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We study long-time existence and asymptotic behavior for the L^2 -gradient flow of the Willmore energy, under the condition that the initial datum is a torus of revolution. We show that if an initial datum has Willmore energy below 8π then the solution of the Willmore flow converges for $t \rightarrow \infty$ to the Clifford torus, possibly rescaled and translated. The energy threshold of 8π turns out to be optimal for such a convergence result. We give an application to the conformally constrained Willmore minimization problem.

1. Introduction

Let $f : \Sigma \rightarrow \mathbb{R}^3$ be a smooth immersion of a two-dimensional manifold without boundary. Its *Willmore energy* is

$$\mathcal{W}(f) = \frac{1}{4} \int_{\Sigma} |\vec{H}|^2 d\mu, \quad (1-1)$$

where \vec{H} denotes the mean curvature vector and $d\mu$ the induced Riemannian measure. Its critical points are called *Willmore immersions* and satisfy

$$\Delta \vec{H} + Q(\mathring{A})\vec{H} = 0, \quad (1-2)$$

where Δ denotes the Laplace–Beltrami operator, \mathring{A} is the trace-free second fundamental form and Q is quadratic in \mathring{A} (see (2-3)). If $f(\Sigma)$ is orientable (or two-sided, which is equivalent in \mathbb{R}^3) then $\vec{H} = (\kappa_1 + \kappa_2)\vec{N}$, with κ_1, κ_2 the principal curvatures of $f(\Sigma)$ and \vec{N} a smooth normal vector field. The L^2 -gradient flow of the Willmore functional with given initial datum f_0 , a smooth immersion, is

$$\partial_t f = -(\Delta \vec{H} + Q(\mathring{A})\vec{H}), \quad (1-3)$$

with $f(t=0) = f_0$. This fourth-order quasilinear geometric evolution equation has been extensively studied in [Kuwert and Schätzle 2001; 2002], where a blow-up criterion is formulated. With the aid of this criterion the same authors proved in [Kuwert and Schätzle 2004] long-time existence and convergence for the *flow of spherical immersions* under the assumption that the initial immersion $f_0 : \mathbb{S}^2 \rightarrow \mathbb{R}^3$ satisfies $\mathcal{W}(f_0) < 8\pi$. The energy threshold of 8π is shown to be sharp in [Blatt 2009] for the convergence of spherical immersions.

In the classical work [Mayer and Simonett 2002] the Willmore flow is studied numerically, not only for spheres but also for surfaces of different genus, such as tori. See also [Barrett et al. 2019] for other numerical examples. In [Mayer and Simonett 2002, Section 8.1] it is stated that the flow converges for

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all tori that the authors looked at, which was astounding as this behavior differs fundamentally from the surface diffusion flow, where the hole of all initial tori seems to close and the curvature blows up; see [Mayer 2001; Barrett et al. 2019]. Our goal is to understand analytically what happens to tori along the Willmore flow. In this article we only look at the special case of tori of revolution.

Definition 1.1. In the sequel we identify $\mathbb{S}^1 = \mathbb{R}/\mathbb{Z}$ and set $\mathbb{H}^2 := \mathbb{R} \times (0, \infty)$. We call an immersion $f : \mathbb{S}^1 \times \mathbb{S}^1 \rightarrow \mathbb{R}^3$ a *torus of revolution* if there exists an immersed curve $\gamma \in C^\infty(\mathbb{S}^1, \mathbb{H}^2)$, $\gamma = (\gamma^{(1)}, \gamma^{(2)})$, such that

$$f(u, v) = \begin{pmatrix} \gamma^{(1)}(u) \\ \gamma^{(2)}(u) \cos(2\pi v) \\ \gamma^{(2)}(u) \sin(2\pi v) \end{pmatrix}. \quad (1-4)$$

We call γ *profile curve* and we will frequently denote f as in (1-4) by F_γ .

An essential element in our argument is that the property of being a torus of revolution is preserved along the Willmore flow. Hence the evolution by Willmore flow can also be regarded as a time evolution of the profile curves. In the arguments to come we will take advantage of an interplay between the revolution symmetry and the blow-up-criterion developed in [Kuwert and Schätzle 2001; 2002]. With this technique we have identified a geometric quantity whose boundedness ensures convergence. This quantity is the *hyperbolic length* of the profile curves given by

$$\mathcal{L}_{\mathbb{H}^2}(\gamma) := \int_{\mathbb{S}^1} \frac{|\gamma'(x)|}{\gamma^{(2)}(x)} dx, \quad \gamma \in C^\infty(\mathbb{S}^1, \mathbb{R} \times (0, \infty)).$$

Strikingly, the *hyperbolic geometry* of the curve evolution is decisive for the convergence behavior. We recall that the hyperbolic plane $\mathbb{H}^2 = \mathbb{R} \times (0, \infty)$ is endowed with the metric $g_{(x,y)} = y^{-2} dx dy$.

Now we can state our main convergence criterion:

Theorem 1.2. *Let $f : [0, T) \times \mathbb{S}^1 \times \mathbb{S}^1 \rightarrow \mathbb{R}^3$ be a maximal evolution by Willmore flow such that $f(0)$ is a torus of revolution. Then $f(t)$ is a torus of revolution for all $t \in [0, T)$. Suppose that $(\gamma(t))_{t \in [0, T)}$ is a collection of profile curves of $f(t)$. If*

$$\liminf_{t \rightarrow T} \mathcal{L}_{\mathbb{H}^2}(\gamma(t)) < \infty, \quad (1-5)$$

then $T = \infty$ and the Willmore flow converges (up to reparametrizations) in C^k for all k to a Willmore torus of revolution f_∞ .

We remark that the concept of C^k -convergence that we impose is a *geometric* one; see Appendix C (Definition C.7) for details. From now on, the term C^k -convergence is understood up to reparametrizations as in Definition C.7.

That the *hyperbolic geometry* of the profile curve plays a role is not surprising — there is an interesting correspondence between the Willmore energy of tori of revolution and the hyperbolic elastic energy of curves, observed in [Langer and Singer 1984a]. With this correspondence one can for example show the Willmore conjecture for tori of revolution; see [Langer and Singer 1984b]. Other applications of this relationship include [Dall'Acqua et al. 2008; Mandel 2018]. To the authors' knowledge, this is the first time that this correspondence is used in a problem depending on time.

The main question now is to identify which initial data generate evolutions with bounded hyperbolic length. It turns out that the same energy threshold of 8π needed for spherical immersions (see [Kuwert and Schätzle 2004]) is needed in the case of tori of revolution.

Theorem 1.3. *Let $f_0 : \mathbb{S}^1 \times \mathbb{S}^1 \rightarrow \mathbb{R}^3$ be a torus of revolution satisfying $\mathcal{W}(f_0) \leq 8\pi$. Let $f : [0, T) \times (\mathbb{S}^1 \times \mathbb{S}^1) \rightarrow \mathbb{R}^3$ evolve by the Willmore flow with initial datum f_0 . Then $T = \infty$ and f converges in C^k for all $k \in \mathbb{N}$ to the Clifford torus, possibly rescaled and translated in the direction $(1, 0, 0)$.*

Here the Clifford torus is the surface of revolution given by

$$(u, v) \mapsto \left(\frac{1}{\sqrt{2}} \sin(2\pi u), \left(1 + \frac{1}{\sqrt{2}} \cos(2\pi u)\right) \cos(2\pi v), \left(1 + \frac{1}{\sqrt{2}} \cos(2\pi u)\right) \sin(2\pi v) \right). \quad (1-6)$$

Notice that it is not important which parametrization we choose since C^k -convergence is a geometric concept. The Clifford torus arises from stereographic projection of the minimal surface $\frac{1}{\sqrt{2}}(\mathbb{S}^1 \times \mathbb{S}^1) \subset \mathbb{S}^3$. From the solution [Marques and Neves 2014] of the famous Willmore conjecture we know that the Clifford torus is the global minimum of the Willmore energy among tori in \mathbb{R}^3 and the unique minimum modulo smooth conformal transformations (of \mathbb{R}^3) and reparametrizations. Our method relies on a *gap theorem* for Willmore tori of revolution, which is a consequence of [Müller and Spener 2020]; see Proposition 2.4. This relates to the findings in [Mondino and Nguyen 2014].

The convergence result in Theorem 1.3 holds up to surprisingly little invariances. It is often expected that such convergence results can only be shown up to invariances of the Willmore energy, i.e., reparametrizations and conformal transformations. The fact that we do not have to apply conformal transformations along the flow to achieve convergence is explained by the use of a Łojasiewicz–Simon gradient inequality. This inequality is a purely analytical tool, so the invariances will not play a role. For the limit immersion, we can rule out all conformal transformations that break the rotational symmetry and even more — symmetry-preserving Möbius inversions can also be ruled out due to the fact that they are not invariances of the Willmore flow equation. What remains is just scaling and translation in the direction $(1, 0, 0)$. This is not surprising since both transformations preserve the symmetry we consider and also preserve solutions of the Willmore flow equation, possibly rescaling appropriately in time.

We also prove that the energy threshold of 8π is sharp by constructing explicit nonconvergent evolutions with initial data f_0 satisfying $\mathcal{W}(f_0) > 8\pi$. There are multiple reasons why this number could be a universal threshold for any genus. The most striking is the inequality of Li and Yau that shows that immersions of Willmore energy below 8π are embeddings; see [Li and Yau 1982, Theorem 6]. Another property is that the metric of tori of energy $\leq 8\pi - \delta$, $\delta > 0$, is uniformly controlled up to Möbius transformations and reparametrizations; see [Schätzle 2013, Theorem 1.1] for details. As pointed out in [Simon 1993, p. 282; Kuwert et al. 2010], there exist surfaces of arbitrary genus with energy below 8π .

As already announced, we also show optimality of the energy bound of 8π .

Theorem 1.4. *For any $\varepsilon > 0$ there exists a torus of revolution $f_0 : \mathbb{S}^1 \times \mathbb{S}^1 \rightarrow \mathbb{R}^3$ such that $\mathcal{W}(f_0) < 8\pi + \varepsilon$ and the maximal Willmore flow $(f(t))_{t \in [0, T)}$ develops a singularity (in finite or infinite time). More precisely, one of the following phenomena occurs:*

- (1) (*Concentration of curvature*) The second fundamental form $(\|A(t)\|_{L^\infty(\Sigma)})_{t \in [0, T]}$ is unbounded. This singularity can occur in finite or infinite time.
- (2) (*Diameter blow-up in infinite time*) $T = \infty$ and $\lim_{t \rightarrow \infty} \text{diam}(f(t))(\mathbb{S}^1 \times \mathbb{S}^1) = \infty$.

In both cases the Willmore flow cannot converge in C^2 .

The singular behavior as described in [Theorem 1.4](#) will actually occur for each initial immersion F_γ , as in [Definition 1.1](#), with γ a curve of *vanishing total curvature*; see (3-20). This gives a class of singular examples for the Willmore flow. The total curvature also plays a significant role in earlier constructions of singular examples; see [\[Blatt 2009\]](#) for $\Sigma = \mathbb{S}^2$.

As a consequence of our main result we are able to show that each rectangular conformal class contains a torus of revolution of energy below 8π . This result has far-reaching consequences for the minimization of the Willmore energy with fixed conformal class, studied for example in [\[Kuwert and Schätzle 2013\]](#). In this article the authors show that minimizers in a given conformal class exist under the condition that the class contains an element of Willmore energy below 8π . By our result this condition is satisfied for every *rectangular* conformal class.

This paper is organized as follows. In [Section 2](#) we fix the notation and collect some useful facts on elastic curves in the hyperbolic plane and on tori of revolution. [Section 3](#) exploits the consequences of the initial datum being a torus of revolution for the symmetry properties of the evolution, for the possible singularities and the limit. It also contains the proofs of the main results and of the optimality results. In the last section we give the application on existence of tori of revolution with energy below 8π in each conformal class. Some useful results on smooth convergence (see [Definition 2.1](#) below) and the Willmore flow are collected in the [Appendix](#).

2. Geometric preliminaries

2.1. Notation. We first recall some basic definitions from differential geometry. Let Σ be a two-dimensional smooth manifold and $f : \Sigma \rightarrow \mathbb{R}^n$ be a smooth immersion. In this paper all manifolds are assumed to have no boundary. If we talk about tori of revolution, we need to impose the restriction that $n = 3$, but we will also discuss some results on the Willmore flow that remain valid in any codimension, i.e., for all $n \geq 3$. Let g be the induced Riemannian metric and ∇ the Levi-Civita connection on Σ , and denote the set of smooth vector fields on Σ by $\mathcal{V}(\Sigma)$. For $X \in \mathcal{V}(\Sigma)$ and $h \in C^\infty(\Sigma, \mathbb{R}^n)$ we define $D_X h \in C^\infty(\Sigma, \mathbb{R}^n)$ as

$$D_X h := \sum_{i=1}^n X(h_i)\vec{e}_i, \quad \text{whenever } h = \sum_{i=1}^n h_i \vec{e}_i \in C^\infty(M; \mathbb{R}^n),$$

and $\{\vec{e}_1, \vec{e}_2, \vec{e}_3, \dots, \vec{e}_n\}$ is the canonical basis of \mathbb{R}^n (see also [Appendix B](#)). The second fundamental form of Σ is $A : \mathcal{V}(\Sigma) \times \mathcal{V}(\Sigma) \rightarrow C^\infty(\Sigma, \mathbb{R}^n)$, given by

$$A(X, Y) := D_X(D_Y f) - D_{\nabla_X Y} f. \tag{2-1}$$

We remark that for all $p \in \Sigma$ one has $A_p(X, Y) \in df_p(T_p\Sigma)^\perp$; we say it takes values in the *normal bundle*. Moreover $A_p(X, Y)$ only depends on $X(p), Y(p)$. Its trace-free part \mathring{A} is given by

$$\mathring{A}(X, Y) := A(X, Y) - \frac{1}{2}g(X, Y)\vec{H},$$

where the mean curvature vector \vec{H} is the trace of the bilinear form (2-1) and can be computed by

$$\vec{H}(p) = A(e_1, e_1) + A(e_2, e_2),$$

with $\{e_1, e_2\}$ being an orthonormal basis of $T_p\Sigma$. Similarly (see [Appendix A](#) for details) we have

$$|A|^2 = \sum_{i,j=1}^2 \langle A(e_i, e_j), A(e_i, e_j) \rangle_{\mathbb{R}^n}.$$

With these definitions we may introduce the Willmore flow of a smooth immersion $f_0 : \Sigma \rightarrow \mathbb{R}^n$. We say that a smooth family of smooth immersions $f : [0, T) \times \Sigma \rightarrow \mathbb{R}^n$, where $T > 0$, evolves by the *Willmore flow* with initial datum f_0 if f satisfies

$$\partial_t f = -(\Delta \vec{H} + Q(\mathring{A})\vec{H}) \quad \text{in } (0, T) \times \Sigma, \tag{2-2}$$

with $f(t = 0) = f_0$. Here, Δ denotes the *normal Laplacian*, i.e., for an orthonormal basis $\{e_1, e_2\}$ that is a basis of $T_p\Sigma$ with respect to $f(t, \cdot)^*g_{\mathbb{R}^n}$ one has

$$\Delta \vec{H} = \sum_{i=1}^2 (\nabla^\perp)^2 \vec{H}(e_i, e_i),$$

where $\nabla_X^\perp Y = (D_X Y)^\perp$ (see [\(B-2\)](#), [\(B-3\)](#) for details). With the same notation as above, the quadratic operator Q is given by

$$(Q(\mathring{A})\vec{H})(t, p) = \sum_{i,j=1}^2 \langle \mathring{A}(e_i, e_j), \vec{H} \rangle_{\mathbb{R}^n} \mathring{A}(e_i, e_j). \tag{2-3}$$

Since (2-2) is well-posed for smooth initial immersions f_0 (see [\[Kuwert and Schätzle 2002, Proposition 1.1\]](#)) we will always assume that the evolution is maximal, i.e., nonextendable in the class of smooth immersions.

To study the behavior of $f(t)$ as $t \rightarrow T$ we use the following notion of smooth convergence on compact sets from [\[Kuwert and Schätzle 2001, Theorem 4.2\]](#); see also [\[Breuning 2015\]](#) and [Appendix C](#).

Definition 2.1 (Smooth convergence of immersions). Let Σ and $\widehat{\Sigma}$ be smooth two-dimensional manifolds and $(f_j)_{j=1}^\infty : \Sigma \rightarrow \mathbb{R}^n$ and $\hat{f} : \widehat{\Sigma} \rightarrow \mathbb{R}^n$ be smooth immersions. Define

$$\widehat{\Sigma}(m) := \{p \in \widehat{\Sigma} : |\hat{f}(p)| < m\}, \quad m \in \mathbb{N}. \tag{2-4}$$

We say that f_j converges to \hat{f} smoothly on compact subsets of \mathbb{R}^n if for each $j \in \mathbb{N}$ there exists a diffeomorphism $\phi_j : \widehat{\Sigma}(j) \rightarrow U_j$ for some open $U_j \subset \Sigma$, and a normal vector field $u_j \in C^\infty(\widehat{\Sigma}(j), \mathbb{R}^n)$ satisfying

$$f_j \circ \phi_j = \hat{f} + u_j \quad \text{on } \widehat{\Sigma}(j), \tag{2-5}$$

as well as $\|(\hat{\nabla}^\perp)^k u_j\|_{L^\infty(\hat{\Sigma}(j))} \rightarrow 0$ as $j \rightarrow \infty$ for all $k \in \mathbb{N}_0$. Here $\hat{\nabla}$ is the Levi-Civita connection on $(\hat{\Sigma}, g_{\hat{f}})$ and $(\hat{\nabla}^\perp)^k u_j$ is defined as in [Appendix B](#). Additionally, we require that for each $R > 0$ there exists $j(R) \in \mathbb{N}$ such that $j \geq j(R)$ implies that $f_j^{-1}(B_R(0)) \subset U_j$.

We exploit a fundamental correspondence between the Willmore energy of tori and the elastic energy of curves in the hyperbolic plane already used in several works since its observation in [\[Langer and Singer 1984a\]](#).

2.2. Curves in the hyperbolic plane. We consider the hyperbolic half-plane $\mathbb{H}^2 = \{(x^{(1)}, x^{(2)}) \in \mathbb{R} \times (0, \infty)\}$ endowed with the metric

$$g_{\mathbb{H}^2}(v, w) = \frac{1}{z^2} \langle v, w \rangle_{\mathbb{R}^2}, \quad v, w \in T_z \mathbb{H}^2,$$

and define $|v|_{\mathbb{H}^2} = \sqrt{g_{\mathbb{H}^2}(v, v)}$, $v \in T_z \mathbb{H}^2$. For a smooth immersed curve $\gamma = (\gamma^{(1)}, \gamma^{(2)})$ in \mathbb{H}^2 , $\gamma \in C^\infty(\mathbb{S}^1, \mathbb{H}^2)$, the length is as in the [Introduction](#) given by

$$\mathcal{L}_{\mathbb{H}^2}(\gamma) := \int_0^1 \frac{|\gamma'(x)|_{\mathbb{R}^2}}{\gamma^{(2)}(x)} dx = \int_0^1 ds, \tag{2-6}$$

where $ds = |\partial_x \gamma|_{\mathbb{H}^2} dx$ denotes the arc length parameter, and the derivative with respect to x is abbreviated with the prime. As usual, $\partial_s = \partial_x / |\partial_x \gamma|_{\mathbb{H}^2}$ denotes the arc length derivative. The curvature vector field of γ is given by

$$\kappa[\gamma] = \nabla_s \partial_s \gamma = \left(\begin{array}{c} \partial_s^2 \gamma^{(1)} - (2/\gamma^{(2)}) \partial_s \gamma^{(1)} \partial_s \gamma^{(2)} \\ \partial_s^2 \gamma^{(2)} + (1/\gamma^{(2)}) ((\partial_s \gamma^{(1)})^2 - (\partial_s \gamma^{(2)})^2) \end{array} \right) \tag{2-7}$$

as an element of $T_z \mathbb{H}^2$ [\[Dall'Acqua and Spener 2017, \(12\)\]](#). Here ∇_s denotes the covariant derivative along γ with respect to the Levi-Civita connection on \mathbb{H}^2 . We write $\kappa = \kappa[\gamma]$ if the curve is clear from the context. The *elastic energy* \mathcal{E} of γ is then defined to be

$$\mathcal{E}(\gamma) := \int_\gamma |\kappa|_{\mathbb{H}^2}^2 ds.$$

Its critical points are called *free hyperbolic elastica* and satisfy

$$(\nabla_s^\perp)^2 \kappa + \frac{1}{2} |\kappa|_{\mathbb{H}^2}^2 \kappa - \kappa = 0,$$

where $\nabla_s^\perp \eta = \nabla_s \eta - \langle \nabla_s \eta, \partial_s \gamma \rangle_{\mathbb{H}^2} \partial_s \gamma$ is the covariant derivative on the normal bundle of γ .

We collect some results connecting the length and the elastic energy of smooth closed curves in the hyperbolic plane.

Theorem 2.2 [\[Müller and Spener 2020, Theorem 5.3\]](#). *For each $\varepsilon > 0$ there exists $c(\varepsilon) > 0$ such that*

$$\frac{\mathcal{E}(\gamma)}{\mathcal{L}_{\mathbb{H}^2}(\gamma)} \geq c(\varepsilon)$$

for all immersed and closed curves $\gamma \in C^\infty(\mathbb{S}^1, \mathbb{H}^2)$ such that $\mathcal{E}(\gamma) \leq 16 - \varepsilon$.

Note that the energy threshold of 16 is sharp for this result; see [Müller and Spener 2020].

We also fix the notion of the Euclidean length of the curve $\gamma : \mathbb{S}^1 \rightarrow \mathbb{H}^2 \subset \mathbb{R}^2$, which is given by $\mathcal{L}_{\mathbb{R}^2}(\gamma)$. We also consider the *Euclidean curvature* of $\gamma : \mathbb{S}^1 \rightarrow \mathbb{R}^2$, which we will denote by

$$\bar{\kappa}_{\text{euc}}[\gamma] := \frac{1}{|\gamma'|} \frac{d}{dt} \frac{\gamma'}{|\gamma'|},$$

and the *Euclidean scalar curvature* $\kappa_{\text{euc}}[\gamma] := (1/|\gamma'|^2)\langle \gamma'', n \rangle_{\mathbb{R}^2}$. To finish this section we discuss some relations between Euclidean and hyperbolic length.

Lemma 2.3. *Let $\gamma \in C^\infty(\mathbb{S}^1, \mathbb{H}^2)$ and $a, b \in [0, 1]$. Then*

$$\gamma^{(2)}(b)e^{-\mathcal{L}_{\mathbb{H}^2}(\gamma)} \leq \gamma^{(2)}(a) \leq \gamma^{(2)}(b)e^{\mathcal{L}_{\mathbb{H}^2}(\gamma)} \tag{2-8}$$

and

$$\mathcal{L}_{\mathbb{H}^2}(\gamma) \geq \frac{\mathcal{L}_{\mathbb{R}^2}(\gamma)}{\sup_{\mathbb{S}^1} \gamma^{(2)}}. \tag{2-9}$$

Proof. For γ, a, b as in the statement, we find by (2-6)

$$\mathcal{L}_{\mathbb{H}^2}(\gamma) \geq \int_a^b \frac{|(\gamma^{(2)})'|}{\gamma^{(2)}} dx \geq |\log \gamma^{(2)}(b) - \log \gamma^{(2)}(a)|,$$

and therefore $\log \gamma^{(2)}(b) - \mathcal{L}_{\mathbb{H}^2}(\gamma) \leq \log \gamma^{(2)}(a) \leq \log \gamma^{(2)}(b) + \mathcal{L}_{\mathbb{H}^2}(\gamma)$. Taking exponentials (2-8) follows. For (2-9) we simply estimate

$$\mathcal{L}_{\mathbb{H}^2}(\gamma) = \int_{\mathbb{S}^1} \frac{|\gamma'(u)|}{\gamma^2(u)} du \geq \frac{1}{\sup_{\mathbb{S}^1} \gamma^{(2)}} \int_{\mathbb{S}^1} |\gamma'(u)| du = \frac{\mathcal{L}_{\mathbb{R}^2}(\gamma)}{\sup_{\mathbb{S}^1} \gamma^{(2)}}. \quad \square$$

2.3. Tori of revolution in \mathbb{R}^3 . Here we collect some basic facts about tori of revolution. More precisely we express some geometric quantities associated to tori of revolution using only their profile curves. If $F_\gamma : \mathbb{S}^1 \times \mathbb{S}^1 \rightarrow \mathbb{R}^3$ is chosen as in Definition 1.1 we can compute the first fundamental form with respect to the local coordinates (u, v) of $\mathbb{S}^1 \times \mathbb{S}^1$. This yields the associated surface measure on the Riemannian manifold $(\mathbb{S}^1 \times \mathbb{S}^1, g = F_\gamma^* g_{\mathbb{R}^3})$ given by

$$d\mu_g = 2\pi \gamma^{(2)}(u) |\gamma'(u)|_{\mathbb{R}^2} du dv. \tag{2-10}$$

As we have already announced, the Willmore energy of F_γ can also be expressed only in terms of γ using the fundamental relationship

$$\mathcal{W}(F_\gamma) = \frac{\pi}{2} \mathcal{E}(\gamma); \tag{2-11}$$

see [Langer and Singer 1984a; Dall’Acqua and Spener 2018, Theorem 4.1]. Moreover, let κ be the hyperbolic curvature vector field of γ in \mathbb{H}^2 . Then

$$-\langle (\nabla_s^\perp)^2 \kappa + \frac{1}{2} |\kappa|_{\mathbb{H}^2}^2 \kappa - \kappa, n \rangle_{\mathbb{H}^2} = 2(\gamma^{(2)})^4 \left(\Delta H + 2H \left(\frac{1}{4} H^2 - K \right) \right), \tag{2-12}$$

where $n = (-\partial_s \gamma^{(2)}, \partial_s \gamma^{(1)})$ is the normal vector field along γ (see [Dall’Acqua and Spener 2018, Theorem 4.1]). In particular, F_γ is a Willmore torus of revolution if and only if γ is a hyperbolic elastica. In Appendix A we discuss the relationship between (2-12) and (1-2).

