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**NONLOCAL OPERATORS RELATED TO NONSYMMETRIC
FORMS
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NONLOCAL OPERATORS RELATED TO NONSYMMETRIC FORMS II: HARNACK INEQUALITIES

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Local boundedness and Harnack inequalities are studied for solutions to parabolic and elliptic integrodifferential equations whose governing nonlocal operators are associated with nonsymmetric forms. We present two independent proofs, one based on the De Giorgi iteration and the other on the Moser iteration technique. This article is a continuation of work of Kassmann and Weidner (2022), where Hölder regularity and a weak Harnack inequality are proved in a similar setup.

1. Introduction	3189
2. Preliminaries	3195
3. Local boundedness via De Giorgi iteration	3204
4. Local boundedness via Moser iteration	3214
5. Local tail estimate	3226
6. Harnack inequalities	3233
Appendix	3240
References	3247

1. Introduction

The aim of this work is to prove local boundedness estimates and a Harnack inequality for weak solutions to parabolic equations of type

$$\partial_t u - Lu = f \quad \text{in } I_R(t_0) \times B_{2R} \subset \mathbb{R}^{d+1}, \quad (\text{PDE})$$

where $B_{2R} \subset \Omega$ is some ball, $I_R(t_0) := (t_0 - R^\alpha, t_0 + R^\alpha) \subset \mathbb{R}$, and $f \in L^\infty(I_R(t_0) \times B_{2R})$. Equation (PDE) is governed by a linear nonlocal operator of the form

$$-Lu(x) = 2 \text{ p.v. } \int_{\mathbb{R}^d} (u(x) - u(y))K(x, y) \, dy. \quad (1-1)$$

Such operators are determined by jumping kernels $K : \mathbb{R}^d \times \mathbb{R}^d \rightarrow [0, \infty]$, which are allowed to be nonsymmetric. We also investigate solutions to the equation

$$\partial_t u - \widehat{L}u = f \quad \text{in } I_R(t_0) \times B_{2R} \subset \mathbb{R}^{d+1}, \quad (\widehat{\text{PDE}})$$

which is driven by the dual operator \widehat{L} associated with L .

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In this work, we prove local boundedness of weak solutions to (PDE) and ($\widehat{\text{PDE}}$) via an adaptation of the De Giorgi method to nonlocal operators with nonsymmetric jumping kernels. We also provide an alternative proof of local boundedness via the Moser iteration. Finally, combined with the weak Harnack inequality from [Kassmann and Weidner 2022], we obtain a full Harnack inequality.

The novelty of our result consists in the lack of symmetry of the underlying operator. Let us write the decomposition $K = K_s + K_a$, where the symmetric part K_s and antisymmetric part K_a are given by

$$K_s(x, y) = \frac{1}{2}K(x, y) + K(y, x), \quad K_a(x, y) = \frac{1}{2}K(x, y) - K(y, x), \quad x, y \in \mathbb{R}^d.$$

Note that the nonnegativity of K implies

$$|K_a(x, y)| \leq K_s(x, y). \tag{1-2}$$

We can write for the nonsymmetric bilinear form associated with L

$$\mathcal{E}(u, v) := 2 \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} (u(x) - u(y))v(x)K(x, y) \, dy \, dx =: \mathcal{E}^{K_s}(u, v) + \mathcal{E}^{K_a}(u, v),$$

where

$$\begin{aligned} \mathcal{E}^{K_s}(u, v) &= \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} (u(x) - u(y))(v(x) - v(y))K_s(x, y) \, dy \, dx, \\ \mathcal{E}^{K_a}(u, v) &= \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} (u(x) - u(y))(v(x) + v(y))K_a(x, y) \, dy \, dx. \end{aligned}$$

In order to treat the antisymmetric part of the bilinear form, a refinement of the existing techniques for symmetric operators is required.

We have in mind the following three prototypes of kernels K for $\alpha \in (0, 2)$:

$$K_1(x, y) = g(x, y)|x - y|^{-d-\alpha}, \tag{1-3}$$

where $g : \mathbb{R}^d \times \mathbb{R}^d \rightarrow [\lambda, \Lambda]$ is a suitable nonsymmetric function for $0 < \lambda \leq \Lambda < \infty$,

$$K_2(x, y) = |x - y|^{-d-\alpha} + (V(x) - V(y))\mathbb{1}_{\{|x-y|\leq L\}}(x, y)|x - y|^{-d-\alpha}, \tag{1-4}$$

where $L \in (0, \infty]$ and $V : \mathbb{R}^d \rightarrow \mathbb{R}^d$ is a suitable function, and

$$K_3(x, y) = |x - y|^{-d-\alpha}\mathbb{1}_D(x - y) + |x - y|^{-d-\beta}\mathbb{1}_C(x - y), \tag{1-5}$$

where $C \subset \mathbb{R}^d$ is a cone, $D \subset \mathbb{R}^d$ is a double-cone such that $C \cap D = \emptyset$, and $0 < \beta < \frac{1}{2}\alpha$.

1.1. Main results. Our first main result is the following Harnack inequality for weak solutions to (PDE). We state and discuss our assumptions in Section 2.

Theorem 1.1. *Assume (K2), (cutoff), (K_{loc}^{\leq}), (Sob), and (Poinc) for some $\alpha \in (0, 2)$. Let $f \in L^\infty(I \times \Omega)$.*

(i) *Assume that (K1_{loc}) holds for some $\theta \in [d/\alpha, \infty]$. Then there exist $c > 0$ and $0 < c_1 < c_2 < c_3 < c_4 \leq 1$ such that, for every $0 < R \leq 1$ and every nonnegative, weak solution u to (PDE) in $I_R(t_0) \times B_{2R}$,*

$$\sup_{(t_0-c_2R^\alpha, t_0-c_1R^\alpha) \times B_{R/4}} u \leq c \inf_{(t_0+c_1R^\alpha, t_0+c_4R^\alpha) \times B_{R/2}} u + c \sup_{(t_0-c_3R^\alpha, t_0-c_1R^\alpha)} \text{Tail}_{K, \alpha}(u, R) + cR^\alpha \|f\|_{L^\infty}, \tag{1-6}$$

where $B_{2R} \subset \Omega \subset \mathbb{R}^d$.

(ii) Assume that $(K1_{\text{glob}})$ holds for some $\theta \in (d/\alpha, \infty]$. Then there exist $c > 0$ and $0 < c_1 < c_2 < c_3 < c_4 \leq 1$ such that, for every $0 < R \leq 1$ and every nonnegative, weak solution u to $(\widehat{\text{PDE}})$ in $I_R(t_0) \times B_{2R}$,

$$\sup_{(t_0 - c_2 R^\alpha, t_0 - c_1 R^\alpha) \times B_{R/4}} u \leq c \inf_{(t_0 + c_1 R^\alpha, t_0 + c_4 R^\alpha) \times B_{R/2}} u + c \sup_{(t_0 - c_3 R^\alpha, t_0 - c_1 R^\alpha)} \widehat{\text{Tail}}_{K, \alpha}(u, R) + c R^\alpha \|f\|_{L^\infty}. \quad (1-7)$$

The aforementioned Harnack inequality for nonnegative weak solutions u to (PDE) is a direct consequence of a weak Harnack inequality as it was proved in [Kassmann and Weidner 2022] (see Theorem 6.3) and an L^∞ - L^1 -estimate of the form (see Theorem 3.6 or Theorem 4.8)

$$\sup_{(t_0 - (R/8)^\alpha, t_0) \times B_{R/2}} u \leq c \left(\int_{(t_0 - (R/4)^\alpha, t_0) \times B_R} u + \sup_{(t_0 - (R/4)^\alpha, t_0)} \text{Tail}_{K, \alpha}(u, R) + R^\alpha \|f\|_{L^\infty} \right). \quad (1-8)$$

Therefore large parts of this paper are dedicated to proving (1-8). Given $0 < R \leq 1$, the nonlocal tail term is defined as

$$\text{Tail}_{K, \alpha}(v, R, x_0) := R^\alpha \int_{B_{2R}(x_0) \setminus B_{R/2}(x_0)} \frac{|v(y)|}{|x_0 - y|^{d+\alpha}} dy + \sup_{x \in B_{3R/2}(x_0)} \int_{B_{2R}(x_0)^c} |v(y)| K(x, y) dy.$$

For a detailed discussion of nonlocal tail terms, we refer the reader to Section 2.3.

Remark 1.2 (time-inhomogeneous kernels). It is possible to extend Theorem 1.1 to time-inhomogeneous jumping kernels $k : I \times \mathbb{R}^d \times \mathbb{R}^d \rightarrow [0, \infty]$ by following an approach similar to that in [Kassmann and Weidner 2022]. For k_s , we may assume pointwise comparability with a time-homogeneous jumping kernel satisfying (cutoff), (\mathcal{E}_\geq) , and $(K1_{\text{loc}}^\leq)$. In place of the first estimate in $(K1_{\text{loc}})$, we need

$$\left\| \int_{B_{2r}} \frac{|k_a(\cdot; \cdot, y)|^2}{J(\cdot, y)} dy \right\|_{L_{t,x}^{\mu, \theta}(I_r \times B_{2r})} \leq C$$

for a suitable symmetric jumping kernel $J : \mathbb{R}^d \times \mathbb{R}^d \rightarrow [0, \infty]$. The parameters (μ, θ) have to satisfy the compatibility condition

$$\frac{d}{\alpha\theta} + \frac{1}{\mu} < 1. \quad (\text{CP})$$

Then, if suitable time-inhomogeneous analogs to $(K2)$ and (UJS) , or $(\widehat{\text{UJS}})$, hold, we can prove a Harnack inequality of the form (1-6) and (1-7) for nonnegative, weak solutions to the corresponding parabolic equations (PDE) and $(\widehat{\text{PDE}})$, respectively. For solutions to (PDE) we can also allow for equality in (CP) if $\theta > d/\alpha$. The range of exponents prescribed by (CP) align with the important classical results from the local theory; see [Aronson and Serrin 1967; Ivanov et al. 1966; Ladyzhenskaya et al. 1968]. Note that, by scaling arguments, one can see $d/(\alpha\theta) + 1/\mu = 1$ is the limit case for regularity results in Hölder spaces.

Remark 1.3. We observe that there is a positive distance of size $2(1 - 2^{-\alpha})R^\alpha$ between the two time intervals in the estimates (1-6) and (1-7). The existence of such time delay in the parabolic Harnack inequality comes from the method of proof we employ; see [Moser 1964]. For nonlocal equations, as for example the fractional heat equation, it can be neglected; see [Bonforte et al. 2017; Dier et al. 2020].

The second main result of this article concerns the corresponding stationary problems

$$-Lu = f \quad \text{in } B_{2R}, \quad (\text{ell-PDE})$$

$$-\widehat{L}u = f \quad \text{in } B_{2R}, \quad (\text{ell-}\widehat{\text{PDE}})$$

where $f \in L^\infty(B_{2R})$. We obtain the following elliptic Harnack inequality for weak solutions.

Theorem 1.4. *Assume (K2), (cutoff), and (\mathcal{E}_\geq) for some $\alpha \in (0, 2)$. Let $f \in L^\infty(\Omega)$.*

- (i) *Assume that $(K1_{\text{loc}})$ and (UJS) hold for some $\theta \in [d/\alpha, \infty]$. Then there exists $c > 0$ such that, for every $0 < R \leq 1$ and every nonnegative, weak solution u to (ell-PDE) in B_{2R} ,*

$$\sup_{B_{R/4}} u \leq c \left(\inf_{B_{R/2}} u + R^\alpha \|f\|_{L^\infty} \right), \quad (1-9)$$

where $B_{2R} \subset \Omega \subset \mathbb{R}^d$.

- (ii) *Assume that $(K1_{\text{glob}})$ and $(\widehat{\text{UJS}})$ hold for some $\theta \in (d/\alpha, \infty]$. Then there exists $c > 0$ such that, for every $0 < R \leq 1$ and every nonnegative, weak solution u to $(\text{ell-}\widehat{\text{PDE}})$ in B_{2R} , estimate (1-9) holds.*

As in (1-9), for elliptic equations, we are able to estimate the supremum of u by local quantities only. To this end, we prove a suitable estimate of the nonlocal tail term (see Corollary 5.3).

In the parabolic case, the situation is more complicated since we require the tail estimate to be uniform in t . The same difficulty occurs in the symmetric case. We comment on possible corresponding extensions of Theorem 1.1 in Section 6.3.

Remark 1.5. All constants in Theorems 1.1 and 1.4 depend only on d , α , θ and the constants in $(K1_{\text{loc}})$, (K2), (cutoff), (Poinc), (Sob), (UJS), (K_{loc}^\leq) , (\mathcal{E}_\geq) .

Remark 1.6. Theorems 1.1 and 1.4 remain valid for solutions u to (PDE), $(\widehat{\text{PDE}})$, and (ell-PDE), $(\text{ell-}\widehat{\text{PDE}})$ if $f \in L^\infty(I_R(t_0); L^\Theta(B_{2R}))$ and $f \in L^\Theta(B_{2R})$, respectively, for some $\theta \in (d/\alpha, \infty)$ with only marginal manipulations in the proofs. We exclude more general source terms in this work.

The contributions of this work can be summarized as follows:

- (i) The main accomplishment is the extension of elliptic and parabolic regularity results—including full Harnack inequalities—for nonlocal problems to operators with *nonsymmetric* jumping kernels. In light of example (1-4), the operators under consideration include nonlocal counterparts of second-order differential operators in divergence form with a drift term

$$-\mathcal{L}u = -\partial_i(a_{i,j}\partial_j u) + b_i\partial_i u \quad \text{and} \quad -\widehat{\mathcal{L}}u = -\partial_i(a_{i,j}\partial_j u + b_i u),$$

respectively. Our results align with the corresponding theory for local operators; see [Aronson and Serrin 1967; Gilbarg and Trudinger 1983; Ladyzhenskaya et al. 1968; Stampacchia 1965].

- (ii) As nonsymmetric kernels require a careful treatment, several parts of the energy methods for nonlocal operators are refined in this work. For instance, we give a new proof of local boundedness using the Moser iteration for positive exponents (see Section 4).

Moreover, as illustrated in example (1-5), nonsymmetric jumping kernels might naturally involve terms of lower-order, causing a difference between the growth behavior at zero and infinity. We introduce tail terms which take into account this phenomenon (see Section 2.3).

(iii) Technical issues of minor importance in other works are clarified, e.g., the treatment of Steklov averages (see the Appendix).

1.2. Related literature. The study of Harnack inequalities for symmetric nonlocal operators has become an active field of research in the past 20 years. It has been observed that a classical elliptic Harnack inequality of the form

$$\sup_{B_r} u \leq c \inf_{B_r} u \quad (1-10)$$

fails even for harmonic functions u with respect to the fractional Laplacian $(-\Delta)^{\alpha/2}$ in B_{2r} if one merely assumes u to be nonnegative in the solution domain B_{2r} ; see [Kassmann 2007]. Indeed, due to the nonlocality it is necessary either to assume u to be globally nonnegative — as in [Riesz 1938] and in this article — or to add the nonlocal tail of u_- to the right-hand side of (1-10). Such an estimate was proposed in [Kassmann 2011]. We refer to both estimates as a *Harnack inequality* in the context of this article.

A lot of research activity has centered around the challenge to establish a Harnack inequality for a larger class of nonlocal operators. First, we comment on corresponding elliptic regularity results for symmetric nonlocal operators related to energy forms. A Harnack inequality and Hölder estimates were proved in [Di Castro et al. 2014; 2016] for operators with a jumping kernel that is pointwise comparable to the kernel of the fractional p -Laplacian by a nonlocal De Giorgi-type iteration. This method was refined in [Cozzi 2017] to allow for more general nonlinearities. [Schulze 2019] considers a class of linear integrodifferential operators governed by jumping kernels satisfying an average integral bound instead of a pointwise lower bound.

However, it is well known that for the deduction of interior Hölder regularity estimates a weak Harnack inequality (see Theorem 6.3) is sufficient. Such inequalities hold for a much larger class of operators. In fact, only comparability of the energy forms to the $H^{\alpha/2}$ -seminorm on small scales and a suitable upper bound for the probability of large jumps are required; see [Dyda and Kassmann 2020]. That is why operators with singular jumping measures that may be anisotropic (see [Chaker and Kassmann 2020; Chaker et al. 2019]) also satisfy Hölder regularity estimates. However, the Harnack inequality may fail for singular operators as was already observed in [Bogdan and Sztonyk 2005]. Hence it is an exciting (and still open) question to find equivalent conditions on the jumping kernel for which a (weak) elliptic Harnack inequality will hold. For α -stable translation-invariant operators, conditions on the jumping kernel are established in [Bogdan and Sztonyk 2005] that are equivalent to a Harnack inequality.

Second, we comment on parabolic Harnack inequalities of the form

$$\sup_{I_r^\ominus \times B_r} u \leq c \inf_{I_r^\oplus \times B_r} u \quad (1-11)$$

for globally nonnegative solutions u to (PDE). Note that such results imply corresponding estimates for weak solutions to the stationary equation (ell-PDE). So far, parabolic Harnack inequalities have not been

obtained via purely analytic methods, not even in the symmetric case. A major challenge in the parabolic case seems to be the correct treatment of the time-dependence in the nonlocal tail terms. For a discussion of this issue, we refer the reader to Section 2 and Section 6.3.

Parabolic Hölder estimates and local boundedness have been obtained via an adaptation of the nonlocal De Giorgi method in [Ding et al. 2021; Kim 2019; 2020; Strömqvist 2019b]. A proof of Hölder estimates based on Moser's technique can be found in [Felsinger and Kassmann 2013].

