cap(0,1)$ such that $|K|r^2<\varepsilon<4$ and

$$\mathsf{d}_{\mathrm{pmGH}}((\mathsf{X},r^{-1}\mathsf{d},\mathsf{m}_x^r,x),(\mathbb{R}^n,\mathsf{d}_e,\underline{\mathcal{L}}^n,0))<\varepsilon\quad\text{and}\quad\frac{r|\mathrm{D}\chi_E|(B_{r/4}(x))}{\mathsf{m}(B_{r/4}(x))}>2\frac{\omega_{n-1}}{\omega_n}.$$

By density of S and thanks to an easy continuity argument, we deduce that, for some point $y=y(x)\in S\cap B_{r/2}(x)$,

$$\mathsf{d}_{\mathrm{pmGH}}((\mathsf{X},r^{-1}\mathsf{d},\mathsf{m}_y^r,y),(\mathbb{R}^n,\mathsf{d}_e,\underline{\mathcal{L}}^n,0))<\varepsilon\quad\text{and}\quad\frac{r|\mathrm{D}\chi_E|(B_{r/4}(y))}{\mathsf{m}(B_{r/4}(y))}>2\frac{\omega_{n-1}}{\omega_n}. \tag{2-14}$$

By the discussion above (that is, [Bruè et al. 2023b, Corollary 3.10]), we obtain a δ-splitting map $u_{y,r}:B_{5r}(y)\to\mathbb{R}^n$ for some $\delta=\delta(\varepsilon)$ (which can be made arbitrarily small, taking ε small enough). By [Bruè et al. 2023b, Corollary 3.12], $u_{y,r}$ is a $C_N\delta^{1/4}$-splitting map on $B_s(x)$ for any $x\in D_{y,r}^\varepsilon\subseteq B_{(5/4)r}(y)$ and $s\in\left(0,\tfrac{5}{4}r\right)$, where

$$\mathcal{H}_5^h(B_{(5/4)r}(y)\setminus D_{y,r}^\varepsilon)\le C_N\delta^{1/2}\frac{\mathsf{m}(B_{(5/2)r}(x))}{\frac{5}{2}r}.$$

Therefore, $D_{y,r}^\varepsilon\subseteq D_{y,r}$ if $C_N\delta^{1/4}<\eta$.

We apply the Vitali covering lemma to the family $\{B_{r(x)/4}(y(x))\}_{x\in\mathcal{F}E}$ constructed as above, and we obtain a sequence of disjoint balls $\{B_{r(x_i)/4}(y(x_i))\}_i$ such that

$$\mathcal{F}E \subseteq \bigcup_i B_{(5/4)r(x_i)}(y(x_i)).$$

Set

$$D^\varepsilon := \bigcup_i D^\varepsilon_{y(x_i),r(x_i)}.$$

Following the computations in the proof of [Bruè et al. 2023b, Theorem 3.2], we obtain

$$\begin{aligned}\mathcal{H}^h_5(\mathcal{F}E\setminus D^\varepsilon) &\le \sum_{i\in\mathbb{N}}\mathcal{H}^h_5(B_{(5/4)r(x_i)}(y(x_i))\setminus D^\varepsilon_{y(x_i),r(x_i)}) \le C_N\delta^{1/2}\sum_{i\in\mathbb{N}}\frac{\mathsf{m}(B_{(5/2)r(x_i)}(y(x_i)))}{\frac{5}{2}r(x_i)}\\ &\le C_N\delta^{1/2}\sum_{i\in\mathbb{N}}\frac{\mathsf{m}(B_{r(x_i)/4}(y(x_i)))}{\frac{1}{4}r(x_i)} \le C_N\delta^{1/2}|\mathrm{D}\chi_E|(\mathsf{X}), \end{aligned}\tag{2-15}$$

where the constants C_N may change from line to line, in the third inequality we are using the doubling property together with the fact that $r(x_i)$ is sufficiently small, and in the last inequality we are using (2-14) together with the fact that the $\{B_{r(x_i)/4}(y(x_i))\}$ are disjoint. Let now $\{\varepsilon_i\}_i$ with $\varepsilon_i \searrow 0$ be such that the corresponding $\{\delta_i\}_i$ satisfy both $\delta_i^{1/2}\le 2^{-i}$ and $C_N\delta_i^{1/4}<\eta$, and set

$$G := \bigcup_i D^{\varepsilon_i} \subseteq D_{y,r}.$$

Then $\mathcal{H}^h_5(\mathcal{F}E\setminus G)=0$, which contradicts (2-13).

Finally, item (iii) is a direct consequence of the fact that, since u^ℓ_k is harmonic for every $\ell=1,\dots,n$ and $k\in\mathbb{N}$, one can give a pointwise meaning to $\nabla u^\ell_k(x)\cdot\nabla u^j_k(x)$, compare with [Bruè et al. 2023a, Remark 2.10]. □

Definition 2.29. Let $(\mathsf{X},\mathsf{d},\mathsf{m})$ be an $\mathrm{RCD}(K,N)$ space having essential dimension n. Then by a *good collection of splitting maps* on X we mean a family $\{\boldsymbol{u}_\eta : \eta\in(0,n^{-1})\cap\mathbb{Q}\}$ of sequences $\boldsymbol{u}_\eta=(u_{\eta,k})_{k\in\mathbb{N}}$ of maps

$$u_{\eta,k}=(u^1_{\eta,k},\dots,u^n_{\eta,k})\colon B_{r_{\eta,k}}(x_{\eta,k})\to\mathbb{R}^n$$

as in Lemma 2.28. We will denote by $D_{\eta,k}\subseteq B_{r_{\eta,k}}(x_{\eta,k})$ the sets associated to $\boldsymbol{u}_\eta$ as in Lemma 2.28. We define

$$D_\eta := \bigcup_{k=1}^\infty D_{\eta,k},$$

and by $k_\eta(x): D_\eta\to\mathbb{N}$ we denote the unique index satisfying $x\in D_{\eta,k_\eta(x)}$. For every $x\in D_{\eta,k}$ we define a matrix $A_\eta(x)\in\mathbb{R}^{n\times n}$ such that, with the same notation of Lemma 2.28, $A_\eta(x)M_\eta(x)A_\eta(x)^T=\mathrm{Id}_{n\times n}$. The existence of such a matrix follows from the choice of $\bar\eta_n$. Indeed, from the construction of the symmetric matrix $B_\eta(x)$, it follows that $\|\mathrm{Id}-M_\eta(x)\|_{L^\infty}<n^{-1}$, thus $\|\mathrm{Id}-M_\eta(x)\|_{\mathrm{op}}<1$, so that the conclusion follows from the spectral theorem.

Notice that, for every $f \in \mathrm{BV}(\mathsf{X})$, we have $|\mathrm{D}f|(\mathsf{X} \setminus D_\eta) = 0$. Let us fix $\eta \in (0, n^{-1}) \cap \mathbb{Q}$. Since for every $x \in D_\eta$ there exists a unique $k_\eta(x)$ such that $x \in D_{\eta, k_\eta(x)}$, and since there exists also a splitting map $u_{\eta, k_\eta(x)}$ on some ball around x, one has that the limit

$$\lim_{r \to 0} \fint_{B_r(x)} \nabla u^\ell_{\eta, k_{\eta(x)}} \cdot \nabla u^j_{\eta, k_{\eta(x)}} \, \mathrm{dm}$$

exists for every $\ell, j \in \{1, \dots, n\}$, compare the end of the proof of Lemma 2.28 and [Bruè et al. 2023a, Remark 2.10]. Hence, for every $\eta \in (0, n^{-1}) \cap \mathbb{Q}$, one can give a pointwise meaning to the $\mathbb{R}^{n \times n}$-valued map

$$M_\eta : x \in D_\eta \mapsto (\nabla u^\ell_{\eta, k_{\eta(x)}} \cdot \nabla u^j_{\eta, k_{\eta(x)}})_{\ell, j \in \{1, \dots, n\}}(x) \tag{2-16}$$

such that (2-12) holds.

3. Main results

3A. ***Representation formula for the perimeter.*** In this section we prove, by exploiting [Deng 2020] and the same argument of [Bruè et al. 2023a], that the total variation of every BV function is concentrated on $\mathcal{R}_n^*$. We use the latter information to deduce that the perimeter measure of every set of locally finite perimeter is mutually absolutely continuous with respect to $\mathcal{H}^{n-1}$. We will be using the following theorem, which is proved in [Deng 2020, Theorem 1.3].

Theorem 3.1. *Let* $(\mathsf{X}, \mathsf{d}, \mathsf{m})$ *be an* $\mathrm{RCD}(K, N)$ *space, with* $K \in \mathbb{R}$ *and* $N \geq 1$, *and* $\mathrm{spt}(\mathsf{m}) = \mathsf{X}$. *Then* $(\mathsf{X}, \mathsf{d}, \mathsf{m})$ *is nonbranching, i.e., if* $\gamma, \sigma : [0, L] \to \mathsf{X}$ *are two unit speed geodesics satisfying* $\gamma(0) = \sigma(0)$ *and* $\gamma(t_0) = \sigma(t_0)$ *for some* $t_0 \in (0, L)$, *then* $\gamma = \sigma$.

Proposition 3.2. *Let* $(\mathsf{X}, \mathsf{d}, \mathsf{m})$ *be an* $\mathrm{RCD}(K, N)$ *space having essential dimension n. Suppose that* $\gamma : [0, 1] \to \mathsf{X}$ *is a geodesic satisfying* $\gamma_t \in \mathcal{R}_n^*$ *for a dense family of* $t \in (0, 1)$. *Then* $\gamma_t \in \mathcal{R}_n^*$ *for every* $t \in (0, 1)$.

Proof. Let $\delta \in \left(0, \frac{1}{20}\right)$ be fixed. Theorem 3.1 ensures that the constant-speed reparametrization of $\gamma|_{[\delta/2, 1-\delta/2]}$ on $[0, 1]$ is the unique geodesic between its endpoints. Then [Deng 2020, (166)] gives $\varepsilon = \varepsilon(N, \delta) > 0$, $\bar{r} = \bar{r}(N, \delta) > 0$, and $C = C(N, \delta) > 0$ such that

$$\left| \frac{\mathsf{m}(B_r(\gamma_s))}{\mathsf{m}(B_r(\gamma_{s'}))} - 1 \right| \leq C|s - s'|^{\frac{1}{2(1+2N)}} \quad \text{for every } r \in (0, \bar{r}) \text{ and } s, s' \in [\delta, 1 - \delta] \text{ with } |s - s'| < \varepsilon.$$

In particular, for any $s, s' \in [\delta, 1 - \delta]$ with $|s - s'| < \varepsilon$, we have

$$\left| \frac{\mathsf{m}(B_r(\gamma_s))}{\omega_n r^n} \left(\frac{\mathsf{m}(B_r(\gamma_{s'}))}{\omega_n r^n} \right)^{-1} - 1 \right| \leq C|s - s'|^{\frac{1}{2(1+2N)}} \quad \text{for every } r \in (0, \bar{r}). \tag{3-1}$$

Now let $t \in [\delta, 1 - \delta]$ be fixed, and choose a sequence $(t_i)_{i \in \mathbb{N}} \subseteq \gamma^{-1}(\mathcal{R}_n^*) \cap [\delta, 1 - \delta] \cap (t - \varepsilon, t + \varepsilon)$ such that $t_i \to t$. Up to a not relabeled subsequence, we can assume that $\Theta_n(\mathsf{m}, \gamma_{t_i}) \to \lambda$ for some $\lambda \in [0, +\infty]$. Pick sequences $(r_j)_{j \in \mathbb{N}}, (\tilde{r}_j)_{j \in \mathbb{N}} \subseteq (0, \bar{r})$ such that

$$\frac{\mathsf{m}(B_{r_j}(\gamma_t))}{\omega_n r_j^n} \to \overline{\Theta}_n(\mathsf{m}, \gamma_t) \quad \text{and} \quad \frac{\mathsf{m}(B_{\tilde{r}_j}(\gamma_t))}{\omega_n \tilde{r}_j^n} \to \underline{\Theta}_n(\mathsf{m}, \gamma_t).$$

Plugging $(s, s', r) = (t, t_i, r_j)$ or $(s, s', r) = (t, t_i, \tilde{r}_j)$ into (3-1) and letting $j \to \infty$, we deduce that $\overline{\Theta}_n(\mathsf{m}, \gamma_t) < +\infty$ and

$$\left|\frac{\overline{\Theta}_n(\mathsf{m}, \gamma_t)}{\Theta_n(\mathsf{m}, \gamma_{t_i})} - 1\right|, \left|\frac{\underline{\Theta}_n(\mathsf{m}, \gamma_t)}{\Theta_n(\mathsf{m}, \gamma_{t_i})} - 1\right| \le C|t - t_i|^{\frac{1}{2(1+2N)}} \quad \text{for every } i \in \mathbb{N}. \tag{3-2}$$

Similarly, plugging $(s, s', r) = (t_i, t, r_j)$ or $(s, s', r) = (t_i, t, \tilde{r}_j)$ into (3-1) and letting $j \to \infty$, we deduce that $\underline{\Theta}_n(\mathsf{m}, \gamma_t) > 0$ and

$$\left|\frac{\Theta_n(\mathsf{m}, \gamma_{t_i})}{\overline{\Theta}_n(\mathsf{m}, \gamma_t)} - 1\right|, \left|\frac{\Theta_n(\mathsf{m}, \gamma_{t_i})}{\underline{\Theta}_n(\mathsf{m}, \gamma_t)} - 1\right| \le C|t - t_i|^{\frac{1}{2(1+2N)}} \quad \text{for every } i \in \mathbb{N}. \tag{3-3}$$

Observe that (3-2) and (3-3) imply, respectively, that, for every $i \in \mathbb{N}$,

$$|\overline{\Theta}_n(\mathsf{m}, \gamma_t) - \underline{\Theta}_n(\mathsf{m}, \gamma_t)| \le 2C|t - t_i|^{\frac{1}{2(1+2N)}}\, \Theta_n(\mathsf{m}, \gamma_{t_i}), \tag{3-4a}$$

$$|\overline{\Theta}_n(\mathsf{m}, \gamma_t) - \underline{\Theta}_n(\mathsf{m}, \gamma_t)| \le 2C|t - t_i|^{\frac{1}{2(1+2N)}}\, \frac{\overline{\Theta}_n(\mathsf{m}, \gamma_t)\underline{\Theta}_n(\mathsf{m}, \gamma_t)}{\Theta_n(\mathsf{m}, \gamma_{t_i})}. \tag{3-4b}$$

Hence, we can conclude that $\overline{\Theta}_n(\mathsf{m}, \gamma_t) = \underline{\Theta}_n(\mathsf{m}, \gamma_t)$ by letting $i \to \infty$ in (3-4a) if $\lambda < +\infty$, or in (3-4b) if $\lambda = +\infty$. This shows that $\gamma_t \in \mathcal{R}_n^*$ for every $t \in [\delta, 1 - \delta]$. Thanks to the arbitrariness of δ, we proved that $\gamma_t \in \mathcal{R}_n^*$ for every $t \in (0, 1)$, as desired. □

Theorem 3.3. *Let* $(\mathsf{X}, \mathsf{d}, \mathsf{m})$ *be an* $\mathrm{RCD}(K, N)$ *space having essential dimension* n. *Then*

$$|\mathrm{D}f|(\mathsf{X} \setminus \mathcal{R}_n^*) = 0 \quad \textit{for every } f \in \mathrm{BV}_{\mathrm{loc}}(\mathsf{X}).$$

Proof. The statement can be achieved by repeating verbatim the proof of [Bruè et al. 2023a, Theorem 3.1], using $\mathcal{R}_n^*$ instead of $\mathcal{R}_n$ and Proposition 3.2 instead of [Bruè et al. 2023a, Proposition 2.14]. □

The following theorem answers [Semola 2020, Conjecture 5.32] in the affirmative.

Theorem 3.4 (representation formula for the perimeter). *Let* $(\mathsf{X}, \mathsf{d}, \mathsf{m})$ *be an* $\mathrm{RCD}(K, N)$ *space having essential dimension* n. *Let* $E \subseteq \mathsf{X}$ *be a set of locally finite perimeter. Then*

$$|\mathrm{D}\chi_E| = \Theta_n(\mathsf{m}, \cdot)\mathcal{H}^{n-1} \llcorner \mathcal{F}E. \tag{3-5}$$

In particular, $\Theta_{n-1}(|\mathrm{D}\chi_E|, x) = \Theta_n(\mathsf{m}, x)$ *for* $\mathcal{H}^{n-1}$*-a.e.* $x \in \mathcal{F}E$.

Proof. Up to a standard localization argument, we can suppose that E is of finite perimeter. Define $R_j := \{x \in \mathcal{R}_n^* : 2^j \le \Theta_n(\mathsf{m}, x) < 2^{j+1}\}$ for all $j \in \mathbb{Z}$. Notice that $\{R_j\}_{j \in \mathbb{Z}}$ is a measurable partition of $\mathcal{R}_n^*$. Given $j \in \mathbb{Z}$ and $B \subseteq \mathsf{X}$ Borel, for any $x \in B \cap R_j \cap \mathcal{F}E$, there exists

$$\Theta_{n-1}(|\mathrm{D}\chi_E|, x) = \frac{\omega_n}{\omega_{n-1}} \lim_{r \searrow 0} \frac{r|\mathrm{D}\chi_E|(B_r(x))}{\mathsf{m}(B_r(x))} \frac{\mathsf{m}(B_r(x))}{\omega_n r^n} = \Theta_n(\mathsf{m}, x) \in [2^j, 2^{j+1}). \tag{3-6}$$

Therefore, an application of [Ambrosio and Tilli 2004, Theorem 2.4.3] yields, for any $B \subseteq \mathsf{X}$ Borel,

$$2^j \mathcal{H}^{n-1}(B \cap R_j \cap \mathcal{F}E) \le |\mathrm{D}\chi_E|(B \cap R_j) \le 2^{j+n} \mathcal{H}^{n-1}(B \cap R_j \cap \mathcal{F}E),$$

whence $2^j\mathcal{H}^{n-1}\llcorner(R_j\cap\mathcal{F}E)\le|\mathrm{D}\chi_E|\llcorner R_j\le 2^{j+n}\mathcal{H}^{n-1}\llcorner(R_j\cap\mathcal{F}E)$. Thanks to Theorem 3.3, we deduce that $\mu_E:=\mathcal{H}^{n-1}\llcorner\mathcal{F}E$ is a σ-finite Borel measure on X satisfying $|\mathrm{D}\chi_E|\ll\mu_E\ll|\mathrm{D}\chi_E|$. In particular, we know from [Bruè et al. 2023b, Theorem 4.1] that the set $\mathcal{F}E$ is countably $\mathcal{H}^{n-1}$-rectifiable, so that [Ambrosio and Kirchheim 2000, Theorem 5.4] and the computation in (3-6) ensure that

$$\frac{\mathrm{d}|\mathrm{D}\chi_E|}{\mathrm{d}\mu_E}(x)=\lim_{r\searrow 0}\frac{|\mathrm{D}\chi_E|(B_r(x))}{\omega_{n-1}r^{n-1}}=\Theta_n(\mathsf{m},x)$$

is satisfied for $\mathcal{H}^{n-1}$-a.e. $x\in\mathcal{F}E$. Therefore, the identity stated in (3-5) is achieved. □

Remark 3.5. Notice that, as a consequence of [Bruè et al. 2023a, Corollary 3.2], for any set E of locally finite perimeter in an $\mathrm{RCD}(K,N)$ space $(\mathsf{X},\mathsf{d},\mathsf{m})$ of essential dimension n, we have

$$|D\chi_E|=\frac{\omega_{n-1}}{\omega_n}\mathcal{H}^h\llcorner\mathcal{F}E.$$

Hence, taking also (3-5) into account, we conclude that the measure $\mathcal{H}^h$ and $\mathcal{H}^{n-1}$ are mutually absolutely continuous on the reduced boundary $\mathcal{F}E$.

3B. *Auxiliary results.* Let $(\mathsf{X},\mathsf{d},\mathsf{m})$ be an $\mathrm{RCD}(K,N)$ space of essential dimension n. Notice that if a given function $u:B_r(\bar{x})\to\mathbb{R}$ is harmonic, then ∇u admits a quasicontinuous representative in a localization of $L^0_{\mathrm{Cap}}(T\mathsf{X})$. Also, by tensorization of the energy, if $k\in\mathbb{N}$, then the function

$$\mathsf{X}\times\mathbb{R}^k\supseteq B_r(\bar{x})\times\mathbb{R}^k\ni(x,y)\mapsto u(x)$$

is harmonic, and hence it admits a quasicontinuous representative in a localization of $L^0_{\mathrm{Cap}}(T(\mathsf{X}\times\mathbb{R}^k))$ with respect to the relevant capacity. Therefore, the following definition is meaningful.

Definition 3.6. Let $(\mathsf{X},\mathsf{d},\mathsf{m})$ be an $\mathrm{RCD}(K,N)$ space having essential dimension n. Let $f\in\mathrm{BV}(\mathsf{X})$ be given. Fix a good collection $\{\boldsymbol{u}_\eta\}_\eta$ of splitting maps on X. Then, given any $\eta\in(0,n^{-1})\cap\mathbb{Q}$, the $|\mathrm{D}f|$-measurable map $\nu_f^{\boldsymbol{u}_\eta}:\mathsf{X}\to\mathbb{R}^n$ is defined at $|\mathrm{D}f|$-a.e. $x\in\mathsf{X}$ as

$$\nu_f^{\boldsymbol{u}_\eta}(x):=((\nu_f\cdot\nabla u^1_{\eta,k_\eta(x)})(x),\dots,(\nu_f\cdot\nabla u^n_{\eta,k_\eta(x)})(x)).$$

The $|\mathrm{D}\chi_{\mathcal{G}_f}|$-measurable map $\nu_{\mathcal{G}_f}^{\boldsymbol{u}_\eta}:\mathsf{X}\times\mathbb{R}\to\mathbb{R}^{n+1}$ is defined at $|\mathrm{D}\chi_{\mathcal{G}_f}|$-a.e. $p=(x,t)\in\mathsf{X}\times\mathbb{R}$ as

$$\nu_{\mathcal{G}_f}^{\boldsymbol{u}_\eta}(p):=((\nu_{\mathcal{G}_f}\cdot\nabla u^1_{\eta,k_\eta(x)})(p),\dots,(\nu_{\mathcal{G}_f}\cdot\nabla u^n_{\eta,k_\eta(x)})(p),(\nu_{\mathcal{G}_f}\cdot\nabla\pi^2)(p));$$

notice that $|\mathrm{D}\chi_{\mathcal{G}_f}|$-a.e. $p=(x,t)$ satisfies $x\in D_\eta$ as a consequence of Lemma 2.28(i), Proposition 2.13, and the existence of functions of locally bounded variation whose total variation equals m.

In view of the following proposition, recall the definition of the reduced boundary in use in this note, Definition 2.21. In particular, notice that, by definition, $\mathcal{F}\mathcal{G}_f\subseteq\mathcal{R}^*_{n+1}(\mathsf{X}\times\mathbb{R})$, and we will use this inclusion throughout (in particular, recall the properties stated in Remark 2.22). Notice finally that the matrix valued maps $C_f\ni x\mapsto A_\eta(x)$ in the proposition below are independent of f (up to the choice of their domain).

Proposition 3.7. *Let* $(\mathsf{X},\mathsf{d},\mathsf{m})$ *be an* $\mathrm{RCD}(K,N)$ *space having essential dimension* n. *Let* $f\in \mathrm{BV}(\mathsf{X})$ *be given. Let* $\{\boldsymbol{u}_\eta\}_\eta$ *be a good collection of splitting maps on* X. *Then there exists a Borel set* $C_f\subseteq \mathsf{X}$ *satisfying the following properties*:

(i) $|\mathrm{D}f|^c=|\mathrm{D}f|\llcorner C_f$ *and* $\mathsf{m}(C_f)=0$.

(ii) $C_f\subseteq \mathcal{R}_n^*(\mathsf{X})\setminus J_f$ *and* $\mathcal{F}\mathcal{G}_f\cap(C_f\times\mathbb{R})=(\mathrm{id}_{\mathsf{X}},\bar f)(C_f)$.

(iii) *Given any* $\eta\in(0,n^{-1})\cap\mathbb{Q}$ *and* $x\in C_f$, *for* $A_\eta(x)\in\mathbb{R}^{n\times n}$ *as in Definition 2.29*, $(A_\eta(x)u_{\eta,k(x)},\pi^2)$ *is a set of good coordinates for* $\mathcal{G}_f$ *at* $(x,\bar f(x))$.

(iv) *If* $u=(u^1,\dots,u^{n+1})\colon B_{r_x}(x,\bar f(x))\to\mathbb{R}^{n+1}$ *is a system of good coordinates for* $\mathcal{G}_f$ *at* $(x,\bar f(x))$ *for some* $x\in C_f$, *and the coordinates* (x_ℓ) *on the* (*Euclidean*) *tangent space to* $\mathsf{X}\times\mathbb{R}$ *at* $(x,\bar f(x))$ *are chosen so that the maps* (u^ℓ) *converge to* $(x_\ell)\colon\mathbb{R}^{n+1}\to\mathbb{R}^{n+1}$ (*when properly rescaled, see Remark 2.25*), *then the blow-up of* $\mathcal{G}_f$ *at* $(x,\bar f(x))$ *can be written as*

$$H:=\{y\in\mathbb{R}^{n+1}\mid y\cdot\nu(x,\bar f(x))\ge 0\},$$

where the unit vector $\nu(x,\bar f(x)):=(\nu^1(x,\bar f(x)),\dots,\nu^{n+1}(x,\bar f(x)))$ *is given by* (2-11).

(v) *If* $p=(x,\bar f(x))\in C_f\times\mathbb{R}$, *then, for every* $\eta\in(0,n^{-1})\cap\mathbb{Q}$, *we have* $x\in D_{\eta,k_\eta(x)}$ *for some* $k_\eta(x)$ *and* p *is a point of density* 1 *of* $D_{\eta,k_\eta(x)}\times\mathbb{R}$ *for* $|\mathrm{D}\chi_{\mathcal{G}_f}|$.

Proof. Let us start this proof by defining several sets whose intersection will define C_f. Hence we will define C_f in (3-11), and we will verify each item separately.

For every $\eta\in(0,n^{-1})\cap\mathbb{Q}$ and every $k\in\mathbb{N}$, take $\mathcal{D}_{\eta,k}$ to be the set of points of density 1 in $(D_{\eta,k}\times\mathbb{R})\cap\mathcal{F}\mathcal{G}_f$ with respect to $|\mathrm{D}\chi_{\mathcal{G}_f}|$. We thus have that $\bigcup_{k\in\mathbb{N}}\mathcal{D}_{\eta,k}$ covers $|\mathrm{D}\chi_{\mathcal{G}_f}|$-almost all $D_\eta\times\mathbb{R}$. Hence, by Proposition 2.13 and Lemma 2.28, the set $\pi^1\big(\bigcup_{k\in\mathbb{N}}\mathcal{D}_{\eta,k}\big)$ covers $|\mathrm{D}f|$-almost all X for every $\eta\in(0,n^{-1})\cap\mathbb{Q}$. As a consequence, if we denote $\mathcal{D}:=\bigcap_{\eta\in(0,n^{-1})\cap\mathbb{Q}}\pi^1\big(\bigcup_{k\in\mathbb{N}}\mathcal{D}_{\eta,k}\big)$, then

$$|\mathrm{D}f|(\mathsf{X}\setminus\mathcal{D})=0. \tag{3-7}$$

Let $\mathcal{A}\subseteq\mathsf{X}\times\mathbb{R}$ be the set of points $(x,t)\in\mathsf{X}\times\mathbb{R}$ such that, if $u=(u^1,\dots,u^{n+1}):B_{r_{(x,t)}}(x,t)\to\mathbb{R}^{n+1}$ is a system of good coordinates for $\mathcal{G}_f$ at (x,t), and the coordinates (x_ℓ) on the (Euclidean) tangent space to $\mathsf{X}\times\mathbb{R}$ at (x,t) are chosen so that the maps (u^ℓ) converge to $(x_\ell)\colon\mathbb{R}^{n+1}\to\mathbb{R}^{n+1}$ (when properly rescaled), then the blow-up of $\mathcal{G}_f$ at (x,t) can be written as

$$\{y\in\mathbb{R}^{n+1}\mid y\cdot\nu(x,t)\ge 0\},$$

where the unit vector $\nu(x,t):=(\nu^1(x,t),\dots,\nu^{n+1}(x,t))$ is given by (2-11). Then, by Proposition 2.26, we have also that

$$|\mathrm{D}\chi_{\mathcal{G}_f}|((\mathsf{X}\times\mathbb{R})\setminus\mathcal{A})=0. \tag{3-8}$$

Let $\eta\in(0,n^{-1})\cap\mathbb{Q}$, and let $\mathcal{T}_\eta$ be the Lebesgue points of $\nu_{\mathcal{G}_f}^{\boldsymbol{u}_\eta}$ (defined in Definition 3.6) with respect to $|\mathrm{D}\chi_{\mathcal{G}_f}|$. Let $\mathcal{T}:=\bigcap_{\eta\in(0,n^{-1})\cap\mathbb{Q}}\mathcal{T}_\eta$, and notice that

$$|\mathrm{D}\chi_{\mathcal{G}_f}|((\mathsf{X}\times\mathbb{R})\setminus\mathcal{T})=0. \tag{3-9}$$

Let us fix $\eta \in (0, n^{-1}) \cap \mathbb{Q}$ and $k \in \mathbb{N}$. Let $\widetilde{M}_\eta := (M_\eta, \pi^2)$ be defined on $\mathsf{X} \times \mathbb{R}$, where M_η is defined in (2-16). Notice that $\widetilde{M}_\eta$ is $|\mathrm{D}\chi_{\mathcal{G}_f}|$-measurable. Let $\mathcal{S}_\eta$ be the Lebesgue points of $\widetilde{M}_\eta$ with respect to $|\mathrm{D}\chi_{\mathcal{G}_f}|$, and let $\mathcal{S} := \bigcap_{\eta \in (0,n^{-1})} \mathcal{S}_\eta$. Notice that

$$|\mathrm{D}\chi_{\mathcal{G}_f}|((\mathsf{X} \times \mathbb{R}) \setminus \mathcal{S}) = 0. \tag{3-10}$$

Let $S \subseteq \mathsf{X}_f$ with $\mathsf{m}(S) = 0$ be such that $|\mathrm{D}f|^s$ is concentrated on S (recall (2-5)). Let us now define

$$C_f := S \cap (\mathcal{R}_n^*(\mathsf{X}) \setminus J_f) \cap \Bigg(\bigcap_{\eta \in (0,n^{-1}) \cap \mathbb{Q}} D_\eta \Bigg) \cap \pi^1(\mathcal{A} \cap \mathcal{T} \cap \mathcal{S} \cap \mathcal{F}\mathcal{G}_f) \cap \mathcal{D}, \tag{3-11}$$

where D_η is defined in Definition 2.29, J_f is the jump set of f, $\mathcal{F}\mathcal{G}_f$ is the reduced boundary of $\mathcal{G}_f$, and $\mathcal{A}$, $\mathcal{T}$, $\mathcal{S}$ are defined above. Let us verify each item separately.

Item (i). Notice that $|\mathrm{D}f|^c$ is concentrated on S. Moreover, $|\mathrm{D}f|^c$ is concentrated on $\mathsf{X} \setminus J_f$, and, due to Lemma 2.28, $|\mathrm{D}f|^c$ is concentrated on $\bigcap_{\eta \in (0,n^{-1}) \cap \mathbb{Q}} D_\eta$ as well. Due to (3-7), $|\mathrm{D}f|$ is concentrated on $\mathcal{D}$. Furthermore, $|\mathrm{D}\chi_{\mathcal{G}_f}|$ is concentrated on $\mathcal{A} \cap \mathcal{T} \cap \mathcal{S} \cap \mathcal{F}\mathcal{G}_f$ due to (3-8)–(3-10) and to the definition of reduced boundary, see Definition 2.21. Thus, due to Proposition 2.13, $|\mathrm{D}f|$ is concentrated on $\pi^1(\mathcal{A} \cap \mathcal{T} \cap \mathcal{S} \cap \mathcal{F}\mathcal{G}_f)$. Putting this all together, we get that $|\mathrm{D}f|^c$ is concentrated on C_f.

Item (ii). By Lemma 2.11, one has that if $x \in C_f \setminus J_f$, then $\mathcal{F}\mathcal{G}_f \cap (\{x\} \times \mathbb{R}) = \{(x, \bar{f}(x))\}$. Indeed, $x \in C_f \subseteq \pi^1(\mathcal{F}\mathcal{G}_f)$, and then $\mathcal{F}\mathcal{G}_f \cap (\{x\} \times \mathbb{R})$ is nonempty. Hence $\mathcal{F}\mathcal{G}_f \cap (C_f \times \mathbb{R}) = (\mathrm{id}_{\mathsf{X}}, \bar{f})(C_f)$.

Item (iii). Let $x \in C_f$. Hence, by item (ii) and by definition of C_f, we have that $x = \pi^1(x, \bar{f}(x))$ and $(x, \bar{f}(x)) \in \mathcal{T} \cap \mathcal{S}$.

Let $\eta \in (0, n^{-1}) \cap \mathbb{Q}$. We have that there exists $k_\eta(x)$ such that $x \in D_{\eta, k_\eta(x)}$. By Lemma 2.28(iii), compare with (2-16), we get the existence of a matrix $M(x) \in \mathbb{R}^{n \times n}$ such that, for every $\ell, j = 1, \dots, n$,

$$\lim_{r \searrow 0} \fint_{B_r(x)} |\nabla u^\ell_{\eta,k_\eta(x)} \cdot \nabla u^j_{\eta,k_\eta(x)} - M(x)_{\ell,j}| \,\mathrm{d}\mathsf{m} = 0.$$

Hence, taking the matrix $A_\eta(x)$ from Definition 2.29, we conclude that, calling $v_{\eta,k_\eta(x)} := A_\eta(x) u_{\eta,k_\eta(x)}$, we have, for every $\ell, j = 1, \dots, n$,

$$\lim_{r \searrow 0} \fint_{B_r(x)} |\nabla v^\ell_{\eta,k_\eta(x)} \cdot \nabla v^j_{\eta,k_\eta(x)} - \delta_{\ell j}| \,\mathrm{d}\mathsf{m} = 0.$$

Hence, as a consequence of the previous equality, the independence of the coordinates in $\mathsf{X} \times \mathbb{R}$, and Fubini's theorem, calling $\tilde{v}_{\eta,k_\eta(x)} := (v_{\eta,k_\eta(x)}, \pi^2)$, we get that the following holds for every $\ell, j = 1, \dots, n+1$:

$$\lim_{r \searrow 0} \fint_{B_r(x, \bar{f}(x))} |\nabla \tilde{v}^\ell_{\eta,k_\eta(x)} \cdot \nabla \tilde{v}^j_{\eta,k_\eta(x)} - \delta_{\ell j}| \,\mathrm{d}(\mathsf{m} \otimes \mathcal{H}^1) = 0. \tag{3-12}$$

Now, since $(x, \bar{f}(x)) \in \mathcal{S}_\eta$ and $\mathcal{S}_\eta$ is the set of the Lebesgue points of (M_η, π^2) (see (2-16)) with respect to $|\mathrm{D}\chi_{\mathcal{G}_f}|$, we also get, for every $\ell, j = 1, \dots, n+1$,

$$\lim_{r \searrow 0} \fint_{B_r(x, \bar{f}(x))} |\nabla \tilde{v}^\ell_{\eta,k_\eta(x)} \cdot \nabla \tilde{v}^j_{\eta,k_\eta(x)} - \delta_{\ell j}| \,\mathrm{d}|\mathrm{D}\chi_{\mathcal{G}_f}| = 0. \tag{3-13}$$

Finally, notice that $(x, \bar f(x)) \in \mathcal{T}_\eta$ and $\mathcal{T}_\eta$ are the Lebesgue points of $\nu_{\mathcal{G}_f}^{\boldsymbol{u}_\eta}$ with respect to $|\mathrm{D}\chi_{\mathcal{G}_f}|$. Hence, $(x, \bar f(x))$ is also a Lebesgue point of the $|\mathrm{D}\chi_{\mathcal{G}_f}|$-measurable map defined for $p = (y, t)$ as

$$\tilde\nu_{\mathcal{G}_f}^{\boldsymbol{u}_\eta}(p) := ((\nu_{\mathcal{G}_f} \cdot \nabla A_\eta(x) u^1_{\eta, k_\eta(x)})(p), \dots, (\nu_{\mathcal{G}_f} \cdot \nabla A_\eta(x) u^n_{\eta, k_\eta(x)})(p), (\nu_{\mathcal{G}_f} \cdot \nabla \pi^2)(p)). \tag{3-14}$$

Arguing as in the last part of [Bruè et al. 2023a, Proposition 3.6], we get that the norm of the $|\mathrm{D}\chi_{\mathcal{G}_f}|$-Lebesgue representative of $\tilde\nu_{\mathcal{G}_f}^{\boldsymbol{u}_\eta}$ at $(x, \bar f(x))$ is 1. Hence the last information, together with (3-12) and (3-13), give that $\tilde v_{\eta, k_\eta(x)}$ is a set of good coordinates for $\mathcal{G}_f$ at $(x, \bar f(x))$.

Item (iv). It follows from item (ii) and the definition of $\mathcal{A}$.

Item (v). It follows from item (ii) and the definition of $\mathcal{D}$. □

Theorem 3.8. *Let* $(\mathsf{X}, \mathsf{d}, \mathsf{m})$ *be an* $\mathrm{RCD}(K, N)$ *space having essential dimension* n. *Fix a function* $f \in \mathrm{BV}(\mathsf{X})$ *and a good collection* $\{\boldsymbol{u}_\eta\}_\eta$ *of splitting maps on* X. *Let* $C_f \subseteq \mathsf{X}$ *be as in Proposition 3.7. Then, for any given* $\eta \in (0, n^{-1}) \cap \mathbb{Q}$,

$$(\nu_{\mathcal{G}_f}^{\boldsymbol{u}_\eta})_{n+1}(p) = 0 \quad \text{for } \mathcal{H}^n\text{-a.e. } p \in \mathcal{F}\mathcal{G}_f \cap (C_f \times \mathbb{R}).$$

Proof. We recall from Proposition 2.13 that $\pi^1_*(|\mathrm{D}\chi_{\mathcal{G}_f}| \llcorner (\mathcal{F}\mathcal{G}_f \cap (C_f \times \mathbb{R}))) = |\mathrm{D}f|^c$. Moreover, Lemma 2.11 ensures that the mapping $\pi^1 \colon \mathcal{F}\mathcal{G}_f \cap (C_f \times \mathbb{R}) \to C_f$ is the inverse of $(\mathrm{id}_{\mathsf{X}}, \bar f) \colon C_f \to \mathcal{F}\mathcal{G}_f \cap (C_f \times \mathbb{R})$. Given any $k, j \in \mathbb{N}$ and $\alpha \in (0, 1) \cap \mathbb{Q}$, we define

$$C_f^{k,\alpha,j} := \left\{ x \in C_f \cap D_{\eta,k} \;\middle|\; |(\nu_{\mathcal{G}_f}^{\boldsymbol{u}_\eta})_{n+1}(x, \bar f(x))| \geq \alpha,\ j^{-1} \leq \Theta_{n+1}(\mathsf{m} \otimes \mathcal{L}^1, (x, \bar f(x))) \leq j \right\}.$$

Notice that the sets $C_f^{k,\alpha,j}$ obviously depend on η, but, as we are working with a fixed $\eta \in (0, n^{-1}) \cap \mathbb{Q}$, we do not make this dependence explicit. Recalling Theorem 3.3, we see that

$$\{x \in C_f \mid (\nu_{\mathcal{G}_f}^{\boldsymbol{u}_\eta})_{n+1}(x, \bar f(x)) \neq 0\} = \bigcup_{k,\alpha,j} C_f^{k,\alpha,j} \quad \text{up to } |\mathrm{D}f|\text{-null sets.}$$

Hence, proving the statement amounts to showing that each set $\mathcal{F}\mathcal{G}_f \cap (C_f^{k,\alpha,j} \times \mathbb{R})$ is $\mathcal{H}^n$-negligible. Given any $\varepsilon > 0$, by Lusin's theorem we can find $\Sigma \subseteq \mathcal{F}\mathcal{G}_f \cap (C_f^{k,\alpha,j} \times \mathbb{R})$ Borel such that $\bar f$ is continuous on $\pi^1(\Sigma)$ and $\mathcal{H}^n((\mathcal{F}\mathcal{G}_f \cap (C_f^{k,\alpha,j} \times \mathbb{R})) \setminus \Sigma) < \varepsilon$.

Our aim is to show that

$$\mathcal{H}^n(\Sigma) = 0 \tag{3-15}$$

since this would imply $\mathcal{H}^n(\mathcal{F}\mathcal{G}_f \cap (C_f^{k,\alpha,j} \times \mathbb{R})) = 0$ by the arbitrariness of $\varepsilon > 0$. Up to discarding an $\mathcal{H}^n$-null set from Σ, we can also assume (thanks to Remark 2.19 and Theorem 3.4) that $\Theta_n(|\mathrm{D}\chi_{\mathcal{G}_f}| \llcorner \Sigma, p) = \Theta_{n+1}(\mathsf{m} \otimes \mathcal{L}^1, p)$ for every $p \in \Sigma$. Now we claim that

$$\lim_{r \searrow 0} \frac{|\mathrm{D}\chi_{\mathcal{G}_f}|((\Sigma \cap B_r(p)) \setminus (\mathsf{X} \times B_{\beta r}(t)))}{r^n} = 0 \quad \text{for every } p = (x, t) \in \Sigma, \tag{3-16}$$

where we set $\beta = \beta(\alpha) := \sqrt{1 - \alpha^2} \in (0, 1)$. The role played by α will be made clear in what follows. To show the claim, fix $p = (x, t) \in \Sigma$ and take any sequence $\{r_i\}_i \subseteq (0, +\infty)$ with $r_i \searrow 0$. Since $x \in \mathcal{R}_n(\mathsf{X})$, one has that

$$(\mathsf{X}, r_i^{-1}\mathsf{d}, \mathsf{m}_x^{r_i}, x) \to (\mathbb{R}^n, \mathsf{d}_e, \underline{\mathcal{L}}^n, 0) \quad \text{in the pmGH topology.}$$

Let $(\mathsf{Z}, \mathsf{d}_{\mathsf{X}})$ be a realization of such convergence. Then $(\mathsf{Z} \times \mathbb{R}, \mathsf{d}_{\mathsf{Z}} \times \mathsf{d}_e)$ is a realization of

$$(\mathsf{X} \times \mathbb{R}, r_i^{-1}(\mathsf{d} \times \mathsf{d}_e), (\mathsf{m} \otimes \mathcal{L}^1)_p^{r_i}, p, \mathcal{G}_f) \to (\mathbb{R}^{n+1}, \mathsf{d}_e, \underline{\mathcal{L}}^{n+1}, 0, H),$$

where $H \subseteq \mathbb{R}^{n+1}$ is a halfspace. We also know from Proposition 3.7(v) that, up to passing to a not relabeled subsequence, the rescaled perimeters $|\mathrm{D}\chi_{\mathcal{G}_f}|$ weakly converge to $\mathcal{H}^n \llcorner \partial H$ in duality with $C_{\mathrm{bs}}(\mathsf{Z})$. Moreover, by Proposition 3.7(iv), ∂H is normal to $\nu_{\mathcal{G}_f}^{\boldsymbol{u}_\eta}(p)$. Thus, since $(\nu_{\mathcal{G}_f}^{\boldsymbol{u}_\eta})_{n+1}(p) \geq \alpha$, we have $\partial H \cap B_1(0) \subseteq B_1(0) \times B_\beta(0)$ by our choice of β. From the latter the claim (3-16) follows, taking into account also (2-10). For $\gamma \in (0, +\infty)$ and $(x, t) \in \mathsf{X} \times \mathbb{R}$, we define the cone

$$C_\gamma(x, t) := \{(y, s) \in \mathsf{X} \times \mathbb{R} \mid \gamma \mathsf{d}(y, x) \geq |s - t|\}.$$

Now take $\gamma = \gamma(\beta) = \sqrt{(1+\beta)/(1-\beta)} \in (1, +\infty)$. Notice that $\gamma^2 > \beta/(1-\beta)$. Next we claim that

$$\lim_{r \searrow 0} \frac{|\mathrm{D}\chi_{\mathcal{G}_f}|((\Sigma \cap B_r(p)) \setminus C_\gamma(p))}{r^n} = 0 \quad \text{for every } p = (x, t) \in \Sigma. \tag{3-17}$$

In order to prove it, fix $\delta > 0$. By virtue of (3-16), we can take $r_0 > 0$ small enough that

$$\sup_{r \in (0, r_0)} \frac{|\mathrm{D}\chi_{\mathcal{G}_f}|((\Sigma \cap B_r(p)) \setminus (\mathsf{X} \times B_{\beta r}(t)))}{r^n} \leq \delta. \tag{3-18}$$

Notice that

$$B_{r_0}(p) \setminus C_\gamma(p) \subseteq \bigcup_i B_{r_i}(p) \setminus (\mathsf{X} \times B_{\beta r_i}(t)), \tag{3-19}$$

where, for any $i \in \mathbb{N}$ with $i \geq 1$, we define

$$r_i := \beta \sqrt{\frac{\gamma^2+1}{\gamma^2}} r_{i-1} = \left(\beta \sqrt{\frac{\gamma^2+1}{\gamma^2}}\right)^i r_0.$$

Given that

$$|\mathrm{D}\chi_{\mathcal{G}_f}|((\Sigma \cap B_{r_i}(p)) \setminus (\mathsf{X} \times B_{\beta r_i}(p))) \overset{\text{(3-18)}}{\leq} \delta r_i^n = \delta \left(\beta \sqrt{\frac{\gamma^2+1}{\gamma^2}}\right)^{ni} r_0^n,$$

it follows from the inclusion in (3-19) that

$$\frac{|\mathrm{D}\chi_{\mathcal{G}_f}|((\Sigma \cap B_{r_0}(p)) \setminus C_\gamma(p))}{r_0^n} \leq \delta \sum_i \left(\beta \sqrt{\frac{\gamma^2+1}{\gamma^2}}\right)^{ni}.$$

Thanks to the arbitrariness of $\delta > 0$ and the finiteness of $\sum_i (\beta \sqrt{(\gamma^2+1)/\gamma^2})^{ni}$, (3-17) is proved.

Let now $\varepsilon' > 0$. We wish to show that there exists a set $\Sigma' \subseteq \Sigma$ with $\mathcal{H}^n(\Sigma \setminus \Sigma') < \varepsilon'$ such that there exists $r_0 \in (0, 1)$ satisfying

$$(\Sigma' \cap B_{r_0}(p)) \setminus C_{2\gamma}(p) = \varnothing \quad \text{for every } p \in \Sigma'. \tag{3-20}$$

We do it using a standard argument, see, e.g., the proof of [Simon 1983, Theorem 1.6]. By Egorov's theorem, we can choose $\Sigma' \subseteq \Sigma$ Borel with $\mathcal{H}^n(\Sigma \setminus \Sigma') < \varepsilon'$ such that, for any given $\delta' > 0$, there exists

$r_0 \in (0,1)$ such that, for every $r \in (0, 2r_0)$ and $p \in \Sigma'$,

$$\frac{|\mathrm{D}\chi_{\mathcal{G}_f}|(\Sigma \cap B_r(p))}{\Theta_{n+1}(\mathsf{m} \otimes \mathcal{L}^1, p)\omega_n r^n} \geq 1 - \delta', \tag{3-21a}$$

$$\frac{|\mathrm{D}\chi_{\mathcal{G}_f}|((\Sigma \cap B_r(p)) \setminus C_\gamma(p))}{\Theta_{n+1}(\mathsf{m} \otimes \mathcal{L}^1, p)\omega_n r^n} \leq \delta'; \tag{3-21b}$$

the former follows from the fact that $\Theta_n(|\mathrm{D}\chi_{\mathcal{G}_f}| \llcorner \Sigma, p) = \Theta_{n+1}(\mathsf{m} \otimes \mathcal{L}^1, p)$, the latter from (3-17). We aim to show that if $\delta' > 0$ is small enough, then this choice of Σ' and r_0 satisfies (3-20). Assume now that there exists $q \in (\Sigma' \cap B_{r_0}(p)) \setminus C_{2\gamma}(p)$ for some $p \in \Sigma'$. Then

$$B_\rho(q) \subseteq B_{\tilde{\mathsf{d}}(p,q)+\rho}(p) \setminus C_\gamma(p), \quad \text{where } \rho := \tilde{\mathsf{d}}(p,q)\sin(\arctan(2\gamma) - \arctan(\gamma)), \tag{3-22}$$

where we write $\tilde{\mathsf{d}} := \mathsf{d} \times \mathsf{d}_e$ for brevity. Therefore, we can estimate

$$\begin{aligned}
\delta' &\overset{(3\text{-}21\text{b})}{\geq} \frac{|\mathrm{D}\chi_{\mathcal{G}_f}|((\Sigma \cap B_{\tilde{\mathsf{d}}(p,q)+\rho}(p)) \setminus C_\gamma(p))}{\Theta_{n+1}(\mathsf{m} \otimes \mathcal{L}^1, p)\omega_n(\tilde{\mathsf{d}}(p,q)+\rho)^n} \\
&\overset{(3\text{-}22)}{\geq} \frac{|\mathrm{D}\chi_{\mathcal{G}_f}|(\Sigma \cap B_\rho(q))}{\Theta_{n+1}(\mathsf{m} \otimes \mathcal{L}^1, p)\omega_n(\tilde{\mathsf{d}}(p,q)+\rho)^n} \\
&\overset{(3\text{-}21\text{b})}{\geq} (1-\delta')\frac{\rho^n}{(\tilde{\mathsf{d}}(p,q)+\rho)^n} = (1-\delta')\frac{(\sin(\arctan(2\gamma) - \arctan(\gamma)))^n}{(1+\sin(\arctan(2\gamma) - \arctan(\gamma)))^n},
\end{aligned}$$

which leads to a contradiction provided $\delta' > 0$ was chosen small enough, proving (3-20).

Finally, our aim is to show that

$$|\mathrm{D}\chi_{\mathcal{G}_f}|(\Sigma') = 0 \tag{3-23}$$

since this, by the arbitrariness of $\varepsilon' > 0$, would imply (3-16) and accordingly the statement. Take $p = (x,t) \in \Sigma'$. Since $\bar{f}$ is continuous on $\pi^1(\Sigma')$, there exists $r_1 \in (0, r_0/\sqrt{2})$ such that $|\bar{f}(y) - \bar{f}(x)| < r_0/\sqrt{2}$ for all $y \in B_{r_1}(x) \cap \pi^1(\Sigma')$. As $\Sigma' \subseteq \{(x,t) \in \mathrm{X} \times \mathbb{R} : t = \bar{f}(x)\}$, we see that

$$\Sigma' \cap (B_{r_1}(x) \times \mathbb{R}) \subseteq \Sigma' \cap B_{r_0}(p) \subseteq C_{2\gamma}(p)$$

by (3-20), so that, setting $\lambda := \sqrt{1+4\gamma^2}$,

$$\Sigma' \cap (B_r(x) \times \mathbb{R}) \subseteq \Sigma' \cap B_{\lambda r}(p) \quad \text{for every } r \in (0, r_1). \tag{3-24}$$

It follows that, for every $p = (x,t) \in \Sigma'$, we have

$$\begin{aligned}
\overline{\Theta}_n(\pi^1_*(|\mathrm{D}\chi_{\mathcal{G}_f}| \llcorner \Sigma'), x) &= \varlimsup_{r \searrow 0} \frac{|\mathrm{D}\chi_{\mathcal{G}_f}|(\Sigma' \cap (B_r(x) \times \mathbb{R}))}{\omega_n r^n} \overset{(3\text{-}24)}{\leq} \varlimsup_{r \searrow 0} \frac{|\mathrm{D}\chi_{\mathcal{G}_f}|(\Sigma \cap B_{\lambda r}(p))}{\omega_n r^n} \\
&= \lambda^n \Theta_n(|\mathrm{D}\chi_{\mathcal{G}_f}| \llcorner \Sigma, p) = \lambda^n \Theta_{n+1}(\mathsf{m} \otimes \mathcal{L}^1, p) \leq \lambda^n j,
\end{aligned}$$

where the last inequality stems from the inclusion $\Sigma' \subseteq \mathcal{F}\mathcal{G}_f \cap (C_f^{k,\alpha,j} \times \mathbb{R})$. Therefore, by applying [Ambrosio and Tilli 2004, Theorem 2.4.3] and using the fact that $\pi^1(\Sigma') \subseteq C_f$, we can conclude that

$$|\mathrm{D}\chi_{\mathcal{G}_f}|(\Sigma') = \pi^1_*(|\mathrm{D}\chi_{\mathcal{G}_f}| \llcorner \Sigma')(\pi^1(\Sigma')) \leq (2\lambda)^n j \mathcal{H}^n(\pi^1(\Sigma')) \leq (2\lambda)^n j \mathcal{H}^n(C_f) = 0,$$

thus obtaining (3-23). Consequently, the statement is achieved. □

Lemma 3.9. *Let* $(\mathsf{X}, \mathsf{d}, \mathsf{m})$ *be an* $\mathrm{RCD}(K, N)$ *space of essential dimension* n. *Fix a function* $f \in \mathrm{BV}(\mathsf{X})$ *and a good collection* $\{\boldsymbol{u}_\eta\}_\eta$ *of splitting maps on* X. *Let* C_f *be as in the statement of Proposition 3.7. Then, for any* $\eta \in (0, n^{-1}) \cap \mathbb{Q}$,

$$\nu_f^{\boldsymbol{u}_\eta}(x) = (\nu_{\mathcal{G}_f}^{\boldsymbol{u}_\eta}(x, \bar{f}(x)))_{1,\dots,n} \quad \textit{for } |\mathrm{D}f| \llcorner C_f\textit{-a.e. } x \in \mathsf{X}.$$

Proof. Recall that $|\mathrm{D}f| \llcorner C_f = \pi^1_*(|\mathrm{D}\chi_{\mathcal{G}_f}| \llcorner (C_f \times \mathbb{R}))$, so that the statement makes sense. By the coarea formula, it is enough to show that, for a.e. t, we have $\nu_f^{\boldsymbol{u}_\eta}(x) = (\nu_{\mathcal{G}_f}^{\boldsymbol{u}_\eta}(x, \bar{f}(x)))_{1,\dots,n}$ for $\mathcal{H}^{n-1}$-a.e. $x \in \mathcal{F}E_t \cap C_f$, where we define $E_t := \{f > t\}$. Taking [Brena and Gigli 2024, Lemma 3.27] into account, we see that it is sufficient to prove that, for a.e. t and for every $k \in \mathbb{N}$,

$$\nu_{\chi_{E_t}}^{\boldsymbol{u}_\eta}(x) = (\nu_{\mathcal{G}_f}^{\boldsymbol{u}_\eta}(x, \bar{f}(x)))_{1,\dots,n} \quad \text{for } \mathcal{H}^{n-1}\text{-a.e. } x \in \mathcal{F}E_t \cap C_f \cap D_{\eta,k}. \tag{3-25}$$

Let $x \in \mathcal{F}E_t \cap C_f \cap D_{\eta,k}$ be a given point where the conclusions of Proposition 2.26 hold with $E = E_t$; notice that $\mathcal{H}^{n-1}$-a.e. point of $\mathcal{F}E_t \cap C_f \cap D_{\eta,k}$ has this property. We aim to show that the identity in (3-25) is verified at x. Write $p := (x, \bar{f}(x))$ for brevity. Thanks to Remark 2.22(i) and Proposition 3.7(v), we can find a sequence $r_i \searrow 0$, halfspaces $H \subseteq \mathbb{R}^{n+1}$ and $H' \subseteq \mathbb{R}^n$, and a proper metric space $(\mathsf{Z}, \mathsf{d}_\mathsf{Z})$ such that

$$(\mathsf{X}, r_i^{-1}\mathsf{d}, \mathsf{m}_x^{r_i}, x, E_t) \to (\mathbb{R}^n, \mathsf{d}_e, \mathcal{L}^n, 0, H'), \tag{3-26a}$$

$$(\mathsf{X} \times \mathbb{R}, r_i^{-1}\mathsf{d}_{\mathsf{X}\times\mathbb{R}}, (\mathsf{m} \otimes \mathcal{H}^1)_p^{r_i}, p, \mathcal{G}_f) \to (\mathbb{R}^{n+1}, \mathsf{d}_e, \mathcal{L}^{n+1}, 0, H) \tag{3-26b}$$

in the realizations Z and $\mathsf{Z} \times \mathbb{R}$, respectively. Notice also that

$$\{(y, s) \in \mathsf{X} \times \mathbb{R} \mid s < t\} \to \{(y, s) \in \mathbb{R}^n \times \mathbb{R} \mid s < 0\} \quad \text{in } L^1_{\mathrm{loc}} \tag{3-27}$$

in the realization $\mathsf{Z} \times \mathbb{R}$. Therefore, by stability, we deduce from (3-26b) and (3-27) that

$$\{(y, s) \in \mathsf{X} \times \mathbb{R} \mid s < f(y),\ s < t\} \to H \cap \{(y, s) \in \mathbb{R}^n \times \mathbb{R} \mid s < 0\} \quad \text{in } L^1_{\mathrm{loc}}.$$

Recalling (3-26a) and using Fubini's theorem and dominated convergence, we see that

$$E_t \times (-\infty, t) \to H' \times (-\infty, 0) \quad \text{in } L^1_{\mathrm{loc}}.$$

Given that $E_t \times (-\infty, t) \subseteq \{(y, s) \in \mathsf{X} \times \mathbb{R} : s < f(y),\ s < t\}$, we obtain that

$$H' \times (-\infty, 0) \subseteq H \cap \{(y, s) \in \mathbb{R}^n \times \mathbb{R} \mid s < 0\}.$$

Thanks to our choice of x and to items (iv) and (v) of Proposition 3.7, we can see that $\nu_{\chi_{E_t}}^{\boldsymbol{u}_\eta}(x)$ and $(\nu_{\mathcal{G}_f}^{\boldsymbol{u}_\eta}(p))_{1,\dots,n}$ have the same direction, namely there exists $\lambda(x) \in [0, 1]$ such that

$$\nu_{\chi_{E_t}}^{\boldsymbol{u}_\eta}(x) = \lambda(x)(\nu_{\mathcal{G}_f}^{\boldsymbol{u}_\eta}(p))_{1,\dots,n}.$$

Now notice that the conclusion of Theorem 3.8 forces $\lambda(x)$ to equal 1, up to discarding a $|\mathrm{D}f| \llcorner C_f$-negligible set. □

3C. *Rank-one theorem.* In this final subsection we prove Theorem 1.3. We first start with an auxiliary definition and a technical result taken from [Bruè et al. 2023b].

Let $(\mathsf{X}, \mathsf{d}, \mathsf{m})$ be an $\mathrm{RCD}(K, N)$ space of essential dimension n and $E \subseteq \mathsf{X}$ a set of locally finite perimeter. Let $\varepsilon > 0$ and $r > 0$ be given. Then, following [Bruè et al. 2023b, Definition 4.6], we define $(\mathcal{F}_n E)_{r,\varepsilon}$ as the set of all points $x \in \mathcal{F}_n E$ such that

$$\mathsf{d}_{\mathrm{pmGH}}((\mathsf{X}, s^{-1}\mathsf{d}, \mathsf{m}_x^s, x), (\mathbb{R}^n, \mathsf{d}_e, \mathcal{L}^n, 0)) < \varepsilon,$$
$$\left|\frac{\mathsf{m}(E \cap B_s(x))}{\mathsf{m}(B_s(x)} - \frac{1}{2}\right| + \left|\frac{s|\mathrm{D}\chi_E|(B_s(x))}{\mathsf{m}(B_s(x))} - \frac{\omega_{n-1}}{\omega_n}\right| < \varepsilon$$

for every $s \in (0, r)$. We remark that, for every $x \in \mathcal{F}_n E$ and for every $\varepsilon > 0$, there exists $r > 0$ such that $x \in (\mathcal{F}_n E)_{r,\varepsilon}$. We now recall the following result, which was proved in [Bruè et al. 2023b, Proposition 4.7].

Proposition 3.10. *Let $(\mathsf{X}, \mathsf{d}, \mathsf{m})$ be an $\mathrm{RCD}(K, N)$ space of essential dimension n. Let $E \subseteq \mathsf{X}$ be a set of locally finite perimeter. Then, for any $\eta > 0$, there exists $\varepsilon = \varepsilon(N, \eta) > 0$ such that the following property is satisfied: if $p \in (\mathcal{F}_n E)_{2r,\varepsilon}$ for some $0 < r < |K|^{-1/2}$ and there exists an ε-splitting map $u\colon B_{2r}(p) \to \mathbb{R}^{n-1}$ such that*

$$\frac{r}{\mathsf{m}(B_{2r}(p))} \int_{B_{2r}(p)} |\nu_E \cdot \nabla u^\ell| \,\mathrm{d}|\mathrm{D}\chi_E| < \varepsilon \quad \textit{for every } \ell = 1, \dots, n-1,$$

then there exists a Borel set $G \subseteq B_r(p)$ with $\mathcal{H}_5^h(B_r(p) \setminus G) \le C_N \eta \mathsf{m}(B_r(p))/r$ such that

$$u\colon G \cap (\mathcal{F}_n E)_{2r,\varepsilon} \to \mathbb{R}^{n-1} \quad \textit{is bi-Lipschitz onto its image.}$$

We pass to the following lemma, which is the technical core of the proof of Theorem 1.3.

Lemma 3.11. *Let $(\mathsf{X}, \mathsf{d}, \mathsf{m})$ be an $\mathrm{RCD}(K, N)$ space of essential dimension n. Fix any two functions $f, g \in \mathrm{BV}(\mathsf{X})$. Let $\{\boldsymbol{u}_\eta\}_\eta$ be a good collection of splitting maps on X. Let us consider the sets $C_f, C_g \subseteq \mathsf{X}$ given by Proposition 3.7. Let τ be the inversion map defined in (2-1), and let*

$$\Sigma_f := \mathcal{FG}_f \cap (C_f \times \mathbb{R}), \quad \widetilde{\Sigma}_f := \Sigma_f \times \mathbb{R},$$
$$\Sigma_g := \mathcal{FG}_g \cap (C_g \times \mathbb{R}), \quad \widetilde{\Sigma}_g := \tau(\Sigma_g \times \mathbb{R}).$$

Moreover, let us set $R := \pi^1(\widetilde{R}) \subseteq \mathsf{X}$, where the set $\widetilde{R} \subseteq \mathsf{X} \times \mathbb{R}^2$ is defined as

$$\bigcap_{\substack{\eta \in \mathbb{Q},\\ 0<\eta<n^{-1}}} \{(x, t, s) \in \widetilde{\Sigma}_f \cap \widetilde{\Sigma}_g \mid \nu_{\mathcal{G}_f}^{\boldsymbol{u}_\eta}(x, t) \ne \pm\nu_{\mathcal{G}_g}^{\boldsymbol{u}_\eta}(x, s),\ (\nu_{\mathcal{G}_f}^{\boldsymbol{u}_\eta}(x, t))_{n+1} = (\nu_{\mathcal{G}_g}^{\boldsymbol{u}_\eta}(x, s))_{n+1} = 0\}. \tag{3-28}$$

Then

$$(|\mathrm{D}f| \wedge |\mathrm{D}g|)(R) = 0.$$

Proof. Let us fix a ball $\bar{B}$ in X, set

$$\Omega_f := (C_f \times \mathbb{R}) \cap (\bar{B} \times \mathbb{R}) \cap \mathcal{FG}_f,$$

and define similarly Ω_g.

For $i \in \mathbb{N}$, set $\eta_i := 2^{-i}\eta_0$. Here $\eta_0 \in (0, n^{-1}) \cap \mathbb{Q}$ satisfies $\eta_0 C_N < 1$, where C_N is given in Proposition 3.10. We claim that, for every i, there exists a decomposition of the kind

$$\Omega_f = G_i(f) \cup M_i(f) \cup R_i(f),$$

and similarly for g, for which the following hold:

- We have the inequality
$$\mathcal{H}^h_5(M_i(f)) + |\mathrm{D}\chi_{\mathcal{G}_f}|(R_i(f)) \le C_{K,N}\eta_i(|\mathrm{D}\chi_{\mathcal{G}_f}|(\bar{B} \times \mathbb{R}) + 1), \tag{3-29}$$
and similarly for g, where $C_{K,N}$ is, in particular, independent of i.
- Set $\widehat{G}_i(f) := \pi^1(G_i(f))$ and $\widehat{G}_i(g) := \pi^1(G_i(g))$. Define similarly $\widehat{M}_i(f)$, $\widehat{M}_i(g)$, $\widehat{R}_i(f)$, and $\widehat{R}_i(g)$. Then
$$(|\mathrm{D}f| \wedge |\mathrm{D}g|)(R \cap \widehat{G}_i(f) \cap \widehat{G}_i(g)) = 0. \tag{3-30}$$

We show now how this decomposition allows us to conclude the proof of the lemma. We set

$$\widehat{G} := \bigcup_{i \in \mathbb{N}} \widehat{G}_i(f) \cap \widehat{G}_i(g).$$

As (3-30) implies that

$$(|\mathrm{D}f| \wedge |\mathrm{D}g|)(R \cap \widehat{G}) = 0,$$

it suffices to show (recall that $R \subseteq C_f \cap C_g$)

$$(|\mathrm{D}f| \wedge |\mathrm{D}g|)((C_f \cap C_g \cap \bar{B}) \setminus \widehat{G}) = 0,$$

as the ball $\bar{B}$ was arbitrary.

Let us go through the proof of the last equality. Notice that, for every i,

$$(|\mathrm{D}f| \wedge |\mathrm{D}g|)((C_f \cap C_g \cap \bar{B}) \setminus \widehat{G}) \le |\mathrm{D}f|(\widehat{M}_i(f) \cup \widehat{R}_i(f)) + |\mathrm{D}g|(\widehat{M}_i(g) \cup \widehat{R}_i(g)).$$

Therefore, it is enough to show that (as a similar statement will hold for g),

$$\lim_{i\to\infty} |\mathrm{D}f|(\widehat{M}_i(f) \cup \widehat{R}_i(f)) = 0,$$

so that, recalling Proposition 2.13 and that $\pi^1|_{\mathcal{F}\mathcal{G}_f}$ is injective on $C_f \times \mathbb{R}$, we can just show

$$\lim_{i\to\infty} |\mathrm{D}\chi_{\mathcal{G}_f}|\Bigl(\bigcup_{j\ge i} M_j(f)\Bigr) + |\mathrm{D}\chi_{\mathcal{G}_f}|(R_i(f)) = 0,$$

which follows from (3-29), since (3-29) again and the definition of η_i imply that

$$\mathcal{H}^h_5\Bigl(\bigcap_{i\in\mathbb{N}} \bigcup_{j\ge i} M^j(f)\Bigr) = 0.$$

For the sake of clarity, we subdivide the rest of the proof into five steps. In Step 1 we construct a candidate decomposition as above in such a way that (3-29) is satisfied. The remaining steps are to prove (3-30) for the decomposition obtained in Step 1. Step 2 and Step 4 are used to obtain technical estimates, whereas Step 3 is the most important and proves a σ-finiteness property via transverse intersection. With these results in mind, we conclude the proof in Step 5. In the rest of the proof, we are going to use heavily all the conditions ensured by the membership to C_f and C_g without pointing it out every time. In other words, we are morally partitioning X into good sets, up to an almost negligible set. These sets are good in the sense that $\widetilde{\Sigma}_f$ and $\widetilde{\Sigma}_g$, restricted to the preimage of these sets with respect to the projection onto X, are bi-Lipschitz equivalent to $(n+1)$-rectifiable subsets of $\mathbb{R}^{n+2}$, via the same chart maps. Then, as explained in the introduction, the task is to prove transversality of these two subsets of $\mathbb{R}^{n+2}$, and this is done via a blow-up argument, taking advantage of the fact that we are using the same chart maps.

Step 1: Construction of the decomposition. Let $\varepsilon_i \in (0, n^{-1}) \cap (0, \omega_n/(2\omega_{n+1})) \cap \mathbb{Q}$ be given by Proposition 3.10 applied to $E = \mathcal{G}_f$, with η_i in place of η. Using the good collection of splitting maps, consider

$$\boldsymbol{u}_i = \{u_{i,k}\}_k := \boldsymbol{u}_{\varepsilon_i/(n+1)}, \quad \{D_{i,k}\}_k := \{D_{\varepsilon_i/(n+1),k}\}_k, \quad k_i := k_{\varepsilon_i/(n+1)}, \quad A_i := A_{\varepsilon_i/(n+1)},$$

where we recall that k and A have been defined in Definition 2.29.

We only consider the case of the function f, the construction for g being the same, and we concentrate on a fixed i. Therefore, we do not indicate the dependence on f for what remains of Step 1.

We refer to the discussion at the beginning of Section 3C for the definition (and the basic properties) of the auxiliary set $(\mathcal{F}_{n+1}\mathcal{G}_f)_{r,\varepsilon}$. Let

$$r_i \in (0, |K|^{-1})$$

be small enough that, setting

$$R_i^1 := \Omega_f \setminus (\mathcal{F}_{n+1}\mathcal{G}_f)_{2r_i,\varepsilon_i},$$

we have

$$|\mathrm{D}\chi_{\mathcal{G}_f}|(R_i^1) < \eta_i.$$

Let also $c = c_i \in (0, 1)$ be small enough that, setting

$$R_i^2 := \Omega_f \setminus \{p \in \mathcal{F}\mathcal{G}_f \mid c < \Theta_n(|\mathrm{D}\chi_{\mathcal{G}_f}|, p) < c^{-1}\},$$

we have

$$|\mathrm{D}\chi_{\mathcal{G}_f}|(R_i^2) < \eta_i.$$

Take now $p = (x, \bar{f}(x)) \in \Omega_f \setminus R_i^1$, so that $x \in D_{i,k}$ for $k = k_i(x)$, see item (v) of Proposition 3.7, and we have an associated invertible matrix $A = A_i(x)$, compare with item (iii) of Proposition 3.7, and the discussion in Definition 2.29. Set $v := (u_{i,k}, \pi^2)$ and $z := (Au_{i,k}, \pi^2)$. Notice, by the fact that $x \in D_{i,k}$, we have that $u_{i,k}$ is ε_i-splitting on a small ball around x. Hence, by tensorization, v is ε_i-splitting on a small ball around p. Recall, moreover, that, by item (iii) of Proposition 3.7, we have that z is a set of good coordinates at $(x, \bar{f}(x))$, see Definition 2.23. Hence, we have that, for some $\nu \in \mathbb{S}^n$,

$$\lim_{r \searrow 0} \fint_{B_r(p)} |\nu^j - \nu_{\mathcal{G}_f} \cdot \nabla z^j| \, \mathrm{d}|\mathrm{D}\chi_{\mathcal{G}_f}| = 0 \quad \text{for every } j = 1, \dots, n+1,$$

so that, for some $\mu \in \mathbb{R}^{n+1} \setminus \{0\}$,

$$\lim_{r \searrow 0} \fint_{B_r(p)} |\mu^j - \nu_{\mathcal{G}_f} \cdot \nabla v^j| \, \mathrm{d}|\mathrm{D}\chi_{\mathcal{G}_f}| = 0 \quad \text{for every } j = 1, \dots, n+1.$$

It follows that, for some $B \in \mathrm{SO}(n+1)$, setting $w = Bv$, we have

$$\lim_{r \searrow 0} \fint_{B_r(p)} |\nu_{\mathcal{G}_f} \cdot \nabla w^j| \, \mathrm{d}|\mathrm{D}\chi_{\mathcal{G}_f}| = 0 \quad \text{for every } j = 1, \dots, n.$$

Indeed, it suffices to take $B \in \mathrm{SO}(n+1)$ such that $B\mu = (0^n, \|\mu\|_{\mathbb{R}^{n+1}})$. The equation above and the membership $p \in \mathcal{F}\mathcal{G}_f$ imply that

$$\lim_{r \searrow 0} \frac{r}{\mathsf{m}\otimes\mathcal{H}^1(B_{2r}(p))} \int_{B_{2r}(p)} |\nu_{\mathcal{G}_f} \cdot \nabla w^j| \, \mathrm{d}|\mathrm{D}\chi_{\mathcal{G}_f}| = 0 \quad \text{for every } j = 1, \dots, n.$$

Take then $\tilde{r} = \tilde{r}_{i,p} \in (0, r_i)$ small enough that w is an ε_i-splitting map on $B_{2\tilde{r}}(p)$ (this is possible thanks to our choice of $\boldsymbol{u}_i$, the fact that v is ε_i-splitting on a small ball around p, and that $B \in \mathrm{SO}(n+1)$[1]), moreover

$$\frac{\tilde{r}}{\mathsf{m}\otimes\mathcal{H}^1(B_{2\tilde{r}}(p))} \int_{B_{2\tilde{r}}(p)} |\nu_{\mathcal{G}_f} \cdot \nabla w^j| \, \mathrm{d}|\mathrm{D}\chi_{\mathcal{G}_f}| < \varepsilon_i \quad \text{for every } j = 1, \dots, n,$$

and finally, using also that $|\mathrm{D}\chi_{\mathcal{G}_f}|$ is asymptotically doubling at p,

$$|\mathrm{D}\chi_{\mathcal{G}_f}|(B_{\tilde{r}}(p) \setminus (D_{i,k} \times \mathbb{R})) < \eta_i |\mathrm{D}\chi_{\mathcal{G}_f}|(B_{\tilde{r}/5}(p)),$$

where we recall that for deducing the last information we are using item (v) of Proposition 3.7. We can also assume that $B_{\tilde{r}}(x) \subseteq \bar{B}$, which will be useful below. Note that $p \in (\mathcal{F}_{n+1}\mathcal{G}_f)_{2r_i,\varepsilon_i} \subseteq (\mathcal{F}_{n+1}\mathcal{G}_f)_{2\tilde{r},\varepsilon_i}$. We can thus apply Proposition 3.10 and obtain a set $G = G_{i,p} \subseteq B_{\tilde{r}}(p)$ such that

$$\mathcal{H}^h_5(B_{\tilde{r}}(p) \setminus G) \le C_N \eta_i \frac{\mathsf{m}\otimes\mathcal{H}^1(B_{\tilde{r}}(p))}{\tilde{r}}$$

and $(w^1, \dots, w^n) : G \cap (\mathcal{F}_{n+1}\mathcal{G}_f)_{2\tilde{r},\varepsilon_i} \to \mathbb{R}^n$ is bi-Lipschitz onto its image. Here C_N depends only on N. Clearly, also $v : G \cap (\mathcal{F}_{n+1}\mathcal{G}_f)_{2\tilde{r},\varepsilon_i} \to \mathbb{R}^{n+1}$ is bi-Lipschitz onto its image, so that the image of v is n-rectifiable, due to the fact that $\mathcal{F}_{n+1}\mathcal{G}_f$ is n-rectifiable.

To sum up, for i fixed, for every $p = (x,t) \in \Omega_f \setminus R^1_i$, we have shown that

$$v_{i,p} := (u_{i,k_i(x)}, \pi^2) : G_{i,p} \cap (\mathcal{F}_{n+1}\mathcal{G}_f)_{2r_i,\varepsilon_i} \to \mathbb{R}^{n+1}$$

is bi-Lipschitz onto its image for some set $G_{i,p} \subseteq B_{\tilde{r}_{i,p}}(p)$, that

$$\mathcal{H}^h_5(B_{\tilde{r}_{i,p}}(p) \setminus G_{i,p}) \le C_N \eta_i \frac{\mathsf{m}\otimes\mathcal{H}^1(B_{\tilde{r}_{i,p}}(p))}{\tilde{r}_{i,p}}, \tag{3-31}$$

and finally that

$$|\mathrm{D}\chi_{\mathcal{G}_f}|(B_{\tilde{r}_{i,p}}(p) \setminus (D_{i,k_i(x)} \times \mathbb{R})) < \eta_i |\mathrm{D}\chi_{\mathcal{G}_f}|(B_{\tilde{r}_{i,p}/5}(p)). \tag{3-32}$$

We apply Vitali's covering lemma to find a sequence of balls $\{B^j_i\}_j$ where, for every j, we have that $B^j_i = B_{r^j_i}(p^j_i) = B_{\tilde{r}_{i,p}}(p)$ for some $p = p^j_i \in \Omega_f \setminus R^1_i$ such that

$$\bigcup_{j\in\mathbb{N}} B^j_i \supseteq \Omega_f \setminus R^i_1$$

and $\{5^{-1}B^j_i\}_j$ are pairwise disjoint; here $5^{-1}B^j_i$ stands for the ball $B_{r^j_i/5}(p^j_i)$. Clearly, to each B^j_i are associated in a natural way the sets G^j_i and D^j_i and maps $v^j_i : G^j_i \cap (\mathcal{F}_{n+1}\mathcal{G}_f)_{2r_i,\varepsilon_i} \to \mathbb{R}^{n+1}$. We set then

$$M_i := \Omega_f \cap \bigcup_{j\in\mathbb{N}} (B^j_i \setminus G^j_i) \quad \text{and} \quad R^3_i := \Omega_f \cap \bigcup_{j\in\mathbb{N}} (B^j_i \setminus (D^j_i \times \mathbb{R})).$$

[1]Notice that the operator norm of B is bounded above by a function of n, hence the Lipschitz constant of w might increase by at most such a factor, but this is clearly not a problem.

Using (3-31) for the first chain of inequalities and (3-32) for the second chain of inequalities, we have

$$\begin{aligned}\mathcal{H}^h_5(M_i) &\le \sum_{j\in\mathbb{N}} \mathcal{H}^h_5(B_i^j\setminus G_i^j) \le C_N\eta_i \sum_{j\in\mathbb{N}} \frac{\mathsf{m}\otimes\mathcal{H}^1(B_i^j)}{r_i^j} \le C_{K,N}\eta_i \sum_{j\in\mathbb{N}} \frac{\mathsf{m}\otimes\mathcal{H}^1(5^{-1}B_i^j)}{\frac{1}{5}r_i^j} \\ &\le C_{K,N}\eta_i \sum_{j\in\mathbb{N}} |\mathrm{D}\chi_{\mathcal{G}_f}|(5^{-1}B_i^j) \le C_{K,N}\eta_i |\mathrm{D}\chi_{\mathcal{G}_f}|(\bar{B}\times\mathbb{R}).\end{aligned}$$

We stress that in the fourth inequality above we are using that $p_i^j \in (\mathcal{F}_{n+1}\mathcal{G}_f)_{2r_i,\varepsilon_i}$ and

$$|\mathrm{D}\chi_{\mathcal{G}_f}|(R_i^3) \le \sum_{j\in\mathbb{N}} |\mathrm{D}\chi_{\mathcal{G}_f}|(B_i^j\setminus(D_i^j\times\mathbb{R})) \le \eta_i \sum_{j\in\mathbb{N}} |\mathrm{D}\chi_{\mathcal{G}_f}|(5^{-1}B_i^j) \le \eta_i |\mathrm{D}\chi_{\mathcal{G}_f}(\bar{B}\times\mathbb{R})|.$$

Now set

$$S_i^j := v_i^j((\Omega_f\cap G_i^j\cap(\mathcal{F}_{n+1}\mathcal{G}_f)_{2r_i,\varepsilon_i})\setminus(R_i^1\cup R_i^2\cup R_i^3)) \subseteq \mathbb{R}^{n+1},$$

and recall that S_i^j is n-rectifiable. For every $j\in\mathbb{N}$, there exists a countable family $\{S_i^{j,\ell}\}_{\ell\in\mathbb{N}}$ of C^1-hypersurfaces in $\mathbb{R}^{n+1}$ such that

$$\mathcal{H}^n\Big(S_i^j\setminus\bigcup_{\ell\in\mathbb{N}} S_i^{j,\ell}\Big)=0.$$

Define

$$\widehat{S}_i^{j,\ell} := \left\{ y\in S_i^{j,\ell}\cap S_i^j \;\middle|\; \lim_{r\searrow 0}\frac{\mathcal{H}^n(B_r(y)\cap S_i^{j,\ell}\cap S_i^j)}{\omega_n r^n}=1\right\}$$

and

$$R_i^4 := \bigcup_{j\in\mathbb{N}}\bigcap_{\ell\in\mathbb{N}}(S_i^j\setminus(v_i^j)^{-1}(\widehat{S}_i^{j,\ell})) \subseteq \Omega_f,$$

and notice that $\mathcal{H}^n(R_i^4)=0$, so that $|\mathrm{D}\chi_{\mathcal{G}_f}|(R_i^4)=0$. We set also

$$Q_i^{j,\ell} := (v_i^j)^{-1}(\widehat{S}_i^{j,\ell}) \subseteq \Omega_f,$$

and notice that

$$\text{if } v_i^j=(u_{i,k},\pi^2),\ \text{ then } Q_i^{j,\ell}\subseteq D_{i,k}\times\mathbb{R} \text{ for every } \ell\in\mathbb{N}. \tag{3-33}$$

Now define

$$R_i^5 := \bigcup_{j,\ell\in\mathbb{N}}\left(Q_i^{j,\ell}\setminus\left\{p\in Q_i^{j,\ell}\;\middle|\;\lim_{r\searrow 0}\frac{|\mathrm{D}\chi_{\mathcal{G}_f}|(B_r(p)\cap Q_i^{j,\ell})}{|\mathrm{D}\chi_{\mathcal{G}_f}|(B_r(p))}=1\right\}\right).$$

We then set

$$R_i := R_i^1\cup R_i^2\cup R_i^3\cup R_i^4\cup R_i^5,$$

and finally

$$G_i := \Omega_f\setminus(M_i\cup R_i) \subseteq \bigcup_{j,\ell\in\mathbb{N}} Q_i^{j,\ell}.$$

It is immediate to check that the sets we constructed satisfy (3-29). The rest of the proof shows that they also satisfy (3-30).

<u>Step 2</u>: Almost one-sided Kuratowski convergence. For any i, let

$$p\in\Omega_f\setminus R_i^1(f),$$

and let $\rho_k\searrow 0$ be such that

$$(\mathsf{X}\times\mathbb{R},\ \rho_k^{-1}\mathsf{d}_{\mathsf{X}\times\mathbb{R}},\ (\mathsf{m}\otimes\mathcal{H}^1)_p^{\rho_k},\ p,\ \mathcal{G}_f) \to (\mathbb{R}^{n+1},\ \mathsf{d}_e,\ \underline{\mathcal{L}}^{n+1},\ 0,\ H),$$

where $H \subseteq \mathbb{R}^{n+1}$ is a halfspace. Fix also $\rho > 0$. Assume the convergence is realized in a proper metric space $(\mathsf{Z}, \mathsf{d}_\mathsf{Z})$. We show that, for every $\varepsilon > 0$, there exists $k_0 \in \mathbb{N}$ such that

$$B^{\mathsf{Z}}_\rho(p^k) \cap (\Omega_f \setminus R^1_i(f))^k \subseteq B^{\mathsf{Z}}_\varepsilon(\partial H) \quad \text{if } k \geq k_0.$$

Here the superscript k denotes the isometric image in Z through the embedding of the ρ_k-rescaled space.

We argue by contradiction. Up to taking a not relabeled subsequence, by the contradiction assumption, there exist $\{q^k\}_k$ such that, for every k,

$$q^k \in (B^{\mathsf{Z}}_\rho(p^k) \cap (\Omega_f \setminus R^1_i(f))^k) \setminus B^{\mathsf{Z}}_\varepsilon(\partial H).$$

Up to a not relabeled subsequence, $q^k \to q \in \mathsf{Z}$, with $\mathsf{d}_\mathsf{Z}(q, \partial H) \geq \frac{1}{2}\varepsilon$. It is easy to see that $q \in \mathbb{R}^{n+1}$. By weak convergence of measures,

$$\lim_{k\to\infty} \frac{\rho_k |\mathrm{D}\chi_{\mathcal{G}_f}|(B_{\varepsilon\rho_k/2}(q^k))}{C_p^{\rho_k}} = 0.$$

On the other hand, recalling that $\{q^k\}_k \subseteq (\mathcal{F}_{n+1}\mathcal{G}_f)_{2r_i,\varepsilon_i}$ and using again the weak convergence of measures,

$$\varliminf_{k\to\infty} \frac{\rho_k |\mathrm{D}\chi_{\mathcal{G}_f}|(B_{\varepsilon\rho_k/2}(q^k))}{C_p^{\rho_k}} = \varliminf_{k\to\infty} \frac{\rho_k |\mathrm{D}\chi_{\mathcal{G}_f}|(B_{\varepsilon\rho_k/2}(q^k))}{\mathsf{m}(B_{\varepsilon\rho_k/2}(q^k))} \frac{\mathsf{m}(B_{\varepsilon\rho_k/2}(q^k))}{C_p^{\rho_k}} \geq \frac{\omega_n}{2\omega_{n+1}} \mathcal{L}^{n+1}(B_{\varepsilon/2}(q)) > 0,$$

which is a contradiction.

Step 3: Proof of the σ-finiteness claim. We use the same notation as in Step 1. We claim that, for every i,

$$\mathcal{H}^n \llcorner \{(x,t,s) \in \widetilde{R} \mid x \in \widehat{G}_i(f) \cap \widehat{G}_i(g)\}$$

is σ-finite. To show this, it is enough to prove that, for every $i, j, k, \ell, m, \xi \in \mathbb{N}$,

$$\mathcal{H}^n \llcorner \widetilde{T}_{i,j,k,\ell,m,\xi}$$

is σ-finite, where we set

$$\widetilde{T}_{i,j,k,\ell,m,\xi} := \{(x,t,s) \in \widetilde{R} \mid x \in \widehat{G}_i(f) \cap \widehat{G}_i(g) \cap D_{i,k},\ (x,t) \in Q_i^{j,m}(f),\ (x,s) \in Q_i^{\ell,\xi}(g)\}.$$

Fix then $i, j, k, \ell, m, \xi \in \mathbb{N}$, and set for simplicity $\widetilde{T} = \widetilde{T}_{i,j,k,\ell,m,\xi}$. Now define

$$v := (u_{i,k}, \pi^2, \pi^3) : (Q_i^{j,m}(f) \times \mathbb{R}) \cup \tau(Q_i^{\ell,\xi}(g) \times \mathbb{R}) \to \mathbb{R}^{n+2}.$$

By the construction in Step 1,

$$v|_{Q_i^{j,m}(f)\times\mathbb{R}} \quad \text{and} \quad v|_{\tau(Q_i^{\ell,\xi}(g)\times\mathbb{R})} \tag{3-34}$$

are bi-Lipschitz onto their image. Therefore, as $\widetilde{T} \subseteq (Q_i^{j,m}(f) \times \mathbb{R}) \cap \tau(Q_i^{\ell,\xi}(g) \times \mathbb{R})$, it is enough to show that

$$\mathcal{H}^n \llcorner v(\widetilde{T})$$

is σ-finite. Here a central point is that $\widetilde{T} \subseteq D_{i,k} \times \mathbb{R} \times \mathbb{R}$, so that, by the construction in Step 1, the map v as above will be suitable both for the part concerning f and the part concerning g (see (3-33)). Now notice that

$$v(\widetilde{T}) \subseteq (\widehat{S}_i^{j,m}(f) \times \mathbb{R}) \cap \tau(\widehat{S}_i^{\ell,\xi}(g) \times \mathbb{R}),$$

so that, by a standard result of geometric measure theory on Euclidean spaces, we can simply show that, at every $p = (x,t,s) \in \widetilde{T}$, we have that $\widehat{S}_i^{j,m}(f) \times \mathbb{R}$ and $\tau(\widehat{S}_i^{\ell,\xi}(g) \times \mathbb{R})$ intersect transversally at $v(p)$, or, equivalently, that $\widehat{S}_i^{j,m}(f)\times\mathbb{R}$ and $\tau(\widehat{S}_i^{\ell,\xi}(g)\times\mathbb{R})$ have different tangent spaces at $v(p)$. We can, and will, assume that $v(p)=0$.

By our assumptions, compare with items (iii) and (iv) of Proposition 3.7, we know that there exists a sequence $\rho_k\searrow 0$ and a proper metric space $(\mathsf{Z},\mathsf{d}_\mathsf{Z})$ such that $(\mathsf{Z}\times\mathbb{R}\times\mathbb{R},\mathsf{d}_{\mathsf{Z}\times\mathbb{R}\times\mathbb{R}})$ realizes both the convergence

$$(\mathsf{X}\times\mathbb{R}\times\mathbb{R},\rho_k^{-1}\mathsf{d}_{\mathsf{X}\times\mathbb{R}\times\mathbb{R}},(\mathsf{m}\otimes\mathcal{H}^1\otimes\mathcal{H}^1)_p^{\rho_k},p,\mathcal{G}_f\times\mathbb{R})\to(\mathbb{R}^n\times\mathbb{R}\times\mathbb{R},\mathsf{d}_e,\underline{\mathcal{L}}^{n+2},0,H\times\mathbb{R}\times\mathbb{R}) \quad (3\text{-}35)$$

and the convergence

$$(\mathsf{X}\times\mathbb{R}\times\mathbb{R},\rho_k^{-1}\mathsf{d}_{\mathsf{X}\times\mathbb{R}\times\mathbb{R}},(\mathsf{m}\otimes\mathcal{H}^1\otimes\mathcal{H}^1)_p^{\rho_k},p,\tau(\mathcal{G}_g\times\mathbb{R}))\to(\mathbb{R}^n\times\mathbb{R}\times\mathbb{R},\mathsf{d}_e,\underline{\mathcal{L}}^{n+2},0,H'\times\mathbb{R}\times\mathbb{R}), \quad (3\text{-}36)$$

where H and H' are halfspaces in $\mathbb{R}^n$. Notice that this can be done since the $(n+1)$-coordinate of the ν's are zero, see the definition of $\widetilde{R}$. We have endowed $\mathbb{R}^n\times\mathbb{R}\times\mathbb{R}$ with the coordinates given by the (locally uniform) limits of appropriate rescalings of the components of z, where

$$z:=(A_i(x)u_{i,k},\pi^2,\pi^3)\colon B_\rho(p)\to\mathbb{R}^{n+2}$$

for some $\rho>0$ (see Remark 2.25). To do so, we needed to take a not relabeled subsequence of $\{\rho_k\}_k$, but this will make no difference. Hence, recalling also the definition of $\widetilde{R}$, it follows that $H\neq H'$.

Fix $D\geq 5$ greater than the bi-Lipschitz constants of the maps in (3-34) and such that

$$|(A_i(x),\pi^1,\pi^2)c|\leq(D-4)|c| \quad \text{for every } c\in\mathbb{R}^{n+2}. \quad (3\text{-}37)$$

Let $\delta\in(0,D^{-1})$ be small enough that we can find $a\in(\partial H\times\mathbb{R}\times\mathbb{R})\cap B_1(0)\subseteq\mathbb{R}^{n+2}$ such that $B_{D\delta}(a)\cap(\partial H'\times\mathbb{R}\times\mathbb{R})=\varnothing$.

As a consequence of the density assumption made by removing R_i^5, we can find a sequence $\{a^k\}_k\subseteq\mathsf{X}\times\mathbb{R}\times\mathbb{R}$ with

$$a^k\in(Q_i^{j,m}(f)\times\mathbb{R})\cap B_{\rho_k}(p) \quad \text{for every } k\in\mathbb{N}$$

and $a^k\to a$ in $\mathsf{Z}\times\mathbb{R}\times\mathbb{R}$, where here and below the superscript k denotes the isometric image in $\mathsf{Z}\times\mathbb{R}\times\mathbb{R}$ through the embedding of the ρ_k-rescaled space.

By weak convergence of measures,

$$\varliminf_{k\to\infty}\frac{\rho_k|\mathrm{D}\chi_{\mathcal{G}_f\times\mathbb{R}}|(B_{D^{-1}\delta\rho_k}(a^k))}{C_p^{\rho_k}}>0,$$
$$\varlimsup_{k\to\infty}\frac{\rho_k|\mathrm{D}\chi_{\tau(\mathcal{G}_g\times\mathbb{R})}|(B_{D\delta\rho_k}(a^k))}{C_p^{\rho_k}}=0.$$

Recalling again the density assumption made by removing R_i^5 together with the bounds on $\Theta_n(|\mathrm{D}\chi_{\mathcal{G}_f}|,\cdot)$ by removing R_i^2, and finally the weak convergence of measures, this reads as

$$\varliminf_{k\to\infty}\rho_k^{-n-1}\mathcal{H}^{n+1}(B_{D^{-1}\delta\rho_k}(a^k)\cap(Q_i^{j,m}\times\mathbb{R}))>0, \quad (3\text{-}38)$$
$$\varlimsup_{k\to\infty}\rho_k^{-n-1}\mathcal{H}^{n+1}(B_{D\delta\rho_k}(a^k)\cap\tau(Q_i^{\ell,\xi}\times\mathbb{R}))=0. \quad (3\text{-}39)$$

It is easy to verify by contradiction that (3-38) implies, by our choice of D, that

$$\varliminf_{k\to\infty} \rho_k^{-n-1}\mathcal{H}^{n+1}(B_{\delta\rho_k}(v(a^k))\cap(\widehat{S}_i^{j,m}(f)\times\mathbb{R})\cap B_{2D\rho_k}(0))>0. \tag{3-40}$$

Now we show

$$\varlimsup_{k\to\infty} \rho_k^{-n-1}\mathcal{H}^{n+1}(B_{\delta\rho_k}(v(a^k))\cap\tau(\widehat{S}_i^{\ell,\xi}(g)\times\mathbb{R})\cap B_{2D\rho_k}(0))=0. \tag{3-41}$$

By Step 2, we get that, for $\varepsilon\in(0,\delta)$, there exists k_0 such that if $k\geq k_0$, then, for every $b\in(B_{2D^2\rho_k}(p)\setminus B_{D\delta\rho_k}(a^k)\cap\tau(Q_i^{\ell,\xi}\times\mathbb{R}))^k$ there exists $b'\in\partial H'\times\mathbb{R}\times\mathbb{R}$ such that

$$\mathsf{d}_{\mathsf{Z}\times\mathbb{R}\times\mathbb{R}}(b,b')<\varepsilon.$$

Up to increasing k_0, we may assume that, for every $k\geq k_0$,

$$\mathsf{d}_{\mathsf{Z}\times\mathbb{R}\times\mathbb{R}}(a,a^k)<\varepsilon.$$

Notice that if b is as above, then

$$|b'-a|\geq D\delta-2\varepsilon$$

and, by local uniform convergence, up to enlarging k_0 and provided $\varepsilon>0$ is small enough,

$$|\rho_k^{-1}z(b)-\rho_k^{-1}z(a^k)|\geq|b'-a|-2\delta,$$

so that

$$|z(b)-z(a^k)|\geq((D-2)\delta-2\varepsilon)\rho_k\geq(D-4)\delta\rho_k,$$

which implies, recalling (3-37),

$$|v(b)-v(a^k)|\geq\delta\rho_k.$$

Notice that the above inequality does *not* follow from the fact that the maps in (3-34) are D-bi-Lipschitz, but implies that (3-41) follows from (3-39) by the choice of D.