An immediate consequence of [Müller and Spener 2020, Proposition 6.5] (that builds on findings in [Langer and Singer 1984b]) is the following.

Proposition 2.4 (A gap theorem for Willmore tori of revolution). *Let $f : \mathbb{S}^1 \times \mathbb{S}^1 \rightarrow \mathbb{R}^3$ be a Willmore torus of revolution that satisfies $\mathcal{W}(f) \leq 8\pi$. Then f is, up to reparametrization, the Clifford torus possibly rescaled and translated in the direction $(1, 0, 0)^T$.*

Proof. Let $f = F_\gamma$ be as in the statement with profile curve $\gamma \in C^\infty(\mathbb{S}^1, \mathbb{H}^2)$. From (2-12) we know that γ is a hyperbolic elastica. From (2-11) we can conclude that $\mathcal{E}(\gamma) \leq 16$. By [Müller and Spener 2020, Proposition 6.5] we obtain that γ has to coincide (up to reparametrization) with the profile curve of the Clifford torus up to isometries of \mathbb{H}^2 . This however implies that f is, up to reparametrization, the Clifford torus possibly rescaled and translated in the direction $(1, 0, 0)$. \square

Another important quantity for our discussion is the second fundamental form $A[F_\gamma]$, which we will also express in terms of γ . A property which we will later make extensive use of is the fact that for a torus of revolution $f = F_\gamma$, $|A[F_\gamma]|^2 \in C^\infty(\mathbb{S}^1 \times \mathbb{S}^1)$ is a function that depends only on u (a parameter that describes the profile curve) and not on v (a parameter that describes the revolution). This is the reason why curvature concentration is “passed along” the revolution. We will describe this more precisely in Section 3.4. For this section it is enough to observe by a direct computation (see [Dall’Acqua and Spener 2018, p. 118]) that with respect to the normal $N_{F_\gamma} = (\partial_u F_\gamma \times \partial_v F_\gamma) / |\partial_u F_\gamma \times \partial_v F_\gamma|$ the principal curvatures are given by

$$\kappa_1[F_\gamma](u, v) = -\kappa_{\text{euc}}[\gamma](u) \quad \text{and} \quad \kappa_2[F_\gamma](u, v) = \frac{(\gamma^{(1)})'(u)}{|\gamma'(u)|\gamma^{(2)}(u)}.$$

With this at hand, one can derive a useful bound for the length of the profile curve in terms of surface quantities.

Lemma 2.5. *Suppose that $f = F_\gamma : \mathbb{S}^1 \times \mathbb{S}^1 \rightarrow \mathbb{R}^3$ is a torus of revolution with profile curve γ . Then*

$$\mathcal{L}_{\mathbb{R}^2}(\gamma) \leq \mu_{g_f}(\mathbb{S}^1 \times \mathbb{S}^1)^{1/2} \mathcal{W}(f)^{1/2}.$$

Proof. We may without loss of generality assume that γ is parametrized with constant velocity, i.e., $|\gamma'| = \mathcal{L}_{\mathbb{R}^2}(\gamma) =: L$. Recall from Appendix A that $\vec{H}(u, v) = (\kappa_1(u, v) + \kappa_2(u, v))N_f(u, v)$, where

$$N_f(u, v) = \frac{1}{\mathcal{L}_{\mathbb{R}^2}(\gamma)} \begin{pmatrix} (\gamma^{(2)})'(u) \\ -(\gamma^{(1)})'(u) \cos(2\pi v) \\ -(\gamma^{(1)})'(u) \sin(2\pi v) \end{pmatrix} \quad \text{with } u, v \in \mathbb{S}^1.$$

We show next that

$$-2L = \int_{\mathbb{S}^1 \times [0, 1/2]} \vec{H} \cdot e_3 \, d\mu_{g_f}. \tag{2-13}$$

Plugging in the quantities characterized in this section and using

$$(\gamma^{(1)})'^2 + (\gamma^{(2)})'^2 = L^2 \quad \text{and} \quad (\gamma^{(1)})''(\gamma^{(1)})' + (\gamma^{(2)})''(\gamma^{(2)})' = 0$$

we obtain

$$\begin{aligned}
 \int_{\mathbb{S}^1 \times [0, 1/2]} \vec{H} \cdot e_3 \, d\mu_{g_f} &= 2\pi \int_0^1 \int_0^{1/2} (\kappa_1(u, v) + \kappa_2(u, v))(N_f(u, v) \cdot e_3) |\gamma'(u)| \gamma^{(2)}(u) \, dv \, du \\
 &= 2\pi \int_0^1 \int_0^{1/2} \left(-\kappa_{\text{euc}}[\gamma](u) + \frac{(\gamma^{(1)})'(u)}{L\gamma^{(2)}(u)} \right) [-(\gamma^{(1)})'(u) \sin(2\pi v)] \gamma^{(2)}(u) \, dv \, du \\
 &= -[-\cos(2\pi v)]_0^{1/2} \int_0^1 \left(\frac{(\gamma^{(1)})''(\gamma^{(2)})' - (\gamma^{(2)})''(\gamma^{(1)})'}{L^3} + \frac{(\gamma^{(1)})'}{L\gamma^{(2)}} \right) (\gamma^{(1)})' \gamma^{(2)} \, du \\
 &= -2 \frac{1}{L^3} \int_0^1 ((\gamma^{(1)})''(\gamma^{(1)})'(\gamma^{(2)})' - (\gamma^{(2)})''(\gamma^{(1)})'^2) \gamma^{(2)} \, du - \frac{2}{L} \int_0^1 (\gamma^{(1)})'^2 \, du \\
 &= -2 \frac{1}{L^3} \int_0^1 (-(\gamma^{(2)})''(\gamma^{(2)})'^2 - (\gamma^{(2)})''(\gamma^{(1)})'^2) \gamma^{(2)} \, du - \frac{2}{L} \int_0^1 (\gamma^{(1)})'^2 \, du \\
 &= \frac{2}{L^3} \int_0^1 (\gamma^{(2)})'' L^2 \gamma^{(2)} \, du - \frac{2}{L} \int_0^1 (\gamma^{(1)})'^2 \, du \\
 &= -\frac{2}{L} \int_0^1 (\gamma^{(2)})'^2 \, du - \frac{2}{L} \int_0^1 (\gamma^{(1)})'^2 \, du,
 \end{aligned}$$

where we have used integration by parts in the last step. Adding up the integrands and once again using $(\gamma^{(1)})'^2 + (\gamma^{(2)})'^2 = L^2$, we obtain (2-13). From (2-13) and the Cauchy–Schwarz inequality we also conclude

$$2L \leq \int_{\mathbb{S}^1 \times \mathbb{S}^1} |\vec{H}| \, d\mu_{g_f} \leq 2\mathcal{W}(f)^{1/2} \mu_{g_f}(\mathbb{S}^1 \times \mathbb{S}^1)^{1/2}. \quad \square$$

A quantity which we will also study is the diameter.

Lemma 2.6. *Let $f = F_\gamma : \mathbb{S}^1 \times \mathbb{S}^1 \rightarrow \mathbb{R}^3$ be a torus of revolution with profile curve γ . Then,*

$$\text{diam}(F_\gamma(\mathbb{S}^1 \times \mathbb{S}^1)) \leq \frac{1}{2} \mathcal{L}_{\mathbb{R}^2}(\gamma) + 2\|\gamma^{(2)}\|_{L^\infty}.$$

Proof. Let $(u, v), (u', v') \in \mathbb{S}^1 \times \mathbb{S}^1$ and $f = F_\gamma$ be as in the statement. Without loss of generality we can assume that $\gamma^{(2)}(u) \leq \gamma^{(2)}(u')$. We start proving

$$|f(u, v) - f(u', v')| \leq |\gamma(u) - \gamma(u')| + \sqrt{2\gamma^{(2)}(u)} \sqrt{1 - \cos(2\pi(v - v'))}.$$

First observe that $|f(u, v) - f(u', v')| \leq |f(u', v') - f(u, v')| + |f(u, v') - f(u, v)|$. Using the definition of the Euclidean distance we find $|f(u', v') - f(u, v')| = |\gamma(u) - \gamma(u')|$. Similarly,

$$\begin{aligned}
 |f(u, v') - f(u, v)| &= \gamma^{(2)}(u) \sqrt{(\cos(2\pi v) - \cos(2\pi v'))^2 + (\sin(2\pi v) - \sin(2\pi v'))^2} \\
 &= \gamma^{(2)}(u) \sqrt{2 - 2\cos(2\pi(v - v'))}.
 \end{aligned}$$

Both computations imply the desired estimate, and the asserted diameter bound follows immediately. \square

3. The Willmore flow of tori of revolution

In this section we understand the interplay between the rotational symmetry and the *curvature concentration criterion*, which is able to detect *singularities* of the Willmore flow. This gives us a better understanding of the singularities that can arise in our symmetric setting. We will then prove the main theorems by excluding those singularities in certain circumstances.

3.1. Singularities of the Willmore flow. In this section we summarize what singularities of the Willmore flow look like. The following result summarizes a list of results that have been obtained previously in other articles on the Willmore flow. It exposes the diameter of appropriate parabolic rescalings as a quantity whose control is sufficient for convergence. The appropriate rescaling is given by a *concentration property* of the Willmore flow; see [Appendix D](#). In the following discussion we will use the two parameters ε_0 and c_0 which have been introduced in [Theorem D.1](#).

Theorem 3.1 (Convergence criterion of the Willmore flow; proof in [Appendix D](#)). *Let Σ be a compact two-dimensional manifold without boundary and let $f : [0, T) \times \Sigma \rightarrow \mathbb{R}^n$ be a maximal evolution by the Willmore flow with initial datum f_0 . Consider an arbitrary sequence $(t_j)_{j \in \mathbb{N}} \subset (0, T)$ with $t_j \rightarrow T$. Then, the **concentration radii***

$$r_j := \sup \left\{ r > 0 : \text{for all } x \in \mathbb{R}^n \text{ one has } \int_{f(t_j)^{-1}(B_r(x))} |A(t_j)|^2 d\mu_{g_{f(t_j)}} \leq \varepsilon_0 \right\}, \quad (3-1)$$

$j \in \mathbb{N}$, satisfy $t_j + c_0 r_j^4 < T$ for all $j \in \mathbb{N}$. Further, the maps

$$\tilde{f}_{j,c_0} : \Sigma \rightarrow \mathbb{R}^3, \quad \tilde{f}_{j,c_0} := \frac{f(t_j + c_0 r_j^4)}{r_j},$$

are called **concentration rescalings** and one of the following alternatives occurs

Case 1: *convergent evolution.* There exists $\delta > 0$ such that $\delta < r_j < 1/\delta$. Then $T = \infty$. If additionally $(\text{diam}(\tilde{f}_{j,c_0}))_{j \in \mathbb{N}}$ is uniformly bounded then the Willmore flow converges to a Willmore immersion. More precisely there exists a Willmore immersion $f_\infty : \Sigma \rightarrow \mathbb{R}^n$ such that $f(t) \rightarrow f_\infty$ in C^k for all $k \in \mathbb{N}$ as $t \rightarrow \infty$.

Case 2: *blow-up or blow-down.* A subsequence of $(r_j)_{j \in \mathbb{N}}$ goes either to zero or to infinity. In this case one has $\text{diam}(\tilde{f}_{j,c_0}) \rightarrow \infty$ as $j \rightarrow \infty$.

In particular, if $(\text{diam}(\tilde{f}_{j,c_0}))_{j \in \mathbb{N}}$ is uniformly bounded, then $T = \infty$ and the Willmore flow converges to a Willmore immersion $f_\infty : \Sigma \rightarrow \mathbb{R}^n$ in C^k for all $k \in \mathbb{N}$.

In the coming sections we will study the relation between the diameter of the concentration rescalings and the hyperbolic length of the profile curves. Having understood this we will finally be able to obtain [Theorems 1.2](#) and [1.3](#).

3.2. Dimension reduction. We have already announced that the rotational symmetry is preserved along the flow. This section is devoted to the proof of this fact, see [Lemma 3.3](#). In the proof of [Lemma 3.3](#) we will make use of an alternative characterization of tori of revolution, see [Definition 1.1](#), which we state next.

Proposition 3.2. *Let $f : \mathbb{S}^1 \times \mathbb{S}^1 \rightarrow \mathbb{R}^3$ be a smooth immersion. Then, f is a torus of revolution if and only if*

$$\text{for all } \phi \in \mathbb{S}^1 \quad f(u, v + \phi) = R_{2\pi\phi} f(u, v), \quad \text{where } R_z = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos z & -\sin z \\ 0 & \sin z & \cos z \end{pmatrix}, \quad (3-2)$$

$$\text{for all } u \in \mathbb{S}^1 \quad f^{(3)}(u, 0) = 0 \quad \text{and} \quad f^{(2)}(u_0, 0) \geq 0 \quad \text{for one value } u_0 \in \mathbb{S}^1. \quad (3-3)$$

Proof. If f is a torus of revolution then (3-2) and (3-3) can be checked by direct computation. If (3-2) and (3-3) hold for some immersion $f : \mathbb{S}^1 \times \mathbb{S}^1 \rightarrow \mathbb{R}^3$ then one can define a smooth curve $\gamma : \mathbb{S}^1 \rightarrow \mathbb{R}^2$ by $\gamma(u) := (f^{(1)}(u, 0), f^{(2)}(u, 0))$. Equation (1-4) is then easy to check, but it also needs to be shown that $\gamma(u) \in \mathbb{H}^2$ for all $u \in \mathbb{S}^1$. So far we have

$$f(u, v) = (\gamma^{(1)}(u), \gamma^{(2)}(u) \cos(2\pi v), \gamma^{(2)}(u) \sin(2\pi v)) \quad \text{for all } (u, v) \in \mathbb{S}^1 \times \mathbb{S}^1.$$

If now there exists a point $u_0 \in \mathbb{S}^1 \times \mathbb{S}^1$ such that $\gamma^{(2)}(u_0) = 0$ then one can compute

$$\partial_v f(u_0, v) = (0, 0, 0)^T \quad \text{for all } v \in \mathbb{S}^1,$$

which is a contradiction to the fact that f is an immersion. Hence $\gamma^{(2)}$ may not change sign or attain the value zero. As a consequence, $\gamma^{(2)} > 0$ and the claim follows. □

In particular, given a torus of revolution its profile curve is given by $\gamma(u) := (f^{(1)}(u, 0), f^{(2)}(u, 0))$. Note that — by inspection of the previous proof — each immersion $f : \mathbb{S}^1 \times \mathbb{S}^1 \rightarrow \mathbb{R}^3$ that fulfills (3-2), as well as $f^{(3)}(u, 0) = 0$ for all $u \in \mathbb{S}^1$, must satisfy $f^{(2)}(\cdot, 0) \neq 0$. In particular it cannot change sign. Thus, either $f^{(2)}(\cdot, 0) > 0$ or $f^{(2)}(\cdot, 0) < 0$. In the latter case $f(\cdot, \cdot + 1)$ defines a torus of revolution. This shows also consistency of our definition with [Blatt 2009, Definition 2.2], whose results we will need later.

When it comes to evolutions $(f(t))_{t \geq 0}$, we however want to work without reparametrizations of $f(t)$ along the flow and hence we specify $\gamma^{(2)} = f^{(2)}(\cdot, 0) > 0$ (and we check that this remains satisfied along the flow).

Lemma 3.3. *Let $f_0 : \mathbb{S}^1 \times \mathbb{S}^1 \rightarrow \mathbb{R}^3$ be a torus of revolution and let $(f(t))_{t \in [0, T]} : \mathbb{S}^1 \times \mathbb{S}^1 \rightarrow \mathbb{R}^3$ evolve by the Willmore flow with initial datum f_0 . Then $(f(t))_{t \in [0, T]}$ is a torus of revolution for all $t \in [0, T]$.*

Proof. We prove that $(f(t))_{t \in [0, T]}$ satisfies (3-2) and (3-3) for all $t \in [0, T]$ so that the claim follows from Proposition 3.2.

Let $\phi \in \mathbb{S}^1$. We observe that $R_{2\pi\phi}$ is an isometry in \mathbb{R}^3 and $(u, v) \mapsto (u, v + \phi)$ is a diffeomorphism. Hence $(R_{2\pi\phi}^{-1} f(t)(\cdot, \cdot + \phi))_{t \in [0, T]} : \mathbb{S}^1 \times \mathbb{S}^1 \rightarrow \mathbb{R}^3$ is an evolution by Willmore flow with initial value $R_{2\pi\phi}^{-1} f_0(\cdot, \cdot + \phi)$. Recall now that f_0 satisfies (3-2), i.e., $R_{2\pi\phi}^{-1} f_0(\cdot, \cdot + \phi) = f_0$. By the uniqueness result for the Willmore flow, see [Kuwert and Schätzle 2002, Proposition 1.1], we obtain that

$$R_{2\pi\phi}^{-1} f(t)(u, v + \phi) = f(t)(u, v) \quad \text{for all } (u, v) \in \mathbb{S}^1 \times \mathbb{S}^1,$$

that is, (3-2). In particular, there exist smooth functions $x, y, z : [0, T) \times \mathbb{S}^1$ such that

$$f(t)(u, v) = R_{2\pi v}(f(t)(u, 0)) = R_{2\pi v} \begin{pmatrix} x(t, u) \\ y(t, u) \\ z(t, u) \end{pmatrix}. \tag{3-4}$$

As an intermediate step for (3-3) we show that $f(t)^{(3)}(u, 0) = 0$ for all $t > 0$ and $u \in \mathbb{S}^1$, i.e., $z \equiv 0$ on $[0, T) \times \mathbb{S}^1$. Set

$$S := \sup\{s \in [0, T) : f(t) \text{ is a torus of revolution for all } t \in [0, s]\}.$$

We show that $S = T$. If $S < T$ then observe that $z(S, u) = 0$ for all $u \in \mathbb{S}^1$ by smoothness of $(f(t))_{t \in [0, T)}$ and the fact that $f(t)^{(3)}(u, 0) = 0$ for all $t \in [0, S)$ and $u \in \mathbb{S}^1$. As additionally $y(S, \cdot)$ is nonnegative and $f(S)$ is an immersion, $f(S)$ is a torus of revolution by Proposition 3.2.

Restart the flow with $\tilde{f}_0 := f(S)$ (if $S = 0$ there is no need to restart). Choose now c_0, ρ for \tilde{f}_0 to be as in Theorem D.1 and consider the time interval $I := [S, S + (1/c_0)\rho^4]$. The Willmore flow equation in the local coordinates (u, v) of $\mathbb{S}^1 \times \mathbb{S}^1$ reads

$$\partial_t f(t) = P(A(t), \nabla^\perp A(t), (\nabla^\perp)^2 A(t)) \vec{N}_{f(t)},$$

where

$$\vec{N}_{f(t)} := \frac{\partial_u f(t) \times \partial_v f(t)}{|\partial_u f(t) \times \partial_v f(t)|}$$

and $P(A, \nabla^\perp A, (\nabla^\perp)^2 A)$ is a scalar quantity that can be bounded in terms of $\|g\|_{L^\infty(\mathbb{S}^1 \times \mathbb{S}^1)}$ and $\|(\nabla^\perp)^k A\|_{L^\infty(\mathbb{S}^1 \times \mathbb{S}^1)}$ ($k = 0, 1, 2$). All of those remain bounded in I by (D-1) and the explanation afterwards. The idea now is to consider the evolution equation satisfied by $z(t, u)^2$. Since

$$\vec{N}_{f(t)}(u, v) = \frac{1}{\sqrt{\det(g(t))}} R_{2\pi v} \begin{pmatrix} y(t, u) \partial_u y(t, u) + \partial_u z(t, u) z(t, u) \\ -y(t, u) \partial_u x(t, u) \\ -z(t, u) \partial_u x(t, u) \end{pmatrix},$$

we find

$$\begin{aligned} \partial_t(z(t, u)^2) &= 2z(t, u) \partial_t z(t, u) = 2z(t, u) P(A(t), \nabla^\perp A(t), (\nabla^\perp)^2 A(t)) \vec{N}_{f(t)}^{(3)}(u, 0) \\ &= -2 \frac{1}{\sqrt{\det(g(t))}} P(A(t), \nabla^\perp A(t), (\nabla^\perp)^2 A(t)) \partial_u x(t, u) z(t, u)^2. \end{aligned}$$

By Theorem D.1 for fixed $u \in \mathbb{S}^1$ we have obtained

$$\begin{cases} \partial_t(z(t, u)^2) \leq Cz(t, u)^2, & t \in I, \\ z(S, u)^2 = 0, \end{cases}$$

and hence $z(t, u) = 0$ for all $t \in I$ and all $u \in \mathbb{S}^1$, as u was chosen arbitrarily. Similar to before, again by Proposition 3.2 and the discussion afterwards it can be shown that $y(t, \cdot) > 0$ for all $t \in I$. This is finally a contradiction to the choice of S and thus $S = T$. The claim follows. \square

The previous lemma implies that for each Willmore evolution $(f(t))_{t \geq 0}$ starting at a torus of revolution $f_0 : \mathbb{S}^1 \times \mathbb{S}^1 \rightarrow \mathbb{R}^3$ there exists a unique smooth evolution of curves $(\gamma(t))_{t \in [0, T)} \subset C^\infty(\mathbb{S}^1, \mathbb{H}^2)$, $\gamma(t)(u) = f(t)(u, 0)$ such that

$$f(t)(u, v) = \begin{pmatrix} \gamma^{(1)}(t)(u) \\ \gamma^{(2)}(t)(u) \cos(2\pi v) \\ \gamma^{(2)}(t)(u) \sin(2\pi v) \end{pmatrix}, \tag{3-5}$$

whereupon the flow can also be seen as an evolution of $(\gamma(t))_{t \in [0, T)}$.

3.3. Symmetry of the limit immersion. Theorem 3.1 provides us with a general convergence criterion for the Willmore flow and yields a smooth limit immersion f_∞ , which is a Willmore immersion. In this section we need to check that the revolution symmetry is passed along to the limit; i.e., we will prove that under certain conditions the limit immersion f_∞ is a (Willmore) torus of revolution. Let us stress that this not trivial because the notion of convergence is *geometric*, i.e., invariant with respect to reparametrization. Hence classical results about pointwise convergence cannot be applied.