Using the corresponding Hunt process and its heat kernel, parabolic Harnack inequalities of the form (1-11) were first proved for symmetric Dirichlet forms with jumping measures pointwise comparable to the α -stable kernel in [Bass and Levin 2002; Chen and Kumagai 2003]. The authors also obtain two-sided heat kernel bounds. Numerous articles have analyzed the exact relationship between parabolic and elliptic Harnack inequalities, heat kernel bounds, and Hölder regularity estimates for nonlocal operators in connection to the geometry of the underlying metric measure space. Such a program was carried out in a series of papers [Chen et al. 2019; 2020; Grigor'yan et al. 2014; 2015; 2018]. On \mathbb{R}^d it turns out that (1-11) is equivalent to a Poincaré inequality (see (Poinc)), a pointwise upper bound of the jumping kernel, and (UJS).

In contrast to the symmetric case, for nonlocal operators associated with nonsymmetric forms, pointwise estimates have not yet been studied systematically. Some results have been obtained making use of a sector-type condition. Well-posedness of the Dirichlet problem is proved in [Felsinger et al. 2015]. In the present article and in [Kassmann and Weidner 2022], we provide Harnack inequalities and interior Hölder regularity estimates for nonlocal operators that contain a nonlocal drift term of lower-order. These results can be regarded as nonlocal counterparts of the famous regularity results for local equations by Aronson and Serrin [1967] and Ladyzhenskaya, Solonnikov, and Ural'tceva [Ladyzhenskaya et al. 1968] in the linear case. Hölder estimates for kinetic integrodifferential equations including certain nonlocal operators with nonsymmetric jumping kernels are established in [Imbert and Silvestre 2020] using an adaptation of the De Giorgi iteration. The class of nonsymmetric kernels in their work does not contain the class of kernels in our work, and vice versa.

Note that, as an application of the regularity estimates in [Kassmann and Weidner 2022], it is possible to establish Markov chain approximation results not only for diffusion processes with drift terms, but also for certain nonsymmetric jump processes; see [Weidner 2023]. In light of [Chen et al. 2020] and [Grigor'yan et al. 2018], we consider it an interesting problem to establish heat kernel estimates for nonlocal operators associated with nonsymmetric forms, and to investigate their stability on general doubling metric measure spaces, as well as their connection to Harnack inequalities.

1.3. Outline. This article is structured as follows: In Section 2 we state and discuss our assumptions and the notion of a weak solution to (PDE) and $(\widehat{\text{PDE}})$. A Caccioppoli-type estimate for nonsymmetric forms and an a priori L^∞ - L^2 -estimate involving the nonlocal tail is proved in Section 3 using a nonsymmetric version of the De Giorgi iteration. An analogous result is established in Section 4 using a nonlocal adaptation of the Moser iteration technique for large positive exponents. Note that Sections 3 and 4 are fully independent of one another. In Section 5 we establish an upper bound for the nonlocal tails of supersolutions to (PDE) and $(\widehat{\text{PDE}})$. Our two main results, Theorems 1.1 and 1.4, are proved in Section 6.

2. Preliminaries

In this section we state and discuss the assumptions in our main results (see Section 2.1). Moreover, we provide the notion of a super- or subsolution to (PDE) and ($\widehat{\text{PDE}}$), as well as the corresponding stationary equations (ell-PDE) and (ell- $\widehat{\text{PDE}}$) (see Section 2.2). Another goal of this section is to introduce nonlocal tail terms which suit the class of nonsymmetric operators under consideration and are designed in such a way that they are compatible with the iteration techniques carried out in the remainder of this article (see Section 2.3).

We introduce the following notation: First of all, given $a, b \in \mathbb{R}$, we write $a \wedge b = \min\{a, b\}$ and $a \vee b := \max\{a, b\}$. Moreover, given a set $M \subset \mathbb{R}^d \times \mathbb{R}^d$, we write

$$\mathcal{E}_M(u, v) := \iint_M (u(x) - u(y))v(x)K(x, y) \, dx \, dy.$$

Analogously, we define $\mathcal{E}_M^{K_s}$ and $\mathcal{E}_M^{K_a}$. If $M := B_r \times B_r$ for a ball $B_r \subset \mathbb{R}^d$, we write $\mathcal{E}_{B_r} = \mathcal{E}_{B_r \times B_r}$.

2.1. Discussion of main assumptions. In this section, we list and discuss the assumptions which are imposed on the jumping kernels K in the course of this article. Except for (UJS), all other assumptions have already been discussed in detail in [Kassmann and Weidner 2022].

First, we assume throughout this article that K_s satisfies the Lévy-integrability condition

$$\left(x \mapsto \int_{\mathbb{R}^d} (|x - y|^2 \wedge 1) K_s(x, y) \, dy\right) \in L^1_{\text{loc}}(\mathbb{R}^d). \quad (2-1)$$

In the following, let $\Omega \subset \mathbb{R}^d$ be an open set. Let us now fix $\alpha \in (0, 2)$ and $\theta \in [d/\alpha, \infty]$. The first two assumptions were introduced and discussed in [Kassmann and Weidner 2022].

Assumption (K1). Let $J : \mathbb{R}^d \times \mathbb{R}^d \rightarrow [0, \infty]$ be a symmetric jumping kernel satisfying (cutoff) and let $\theta \in [d/\alpha, \infty]$.

- K satisfies (K1_{loc}) if there is $C > 0$ such that, for every ball $B_{2r} \subset \Omega$ with $r \leq 1$,

$$\left\| \int_{B_{2r}} \frac{|K_a(\cdot, y)|^2}{J(\cdot, y)} \, dy \right\|_{L^\theta(B_{2r})} \leq C, \quad \mathcal{E}_{B_{2r}}^J(v, v) \leq C \mathcal{E}_{B_{2r}}^{K_s}(v, v) \quad \text{for all } v \in L^2(B_{2r}). \quad (\text{K1}_{\text{loc}})$$

- K satisfies (K1_{glob}) if there is $C > 0$ such that, for every ball $B_{2r} \subset \Omega$ with $r \leq 1$,

$$\left\| \int_{\mathbb{R}^d} \frac{|K_a(\cdot, y)|^2}{J(\cdot, y)} \, dy \right\|_{L^\theta(\mathbb{R}^d)} \leq C, \quad \mathcal{E}_{B_{2r}}^J(v, v) \leq C \mathcal{E}_{B_{2r}}^{K_s}(v, v) \quad \text{for all } v \in L^2(B_{2r}). \quad (\text{K1}_{\text{glob}})$$

Assumption (K2). There exist $C > 0$, $D < 1$, and a symmetric jumping kernel j such that, for every ball $B_{2r} \subset \Omega$ with $r \leq 1$ and every $v \in L^2(B_{2r})$ with $\mathcal{E}_{B_{2r}}^{K_s}(v, v) < \infty$,

$$K(x, y) \geq (1 - D)j(x, y) \quad \text{for all } x, y \in B_{2r}, \quad \mathcal{E}_{B_{2r}}^{K_s}(v, v) \leq C \mathcal{E}_{B_{2r}}^j(v, v). \quad (\text{K2})$$

Remark 2.1. (i) (K1_{loc}) ensures that the quantities in (2-6) and (PDE) are well defined (see Lemma 2.9) and simultaneously requires that \mathcal{E}^{K_a} is a term of lower-order. It gives rise to a nonlocal drift, analogous to $(b, \nabla u)$, where $b \in L^{2\theta}(\mathbb{R}^d)$ with $\theta \in [\frac{1}{2}d, \infty]$.

- (ii) (K2) is only needed in the proof of the weak Harnack inequality (see Theorem 6.3). It ensures that the symmetric kernel $K_s - |K_a|$ is locally coercive with respect to \mathcal{E}^{K_s} .
- (iii) For a detailed discussion of (K1_{loc}) and (K2) including their redundancy, we refer the reader to [Kassmann and Weidner 2022]. Equations (K1_{loc}) and (K2) are verified for the examples K_1 , K_2 , and K_3 from above in Section 8 of that paper.
- (iv) In the simplest case, (K1_{loc}) (and (K1_{glob})) and (K2) hold with $J = j = K_s$. However, allowing for general symmetric kernels J and j significantly increases the class of admissible operators.

The following two assumptions on K only depend on the symmetric part. They are standard in the regularity for nonlocal operators associated with symmetric forms.

Assumption (cutoff). There is $c > 0$ such that, for every $0 < \rho \leq r \leq 1$ and $z \in \Omega$ such that $B_{r+\rho}(z) \subset \Omega$, there is a radially decreasing function $\tau = \tau_{z,r,\rho}$ centered at $z \in \mathbb{R}^d$ with $\text{supp}(\tau) \subset \overline{B_{r+\rho}(z)}$, $0 \leq \tau \leq 1$, $\tau \equiv 1$ on $B_r(z)$, $|\nabla \tau| \leq \frac{3}{2}\rho^{-1}$, and

$$\sup_{x \in B_{r+\rho}(z)} \Gamma^{K_s}(\tau, \tau)(x) \leq c\rho^{-\alpha}, \tag{cutoff}$$

where

$$\Gamma^{K_s}(\tau, \tau)(x) := \int_{\mathbb{R}^d} (\tau(x) - \tau(y))^2 K_s(x, y) \, dy$$

is the carré du champ associated with \mathcal{E}^{K_s} .

Note that $\Gamma^{K_s}(\tau, \tau)$ can be interpreted as the density of the energy $\mathcal{E}^{K_s}(\tau, \tau)$. Such an object is often called ‘‘carré du champ’’ in the literature.

Assumption (\mathcal{E}_{\geq}). There exists $c > 0$ such that, for every ball $B_{2r} \subset \Omega$ and every $v \in L^2(B_{2r})$,

$$\mathcal{E}_{B_{2r}}^{K_s}(v, v) \geq c[v]_{H^{\alpha/2}(B_{2r})}^2. \tag{\mathcal{E}_{\geq}}$$

Remark 2.2. (i) A sufficient condition for (cutoff) to hold for every $\tau_{z,r,\rho}$ is (see [Kassmann and Weidner 2022]): there is $c > 0$ such that, for every $0 < \zeta \leq \rho \leq r \leq 1$ and $z \in \mathbb{R}^d$ with $B_{r+\rho}(z) \subset \Omega$,

$$\sup_{x \in B_{r+\rho}(z)} \left(\int_{\mathbb{R}^d \setminus B_{\zeta}(x)} K_s(x, y) \, dy \right) \leq c\zeta^{-\alpha}. \tag{2-2}$$

- (ii) (\mathcal{E}_{\geq}) is a classical coercivity condition on K_s . It is significantly weaker than a pointwise lower bound of the form $K_s(x, y) \geq c|x - y|^{-d-\alpha}$ since it allows for non-fully-supported kernels such as K_3 (see (1-5)).
- (iii) Under (\mathcal{E}_{\geq}), we have the following Poincaré and Sobolev inequalities: there is $c > 0$ such that, for every ball $B_{r+\rho} \subset \Omega$ with $0 < \rho \leq r \leq 1$ and $v \in L^2(B_{r+\rho})$,

$$\|v^2\|_{L^{d/(d-\alpha)}(B_r)} \leq c\mathcal{E}_{B_{r+\rho}}^{K_s}(v, v) + c\rho^{-\alpha}\|v^2\|_{L^1(B_{r+\rho})}, \tag{Sob}$$

$$\int_{B_r} (v(x) - [v]_{B_r})^2 \, dx \leq cr^\alpha \mathcal{E}_{B_r}^{K_s}(v, v), \tag{Poinc}$$

where $[v]_{B_r} = \int_{B_r} v(x) dx$. Equation (Poinc) is not explicitly needed in any of the proofs of this article. Nevertheless it is required for Theorem 6.3 to hold and therefore appears in the assumptions of our main result Theorem 1.1.

The following assumption did not appear in [Kassmann and Weidner 2022] and is designed to estimate nonlocal tails of supersolutions to (PDE) from above. It is required for the proof of the Harnack inequality.

Assumption (UJS). • K satisfies (UJS) if there exists $c > 0$ such that, for every $x, y \in \mathbb{R}^d$ and every $r \leq (\frac{1}{4} \wedge \frac{1}{4}|x - y|)$ with $B_r(x) \subset \Omega$,

$$K(x, y) \leq c \int_{B_r(x)} K(z, y) dz. \quad (\text{UJS})$$

• K satisfies $(\widehat{\text{UJS}})$ if there exists $c > 0$ such that, for every $x, y \in \mathbb{R}^d$ and every $r \leq (\frac{1}{4} \wedge \frac{1}{4}|x - y|)$ with $B_r(x) \subset \Omega$,

$$K(y, x) \leq c \int_{B_r(x)} K(y, z) dz. \quad (\widehat{\text{UJS}})$$

Remark 2.3. (i) If K satisfies both conditions $(\widehat{\text{UJS}})$ and (UJS), then K_s satisfies (UJS).

(ii) Also for symmetric kernels the conditions (cutoff), (Poinc), and (Sob) are known to be insufficient for a Harnack inequality to hold; see [Bogdan and Sztonyk 2005].

(iii) Analogs to (UJS) for symmetric jumping kernels appeared in [Chen et al. 2020; Schulze 2019]. A pointwise version of (UJS) was considered in [Bass and Kassmann 2005].

Remark 2.4. (i) (UJS) clearly holds if $K(x, y)$ is pointwise comparable to $|x - y|^{-d-\alpha}$ for every $x, y \in \mathbb{R}^d$. However, (UJS) neither implies nor is implied by (\mathcal{E}_{\geq}) .

(ii) Assume a global version of (K2), namely

$$|K_a(x, y)| \leq DK_s(x, y) \quad \text{for all } x \in \Omega, \quad y \in \mathbb{R}^d. \quad (2-3)$$

Then $(1 - D)K_s \leq K \leq 2K_s$, and therefore (UJS) is equivalent to

$$K_s(x, y) \leq c \int_{B_r(x)} K_s(z, y) dz$$

for $x, y \in \mathbb{R}^d$ and $r \leq (\frac{1}{4} \wedge \frac{1}{4}|x - y|)$ with $B_r(x) \subset \Omega$, i.e., it remains to verify (UJS) for K_s .

(iii) In [Schulze 2019] it was proved that kernels of the form

$$K_s(x, y) = \mathbb{1}_S(x - y)|x - y|^{-d-\alpha}$$

satisfy (UJS) if $S = -S$, and there exists $c > 0$ such that, for every $x \in S$ and $r \leq (\frac{1}{4}|x| \wedge \frac{1}{4})$, we have that $|B_r(x)| \leq c|B_r(x) \cap S|$.

We provide sufficient conditions for (UJS) to hold for the examples K_1, K_2, K_3 in (1-3)–(1-5).

Example 2.5. (i) Let $K_1(x, y) = g(x, y)|x - y|^{-d-\alpha}$ be as in (1-3). It was shown in [Kassmann and Weidner 2022] that (2-3) holds for K_1 with $D = (\Lambda - \lambda)/(\Lambda + \lambda) < 1$. As

$$2K_s(x, y) = (g(x, y) + g(y, x))|x - y|^{-d-\alpha},$$

it follows that (UJS) holds for K .

(ii) Let K_2 be as in (1-4). Then, the antisymmetric part of K_2 is given by

$$K_a(x, y) = (V(x) - V(y))\mathbb{1}_{\{|x-y|\leq L\}}(x, y)|x - y|^{-d-\alpha} \leq K_s(x, y) = |x - y|^{-d-\alpha}.$$

Therefore, (UJS) holds if there exists $c > 0$ such that, for every $x, y \in \mathbb{R}^d$ and $r \leq (\frac{1}{4}|x - y| \wedge \frac{1}{4})$ with $B_r(x) \subset \Omega$,

$$1 + (V(x) - V(y))\mathbb{1}_{\{|x-y|\leq L\}} \leq c \int_{B_r(x)} 1 + (V(z) - V(y))\mathbb{1}_{\{|z-y|\leq L\}} dz. \tag{2-4}$$

(iii) We claim that (UJS) holds for K_3 . Let us prove the following more general statement: Let $S \subset \mathbb{R}^d$ with $0 \in S$ and $c > 0$ such that, for every $x \in S$ and $r < \frac{1}{4}$, we have $|S \cap B_r(x)|/r^d \geq c$. Then,

$$K(x, y) = \mathbb{1}_S(x - y)|x - y|^{-d-\alpha}$$

satisfies (UJS) and $(\widehat{\text{UJS}})$.

In fact it suffices to prove that

$$\mathbb{1}_S(x - y) \leq c \int_{B_r(x)} \mathbb{1}_S(z - y) dz \tag{2-5}$$

in order to deduce (UJS). Note that $(\widehat{\text{UJS}})$ follows by consideration of $-S$. We compute

$$\mathbb{1}_S(x - y) \leq c \frac{|B_r(x - y) \cap S|}{r^d} = c \frac{|B_r(x) \cap (y + S)|}{r^d} = c \int_{B_r(x)} \mathbb{1}_S(z - y) dz.$$

Finally, we introduce the assumption of an upper bound of the jumping kernel which will be used only to prove an L^∞ - L^2 + Tail estimate (see Theorem 3.6) and is not required for the proof of the main theorems. However it follows from (UJS) and (cutoff).

Assumption (K_{loc}^{\leq}). There exists $c > 0$ such that, for every ball $B_{2r} \subset \Omega$ with $r \leq 1$ and every $x, y \in B_{2r}$,

$$K(x, y) \leq c|x - y|^{-d-\alpha}. \tag{K_{\text{loc}}^{\leq}}$$

Remark 2.6. Note that (K_{loc}^{\leq}) follows from (UJS) and (cutoff). Indeed, for any $x, y \in \mathbb{R}^d$ with $|x - y| \leq 4$ and $r = \frac{1}{16}|x - y| \leq (\frac{1}{4} \wedge \frac{1}{4}|x - y|)$, we have $B_r(x) \subset B_r(y)^c$, and therefore

$$K(x, y) \leq c_1 \int_{B_r(x)} K(z, y) dz \leq c_2 r^{-d} \int_{B_r(y)^c} K(z, y) dz \leq c_3 r^{-d-\alpha} \leq c_4 |x - y|^{-d-\alpha}$$

for some constants $c_1, c_2, c_3, c_4 > 0$.

