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OPTIMAL PRANDTL EXPANSION AROUND A CONCAVE BOUNDARY LAYER

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We show an optimal stability result for boundary layer solutions of the Navier–Stokes equation in a half-plane, under a mild concavity condition on the boundary layer profile. The key point is the derivation of sharp Gevrey estimates for the linearized Navier–Stokes equation in vorticity form, on a time interval uniform in ν. As the nonlocal boundary condition on the vorticity prevents us from deriving direct estimates, we use a novel iteration scheme, similar to a splitting method in numerical analysis. Our result is a big step forward compared to our previous work (*Duke Math. J.* **167** (2018), 2531–2631), where we proved stability of boundary layer expansions of shear flow type. Indeed, the approach of the present paper is much more robust than the one in that previous work, which was based on the Fourier transform and hence only adapted to expansions independent of the tangential variable. Moreover, we are now able to relax the assumption of strict concavity made in our previous work to obtain the optimal Gevrey $\frac{3}{2}$ stability, which was not satisfied by generic boundary layer expansions. We provide in this way the first justification of unsteady boundary layer theory outside the analytic setting.

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1. Introduction

We are interested in the high Reynolds number dynamics of the Navier–Stokes equation in a half-plane:

$$\begin{aligned} \partial_t u^\nu - \nu\Delta u^\nu + \nabla p^\nu + u^\nu\cdot\nabla u^\nu = 0, &\quad t>0,\ x\in\mathbb{T},\ y>0,\\ \nabla\cdot u^\nu = 0, &\quad t\ge 0,\ x\in\mathbb{T},\ y>0,\\ u^\nu|_{y=0}=0, \quad u^\nu|_{t=0}=u_0, & \end{aligned} \tag{1-1}$$

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where ν stands for the inverse Reynolds number. Note that we consider periodic boundary conditions in x, but could consider decay conditions as well. As is well known, the Navier–Stokes solution u^ν exhibits a boundary layer near $y=0$, that is a region of high velocity gradients generated by the no-slip condition. A famous modeling of this boundary layer was provided by Prandtl. In modern language, he provided approximate solutions of Navier–Stokes equations in the form of multiscale asymptotic expansions:

$$v = \sum_{i=0}^{N} \sqrt{\nu}^i U^{E,i}(t,x,y) + \sum_{i=0}^{N} \sqrt{\nu}^i \big(V_1^{\mathrm{bl},i}(t,x,y/\sqrt{\nu}), \sqrt{\nu} V_2^{\mathrm{bl},i}(t,x,y/\sqrt{\nu})\big), \tag{1-2}$$

where the profiles $U^{E,i} = U^{E,i}(t,x,y)$ describe the flow away from the boundary, and the profiles $V^{\mathrm{bl},i} = V^{\mathrm{bl},i}(t,x,Y)$ are boundary layer correctors that go to zero exponentially fast in variable $Y = y/\sqrt{\nu}$. We stress that there is a factor $\sqrt{\nu}$ between the amplitudes of the horizontal and vertical components of the boundary layer profiles: this is consistent with the divergence-free condition. In particular, the leading order term $U^E := U^{E,0}$ solves the Euler equation, while the leading order boundary corrector $V^{\mathrm{bl}} := V^{\mathrm{bl},0}$ solves the modified Prandtl equation

$$\begin{aligned}
\partial_t V_1^{\mathrm{bl}} + (U_1^E|_{y=0} + V_1^{\mathrm{bl}})\partial_x V^{\mathrm{bl},1} + V_1^{\mathrm{bl}} \partial_x U_1^E|_{y=0} + (Y\partial_y U_2^E|_{y=0} + V_2^{\mathrm{bl}})\partial_Y V_1^{\mathrm{bl}} - \partial_Y^2 V_1^{\mathrm{bl}} &= 0,\\
\partial_x V^{\mathrm{bl},1} + \partial_Y V_2^{\mathrm{bl}} &= 0,\\
V_1^{\mathrm{bl}}|_{Y=0} = -U_1^E|_{y=0}, \quad V^{\mathrm{bl}} \to 0, \quad Y \to +\infty.&
\end{aligned}$$

Prandtl boundary layer theory has revealed much about the mechanism of vorticity generation in fluids and has contributed to the quantitative understanding of some model problems, notably the description of the Blasius flow near a flat plate. It can moreover be rigorously justified under strong symmetry conditions on the flow and its perturbations; see for instance [Lopes Filho et al. 2008; Mazzucato and Taylor 2008]. Still, under generic perturbations, Navier–Stokes flows of type (1-2) are known to experience instabilities, due to two main mechanisms:

- Boundary layer separation, which corresponds to a loss of monotonicity and concavity of the boundary layer profile V_1^{bl}, under an adverse pressure gradient. Mathematically, it corresponds to some ill-posedness or blow-up of the Prandtl model.
- Hydrodynamic instabilities of Tollmien–Schlichting-type, experienced by concave boundary layer flows.

These phenomena have crucial consequences in hydrodynamics and aerodynamics. From the mathematical point of view, describing the stability/instability properties of flows v of type (1-2) is a difficult topic. The evolution of the perturbation $w = u^\nu - v$ obeys the perturbed Navier–Stokes system

$$\begin{aligned}
\partial_t w - \nu\Delta w + \nabla q + v\cdot\nabla w + w\cdot\nabla v &= -w\cdot\nabla w + r, && t>0,\ x\in\mathbb{T},\ y>0,\\
\nabla\cdot w &= 0, && t\ge 0,\ x\in\mathbb{T},\ y>0,\\
w|_{y=0} = 0, \quad w|_{t=0} &= w_0.
\end{aligned} \tag{1-3}$$

Here, r represents a remainder term due to the approximation v, while w_0 is a given initial perturbation of the velocity. We will assume that r and w_0 are of the order $O(\nu^n)$ in some norm with $n \gg 1$. In the case of r, this is realized by taking N large enough in (1-2). More precisely, one has to consider functional frameworks such that the equations of both Prandtl-type and Euler-type are uniquely solvable at least locally in time. Then, the point is to understand under which conditions one can obtain uniform (in ν) estimates of w in a suitable norm, that is justification of the Prandtl theory.

An important result in this direction is due to Sammartino and Caflisch [1998a; 1998b], who proved local well-posedness of Euler and Prandtl equations, as well as stability results for (1-3) in the case of analytic data. This stability result is then extended by [Fei et al. 2018; Kukavica et al. 2020; 2022; Maekawa 2014; Wang and Wang 2020; Wang et al. 2017], all of which require the analyticity near the boundary. This general analytic stability result is somehow optimal, in view of [Grenier 2000a]; see also [Grenier and Nguyen 2019]. Grenier studied the case where the Prandtl expansion v in (1-2) is a shear flow: this means that

$$v = (V_1^{\mathrm{bl}}(t, x, y/\sqrt{\nu}), 0), \tag{1-4}$$

where V_1^{bl} solves the heat equation

$$\partial_t V_1^{\mathrm{bl}} - \partial_Y^2 V_1^{\mathrm{bl}} = 0, \qquad V_1^{\mathrm{bl}}|_{Y=0} = 0. \tag{1-5}$$

He proved that for some profiles V_1^{bl} that have initially inflection points, the linearized version of (1-3) admits growing perturbations of the form

$$w^\nu(t, x, y) \approx e^{\alpha t/\nu^{1/2}} e^{ix/\nu^{1/2}} \widetilde{w}^\nu(y),$$

with fixed $\alpha > 0$. This shows that high frequencies $k \approx 1/\nu^{1/2}$ in variable x may be amplified by $e^{\alpha k t}$. In other words, to obtain a bound independent of ν over a time $T = O(1)$ will only be possible if those modes k have amplitude less than $e^{-\delta k}$, with $\delta \leq \alpha T$. This necessary exponential decay of the frequency spectrum corresponds to analytic perturbations. Let us note that the result of Grenier relies on the so-called Rayleigh instability, which is an inviscid instability mechanism for shear flows with inflection points. In terms of hydrodynamics of the boundary layer, the appearance of inflection points corresponds to the separation phenomenon. Hence, it is a framework in which various negative results exist for the Prandtl equation itself [E and Engquist 1997; Gérard-Varet and Dormy 2010; Gérard-Varet and Nguyen 2012; Kukavica et al. 2017].

The case without inflection points, corresponding to the nicer situation where the boundary layer profile V_1^{bl} is concave in variable Y, is much more involved. Again, the natural first step is to consider the shear flow situation (1-4). The stability of shear flows within the Navier–Stokes equation is an old topic of hydrodynamics, notably studied by Tollmien and Schlichting. See [Drazin and Reid 2004] for a detailed account. They showed that generic concave shear flows, although stable in the Euler evolution, exhibit instability in the Navier–Stokes one (albeit with a growth rate vanishing with viscosity). This is the so-called Tollmien–Schlichting instability, revisited on a rigorous basis by Grenier, Guo and Nguyen [Grenier et al. 2016]. Roughly, by using a proper rescaling of these unstable eigenmodes, one can construct for the linearization of (1-3) solutions of the type

$$w^\nu(t, x, y) \approx e^{\alpha t/\nu^{1/4}} e^{ix/\nu^{3/8}} \widetilde{w}^\nu(y).$$

This time, high frequencies $k \approx 1/\nu^{3/8}$ may be amplified by $e^{\alpha k^{2/3} t}$. This is still not compatible with Sobolev uniform bounds. More precisely, under the assumption that the spectral radius of the linearized Navier–Stokes operator is given by the growth rate of the Tollmien–Schlichting instability, one can obtain exponential bounds on the semigroup and from there show nonlinear Sobolev instability of Prandtl expansions of shear flow type; see [Grenier and Nguyen 2017; 2024].

Nevertheless, in the setting of concave boundary layer flows, the class of data w_0 for which one can hope to have uniform (in ν) local (in time) control of w is larger than analytic: namely, one may expect control for data whose Fourier spectrum in x decays like $O(e^{-k^{2/3}})$. This corresponds to the so-called Gevrey class of exponent $\frac{3}{2}$.

To show such optimal stability result for general "concave" Prandtl expansions is the main goal of the present paper. It goes much beyond our result [Gérard-Varet et al. 2018], limited to the case when the boundary layer is of shear type (1-4). See also the recent development [Chen et al. 2022], still on shear flow expansions. Precise statements will be given in Section 2. Three preliminary remarks are in order:

- The approach in [Gérard-Varet et al. 2018] was very much based on the Fourier transform in x, made easy because (1-4) is independent of x. It does not adapt to general Prandtl expansions. The approach in the present paper relies on very different ideas.

- The main step in our approach is the derivation of stability estimates for the linearized equations

$$\begin{aligned} \partial_t w - \nu \Delta w + \nabla q + v \cdot \nabla w + w \cdot \nabla v = f, \quad t > 0, \quad x \in \mathbb{T}, \quad y > 0, \\ \nabla \cdot w = 0, \quad t \geq 0, \quad x \in \mathbb{T}, \quad y > 0, \\ w|_{y=0} = 0, \quad w|_{t=0} = w_0. \end{aligned} \tag{1-6}$$

But to derive such bounds, we do not make any assumption on the spectral radius of the linearized operator, in contrast with the works [Grenier and Nguyen 2017; 2024].

- A strong point of our analysis is that it applies to boundary layer profiles V_1^{bl} that are concave in Y but not necessarily strictly concave. See Section 2 for detailed hypotheses. This is important for applications, as can be seen from (1-5): there, $\partial_Y^2 V_1^{\text{bl}}$ vanishes at the boundary for $Y = 0$ at positive times. Despite such possible degeneracies, we are able to reach Gevrey $\frac{3}{2}$ stability: this was not the case in our previous paper [Gérard-Varet et al. 2018], where our Gevrey exponent for stability was less than $\frac{3}{2}$ for nonstrictly concave flows.

The outline of the paper is as follows. Section 2 contains our main assumptions and stability results. We notably explain how our assumptions are adapted to generic boundary layer expansions. Section 3 gives an overview of our proof. The key point is the analysis of system (1-6), expressed in vorticity form. While this form allows to get rid of the pressure term, we face the difficulty that the vorticity $\omega = \operatorname{curl} v$ satisfies an intricate nonlocal condition, which forbids good direct stability estimates. To overcome this issue, we construct (and estimate) ω through an iteration scheme, where each step of the iteration can be split in two:

- In a first substep, we solve the linearized equation but with an artificial Neumann boundary condition on ω. This change in boundary condition allows to obtain stability estimates through the use of weighted norms inspired by the analysis of the hydrostatic Euler equations and subsequent works [Brenier 1999; Grenier 2000b]. This is where concavity is involved. These estimates, which would be wrong with the Dirichlet conditions for the velocity, remain valid under this modified boundary condition.

- In the next substep, we solve the linearized equation with zero source term or initial data but with a inhomogeneous Dirichlet condition on the velocity, correcting the error of the previous substep. This time, as the forcing is only through the boundary, the corresponding solution is more localized and of a parabolic nature. This allows for stabilizing effects.

More elements of the strategy are provided in Section 3. Afterwards, Section 4 details the estimates useful for the first substep of the iteration scheme, and Section 5 details the construction of the boundary corrector of the second substep. Eventually, Sections 6 and 7 provide the final linear and nonlinear estimates respectively.

2. Statements of the results

To state our stability result, we first introduce our functional framework. Let $p \in [1, \infty]$, $K \geq 1$, and $\nu \in (0, 1]$. For simplicity we assume $\nu^{-1/2} \in \mathbb{N}$, but it is not at all essential to our argument. We set

$$\|f\|_{G^p_{3/2}} = \sum_{j=0}^{\nu^{-1/2}} \frac{1}{(j!)^{3/2}} \sup_{j_2=0,\dots,j} \|e^{-Kt(j+1)} \beta_{j_2} \partial_x^{j-j_2} f\|_{L^p_t(0,1/K;L^2_{x,y})}, \tag{2-1}$$

where

$$\beta_{j_2} = \chi^{j_2} \partial_y^{j_2}, \quad \chi(y) = 1 - e^{-\kappa y}. \tag{2-2}$$

Here $\kappa \in (0, 1]$ is a fixed number, which will be taken small enough. We note that $\|f\|_{G^p_{3/2}}$ depends on $\nu, \kappa \in (0, 1]$ and $K \geq 1$, though we drop this dependence to simplify the notation. Note that for each fixed ν the norm $\|f\|_{G^p_{3/2}}$ is of Sobolev-type, but if $\|f\|_{G^p_{3/2}}$ is uniformly bounded in ν, it implies a usual Gevrey $\frac{3}{2}$ regularity for the C^∞ function f. The reason we can restrict to $j \leq \nu^{-1/2}$ in the sum above is that, in (1-3), the stretching term $\nabla v = O(\nu^{-1/2})$ creates at most an amplification $O(e^{C\nu^{-1/2}t})$. For $j \sim \nu^{-1/2}$, it is therefore balanced by the factor $e^{-Kt(j+1)}$ for large enough K. This means that we will be able to close an estimate considering only derivatives up to order $\nu^{-1/2}$.

Our main theorem is the following. Let us set $H^1_{0,\sigma}(\mathbb{T}\times\mathbb{R}_+) = \{f \in H^1_0(\mathbb{T}\times\mathbb{R}_+)^2 \mid \operatorname{div} f = 0 \text{ in } \mathbb{T}\times\mathbb{R}_+\}$, the space of all H^1 solenoidal vector fields satisfying the no-slip boundary condition at $Y = 0$.

Theorem 2.1 (nonlinear stability of concave Prandtl expansions). *Let $v = v(t, x, y)$ be a divergence-free vector field that fulfills the regularity and concavity conditions gathered in the Assumptions below but is not necessarily of type (1-2). There exists $\kappa_0 > 0$ such that the following statement holds for any $\kappa \in (0, \kappa_0]$: there exist $C > 0$, $K > 0$, $\delta_0 > 0$ such that, for all $\nu \leq K^{-2}$, if $r \in L^2(0, 1/K; L^2(\mathbb{T}\times\mathbb{R}_+)^2)$ and $w_0 \in H^1_{0,\sigma}(\mathbb{T}\times\mathbb{R}_+)$ satisfy*

$$[|w_0|]_{G_{3/2}} + [|\operatorname{rot} w_0|]_{G_{3/2}} \leq \delta_0 \nu^{\frac{9}{4}}, \quad \|r\|_{G^2_{3/2}} \leq \delta_0 \nu^{\frac{11}{4}}, \tag{2-3}$$

then the system (1-3) *has a unique solution* $w \in C([0,1/K], H^1_{0,\sigma}(\mathbb{T}\times\mathbb{R}_+))$ *satisfying*

$$\|w\|_{G^\infty_{3/2}} + \nu^{\frac12}\|\mathrm{rot}\, w\|_{G^\infty_{3/2}} \le C\nu^{-\frac12}([|w_0|]_{G_{3/2}} + [|\mathrm{rot}\, w_0|]_{G_{3/2}} + \nu^{-\frac12}\|r\|_{G^2_{3/2}}). \tag{2-4}$$

Here $\mathrm{rot}\, w = \partial_x w_2 - \partial_y w_1$ *and*

$$[|w_0|]_{G_{3/2}} = \sum_{j=0}^{\nu^{-1/2}} \frac{1}{(j!)^{3/2}} \sup_{j_2=0,\dots,j} \|\beta_{j_2}\partial_x^{j-j_2} w_0\|_{L^2_{x,y}}.$$

To complete the statement of our theorem, it remains to describe the set of assumptions on v that yield Theorem 2.1. Of course, these assumptions are designed to be satisfied by Prandtl expansions of type (1-2), when V_1^{bl} has some mild concavity. Due to the boundary layer variable Y, it is more convenient to work with rescaled variables $(\tau, X, Y) := \nu^{-1/2}(t,x,y)$. Accordingly, we shall express our assumptions directly on

$$V(\tau, X, Y) := v(t,x,y), \quad \tau > 0, \quad X \in \mathbb{T}_\nu, \quad Y > 0.$$

Here, $\mathbb{T}_\nu := \nu^{-1/2}\mathbb{T}$. We set

$$\Omega = \partial_X V_2 - \partial_Y V_1, \tag{2-5}$$

which describes the vorticity field of the approximation in the rescaled variables. We also set

$$\chi_\nu = \chi(\nu^{\frac12} Y) = 1 - e^{-\kappa\nu^{1/2} Y}. \tag{2-6}$$

Note that $\kappa \in (0,1]$ is fixed but taken small enough. Also, in the rescaled variables, our almost Gevrey norm $\|\cdot\|_{G^p_{3/2}}$ becomes

$$|||F|||_p = \sum_{j=0}^{\nu^{-1/2}} \frac{1}{(j!)^{3/2}\nu^{j/2}} \sup_{j_2=0,\dots,j} \|e^{-K\tau\nu^{1/2}(j+1)} B_{j_2}\partial_X^{j-j_2} F\|_{L^p_\tau(0,1/(K\nu^{1/2}); L^2_{X,Y})}, \qquad B_{j_2} = \chi_\nu^{j_2}\partial_Y^{j_2}. \tag{2-7}$$

We state our key assumptions in terms of V and Ω.

Assumptions. (i) *Divergence-free and Dirichlet condition on* V:

$$\partial_X V_1 + \partial_Y V_2 = 0, \qquad V|_{Y=0} = 0. \tag{2-8}$$

Moreover, there exist constants $C_* \ge 1$ *and* $C_0^*, C_1^*, C_2^* > 0$ *such that the following statements hold for any* $\nu \in (0,1]$ *and* $K \ge 1$:

(ii) *Almost Gevrey* L^∞ *bounds for* V *and* $\nabla\Omega$: *For any* $\kappa \in (0,1]$, *we have*

$$\begin{aligned}\sum_{j=0}^{\nu^{-1/2}} \frac{1}{(j!)^{3/2}\nu^{j/2}} \sup_{j_2=0,\dots,j} \Bigg(&\|e^{-K\tau\nu^{1/2}j} B_{j_2}\partial_X^{j-j_2} V_1\|_{L^\infty_{\tau,X,Y}} + \kappa\left\| e^{-K\tau\nu^{1/2}j}\frac{\partial_X^j V_2}{\chi_\nu}\right\|_{L^\infty_{\tau,X,Y}} \\ &+ \nu^{-\frac12}(j+1)^{\frac12}\|e^{-K\tau\nu^{1/2}j} B_{j_2}\partial_X^{j-j_2}\partial_X V_1\|_{L^\infty_{\tau,X,Y}} + (j+1)^{\frac12}\|e^{-K\tau\nu^{1/2}j} B_{j_2}\partial_X^{j-j_2}\partial_Y V_1\|_{L^\infty_{\tau,X,Y}} \\ &+ \nu^{-\frac12}\left\|\frac{1+Y}{1+\nu^{1/2}Y} e^{-K\tau\nu^{1/2}j} B_{j_2}\partial_X^{j-j_2}\partial_X\Omega\right\|_{L^\infty_{\tau,X,Y}} + \left\|\left(\frac{1+Y}{1+\nu^{1/2}Y}\right)^2 e^{-K\tau\nu^{1/2}j} B_{j_2}\partial_X^{j-j_2}\partial_Y\Omega\right\|_{L^\infty_{\tau,X,Y}}\Bigg) \le C_0^*.\end{aligned}$$

Here $L^\infty_{\tau,X,Y} = L^\infty_\tau(0, 1/(K\nu^{1/2}); L^\infty_{X,Y})$.

(iii) *Derivative bounds for V and Ω*: *We have*

$$\|V\|_{L^\infty_{\tau,X,Y}} + \nu^{-\frac12}\|\partial_X V\|_{L^\infty_{\tau,X,Y}} + \left\|\frac{1+Y}{1+\nu^{1/2}Y}\partial_Y V_1\right\|_{L^\infty_{\tau,X,Y}} + \nu^{-\frac12}\left\|\frac{1+Y}{1+\nu^{1/2}Y}\partial_X\Omega\right\|_{L^\infty_{\tau,X,Y}} + \left\|\left(\frac{1+Y}{1+\nu^{1/2}Y}\right)^2\partial_Y\Omega\right\|_{L^\infty_{\tau,X,Y}} + \nu^{-\frac12}\left\|\left(\frac{Y}{1+\nu^{1/2}Y}\right)^2\partial_\tau\partial_Y\Omega\right\|_{L^\infty_{\tau,X,Y}} + \nu^{-\frac12}\left\|\frac{Y(1+Y)}{(1+\nu^{1/2}Y)^2}\partial^2_{XY}\Omega\right\|_{L^\infty_{\tau,X,Y}} + \left\|\frac{Y(1+Y)^2}{(1+\nu^{1/2}Y)^3}\partial^2_Y\Omega\right\|_{L^\infty_{\tau,X,Y}} \le C_1^*. \tag{2-9}$$

(iv) *Monotonicity of Ω*: *Set $\rho(Y) = C_*((1+Y/\nu^{1/4})^{-2} + \nu^{1/2}(1+Y)^{-2} + \nu)$. Then we have*

$$\partial_Y\Omega + \rho \ge 0 \tag{2-10}$$

and

$$\nu^{-\frac12}\left\|\frac{Y}{1+\nu^{1/2}Y}\frac{\partial^2_{XY}\Omega}{\sqrt{\partial_Y\Omega+2\rho}}\right\|_{L^\infty_{\tau,X,Y}} + \left\|\frac{Y(1+Y)}{(1+\nu^{1/2}Y)^2}\frac{\partial^2_Y\Omega}{\sqrt{\partial_Y\Omega+2\rho}}\right\|_{L^\infty_{\tau,X,Y}} \le C_2^*. \tag{2-11}$$

Remark 2.2 (link between the Prandtl expansions and the Assumptions). Let us explain how the set of assumptions above relates to the Prandtl expansions as given in (1-2).

(i) The divergence-free and Dirichlet conditions are satisfied by Prandtl expansions of type (1-2). Fields $U^{E,i}$ solve Euler or linearized Euler equations, while fields $V^{\mathrm{bl},i}$ solve Prandtl or linearized Prandtl equations: in both cases, they are divergence-free. Moreover, they are constructed alternatively in order to satisfy the Dirichlet boundary condition: once $U^{E,i}$ is constructed, $V^{\mathrm{bl},i}$ is constructed so that

$$U_1^{E,i}|_{y=0} + V_1^{\mathrm{bl},i}|_{Y=0} = 0.$$

Then, $U^{E,i+1}$ is constructed by solving an Euler-type equation with the nonpenetration condition

$$U_2^{E,i+1}|_{y=0} + V_2^{\mathrm{bl},i}|_{Y=0} = 0.$$

More precisely, one can construct $(U^{E,i}, V^{\mathrm{bl},i})$ in this way for $i \le N-1$ and conclude with

$$U^{E,N}(t,x,y) := (0, -V_2^{\mathrm{bl},N-1}(t,x,0)), \qquad V^{\mathrm{bl},N} := 0.$$

(ii) Assumption (ii) amounts essentially to a Gevrey $\frac32$ bound on solutions $U^{E,i}$ and $V^{\mathrm{bl},i}$ of Euler-like and Prandtl-like equations, respectively. Such solutions exist locally in time. For the Euler equations, we refer to [Kukavica and Vicol 2011]. For the Prandl equations, as mentioned before, the works [Kukavica and Vicol 2013; Sammartino and Caflisch 1998a] provide local-in-time solutions for analytic data. These local solutions being analytic, they belong to the Gevrey class $\frac32$. More recently, Gevrey local-in-time well-posedness of the Prandtl equation has been established in [Dietert and Gérard-Varet 2019] (see [Gerard-Varet and Masmoudi 2015; Li and Yang 2020] for preliminary partial results). Also, if v is given by (1-2), as $V_2(\tau, X, Y) = v_2(t, x, y)$ is zero at the boundary $Y = 0$, we can write

$$V_2 = \int_0^Y \partial_Y V_2 \approx \int_0^Y (\nu^{\frac12}(\partial_y V_2^{E,0} + \partial_Y V_2^{\mathrm{bl},0}) + \cdots) = O(\nu^{\frac12}Y) = O\Big(\frac{1}{\kappa}\chi_\nu(Y)\Big) \quad \text{at } Y = 0,$$

so that $(1/\kappa)(V_2/\chi_\nu)$ is under control as required in (ii).

(iii) Again, Assumption (iii) is satisfied by classical Prandtl expansions of type (1-2). To check that, one has to keep in mind that $\partial_\tau \sim \nu^{1/2}\partial_t$, $\partial_X \sim \nu^{1/2}\partial_x$, so that for Prandtl expansions, which depend smoothly on t and x, any τ- or X-derivative allows to gain $\nu^{1/2}$. This explains for instance the factor $\nu^{-1/2}$ in front of the second and fourth terms of (2-9), related to $\partial_X V$ and $\partial_X \Omega$. In the same spirit, as $\partial_Y \sim \nu^{1/2}\partial_y$, for the Euler part of the Prandtl expansion (which depends smoothly on y), any Y-derivative allows to gain $\nu^{1/2}$. This remark does not apply to the boundary layer part of the expansion, as it depends genuinely on Y. Still, this part has good decay in Y (typically like e^{-Y} or $(1+Y)^{-N}$ for large Y). This is coherent with the weights $(1+Y)/(1+\nu^{1/2}Y)$ or $Y/(1+\nu^{1/2}Y)$ that can be found in (2-9) in front of terms with Y derivatives: outside the boundary layer ($Y \gg 1$), it yields a gain of $\nu^{1/2}$, but in the boundary layer ($Y \sim 1$), it yields some decay information on the boundary layer terms.

(iv) In the case when v is given by Prandtl expansions of type (1-2),

$$\partial_Y \Omega = \partial^2_{XY} V_2 - \partial^2_Y V_1 = -\partial^2_Y V_1^{\mathrm{bl}} + O(\nu) + O(\sqrt{\nu}(1+Y)^{-2})$$

Here, the $O(\nu)$ comes from the Euler part of the Prandtl expansion. The $O(\sqrt{\nu}(1+Y)^{-2})$ corresponds to the boundary layer profiles $V^{\mathrm{bl},i}$, $i \geq 1$. The last two terms in the definition of the weight ρ allow to control them for C_* large enough. Hence, condition (2-10) is essentially a (nonstrict) concavity condition on the leading term of the Prandtl boundary layer, $V^{\mathrm{bl}} := V^{\mathrm{bl},0}$. Moreover, by the addition of the sublayer term $(1+(Y/\nu^{1/4}))^{-2}$ in the definition of ρ, we allow any sign for $\partial^2_Y V^P_{0,1}$ in the sublayer $0 \leq Y \leq O(\nu^{1/4})$, and the concavity is only needed for $Y \geq O(\nu^{1/4})$. In the original variables this sublayer is of the order $O(\nu^{3/4})$, which is typical order of Kolmogorov dissipation length in the theory of turbulence.

As regards (2-11), we notice that for Prandtl expansions:

$$\partial^2_{XY}\Omega = -\partial_X \partial^2_Y V_1^{\mathrm{bl}} + O(\nu^{\frac{3}{2}}) + O(\nu(1+Y)^{-2}) \quad \text{and} \quad \partial^2_Y \Omega = -\partial^3_Y V_1^{\mathrm{bl}} + + O(\nu^{\frac{3}{2}}) + O(\nu^{\frac{1}{2}}(1+Y)^{-2}).$$

Hence, by taking into account the bound $1/\sqrt{\partial_Y \Omega + 2\rho} \leq 1/(C_* \nu^{1/2})$, the condition (2-11) is essentially verified if V_1^{bl} satisfies

$$\nu^{-\frac{1}{2}} \left\| \frac{Y \partial_X \partial^2_Y V_1^{\mathrm{bl}}}{\sqrt{-\partial^2_Y V_1^{\mathrm{bl}} + 2C_*(1+Y/\nu^{1/4})^{-2}}} \right\|_{L^\infty_{\tau,X,Y}} + \left\| \frac{Y(1+Y)\partial^3_Y V_1^{\mathrm{bl}}}{\sqrt{-\partial^2_Y V_1^{\mathrm{bl}} + 2C_*(1+Y/\nu^{1/4})^{-2}}} \right\|_{L^\infty_{\tau,X,Y}} \leq C < \infty.$$

In the next section, we will explain the general strategy for the proof of our main stability theorem. More precisely, we will briefly describe our stability analysis of the linearized equation (1-6) for f a given force. This is the core of our paper: the transition from linear to nonlinear stability is more standard. As explained before, we shall work with the rescaled variables (τ, X, Y). We set

$$W(\tau, X, Y) := w(t, x, y), \quad F(\tau, X, Y) := \sqrt{\nu} f(t, x, y), \quad W_0(X, Y) := w_0(x, y)$$

(and still $V(\tau, X, Y) = v(t, x, y)$). System (1-6) becomes

$$\begin{aligned} \partial_\tau W - \nu^{\frac{1}{2}} \Delta W + \nabla Q + V \cdot \nabla W + W \cdot \nabla V = F, &\quad \tau > 0, \quad X \in \mathbb{T}_\nu, \quad Y > 0, \\ \nabla \cdot W = 0, &\quad \tau \geq 0, \quad X \in \mathbb{T}_\nu, \quad Y > 0, \\ W|_{Y=0} = 0, \quad W|_{\tau=0} = W_0. & \end{aligned} \tag{2-12}$$

The main result on this linear system is:

Theorem 2.3. *Suppose that the Assumptions hold. Then there exists* $\kappa_0 \in (0, 1]$ *such that the following statement holds for any* $\kappa \in (0, \kappa_0]$. *There exists* $K_0 = K_0(\kappa, C_*, C_j^*) \geq 1$ *such that if* $K \geq K_0$ *then the system (2-12) admits a unique solution* $W \in C([0, 1/(K\nu^{1/2})]; H^1_{0,\sigma}(\mathbb{T}_\nu \times \mathbb{R}_+))$ *satisfying*

$$|||W|||_\infty + |||\mathrm{rot}\, W|||_\infty \leq C((\nu^{-\frac{1}{2}} + K^{\frac{1}{2}}\nu^{-\frac{1}{4}})[\|W_0\|] + \nu^{-1}[\|\mathrm{rot}\, W_0\|] + \nu^{-\frac{5}{4}}|||F|||_2). \tag{2-13}$$

Here $\mathrm{rot}\, W = \partial_X W_2 - \partial_Y W_1$ *and*

$$[\|W_0\|] = \sum_{j=0}^{\nu^{-1/2}} \frac{1}{(j!)^{3/2}\nu^{j/2}} \sup_{j_2=0,\dots,j} \|B_{j_2}\partial_X^{j-j_2} W_0\|_{L^2_{X,Y}},$$

and C *is a universal constant.*

As a consequence, we have the following result in the original variables. Note that, from $F(\tau, X, Y) = \nu^{1/2} f(t, x, y)$, we have $\nu^{-5/4}|||F|||_2 = \nu^{-3/2}\|f\|_{G^2_{3/2}}$.

Theorem 2.4. *Suppose that the Assumptions hold. Then there exists* $\kappa_0 \in (0, 1]$ *such that the following statement holds for any* $\kappa \in (0, \kappa_0]$. *There exists* $K_0 = K_0(\kappa, C_*, C_j^*) \geq 1$ *such that if* $K \geq K_0$ *then the system (1-6) admits a unique solution* $w \in C([0, 1/K]; H^1_{0,\sigma}(\mathbb{T} \times \mathbb{R}_+))$ *satisfying*

$$\|w\|_{G^\infty_{3/2}} + \nu^{\frac{1}{2}}\|\mathrm{rot}\, w\|_{G^\infty_{3/2}} \leq C\nu^{-\frac{1}{2}}((1 + K^{\frac{1}{2}}\nu^{\frac{1}{4}})[|w_0|]_{G_{3/2}} + [|\mathrm{rot}\, w_0|]_{G_{3/2}} + \nu^{-\frac{1}{2}}\|f\|_{G^2_{3/2}}). \tag{2-14}$$

Here $\mathrm{rot}\, w = \partial_x w_2 - \partial_y w_1$ *and*

$$[|w_0|]_{G_{3/2}} = \sum_{j=0}^{\nu^{-1/2}} \frac{1}{(j!)^{3/2}} \sup_{j_2=0,\dots,j} \|\beta_{j_2}\partial_x^{j-j_2} w_0\|_{L^2_{x,y}},$$

and C *is a universal constant.*

3. General strategy

Estimates on system (2-12) will be performed at the level of the vorticity field $\omega = \mathrm{rot}\, W := \partial_X W_2 - \partial_Y W_1$:

$$\begin{gathered}(\partial_\tau + V\cdot\nabla - \nu^{\frac{1}{2}}\Delta)\omega + W\cdot\nabla\Omega = \mathrm{rot}\, F, \quad \nabla\cdot W = 0, \quad \tau > 0, \ X \in \mathbb{T}_\nu, \ Y > 0, \\ W|_{Y=0} = 0.\end{gathered} \tag{3-1}$$

We recall that $\tau = \nu^{-1/2}t$: the point is to get estimates that are valid over time intervals of size $\nu^{-1/2}$, which is difficult due to the stretching term $W\cdot\nabla\Omega$. Classical estimates and Gronwall's lemma would only yield a control on time intervals $O(1)$. We have to use both our Gevrey functional framework and concavity condition.

Actually, several difficulties are already captured by the toy model

$$\begin{gathered}(\partial_\tau - \nu^{\frac{1}{2}}\Delta)\omega + W_2\partial_Y\Omega = 0, \quad \omega = \mathrm{rot}\, W, \quad \nabla\cdot W = 0, \quad \tau > 0, \ X \in \mathbb{T}_\nu, \ Y > 0, \\ W|_{Y=0} = 0,\end{gathered} \tag{3-2}$$

where $\Omega = \Omega(Y)$ (for simplicity, we assume no dependence on τ and X). We shall stick to this model for what follows.

In the case of the inviscid equation

$$\partial_\tau \omega + W_2 \partial_Y \Omega = 0, \quad \omega = \mathrm{rot}\, W, \quad \nabla \cdot W = 0, \quad W_2|_{Y=0} = 0$$

under the strict sign condition $\partial_Y \Omega \geq C > 0$, a trick that goes back to [Grenier 2000b] is to test the equation against $\omega/(\partial_Y \Omega)$. By the cancellation

$$\int W_2 \partial_Y \Omega \frac{\omega}{\partial_Y \Omega} = \int W_2 \,\mathrm{rot}\, W = -\frac{1}{2} \int \partial_X |W|^2 = 0,$$

one can obtain a uniform-in-time control on the weighted quantity $\|\omega/\sqrt{\partial_Y \Omega}\|_{L^2} \sim \|\omega\|_{L^2}$. However, back to the model (3-2), we are facing two difficulties:

(1) Inspired by the case of Prandtl layers, we must consider situations where $\partial_Y \Omega$ vanishes or even becomes slightly negative; see Assumption (iv).

(2) Even in the simpler case $\partial_Y \Omega \geq C > 0$, the weighted estimate above is not compatible with the introduction of viscosity and no-slip conditions.

We recall that these difficulties are not purely technical, as no uniform-in-ν stability estimate is expected below Gevrey $\frac{3}{2}$ regularity. To overcome these issues, we shall proceed in two steps.