The arguments in this section make frequent use of the fact that to each torus of revolution $f = F_\gamma : \mathbb{S}^1 \times \mathbb{S}^1 \rightarrow \mathbb{R}^3$ one can easily associate a smooth orthonormal frame with respect to g_f , given by

$$E_1(u, v) := \frac{1}{|\gamma'(u)|} \frac{\partial}{\partial u}, \quad E_2(u, v) := \frac{1}{2\pi \gamma^{(2)}(u)} \frac{\partial}{\partial v}. \tag{3-6}$$

This orthonormal frame also has some further interesting properties, for example that it diagonalizes the second fundamental form $A[f]$, and hence yields the principal curvatures of f . The first principal curvature

$$\kappa_1[f] = \langle A[f]_{(u,v)}(E_1, E_1), N_f \rangle_{\mathbb{R}^3} = -\kappa_{\text{euc}}[\gamma](u)$$

coincides up to a sign with the Euclidean scalar curvature of the profile curve, while the second principal curvature

$$\kappa_2[f] = \langle A[f]_{(u,v)}(E_2, E_2), N_f \rangle_{\mathbb{R}^3} = \frac{(\gamma^{(1)})'(u)}{|\gamma'(u)|\gamma^{(2)}(u)}$$

depends heavily on the distance of the profile curve to the revolution axis. This will be of great use when it comes to explicit estimates involving the second fundamental form.

Lemma 3.4 (Revolution symmetry of the limit). *Suppose that $f : [0, \infty) \times (\mathbb{S}^1 \times \mathbb{S}^1) \rightarrow \mathbb{R}^3$ is a global evolution by Willmore flow, convergent to some Willmore immersion $f_\infty : \mathbb{S}^1 \times \mathbb{S}^1 \rightarrow \mathbb{R}^3$ in C^k for all $k \in \mathbb{N}$. Suppose further that $f(0)$ is a torus of revolution and $(\gamma(t))_{t \in [0, \infty)} \subset C^\infty(\mathbb{S}^1, \mathbb{R}^2)$ is as in (3-5). Then f_∞ is (up to reparametrization) a Willmore torus of revolution. A profile curve γ_∞ of f_∞ can be obtained by a $C^m(\mathbb{S}^1, \mathbb{R}^2)$ -limit of appropriate reparametrizations of a sequence $(\gamma(t_j))_{j \in \mathbb{N}}$, $t_j \rightarrow \infty$. Here $m \in \mathbb{N}$ is arbitrary. In particular $\gamma_\infty \in C^\infty(\mathbb{S}^1, \mathbb{H}^2)$ is a hyperbolic elastica.*

Proof. Let $(t_j)_{j \in \mathbb{N}} \subset [0, \infty)$ be an arbitrary sequence such that $t_j \rightarrow \infty$.

Step 1: bounds for the profile curves. After reparametrization we may assume without loss of generality that $(\gamma(t_j))_{j \in \mathbb{N}}$ is parametrized with constant Euclidean speed.

Now fix $m \in \mathbb{N}$ arbitrary. To bound the $W^{m,2}$ -norm of $(\gamma(t_j))_{j \in \mathbb{N}}$ we first bound $\|\gamma(t_j)\|_{L^\infty(\mathbb{S}^1, \mathbb{R}^2)}$. To this end we observe by (3-5) that

$$\|\gamma(t_j)\|_{L^\infty(\mathbb{S}^1, \mathbb{R}^2)} = \|f(t_j)\|_{L^\infty(\mathbb{S}^1 \times \mathbb{S}^1, \mathbb{R}^3)}.$$

Now $\|f(t_j)\|_{L^\infty}$ is uniformly bounded because it converges in C^k for all $k \in \mathbb{N}$ to f_∞ , whose image is a compact subset of \mathbb{R}^3 . Note that we have used here that the L^∞ -norm is not affected by reparametrization. Next we bound $\mathcal{L}_{\mathbb{R}^2}(\gamma(t_j)) = \|\partial_u \gamma(t_j)\|_{L^\infty}$. We use Lemmas 2.5 and D.7 to compute

$$\mathcal{L}_{\mathbb{R}^2}(\gamma(t_j)) \leq \mathcal{W}(f(t_j))^{1/2} \mu_{g_{f(t_j)}}(\mathbb{S}^1 \times \mathbb{S}^1)^{1/2} \leq \text{diam}(f(t_j)(\mathbb{S}^1 \times \mathbb{S}^1)) \mathcal{W}(f(t_j)).$$

Notice that $\text{diam}(f(t_j)(\mathbb{S}^1 \times \mathbb{S}^1)) \leq 2\|f(t_j)\|_{L^\infty(\mathbb{S}^1 \times \mathbb{S}^1, \mathbb{R}^3)}$, which is uniformly bounded in j . By Lemma C.5 and the fact that $\mathbb{S}^1 \times \mathbb{S}^1$ is compact we infer that $\mathcal{W}(f(t_j)) \rightarrow \mathcal{W}(f_\infty)$ and hence $(\mathcal{W}(f(t_j)))_{j \in \mathbb{N}}$ is also uniformly bounded. We conclude the boundedness of $(\mathcal{L}_{\mathbb{R}^2}(\gamma(t_j)))_{j \in \mathbb{N}}$.

Further, we bound second derivatives uniformly in j . To this end we introduce the following notation. For a torus of revolution $f : \mathbb{S}^1 \times \mathbb{S}^1 \rightarrow \mathbb{R}^3$ with profile curve $\gamma \in C^\infty(\mathbb{S}^1, \mathbb{H}^2)$ we introduce the vector field on $\mathbb{S}^1 \times \mathbb{S}^1$

$$\partial_s|_{(u,v)} = \frac{1}{|\partial_u \gamma(u)|_{\text{euc}}} \frac{\partial}{\partial u} \Big|_{(u,v)}.$$

One easily checks that $g_f(\partial_s, \partial_s) = 1$ and

$$\begin{pmatrix} -\vec{\kappa}_{\text{euc}}[\gamma](u) \\ 0 \end{pmatrix} = A_{(u,0)}[f](\partial_s, \partial_s) \quad \text{for all } u \in \mathbb{S}^1.$$

By Remark D.4, $\|A[f(t_j)]\|_{L^\infty}$ is uniformly bounded in j . This is why

$$\|\vec{\kappa}_{\text{euc}}[\gamma(t_j)]\|_{L^\infty} \leq \|A[f(t_j)]\|_{L^\infty} \|g_{f(t_j)}(\partial_s, \partial_s)\|_{L^\infty}^2$$

is also uniformly bounded in j . We next control all higher-order arclength derivatives of the curvature of $\gamma(t_j)$ uniformly in j . Easy tensor calculus and $\partial_s = \partial_u / |\partial_u \gamma(t_j)(u)|$ implies with (B-4)

$$\begin{aligned} \frac{1}{|\partial_u \gamma(t_j)(u)|} \begin{pmatrix} -\partial_u \vec{\kappa}_{\text{euc}}[\gamma(t_j)](u) \\ 0 \end{pmatrix} &= -D_{\partial_s} \begin{pmatrix} \vec{\kappa}_{\text{euc}}[\gamma(t_j)](u) \\ 0 \end{pmatrix} = D_{\partial_s} A[f(t_j)](\partial_s, \partial_s) \\ &= \nabla_{\partial_s}^\perp A(\partial_s, \partial_s) - \sum_{i=1}^2 \langle A(\partial_s, \partial_s), A(\partial_s, E_i) \rangle_{\mathbb{R}^3} D_{E_i}[f(t_j)], \end{aligned} \tag{3-7}$$

where $\{E_1, E_2\}$ is an arbitrary orthonormal basis of $T_{(u,0)}(\mathbb{S}^1 \times \mathbb{S}^1)$ with respect to $g_{f(t_j)}$ and we have used the (slightly ambiguous) shorthand notation A for $A[f(t_j)]$. Choosing $E_1 = \partial_s$ and

$$E_2(u, v) = \frac{1}{\gamma(t_j)^{(2)}(u)} \frac{\partial}{\partial v} \Big|_{(u,v)},$$

we obtain with (B-3)

$$\begin{aligned} \frac{1}{|\partial_u \gamma(t_j)(u)|} \begin{pmatrix} -\partial_u \vec{\kappa}_{\text{euc}}[\gamma(t_j)](u) \\ 0 \end{pmatrix} &= \nabla_{\partial_s}^\perp A(\partial_s, \partial_s) - |A(\partial_s, \partial_s)|^2 D_{\partial_s} f(t_j) \\ &= \nabla^\perp A(\partial_s, \partial_s, \partial_s) + A(\nabla_{\partial_s} \partial_s, \partial_s) + A(\partial_s, \nabla_{\partial_s} \partial_s) - |A(\partial_s, \partial_s)|^2 D_{\partial_s} f(t_j) \\ &= \nabla^\perp A(\partial_s, \partial_s, \partial_s) - |A(\partial_s, \partial_s)|^2 D_{\partial_s} f(t_j), \end{aligned} \tag{3-8}$$

where we have used in the last step that $\nabla_{\partial_s} \partial_s = 0$, which is an immediate consequence of the formula $df_p(\nabla_X Y) = \nabla_{df_p(X)}^{R^3}(df(\cdot)(Y))$ applied with $f = f(t_j)$. Note that

$$D_{\partial_s} f(u, 0) = \frac{1}{|\gamma'(u)|} D_{\partial_u} f = \frac{1}{|\gamma'(u)|} (\partial_u f)$$

has Euclidean norm equal to 1. We obtain, since $g_{f(t_j)}(\partial_s, \partial_s) \leq 1$, that

$$\frac{|\partial_u \vec{\kappa}_{\text{euc}}[\gamma(t_j)](u)|}{|\partial_u \gamma(t_j)(u)|} \leq \|\nabla^\perp A[f(t_j)]\|_{L^\infty} + \|A\|_{L^\infty}^2.$$

If we introduce the differential operator $\partial^{\text{arc}} := \partial_u / |\partial_u \gamma(t_j)|$ on \mathbb{S}^1 , we have obtained

$$\|\partial^{\text{arc}} \vec{\kappa}_{\text{euc}}[\gamma(t_j)]\|_{L^\infty} \leq \|\nabla^\perp A[f(t_j)]\|_{L^\infty} + \|A\|_{L^\infty}^2. \tag{3-9}$$

Next we obtain by differentiating (3-8) and using the shorthand notation $f = f(t_j)$, as well as $\nabla_{\partial_s} \partial_s = 0$, again proceeding as in (3-7) and (3-8)

$$\begin{aligned} & \begin{pmatrix} -(\partial^{\text{arc}})^2 \vec{\kappa}_{\text{euc}}[\gamma(t_j)](u) \\ 0 \end{pmatrix} \\ &= D_{\partial_s} [\nabla^\perp A(\partial_s, \partial_s, \partial_s) - |A(\partial_s, \partial_s)|^2 D_{\partial_s} f] \\ &= D_{\partial_s} \nabla^\perp A(\partial_s, \partial_s, \partial_s) - \partial_s (|A(\partial_s, \partial_s)|^2) D_{\partial_s} f - |A(\partial_s, \partial_s)|^2 D_{\partial_s} D_{\partial_s} f \\ &= \nabla_{\partial_s}^\perp \nabla^\perp A(\partial_s, \partial_s, \partial_s) - (\nabla^\perp A(\partial_s, \partial_s, \partial_s), A(\partial_s, \partial_s)) D_{\partial_s} f - \partial_s (|A(\partial_s, \partial_s)|^2) D_{\partial_s} f - |A(\partial_s, \partial_s)|^2 D_{\partial_s} D_{\partial_s} f \\ &= (\nabla^\perp)^2 A(\partial_s, \partial_s, \partial_s, \partial_s) - (\nabla^\perp A(\partial_s, \partial_s, \partial_s), A(\partial_s, \partial_s)) D_{\partial_s} f - \partial_s (|A(\partial_s, \partial_s)|^2) D_{\partial_s} f - |A(\partial_s, \partial_s)|^2 D_{\partial_s} D_{\partial_s} f. \end{aligned}$$

Note that since A is normal and $\nabla_{\partial_s} \partial_s = 0$ we have

$$\begin{aligned} \partial_s |A(\partial_s, \partial_s)|^2 &= 2(D_{\partial_s} A(\partial_s, \partial_s), A(\partial_s, \partial_s)) \\ &= 2(\nabla_{\partial_s}^\perp A(\partial_s, \partial_s), A(\partial_s, \partial_s)) = 2(\nabla^\perp A(\partial_s, \partial_s, \partial_s), A(\partial_s, \partial_s)). \end{aligned}$$

Moreover we have

$$D_{\partial_s} D_{\partial_s} f = (D_{\partial_s} D_{\partial_s} f)^T + A(\partial_s, \partial_s).$$

An easy computation¹ now reveals that $(D_{\partial_s} D_{\partial_s} f)^T = 0$ and we obtain

$$\begin{aligned} & \begin{pmatrix} -(\partial^{\text{arc}})^2 \vec{\kappa}_{\text{euc}}[\gamma(t_j)](u) \\ 0 \end{pmatrix} \\ &= (\nabla^\perp)^2 A(\partial_s, \partial_s, \partial_s, \partial_s) - 3(\nabla^\perp A(\partial_s, \partial_s, \partial_s), A(\partial_s, \partial_s)) D_{\partial_s} f - |A(\partial_s, \partial_s)|^2 A(\partial_s, \partial_s). \end{aligned}$$

For short we write

$$\begin{pmatrix} -(\partial^{\text{arc}})^2 \vec{\kappa}_{\text{euc}}[\gamma(t_j)](u) \\ 0 \end{pmatrix} = (\nabla^\perp)^2 A + \nabla^\perp A * A * D_{\partial_s} f + A * A * A,$$

which implies

$$\|(\partial^{\text{arc}})^2 \vec{\kappa}_{\text{euc}}\|_{L^\infty} \leq C[\|(\nabla^\perp)^2 A\|_{L^\infty} + \|\nabla A\|_{L^\infty} \|A\|_{L^\infty} + \|A\|_{L^\infty}^3].$$

¹Recall that the normal to the curve γ coincides up to a sign with the normal to $f(\mathbb{S}^1)$.

Inductively one shows that for all $m \in \mathbb{N}$

$$\begin{aligned} & \begin{pmatrix} -(\partial^{\text{arc}})^m \vec{\kappa}_{\text{euc}}[\gamma(t_j)](u) \\ 0 \end{pmatrix} \\ &= (\nabla^\perp)^m A + P_1(A, \nabla^\perp A, \dots, (\nabla^\perp)^{m-1} A) * D_{\partial_s} f + P_2(A, \nabla^\perp A, \dots, (\nabla^\perp)^{m-2} A), \end{aligned} \tag{3-10}$$

where P_1 is a real-valued polynomial of degree ≤ 2 and P_2 is an \mathbb{R}^3 -valued polynomial of degree ≤ 3 .

We conclude from (3-10) that for all $m \in \mathbb{N}$

$$\|\partial_u^m \gamma(t_j)\|_{L^\infty} \leq C(m) \mathcal{L}_{\mathbb{R}^2}(\gamma(t_j))^m \left[\|(\nabla^\perp)^m A\|_{L^\infty} + \sum_{i=0}^{m-1} \|(\nabla^\perp)^i A\|_{L^\infty}^3 \right]. \tag{3-11}$$

Hence for each fixed $m \in \mathbb{N}$ we can bound $(\gamma(t_j))_{j \in \mathbb{N}}$ uniformly in $W^{m+1, \infty}(\mathbb{S}^1, \mathbb{R}^2)$ and hence obtain a convergent subsequence in $C^m(\mathbb{S}^1, \mathbb{R}^2)$ for any m .

Step 2: the limit curve is a profile curve. By a diagonal argument we can also obtain a sequence $t_j \rightarrow \infty$ (no relabeling) and $\gamma_\infty \in C^\infty(\mathbb{S}^1, \mathbb{R}^2)$ such that $\gamma(t_j)$ converges to γ_∞ in $C^m(\mathbb{S}^1, \mathbb{R}^2)$ for all $m \in \mathbb{N}$ (classical convergence). Note also that γ_∞ is parametrized with constant Euclidean speed and $\gamma_\infty^{(2)} \geq 0$ on \mathbb{S}^1 . We next show that $\gamma_\infty \in C^\infty(\mathbb{S}^1, \mathbb{H}^2)$, i.e., $\inf_{\mathbb{S}^1} \gamma_\infty^{(2)} > 0$. Indeed, assume the opposite, i.e., there exists $u_0 \in \mathbb{S}^1$ such that $\gamma_\infty^{(2)}(u_0) = 0$. Notice that this and $\gamma_\infty^{(2)} \geq 0$ also yield $(\gamma_\infty^{(2)})'(u_0) = 0$. As a consequence, we infer that there exist $C > 0$ and $\delta_0 > 0$ such that $0 \leq \gamma_\infty^{(2)}(u) \leq C|u - u_0|^2$ for all $u \in (u_0 - \delta_0, u_0 + \delta_0)$. The fact that γ_∞ is parametrized with constant Euclidean velocity also yields that $|(\gamma_\infty^{(1)})'(u_0)| = \mathcal{L}_{\mathbb{R}^2}(\gamma_\infty) > 0$. With this information we now estimate the following quantity for arbitrary $\delta \in (0, \delta_0)$:

$$Q := \int_0^1 \frac{|(\gamma_\infty^{(1)})'(u)|^2}{\gamma_\infty^{(2)}(u)} du \geq \int_{u_0-\delta}^{u_0+\delta} \frac{|(\gamma_\infty^{(1)})'(u)|^2}{C|u - u_0|^2} du \geq \frac{1}{C\delta^2} \int_{u_0-\delta}^{u_0+\delta} |(\gamma_\infty^{(1)})'(u)|^2 du.$$

Taking the limit $\delta \rightarrow 0+$ yields infinity on the right-hand side, since

$$\frac{1}{2\delta} \int_{u_0-\delta}^{u_0+\delta} |(\gamma_\infty^{(1)})'(u)|^2 du \rightarrow |(\gamma_\infty^{(1)})'(u_0)|^2 = \mathcal{L}_{\mathbb{R}^2}(\gamma_\infty)^2 > 0.$$

We infer that $Q = \infty$. On the other hand, Fatou's lemma and the explicit formula for the second principal curvature κ_2 of a surface imply that

$$\begin{aligned} Q &\leq \liminf_{j \rightarrow \infty} \int_0^1 \frac{|(\gamma(t_j)^{(1)})'(u)|^2}{\gamma(t_j)^{(2)}(u)} du = \liminf_{j \rightarrow \infty} \frac{\mathcal{L}_{\mathbb{R}^2}(\gamma(t_j))}{2\pi} \int_0^1 \int_0^1 \frac{2\pi |(\gamma(t_j)^{(1)})'(u)|^2}{\gamma(t_j)^{(2)}(u) \mathcal{L}_{\mathbb{R}^2}(\gamma(t_j))} du dv \\ &\leq \liminf_{j \rightarrow \infty} \frac{\mathcal{L}_{\mathbb{R}^2}(\gamma(t_j))}{2\pi} \int_0^1 \int_0^1 \kappa_2[F_{\gamma(t_j)}]^2 \{2\pi \gamma(t_j)^{(2)} \mathcal{L}_{\mathbb{R}^2}(\gamma(t_j))\} du dv \\ &\leq \liminf_{j \rightarrow \infty} \frac{\mathcal{L}_{\mathbb{R}^2}(\gamma(t_j))}{2\pi} \int_{\mathbb{S}^1 \times \mathbb{S}^1} |A[F_{\gamma(t_j)}]|^2 d\mu_{F_{\gamma(t_j)}} \\ &= \liminf_{j \rightarrow \infty} \frac{\mathcal{L}_{\mathbb{R}^2}(\gamma(t_j))}{2\pi} \int_{\mathbb{S}^1 \times \mathbb{S}^1} |A[f(t_j)]|^2 d\mu_{f(t_j)} = \liminf_{j \rightarrow \infty} \frac{2\mathcal{L}_{\mathbb{R}^2}(\gamma(t_j))}{\pi} \mathcal{W}(f(t_j)), \end{aligned}$$

where the last identity is due to the Gauss–Bonnet theorem; see (A-4). Recall from estimates in Step 1 that $\mathcal{L}_{\mathbb{R}^2}(\gamma(t_j))$ is uniformly bounded. As a consequence of this one infers that $Q < \infty$, a contradiction. We obtain therefore that $\gamma_\infty \in C^\infty(\mathbb{S}^1, \mathbb{H}^2)$.

Step 3: convergence of the associated surfaces. By the following proposition (Proposition 3.5), the tori of revolution $F_{\gamma(t_j)}$ converge to F_{γ_∞} classically in C^k for all k . Since $F_{\gamma(t_j)}$ is a reparametrization of $f(t_j)$ for all $j \in \mathbb{N}$, also $f(t_j)$ converges to F_{γ_∞} in C^k for all k . By assumption however, $f(t_j)$ also converges to f_∞ in C^k for all k (in general not anymore classically, but in the sense of Definition C.7). Applying Corollary C.12 we infer that f_∞ coincides up to reparametrization with F_{γ_∞} . In particular f_∞ is (up to reparametrization) a torus of revolution. Since f_∞ is also a Willmore immersion it must (up to reparametrization) be a Willmore torus of revolution. By (2-12) we infer also that γ_∞ is a hyperbolic elastica. □

The following proposition is needed to complete the proof of the previous lemma.

Proposition 3.5. *Let $m \geq 1$ and suppose that $(\gamma_j)_{j \in \mathbb{N}} \subset C^\infty(\mathbb{S}^1, \mathbb{H}^2)$ converges in $C^m(\mathbb{S}^1, \mathbb{R}^2)$ (classically) to some immersed curve $\gamma \in C^m(\mathbb{S}^1, \mathbb{H}^2)$. Then F_{γ_j} converges classically to F_γ in $C^m(\mathbb{S}^1 \times \mathbb{S}^1)$.*

Proof. We will use without further notice the characterization of C^m -convergence in Proposition C.9. We show the claim only for $m = 1$, the other cases follow by induction. We define $w_j : \mathbb{S}^1 \times \mathbb{S}^1 \rightarrow \mathbb{R}^3$ via

$$w_j(u, v) := F_{\gamma_j}(u, v) - F_\gamma(u, v) = \begin{pmatrix} \gamma_j^{(1)}(u) - \gamma^{(1)}(u) \\ (\gamma_j^{(2)}(u) - \gamma^{(2)}(u)) \cos(2\pi v) \\ (\gamma_j^{(2)}(u) - \gamma^{(2)}(u)) \sin(2\pi v) \end{pmatrix} \tag{3-12}$$

and we show that $\|w_j\|_{L^\infty(\mathbb{S}^1 \times \mathbb{S}^1, \hat{g})}, \|Dw_j\|_{L^\infty(\mathbb{S}^1 \times \mathbb{S}^1, \hat{g})} \rightarrow 0$ as $j \rightarrow \infty$. Here $\hat{g} = F_\gamma^* g_{\mathbb{R}^3}$ is the metric induced by F_γ . The fact that $\|w_j\|_{L^\infty} \rightarrow 0$ follows directly from (3-12) by the estimate

$$\|w_j\|_{L^\infty} \leq \|\gamma_j - \gamma\|_{L^\infty} \rightarrow 0.$$

Let E_1, E_2 be the orthonormal frame as in (3-6). Then

$$\|Dw_j\|_{L^\infty} = \sup_{\mathbb{S}^1 \times \mathbb{S}^1} \sup_{g(X, X) \leq 1} |Dw_j(X)| = \sup_{\mathbb{S}^1 \times \mathbb{S}^1} \sup_{\theta_1^2 + \theta_2^2 \leq 1} |Dw_j(\theta_1 E_1 + \theta_2 E_2)|, \tag{3-13}$$

and

$$\begin{aligned} |Dw_j(E_1)| &= \frac{1}{|\gamma'(u)|} \left| \frac{\partial w_j}{\partial u} \right| \leq \frac{1}{|\gamma'(u)|} \|\gamma_j' - \gamma'\|_\infty \leq \frac{1}{\inf_{\mathbb{S}^1} |\gamma'|} \|\gamma_j' - \gamma'\|_\infty, \\ |Dw_j(E_2)| &= \frac{1}{2\pi \gamma^{(2)}(u)} \left| \frac{\partial w_j}{\partial v} \right| \leq \frac{1}{\inf_{\mathbb{S}^1} \gamma^{(2)}} \|\gamma_j - \gamma\|_{L^\infty}. \end{aligned}$$

Note that $\inf_{\mathbb{S}^1} |\gamma'| > 0$ as γ is immersed and $\inf_{\mathbb{S}^1} \gamma^{(2)} > 0$ since $\gamma \in C^\infty(\mathbb{S}^1, \mathbb{H}^2)$ and \mathbb{S}^1 is compact. The claim follows from (3-13) since $\gamma_j \rightarrow \gamma$ in C^1 . □

3.4. Rotational symmetry and concentration. In this section we will prove a lemma that controls the distance of the concentration points to the axis of revolution. Here the revolution symmetry will play an important role. The following lemma is the main observation that rules out Case 2 in Theorem 3.1.