2.2. Weak solution concept. We introduce the following function spaces for $\Omega \subset \mathbb{R}^d$:

$$V(\Omega|\mathbb{R}^d) = \{v : \mathbb{R}^d \rightarrow \mathbb{R} \text{ s.t. } v|_\Omega \in L^2(\Omega) \text{ and } (v(x) - v(y))K_s^{1/2}(x, y) \in L^2(\Omega \times \mathbb{R}^d)\},$$

$$H_\Omega(\mathbb{R}^d) = \{v \in V(\mathbb{R}^d|\mathbb{R}^d) \text{ s.t. } v = 0 \text{ on } \mathbb{R}^d \setminus \Omega\}$$

equipped with

$$\|v\|_{V(\Omega|\mathbb{R}^d)}^2 = \|v\|_{L^2(\Omega)}^2 + \int_\Omega \int_{\mathbb{R}^d} (v(x) - v(y))^2 K_s(x, y) \, dy \, dx,$$

$$\|v\|_{H_\Omega(\mathbb{R}^d)}^2 = \|v\|_{L^2(\mathbb{R}^d)}^2 + \mathcal{E}^{K_s}(v, v).$$

We emphasize that both spaces are completely determined by the symmetric part of the jumping kernel K_s . Moreover, for $\alpha \in (0, 2)$, we define $V^\alpha(\Omega|\mathbb{R}^d)$ and $H_\Omega^\alpha(\mathbb{R}^d)$ as the corresponding function spaces associated with $K_s(x, y) = |x - y|^{-d-\alpha}$.

We are ready to define the notion of a weak solution to (PDE) and $(\widehat{\text{PDE}})$. Let us define $\theta' := \theta/(\theta - 1)$ as the Hölder conjugate exponent of θ .

Definition 2.7. Let $\Omega \subset \mathbb{R}^d$ be a bounded domain, $I \subset \mathbb{R}$ a finite interval, and $f \in L^\infty(I \times \Omega)$.

- (i) We say that $u \in L^2_{\text{loc}}(I; V(\Omega|\mathbb{R}^d))$ is a weak supersolution to (PDE) in $I \times \Omega$ if the weak $L^2(\Omega)$ -derivative $\partial_t u$ exists, $\partial_t u \in L^1_{\text{loc}}(I; L^2(\Omega))$, and

$$(\partial_t u(t), \phi) + \mathcal{E}(u(t), \phi) \leq (f(t), \phi) \quad \text{for all } t \in I \text{ and for all } \phi \in H_\Omega(\mathbb{R}^d) \text{ with } \phi \leq 0. \quad (2-6)$$

We call u a weak subsolution if (2-6) holds for every $\phi \geq 0$. We call u a weak solution, if it is a supersolution and a subsolution.

- (ii) We say that $u \in L^2_{\text{loc}}(I; V(\Omega|\mathbb{R}^d) \cap L^{2\theta'}(\mathbb{R}^d))$ is a weak supersolution to $(\widehat{\text{PDE}})$ in $I \times \Omega$ if the weak $L^2(\Omega)$ -derivative $\partial_t u$ satisfies the same properties as before and

$$(\partial_t u(t), \phi) + \widehat{\mathcal{E}}(u(t), \phi) \leq (f(t), \phi) \quad \text{for all } t \in I \text{ and for all } \phi \in H_\Omega(\mathbb{R}^d) \text{ with } \phi \leq 0.$$

Weak (sub-)solutions to $(\widehat{\text{PDE}})$ are defined in analogy with (i).

Next, we introduce the solution concept for stationary equations.

Definition 2.8. Let $\Omega \subset \mathbb{R}^d$ be a bounded domain and $f \in L^\infty(\Omega)$.

- (i) We say that $u \in V(\Omega|\mathbb{R}^d)$ is a weak supersolution to (ell-PDE) in Ω if

$$\mathcal{E}(u, \phi) \leq (f, \phi) \quad \text{for all } \phi \in H_\Omega(\mathbb{R}^d) \text{ with } \phi \leq 0. \quad (2-7)$$

We call u a weak subsolution if (2-7) holds for every $\phi \geq 0$. We call u a weak solution if it is a supersolution and a subsolution.

- (ii) We say that $u \in V(\Omega|\mathbb{R}^d) \cap L^{2\theta'}(\mathbb{R}^d)$ is a weak supersolution to (ell- $\widehat{\text{PDE}}$) in Ω if

$$\widehat{\mathcal{E}}(u, \phi) \leq (f, \phi) \quad \text{for all } \phi \in H_\Omega(\mathbb{R}^d) \text{ with } \phi \leq 0.$$

(Sub)solutions to (ell- $\widehat{\text{PDE}}$) are defined in analogy with (i).

Let us point out that the solution concept also makes sense under much weaker assumptions on u without any change in the proofs being needed; see [Felsinger and Kassmann 2013]. In particular, one can drop the condition that the weak time derivative $\partial_t u$ exists.

We will only consider solutions on special time-space cylinders $I_R(t_0) \times B_{2R}$, where $B_{2R} \subset \Omega$ is a ball, $I_R(t_0) = (t_0 - R^\alpha, t_0 + R^\alpha)$, $0 < R \leq 1$, and $t_0 \in \mathbb{R}$. Moreover,

$$I_R^\ominus(t_0) := (t_0 - R^\alpha, t_0), \quad I_R^\oplus(t_0) := (t_0, t_0 + R^\alpha).$$

Recall the following lemma, which was proved in [Kassmann and Weidner 2022]. It ensures that the expressions in Definitions 2.7 and 2.8 are well defined.

Lemma 2.9 [Kassmann and Weidner 2022, Lemma 2.2]. *Let $0 < \rho \leq r \leq 1$ and $B_{2r} \subset \Omega$.*

(i) *Assume that one of the following is true:*

- $(K1_{\text{loc}})$ holds with $\theta = \infty$,
- $(K1_{\text{loc}})$ holds with $\theta \in [d/\alpha, \infty)$ and (Sob) holds.

Then $\mathcal{E}(u, \phi)$ is well defined for $u \in V(B_{r+\rho} | \mathbb{R}^d)$ and $\phi \in H_{B_{r+\rho}}(\mathbb{R}^d)$.

(ii) *Assume that $(K1_{\text{glob}})$ holds with $\theta \in [d/\alpha, \infty]$. Then $\widehat{\mathcal{E}}(u, \phi)$ is well defined for $\phi \in H_{B_{r+\rho/2}}(\mathbb{R}^d)$ and $u \in V(B_{r+\rho/2} | \mathbb{R}^d) \cap L^{2\theta'}(\mathbb{R}^d)$.*

The following lemma is of central importance in the proofs of the Caccioppoli estimates for nonsymmetric nonlocal operators. Note that the proof in the special case $\theta = \infty$ is trivial.

Lemma 2.10 [Kassmann and Weidner 2022, Lemma 2.4]. (i) *Assume that $(K1_{\text{loc}})$ holds for some $\theta \in [d/\alpha, \infty]$. Moreover, assume (Sob) if $\theta < \infty$. Then, there exists $c_1 > 0$ such that, for every $\delta > 0$, there is $C(\delta) > 0$ such that, for every $v \in L^2(B_{r+\rho})$ with $\text{supp}(v) \subset B_{r+\rho/2}$ and every ball $B_{2r} \subset \Omega$ with $0 < \rho \leq r \leq 1$, we have*

$$\int_{B_{r+\rho}} v^2(x) \left(\int_{B_{r+\rho}} \frac{|K_a(x, y)|^2}{J(x, y)} dy \right) dx \leq \delta \mathcal{E}_{B_{r+\rho}}^{K_s}(v, v) + c_1(C(\delta) + \delta \rho^{-\alpha}) \|v\|_{L^1(B_{r+\rho})}^2. \tag{2-8}$$

Moreover, if $\theta \in (d/\alpha, \infty]$, the constant $C(\delta)$ has the following form:

$$C(\delta) = \begin{cases} \|W\|_{L^\infty(B_{r+\rho})}, & \theta = \infty, \\ \delta^{d/(d-\theta\alpha)} \|W\|_{L^\theta(B_{r+\rho})}^{\theta\alpha/(\theta\alpha-d)}, & \theta \in (d/\alpha, \infty), \end{cases} \quad \text{where } W(x) := \int_{B_{r+\rho}} \frac{|K_a(x, y)|^2}{J(x, y)} dy. \tag{2-9}$$

(ii) *Assume that $(K1_{\text{glob}})$ holds for some $\theta \in [d/\alpha, \infty]$. Moreover, assume (Sob) if $\theta < \infty$. Then (2-8) and (2-9) hold with*

$$\int_{\mathbb{R}^d} \frac{|K_a(x, y)|^2}{J(x, y)} dy \quad \text{instead of} \quad \int_{B_{r+\rho}} \frac{|K_a(x, y)|^2}{J(x, y)} dy.$$

2.3. Nonlocal tail terms. Due to the nonlocality of the problems under consideration, certain nonlocal tail terms naturally enter the picture. For references concerning the treatment of tail terms in the study of symmetric nonlocal operators, we refer the reader to [Chen et al. 2020; Di Castro et al. 2014; 2016]. It is crucial for our analysis to make sure that the respective tail terms are finite for any weak solution under

reasonable assumptions on K and that the tail terms are compatible with the iteration techniques carried out in the remainder of this article.

Given any ball $B_{2r}(x_0) \subset \Omega$, a function $v \in V(B_{2r}(x_0) | \mathbb{R}^d)$, and $0 < r_1 < r_2 \leq 2r$, we define

$$\begin{aligned} \text{Tail}_K(v, r_1, r_2, x_0) &:= \sup_{x \in B_{r_1}(x_0)} \int_{B_{r_2}(x_0)^c} |v(y)| K(x, y) \, dy, \\ \widehat{\text{Tail}}_K(v, r_1, r_2, x_0) &:= \sup_{x \in B_{r_1}(x_0)} \int_{B_{r_2}(x_0)^c} |v(y)| K(y, x) \, dy. \end{aligned}$$

Remark 2.11. (i) For $0 < \rho_1 \leq r_1$ and $0 < \rho_2 \leq r_2$, we have $\text{Tail}_K(v, \rho_1, r_2) \leq \text{Tail}_K(v, r_1, \rho_2)$.
 (ii) Note that Tail_K has been introduced in [Schulze 2019] for symmetric kernels.

We would like to point out that Tail_K will naturally appear in the proofs of the Caccioppoli estimates in Sections 3 and 4. However, it is not suitable for De Giorgi-type and Moser-type iteration arguments. Therefore, we introduce another nonlocal tail term defined as follows:

$$\begin{aligned} \text{Tail}_{K,\alpha}(u, R, x_0) &:= R^\alpha \int_{B_{2R}(x_0) \setminus B_{R/2}(x_0)} \frac{|u(y)|}{|x_0 - y|^{d+\alpha}} \, dy + \sup_{x \in B_{3R/2}(x_0)} \int_{B_{2R}(x_0)^c} |u(y)| K(x, y) \, dy, \\ \widehat{\text{Tail}}_{K,\alpha}(u, R, x_0) &:= R^\alpha \int_{B_{2R}(x_0) \setminus B_{R/2}(x_0)} \frac{|u(y)|}{|x_0 - y|^{d+\alpha}} \, dy + \sup_{x \in B_{3R/2}(x_0)} \int_{B_{2R}(x_0)^c} |u(y)| K(y, x) \, dy. \end{aligned}$$

$\text{Tail}_{K,\alpha}$ can be regarded as a hybrid between a tail term for general kernels introduced in [Schulze 2019] and a tail term for rotationally symmetric kernels as in [Chen et al. 2020; Di Castro et al. 2016].

The advantage of $\text{Tail}_{K,\alpha}$ is that it fits the iteration schemes, since, for short connections, the weight is a radial function. Moreover, it still takes into account the correct decay of the jumping kernel K for long jumps, which might be of lower-order due to the presence of a nonlocal drift term (see K_3 in (1-5)). Since we do not want to impose any pointwise upper bound on K for long jumps, the second summand contains the supremum in x .

We have the following connection between Tail_K and $\text{Tail}_{K,\alpha}$.

Lemma 2.12. Assume (K_{loc}^{\leq}) . Let $0 < \rho \leq r \leq r + \rho \leq R \leq 1$, $x_0 \in \mathbb{R}^d$, and $v \in V(B_R(x_0) | \mathbb{R}^d)$. Then we have

$$\text{Tail}_K(v, r, r + \rho, x_0) \leq c \rho^{-\alpha} \left(\frac{r + \rho}{\rho} \right)^d \text{Tail}_{K,\alpha}(u, R, x_0), \tag{2-10}$$

$$\widehat{\text{Tail}}_K(v, r, r + \rho, x_0) \leq c \rho^{-\alpha} \left(\frac{r + \rho}{\rho} \right)^d \widehat{\text{Tail}}_{K,\alpha}(u, R, x_0). \tag{2-11}$$

Proof. We use that, for

$$x \in B_r(x_0), \quad y \in B_{r+\rho}(x_0)^c \cap B_{2R}(x_0), \quad \text{and} \quad z \in B_{r+\rho}(x_0)^c \cap B_{2R}(x_0)^c = B_{2R}(x_0)^c,$$

we have

$$|y - x_0| \leq \frac{r + \rho}{\rho} |y - x|, \quad |z - x_0| \leq 2|z - x|,$$

which implies upon (K_{loc}^{\leq}) that, for every $x \in B_r(x_0)$,

$$\begin{aligned} & \int_{B_{r+\rho}(x_0)^c} v(y)K(x, y) \, dy \\ & \leq \int_{B_{r+\rho}(x_0)^c \cap B_{2R}(x_0)} v(y)K(x, y) \, dy + \int_{B_{2R}(x_0)^c} v(z)K(x, z) \, dz \\ & \leq c_1 \left(\frac{r+\rho}{\rho}\right)^{d+\alpha} \int_{B_{r+\rho}(x_0)^c \cap B_{2R}(x_0)} v(y)|x_0 - y|^{-d-\alpha} \, dy + c_1 \int_{B_{2R}(x_0)^c} v(z)K(x, z) \, dz \\ & \leq c_2 \rho^{-\alpha} \left(\frac{r+\rho}{\rho}\right)^d \text{Tail}_{K,\alpha}(v, R), \end{aligned}$$

where $c_1, c_2 > 0$. This proves (2-10), as desired. The proof of (2-11) works in the same way. \square

Moreover, $\text{Tail}_{K,\alpha}(u, R, x_0)$ and $\widehat{\text{Tail}}_{K,\alpha}(u, R, x_0)$ are finite for any $u \in V(B_{2R}(x_0) | \mathbb{R}^d)$ under natural and nonrestrictive assumptions on K . This property is of some importance to us since it allows us to work with the natural function space $V(B_{2R}(x_0) | \mathbb{R}^d)$ associated with K .

Lemma 2.13. *Assume (cutoff) and (\mathcal{E}_{\geq}) .*

- (i) *If (UJS) holds, then $\text{Tail}_{K,\alpha}(u, R, x_0) < \infty$ for every $u \in V(B_{2R}(x_0) | \mathbb{R}^d)$,*
- (ii) *If $(\widehat{\text{UJS}})$ holds, then $\widehat{\text{Tail}}_{K,\alpha}(u, R, x_0) < \infty$ for every $u \in V(B_{2R}(x_0) | \mathbb{R}^d)$.*

Proof. We restrict ourselves to proving (i). The proof of (ii) follows via analogous arguments. By (cutoff), it clearly suffices to prove that

$$\int_{B_{2R}(x_0) \setminus B_{R/2}(x_0)} |u(y)|^2 |x_0 - y|^{-d-\alpha} \, dy + \sup_{x \in B_{3R/2}(x_0)} \int_{B_{2R}(x_0)^c} |u(y)|^2 K(x, y) \, dy < \infty. \tag{2-12}$$

We start by proving finiteness of the first summand. This can be achieved by the same argument as in the proof of Proposition 12 in [Dyda and Kassmann 2019]. Since $|x - y| \leq 3|x_0 - y|$ for every $x \in B_{R/4}(x_0)$ and $y \in \mathbb{R}^d \setminus B_{R/4}(x_0)$, we compute

$$\begin{aligned} & \int_{B_{2R}(x_0) \setminus B_{R/2}(x_0)} |u(y)|^2 |x_0 - y|^{-d-\alpha} \, dy \\ & \leq 3 \int_{B_{2R}(x_0) \setminus B_{R/2}(x_0)} \int_{B_{R/4}(x_0)} |u(y)|^2 |x - y|^{-d-\alpha} \, dx \, dy \\ & \leq c \int_{B_{2R}(x_0)} \int_{B_{R/4}(x_0)} |u(y) - u(x)|^2 |x - y|^{-d-\alpha} \, dx \, dy \\ & \quad + c \int_{B_{R/4}(x_0)} |u(x)|^2 \left(\int_{B_{2R}(x_0) \setminus B_{R/2}(x_0)} |x - y|^{-d-\alpha} \, dy \right) dx \\ & \leq cR^{-d} \mathcal{E}_{B_{2R}(x_0)}^\alpha(u, u) + c \int_{B_{R/4}(x_0)} |u(x)|^2 \left(\int_{B_{R/4}(x_0)^c} |x - y|^{-d-\alpha} \, dy \right) dx \\ & \leq cR^{-d} \mathcal{E}_{B_{2R}(x_0)}^\alpha(u, u) + cR^{-d-\alpha} \|u\|_{L^2(B_{R/4}(x_0))}^2 < \infty. \end{aligned}$$

Finiteness of the quantity on the right follows from (\mathcal{E}_{\geq}) and since $u \in V(B_{2R}(x_0) | \mathbb{R}^d)$.