We can now conclude the proof of Step 3, as by (3-40) and (3-41) it follows easily that $\widehat{S}_i^{j,m}(f)\times\mathbb{R}$ and $\tau(\widehat{S}_i^{\ell,\xi}(g)\times\mathbb{R})$ have different tangent spaces at 0.

Step 4: A technical estimate. For some $i\in\mathbb{N}$, let us assume $\widetilde{R}'$ is such that

$$\widetilde{R}'\subseteq\widetilde{R}\cap(\widehat{G}_i(f)\times\mathbb{R}\times\mathbb{R})\cap(\widehat{G}_i(g)\times\mathbb{R}\times\mathbb{R})$$

and that $\widetilde{R}'$ has finite $\mathcal{H}^n$-measure. Let $p\in\widetilde{R}'$ be fixed. We claim that

$$\lim_{r\searrow 0}\frac{\mathcal{H}^n_5(\pi^{1,2}(\widetilde{R}'\cap B_r(p)))}{r^n}=0.$$

Let us prove the claim. Take a sequence $\rho_k\searrow 0$. We recall that, with the same notation as above, up to a not relabeled subsequence, (3-35) and (3-36) hold. Let

$$I:=I((\partial H\cap\partial H')\times\mathbb{R}\times\mathbb{R})$$

be a neighborhood (in $\mathsf{Z}\times\mathbb{R}\times\mathbb{R}$) of $((\partial H\cap\partial H')\times\mathbb{R}\times\mathbb{R})\cap B_2(0)$ that satisfies

$$\mathcal{H}^n_5(\pi^{1,2}(I))<\varepsilon.$$

As a consequence of Step 2, there exists $k_0 \in \mathbb{N}$ such that

$$B_1^{\mathsf{Z}\times\mathbb{R}\times\mathbb{R}}(p^k) \cap \widetilde{R}' \subseteq I \quad \text{for every } k \geq k_0,$$

from which, taking the projection $\pi^{1,2}$, the claim follows.

Step 5: Conclusion. Let us finally prove (3-30). By Step 3, it is enough to show that

$$(|\mathrm{D}f| \wedge |\mathrm{D}g|)(\pi^1(\widetilde{R}')) = 0,$$

where $\widetilde{R}'$ is as in Step 4. Fix $\varepsilon > 0$. For every $j \in \mathbb{N}$, $j \geq 1$ we consider the sets

$$\widetilde{R}'_j := \left\{ p \in \widetilde{R}' \;\middle|\; \frac{\mathcal{H}^n_5(\pi^{1,2}(\widetilde{R}' \cap B_r(p)))}{r^n} < \varepsilon \text{ for every } r \in (0, j^{-1}) \right\}$$

and

$$\widetilde{R}''_j := \widetilde{R}'_j \setminus \bigcup_{i<j} \widetilde{R}'_i.$$

Notice that, by Step 4,

$$\widetilde{R}' = \bigcup_{j\geq 1} \widetilde{R}''_j$$

and, by construction, this union is disjoint. For every $j \geq 1$, we take a countable family of balls $\{B_{r_i^j}(p_i^j)\}_i$ such that, for every $i \in \mathbb{N}$, we have $r_i^j < j^{-1}$ and $p_i^j \in \widetilde{R}''_j$, as well as

$$\widetilde{R}''_j \subseteq \bigcup_{i\in\mathbb{N}} B_{r_i^j}(p_i^j) \quad \text{and} \quad \sum_{i\in\mathbb{N}} (r_i^j)^n \leq 2^n \mathcal{H}^n(\widetilde{R}''_j) + 2^{-j}. \tag{3-42}$$

We can compute, recalling the definition of $\widetilde{R}''_j$ and (3-42),

$$\mathcal{H}^n_5(\pi^{1,2}(\widetilde{R}''_j)) \leq \mathcal{H}^n_5\Bigl(\pi^{1,2}\Bigl(\widetilde{R}'' \cap \bigcup_{i\in\mathbb{N}} B_{r_i^j}(p_i^j)\Bigr)\Bigr) \leq \sum_{i\in\mathbb{N}} \varepsilon (r_i^j)^n \leq \varepsilon(2^n \mathcal{H}^n(\widetilde{R}''_j) + 2^{-j}).$$

Therefore,

$$\mathcal{H}^n_5(\pi^{1,2}(\widetilde{R}')) \leq \varepsilon(2^n \mathcal{H}^n(\widetilde{R}') + 1)$$

and, $\varepsilon > 0$ being arbitrary, $|\mathrm{D}\chi_{\mathcal{G}_f}|(\pi^{1,2}(\widetilde{R}')) = 0$, whence the result follows due to Proposition 2.13. □

Lemma 3.12. *Let* $(\mathsf{X}, \mathsf{d}, \mathsf{m})$ *be an* $\mathrm{RCD}(K, N)$ *space of essential dimension* n, *and let* $f, g \in \mathrm{BV}(\mathsf{X})$. *Choose two* Cap*-vector field representatives for* ν_f *and* ν_g. *Then*

$$\nu_f = \pm \nu_g \quad (|\mathrm{D}f| \wedge |\mathrm{D}g|)\text{-a.e. on } C_f \cap C_g.$$

Proof. From Lemmas 3.9 and 3.11 together with Theorem 3.8 we have that, for $(|\mathrm{D}f| \wedge |\mathrm{D}g|)$-a.e. $x \in C_f \cap C_g$, there exists $\eta = \eta(x) \in (0, n^{-1}) \cap \mathbb{Q}$ such that

$$\nu_f^{\boldsymbol{u}_\eta}(x) = \pm \nu_g^{\boldsymbol{u}_\eta}(x).$$

It remains to show that if, for some $\eta \in (0, n^{-1}) \cap \mathbb{Q}$, it holds that $\nu_f^{\boldsymbol{u}_\eta} = \pm \nu_g^{\boldsymbol{u}_\eta}$ Cap-a.e. on a Borel set A, then $\nu_f = \pm \nu_g$ Cap-a.e. on A. This follows since the gradients of the functions in $\boldsymbol{u}_{\eta,k}$ are a generating subspace of $L^0_{\mathrm{Cap}}(T\mathsf{X})$ on $D_{\eta,k}$ since the $L^0_{\mathrm{Cap}}(T\mathsf{X})$ module has local dimension at most n. Indeed, if $h_1, \ldots, h_{n+1} \in \mathrm{TestF}(\mathsf{X})$ then $\det(\nabla h_i \cdot \nabla h_j)_{i,j} = 0$ m-a.e. hence Cap-a.e., so that it is now easy to bound the local dimension of $L^0_{\mathrm{Cap}}(T\mathsf{X})$. □

The following lemma is extracted from [Brena and Gigli 2024, Proposition 3.30].

Lemma 3.13. *Let* $(\mathsf{X}, \mathsf{d}, \mathsf{m})$ *be an* $\mathrm{RCD}(K, N)$ *space of essential dimension* n, *and let* $f, g \in \mathrm{BV}(\mathsf{X})$. *Choose two* Cap-*vector field representatives for* ν_f *and* ν_g. *Then*

$$\nu_f = \pm\nu_g \quad (|\mathrm{D}f| \wedge |\mathrm{D}g|)\text{-a.e. on } J_f \cap J_g.$$

Proof of Theorem 1.3. We first notice that, for every $i = 1, \dots, k$,

$$(\nu_F)_i = \frac{\mathrm{d}|\mathrm{D}F_i|}{\mathrm{d}|\mathrm{D}F|}\nu_{F_i} \quad |\mathrm{D}F|\text{-a.e.}$$

The conclusion on the jump part is given by Lemma 3.13 applied to every pair of components of F together with the well-known fact that, for every $i = 1, \dots, k$, we have $|\mathrm{D}F_i|(J_F \setminus J_{F_i}) = 0$. On the Cantor part, the result follows from Lemma 3.12 applied to every pair of components of F. □

Appendix: Rectifiability of the reduced boundary

In this appendix, we give an alternative proof of the known fact that reduced boundaries of sets of finite perimeter in finite-dimensional RCD spaces are rectifiable. Roughly speaking, this is a consequence of the rectifiability result of [Bate 2022] and the uniqueness of tangents to sets of finite perimeter proved in [Bruè et al. 2023b], once one takes into account the regularity result Theorem 3.3.

Let us recall part of the statement of [Bate 2022, Theorem 1.2].

Theorem A.1. *Let* (X, d) *be a complete metric space*, $k \in \mathbb{N}$, *and* $S \subseteq \mathsf{X}$ *such that* $\mathcal{H}^k(S) < \infty$. *Hence the following are equivalent*:

(1) *S is k-rectifiable.*

(2) *For $\mathcal{H}^k$-almost every $x \in S$, we have $\underline{\Theta}_k(S, x) > 0$ and the existence of a k-dimensional Banach space $(\mathbb{R}^k, \|\cdot\|_k)$ such that*

$$\mathrm{Tan}_x(\mathsf{X}, \mathsf{d}, \mathcal{H}^k \llcorner S) = \{(\mathbb{R}^k, \|\cdot\|_x, \mathcal{H}^k, 0)\}. \tag{A-1}$$

Let us fix $(\mathsf{X}, \mathsf{d}, \mathsf{m})$ an $\mathrm{RCD}(K, N)$ space of essential dimension n. Let $E \subseteq \mathsf{X}$ be a set of locally finite perimeter. Now by Theorem 3.3 and the first part of the argument of Theorem 3.4, we have:

(1) $|\mathrm{D}\chi_E|(\mathsf{X} \setminus \mathcal{R}_n^*) = 0$, and hence $|\mathrm{D}\chi_E|$ is concentrated on $\mathcal{F}E$.

(2) $\mathcal{H}^{n-1} \llcorner \mathcal{F}E$ is a σ-finite Borel measure that is mutually absolutely continuous with respect to $|\mathrm{D}\chi_E|$. Notice that, for the precise computation of the density of $|\mathrm{D}\chi_E|$ with respect to $\mathcal{H}^{n-1} \llcorner \mathcal{F}E$ in Theorem 3.4, we needed the rectifiability of $\mathcal{F}E$, which we will not use in the following argument.

Hence let us call $f \in L^1_{\mathrm{loc}}(|\mathrm{D}\chi_E|)$ the function such that $\mathcal{H}^{n-1} \llcorner \mathcal{F}E = f|\mathrm{D}\chi_E|$, and let $\mathcal{D} \subseteq \mathcal{F}E$ be the set of the Lebesgue points of f with respect to the asymptotically doubling measure $|\mathrm{D}\chi_E|$ that are also differentiability points of $\mathcal{H}^{n-1} \llcorner \mathcal{F}E$ with respect to $|\mathrm{D}\chi_E|$, i.e., for every $x \in \mathcal{D}$,

$$\lim_{r \to 0} \fint_{B_r(x)} |f - f(x)| \, \mathrm{d}|\mathrm{D}\chi_E| = 0 \tag{A-2}$$

and

$$f(x) = \lim_{r \to 0} \frac{\mathcal{H}^{n-1} \llcorner \mathcal{F}E(B_r(x))}{|\mathrm{D}\chi_E|(B_r(x))}. \tag{A-3}$$

Notice that $|D\chi_E|(X\setminus\mathcal{D})=\mathcal{H}^{n-1}(\mathcal{F}E\setminus\mathcal{D})=0$ due to the Lebesgue differentiation theorem [Heinonen et al. 2015, p. 77], and the Lebesgue–Radon–Nikodým theorem [Heinonen et al. 2015, p. 81 and Remark 3.4.29]. Notice, moreover, that since $|D\chi_E|$ is mutually absolutely continuous with respect to $\mathcal{H}^{n-1}\llcorner\mathcal{F}E$, we have $f(x)>0$ for $|D\chi_E|$-almost every $x\in\mathsf{X}$, or equivalently for $\mathcal{H}^{n-1}\llcorner\mathcal{F}E$-almost every $x\in\mathsf{X}$.

Let us now prove that $\mathcal{F}E$ is $(n-1)$-rectifiable by exploiting Theorem A.1. Let us verify item (2) there. By the third line in (2-10) together with the fact that $x\in\mathcal{R}_n^*$ and (A-3), we get that $\underline{\Theta}_{n-1}(\mathcal{F}E,x)>0$ for every $x\in\mathcal{D}$, and hence for $\mathcal{H}^{n-1}$-almost every $x\in\mathcal{F}E$. Let us now verify the second part of item (2). Let us fix $x\in\mathcal{D}$, and let us take an arbitrary sequence $r_i\to 0$. We have that, up to subsequences,

$$\mathsf{X}_i:=(\mathsf{X},r_i^{-1}\mathsf{d},\mathsf{m}_x^{r_i},x,E)\to(\mathbb{R},\mathsf{d}_e,\underline{\mathcal{L}}^n,0,\{x_n>0\})$$

and, in a realization of the previous convergence, we have that the $|D\chi_E|_{\mathsf{X}_i}$ weakly converge to $|D\chi_{\{x_n>0\}}|$. For the sake of clarity, we denoted by $|D\chi_E|_{\mathsf{X}_i}$ the perimeter measure of E in the rescaled space X_i. Notice that $|D\chi_E|_{\mathsf{X}_i}=(r_i/C_x^{r_i})|D\chi_E|$, where $|D\chi_E|$ is the perimeter measure on X. Let $g\in C_{\mathrm{bs}}(\mathsf{Z})$, where Z is a realization of the previous convergence. Hence we have

$$\int_{\mathsf{X}_i}g\,\mathrm{d}\frac{r_i\mathcal{H}^{n-1}\llcorner FE}{C_x^{r_i}}=\int_{\mathsf{X}_i}gf\,\mathrm{d}|D\chi_E|_{\mathsf{X}_i}=\int_{\mathsf{X}_i}g(y)f(x)\,\mathrm{d}|D\chi_E|_{\mathsf{X}_i}(y)+\int_{\mathsf{X}_i}g(y)(f(y)-f(x))\,\mathrm{d}|D\chi_E|_{\mathsf{X}_i}(y),$$

and hence, by using (A-2) and the fact that

$$|D\chi_E|(B_{r_i}(x))\sim\frac{(n+1)\omega_{n-1}}{\omega_n}\frac{C_x^{r_i}}{r_i}$$

as a consequence of the second and third line of (2-10), we conclude that[2]

$$\frac{r_i\mathcal{H}^{n-1}\llcorner\mathcal{F}E}{C_x^{r_i}}\rightharpoonup f(x)|D\chi_{\{x_n>0\}}| \tag{3-4}$$

in the realization Z. This immediately implies that

$$\frac{\mathcal{H}^{n-1}\llcorner\mathcal{F}E}{\mathcal{H}^{n-1}\llcorner\mathcal{F}E(B_{r_i}(x))}\rightharpoonup\mathcal{H}^{n-1}\llcorner\{x_n=0\}$$

because $\mathcal{H}^{n-1}\llcorner\{x_n=0\}$ is the surface measure on $\{x_n=0\}$ that gives measure 1 to the unit ball.

Hence we have shown that, for every $x\in\mathcal{D}$ and every sequence $r_i\to 0$, there is a realization Z in which one has the convergence

$$\left(\mathsf{X},\frac{\mathsf{d}}{r_i},\frac{\mathcal{H}^{n-1}\llcorner\mathcal{F}E}{\mathcal{H}^{n-1}\llcorner\mathcal{F}E(B_{r_i}(x))},x\right)\to(\mathbb{R}^{n-1},\mathsf{d}_e,\mathcal{H}^{n-1},0),$$

which is exactly what one needed to show in order to verify (A-1) (recall [Bate 2022, Proposition 2.13]). Hence the application of Theorem A.1 gives the $(n-1)$-rectifiability of $\mathcal{F}E$.

[2] Notice that in the following equation we are considering $\mathcal{H}^{n-1}\llcorner\mathcal{F}E$ in the original space X and not in the rescaled space

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A NEW APPROACH TO THE FOURIER EXTENSION PROBLEM FOR THE PARABOLOID

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Dedicated to the memory of Robert S. Strichartz

We propose a new approach to the Fourier restriction conjectures. It is based on a discretization of the Fourier extension operators in terms of quadratically modulated wave packets. Using this new point of view, and by combining natural scalar and mixed norm quantities from appropriate level sets, we prove that all the L^2-based k-linear extension conjectures are true up to the endpoint for every $1 \leq k \leq d+1$ if one of the functions involved is a full tensor. We also introduce the concept of *weak transversality*, under which we show that all conjectured L^2-based multilinear extension estimates are still true up to the endpoint, provided that one of the functions involved has a weaker tensor structure, and we prove that this result is sharp. Under additional tensor hypotheses, we show that one can improve the conjectured threshold of these problems in some cases. In general, the largely unknown multilinear extension theory beyond L^2 inputs remains open even in the bilinear case; with this new point of view, and still under the previous tensor hypothesis, we obtain the near-restriction target for the k-linear extension operator if the inputs are in a certain L^p space for p sufficiently large. The proof of this result is adapted to show that the k-fold product of linear extension operators (no transversality assumed) also "maps near restriction" if one input is a tensor. Finally, we exploit the connection between the geometric features behind the results of this paper and the theory of Brascamp–Lieb inequalities, which allows us to verify a special case of a conjecture by Bennett, Bez, Flock and Lee.

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1. Introduction

Given a compact submanifold $S \subset \mathbb{R}^{d+1}$ and a function $f : \mathbb{R}^{d+1} \mapsto \mathbb{R}$, the *Fourier restriction problem* asks for which pairs (p, q) one has

$$\|\hat{f}|_S\|_{L^q(S)} \lesssim \|f\|_{L^p(\mathbb{R}^{d+1})},$$

where $\hat{f}|_S$ is the restriction of the Fourier transform $\hat{f}$ to S. This problem arises naturally in the study of certain Fourier summability methods and is known to be connected to questions in geometric measure theory and in nonlinear dispersive PDEs. The interaction between curvature and the Fourier transform has been exploited in a variety of contexts since the works [Hörmander 1973; Fefferman 1971; Stein and Wainger 1978] in the study of oscillatory integrals. For a more detailed description of the restriction problem we refer the reader to the classical survey [Tao 2004]. In this paper we work with the equivalent dual formulation of the question above (known as the *Fourier extension problem*), and specialize to the case where S is the compact piece of the paraboloid parametrized by $\Gamma(x) = (x, |x|^2) \subset \mathbb{R}^{d+1}$ with $x \in [0, 1]^d$. In this setting, the *Fourier extension operator* is initially defined on $C([0, 1]^d)$ by

$$\mathcal{E}_d g(x_1, \dots, x_d, t) = \int_{[0,1]^d} g(\xi_1, \dots, \xi_d) e^{-2\pi i(\xi_1 x_1 + \cdots + \xi_d x_d)} e^{-2\pi i t(\xi_1^2 + \cdots + \xi_d^2)}\, d\xi. \tag{1}$$

E. Stein [1993, Chapter IX] proposed the following conjecture:

Conjecture 1.1. *The inequality*

$$\|\mathcal{E}_d g\|_{L^q(\mathbb{R}^{d+1})} \lesssim_{p,q,d} \|g\|_{L^p([0,1]^d)} \tag{2}$$

holds if and only if $q > \frac{2(d+1)}{d}$ *and* $q \geq \frac{(d+2)}{d} p'$.

Multilinear variants[1] of Conjecture 1.1 arose naturally from the works [Klainerman and Machedon 1993; 1995; 1996] on wellposedness of certain PDEs. Given $2 \leq k \leq d+1$ compact and connected domains $U_j \subset \mathbb{R}^d$, $1 \leq j \leq k$, define

$$\mathcal{E}_{U_j} g(x, t) := \int_{U_j} g(\xi) e^{-2\pi i x \cdot \xi} e^{-2\pi i t |\xi|^2}\, d\xi, \quad (x, t) \in \mathbb{R}^d \times \mathbb{R}. \tag{3}$$

Taking the product of all k such operators associated to a set of *transversal* U_j leads to the following conjecture (see Appendix A):

Conjecture 1.2 [Bennett 2014]. *If the caps parametrized by U_j are transversal, then*

$$\Big\| \prod_{j=1}^{k} \mathcal{E}_{U_j} g_j \Big\|_p \lesssim \prod_{j=1}^{k} \|g_j\|_2 \quad \textit{for all } p \geq \frac{2(d+k+1)}{k(d+k-1)}.$$

Roughly, *transversality* means that any choice of one normal vector per cap is a set of linearly independent vectors, as shown in Figure 1.

Remark 1.3. From now on, we shall refer to Conjecture 1.1 as *the case* $k = 1$. It was settled only for $d = 1$ in [Fefferman 1970; Zygmund 1974]. In higher dimensions we highlight the case $p = 2$ solved in [Strichartz 1977], which is equivalent to the Tomas–Stein theorem [Tomas 1975] in the restriction setting.

[1] Multilinear extension estimates also play a fundamental role in Bourgain and Demeter's decoupling theory [2015].

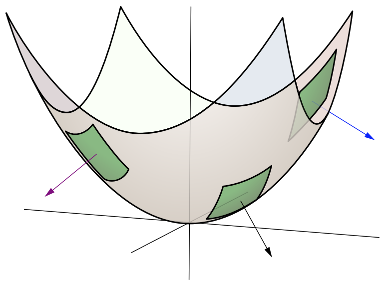

Figure 1. A choice of normal vectors to the caps parametrized by U_j via $x \mapsto |x|^2$.

Progress beyond these two results was made in many works over the last decades through a diverse set of techniques: localization, bilinear estimates, wave-packet decompositions and more recently polynomial methods. We mention [Bourgain 1991; Tao and Vargas 2000; Tao 2003; Moyua et al. 1996; Wang 2018; Guth 2018; Hickman and Rogers 2019]. Analogous problems for other manifolds were studied in [Wolff 2001; Strichartz 1977; Ou and Wang 2022].

Remark 1.4. Guth [2018] proved a weaker version of Conjecture 1.2 for all $2 \le k \le d+1$ and up to the endpoint, which is known as the k-broad restriction inequality. This estimate plays a central role in his argument in [Guth 2018] to improve the range for which Conjecture 1.1 is known. In Lemma A.3 of [Bourgain and Guth 2011], the authors proved an L^2-based k-linear estimate for an exponent p slightly larger than the conjectured threshold in Conjecture 1.2.

Only three cases of Conjecture 1.2 are well understood:

(i) Tao [2003] settled the case $k = 2$ up to the endpoint inspired by [Wolff 2001] for the cone. Lee [2021] obtained the endpoint for $k = 2$.

(ii) Bennett, Carbery and Tao [Bennett et al. 2006] settled the case $k = d + 1$ up to the endpoint.

(iii) Bejenaru [2022] settled the case $k = d$ up to the endpoint.

The goal of this paper is to propose a new approach to these problems based on a natural discretization of the operators in terms of scalar products against quadratically modulated wave-packets. Our main theorem reads as follows:

Theorem 1.5. *Conjectures 1.1 and 1.2 hold up to the endpoint if one (any) of the functions involved is a full tensor.*[2]

Remark 1.6. The endpoint $(p,q) = \left(\frac{2(d+1)}{d}, \frac{2(d+1)}{d}\right)$ is not included in the range where (2) is supposed to hold; therefore our main theorem implies the case $k = 1$ when g is a full tensor.

Remark 1.7. For $2 \le k \le d+1$, Theorem 1.5 can be proved if the caps are assumed to be *weakly transversal*, which is defined in Section 3. We will prove that transversality implies weak transversality (up to dividing the caps into finitely many pieces), the latter being what is actually exploited in this paper.

[2]A function g in d variables is a *full tensor* if it can be written as $g(x_1, \dots, x_d) = g_1(x_1) \cdot \dots \cdot g_d(x_d)$. We refer the reader to [Igari 1986; Tanaka 2001] for other results related to the restriction problem involving tensors, and we thank Terence Tao for pointing these papers out to us.

Under weak transversality, Theorem 1.5 holds if one (any) of the functions has a weaker tensor structure. This will be made precise in Section 9.

Remark 1.8. For $2 \le k \le d+1$, Theorem 1.5 is sharp under weak transversality in the following sense: if all functions $g_1, \dots, g_k$ are generic, it does not hold if the caps are assumed to be weakly transversal. This is explained in Appendix A.

Remark 1.9. For $2 \le k \le d+1$ we do not use the tensor structure explicitly. It is used in an implicit way when comparing the sizes of natural scalar and mixed norm quantities that appear in the proofs.

Remark 1.10. For $2 \le k \le d$, if all functions involved are full tensors, one has more estimates than those predicted by Conjecture 1.2 assuming extra *degrees of transversality*, as proven in Section 11.

It is natural to try to generalize the statement of Conjecture 1.2 for L^p inputs rather than just L^2. A motivation for that is to deeply understand the role played by transversality; as we will see, the farther our inputs are from L^2, the less impact the configuration of the caps on the paraboloid has in the best possible estimate (with a single exception to be detailed soon). The general statement of the k-linear extension conjecture for the paraboloid is (as in [Bennett 2014]):

Conjecture 1.11. *Let $k \ge 2$ and suppose that $U_1, \dots, U_k$ parametrize transversal caps of the paraboloid $x \mapsto |x|^2$ in $\mathbb{R}^{d+1}$. If*

$$\frac{1}{q} < \frac{d}{2(d+1)}, \quad \frac{1}{q} \le \frac{d+k-1}{d+k+1}\frac{1}{p'} \quad \text{and} \quad \frac{1}{q} \le \frac{d-k+1}{d+k+1}\frac{1}{p'} + \frac{k-1}{k+d+1},$$

then

$$\Big\| \prod_{j=1}^{k} \mathcal{E}_{U_j} g_j \Big\|_{L^{q/k}(\mathbb{R}^{d+1})} \lesssim_{p,q} \prod_{j=1}^{k} \|g_j\|_{L^p(U_j)}.$$

For $2 \le k < d+1$, to recover the interior of the conjectured range, it is enough[3] to prove Conjecture 1.2 and

$$\Big\| \prod_{j=1}^{k} \mathcal{E}_{U_j} g_j \Big\|_{L^{2(d+1)/(kd)+\varepsilon}(\mathbb{R}^{d+1})} \lesssim_{\varepsilon} \prod_{j=1}^{k} \|g_j\|_{L^{2(d+1)/d}(U_j)} \tag{4}$$

for all $\varepsilon > 0$.

Remark 1.12. Observe that (4) covers the case $(p,q) = \left(\frac{2(d+1)}{d}, \frac{2(d+1)}{d} + \varepsilon\right)$ of Conjecture 1.11. Notice also that this case would follow from the case $(p,q) = \left(\frac{2(d+1)}{d}, \frac{2(d+1)}{d} + \varepsilon\right)$ of the *linear* extension of Conjecture 1.1 and Hölder's inequality. This means that the closer we get to the endpoint extension exponent, the fewer improvements transversality yields in the multilinear theory. The exception to this is the $k = d+1$ case, for which L^2 functions give the best possible output for the corresponding multilinear operator (rather than $L^{2(d+1)/d}$). Indeed, when one function is a tensor, the best result in this case is obtained in Section 10.

By adapting the argument that shows the case $2 \le k \le d+1$ of Theorem 1.5, we are able to prove the following weaker version of (4):

[3]The interior of the full range of estimates follows by interpolation between these two cases and the trivial bound $(p,q) = (1,\infty)$.

Theorem 1.13. *Let $2 \leq k < d+1$. If g_1 is a tensor in addition to the hypotheses of Conjecture 1.11, the following estimate holds*:

$$\Big\| \prod_{j=1}^{k} \mathcal{E}_{U_j} g_j \Big\|_{L^{2(d+1)/(kd)+\varepsilon}(\mathbb{R}^{d+1})} \lesssim_\varepsilon \prod_{j=1}^{k} \|g_j\|_{L^{p(k,d)}(U_j)} \tag{5}$$

for all $\varepsilon > 0$, where

$$p(k,d) = \begin{cases} \dfrac{4(d+1)}{d+k+1} & \text{if } 2 \leq k < \dfrac{d}{2}, \\ \dfrac{4(d+1)}{2d-k+1} & \text{if } \dfrac{d}{2} \leq k < d+1. \end{cases}$$

Remark 1.14. Notice that $\frac{2(d+1)}{d} < p(k,d)$, so Theorem 1.13 is not optimal on the space of the input functions. On the other hand, the output $L^{2(d+1)/(kd)+\varepsilon}$ (for all $\varepsilon > 0$) is the best to which one can hope to map the multilinear operator on the left-hand side. The case $k = d+1$ of the theorem above coincides with the case $k = d+1$ of the L^2-based theory, which is covered in Section 10.

Remark 1.15. Bounds such as the one from Theorem 1.13, i.e., in which one needs p big enough (and not sharp) to map L^p inputs to a fixed L^q, are common in linear extension theory. For example, Wang [2018] showed that $\mathcal{E}_2$ maps $L^\infty([-1,1]^2)$ to $L^q(\mathbb{R}^3)$ for $q > 3 + \frac{3}{13}$. As mentioned in [Wang 2018], this implies the (seemingly stronger) bound

$$\|\mathcal{E}_2 g\|_{L^q(\mathbb{R}^3)} \lesssim_q \|g\|_{L^q([-1,1]^2)}$$

for $q > 3 + \frac{3}{13}$ via the factorization theory of Nikishin and Pisier (see [Bourgain 1991]).

Remark 1.16. The multilinear extension theory for inputs near $L^{2(d+1)/d}$ remains largely unknown in general (except for the almost optimal result in the $k = d+1$ case in [Bennett et al. 2006]). In fact, it is not fully settled even in the $k = 2$, $d > 1$ case (whose L^2-based analogue is known). We refer the reader to [Oh 2023] for partial results in this direction.

Remark 1.17. As the reader may expect, any function can be taken to be the tensor in the statement of Theorem 1.13.

The linear and multilinear theories studied in this paper meet very naturally once more in the context of the techniques we use: the simplest multilinear variant of a linear operator T is given by the product of a certain number of identical copies of it:

$$T_{(k)}(g_1, \ldots, g_k) := \prod_{j=1}^{k} T g_j.$$

Proving that T maps $L^p(U)$ to $L^q(V)$ is equivalent to proving that $T_{(k)}$ maps $L^p(U)$ to $L^{q/k}(V)$, as one can easily check with Hölder's inequality. Multilinearizing $\mathcal{E}_d$ without any regard to transversality yields the operator

$$\mathcal{E}_{d,(k)}(g_1, \ldots, g_k) := \prod_{j=1}^{k} \mathcal{E}_d g_j. \tag{6}$$

Combining the previous observation with the factorization theory of Nikishin and Pisier, Conjecture 1.1 follows from the bound

$$\Big\| \prod_{j=1}^{k} \mathcal{E}_d g_j \Big\|_{L^{2(d+1)/(kd)+\varepsilon}} \lesssim_\varepsilon \prod_{j=1}^{k} \|g_j\|_{L^\infty([0,1]^d)}. \tag{7}$$

The proof of Theorem 1.13 can be adapted to show the following:

Theorem 1.18. *Let $2 \le k \le d+1$. If g_1 is a tensor, the inequality*

$$\Big\| \prod_{j=1}^{k} \mathcal{E}_d g_j \Big\|_{L^{2(d+1)/(kd)+\varepsilon}(\mathbb{R}^{d+1})} \lesssim_\varepsilon \prod_{j=1}^{k} \|g_j\|_{L^4([0,1]^d)} \tag{8}$$

holds for all $\varepsilon > 0$.

Remark 1.19. Since the inputs g_j are compactly supported, Theorem 1.18 implies (7).

Remark 1.20. Given that the proof of Theorem 1.18 has the L^4-$L^{4+\varepsilon}$ bound for $\mathcal{E}_1$ as its main building block, it is not surprising that we have a product of L^4 norms in the right-hand side of the statement above.

We finish this introduction by highlighting the close connection between our results and the theory of linear and nonlinear Brascamp–Lieb inequalities. The concept of *weak transversality* that we introduce can be characterized in terms of certain Brascamp–Lieb data, and by exploiting the geometric features arising from this fact we are able to verify a special case of a conjecture by Bennett, Bez, Flock and Lee.

The paper is organized as follows: in Section 2 we present the linear and multilinear models that we will work with in the proof of Theorem 1.5. We also highlight the main differences between the linearized models that are used in most recent approaches and ours. In Section 3 we define the concepts of transversality and weak transversality, and state in what sense the former implies the latter. Section 4 presents what we refer to as the *building blocks* of our approach. Sections 5, 6 and 7 establish these building blocks: in Section 5 we revisit the case $k = 1$ and $p = 2$ for our model, in Section 6 we revisit Zygmund's argument and recover the case $k = 1$ for $d = 1$, and in Section 7 we deal with the case $k = 2$ and $d = 1$. In Section 8 we settle the case $k = 1$ of Theorem 1.5, and in Section 9 we show the cases $2 \le k \le d+1$. Section 10 covers the endpoint estimate of the case $k = d+1$. In Section 11 we discuss how one can improve the bounds of Conjecture 1.2 under extra transversality and tensor hypotheses. Theorem 1.13 (our partial result beyond the L^2-based k-linear theory) is presented in Section 12 along with its "nontransversal" counterpart Theorem 1.18. In Section 13 we establish a connection between the classical theory of Brascamp–Lieb inequalities and our results, and give an application of this link to a conjecture made in [Bennett et al. 2018]. In Section 14 we make a few additional remarks. Appendix A contains examples that show that the range of p in Conjecture 1.2 is sharp, and also that one cannot obtain this range in general under a condition that is strictly weaker than transversality. Appendix B contains technical results used throughout the paper.

2. Discrete models

A common first step of the earlier works is to *linearize* the contribution of the quadratic phase $x \mapsto |x|^2$. One starts by studying $\mathcal{E}_d g$ on a ball of radius R (hence $|(x,t)| \le R$) and splits the domain of g into balls θ_k of radius $R^{-1/2}$. Let us consider $d = 1$ here for simplicity. If

$$g_{\theta_k} := g \cdot \varphi_{\theta_k},$$

where φ_{θ_k} is a bump adapted to $[kR^{-1/2}, (k+1)R^{-1/2}]$, the quadratic exponential

$$e_{x,t}(\xi) = e^{2\pi i x\xi} e^{2\pi i t\xi^2} \tag{9}$$

behaves in a similar way to a linear exponential $e^{i\#\xi}$ when restricted to this interval. Indeed, the phase-space portrait of $e_{x,t}$ is the (oblique if $t \neq 0$) line

$$u \mapsto x + 2tu,$$

as is explained in more detail in Chapter 1 of [Muscalu and Schlag 2013b]. When we evaluate this line at the endpoints of the support of g_{θ_k} (taking into account that $|t| \leq R$), we see that the phase-space portrait of

$$\varphi_{\theta_k} \cdot e_{x,t}$$

is a parallelogram that essentially coincides with the rectangle

$$I \times J = [kR^{-\frac{1}{2}}, (k+1)R^{-\frac{1}{2}}] \times [x + 2tkR^{-\frac{1}{2}}, x + 2tkR^{-\frac{1}{2}} + R^{\frac{1}{2}}]. \tag{10}$$

Observe that $I \times J$ has area 1. On the other hand, the phase-space portrait of φ_{θ_k} is a Heisenberg box of sizes $R^{-1/2}$ and $R^{1/2}$, and the linear modulation

$$e^{2\pi i \xi(x + 2tkR^{-1/2})} \tag{11}$$

shifts it in frequency to J. The conclusion is that the phase-space portrait of

$$\varphi_{\theta_k} \cdot e^{2\pi i \xi(x + 2tkR^{-1/2})}$$

is the Heisenberg box (10); hence the effect of the quadratic modulation $e_{x,t}$ in this setting is essentially the same as the linear one in (11).

Using bumps such as φ_θ to decompose the domain of g and expanding each g_θ into Fourier series allows us to write

$$g(x) = \sum_{\theta \in R^{-1/2}\mathbb{Z}^d \cap [0,1]^d} \overbrace{g(x)\varphi_\theta(x)}^{g_\theta(x)} \tilde{\varphi}_\theta(x) = \sum_{\theta \in R^{-1/2}\mathbb{Z}^d \cap [0,1]^d} \sum_{\nu \in R^{1/2}\mathbb{Z}^d} \overbrace{c_{\nu,\theta} e^{2\pi i x \cdot \nu} \tilde{\varphi}_\theta(x)}^{g_{\theta,\nu}(x)},$$

where $\tilde{\varphi}_\theta$ is $\equiv 1$ on the support of φ_θ and decays very fast away from it. Applying $\mathcal{E}_d$ and using the previous intuition gives rise to the *wave packet decomposition*

$$\mathcal{E}_d g = \sum_{(\theta,\nu) \in R^{-1/2}\mathbb{Z}^d \cap [0,1]^d \times R^{1/2}\mathbb{Z}^d} \mathcal{E}_d(g_{\theta,\nu}),$$

where $\mathcal{E}_d(g_{\theta,\nu})$ is essentially supported on a tube in $\mathbb{R}^{d+1}$ of size $R^{1/2} \times \cdots \times R^{1/2} \times R$ whose direction is determined by θ and that is translated by a parameter depending on ν. With this linearized model at hand, one can study the interference between these tubes pointing in different directions (both in the linear and multilinear settings) and take advantage of orthogonality both in space and in frequency. This leads to local estimates of type

$$\|\mathcal{E}_d g\|_{L^q(B(0,R))} \lesssim_\varepsilon R^\varepsilon \|f\|_p \quad \text{for all } \varepsilon > 0$$

and multilinear analogues of it that are later used to obtain global estimates via ε-removal arguments (as in [Tao 1999]). The reader is referred to [Guth 2016] for the details of the decomposition above. This approach has given the current best L^p bounds for $\mathcal{E}_d$.

In our case, we do not linearize the contribution of the quadratic phase. Instead, we consider a discrete model that keeps the quadratic nature of $\mathcal{E}_d$ intact.

2A. ***The linear model (k = 1).*** We consider $d = 1$ for simplicity, but the discretization process is analogous for all $d > 1$. Recall that the extension operator for the parabola defined for functions supported on $[0, 1]$ is given by

$$\mathcal{E}_1 g(x,t) = \int_0^1 g(\xi) e^{-2\pi i x\xi} e^{-2\pi i t\xi^2}\, \mathrm{d}\xi. \tag{12}$$

We can insert a bump φ in the integrand that is equal to 1 on $[0, 1]$ and supported in a small neighborhood of this interval. Tiling $\mathbb{R}^2$ with unit squares with vertices in $\mathbb{Z}^2$ and rewriting $\mathcal{E}_1$,

$$\mathcal{E}_1 g(x,t) = \sum_{n,m\in\mathbb{Z}} \left[\int g(u)\varphi(u) e^{-2\pi i x u} e^{-2\pi i t u^2}\, \mathrm{d}u \right] \chi_n(x)\chi_m(t),$$

where $\chi_n := \chi_{[n,n+1)}$. For a fixed (x, t), one can write

$$\begin{aligned} e^{-2\pi i x\xi} e^{-2\pi i t\xi^2} \varphi(\xi) &= e^{-2\pi i n\xi} e^{-2\pi i m\xi^2} \cdot e^{-2\pi i (x-n)\xi} e^{-2\pi i (t-m)\xi^2} \varphi(\xi) \\ &= e^{-2\pi i n\xi} e^{-2\pi i m\xi^2} \cdot \sum_{u\in\mathbb{Z}} \langle e^{-2\pi i (x-n)(\cdot)} e^{-2\pi i (t-m)(\cdot)^2}, \varphi^u_{[0,1]} \rangle \cdot \varphi^u_{[0,1]}(\xi) \\ &= e^{-2\pi i n\xi} e^{-2\pi i m\xi^2} \cdot \sum_{u\in\mathbb{Z}} C_u^{n,m,x,t} \cdot \varphi^u_{[0,1]}(\xi), \end{aligned}$$

where we expanded $e^{-2\pi i (x-n)\xi} e^{-2\pi i (t-m)\xi^2}$ as a Fourier series at scale 1,

$$\begin{aligned} C_u^{n,m,x,t} &:= \langle e^{-2\pi i (x-n)(\cdot)} e^{-2\pi i (t-m)(\cdot)^2}, \varphi^u_{[0,1]} \rangle, \\ \varphi^u_{[0,1]}(\xi) &:= \varphi_{[0,1]}(\xi) \cdot e^{-2\pi i u\cdot\xi} \end{aligned}$$

and $\varphi_{[0,1]}$ is a bump adapted to $[0, 1]$ (and compactly supported) just like[4] φ. Plugging this in (12),

$$\begin{aligned} \mathcal{E}_1 g(x,t) &= \sum_{n,m\in\mathbb{Z}} \left[\int g(\xi)\varphi(\xi) e^{-2\pi i x\xi} e^{-2\pi i t\xi^2}\, \mathrm{d}\xi \right] \chi_n(x)\chi_m(t) \\ &= \sum_{n,m\in\mathbb{Z}} \left[\int g(\xi) \left(e^{-2\pi i n\xi} e^{-2\pi i m\xi^2} \cdot \sum_{u\in\mathbb{Z}} C_u^{n,m,x,t} \cdot \varphi^u(\xi) \right) \mathrm{d}\xi \right] \chi_n(x)\chi_m(t) \\ &= \sum_{u\in\mathbb{Z}} \sum_{n,m\in\mathbb{Z}} C_u^{n,m,x,t} \cdot \left[\int g(\xi) e^{-2\pi i n\xi} e^{-2\pi i m\xi^2} \cdot \varphi^u(\xi)\, \mathrm{d}\xi \right] \chi_n(x)\chi_m(t). \end{aligned}$$

[4]We will not distinguish between $\varphi_{[0,1]}$ and φ from now on.

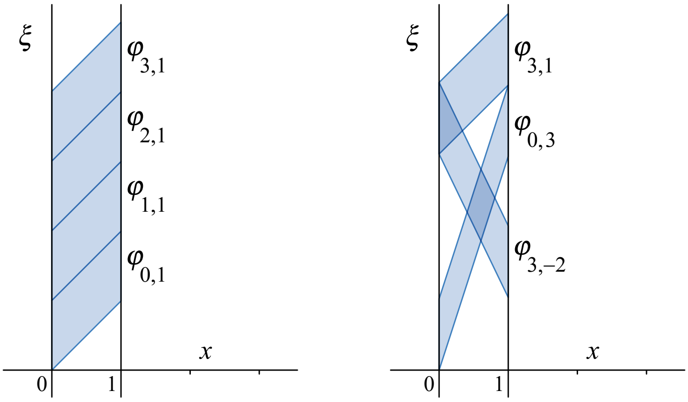

Figure 2. The phase-space portrait of $\varphi_{n,m}$.

For the expression defining $\mathcal{E}_1$ to be nonzero, (n,m) must satisfy $|x-n| \le 1$ and $|t-m| \le 1$; hence the Fourier coefficients $C_u^{n,m,x,t}$ decay like $O(|u|^{-100})$. In addition, the extra factor φ^u in the integral simply shifts the integrand in frequency, and this does not affect in any way the arguments that follow. In order to obtain the final form of our linear model, let us introduce the following notation: if φ is a compactly supported bump (say, in a very small open neighborhood of $[0,1]^d$) with $\varphi \equiv 1$ on $[0,1]^d$, we set

$$\varphi_{\vec{n},m}(x) := \varphi(x) e^{2\pi i x\cdot\vec{n}} e^{2\pi i |x|^2 m}. \tag{13}$$

Due to the fast decay of $C_u^{n,x}$ and $C_v^{m,t}$, it is then enough to bound the $u = v = 0$ piece of the sum above, which leads to the discretized model:[5]

$$E_1(g) = \sum_{(n,m)\in\mathbb{Z}^2} \langle g, \varphi_{n,m}\rangle (\chi_n \otimes \chi_m).$$

With the appropriate adaptations, one proceeds in the exact same way in dimension d to reduce matters to the study of the following model operator:

Definition 2.1. Let E_d be defined on $C([0,1]^d)$ given by

$$E_d(g) = \sum_{\vec{n}\in\mathbb{Z}^d,\, m\in\mathbb{Z}} \langle g, \varphi_{\vec{n},m}\rangle (\chi_{\vec{n}} \otimes \chi_m),$$

where $\chi_{\vec{n}}$ and χ_m are the characteristic functions of the boxes $[n_1, n_1+1) \times \cdots \times [n_d, n_d+1)$ and $[m, m+1)$, respectively.[6]

The wave packets (13) have a natural phase-space portrait that consist of parallelograms in the phase plane. See Figure 2.

[5] There is a slight abuse of notation here: observe that $\tilde{\chi}_n(x)\tilde{\chi}_m(t) := C_0^{n,m,x,t} \cdot \chi_n(x)\chi_m(t)$ is a smooth function supported in $[n, n+1) \times [m, m+1)$, which is all that is needed in the proof. We will continue to call it $\chi_n(x)\chi_m(t)$ to lighten the notation.

[6] Morally speaking, the discrete model and the original operator are "comparable", but we were not able to prove that rigorously. For that reason we included the proof of known extension estimates for E_d.

By keeping the quadratic nature of E_d intact we take advantage of orthogonality in different ways. For example, for a fixed m the wave packets $\varphi_{n,m}$ are almost orthogonal, as suggested by the fact that the corresponding parallelograms are (almost) disjoint.

2B. ***The multilinear model*** **($2 \le k \le d+1$).** We recall the definition of the k-linear extension operator:

Definition 2.2. For $\mathcal{Q}=\{Q_1,\dots,Q_k\}$ a transversal set of cubes, the k-linear extension operator is given by

$$\mathcal{ME}_{k,d}(g_1,\dots,g_k) := \prod_{j=1}^{k} \mathcal{E}_{Q_j} g_j, \tag{14}$$

where

$$\mathcal{E}_{Q_j} g_j(x,t) = \int_{Q_j} g_j(\xi) e^{-2\pi i x\cdot\xi} e^{-2\pi i t|\xi|^2}\, d\xi, \quad (x,t)\in\mathbb{R}^d\times\mathbb{R}.$$

By an analogous argument to the one we showed in Section 2A, it is enough to prove the corresponding bounds for the following model operator:

Definition 2.3. Let $\mathrm{ME}_{k,d}$ be defined on $C(Q_1)\times\cdots\times C(Q_k)$ by

$$\mathrm{ME}_{k,d}(g_1,\dots,g_k) := \sum_{(\vec{n},m)\in\mathbb{Z}^{d+1}} \prod_{j=1}^{k} \langle g_j, \varphi^j_{\vec{n},m}\rangle (\chi_{\vec{n}}\otimes\chi_m).$$

where

$$\varphi^j_{\vec{n},m} = \bigotimes_{l=1}^{d} \varphi^{l,j}_{n_l,m}, \quad \varphi^{l,j}_{n_l,m}(x_l) = \varphi^{l,j}(x_l) e^{2\pi i n_l x_l} e^{2\pi i m x_l^2}$$

and $\varphi^{l,j}(x)$ is $\equiv 1$ on the l-coordinate projection of the domain of g_j defined above and decays fast away from it.

Remark 2.4. It is clear that the discretization process does not depend on whether the collection $\mathcal{Q}$ is made of transversal cubes or not. In particular, it will be of interest in Section 12B to study the operator given by the right-hand side of (14), but *without* the assumption that the cubes Q_j are transversal. The model for such operator is also given by $\mathrm{ME}_{k,d}$, but without that hypothesis.

3. Transversality versus weak transversality

We recall the following definition from [Bennett 2014]:

Definition 3.1. Let $2\le k\le d+1$ and $c>0$. A k-tuple $S_1,\dots,S_k$ of smooth codimension-1 submanifolds of $\mathbb{R}^{d+1}$ is *c-transversal* if

$$|v_1\wedge\cdots\wedge v_k| \ge c$$

for all choices $v_1,\dots,v_k$ of unit normal vectors to $S_1,\dots,S_k$, respectively. We say that $S_1,\dots,S_k$ are *transversal* if they are c-transversal for some $c>0$.

In other words, if the k-dimensional volume of the parallelepiped generated by $v_1,\dots,v_k$ is bounded below by some absolute constant for any choice of normal vectors v_j, the submanifolds are transversal.

From now on, we will say that a collection of k cubes in $\mathbb{R}^d$ is *transversal* if the associated caps defined by them on the paraboloid are transversal in the sense of Definition 3.1.

One can assume without loss of generality that the U_j in the statements of Conjecture 1.2 are cubes that parametrize transversal caps on $\mathbb{P}^d$ via the map $x \mapsto |x|^2$. Even though these conjectures are known to fail in general if one does not assume transversality between the caps (see Section AB), the theorem that we will prove holds under a weaker condition, since one of the functions is a tensor.

Definition 3.2. Let $\mathcal{Q} = \{Q_1, \dots, Q_k\}$ be a collection of k (open or closed) cubes[7] in $\mathbb{R}^d$. The collection $\mathcal{Q}$ is said to be *weakly transversal with pivot* Q_j if there is a set of $k-1$ distinct directions $\mathscr{E}_j = \{e_{i_1}, \dots, e_{i_{k-1}}\}$ (depending on j) of the canonical basis such that

$$\begin{cases} \overline{\pi_{i_1}(Q_j)} \cap \overline{\pi_{i_1}(Q_1)} = \varnothing, \\ \quad\vdots \\ \overline{\pi_{i_{j-1}}(Q_j)} \cap \overline{\pi_{i_{j-1}}(Q_{j-1})} = \varnothing, \\ \overline{\pi_{i_j}(Q_j)} \cap \overline{\pi_{i_j}(Q_{j+1})} = \varnothing, \\ \quad\vdots \\ \overline{\pi_{i_{k-1}}(Q_j)} \cap \overline{\pi_{i_{k-1}}(Q_k)} = \varnothing, \end{cases} \tag{15}$$

where π_l is the projection onto e_l. We say that $\mathcal{Q}$ is *weakly transversal* if it is weakly transversal with pivot Q_j for all $1 \le j \le k$.[8]

Remark 3.3. For each $1 \le j \le k$, from now on we will refer to a set[9] $\mathscr{E}_j$ above as *a set of directions associated to* Q_j. Notice that there could be many of such sets for a single j. Also, if $j_1 \neq j_2$, it could be the case that no set of directions associated to Q_{j_1} is associated to Q_{j_2}.

Let us give a few examples to distinguish between Definitions 3.1 and 3.2. Consider the case $d = 2$, $k = 3$, $Q_1 = [0,1]^2$, $Q_2 = [2,3]^2$, and $Q_3 = [4,5]^2$. The line $y = x$ intersects Q_1, Q_2 and Q_3; then it follows from Definition 3.1 that they are not transversal. However, observe that

$$\begin{cases} \pi_1(Q_1) \cap \pi_1(Q_2) = \varnothing, \\ \pi_2(Q_1) \cap \pi_2(Q_3) = \varnothing, \end{cases}$$

so $\{e_1, e_2\}$ is a set associated to Q_1 (and similarly one can verify that it is also associated to Q_2 and Q_3). This shows that the collection defined by Q_1, Q_2 and Q_3 is weakly transversal.

Consider now the cubes $K_1 = [0,1]^2$, $K_2 = [4,5] \times [0,1]$ and $K_3 = [2,3]^2$. Not only are they transversal in the sense of Definition 3.1, but also weakly transversal.

This is not by chance: a given transversal collection of k cubes can be "decomposed" into finitely many collections of k cubes that are *also* weakly transversal.

[7]The word *cube* will be used throughout the paper to refer to any rectangular box in $\mathbb{R}^d$, regardless of the sizes of its edges, and they always refer to the supports of the input functions of our linear and multilinear operators. In this paper, it will not be relevant whether the sides of a box have the same length or not; therefore this slight abuse of terminology is harmless.

[8]The estimates that we will prove depend on the separation of the projections in Definition 3.2, just as they depend on the behavior of c from Definition 3.1 in the general case for transversal caps.

[9]The typeface $\mathscr{E}_j$ is being used to distinguish this concept from the previously defined operators $\mathcal{E}_d$ and E_d.

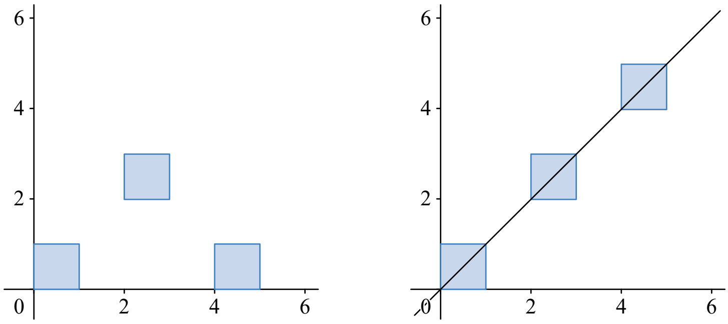

Figure 3. Transversality versus weak transversality.

Claim 3.4. *Given a collection* $\mathcal{Q} = \{Q_1, \dots, Q_k\}$ *of transversal cubes, each* $Q_l \in \mathcal{Q}$ *can be partitioned into* $O(1)$ *many subcubes*

$$Q_l = \bigcup_i Q_{l,i}$$

so that all collections $\widetilde{\mathcal{Q}}$ *made of picking one subcube* $Q_{l,i}$ *per* Q_l

$$\widetilde{\mathcal{Q}} = \{\widetilde{Q}_1, \dots, \widetilde{Q}_k\}, \quad \widetilde{Q}_l \in \{Q_{l,i}\}_i,$$

are weakly transversal.

Proof. See Claim B.4 in Appendix B. □

As a consequence of Claim 3.4, to prove the case $2 \le k \le d+1$ of Theorem 1.5 it suffices to show it for weakly transversal collections. To simplify the exposition, we will present our results for the cubes

$$Q_1 = [0,1]^d,$$
$$Q_j = [2,3]^{j-2} \times [4,5] \times [0,1]^{d-j+1}, \quad 2 \le j \le k.$$

The associated directions to Q_1 are $\{e_1, \dots, e_{k-1}\}$, and we will use it as the pivot. Any other weakly transversal collection of cubes can be dealt with in the same way.

4. Our approach and its building blocks

Notice that the operators $\mathcal{E}_d$ and $\mathcal{ME}_{k,d}$ are pointwise bounded by E_d and $\mathrm{ME}_{k,d}$, respectively; therefore we cannot directly conclude any result about the models from the fact that they hold for the original operators. Some of these results will be reproven for the models in this paper, and they will act as *building blocks* in the proof of Theorem 1.5, which is presented in Sections 8 and 9. More precisely, Theorem 1.5 relies on the following:

(1) *Mixed norm Strichartz/Tomas–Stein* ($k = 1$, $p = 2$). In Section 5 we show the following:

Proposition 4.1. *For all* $p > \frac{2(d+2)}{d}$,

$$\|E_d g\|_p \lesssim_p \|g\|_2.$$

As a consequence, we have:

Corollary 4.2. *For all* $\varepsilon > 0$,

$$\|E_d(g)\|_{L^{2(d-l+2)/(d-l)+\varepsilon}_{x_{l+1},\dots,x_d,t} L^2_{x_1,\dots,x_l}} \lesssim_\varepsilon \|g\|_2. \tag{16}$$

Proof. Apply Minkowski's inequality and Proposition 4.1 in dimension $d-l$. Notice that, after taking L^2 norm in the first l variables, we can use Bessel to bound the left-hand side of (16) by

$$\left[\sum_{n_{l+1},\dots,n_d,m}\left(\sum_{n_1,\dots,n_l}|\langle\langle g,\varphi_{n_{l+1},\dots,n_d,m}\rangle,\varphi_{n_1,\dots,n_l,m}\rangle|^2\right)^{\frac{p_0}{2}}\right]^{\frac{1}{p_0}}$$
$$\lesssim\left[\sum_{n_{l+1},\dots,n_d,m}\|\langle g,\varphi_{n_{l+1},\dots,n_d,m}\rangle\|_2^{p_0}\right]^{\frac{1}{p_0}}, \quad \text{where } p_0=\frac{2(d-l+2)}{d-l}+\varepsilon.$$

This is how we will use Corollary 4.2 in (56). □

We will use Corollary 4.2 in Conjecture 1.2 to prove Theorem 1.5 for $2 \le k \le d+1$. It will not be needed when $k = d+1$.

(2) *Extension conjecture for the parabola* ($k = 1$, $d = 1$, $p = 4$). In Section 6 we prove the following:

Proposition 4.3. *For all* $\varepsilon > 0$,

$$\|E_1 g\|_{4+\varepsilon} \lesssim_\varepsilon \|g\|_4. \tag{17}$$

One can show by interpolation that Proposition 4.3 implies Conjecture 1.1 for $d = 1$. We will use it in Section 8 to settle the case $k = 1$ of Theorem 1.5.

(3) *Bilinear extension conjecture for the parabola* ($k = 2$, $d = 1$). In Section 7 we show that the model $\mathrm{ME}_{2,1}$ in Definition 2.3 maps $L^2([0,1]) \times L^2([4,5])$ to $L^2(\mathbb{R}^2)$.

Proposition 4.4. *The following estimate holds*:

$$\|\mathrm{ME}_{2,1}(f,g)\|_2 \lesssim \|f\|_2 \cdot \|g\|_2. \tag{18}$$

Transversality will be captured in Section 9 through (18).

By combining scalar and mixed norm stopping times[10] performed simultaneously, we are able to put together the key estimates (16), (17) and (18). In the $2 \le k \le d+1$ case, the tensor structure is used in an implicit way to allow us to better relate these scalar and mixed norm stopping times.

Remark 4.5. The tensor structure $g = g_1 \otimes \cdots \otimes g_d$ in the $k = 1$ case allows us to write

$$\langle g, \varphi_{\vec{n},m}\rangle = \prod_{j=1}^{d}\langle g_j, \varphi_{n_j,m}\rangle. \tag{19}$$

[10]This is not meant in a literal probabilistic sense; strictly speaking, the argument combines the level sets of various scalar and mixed norm quantities that appear naturally in our analysis.

We then obtain the following multilinear form by dualization:

$$\Lambda_d(g_1,\dots,g_d,h) = \langle E_d(g),h\rangle = \sum_{\vec{\boldsymbol{n}}\in\mathbb{Z}^d,\, m\in\mathbb{Z}} \prod_{j=1}^{d} \langle g_j,\varphi_{n_j,m}\rangle\cdot\langle h,\chi_{\vec{\boldsymbol{n}}}\otimes\chi_m\rangle, \tag{20}$$

The goal in the $k=1$ case is to show that

$$|\Lambda_d(g_1,\dots,g_d,h)| \lesssim \|h\|_q\cdot\prod_{j=1}^{d}\|g_j\|_{p_j}$$

for appropriate exponents p_j and q. Interpolation theory shows that it suffices to obtain

$$|\Lambda_d(g_1,\dots,g_d,h)| \lesssim_\varepsilon |F|^{\gamma_{d+1}}\cdot\prod_{j=1}^{d}|E_j|^{\gamma_j} \tag{21}$$

for all $\varepsilon>0$, $|g_j|\le\chi_{E_j}$, $|h|\le\chi_F$,[11] $E_j\subset[0,1]$ and $F\subset\mathbb{R}^3$ measurable sets such that γ_j $(1\le j\le d)$ and γ_{d+1} are in a small neighborhood of $\frac{d}{2(d+1)}$ and $\frac{d+2}{2(d+1)}+\varepsilon$, respectively.[12] We refer the reader to [Thiele 2006, Chapter 3] for a detailed account of multilinear interpolation theory. To keep the notation simple, all restricted weak-type estimates we will prove in this paper will be for the centers of such neighborhoods. For example, we will show that

$$|\Lambda_d(g_1,\dots,g_d,h)| \lesssim_\varepsilon |F|^{\frac{d+2}{2(d+1)}+\varepsilon}\cdot\prod_{j=1}^{d}|E_j|^{\frac{d}{2(d+1)}} \tag{22}$$

for all $\varepsilon>0$, but it will be clear from the arguments that as long as we give this $\varepsilon>0$ away, a slightly different choice of interpolation parameters yields (21). The restricted weak-type estimates that we will prove in the $2\le k\le d+1$ case will also be for the centers of the corresponding neighborhoods.

5. Proof of Proposition 4.1: Strichartz/Tomas–Stein for E_d ($k=1$, $p=2$)

Our proof is inspired by the classical TT^* argument. It is possible to prove the endpoint estimate directly for the model E_d by repeating the steps of this argument (see for example [Muscalu and Schlag 2013a, Section 11.2.2]), but we chose the following approach because of its similarity with the one we will use to prove Theorem 1.5. By interpolation with the trivial bound for $q=\infty$, it is enough to prove the bound

$$\|E_d g\|_{\frac{2(d+2)}{d}+\varepsilon} \lesssim_\varepsilon \|g\|_2$$

for all $\varepsilon>0$.

We start by dualizing E_d to obtain a bilinear form Λ_d:

$$\Lambda_d(g,h) = \langle E_d(g),h\rangle = \sum_{\vec{\boldsymbol{n}}\in\mathbb{Z}^d,\, m\in\mathbb{Z}} \langle g,\varphi_{\vec{\boldsymbol{n}},m}\rangle\cdot\langle h,\chi_{\vec{\boldsymbol{n}}}\otimes\chi_m\rangle.$$

[11] There is an overlap of classical notation here that we hope will not compromise the comprehension of the paper: we chose the typeface E_d to represent the discrete model of the official extension operator $\mathcal{E}$. On the other hand, the classical theory of restricted weak-type multilinear interpolation usually labels the measurable sets involved in the problems by E_j or F_j. The context will make it clear which object we are referring to.

[12] Rigorously, this only verifies the case $k=1$ near the endpoint $\left(\frac{2(d+1)}{d},\frac{2(d+1)}{d}\right)$, but this is known to imply the desired estimates in the full range. For details, see [Mattila 2015, Theorem 19.8].

Let $E_1 \subset \mathbb{R}^d$ and $E_2 \subset \mathbb{R}^{d+1}$ be measurable sets of finite measure with $|g| \le \chi_{E_1}$ and $|h| \le \chi_{E_2}$. Split $\mathbb{Z}^{d+1}$ in two ways:

$$\mathbb{Z}^{d+1} = \bigcup_{l_1 \in \mathbb{Z}} \mathbb{A}^{l_1}, \quad \text{where } (\vec{\boldsymbol{n}}, m) \in \mathbb{A}^{l_1} \iff |\langle g, \varphi_{\vec{\boldsymbol{n}},m} \rangle| \approx 2^{-l_1},$$

$$\mathbb{Z}^{d+1} = \bigcup_{l_2 \in \mathbb{Z}} \mathbb{B}^{l_2}, \quad \text{where } (\vec{\boldsymbol{n}}, m) \in \mathbb{B}^{l_2} \iff |\langle h, \chi_{\vec{\boldsymbol{n}}} \otimes \chi_m \rangle| \approx 2^{-l_2}.$$

Define $\mathbb{X}^{l_1,l_2} := \mathbb{A}^{l_1} \cap \mathbb{B}^{l_2}$ and observe that

$$|\Lambda_d(g,h)| \lesssim \sum_{l_1,l_2 \in \mathbb{Z}} 2^{-l_1} 2^{-l_2} \, \#\mathbb{X}^{l_1,l_2}.$$

Notice that, for all $(\vec{\boldsymbol{n}}, m) \in \mathbb{X}^{l_1,l_2}$,

$$2^{-l_1} \lesssim \int_{\mathbb{R}^d} |g(x)||\varphi_{\vec{\boldsymbol{n}},m}(x)| \, \mathrm{d}x \le \min\{|E_1|, 1\},$$

$$2^{-l_2} \lesssim \int_{\mathbb{R}^d} |h(x)||\chi_{\vec{\boldsymbol{n}}} \otimes \chi_m(x)| \, \mathrm{d}x \le \min\{|E_2|, 1\}.$$

In particular, $l_1, l_2 \ge 0$ in the sum above. Now we bound $\#\mathbb{X}^{l_1,l_2}$ in two different ways and interpolate between them:

(a) L^1*-type bound*: Exploit h:

$$\#\mathbb{X}^{l_1,l_2} \le \#\mathbb{B}^{l_2} \lesssim 2^{l_2} \sum_{(\vec{\boldsymbol{n}},m) \in \mathbb{B}^{l_2}} |\langle h, \chi_{\vec{\boldsymbol{n}}} \otimes \chi_m \rangle| \lesssim 2^{l_2} \sum_{(\vec{\boldsymbol{n}},m) \in \mathbb{Z}^{d+1}} \int_{Q_{\vec{\boldsymbol{n}},m}} |h| = 2^{l_2} \|h\|_1 \le 2^{l_2} |E_2|, \tag{23}$$

where $Q_{\vec{\boldsymbol{n}},m} := \Pi_{i=1}^d [n_i, n_i+1] \times [m, m+1]$, $\vec{\boldsymbol{n}} = (n_1, \dots, n_d)$.

(b) L^2*-type bound*: Exploit g:

$$\begin{aligned} \#\mathbb{X}^{l_1,l_2} &\lesssim 2^{2l_1} \sum_{(\vec{\boldsymbol{n}},m) \in \mathbb{X}^{l_1,l_2}} |\langle g, \varphi_{\vec{\boldsymbol{n}},m} \rangle|^2 \\ &= 2^{2l_1} \left| \left\langle \sum_{(\vec{\boldsymbol{n}},m) \in \mathbb{X}^{l_1,l_2}} \langle g, \varphi_{\vec{\boldsymbol{n}},m} \rangle \varphi_{\vec{\boldsymbol{n}},m}, g \right\rangle \right| \\ &\le 2^{2l_1} |E_1|^{\frac{1}{2}} \underbrace{\left\| \sum_{(\vec{\boldsymbol{n}},m) \in \mathbb{X}^{l_1,l_2}} \langle g, \varphi_{\vec{\boldsymbol{n}},m} \rangle \varphi_{\vec{\boldsymbol{n}},m} \right\|_2}_{(*)}. \end{aligned} \tag{24}$$

For each set $\mathbb{X}^{l_1,l_2}$ define $\pi_m := \{\vec{\boldsymbol{n}} \in \mathbb{Z}^d; (\vec{\boldsymbol{n}}, m) \in \mathbb{X}^{l_1,l_2}\}$. Observe that

$$(*)^2 = \sum_{m:\, \pi_m \ne \varnothing} \sum_{\tilde{m}:\, \pi_{\tilde{m}} \ne \varnothing} \underbrace{\sum_{\vec{\boldsymbol{n}} \in \pi_m} \sum_{\vec{\boldsymbol{k}} \in \pi_{\tilde{m}}} \langle g, \varphi_{\vec{\boldsymbol{n}},m} \rangle \overline{\langle g, \varphi_{\vec{\boldsymbol{k}},\tilde{m}} \rangle} \langle \varphi_{\vec{\boldsymbol{n}},m}, \varphi_{\vec{\boldsymbol{k}},\tilde{m}} \rangle}_{U((\langle g, \varphi_{\vec{\boldsymbol{n}},m} \rangle)_{\vec{\boldsymbol{n}} \in \pi_m}, (\langle g, \varphi_{\vec{\boldsymbol{k}},\tilde{m}} \rangle)_{\vec{\boldsymbol{k}} \in \pi_{\tilde{m}}})}$$

We will estimate U in two ways. Let $a_{\vec{n},m} := \langle g, \varphi_{\vec{n},m}\rangle$. First, by the triangle inequality and the stationary phase Theorem B.3

$$|U((a_{\vec{n},m})_{\vec{n}\in\pi_m}, (a_{\vec{k},\tilde{m}})_{\vec{k}\in\pi_{\tilde{m}}})| \le \sum_{\vec{n}\in\pi_m}\sum_{\vec{k}\in\pi_{\tilde{m}}} |\langle g,\varphi_{\vec{n},m}\rangle|\cdot|\langle g,\varphi_{\vec{k},\tilde{m}}\rangle| \frac{1}{\langle m-\tilde{m}\rangle^{\frac{d}{2}}}$$
$$= \frac{\|\langle g,\varphi_{\cdot,m}\rangle\|_{\ell^1(\pi_m)}\cdot\|\langle g,\varphi_{\cdot,\tilde{m}}\rangle\|_{\ell^1(\pi_{\tilde{m}})}}{\langle m-\tilde{m}\rangle^{\frac{d}{2}}}.$$

Another possibility is

$$|U((a_{\vec{n},m})_{\vec{n}\in\pi_m}, (a_{\vec{k},\tilde{m}})_{\vec{k}\in\pi_{\tilde{m}}})|$$
$$\le \left|\int_{\mathbb{R}^d}\Big(\sum_{\vec{n}\in\pi_m}\langle g,\varphi_{\vec{n},m}\rangle e^{2\pi i\vec{n}\cdot x}\Big)\Big(\sum_{\vec{k}\in\pi_{\tilde{m}}}\langle g,\varphi_{\vec{k},m}\rangle e^{2\pi i\vec{k}\cdot x}\Big)\varphi(x)\varphi(x)e^{2\pi i(m-\tilde{m})|x|^2}\,\mathrm{d}x\right|$$
$$\lesssim \|\langle g,\varphi_{\cdot,m}\rangle\|_{\ell^2(\pi_m)}\cdot\|\langle g,\varphi_{\cdot,\tilde{m}}\rangle\|_{\ell^2(\pi_{\tilde{m}})}$$

by Cauchy–Schwarz and orthogonality on the sets π_m and $\pi_{\tilde{m}}$ (recall that m and $\tilde{m}$ are fixed). Interpolating between these bounds for $1\le p\le 2$,

$$|U((a_{\vec{n},m})_{\vec{n}\in\pi_m}, (a_{\vec{k},\tilde{m}})_{\vec{k}\in\pi_{\tilde{m}}})| \lesssim \frac{\|\langle g,\varphi_{\cdot,m}\rangle\|_{\ell^p(\pi_m)}\cdot\|\langle g,\varphi_{\cdot,\tilde{m}}\rangle\|_{\ell^p(\pi_{\tilde{m}})}}{\langle m-\tilde{m}\rangle^{\frac{d}{2}(\frac{1}{p}-\frac{1}{p'})}}.$$

Back to $(*)$:

$$(*)^2 \lesssim \sum_{m:\pi_m\neq\varnothing}\sum_{\tilde{m}:\pi_{\tilde{m}}\neq\varnothing}\frac{\|\langle g,\varphi_{\cdot,m}\rangle\|_{\ell^p(\pi_m)}\cdot\|\langle g,\varphi_{\cdot,\tilde{m}}\rangle\|_{\ell^p(\pi_{\tilde{m}})}}{\langle m-\tilde{m}\rangle^{\frac{d}{2}(\frac{1}{p}-\frac{1}{p'})}}$$
$$= \sum_{m:\pi_m\neq\varnothing}\|\langle g,\varphi_{\cdot,m}\rangle\|_{\ell^p(\pi_m)}\sum_{\tilde{m}:\pi_{\tilde{m}}\neq\varnothing}\frac{\|\langle g,\varphi_{\cdot,\tilde{m}}\rangle\|_{\ell^p(\pi_{\tilde{m}})}}{\langle m-\tilde{m}\rangle^{\frac{d}{2}(\frac{1}{p}-\frac{1}{p'})}}$$
$$\le \big\|\|\langle g,\varphi_{\cdot,m}\rangle\|_{\ell^p(\pi_m)}\big\|_{\ell^p(\mathbb{Z})}\left\|\sum_{\tilde{m}:\pi_{\tilde{m}}\neq\varnothing}\frac{\|\langle g,\varphi_{\cdot,\tilde{m}}\rangle\|_{\ell^p(\pi_{\tilde{m}})}}{\langle m-\tilde{m}\rangle^{\frac{d}{2}(\frac{1}{p}-\frac{1}{p'})}}\right\|_{\ell^{p'}(\mathbb{Z})}$$
$$\le \big\|\|\langle g,\varphi_{\cdot,m}\rangle\|_{\ell^p(\pi_m)}\big\|_{\ell^p(\mathbb{Z})}\cdot\big\|\|\langle g,\varphi_{\cdot,\tilde{m}}\rangle\|_{\ell^p(\pi_{\tilde{m}})}\big\|_{\ell^p(\mathbb{Z})}$$
$$= \big\|\|\langle g,\varphi_{\cdot,m}\rangle\|_{\ell^p(\pi_m)}\big\|^2_{\ell^p(\mathbb{Z})},$$

as long as

$$\frac{1}{p}-\frac{1}{p'} = 1-\frac{d}{2}\Big(\frac{1}{p}-\frac{1}{p'}\Big) \iff \frac{1}{p}-\frac{1}{p'}=\frac{2}{d+2} \iff \frac{2}{p'}=\frac{d}{d+2} \iff p'=\frac{2d+4}{d},$$

by discrete fractional integration. Plugging this back in (24),

$$\#\mathbb{X}^{l_1,l_2} \lesssim 2^{2l_1}|E_1|^{\frac{1}{2}}\big\|\|\langle g,\varphi_{\cdot,m}\rangle\|_{\ell^p(\pi_m)}\big\|_{\ell^p(\mathbb{Z})}$$
$$= 2^{2l_1}|E_1|^{\frac{1}{2}}\Big(\sum_{(\vec{n},m)\in\mathbb{X}^{l_1,l_2}}|\langle g,\varphi_{\vec{n},m}\rangle|^p\Big)^{\frac{1}{p}} \lesssim 2^{2l_1}|E_1|^{\frac{1}{2}}(2^{-pl_1}\#\mathbb{X}^{l_1,l_2})^{\frac{1}{p}},$$

which implies

$$\#\mathbb{X}^{l_1,l_2} \lesssim 2^{(2+\frac{4}{d})l_1}|E_1|^{1+\frac{2}{d}}. \tag{25}$$

Interpolating between (23) and (25):

$$
\begin{aligned}
|\Lambda_d(g,h)| &\lesssim \sum_{l_1,l_2\geq 0} 2^{-l_1}2^{-l_2}(2^{(2+\frac{4}{d})l_1}|E_1|^{1+\frac{2}{d}})^{\theta_1}(2^{l_2}|E_2|)^{\theta_2} \\
&= \Big(\sum_{l_1\geq 0} 2^{-l_1(1-(2+\frac{4}{d})\theta_1)}\Big)\Big(\sum_{l_2\geq 0} 2^{-l_2(1-\theta_2)}\Big)|E_1|^{(1+\frac{2}{d})\theta_1}|E_2|^{\theta_2} \\
&\lesssim 2^{-\tilde{l}_1(1-(2+\frac{4}{d})\theta_1)}2^{-\tilde{l}_2(1-\theta_2)}|E_1|^{(1+\frac{2}{d})\theta_1}|E_2|^{\theta_2} \\
&\lesssim \min\{|E_1|^{(1-(2+\frac{4}{d})\theta_1)},1\}\min\{|E_2|^{1-\theta_2},1\}|E_1|^{(1+\frac{2}{d})\theta_1}|E_2|^{\theta_2} \\
&\lesssim |E_1|^{\alpha_1(1-(2+\frac{4}{d})\theta_1)+(1+\frac{2}{d})\theta_1}|E_2|^{\alpha_2(1-\theta_2)+\theta_2}
\end{aligned}
\tag{26}
$$

for all $0\leq\alpha_1,\alpha_2\leq 1,\ \theta_1+\theta_2=1,$ with $0\leq\big(2+\frac{4}{d}\big)\theta_1<1,\ 0\leq\theta_2<1$, where $\tilde{l}_1$ is the smallest possible value of l_1 for which $\mathbb{A}^{l_1}\neq\varnothing$ and $\tilde{l}_2$ is defined analogously. Picking $\alpha_1=\frac{1}{2},\ \alpha_2=0,\ \theta_1=\frac{d}{2(d+2)}-\varepsilon$ and $\theta_2=\frac{d+4}{2(d+2)}+\varepsilon$ gives

$$|\Lambda_d(g,h)|\lesssim_\varepsilon |E_1|^{\frac{1}{2}}\cdot|E_2|^{\frac{d+4}{2(d+2)}+\varepsilon}$$

for all $\varepsilon>0$, which proves the proposition by restricted weak-type interpolation.

6. Proof of Proposition 4.3-Conjecture 1.1 for E_1 ($k=1, d=1, p=4$)

The following argument is inspired by Zygmund's original proof of this case. Define

$$\Phi_{n,m}(s,t):=|t-s|^{\frac{1}{2}}\varphi(s)\varphi(t)e^{2\pi i(s-t)n}e^{2\pi i(s^2-t^2)m}.$$

Claim 6.1.

$$\langle\Phi_{n,m},\Phi_{\tilde{n},\tilde{m}}\rangle=O_N\Big(\frac{1}{|(n-\tilde{n})(m-\tilde{m})|^N}\Big)$$

for any natural N if $n\neq\tilde{n}$ and $m\neq\tilde{m}$.

Proof. We have

$$
\begin{aligned}
\langle\Phi_{n,m},\Phi_{\tilde{n},\tilde{m}}\rangle &= \iint_{[0,1]^2}|t-s|\,|\varphi(s)|^2|\varphi(t)|^2e^{2\pi i(s-t)(n-\tilde{n})}e^{2\pi i(s^2-t^2)(m-\tilde{m})}\,\mathrm{d}s\,\mathrm{d}t \\
&= \iint_R \frac{|u|}{|u|}\psi(u,v)e^{2\pi iu(n-\tilde{n})}e^{2\pi iv(m-\tilde{m})}\,\mathrm{d}u\,\mathrm{d}v,
\end{aligned}
$$

where R is the region that we obtain after making the change of variables $s-t=u,\ s^2-t^2=v$, and

$$\psi(u,v)=\varphi\otimes\varphi\Big(\frac{v+u^2}{u},\frac{v-u^2}{u}\Big).$$

The claim follows by the nonstationary phase Theorem B.2. □

We now prove the following:

Lemma 6.2. *For G smooth supported on* $[0,1]\times[0,1]$,

$$\Big\|\sum_{n,m\in\mathbb{Z}}\langle G,\varphi_{n,m}\otimes\bar{\varphi}_{n,m}\rangle(\chi_n\otimes\chi_m)\Big\|_2\lesssim\Big(\iint_{[0,1]^2}\frac{|G(s,t)|^2}{|s-t|}\,\mathrm{d}s\,\mathrm{d}t\Big)^{\frac{1}{2}}.$$

Proof. Define

$$\widetilde{G}(s,t) = \frac{G(s,t)}{|s-t|^{\frac{1}{2}}}$$

on $[0,1]^2 \backslash \{(x,x); 0 \le x \le 1\}$. Observe that

$$\Big\| \sum_{n,m\in\mathbb{Z}} \langle G, \varphi_{n,m} \otimes \bar{\varphi}_{n,m} \rangle (\chi_n \otimes \chi_m) \Big\|_2^2 = \sum_{n,m\in\mathbb{Z}} |\langle G, \varphi_{n,m} \otimes \bar{\varphi}_{n,m} \rangle|^2 = \sum_{n,m\in\mathbb{Z}} |\langle \widetilde{G}, \Phi_{n,m} \rangle|^2 \lesssim \|\widetilde{G}\|_2^2,$$

by the almost orthogonality of the $\Phi_{n,m}$ proved in the previous claim. ☐

Remark 6.3. By the triangle inequality,

$$\Big\| \sum_{n,m\in\mathbb{Z}} \langle G, \varphi_{n,m} \otimes \bar{\varphi}_{n,m} \rangle (\chi_n \otimes \chi_m) \Big\|_\infty \lesssim \iint_{[0,1]^2} |G(s,t)| \, \mathrm{d}s \, \mathrm{d}t.$$

Hence by interpolation we obtain

$$\Big\| \sum_{n,m\in\mathbb{Z}} \langle G, \varphi_{n,m} \otimes \bar{\varphi}_{n,m} \rangle (\chi_n \otimes \chi_m) \Big\|_p \lesssim \left(\iint_{[0,1]^2} \frac{|G(s,t)|^{p'}}{|s-t|^{p'-1}} \, \mathrm{d}s \, \mathrm{d}t \right)^{\frac{1}{p'}} \tag{27}$$

for $2 \le p \le \infty$.

Let $E \subset \mathbb{R}^d$ be a measurable set of finite measure with $|g| \le \chi_E$. Using Remark 6.3 and Lemma 6.2 for $G = g \otimes \bar{g}$, we have

$$\begin{aligned}
\Big[\sum_{n,m\in\mathbb{Z}} |\langle g, \varphi_{n,m} \rangle|^{4+\varepsilon} \Big]^{\frac{2}{4+\varepsilon}} &= \Big[\int_{\mathbb{R}^2} \Big(\sum_{n,m\in\mathbb{Z}} |\langle g, \varphi_{n,m} \rangle|^{4+\varepsilon} (\chi_n \otimes \chi_m) \Big) \Big]^{\frac{2}{4+\varepsilon}} \\
&\le \Big[\int_{\mathbb{R}^2} \Big(\sum_{n,m\in\mathbb{Z}} |\langle g, \varphi_{n,m} \rangle|^2 (\chi_n \otimes \chi_m) \Big)^{\frac{4+\varepsilon}{2}} \Big]^{\frac{2}{4+\varepsilon}} \\
&= \Big\| \sum_{n,m\in\mathbb{Z}} |\langle g, \varphi_{n,m} \rangle|^2 (\chi_n \otimes \chi_m) \Big\|_{2+\frac{\varepsilon}{2}} \\
&\lesssim \left(\iint_{[0,1]^2} \frac{|g(s)|^{p'} |g(t)|^{p'}}{|s-t|^{p'-1}} \, \mathrm{d}s \, \mathrm{d}t \right)^{\frac{1}{p'}}, \quad \text{where } p' = \frac{4+\varepsilon}{2+\varepsilon}.
\end{aligned}$$

To bound this last integral, we proceed as follows:

$$\begin{aligned}
\int_0^1 \int_0^1 \frac{|\rho(s)| \cdot |\rho(t)|}{|s-t|^\gamma} \, \mathrm{d}s \, \mathrm{d}t &= \int_0^1 |\rho(t)| \int_0^1 \frac{|\rho(s)|}{|s-t|^\gamma} \, \mathrm{d}s \, \mathrm{d}t = \int_0^1 |\rho(t)| \cdot \Big(|\rho| * \frac{1}{|s|^\gamma} \Big)(t) \, \mathrm{d}t \\
&= \Big\| |\rho| \Big(|\rho| * \frac{1}{|s|^\gamma} \Big) \Big\|_{L^1(\mathrm{d}t)} \le \|\rho\|_{L^q(\mathrm{d}t)} \Big\| |\rho| * \frac{1}{|s|^\gamma} \Big\|_{L^{p'}(\mathrm{d}t)} \lesssim_\varepsilon \|\rho\|_p^2
\end{aligned}$$

if $\frac{1}{p'} = \frac{1}{p} - (1-\gamma)$, by Theorem B.1. In our case, $\rho = |g|^{p'}$, $\gamma = p' - 1$ and

$$pp' = \frac{(4+\varepsilon)^2}{2(2+\varepsilon)} > 4.$$

Then

$$\left(\int_0^1\int_0^1 \frac{|g(s)|^{p'}\cdot|g(t)|^{p'}}{|s-t|^{p'-1}}\,\mathrm{d}s\,\mathrm{d}t\right)^{\frac{1}{p'}} \lesssim \left(\int_0^1 |g(t)|^{pp'}\,\mathrm{d}t\right)^{\frac{2}{pp'}} = \left(\int_0^1 |g(t)|^{4+\left(\frac{(4+\varepsilon)^2}{2(2+\varepsilon)}-4\right)}\,\mathrm{d}t\right)^{\frac{4(2+\varepsilon)}{(4+\varepsilon)^2}}$$

$$\lesssim \left(\int_0^1 |g(t)|^4\,\mathrm{d}t\right)^{\frac{4(2+\varepsilon)}{(4+\varepsilon)^2}} = |E|^{\frac{4(2+\varepsilon)}{(4+\varepsilon)^2}}.$$

Observed that in the second line of the chain of inequalities above we used the fact that $|g|\le 1$. Finally,

$$\|E_1 g\|_{4+\varepsilon} = \left[\sum_{n,m\in\mathbb{Z}} |\langle g,\varphi_{n,m}\rangle|^{4+\varepsilon}\right]^{\frac{1}{4+\varepsilon}} \lesssim |E|^{\frac{2(2+\varepsilon)}{(4+\varepsilon)^2}} \le |E|^{\frac{1}{4}}.$$

This shows that E_1 maps $L^4([0,1])$ to $L^q(\mathbb{R}^2)$ for any $q>4$ by restricted weak-type interpolation.

7. Proof of Proposition 4.4-Conjecture 1.2 for $\mathrm{ME}_{2,1}$ ($k=2$, $d=1$)

The model to be treated is

$$\mathrm{ME}_{2,1}(f,g) := \sum_{(n,m)\in\mathbb{Z}^2} \langle f,\varphi^1_{n,m}\rangle\cdot\langle g,\varphi^2_{n,m}\rangle(\chi_n\otimes\chi_m).$$

Since $d=1$, we do not have to deal with the multivariable quantity

$$\varphi^j_{\vec{\boldsymbol{n}},m} = \bigotimes_{l=1}^{d}\varphi^{l,j}_{n_l,m}$$

from Definition 2.3, so we will simplify the notation by taking $\varphi^1_{n,m} := \varphi^{1,1}_{n,m}$ and $\varphi^2_{n,m} := \varphi^{1,2}_{n,m}$. We also replaced (g_1,g_2) by (f,g) here to reduce the number of indices carried through the section.

We provide a simple argument involving Bessel's inequality. After a change of variables to move the domain of φ^2 to be the same as the one of φ^1, we have

$$|\,\mathrm{ME}_{2,1}(f,g)| \lesssim \sum_{(n,m)\in\mathbb{Z}^2} |\langle f,\varphi^1_{n,m}\rangle|\,|\overline{\langle (g)_{-4},\varphi^1_{n+8m,m}\rangle}|(\chi_n\otimes\chi_m)$$

$$= \sum_{(n,m)\in\mathbb{Z}^2} |\langle f\otimes(g)_{-4},\varphi^1_{n,m}\otimes\bar{\varphi}^1_{n+8m,m}\rangle|(\chi_n\otimes\chi_m),$$

where[13] $(g)_{-4}(y)=g(y+4)$. Observe that

$$\langle f\otimes(g)_{-4},\varphi^1_{n,m}\otimes\bar{\varphi}^1_{n+8m,m}\rangle$$

$$= \iint f(x)g(y+4)\varphi^1(x)\varphi^1(y)e^{-2\pi i n x}e^{-2\pi i m x^2}e^{2\pi i(n+8m)y}e^{2\pi i m y^2}\,\mathrm{d}x\,\mathrm{d}y$$

$$= \iint f(x)g(y+4)e^{2\pi i n(y-x)}e^{2\pi i m(y-x)(y+x)}e^{16\pi i m y}\,\mathrm{d}x\,\mathrm{d}y$$

$$\approx \int \underbrace{\left[\int f\left(\frac{v-u}{2}\right)g\left(\frac{v+u}{2}+4\right)e^{2\pi i m u v}e^{8\pi i m(u+v)}\,\mathrm{d}v\right]}_{H_m(u)} e^{2\pi i n u}\,\mathrm{d}u = \widehat{H}_m(-n)$$

[13] This was done to bring the support of $\varphi^2_{n,m}$ to the one of $\varphi^1_{n+8m,m}$. The price to pay is the $+4m$ shift in the linear modulation index of the bump.

Hence

$$\|\operatorname{ME}_{2,1}(f,g)\|_2^2 \lesssim \sum_{m\in\mathbb{Z}}\sum_{n\in\mathbb{Z}} |\widehat{H}_m(-n)|^2 = \sum_{m\in\mathbb{Z}} \|H_m\|_2^2,$$

by Bessel. On the other hand,

$$\begin{aligned}\|H_m\|_2^2 &= \int\left|\int f\left(\frac{v-u}{2}\right)g\left(\frac{v+u}{2}+4\right)e^{2\pi i muv}e^{8\pi i m(u+v)}\,\mathrm{d}v\right|^2\mathrm{d}u\\ &= \int\left|\int \underbrace{f\left(\frac{v-u}{2}\right)g\left(\frac{v+u}{2}+4\right)}_{\widetilde{H}_u(v)} e^{2\pi i mv(u+4)}\,\mathrm{d}v\right|^2\mathrm{d}u = \int |\widehat{\widetilde{H}}_u(m(u+4))|^2\,\mathrm{d}u.\end{aligned}$$

Transversality enters the picture here through the factor $(u+4)$ above: the $+4$ shift in u comes from the fact that the supports of φ^1 and φ^2 are disjoint and far enough from each other; hence $u+4\geq c>0$. This way,

$$\begin{aligned}\|\operatorname{ME}_{2,1}(f,g)\|_2^2 &\lesssim \int\left(\sum_{m\in\mathbb{Z}} |\widehat{\widetilde{H}}_u(m(u+4))|^2\right)\mathrm{d}u\\ &\lesssim \iint |\widetilde{H}_u(v)|^2\,\mathrm{d}v\,\mathrm{d}u \lesssim \|f\|_2^2\|g\|_2^2,\end{aligned}$$

by Bessel again.

8. Case $k=1$ of Theorem 1.5

In this section we start the proof of Theorem 1.5. There are two main ingredients in the argument for the case $k=1$: Proposition 4.3 and the fact that the wave packets

$$\varphi_{\vec{n},m}(x) := \varphi(x_1)\cdot\dots\cdot\varphi(x_d)e^{2\pi i x\cdot\vec{n}}e^{2\pi i|x|^2 m}$$

are almost orthogonal for a fixed m and $\vec{n}$ varying in $\mathbb{Z}^d$. The latter fact will be exploited through Bessel's inequality whenever possible. Recall from Remark 4.5 that, since $g=g_1\otimes\cdots\otimes g_d$, it suffices to study the multilinear form

$$\Lambda_d(g_1,\dots,g_d,h) = \sum_{\vec{n}\in\mathbb{Z}^d,\,m\in\mathbb{Z}}\prod_{j=1}^{d}\langle g_j,\varphi_{n_j,m}\rangle\cdot\langle h,\chi_{\vec{n}}\otimes\chi_m\rangle,$$

Now we focus on obtaining (22). Let $E_j\subset[0,1]$, $1\leq j\leq d$, and $F\subset\mathbb{R}^{d+1}$ be measurable sets for which $|g_j|\leq\chi_{E_j}$ and $|h|\leq\chi_F$. Define the sets

$$\begin{aligned}\mathbb{A}_j^{l_j} &:= \{(n_j,m)\in\mathbb{Z}^2 : |\langle g_j,\varphi_{n_j,m}\rangle|\approx 2^{-l_j}\},\quad 1\leq j\leq d.\\ \mathbb{B}^{l_{d+1}} &:= \{(\vec{n},m)\in\mathbb{Z}^{d+1} : |\langle h,\chi_{\vec{n}}\otimes\chi_m\rangle|\approx 2^{-l_{d+1}}\},\\ \mathbb{X}^{l_1,\dots,l_{d+1}} &:= \{(\vec{n},m)\in\mathbb{Z}^{d+1} : (n_j,m)\in\mathbb{A}_j^{l_j},\,1\leq j\leq d\}\cap\mathbb{B}^{l_{d+1}}.\end{aligned}$$

Hence,

$$|\Lambda_d(g_1,\dots,g_d,h)| \lesssim \sum_{l_1,\dots,l_{d+1}\in\mathbb{Z}} 2^{-l_1}\cdot\dots\cdot 2^{-l_{d+1}}\#\mathbb{X}^{l_1,\dots,l_{d+1}}.$$

As in Section 5, we know that $l_1, \dots, l_{d+1} \geq 0$. We can estimate $\#\mathbb{X}^{l_1,\dots,l_{d+1}}$ using the function h:

$$\#\mathbb{X}^{l_1,\dots,l_{d+1}} \lesssim 2^{l_{d+1}} \sum_{(\vec{\boldsymbol{n}},m)\in\mathbb{Z}^{d+1}} |\langle h, \chi_{\vec{\boldsymbol{n}}} \otimes \chi_m \rangle| \lesssim 2^{l_{d+1}}|F|. \tag{28}$$

Alternatively, many bounds for $\#\mathbb{X}^{l_1,\dots,l_{d+1}}$ can be obtained using the input functions $g_1, \dots, g_d$:

$$\begin{aligned} \#\mathbb{X}^{l_1,\dots,l_{d+1}} &\lesssim \sum_{(\vec{\boldsymbol{n}},m)\in\mathbb{Z}^{d+1}} \mathbb{1}_{\mathbb{A}_1^{l_1}}(n_1,m)\cdot\dots\cdot\mathbb{1}_{\mathbb{A}_d^{l_d}}(n_d,m) \\ &= \sum_{m\in\mathbb{Z}}\sum_{n_1\in\mathbb{Z}}\cdots\sum_{n_{d-1}\in\mathbb{Z}} \mathbb{1}_{\mathbb{A}_1^{l_1}}(n_1,m)\cdot\dots\cdot\mathbb{1}_{\mathbb{A}_{d-1}^{l_{d-1}}}(n_{d-1},m) \underbrace{\sum_{n_d\in\mathbb{Z}} \mathbb{1}_{\mathbb{A}_d^{l_d}}(n_d,m)}_{\alpha_{d,m}} \end{aligned} \tag{29}$$

Observe that $\alpha_{d,m} = \#\{n; (n,m) \in \mathbb{A}_d^{l_d}\}$ and $(n,m) \in \mathbb{A}_d^{l_d} \Rightarrow 1 \lesssim 2^{2l_d}|\langle g_d, \varphi_{n,m}\rangle|^2$. Adding up in n,

$$\alpha_{d,m} \lesssim 2^{2l_d} \sum_{n:\,(n,m)\in\mathbb{A}_d^{l_d}} |\langle g_d, \varphi_{n,m}\rangle|^2 \lesssim 2^{2l_d}|E_d|$$

by orthogonality. Notice that this quantity does not depend on m; therefore we can iterate this argument for $d-2$ of the remaining $d-1$ characteristic functions:

$$\begin{aligned} \#\mathbb{X}^{l_1,\dots,l_{d+1}} &\lesssim 2^{2l_d}|E_d| \sum_{m\in\mathbb{Z}}\sum_{n_1\in\mathbb{Z}} \mathbb{1}_{\mathbb{A}_1^{l_1}}(n_1,m)\cdot\dots\cdot\mathbb{1}_{\mathbb{A}_{d-2}^{l_{d-2}}}(n_{d-2},m) \underbrace{\sum_{n_{d-1}\in\mathbb{Z}} \mathbb{1}_{\mathbb{A}_{d-1}^{l_{d-1}}}(n_{d-1},m)}_{\alpha_{d-1,m}} \\ &\lesssim 2^{2l_d}|E_d|2^{2l_{d-1}}|E_{d-1}| \sum_{m\in\mathbb{Z}}\sum_{n_1\in\mathbb{Z}} \mathbb{1}_{\mathbb{A}_1^{l_1}}(n_1,m)\cdot\dots\cdot\mathbb{1}_{\mathbb{A}_{d-3}^{l_{d-3}}}(n_{d-3},m) \sum_{n_{d-2}\in\mathbb{Z}} \mathbb{1}_{\mathbb{A}_{d-2}^{l_{d-2}}}(n_{d-1},m) \\ &\lesssim 2^{2l_d}2^{2l_{d-1}}\dots 2^{2l_2}|E_d|\cdots|E_2| \underbrace{\sum_{m\in\mathbb{Z}}\sum_{n_1\in\mathbb{Z}} \mathbb{1}_{\mathbb{A}_1^{l_1}}(n_1,m)}_{\#\mathbb{A}_1^{l_1}}. \end{aligned} \tag{30}$$

To bound $\#\mathbb{A}_1^{l_1}$ we can use Proposition 4.3. For $\varepsilon > 0$ we have

$$\begin{aligned} (n,m) \in \mathbb{A}_1^{l_1} \quad &\Longrightarrow \quad 1 \lesssim 2^{(4+\varepsilon)l_1}|\langle g_1, \varphi_{n,m}\rangle|^{4+\varepsilon} \\ &\Longrightarrow \quad \#\mathbb{A}_1^{l_1} \lesssim 2^{(4+\varepsilon)l_1} \sum_{(n,m)\in\mathbb{A}_1^{l_1}} |\langle g_1, \varphi_{n,m}\rangle|^{4+\varepsilon} \lesssim_\varepsilon 2^{(4+\varepsilon)l_1}|E_1|. \end{aligned}$$

Using this above,

$$\#\mathbb{X}^{l_1,\dots,l_{d+1}} \lesssim_\varepsilon 2^{2l_d}2^{2l_{d-1}}\dots 2^{2l_2}2^{(4+\varepsilon)l_1}|E_d|\cdots|E_2||E_1|. \tag{31}$$

We could have used the L^4-$L^{4+\varepsilon}$ bound for any g_j and a Bessel bound for the remaining ones. More precisely, if $\sigma \in S_d$ is a permutation, we have

$$\#\mathbb{X}^{l_1,\dots,l_{d+1}} \lesssim_\varepsilon 2^{2l_{\sigma(d)}}2^{2l_{\sigma(d-1)}}\dots 2^{2l_{\sigma(2)}}2^{(4+\varepsilon)l_{\sigma(1)}}|E_{\sigma(d)}|\cdots|E_{\sigma(2)}||E_{\sigma(1)}|. \tag{32}$$

This amounts to exactly d different estimates. Interpolating between all of them with equal weight $\frac{1}{d}$, we obtain

$$\begin{aligned}\#\mathbb{X}^{l_1,\dots,l_{d+1}} &\lesssim_\varepsilon 2^{\frac{2(d-1)+4+\varepsilon}{d}l_1}\dots 2^{\frac{2(d-1)+4+\varepsilon}{d}l_d}|E_1|\cdots|E_d| \\ &= 2^{(2+\frac{2}{d}+\frac{\varepsilon}{d})l_1}\dots 2^{(2+\frac{2}{d}+\frac{\varepsilon}{d})l_d}|E_1|\cdots|E_d|. \end{aligned} \tag{33}$$

Finally, we interpolate between bounds (28) and (33):

$$\begin{aligned}|\Lambda_d(g_1,\dots,g_d,h)| &\lesssim \sum_{l_1,\dots,l_{d+1}\in\mathbb{Z}_+} 2^{-l_1}\cdot\dots\cdot 2^{-l_{d+1}}\#\mathbb{X}^{l_1,\dots,l_{d+1}} \\ &\lesssim \sum_{l_1,\dots,l_{d+1}\in\mathbb{Z}_+} 2^{-l_1}\cdot\dots\cdot 2^{-l_{d+1}}(2^{(2+\frac{2}{d}+\frac{\varepsilon}{d})l_1}\dots 2^{(2+\frac{2}{d}+\frac{\varepsilon}{d})l_d}|E_1|\cdots|E_d|)^{\theta_1}(2^{l_{d+1}}|F|)^{\theta_2} \\ &\lesssim \Big(\sum_{l_{d+1}\geq 0} 2^{-(1-\theta_2)l_{d+1}}|F|^{\theta_2}\Big)\prod_{j=1}^{d}\sum_{l_j\geq 0} 2^{-(1-(2+\frac{2}{d}+\frac{\varepsilon}{d})\theta_1)l_j}|E_j|^{\theta_1} \\ &\lesssim |E_1|^{\alpha(1-(2+\frac{2}{d}+\frac{\varepsilon}{d})\theta_1)+\theta_1}\dots|E_d|^{\alpha(1-(2+\frac{2}{d}+\frac{\varepsilon}{d})\theta_1)+\theta_1}|F|^{\theta_2}\end{aligned}$$

for any $0\leq\alpha\leq 1$. On the other hand, for several of the series above to converge we need $\big(2+\frac{2}{d}+\frac{\varepsilon}{d}\big)\theta_1>1$. By choosing the appropriate α and θ_1 close to $\big(2+\frac{2}{d}\big)^{-1}$, one concludes this case.