3A. ***First step*: *Gevrey estimates for artificial boundary conditions.*** The first step consists in deriving Gevrey bounds for the same equation, but with pure slip instead of no-slip conditions. For the real vorticity equation, this will be performed in Section 4. For our toy model, this means that we consider

$$\begin{gathered}(\partial_\tau - \nu^{\frac{1}{2}} \Delta)\omega + W_2 \partial_Y \Omega = 0, \quad \omega = \mathrm{rot}\, W, \quad \nabla \cdot W = 0, \quad \tau > 0, \quad X \in \mathbb{T}_\nu, \quad Y > 0, \\ W_2|_{Y=0} = \omega|_{Y=0} = 0. \end{gathered} \tag{3-3}$$

The main point in this change of boundary conditions is that difficulty (2) mentioned above disappears: the Dirichlet condition on ω goes well with integration by parts, and in the case $\partial_Y \Omega \geq C > 0$, one can achieve again some good control on $\|\omega/\sqrt{\partial_Y \Omega}\|_{L^2}$. Still, we have to explain how to obtain stability under the less stringent condition in Assumption (iv). Here, we need Gevrey regularity. Let us for simplicity forget about Y-derivatives, which are not important for the toy model, and set

$$\omega^j := e^{-K\tau\nu^{1/2}(j+1)} \partial_X^j \omega, \quad W^j := e^{-K\tau\nu^{1/2}(j+1)} \partial_X^j W.$$

The point is to obtain a bound on

$$\sum_{j \leq \nu^{-1/2}} \frac{1}{(j!)^{3/2} \nu^{j/2}} \|\omega^j\|_{L^2_{X,Y}}.$$

As $\Omega = \Omega(Y)$, the equation satisfied by ω^j is

$$(K\nu^{1/2}(j+1) + \partial_\tau - \nu^{\frac{1}{2}} \Delta)\omega^j + W_2^j \partial_Y \Omega = 0. \tag{3-4}$$

Roughly, the idea is to control a weighted Gevrey norm of the form

$$\sum_{j \leq \nu^{-1/2}} \frac{1}{(j!)^{3/2} \nu^{j/2}} \left\| \frac{\omega^j}{\sqrt{\partial_Y \Omega + 2\rho_j}} \right\|_{L^2_{X,Y}},$$

where ρ_j is added to compensate for possible degeneracies of $\partial_Y \Omega$. Testing (3-4) against $\omega_j/(\partial_Y\Omega+2\rho_j)$, we find

$$K\nu^{\frac12}(j+1)\left\|\frac{\omega^j}{\sqrt{\partial_Y\Omega+2\rho_j}}\right\|_{L^2}^2+\frac12\frac{d}{d\tau}\left\|\frac{\omega^j}{\sqrt{\partial_Y\Omega+2\rho_j}}\right\|_{L^2}^2+\nu^{\frac12}\left\|\frac{\nabla\omega^j}{\sqrt{\partial_Y\Omega+2\rho_j}}\right\|_{L^2}^2$$
$$=-\nu^{\frac12}\int\nabla\frac{1}{\partial_Y\Omega+2\rho_j}\cdot\nabla\omega^j\omega^j-\int W_2^j\partial_Y\Omega\frac{\omega^j}{\partial_Y\Omega+2\rho_j}$$
$$=\nu^{\frac12}\int\frac{\nabla\partial_Y\Omega}{(\partial_Y\Omega+2\rho_j)^2}\cdot\nabla\omega^j\omega^j+\nu^{\frac12}\int\frac{\nabla\rho_j}{(\partial_Y\Omega+2\rho_j)^2}\cdot\nabla\omega^j\omega^j+\int W_2^j\frac{2\rho_j}{\partial_Y\Omega+2\rho_j}\omega^j, \quad (3\text{-}5)$$

where we used again the cancellation property $\int W_2^j\omega^j=0$. One must then choose ρ_j so that the three terms at the right are controlled by the left-hand side for K large enough. Roughly, this can be achieved by taking ρ_j in the form $\rho_j(Y)\approx\rho+(1+\lambda_jY)^{-2}$, $\lambda_j:=(j+1)^{1/2}$. To give an idea of why it works, let us consider the first and last terms. As regards the first one, we write

$$\nu^{\frac12}\int\frac{\nabla\partial_Y\Omega}{(\partial_Y\Omega+2\rho_j)^2}\cdot\nabla\omega^j\omega^j=\nu^{\frac12}\int_{\{Y\ge1/\lambda_j\}}\frac{1}{Y\sqrt{\partial_Y\Omega+2\rho_j}}\frac{Y\nabla\partial_Y\Omega}{\sqrt{\partial_Y\Omega+2\rho_j}}\cdot\frac{\nabla\omega^j}{\sqrt{\partial_Y\Omega+2\rho_j}}\frac{\omega^j}{\sqrt{\partial_Y\Omega+2\rho_j}}$$
$$+\nu^{\frac12}O\left(\left\|\frac{\nabla\omega^j}{\sqrt{\partial_Y\Omega+2\rho_j}}\right\|_{L^2}\left\|\frac{\omega^j}{\sqrt{\partial_Y\Omega+2\rho_j}}\right\|_{L^2}\right).$$

The second term on the right side corresponds to the contribution of the region $Y\le1/\lambda_j$, for which the weight $\partial_Y\Omega+2\rho_j$ is bounded from below and raises no issue (we further assumed here that $\partial_Y\nabla\Omega$ for the sake of brevity). As regards the first term on the right side, for all $Y\ge1/\lambda_j$, we use the bounds

$$\frac{1}{Y\sqrt{\partial_Y\Omega+2\rho_j}}\le\frac{1}{Y\sqrt{2\rho_j}}\le C\lambda_j\quad\text{and}\quad\frac{|Y\nabla\partial_Y\Omega|}{\sqrt{\partial_Y\Omega+2\rho_j}}\le\frac{|Y\nabla\partial_Y\Omega|}{\sqrt{\partial_Y\Omega+2\rho}}\le C,$$

where we used Assumption (iv). We end up with

$$\nu^{\frac12}\int\frac{\nabla\partial_Y\Omega}{(\partial_Y\Omega+2\rho_j)^2}\cdot\nabla\omega^j\omega^j\le C\nu^{\frac12}\lambda_j\left\|\frac{\nabla\omega^j}{\sqrt{\partial_Y\Omega+2\rho_j}}\right\|_{L^2}\left\|\frac{\omega^j}{\sqrt{\partial_Y\Omega+2\rho_j}}\right\|_{L^2},$$

which is absorbed by the left-hand side under the constraint $\lambda_j\lesssim(j+1)^{1/2}$. As regards the third term on the right side of (3-5), we use the inequality

$$\frac{\rho_j}{\sqrt{\partial_Y\Omega+2\rho_j}}\le\frac{\sqrt{\rho_j}}{\sqrt2}\le C\left(\sqrt\nu+\frac{1}{\lambda_jY}\right)$$

to obtain

$$\int W_2^j\frac{2\rho_j}{\partial_Y\Omega+2\rho_j}\omega^j\le C\sqrt\nu\|W_2^j\|_{L^2}\left\|\frac{\omega^j}{\sqrt{\partial_Y\Omega+2\rho_j}}\right\|_{L^2}+\frac{C}{\lambda_j}\left\|\frac{W_2^j}{Y}\right\|_{L^2}\left\|\frac{\omega^j}{\sqrt{\partial_Y\Omega+2\rho_j}}\right\|_{L^2}$$
$$\le C\left(\sqrt\nu\|W_2^j\|_{L^2}+\frac{1}{\lambda_j}\|\partial_YW_2^j\|_{L^2}\right)\left\|\frac{\omega^j}{\sqrt{\partial_Y\Omega+2\rho_j}}\right\|_{L^2},$$

where the second line comes from Hardy's inequality. Using that $\|\partial_Y W_2^j\|_{L^2} = \|\partial_X W_1^j\|_{L^2} \approx \|W_1^{j+1}\|_{L^2}$, we have, for any sequence (a_j),

$$\sum_j \frac{1}{(j!)^{3/2}\nu^{j/2}} a_j \|\partial_Y W_2^j\|_{L^2} \approx \sum_j \frac{1}{(j!)^{3/2}\nu^{j/2}} \nu^{\frac{1}{2}}(j+1)^{\frac{3}{2}} a_{j-1}\|W_1^j\|_{L^2}.$$

In other words, at Gevrey $\frac{3}{2}$ regularity, $a_j\|\partial_Y W_2^j\|_{L^2}$ behaves like $\nu^{1/2}(j+1)^{3/2}a_{j-1}\|W_1^j\|_{L^2}$. Combining this with a control of $\|W^j\|_{L^2}$ by $\|\omega_j/\sqrt{\partial_Y\Omega+2\rho_j}\|_{L^2}$ and with a precise statement to be given in Section 4, the previous bound is in the same spirit as

$$\int W_2^j \frac{2\rho_j}{\partial_Y\Omega+2\rho_j}\omega^j \le C\frac{\nu^{1/2}(j+1)^{3/2}}{\lambda_{j-1}}\left\|\frac{\omega^j}{\sqrt{\partial_Y\Omega+2\rho_j}}\right\|_{L^2}^2,$$

which allows a control by the left-hand side of (3-5) as soon as $(j+1)^{1/2} \lesssim \lambda_j$. Hence the choice $\lambda_j = (j+1)^{1/2}$.

Of course, the elements above provide only glimpses of the approach carried out in the first step of our stability study. The full study of the vorticity equation with artificial boundary conditions is given in Section 4.

3B. ***Recovery of the right boundary conditions.*** We give again a few elements on the toy model (3-2). The analysis of the complete model is carried in Section 5. After the first step, one has a solution of system (3-3), with the same initial condition and same boundary condition $W_2|_{Y=0} = 0$ as in (3-2) but not the same boundary condition on the tangential velocity: $h := W_1|_{Y=0} \neq 0$. Note that by the first step and the trace theorem, one is able to get a Gevrey bound for h: as shown rigorously in the next sections, one may get an estimate of the form

$$|||h|||_{\mathrm{bc}} := \sum_{j\le\nu^{-1/2}} \frac{1}{(j!)^{3/2}\nu^{j/2}}\|h^j\|_{L^2((0,1/(K\nu^{1/2}));L^2_X)} \le \frac{C}{K^{1/4}}\left(\|W_0\|_{L^2} + C\sum_{j\le\nu^{-1/2}}\frac{1}{(j!)^{3/2}\nu^{j/2}}\|\omega_0^j\|_{L^2_{X,Y}}\right),$$

where W_0 and $\omega_0 := \operatorname{rot} W_0$ are the initial data for the velocity and vorticity, respectively.

Working in Gevrey regularity, the point is then to solve

$$\begin{aligned}(\partial_\tau - \nu^{\frac{1}{2}}\Delta)\omega + W_2\partial_Y\Omega = 0, \quad \omega = \operatorname{rot} W, \quad \nabla\cdot W = 0, \quad \tau > 0, \quad X\in\mathbb{T}_\nu, \quad Y>0,\\ W_2|_{Y=0} = 0, \quad W_1|_{Y=0} = h, \quad W|_{t=0} = 0.\end{aligned} \tag{3-6}$$

The main idea is to use the following scheme:

Step (a): We solve the approximate Stokes equation

$$\begin{aligned}(\partial_\tau - \nu^{\frac{1}{2}}\Delta)\omega = 0, \quad \omega = \operatorname{rot} W, \quad \nabla\cdot W = 0,\\ W_2|_{Y=0} = 0, \quad W_1|_{Y=0} = h, \quad W|_{t=0} = 0\end{aligned} \tag{3-7}$$

and obtain in this way a solution $W_a = (W_{a,1}, W_{a,2}) = W_a[h]$.

Step (b): We correct the stretching term created by the previous approximation by considering the full equation with artificial boundary condition:

$$\begin{aligned}(\partial_\tau - \nu^{\frac{1}{2}}\Delta)\omega + W_2\partial_Y\Omega = -W_{a,2}\partial_Y\Omega, \quad \omega = \operatorname{rot} W, \quad \nabla\cdot W = 0,\\ W_2|_{Y=0} = 0, \quad \omega|_{Y=0} = 0, \quad W|_{t=0} = 0.\end{aligned} \tag{3-8}$$

We denote by $W_b = W_b[h]$ the solution of such a system. It can be seen as a functional of h through W_a.

Step (c): At the end of the Steps (a) and (b), the function $W - W_a - W_b$ solves formally the same system as W, replacing h by $R_{\mathrm{bc}}[h] := -W_{b,1}[h]|_{Y=0}$. The point is to show that, for K large enough,

$$|||R_{\mathrm{bc}}[h]|||_{\mathrm{bc}} \le \tfrac{1}{2}|||h|||_{\mathrm{bc}}, \tag{3-9}$$

which allows us to solve (3-6) by iteration.

Obviously, to establish (3-9), one must have careful Gevrey stability estimates for systems (3-7) and (3-8). The estimates for (3-8) follow from the same ideas as those described in Section 3A to treat (3-3) (the initial condition is just replaced by a source term). As regards (3-7), the initial data being zero, one can take the Laplace transform in τ and the Fourier transform in X and solve explicitly the resulting ordinary differential equation in Y. It leads to sharp L^2 estimates on W and its derivatives on the Fourier–Laplace side, which transfer to L^2 estimates in the physical space by the Plancherel theorem.

All the analysis in the framework of the vorticity equation is provided in Section 5. In this setting, the iteration scheme mentioned above has to be modified, because the advection term creates extra difficulties. Namely, one has to add an intermediate step between Steps (a) and (b) above; see Section 5 for details.

Of course, we have indicated here key ideas for the stability analysis of the linearized system (1-6). One has then to go from these estimates to the nonlinear Theorem 2.1. This will be achieved in Section 7. Finally we introduce the simplified notation

$$\|f\| = \|f\|_{L^2_{X,Y}}, \quad \langle f, g\rangle = \langle f, g\rangle_{L^2_{X,Y}}$$

for convenience.

4. Vorticity estimate under artificial boundary condition

In accordance with the strategy described in the previous section, we consider here the solution to the system

$$\begin{aligned}-\nu^{\frac{1}{2}}\Delta\omega + \partial_\tau\omega + V\cdot\nabla\omega + W\cdot\nabla\Omega = \operatorname{rot} F + G, \quad \omega = \operatorname{rot} W, \quad \nabla\cdot W = 0,\\ \tau > 0, \quad X \in \mathbb{T}_\nu, \quad Y > 0,\\ W_2|_{Y=0} = \omega|_{Y=0} = 0, \quad W|_{\tau=0} = W_0.\end{aligned} \tag{4-1}$$

Here a given force term $G \in L^2(0, 1/(K\nu^{1/2}); L^2 \cap \dot{H}^{-1})$, where $\dot{H}^{-1}$ is the dual space of the homogeneous Sobolev space $\dot{H}^1_0(\mathbb{T}_\nu \times \mathbb{R}_+)$ (the subscript 0 means the zero boundary trace), is also introduced for later use. As usual, the velocity W is given in terms of the stream function ϕ, i.e.,

$$W = \nabla^\perp \phi = \begin{pmatrix} \partial_Y \phi \\ -\partial_X \phi \end{pmatrix}, \tag{4-2}$$

and $\phi \in \dot{H}^1_0(\mathbb{T} \times \mathbb{R}_+)$ is the unique solution to the Poisson equation $-\Delta\phi = \omega$ with the zero Dirichlet boundary condition $\phi|_{Y=0} = 0$. This formulation is well defined, and the unique solvability of (4-1) in the class $\omega \in C([0, 1/(K\nu^{1/2})]; L^2 \cap \dot{H}^{-1}) \cap L^2(0, 1/(K\nu^{1/2}); H^1_0)$ is shown without difficulty (under the regularity condition we impose on V, Ω, and the forces). The reason why the regularity $\omega \in C([0, 1/(K\nu^{1/2})]; \dot{H}^{-1})$ is preserved is that the term $-V \cdot \nabla\omega - W \cdot \nabla\Omega + \operatorname{rot} F + G$ has a bound in $\dot{H}^{-1}$ (in space) such as

$$\|-V \cdot \nabla\omega - W \cdot \nabla\Omega + \operatorname{rot} F + G\|_{L^2\dot{H}^{-1}} \le \|V\|_{L^\infty}\|\omega\|_{L^2L^2} + \|\Omega\|_{L^\infty}\|W\|_{L^2L^2} + \|F\|_{L^2L^2} + \|G\|_{L^2\dot{H}^{-1}}$$

and also $\|\operatorname{rot} W_0\|_{\dot{H}^{-1}} \le \|W_0\|_{L^2}$ for the initial vorticity. Hence the space $C([0, 1/(K\nu^{1/2})]; \dot{H}^{-1})$ for the vorticity field and the regularity $\phi(\tau, \cdot) \in \dot{H}^1_0(\mathbb{T} \times \mathbb{R}_+)$ for the stream function are compatible in our setting. By the parabolic regularity of the system, the ν-dependent estimates for the higher-order derivatives are easily obtained, and thus, our main interest here is the *uniform* estimate in time and ν. To this end, for $\boldsymbol{j} = (j_1, j_2)$ with $j_1 + j_2 = j$, we set

$$\omega^{\boldsymbol{j}} = e^{-K\tau\nu^{1/2}(j+1)} B_{j_2}\partial_X^{j_1}\omega, \quad (\nabla\phi)^{\boldsymbol{j}} = e^{-K\tau\nu^{1/2}(j+1)} B_{j_2}\partial_X^{j_1}\nabla\phi, \tag{4-3}$$

and similarly, $(\Delta\omega)^{\boldsymbol{j}} = e^{-K\tau\nu^{1/2}(j+1)} B_{j_2}\partial_X^{j_1}\Delta\omega$. We also set

$$V^{\boldsymbol{j}} = e^{-K\tau\nu^{1/2}j} B_{j_2}\partial_X^{j_1} V, \quad (\nabla\Omega)^{\boldsymbol{j}} = e^{-K\tau\nu^{1/2}j} B_{j_2}\partial_X^{j_1}\nabla\Omega. \tag{4-4}$$

From the first equation of (4-1), we observe that $\omega^{\boldsymbol{j}}$ satisfies, by setting $\boldsymbol{l} = (l - l_2, l_2)$,

$$\begin{aligned} -\nu^{\frac12}(\Delta\omega)^{\boldsymbol{j}} &+ (\partial_\tau + K\nu^{\frac12}(j+1) + V\cdot\nabla)\omega^{\boldsymbol{j}} + (\nabla^\perp\phi)^{\boldsymbol{j}} \cdot \nabla\Omega \\ &= -V_2[B_{j_2}, \partial_Y] e^{-K\tau\nu^{1/2}(j+1)}\partial_X^{j_1}\omega \\ &\quad - \sum_{l=0}^{j-1} \sum_{\max\{0, l+j_2-j\} \le l_2 \le \min\{l, j_2\}} \binom{j_2}{l_2}\binom{j-j_2}{l-l_2} V^{\boldsymbol{j}-\boldsymbol{l}} \cdot (\nabla\omega)^{\boldsymbol{l}} \\ &\quad - \sum_{l=0}^{j-1} \sum_{\max\{0, l+j_2-j\} \le l_2 \le \min\{l, j_2\}} \binom{j_2}{l_2}\binom{j-j_2}{l-l_2} (\nabla^\perp\phi)^{\boldsymbol{l}} \cdot (\nabla\Omega)^{\boldsymbol{j}-\boldsymbol{l}} \\ &\quad + \operatorname{rot} F^{\boldsymbol{j}} - [B_{j_2}, \partial_Y]\partial_X^{j_1} e^{-K\tau\nu^{1/2}(j+1)} F_1 + G^{\boldsymbol{j}}. \end{aligned} \tag{4-5}$$

Here the sum $\sum_{l=0}^{j-1}$ is defined to be 0 for $j = 0$, and the definitions of $F^{\boldsymbol{j}}$ and $G^{\boldsymbol{j}}$ are straightforward.

To simplify notations let us introduce weighted seminorms; for a given nonnegative smooth function $\xi_j = \xi_j(\tau, X, Y)$, we set

$$M_{p,j,\xi_j}[\omega] = \sup_{j_2 = 0,\dots,j} \|\xi_j \omega^{(j-j_2, j_2)}\|_{L^p_\tau(0, 1/(K\nu^{1/2}); L^2_{X,Y})} \tag{4-6}$$

and also set, with the definition $\boldsymbol{\xi} = (\xi_j)_{j=0}^{\infty}$,

$$|||F|||'_{p,\boldsymbol{\xi}} = \sum_{j=0}^{\nu^{-1/2}} \frac{\nu^{1/(2p)}(j+1)^{1/p}}{(j!)^{3/2}\nu^{j/2}} M_{p,j,\xi_j}[F]. \tag{4-7}$$

Note that

$$|||F|||'_{\infty,\mathbf{1}} = |||F|||_{\infty}, \quad \mathbf{1} = (1, 1, \dots). \tag{4-8}$$

The choice of ξ_j is essential in the stability estimate for ω^j. We will take

$$\xi_j = \frac{1}{\sqrt{\partial_Y \Omega + 2\rho_j}}, \tag{4-9}$$

where

$$\rho_j = K^{\frac{1}{4}} C_*(1+(j+1)^{\frac{1}{2}} Y)^{-2} + C_*\left(\left(1+\frac{Y}{\nu^{1/4}}\right)^{-2} + \nu^{\frac{1}{2}}(1+Y)^{-2} + \nu\right). \tag{4-10}$$

See Section 3 for more on the origin of this weight. We also introduce the norm of the boundary trace as

$$|||\partial_Y \phi|_{Y=0}|||_{\mathrm{bc}} = \sum_{j=0}^{\nu^{-1/2}} \frac{\nu^{1/4}(j+1)^{1/2}}{(j!)^{3/2}\nu^{j/2}} \|e^{-K\tau\nu^{1/2}(j+1)} \partial_X^j \partial_Y \phi|_{Y=0}\|_{L^2(0,1/(K\nu^{1/2});L^2_X)}. \tag{4-11}$$

The main result of this section is:

Proposition 4.1. *There exists $\kappa_1 \in (0, 1]$ such that the following statement holds for any $\kappa \in (0, \kappa_1]$. There exists $K_1 = K_1(\kappa, C_*, C_j^*) \geq 1$ such that if $K \geq K_1$ then the system (4-1) admits a unique solution $\omega \in C([0, 1/(K\nu^{1/2})]; L^2 \cap \dot{H}^{-1}) \cap L^2(0, 1/(K\nu^{1/2}); H_0^1)$ satisfying*

$$\begin{aligned} &|||\omega|||'_{\infty,\boldsymbol{\xi}} + K^{\frac{1}{2}}|||\omega|||'_{2,\boldsymbol{\xi}} + K^{\frac{1}{4}}|||\nabla\phi|||'_{2,\mathbf{1}} + K^{\frac{1}{4}}|||\partial_Y\phi|_{Y=0}|||_{\mathrm{bc}} \\ &\quad \leq C\biggl(\|W_0\|_{L^2_{X,Y}} + \nu^{-}[\|\mathrm{rot}\, W_0\|] + (C_2^*+1)\nu^{-\frac{1}{2}}|||F|||'_{2,\tilde{\xi}^{(1)}} \\ &\qquad\qquad + \frac{1}{K^{1/2}\nu^{1/2}}|||G|||'_{2,\tilde{\xi}^{(2)}} + \frac{1}{K^{1/2}\nu^{1/4}}\|G\|_{L^2(0,1/(K\nu^{1/2});\dot{H}^{-1})}\biggr). \end{aligned} \tag{4-12}$$

Here $C > 0$ is a universal constant, while the weight $\tilde{\xi}^{(k)}$ is defined as

$$\tilde{\xi}^{(k)} = \left(\frac{\xi_j}{(j+1)^{k/2}}\right)_{j=0}^{\infty}.$$

Remark 4.2. (1) From the bound $1/\xi_j \leq (C_1^* + 8K^{1/4}C_*)^{1/2}$ in (4-18) below, we have

$$K^{\frac{3}{16}}|||\omega|||'_{2,\mathbf{1}} \leq K^{\frac{3}{16}}(C_1^* + 8K^{\frac{1}{4}}C_*)^{\frac{1}{2}}|||\omega|||'_{2,\boldsymbol{\xi}} \leq K^{\frac{1}{2}}|||\omega|||'_{2,\boldsymbol{\xi}} \tag{4-13}$$

if K is large enough further depending only on C_1^* and C_*. Estimates (4-13) and (4-12) gives the estimate of $K^{3/16}|||\omega|||'_{2,\mathbf{1}}$.

(2) By the definition of (4-7), we have

$$\nu^{-\frac{1}{2}}|||F|||'_{2,\tilde{\xi}^{(1)}} = \nu^{-\frac{1}{4}} \sum_{j=0}^{\nu^{-1/2}} \frac{M_{2,j,\xi_j}[F]}{(j!)^{3/2}\nu^{j/2}}, \quad \nu^{-\frac{1}{2}}|||G|||'_{2,\tilde{\xi}^{(2)}} = \nu^{-\frac{1}{4}} \sum_{j=0}^{\nu^{-1/2}} \frac{M_{2,j,\xi_j}[G]}{(j!)^{3/2}\nu^{j/2}(j+1)^{1/2}}.$$

Since $\xi_j \le 1/\sqrt{\rho_j} \le 1/(C_*\nu^{1/2})$ by the definitions (4-9)–(4-10) with the monotonicity condition (2-10), we have

$$\nu^{-\frac{1}{2}}|||F|||'_{2,\tilde{\xi}^{(1)}} \le \frac{|||F|||_2}{C_*\nu^{3/4}}. \tag{4-14}$$

Before going into the details of the proof of Proposition 4.1, let us give a lemma for the weight ξ_j and ρ_j, which will be used frequently. By the concavity condition on $\partial_Y\Omega$ in Assumption (iv) and the definition of ρ_j we have:

Lemma 4.3. *There exists $C > 0$ such that the following estimates hold for any $j \ge 0$:*

$$\begin{aligned} &\xi_j^2 \le \frac{1}{\rho_j} \le \frac{1}{C_* \max\{K^{1/4}(1+(j+1)^{1/2}Y)^{-2}, \nu\}} && \text{for } Y \ge 0, \\ &\frac{1}{\rho_j} \le \frac{4}{K^{1/4}C_*} && \text{for } 0 \le Y \le (j+1)^{-\frac{1}{2}}. \end{aligned} \tag{4-15}$$

In particular,

$$\left\| \frac{1+\nu^{1/2}Y}{1+Y}\xi_j \right\|_{L^\infty} + \left\| \frac{1+\nu^{1/2}Y}{Y}\xi_j \right\|_{L^\infty(\{Y \ge (j+1)^{-1/2}\})} \le C(j+1)^{\frac{1}{2}}. \tag{4-16}$$

Moreover,

$$\|\rho_j\|_{L^\infty} \le 4K^{\frac{1}{4}}C_*, \quad \left\| \frac{Y\partial_Y\rho_j}{\rho_j} \right\|_{L^\infty} \le 2 \tag{4-17}$$

and

$$\left\| \frac{1}{\xi_j} \right\|_{L^\infty} \le (C_1^* + 8K^{\frac{1}{4}}C_*)^{\frac{1}{2}}, \quad \sup_{j\ge 1} \left\| \frac{\xi_j}{\xi_{j-1}} \right\|_{L^\infty} \le C. \tag{4-18}$$

The proof of Lemma 4.3 is a straightforward consequence of the definitions of ξ_j and ρ_j, so we omit the details.

4A. ***Vorticity estimate for the modified system.*** In this subsection we collect lemmas for the solution to (4-5) and give the estimate for the vorticity. The main result of this subsection is as follows.

Proposition 4.4. *There exists $\kappa_1' \in (0, 1]$ such that the following statement holds for any $\kappa \in (0, \kappa_1']$. There exists $K_1' = K_1'(\kappa, C_*, C_j^*) \ge 1$ such that if $K \ge K_1'$ then the system (4-1) admits a unique solution $\omega \in C([0, 1/(K\nu^{1/2})]; L^2 \cap \dot{H}^{-1}) \cap L^2(0, 1/(K\nu^{1/2}); H_0^1)$ satisfying*

$$\begin{aligned} &|||\nabla\omega|||'_{2,\tilde{\xi}^{(1)}} + |||\omega|||'_{2,\boldsymbol{\xi}} + K^{\frac{1}{2}}|||\omega|||'_{2,\boldsymbol{\xi}} \\ &\qquad \le C\left(\nu^{-\frac{1}{2}}[\|\mathrm{rot}\, W_0\|] + \frac{C_2^*+1}{\nu^{1/2}}|||F|||'_{2,\tilde{\xi}^{(1)}} + \frac{1}{K^{1/2}\nu^{1/2}}|||G|||'_{2,\tilde{\xi}^{(2)}} + |||W|||'_{2,\mathbf{1}} \right). \end{aligned} \tag{4-19}$$

Here $C > 0$ is a universal constant.

Since the unique solvability of the linear system (4-1) itself follows from the standard theory of parabolic equations, we focus on establishing the estimate (4-19). Then the core part of the proof of Proposition 4.4 consists of the calculation of the inner product for each term in (4-5) with $\xi_j^2\omega^{\boldsymbol{j}}$, where $\boldsymbol{j}=(j_1,j_2)$ with $j_1+j_2=j$ and the weight ξ_j is defined as in (4-9). Let us start from the following lemma. The number $\tau_0\in(0,1/(K\nu^{1/2})]$ is taken arbitrarily below.

Lemma 4.5. *There exists $K_{1,1}=K_{1,1}(C_1^*,C_*)\ge 1$ such that if $K\ge K_{1,1}$ then we have*

$$\int_0^{\tau_0}\langle -\nu^{\frac12}(\Delta\omega)^{\boldsymbol{j}},\xi_j^2\omega^{\boldsymbol{j}}\rangle\,d\tau$$
$$\ge \tfrac12\nu^{\frac12}\|\xi_j(\nabla\omega)^{\boldsymbol{j}}\|^2_{L^2(0,\tau_0;L^2_{X,Y})}-C\nu^{\frac12}(\kappa\nu^{\frac12}j_2)^2M_{2,j-1,\xi_{j-1}}[\partial_Y\omega]^2-C(C_2^*+1)\nu^{\frac12}(j+1)\|\xi_j\omega^{\boldsymbol{j}}\|^2_{L^2(0,\tau_0;L^2_{X,Y})}.$$

Here $C>0$ is a universal constant.

Proof. Let us write $\chi'_\nu=(\chi')(\nu^{1/2}Y)=\kappa e^{-\kappa\nu^{1/2}Y}$. We will frequently use the identity

$$[B_{j_2},\partial_Y]=-\nu^{\frac12}j_2\chi'_\nu B_{j_2-1}\partial_Y=-\frac{\nu^{1/2}j_2\chi'_\nu}{\chi_\nu}B_{j_2}. \tag{4-20}$$

Then we observe that

$$(\Delta\omega)^{\boldsymbol{j}}=e^{-K\tau\nu^{1/2}(j+1)}B_{j_2}\partial_X^{j_1}\Delta\omega=\nabla\cdot(\nabla\omega)^{\boldsymbol{j}}-\frac{\nu^{1/2}j_2\chi'_\nu}{\chi_\nu}(\partial_Y\omega)^{\boldsymbol{j}} \tag{4-21}$$

and

$$\nabla\omega^{\boldsymbol{j}}=(\nabla\omega)^{\boldsymbol{j}}+\nu^{\frac12}j_2\chi'_\nu e^{-K\tau\nu^{1/2}}(\partial_Y\omega)^{(j_1,j_2-1)}\boldsymbol{e}_2,\quad \omega^{\boldsymbol{j}}=\chi_\nu e^{-K\tau\nu^{1/2}}(\partial_Y\omega)^{(j_1,j_2-1)}. \tag{4-22}$$

Here $\boldsymbol{e}_2=(0,1)$. Hence integration by parts gives

$$\begin{aligned}&\int_0^{\tau_0}-\nu^{\frac12}\langle(\Delta\omega)^{\boldsymbol{j}},\xi_j^2\omega^{\boldsymbol{j}}\rangle\,d\tau\\
&=\nu^{\frac12}\int_0^{\tau_0}\Big(\|\xi_j(\nabla\omega)^{\boldsymbol{j}}\|^2+2\nu^{\frac12}j_2e^{-K\tau\nu^{1/2}}\langle\xi_j(\nabla\omega)^{\boldsymbol{j}},\chi'_\nu\xi_j(\partial_Y\omega)^{(j_1,j_2-1)}\rangle+\langle(\nabla\omega)^{\boldsymbol{j}}\cdot\nabla(\xi_j^2),\omega^{\boldsymbol{j}}\rangle\Big)\,d\tau\\
&\ge\tfrac34\nu^{\frac12}\|\xi_j(\nabla\omega)^{\boldsymbol{j}}\|^2_{L^2(0,\tau_0;L^2)}-C\nu^{\frac12}(\kappa\nu^{\frac12}j_2)^2\|\xi_{j-1}(\partial_Y\omega)^{(j_1,j_2-1)}\|^2_{L^2(0,\tau_0;L^2)}\\
&\qquad-\nu^{\frac12}\int_0^{\tau_0}|\langle(\nabla\omega)^{\boldsymbol{j}}\cdot\nabla(\xi_j^2),\omega^{\boldsymbol{j}}\rangle|\,d\tau.\end{aligned}$$

Here we have used $\|\xi_j/\xi_{j-1}\|_{L^\infty}\le C$ in the last line as stated in Lemma 4.3. When $j_2=0$, the term $(\partial_Y\omega)^{(j_1,j_2-1)}$ is defined as 0 for convenience. It suffices to estimate $\langle(\nabla\omega)^{\boldsymbol{j}}\cdot\nabla(\xi_j^2),\omega^{\boldsymbol{j}}\rangle$. We have

$$\nabla(\xi_j^2)=-\frac{\nabla\partial_Y\Omega+2\nabla\rho_j}{\sqrt{\partial_Y\Omega+2\rho_j}}\xi_j^3, \tag{4-23}$$

which yields

$$|\langle(\nabla\omega)^{\boldsymbol{j}}\cdot\nabla(\xi_j^2),\omega^{\boldsymbol{j}}\rangle|\le\|\xi_j(\nabla\omega)^{\boldsymbol{j}}\|\bigg(\bigg\|\frac{\nabla\partial_Y\Omega}{\sqrt{\partial_Y\Omega+2\rho_j}}\xi_j^2\omega^{\boldsymbol{j}}\bigg\|+\bigg\|\frac{2\partial_Y\rho_j}{\sqrt{\partial_Y\Omega+2\rho_j}}\xi_j^2\omega^{\boldsymbol{j}}\bigg\|\bigg).$$

To estimate $\|(\nabla\partial_Y\Omega/\sqrt{\partial_Y\Omega+2\rho_j})\xi_j^2\omega^{\boldsymbol{j}}\|$, we decompose the integral about Y into $0\le Y\le (j+1)^{-1/2}$ and $Y\ge (j+1)^{-1/2}$. Then we see from Lemma 4.3 with $\xi_j^2/\sqrt{\partial_Y\Omega+2\rho_j}=\xi_j^3\le 1/\rho_j^{3/2}$,

$$\begin{aligned}\left\|\frac{\nabla\partial_Y\Omega}{\sqrt{\partial_Y\Omega+2\rho_j}}\xi_j^2\omega^{\boldsymbol{j}}\right\|_{L^2(\{0<Y<(j+1)^{-1/2}\})} &\le \left\|\frac{1}{\rho_j^{3/2}}\right\|_{L^\infty(\{0<Y<(j+1)^{-1/2}\})}\|\nabla\partial_Y\Omega\omega^{\boldsymbol{j}}\|_{L^2(\{0<Y<(j+1)^{-1/2}\})}\\ &\le \frac{2}{(K^{1/4}C_*)^{3/2}}\left\|\frac{Y}{1+\nu^{1/2}Y}\nabla\partial_Y\Omega\right\|_{L^\infty}\left\|\frac{\omega^{\boldsymbol{j}}}{Y}\right\|\\ &\le \frac{CC_1^*}{(K^{1/4}C_*)^{3/2}}\|\partial_Y\omega^{\boldsymbol{j}}\|.\end{aligned}$$

Here we have used Assumption (iii) and the Hardy inequality $\|\omega^{\boldsymbol{j}}/Y\|\le 4\|\partial_Y\omega^{\boldsymbol{j}}\|$. Then by using (4-22) for $\partial_Y\omega^{\boldsymbol{j}}$ and (4-18) we have

$$\begin{aligned}\|\partial_Y\omega^{\boldsymbol{j}}\| &\le \|(\partial_Y\omega)^{\boldsymbol{j}}\|+\kappa\nu^{\frac12}j_2\|(\partial_Y\omega)^{(j_1,j_2-1)}\|\\ &\le \left\|\frac{1}{\xi_j}\right\|_{L^\infty}\|\xi_j(\partial_Y\omega)^{\boldsymbol{j}}\|+\kappa\nu^{\frac12}j_2\left\|\frac{1}{\xi_{j-1}}\right\|_{L^\infty}\|\xi_{j-1}(\partial_Y\omega)^{(j_1,j_2-1)}\|\\ &\le C(C_1^*+K^{\frac14}C_*)^{\frac12}(\|\xi_j(\partial_Y\omega)^{\boldsymbol{j}}\|+\kappa\nu^{\frac12}j_2\|\xi_{j-1}(\partial_Y\omega)^{(j_1,j_2-1)}\|).\end{aligned}\tag{4-24}$$

On the other hand, we have from Assumption (iv) and (4-16) in Lemma 4.3,

$$\begin{aligned}\left\|\frac{\nabla\partial_Y\Omega}{\sqrt{\partial_Y\Omega+2\rho_j}}\xi_j^2\omega^{\boldsymbol{j}}\right\|_{L^2(\{Y\ge(j+1)^{-1/2}\})} &\le \left\|\frac{Y\nabla\partial_Y\Omega}{(1+\nu^{1/2}Y)\sqrt{\partial_Y\Omega+2\rho_j}}\right\|\left\|\frac{1+\nu^{1/2}Y}{Y}\xi_j\right\|_{L^\infty(\{Y\ge(j+1)^{-1/2}\})}\|\xi_j\omega^{\boldsymbol{j}}\|\\ &\le CC_2^*(j+1)^{\frac12}\|\xi_j\omega^{\boldsymbol{j}}\|.\end{aligned}$$

Next we estimate the term $\|(2\partial_Y\rho_j/\sqrt{\partial_Y\Omega+2\rho_j})\xi_j^2\omega^{\boldsymbol{j}}\|$. To this end we observe that

$$\begin{aligned}|\partial_Y\rho_j| &\le 2(j+1)^{\frac12}K^{\frac14}C_*(1+(j+1)^{\frac12}Y)^{-3}+2C_*\nu^{\frac12}(1+Y)^{-3}+2C_*\nu^{-\frac14}\left(1+\frac{Y}{\nu^{1/4}}\right)^{-3}\\ &\le \begin{cases}2(j+1)^{\frac12}\rho_j+2C_*/Y, & 0<Y<(j+1)^{-\frac12},\\ 2(j+1)^{\frac12}\rho_j+2\rho_j/Y, & Y\ge(j+1)^{-\frac12},\end{cases}\end{aligned}$$

which gives, from Lemma 4.3,

$$\begin{aligned}\left\|\frac{2\partial_Y\rho_j}{\sqrt{\partial_Y\Omega+2\rho_j}}\xi_j^2\omega^{\boldsymbol{j}}\right\|_{L^2} &\le 4(j+1)^{\frac12}\|\xi_j\omega^{\boldsymbol{j}}\|+2C_*\left\|\frac{\xi_j^3\omega^{\boldsymbol{j}}}{Y}\right\|_{L^2(\{0<Y<(j+1)^{-1/2}\})}+2\left\|\frac{\rho_j\xi_j^3\omega^{\boldsymbol{j}}}{Y}\right\|_{L^2(\{Y\ge(j+1)^{-1/2}\})}\\ &\le 4(j+1)^{\frac12}\|\xi_j\omega^{\boldsymbol{j}}\|+\frac{2C_*}{(K^{1/4}C_*)^{3/2}}\left\|\frac{\omega^{\boldsymbol{j}}}{Y}\right\|+2(j+1)^{\frac12}\|\xi_j\omega^{\boldsymbol{j}}\|.\end{aligned}$$

Then we apply the Hardy inequality $\|\omega^{\boldsymbol{j}}/Y\|\le 4\|\partial_Y\omega^{\boldsymbol{j}}\|$ and then use (4-24). Collecting these, we obtain

$$\begin{aligned}&|\langle(\nabla\omega)^{\boldsymbol{j}}\cdot\nabla(\xi_j^2),\omega^{\boldsymbol{j}}\rangle|\\ &\quad\le \|\xi_j(\nabla\omega)^{\boldsymbol{j}}\|\biggl(\frac{C(C_1^*+1)(C_1^*+K^{1/4}C_*)^{1/2}}{(K^{1/4}C_*)^{3/2}}(\|\xi_j(\partial_Y\omega)^{\boldsymbol{j}}\|+\kappa\nu^{\frac12}j_2\|\xi_{j-1}(\partial_Y\omega)^{(j_1,j_2-1)}\|)\\ &\qquad\qquad +C(C_2^*+1)(j+1)^{\frac12}\|\xi_j\omega^{\boldsymbol{j}}\|\biggr).\end{aligned}$$

Thus, by taking K large enough depending only on C_1^* and C_*, we obtain the desired estimate as stated in Lemma 4.6. □

Lemma 4.6. *There exists $K_{1,2} = K_{1,2}(C_1^*, C_*) \ge 1$ such that if $K \ge K_{1,2}$ then we have*

$$\int_0^{\tau_0} \langle (\partial_\tau + K\nu^{\frac12}(j+1) + V\cdot\nabla)\omega^{\boldsymbol{j}}, \xi_j^2\omega^{\boldsymbol{j}}\rangle\, d\tau \ge \tfrac12\|\xi_j\omega^{\boldsymbol{j}}(\tau_0)\|^2_{L^2_{X,Y}} - \tfrac12\|\xi_j\omega^{\boldsymbol{j}}(0)\|^2_{L^2_{X,Y}} + \tfrac12 K\nu^{\frac12}(j+1)\|\xi_j\omega^{\boldsymbol{j}}\|^2_{L^2(0,\tau_0;L^2_{X,Y})} - \frac{CC_1^*\nu^{1/2}}{K^{1/4}C_*}\big(\|\xi_j(\partial_Y\omega)^{\boldsymbol{j}}\|^2_{L^2(0,\tau_0;L^2_{X,Y})} + (\kappa\nu^{\frac12}j)^2 M_{2,j-1,\xi_{j-1}}[\partial_Y\omega]^2\big).$$

Here $C > 0$ is a universal constant.