For the second summand in (2-12), we estimate using (UJS) and (cutoff) that, for every $x \in B_{3R/2}(x_0)$,

$$\begin{aligned} & \int_{B_{2R}(x_0)^c} |u(y)|^2 K(x, y) \, dy \\ & \leq \int_{B_{2R}(x_0)^c} \int_{B_{R/4}(x)} |u(y)|^2 K(z, y) \, dz \, dy \\ & \leq cR^{-d} \int_{B_{2R}(x_0)^c} \int_{B_{2R}(x_0)} |u(y) - u(z)|^2 K_s(z, y) \, dz \, dy + 2 \int_{B_{R/4}(x)} |u(z)|^2 \left(\int_{B_{2R}(x_0)^c} K_s(z, y) \, dy \right) dz \\ & \leq cR^{-d} [u]_{V(B_{2R}(x_0)|\mathbb{R}^d)}^2 + c \int_{B_{R/4}(x)} |u(z)|^2 \left(\int_{B_{R/4}(z)^c} K_s(z, y) \, dy \right) dz \\ & \leq cR^{-d} [u]_{V(B_{2R}(x_0)|\mathbb{R}^d)}^2 + cR^{-d-\alpha} \|u\|_{L^2(B_{2R}(x_0))}^2 < \infty. \end{aligned}$$

Here we used that $B_{R/4}(x) \subset B_{2R}(x_0)$ for every $x \in B_{3R/2}(x_0)$. □

Remark 2.14. Note that (UJS) and $(\widehat{\text{UJS}})$ are not necessary for $\text{Tail}_{K,\alpha}(u, R, x_0)$ and $\widehat{\text{Tail}}_{K,\alpha}(u, R, x_0)$ to be finite, respectively. Consider for example a jumping kernel K whose symmetric part satisfies global versions of (\mathcal{E}_{\geq}) and (K_{loc}^{\leq}) , namely;

$$\mathcal{E}^{K_s}(u, u) \geq c[u]_{H^{\alpha/2}(B_r)}^2 \quad \text{for all } v \in L^2(B_r), \, r > 0, \quad K(x, y) \leq c|x - y|^{-d-\alpha} \quad \text{for all } x, y \in \mathbb{R}^d,$$

then we have that $V(B_{2R}|\mathbb{R}^d) = V^\alpha(B_{2R}|\mathbb{R}^d)$. Therefore,

$$\text{Tail}_{K,\alpha}(u, R, x_0) \leq c \widehat{\text{Tail}}_\alpha(u, R, x_0) = R^\alpha \int_{B_{R/2}(x_0)} |u(y)| |x_0 - y|^{-d-\alpha} \, dy < \infty \quad \text{for all } u \in V(B_{2R}|\mathbb{R}^d).$$

Remark 2.15. (i) Later, we will require finiteness of $\text{Tail}_{K,\alpha}(u, R, x_0)$ and $\widehat{\text{Tail}}_{K,\alpha}(u, R, x_0)$ in order to deduce local boundedness of weak solutions to (ell-PDE) and (ell-PDE) from Theorem 3.6 and Theorem 4.8, respectively. The above lemma shows that under the natural assumptions (cutoff), (\mathcal{E}_{\geq}) , and (UJS) or $(\widehat{\text{UJS}})$, finiteness of the tail terms for weak solutions follows already from the solution concept.

(ii) For parabolic equations, the aforementioned assumptions merely imply finiteness of

$$\text{Tail}_{K,\alpha}(u(t), R, x_0) \quad \text{and} \quad \widehat{\text{Tail}}_{K,\alpha}(u(t), R, x_0)$$

for a.e. t , but do not yield a uniform upper bound in t .

(iii) Since parabolic tails of the form $\sup_{t \in I} \text{Tail}_K(u(t), r, r + \rho, x_0)$ and $\sup_{t \in I} \widehat{\text{Tail}}_K(u(t), r, r + \rho, x_0)$ naturally appear in the analysis of solutions to (PDE) and $(\widehat{\text{PDE}})$, respectively, it is an important research question to investigate these quantities and to derive suitable estimates. First results have been obtained in [Strömquist 2019b], where an estimate for $\sup_{t \in I} \widehat{\text{Tail}}_K(u(t), r, r + \rho, x_0)$ is derived for global solutions u to (PDE) in the symmetric case under pointwise bounds for K . Another attempt has been made in [Kim 2019] for solutions to a parabolic boundary value problem with given continuous, bounded data. However, the proof of [Kim 2019, Lemma 5.3] is not complete.

3. Local boundedness via De Giorgi iteration

The goal of this section is to prove that the supremum of a weak subsolution u to (PDE), or to $(\widehat{\text{PDE}})$, can locally be estimated from above by the L^2 -norm of u and a nonlocal tail term (see Theorem 3.6). Under the assumption that the tail term is finite, this result is the key to proving the Harnack inequality. The strategy of proof is based on the De Giorgi iteration for nonlocal operators, as adopted in [Cozzi 2017; Di Castro et al. 2014; 2016].

3.1. Caccioppoli estimates. In this section nonlocal Caccioppoli estimates are established. They are derived by testing the weak formulation of (PDE), or of $(\widehat{\text{PDE}})$, with a test function of the form $\tau^2(u - k)_+$. The lack of symmetry of the jumping kernel K calls for a refinement of the existing proofs for symmetric operators. The main technical ingredient is Lemma 2.10. Such estimates will be used in Section 3.2 to set up a De Giorgi-type iteration scheme which allows us to prove Theorem 3.6.

The following lemma can be regarded as a generalization of Proposition 8.5 in [Cozzi 2017] to nonsymmetric jumping kernels.

Lemma 3.1. *Assume that $(K1_{\text{loc}})$ and (cutoff) hold for some $\theta \in [d/\alpha, \infty]$. Moreover, assume (Sob) if $\theta < \infty$. Then there exist $c_1, c_2 > 1$ such that, for every $0 < \rho \leq r \leq 1$, every $l \in \mathbb{R}$, and every function $u \in V(B_{r+\rho} | \mathbb{R}^d)$, we have*

$$\begin{aligned} & \mathcal{E}_{B_{r+\rho}}^{K_s}(\tau w_+, \tau w_+) - \mathcal{E}_{B_{r+\rho}}(w_-, \tau w_+) \\ & \leq c_1 \mathcal{E}(u, \tau^2 w_+) + c_2 \rho^{-\alpha} \|w_+^2\|_{L^1(B_{r+\rho})} + c_2 \|w_+\|_{L^1(B_{r+\rho})} \text{Tail}_K(w_+, r + \frac{1}{2}\rho, r + \rho), \end{aligned} \quad (3-1)$$

where $B_{2r} \subset \Omega$, $w = u - l$, and $\tau = \tau_{r, \rho/2}$.

Proof. Step 1: We claim that there exists a constant $c > 0$ such that

$$\mathcal{E}_{B_{r+\rho}}^{K_s}(\tau w_+, \tau w_+) - \mathcal{E}_{B_{r+\rho}}^{K_s}(w_-, \tau w_+) \leq \mathcal{E}_{B_{r+\rho}}^{K_s}(u, \tau^2 w_+) + c \rho^{-\alpha} \|w_+^2\|_{L^1(B_{r+\rho})}. \quad (3-2)$$

Observe that by the algebraic identities

$$\begin{aligned} a - b &= ((a - l)_+ - (b - l)_+) - ((a - l)_- - (b - l)_-), \\ (w_1 - w_2)(\tau_1^2 w_1 - \tau_2^2 w_2) &= (\tau_1 w_1 - \tau_2 w_2)^2 - w_1 w_2 (\tau_1 - \tau_2)^2, \end{aligned}$$

we have that

$$\begin{aligned} & \mathcal{E}_{B_{r+\rho}}^{K_s}(\tau w_+, \tau w_+) - \mathcal{E}_{B_{r+\rho}}^{K_s}(w_-, \tau w_+) \\ & = \mathcal{E}_{B_{r+\rho}}^{K_s}(u, \tau^2 w_+) + \int_{B_{r+\rho}} \int_{B_{r+\rho}} w_+(x) w_+(y) (\tau(x) - \tau(y))^2 K_s(x, y) \, dy \, dx. \end{aligned}$$

Thus, (3-2) follows immediately from (cutoff).

Step 2: For every $\delta > 0$, there exists $c > 0$ such that

$$\mathcal{E}_{B_{r+\rho}}^{K_a}(u, \tau^2 w_+) \geq -\mathcal{E}_{B_{r+\rho}}^{K_a}(w_-, \tau^2 w_+) - \delta \mathcal{E}_{B_{r+\rho}}^{K_s}(\tau w_+, \tau w_+) - c \rho^{-\alpha} \|w_+^2\|_{L^1(B_{r+\rho})}. \quad (3-3)$$

For the proof, we first observe the algebraic identity

$$(w_1 - w_2)(\tau_1^2 w_1 + \tau_2^2 w_2) = (\tau_1^2 w_1^2 - \tau_2^2 w_2^2) + w_1 w_2 (\tau_2^2 - \tau_1^2).$$

Thus, we obtain

$$\begin{aligned} \mathcal{E}_{B_{r+\rho}}^{K_a}(u, \tau^2 w_+) &= -\mathcal{E}_{B_{r+\rho}}^{K_a}(w_-, \tau^2 w_+) + \int_{B_{r+\rho}} \int_{B_{r+\rho}} (\tau^2 w_+^2(x) - \tau^2 w_+^2(y)) K_a(x, y) dy dx \\ &\quad + \int_{B_{r+\rho}} \int_{B_{r+\rho}} w_+(x) w_+(y) (\tau^2(y) - \tau^2(x)) K_a(x, y) dy dx =: I_1 + I_2 + I_3. \end{aligned}$$

For I_2 , we estimate, using (K1_{loc}) and (2-8),

$$\begin{aligned} I_2 &\geq -\frac{1}{2} \delta \mathcal{E}_{B_{r+\rho}}^{K_s}(\tau w_+, \tau w_+) - c \int_{B_{r+\rho}} \tau^2(x) w_+^2(x) \left(\int_{B_{r+\rho}} \frac{|K_a(x, y)|^2}{J(x, y)} dy \right) dx \\ &\geq -\delta \mathcal{E}_{B_{r+\rho}}^{K_s}(\tau w_+, \tau w_+) - c \rho^{-\alpha} \|w_+^2\|_{L^1(B_{r+\rho})}. \end{aligned}$$

For I_3 , using the standard estimate

$$(\tau^2(x) - \tau^2(y)) \leq 2(\tau(x) - \tau(y))^2 + 2(\tau(x) - \tau(y))(\tau(x) \wedge \tau(y)), \quad (3-4)$$

estimate (1-2), (cutoff), and (K1_{loc}), we get

$$\begin{aligned} I_3 &\geq -2 \int_{B_{r+\rho}} \int_{B_{r+\rho}} (w_+^2(x) \vee w_+^2(y)) (\tau(x) - \tau(y))^2 K_s(x, y) dy dx \\ &\quad - 2 \int_{B_{r+\rho}} \int_{B_{r+\rho}} (w_+^2(x) \vee w_+^2(y)) (\tau(x) \wedge \tau(y)) |\tau(x) - \tau(y)| |K_a(x, y)| dy dx \\ &\geq -c \rho^{-\alpha} \|w_+^2\|_{L^1(B_{r+\rho})} - \int_{B_{r+\rho}} \tau^2(x) w_+^2(x) \left(\int_{B_{r+\rho}} \frac{|K_a(x, y)|^2}{J(x, y)} dy \right) dx \\ &\geq -c \rho^{-\alpha} \|w_+^2\|_{L^1(B_{r+\rho})} - \delta \mathcal{E}_{B_{r+\rho}}^{K_s}(\tau w_+, \tau w_+). \end{aligned}$$

This proves (3-3).

Step 3: Next, let us show how to prove

$$-\mathcal{E}_{(B_{r+\rho} \times B_{r+\rho})^c}(u, \tau^2 w_+) \leq 2 \left(\int_{B_{r+\rho}} w_+(x) dx \right) \text{Tail}_K(w_+, r + \frac{1}{2}\rho, r + \rho). \quad (3-5)$$

We estimate

$$\begin{aligned} -\mathcal{E}_{(B_{r+\rho} \times B_{r+\rho})^c}(u, \tau^2 w_+) &= 2 \int_{B_{r+\rho/2}} \int_{B_{r+\rho}^c} (u(y) - u(x)) \tau^2 w_+(x) K(x, y) dy dx \\ &\leq 2 \int_{B_{r+\rho/2}} \int_{B_{r+\rho}^c} (u(y) - u(x))_+ \tau^2 w_+(x) K(x, y) dy dx \\ &\leq 2 \int_{B_{r+\rho/2}} \int_{B_{r+\rho}^c} (u(y) - l)_+ \tau^2 w_+(x) K(x, y) dy dx \\ &\leq 2 \int_{B_{r+\rho/2}} w_+(x) \sup_{z \in B_{r+\rho/2}} \left(\int_{B_{r+\rho}^c} w_+(y) K(z, y) dy \right) dx, \end{aligned}$$

where we used that K is nonnegative and $\tau \equiv 0$ in $B_{r+\rho/2}^c$.

Step 4: We will now combine (3-2), (3-3), and (3-5). Observe

$$\mathcal{E}_{B_{r+\rho}}^{K_s}(u, \tau^2 w_+) = \mathcal{E}(u, \tau^2 w_+) - \mathcal{E}_{B_{r+\rho}}^{K_a}(u, \tau^2 w_+) - \mathcal{E}_{(B_{r+\rho} \times B_{r+\rho})^c}(u, \tau^2 w_+).$$

Altogether, we immediately obtain the desired result by choosing $\delta > 0$ from Step 2 small enough. \square

Note that $-\mathcal{E}_{B_{r+\rho}}(w_-, \tau^2 w_+) \geq 0$ since $K \geq 0$. Thus, we have the following corollary of Lemma 3.1.

Corollary 3.2. *Assume that (K1_{loc}) and (cutoff) hold for some $\theta \in [d/\alpha, \infty]$. Moreover, assume (Sob) if $\theta < \infty$. Then there exist $c_1, c_2 > 0$ such that, for every $0 < \rho \leq r \leq 1$, every $l \in \mathbb{R}$, and every function $u \in V(B_{r+\rho} | \mathbb{R}^d)$, we have*

$$\begin{aligned} \mathcal{E}_{B_{r+\rho}}^{K_s}(\tau w_+, \tau w_+) \\ \leq c_1 \mathcal{E}(u, \tau^2 w_+) + c_2 \rho^{-\alpha} \|w_+^2\|_{L^1(B_{r+\rho})} + c_2 \|w_+\|_{L^1(B_{r+\rho})} \text{Tail}_K(w_+, r + \frac{1}{2}\rho, r + \rho), \end{aligned} \quad (3-6)$$

where $B_{2r} \subset \Omega$, $w = u - l$, and $\tau = \tau_{r,\rho/2}$.

Remark 3.3. Let us point out that both Caccioppoli-type inequalities (3-1) and (3-6) appear in the literature for symmetric jumping kernels. Inequality (3-1) was introduced in [Cozzi 2017] (see also [Caffarelli et al. 2011; Cozzi 2019]) and is used to prove Hölder estimates for small α . For our purposes, inequality (3-6) is sufficient.

Next, we present a Caccioppoli inequality that is tailored to subsolutions to (PDE). Due to the different shape of the bilinear form, we obtain an additional summand on the right-hand side of the estimate.

Lemma 3.4. *Assume that (K1_{loc}) and (cutoff) hold for some $\theta \in [d/\alpha, \infty]$. Moreover, assume (Sob) if $\theta < \infty$. Then there exist $c_1, c_2 > 1$ such that, for every $0 < \rho \leq r \leq 1$, every $l \in \mathbb{R}$, and every function $u \in V(B_{r+\rho} | \mathbb{R}^d)$, we have*

$$\begin{aligned} \mathcal{E}_{B_{r+\rho}}^{K_s}(\tau w_+, \tau w_+) - \mathcal{E}_{B_{r+\rho}}(w_-, \tau w_+) \\ \leq c_1 \widehat{\mathcal{E}}(u, \tau^2 w_+) + c \rho^{-\alpha} \|w_+^2\|_{L^1(B_{r+\rho})} + c_2 l^2 \rho^{-\alpha} \left[|A(l, r + \rho)| + |B_{r+\rho}| \left(\frac{|A(l, r + \rho)|}{|B_{r+\rho}|} \right)^{1/\theta'} \right] \\ + c_2 \|w_+\|_{L^1(B_{r+\rho})} \widehat{\text{Tail}}_K(u, r + \frac{1}{2}\rho, r + \rho), \end{aligned} \quad (3-7)$$

where $B_{2r} \subset \Omega$, $w = u - l$, $\tau = \tau_{r,\rho/2}$, and $A(l, r + \rho) = \{x \in B_{r+\rho} : w_+ > 0\}$.

Proof. The proof follows the structure of the proof of Lemma 3.1.