9. Case $2\leq k\leq d+1$ of Theorem 1.5

Recall that we fixed a set of weakly transversal cubes $\mathcal{Q}=\{Q_1,\dots,Q_k\}$ in Section 3 and let g_j be supported on Q_j. The averaged k-linear extension operator[14] in $\mathbb{R}^d$ is given by

$$\mathrm{ME}^{\frac{1}{k}}_{k,d}(g_1,\dots,g_k)=\sum_{(\vec{n},m)\in\mathbb{Z}^{d+1}}\Big(\prod_{j=1}^{k}|\langle g_j,\varphi^j_{\vec{n},m}\rangle|\Big)^{\frac{1}{k}}(\chi_{\vec{n}}\otimes\chi_m).$$

The conjectured bounds for it are

$$\|\mathrm{ME}^{\frac{1}{k}}_{k,d}(g_1,\dots,g_k)\|_{L^p(\mathbb{R}^{d+1})}\lesssim\prod_{j=1}^{k}\|g_j\|^{\frac{1}{k}}_{L^2(Q_j)}\quad\text{for all } p\geq\frac{2(d+k+1)}{(d+k-1)}. \tag{34}$$

As done in the case $k=1$, it's enough to prove certain restricted weak-type bounds for its associated form

$$\tilde{\tilde{\Lambda}}_{k,d}(g,h):=\sum_{(\vec{n},m)\in\mathbb{Z}^{d+1}}\Big(\prod_{i=1}^{k}|\langle g_j,\varphi^j_{\vec{n},m}\rangle|\Big)^{\frac{1}{k}}\langle h,\chi_{\vec{n}}\otimes\chi_m\rangle, \tag{35}$$

where $g:=(g_1,\dots,g_k)$ by a slight abuse of notation.

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[14]We consider this averaged version of $\mathrm{ME}_{k,d}$ for technical reasons. The conjectured bounds for it have a Banach space as target, as opposed to the quasi-Banach space (for most k and d) $L^{2(d+k+1)/(k(d+k-1))}$ that is the target of Conjecture 1.2. The fact that L^p for $p\geq 2(d+k+1)/(d+k-1)$ is Banach lets us use (49) effectively in the interpolation argument, since it forces the final power γ on $|F|^\gamma$ to be positive.

When $k=d=2$, Conjecture 1.2 has $L^{5/3}$ as target. We will discuss this case first to help digest the main ideas of the general argument, and since this space is Banach, we can work directly with $\mathrm{ME}_{2,2}$ instead of considering the averaged operator $\mathrm{ME}^{1/2}_{2,2}$.

Remark 9.1. We will prove (34) up to the endpoint assuming that g_1 is a full tensor, but the argument can be repeated if any other g_j is assumed to be of this type. As the reader will notice, the proof depends on the fact that we can find $k-1$ canonical directions associated to Q_j, which is the defining property of a weakly transversal collection of cubes with pivot Q_j. In what follows, we are taking $\{e_1, \ldots, e_{k-1}\}$ to be the set of directions associated to Q_1.

Remark 9.2. As we mentioned in Remark 1.7, under weak transversality alone we do not need g_1 to be a full tensor to prove the case $2 \le k \le d$ of Theorem 1.5. In fact, the following structure is enough in this section:

$$g_1(x_1, \ldots, x_d) = g_{1,1}(x_1) \cdot g_{1,2}(x_2) \cdot \cdots \cdot g_{1,k-1}(x_{k-1}) \cdot g_{1,k}(x_k, \ldots, x_d).$$

Notice that we have $k-1$ single-variable functions and one function in $d-k+1$ variables. The single-variable ones are defined along $k-1$ canonical directions $\{e_1, \ldots, e_{k-1}\}$ associated to Q_1, and $g_{1,k}$ is a function in the remaining variables.

In general, if we are given a weakly transversal collection $\widetilde{\mathcal{Q}}$, for a fixed $1 \le j \le k-1$ we have a set of associated directions $\mathscr{E}_j = \{e_{i_1}, \ldots, e_{i_{k-1}}\}$ (see Definition 3.2). Denote by $x_{\mathscr{E}_j^c}$ the vector of $d-k+1$ entries obtained after removing $x_{i_1}, \ldots, x_{i_{k-1}}$ from $(x_1, \ldots, x_d)$. Assuming that the functions g_l for $l \ne j$ are generic and that g_j has the weaker tensor structure

$$g_j(x_1, \ldots, x_d) = g_{j,1}(x_{i_1}) \cdot \cdots \cdot g_{j,k-1}(x_{i_{k-1}}) \cdot g_{\mathscr{E}_j^c}(x_{\mathscr{E}_j^c}) \tag{36}$$

will suffice to conclude Theorem 1.5 for $\widetilde{\mathcal{Q}}$ through the argument that we will present in this section.

Remark 9.3. As a consequence of Claim 3.4, a collection $\mathcal{Q} = \{Q_1, \ldots, Q_k\}$ of transversal cubes generates finitely many subcollections $\widetilde{\mathcal{Q}}$ of weakly transversal ones (after partitioning each Q_l into small enough cubes and defining new collections with them). However, for a fixed $1 \le j \le k$, the associated $k-1$ directions in $\mathscr{E}_j$ can potentially change from one such weakly transversal subcollection to another, and this is why we assume g_j to be a full tensor under the transversality assumption.

In this section we will use the following conventions:

- The variables of g_j are $x_1, x_2, \ldots, x_d$, but we will split them in two groups: $k-1$ blocks of one variable represented by x_i, $1 \le i \le k-1$, and one block of $d-k+1$ variables $\vec{x_k} = (x_k, x_{k+1}, \ldots, x_{d-1}, x_d)$.
- The index x_i in $\langle \cdot, \cdot \rangle_{x_i}$ indicates that the inner product is an integral in the variable x_i only. For instance,

$$\langle g_j, \varphi \rangle_{x_1} := \int_{\mathbb{R}} g_j(x_1, \ldots, x_d) \cdot \bar{\varphi}(x_1, \ldots, x_d)\, \mathrm{d}x_1 \tag{37}$$

is now a function of the variables $x_2, \ldots, x_d$. The vector index $\vec{x_k}$ in $\langle \cdot, \cdot \rangle_{\vec{x_k}}$ is understood analogously:

$$\langle g_j, \varphi \rangle_{\vec{x_k}} := \int_{\mathbb{R}^{d-k+1}} g_j(x_1, \ldots, x_d) \cdot \bar{\varphi}(x_1, \ldots, x_d)\, \mathrm{d}\vec{x_k} \tag{38}$$

- The expression $\|\langle g_j,\cdot\rangle_{x_i}\|_2$ is the L^2 norm of a function in the variables $x_l,\ 1\le l\le k-1,\ l\ne i$. To illustrate using (37),

$$\|\langle g_j,\varphi\rangle_{x_1}\|_2=\left[\int_{\mathbb{R}^{d-1}}\left|\int_{\mathbb{R}} g_j(x_1,\dots,x_d)\cdot\bar{\varphi}(x_1,\dots,x_d)\,\mathrm{d}x_1\right|^2\mathrm{d}x_2\cdots\mathrm{d}x_d\right]^{\frac{1}{2}}.$$

The quantity $\|\langle g_j,\cdot\rangle_{\vec{x_k}}\|_2$ is defined analogously as

$$\|\langle g_j,\varphi\rangle_{\vec{x_k}}\|_2=\left[\int_{\mathbb{R}^{k-1}}\left|\int_{\mathbb{R}^{d-k+1}} g_j(x_1,\dots,x_d)\cdot\bar{\varphi}(x_1,\dots,x_d)\,\mathrm{d}\vec{x_k}\right|^2\mathrm{d}x_1\cdots\mathrm{d}x_{k-1}\right]^{\frac{1}{2}}.$$

- For $\vec{\boldsymbol{n}}=(n_1,\dots,n_d)$, define the vector

$$\hat{n}_i:=(n_1,\dots,n_{i-1},n_{i+1},\dots,n_d).$$

In other words, the hat on $\hat{n}_i$ indicates that n_i was removed from the vector $\vec{\boldsymbol{n}}$. For $f:\mathbb{Z}^d\to\mathbb{C}$, define

$$\|f(\vec{\boldsymbol{n}})\|_{\ell^1_{\hat{n}_i}}:=\sum_{\hat{n}_i\in\mathbb{Z}^{d-1}}|f(\vec{\boldsymbol{n}})|.$$

That is, $\|f(\vec{\boldsymbol{n}})\|_{\ell^1_{\hat{n}_i}}$ is the ℓ^1 norm of f over all $n_1,\dots,n_d$, except for n_i. Hence $\|f(\vec{\boldsymbol{n}})\|_{\ell^1_{\hat{n}_i}}$ is a function of the remaining variable n_i. The quantity $\|f(\vec{\boldsymbol{n}})\|_{\ell^1_{\hat{\hat{n}}_k}}$ is defined analogously as

$$\|f(\vec{\boldsymbol{n}})\|_{\ell^1_{\hat{\hat{n}}_k}}:=\sum_{(n_1,\dots,n_{k-1})\in\mathbb{Z}^{k-1}}|f(\vec{\boldsymbol{n}})|.$$

Finally, the integral $\int g\,\mathrm{d}\hat{x}_i$ means

$$\int g(x_1,\dots,x_d)\,\mathrm{d}\hat{x}_i:=\int g(x_1,\dots,x_d)\,\mathrm{d}x_1\cdots\mathrm{d}x_{i-1}\,\mathrm{d}x_{i+1}\cdots\mathrm{d}x_d.$$

In what follows, let $E_{1,1},\dots,E_{1,k-1}\subset[0,1],\ E_{1,k}\subset[0,1]^{d-k+1},\ E_j\subset Q_j\ (2\le j\le k)$ and $F\subset\mathbb{R}^{d+1}$ be measurable sets such that $|g_{1,l}|\le\chi_{E_{1,l}}$ for $1\le l\le k-1,\ |g_{1,k}|\le\chi_{E_{1,k}},\ |g_j|\le\chi_{E_j}$ for $2\le j\le k$ and $|h|\le\chi_F$. Furthermore, $E_1:=E_{1,1}\times\cdots\times E_{1,k-1}\times E_{1,k}$.

A rough description of the argument in one sentence is: *the proof is a combination of Strichartz in some variables and bilinear extension in many pairs of the other variables*. In order to illustrate that, we will first present the simplest case in an informal way, which means that we will avoid the purely technical aspects in this preliminary part. Once this is understood, it will be clear how to rigorously extend the argument in general.

9A. ***Understanding the core ideas in the $k=d=2$ case.*** Consider the model

$$\mathrm{ME}_{2,2}(g_1,g_2)=\sum_{(\vec{\boldsymbol{n}},m)\in\mathbb{Z}^3}\langle g_1,\varphi^1_{\vec{\boldsymbol{n}},m}\rangle\langle g_2,\varphi^2_{\vec{\boldsymbol{n}},m}\rangle(\chi_{\vec{\boldsymbol{n}}}\otimes\chi_m)$$

and its associated trilinear form[15]

$$\tilde{\tilde{\Lambda}}_{2,2}(g_1,g_2,h)=\sum_{(\vec{\boldsymbol{n}},m)\in\mathbb{Z}^3}\langle g_1,\varphi^1_{\vec{\boldsymbol{n}},m}\rangle\langle g_2,\varphi^2_{\vec{\boldsymbol{n}},m}\rangle(\chi_{\vec{\boldsymbol{n}}}\otimes\chi_m).$$

Assuming that $g_1=g_{1,1}\otimes g_{1,2}$, we want to prove that

$$|\tilde{\tilde{\Lambda}}_{2,2}(g_1,g_2)|\lesssim_\varepsilon |E_1|^{\frac{1}{2}}\cdot|E_2|^{\frac{1}{2}}\cdot|F|^{\frac{2}{5}+\varepsilon}$$

for all $\varepsilon>0$. The $L^2\times L^2\mapsto L^{5/3+\varepsilon}$ bound will then follow by multilinear interpolation and Remark 4.5. Given the expository character of this subsection, we adopt the informal convention

$$\begin{cases} x^+ \text{ means } x+\delta, \text{ where } \delta>0 \text{ is arbitrarily small,}\\ x^- \text{ means } x-\delta, \text{ where } \delta>0 \text{ is arbitrarily small.}\end{cases}$$

We will always be able to control how small the δ above is, so we do not worry about making it precise for now.

The first step is to define the level sets of the scalar products appearing in $\mathrm{ME}_{2,2}$:

$$\mathbb{A}_1^{l_1}=\{(\vec{\boldsymbol{n}},m):|\langle g_1,\varphi^1_{\vec{\boldsymbol{n}},m}\rangle|\approx 2^{-l_1}\},$$
$$\mathbb{A}_2^{l_2}=\{(\vec{\boldsymbol{n}},m):|\langle g_2,\varphi^2_{\vec{\boldsymbol{n}},m}\rangle|\approx 2^{-l_2}\}.$$

Transversality will be captured by exploiting the sizes of "lower-dimensional" information: in fact, we want to make the operator $\mathrm{ME}_{2,1}$ appear, and this will be possible thanks to the interaction between the quantities associated to the level sets

$$\mathbb{B}_1^{r_1}=\{(n_1,m):\|\langle g_1,\varphi^{1,1}_{n_1,m}\rangle_{x_1}\|_2\approx 2^{-r_1}\},$$
$$\mathbb{C}_1^{s_1}=\{(n_1,m):\|\langle g_2,\varphi^{1,2}_{n_1,m}\rangle_{x_1}\|_2\approx 2^{-s_1}\}.$$

Since there is only one direction along which one can exploit transversality, we will use the L^2 theory for E_1 (i.e., Strichartz) along the remaining one. In order to do that, the following level sets will be used:

$$\mathbb{B}_2^{r_2}=\{(n_2,m):\|\langle g_1,\varphi^{2,1}_{n_2,m}\rangle_{x_2}\|_2\approx 2^{-r_2}\},$$
$$\mathbb{C}_2^{s_2}=\{(n_2,m):\|\langle g_2,\varphi^{2,2}_{n_2,m}\rangle_{x_2}\|_2\approx 2^{-s_2}\}.$$

The size of the scalar product involving h will be captured by the set

$$\mathbb{D}^k=\{(\vec{\boldsymbol{n}},m):|\langle H,\chi_{\vec{\boldsymbol{n}}}\otimes\chi_m\rangle|\approx 2^{-k}\}.$$

We will also need to organize all the information above in appropriate "slices" and in a major set that takes everything into account. The sets that do that are

$$\mathbb{X}^{l_2,s_1}:=\mathbb{A}_2^{l_2}\cap\{(\vec{\boldsymbol{n}},m);(n_1,m)\in\mathbb{C}_1^{s_1}\},$$
$$\mathbb{X}^{l_2,s_2}:=\mathbb{A}_2^{l_2}\cap\{(\vec{\boldsymbol{n}},m);(n_2,m)\in\mathbb{C}_2^{s_2}\},$$
$$\mathbb{X}^{\vec{l},\vec{r},\vec{s},k}=\mathbb{A}_1^{l_1}\cap\mathbb{A}_2^{l_2}\cap\{(\vec{\boldsymbol{n}},m):(n_1,m)\in\mathbb{B}_1^{r_1}\cap\mathbb{C}_1^{s_1},(n_2,m)\in\mathbb{B}_2^{r_2}\cap\mathbb{C}_2^{s_2}\}\cap\mathbb{D}^k,$$

[15]There is a slight abuse of notation here: we are using $\tilde{\tilde{\Lambda}}_{2,2}$ for the form associated to $\mathrm{ME}_{2,2}$ and not for its averaged version $\mathrm{ME}_{2,2}$, as established in the beginning of this section.

where we are using the abbreviations $\vec{l} = (l_1, l_2)$, $\vec{r} = (r_1, r_2)$ and $\vec{s} = (s_1, s_2)$. This gives us

$$|\tilde{\tilde{\Lambda}}_{2,2}(g_1, g_2, h)| \lesssim \sum_{\vec{l},\vec{r},\vec{s},k} 2^{-l_1} 2^{-l_2} 2^{-k} \# \mathbb{X}^{\vec{l},\vec{r},\vec{s},k}.$$

For the sake of simplicity, let us assume that $g_1 = \mathbb{1}_{E_{1,1}} \otimes \mathbb{1}_{E_{1,2}}$, $g_2 = \mathbb{1}_{E_2}$ and $h = \mathbb{1}_F$.[16] We will need efficient ways of relating the scalar and mixed-norm quantities above. A direct computation (using the definition of $\mathbb{X}^{\vec{l},\vec{r},\vec{s},k}$) shows that

$$2^{-l_1} = \frac{2^{-r_1} \cdot 2^{-r_2}}{|E_1|^{\frac{1}{2}}}. \tag{39}$$

Using Bessel along a direction, for $(n_1, n_2, m) \in \mathbb{X}^{l_2,s_1}$ we have

$$\begin{aligned} 1 \approx 2^{2l_2} |\langle g_2, \varphi^2_{\vec{\boldsymbol{n}},m}\rangle|^2 \quad &\Longrightarrow \quad \# \mathbb{X}^{l_2,s_1}_{(n_1,m)} \approx 2^{2l_2} \sum_{n_2 \in \mathbb{X}^{l_2,s_1}_{(n_1,m)}} |\langle g_2, \varphi^2_{\vec{\boldsymbol{n}},m}\rangle|^2 \\ &\Longrightarrow \quad \# \mathbb{X}^{l_2,s_1}_{(n_1,m)} \lesssim 2^{2l_2} \|\langle g_2, \varphi^{1,2}_{n_1,m}\rangle\|_2^2 \\ &\Longrightarrow \quad 2^{-l_2} \lesssim \frac{2^{-s_1}}{(\# \mathbb{X}^{l_2,s_1}_{(n_1,m)})^{\frac{1}{2}}} \quad \Longrightarrow \quad 2^{-l_2} \lesssim \frac{2^{-s_1}}{\|\mathbb{1}_{\mathbb{X}^{l_2,s_1}}\|^{\frac{1}{2}}_{\ell^\infty_{n_1,m} \ell^1_{n_2}}}, \end{aligned} \tag{40}$$

by taking the supremum in (n_1, m). Analogously,

$$2^{-l_2} \lesssim \frac{2^{-s_2}}{\|\mathbb{1}_{\mathbb{X}^{l_2,s_2}}\|^{\frac{1}{2}}_{\ell^\infty_{n_2,m} \ell^1_{n_1}}}. \tag{41}$$

Relations (39), (40) and (41) play a major role in the proof. The last major piece is a way of bounding $\# \mathbb{X}^{\vec{l},\vec{r},\vec{s},k}$ that allows us to exploit transversality and Strichartz along the right directions, as well as the dual function h. We start with the simplest one of them:

$$\# \mathbb{X}^{\vec{l},\vec{r},\vec{s},k} \lesssim 2^k \sum_{(\vec{\boldsymbol{n}},m) \in \mathbb{Z}^3} |\langle h, \chi_{\vec{\boldsymbol{n}}} \otimes \chi_m \rangle| = 2^k |F|. \tag{42}$$

By dropping most of the indicator functions in the definition of $\mathbb{X}^{\vec{l},\vec{r},\vec{s},k}$ and using Hölder, we obtain

$$\# \mathbb{X}^{\vec{l},\vec{r},\vec{s},k} \le \sum_{(\vec{\boldsymbol{n}},m) \in \mathbb{Z}^3} \mathbb{1}_{\mathbb{X}^{l_2,s_1}}(\vec{\boldsymbol{n}}, m) \cdot \mathbb{1}_{\mathbb{B}_1^{r_1} \cap \mathbb{C}_1^{s_1}}(n_1, m) \le \|\mathbb{1}_{\mathbb{X}^{l_2,s_1}}\|_{\ell^\infty_{n_1,m} \ell^1_{n_2}} \cdot \|\mathbb{1}_{\mathbb{B}_1^{r_1} \cap \mathbb{C}_1^{s_1}}\|_{\ell^1_{n_1,m}}.$$

The second factor of the inequality above will be bounded by the one-dimensional bilinear theory:

$$\begin{aligned} \# \mathbb{B}_1^{r_1} \cap \mathbb{C}_1^{s_1} &\lesssim 2^{2r_1 + 2s_1} \sum_{n_1, m \in \mathbb{Z}} \|\langle g_1, \varphi^{1,1}_{n_1,m}\rangle_{x_1}\|_2^2 \cdot \|\langle g_2, \varphi^{1,2}_{n_1,m}\rangle_{x_1}\|_2^2 \\ &= 2^{2r_1 + 2s_1} \iint \left(\sum_{n_1, m \in \mathbb{Z}} |\langle g_1, \varphi^{1,1}_{n_1,m}\rangle_{x_1}|^2 \cdot |\langle g_2, \varphi^{1,2}_{n_1,m}\rangle_{x_1}|^2 \right) \mathrm{d}x_2 \, \mathrm{d}\tilde{x}_2 \\ &= 2^{2r_1 + 2s_1} \iint \|g_1\|^2_{L^2_{x_1}} \cdot \|g_2\|^2_{L^2_{x_1}} \, \mathrm{d}x_2 \, \mathrm{d}\tilde{x}_2 \le 2^{2r_1 + 2s_1} \|g_1\|_2^2 \cdot \|g_2\|_2^2, \end{aligned}$$

[16] These indicator functions actually *bound* g_1 and g_2, but this does not affect the core of the argument.

by Proposition 4.4 since the supports of $\varphi^{1,1}$ and $\varphi^{1,2}$ are disjoint (this is equivalent to transversality in dimension one). This gives us

$$\#\mathbb{X}^{\vec{l},\vec{r},\vec{s},k} \le \|\mathbb{1}_{\mathbb{X}^{l_2,s_1}}\|_{\ell^\infty_{n_1,m}\ell^1_{n_2}} \cdot 2^{2r_1+2s_1} \cdot |E_1| \cdot |E_2|. \tag{43}$$

Alternatively,

$$\begin{aligned}\#\mathbb{X}^{\vec{l},\vec{r},\vec{s},k} &\le \sum_{(n_2,m)\in\mathbb{Z}^2} \mathbb{1}_{\mathbb{B}_2^{r_2}\cap\mathbb{C}_2^{s_2}}(n_2,m) \sum_{n_1\in\mathbb{Z}} \mathbb{1}_{\mathbb{X}^{l_2,s_2}}(\vec{\boldsymbol{n}},m)\cdot \mathbb{1}_{\mathbb{B}_1^{r_1}}(n_1,m)\\ &\le \|\mathbb{1}_{\mathbb{X}^{l_2,s_2}}\|^{\frac12}_{\ell^\infty_{n_2,m}\ell^1_{n_1}} \cdot \|\mathbb{1}_{\mathbb{B}_1^{r_1}}\|^{\frac12}_{\ell^\infty_m\ell^1_{n_1}} \cdot \|\mathbb{1}_{\mathbb{B}_2^{r_2}\cap\mathbb{C}_2^{s_2}}\|_{\ell^1_{n_2,m}}.\end{aligned}$$

We can treat the last two factors appearing in the right-hand side above as follows: For a fixed $m \in \mathbb{Z}$,

$$\sum_{n_1\in\mathbb{Z}} \mathbb{1}_{\mathbb{B}^{r_1}}(n_1,m) \lesssim 2^{2r_1} \sum_{n_1\in\mathbb{Z}} \|\langle g_1, \varphi^{1,1}_{n_1,m}\rangle_{x_1}\|_2^2 \le 2^{2r_1}\cdot\|g_1\|_2^2$$

by Bessel (recall that the modulated bumps $\varphi^{1,1}_{n_1,m}$ are almost-orthogonal if n_1 varies and m is fixed), and then we take the supremum in m. As for the other factor, observe that[17]

$$\begin{aligned}\#\mathbb{B}_2^{r_2}\cap\mathbb{C}_2^{s_2} &\lesssim 2^{5r_2+s_2} \sum_{n_2,m\in\mathbb{Z}} \|\langle g_1,\varphi^{2,1}_{n_2,m}\rangle_{x_2}\|_2^5 \cdot \|\langle g_2,\varphi^{2,2}_{n_2,m}\rangle_{x_2}\|_2\\ &\lesssim 2^{5r_2+s_2}\Bigl(\sum_{n_2,m\in\mathbb{Z}} \|\langle g_1,\varphi^{2,1}_{n_2,m}\rangle_{x_2}\|_2^6\Bigr)^{\frac56}\Bigl(\sum_{n_2,m\in\mathbb{Z}} \|\langle g_2,\varphi^{2,2}_{n_2,m}\rangle_{x_2}\|_2^6\Bigr)^{\frac16}\\ &\lesssim 2^{5r_2+s_2}\|g_1\|_2^5\cdot\|g_2\|_2\end{aligned}$$

by Corollary 4.2. These last to estimates give the following bound on $\#\mathbb{X}^{\vec{l},\vec{r},\vec{s},k}$:

$$\#\mathbb{X}^{\vec{l},\vec{r},\vec{s},k} \lesssim \|\mathbb{1}_{\mathbb{X}^{l_2,s_2}}\|^{\frac12}_{\ell^\infty_{n_2,m}\ell^1_{n_1}} \cdot 2^{r_1}\cdot|E_1|^{\frac12}\cdot 2^{5r_2+s_2}\cdot|E_1|^{\frac52}\cdot|E_2|^{\frac12}. \tag{44}$$

In what follows, we interpolate between (43), (44) and (42) with weights $\frac25^-$, $\frac15^-$ and $\frac25^+$, respectively. We also take an appropriate of combination between (40) and (41), and use (39):

$$\begin{aligned}|\tilde{\tilde{\Lambda}}_{2,2}(g_1,g_2,h)| \text{“}\lesssim\text{”} &\sum_{\vec{r},\vec{s},k} \frac{2^{-r_1}\cdot 2^{-r_2}}{|E_1|^{\frac12}} \cdot \frac{2^{-\frac45 s_1}}{\|\mathbb{1}_{\mathbb{X}^{l_2,s_1}}\|^{\frac25}_{\ell^\infty_{n_1,m}\ell^1_{n_2}}} \cdot \frac{2^{-\frac15 s_2}}{\|\mathbb{1}_{\mathbb{X}^{l_s,s_2}}\|^{\frac{1}{10}}_{\ell^\infty_{n_2,m}\ell^1_{n_1}}}\cdot 2^{-k}\\ &\cdot\bigl(\|\mathbb{1}_{\mathbb{X}^{l_2,s_1}}\|_{\ell^\infty_{n_1,m}\ell^1_{n_2}}\cdot 2^{2r_1+2s_1}\cdot|E_1|\cdot|E_2|\bigr)^{\frac25^-}\\ &\cdot\bigl(\|\mathbb{1}_{\mathbb{X}^{l_2,s_2}}\|^{\frac12}_{\ell^\infty_{n_2,m}\ell^1_{n_1}}\cdot 2^{r_1}\cdot|E_1|^{\frac12}\cdot 2^{5r_2+s_2}\cdot|E_1|^{\frac52}\cdot|E_2|^{\frac12}\bigr)^{\frac15^-}\cdot(2^k|F|)^{\frac25^-}\\ \lesssim{}& |E_1|^{\frac12}\cdot|E_2|^{\frac12}\cdot|F|^{\frac25^+},\end{aligned}$$

which is the estimate that we were looking for.[18]

[17] Here we are also ignoring the fact that we do not prove the endpoint L^2-L^6 estimate for the model E_1. It will not compromise this preliminary exposition.

[18] This bound on $\tilde{\tilde{\Lambda}}_{2,2}$ is of course informal, which is why we wrote "$\lesssim$". Observe that we also removed the sum in $\vec{l}$; it contributes with a term that depends on ε in the formal argument. Later in the text we will see why we can assume $\vec{r},\vec{s},k\ge 0$ in the sum above.

9B. ***The general argument.*** Roughly, this is a one-paragraph outline of the proof: we split the sum in (35) into certain level sets, find good upper bounds for how many points $(\vec{n}, m)$ are in each level set using the weak transversality and Strichartz information, and then average all this data appropriately.

First we will prove the bound

$$\| \mathrm{ME}_{k,d}^{\frac{1}{k}}(g)\|_{L^{2(d+k+1)/(d+k-1)+\varepsilon}(\mathbb{R}^{d+1})} \lesssim_\varepsilon \prod_{l=1}^{k} |E_{1,l}|^{\frac{1}{2k}} \cdot \prod_{j=2}^{k} |E_j|^{\frac{1}{2k}} \tag{45}$$

for every $\varepsilon > 0$. As we remarked at the end of Section 4, this is the restricted weak-type bound that will be proved directly; all the other ones that are necessary for multilinear interpolation can be proved in a similar way, as the reader will notice.

We will define several level sets that encode the sizes of many quantities that will play a role in the proof. We start with the ones involving the scalar products in the multilinear form above:

$$\mathbb{A}_j^{l_j} := \{(\vec{n}, m) \in \mathbb{Z}^{d+1} : |\langle g_j, \varphi^j_{\vec{n},m}\rangle| \approx 2^{-l_j}\}, \quad 1 \le j \le k.$$

The sizes of the $\langle g_j, \varphi_{\vec{n},m}\rangle$ are not the only information that we will need to control. As in the previous subsection, some mixed-norm quantities appear naturally after using Bessel's inequality along certain directions, and we will need to capture these as well:

$$\begin{aligned}
\mathbb{B}_{i,1}^{r_{i,1}} &:= \{(n_i, m) \in \mathbb{Z}^2 : \|\langle g_1, \varphi^{i,1}_{n_i,m}\rangle_{x_i}\|_2 \approx 2^{-r_{i,1}}\}, && 1 \le i \le k-1,\\
\mathbb{B}_{i,i+1}^{r_{i,i+1}} &:= \{(n_i, m) \in \mathbb{Z}^2 : \|\langle g_{i+1}, \varphi^{i,i+1}_{n_i,m}\rangle_{x_i}\|_2 \approx 2^{-r_{i,i+1}}\}, && 1 \le i \le k-1,\\
\mathbb{B}_{k,j}^{r_{k,j}} &:= \{(\vec{n}_k, m) \in \mathbb{Z}^{d-k+2} : \|\langle g_j, \varphi^{k,j}_{\vec{n}_k,m}\rangle_{\vec{x}_k}\|_2 \approx 2^{-r_{k,j}}\}, && 1 \le j \le k.
\end{aligned}$$

Set $\mathbb{B}_{i,j}^{r_{i,j}} := \varnothing$ for any other pair (i, j) not included in the above definitions. Observe that g_1 (the function that has a tensor structure) has k sets $\mathbb{B}$ associated to it: $k-1$ sets $\mathbb{B}_{i,1}^{r_{i,1}}$ and one set $\mathbb{B}_{k,1}^{r_{k,1}}$. The other functions g_j, $j \ne 1$, have only two: one set $\mathbb{B}_{j-1,j}^{r_{j-1,j}}$ and one set $\mathbb{B}_{k,j}^{r_{k,j}}$ for each $1 \le j \le k$. The idea behind the sets $\mathbb{B}_{i,1}^{r_{i,1}}$ and $\mathbb{B}_{i,i+1}^{r_{i,i+1}}$ is to isolate the "piece" of each function that encodes the weak transversality information from the part that captures the Strichartz/Tomas–Stein behavior, which is in the set $\mathbb{B}_{k,j}^{r_{k,j}}$. For each $1 \le i \le k-1$, we will pair the information of the sets $\mathbb{B}_{i,1}^{r_{i,1}}$ and $\mathbb{B}_{i,i+1}^{r_{i,i+1}}$ and use Proposition 4.4 to extract the gain yielded by weak transversality. The information contained in the sets $\mathbb{B}_{k,j}^{r_{k,j}}$ will be exploited via Corollary 4.2.

The last quantity we have to control is the one arising from the dualizing function h:

$$\mathbb{C}^t := \{(\vec{n}, m) \in \mathbb{Z}^{d+1} : |\langle h, \chi_{\vec{n}} \otimes \chi_m\rangle| \approx 2^{-t}\}.$$

In order to prove some crucial bounds, at some point we will have to isolate the previous information for only one of the functions g_j. This will be done in terms of the following set:[19]

$$\mathbb{X}^{l_j; r_{i,j}} = \mathbb{A}_j^{l_j} \cap \{(\vec{n}, m) \in \mathbb{Z}^{d+1} : (n_i, m) \in \mathbb{B}_{i,j}^{r_{i,j}}\}.$$

[19]Many of these sets are empty since we set $\mathbb{B}_{i,j}^{r_{i,j}} = \varnothing$ for most (i, j), but only the nonempty ones will appear in the argument.

In other words, $\mathbb{X}^{l_j;r_{i,j}}$ contains all the $(n_1,\dots,n_d,m)$ whose corresponding scalar product $\langle g_j,\varphi_{\vec{\boldsymbol{n}},m}\rangle$ has size about 2^{-l_j} and with (n_i,m) being such that $\|\langle g_j,\varphi^{i,j}_{n_i,m}\rangle_{x_i}\|_2$ has size about $2^{-r_{i,j}}$.

Finally, it will also be important to encode all the previous information into one single set. This will be done with

$$\mathbb{X}^{\vec{l},R,t} := \bigcap_{1\le j\le k} \mathbb{A}_j^{l_j} \cap \left\{ (\vec{\boldsymbol{n}},m)\in\mathbb{Z}^{d+1} : (n_i,m)\in\bigcap_j \mathbb{B}^{r_{i,j}}_{i,j},\ 1\le i\le d \right\} \cap \mathbb{C}^t,$$

where we are using the abbreviations $\vec{l}=(l_1,\dots,l_k)$ and $R:=(r_{i,j})_{i,j}$. Hence we can bound the form $\tilde{\tilde{\Lambda}}_{k,d}$ as follows:

$$|\tilde{\tilde{\Lambda}}_{k,d}(g,h)| \lesssim \sum_{\vec{l},R,t\ge 0} 2^{-t}\prod_{j=1}^{k} 2^{-\frac{l_j}{k}}\#\mathbb{X}^{\vec{l},R,t}. \tag{46}$$

Observe that we are assuming without loss of generality that $l_j, r_{i,j}, t\ge 0$. Indeed,

$$2^{-l_j}\lesssim |\langle g_j,\varphi^j_{\vec{\boldsymbol{n}},m}\rangle| \le \|g_j\|_\infty\cdot\|\varphi\|_1\lesssim 1,$$

so l_j is at least as big as a universal integer. The argument for the remaining indices is the same.

The following two lemmas play a crucial role in the argument by relating the scalar and mixed-norm quantities involved in the stopping-time above. Lemma 9.4 allows us to do that for the quantities associated to g_1, the function that has a tensor structure. We remark that this is the only place in the proof where the tensor structure is used.

Lemma 9.4. *If* $\mathbb{X}^{\vec{l},R,t}\neq\varnothing$, *then*

$$2^{-l_1}\approx\frac{2^{-r_{1,1}}\cdot\dots\cdot 2^{-r_{k,1}}}{\|g_1\|_2^{k-1}},$$

Proof. Observe that

$$\begin{aligned}2^{-r_{1,1}}\cdot\dots\cdot 2^{-r_{k,1}} &\approx \prod_{i=1}^k \|\langle g_1,\varphi^{i,1}_{n_i,m}\rangle_{x_i}\|_2 = \prod_{i=1}^k \|\langle g_{1,1}\otimes\dots\otimes g_{1,k},\varphi^{i,1}_{n_i,m}\rangle_{x_i}\|_2\\ &= \prod_{i=1}^k |\langle g_{1,i},\varphi^{i,1}_{n_i,m}\rangle_{x_i}|\cdot\|g_{1,1}\otimes\dots\otimes\hat{g}_{1,i}\otimes\dots\otimes g_{1,k}\|_2\\ &= |\langle g_1,\varphi^1_{\vec{\boldsymbol{n}},m}\rangle|\cdot\|g_1\|_2^{k-1}\approx 2^{-l_1}\cdot\|g_1\|_2^{k-1},\end{aligned}$$

and this proves the lemma. □

Lemma 9.5 gives us an alternative way of relating the quantities previously defined for the generic functions $g_2,\dots,g_k$.

Lemma 9.5. *If* $\mathbb{X}^{\vec{l},R,t}\neq\varnothing$, *then*

$$2^{-l_{i+1}}\lesssim\frac{2^{-r_{i,i+1}}}{\|\mathbb{1}_{\mathbb{X}^{l_{i+1};r_{i,i+1}}}\|^{\frac12}_{\ell^\infty_{n_i,m}\ell^1_{\hat{n}_i}}}, \tag{47}$$

$$2^{-l_{i+1}}\lesssim\frac{2^{-r_{k,i+1}}}{\|\mathbb{1}_{\mathbb{X}^{l_{i+1};r_{k,i+1}}}\|^{\frac12}_{\ell^\infty_{\vec{n}_k,m}\ell^1_{\hat{\hat{n}}_k}}} \tag{48}$$

for all $1\le i\le k-1$.

Proof. Inequality (47) is a consequence of orthogonality: for a fixed (n_i, m), define

$$\mathbb{X}_{(n_i,m)}^{l_{i+1};r_{i,i+1}} := \{\hat{n}_i : (\vec{\boldsymbol{n}}, m) \in \mathbb{X}^{l_{i+1},r_{i,i+1}}\}.$$

This way,

$$\begin{aligned}
\#\mathbb{X}_{(n_i,m)}^{l_{i+1};r_{i,i+1}} &\approx 2^{2l_{i+1}} \sum_{\hat{n}_i \in \mathbb{X}_{(n_i,m)}^{l_{i+1};r_{i,i+1}}} |\langle g_{i+1}, \varphi^{i+1}_{\vec{\boldsymbol{n}},m}\rangle|^2 \\
&\leq 2^{2l_{i+1}} \sum_{\hat{n}_i} \left| \int \langle g_{i+1}, \varphi^{i,i+1}_{n_i,m}\rangle_{x_i} \cdot e^{-2\pi i m (\sum_{j\neq i} x_j^2)} \cdot \prod_{j \neq i} e^{-2\pi i n_j x_j} \, \mathrm{d}\hat{x}_i \right|^2 \\
&\leq 2^{2l_{i+1}} \int \| \langle g_{i+1}, \varphi^{i,i+1}_{n_i,m}\rangle_{x_i}|^2 \, \mathrm{d}\hat{x}_i \\
&\approx 2^{2l_{i+1}} \cdot 2^{-2r_{i,i+1}},
\end{aligned}$$

where we used Bessel's inequality from the second to the third line. The lemma follows by taking the supremum in (n_i, m). Equation (48) is proven analogously. □

The following corollary gives a convex combination of the relations in Lemma 9.5 that will be used in the proof.

Corollary 9.6. *For* $1 \leq i \leq k-1$ *we have*

$$2^{-l_{i+1}} \lesssim \frac{2^{-\frac{2k}{d+k+1}\cdot r_{i,i+1}}}{\|\mathbb{1}_{\mathbb{X}^{l_{i+1};r_{i,i+1}}}\|^{\frac{k}{d+k+1}}_{\ell^\infty_{n_i,m}\ell^1_{\hat{n}_i}}} \cdot \frac{2^{-\frac{(d-k+1)}{(d+k+1)}\cdot r_{k,i+1}}}{\|\mathbb{1}_{\mathbb{X}^{l_{i+1};r_{k,i+1}}}\|^{\frac{(d-k+1)}{2(d+k+1)}}_{\ell^\infty_{\vec{\boldsymbol{n}}_k,m}\ell^1_{\hat{\hat{n}}_k}}}.$$

Proof. Interpolate between the bounds of Lemma 9.5 with weights

$$\frac{2k}{d+k+1} \quad \text{and} \quad \frac{d-k+1}{d+k+1},$$

respectively. □

We now concentrate on estimating the right-hand side of (46) by finding good bounds for $\#\mathbb{X}^{\vec{l},R,t}$. The following bound follows immediately from the disjointness of the supports of $\chi_{\vec{\boldsymbol{n}}} \otimes \chi_m$:

$$\#\mathbb{X}^{\vec{l},R,t} \lesssim \sum_{(\vec{\boldsymbol{n}},m)\in\mathbb{Z}^{d+1}} |\langle h, \chi_{\vec{\boldsymbol{n}}} \otimes \chi_m\rangle| \lesssim 2^t |F|. \tag{49}$$

By definition of the set $\mathbb{X}^{\vec{l},R,t}$,

$$\#\mathbb{X}^{\vec{l},R,t} \leq \sum_{(\vec{\boldsymbol{n}},m)\in\mathbb{Z}^{d+1}} \prod_{j=1}^{k} \mathbb{1}_{\mathbb{A}_j^{l_j}}(\vec{\boldsymbol{n}}, m) \cdot \prod_{i,j,\ \mathbb{B}_{i,j}^{r_{i,j}} \neq \varnothing} \mathbb{1}_{\mathbb{B}_{i,j}^{r_{i,j}}}(n_i, m). \tag{50}$$

We will manipulate (50) in k different ways: $k-1$ of them will exploit orthogonality (through the one-dimensional bilinear theory after combining the sets $\mathbb{B}_{i,1}^{r_{i,1}}$ and $\mathbb{B}_{i,i+1}^{r_{i,i+1}}$, $1 \leq i \leq k-1$) and the last one will reflect Strichartz/Tomas–Stein in an appropriate dimension. The following lemma gives us estimates for the cardinality of $\mathbb{X}^{\vec{l},R,t}$ based on the sizes of some of its slices along canonical directions.[20]

Lemma 9.7. *The bounds above imply*:

(a) *The orthogonality-type bounds*:[21]

$$\#\mathbb{X}^{\vec{l},R,t} \lesssim \|\mathbb{1}_{\mathbb{X}^{l_{i+1};r_{i,i+1}}}\|_{\ell^{\infty}_{n_i,m}\ell^1_{\hat{n}_i}} \cdot 2^{2r_{i,1}+2r_{i,i+1}} \cdot \|g_1\|_2^2 \cdot \|g_{i+1}\|_2^2, \quad 1 \le i \le k-1. \tag{51}$$

(b) *The Strichartz-type bound*:

$$\#\mathbb{X}^{\vec{l},R,t} \lesssim \prod_{j=2}^{k} \|\mathbb{1}_{\mathbb{X}^{l_j;r_{k,j}}}\|^{\frac{1}{k}}_{\ell^{\infty}_{\vec{n}_k,m}\ell^1_{\hat{\hat{n}}_k}} \cdot 2^{\frac{2}{k}\sum_{i=1}^{k-1} r_{i,1}} \cdot \|g_1\|_2^{\frac{2(k-1)}{k}} \cdot 2^{\alpha\cdot r_{k,1}+\sum_{l=2}^{k}\beta\cdot r_{k,l}} \cdot \|g_1\|_2^{\alpha} \cdot \prod_{l=2}^{k} \|g_l\|_2^{\beta}, \tag{52}$$

where

$$\alpha := \frac{2(d+k+1)}{k(d-k+1)} + \delta\cdot\frac{(d+k+1)}{k(d-k+3)},$$
$$\beta := \frac{2}{k} + \tilde{\delta}\cdot\frac{(d-k+1)}{k(d-k+3)},$$

with $\delta, \tilde{\delta} > 0$ being arbitrarily small parameters to be chosen later.[22]

Proof. For each $1 \le i \le k-1$ we bound most of the indicator functions in (50) by 1 and obtain

$$\begin{aligned}
\#\mathbb{X}^{\vec{l},R,t} &\le \sum_{(\vec{n},m)\in\mathbb{Z}^{d+1}} \mathbb{1}_{\mathbb{A}^{l_{i+1}}_{i+1}}(\vec{n},m)\cdot\mathbb{1}_{\mathbb{B}^{r_{i,1}}_{i,1}}(n_i,m)\cdot\mathbb{1}_{\mathbb{B}^{r_{i,i+1}}_{i,i+1}}(n_i,m)\\
&= \sum_{(\vec{n},m)\in\mathbb{Z}^{d+1}} \mathbb{1}_{\mathbb{X}^{l_{i+1};r_{i,i+1}}}(\vec{n},m)\cdot\mathbb{1}_{\mathbb{B}^{r_{i,1}}_{i,1}\cap\mathbb{B}^{r_{i,i+1}}_{i,i+1}}(n_i,m)\\
&= \sum_{n_i,m} \mathbb{1}_{\mathbb{B}^{r_{i,1}}_{i,1}\cap\mathbb{B}^{r_{i,i+1}}_{i,i+1}}(n_i,m)\sum_{\hat{n}_i}\mathbb{1}_{\mathbb{X}^{l_{i+1};r_{i,i+1}}}(\vec{n},m)\\
&\le \|\mathbb{1}_{\mathbb{X}^{l_{i+1};r_{i,i+1}}}\|_{\ell^{\infty}_{n_i,m}\ell^1_{\hat{n}_i}}\cdot\|\mathbb{1}_{\mathbb{B}^{r_{i,1}}_{i,1}\cap\mathbb{B}^{r_{i,i+1}}_{i,i+1}}\|_{\ell^1_{n_i,m}}.
\end{aligned} \tag{53}$$

Transversality is exploited now: the cube Q_1 with $\{e_1,\dots,e_{k-1}\}$ as associated set of directions satisfies (15), which allows us to apply Proposition 4.4 for each $1 \le i \le k-1$ since weak transversality is equivalent to transversality in dimension $d = 1$. By definition of the sets $\mathbb{B}^{r_{i,1}}_{i,1}$ and $\mathbb{B}^{r_{i,i+1}}_{i,i+1}$, Fubini and Proposition 4.4

[20] The reader may associate this idea to certain discrete Loomis–Whitney or Brascamp–Lieb inequalities. While reducing matters to lower-dimensional theory is at the core of our paper, we do not yet have a genuine "Brascamp–Lieb way" of bounding $\#\mathbb{X}^{\vec{l},R,t}$ for which our methods work. For instance, no "slice" of $\mathbb{X}^{\vec{l},R,t}$ given by fixing a few (or all) n_j and summing over m appears in our estimates, which breaks the Loomis–Whitney symmetry.

[21] Weak transversality enters the picture here.

[22] One should think of δ and $\tilde{\delta}$ as being "morally zero". They will be chosen as a function of the initially given $\varepsilon > 0$, and the only reason we introduce them is to make the appropriate up to the endpoint Strichartz exponent appear in (56). The main terms of α and β are also chosen with that in mind.

we have

$$\begin{aligned}\|\mathbb{1}_{\mathbb{B}_{i,1}^{r_{i,1}}\cap\mathbb{B}_{i,i+1}^{r_{i,i+1}}}\|_{\ell^1_{n_i,m}} &\lesssim 2^{2r_{i,1}+2r_{i,i+1}}\sum_{(n_i,m)\in\mathbb{B}_{i,1}^{r_{i,1}}\cap\mathbb{B}_{i,i+1}^{r_{i,i+1}}}\|\langle g_1,\varphi^{i,1}_{n_i,m}\rangle_{x_i}\|_2^2\cdot\|\langle g_{i+1},\varphi^{i,i+1}_{n_i,m}\rangle_{y_i}\|_2^2\\ &\le 2^{2r_{i,1}+2r_{i,i+1}}\iint\Big(\sum_{(n_i,m)\in\mathbb{Z}^2}|\langle g_1,\varphi^{i,1}_{n_i,m}\rangle_{x_i}|^2\cdot|\langle g_{i+1},\varphi^{i,i+1}_{n_i,m}\rangle_{y_i}|^2\Big)\,\mathrm{d}\hat{x}_i\,\mathrm{d}\hat{y}_i\\ &\le 2^{2r_{i,1}+2r_{i,i+1}}\int\int\|g_1\|^2_{L^2_{x_i}}\cdot\|g_{i+1}\|^2_{L^2_{y_i}}\,\mathrm{d}\hat{x}_i\,\mathrm{d}\hat{y}_i\\ &= 2^{2r_{i,1}+2r_{i,i+1}}\cdot\|g_1\|_2^2\cdot\|g_{i+1}\|_2^2.\end{aligned}$$

Using this in (53) gives (a). As for (b), bound $\#\mathbb{X}^{\vec{l},R,t}$ as follows:

$$\begin{aligned}\#\mathbb{X}^{\vec{l},R,t} &= \sum_{(\vec{\boldsymbol{n}},m)\in\mathbb{Z}^{d+1}}\mathbb{1}_{\mathbb{X}^{\vec{l},R,t}}(\vec{\boldsymbol{n}},m)\\ &\le \sum_{(\vec{\boldsymbol{n}},m)\in\mathbb{Z}^{d+1}}\prod_{j=2}^{k}\mathbb{1}_{\mathbb{X}^{l_j;r_{k,j}}}(\vec{\boldsymbol{n}},m)\prod_{i=1}^{k-1}\mathbb{1}_{\mathbb{B}_{i,1}^{r_{i,1}}}(n_i,m)\cdot\prod_{l=1}^{k}\mathbb{1}_{\mathbb{B}_{k,l}^{r_{k,l}}}(\vec{\boldsymbol{n}}_k,m)\\ &= \sum_{\vec{\boldsymbol{n}}_k,m}\prod_{l=1}^{k}\mathbb{1}_{\mathbb{B}_{k,l}^{r_{k,l}}}(\vec{\boldsymbol{n}}_k,m)\sum_{n_1,\dots,n_{k-1}}\prod_{j=2}^{k}\mathbb{1}_{\mathbb{X}^{l_j;r_{k,j}}}(\vec{\boldsymbol{n}},m)\prod_{i=1}^{k-1}\mathbb{1}_{\mathbb{B}_{i,1}^{r_{i,1}}}(n_i,m)\\ &\le \sum_{\vec{\boldsymbol{n}}_k,m}\prod_{l=1}^{k}\mathbb{1}_{\mathbb{B}_{k,l}^{r_{k,l}}}(\vec{\boldsymbol{n}}_k,m)\prod_{j=2}^{k}\|\mathbb{1}_{\mathbb{X}^{l_j;r_{k,j}}}(\vec{\boldsymbol{n}},m)\|^{\frac{1}{k}}_{\ell^1_{\hat{\vec{n}}_k}}\cdot\Big\|\prod_{i=1}^{k-1}\mathbb{1}_{\mathbb{B}_{i,1}^{r_{i,1}}}(n_i,m)\Big\|^{\frac{1}{k}}_{\ell^1_{\hat{\vec{n}}_k}}\\ &\le \prod_{j=2}^{k}\|\mathbb{1}_{\mathbb{X}^{l_j;r_{k,j}}}\|^{\frac{1}{k}}_{\ell^\infty_{\vec{n}_k,m}\ell^1_{\hat{\vec{n}}_k}}\cdot\prod_{i=1}^{k-1}\|\mathbb{1}_{\mathbb{B}_{i,1}^{r_{i,1}}}\|^{\frac{1}{k}}_{\ell^\infty_m\ell^1_{n_i}}\cdot\Big\|\prod_{l=1}^{k}\mathbb{1}_{\mathbb{B}_{k,l}^{r_{k,l}}}\Big\|_{\ell^1_{\vec{n}_k,m}},\end{aligned}\tag{54}$$

where we used Hölder's inequality from the third to fourth line. Next, notice that

$$\begin{aligned}\|\mathbb{1}_{\mathbb{B}_{i,1}^{r_{i,1}}}\|_{\ell^\infty_m\ell^1_{n_i}} &\lesssim \sup_m 2^{2r_{i,1}}\sum_{n_i}\|\langle g_1,\varphi^{i,1}_{n_i,m}\rangle_{x_i}\|_2^2\\ &= \sup_m 2^{2r_{i,1}}\int\sum_{n_i}|\langle g_1,\varphi^{i,1}_{n_i,m}\rangle_{x_i}|^2\,\mathrm{d}\hat{x}_i\\ &\lesssim 2^{2r_{i,1}}\cdot\|g_1\|_2^2\end{aligned}\tag{55}$$

by orthogonality. Now let

$$p_{k,1} := \frac{k(d-k+3)}{(d+k+1)},\qquad p_{k,l} := \frac{k(d-k+3)}{(d-k+1)}\quad\text{for all } 2\le l\le k$$

and notice that

$$\sum_{l=1}^{k}\frac{1}{p_{k,l}} = 1.$$

This way, by definition of $\mathbb{B}_{k,l}^{r_{k,l}}$ and by Hölder's inequality with these $p_{k,l}$ we have

$$
\begin{aligned}
&\Big\|\prod_{l=1}^{k}\mathbb{1}_{\mathbb{B}_{k,l}^{r_{k,l}}}\Big\|_{\ell^1_{\vec{n}_k,m}}\\
&\lesssim 2^{\alpha\cdot r_{k,1}+\sum_{l=2}^{k}\beta\cdot r_{k,l}}\sum_{(\vec{n}_k,m)}\|\langle g_1,\varphi_{\vec{n}_k,m}^{k,1}\rangle_{\vec{x}_k}\|_2^{\alpha}\cdot\prod_{l=2}^{k}\|\langle g_l,\varphi_{\vec{n}_k,m}^{k,l}\rangle_{\vec{x}_k}\|_2^{\beta}\\
&\le 2^{\alpha\cdot r_{k,1}+\sum_{l=2}^{k}\beta\cdot r_{k,l}}\Big(\sum_{(\vec{n}_k,m)}\|\langle g_1,\varphi_{\vec{n}_k,m}^{k,1}\rangle_{\vec{x}_k}\|_2^{\alpha\cdot p_{k,1}}\Big)^{\frac{1}{p_{k,1}}}\cdot\prod_{l=2}^{k}\Big(\sum_{(\vec{n}_k,m)}\|\langle g_l,\varphi_{\vec{n}_k,m}^{k,l}\rangle_{\vec{x}_k}\|_2^{\beta\cdot p_{k,l}}\Big)^{\frac{1}{p_{k,l}}}\\
&= 2^{\alpha\cdot r_{k,1}+\sum_{l=2}^{k}\beta\cdot r_{k,l}}\Big(\sum_{(\vec{n}_k,m)}\|\langle g_1,\varphi_{\vec{n}_k,m}^{k,1}\rangle_{\vec{x}_k}\|_2^{\frac{2(d-k+3)}{(d-k+1)}+\delta}\Big)^{\frac{1}{p_{k,1}}}\cdot\prod_{l=2}^{k}\Big(\sum_{(\vec{n}_k,m)}\|\langle g_l,\varphi_{\vec{n}_k,m}^{k,l}\rangle_{\vec{x}_k}\|_2^{\frac{2(d-k+3)}{(d-k+1)}+\tilde{\delta}}\Big)^{\frac{1}{p_{k,l}}}\\
&\le 2^{\alpha\cdot r_{k,1}+\sum_{l=2}^{k}\beta\cdot r_{k,l}}\cdot\|g_1\|_2^{\alpha}\cdot\prod_{l=2}^{k}\|g_l\|_2^{\beta},
\end{aligned}
\tag{56}
$$

by the up to the endpoint mixed-norm Strichartz bound in Corollary 4.2.[23] Using (55) and (56) in (54) yields (b). □

Given $\varepsilon>0$ small,[24] we interpolate between $k+1$ bounds for $\#\mathbb{X}^{\vec{l},R,t}$ with the following weights:[25]

$$
\begin{cases}
\theta_l=\dfrac{1}{d+k+1}-\dfrac{\varepsilon}{k}, \quad 1\le l\le k-1, & \text{for (51)},\\
\theta_k=\dfrac{(d-k+1)}{2(d+k+1)}-\dfrac{\varepsilon}{k} & \text{for (52)},\\
\theta_{k+1}=\Big[1-\dfrac{(d+k-1)}{2(d+k+1)}\Big]+\varepsilon & \text{for (49)},
\end{cases}
$$

which leads to

$$
\begin{aligned}
&|\tilde{\tilde{\Lambda}}_{k,d}(g,h)|\\
&\lesssim\sum_{\vec{l},R,t\ge 0}2^{-t}\times\prod_{j=1}^{k}2^{-\frac{l_j}{k}}\times\prod_{l=1}^{k-1}\Big(\|\mathbb{1}_{\mathbb{X}^{l_{l+1};r_{l,l+1}}}\|_{\ell^\infty_{\hat{n}_l,m}\ell^1_{\hat{n}_l}}\cdot 2^{2r_{l,1}+2r_{l,l+1}}\cdot\|g_1\|_2^2\cdot\|g_{l+1}\|_2^2\Big)^{\frac{1}{d+k+1}-\frac{\varepsilon}{k}}\\
&\times\Big(\prod_{j=2}^{k}\|\mathbb{1}_{\mathbb{X}^{l_j;r_{k,j}}}\|^{\frac{1}{k}}_{\ell^\infty_{\vec{n}_k,m}\ell^1_{\hat{\hat{n}}_k}}\cdot 2^{\frac{2}{k}\sum_{i=1}^{k-1}r_{i,1}}\cdot\|g_1\|_2^{\frac{2(k-1)}{k}}\cdot 2^{\alpha\cdot r_{k,1}+\sum_{l=2}^{k}\beta\cdot r_{k,l}}\cdot\|g_1\|_2^{\alpha}\cdot\prod_{l=2}^{k}\|g_l\|_2^{\beta}\Big)^{\frac{(d-k+1)}{2(d+k+1)}-\frac{\varepsilon}{k}}\\
&\times(2^t|F|)^{[1-\frac{(d+k-1)}{2(d+k+1)}]+\varepsilon},
\end{aligned}
$$

[23] See the footnote related to Corollary 4.2.

[24] Perhaps it is helpful for the reader to think of ε, δ and $\tilde{\delta}$ as equal to zero to focus on the important parts of the proof. The presence of these parameters here is a mere technicality, except of course for the fact that $\varepsilon>0$ makes us lose the endpoint in this case.

[25] Observe that $\sum_{l=1}^{k+1}\theta_l=1$. These weights are chosen so that the correct powers of the measures $|E_j|$ and $|F|$ appear in (58).

Using Lemma 9.4 and Corollary 9.6 to bound the 2^{-l_j} in the form $\tilde{\tilde{\Lambda}}_{k,d}$ yields

$$
\begin{aligned}
&|\tilde{\tilde{\Lambda}}_{k,d}(g,h)| \\
&\lesssim \sum_{\vec{l},R,t\geq 0} 2^{-t}\times 2^{-\frac{\varepsilon}{k^2}l_1}\times\Big(\frac{1}{\|g_1\|_2^{k-1}}\prod_{j=1}^{k}2^{-r_{j,1}}\Big)^{\frac{1}{k}-\frac{\varepsilon}{k^2}} \\
&\times\prod_{i=1}^{k-1}2^{-\frac{\varepsilon}{k^2}l_{i+1}}\times\prod_{i=1}^{k-1}\Bigg[\frac{2^{-\frac{2}{d+k+1}\cdot r_{i,i+1}}}{\|\mathbb{1}_{\mathbb{X}^{l_{i+1};r_{i,i+1}}}\|_{\ell^\infty_{n_i,m}\ell^1_{\hat{n}_i}}^{\frac{1}{d+k+1}}}\cdot\frac{2^{-\frac{(d-k+1)}{k(d+k+1)}\cdot r_{k,i+1}}}{\|\mathbb{1}_{\mathbb{X}^{l_{i+1};r_{k,i+1}}}\|_{\ell^\infty_{\vec{n}_k,m}\ell^1_{\hat{\vec{n}}_k}}^{\frac{(d-k+1)}{2k(d+k+1)}}}\Bigg]^{1-\frac{\varepsilon}{k}} \\
&\times\prod_{l=1}^{k-1}\big(\|\mathbb{1}_{\mathbb{X}^{l_{l+1};r_{l,l+1}}}\|_{\ell^\infty_{n_l,m}\ell^1_{\hat{n}_l}}\cdot 2^{2r_{l,1}+2r_{l,l+1}}\cdot\|g_1\|_2^2\cdot\|g_{l+1}\|_2^2\big)^{\frac{1}{d+k+1}-\frac{\varepsilon}{k}} \\
&\times\Bigg(\prod_{j=2}^{k}\|\mathbb{1}_{\mathbb{X}^{l_j;r_{k,j}}}\|_{\ell^\infty_{\vec{n}_k,m}\ell^1_{\hat{\vec{n}}_k}}^{\frac{1}{k}}\cdot 2^{\frac{2}{k}\sum_{i=1}^{k-1}r_{i,1}}\cdot\|g_1\|_2^{\frac{2(k-1)}{k}}\cdot 2^{\alpha\cdot r_{k,1}+\sum_{l=2}^{k}\beta\cdot r_{k,l}}\cdot\|g_1\|_2^{\alpha}\cdot\prod_{l=2}^{k}\|g_l\|_2^{\beta}\Bigg)^{\frac{(d-k+1)}{2(d+k+1)}-\frac{\varepsilon}{k}} \\
&\times(2^t|F|)^{[1-\frac{(d+k-1)}{2(d+k+1)}]+\varepsilon},
\end{aligned}
$$

Developing the expression above,

$$
\begin{aligned}
|\tilde{\tilde{\Lambda}}_{k,d}(g,h)| \lesssim &\sum_{\vec{l},R,t\geq 0} 2^{-t}\times 2^{-\frac{\varepsilon}{k^2}l_1}\times\Big(\prod_{j=1}^{k}2^{-r_{j,1}}\Big)^{\frac{1}{k}-\frac{\varepsilon}{k^2}}\cdot\|g_1\|_2^{\frac{(k-1)}{k^2}\varepsilon-\frac{(k-1)}{k}} \\
&\times\prod_{i=1}^{k-1}2^{-\frac{\varepsilon}{k^2}l_{i+1}}\times\prod_{i=1}^{k-1}[2^{-\frac{2}{d+k+1}\cdot r_{i,i+1}}\cdot 2^{-\frac{(d-k+1)}{k(d+k+1)}\cdot r_{k,i+1}}]^{1-\frac{\varepsilon}{k}} \\
&\times\prod_{i=1}^{k-1}\Big[\|\mathbb{1}_{\mathbb{X}^{l_{i+1};r_{i,i+1}}}\|_{\ell^\infty_{n_i,m}\ell^1_{\hat{n}_i}}^{\frac{1}{d+k+1}\cdot(\frac{\varepsilon}{k}-1)}\cdot\|\mathbb{1}_{\mathbb{X}^{l_{i+1};r_{k,i+1}}}\|_{\ell^\infty_{\vec{n}_k,m}\ell^1_{\hat{\vec{n}}_k}}^{\frac{(d-k+1)}{2k(d+k+1)}\cdot(\frac{\varepsilon}{k}-1)}\Big] \\
&\times\Bigg[\prod_{l=1}^{k-1}\|\mathbb{1}_{\mathbb{X}^{l_{l+1};r_{l,l+1}}}\|_{\ell^\infty_{n_l,m}\ell^1_{\hat{n}_l}}^{\frac{1}{d+k+1}-\frac{\varepsilon}{k}}\Bigg]\times\Bigg[\prod_{l=1}^{k-1}(2^{r_{l,1}+r_{l,l+1}})^{\frac{2}{d+k+1}-\frac{2\varepsilon}{k}}\Bigg] \\
&\times\|g_1\|_2^{\frac{2(k-1)}{d+k+1}-\frac{2(k-1)\varepsilon}{k}}\cdot\prod_{l=1}^{k-1}\|g_{l+1}\|_2^{\frac{2}{d+k+1}-\frac{2\varepsilon}{k}} \\
&\times\prod_{j=2}^{k}\|\mathbb{1}_{\mathbb{X}^{l_j;r_{k,j}}}\|_{\ell^\infty_{\vec{n}_k,m}\ell^1_{\hat{\vec{n}}_k}}^{\frac{1}{k}\cdot(\frac{(d-k+1)}{2(d+k+1)}-\frac{\varepsilon}{k})}\cdot(2^{\frac{2}{k}\sum_{i=1}^{k-1}r_{i,1}}\cdot 2^{\alpha\cdot r_{k,1}+\sum_{l=2}^{k}\beta\cdot r_{k,l}})^{\frac{(d-k+1)}{2(d+k+1)}-\frac{\varepsilon}{k}} \\
&\times\|g_1\|_2^{(\frac{2(k-1)}{k}+\alpha)\cdot(\frac{(d-k+1)}{2(d+k+1)}-\frac{\varepsilon}{k})}\cdot\prod_{l=2}^{k}\|g_l\|_2^{\beta\cdot(\frac{(d-k+1)}{2(d+k+1)}-\frac{\varepsilon}{k})} \\
&\times(2^t|F|)^{[1-\frac{(d+k-1)}{2(d+k+1)}]+\varepsilon}.
\end{aligned}
$$

At this point we set the values of δ and $\tilde{\delta}$ (as functions of ε) to be such that[26]

$$
\delta\cdot\Big[\frac{(d-k+1)}{k(d-k+3)}-\frac{(d+k+1)\varepsilon}{k^2(d-k+3)}\Big]=\frac{1}{2}\Big[-\frac{\varepsilon}{k^2}+\frac{2(d+k+1)\varepsilon}{k^2(d-k+1)}\Big],
$$

$$
\tilde{\delta}\cdot\Big[\frac{(d-k+1)^2}{2k(d+k+1)(d-k+3)}-\frac{(d-k+1)\varepsilon}{k^2(d-k+3)}\Big]=\frac{1}{2}\Big[\frac{2\varepsilon}{k^2}-\frac{(d-k+1)\varepsilon}{k^2(d+k+1)}\Big].
$$

[26]We emphasize that these particular choices are just for computational convenience, and we have not developed the expressions because this is exactly how we use them to simplify the previous calculations.

Simplifying (and using the expressions that define α and β in Lemma 9.7),

$$
\begin{aligned}
|\tilde{\tilde{\Lambda}}_{k,d}(g,h)| \lesssim & \Big[\sum_{l_1\geq 0} 2^{-\frac{\varepsilon}{k^2}l_1}\Big]\times\Big[\prod_{j=1}^{k-1}\Big(\sum_{r_{j,1}\geq 0} 2^{-(\frac{2\varepsilon}{k}+\frac{\varepsilon}{k^2})r_{j,1}}\Big)\Big]\cdot\Big[\sum_{r_{k,1}\geq 0} 2^{-r_{k,1}(-\frac{\varepsilon}{2k^2}+\frac{(d+k+1)}{(d-k+1)}\frac{\varepsilon}{k^2})}\Big] \\
&\times\Big[\prod_{i=1}^{k-1}\Big(\sum_{l_{i+1}\geq 0} 2^{-\frac{\varepsilon}{k^2}l_{i+1}}\Big)\Big]\times\Big[\sum_{t\geq 0} 2^{-t(\frac{(d+k-1)}{2(d+k+1)}-\varepsilon)}\Big] \\
&\times\Big[\prod_{i=1}^{k-1}\Big(\sum_{r_{i,i+1}\geq 0} 2^{-\frac{2\varepsilon}{k}(1-\frac{1}{d+k+1})r_{i,i+1}}\Big)\Big]\times\Big[\prod_{i=1}^{k-1}\Big(\sum_{r_{k,i+1}\geq 0} 2^{-\frac{\varepsilon}{k^2}(1-\frac{(d-k+1)}{2(d+k+1)})r_{k,i+1}}\Big)\Big] \\
&\times\prod_{i=1}^{k-1}\Big[\sup_{l_{i+1},r_{i,i+1}} \|\mathbb{1}_{\mathbb{X}^{l_{i+1};r_{i,i+1}}}\|_{\ell^\infty_{n_i,m}\ell^1_{\hat{n}_i}}^{-\frac{\varepsilon}{k}(1-\frac{1}{d+k+1})}\cdot \sup_{l_{i+1},r_{k,i+1}} \|\mathbb{1}_{\mathbb{X}^{l_{i+1};r_{k,i+1}}}\|_{\ell^\infty_{\vec{n}_k,m}\ell^1_{\hat{n}_k}}^{-\frac{\varepsilon}{k^2}(1-\frac{(d-k+1)}{2(d+k+1)})}\Big] \\
&\times\|g_1\|_2^{\frac{1}{k}-\frac{4\varepsilon}{k(d-k+1)}+\frac{\varepsilon}{k}-\frac{\varepsilon}{k^2}-2\varepsilon+\frac{1}{2}(-\frac{\varepsilon}{k^2}+\frac{2(d+k+1)}{(d-k+1)}\frac{\varepsilon}{k^2})}\times\prod_{l=2}^{k}\|g_l\|_2^{\frac{1}{k}-\frac{2\varepsilon}{k}+\frac{\varepsilon}{k^2}(\frac{(d-k+1)}{2(d+k+1)}-1)} \\
&\times|F|^{[1-\frac{(d+k-1)}{2(d+k+1)}]+\varepsilon}.
\end{aligned}
$$

Observe that

$$\sum_{l_1\geq 0} 2^{-\frac{\varepsilon}{k^2}l_1} \lesssim_\varepsilon 2^{-\frac{\varepsilon}{k^2}\tilde{l}_1},$$

where $\tilde{l}_1$ is the smallest index l_1 such that $\mathbb{X}^{\vec{l},R,t}\neq\varnothing$. Hence there exists some $(\vec{\boldsymbol{k}},\tilde{m})$ such that

$$2^{-\tilde{l}_1} \approx |\langle g_1,\varphi^1_{\vec{\boldsymbol{k}},\tilde{m}}\rangle| \leq |E_1|.$$

Therefore

$$\sum_{l_1\geq 0} 2^{-\frac{\varepsilon}{k^2}l_1} \lesssim_\varepsilon |E_1|^{\frac{\varepsilon}{k^2}}.$$

Notice also that

$$\sum_{r_{j,1}\geq 0} 2^{-(\frac{2\varepsilon}{k}+\frac{\varepsilon}{k^2})r_{j,1}} \lesssim_\varepsilon 2^{-(\frac{2\varepsilon}{k}+\frac{\varepsilon}{k^2})\tilde{r}_{j,1}},$$

where $\tilde{r}_{j,1}$ is defined analogously. We can then find (n_j,m) such that

$$2^{-r_{j,1}} \lesssim \|\langle g_1,\varphi^{j,1}_{n_j,m}\rangle_{x_j}\|_2 \leq |E_1|^{\frac{1}{2}}.$$

Therefore

$$\sum_{r_{j,1}\geq 0} 2^{-(\frac{2\varepsilon}{k}+\frac{\varepsilon}{k^2})r_{j,1}} \lesssim_\varepsilon |E_1|^{\frac{\varepsilon}{k}+\frac{\varepsilon}{2k^2}}.$$

We can estimate all other sums in the bound above analogously. Observe that since the cardinalities appearing in

$$\prod_{i=1}^{k-1}\Big[\sup_{l_{i+1},r_{i,i+1}} \|\mathbb{1}_{\mathbb{X}^{l_{i+1};r_{i,i+1}}}\|_{\ell^\infty_{n_i,m}\ell^1_{\hat{n}_i}}^{-\frac{\varepsilon}{k}(1-\frac{1}{d+k+1})}\cdot \sup_{l_{i+1},r_{k,i+1}} \|\mathbb{1}_{\mathbb{X}^{l_{i+1};r_{k,i+1}}}\|_{\ell^\infty_{\vec{n}_k,m}\ell^1_{\hat{n}_k}}^{-\frac{\varepsilon}{k^2}(1-\frac{(d-k+1)}{2(d+k+1)})}\Big] \tag{57}$$

are integers, the whole expression (57) is $O(1)$. Using these observations and the fact that $|E_j| < 1$ gives us

$$|\tilde{\tilde{\Lambda}}_{k,d}(g,h)| \lesssim_\varepsilon |F|^{1-\frac{(d+k-1)}{2(d+k+1)}+\varepsilon} \cdot \prod_{j=1}^{k} |E_j|^{\frac{1}{2k}}. \tag{58}$$

To simplify our notation, set $g := (g_{1,1}, g_{1,2}, \dots, g_{1,k-1}, g_{1,k}, g_2, \dots, g_k)$. To rigorously use multilinear interpolation theory, one can run the argument above for the following averaged multilinearized version of $\mathrm{ME}_{k,d}$:

$$\widetilde{ME}^{\frac{1}{k}}_{k,d}(g) := \sum_{(\vec{n},m)\in\mathbb{Z}^{d+1}} \left(\prod_{l=1}^{k-1} |\langle g_{1,l}, \varphi^{l,1}_{n_l,m}\rangle|\right)^{\frac{1}{k}} \cdot |\langle g_{1,k}, \varphi^{k,1}_{\vec{n}_k,m}\rangle|^{\frac{1}{k}} \cdot \left(\prod_{j=2}^{k} |\langle g_j, \varphi^{j}_{\vec{n},m}\rangle|\right)^{\frac{1}{k}} (\chi_{\vec{n}} \otimes \chi_m),$$

with associated dual form[27]

$$\tilde{\Lambda}_{k,d}(g,h) := \sum_{(\vec{n},m)\in\mathbb{Z}^{d+1}} \left(\prod_{l=1}^{k-1} |\langle g_{1,l}, \varphi^{l,1}_{n_l,m}\rangle|\right)^{\frac{1}{k}} \cdot |\langle g_{1,k}, \varphi^{k,1}_{\vec{n}_k,m}\rangle|^{\frac{1}{k}} \left(\prod_{j=2}^{k} |\langle g_j, \varphi^{j}_{\vec{n},m}\rangle|\right)^{\frac{1}{k}} \langle h, \chi_{\vec{n}} \otimes \chi_m\rangle.$$

Hence (58) gives us

$$\| \widetilde{ME}^{\frac{1}{k}}_{k,d}(g)\|_{L^{2(d+k+1)/(d+k-1)+\varepsilon}(\mathbb{R}^{d+1})} \lesssim_\varepsilon \prod_{l=1}^{k} |E_{1,l}|^{\frac{1}{2k}} \cdot \prod_{j=2}^{k} |E_j|^{\frac{1}{2k}}, \tag{59}$$

which is (45) for $\widetilde{ME}_{k,d}$. Finally, observe that

$$\begin{aligned}
&\| \widetilde{ME}_{k,d}(g)\|_{L^{2(d+k+1)/(k(d+k-1))+\varepsilon}(\mathbb{R}^{d+1})} \\
&\le \overbrace{\| \widetilde{ME}_{k,d}(g)^{\frac{1}{k}}\|_{L^{2(d+k+1)/(d+k-1)+k\varepsilon}(\mathbb{R}^{d+1})} \cdot \dots \cdot \| \widetilde{ME}_{k,d}(g)^{\frac{1}{k}}\|_{L^{2(d+k+1)/(d+k-1)+k\varepsilon}(\mathbb{R}^{d+1})}}^{k \text{ times}} \\
&\lesssim \left[\prod_{l=1}^{k} |E_{1,l}|^{\frac{1}{2k}} \cdot \prod_{j=2}^{k} |E_j|^{\frac{1}{2k}}\right]^k = \prod_{l=1}^{k} |E_{1,l}|^{\frac{1}{2}} \cdot \prod_{j=2}^{k} |E_j|^{\frac{1}{2}},
\end{aligned} \tag{60}$$

which finishes the proof of the case $2 \le k \le d+1$ by restricted weak-type interpolation.

10. The endpoint estimate of the case $k = d+1$ of Theorem 1.5

Let $g_1 : Q_1 \to \mathbb{R}$, $g_j : Q_j \to \mathbb{R}$ for $2 \le j \le d+1$ be continuous functions. Recall that the multilinear model for $k = d+1$ is given in Section 2 by

$$\mathrm{ME}_{d+1,d}(g_1, \dots, g_{d+1}) := \sum_{(\vec{n},m)\in\mathbb{Z}^{d+1}} \prod_{j=1}^{d+1} \langle g_j, \varphi^{j}_{\vec{n},m}\rangle (\chi_{\vec{n}} \otimes \chi_m),$$

[27] There is a slight difference between the forms $\tilde{\tilde{\Lambda}}_{k,d}$ and $\tilde{\Lambda}_{k,d}$: the latter is $2(k-1)$-linear, whereas the former is k-linear. We cannot apply multilinear interpolation theory with inequality (58) directly, because all we proved is that it holds when g_1 is a tensor. In order to correctly place our estimates in the context of multilinear interpolation, we need to consider a form that has the appropriate level of multilinearity, which is $\tilde{\Lambda}_{k,d}$.

where

$$\varphi^j_{\vec{\boldsymbol{n}},m} = \bigotimes_{l=1}^{d} \varphi^{l,j}_{n_l,m}, \quad \varphi^{l,j}_{n_l,m}(x_l) = \varphi^{l,j}(x_l) e^{2\pi i n_l x_l} e^{2\pi i m x_l^2},$$

and $\varphi^{l,j}(x)$ was defined in Section 2. From now on, we will assume without loss of generality that g_1 is the full tensor. To simplify our notation, set $g := (g_{1,1}, \dots, g_{1,d}, g_2, \dots, g_{d+1})$. Define

$$\widetilde{ME}_{d+1,d}(g) := \sum_{(\vec{\boldsymbol{n}},m)\in\mathbb{Z}^{d+1}} \prod_{l=1}^{d} \langle g_{1,l}, \varphi^{l,1}_{n_l,m}\rangle \prod_{j=2}^{d+1} \langle g_j, \varphi^j_{\vec{\boldsymbol{n}},m}\rangle (\chi_{\vec{\boldsymbol{n}}} \otimes \chi_m).$$

We will show that $\widetilde{ME}_{d+1,d}$ maps

$$\underbrace{L^2([0,1]) \times \cdots \times L^2([0,1]) \times L^2(Q_2) \times \cdots \times L^2(Q_{d+1})}_{2d \text{ times}}$$

to $L^{2/d}$, which implies the endpoint estimate of the case $k = d+1$ in Theorem 1.5.

Endpoint estimate of the case $k = d+1$. Notice that we have d factors in the first product and d factors in the second. We will pair them in the following way:

$$\widetilde{ME}_{d+1,d}(g) := \sum_{(\vec{\boldsymbol{n}},m)\in\mathbb{Z}^{d+1}} \prod_{j=2}^{d+1} \langle g_j, \varphi^j_{\vec{\boldsymbol{n}},m}\rangle \cdot \langle g_{1,j-1}, \varphi^{1,j-1}_{n_{j-1},m}\rangle (\chi_{\vec{\boldsymbol{n}}} \otimes \chi_m)$$

Now observe that

$$\begin{aligned} \| \widetilde{ME}_{d+1,d}(g)\|^{\frac{2}{d}}_{\frac{2}{d}} &= \sum_{(\vec{\boldsymbol{n}},m)\in\mathbb{Z}^{d+1}} \prod_{j=2}^{d+1} |\langle g_j \otimes \bar{g}_{1,j-1}, \varphi^j_{\vec{\boldsymbol{n}},m} \otimes \bar{\varphi}^{1,1}_{n_{j-1},m}\rangle|^{\frac{2}{d}} \\ &\le \prod_{j=2}^{d+1} \Bigg(\sum_{(\vec{\boldsymbol{n}},m)\in\mathbb{Z}^{d+1}} |\langle g_j \otimes \bar{g}_{1,j-1}, \varphi^j_{\vec{\boldsymbol{n}},m} \otimes \bar{\varphi}^{1,1}_{n_{j-1},m}\rangle|^2 \Bigg)^{\frac{1}{d}}. \end{aligned} \tag{61}$$

Let us analyze the $j = 2$ scalar product inside the parentheses (the others are dealt with in a similar way): $\langle g_j \otimes \bar{g}_{1,1}, \varphi^2_{\vec{\boldsymbol{n}},m} \otimes \bar{\varphi}^{1,1}_{n_1,m}\rangle$

$$\begin{aligned} &= \int_{\mathbb{R}^{d-1}} \langle g_{2,1} \otimes \bar{g}_{1,1}, \varphi^{1,2}_{n_1,m} \otimes \bar{\varphi}^{1,1}_{n_1,m}\rangle \Bigg(\prod_{u\ge 2} \varphi^{u,2}(x_u) \Bigg) e^{-2\pi i m (\sum_{l\ge 2} x_l^2)} e^{-2\pi i (\sum_{l\ge 2} n_l x_l)} \widehat{\mathrm{d}x_1} \\ &= \widehat{H}_{n_1,m}(n_2, \dots, n_d), \end{aligned}$$

where

$$H_{n_1,m}(x_2, \dots, x_d) := \langle g_{2,1} \otimes \bar{g}_{1,1}, \varphi^{1,2}_{n_1,m} \otimes \bar{\varphi}^{1,1}_{n_1,m}\rangle \Bigg(\prod_{u\ge 2} \varphi^{u,2}(x_u) \Bigg) e^{-2\pi i m (\sum_{l\ge 2} x_l^2)}.$$

We can then use Plancherel if we sum over $n_2, \dots, n_d$ first:

$$\begin{aligned} &\sum_{(\vec{\boldsymbol{n}},m)\in\mathbb{Z}^{d+1}} |\langle g_j \otimes \bar{g}_{1,j-1}, \varphi^j_{\vec{\boldsymbol{n}},m} \otimes \bar{\varphi}^{1,1}_{n_{j-1},m}\rangle|^2 \\ &\qquad = \sum_{n_1,m} \sum_{n_2,\dots,n_d} |\widehat{H}_{n_1,m}(n_2, \dots, n_d)|^2 = \sum_{n_1,m} \|H_{n_1,m}\|_2^2 \\ &\qquad = \int_{\mathbb{R}^{d-1}} \Bigg(\prod_{u\ge 2} \varphi^{u,2}(x_u) \Bigg) \Bigg(\sum_{n_1,m} |\langle g_2 \otimes \bar{g}_{1,1}, \varphi^{1,2}_{n_1,m} \otimes \bar{\varphi}^{1,1}_{n_1,m}\rangle|^2 \Bigg) \widehat{\mathrm{d}x_1}. \end{aligned}$$

By our initial choice of cubes, $\mathrm{supp}(\varphi^{1,1}_{n_1,m}) \cap \mathrm{supp}(\varphi^{1,2}_{n_1,m}) = \varnothing$, so the sum in (n_1, m) is actually $M_{2,1}$ (we are freezing $d-1$ variables of g_2 in this sum). Our results from Section 7 imply

$$\sum_{(\vec{\boldsymbol{n}},m)\in\mathbb{Z}^{d+1}} |\langle g_j \otimes \bar{g}_{1,j-1}, \varphi^{j}_{\vec{\boldsymbol{n}},m} \otimes \bar{\varphi}^{1,1}_{n_{j-1},m}\rangle|^2 = \|g_2 \otimes \bar{g}_{1,1}\|_2^2.$$

Arguing like that for all $2 \le j \le d+1$, (61) gives us

$$\|\widetilde{ME}_{d+1,d}(g)\|_{\frac{2}{d}}^{\frac{2}{d}} \le \prod_{j=2}^{d+1} \|g_2 \otimes \bar{g}_{1,j-1}\|_2^{\frac{2}{d}} = \prod_{j=1}^{d+1} \|g_j\|_2^{\frac{2}{d}}$$

and the result follows. □

11. Improved k-linear bounds for tensors

In this section we investigate the following question: *can one obtain better bounds than those of Conjecture 1.2 if one is restricted to the class of tensors?*[28] The answer depends on the concept of *degree of transversality*. The extra information that the input functions are supported on cubes that have disjoint projections along many directions leads to new transversality conditions, and we can take advantage of it in the full tensor case. This is the content of Theorem 11.2.

Let $\{e_j\}_{1\le j\le d}$ be the canonical basis of $\mathbb{R}^d$. If $Q \subset \mathbb{R}^d$ is a cube, $\pi_j(Q)$ represents the projection of Q along the e_j direction.

Definition 11.1. Let $\{Q_1, \dots, Q_k\}$ be a collection of k closed unit cubes in $\mathbb{R}^d$ with vertices in $\mathbb{Z}^d$. We associate to this collection its *transversality vector*

$$\tau = (\tau_1, \dots, \tau_d),$$

where $\tau_j = 1$ if there are at least two distinct intervals among the projections $\pi_j(Q_l)$, $1 \le l \le k$, and $\tau_j = 0$ otherwise. The *total degree of transversality* of the collection $\{Q_1, \dots, Q_k\}$ is

$$|\tau| := \sum_{1\le l\le d} \tau_l.$$

The k-linear extension model for a set of cubes $\{Q_l\}_{1\le l\le k}$ as in Definition 11.1 is initially given on $C(Q_1) \times \cdots \times C(Q_k)$ by

$$\mathrm{ME}^{Q_1,\dots,Q_k}_{k,d}(g_1, \dots, g_k) := \sum_{(\vec{\boldsymbol{n}},m)\in\mathbb{Z}^{d+1}} \prod_{j=1}^{k} \langle g_j, \varphi^{j}_{\vec{\boldsymbol{n}},m}\rangle (\chi_{\vec{\boldsymbol{n}}} \otimes \chi_m), \tag{62}$$

where the bumps $\varphi^{j}_{\vec{\boldsymbol{n}},m}$ are analogous to the ones in Section 9, but now adapted to the cubes Q_k.

From now on we will assume that g_j is a full tensor $g_j^1 \otimes \cdots \otimes g_j^d$ for $1 \le j \le k$ and that the transversality vector of the collection $\{Q_1, \dots, Q_k\}$ is τ. To simplify the notation, we will replace the superscripts Q_j in (62) with τ and define

$$g := (g_1^1, \dots, g_1^d, \dots, g_j^1, \dots, g_j^d, \dots, g_k^1, \dots, g_k^d).$$

[28] Extension estimates beyond the conjectured range have been verified in [Mandel and Oliveira e Silva 2023] for a certain class of functions when the underlying submanifold is $\mathbb{S}^{d-1}$; [Shao 2009] also contains results of this kind for the paraboloid.