Lemma 3.6 (Distance control for concentration points). *Let $f : [0, T) \times (\mathbb{S}^1 \times \mathbb{S}^1) \rightarrow \mathbb{R}^3$ be a maximal evolution by Willmore flow such that $f(0)$ is a torus of revolution. Suppose that $t_j \rightarrow T$. Let $(r_j)_{j \in \mathbb{N}}$ be as in Theorem 3.1 and let $x_j \in \mathbb{R}^3$ be such that*

$$\int_{f(t_j)^{-1}(\overline{B_{r_j}(x_j)})} |A[f(t_j)]|^2 d\mu_{g_{f(t_j)}} \geq \varepsilon_0. \tag{3-14}$$

Let $h_j \in \mathbb{R}$, $\rho_j > 0$ and $\sigma_j \in \mathbb{S}^1$ such that x_j/r_j is expressed in cylindrical coordinates by $x_j/r_j = (h_j, \rho_j \sigma_j)$. Then $(\rho_j)_{j \in \mathbb{N}}$ is bounded.

Proof. We first use scaling properties to obtain that

$$\int_{(f(t_j)/r_j)^{-1}(\overline{B_1(x_j/r_j)})} \left| A \left[\frac{f(t_j)}{r_j} \right] \right|^2 d\mu_{g_{f(t_j)/r_j}} \geq \varepsilon_0. \tag{3-15}$$

Now write $x_j/r_j = (h_j, \rho_j \sigma_j)$ as in the statement. Since $f(t_j)/r_j$ has a revolution symmetry (see Lemma 3.3), we conclude from (3-15) that the curvature concentration does not only happen at points but actually on circles. More precisely,

$$\int_{(f(t_j)/r_j)^{-1}(\overline{B_1(h_j, \rho_j \sigma)})} \left| A \left[\frac{f(t_j)}{r_j} \right] \right|^2 d\mu_{g_{f(t_j)/r_j}} \geq \varepsilon_0 \quad \text{for all } \sigma \in \mathbb{S}^1. \tag{3-16}$$

Next, we define for each $\rho > 0$ the maximal number of disjoint closed balls of radius 1 needed to cover the circle $(0, \rho \mathbb{S}^1) \subset \mathbb{R}^3$

$$N(\rho) := \max \left\{ l \in \mathbb{N} : \text{there exist } \omega_1, \dots, \omega_l \in \mathbb{S}^1 \text{ such that } \overline{B_1((0, \rho \omega_1))}, \dots, \overline{B_1((0, \rho \omega_l))} \text{ are pairwise disjoint} \right\}.$$

This number depends only on the radius of the circle and not on its position in \mathbb{R}^3 . By compactness of \mathbb{S}^1 , $N(\rho)$ is well-defined and finite. Moreover, using (3-16) on $N(\rho_j)$ disjoint balls that cover $(h_j, \rho_j \mathbb{S}^1)$ and that preimages of disjoint sets are always disjoint, we infer

$$\int_{\mathbb{S}^1 \times \mathbb{S}^1} \left| A \left[\frac{f(t_j)}{r_j} \right] \right|^2 d\mu_{g_{f(t_j)/r_j}} \geq N(\rho_j) \varepsilon_0.$$

Note that this implies by scaling properties and the Gauss–Bonnet theorem that

$$N(\rho_j) \leq \frac{1}{\varepsilon_0} \int_{\mathbb{S}^1 \times \mathbb{S}^1} |A[f(t_j)]|^2 d\mu_{g_{f(t_j)}} = \frac{1}{\varepsilon_0} \mathcal{W}(f(t_j)) \leq \frac{\mathcal{W}(f_0)}{\varepsilon_0}.$$

To infer that ρ_j is bounded it suffices now to show that $N(\rho) \rightarrow \infty$ as $\rho \rightarrow \infty$. To this end we prove that

$$N(\rho) \geq \frac{\pi}{4 \arccos(1 - 8/\rho^2)} \quad \text{for } \rho \geq 4. \tag{3-17}$$

² That is, $h_j = x_j^{(1)}/r_j \in \mathbb{R}$, $\rho_j = \sqrt{(x_j^{(2)})^2 + (x_j^{(3)})^2}/r_j \geq 0$ and $\sigma_j = (x_j^{(2)}, x_j^{(3)})/(\rho_j r_j) \in \mathbb{S}^1$. We consider a cylinder with axis in the direction $(1, 0, 0)$.

Let us first fix $\rho \geq 4$. Note first that the squared Euclidean distance in \mathbb{R}^3 between $(0, \rho \cos(\alpha), \rho \sin(\alpha))$ and $(0, \rho \cos(\beta), \rho \sin(\beta))$ is given by

$$d_{\alpha,\beta}^2 := 2\rho^2(1 - \cos(\alpha - \beta)).$$

Also observe that the balls $\overline{B_1((0, \rho \cos(\alpha), \rho \sin(\alpha)))}$, $\overline{B_1((0, \rho \cos(\beta), \rho \sin(\beta)))}$ are disjoint if and only if $d_{\alpha,\beta}^2 > 4$. Hence it suffices to find distinct values $\alpha_1, \dots, \alpha_{\tilde{N}} \in [0, 2\pi)$ such that for all $i, j \in \{1, \dots, \tilde{N}\}$ one has

$$d_{\alpha_i,\alpha_j}^2 \geq 16 \quad \text{for all } i, j \in \{1, \dots, \tilde{N}\}.$$

We claim that the choice of $\alpha_j := j \arccos(1 - 8/\rho^2)$, $j = 1, \dots, \tilde{N}$, with

$$\tilde{N} = \left\lfloor \frac{\pi}{4 \arccos(1 - 8/\rho^2)} \right\rfloor$$

has the desired properties. Indeed, note that $\alpha_1, \dots, \alpha_{\tilde{N}} \in [0, \frac{\pi}{4}]$ which implies that $|\alpha_i - \alpha_j| \in [0, \frac{\pi}{2}]$ for all i, j . Using evenness of \cos and monotonicity of \cos in $[0, \frac{\pi}{2}]$ we obtain for all $i, j \in \{1, \dots, \tilde{N}\}$

$$\begin{aligned} d_{\alpha_i,\alpha_j}^2 &= 2\rho^2(1 - \cos(\alpha_i - \alpha_j)) = 2\rho^2\left(1 - \cos\left(|i - j| \arccos\left(1 - \frac{8}{\rho^2}\right)\right)\right) \\ &\geq 2\rho^2\left(1 - \cos\left(1 \cdot \arccos\left(1 - \frac{8}{\rho^2}\right)\right)\right) = 16. \end{aligned}$$

We have thus shown (3-17) and thus the claim follows. □

Remark 3.7. The lemma reveals an interesting property of the Willmore flow of tori of revolution. Suppose that $T < \infty$. Then by Theorem 3.1 and in particular the property $t_j + c_0 r_j^4 < T$, necessarily $r_j \rightarrow 0$. Now let $(x_j)_{j \in \mathbb{N}}$ be a collection of *points of concentration*, i.e., points where (3-14) holds true. From the previous lemma we know that the distance of x_j/r_j to the x -axis is bounded. Hence the distance $(x_j)_{j \in \mathbb{N}}$ to the x -axis tends to zero. In other words, finite-time-concentration may only happen close to the x -axis.

3.5. Proof of Theorems 1.2 and 1.3.

Proof of Theorem 1.2. Let $f : [0, T) \times \mathbb{S}^1 \times \mathbb{S}^1 \rightarrow \mathbb{R}^3$ be as in the statement. That $f(t)$ is a torus of revolution for all $t \in [0, T)$ follows from Lemma 3.3. Thus we can actually choose $(\gamma(t))_{t \in [0, T)}$ as in the statement; see also the discussion after Lemma 3.3. Let $t_j \rightarrow T$ be such that $\mathcal{L}_{\mathbb{H}^2}(\gamma(t_j)) \leq M$ for some $M > 0$ and let $r_j > 0$ and \tilde{f}_{j,c_0} be as in Theorem 3.1. By Theorem 3.1 it is sufficient for the convergence of the Willmore flow that $(\text{diam}(\tilde{f}_{j,c_0}))_{j \in \mathbb{N}}$ is bounded. Notice that we assume a bound on $\mathcal{L}_{\mathbb{H}^2}$ at t_j and we want a bound on the diameter at $t_j + c_0 r_j^4$. To this end we define $\tilde{f}_{j,0} := f(t_j)/r_j$ and choose for all $j \in \mathbb{N}$, x_j as in (3-14). Such a choice of x_j exists due to the definition of r_j in Theorem 3.1. We write $x_j/r_j = (h_j, \rho_j \sigma_j)$, $\rho_j > 0$ and $\sigma_j \in \mathbb{S}^1$ as in Lemma 3.6 and infer from Lemma 3.6 that $(\rho_j)_{j \in \mathbb{N}}$ is bounded, say $\rho_j \leq C$ for all $j \in \mathbb{N}$. Note that by the choice of x_j , in particular (3-15), for all $j \in \mathbb{N}$ one has $\text{dist}(x_j/r_j, \tilde{f}_{j,0}(\mathbb{S}^1 \times \mathbb{S}^1)) \leq 1$. Now we look at $\tilde{\gamma}_j = \gamma(t_j)/r_j$, which is clearly a profile curve of $\tilde{f}_{j,0}$ and satisfies also $\mathcal{L}_{\mathbb{H}^2}(\tilde{\gamma}_j) \leq M$ by scaling invariance of the hyperbolic length. By the distance estimate we can find $u_j, v_j \in \mathbb{S}^1$ such that

$$\left| \frac{1}{r_j} [(x_j^{(2)}, x_j^{(3)}) - \gamma_j^{(2)}(u_j)(\cos(2\pi v_j), \sin(2\pi v_j))] \right| \leq 1.$$

Hence we infer that

$$\tilde{\gamma}_j^{(2)}(u_j) \leq 1 + \left| \frac{1}{r_j}(x_j^{(2)}, x_j^{(3)}) \right| \leq 1 + \rho_j \leq 1 + C.$$

From the bounded hyperbolic length and (2-8) we infer that

$$\sup_{\mathbb{S}^1} \tilde{\gamma}_j^{(2)} \leq \tilde{\gamma}_j^{(2)}(u_j) e^{\mathcal{L}_{\mathbb{H}^2}(\tilde{\gamma}_j)} \leq (1 + C)e^M.$$

This implies also by (2-9) that

$$\mathcal{L}_{\mathbb{R}^2}(\tilde{\gamma}_j) \leq \sup_{\mathbb{S}^1} \tilde{\gamma}_j^{(2)} \mathcal{L}_{\mathbb{H}^2}(\tilde{\gamma}_j) \leq M(1 + C)e^M,$$

and from Lemma 2.6 we now infer

$$\text{diam}(f_{j,0}) \leq \frac{1}{2} \mathcal{L}_{\mathbb{R}^2}(\tilde{\gamma}_j) + 2 \sup_{\mathbb{S}^1} \tilde{\gamma}_j^{(2)} \leq D \tag{3-18}$$

for some constant $D \geq 0$. We now define $\tilde{f}_j(s) := f(t_j + sr_j^4)/r_j$, $s \in [0, c_0]$, taking into account the parabolic scaling. It is easy to see that then \tilde{f}_j is a solution of the Willmore flow equation and $\tilde{f}_j(0) = \tilde{f}_{j,0}$ and $\tilde{f}_j(c_0) = \tilde{f}_{j,c_0}$. Hence we can estimate by Lemma D.6

$$\text{diam}(f_{j,c_0}) \leq C(\mathcal{W}(\tilde{f}_{j,0}))(\text{diam}(\tilde{f}_{j,0}) + c_0^{1/4}).$$

Using that by scaling invariance $\mathcal{W}(\tilde{f}_{j,0}) = \mathcal{W}(f(t_j)) \leq \mathcal{W}(f_0)$ and (3-18) we obtain

$$\text{diam}(f_{j,c_0}) \leq C(\mathcal{W}(f_0))(D + c_0^{1/4}). \tag{3-19}$$

By Theorem 3.1 this implies that $T = \infty$ and $(f(t))_{t \in [0, \infty)}$ is a convergent evolution. It only remains to show that the limit is a torus of revolution. This is however a direct consequence of Lemma 3.4. \square

Proof of Theorem 1.3. Let $(f(t))_{t \in [0, T)}$ and $(\gamma(t))_{t \in [0, T)}$ be as in the statement. We distinguish two cases.

Case 1: $\mathcal{W}(f_0) < 8\pi$. To show long-time existence and convergence of the evolution we apply Theorem 1.2. To this end we need to show that

$$\liminf_{t \rightarrow T} \mathcal{L}_{\mathbb{H}^2}(\gamma(t)) < \infty.$$

First we observe that $(\gamma(t))_{t \in [0, T)}$ satisfies

$$\mathcal{E}(\gamma(t)) = \frac{2}{\pi} \mathcal{W}(F_{\gamma(t)}) = \frac{2}{\pi} \mathcal{W}(f(t)) \leq \frac{2}{\pi} \mathcal{W}(f_0) < 16.$$

We apply Theorem 2.2 with $\varepsilon := 16 - \frac{2}{\pi} \mathcal{W}(f_0)$ to find that for each $t \in [0, T)$ one has

$$\mathcal{L}_{\mathbb{H}^2}(\gamma(t)) \leq \frac{1}{c(\varepsilon)} \mathcal{E}(\gamma(t)) = \frac{2}{\pi c(\varepsilon)} \mathcal{W}(f(t)) \leq \frac{2}{\pi c(\varepsilon)} \mathcal{W}(f_0),$$

and hence the hyperbolic length is uniformly bounded for $t \in [0, T)$. By Theorem 1.2 the evolution converges in C^k for all k and the limit, say $f_\infty : \mathbb{S}^1 \times \mathbb{S}^1 \rightarrow \mathbb{R}^3$, is a Willmore torus of revolution. By the gradient flow properties of the Willmore flow and Lemma C.3 we obtain that $\mathcal{W}(f_\infty) \leq \mathcal{W}(f_0) < 8\pi$.

We obtain from Proposition 2.4 that f_∞ is, up to reparametrization, a Clifford torus, possibly rescaled and translated in the direction $(1, 0, 0)^T$. The claim follows.

Case 2: $\mathcal{W}(f_0) = 8\pi$. We first claim that f_0 is not a Willmore surface. Indeed, if it were then it would by Proposition 2.4 be a rescaled and translated reparametrization of a Clifford torus. But the Willmore energy of the Clifford torus is $2\pi^2$, contradicting $\mathcal{W}(f_0) = 8\pi$. Hence

$$\frac{d}{dt}\mathcal{W}(f(t))\Big|_{t=0} = -\|\nabla_{L^2}\mathcal{W}(f_0)\|_{L^2(\Sigma)}^2 < 0,$$

which implies that there exists $t_0 > 0$ such that $\mathcal{W}(f(t_0)) < 8\pi$. We restart the Willmore flow with $f(t_0)$ which satisfies the assumptions of Case 1 and hence converges to a reparametrization of the Clifford torus, possibly rescaled and translated in the direction $(1, 0, 0)^T$. The claim follows. \square

3.6. Optimality. We show that the upper bound of 8π on the Willmore energy of the initial datum in Theorem 1.3 is sharp by proving Theorem 1.4. In the statement of this theorem, the geometric quantities that may possibly degenerate along the flow are the second fundamental form or the diameter. On contrary, the statement of Theorem 1.2 suggests another quantity which must degenerate — the hyperbolic length. In the following we will construct the nonconvergent evolutions and study the relation between the degenerating quantities.

Lemma 3.8 (The singular evolutions). *For any $\varepsilon > 0$ there exists a torus of revolution $f_0 : \mathbb{S}^1 \times \mathbb{S}^1 \rightarrow \mathbb{R}^3$ such that $\mathcal{W}(f_0) < 8\pi + \varepsilon$, and the maximal Willmore flow $(f(t))_{t \in [0, T]}$ starting at f_0 satisfies*

$$\lim_{t \rightarrow T} \mathcal{L}_{\mathbb{H}^2}(\gamma(t)) = \infty.$$

The main idea is to start the flow with an immersed curve that has *total curvature*

$$T[\gamma] := \frac{1}{2\pi} \int_{\gamma} \kappa_{\text{euc}}[\gamma] \, ds \tag{3-20}$$

equal to zero. This quantity $T[\cdot]$ turns out to be a flow invariant and can hence be helpful to classify possible limits of convergent evolution. This in turn can also be used to show that some evolutions cannot be convergent.

Lemma 3.9. *The total curvature T , defined on curves in $W^{2,2}(\mathbb{S}^1, \mathbb{R}^2)_{\text{imm}} := \{\gamma \in W^{2,2}(\mathbb{S}^1, \mathbb{R}^2) : \gamma \text{ immersed}\}$ is integer-valued and weakly continuous in the relative topology of $W^{2,2}(\mathbb{S}^1, \mathbb{R}^2)_{\text{imm}}$. Moreover it is a flow invariant for the Willmore flow of tori of revolution; i.e., if $(f(t))_{t \in [0, T]}$ is an evolution by the Willmore flow with profile curve $(\gamma(t))_{t \in [0, T]}$ then $T[\gamma(t)] = T[\gamma(0)]$ for all $t \in [0, T]$.*

Proof. The fact that $T[\cdot]$ is integer-valued and an invariant with respect to regular homotopies is very classical and follows from the Whitney–Graustein theorem. Since $\gamma(t) = f(t)(u, 0)$ (see (3-5)) and $t \mapsto f(t)$ is a regular homotopy, so is $t \mapsto \gamma(t)$. Hence we can also conclude that it is a Willmore flow invariant. The weak $W^{2,2}$ -continuity follows immediately from the formula

$$T[\gamma] := \frac{1}{2\pi} \int_0^1 \frac{1}{|\gamma'|} ((\gamma^{(2)})''(\gamma^{(1)})' - (\gamma^{(1)})''(\gamma^{(2)})') \, dx$$

and the compact embedding $W^{2,2} \hookrightarrow C^1$. \square

Proof of Lemma 3.8. Fix $\varepsilon > 0$. By [Müller and Spener 2020, Corollary 6.4] there exists a curve γ_ε such that $16 \leq \mathcal{E}(\gamma_\varepsilon) < 16 + \varepsilon$ and $T[\gamma_\varepsilon] = 0$, where $T[\cdot]$ is given as in (3-20). Now start the flow with $f_0 = F_{\gamma_\varepsilon} : \mathbb{S}^1 \times \mathbb{S}^1 \rightarrow \mathbb{R}^3$ defined as in (1-4) with profile curve γ_ε and let $(f(t))_{t \in [0, T]}$ be the corresponding evolution by the Willmore flow. Assume that for $(\gamma(t))_{t \in [0, \infty)}$ as in (3-5) one has

$$\liminf_{t \rightarrow T} \mathcal{L}_{\mathbb{H}^2}(\gamma(t)) < \infty.$$

By Theorem 1.2 we obtain that then $T = \infty$ and $(f(t))_{t \in [0, \infty)}$ is convergent to a Willmore torus of revolution f_∞ . Let now $t_j \rightarrow \infty$ be a sequence such that $\mathcal{L}_{\mathbb{H}^2}(\gamma(t_j)) \leq M < \infty$ for all $j \in \mathbb{N}$. By Lemma 3.4 we obtain that an appropriate reparametrization of $\gamma(t_j)$ converges in $C^k(\mathbb{S}^1, \mathbb{R}^2)$ to some $\gamma_\infty \in C^\infty(\mathbb{S}^1, \mathbb{H}^2)$, which is a profile curve of f_∞ , i.e., up to a reparametrization one has $f_\infty = F_{\gamma_\infty}$. By (2-12) we infer that γ_∞ is a hyperbolic elastica.

Now we choose $\phi_j \in C^4(\mathbb{S}^1, \mathbb{S}^1)$ such that $\gamma(t_j) \circ \phi_j$ converges to γ_∞ classically in $C^4(\mathbb{S}^1, \mathbb{R}^2)$. Then, by the previous lemma

$$T[\gamma_\infty] = \lim_{j \rightarrow \infty} T[\gamma(t_j)] = T[\gamma(0)] = 0.$$

Hence γ_∞ is a hyperbolic elastica with vanishing Euclidean total curvature. By [Müller and Spener 2020, Corollary 5.8] there exist no hyperbolic elastica of vanishing total curvature. We obtain a contradiction and the claim follows. □

As an important ingredient for case (2) in Theorem 1.4, we need to show that global evolutions under the Willmore flow of tori of revolution with unbounded hyperbolic length and no curvature concentration must have unbounded diameter.

Lemma 3.10 (Diameter blow-up). *Let $f_0 : \mathbb{S}^1 \times \mathbb{S}^1 \rightarrow \mathbb{R}^3$ be a torus of revolution and let $(f(t))_{t \in [0, \infty)}$ evolve by the Willmore flow with initial datum f_0 . Let $\gamma(t) = f(t)(\cdot, 0)$ be the profile curve of $f(t)$ for all $t \geq 0$. Assume that $(A(t))_{t \in [0, \infty)}$ is bounded in $L^\infty(\Sigma)$ and $\lim_{t \rightarrow \infty} \mathcal{L}_{\mathbb{H}^2}(\gamma(t)) = \infty$. Then*

$$\lim_{t \rightarrow \infty} \text{diam}(f(t)(\mathbb{S}^1 \times \mathbb{S}^1)) = \infty.$$

Proof. We first introduce the constant $D := \sup_{t \in [0, \infty)} \|A(t)\|_{L^\infty} < \infty$. Next we assume for a contradiction that there exists some $t_j \rightarrow T = \infty$ such that $\text{diam}(f(t_j)(\mathbb{S}^1 \times \mathbb{S}^1)) \leq M < \infty$ for all $j \in \mathbb{N}$. Let $(r_j)_{j \in \mathbb{N}} \subset (0, \infty)$ be as in Theorem 3.1. Note that there exists $x_j \in \mathbb{R}^3$

$$\varepsilon_0 \leq \int_{f(t_j)^{-1}(\overline{B_{r_j}(x_j)})} |A[f(t_j)]|^2 \, d\mu_{g_{f(t_j)}} \leq D^2 \mu_{g_{f(t_j)}}(f(t_j)^{-1}(\overline{B_{r_j}(x_j)})).$$

By (D-2) we have that

$$\mu_{g_{f(t_j)}}(f(t_j)^{-1}(\overline{B_{r_j}(x_j)})) \leq CW(f(t_j))r_j^2 \leq CW(f_0)r_j^2.$$

In particular we find by the previous two equations

$$r_j^2 \geq \frac{\varepsilon_0}{D^2 CW(f_0)}; \tag{3-21}$$

i.e., there exists $\delta > 0$ such that $r_j \geq \delta$ for all $j \in \mathbb{N}$. Since we have assumed that $\text{diam}(f(t_j)(\mathbb{S}^1 \times \mathbb{S}^1)) \leq M$ we obtain that

$$\text{diam}\left(\frac{f(t_j)}{r_j}(\mathbb{S}^1 \times \mathbb{S}^1)\right) \leq \frac{1}{r_j} \text{diam}(f(t_j)(\mathbb{S}^1 \times \mathbb{S}^1)) \leq \frac{1}{\delta} M.$$

Now recall that $\tilde{f}_j(s) := f(t_j + sr_j^4)/r_j$, $s \in [0, c_0]$, defines a solution of the Willmore flow, with $\tilde{f}_j(0) = f(t_j)/r_j$ and $\tilde{f}_j(c_0) = \tilde{f}_{j,c_0}$, defined as in [Theorem 3.1](#). With [Lemma D.6](#) we obtain thus that

$$\text{diam}(\tilde{f}_{j,c_0}) \leq C\left(\mathcal{W}\left(\frac{f(t_j)}{r_j}\right)\right)\left(\text{diam}\left(\frac{f(t_j)}{r_j}\right) + c_0^{1/4}\right) \leq C(\mathcal{W}(f_0))\left(\frac{M}{\delta} + c_0^{1/4}\right),$$

which is uniformly bounded in j . This implies by [Theorem 3.1](#) that there exists a Willmore immersion $f_\infty : \mathbb{S}^1 \times \mathbb{S}^1 \rightarrow \mathbb{R}^3$ such that $f(t) \rightarrow f_\infty$ in C^k for all $k \in \mathbb{N}$. By [Lemma 3.4](#), f_∞ is a Willmore torus of revolution. In particular, up to reparametrization one has $f_\infty = F_{\gamma_\infty}$ for some $\gamma_\infty \in C^\infty(\mathbb{S}^1, \mathbb{H}^2)$. We next claim that there exists $\delta > 0$ such that $\inf_{\mathbb{S}^1} \gamma(t)^{(2)} > \delta$ for all $t \in [0, \infty)$. To this end observe

$$\begin{aligned} \liminf_{t \rightarrow \infty} \inf_{\mathbb{S}^1} \gamma(t)^{(2)} &= \lim_{t \rightarrow \infty} \inf_{\mathbb{S}^1 \times \mathbb{S}^1} \sqrt{(f(t)^{(2)})^2 + (f(t)^{(3)})^2} \\ &= \inf_{\mathbb{S}^1 \times \mathbb{S}^1} \sqrt{(f_\infty^{(2)})^2 + (f_\infty^{(3)})^2} = \inf_{\mathbb{S}^1} \gamma_\infty^{(2)} > 0, \end{aligned}$$

since $\gamma_\infty^{(2)}(u) > 0$ for all $u \in \mathbb{S}^1$ and \mathbb{S}^1 is compact. Note that we have used here that the infimum expression is independent of the parametrization of $f(t)$. This and the fact that $(f(t))_{t \in [0, \infty)}$ is a smoothly evolving family of tori of revolution implies $\inf_{\mathbb{S}^1} \gamma(t)^{(2)} > \delta$ for all $t \in [0, \infty)$. Next we look at the surface area of $f(t)$, i.e.,

$$\mu_{g_{f(t)}}(\mathbb{S}^1 \times \mathbb{S}^1) = 2\pi \int_0^1 |\gamma(t)'(u)| \gamma^{(2)}(t)(u) \, du,$$

and infer

$$\mu_{g_{f(t_j)}}(\mathbb{S}^1 \times \mathbb{S}^1) \geq 2\pi \delta^2 \mathcal{L}_{\mathbb{H}^2}(\gamma(t_j)) \rightarrow \infty.$$

With [Lemma D.7](#) it follows

$$M \geq \text{diam}(f(t_j)(\mathbb{S}^1 \times \mathbb{S}^1)) \geq \sqrt{\frac{\mu_{g_{f(t_j)}}(\mathbb{S}^1 \times \mathbb{S}^1)}{\mathcal{W}(f(t_j))}} \geq \sqrt{\frac{\mu_{g_{f(t_j)}}(\mathbb{S}^1 \times \mathbb{S}^1)}{\mathcal{W}(f_0)}} \rightarrow \infty.$$

A contradiction. We infer that $\lim_{t \rightarrow \infty} \text{diam}(f(t)(\mathbb{S}^1 \times \mathbb{S}^1)) = \infty$. □

In the proof we have used without further notice that the concept of tori of revolution in [[Blatt 2009](#), Definition 2.2] coincides with our definition in [Definition 1.1](#), at least up to reparametrization. For details recall [Proposition 3.2](#) and the discussion afterwards.