Proof. Integration by parts yields

$$\int_0^{\tau_0} \langle (\partial_\tau + K\nu^{\frac12}(j+1) + V\cdot\nabla)\omega^{\boldsymbol{j}}, \xi_j^2\omega^{\boldsymbol{j}}\rangle\, d\tau = \tfrac12\|\xi_j\omega^{\boldsymbol{j}}(\tau_0)\|^2_{L^2_{X,Y}} - \tfrac12\|\xi_j\omega^{\boldsymbol{j}}(0)\|^2_{L^2_{X,Y}} + K\nu^{\frac12}(j+1)\|\xi_j\omega^{\boldsymbol{j}}\|^2_{L^2(0,\tau_0;L^2_{X,Y})} - \frac12\int_0^{\tau_0}\langle \partial_\tau(\xi_j^2) + V\cdot\nabla(\xi_j^2), (\omega^{\boldsymbol{j}})^2\rangle\, d\tau.$$

As for the term $\langle \partial_\tau(\xi_j^2), (\omega^{\boldsymbol{j}})^2\rangle$, we decompose the integral about Y into $\{0 < Y < (j+1)^{-1/2}\}$ and $\{Y \ge (j+1)^{-1/2}\}$ and compute as follows:

$$\begin{aligned} &|\langle \partial_\tau(\xi_j^2), (\omega^{\boldsymbol{j}})^2\rangle| \\ &\le \left\|\left(\frac{Y}{1+\nu^{1/2}Y}\right)^2 \partial_\tau\partial_Y\Omega\right\|_{L^\infty}\left\|\left(\frac{1+\nu^{1/2}Y}{Y}\right)\xi_j^2\omega^{\boldsymbol{j}}\right\|^2 \\ &\le C_1^*\nu^{\frac12}\left(\|(1+\nu^{\frac12}Y)\xi_j^2\|^2_{L^\infty(\{0<Y<(j+1)^{-1/2}\})}\left\|\frac{\omega^{\boldsymbol{j}}}{Y}\right\|^2 + \left\|\left(\frac{1+\nu^{1/2}Y}{Y}\right)\xi_j\right\|^2_{L^\infty(\{Y\ge(j+1)^{-1/2}\})}\|\xi_j\omega^{\boldsymbol{j}}\|^2\right) \\ &\le C_1^*\nu^{\frac12}\left(\frac{C}{(K^{1/4}C_*)^2}\|\partial_Y\omega^{\boldsymbol{j}}\|^2 + C(j+1)\|\xi_j\omega^{\boldsymbol{j}}\|^2\right) \quad \text{(by the Hardy inequality and Lemma 4.3).} \end{aligned} \tag{4-25}$$

Next we have

$$|\langle V\cdot\nabla(\xi_j^2), (\omega^{\boldsymbol{j}})^2\rangle| \le \left\|\frac{V\cdot\nabla(\partial_Y\Omega + 2\rho_j)}{\partial_Y\Omega + 2\rho_j}\right\|_{L^\infty}\|\xi_j\omega^{\boldsymbol{j}}\|^2.$$

Then we have from Assumption (iii) and Lemma 4.3,

$$\begin{aligned} \left\|\frac{V_1\partial_Y\partial_X\Omega}{\partial_Y\Omega + 2\rho_j}\right\|_{L^\infty} &\le \left\|\frac{Y(1+Y)}{(1+\nu^{1/2}Y)^2}\partial_X\partial_Y\Omega\right\|_{L^\infty}\left\|\frac{V_1(1+\nu^{1/2}Y)^2}{Y(1+Y)\rho_j}\right\|_{L^\infty} \\ &\le C_1^*\nu^{\frac12}\left(2\left\|\frac{V_1}{Y(1+Y)\rho_j}\right\|_{L^\infty} + 2\|V_1\|_{L^\infty}\left\|\frac{(\nu^{1/2}Y)^2}{Y(1+Y)\rho_j}\right\|_{L^\infty}\right) \le C(C_1^*)^2\nu^{\frac12}(j+1). \end{aligned}$$

Here we have computed, using $V_1|_{Y=0}=0$,

$$\left\|\frac{V_1}{Y(1+Y)\rho_j}\right\|_{L^\infty} \le \|\partial_Y V_1\|_{L^\infty}\left\|\frac{1}{\rho_j}\right\|_{L^\infty(\{0<Y<(j+1)^{-1/2}\})} + \|V_1\|_{L^\infty}\left\|\frac{1}{Y(1+Y)\rho_j}\right\|_{L^\infty(\{Y\ge(j+1)^{-1/2}\})} \le C_1^*(j+1).$$

Similarly,

$$\begin{aligned}\left\|\frac{V_2(\partial_Y^2\Omega+2\partial_Y\rho_j)}{\partial_Y\Omega+2\rho_j}\right\|_{L^\infty} &\le \left\|\frac{Y(1+Y)^2}{(1+\nu^{1/2}Y)^3}\partial_Y^2\Omega\right\|_{L^\infty}\left\|\frac{V_2(1+\nu^{1/2}Y)^3}{Y(1+Y)^2\rho_j}\right\|_{L^\infty} + \left\|\frac{V_2}{Y}\right\|_{L^\infty}\left\|\frac{Y\partial_Y\rho_j}{\rho_j}\right\|_{L^\infty}\\ &\le CC_1^*\left(\left\|\frac{V_2}{Y(1+Y)^2\rho_j}\right\|_{L^\infty} + \|V_2\|_{L^\infty}\left\|\frac{(\nu^{1/2}Y)^3}{Y(1+Y)^2\rho_j}\right\|_{L^\infty}\right) + 2C_1^*\nu^{\frac12}\\ &\le CC_1^*\left(\|\partial_Y V_2\|_{L^\infty}\left\|\frac{1}{(1+Y)^2\rho_j}\right\|_{L^\infty} + \|V_2\|_{L^\infty}\nu^{\frac32}\left\|\frac{1}{\rho_j}\right\|_{L^\infty}\right) + 2C_1^*\nu^{\frac12}\\ &\le CC_1^*(C_1^*+1)\nu^{\frac12}(j+1). \qquad \text{(by Lemma 4.3)}.\end{aligned}$$

Note that we have also used $\|\partial_Y V_2\|_{L^\infty}=\|\partial_X V_1\|_{L^\infty}\le C_1^*\nu^{1/2}$. Collecting these and applying the identity (4-22) for $\partial_Y\omega^j$ in (4-25) (that is, we use (4-24)), we obtain the desired estimate by taking K large enough depending only on C_1^* and C_*. □

Lemma 4.7. *It follows that*

$$\int_0^{\tau_0}|\langle(\nabla^\perp\phi)^j\cdot\nabla\Omega,\,\xi_j^2\omega^j\rangle|\,d\tau \le \frac{C(R_{j,\text{Lemma 4.7}}[\nabla\phi])^2}{\nu^{1/2}(j+1)} + \tfrac18 K\nu^{\frac12}(j+1)\|\xi_j\omega^j\|^2_{L^2(0,\tau_0;L^2_{X,Y})}, \tag{4-26}$$

where

$$R_{j,\text{Lemma 4.7}}[\nabla\phi] := \left(\frac{C_1^*}{K^{1/2}} + \frac{(K^{1/4}C_*)^{1/2}}{K^{1/2}} + \kappa^{\frac12}\right)\nu^{\frac12}(j+1)M_{2,j}[\nabla\phi] + \frac{(K^{1/2}C_*)^{1/2}}{K^{1/2}}\delta_{j\le\nu^{-1/2}-1}\frac{M_{2,j+1}[\partial_Y\phi]}{(j+1)^{1/2}}.$$

Here

$$\delta_{j\le\nu^{-1/2}-1} = \begin{cases}1 & \text{for } 0\le j\le\nu^{-1/2}-1,\\ 0 & \text{for } j=\nu^{-1/2}.\end{cases}$$

Moreover, there exists $K_{1,3}=K_{1,3}(C_1^*,C_*)\ge1$ *such that if* $K\ge K_{1,3}$ *then*

$$\sum_{j=0}^{\nu^{-1/2}}\frac{R_{j,\text{Lemma 4.7}}[\nabla\phi]}{(j!)^{3/2}\nu^{j/2}\nu^{1/4}(j+1)^{1/2}} \le C|||\nabla\phi|||'_{2,\mathbf{1}}. \tag{4-27}$$

Here $C>0$ *is a universal constant.*

Proof. It suffices to show

$$\int_0^{\tau_0}|\langle(\partial_X\phi)^j,\omega^j\rangle|\,d\tau \le 2\kappa\nu^{\frac12}j_2(M_{2,j}[\nabla\phi])^2, \tag{4-28}$$

$$\int_0^{\tau_0} |\langle \rho_j (\partial_X \phi)^{\boldsymbol{j}}, \xi_j^2 \omega^{\boldsymbol{j}} \rangle| \, d\tau$$
$$\leq \begin{cases} C(K^{\frac14} C_*)^{\frac12} \Big(\frac{M_{2,j+1}[\partial_Y \phi]}{(j+1)^{1/2}} + \kappa \nu^{\frac12} (j+1)^{\frac12} M_{2,j}[\nabla \phi] \Big) \|\xi_j \omega^{\boldsymbol{j}}\|_{L^2(0,\tau_0; L^2_{X,Y})}, & 0 \leq j \leq \nu^{-\frac12} - 1, \\ C(K^{\frac14} C_*)^{\frac12} M_{2,j}[\partial_X \phi] \|\xi_j \omega^{\boldsymbol{j}}\|_{L^2(0,\tau_0; L^2_{X,Y})}, & j = \nu^{-\frac12}, \end{cases} \tag{4-29}$$

and

$$\int_0^{\tau_0} |\langle (\partial_Y \phi)^{\boldsymbol{j}} \partial_X \Omega, \xi_j^2 \omega^{\boldsymbol{j}} \rangle| \, d\tau \leq C C_1^* \nu^{\frac12} (j+1)^{\frac12} M_{2,j}[\partial_Y \phi] \|\xi_j \omega^{\boldsymbol{j}}\|_{L^2(0,\tau_0; L^2_{X,Y})}. \tag{4-30}$$

Let us start from (4-28). To compute $\langle (\partial_X \phi)^{\boldsymbol{j}}, \omega^{\boldsymbol{j}} \rangle$, we first observe that

$$\omega^{\boldsymbol{j}} = \nabla \cdot (\nabla \phi)^{\boldsymbol{j}} - \frac{\nu^{1/2} j_2 \chi_\nu'}{\chi_\nu} (\partial_Y \phi)^{\boldsymbol{j}}. \tag{4-31}$$

Then we have, from integration by parts and $[B_{j_2}, \partial_Y] = -((\nu^{1/2} j_2 \chi_\nu')/\chi_\nu) B_{j_2}$,

$$\begin{aligned} \langle (\partial_X \phi)^{\boldsymbol{j}}, \omega^{\boldsymbol{j}} \rangle &= -\langle \nabla (\partial_X \phi)^{\boldsymbol{j}}, (\nabla \phi)^{\boldsymbol{j}} \rangle - \nu^{\frac12} j_2 \langle (\partial_Y \phi)^{(j_1+1, j_2-1)}, \chi_\nu' (\partial_Y \phi)^{\boldsymbol{j}} \rangle \\ &= -\langle \partial_X (\nabla \phi)^{\boldsymbol{j}}, (\nabla \phi)^{\boldsymbol{j}} \rangle - 2\nu^{\frac12} j_2 \langle \chi_\nu' (\partial_Y \phi)^{(j_1+1, j_2-1)}, (\partial_Y \phi)^{\boldsymbol{j}} \rangle \\ &= -2\nu^{\frac12} j_2 \langle \chi_\nu' (\partial_Y \phi)^{(j_1+1, j_2-1)}, (\partial_Y \phi)^{\boldsymbol{j}} \rangle. \end{aligned}$$

Hence we have, from $\|\chi_\nu'\|_{L^\infty} = \kappa$,

$$\int_0^{\tau_0} |\langle (\partial_X \phi)^{\boldsymbol{j}}, \omega^{\boldsymbol{j}} \rangle| \, d\tau \leq 2\kappa \nu^{\frac12} j_2 M_{2,j}[\partial_Y \phi]^2. \tag{4-32}$$

To estimate $\int_0^{\tau_0} |\langle \rho_j (\partial_X \phi)^{\boldsymbol{j}}, \xi_j^2 \omega^{\boldsymbol{j}} \rangle| \, d\tau$, the key inequality from the definition (4-10) is

$$\xi_j \rho_j \leq \sqrt{\rho_j} \leq C(K^{\frac14} C_*)^{\frac12} (1 + (j+1)^{\frac12} Y)^{-1} + C\nu^{\frac12}, \tag{4-33}$$

where $\nu^{1/2}(j+1) \leq 2$ is used. Thus we have from the Hardy inequality

$$\begin{aligned} &\int_0^{\tau_0} |\langle \rho_j (\partial_X \phi)^{\boldsymbol{j}}, \xi_j^2 \omega^{\boldsymbol{j}} \rangle| \, d\tau \\ &\quad \leq \int_0^{\tau_0} \|\xi_j \rho_j (\partial_X \phi)^{\boldsymbol{j}}\| \|\xi_j \omega^{\boldsymbol{j}}\| \, d\tau \\ &\quad \leq \frac{C(K^{1/4} C_*)^{1/2}}{(j+1)^{1/2}} \int_0^{\tau_0} \left\| \frac{(\partial_X \phi)^{\boldsymbol{j}}}{Y} \right\| \|\xi_j \omega^{\boldsymbol{j}}\| \, d\tau + C\nu^{\frac12} \|(\partial_X \phi)^{\boldsymbol{j}}\|_{L^2(0,\tau_0;L^2)} \|\xi_j \omega^{\boldsymbol{j}}\|_{L^2(0,\tau_0;L^2)} \\ &\quad \leq \frac{C(K^{1/4} C_*)^{1/2}}{(j+1)^{1/2}} \|\partial_Y (\partial_X \phi)^{\boldsymbol{j}}\|_{L^2(0,\tau_0;L^2)} \|\xi_j \omega^{\boldsymbol{j}}\|_{L^2(0,\tau_0;L^2)} + C\nu^{\frac12} \|(\partial_X \phi)^{\boldsymbol{j}}\|_{L^2(0,\tau_0;L^2)} \|\xi_j \omega^{\boldsymbol{j}}\|_{L^2(0,\tau_0;L^2)}. \end{aligned}$$

Then the desired estimate for $0 \leq j \leq \nu^{-1/2} - 1$ follows from $K\tau\nu^{1/2} \leq 1$ and

$$\partial_Y (\partial_X \phi)^{\boldsymbol{j}} = e^{K\tau\nu^{1/2}} (\partial_Y \phi)^{(j_1+1, j_2)} + \nu^{\frac12} j_2 \chi_\nu' (\partial_Y \phi)^{(j_1+1, j_2-1)}. \tag{4-34}$$

On the other hand, the estimate for $j = \nu^{-1/2}$ easily follows from

$$\|\xi_j \rho_j (\partial_X \phi)^{\boldsymbol{j}}\| \leq \|\sqrt{\rho_j}\|_{L^\infty} \|(\partial_X \phi)^{\boldsymbol{j}}\| \leq C(K^{\frac14} C_*)^{\frac12} \|(\partial_X \phi)^{\boldsymbol{j}}\|. \tag{4-35}$$

Finally we have, from Assumption (iii) and Lemma 4.3,

$$\|\xi_j(\partial_Y\phi)^j\partial_X\Omega\| \le \left\|\frac{1+Y}{1+\nu^{1/2}Y}\partial_X\Omega\right\|_{L^\infty}\left\|\frac{1+\nu^{1/2}Y}{1+Y}\xi_j\right\|_{L^\infty}\|(\partial_Y\phi)^j\| \le CC_1^*\nu^{\frac12}(j+1)^{\frac12}\|(\partial_Y\phi)^j\|,$$

which gives

$$\int_0^{\tau_0}|\langle(\partial_Y\phi)^j\partial_X\Omega,\xi_j^2\omega^j\rangle|\,d\tau \le CC_1^*\nu^{\frac12}(j+1)^{\frac12}M_{2,j}[\partial_Y\phi]\|\xi_j\omega^j\|_{L^2(0,\tau_0;L^2_{X,Y})}.$$

Collecting these, we obtain (4-26), for the identity

$$\partial_Y\Omega\xi_j^2 = \frac{\partial_Y\Omega}{\partial_Y\Omega+2\rho_j} = 1-2\rho_j\xi_j^2$$

holds. The estimate (4-27) is verified from the definition

$$|||\nabla\phi|||'_{2,\mathbf{1}} = \sum_{j=0}^{\nu^{-1/2}}\frac{\nu^{1/4}(j+1)^{1/2}}{(j!)^{3/2}\nu^{j/2}}M_{2,j}[\nabla\phi]$$

and

$$\sum_{j=0}^{\nu^{-1/2}-1}\frac{M_{2,j+1}[\partial_Y\phi]}{(j!)^{3/2}\nu^{j/2}\nu^{1/4}(j+1)} = \sum_{j=0}^{\nu^{-1/2}-1}\frac{\nu^{1/2}(j+1)^{3/2}M_{2,j+1}[\nabla\phi]}{((j+1)!)^{3/2}\nu^{(j+1)/2}\nu^{1/4}(j+1)}$$
$$\le \sum_{j=1}^{\nu^{-1/2}}\frac{\nu^{1/4}j^{1/2}M_{2,j}[\nabla\phi]}{(j!)^{3/2}\nu^{j/2}} \le |||\nabla\phi|||'_{2,\mathbf{1}}. \qquad \square$$

Lemma 4.8. *Let $j_2\ge 1$. Then it follows that*

$$\int_0^{\tau_0}|\langle V_2[B_{j_2},\partial_Y]e^{-K\tau\nu^{1/2}(j+1)}\partial_X^{j_1}\omega,\xi_j^2\omega^j\rangle|\,d\tau \le CC_1^*\nu^{\frac12}j_2\|\xi_j\omega^j\|^2_{L^2(0,\tau_0;L^2_{X,Y})}. \tag{4-36}$$

Here $C>0$ is a universal constant.

Proof. The estimate directly follows from (4-20) and

$$|V_2\chi'_\nu| \le \left\|\frac{V_2}{Y}\right\|_{L^\infty}|Y\chi'_\nu| \le \|\partial_X V_1\|_{L^\infty}|Y\chi'_\nu| \le C_1^*\nu^{\frac12}|Y\chi'_\nu| \le CC_1^*\chi_\nu$$

by Assumption (iii) and $\kappa\nu^{1/2}Ye^{-\kappa\nu^{1/2}Y}\le C\chi_\nu$ for a universal constant $C>0$. $\square$

Lemma 4.9. *Let $j\ge1$. It follows that*

$$\int_0^{\tau_0}\left|\left\langle\sum_{l=0}^{j-1}\sum_{\max\{0,l+j_2-j\}\le l_2\le\min\{l,j_2\}}\binom{j_2}{l_2}\binom{j-j_2}{l-l_2}V^{j-l}\cdot(\nabla\omega)^l,\xi_j^2\omega^j\right\rangle\right|d\tau$$
$$\le \frac{C}{\kappa}R_{j,\text{Lemma 4.9}}[\omega]\|\xi_j\omega^j\|_{L^2(0,\tau_0;L^2_{X,Y})},$$

where

$$R_{j,\text{Lemma 4.9}}[\omega] := \sum_{l=0}^{j-1}(j-l+1)^{\frac12}\min\{l+1,j-l+1\}\binom{j}{l}N_{\infty,j-l}[V]M_{2,l+1,\xi_l}[\omega],$$

and

$$N_{\infty,j}[V] := \sup_{j_2=0,\dots,j}\left(\|B_{j_2}\partial_X^{j-j_2}V_1\|_{L^\infty(0,1/(K\nu^{1/2});L^\infty_{X,Y})} + \kappa\left\|\frac{\partial_X^j V_2}{\chi_\nu}\right\|_{L^\infty(0,1/(K\nu^{1/2});L^\infty_{X,Y})}\right).$$

Moreover,

$$\sum_{j=0}^{\nu^{-1/2}} \frac{R_{j,\text{Lemma 4.9}}[\omega]}{(j!)^{3/2}\nu^{j/2}\nu^{1/4}(j+1)^{1/2}} \le CC_0^*|||\omega|||'_{2,\xi}. \tag{4-37}$$

Here $C>0$ is a universal constant.

Proof. We first observe that

$$\binom{j_2}{l_2}\binom{j-j_2}{l-l_2} \le \binom{j}{l}, \quad 0\le j_2\le l_2\le l\le j, \tag{4-38}$$

and

$$\#\{l_2\in\mathbb{N}\cup\{0\} \mid \max\{0, l+j_2-j\}\le l_2\le \min\{l, j_2\}\} \le \min\{l+1, j-l+1\}. \tag{4-39}$$

Hence we have

$$\int_0^{\tau_0}\left|\left\langle\sum_{l=0}^{j-1}\sum_{\max\{0,l+j_2-j\}\le l_2\le\min\{l,j_2\}}\binom{j_2}{l_2}\binom{j-j_2}{l-l_2}V^{\boldsymbol{j-l}}\cdot(\nabla\omega)^{\boldsymbol{l}}, \xi_j^2\omega^{\boldsymbol{j}}\right\rangle\right| d\tau$$
$$\le \sum_{l=0}^{j-1}\binom{j}{l}\min\{l+1, j-l+1\}\|\xi_j V^{\boldsymbol{j-l}}\cdot(\nabla\omega)^{\boldsymbol{l}}\|_{L^2(0,\tau_0;L^2)}\|\xi_j\omega^{\boldsymbol{j}}\|_{L^2(0,\tau_0;L^2)}.$$

From the definition of ξ_j, we see, for $0\le l\le j-1$,

$$\frac{\xi_j}{\xi_l} \le \sqrt{1+\frac{(1+(j+1)^{1/2}Y)^{-2}}{(1+(l+1)^{1/2}Y)^{-2}}} \le C(j+l-1)^{\frac12},$$

where $C>0$ is a universal constant, and thus,

$$\|\xi_j V^{\boldsymbol{j-l}}\cdot(\nabla\omega)^{\boldsymbol{l}}\|_{L^2(0,\tau_0;L^2)} \le C(j+l-1)^{\frac12}\|\xi_l V^{\boldsymbol{j-l}}\cdot(\nabla\omega)^{\boldsymbol{l}}\|_{L^2(0,\tau_0;L^2)}.$$

Next we have

$$\|\xi_l V_1^{\boldsymbol{j-l}}(\partial_X\omega)^{\boldsymbol{l}}\|_{L^2(0,\tau_0;L^2)} \le \left\|\frac{\xi_l}{\xi_{l+1}}\right\|_{L^\infty}\|V_1^{\boldsymbol{j-l}}\|_{L^\infty}\|\xi_{l+1}\omega^{(l_1+1,l_2)}\|_{L^2(0,\tau_0;L^2)}$$
$$\le CN_{\infty,j-l}[V]M_{2,l+1,\xi_{l+1}}[\omega],$$

and similarly,

$$\|\xi_l V_2^{\boldsymbol{j-l}}(\partial_Y\omega)^{\boldsymbol{l}}\|_{L^2(0,\tau_0;L^2)} \le \left\|\frac{\xi_l}{\xi_{l+1}}\right\|_{L^\infty}\left\|\frac{V_2^{\boldsymbol{j-l}}}{\chi_\nu}\right\|_{L^\infty}\|\xi_{l+1}\omega^{(l_1,l_2+1)}\|_{L^2(0,\tau_0;L^2)}$$
$$\le \frac{C}{\kappa}N_{\infty,j-l}[V]M_{2,l+1,\xi_{l+1}}[\omega],$$

Here we have used from $\partial_X V_1 + \partial_Y V_2 = 0$ that $V_2^{j-l}/\chi_\nu = (\partial_Y V_2)^{(j_1-l_1,\, j_2-l_2-1)} = -V_1^{(j_1-l_1+1,\, j_2-l_2-1)}$ for $j_2 - l_2 \geq 1$, which satisfies $\|V_2^{j-l}/\chi_\nu\|_{L^\infty} \leq C N_{\infty, j-l}[V]$. The estimate (4-37) follows from

$$\begin{aligned}
&\sum_{j=0}^{\nu^{-1/2}} \frac{1}{(j!)^{3/2}\nu^{j/2}\nu^{1/4}(j+1)^{1/2}} \sum_{l=0}^{j-1} (j-l+1)^{\frac{1}{2}} \min\{l+1,\, j-l+1\} \binom{j}{l} \{(j-l)!\,(l+1)!\}^{\frac{3}{2}} \nu^{(j+1)/2} \\
&\qquad\qquad \times \frac{N_{\infty, j-l}[V]}{((j-l)!)^{3/2}\nu^{(j-l)/2}} \frac{M_{2, l+1, \xi_l}[\omega]}{((l+1)!)^{3/2}\nu^{(l+1)/2}} \\
&\leq \sum_{j=0}^{\nu^{-1/2}} \sum_{l=0}^{j-1} (j-l+1)^{\frac{1}{2}} \min\{l+1,\, j-l+1\} \frac{(l+1)^{3/2}}{(j+1)^{1/2}(l+2)^{1/2}} \left(\frac{(j-l)!\, l!}{j!}\right)^{\frac{1}{2}} \\
&\qquad\qquad \times \frac{N_{\infty, j-l}[V]}{((j-l)!)^{3/2}\nu^{(j-l)/2}} \frac{\nu^{1/4}(l+2)^{1/2} M_{2, l+1, \xi_l}[\omega]}{((l+1)!)^{3/2}\nu^{(l+1)/2}} \\
&\leq C \sum_{j=0}^{\nu^{-1/2}} \sum_{l=0}^{j-1} \frac{N_{\infty, j-l}[V]}{((j-l)!)^{3/2}\nu^{(j-l)/2}} \frac{\nu^{1/4}(l+2)^{1/2} M_{2, l+1, \xi_l}[\omega]}{((l+1)!)^{3/2}\nu^{(l+1)/2}}.
\end{aligned}$$

Here we have used, for $j \geq 1$,

$$(j-l+1)^{\frac{1}{2}} \min\{l+1,\, j-l+1\} \frac{(l+1)^{3/2}}{(j+1)^{1/2}(l+2)^{1/2}} \left(\frac{(j-l)!\, l!}{j!}\right)^{\frac{1}{2}} \leq C, \quad 0 \leq l \leq j-1, \tag{4-40}$$

with a universal constant $C > 0$. Here the key is the following estimate for each $k = 0, 1, 2, 3$:

$$\frac{(j-l)!\, l!}{j!} \leq \frac{C}{(j+1)^{1+k}} \quad \text{for } 1+k \leq l \leq j-1-k. \tag{4-41}$$

Then we obtain (4-37) from the Young inequality by convolution in the l^1 space. □

Lemma 4.10. *Let $j \geq 1$. It follows that*

$$\begin{aligned}
\int_0^{\tau_0} \bigg| \bigg\langle \sum_{l=0}^{j-1} \sum_{\max\{0, l+j_2-j\} \leq l_2 \leq \min\{l, j_2\}} \binom{j_2}{l_2}\binom{j-j_2}{l-l_2} (\nabla^\perp \phi)^l \cdot (\nabla\Omega)^{j-l}, \xi_j^2 \omega^j \bigg\rangle \bigg| \, d\tau \\
\leq C R_{j, \text{Lemma 4.10}}[\nabla\phi] \|\xi_j \omega^j\|_{L^2(0,\tau_0; L^2_{X,Y})},
\end{aligned}$$

where

$$\begin{aligned}
&R_{j, \text{Lemma 4.10}}[\nabla\phi] \\
&\quad := C_2^* \nu^{\frac{1}{2}} j (M_{2,j}[\nabla\phi] + \nu^{\frac{1}{2}} j M_{2,j-1}[\nabla\phi]) \\
&\qquad\qquad + (j+1)^{\frac{1}{2}} \sum_{l=0}^{j-2} \min\{l+1,\, j-l+1\} \binom{j}{l} N_{\infty, j-l}[\nabla\Omega] (M_{2,l+1}[\partial_Y\phi] + \nu^{\frac{1}{2}}(l+1) M_{2,l}[\nabla\phi]) \\
&\qquad\qquad\qquad + \nu^{\frac{1}{2}} (j+1)^{\frac{3}{2}} N_{\infty,1}[\nabla\Omega] M_{2,j-1}[\partial_Y\phi]
\end{aligned}$$

and

$$\begin{aligned}
&N_{\infty, j-l}[\nabla\Omega] \\
&\quad := \sup_{j_2=0,\ldots,j} \left(\left\| \left(\frac{1+Y}{1+\nu^{1/2}Y}\right)^2 (\partial_Y\Omega)^j \right\|_{L^2(0,1/(K\nu^{1/2}); L^2_{X,Y})} + \nu^{-\frac{1}{2}} \left\| \frac{1+Y}{1+\nu^{1/2}Y} (\partial_X\Omega)^j \right\|_{L^2(0,1/(K\nu^{1/2}); L^2_{X,Y})} \right).
\end{aligned}$$

Here the second term on the right-hand side is defined as zero when $j = 1$. Moreover,

$$\sum_{j=0}^{\nu^{-1/2}} \frac{R_{j,\text{Lemma 4.10}}[\nabla\phi]}{(j!)^{3/2}\nu^{j/2}\nu^{1/4}(j+1)^{1/2}} \le C(C_0^* + C_2^*)\|\!|\nabla\phi\|\!|'_{2,\mathbf{1}}. \tag{4-42}$$

Proof. As in the proof of Lemma 4.9, we have, from (4-38) and (4-39),

$$\begin{aligned}\int_0^{\tau_0}\left|\left\langle \sum_{l=0}^{j-1}\sum_{\max\{0,l+j_2-j\}\le l_2\le\min\{l,j_2\}}\binom{j_2}{l_2}\binom{j-j_2}{l-l_2}(\nabla^\perp\phi)^{\boldsymbol{l}}\cdot(\nabla\Omega)^{\boldsymbol{j-l}}, \xi_j^2\omega^{\boldsymbol{j}}\right\rangle\right| d\tau \\ \le \sum_{l=0}^{j-1}\binom{j}{l}\min\{l+1, j-l+1\}\|\xi_j(\nabla^\perp\phi)^{\boldsymbol{l}}\cdot(\nabla\Omega)^{\boldsymbol{j-l}}\|_{L^2(0,\tau_0;L^2)}\|\xi_j\omega^{\boldsymbol{j}}\|_{L^2(0,\tau_0;L^2)}.\end{aligned}$$

Then we have, from Lemma 4.3,

$$\begin{aligned}\|\xi_j(\partial_Y\phi)^{\boldsymbol{l}}(\partial_X\Omega)^{\boldsymbol{j-l}}\|_{L^2(0,\tau_0;L^2)} &\le \left\|\frac{(1+\nu^{1/2}Y)\xi_j}{1+Y}\right\|_{L^\infty}\left\|\frac{1+Y}{1+\nu^{1/2}Y}(\partial_X\Omega)^{\boldsymbol{j-l}}\right\|_{L^\infty}\|(\partial_Y\phi)^{\boldsymbol{l}}\|_{L^2(0,\tau_0;L^2)} \\ &\le C\nu^{\frac12}(j+1)^{\frac12}N_{\infty,j-l}[\nabla\Omega]M_{2,l}[\partial_Y\phi].\end{aligned}$$

Let $j \ge 2$ and $0 \le l \le j-2$. Then,

$$\begin{aligned}&\|\xi_j(\partial_X\phi)^{\boldsymbol{l}}(\partial_Y\Omega)^{\boldsymbol{j-l}}\|_{L^2(0,\tau_0;L^2)} \\ &\quad\le \left\|\frac{(1+\nu^{1/2}Y)\xi_j}{1+Y}\right\|_{L^\infty}\left\|\left(\frac{1+Y}{1+\nu^{1/2}Y}\right)^2(\partial_Y\Omega)^{\boldsymbol{j-l}}\right\|_{L^\infty}\left\|\frac{1+\nu^{1/2}Y}{1+Y}(\partial_X\phi)^{\boldsymbol{l}}\right\|_{L^2(0,\tau_0;L^2)} \\ &\quad\le C(j+1)^{\frac12}N_{\infty,j-l}[\nabla\Omega](\|\partial_Y(\partial_X\phi)^{\boldsymbol{l}}\|_{L^2(0,\tau_0;L^2)} + \nu^{\frac12}\|(\partial_X\phi)^{\boldsymbol{l}}\|_{L^2(0,\tau_0;L^2)}),\end{aligned}$$

where the Hardy inequality is applied in the last line. Then (4-34) gives

$$\begin{aligned}&\|\xi_j(\partial_X\phi)^{\boldsymbol{l}}(\partial_Y\Omega)^{\boldsymbol{j-l}}\|_{L^2(0,\tau_0;L^2)} \\ &\quad\le C(j+1)^{\frac12}N_{\infty,j-l}[\nabla\Omega](M_{2,l+1}[\partial_Y\phi] + \kappa\nu^{\frac12}(l+1)M_{2,l}[\nabla\phi]), \quad 0\le l\le j-2.\end{aligned}$$

As for the case $l = j-1$, by recalling $\xi_j \le 1/\sqrt{\partial_Y\Omega + 2\rho}$, we compute

$$\begin{aligned}\|\xi_j(\partial_X\phi)^{\boldsymbol{l}}(\partial_Y\Omega)^{\boldsymbol{j-l}}\|_{L^2(0,\tau_0;L^2)} &\le \left\|\frac{Y}{1+\nu^{1/2}Y}\xi_j(\partial_Y\Omega)^{\boldsymbol{j-l}}\right\|_{L^\infty}\left\|\frac{1+\nu^{1/2}Y}{Y}(\partial_X\phi)^{\boldsymbol{l}}\right\|_{L^2(0,\tau_0;L^2)} \\ &\le C\left(\left\|\frac{Y}{1+\nu^{1/2}Y}\frac{\partial_{XY}^2\Omega}{\sqrt{\partial_Y\Omega+2\rho}}\right\|_{L^\infty} + \left\|\frac{Y}{1+\nu^{1/2}Y}\frac{\chi_\nu\partial_Y^2\Omega}{\sqrt{\partial_Y\Omega+2\rho}}\right\|_{L^\infty}\right) \\ &\qquad\times(\|\partial_Y(\partial_X\phi)^{\boldsymbol{l}}\|_{L^2(0,\tau_0;L^2)} + \nu^{\frac12}\|(\partial_X\phi)^{\boldsymbol{l}}\|_{L^2(0,\tau_0;L^2)}).\end{aligned}$$

Here we have used the Hardy inequality and that, when $l = j-1$, either $(\partial_Y\Omega)^{\boldsymbol{j-l}} = \partial_{XY}^2\Omega$ or $\chi_\nu\partial_Y^2\Omega$. Then, by using $\|((1+\nu^{1/2}Y)/Y)\chi_\nu\|_{L^\infty} \le C\nu^{1/2}$, Assumption (iii), and (4-34), we have

$$\|\xi_j(\partial_X\phi)^{\boldsymbol{l}}(\partial_Y\Omega)^{\boldsymbol{j-l}}\|_{L^2(0,\tau_0;L^2)} \le CC_2^*\nu^{\frac12}(M_{2,l+1}[\partial_Y\phi] + \kappa\nu^{\frac12}(l+1)M_{2,l}[\nabla\phi]), \quad l = j-1.$$

Collecting these, we obtain the term $R_{j,\text{Lemma 4.10}}[\nabla\phi]$ by noticing ${}_jC_l = j$ for $l = j-1$, as desired. The estimate (4-42) is proved as in (4-37) but by also using the Young inequality for convolution in the l^1 space together with the following estimates for $j \ge 2$:

$$(j+1)^{\frac12}\min\{l+1,\, j-l+1\}\frac{(l+1)^{3/2}}{(j+1)^{1/2}(l+2)^{1/2}}\Big(\frac{(j-l)!\,l!}{j!}\Big)^{\frac12} \le C, \quad 0 \le l \le j-2,$$

$$(j+1)^{\frac12}\min\{l+1,\, j-l+1\}\frac{l+1}{(j+1)^{1/2}(l+1)^{1/2}}\Big(\frac{(j-l)!\,l!}{j!}\Big)^{\frac12} \le C, \quad 0 \le l \le j-2.$$

Note that the condition $l \le j-2$ is crucial here, for we apply (4-41). We omit the details. □

Lemma 4.11. *There exists $K_{1,4} = K_{1,4}(C_1^*, C_*) \ge 1$ such that, for $K \ge K_{1,4}$,*

$$\int_0^{\tau_0} \langle \operatorname{rot} F^{\boldsymbol j} - [B_{j_2}, \partial_Y]\partial_X^{j_1} e^{-K\tau\nu^{1/2}(j+1)} F_1, \xi_j^2 \omega^{\boldsymbol j}\rangle\, d\tau$$
$$\le C(C_2^*+1) M_{2,j,\xi_j}[F]\big(\|\xi_j(\nabla\omega)^{\boldsymbol j}\|_{L^2(0,\tau_0;L^2_{X,Y})} + \kappa\nu^{\frac12} j M_{2,j-1,\xi_{j-1}}[\partial_Y\omega] + (j+1)^{\frac12}\|\xi_j\omega^{\boldsymbol j}\|_{L^2(0,\tau_0;L^2_{X,Y})}\big)$$

and

$$\int_0^{\tau_0} \langle G^{\boldsymbol j}, \xi_j^2\omega^{\boldsymbol j}\rangle\, d\tau \le M_{2,j,\xi_j}[G]\|\xi_j\omega^{\boldsymbol j}\|_{L^2(0,\tau_0;L^2_{X,Y})}.$$

Here $C > 0$ is a universal constant.