We are then led to consider

$$\mathrm{ME}^{\tau}_{k,d}(g) := \sum_{(\vec{\boldsymbol{n}},m)\in\mathbb{Z}^{d+1}} \prod_{j=1}^{k}\prod_{l=1}^{d} \langle g^l_j, \varphi^{l,j}_{n_l,m}\rangle (\chi_{\vec{\boldsymbol{n}}}\otimes \chi_m), \tag{63}$$

where

$$\varphi^{l,j}_{n_l,m}(x) = \varphi^{l,j}(x) e^{2\pi i n_l x} e^{2\pi i m x^2}, \quad \operatorname{supp}(\varphi^{l,j}) \subset \pi_l(Q_j).$$

As was the case in Section 9, we will deal first with an averaged version of $\mathrm{ME}^{\tau}_{k,d}$ for technical reasons. Define

$$\widetilde{ME}^{\tau}_{k,d}(g) := \sum_{(\vec{\boldsymbol{n}},m)\in\mathbb{Z}^{d+1}} \prod_{j=1}^{k}\prod_{l=1}^{d} |\langle g^l_j, \varphi^{l,j}_{n_l,m}\rangle|^{\frac{1}{k}} (\chi_{\vec{\boldsymbol{n}}}\otimes \chi_m), \tag{64}$$

and consider its dual form

$$\widetilde{\Lambda}^{\tau}_{k,d}(g,h) := \sum_{(\vec{\boldsymbol{n}},m)\in\mathbb{Z}^{d+1}} \prod_{j=1}^{k}\prod_{l=1}^{d} |\langle g^l_j, \varphi^{l,j}_{n_l,m}\rangle|^{\frac{1}{k}} \cdot \langle h, \chi_{\vec{\boldsymbol{n}}}\otimes \chi_m\rangle.$$

Let $E_{j,l}$, $1 \le j \le k$ and $1 \le l \le d$, be measurable sets such that $|g^l_j| \le \chi_{E_{j,l}}$. Let $F \subset \mathbb{R}^{d+1}$ be a measurable set such that $|h| \le \chi_F$. Under these conditions we have the following result:

Theorem 11.2. *$ME^{\tau}_{k,d}$ satisfies*

$$\|ME^{\tau}_{k,d}(g)\|_{L^p(\mathbb{R}^{d+1})} \lesssim_p \prod_{j=1}^{k}\prod_{l=1}^{d} \|g^l_j\|_2 \quad \textit{for all } p > p_\tau := \frac{2(d+|\tau|+2)}{k(d+|\tau|)}.$$

Proof. It is enough to prove that

$$\|\widetilde{ME}^{\tau}_{k,d}(g)\|_{L^p(\mathbb{R}^{d+1})} \lesssim_p \prod_{j=1}^{k}\prod_{l=1}^{d} |E_{j,l}|^{\frac{1}{2k}},$$

holds for every

$$p > \frac{2(d+|\tau|+2)}{(d+|\tau|)}.$$

Define the level sets

$$\mathbb{A}^{r_{j,l}}_{j,l} := \{(n_l,m)\in\mathbb{Z}^2 : |\langle g^l_j, \varphi^{l,j}_{n_l,m}\rangle| \approx 2^{-r_{j,l}}\},$$

$$\mathbb{B}^t := \{(\vec{\boldsymbol{n}},m)\in\mathbb{Z}^{d+1} : |\langle h, \chi_{\vec{\boldsymbol{n}}}\otimes\chi_m\rangle| \approx 2^{-t}\}.$$

Set $R := (r_{i,j})_{i,j}$ and

$$\mathbb{X}^{R,t} := \left\{(\vec{\boldsymbol{n}},m)\in\mathbb{Z}^{d+1} : (n_l,m) \in \bigcap_{j=1}^{k} \mathbb{A}^{r_{j,l}}_{j,l}, \quad 1\le l\le d\right\} \cap \mathbb{B}^t.$$

We then have

$$|\widetilde{\Lambda}^{\tau}_{k,d}(g,h)| \lesssim \sum_{R,t\ge 0} 2^{-t}\cdot \prod_{j=1}^{k}\prod_{l=1}^{d} 2^{-\frac{r_{j,l}}{k}} \cdot \#\mathbb{X}^{R,t}.$$

As in the previous section, we can assume without loss of generality that $r_{j,l}, t \geq 0$. We can estimate $\#\mathbb{X}^{R,t}$ using the function h:

$$\#\mathbb{X}^{R,t} \lesssim 2^t \sum_{(\vec{n},m)\in\mathbb{Z}^{d+1}} |\langle h, \chi_{\vec{n}} \otimes \chi_m\rangle| \lesssim 2^t |F|. \tag{65}$$

Alternatively, by the definition of $\mathbb{X}^{R,t}$,

$$\#\mathbb{X}^{R,t} \leq \sum_{(\vec{n},m)\in\mathbb{Z}^{d+1}} \prod_{j=1}^{k} \prod_{l=1}^{d} \mathbb{1}_{\mathbb{A}_{j,l}^{r_{j,l}}}(n_l, m) \tag{66}$$

There are many ways to estimate the right-hand side above. We will obtain d different bounds for it, each one arising from summing in a different order. Fix $1 \leq l \leq d$ and leave the sum over (n_l, m) for last:

$$\begin{aligned}\#\mathbb{X}^{R,t} &= \sum_{(n_l,m)\in\mathbb{Z}^2} \Bigg[\prod_{j=1}^{k} \mathbb{1}_{\mathbb{A}_{j,l}^{r_{j,l}}}(n_l, m)\Bigg] \cdot \prod_{\tilde{l}=1,\tilde{l}\neq l}^{d} \Bigg[\sum_{n_{\tilde{l}}} \prod_{\tilde{j}=1}^{k} \mathbb{1}_{\mathbb{A}_{\tilde{j},\tilde{l}}^{r_{\tilde{j},\tilde{l}}}}(n_{\tilde{l}}, m)\Bigg] \\ &\leq \sum_{(n_l,m)\in\mathbb{Z}^2} \Bigg[\prod_{j=1}^{k} \mathbb{1}_{\mathbb{A}_{j,l}^{r_{j,l}}}(n_l, m)\Bigg] \cdot \prod_{\tilde{l}=1,\tilde{l}\neq l}^{d} \prod_{\tilde{j}=1}^{k} \Bigg[\sum_{n_{\tilde{l}}} \mathbb{1}_{\mathbb{A}_{\tilde{j},\tilde{l}}^{r_{\tilde{j},\tilde{l}}}}(n_{\tilde{l}}, m)\Bigg]^{\gamma_{l,\tilde{j},\tilde{l}}},\end{aligned} \tag{67}$$

where we used Hölder's inequality in the last line and $\gamma_{l,\tilde{j},\tilde{l}}$ are generic parameters such that

$$\sum_{\tilde{j}=1}^{k} \gamma_{l,\tilde{j},\tilde{l}} = 1 \tag{68}$$

for all $1 \leq l, \tilde{l} \leq d$ with $l \neq \tilde{l}$ fixed. Let us briefly explain the labels in these parameters that we just introduced:

$$\gamma_{l,\tilde{j},\tilde{l}} \longrightarrow \begin{cases} l \text{ indicates that the last variables to be summed are } (n_l, m), \\ \tilde{j} \text{ corresponds to the } \tilde{j}\text{-th function } g_{\tilde{j}}, \\ \tilde{l} \neq l \text{ corresponds to the } \tilde{l}\text{-th variable } n_{\tilde{l}}. \end{cases}$$

We will not make any specific choice for the $\gamma_{l,\tilde{j},\tilde{l}}$ since condition (68) will suffice. Now observe that for a fixed $m \in \mathbb{Z}$ we have

$$\sum_{n_{\tilde{l}}} \mathbb{1}_{\mathbb{A}_{\tilde{j},\tilde{l}}^{r_{\tilde{j},\tilde{l}}}}(n_{\tilde{l}}, m) \leq 2^{2r_{\tilde{j},\tilde{l}}} \sum_{n_{\tilde{l}}} |\langle g_{\tilde{j}}^{\tilde{l}}, \varphi_{n_{\tilde{l}},m}^{\tilde{l},\tilde{j}}\rangle|^2 \leq 2^{2r_{\tilde{j},\tilde{l}}} \cdot |E_{\tilde{j},\tilde{l}}| \tag{69}$$

by Bessel's inequality. Using (69) back in (67):

$$\begin{aligned}\#\mathbb{X}^{R,t} &\leq \prod_{\tilde{l}=1,\tilde{l}\neq l}^{d} \prod_{\tilde{j}=1}^{k} 2^{2\gamma_{l,\tilde{j},\tilde{l}} r_{\tilde{j},\tilde{l}}} \cdot |E_{\tilde{j},\tilde{l}}|^{\gamma_{l,\tilde{j},\tilde{l}}} \cdot \sum_{(n_l,m)\in\mathbb{Z}^2} \Bigg[\prod_{j=1}^{k} \mathbb{1}_{\mathbb{A}_{j,l}^{r_{j,l}}}(n_l, m)\Bigg], \\ &= \prod_{\tilde{l}=1,\ \tilde{l}\neq l}^{d} \prod_{\tilde{j}=1}^{k} 2^{2\gamma_{l,\tilde{j},\tilde{l}} r_{\tilde{j},\tilde{l}}} \cdot |E_{\tilde{j},\tilde{l}}|^{\gamma_{l,\tilde{j},\tilde{l}}} \cdot \sum_{(n_l,m)\in\mathbb{Z}^2} \Bigg[\prod_{(j_1,j_2),j_1\neq j_2} \mathbb{1}_{\mathbb{A}_{j_1,l}^{r_{j_1,l}}}(n_l, m) \cdot \mathbb{1}_{\mathbb{A}_{j_2,l}^{r_{j_2,l}}}(n_l, m)\Bigg].\end{aligned} \tag{70}$$

We simply used the fact that $\mathbb{1}^2 = \mathbb{1}$ in the last line above. Our goal is to pair the scalar products in (63) corresponding to the functions $g_{j_1}^{l}$ and $g_{j_2}^{l}$. There are two kinds of such pairs:

(a) A pair (j_1, j_2) with $j_1 \neq j_2$ is *l-transversal* if $\operatorname{supp}(\varphi^{l,j_1}) \cap \operatorname{supp}(\varphi^{l,j_2}) = \varnothing$.

(b) A pair (j_1, j_2) with $j_1 \neq j_2$ is *non-l-transversal* along the direction e_l if $\operatorname{supp}(\varphi^{l,j_1}) \cap \operatorname{supp}(\varphi^{l,j_2}) \neq \varnothing$.

Thus we have by Hölder's inequality for generic parameters α_{l,j_1,j_2} and β_{l,j_1,j_2},

$$\#\mathbb{X}^{R,t} \leq \prod_{\substack{\tilde{l}=1\\ \tilde{l}\neq l}}^{d} \prod_{\tilde{j}=1}^{k} 2^{2\gamma_{l,\tilde{j},\tilde{l}} r_{\tilde{j},\tilde{l}}} \cdot |E_{\tilde{j},\tilde{l}}|^{\gamma_{l,\tilde{j},\tilde{l}}} \cdot \prod_{\substack{(j_1,j_2)\\ l\text{-transversal, } j_1\neq j_2}} \Bigg(\sum_{(n_l,m)\in\mathbb{Z}^2} \mathbb{1}_{\mathbb{A}_{j_1,l}^{r_{j_1,l}}}(n_l,m) \cdot \mathbb{1}_{\mathbb{A}_{j_2,l}^{r_{j_2,l}}}(n_l,m) \Bigg)^{\alpha_{l,j_1,j_2}}$$
$$\times \prod_{\substack{(j_1,j_2)\\ \text{non-}l\text{-transversal, } j_1\neq j_2}} \Bigg(\sum_{(n_l,m)\in\mathbb{Z}^2} \mathbb{1}_{\mathbb{A}_{j_1,l}^{r_{j_1,l}}}(n_l,m) \cdot \mathbb{1}_{\mathbb{A}_{j_2,l}^{r_{j_2,l}}}(n_l,m) \Bigg)^{\beta_{l,j_1,j_2}}. \tag{71}$$

Define

$$\alpha_{l,j_1,j_2} = 0 \quad \text{if } (j_1, j_2) \text{ is non-}l\text{-transversal},$$
$$\beta_{l,j_1,j_2} = 0 \quad \text{if } (j_1, j_2) \text{ is } l\text{-transversal}.$$

Hence Hölder's condition is

$$\sum_{\substack{(j_1,j_2)\\ 1\leq j_1,j_2\leq k\\ j_1\neq j_2}} \alpha_{l,j_1,j_2} + \beta_{l,j_1,j_2} = 2, \tag{72}$$

since we are counting each α_{l,j_1,j_2} and β_{l,j_1,j_2} twice, for all $1 \leq l \leq d$. The labels in the parameters α and β track the following information:

$$\alpha_{l,j_1,j_2} \text{ and } \beta_{l,j_1,j_2} \longrightarrow \begin{cases} l \text{ indicates that we are summing over } (n_l, m), \\ j_1 \text{ and } j_2 \text{ correspond to two distinct functions } g_{j_1} \text{ and } g_{j_2}. \end{cases}$$

We can then use Proposition 4.4 for the transversal pairs and a combination of one-dimensional Strichartz/Tomas–Stein with Hölder for the nontransversal ones:

$$\#\mathbb{X}^{R,t} \leq \prod_{\substack{\tilde{l}=1\\ \tilde{l}\neq l}}^{d} \prod_{\tilde{j}=1}^{k} 2^{2\gamma_{l,\tilde{j},\tilde{l}} r_{\tilde{j},\tilde{l}}} \cdot |E_{\tilde{j},\tilde{l}}|^{\gamma_{l,\tilde{j},\tilde{l}}} \cdot \prod_{\substack{(j_1,j_2)\\ l\text{-transversal, } j_1\neq j_2}} (2^{2\alpha_{l,j_1,j_2}(r_{j_1,l}+r_{j_2,l})} \cdot |E_{j_1,l}|^{\alpha_{l,j_1,j_2}} \cdot |E_{j_2,l}|^{\alpha_{l,j_1,j_2}})$$
$$\times \prod_{\substack{(j_1,j_2)\\ \text{non-}l\text{-transversal, } j_1\neq j_2}} (2^{3\beta_{l,j_1,j_2}(r_{j_1,l}+r_{j_2,l})} \cdot |E_{j_1,l}|^{\frac{3}{2}\beta_{l,j_1,j_2}} \cdot |E_{j_2,l}|^{\frac{3}{2}\beta_{l,j_1,j_2}}). \tag{73}$$

As mentioned earlier in this section, we have d estimates like (73). We will interpolate between them with weights θ_l:

$$\#\mathbb{X}^{R,t} = \prod_{l=1}^{d} (\#\mathbb{X}^{R,t})^{\theta_l},$$

with

$$\sum_{l=1}^{d} \theta_l = 1. \tag{74}$$

This yields

$$\#\mathbb{X}^{R,t} \lesssim \prod_{j=1}^{k} \prod_{l=1}^{d} 2^{\#_{j,l} \cdot r_{j,l}} \cdot |E_{j,l}|^{\frac{\#_{j,l}}{2}}, \tag{75}$$

where

$$\#_{j,l} = \Big[\sum_{j_1 \neq j} (2\alpha_{l,j,j_1} + 3\beta_{l,j,j_1})\Big] \cdot \theta_l + \sum_{\tilde{l} \neq l} 2\gamma_{\tilde{l},j,l} \cdot \theta_{\tilde{l}}.$$

In order to prove an estimate like $L^2 \times \cdots \times L^2 \mapsto L^p$, we will need all these coefficients $\#_{j,l}$ to be equal. Let us call them all X for now and sum over j:

$$\sum_{j=1}^{k} X = \Big[\sum_{j=1}^{k} \sum_{j_1 \neq j} (2\alpha_{l,j,j_1} + 3\beta_{l,j,j_1})\Big] \cdot \theta_l + \sum_{\tilde{l} \neq l} 2\Big[\sum_{j=1}^{k} \gamma_{\tilde{l},j,l}\Big] \cdot \theta_{\tilde{l}}$$

By (68) and (72)

$$X = \frac{1}{k}\Big[6 - \sum_{j=1}^{k} \sum_{j_1 \neq j} \alpha_{l,j,j_1}\Big] \cdot \theta_l + \sum_{\tilde{l} \neq l} \frac{2}{k} \cdot \theta_{\tilde{l}} \tag{76}$$

for all $1 \leq l \leq d$. Together with (74), (76) gives us a linear system of d equations in the d variables $\theta_1, \ldots, \theta_d$. The solution is

$$\theta_l = \Big[\sum_{\tilde{l}=1}^{d} \frac{4 - \sum_{j=1}^{k} \sum_{j_1 \neq j} \alpha_{l,j,j_1}}{4 - \sum_{j=1}^{k} \sum_{j_1 \neq j} \alpha_{\tilde{l},j,j_1}}\Big]^{-1}. \tag{77}$$

Plugging (77) back in (76) gives us

$$X = \frac{2}{k}\Big[1 + \Big(\sum_{\tilde{l}=1} \frac{1}{[4 - \sum_{j=1}^{k} \sum_{j_1 \neq j} \alpha_{\tilde{l},j,j_1}]}\Big)^{-1}\Big]. \tag{78}$$

To minimize X we must maximize

$$\sum_{j=1}^{k} \sum_{j_1 \neq j} \alpha_{\tilde{l},j,j_1}.$$

This is achieved by choosing $\beta_{l,j_1,j_2} = 0$ for all (j_1, j_2) if there is at least one l-transversal pair (j_1, j_2). In other words, choose

$$\beta_{l,j_1,j_2} = 0 \quad \text{for all } (j_1, j_2) \text{ if } \tau_l = 1.$$

Hence by (72),

$$\sum_{j=1}^{k} \sum_{j_1 \neq j} \alpha_{\tilde{l},j,j_1} = \begin{cases} 2 & \text{if } \tau_{\tilde{l}} = 1, \\ 0 & \text{if } \tau_{\tilde{l}} = 0. \end{cases}$$

This choice of parameters gives us

$$X = \frac{2(d + |\tau| + 2)}{k(d + |\tau|)},$$

which implies the following estimate for $\#\mathbb{X}^{R,t}$:

$$\#\mathbb{X}^{R,t} \lesssim \prod_{j=1}^{k} \prod_{l=1}^{d} 2^{X \cdot r_{j,l}} \cdot |E_{j,l}|^{\frac{X}{2}}, \tag{79}$$

Finally, we interpolate between (79) with weight $\frac{1}{k\cdot X}-\varepsilon$ and (65) with weight $\left(1-\frac{1}{k\cdot X}\right)+\varepsilon$ to bound the form $\Lambda^{\tau}_{k,d}$:

$$|\Lambda^{\tau}_{k,d}(g,h)| \lesssim \sum_{R,t\geq 0} 2^{-t}\cdot\prod_{j=1}^{k}\prod_{l=1}^{d}2^{-\frac{r_{j,l}}{k}}\times\left[\prod_{j=1}^{k}\prod_{l=1}^{d}2^{X\cdot r_{j,l}}\cdot|E_{j,l}|^{\frac{X}{2}}\right]^{\frac{1}{k\cdot X}-\varepsilon}\cdot[2^t|F|]^{(1-\frac{1}{k\cdot X})+\varepsilon}.$$

Developing the right-hand side:

$$|\Lambda^{\tau}_{k,d}(g,h)| \lesssim \left(\sum_{t\geq 0}2^{-(\frac{1}{k\cdot X}-\varepsilon)t}\right)\prod_{j=1}^{k}\prod_{l=1}^{d}\left(\sum_{r_{j,l}\geq 0}2^{-\varepsilon X\cdot r_{j,l}}\right)\times\left[\prod_{j=1}^{k}\prod_{l=1}^{d}|E_{j,l}|^{\frac{1}{2k}-\frac{\varepsilon X}{2}}\right]\cdot|F|^{(1-\frac{1}{k\cdot X})+\varepsilon}.$$

As in the previous section, these series are summable. We have

$$\sum_{r_{j,l}\geq 0}2^{-\varepsilon X\cdot r_{j,l}} \lesssim_{\varepsilon} |E_{j,l}|^{\varepsilon X}.$$

For the series in t we can just bound it by an absolute constant depending on ε. This leads to

$$|\Lambda^{\tau}_{k,d}(g,h)| \lesssim_{\varepsilon} \left[\prod_{j=1}^{k}\prod_{l=1}^{d}|E_{j,l}|^{\frac{1}{2k}+\frac{\varepsilon X}{2}}\right]\cdot|F|^{(1-\frac{1}{k\cdot X})+\varepsilon} \lesssim \left[\prod_{j=1}^{k}\prod_{l=1}^{d}|E_{j,l}|^{\frac{1}{2k}}\right]\cdot|F|^{(1-\frac{1}{k\cdot X})+\varepsilon},$$

since $|E_{j,l}|\leq 1$, which finishes the proof by multilinear interpolation. □

Remark 11.3. If $\tau_l=0$ for $1\leq l\leq d$, then

$$p_\tau=\frac{2(d+2)}{kd},$$

which could have been proven in general with Hölder and Strichartz/Tomas–Stein. This is because there is no transversality to exploit; therefore the best bounds we can hope for in the multilinear setting come from the linear one.

Remark 11.4. If there are exactly $k-1$ indices l such that $\tau_l=1$, then

$$p_\tau=\frac{2(d+k+1)}{k(d+k-1)},$$

which is consistent with Theorem 1.5.

Remark 11.5. Finally, if one has more than $k-1$ indices l such that $\tau_l=1$, then

$$p_\tau<\frac{2(d+k+1)}{k(d+k-1)},$$

which clearly illustrates the point of this section. The extreme case is when $\tau_l=1$ for $1\leq l\leq d$, which gives

$$p_\tau=\frac{2(d+1)}{kd}.$$

This can be seen as an improvement upon the linear extension conjecture itself in the following sense: if we take the product of k extensions $E_{U_j}g_j$, $1\leq j\leq k$, and combine the linear extension conjecture with Hölder's inequality, we obtain an operator that maps $L^{2(d+1)/d}\times\cdots\times L^{2(d+1)/d}$ to $L^{2(d+1)/(kd)+\varepsilon}$.

On the other hand, if we are in a situation in which we have as much transversality as possible and all g_j are full tensors, we obtain $L^2 \times \cdots \times L^2$ to $L^{2(d+1)/(kd)+\varepsilon}$.

12. Beyond the L^2-based k-linear theory with and without transversality

Given a collection $\mathcal{Q} = \{Q_1, \dots, Q_k\}$ of cubes, the purpose of this section is to investigate *near-restriction* k-linear estimates associated to $\mathcal{Q}$. In other words, we study bounds of the form

$$\Big\| \prod_{j=1}^{k} \mathcal{E}_{Q_j} g_j \Big\|_{L^{2(d+1)/(kd)+\varepsilon}(\mathbb{R}^{d+1})} \lesssim_\varepsilon \prod_{j=1}^{k} \|g_j\|_{L^p(Q_j)} \tag{80}$$

for all $\varepsilon > 0$ and for some $p > 1$. There are two cases of interest here:

- $\mathcal{Q}$ is a collection of transversal cubes.
- All cubes in $\mathcal{Q}$ are the same.

It will be clear that all cases in between these two can be studied in the same framework that we now present.

12A. ***Near-restriction estimates with transversality.*** We start by restating (4). For $2 \leq k < d+1$, to recover the whole range of the generalized k-linear extension conjecture, it is enough to prove Conjecture 1.2 and

$$\Big\| \prod_{j=1}^{k} \mathcal{E}_{U_j} g_j \Big\|_{L^{2(d+1)/(kd)+\varepsilon}(\mathbb{R}^{d+1})} \lesssim_\varepsilon \prod_{j=1}^{k} \|g_j\|_{L^{2(d+1)/d}(U_j)} \tag{81}$$

for all $\varepsilon > 0$.

Let $\mathcal{Q} = \{Q_1, \dots, Q_k\}$ be our initially fixed set of cubes.[29] In what follows, we recast the statement of Theorem 1.13 in terms of this set:

Theorem 12.1. *If $\mathcal{Q}$ is a collection of transversal cubes and g_1 is a tensor, the operator $\mathcal{ME}_{k,d}(g_1, \dots, g_k)$ satisfies*

$$\|\mathcal{ME}_{k,d}(g_1, \dots, g_k)\|_{L^{2(d+1)/(kd)+\varepsilon}(\mathbb{R}^{d+1})} \lesssim_\varepsilon \prod_{j=1}^{k} \|g_j\|_{L^{p(k,d)}(Q_j)}, \tag{82}$$

where

$$p(k,d) = \begin{cases} \dfrac{4(d+1)}{d+k+1} & \text{if } 2 \leq k < \dfrac{d}{2}, \\[2ex] \dfrac{4(d+1)}{2d-k+1} & \text{if } \dfrac{d}{2} \leq k < d+1. \end{cases}$$

As anticipated in the Introduction, we prove it by adapting the argument from Section 9.

Remark 12.2. As in Section 9, the theorem above holds under the assumption that the given set of cubes is weakly transversal and any other g_j, $j \neq 1$, can be assumed to be the tensor.

[29] See Section 3.

Remark 12.3. Roughly speaking, the difference between the proof of Theorem 12.1 and the one done in Section 9 is in the building blocks we use: instead of Strichartz/Tomas–Stein (in the form of Corollary 4.2), we will use the best extension bound for the parabola (in the form of Proposition 4.3). One can think of the argument in this section as a rigorous way of replacing the former piece by the latter in our machinery.

Proof of Theorem 12.1. We work in the same setting as in Section 9. Even though there are some slight differences between the level sets from that section and the ones that we will define here, the approach is very similar.

It is convenient to recall a few important points from Section 9:

- The form of interest here is (in its averaged form):

$$\tilde{\tilde{\Lambda}}_{k,d}(g,h) := \sum_{(\vec{\boldsymbol{n}},m)\in\mathbb{Z}^{d+1}} \left(\prod_{i=1}^{k} |\langle g_j, \varphi^j_{\vec{\boldsymbol{n}},m}\rangle|\right)^{\frac{1}{k}} \langle h, \chi_{\vec{\boldsymbol{n}}} \otimes \chi_m\rangle. \tag{83}$$

- The tensor g_1 has the structure $g_1 = g_{1,1} \otimes \cdots \otimes g_{1,d}$.
- $E_{1,1}, \ldots, E_{1,d} \subset [0,1]$, $E_j \subset Q_j$ $(2 \le j \le k)$ and $F \subset \mathbb{R}^{d+1}$ are measurable sets such that $|g_{1,l}| \le \chi_{E_{1,l}}$ for $1 \le l \le d$, $|g_j| \le \chi_{E_j}$ for $2 \le j \le k$ and $|h| \le \chi_F$. Furthermore, $E_1 := E_{1,1} \times \cdots \times E_{1,d}$.

We start by encoding the sizes of the scalar products appearing in (83):

$$\mathbb{A}_j^{l_j} := \{(\vec{\boldsymbol{n}},m) \in \mathbb{Z}^{d+1} : |\langle g_j, \varphi_{\vec{\boldsymbol{n}},m}\rangle| \approx 2^{-l_j}\}, \quad 1 \le j \le k.$$

Now we see the first difference between the argument in this section and the one in Section 9: the mixed-norm quantities here are all of the same kind, in the sense that the inner products inside the L^2 norms are all one-dimensional:

$$\mathbb{B}_{i,j}^{r_{i,j}} := \{(n_i,m) \in \mathbb{Z}^2 : \|\langle g_j, \varphi^{i,j}_{n_i,m}\rangle_{x_i}\|_2 \approx 2^{-r_{i,j}}\}, \quad 1 \le i \le d, \quad 1 \le j \le k,$$

The remaining sets are defined just as in Section 9, and with the exact same purpose:

$$\begin{aligned}
\mathbb{C}^t &:= \{(\vec{\boldsymbol{n}},m) \in \mathbb{Z}^{d+1} : |\langle h, \chi_{\vec{\boldsymbol{n}}} \otimes \chi_m\rangle| \approx 2^{-t}\},\\
\mathbb{X}^{l_j;r_{i,j}} &= \mathbb{A}_j^{l_j} \cap \{(\vec{\boldsymbol{n}},m) \in \mathbb{Z}^{d+1} : (n_i,m) \in \mathbb{B}_{i,j}^{r_{i,j}}\},\\
\mathbb{X}^{\vec{l},R,t} &:= \bigcap_{1\le j\le k} \mathbb{A}_j^{l_j} \cap \left\{(\vec{\boldsymbol{n}},m) \in \mathbb{Z}^{d+1} : (n_i,m) \in \bigcap_{1\le j\le k} \mathbb{B}_{i,j}^{r_{i,j}},\ 1 \le i \le d\right\} \cap \mathbb{C}^t,
\end{aligned}$$

where we are using the abbreviations $\vec{l} = (l_1, \ldots, l_k)$ and $R := (r_{i,j})_{i,j}$. Hence,

$$|\tilde{\tilde{\Lambda}}_{k,d}(g,h)| \lesssim \sum_{\vec{l},R,t} 2^{-t} \prod_{j=1}^{k} 2^{-\frac{l_j}{k}} \#\mathbb{X}^{\vec{l},R,t}.$$

The analogue of Lemma 9.4 is the bound

$$2^{-l_1} \approx \frac{2^{-r_{1,1}} \cdot \ldots \cdot 2^{-r_{d,1}}}{\|g_1\|_2^{d-1}}, \tag{84}$$

which is proven in the same way. By an argument entirely analogous to that of Lemma 9.5, we can show

$$2^{-l_j} \lesssim \frac{2^{-r_{i,j}}}{\|\mathbb{1}_{\mathbb{X}^{l_j;r_{i,j}}}\|^{\frac{1}{2}}_{\ell^\infty_{n_i,m}\ell^1_{\hat{n}_i}}} \quad \text{for all } 1 \le i \le d,\ 2 \le j \le k. \tag{85}$$

The following corollary of the estimates above will give us the appropriate convex combination of such relations:[30]

Corollary 12.4. *For* $1 \le i \le k-1$ *we have*

$$2^{-l_{i+1}} \lesssim \frac{2^{-\frac{k}{d+1}\cdot r_{i,i+1}}}{\|\mathbb{1}_{\mathbb{X}^{l_{i+1};r_{i,i+1}}}\|^{\frac{1}{2(d+1)}}_{\ell^\infty_{n_i,m}\ell^1_{\hat{n}_i}}} \cdot \prod_{u=k}^{d} \frac{2^{-\frac{1}{d+1}\cdot r_{u,i+1}}}{\|\mathbb{1}_{\mathbb{X}^{l_{i+1};r_{u,i+1}}}\|^{\frac{1}{2k(d+1)}}_{\ell^\infty_{n_u,m}\ell^1_{\hat{n}_u}}}.$$

Proof. Interpolate between the bounds in (85) with one weight equal to $\frac{k}{d+1}$ for $(i,j) := (i, i+1)$ and $d-k+1$ weights $\frac{1}{d+1}$ for $(i,j) := (u, i+1)$, $k \le u \le d$. □

We can estimate $\#\mathbb{X}^{\vec{l},R,t}$ using the function h:

$$\#\mathbb{X}^{\vec{l},R,t} \lesssim \sum_{(\vec{\boldsymbol{n}},m)\in\mathbb{Z}^{d+1}} |\langle h, \chi_{\vec{\boldsymbol{n}}} \otimes \chi_m\rangle| \lesssim 2^t|F|. \tag{86}$$

Alternatively,

$$\#\mathbb{X}^{\vec{l},R,t} \le \sum_{(\vec{\boldsymbol{n}},m)\in\mathbb{Z}^{d+1}} \prod_{j=1}^{k} \mathbb{1}_{\mathbb{A}_j^{l_j}}(\vec{\boldsymbol{n}},m) \prod_{i=1}^{d}\prod_{j=1}^{k} \mathbb{1}_{\mathbb{B}_{i,j}^{r_{i,j}}}(n_i,m). \tag{87}$$

Similarly to what was done in Section 9, we will manipulate the inequality above in d ways: $k-1$ of them will exploit orthogonality (from the combination of the sets $\mathbb{B}_{i,1}^{r_{i,1}}$ and $\mathbb{B}_{i,i+1}^{r_{i,i+1}}$, $1 \le i \le k-1$), but now the other $d-k+1$ ones will reflect the linear extension problem in dimension 1. The following lemma is the appropriate analogue of Lemma 9.7 in this section:

Lemma 12.5. *The bounds above imply*:

(a) *The orthogonality-type bounds*: *for all* $1 \le i \le k-1$,

$$\#\mathbb{X}^{\vec{l},R,t} \lesssim \|\mathbb{1}_{\mathbb{X}^{l_{i+1};r_{i,i+1}}}\|_{\ell^\infty_{n_i,m}\ell^1_{\hat{n}_i}} \cdot 2^{2r_{i,1}+2r_{i,i+1}} \cdot \|g_1\|_2^2 \cdot \|g_{i+1}\|_2^2. \tag{88}$$

(b) *The extension-type bounds*: *for all* $k \le u \le d$,

$$\begin{aligned}\#\mathbb{X}^{\vec{l},R,t} \lesssim{}& \prod_{j=2}^{k} \|\mathbb{1}_{\mathbb{X}^{l_j;r_{u,j}}}\|^{\frac{1}{k}}_{\ell^\infty_{n_u,m}\ell^1_{\hat{n}_u}} \cdot 2^{\frac{2}{k}\sum_{i\neq u} r_{i,1}} \cdot \|g_1\|_2^{\frac{2(d-1)}{k}} \\ &\times 2^{\alpha\cdot r_{u,1}+\sum_{l=2}^{k}\beta\cdot r_{u,l}} \cdot \Big(\prod_{j\neq u}\|g_{1,j}\|_2\Big)^{\alpha} \cdot \|g_{1,u}\|_4^{\alpha} \cdot \prod_{l=2}^{k}\|g_l\|_4^{\beta},\end{aligned} \tag{89}$$

where

$$\alpha := \frac{2(k+1)}{k} + \delta \cdot \frac{(k+1)}{2k}, \qquad \beta := \frac{2}{k} + \tilde{\delta}\cdot\frac{1}{2k},$$

with $\delta, \tilde{\delta} > 0$ *being arbitrarily small parameters to be chosen later.*

[30] Notice that instead of using just two mixed quantities for each scalar one (as in Corollary 9.6), we are using $d-k+2$ many of them here.

Proof. Part (a) is the same as in Lemma 9.7(a). As for (b), fix $k \le u \le d$ and bound $\#\mathbb{X}^{\vec{l},R,t}$ as follows:

$$\begin{aligned}
\#\mathbb{X}^{\vec{l},R,t} &= \sum_{(\vec{\boldsymbol{n}},m)\in\mathbb{Z}^{d+1}} \mathbb{1}_{\mathbb{X}^{\vec{l},R,t}}(\vec{\boldsymbol{n}},m) \\
&\le \sum_{(\vec{\boldsymbol{n}},m)\in\mathbb{Z}^{d+1}} \prod_{j=2}^{k} \mathbb{1}_{\mathbb{X}^{l_j;r_{u,j}}}(\vec{\boldsymbol{n}},m) \prod_{i\neq u} \mathbb{1}_{\mathbb{B}_{i,1}^{r_{i,1}}}(n_i,m)\cdot \prod_{l=1}^{k} \mathbb{1}_{\mathbb{B}_{u,l}^{r_{u,l}}}(n_u,m) \\
&= \sum_{n_u,m} \prod_{l=1}^{k} \mathbb{1}_{\mathbb{B}_{u,l}^{r_{u,l}}}(n_u,m) \sum_{\hat{n}_u} \prod_{j=2}^{k} \mathbb{1}_{\mathbb{X}^{l_j;r_{u,j}}}(\vec{\boldsymbol{n}},m) \prod_{i\neq u} \mathbb{1}_{\mathbb{B}_{i,1}^{r_{i,1}}}(n_i,m) \\
&\le \sum_{n_u,m} \prod_{l=1}^{k} \mathbb{1}_{\mathbb{B}_{u,l}^{r_{u,l}}}(n_u,m) \prod_{j=2}^{k} \|\mathbb{1}_{\mathbb{X}^{l_j;r_{u,j}}}(\vec{\boldsymbol{n}},m)\|_{\ell^1_{\hat{n}_u}}^{\frac{1}{k}} \cdot \Big\| \prod_{i\neq u} \mathbb{1}_{\mathbb{B}_{i,1}^{r_{i,1}}}(n_i,m)\Big\|_{\ell^1_{\hat{n}_u}}^{\frac{1}{k}} \\
&\le \prod_{j=2}^{k} \|\mathbb{1}_{\mathbb{X}^{l_j;r_{u,j}}}\|_{\ell^\infty_{n_u,m}\ell^1_{\hat{n}_u}}^{\frac{1}{k}} \cdot \prod_{i\neq u} \|\mathbb{1}_{\mathbb{B}_{i,1}^{r_{i,1}}}\|_{\ell^\infty_m \ell^1_{n_i}}^{\frac{1}{k}} \cdot \Big\| \prod_{l=1}^{k} \mathbb{1}_{\mathbb{B}_{u,l}^{r_{u,l}}}\Big\|_{\ell^1_{n_u,m}},
\end{aligned} \tag{90}$$

where we used Hölder's inequality from the third to fourth line. Next, notice that

$$\begin{aligned}
\|\mathbb{1}_{\mathbb{B}_{i,1}^{r_{i,1}}}\|_{\ell^\infty_m\ell^1_{n_i}} &\lesssim \sup_m 2^{2r_{i,1}} \sum_{n_i} \|\langle g_1, \varphi^{i,1}_{n_i,m}\rangle_{x_i}\|_2^2 \\
&= \sup_m 2^{2r_{i,1}} \int \sum_{n_i} |\langle g_1, \varphi^{i,1}_{n_i,m}\rangle_{x_i}|^2\, \mathrm{d}\hat{x}_i \lesssim 2^{2r_{i,1}}\cdot \|g_1\|_2^2
\end{aligned} \tag{91}$$

by orthogonality. Now let

$$\begin{aligned}
p_{u,1} &:= \frac{2k}{(k+1)}, \\
p_{u,l} &:= 2k \quad \text{for all } 2\le l\le k
\end{aligned}$$

and notice that

$$\sum_{l=1}^{k} \frac{1}{p_{u,l}} = 1.$$

This way, by the definition of $\mathbb{B}_{u,l}^{r_{u,l}}$ and by Hölder's inequality with these $p_{u,l}$ we have

$$\begin{aligned}
&\Big\| \prod_{l=1}^{k} \mathbb{1}_{\mathbb{B}_{u,l}^{r_{u,l}}}\Big\|_{\ell^1_{n_u,m}} \\
&\lesssim 2^{\alpha\cdot r_{u,1}+\sum_{l=2}^{k}\beta\cdot r_{u,l}} \sum_{(n_u,m)} \|\langle g_1, \varphi^{u,1}_{n_u,m}\rangle_{x_u}\|_2^{\alpha}\cdot \prod_{l=2}^{k} \|\langle g_l, \varphi^{u,l}_{n_u,m}\rangle_{x_u}\|_2^{\beta} \\
&\le 2^{\alpha\cdot r_{u,1}+\sum_{l=2}^{k}\beta\cdot r_{u,l}} \Big(\sum_{(n_u,m)} \|\langle g_1, \varphi^{u,1}_{n_u,m}\rangle_{x_u}\|_2^{\alpha\cdot p_{u,1}}\Big)^{\frac{1}{p_{u,1}}} \cdot \prod_{l=2}^{k}\Big(\sum_{(n_u,m)} \|\langle g_l, \varphi^{u,l}_{n_u,m}\rangle_{x_u}\|_2^{\beta\cdot p_{u,l}}\Big)^{\frac{1}{p_{u,l}}} \\
&= 2^{\alpha\cdot r_{u,1}+\sum_{l=2}^{k}\beta\cdot r_{u,l}} \Big(\sum_{(n_u,m)} \|\langle g_1, \varphi^{u,1}_{n_u,m}\rangle_{x_u}\|_2^{4+\delta}\Big)^{\frac{1}{p_{u,1}}} \cdot \prod_{l=2}^{k}\Big(\sum_{(n_u,m)} \|\langle g_l, \varphi^{u,l}_{n_u,m}\rangle_{x_u}\|_2^{4+\tilde{\delta}}\Big)^{\frac{1}{p_{u,l}}}.
\end{aligned} \tag{92}$$

At this point we see another difference between this proof and the argument in Section 9: We do not obtain a pure L^p norm when using the near-L^4 extension analogue of Corollary 4.2 for $l = d-1$. Alternatively, we use Hölder in the term involving g_1 once more:

$$\|\langle g_1, \varphi^{u,1}_{n_u,m}\rangle_{x_u}\|_2^{4+\delta} = \left[\int \left(\prod_{j\neq u} |g_{1,j}|^2(x_j)\right) \cdot |\langle g_{1,u}, \varphi^{u,1}_{n_u,m}\rangle_{x_u}|^2 \, \widehat{\mathrm{d}x_u}\right]^{\frac{4+\delta}{2}}$$
$$\leq \left(\prod_{j\neq u} \|g_{1,j}\|_2\right)^{4+\delta} \cdot |\langle g_{1,u}, \varphi^{u,1}_{n_u,m}\rangle_{x_u}|^{4+\delta}.$$

For the remaining g_l we simply use Hölder and the fact that they are compactly supported:[31]

$$\|\langle g_l, \varphi^{u,l}_{n_u,m}\rangle_{x_u}\|_2^{4+\tilde{\delta}} \lesssim \|\langle g_l, \varphi^{u,l}_{n_u,m}\rangle_{x_u}\|_4^{4+\tilde{\delta}}.$$

These observations imply

$$\left\|\prod_{l=1}^{k} \mathbb{1}_{\mathbb{B}^{r_{u,l}}_{u,l}}\right\|_{\ell^1_{n_u,m}} \lesssim 2^{\alpha\cdot r_{u,1}+\sum_{l=2}^{k}\beta\cdot r_{u,l}} \cdot \left(\prod_{j\neq u}\|g_{1,j}\|_2\right)^{\frac{4+\delta}{p_{u,1}}} \cdot \left(\sum_{(n_u,m)} |\langle g_{1,u}, \varphi^{u,1}_{n_u,m}\rangle_{x_u}|^{4+\delta}\right)^{\frac{1}{p_{u,1}}}$$
$$\cdot \prod_{l=2}^{k}\left(\sum_{(n_u,m)} \|\langle g_l, \varphi^{u,l}_{n_u,m}\rangle_{x_u}\|_4^{4+\tilde{\delta}}\right)^{\frac{1}{p_{u,l}}}$$
$$\leq 2^{\alpha\cdot r_{u,1}+\sum_{l=2}^{k}\beta\cdot r_{u,l}} \cdot \left(\prod_{j\neq u}\|g_{1,j}\|_2\right)^{\alpha} \cdot \|g_{1,u}\|_4^{\alpha} \cdot \prod_{l=2}^{k}\|g_l\|_4^{\beta}, \tag{93}$$

where we used Minkowski for norms and the L^4-$L^{4+\tilde{\delta}}$ one-dimensional extension estimate from the second to third line above. Part (b) follows from applying (91) and (93) to (90). □

Given $\varepsilon > 0$, we bound the multilinear form $\tilde{\tilde{\Lambda}}_{k,d}$ using the estimates from (84) and Corollary 12.4 (with the appropriate ε-losses for later convenience), and the ones from Lemma 12.5 with the following weights:

$$\begin{cases} \theta_l = \dfrac{1}{2(d+1)} - \dfrac{\varepsilon}{d}, \quad 1 \leq l \leq d & \text{for the } d \text{ estimates in (88) and (89)}, \\ \theta_{d+1} = 1 - \dfrac{d}{2(d+1)} + \varepsilon & \text{for (86)}. \end{cases}$$

Hence,

$$|\tilde{\tilde{\Lambda}}_{k,d}(g,h)| \lesssim \sum_{\vec{l},R,t\geq 0} 2^{-t} \times 2^{-\frac{(d+1)\varepsilon}{2kd}l_1} \times \left(\frac{1}{\|g_1\|_2^{d-1}}\prod_{j=1}^{d} 2^{-r_{j,1}}\right)^{\frac{1}{k}-\frac{(d+1)\varepsilon}{2kd}}$$
$$\times \prod_{i=1}^{k-1} 2^{-\frac{(d+1)\varepsilon}{2kd}l_{i+1}} \times \prod_{i=1}^{k-1}\left[\frac{2^{-\frac{1}{d+1}\cdot r_{i,i+1}}}{\|\mathbb{1}_{\mathbb{X}^{l_{i+1};r_{i,i+1}}}\|^{\frac{1}{2(d+1)}}_{\ell^\infty_{n_i,m}\ell^1_{\hat{n}_i}}} \cdot \prod_{u=k}^{d} \frac{2^{-\frac{1}{k(d+1)}\cdot r_{u,i+1}}}{\|\mathbb{1}_{\mathbb{X}^{l_{i+1};r_{u,i+1}}}\|^{\frac{1}{2k(d+1)}}_{\ell^\infty_{n_u,m}\ell^1_{\hat{n}_u}}}\right]^{1-\frac{(d+1)\varepsilon}{2d}}$$

[31] We use this crude estimate for the remaining g_l because they do not have the same structure that allows "pulling out" the one-dimensional functions $g_{1,j}$, like g_1 does. There is a clear loss here and it is reflected in the fact that $p(k,d)$ is not the best exponent for which (82) holds.

$$\times \prod_{l=1}^{k-1} \left(\|\mathbb{1}_{\mathbb{X}^{l_{l+1};r_{l,l+1}}}\|_{\ell^\infty_{n_l,m}\ell^1_{\hat{n}_l}} \cdot 2^{2r_{l,1}+2r_{l,l+1}} \cdot \|g_1\|_2^2 \cdot \|g_{l+1}\|_2^2 \right)^{\frac{1}{2(d+1)}-\frac{\varepsilon}{d}}$$
$$\times \prod_{k\le u\le d} \left(\prod_{j=2}^{k} \|\mathbb{1}_{\mathbb{X}^{l_j;r_{u,j}}}\|^{\frac{1}{k}}_{\ell^\infty_{n_u,m}\ell^1_{\hat{n}_u}} \cdot 2^{\frac{2}{k}\sum_{i\neq u} r_{i,1}} \cdot \|g_1\|_2^{\frac{2(d-1)}{k}} \right)^{\frac{1}{2(d+1)}-\frac{\varepsilon}{d}}$$
$$\times \prod_{k\le u\le d} \left(2^{\alpha\cdot r_{u,1}+\sum_{l=2}^{k}\beta\cdot r_{u,l}} \cdot \left(\prod_{j\neq u}\|g_{1,j}\|_2\right)^{\alpha} \cdot \|g_{1,u}\|_4^{\alpha} \cdot \prod_{l=2}^{k}\|g_l\|_4^{\beta} \right)^{\frac{1}{2(d+1)}-\frac{\varepsilon}{d}}$$
$$\times (2^t|F|)^{1-\frac{d}{2(d+1)}+\varepsilon}.$$

Developing the expression above,

$$\begin{aligned}
&|\tilde{\tilde{\Lambda}}_{k,d}(g,h)| \\
&\lesssim \sum_{\vec{l},R,t\ge 0} 2^{-t}\times 2^{-\frac{(d+1)\varepsilon}{2kd}l_1}\times\left(\prod_{j=1}^{d}2^{-r_{j,1}}\right)^{\frac{1}{k}-\frac{(d+1)\varepsilon}{2kd}} \times \|g_1\|_2^{\frac{(d+1)(d-1)\varepsilon}{2kd}-\frac{(d-1)}{k}} \\
&\times \prod_{i=1}^{k-1} 2^{-\frac{(d+1)\varepsilon}{2kd}l_{i+1}} \times \prod_{i=1}^{k-1}\left[2^{-\frac{1}{d+1}\cdot r_{i,i+1}}\cdot\prod_{u=k}^{d}2^{-\frac{1}{k(d+1)}\cdot r_{u,i+1}}\right]^{1-\frac{(d+1)\varepsilon}{2d}} \\
&\times \prod_{i=1}^{k-1}\left[\|\mathbb{1}_{\mathbb{X}^{l_{i+1};r_{i,i+1}}}\|_{\ell^\infty_{n_i,m}\ell^1_{\hat{n}_i}}^{\frac{1}{2(d+1)}\cdot\left(\frac{(d+1)\varepsilon}{2d}-1\right)}\cdot\prod_{u=k}^{d}\|\mathbb{1}_{\mathbb{X}^{l_{i+1};r_{u,i+1}}}\|_{\ell^\infty_{n_u,m}\ell^1_{\hat{n}_u}}^{\frac{1}{2k(d+1)}\left(\frac{(d+1)\varepsilon}{2d}-1\right)}\right] \\
&\times\left[\prod_{l=1}^{k-1}\|\mathbb{1}_{\mathbb{X}^{l_{l+1};r_{l,l+1}}}\|_{\ell^\infty_{n_l,m}\ell^1_{\hat{n}_l}}^{\frac{1}{2(d+1)}-\frac{\varepsilon}{d}}\right]\times\left[\prod_{l=1}^{k-1}\left(2^{r_{l,1}+r_{l,l+1}}\right)^{\frac{1}{d+1}-\frac{2\varepsilon}{d}}\right]\times\|g_1\|_2^{\frac{(k-1)}{d+1}-\frac{2(k-1)\varepsilon}{d}}\cdot\prod_{l=1}^{k-1}\|g_{l+1}\|_2^{\frac{1}{d+1}-\frac{2\varepsilon}{d}} \\
&\times\prod_{u=k}^{d}\left[\left(\prod_{j=2}^{k}\|\mathbb{1}_{\mathbb{X}^{l_j;r_{u,j}}}\|_{\ell^\infty_{n_u,m}\ell^1_{\hat{n}_u}}^{\frac{1}{k}\cdot\left(\frac{1}{2(d+1)}-\frac{\varepsilon}{d}\right)}\right)\cdot\left(2^{\frac{2}{k}\sum_{i\neq u}r_{i,1}}\cdot 2^{\alpha\cdot r_{u,1}+\sum_{l=2}^{k}\beta\cdot r_{u,l}}\right)^{\frac{1}{2(d+1)}-\frac{\varepsilon}{d}}\right] \\
&\times\|g_1\|_2^{\frac{2(d-k+1)(d-1)}{k}\left(\frac{1}{2(d+1)}-\frac{\varepsilon}{d}\right)}\cdot\prod_{k\le u\le d}\left[\left(\|g_{1,u}\|_4\cdot\prod_{j\neq u}\|g_{1,j}\|_2\right)^{\alpha\left(\frac{1}{2(d+1)}-\frac{\varepsilon}{d}\right)}\right] \\
&\times\prod_{l=2}^{k}\|g_l\|_4^{\beta(d-k+1)\left(\frac{1}{2(d+1)}-\frac{\varepsilon}{d}\right)}\times(2^t|F|)^{\left[1-\frac{d}{2(d+1)}\right]+\varepsilon}.
\end{aligned} \tag{94}$$

Observe that the product of the blue factors above (for $k\le u\le d$) is[32]

$$\begin{aligned}
\prod_{k\le u\le d}\left(\|g_{1,u}\|_4\cdot\prod_{j\neq u}\|g_{1,j}\|_2\right) &= \left[\prod_{l=1}^{k-1}\|g_{1,l}\|_2^{d-k+1}\right]\cdot\prod_{u=k}^{d}\left[\|g_{1,u}\|_2^{d-k}\cdot\|g_{1,u}\|_4\right] \\
&= \left[\prod_{j=1}^{d}\|g_{1,j}\|_2^{d-k}\right]\cdot\left[\prod_{l=1}^{k-1}\|g_{1,l}\|_2\right]\cdot\left[\prod_{u=k}^{d}\|g_{1,u}\|_4\right]\le\|g_1\|_2^{d-k}\cdot|E_1|^{\frac{1}{4}}.
\end{aligned}$$

[32] Recall that $|g_1| = |g_{1,1}\otimes\cdots\otimes g_{1,d}|\le \mathbb{1}_{E_{1,1}}\otimes\cdots\otimes\mathbb{1}_{E_{1,d}}\le\mathbb{1}_{E_1}$.

Notice that the previous step was lossy, which also reflects in the suboptimal final exponent $p(k,d)$. Now we set the values of δ and $\tilde{\delta}$ (as functions of ε) to be such that

$$\delta\cdot\frac{(k+1)}{2k}\left(\frac{1}{2(d+1)}-\frac{\varepsilon}{d}\right)=\frac{(d+1)\varepsilon}{kd},$$
$$\tilde{\delta}\cdot\frac{1}{2k}\left(\frac{1}{2(d+1)}-\frac{\varepsilon}{d}\right)=\frac{\varepsilon}{kd}.$$

Simplifying the expression above with this choice of δ and $\tilde{\delta}$,

$$\begin{aligned}
&|\tilde{\tilde{\Lambda}}_{k,d}(g,h)| \\
&\quad\lesssim\Bigg[\sum_{l_1\geq 0}2^{-\frac{(d+1)\varepsilon}{2kd}\cdot l_1}\Bigg]\times\Bigg[\prod_{j=1}^{k-1}\Bigg(\sum_{r_{j,1}\geq 0}2^{-\frac{3(d+1)\varepsilon}{2kd}\cdot r_{j,1}}\Bigg)\Bigg]\times\Bigg[\prod_{u=k}^{d}\Bigg(\sum_{r_{u,1}\geq 0}2^{-\frac{(d+1)\varepsilon}{2kd}\cdot r_{u,1}}\Bigg)\Bigg] \\
&\qquad\times\Bigg[\prod_{i=1}^{k-1}\Bigg(\sum_{l_{i+1}\geq 0}2^{-\frac{(d+1)\varepsilon}{2kd}\cdot l_{i+1}}\Bigg)\Bigg]\times\Bigg[\sum_{t\geq 0}2^{-t\left(\frac{d}{2(d+1)}-\varepsilon\right)}\Bigg] \\
&\qquad\times\prod_{i=1}^{k-1}\Bigg[\Bigg(\sum_{r_{i,i+1}\geq 0}2^{-\frac{3\varepsilon}{2d}\cdot r_{i,i+1}}\Bigg)\cdot\prod_{u=k}^{d}\Bigg(\sum_{r_{u,i+1}\geq 0}2^{-\frac{\varepsilon}{2kd}\cdot r_{u,i+1}}\Bigg)\Bigg] \\
&\qquad\times\prod_{i=1}^{k-1}\Bigg[\sup_{l_{i+1},r_{i,i+1}}\|\mathbb{1}_{\mathbb{X}^{l_{i+1};r_{i,i+1}}}\|_{\ell^\infty_{n_i,m}\ell^1_{\hat{n}_i}}^{-\frac{3\varepsilon}{4d}}\cdot\prod_{u=k}^{d}\sup_{l_{i+1},r_{u,i+1}}\|\mathbb{1}_{\mathbb{X}^{l_{i+1};r_{u,i+1}}}\|_{\ell^\infty_{n_u,m}\ell^1_{\hat{n}_u}}^{-\frac{3\varepsilon}{4kd}}\Bigg] \\
&\qquad\times\|g_1\|_2^{\frac{(d-k)}{k(d+1)}-\frac{2(d-k)(k+1)\varepsilon}{kd}+\frac{(d-k)(d+1)\varepsilon}{kd}+\frac{(d+1)(d-1)\varepsilon}{2kd}-\frac{2(k-1)\varepsilon}{d}-\frac{2(d-k+1)(d-1)\varepsilon}{kd}}\cdot|E_1|^{\frac{(k+1)}{4k(d+1)}+\frac{(d+1)\varepsilon}{4kd}} \\
&\qquad\times\prod_{l=1}^{k-1}|E_{l+1}|^{\frac{(d+k+1)}{4k(d+1)}-\frac{\varepsilon}{d}-\frac{(d-k+1)\varepsilon}{2kd}+\frac{(d-k+1)\varepsilon}{4kd}}\times|F|^{\left[1-\frac{d}{2(d+1)}\right]+\varepsilon}.
\end{aligned}\tag{95}$$

By considerations identical to the ones in the end of Section 9, this implies

$$|\tilde{\tilde{\Lambda}}_{k,d}(g,h)|\lesssim_\varepsilon|F|^{1-\frac{d}{2(d+1)}+\varepsilon}\cdot|E_1|^{\frac{2d-k+1}{4k(d+1)}}\prod_{l=1}^{k-1}|E_{l+1}|^{\frac{d+k+1}{4k(d+1)}}.\tag{96}$$

To make all exponents of $|E_j|$ $(1\leq j\leq k)$ the same, we have to take

$$\frac{1}{\tilde{p}(k,d)}=\min\left\{\frac{2d-k+1}{4k(d+1)},\frac{d+k+1}{4k(d+1)}\right\}.$$

Again by the same considerations from Section 9, (96) implies[33] Theorem 12.1. □

12B. ***Near-restriction estimates without transversality.*** To make the notation lighter, let us omit the index Q and set $\mathcal{E}_d$ be the extension operator associated to a fixed cube $Q\subset\mathbb{R}^d$. Recall the k-product

[33] Notice that we obtain something slightly better than Theorem 12.1 if one is looking for *asymmetric estimates*: (96) implies a bound of type $L^{p_1}\times L^{p_2}\times L^{p_2}\times\cdots\times L^{p_2}\to L^{2(d+1)/(kd)+\varepsilon}$, $p_1\neq p_2$ and $p_1,p_2\leq p(k,d)$, if g_1 is a tensor.

operator obtained from $\mathcal{E}_d$ defined in (6)

$$\mathcal{E}_{d,(k)}(g_1,\ldots,g_k)=\prod_{j=1}^{k}\mathcal{E}_d g_j.$$

In this subsection we prove Theorem 1.18, which we restate here for the convenience of the reader.

Theorem 12.6. *Let $2\le k\le d+1$. If g_1 is a tensor, the inequality*

$$\Big\|\prod_{j=1}^{k}\mathcal{E}_d g_j\Big\|_{L^{2(d+1)/(kd)+\varepsilon}(\mathbb{R}^{d+1})}\lesssim_{Q,\varepsilon}\prod_{j=1}^{k}\|g_j\|_{L^4(Q)} \tag{97}$$

holds for all $\varepsilon>0$.

Remark 12.7. As in the previous subsection, the difference between the proof of Theorem 12.6 and the one done in Section 9 is in the building blocks used: since there is no transversality to be exploited, we only use the best extension bound for the parabola (in the form of Proposition 4.3).

Proof of Theorem 12.6. The framework is the exact same as in the proof of Theorem 12.1. We have to bound $\#\mathbb{X}^{\vec{l},R,t}$ to effectively estimate[34]

$$|\tilde{\tilde{\Lambda}}_{k,d}(g,h)|\lesssim\sum_{\vec{l},R,t}2^{-t}\prod_{j=1}^{k}2^{-\frac{l_j}{k}}\#\mathbb{X}^{\vec{l},R,t}$$

in terms of the measures of the sets $E_{1,\ell}$, $1\le\ell\le d$, E_j, $2\le j\le k$, and F. This will be done by the following analogue of Lemma 12.5:

Lemma 12.8. *The two following extension-type bounds for the cardinality $\#\mathbb{X}^{\vec{l},R,t}$ hold:*

(a) *For all $1\le i\le k-1$ and all*[35] $\lambda>0$,

$$\#\mathbb{X}^{\vec{l},R,t}\lesssim\|\mathbb{1}_{\mathbb{X}^{l_{i+1};r_{i,i+1}}}\|_{\ell^\infty_{n_i,m}\ell^1_{\hat{n}_i}}\cdot 2^{(2+\lambda)(r_{i,1}+r_{i,i+1})}\cdot\|g_{1,i}\|_4^{2+\lambda}\cdot\Big(\prod_{\ell\neq i}\|g_{1,\ell}\|_{2+\lambda}^{2+\lambda}\Big)\cdot\|g_{i+1}\|_4^{2+\lambda}. \tag{98}$$

(b) *If $k<d+1$, for all $k\le u\le d$,*

$$\begin{aligned}\#\mathbb{X}^{\vec{l},R,t}\lesssim\prod_{j=2}^{k}\|\mathbb{1}_{\mathbb{X}^{l_j;r_{u,j}}}\|^{\frac{1}{k}}_{\ell^\infty_{n_u,m}\ell^1_{\hat{n}_u}}&\cdot 2^{\frac{2}{k}\sum_{i\neq u}r_{i,1}}\cdot\|g_1\|_2^{\frac{2(d-1)}{k}}\\ &\times 2^{\alpha\cdot r_{u,1}+\sum_{l=2}^{k}\beta\cdot r_{u,l}}\cdot\Big(\prod_{j\neq u}\|g_{1,j}\|_2\Big)^{\alpha}\cdot\|g_{1,u}\|_4^{\alpha}\cdot\prod_{l=2}^{k}\|g_l\|_4^{\beta},\end{aligned} \tag{99}$$

where

$$\alpha:=\frac{2(k+1)}{k}+\delta\cdot\frac{(k+1)}{2k},\qquad\beta:=\frac{2}{k}+\tilde{\delta}\cdot\frac{1}{2k},$$

with $\delta,\tilde{\delta}>0$ being arbitrarily small parameters to be chosen later.

Remark 12.9. We highlight that (99) is only going to be used if $k<d+1$. The argument that follows will make it clear what changes in the case $k=d+1$ if we only use (98).

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[34]Rigorously, we are dealing with a different operator here, but we will keep the notation unchanged for simplicity.

[35]The parameter λ will be chosen later. It should be regarded as morally zero, and we only introduce it to be able to use Proposition 4.3 since it does not hold at the endpoint.

Proof. We only prove (98), since (99) is identical to (89). From (53),

$$\#\mathbb{X}^{\vec{l},R,t} \leq \|\mathbb{1}_{\mathbb{X}^{l_{i+1};r_{i,i+1}}}\|_{\ell^{\infty}_{n_i,m}\ell^1_{\hat{n}_i}} \cdot \|\mathbb{1}_{\mathbb{B}^{r_{i,1}}_{i,1}\cap\mathbb{B}^{r_{i,i+1}}_{i,i+1}}\|_{\ell^1_{n_i,m}}.$$

We bound the second factor in the right-hand side above as follows:

$$\begin{aligned}
&\|\mathbb{1}_{\mathbb{B}^{r_{i,1}}_{i,1}\cap\mathbb{B}^{r_{i,i+1}}_{i,i+1}}\|_{\ell^1_{n_i,m}}\\
&\lesssim 2^{(2+\lambda)(r_{i,1}+r_{i,i+1})} \sum_{(n_i,m)\in\mathbb{B}^{r_{i,1}}_{i,1}\cap\mathbb{B}^{r_{i,i+1}}_{i,i+1}} \|\langle g_1,\varphi^{i,1}_{n_i,m}\rangle_{x_i}\|_2^{2+\lambda}\cdot\|\langle g_{i+1},\varphi^{i,i+1}_{n_i,m}\rangle_{y_i}\|_2^{2+\lambda}\\
&\leq 2^{(2+\lambda)(r_{i,1}+r_{i,i+1})} \sum_{(n_i,m)\in\mathbb{B}^{r_{i,1}}_{i,1}\cap\mathbb{B}^{r_{i,i+1}}_{i,i+1}} \|\langle g_1,\varphi^{i,1}_{n_i,m}\rangle_{x_i}\|_{2+\lambda}^{2+\lambda}\cdot\|\langle g_{i+1},\varphi^{i,i+1}_{n_i,m}\rangle_{y_i}\|_{2+\lambda}^{2+\lambda}\\
&\leq 2^{(2+\lambda)(r_{i,1}+r_{i,i+1})} \iint\Big(\sum_{(n_i,m)\in\mathbb{Z}^2} |\langle g_1,\varphi^{i,1}_{n_i,m}\rangle_{x_i}|^{2+\lambda}\cdot|\langle g_{i+1},\varphi^{i,i+1}_{n_i,m}\rangle_{y_i}|^{2+\lambda}\Big)\,\mathrm{d}\hat{x}_i\,\mathrm{d}\hat{y}_i\\
&\leq 2^{(2+\lambda)(r_{i,1}+r_{i,i+1})} \iint\Big(\sum_{(n_i,m)\in\mathbb{Z}^2} |\langle g_1,\varphi^{i,1}_{n_i,m}\rangle_{x_i}|^{4+2\lambda}\Big)^{\frac{1}{2}}\cdot\Big(\sum_{(n_i,m)\in\mathbb{Z}^2} |\langle g_{i+1},\varphi^{i,i+1}_{n_i,m}\rangle_{y_i}|^{4+2\lambda}\Big)^{\frac{1}{2}}\,\mathrm{d}\hat{x}_i\,\mathrm{d}\hat{y}_i\\
&\lesssim_\lambda 2^{(2+\lambda)(r_{i,1}+r_{i,i+1})} \iint \|g_1\|^{2+\lambda}_{L^4_{x_i}}\cdot\|g_{i+1}\|^{2+\lambda}_{L^4_{y_i}}\,\mathrm{d}\hat{x}_i\,\mathrm{d}\hat{y}_i\\
&\lesssim 2^{(2+\lambda)(r_{i,1}+r_{i,i+1})}\cdot\|g_{1,i}\|_4^{2+\lambda}\cdot\Big(\prod_{\ell\neq i}\|g_{1,\ell}\|^{2+\lambda}_{2+\lambda}\Big)\cdot\|g_{i+1}\|_4^{2+\lambda},
\end{aligned}$$

where we used Hölder's inequality from the second to third lines, Fubini from the third to fourth, Hölder again twice, Proposition 4.3 and the fact that g_1 is a tensor. This finishes the proof of the lemma. □

As in the previous subsection, given $\varepsilon > 0$, we bound $\tilde{\tilde{\Lambda}}_{k,d}$ using the estimates from (84) and Corollary 12.4, and the ones from Lemma 12.8 with the exact same weights[36] we used in the proof of Theorem 12.1:

$$\begin{cases}\theta_l = \dfrac{1}{2(d+1)} - \dfrac{\varepsilon}{d}, \quad 1\leq l\leq d, & \text{for the } d \text{ estimates in (98) and (99)},\\[2mm] \theta_{d+1} = 1-\dfrac{d}{2(d+1)}+\varepsilon & \text{for (86).}\end{cases}$$

Hence,

$$\begin{aligned}
|\tilde{\tilde{\Lambda}}_{k,d}(g,h)| \lesssim& \sum_{\vec{l},R,t\geq 0} 2^{-t}\times 2^{-\frac{(d+1)\varepsilon}{2kd}l_1}\times\Big(\frac{1}{\|g_1\|_2^{d-1}}\prod_{j=1}^{d}2^{-r_{j,1}}\Big)^{\frac{1}{k}-\frac{(d+1)\varepsilon}{2kd}}\\
&\times\prod_{i=1}^{k-1}2^{-\frac{(d+1)\varepsilon}{2kd}l_{i+1}}\times\prod_{i=1}^{k-1}\Bigg[\frac{2^{-\frac{1}{d+1}\cdot r_{i,i+1}}}{\|\mathbb{1}_{\mathbb{X}^{l_{i+1};r_{i,i+1}}}\|^{\frac{1}{2(d+1)}}_{\ell^\infty_{n_i,m}\ell^1_{\hat{n}_i}}}\cdot\prod_{u=k}^{d}\frac{2^{-\frac{1}{k(d+1)}\cdot r_{u,i+1}}}{\|\mathbb{1}_{\mathbb{X}^{l_{i+1};r_{u,i+1}}}\|^{\frac{1}{2k(d+1)}}_{\ell^\infty_{n_u,m}\ell^1_{\hat{n}_u}}}\Bigg]^{1-\frac{(d+1)\varepsilon}{2d}}
\end{aligned}$$

[36] If $k=d+1$, we give weight $\frac{1}{2(d+1)}-\frac{\varepsilon}{d}$ to each one of the d estimates in (98) only.

$$\times \prod_{l=1}^{k-1} \left(2^{(2+\lambda)(r_{l,1}+r_{l,l+1})} \cdot \|g_{1,l}\|_4^{2+\lambda} \cdot \left(\prod_{\ell \neq l} \|g_{1,\ell}\|_{2+\lambda}^{2+\lambda} \right) \cdot \|g_{l+1}\|_4^{2+\lambda} \right)^{\frac{1}{2(d+1)} - \frac{\varepsilon}{d}}$$
$$\times \prod_{k \le u \le d} \left(\prod_{j=2}^{k} \|\mathbb{1}_{\mathbb{X}^{l_j; r_{u,j}}}\|_{\ell^\infty_{\hat{n}_u, m}\ell^1_{\hat{n}_u}}^{\frac{1}{k}} \cdot 2^{\frac{2}{k}\sum_{i \neq u} r_{i,1}} \cdot \|g_1\|_2^{\frac{2(d-1)}{k}} \right)^{\frac{1}{2(d+1)} - \frac{\varepsilon}{d}}$$
$$\times \prod_{k \le u \le d} \left(2^{\alpha \cdot r_{u,1} + \sum_{l=2}^{k} \beta \cdot r_{u,l}} \cdot \left(\prod_{j \neq u} \|g_{1,j}\|_2 \right)^{\alpha} \cdot \|g_{1,u}\|_4^{\alpha} \cdot \prod_{l=2}^{k} \|g_l\|_4^{\beta} \right)^{\frac{1}{2(d+1)} - \frac{\varepsilon}{d}}$$
$$\times (2^t |F|)^{1 - \frac{d}{2(d+1)} + \varepsilon}.$$

Developing the expression above[37],

$$|\tilde{\tilde{\Lambda}}_{k,d}(g,h)|$$
$$\lesssim \sum_{\vec{l},R,t \ge 0} 2^{-t} \times 2^{-\frac{(d+1)\varepsilon}{2kd} l_1} \times \left(\prod_{j=1}^{d} 2^{-r_{j,1}} \right)^{\frac{1}{k} - \frac{(d+1)\varepsilon}{2kd}} \times \|g_1\|_2^{\frac{(d+1)(d-1)\varepsilon}{2kd} - \frac{(d-1)}{k}}$$
$$\times \prod_{i=1}^{k-1} 2^{-\frac{(d+1)\varepsilon}{2kd} l_{i+1}} \times \prod_{i=1}^{k-1} \left[2^{-\frac{1}{d+1} \cdot r_{i,i+1}} \cdot \prod_{u=k}^{d} 2^{-\frac{1}{k(d+1)} \cdot r_{u,i+1}} \right]^{1 - \frac{(d+1)\varepsilon}{2d}}$$
$$\times \prod_{i=1}^{k-1} \left[\|\mathbb{1}_{\mathbb{X}^{l_{i+1}; r_{i,i+1}}}\|_{\ell^\infty_{\hat{n}_i, m}\ell^1_{\hat{n}_i}}^{\frac{1}{2(d+1)} \cdot \left(\frac{(d+1)\varepsilon}{2d} - 1 \right)} \cdot \prod_{u=k}^{d} \|\mathbb{1}_{\mathbb{X}^{l_{i+1}; r_{u,i+1}}}\|_{\ell^\infty_{\hat{n}_u, m}\ell^1_{\hat{n}_u}}^{\frac{1}{2k(d+1)} \left(\frac{(d+1)\varepsilon}{2d} - 1 \right)} \right]$$
$$\times \left[\prod_{l=1}^{k-1} \|\mathbb{1}_{\mathbb{X}^{l_{l+1}; r_{l,l+1}}}\|_{\ell^\infty_{\hat{n}_l, m}\ell^1_{\hat{n}_l}}^{\frac{1}{2(d+1)} - \frac{\varepsilon}{d}} \right] \times \left[\prod_{l=1}^{k-1} (2^{r_{l,1} + r_{l,l+1}})^{(2+\lambda) \cdot \left(\frac{1}{2(d+1)} - \frac{\varepsilon}{d} \right)} \right]$$
$$\times \left[\prod_{l=1}^{k-1} |E_{1,l}|^{\left(\frac{2+\lambda}{4} + (k-2) \right) \cdot \left(\frac{1}{2(d+1)} - \frac{\varepsilon}{d} \right)} \right] \cdot \left[\prod_{u=k}^{d} |E_{1,u}|^{\left(\frac{1}{2(d+1)} - \frac{\varepsilon}{d} \right) \cdot (k-1)} \right] \cdot \left[\prod_{l=1}^{k-1} |E_{l+1}|^{\frac{(2+\lambda)}{4} \cdot \left(\frac{1}{2(d+1)} - \frac{\varepsilon}{d} \right)} \right]$$
$$\times \prod_{u=k}^{d} \left[\left(\prod_{j=2}^{k} \|\mathbb{1}_{\mathbb{X}^{l_j; r_{u,j}}}\|_{\ell^\infty_{\hat{n}_u, m}\ell^1_{\hat{n}_u}}^{\frac{1}{k} \cdot \left(\frac{1}{2(d+1)} - \frac{\varepsilon}{d} \right)} \right) \cdot \left(2^{\frac{2}{k}\sum_{i \neq u} r_{i,1}} \cdot 2^{\alpha \cdot r_{u,1} + \sum_{l=2}^{k} \beta \cdot r_{u,l}} \right)^{\frac{1}{2(d+1)} - \frac{\varepsilon}{d}} \right]$$
$$\times \|g_1\|_2^{\frac{2(d-k+1)(d-1)}{k} \left(\frac{1}{2(d+1)} - \frac{\varepsilon}{d} \right)} \cdot \prod_{k \le u \le d} \left[\left(\|g_{1,u}\|_4 \cdot \prod_{j \neq u} \|g_{1,j}\|_2 \right)^{\alpha \left(\frac{1}{2(d+1)} - \frac{\varepsilon}{d} \right)} \right]$$
$$\times \prod_{l=2}^{k} \|g_l\|_4^{\beta(d-k+1) \left(\frac{1}{2(d+1)} - \frac{\varepsilon}{d} \right)} \times (2^t |F|)^{\left[1 - \frac{d}{2(d+1)} \right] + \varepsilon}. \tag{100}$$

Observe that we highlighted a few factors in red in (100); this is just to compare them to the red terms in (94): the red terms are the only ones that differ in the right-hand sides of (94) and (100). On the other hand, we will bound the product of the blue factors[38] in (100) in a slightly better way than we did in the

[37] The products in the fourth and fifth lines above are void if $k = d + 1$. We can think of them as being 1.

[38] The seventh and eighth lines are void if $k = d + 1$, hence the blue factors do not contribute at all in this case.

proof of Theorem 12.1:

$$\prod_{k\le u\le d}\Big(\|g_{1,u}\|_4\cdot\prod_{j\ne u}\|g_{1,j}\|_2\Big)=\Big[\prod_{j=1}^{d}\|g_{1,j}\|_2^{d-k}\Big]\cdot\Big[\prod_{l=1}^{k-1}\|g_{1,l}\|_2\Big]\cdot\Big[\prod_{u=k}^{d}\|g_{1,u}\|_4\Big]$$
$$\le\Big[\prod_{l=1}^{k-1}|E_{1,l}|^{\frac{d-k+1}{2}}\Big]\cdot\Big[\prod_{u=k}^{d}|E_{1,u}|^{\frac{d-k}{2}+\frac14}\Big]. \tag{101}$$

Setting δ and $\tilde{\delta}$ exactly as in the previous subsection and using the observations we just made, we conclude that the final bound for $|\widetilde{\widetilde{\Lambda}}_{k,d}(g,h)|$ compares to (96) exactly as follows:

- The coefficients of the "$r_{j,1}$ power" is now

$$2^{[-\frac{3(d+1)\varepsilon}{2kd}+\lambda(\frac{1}{2(d+1)}-\frac{\varepsilon}{d})]r_{j,1}},$$

whereas in (96) it was

$$2^{(-\frac{3(d+1)\varepsilon}{2kd})r_{j,1}}.$$

- For $1\le l\le k-1$, (101) gives $|E_{1,l}|$ an extra power of[39]

$$\Big(\frac12+\frac{1}{2k}\Big)\cdot\Big(\frac{1}{2(d+1)}-\frac{\varepsilon}{d}\Big)+\frac{(d+1)\varepsilon}{4kd}.$$

On the other hand, still for $1\le l\le k-1$, the red factors in (100) produce a power of $|E_{1,l}|$ that is exactly

$$\frac{(2-\lambda)}{4}\cdot\Big(\frac{1}{2(d+1)}-\frac{\varepsilon}{d}\Big) \tag{102}$$

less than the one produced by the corresponding red factors in (94). If $k<d+1$, these provide a *net gain* of

$$\Big(\frac{1}{2k}-\frac{\lambda}{4}\Big)\cdot\Big(\frac{1}{2(d+1)}-\frac{\varepsilon}{d}\Big)+\frac{(d+1)\varepsilon}{4kd}$$

in the final power of $|E_{1,l}|$. If $k=d+1$, we just lose (compared to (96)) (102) in the final power of $|E_{1,l}|$.

- For $k\le u\le d$, the powers of the measures $|E_{1,u}|$ are exactly the same in both (94) and in (100).
- For $2\le l\le k$, the red factors in (100) produce a power of $|E_l|$ that is exactly

$$\frac{(2-\lambda)}{4}\Big(\frac{1}{2(d+1)}-\frac{\varepsilon}{d}\Big)$$

less than the one produced by the corresponding red factors in (94).

- All other factors are precisely the same.

[39] Here we are using the explicit choice of δ.

By choosing λ small enough compared to ε and by the same considerations made in the end of Section 9, this implies

$$|\tilde{\tilde{\Lambda}}_{k,d}(g,h)| \lesssim_\varepsilon |F|^{1-\frac{d}{2(d+1)}+\varepsilon} \cdot |E_1|^{\frac{2d-k+2}{4k(d+1)}} \prod_{l=1}^{k-1} |E_{l+1}|^{\frac{1}{4k}}$$

for $k < d+1$ and

$$|\tilde{\tilde{\Lambda}}_{k,d}(g,h)| \lesssim_\varepsilon |F|^{1-\frac{d}{2(d+1)}+\varepsilon} \cdot \prod_{l=1}^{k} |E_l|^{\frac{1}{4k}}$$

for $k = d+1$. Again by the same considerations from Section 9, these imply Theorem 12.6. □

13. Weak transversality, Brascamp–Lieb and an application

We were recently asked by Jonathan Bennett if there was a link between our results and the theory of Brascamp–Lieb inequalities. The motivation for that comes from the fact that, assuming $g_1 = g_{1,1} \otimes \cdots \otimes g_{1,d}$, one can see the operator $\mathcal{ME}_{d+1,d}$ as the $2d$-linear object

$$T(g_{1,1}, \ldots, g_{1,d}, g_2, \ldots, g_{d+1}) := \mathcal{ME}_{d+1,d}(g_{1,1} \otimes \cdots \otimes g_{1,d}, g_2, \ldots, g_{d+1}),$$

and given that such a link exists in the theory of $\mathcal{ME}_{d+1,d}$ (see [Bennett 2014]), it is natural to wonder if boundedness for T is related somehow to the finiteness condition of certain Brascamp–Lieb constants $\mathrm{BL}(\boldsymbol{L}, \boldsymbol{p})$.

The purposes of this section are to make this connection clear and to give a modest application of our results to the theory of *restriction-Brascamp–Lieb inequalities*.

13A. ***A link between weak transversality and Brascamp–Lieb inequalities.*** We start with some classical background. Let $L_j : \mathbb{R}^n \to \mathbb{R}^{n_j}$ be linear maps and $p_j \geq 0$, $1 \leq j \leq m$. Inequalities of the form

$$\int_{\mathbb{R}^n} \prod_{j=1}^{m} (f_j \circ L_j)^{p_j}(v)\,\mathrm{d}v \leq C \prod_{j=1}^{m} \left(\int_{\mathbb{R}^{n_j}} f_j(y_j)\,\mathrm{d}y_j \right)^{p_j} \tag{103}$$

are called *Brascamp–Lieb inequalities*. Bennett, Carbery, Christ and Tao [Bennett et al. 2008] established for which *Brascamp–Lieb data* $(\boldsymbol{L}, \boldsymbol{p})$ the inequality above holds, where $\boldsymbol{L} = (L_1, \ldots, L_m)$ and $\boldsymbol{p} = (p_1, \ldots, p_m)$. The best constant for which (103) holds for all nonnegative input functions $f_j \in L^1(\mathbb{R}^{n_j})$ is denoted by $\mathrm{BL}(\boldsymbol{L}, \boldsymbol{p})$.

Theorem 13.1 [Bennett et al. 2008]. *The constant* $\mathrm{BL}(\boldsymbol{L}, \boldsymbol{p})$ *in* (103) *is finite if and only if for all subspaces* $V \subset \mathbb{R}^n$

$$\dim(V) \leq \sum_{j=1}^{m} p_j \dim(L_j V) \tag{104}$$

and

$$\sum_{j=1}^{m} p_j n_j = n. \tag{105}$$

Remark 13.2. By taking $V = \mathbb{R}^n$ in (104) it follows that each L_j must be surjective for (105) to hold as well.

We will work with explicit maps L_j and use Theorem 13.1 to establish a link between the concept of weak transversality and inequalities such as (103).[40] These maps will be associated to the submanifolds relevant to the problem at hand: the d-dimensional paraboloid $\mathbb{P}^d$ in $\mathbb{R}^{d+1}$ and some "canonical" two-dimensional parabolas.

In order to define L_j, we fix standard parametrizations for the submanifolds mentioned above. Let

$$\Gamma : \mathbb{R}^d \longrightarrow \mathbb{R}^{d+1}, \tag{106}$$

$$(x_1, \dots, x_d) \longmapsto \left(x_1, \dots, x_d, \textstyle\sum_{i=1}^d x_i^2\right), \tag{107}$$

parametrize $\mathbb{P}^d$ and

$$\gamma_j : \mathbb{R} \longrightarrow \mathbb{R}^{d+1}, \tag{108}$$

$$x \longmapsto (x \cdot \delta_{1j}, \dots, x \cdot \delta_{dj}, x^2), \tag{109}$$

parametrize a parabola in the two-dimensional canonical subspace generated by e_j and e_{d+1} (δ_{ij} is the Kronecker delta). Their differentials are given by

$$\mathrm{d}\Gamma : \mathbb{R}^d \longrightarrow M_{(d+1)\times d}, \quad (x_1, \dots, x_d) \longmapsto \begin{bmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 \\ 2x_1 & 2x_2 & \dots & 2x_d \end{bmatrix}$$

and

$$\mathrm{d}\gamma_j : \mathbb{R} \longrightarrow M_{(d+1)\times 1}, \quad x \longmapsto [\delta_{1j} \;\; \delta_{2j} \;\; \cdots \;\; \delta_{dj} \;\; 2x]^\top.$$

For $d+1$ points $x^j = (x_1^j, \dots, x_d^j) \in \mathbb{R}^d$, $1 \le j \le d+1$, define the linear maps[41]

$$\begin{aligned} L_\ell^{x_\ell^1} &:= (\mathrm{d}\gamma_\ell(x_\ell^1))^* && \text{for all } 1 \le \ell \le d, \\ L_{d+\ell}^{x^{\ell+1}} &:= (\mathrm{d}\Gamma(x_1^{\ell+1}, \dots, x_d^{\ell+1}))^* && \text{for all } 1 \le \ell \le d. \end{aligned} \tag{110}$$

It is important to emphasize that $L_{d+\ell}$ *depends on* $x^{\ell+1}$ (and similarly, L_ℓ depends on x_ℓ^1). The main result of this subsection is:

Theorem 13.3. *Let $\mathcal{Q} = \{Q_1, \dots, Q_{d+1}\}$ be a collection of closed cubes in $\mathbb{R}^d$. If $\mathcal{Q}$ is weakly transversal with pivot Q_1, then for any choice of points $x^j = (x_1^j, \dots, x_d^j) \in Q_j$, the linear maps in (110) satisfy*

$$\mathrm{BL}(\boldsymbol{L}(x), \boldsymbol{p}) < \infty \quad \textit{for } \boldsymbol{L}(x) = (L_1^{x_1^1}, \dots, L_{2d}^{x^{d+1}}) \textit{ and } \boldsymbol{p} = \left(\frac{1}{d}, \dots, \frac{1}{d}\right). \tag{111}$$

Conversely, if (111) is satisfied by the linear maps in (110) for any choice of points $x^j = (x_1^j, \dots, x_d^j) \in Q_j$, then $\mathcal{Q}$ can be decomposed into $O(1)$ weakly transversal collections $\mathcal{Q}'$ of $d+1$ cubes, each one having a cube $Q_1' \subset Q_1$ as pivot.

[40] From now on, we will replace n by $d+1$ when referring to the dimension of the euclidean space.

[41] We highlight that the *superscript* j in x_i^j denotes the *point*, whereas the *subscript* i denotes the i*-coordinate* of the corresponding point. Notice also that we are identifying the adjoint operator T^* with the transpose of the matrix that represents T in the canonical basis.

Remark 13.4. If $\mathcal{Q}$ can be decomposed into $O(1)$ weakly transversal collections $\mathcal{Q}'$ of $d+1$ cubes (in the sense of Claim 3.4), each one having a cube $Q_1' \subset Q_1$ as pivot, then the conclusion of the first part of the theorem above also holds for $\mathcal{Q}$. Some important examples to keep in mind are the ones of transversal configurations that are *not* weakly transversal by themselves, but that are decomposable into such: for instance, $\{Q_1, Q_2, Q_3\}$, where $Q_1 = [1,4] \times [2,3]$, $Q_2 = [0,2] \times [0,1]$ and $Q_3 = [3,5] \times [0,1]$ is a transversal collection of cubes in $\mathbb{R}^2$, but not weakly transversal with pivot Q_1 since $\pi_1(Q_1)$ intersects both $\pi_1(Q_2)$ and $\pi_1(Q_3)$.

Remark 13.5. We can of course obtain a similar statement if $\mathcal{Q}$ is weakly transversal with any other pivot Q_j, $j \neq 1$. The linear maps L_ℓ and $L_{d+\ell}$ would have to be changed accordingly.

Proof of Theorem 13.3. Suppose that $\mathcal{Q}$ is weakly transversal with pivot Q_1. We can then assume without loss of generality that

$$\begin{cases} \pi_1(Q_1) \cap \pi_1(Q_2) = \varnothing, \\ \quad \vdots \\ \pi_d(Q_1) \cap \pi_d(Q_{d+1}) = \varnothing. \end{cases} \tag{112}$$

The strategy is to apply Theorem 13.1. Condition (105) is trivially satisfied, so we just have to check (104). Fix the points $x^j = (x_1^j, \dots, x_d^j) \in Q_j$, $1 \le j \le d$. To avoid heavy notation, we will omit the superscripts x_ℓ^1 and $x^{\ell+1}$ when referring to $L_\ell^{x_\ell^1}$ and $L_{d+\ell}^{x^{\ell+1}}$, respectively, but these points will be referenced whenever they play an important role. We emphasize that the maps L_ℓ, $1 \le \ell \le d$, are being identified with the row vector

$$[\delta_{1\ell} \;\; \delta_{2\ell} \;\; \dots \;\; \delta_{d\ell} \;\; 2x_\ell^1],$$

whereas the maps $L_{d+\ell}$, $1 \le \ell \le d$, are identified with the $d \times (d+1)$ matrix

$$\begin{bmatrix} 1 & 0 & \dots & 0 & 2x_1^{\ell+1} \\ 0 & 1 & \dots & 0 & 2x_2^{\ell+1} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 1 & 2x_d^{\ell+1} \end{bmatrix}.$$

If $V \subset \mathbb{R}^{d+1}$ is a subspace of dimension k, we have to verify that

$$dk \le \sum_{j=1}^{d} \dim(L_j V) + \sum_{\ell=1}^{d} \dim(L_{d+\ell} V). \tag{113}$$

Suppose that there are exactly $m \ge 0$ indices $j \in \{1, \dots, d\}$ such that $\dim(L_j V) = 0$. If $m = 0$, we must have $L_j V = \mathbb{R}$ for all $1 \le j \le d$; hence

$$\sum_{j=1}^{d} \dim(L_j V) = d. \tag{114}$$

Surjectivity of $L_{d+\ell}$, $1 \le \ell \le d$, implies $\dim(\ker(L_{d+\ell})) = 1$, which gives the lower bound $\dim(L_{d+\ell} V) \ge k-1$. We then obtain

$$\sum_{\ell=1}^{d} \dim(L_{d+\ell} V) \ge d(k-1). \tag{115}$$

It is clear that (114) and (115) together verify (113) in the $m = 0$ case. If $m \geq 1$, assume without loss of generality that

$$L_1 V = \cdots = L_m V = 0, \tag{116}$$

$$L_{m+1} V = \ldots = L_d V = \mathbb{R}. \tag{117}$$

This gives us

$$\sum_{j=1}^{d} \dim(L_j V) = d - m. \tag{118}$$

We will show that

$$\sum_{\ell=1}^{d} \dim(L_{d+\ell} V) \geq (d-m)(k-1) + mk. \tag{119}$$

Observe that (118) and (119) together verify (113) in the $m \geq 1$ case.

We claim that there are at least m maps L_{ℓ_j} among $L_{\ell+1}, \ldots, L_{2d}$ such that $\dim(L_{\ell_j} V) = k$. If not, there are $d - m + 1$ maps $L_{\ell_1}, \ldots, L_{\ell_{d-m+1}}$ with $\dim(L_{\ell_j} V) \leq k - 1$. Since $\dim V = k$, the rank-nullity theorem implies the existence of

$$0 \neq v^{\ell_j} \in \ker(L_{\ell_j}) \cap V, \quad 1 \leq j \leq d - m + 1. \tag{120}$$

By (116),

$$L_r v^{\ell_j} = v_r^{\ell_j} + 2x_r^1 v_{d+1}^{\ell_j} = 0, \quad 1 \leq r \leq m, \tag{121}$$

and by (120) we have

$$L_{\ell_j} v^{\ell_j} = \begin{bmatrix} 1 & 0 & \ldots & 0 & 2x_1^{\ell_j - d + 1} \\ 0 & 1 & \ldots & 0 & 2x_2^{\ell_j - d+1} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \ldots & 1 & 2x_d^{\ell_j - d+1} \end{bmatrix} \cdot \begin{bmatrix} v_1^{\ell_j} \\ v_2^{\ell_j} \\ \vdots \\ v_{d+1}^{\ell_j} \end{bmatrix} = \begin{bmatrix} v_1^{\ell_j} + 2x_1^{\ell_j - d+1} v_{d+1}^{\ell_j} \\ v_2^{\ell_j} + 2x_2^{\ell_j - d+1} v_{d+1}^{\ell_j} \\ \vdots \\ v_d^{\ell_j} + 2x_d^{\ell_j - d+1} v_{d+1}^{\ell_j} \end{bmatrix} = 0 \tag{122}$$

for $1 \leq j \leq d - m + 1$. For each $1 \leq r \leq m$, combining the information from (121) and (122) gives us

$$v_{d+1}^{\ell_j} \cdot (x_r^1 - x_r^{\ell_j - d+1}) = 0.$$

If $v_{d+1}^{\ell_j} = 0$, then (122) also implies $v_n^{\ell_j} = 0$ for all $n \in \{1, \ldots, d\}$; thus $v^{\ell_j} = 0$, which contradicts (120). Then we must have

$$x_r^1 = x_r^{\ell_j - d+1}, \quad 1 \leq r \leq m.$$

Let us now see why this cannot happen. We have just shown that there are $d - m + 1$ values of α for which

$$\begin{cases} \pi_1(Q_1) \cap \pi_1(Q_\alpha) \neq \varnothing, \\ \quad \vdots \\ \pi_m(Q_1) \cap \pi_m(Q_\alpha) \neq \varnothing. \end{cases} \tag{123}$$

On the other hand, (112) tells us that $\alpha \notin \{2, 3, \ldots, m+1\}$; hence there are at most $d - m$ possible values for α (we cannot have $\alpha = 1$ either), which is a contradiction.

Hence there are at least m maps L_{ℓ_j} among $L_{\ell+1}, \ldots, L_{2d}$ such that $\dim(L_{\ell_j} V) = k$. The remaining $d-m$ maps have kernels of dimension 1, so the image of V through them has dimension at least $k-1$ (again by surjectivity of L_{ℓ_j} and the rank-nullity theorem). This verifies (119).

For the converse implication, suppose that (111) is satisfied by the linear maps in (110) for any choice of points $(x_1^j, \ldots, x_d^j) \in Q_j$. As a consequence of the proof of Claim B.4, each $Q_l \in \mathcal{Q}$ can be partitioned into $O(1)$ subcubes

$$Q_l = \bigcup_i Q_{l,i}$$

so that all collections $\widetilde{\mathcal{Q}}$ made of picking one subcube $Q_{l,i}$ per Q_l

$$\widetilde{\mathcal{Q}} = \{\widetilde{Q}_1, \ldots, \widetilde{Q}_{d+1}\}, \quad \widetilde{Q}_l \in \{Q_{l,i}\}_i,$$

satisfy the following:

(a) For any two $\widetilde{Q}_r, \widetilde{Q}_s \in \widetilde{\mathcal{Q}}$, either $\pi_j(\widetilde{Q}_r) \cap \pi_j(\widetilde{Q}_s) = \varnothing$, or $\pi_j(\widetilde{Q}_r) = \pi_j(\widetilde{Q}_s)$, or $\pi_j(\widetilde{Q}_r) \cap \pi_j(\widetilde{Q}_s) = \{p_{r,s}\}$, where $p_{r,s}$ is an endpoint of both $\pi_j(\widetilde{Q}_r)$ and $\pi_j(\widetilde{Q}_s)$.

(b) All $\pi_j(\widetilde{Q}_s)$ that intersect a given $\pi_j(\widetilde{Q}_r)$ (but distinct from it) do so at the same endpoint.[42]

By a slight abuse of notation, let $\mathcal{Q}$ denote one such subcollection that has the two properties above. Suppose, by contradiction, that $\mathcal{Q}$ is not weakly transversal with pivot Q_1 (recall that this is a cube obtained from the original Q_1). The strategy now is to construct a subspace $V \subset \mathbb{R}^{d+1}$ that contradicts (104) for a certain choice of one point per cube in $\mathcal{Q}$. This construction will exploit a certain feature of a special subset of $\mathcal{Q}$, which is the content of Claim 13.6.

For simplicity of future references, let us say that a subset $\mathcal{A} \subset \mathcal{Q}$ has the *property* (P) if:

(1) $Q_1 \in \mathcal{A}$.

(2) $\mathcal{A}$ is not weakly transversal with pivot Q_1.

We say that a subset $\mathcal{A} \subset \mathcal{Q}$ is *minimal* if $\mathcal{A}' \subset \mathcal{A}$ has the property (P) if and only if $\mathcal{A}' = \mathcal{A}$. It is clear that, since $\mathcal{Q}$ has the property (P) itself, it must contain a minimal subset of cardinality at least 2.

Claim 13.6. *Let $\mathcal{A} = \{Q_1, K_2, \ldots, K_n\}$ be a minimal set of n cubes.*[43] *There is a set D of $d-n+2$ canonical directions v for which*

$$\pi_v(Q_1) \cap \pi_v(K_j) \neq \varnothing \quad \textit{for all } 2 \leq j \leq n. \tag{124}$$

Proof of Claim 13.6. See Claim B.6 in Appendix B. □

We know that $\mathcal{Q}$ has a minimal subset of cardinality $2 \leq n \leq d+1$. By the previous claim and by conditions (a) and (b) of our initial reductions, if $\mathcal{A}' = \{Q_1, K_2, \ldots, K_n\}$ is a minimal subset of $\mathcal{Q}$, for

[42] In other words, all $\pi_j(\widetilde{Q}_s)$ that intersect a given $\pi_j(\widetilde{Q}_r)$ (but distinct from it) do so on the same side. In short notation, let $\mathcal{S}_{j,r}$ be the set of s for which $\pi_j(\widetilde{Q}_r) \cap \pi_j(\widetilde{Q}_s) \neq \varnothing$. The conclusion is that there is some real number γ_j such that $\gamma_j \in \pi_j(Q_r) \cap \bigcap_{s \in \mathcal{S}_{j,r}} \pi_j(Q_s)$.

[43] Observe that Q_1 is the only "Q" cube in this collection. The others are labeled by K_j.

every $v \in D$ there is a number γ_v such that

$$\gamma_v \in \pi_v(Q_1) \cap \bigcap_{j=2}^{n} \pi_v(K_j).$$

Indeed, $\pi_v(Q_1)$ intersects each $\pi_v(Q_j)$ "on the same side", so the intersection above must be nonempty (the existence of these γ_v is the only reason why we may need to decompose the initial collection $\mathcal{Q}$ into subcollections that satisfy (a) and (b)).

For simplicity and without loss of generality, assume that $\mathcal{A} = \{Q_1, Q_2, \dots, Q_n\}$ is minimal[44] and $D = \{e_1, \dots, e_{d-n+2}\}$. Consider the points

$$(\gamma_1, \dots, \gamma_{d-n+2}, x^j_{d-n+3}, \dots, x^j_d) \in Q_j, \quad 1 \le j \le n,$$
$$(x^l_1, \dots, x^l_d) \in Q_l, \quad n+1 \le l \le d+1.$$

By hypothesis, $\mathrm{BL}(\boldsymbol{L}(x), \boldsymbol{p}) < \infty$ for the following collection of linear maps and exponents:

$$L^{\gamma_r}_r(v_1, \dots, v_{d+1}) = v_r + 2\gamma_r v_{d+1}, \quad 1 \le r \le d-n+2,$$
$$L^{x^1_s}_s(v_1, \dots, v_{d+1}) = v_s + 2x^1_s v_{d+1}, \quad d-n+3 \le s \le d,$$

$$L^{(\gamma_1,\dots,\gamma_{d-n+2},x^{r+1}_{d-n+3},\dots,x^{r+1}_d)}_{d+r}(v_1, \dots, v_{d+1}) = \begin{bmatrix} v_1 + 2\gamma_1 v_{d+1} \\ \vdots \\ v_{d-n+2} + 2\gamma_{d-n+2} v_{d+1} \\ v_{d-n+3} + 2x^{r+1}_{d-n+3} v_{d+1} \\ \vdots \\ v_d + 2x^{r+1}_d v_{d+1} \end{bmatrix}, \quad 1 \le r \le n-1,$$

$$L^{x^{l+1}}_{d+l} = \begin{bmatrix} v_1 + 2x^{l+1}_1 v_{d+1} \\ \vdots \\ v_d + 2x^{l+1}_d v_{d+1} \end{bmatrix}, \quad n \le l \le d, \qquad \boldsymbol{p} = \left(\frac{1}{d}, \dots, \frac{1}{d}\right).$$

Define

$$V := \bigcap_{r=1}^{d-n+2} \ker(L^{\gamma_r}_r).$$

Observe that $\dim(V) = n-1$. Indeed, if we start with a vector $v = (v_1, \dots, v_{d+1})$ of $d+1$ "free coordinates", we lose one degree of freedom for each kernel in the intersection above, since $L^{\gamma_r}_r(v) = 0$ gives a relation between v_r and v_{d+1}. We have $d-n+2$ many of them; hence the total degree of freedom is $(d+1)-(d-n+2) = n-1$, which is the dimension of V. On the other hand, for every $v \in V$ we have by definition

$$L^{\gamma_r}_r(v) = 0, \quad 1 \le r \le d-n+2.$$

Hence

$$\sum_{j=1}^{d} \dim(L_j V) \le n-2.$$

[44]Here we are assuming $K_j = Q_j$, $2 \le j \le n$.