Step 1: As before, there exists a constant $c > 0$ such that

$$\mathcal{E}_{B_{r+\rho}}^{K_s}(\tau w_+, \tau w_+) - \mathcal{E}_{B_{r+\rho}}^{K_s}(w_-, \tau w_+) \leq \mathcal{E}_{B_{r+\rho}}^{K_s}(u, \tau^2 w_+) + c \rho^{-\alpha} \|w_+^2\|_{L^1(B_{r+\rho})}. \quad (3-8)$$

Step 2: We claim that, for every $\delta > 0$, there exists $c > 0$ such that

$$\begin{aligned} \widehat{\mathcal{E}}_{B_{r+\rho}}^{K_a}(u, \tau^2 w_+) \geq -\widehat{\mathcal{E}}_{B_{r+\rho}}^{K_a}(w_-, \tau^2 w_+) - \delta \mathcal{E}_{B_{r+\rho}}^{K_s}(\tau w_+, \tau w_+) - c \rho^{-\alpha} \|w_+^2\|_{L^1(B_{r+\rho})} \\ - c l^2 \rho^{-\alpha} \left[|A(l, r + \rho)| + |B_{r+\rho}| \left(\frac{|A(l, r + \rho)|}{|B_{r+\rho}|} \right)^{1/\theta'} \right]. \end{aligned} \quad (3-9)$$

This is the main part of the proof, and it differs from Step 2 in Lemma 3.1. First, we observe

$$\widehat{\mathcal{E}}_{B_{r+\rho}}^{K_a}(u, \tau^2 w_+) = \mathcal{E}_{B_{r+\rho}}^{K_a}(\tau^2 w_+, u) = -\mathcal{E}_{B_{r+\rho}}^{K_a}(\tau^2 w_+, w_-) + \mathcal{E}_{B_{r+\rho}}^{K_a}(\tau^2 w_+, w_+) + \mathcal{E}_{B_{r+\rho}}^{K_a}(\tau^2 w_+, l).$$

To estimate the second term, observe

$$(\tau_1^2 w_1 - \tau_2^2 w_2)(w_1 + w_2) = (\tau_1^2 w_1^2 - \tau_2^2 w_2^2) + w_1 w_2 (\tau_1^2 - \tau_2^2).$$

Thus, we note that, for every $\delta > 0$, there exists $c > 0$ such that

$$\begin{aligned} \mathcal{E}_{B_{r+\rho}}^{K_a}(\tau^2 w_+, w_+) &= \int_{B_{r+\rho}} \int_{B_{r+\rho}} (\tau^2 w_+^2(x) - \tau^2 w_+^2(y)) K_a(x, y) \, dy \, dx \\ &\quad + \int_{B_{r+\rho}} \int_{B_{r+\rho}} w_+(x) w_+(y) (\tau^2(x) - \tau^2(y)) K_a(x, y) \, dy \, dx \\ &\geq -\delta \mathcal{E}_{B_{r+\rho}}^{K_s}(\tau w_+, \tau w_+) - c \rho^{-\alpha} \|w_+\|_{L^1(B_{r+\rho})}^2. \end{aligned}$$

The estimate in the last step works exactly as in the estimation of I_2 and I_3 in the proof of Lemma 3.1.

The estimate of the remaining term $\mathcal{E}_{B_{r+\rho}}^{K_a}(\tau^2 w_+, l)$ goes as follows:

$$\begin{aligned} \mathcal{E}_{B_{r+\rho}}^{K_a}(\tau^2 w_+, l) &= 2l \int_{B_{r+\rho}} \int_{B_{r+\rho}} (\tau^2 w_+(x) - \tau^2 w_+(y)) K_a(x, y) \, dy \, dx \\ &= 2l \int_{B_{r+\rho}} \int_{B_{r+\rho}} (\tau(x) - \tau(y)) (\tau w_+(x) + \tau w_+(y)) K_a(x, y) \, dy \, dx \\ &\quad + 2l \int_{B_{r+\rho}} \int_{B_{r+\rho}} (\tau(x) + \tau(y)) (\tau w_+(x) - \tau w_+(y)) K_a(x, y) \, dy \, dx \\ &=: J_1 + J_2. \end{aligned}$$

To estimate J_1 , we apply (cutoff) and (2-8):

$$\begin{aligned} J_1 &\geq -4l \int_{A(l, r+\rho)} \int_{B_{r+\rho}} |\tau(x) - \tau(y)| \tau w_+(x) |K_a(x, y)| \, dy \, dx \\ &\geq -cl^2 \int_{A(l, r+\rho)} \Gamma^J(\tau, \tau)(x) \, dx - c \int_{A(l, r+\rho)} \int_{B_{r+\rho}} \tau^2 w_+^2(x) \frac{|K_a(x, y)|^2}{J(x, y)} \, dy \, dx \\ &\geq -c \rho^{-\alpha} l^2 |A(l, r+\rho)| - \delta \mathcal{E}_{B_{r+\rho}}^{K_s}(\tau w_+, \tau w_+) - c \rho^{-\alpha} \|w_+\|_{L^1(B_{r+\rho})}^2. \end{aligned}$$

J_2 can also be estimated with the help of (cutoff) and (K1_{loc}):

$$\begin{aligned} J_2 &\geq -4l \int_{A(l, r+\rho)} \int_{B_{r+\rho}} (\tau(x) + \tau(y)) |\tau w_+(x) - \tau w_+(y)| |K_a(x, y)| \, dy \, dx \\ &\geq -8l \int_{A(l, r+\rho)} \int_{B_{r+\rho}} |\tau(x) - \tau(y)| |\tau w_+(x) - \tau w_+(y)| |K_s(x, y)| \, dy \, dx \\ &\quad - 8l \int_{A(l, r+\rho)} \int_{B_{r+\rho}} (\tau(x) \wedge \tau(y)) |\tau w_+(x) - \tau w_+(y)| |K_a(x, y)| \, dy \, dx \\ &\geq -cl^2 \int_{A(l, r+\rho)} \Gamma^{K_s}(\tau, \tau)(x) \, dx - \delta \mathcal{E}_{B_{r+\rho}}^{K_s}(\tau w_+, \tau w_+) \\ &\quad - \delta \mathcal{E}_{B_{r+\rho}}^J(\tau w_+, \tau w_+) - cl^2 \int_{A(l, r+\rho)} \int_{B_{r+\rho}} (\tau^2(x) \wedge \tau^2(y)) \frac{|K_a(x, y)|^2}{J(x, y)} \, dy \, dx \\ &\geq -c \delta \mathcal{E}_{B_{r+\rho}}^{K_s}(\tau w_+, \tau w_+) - cl^2 \rho^{-\alpha} |A(l, r+\rho)| - cl^2 \rho^{-\alpha} |B_{r+\rho}| \left(\frac{|A(l, r+\rho)|}{|B_{r+\rho}|} \right)^{1/\theta'}. \end{aligned}$$

Here, we used that, by $(K1_{\text{loc}})$ and Hölder’s inequality,

$$\begin{aligned} l^2 \int_{A(l,r+\rho)} \int_{B_{r+\rho}} (\tau^2(x) \wedge \tau^2(y)) \frac{|K_a(x,y)|^2}{J(x,y)} dy dx &\leq l^2 \int_{A(l,r+\rho)} \tau^2(x) \left(\int_{B_{r+\rho}} \frac{|K_a(x,y)|^2}{J(x,y)} dy \right) dx \\ &\leq cl^2 \|\tau^2\|_{L^{\theta'}(A(l,r+\rho))} \\ &\leq cl^2 \rho^{-\alpha} |B_{r+\rho}| \left(\frac{|A(l,r+\rho)|}{|B_{r+\rho}|} \right)^{1/\theta'}, \end{aligned}$$

since

$$1 \leq c|B_{r+\rho}|^{-\alpha/d+1-1/\theta'} \leq c\rho^{-\alpha} |B_{r+\rho}|^{1-1/\theta'}$$

for some constant $c > 0$ because $\theta \geq d/\alpha$, which implies that

$$-\frac{\alpha}{d} + 1 - \frac{1}{\theta'} \in \left[-\frac{\alpha}{d}, 0 \right) \quad \text{and} \quad \rho \leq r \leq 1.$$

Step 3: Next, let us demonstrate how to prove

$$-\widehat{\mathcal{E}}_{(B_{r+\rho} \times B_{r+\rho})^c}(u, \tau^2 w_+) \leq 2 \left(\int_{B_{r+\rho}} w_+(x) dx \right) \widehat{\text{Tail}}_K(u, r + \frac{1}{2}\rho, r + \rho). \tag{3-10}$$

We estimate

$$\begin{aligned} &-\widehat{\mathcal{E}}_{(B_{r+\rho} \times B_{r+\rho})^c}(u, \tau^2 w_+) \\ &= 2 \int_{B_{r+\rho}^c} \int_{B_{r+\rho/2}} \tau^2 w_+(y) u(x) K(x, y) dy dx - 2 \int_{B_{r+\rho/2}} \int_{B_{r+\rho}^c} \tau^2 w_+(x) u(x) K(x, y) dy dx \\ &\leq 2 \int_{B_{r+\rho/2}} w_+(y) \left(\int_{\mathbb{R}^d \setminus B_{r+\rho}} u(x) K(x, y) dx \right) dy, \end{aligned}$$

where we used that K is nonnegative and $\tau \equiv 0$ in $B_{r+\rho/2}^c$. Note that the second summand in the first step is negative since $w_+(x)u(x) \geq 0$, and can therefore be neglected.

Step 4: We will now combine (3-8), (3-9), and (3-10). Observe that

$$\mathcal{E}_{B_{r+\rho}}^{K_s}(u, \tau^2 w_+) = \widehat{\mathcal{E}}(u, \tau^2 w_+) - \widehat{\mathcal{E}}_{B_{r+\rho}}^{K_a}(u, \tau^2 w_+) - \widehat{\mathcal{E}}_{(B_{r+\rho} \times B_{r+\rho})^c}(u, \tau^2 w_+).$$

Altogether, we immediately obtain the desired result by choosing $\delta > 0$ from Step 2 small enough. \square

Corollary 3.5. *Assume that $(K1_{\text{glob}})$ and (cutoff) hold for some $\theta \in [d/\alpha, \infty]$. Moreover, assume (Sob) if $\theta < \infty$. Then there exist $c_1, c_2 > 0$ such that, for every $0 < \rho \leq r \leq 1$, every $l \in \mathbb{R}$, and every function $u \in V(B_{r+\rho}|\mathbb{R}^d)$, we have*

$$\begin{aligned} &\mathcal{E}_{B_{r+\rho}}^{K_s}(\tau w_+, \tau w_+) \\ &\leq c_1 \widehat{\mathcal{E}}(u, \tau^2 w_+) + c_2 \rho^{-\alpha} \|w_+\|_{L^1(B_{r+\rho})}^2 + c_2 l^2 \rho^{-\alpha} \left[|A(l, r + \rho)| + |B_{r+\rho}| \left(\frac{|A(l, r + \rho)|}{|B_{r+\rho}|} \right)^{1/\theta'} \right] \\ &\quad + c_2 \|w_+\|_{L^1(B_{r+\rho})} \widehat{\text{Tail}}_K(w_+, r + \frac{1}{2}\rho, r + \rho), \end{aligned} \tag{3-11}$$

where $B_{2r} \subset \Omega$, $w = u - l$, and $\tau = \tau_{r,\rho/2}$.

3.2. Local boundedness. The following theorem is the main result of this section. It yields a priori local boundedness of subsolutions to (PDE), or to $(\widehat{\text{PDE}})$, if the nonlocal tail is finite.

Theorem 3.6. *Assume that (K_{loc}^{\leq}) , (cutoff), and (Sob) hold.*

- (i) *Assume that $(K1_{\text{loc}})$ holds for some $\theta \in [d/\alpha, \infty]$. Then there exists $c > 0$ such that, for every $0 < R \leq 1$, every $\delta \in (0, 1]$, and every nonnegative, weak subsolution u to (PDE) in $I_R^\ominus(t_0) \times B_{2R}$,*

$$\sup_{I_{R/8}^\ominus \times B_{R/2}} u \leq c \delta^{-(d+\alpha)/(2\alpha)} \left(\int_{I_{R/4}^\ominus} \int_{B_R} u^2(t, x) \, dx \, dt \right)^{1/2} + \delta \sup_{t \in I_{R/4}^\ominus} \text{Tail}_{K, \alpha}(u(t), R) + \delta R^\alpha \|f\|_{L^\infty},$$

where $B_{2R} \subset \Omega$.

- (ii) *Assume that $(K1_{\text{glob}})$ holds for some $\theta \in (d/\alpha, \infty]$. Then there exists $c > 0$ such that, for every $0 < R \leq 1$, every $\delta \in (0, 1]$, and every nonnegative, weak subsolution u to $(\widehat{\text{PDE}})$ in $I_R^\ominus(t_0) \times B_{2R}$,*

$$\sup_{I_{R/8}^\ominus \times B_{R/2}} u \leq c \delta^{-\tilde{\kappa}'/2} \left(\int_{I_{R/4}^\ominus} \left(\int_{B_R} u^{2\theta'}(t, x) \, dx \right)^{1/\theta'} dt \right)^{1/2} + \delta \sup_{t \in I_{R/4}^\ominus} \widehat{\text{Tail}}_{K, \alpha}(u(t), R) + \delta R^\alpha \|f\|_{L^\infty},$$

where $B_{2R} \subset \Omega$ and $\tilde{\kappa} = 1 + \alpha/d - 1/\theta > 1$.

Proof. We first explain how to prove (i). Let $l > 0$, and define $w_l := (u - l)_+$. Let $r, \rho > 0$ such that $\frac{1}{2}R \leq r \leq R$ and $\rho \leq r \leq r + \rho \leq R$. Let $\tau = \tau_{r, \rho/2}$. Moreover, we define $\chi \in C^1(\mathbb{R})$ to be a function satisfying

$$0 \leq \chi \leq 1, \quad \|\chi'\|_\infty \leq 16((r + \rho)^\alpha - r^\alpha)^{-1}, \quad \chi(t_0 - ((r + \rho)/4)^\alpha) = 0, \quad \chi \equiv 1 \quad \text{in } I_{r/4}^\ominus(t_0).$$

Since u is a weak subsolution to (PDE), Lemma A.1 yields, for any $t \in I_{r/4}^\ominus(t_0)$,

$$\begin{aligned} & \int_{B_{r+\rho}} \chi^2(t) \tau^2(x) w_l^2(t, x) \, dx + \int_{t_0 - ((r+\rho)/4)^\alpha}^t \chi^2(s) \mathcal{E}(u(s), \tau^2 w_l(s)) \, ds \\ & \leq \int_{t_0 - ((r+\rho)/4)^\alpha}^t \chi^2(s) (f(s), \tau^2 w_l(s)) \, ds + 2 \int_{t_0 - ((r+\rho)/4)^\alpha}^t \chi(s) |\chi'(s)| \int_{B_{r+\rho}} \tau^2(x) w_l^2(s, x) \, dx \, ds \\ & \leq \|f\|_{L^\infty} \int_{I_{(r+\rho)/4}^\ominus} \|w_l(s)\|_{L^1(B_{r+\rho})} \, ds + c_1 ((r + \rho)^\alpha - r^\alpha)^{-1} \int_{I_{(r+\rho)/4}^\ominus} \|w_l^2(s)\|_{L^1(B_{r+\rho})} \, ds \end{aligned}$$

for some constant $c_1 > 0$. Applying Corollary 3.2, we obtain

$$\begin{aligned} & \sup_{t \in I_{r/4}^\ominus} \int_{B_r} w_l^2(t, x) \, dx + \int_{I_{r/4}^\ominus} \mathcal{E}_{B_{r+\rho}}^{K_s}(\tau w_l(s), \tau w_l(s)) \, ds \\ & \leq c_2 (\rho^{-\alpha} \vee ((r + \rho)^\alpha - r^\alpha)^{-1}) \int_{I_{(r+\rho)/4}^\ominus} \|w_l^2(s)\|_{L^1(B_{r+\rho})} \, ds \\ & \quad + c_2 \|w_l\|_{L^1(I_{(r+\rho)/4}^\ominus \times B_{r+\rho})} \left(\sup_{t \in I_{(r+\rho)/4}^\ominus} \text{Tail}_K(u(t), r + \frac{1}{2}\rho, r + \rho) + \|f\|_{L^\infty} \right) \quad (3-12) \end{aligned}$$

for some $c_2 > 0$. Recall $\kappa = 1 + \alpha/d > 1$. Hölder interpolation and the Sobolev inequality (Sob) yield

$$\begin{aligned} \|w_l^2\|_{L^\kappa(I_{r/4}^\ominus \times B_r)} &\leq \left(\sup_{t \in I_{r/4}^\ominus} \|w_l^2(t)\|_{L^1(B_r)}^{\kappa-1} \int_{I_{r/4}^\ominus} \|w_l^2(s)\|_{L^{d/(d-\alpha)}(B_r)} ds \right)^{1/\kappa} \\ &\leq c_3 \sigma(r, \rho) \|w_l^2\|_{L^1(I_{(r+\rho)/4}^\ominus \times B_{r+\rho})} \\ &\quad + c_3 \|w_l\|_{L^1(I_{(r+\rho)/4}^\ominus \times B_{r+\rho})} \left(\sup_{t \in I_{(r+\rho)/4}^\ominus} \text{Tail}_K(u(t), r + \frac{1}{2}\rho, r + \rho) + \|f\|_{L^\infty} \right), \end{aligned} \quad (3-13)$$

where $c_3 > 0$ and we used that there is $c > 0$ such that

$$(\rho^{-\alpha} \vee ((r + \rho)^\alpha - r^\alpha)^{-1}) \leq c \rho^{-(\alpha \vee 1)} (r + \rho)^{(\alpha \vee 1) - \alpha} =: \sigma(r, \rho).$$

Furthermore, set

$$|A(l, r)| := \int_{I_{r/4}^\ominus} |\{x \in B_r : u(s, x) > l\}| ds.$$

Then, by application of Hölder's inequality, with κ and $\kappa/(\kappa - 1)$ both in time and in space, and (3-13),

$$\begin{aligned} \|w_l^2\|_{L^1(I_{r/4}^\ominus \times B_r)} &\leq |A(l, r)|^{1/\kappa'} \|w_l^2\|_{L^\kappa(I_{r/4}^\ominus \times B_r)} \\ &\leq c_4 |A(l, r)|^{1/\kappa'} \left[\sigma(r, \rho) \|w_l^2\|_{L^1(I_{(r+\rho)/4}^\ominus \times B_{r+\rho})} \right. \\ &\quad \left. + \|w_l\|_{L^1(I_{(r+\rho)/4}^\ominus \times B_{r+\rho})} \left(\sup_{t \in I_{(r+\rho)/4}^\ominus} \text{Tail}_K(u(t), r + \frac{1}{2}\rho, r + \rho) + \|f\|_{L^\infty} \right) \right], \end{aligned} \quad (3-14)$$

where $c_4 > 0$ is a constant. Let now $0 < k < l$ be arbitrary. Then the following hold:

$$\begin{aligned} \|w_l^2\|_{L^1(I_{(r+\rho)/4}^\ominus \times B_{r+\rho})} &\leq \|w_k^2\|_{L^1(I_{(r+\rho)/4}^\ominus \times B_{r+\rho})}, \\ \|w_l\|_{L^1(I_{(r+\rho)/4}^\ominus \times B_{r+\rho})} &\leq \frac{\|w_k^2\|_{L^1(I_{(r+\rho)/4}^\ominus \times B_{r+\rho})}}{l - k}, \\ |A(l, r)| &\leq \frac{\|w_k^2\|_{L^1(I_{(r+\rho)/4}^\ominus \times B_{r+\rho})}}{(l - k)^2}. \end{aligned} \quad (3-15)$$

By combining (3-14) and (3-15), we obtain

$$\begin{aligned} \|w_l^2\|_{L^1(I_{r/4}^\ominus \times B_r)} &\leq c_5 |A(l, r)|^{1/\kappa'} \left(\sigma(r, \rho) + \frac{\sup_{t \in I_{(r+\rho)/4}^\ominus} \text{Tail}_K(u(t), r + \frac{1}{2}\rho, r + \rho) + \|f\|_{L^\infty}}{l - k} \right) \|w_k^2\|_{L^1(I_{(r+\rho)/4}^\ominus \times B_{r+\rho})} \\ &\leq c_6 (l - k)^{-2/\kappa'} \left(\sigma(r, \rho) + \frac{\sup_{t \in I_{(r+\rho)/4}^\ominus} \text{Tail}_K(u(t), r + \frac{1}{2}\rho, r + \rho) + \|f\|_{L^\infty}}{l - k} \right) \|w_k^2\|_{L^1(I_{(r+\rho)/4}^\ominus \times B_{r+\rho})}^{1+1/\kappa'} \end{aligned}$$

for some $c_5, c_6 > 0$. The plan for the remainder of the proof is to iterate the above estimate. Recall (2-10), which we will apply in the sequel. Let us now set up the iteration scheme. For this purpose, we define two

sequences $l_i = M(1 - 2^{-i})$ and $\rho_i = 2^{-i-1}R$, $i \in \mathbb{N}$, where $M > 0$ is to be determined later. We also set

$$r_0 = R, \quad r_{i+1} = r_i - \rho_{i+1} = \frac{1}{2}R\left(1 + \left(\frac{1}{2}\right)^{i+1}\right), \quad \text{and} \quad l_0 = 0.$$

Then $r_i \searrow \frac{1}{2}R$ and $l_i \nearrow M$ as $i \rightarrow \infty$.