Proof. The estimate about $G^{\boldsymbol j}$ is straightforward and we focus on the estimate about $F^{\boldsymbol j}$. Integration by parts and also (4-20) yield

$$\int_0^{\tau_0} \langle \operatorname{rot} F^{\boldsymbol j} - [B_{j_2}, \partial_Y]\partial_X^{j_1} e^{-K\tau\nu^{1/2}(j+1)} F_1, \xi_j^2 \omega^{\boldsymbol j}\rangle\, d\tau$$
$$= \int_0^{\tau_0} \langle F^{\boldsymbol j}, \nabla^\perp(\xi_j^2\omega^{\boldsymbol j})\rangle + \nu^{\frac12} j_2 \langle \chi_\nu' F_1^{\boldsymbol j}, \xi_j^2 e^{-K\tau\nu^{1/2}}(\partial_Y\omega)^{(j_1,j_2-1)}\rangle\, d\tau.$$

The second term is bounded from above by $C\kappa\nu^{1/2} j_2\|\xi_j F_1^{\boldsymbol j}\|_{L^2(0,\tau_0;L^2)} M_{2,j-1,\xi_{j-1}}[\partial_Y\omega]$, and thus we focus on the first term:

$$\int_0^{\tau_0}\langle F^{\boldsymbol j}, \nabla^\perp(\xi_j^2\omega^{\boldsymbol j})\rangle\, d\tau$$
$$= \int_0^{\tau_0} \langle F^{\boldsymbol j}\cdot\nabla^\perp(\xi_j^2), \omega^{\boldsymbol j}\rangle + \langle F^{\boldsymbol j}, \xi_j^2(\nabla^\perp\omega)^{\boldsymbol j}\rangle + \nu^{\frac12} j_2\langle F_1^{\boldsymbol j}, \xi_j^2\chi_\nu' e^{-K\tau\nu^{1/2}}(\partial_Y\omega)^{(j_1,j_2-1)}\rangle\, d\tau$$
$$\le \int_0^{\tau_0} \langle F^{\boldsymbol j}\cdot\nabla^\perp(\xi_j^2), \omega^{\boldsymbol j}\rangle\, d\tau + M_{2,j,\xi_j}[F]\|\xi_j(\nabla\omega)^{\boldsymbol j}\|_{L^2(0,\tau_0;L^2)} + C\kappa\nu^{\frac12} j_2 M_{2,j,\xi_j}[F] M_{2,j-1,\xi_{j-1}}[\partial_Y\omega].$$

Then, from Assumption (iv) and Lemma 4.3, and by recalling

$$\nabla^\perp(\xi_j^2) = -\frac{\nabla^\perp(\partial_Y\Omega + 2\rho_j)}{\sqrt{\partial_Y\Omega + 2\rho_j}}\xi_j^3 = -\frac{\nabla^\perp\partial_Y\Omega}{\sqrt{\partial_Y\Omega+2\rho_j}}\xi_j^3 - 2(\nabla^\perp\rho_j)\xi_j^4,$$

we have

$$
\begin{aligned}
&\langle F^j \cdot \nabla^{\perp}(\xi_j^2), \omega^j \rangle \\
&\quad \le \|\xi_j F^j\| \Biggl(\biggl\| \frac{Y\nabla(\partial_Y\Omega + 2\rho_j)}{(1+\nu^{1/2}Y)\sqrt{\partial_Y\Omega + 2\rho_j}} \xi_j^2 \biggr\|_{L^\infty(\{0<Y<(j+1)^{-1/2}\})} \biggl\| \frac{1+\nu^{1/2}Y}{Y} \omega^j \biggr\|_{L^2(\{0<Y<(j+1)^{-1/2}\})} \\
&\qquad + \biggl\| \frac{Y\nabla\partial_Y\Omega}{(1+\nu^{1/2}Y)\sqrt{\partial_Y\Omega + 2\rho_j}} \biggr\|_{L^\infty(\{Y\ge(j+1)^{-1/2}\})} \biggl\| \frac{1+\nu^{1/2}Y}{Y} \xi_j \biggr\|_{L^\infty(\{Y\ge(j+1)^{-1/2}\})} \|\xi_j\omega^j\| \\
&\qquad + \|Y\partial_Y\rho_j\xi_j^2\|_{L^\infty} \biggl\| \frac{1}{Y} \biggr\|_{L^\infty(\{Y\ge(j+1)^{-1/2}\})} \|\xi_j\omega^j\| \Biggr) \\
&\quad \le C\|\xi_j F^j\| \Biggl(\Bigl(C_2^* \|\xi_j^2\|_{L^\infty(\{0<Y<(j+1)^{-1/2}\})} + \|Y\partial_Y\rho_j\xi_j^2\|_{L^\infty} \|\xi_j\|_{L^\infty(\{0<Y<(j+1)^{-1/2}\})} \Bigr) \biggl\| \frac{\omega^j}{Y} \biggr\| \\
&\qquad + (C_2^*+1)(j+1)^{\frac{1}{2}} \|\xi_j\omega^j\| \Biggr) \\
&\quad \le C\|\xi_j F^j\| \Biggl(\Bigl(\frac{C_2^*}{K^{1/4}C_*} + \frac{1}{(K^{1/4}C_*)^{1/2}} \Bigr) \|\partial_Y\omega^j\| + (C_2^*+1)(j+1)^{\frac{1}{2}} \|\xi_j\omega^j\| \Biggr).
\end{aligned}
$$

Thus, the estimate (4-24) for $\partial_Y\omega^j$ yields the desired estimate by taking K large enough depending only on C_1^* and C_*. □

Proof of Proposition 4.4. We are now in position to prove Proposition 4.4. Lemmas 4.5–4.11 imply that, by taking the supremum over $j_2 = 0, \dots, j$,

$$
\begin{aligned}
&\nu^{\frac{1}{4}} M_{2,j,\xi_j}[\nabla\omega] + M_{\infty,j,\xi_j}[\omega] + (K\nu^{\frac{1}{2}}(j+1))^{\frac{1}{2}} M_{2,j,\xi_j}[\omega] \\
&\le C\Biggl(\sup_{j_2=0,\dots,j} \|\xi_j\omega^j(0)\| + \kappa\nu^{\frac{1}{4}}\nu^{\frac{1}{2}} j M_{2,j-1,\xi_{j-1}}[\nabla\omega] \\
&\quad + \frac{R_{j,\text{Lemma 4.7}}[\nabla\phi]}{\nu^{1/4}(j+1)^{1/2}} + \frac{\kappa^{-1}R_{j,\text{Lemma 4.9}}[\omega] + R_{j,\text{Lemma 4.10}}[\nabla\phi] + M_{2,j,\xi_j}[G]}{(K\nu^{1/2}(j+1))^{1/2}} + (C_2^*+1)\nu^{-\frac{1}{4}} M_{2,j,\xi_j}[F] \Biggr)
\end{aligned}
$$

for $j = 0, 1, \dots, \nu^{-1/2}$. Here $K \ge 1$ is taken large enough depending only on C_* and C_j^*, while $C > 0$ is a universal constant. Hence, by taking the sum $\sum_{j=0}^{\nu^{-1/2}}$ with the factor $1/((j!)^{3/2}\nu^{j/2})$, we obtain

$$
\begin{aligned}
&|||\nabla\omega|||'_{2,\tilde{\xi}^{(1)}} + |||\omega|||'_{\infty,\xi} + K^{\frac{1}{2}} |||\omega|||'_{2,\xi} \\
&\quad \le C\Biggl(\sum_{j=0}^{\nu^{-1/2}} \frac{1}{(j!)^{3/2}\nu^{j/2}} \sup_{j_2=0,\dots,j} \|\xi_j\omega^j(0)\| + \kappa |||\nabla\omega|||'_{2,\tilde{\xi}^{(1)}} + \frac{C_0^*}{K^{1/2}\kappa} |||\omega|||'_{2,\xi} \\
&\qquad + \Bigl(1 + \frac{C_0^*+C_2^*}{K^{1/2}}\Bigr) |||\nabla\phi|||'_{2,\mathbf{1}} + \frac{1}{K^{1/2}\nu^{1/2}} |||G|||'_{2,\tilde{\xi}^{(2)}} + \frac{C_2^*+1}{\nu^{1/2}} |||F|||'_{2,\tilde{\xi}^{(1)}} \Biggr).
\end{aligned}
$$

Thus we obtain (4-19) by first taking $\kappa > 0$ small enough and then by taking K large enough, and also by using $\xi_j \le 1/(C_*\nu^{1/2}) \le 1/\nu^{1/2}$ to bound $\|\xi_j\omega^j(0)\|$. Note that the required smallness on κ is independent of ν, K, C_*, and C_j^*, while the required largeness of K depends only on κ, C_*, and C_j^*. The proof of Proposition 4.4 is complete. □

4B. ***Estimate for the velocity in terms of the vorticity.*** In this subsection we give the estimate of the stream function ϕ in terms of the vorticity ω. We remind the reader that $\omega = -\Delta\phi$ with the boundary condition $\phi|_{Y=0} = 0$.

Proposition 4.12. *There exists $\kappa_2 \in (0, 1]$ such that, for any $K \geq 1$, $\kappa \in (0, \kappa_2]$, and $p \in [1, \infty]$,*

$$\|\!|\nabla\phi|\!\|'_{p,\mathbf{1}} \leq C(K^{\frac{1}{4}}C_* + C_1^*)^{\frac{1}{2}} \|\!|\omega|\!\|'_{p,\xi} + C\nu^{1/(2p)} \|\nabla\phi^{(0,0)}\|_{L^p(0,1/(K\nu^{1/2});L^2_{X,Y})}.$$

Here $C > 0$ is a universal constant.

Proof. It suffices to show

$$\sum_{j=1}^{\nu^{-1/2}} \frac{\nu^{1/(2p)}(j+1)^{1/p}}{(j!)^{3/2}\nu^{j/2}} M_{p,j,1}[\nabla\phi] \leq C(K^{\frac{1}{4}}C_* + C_1^*)^{\frac{1}{2}} \|\!|\omega|\!\|'_{p,\xi} + C\nu^{1/(2p)} \|\nabla\phi^{(0,0)}\|_{L^p(0,1/(K\nu^{1/2});L^2_{X,Y})}. \tag{4-43}$$

Let $j \geq 1$, and let us recall that $\omega^{\boldsymbol{j}} = e^{-K\tau\nu^{1/2}(j+1)} B_{j_2} \partial_X^{j-j_2}\omega$ with $\omega = -\Delta\phi$. Computations similar to those in (4-21) imply

$$\omega^{\boldsymbol{j}} = -\nabla\cdot(\nabla\phi)^{\boldsymbol{j}} + \frac{\nu^{1/2} j_2 \chi'_\nu}{\chi_\nu}(\partial_Y\phi)^{\boldsymbol{j}}.$$

Then integration by parts together with the identity

$$\nabla\phi^{\boldsymbol{j}} = (\nabla\phi)^{\boldsymbol{j}} + \nu^{\frac{1}{2}} j_2 \chi'_\nu e^{-K\tau\nu^{1/2}} (\partial_Y\phi)^{(j-j_2,j_2-1)} \boldsymbol{e}_2$$

yields

$$\langle\omega^{\boldsymbol{j}}, \phi^{\boldsymbol{j}}\rangle = \|(\nabla\phi)^{\boldsymbol{j}}\|^2 + 2\nu^{\frac{1}{2}} j_2 e^{-K\tau\nu^{1/2}} \langle\chi'_\nu(\partial_Y\phi)^{\boldsymbol{j}}, (\partial_Y\phi)^{(j-j_2,j_2-1)}\rangle. \tag{4-44}$$

Then $\langle\omega^{\boldsymbol{j}}, \phi^{\boldsymbol{j}}\rangle \leq \|\xi_j\omega^{\boldsymbol{j}}\| \|\phi^{\boldsymbol{j}}/\xi_j\|$, and the definition of ξ_j in (4-9) gives

$$\left\|\frac{\phi^{\boldsymbol{j}}}{\xi_j}\right\| = \|\sqrt{\partial_Y\Omega + 2\rho_j}\,\phi^{\boldsymbol{j}}\| \leq \left\|\left(\frac{1+Y}{1+\nu^{1/2}Y}\right)^2 \partial_Y\Omega\right\|_{L^\infty}^{1/2} \left\|\frac{1+\nu^{1/2}Y}{1+Y}\phi^{\boldsymbol{j}}\right\| + \sqrt{2}\|\sqrt{\rho_j}\,\phi^{\boldsymbol{j}}\|$$
$$\leq (C_1^*)^{\frac{1}{2}}(C\|\partial_Y\phi^{\boldsymbol{j}}\| + \nu^{\frac{1}{2}}\|\phi^{\boldsymbol{j}}\|) + \sqrt{2}\|\sqrt{\rho_j}\,\phi^{\boldsymbol{j}}\|.$$

Here we have used Assumption (iii) and the Hardy inequality. Next the definition of ρ_j in (4-10) implies

$$\sqrt{\rho_j} \leq K^{\frac{1}{8}} C_*^{\frac{1}{2}} (1 + (j+1)^{\frac{1}{2}} Y)^{-1} + C_*^{\frac{1}{2}} \left(\left(1 + \frac{Y}{\nu^{1/4}}\right)^{-1} + \nu^{\frac{1}{4}}(1+Y)^{-1} + \nu^{\frac{1}{2}}\right),$$

which gives, from the Hardy inequality, $\nu^{1/2}(j+1) \leq 2$, and $K \geq 1$,

$$\|\sqrt{\rho_j}\,\phi^{\boldsymbol{j}}\| \leq CK^{\frac{1}{8}} C_*^{\frac{1}{2}} (j+1)^{-\frac{1}{2}} \|\partial_Y\phi^{\boldsymbol{j}}\| + C_*^{\frac{1}{2}} \nu^{\frac{1}{2}} \|\phi^{\boldsymbol{j}}\|.$$

Thus we have

$$\left\|\frac{\phi^{\boldsymbol{j}}}{\xi_j}\right\| \leq C(C_1^* + K^{\frac{1}{4}}C_*)^{\frac{1}{2}} \|\partial_Y\phi^{\boldsymbol{j}}\| + C(C_1^* + C_*)^{\frac{1}{2}} \nu^{\frac{1}{2}} \|\phi^{\boldsymbol{j}}\|.$$

Thus (4-44) and the identity $\partial_Y \phi^{j} = (\partial_Y \phi)^{j} + \nu^{1/2} j_2 \chi_\nu' e^{-K\tau\nu^{1/2}} (\partial_Y \phi)^{(j-j_2, j_2-1)}$ finally give

$$\|(\nabla\phi)^{j}\| \le C(C_1^* + K^{\frac14} C_*)^{\frac12} \|\xi_j \omega^{j}\| + C\kappa \nu^{\frac12} j_2 \|(\partial_Y \phi)^{(j-j_2, j_2-1)}\| + \tfrac{1}{16} \nu^{\frac12} \|\phi^{j}\|.$$

Here $C > 0$ is a universal constant. Taking the supremum over $j_2 = 0, \dots, j$ yields

$$M_{p,j,1}[\nabla\phi] \le C(C_1^* + K^{\frac14} C_*)^{\frac12} M_{p,j,\xi_j}[\omega] + C\kappa \nu^{\frac12} j M_{p,j-1,1}[\nabla\phi] + \tfrac{1}{16} \nu^{\frac12} M_{p,j,1}[\phi].$$

Thus we have, from $M_{p,j,1}[\phi] \le M_{p,j-1,1}[\nabla\phi]$ and $(j+1)/j \le 2$ for $j \ge 1$,

$$\sum_{j=1}^{\nu^{-1/2}} \frac{\nu^{1/(2p)} (j+1)^{1/p}}{(j!)^{3/2} \nu^{j/2}} M_{p,j,1}[\nabla\phi] \le C(K^{\frac14} C_* + C_1^*)^{\frac12} \|\|\omega\|\|'_{p,\xi} + \Big(C\kappa + \frac18\Big) \sum_{j=0}^{\nu^{-1/2}} \frac{\nu^{1/(2p)} (j+1)^{1/p}}{(j!)^{3/2} \nu^{j/2}} M_{p,j,1}[\nabla\phi].$$

Here $C > 0$ is a universal constant. By taking κ small enough we obtain (4-43). □

In view of the estimate in Proposition 4.12, our next task is to show the estimate of the zeroth-order term $\nabla\phi^{(0,0)}$.

Proposition 4.13. *Let $\kappa_2 \in (0, 1]$ be the number in Proposition 4.12. There exists $K_2 = K_2(C_*, C_1^*) \ge 1$ such that, for any $K \ge K_2$ and $\kappa \in (0, \kappa_2]$,*

$$\nu^{\frac14} \|\omega^{(0,0)}\|_{L^2(0,1/(K\nu^{1/2}); L^2_{X,Y})} + \|\nabla\phi^{(0,0)}\|_{L^\infty(0,1/(K\nu^{1/2}); L^2_{X,Y})} + K^{\frac12} \nu^{\frac14} \|\nabla\phi^{(0,0)}\|_{L^2(0,1/(K\nu^{1/2}); L^2_{X,Y})}$$
$$\le C\bigg(\|W_0\|_{L^2_{X,Y}} + \frac{1}{K^{1/2}\nu^{1/4}} \|F\|_{L^2(0,1/(K\nu^{1/2}); L^2_{X,Y})} + \frac{1}{K^{1/2}\nu^{1/4}} \|G\|_{L^2(0,1/(K\nu^{1/2}); \dot H^{-1})} + \|\|\omega\|\|'_{2,\xi}\bigg). \tag{4-45}$$

Here $C > 0$ is a universal constant.

Proof. It suffices to show

$$\nu^{\frac14} \|\omega^{(0,0)}\|_{L^2(0,1/(K\nu^{1/2}); L^2_{X,Y})} + \|\nabla\phi^{(0,0)}\|_{L^\infty(0,1/(K\nu^{1/2}); L^2_{X,Y})} + K^{\frac12} \nu^{\frac14} \|\nabla\phi^{(0,0)}\|_{L^2(0,1/(K\nu^{1/2}); L^2_{X,Y})}$$
$$\le C\bigg(\|W_0\|_{L^2_{X,Y}} + \frac{1}{K^{1/2}\nu^{1/4}} \|F\|_{L^2(0,1/(K\nu^{1/2}); L^2_{X,Y})} + \frac{1}{K^{1/2}\nu^{1/4}} \|G\|_{L^2(0,1/(K\nu^{1/2}); \dot H^{-1})} + \frac{C_1^*}{K^{1/2}} \|\|\partial_Y \phi\|\|'_{2,\mathbf{1}}\bigg). \tag{4-46}$$

Indeed, estimate (4-45) is a direct consequence of (4-46) and Proposition 4.12 by taking K large enough depending only on C_1^* and C_*. To prove (4-45), let us go back to (4-1) and take the inner product with $\eta_R \phi$ for (4-1), where $\eta_R = \eta(Y/R)$ with a smooth cut-off η such that $\eta = 1$ for $0 \le Y \le 1$ and $\eta = 0$ for $Y \ge 1$. Then, taking the limit $R \to \infty$ after integration by parts verifies the identity

$$\nu^{\frac12} \|\omega^{(0,0)}\|^2 + \frac12 \frac{d}{d\tau} \|\nabla\phi^{(0,0)}\|^2 + K\nu^{\frac12} \|\nabla\phi^{(0,0)}\|^2 = -\langle \Delta\phi^{(0,0)}, V \cdot \nabla\phi^{(0,0)} \rangle + \langle F^{(0,0)}, \nabla^\perp \phi^{(0,0)} \rangle + \langle G^{(0,0)}, \phi^{(0,0)} \rangle, \quad \tau > 0. \tag{4-47}$$

Note that $|\langle F^{(0,0)}, \nabla^\perp \phi^{(0,0)}\rangle| \le \|F\|\|\nabla\phi^{(0,0)}\|$ and $|\langle G^{(0,0)}, \phi^{(0,0)}\rangle| \le \|G\|_{\dot H^{-1}}\|\nabla\phi^{(0,0)}\|$. Thus it suffices to focus on the term $-\langle \Delta\phi^{(0,0)}, V\cdot\nabla\phi^{(0,0)}\rangle$. Integration by parts and $\nabla\cdot V = 0$ imply

$$
\begin{aligned}
-\langle \Delta\phi^{(0,0)}, V\cdot\nabla\phi^{(0,0)}\rangle
&= \langle \partial_X\phi^{(0,0)}, (\partial_X V)\cdot\nabla\phi^{(0,0)}\rangle + \langle \partial_Y\phi^{(0,0)}, (\partial_Y V)\cdot\nabla\phi^{(0,0)}\rangle\\
&= \langle \partial_X\phi^{(0,0)}, (\partial_X V)\cdot\nabla\phi^{(0,0)}\rangle - \langle \partial_Y\phi^{(0,0)}, (\partial_X V_2)\partial_Y\phi^{(0,0)}\rangle + \langle \partial_Y\phi^{(0,0)}, (\partial_Y V_1)\partial_X\phi^{(0,0)}\rangle\\
&\le 2C_1^*\nu^{\frac12}\|\nabla\phi^{(0,0)}\|^2 + \langle \partial_Y\phi^{(0,0)}, (\partial_Y V_1)\partial_X\phi^{(0,0)}\rangle.
\end{aligned}
$$

Here we have used Assumption (ii). Then the last term is estimated as

$$
\begin{aligned}
\langle \partial_Y\phi^{(0,0)}, (\partial_Y V_1)\partial_X\phi^{(0,0)}\rangle &\le \left\| \frac{Y}{1+\nu^{1/2}Y}\partial_Y V_1\right\|_{L^\infty} \|\partial_Y\phi^{(0,0)}\| \left\| \frac{1+\nu^{1/2}Y}{Y}\partial_X\phi^{(0,0)}\right\|\\
&\le C_1^*\|\partial_Y\phi^{(0,0)}\|(C\|\partial_{XY}^2\phi^{(0,0)}\| + \nu^{\frac12}\|\partial_X\phi^{(0,0)}\|).
\end{aligned}
$$

Here we have used Assumption (ii) and the Hardy inequality. Hence, by taking K large enough depending only on C_1^*, we obtain

$$
\nu^{\frac12}\|\omega^{(0,0)}\|^2 + \frac12\frac{d}{d\tau}\|\nabla\phi^{(0,0)}\|^2 + K\nu^{\frac12}\|\nabla\phi^{(0,0)}\|^2 \le \frac{C(C_1^*)^2}{K\nu^{1/2}}\|\partial_X\partial_Y\phi^{(0,0)}\|^2 + C(\|F\|^2 + \|G\|_{\dot H^{-1}}^2).
$$

Integrating about τ shows (4-46), for $\nu^{-1/2}\|\partial_X\partial_Y\phi^{(0,0)}\|^2_{L^2(0,1/(K\nu^{1/2});L^2_{X,Y})} \le (|||\partial_Y\phi^{(0,0)}|||'_{2,\mathbf{1}})^2$ holds. $\square$

4C. *Proof of Proposition 4.1.* Propositions 4.12 and 4.13 yield

$$
K^{\frac14}|||\nabla\phi|||'_{2,\mathbf{1}} \le C\biggl(K^{\frac14}(K^{\frac14}C_* + C_1^*)^{\frac12}|||\omega|||'_{2,\boldsymbol{\xi}} + \|W_0\|_{L^2_{X,Y}} + \frac{1}{K^{1/2}\nu^{1/4}}\|F\|_{L^2(0,1/(K\nu^{1/2});L^2_{X,Y})} + \frac{1}{K^{1/2}\nu^{1/4}}\|G\|_{L^2(0,1/(K\nu^{1/2});\dot H^{-1})}\biggr). \tag{4-48}
$$

Then (4-48) and Proposition 4.4 give

$$
\begin{aligned}
&|||\omega|||'_{\infty,\xi} + K^{\frac12}|||\omega|||'_{2,\xi} + K^{\frac14}|||\nabla\phi|||'_{2,\mathbf{1}}\\
&\quad\le C\biggl(\|W_0\|_{L^2_{X,Y}} + \nu^-[\|\mathrm{rot}\, W_0\|] + (C_2^*+1)\nu^{-\frac12}|||F|||'_{2,\tilde\xi^{(1)}} + \frac{1}{K^{1/2}\nu^{1/2}}|||G|||'_{2,\tilde\xi^{(2)}} + \frac{1}{K^{1/2}\nu^{1/4}}\|G\|_{L^2(0,1/(K\nu^{1/2});\dot H^{-1})}\biggr).
\end{aligned} \tag{4-49}
$$

It remains to estimate the boundary trace $|||\partial_Y\phi|_{Y=0}|||_{\mathrm{bc}}$. By the interpolation inequality we have

$$
|\partial_X^j\partial_Y\phi(\tau,X,0)| \le C\|\partial_X^j\partial_Y^2\phi(\tau,X,\cdot)\|_{L^2_Y}^{1/2}\|\partial_X^j\partial_Y\phi(\tau,X,\cdot)\|_{L^2_Y}^{1/2},
$$

which implies

$$
\begin{aligned}
&K^{\frac14}\|\partial_Y\phi^{(j,0)}|_{Y=0}\|_{L^2(0,1/(K\nu^{1/2});L^2_X)}\\
&\quad\le CK^{\frac14}\|\partial_Y^2\phi^{(j,0)}\|^{\frac12}_{L^2(0,1/(K\nu^{1/2});L^2_{X,Y})}\|\partial_Y\phi^{(j,0)}\|^{\frac12}_{L^2(0,1/(K\nu^{1/2});L^2_{X,Y})}\\
&\quad\le C(K^{\frac14}\|\omega^{(j,0)}\|_{L^2(0,1/(K\nu^{1/2});L^2_{X,Y})})^{\frac12}(K^{\frac14}\|\partial_Y\phi^{(j,0)}\|_{L^2(0,1/(K\nu^{1/2});L^2_{X,Y})})^{\frac12}.
\end{aligned} \tag{4-50}
$$

Here we used the Calderón–Zygmund inequality. Since (4-18) yields

$$\|\omega^{(j,0)}\|_{L^2(0,1/(K\nu^{1/2});L^2_{X,Y})} \le (C_1^* + 8K^{\frac14}C_*)^{\frac12} M_{2,j,\cdot\xi_j}[\omega],$$

we have from (4-49) that, by taking K further large enough if necessary,

$$\begin{aligned} K^{\frac14}|||\partial_Y\phi|_{Y=0}|||_{\mathrm{bc}} &\le C(K^{\frac12}|||\omega|||'_{2,\xi})^{\frac12}(K^{\frac14}|||\nabla\phi|||'_{2,\mathbf{1}})^{\frac12} \\ &\le C\biggl(\|W_0\|_{L^2_{X,Y}} + \nu^{-}[\|\mathrm{rot}\, W_0\|] + (C_2^*+1)\nu^{-\frac12}|||F|||'_{2,\tilde\xi^{(1)}} \\ &\qquad + \frac{1}{K^{1/2}\nu^{1/2}}|||G|||'_{2,\tilde\xi^{(2)}} + \frac{1}{K^{1/2}\nu^{1/4}}\|G\|_{L^2(0,1/(K\nu^{1/2});\dot H^{-1})}\biggr). \end{aligned}$$

The proof of Proposition 4.1 is complete. □

5. Construction of the boundary corrector

In the previous section, we constructed a solution to the vorticity equation with arbitrary initial data but artificial boundary conditions: we replaced condition $W_1|_{Y=0} = 0$ by $\omega|_{Y=0} = 0$. Hence, to prove Theorem 2.3, we still need to understand how to correct the Neumann condition, that is how to construct solutions for systems of the following type:

$$\begin{aligned} -\nu^{\frac12}\Delta\omega + \partial_\tau\omega + V\cdot\nabla\omega + \nabla^\perp\phi\cdot\nabla\Omega = 0, &\quad \tau>0,\quad X\in\mathbb{T}_\nu,\quad Y>0, \\ \phi|_{Y=0} = 0,\quad \partial_Y\phi|_{Y=0} = h,&\quad \phi|_{\tau=0}=0. \end{aligned} \tag{5-1}$$

Here $\phi(\tau,\cdot)$ is the stream function associated with the vorticity $\omega(\tau,\cdot)$, i.e., $\phi\in\dot H^1_0(\mathbb{T}_\nu\times\mathbb{R}_+)$ is the unique solution to $-\Delta\phi=\omega$ subject to the zero Dirichlet boundary condition. Such a construction will be performed through an iteration, with first approximation given by the Stokes equation.

5A. ***Stokes estimate.*** In this subsection we consider the solution to the Stokes equations (in terms of the stream function):

$$\begin{aligned} -\nu^{\frac12}\Delta\omega + \partial_\tau\omega = 0, &\quad \tau>0,\quad X\in\mathbb{T}_\nu,\quad Y>0, \\ \phi|_{Y=0}=0,\quad \partial_Y\phi|_{Y=0}=h,&\quad \phi|_{\tau=0}=0. \end{aligned} \tag{5-2}$$

Here $\phi\in\dot H^1_0(\mathbb{T}_\nu\times\mathbb{R}_+)$ is the stream function associated with ω, and h is a given boundary data satisfying $h(\tau)=0$ for $\tau=0$ and $\tau\ge 1/(K\nu^{1/2})$, and the norm $|||h|||_{\mathrm{bc}}$ is defined as

$$|||h|||_{\mathrm{bc}} = \sum_{j=0}^{\nu^{-1/2}} \frac{\nu^{1/4}(j+1)^{1/2}}{(j!)^{3/2}\nu^{j/2}} \|e^{-K\tau\nu^{1/2}(j+1)}\partial_X^j h\|_{L^2(0,1/(K\nu^{1/2});L^2_X)} < \infty. \tag{5-3}$$

Set $\psi = e^{-K\tau\nu^{1/2}(j+1)}\partial_X^{j_1}\phi$, $0\le j_1\le j$, with the zero extension for $\tau\le 0$, and let $\hat\psi=\hat\psi(\lambda,\alpha,Y)$ be the Fourier (in X and τ) transform of ψ. Then, since $-\Delta\phi=\omega$, the function $\hat\psi$ obeys the ODE

$$\begin{aligned} \nu^{\frac12}(\partial_Y^2-\alpha^2)^2\hat\psi - (i\lambda + K\nu^{\frac12}(j+1))(\partial_Y^2-\alpha^2)\hat\psi = 0, &\quad Y>0, \\ \hat\psi|_{Y=0}=0,\quad \partial_Y\hat\psi|_{Y=0} = \hat g^{(j_1)}, \end{aligned} \tag{5-4}$$

where $\lambda \in \mathbb{R}$ and $\hat{g}^{(j_1)}$ is the Fourier transform of $g^{(j_1)} := e^{-K\tau\nu^{1/2}(j+1)}\partial_X^{j_1} h$. We note that

$$\alpha = \nu^{\frac{1}{2}} n, \tag{5-5}$$

where n is the n-th Fourier mode in the original variable $x \in \mathbb{T}$. Assuming the decay of $(|\alpha|\hat{\psi}, \partial_Y \hat{\psi})$ and the boundedness of $\hat{\psi}$, we obtain the formula

$$\begin{aligned} \hat{\psi}(\lambda, \alpha, Y) &= -\frac{e^{-\gamma Y} - e^{-|\alpha| Y}}{\gamma - |\alpha|} \hat{g}^{(j_1)}(\lambda, \alpha), \\ \gamma = \gamma_j(\lambda, \alpha, \nu, K) &= \sqrt{\alpha^2 + K(j+1) + \frac{i\lambda}{\nu^{1/2}}}, \end{aligned} \tag{5-6}$$

where the square root is taken so that the real part is positive, and it follows that

$$|\alpha| \le \sqrt{\alpha^2 + K(j+1)} \le \operatorname{Re}(\gamma) \le |\gamma| \le \sqrt{2}\operatorname{Re}(\gamma). \tag{5-7}$$

This inequality will be freely used. We can also check the identity

$$\partial_Y \hat{\psi}(\lambda, \alpha, Y) = -e^{-\gamma Y} \hat{g}^{(j_1)}(\lambda, \alpha) + \operatorname{sgn}(\alpha)\alpha \hat{\psi}(\lambda, \alpha, Y). \tag{5-8}$$

We also have, from (5-6),

$$-(\partial_Y^2 - \alpha^2)\hat{\psi} = (\gamma + |\alpha|) e^{-\gamma Y} \hat{g}^{(j_1)}. \tag{5-9}$$

This formula will be used in estimating the vorticity field.

Lemma 5.1. *There exists $\kappa' \in (0, 1]$ such that the following statement holds for any $\kappa \in (0, \kappa']$. Let $j_1 = 0, \dots, j$ and $j_2 = j - j_1$. Then*

$$|B_{j_2} i\alpha \hat{\psi}(\lambda, \alpha, Y)| \le \frac{C\nu^{j_2/2} j_2! |\alpha \hat{g}^{(j_1)}|}{j_2 + 1} \left(Y e^{-\operatorname{Re}(\gamma)Y/2} + e^{-|\alpha|Y/2} \left| \frac{1 - e^{-(\gamma - |\alpha|)Y}}{\gamma - |\alpha|} \right| \right), \tag{5-10}$$

$$|B_{j_2} \partial_Y \hat{\psi}(\lambda, \alpha, Y)| \le \frac{C\nu^{j_2/2} j_2! |\hat{g}^{(j_1)}|}{j_2 + 1} e^{-\operatorname{Re}(\gamma)Y/2}. \tag{5-11}$$

As a consequence,

$$\left(\sum_{\alpha \in \nu^{1/2}\mathbb{Z}} \|B_{j_2} i\alpha \hat{\psi}(\cdot, \alpha, \cdot)\|^2_{L^2_{\lambda,Y}} + \|B_{j_2} \partial_Y \hat{\psi}(\cdot, \alpha, \cdot)\|^2_{L^2_{\lambda,Y}} \right)^{\frac{1}{2}} \le \frac{C\nu^{j_2/2} j_2!}{K^{1/4}(j+1)^{1/4}(j_2+1)} \left(\sum_{\alpha \in \nu^{1/2}\mathbb{Z}} \|\hat{g}^{(j_1)}(\cdot, \alpha)\|^2_{L^2_\lambda} \right)^{\frac{1}{2}}. \tag{5-12}$$

We also have

$$\left(\sum_{\alpha \in \nu^{1/2}\mathbb{Z}} \left\| \frac{1}{1+Y} B_{j_2} i\alpha \hat{\psi}(\cdot, \alpha, \cdot) \right\|^2_{L^2_{\lambda,Y}} \right)^{\frac{1}{2}} \le \frac{C\nu^{j_2/2} j_2!}{K^{1/2}(j+1)^{1/2}(j_2+1)} \left(\sum_{\alpha \in \nu^{1/2}\mathbb{Z}} \|\alpha \hat{g}^{(j_1)}(\cdot, \alpha)\|^2_{L^2_\lambda} \right)^{\frac{1}{2}}. \tag{5-13}$$

Here $C > 0$ is a universal constant.