Also,

$$L_{d+r}^{(\gamma_1,\dots,\gamma_{d-n+2},x_{d-n+3}^{r+1},\dots,x_d^{r+1})}(v)=\begin{bmatrix}0\\ \vdots\\ 0\\ v_{d-n+3}+2x_{d-n+3}^{r+1}v_{d+1}\\ \vdots\\ v_d+2x_d^{r+1}v_{d+1}\end{bmatrix},\quad 1\le r\le n-1.$$

Thus

$$\dim(L_{d+r}V)\le n-2,\quad 1\le r\le n-1.$$

Since $\dim(V)=n-1$, we have the trivial bound

$$\dim(L_{d+l}V)\le n-1,\quad n\le l\le d.$$

Altogether, these bounds imply

$$\begin{aligned}\frac{1}{d}\biggl(\sum_{j=1}^{d}\dim(L_jV)+\sum_{\ell=1}^{d}\dim(L_{d+\ell}V)\biggr)&\le\frac{1}{d}[(n-2)+(n-1)(n-2)+(d-n+1)(n-1)]\\&=\frac{1}{d}[(n-1)d-1]<n-1=\dim(V).\end{aligned}$$

Our initial hypothesis, however, is that $\mathrm{BL}(\boldsymbol{L}(x),\boldsymbol{p})<\infty$; therefore by Theorem 13.1 we must have

$$\dim(V)\le\frac{1}{d}\biggl(\sum_{j=1}^{d}\dim(L_jV)+\sum_{\ell=1}^{d}\dim(L_{d+\ell}V)\biggr),$$

which gives a contradiction. We conclude that $\mathcal{Q}$ is weakly transversal with pivot Q_1. □

13B. ***An application to Restriction-Brascamp–Lieb inequalities.*** The following conjecture was proposed in Bennett, Bez, Flock and Lee [Bennett et al. 2018]:

Conjecture 13.7. *Suppose that, for each* $1\le j\le m$, $\Sigma_j:U_j\mapsto\mathbb{R}^n$ *is a smooth parametrization of a* n_j*-dimensional submanifold* S_j *of* $\mathbb{R}^n$ *by a neighborhood* U_j *of the origin in* $\mathbb{R}^{n_j}$. *Let*

$$\mathcal{E}_jg_j(\xi):=\int_{U_j}e^{-2\pi i\xi\cdot\Sigma_j(x)}g_j(x)\,\mathrm{d}x$$

be the associated (parametrized) extension operator. If the Brascamp–Lieb constant $\mathrm{BL}(\boldsymbol{L},\boldsymbol{p})$ *is finite for the linear maps* $L_j:=(\mathrm{d}\Sigma_j(0))^*:\mathbb{R}^n\mapsto\mathbb{R}^{n_j}$, *then provided the neighborhoods* U_j *of* 0 *are chosen to be small enough, the inequality*

$$\int_{\mathbb{R}^n}\prod_{j=1}^{m}|\mathcal{E}_jg_j|^{2p_j}\lesssim\prod_{j=1}^{m}\|g_j\|_{L^2(U_j)}^{2p_j}\tag{125}$$

holds for all $g_j\in L^2(U_j)$, $1\le j\le m$.

Remark 13.8. The weaker inequality

$$\int_{B(0,R)}\prod_{j=1}^{m}|\mathcal{E}_jg_j|^{2p_j}\lesssim_\varepsilon R^\varepsilon\prod_{j=1}^{m}\|g_j\|_{L^2(U_j)}^{2p_j}\tag{126}$$

involving an arbitrary $\varepsilon>0$ loss was established in [Bennett et al. 2018].

Remark 13.9. Very few cases of Conjecture 13.7 are fully understood.[45] Recently, Bennett, Nakamura and Shiraki settled the *rank*-1 *case* $n_1 = \cdots = n_m = 1$ as an application of their results on *tomographic Fourier analysis*.[46]

Given their hybrid nature, estimates such as (125) are called *restriction-Brascamp–Lieb inequalities*.

Our goal here is to verify Conjecture 13.7 in a special case. We chose to state the main result of this subsection in a way that does not emphasize the origin in the domains of Σ_j. The reason for this choice is that it brings to light key geometric features of the problem.

We will need a result from [Bennett et al. 2018] on the stability of Brascamp–Lieb constants[47]:

Theorem 13.10 [Bennett et al. 2018]. *Suppose that* $(\boldsymbol{L}^0, \boldsymbol{p})$ *is a Brascamp–Lieb datum for which* $\mathrm{BL}(\boldsymbol{L}^0, \boldsymbol{p}) < \infty$. *Then there exists* $\delta > 0$ *and a constant* $C < \infty$ *such that*

$$\mathrm{BL}(\boldsymbol{L}, \boldsymbol{p}) \le C$$

whenever $\|\boldsymbol{L} - \boldsymbol{L}^0\| < \delta$.

Now we are ready to state and prove our result:

Theorem 13.11. *Let* Γ *and* γ_j *be the parametrizations from* (106) *and* (108), *respectively. If, for* $x^j = (x_1^j, \dots, x_d^j) \in \mathbb{R}^d$, *the linear maps in* (110) *satisfy*

$$\mathrm{BL}(\boldsymbol{L}(x), \boldsymbol{p}) < \infty \quad \text{for } \boldsymbol{L}(x) = (L_1^{x_1^1}, \dots, L_{2d}^{x^{d+1}}) \text{ and } \boldsymbol{p} = \left(\frac{1}{d}, \dots, \frac{1}{d}\right), \tag{127}$$

then there are small enough cube-neighborhoods $U_i \subset \mathbb{R}$ $(1 \le i \le d)$ *of* x_i^1 *and* $V_\ell \subset \mathbb{R}^d$ *of* x^ℓ $(2 \le \ell \le d+1)$ *for which* (125) *holds.*

Remark 13.12. Rephrasing Theorem 13.11 in terms of the original statement, it says that Conjecture 13.7 holds for[48]

$$\Sigma_i = \gamma_i - (\delta_{1i} \cdot x_i^1, \dots, \delta_{di} \cdot x_i^1, 0), \quad 1 \le i \le d.$$
$$\Sigma_\ell = \Gamma - (x^{\ell-d+1}, 0), \quad d+1 \le \ell \le 2d.$$
$$m = 2d, \quad \boldsymbol{p} = \left(\frac{1}{d}, \dots, \frac{1}{d}\right).$$

Proof of Theorem 13.11. The argument is just a matter of putting the pieces together. By (127) and Theorem 13.10, there are small enough cube-neighborhoods $U_i \subset \mathbb{R}$ $(1 \le i \le d)$ of x_i^1 and $V_\ell \subset \mathbb{R}^d$ of x^ℓ $(2 \le \ell \le d+1)$ for which (127) still holds[49]. Define

$$Q_1 := \overline{U_1 \times \cdots \times U_d}, \qquad Q_\ell := \overline{V}_\ell, \quad 2 \le \ell \le d+1.$$

[45] Most of them being very elementary situations, as mentioned in [Bennett et al. 2018].

[46] See [Bennett and Nakamura 2021] for a more detailed exposition of this approach.

[47] Theorem 13.10 says that the map $\boldsymbol{L} \mapsto \mathrm{BL}(\boldsymbol{L}, \boldsymbol{p})$ is *locally bounded* for a fixed $\boldsymbol{p}$, and this is enough for our purposes. On the other hand, it was shown in [Bennett et al. 2017] that the Brascamp–Lieb constant is *continuous* in $\boldsymbol{L}$. It was later shown in [Bennett et al. 2020] that $\mathrm{BL}(\boldsymbol{L}, \boldsymbol{p})$ is in fact *locally Hölder continuous* in $\boldsymbol{L}$.

[48] Observe that we are just translating the domain of the Σ's back to the origin.

[49] Our maps L_j are sufficiently smooth for the stability theorem to be applied. The entries of the matrices that represent them are polynomials.

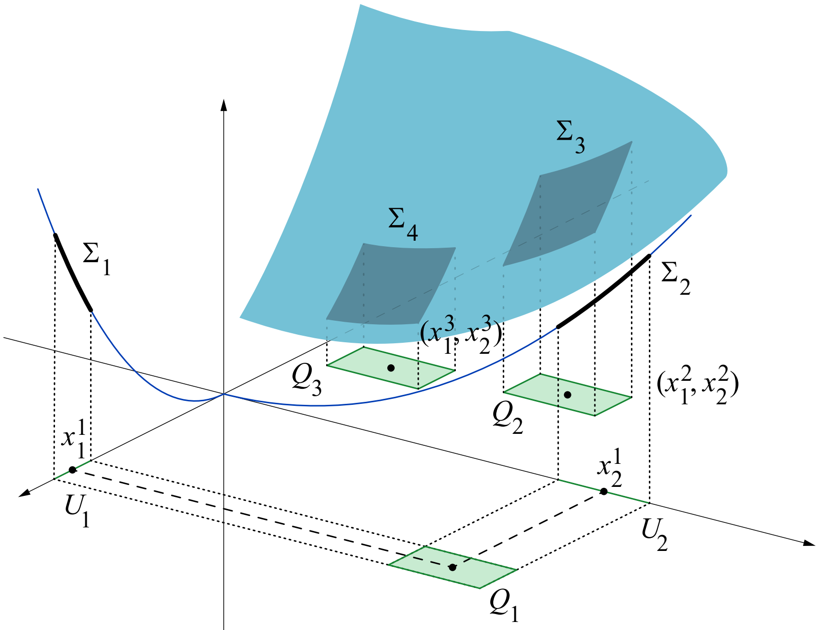

Figure 4. Unveiling the geometric features of the problem when $d = 2$. The cubes we find from Theorem 13.10 are weakly transversal, which gives us access to our earlier results.

Now we apply Theorem 13.3 to conclude that the collection $\mathcal{Q} = \{Q_1, \ldots, Q_{d+1}\}$ can be decomposed into $O(1)$ weakly transversal collections $\mathcal{Q}'$ of $d+1$ cubes, each one having a cube $Q_1' \subset Q_1$ as pivot.

To each such subcollection we apply the endpoint estimate from Section 10 (all we need to apply it is weak transversality), which finishes the proof. □

14. Further remarks

Remark 14.1. It was pointed out to us by Jonathan Bennett that the d-dimensional estimates (2) for tensors are equivalent to certain one-dimensional mixed norm bounds. We present this remark in the following proposition:

Proposition 14.2 (Bennett). *For all $p, q \geq 1$, the estimate*

$$\|\mathcal{E}_d g\|_{L^q_{\xi_1,\ldots,\xi_{d+1}}} \lesssim \|g\|_p \tag{128}$$

holds for tensors $g(x) = g_1(x_1) \cdot \cdots \cdot g_d(x_d)$ if and only if

$$\|\mathcal{E}_1 f\|_{L^{dq}_{\xi_2} L^q_{\xi_1}} \lesssim \|f\|_p. \tag{129}$$

holds.

Proof. Assume first that (128) holds for tensors. Then

$$\begin{aligned}\|\mathcal{E}_1 f\|_{L^{dq}_{\xi_2} L^q_{\xi_1}} &= \Big[\int\Big[\int |\mathcal{E}_1 f(\xi_1,\xi_2)|^q \,\mathrm{d}\xi_1\Big]^d \mathrm{d}\xi_2\Big]^{\frac{1}{dq}} \\ &= \Big[\int \prod_{j=1}^d \Big[\int |\mathcal{E}_1 f(\eta_j,\xi_2)|^q \,\mathrm{d}\eta_j\Big] \mathrm{d}\xi_2\Big]^{\frac{1}{dq}} \\ &= \Big[\int \prod_{j=1}^d \int |\mathcal{E}_d(f\otimes\cdots\otimes f)(\eta_1,\dots,\eta_d)|^q \,\mathrm{d}\vec{\eta}\,\mathrm{d}\xi_2\Big]^{\frac{1}{dq}} \\ &= \|\mathcal{E}_d(f\otimes\cdots\otimes f)\|_q^{\frac{1}{d}} \lesssim \|f\otimes\cdots\otimes f\|_p^{\frac{1}{d}} \lesssim \|f\|_p,\end{aligned}$$

which proves (129). Conversely, assuming that (129) holds for all $f\in L^p([0,1])$ yields

$$\begin{aligned}\|\mathcal{E}_d(g_1\otimes\cdots\otimes g_d)\|_q^q &= \int |\mathcal{E}_1 g_1(\xi_1,\xi_{d+1})|^q \cdot\dots\cdot |\mathcal{E}_1 g_d(\xi_d,\xi_{d+1})|^q \,\mathrm{d}\xi_1\cdot\dots\cdot\mathrm{d}\xi_{d+1} \\ &= \int \prod_{j=1}^d \Big[\int |\mathcal{E}_1 g_j(\xi_j,\xi_{d+1})|^q \,\mathrm{d}\xi_j\Big] .\mathrm{d}\xi_{d+1} \\ &\le \prod_{j=1}^d \Big[\int\Big[\int |\mathcal{E}_1 g_j(\xi_j,\xi_{d+1})|^q \,\mathrm{d}\xi_j\Big]^d \mathrm{d}\xi_{d+1}\Big]^{\frac{1}{d}} \\ &= \prod_j \|\mathcal{E}_1 g_j\|^q_{L^{dq}_{\xi_{d+1}} L^q_{\xi_j}} \lesssim \prod_{j=1}^d \|g_j\|_p^q = \|g\|_p^q. \qquad \square\end{aligned}$$

Estimates such as (129) can be verified directly by interpolation. Taking sup in ξ_2 gives

$$\|\mathcal{E}_1 f\|_{L^\infty_{\xi_2} L^2_{\xi_1}} \lesssim_\varepsilon \|f\|_{L^2([0,1])}. \tag{130}$$

Conjecture 1.1 for $d=1$ follows from

$$\|\mathcal{E}_1 f\|_{L^{4+\varepsilon}_{\xi_2,\xi_1}} \lesssim_\varepsilon \|f\|_{L^4([0,1])} \tag{131}$$

for all $\varepsilon>0$. Using mixed-norm Riesz-Thorin interpolation with weights $\approx \frac{d-1}{d+1}$ for (130) and $\approx \frac{2}{d+1}$ for (131), one obtains (129) for $p=\frac{2(d+1)}{d}$ and $q=\frac{2(d+1)}{d}+\varepsilon'$, which shows (128) by the previous claim.

The reader will notice that our proof for the case $k=1$ of Theorem 1.5 has a similar idea in its core: we interpolate (at the level of the sets $\mathbb{X}^{l_1,\dots,l_d}$) between two estimates similar to (130) and (131). On the other hand, we have not found an extension of Bennett's remark to the case $2\le k\le d+1$, in which we still need to interpolate locally instead of globally and assume that only one function has a tensor structure.

Remark 14.3. In [Tao et al. 1998] the authors obtain the following off-diagonal type bounds:

Theorem [Tao et al. 1998]. *$\mathcal{ME}_{2,d}$ satisfies*

$$\|\mathcal{ME}_{2,d}(g_1,g_2)\|_2 \lesssim \|g_1\|_2\cdot\|g_2\|_{\frac{d+1}{d}},$$
$$\|\mathcal{ME}_{2,d}(g_1,g_2)\|_2 \lesssim \|f\|_{\frac{d+1}{d}}\cdot\|g\|_2.$$

In general, under the extra hypothesis that either g_1 or g_2 is a full tensor, one can obtain all k-linear off-diagonal type bounds like $L^{p_1} \times \cdots \times L^{p_k} \mapsto L^2$ by a straightforward adaptation of the argument presented in Section 9. We chose not to include them in this manuscript.

Remark 14.4. Under the assumption that g_j are *full tensors*

$$g_j(x_1, \ldots, x_d) = g_{j,1}(x_1) \cdot \cdots \cdot g_{j,d}(x_d), \quad 1 \le j \le k,$$

the methods of this work allow to prove Conjecture 1.11. We will not cover the details of this result here, but the idea is simply to interpolate between the $p = 2$ result and the case $k = 1$ for tensors.

Appendix A: Sharp examples

The goal of this first appendix is to discuss the sharpness of Theorems 1.5 and 11.2. We remark that sharp examples already exist in the literature, notably in the context of the bilinear problem for the sphere in [Foschi and Klainerman 2000], and in the multilinear case for surfaces of any signature in [Hickman and Iliopoulou 2022]. Our examples, however, exploit different ideas than those present in those works in the sense that they are robust enough to address weakly transversal configurations of caps and give sharp results in such cases as well.

The first part of this appendix is about Theorem 11.2, whereas in the second one we prove that, to attain the sharp range of Conjecture 1.2 in general, transversality cannot be replaced by the concept of weak transversality that we introduce.

AA. ***Range optimality.*** The main result of this subsection is the following:

Proposition A.1. *The condition*

$$p \ge \frac{2(d + |\tau| + 2)}{k(d + |\tau|)}$$

is necessary for Theorem 11.2 to hold.

Our examples are constructed based on one-dimensional considerations. For the benefit of simplifying the notation, smoothing the exposition to the reader and to establish a clear link with Conjecture 1.2, we present them in the $|\tau| = k - 1$ case, which is the smallest possible value for the corresponding $|\tau|$ of a given collection of transversal cubes (up to decomposing it into weakly transversal collections, see Claim B.4). It will be clear, however, how to work out the general case of arbitrary $|\tau|$, and we will point that out along the proof of Claim A.3.

Consider the caps that project onto the following transversal domains via $x \mapsto |x|^2$:

$$U_1 = [0,1]^d,$$
$$U_j = [2,3]^{j-2} \times [4,5] \times [0,1]^{d-j+1}, \quad 2 \le j \le k.$$

Observe that these caps are transversal as well;[50] therefore the following argument for the case $|\tau| = k - 1$ of Proposition A.1 also shows that the range of Conjecture 1.2 is necessary.

[50] For general $|\tau|$ we would have to start with a different collection of cubes with the appropriate total degree of transversality.

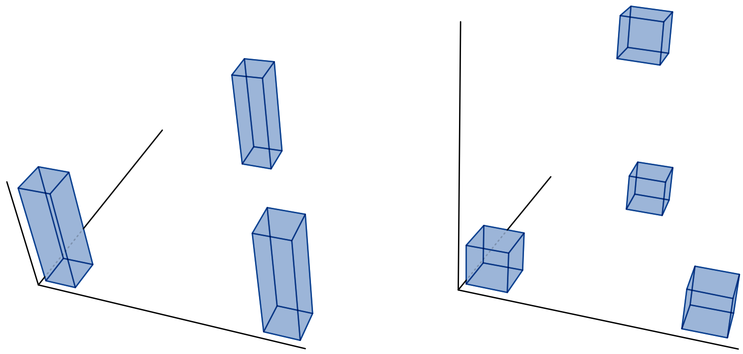

Figure 5. Cases $k = 3$ and $k = 4$ when $d = 3$.

We present the examples separately to distinguish their features. For $k = d+1$ we will take appropriately placed cubes, whereas for $2 \le k \le d$ we will take slabs (boxes with edges of two different scales).

Claim A.2. *Let $k = d + 1$, $\delta > 0$ small and let A_j^δ be given by*

$$A_1^\delta = [0,\delta]^d,$$
$$A_j^\delta = [2, 2+\delta]^{j-2} \times [4, 4+\delta] \times [0,\delta]^{d-j+1}, \quad 2 \le j \le d+1.$$

Define $f_j^\delta := \mathbb{1}_{A_j^\delta}$. Then

$$\frac{\|\prod_{j=1}^{d+1} \mathcal{E}_{U_j} f_j^\delta\|_p}{\prod_{j=1}^{d+1} \|f_j^\delta\|_2} \gtrsim \delta^{\frac{d(d+1)}{2} - \frac{1}{p}(d+1)}.$$

Therefore, letting $\delta \to 0$ implies $p \ge \frac{2}{d}$ is a necessary condition for the $(d+1)$-linear extension conjecture to hold for this choice of the U_j and for all f_j that are full tensors.

Claim A.3. *Let $2 \le k < d+1$, $\delta > 0$ small and let B_j^δ be given by*

$$B_1^\delta = [0,\delta^2]^{k-1} \times [0,\delta]^{d-k+1},$$
$$B_j^\delta = [2, 2+\delta^2]^{j-2} \times [4, 4+\delta^2] \times [0,\delta^2]^{k-j} \times [0,\delta]^{d-k+1}, \quad 2 \le j \le k.$$

Define $g_j^\delta := \mathbb{1}_{B_j^\delta}$. Then

$$\frac{\|\prod_{j=1}^{k} \mathcal{E}_{U_j} g_j^\delta\|_p}{\prod_{j=1}^{k} \|g_j^\delta\|_2} \gtrsim \delta^{\frac{k}{2}(d+k-1) - \frac{1}{p}(d+k+1)}.$$

Therefore, letting $\delta \to 0$ implies

$$p \ge \frac{2(d+k+1)}{k(d+k-1)}$$

is a necessary condition for the k-linear extension conjecture to hold for this choice of the U_j and for all g_j that are full tensors.

Before proving the claims, we need the following lemma:

Lemma A.4 (scale-1 phase-space portrait of $e^{2\pi i x^2}$). *There exists a sequence of smooth bumps* $(\varphi_n)_{n\in\mathbb{Z}}$ *such that*

(i) $\operatorname{supp}(\varphi_n) \subset [n-1, n+1]$, $n \in \mathbb{Z}$,

(ii) $|\varphi_n^{(\ell)}(x)| \le C_\ell$ *uniformly in* $n \in \mathbb{Z}$ *and such that*

$$e^{2\pi i x^2} = \sum_{n\in\mathbb{Z}} e^{4\pi i n x}\varphi_n(x).$$

Proof. See [Muscalu and Schlag 2013b, Proposition 1.10, page 23]. □

Rescaling with $t > 0$, the corresponding phase space portrait of $e^{2\pi i t x^2}$ is

$$e^{2\pi i t x^2} = e^{2\pi i(\sqrt{t}x)^2} = \sum_{n\in\mathbb{Z}} e^{4\pi i n\sqrt{t}x}\varphi_n(\sqrt{t}x).$$

Observe that $\tilde{\varphi}_t(x) = \varphi_n(\sqrt{t}x)$ is adapted to the Heisenberg box $\left[\frac{n}{\sqrt{t}}, \frac{n+1}{\sqrt{t}}\right] \times [0, \sqrt{t}]$, but strictly supported on $\left[\frac{n-1}{\sqrt{t}}, \frac{n+1}{\sqrt{t}}\right]$. This way, we can write

$$e^{2\pi i t x^2} = \sum_{n\in\mathbb{Z}} \Phi_{n,t}(x), \tag{132}$$

where $\Phi_{n,t}$ is adapted to the Heisenberg box $\left[\frac{n}{\sqrt{t}}, \frac{n+1}{\sqrt{t}}\right] \times [2n\sqrt{t}, (2n+1)\sqrt{t}]$.

Proof of Claim A.2. Motivated by the uncertainty principle, the first step is to analyze the behavior of the extension operator $\mathcal{E}_{U_j}$ applied to f_j^δ on a box whose sizes are reciprocal to the ones of $\operatorname{supp}(f_j^\delta)$. More precisely, we will show that $|E_{U_j}(f_j^\delta)| \gtrsim \delta^d$ on such boxes.

If $\delta < \frac{1}{\sqrt{t}}$,

$$\begin{aligned}\mathcal{E}_{U_1}(f_1^\delta)(\xi_1, \dots, \xi_d, t) &= \prod_{j=1}^d \left[\int_0^\delta e^{-2\pi i \xi_j x_j} e^{-2\pi i t x_j^2}\, \mathrm{d}x_j\right] \\ &= \prod_{j=1}^d \left[\int_0^\delta e^{-2\pi i \xi_j x_j} \cdot [\Phi_{0,t}(x_j) + \Phi_{1,t}(x_j)]\, \mathrm{d}x_j\right],\end{aligned}$$

since $\operatorname{supp}(\Phi_{n,t}) \cap [0, \delta] = \varnothing$ if $n \in \mathbb{Z}\backslash\{0, 1\}$. If $|\xi_j x_j| < \frac{1}{N}$ (N is a big number to be chosen later), we then have

$$\begin{aligned}&|\mathcal{E}_{U_1}(f_1^\delta)(\xi_1, \dots, \xi_d, t)| \\ &\quad= \prod_{j=1}^d \left|\int_0^\delta e^{-2\pi \xi_j x_j} \cdot [\Phi_{0,t}(x_j) + \Phi_{1,t}(x_j)]\, \mathrm{d}x_j\right|, \\ &\quad\ge \prod_{j=1}^d \left(\left|\int_0^\delta [\Phi_{0,t}(x_j) + \Phi_{1,t}(x_j)]\, \mathrm{d}x_j\right| - \left|\int_0^\delta [e^{-2\pi \xi_j x_j} - 1] \cdot [\Phi_{0,t}(x_j) + \Phi_{1,t}(x_j)]\right|\right),\end{aligned} \tag{133}$$

where N is picked so that $[e^{-2\pi \xi_j x_j} - 1]$ is close enough to zero to make

$$A_j := \left|\int_0^\delta [\Phi_{0,t}(x_j) + \Phi_{1,t}(x_j)]\, \mathrm{d}x_j\right|$$

dominate each factor above. Since $A_j \gtrsim \delta$ (recall that $\Phi_{0,t}$ and $\Phi_{1,t}$ are adapted to Heisenberg boxes of size $\frac{1}{\sqrt{t}} \times \sqrt{t}$ and $\delta < \frac{1}{\sqrt{t}}$), we conclude that if $|\xi_j| \lesssim \frac{1}{\delta}$ for $1 \le j \le d$ and $|t| < \frac{1}{\delta^2}$, then

$$|\mathcal{E}_{U_1}(f_1^\delta)(\xi_1, \dots, \xi_d, t)| \ge \delta^d.$$

If ϕ is a bump supported on $[-1, 1]$, we have just proved that

$$|\mathcal{E}_{U_1}(f_1^\delta)(\xi_1, \dots, \xi_d, t)| \gtrsim \delta^d \phi_\delta(\xi_1) \cdot \dots \cdot \phi_\delta(\xi_d) \phi_{\delta^2}(t), \tag{134}$$

where $\phi_\delta(\xi) := \phi(\delta x)$. Analogously, if $\delta < \frac{1}{\sqrt{t}}$,

$$\begin{aligned}
&\mathcal{E}_{U_2}(f_2^\delta)(\xi_1, \dots, \xi_d, t) \\
&\quad = \Big[\int_4^{4+\delta} e^{-2\pi i \xi_1 x_1} e^{-2\pi i t x_1^2}\, \mathrm{d}x_1\Big] \cdot \prod_{j=2}^{d} \Big[\int_0^\delta e^{-2\pi i \xi_j x_j} e^{-2\pi i t x_j^2}\, \mathrm{d}x_j\Big] \\
&\quad = \underbrace{\Big[\int_4^{4+\delta} e^{-2\pi i \xi_1 x_1} \Big(\sum_{n\in\mathbb{Z}} \Phi_{n,t}(x_1)\Big) \mathrm{d}x_1\Big]}_{I_1} \cdot \prod_{j=2}^{d} \underbrace{\Big[\int_0^\delta e^{-2\pi i \xi_j x_j} \cdot [\Phi_{0,t}(x_j) + \Phi_{1,t}(x_j)]\, \mathrm{d}x_j\Big]}_{I_j}.
\end{aligned}$$

There are at most $O(1)$ integers n such that $\mathrm{supp}(\Phi_{n,t}) \cap [4, 4+\delta] \ne \varnothing$, and they cluster around $\lfloor 4\sqrt{t} \rfloor$. Without loss of generality, one can assume that $n = 4\sqrt{t}$ so that the main contribution for I_1 comes from $\Phi_{4\sqrt{t},t}$ whose Heisenberg box is $\big[4, 4 + \frac{1}{\sqrt{t}}\big] \times [8t, 8t + \sqrt{t}]$. The modulation $e^{-2\pi i \xi_i x_i}$ shifts this box vertically by $-\xi_1$, and I_1 is negligible if the boxes $\big[4, 4 + \frac{1}{\sqrt{t}}\big] \times [8t - \xi_1, 8t + \sqrt{t} - \xi_1]$ and $[0, \delta] \times \big[0, \frac{1}{\delta}\big]$ are disjoint in frequency, so we need $|\xi_1 - 8t| \lesssim \frac{1}{\delta}$ to have a significant contribution to I_1. In that case,

$$|I_1| \gtrsim \Big| \int_4^{4+\delta} e^{-2\pi i \xi_1 x_1} \Phi_{4\sqrt{t},t}(x_1)\, \mathrm{d}x_1 \Big| \gtrsim \delta.$$

The analysis of I_j for $j \ge 2$ is the same as the one for the factors of $E_{U_1}(f_1^\delta)$. We conclude that if $|\xi_1 - 8t| \lesssim \frac{1}{\delta}$, $|\xi_j| \lesssim \frac{1}{\delta}$ for $2 \le j \le d$ and $|t| \le \frac{1}{\delta^2}$, then

$$|\mathcal{E}_{U_2}(f_2^\delta)(\xi_1, \dots, \xi_d, t)| \ge \delta^d.$$

As before,

$$|\mathcal{E}_{U_2}(f_2^\delta)(\xi_1, \dots, \xi_d, t)| \gtrsim \delta^d \phi_\delta(\xi_1 - 8t) \cdot \phi_\delta(\xi_2) \cdot \dots \cdot \phi_\delta(\xi_d) \phi_{\delta^2}(t).$$

The extensions $\mathcal{E}_{U_j}(f_j^\delta)$ for $3 \le j \le d+1$ are treated in the same way we treated $\mathcal{E}_{U_2}(f_2^\delta)$. The conclusion is that

$$\begin{aligned}
&|\mathcal{E}_{U_j}(f_j^\delta)(\xi_1, \dots, \xi_d, t)| \\
&\qquad \gtrsim \delta^d \phi_\delta(\xi_1 - 4t) \cdot \dots \cdot \phi_\delta(\xi_{j-2} - 4t) \cdot \phi_\delta(\xi_{j-1} - 8t) \cdot \phi_\delta(\xi_j) \cdot \dots \cdot \phi_\delta(\xi_d) \phi_{\delta^2}(t)
\end{aligned} \tag{135}$$

for all $2 \le j \le d+1$.

Let $\xi = (\xi_1, \dots, \xi_d)$. From (134) and (135) we obtain

$$\prod_{j=1}^{d+1} |\mathcal{E}_{U_j}(f_j^\delta)(\xi,t)| \gtrsim \delta^{d(d+1)} \Big[\phi_{\delta^2}(t) \prod_{l=1}^{d} \phi_\delta(\xi_l)\Big] \times \Big[\prod_{j=2}^{d} \phi_\delta(\xi_1 - 4t) \cdot \dots \cdot \phi_\delta(\xi_{j-2} - 4t) \cdot \phi_\delta(\xi_{j-1} - 8t) \cdot \phi_\delta(\xi_j) \cdot \dots \cdot \phi_\delta(\xi_d) \phi_{\delta^2}(t)\Big]. \quad (136)$$

Now we analyze the support of the product of the right-hand side of (136). Notice that we have at least one bump like $\phi_\delta(\xi_j)$ for every $1 \le j \le d+1$, so $|\xi_j| \lesssim \frac{1}{\delta}$ is a necessary condition for the product not to be zero. On the other hand, the conditions

$$|\xi_j| \lesssim \frac{1}{\delta}, \quad |\xi_j - 8t| \lesssim \frac{1}{\delta}$$

together imply $|t| \lesssim \frac{1}{\delta}$, which is much more restrictive than the $|t| \lesssim \frac{1}{\delta^2}$ that comes from the support of the bump $\phi_{\delta^2}(t)$. We conclude that the right-hand side of (136) is supported on the box

$$R_\delta^* = \Big\{(\xi_1, \dots, \xi_d, t) \in \mathbb{R}^{d+1} : |t| \lesssim \frac{1}{\delta},\ |\xi_j| \lesssim \frac{1}{\delta},\ 1 \le j \le d\Big\}.$$

Finally,

$$\frac{\|\prod_{j=1}^{d+1} \mathcal{E}_{U_j} f_j^\delta\|_p}{\prod_{j=1}^{d+1} \|f_j^\delta\|_2} \gtrsim \frac{\delta^{d(d+1)} \cdot |R_\delta^*|^{\frac{1}{p}}}{\delta^{\frac{d(d+1)}{2}}} \gtrsim \delta^{\frac{d(d+1)}{2} - \frac{1}{p}(d+1)} \quad (137)$$

and the claim follows. □

Proof of Claim A.3. The outline of the following argument is the same as the one used in previous proof. Let $\xi = (\xi_1, \dots, \xi_d)$. If $\delta^2 < \frac{1}{\sqrt{t}}$,

$$\begin{aligned}\mathcal{E}_{U_1}(g_1^\delta)(\xi,t) &= \prod_{j=1}^{k-1}\Big[\int_0^{\delta^2} e^{-2\pi i \xi_j x_j} e^{-2\pi i t x_j^2}\, \mathrm{d}x_j\Big] \cdot \prod_{l=k}^{d}\Big[\int_0^{\delta} e^{-2\pi i \xi_l x_l} e^{-2\pi i t x_l^2}\, \mathrm{d}x_l\Big] \\ &= \prod_{j=1}^{k-1}\Big[\int_0^{\delta^2} e^{-2\pi i \xi_j x_j} [\Phi_{0,t}(x_j) + \Phi_{1,t}(x_j)]\, \mathrm{d}x_j\Big] \cdot \prod_{l=k}^{d} \underbrace{\Big[\int_0^{\delta} e^{-2\pi i \xi_l x_l} \Big(\sum_{n\in\mathbb{Z}} \Phi_{n,t}(x_l)\Big) \mathrm{d}x_l\Big]}_{(*)},\end{aligned}$$

since $\mathrm{supp}(\Phi_{n,t}) \cap [0, \delta^2] = \varnothing$ if $n \in \mathbb{Z}\backslash\{0,1\}$. If $\delta < \frac{1}{\sqrt{t}}$ (which is stronger than the previous condition $\delta^2 < \frac{1}{\sqrt{t}}$), we can eliminate most $\Phi_{n,t}$ in $(*)$ as well:

$$\begin{aligned}&\mathcal{E}_{U_1}(g_1^\delta)(\xi,t) \\ &\quad = \prod_{j=1}^{k-1}\Big[\int_0^{\delta^2} e^{-2\pi i \xi_j x_j} [\Phi_{0,t}(x_j) + \Phi_{1,t}(x_j)]\, \mathrm{d}x_j\Big] \cdot \prod_{l=k}^{d}\Big[\int_0^{\delta} e^{-2\pi i \xi_l x_l} \cdot [\Phi_{0,t}(x_l) + \Phi_{1,t}(x_l)]\, \mathrm{d}x_l\Big],\end{aligned}$$

If $|\xi_j x_j| < \frac{1}{N}$ (for N big enough), we then have

$$|\mathcal{E}_{U_1}(g_1^\delta)(\xi,t)| = \prod_{j=1}^{k-1}\left|\int_0^{\delta^2} e^{-2\pi\xi_j x_j}[\Phi_{0,t}(x_j)+\Phi_{1,t}(x_j)]\,\mathrm{d}x_j\right|\cdot\prod_{l=k}^{d}\left|\int_0^{\delta} e^{-2\pi\xi_l x_l}\cdot[\Phi_{0,t}(x_l)+\Phi_{1,t}(x_l)]\,\mathrm{d}x_l\right|, \gtrsim \delta^{2(k-1)+(d-k+1)} = \delta^{d+k-1},$$

by the same argument presented when we analyzed (133). We conclude that if $|\xi_j| \lesssim \frac{1}{\delta^2}$ for $1 \le j \le k-1$, $|\xi_l| \lesssim \frac{1}{\delta}$ for $k \le l \le d$ and $|t| < \frac{1}{\delta^2}$, then[51]

$$|\mathcal{E}_{U_1}(g_1^\delta)(\xi,t)| \gtrsim \delta^{d+k-1}.$$

Using the same notation from the proof of Claim A.2, we have just proved that

$$|\mathcal{E}_{U_1}(g_1^\delta)(\xi,t)| \gtrsim \delta^d \phi_{\delta^2}(\xi_1)\cdot\dots\cdot\phi_{\delta^2}(\xi_{k-1})\phi_\delta(x_k)\cdot\dots\cdot\phi_\delta(x_d)\cdot\phi_{\delta^2}(t), \tag{138}$$

where $\phi_\delta(\xi) := \phi(\delta x)$ and ϕ is a bump supported on $[-1,1]$. Analogously, if $\delta < \frac{1}{\sqrt{t}}$,

$$\begin{aligned}\mathcal{E}_{U_2}(g_2^\delta)(\xi,t) &= \left[\int_4^{4+\delta^2} e^{-2\pi i\xi_1 x_1} e^{-2\pi i t x_1^2}\,\mathrm{d}x_1\right]\cdot\prod_{j=2}^{k-1}\left[\int_0^{\delta^2} e^{-2\pi i\xi_j x_j} e^{-2\pi i t x_j^2}\mathrm{d}x_j\right]\cdot\prod_{l=k}^{d}\left[\int_0^{\delta} e^{-2\pi i\xi_l x_l} e^{-2\pi i t x_l^2}\,\mathrm{d}x_l\right]\\ &= \underbrace{\left[\int_4^{4+\delta^2} e^{-2\pi i\xi_1 x_1}\left(\sum_{n\in\mathbb{Z}}\Phi_{n,t}(x_1)\right)\mathrm{d}x_1\right]}_{M_1}\cdot\prod_{j=2}^{k-1}\underbrace{\left[\int_0^{\delta^2} e^{-2\pi i\xi_j x_j}\cdot[\Phi_{0,t}(x_j)+\Phi_{1,t}(x_j)]\,\mathrm{d}x_j\right]}_{M_j}\\ &\quad\times\prod_{l=k}^{d}\underbrace{\left[\int_0^{\delta} e^{-2\pi i\xi_l x_l}\cdot[\Phi_{0,t}(x_l)+\Phi_{1,t}(x_l)]\,\mathrm{d}x_l\right]}_{M_l}.\end{aligned} \tag{139}$$

As in the proof of Claim A.2, the main contribution for M_1 comes from $\Phi_{4\sqrt{t},t}$, whose Heisenberg box is $\left[4,4+\frac{1}{\sqrt{t}}\right]\times[8t,8t+\sqrt{t}]$. The modulation $e^{-2\pi i\xi_i x_i}$ shifts this box vertically by $-\xi_1$, and M_1 is negligible if the boxes $\left[4,4+\frac{1}{\sqrt{t}}\right]\times[8t-\xi_1,8t+\sqrt{t}-\xi_1]$ and $[0,\delta^2]\times\left[0,\frac{1}{\delta^2}\right]$ are disjoint in frequency, so we need $|\xi_1-8t| \lesssim \frac{1}{\delta^2}$ to have a significant contribution to M_1. In that case,

$$|M_1| \gtrsim \left|\int_4^{4+\delta^2} e^{-2\pi i\xi_1 x_1}\Phi_{2\sqrt{t},t}(x_1)\,\mathrm{d}x_1\right| \gtrsim \delta^2.$$

The analyses of M_j for $2 \le j \le k-1$ and of M_l for $k \le l \le d-k+1$ are the same as the one for the factors of $E_{U_1}(g_1^\delta)$. We conclude that if $|\xi_1-8t| \lesssim \frac{1}{\delta^2}$, $|\xi_j| \lesssim \frac{1}{\delta^2}$ for $2 \le j \le k-1$, $|\xi_l| \lesssim \frac{1}{\delta}$ for $k \le l \le d$ and $|t| \le \frac{1}{\delta^2}$, then

$$|\mathcal{E}_{U_2}(g_2^\delta)(\xi,t)| \ge \delta^{d+k-1}.$$

[51] For general $|\tau|$, we would have $|\tau|$ conditions of type $|\xi_j| \lesssim \frac{1}{\delta^2}$ and $(d-|\tau|)$ like $|\xi_l| \lesssim \frac{1}{\delta}$.

As before,

$$|\mathcal{E}_{U_2}(g_2^\delta)(\xi,t)| \gtrsim \delta^d \phi_\delta(\xi_1-8t)\cdot\phi_{\delta^2}(\xi_2)\cdot\dots\cdot\phi_{\delta^2}(\xi_{k-1})\cdot\phi_\delta(\xi_k)\cdot\dots\cdot\phi_\delta(\xi_d)\phi_{\delta^2}(t).$$

The extensions $\mathcal{E}_{U_j}(g_j^\delta)$ for $3\le j\le k$ are treated in the same way. The conclusion is that

$$|\mathcal{E}_{U_j}(g_j^\delta)(\xi,t)| \gtrsim \delta^d \phi_\delta(\xi_1-4t)\cdot\dots\cdot\phi_\delta(\xi_{j-2}-4t)\cdot\phi_\delta(\xi_{j-1}-8t)\cdot\phi_\delta(\xi_j)\cdot\dots\cdot\phi_\delta(\xi_d)\phi_{\delta^2}(t) \quad (140)$$

for all $2\le j\le k$. From (138) and (140) we obtain

$$\begin{aligned}\prod_{j=1}^k &|\mathcal{E}_{U_j}(g_j^\delta)(\xi,t)|\\ &\gtrsim \delta^{k(d+k-1)}\Big[\phi_{\delta^2}(t)\prod_{l=1}^{k-1}\phi_{\delta^2}(\xi_l)\cdot\prod_{n=k}^{d}\phi_\delta(\xi_n)\Big]\\ &\quad\times\Big[\prod_{j=2}^{d}\Big(\prod_{n=1}^{j-2}\phi_{\delta^2}(\xi_n-4t)\Big)\cdot\phi_{\delta^2}(\xi_{j-1}-8t)\cdot\Big(\prod_{m=j}^{k-1}\phi_{\delta^2}(\xi_m)\Big)\cdot\Big(\prod_{r=k}^{d}\phi_\delta(\xi_r)\Big)\cdot\phi_{\delta^2}(t)\Big]. \quad (141)\end{aligned}$$

Notice that we have at least one bump like $\phi_{\delta^2}(\xi_j)$ for every $1\le j\le k-1$ and at least one $\phi_\delta(\xi_l)$ for $k\le l\le d$, so $|\xi_j|\lesssim\frac{1}{\delta^2}$ and $|\xi_l|\lesssim\frac{1}{\delta}$ are necessary conditions for the product not to be zero. On the other hand, the conditions

$$|\xi_j|\lesssim\frac{1}{\delta^2},\quad |\xi_j-8t|\lesssim\frac{1}{\delta^2}$$

together imply $|t|\lesssim\frac{1}{\delta^2}$, which does not add any new information compared to the one coming from the bump $\phi_{\delta^2}(t)$ (this is the main difference between the analysis in Claims A.2 and A.3). We conclude that the right-hand side of (141) is supported on the box

$$S_\delta^* = \Big\{(\xi_1,\dots,\xi_d,t)\in\mathbb{R}^{d+1} : |t|\lesssim\frac{1}{\delta^2};\quad |\xi_j|\lesssim\frac{1}{\delta^2},\quad 1\le j\le k-1;\quad |\xi_l|\lesssim\frac{1}{\delta},\quad k\le l\le d\Big\}.$$

Finally,

$$\frac{\|\prod_{j=1}^k \mathcal{E}_{U_j} g_j^\delta\|_p}{\prod_{j=1}^{d+1}\|g_j^\delta\|_2} \gtrsim \frac{\delta^{(d+k-1)k}\cdot|S_\delta^*|^{\frac{1}{p}}}{\delta^{\frac{(d+k-1)k}{2}}} \gtrsim \delta^{\frac{(d+k-1)k}{2}-\frac{(d+k+1)}{p}} \quad (142)$$

and the claim follows. □

AB. ***Transversality as a necessary condition in general.*** A natural question is: given k cubes U_j, $1\le j\le k$, is it possible to prove

$$\Big\|\prod_{j=1}^k \mathcal{E}_{U_j} g_j\Big\|_p \lesssim \prod_{j=1}^k \|g_j\|_2$$

for

$$p\ge\frac{2(d+k+1)}{k(d+k-1)}$$

and all $g_j\in L^2(U_j)$if the U_j are assumed to be weakly transversal?

The answer is no and we will address it in this second part of the first appendix. As a consequence, we conclude that Theorem 1.5 is sharp under weak transversality, as observed in Remark 1.8.

We will treat the case $k = 3$ and $d = 2$ for simplicity, but a similar construction holds in general. If three boxes $U_1, U_2, U_3 \subset \mathbb{R}^2$ are not transversal, there is a line that crosses them. Assume without loss of generality that $U_1 = [0,1]^2$, $U_2 = [2,3]^2$ and $U_3 = [4,5]^2$. We will show that

$$\|E_{U_1}(h_1)\cdot E_{U_2}(h_2)\cdot E_{U_3}(h_3)\|_p \lesssim \|h_1\|_2\cdot\|h_2\|_2\cdot\|h_3\|_2$$

only if $p \geq \frac{10}{9}$. The trilinear extension conjecture for $d = 2$ states that $p \geq 1$ is the sharp range under the transversality hypothesis.

Claim A.5. *Define the sets* D_j^δ *by*

$$D_1^\delta = \left[\frac{\sqrt{2}-\delta^2}{2}, \frac{\sqrt{2}+\delta^2}{2}\right]\times\left[-\frac{\delta}{2}, \frac{\delta}{2}\right],$$
$$D_2^\delta = \left[\frac{5\sqrt{2}-\delta^2}{2}, \frac{5\sqrt{2}+\delta^2}{2}\right]\times\left[-\frac{\delta}{2}, \frac{\delta}{2}\right],$$
$$D_3^\delta = \left[\frac{9\sqrt{2}-\delta^2}{2}, \frac{9\sqrt{2}+\delta^2}{2}\right]\times\left[-\frac{\delta}{2}, \frac{\delta}{2}\right].$$

Define $h_j^\delta := \mathbb{1}_{D_j^\delta}$. *Then*

$$\frac{\|\prod_{j=1}^3 \mathcal{E}_{D_j} h_j^\delta\|_p}{\prod_{j=1}^3 \|h_j^\delta\|_2} \gtrsim \delta^{\frac{9}{2}-\frac{5}{p}}.$$

Proof. The proof is analogous to the ones of Claims A.2 and A.3. □

Let the rhombuses $\widetilde{D}_j$ be given by

$$\widetilde{D}_1 = \operatorname{Conv}\left((0,0); \left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right); \left(\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}\right); (\sqrt{2}, 0)\right),$$
$$\widetilde{D}_2 = \operatorname{Conv}\left((2\sqrt{2},0); \left(\frac{5\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right); \left(\frac{5\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}\right); (3\sqrt{2}, 0)\right),$$
$$\widetilde{D}_3 = \operatorname{Conv}\left((4\sqrt{2},0); \left(\frac{9\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right); \left(\frac{9\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}\right); (5\sqrt{2}, 0)\right).$$

Observe that $D_j^\delta \subset \widetilde{D}_j$ for $\delta > 0$ small enough. Extend the domain of h_j^δ to $\widetilde{D}_j$ so that it is 0 on $\widetilde{D}_j \setminus D_j^\delta$. Let T be a $\frac{\pi}{4}$ counterclockwise rotation and let

$$H_j^\delta(x) := h_j^\delta \circ T^{-1}(x).$$

Notice that T takes $\widetilde{D}_j$ to U_j, as shown in the picture below.

Since L^p norms are invariant under rotations, we have

$$\frac{\|\prod_{j=1}^3 \mathcal{E}_{U_j} H_j^\delta\|_p}{\prod_{j=1}^3 \|H_j^\delta\|_2} \gtrsim \delta^{\frac{9}{2}-\frac{5}{p}}$$

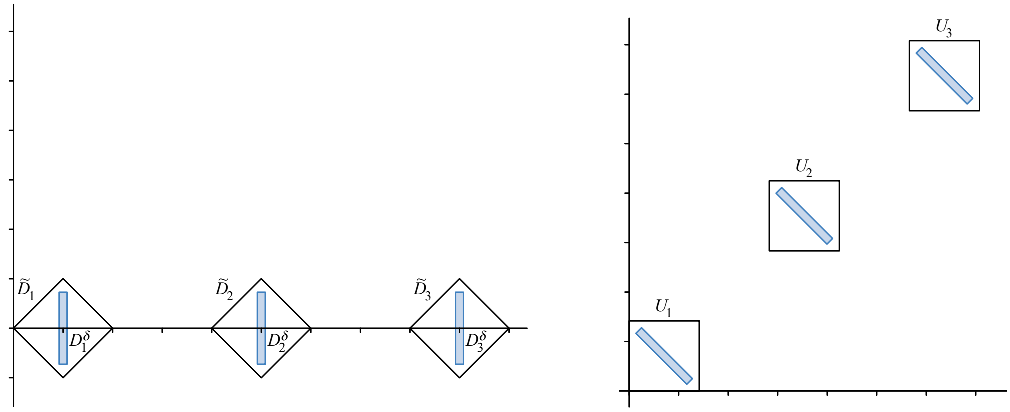

Figure 6. Left: The function h_j^δ. Right: H_j^δ is supported on U_j.

from Claim A.5. Letting $\delta \to 0$ shows that we need $p \geq \frac{10}{9}$, so the sharp range $p \geq 1$ cannot be obtained if the boxes U_1, U_2, U_3 are not transversal.

Remark A.6. As expected, the functions H_j^δ do not have a tensor structure with respect to the canonical basis. If this was the case, our methods would have allowed us to prove that the corresponding trilinear extension operator maps $L^2 \times L^2 \times L^2$ to L^1.

Appendix B: Technical results

Here we collect a few technical results used throughout the paper.

Theorem B.1. *For* $0 < \gamma < d,\ 1 < p < q < \infty$, *and* $\frac{1}{q} = \frac{1}{p} - \frac{d-\gamma}{d}$, *we have*

$$\| f * (|y|^{-\gamma}) \|_{L^q(\mathbb{R}^d)} \leq A_{p,q} \cdot \| f \|_{L^p(\mathbb{R}^d)}. \tag{143}$$

Proof. See Proposition 7.8 in [Muscalu and Schlag 2013a]. □

Theorem B.2 (nonstationary phase). *Let* $a \in C_0^\infty$ *and*

$$I(\lambda) = \int_{\mathbb{R}^d} e^{2\pi i \lambda \phi(\xi)} a(\xi)\, \mathrm{d}\xi.$$

If $\nabla \phi \neq 0$ *on supp*(a), *then*

$$|I(\lambda)| \leq C(N, a, \phi) \lambda^{-N}$$

as $\lambda \to \infty$ *for arbitrary* $N \geq 1$.

Proof. See Lemma 4.14 in [Muscalu and Schlag 2013a]. □

Theorem B.3 (stationary phase). *If* $\nabla \phi(\xi_0) = 0$ *for some* $\xi_0 \in \operatorname{supp}(a)$, $\nabla \phi \neq 0$ *away from* ξ_0 *and the Hessian of* ϕ *at the stationary point* ξ_0 *is nondegenerate*, *i.e.*, $\det D^2 \phi(\xi_0) \neq 0$, *then for all* $\lambda \geq 1$

$$|I(\lambda)| \leq C(N, a, \phi) \lambda^{-\frac{d}{2}}.$$

Proof. See Lemma 4.15 in [Muscalu and Schlag 2013a]. □

We now restate and prove the main claim from Section 3:

Claim B.4. *Given a collection* $\mathcal{Q} = \{Q_1, \dots, Q_k\}$ *of transversal cubes, each* $Q_l \in \mathcal{Q}$ *can be partitioned into* $O(1)$ *many subcubes*

$$Q_l = \bigcup_i Q_{l,i}$$

so that all collections $\widetilde{\mathcal{Q}}$ *made of picking one subcube* $Q_{l,i}$ *per* Q_l

$$\widetilde{\mathcal{Q}} = \{\widetilde{Q}_1, \dots, \widetilde{Q}_k\}, \quad \widetilde{Q}_l \in \{Q_{l,i}\}_i,$$

are weakly transversal.

Proof. For each $1 \leq j \leq d$, consider the set A_j of endpoints of the intervals $\pi_j(Q_1), \dots, \pi_j(Q_k)$. Using these endpoints to partition this collection of intervals, one can assume that there are three cases for two cubes Q_r and Q_s:

(1) $\pi_j(Q_r) \cap \pi_j(Q_s) = \varnothing$.

(2) $\pi_j(Q_r) = \pi_j(Q_s)$.

(3) $\pi_j(Q_r) \cap \pi_j(Q_s) = \{p_{r,s}\}$, where $p_{r,s}$ is an endpoint of both $\pi_j(Q_r)$ and $\pi_j(Q_s)$.

We can go one step further and assume that all $\pi_j(Q_s)$ that intersect a given $\pi_j(Q_r)$ (but distinct from it) do so at the same endpoint. Indeed, if $\pi_j(Q_{s_1}) \cap \pi_j(Q_r) = \{p\}$, $\pi_j(Q_{s_2}) \cap \pi_j(Q_r) = \{q\}$ and $\pi_j(Q_r) = [p, q]$, we can simply split $\pi_j(Q_r)$ in half and obtain intervals that satisfy this property.

Now we choose a point $x_{j,r}$ in every interval $\pi_j(Q_r)$:

(1) If $\pi_j(Q_r) \cap \pi_j(Q_s) = \varnothing$ for all $s \neq r$, let $x_{j,r}$ be $c_{j,r}$, the center of $\pi_j(Q_r)$.

(2) If $\pi_j(Q_r)$ intersects some $\pi_j(Q_{s_1})$ at p, any other $\pi_j(Q_{s_2})$ that intersects $\pi_j(Q_r)$ also does it at p. In this case choose $x_{j,r} = x_{j,s} = p$ for all s such that $\pi_j(Q_r) \cap \pi_j(Q_s) \neq \varnothing$.

Let us now show that, after the reductions above, the transversal set of cubes $\mathcal{Q}$ is weakly transversal. More precisely, for a fixed $1 \leq l \leq k$, we will show that there is a set of $k-1$ canonical directions that together with Q_l satisfy (15). Let $\vec{x}_i \in Q_i$ for $1 \leq i \leq k$ be given in coordinates by

$$\vec{x}_i = (x_{1,i}, x_{2,i}, \dots, x_{d,i}).$$

The normal vector to $\mathbb{P}^d$ at $\vec{x}_i$ is

$$\vec{v}_i = (-2x_{1,i}, -2x_{2,i}, \dots, -2x_{d,i}, 1).$$

Then the cubes in $\mathcal{Q}$ are transversal if and only if the matrix

$$\begin{pmatrix} -2x_{1,1} & -2x_{1,2} & \cdots & -2x_{1,k} \\ -2x_{2,1} & -2x_{2,2} & \cdots & -2x_{2,k} \\ \vdots & \vdots & \ddots & \vdots \\ -2x_{d,1} & -2x_{d,2} & \cdots & -2x_{d,k} \\ 1 & 1 & \cdots & 1 \end{pmatrix}$$

has rank k for all $x_{j,i} \in \pi_j(Q_i)$, $1 \leq j \leq d$, $1 \leq i \leq k$.

By Lemma B.5 (proven at the end of this appendix), there are $k-1$ rows

$$R_{i_n} = (-2x_{i_n,1}, \dots, -2x_{i_n,k})$$

of the above matrix, $1 \le n \le k-1$, such that

$$\begin{cases} x_{i_1,l} \neq x_{i_1,1} \\ \quad\vdots \\ x_{i_{l-1},l} \neq x_{i_{l-1},l-1} \\ x_{i_l,l} \neq x_{i_l,l+1} \\ \quad\vdots \\ x_{i_{k-1},l} \neq x_{i_{k-1},k}. \end{cases}$$

Because of the choices we made, $x_{i_n,l} \neq x_{i_n,r}$ implies

$$\pi_{i_n}(Q_l) \cap \pi_{i_n}(Q_r) = \varnothing,$$

which finishes the proof. □

Finally, we state and prove the auxiliary linear algebra lemma used in the proof of Claim B.4.

Lemma B.5. *Let M be the $(d+1)\times k$ matrix*

$$M = \begin{pmatrix} a_{1,1} & a_{1,2} & \cdots & a_{1,k} \\ a_{2,1} & a_{2,2} & \cdots & a_{2,k} \\ \vdots & \vdots & \ddots & \vdots \\ a_{d,1} & a_{d,2} & \cdots & a_{d,k} \\ 1 & 1 & \cdots & 1 \end{pmatrix}$$

and assume that it has rank k. For each column $C_j = (a_{1,j}, \dots, a_{d,j}, 1)$ there are $k-1$ rows $R_{i_l} = (a_{i_l,1}, \dots, a_{i_l,k})$, $1 \le l \le k-1$, such that

$$\begin{cases} a_{i_1,j} \neq a_{i_1,l_1} \\ a_{i_2,j} \neq a_{i_2,l_2} \\ \quad\vdots \\ a_{i_{k-1},j} \neq a_{i_{k-1},l_{k-1}}, \end{cases}$$

where $(l_1, l_2, \dots, l_{k-1})$ is some permutation of $(1, 2, \dots, j-1, j+1, \dots, k)$.

Proof. Let us first consider the case $k = d+1$. We have to show that for all columns C_j the first $k-1$ rows satisfy the property of the lemma. Observe that the product

$$MA = \begin{pmatrix} a_{1,1} & a_{1,2} & \cdots & a_{1,k} \\ a_{2,1} & a_{2,2} & \cdots & a_{2,k} \\ \vdots & \vdots & \ddots & \vdots \\ a_{k-1,1} & a_{k-1,2} & \cdots & a_{k-1,k} \\ 1 & 1 & \cdots & 1 \end{pmatrix} \cdot \underbrace{\begin{pmatrix} 1 & 1 & \cdots & 1 & 1 & 1 \\ -1 & 0 & \cdots & 0 & 0 & 0 \\ 0 & -1 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & -1 & 0 & 0 \\ 0 & 0 & \cdots & 0 & -1 & 0 \end{pmatrix}}_{k\times k \text{ matrix } A}$$

is a rank k matrix equal to

$$\begin{pmatrix} (a_{1,1}-a_{1,2}) & (a_{1,1}-a_{1,3}) & \cdots & (a_{1,1}-a_{1,k-1}) & (a_{1,1}-a_{1,k}) & a_{1,1} \\ (a_{2,1}-a_{2,2}) & (a_{2,1}-a_{2,3}) & \cdots & (a_{2,1}-a_{2,k-1}) & (a_{2,1}-a_{2,k}) & a_{2,1} \\ (a_{3,1}-a_{3,2}) & (a_{3,1}-a_{3,3}) & \cdots & (a_{3,1}-a_{3,k-1}) & (a_{3,1}-a_{3,k}) & a_{3,1} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ (a_{k-1,1}-a_{k-1,2}) & (a_{k-1,1}-a_{k-1,3}) & \cdots & (a_{k-1,1}-a_{k-1,k-1}) & (a_{k-1,1}-a_{k-1,k}) & a_{k-1,1} \\ 0 & 0 & \cdots & 0 & 0 & 1 \end{pmatrix}.$$

By computing the Laplace expansion with respect to the last row, we conclude that $\det(MA)$ is equal to

$$\det\begin{pmatrix} (a_{1,1}-a_{1,2}) & (a_{1,1}-a_{1,3}) & \cdots & (a_{1,1}-a_{1,k-1}) & (a_{1,1}-a_{1,k}) \\ (a_{2,1}-a_{2,2}) & (a_{2,1}-a_{2,3}) & \cdots & (a_{2,1}-a_{2,k-1}) & (a_{2,1}-a_{2,k}) \\ (a_{3,1}-a_{3,2}) & (a_{3,1}-a_{3,3}) & \cdots & (a_{3,1}-a_{3,k-1}) & (a_{3,1}-a_{3,k}) \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ (a_{k-1,1}-a_{k-1,2}) & (a_{k-1,1}-a_{k-1,3}) & \cdots & (a_{k-1,1}-a_{k-1,k-1}) & (a_{k-1,1}-a_{k-1,k}) \end{pmatrix}.$$

The entries of this matrix are

$$x_{i,j} := a_{i,1}-a_{i,j+1}, \quad 1 \le i, j \le k-1.$$

The column C_1 has the property of the lemma if and only if there is some permutation π of $(1, 2, \ldots, k-1)$ such that

$$\begin{cases} x_{1,\pi(1)} = a_{1,1}-a_{1,\pi(1)+1} \neq 0 \\ x_{2,\pi(2)} = a_{2,1}-a_{2,\pi(2)+1} \neq 0 \\ \qquad\vdots \\ x_{k-1,\pi(k-1)} = a_{k-1,1}-a_{k-1,\pi(k-1)+1} \neq 0. \end{cases}$$

If this was not the case, for all such permutations π of $(1, 2, \ldots, k-1)$ at least one among $x_{1,\pi(1)}$, $x_{2,\pi(2)}, \ldots, x_{k-1,\pi(k-1)}$ would be zero. Hence

$$\det(MA) = \sum_{\pi \in S_{k-1}} \operatorname{sgn}(\pi) \cdot x_{1,\pi(1)} \cdot \cdots \cdot x_{k-1,\pi(k-1)} = 0,$$

a contradiction. A similar argument shows that any other column also has this property.

The case $k < d+1$ can be reduced to the previous one. Indeed, the rank-k condition guarantees that there is a $k \times k$ minor of M that has rank k. There are two possibilities:

(1) *There is a $k \times k$ minor of rank k that has a row of* 1*s.* This is identical to the case $k = d+1$ and we conclude that the rows that generate this minor are the ones that satisfy the property of the lemma.

(2) *No $k \times k$ minor of rank k has a row of* 1*'s.* Here the rows of all nonsingular minors are among the first d ones of M. Let R_{i_l}, $1 \leq l \leq k$, be k rows of M that generate such a minor $\widetilde{M}$:

$$\widetilde{M} = \begin{pmatrix} a_{i_1,1} & a_{i_1,2} & \cdots & a_{i_1,k} \\ a_{i_2,1} & a_{i_2,2} & \cdots & a_{i_2,k} \\ \vdots & \vdots & \ddots & \vdots \\ a_{i_{k-1},1} & a_{i_{k-1},2} & \cdots & a_{i_{k-1},k} \\ a_{i_k,1} & a_{i_k,2} & \cdots & a_{i_k,k} \end{pmatrix}.$$

Proceed as in the case $k = d+1$ and multiply $\widetilde{M}$ by the matrix A to obtain

$$\widetilde{M}A = \begin{pmatrix} (a_{i_1,1} - a_{i_1,2}) & (a_{i_1,1} - a_{i_1,3}) & \cdots & (a_{i_1,1} - a_{i_1,k}) & a_{i_1,1} \\ (a_{i_2,1} - a_{i_2,2}) & (a_{i_2,1} - a_{i_2,3}) & \cdots & (a_{i_2,1} - a_{i_2,k}) & a_{i_2,1} \\ (a_{i_3,1} - a_{i_3,2}) & (a_{i_3,1} - a_{i_3,3}) & \cdots & (a_{i_3,1} - a_{i_3,k}) & a_{i_3,1} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ (a_{i_{k-1},1} - a_{i_{k-1},2}) & (a_{i_{k-1},1} - a_{i_{k-1},3}) & \cdots & (a_{i_{k-1},1} - a_{i_{k-1},k}) & a_{i_{k-1},1} \\ (a_{i_k,1} - a_{i_k,2}) & (a_{i_k,1} - a_{i_k,3}) & \cdots & (a_{i_k,1} - a_{i_k,k}) & a_{i_k,1} \end{pmatrix}.$$

By computing the Laplace expansion along the last column of $\widetilde{M}A$, we conclude that at least one $(k-1) \times (k-1)$ minor obtained from the first $k-1$ columns of $\widetilde{M}A$ is nonsingular. We argue again as in the $k = d+1$ case to find the $k-1$ rows that satisfy the property of the lemma for the column C_1. An analogous argument works for any other column of M, but these $k-1$ special rows may vary from column to column. □

Let us recall some of the terminology from the proof of Theorem 13.3 in Section 13. A subset $\mathcal{A} \subset \mathcal{Q}$ has the *property* (P) if:

(1) $Q_1 \in \mathcal{A}$.

(2) $\mathcal{A}$ is not weakly transversal with pivot Q_1.

We say that $\mathcal{A} \subset \mathcal{Q}$ is *minimal* if $\mathcal{A}' \subset \mathcal{A}$ has the property (P) if and only if $\mathcal{A}' = \mathcal{A}$. Since $\mathcal{Q}$ itself has the property (P), it must contain a minimal subset of cardinality at least 2.

Claim B.6. *Let $\mathcal{A} = \{Q_1, K_2, \ldots, K_n\}$ be a minimal set of n cubes.*[52] *There is a set D of $d-n+2$ canonical directions v for which*

$$\pi_v(Q_1) \cap \pi_v(K_j) \neq \varnothing \quad \text{for all } 2 \leq j \leq n. \tag{144}$$

Proof of Claim B.6. If $n = 2$, then $Q_1 \cap K_2 \neq \varnothing$ and the claim follows directly. If $n > 2$, observe that $\mathcal{A}' = \{Q_1, K_2, \ldots, K_{n-1}\}$ is weakly transversal with pivot Q_1; otherwise $\mathcal{A}$ would not be minimal. Hence there are $1 \leq j_1, \ldots, j_{n-2} \leq d$ *distinct* such that

$$\begin{cases} \pi_{j_1}(Q_1) \cap \pi_{j_1}(K_2) = \varnothing, \\ \quad\vdots \\ \pi_{j_{n-2}}(Q_1) \cap \pi_{j_{n-2}}(K_{n-1}) = \varnothing. \end{cases} \tag{145}$$

[52]Observe that Q_1 is the only "Q" cube in this collection. The others are labeled by K_j.

Let $D := \{e_1, \dots, e_d\} \backslash \{e_{j_1}, \dots, e_{j_{n-2}}\}$. In what follows, we will show that (144) holds for this set of directions. Notice that if

$$\pi_l(Q_1) \cap \pi_l(K_n) = \varnothing \tag{146}$$

for some $l \in D$, then $\mathcal{A}$ would be weakly transversal with pivot Q_1 (because (145) together with (146) verify the definition of weak transversality), which is false by hypothesis. Hence (144) holds for $j = n$.

Let us argue by induction that, if (144) holds for $1 \le m < n-1$ cubes $K_n, K_{\alpha_1}, \dots, K_{\alpha_{m-1}}$, then it's possible to find a new one K_{α_m} for which (144) also holds[53] This will be achieved by the following algorithm: consider the set

$$\mathcal{A}'' := \{Q_1, K_n, K_{\alpha_1}, \dots, K_{\alpha_{m-1}}\}.$$

By the minimality of $\mathcal{A}$, we know $\mathcal{A}''$ is weakly transversal with pivot Q_1; hence there are $1 \le r_1, \dots, r_m \le d$ distinct such that

$$\begin{cases} \pi_{r_1}(Q_1) \cap \pi_{r_1}(K_n) = \varnothing, \\ \pi_{r_2}(Q_1) \cap \pi_{r_2}(K_{\alpha_1}) = \varnothing, \\ \qquad \vdots \\ \pi_{r_m}(Q_1) \cap \pi_{r_m}(K_{\alpha_{m-1}}) = \varnothing. \end{cases} \tag{147}$$

Property (P) for $\mathcal{A}$ implies $r_1 \in \{j_1, \dots, j_{n-2}\}$.[54] Then there is j_{β_1} such that $r_1 = j_{\beta_1}$; therefore

$$\begin{cases} \pi_{j_{\beta_1}}(Q_1) \cap \pi_{j_{\beta_1}}(K_{\beta_1+1}) = \varnothing, \\ \pi_{j_{\beta_1}}(Q_1) \cap \pi_{j_{\beta_1}}(K_n) = \varnothing. \end{cases} \tag{148}$$

Since K_{β_1+1} appears in (145), it is one among $K_2, \dots, K_{n-1}$; hence $K_{\beta_1+1} \ne K_n$. We are done if $K_{\beta_1+1} \notin \mathcal{A}''$: indeed, if

$$\pi_l(Q_1) \cap \pi_l(K_{\beta_1+1}) = \varnothing \tag{149}$$

for some $l \in D$, then

$$\begin{cases} \pi_{j_1}(Q_1) \cap \pi_{j_1}(K_2) = \varnothing, \\ \qquad \vdots \\ \pi_{j_{\beta_1-1}}(Q_1) \cap \pi_{j_{\beta_1-1}}(K_{\beta_1}) = \varnothing, \\ \pi_l(Q_1) \cap \pi_l(K_{\beta_1+1}) = \varnothing, \\ \pi_{j_{\beta_1+1}}(Q_1) \cap \pi_{j_{\beta_1+1}}(K_{\beta_1+2}) = \varnothing, \\ \qquad \vdots \\ \pi_{j_{n-2}}(Q_1) \cap \pi_{j_{n-2}}(K_{n-1}) = \varnothing. \\ \pi_{j_{\beta_1}}(Q_1) \cap \pi_{j_{\beta_1}}(K_n) = \varnothing, \end{cases} \tag{150}$$

and $\mathcal{A}$ would be weakly transversal with pivot Q_1 (by definition again), which contradicts property (P). This way, we would find a new (*not* in $\mathcal{A}''$) cube K_{β_1+1} for which (144) also holds.

[53] We are done if there are $m = n-1$ for which (144) holds, therefore we assume the strict inequality $m < n-1$.

[54] Otherwise we face the same problem that appeared in (146).

On the other hand, if $K_{\beta_1+1} = K_{\alpha_{q_1}}$ for some $K_{\alpha_{q_1}} \in \mathcal{A}''\backslash\{K_n\}$, then we simply switch the projections $\pi_{j_{\beta_1}}$ and $\pi_{r_{q_1+1}}$ in (145) (they are distinct because $j_{\beta_1} = r_1 \neq r_{q_1+1}$) and consider the conditions

$$\begin{cases} \pi_{j_1}(Q_1) \cap \pi_{j_1}(K_2) = \varnothing, \\ \qquad \vdots \\ \pi_{j_{\beta_1-1}}(Q_1) \cap \pi_{j_{\beta_1-1}}(K_{\beta_1}) = \varnothing, \\ \pi_{r_{q_1+1}}(Q_1) \cap \pi_{r_{q_1+1}}(K_{\alpha_{q_1}}) = \varnothing, \\ \pi_{j_{\beta_1+1}}(Q_1) \cap \pi_{j_{\beta_1+1}}(K_{\beta_1+2}) = \varnothing, \\ \qquad \vdots \\ \pi_{j_{n-2}}(Q_1) \cap \pi_{j_{n-2}}(K_{n-1}) = \varnothing \\ \pi_{j_{\beta_1}}(Q_1) \cap \pi_{j_{\beta_1}}(K_n) = \varnothing, \end{cases} \tag{151}$$

where the last condition is taken from (148). Since $j_{\beta_1} \neq r_{q_1+1}$, property (P) for $\mathcal{A}$ again implies that $r_{q_1+1} = j_{\beta_2}$. Notice that $\beta_2 \neq \beta_1$ because $r_1 = j_{\beta_1}$ and $r_1 \neq r_{q_1+1}$. This way, from (145),

$$\begin{cases} \pi_{j_{\beta_2}}(Q_1) \cap \pi_{j_{\beta_2}}(K_{\beta_2+1}) = \varnothing, \\ \pi_{j_{\beta_2}}(Q_1) \cap \pi_{j_{\beta_2}}(K_{\alpha_{q_1}}) = \varnothing. \end{cases} \tag{152}$$

The index j_{β_2} is one of the elements in the set $\{j_1, \dots, j_{\beta_1-1}, j_{\beta_1+1}, \dots, j_{n-2}\}$; hence K_{β_2+1} is in the set $\{K_2, \dots, K_{\beta_1}, K_{\beta_1+2}, \dots, K_{n-1}\}$. As before, we are done if $K_{\beta_2+1} \notin \mathcal{A}''$. If not, $K_{\beta_2+1} = K_{\alpha_{q_2}}$ for some $K_{\alpha_{q_2}} \in \mathcal{A}''\backslash\{K_n, K_{\alpha_{q_1}}\}$ and we switch the projections $\pi_{j_{\beta_2}}$ and $\pi_{r_{q_2+1}}$ in (151) to find some $\beta_3 \notin \{\beta_1, \beta_2\}$ such that

$$\begin{cases} \pi_{j_{\beta_3}}(Q_1) \cap \pi_{j_{\beta_3}}(K_{\beta_3+1}) = \varnothing, \\ \pi_{j_{\beta_3}}(Q_1) \cap \pi_{j_{\beta_3}}(K_{\alpha_{q_2}}) = \varnothing. \end{cases} \tag{153}$$

We keep doing that until we find some $K_{\beta_\ell+1} \notin \mathcal{A}''$. This is guaranteed to happen since there are $n-1$ cubes K_j, but only $m < n-1$ of them in $\mathcal{A}''$. The conclusion is that

$$m < n-1 \text{ cubes } K_j \text{ satisfy (144)} \quad \Longrightarrow \quad m+1 \text{ cubes } K_j \text{ satisfy (144)};$$

therefore (144) holds for $2 \leq j \leq n$. □

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A POINCARÉ–STEKLOV MAP FOR THE MIT BAG MODEL

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The purpose of this paper is to introduce and study Poincaré–Steklov (PS) operators associated to the Dirac operator D_m with the so-called MIT bag boundary condition. In a domain $\Omega \subset \mathbb{R}^3$, for a complex number z and for U_z a solution of $(D_m - z)U_z = 0$, the associated PS operator maps the value of $\Gamma_- U_z$ — the MIT bag boundary value of U_z — to $\Gamma_+ U_z$, where $\Gamma_\pm$ are projections along the boundary $\partial\Omega$ and $(\Gamma_- + \Gamma_+) = t_{\partial\Omega}$ is the trace operator on $\partial\Omega$.

In the first part of this paper, we show that the PS operator is a zeroth-order pseudodifferential operator and give its principal symbol. In the second part, we study the PS operator when the mass m is large, we prove that it fits into the framework of $1/m$-pseudodifferential operators, and we derive some important properties, especially its semiclassical principal symbol. Subsequently, we apply these results to establish a Krein-type resolvent formula for the Dirac operator $H_M = D_m + M\beta 1_{\mathbb{R}^3 \setminus \overline{\Omega}}$ for large masses $M > 0$ in terms of the resolvent of the MIT bag operator on Ω. With its help, the large coupling convergence with a convergence rate of $\mathcal{O}(M^{-1})$ is shown.

1. Introduction

Motivation. Boundary integral operators have played a key role in the study of many boundary value problems for partial differential equations arising in various areas of mathematical physics, such as electromagnetism, elasticity, and potential theory. In particular, they are used as a tool for proving the existence of solutions as well as for their construction by means of integral equation methods; see, e.g., [Fabes et al. 1978; Jerison and Kenig 1981a; 1981b; Verchota 1984].

The study of boundary integral operators has also been the motivation for the development of various tools and branches of mathematics, e.g., Fredholm theory and singular integral and pseudodifferential operators. Moreover, it turned out that the functional analytic and spectral properties of some of these

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operators are strongly related to the regularity and geometric properties of surfaces; see for example [Hofmann et al. 2009; 2010]. A typical and well-known example which occurs in many applications is the Dirichlet-to-Neumann (DtN) operator. In the classical setting of a bounded domain $\Omega \subset \mathbb{R}^d$ with a smooth boundary, the DtN operator, $\mathcal{N}$, is defined by

$$\mathcal{N} : H^{1/2}(\partial\Omega) \to H^{-1/2}(\partial\Omega), \quad g \mapsto \mathcal{N} g = \Gamma_N U(g),$$

where $U(g)$ is the harmonic extension of g (i.e., $\Delta U(g) = 0$ in Ω and $\Gamma_D U = g$ on $\partial\Omega$). Here Γ_D and Γ_N denote the Dirichlet and the Neumann traces, respectively. In this setting, it is well known that the DtN operator fits into the framework of pseudodifferential operators; see, e.g., [Taylor 1996]. Moreover, from the point of view of the spectral theory, several geometric properties of the eigenvalue problem for the DtN operator (such as isoperimetric inequalities, spectral asymptotics, and geometric invariants) are closely related to the theory of minimal surfaces [Fraser and Schoen 2016] as well as the problem of determining a complete Riemannian manifold with boundary from the Cauchy data of harmonic functions; see [Lassas et al. 2003] (see also the survey [Girouard and Polterovich 2017] for further details).

The main goal of this paper is to introduce a Poincaré–Steklov map for the Dirac operator (i.e., an analogue of the DtN map for the Laplace operator) and to study its (semiclassical) pseudodifferential properties. Our main motivation for considering this operator is that it arises naturally in the study of the well-known Dirac operator with the MIT bag boundary condition, $H_{\mathrm{MIT}}(m)$, defined rigorously below.

Description of main results. In order to give a rigorous definition of the operator we are dealing with in this paper and to go more into details, we need to introduce some notation. Given $m > 0$, the free Dirac operator D_m on $\mathbb{R}^3$ is defined by $D_m := -i\alpha \cdot \nabla + m\beta$, where

$$\alpha_j = \begin{pmatrix} 0 & \sigma_j \\ \sigma_j & 0 \end{pmatrix} \text{ for } j = 1, 2, 3, \quad \beta = \begin{pmatrix} I_2 & 0 \\ 0 & -I_2 \end{pmatrix}, \quad I_2 := \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$
$$\text{and} \quad \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

are the family of Dirac and Pauli matrices. We use the notation $\alpha \cdot x = \sum_{j=1}^{3} \alpha_j x_j$ for $x = (x_1, x_2, x_3) \in \mathbb{R}^3$. We refer to the Appendix, where we recall some important properties of Dirac matrices for the convenience of the reader. We recall that D_m is self-adjoint in $L^2(\mathbb{R}^3)^4$ with $\mathrm{dom}(D_m) = H^1(\mathbb{R}^3)^4$ (see, e.g., [Thaller 1992, Section 1.4]), and for the spectrum and the continuous spectrum, we have

$$\mathrm{Sp}(D_m) = \mathrm{Sp}_{\mathrm{cont}}(D_m) = (-\infty, -m] \cup [m, +\infty).$$

Let $\Omega \subset \mathbb{R}^3$ be a domain with a compact smooth boundary $\partial\Omega$, let n be the outward unit normal to Ω, and let $\Gamma_\pm$ and $P_\pm$ be the trace mappings and the orthogonal projections, respectively, defined by

$$\Gamma_\pm = P_\pm \Gamma_D : H^1(\Omega)^4 \to P_\pm H^{1/2}(\partial\Omega)^4 \qquad \text{and} \qquad P_\pm := \tfrac{1}{2}(I_4 \mp i\beta(\alpha \cdot n(x))), \quad x \in \partial\Omega.$$

In the present paper, we investigate the specific case of the Poincaré–Steklov (PS for short) operator, $\mathscr{A}_m$, defined by

$$\mathscr{A}_m : P_- H^{1/2}(\partial\Omega)^4 \to P_+ H^{1/2}(\partial\Omega)^4, \quad g \mapsto \mathscr{A}_m(g) = \Gamma_+ U_z,$$

where z belongs to the resolvent set of the MIT bag operator on Ω (i.e., $z \in \rho(H_{\mathrm{MIT}}(m))$) and $U_z \in H^1(\Omega)^4$ is the unique solution to the elliptic boundary problem

$$\begin{cases} (D_m - z)U_z = 0 & \text{in } \Omega, \\ \Gamma_- U_z = g & \text{on } \partial\Omega. \end{cases} \tag{1-1}$$

Here and also in what follows, z or any complex number stands for zI, with I the identity.

We point out that, in the R-matrix theory and the embedding method for the Dirac equation, similar operators linking on $\partial\Omega$ values of the upper and lower components of the spinor wave functions have been studied in [Agranovich 2001; Agranovich and Rozenblum 2004; Bielski and Szmytkowski 2006; Szmytkowski 1998]. There it corresponds to a different boundary condition (the trace of the upper/lower components) which is not necessarily elliptic. As far as we know, such operators for the MIT bag boundary condition have not been studied yet.

Let us now briefly describe the content of the present paper. Our results are mainly concerned with the pseudodifferential properties of $\mathscr{A}_m$ and their applications. Thus, our first goal is to show that $\mathscr{A}_m$ fits into the framework of pseudodifferential operators. In Section 4, we show that, when the mass m is fixed and $z \in \rho(D_m)$, the Poincaré–Steklov operator $\mathscr{A}_m$ is a classical homogeneous pseudodifferential operator of order 0, and that

$$\mathscr{A}_m = S \cdot \left(\frac{\nabla_{\partial\Omega} \wedge n}{\sqrt{-\Delta_{\partial\Omega}}} \right) P_- \mod \operatorname{Op} \mathcal{S}^{-1}(\partial\Omega),$$

where $S = \frac{1}{2} i (\alpha \wedge \alpha)$ is the spin angular momentum, $\nabla_{\partial\Omega}$ and $\Delta_{\partial\Omega}$ are the surface gradient and the Laplace–Beltrami operator on $\partial\Omega$ (equipped with the Riemann metric induced by the Euclidean one in $\mathbb{R}^3$), respectively, and $\operatorname{Op} \mathcal{S}^{-1}$ is the classical class of pseudodifferential operators of order -1 (see Theorem 4.5 for details). For $D_{\partial\Omega}$ — the extrinsically defined Dirac operator introduced in Section 2D — we also have

$$\mathscr{A}_m = D_{\partial\Omega} (-\Delta_{\partial\Omega})^{-1/2} P_- \mod \operatorname{Op} \mathcal{S}^{-1}(\partial\Omega).$$

The proof of the above result is based on the fact that we have an explicit solution of the system (1-1) for any $z \in \rho(D_m)$, and in this case the PS operator takes the following layer potential form:

$$\mathscr{A}_m = -P_+ \beta \left(\tfrac{1}{2} \beta + \mathscr{C}_{z,m} \right)^{-1} P_-, \tag{1-2}$$

where $\mathscr{C}_{z,m}$ is the Cauchy operator associated with $(D_m - z)$ defined on $\partial\Omega$ in the principal value sense (see Section 2B for the precise definition). So the starting point of the proof is to analyze the pseudodifferential properties of the Cauchy operator. In this sense, we show that $2\mathscr{C}_{z,m}$ is equal, modulo $\operatorname{Op} \mathcal{S}^{-1}(\partial\Omega)$, to $\alpha \cdot (\nabla_{\partial\Omega} (-\Delta_{\partial\Omega})^{-1/2})$. Using this, the explicit layer potential description of $\mathscr{A}_m$, and the symbol calculus, we then prove that $\mathscr{A}_m$ is a pseudodifferential operator and catch its principal symbol (see Theorem 4.5).

The above strategy allows us to capture the pseudodifferential character of $\mathscr{A}_m$, but unfortunately it does not allow us to trace the dependence on the parameter m, and it also imposes a restriction on the spectral parameter z (i.e., $z \in \rho(D_m)$), whereas $\mathscr{A}_m$ is well defined for any $z \in \rho(H_{\mathrm{MIT}}(m))$. In Section 5, we address the m-dependence of the pseudodifferential properties of $\mathscr{A}_m$ for any $z \in \rho(H_{\mathrm{MIT}}(m))$. Since we are mainly concerned with large masses m in our application, we treat this problem from the semiclassical point of view, where $h = 1/m \in (0, 1]$ is the semiclassical parameter. In fact, we show in Theorem 5.1 that $\mathscr{A}_{1/h}$ admits a semiclassical approximation, and that

$$\mathscr{A}_{1/h} = \frac{h D_{\partial\Omega}}{\sqrt{-h^2 \Delta_{\partial\Omega} + I} + I} P_- \mod h \operatorname{Op}^h \mathcal{S}^{-1}(\partial\Omega).$$

The main idea of the proof is to use the system (1-1) instead of the explicit formula (1-2), and it is based on the following two steps. The first step is to construct a local approximate solution for the pushforward of the system (1-1) of the form

$$U^h(\tilde{x}, x_3) = \operatorname{Op}^h(A^h(\cdot\,,\cdot\,, x_3))g = \frac{1}{(2\pi)^2} \int_{\mathbb{R}^2} A^h(\tilde{x}, h\xi, x_3) e^{iy\cdot\xi} \hat{g}(\xi)\, \mathrm{d}\xi, \qquad (\tilde{x}, x_3) \in \mathbb{R}^2 \times [0, \infty),$$

where A^h belongs to a specific symbol class and has the asymptotic expansion

$$A^h(\tilde{x}, \xi, x_3) \sim \sum_{j \geq 0} h^j A_j(\tilde{x}, \xi, x_3).$$

The second step is to show that, when applying the trace mapping Γ_+ to the pullback of $U^h(\cdot\,, 0)$, it coincides locally with $\mathscr{A}_{1/h}$ modulo a regularizing and negligible operator. At this point, the properties of the MIT bag operator become crucial, in particular the regularization property of its resolvent which allows us to achieve this second step, as we will see in Section 5. The MIT bag operator on Ω is the Dirac operator on $L^2(\Omega)^4$ defined by

$$H_{\mathrm{MIT}}(m)\psi = D_m\psi \quad \text{for all } \psi \in \operatorname{dom}(H_{\mathrm{MIT}}(m)) := \{\psi \in H^1(\Omega)^4 : \Gamma_-\psi = 0 \text{ on } \partial\Omega\}.$$

It is well known that $(H_{\mathrm{MIT}}(m), \operatorname{dom}(H_{\mathrm{MIT}}(m))$ is self-adjoint when Ω is smooth; see, e.g., [Ourmières-Bonafos and Vega 2018]. In Section 3, we briefly discuss the basic spectral properties of $H_{\mathrm{MIT}}(m)$, when Ω is a domain with compact Lipschitz boundary (see Theorem 3.1). Moreover, in Theorem 3.4 we establish regularity results concerning the regularization property of the resolvent and the Sobolev regularity of the eigenfunctions of H_{MIT}. In particular, we prove that $(H_{\mathrm{MIT}}(m) - z)^{-1}$ is bounded from $H^n(\Omega)^4$ into $H^{n+1}(\Omega)^4 \cap \operatorname{dom}(H_{\mathrm{MIT}}(m))$ for all $n \geqslant 1$.

Motivated by the natural way in which the PS operator is related to the MIT bag operator and to illustrate its usefulness, we consider in Section 6 the large mass problem for the self-adjoint Dirac operator $H_M = D_m + M\beta 1_{\mathcal{U}}$, where $\mathcal{U} = \mathbb{R}^3 \setminus \overline{\Omega}$. Indeed, it is known that, in the limit $M \to \infty$, every eigenvalue of $H_{\mathrm{MIT}}(m)$ is a limit of eigenvalues of H_M; see [Arrizabalaga et al. 2019; Moroianu et al. 2020] (see also [Barbaroux et al. 2019; Benhellal 2019; Stockmeyer and Vugalter 2019] for the two-dimensional setting). Moreover, it is shown in [Barbaroux et al. 2019; Benhellal 2019] that the two-dimensional analogue of H_M converges to the two-dimensional analogue of $H_{\mathrm{MIT}}(m)$ in the norm resolvent sense with a convergence rate of $\mathcal{O}(M^{-1/2})$.

The main goal of Section 6 is to address the following question: Let $M_0 > 0$ be large enough and fix $M \geqslant M_0$ and $z \in \rho(H_{\mathrm{MIT}}(m)) \cap \rho(H_M)$. Given $f \in L^2(\mathbb{R}^3)^4$ such that $f = 0$ in $\mathbb{R}^3 \setminus \overline{\Omega}$ and $U \in H^1(\mathbb{R}^3)^4$, what is the boundary value problem on Ω whose solutions closely approximate those of $(D_m + M\beta 1_{\mathbb{R}^3\setminus\overline{\Omega}} - z)U = f$?

It is worth noting that the answer to this question becomes trivial if one establishes an explicit formula for the resolvent of H_M. Having in mind the connection between the Dirac operators H_M and $H_{\mathrm{MIT}}(m)$, this leads us to address the following question: *for M sufficiently large, is it possible to relate the resolvents of H_M and H_{MIT} via a Krein-type resolvent formula*? In Theorem 6.2, which is the main result of Section 6, we establish a Krein-type resolvent formula for H_M in terms of the resolvent of $H_{\mathrm{MIT}}(m)$. The key point to establish this result is to treat the elliptic problem $(H_M - z)U = f \in L^2(\mathbb{R}^3)^4$ as a transmission problem (where $\Gamma_\pm U_{|\Omega} = \Gamma_\pm U_{|\mathbb{R}^3\setminus\Omega}$ are the transmission conditions) and to use the semiclassical properties of the Poincaré–Steklov operators in order to invert the auxiliary operator $\Psi_M(z)$ acting on the boundary $\partial\Omega$ (see Theorem 6.2 for the precise definition). In addition, we prove an adapted Birman–Schwinger principle relating the eigenvalues of H_M in the gap $(-(m+M), m+M)$ with a spectral property of $\Psi_M(z)$. With their help, we show in Corollary 6.5 that the restriction of U on Ω satisfies the elliptic problem

$$\begin{cases} (D_m - z)U_{|\Omega} = f & \text{in } \Omega, \\ \Gamma_- U_{|\Omega} = \mathscr{B}_M \Gamma_+ R_{\mathrm{MIT}}(z) f & \text{on } \partial\Omega, \\ \Gamma_+ U_{|\Omega} = \Gamma_+ R_{\mathrm{MIT}}(z) f + \mathscr{A}_m \Gamma_- v & \text{on } \partial\Omega, \end{cases}$$

where $\mathscr{B}_M$ is a semiclassical pseudodifferential operator of order 0. Here, the semiclassical parameter is $1/M$. Moreover, we show that the convergence of H_M to $H_{\mathrm{MIT}}(m)$ in the norm resolvent sense indeed holds with a convergence rate of $\mathcal{O}(M^{-1})$, which improves previous works; see Proposition 6.9. The most important ingredient in proving these results is the use of the Krein formula relating the resolvents of H_M and $H_{\mathrm{MIT}}(m)$, as well as regularity estimates for the PS operators (see Proposition 6.4) and layer potential operators (see Lemma 6.10 for details).

Organization of the paper. The paper is organized as follows. Sections 2 and 3 are devoted to preliminaries for the sake of completeness and self-containedness of the paper. In Section 2, we set up some notation, and we recall some basic properties of boundary integral operators associated with $(D_m - z)$. Section 3 is devoted to the study of the MIT bag operator, where we gather its basic properties in Theorem 3.1 and we establish the regularization property of its resolvent in Theorem 3.4. In Section 4 we establish Theorem 4.5, proving that the PS operator is a classical pseudodifferential operator. Then, in Section 5, we study the PS operator from the point of view of semiclassical pseudodifferential operators, the main result being Theorem 5.1. Finally, Section 6 is devoted to the study of the large mass problem for the operator H_M. There, we prove Theorem 6.2 regarding the Krein-type resolvent formula and we solve the large mass problem, and we prove Proposition 6.9 on the resolvent convergence.

2. Preliminaries

In this section we gather some well-known results about boundary integral operators. We also recall some properties of symbol classes and their associated pseudodifferential operators. Before proceeding further, however, we need to introduce some notation that we will use in what follows.

2A. ***Notations.*** Throughout this paper we will write $a \lesssim b$ if there is $C > 0$ such that $a \leqslant Cb$. As usual, the letter C stands for some constant which may change its value at different occurrences.

For a bounded or unbounded Lipschitz domain $\Omega \subset \mathbb{R}^3$, we write $\partial\Omega$ for its boundary, and we denote by n and σ the outward-pointing normal to Ω and the surface measure on $\partial\Omega$, respectively. By $L^2(\mathbb{R}^3)^4 := L^2(\mathbb{R}^3; \mathbb{C}^4)$ and $L^2(\Omega)^4 := L^2(\Omega, \mathbb{C}^4)$, we denote the usual L^2-space over $\mathbb{R}^3$ and Ω, respectively, and we let $r_\Omega : L^2(\mathbb{R}^3)^4 \to L^2(\Omega)^4$ be the restriction operator on Ω and $e_\Omega : L^2(\Omega)^4 \to L^2(\mathbb{R}^3)^4$ be its adjoint operator, i.e., the extension by zero outside of Ω.

For a function $u \in L^2(\mathbb{R}^d)$, its Fourier transform is defined by the formula

$$\hat{u}(\xi) = \int_{\mathbb{R}^d} e^{-ix\cdot\xi} u(x)\,\mathrm{d}x \quad \text{for all } \xi \in \mathbb{R}^d.$$

For $s \in [0, 1]$, we define the usual Sobolev space $H^s(\mathbb{R}^d)^4$ as

$$H^s(\mathbb{R}^d)^4 := \left\{ u \in L^2(\mathbb{R}^d)^4 : \int_{\mathbb{R}^d} (1 + |\xi|^2)^s |\hat{u}(\xi)|^2\,\mathrm{d}\xi < \infty \right\},$$

and we shall designate by $H^s(\Omega)^4$ the standard L^2-based Sobolev space of order s. We denote the usual L^2-space over $\partial\Omega$ by $L^2(\partial\Omega)^4 := L^2(\partial\Omega, \mathrm{d}\sigma)^4$. If Ω is a C^2-smooth domain with compact boundary $\partial\Omega$, then the Sobolev space of order $s \in (0, 1]$ along the boundary, $H^s(\partial\Omega)^4$, is defined using a local coordinate representation on the surface $\partial\Omega$. As usual, we use the symbol $H^{-s}(\partial\Omega)^4$ to denote the dual space of $H^s(\partial\Omega)^4$. We denote by $t_{\partial\Omega} : H^1(\Omega)^4 \to H^{1/2}(\partial\Omega)^4$ the classical trace operator, and by $\mathcal{E}_\Omega : H^{1/2}(\partial\Omega)^4 \to H^1(\Omega)^4$ the extension operator, that is,

$$t_{\partial\Omega}\mathcal{E}_\Omega[f] = f \quad \text{for all } f \in H^{1/2}(\partial\Omega)^4.$$

Throughout the current paper, we denote by $P_\pm$ the orthogonal projections defined by

$$P_\pm := \tfrac{1}{2}(I_4 \mp i\beta(\alpha \cdot n(x))), \quad x \in \partial\Omega. \tag{2-1}$$

We use the symbol $H(\alpha, \Omega)$ for the Dirac–Sobolev space on a smooth domain Ω defined as

$$H(\alpha, \Omega) = \{\varphi \in L^2(\Omega)^4 : (\alpha \cdot \nabla)\varphi \in L^2(\Omega)^4\}, \tag{2-2}$$

which is a Hilbert space (see [Ourmières-Bonafos and Vega 2018, Section 2.3]) endowed with the scalar product

$$\langle \varphi, \psi \rangle_{H(\alpha,\Omega)} = \langle \varphi, \psi \rangle_{L^2(\Omega)^4} + \langle (\alpha \cdot \nabla)\varphi, (\alpha \cdot \nabla)\psi \rangle_{L^2(\Omega)^4}, \quad \varphi, \psi \in H(\alpha, \Omega).$$

We also recall that the trace operator $t_{\partial\Omega}$ extends into a continuous map $t_{\partial\Omega} : H(\alpha, \Omega) \to H^{-1/2}(\partial\Omega)^4$. Moreover, if $v \in H(\alpha, \Omega)$ and $t_{\partial\Omega} v \in H^{1/2}(\partial\Omega)^4$, then $v \in H^1(\Omega)^4$; see [Ourmières-Bonafos and Vega 2018, Propositions 2.1 and 2.16].

2B. *Boundary integral operators.* The aim of this part is to introduce boundary integral operators associated with the fundamental solution of the free Dirac operator D_m and to summarize some of their well-known properties.