Proof of Theorem 1.4. Let $\varepsilon > 0$ be as in the statement and f_0 be as in [Lemma 3.8](#). Then the evolution $(f(t))_{t \in [0, T]}$ satisfies $\lim_{t \rightarrow T} \mathcal{L}_{\mathbb{H}^2}(\gamma(t)) = \infty$. Next let $t_j \uparrow T$ be a sequence. Let $\varepsilon_0 > 0$, $c_0 > 0$ and $(r_j)_{j \in \mathbb{N}}$ be as in [Theorem 3.1](#). We distinguish now two cases.

Case 1: there exists a subsequence of r_j that converges to zero. We claim that then condition (1) in the statement occurs. To this end assume that $(\|A(t)\|_{L^\infty})_{t \in [0, T]}$ is bounded, say $D := \sup_{t \in [0, T]} \|A(t)\|_{L^\infty} < \infty$.

Then one has by (3-1) that for all $j \in \mathbb{N}$ there exists $x_j \in \mathbb{R}^3$ such that

$$\varepsilon_0 \leq \int_{f(t_j)^{-1}(\overline{B_{r_j}(x_j)})} |A(t_j)|^2 d\mu_{g_{f(t_j)}} \leq D^2 \mu_{g_{f(t_j)}}(f(t_j)^{-1}(\overline{B_{r_j}(x_j)})).$$

Using (D-2) we find that $\varepsilon_0 \leq c\mathcal{W}(f_0)D^2r_j^2$. This is a contradiction to the condition that up to a subsequence $r_j \rightarrow 0$. Hence we have shown that $(\|A(t)\|_{L^\infty(\Sigma)})_{t \in [0, T]}$ is unbounded.

Case 2: there exists $\delta > 0$ such that $r_j \geq \delta$ for all $j \in \mathbb{N}$. First observe that in this case $T = \infty$ since $t_j + c_0r_j^4 < T$ by Theorem 3.1. If condition (1) in the statement holds true, i.e., $(\|A(t)\|_{L^\infty(\Sigma)})_{t \geq 0}$ is unbounded, there is nothing to prove. Hence we may assume that $(\|A(t)\|_{L^\infty(\Sigma)})_{t \geq 0}$ is bounded. Since $\lim_{t \rightarrow \infty} \mathcal{L}_{\mathbb{H}^2}(\gamma(t)) = \infty$, by Lemma 3.10 we find that $\lim_{t \rightarrow \infty} \text{diam}(f(t))(\mathbb{S}^1 \times \mathbb{S}^1) = \infty$ and hence condition (2) occurs. This proves the claim. \square

4. An application: energy minimization among conformal constraints

A very vivid field of research is the minimization of the Willmore energy among all tori that are *conformally equivalent* to a reference torus. Being conformally equivalent means that the surface can be parametrized with a *conformal* immersion of the reference torus. Taking a reference torus of the form $\mathbb{C}/(\mathbb{Z} + \omega\mathbb{Z})$ one can also associate to every torus its *conformal class*, defined as follows.

Definition 4.1 (Conformal class; see [Ndiaye and Schätzle 2015, p. 293]). Let $S \subset \mathbb{R}^3$ be a smooth torus. Then there exists a unique $\omega \in \mathbb{C}$ satisfying $|\omega| \geq 1$, $\text{Im}(\omega) > 0$ and $\text{Re}(\omega) \in [0, \frac{1}{2}]$ such that there exists a conformal smooth immersion

$$F : \frac{\mathbb{C}}{\mathbb{Z} + \omega\mathbb{Z}} \rightarrow S,$$

i.e.,

$$g_{i,j}^F = e^{2u} \delta_{i,j} \quad \text{for some } u \in C^\infty\left(\frac{\mathbb{C}}{\mathbb{Z} + \omega\mathbb{Z}}\right). \tag{4-1}$$

The value $\omega = \omega(S) \in \mathbb{C}$ is then called the *conformal class* of S . If ω is purely imaginary, we call the torus *rectangular*.

As it turns out, all tori of revolution are rectangular (see also [Langer and Singer 1984a, Proposition 7]).

Proposition 4.2. *Suppose that $\gamma \in C^\infty(\mathbb{S}^1, \mathbb{H}^2)$. Then $F_\gamma(\mathbb{S}^1 \times \mathbb{S}^1)$, the torus with profile curve γ , has conformal class*

$$\omega(F_\gamma(\mathbb{S}^1 \times \mathbb{S}^1)) = \begin{cases} i\mathcal{L}_{\mathbb{H}^2}(\gamma)/(2\pi), & \mathcal{L}_{\mathbb{H}^2}(\gamma) \geq 2\pi, \\ i2\pi/\mathcal{L}_{\mathbb{H}^2}(\gamma), & \mathcal{L}_{\mathbb{H}^2}(\gamma) < 2\pi. \end{cases}$$

In particular, each torus of revolution is rectangular and $\omega(F_\gamma(\mathbb{S}^1 \times \mathbb{S}^1))$ is a continuous function of $\mathcal{L}_{\mathbb{H}^2}(\gamma)$.

Proof. Let $\bar{\gamma} : \mathbb{R} \rightarrow \mathbb{R}$ be the $\frac{1}{2\pi}\mathcal{L}_{\mathbb{H}^2}(\gamma)$ -periodic reparametrization of γ with constant hyperbolic velocity 2π . If $\mathcal{L}_{\mathbb{H}^2}(\gamma) \geq 2\pi$ we choose the smooth immersion

$$F : \frac{\mathbb{C}}{\mathbb{Z} + (i\mathcal{L}_{\mathbb{H}^2}(\gamma)/(2\pi))\mathbb{Z}} \rightarrow F_\gamma(\mathbb{S}^1 \times \mathbb{S}^1)$$

by

$$F(s + it) = \begin{pmatrix} \bar{\gamma}^1(t) \\ \bar{\gamma}^2(t) \cos(2\pi s) \\ \bar{\gamma}^2(t) \sin(2\pi s) \end{pmatrix}. \tag{4-2}$$

An easy computation shows $g_{1,2}^F = g_{2,1}^F = 0$ and

$$g_{1,1}^F = (\bar{\gamma}^1)'{}^2 + (\bar{\gamma}^2)'{}^2, \quad g_{2,2}^F = 4\pi^2(\bar{\gamma}^2)^2.$$

Therefore by our choice of parametrization

$$\frac{g_{1,1}^F}{g_{2,2}^F} = \frac{(\bar{\gamma}^1)'{}^2 + (\bar{\gamma}^2)'{}^2}{4\pi^2(\bar{\gamma}^2(t))^2} = 1.$$

Hence (4-1) is satisfied and F is a conformal immersion. Moreover one readily checks that $\omega = i\mathcal{L}_{\mathbb{H}^2}(\gamma)/(2\pi)$ meets the requirements of Definition 4.1.

If $\mathcal{L}_{\mathbb{H}^2}(\gamma) < 2\pi$ we choose

$$\tilde{F} : \frac{\mathbb{C}}{\mathbb{Z} + i(2\pi/\mathcal{L}_{\mathbb{H}^2}(\gamma))\mathbb{Z}} \rightarrow F_\gamma(\mathbb{S}^1 \times \mathbb{S}^1)$$

to be given by

$$\tilde{F}(s + it) := F\left(\frac{\mathcal{L}_{\mathbb{H}^2}(\gamma)}{2\pi}t + i\frac{\mathcal{L}_{\mathbb{H}^2}(\gamma)}{2\pi}s\right),$$

where F is as in (4-2) and the claim follows also in this case arguing as before. □

Remark 4.3. The conformal class of the Clifford torus is $\omega = i$. Indeed, its defining curve is

$$\gamma(t) = \begin{pmatrix} 0 \\ 1 \end{pmatrix} + \frac{1}{\sqrt{2}} \begin{pmatrix} \cos(t) \\ \sin(t) \end{pmatrix}, \quad t \in [-\pi, \pi).$$

From this we conclude with the residue theorem (more precisely [Freitag and Busam 2005, Proposition III.7.10]) that

$$\begin{aligned} \mathcal{L}_{\mathbb{H}^2}(\gamma) &= \int_{-\pi}^{\pi} \frac{1}{\sqrt{2} + \sin(t)} dt = \int_{-\pi}^{\pi} \frac{1}{\sqrt{2} + 2 \cos(t/2) \sin(t/2)} dt \\ &= \int_{-\pi}^{\pi} \frac{1}{\sqrt{2} + 2 \frac{\tan(t/2)}{1 + \tan^2(t/2)}} dt = \int_{-\pi}^{\pi} \frac{1 + \tan^2(t/2)}{\sqrt{2}(1 + \tan^2(t/2)) + 2 \tan(t/2)} dt \\ &= 2 \int_{-\infty}^{\infty} \frac{1}{\sqrt{2}(1 + z^2) + 2z} dz = 2 \int_{-\infty}^{\infty} \frac{1}{\sqrt{2}(z - \frac{-1+i}{\sqrt{2}})(z - \frac{-1-i}{\sqrt{2}})} \\ &= 2(2\pi i) \sum_{a: \text{Im}(a) > 0} \text{Res} \left(\frac{-1}{\sqrt{2}(z - \frac{-1+i}{\sqrt{2}})(z - \frac{-1-i}{\sqrt{2}})}, a \right) = 4\pi i \frac{1}{\sqrt{2}(\frac{-1+i}{\sqrt{2}} - \frac{-1-i}{\sqrt{2}})} = 2\pi. \end{aligned}$$

An interesting problem is the minimization of the Willmore functional in each conformal class.

Definition 4.4 (Conformally constrained Willmore minimization). For ω as in Definition 4.1 we set

$$M_{3,1}(\omega) := \inf \left\{ \mathcal{W}(f) : f : \frac{\mathbb{C}}{\mathbb{Z} + \omega\mathbb{Z}} \rightarrow \mathbb{R}^3 \text{ conformal immersion} \right\}.$$

In [Ndiaye and Schätzle 2015, Proposition D.1] the authors show that there exists some $b_0 \geq 1$ such that $b \geq b_0$ implies $M_{3,1}(ib) < 8\pi$. Our first contribution in this context is the new insight that $b_0 = 1$. We prove the existence of tori of revolution with Willmore energy smaller than 8π in each conformal class $\omega = ib$, $b \geq 1$, via the Willmore flow studied in Theorem 1.3. Note that $\mathbb{C}/(\mathbb{Z} + ib\mathbb{Z})$ and $\mathbb{S}^1 \times \mathbb{S}^1$ are diffeomorphic with diffeomorphism $\phi : \mathbb{S}^1 \times \mathbb{S}^1 \rightarrow \mathbb{C}/(\mathbb{Z} + ib\mathbb{Z})$ being given by $\phi(u, v) = u + ibv$. Hence the results about the Willmore flow in Theorem 1.3 apply also for surfaces defined on $\mathbb{C}/(\mathbb{Z} + ib\mathbb{Z})$.

Theorem 4.5. For each $b \geq 1$ there exists a torus of revolution T_b such that $\omega(T_b) = ib$ and $\mathcal{W}(T_b) < 8\pi$.

Proof. From the construction in the proof of [Ndiaye and Schätzle 2015, Proposition D.1] follows that there exists $b_0 > 1$ such that for all $b \geq b_0$ there exists a torus T_b as in the statement. Note that actually the authors construct only a $C^{1,1}$ -torus of revolution T_b , but by mollification of the profile curve one can easily obtain a smooth torus of revolution that satisfies the same requirements and differs not too much in the conformal class as the hyperbolic length depends continuously on γ .

It remains to prove the claim for $b \in [1, b_0)$. For this choose $f_0 : \mathbb{C}/(\mathbb{Z} + ib_0\mathbb{Z}) \rightarrow \mathbb{R}^3$ to be a smooth conformal parametrization of T_{b_0} and let $(f(t))_{t \in (0, \infty)}$ be the evolution of f_0 by the Willmore flow, which is global and smoothly convergent to the Clifford torus (possibly rescaled and translated in the direction $(1, 0, 0)$) by Theorem 1.3. Moreover, $f(t)$ is a torus of revolution for all $t \geq 0$. Let $\gamma(t) = f(t)(\cdot, 0)$ be the profile curve of $f(t)$ for all $t \geq 0$, i.e., $f(t) = F_{\gamma(t)}$. By (3-5), $t \mapsto \gamma(t)$ is a smooth family of curves for $t \geq 0$ and in particular $\mathcal{L}_{\mathbb{H}^2}(\gamma(t))$ depends smoothly on t . By Proposition 4.2 one obtains that $t \mapsto (1/i)\omega(F_{\gamma(t)})$ is real-valued and depends continuously on t . We show next that along a subsequence $t \mapsto (1/i)\omega(F_{\gamma(t)})$ tends to 1 as $t \rightarrow \infty$. By Lemma 3.4 we obtain that there exists some $t_j \rightarrow \infty$ such that an appropriate reparametrization of $\gamma(t_j)$ converges in $C^2(\mathbb{S}^1, \mathbb{R}^2)$ to $\gamma_\infty \in C^\infty(\mathbb{S}^1, \mathbb{H}^2)$, a profile curve of the Clifford torus (possibly rescaled and translated in the direction $(1, 0, 0)$). Thus we have

$$2\pi = \mathcal{L}_{\mathbb{H}^2}(\gamma_\infty) = \lim_{j \rightarrow \infty} \mathcal{L}_{\mathbb{H}^2}(t_j), \tag{4-3}$$

i.e., $(1/i)\omega(F_{\gamma(t_j)}) \rightarrow 1$ as $j \rightarrow \infty$. Since $(1/i)\omega(F_{\gamma(0)}) = b_0$, each value between 1 and b_0 is attained by the intermediate value theorem. From this the existence of a torus of revolution T_b for each $b \in [1, b_0)$ follows. □

Remark 4.6. Theorem 4.5 can also be proven using the results in [Müller and Spener 2020] concerning the elastic flow in \mathbb{H}^2 (which also dissipates the Willmore energy).

In [Kuwert and Schätzle 2013] the authors prove that the infimum in a conformal class ω is attained once one can find a competitor with energy below 8π . For $\omega = ib$ our small energy tori serve as such competitors and show that the infimum is attained.

Corollary 4.7. For each $b \geq 1$ the infimum $M_{3,1}(ib)$ is attained and the map $b \rightarrow M_{3,1}(ib)$ is continuous on $[1, \infty)$.

Proof. Theorem 7.3 and Proposition 5.1 in [Kuwert and Schätzle 2013] show that each $b \geq 1$ where $M_{3,1}(ib) < 8\pi$ is a point of continuity of $b \mapsto M_{3,1}(ib)$ and a point where the infimum in the definition of $M_{3,1}$ is attained. The claim then follows directly from this result and Theorem 4.5. \square

The symmetries of the Willmore energy might suggest that the infimum of the Willmore energy in each class of rectangular tori (i.e., $\omega = ib$) is attained at a torus of revolution. This is in general still open. Far reaching results are obtained using a formulation of the Willmore energy in \mathbb{S}^3 by means of the *stereographic projection*. Since the stereographic projection is conformal it does also not change the conformal class. Looking at the Willmore energy in \mathbb{S}^3 one can find tori with a lot of symmetries: For $\alpha \in (0, 1)$ one can look at $\alpha\mathbb{S}^1 + \sqrt{1 - \alpha^2}\mathbb{S}^1$. The stereographic projections of all of these are tori of revolution. In particular, these are good candidates for minimizers in their conformal classes $\omega = i\sqrt{1 - \alpha^2}/\alpha$. For $\alpha = \frac{1}{\sqrt{2}}$ we obtain the Clifford torus which is the global minimizer and hence surely the minimizer in its conformal class. In [Ndiaye and Schätzle 2014; 2015] the authors show that for conformal classes close to the Clifford torus one still gets minimizers of the form $\alpha\mathbb{S}^1 \times \sqrt{1 - \alpha^2}\mathbb{S}^1$. More precisely, the result [Ndiaye and Schätzle 2015, Theorem 3.1] shows that there exists $b_1 > 1$ such that for all $b \leq b_1$ one has that $M_{3,1}(b)$ is attained by

$$\Sigma_b := P\left(\frac{1}{\sqrt{1+b^2}}\mathbb{S}^1 \times \frac{b}{\sqrt{1+b^2}}\mathbb{S}^1\right),$$

where $P : \mathbb{S}^3 \rightarrow \mathbb{R}^3$ denotes the stereographic projection. The authors also obtain that $b_1 < \infty$. The critical value b_1 can be understood as a point where a symmetry of the minimizers breaks down. They also note that this property has to break down for large conformal classes; see [Ndiaye and Schätzle 2015, p. 293–294]. In the following we will be able to find an explicit upper bound on the symmetry-breaking value b_1 . This result is now obtained by energy comparison. There are other (sharper) results using a stability discussion of Σ_b in \mathbb{S}^3 ; see [Kuwert and Lorenz 2013].

Corollary 4.8. *Let $b_1 \geq 1$ be such that for $b \leq b_1$ the minimizer for $M_{3,1}(b)$ is attained by*

$$\Sigma_b := P\left(\frac{1}{\sqrt{1+b^2}}\mathbb{S}^1 \times \frac{b}{\sqrt{1+b^2}}\mathbb{S}^1\right),$$

where $P : \mathbb{S}^3 \rightarrow \mathbb{R}^3$ denotes the stereographic projection. Then

$$b_1 < \frac{4}{\pi} + \sqrt{\frac{16}{\pi^2} - 1} \simeq 2.06136. \tag{4-4}$$

Proof. Let $b > 1$ be such that Σ_b is a minimizer and let T_b be the torus constructed in Theorem 4.5. Then, necessarily, $\mathcal{W}(\Sigma_b) \leq \mathcal{W}(T_b) < 8\pi$. This inequality implies the claim once we have shown that $\mathcal{W}(\Sigma_b) = \pi^2(b + 1/b)$.

For this according to [Topping 2000, equation (9)] for all $f : \Sigma \rightarrow \mathbb{R}^3$

$$\mathcal{W}(f) = \int_{\Sigma} \left(\frac{1}{4}|\tilde{H}|_{P^{-1}(f)}^2 + 1\right) d\mu_f,$$

where \tilde{H} denotes the mean curvature of $P^{-1}(f)$ in \mathbb{S}^3 and μ_f denotes the surface measure in \mathbb{S}^3 . By [Ndiaye and Schätzle 2014, equation (2.3)] we have $|\vec{H}_{\mathbb{R}^4}|^2 = |\tilde{H}|^2 + 4$ and hence we obtain

$$\mathcal{W}(f) = \frac{1}{4} \int_{\Sigma} |\vec{H}_{\mathbb{R}^4}(P^{-1}(f))|^2 d\mu_f.$$

Having now arrived in \mathbb{R}^4 and using that

$$P^{-1}(\Sigma_b) = \frac{1}{\sqrt{1+b^2}} \mathbb{S}^1 \times \frac{b}{\sqrt{1+b^2}} \mathbb{S}^1,$$

we can define $r := 1/\sqrt{1+b^2}$ and use the parametrization

$$F : \mathbb{S}^1 \times \mathbb{S}^1 \ni (\phi, \theta) \mapsto \begin{pmatrix} r \cos(2\pi\phi) \\ r \sin(2\pi\phi) \\ \sqrt{1-r^2} \cos(2\pi\theta) \\ \sqrt{1-r^2} \sin(2\pi\theta) \end{pmatrix} \in \mathbb{R}^4.$$

A computation reveals that

$$g = 4\pi^2 \begin{pmatrix} r^2 & 0 \\ 0 & 1-r^2 \end{pmatrix}.$$

We obtain that

$$\left\{ \frac{1}{2\pi r} \frac{\partial}{\partial \phi}, \frac{1}{2\pi \sqrt{1-r^2}} \frac{\partial}{\partial \theta} \right\}$$

is an orthonormal basis of $T_{(\phi,\theta)}(\mathbb{S}^1, \mathbb{S}^1)$ and hence

$$\vec{H}_{\mathbb{R}^4}(F) = \frac{1}{4\pi^2 r^2} \frac{\partial^2 F}{\partial \phi^2} + \frac{1}{4\pi^2 (1-r^2)} \frac{\partial^2 F}{\partial \theta^2},$$

which implies that

$$|\vec{H}_{\mathbb{R}^4}(F)|^2 = \frac{1}{r^2} + \frac{1}{1-r^2}.$$

Also note that $\sqrt{\det(g)} = 4\pi^2 r \sqrt{1-r^2}$. The Willmore energy then reads

$$\mathcal{W}(\Sigma_b) = \frac{1}{4} \left(\frac{1}{r^2} + \frac{1}{1-r^2} \right) 4\pi^2 r \sqrt{1-r^2} = \pi^2 \left(\frac{\sqrt{1-r^2}}{r} + \frac{r}{\sqrt{1-r^2}} \right)$$

and the claim follows using that by definition of r one has $r = 1/\sqrt{1+b^2}$. □

Appendix A: Consistency between extrinsic and intrinsic view

In literature there are multiple ways to define geometric quantities like curvature. This also leads to different notions of the Willmore energy and its gradient flow. Here we want to convince the reader that all those notions are consistent with the one we chose. For this we first have to do some computations in local coordinates. Let M be a smooth two-dimensional manifold, $f : M \rightarrow \mathbb{R}^3$ be an immersion

and $\psi : M \rightarrow \mathbb{R}^2$ be a chart for M with coordinates (u^1, u^2) . Given vector field $X = x^i(\partial/\partial u^i)$ and $Y = y^j(\partial/\partial u^j)$ then

$$A(X, Y) = x^i y^j \left(\frac{\partial^2 f}{\partial u^i \partial u^j} - \Gamma_{i,j}^k \frac{\partial f}{\partial u^k} \right), \tag{A-1}$$

where $\Gamma_{i,j}^k$ are the Christoffel symbols defined using the metric $g_{ij} = \langle (\partial/\partial u^i) f, (\partial/\partial u^j) f \rangle$. In particular, we see that the second fundamental form is symmetric.