Proof. We first show (5-10) for $B_{j_2} i\alpha\hat{\psi}$. It suffices to consider the case $j_2 \ge 1$, for the case $j_2 = 0$ is trivial from (5-6). We observe from (5-6) that

$$B_{j_2}\hat{\psi} = -\frac{\hat{g}^{(j_1)}\chi_\nu^{j_2}}{\gamma - |\alpha|}((-\gamma)^{j_2} e^{-\gamma Y} - (-|\alpha|)^{j_2} e^{-|\alpha|Y})$$
$$= -\frac{(-\gamma)^{j_2} - (-|\alpha|)^{j_2}}{\gamma - |\alpha|}\chi_\nu^{j_2} e^{-\gamma Y}\hat{g}^{(j_1)} + (-|\alpha|)^{j_2}\chi_\nu^{j_2} e^{-|\alpha|Y}\hat{g}^{(j_1)}\frac{1 - e^{-(\gamma - |\alpha|)Y}}{\gamma - |\alpha|}. \tag{5-14}$$

Since

$$(-\gamma)^{j_2} - (-|\alpha|)^{j_2} = (-1)^{j_2}\sum_{l_2=0}^{j_2-1}\binom{j_2}{l_2}(\gamma - |\alpha|)^{j_2-l_2}|\alpha|^{l_2},$$

we have, from $\binom{j_2}{l_2} \le j_2\binom{j_2-1}{l_2}$ for $0 \le l_2 \le j_2 - 1$,

$$\left|\frac{(-\gamma)^{j_2} - (-|\alpha|)^{j_2}}{\gamma - |\alpha|}\right| \le \sum_{l_2=0}^{j_2-1}\binom{j_2}{l_2}\big|\gamma - |\alpha|\big|^{j_2-l_2-1}|\alpha|^{l_2} \le j_2\sum_{l_2=0}^{j_2-1}\binom{j_2-1}{l_2}\big|\gamma - |\alpha|\big|^{j_2-l_2-1}|\alpha|^{l_2}$$
$$= j_2\big(\big|\gamma - |\alpha|\big| + |\alpha|\big)^{j_2-1} \le j_2(3|\gamma|)^{j_2-1}.$$

Here we have used $|\alpha| \le |\gamma|$ by (5-7). Then the inequality $\chi_\nu = 1 - e^{-\kappa\nu^{1/2}Y} \le \kappa\nu^{1/2}Y$ implies

$$\left|\frac{(-\gamma)^{j_2} - (-|\alpha|)^{j_2}}{\gamma - |\alpha|}\chi_\nu^{j_2} e^{-\gamma Y}\right| \le j_2\kappa\nu^{\frac{1}{2}}Y(3\kappa\nu^{\frac{1}{2}}|\gamma|Y)^{j_2-1}e^{-\operatorname{Re}(\gamma)Y}$$
$$\le j_2\kappa\nu^{j_2/2}Y(3\sqrt{2}\kappa\operatorname{Re}(\gamma)Y)^{j_2-1}e^{-\operatorname{Re}(\gamma)Y} \quad \text{(by (5-7))}.$$

From the bound $r^k e^{-r} \le (k/e)^k$ and the Stirling bound $(k/e)^k \le (2\pi)^{-1/2}k^{-1/2}k!$ for $k \in \mathbb{N}$, we have

$$\big(\tfrac{1}{2}\operatorname{Re}(\gamma)Y\big)^{j_2-1}e^{-\operatorname{Re}(\gamma)Y/2} \le \frac{(j_2-1)!}{\sqrt{2\pi}(j_2-1)^{1/2}}, \quad j_2 \ge 2.$$

This gives, when $6\sqrt{2}\kappa \le \frac{1}{2}$,

$$\left|\frac{(-\gamma)^{j_2} - (-|\alpha|)^{j_2}}{\gamma - |\alpha|}\chi_\nu^{j_2} e^{-\gamma Y}\right| \le \frac{\nu^{j_2/2}j_2!}{(j_2+1)}Ye^{-\operatorname{Re}(\gamma)Y/2}, \quad j_2 \ge 1.$$

Similarly, we have, for $j_2 \ge 1$,

$$|(-|\alpha|)^{j_2}\chi_\nu^{j_2}e^{-|\alpha|Y}| \le \frac{\nu^{j_2/2}j_2!}{j_2+1}e^{-|\alpha|Y/2}.$$

Hence (5-10) for $B_{j_2} i\alpha\hat{\psi}$ follows by collecting these with (5-14). The estimate for $B_{j_2}\partial_Y\hat{\psi}$ is proved in the same manner in view of (5-8), and we omit the details. Estimate (5-12) follows from (5-10) and the Plancherel theorem, by observing the estimates for the multipliers

$$\|\alpha Ye^{-\operatorname{Re}(\gamma)Y/2}\|_{L^2_Y} \le \frac{C}{K^{1/4}(j+1)^{1/4}}, \tag{5-15}$$

$$\left\|\alpha e^{-|\alpha|Y/2}\left|\frac{1 - e^{-(\gamma - |\alpha|)Y}}{\gamma - |\alpha|}\right|\right\|_{L^2_Y} \le \frac{C}{K^{1/4}(j+1)^{1/4}}. \tag{5-16}$$

Here $C > 0$ is a universal constant. Estimate (5-15) is a consequence of (5-7). As for (5-16), we divide into two cases. (i) The case $|\alpha| \le \frac{1}{2}K^{1/2}(j+1)^{1/2}$: in this case we have, from (5-7),

$$|\gamma - |\alpha|| \ge |\gamma| - |\alpha| \ge \frac{|\alpha| + K^{1/2}(j+1)^{1/2}}{C}$$

with a universal constant $C > 0$, which gives

$$\begin{aligned}\left\| \alpha e^{-|\alpha|Y/2} \left| \frac{1-e^{-(\gamma-|\alpha|)Y}}{\gamma - |\alpha|} \right| \right\|_{L^2_Y} &\le \frac{C}{|\alpha| + K^{1/2}(j+1)^{1/2}} \|\alpha e^{-|\alpha|Y/2}\|_{L^2_Y} \\ &\le \frac{C|\alpha|^{1/2}}{|\alpha| + K^{1/2}(j+1)^{1/2}} \le \frac{C}{K^{1/4}(j+1)^{1/4}}.\end{aligned}$$

(ii) The case $|\alpha| \ge \frac{1}{2}K^{1/2}(j+1)^{1/2}$: in this case we used the bound

$$\sup_{\mathrm{Re}(z)>0} \left| \frac{1-e^{-z}}{z} \right| \le C,$$

which gives

$$\left\| \alpha e^{-|\alpha|Y/2} \left| \frac{1-e^{-(\gamma-|\alpha|)Y}}{\gamma - |\alpha|} \right| \right\|_{L^2_Y} \le C \|\alpha Y e^{-|\alpha|Y/2}\|_{L^2_Y} \le \frac{C}{|\alpha|^{1/2}} \le \frac{C}{K^{1/4}(j+1)^{1/4}}.$$

The proof of (5-16) is complete, and (5-12) is proved. Estimate (5-13) is proved similarly by using (5-10), the Plancherel theorem, and

$$\left\| \frac{Y}{1+Y} e^{-\mathrm{Re}(\gamma)Y/2} \right\|_{L^2_Y} \le \frac{C}{K^{3/4}(j+1)^{3/4}}, \tag{5-17}$$

$$\left\| \frac{1}{1+Y} e^{-|\alpha|Y/2} \left| \frac{1-e^{-(\gamma-|\alpha|)Y}}{\gamma - |\alpha|} \right| \right\|_{L^2_Y} \le \frac{C}{K^{1/2}(j+1)^{1/2}}. \tag{5-18}$$

Here $C > 0$ is a universal constant. Indeed, (5-17) is straightforward, while in (5-18), the estimate becomes worse due to the case $|\alpha| \le \frac{1}{2}K^{1/2}(j+1)^{1/2}$ with $|\alpha| \ll 1$, where we compute

$$\left\| \frac{1}{1+Y} e^{-|\alpha|Y/2} \left| \frac{1-e^{-(\gamma-|\alpha|)Y}}{\gamma - |\alpha|} \right| \right\|_{L^2_Y} \le \frac{C}{|\alpha| + K^{1/2}(j+1)^{1/2}} \left\| \frac{1}{1+Y} e^{-|\alpha|Y/2} \right\|_{L^2_Y} \le \frac{C}{K^{1/2}(j+1)^{1/2}}.$$

Here we essentially use the factor $1/(1+Y)$ to obtain the uniform estimate in α. □

In Propositions 5.2 and 5.4 below we give estimates for the solution to (5-1) given by the formula as above in terms of the Fourier transform. We always take κ small enough such that $\kappa \in (0, \kappa']$ as in Lemma 5.1.

Proposition 5.2 (estimate for velocity). *It follows that*

$$\sum_{j=0}^{\nu^{-1/2}} \frac{\nu^{1/4}(j+1)^{3/4}}{(j!)^{3/2}\nu^{j/2}} M_{2,j,1}[\nabla\phi] + \sum_{j=0}^{\nu^{-1/2}} \frac{1}{(j!)^{3/2}\nu^{j/2+1/4}(j+1)^{1/2}} M_{2,j,1/(1+Y)}[\partial_X \phi] \le \frac{C}{K^{1/4}} |||h|||_{\mathrm{bc}}. \tag{5-19}$$

Here $C > 0$ is a universal constant.

Proof. Assume that $M_{2,j,1}[\nabla\phi] = \|(\nabla\phi)^{\boldsymbol{j}}\|_{L^2(0,1/(K\nu^{1/2});L^2_{X,Y})}$ for some $\boldsymbol{j} = (j_1, j_2)$ with $j_1 + j_2 = j$. Note that this j_1 depends on j, and we write $j_1[j]$ if necessary. By the Plancherel theorem the estimate (5-12) implies

$$\|(\nabla\phi)^{\boldsymbol{j}}\|_{L^2(0,1/(K\nu^{1/2});L^2_{X,Y})} \le \frac{C\nu^{j-j_1[j]/2}(j-j_1[j])!}{K^{1/4}(j+1)^{1/4}(j-j_1[j]+1)}\|h^{(j_1)}\|_{L^2(0,1/(K\nu^{1/2});L^2_X)},$$
$$h^{(j_1)} = e^{-K\tau\nu^{1/2}(j_1+1)}\partial_X^{j_1}h.$$

Thus we have

$$\begin{aligned}
&\sum_{j=0}^{\nu^{-1/2}} \frac{\nu^{1/4}(j+1)^{3/4}}{(j!)^{3/2}\nu^{j/2}} M_{2,j,1}[\nabla\phi] \\
&\quad\le \frac{C}{K^{1/4}}\sum_{j=0}^{\nu^{-1/2}} \binom{j}{j-j_1[j]}^{-1}\frac{1}{(j-j_1[j]+1)}\Big(\frac{j_1[j]!}{j!}\Big)^{\frac12}\Big(\frac{j+1}{j_1[j]+1}\Big)^{\frac12} \\
&\qquad\qquad\times\Big(\frac{\nu^{1/4}(j_1[j]+1)^{1/2}}{(j_1[j]!)^{3/2}\nu^{j_1[j]/2}}\|h^{(j_1[j])}\|_{L^2(0,1/(K\nu^{1/2});L^2_X)}\Big).
\end{aligned}$$

We decompose the summation in the right-hand side as $\sum_{j_1[j]=j}$ (i.e., j's such that $0 \le j \le \nu^{-1/2}$ and $j_1[j] = j$) and $\sum_{j_1[j]\le j-1}$ (i.e., j's such that $0 \le j \le \nu^{-1/2}$ and $j_1[j] \le j-1$). Then the sum of $\sum_{j_1[j]=j}$ is bounded from above by $|||h|||_{\mathrm{bc}}$, while the sum of $\sum_{j_1[j]\le j-1}$ is bounded as

$$\begin{aligned}
&\sum_{j_1[j]\le j-1} \binom{j}{j-j_1[j]}^{-1}\frac{1}{(j-j_1[j]+1)}\Big(\frac{j_1[j]!}{j!}\Big)^{\frac12}\Big(\frac{j+1}{j_1[j]+1}\Big)^{\frac12} \\
&\qquad\qquad\times\Big(\frac{\nu^{1/4}(j_1[j]+1)^{1/2}}{(j_1[j]!)^{3/2}\nu^{j_1[j]/2}}\|h^{(j_1[j])}\|_{L^2(0,1/(K\nu^{1/2});L^2_X)}\Big) \\
&\quad\le \sum_{j_1[j]\le j-1} \binom{j}{j-j_1[j]}^{-1}\frac{1}{(j-j_1[j]+1)}\Big(\frac{j_1[j]!}{j!}\Big)^{\frac12}\Big(\frac{j+1}{j_1[j]+1}\Big)^{\frac12}\sup_{0\le k\le\nu^{-1/2}} \\
&\qquad\qquad\times\Big(\frac{\nu^{1/4}(k+1)^{1/2}}{(k!)^{3/2}\nu^{k/2}}\|h^{(k)}\|_{L^2(0,1/(K\nu^{1/2});L^2_X)}\Big) \\
&\quad\le C|||h|||_{\mathrm{bc}}.
\end{aligned}$$

Indeed, it suffices to use

$$\sum_{j_1[j]\le j-1} \binom{j}{j-j_1[j]}^{-1}\frac{1}{(j-j_1[j]+1)}\Big(\frac{j_1[j]!}{j!}\Big)^{\frac12}\Big(\frac{j+1}{j_1[j]+1}\Big)^{\frac12} \le C\sum_{j_1[j]\le j-1}(j+1)^{-\frac32} \le C. \tag{5-20}$$

Next we prove the estimate about $M_{2,j,1/(1+Y)}[\partial_X\phi]$. Arguing as above, we have from (5-13) that, for $0 \le j \le \nu^{-1/2} - 1$,

$$M_{2,j,1/(1+Y)}[\partial_X\phi] \le \frac{C\nu^{(j-j_1[j])/2}(j-j_1[j])!}{K^{1/2}(j+1)^{1/2}(j-j_1[j]+1)}\|\partial_X h^{(j_1[j])}\|_{L^2(0,1/(K\nu^{1/2});L^2_X)},$$

where $j_1[j]$ is taken similarly as in the above argument. Thus we have

$$\begin{aligned}
&\sum_{j=0}^{\nu^{-1/2}} \frac{1}{(j!)^{3/2}\nu^{j/2+1/4}(j+1)^{1/2}} M_{2,j,1/(1+Y)}[\partial_X\phi] \\
&\le \frac{C}{K^{1/2}} \sum_{j=0}^{\nu^{-1/2}-1} \frac{(j-j_1[j])!}{(j!)^{3/2}\nu^{j_1/2+1/4}(j+1)(j-j_1[j]+1)} \|\partial_X h^{(j_1)}\|_{L^2(0,1/(K\nu^{1/2});L^2_X)} \\
&\qquad\qquad + \frac{M_{2,j,1/(1+Y)}[\partial_X\phi]}{(j!)^{3/2}\nu^{j/2+1/4}(j+1)^{1/2}}\bigg|_{j=\nu^{-1/2}} \\
&\le \frac{C}{K^{1/2}} \sum_{j=0}^{\nu^{-1/2}-1} \frac{(j-j_1[j])!}{(j!)^{3/2}\nu^{j_1/2+1/4}(j+1)(j-j_1[j]+1)} \|h^{(j_1+1)}\|_{L^2(0,1/(K\nu^{1/2});L^2_X)} \\
&\qquad\qquad + \frac{\nu^{1/4}(j+1)^{3/4} M_{2,j,1}[\partial_X\phi]}{(j!)^{3/2}\nu^{j/2}}\bigg|_{j=\nu^{-1/2}}.
\end{aligned}$$

The second term is bounded from above by $(C/K^{1/4})|||h|||_{\mathrm{bc}}$, as we have shown above. As for the first term, we again decompose the summation $\sum_{j=0}^{\nu^{-1/2}-1}$ into $\sum_{j_1[j]=j}$ and $\sum_{j_1[j]\le j-1}$, as we have done previously. Then the sum of $\sum_{j_1[j]=j}$ is bounded from above by $C|||h|||_{\mathrm{bc}}$, while the sum of $\sum_{j_1[j]\le j-1}$ is estimated as

$$\begin{aligned}
&\sum_{j_1[j]\le j-1} \frac{(j-j_1[j])!}{(j!)^{3/2}\nu^{j_1[j]/2+1/4}(j+1)(j-j_1[j]+1)} \|h^{(j_1[j]+1)}\|_{L^2(0,1/(K\nu^{1/2});L^2_X)} \\
&\le \sum_{j_1[j]\le j-1} \frac{(j-j_1[j])!\,(j_1[j]+1)!}{j!}\left(\frac{(j_1[j]+1)!}{j!}\right)^{\frac12} \frac{1}{(j+1)(j_1[j]+1)^{1/2}(j-j_1[j]+1)} \\
&\qquad\qquad \times \sup_{0\le k\le \nu^{-1/2}} \left(\frac{\nu^{1/4}(k+1)^{1/2}}{(k!)^{3/2}\nu^{k/2}} \|h^{(k)}\|_{L^2(0,1/(K\nu^{1/2});L^2_X)}\right) \\
&\le C\sum_{j=0}^{\nu^{-1/2}} \frac{1}{(j+1)^{3/2}} \sup_{0\le k\le \nu^{-1/2}} \left(\frac{\nu^{1/4}(k+1)^{1/2}}{(k!)^{3/2}\nu^{k/2}} \|h^{(k)}\|_{L^2(0,1/(K\nu^{1/2});L^2_X)}\right) \le C|||h|||_{\mathrm{bc}}. \qquad \square
\end{aligned}$$

Next we show the estimate for the vorticity field. The argument is similar to the one for the velocity.

Lemma 5.3. *There exists $\kappa'' \in (0,1]$ such that the following statement holds for any $\kappa \in (0,\kappa'']$. Let $j_1 = 0,\dots,j$ and $j_2 = j - j_1$. Then*

$$|B_{j_2}(\partial_Y^2-\alpha^2)\hat\psi(\lambda,\alpha,Y)| + |YB_{j_2}\partial_Y(\partial_Y^2-\alpha^2)\hat\psi(\lambda,\alpha,Y)| \le \frac{C\nu^{j_2/2}j_2!}{j_2+1}|\gamma|e^{-\mathrm{Re}(\gamma)Y/2}|\hat g^{(j_1)}|. \tag{5-21}$$

As a consequence, for $\theta' \in \left[-\frac12, 2\right]$,

$$\begin{aligned}
&\biggl(\sum_{\alpha\in\nu^{1/2}\mathbb{Z}} \|Y^{1+\theta'}B_{j_2}(\partial_Y^2-\alpha^2)\hat\psi(\,\cdot\,,\alpha,\cdot\,)\|^2_{L^2_{\lambda,Y}} + \|Y^{2+\theta'}B_{j_2}\alpha(\partial_Y^2-\alpha^2)\hat\psi(\,\cdot\,,\alpha,\cdot\,)\|^2_{L^2_{\lambda,Y}} \\
&\qquad\qquad + \|Y^{2+\theta'}B_{j_2}\partial_Y(\partial_Y^2-\alpha^2)\hat\psi(\,\cdot\,,\alpha,\cdot\,)\|^2_{L^2_{\lambda,Y}}\biggr)^{\frac12} \\
&\le \frac{C\nu^{j_2/2}j_2!}{K^{\theta'/2+1/4}(j+1)^{\theta'/2+1/4}(j_2+1)}\biggl(\sum_{\alpha\in\nu^{1/2}\mathbb{Z}} \|\hat g^{(j_1)}(\,\cdot\,,\alpha)\|^2_{L^2_\lambda}\biggr)^{\frac12}.
\end{aligned} \tag{5-22}$$

Here $C > 0$ is a universal constant.

Proof. Estimate (5-21) follows from (5-9) by arguing as in the proof of (5-10). Estimate (5-22) then follows from (5-21), the Plancherel theorem, and

$$\|Y^{1+m}|\gamma|e^{-\operatorname{Re}(\gamma)Y/2}\|_{L^2_Y} \le \frac{C}{(\operatorname{Re}(\gamma))^{m+1/2}} \le \frac{C}{(|\alpha|+K^{1/2}(j+1)^{1/2})^{m+1/2}} \quad \text{(by (5-7))}$$

for $m \in \left[-\frac{1}{2}, 3\right]$. The details are omitted here. □

Proposition 5.4 (estimate for vorticity). *Let* $\theta \in [0, 2]$. *It follows that*

$$\sum_{j=0}^{\nu^{-1/2}} \frac{(j+1)^{1/4}}{(j!)^{3/2}\nu^{j/2}} \nu^{\frac{1}{4}}(j+1)^{\frac{1}{2}}(M_{2,j,Y}[\omega]+M_{2,j,Y^2}[\nabla\omega]) \le \frac{C}{K^{1/4}}|||h|||_{\mathrm{bc}} \tag{5-23}$$

and

$$\sum_{j=0}^{\nu^{-1/2}} \frac{(j+1)^{(\theta-1)/2}}{(j!)^{3/2}\nu^{j/2+1/4}} (M_{2,j,Y^{3/2+\theta}}[\partial_X\omega]+\nu^{\frac{1}{2}}M_{2,j,Y^{3/2+\theta}}[\partial_Y\omega]) \le \frac{C}{K^{\theta/2}}|||h|||_{\mathrm{bc}}. \tag{5-24}$$

Here $C > 0$ *is a universal constant.*

Proof. Estimate (5-23) is a consequence of (5-22) with $\theta' = 0$, by introducing $j_1[j]$ as in the proof of Proposition 5.2. As for (5-24), we have from (5-22) with $\theta' = \theta - \frac{1}{2}$ that

$$\sum_{j=0}^{\nu^{-1/2}} \frac{\nu^{1/4}(j+1)^{(\theta-1)/2}}{(j!)^{3/2}\nu^{j/2}} M_{2,j,Y^{3/2+\theta}}[\partial_Y\Delta\phi] \le \frac{C}{K^{\theta/2}}|||h|||_{\mathrm{bc}}.$$

Next we have from $M_{2,j,Y^{3/2+\theta}}[\partial_X\Delta\phi] \le C M_{2,j+1,Y^{3/2+\theta}}[\Delta\phi]$ that

$$\begin{aligned}\sum_{j=0}^{\nu^{-1/2}-1} \frac{(j+1)^{(\theta-1)/2}}{(j!)^{3/2}\nu^{j/2+1/4}} M_{2,j,Y^{3/2+\theta}}[\partial_X\Delta\phi] &\le C\sum_{j=0}^{\nu^{-1/2}-1} \frac{(j+1)^{(\theta-1)/2}}{(j!)^{3/2}\nu^{j/2+1/4}} M_{2,j+1,Y^{3/2+\theta}}[\Delta\phi] \\ &= C\sum_{j=0}^{\nu^{-1/2}-1} \frac{\nu^{1/4}(j+1)^{3/2+(\theta-1)/2}}{((j+1)!)^{3/2}\nu^{(j+1)/2}} M_{2,j+1,Y^{3/2+\theta}}[\Delta\phi] \\ &= C\sum_{j=1}^{\nu^{-1/2}} \frac{\nu^{1/4}j^{\theta/2+1}}{(j!)^{3/2}\nu^{j/2}} M_{2,j,Y^{3/2+\theta}}[\Delta\phi].\end{aligned}$$

By arguing as in the proof of Proposition 5.2, the application of (5-22) gives

$$C\sum_{j=1}^{\nu^{-1/2}} \frac{\nu^{1/4}j^{\theta/2+1}}{(j!)^{3/2}\nu^{j/2}} M_{2,j,Y^{3/2+\theta}}[\Delta\phi] \le \frac{C}{K^{\theta+1/2}} \sum_{j=1}^{\nu^{-1/2}} \frac{\nu^{1/4}(j+1)^{1/2}}{(j!)^{3/2}\nu^{j/2}} \|e^{-K\tau\nu^{1/2}(j+1)}\partial_X^j h\|_{L^2(0,1/(K\nu^{1/2});L^2_X)},$$

where the smoothing factor $(j+1)^{-\theta'/2-1/4}$ with $\theta'=\theta+\frac{1}{2}$ in (5-22) plays a key role. When $j=\nu^{-1/2}$, we have

$$\begin{aligned}
&\frac{(j+1)^{(\theta-1)/2}}{(j!)^{3/2}\nu^{j/2+1/4}} M_{2,j,Y^{3/2+\theta}}[\partial_X \Delta\phi]\Big|_{j=\nu^{-1/2}} \\
&\quad \le \frac{\nu^{1/4}(j+1)^{(\theta+1)/2}}{(j!)^{3/2}\nu^{j/2}} M_{2,j,Y^{3/2+\theta}}[\partial_X \Delta\phi]\Big|_{j=\nu^{-1/2}} \\
&\quad \le \frac{C}{K^{\theta/2}} \sum_{j=0}^{\nu^{-1/2}} \frac{\nu^{1/4}(j+1)^{1/2}}{(j!)^{3/2}\nu^{j/2}} \|e^{-K\tau\nu^{1/2}(j+1)}\partial_X^j h\|_{L^2(0,1/(K\nu^{1/2});L^2_X)} \quad \big(\text{by (5-22) with } \theta'=\theta-\tfrac{1}{2}\big) \\
&\quad \le \frac{C}{K^{\theta/2}} |||h|||_{\mathrm{bc}}. \qquad \square
\end{aligned}$$

5B. ***Vorticity transport estimate.*** Propositions 5.2 and 5.4 of the previous paragraph reflect a strong difference between the weighted fields $(\nabla\phi)^j$ and $(\Delta\phi)^j$ associated to the Stokes solution ϕ of (5-1): the former is not localized near the boundary, while the latter is, at scale $(K(j+1))^{-1/2}$. This is due to a harmonic nonlocalized part in ϕ, see expression (5-6). As a consequence, as shown in Proposition 5.4, for the vorticity field the weight Y^θ gives a gain of $(j+1)^{-\theta/2}$. In particular, the transport term $V\cdot\nabla\Delta\phi$ shares similar properties. When working in the Gevrey class $\frac{3}{2}$, this term can be seen to be formally of the same size as the Stokes term $\nu^{1/2}\Delta^2\phi-\partial_\tau\Delta\phi$. Hence, we need to add one step to our iteration in which we solve the heat-transport equations

$$\begin{aligned}
-\nu^{\frac{1}{2}}\Delta\omega+\partial_\tau\omega+V\cdot\nabla\omega=H, &\quad \tau>0,\ \ X\in\mathbb{T}_\nu,\ \ Y>0, \\
\phi|_{Y=0}=\omega|_{Y=0}=0, &\quad \phi|_{\tau=0}=0.
\end{aligned} \tag{5-25}$$

Here $\phi\in\dot H^1_0(\mathbb{T}_\nu\times\mathbb{R}_+)$ is the stream function associated with ω, and the source term $H\in L^2\dot H^{-1}$ will be the transport term created by the Stokes approximation. A key point in dealing with this equation rather than with the full vorticity equation is that we will be able to propagate weighted estimates with weight Y^θ, which is crucial to have sharp bounds. In the last step of our iteration, we will correct nonlocal stretching terms using the vorticity equation with artificial boundary conditions, using the bounds of Section 4. The main result of this paragraph is:

Proposition 5.5. *There exists $K_3=K_3(C_1^*)\ge 1$ such that if $K\ge K_3$ then the system (5-25) admits a unique solution $\omega\in C([0,1/(K\nu^{1/2})];L^2\cap\dot H^{-1})\cap L^2(0,1/(K\nu^{1/2});H^1_0)$ satisfying, for $0\le j\le\nu^{-1/2}$, $\kappa\in(0,1]$, and $\theta=0,1,2$,*

$$\begin{aligned}
&\nu^{\frac{1}{4}}M_{2,j,Y^\theta}[\nabla\omega]+M_{\infty,j,Y^\theta}[\omega]+K^{\frac{1}{2}}\nu^{\frac{1}{4}}(j+1)^{\frac{1}{2}}M_{2,j,Y^\theta}[\omega] \\
&\quad \le C\bigg(\kappa\nu^{\frac{3}{4}}jM_{2,j-1,Y^\theta}[\nabla\omega]+\nu^{\frac{1}{4}}\theta M_{2,j,Y^{\theta-1}}[\omega]+\frac{1}{K^{1/4}\nu^{1/4}(j+1)^{1/4}}M_{2,j,Y^{\theta+1/2}}[H] \\
&\qquad +\frac{1}{\kappa K^{1/2}\nu^{1/4}(j+1)^{1/2}}\sum_{l=0}^{j-1}\min\{l+1,j-l+1\}\binom{j}{l}N_{\infty,j-l}[V]M_{2,l+1,Y^\theta}[\omega]\bigg).
\end{aligned} \tag{5-26}$$

Here $C>0$ is a universal constant.

Remark 5.6. The solution ω to (5-25) in Proposition 5.5 has the regularity

$$(\partial_\tau - \nu^{\frac{1}{2}}\Delta)Y^\theta\omega \in L^2_{\text{loc}}([0,\infty); L^2(\mathbb{T}_\nu \times \mathbb{R}_+)), \quad \theta = 0, 1, 2,$$

with the Dirichlet boundary condition. Hence, the maximal regularity for the heat equation implies

$$\partial_\tau Y^\theta\omega,\ \Delta(Y^\theta\omega) \in L^2_{\text{loc}}([0,\infty); L^2(\mathbb{T}_\nu \times \mathbb{R}_+)).$$

To prove Proposition 5.5 let us recall that $\omega^j = e^{-K\tau\nu^{1/2}(j+1)}B_{j_2}\partial_X^{j-j_2}\omega$ satisfies

$$\begin{aligned}&-\nu^{\frac{1}{2}}(\Delta\omega)^j + \partial_\tau\omega^j + K\nu^{\frac{1}{2}}(j+1)\omega^j + V\cdot\nabla\omega^j \\ &= -V_2[B_{j_2},\partial_Y]e^{-K\tau\nu^{1/2}(j+1)}\partial_X^{j_1}\omega - \sum_{l=0}^{j-1}\sum_{\max\{0,l+j_2-j\}\le l_2\le\min\{l,j_2\}}\binom{j_2}{l_2}\binom{j-j_2}{l-l_2}V^{j-l}\cdot(\nabla\omega)^l + H^j. \quad (5\text{-}27)\end{aligned}$$

Then (5-26) is proved by taking the inner product in (5-27) with $Y^{2\theta}\omega^j$ for each $\theta = 0, 1, 2$, and then by taking the supremum over $j_2 = 0, \dots, j$ and about $\tau_0 \in (0, 1/(K\nu^{1/2})]$. Hence the proof proceeds as in the proof of Proposition 4.4.

Lemma 5.7. *There exists $C > 0$ such that, for any $K \ge 1$ and $\kappa \in (0,1]$,*

$$\begin{aligned}\int_0^{\tau_0}\langle -\nu^{\frac{1}{2}}(\Delta\omega)^j, Y^{2\theta}\omega^j\rangle\,d\tau \ge \tfrac{3}{4}\nu^{\frac{1}{2}}\|Y^\theta(\nabla\omega)^j\|^2_{L^2(0,\tau_0;L^2_{X,Y})} - C\nu^{\frac{1}{2}}(\kappa\nu_2^j)^2 M_{2,j-1,Y^\theta}[\partial_Y\omega]^2 \\ - C\theta^2\nu^{\frac{1}{2}}M_{2,j,Y^{\theta-1}}[\omega]^2.\end{aligned}$$

Proof. The proof is similar to (and much simpler than) the one of Lemma 4.5. Indeed, the only difference is the presence of the weight $Y^{2\theta}$ with $\theta = 0, 1, 2$, which creates the term

$$2\theta\nu^{\frac{1}{2}}\int_0^{\tau_0}\langle Y^\theta(\partial_Y\omega)^j, Y^{\theta-1}\omega^j\rangle\,d\tau$$

after integration by parts. This is responsible for the last term in the estimate of this lemma. The details are omitted. □

Lemma 5.8. *There exists $K_{3,2} = K_{3,2}(C_1^*) \ge 1$ such that if $K \ge K_{3,2}$ then*

$$\int_0^{\tau_0}\langle\partial_\tau\omega^j + K\nu^{\frac{1}{2}}(j+1)\omega^j + V\cdot\nabla\omega^j, Y^{2\theta}\omega^j\rangle\,d\tau \ge \tfrac{1}{2}\|Y^\theta\omega^j(\tau_0)\|^2 + \tfrac{3}{4}K\nu^{\frac{1}{2}}(j+1)\|Y^\theta\omega^j\|^2_{L^2(0,\tau_0;L^2_{X,Y})}.$$

Proof. The proof is a simple modification of the one of Lemma 4.6. We note that the initial data is taken as zero, and integration by parts gives

$$\int_0^{\tau_0}\langle V\cdot\nabla\omega^j, Y^{2\theta}\omega^j\rangle\,d\tau \le \theta\left\|\frac{V_2}{Y}\right\|_{L^\infty}\|Y^\theta\omega^j\|^2_{L^2(0,\tau_0;L^2)}.$$

Then the desired estimate follows by taking K large enough depending only on C_1^* for $\|V_2/Y\|_{L^\infty} \le \|\partial_Y V_2\|_{L^\infty} = \|\partial_X V_1\|_{L^\infty} \le C_1^*\nu^{1/2}$. The details are omitted. □

Lemma 5.9. *Let $j_2 \geq 1$. It follows that*

$$\int_0^{\tau_0} \langle -V_2[B_{j_2}, \partial_Y] e^{-K\tau\nu^{1/2}(j+1)} \partial_X^{j_1}\omega, Y^{2\theta}\omega^{j}\rangle \, d\tau \leq C C_1^* \nu^{\frac{1}{2}} j_2 \|Y^\theta \omega^{j}\|^2_{L^2(0,\tau_0; L^2_{X,Y})}.$$

Here $C > 0$ is a universal constant.

Proof. The proof is similar to the one of Lemma 4.7. The details are omitted here. □

Lemma 5.10. *Let $j \geq 1$. It follows that*

$$\int_0^{\tau_0} \Big\langle -\sum_{l=0}^{j-1} \sum_{\max\{0, l+j_2-j\} \leq l_2 \leq \min\{l, j_2\}} \binom{j_2}{l_2}\binom{j-j_2}{l-l_2} V^{j-l} \cdot (\nabla\omega)^{l}, Y^{2\theta}\omega^{j} \Big\rangle d\tau \leq \frac{C}{\kappa} R_{j,\text{Lemma 5.10}}[\omega] M_{2,j,Y^\theta}[\omega],$$

where

$$R_{j,\text{Lemma 5.10}}[\omega] = \sum_{l=0}^{j-1} \min\{l+1, j-l+1\} \binom{j}{l} N_{\infty, j-l}[V] M_{2,l+1,Y^\theta}[\omega].$$

Here $C > 0$ is a universal constant, and $N_{\infty,j-l}[V]$ is defined as in Lemma 4.9.

Proof. The proof is similar to the one of Lemma 4.9. The details are omitted here. □

Lemma 5.11. *It follows that*

$$\int_0^{\tau_0} \langle H^{j}, Y^{2\theta}\omega^{j}\rangle \, d\tau \leq \begin{cases} C M_{2,j,Y^{\theta+1/2}}[H] (M_{2,j,Y^\theta}[\partial_Y\omega] + \kappa\nu^{\frac{1}{2}} j M_{2,j-1,Y^\theta}[\nabla\omega])^{\frac{1}{2}} (M_{2,j,Y^\theta}[\omega])^{\frac{1}{2}}, & \theta = 0, \\ C M_{2,j,Y^{\theta+1/2}}[H] (M_{2,j,Y^{\theta-1}}[\omega])^{\frac{1}{2}} (M_{2,j,Y^\theta}[\omega])^{\frac{1}{2}}, & \theta = 1, 2. \end{cases}$$

Here $C > 0$ is a universal constant.

Proof. The estimate follows from the inequality

$$\langle H^{j}, Y^{2\theta}\omega^{j}\rangle \leq \|Y^{\theta+\frac{1}{2}} H^{j}\| \|Y^{\theta-\frac{1}{2}}\omega^{j}\| \leq \|Y^{\theta+\frac{1}{2}} H^{j}\| \|Y^{\theta-1}\omega^{j}\|^{\frac{1}{2}} \|Y^\theta\omega^{j}\|^{\frac{1}{2}}$$

and the Hardy inequality for $\theta = 0$:

$$\|Y^{-1}\omega^{j}\| \leq C\|\partial_Y\omega^{j}\| \leq C(\|(\partial_Y\omega)^{j}\| + \kappa\nu^{\frac{1}{2}} j_2 \|(\partial_Y\omega)^{(j_1,j_2-1)}\|).$$ □

Proof of Proposition 5.5. It suffices to show the estimate (5-26), but it follows from Lemmas 5.7–5.11 by considering the cases $\theta = 0$ and $\theta = 1, 2$ separately. The details are omitted here. □

Corollary 5.12. *There exists $\kappa_3 \in (0, 1]$ such that the following statement holds for any $\kappa \in (0, \kappa_3]$. There exists $K_3' = K_3'(\kappa, C_0^*, C_1^*) \geq 1$ such that if $K \geq K_3'$ then the system (5-25) admits a unique solution $\omega \in C([0, 1/(K\nu^{1/2})]; L^2 \cap \dot{H}^{-1}) \cap L^2(0, 1/(K\nu^{1/2}); H_0^1)$ satisfying, for $\theta = 0, 1, 2$,*

$$\sum_{j=0}^{\nu^{-1/2}} \frac{(j+1)^{\theta/2-1/4}}{(j!)^{3/2}\nu^{j/2}} (\nu^{\frac{1}{4}} M_{2,j,Y^\theta}[\nabla\omega] + M_{\infty,j,Y^\theta}[\omega] + K^{\frac{1}{2}}\nu^{\frac{1}{4}}(j+1)^{\frac{1}{2}} M_{2,j,Y^\theta}[\omega]) \leq \frac{C}{K^{1/4}} \sum_{\theta'=0}^{\theta} \sum_{j=0}^{\nu^{-1/2}} \frac{1}{(j!)^{3/2}\nu^{j/2+1/4}(j+1)^{(1-\theta')/2}} M_{2,j,Y^{\theta'+1/2}}[H], \quad (5\text{-}28)$$

and

$$|||\nabla\phi|||_{2,\mathbf{1}}' + |||\partial_Y\phi|_{Y=0}|||_{\mathrm{bc}} \le \frac{C}{K^{3/4}} \sum_{\theta'=0}^{1} \sum_{j=0}^{\nu^{-1/2}} \frac{1}{(j!)^{3/2}\nu^{j/2+1/4}(j+1)^{(1-\theta')/2}} M_{2,j,Y^{\theta'+1/2}}[H]. \tag{5-29}$$

Here $C > 0$ is a universal constant.

Proof. Let us first show (5-28). By virtue of Proposition 5.5 we have, for $\theta = 0, 1, 2$,

$$\begin{aligned}
&\sum_{\theta'=0}^{\theta} \sum_{j=0}^{\nu^{-1/2}} \frac{(j+1)^{\theta'/2-1/4}}{(j!)^{3/2}\nu^{j/2}} \big(\nu^{\frac14} M_{2,j,Y^{\theta'}}[\nabla\omega] + M_{\infty,j,Y^{\theta'}}[\omega] + K^{\frac12}\nu^{\frac14}(j+1)^{\frac12} M_{2,j,Y^{\theta'}}[\omega]\big)\\
&\quad\le C \sum_{\theta'=0}^{\theta} \sum_{j=0}^{\nu^{-1/2}} \frac{(j+1)^{\theta'/2-1/4}}{(j!)^{3/2}\nu^{j/2}}\\
&\qquad\quad \times \Big(\kappa\nu^{\frac34} j M_{2,j-1,Y^{\theta'}}[\nabla\omega] + \nu^{\frac14}\theta' M_{2,j,Y^{\theta'-1}}[\omega] + \frac{1}{K^{1/4}\nu^{1/4}(j+1)^{1/4}} M_{2,j,Y^{\theta'+1/2}}[H]\\
&\qquad\qquad + \frac{1}{\kappa K^{1/2}\nu^{1/4}(j+1)^{1/2}} \sum_{l=0}^{j-1} \min\{l+1,\, j-l+1\}\binom{j}{l} N_{\infty,j-l}[V] M_{2,l+1,Y^{\theta'}}[\omega]\Big)\\
&\quad\le C\kappa \sum_{\theta'=0}^{\theta} \sum_{j=0}^{\nu^{-1/2}-1} \frac{(j+1)^{\theta'/2-1/4}}{(j!)^{3/2}\nu^{j/2}} \nu^{\frac14} M_{2,j,Y^{\theta'}}[\nabla\omega]\\
&\qquad + C \sum_{\theta'=0}^{\theta} \theta' \sum_{j=0}^{\nu^{-1/2}} \frac{(j+1)^{\theta'-1/2-1/4}}{(j!)^{3/2}\nu^{j/2}} \nu^{\frac14}(j+1)^{\frac12} M_{2,j,Y^{\theta'-1}}[\omega]\\
&\qquad + C \sum_{\theta'=0}^{\theta} \sum_{j=0}^{\nu^{-1/2}} \frac{(j+1)^{\theta'/2-1/4}}{(j!)^{3/2}\nu^{j/2}} \frac{1}{K^{1/4}\nu^{1/4}(j+1)^{1/4}} M_{2,j,Y^{\theta'+1/2}}[H]\\
&\qquad + C \sum_{\theta'=0}^{\theta} \sum_{j=0}^{\nu^{-1/2}} \frac{(j+1)^{\theta'/2-1/4}}{(j!)^{3/2}\nu^{j/2}} \frac{1}{\kappa K^{1/2}\nu^{1/4}(j+1)^{1/2}}\\
&\qquad\qquad \times \sum_{l=0}^{j-1} \min\{l+1,\, j-l+1\}\binom{j}{l} N_{\infty,j-l}[V] M_{2,l+1,Y^{\theta'}}[\omega].
\end{aligned} \tag{5-30}$$

Here $C > 0$ is a universal constant. As for the last term in (5-30), arguing as at the end of the proof of Lemma 4.9, we find that

$$\begin{aligned}
&\sum_{j=0}^{\nu^{-1/2}} \frac{(j+1)^{\theta'/2-1/4}}{(j!)^{3/2}\nu^{j/2}} \frac{1}{K^{1/2}\nu^{1/4}(j+1)^{1/2}} \sum_{l=0}^{j-1} \min\{l+1,\, j-l+1\}\binom{j}{l} N_{\infty,j-l}[V] M_{2,l+1,Y^{\theta'}}[\omega]\\
&\qquad\qquad \le \frac{CC_0^*}{\kappa K^{1/2}} \sum_{j=0}^{\nu^{-1/2}} \frac{(j+1)^{\theta'/2-1/4}}{(j!)^{3/2}\nu^{j/2}} \nu^{\frac14}(j+1)^{\frac12} M_{2,j,Y^{\theta'}}[\omega].
\end{aligned}$$

Hence (5-28) follows by taking κ small enough that $C\kappa \le \frac12$, and then by taking K large enough that $CC_0^*/(\kappa K) \le \frac12$.