For $z \in \rho(D_m)$, with the convention that $\operatorname{Im}\sqrt{z^2 - m^2} > 0$, the fundamental solution of $(D_m - z)$ is

$$\phi_m^z(x) = \frac{e^{i\sqrt{z^2-m^2}|x|}}{4\pi|x|}\left(z + m\beta + (1 - i\sqrt{z^2 - m^2}|x|)i\alpha \cdot \frac{x}{|x|^2} \right) \quad \text{for all } x \in \mathbb{R}^3 \setminus \{0\}. \tag{2-3}$$

We define the potential operator $\Phi^{\Omega}_{z,m} : L^2(\partial\Omega)^4 \to L^2(\Omega)^4$ by

$$\Phi^{\Omega}_{z,m}[g](x) = \int_{\partial\Omega} \phi^z_m(x-y)g(y)\,\mathrm{d}\sigma(y) \quad \text{for all } x \in \Omega \tag{2-4}$$

and the Cauchy operator $\mathscr{C}_{z,m} : L^2(\partial\Omega)^4 \to L^2(\partial\Omega)^4$ as the singular integral operator acting as

$$\mathscr{C}_{z,m}[f](x) = \lim_{\rho\searrow 0} \int_{|x-y|>\rho} \phi^z_m(x-y) f(y)\,\mathrm{d}\sigma(y) \quad \text{for } \sigma\text{-a.e. } x \in \partial\Omega,\ f \in L^2(\partial\Omega)^4. \tag{2-5}$$

It is well known that $\Phi^{\Omega}_{z,m}$ and $\mathscr{C}_{z,m}$ are bounded and everywhere defined (see, for instance, [Arrizabalaga et al. 2014, Section 2]) and that

$$((\alpha\cdot n)\mathscr{C}_{z,m})^2 = (\mathscr{C}_{z,m}(\alpha\cdot n))^2 = -\tfrac{1}{4} \quad \text{for all } z \in \rho(D_m) \tag{2-6}$$

holds in $L^2(\partial\Omega)^4$; see [Arrizabalaga et al. 2015, Lemma 2.2]. In particular, the inverse

$$\mathscr{C}^{-1}_{z,m} = -4(\alpha\cdot n)\mathscr{C}_{z,m}(\alpha\cdot n)$$

exists and is bounded and everywhere defined. Since we have $\phi^z_m(y-x)^* = \phi^{\bar z}_m(x-y)$ for all $z \in \rho(D_m)$, it follows that $\mathscr{C}^*_{z,m}$ and $\mathscr{C}_{\bar z,m}$ are equal as operators in $L^2(\partial\Omega)^4$. In particular, $\mathscr{C}_{z,m}$ is self-adjoint in $L^2(\partial\Omega)^4$ for all $z \in (-m, m)$.

Next, recall that the trace of the single layer operator S_z associated with the Helmholtz operator $(-\Delta + m^2 - z^2)I_4$ is defined, for every $f \in L^2(\partial\Omega)^4$ and $z \in \rho(D_m)$, by

$$S_z[f](x) := \int_{\partial\Omega} \frac{e^{i\sqrt{z^2-m^2}|x-y|}}{4\pi|x-y|} f(y)\,\mathrm{d}\sigma(y) \quad \text{for } x \in \partial\Omega.$$

It is well known that S_z is bounded from $L^2(\partial\Omega)^4$ into $H^{1/2}(\partial\Omega)^4$ and it is a positive operator in $L^2(\partial\Omega)^4$ for all $z \in (-m, m)$; see [Arrizabalaga et al. 2015, Lemma 4.2]. Now we define the operator Λ^z_m by

$$\Lambda^z_m = \tfrac{1}{2}\beta + \mathscr{C}_{z,m} \qquad \text{for all } z \in \rho(D_m),$$

which is clearly a bounded operator from $L^2(\partial\Omega)^4$ into itself.

In the next lemma we collect the main properties of the operators $\Phi^{\Omega}_{z,m}$, $\mathscr{C}_{z,m}$, and Λ^z_m.

Lemma 2.1. *Assume that Ω is C^2-smooth. Given $z \in \rho(D_m)$, let $\Phi^{\Omega}_{z,m}$, $\mathscr{C}_{z,m}$, and Λ^z_m be as above. Then the following hold*:

(i) *The operator $\Phi^{\Omega}_{z,m}$ is bounded from $H^{1/2}(\partial\Omega)^4$ to $H^1(\Omega)^4$ and extends into a bounded operator from $H^{-1/2}(\partial\Omega)^4$ to $H(\alpha,\Omega)$. Moreover,*

$$t_{\partial\Omega}\Phi^{\Omega}_{z,m}[f] = \left(-\tfrac{1}{2}i(\alpha\cdot n) + \mathscr{C}_{z,m}\right)[f] \quad \textit{for all } f \in H^{1/2}(\partial\Omega)^4. \tag{2-7}$$

(ii) *The operator $\mathscr{C}_{z,m}$ gives rise to a bounded operator $\mathscr{C}_{z,m} : H^{1/2}(\partial\Omega)^4 \to H^{1/2}(\partial\Omega)^4$.*

(iii) *The operator $\Lambda^z_m : H^{1/2}(\partial\Omega)^4 \to H^{1/2}(\partial\Omega)^4$ is bounded invertible for all $z \in \rho(D_m)$.*

Proof. (i) The proof of the boundedness of $\Phi^{\Omega}_{z,m}$ from $H^{1/2}(\partial\Omega)^4$ into $H^1(\Omega)^4$ is contained in [Behrndt and Holzmann 2020, Proposition 4.2], and the jump formula (2-7) is proved in [Arrizabalaga et al. 2014, Lemma 3.3] in terms of the nontangential limit which coincides (almost everywhere in $\partial\Omega$) with the trace operator for functions in $H^1(\Omega)^4$. The boundedness of $\Phi^{\Omega}_{z,m}$ from $H^{-1/2}(\partial\Omega)^4$ to $H(\alpha,\Omega)$ is established in [Ourmières-Bonafos and Vega 2018, Theorem 2.2].

Since n is smooth, it is clear from (i) that $\mathscr{C}_{z,m}$ is bounded from $H^{1/2}(\partial\Omega)^4$ into itself, which proves (ii). As consequence we also obtain that Λ^z_m is bounded from $H^{1/2}(\partial\Omega)^4$ into itself. Now, the invertibility of Λ^z_m in $H^{1/2}(\partial\Omega)^4$ for $z\in\mathbb{C}\setminus\mathbb{R}$ is shown in [Behrndt et al. 2019, Lemma 3.3 (iii)]; see also [Behrndt et al. 2020, Lemma 3.12]. To complete the proof of (iii), note that if $f\in L^2(\partial\Omega)^4$ is such that $\Lambda^z_m[f]\in H^{1/2}(\partial\Omega)^4$, then a simple computation shows that

$$H^{1/2}(\partial\Omega)^4 \ni (\Lambda^z_m)^2[f] = \big(\tfrac{1}{4} + (\mathscr{C}_{z,m})^2 + (m+z\beta)S_z\big)[f],$$

which means that $f\in H^{1/2}(\partial\Omega)^4$. From the above computation, we see that Λ^z_m is invertible from $H^{1/2}(\partial\Omega)^4$ into itself for all $z\in(-m,m)$, since $((\mathscr{C}_{z,m})^2+(m+z\beta)S_z)$ is a positive operator. □

Remark 2.2. Note that if Ω is a Lipschitz domain with a compact boundary, then, for all $z\in\rho(D_m)$, the operators $\mathscr{C}_{z,m}$ and Λ^z_m are bounded from $L^2(\partial\Omega)^4$ into itself (see, e.g, [Arrizabalaga et al. 2014, Lemma 3.3]), and since Λ^z_m is an injective Fredholm operator (see the proof of [Benhellal 2022a, Theorem 4.5]), it follows that it is also invertible in $L^2(\partial\Omega)^4$. Note also that, thanks to [Behrndt et al. 2021, Lemmas 5.1 and 5.2], we know the mapping $\Phi^{\Omega}_{z,m}$ defined by (2-4) is bounded from $L^2(\partial\Omega)^4$ to $H^{1/2}(\Omega)^4$, $t_{\partial\Omega}\Phi^{\Omega}_{z,m}[g]\in L^2(\partial\Omega)^4$, and the formula (2-7) still holds true for all $g\in L^2(\partial\Omega)^4$.

2C. ***Symbol classes and pseudodifferential operators.*** We recall here the basic facts concerning the classes of pseudodifferential operators that will serve in the rest of the paper.

Let $\mathscr{M}_4(\mathbb{C})$ be the set of 4×4 matrices over $\mathbb{C}$. For $d\in\mathbb{N}^*$, we let $S^m(\mathbb{R}^d\times\mathbb{R}^d)$ be the standard symbol class of order $m\in\mathbb{R}$ whose elements are matrix-valued functions a in the space $C^\infty(\mathbb{R}^d\times\mathbb{R}^d;\mathscr{M}_4(\mathbb{C}))$ such that

$$|\partial^\alpha_x\,\partial^\beta_\xi a(x,\xi)| \leqslant C_{\alpha\beta}(1+|\xi|^2)^{m-|\beta|} \quad \text{for all } (x,\xi)\in\mathbb{R}^d\times\mathbb{R}^d,\ \text{for all } \alpha\in\mathbb{N}^d,\ \text{for all } \beta\in\mathbb{N}^d.$$

Let $\mathscr{S}(\mathbb{R}^d)$ be the Schwartz class of functions. Then, for each $a\in S^m(\mathbb{R}^d\times\mathbb{R}^d)$ and any $h\in(0,1]$, we associate to it a semiclassical pseudodifferential operator $\mathrm{Op}^h(a):\mathscr{S}(\mathbb{R}^d)^4\to\mathscr{S}(\mathbb{R}^d)^4$ via the standard formula

$$\mathrm{Op}^h(a)u(x) = \frac{1}{(2\pi)^d}\int_{\mathbb{R}^d} e^{i\xi\cdot x}a(x,h\xi)\hat{u}(\xi)\,\mathrm{d}\xi \quad \text{for all } u\in\mathscr{S}(\mathbb{R}^d)^4.$$

If $a\in S^0(\mathbb{R}^d\times\mathbb{R}^d)$, then the Calderón–Vaillancourt theorem (see, e.g., [Calderón and Vaillancourt 1972]) yields that $\mathrm{Op}^h(a)$ extends to a bounded operator from $L^2(\mathbb{R}^d)^4$ into itself, and there exists $C, N_C>0$ such that

$$\|\mathrm{Op}^h(a)\|_{L^2\to L^2} \leqslant C \max_{|\alpha+\beta|\leqslant N_C} \|\partial^\alpha_x\,\partial^\beta_\xi a\|_{L^\infty}. \tag{2-8}$$

By definition, a semiclassical pseudodifferential operator $\mathrm{Op}^h(a)$, with $a\in S^0(\mathbb{R}^d\times\mathbb{R}^d)$, can also be considered as a classical pseudodifferential operator $\mathrm{Op}^1(a_h)$, with $a_h=a(x,h\xi)$, which is bounded with respect to $h \in (0, h_0)$, where $h_0 > 0$ is fixed. Thus the Calderón–Vaillancourt theorem also provides the boundedness of these operators in Sobolev spaces $H^s(\mathbb{R}^d)^4 = \langle D_x \rangle^{-s} L^2(\mathbb{R}^d)^4$, where $\langle D_x \rangle = \sqrt{-\Delta + I}$. Indeed, we have

$$\|\mathrm{Op}^1(a_h)\|_{H^s \to H^s} = \|\langle D_x \rangle^s \mathrm{Op}^1(a_h) \langle D_x \rangle^{-s}\|_{L^2 \to L^2}, \tag{2-9}$$

and since $\langle D_x \rangle^s \mathrm{Op}^1(a_h) \langle D_x \rangle^{-s}$ is a classical pseudodifferential operator with a uniformly bounded symbol in $\mathcal{S}^0$, we deduce that $\mathrm{Op}^h(a)$ is uniformly bounded with respect to h from H^s into itself.

Consider a C^∞-smooth domain $\Omega \subset \mathbb{R}^3$ with a compact boundary $\Sigma = \partial\Omega$. Then Σ is a 2-dimensional parametrized surface, which, in the sense of differential geometry, can also be viewed as a smooth 2-dimensional manifold immersed into $\mathbb{R}^3$. Thus Σ can be covered by an atlas (i.e., a collection of smooth charts)

$$\mathbb{A} = \{(U_j, V_j, \varphi_j) : j \in \{1, \dots, N\}\}, \quad \text{where } N \in \mathbb{N}^*.$$

That is

$$\Sigma = \bigcup_{j=1}^{N} U_j,$$

and for each $j \in \{1, \dots, N\}$, we have that U_j is an open set of Σ, $V_j \subset \mathbb{R}^2$ is an open set of the parametric space $\mathbb{R}^2$, and $\varphi_j : U_j \to V_j$ is a C^∞-diffeomorphism. Moreover, by the definition of a smooth manifold, if $U_j \cap U_k \neq \varnothing$ then

$$\varphi_k \circ (\varphi_j)^{-1} \in C^\infty(\varphi_j(U_j \cap U_k); \varphi_k(U_j \cap U_k)).$$

As usual, the pullback $(\varphi_j^{-1})^*$ and the pushforward φ_j^* are defined by

$$(\varphi_j^{-1})^* u = u \circ \varphi_j^{-1} \quad \text{and} \quad \varphi_j^* v = v \circ \varphi_j$$

for u and v functions on U_j and V_j, respectively. We also recall that a function u on Σ is said to be in the class $C^k(\Sigma)$ if, for every chart, the pushforward has the property $(\varphi_j^{-1})^* u \in C^k(V_j)$.

Following [Zworski 2012, Part 4], we define pseudodifferential operators on the boundary Σ as follows.

Definition 2.3. Let $\mathscr{A} : C^\infty(\Sigma)^4 \to C^\infty(\Sigma)^4$ be a continuous linear operator. Then $\mathscr{A}$ is said to be a h-pseudodifferential operator of order $m \in \mathbb{R}$ on Σ, and we write $\mathscr{A} \in \mathrm{Op}^h \mathcal{S}^m(\Sigma)$, if,

(1) for every chart (U_j, V_j, φ_j), there exists a symbol $a \in \mathcal{S}^m$ such that

$$\psi_1 \mathscr{A}(\psi_2 u) = \psi_1 \varphi_j^* \mathrm{Op}^h(a) (\varphi_j^{-1})^* (\psi_2 u)$$

for any $\psi_1, \psi_2 \in C_0^\infty(U_j)$ and $u \in C^\infty(\Sigma)^4$,

(2) for all $\psi_1, \psi_2 \in C^\infty(\Sigma)$ such that $\mathrm{supp}(\psi_1) \cap \mathrm{supp}(\psi_2) = \varnothing$ and for all $N \in \mathbb{N}$, we have

$$\|\psi_1 \mathscr{A} \psi_2\|_{H^{-N}(\Sigma)^4 \to H^N(\Sigma)^4} = \mathcal{O}(h^\infty).$$

For h fixed (for example $h = 1$), $\mathscr{A}$ is called a pseudodifferential operator.

Since the study of a given pseudodifferential operator on Σ reduces to a local study on local charts, we recall below the specific local coordinates and surface geometry notation used in the rest of the paper.

We always fix an open set $U \subset \Sigma$, and we let $\chi : V \to \mathbb{R}$ be a C^∞-function (where $V \subset \mathbb{R}^2$ is open) such that its graph coincides with U. Here and in the following, we omit the possible composition with a rotation that allows this, since changes of variables take h-pseudodifferential operators to h-pseudodifferential operators modulo smoothing operators and leave the principal symbol invariant. Set $\varphi(\tilde{x}) = (\tilde{x}, \chi(\tilde{x}))$. Then for $x \in U$ we write $x = \varphi(\tilde{x})$ with $\tilde{x} \in V$. Here and also in what follows, $\partial_1\chi$ and $\partial_2\chi$ stand for the partial derivatives $\partial_{\tilde{x}_1}\chi$ and $\partial_{\tilde{x}_2}\chi$, respectively. Recall that the first fundamental form, I, and the metric tensor $G(\tilde{x}) = (g_{jk}(\tilde{x}))$, have the following forms:

$$\mathrm{I} = g_{11}\, d\tilde{x}_1^2 + 2g_{12}\, d\tilde{x}_1\, d\tilde{x}_2 + g_{22}\, d\tilde{x}_2^2,$$
$$G(\tilde{x}) = (g_{jk}(\tilde{x})) = \begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix}(\tilde{x}) := \begin{pmatrix} 1+|\partial_1\chi|^2 & \partial_1\chi\,\partial_2\chi \\ \partial_1\chi\,\partial_2\chi & 1+|\partial_2\chi|^2 \end{pmatrix}(\tilde{x}).$$

As $G(\tilde{x})$ is symmetric, it follows that it is diagonalizable by an orthogonal matrix. Indeed, let

$$Q(\tilde{x}) := \begin{pmatrix} \dfrac{|\partial_2\chi|}{|\nabla\chi|} & \dfrac{\partial_1\chi\,\partial_2\chi}{|\partial_2\chi||\nabla\chi|} \\ -\dfrac{\partial_1\chi\,\partial_2\chi}{|\partial_2\chi||\nabla\chi|} & \dfrac{|\partial_2\chi|}{|\nabla\chi|} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & g^{-1/2} \end{pmatrix}(\tilde{x}), \tag{2-10}$$

where g stands for the determinant of G. Then, it is straightforward to check that

$$Q^{\mathrm{T}} G Q(\tilde{x}) = I_2, \quad Q Q^{\mathrm{T}}(\tilde{x}) = G(\tilde{x})^{-1} =: (g^{jk}(\tilde{x})), \quad \det(Q) = \det(Q^{\mathrm{T}}) = g^{-1/2}. \tag{2-11}$$

2D. ***Operators on the boundary $\Sigma = \partial\Omega$.*** As above, we consider $\Sigma = \partial\Omega$ the boundary of a smooth bounded domain Ω. On Σ equipped with the Riemann metric induced by the Euclidean one in $\mathbb{R}^3$, we consider the Laplace–Beltrami operator $-\Delta_\Sigma$ and the surface gradient $\nabla_\Sigma = \nabla - n(n\cdot\nabla)$, where n is the unit normal to the surface pointing outside Ω. Note that, for (e_1, e_2) an orthonormal basis of the tangent space, $\nabla_\Sigma = e_1\nabla_{e_1} + e_2\nabla_{e_2}$, where ∇_{e_j} stands for the tangential derivative in the direction e_j. With the notation of the previous section, in local coordinates, $-\Delta_\Sigma$ and ∇_Σ are pseudodifferential operators with respective principal symbols

$$p_{-\Delta_\Sigma}(\tilde{x}, \xi) = \langle G(\tilde{x})^{-1}\xi, \xi\rangle, \qquad p_{\nabla_\Sigma}(\tilde{x}, \xi) = \xi_G := \begin{pmatrix} G(\tilde{x})^{-1}\xi \\ \langle\nabla\chi(\tilde{x}), G(\tilde{x})^{-1}\xi\rangle \end{pmatrix}. \tag{2-12}$$

Let us now introduce D_Σ, the extrinsically defined Dirac operator. To any $x \in \mathbb{R}^3$ we associate the matrix $\alpha(x) = \alpha\cdot x$, where $\alpha = (\alpha_1, \alpha_2, \alpha_3)$. For H_1, the mean curvature of Σ, D_Σ, is given by

$$D_\Sigma = -\alpha(n)\alpha(\nabla_\Sigma) + \tfrac{1}{2}H_1$$

(for more details see [Moroianu et al. 2020, Appendix B]). It is a pseudodifferential operator with principal symbol

$$p_{D_\Sigma}(\tilde{x}, \xi) = -i\alpha(n^\varphi(\tilde{x}))\alpha(\xi_G),$$

where $n^\varphi = \varphi^* n$. We now define the spin angular momentum S as

$$S\cdot X = -\gamma_5(\alpha\cdot X) \quad \text{for all } X \in \mathbb{R}^3, \qquad \text{where } \gamma_5 := -i\alpha_1\alpha_2\alpha_3 = \begin{pmatrix} 0 & I_2 \\ I_2 & 0 \end{pmatrix}. \tag{2-13}$$

Using properties (A-1) and (A-2) and the fact that $n \cdot \xi_G = 0$, we then have

$$p_{D_\Sigma}(\tilde{x}, \xi) = -i\alpha \cdot n^\varphi(\tilde{x})\alpha \cdot \xi_G = S \cdot (\xi_G \wedge n^\varphi(\tilde{x})).$$

Moreover, for $\bar{\xi} := \binom{\xi}{0}$, we have $\bar{\xi} = \xi_G + (\bar{\xi} \cdot n^\varphi)n^\varphi$. Thus, in local coordinates, the principal symbol of D_Σ is also

$$p_{D_\Sigma}(\tilde{x}, \xi) = S \cdot (\bar{\xi} \wedge n^\varphi(\tilde{x})). \tag{2-14}$$

Let us also point out the relationship between the principal symbols of Δ_Σ and D_Σ:

$$|\bar{\xi} \wedge n^\varphi(\tilde{x})|^2 = \langle G(\tilde{x})^{-1}\xi, \xi\rangle. \tag{2-15}$$

3. Basic properties of the MIT bag model

In this section, we give a brief review of the basic spectral properties of the Dirac operator with the MIT bag boundary condition on Lipschitz domains. Then, we establish some results concerning the regularization properties of the resolvent and the Sobolev regularity of the eigenfunctions in the case of smooth domains.

Let $\mathcal{U} \subset \mathbb{R}^3$ be a Lipschitz domain with a compact boundary $\partial\mathcal{U}$. Then, for $m > 0$, the Dirac operator with the MIT bag boundary condition on $\mathcal{U}$, $(H_{\mathrm{MIT}}(m), \mathrm{dom}(H_{\mathrm{MIT}}(m)))$, or simply the MIT bag operator, is defined on the domain

$$\mathrm{dom}(H_{\mathrm{MIT}}(m)) := \{\psi \in H^{1/2}(\mathcal{U})^4 : (\alpha \cdot \nabla)\psi \in L^2(\mathcal{U})^4 \text{ and } P_- t_{\partial\mathcal{U}}\psi = 0 \text{ on } \partial\mathcal{U}\}$$

by $H_{\mathrm{MIT}}(m)\psi = D_m\psi$ for all $\psi \in \mathrm{dom}(H_{\mathrm{MIT}}(m))$ and where the boundary condition holds in $L^2(\partial\mathcal{U})^4$. Here $P_\pm$ are the orthogonal projections defined by (2-1).

The following theorem gathers the basic properties of the MIT bag operator. We mention that some of these properties are well known in the case of smooth domains; see, e.g., [Arrizabalaga et al. 2017; 2019; 2023; Behrndt et al. 2020; Ourmières-Bonafos and Vega 2018].

Theorem 3.1. *The operator* $(H_{\mathrm{MIT}}(m), \mathrm{dom}(H_{\mathrm{MIT}}(m)))$ *is self-adjoint, and we have*

$$(H_{\mathrm{MIT}}(m) - z)^{-1} = r_{\mathcal{U}}(D_m - z)^{-1}e_{\mathcal{U}} - \Phi^{\mathcal{U}}_{z,m}(\Lambda^z_m)^{-1}t_{\partial\mathcal{U}}(D_m - z)^{-1}e_{\mathcal{U}} \quad \text{for all } z \in \rho(D_m). \tag{3-1}$$

Moreover, the following statements hold:

(i) *If $\mathcal{U}$ is bounded, then* $\mathrm{Sp}(H_{\mathrm{MIT}}(m)) = \mathrm{Sp}_{\mathrm{disc}}(H_{\mathrm{MIT}}(m)) \subset \mathbb{R} \setminus [-m, m]$.

(ii) *If $\mathcal{U}$ is unbounded, then* $\mathrm{Sp}(H_{\mathrm{MIT}}(m)) = \mathrm{Sp}_{\mathrm{ess}}(H_{\mathrm{MIT}}(m)) = (-\infty, -m] \cup [m, +\infty)$. *Moreover, if $\mathcal{U}$ is connected, then* $\mathrm{Sp}(H_{\mathrm{MIT}}(m))$ *is purely continuous.*

(iii) *Let $z \in \rho(H_{\mathrm{MIT}}(m))$ be such that $2|z| < m$. Then, for all $f \in L^2(\mathcal{U})^4$,*

$$\|(H_{\mathrm{MIT}}(m) - z)^{-1} f\|_{L^2(\mathcal{U})^4} \lesssim \frac{1}{m}\|f\|_{L^2(\mathcal{U})^4}.$$

Proof. Let $\varphi, \psi \in \mathrm{dom}(H_{\mathrm{MIT}}(m))$. Then by density arguments we get the Green formula

$$\langle(-i\alpha \cdot \nabla)\varphi, \psi\rangle_{L^2(\mathcal{U})^4} - \langle\varphi, (-i\alpha \cdot \nabla)\psi\rangle_{L^2(\mathcal{U})^4} = \langle(-i\alpha \cdot n)t_{\partial\mathcal{U}}\varphi, t_{\partial\mathcal{U}}\psi\rangle_{L^2(\partial\mathcal{U})^4}. \tag{3-2}$$

Since $P_- t_{\partial\mathcal{U}}\varphi = P_- t_{\partial\mathcal{U}}\psi = 0$ and $P_\pm(\alpha\cdot n) = (\alpha\cdot n)P_\mp$ (see Lemma A.3), it follows that

$$\langle(-i\alpha\cdot\nabla)\varphi, \psi\rangle_{L^2(\mathcal{U})^4} - \langle\varphi, (-i\alpha\cdot\nabla)\psi\rangle_{L^2(\mathcal{U})^4} = \langle P_+(-i\alpha\cdot n)P_+ t_{\partial\mathcal{U}}\varphi, P_+ t_{\partial\mathcal{U}}\psi\rangle_{L^2(\partial\mathcal{U})^4} = 0.$$

Consequently, we obtain

$$\begin{aligned}\langle H_{\mathrm{MIT}}(m)\varphi, \psi\rangle_{L^2(\mathcal{U})^4} - \langle\varphi, H_{\mathrm{MIT}}(m)\psi\rangle_{L^2(\mathcal{U})^4} &= \langle D_m\varphi, \psi\rangle_{L^2(\mathcal{U})^4} - \langle\varphi, D_m\psi\rangle_{L^2(\mathcal{U})^4}\\ &= \langle(-i\alpha\cdot\nabla)\varphi, \psi\rangle_{L^2(\mathcal{U})^4} - \langle\varphi, (-i\alpha\cdot\nabla)\psi\rangle_{L^2(\mathcal{U})^4} = 0.\end{aligned}$$

Therefore $(H_{\mathrm{MIT}}(m), \mathrm{dom}(H_{\mathrm{MIT}}(m)))$ is symmetric. Now, thanks to [Benhellal 2022a, Proposition 4.3], we know that the MIT bag operator defined on the domain

$$\mathscr{D} = \{\psi = u + \Phi^{\mathcal{U}}_{0,m}[g],\ u \in H^1(\mathcal{U})^4,\ g \in L^2(\partial\mathcal{U})^4 : P_- t_{\partial\mathcal{U}}\psi = 0 \text{ on } \partial\mathcal{U}\} \tag{3-3}$$

by $H_{\mathrm{MIT}}(m)(u + \Phi^{\mathcal{U}}_{0,m}[g]) = D_m u$ for all $(u + \Phi^{\mathcal{U}}_{0,m}[g]) \in \mathscr{D}$ is a self-adjoint operator. Since $H_{\mathrm{MIT}}(m)$ is symmetric on $\mathrm{dom}(H_{\mathrm{MIT}}(m))$, we can deduce that $\mathrm{dom}(H_{\mathrm{MIT}}(m)) \subset \mathscr{D}$. Now, from Remark 2.2, we also get that $\mathscr{D} \subset \mathrm{dom}(H_{\mathrm{MIT}}(m))$, which proves the equality $\mathscr{D} = \mathrm{dom}(H_{\mathrm{MIT}}(m))$, and thus that $(H_{\mathrm{MIT}}(m), \mathrm{dom}(H_{\mathrm{MIT}}(m)))$ is self-adjoint. Next, we check the resolvent formula (3-1). Let $f \in L^2(\mathcal{U})^4$ and $z \in \rho(D_m)$, and set

$$\psi = r_{\mathcal{U}}(D_m - z)^{-1} e_{\mathcal{U}} f - \Phi^{\mathcal{U}}_{z,m}(\Lambda^z_m)^{-1} t_{\partial\mathcal{U}}(D_m - z)^{-1} e_{\mathcal{U}} f.$$

Since $(D_m - z)^{-1} e_{\mathcal{U}}$ is bounded from $L^2(\mathcal{U})^4$ into $H^1(\mathbb{R}^3)^4$ and $(\Lambda^z_m)^{-1}$ is well defined by Remark 2.2, it follows that

$$u := r_{\mathcal{U}}(D_m - z)^{-1} e_{\mathcal{U}} f \in H^1(\mathcal{U})^4 \quad \text{and} \quad g := -(\Lambda^z_m)^{-1} t_{\partial\mathcal{U}}(D_m - z)^{-1} e_{\mathcal{U}} f \in L^2(\partial\mathcal{U})^4,$$

which gives that $\psi \in H^{1/2}(\mathcal{U})^4$ and that $(\alpha\cdot\nabla)\psi \in L^2(\mathcal{U})^4$. Next, using Lemma 2.1(i) and Remark 2.2, we easily get

$$\begin{aligned} t_{\partial\mathcal{U}}\psi &= t_{\partial\mathcal{U}}(D_m - z)^{-1} e_{\mathcal{U}} f + \big(\tfrac12 i(\alpha\cdot n) - \mathscr{C}_{z,m}\big)\big(\tfrac12\beta + \mathscr{C}_{z,m}\big)^{-1} t_{\partial\mathcal{U}}(D_m - z)^{-1} e_{\mathcal{U}} f\\ &= P_+\beta(\Lambda^z_m)^{-1} t_{\partial\mathcal{U}}(D_m - z)^{-1} e_{\mathcal{U}} f,\end{aligned}$$

thus $P_- t_{\partial\mathcal{U}}\psi = 0$ on $\partial\mathcal{U}$, which means that $\psi \in \mathrm{dom}(H_{\mathrm{MIT}}(m))$. Since $(D_m - z)\Phi^{\mathcal{U}}_{z,m}[g] = 0$ in $\mathcal{U}$, it follows that $(H_{\mathrm{MIT}}(m) - z)\psi = f$, and formula (3-1) is proved.

We are now going to prove assertions (i) and (ii). First, note that, for $\psi \in \mathrm{dom}(H_{\mathrm{MIT}}(m))$, a straightforward application of the Green formula (3-2) yields

$$\|H_{\mathrm{MIT}}(m)\psi\|^2_{L^2(\mathcal{U})^4} = \|(\alpha\cdot\nabla)\psi\|^2_{L^2(\mathcal{U})^4} + m^2\|\psi\|^2_{L^2(\mathcal{U})^4} + m\|P_+ t_{\partial\mathcal{U}}\psi\|^2_{L^2(\partial\mathcal{U})^4}. \tag{3-4}$$

Thus $\|H_{\mathrm{MIT}}(m)\psi\|^2_{L^2(\mathcal{U})^4} \geqslant m^2\|\psi\|^2_{L^2(\mathcal{U})^4}$, which yields $\mathrm{Sp}(H_{\mathrm{MIT}}(m)) \subset (-\infty, -m] \cup [m, +\infty)$. Note that this can be seen immediately from (3-1). Next, we show that $\{-m, m\} \notin \mathrm{Sp}_{\mathrm{disc}}(H_{\mathrm{MIT}}(m))$. Assume that there is $0 \neq \psi \in \mathrm{dom}(H_{\mathrm{MIT}}(m))$ such that $(H_{\mathrm{MIT}}(m) - m)\psi = 0$ in $\mathcal{U}$. Then, from (3-4), we have

$$\|(-i\alpha\cdot\nabla)\psi\|^2_{L^2(\mathcal{U})^4} + m\|P_+ t_{\partial\mathcal{U}}\psi\|^2_{L^2(\partial\mathcal{U})^4} = 0.$$

Since $m > 0$, it follows that $P_+ t_{\partial\mathcal{U}}\psi = 0$ and thus that $t_{\partial\mathcal{U}}\psi = 0$. Using this and the above equation, an integration by parts (using density arguments) gives

$$\|\nabla\psi\|_{L^2(\mathcal{U})^4} = \|(-i\alpha\cdot\nabla)\psi\|_{L^2(\mathcal{U})^4} = 0.$$

From this we conclude that ψ vanishes identically, which contradicts the fact that $\psi \neq 0$, and thus $m \notin \mathrm{Sp}_{\mathrm{disc}}(H_{\mathrm{MIT}}(m))$. Following the same lines as above we also get that $-m \notin \mathrm{Sp}_{\mathrm{disc}}(H_{\mathrm{MIT}}(m))$. Thus, if $\mathcal{U}$ is bounded, then the above considerations and the fact that $\mathrm{dom}(H_{\mathrm{MIT}}(m)) \subset H^{1/2}(\mathcal{U})^4$ is compactly embedded in $L^2(\mathcal{U})^4$ yield $\mathrm{Sp}(H_{\mathrm{MIT}}(m)) = \mathrm{Sp}_{\mathrm{disc}}(H_{\mathrm{MIT}}(m)) \subset \mathbb{R}\setminus[-m, m]$, which shows assertion (i).

To finish the proof of (ii), suppose $\mathcal{U}$ is unbounded. We first show $(-\infty, -m]\cup[m, +\infty)$ is contained in $\mathrm{Sp}_{\mathrm{ess}}(H_{\mathrm{MIT}}(m))$ by constructing Weyl sequences as in the case of half-space; see [Benhellal 2022b, Theorem 4.1]. As $\mathcal{U}$ is unbounded, there is $R_1 > 0$ such that the half-space $\{x = (x_1, x_2, x_3) \in \mathbb{R}^3 : x_3 > R_1\}$ is strictly contained in $\mathcal{U}$ and $\mathbb{R}^3 \setminus \overline{\mathcal{U}} \subset B(0, R_1)$. Fix $\lambda \in (-\infty, -m)\cup(m, +\infty)$, and let $\xi = (\xi_1, \xi_2)$ be such that $|\xi|^2 = \lambda^2 - m^2$. We define the function $\varphi : \mathbb{R}^3 \to \mathbb{C}^4$ by

$$\varphi(\bar{x}, x_3) = \left(\frac{\xi_1 - i\xi_2}{\lambda - m}, 0, 0, 1\right)^t e^{i\xi\cdot\bar{x}}, \quad \text{with } \bar{x} = (x_1, x_2).$$

Clearly we have $(D_m - \lambda)\varphi = 0$. Now, fix $R_2 > R_1$, and let $\eta \in C_0^\infty(\mathbb{R}^2, \mathbb{R})$ and $\chi \in C_0^\infty(\mathbb{R}, \mathbb{R})$ be such that $\mathrm{supp}(\chi) \subset [R_1, R_2]$. For $n \in \mathbb{N}^\star$, we define the sequences of functions

$$\varphi_n(\bar{x}, x_3) = n^{-3/2}\varphi(\bar{x}, x_3)\eta(\bar{x}/n)\chi(x_3/n) \quad \text{for } (\bar{x}, x_3) \in \mathcal{U}.$$

Then, it is easy to check that $\varphi_n \in H_0^1(\mathcal{U}) \subset \mathrm{dom}(H_{\mathrm{MIT}}(m))$, $(\varphi_n)_{n\in\mathbb{N}^\star}$ converges weakly to zero, and

$$\|\varphi_n\|^2_{L^2(\mathcal{U})^4} = \frac{2\lambda}{\lambda - m}\|\eta\|^2_{L^2(\mathbb{R}^2)}\|\chi\|^2_{L^2(\mathbb{R})} > 0, \qquad \frac{\|(D_m - \lambda)\varphi_n\|_{L^2(\mathcal{U})^4}}{\|\varphi_n\|_{L^2(\mathcal{U})^4}} \xrightarrow[n\to\infty]{} 0;$$

for more details see the proof of [Benhellal 2022b, Theorem 4.1]. Therefore, Weyl's criterion yields

$$(-\infty, -m)\cup(m, +\infty) \subset \mathrm{Sp}_{\mathrm{ess}}(H_{\mathrm{MIT}}(m)).$$

Since the spectrum of a self-adjoint operator is closed, we then get the first statement of (ii). Now, if we assume in addition that $\mathcal{U}$ is connected, then using the same arguments as in the proof of [Arrizabalaga et al. 2015, Theorem 3.7] (i.e., using Rellich's lemma and the unique continuation property), one can verify that $H_{\mathrm{MIT}}(m)$ has no eigenvalues in $\mathbb{R}\setminus[-m, m]$. As $\{-m, m\} \notin \mathrm{Sp}_{\mathrm{disc}}(H_{\mathrm{MIT}}(m))$, it follows that $H_{\mathrm{MIT}}(m)$ has a purely continuous spectrum.

Now we prove (iii). Let $\psi \in \mathrm{dom}(H_{\mathrm{MIT}}(m))$. Then (3-4) yields that $\|H_{\mathrm{MIT}}(m)\psi\|^2_{L^2(\Omega)^4} \geqslant m^2\|\psi\|^2_{L^2(\Omega)^4}$, and thus

$$m\|\psi\|_{L^2(\mathcal{U})^4} \leqslant \|H_{\mathrm{MIT}}(m)\psi\|_{L^2(\mathcal{U})^4} \leqslant \|(H_{\mathrm{MIT}}(m) - z)\psi\|_{L^2(\mathcal{U})^4} + |z|\|\psi\|_{L^2(\mathcal{U})^4}.$$

Therefore, for $2|z| < m$ with $z \in \rho(H_{\mathrm{MIT}}(m))$, we get that $\|\psi\|_{L^2(\mathcal{U})^4} \leqslant 2m^{-1}\|(H_{\mathrm{MIT}}(m) - z)\psi\|_{L^2(\mathcal{U})^4}$. Thus, (iii) follows by taking $\psi = (H_{\mathrm{MIT}}(m) - z)^{-1} f$. □

Remark 3.2. We mention that the above statement on the self-adjointness can also be deduced from [Behrndt et al. 2021, Theorem 5.4]. We also mention that the MIT bag operator defined on the domain $\mathscr{D}$ given by (3-3) is still self-adjoint for less regular domains; see [Benhellal 2022a] for more details.

Remark 3.3. Note that if $\mathcal{U}$ is in the class of Hölder's domains $C^{1,\omega}$, with $\omega \in \left(\frac{1}{2}, 1\right)$, then $H_{\mathrm{MIT}}(m)$ is self-adjoint and $\mathrm{dom}(H_{\mathrm{MIT}}(m)) := \{\psi \in H^1(\mathcal{U})^4 : P_- t_{\partial\mathcal{U}}\psi = 0 \text{ on } \partial\mathcal{U}\}$; see [Benhellal 2022a, Theorem 4.3] for example.

Now we establish regularity results concerning the regularization property of the resolvent and the Sobolev regularity of the eigenfunctions of $H_{\mathrm{MIT}}(m)$. The first statement of the following theorem will be crucial in Section 5 when studying the semiclassical pseudodifferential properties of the Poincaré–Steklov operator.

Theorem 3.4. *Let $k \geqslant 1$ be an integer and assume that $\mathcal{U}$ is C^{2+k}-smooth. Then the following statements hold*:

(i) *The mapping $(H_{\mathrm{MIT}}(m)-z)^{-1} : H^k(\mathcal{U})^4 \to H^{k+1}(\mathcal{U})^4 \cap \mathrm{dom}(H_{\mathrm{MIT}}(m))$ is well defined and bounded for all $m > 0$ and all $z \in \rho(H_{\mathrm{MIT}}(m))$. Moreover, for any compact set $K \subset \mathbb{C}$, there exist $m_0, C > 0$ such that, for all $m \geqslant m_0$ and $z \in K$,*

$$\|(H_{\mathrm{MIT}}(m) - z)^{-1}\|_{H^{k-1}(\mathcal{U})^4 \to H^k(\mathcal{U})^4} \leq C m^{k-1}.$$

(ii) *If ϕ is an eigenfunction associated with an eigenvalue $z \in \mathrm{Sp}(H_{\mathrm{MIT}}(m))$, i.e., $(H_{\mathrm{MIT}}(m) - z)\phi = 0$, then $\phi \in H^{1+k}(\mathcal{U})^4$. In particular, if $\mathcal{U}$ is C^∞-smooth, then $\phi \in C^\infty(\mathcal{U})^4$.*

To prove this theorem we need the following classical regularity result.

Proposition 3.5. *Let k be a nonnegative integer. Assume that $\mathcal{U}$ is C^{3+k}-smooth and $u \in H^1(\mathcal{U})$. If u solves the Neumann problem*

$$-\Delta u = f \in H^k(\mathcal{U}) \quad \textit{and} \quad \partial_n u = g \in H^{1/2+k}(\partial\mathcal{U}),$$

then $u \in H^{2+k}(\mathcal{U})$.

Proof. First, assume that $k = 0$. As $\mathcal{U}$ is C^3-smooth we know the Neumann trace $\partial_n : H^2(\mathcal{U}) \to H^{1/2}(\partial\mathcal{U})$ is surjective. Thus, there is $G \in H^2(\mathcal{U})$ such that $\partial_n G = g$ in $\partial\mathcal{U}$. Note that the function $\tilde{u} = u - G$ satisfies the homogeneous Neumann problem

$$-\Delta\tilde{u} = f + \Delta G \quad \text{in } \mathcal{U} \qquad \text{and} \qquad \partial_n \tilde{u} = 0 \quad \text{on } \partial\mathcal{U}.$$

Therefore, $\tilde{u} \in H^2(\mathcal{U})$ by [Mikhailov 1978, Theorem 5, p. 217], which implies that $u \in H^2(\mathcal{U})$, and this proves the result for $k = 0$. If $k \geqslant 1$, then the result follows by [Grisvard 1985, Theorem 2.5.1.1]. □

Proof of Theorem 3.4. We prove the theorem by induction on k. First, we show (i), so fix $z \in \rho(H_{\mathrm{MIT}}(m))$ and assume that $k = 1$. Let $\phi = (\phi_1, \phi_2)^\top \in \mathrm{dom}(H_{\mathrm{MIT}}(m))$ be such that $(D_m - z)\phi = f$ in $\mathcal{U}$, with $f = (f_1, f_2)^\top \in H^1(\mathcal{U})^4$. By assumption we have $(\Delta + m^2 - z^2)\phi = (D_m + z) f$ in $\mathcal{D}'(\mathcal{U})^4$, and then also in $L^2(\mathcal{U})^4$. We next prove that $\partial_n \phi \in H^{1/2}(\partial\mathcal{U})^4$. To this end, consider $\mathcal{U}_\epsilon := \{x \in \mathbb{R}^3 : \mathrm{dist}(x, \partial\mathcal{U}) < \epsilon\}$ for $\epsilon > 0$. Then, for $\delta > 0$ small enough and $0 < \epsilon \leqslant \delta$, the mapping $\Psi : \partial\mathcal{U} \times (-\epsilon, \epsilon) \to \mathcal{U}_\epsilon$, defined by

$$\Psi(x_{\partial\mathcal{U}}, t) = x_{\partial\mathcal{U}} + t n(x_{\partial\mathcal{U}}), \qquad x_{\partial\mathcal{U}} \in \partial\mathcal{U}, \quad t \in (-\epsilon, \epsilon), \tag{3-5}$$

is a C^2-diffeomorphism and $\mathcal{U}_\epsilon := \{x + t n(x) : x \in \partial\mathcal{U}, t \in (-\epsilon, \epsilon)\}$.

Let $\widetilde{P}_- : L^2(\mathcal{U}_\epsilon \cap \mathcal{U})^4 \to L^2(\mathcal{U}_\epsilon \cap \mathcal{U})^4$ be the bounded operator defined by

$$\widetilde{P}_- \varphi(\Psi(x,t)) = \tfrac{1}{2}(1 + i\beta(\alpha \cdot n(x)))\varphi(\Psi(x,t)), \quad \Psi(x,t) \in \mathcal{U}_\epsilon \cap \mathcal{U}.$$

Let $x^0_{\partial\mathcal{U}}$ be an arbitrary point on the boundary $\partial\mathcal{U}$, fix $0 < r < \frac{1}{2}\epsilon$, and let $\zeta : \mathbb{R}^3 \to [0,1]$ be a C^∞-smooth and compactly supported function such that $\zeta = 1$ on $B(x^0_{\partial\mathcal{U}}, r)$ and $\zeta = 0$ on $\mathbb{R}^3 \setminus B(x^0_{\partial\mathcal{U}}, 2r)$. We claim that $\widetilde{P}_- \zeta\phi$ satisfies the elliptic problem

$$\begin{cases} -\Delta(\widetilde{P}_-\zeta\phi) = g & \text{in } \mathcal{U}, \\ t_{\partial\mathcal{U}}(\widetilde{P}_-\zeta\phi) = 0 & \text{on } \partial\mathcal{U}, \end{cases}$$

with $g \in L^2(\mathcal{U})^4$. Indeed, set $\mathcal{B}(x) = i\beta(\alpha \cdot n(x))$ for $x \in \partial\mathcal{U}$, and observe that

$$(D_m - z)(\widetilde{P}_-\zeta\phi) = \big(\widetilde{P}_-\zeta f + \tfrac{1}{2}[D_m, \zeta]\phi\big) + \tfrac{1}{2}[D_m, \zeta\mathcal{B}]\phi =: I(\phi, f) + \tfrac{1}{2}[D_m, \zeta\mathcal{B}]\phi.$$

Since n is C^2-smooth, ζ is an infinitely differentiable scalar function, and $\phi, f \in H^1(\mathcal{U})^4$, it is clear that $I(\phi, f) \in H^1(\mathcal{U})^4$ and $[D_m, \zeta\mathcal{B}]\phi \in L^2(\mathcal{U})^4$. Now, applying $(D_m + z)$ to the above equation yields $-\Delta(\widetilde{P}_-\zeta\phi) = g$, with

$$g := (z^2 - m^2)\widetilde{P}_-\zeta\phi + (D_m + z)I(\phi, f) + \tfrac{1}{2}z[D_m, \zeta\mathcal{B}]\phi + \tfrac{1}{2}D_m[D_m, \zeta\mathcal{B}]\phi.$$

As before, it is clear that the first three terms are square integrable. Next, observe that

$$D_0[D_0, \zeta\mathcal{B}]\phi = \{D_0, [D_0, \zeta\mathcal{B}]\}\phi - [D_0, \zeta\mathcal{B}]D_0\phi = [-\Delta, \zeta\mathcal{B}]\phi - [D_0, \zeta\mathcal{B}]((D_m - z)\phi - (m\beta - z)\phi),$$

where $\{A, B\} =: AB + BA$ is the anticommutator bracket. Using this, the smoothness assumption on n, the facts that $(D_m - z)\phi = f \in H^1(\mathcal{U})^4$ and that $[D_0, \zeta\mathcal{B}]$ and $[-\Delta, \zeta\mathcal{B}]$ are first-order differential operators, we easily see that $D_0[D_0, \zeta\mathcal{B}]\phi \in L^2(\mathcal{U})^4$. Hence, $D_m[D_m, \zeta\mathcal{B}]\phi$ is square integrable, which means that $g \in L^2(\mathcal{U})^4$. As $P_- t_{\partial\mathcal{U}}\phi = 0$ and $t_{\partial\mathcal{U}}(\widetilde{P}_-\zeta\phi) = t_{\partial\mathcal{U}}\zeta P_- t_{\partial\mathcal{U}}\phi = 0$ on $\partial\mathcal{U}$, by [Gilbarg and Trudinger 1983, Theorem 8.12], it follows that $\widetilde{P}_-\zeta\phi \in H^2(\mathcal{U}_\epsilon \cap \mathcal{U})^4$, which implies

$$\zeta(\phi_1 + i(\sigma \cdot n)\phi_2) \in H^2(B(x^0_{\partial\mathcal{U}}, 2r) \cap \mathcal{U})^2 \quad \text{and} \quad \zeta(-i(\sigma \cdot n)\phi_1 + \phi_2) \in H^2(B(x^0_{\partial\mathcal{U}}, 2r) \cap \mathcal{U})^2.$$

Consequently, we get

$$\phi_1 + i(\sigma \cdot n)\phi_2 \in H^2(B(x^0_{\partial\mathcal{U}}, r) \cap \mathcal{U})^2 \quad \text{and} \quad -i(\sigma \cdot n)\phi_1 + \phi_2 \in H^2(B(x^0_{\partial\mathcal{U}}, r) \cap \mathcal{U})^2. \tag{3-6}$$

Since $-i(\sigma \cdot \nabla)\phi_2 = (z - m)\phi_1 + f_1$ and $-i(\sigma \cdot \nabla)\phi_1 = (z + m)\phi_2 + f_2$ hold in $H^1(\mathcal{U})^2$, it follows from (3-6) that

$$(\sigma \cdot \nabla)\phi_j \in H^1(B(x^0_{\partial\mathcal{U}}, r))^2 \quad \text{and} \quad (\sigma \cdot \nabla)(\sigma \cdot n)\phi_j \in H^1(B(x^0_{\partial\mathcal{U}}, r))^2, \quad j = 1, 2.$$

Using this and the fact that n is C^2-smooth, we easily get

$$(\sigma \cdot n)(\sigma \cdot \nabla)\phi_j + (\sigma \cdot \nabla)(\sigma \cdot n)\phi_j = (n \cdot \nabla)\phi_j + F_j \in H^1(B(x^0_{\partial\mathcal{U}}, r))^2,$$

with $F_j \in H^1(B(x^0_{\partial\mathcal{U}}, r) \cap \mathcal{U})^2$. As a consequence, we get $(n \cdot \nabla)\phi_j \in H^1(B(x^0_{\partial\mathcal{U}}, r) \cap \mathcal{U})^2$. Since this holds true for all $x^0_{\partial\mathcal{U}} \in \partial\mathcal{U}$, using the compactness of $\partial\mathcal{U}$, it follows that $\partial_n\phi \in H^{1/2}(\partial\mathcal{U})^4$. Therefore, Proposition 3.5 yields $\phi \in H^2(\mathcal{U})^4$.

Next, assume $k \geqslant 2$, $\mathcal{U}$ is C^{2+k}-smooth, and $\phi, f \in H^k(\mathcal{U})^4$. Since n is C^{1+k}-smooth and Ψ defined by (3-5) is a C^{1+k}-diffeomorphism, following the same arguments as above we then conclude that $\partial_n \phi \in H^{k-1/2}(\partial\mathcal{U})^4$. Note also that $-\Delta\phi = (z^2 - m^2)\phi + (D_m - z) f \in H^{k-1}(\mathcal{U})^4$. Therefore, thanks to Proposition 3.5, we conclude that $\phi \in H^{k+1}(\mathcal{U})^4$, which proves the first statement of (i).

Now, the second statement of (i) is a consequence of the first one, Theorem 3.1(iii), and the Gårding-type inequality

$$\|\varphi\|^2_{H^{k+1}(\mathcal{U})^4} \lesssim \|\varphi\|^2_{H^k(\mathcal{U})^4} + \|D_0\varphi\|^2_{H^k(\mathcal{U})^4}, \tag{3-7}$$

which holds for any $\varphi \in \mathrm{dom}(H_{\mathrm{MIT}}(m)) \cap H^{k+1}(\mathcal{U})^4$, $k \in \mathbb{N}$. Indeed, suppose for instance that (3-7) holds true. Fix a compact set $K \subset \mathbb{C}$, and let $z \in K$. Note that if $z \in \rho(H_{\mathrm{MIT}}(m))$ then, for $\psi \in H^k(\mathcal{U})^4$, $k \geq 0$, we have

$$\|D_0(H_{\mathrm{MIT}}(m) - z)^{-1}\psi\|_{H^k(\mathcal{U})^4} \leq \|\psi\|_{H^k(\mathcal{U})^4} + (m + |z|)\|(H_{\mathrm{MIT}}(m) - z)^{-1}\psi\|_{H^k(\mathcal{U})^4}. \tag{3-8}$$

Let us also remark that Theorem 3.1(iii) gives that there is $m_0 > 0$ such that $z \in \rho(H_{\mathrm{MIT}}(m))$ for any $m \geq m_0$ and, for any $\psi \in H^k(\mathcal{U})^4$, $k \geq 0$,

$$\|D_0(H_{\mathrm{MIT}}(m) - z)^{-1}\psi\|_{L^2(\mathcal{U})^4} \lesssim \|\psi\|_{L^2(\mathcal{U})^4} \leq \|\psi\|_{H^k(\mathcal{U})^4}. \tag{3-9}$$

Hence, by iterating the Gårding inequality and taking into account (3-8) and (3-9), we get

$$\|D_0(H_{\mathrm{MIT}}(m) - z)^{-1}\psi\|_{H^k(\mathcal{U})^4} \lesssim m^k \|\psi\|_{H^k(\mathcal{U})^4},$$

and the conclusion follows by applying again the Gårding inequality. We now return to the proof of (3-7). Let $\varphi \in \mathrm{dom}(H_{\mathrm{MIT}}(m))$. Then [Arrizabalaga et al. 2017, Theorem 1.5] yields

$$\|D_0\varphi\|^2_{L^2(\mathcal{U})^4} = \|\nabla\varphi\|^2_{L^2(\mathcal{U})^4} + \int_{\partial\mathcal{U}} H_1 |t_{\partial\mathcal{U}}\varphi|^2 \, d\sigma, \tag{3-10}$$

where we recall that $H_1(x)$ is the mean curvature at $x \in \partial\mathcal{U}$. Recall that, for any $\epsilon > 0$, there is $C_\epsilon > 0$ such that

$$\|t_{\partial\mathcal{U}}\varphi\|_{L^2(\partial\mathcal{U})^4} \leq \epsilon\|\nabla\varphi\|^2_{L^2(\mathcal{U})^4} + C_\epsilon\|\varphi\|^2_{L^2(\mathcal{U})^4} \quad \text{for all } \varphi \in H^1(\mathcal{U})^4;$$

see [Barbaroux et al. 2019, Remark 1]. Using this inequality with ϵ sufficiently small and estimating (3-10) we get, for all $\varphi \in H^1(\mathcal{U})^4$,

$$\|\varphi\|^2_{H^1(\mathcal{U})^4} = \|\varphi\|^2_{L^2(\mathcal{U})^4} + \|\nabla\varphi\|^2_{L^2(\mathcal{U})^4} \lesssim \|\varphi\|^2_{L^2(\mathcal{U})^4} + \|D_0\varphi\|^2_{L^2(\mathcal{U})^4},$$

which shows (3-7) for $k = 0$. Note that by local arguments one has

$$\|\varphi\|^2_{H^{k+1}(\mathcal{U})^4} \lesssim \|\varphi\|^2_{L^2(\mathcal{U})^4} + \sum_j \|\partial_j\varphi\|^2_{H^k(\mathcal{U})^4},$$

and since $[\partial_j, D_0] = 0$, (3-7) easily follows by induction for any $k \geq 1$.

Finally, the proof of the first statement of (ii) follows the same lines as the one of (i). In particular, if $\mathcal{U}$ is C^∞-smooth, we then get $\phi \in H^{k+1}(\mathcal{U})^4$ for any $k \geqslant 0$, which implies that ϕ is infinitely differentiable in $\mathcal{U}$, and the theorem is proved. □

Remark 3.6. Note that the estimate in Theorem 3.4(i) is certainly not sharp, but it will be enough for our purposes.

4. Poincaré–Steklov operators as pseudodifferential operators

The main purpose of this section is to introduce the Poincaré–Steklov operator $\mathscr{A}_m$ associated with the MIT bag operator and to prove that it fits into the framework of pseudodifferential operators.

Throughout this section, let Ω be a smooth domain with a compact boundary Σ, and let $P_\pm$ be as in (2-1). Let us start by giving the rigorous definition of the Poincaré–Steklov operator, which is the main subject of this paper.

Definition 4.1 (PS operator). Let $z \in \rho(H_{\mathrm{MIT}}(m))$ and $g \in P_- H^{1/2}(\Sigma)^4$. We denote by $E_m^\Omega(z) : P_- H^{1/2}(\Sigma)^4 \to H^1(\Omega)^4$ the lifting operator associated with the elliptic problem

$$\begin{cases} (D_m - z)U_z = 0 & \text{in } \Omega, \\ P_- t_\Sigma U_z = g & \text{on } \Sigma. \end{cases} \tag{4-1}$$

That is, $E_m^\Omega(z)g$ is the unique function in $H^1(\Omega)^4$ satisfying the equations $(D_m - z)E_m^\Omega(z)g = 0$ in Ω and $P_- t_\Sigma E_m^\Omega(z)g = g$ on Σ. Then, the Poincaré–Steklov (PS) operator $\mathscr{A}_m : P_- H^{1/2}(\Sigma)^4 \to P_+ H^{1/2}(\Sigma)^4$ associated with the system (4-1) is defined by

$$\mathscr{A}_m(g) = P_+ t_\Sigma E_m^\Omega(z)g.$$

Recall the definitions of $\Phi_{z,m}^\Omega$ and Λ_m^z from Section 2B. Then, the following proposition justifies the existence and the uniqueness of the solution to the elliptic problem (4-1), and gives in particular the explicit formula of the PS operator in terms of the operator $(\Lambda_m^z)^{-1}$ when $z \in \rho(D_m)$. The second assertion of the proposition will be particularly important in Section 5 when studying the PS operator from the semiclassical point of view. In the last statement, we use the notation $\mathscr{A}_m(z)$ to highlight the dependence on the parameter $z \in \rho(H_{\mathrm{MIT}}(m))$.

Proposition 4.2. *For any $z \in \rho(H_{\mathrm{MIT}}(m))$ and $g \in P_- H^{1/2}(\Sigma)^4$, the elliptic problem (4-1) has a unique solution $E_m^\Omega(z)[g] \in H^1(\Omega)^4$. Moreover, the following hold*:

(i) $(E_m^\Omega(z))^* = -\beta P_+ t_\Sigma (H_{\mathrm{MIT}}(m) - \bar{z})^{-1}$.

(ii) *For any compact set $K \subset \mathbb{C}$, there is $m_0 > 0$ such that, for all $m \geqslant m_0$, we have $K \subset \rho(H_{\mathrm{MIT}}(m))$ and, for all $z \in K$, we have*

$$\|E_m^\Omega(z)g\|_{L^2(\Omega)^4} \lesssim \frac{1}{\sqrt{m}} \|g\|_{L^2(\Sigma)^4} \quad \textit{for all } g \in P_- H^{1/2}(\Sigma)^4.$$

(iii) *If $z \in \rho(D_m)$, then $E_m^\Omega(z)$ and $\mathscr{A}_m$ are explicitly given by*

$$E_m^\Omega(z) = \Phi_{z,m}^\Omega (\Lambda_m^z)^{-1} P_- \quad \textit{and} \quad \mathscr{A}_m = -P_+ \beta (\Lambda_m^z)^{-1} P_-. \tag{4-2}$$

(iv) *Let $z \in \rho(H_{\mathrm{MIT}}(m))$, and let $E_m^\Omega(z)$ be as above. Then, for any $\xi \in \rho(H_{\mathrm{MIT}}(m))$, the operator $E_m^\Omega(\xi)$ has the representation*

$$E_m^\Omega(\xi) = (I_4 + (\xi - z)(H_{\mathrm{MIT}}(m) - \xi)^{-1}) E_m^\Omega(z). \tag{4-3}$$

In particular, we have

$$\mathscr{A}_m(\xi) - \mathscr{A}_m(z) = (z-\xi)\beta(E_m^\Omega(\bar{\xi}))^* E_m^\Omega(z). \tag{4-4}$$

(v) *For any $z \in \rho(H_{\mathrm{MIT}}(m))$, the operator $E_m^\Omega(z)$ extends into a bounded operator from $P_- H^{-1/2}(\Sigma)^4$ to $H(\alpha, \Omega)$.*

Proof. We first show that the boundary value problem (4-1) has a unique solution. For this, assume that u_1 and u_2 are both solutions of (4-1). Then $(D_m - z)(u_1 - u_2) = 0$ in Ω and $P_- t_\Sigma(u_1 - u_2) = 0$ on Σ. Thus, $(u_1 - u_2) \in \mathrm{dom}(H_{\mathrm{MIT}}(m))$ holds by Remark 3.3, and since $H_{\mathrm{MIT}}(m)$ is injective by Theorem 3.1 it follows that $u_1 = u_2$, which proves the uniqueness. Next, observe that the function

$$v_g = \mathcal{E}_\Omega(P_- g) - (H_{\mathrm{MIT}}(m) - z)^{-1}(D_m - z)\mathcal{E}_\Omega(P_- g)$$

is a solution to (4-1). Indeed, we have $\mathcal{E}_\Omega(P_- g) \in H^1(\Omega)^4$ and thus $v_g \in H^1(\Omega)^4$, moreover, we clearly have that $P_- t_\Sigma v_g = g$ and $(D_m - z)v_g = 0$. Since we already know that the solution to (4-1) is unique, it follows that v_g is independent of the extension operator $\mathcal{E}_\Omega$, and hence there is a unique solution in $H^1(\Omega)^4$ to the elliptic problem (4-1).

Let us show the assertion (i). Let $\psi \in P_- H^{1/2}(\Sigma)^4$ and $f \in L^2(\Omega)^4$. Then, using Green's formula and the fact that $P_+(-i\alpha \cdot n) = (-i\alpha \cdot n)P_- = -\beta P_-$, we get

$$\begin{aligned}
&\langle E_m^\Omega(z)\psi, f\rangle_{L^2(\Omega)^4} \\
&\quad = \langle E_m^\Omega(z)\psi, (H_{\mathrm{MIT}}(m) - \bar{z})(H_{\mathrm{MIT}}(m) - \bar{z})^{-1} f\rangle_{L^2(\Omega)^4} \\
&\quad = \langle E_m^\Omega(z)\psi, (D_m - \bar{z})(H_{\mathrm{MIT}}(m) - \bar{z})^{-1} f\rangle_{L^2(\Omega)^4} \\
&\quad = \langle (D_m - z)E_m^\Omega(z)\psi, (H_{\mathrm{MIT}}(m) - \bar{z})^{-1} f\rangle_{L^2(\Omega)^4} + \langle (-i\alpha\cdot n)t_\Sigma E_m^\Omega(z)\psi, t_\Sigma(H_{\mathrm{MIT}}(m) - \bar{z})^{-1} f\rangle_{L^2(\Sigma)^4} \\
&\quad = \langle (-i\alpha\cdot n)P_- t_\Sigma E_m^\Omega(z)\psi, P_+ t_\Sigma(H_{\mathrm{MIT}}(m) - \bar{z})^{-1} f\rangle_{L^2(\Sigma)^4} \\
&\quad = \langle \psi, -\beta P_+ t_\Sigma(H_{\mathrm{MIT}}(m) - \bar{z})^{-1} f\rangle_{L^2(\Sigma)^4},
\end{aligned}$$

which gives that $-\beta P_+ t_\Sigma(H_{\mathrm{MIT}}(m) - \bar{z})^{-1}$ is the adjoint of $E_m^\Omega(z)$ and proves (i).

Now we are going to show assertion (ii). So, let K be a compact set of $\mathbb{C}$, and note that, for all $m > \sup\{|\mathrm{Re}(z)| : z \in K\}$, we have that $K \subset \rho(D_m) \subset \rho(H_{\mathrm{MIT}}(m))$. Hence $v := E_m^\Omega(z)g$ is well defined for any $z \in K$ and $g \in P_- H^{1/2}(\Sigma)^4$. Then a straightforward application of Green's formula yields

$$\begin{aligned}
0 &= \|(D_m - z)v\|^2_{L^2(\Omega)^4} \\
&= \|(i\alpha\cdot\nabla - z)v\|^2_{L^2(\Omega)^4} + m^2\|v\|^2_{L^2(\Omega)^4} + m(\langle -i(\alpha\cdot n)t_\Sigma v, \beta t_\Sigma v\rangle_{L^2(\Sigma)^4} - 2\,\mathrm{Re}(z)\langle v, \beta v\rangle_{L^2(\Omega)^4}).
\end{aligned} \tag{4-5}$$

Observe that

$$\langle -i(\alpha\cdot n)t_\Sigma v, \beta t_\Sigma v\rangle_{L^2(\Sigma)^4} = \langle (P_+ - P_-)t_\Sigma v, t_\Sigma v\rangle_{L^2(\Sigma)^4} = \|P_+ t_\Sigma v\|^2_{L^2(\Sigma)^4} - \|P_- t_\Sigma v\|^2_{L^2(\Sigma)^4}.$$

Since $P_- t_\Sigma v = g$ and $P_+ t_\Sigma v = \mathscr{A}_m(g)$ hold by definition and

$$-\mathrm{Re}(z)\langle v, \beta v\rangle_{L^2(\Omega)^4} \geqslant -|\mathrm{Re}(z)|\,\|v\|^2_{L^2(\Omega)^4}$$

holds by the Cauchy–Schwarz inequality, it follows from (4-5) that

$$\|g\|^2_{L^2(\Sigma)^4} \geqslant m\|v\|^2_{L^2(\Omega)^4} - 2|\mathrm{Re}(z)|\|v\|^2_{L^2(\Omega)^4} + \|\mathscr{A}_m(g)\|^2_{L^2(\Sigma)^4}.$$

Thus, if we take $m_0 \geqslant 4\sup\{|\mathrm{Re}(z)| : z \in K\}$, then

$$\|\mathscr{A}_m(g)\|^2_{L^2(\Sigma)^4} + \tfrac{1}{2}m\|v\|^2_{L^2(\Omega)^4} \leqslant \|g\|^2_{L^2(\Sigma)^4}$$

for any $m \geqslant m_0$, which proves the desired estimate for $E^\Omega_m(z)$.

Let us now show the assertion (iii). Let $z \in \rho(D_m)$, and recall that $\Phi^\Omega_{z,m}(\Lambda^z_m)^{-1} : H^{1/2}(\Sigma)^4 \to H^1(\Omega)^4$ is well defined and bounded by Lemma 2.1. Since ϕ^z_m is a fundamental solution of $(D_m - z)$,

$$(D_m - z)\Phi^\Omega_{z,m}(\Lambda^z_m)^{-1}[g] = 0 \quad \text{in } L^2(\Omega)^4 \qquad \text{for all } g \in H^{1/2}(\Sigma)^4.$$

Now, observe that if $g \in P_- H^{1/2}(\Sigma)^4$, then a direct application of the identity (2-7) yields

$$t_\Sigma \Phi^\Omega_{z,m}(\Lambda^z_m)^{-1}[g] = \big(-\tfrac{1}{2}i(\alpha\cdot n) + \mathscr{C}_{z,m}\big)(\Lambda^z_m)^{-1}[g] = g - P_+\beta(\Lambda^z_m)^{-1}[g].$$

Consequently, we get

$$P_- t_\Sigma \Phi^\Omega_{z,m}(\Lambda^z_m)^{-1}[g] = g \quad \text{and} \quad P_+ t_\Sigma \Phi^\Omega_{z,m}(\Lambda^z_m)^{-1}[g] = -P_+\beta(\Lambda^z_m)^{-1}[g],$$

which means that $E^\Omega_m(z)[g] = \Phi^\Omega_{z,m}(\Lambda^z_m)^{-1}[g]$ is the unique solution to the boundary value problem (4-1) and proves the identity $\mathscr{A}_m = -P_+\beta(\Lambda^z_m)^{-1}P_-$.

We now prove assertion (iv). Fix $z, \xi \in \rho(H_{\mathrm{MIT}}(m))$, and let $g \in P_- H^{1/2}(\Sigma)^4$. Then, by the definition of $E^\Omega_m(z)$, we have

$$\begin{aligned}(D_m - \xi)&(1 + (\xi - z)(H_{\mathrm{MIT}}(m) - \xi)^{-1})E^\Omega_m(z)g \\ &= (D_m - z)E^\Omega_m(z)g - (\xi - z)E^\Omega_m(z)g + (\xi - z)(D_m - \xi)(H_{\mathrm{MIT}}(m) - \xi)^{-1}E^\Omega_m(z)g \\ &= (\xi - z)E^\Omega_m(z)g - (\xi - z)E^\Omega_m(z)g = 0.\end{aligned}$$

Since $(H_{\mathrm{MIT}}(m) - \xi)^{-1}E^\Omega_m(z)g \in \mathrm{dom}(H_{\mathrm{MIT}}(m))$, and hence $P_- t_\Sigma(H_{\mathrm{MIT}}(m) - \xi)^{-1}E^\Omega_m(z)g = 0$, it follows that $P_- t_\Sigma(1 + (\xi - z)(H_{\mathrm{MIT}}(m) - \xi)^{-1})E^\Omega_m(z)g = P_- t_\Sigma E^\Omega_m(z)g = g$, which prove identity (4-3). Now, (4-4) follows by applying $P_+ t_\Sigma$ to the representation (4-3) and using assertion (i).

It remains to prove item (v). We first consider the case $z \in \rho(D_m)$. For $z \in \rho(H_{\mathrm{MIT}}(m)) \setminus \rho(D_m)$, the claim follows by the representation formula (4-3). Fix $z \in \rho(D_m)$, and recall that the operators $\mathscr{C}_{z,m}$ and Λ^z_m are bounded invertible in $H^{1/2}(\Sigma)^4$ by Lemma 2.1(ii)–(iii) and (2-6). Since $\mathscr{C}^*_{z,m} = \mathscr{C}_{\bar{z},m}$, by duality it follows that Λ^z_m admits a bounded and everywhere defined inverse in $H^{-1/2}(\Sigma)^4$. This together with Lemma 2.1(i) and item (iii) of this proposition show that $E^\Omega_m(z)$ admits a continuous extension from $P_- H^{-1/2}(\Sigma)^4$ to $H(\alpha, \Omega)$. This completes the proof of the proposition. □

Remark 4.3. The proof above gives more, namely that, for all $m_0 > 0$, $K \subset \rho(D_{m_0})$ a compact set, and $z \in K$, there is $m_1 \gg 1$ such that

$$\sup_{m \geqslant m_1} \|\mathscr{A}_m\|_{P_- H^{1/2}(\Sigma)^4 \to P_+ L^2(\Sigma)^4} \lesssim 1.$$

Remark 4.4. Thanks to Theorem 3.1 and Remark 2.2, if Ω is a Lipschitz domain, then $E_m^{\Omega}(z)$ is the unique solution in $H^{1/2}(\Omega)^4$ to the system (4-1) for datum in $L^2(\Sigma)^4$. Moreover, the PS operator $\mathscr{A}_m = -P_+\beta(\Lambda_m^z)^{-1}P_-$ is well defined and bounded as an operator from $P_-L^2(\Sigma)^4$ to $P_+L^2(\Sigma)^4$.

In the rest of this section, we will only address the case $z \in \rho(D_m)$, and we show that the Poincaré–Steklov operator $\mathscr{A}_m$ from Definition 4.1 is a homogeneous pseudodifferential operators of order 0 and capture its principal symbol in local coordinates. To this end, we first study the pseudodifferential properties of the Cauchy operator $\mathscr{C}_{z,m}$. Once this is done, we use the explicit formula of $\mathscr{A}_m$ given by (4-2) and the symbol calculus to obtain the principal symbol of $\mathscr{A}_m$.

Recall the definition of ϕ_m^z from (2-3), and observe that

$$\phi_m^z(x-y) = k^z(x-y) + w(x-y),$$

where

$$k^z(x-y) = \frac{e^{i\sqrt{z^2-m^2}|x-y|}}{4\pi|x-y|}\left(z + m\beta + \sqrt{z^2-m^2}\,\alpha\cdot\frac{x-y}{|x-y|}\right) + i\,\frac{e^{i\sqrt{z^2-m^2}|x-y|}-1}{4\pi|x-y|^3}\alpha\cdot(x-y),$$
$$w(x-y) = \frac{i}{4\pi|x-y|^3}\alpha\cdot(x-y).$$

Using this, it follows that

$$\begin{aligned}\mathscr{C}_{z,m}[f](x) &= \lim_{\rho\searrow 0}\int_{|x-y|>\rho} w(x-y)f(y)\,\mathrm{d}\sigma(y) + \int_{\Sigma} k^z(x-y)f(y)\,\mathrm{d}\sigma(y)\\ &= W[f](x) + K[f](x).\end{aligned} \tag{4-6}$$

As $|k^z(x-y)| = \mathcal{O}(|x-y|^{-1})$ when $|x-y|\to 0$, using the standard layer potential techniques (see, e.g., [Taylor 2000, Chapter 3, Section 4] and [Taylor 1996, Chapter 7, Section 11]), it is not hard to prove that the integral operator K gives rise to a pseudodifferential operator of order -1, i.e., $K \in \mathrm{Op}\,\mathcal{S}^{-1}(\Sigma)$. Thus, we can (formally) write

$$\mathscr{C}_{z,m} = W \mod \mathrm{Op}\,\mathcal{S}^{-1}(\Sigma), \tag{4-7}$$

which means that the operator W encodes the main contribution in the pseudodifferential character of $\mathscr{C}_{z,m}$. So we only need to focus on the study of the pseudodifferential properties of W. The following theorem makes this heuristic more rigorous. Its proof follows similar arguments as in [Ando et al. 2019; Miyanishi 2022; Miyanishi and Rozenblum 2019].

Theorem 4.5. *Let $\mathscr{C}_{z,m}$ be as in (2-5), W as in (4-6), and $\mathscr{A}_m$ as in Definition 4.1. Then $\mathscr{C}_{z,m}$, W and $\mathscr{A}_m$ are homogeneous pseudodifferential operators of order* 0, *and we have*

$$\mathscr{C}_{z,m} = \frac{1}{2}\alpha\cdot\frac{\nabla_\Sigma}{\sqrt{-\Delta_\Sigma}} \mod \mathrm{Op}\,\mathcal{S}^{-1}(\Sigma),$$
$$\mathscr{A}_m = \frac{1}{\sqrt{-\Delta_\Sigma}}S\cdot(\nabla_\Sigma\wedge n)P_- \mod \mathrm{Op}\,\mathcal{S}^{-1}(\Sigma) = \frac{D_\Sigma}{\sqrt{-\Delta_\Sigma}}P_- \mod \mathrm{Op}\,\mathcal{S}^{-1}(\Sigma).$$

Proof. We first deal with the operator W. Let $\psi_k : \Sigma \to \mathbb{R}$, $k = 1, 2$, be a C^∞-smooth function. Clearly, if $\operatorname{supp}(\psi_2) \cap \operatorname{supp}(\psi_1) = \varnothing$, then $\psi_2 W \psi_1$ gives rise to a bounded operator from $H^{-j}(\Sigma)^4$ into $H^j(\Sigma)^4$ for all $j \geqslant 0$.

Now, fix a local chart (U, V, φ) as in Section 2C, and recall the definition of the first fundamental form I and the metric tensor $G(\tilde{x})$. That is, up to a rotation, for all $x \in U$, we have $x = \varphi(\tilde{x}) = (\tilde{x}, \chi(\tilde{x}))$ with $\tilde{x} \in V$, where the graph of $\chi : V \to \mathbb{R}$ coincides with U. Notice that if we assume that ψ_k is compactly supported with $\operatorname{supp}(\psi_k) \subset U$, then, in this setting, the operator $\psi_2 W \psi_1$ has the form

$$\begin{aligned}\psi_2 W[\psi_1 f](x) &= \psi_2(x)\, \text{p.v.}\!\int_V i\alpha \cdot \frac{\varphi(\tilde{x}) - \varphi(\tilde{y})}{4\pi|\varphi(\tilde{x}) - \varphi(\tilde{y})|^3} \psi_1(\varphi(\tilde{y})) f(\varphi(\tilde{y}))\sqrt{g(\tilde{y})}\, d\tilde{y} \\ &= \psi_2(x)\sqrt{g(\tilde{x})}\, \text{p.v.}\!\int_V i\alpha \cdot \frac{\varphi(\tilde{x}) - \varphi(\tilde{y})}{4\pi|\varphi(\tilde{x}) - \varphi(\tilde{y})|^3} \psi_1(\varphi(\tilde{y})) f(\varphi(\tilde{y}))\, d\tilde{y} \\ &\qquad + \psi_2(x) \int_V i\alpha \cdot \frac{\varphi(\tilde{x}) - \varphi(\tilde{y})}{4\pi|\varphi(\tilde{x}) - \varphi(\tilde{y})|^3} f(\varphi(\tilde{y}))(\sqrt{g(\tilde{y})} - \sqrt{g(\tilde{x})})\, d\tilde{y}, \end{aligned} \tag{4-8}$$

where g is the determinant of the metric tensor G. Since $g(\,\cdot\,)$ is smooth, it follows that

$$|\sqrt{g(\tilde{y})} - \sqrt{g(\tilde{x})}| \lesssim |\tilde{x} - \tilde{y}|.$$

Therefore, the last integral operator on the right-hand side of (4-8) has a nonsingular kernel and does not require us to write it as an integral operator in the principal value sense. Thus, a simple computation using Taylor's formula shows

$$|x - y|^2 = |\varphi(\tilde{x}) - \varphi(\tilde{y})|^2 = \langle \tilde{x} - \tilde{y}, G(\tilde{x})(\tilde{x} - \tilde{y})\rangle(1 + \mathcal{O}|\tilde{x} - \tilde{y}|),$$

where the definition of I was used in the last equality. It follows from the above computations that

$$|x - y|^{-3} = \frac{1}{\langle \tilde{x} - \tilde{y}, G(\tilde{x})(\tilde{x} - \tilde{y})\rangle^{3/2}} + k_1(\tilde{x}, \tilde{y}),$$

where the kernel k_1 satisfies $|k_1(\tilde{x}, \tilde{y})| = \mathcal{O}(|\tilde{x} - \tilde{y}|^{-2})$ when $|\tilde{x} - \tilde{y}| \to 0$. Consequently, we get

$$\frac{x_j - y_j}{|x - y|^3} = \begin{cases} \dfrac{\tilde{x}_j - \tilde{y}_j}{\langle \tilde{x} - \tilde{y}, G(\tilde{x})(\tilde{x} - \tilde{y})\rangle^{3/2}} + (\tilde{x}_j - \tilde{y}_j)k_1(\tilde{x}, \tilde{y}) & \text{for } j = 1, 2, \\ \dfrac{\langle \nabla\chi, \tilde{x} - \tilde{y}\rangle}{\langle \tilde{x} - \tilde{y}, G(\tilde{x})(\tilde{x} - \tilde{y})\rangle^{3/2}} + k_2(\tilde{x}, \tilde{y}) & \text{for } j = 3, \end{cases}$$

with $|k_2(\tilde{x}, \tilde{y})| = \mathcal{O}(|\tilde{x} - \tilde{y}|^{-1})$ when $|\tilde{x} - \tilde{y}| \to 0$. Note that this implies

$$\alpha \cdot \left(\frac{x - y}{|x - y|^3}\right) = \alpha \cdot \frac{(\tilde{x} - \tilde{y}, \langle \nabla\chi, \tilde{x} - \tilde{y}\rangle)}{\langle \tilde{x} - \tilde{y}, G(\tilde{x})(\tilde{x} - \tilde{y})\rangle^{3/2}} + \mathcal{O}(|\tilde{x} - \tilde{y}|^{-1}).$$

Combining the above computations and (4-8), we deduce that

$$\begin{aligned}&\psi_2 W[\psi_1 f](x) \\ &\quad = \psi_2(x)\sqrt{g(\tilde{x})}\, \text{p.v.}\!\int_V i\alpha \frac{(\tilde{x} - \tilde{y}, \langle \nabla\chi, \tilde{x} - \tilde{y}\rangle)}{4\pi\langle \tilde{x} - \tilde{y}, G(\tilde{x})(\tilde{x} - \tilde{y})\rangle^{3/2}} f(\varphi(\tilde{y}))\, d\tilde{y} + \psi_2(x)L[\psi_1 f](x), \end{aligned} \tag{4-9}$$

where L is an integral operator with a kernel $l(x, y)$ satisfying

$$|l(x, y)| = \mathcal{O}(|x-y|^{-1}) \quad \text{when } |x-y| \to 0.$$

Thus, similar arguments as the ones in [Taylor 1996, Chapter 7, Section 11] yield that L is a pseudodifferential operator of order -1. Now, for $h \in L^2(\mathbb{R}^2)$ and $k = 1, 2$, observe that if we set

$$R_k[h](\tilde{x}) = \frac{i\sqrt{g(\tilde{x})}}{4\pi} \int_{\mathbb{R}^2} r_k(\tilde{x}, \tilde{x}-\tilde{y}) h(\tilde{y}) \, \mathrm{d}(\tilde{y}),$$

where, for $(\tilde{x}, \tau) \in \mathbb{R}^2 \times \mathbb{R}^2 \setminus \{0\}$,

$$r_k(\tilde{x}, \tau) = \frac{\tau_k}{\langle \tau, G(\tilde{x})\tau \rangle^{3/2}}.$$

Then the standard formula connecting a pseudodifferential operator and its symbol yields

$$R_k[h](\tilde{x}) = \frac{1}{(2\pi)^2} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} e^{i(\tilde{x}-\tilde{y})\cdot\xi} q_k(\tilde{x}, \xi) h(\tilde{y}) \, \mathrm{d}\xi \, \mathrm{d}\tilde{y},$$

where

$$q_k(\tilde{x}, \xi) = \frac{i\sqrt{g(\tilde{x})}}{4\pi} \int_{\mathbb{R}^2} e^{-i\omega\cdot\xi} r_k(\tilde{x}, \omega) \, \mathrm{d}\omega.$$

Recall the definition of Q from (2-10) and set $\omega = Q(\tilde{x})\tau$. Also recall that

$$\int_{\mathbb{R}^2} e^{-i\omega\cdot\xi} \frac{\omega_k}{|\omega|^3} \, \mathrm{d}\omega = -2\pi i \frac{\xi_k}{|\xi|}, \quad k = 1, 2. \tag{4-10}$$

Thus, the above change of variables together with the properties (2-11) and (4-10) yield

$$q_k(\tilde{x}, \xi) = \frac{i}{4\pi} \int_{\mathbb{R}^2} e^{-i(Q(\tilde{x})\tau)\cdot\xi} \frac{(Q(\tilde{x})\tau)_k}{|\tau|^3} \, \mathrm{d}\tau = \frac{(G^{-1}(\tilde{x})\xi)_k}{2\langle G^{-1}(\tilde{x})\xi, \xi\rangle^{1/2}} = \frac{g_{k1}\xi_1 + g_{k2}\xi_2}{2\langle G^{-1}(\tilde{x})\xi, \xi\rangle^{1/2}},$$

which means that $q_k(\tilde{x}, \xi)$ is homogeneous of degree 0 in ξ. Therefore, R_k is a homogeneous pseudodifferential operators of degree 0. From the above observation and (4-9) it follows that

$$\psi_2 W \psi_1 = \psi_2 \alpha \cdot (R_1, R_2, \partial_1\chi(\tilde{x})R_1 + \partial_2\chi(\tilde{x})R_2)\psi_1 + \psi_2 L \psi_1.$$

Since L is a pseudodifferential operator of order -1, we deduce that W is a homogeneous pseudodifferential operator of order 0, and exploiting (2-12), we obtain

$$W = \frac{1}{2}\alpha \cdot \frac{\nabla_\Sigma}{\sqrt{-\Delta_\Sigma}} \mod \mathrm{Op}\,\mathcal{S}^{-1}(\Sigma). \tag{4-11}$$

Thanks to (4-7) and (4-11), we deduce that the Cauchy operator $\mathscr{C}_{z,m}$ has the same principal symbol as the operator W.

Now we are going to deal with the operator $\mathscr{A}_m$. Note that we have

$$\frac{1}{2}\left(\beta + \alpha \cdot \frac{\nabla_\Sigma}{\sqrt{-\Delta_\Sigma}}\right)^2 = I_4 \tag{4-12}$$

and, as $\mathscr{A}_m$ is given by the formula

$$\mathscr{A}_m = -P_+\beta\big(\tfrac{1}{2}\beta + \mathscr{C}_{z,m}\big)^{-1} P_-,$$

using (4-12) and the standard mollification arguments, it follows from the product formula for calculus of pseudodifferential operators that, in local coordinates, the symbol of $\mathscr{A}_m$ denoted by $q_{\mathscr{A}_m}$ has the form

$$q_{\mathscr{A}_m}(\tilde{x},\xi) = -P_+\beta\left(\beta+\alpha\cdot\left(\frac{\xi_G}{\langle G^{-1}\xi,\xi\rangle^{1/2}}\right)\right)P_- + p(\tilde{x},\xi),$$

where $p \in \mathcal{S}^{-1}(\Sigma)$ and ξ_G defined in (2-12) is the principal symbol of ∇_Σ. Therefore, we get

$$q_{\mathscr{A}_m}(\tilde{x},\xi) = -P_+\beta\alpha\cdot\xi_G\langle G^{-1}\xi,\xi\rangle^{-1/2}P_- + p(\tilde{x},\xi).$$

Hence, using the fact that $P_\pm$ are projectors and Lemma A.3, we obtain

$$q_{\mathscr{A}_m}(\tilde{x},\xi) = -i\alpha\cdot n^{\varphi}(\tilde{x})\alpha\cdot\xi_G\langle G^{-1}\xi,\xi\rangle^{-1/2}P_- + p(\tilde{x},\xi).$$

Finally, from results of Section 2D, we deduce

$$q_{\mathscr{A}_m}(\tilde{x},\xi) = S\cdot\left(\frac{\xi_G\wedge n^{\varphi}(\tilde{x})}{\langle G^{-1}\xi,\xi\rangle}\right)P_- + p(\tilde{x},\xi)$$

and

$$\mathscr{A}_m = \frac{D_\Sigma}{\sqrt{-\Delta_\Sigma}}P_- \mod \operatorname{Op}\mathcal{S}^{-1}(\Sigma) = \frac{1}{\sqrt{-\Delta_\Sigma}}S\cdot(\nabla_\Sigma\wedge n)P_- \mod \operatorname{Op}\mathcal{S}^{-1}(\Sigma).$$

This satisfies the claim that $\mathscr{A}_m$ is a homogeneous pseudodifferential operator of order 0 and completes the proof of the theorem. □

5. Approximation of the Poincaré–Steklov operators for large masses

The technique used in the last section allows us to treat the layer potential operator $\mathscr{A}_m$ as a pseudodifferential operator and to derive its principal symbol. However, it does not allow us to capture the dependence on m. The main goal of this section is to study the Poincaré–Steklov operator, $\mathscr{A}_m$, as an m-dependent pseudodifferential operator when m is large enough. For this purpose, we consider $h = 1/m$ as a semiclassical parameter (for $m \gg 1$) and use the system (4-1) instead of the layer potential formula of $\mathscr{A}_m$. Roughly speaking, we will look for a local approximate formula for the solution of (4-1). Once this is done, we use the regularization property of the resolvent of the MIT bag operator to catch the semiclassical principal symbol of $\mathscr{A}_m$.

Throughout this section, we assume that $m > 1$, $z \in \rho(H_{\mathrm{MIT}}(m))$, and that Ω is smooth with a compact boundary $\Sigma := \partial\Omega$. Next, we introduce the semiclassical parameter $h = m^{-1} \in (0,1]$, and we set $\mathscr{A}^h := \mathscr{A}_m$. The following theorem is the main result of this section; it ensures that $\mathscr{A}^h$ is an h-pseudodifferential operator of order 0 and gives its semiclassical principal symbol.

Theorem 5.1. *Let $h \in (0,1]$ and $z \in \rho(H_{\mathrm{MIT}}(m))$, and let $\mathscr{A}^h$ be as above. Then, for any $N \in \mathbb{N}$, there exists an h-pseudodifferential operator of order* 0, $\mathscr{A}^h_N \in \operatorname{Op}^h\mathcal{S}^0(\Sigma)$, *such that, for h sufficiently small and any $0 \le l \le N + \frac{1}{2}$,*

$$\|\mathscr{A}^h - \mathscr{A}^h_N\|_{H^{1/2}(\Sigma)\to H^{N+3/2-l}(\Sigma)} = \mathcal{O}(h^{2l-1/2}),$$

and

$$\mathscr{A}^h_N = \frac{hD_\Sigma}{\sqrt{-h^2\Delta_\Sigma + I} + I}P_- \mod h\operatorname{Op}^h\mathcal{S}^{-1}(\Sigma).$$

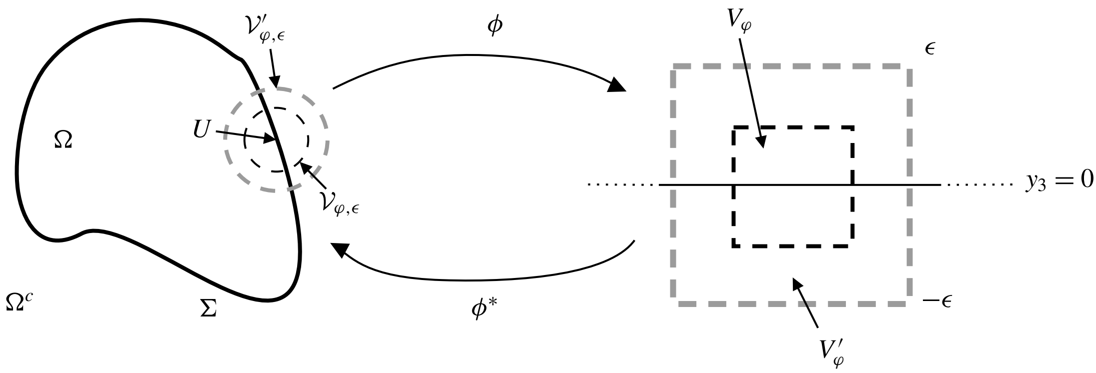

Figure 1. Change of coordinates

Let us consider $\mathbb{A} = \{(U_{\varphi_j}, V_{\varphi_j}, \varphi_j) : j \in \{1, \cdots, N\}\}$ an atlas of Σ and $(U_\varphi, V_\varphi, \varphi) \in \mathbb{A}$. As previously, without loss of generality, we consider only the case where U_φ is the graph of a smooth function χ, and we assume that Ω corresponds locally to the side $x_3 > \chi(x_1, x_2)$ (see Figure 1). Then, for

$$U_\varphi = \{(x_1, x_2, \chi(x_1, x_2)) : (x_1, x_2) \in V_\varphi\}, \quad \varphi((x_1, x_2, \chi(x_1, x_2)) = (x_1, x_2),$$
$$\mathcal{V}_{\varphi,\varepsilon} := \{(y_1, y_2, y_3 + \chi(y_1, y_2)) : (y_1, y_2, y_3) \in V_\varphi \times (0, \varepsilon)\} \subset \Omega,$$

with ε sufficiently small, we have the homeomorphism

$$\phi : \mathcal{V}_{\varphi,\varepsilon} \to V_\varphi \times (0, \varepsilon), \quad (x_1, x_2, x_3) \mapsto (x_1, x_2, x_3 - \chi(x_1, x_2)).$$

Then the pullback

$$\phi^* : C^\infty(V_\varphi \times (0, \varepsilon)) \to C^\infty(\mathcal{V}_{\varphi,\varepsilon}), \quad v \mapsto \phi^* v := v \circ \phi$$

transforms the differential operator D_m restricted on $\mathcal{V}_{\varphi,\varepsilon}$ into the following operator on $V_\varphi \times (0, \varepsilon)$:

$$\begin{aligned}\widetilde{D}_m^\varphi := (\phi^{-1})^* D_m (\phi)^* &= -i(\alpha_1 \partial_{y_1} + \alpha_2 \partial_{y_2} - (\alpha_1 \partial_{x_1} \chi + \alpha_2 \partial_{x_2} \chi - \alpha_3) \partial_{y_3}) + m\beta \\ &= -i(\alpha_1 \partial_{y_1} + \alpha_2 \partial_{y_2}) + \sqrt{1 + |\nabla \chi|^2} (i\alpha \cdot n^\varphi)(\tilde{y}) \partial_{y_3} + m\beta,\end{aligned}$$

where $\tilde{y} = (y_1, y_2)$ and $n^\varphi = (\varphi^{-1})^* n$ is the pullback of the outward-pointing normal to Ω restricted on V_φ:

$$n^\varphi(\tilde{y}) = \frac{1}{\sqrt{1 + |\nabla \chi|^2}} \begin{pmatrix} \partial_{x_1} \chi \\ \partial_{x_2} \chi \\ -1 \end{pmatrix} (y_1, y_2).$$

For the projectors $P_\pm$, we have

$$P_\pm^\varphi := (\varphi^{-1})^* P_\pm (\varphi)^* = \tfrac{1}{2}(I_4 \mp i\beta\alpha \cdot n^\varphi(\tilde{y})).$$

Thus, in the variable $y \in V_\varphi \times (0, \varepsilon)$, equation (4-1) becomes

$$\begin{cases} (\widetilde{D}_m^\varphi - z)u = 0 & \text{in } V_\varphi \times (0, \varepsilon), \\ \Gamma_-^\varphi u = g^\varphi = g \circ \varphi^{-1} & \text{on } V_\varphi \times \{0\}, \end{cases} \tag{5-1}$$

where $\Gamma_\pm^\varphi = P_\pm^\varphi t_{\{y_3=0\}}$.

By isolating the derivative with respect to y_3 and using that $(i\alpha\cdot n^\varphi)^{-1} = -i\alpha\cdot n^\varphi$, the system (5-1) becomes

$$\begin{cases}\partial_{y_3}u = \dfrac{i\alpha\cdot n^\varphi(\tilde y)}{\sqrt{1+|\nabla\chi(\tilde y)|^2}}(-i\alpha_1\partial_{y_1} - i\alpha_2\partial_{y_2} + m\beta - z)u & \text{in } V_\varphi\times(0,\varepsilon),\\ \Gamma_-^\varphi u = g^\varphi & \text{on } V_\varphi\times\{0\}.\end{cases} \tag{5-2}$$

Let us now introduce the matrix-valued symbols

$$L_0(\tilde y,\xi) := \frac{i\alpha\cdot n^\varphi(\tilde y)}{\sqrt{1+|\nabla\chi(\tilde y)|^2}}(\alpha\cdot\xi+\beta),\quad L_1(\tilde y) := \frac{-iz\alpha\cdot n^\varphi(\tilde y)}{\sqrt{1+|\nabla\chi(\tilde y)|^2}}, \tag{5-3}$$

with $\xi = (\xi_1,\xi_2)$ identified with $(\xi_1,\xi_2,0)$. Then, for $h = m^{-1}$, (5-2) is equivalent to

$$\begin{cases} h\,\partial_{y_3}u = L_0(\tilde y, hD_{\tilde y})u + hL_1(\tilde y)u & \text{in } V_\varphi\times(0,\varepsilon),\\ \Gamma_-^\varphi u = g^\varphi & \text{on } V_\varphi\times\{0\}.\end{cases} \tag{5-4}$$

Before constructing an approximate solution of the system (5-4), let us give some properties of L_0.