Note that $\sigma(r_i, \rho_i) \leq c_7 R^{-\alpha} 2^{2i}$ for some $c_7 > 0$. Define $A_i = \|w_{l_i}^2\|_{L^1(I_{r_i/4}^\ominus \times B_{r_i})}$. Then

$$\begin{aligned} A_i &\leq c_8 \frac{1}{(l_i - l_{i-1})^{2/\kappa'}} \left(\sigma(r_i, \rho_i) + \frac{\sup_{t \in I_{r_i/4}^\ominus} \text{Tail}_K(u(t), r_i + \frac{1}{2}\rho_i, r_i + \rho_i) + \|f\|_{L^\infty}}{l_i - l_{i-1}} \right) A_{i-1}^{1+1/\kappa'} \\ &\leq c_9 \frac{1}{(l_i - l_{i-1})^{2/\kappa'}} \left(\sigma(r_i, \rho_i) + \rho_i^{-\alpha} \left(\frac{r_i}{\rho_i}\right)^d \frac{\sup_{t \in I_{R/4}^\ominus} \text{Tail}_{K,\alpha}(u(t), R) + R^\alpha \|f\|_{L^\infty}}{l_i - l_{i-1}} \right) A_{i-1}^{1+1/\kappa'} \\ &\leq c_{10} \frac{2^{2i/\kappa'}}{M^{2/\kappa'}} \left(\frac{2^{2i}}{R^\alpha} + \frac{2^{(1+\alpha+d)i} \sup_{t \in I_{R/4}^\ominus} \text{Tail}_{K,\alpha}(u(t), R) + R^\alpha \|f\|_{L^\infty}}{M} \right) A_{i-1}^{1+1/\kappa'} \\ &\leq \frac{c_{11}}{R^\alpha M^{2/\kappa'}} 2^{\gamma i} \left(1 + \frac{\sup_{t \in I_{R/4}^\ominus} \text{Tail}_{K,\alpha}(u(t), R) + R^\alpha \|f\|_{L^\infty}}{M} \right) A_{i-1}^{1+1/\kappa'} \end{aligned} \tag{3-16}$$

for $c_8, c_9, c_{10}, c_{11} > 0$, $\gamma > 1$. Note that here we also applied (2-10). If, given $\delta \in (0, 1]$, we choose

$$M \geq \delta \left(\sup_{t \in I_{R/4}^\ominus} \text{Tail}_{K,\alpha}(u(t), R) + R^\alpha \|f\|_{L^\infty} \right)$$

then,

$$A_i \leq \frac{c_{12}}{\delta R^\alpha M^{2/\kappa'}} C^i A_{i-1}^{1+1/\kappa'},$$

where $C := 2^{2/\kappa'+2} > 1$ and $c_{12} > 0$. We choose

$$M := \delta \left(\sup_{t \in I_{R/4}^\ominus} \text{Tail}_{K,\alpha}(u(t), R) + R^\alpha \|f\|_{L^\infty} \right) + C^{\kappa'/2} c_{12}^{\kappa'/2} \delta^{-\kappa'/2} R^{-\alpha\kappa'/2} A_0^{1/2}.$$

It follows that

$$A_0 \leq c_{12}^{-\kappa'} \delta^{\kappa'} R^{\alpha\kappa'} M^2 C^{-\kappa'^2} = \left(\frac{c_{12}}{\delta R^\alpha M^{2/\kappa'}} \right)^{-\kappa'} C^{-\kappa'^2},$$

and therefore we know from Lemma 7.1 in [Giusti 2003] that $A_i \searrow 0$ as $i \rightarrow \infty$, i.e.,

$$\begin{aligned} \sup_{I_{R/8}^\ominus \times B_{R/2}} u &\leq M = \delta \left(\sup_{t \in I_{R/4}^\ominus} \text{Tail}_{K,\alpha}(u(t), R) + R^\alpha \|f\|_{L^\infty} \right) + C^{\kappa'/2} c_{12}^{\kappa'/2} \delta^{-\kappa'/2} R^{-\alpha\kappa'/2} A_0^{1/2} \\ &= \delta \left(\sup_{t \in I_{R/4}^\ominus} \text{Tail}_{K,\alpha}(u(t), R) + R^\alpha \|f\|_{L^\infty} \right) + c_{13} \delta^{-\kappa'/2} \left(R^{-\alpha\kappa'} \int_{I_{R/4}^\ominus} \int_{B_R} u^2(t, x) \, dx \, dt \right)^{1/2} \end{aligned}$$

for $c_{13} > 0$. Note that, by the definition of κ , we have $\alpha\kappa' = \alpha + d$. Therefore,

$$\sup_{I_{R/8}^\ominus \times B_{R/2}} u \leq \delta \sup_{t \in I_{R/4}^\ominus} \text{Tail}_{K,\alpha}(u(t), R) + \delta R^\alpha \|f\|_{L^\infty} + c_{14} \delta^{-\kappa'/2} \left(\int_{I_{R/4}^\ominus} \int_{B_R} u^2(t, x) \, dx \, dt \right)^{1/2}$$

for some $c_{14} > 0$. This proves (i).

To prove (ii), observe that, instead of (3-12), applying Corollary 3.5 to a weak subsolution u to $(\widehat{\text{PDE}})$ yields

$$\begin{aligned} & \sup_{t \in I_{r/4}^\ominus} \int_{B_r} w_l^2(t, x) \, dx + \int_{I_{r/4}^\ominus} \mathcal{E}_{B_{r+\rho}}^{K_s} (\tau w_l(t), \tau w_l(t)) \, dt \\ & \leq c_1 \sigma(r, \rho) \|w_l^2(t)\|_{L^1(I_{(r+\rho)/4}^\ominus \times B_{r+\rho})} + c_1 l^2 \rho^{-\alpha} \left[|A(l, r+\rho)| + |B_{r+\rho}|^{1/\theta} \int_{I_{(r+\rho)/4}^\ominus} |B_{r+\rho} \cap \{u(t, x) > l\}|^{1/\theta'} \, dt \right] \\ & \quad + c_1 \|w_l(t)\|_{L^1(I_{(r+\rho)/4}^\ominus \times B_{r+\rho})} \left(\sup_{t \in I_{(r+\rho)/4}^\ominus} \widehat{\text{Tail}}_K(u(t), r + \frac{1}{2}\rho, r + \rho) + \|f\|_{L^\infty} \right) \end{aligned}$$

for some $c_1 > 0$. Proceeding as in the proof of (i), we derive the following estimate as a replacement of (3-14), where $\tilde{\kappa} := \kappa - 1/\theta > 1$:

$$\begin{aligned} & \int_{I_{r/4}^\ominus} \|w_l^2(t)\|_{L^{\theta'}(B_r)} \, dt \\ & \leq |A(l, r)|^{1/\tilde{\kappa}'} \left(\int_{I_{r/4}^\ominus} \|w_l^2(t)\|_{L^{\tilde{\kappa}\theta'}(B_r)}^{\tilde{\kappa}} \, dt \right)^{1/\tilde{\kappa}} \\ & \leq |A(l, r)|^{1/\tilde{\kappa}'} \left(\sup_{t \in I_{r/4}^\ominus} \|w_l^2(t)\|_{L^1(B_r)}^{\tilde{\kappa}-1} \int_{I_{r/4}^\ominus} \|w_l^2(s)\|_{L^{d/(d-\alpha)}(B_r)} \, ds \right)^{1/\tilde{\kappa}} \\ & \leq c_2 |A(l, r + \rho)|^{1/\tilde{\kappa}'} \left[\sigma(r, \rho) \|w_l^2\|_{L^1(I_{(r+\rho)/4}^\ominus \times B_{r+\rho})} \right. \\ & \quad \left. + l^2 \rho^{-\alpha} \left[|A(l, r + \rho)| + |B_{r+\rho}|^{1/\theta} \int_{I_{(r+\rho)/4}^\ominus} |B_{r+\rho} \cap \{u(t, x) > l\}|^{1/\theta'} \, dt \right] \right. \\ & \quad \left. + \|w_l\|_{L^1(I_{(r+\rho)/4}^\ominus \times B_{r+\rho})} \left(\sup_{t \in I_{(r+\rho)/4}^\ominus} \text{Tail}_K(u(t), r + \frac{1}{2}\rho, r + \rho) + \|f\|_{L^\infty} \right) \right] \\ & \leq c_3 |B_{r+\rho}|^{1/\theta} |A(l, r + \rho)|^{1/\tilde{\kappa}'} \left[\sigma(r, \rho) \left(1 + \left(\frac{l}{l-k} \right)^2 \right) + \frac{\sup_{t \in I_{(r+\rho)/4}^\ominus} \text{Tail}_K(u(t), r + \frac{1}{2}\rho, r + \rho) + \|f\|_{L^\infty}}{l-k} \right. \\ & \quad \left. \times \int_{I_{(r+\rho)/4}^\ominus} \|w_k^2(t)\|_{L^{\theta'}(B_{r+\rho})} \, dt \right] \\ & \leq c_4 \frac{|B_{r+\rho}|^{1/\theta}}{(l-k)^{2/\tilde{\kappa}'}} \left[\sigma(r, \rho) \left(1 + \left(\frac{l}{l-k} \right)^2 \right) + \frac{\sup_{t \in I_{(r+\rho)/4}^\ominus} \text{Tail}_K(u(t), r + \frac{1}{2}\rho, r + \rho) + \|f\|_{L^\infty}}{l-k} \right. \\ & \quad \left. \times \left(\int_{I_{(r+\rho)/4}^\ominus} \|w_k^2(t)\|_{L^{\theta'}(B_{r+\rho})} \, dt \right)^{1+1/\tilde{\kappa}'} \right] \end{aligned}$$

for some $c_2, c_3, c_4 > 0$, and we used

$$\int_{I_{(r+\rho)/4}^\ominus} |B_{r+\rho} \cap \{u(t, x) > l\}|^{1/\theta'} \, dt \leq (l-k)^{-2} \int_{I_{(r+\rho)/4}^\ominus} \|w_k^2(t)\|_{L^{\theta'}(B_{r+\rho})} \, dt \tag{3-17}$$

and applied (3-15). From here, the proof basically proceeds as before. We define sequences (l_i) , (ρ_i) , and (r_i) as before, write $A_i = \int_{I_{r_i/4}^\ominus} \|w_{l_i}(t)\|_{L^{\theta'}(B_{r_i})} \, dt$, and deduce that, for any $\delta \in (0, 1]$, by choosing

$$M \geq \delta \left(\sup_{t \in I_{R/4}^\ominus} \text{Tail}_{K,\alpha}(u(t), R) + R^\alpha \|f\|_{L^\infty} \right),$$

we deduce that

$$A_i \leq \frac{c_5}{\delta R^{\alpha-d/\theta} M^{2/\tilde{\kappa}'}} C^i A_{i-1}^{1+1/\tilde{\kappa}'},$$

where $C > 1$ and $c_5 > 0$ are constants. We choose

$$M := \delta \left(\sup_{t \in I_{R/4}^\ominus} \text{Tail}_{K,\alpha}(u(t), R) + R^\alpha \|f\|_{L^\infty} \right) + C^{\tilde{\kappa}'/2} c_5^{\tilde{\kappa}'/2} \delta^{-\tilde{\kappa}'/2} R^{-(\alpha-d/\theta)\tilde{\kappa}'/2} A_0^{1/2}.$$

It follows that

$$A_0 \leq c_5^{-\tilde{\kappa}'} \delta^{\tilde{\kappa}'} R^{(\alpha-d/\theta)\tilde{\kappa}'} M^2 C^{-\tilde{\kappa}'^2} = \left(\frac{c_5}{\delta R^{\alpha-d/\theta} M^{d/\tilde{\kappa}'}} \right)^{-\tilde{\kappa}'} C^{-\tilde{\kappa}'^2},$$

and therefore we know from Lemma 7.1 in [Giusti 2003] that $A_i \searrow 0$ as $i \rightarrow \infty$, i.e.,

$$\begin{aligned} \sup_{I_{R/8}^\ominus \times B_{R/2}} u &\leq M = \delta \left(\sup_{t \in I_{R/4}^\ominus} \text{Tail}_{K,\alpha}(u(t), R) + R^\alpha \|f\|_{L^\infty} \right) + C^{\tilde{\kappa}'/2} c_5^{\tilde{\kappa}'/2} \delta^{-\tilde{\kappa}'/2} R^{-(\alpha-d/\theta)\tilde{\kappa}'/2} A_0^{1/2} \\ &= \delta \left(\sup_{t \in I_{R/4}^\ominus} \text{Tail}_{K,\alpha}(u(t), R) + R^\alpha \|f\|_{L^\infty} \right) + c_6 \delta^{-\tilde{\kappa}'/2} \left(\int_{I_{R/4}^\ominus} \left(\int_{B_R} u^{2\theta'}(t, x) dx \right)^{1/\theta'} dt \right)^{1/2} \end{aligned}$$

for $c_6 > 0$, where we used $(\alpha - d/\theta)\tilde{\kappa}' = \alpha + d/\theta'$. □

Remark 3.7. Let us comment on the appearance of the $L_{t,x}^{2,2\theta'}$ -norm of u in the estimate (ii) for subsolutions to $(\widehat{\text{PDE}})$. In fact, this term appears since we iterate the $L_{t,x}^{2,2\theta'}$ -norms of w_i in the proof of (ii). In fact, upon estimating

$$|B_{r+\rho}|^{1/\theta} \int_{I_{(r+\rho)/4}^\ominus} |B_{r+\rho} \cap \{u(t, x) > l\}|^{1/\theta'} dt \leq c |I_{(r+\rho)/4}^\ominus| \times |B_{r+\rho}|^{1/\theta} |A(l, r + \rho)|^{1/\theta'},$$

instead of (3-17), we could iterate the $L^{2,2}$ -norms of w_i as in the proof of (i), however, only as long as

$$\mu := \frac{1}{\kappa'} - \frac{1}{\theta} = \frac{\alpha}{d + \alpha} - \frac{1}{\theta} > 0.$$

This means that we would have to restrict ourselves to the suboptimal range $\theta \in ((d + \alpha)/\alpha, \infty]$. In the local case, an analogous phenomenon appears in Chapter VI.13 in [Lieberman 1996].

Note that, for subsolutions (ell-PDE), the analogous condition reads $\mu := \alpha/d - 1/\theta > 0$, which allows us to estimate the supremum of u by the L^2 -norm, as expected for the full range $\theta \in (d/\alpha, \infty]$.

We now state the analog to Theorem 3.6 for stationary solutions.

Theorem 3.8. Assume that (K_{loc}^\leq) , (cutoff), and (Sob) hold.

- (i) Assume that $(K1_{\text{loc}})$ holds for some $\theta \in [d/\alpha, \infty]$. Then there exists $c > 0$ such that, for every $0 < R \leq 1$, every $\delta \in (0, 1]$, and every nonnegative, weak subsolution u to (ell-PDE) in $B_{2R} \subset \Omega$,

$$\sup_{B_{R/2}} u \leq c \delta^{-d/(2\alpha)} \left(\int_{B_R} u^2(x) dx \right)^{1/2} + \delta \text{Tail}_{K,\alpha}(u, R) + R^\alpha \|f\|_{L^\infty}. \tag{3-18}$$

(ii) Assume that $(K1_{\text{glob}})$ holds for some $\theta \in (d/\alpha, \infty]$. Then there exists $c > 0$ such that, for every $0 < R \leq 1$, every $\delta \in (0, 1]$, and every nonnegative, weak subsolution u to (ell-PDE) in $B_{2R} \subset \Omega$,

$$\sup_{B_{R/2}} u \leq c\delta^{-1/(2\mu)} \left(\int_{B_R} u^2(x) \, dx \right)^{1/2} + \delta \widehat{\text{Tail}}_{K,\alpha}(u, R) + R^\alpha \|f\|_{L^\infty},$$

where $\mu := \alpha/d - 1/\theta \in (0, \alpha/d]$.