To show (5-29) let ϕ be the stream function associated to ω, and it suffices to prove the embedding inequality

$$\|\!|\nabla\phi|\!\|'_{2,\mathbf{1}} \le \sum_{j=0}^{\nu^{-1/2}} \frac{(j+1)^{1/4}}{(j!)^{3/2}\nu^{j/2}} \nu^{\frac14}(j+1)^{\frac12} M_{2,j,1}[\nabla\phi]$$
$$\le C\sum_{j=0}^{\nu^{-1/2}} \frac{(j+1)^{1/4}}{(j!)^{3/2}\nu^{j/2}} \nu^{\frac14}(j+1)^{\frac12} M_{2,j,Y}[\omega] \tag{5-31}$$

and the interpolation inequality

$$\|\!|\partial_Y\phi|_{Y=0}|\!\|_{\mathrm{bc}} := \sum_{j=0}^{\nu^{-1/2}} \frac{\nu^{1/4}(j+1)^{1/2}}{(j!)^{3/2}\nu^{j/2}} \|e^{-K\tau\nu^{1/2}(j+1)}\partial_X^j\partial_Y\phi|_{Y=0}\|_{L^2(0,1/(K\nu^{1/2});L^2_X)}$$
$$\le C\Big(\sum_{j=0}^{\nu^{-1/2}} \frac{(j+1)^{-1/4}}{(j!)^{3/2}\nu^{j/2}} \nu^{\frac14}(j+1)^{\frac12} M_{2,j,1}[\omega]\Big)^{\frac12}\Big(\sum_{j=0}^{\nu^{-1/2}} \frac{(j+1)^{1/4}}{(j!)^{3/2}\nu^{j/2}} \nu^{\frac14}(j+1)^{\frac12} M_{2,j,Y}[\omega]\Big)^{\frac12}. \tag{5-32}$$

Then (5-29) follows from (5-28) with (5-31) and (5-32). The proof of (5-31) proceeds as in the proof of Proposition 4.12. Indeed, from

$$\omega^{\mathbf{j}} = -\nabla\cdot(\nabla\phi)^{\mathbf{j}} + \frac{\nu^{1/2} j_2\chi'_\nu}{\chi_\nu}(\partial_Y\phi)^{\mathbf{j}}$$

and integration by parts, we have

$$\|(\nabla\phi)^{\mathbf{j}}\|^2 = \langle\omega^{\mathbf{j}},\phi^{\mathbf{j}}\rangle - 2\nu^{\frac12} j_2 e^{-K\tau\nu^{1/2}}\langle\chi'_\nu(\partial_Y\phi)^{\mathbf{j}},(\partial_Y\phi)^{(j-j_2,j_2-1)}\rangle$$
$$\le \|Y\omega^{\mathbf{j}}\|\Big\|\frac{\phi^{\mathbf{j}}}{Y}\Big\| + 2\nu^{\frac12} j_2\kappa\|(\partial_Y\phi)^{\mathbf{j}}\|\|(\partial_Y\phi)^{(j-j_2,j_2-1)}\|$$
$$\le C\|Y\omega^{\mathbf{j}}\|\|\partial_Y\phi^{\mathbf{j}}\| + 2\nu^{\frac12} j_2\kappa\|(\partial_Y\phi)^{\mathbf{j}}\|\|(\partial_Y\phi)^{(j-j_2,j_2-1)}\|.$$

Here the Hardy inequality is used in the last line. Then the identity

$$\partial_Y\phi^{\mathbf{j}} = (\partial_Y\phi)^{\mathbf{j}} + \nu^{\frac12} j_2\chi'_\nu e^{-K\tau\nu^{1/2}}(\partial_Y\phi)^{(j-j_2,j_2-1)}$$

yields

$$\|(\nabla\phi)^{\mathbf{j}}\| \le C(\|Y\omega^{\mathbf{j}}\| + \nu^{\frac12} j_2\kappa\|(\partial_Y\phi)^{(j-j_2,j_2-1)}\|).$$

This estimate gives

$$\sum_{j=0}^{\nu^{-1/2}} \frac{(j+1)^{1/4}}{(j!)^{3/2}\nu^{j/2}} \nu^{\frac14}(j+1)^{\frac12} M_{2,j,1}[\nabla\phi]$$
$$\le C\sum_{j=0}^{\nu^{-1/2}} \frac{(j+1)^{1/4}}{(j!)^{3/2}\nu^{j/2}} \nu^{\frac14}(j+1)^{\frac12}(M_{2,j,Y}[\omega] + \nu^{\frac12} j\kappa M_{2,j-1,1}[\nabla\phi])$$
$$\le C\sum_{j=0}^{\nu^{-1/2}} \frac{(j+1)^{1/4}}{(j!)^{3/2}\nu^{j/2}} \nu^{\frac14}(j+1)^{\frac12} M_{2,j,Y}[\omega] + C\kappa\sum_{j=0}^{\nu^{-1/2}} \frac{(j+1)^{1/4}}{(j!)^{\gamma}\nu^{j/2}} \nu^{\frac14}(j+1)^{\frac12} M_{2,j,1}[\nabla\phi],$$

where $C>0$ is a universal constant. This proves (5-31) if κ is small enough that $C\kappa \le \frac{1}{2}$. As for (5-32), we observe from (4-50) that

$$\|e^{-K\tau\nu^{1/2}(j+1)}\partial_X^j \partial_Y\phi|_{Y=0}\|_{L^2(0,1/(K\nu^{1/2});L^2_X)} \le C((j+1)^{-\frac{1}{4}}\|\omega^{(j,0)}\|_{L^2(0,1/(K\nu^{1/2});L^2_X)})^{\frac{1}{2}}((j+1)^{\frac{1}{4}}\|\partial_Y\phi^{(j,0)}\|_{L^2(0,1/(K\nu^{1/2});L^2_X)})^{\frac{1}{2}},$$

which implies, from the Schwarz inequality,

$$|||\partial_Y\phi|_{Y=0}|||_{\mathrm{bc}} \le C\biggl(\sum_{j=0}^{\nu^{-1/2}}\frac{(j+1)^{-1/4}}{(j!)^\gamma\nu^{j/2}}\nu^{\frac{1}{4}}(j+1)^{\frac{1}{2}}M_{2,j,1}[\omega]\biggr)^{\frac{1}{2}}\biggl(\sum_{j=0}^{\nu^{-1/2}}\frac{(j+1)^{1/4}}{(j!)^\gamma\nu^{j/2}}\nu^{\frac{1}{4}}(j+1)^{\frac{1}{2}}M_{2,j,1}[\nabla\phi]\biggr)^{\frac{1}{2}}.$$

Then (5-31) shows (5-32). □

Corollary 5.13. *In Corollary 5.12, let $H=-V\cdot\nabla\omega_{1,1}[h]$, where $\omega_{1,1}[h]$ is the solution to (5-2) in Propositions 5.2 and 5.4. Then*

$$\sum_{j=0}^{\nu^{-1/2}}\frac{(j+1)^{\theta/2-1/4}}{(j!)^{3/2}\nu^{j/2}}(\nu^{\frac{1}{4}}M_{2,j,Y^\theta}[\nabla\omega]+M_{\infty,j,Y^\theta}[\omega]+K^{\frac{1}{2}}\nu^{\frac{1}{4}}(j+1)^{\frac{1}{2}}M_{2,j,Y^\theta}[\omega]) \le \frac{CC_0^*}{K^{1/4}}|||h|||_{\mathrm{bc}} \tag{5-33}$$

and

$$|||\nabla\phi|||'_{2,\mathbf{1}}+|||\partial_Y\phi|_{Y=0}|||_{\mathrm{bc}} \le \frac{CC_0^*}{K^{3/4}}|||h|||_{\mathrm{bc}}. \tag{5-34}$$

Moreover, we have

$$\sum_{j=0}^{\nu^{-1/2}}\frac{1}{(j!)^{3/2}\nu^{j/2+1/4}(j+1)^{1/2}}M_{2,j,1/(1+Y)}[\partial_X\phi] \le \frac{CC_0^*}{K^{3/4}}|||h|||_{\mathrm{bc}}. \tag{5-35}$$

Here $C>0$ is a universal constant.

Proof. To show (5-33) and (5-34), it suffices to prove, for $\theta'=0,1,2$,

$$\sum_{j=0}^{\nu^{-1/2}}\frac{1}{(j!)^{3/2}\nu^{j/2}(j+1)^{(1-\theta')/2}}M_{2,j,Y^{\theta'+1/2}}[H] \le CC_0^*\sum_{j=0}^{\nu^{-1/2}}\frac{1}{(j!)^{3/2}\nu^{j/2}(j+1)^{(1-\theta')/2}}(M_{2,j,Y^{3/2+\theta'}}[\partial_X\omega_{1,1}]+\nu^{\frac{1}{2}}M_{2,j,Y^{3/2+\theta'}}[\partial_Y\omega_{1,1}]). \tag{5-36}$$

Then (5-33) and (5-34) follow from (5-28), (5-29), (5-24) and (5-36). To show (5-36), we observe that

$$H^j=-\sum_{l=0}^{j}\sum_{\max\{0,l+j_2-j\}\le l_2\le\min\{l,j_2\}}\binom{j_2}{l_2}\binom{j-j_2}{l-l_2}V^{j-l}\cdot(\nabla\omega_{1,1})^l.$$

Thus we have

$$\|Y^{\theta'+\frac{1}{2}}H^j\| \le \sum_{l=0}^{j}\binom{j}{l}\sum_{\max\{0,l+j_2-j\}\le l_2\le\min\{l,j_2\}}(\|\partial_Y V_1^{j-l}\|_{L^\infty}\|Y^{\frac{3}{2}+\theta'}(\partial_X\omega_{1,1})^l\|+\|\partial_Y V_2^{j-l}\|_{L^\infty}\|Y^{\frac{3}{2}+\theta'}(\partial_Y\omega_{1,1})^l\|).$$

Set

$$N_{\infty,j}[\nabla V_1] = (j+1)^{\frac{1}{2}} \sup_{j_2=0,\dots,j} (\nu^{-\frac{1}{2}} \|(\partial_X V_1)^{\boldsymbol{j}}\|_{L^\infty_{\tau,X,Y}} + \|(\partial_Y V_1)^{\boldsymbol{j}}\|_{L^\infty_{\tau,X,Y}}). \tag{5-37}$$

Since

$$\begin{aligned}\|\partial_Y V_1^{\boldsymbol{j-l}}\|_{L^\infty} &\le \|(\partial_Y V_1)^{\boldsymbol{j-l}}\|_{L^\infty} + \kappa \nu^{\frac{1}{2}} (j_2 - l_2) \|(\partial_Y V_1)^{(j_1-l_1, j_2-l_2-1)}\|_{L^\infty} \\ &\le (j-l+1)^{-\frac{1}{2}} N_{\infty,j-l}[\nabla V_1] + \kappa \nu^{\frac{1}{2}} (j-l)^{\frac{1}{2}} N_{\infty,j-l-1}[\nabla V_1]\end{aligned}$$

and similarly

$$\begin{aligned}\|\partial_Y V_2^{\boldsymbol{j-l}}\|_{L^\infty} &\le \|(\partial_Y V_2)^{\boldsymbol{j-l}}\|_{L^\infty} + \kappa \nu^{\frac{1}{2}} (j_2 - l_2) \|(\partial_Y V_2)^{(j_1-l_1, j_2-l_2-1)}\|_{L^\infty} \\ &= \|(\partial_X V_1)^{\boldsymbol{j-l}}\|_{L^\infty} + \kappa \nu^{\frac{1}{2}} (j_2 - l_2) \|(\partial_X V_1)^{(j_1-l_1, j_2-l_2-1)}\|_{L^\infty} \\ &\le \nu^{\frac{1}{2}} ((j-l+1)^{-\frac{1}{2}} N_{\infty,j-l}[\nabla V_1] + \kappa \nu^{\frac{1}{2}} (j-l)^{\frac{1}{2}} N_{\infty,j-l-1}[\nabla V_1]),\end{aligned}$$

we obtain

$$\begin{aligned}M_{2,j,Y^{\theta'+1/2}}[H] \le \sum_{l=0}^{j} \binom{j}{l} \min\{l+1,\, j-l+1\} ((j-l+1)^{-\frac{1}{2}} N_{\infty,j-l}[\nabla V_1] + \kappa \nu^{\frac{1}{2}} (j-l)^{\frac{1}{2}} N_{\infty,j-l-1}[\nabla V_1]) \\ \times (M_{2,l,Y^{3/2+\theta'}}[\partial_X \omega_{1,1}] + \nu^{\frac{1}{2}} M_{2,l,Y^{3/2+\theta'}}[\partial_Y \omega_{1,1}]).\end{aligned}$$

Then (5-36) follows from the Young inequality for convolution in the l^1 space. For example, using

$$\frac{(l+1)^{(1-\theta')/2}}{(j+1)^{(1-\theta')/2} (j-l+1)^{1/2}} \le C \quad \text{for } \theta' = 0, 1, 2$$

and

$$\left(\frac{(j-l)!\, l!}{j!} \right)^{\frac{1}{2}} \min\{l+1,\, j-l+1\} \le C,$$

we have

$$\begin{aligned}&\sum_{j=0}^{\nu^{-1/2}} \sum_{l=0}^{j} \frac{1}{(j+1)^{1-\theta'/2}} \left(\frac{(j-l)!\, l!}{j!} \right)^{\frac{1}{2}} \min\{l+1,\, j-l+1\} (j-l+1)^{-\frac{1}{2}} (l+1)^{(1-\theta')/2} \\ &\qquad \times \left(\frac{1}{((j-l)!)^{3/2} \nu^{(j-l)/2}} N_{\infty,j-l}[\nabla V_1] \right) \left(\frac{1}{(l!)^{3/2} \nu^{l/2} (l+1)^{(1-\theta')/2}} M_{2,l,Y^{3/2+\theta'}}[\partial_X \omega_{1,1}] \right) \\ &\le C \sum_{j=0}^{\nu^{-1/2}} \sum_{l=0}^{j} \left(\frac{1}{((j-l)!)^{3/2} \nu^{(j-l)/2}} N_{\infty,j-l}[\nabla V_1] \right) \left(\frac{1}{(l!)^{3/2} \nu^{l/2} (l+1)^{(1-\theta')/2}} M_{2,l,Y^{3/2+\theta'}}[\partial_X \omega_{1,1}] \right) \\ &\le C C_0^* \sum_{j=0}^{\nu^{-1/2}} \frac{1}{(j!)^{3/2} \nu^{j/2} (j+1)^{(1-\theta')/2}} M_{2,j,Y^{3/2+\theta'}}[\partial_X \omega_{1,1}].\end{aligned}$$

The other terms are handled in the same manner and we omit the details. The proof of (5-33)–(5-34) is complete. Finally let us prove (5-35). The key is to apply the interpolation-type inequality proved in

Proposition A.2. Indeed, Proposition A.2 implies, for the stream function ϕ associated with ω,

$$
\begin{aligned}
&\sum_{j=0}^{\nu^{-1/2}} \frac{1}{(j!)^{3/2}\nu^{j/2+1/4}(j+1)^{1/2}} M_{2,j,1/(1+Y)}[\partial_X \phi] \\
&\le C \sum_{\theta=0}^{1} \sum_{j=0}^{\nu^{-1/2}-1} \frac{1}{(j!)^{3/2}\nu^{j/2+1/4}(j+1)^{1/2}} (j+1)^{\theta/2-\frac{1}{4}} M_{2,j+1,Y^{1+\theta}}[\omega] \\
&\qquad + C \sum_{j=0}^{\nu^{-1/2}-1} \frac{1}{(j!)^{3/2}\nu^{j/2+1/4}(j+1)^{1/2}} \kappa \nu^{\frac{1}{2}} j (M_{2,j-1,Y}[\omega] + M_{2,j-1,1}[\nabla\phi]) \\
&\qquad\qquad + \frac{1}{(j!)^{3/2}\nu^{j/2+1/4}(j+1)^{1/2}} M_{2,j,1/(1+Y)}[\partial_X \phi] \Big|_{j=\nu^{-1/2}} \\
&\le C \sum_{\theta=0}^{1} \sum_{j=0}^{\nu^{-1/2}} \frac{\nu^{1/4}(j+1)^{\theta/2+3/4}}{(j!)^{3/2}\nu^{j/2}} M_{2,j,Y^{1+\theta}}[\omega] + C \sum_{j=0}^{\nu^{-1/2}} \frac{\nu^{1/4}(j+1)^{1/2}}{(j!)^{3/2}\nu^{j/2}} M_{2,j,Y}[\omega] + C|||\nabla\phi|||'_{2,\mathbf{1}} \\
&\le \frac{CC_\kappa}{K^{3/4}} |||h|||_{\mathrm{bc}}.
\end{aligned}
$$

Here we have used (5-33) and (5-34) in the last line. □

5C. *Full construction of boundary corrector.* We set

$$\omega_{\mathrm{app},1} = \omega_{\mathrm{app},1}[h] = \omega_{1,1}[h] + \omega_{1,2}[h],$$

where $\omega_{1,1}[h]$ is the solution to (5-2) in Propositions 5.2–5.4, and $\omega_{1,2}[h]$ is the solution to (5-25) with $H = -V \cdot \nabla\omega_{1,1}[h]$ as in Corollary 5.13. Then the approximate solution ω_{app} to the full system (5-1) is constructed in the form

$$\omega_{\mathrm{app}} = \omega_{\mathrm{app},1} + \tilde{\omega}_1,$$

which leads to the equations for $\tilde{\omega}_1 = \tilde{\omega}_1[h]$, as

$$
\begin{aligned}
-\nu^{\frac{1}{2}}\Delta\tilde{\omega}_1 + \partial_\tau\tilde{\omega}_1 + V\cdot\nabla\tilde{\omega}_1 + \nabla^\perp\tilde{\phi}_1\cdot\nabla\Omega &= -\nabla^\perp\phi_{\mathrm{app},1}\cdot\nabla\Omega, \quad \tau>0,\quad X\in\mathbb{T}_\nu,\quad Y>0, \\
\tilde{\phi}_1|_{Y=0} = \tilde{\omega}_1|_{Y=0} &= 0, \quad \tilde{\omega}_1|_{\tau=0} = 0.
\end{aligned} \tag{5-38}
$$

Here $\tilde{\phi}_1$ and $\phi_{\mathrm{app},1}$ are the stream functions associated with $\tilde{\omega}_1$ and $\omega_{\mathrm{app},1}$, respectively. Let us first give the estimate for the force term $-\nabla^\perp\phi_{\mathrm{app},1}\cdot\nabla\Omega$.

Proposition 5.14. *Let $\kappa_3 \in (0,1]$ be the number in Corollary 5.12. For any $\kappa \in (0,\kappa_3]$ there exists $K_3' = K_3'(\kappa, C_*, C_j^*) \ge 1$ such that, for any $K \ge K_3'$,*

$$
\begin{aligned}
&\frac{1}{K^{1/2}\nu^{1/2}} |||\nabla^\perp\phi_{\mathrm{app},1}\cdot\nabla\Omega|||'_{2,\tilde{\xi}^{(2)}} + \frac{1}{K^{1/2}\nu^{1/4}} \|\nabla^\perp\phi_{\mathrm{app},1}\cdot\nabla\Omega\|_{L^2(0,1/(K\nu^{1/2});\dot{H}^{-1})} \\
&\qquad \le \frac{1}{K^{1/4}} \bigg(\sum_{j=0}^{\nu^{-1/2}} \frac{1}{(j!)^{3/2}\nu^{j/2+1/4}(j+1)^{1/2}} M_{2,j,1/(1+Y)}[\partial_X\phi_{\mathrm{app},1}] + 2|||\nabla\phi_{\mathrm{app},1}|||'_{2,\mathbf{1}} \bigg).
\end{aligned} \tag{5-39}
$$

Proof. Let us recall that

$$\frac{1}{\nu^{1/2}}|||\nabla^\perp\phi_{\mathrm{app},1}\cdot\nabla\Omega|||'_{2,\tilde{\xi}^{(2)}} = \sum_{j=0}^{\nu^{-1/2}} \frac{1}{(j!)^{3/2}\nu^{j/2}\nu^{1/4}(j+1)^{1/2}} \sup_{j_2=0,\dots,j} \|\xi_j e^{-K\tau\nu^{1/2}(j+1)} B_{j_2}\partial_X^{j-j_2}(\nabla^\perp\phi_{\mathrm{app},1}\cdot\nabla\Omega)\|_{L^2(0,1/(K\nu^{1/2});L^2_{X,Y})}.$$

Thus we consider the estimate of

$$e^{-K\tau\nu^{1/2}(j+1)} B_{j_2}\partial_X^{j-j_2}(\nabla^\perp\phi_{\mathrm{app},1}\cdot\nabla\Omega) = (\nabla^\perp\phi_{\mathrm{app},1})^{\boldsymbol{j}}\cdot\nabla\Omega + \sum_{l=0}^{j-1}\sum_{\max\{0,l+j_2-j\}\le l_2\le\min\{l,j_2\}} \binom{j_2}{l_2}\binom{j-j_2}{l-l_2}(\nabla^\perp\phi_{\mathrm{app},1})^{\boldsymbol{l}}\cdot(\nabla\Omega)^{\boldsymbol{j}-\boldsymbol{l}},$$

where $\boldsymbol{j}=(j-j_2,j_2)$ and $\boldsymbol{l}=(l-l_2,l_2)$. We observe that, from the definition of ρ_j in (4-10), Assumption (iii), and $K\ge 1$,

$$\begin{aligned}
\|\xi_j\partial_X\phi^{\boldsymbol{j}}_{\mathrm{app},1}\partial_Y\Omega\| &= \left\|\frac{\partial_Y\Omega}{\sqrt{\partial_Y\Omega+2\rho_j}}\partial_X\phi^{\boldsymbol{j}}_{\mathrm{app},1}\right\| \\
&\le C\|(|\partial_Y\Omega|^{\frac12}+\sqrt{\rho_j})\partial_X\phi^{\boldsymbol{j}}_{\mathrm{app},1}\| \\
&\le C\left\|\left(\frac{1+Y}{1+\nu^{1/2}Y}\right)^2\partial_Y\Omega\right\|_{L^\infty}^{\frac12}\left\|\frac{1+\nu^{1/2}Y}{1+Y}\partial_X\phi^{\boldsymbol{j}}_{\mathrm{app},1}\right\| \\
&\qquad + C(K^{1/4}C_*)^{\frac12}\left\|\frac{1}{1+Y}\partial_X\phi^{\boldsymbol{j}}_{\mathrm{app},1}\right\| + CC_*^{\frac12}\nu^{\frac12}\|\partial_X\phi^{\boldsymbol{j}}_{\mathrm{app},1}\| \\
&\le C(C_1^*+K^{1/4}C_*)^{\frac12}\left\|\frac{1}{1+Y}\partial_X\phi^{\boldsymbol{j}}_{\mathrm{app},1}\right\| + C(C_1^*+C_*)^{\frac12}\nu^{\frac12}\|\partial_X\phi^{\boldsymbol{j}}_{\mathrm{app},1}\|.
\end{aligned}$$

On the other hand, we have

$$\begin{aligned}
\|\xi_j(\partial_Y\phi_{\mathrm{app},1})^{\boldsymbol{j}}\partial_X\Omega\| &\le \left\|\frac{1+Y}{1+\nu^{1/2}Y}\partial_X\Omega\right\|_{L^\infty}\left\|\frac{1+\nu^{1/2}Y}{1+Y}\xi_j\right\|_{L^\infty}\|(\partial_Y\phi_{\mathrm{app},1})^{\boldsymbol{j}}\| \\
&\le CC_1^*\nu^{\frac12}(j+1)^{\frac12}\|(\partial_Y\phi_{\mathrm{app},1})^{\boldsymbol{j}}\|.
\end{aligned}$$

Here we have used (4-16) and Assumption (iii). Thus we have, from $C_*\ge 1$,

$$\|\xi_j(\nabla^\perp\phi_{\mathrm{app},1})^{\boldsymbol{j}}\cdot\nabla\Omega\|_{L^2(0,1/(K\nu^{1/2});L^2_{X,Y})} \le C(C_1^*+K^{\frac14}C_*)^{\frac12}M_{2,j,1/(1+Y)}[\partial_X\phi_{\mathrm{app},1}] + C(C_1^*+C_*)\nu^{\frac12}(j+1)^{\frac12}M_{2,j,1}[\partial_Y\phi_{\mathrm{app},1}]. \tag{5-40}$$

Next we see

$$\left\|\xi_j\sum_{l=0}^{j-1}\sum_{\max\{0,l+j_2-j\}\le l_2\le\min\{l,j_2\}}\binom{j_2}{l_2}\binom{j-j_2}{l-l_2}(\nabla^\perp\phi_{\mathrm{app},1})^{\boldsymbol{l}}\cdot(\nabla\Omega)^{\boldsymbol{j}-\boldsymbol{l}}\right\| \le \sum_{l=0}^{j-1}\binom{j}{l}\sum_{\max\{0,l+j_2-j\}\le l_2\le\min\{l,j_2\}}\|\xi_j(\nabla^\perp\phi_{\mathrm{app},1})^{\boldsymbol{l}}\cdot(\nabla\Omega)^{\boldsymbol{j}-\boldsymbol{l}}\|$$

and

$$
\begin{aligned}
&\|\xi_j(\nabla^\perp\phi_{\mathrm{app},1})^{\boldsymbol{l}}\cdot(\nabla\Omega)^{\boldsymbol{j}-\boldsymbol{l}}\| \\
&\le \left\|\left(\frac{1+Y}{1+\nu^{1/2}Y}\right)^2(\partial_Y\Omega)^{\boldsymbol{j}-\boldsymbol{l}}\right\|_{L^\infty}\left\|\frac{1+\nu^{1/2}Y}{1+Y}\xi_j\right\|_{L^\infty}\left\|\frac{1+\nu^{1/2}Y}{1+Y}\partial_X\phi^{\boldsymbol{l}}_{\mathrm{app},1}\right\| \\
&\qquad+\left\|\frac{1+Y}{1+\nu^{1/2}Y}(\partial_X\Omega)^{\boldsymbol{j}-\boldsymbol{l}}\right\|_{L^\infty}\left\|\frac{1+\nu^{1/2}Y}{1+Y}\xi_j\right\|_{L^\infty}\|(\partial_Y\phi_{\mathrm{app},1})^{\boldsymbol{l}}\| \\
&\le C(j+1)^{\frac12}N_{\infty,j-l,((1+Y)/(1+\nu^{1/2}Y))^2}[\partial_Y\Omega]\left\|\frac{1}{1+Y}\partial_X\phi^{\boldsymbol{l}}_{\mathrm{app},1}\right\|+C\nu^{\frac12}(j+1)^{\frac12}N_{\infty,j-l}[\nabla\Omega]\|(\nabla\phi_{\mathrm{app},1})^{\boldsymbol{l}}\|.
\end{aligned}
$$

Thus we have

$$
\begin{aligned}
&\left\|\xi_j\sum_{l=0}^{j-1}\sum_{\max\{0,l+j_2-j\}\le l_2\le\min\{l,j_2\}}\binom{j_2}{l_2}\binom{j-j_2}{l-l_2}(\nabla^\perp\phi_{\mathrm{app},1})^{\boldsymbol{l}}\cdot(\nabla\Omega)^{\boldsymbol{j}-\boldsymbol{l}}\right\|_{L^2(0,1/(K\nu^{1/2});L^2_{X,Y})} \\
&\le C(j+1)^{\frac12}\sum_{l=0}^{j-1}\min\{l+1,\,j-l+1\}\binom{j}{l}N_{\infty,j-l}[\nabla\Omega] \\
&\qquad\times\left(M_{2,l,1/(1+Y)}[\partial_X\phi_{\mathrm{app},1}]+\nu^{\frac12}M_{2,l,1}[\nabla\phi_{\mathrm{app},1}]\right). \qquad (5\text{-}41)
\end{aligned}
$$

We note that

$$
(j+1)^{\frac12}\min\{l+1,\,j-l+1\}\left(\frac{(j-l)!\,l!}{j!}\right)^{\frac12}\le C,\qquad 1\le l\le j-1.
$$

Taking into account this uniform bound — by decomposing the sum $\sum_{l=0}^{j-1}$ into the "$l=0$" term and the sum $\sum_{l=1}^{j-1}$ — and collecting (5-40) and (5-41), we obtain, from the Young inequality for convolution in the l^1 space,

$$
\begin{aligned}
&\frac{1}{K^{1/2}\nu^{1/2}}|||\nabla^\perp\phi_{\mathrm{app},1}\cdot\nabla\Omega|||'_{2,\tilde\xi^{(2)}} \\
&\qquad\le\frac{1}{K^{1/4}}\left(\sum_{j=0}^{\nu^{-1/2}}\frac{1}{(j!)^{3/2}\nu^{j/2+1/4}(j+1)^{1/2}}M_{2,j,1/(1+Y)}[\partial_X\phi_{\mathrm{app},1}]+|||\nabla\phi_{\mathrm{app},1}|||'_{2,\mathbf{1}}\right), \qquad (5\text{-}42)
\end{aligned}
$$

where K has been taken large enough depending on C_*, C_1^*, and C_κ. As for the estimate of

$$
\|\nabla^\perp\phi_{\mathrm{app},1}\cdot\nabla\Omega\|_{L^2(0,1/(K\nu^{1/2});\dot H^{-1})},
$$

let us take any $\eta\in\dot H^1_0(\mathbb{T}\times\mathbb{R}_+)$. Then we have

$$
\begin{aligned}
\langle\nabla^\perp\phi_{\mathrm{app},1}\cdot\nabla\Omega,\eta\rangle&=\left\langle\frac{1+Y}{1+\nu^{1/2}Y}\nabla^\perp\phi_{\mathrm{app},1}\cdot\nabla\Omega,\,\frac{\eta}{1+Y}\right\rangle+\left\langle\nabla^\perp\phi_{\mathrm{app},1}\cdot\nabla\Omega,\,\frac{\nu^{1/2}Y\eta}{1+\nu^{1/2}Y}\right\rangle \\
&=\left\langle\frac{1+Y}{1+\nu^{1/2}Y}\nabla^\perp\phi_{\mathrm{app},1}\cdot\nabla\Omega,\,\frac{\eta}{1+Y}\right\rangle-\left\langle\Omega,\,\nabla^\perp\phi_{\mathrm{app},1}\cdot\nabla\left(\frac{\nu^{1/2}Y\eta}{1+\nu^{1/2}Y}\right)\right\rangle.
\end{aligned}
$$

This implies

$$
\begin{aligned}
&|\langle \nabla^\perp \phi_{\mathrm{app},1} \cdot \nabla\Omega, \eta\rangle| \\
&\qquad \le \left\| \frac{1+Y}{1+\nu^{1/2}Y} \nabla^\perp \phi_{\mathrm{app},1} \cdot \nabla\Omega \right\| \left\| \frac{\eta}{1+Y} \right\| + \left\| \frac{1+Y}{1+\nu^{1/2}Y} \Omega \nabla^\perp \phi_{\mathrm{app},1} \right\| \left\| \frac{1+\nu^{1/2}Y}{1+Y} \nabla\Big(\frac{\nu^{1/2}Y\eta}{1+\nu^{1/2}Y}\Big) \right\| \\
&\qquad \le C \left\| \frac{1+Y}{1+\nu^{1/2}Y} \nabla^\perp \phi_{\mathrm{app},1} \cdot \nabla\Omega \right\| \|\partial_Y \eta\| + C\nu^{\frac12} \left\| \frac{1+Y}{1+\nu^{1/2}Y} \Omega \nabla^\perp \phi_{\mathrm{app},1} \right\| \|\nabla \eta\|,
\end{aligned}
$$

where the Hardy inequality was used several times. Hence we obtain

$$
\begin{aligned}
&\|\nabla^\perp \phi_{\mathrm{app},1} \cdot \nabla\Omega\|_{\dot H^{-1}} \\
&\quad \le C \left\| \frac{1+Y}{1+\nu^{1/2}Y} \nabla^\perp \phi_{\mathrm{app},1} \cdot \nabla\Omega \right\| + C\nu^{\frac12} \left\| \frac{1+Y}{1+\nu^{1/2}Y} \Omega \nabla^\perp \phi_{\mathrm{app},1} \right\| \\
&\quad \le C \left\| \frac{1+Y}{1+\nu^{1/2}Y} \partial_X \Omega \right\|_{L^\infty} \|\partial_Y \phi_{\mathrm{app},1}\| + C \left\| \Big(\frac{1+Y}{1+\nu^{1/2}Y}\Big)^2 \partial_Y \Omega \right\|_{L^\infty} \left\| \frac{1+\nu^{1/2}Y}{1+Y} \partial_X \phi_{\mathrm{app},1} \right\| \\
&\qquad\qquad + C\nu^{\frac12} \left\| \frac{1+Y}{1+\nu^{1/2}Y} \Omega \right\|_{L^\infty} \|\nabla \phi_{\mathrm{app},1}\| \\
&\quad \le C C_1^* (\nu^{\frac12} \|\nabla \phi_{\mathrm{app},1}\| + \|\partial_Y \partial_X \phi_{\mathrm{app},1}\|).
\end{aligned}
$$

Then

$$
\begin{aligned}
&\frac{1}{K^{1/2}\nu^{1/4}} \|\nabla^\perp \phi_{\mathrm{app},1} \cdot \nabla\Omega\|_{L^2(0,1/(K\nu^{1/2}); \dot H^{-1})} \\
&\le \frac{CC_1^*}{K^{1/2}\nu^{1/4}} (\nu^{\frac12} \|\nabla \phi_{\mathrm{app},1}\|_{L^2(0,1/(K\nu^{1/2}); L^2_{X,Y})} + \|\partial_X \partial_Y \phi_{\mathrm{app},1}\|_{L^2(0,1/(K\nu^{1/2}); L^2_{X,Y})}) \le \frac{1}{K^{1/4}} |||\nabla \phi_{\mathrm{app},1}|||'_{2,\mathbf{1}}. \qquad \square
\end{aligned}
$$

Propositions 4.1 and 5.14 yield:

Corollary 5.15. *There exists $\kappa_4 \in (0, 1]$ such that the following statement holds for any $\kappa \in (0, \kappa_4]$. There exists $K_4 = K_4(\kappa, C_*, C_j^*) \ge 1$ such that if $K \ge K_4$ then the system (5-38) admits a unique solution $\tilde\omega_1 \in C([0, 1/(K\nu^{1/2})]; L^2 \cap \dot H^{-1}) \cap L^2(0, 1/(K\nu^{1/2}); H_0^1)$ satisfying*

$$
|||\tilde\omega_1|||'_{\infty,\xi} + K^{\frac12} |||\tilde\omega_1|||'_{2,\xi} + K^{\frac14} |||\nabla\tilde\phi_1|||'_{2,\mathbf{1}} + K^{\frac14} |||\partial_Y \tilde\phi_1|_{Y=0}|||_{\mathrm{bc}} \le \frac{1}{K^{1/2}} |||h|||_{\mathrm{bc}}. \tag{5-43}
$$

Proof. Propositions 4.1 and 5.14 give

$$
\begin{aligned}
&|||\tilde\omega_1|||'_{\infty,\xi} + K^{\frac12} |||\tilde\omega_1|||'_{2,\xi} + K^{\frac14} |||\nabla\tilde\phi_1|||'_{2,\mathbf{1}} + K^{\frac14} |||\partial_Y \tilde\phi_1|_{Y=0}|||_{\mathrm{bc}} \\
&\qquad\qquad \le \frac{C}{K^{1/4}} \bigg(\sum_{j=0}^{\nu^{-1/2}} \frac{1}{(j!)^{3/2}\nu^{j/2+1/4}(j+1)^{1/2}} M_{2,j,1/(1+Y)}[\partial_X \phi_{\mathrm{app},1}] + |||\nabla\phi_{\mathrm{app},1}|||'_{2,\mathbf{1}} \bigg).
\end{aligned}
$$

Here $C > 0$ is a universal constant. Recall that $\phi_{\mathrm{app},1}[h] = \phi_{1,1}[h] + \phi_{1,2}[h]$, where $\phi_{1,j}[h]$ is the stream function associated with $\omega_{1,j}[h]$. Then the assertion follows from Proposition 5.2 for $\phi_{1,1}[h]$ and Corollary 5.13 for $\phi_{1,2}[h]$. $\square$

From the construction, the vorticity $\omega_{\rm app} = \omega_{\rm app}[h] = \omega_{{\rm app},1}[h] + \tilde{\omega}_1[h]$ satisfies

$$\begin{aligned} -\nu^{\frac{1}{2}}\Delta\omega_{\rm app} + \partial_\tau\omega_{\rm app} + V\cdot\nabla\omega_{\rm app} + \nabla^\perp\phi_{\rm app}\cdot\nabla\Omega = 0, \quad \tau>0,\ \ X\in\mathbb{T}_\nu,\ \ Y>0,\\ \phi_{\rm app}|_{Y=0}=0, \quad \partial_Y\phi_{\rm app}|_{Y=0} = h + R_{\rm bc}[h], \quad \phi_{\rm app}|_{\tau=0}=0. \end{aligned} \tag{5-44}$$

Here $\phi_{\rm app}$ is the stream function associated with $\omega_{\rm app}$, and $R_{\rm bc}[h]$ is the linear operator defined as

$$R_{\rm bc}[h] = \partial_Y\phi_{1,2}[h]|_{Y=0} + \partial_Y\tilde{\phi}_1[h]|_{Y=0}. \tag{5-45}$$

We note that the operator $R_{\rm bc}$ is well defined on the Banach space

$$Z_{\rm bc} = \{h\in L^2(0,1/(K\nu^{\frac{1}{2}}); L^2_X) \mid \|h\|_{Z_{\rm bc}} := |||h|||_{\rm bc} < \infty\}. \tag{5-46}$$

Proposition 5.16. *There exists $\kappa_5\in(0,1]$ such that the following statement holds for any $\kappa\in(0,\kappa_5]$. There exists $K_5 = K_5(\kappa, C_*, C_j^*)\ge 1$ such that if $K\ge K_5$ then the map $R_{\rm bc}: Z_{\rm bc}\to Z_{\rm bc}$ defined by (5-45) satisfies*

$$|||R_{\rm bc}[h]|||_{\rm bc} \le \tfrac{1}{2}|||h|||_{\rm bc}. \tag{5-47}$$

Hence, the operator $I+R_{\rm bc}$ is invertible in $Z_{\rm bc}$, and the map

$$\Phi_{\rm bc}[h] := \phi_{\rm app}[(I+R_{\rm bc})^{-1}h], \quad h\in Z_{\rm bc}, \tag{5-48}$$

gives the solution to (5-1) and satisfies

$$|||\nabla\Phi_{\rm bc}[h]|||'_{2,\mathbf{1}} \le C|||h|||_{\rm bc}. \tag{5-49}$$

Here $C>0$ is a universal constant.