5A. *Properties of L_0.* The following proposition will be used in the sequel; it gathers some useful spectral properties of the matrix-valued symbol $L_0(\tilde y,\xi)$ introduced in (5-3). The spectral properties of $l_0(n,\xi) = i(\alpha\cdot n)(\alpha\cdot\xi+\beta)$ given in Proposition A.2 (from the Appendix) provides the following properties for

$$L_0(\tilde y,\xi) = \frac{1}{\sqrt{1+|\nabla\chi(\tilde y)|^2}}l_0(n^\varphi(\tilde y),\xi).$$

Proposition 5.2. *Let $L_0(\tilde y,\xi)$ be as in (5-3). Then we have*

$$\begin{aligned}L_0(\tilde y,\xi) &= \frac{1}{\sqrt{1+|\nabla\chi(\tilde y)|^2}}(i\xi\cdot n^\varphi(\tilde y) + S\cdot(n^\varphi(\tilde y)\wedge\xi) - i\beta(\alpha\cdot n^\varphi(\tilde y)))\\ &= i\xi\cdot\tilde n^\varphi(\tilde y) + \frac{\lambda(\tilde y,\xi)}{\sqrt{1+|\nabla\chi(\tilde y)|^2}}\Pi_+(\tilde y,\xi) - \frac{\lambda(\tilde y,\xi)}{\sqrt{1+|\nabla\chi(\tilde y)|^2}}\Pi_-(\tilde y,\xi),\end{aligned}$$

where

$$\begin{aligned}\lambda(\tilde y,\xi) &:= \sqrt{|n^\varphi(\tilde y)\wedge\xi|^2+1} = \sqrt{\langle G(\tilde y)^{-1}\xi,\xi\rangle+1},\\ \tilde n^\varphi(\tilde y) &:= \frac{1}{\sqrt{1+|\nabla\chi|^2}}n^\varphi(\tilde y),\\ \Pi_\pm(\tilde y,\xi) &:= \frac12\left(I_4 \pm \frac{S\cdot(n^\varphi(\tilde y)\wedge\xi) - i\beta(\alpha\cdot n^\varphi(\tilde y))}{\lambda(\tilde y,\xi)}\right),\end{aligned} \tag{5-5}$$

with G the induced metric defined in Section 2C.

In particular, the symbol $L_0(\tilde y,\xi)$ is elliptic in $\mathcal S^1$ and it admits two eigenvalues $\rho_\pm(\cdot,\cdot)\in\mathcal S^1$ of multiplicity two, which are given by

$$\rho_\pm(\tilde y,\xi) = \frac{in^\varphi(\tilde y)\cdot\xi \pm \lambda(\tilde y,\xi)}{\sqrt{1+|\nabla\chi|^2}}, \tag{5-6}$$

and for which there exists $c > 0$ such that

$$\frac{(\rho_+ - \rho_-)(\tilde{y}, \xi)}{2} = \pm \operatorname{Re} \rho_\pm(\tilde{y}, \xi) > c\langle \xi \rangle \tag{5-7}$$

uniformly with respect to $\tilde{y}$. Moreover, $\Pi_\pm(\tilde{y}, \xi)$, the projections onto $\operatorname{Kr}(L_0(\tilde{y}, \xi) - \rho_\pm(\tilde{y}, \xi) I_4)$, belong to the symbol class $\mathcal{S}^0$ and satisfy

$$P_\pm^\varphi \Pi_\pm(\tilde{y}, \xi) P_\pm^\varphi = k_+^\varphi(\tilde{y}, \xi) P_\pm^\varphi \quad \textit{and} \quad P_\pm^\varphi \Pi_\mp(\tilde{y}, \xi) P_\mp^\varphi = \mp \Theta^\varphi(\tilde{y}, \xi) P_\mp^\varphi, \tag{5-8}$$

with

$$k_\pm^\varphi(\tilde{y}, \xi) = \frac{1}{2}\left(1 \pm \frac{1}{\lambda(\tilde{y}, \xi)}\right), \quad \Theta^\varphi(\tilde{y}, \xi) = \frac{1}{2\lambda(\tilde{y}, \xi)}(S \cdot (n^\varphi(\tilde{y}) \wedge \xi)). \tag{5-9}$$

That is, k_+^φ is a positive function of $\mathcal{S}^0$, $(k_+^\varphi)^{-1} \in \mathcal{S}^0$, and $\Theta^\varphi \in \mathcal{S}^0$.

Remark 5.3. Thanks to property (5-8), a 4×4-matrix A is uniquely determined by $P_-^\varphi A$ and $\Pi_+ A$, and we have

$$A = P_-^\varphi A + P_+^\varphi A = P_-^\varphi A + \frac{1}{k_+^\varphi} P_+^\varphi \Pi_+ P_+^\varphi A = \left(I - \frac{P_+^\varphi \Pi_+}{k_+^\varphi}\right) P_-^\varphi A + \frac{P_+^\varphi}{k_+^\varphi} \Pi_+ A.$$

Proof of Proposition 5.2. By definition it is clear that $L_0(\tilde{y}, \xi)$ belongs to the symbol class $\mathcal{S}^1$, and all the formulas follow from those of $l_0(n, \xi)$ proved in the Appendix (see Proposition A.2 and Lemma A.3), mainly taking $n = n^\varphi(\tilde{y})$ and multiplying by $1/\sqrt{1 + |\nabla \chi(\tilde{y})|^2}$. Next, using (2-15),

$$\pm \operatorname{Re} \rho_\pm(\tilde{y}, \xi) = \frac{\sqrt{|n^\varphi \wedge \xi|^2 + 1}}{\sqrt{1 + |\nabla \chi|^2}} = \frac{\sqrt{\langle G(\tilde{y})^{-1} \xi, \xi \rangle + 1}}{\sqrt{1 + |\nabla \chi|^2}} \geqslant c(1 + |\xi|),$$

which gives (5-7) and shows that $\rho_\pm$ are elliptic in $\mathcal{S}^1$. Consequently, we also get that $L_0(\tilde{y}, \xi)$ is elliptic in $\mathcal{S}^1$ and that the functions $\Pi_\pm$, k_+^φ, $(k_+^\varphi)^{-1}$ and Θ^φ belong to the symbol class $\mathcal{S}^0$. ☐

5B. *Semiclassical parametrix for the boundary problem.* In this section, we construct the approximate solution of the system (1-1) mentioned in the introduction. For simplicity of notation, in the sequel we will use y and $P_\pm$ instead of $\tilde{y}$ and $P_\pm^\varphi$, respectively.

We are going to construct a local approximate solution of the first order system

$$\begin{cases} h\,\partial_\tau u^h = L_0(y, hD_y) u^h + hL_1(y) u^h & \text{in } \mathbb{R}^2 \times (0, +\infty), \\ P_- u^h = f & \text{on } \mathbb{R}^2 \times \{0\}. \end{cases}$$

To be precise, we will look for a solution u^h in the form

$$u^h(y, \tau) = \operatorname{Op}^h(A^h(\cdot, \cdot, \tau)) f = \frac{1}{(2\pi)^2} \int_{\mathbb{R}^2} A^h(y, h\xi, \tau) e^{iy \cdot \xi} \hat{f}(\xi)\, d\xi, \tag{5-10}$$

with $A^h(\cdot, \cdot, \tau) \in \mathcal{S}^0$ for any $\tau > 0$ constructed inductively in the form

$$A^h(y, \xi, \tau) \sim \sum_{j \geq 0} h^j A_j(y, \xi, \tau).$$

The action of $h\,\partial_\tau - L_0(y, hD_y) - hL_1(y)$ on $A^h(y, hD_y, \tau)f$ is given by $T^h(y, hD_y, \tau)f$, with

$$T^h(y,\xi,\tau) = h(\partial_\tau A)(y,\xi,\tau) - L_0(y,\xi)A(y,\xi,\tau) - h(L_1(y)A(y,\xi,\tau) - i\,\partial_\xi L_0(y,\xi)\cdot\partial_y A(y,\xi,\tau)).$$

Here we exploited the particular form of L_1 (independent of ξ) and of L_0 (first order polynomial in ξ).

Then we look for A_0 satisfying

$$\begin{cases} h\,\partial_\tau A_0(y,\xi,\tau) = L_0(y,\xi)A_0(y,\xi,\tau), \\ P_-(y)A_0(y,\xi,\tau) = P_-(y), \end{cases} \tag{5-11}$$

and, for $j \geq 1$,

$$\begin{cases} h\,\partial_\tau A_j(y,\xi,\tau) = L_0(y,\xi)A_j(y,\xi,\tau)+L_1(y)A_{j-1}(y,\xi,\tau)-i\,\partial_\xi L_0(y,\xi)\cdot\partial_y A_{j-1}(y,\xi,\tau), \\ P_-(y)A_j(y,\xi,\tau) = 0. \end{cases} \tag{5-12}$$

Let us introduce a class of parametrized symbols in which we will construct the family A_j:

$$\mathcal{P}_h^m := \{b(\cdot,\cdot,\tau) \in \mathcal{S}^m : \forall (k,l) \in \mathbb{N}^2,\ \tau^k\,\partial_\tau^l b(\cdot,\cdot,\tau) \in h^{k-l}\mathcal{S}^{m-k+l}\}, \quad m \in \mathbb{Z}.$$

More precisely, $b \in \mathcal{P}_h^m$ means that, for all $(k,l) \in \mathbb{N}^2$, the function $(\tau, h) \mapsto (h^{-1}\tau)^k(h\,\partial_\tau)^l b(\cdot,\cdot,\tau)$ is uniformly bounded with respect to $(\tau,h) \in (0,+\infty)\times(0,1)$ in $\mathcal{S}^{m-k+l}$.

Proposition 5.4. *There exists $A_0 \in \mathcal{P}_h^0$ a solution of (5-11) given by*

$$A_0(y,\xi,\tau) = \frac{\Pi_-(y,\xi)P_-(y)}{k_+^\varphi(y,\xi)} e^{h^{-1}\tau\rho_-(y,\xi)}.$$

Proof. The solutions of the differential system $h\,\partial_\tau A_0 = L_0 A_0$ are $A_0(y,\xi,\tau) = e^{h^{-1}\tau L_0(y,\xi)}A_0(y,\xi,0)$. By definition of $\rho_\pm$ and $\Pi_\pm$, we have

$$e^{h^{-1}\tau L_0(y,\xi)} = e^{h^{-1}\tau\rho_-(y,\xi)}\Pi_-(y,\xi) + e^{h^{-1}\tau\rho_+(y,\xi)}\Pi_+(y,\xi). \tag{5-13}$$

It follows from (5-7) that A_0 belongs to $\mathcal{S}^0$ for any $\tau > 0$ if and only if $\Pi_+(y,\xi)A_0(y,\xi,0) = 0$. Moreover, the boundary condition $P_-A_0 = P_-$ implies $P_-(y)A_0(y,\xi,0) = P_-(y)$. Thus, thanks to Remark 5.3, we deduce that

$$A_0(y,\xi,0) = P_-(y) - \frac{P_+\Pi_+P_-}{k_+^\varphi}(y,\xi) = P_-(y) + \frac{P_+\Pi_-P_-}{k_+^\varphi}(y,\xi) = \frac{\Pi_-P_-}{k_+^\varphi}(y,\xi).$$

The properties of ρ_-, Π_-, P_-, and k_+ given in Proposition 5.2, imply that $(k_+^\varphi)^{-1}\Pi_-P_- \in \mathcal{S}^0$ and that $e^{h^{-1}\tau\rho_-(y,\xi)} \in \mathcal{P}_h^0$. This concludes the proof of Proposition 5.4. ☐

For the other terms A_j, $j \geq 1$, we have the following.

Proposition 5.5. *Let A_0 be defined by Proposition 5.4. Then, for any $j \geq 1$, there exists $A_j \in h^j\mathcal{P}_h^{-j}$ a solution of (5-12) which has the form*

$$A_j(y,\xi,\tau) = e^{h^{-1}\tau\rho_-(y,\xi)} \sum_{k=0}^{2j} (h^{-1}\tau\langle\xi\rangle)^k B_{j,k}(y,\xi), \tag{5-14}$$

with $B_{j,k} \in h^j\mathcal{S}^{-j}$.

Proof. Let us prove the result by induction. Thanks to Proposition 5.4, the claimed property holds for $j=0$. Now, assume that there exists $A_j \in h^j \mathcal{P}_h^{-j}$, a solution of (5-12) satisfying the above property, and let us prove that the same holds for A_{j+1}. In order to be a solution of the differential system $h\,\partial_\tau A_{j+1} = L_0 A_{j+1} + L_1 A_j - i\,\partial_\xi L_0 \cdot \partial_y A_j$, for A_{j+1} we have

$$A_{j+1} = e^{h^{-1}\tau L_0} A_{j+1|\tau=0} + e^{h^{-1}\tau L_0} \int_0^\tau e^{-h^{-1}sL_0} (L_1 A_j - i\,\partial_\xi L_0 \cdot \partial_y A_j)\,\mathrm{d}s, \tag{5-15}$$

where $L_1 A_j$ has still the form (5-14), and we have

$$\partial_y A_j = e^{h^{-1}\tau\rho_-}(h^{-1}\tau\,\partial_y\rho_- + \partial_y) \sum_{k=0}^{2j} (h^{-1}\tau\langle\xi\rangle)^k B_{j,k}.$$

Thus, thanks to the properties of ρ_- and $B_{j,k}$, the quantity $(L_1 A_j - i\,\partial_\xi L_0 \cdot \partial_y A_j)(y,\xi,s)$ has the form

$$e^{h^{-1}s\rho_-(y,\xi)} \sum_{k=0}^{2j+1} (h^{-1}s\langle\xi\rangle)^k \widetilde{B}_{j,k}(y,\xi), \tag{5-16}$$

with $\widetilde{B}_{j,k} \in h^j \mathcal{S}^{-j}$. So, using the decomposition (5-13), for the second term of the right-hand side of (5-15), we have

$$e^{h^{-1}\tau L_0} \int_0^\tau e^{-h^{-1}sL_0} (L_1 A_j - i\,\partial_\xi L_0 \cdot \partial_y A_j)\,\mathrm{d}s = e^{h^{-1}\tau\rho_-}\Pi_- I_-^j(\tau) + e^{h^{-1}\tau\rho_+}\Pi_+ I_+^j(\tau) \tag{5-17}$$

with

$$I_\pm^j(\tau) = \int_0^\tau e^{h^{-1}s(\rho_- - \rho_\pm)} \sum_{k=0}^{2j+1} (h^{-1}s\langle\xi\rangle)^k \widetilde{B}_{j,k}\,\mathrm{d}s.$$

For I_-^j, the exponential term is equal to 1, and by integration of s^k, we obtain

$$I_-^j(\tau) = \sum_{k=0}^{2j+1} (h^{-1}\tau\langle\xi\rangle)^{k+1} \frac{h\langle\xi\rangle^{-1}}{k+1} \widetilde{B}_{j,k}. \tag{5-18}$$

For I_+^j, let us introduce P_k, the polynomial of degree k such that

$$\int_0^\tau e^{\lambda s} s^k\,\mathrm{d}s = \frac{1}{\lambda^{k+1}}(e^{\tau\lambda}P_k(\tau\lambda) - P_k(0))$$

for any $\lambda \in \mathbb{C}^*$. With this notation in hand, we easily see that the term $e^{\tau^h\rho_+}\Pi_+ I_+^j(\tau)$ has the form

$$e^{\tau^h\rho_+}\Pi_+ I_+^j(\tau) = \Pi_+ \sum_{k=0}^{2j+1} \frac{h\langle\xi\rangle^k}{(\rho_- - \rho_+)^{k+1}} \widetilde{B}_{j,k}(e^{\tau^h\rho_-}P_k(\tau^h(\rho_- - \rho_+)) - e^{\tau^h\rho_+}P_k(0)), \tag{5-19}$$

where $\tau^h := h^{-1}\tau$. Thus, combining (5-18) and (5-19) with (5-15), (5-17) and (5-13) yields

$$A_{j+1} = e^{h^{-1}\tau\rho_+}(\Pi_+ A_{j+1|\tau=0} - \widetilde{B}_{j+1}^+) + e^{h^{-1}\tau\rho_-}\left(\Pi_- A_{j+1|\tau=0} + \sum_{k=0}^{2(j+1)} (h^{-1}\tau\langle\xi\rangle)^k \widetilde{B}_{j+1,k}^-\right),$$

where

$$\widetilde{B}^+_{j+1} = \Pi_+ \sum_{k=0}^{2j+1} \frac{h\langle\xi\rangle^k}{(\rho_- - \rho_+)^{k+1}} P_k(0)\widetilde{B}_{j,k} \in h^{j+1}\mathcal{S}^{-j-1}$$

and $\widetilde{B}^-_{j+1,k} \in h^{j+1}\mathcal{S}^{-j-1}$ as a linear combination of products of $\Pi_- \in \mathcal{S}^0$, of $h\langle\xi\rangle^{-1}$ (or $h\langle\xi\rangle^k(\rho_- - \rho_+)^{-k-1}$) belonging to $h\mathcal{S}^{-1}$, and of $\widetilde{B}_{j,k} \in h^j\mathcal{S}^{-j}$.

Now, in order to have $A_{j+1} \in \mathcal{S}^0$, we let the contribution of the exponentially growing term vanish by choosing

$$\Pi_+ A_{j+1}(y, \xi, 0) = \widetilde{B}^+_{j+1}(y, \xi).$$

Then, thanks to Remark 5.3, the boundary condition $P_-(y)A_{j+1}(y, \xi, 0) = 0$ gives

$$A_{j+1}(y, \xi, 0) = \frac{P_+\Pi_+}{k_+^\varphi}\widetilde{B}^+_{j+1}(y, \xi).$$

Finally, we have

$$A_{j+1}(y, \xi, \tau) = e^{h^{-1}\tau\rho_-(y,\xi)}\Bigg(\frac{\Pi_- P_+\Pi_+}{k_+^\varphi}\widetilde{B}^+_{j+1}(y, \xi) + \sum_{k=0}^{2(j+1)} (h^{-1}\tau\langle\xi\rangle)^k \widetilde{B}^-_{j+1,k}(y, \xi)\Bigg),$$

and Proposition 5.5 is proven with

$$B_{j+1,0} = \frac{\Pi_- P_+\Pi_+}{k_+^\varphi}\widetilde{B}^+_{j+1} + \widetilde{B}^-_{j+1,0},$$

and, for $k \geq 1$, $B_{j+1,k} = \widetilde{B}^-_{j+1,k}$. □

Remark 5.6. The computation of each term $B_{j,0}$ can be done recursively, but this leads to complicated calculations. For example $B_{1,0}$ has the form

$$B_{1,0}(y, \xi) = h\left[\Pi_+ a_0 + \frac{\Pi_- P_+\Pi_+ a_0}{k_+^\varphi}\right]\left(\frac{(z + i\alpha\cdot\partial_y)}{2\lambda} + \frac{i\alpha\cdot\partial_y\rho_-}{4\lambda^2}\right)\Pi_- A_0(y, \xi),$$

with $a_0(\tilde{y}) = i\alpha\cdot\tilde{n}^\varphi(\tilde{y})$.

Thanks to the relation (5-10), to any $A^h \in \mathcal{P}_h^0$ we can associate a bounded operator from $L^2(\mathbb{R}^2)$ into $L^2(\mathbb{R}^2 \times (0, +\infty))$. The boundedness in the variable $y \in \mathbb{R}^2$ is a consequence of the Calderon–Vaillancourt theorem (see (2-8)), and in the variable $\tau \in (0, +\infty)$, it is essentially multiplication by an L^∞-function. Moreover, for A_j of the form (5-14), we have the following mapping property which captures the Sobolev space regularity.

Proposition 5.7. *Let A_j, $j \geq 0$, be of the form (5-14). Then, for any $s \geq -j - \frac{1}{2}$, the operator $\mathcal{A}_j$ defined by*

$$\mathcal{A}_j : f \mapsto (\mathcal{A}_j f)(y, y_3) = \frac{1}{(2\pi)^2}\int_{\mathbb{R}^2} A_j(y, h\xi, y_3)e^{iy\cdot\xi}\hat{f}(\xi)\,d\xi$$

gives rise to a bounded operator from $H^s(\mathbb{R}^2)$ into $H^{s+j+1/2}(\mathbb{R}^2 \times (0, +\infty))$. Moreover, for any $l \in [0, j + \frac{1}{2}]$ we have

$$\|\mathcal{A}_j\|_{H^s \to H^{s+j+1/2-l}} = \mathcal{O}(h^{l-|s|}). \tag{5-20}$$

Proof. First, let us prove the result for $s = k - j - \frac{1}{2},\ k \in \mathbb{N}$, between the semiclassical Sobolev spaces

$$H^s_{\mathrm{scl}}(\mathbb{R}^2) := \langle hD_y\rangle^{-s} L^2(\mathbb{R}^2),$$
$$H^k_{\mathrm{scl}}(\mathbb{R}^2\times(0,+\infty)) := \{u \in L^2 : \langle hD_y\rangle^{k_1}(h\,\partial_{y_3})^{k_2}u \in L^2 \text{ for } (k_1,k_2)\in\mathbb{N}^2,\ k_1+k_2=k\},$$

where $\langle hD_y\rangle = \sqrt{-h^2\Delta_{\mathbb{R}^2}+I}$. Then, for $f \in H^s(\mathbb{R}^2)^4$, we have

$$\|\mathcal{A}_j f\|^2_{H^k_{\mathrm{scl}}(\mathbb{R}^2\times(0,+\infty))} = \sum_{k_1+k_2=k} \|\langle hD_y\rangle^{k_1}(h\,\partial_{y_3})^{k_2}\mathcal{A}_j f\|^2_{L^2(\mathbb{R}^2\times(0,+\infty))}$$
$$= \sum_{k_1+k_2=k}\int_0^{+\infty} \|\langle hD_y\rangle^{k_1}(h\,\partial_{y_3})^{k_2}(\mathcal{A}_j f)(\,\cdot\,,y_3)\|^2_{L^2(\mathbb{R}^2)}\,dy_3. \tag{5-21}$$

Thanks to the ellipticity property (5-7), for A_j given by Proposition 5.5, we have

$$(h\,\partial_{y_3})^{k_2}A_j(y,\xi,y_3) = h^j b_j(y,\xi;y_3)e^{-h^{-1}y_3c\langle\xi\rangle/2}\langle\xi\rangle^{k_2-j},$$

with b_j satisfying the following: for any $(\alpha,\beta)\in\mathbb{N}^2\times\mathbb{N}^2$ there exists $C_{\alpha,\beta}>0$ such that

$$|\partial_y^\alpha\,\partial_\xi^\beta b_j(y,\xi;y_3)| \le C_{\alpha,\beta} \quad \text{for all } (y,\xi;y_3)\in\mathbb{R}^2\times\mathbb{R}^2\times(0,+\infty).$$

Consequently, thanks to the Calderón–Vaillancourt theorem (see (2-8)), we can write

$$\langle hD_y\rangle^{k_1}(h\,\partial_{y_3})^{k_2}\mathcal{A}_j = h^j\mathcal{B}_j(y_3)\langle hD_y\rangle^{k_1+k_2-j}e^{-h^{-1}y_3c\langle hD_y\rangle/2},$$

with $(\mathcal{B}_j(y_3))_{y_3>0}$ a family of bounded operators on $L^2(\mathbb{R}^2)$, uniformly bounded with respect to $y_3>0$. Then, for $f\in H^s(\mathbb{R}^2)^4$, we have

$$\|\langle hD_y\rangle^{k_1}(h\,\partial_{y_3})^{k_2}(\mathcal{A}_j f)(\,\cdot\,,y_3)\|^2_{L^2(\mathbb{R}^2)} \lesssim h^j\|\langle hD_y\rangle^{k_1+k_2-j}e^{-h^{-1}y_3c\langle hD_y\rangle/2}f\|^2_{L^2(\mathbb{R}^2)},$$

and from (5-21) we deduce that

$$\|\mathcal{A}_j f\|^2_{H^k_{\mathrm{scl}}(\mathbb{R}^2\times(0,+\infty))} \lesssim h^{2j+1}\|\langle hD_y\rangle^{k-j-1/2}f\|^2_{L^2(\mathbb{R}^2)} = h^{2j+1}\|f\|^2_{H^{k-j-1/2}_{\mathrm{scl}}(\mathbb{R}^2)},$$

where we used that, for any $l\in\mathbb{N}$ and $f\in H^{l-1/2}_{\mathrm{scl}}(\mathbb{R}^2)$,

$$\|\langle hD_y\rangle^l e^{-h^{-1}y_3c\langle hD_y\rangle/2}f\|^2_{L^2(\mathbb{R}^2)} = \langle e^{-h^{-1}y_3c\langle hD_y\rangle}\langle hD_y\rangle^l f,\langle hD_y\rangle^l f\rangle_{L^2}$$
$$= -\frac{h}{c}\frac{\partial}{\partial_{y_3}}\langle e^{-h^{-1}y_3c\langle hD_y\rangle}\langle hD_y\rangle^{l-1}f,\langle hD_y\rangle^l f\rangle_{L^2}.$$

By interpolation arguments we thus deduce that, for any $j\in\mathbb{N}$ and $s\ge -j-\frac{1}{2}$,

$$\|\mathcal{A}_j\|_{H^s_{\mathrm{scl}}\to H^{s+j+1/2}_{\mathrm{scl}}} = \mathcal{O}(h^{j+1/2}).$$

This means that, for $\bar{y} := (y,y_3)$,

$$\|\langle hD_{\bar{y}}\rangle^{s+j+1/2}\mathcal{A}_j\langle hD_y\rangle^{-s}\|_{L^2(\mathbb{R}^2)\to L^2(\mathbb{R}^2\times(0,+\infty))} = \mathcal{O}(h^{j+1/2}). \tag{5-22}$$

In order to prove (5-20) (in *classical* Sobolev spaces), let us estimate $\langle D_{\bar y}\rangle^{s+j+1/2-l}\mathcal{A}_j\langle D_y\rangle^{-s}$ from $L^2(\mathbb{R}^2)$ into $L^2(\mathbb{R}^2\times(0,+\infty))$. The inequalities, for all $\xi\in\mathbb{R}^d$, $d=2,3$, and $h\in(0,1)$,

$$1\le\langle\xi\rangle\le h^{-1}\langle h\xi\rangle,\quad \langle\xi\rangle^{-1}\le\langle h\xi\rangle^{-1},\quad \langle\xi\rangle^{-1}\le 1$$

imply, for $j+\frac{1}{2}\ge l$, $s_+=\max(s,0)$, and $s_-=s-s_+$, the estimates

$$\langle\xi\rangle^{s+j+1/2-l}\le h^{-j-1/2+l}h^{-s_+}\langle h\xi\rangle^{s+j+1/2},\quad \langle\xi\rangle^{-s}\le h^{s_-}\langle h\xi\rangle^{-s}.$$

We deduce

$$\|\langle D_{\bar y}\rangle^{s+j+1/2-l}\mathcal{A}_j\langle D_y\rangle^{-s}\|_{L^2\to L^2}\lesssim h^{-j-1/2+l}h^{-s_+}h^{s_-}\|\langle hD_{\bar y}\rangle^{s+j+1/2}\mathcal{A}_j\langle hD_y\rangle^{-s}\|_{L^2\to L^2}.$$

Then estimate (5-20) follows from (5-22) using $s_+-s_-=|s|$. □

Proposition 5.8. *Let $f\in H^s(\mathbb{R}^2)$ and A_j, $j\ge 0$, be as in Propositions 5.4 and 5.5. Then, for any $N\ge -s-\frac{1}{2}$, the function $u_N^h=\sum_{j=0}^N h^j\mathcal{A}_jf$ satisfies*

$$\begin{cases} h\,\partial_\tau u_N^h-L_0(y,hD_y)u_N^h-hL_1(y)u_N^h=h^{N+1}\mathcal{R}_N^hf & \text{in } \mathbb{R}^2\times(0,+\infty),\\ P_-u_N^h=f & \text{on } \mathbb{R}^2\times\{0\},\end{cases}\tag{5-23}$$

with

$$\mathcal{R}_N^h: f\mapsto\frac{-1}{(2\pi)^2}\int_{\mathbb{R}^2}(L_1A_N-i\,\partial_\xi L_0\cdot\partial_yA_N)(y,h\xi,\tau)e^{iy\cdot\xi}\hat f(\xi)\,\mathrm{d}\xi$$

a bounded operator from $H^s(\mathbb{R}^2)$ into $H^{s+N+1/2}(\mathbb{R}^2\times(0,+\infty))$ satisfying, for any $l\in\left[0,N+\frac{1}{2}\right]$,

$$\|\mathcal{R}_N^h\|_{H^s\to H^{s+N+1/2-l}}=\mathcal{O}(h^{l-|s|}).\tag{5-24}$$

Proof. By construction of the sequence $(A_j)_{j\in\{0,\dots,N-1\}}$, we have the system (5-23) with

$$\mathcal{R}_N^h=\mathrm{Op}^h(r_N^h(\cdot,\cdot,\tau))\quad\text{and}\quad r_N^h(y,\xi,\tau)=-(L_1A_N-i\,\partial_\xi L_0\cdot\partial_yA_N)(y,\xi,\tau)$$

(see the beginning of Section 5B). As in the proof of Proposition 5.5, r_N^h has the form (5-16) (with $j=N$). Then, as in the proof of Proposition 5.7 we obtain the estimate (5-24). □

5C. *Proof of Theorem 5.1.*

In this section, we apply the above construction in order to prove Theorem 5.1.

Let $g\in P_-H^{1/2}(\partial\Omega)^4$, let $(U_\varphi,V_\varphi,\varphi)$ be a chart of the atlas $\mathbb{A}$, and let $\psi_1,\psi_2\in C_0^\infty(U_\varphi)$. Then $f:=(\varphi^{-1})^*(\psi_2g)$ is a function of $H^{1/2}(V_\varphi)^4$ which can be extended by 0 to a function of $H^{1/2}(\mathbb{R}^2)^4$. Then, for $h=1/m$ and any $N\in\mathbb{N}$, the previous construction provides a function $u_N^h\in H^1(\mathbb{R}^2\times(0,+\infty))^4$ satisfying

$$\begin{cases}(\widetilde{D}_m^\varphi-z)u_N^h=h^{N+1}\mathcal{R}_N^hf & \text{in } \mathbb{R}^2\times(0,\varepsilon),\\ \Gamma_-u_N^h=f & \text{on } \mathbb{R}^2\times\{0\},\end{cases}$$

with $u_N^h=\sum_{j=0}^N h^j\mathcal{A}_jf$ (see Proposition 5.7) and $\mathcal{R}_N^hf\in H^{N+1}(\mathbb{R}^2\times(0,\varepsilon))$ with norm in H^{N+1-l}, $l\in\left[0,N+\frac{1}{2}\right]$, bounded by $\mathcal{O}(h^{l-1/2})$. Consequently, $v_N^h:=\phi^*u_N^h$, defined on $\mathcal{V}_{\varphi,\varepsilon}$, satisfies

$$\begin{cases}(D_m-z)v_N^h=h^{N+1}\phi^*(\mathcal{R}_N^hf) & \text{in } \mathcal{V}_{\varphi,\varepsilon},\\ \Gamma_-v_N^h=\psi_2g & \text{on } U_\varphi.\end{cases}$$

Now, let $E_m^\Omega(z)[\psi_2 g] \in H^1(\Omega)^4$ be as in Definition 4.1. Since $\Gamma_- v_N^h = \Gamma_- E_m^\Omega(z)[\psi_2 g] = \psi_2 g$, the following equality holds in $\mathcal{V}_{\varphi,\varepsilon}$:

$$v_N^h - E_m^\Omega(z)[\psi_2 g] = h^{N+1}(H_{\mathrm{MIT}}(m) - z)^{-1}\phi^*(\mathcal{R}_N^h(\varphi^{-1})^*(\psi_2 g)).$$

From this, we deduce that

$$\psi_1 \mathscr{A}_m \psi_2(g) := \psi_1 \Gamma_+ E_m^\Omega(z)[\psi_2 g] = \psi_1 \Gamma_+ v_N^h - h^{N+1}\psi_1 \Gamma_+ (H_{\mathrm{MIT}} - z)^{-1}\phi^*(\mathcal{R}_N^h(\varphi^{-1})^*(\psi_2 g)).$$

Since $\phi\restriction_{U_\varphi} = \varphi$, for any $u \in H^1(V_\varphi \times (0,\varepsilon))^4$, we have that

$$\Gamma_+ \phi^*(u) = \varphi^*(P_+ u\restriction_{V_\varphi \times \{0\}}), \qquad \psi_1 \Gamma_+ v_N^h = \psi_1 \varphi^* \operatorname{Op}^h(a_N^h)(\varphi^{-1})^* \psi_2 g,$$

with

$$a_N^h(\tilde{y}, \xi) = \sum_{j=0}^{N} h^j P_+ A_j(y, \xi, 0) = \sum_{j=0}^{N} h^j P_+ B_{j,0}(y, \xi),$$

where $B_{j,0} \in h^j \mathcal{S}^{-j}$ are introduced in Proposition 5.5. Thus, from Proposition 5.4, in local coordinates, the principal semiclassical symbol of $\mathscr{A}_m$ is given by

$$P_+ B_{0,0}(y, \xi) = P_+ A_0(y, \xi, 0) = \frac{P_+ \Pi_- P_-}{k_+^\varphi}(y, \xi).$$

Thanks to property (5-8) it is equal to

$$-\Theta^\varphi P_-(y, \xi) = \frac{S \cdot (\xi \wedge n^\varphi(y))}{\sqrt{\langle G(y)^{-1}\xi, \xi\rangle + 1} + 1} P_-(y, \xi).$$

We conclude the proof of Theorem 5.1 from results of Section 2D and by proving the following lemma which is a consequence of the above considerations, the regularity estimates from Theorem 3.1(iii), Theorem 3.4(i), and Proposition 4.2.

Lemma 5.9. *Let $\psi_1, \psi_2 \in C^\infty(\Sigma)$ be such that* $\operatorname{supp}(\psi_1) \cap \operatorname{supp}(\psi_2) = \varnothing$. *Then, for $m_0 > 0$ sufficiently large, $m \geqslant m_0$, and for any $(k, N) \in \mathbb{N}^* \times \mathbb{N}^*$,*

$$\|\psi_1 \mathscr{A}_m \psi_2\|_{P_- H^{1/2}(\Sigma)^4 \to P_+ H^k(\Sigma)^4} = \mathcal{O}(m^{-N}).$$

Proof. Let $\psi_1, \psi_2 \in C^\infty(\Sigma)$ with disjoint supports. Thanks to Theorem 3.1(iii) and Theorem 3.4(i), to prove the lemma it suffices to show that, for any $(N_1, N_2) \in \mathbb{N}^2$, there exists C_{N_1,N_2} such that, for $g \in P_- H^{1/2}(\Sigma)^4$,

$$\|(\psi_1 \mathscr{A}_m \psi_2) g\|_{P_+ H^{N_2+1/2}(\Sigma)^4} \leq \frac{C_{N_1,N_2}}{\sqrt{m}} (\Pi_{i=0}^{N_2} \|(H_{\mathrm{MIT}}(m) - z)^{-1}\|_{H^i(\Omega)^4 \to H^{i+1}(\Omega)^4}) \times \|(H_{\mathrm{MIT}}(m) - z)^{-1}\|_{L^2(\Omega)^4 \to L^2(\Omega)^4}^{N_1} \|g\|_{P_- H^{1/2}(\Sigma)^4}. \tag{5-25}$$

For this, let us introduce $\Phi_1 \in C_0^\infty(\overline{\Omega})$ such that $\Phi_1 = 1$ near $\operatorname{supp}(\psi_1)$ and $\Phi_1 = 0$ near $\operatorname{supp}(\psi_2)$. Thus for $g \in P_- H^{1/2}(\Sigma)^4$ and $E_m^\Omega(z)[\psi_2 g] \in H^1(\Omega)$ as in Definition 4.1, the function $u_{1,2} := \Phi_1 E_m^\Omega(z)[\psi_2 g]$ satisfies

$$\begin{cases} (D_m - z) u_{1,2} = [D_0, \Phi_1] E_m^\Omega(z)[\psi_2 g] & \text{in } \Omega, \\ \Gamma_- u_{1,2} = \Phi_{1\restriction_\Sigma} \psi_2 g = 0 & \text{on } \Sigma. \end{cases}$$

Then, $u_{1,2} = (H_{\mathrm{MIT}}(m) - z)^{-1}[D_0, \Phi_1]E_m^{\Omega}(z)[\psi_2 g]$, and, for any $\widetilde{\Phi}_1 \in C_0^{\infty}(\overline{\Omega})$ equal to 1 near $\mathrm{supp}(\psi_1)$, we have

$$\psi_1 \mathscr{A}_m \psi_2(g) = \psi_1 \Gamma_+ \widetilde{\Phi}_1 (H_{\mathrm{MIT}}(m) - z)^{-1}[D_0, \Phi_1]E_m^{\Omega}(z)[\psi_2 g].$$

Moreover, by choosing $\widetilde{\Phi}_1$ such that $\widetilde{\Phi}_1 \prec \Phi_1$, that is $\Phi_1 = 1$ on $\mathrm{supp}(\widetilde{\Phi}_1)$, both functions $\widetilde{\Phi}_1$ and $[D_0, \Phi_1]$ have disjoint supports, and we can then apply the telescopic formula

$$\widetilde{\Phi}_1 (H_{\mathrm{MIT}}(m) - z)^{-1}(1 - \chi_1) = \widetilde{\Phi}_1 (H_{\mathrm{MIT}}(m) - z)^{-1}[D_0, \chi_J] \cdots (H_{\mathrm{MIT}}(m) - z)^{-1}[D_0, \chi_2] \times (H_{\mathrm{MIT}}(m) - z)^{-1}(1 - \chi_1)$$

for $(\chi_i)_{1 \le i \le J}$ a family of compactly supported smooth functions such that $\widetilde{\Phi}_1 \prec \chi_J \prec \chi_{J-1} \prec \cdots \prec \chi_1 \prec \Phi_1$, $J = N_1 + N_2$. Since $[D_0, \Phi_1] = (1 - \chi_1)[D_0, \Phi_1]$, the above telescopic formula allows us to write $\psi_1 \mathscr{A}_m \psi_2(g)$ as a product of J cutoff resolvents of $H_{\mathrm{MIT}}(m)$. Now, by Proposition 4.2, we have

$$\| E_m^{\Omega}(z)[\psi_2 g] \|_{L^2(\Omega)^4} \lesssim \frac{1}{\sqrt{m}} \| g \|_{L^2(\Sigma)^4}.$$

Thus, using the continuity of Γ_+ from $H^{N_2+1}(\Omega)$ to $H^{N_2+1/2}(\Sigma)$, we then get the estimation (5-25), finishing the proof of the lemma taking $N_2 = k$ and N_1 such that $N_1 \ge N + \frac{1}{2}N_2(N_2 - 1)$. □

Remark 5.10. Note that, for any $m > 0$ and $z \in \rho(H_{\mathrm{MIT}}(m))$, the parametrix we have constructed for $\mathscr{A}_m$ is valid from the classical pseudodifferential point of view. Actually, Lemma 5.9 is the only result where the assumption that m is big enough has been assumed, and it is exclusively required to ensure that away from the diagonal the operator $\mathscr{A}_m$ is negligible in $1/m$. In the same vein, if m is fixed then the proof of Lemma 5.9 still ensures that away from the diagonal $\mathscr{A}_m$ is regularizing. Consequently, we deduce that, for any $m > 0$ and $z \in \rho(H_{\mathrm{MIT}}(m))$, the operator $\mathscr{A}_m$ is a homogeneous pseudodifferential operator of order 0, and that

$$\mathscr{A}_m = \frac{D_\Sigma}{\sqrt{-\Delta_\Sigma}} P_- \mod \mathrm{Op}\,\mathcal{S}^{-1}(\Sigma),$$

which is in accordance with Theorem 4.5.

Remark 5.11. If Ω is the upper half-plane $\{(x_1, x_2, x_3) \in \mathbb{R}^3 : x_3 > 0\}$, we easily obtain that $\mathscr{A}_m$ is a Fourier multiplier with symbol

$$a_m(\xi) = -\frac{i\alpha_3(\alpha \cdot \xi - z)}{\sqrt{|\xi|^2 + m^2} + m} P_-.$$

6. Resolvent convergence to the MIT bag model

In the whole section, $\Omega \subset \mathbb{R}^3$ denotes a bounded smooth domain, we set

$$\Omega_i = \Omega, \quad \Omega_e = \mathbb{R}^3 \setminus \overline{\Omega}, \quad \text{and} \quad \Sigma = \partial\Omega,$$

and we let n be the outward (with respect to Ω_i) unit normal vector field on Σ.

Fix $m > 0$, and let $M > 0$. Consider the perturbed Dirac operator

$$H_M \varphi = (D_m + M\beta 1_{\Omega_e})\varphi \quad \text{for all } \varphi \in \mathrm{dom}(H_M) := H^1(\mathbb{R}^3)^4,$$

where 1_{Ω_e} is the characteristic function of Ω_e. Using the Kato–Rellich theorem and Weyl's theorem, it is easy to see that $(H_M, \mathrm{dom}(H_M))$ is self-adjoint and that

$$\mathrm{Sp}_{\mathrm{ess}}(H_M) = (-\infty, -(m+M)] \cup [m+M, +\infty)$$

and

$$\mathrm{Sp}(H_M) \cap (-(m+M), m+M) \text{ is purely discrete.}$$

Now, let $H_{\mathrm{MIT}}(m)$ be the MIT bag operator acting on $L^2(\Omega_i)^4$, that is

$$H_{\mathrm{MIT}}(m)v = D_m v \quad \text{for all } v \in \mathrm{dom}(H_{\mathrm{MIT}}(m)) := \{v \in H^1(\Omega_i)^4 : P_- t_\Sigma v = 0 \text{ on } \Sigma\},$$

where t_Σ and $P_\pm$ are the trace operator and the orthogonal projection from Section 2A.

The aim of this section is to use the properties of the Poincaré–Steklov operators carried out in the previous sections to study the resolvent of H_M when M is large enough. Namely, we give a Krein-type resolvent formula in terms of the resolvent of $H_{\mathrm{MIT}}(m)$, and we show that the convergence of H_M toward $H_{\mathrm{MIT}}(m)$ holds in the norm resolvent sense with a convergence rate of $\mathcal{O}(1/M)$, which improves the result of [Barbaroux et al. 2019].

Before stating the main results of this section, we need to introduce some notation and definitions. First, we introduce the Dirac auxiliary operator

$$\widetilde{H}_M u = D_{m+M} u \quad \text{for all } u \in \mathrm{dom}(\widetilde{H}_M) := \{u \in H^1(\Omega_e)^4 : P_+ t_\Sigma u = 0 \text{ on } \Sigma\}.$$

Notice that $\widetilde{H}_M$ is the MIT bag operator on Ω_e (the boundary condition is with P_+ because the normal n is incoming for Ω_e). Since Ω_e is unbounded, Theorem 3.1 together with Remark 3.2 imply that $(\widetilde{H}_M, \mathrm{dom}(\widetilde{H}_M))$ is self-adjoint and that

$$\mathrm{Sp}(\widetilde{H}_M) = \mathrm{Sp}_{\mathrm{ess}}(\widetilde{H}_M) = (-\infty, -(m+M)] \cup [m+M, +\infty).$$

In particular, $\rho(H_M) \subset \rho(\widetilde{H}_M))$. Let $z \in \rho(H_{\mathrm{MIT}}(m)) \cap \rho(\widetilde{H}_M)$, $g \in P_- H^{1/2}(\Sigma)^4$, and $h \in P_+ H^{1/2}(\Sigma)^4$. We denote by $E_m^{\Omega_i}(z) : P_- H^{1/2}(\Sigma)^4 \to H^1(\Omega_i)^4$ the unique solution of the boundary value problem

$$\begin{cases} (D_m - z)v = 0 & \text{in } \Omega_i, \\ P_- t_\Sigma v = g & \text{in } \Sigma. \end{cases} \tag{6-1}$$

Similarly, we denote by $E_{m+M}^{\Omega_e}(z) : P_+ H^{1/2}(\Sigma)^4 \to H^1(\Omega_e)^4$ the unique solution of the boundary value problem

$$\begin{cases} (D_{m+M} - z)u = 0 & \text{in } \Omega_e, \\ P_+ t_\Sigma u = h & \text{in } \Sigma. \end{cases} \tag{6-2}$$

Define the Poincaré–Steklov operators associated to the above problems by

$$\mathscr{A}_m^i = P_+ t_\Sigma E_m^{\Omega_i}(z) P_- \quad \text{and} \quad \mathscr{A}_{m+M}^e = P_- t_\Sigma E_{m+M}^{\Omega_e}(z) P_+.$$

Notation 6.1. In the sequel we shall denote by $R_M(z)$, $\widetilde{R}_M(z)$, and $R_{\mathrm{MIT}}(z)$ the resolvent of H_M, $\widetilde{H}_M$, and $H_{\mathrm{MIT}}(m)$, respectively. We also use the notation

- $\Gamma_\pm = P_\pm t_\Sigma$ and $\Gamma = \Gamma_+ r_{\Omega_i} + \Gamma_- r_{\Omega_e}$,
- $E_M(z) = e_{\Omega_i} E_m^{\Omega_i}(z) P_- + e_{\Omega_e} E_{m+M}^{\Omega_e}(z) P_+$,
- $\widetilde{R}_{\mathrm{MIT}}(z) = e_{\Omega_i} R_{\mathrm{MIT}}(z) r_{\Omega_i} + e_{\Omega_e} \widetilde{R}_M(z) r_{\Omega_e}$.

With these notations in hand, we can state the main results of this section. The following theorem is the main tool to show the large coupling convergence with a rate of convergence of $\mathcal{O}(1/M)$.

Theorem 6.2. *There is $M_0 > 0$ such that, for all $M > M_0$ and all $z \in \rho(H_{\mathrm{MIT}}(m)) \cap \rho(H_M)$, the operator $\Psi_M(z) := (I - \mathscr{A}_m^i - \mathscr{A}_{m+M}^e)$ is bounded invertible in $H^{1/2}(\Sigma)^4$, the inverse is given by*

$$\Psi_M^{-1}(z) = (I_4 - \mathscr{A}_m^i \mathscr{A}_{m+M}^e - \mathscr{A}_{m+M}^e \mathscr{A}_m^i)^{-1}(I + \mathscr{A}_m^i + \mathscr{A}_{m+M}^e),$$

and the following resolvent formula holds:

$$R_M(z) = \widetilde{R}_{\mathrm{MIT}}(z) + E_M(z)\Psi_M^{-1}(z)\Gamma \widetilde{R}_{\mathrm{MIT}}(z). \tag{6-3}$$

Remark 6.3. By Proposition 4.2(i), we have that

$$(E_m^{\Omega_i}(z))^* = -\beta\Gamma_+ R_{\mathrm{MIT}}(\bar{z}) \quad \text{and} \quad (E_{m+M}^{\Omega_e}(z))^* = -\beta\Gamma_- \widetilde{R}_M(\bar{z})$$

for any $z \in \rho(H_{\mathrm{MIT}}(m)) \cap \rho(H_M)$. Thus, the resolvent formula (6-3) can be written in the form

$$R_M(z) = \widetilde{R}_{\mathrm{MIT}}(z) - (\beta\Gamma \widetilde{R}_{\mathrm{MIT}}(\bar{z}))^* \Psi_M^{-1}(z)\Gamma \widetilde{R}_{\mathrm{MIT}}(z).$$

Before going through the proof of Theorem 6.2, we first establish a regularity result that will play a crucial role in the rest of this section. It concerns the dependence on the parameter M of the norm of an auxiliary operator which involves the composition of the operators $\mathscr{A}_m^i$ and $\mathscr{A}_{m+M}^e$.

Proposition 6.4. *Let $\mathscr{A}_m^i$ and $\mathscr{A}_{m+M}^e$ be as above. Then, there is $M_0 > 0$ such that, for every $M > M_0$ and all $z \in \rho(H_{\mathrm{MIT}}(m)) \cap \rho(H_M)$, the following hold*:

(i) *For any $s \in \mathbb{R}$, the operator $\Xi_M(z) : H^s(\Sigma)^4 \to H^s(\Sigma)^4$ defined by*

$$\Xi_M(z) = (I_4 - \mathscr{A}_m^i \mathscr{A}_{m+M}^e - \mathscr{A}_{m+M}^e \mathscr{A}_m^i)^{-1} \tag{6-4}$$

is everywhere defined and uniformly bounded with respect to M.

(ii) *The Poincaré–Steklov operator, $\mathscr{A}_{m+M}^e$, satisfies the estimate*

$$\|\mathscr{A}_{m+M}^e\|_{P_+ H^{s+1}(\Sigma)^4 \to P_- H^s(\Sigma)^4} \lesssim M^{-1} \quad \textit{for all } s \in \mathbb{R}.$$

Proof. (i) Set $\tau := (m + M)$. Then the result essentially follows from the fact that $\Xi_M(z)$ is a $1/\tau$-pseudodifferential operator of order 0. Indeed, fix $z \in \rho(H_{\mathrm{MIT}}(m)) \cap \rho(H_M)$ and set $h = \tau^{-1}$. Then, from Theorem 4.5 and Remark 5.10, we know that $\mathscr{A}_m^i$ is a homogeneous pseudodifferential operator of order 0. Thus $\mathscr{A}_m^i$ can also be viewed as a h-pseudodifferential operators of order 0. That is, $\mathscr{A}_m^i \in \mathrm{Op}^h \mathcal{S}^0(\Sigma)$, and, in local coordinates, its semiclassical principal symbol is given by

$$p_{h,\mathscr{A}_m^i}(x,\xi) = \frac{S \cdot (\xi \wedge n(x)) P_-}{|\xi \wedge n(x)|},$$

where we identify $\xi \in \mathbb{R}^2$ with $\bar{\xi} = (\xi_1, \xi_2, 0)^t \in \mathbb{R}^3$, and, for $x = \varphi(\tilde{x}) \in \Sigma$, we let $n(x)$ stand for $n^{\varphi}(\tilde{x})$. Similarly, thanks to Theorem 5.1, for h_0 sufficiently small (and hence M_0 big enough) and all $h < h_0$, we also know that $\mathscr{A}^e_{m+M}$ is a h-pseudodifferential operator and that

$$\mathscr{A}^e_{m+M} \in \operatorname{Op}^h \mathcal{S}^0(\Sigma), \quad p_{h,\mathscr{A}^e_{m+M}}(x,\xi) = -\frac{S \cdot (\xi \wedge n(x)) P_+}{\sqrt{|\xi \wedge n(x)|^2 + 1} + 1}.$$

Therefore, the symbol calculus yields, for all $h < h_0$, that $(I_4 - \mathscr{A}^i_m \mathscr{A}^e_{m+M} - \mathscr{A}^e_{m+M} \mathscr{A}^i_m)$ is a $1/\tau$-pseudodifferential operator of order 0. Now, Lemmas A.3 and A.1 yield

$$\frac{S \cdot (\xi \wedge n(x)) P_\pm S \cdot (\xi \wedge n(x)) P_\mp}{|\xi \wedge n(x)|(\sqrt{|\xi \wedge n(x)|^2 + 1} + 1)} = \frac{|\xi \wedge n(x)| P_\mp}{\sqrt{|\xi \wedge n(x)|^2 + 1} + 1}.$$

Thus

$$\begin{aligned} I_4 - p_{h,\mathscr{A}^i_m}(x,\xi) p_{h,\mathscr{A}^e_{m+M}}(x,\xi) - p_{h,\mathscr{A}^e_{m+M}}(x,\xi) p_{h,\mathscr{A}^i_m}(x,\xi) \\ = I_4 + \frac{|\xi \wedge n(x)|}{\sqrt{|\xi \wedge n(x)|^2 + 1} + 1} = \frac{\sqrt{|\xi \wedge n(x)|^2 + 1} + 1 + |\xi \wedge n(x)|}{\sqrt{|\xi \wedge n(x)|^2 + 1} + 1} \gtrsim 1. \end{aligned}$$

From this, we deduce that $(I_4 - \mathscr{A}^i_m \mathscr{A}^e_{m+M} - \mathscr{A}^e_{m+M} \mathscr{A}^i_m)$ is elliptic in $\operatorname{Op}^h \mathcal{S}^0(\Sigma)$. Thus, $\Xi_M(z) \in \operatorname{Op}^h \mathcal{S}^0(\Sigma)$, and, in local coordinates, its semiclassical principal symbol is given by

$$p_{h,\Xi_M(z)}(x,\xi) = \frac{\sqrt{|\xi \wedge n(x)|^2 + 1} + 1}{\sqrt{|\xi \wedge n(x)|^2 + 1} + 1 + |\xi \wedge n(x)|}.$$

As $\Xi_M(z)$ is an h-pseudodifferential operator of order 0, it follows from the Calderón–Vaillancourt theorem (see (2-9)) that $\Xi_M(z) : H^s(\Sigma)^4 \to H^s(\Sigma)^4$ is well defined and uniformly bounded with respect to M for any $s \in \mathbb{R}$ proving assertion (i) of the theorem.

The proof of assertion (ii) exploits also the Calderón–Vaillancourt theorem which shows that, for any $s \in \mathbb{R}$, any operator in $h \operatorname{Op}^h \mathcal{S}^0(\Sigma)$ is uniformly bounded by $\mathcal{O}(h)$, with respect to $h = \tau^{-1} \in (0,1)$, from $H^{s+1}(\Sigma)^4$ into $H^s(\Sigma)^4$ (see (2-9)). Thus, for any $s \in \mathbb{R}$,

$$\left\| \mathscr{A}^e_\tau - \frac{1}{\tau} D_\Sigma (\sqrt{-\tau^{-2} \Delta_\Sigma + I} + I)^{-1} P_+ \right\|_{H^{s+1}(\Sigma)^4 \to H^s(\Sigma)^4} \lesssim \tau^{-1},$$

uniformly with respect to τ large enough. Then we conclude the proof of assertion (ii) by using that $(\sqrt{-\tau^{-2}\Delta_\Sigma + I} + I)^{-1}$ is uniformly bounded from $H^{s+1}(\Sigma)^4$ into itself and that D_Σ is bounded from $H^{s+1}(\Sigma)^4$ into $H^s(\Sigma)^4$ (as a first order differential operator). □

We can now give the proof of Theorem 6.2.

Proof of Theorem 6.2. Let M_0 be as in Proposition 6.4 and $M > M_0$. Fix $z \in \rho(H_{\mathrm{MIT}}(m)) \cap \rho(H_M)$, and let $f \in L^2(\mathbb{R}^3)^4$. We set

$$v = r_{\Omega_i} R_M(z) f \quad \text{and} \quad u = r_{\Omega_e} R_M(z) f.$$

Then u and v satisfy the system

$$\begin{cases} (D_m - z)v = f & \text{in } \Omega_i, \\ (D_{m+M} - z)u = f & \text{in } \Omega_e, \\ P_- t_\Sigma v = P_- t_\Sigma u & \text{on } \Sigma, \\ P_+ t_\Sigma v = P_+ t_\Sigma u & \text{on } \Sigma. \end{cases}$$

Since $E_m^{\Omega_i}(z)$ and $E_{m+M}^{\Omega_e}(z)$ give the unique solution to the boundary value problem (6-1) and (6-2), respectively, and

$$\Gamma_- R_{\mathrm{MIT}}(z) r_{\Omega_i} f = 0 \quad \text{and} \quad \Gamma_+ \widetilde{R}_M(z) r_{\Omega_e} f = 0,$$

if we let

$$\varphi = \Gamma_- u \quad \text{and} \quad \psi = \Gamma_+ v,$$

then it is easy to check that

$$\begin{cases} v = R_{\mathrm{MIT}}(z) r_{\Omega_i} f + E_m^{\Omega_i}(z)\varphi, \\ u = \widetilde{R}_M(z) r_{\Omega_e} f + E_{m+M}^{\Omega_e}(z)\psi. \end{cases} \tag{6-5}$$

Hence, to get an explicit formula for $R_M(z)$, it remains to find the unknowns φ and ψ. For this, note that from (6-5) we have

$$\begin{cases} \psi = \Gamma_+ r_{\Omega_i} R_M(z) f = \Gamma_+ R_{\mathrm{MIT}}(z) r_{\Omega_i} f + \Gamma_+ E_m^{\Omega_i}(z)[\varphi], \\ \varphi = \Gamma_- r_{\Omega_e} R_M(z) f = \Gamma_- \widetilde{R}_M(z) r_{\Omega_e} f + \Gamma_- E_{m+M}^{\Omega_e}(z)[\psi]. \end{cases} \tag{6-6}$$

Substituting the values of ψ and φ (from (6-6)) into the system (6-5), we obtain

$$\begin{aligned} R_M(z) &= e_{\Omega_i} R_{\mathrm{MIT}}(z) r_{\Omega_i} + e_{\Omega_e} \widetilde{R}_M(z) r_{\Omega_e} + (e_{\Omega_i} E_m^{\Omega_i}(z)\Gamma_- r_{\Omega_e} + e_{\Omega_e} E_{m+M}^{\Omega_e}(z)\Gamma_+ r_{\Omega_i}) R_M(z) \\ &= \widetilde{R}_{\mathrm{MIT}}(z) + E_M(z)\Gamma R_M(z). \end{aligned} \tag{6-7}$$

Note that, by definition of the Poincaré–Steklov operators, (6-6) is equivalent to

$$\begin{cases} \psi = \Gamma_+ R_{\mathrm{MIT}}(z) r_{\Omega_i} f + \mathscr{A}_m^i(\varphi), \\ \varphi = \Gamma_- \widetilde{R}_M(z) r_{\Omega_e} f + \mathscr{A}_{m+M}^e(\psi). \end{cases} \tag{6-8}$$

Thus, applying Γ to the identity (6-7) yields

$$\Gamma \widetilde{R}_{\mathrm{MIT}}(z) = (I - \mathscr{A}_m^i - \mathscr{A}_{m+M}^e)\Gamma R_M(z) = \Psi_M(z)\Gamma R_M(z).$$

Now, we apply $(I + \mathscr{A}_m^i + \mathscr{A}_{m+M}^e)$ to the last identity and get

$$(I + \mathscr{A}_m^i + \mathscr{A}_{m+M}^e)\Gamma \widetilde{R}_{\mathrm{MIT}}(z) = (I - \mathscr{A}_m^i \mathscr{A}_{m+M}^e - \mathscr{A}_{m+M}^e \mathscr{A}_m^i)\Gamma R_M(z) =: (\Xi_M(z))^{-1}\Gamma R_M(z),$$

where $\Xi_M(z)$ is given by (6-4). Then, thanks to Proposition 6.4, we know that, for $M > M_0$, the operator $(\Xi_M(z))^{-1}$ is bounded invertible from $H^{1/2}(\Sigma)^4$ into itself, which actually means that Ψ_M is bounded invertible from $H^{1/2}(\Sigma)^4$ into itself, and that

$$\Psi_M^{-1} = \Xi_M(z)(I + \mathscr{A}_m^i + \mathscr{A}_{m+M}^e).$$

From this, it follows that

$$\Gamma R_M(z) = \Psi_M^{-1}(z)\Gamma \widetilde{R}_{\mathrm{MIT}}(z).$$

Substituting this into formula (6-7) yields

$$R_M(z) = \widetilde{R}_{\mathrm{MIT}}(z) + E_M(z)\Psi_M^{-1}(z)\Gamma \widetilde{R}_{\mathrm{MIT}}(z),$$

which achieves the proof of the theorem. □

As an immediate consequence of Theorem 6.2 and Proposition 6.4 we have the following.

Corollary 6.5. *There is $M_0 > 0$ such that, for every $M > M_0$ and all $z \in \rho(H_{\text{MIT}}(m)) \cap \rho(H_M)$, the operators $\Xi_M^{\pm}(z) : P_{\pm} H^s(\Sigma)^4 \to P_{\pm} H^s(\Sigma)^4$ defined by*

$$\Xi_M^{+}(z) = (I - \mathscr{A}_m^i \mathscr{A}_{m+M}^e)^{-1} \quad \text{and} \quad \Xi_M^{-}(z) = (I - \mathscr{A}_{m+M}^e \mathscr{A}_m^i)^{-1}$$

are everywhere defined and bounded for any $s \in \mathbb{R}$, and

$$\| \Xi_M^{\pm}(z) \|_{P_{\pm} H^s(\Sigma)^4 \to P_{\pm} H^s(\Sigma)^4} \lesssim 1$$

uniformly with respect to $M > M_0$.

Moreover, if $v \in H^1(\mathbb{R}^3)^4$ solves $(D_m + M\beta 1_{\Omega_e} - z)v = e_{\Omega_i} f$, for some $f \in L^2(\Omega_i)^4$, then $r_{\Omega_i} v$ satisfies the boundary value problem

$$\begin{cases} (D_m - z) r_{\Omega_i} v = f & \text{in } \Omega_i, \\ \Gamma_- v = \Xi_M^{-}(z) \mathscr{A}_{m+M}^e \Gamma_+ R_{\text{MIT}}(z) f & \text{on } \Sigma, \\ \Gamma_+ v = \Gamma_+ R_{\text{MIT}}(z) f + \mathscr{A}_m^i \Gamma_- v & \text{on } \Sigma. \end{cases} \tag{6-9}$$

Proof. We first note that $\Xi_M^{\pm}(z) = P_{\pm} \Xi_M(z) P_{\pm}$. Thus, the first statement follows immediately from Proposition 6.4 . Now, let $f \in L^2(\Omega_i)^4$, and suppose that $v \in H^1(\mathbb{R}^3)^4$ solves $(D_m + M\beta 1_{\Omega_e} - z)v = e_{\Omega_i} f$. Thus $(D_m - z) r_{\Omega_i} v = f$ in Ω_i, and if we set

$$\varphi = P_- t_\Sigma v \quad \text{and} \quad \psi = P_+ t_\Sigma v,$$

then, from (6-8), we easily get

$$\varphi = \Xi_M^{-}(z) \mathscr{A}_{m+M}^e \Gamma_+ R_{\text{MIT}}(z) f \quad \text{and} \quad \psi = \Gamma_+ R_{\text{MIT}}(z) f + \mathscr{A}_m^i \varphi,$$

which means that $r_{\Omega_i} v$ satisfies (6-9). □

Remark 6.6. Notice, from (6-8) and Corollary 6.5, we have

$$\begin{pmatrix} \Gamma_+ r_{\Omega_i} R_M(z) f \\ \Gamma_- r_{\Omega_e} R_M(z) f \end{pmatrix} = \begin{pmatrix} \Xi_M^{+}(z) & 0 \\ 0 & \Xi_M^{-}(z) \end{pmatrix} \begin{pmatrix} I_4 & \mathscr{A}_m^i \\ \mathscr{A}_{m+M}^e & I_4 \end{pmatrix} \begin{pmatrix} \Gamma_+ R_{\text{MIT}}(z) r_{\Omega_i} f \\ \Gamma_- \widetilde{R}_M(z) r_{\Omega_e} f \end{pmatrix}.$$

With this observation, we remark that the resolvent formula (6-3) can also be written in the following matrix form:

$$\begin{pmatrix} r_{\Omega_i} R_M(z) \\ r_{\Omega_e} R_M(z) \end{pmatrix} = \begin{pmatrix} R_{\text{MIT}}(z) r_{\Omega_i} \\ \widetilde{R}_M(z) r_{\Omega_e} \end{pmatrix} + \begin{pmatrix} E_m^{\Omega_i}(z) \Xi_M^{-}(z) \mathscr{A}_{m+M}^e & E_m^{\Omega_i}(z) \Xi_M^{-}(z) \\ E_{m+M}^{\Omega_e}(z) \Xi_M^{+}(z) & E_{m+M}^{\Omega_e}(z) \Xi_M^{+}(z) \mathscr{A}_m^i \end{pmatrix} \begin{pmatrix} \Gamma_+ R_{\text{MIT}}(z) r_{\Omega_i} \\ \Gamma_- \widetilde{R}_M(z) r_{\Omega_e} \end{pmatrix}.$$

An inspection of the proof of Theorem 6.2 shows that, for any $M > 0$, $z \in \rho(H_{\text{MIT}}(m)) \cap \rho(H_M)$, and $f \in L^2(\mathbb{R}^3)^4$, one has

$$\Gamma \widetilde{R}_{\text{MIT}}(z) f = \Psi_M(z) \Gamma R_M(z) f. \tag{6-10}$$

When f runs through the whole space $L^2(\mathbb{R}^3)^4$, then the values of $\Gamma \widetilde{R}_{\text{MIT}}(z) f$ and $\Gamma R_M(z) f$ cover the whole space $H^{1/2}(\Sigma)^4$, which means that $\text{Rn}(\Psi_M(z)) = H^{1/2}(\Sigma)^4$. Hence, if one proves that $\text{Kr}(\Psi_M(z)) = \{0\}$, then $\Psi_M(z)$ would be boundedly invertible in $H^{1/2}(\Sigma)^4$, and thus (6-3) holds without restriction on $M > 0$. The following theorem provides a Birman–Schwinger-type principle relating $\text{Kr}(H_M - z)$ with $\text{Kr}(\Psi_M(z))$ and allows us to recover the resolvent formula (6-3) for any $M > 0$.

Theorem 6.7. *Let* $M > 0$, *and let* Ψ_M *be as in Theorem 6.2. Then, the following hold*:

(i) *For any* $a \in (-(m+M), m+M) \cap \rho(H_{\mathrm{MIT}}(m))$, *we have* $a \in \mathrm{Sp}_p(H_M) \Leftrightarrow 0 \in \mathrm{Sp}_p(\Psi_M(a))$ *and*

$$\mathrm{Kr}(H_M - a) = \{E_M(a)g : g \in \mathrm{Kr}(\Psi_M(a))\}.$$

In particular, $\dim \mathrm{Kr}(H_M - a) = \dim \mathrm{Kr}(\Psi_M(a))$ *for all* $a \in (-(m+M), m+M) \cap \rho(H_{\mathrm{MIT}}(m))$.

(ii) *The operator* $\Psi_M(z)$ *is boundedly invertible in* $H^{1/2}(\Sigma)^4$ *for all* $z \in \rho(H_{\mathrm{MIT}}(m)) \cap \rho(H_M)$, *and the following resolvent formula holds*:

$$R_M(z) = \widetilde{R}_{\mathrm{MIT}}(z) + E_M(z)\Psi_M^{-1}(z)\Gamma \widetilde{R}_{\mathrm{MIT}}(z). \tag{6-11}$$

Proof. (i) Let us first prove the implication ($\Rightarrow$). Let $a \in (-(m+M), m+M) \cap \rho(H_{\mathrm{MIT}}(m))$ be such that $(H_M - a)\varphi = 0$ for some $0 \neq \varphi \in H^1(\mathbb{R}^3)^4$. Set $\varphi_+ = \varphi_{|\Omega_i}$ and $\varphi_- = \varphi_{|\Omega_e}$. Then, it is clear that φ_+ solves the system (6-1) for $z = a$ with $g = \Gamma_-\varphi$, and φ_- solves the system (6-2) with $h = \Gamma_+\varphi$. Thus, $\varphi_+ = E_m^{\Omega_i}(a)\Gamma_-\varphi$ and $\varphi_- = E_{m+M}^{\Omega_e}(a)\Gamma_+\varphi$. Hence, $\varphi = E_M(a)t_\Sigma\varphi$ and $\Gamma_\pm\varphi \neq 0$, as otherwise φ would be zero. Using this and the definition of the Poincaré–Steklov operators, we obtain

$$(I_4 + \mathscr{A}_m^i)\Gamma_-\varphi =: t_\Sigma\varphi_+ = t_\Sigma\varphi = t_\Sigma\varphi_- := (I_4 + \mathscr{A}_{m+M}^e)\Gamma_+\varphi,$$

and, since $t_\Sigma\varphi \neq 0$, it follows that

$$\Psi_M(a)t_\Sigma\varphi = (I_4 - \mathscr{A}_m^i - \mathscr{A}_{m+M}^e)t_\Sigma\varphi = 0,$$

which means that $0 \in \mathrm{Sp}_p(\Psi_M(a))$ and proves the inclusion $\mathrm{Kr}(H_M - a) \subset \{E_M(a)g : g \in \mathrm{Kr}(\Psi_M(a))\}$.

Now, we turn to the proof of the implication ($\Leftarrow$). Let $a \in (-(m+M), m+M) \cap \rho(H_{\mathrm{MIT}}(m))$ and assume that 0 is an eigenvalue of $\Psi_M(a)$. Then, there is $g \in H^{1/2}(\Sigma)^4 \setminus \{0\}$ such that $\Psi_M(a)g = 0$ on Σ. Note that this is equivalent to

$$(P_- + \mathscr{A}_m^i)g = (P_+ + \mathscr{A}_{m+M}^e)g. \tag{6-12}$$

Since $a \in (-(m+M), m+M) \cap \rho(H_{\mathrm{MIT}}(m))$, the operators $E_m^{\Omega_i}(a) : P_- H^{1/2}(\Sigma)^4 \to H^1(\Omega_i)^4$ and $E_{m+M}^{\Omega_e}(a) : P_+ H^{1/2}(\Sigma)^4 \to H^1(\Omega_e)^4$ are well defined and bounded. Thus, if we let $\varphi = E_M(a)g = (E_m^{\Omega_i}(a)P_-g, E_{m+M}^{\Omega_e}(a)P_+g)$, then $\varphi \neq 0$ and we have that $(D_m - a)\varphi = 0$ in Ω_i and that $(D_{m+M} - a)\varphi = 0$ in Ω_e. Hence, it remains to show that $\varphi \in H^1(\mathbb{R}^3)^4$. For this, observe that, by (6-12), we have

$$t_\Sigma E_m^{\Omega_i}(a)P_-g = (P_- + \mathscr{A}_m^i)g = (P_+ + \mathscr{A}_{m+M}^e)g = t_\Sigma E_{m+M}^{\Omega_e}(a)P_+g.$$

Thanks to the boundedness properties of $E_m^{\Omega_i}(a)$ and $E_{m+M}^{\Omega_e}(a)$, it follows from the above computations that $\varphi = E_M(a)g \in H^1(\mathbb{R}^3)^4 \setminus \{0\}$ and φ satisfies the equation $(H_M - a)\varphi = 0$. Therefore, $a \in \mathrm{Sp}_p(H_M)$, and the inclusion $\{E_M(a)g : g \in \mathrm{Kr}(\Psi_M(a))\} \subset \mathrm{Kr}(H_M - a)$ holds, which completes the proof of (i).

(ii) Let $z \in \rho(H_{\mathrm{MIT}}(m)) \cap \rho(H_M)$, and note that the self-adjointness of H_M together with assertion (i) imply that $\mathrm{Kr}(\Psi_M(z)) = \{0\}$, as otherwise $\mathrm{Kr}(H_M - z) \neq \{0\}$. Since $\mathrm{Rn}(\Psi_M(z)) = H^{1/2}(\Sigma)^4$ holds for all $z \in \rho(H_{\mathrm{MIT}}(m)) \cap \rho(H_M)$, it follows that $\Psi_M(z)$ admits a bounded and everywhere defined inverse in $H^{1/2}(\Sigma)^4$. Therefore, (6-10) yields $\Gamma R_M(z) = \Psi_M^{-1}(z)\Gamma\widetilde{R}_{\mathrm{MIT}}(z)$, and the resolvent formula (6-11) follows from this and (6-7). □

Remark 6.8. Note the different nature of Theorems 6.2 and 6.7: the second ensures the invertibility of Ψ_M and yields the resolvent formula (6-11) without assumption, while the first is based on a largeness assumption that allows us (thanks to the semiclassical properties of PS operators) to obtain the explicit formula of the operator $(\Psi_M)^{-1}$. Note that in Theorem 6.7 we do not know a priori whether $(\Psi_M)^{-1}$ is uniformly bounded when M is large, and hence (6-11) is not suitable for studying the large coupling convergence.

In the next proposition we prove the norm convergence of $R_M(z)$ toward $R_{\mathrm{MIT}}(z)$ and estimate the rate of convergence.

Proposition 6.9. *For any compact set $K \subset \rho(H_{\mathrm{MIT}}(m))$, there is $M_0 > 0$ such that, for all $M > M_0$, we have $K \subset \rho(H_M)$ and, for all $z \in K$, the resolvent R_M admits an asymptotic expansion in $\mathcal{L}(L^2(\mathbb{R}^3)^4)$ of the form*

$$R_M(z) = e_{\Omega_i} R_{\mathrm{MIT}}(z) r_{\Omega_i} + \frac{1}{M}(K_M(z) + L_M(z)), \tag{6-13}$$

where $K_M(z), L_M(z) : L^2(\mathbb{R}^3)^4 \to L^2(\mathbb{R}^3)^4$ are uniformly bounded with respect to M and satisfy

$$r_{\Omega_i} K_M(z) e_{\Omega_i} = 0 = r_{\Omega_e} K_M(z) e_{\Omega_e}.$$

In particular,

$$\| R_M(z) - e_{\Omega_i} R_{\mathrm{MIT}}(z) r_{\Omega_i} \|_{L^2(\mathbb{R}^3)^4 \to L^2(\mathbb{R}^3)^4} = \mathcal{O}\Big(\frac{1}{M}\Big). \tag{6-14}$$

Before giving the proof, we need the following estimates.

Lemma 6.10. *Let $K \subset \mathbb{C}$ be a compact set. Then, there is $M_0 > 0$ such that, for all $M > M_0$, we have $K \subset \rho(\widetilde{H}_M)$ and, for every $z \in K$, the following estimates hold:*

$$\begin{aligned}
\|\widetilde{R}_M(z) f\|_{L^2(\Omega_e)^4} + \frac{1}{\sqrt{M}} \|\Gamma_- \widetilde{R}_M(z) f\|_{L^2(\Sigma)^4} &\lesssim \frac{1}{M} \|f\|_{L^2(\Omega_e)^4} && \text{for all } f \in L^2(\Omega_e)^4, \\
\|\Gamma_- \widetilde{R}_M(z) f\|_{H^{-1/2}(\Sigma)^4} &\lesssim \frac{1}{M} \|f\|_{L^2(\Omega_e)^4} && \text{for all } f \in L^2(\Omega_e)^4, \\
\|E^{\Omega_e}_{m+M}(z)\psi\|_{L^2(\Omega_e)^4} &\lesssim \frac{1}{\sqrt{M}} \|\psi\|_{L^2(\Sigma)^4} && \text{for all } \psi \in P_+ L^2(\Sigma)^4, \\
\|E^{\Omega_e}_{m+M}(z)\psi\|_{L^2(\Omega_e)^4} &\lesssim \frac{1}{M} \|\psi\|_{H^{1/2}(\Sigma)^4} && \text{for all } \psi \in P_+ H^{1/2}(\Sigma)^4.
\end{aligned}$$

Proof. Fix a compact set $K \subset \mathbb{C}$, and note that, for $M_1 > \sup_{z \in K}\{|\mathrm{Re}(z)| - m\}$, we have $K \subset \rho(D_{m+M_1})$, and hence, $K \subset \rho(\widetilde{H}_M)$ for all $M > M_1$. We next show the claimed estimates for $\widetilde{R}_M(z)$ and $\Gamma_- \widetilde{R}_M(z)$. For this, let $z \in K$, and assume that $M > M_1$. Let $\varphi \in \mathrm{dom}(\widetilde{H}_M)$. Then a straightforward application of Green's formula yields

$$\|\widetilde{H}_M \varphi\|^2_{L^2(\Omega_e)^4} = \|(\alpha \cdot \nabla)\varphi\|^2_{L^2(\Omega_e)^4} + (m+M)^2 \|\varphi\|^2_{L^2(\Omega_e)^4} + (m+M)\|P_- t_\Sigma \varphi\|^2_{L^2(\Sigma)^4}.$$

Using this and the Cauchy–Schwarz inequality we obtain

$$\begin{aligned}
\|(\widetilde{H}_M - z)\varphi\|^2_{L^2(\Omega_e)^4} &= \|\widetilde{H}_M \varphi\|^2_{L^2(\Omega_e)^4} + |z|^2 \|\varphi\|^2_{L^2(\Omega_e)^4} - 2\,\mathrm{Re}(z) \langle \widetilde{H}_M \varphi, \varphi \rangle_{L^2(\Omega_e)^4} \\
&\geqslant \|\widetilde{H}_M \varphi\|^2_{L^2(\Omega_e)^4} + |z|^2 \|\varphi\|^2_{L^2(\Omega_e)^4} - \tfrac{1}{2} \|\widetilde{H}_M \varphi\|^2_{L^2(\Omega_e)^4} - 2|\mathrm{Re}(z)|^2 \|\varphi\|^2_{L^2(\Omega_e)^4} \\
&\geqslant \big(\tfrac{1}{2}(m+M)^2 + |\mathrm{Im}(z)|^2 - |\mathrm{Re}(z)|^2\big) \|\varphi\|^2_{L^2(\Omega_e)^4} + \tfrac{1}{2} M \|P_- t_\Sigma \varphi\|^2_{L^2(\Sigma)^4}.
\end{aligned}$$

Therefore, taking $\widetilde{R}_M(z)f=\varphi$ and $M\geqslant M_2\geqslant \sup_{z\in K}\{\sqrt{|\mathrm{Re}(z)|^2-|\mathrm{Im}(z)|^2}-m\}$, we obtain the inequality

$$\|\widetilde{R}_M(z)f\|_{L^2(\Omega_e)^4}+\frac{1}{\sqrt{M}}\|\Gamma_-\widetilde{R}_M(z)f\|_{L^2(\Sigma)^4}\lesssim\frac{1}{M}\|f\|_{L^2(\Omega_e)^4}.$$

Since Γ_- is bounded from $L^2(\Omega_e)^4$ into $H^{-1/2}(\Sigma)^4$, it follows from the above inequality that

$$\|\Gamma_-\widetilde{R}_M(z)f\|_{H^{-1/2}(\Sigma)^4}\lesssim\|\Gamma_-\|_{L^2(\Omega_e)^4\to H^{-1/2}(\Sigma)^4}\|\widetilde{R}_M(z)f\|_{L^2(\Omega_e)^4}\lesssim\frac{1}{M}\|f\|_{L^2(\Omega_e)^4}$$

for any $f\in L^2(\Omega_e)^4$, which gives the second inequality.

Let us now turn to the proof of the claimed estimates for $E^{\Omega_e}_{m+M}(z)$. Let $\psi\in P_+L^2(\Sigma)^4$. Then, from the proof of Proposition 4.2, we have

$$\|\psi\|^2_{L^2(\Sigma)^4}\geqslant(m+M)\|E^{\Omega_e}_{m+M}(z)\psi\|^2_{L^2(\Omega_e)^4}-2|\mathrm{Re}(z)|\,\|E^{\Omega_e}_{m+M}(z)\psi\|^2_{L^2(\Omega_e)^4}.$$

Thus, for any $M\geqslant M_3\geqslant\sup_{z\in K}\{4|\mathrm{Re}(z)|-m\}$, we get

$$M\|E^{\Omega_e}_{m+M}(z)\psi\|^2_{L^2(\Omega_e)^4}\leqslant 2\|\psi\|^2_{L^2(\Sigma)^4},$$

and this proves the first estimate for $E^{\Omega_e}_{m+M}(z)$. Finally, the last inequality is a consequence of the first one and Proposition 4.2. Indeed, from Proposition 4.2(ii), we know that $\beta\Gamma_-\widetilde{R}_M(\bar z)$ is the adjoint of the operator $E^{\Omega_e}_{m+M}(z):P_+H^{1/2}(\Sigma)^4\to L^2(\Omega_e)^4$. Using this and the estimate fulfilled by $\Gamma_-\widetilde{R}_M(\bar z)$, we obtain

$$\begin{aligned}|\langle f,E^{\Omega_e}_{m+M}(z)\psi\rangle_{L^2(\Omega_e)^4}|&=|\langle\Gamma_-\widetilde{R}_M(\bar z)f,\beta\psi\rangle_{H^{-1/2}(\Sigma)^4,H^{1/2}(\Sigma)^4}|\\&\leqslant\|\Gamma_-\widetilde{R}_M(z)f\|_{H^{-1/2}(\Sigma)^4}\|\psi\|_{H^{1/2}(\Sigma)^4}\lesssim\frac{1}{M}\|f\|_{L^2(\Omega_e)^4}\|\psi\|_{H^{1/2}(\Sigma)^4}.\end{aligned}$$

Since this is true for all $f\in L^2(\Omega_e)^4$, by duality arguments, it follows that

$$\|E^{\Omega_e}_{m+M}(z)\psi\|_{L^2(\Omega_e)^4}\lesssim\frac{1}{M}\|\psi\|_{H^{1/2}(\Sigma)^4}\quad\text{for all }\psi\in P_+H^{1/2}(\Sigma)^4,$$

which proves the last inequality. Hence, the lemma follows by taking $M_0=\max\{M_1,M_2,M_3\}$. □

Proof of Proposition 6.9. We first show (6-14) for some $M_0'>0$ and any $z\in\mathbb{C}\setminus\mathbb{R}$. So, let us fix such a z, and let $f\in L^2(\mathbb{R}^3)^4$. Then, it is clear that $z\in\rho(H_{\mathrm{MIT}}(m))\cap\rho(H_M)$, and, from Theorem 6.2 and Remark 6.6, we know that there is $M_0'>0$ such that, for all $M>M_0'$,

$$\begin{aligned}&\|(R_M(z)-e_{\Omega_i}R_{\mathrm{MIT}}(z)r_{\Omega_i})f\|_{L^2(\mathbb{R}^3)^4}\\&\quad\leqslant\|E^{\Omega_i}_m(z)\Xi^-_M(z)\mathscr{A}^e_{m+M}\Gamma_+R_{\mathrm{MIT}}(z)r_{\Omega_i}f\|_{L^2(\Omega_i)^4}+\|E^{\Omega_i}_m(z)\Xi^-_M(z)\Gamma_-\widetilde{R}_M(z)r_{\Omega_e}f\|_{L^2(\Omega_i)^4}\\&\qquad+\|E^{\Omega_e}_{m+M}(z)\Xi^+_M(z)\Gamma_+R_{\mathrm{MIT}}(z)r_{\Omega_i}f\|_{L^2(\Omega_e)^4}+\|E^{\Omega_e}_{m+M}(z)\Xi^+_M(z)\mathscr{A}^i_m\Gamma_-\widetilde{R}_M(z)r_{\Omega_e}f\|_{L^2(\Omega_e)^4}\\&\qquad+\|\widetilde{R}_M(z)r_{\Omega_e}f\|_{L^2(\Omega_e)^4}\\&\quad=:J_1+J_2+J_3+J_4+J_5.\end{aligned}$$

From Lemma 6.10 we immediately get $J_5\lesssim M^{-1}\|f\|$. Now notice that $\Gamma_+R_{\mathrm{MIT}}(z):L^2(\Omega_i)^4\to H^{1/2}(\Sigma)^4$, $\mathscr{A}^i_m:H^{1/2}(\Sigma)^4\to H^{1/2}(\Sigma)^4$ and $E^{\Omega_i}_m(z):H^{-1/2}(\Sigma)^4\to H(\alpha,\Omega_i)\subset L^2(\Omega_i)^4$ (where $H(\alpha,\Omega_i)$ is defined by (2-2)) are bounded operators and do not depend on M. Moreover, thanks to Corollary 6.5, we know that, for all $s \in \mathbb{R}$, there is $C > 0$ independent of M such that

$$\|\Xi_M^{\pm}(z)\|_{P_\pm H^s(\Sigma)^4 \to P_\pm H^s(\Sigma)^4} \leqslant C.$$

Using this and the above observation, for $j \in \{1,2,3,4\}$, we can estimate J_k as follows:

$$\begin{aligned}
J_1 &\lesssim \|E_m^{\Omega_i}(z)\|_{P_- H^{-1/2}(\Sigma)^4 \to L^2(\Omega_i)^4} \|\mathscr{A}_{m+M}^e\|_{H^{1/2}(\Sigma)^4 \to H^{-1/2}(\Sigma)^4} \|\Gamma_+ R_{\mathrm{MIT}}(z) r_{\Omega_i} f\|_{H^{1/2}(\Sigma)^4},\\
J_2 &\lesssim \|E_m^{\Omega_i}(z)\|_{H^{-1/2}(\Sigma)^4 \to L^2(\Omega_i)^4} \|\Gamma_- \widetilde{R}_M(z) r_{\Omega_e} f\|_{H^{-1/2}(\Sigma)^4},\\
J_3 &\lesssim \|E_{m+M}^{\Omega_e}(z)\|_{H^{1/2}(\Sigma)^4 \to L^2(\Omega_e)^4} \|\Gamma_+ R_{\mathrm{MIT}}(z) r_{\Omega_i} f\|_{H^{1/2}(\Sigma)^4},\\
J_4 &\lesssim \|E_{m+M}^{\Omega_e}(z)\|_{L^2(\Sigma)^4 \to L^2(\Omega_e)^4} \|\mathscr{A}_m^i\|_{L^2(\Sigma)^4 \to L^2(\Sigma)^4} \|\Gamma_- \widetilde{R}_M(z) r_{\Omega_e} f\|_{L^2(\Sigma)^4}.
\end{aligned}$$

Therefore, Proposition 6.4(ii) together with Lemma 6.10 yield

$$J_k \lesssim \frac{1}{M}\|f\|_{L^2(\mathbb{R}^3)^4} \quad \text{for any } j \in \{1,2,3,4\}.$$

Thus, we obtain the estimate

$$\|(R_M(z) - e_{\Omega_i} R_{\mathrm{MIT}}(z) r_{\Omega_i}) f\|_{L^2(\mathbb{R}^3)^4} \leqslant \frac{C}{M}\|f\|_{L^2(\mathbb{R}^3)^4}. \tag{6-15}$$

Moreover, the asymptotic expansion (6-13) holds with

$$\begin{aligned}
L_M(z) = M\big(e_{\Omega_e}\widetilde{R}_M(z) r_{\Omega_e} &+ e_{\Omega_i} E_m^{\Omega_i}(z)\,\Xi_M^-(z)\mathscr{A}_{m+M}^e \Gamma_+ R_{\mathrm{MIT}}(z) r_{\Omega_i}\\
&+ e_{\Omega_e} E_{m+M}^{\Omega_e}(z)\,\Xi_M^+(z)\mathscr{A}_m^i \Gamma_- \widetilde{R}_M(z) r_{\Omega_e}\big),
\end{aligned}$$

and

$$K_M(z) = M\big(e_{\Omega_i} E_m^{\Omega_i}(z)\,\Xi_M^-(z)\Gamma_- \widetilde{R}_M(z) r_{\Omega_e} + e_{\Omega_e} E_{m+M}^{\Omega_e}(z)\,\Xi_M^+(z)\Gamma_+ R_{\mathrm{MIT}}(z) r_{\Omega_i}\big),$$

and we clearly see that $r_{\Omega_i} K_M(z) e_{\Omega_i} = 0 = r_{\Omega_e} K_M(z) e_{\Omega_e}$.

Finally, since (6-15) holds true for every $z \in \mathbb{C} \setminus \mathbb{R}$, for any fixed compact subset $K \subset \rho(H_{\mathrm{MIT}}(m))$, one can show by arguments similar to those in the proof of [Barbaroux et al. 2019, Lemma A.1] that there is $M_0 > M_0'$ such that $K \subset \rho(H_M)$. The proposition follows from the same arguments as before. □

6A. *Comments and further remarks.* In this part we discuss possible generalizations of our results and comment on the usefulness of the pseudodifferential properties of the Poincaré–Steklov operators.

(1) First note that all the results in this article which are proved without the use of the (semi) classical properties of the Poincaré–Steklov operator are valid when Σ is just $C^{1,\omega}$-smooth with $\omega \in \left(\frac{1}{2}, 1\right)$, and can also be generalized without difficulty to the case of local deformation of the plane $\mathbb{R}^2 \times \{0\}$ (see [Benhellal 2022b] where the self-adjointness of $H_{\mathrm{MIT}}(m)$ and the regularity properties of $\Phi_{z,m}^{\Omega}$, $\mathscr{C}_{z,m}$, and Λ_m^z were shown for this case). We mention, however, that in the latter case the spectrum of the MIT bag operator is equal to that of the free Dirac operator; see [Benhellal 2022b, Theorem 4.1].

(2) It should also be noted that there are several boundary conditions that lead to self-adjoint realizations of the Dirac operator on domains (see, e.g., [Arrizabalaga et al. 2023; Behrndt et al. 2020; Benhellal 2022a]) and for which the associated PS operators can be analyzed in a similar way as for the MIT

bag model. In particular, one can consider the PS operator $\mathscr{B}_m(z)$ associated with the self-adjoint Dirac operator

$$\widetilde{H}_{\mathrm{MIT}}(m)v = D_m v \quad \text{for all } v \in \mathrm{dom}(\widetilde{H}_{\mathrm{MIT}}(m)) := \{v \in H^1(\Omega_i)^4 : P_+ t_\Sigma v = 0 \text{ on } \Sigma\}.$$

According to the previous considerations, this operator can be viewed as an analogue of the Neumann-to-Dirichlet map for the Dirac operator. Moreover, the same arguments as in the proof of Theorem 4.5 show that

$$\mathscr{B}_m(z) = \frac{1}{\sqrt{-\Delta_\Sigma}} S \cdot (\nabla_\Sigma \wedge n) P_+ \mod \mathrm{Op}\,\mathcal{S}^{-1}(\Sigma) = \frac{D_\Sigma}{\sqrt{-\Delta_\Sigma}} P_+ \mod \mathrm{Op}\,\mathcal{S}^{-1}(\Sigma)$$

for all $z \in \rho(D_m) \cap \rho(\widetilde{H}_{\mathrm{MIT}}(m))$.

(3) As already mentioned in the introduction, in [Barbaroux et al. 2019], it was shown that (in the two-dimensional massless case) the norm resolvent convergence of H_M to $H_{\mathrm{MIT}}(m)$ holds with a convergence rate of $M^{-1/2}$. Their proof is based on two main ingredients: the first is a resolvent identity (see [Barbaroux et al. 2019, Lemma 2.2] for the exact formula), and the second is the inequality

$$\|\Gamma_- R_M(z) f\|_{L^2(\Sigma)^4} \lesssim \frac{1}{\sqrt{M}} \|f\|_{L^2(\mathbb{R}^3)^4}, \tag{6-16}$$

which is a consequence of the lower bound

$$\|\nabla\psi\|^2_{L^2(\Omega_e)^4} + M^2\|\psi\|^2_{L^2(\Omega_e)^4} \geqslant (M - C)\|t_\Sigma \psi\|^2_{L^2(\Sigma)^4},$$

which holds for all $\psi \in H^1(\mathbb{R}^3)^4$ and M large enough (see [Stockmeyer and Vugalter 2019, Lemma 4] for the proof in the two-dimensional case, and [Arrizabalaga et al. 2019, Proposition 2.1(i)] for the three-dimensional case). Note that the resolvent formula (6-7) together with (6-16) yield the same result. Indeed, from (6-6) and (6-16), we easily get the inequality

$$\|\Gamma_+ R_M(z) f\|_{L^2(\Sigma)^4} \lesssim \|f\|_{L^2(\mathbb{R}^3)^4}.$$

This together with (6-7) and Lemma 6.10 yield

$$\begin{aligned}
&\|(R_M(z) - e_{\Omega_i} R_{\mathrm{MIT}}(z) r_{\Omega_i}) f\|_{L^2(\mathbb{R}^3)^4} \\
&\qquad \leqslant \|E_m^{\Omega_i}(z)\Gamma_- r_{\Omega_e} R_M(z) f\|_{L^2(\Omega_i)^4} + \|\widetilde{R}_M(z) r_{\Omega_e} f\|_{L^2(\Omega_e)^4} + \|E_{m+M}^{\Omega_e}(z)\Gamma_+ r_{\Omega_i} R_M(z) f\|_{L^2(\Omega_e)^4} \\
&\qquad \lesssim \frac{1}{\sqrt{M}} \|f\|_{L^2(\mathbb{R}^3)^4}.
\end{aligned}$$

(4) Finally, let us point out that a first order asymptotic expansion of the eigenvalues of H_M in terms of the eigenvalues of $H_{\mathrm{MIT}}(m)$ was established in [Arrizabalaga et al. 2019] when $M \to \infty$. In their proof, the authors used the min-max characterization and optimization techniques. Note that it is also possible to obtain such a result using the properties of the PS operator, the Krein formula from Theorem 6.2, and the finite-dimensional perturbation theory (see [Kato 1966] for example); see, e.g., [Benhellal 2019; Bruneau and Carbou 2002] for similar arguments. Note also that the asymptotic expansion of the eigenvalues of H_M depends only on the term $E_m^{\Omega_i}(z)\Xi_M^-(z)\mathscr{A}_{m+M}^e \Gamma_+ R_{\mathrm{MIT}}(z) r_{\Omega_i}$. Indeed, let λ_{MIT} be

an eigenvalue of $H_{\mathrm{MIT}}(m)$ with multiplicity l, and let $(f_1, \dots, f_l)$ be an $L^2(\Omega_i)^4$-orthonormal basis of $\mathrm{Kr}(H_{\mathrm{MIT}}(m) - \lambda_{\mathrm{MIT}} I_4)$. Then, using the explicit resolvent formula from Remark 6.6, we see that

$$\begin{aligned}\langle R_M(z)e_{\Omega_i} f_k, e_{\Omega_i} f_j\rangle_{L^2(\mathbb{R}^3)^4} &= \langle E_m^{\Omega_i}(z)\,\Xi_M^-(z)\mathscr{A}_{m+M}^e\Gamma_+ R_{\mathrm{MIT}}(z) f_k, f_j\rangle_{L^2(\Omega_i)^4}\\ &= \langle \Xi_M^-(z)\mathscr{A}_{m+M}^e\Gamma_+ R_{\mathrm{MIT}}(z) f_k, -\beta\Gamma_+ R_{\mathrm{MIT}}(\bar z) f_j\rangle_{L^2(\Sigma)^4}\\ &= \frac{1}{(z-\lambda_{\mathrm{MIT}})^2}\langle \Xi_M^-(z)\mathscr{A}_{m+M}^e\Gamma_+ f_k, -\beta\Gamma_+ f_j\rangle_{L^2(\Sigma)^4},\end{aligned}$$

which means that $E_m^{\Omega_i}(z)\,\Xi_M^-(z)\mathscr{A}_{m+M}^e\Gamma_+ R_{\mathrm{MIT}}(z) r_{\Omega_i}$ is the only term that intervenes in the asymptotic expansion of the eigenvalues of H_M. Besides, recall that the principal symbol of $\Xi_M^-(z)\mathscr{A}_{m+M}^e$ is given by

$$q_M(x,\xi) = -\frac{S\cdot(\xi\wedge n(x))P_+}{\sqrt{|\xi\wedge n(x)|^2+(m+M)^2}+|\xi\wedge n(x)|+(m+M)},$$

and, for $M > 0$ large enough, one has

$$q_M(x,\xi) = -\frac{1}{2M} S\cdot(\xi\wedge n(x))P_+ \sum_{l=1}^{\infty}\frac{1}{M^{l+1}} p_l(x,\xi)P_+, \qquad p_l \in \mathcal{S}^{-l}.$$

Using this, we formally deduce that, for sufficiently large M, H_M has exactly l eigenvalues $(\lambda_k^M)_{1\leqslant k\leqslant l}$ counted according to their multiplicities (in $B(\lambda_{\mathrm{MIT}}, \eta)$, with $B(\lambda_{\mathrm{MIT}}, \eta)\cap \mathrm{Sp}(H_{\mathrm{MIT}}(m)) = \{\lambda_{\mathrm{MIT}}\}$) and these eigenvalues admit an asymptotic expansion of the form

$$\lambda_k^M = \lambda_{\mathrm{MIT}} + \frac{1}{M}\mu_k + \sum_{j=2}^{N}\frac{1}{M^j}\mu_k^j + \mathcal{O}(M^{-(N+1)}), \tag{6-17}$$

where $(\mu_k)_{1\leqslant k\leqslant l}$ are the eigenvalues of the matrix $\mathcal{M}$ with coefficients

$$m_{kj} = \tfrac{1}{2}\langle \beta\,\mathrm{Op}(S\cdot(\xi\wedge n(x)))\Gamma_+ f_k, \Gamma_+ f_j\rangle_{L^2(\Sigma)^4}.$$

Appendix: Dirac algebra and applications

In this appendix, we recall the anticommutation relations of Dirac matrices and give formulas used in the paper. Consider the 4×4-Hermitian Dirac matrices α_j, $j = 1, 2, 3$, and β, whose possible representation is given at the beginning of the paper. These Dirac matrices satisfy the anticommutation relations

$$\{\alpha_j, \alpha_k\} = 2\delta_{jk} I_4, \quad \{\alpha_j, \beta\} = 0, \quad \beta^2 = I_4, \quad j, k \in \{1, 2, 3\}, \tag{A-1}$$

where we recall that $\{\cdot,\cdot\}$ is the anticommutator bracket.