The first estimate can be read off from Theorem 3.6 (i). The proof of (ii) works similar to the proof of Theorem 3.6 (ii) up to small modifications in the sense of the aforementioned remark. The factor $\delta^{-d/(2\alpha)}$ in (3-18) stems from defining $\kappa = d/(d - \alpha)$ and $\kappa' = d/\alpha$ in the stationary case.

4. Local boundedness via Moser iteration

The goal of this section is to give another proof of Theorem 3.6 via the Moser iteration for positive exponents (see Theorem 4.8). For our main result there is no need of a second proof. However, we consider this independent approach interesting due to the wide range of applicability of the Moser iteration. While local boundedness for symmetric nonlocal operators has been established in numerous works by the De Giorgi iteration technique, the following proof of local boundedness (see Theorem 4.8) using a Moser iteration scheme seems to be new.

The Moser iteration for positive exponents is arguably more complicated than for negative exponents for the following two reasons: Roughly speaking, one would like to use test-functions of the form $\phi = \tau^2 u^{2q-1}$ for $q > 1$. Unfortunately, ϕ a priori does not belong to the correct function space unless u is bounded. Since boundedness of u is one of the main goals of this section, such an assumption is illegal. Instead, we truncate the monomial u^{2q-1} in an adequate way, similar to [Aronson and Serrin 1967]. The second reason concerns the appearance of nonlocal tail terms (see Section 3) due to the nonlocality of the equation. These quantities require special treatment in order to make the iteration work.

Note that Sections 3 and 4 are fully independent of each other.

4.1. Algebraic estimates. The first step is to establish suitable algebraic estimates, which can be seen and will be used as nonlocal analogs to the chain rule. Note that an estimate similar to (4-1) was established in [Brasco and Parini 2016]. We also refer to [Kassmann and Weidner 2022], where the Moser iteration schemes were established for negative and small positive exponents for the same class of nonsymmetric nonlocal operators.

Lemma 4.1. *Let $g : [0, \infty) \rightarrow [0, \infty)$ be continuously differentiable. Assume that g is increasing and that $g(0) = 0$. Set $G(t) := \int_0^t g'(\tau)^{1/2} \, d\tau$. Then, for every $s, t \geq 0$,*

$$(t - s)(g(t) - g(s)) \geq (G(t) - G(s))^2, \quad (4-1)$$

$$\frac{(g(t) \wedge g(s))|t - s|}{|G(t) - G(s)|} \leq G(t) \wedge G(s), \quad (4-2)$$

$$\frac{|g(t) - g(s)|}{|G(t) - G(s)|} \leq g'(t \vee s)^{1/2}. \quad (4-3)$$

Proof. Note that, by assumption, $t \mapsto G'(t) = g'(t)^{1/2}$ is nonnegative. Let us assume without loss of generality that $s \leq t$. First, we compute, with the help of Jensen's inequality,

$$(t - s)(g(t) - g(s)) = (t - s) \int_s^t g'(\tau) \, d\tau = (t - s) \int_s^t G'(\tau)^2 \, d\tau \geq \left(\int_s^t G'(\tau) \, d\tau \right)^2 = (G(t) - G(s))^2,$$

which proves (4-1). Next,

$$\frac{|G(t) - G(s)|}{|t - s|} = \int_s^t G'(\tau) \, d\tau \geq G'(s).$$

Moreover, we compute

$$g(s) = \int_0^s g'(\tau) \, d\tau \leq g'(s)^{1/2} \int_0^s g'(\tau)^{1/2} \, d\tau = G'(s)G(s).$$

This implies

$$\frac{|G(t) - G(s)|}{|t - s|} \geq \frac{g(s)}{G(s)},$$

which proves (4-2). For (4-3), we compute, using the chain rule and again that $G'(t) = g'(t)^{1/2}$ is nondecreasing,

$$\frac{|g(t) - g(s)|}{|G(t) - G(s)|} = \left| \int_{G(s)}^{G(t)} [g \circ G^{-1}]'(\tau) \, d\tau \right| = \int_{G(s)}^{G(t)} g'(G^{-1}(\tau))^{1/2} \, d\tau \leq g'(t)^{1/2}. \quad \square$$

The following lemma has already been established and applied in [Kassmann and Weidner 2022] (see Lemma 3.2 therein).

Lemma 4.2. *Let $G : [0, \infty) \rightarrow \mathbb{R}$. Then, for any $\tau_1, \tau_2 \geq 0$ and $t, s > 0$,*

$$(\tau_1^2 \wedge \tau_2^2)|G(t) - G(s)|^2 \geq \frac{1}{2}|\tau_1 G(t) - \tau_2 G(s)|^2 - (\tau_1 - \tau_2)^2(G^2(t) \vee G^2(s)), \tag{4-4}$$

$$(\tau_1^2 \vee \tau_2^2)|G(t) - G(s)|^2 \leq 2|\tau_1 G(t) - \tau_2 G(s)|^2 + 2(\tau_1 - \tau_2)^2(G^2(t) \vee G^2(s)). \tag{4-5}$$

From now on, let us define the functions $g : [0, \infty) \rightarrow [0, \infty)$ and $G(t) = \int_0^t g'(s)^{1/2} \, ds$ for $M > 0$ and $q \geq 1$ via

$$g(t) = \begin{cases} t^{2q-1}, & t \leq M, \\ M^{2q-1} + (2q - 1)M^{2q-2}(t - M), & t > M, \end{cases}$$

$$G(t) = \begin{cases} \frac{\sqrt{2q-1}}{q} t^q, & t \leq M, \\ \frac{\sqrt{2q-1}}{q} M^q + \sqrt{2q-1}(t - M)M^{q-1}, & t > M. \end{cases}$$

One easily checks that g is continuously differentiable, increasing, and satisfies $g(0) = 0$. Therefore g satisfies the assumptions of Lemma 4.1. Moreover, note that g is convex.

The following lemma is a direct consequence of the definition of g .

Lemma 4.3. For every $t \geq 0$,

$$G'(t) = g'(t)^{1/2} \leq q \frac{G(t)}{t}, \quad (4-6)$$

$$g(t)t \leq \frac{q^2}{2q-1} G^2(t). \quad (4-7)$$

Proof. Let us start by proving the first estimate. In the case $t \leq M$, a direct computation shows,

$$g'(t)^{1/2} = \sqrt{2q-1} t^{q-1} = q \frac{\sqrt{2q-1}}{q} t^{q-1} = q \frac{G(t)}{t}.$$

For $t > M$, we use $\sqrt{2q-1} \leq q$ to compute

$$\begin{aligned} g'(t)^{1/2} &= \sqrt{2q-1} M^{q-1} = q \frac{\frac{\sqrt{2q-1}}{q} (M^q + (t-M)M^{q-1})}{t} \\ &\leq q \frac{\frac{\sqrt{2q-1}}{q} M^q + \sqrt{2q-1} (t-M)M^{q-1}}{t} = q \frac{G(t)}{t}. \end{aligned}$$

This proves (4-6). For (4-7), in the case $t \leq M$, we compute

$$g(t)t = t^{2q} = \frac{q^2}{2q-1} G^2(t).$$

In the case $t > M$, we use $\sqrt{2q-1} \leq q$ to compute

$$g(t)t = t^2 M^{2q-2} = \frac{q^2}{2q-1} \left(\frac{\sqrt{2q-1}}{q} (M^q + (t-M)M^{q-1}) \right)^2 \leq \frac{q^2}{2q-1} G^2(t). \quad \square$$

Remark 4.4. Note that (4-6) already implies a slightly weaker version of the estimate in (4-7). Indeed, by (4-6),

$$q^2 G^2(t) \geq (G'(t)t)^2 = g'(t)t^2 \geq g(t)t,$$

where we used convexity and $g(0) = 0$ in the last estimate.

Lemma 4.5. Let $q \geq 1$. Then, for every $s, t \geq 0$, we have

$$(G(t) - G(s))^2 \nearrow \frac{2q-1}{q^2} (t^q - s^q)^2 \quad \text{as } M \nearrow \infty.$$

Proof. Clearly,

$$(G(t) - G(s))^2 \rightarrow \frac{2q-1}{q^2} (t^q - s^q)^2$$

as $M \rightarrow \infty$, since, for $t, s < M$, we already have

$$(G(t) - G(s))^2 = \frac{2q-1}{q^2} (t^q - s^q)^2.$$

It remains to prove that the convergence is monotone. Let us fix $t > s > 0$. First, we observe that $M \mapsto (G(t) - G(s))^2$ is continuous. Now, clearly, for $M < t < s$, we have

$$(G(t) - G(s))^2 = (2q-1)M^{2q-2} (t-s)^2,$$

which is increasing in M . In the case $s < M < t$,

$$(G(t) - G(s))^2 = \left(\frac{\sqrt{2q-1}}{q} M^q + (t-M)\sqrt{2q-1} M^{q-1} - \frac{\sqrt{2q-1}}{q} s^q \right)^2.$$

This expression is clearly monotone in M as long as $t > M$, since

$$\frac{d}{dM} \frac{\sqrt{2q-1}}{q} M^q + (t-M)\sqrt{2q-1} M^{q-1} = (q-1)\sqrt{2q-1}(t-M)M^{q-2} \geq 0.$$

This proves the desired result. □

4.2. Caccioppoli estimates. Now, we are in the position to prove the following Caccioppoli-type estimate. We emphasize that $\tau^2 g(\tilde{u}) \in H_{B_{r+\rho}}(\mathbb{R}^d)$ in the lemma below, where $\tilde{u} = u + R^\alpha \|f\|_{L^\infty}$, whenever $u \in V(B_{r+\rho}|\mathbb{R}^d)$. This is a direct consequence of the definition of g .

Lemma 4.6. *Assume that (K1_{loc}) and (cutoff) hold for some $\theta \in [d/\alpha, \infty]$. Moreover, assume (Sob) if $\theta < \infty$. Then there exist $c_1, c_2 > 0$ such that, for every $0 < \rho \leq r \leq 1$, every nonnegative function $u \in V(B_{r+\rho}|\mathbb{R}^d)$, and every $q \geq 1$, we have*

$$\begin{aligned} \mathcal{E}_{B_{r+\rho}}^{K_s}(\tau G(\tilde{u}), \tau G(\tilde{u})) \\ \leq c_1 \mathcal{E}(u, \tau^2 g(\tilde{u})) + c_2 \rho^{-\alpha} \|G(\tilde{u})^2\|_{L^1(B_{r+\rho})} + c_2 \|g(\tilde{u})\|_{L^1(B_{r+\rho})} \text{Tail}_K(u, r + \frac{1}{2}\rho, r + \rho), \end{aligned}$$

where $B_{2r} \subset \Omega$, $\tau = \tau_{r,\rho/2}$, and $\tilde{u} = u + R^\alpha \|f\|_{L^\infty}$.

Proof. We define

$$M := \{(x, y) \in B_{r+\rho} \times B_{r+\rho} : u(x) > u(y)\}.$$

Note that, for $(x, y) \in M$, we have $g(u(x)) \geq g(u(y))$ and $G(u(x)) \geq G(u(y))$. The proof is divided into several steps.

Step 1: First, we claim that, for some $c_1, c_2 > 0$,

$$\mathcal{E}_{B_{r+\rho}}(u, \tau^2 g(\tilde{u})) \geq c_1 \mathcal{E}_{B_{r+\rho}}^{K_s}(\tau G(\tilde{u}), \tau G(\tilde{u})) - c_2 \rho^{-\alpha} \|G(\tilde{u})^2\|_{L^1(B_{r+\rho})}. \tag{4-8}$$

For the symmetric part, we compute the following using the symmetry of K_s (see also Lemma 2.3 in [Kassmann and Weidner 2022]):

$$\begin{aligned} \mathcal{E}_{B_{r+\rho}}^{K_s}(u, \tau^2 g(\tilde{u})) &= 2 \iint_M (\tilde{u}(x) - \tilde{u}(y)) (\tau^2(x)g(\tilde{u}(x)) - \tau^2(y)g(\tilde{u}(y))) K_s(x, y) \, dy \, dx \\ &= 2 \iint_M (\tilde{u}(x) - \tilde{u}(y)) (g(\tilde{u}(x)) - g(\tilde{u}(y))) \tau^2(x) K_s(x, y) \, dy \, dx \\ &\quad + 2 \iint_M (\tilde{u}(x) - \tilde{u}(y)) g(\tilde{u}(y)) (\tau^2(x) - \tau^2(y)) K_s(x, y) \, dy \, dx \\ &= I_s + J_s. \end{aligned}$$

For the nonsymmetric part, we compute, using the antisymmetry of K_a and with the help of Lemma 2.3 in [Kassmann and Weidner 2022],

$$\begin{aligned} \mathcal{E}_{B_{r+\rho}}^{K_a}(u, \tau^2 g(\tilde{u})) &= 2 \iint_M (\tilde{u}(x) - \tilde{u}(y))(\tau^2(x)g(\tilde{u}(x)) + \tau^2(y)g(\tilde{u}(y)))K_a(x, y) \, dy \, dx \\ &= 2 \iint_M (\tilde{u}(x) - \tilde{u}(y))(g(\tilde{u}(x)) - g(\tilde{u}(y)))\tau^2(x)K_a(x, y) \, dy \, dx \\ &\quad + 2 \iint_M (\tilde{u}(x) - \tilde{u}(y))g(\tilde{u}(y))(\tau^2(x) + \tau^2(y))K_a(x, y) \, dy \, dx \\ &= I_a + J_a. \end{aligned}$$

By adding $I_s + I_a$ and using (4-1), (4-4), as well as (cutoff), we obtain

$$\begin{aligned} I_s + I_a &= 2 \iint_M (\tilde{u}(x) - \tilde{u}(y))(g(\tilde{u}(x)) - g(\tilde{u}(y)))\tau^2(x)K(x, y) \, dy \, dx \\ &\geq \iint_M (G(\tilde{u}(x)) - G(\tilde{u}(y)))^2(\tau^2(x) \wedge \tau^2(y))K(x, y) \, dy \, dx \\ &\geq \frac{1}{2}\mathcal{E}_{B_{r+\rho}}^{K_s}(\tau G(\tilde{u}), \tau G(\tilde{u})) - c\rho^{-\alpha}\|G(\tilde{u})^2\|_{L^1(B_{r+\rho})} \\ &\quad - \frac{1}{2} \int_{B_{r+\rho}} \int_{B_{r+\rho}} (\tau G(\tilde{u}(x)) - \tau G(\tilde{u}(y)))^2 |K_a(x, y)| \, dy \, dx. \end{aligned}$$

For the nonsymmetric part, using $(K1)_{\text{loc}}$ and (2-8), we find that, for every $\varepsilon > 0$, there is $c > 0$ such that

$$\begin{aligned} \int_{B_{r+\rho}} \int_{B_{r+\rho}} (\tau G(\tilde{u}(x)) - \tau G(\tilde{u}(y)))^2 |K_a(x, y)| \, dy \, dx \\ \leq \varepsilon \mathcal{E}_{B_{r+\rho}}^{K_s}(\tau G(\tilde{u}), \tau G(\tilde{u})) + c \int_{B_{r+\rho}} \tau^2(x)G^2(\tilde{u}(x)) \left(\int_{B_{r+\rho}} \frac{|K_a(x, y)|^2}{J(x, y)} \, dy \right) dx \\ \leq 2\varepsilon \mathcal{E}_{B_{r+\rho}}^{K_s}(\tau G(\tilde{u}), \tau G(\tilde{u})) + c\rho^{-\alpha}\|G(\tilde{u})^2\|_{L^1(B_{r+\rho})}. \end{aligned}$$

Consequently,

$$I_s + I_a \geq \frac{1}{4}\mathcal{E}_{B_{r+\rho}}^{K_s}(\tau G(\tilde{u}), \tau G(\tilde{u})) - c\rho^{-\alpha}\|G(\tilde{u})^2\|_{L^1(B_{r+\rho})}.$$

For J_s , we use (4-2), (4-5), and (cutoff) to prove that, for every $\varepsilon > 0$, there exists $c > 0$ such that

$$\begin{aligned} J_s &\geq - \iint_M |G(\tilde{u}(x)) - G(\tilde{u}(y))|G(\tilde{u}(y))|\tau(x) - \tau(y)|(\tau(x) \vee \tau(y))K_s(x, y) \, dy \, dx \\ &\geq -\varepsilon \iint_M (G(\tilde{u}(x)) - G(\tilde{u}(y)))^2(\tau^2(x) \vee \tau^2(y))K_s(x, y) \, dy \, dx - c\rho^{-\alpha}\|G(\tilde{u})^2\|_{L^1(B_{r+\rho})} \\ &\geq -\varepsilon \mathcal{E}_{B_{r+\rho}}^{K_s}(\tau G(\tilde{u}), \tau G(\tilde{u})) - c\rho^{-\alpha}\|G(\tilde{u})^2\|_{L^1(B_{r+\rho})}. \end{aligned}$$