Proof. By the definition of $R_{\rm bc}$ in (5-45), estimate (5-47) is a consequence of Corollaries 5.13 and 5.15, by taking κ small first and then K large enough depending only on C_*, C_j^*, and C_κ. In particular, we have

$$|||(I+R_{\rm bc})^{-1}h|||_{\rm bc} \le 2|||h|||_{\rm bc}, \quad h\in Z_{\rm bc}. \tag{5-50}$$

Then Proposition 5.2 and Corollaries 5.13–5.15 give (5-49). □

6. Full estimate for linearization

We have constructed the solution to (2-12) of the form

$$W = \nabla^\perp\phi = \nabla^\perp\Phi_{\rm slip} + \nabla^\perp\Phi_{\rm bc}[h], \quad h = -\partial_Y\Phi_{\rm slip}|_{Y=0}\in Z_{\rm bc}, \tag{6-1}$$

where $\nabla^\perp\Phi_{\rm slip}$ is the velocity field associated with the solution to (4-1) and

$$\Phi_{\rm bc}[h] = \phi_{{\rm app},1}[(I+R_{\rm bc})^{-1}h] + \tilde{\phi}_1[(I+R_{\rm bc})^{-1}h], \quad \phi_{{\rm app},1} = \phi_{1,1}+\phi_{1,2}.$$

To simplify the notation we will write $\phi_{{\rm app},1}$ for $\phi_{{\rm app},1}[(I+R_{\rm bc})^{-1}h]$ below. So far we have the bound of $\nabla^\perp\phi_{1,1}$ only in the norm $|||\cdot|||'_{2,\mathbf{1}}$. To obtain the estimates of $|||\nabla\phi|||_\infty$ and $|||\omega|||_\infty$ we need the extra work.

Proposition 6.1. *There exists $\kappa_6 \in (0,1]$ such that the following statement holds for any $\kappa \in (0,\kappa_6]$. There exists $K_6 = K_6(C_0^*, C_1^*) \geq 1$ such that if $K \geq K_6$ then the solution to (2-12) constructed as in (6-1) satisfies*

$$\nu^{\frac14} |||\omega|||_\infty + K^{\frac12}\nu^{\frac14}|||\nabla\phi|||_\infty \leq \frac{C(C_0^*+C_1^*)}{\nu^{1/4}}(|||\nabla\phi|||'_{2,\mathbf{1}} + |||\Delta(\phi-\phi_{\mathrm{app},1})|||'_{2,\mathbf{1}} + |||\Delta\phi_{\mathrm{app},1}|||'_{2,Y}) + C(K^{\frac12}[\|W_0\|] + \nu^{\frac14}[\|\mathrm{rot}\, W_0\|] + |||F|||_2).$$

Here $C>0$ is a universal constant.

The proof of Proposition 6.1 is similar to the one of Proposition 4.4, and we postpone it to Appendix B. Admitting Proposition 6.1, we will now complete the proof of Theorem 2.3. Let us recall (6-1). We first observe from Proposition 4.1 and Remark 4.2 that

$$|||\Delta\Phi_{\mathrm{slip}}|||'_{2,\mathbf{1}} + |||\nabla\Phi_{\mathrm{slip}}|||'_{2,\mathbf{1}} + |||\partial_Y\Phi_{\mathrm{slip}}|_{Y=0}|||_{\mathrm{bc}} \leq \frac{1}{K^{1/8}}(\|W_0\|_{L^2_{X,Y}} + \nu^{-\frac12}[\|\mathrm{rot}\, W_0\|] + \nu^{-\frac34}|||F|||_2) \quad (6\text{-}2)$$

by taking K large enough. On the other hand, Proposition 5.16 (for $\nabla\Phi_{\mathrm{bc}}$), Corollary 5.15 and Remark 4.2(1) (for $\Delta(\Phi_{\mathrm{bc}}-\phi_{\mathrm{app},1}) = \Delta\tilde{\phi}_1$), Proposition 5.4 and Corollary 5.13 (for $\Delta\phi_{\mathrm{app},1} = \Delta\phi_{1,1}+\Delta\phi_{1,2}$), and (6-2) give

$$\begin{aligned} |||\nabla\Phi_{\mathrm{bc}}|||'_{2,1} + |||\Delta(\Phi_{\mathrm{bc}}-\phi_{\mathrm{app},1})|||'_{2,\mathbf{1}} + |||\Delta\phi_{\mathrm{app},1}|||'_{2,Y} &\leq C|||\partial_Y\Phi_{\mathrm{slip}}|_{Y=0}|||_{\mathrm{bc}} \\ &\leq \frac{C}{K^{1/8}}(\|W_0\|_{L^2_{X,Y}} + \nu^{-\frac12}[\|\mathrm{rot}\, W_0\|] + \nu^{-\frac34}|||F|||_2). \end{aligned} \quad (6\text{-}3)$$

Here $C>0$ is a universal constant. By applying the estimate in Proposition 6.1 and by taking K large enough, the proof of Theorem 2.3 is complete. □

7. Nonlinear stability: proof of Theorem 2.1

Let us recall the nonlinear system (1-3). Theorem 2.1 is a consequence of Theorem 2.4 for the linear system (1-6) and the bilinear estimate in Lemma 7.1 stated below. We observe that

$$-w\cdot\nabla w = w\,\mathrm{rot}\, w + \nabla\tilde{q}$$

for any solenoidal vector field w, so the bilinear term we consider here is of the form $f\,\mathrm{rot}\, g$. To this end we fix $K\geq 1$ and $\nu\in(0,1]$, and let X be the Banach space of solenoidal vector fields $f=(f_1,f_2)$ on $[0,1/K]\times\mathbb{R}^2_+$ defined as

$$X = \left\{ f \in C\left(\left[0,\frac1K\right]; H^1_{0,\sigma}(\mathbb{T}\times\mathbb{R}_+)\right) \,\middle|\, \|f\|_X = \|f\|_{G^\infty_{3/2}} + \nu^{\frac12}\|\mathrm{rot}\, f\|_{G^\infty_{3/2}} < \infty \right\},$$

where $\|\cdot\|_{G^\infty_{3/2}}$ is defined in (2-1) with $p=\infty$.

Lemma 7.1. *There exists a universal constant $C>0$ such that, for any $f,g\in X$,*

$$\|f\,\mathrm{rot}\, g\|_{G^2_{3/2}} \leq \frac{C}{K^{1/2}}\nu^{-\frac34}\|f\|_X\|g\|_X. \quad (7\text{-}1)$$

Proof. We compute

$$\begin{aligned}\|f\operatorname{rot}g\|_{G^2_{3/2}} &\le C\sum_{j=0}^{\nu^{-1/2}}\frac{1}{j!^{3/2}}\sup_{|\boldsymbol{j}|=j}\sum_{\boldsymbol{l}\le\boldsymbol{j}}\binom{\boldsymbol{j}}{\boldsymbol{l}}\|f^{\boldsymbol{l}}(\operatorname{rot}g)^{\boldsymbol{j}-\boldsymbol{l}}\|_{L^2(0,1/K;L^2_{x,y})}\\ &\le\frac{C}{K^{1/2}}\sum_{j=0}^{\nu^{-1/2}}\frac{1}{j!^{3/2}}\sup_{|\boldsymbol{j}|=j}\sum_{\boldsymbol{l}\le\boldsymbol{j}}\binom{\boldsymbol{j}}{\boldsymbol{l}}\|f^{\boldsymbol{l}}(\operatorname{rot}g)^{\boldsymbol{j}-\boldsymbol{l}}\|_{L^\infty(0,1/K;L^2_{x,y})}.\end{aligned}$$

As $\binom{\boldsymbol{j}}{\boldsymbol{l}}\le\binom{|\boldsymbol{j}|}{|\boldsymbol{l}|}$ and as, for all $l\in\mathbb{N}_0$,

$$\sharp\{\boldsymbol{l},\ |\boldsymbol{l}|=l,\ \boldsymbol{l}\le\boldsymbol{j}\}=\sharp\{l_2,\ \max(0,\ l-j+j_2)\le l_2\le\min(j_2,\ l)\}\le\min(l+1,\ j-l+1),$$

we end up with

$$\begin{aligned}&\|f\operatorname{rot}g\|_{G^2_{3/2}}\\ &\quad\le\frac{C}{K^{1/2}}\sum_{j=0}^{\nu^{-1/2}}\frac{1}{j!^{3/2}}\sum_{l=0}^{j}\min(l+1,\ j-l+1)\binom{j}{l}\sup_{|\boldsymbol{l}|=l}\sup_{|\boldsymbol{k}|=j-l}\|f^{\boldsymbol{l}}(\operatorname{rot}g)^{\boldsymbol{k}}\|_{L^\infty_tL^2_{x,y}}\\ &\quad\le\frac{C}{K^{1/2}}\sum_{j=0}^{\nu^{-1/2}}\sum_{0\le l\le j/2}(l+1)\binom{j}{l}^{-\frac12}\frac{1}{l!^{3/2}}\sup_{|\boldsymbol{l}|=l}\|f^{\boldsymbol{l}}\|_{L^\infty_{t,x,y}}\frac{1}{(j-l)!^{3/2}}\sup_{|\boldsymbol{k}|=j-l}\|(\operatorname{rot}g)^{\boldsymbol{k}}\|_{L^\infty_tL^2_{x,y}}\\ &\qquad+\frac{C}{K^{1/2}}\sum_{j=0}^{\nu^{-1/2}}\sum_{j/2<l\le j}(j-l+1)\binom{j}{l}^{-\frac12}\frac{1}{l!^{3/2}}\sup_{|\boldsymbol{l}|=l}\|f^{\boldsymbol{l}}\|_{L^\infty_tL^2_xL^\infty_y}\frac{1}{(j-l)!^{3/2}}\sup_{|\boldsymbol{k}|=j-l}\|(\operatorname{rot}g)^{\boldsymbol{k}}\|_{L^\infty_tL^\infty_xL^2_y}\\ &\quad\le\frac{C}{K^{1/2}}\sum_{j=0}^{\nu^{-1/2}}\sum_{0\le l\le j/2}(l+1)^{\frac52}\binom{j}{l}^{-\frac12}\frac{1}{(l+1)!^{3/2}}\\ &\qquad\qquad\times\sup_{|\boldsymbol{l}|=l}(\|\partial_xf^{\boldsymbol{l}}\|_{L^\infty_tL^2_xL^2_y}+\|f^{\boldsymbol{l}}\|_{L^\infty_tL^2_xL^2_y})^{\frac12}(\|\partial_x\partial_yf^{\boldsymbol{l}}\|_{L^\infty_tL^2_xL^2_y}+\|\partial_yf^{\boldsymbol{l}}\|_{L^\infty_tL^2_xL^2_y})^{\frac12}\\ &\qquad\qquad\times\frac{1}{(j-l)!^{3/2}}\sup_{|\boldsymbol{k}|=j-l}\|(\operatorname{rot}g)^{\boldsymbol{k}}\|_{L^\infty_tL^2_{x,y}}\\ &\qquad+\frac{C}{K^{1/2}}\sum_{j=0}^{\nu^{-1/2}}\sum_{j/2<l\le j}(j-l+1)^{\frac52}\binom{j}{l}^{-\frac12}\frac{1}{l!^{3/2}}\sup_{|\boldsymbol{l}|=l}\|f^{\boldsymbol{l}}\|^{\frac12}_{L^\infty_tL^2_xL^2_y}\|\partial_yf^{\boldsymbol{l}}\|^{\frac12}_{L^\infty_tL^2_xL^2_y}\\ &\qquad\qquad\times\frac{1}{(j-l+1)!^{3/2}}\sup_{|\boldsymbol{k}|=j-l}(\|\partial_x(\operatorname{rot}g)^{\boldsymbol{k}}\|_{L^\infty_tL^\infty_xL^2_y}+\|(\operatorname{rot}g)^{\boldsymbol{k}}\|_{L^\infty_tL^\infty_xL^2_y}).\end{aligned}$$

Here we have used the Sobolev embedding type inequality. By using the bound

$$\begin{aligned}&\sup_{|\boldsymbol{l}|=l}(\|\partial_xf^{\boldsymbol{l}}\|_{L^\infty_tL^2_xL^2_y}+\|f^{\boldsymbol{l}}\|_{L^\infty_tL^2_xL^2_y})^{\frac12}(\|\partial_x\partial_yf^{\boldsymbol{l}}\|_{L^\infty_tL^2_xL^2_y}+\|\partial_yf^{\boldsymbol{l}}\|_{L^\infty_tL^2_xL^2_y})^{\frac12}\\ &\qquad\le\nu^{-\frac14}\sup_{l\le|\boldsymbol{l}|\le l+1}\|f^{\boldsymbol{l}}\|_{L^\infty_tL^2_{x,y}}+\nu^{\frac14}\sup_{l\le|\boldsymbol{l}|\le l+1}\|\partial_yf^{\boldsymbol{l}}\|_{L^\infty_tL^2_{x,y}}\end{aligned}$$

and by observing that there exists $C>0$ such that, for $\binom{j}{l}^{-1/2}(l+1)^{5/2}\le C$ for $0\le l\le \frac12 j$, we have

$$\begin{aligned}
&\frac{C}{K^{1/2}}\sum_{j=0}^{\nu^{-1/2}}\sum_{0\le l\le j/2}(l+1)^{\frac52}\binom{j}{l}^{-\frac12}\frac{1}{(l+1)!^{3/2}}\\
&\qquad\times\sup_{|\boldsymbol{l}|=l}(\|\partial_x f^{\boldsymbol{l}}\|_{L^\infty_t L^2_x L^2_y}+\|f^{\boldsymbol{l}}\|_{L^\infty_t L^2_x L^2_y})^{\frac12}(\|\partial_x\partial_y f^{\boldsymbol{l}}\|_{L^\infty_t L^2_x L^2_y}+\|\partial_y f^{\boldsymbol{l}}\|_{L^\infty_t L^2_x L^2_y})^{\frac12}\\
&\qquad\qquad\times\frac{1}{(j-l)!^{3/2}}\sup_{|\boldsymbol{k}|=j-l}\|(\operatorname{rot} g)^{\boldsymbol{k}}\|_{L^\infty_t L^2_{x,y}}\\
&\le\frac{C}{K^{1/2}\nu^{1/4}}\sum_{j=0}^{\nu^{-1/2}}\sum_{0\le l\le j/2}\frac{1}{(l+1)!^{3/2}}\sup_{l\le|\boldsymbol{l}|\le l+1}\|f^{\boldsymbol{l}}\|_{L^\infty_t L^2_{x,y}}\frac{1}{(j-l)!^{3/2}}\sup_{|\boldsymbol{k}|=j-l}\|(\operatorname{rot} g)^{\boldsymbol{k}}\|_{L^\infty_t L^2_{x,y}}\\
&\quad+\frac{C\nu^{1/4}}{K^{1/2}}\sum_{j=0}^{\nu^{-1/2}}\sum_{0\le l\le j/2}\frac{1}{(l+1)!^{3/2}}\sup_{l\le|\boldsymbol{l}|\le l+1}\|\partial_y f^{\boldsymbol{l}}\|_{L^\infty_t L^2_{x,y}}\frac{1}{(j-l)!^{3/2}}\sup_{|\boldsymbol{k}|=j-l}\|(\operatorname{rot} g)^{\boldsymbol{k}}\|_{L^\infty_t L^2_{x,y}}\\
&\le\frac{C}{K^{1/2}\nu^{1/4}}\|f\|_{G^\infty_{3/2}}\|\operatorname{rot} g\|_{G^\infty_{3/2}}+\frac{C\nu^{1/4}}{K^{1/2}}\|\partial_y f\|_{G^\infty_{3/2}}\|\operatorname{rot} g\|_{G^\infty_{3/2}},
\end{aligned}$$

where the discrete Young's convolution inequality is applied in the last line together with the estimate

$$\sum_{j=0}^{\nu^{-1/2}}\frac{1}{j!^{3/2}}\sup_{|\boldsymbol{j}|=j}\|\partial_y f^{\boldsymbol{j}}\|_{L^\infty_t L^2_{x,y}}\le C\|\partial_y f\|_{G^\infty_{3/2}}.$$

Similarly, since $(j-l+1)^{5/2}\binom{j}{l}^{-1/2}\le C$ for $\frac12 j\le l\le j$, we have

$$\begin{aligned}
&\frac{C}{K^{1/2}}\sum_{j=0}^{\nu^{-1/2}}\sum_{j/2<l\le j}(j-l+1)^{\frac52}\binom{j}{l}^{-\frac12}\frac{1}{l!^{3/2}}\sup_{|\boldsymbol{l}|=l}\|f^{\boldsymbol{l}}\|^{\frac12}_{L^\infty_t L^2_x L^2_y}\|\partial_y f^{\boldsymbol{l}}\|^{\frac12}_{L^\infty_t L^2_x L^2_y}\\
&\qquad\times\frac{1}{(j-l+1)!^{3/2}}\sup_{|\boldsymbol{k}|=j-l}(\|\partial_x(\operatorname{rot} g)^{\boldsymbol{k}}\|_{L^\infty_t L^\infty_x L^2_y}+\|(\operatorname{rot} g)^{\boldsymbol{k}}\|_{L^\infty_t L^\infty_x L^2_y})\\
&\le\frac{C}{K^{1/2}}\sum_{j=0}^{\nu^{-1/2}}\sum_{j/2<l\le j}\frac{1}{l!^{3/2}}\sup_{|\boldsymbol{l}|=l}(\nu^{-\frac14}\|f^{\boldsymbol{l}}\|_{L^\infty_t L^2_x L^2_y}+\nu^{\frac14}\|\partial_y f^{\boldsymbol{l}}\|_{L^\infty_t L^2_x L^2_y})\\
&\qquad\times\frac{1}{(j-l+1)!^{3/2}}\sup_{j-l\le|\boldsymbol{k}|\le j-l+1}\|(\operatorname{rot} g)^{\boldsymbol{k}}\|_{L^\infty_t L^\infty_x L^2_y}\\
&\le\frac{C}{K^{1/2}\nu^{1/4}}\|f\|_{G^\infty_{3/2}}\|\operatorname{rot} g\|_{G^\infty_{3/2}}+\frac{C\nu^{1/4}}{K^{1/2}}\|\partial_y f\|_{G^\infty_{3/2}}\|\operatorname{rot} g\|_{G^\infty_{3/2}}.
\end{aligned}$$

Hence the result follows from Lemma C.1. □

Proof of Theorem 2.1. Let C be the universal constant in Theorem 2.4. Then the standard fixed-point theorem in the closed convex set

$$X_R=\Big\{f\in C\Big(\Big[0,\frac1K\Big];H^1_{0,\sigma}(\mathbb{T}\times\mathbb{R}_+)\Big)\;\Big|\;\|f\|_X\le R\Big\},\qquad R=4C\delta_0\nu^{\frac74},$$

is applied by using Theorem 2.4 and Lemma 7.1, if $\nu \le K^{-2}$ holds and if δ_0 is sufficiently small. We note that the smallness condition $[|w_0|] + [|\mathrm{rot}\, w_0|] \le \delta_0 \nu^{9/4}$, $\|r\|_{G^2_{3/2}} \le \delta_0 \nu^{11/4}$, is needed to close the estimate. Since the argument is standard we omit the details. □

Appendix A: Interpolation estimate for solutions to the Poisson equation

Lemma A.1. *Assume that $Y^k \omega \in L^2(\mathbb{T}_\nu \times \mathbb{R}_+)$ for $k = 0, 1, 2$. Let $\phi \in \dot{H}^1_0(\mathbb{T}_\nu \times \mathbb{R}_+)$ be the solution to the Poisson equation $-\Delta\phi = \omega$ in $\mathbb{T}_\nu \times \mathbb{R}_+$ with $\phi|_{Y=0} = 0$. Then there exists $C > 0$ such that, for any $j \ge 0$, we have*

$$\sup_{Y>0} \|\phi(\,\cdot\,, Y)\|_{L^2(\mathbb{T}_\nu)} \le C((j+1)^{-\frac{1}{4}} \|Y\omega\|_{L^2(\mathbb{T}_\nu\times\mathbb{R}_+)} + (j+1)^{\frac{1}{4}} \|Y^2\omega\|_{L^2(\mathbb{T}_\nu\times\mathbb{R}_+)}). \tag{A-1}$$

Proof. The solution is given by the formula

$$\phi(X, Y) = \int_0^Y e^{-(Y-Y')(-\partial_X^2)^{1/2}} \int_{Y'}^\infty e^{-(Y''-Y')(-\partial_X^2)^{1/2}} \omega(\,\cdot\,, Y'')\, dY''\, dY'.$$

Here $e^{-Y(-\partial_X^2)^{1/2}}$ is the Poisson semigroup. Then we have

$$\|\phi(\,\cdot\,, Y)\|_{L^2(\mathbb{T}_\nu)} \le \int_0^Y \int_{Y'}^\infty \|\omega(Y'')\|_{L^2(\mathbb{T}_\nu)}\, dY''\, dY'.$$

By decomposing the integral $\int_0^Y$ into $\int_0^{\min\{Y,(j+1)^{-1/2}\}}$ and $\int_{\min\{Y,(j+1)^{-1/2}\}}^Y$, we have, from the Hölder inequality,

$$\sup_{Y>0} \|\phi(\,\cdot\,, Y)\|_{L^2(\mathbb{T}_\nu)} \le C(j+1)^{-\frac{1}{4}} \|Y\omega\|_{L^2(\mathbb{T}_\nu\times\mathbb{R})} + C(j+1)^{\frac{1}{4}} \|Y^2\omega\|_{L^2(\mathbb{T}_\nu\times\mathbb{R}_+)}. \qquad \square$$

Lemma A.1 yields the following:

Proposition A.2. *Let $\phi \in \dot{H}^1_0(\mathbb{T}_\nu \times \mathbb{R}_+)$ be the solution to the Poisson equation $-\Delta\phi = \omega$ in $\mathbb{T}_\nu \times \mathbb{R}_+$ with $\phi|_{Y=0} = 0$. Then, for any $j \ge 0$, we have*

$$\begin{aligned} &M_{2,j,1/(1+Y)}[\partial_X \phi] \\ &\le C(j+1)^{-\frac{1}{4}} M_{2,j+1,Y}[\omega] + C(j+1)^{\frac{1}{4}} M_{2,j+1,Y^2}[\omega] + C\kappa \nu^{\frac{1}{2}} j (M_{2,j-1,Y}[\omega] + M_{2,j-1,1}[\nabla\phi]). \end{aligned} \tag{A-2}$$

Here $C > 0$ is a universal constant.

Proof. Since $-\Delta\partial_X\phi = \partial_X\omega$, we have $-(\Delta\partial_X\phi)^j = \partial_X\omega^j$. Then we use the commutator relation

$$-(\Delta\phi)^j = -\nabla\cdot(\nabla\phi)^j + \nu^{\frac{1}{2}} j 2 \frac{\chi_\nu'}{\chi_\nu} (\partial_Y\phi)^j = -\Delta\phi^j + \partial_Y\left(\nu^{\frac{1}{2}} j 2 \frac{\chi_\nu'}{\chi_\nu} \phi^j\right) + \nu^{\frac{1}{2}} j 2 \frac{\chi_\nu'}{\chi_\nu} (\partial_Y\phi)^j.$$

Thus we have the following Poisson equation for ϕ^j:

$$-\Delta\phi^j = \omega^j - \partial_Y\left(\nu^{\frac{1}{2}} j 2 \frac{\chi_\nu'}{\chi_\nu} \phi^j\right) - \nu^{\frac{1}{2}} j 2 \frac{\chi_\nu'}{\chi_\nu} (\partial_Y\phi)^j.$$

Then we decompose $\phi^{\boldsymbol{j}}$ into $\phi_1+\phi_{2,1}+\phi_{2,2}$, so that

$$-\Delta\phi_1=\omega^{\boldsymbol{j}},\quad -\Delta\phi_{2,1}=-\partial_Y\left(\nu^{\frac{1}{2}}j_2\frac{\chi_\nu'}{\chi_\nu}\phi^{\boldsymbol{j}}\right),\quad -\Delta\phi_{2,2}=-\nu^{\frac{1}{2}}j_2\frac{\chi_\nu'}{\chi_\nu}(\partial_Y\phi)^{\boldsymbol{j}},$$

subject to the Dirichlet boundary condition. Then Lemma A.1 implies, for $\partial_X\phi_1$,

$$\sup_{Y>0}\|\partial_X\phi_1(\cdot,Y)\|_{L^2(\mathbb{T}_\nu)}\le C((j+1)^{-\frac{1}{4}}\|Y\partial_X\omega^{\boldsymbol{j}}\|_{L^2(\mathbb{T}_\nu\times\mathbb{R}_+)}+(j+1)^{\frac{1}{4}}\|Y^2\partial_X\omega^{\boldsymbol{j}}\|_{L^2(\mathbb{T}_\nu\times\mathbb{R}_+)}).\tag{A-3}$$

On the other hand, the simple energy estimate gives

$$\|\nabla\phi_{2,1}\|\le\nu^{\frac{1}{2}}j_2\left\|\frac{\chi_\nu'}{\chi_\nu}\phi^{\boldsymbol{j}}\right\|\le\kappa\nu^{\frac{1}{2}}j_2\|(\partial_Y\phi)^{(j_1,j_2-1)}\|.$$

As for $\phi_{2,2}$, from

$$\frac{1}{\chi_\nu}(\partial_Y\phi)^{\boldsymbol{j}}=e^{-K\tau\nu^{1/2}}(\partial_Y^2\phi)^{(j_1,j_2-1)}=e^{-K\tau\nu^{1/2}}(-\omega^{(j_1,j_2-1)}-\partial_X^2\phi^{(j_1,j_2-1)}),$$

the Hardy inequality, and integration by parts, we have

$$\|\nabla\phi_{2,2}\|\le C\kappa\nu^{\frac{1}{2}}j_2(\|Y\omega^{(j_1,j_2-1)}\|+\|\partial_X\phi^{(j_1,j_2-1)}\|).$$

Hence we obtain the desired estimate by taking the L^2 norm in time and by taking the supremum over $\boldsymbol{j}$ such that $|\boldsymbol{j}|=j$. □

Appendix B: Proof of Proposition 6.1

Let us go back to (4-1) with $G=0$, but now we impose the no-slip boundary condition $\phi|_{Y=0}=\partial_Y\phi|_{Y=0}=0$ in this appendix. Then we have

$$-\nu^{\frac{1}{2}}(\Delta\omega)^{\boldsymbol{j}}+(\partial_\tau+K\nu^{\frac{1}{2}}(j+1))\omega^{\boldsymbol{j}}=-(V\cdot\nabla\omega)^{\boldsymbol{j}}-(\nabla^\perp\phi\cdot\nabla\Omega)^{\boldsymbol{j}}+(\mathrm{rot}\,F)^{\boldsymbol{j}}=(\mathrm{div}\,H)^{\boldsymbol{j}},\tag{B-1}$$

where

$$H=-V\omega-\Omega\nabla^\perp\phi+(F_2,-F_1).$$

The idea is to take the L^2 inner product with $\partial_\tau\phi^{\boldsymbol{j}}$, which gives the estimates of $|||\nabla\phi|||_\infty$ and $|||\Delta\phi|||_\infty$ in terms of $|||\nabla\phi|||'_{2,\mathbf{1}}$. The most technical part is the computation of the viscous term $\langle(\Delta\omega)^{\boldsymbol{j}},\partial_\tau\phi^{\boldsymbol{j}}\rangle$ when $j_2\ne0$, for which one needs to convert the vertical derivative $\partial_Y^2\omega$ into the tangential ones by using equation (B-1).

Lemma B.1. *For any $\kappa\in(0,1]$ and $K\ge1$, we have*

$$\begin{aligned}\int_0^{\tau_0}&\langle(\partial_\tau+K\nu^{\frac{1}{2}}(j+1))\omega^{\boldsymbol{j}},\partial_\tau\phi^{\boldsymbol{j}}\rangle\,d\tau\\ &\ge\tfrac{1}{2}\|\partial_\tau(\nabla\phi)^{\boldsymbol{j}}\|^2_{L^2(0,\tau_0;L^2_{X,Y})}+\tfrac{1}{2}K\nu^{\frac{1}{2}}(j+1)(\|(\nabla\phi)^{\boldsymbol{j}}(\tau_0)\|^2-\|(\nabla\phi)^{\boldsymbol{j}}(0)\|^2)\\ &\qquad-C\kappa^2K\nu^{\frac{1}{2}}j(\nu^{\frac{1}{2}}j^{\frac{3}{2}})^2M_{\infty,j-1,1}[\nabla\phi]^2-C(\kappa\nu^{\frac{1}{2}}j)^2M_{2,j-1,1}[\partial_\tau\nabla\phi]^2.\end{aligned}$$

Here C is a universal constant.

Proof. Let us recall the identity

$$\omega^{\boldsymbol{j}} = -(\Delta\phi)^{\boldsymbol{j}} = -\nabla\cdot(\nabla\phi)^{\boldsymbol{j}} + \nu^{\frac12} j_2 \frac{\chi_\nu'}{\chi_\nu}(\partial_Y\phi)^{\boldsymbol{j}}, \tag{B-2}$$

which implies

$$\langle(\partial_\tau + K\nu^{\frac12}(j+1))\omega^{\boldsymbol{j}}, \partial_\tau\phi^{\boldsymbol{j}}\rangle = \|\partial_\tau(\nabla\phi)^{\boldsymbol{j}}\|^2 + 2\nu^{\frac12} j_2\left\langle \frac{\chi_\nu'}{\chi_\nu}\partial_\tau(\partial_Y\phi)^{\boldsymbol{j}}, \partial_\tau\phi^{\boldsymbol{j}}\right\rangle + \tfrac12 K\nu^{\frac12}(j+1)\partial_\tau\|(\nabla\phi)^{\boldsymbol{j}}\|^2 + 2\nu^{\frac12} j_2 K\nu^{\frac12}(j+1)\left\langle\frac{\chi_\nu'}{\chi_\nu}(\partial_Y\phi)^{\boldsymbol{j}}, \partial_\tau\phi^{\boldsymbol{j}}\right\rangle.$$

Then, from $\partial_\tau\phi^{\boldsymbol{j}} = \chi_\nu\partial_\tau(e^{-K\tau\nu^{1/2}}(\partial_Y\phi)^{(j_1,j_2-1)})$ for $j_2\ge 1$, we have

$$\int_0^{\tau_0} 2\nu^{\frac12} j_2\left\langle\frac{\chi_\nu'}{\chi_\nu}\partial_\tau(\partial_Y\phi)^{\boldsymbol{j}}, \partial_\tau\phi^{\boldsymbol{j}}\right\rangle d\tau \ge -\tfrac14\|\partial_\tau(\nabla\phi)^{\boldsymbol{j}}\|^2_{L^2(0,\tau_0;L^2)} - C(\kappa\nu^{\frac12} j)^2(M_{2,j-1,1}[\partial_\tau\nabla\phi]^2 + (K\nu^{\frac12})^2 M_{2,j-1,1}[\nabla\phi]^2),$$

while we have, from integration by parts in time,

$$\begin{aligned}\int_0^{\tau_0} & 2\nu^{\frac12} j_2 K\nu^{\frac12}(j+1)\left\langle\frac{\chi_\nu'}{\chi_\nu}(\partial_Y\phi)^{\boldsymbol{j}}, \partial_\tau\phi^{\boldsymbol{j}}\right\rangle d\tau \\ &= 2\nu^{\frac12} j_2 K\nu^{\frac12}(j+1)(e^{-K\tau_0\nu^{1/2}}\langle\chi_\nu'(\partial_Y\phi)^{\boldsymbol{j}}, (\partial_Y\phi)^{(j_1,j_2-1)}\rangle(\tau_0) - \langle\chi_\nu'(\partial_Y\phi)^{\boldsymbol{j}}, (\partial_Y\phi)^{(j_1,j_2-1)}\rangle(0)) \\ &\qquad - 2\nu^{\frac12} j_2 K\nu^{\frac12}(j+1)\int_0^{\tau_0} e^{-K\tau\nu^{1/2}}\langle\partial_\tau(\partial_Y\phi)^{\boldsymbol{j}}, \chi_\nu'(\partial_Y\phi)^{(j_1,j_2-1)}\rangle\, d\tau \\ &\ge 2\nu^{\frac12} j_2 K\nu^{\frac12}(j+1)(e^{-K\tau_0\nu^{1/2}}\langle\chi_\nu'(\partial_Y\phi)^{\boldsymbol{j}}(\tau_0), (\partial_Y\phi)^{(j_1,j_2-1)}(\tau_0)\rangle - \langle\chi_\nu'(\partial_Y\phi)^{\boldsymbol{j}}(0), (\partial_Y\phi)^{(j_1,j_2-1)}(0)\rangle) \\ &\qquad - \tfrac14\|\partial_\tau(\partial_Y\phi)^{\boldsymbol{j}}\|^2_{L^2(0,\tau_0;L^2)} - C(K\kappa\nu j^2)^2 M_{2,j-1,1}[\nabla\phi]^2.\end{aligned}$$

We also observe that, for $j_2\ge 1$,

$$\begin{aligned}\langle\chi_\nu'(\partial_Y\phi)^{\boldsymbol{j}}, & (\partial_Y\phi)^{(j_1,j_2-1)}\rangle \\ &= e^{-K\tau\nu^{1/2}}\langle\chi_\nu'\chi_\nu(\partial_Y\partial_Y\phi)^{(j_1,j_2-1)}, (\partial_Y\phi)^{(j_1,j_2-1)}\rangle \\ &= -\tfrac12 e^{-K\tau\nu^{1/2}}\langle\partial_Y(\chi_\nu'\chi_\nu)(\partial_Y\phi)^{(j_1,j_2-1)}, (\partial_Y\phi)^{(j_1,j_2-1)}\rangle - e^{-K\tau\nu^{1/2}}\nu^{\frac12}(j_2-1)\|\chi_\nu'(\partial_Y\phi)^{(j_1,j_2-1)}\|^2.\end{aligned}$$

Thus we conclude also from $K\tau\nu^{1/2}\le 1$ that

$$\begin{aligned}\int_0^{\tau_0} & 2\nu^{\frac12} j_2 K\nu^{\frac12}(j+1)\left\langle\frac{\chi_\nu'}{\chi_\nu}(\partial_Y\phi)^{\boldsymbol{j}}, \partial_\tau\phi^{\boldsymbol{j}}\right\rangle d\tau \\ &\ge -CK\nu^{\frac12}(\kappa\nu^{\frac12} j)^2(j\|(\partial_Y\phi)^{(j_1,j_2-1)}(\tau_0)\|^2 + \|(\partial_Y\phi)^{(j_1,j_2-1)}(0)\|^2) \\ &\qquad - \tfrac14\|\partial_\tau(\partial_Y\phi)^{\boldsymbol{j}}\|^2_{L^2(0,\tau_0;L^2)} - C(K\kappa\nu j^2)^2 M_{2,j-1,1}[\nabla\phi]^2.\end{aligned}$$

Combining the above and $M_{2,j-1,1}[\nabla\phi]^2 \le (K\nu^{1/2})^{-1}M_{\infty,j-1,1}[\nabla\phi]^2$, we obtain the desired estimate. □

Lemma B.2. *For any* $\kappa \in (0,1]$ *and* $K \geq 1$, *we have*

$$\int_0^{\tau_0} \langle -\nu^{\frac{1}{2}}(\Delta\omega)^{\boldsymbol{j}}, \partial_\tau \phi^{\boldsymbol{j}}\rangle \, d\tau \geq \tfrac{1}{2}\nu^{\frac{1}{2}}(\|\omega^{\boldsymbol{j}}(\tau_0)\|^2 - \|\omega^{\boldsymbol{j}}(0)\|^2) - \tfrac{1}{4}M_{2,j,1}[\partial_\tau \nabla\phi]^2$$
$$- C(\kappa\nu^{\frac{1}{2}} j)^2 (M_{2,j-1,1}[\partial_\tau\nabla\phi]^2 + (\nu^{\frac{1}{2}}(j-1))^2 M_{2,j-2,1}[\partial_\tau\nabla\phi]^2)$$
$$- C\kappa^2\nu^{\frac{1}{2}}(M_{\infty,j,1}[\omega]^2 + (\nu^{\frac{1}{2}} j)^2 M_{\infty,j-1,1}[\omega])$$
$$- CK\nu^{\frac{1}{2}} j (\kappa\nu^{\frac{1}{2}} j^{\frac{3}{2}})^2 (M_{\infty,j-1,1}[\nabla\phi]^2 + (\nu^{\frac{1}{2}}(j-1))^2 M_{\infty,j-2,1}[\nabla\phi]^2)$$
$$- C(M_{2,j,1}[H]^2 + (\nu^{\frac{1}{2}} j)^2 M_{2,j-1,1}[H]^2).$$

Here C *is a universal constant.*

Proof. We observe from

$$(\Delta\omega)^{\boldsymbol{j}} = \nabla\cdot(\nabla\omega)^{\boldsymbol{j}} - \nu^{\frac{1}{2}} j_2 \frac{\chi_\nu'}{\chi_\nu}(\partial_Y\omega)^{\boldsymbol{j}}, \qquad \chi_\nu' = \kappa e^{-\kappa\nu^{1/2}Y},$$
$$\nabla\partial_\tau\phi^{\boldsymbol{j}} = \partial_\tau(\nabla\phi)^{\boldsymbol{j}} + \nu^{\frac{1}{2}} j_2 \frac{\chi_\nu'}{\chi_\nu}\partial_\tau\phi^{\boldsymbol{j}}\boldsymbol{e}_2, \tag{B-3}$$

and integration by parts that

$$\langle -\nu^{\frac{1}{2}}(\Delta\omega)^{\boldsymbol{j}}, \partial_\tau\phi^{\boldsymbol{j}}\rangle = \nu^{\frac{1}{2}}\langle(\nabla\omega)^{\boldsymbol{j}}, \partial_\tau(\nabla\phi)^{\boldsymbol{j}}\rangle + 2\nu j_2 \left\langle \frac{\chi_\nu'}{\chi_\nu}(\partial_Y\omega)^{\boldsymbol{j}}, \partial_\tau\phi^{\boldsymbol{j}}\right\rangle.$$

Then the similar identities

$$(\nabla\omega)^{\boldsymbol{j}} = \nabla\omega^{\boldsymbol{j}} - \nu^{\frac{1}{2}} j_2 \frac{\chi_\nu'}{\chi_\nu}\omega^{\boldsymbol{j}}\boldsymbol{e}_2,$$
$$\nabla\cdot\partial_\tau(\nabla\phi)^{\boldsymbol{j}} = \partial_\tau(\Delta\phi)^{\boldsymbol{j}} + \nu^{\frac{1}{2}} j_2 \frac{\chi_\nu'}{\chi_\nu}\partial_\tau(\partial_Y\phi)^{\boldsymbol{j}}, \tag{B-4}$$

together with integration by parts, yield

$$\langle -\nu^{\frac{1}{2}}(\Delta\omega)^{\boldsymbol{j}}, \partial_\tau\phi^{\boldsymbol{j}}\rangle = \nu^{\frac{1}{2}}\langle\omega^{\boldsymbol{j}}, \partial_\tau\omega^{\boldsymbol{j}}\rangle - 2\nu j_2\left\langle\frac{\chi_\nu'}{\chi_\nu}\omega^{\boldsymbol{j}}, \partial_\tau(\partial_Y\phi)^{\boldsymbol{j}}\right\rangle + 2\nu j_2\left\langle\frac{\chi_\nu'}{\chi_\nu}(\partial_Y\omega)^{\boldsymbol{j}}, \partial_\tau\phi^{\boldsymbol{j}}\right\rangle. \tag{B-5}$$