If $f : M \rightarrow \mathbb{R}^3$ is an isometric immersion then for each local chart (u^1, u^2) of M one can define a unit normal field

$$\vec{N} = \frac{\partial_{u^1} f \times \partial_{u^2} f}{|\partial_{u^1} f \times \partial_{u^2} f|}$$

for (u^1, u^2) and rewrite

$$A(X, Y) = x^i y^j \left(\frac{\partial^2 f}{\partial u^i \partial u^j} - g^{kl} \left\langle \frac{\partial^2 f}{\partial u^i \partial u^j}, \frac{\partial f}{\partial u^l} \right\rangle_{\mathbb{R}^3} \frac{\partial f}{\partial u^k} \right) = x^i y^j \left\langle \frac{\partial^2 f}{\partial u^i \partial u^j}, \vec{N} \right\rangle_{\mathbb{R}^3} \vec{N}. \tag{A-2}$$

If $f : M \rightarrow f(M) \subset \mathbb{R}^3$ is now an isometric embedding and $f(M)$ is orientable, \vec{N} is independent of the chosen chart and (A-2) coincides with the usual definition of the second fundamental form.

Let us now choose normal coordinates (u^1, u^2) and fix $e_1 = \partial f/\partial u^1$ and $e_2 = \partial f/\partial u^2$. Then by (A-2) we find

$$A(e_i, e_j) = h_i^j \vec{N},$$

where h_i^j denote the usual coefficients of the Weingarten map. Then, the mean curvature (vector) and Gauss curvature are given by

$$\vec{H} = A(e_1, e_1) + A(e_2, e_2) = (h_1^1 + h_2^2) \vec{N} = H \vec{N}, \tag{A-3}$$

$$K := \langle A(e_1, e_1), A(e_2, e_2) \rangle_{\mathbb{R}^3} - \langle A(e_1, e_2), A(e_2, e_1) \rangle_{\mathbb{R}^3} = h_1^1 h_2^2 - (h_2^1)^2,$$

where H denotes the *scalar* mean curvature. For $Q(\mathring{A})H$, the ‘‘cubic’’-term in the Willmore equation, one easily derives

$$Q(\mathring{A})\vec{H} = \frac{1}{2} H(H^2 - 4K).$$

With similar computations,

$$|A|^2 = |H|^2 - 2K = \sum_{i,j=1}^2 \langle A(e_i, e_j), A(e_i, e_j) \rangle_{\mathbb{R}^3},$$

and hence for each toroidal immersion $f : \mathbb{S}^1 \times \mathbb{S}^1 \rightarrow \mathbb{R}^3$ one has by the Gauss–Bonnet theorem

$$\int_{\Sigma} |A|^2 d\mu_f = 4\mathcal{W}(f). \tag{A-4}$$

Similarly, again in the case of tori, an easy computation shows that $|\mathring{A}|^2 = \frac{1}{2} H^2 - 2K$ and

$$\int_{\Sigma} |\mathring{A}|^2 d\mu_f = 2\mathcal{W}(f).$$

Also, note that $|\mathring{A}|^2 \leq |A|^2$.

Appendix B: Tensor calculus

Throughout the article, we use a nonstandard notation for some differential geometric concepts involving connections, derivatives and tensors. We discuss here that our notation is consistent with that used in [Kuwert and Schätzle 2001; 2002; 2004], since many results cited there are used. Here we shall briefly introduce these concepts and clarify their meaning. Let M be a smooth two-dimensional manifold and $f \in C^\infty(M; \mathbb{R}^n)$ be an immersion. Moreover, let ∇ be the Levi-Civita connection on M . For a vector field $X \in \mathcal{V}(M)$ we define the *full derivative* $D_X : C^\infty(M; \mathbb{R}^n) \rightarrow C^\infty(M; \mathbb{R}^n)$ via

$$D_X G := \sum_{i=1}^n X(G_i)\vec{e}_i, \quad \text{whenever } G = \sum_{i=1}^n G_i\vec{e}_i \in C^\infty(M; \mathbb{R}^n), \tag{B-1}$$

and $\{\vec{e}_1, \vec{e}_2, \dots, \vec{e}_n\}$ is the canonical basis of \mathbb{R}^n . We say that $G \in C^\infty(M; \mathbb{R}^n)$ is a *normal vector field* if $G(p) \perp df_p(T_p M)$ for all $p \in M$. We define for short $N_p M := df_p(T_p M)^\perp$ and $NM := \bigsqcup_{p \in M} N_p M$ the *normal bundle*. For such a normal vector field $G \in C^\infty(M, NM)$ we define the *normal connection* of G to be

$$\nabla_X^\perp G|_p := \pi_{N_p M}(D_X G|_p) = D_X G^\perp, \tag{B-2}$$

where π_U denotes the orthogonal projection on U . A normal vector field that will be used very frequently is $Y = A(Z, W)$ for some $Z, W \in \mathcal{V}(M)$. This is however not just a normal vector field but each of its components is also a $(2, 0)$ -tensor — we may think of $p \rightarrow A_p(Z, W)$ as a $(2, 0)$ -tensor on M with values in the normal bundle NM , i.e., a for each $p \in M$ it is a multilinear map from $T_p M^2$ to $N_p M$. If we do so, the standard concept of *tensorial connections* (see [Lee 2018, Lemma 4.6]) is not applicable, since it is needed that the tensor takes values in \mathbb{R} . One can however overcome this by using two *different connections*, namely ∇ and ∇^\perp . More precisely, for a $(k, 0)$ -tensor $F : p \mapsto (F_p : T_p M^k \rightarrow N_p M)$ on M with values in the normal bundle NM we can define a $(k + 1, 0)$ -tensor $\nabla^\perp F$ via

$$\nabla^\perp F(X_1, \dots, X_{k+1}) := \nabla_{X_1}^\perp F(X_2, \dots, X_{k+1}) - \sum_{j=2}^{k+1} F(X_2, \dots, \nabla_{X_1} X_j, \dots, X_{k+1}) \tag{B-3}$$

for $X_1, \dots, X_{k+1} \in \mathcal{V}(M)$. It can easily be checked that $\nabla^\perp F$ is indeed a $(k+1)$ -tensor, i.e., $\nabla^\perp F_p$ depends only on $X_1(p), \dots, X_{k+1}(p)$. Moreover, if F is a $(0, 0)$ -tensor on M with values in NM , i.e., $F \in C^\infty(M; NM)$, then the notation of $\nabla^\perp F$ coincides with the previous definition in (B-2). We remark that in [Kuwert and Schätzle 2001; 2002; 2004], ∇^\perp and ∇ are both denoted by ∇ . The $L^\infty(M)$ -norm of a $(k, 0)$ -tensor F on M with values in NM is defined to be

$$\|F\|_{L^\infty(M)} := \sup_{p \in M} \sup_{\{E_1, E_2\} \text{ orthonormal basis of } T_p M} \sum_{i_1, \dots, i_k=1}^2 |F(E_{i_1}, \dots, E_{i_k})|,$$

where $|\cdot|$ denotes the norm in \mathbb{R}^n . We will also use very frequently [Kuwert and Schätzle 2002, equation (2.7)], which we state here for the reader's convenience. Let $f \in C^\infty(M; \mathbb{R}^n)$ be an immersion with second fundamental form A and normal bundle NM . Then for each $G \in C^\infty(M; NM)$ and $X \in \mathcal{V}(M)$

one has

$$D_X G = \nabla_X^\perp G - \sum_{i=1}^2 \langle G, A(X, E_i) \rangle_{\mathbb{R}^n} D_{E_i} f, \tag{B-4}$$

where $\{E_1, E_2\}$ is an arbitrary orthonormal basis of $T_p M$ with respect to $g_f := f^* g_{\mathbb{R}^n}$. We also remark that we can define a tensorial version of D , treated as a tensor on M with values in \mathbb{R}^n . The transformation law we prescribe here is analogous to (B-3), namely if F is a $(k, 0)$ -tensor on M with values in \mathbb{R}^n , we define for $X_1, \dots, X_{k+1} \in \mathcal{V}(M)$

$$DF(X_1, \dots, X_{k+1}) := D_{X_1} F(X_2, \dots, X_{k+1}) - \sum_{j=2}^{k+1} F(X_2, \dots, \nabla_{X_1} X_j, \dots, X_{k+1}).$$

As an important special case we obtain for $f \in C^\infty(M, \mathbb{R}^n)$

$$D^2 f(X, Y) = D_X D_Y f - D_{\nabla_X Y} f.$$

If f is additionally an immersion, this formula yields exactly the second fundamental form (see (2-1)). Hence one could also write $A[f] = D^2 f$.

Appendix C: On the smooth convergence of surfaces

Here we present some useful results concerning smooth convergence on compact subsets of \mathbb{R}^n , which we will simply call smooth convergence.

We remark that smooth convergence, see Definition 2.1, actually takes place in the equivalence class of surfaces that coincide up to reparametrization, more precisely

Remark C.1. Consider a sequence of immersions $(f_j)_{j \in \mathbb{N}}, f_j : \Sigma \rightarrow \mathbb{R}^n$, that converges to \hat{f} smoothly on compact subsets of \mathbb{R}^n and a sequence of diffeomorphisms $(\Psi_j)_{j \in \mathbb{N}}, \Psi_j : \Sigma_j \rightarrow \Sigma$, with Σ_j a smooth manifold without boundary. Then it follows from the definition of smooth convergence that $f_j \circ \Psi_j$ converges to \hat{f} smoothly on compact subsets of \mathbb{R}^n . Moreover if $\Psi : \tilde{\Sigma} \rightarrow \hat{\Sigma}$ is yet another diffeomorphism then f_j also converges to $f \circ \Psi$ smoothly on compact subsets of \mathbb{R}^n .

Remark C.2. In general, smooth convergence is not *topology-preserving*, i.e., the topologies of $\hat{\Sigma}$ and Σ need not coincide; see [Breuning 2015, Figure 6]. The situation is better if Σ is connected and $\hat{\Sigma}$ has a compact component C . Lemma 4.3 in [Kuwert and Schätzle 2001] gives that $\Sigma, \hat{\Sigma}$ are diffeomorphic. By the previous remark they can then also chosen to be equal.

Next we examine how relevant geometric quantities behave with respect to smooth convergence, for instance the diameter.

Lemma C.3. Suppose that $(f_j)_{j=1}^\infty : \Sigma \rightarrow \mathbb{R}^n$ is a sequence that converges smoothly on compact subsets of \mathbb{R}^n to $\hat{f} : \hat{\Sigma} \rightarrow \mathbb{R}^n$. Then

$$\text{diam } \hat{f}(\hat{\Sigma}) \leq \liminf_{j \rightarrow \infty} \text{diam } f_j(\Sigma).$$

Proof. Suppose that $(\hat{f}(p_k))_{k=1}^\infty, (\hat{f}(q_k))_{k=1}^\infty \subset \hat{f}(\widehat{\Sigma})$ are sequences such that

$$|\hat{f}(p_k) - \hat{f}(q_k)| \rightarrow \text{diam } \hat{f}(\widehat{\Sigma}).$$

Then, by [Definition 2.1](#) for each $k \in \mathbb{N}$ there exists $j(k) \in \mathbb{N}$ such that $p_k, q_k \in \widehat{\Sigma}(j)$ for all $j \geq j(k)$. Now [\(2-5\)](#) implies that for all $j \geq j(k)$

$$\begin{aligned} |\hat{f}(p_k) - \hat{f}(q_k)| &\leq |f_j \circ \phi_j(p_k) - f_j \circ \phi_j(q_k)| + |u_j(p_k) - u_j(q_k)| \\ &\leq \text{diam } f_j(\Sigma) + 2\|u_j\|_{L^\infty(\widehat{\Sigma}(j))}. \end{aligned}$$

Letting first $j \rightarrow \infty$ and then $k \rightarrow \infty$ we obtain the claim. □

Now we study the lower semicontinuity with respect to smooth convergence of the Willmore energy. As a first step we prove the following result.

Lemma C.4. *Let $(f_j)_{j \in \mathbb{N}}, f_j : \Sigma \rightarrow \mathbb{R}^n$ be a sequence of immersions that converges smoothly on compact subsets of \mathbb{R}^n to an immersion $\hat{f} : \widehat{\Sigma} \rightarrow \mathbb{R}^n$. Let (U, ψ) be a chart for $\widehat{\Sigma}$ such that $U \subset \widehat{\Sigma}(J)$ for some $J \in \mathbb{N}$ and $\hat{g}_{i\tau} \circ \psi^{-1} \in C^1(\overline{\psi(U)})$, $\widehat{\Gamma}_{i\tau}^\alpha \circ \psi^{-1} \in C^0(\overline{\psi(U)})$ for all i, τ, α , and $\det(\hat{g}), \hat{g}_{11}$ are bounded from below by some positive $\delta > 0$, where $\hat{g}_{i\tau}$ and $\widehat{\Gamma}_{i\tau}^\alpha$ denote the metric and Christoffel's symbols induced by \hat{f} on $\widehat{\Sigma}$. Moreover we require that $\|D^2 \hat{f}\|_{L^\infty(U, g_{\hat{f}})}, \|A[\hat{f}]\|_{L^\infty(U, g_{\hat{f}})}, \|DA[\hat{f}]\|_{L^\infty(U, g_{\hat{f}})} < \infty$. Let $(\phi_j)_{j=1}^\infty, \phi_j : \widehat{\Sigma} \rightarrow \Sigma$, be a sequence of diffeomorphisms as in [Definition 2.1](#). Let $\hat{g}(m)$ be the first fundamental form induced by $f_m \circ \phi_m$ on U with respect to the chart (U, ψ) and $H(m) := H_{f_m \circ \phi_m}$ be the mean curvature of $f_m \circ \phi_m$.*

Then, $\hat{g}(m) \circ \psi^{-1}$ converges to $\hat{g} \circ \psi^{-1}$ uniformly in $\psi(U)$ and $H(m) \circ \psi^{-1}$ converges to $H_{\hat{f}} \circ \psi^{-1}$ uniformly in $\psi(U)$.

Proof. For $m > J$ let u_m be as in [Definition 2.1](#) such that on $\widehat{\Sigma}(m)$ one has

$$f_m \circ \phi_m + u_m = \hat{f} \quad \text{and} \quad \|(\widehat{\nabla}^\perp)^k u_m\|_{L^\infty(\widehat{\Sigma}(m))} \rightarrow 0, \quad m \rightarrow \infty. \tag{C-1}$$

Let (y^1, y^2) be the local coordinates induced by (U, ψ) ; in particular for all $h \in C^\infty(\Sigma; \mathbb{R}^d)$, $d \in \mathbb{N}$, in particular observe that $\partial h / \partial y^i = (\partial(h \circ \psi^{-1}) / \partial e_i) \circ \psi$ for all $h \in C^\infty(\Sigma; \mathbb{R}^d)$, $d \in \mathbb{N}$. Our first intermediate claim is that $\partial u_m / \partial y^i$ and $\partial^2 u_m / (\partial y^i \partial y^\tau)$ converge to zero uniformly in U for all i, τ .

In the following we let $E_1, E_2 \in \mathcal{V}(U)$ be the smooth orthonormal frame on $(U, g_{\hat{f}})$ which we obtain by applying the Gram-Schmidt procedure on $\{\partial / \partial y^1, \partial / \partial y^2\}$, i.e.,

$$E_1 = \frac{1}{\sqrt{\hat{g}_{1,1}}} \frac{\partial}{\partial y^1} \quad \text{and} \quad E_2 = \frac{1}{\sqrt{\hat{g}_{1,1} \det(\hat{g})}} \left(\hat{g}_{11} \frac{\partial}{\partial y^2} - \hat{g}_{12} \frac{\partial}{\partial y^1} \right).$$

Note that by [\(B-4\)](#)

$$\frac{\partial u_m}{\partial y^i} = D_{(\partial / \partial y^i)} u_m = \widehat{\nabla}_{(\partial / \partial y^i)}^\perp u_m - \sum_{j=1}^2 \left\langle u_m, A[\hat{f}] \left(\frac{\partial}{\partial y^i}, E_j \right) \right\rangle_{\mathbb{R}^3} D_{E_j} \hat{f}$$

and hence on U we have

$$\left| \frac{\partial u_m}{\partial y^i} \right| \leq \|\widehat{\nabla}^\perp u_m\|_{L^\infty(U)} + 2\|A[\hat{f}]\|_{L^\infty(U)} |\hat{g}_{i,i}|^{1/2} \|u_m\|_{L^\infty(U)}. \tag{C-2}$$

Estimating

$\|\hat{\nabla}^\perp u_m\|_{L^\infty(U)} \leq \|\hat{\nabla}^\perp u_m\|_{L^\infty(\widehat{\Sigma}(m))} \rightarrow 0, \quad \|u_m\|_{L^\infty(U)} \leq \|u_m\|_{L^\infty(\widehat{\Sigma}(m))} \rightarrow 0, \quad |\hat{g}_{i,i}| \leq \|\hat{g}_{i,i} \circ \psi^{-1}\|_{L^\infty(\psi(U))},$

we infer that $\partial u_m / \partial y^i$ converges to zero uniformly on U . Next we compute for all i, τ , writing for short $A = A[\hat{f}]$,

$$\begin{aligned} \frac{\partial^2 u_m}{\partial y^\tau \partial y^i} &= D_{\partial/\partial y^\tau} D_{\partial/\partial y^i} u_m = D_{\partial/\partial y^\tau} \left(\hat{\nabla}_{\partial/\partial y^i}^\perp u_m - \sum_{j=1}^2 \left\langle u_m, A\left(\frac{\partial}{\partial y^i}, E_j\right) \right\rangle_{\mathbb{R}^n} D_{E_j} f \right) \\ &= D_{\partial/\partial y^\tau} \hat{\nabla}_{\partial/\partial y^i}^\perp u_m - \sum_{j=1}^2 D_{\partial/\partial y^\tau} \left[\left\langle u_m, A\left(\frac{\partial}{\partial y^i}, E_j\right) \right\rangle_{\mathbb{R}^n} D_{E_j} f \right] \\ &= \hat{\nabla}_{\partial/\partial y^\tau}^\perp \hat{\nabla}_{\partial/\partial y^i}^\perp u_m - \sum_{l=1}^2 \left\langle \hat{\nabla}_{\partial/\partial y^i}^\perp u_m, A\left(\frac{\partial}{\partial y^\tau}, E_l\right) \right\rangle_{\mathbb{R}^n} D_{E_l} f \\ &\quad - \sum_{j=1}^2 \left\langle \frac{\partial u_m}{\partial y^\tau}, A\left(\frac{\partial}{\partial y^i}, E_j\right) \right\rangle_{\mathbb{R}^n} D_{E_j} \hat{f} - \sum_{j=1}^2 \left\langle u_m, D_{\partial/\partial y^\tau} A\left(\frac{\partial}{\partial y^i}, E_j\right) \right\rangle_{\mathbb{R}^n} D_{E_j} f \\ &\quad - \sum_{j=1}^2 \left\langle u_m, A\left(\frac{\partial}{\partial y^i}, E_j\right) \right\rangle_{\mathbb{R}^n} D_{\partial/\partial y^\tau} D_{E_j} \hat{f} \\ &= (\hat{\nabla}^\perp)^2 u_m \left(\frac{\partial}{\partial y^\tau}, \frac{\partial}{\partial y^i} \right) + \hat{\nabla}^\perp u_m \left(\hat{\nabla}_{\partial/\partial y^\tau} \frac{\partial}{\partial y^i} \right) \\ &\quad - \sum_{l=1}^2 \left\langle \hat{\nabla}^\perp u_m \left(\frac{\partial}{\partial y^i} \right), A\left(\frac{\partial}{\partial y^\tau}, E_l\right) \right\rangle_{\mathbb{R}^n} D_{E_l} f - \sum_{j=1}^2 \left\langle \frac{\partial u_m}{\partial y^\tau}, A\left(\frac{\partial}{\partial y^i}, E_j\right) \right\rangle_{\mathbb{R}^n} D_{E_j} \hat{f} \\ &\quad - \sum_{j=1}^2 \left\langle u_m, DA\left(\frac{\partial}{\partial y^\tau}, \frac{\partial}{\partial y^i}, E_j\right) + A\left(\hat{\nabla}_{\partial/\partial y^\tau} \frac{\partial}{\partial y^i}, E_j\right) + A\left(\frac{\partial}{\partial y^i}, \hat{\nabla}_{\partial/\partial y^\tau} E_j\right) \right\rangle_{\mathbb{R}^n} D_{E_j} \hat{f} \\ &\quad - \sum_{j=1}^2 \left\langle u_m, A\left(\frac{\partial}{\partial y^i}, E_j\right) \right\rangle_{\mathbb{R}^n} \left[D^2 \hat{f} \left(\frac{\partial}{\partial y^\tau}, E_j \right) + D \hat{f} \left(\hat{\nabla}_{\partial/\partial y^\tau} E_j \right) \right]. \end{aligned}$$

All terms that appear here as arguments of tensors can be bounded in L^∞ -norm with quantities that we assumed to be bounded. Notice that a bound on $\hat{\nabla}_{\partial/\partial y^\tau} \partial/\partial y^i$ needs the fact that the Christoffel symbols lie in $C^0(\overline{\psi(U)})$. Bounding $\hat{\nabla}_{\partial/\partial y^\tau} E_j$ in terms of the given quantities needs the explicit representation of E_j that we discussed above. Here we also need that $\det(\hat{g}), \hat{g}_{11}$ are bounded from below uniformly in U . We obtain with a straightforward computation that $\partial^2 u_m / (\partial y^\tau \partial y^i)$ converges to zero uniformly in U .

We now show that $\hat{g}(m)$ converges to \hat{g} uniformly on U , which implies the convergence claimed in the statement. First note that by (C-1) and (C-2)

$$\frac{\partial(f_m \circ \phi_m)}{\partial y^\tau} = \frac{\partial \hat{f}}{\partial y^\tau} + o(1),$$

where $\partial \hat{f} / \partial y^\tau$ are bounded by assumption. Hence, $\partial(f_m \circ \phi_m) / \partial y^\tau$ and $\hat{g}(m)$ are uniformly bounded. Now we can compute using (C-1)

$$\hat{g}_{i\tau} = \left\langle \frac{\partial \hat{f}}{\partial y^i}, \frac{\partial \hat{f}}{\partial y^\tau} \right\rangle_{\mathbb{R}^n} = \hat{g}_{i\tau}(m) + \left\langle \frac{\partial(f_m \circ \phi_m)}{\partial y^i}, \frac{\partial u_m}{\partial y^\tau} \right\rangle_{\mathbb{R}^n} + \left\langle \frac{\partial(f_m \circ \phi_m)}{\partial y^\tau}, \frac{\partial u_m}{\partial y^i} \right\rangle_{\mathbb{R}^n} + \left\langle \frac{\partial u_m}{\partial y^\tau}, \frac{\partial u_m}{\partial y^i} \right\rangle_{\mathbb{R}^n}.$$

By the arguments above, the last three terms are uniformly convergent to zero and so convergence of the first fundamental form is shown. Note in particular that also $\hat{g}^{-1}(m)$ converges to \hat{g}^{-1} since we assumed that $\det(\hat{g})$ is strictly bounded from below.

Observe now that by (A-1) and (A-3)

$$\begin{aligned} \vec{H}_{\hat{f}} &= \hat{g}^{i\tau} \left(\frac{\partial^2 \hat{f}}{\partial y^i \partial y^\tau} - \hat{\Gamma}_{i\tau}^\alpha \frac{\partial \hat{f}}{\partial y^\alpha} \right), \\ \vec{H}(m) &= \hat{g}^{i\tau}(m) \left(\frac{\partial^2 (f_m \circ \phi_m)}{\partial y^i \partial y^\tau} - \hat{\Gamma}_{i\tau}^\alpha(m) \frac{\partial (f_m \circ \phi_m)}{\partial y^\alpha} \right), \end{aligned}$$

where $\hat{\Gamma}_{i\tau}^\alpha(m)$ denotes the Christoffel symbols of the immersion $f_m \circ \phi_m$ with respect to the chart (U, ψ) . We have already discussed the uniform convergence of all terms that $H(m)$ consists of except for the Christoffel symbols. The convergence of those however follows analogously to the convergence of $\hat{g}(m)$ from the classical formula

$$\hat{\Gamma}_{i\tau}^\alpha(m) = g^{\alpha\beta}(m) \left\langle \frac{\partial^2 (f_m \circ \phi_m)}{\partial y^i \partial y^\tau}, \frac{\partial (f_m \circ \phi_m)}{\partial y^\beta} \right\rangle_{\mathbb{R}^n}. \quad \square$$

Lemma C.5. *Suppose that $(f_j)_{j=1}^\infty : \Sigma \rightarrow \mathbb{R}^n$ is a sequence of immersions that converges smoothly on compact subsets of \mathbb{R}^n to an immersion $\hat{f} : \hat{\Sigma} \rightarrow \mathbb{R}^n$. Then*

$$\mathcal{W}(\hat{f}) \leq \liminf_{j \rightarrow \infty} \mathcal{W}(f_j).$$

Additionally, if $\hat{\Sigma}$ is compact then $\mathcal{W}(\hat{f}) = \lim_{j \rightarrow \infty} \mathcal{W}(f_j)$.