Recall the definition of the spin angular momentum S and the matrix γ_5 (see (2-13)), and note that, by (A-1), we have $S = (i\alpha_2\alpha_3, -i\alpha_1\alpha_3, i\alpha_1\alpha_2)$.

Using the anticommutation relations (A-1), we easily get the following identities for all $X, Y \in \mathbb{R}^3$:

$$\begin{aligned} i(\alpha\cdot X)(\alpha\cdot Y) &= iX\cdot Y + S\cdot(X\wedge Y), & [\gamma_5, \alpha\cdot X] &= 0,\\ \{S\cdot X, \alpha\cdot Y\} &= -2(X\cdot Y)\gamma_5, & [S\cdot X, \beta] &= 0.\end{aligned} \tag{A-2}$$

Let us now give some relations we have used for n, a normal vector field to a smooth domain $\Omega\subset\mathbb{R}^3$, and for τ, a tangent vector, in particular for $\tau = n\wedge\xi$, where ξ is a Fourier variable.

Lemma A.1. *Let $n \in \mathbb{R}^3$, and let $\tau \in \mathbb{R}^3$ be such that $\tau \perp n$. Then the following identity holds*:

$$(S \cdot \tau + i(\alpha \cdot n)\beta)^2 = (|\tau|^2 + |n|^2)I_4.$$

Proof. Using the relations (A-1) and (A-2), we get

$$(S \cdot \tau)^2 = \gamma_5(\alpha \cdot \tau)\gamma_5(\alpha \cdot \tau) = (\gamma_5)^2(\alpha \cdot \tau)^2 = |\tau|^2 I_4.$$

Then we have

$$(S \cdot \tau + i(\alpha \cdot n)\beta)^2 = |\tau|^2 I_4 - ((\alpha \cdot n)\beta)^2 + i\{S \cdot \tau, (\alpha \cdot n)\beta\} = (|\tau|^2 + |n|^2)I_4 + i\{S \cdot \tau, (\alpha \cdot n)\beta\},$$

and since $\tau \cdot n = 0$, by (A-2), we obtain

$$\{S \cdot \tau, (\alpha \cdot n)\beta\} = \{S \cdot \tau, \alpha \cdot n\}\beta + \alpha \cdot n[S \cdot \tau, \beta] = 0,$$

and the conclusion follows. □

Proposition A.2. *For $\xi \in \mathbb{R}^3$ and $n \in \mathbb{R}^3$ such that $|n| = 1$, we define the matrix-valued function*

$$l_0(n, \xi) = i(\alpha \cdot n)(\alpha \cdot \xi + \beta).$$

Then $l_0(n, \xi)$ has two eigenvalues given by

$$\rho_\pm(n, \xi) := in \cdot \xi \pm \lambda(n, \xi), \quad \text{with } \lambda(n, \xi) = \sqrt{|n \wedge \xi|^2 + 1}.$$

The associated eigenprojections (*onto* $\mathrm{Kr}(l_0(n, \xi) - \rho_\pm(n, \xi)I_4)$) *are given by*

$$\Pi_\pm(n, \xi) := \frac{1}{2}\left(I_4 \pm \frac{S \cdot (n \wedge \xi) + i(\alpha \cdot n)\beta}{\lambda(n, \xi)}\right).$$

Proof. By applying (A-2) for $(X, Y) = (n, \xi)$, we get

$$l_0(n, \xi) = in \cdot \xi I_4 + S \cdot (n \wedge \xi) + i(\alpha \cdot n)\beta.$$

Thanks to Lemma A.1, the Hermitian matrix $h(n, \xi) := S \cdot (n \wedge \xi) + i(\alpha \cdot n)\beta$ satisfies

$$h(n, \xi)^2 = (|n \wedge \xi|^2 + 1)I_4 = \lambda(n, \xi)^2 I_4.$$

Therefore, $h(n, \xi)$ has the eigenvalues $\pm\lambda(n, \xi)$, and the associated eigenprojections are given by

$$\Pi_\pm(n, \xi) = \frac{1}{2}\left(I_4 \pm \frac{h(n, \xi)}{\lambda(n, \xi)}\right),$$

which proves the claimed results since $l_0(n, \xi) = in \cdot \xi I_4 + h(n, \xi)$. □

Lemma A.3. *Given $n \in \mathbb{R}^3$ such that $|n| = 1$, let $P_\pm = \Pi_\pm(n, 0) = \frac{1}{2}(I_4 \pm i(\alpha \cdot n)\beta)$ be the eigenprojections onto* $\mathrm{Kr}(i(\alpha \cdot n)\beta \mp I_4)$. *The following properties hold*:

(i) *For any $\tau \in \mathbb{R}^3$ such that $\tau \perp n$, we have*

$$P_\pm(S \cdot \tau) = (S \cdot \tau)P_\mp, \quad P_\pm(\alpha \cdot n) = (\alpha \cdot n)P_\mp \quad \text{and} \quad P_\pm\beta = \beta P_\mp.$$

(ii) *For any* $\xi \in \mathbb{R}^3$, *the projections* $\Pi_\pm(n,\xi)$ *defined in Proposition A.2 satisfy*

$$P_\pm \Pi_\pm P_\pm = k_+ P_\pm, \quad P_\mp \Pi_\pm P_\mp = k_- P_\pm \quad \textit{and} \quad P_\pm \Pi_\mp P_\mp = \mp \Theta P_\mp, \tag{A-3}$$

with

$$k_\pm(n,\xi) = \frac{1}{2}\left(1 \pm \frac{1}{\lambda(n,\xi)}\right), \quad \Theta(n,\xi) = \frac{1}{2\lambda(n,\xi)} S\cdot(n\wedge\xi). \tag{A-4}$$

Proof. The relations of (i) follow from (A-2). For the proof of (ii), let us write $\Pi_\pm(n,\xi)$ as

$$\Pi_\pm(n,\xi) = P_\pm \pm \frac{1}{2\lambda(x,\xi)} S\cdot(n\wedge\xi)P_\mp \pm \frac{i}{2}(\alpha\cdot n)\beta\left(\frac{1}{\lambda(n,\xi)} - 1\right).$$

Then, using item (i) of this lemma (with $\tau = n\wedge\xi$) and the fact that $P_\pm i(\alpha\cdot n)\beta = \pm P_\pm$, we get

$$P_\pm \Pi_\pm = P_\pm \pm \frac{1}{2\lambda} S\cdot(n\wedge\xi)P_\mp + \frac{1}{2}\left(\frac{1}{\lambda} - 1\right)P_\pm = k_+ P_\pm \pm \Theta P_\mp,$$

$$P_\mp \Pi_\pm = \pm\frac{1}{2\lambda} S\cdot(n\wedge\xi)P_\pm - \frac{1}{2}\left(\frac{1}{\lambda} - 1\right)P_\mp = k_- P_\mp \pm \Theta P_\pm,$$

with $k_\pm$ and Θ as in (A-4). Hence, (A-3) directly follows from the above formulas and the fact that $P_\pm$ are orthogonal projections. □

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THE WEAK NULL CONDITION ON KERR BACKGROUNDS

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We study a system of semilinear wave equations on Kerr backgrounds that satisfies the weak null condition. Under the assumption of small initial data, we prove global existence and pointwise decay estimates.

1. Introduction

The semilinear system of wave equations in $\mathbb{R}^{1+3}$

$$\Box\phi = Q[\partial\phi, \partial\phi], \quad \phi|_{t=0} = \phi_0, \quad \partial_t\phi|_{t=0} = \phi_1,$$

where Q is a quadratic form, for small initial data has been studied extensively. For the scalar equation, it is known that the solution can blow up in finite time for $\Box\phi = (\partial_t\phi)^2$; see [John 1979]. On the other hand, if the nonlinearity satisfies the null condition by Klainerman [1984], e.g., $\Box\phi = (\partial_t\phi)^2 - |\partial_x\phi|^2$, it was shown independently in [Christodoulou 1986] and [Klainerman 1986] that the solution exists globally. This result was extended to quasilinear systems with multiple speeds, as well as the case of exterior domains; see, for instance, [Metcalfe et al. 2005; Metcalfe and Sogge 2005; 2007; Hidano 2004; Lindblad et al. 2013; Klainerman and Sideris 1996; Alinhac 2003; Lindblad 1992; 2008; Sideris and Tu 2001; Facci and Metcalfe 2022]. There have also been many works for small data in the variable coefficient case. Almost global existence for nontrapping metrics was shown in [Bony and Häfner 2010; Sogge and Wang 2010]. Global existence for stationary, small perturbations of Minkowski was shown in [Wang and Yu 2014], for nonstationary, compactly supported perturbations in [Yang 2013], and for large, asymptotically flat perturbations that satisfy the strong local energy decay estimates in [Looi and Tohaneanu 2022]. In the context of black holes, global existence was shown in [Luk 2013] for Kerr space-times with small angular momentum, and in [Angelopoulos et al. 2020] for the Reissner–Nordström backgrounds.

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Written in harmonic coordinates, the Einstein equations take the form

$$\Box_g g_{\mu\nu} = P[\partial_\mu g, \partial_\nu g] + Q_{\mu\nu}[\partial g, \partial g],$$

where $\Box_g$ is the wave operator on the background of the Lorentzian metric g, and P and $Q_{\mu\nu}$ are quadratic forms with coefficients depending on the metric. Unfortunately the nonlinear terms do not satisfy the null condition. Yet Christodoulou and Klainerman [1993] were able to prove global existence for Einstein vacuum equations $R_{\mu\nu} \equiv 0$ for small asymptotically flat initial data. Their proof avoids using coordinates since it was believed the metric in harmonic coordinates would blow up for large times. However, later Lindblad and Rodnianski [2003] noticed that Einstein's equations in harmonic coordinates satisfy a weak null condition, and subsequently used it to prove stability of Minkowski in harmonic coordinates [Lindblad and Rodnianski 2005; 2010]. Whereas it is still unknown if general equations satisfying the weak null condition have global existence for small initial data, there have been a number of results in that direction, including detailed asymptotics of the solution; see for example [Alinhac 2003; Lindblad 1992; 2008; 2017; Keir 2018; Deng and Pusateri 2020; Yu 2021a; 2021b].

There has recently been a lot of activity in proving asymptotic stability of black holes. As a first step people have proved decay of solutions to wave equations on Schwarzschild and Kerr background [Blue and Soffer 2003; 2005; Blue and Sterbenz 2006; Marzuola et al. 2010; Dafermos and Rodnianski 2009; Tataru and Tohaneanu 2011; Dafermos et al. 2016; Andersson and Blue 2015]. People have also studied semilinear perturbations [Luk 2013; Ionescu and Klainerman 2015] satisfying the null condition, but apart from our recent papers [Lindblad and Tohaneanu 2018; 2020], and a global existence result for the Maxwell–Born–Infeld system on a Schwarzschild background [Pasqualotto 2019], little is known about quasilinear perturbations or semilinear perturbations satisfying the weak null condition. There has more recently been progress on the nonlinear stability of Schwarzschild and Kerr [Klainerman and Szeftel 2022a; 2022b; 2023; Dafermos et al. 2021; Giorgi et al. 2022]. These proofs are very long, using sophisticated geometric constructions. We hope that studying models of Einstein's equations in wave coordinates will simplify the proofs and lead to a better understanding and extensions as it did for the stability of Minkowski space.

Finally we remark that there are several recent works on the cosmological case. Hintz and Vasy [2018] proved the stability of Kerr–de Sitter with small angular momentum; see also [Fang 2021; 2022] for an alternative proof. More recently there have been works on the wave equation on Kerr–de Sitter background for large angular momentum assuming there are no growing modes [Peterson and Vasy 2021; Mavrogiannis 2022].

1.0.1. *The semilinear Einstein model.* An example of a simple semilinear system satisfying the weak null condition, but not the classical null condition, is the system

$$\Box \phi_1 = (\partial_t \phi_2)^2, \quad \Box \phi_2 = 0.$$

It is trivial to see that this has global solutions, and moreover that ϕ_1 decays slower than $1/t$. A less trivial example is the semilinear system

$$\Box \phi_1 = (\partial_t \phi_2)^2 + Q_1[\partial \phi, \partial \phi], \quad \Box \phi_2 = Q_2[\partial \phi, \partial \phi],$$

where Q_j are null forms. These systems have the advantage that the components ϕ_1 and ϕ_2 decouple to highest order. For Einstein's equations there is the additional difficulty that this decoupling can only be seen in a null frame, and contractions with the frame do not commute with the wave operator as far as the L^2 estimate. Hence a more realistic model is the system is

$$\Box\phi_{\mu\nu} = P[\partial_\mu\phi, \partial_\nu\phi] + Q_{\mu\nu}[\partial\phi, \partial\phi],$$

where P is assumed to have a certain weak null structure. Contracting with a nullframe this resembles the decoupled systems with $\phi_{\underline{L}\underline{L}}$ in place of ϕ_1, where $\underline{L}^\mu\partial_\mu = \partial_t - \partial_r$, and ϕ_2 replaced by the other components ϕ_{TU}, where T is tangential to the outgoing light cones. The only really bad component is $\partial\phi_{\underline{L}\underline{L}}$ but this one does not show up quadratically in P for Einstein's equations. It shows up linearly but multiplied with a component $\partial\phi_{LL}$ that has vanishing radiation field due to the wave coordinate condition.

With the goal of understanding Einstein's equations in (generalized) harmonic coordinates close to Kerr with small angular momentum, we will focus on the following system, which resembles the semilinear part of Einstein's equations:

$$\Box_K\phi_{\mu\nu} = P[\partial_\mu\phi, \partial_\nu\phi] + Q_{\mu\nu}[\partial\phi, \partial\phi], \quad \tilde{t} \geq 0, \quad \phi|_{\tilde{t}=0} = \phi_0, \quad \widetilde{T}\phi|_{\tilde{t}=0} = \phi_1. \tag{1-1}$$

Here $\Box_K$ denotes the d'Alembertian with respect to the Kerr metric, and $\widetilde{T}$ is a smooth, everywhere timelike vector field that equals ∂_t away from the black hole. The coordinate $\tilde{t}$ is chosen so that the slices $\tilde{t} = \text{const.}$ are space-like and $\tilde{t} = t$ away from the black hole. For simplicity we will consider compactly supported smooth initial data, but suitably weighted Sobolev spaces of large enough order would suffice. Moreover, $Q_{\mu\nu}$ are null forms and P is a symmetric quadratic form:

$$P[\phi, \psi] = P^{\alpha\beta\gamma\delta}(x/\tilde{t})\phi_{\alpha\beta}\psi_{\gamma\delta},$$

with coefficients with a certain weak null structure. We remove the component $\partial\phi_{\underline{L}\underline{L}}$ by imposing the condition

$$P^{\underline{L}\underline{L}\alpha\beta}(x/\tilde{t}) = P^{\alpha\beta\underline{L}\underline{L}}(x/\tilde{t}) = 0.$$

For this system we cannot have different energy estimates for different components because the null structure is only seen in a null frame and contractions with the frame do not commute with the wave operator. Because of this, one cannot get the decay estimates directly from the L^2 estimates but one has to use the equations again to get improved decay estimates. As a result, the proof is more involved. The method we develop avoids boosts vector fields and combines local energy decay in a compact set with estimates in characteristic coordinates at the light cone. It gives an essentially optimal decay of almost $\tilde{t}^{-1}$, which is an improvement over $\tilde{t}^{-1/2}$, which can be obtained more easily from energy estimates. The method in particular works close to Minkowski where it gives the optimal decay without using boosts.

Finally we remark that this system can be combined with the quasilinear system that we previously studied [Lindblad and Tohaneanu 2018; 2020] (see also [Looi 2022] for improved pointwise bounds) to resemble also the quasilinear part of Einstein's equations

$$\Box_{g[\phi]}\phi_{\mu\nu} = P[\partial_\mu\phi, \partial_\nu\phi] + Q_{\mu\nu}[\partial\phi, \partial\phi],$$

where

$$g^{\alpha\beta}[\phi] = K^{\alpha\beta} + H^{\alpha\beta}[\phi], \quad \text{where } H^{\alpha\beta}[\phi] = H^{\alpha\beta\mu\nu}(x/\tilde{t})\phi_{\mu\nu} \text{ and } H^{\underline{L}\underline{L}\mu\nu}(x/\tilde{t}) = 0.$$

1.0.2. *Statement of the results.* We are now ready to state our main result. We define $\tilde{r}$ to be some function that equals r near the event horizon, and r_K^* away from it; see Section 2 for more details. We also fix r_e satisfying $r_- < r_e < r_+$, and define $\langle x \rangle = (2 + |x|^2)^{1/2}$.

Theorem 1.1. *Let $R_0 > r_e$, and assume that ϕ_0, ϕ_1 are smooth and compactly supported in $\tilde{r} \le R_0$. Then there exists a global classical solution to the system* (1-1) (*on a Kerr metric with $|a| \ll M$*) *provided that, for a certain $\epsilon_0 \ll 1$ and large enough N, we have*

$$\mathcal{E}_N(0) = \|(\phi_0, \phi_1)\|_{H^{N+1} \times H^N} \le \epsilon_0.$$

Moreover, for some fixed positive integer m, independent of N, we have for any $\delta > 0$

$$|\phi_{\le N-m}| \lesssim \frac{\mathcal{E}_N(0)\langle \ln(\langle \tilde{t}\rangle / \langle \tilde{t} - \tilde{r}\rangle)\rangle}{\langle \tilde{t}\rangle}, \quad |\partial\phi_{\le N-m}| \lesssim \frac{\mathcal{E}_N(0)\langle \ln(\langle \tilde{t}\rangle / \langle \tilde{t} - \tilde{r}\rangle)\rangle}{r\langle \tilde{t} - \tilde{r}\rangle},$$
$$|(\partial\phi_{TU})_{\le N-m}| \lesssim \frac{\mathcal{E}_N(0)}{r\langle \tilde{t} - \tilde{r}\rangle^{1-\delta}}.$$

Note that is an improvement of the decay estimates we previously proved essentially by a factor of $\tilde{t}^{-1/2}$. Note also the structure here, that a derivative decreases the homogeneity, but because the homogeneous vector fields we can use together with the wave operator do not span the tangent space at the origin or at the light cone, a derivative only improves by a power of r close to the origin and a power of $\tilde{t} - \tilde{r}$ close to the light cone. Note also that close to the light cone we have a better estimate for the good components which is due to the weak null structure.

1.0.3. *Structure of the proof.* The starting point is the local energy estimate in Section 2. The local energy scales like the energy, which is consistent with a decay $\tilde{t}^{-1/2}$ of order $-\frac{1}{2}$ for ϕ and $-\frac{3}{2}$ for the derivatives, and this is also the decay we were able to obtain in our previous paper from a bound of the local energy applied to scaling and rotation vector fields; see Section 3. Assuming these decay estimates, one can go back into the equation and get improved decay estimates. In fact from these decay estimates the total decay of the inhomogeneous term would be -3, which would be consistent with a solution of the wave equation with decay of order -1. We prove this using L^∞ estimates for the wave operator from Section 5. However the first improved estimates we obtain have the improved decay in r or $\tilde{t} - \tilde{r}$ and we need improved decay in $\tilde{t}$. For this we have other estimates turn decay in r or $\tilde{t} - \tilde{r}$ into decay in $\tilde{t}$; see Section 4. The whole argument is put together in the last section.

The paper is structured as follows. In Section 2 we introduce the Kerr metric, the vector fields we will use, and the local energy estimates which will play a key role in the proof. Sections 3, 4, 5, and 6 contain various estimates that will allow us to extract the necessary pointwise bounds for (vector fields applied to) the solution. Finally, Section 7 contains the bootstrap argument.

2. The Kerr metric and local energy estimates

2.1. ***The Kerr metric.*** The Kerr geometry in Boyer–Lindquist coordinates is given by

$$ds^2 = g^K_{tt}dt^2 + g_{t\phi}dtd\phi + g^K_{rr}dr^2 + g^K_{\phi\phi}d\phi^2 + g^K_{\theta\theta}d\theta^2,$$

where $t \in \mathbb{R}$, $r > 0$, (ϕ, θ) are the spherical coordinates on $\mathbb{S}^2$ and

$$g^K_{tt} = -\frac{\Delta - a^2\sin^2\theta}{\rho^2}, \quad g^K_{t\phi} = -2a\frac{2Mr\sin^2\theta}{\rho^2}, \quad g^K_{rr} = \frac{\rho^2}{\Delta},$$
$$g^K_{\phi\phi} = \frac{(r^2+a^2)^2 - a^2\Delta\sin^2\theta}{\rho^2}\sin^2\theta, \quad g^K_{\theta\theta} = \rho^2,$$

with

$$\Delta = r^2 - 2Mr + a^2, \quad \rho^2 = r^2 + a^2\cos^2\theta.$$

Here M represents the mass of the black hole, and aM its angular momentum.

A straightforward computation gives us the inverse of the metric:

$$g^{tt}_K = -\frac{(r^2+a^2)^2 - a^2\Delta\sin^2\theta}{\rho^2\Delta}, \quad g^{t\phi}_K = -a\frac{2Mr}{\rho^2\Delta}, \quad g^{rr}_K = \frac{\Delta}{\rho^2},$$
$$g^{\phi\phi}_K = \frac{\Delta - a^2\sin^2\theta}{\rho^2\Delta\sin^2\theta}, \quad g^{\theta\theta}_K = \frac{1}{\rho^2}.$$

The case $a = 0$ corresponds to the Schwarzschild space-time. We shall subsequently assume that a is small $0 < a \ll M$, so that the Kerr metric is a small perturbation of the Schwarzschild metric. Note also that the coefficients depend only r and θ but are independent of ϕ and t. We denote the d'Alembertian associated to the Kerr metric by $\Box_K$.

In the above coordinates the Kerr metric has singularities at $r = 0$, on the equator $\theta = \frac{\pi}{2}$, and at the roots of Δ, namely $r_\pm = M \pm \sqrt{M^2 - a^2}$. To remove the singularities at $r = r_\pm$ we introduce functions $r^*_K = r^*_K(r)$, $v_+ = t + r^*_K$ and $\phi_+ = \phi_+(\phi, r)$ so that (see [Hawking and Ellis 1973])

$$dr^*_K = (r^2+a^2)\Delta^{-1}\,dr, \quad dv_+ = dt + dr^*_K, \quad d\phi_+ = d\phi + a\Delta^{-1}\,dr.$$

Note that when $a = 0$ the r^*_K-coordinate becomes the Schwarzschild Regge–Wheeler coordinate

$$r^* = r + 2M\log(r - 2M).$$

The Kerr metric can be written in the new coordinates (v_+, r, ϕ_+, θ),

$$ds^2 = -\left(1 - \frac{2Mr}{\rho^2}\right)dv_+^2 + 2drdv_+ - 4a\rho^{-2}Mr\sin^2\theta dv_+d\phi_+ - 2a\sin^2\theta drd\phi_+ + \rho^2d\theta^2 + \rho^{-2}[(r^2+a^2)^2 - \Delta a^2\sin^2\theta]\sin^2\theta\,d\phi_+^2,$$

which is smooth and nondegenerate across the event horizon up to but not including $r = 0$. We introduce the function

$$\tilde{t} = v_+ - \mu(r),$$

where μ is a smooth function of r. In the $(\tilde{t}, r, \phi_+, \theta)$-coordinates the metric has the form

$$ds^2 = \left(1 - \frac{2Mr}{\rho^2}\right) d\tilde{t}^2 + 2\left(1 - (1 - \frac{2Mr}{\rho^2})\mu'(r)\right) d\tilde{t}\, dr - 4a\rho^{-2}Mr\sin^2\theta\, d\tilde{t}\, d\phi_+ + \left(2\mu'(r) - \left(1 - \frac{2Mr}{\rho^2}\right)(\mu'(r))^2\right) dr^2 - 2a(1 + 2\rho^{-2}Mr\mu'(r))\sin^2\theta\, dr\, d\phi_+ + \rho^2\, d\theta^2 + \rho^{-2}[(r^2 + a^2)^2 - \Delta a^2 \sin^2\theta]\sin^2\theta\, d\phi_+^2.$$

On the function μ we impose the following two conditions:

(i) $\mu(r) \geq r_K^*$ for $r > 2M$, with equality for $r > \frac{5}{2}M$.

(ii) The surfaces $\tilde{t} = \text{const.}$ are space-like, i.e.,

$$\mu'(r) > 0, \quad 2 - \left(1 - \frac{2Mr}{\rho^2}\right)\mu'(r) > 0.$$

As long as a is small, we can use the same function μ as in the case of the Schwarzschild space-time in [Marzuola et al. 2010].

We also introduce

$$\tilde{\phi} = \zeta(r)\phi_+ + (1 - \zeta(r))\phi,$$

where ζ is a cutoff function supported near the event horizon.

We fix r_e satisfying $r_- < r_e < r_+$. The choice of r_e is unimportant, and for convenience we may simply use $r_e = M$ for all Kerr metrics with $a/M \ll 1$. Let $\mathcal{M} = \{\tilde{t} \geq 0 : r \geq r_e\}$, $\Sigma(T) = \mathcal{M} \cap \{\tilde{t} = T\}$, and $d\Sigma_K$ be the induced volume element on $\Sigma(T)$.

Let $\tilde{r}$ denote a smooth strictly increasing function (of r) that equals r for $r \leq R$ and r_K^* for $r \geq 2R$ for some large R. We will use the coordinates $(\tilde{t}, x^i)$, where $x^i = \tilde{r}\omega$. Note that, since $r \approx \tilde{r}$, we can use r^k and $\tilde{r}^k$ interchangeably when defining our spaces of functions in what follows.

2.2. ***Vector fields and spaces of functions.*** Our favorite sets of vector fields will be

$$\partial \in \{\partial_{\tilde{t}}, \partial_i\}, \quad \Omega \in \{x^i\partial_j - x^j\partial_i\}, \quad S = \tilde{t}\,\partial_{\tilde{t}} + \tilde{r}\,\partial_{\tilde{r}},$$

namely the generators of translations, rotations and scaling. We set $Z = \{\partial, \Omega, S\}$.

We also denote by $\not\partial$ the angular derivatives,

$$\partial_j = \frac{x^i}{\tilde{r}}\,\partial_{\tilde{r}} + \not\partial_i$$

and let

$$\bar{\partial} \in (\partial_v, \not\partial), \quad \partial_v = \partial_{\tilde{t}} + \partial_{\tilde{r}}$$

denote the tangential derivatives. We also let $\underline{L} = \partial_{\tilde{t}} - \partial_{\tilde{r}}$.

For a triplet $\alpha = (i, j, k)$, we define $|\alpha| = i + 3j + 3k$ and

$$u_\alpha = \partial^i\Omega^j S^k u, \quad u_{\leq m} = (u_\Lambda)_{|\Lambda| \leq m}.$$

The notation is borrowed from [Lindblad and Tohaneanu 2018], and takes into account the loss of derivatives that occurs when applying weak local energy estimates to vector fields.

Given a norm $\|\cdot\|_X$, we write

$$\|u_{\le m}\|_X = \sum_{|\Lambda|\le m} \|u_\Lambda\|_X.$$

We define the classes $S^Z(r^k)$ of functions in $\mathbb{R}^+ \times \mathbb{R}^3$ by

$$f \in S^Z(r^k) \quad\iff\quad |Z^j f(t,x)| \le c_j \langle r\rangle^k, \quad j \ge 0.$$

Given a family of functions $\mathcal{G}$, we will also use the notation

$$f \in S^Z(r^k)\mathcal{G}$$

to mean that

$$f = \sum h_i g_i, \quad h_i \in S^Z(r^k), \quad g_i \in \mathcal{G}.$$

We will also use the notation U for an element of $S^Z(1)Z$, and T for an element of $S^Z(1)\,\bar\partial$.

An important observation is that, since

$$\partial_v = \frac{\tilde t - \tilde r}{\tilde t}\,\partial_{\tilde r} + \frac{1}{\tilde t} S, \quad \not\partial\phi \in S^Z(r^{-1})\Omega\phi,$$

we have

$$|\bar\partial w| \lesssim \frac{\tilde t - \tilde r}{r}|\partial w| + \frac{1}{r}|\Omega w|. \tag{2-1}$$

Moreover, an easy computation gives

$$[\Box_K, \partial]\phi \in S^Z(r^{-2})\,\partial\partial^{\le 1}\phi, \quad [\Box_K, \Omega]\phi \in S^Z(r^{-2})\,\partial\partial^{\le 1}\phi,$$
$$[\Box_K, S]\phi \in S^Z(1)\Box_K\phi + S^Z(r^{-2+})\,\partial\phi + S^Z(r^{-2+})\,\partial\Omega\phi + S^Z(r^{-2})\,\partial\partial^{\le 1}\phi,$$

and thus by induction we obtain that

$$[\Box_K, Z^\alpha]\phi = F_1 + F_2, \quad F_1 \in S^Z(1)(\Box_K\phi)_{\le|\alpha|}, \quad F_2 \in S^Z(r^{-2+})\,\partial\phi_{\le|\alpha|}. \tag{2-2}$$

We now claim that

$$[Z, \bar\partial] \in S^Z(1)\bar\partial + S^Z(r^{-1})\,\partial. \tag{2-3}$$

Indeed, we compute

$$[\partial_{\tilde t}, \bar\partial] = 0, \quad [\partial_i, \partial_v] = [\not\partial_i, \partial_{\tilde r}] \in S^Z(r^{-1})\not\partial, \quad [\partial_i, \not\partial] \in S^Z(r^{-1})\,\partial,$$
$$[\Omega, \partial_v] = 0, \quad [\Omega, \not\partial] \in S^Z(1)\not\partial, \quad [S, \partial_v] = \partial_v, \quad [S, \not\partial] \in S^Z(1)\not\partial.$$

This proves (2-3).

Given vector fields X and Y, we define

$$\phi_{XY} = X^\alpha Y^\beta \phi_{\alpha\beta}.$$

Similarly, we can write the coefficients P with respect to the vector frame $\{\underline{L}, \bar\partial\}$ as

$$P^{\alpha\beta\gamma\delta} = P^{\underline{L}\underline{L}\gamma\delta}\underline{L}^\alpha\underline{L}^\beta + \sum P^{TU\gamma\delta}T^\alpha U^\beta,$$
$$P^{\alpha\beta\gamma\delta} = P^{\alpha\beta\underline{L}\underline{L}}\underline{L}^\gamma\underline{L}^\delta + \sum P^{\alpha\beta TU}T^\gamma U^\delta.$$

The assumptions on the coefficients $P^{\alpha\beta\gamma\delta}$ are the following:

$$P^{\alpha\beta\gamma\delta} \in S^Z(1), \tag{2-4}$$
$$P^{\underline{L}\underline{L}\alpha\beta} = P^{\alpha\beta\underline{L}\underline{L}} = 0. \tag{2-5}$$

Equation (2-5) means that terms like $\underline{L}\phi_{\underline{L}\underline{L}}\,\partial\phi$ do not appear on the right-hand side of (1-1).

The assumption on the null forms $Q_{\mu\nu}$ is that

$$Q_{\mu\nu}[\partial\phi, \partial\phi] \in S^Z(1)\,\partial\phi\bar\partial\phi. \tag{2-6}$$

2.3. *Local energy estimates.* We consider a partition of $\mathbb{R}^3$ into the dyadic sets $A_R = \{R \le \langle\tilde r\rangle \le 2R\}$ for $R \ge 1$.

We now introduce the local energy norm LE

$$\|u\|_{\mathrm{LE}} = \sup_R \|\langle r\rangle^{-1/2}u\|_{L^2(\mathcal{M}\cap\mathbb{R}\times A_R)},$$
$$\|u\|_{\mathrm{LE}[0,1]} = \sup_R \|\langle r\rangle^{-1/2}u\|_{L^2(\mathcal{M}\cap[0,1]\times A_R)},$$

its H^1 counterpart

$$\|u\|_{\mathrm{LE}^1} = \|\partial u\|_{\mathrm{LE}} + \|\langle r\rangle^{-1}u\|_{\mathrm{LE}},$$
$$\|u\|_{\mathrm{LE}^1[0,1]} = \|\partial u\|_{\mathrm{LE}[0,1]} + \|\langle r\rangle^{-1}u\|_{\mathrm{LE}[0,1]},$$

as well as the dual norm

$$\|f\|_{\mathrm{LE}^*} = \sum_R \|\langle r\rangle^{1/2}f\|_{L^2(\mathcal{M}\cap\mathbb{R}\times A_R)},$$
$$\|f\|_{\mathrm{LE}^*[0,1]} = \sum_R \|\langle r\rangle^{1/2}f\|_{L^2(\mathcal{M}\cap[0,1]\times A_R)}.$$

We also define similar norms for higher Sobolev regularity

$$\|u_{\le m}\|_{\mathrm{LE}^1} = \sum_{|\alpha|\le m}\|u_\alpha\|_{\mathrm{LE}^1},$$
$$\|u_{\le m}\|_{\mathrm{LE}^1[0,1]} = \sum_{|\alpha|\le m}\|u_\alpha\|_{\mathrm{LE}^1[0,1]},$$
$$\|u_{\le m}\|_{\mathrm{LE}[0,1]} = \sum_{|\alpha|\le m}\|u_\alpha\|_{\mathrm{LE}[0,1]},$$

respectively,

$$\|f\|_{\mathrm{LE}^{*,k}} = \sum_{|\alpha|\le k}\|\partial^\alpha f\|_{\mathrm{LE}^*},$$
$$\|f\|_{\mathrm{LE}^{*,k}[0,1]} = \sum_{|\alpha|\le k}\|\partial^\alpha f\|_{\mathrm{LE}^*[0,1]}.$$

Finally, we introduce a weaker version of the local energy decay norm

$$\|u\|_{\mathrm{LE}^1_w} = \|(1-\chi_{ps})\,\partial u\|_{\mathrm{LE}} + \|\partial_r u\|_{\mathrm{LE}} + \|\langle r\rangle^{-1}u\|_{\mathrm{LE}},$$
$$\|u\|_{\mathrm{LE}^1_w[0,1]} = \|(1-\chi_{ps})\,\partial u\|_{\mathrm{LE}[0,1]} + \|\partial_r u\|_{\mathrm{LE}[0,1]} + \|\langle r\rangle^{-1}u\|_{\mathrm{LE}[0,1]}.$$

To measure the inhomogeneous term, we define

$$\|f\|_{\mathrm{LE}^*_w} = \inf_{f_1+f_2=f} \|f_1\|_{L^1L^2} + \|(1-\chi_{ps})f_2\|_{\mathrm{LE}^*},$$
$$\|f\|_{\mathrm{LE}^*_w[0,1]} = \inf_{f_1+f_2=f} \|f_1\|_{L^1[0,1]L^2} + \|(1-\chi_{ps})f_2\|_{\mathrm{LE}^*[0,1]}.$$

Here χ_{ps} is a smooth, compactly supported spatial cutoff function that equals 1 in a neighborhood of the trapped set. We also define the higher-order weak norms as above.

We define the (nondegenerate) energy

$$E[u](\tilde{t}) = \left(\int_{\Sigma(\tilde{t})} |\partial u|^2\, d\Sigma_K\right)^{1/2}.$$

We now fix some $\delta_1 \ll 1$, and define, for a large enough constant R_1 (so that in particular $\chi_{ps} = 0$ when $r > R_1$):

$$\mathcal{E}_N(T) = \sup_{0\le\tilde{t}\le T} E[\phi_{\le N}](\tilde{t}) + \|\phi_{\le N}\|_{\mathrm{LE}^1_w[0,T]} + \|\langle \tilde{t}-\tilde{r}\rangle^{(-1-\delta_1)/2}\bar{\partial}\phi_{\le N}\|_{L^2[0,T]L^2(r\ge R_1)}. \tag{2-7}$$

We will need the following local energy estimates for the linear problem:

Lemma 2.1. *Assume that $\Box_K\phi = F$, and N is any nonnegative integer. We then have for any $T \ge 0$ that*

$$\mathcal{E}_N(T) \lesssim \mathcal{E}_N(0) + \|F_{\le N}\|_{\mathrm{LE}^*_w[0,T]}, \tag{2-8}$$

where the implicit constant is independent of T.

Proof. We start by proving the base case $N = 0$, that is,

$$\sup_{0\le\tilde{t}\le T} E[\phi](\tilde{t}) + \|\phi\|_{\mathrm{LE}^1_w[0,T]} + \|\langle \tilde{t}-\tilde{r}\rangle^{(-1-\delta_1)/2}\bar{\partial}\phi\|_{L^2[0,T]L^2(r\ge R_1)} \lesssim E[\phi](0) + \|F\|_{\mathrm{LE}^*_w[0,T]}. \tag{2-9}$$

Theorem 4.1 from [Tataru and Tohaneanu 2011] gives the desired bound for the first two terms on the left-hand side. On the other hand, Lemma 4.3 in [Lindblad and Tohaneanu 2018] and Cauchy–Schwarz yield

$$\|\langle \tilde{t}-\tilde{r}\rangle^{(-1-\delta_1)/2}\bar{\partial}\phi\|_{L^2[0,T]L^2(r\ge R_1)} \lesssim \|\phi\|_{\mathrm{LE}^1_w[0,T]} + \|F\|_{\mathrm{LE}^*_w[0,T]}.$$

We now commute the equation with the vector fields in Z. Applying the base case estimate (2-9) to ϕ_α for some $|\alpha| = N$ yields

$$\sup_{0\le\tilde{t}\le T} E[\phi_\alpha](\tilde{t}) + \|\phi_\alpha\|_{\mathrm{LE}^1_w[0,T]} + \|\langle \tilde{t}-\tilde{r}\rangle^{(-1-\delta_1)/2}\bar{\partial}\phi_\alpha\|_{L^2[0,T]L^2(r\ge R_1)}$$
$$\lesssim \|\phi_\alpha\|_{\mathrm{LE}^1_w[0,T]} + \|F_\alpha\|_{\mathrm{LE}^*_w[0,T]} + \|[\Box_K, Z^\alpha]\phi\|_{\mathrm{LE}^*_w[0,T]}.$$

We are left with bounding the last term on the right-hand side. By (2-2) we have

$$\|[\Box_K, Z^\alpha]\phi\|_{LE_w^*[0,T]} \lesssim \|F_{\le|\alpha|}\|_{LE_w^*[0,T]} + \|r^{-2+}\,\partial\phi_{\le|\alpha|}\|_{LE_w^*[0,T]} \lesssim \|F_{\le|\alpha|}\|_{LE_w^*[0,T]} + \|\phi_{\le|\alpha|}\|_{LE_w^1[0,T]}.$$

We now sum over all $|\alpha| = N$. □

The first estimate of this kind was obtained by Morawetz for the Klein–Gordon equation [Morawetz 1968]. In the Schwarzschild case, similar estimates were shown in [Blue and Soffer 2003; 2005; Blue and Sterbenz 2006; Dafermos and Rodnianski 2009; 2007; Marzuola et al. 2010]. The estimate for Kerr with small angular momentum was proven in [Tataru and Tohaneanu 2011] (see also [Andersson and Blue 2015; Dafermos and Rodnianski 2013] for related works). For large angular momentum, see [Dafermos et al. 2016] ($|a| < M$) and [Aretakis 2012] ($|a| = M$).

3. Pointwise estimates from local energy decay estimates

The goal of this section is to show how to extract (weak) pointwise estimates from local energy norms. These bounds will serve as the starting point in an iteration that will yield strong enough pointwise bounds to close the bootstrap argument in Section 7.

Let

$$C_T = \{T \le \tilde{t} \le 2T : \tilde{r} \le \tilde{t}\}.$$

We use a double dyadic decomposition of C_T with respect to either the size of $\tilde{t} - \tilde{r}$ or the size of r, depending on whether we are close or far from the cone,

$$C_T = \bigcup_{1\le R\le T/4} C_T^R \bigcup \bigcup_{1\le U<T/4} C_T^U,$$

where for $R, U > 1$ we set

$$C_T^R = C_T \cap \{R < r < 2R\}, \quad C_T^U = C_T \cap \{U < \tilde{t} - \tilde{r} < 2U\},$$

while for $R = 1$ and $U = 1$ we have

$$C_T^{R=1} = C_T \cap \{0 < r < 2\}, \quad C_T^{U=1} = C_T \cap \{0 < \tilde{t} - \tilde{r} < 2\}.$$

The sets C_T^R and C_T^U represent the setting in which we apply Sobolev embeddings, which allow us to obtain pointwise bounds from L^2 bounds. Precisely, we have (see Lemma 3.8 from [Metcalfe et al. 2012] and Lemma 6.2 in [Lindblad and Tohaneanu 2018]):

Lemma 3.1. *For any function w and all $T \ge 1$ and $1 \le R, U \le \frac{1}{4}T$ we have*

$$\|w\|_{L^\infty(C_T^R)} \lesssim \frac{1}{T^{1/2}R^{3/2}} \sum_{i\le1, j\le2} \|S^i\Omega^j w\|_{L^2(C_T^R)} + \frac{1}{T^{1/2}R^{1/2}} \sum_{i\le1, j\le2} \|S^i\Omega^j\,\partial w\|_{L^2(C_T^R)}, \tag{3-1}$$

respectively,

$$\|w\|_{L^\infty(C_T^U)} \lesssim \frac{1}{T^{3/2}U^{1/2}} \sum_{i\le1, j\le2} \|S^i\Omega^j w\|_{L^2(C_T^U)} + \frac{U^{1/2}}{T^{3/2}} \sum_{i\le1, j\le2} \|S^i\Omega^j\,\partial w\|_{L^2(C_T^U)}. \tag{3-2}$$

Using the lemma above, we prove the following pointwise bound:

$$\|w\|_{L^\infty(C_T)} \lesssim \langle \tilde{t}\rangle^{-1}\langle \tilde{t}-\tilde{r}\rangle^{1/2}\|w_{\le 12}\|_{\mathrm{LE}^1[T,2T]}. \tag{3-3}$$

Indeed, in the region C_T^R, this is an immediate application of (3-1). On the other hand, in the region C_T^U this follows from (3-2) and Hardy's inequality; see (6.7) in [Lindblad and Tohaneanu 2018].

We also need an L^∞ bound on the derivative that is better than (3-3) for large r. This is the content of the following, which is essentially Proposition 3.5 in [Looi and Tohaneanu 2022]

Proposition 3.2. *Let*

$$\mu := \min(\langle \tilde{t}\rangle, \langle \tilde{t}-\tilde{r}\rangle)^{1/2}.$$

Assume that ϕ solve (1-1) for $t \in [T, 2T]$. Then for any dyadic region $C \in \{C_T^R, C_U^R\}$ and $m \ge 0$ we have

$$\|\partial\phi_{\le m}\|_{L^\infty(C)} \le \bar{C}_m \frac{1}{\mu}\left(\frac{1}{\langle r\rangle} + \|\partial\phi_{\le (m+10)/2}\|_{L^\infty(C)}\right)\|\phi_{\le m+5}\|_{\mathrm{LE}^1[T,2T]}. \tag{3-4}$$

Here the crucial estimate was the following Klainerman–Sideris-type estimate; see Lemma 5.4 in [Lindblad and Tohaneanu 2018] (for Schwarzschild) combined with the remarks after (5.13) in [Lindblad and Tohaneanu 2020]:

Lemma 3.3. *For any w and multiindex Λ we have in the region $r \ge 2R_1$ that*

$$|\partial^2 w_\Lambda| \lesssim \frac{\tilde{t}}{r\langle \tilde{t}-\tilde{r}\rangle}|\partial w_{\le|\Lambda|+3}| + \frac{\tilde{t}}{\langle \tilde{t}-\tilde{r}\rangle}|(\Box_K w)_{\le|\Lambda|}|.$$

We now apply (3-2) to $\partial\phi_\Lambda$ for any $|\Lambda| \le m$. We obtain

$$\begin{aligned}\|\partial\phi_\Lambda\|_{L^\infty(C_T^U)} &\lesssim \frac{1}{T^{3/2}U^{1/2}}\sum_{i\le 1, j\le 2}\|S^i\Omega^j\,\partial\phi_\Lambda\|_{L^2(C_T^U)} + \frac{U^{1/2}}{T^{3/2}}\sum_{i\le 1, j\le 2}\|S^i\Omega^j\,\partial^2\phi_\Lambda\|_{L^2(C_T^U)} \\ &\lesssim \frac{1}{TU^{1/2}}\|\phi_{\le|\Lambda|+13}\|_{\mathrm{LE}^1[T,2T]} + \frac{1}{(TU)^{1/2}}\|(\Box_K\phi)_{\le|\Lambda|+10}\|_{L^2(C_T^U)}.\end{aligned}$$

Since

$$|(\Box_K\phi)_{\le|\Lambda|+10}| \lesssim |\partial\phi_{\le|\Lambda|/2+5}|\,|\partial\phi_{\le|\Lambda|+10}|,$$

the conclusion follows in the region C_T^U. A similar computation yields the result in C_T^R.

4. Improved pointwise bounds

We will use three lemmas that will help us improve our pointwise bounds. The first one is Proposition 3.14 from [Metcalfe et al. 2012], which will allow us to turn r-decay into t-decay in the region $r \le \frac{1}{2}t$.

Lemma 4.1. *The following estimate holds for all $m \ge 0$ and some fixed (m-independent) n:*

$$\|u_{\le m}\|_{\mathrm{LE}^1(C_T^{<T/2})} \lesssim T^{-1}\|\langle r\rangle u_{\le m+n}\|_{\mathrm{LE}^1(C_T^{<T/2})} + \|(\Box_K u)_{\le m+n}\|_{\mathrm{LE}^*(C_T^{<T/2})}.$$

The second lemma is a slight modification of Lemma 3.11 from [Metcalfe et al. 2012], the difference being that we may not enlarge our regions in time. The role of the lemma is to gain a factor of $\tilde t/(r\langle \tilde t-\tilde r\rangle)$ for the derivative.

We let $\widetilde{C}_T^R$ and $\widetilde{C}_T^U$ denote enlargements of C_T^R and C_T^U in space (but not in time) that contain all the integral curves of the scaling vector field S (i.e., if $(t,x)\in C_T^R$ then $(st,sx)\in \widetilde{C}_T^R$ as long as $T\le st\le 2T$ and similarly for C_T^U). More precisely, let

$$\widetilde{C}_T^R=\left\{T\le \tilde t\le 2T:\frac{8}{10}\frac{T}{2R}\le\frac{\tilde t}{\tilde r}\le\frac{12}{10}\frac{2T}{R}\right\},\qquad \widetilde{C}_T^R(\tau)=\widetilde{C}_T^R\cap\{\tilde t=\tau\},$$
$$\widetilde{C}_T^U=\left\{T\le \tilde t\le 2T:\frac{8}{10}\frac{T}{T-2U}\le\frac{\tilde t}{\tilde r}\le\frac{12}{10}\frac{2T}{2T-U}\right\},\quad \widetilde{C}_T^U(\tau)=\widetilde{C}_T^U\cap\{\tilde t=\tau\}.$$

An important observation here is that $\tilde r\approx R$ and $\tilde t-\tilde r\approx U$ in $\widetilde{C}_T^R$ and $\widetilde{C}_T^U$ respectively.

Lemma 4.2. *For* $1\ll U,R\le\frac14 T$ *we have*

$$\|\partial w\|_{L^2(C_T^R)}\lesssim R^{-1}\|w\|_{L^2(\widetilde{C}_T^R)}+T^{-1}(\|Sw\|_{L^2(\widetilde{C}_T^R)}+\|S^2w\|_{L^2(\widetilde{C}_T^R)})+R\|\Box_K w\|_{L^2(\widetilde{C}_T^R)},\tag{4-1}$$

respectively,

$$\|\partial w\|_{L^2(C_T^U)}\lesssim U^{-1}(\|w\|_{L^2(\widetilde{C}_T^U)}+\|Sw\|_{L^2(\widetilde{C}_T^U)}+\|S^2w\|_{L^2(\widetilde{C}_T^U)})+T\|\Box_K w\|_{L^2(\widetilde{C}_T^U)}.\tag{4-2}$$

Proof. The proof is similar to the one in Lemma 3.11 from [Metcalfe et al. 2012], except that we need to estimate the boundary terms at $\tilde t=T$ and $\tilde t=2T$.

To keep the ideas clear we first prove the lemma with $\Box_K$ replaced by $\Box$. We consider a cutoff function χ supported in $\left[\frac{8}{20},\frac{22}{10}\right]$ which equals 1 on $\left[\frac{9}{20},\frac{21}{10}\right]$. Let

$$\beta(\tilde t,\tilde r)=\chi\left(\frac{\tilde r}{\tilde t}\frac{T}{R}\right).$$

Note that $\beta\equiv 1$ on C_T^R, and that the restriction of β to $T\le\tilde t\le 2T$ is supported in $\widetilde{C}_T^R$.

Integrating $\frac12\beta\Box w^2=\beta(w\Box w+m^{\alpha\beta}\,\partial_\alpha w\,\partial_\beta w)$ by parts twice gives

$$\int_T^{2T}\!\!\int\beta(|\partial_x w|^2-|\partial_t w|^2)\,dx\,dt =\int_T^{2T}\!\!\int\Box w\cdot\beta w\,dx\,dt-\frac12\int_T^{2T}\!\!\int(\Box\beta)w^2\,dx\,dt-\int\left(\beta w\partial_t w-\frac12\beta_t w^2\right)dx\Big|_T^{2T}.$$

Since we can write $w_t=(Sw-x^i\partial_i w)/t$ it follows after integration by parts that

$$\int\beta w\partial_t w\,dx=\frac{1}{\tilde t}\int\beta wSw\,dx+\frac{1}{2\tilde t}\int w^2\partial_i(x^i\beta)\,dx.$$

Since $|\partial_i(x^i\beta)|+\tilde t|\partial_t\beta|\le C$ on the support of β, it follows that the boundary terms are bounded by

$$CT^{-1}(\|w(2T,\cdot)\|^2_{L^2(\widetilde{C}_T^R(2T))}+\|Sw(2T,\cdot)\|^2_{L^2(\widetilde{C}_T^R(2T))}+\|w(T,\cdot)\|^2_{L^2(\widetilde{C}_T^R(T))}+\|Sw(T,\cdot)\|^2_{L^2(\widetilde{C}_T^R(T))}).$$

Let $\chi(t/T)$ be another smooth cutoff such that $\chi(2)=1$ and $\chi(1)=0$. We write

$$w(2T,x)^2 = \int_{1/2}^{1} \frac{d}{ds}(\chi w^2)(s2T,sx)\,ds$$
$$= \int_{1/2}^{1} S(\chi w^2)(s2T,sx)\,\frac{ds}{s} = \int_{T}^{2T} S(\chi w^2)\Big(t,t\frac{x}{2T}\Big)\,\frac{dt}{t}$$

and thus

$$\|w(2T,\cdot)\|^2_{L^2(\widetilde{C}^R_T(2T))} \lesssim \frac{1}{T}\|S(\chi w^2)(t,x)\|^2_{L^2(\widetilde{C}^R_T)} \lesssim \frac{1}{T}(\|w\|^2_{L^2(\widetilde{C}^R_T)} + \|Sw\|^2_{L^2(\widetilde{C}^R_T)}).$$

A similar argument holds for $2T$ replaced by T, and for w replaced by Sw. Hence the boundary term can be estimated by

$$\frac{1}{T^2}\sum_{j=0}^{2}\|S^j w\|^2_{L^2(\widetilde{C}^R_T)}.$$

To estimate ∂w we use the pointwise inequality

$$|\partial w|^2 \le \widetilde{C}\frac{1}{(\tilde{t}-\tilde{r})^2}|Sw|^2 + \frac{\tilde{t}}{\tilde{t}-\tilde{r}}(|\partial_x w|^2 - |\partial_t w|^2), \tag{4-3}$$

which is valid inside the cone C for a fixed large constant $\widetilde{C}$. Hence

$$\int \beta|\partial w|^2\,dx\,dt \lesssim \int \frac{1}{(\tilde{t}-\tilde{r})^2}\beta|Sw|^2 + \frac{\tilde{t}}{\tilde{t}-\tilde{r}}|\Box\beta|w^2 + \frac{\tilde{t}}{\tilde{t}-\tilde{r}}\beta|\Box w|\,|w|\,dx\,dt, \tag{4-4}$$

where all weights have a fixed size in the support of β. The function β also satisfies $|\Box\beta| \lesssim R^{-2}$. Then the conclusion of the lemma follows by applying Cauchy–Schwarz to the last term.

The argument for C^U_T is similar. We now consider

$$\beta(\tilde{t},\tilde{r}) = \chi\bigg(\frac{\tilde{t}-\tilde{r}}{\tilde{t}}\frac{T}{U}\bigg).$$

We multiply by βw and integrate by parts as above. The boundary terms are now controlled by

$$CU^{-1}(\|w(2T,\cdot)\|^2_{L^2(\widetilde{C}^U_T(2T))} + \|Sw(2T,\cdot)\|^2_{L^2(\widetilde{C}^U_T(2T))} + \|w(T,\cdot)\|^2_{L^2(\widetilde{C}^U_T(T))} + \|Sw(T,\cdot)\|^2_{L^2(\widetilde{C}^U_T(T))}),$$

which in turn is controlled, by using the scaling S as above, by

$$\frac{1}{TU}\sum_{j=0}^{2}\|S^j w\|^2_{L^2(\widetilde{C}^R_T)}.$$

The estimate now follows from (4-4), using the fact that $|\Box\beta| \lesssim T^{-1}U^{-1}$.

Now consider the above proof but with $\Box$ replaced by $\Box_K$. Integrating

$$\tfrac{1}{2}\beta\Box_K w^2 = \beta(w\Box_K w + g_K^{\alpha\beta}\partial_\alpha w\,\partial_\beta w)$$

by parts twice gives

$$-\int_T^{2T}\int \beta g_K^{\alpha\beta}\,\partial_\alpha w\,\partial_\beta w\sqrt{|g_K|}\,dx\,dt$$
$$=\int_T^{2T}\int\Bigl(\beta w\Box_K w-\frac{1}{2}(\Box_K\beta)w^2\Bigr)\sqrt{|g_K|}\,dx\,dt-\frac{1}{2}\int(\beta g_K^{0\alpha}\,\partial_\alpha w^2-g_K^{\alpha 0}w^2\,\partial_\alpha\beta)\sqrt{|g_K|}\,dx\Big|_T^{2T}.$$

First we estimate the boundary term. The terms with $\alpha=0$ are handled as before and so is the second term with $\alpha>0$. For the first term with $\alpha>0$ we integrate by parts and see that it is bounded by a term of the same form as the second term plus a term of the form

$$\frac{1}{2}\int\beta\partial_\alpha(g_K^{0\alpha}\sqrt{|g_K|})w^2\,dx\lesssim\int\beta r^{-2}w^2\,dx,$$

which can be estimated as above. To estimate the interior term we just note that

$$\sqrt{|g_K|}g_K^{\alpha\beta}\partial_\alpha w\,\partial_\beta w=|\partial_x w|^2-|\partial_t w|^2+O(r^{-1})|\partial w|^2,$$

where the error term can be absorbed in the left of (4-3) for large enough R.

This finishes the proof of (4-1), and (4-2) follows in a similar manner. □

Applying Lemma 4.2 to w_α for some multiindex α, and using (2-2), we obtain the higher-order version of the estimates:

$$\|\partial w_\alpha\|_{L^2(C_T^R)}\lesssim R^{-1}\|w_{|\alpha|+n}\|_{L^2(\tilde{C}_T^R)}+R\|(\Box_K w)_{|\alpha|+n}\|_{L^2(\tilde{C}_T^R)},\tag{4-5}$$
$$\|\partial w_\alpha\|_{L^2(C_T^U)}\lesssim U^{-1}\|w_{|\alpha|+n}\|_{L^2(\tilde{C}_T^U)}+T\|(\Box_K w)_{|\alpha|+n}\|_{L^2(\tilde{C}_T^U)}.\tag{4-6}$$

Combining the two estimates above (4-5) and (4-6) with the Sobolev embeddings from Lemma 3.1 and the pointwise estimate for second-order derivatives in Lemma 3.3 we obtain:

Corollary 4.3. *For all $T\ge 1$ and $1\le R, U\le\frac{1}{4}T$ we have for some n independent of α*

$$\|\partial w_\alpha\|_{L^\infty(C_T^R)}\lesssim\frac{1}{R}\|w_{\le|\alpha|+n}\|_{L^\infty(\tilde{C}_T^R)}+R\|(\Box_K w)_{|\alpha|+n}\|_{L^\infty(\tilde{C}_T^R)},$$

respectively,

$$\|\partial w_\alpha\|_{L^\infty(C_T^U)}\lesssim\frac{1}{U}\|w_{\le|\alpha|+n}\|_{L^\infty(\tilde{C}_T^U)}+T\|(\Box_K w)_{|\alpha|+n}\|_{L^\infty(\tilde{C}_T^U)}.$$

Finally, we will derive a sharp estimate for the bad first-order derivative, following [Lindblad 1990].

Lemma 4.4. *Let $D_t=\{x:0\le t-|x|\le\frac{1}{4}t\}$, $C_t^q=\{x:t-|x|=q\}$, and let $\bar{w}(q)$ be any positive continuous function, where $q=t-r$. Suppose that $\Box\phi=F$. Then the following holds in D_t, $t\ge 1$:*

$$t|\partial\phi(t,x)\bar{w}(q)|\lesssim\sup_{4q\le\tau\le t}\Bigl(\|q\partial\phi(\tau,\cdot)\bar{w}\|_{L^\infty(C_\tau^q)}+\sum\nolimits_{|I|\le 1}\|Z^I\phi(\tau,\cdot)\bar{w}\|_{L^\infty(C_\tau^q)}\Bigr)$$
$$+\int_{4q}^t\Bigl(\langle\tau\rangle\|F(\tau,\cdot)\bar{w}\|_{L^\infty(C_\tau^q)}+\sum\nolimits_{|I|+|J|\le 2}\langle\tau\rangle^{-1}\|\partial^I\Omega^J\phi(\tau,\cdot)\bar{w}\|_{L^\infty(C_\tau^q)}\Bigr)\,d\tau.$$

Proof. We write

$$\Box\phi=-\frac{1}{r}\,\partial_v\,\partial_u(r\phi)+\frac{1}{r^2}\triangle_\omega\phi,$$

where $\partial_u = \partial_t - \partial_r$ and $\partial_v = \partial_t + \partial_r$. Hence in D_t

$$|\partial_v \partial_u (r\phi)| \lesssim |r\Box\phi| + \langle r\rangle^{-1}\sum_{|I|+|J|\le 2}|\partial^I\Omega^J\phi| \lesssim |\langle t\rangle\Box\phi| + \langle t\rangle^{-1}\sum_{|I|+|J|\le 2}|\partial^I\Omega^J\phi|. \tag{4-7}$$

Integrating this along the flow lines of the vector field ∂_v from the boundary of $D = \bigcup_{\tau\ge 0} D_\tau$ to any point inside D_t for $t \ge 1$. Using that $\bar{w}$ is constant along the flow lines, and (4-7), we obtain

$$\begin{aligned}&|\partial_u(r\phi(t,x))\bar{w}(q)|\\ &\quad\lesssim |\partial_u(r\phi)(4q,3q)\bar{w}(q)| + \int_{4q}^t \Big(\langle\tau\rangle\|F(\tau,\cdot)\bar{w}\|_{L^\infty(C^q_\tau)} + \sum_{|I|+|J|\le 2}\langle\tau\rangle^{-1}\|\partial^I\Omega^J\phi(\tau,\cdot)\bar{w}\|_{L^\infty(C^q_\tau)}\Big)\,d\tau.\end{aligned}$$

Moreover

$$t|\partial_u\phi(t,x)\bar{w}(q)| \lesssim |\partial_u(r\phi(t,x))\bar{w}(q)| + |\phi(t,x)\bar{w}(q)|$$

and

$$|\partial_u(r\phi)(4q,3q)\bar{w}(q)| \lesssim |q\partial_u\phi(4q,3q)\bar{w}(q)| + |\phi(4q,3q)\bar{w}(q)|.$$

The last three inequalities yield

$$\begin{aligned}t|\partial_u\phi(t,x)\bar{w}(q)| \lesssim &\sup_{4q\le\tau\le t}\big(\|q\partial\phi(\tau,\cdot)\bar{w}\|_{L^\infty(C^q_\tau)} + \|\phi(\tau,\cdot)\bar{w}\|_{L^\infty(C^q_\tau)}\big)\\ &+ \int_{4q}^t \Big(\langle\tau\rangle\|F(\tau,\cdot)\bar{w}\|_{L^\infty(C^q_\tau)} + \sum_{|I|+|J|\le 2}\langle\tau\rangle^{-1}\|\partial^I\Omega^J\phi(\tau,\cdot)\bar{w}\|_{L^\infty(C^q_\tau)}\Big)\,d\tau.\end{aligned}$$

The lemma follows from also using that $r|\partial\phi| \lesssim |r\partial_q\phi| + |S\phi| + |\Omega\phi|$. □

5. Pointwise estimates from the Minkowski fundamental solution

In this section, we translate pointwise bounds on the inhomogeneous terms into pointwise bounds for the solution by using the fundamental solution of the Minkowski metric.

For any $\beta, \gamma, \eta \in \mathbb{R}$, we define the weighted L^∞ norms

$$\|G\|_{L^\infty_{\beta,\gamma,\eta}} = \|\langle r\rangle^\beta\langle t\rangle^\gamma\langle t-r\rangle^\eta H(t,r)\|_{L^\infty_{t,r}}, \qquad H(t,r) = \sum_0^2 \|\Omega^i G(t,r\omega)\|_{L^2(\mathbb{S}^2)}.$$

We use the following lemma (see Section 6 of [Tohaneanu 2022]).

Lemma 5.1. *Let ψ solve*

$$\Box\psi = G, \qquad \psi(0) = 0, \qquad \partial_t\psi(0) = 0,$$

where G is supported in $\{|x| \le t + R_0\}$. Assume also that $2 \le \beta \le 3$ and $\eta \ge -\frac{1}{2}$. We define, for any arbitrary $\delta > 0$,

$$\tilde{\eta} = \begin{cases}\eta - \delta - 2, & \eta < 1,\\ -1, & \eta > 1.\end{cases}$$

(i) *If $\gamma \ge 0$, we have*

$$r\psi(t,x) \lesssim \frac{1}{\langle t-r\rangle^{\beta+\gamma+\tilde{\eta}-1}}\|G\|_{L^\infty_{\beta,\gamma,\eta}}. \tag{5-1}$$

(ii) *If* $\gamma < 0$, *we have*

$$r\psi(t,x) \lesssim \frac{\langle t\rangle^{-\gamma}}{\langle t-r\rangle^{\beta+\tilde{\eta}-1}} \|G\|_{L^\infty_{\beta,\gamma,\eta}}. \tag{5-2}$$

(iii) *If* $\eta > 1$, *we have*

$$r\psi(t,x) \lesssim \left\langle \ln \frac{\langle t\rangle}{\langle t-r\rangle} \right\rangle \|G\|_{L^\infty_{2,0,\eta}}. \tag{5-3}$$

Proof. Note first that, after a translation in time, we may assume that $R_0 = 0$.

We use the ideas from [Metcalfe et al. 2012]. Define

$$H(t,r) = \sum_0^2 \|\Omega^i G(t, r\omega)\|_{L^2(\mathbb{S}^2)}.$$

By Sobolev embeddings on the sphere, we have $|G| \lesssim H$. Let v be the radial solution to

$$\Box v = H, \quad v[0] = 0.$$

By the positivity of the fundamental solution, we have that $|\psi| \lesssim |v|$. On the other hand, we can write v explicitly:

$$rv(t,r) = \frac{1}{2}\int_{D_{tr}} \rho H(s,\rho)\, ds\, d\rho,$$

where D_{tr} is the rectangle

$$D_{tr} = \{0 \le s-\rho \le t-r,\ t-r \le s+\rho \le t+r\}.$$

We partition the set D_{tr} into a double dyadic manner as

$$D_{tr} = \bigcup_{R\le t} D_{tr}^R, \quad D_{tr}^R = D_{tr} \cap \{R < r < 2R\}$$

and estimate the corresponding parts of the above integral.

We clearly have

$$\int_{D_{tr}^R} \rho H\, ds\, d\rho \lesssim \|G\|_{L^\infty_{\beta,\gamma,\eta}} \int_{D_{tr}^R} \rho^{1-\beta} \langle s\rangle^{-\gamma} \langle s-\rho\rangle^{-\eta}\, d\rho\, ds.$$

We now consider two cases:

(i) $R < \frac{1}{8}(t-r)$. Here we have $\rho \sim R$ and $s \approx s-\rho \approx \langle t-r\rangle$; therefore we obtain

$$\int_{D_{tr}^R} \rho^{1-\beta} \langle s\rangle^{-\gamma} \langle s-\rho\rangle^{-\eta}\, d\rho\, ds \lesssim R^{3-\beta} \langle t-r\rangle^{-\gamma-\eta},$$

and after summation, using that $\beta \le 3$, we obtain

$$\sum_{R<(t-r)/8} \int_{D_{tr}^R} \rho H\, ds d\rho \lesssim \frac{\ln\langle t-r\rangle \langle t-r\rangle^{3-\beta}}{\langle t-r\rangle^{\gamma+\eta}} \lesssim \frac{1}{\langle t-r\rangle^{\beta+\tilde{\eta}}},$$

which is the desired bound in all cases.

(ii) $\frac{1}{8}(t-r) < R < t$. Here we have $\rho \sim R$ and $t \geq s \gtrsim R$. Let $u = s - \rho$.

Assume first that $\gamma \geq 0$; then

$$\int_{D_{tr}^R} \rho^{1-\beta} \langle s \rangle^{-\gamma} \langle s - \rho \rangle^{-\eta} \, d\rho \, ds \lesssim R^{2-\beta-\gamma} \int_0^{t-r} \langle u \rangle^{-\eta} \, du \lesssim R^{2-\beta-\gamma} \langle t - r \rangle^{\mu(\eta)},$$

where

$$\mu(\eta) = \begin{cases} 1-\eta, & \eta < 1, \\ 0, & \eta > 1. \end{cases}$$

If $\beta + \gamma > 2$, we obtain after summation

$$\sum_{R > (t-r)/8} \int_{D_{tr}^R} \rho H \, ds \, d\rho \lesssim \langle t - r \rangle^{2-\beta-\gamma+\mu(\eta)},$$

which is (5-1).

Assume now that $\beta = 2$ and $\gamma = 0$. Equation (5-3) is obvious when $t \leq 1$. When $t \geq 1$, we see that there are $\ln(t/\langle t-r \rangle)$ dyadic regions when $\frac{1}{8}(t-r) < R < t$, so we obtain (5-3) after summation.

Finally, if $\gamma < 0$ we obtain

$$\int_{D_{tr}^R} \rho^{1-\beta} \langle s \rangle^{-\gamma} \langle s - \rho \rangle^{-\eta} \, d\rho \, ds \lesssim R^{2-\beta} \langle t \rangle^{-\gamma} \int_0^{t-r} \langle u \rangle^{-\eta} \, du \lesssim R^{2-\beta} \langle t \rangle^{-\gamma} \langle t - r \rangle^{\mu(\eta)}.$$

Since $\beta \geq 2$, we obtain after summation

$$\sum_{R > (t-r)/8} \int_{D_{tr}^R} \rho H \, ds \, d\rho \lesssim \langle t \rangle^{-\gamma} \langle t - r \rangle^{2-\beta+\mu(\eta)},$$

which is (5-2). □

6. Setup for pointwise estimates

In this section, we will slightly adjust $\Box_K$ to an operator closer to $\Box$ (with respect to the $(\tilde{t}, x)$-coordinates). Indeed, we let

$$P = |g_K|^{1/4} (-g_K^{\tilde{t}\tilde{t}})^{-1/2} \Box_K (-g_K^{\tilde{t}\tilde{t}})^{-1/2} |g_K|^{-1/4}.$$

P is self-adjoint with respect to $d\tilde{t}\, dx$. More importantly, a quick computation yields that

$$P = \partial_\alpha (g_K^{\alpha\beta} (-g_K^{\tilde{t}\tilde{t}}) \partial_\beta) + V, \quad V = |g_K|^{1/4} (-g_K^{\tilde{t}\tilde{t}})^{-1/2} \Box_K ((-g_K^{\tilde{t}\tilde{t}})^{-1/2} |g_K|^{-1/4}).$$

It is easy to see that $V \in S^Z(r^{-3})$.

Let us first consider the Schwarzschild metric. In this case we have that, for large r, $-g_S^{\tilde{t}\tilde{t}} = g_S^{r^* r^*}$ and $g_S^{\tilde{t} r^*} = 0$. We thus have

$$P = \Box + P_{lr},$$

where the long-range spherically symmetric part P_{lr} has the form

$$P_{lr} = g_{lr}(r) \Delta_\omega + V, \quad g_{lr} \in S^Z(r^{-3}), \; V \in S^Z(r^{-3}). \tag{6-1}$$

For the Kerr metric, we use the fact that the metric coefficients have the following properties:

$$g_K^{\alpha\beta} - g_S^{\alpha\beta} \in S^Z(r^{-2}), \tag{6-2}$$

$$\partial g_K \in S^Z(r^{-2}), \quad \partial^2 g_K \in S^Z(r^{-3}). \tag{6-3}$$

Using (6-1) and (6-2) we see that we can write

$$P = \Box + P_{lr} + P_{sr}, \tag{6-4}$$

where the short-range part P_{sr} has the form

$$P_{sr} = \partial_\alpha g_{sr}^{\alpha\beta} \partial_\beta, \quad g_{sr}^{\alpha\beta} \in S^Z(r^{-2}). \tag{6-5}$$

Using (6-3) we see that for any function ϕ we have

$$P\phi = (-g_K^{\tilde{t}\tilde{t}})\Box_K \phi + h_1 \phi + h_2 \partial\phi, \quad h_1 \in S^Z(r^{-3}),\ h_2 \in S^Z(r^{-2}). \tag{6-6}$$

Now pick any multiindex α. After commuting with vector fields, using (6-4), (6-1), and (6-5), we obtain

$$P\phi_\alpha \in S^Z(1)(\Box_K \phi)_{\le|\alpha|} + S^Z(r^{-3})\phi_{\le|\alpha|+6} + S^Z(r^{-2})\partial\phi_{\le|\alpha|+5},$$

which in turn implies, using (6-4),

$$\Box\phi_\alpha \in S^Z(1)(\Box_K \phi)_{\le|\alpha|} + S^Z(r^{-3})\phi_{\le|\alpha|+6} + S^Z(r^{-2})\partial\phi_{\le|\alpha|+5}. \tag{6-7}$$

Moreover, by finite speed of propagation, and the assumption on the support of the initial data, the right-hand side is supported in the forward light cone $\{|x| < \tilde{t} + R_0\}$.

We will use (6-7) in the next section to extract more decay for the solution.

7. The bootstrap argument for the Einstein model

We now prove Theorem 1.1 by using a bootstrap argument. We first write

$$\mathcal{E}_N(0) = \mu_N \epsilon,$$

where $\mu_N > 0$ is a fixed, small N-dependent constant to be determined below (see (7-5), (7-6)).

Let $N_1 = \frac{1}{2}N$. We will assume that the following a priori bounds hold for some large constant $\widetilde{C}$ independent of ϵ and $\tilde{t}$, and a fixed small $\delta > 0$:

$$\mathcal{E}_N(\tilde{t}) \le \widetilde{C}\mu_N \epsilon \langle \tilde{t}\rangle^\delta, \tag{7-1}$$

$$|\phi_{\le N_1+2}| \le \frac{\epsilon\langle \ln(\langle\tilde{t}\rangle/\langle\tilde{t}-\tilde{r}\rangle)\rangle}{\langle\tilde{t}\rangle}, \quad |\partial\phi_{\le N_1+2}| \le \frac{\epsilon\langle \ln(\langle\tilde{t}\rangle/\langle\tilde{t}-\tilde{r}\rangle)\rangle}{r\langle\tilde{t}-\tilde{r}\rangle}, \tag{7-2}$$

$$|(\partial\phi_{TU})_{\le N_1+2}| \le \frac{\epsilon}{\langle\tilde{t}\rangle}. \tag{7-3}$$

Clearly (7-1), (7-2) and (7-3) hold for small times. We assume now that the bounds hold on some time interval $0 \le \tilde{t} \le T$, and we improve the constants by $\frac{1}{2}$. By the continuity method this implies that the solution exists globally, and that the bounds also hold globally.

In order to improve (7-1), we show that, for small enough ϵ, there is C_N independent of T so that

$$\mathcal{E}_N(\tilde{t}) \le C_N \langle \tilde{t} \rangle^{C_N \epsilon} \mathcal{E}_N(0), \quad 0 \le \tilde{t} \le T. \tag{7-4}$$

If we now additionally take $\widetilde{C} = 2C_N$ and $\epsilon < \delta/C_N$, we thus improve the a priori bound for $\mathcal{E}_N(\tilde{t})$ to

$$\mathcal{E}_N(\tilde{t}) \le \tfrac{1}{2}\widetilde{C}\mu_N \epsilon \langle \tilde{t} \rangle^{\delta}.$$

In order to improve the pointwise bounds, we will show that, for some fixed positive integer m, independent of N, we have

$$|\phi_{\le N-m}| \lesssim \frac{\mathcal{E}_N(0)\langle \ln(\langle \tilde{t}\rangle/\langle \tilde{t}-\tilde{r}\rangle)\rangle}{\langle \tilde{t}\rangle}, \quad |\partial\phi_{\le N-m}| \lesssim \frac{\mathcal{E}_N(0)\langle \ln(\langle \tilde{t}\rangle/\langle \tilde{t}-\tilde{r}\rangle)\rangle}{r\langle \tilde{t}-\tilde{r}\rangle}, \tag{7-5}$$

$$|(\partial\phi_{TU})_{\le N-m}| \lesssim \frac{\mathcal{E}_N(0)}{r\langle \tilde{t}-\tilde{r}\rangle^{1-\delta}}. \tag{7-6}$$

We can now pick a small μ_N to improve (7-2) and (7-3).

7.1. ***The energy estimates.*** We will now use assumptions (7-2) and (7-3) to show (7-4) for small enough ϵ. By Gronwall's inequality and (2-8), it is enough to show that

$$\|(\Box_K \phi)_{\le N}\|_{LE^*_w[0,\tilde{t}]} \lesssim \int_0^{\tilde{t}} \frac{\epsilon}{\langle \tau \rangle} \mathcal{E}_N(\tau)\, d\tau + \epsilon \mathcal{E}_N(\tilde{t}). \tag{7-7}$$

We can write, using (2-4), (2-5) and (2-6),

$$\Box_K \phi \in S^Z(1)(\partial\phi_{TU})^2 + S^Z(1)\partial\phi\bar{\partial}\phi.$$

After commuting with vector fields, and using (2-3), we also get that

$$(\Box_K\phi)_{\le N} \lesssim (\partial\phi_{TU})_{\le N_1}(\partial\phi_{TU})_{\le N} + \partial\phi_{\le N_1}\bar{\partial}\phi_{\le N} + \bar{\partial}\phi_{\le N_1}\partial\phi_{\le N} + r^{-1}\partial\phi_{\le N_1}\partial\phi_{\le N-1}. \tag{7-8}$$

The first term is easy. By (7-3) we have

$$\|(\partial\phi_{TU})_{\le N_1}(\partial\phi_{TU})_{\le N}\|_{L^1[0,\tilde{t}]L^2} \lesssim \int_0^{\tilde{t}} \frac{\epsilon}{\langle \tau \rangle} \mathcal{E}_N(\tau)\, d\tau.$$

Similarly, the last term can be estimated in L^1L^2. Indeed, we note that (7-2) implies

$$|r^{-1}\partial\phi_{\le N_1}| \lesssim \frac{\epsilon}{\langle \tilde{t} \rangle},$$

and thus

$$\|r^{-1}\partial\phi_{\le N_1}\partial\phi_{\le N-1}\|_{L^1[0,\tilde{t}]L^2} \lesssim \int_0^{\tilde{t}} \frac{\epsilon}{\langle \tau \rangle} \mathcal{E}_N(\tau)\, d\tau.$$

For the second term, we divide it into two parts. When $r < R_1$ we have by (7-2)

$$\|\partial\phi_{\le N_1}\bar{\partial}\phi_{\le N}\|_{L^1[0,\tilde{t}]L^2(r<R_1)} \lesssim \int_0^{\tilde{t}} \frac{\epsilon}{\langle \tau \rangle} \mathcal{E}_N(\tau)\, d\tau.$$

When $r > R_1$, we use (7-2) and the last term in (2-7):

$$\|\partial\phi_{\le N_1}\bar\partial\phi_{\le N}\|^2_{\mathrm{LE}^*[0,t]} \lesssim \int_0^{\tilde t}\int_{r>R_1}\frac{\epsilon^2\tau^{2\delta}}{r\langle\tau-\tilde r\rangle^{2+2\delta}}|\bar\partial\phi_{\le N}|^2\,dV \lesssim \|\epsilon\langle\tau-\tilde r\rangle^{(-1-\delta_1)/2}\bar\partial\phi_{\le N}\|^2_{L^2[0,\tilde t]L^2(r\ge R_1)} \lesssim (\epsilon\mathcal{E}_N(\tilde t))^2.$$

For the third term, note that (2-1) and (7-2) imply that

$$|\bar\partial\phi_{\le N_1}| \lesssim \frac{\epsilon}{\langle\tilde t\rangle}. \tag{7-9}$$

Using (7-9) gives

$$\|\bar\partial\phi_{\le N_1}\partial\phi_{\le N}\|_{L^1[0,\tilde t]L^2} \lesssim \int_0^{\tilde t}\frac{\epsilon}{\langle\tau\rangle}\mathcal{E}_N(\tau)\,d\tau.$$

Putting all these together we obtain (7-7).

7.2. ***The decay estimates.*** We now show that (7-5) and (7-6) hold.

The proof uses an iteration procedure. The most important part here is to obtain pointwise decay rates of $\tilde t^{-1}$ near the trapped set for all components. We start with a weak decay rate of $\tilde t^{-1/2+C\epsilon}$ given by the slow growth $\tilde t^{C\epsilon}$ combined with the results of Section 3. We then use Lemma 5.1 to improve decay in r, followed by Corollary 4.3 to improve the decay of derivatives. Lemma 4.1 then allows us to turn the r-decay into $\tilde t$-decay. This yields an improved global decay rate of $\tilde t^{-1+C\epsilon}$, which is barely not enough. We then use Lemma 4.4 to improve the decay of the derivative of the good components $\partial\phi_{TU}$ to $\tilde t^{-1}$ near the cone. We can now go back to the iteration procedure, and use the improved bounds combined with Lemma 5.1, Corollary 4.3 and Lemma 4.1 to improve the decay rate of all components to $\tilde t^{-1}$ away from the cone. This finishes the proof.

Let $N_2 = N - 13$. We first note that (3-3) and (3-4), combined with the energy bounds (7-4), yield the weak pointwise bounds

$$|\partial\phi_{\le N_2}| \lesssim \frac{\langle\tilde t\rangle^{C\epsilon}\mathcal{E}_N(0)}{r\langle\tilde t-\tilde r\rangle^{1/2}}, \quad |\phi_{\le N_2}| \lesssim \frac{\langle\tilde t-\tilde r\rangle^{1/2}\mathcal{E}_N(0)}{\langle\tilde t\rangle^{1-C\epsilon}}. \tag{7-10}$$

We now need to improve the decay of $\phi_{\le N-m}$ and $\partial\phi_{\le N-m}$. To that extent, we will use Lemma 5.1, followed by Lemma 4.1 and Corollary 4.3.

We cannot apply Lemma 5.1 directly. On one hand, we have no control on the solution for $r \ll 2M$, and on the other hand, the initial data is not trivial. Instead, let

$$\chi = \chi_1(\tilde r)\chi_2(\tilde t).$$

Here $\chi_1 \equiv 1$ for $\tilde r \ge R \gg M$ and supported in $\tilde r \ge \frac12 R$, while $\chi_2 \equiv 1$ for $\tilde t \ge 1$ and supported in $\tilde t \ge \frac12$.

We now consider $\psi_{\alpha\beta} = \chi\phi_{\alpha\beta}$. Using (6-7), we see that ψ satisfies the system

$$\Box(\psi_{\le n}) = G_n, \quad G_n \in S^Z(r^{-2})\partial\phi_{\le n+5} + S^Z(r^{-3})\phi_{\le n+6} + S^Z(1)(\partial\phi_{\le n})^2,$$

with trivial initial data, and G_n supported in the region $r \geq \frac{1}{2}R$. Using (7-10), we see that, for all $n \leq N_3 := N_2 - 12$, we have

$$G_{n+6} \lesssim \mathcal{E}_N(0)\left(\frac{\langle\tilde{t}\rangle^{C\epsilon}}{r^3\langle\tilde{t}-\tilde{r}\rangle^{1/2}} + \frac{\langle\tilde{t}-\tilde{r}\rangle^{1/2}}{r^3\langle\tilde{t}\rangle^{1-C\epsilon}} + \frac{\langle\tilde{t}\rangle^{C\epsilon}}{r^2\langle\tilde{t}-\tilde{r}\rangle}\right).$$

We now apply Lemma 5.1. The first term on the right-hand side is controlled by the other two terms. For the second term we use (5-1) with $\beta = 3$, $\gamma = 1 - C\epsilon$ and $\eta = -\frac{1}{2}$. For the third term, we use (5-1) with $\beta = 2$, $\gamma = -C\epsilon$ and $\eta = 1 - C\epsilon$. We obtain

$$|\phi_{\leq N_3}| \lesssim \frac{\langle\tilde{t}\rangle^{C\epsilon}}{r}\mathcal{E}_N(0). \tag{7-11}$$

We now plug in the bounds (7-11) and (7-10) into Corollary 4.3. We thus obtain for $N_4 = N_3 - n$ with n from Corollary 4.3:

$$\|\partial\phi_{N_4}\|_{L^\infty(C_T^R)} \lesssim \frac{1}{R}\frac{T^{C\epsilon}}{R}\mathcal{E}_N(0) + R\left(\frac{T^{C\epsilon}}{RT^{1/2}}\mathcal{E}_N(0)\right)^2 \lesssim \frac{T^{C\epsilon}}{R^2}\mathcal{E}_N(0),$$
$$\|\partial\phi_{N_4}\|_{L^\infty(C_T^U)} \lesssim \frac{1}{U}\frac{T^{C\epsilon}}{R}\mathcal{E}_N(0) + T\left(\frac{T^{C\epsilon}}{RU^{1/2}}\mathcal{E}_N(0)\right)^2 \lesssim \frac{T^{C\epsilon}}{RU}\mathcal{E}_N(0).$$

The last two inequalities can be written as

$$|\partial\phi_{\leq N_4}| \lesssim \frac{\langle\tilde{t}\rangle^{1+C\epsilon}}{r^2\langle\tilde{t}-\tilde{r}\rangle}\mathcal{E}_N(0). \tag{7-12}$$

We now use Lemma 4.1. Note that (7-11) and (7-12) yield

$$\|\langle r\rangle\phi_{\leq N_4}\|_{\mathrm{LE}^1(C_T^{<T/2})} \lesssim T^{1/2+C\epsilon}\mathcal{E}_N(0).$$

Moreover, (7-10) implies that

$$\|(\Box_K\phi)_{\leq N_4}\|_{\mathrm{LE}^*(C_T^{<T/2})} \lesssim T^{-1/2+C\epsilon}\mathcal{E}_N(0).$$

The two inequalities above and Lemma 4.1 with $N_5 = N_4 - n$ give us

$$\|\phi_{\leq N_5}\|_{\mathrm{LE}^1(C_T^{<T/2})} \lesssim T^{-1/2+C\epsilon}\mathcal{E}_N(0),$$

which combined with the Sobolev embeddings from Lemma 3.1 give for $N_6 = N_5 - 13$

$$|\phi_{\leq N_6}| \lesssim \langle\tilde{t}\rangle^{-1+C\epsilon}\mathcal{E}_N(0). \tag{7-13}$$

We now plug in the bounds (7-13) and (7-10) into Corollary 4.3. We thus obtain for $N_7 = N_6 - n$

$$\|\partial\phi_{\leq N_7}\|_{L^\infty(C_T^R)} \lesssim \frac{1}{R}\frac{T^{C\epsilon}}{T}\mathcal{E}_N(0) + R\left(\frac{T^{C\epsilon}}{RT^{1/2}}\mathcal{E}_N(0)\right)^2 \lesssim \frac{T^{C\epsilon}}{RT}\mathcal{E}_N(0).$$

Combined with (7-12), this gives

$$|\partial\phi_{\leq N_7}| \lesssim \frac{\langle\tilde{t}\rangle^{C\epsilon}}{r\langle\tilde{t}-\tilde{r}\rangle}\mathcal{E}_N(0). \tag{7-14}$$

Note also that (2-1), (7-13) and (7-14) give

$$|\bar{\partial}\phi_{\le N_7-2}| \lesssim \frac{\langle \tilde{t}\rangle^{C\epsilon}}{r\langle \tilde{t}\rangle}\mathcal{E}_N(0). \tag{7-15}$$

Equations (7-13) and (7-14) almost finish the proof of (7-5), except that we need to replace $\langle \tilde{t}\rangle^{C\epsilon}$ by $\langle \ln(\langle \tilde{t}\rangle/\langle \tilde{t}-\tilde{r}\rangle)\rangle$.

We now prove the fact that ψ_{TU} actually satisfies better decay estimates. Indeed, note first that

$$\Box(T^\alpha U^\beta \phi_{\alpha\beta}) - T^\alpha U^\beta \Box\phi_{\alpha\beta} \in S^Z(r^{-2})\phi_{\le 1}.$$

Using (6-4) and (6-6) we obtain

$$\Box\phi_{TU} \in S^Z(1)(\Box_K\phi)_{TU} + S^Z(r^{-2})\phi_{\le 6}.$$

Moreover,

$$(\Box_K\phi)_{TU} \in S^Z(1)\partial\phi\bar{\partial}\phi,$$

and thus

$$\Box\phi_{TU} \in S^Z(1)\partial\phi\bar{\partial}\phi + S^Z(r^{-2})\phi_{\le 6}.$$

After commuting with vector fields (in particular using (2-3)) and applying the cutoff we thus obtain

$$\Box(\psi_{TU})_{\le m} = H_m, \quad H_m \in S^Z(r^{-2})\phi_{\le m+6} + S^Z(1)\partial\phi_{\le m}\bar{\partial}\phi_{\le m} + S^Z(r^{-1})(\partial\phi_{\le m})^2.$$

Using (7-13), (7-14) and (7-15), we see that

$$|H_m| \lesssim \frac{\mathcal{E}_N(0)}{r^2\langle \tilde{t}\rangle^{1-C\epsilon}}, \quad m \le N_7-2. \tag{7-16}$$

Let $N_8 = N_7 - 6$. We now apply Lemma 4.4 with $\bar{w}(q) = \langle q\rangle^{1-\delta}$ to $(\psi_{TU})_{\le N_8}$. Note first that, due to (7-14) and (7-13) we have

$$\sup_{4q\le\tau\le\tilde{t}}\left(\|q\,\partial\phi_{\le N_8}(\tau,\cdot)\bar{w}\|_{L^\infty(C^q_\tau)} + \sum_{|I|\le 1}\|Z^I\phi_{\le N_8}(\tau,\cdot)\bar{w}\|_{L^\infty(C^q_\tau)}\right) \lesssim \mathcal{E}_N(0).$$

Moreover, (7-13) implies that

$$\int_{4q}^{\tilde{t}}\sum_{|I|\le 2}\langle\tau\rangle^{-1}\|\Omega^I\phi_{\le N_8}(\tau,\cdot)\bar{w}\|_{L^\infty(C^q_\tau)}\,d\tau \lesssim \int_{4q}^{\tilde{t}}\langle\tau\rangle^{-1}\frac{\langle q\rangle^{1-\delta}\mathcal{E}_N(0)}{\langle\tau\rangle^{1-C\epsilon}}\,d\tau \lesssim \mathcal{E}_N(0).$$

Finally, we obtain by (7-16) that

$$\int_{4q}^{\tilde{t}}\langle\tau\rangle\|H_m(\tau,\cdot)\bar{w}\|_{L^\infty(C^q_\tau)} \lesssim \int_{4q}^{\tilde{t}}\langle\tau\rangle\frac{\langle q\rangle^{1-\delta}\mathcal{E}_N(0)}{\langle\tau\rangle^{3-C\epsilon}}\,d\tau \lesssim \mathcal{E}_N(0).$$

Lemma 4.4 thus implies, in conjunction with (7-14), that

$$|\partial(\psi_{TU})_{\le N_8}| \lesssim \frac{\mathcal{E}_N(0)}{r\langle \tilde{t}-\tilde{r}\rangle^{1-\delta}}. \tag{7-17}$$

This finishes the proof of (7-6).

Finally, to obtain a decay rate of $1/\tilde{t}$ in the interior, we see that, using (6-7) and (7-8), we can write our system as

$$\Box(\psi_{\le m}) = J_m,$$
$$J_m \in S^Z(r^{-2})\partial\phi_{\le m+5} + S^Z(r^{-3})\phi_{\le m+6} + S^Z(1)(\partial\phi_{TU})^2_{\le m} + S^Z(1)\partial\phi_{\le m}\bar{\partial}\phi_{\le m} + S^Z(r^{-1})(\partial\phi_{\le m})^2,$$

and J_m is supported in the region $\{\tilde{t} \ge \frac{1}{2},\ \tilde{r} \ge \frac{1}{2}R\}$. Due to the improved bounds (7-13), (7-14) and (7-17) we obtain

$$|J_{m+6}| \lesssim \mathcal{E}_N(0)\left(\frac{\langle\tilde{t}\rangle^{C\epsilon}}{r^3\langle\tilde{t}-\tilde{r}\rangle} + \frac{1}{r^2\langle\tilde{t}-\tilde{r}\rangle^{2-2\delta}}\right), \qquad m \le N_9 := N_8 - 8.$$

We now apply Lemma 5.1 and in particular (5-3) to control the last term. We obtain

$$|\psi_{\le N_9}| \lesssim \frac{\langle \ln(\langle\tilde{t}\rangle/\langle\tilde{t}-\tilde{r}\rangle)\rangle}{r}\mathcal{E}_N(0). \tag{7-18}$$

Corollary 4.3 thus implies, with $N_{10} = N_9 - n$,

$$|\partial\psi_{\le N_{10}}| \lesssim \frac{\langle \ln(\langle\tilde{t}\rangle/\langle\tilde{t}-\tilde{r}\rangle)\rangle}{r\langle\tilde{t}-\tilde{r}\rangle}\mathcal{E}_N(0). \tag{7-19}$$

Equations (7-18) and (7-19) finish the proof of (7-5) when $\tilde{r} \ge \frac{1}{2}\tilde{t}$.

All that is left is to replace r by $\tilde{t}$ in the region $\tilde{r} \le \frac{1}{2}\tilde{t}$. Note first that (7-18) and (7-19), combined with (7-13) and (7-14), yield the (relatively weak) bound

$$|\phi_{\le N_{10}}| \lesssim \frac{1}{r}\mathcal{E}_N(0), \quad \partial\phi_{\le N_{10}} \lesssim \frac{1}{r^2}\mathcal{E}_N(0), \quad \tilde{r} < \frac{3\tilde{t}}{4}. \tag{7-20}$$

We now use Lemma 4.1. Note that (7-20) gives

$$\|\langle r\rangle\phi_{\le N_{10}}\|_{LE^1(C_T^{<T/2})} \lesssim T^{1/2}\mathcal{E}_N(0).$$

Moreover, (7-14) implies that

$$\|(\Box_K\phi)_{\le N_{10}}\|_{LE^*(C_T^{<T/2})} \lesssim T^{-1/2}\mathcal{E}_N(0).$$

The two inequalities above and Lemma 4.1 give us, for $N_{11} = N_{10} - n$,

$$\|\phi_{\le N_{11}}\|_{LE^1(C_T^{<T/2})} \lesssim T^{-1/2}\mathcal{E}_N(0),$$

which, combined with the Sobolev embeddings from Lemma 3.1 with $N_{12} = N_{11} - 13$, gives

$$|\phi_{\le N_{12}}| \lesssim \frac{\mathcal{E}_N(0)}{\langle\tilde{t}\rangle}, \quad \tilde{r} \le \frac{\tilde{t}}{2}.$$

Finally, one last application of Corollary 4.3 with $N_{13} = N_{12} - n$ gives

$$|\partial\phi_{\le N_{13}}| \lesssim \frac{\mathcal{E}_N(0)}{r\langle\tilde{t}\rangle}, \quad \tilde{r} \le \frac{\tilde{t}}{2}.$$

This finishes the proof of (7-5) if we pick N large enough so that $N_{13} \ge N_1$.

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