Again from the above identities about the commutators we have, for $j_2 \geq 1$,

$$\left\langle\frac{\chi_\nu'}{\chi_\nu}\omega^{\boldsymbol{j}}, \partial_\tau(\partial_Y\phi)^{\boldsymbol{j}}\right\rangle = -\left\langle\frac{\chi_\nu'}{\chi_\nu}(\partial_Y\omega)^{\boldsymbol{j}}, \partial_\tau\phi^{\boldsymbol{j}}\right\rangle - \nu^{\frac{1}{2}}\left\langle\frac{\chi_\nu''}{\chi_\nu}\omega^{\boldsymbol{j}}, \partial_\tau\phi^{\boldsymbol{j}}\right\rangle - \nu^{\frac{1}{2}}(2j_2-1)\left\langle\left(\frac{\chi_\nu'}{\chi_\nu}\right)^2\omega^{\boldsymbol{j}}, \partial_\tau\phi^{\boldsymbol{j}}\right\rangle.$$

Here $\chi_\nu'' = -\kappa^2 e^{-\kappa\nu^{1/2}Y}$. Thus (B-5) is written as

$$\langle -\nu^{\frac{1}{2}}(\Delta\omega)^{\boldsymbol{j}}, \partial_\tau\phi^{\boldsymbol{j}}\rangle = \nu^{\frac{1}{2}}\langle\omega^{\boldsymbol{j}}, \partial_\tau\omega^{\boldsymbol{j}}\rangle + 4\nu j_2\left\langle\frac{\chi_\nu'}{\chi_\nu}(\partial_Y\omega)^{\boldsymbol{j}}, \partial_\tau\phi^{\boldsymbol{j}}\right\rangle$$
$$+ 2\nu^{\frac{3}{2}} j_2\left\langle\frac{\chi_\nu''}{\chi_\nu}\omega^{\boldsymbol{j}}, \partial_\tau\phi^{\boldsymbol{j}}\right\rangle + 2\nu^{\frac{3}{2}} j_2(2j_2-1)\left\langle\left(\frac{\chi_\nu'}{\chi_\nu}\right)^2\omega^{\boldsymbol{j}}, \partial_\tau\phi^{\boldsymbol{j}}\right\rangle. \tag{B-6}$$

Let us compute the term $\langle(\chi_\nu'/\chi_\nu)(\partial_Y\omega)^{\boldsymbol{j}}, \partial_\tau\phi^{\boldsymbol{j}}\rangle$. From the identity

$$\frac{1}{\chi_\nu}(\partial_Y\omega)^{\boldsymbol{j}} = e^{-K\tau\nu^{1/2}}(\partial_Y^2\omega)^{(j_1,j_2-1)} = e^{-K\tau\nu^{1/2}}((\Delta\omega)^{(j_1,j_2-1)} - \partial_X^2\omega^{(j_1,j_2-1)}),$$

we have

$$\left\langle \frac{\chi_\nu'}{\chi_\nu}(\partial_Y\omega)^{\boldsymbol{j}},\,\partial_\tau\phi^{\boldsymbol{j}}\right\rangle = e^{-K\tau\nu^{1/2}}\langle \chi_\nu'(\Delta\omega)^{(j_1,j_2-1)},\,\partial_\tau\phi^{\boldsymbol{j}}\rangle + \langle \chi_\nu'\omega^{(j_1+1,j_2-1)},\,\partial_\tau\partial_X\phi^{\boldsymbol{j}}\rangle.$$

Since $\nu^{1/2}(\Delta\omega)^{(j_1,j_2-1)} = (\partial_\tau + K\nu^{1/2}j)\omega^{(j_1,j_2-1)} - (\mathrm{div}\, H)^{(j_1,j_2-1)}$, the identity (B-6) is written as

$$\begin{aligned}\langle -\nu^{\frac12}(\Delta\omega)^{\boldsymbol{j}},\,\partial_\tau\phi^{\boldsymbol{j}}\rangle &= \nu^{\frac12}\langle\omega^{\boldsymbol{j}},\,\partial_\tau\omega^{\boldsymbol{j}}\rangle + 4\nu^{\frac12}j_2e^{-K\tau\nu^{1/2}}\langle\chi_\nu'(\partial_\tau + K\nu^{\frac12}j)\omega^{(j_1,j_2-1)},\,\partial_\tau\phi^{\boldsymbol{j}}\rangle\\ &\quad - 4\nu^{\frac12}j_2e^{-K\tau\nu^{1/2}}\langle\chi_\nu'(\mathrm{div}\,H)^{(j_1,j_2-1)},\,\partial_\tau\phi^{\boldsymbol{j}}\rangle + 4\nu j_2\langle\chi_\nu'\omega^{(j_1+1,j_2-1)},\,\partial_\tau\partial_X\phi^{\boldsymbol{j}}\rangle\\ &\quad + 2\nu^{\frac32}j_2\left\langle\frac{\chi_\nu''}{\chi_\nu}\omega^{\boldsymbol{j}},\,\partial_\tau\phi^{\boldsymbol{j}}\right\rangle + 2\nu^{\frac32}j_2(2j_2-1)\left\langle\left(\frac{\chi_\nu'}{\chi_\nu}\right)^2\omega^{\boldsymbol{j}},\,\partial_\tau\phi^{\boldsymbol{j}}\right\rangle. \end{aligned}\tag{B-7}$$

Next we compute the term $\nu^{1/2}j_2e^{-K\tau\nu^{1/2}}\langle\chi_\nu'(\partial_\tau + K\nu^{1/2}j)\omega^{(j_1,j_2-1)},\,\partial_\tau\phi^{\boldsymbol{j}}\rangle$ in (B-7): from the identities in (B-4), we have

$$\begin{aligned}e^{-K\tau\nu^{1/2}}\langle\chi_\nu'(\partial_\tau + K\nu^{\frac12}j)\omega^{(j_1,j_2-1)},\,\partial_\tau\phi^{\boldsymbol{j}}\rangle &= e^{-K\tau\nu^{1/2}}\langle\chi_\nu'(\partial_\tau + K\nu^{\frac12}j)(\nabla\phi)^{(j_1,j_2-1)},\,\partial_\tau(\nabla\phi)^{\boldsymbol{j}}\rangle\\ &\quad + 2\nu^{\frac12}j_2e^{-K\tau\nu^{1/2}}\left\langle\frac{(\chi_\nu')^2}{\chi_\nu}(\partial_\tau + K\nu^{\frac12}j)(\partial_Y\phi)^{(j_1,j_2-1)},\,\partial_\tau\phi^{\boldsymbol{j}}\right\rangle\\ &\quad + \nu^{\frac12}e^{-K\tau\nu^{1/2}}\langle\chi_\nu''(\partial_\tau + K\nu^{\frac12}j)(\partial_Y\phi)^{(j_1,j_2-1)},\,\partial_\tau\phi^{\boldsymbol{j}}\rangle.\end{aligned}$$

By setting $(\nabla\phi)^{\tilde{\boldsymbol{j}}-1} = e^{-K\tau\nu^{1/2}}(\nabla\phi)^{(j_1,j_2-1)}$ for simplicity, we have

$$\begin{aligned}&e^{-K\tau\nu^{1/2}}\langle\chi_\nu'(\partial_\tau + K\nu^{\frac12}j)\omega^{(j_1,j_2-1)},\,\partial_\tau\phi^{\boldsymbol{j}}\rangle\\ &\quad = \langle\chi_\nu'\partial_\tau(\nabla\phi)^{\tilde{\boldsymbol{j}}-1},\,\partial_\tau(\nabla\phi)^{\boldsymbol{j}}\rangle + 2\nu^{\frac12}j_2\left\langle\frac{(\chi_\nu')^2}{\chi_\nu}\partial_\tau(\partial_Y\phi)^{\tilde{\boldsymbol{j}}-1},\,\partial_\tau\phi^{\boldsymbol{j}}\right\rangle + \nu^{\frac12}\langle\chi_\nu''\partial_\tau(\partial_Y\phi)^{\tilde{\boldsymbol{j}}-1},\,\partial_\tau\phi^{\boldsymbol{j}}\rangle\\ &\qquad + K\nu^{\frac12}j\left(\langle\chi_\nu'(\nabla\phi)^{\tilde{\boldsymbol{j}}-1},\,\partial_\tau(\nabla\phi)^{\boldsymbol{j}}\rangle + 2\nu^{\frac12}j_2\left\langle\frac{(\chi_\nu')^2}{\chi_\nu}(\partial_Y\phi)^{\tilde{\boldsymbol{j}}-1},\,\partial_\tau\phi^{\boldsymbol{j}}\right\rangle + \nu^{\frac12}\langle\chi_\nu''(\partial_Y\phi)^{\tilde{\boldsymbol{j}}-1},\,\partial_\tau\phi^{\boldsymbol{j}}\rangle\right).\end{aligned}$$

Since

$$\begin{aligned}\partial_\tau(\nabla\phi)^{\boldsymbol{j}} &= \chi_\nu\partial_\tau(\partial_Y\nabla\phi)^{\tilde{\boldsymbol{j}}-1} = \chi_\nu\partial_Y\partial_\tau(\nabla\phi)^{\tilde{\boldsymbol{j}}-1} - \nu^{\frac12}(j_2-1)\chi_\nu'\partial_\tau(\nabla\phi)^{\tilde{\boldsymbol{j}}-1},\\ \partial_\tau\phi^{\boldsymbol{j}} &= \chi_\nu\partial_\tau(\partial_Y\phi)^{\tilde{\boldsymbol{j}}-1},\end{aligned}$$

we then arrive at

$$\begin{aligned}&\nu^{\frac12}j_2e^{-K\tau\nu^{1/2}}\langle\chi_\nu'(\partial_\tau + K\nu^{\frac12}j)\omega^{(j_1,j_2-1)},\,\partial_\tau\phi^{\boldsymbol{j}}\rangle\\ &= \nu^{\frac12}j_2\Big\{-\tfrac12\langle\partial_Y(\chi_\nu'\chi_\nu)\partial_\tau(\nabla\phi)^{\tilde{\boldsymbol{j}}-1},\,\partial_\tau(\nabla\phi)^{\tilde{\boldsymbol{j}}-1}\rangle - \nu^{\frac12}(j_2-1)\langle(\chi_\nu')^2\partial_\tau(\nabla\phi)^{\tilde{\boldsymbol{j}}-1},\,\partial_\tau(\nabla\phi)^{\tilde{\boldsymbol{j}}-1}\rangle\\ &\qquad + 2\nu^{\frac12}j_2\langle(\chi_\nu')^2\partial_\tau(\partial_Y\phi)^{\tilde{\boldsymbol{j}}-1},\,\partial_\tau(\partial_Y\phi)^{\tilde{\boldsymbol{j}}-1}\rangle + \nu^{\frac12}\langle\chi_\nu''\partial_\tau(\partial_Y\phi)^{\tilde{\boldsymbol{j}}-1},\,\chi_\nu\partial_\tau(\partial_Y\phi)^{\tilde{\boldsymbol{j}}-1}\rangle\\ &\qquad + K\nu^{\frac12}j\big(\langle\chi_\nu'(\nabla\phi)^{\tilde{\boldsymbol{j}}-1},\,\partial_\tau(\nabla\phi)^{\boldsymbol{j}}\rangle + 2\nu^{\frac12}j_2\langle(\chi_\nu')^2(\partial_Y\phi)^{\tilde{\boldsymbol{j}}-1},\,\partial_\tau(\partial_Y\phi)^{\tilde{\boldsymbol{j}}-1}\rangle\\ &\qquad\qquad + \nu^{\frac12}\langle\chi_\nu''(\partial_Y\phi)^{\tilde{\boldsymbol{j}}-1},\,\chi_\nu\partial_\tau(\partial_Y\phi)^{\tilde{\boldsymbol{j}}-1}\rangle\big)\Big\}\\ &\ge -C(\kappa\nu^{\frac12}j_2)^2\|\partial_\tau(\nabla\phi)^{\tilde{\boldsymbol{j}}-1}\|^2\\ &\quad + K\nu j_2 j\big(\langle\chi_\nu'(\nabla\phi)^{\tilde{\boldsymbol{j}}-1},\,\partial_\tau(\nabla\phi)^{\boldsymbol{j}}\rangle + \nu^{\frac12}j_2\partial_\tau\|\chi_\nu'(\partial_Y\phi)^{\tilde{\boldsymbol{j}}-1}\|^2 + \tfrac12\nu^{\frac12}\partial_\tau\langle\chi_\nu''(\partial_Y\phi)^{\tilde{\boldsymbol{j}}-1},\,\chi_\nu(\partial_Y\phi)^{\tilde{\boldsymbol{j}}-1}\rangle\big).\end{aligned}\tag{B-8}$$

Here we have used the fact that it suffices to consider the case $j_2 \geq 1$, and C is a universal constant. Hence, by going back to (B-7), we have

$$\begin{aligned}
&\langle -\nu^{\frac{1}{2}}(\Delta\omega)^{\boldsymbol{j}}, \partial_\tau \phi^{\boldsymbol{j}}\rangle \\
&\quad\geq \nu^{\frac{1}{2}}\langle \omega^{\boldsymbol{j}}, \partial_\tau \omega^{\boldsymbol{j}}\rangle - C(\kappa\nu^{\frac{1}{2}} j_2)^2 \|\partial_\tau(\nabla\phi)^{\tilde{\boldsymbol{j}}-1}\|^2 \\
&\qquad + K\nu j_2 j\big(\langle \chi_\nu'(\nabla\phi)^{\tilde{\boldsymbol{j}}-1}, \partial_\tau(\nabla\phi)^{\boldsymbol{j}}\rangle + \nu^{\frac{1}{2}} j_2 \partial_\tau \|\chi_\nu'(\partial_Y\phi)^{\tilde{\boldsymbol{j}}-1}\|^2 + \tfrac{1}{2}\nu^{\frac{1}{2}}\partial_\tau\langle \chi_\nu''(\partial_Y\phi)^{\tilde{\boldsymbol{j}}-1}, \chi_\nu(\partial_Y\phi)^{\tilde{\boldsymbol{j}}-1}\rangle\big) \\
&\qquad\quad - 4\nu^{\frac{1}{2}} j_2 e^{-K\tau\nu^{1/2}}\langle \chi_\nu'(\operatorname{div} H)^{(j_1,j_2-1)}, \partial_\tau\phi^{\boldsymbol{j}}\rangle + 4\nu j_2\langle \chi_\nu'\omega^{(j_1+1,j_2-1)}, \partial_\tau\partial_X\phi^{\boldsymbol{j}}\rangle \\
&\qquad\qquad + 2\nu^{\frac{3}{2}} j_2\Big\langle \frac{\chi_\nu''}{\chi_\nu}\omega^{\boldsymbol{j}}, \partial_\tau\phi^{\boldsymbol{j}}\Big\rangle + 2\nu^{\frac{3}{2}} j_2(2j_2-1)\Big\langle\Big(\frac{\chi_\nu'}{\chi_\nu}\Big)^2\omega^{\boldsymbol{j}}, \partial_\tau\phi^{\boldsymbol{j}}\Big\rangle. \qquad \text{(B-9)}
\end{aligned}$$

Here C is a universal constant. Next we observe from $\partial_\tau\phi^{\boldsymbol{j}} = \chi_\nu\partial_\tau(\partial_Y\phi)^{\tilde{\boldsymbol{j}}-1}$ that

$$-4\nu^{\frac{1}{2}} j_2 e^{-K\tau\nu^{1/2}}\langle \chi_\nu'(\operatorname{div} H)^{(j_1,j_2-1)}, \partial_\tau\phi^{\boldsymbol{j}}\rangle \geq -C\kappa\nu^{\frac{1}{2}} j_2(\|H_1^{(j_1+1,j_2-1)}\| + \|H_2^{\boldsymbol{j}}\|)\|\partial_\tau(\partial_Y\phi)^{\tilde{\boldsymbol{j}}-1}\| \quad \text{(B-10)}$$

and also

$$4\nu j_2\langle \chi_\nu'\omega^{(j_1+1,j_2-1)}, \partial_\tau\partial_X\phi^{\boldsymbol{j}}\rangle \geq -C\kappa\nu j_2\|\omega^{(j_1+1,j_2-1)}\|\|\partial_\tau\partial_X\phi^{\boldsymbol{j}}\|, \quad \text{(B-11)}$$

$$2\nu^{\frac{3}{2}} j_2\Big\langle \frac{\chi_\nu''}{\chi_\nu}\omega^{\boldsymbol{j}}, \partial_\tau\phi^{\boldsymbol{j}}\Big\rangle \geq -C\kappa^2\nu^{\frac{3}{2}} j_2\|\omega^{\boldsymbol{j}}\|\|\partial_\tau(\partial_Y\phi)^{\tilde{\boldsymbol{j}}-1}\|. \quad \text{(B-12)}$$

Finally let us compute the term $\nu^{1/2}\langle(\chi_\nu'/\chi_\nu)^2\omega^{\boldsymbol{j}}, \partial_\tau\phi^{\boldsymbol{j}}\rangle$ when $j_2 \geq 1$. If $j_2 = 1$ then

$$\begin{aligned}
&\nu^{\frac{1}{2}}\Big\langle\Big(\frac{\chi_\nu'}{\chi_\nu}\Big)^2\omega^{\boldsymbol{j}}, \partial_\tau\phi^{\boldsymbol{j}}\Big\rangle \\
&\quad = \nu^{\frac{1}{2}}\langle(\chi_\nu')^2 e^{-K\tau\nu^{1/2}}(\partial_Y\omega)^{(j_1,0)}, \partial_\tau(e^{-K\tau\nu^{1/2}}(\partial_Y\phi)^{(j_1,0)})\rangle \\
&\quad = \nu^{\frac{1}{2}}\langle e^{-K\tau\nu^{1/2}}\nabla\partial_Y\phi^{(j_1,0)}, \nabla((\chi_\nu')^2\partial_\tau(e^{-K\tau\nu^{1/2}}(\partial_Y\phi)^{(j_1,0)}))\rangle \\
&\quad = \tfrac{1}{2}\nu^{\frac{1}{2}}\partial_\tau\|\chi_\nu' e^{-K\tau\nu^{1/2}}\nabla\partial_Y\phi^{(j_1,0)}\|^2 + 2\nu\langle\chi_\nu''\chi_\nu' e^{-K\tau\nu^{1/2}}\partial_Y^2\phi^{(j_1,0)}, \partial_\tau(e^{-K\tau\nu^{1/2}}(\partial_Y\phi)^{(j_1,0)})\rangle \\
&\quad \geq \tfrac{1}{2}\nu^{\frac{1}{2}}\partial_\tau\|\chi_\nu' e^{-K\tau\nu^{1/2}}\nabla\partial_Y\phi^{(j_1,0)}\|^2 - C\kappa^3\nu\|\omega^{(j_1,0)}\|\|\partial_\tau(\partial_Y\phi)^{\tilde{\boldsymbol{j}}-1}\|. \qquad \text{(B-13)}
\end{aligned}$$

If $j_2 \geq 2$ then

$$\nu^{\frac{1}{2}}\Big\langle\Big(\frac{\chi_\nu'}{\chi_\nu}\Big)^2\omega^{\boldsymbol{j}}, \partial_\tau\phi^{\boldsymbol{j}}\Big\rangle = e^{-2K\tau\nu^{1/2}}\nu^{\frac{1}{2}}\langle(\chi_\nu')^2(\partial_Y^2\omega)^{(j_1,j_2-2)}, \partial_\tau\phi^{\boldsymbol{j}}\rangle, \quad \text{(B-14)}$$

and then by using the identity

$$\nu^{\frac{1}{2}}(\Delta\omega)^{(j_1,j_2-2)} = (\partial_\tau + K\nu^{\frac{1}{2}}(j-1))\omega^{(j_1,j_2-2)} - (\operatorname{div} H)^{(j_1,j_2-2)},$$

we have

$$\begin{aligned}
\nu^{\frac{1}{2}}\Big\langle\Big(\frac{\chi_\nu'}{\chi_\nu}\Big)^2\omega^{\boldsymbol{j}}, \partial_\tau\phi^{\boldsymbol{j}}\Big\rangle &= -\nu^{\frac{1}{2}}\langle(\chi_\nu')^2\omega^{(j_1+2,j_2-2)}, \partial_\tau\phi^{\boldsymbol{j}}\rangle \\
&\quad + e^{-2K\tau\nu^{1/2}}\langle(\chi_\nu')^2(\partial_\tau + K\nu^{\frac{1}{2}}(j-1))\omega^{(j_1,j_2-2)}, \partial_\tau\phi^{\boldsymbol{j}}\rangle \\
&\quad - e^{-2K\tau\nu^{1/2}}\langle(\chi_\nu')^2(\operatorname{div} H)^{(j_1,j_2-2)}, \partial_\tau\phi^{\boldsymbol{j}}\rangle. \qquad \text{(B-15)}
\end{aligned}$$

As for the second term on the right-hand side of (B-15), we have, for $j \geq j_2 \geq 2$,

$$
\begin{aligned}
&e^{-2K\tau\nu^{1/2}}\langle(\chi_\nu')^2(\partial_\tau + K\nu^{\frac12}(j-1))\omega^{(j_1,j_2-2)}, \partial_\tau\phi^{\boldsymbol{j}}\rangle \\
&\quad = e^{-2K\tau\nu^{1/2}}\langle(\chi_\nu')^2(\partial_\tau + K\nu^{\frac12}(j-1))\partial_X\phi^{(j_1,j_2-2)}, \partial_\tau\partial_X\phi^{\boldsymbol{j}}\rangle \\
&\qquad\qquad - e^{-2K\tau\nu^{1/2}}\langle(\chi_\nu')^2(\partial_\tau + K\nu^{\frac12}(j-1))(e^{2K\tau\nu^{1/2}}(\partial_Y\phi)^{\tilde{\boldsymbol{j}}-1}), \partial_\tau(\partial_Y\phi)^{\tilde{\boldsymbol{j}}-1}\rangle \\
&\quad \geq -\kappa^2(\|\partial_\tau\partial_X\phi^{(j_1,j_2-2)}\| + K\nu^{\frac12}j\|\partial_X\phi^{(j_1,j_2-2)}\|)\|\partial_\tau\partial_X\phi^{\boldsymbol{j}}\| \\
&\qquad\qquad - \kappa^2(\|\partial_\tau(\partial_Y\phi)^{\tilde{\boldsymbol{j}}-1}\| + K\nu^{\frac12}j\|(\partial_Y\phi)^{\tilde{\boldsymbol{j}}-1}\|)\|\partial_\tau(\partial_Y\phi)^{\tilde{\boldsymbol{j}}-1}\|.
\end{aligned}
$$

Since it is straightforward to see that

$$
\begin{aligned}
-\nu^{\frac12}\langle(\chi_\nu')^2\omega^{(j_1+2,j_2-2)}, \partial_\tau\phi^{\boldsymbol{j}}\rangle &\geq -\kappa^2\nu^{\frac12}\|\omega^{(j_1+1,j_2-2)}\|\|\partial_\tau(\partial_X\phi)^{\boldsymbol{j}}\|, \\
-e^{-2K\tau\nu^{1/2}}\langle(\chi_\nu')^2(\operatorname{div} H)^{(j_1,j_2-2)}, \partial_\tau\phi^{\boldsymbol{j}}\rangle &\geq -\kappa^2(\|H_1^{(j_1+1,j_2-2)}\| + \|H_2^{(j_1,j_2-1)}\|)\|\partial_\tau(\partial_Y\phi)^{\tilde{\boldsymbol{j}}-1}\|,
\end{aligned}
$$

we obtain, for $j_2 \geq 2$,

$$
\begin{aligned}
&\nu^{\frac12}\left\langle\left(\frac{\chi_\nu'}{\chi_\nu}\right)^2\omega^{\boldsymbol{j}}, \partial_\tau\phi^{\boldsymbol{j}}\right\rangle \\
&\quad \geq -\kappa^2(\|\partial_\tau\partial_X\phi^{(j_1,j_2-2)}\| + K\nu^{\frac12}j\|\partial_X\phi^{(j_1,j_2-2)}\| + \nu^{\frac12}\|\omega^{(j_1+1,j_2-2)}\|)\|\partial_\tau(\partial_X\phi)^{\boldsymbol{j}}\| \\
&\qquad - \kappa^2(\|\partial_\tau(\partial_Y\phi)^{\tilde{\boldsymbol{j}}-1}\| + K\nu^{\frac12}j\|(\partial_Y\phi)^{\tilde{\boldsymbol{j}}-1}\| + \|H_1^{(j_1+1,j_2-2)}\| + \|H_2^{(j_1,j_2-1)}\|)\|\partial_\tau(\partial_Y\phi)^{\tilde{\boldsymbol{j}}-1}\|. \qquad \text{(B-16)}
\end{aligned}
$$

Collecting (B-9)–(B-12) with (B-13) (for $j_2 = 1$) and (B-16) (for $j_2 \geq 2$), we conclude the desired estimate by using the bound

$$
\begin{aligned}
M_{2,j,1}[f]^2 &= \sup_{|\boldsymbol{j}|=j}\|f^{\boldsymbol{j}}\|^2_{L^2(0,1/(K\nu^{1/2});L^2_{X,Y})} \\
&\leq \frac{1}{K\nu^{1/2}}\sup_{|\boldsymbol{j}|=j}\|f^{\boldsymbol{j}}\|^2_{L^\infty(0,1/(K\nu^{1/2});L^2_{X,Y})} \\
&= \frac{1}{K\nu^{1/2}}M_{\infty,j,1}[f]^2. \qquad \square
\end{aligned}
$$

As a consequence of Lemmas B.1 and B.2, we obtain:

Corollary B.3. *There exists* $\kappa_B \in (0,1]$ *such that, for any* $\kappa \in (0,\kappa_B]$ *and* $K \geq 1$,

$$
\begin{aligned}
&\nu^{\frac14}\sum_{j=0}^{\nu^{-1/2}}\frac{1}{(j!)^{3/2}\nu^{j/2}}M_{\infty,j,1}[\omega] + K^{\frac12}\nu^{\frac14}\sum_{j=0}^{\nu^{-1/2}}\frac{(j+1)^{1/2}}{(j!)^{3/2}\nu^{j/2}}M_{\infty,j,1}[\nabla\phi] + \sum_{j=0}^{\nu^{-1/2}}\frac{1}{(j!)^{3/2}\nu^{j/2}}M_{2,j,1}[\partial_\tau\nabla\phi], \\
&\leq C\biggl(\nu^{\frac14}\sum_{j=0}^{\nu^{-1/2}}\frac{1}{(j!)^{3/2}\nu^{j/2}}\|\omega^{\boldsymbol{j}}|_{\tau=0}\| + K^{\frac12}\sum_{j=0}^{\nu^{-1/2}}\frac{\nu^{1/4}(j+1)^{1/2}}{(j!)^{3/2}\nu^{j/2}}\|(\nabla\phi)^{\boldsymbol{j}}|_{\tau=0}\| + \sum_{j=0}^{\nu^{-1/2}}\frac{1}{(j!)^{3/2}\nu^{j/2}}M_{2,j,1}[H]\biggr).
\end{aligned}
$$

Here C *is a universal constant.*

We note that

$$\sum_{j=0}^{\nu^{-1/2}} \frac{\nu^{1/4}(j+1)^{1/2}}{(j!)^{3/2}\nu^{j/2}} \|(\nabla\phi)^{\boldsymbol{j}}|_{\tau=0}\| \le C \sum_{j=0}^{\nu^{-1/2}} \frac{1}{(j!)^{3/2}\nu^{j/2}} \|(\nabla\phi)^{\boldsymbol{j}}|_{\tau=0}\| = C[\|\nabla\phi|_{\tau=0}\|]$$

since $j \le \nu^{-1/2}$. By virtue of Corollary B.3, it remains to estimate

$$\sum_{j=1}^{\nu^{-1/2}} \frac{1}{(j!)^{3/2}\nu^{j/2}} M_{2,j,1}[H].$$

Recall that $H = -V\omega - \Omega\nabla^{\perp}\phi + (F_2, -F_1)$. Hence it suffices to show:

Lemma B.4. *For any $\kappa \in (0, 1]$ and $K \ge 1$, we have*

$$\sum_{j=0}^{\nu^{-1/2}} \frac{1}{(j!)^{3/2}\nu^{j/2}} M_{2,j,1}[\Omega\nabla\phi] \le \frac{C(C_0^* + C_1^*)}{\nu^{1/4}} |||\nabla\phi|||'_{2,\mathbf{1}}, \tag{B-17}$$

$$\sum_{j=0}^{\nu^{-1/2}} \frac{1}{(j!)^{3/2}\nu^{j/2}} M_{2,j,1}[V\omega] \le \frac{C(C_0^* + C_1^*)}{\nu^{1/4}} (|||\Delta(\phi - \phi_{\mathrm{app},1})|||'_{2,\mathbf{1}} + |||\Delta\phi_{\mathrm{app},1}|||'_{2,Y}). \tag{B-18}$$

Here $\phi_{\mathrm{app},1} = (\phi_{1,1} + \phi_{1,2})[(I + R_{\mathrm{bc}})^{-1}h]$ with $h = -\partial_Y \Phi_{\mathrm{slip}}|_{Y=0}$, and C is a universal constant.

Proof. We give a sketch of the proof only for (B-18), for (B-17) is proved in a similar manner. Let $|\boldsymbol{j}| = j$. Then

$$\sum_{j=1}^{\nu^{-1/2}} \frac{1}{j!^{3/2}\nu^{j/2}} M_{2,j,1}[V\omega] \le \sum_{j=0}^{\nu^{-1/2}} \frac{1}{j!^{3/2}\nu^{j/2}} \max_{|\boldsymbol{j}|=j} \sum_{\boldsymbol{l}\le\boldsymbol{j}} \binom{\boldsymbol{j}}{\boldsymbol{l}} \|V^{\boldsymbol{l}}\omega^{\boldsymbol{j}-\boldsymbol{l}}\|_{L^2(0,1/(K\nu^{1/2});L^2)}.$$

Here $V^{\boldsymbol{j}} = e^{-K\tau\nu^{1/2}j} B_{j_2}\partial_X^{j_1} V$, while $\omega^{\boldsymbol{j}} = e^{-K\tau\nu^{1/2}(j+1)} B_{j_2}\partial_X^{j_1}\omega$. Since $\omega = -\Delta(\phi - \phi_{\mathrm{app},1}) - \Delta\phi_{\mathrm{app},1}$ by virtue of the construction, we have

$$\begin{aligned}\|V^{\boldsymbol{l}}\omega^{\boldsymbol{j}-\boldsymbol{l}}\|&_{L^2(0,1/(K\nu^{1/2});L^2)} \\ &\le \|V^{\boldsymbol{l}}\|_{L^\infty} \|(\Delta(\phi - \phi_{\mathrm{app},1}))^{\boldsymbol{j}-\boldsymbol{l}}\|_{L^2(0,1/(K\nu^{1/2});L^2)} + \|\partial_Y V^{\boldsymbol{l}}\|_{L^\infty} \|Y(\Delta\phi_{\mathrm{app},1})^{\boldsymbol{j}-\boldsymbol{l}}\|_{L^2(0,1/(K\nu^{1/2});L^2)}.\end{aligned}$$

By using $\binom{\boldsymbol{j}}{\boldsymbol{l}} \le \binom{j}{l}$ with $l = |\boldsymbol{l}|$, we have

$$\begin{aligned}&\frac{1}{j!^{3/2}\nu^{j/2}} \sum_{\boldsymbol{l}\le\boldsymbol{j}} \binom{\boldsymbol{j}}{\boldsymbol{l}} \|V^{\boldsymbol{l}}\omega^{\boldsymbol{j}-\boldsymbol{l}}\|_{L^2(0,1/(K\nu^{1/2});L^2)} \\ &\le \sum_{\boldsymbol{l}\le\boldsymbol{j}} \left(\frac{l!\,(j-l)!}{j!}\right)^{\frac{1}{2}} \frac{M_{2,j-l,1}[\Delta(\phi - \phi_{\mathrm{app},1})] + M_{2,j-l,Y}[\Delta\phi_{\mathrm{app},1}]}{(j-l)!^{3/2}\nu^{(j-l)/2}} \frac{1}{l!^{3/2}\nu^{l/2}} \max_{|\boldsymbol{l}|=l}(\|V^{\boldsymbol{l}}\|_{L^\infty} + \|\partial_Y V^{\boldsymbol{l}}\|_{L^\infty}).\end{aligned}$$

Next we observe that, for all $l \in \mathbb{N} \cup \{0\}$,

$$\#\{\boldsymbol{l} \mid |\boldsymbol{l}| = l,\ \boldsymbol{l} \le \boldsymbol{j}\} = \#\{l_2,\ \max(0,\ l - j + j_2) \le l_2 \le \min(j_2,\ l)\} \le \min(l+1,\ j-l+1),$$

which gives the bound of the form $\sum_{l\le j} \le \sum_{l=0}^{j} \min(l+1,\, j-l+1)$. Hence we have

$$\begin{aligned}
&\frac{1}{j!^{3/2}\nu^{j/2}}\sum_{\boldsymbol{l}\le \boldsymbol{j}}\binom{\boldsymbol{j}}{\boldsymbol{l}}\|V^{\boldsymbol{l}}\omega^{\boldsymbol{j}-\boldsymbol{l}}\|_{L^2(0,1/(K\nu^{1/2});L^2)}\\
&\quad\le \sum_{l=0}^{j}\min(l+1,\, j-l+1)\Big(\frac{l!\,(j-l)!}{j!}\Big)^{\frac12}\\
&\qquad\qquad\times\frac{M_{2,j-l,1}[\Delta(\phi-\phi_{\mathrm{app},1})]+M_{2,j-l,Y}[\Delta\phi_{\mathrm{app},1}]}{(j-l)!^{3/2}\nu^{(j-l)/2}}\frac{1}{l!^{3/2}\nu^{l/2}}\max_{|\boldsymbol{l}|=l}(\|V^{\boldsymbol{l}}\|_{L^\infty}+\|\partial_Y V^{\boldsymbol{l}}\|_{L^\infty}).
\end{aligned}$$

Since $\min(l+1,\, j-l+1)(l!\,(j-l)!/j!)^{1/2}$ is uniformly bounded about $0\le l\le j$, the Young inequality for l^1 convolution gives the inequality

$$\begin{aligned}
&\sum_{j=0}^{\nu^{-1/2}}\frac{1}{j!^{3/2}\nu^{j/2}}\max_{|\boldsymbol{j}|=j}\sum_{\boldsymbol{l}\le \boldsymbol{j}}\binom{\boldsymbol{j}}{\boldsymbol{l}}\|V^{\boldsymbol{l}}\omega^{\boldsymbol{j}-\boldsymbol{l}}\|_{L^2(0,1/(K\nu^{1/2});L^2)}\\
&\quad\le C\sum_{j=0}^{\nu^{-1/2}}\frac{1}{j!^{3/2}\nu^{j/2}}\max_{|\boldsymbol{j}|=j}(\|V^{\boldsymbol{j}}\|_{L^\infty}+\|\partial_Y V^{\boldsymbol{j}}\|_{L^\infty})\\
&\qquad\qquad\times\sum_{j=0}^{\nu^{-1/2}}\frac{1}{j!^{3/2}\nu^{j/2}}\max_{|\boldsymbol{j}|=j}(M_{2,\boldsymbol{j},1}[\Delta(\phi-\phi_{\mathrm{app},1})]+M_{2,\boldsymbol{j},Y}[\Delta\phi_{\mathrm{app},1}]).
\end{aligned}$$

Then the desired estimate follows by noticing $\partial_Y V^{\boldsymbol{j}}=(\partial_Y V)^{\boldsymbol{j}}+\nu^{1/2}j_2\chi_\nu'(\partial_Y V)^{(j_1,j_2-1)}$ and the bound of the form $|||f|||_2\le \nu^{-1/4}|||f|||_{2,\mathbf{1}}'$. □

Proposition 6.1 follows from Corollary B.3 and Lemma B.4.

Appendix C: Estimate of the Biot–Savart law

Lemma C.1. *The following statement holds if κ is sufficiently small. Assume that*

$$f\in C([0,1/K);\, H^1(\mathbb{T}\times\mathbb{R}_+)^2)$$

satisfies div $f=0$ *for* $y>0$ *and* $f_2|_{y=0}=0$. *Then*

$$\|\nabla f\|_{G^p_{3/2}}\le C\|\mathrm{rot}\, f\|_{G^p_{3/2}},\qquad p\in[1,\infty].$$

Here C is a universal constant.

Proof. We observe that $\partial_y f_1=\mathrm{rot}\, f+\partial_x f_2$ and $\partial_y f_2=-\partial_x f_1$. Hence it suffices to show

$$\|\partial_x f\|_{G^p_{3/2}}\le C\|\mathrm{rot}\, f\|_{G^p_{3/2}}.$$

Since $f=\nabla^\perp\phi$ with the stream function ϕ and $-\Delta\phi=\omega$ with $\omega=\mathrm{rot}\, g$ and $\phi|_{y=0}=0$, we have

$$-(\Delta\partial_x\phi)^{\boldsymbol{j}}=\partial_x\omega^{\boldsymbol{j}},\qquad \omega^{\boldsymbol{j}}=e^{-Kt(j+1)}\chi^{j_2}\partial_y^{j_2}\partial_x^{j_1}\omega,\qquad j_1+j_2=j.$$

By virtue of the identity $-(\Delta\partial_x\phi)^j = -\nabla\cdot(\partial_x\nabla\phi)^j + j_2(\chi'/\chi)(\partial_y\partial_x\phi)^j$, integration by parts gives

$$\|(\nabla\partial_x\phi)^j\|^2 + 2j_2\left\langle\frac{\chi'}{\chi}(\partial_y\partial_x\phi)^j, \partial_x\phi^j\right\rangle = -\langle\omega^j, \partial_x^2\phi^j\rangle.$$

Since $\partial_x\phi^j = e^{-Kt}\chi(\partial_y\partial_x\phi)^{(j_1,j_2-1)}$, we thus have

$$\|(\nabla\partial_x\phi)^j\| \le C(\|\omega^j\| + \kappa j\|(\partial_y\partial_x\phi)^{(j_1,j_2-1)}\|),$$

where C is a universal constant. This estimate implies $\|\partial_x\nabla\phi\|_{G^p_{3/2}} \le C(\|\omega\|_{G^p_{3/2}} + \kappa\|\partial_x\partial_y\phi\|_{G^p_{3/2}})$, and thus, by taking κ small enough, we obtain $\|\partial_x\nabla\phi\|_{G^p_{3/2}} \le C\|\omega\|_{G^p_{3/2}}$. □

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