Proof. We start choosing a cover $\{(U_p, \psi_p)\}_{p \in \hat{\Sigma}}$ of $\hat{\Sigma}$ such that U_p is an open neighborhood of p . Since each p is contained in some $\Sigma(m_p)$ for some $m_p \in \mathbb{N}$ and $\Sigma(m_p)$ is open, we may assume that $U_p \subset \Sigma(m_p)$ by possibly shrinking U_p . Let V_p be a neighborhood of p compactly contained in U_p . Then in each chart (V_p, ψ_p) , \hat{g}_{it} and Γ_{it}^α are bounded and $\det(\hat{g})$ is uniformly bounded from below by some $\delta = \delta(p) > 0$. By second countability there exist countably many points $\{p_\nu\}_{\nu=1}^\infty$ such that $\{(V_{p_\nu}, \psi_{p_\nu})\}_{\nu=1}^\infty$ is a cover of $\hat{\Sigma}$ and there exists a locally finite partition of unity $(\eta_\nu)_{\nu=1}^\infty$ of smooth and compactly supported functions that satisfy $\text{supp}(\eta_\nu) \subset V_{p_\nu}$. Now we infer by Lemma C.4 (taking diffeomorphisms ϕ_m as in (C-1)) and Fatou's lemma

$$\begin{aligned} \int_{\hat{\Sigma}} H_{\hat{f}}^2 d\mu_{\hat{f}} &= \sum_{\nu=1}^\infty \int_{V_{p_\nu}} \eta_\nu H_{\hat{f}}^2 d\mu_{\hat{f}} = \sum_{\nu=1}^\infty \int_{\psi_{p_\nu}(V_{p_\nu})} (\eta_\nu \circ \psi_{p_\nu}^{-1})(H_{\hat{f}} \circ \psi_{p_\nu}^{-1})^2 \sqrt{\det \hat{g}} \circ \psi_{p_\nu}^{-1} dx \\ &= \sum_{\nu=1}^\infty \lim_{m \rightarrow \infty} \int_{\psi_{p_\nu}(V_{p_\nu})} (\eta_\nu \circ \psi_{p_\nu}^{-1})(H_{f_m \circ \phi_m} \circ \psi_{p_\nu}^{-1})^2 \sqrt{\det \hat{g}(m)} \circ \psi_{p_\nu}^{-1} dx \\ &\leq \liminf_{m \rightarrow \infty} \sum_{\nu=1}^\infty \int_{\psi_{p_\nu}(V_{p_\nu})} (\eta_\nu \circ \psi_{p_\nu}^{-1})(H_{f_m \circ \phi_m} \circ \psi_{p_\nu}^{-1})^2 \sqrt{\det \hat{g}(m)} \circ \psi_{p_\nu}^{-1} dx \\ &= \liminf_{m \rightarrow \infty} \sum_{\nu=1}^\infty \int_{V_{p_\nu}} \eta_\nu H_{f_m \circ \phi_m}^2 d\mu_{f_m \circ \phi_m} = \liminf_{m \rightarrow \infty} \int_{\hat{\Sigma}} H_{f_m \circ \phi_m}^2 d\mu_{f_m \circ \phi_m}. \end{aligned}$$

All in all we obtain $\mathcal{W}(\hat{f}) \leq \liminf_{m \rightarrow \infty} \mathcal{W}(f_m \circ \phi_m) = \liminf_{m \rightarrow \infty} \mathcal{W}(f_m)$ as the Willmore energy does not depend on the reparametrization. If $\hat{\Sigma}$ is compact then the partition of unity can be chosen to be finite and the last claim follows then with the same techniques. \square

Lemma C.6 [Breuning 2015, Corollary 1.4]. *Suppose that $f_j : \Sigma \rightarrow \mathbb{R}^n$ and $\hat{f} : \hat{\Sigma} \rightarrow \mathbb{R}^n$ are such that f_j converges to \hat{f} smoothly on compact subsets of \mathbb{R}^n . Then the surface measures $f_j^* \mu_{g_j}$ converge in $C_0(\mathbb{R}^n)'$ to $\hat{f}^* \mu_{\hat{f}}$.*

A second concept of convergence that is related to smooth convergence is the C^l -convergence which we also use throughout the article.

Definition C.7. We say that a sequence of immersions $(f_j)_{j \in \mathbb{N}}$, $f_j : \Sigma \rightarrow \mathbb{R}^n$, defined on a two-dimensional manifold Σ without boundary converges to $\hat{f} : \Sigma \rightarrow \mathbb{R}^n$ in $C^l(\Sigma)$, $l \in \mathbb{N}$, if there exist diffeomorphisms $\phi_j : \Sigma \rightarrow \Sigma$ for all $j \in \mathbb{N}$ and $u_j : \Sigma \rightarrow N\Sigma$ such that $f_j \circ \phi_j + u_j = \hat{f}$ on Σ and $\|(\hat{\nabla}^\perp)^k u_j\|_{L^\infty(\Sigma)} \rightarrow 0$ as $j \rightarrow \infty$ for all $k \in \{0, \dots, l\}$.

Remark C.8. The two concepts of convergence we discussed are obviously related. Indeed, if $f_j : \tilde{\Sigma} \rightarrow \mathbb{R}^n$ is a sequence that converges smoothly on compact subsets to some $\hat{f} : \tilde{\Sigma} \rightarrow \mathbb{R}^n$ and $\tilde{\Sigma}$ is compact, then f_j converges to \hat{f} in C^l for all $l \in \mathbb{N}$. We further say that a family $(f(t))_{t \in [0, \infty)}$ converges to \hat{f} in C^l for all l if for each sequence $t_j \rightarrow \infty$ one has $f(t_j) \rightarrow \hat{f}$ as $j \rightarrow \infty$.

We will now present an alternative characterization of C^l convergence in which we do not need to require that u_j are orthogonal. However we have to pay a price — in this case one needs to control the full derivative. Even though we expect this result to be true even in higher codimension, we formulate it only in the case of $n = 3$ for the sake of simplicity. This will be sufficient for our purposes.

Proposition C.9. *Let Σ be a compact orientable two-dimensional manifold without boundary and $f_j : \Sigma \rightarrow \mathbb{R}^3$ be a sequence of immersions and $k \geq 2$. Then f_j converges to a limit immersion $\hat{f} : \Sigma \rightarrow \mathbb{R}^3$ in C^k if and only if there exist $w_j \in C^k(\Sigma, \mathbb{R}^3)$ and C^k -smooth diffeomorphisms $\psi_j : \Sigma \rightarrow \Sigma$ such that for j large enough*

$$f_j \circ \psi_j = \hat{f} + w_j \quad \text{on } \Sigma$$

and for all $k \in \mathbb{N}$ one has $\|D^k w_j\|_{L^\infty(\Sigma, g_{f_j})} \rightarrow 0$ as $j \rightarrow \infty$.

Proof. First assume that $f_j : \Sigma \rightarrow \mathbb{R}^3$ converges to $\hat{f} : \Sigma \rightarrow \mathbb{R}^3$ in $C^k(\Sigma)$. Then, for j large enough one can find $u_j \in C^k(\Sigma, N\Sigma)$ and C^k -diffeomorphisms $\phi_j : \Sigma \rightarrow \Sigma$ such that

$$f_j \circ \phi_j = \hat{f} + u_j \quad \text{on } \Sigma$$

and for all $k \in \mathbb{N}$ one has $\|(\hat{\nabla}^\perp)^k u_j\|_{L^\infty} \rightarrow 0$ as $j \rightarrow \infty$. Now we choose $\psi_j := \phi_j$ and $w_j := u_j$. It only remains to show that $\|D^k w_j\|_{L^\infty} \rightarrow 0$ as $k \rightarrow \infty$. For $k = 1$ we observe that for each $X \in \mathcal{V}(M)$ one has by (B-4)

$$D_X w_j = \hat{\nabla}_X^\perp w_j - \sum_{i=1}^2 \langle w_j, A[\hat{f}](X, E_i) \rangle_{\mathbb{R}^3} D_{E_i} f = \hat{\nabla}_X^\perp u_j - \sum_{i=1}^2 \langle u_j, A[\hat{f}](X, E_i) \rangle_{\mathbb{R}^3} D_{E_i} f. \quad (\text{C-3})$$

We obtain that

$$\|Dw_j\|_{L^\infty} \leq \|\hat{\nabla}^\perp u_j\|_{L^\infty} + C\|u_j\|_{L^\infty}\|A[\hat{f}]\|_{L^\infty}\|D\hat{f}\|_{L^\infty}.$$

Since Σ is compact, $\|A[\hat{f}]\|_{L^\infty}$ and $\|D\hat{f}\|_{L^\infty}$ are finite and thus $\|Dw_j\|_{L^\infty} \rightarrow 0$ as $j \rightarrow \infty$. The estimates for $k \geq 2$ follow easily by using iterated versions of (C-3).

For the converse, suppose we have diffeomorphisms ψ_j and w_j as in the statement. We denote by $C^k(\Sigma; \mathbb{R})$ the set of all C^k -smooth real-valued maps from Σ of \mathbb{R} equipped with the norm $\|f\|_{C^k(\Sigma; \mathbb{R})} := \sum_{l=1}^k \|\hat{\nabla}^l f\|_{L^\infty}$, where $\hat{\nabla}$ here denotes the tensorial connection with respect to the Levi-Civita connection on $(\Sigma, g_{\hat{f}})$; see [Lee 2018, Lemma 4.6]. We also endow $C^k(\Sigma; \mathbb{R}^3)$ with the norm $\|f\|_{C^k(\Sigma; \mathbb{R}^3)} = \sum_{l=1}^k \|D^l f\|_{L^\infty}$. Moreover we define $\text{Diffeo}^k(\Sigma, \Sigma)$ to be the set of all C^k smooth diffeomorphisms of Σ . Note that $\text{Diffeo}^k(\Sigma, \Sigma)$ is a smooth Banach manifold with the compact-open topology and for all $\phi \in \text{Diffeo}^k(\Sigma)$ the tangent space $T_\phi \text{Diffeo}^k(\Sigma, \Sigma)$ can be identified with $\mathcal{V}(\Sigma)$. This fact follows from [Wittmann 2019; Hirsch 1976, Chapter 2, Theorem 1.7]. Let now $N_{\hat{f}}$ be a smooth unit normal field along \hat{f} . (Here orientability of Σ is needed). We now define for all $k \in \mathbb{N}$ the map

$$F : \text{Diffeo}^k(\Sigma, \Sigma) \times C^k(\Sigma; \mathbb{R}) \rightarrow C^k(\Sigma; \mathbb{R}^3), \quad F(\eta, \beta) := (\hat{f} + \beta N_{\hat{f}}) \circ \eta. \tag{C-4}$$

It is easy to show that for all $X \in \mathcal{V}(\Sigma)$ and $\alpha \in C^k(\Sigma; \mathbb{R})$ one has $d_{(\text{id},0)} F(X, \alpha) = D_X \hat{f} + \alpha N_{\hat{f}}$. Having this formula, one checks that $d_{(\text{id},0)} F : T_{(\text{id},0)}(\text{Diffeo}^k(\Sigma, \Sigma) \times C^k(\Sigma; \mathbb{R})) \rightarrow T_{\hat{f}}(C^k(\Sigma; \mathbb{R}^3)) \simeq C^k(\Sigma; \mathbb{R}^3)$ is an isomorphism. As a consequence one can find a small neighborhood V of $(\text{id}, 0)$ such that $F|_V$ is a diffeomorphism. We conclude that for all $k \in \mathbb{N}$ there exists $\varepsilon > 0$ such that $\|g - \hat{f}\|_{C^k(\Sigma; \mathbb{R}^3)} < \varepsilon$ implies that there exists $\eta \in \text{Diffeo}^k$ and $\beta \in C^k$ such that $g = (\hat{f} + \beta N_{\hat{f}}) \circ \eta$. Next we look at $g = \hat{f} + w_j$. For j large enough one has that there exists $\eta_j \in \text{Diffeo}^k$ and $\beta_j \in C^k$ such that

$$\hat{f} + w_j = (\hat{f} + \beta_j N_{\hat{f}}) \circ \eta_j$$

and thus we infer that

$$f_j \circ \phi_j = (\hat{f} + \beta_j N_{\hat{f}}) \circ \eta_j.$$

We compose with η_j^{-1} to obtain

$$f_j \circ \phi_j \circ \eta_j^{-1} = \hat{f} + \beta_j N_{\hat{f}}.$$

Defining $\psi_j := \phi_j \circ \eta_j^{-1}$ and $u_j := \beta_j N_{\hat{f}}$ we obtain that $f_j \circ \psi_j = \hat{f} + u_j$ and $u_j \in C^k(\Sigma, N\Sigma)$. It remains to show that $\|(\hat{\nabla}^\perp)^l u_j\| \rightarrow 0$ for all $l = 1, \dots, k$. To do so we compute for any $X \in \mathcal{V}(\Sigma)$

$$\hat{\nabla}_X^\perp u_j = \hat{\nabla}_X^\perp(\beta_j N_{\hat{f}}) = X(\beta_j)N_{\hat{f}} + \beta_j \hat{\nabla}_X^\perp N_{\hat{f}}.$$

Note that $X(\beta_j) = \hat{\nabla}_X \beta_j$ and thus

$$\|\hat{\nabla}^\perp u_j\|_{L^\infty} \leq \|\beta_j\|_{C^1(\Sigma, \mathbb{R})}(1 + \|\hat{\nabla}^\perp N_{\hat{f}}\|_{L^\infty}).$$

Observe that $\|\hat{\nabla}^\perp N_{\hat{f}}\|_{L^\infty}$ is finite by the compactness of Σ . Similarly one can show that

$$\|(\hat{\nabla}^\perp)^j u_j\|_{L^\infty} \leq C(k, \Sigma, \hat{f})\|\beta_j\|_{C^j(\Sigma, \mathbb{R})} \quad \text{for all } j = 1, \dots, k. \tag{C-5}$$

Note that $\hat{f} + w_j \rightarrow \hat{f}$ in $C^k(\Sigma; \mathbb{R}^3)$ and the fact that F , defined in (C-4), is a local diffeomorphism implies that (η_j, β_j) converges to $(\text{id}, 0)$ in $\text{Diffeo}^k(\Sigma, \Sigma) \times C^k(\Sigma)$. Thus β_j converges to 0 in $C^k(\widehat{\Sigma})$. This and (C-5) verify Definition C.7 for $l = k$. The claim is shown. \square

Also C^l -convergence is not affected by reparametrizations and Remark C.1 can be formulated also for the C^l -convergence. This implies in particular that limits with respect to C^l -convergence are not unique. In the rest of this section we will however show that, in our setting, C^l -limits are unique up to reparametrizations. Let us first fix what we mean by classical C^l convergence.

Definition C.10. We say that a sequence of immersions $(h_j)_{j=1}^\infty, h_j : \Sigma \rightarrow \mathbb{R}^n$, converges classically in C^l to some immersion $h : \Sigma \rightarrow \mathbb{R}^n$ if $u_j := h - h_j : \Sigma \rightarrow \mathbb{R}^n$ satisfies $\|D^k u_j\|_{L^\infty(\Sigma)} \rightarrow 0$ for all $k = 0, \dots, l$.

Proposition C.11. Let $(f_j)_{j=1}^\infty : \mathbb{S}^1 \times \mathbb{S}^1 \rightarrow \mathbb{R}^3$ be a sequence of smooth immersions and $l \geq 2$. Let $f, h : \mathbb{S}^1 \times \mathbb{S}^1 \rightarrow \mathbb{R}^3$ be such that f_j converges to f in C^l and f_j converges to h classically in C^l . Then f and h coincide up to reparametrization, i.e., there exists a C^l -diffeomorphism $\phi : \mathbb{S}^1 \times \mathbb{S}^1 \rightarrow \mathbb{S}^1 \times \mathbb{S}^1$ such that $h = f \circ \phi$.

Proof. Since f_j converges to f in C^l there exist diffeomorphisms ϕ_j of $\mathbb{S}^1 \times \mathbb{S}^1$ and maps $u_j : \mathbb{S}^1 \times \mathbb{S}^1 \rightarrow \mathbb{R}^3$ such that

$$f_j \circ \phi_j + u_j = f \quad \text{on } \mathbb{S}^1 \times \mathbb{S}^1, \tag{C-6}$$

and $\|u_j\|_{L^\infty}, \|Du_j\|_{L^\infty}$ converge to zero. Moreover there exist v_j such that

$$f_j + v_j = h \quad \text{on } \mathbb{S}^1 \times \mathbb{S}^1, \tag{C-7}$$

and $\|v_j\|_{L^\infty}, \|Dv_j\|_{L^\infty}$ converge to zero.

Step 1: $(\phi_j)_{j=1}^\infty$ converges uniformly to some $\phi \in C^0(\mathbb{S}^1 \times \mathbb{S}^1)$ that satisfies $h = f \circ \phi$. First note that functions on $\mathbb{S}^1 \times \mathbb{S}^1$ can be periodically extended on \mathbb{R}^2 . Doing so and tacitly identifying all the functions we defined above with their unique periodic extensions we infer that (C-6) and (C-7) hold on the whole of \mathbb{R}^2 . From both equations we infer that

$$h \circ \phi_j - v_j \circ \phi_j + u_j = f \quad \text{on } \mathbb{R}^2. \tag{C-8}$$

Since we deal now with functions in $C^1(\mathbb{R}^2; \mathbb{R}^3)$, we can compute derivatives simply using the Jacobi matrix. By the chain rule

$$(Dh(\phi_j) - Dv_j(\phi_j))D\phi_j + Du_j = Df \quad \text{in } \mathbb{R}^2. \tag{C-9}$$

We claim that $\|D\phi_j\|_{L^\infty(\mathbb{R}^{2 \times 2})}$ is bounded. For this assume that a subsequence (which we do not relabel) satisfies $\|D\phi_j\|_{L^\infty} \rightarrow \infty$ and let $p_j \in \mathbb{S}^1 \times \mathbb{S}^1$ be such that $|D\phi_j(p_j)| = \|D\phi_j\|_{L^\infty}$, where $|\cdot|$ is a suitable matrix norm. Evaluating (C-9) at p_j and dividing by $\|D\phi_j\|_{L^\infty}$ we obtain

$$(Dh(\phi_j(p_j)) - Dv_j(\phi_j(p_j))) \frac{D\phi_j(p_j)}{\|D\phi_j\|_{L^\infty}} + \frac{1}{\|D\phi_j\|_{L^\infty}} Du_j(p_j) = \frac{1}{\|D\phi_j\|_{L^\infty}} Df(p_j). \tag{C-10}$$

By the boundedness of $\phi_j : \mathbb{R}^2 \rightarrow \mathbb{S}^1 \times \mathbb{S}^1$ and the choice of p_j one can choose a subsequence such that $(\phi_j(p_j))_{j=1}^\infty$ converges to some $q \in \mathbb{S}^1 \times \mathbb{S}^1$ and $D\phi_j(p_j)/\|D\phi_j\|_{L^\infty}$ converges to some $B \in \mathbb{R}^{2 \times 2}$ that

satisfies $|B| = 1$. Note that by the requirements on u_j, v_j and the fact that the first fundamental forms of f, h with respect to the local coordinates (u, v) are bounded one has $\|Du_j\|_{L^\infty(\mathbb{R}^2, \mathbb{R}^{2 \times 3})}, \|Dv_j\|_{L^\infty(\mathbb{R}^2, \mathbb{R}^{2 \times 3})} \rightarrow 0$ as $j \rightarrow \infty$. Passing to the limit in (C-10) we obtain

$$Dh(q)B = 0.$$

This is a contradiction to h being an immersion and $|B| = 1$. Hence $\|D\phi_j\|_{L^\infty(\mathbb{R}^2, \mathbb{R}^{2,2})}$ is bounded. Note also that $\phi_j : \mathbb{R}^2 \rightarrow \mathbb{S}^1 \times \mathbb{S}^1$ is uniformly bounded as it takes values only in $\mathbb{S}^1 \times \mathbb{S}^1$. By the Arzelà–Ascoli theorem there exists a subsequence (which we do not relabel) and $\phi \in C^0(\mathbb{S}^1 \times \mathbb{S}^1)$ such that $\phi_j \rightarrow \phi$ on $\mathbb{S}^1 \times \mathbb{S}^1$. We can now go back to (C-8) and pass to the limit there to obtain

$$h \circ \phi = f \quad \text{on } \mathbb{S}^1 \times \mathbb{S}^1. \tag{C-11}$$

Step 2: ϕ is a local C^l diffeomorphism; i.e., ϕ is C^l smooth and for all $p \in \mathbb{S}^1 \times \mathbb{S}^1$ there exists an open neighborhood U containing p such that $\phi|_U$ is a diffeomorphism onto its image. To this end fix $p \in \mathbb{S}^1 \times \mathbb{S}^1$ and recall that, being h an immersion, there exists an open neighborhood W of $\phi(p)$ such that $h|_W$ is a diffeomorphism onto its image $V := h(W)$. We denote by $\tilde{h} : V \rightarrow W$ the inverse of $h|_W$. By (C-11) we obtain

$$\phi = \tilde{h} \circ f \quad \text{on } f^{-1}(V). \tag{C-12}$$

Notice that since $\phi(p) \in W$ it follows that $f(p) = h(\phi(p)) \in V$ and hence $p \in f^{-1}(V)$ so that $f^{-1}(V)$ is an open neighborhood of p . Now there exists another open neighborhood G of p such that $f|_G$ is a C^l -diffeomorphism onto its image. Defining $U = G \cap f^{-1}(V)$ we obtain that $\phi|_U$ is a C^l -diffeomorphism as a composition of two diffeomorphisms. Note in particular that $D\phi(p)$ is invertible at each point $p \in \mathbb{S}^1 \times \mathbb{S}^1$. This implies in particular, as $\mathbb{S}^1 \times \mathbb{S}^1$ is connected and $\phi \in C^l$ that $\text{sgn}(\det(D\phi))$ is constant.

Step 3: $\text{deg}(\phi) = \pm 1$. Recall that the *mapping degree* of ϕ is given by

$$\text{deg}(\phi) := \sum_{x \in \phi^{-1}(\{y\})} \text{sgn}(\det(D\phi(x))) \tag{C-13}$$

for any choice of $y \in \mathbb{S}^1 \times \mathbb{S}^1$. See [Outerelo and Ruiz 2009, Chapter 3] or [Guillemin and Pollack 1974, Chapter 3, Section 3] for the well-definedness of deg , e.g., the independence of the definition of the chosen y and finiteness of the sum in the definition. We make use of the degree-integration formula (see [Guillemin and Pollack 1974, p. 188]) to compute $\text{deg}(\phi)$. Since $\phi : \mathbb{S}^1 \times \mathbb{S}^1 \rightarrow \mathbb{S}^1 \times \mathbb{S}^1$ is sufficiently smooth, one has for all differential forms ω on $\mathbb{S}^1 \times \mathbb{S}^1$ that

$$\int_{\mathbb{S}^1 \times \mathbb{S}^1} \phi^* \omega = \text{deg}(\phi) \int_{\mathbb{S}^1 \times \mathbb{S}^1} \omega,$$

where $\phi^* \omega$ is defined as in [Guillemin and Pollack 1974, p. 166]. Let $\eta \in C_0^\infty(\mathbb{R}^3)$ be arbitrary. Take $\omega_\eta(u, v) := \eta(h(u, v))\sqrt{\det Dh^T Dh} du \wedge dv$. Then

$$\int_{\mathbb{S}^1 \times \mathbb{S}^1} \omega_\eta = \int_0^1 \int_0^{2\pi} \eta(h(u, v))\sqrt{\det(Dh^T Dh)} du dv = \int \eta dh^* \mu_h, \tag{C-14}$$

since $Dh^T Dh$ is the first fundamental form of $(\mathbb{S}^1 \times \mathbb{S}^1, g_h)$. Note that by [Lemma C.6](#) $f^* \mu_f$ coincides with $h^* \mu_h$ as both measures are $C_0(\mathbb{R}^n)'$ -limits of $f_j^* \mu_{f_j}$. Hence by [\(C-14\)](#)

$$\int \eta \, df^* \mu_f = \int \eta \, dh^* \mu_h = \int_{\mathbb{S}^1 \times \mathbb{S}^1} \omega_\eta. \tag{C-15}$$

Using now that $f = h \circ \phi$ we can also compute $\int \eta \, df^* \mu_f$ in another way. Since $s := \operatorname{sgn} \det D\phi$ is constant, by definition of $\phi^* \omega_\eta$ we obtain

$$\begin{aligned} \int \eta \, df^* \mu_f &= \int_0^1 \int_0^1 \eta(f(u, v)) \sqrt{\det(Df^T Df)} \, du \, dv \\ &= \int_0^1 \int_0^1 \eta(h(\phi(u, v))) \sqrt{\det(Dh^T Dh)} |\det(D\phi)| \, du \, dv \\ &= s \int_0^1 \int_0^1 \eta(h(\phi(u, v))) \sqrt{\det(Dh^T Dh)} \det(D\phi) \, du \, dv \\ &= s \int_{\mathbb{S}^1 \times \mathbb{S}^1} \phi^* \omega_\eta = s \cdot \operatorname{deg}(\phi) \int_{\mathbb{S}^1 \times \mathbb{S}^1} \omega_\eta. \end{aligned}$$

This and [\(C-15\)](#) yields that $\operatorname{deg}(\phi) = 1/s = \pm 1$.

Conclusion: The fact that $\operatorname{deg}(\phi) = \pm 1$, $\operatorname{sgn}(\det(D\phi))$ is constant together with [\(C-13\)](#) imply that $\phi^{-1}(\{y\})$ must be a singleton for any choice of $y \in \mathbb{S}^1 \times \mathbb{S}^1$. This proves the injectivity of ϕ . Surjectivity follows directly from [\[Outerelo and Ruiz 2009, Chapter 3, Remark 1.5\(2\)\]](#). We finally end up with a surjective and injective local diffeomorphism. By this inverse function theorem, this is also a global diffeomorphism. \square

Corollary C.12. *If $(f_j)_{j=1}^\infty : \mathbb{S}^1 \times \mathbb{S}^1 \rightarrow \mathbb{R}^3$ converges in C^l to some $f : \mathbb{S}^1 \times \mathbb{S}^1 \rightarrow \mathbb{R}^3$ and also to some $h : \mathbb{S}^1 \times \mathbb{S}^1 \rightarrow \mathbb{R}^3$. Then there exists a C^l diffeomorphism $\phi : \mathbb{S}^1 \times \mathbb{S}^1 \rightarrow \mathbb{S}^1 \times \mathbb{S}^1$ such that $f = h \circ \phi$.*

Proof. If f_j converges to h in C^l then by [Proposition C.9](#) there exists a sequence of diffeomorphisms $(\psi_j)_{j=1}^\infty$ of $\mathbb{S}^1 \times \mathbb{S}^1$ such that $f_j \circ \psi_j$ converges to h classically in C^l . Since (nonclassical) C^l convergence is not affected by reparametrizations, we infer that also $f_j \circ \psi_j$ converges to f in C^l . By [Proposition C.11](#) applied to $f_j \circ \psi_j$ we infer that $f = h \circ \phi$ for a C^l -diffeomorphism ϕ of $\mathbb{S}^1 \times \mathbb{S}^1$. \square

Appendix D: On the Willmore flow

Here we mention some previous results on the Willmore flow, which we will use. Since we need the precise formulations and constants we state them here for the readers convenience. We start with a short time existence and uniqueness result. We remark that this result is not the only short time existence result in the literature (see, e.g., [\[Simonett 2001\]](#)), but it is the most useful for the formulation we use.

Theorem D.1 [\[Kuwert and Schätzle 2002, Theorem 1.2\]](#). *Suppose that $f_0 : \Sigma \rightarrow \mathbb{R}^n$ is a smooth immersion. Then there exist constants $\varepsilon_0 > 0$, $c_0 < \infty$ that depend only on n such that for all $\rho > 0$ that satisfy*

$$\sup_{x \in \mathbb{R}^n} \int_{f_0^{-1}(B_\rho(x))} |A[f_0]|^2 \, d\mu_{f_0} \leq \varepsilon_0$$

there exists a unique maximal smooth Willmore flow $(f(t))_{t \in [0, T]}$ starting at f_0 that satisfies $T \geq c_0 \rho^4$. Moreover, for all $m \geq 0$ there exists $C = C(n, m, f_0)$ such that

$$\|(\nabla^\perp)^m A[f(t)]\|_{L^\infty(\Sigma)} \leq C \quad \text{for all } t \in [0, c_0 \rho^4]. \tag{D-1}$$

Note that (D-1) is not in the statement of [Kuwert and Schätzle 2002, Theorem 1.2] but in its proof; see [Kuwert and Schätzle 2002, equation (4.27)]. In fact the bound of the derivatives of the curvature are crucial in the proof of the short time existence theorem. In addition to bounds on the curvature one also needs a bound on the metric. Let us emphasize that this bound is (in finite time) implied by the curvature bounds as part of a more general result; see [Hamilton 1982, Lemma 14.2]. Once short time existence is shown one can look at long time existence. The most important blow up criterion obtained so far is the one discussed in Theorem D.5 below. It says that if $T < \infty$ then the curvature has to concentrate. One can ask what happens to other quantities once the curvature degenerates. By Simon’s monotonicity formula, the “density” will not degenerate. Indeed, in [Simon 1993, equation (1.3)], a local bound for the surface measure is shown. A useful implication stated in [Kuwert and Schätzle 2001, Lemma 4.1] is that there exists $c > 0$ such that for all proper immersions $f : \Sigma \rightarrow \mathbb{R}^n$ (Σ compact and without boundary) one has

$$\frac{\mu_f(f^{-1}(B_\rho(x_0)))}{\rho^2} \leq c \mathcal{W}(f) \quad \text{for all } \rho > 0, \tag{D-2}$$

where we further assume that Σ is a torus so that its Euler characteristic vanishes.

Up to this point, no examples of evolutions where the curvature degenerates are known, even though there exists one candidate for this phenomenon; see [Mayer and Simonett 2002].

Close to local minimizers curvature concentration cannot occur and one deduces convergence with the aid of a Łojasiewicz–Simon gradient inequality.

Theorem D.2 [Chill et al. 2009, Lemma 4.1]. *Let $f_W : \Sigma \rightarrow \mathbb{R}^n$ be a Willmore immersion of a compact manifold Σ without boundary, and let $k \in \mathbb{N}$, $\delta > 0$. Then there exists $\varepsilon = \varepsilon(f_W) > 0$ such that the following is true: suppose that $(f(t))_{t \in [0, T]}$ is a Willmore flow of Σ satisfying*

$$\|f_0 - f_W\|_{W^{2,2} \cap C^1} < \varepsilon$$

and

$$\mathcal{W}(f(t)) \geq \mathcal{W}(f_W) \quad \text{whenever } \|f(t) \circ \Phi(t) - f_W\|_{C^k} \leq \delta, \tag{D-3}$$

for some appropriate diffeomorphisms $\Phi(t) : \Sigma \rightarrow \Sigma$.

Then this Willmore flow exists globally, that is, $T = \infty$, and converges, after reparametrization by appropriate diffeomorphisms $\tilde{\Phi}(t) : \Sigma \rightarrow \Sigma$, smoothly to a Willmore immersion f_∞ . That is,

$$f(t) \circ \tilde{\Phi}(t) \rightarrow f_\infty \quad \text{as } t \rightarrow \infty.$$

Moreover, $\mathcal{W}(f_\infty) = \mathcal{W}(f_W)$ and $\|f_0 - f_W\|_{C^k} < \delta$.

Remark D.3. Notice that ε in the statement does not change if instead of f_W one considers the translated Willmore surface $f_W + \bar{x}$ for $\bar{x} \in \mathbb{R}^n$. Indeed, if f_0 satisfies

$$\|f_0 - (f_W + \bar{x})\|_{W^{2,2} \cap C^1} < \varepsilon = \varepsilon(f_W),$$

then clearly $f_0 - \bar{x}$ satisfies the assumptions on the initial datum stated in [Theorem D.2](#) so that the corresponding Willmore flow $\tilde{f}(t)$ converges. Due to the uniqueness of the solution for the Willmore flow, $\tilde{f}(t) = f(t) - \bar{x}$ with $f(t)$ the solution of the Willmore flow which starts in f_0 . Hence, also $f(t)$ converges.

Remark D.4. We also remark that in case that the Willmore flow converges in C^k for all k one obtains uniform bounds on all derivatives of the second fundamental form, i.e., for all $m \in \mathbb{N}_0$ there exists $C = C(m, f_0)$ such that

$$\|(\nabla^\perp)^m A[f(t)]\|_{L^\infty} \leq C \quad \text{for all } t \in [0, \infty).$$

Not every evolution of the Willmore flow is convergent. What one can however always obtain is a *Willmore concentration limit* of appropriate parabolic rescalings. Below we will introduce the Willmore concentration limit rigorously since we need to examine it for the proof of [Theorem 3.1](#).

Theorem D.5 (Willmore concentration limit [[Kuwert and Schätzle 2001](#), Section 4]). *Let Σ be a compact two-dimensional manifold without boundary and let $f : [0, T) \times \Sigma \rightarrow \mathbb{R}^n$ be immersions evolving by the Willmore flow with initial datum f_0 . Let $\varepsilon_0 > 0$ and c_0 be defined as in [Theorem D.1](#).*

Then for each sequence $(t_j)_{j=1}^\infty \nearrow T$ there exist $(x_j)_{j=1}^\infty \subset \mathbb{R}^n$, $(r_j)_{j=1}^\infty \subset (0, \infty)$ (defined as in (3-1)) and $c_0 > 0$ such that

$$t_j + c_0 r_j^4 < T \quad \text{for all } j \in \mathbb{N} \tag{D-4}$$

and

$$\tilde{f}_j := \frac{1}{r_j} (f(t_j + c_0 r_j^4, \cdot) - x_j) : \Sigma \rightarrow \mathbb{R}^n \tag{D-5}$$

converges smoothly on compact subsets of \mathbb{R}^n to a proper Willmore immersion $\hat{f} : \hat{\Sigma} \rightarrow \mathbb{R}^n$, where $\hat{\Sigma} \neq \emptyset$ is a smooth two-dimensional manifold without boundary. Moreover

$$\liminf_{j \rightarrow \infty} \int_{B_j} |A(t_j + c_0 r_j^4)|^2 d\mu_{g(t_j + c_0 r_j^4)} > 0, \tag{D-6}$$

where $B_j = (f(t_j + c_0 r_j^4))^{-1}(\overline{B_{r_j}(x_j)})$.

Now we are finally ready to prove [Theorem 3.1](#).

Proof of Theorem 3.1. The first part of the statement follows from (D-4). From [Theorem D.5](#) it follows that there exists a sequence $(x_j)_{j \in \mathbb{N}} \subset \mathbb{R}^n$ and a proper Willmore immersion $\hat{f} : \hat{\Sigma} \rightarrow \mathbb{R}^n$ such that

$$\tilde{f}_{j,c_0} - \frac{x_j}{r_j} \rightarrow \hat{f}, \tag{D-7}$$

smoothly as $j \rightarrow \infty$. Now we examine the asymptotics of $(r_j)_{j \in \mathbb{N}}$.

If there exists a subsequence of the radii r_j that tends to zero or infinity. By [[Chill et al. 2009](#), Theorem 1.1], $\hat{\Sigma}$ is not compact. In particular $\text{diam}(\hat{f}(\hat{\Sigma})) = \infty$ since otherwise $\hat{f}(\hat{\Sigma})$ lies in a compact set of \mathbb{R}^n which is a contradiction to the properness of \hat{f} . By lower semicontinuity of the diameter, see

Lemma C.3, we infer

$$\infty = \text{diam}(\hat{f}(\widehat{\Sigma})) \leq \liminf_{j \rightarrow \infty} \text{diam}\left(\tilde{f}_{j,c_0} - \frac{x_j}{r_j}\right) = \liminf_{j \rightarrow \infty} \text{diam}(\tilde{f}_{j,c_0}).$$

Hence we have shown that (2) occurs.

Suppose on contrary that $(r_j)_{j \in \mathbb{N}}$ has no subsequence that tends to zero or infinity. Then there exists $\delta > 0$ such that $\delta < r_j < 1/\delta$ for all $j \in \mathbb{N}$ and Case 1 occurs. Necessarily from (D-4) we see that $T = \infty$.

It remains to show that a bound on the diameter ensures full convergence to a Willmore immersion. Suppose therefore that $\text{diam}(\tilde{f}_{j,c_0}) \leq M$ for all $j \in \mathbb{N}$. Note that then - once again by lower semicontinuity, see Lemma C.3,

$$\text{diam}(\hat{f}(\widehat{\Sigma})) \leq \liminf_{j \rightarrow \infty} \text{diam}\left(f_{j,c_0} - \frac{x_j}{r_j}\right) = \liminf_{j \rightarrow \infty} \text{diam}(f_{j,c_0}) \leq M.$$

Since \hat{f} is proper this ensures that $\widehat{\Sigma}$ is compact. By [Kuwert and Schätzle 2001, Lemma 4.3] we infer that $\widehat{\Sigma} = \mathbb{S}^1 \times \mathbb{S}^1$ and the convergence in (D-7) is actually convergence in C^k for all $k \in \mathbb{N}$. Now we define

$$\tilde{f}_j : [0, c_0] \times \mathbb{S}^1 \times \mathbb{S}^1 \rightarrow \mathbb{R}^n, \quad \tilde{f}_j(s) := \frac{f(t_j + sr_j^4)}{r_j}.$$

Note that by scaling properties of the Willmore gradient \tilde{f}_j solves the Willmore flow equation. By (D-7) we can now fix $j_0 \in \mathbb{N}$ and a smooth diffeomorphism $\Phi : \mathbb{S}^1 \times \mathbb{S}^1 \rightarrow \mathbb{S}^1 \times \mathbb{S}^1$ such that

$$\left\| \tilde{f}_{j_0,c_0} \circ \Phi - \frac{x_{j_0}}{r_{j_0}} - \hat{f} \right\|_{C^2} < \varepsilon = \varepsilon(\hat{f}), \tag{D-8}$$

where $\varepsilon(\hat{f})$ is chosen as in Theorem D.2. By Remark D.3 we also have $\varepsilon(\hat{f}) = \varepsilon(\hat{f} + x_{j_0}/r_{j_0})$. We infer by Theorem D.2 that the Willmore flow starting at $\tilde{f}_{j_0,c_0} \circ \Phi$ exists globally and converges (up to reparametrization) to a Willmore immersion $f_\infty : \mathbb{S}^1 \times \mathbb{S}^1 \rightarrow \mathbb{R}^n$. By geometric uniqueness of Willmore evolutions we infer that $\tilde{f}_{j_0} \circ \Phi$, first defined on $[0, c_0]$, extends to a global evolution, i.e., defined on $[0, \infty)$, and converges (up to reparametrization) to f_∞ . Again by geometric uniqueness we infer that \tilde{f}_{j_0} extends to a global evolution converging (up to reparametrization) to $f_\infty \circ \Phi^{-1}$. Using scaling properties of the Willmore flow we infer that f extends to a global evolution by Willmore flow that converges to $r_{j_0}f_\infty$, which is again a Willmore immersion.

To show the last sentence of the claim we first observe that a uniform bound on the diameter implies that Case 2 may not occur, in particular $r_j \in (\delta, 1/\delta)$ for some $\delta > 0$. Then the fact that $t_j + c_0r_j^4 < T$ for all j and $t_j \rightarrow T$ implies that $T = \infty$. Convergence follows then according to case (1) with the diameter bound. □

With this theorem we have proved that boundedness of $\text{diam}(\tilde{f}_{j,c_0})$ implies convergence. The fact that the \tilde{f}_{j,c_0} need information about $f(t_j + c_0r_j^4)$ and not just about $f(t_j)$ adds a technical difficulty — the time shift might cause geometric quantities to degenerate. Luckily, the diameter is not so much affected by (bounded) time shifts, as we shall see in the following:

Lemma D.6 (Evolution of diameter and area). *Suppose that $f : [0, T) \times \Sigma \rightarrow \mathbb{R}^n$ is a maximal evolution by Willmore flow. Then there exist constants $C_1 = C_1(\mathcal{W}(f(0)))$, $C_2 = C_2(\mathcal{W}(f(0)))$ depending monotonically on $\mathcal{W}(f(0))$ such that*

$$\mu_{g_{f(t)}}(\Sigma) \leq \mu_{g_{f(0)}}(\Sigma) + C_1(\mathcal{W}(f(0)))t^{1/2} \tag{D-9}$$

and

$$\text{diam}(f(t)(\Sigma)) \leq C_2(\mathcal{W}(f(0)))(\text{diam}(f(0)(\Sigma)) + t^{1/4}).$$

Proof. First we remark that, since the Willmore flow is a gradient flow, for all $s \geq 0$

$$\int_0^s \int_{\Sigma} |\partial_t f(t)|^2 d\mu_{g_{f(t)}} = \mathcal{W}(f(0)) - \mathcal{W}(f(s)) \leq \mathcal{W}(f(0)). \tag{D-10}$$

By [Kuwert and Schätzle 2002, equation (2.16)] we have

$$\begin{aligned} \left| \frac{d}{dt} \mu_{g_{f(t)}}(\Sigma) \right| &= \left| \int_{\Sigma} \langle \vec{H}[f(t)], \partial_t f(t) \rangle d\mu_{g_{f(t)}} \right| \\ &\leq \left(\int_{\Sigma} |\vec{H}[f(t)]|^2 d\mu_{g_{f(t)}} \right)^{1/2} \left(\int_{\Sigma} |\partial_t f(t)|^2 d\mu_{g_{f(t)}} \right)^{1/2} \\ &\leq 2\sqrt{\mathcal{W}(f(t))} \left(\int_{\Sigma} |\partial_t f(t)|^2 d\mu_{g_{f(t)}} \right)^{1/2}. \end{aligned}$$

Integrating with respect to t and since $t \mapsto \mathcal{W}(f(t))$ is decreasing we obtain

$$\begin{aligned} |\mu_{g_{f(s)}}(\Sigma) - \mu_{g_{f(0)}}(\Sigma)| &\leq 2\sqrt{\mathcal{W}(f(0))} \int_0^s \left(\int_{\Sigma} |\partial_t f(t)|^2 d\mu_{g_{f(t)}} \right)^{1/2} dt \\ &\leq 2\sqrt{\mathcal{W}(f(0))} s^{1/2} \left(\int_0^s \int_{\Sigma} |\partial_t f(t)|^2 d\mu_{g_{f(t)}} dt \right)^{1/2} \leq 2\mathcal{W}(f(0))s^{1/2}, \end{aligned}$$

using (D-10) in the last step. The estimate in (D-9) follows if we choose $C_1(W) = 2\mathcal{W}(f(0))$. Next we use a generalization of [Simon 1993, Lemma 1.1] (see the following lemma) for immersed surfaces to obtain that there exists $C_S > 0$ such that $\text{diam}(f(\Sigma))^2 \leq C_S \mathcal{W}(f) \mu_{g_f}(\Sigma)$. Using this, (D-9) and Lemma D.7 we obtain

$$\begin{aligned} \text{diam}(f(t)(\Sigma))^2 &\leq C_S \mathcal{W}(f(t)) \mu_{g_{f(t)}}(\Sigma) \leq C_S \mathcal{W}(f(0)) (\mu_{g_{f(0)}}(\Sigma) + 2\mathcal{W}(f(0))t^{1/2}) \\ &\leq C_S \mathcal{W}(f(0)) (\mathcal{W}(f(0)) \text{diam}(f(0))^2 + 2\mathcal{W}(f(0))t^{1/2}). \\ &\leq C_S \mathcal{W}(f(0))^2 (\text{diam}(f(0))^2 + 2t^{1/2}) \\ &\leq 2C_S \mathcal{W}(f(0))^2 (\text{diam}(f(0)) + t^{1/4})^2. \end{aligned}$$

The choice of $C_2(W) := 2C_S W^2$ does the job. □

In this proof we have used the following lemma, which generalizes [Simon 1993, Lemma 1.1].

Lemma D.7 (cf. [Simon 1993, Lemma 1.1]). *There exists $C_S = C_S(n) > 0$ such that for all immersions $f : \Sigma \rightarrow \mathbb{R}^n$ of a compact connected two-dimensional manifold without boundary Σ one has*

$$\frac{\mu_{g_f}(\Sigma)}{\mathcal{W}(f)} \leq \text{diam}(f(\Sigma))^2 \leq C_S(n) \mu_{g_f}(\Sigma) \mathcal{W}(f).$$

Proof. Let Σ be as in the statement. By [Simon 1993, Lemma 1.1] we infer that for all $n \in \mathbb{N}$ there exists $c(n) > 0$ such that for all embeddings $f : \Sigma \rightarrow \mathbb{R}^n$ one has

$$\frac{\mu_{g_f}(\Sigma)}{\mathcal{W}(f)} \leq \text{diam}(f(\Sigma))^2 \leq c(n)\mu_{g_f}(\Sigma)\mathcal{W}(f). \tag{D-11}$$

We need to generalize this result to immersions. Let $N \in \mathbb{N}$ be such that each smooth two-dimensional manifold can be smoothly embedded into \mathbb{R}^N . Such a constant N exists due to Nash's embedding theorem (or alternatively one can derive $N = 4$ explicitly using a handle decomposition). We will show that the desired estimate is satisfied with the constant $C_S(n) := c(n + N)$. To this end let $f : \Sigma \rightarrow \mathbb{R}^n$ be an immersion and $\iota : \Sigma \rightarrow \mathbb{R}^N$ be an embedding. For fixed $\varepsilon > 0$ define $f_\varepsilon : \Sigma \rightarrow \mathbb{R}^{n+N}$ via $f_\varepsilon(p) := (f(p), \varepsilon\iota(p))^T$. It is easy to check that f_ε is an embedding. We infer by (D-11) that

$$\frac{\mu_{g_{f_\varepsilon}}(\Sigma)}{\mathcal{W}(f_\varepsilon)} \leq \text{diam}(f_\varepsilon(\Sigma))^2 \leq c(n + N)\mu_{g_{f_\varepsilon}}(\Sigma)\mathcal{W}(f_\varepsilon). \tag{D-12}$$

Next we pass to the limit as $\varepsilon \rightarrow 0$. First we examine the diameter. Note that for all $x, y \in \Sigma$ one has

$$|f_\varepsilon(x) - f_\varepsilon(y)|^2 = |f(x) - f(y)|^2 + \varepsilon^2|\iota(x) - \iota(y)|^2.$$

From this one easily infers

$$\text{diam}(f(\Sigma))^2 \leq \text{diam}(f_\varepsilon(\Sigma))^2 \leq \text{diam}(f(\Sigma))^2 + \varepsilon^2 \text{diam}(\iota(\Sigma)).$$

Since Σ is compact we find that $\text{diam}(\iota(\Sigma)) < \infty$. Hence

$$\lim_{\varepsilon \rightarrow 0} \text{diam}(f_\varepsilon(\Sigma)) = \text{diam}(f(\Sigma)).$$

One readily checks that $f_\varepsilon \rightarrow (f, 0)$ in C^k for all k . From Lemma C.5 one infers then that $\lim_{\varepsilon \rightarrow 0} \mathcal{W}(f_\varepsilon) = \mathcal{W}((f, 0)) = \mathcal{W}(f)$. That $\mathcal{W}((f, 0)) = \mathcal{W}(f)$ can easily be checked since

$$A[(f, 0)](X, Y) = D^2(f, 0)(X, Y) = (D^2 f(X, Y), 0),$$

where the last identity is due to the fact that D is defined componentwise; see (B-1). Using methods similar to Lemma C.5 one can also check $\lim_{\varepsilon \rightarrow 0} \mu_{g_{f_\varepsilon}}(\Sigma) = \mu_{g_{(f,0)}}(\Sigma) = \mu_{g_f}(\Sigma)$. This being shown, the claim follows from (D-12) letting $\varepsilon \rightarrow 0$. □

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