Next, we estimate J_a and prove, using (3-4), (1-2), (4-2), (cutoff), and (4-5), that, for every $\varepsilon > 0$, there is $c > 0$ such that

$$\begin{aligned} J_a &\geq -8 \iint_M |\tilde{u}(x) - \tilde{u}(y)|g(\tilde{u}(y))(\tau^2(x) \wedge \tau^2(y))|K_a(x, y)| \, dy \, dx \\ &\quad - 8 \iint_M |\tilde{u}(x) - \tilde{u}(y)|g(\tilde{u}(y))(\tau(x) - \tau(y))^2 K_s(x, y) \, dy \, dx \end{aligned}$$

$$\begin{aligned} &\geq -\varepsilon \iint_M (G(\tilde{u}(x)) - G(\tilde{u}(y)))^2 (\tau^2(x) \wedge \tau^2(y)) J(x, y) \, dy \, dx \\ &\quad - c \iint_M G(\tilde{u}(y)) (\tau^2(x) \wedge \tau^2(y)) \frac{|K_a(x, y)|^2}{J(x, y)} \, dy \, dx - c\rho^{-\alpha} \|G(\tilde{u})\|_{L^1(B_{r+\rho})}^2 \\ &\geq -2\varepsilon \mathcal{E}_{B_{r+\rho}}^{K_s}(\tau G(\tilde{u}), \tau G(\tilde{u})) - c\rho^{-\alpha} \|G(\tilde{u})\|_{L^1(B_{r+\rho})}^2, \end{aligned}$$

where we used (K1_{loc}) and (2-8) in the last step to estimate

$$c \iint_M G^2(\tilde{u}(y)) (\tau^2(x) \wedge \tau^2(y)) \frac{|K_a(x, y)|^2}{J(x, y)} \, dy \, dx \leq 2\varepsilon \mathcal{E}_{B_{r+\rho}}^{K_s}(\tau G(\tilde{u}), \tau G(\tilde{u})) + c\rho^{-\alpha} \|G(\tilde{u})\|_{L^1(B_{r+\rho})}^2$$

and used Lemma 2.6 in [Kassmann and Weidner 2022], (K1_{loc}), (4-4), and (cutoff) to estimate

$$\begin{aligned} &\iint_M (G(\tilde{u}(x)) - G(\tilde{u}(y)))^2 (\tau^2(x) \wedge \tau^2(y)) J(x, y) \, dy \, dx \\ &\quad \leq c \int_{B_{r+\rho}} \int_{B_{r+\rho}} (G(\tilde{u}(x)) - G(\tilde{u}(y)))^2 (\tau^2(x) \wedge \tau^2(y)) K_s(x, y) \, dy \, dx \\ &\quad \leq c \mathcal{E}_{B_{r+\rho}}^{K_s}(\tau G(\tilde{u}), \tau G(\tilde{u})) + c\rho^{-\alpha} \|G(\tilde{u})\|_{L^1(B_{r+\rho})}^2. \end{aligned} \tag{4-9}$$

Altogether, we obtain

$$\mathcal{E}_{B_{r+\rho}}(u, \tau^2 g(\tilde{u})) \geq \left[\frac{1}{4} - 2\varepsilon\right] \mathcal{E}_{B_{r+\rho}}^{K_s}(\tau G(\tilde{u}), \tau G(\tilde{u})) - c\rho^{-\alpha} \|G(\tilde{u})\|_{L^1(B_{r+\rho})}^2.$$

The desired estimate (4-8) now follows by choosing $\varepsilon > 0$ small enough.

Step 2: In addition, we claim

$$-\mathcal{E}_{(B_{r+\rho} \times B_{r+\rho})^c}(u, \tau^2 g(\tilde{u})) \leq 2\|g(\tilde{u})\|_{L^1(B_{r+\rho})} \sup_{z \in B_{r+\rho/2}} \left(\int_{B_{r+\rho}^c} u(y) K(z, y) \, dy \right). \tag{4-10}$$

To see this, we compute

$$\begin{aligned} &-\mathcal{E}_{(B_{r+\rho} \times B_{r+\rho})^c}(u, \tau^2 g(\tilde{u})) \\ &= -2 \int_{(B_{r+\rho} \times B_{r+\rho})^c} (u(x) - u(y)) \tau^2(x) g(\tilde{u}(x)) K(x, y) \, dy \, dx \\ &= -2 \int_{B_{r+\rho}} \tau^2(x) u(x) g(\tilde{u}(x)) \left(\int_{B_{r+\rho}^c} K(x, y) \, dy \right) dx + 2 \int_{B_{r+\rho}} \tau^2(x) g(\tilde{u}(x)) \left(\int_{B_{r+\rho}^c} u(y) K(x, y) \, dy \right) dx \\ &\leq 2\|g(\tilde{u})\|_{L^1(B_{r+\rho})} \sup_{z \in B_{r+\rho/2}} \left(\int_{B_{r+\rho}^c} u(y) K(z, y) \, dy \right) \end{aligned}$$

using $u, K \geq 0$ and $\text{supp}(\tau) \subset B_{r+\rho/2}$.

Step 3: Observe that

$$\mathcal{E}_{B_{r+\rho}}^{K_s}(u, \tau^2 g(\tilde{u})) = \mathcal{E}(u, \tau^2 g(\tilde{u})) - \mathcal{E}_{(B_{r+\rho} \times B_{r+\rho})^c}(u, \tau^2 g(\tilde{u})).$$

Therefore, combining (4-8) and (4-10) yields the desired result. □

The following Caccioppoli-type estimate is designed for the dual equation.

Lemma 4.7. *Assume that $(K1_{\text{glob}})$ and (cutoff) hold for some $\theta \in (d/\alpha, \infty]$. Moreover, assume (Sob) if $\theta < \infty$. Then there exist $c_1, c_2, \gamma > 0$ such that, for every $0 < \rho \leq r \leq 1$, every nonnegative function $u \in V(B_{r+\rho} | \mathbb{R}^d) \cap L^{2\theta'}(\mathbb{R}^d)$, and every $q \geq 1$, we have*

$$\begin{aligned} \mathcal{E}_{B_{r+\rho}}^{K_s}(\tau G(\tilde{u}), \tau G(\tilde{u})) &\leq c_1 \widehat{\mathcal{E}}(u, \tau^2 g(\tilde{u})) + c_2 q^\gamma \rho^{-\alpha} \|G(\tilde{u})\|_{L^1(B_{r+\rho})}^2 + c_2 \|g(\tilde{u})\|_{L^1(B_{r+\rho})} \widehat{\text{Tail}}_K(u, r + \frac{1}{2}\rho, r + \rho), \end{aligned}$$

where $B_{2r} \subset \Omega$, $\tau = \tau_{r,\rho/2}$, and $\tilde{u} = u + R^\alpha \|f\|_{L^\infty}$.

Proof. Step 1: We claim that there exists $c > 0$ such that, for some $\gamma \geq 1$,

$$\widehat{\mathcal{E}}_{B_{r+\rho}}(u, \tau^2 g(\tilde{u})) \geq c_1 \mathcal{E}_{B_{r+\rho}}^{K_s}(\tau G(\tilde{u}), \tau G(\tilde{u})) - c_2 q^\gamma \rho^{-\alpha} \|G(\tilde{u})\|_{L^1(B_{r+\rho})}^2. \tag{4-11}$$

Let M be as in the proof of Lemma 4.6. Moreover, we observe the algebraic identity

$$(a + b)(\tau_1^2 g(\tilde{a}) - \tau_2^2 g(\tilde{b})) = (\tilde{a} - \tilde{b})(g(\tilde{a}) - g(\tilde{b}))\tau_1^2 + 2b(g(\tilde{a}) - g(\tilde{b}))\tau_1^2 + (a + b)g(\tilde{b})(\tau_1^2 - \tau_2^2).$$

We use again Lemma 2.3 in [Kassmann and Weidner 2022] to estimate

$$\begin{aligned} \widehat{\mathcal{E}}_M^{K_a}(u, \tau^2 g(\tilde{u})) &= 2 \iint_M (\tilde{u}(x) - \tilde{u}(y))(g(\tilde{u}(x)) - g(\tilde{u}(y)))\tau^2(x)K_a(x, y) \, dy \, dx \\ &\quad + 4 \iint_M u(y)(g(\tilde{u}(x)) - g(\tilde{u}(y)))\tau^2(x)K_a(x, y) \, dy \, dx \\ &\quad + 4 \iint_M (u(x) + u(y))g(\tilde{u}(y))(\tau^2(x) - \tau^2(y))K_a(x, y) \, dy \, dx \\ &\geq 2 \iint_M (\tilde{u}(x) - \tilde{u}(y))(g(\tilde{u}(x)) - g(\tilde{u}(y)))\tau^2(x)K_a(y, x) \, dy \, dx \\ &\quad - 4 \iint_M \tilde{u}(x)|g(\tilde{u}(x)) - g(\tilde{u}(y))|\tau^2(x)|K_a(x, y)| \, dy \, dx \\ &\quad - 8 \iint_M g(\tilde{u}(x))\tilde{u}(x)|\tau^2(x) - \tau^2(y)||K_a(x, y)| \, dy \, dx \\ &= I_a + M_a + N_a, \end{aligned}$$

where we used that $u(x) \geq u(y)$ and $g(u(x)) \geq g(u(y))$ on M , as well as $u \leq \tilde{u}$. As in the proof of Lemma 4.6, we can write the decomposition

$$\mathcal{E}_{B_{r+\rho}}^{K_s}(u, \tau^2 g(\tilde{u})) = I_s + J_s.$$

Then, using (4-1), (2-8), and (cutoff), we estimate

$$\begin{aligned} I_s + I_a &= 2 \iint_M (\tilde{u}(x) - \tilde{u}(y))(g(\tilde{u}(x)) - g(\tilde{u}(y)))\tau^2(x)K(x, y) \, dy \, dx \\ &\geq \frac{1}{4} \mathcal{E}_{B_{r+\rho}}^{K_s}(\tau G(\tilde{u}), \tau G(\tilde{u})) - c\rho^{-\alpha} \|G(\tilde{u})\|_{L^1(B_{r+\rho})}^2. \end{aligned}$$

For M_a , we use (4-3), (4-6), (4-5), and (cutoff) to obtain

$$\begin{aligned} M_a &\geq -4q \iint_M |G(\tilde{u}(x)) - G(\tilde{u}(y))| G(\tilde{u}(x)) \tau^2(x) |K_a(x, y)| \, dy \, dx \\ &\geq -\varepsilon \iint_M (G(\tilde{u}(x)) - G(\tilde{u}(y)))^2 (\tau^2(y) \vee \tau^2(x)) J(x, y) \, dy \, dx \\ &\quad - cq^2 \iint_M \tau^2(x) G^2(\tilde{u}(x)) \frac{|K_a(x, y)|^2}{J(x, y)} \, dy \, dx \\ &\geq -c\varepsilon \mathcal{E}_{B_{r+\rho}}^{K_s}(\tau G(\tilde{u}), \tau G(\tilde{u})) - cq^{\gamma_1} \rho^{-\alpha} \|G(\tilde{u})\|_{L^1(B_{r+\rho})}^2 \end{aligned}$$

for some $\gamma_1 > 0$, where we used that, by (K1_{glob}) and (2-9) applied with some $\delta \leq \varepsilon/(cq^2)$,

$$\begin{aligned} cq^2 \iint_M \tau^2(x) G^2(\tilde{u}(x)) \frac{|K_a(x, y)|^2}{J(x, y)} \, dy \, dx \\ \leq \varepsilon \mathcal{E}_{B_{r+\rho}}^{K_s}(\tau G(\tilde{u}), \tau G(\tilde{u})) + cq^2 (\delta^{-\gamma_2} + \delta) \rho^{-\alpha} \|G(\tilde{u})\|_{L^1(B_{r+\rho})}^2 \end{aligned} \quad (4-12)$$

for some $\gamma_2 > 0$, and moreover, by (3-4), (cutoff), and using the same argument as in (4-9),

$$\begin{aligned} \varepsilon \iint_M (G(\tilde{u}(x)) - G(\tilde{u}(y)))^2 (\tau^2(y) \vee \tau^2(x)) J(x, y) \, dy \, dx \\ \leq 2\varepsilon \int_{B_{r+\rho}} \int_{B_{r+\rho}} (G(\tilde{u}(x)) - G(\tilde{u}(y)))^2 (\tau^2(y) \wedge \tau^2(x)) J(x, y) \, dy \, dx + c\rho^{-\alpha} \|G(\tilde{u})\|_{L^1(B_{r+\rho})}^2 \\ \leq c\varepsilon \mathcal{E}_{B_{r+\rho}}^{K_s}(\tau G(\tilde{u}), \tau G(\tilde{u})) + c\rho^{-\alpha} \|G(\tilde{u})\|_{L^1(B_{r+\rho})}^2. \end{aligned}$$

For N_a , we compute, using (4-7), (3-4), and (1-2),

$$\begin{aligned} N_a &\geq -cq \iint_M G^2(\tilde{u}(x)) (\tau(x) - \tau(y))^2 K_s(x, y) \, dy \, dx \\ &\quad - cq \iint_M G^2(\tilde{u}(x)) (\tau(x) \wedge \tau(y)) |\tau(x) - \tau(y)| |K_a(x, y)| \, dy \, dx \\ &\geq -cq\rho^{-\alpha} \|G(\tilde{u})\|_{L^1(B_{r+\rho})}^2 - cq^2 \int_{B_{r+\rho}} \int_{B_{r+\rho}} \tau^2(x) G^2(\tilde{u}(x)) \frac{|K_a(x, y)|^2}{J(x, y)} \, dy \, dx \\ &\geq -cq^{\gamma_3} \rho^{-\alpha} \|G(\tilde{u})\|_{L^1(B_{r+\rho})}^2 - \varepsilon \mathcal{E}_{B_{r+\rho}}^{K_s}(\tau G(\tilde{u}), \tau G(\tilde{u})) \end{aligned}$$

for some $\gamma_3 > 0$, where we applied (cutoff) and used the same argument as in (4-12) to estimate the second summand in the last step. Altogether, we have shown

$$\widehat{\mathcal{E}}_{B_{r+\rho}}(u, \tau^2 g(\tilde{u})) \geq \left[\frac{1}{4} - 3\varepsilon \right] \mathcal{E}_{B_{r+\rho}}^{K_s}(\tau G(\tilde{u}), \tau G(\tilde{u})) - cq^{\gamma} \rho^{-\alpha} \|G(\tilde{u})\|_{L^1(B_{r+\rho})}^2.$$

Thus, by choosing $\varepsilon > 0$ small enough, we obtain (4-11), as desired.

Step 2: Moreover, we have

$$-\widehat{\mathcal{E}}_{(B_{r+\rho} \times B_{r+\rho})^c}(u, \tau^2 g(\tilde{u})) \leq c \|g(\tilde{u})\|_{L^1(B_{r+\rho})} \sup_{z \in B_{r+\rho/2}} \left(\int_{B_{r+\rho}^c} u(y) K(y, z) \, dy \right). \quad (4-13)$$

The proof works similar to the proof of Step 2 in Lemma 4.6:

$$\begin{aligned}
 & -\widehat{\mathcal{E}}_{(B_{r+\rho} \times B_{r+\rho})^c}(u, \tau^2 g(\tilde{u})) \\
 &= -2 \int_{(B_{r+\rho} \times B_{r+\rho})^c} (\tau^2 g(\tilde{u}(x)) - \tau^2 g(\tilde{u}(y)))u(x)K(x, y) \, dy \, dx \\
 &= -2 \int_{B_{r+\rho}} \tau^2(x)u(x)g(\tilde{u}(x)) \left(\int_{B_{r+\rho}^c} K(x, y) \, dy \right) dx + 2 \int_{B_{r+\rho}} \tau^2(y)g(\tilde{u}(y)) \left(\int_{B_{r+\rho}^c} u(x)K(x, y) \, dx \right) dy \\
 &\leq 2 \|g(\tilde{u})\|_{L^1(B_{r+\rho})} \sup_{z \in B_{r+\rho/2}} \left(\int_{B_{r+\rho}^c} u(y)K(y, z) \, dy \right)
 \end{aligned}$$

using $u, K \geq 0$ and $\text{supp}(\tau) \subset B_{r+\rho/2}$. □

4.3. Local boundedness. Now, we will show how to prove Theorem 3.6 via the Moser iteration. Note that we get a slightly better bound for subsolutions to $(\widehat{\text{PDE}})$ compared to Theorem 3.6 (ii).

Theorem 4.8. *Assume that (K_{loc}^{\leq}) , (cutoff), and (Sob) hold.*

- (i) *Assume $(K1_{\text{loc}})$ holds for some $\theta \in [d/\alpha, \infty]$. Then there exists $c > 0$ such that, for every $0 < R \leq 1$ and every nonnegative, weak subsolution u to (PDE) in $I_R^\ominus(t_0) \times B_{2R}$,*

$$\sup_{I_{R/8}^\ominus \times B_{R/2}} u \leq c \left(\int_{I_{R/4}^\ominus} \int_{B_R} u^2(t, x) \, dx \, dt \right)^{1/2} + c \sup_{t \in I_{R/4}^\ominus} \widehat{\text{Tail}}_{K, \alpha}(u(t), R) + cR^\alpha \|f\|_{L^\infty}, \tag{4-14}$$

where $B_{2R} \subset \Omega$.

- (ii) *Assume $(K1_{\text{glob}})$ holds for some $\theta \in (d/\alpha, \infty]$. Then there exists $c > 0$ such that, for every $0 < R \leq 1$ and every nonnegative, weak subsolution u to $(\widehat{\text{PDE}})$ in $I_R^\ominus(t_0) \times B_{2R}$,*

$$\sup_{I_{R/8}^\ominus \times B_{R/2}} u \leq c \left(\int_{I_{R/4}^\ominus} \int_{B_R} u^2(t, x) \, dx \, dt \right)^{1/2} + c \sup_{t \in I_{R/4}^\ominus} \widehat{\text{Tail}}_{K, \alpha}(u(t), R) + cR^\alpha \|f\|_{L^\infty}, \tag{4-15}$$

where $B_{2R} \subset \Omega$.

Proof. We will only demonstrate the proof of (ii). The proof of (i) follows via the same arguments, but uses Lemma 4.6 instead of Lemma 4.7. Let $0 < \rho \leq r \leq r + \rho \leq R$ and $q \geq 1$. By applying Lemma 4.7, we obtain

$$\begin{aligned}
 & c \int_{B_{r+\rho}} \tau^2(x) \partial_t u(t, x) g(\tilde{u}(t, x)) \, dx + \mathcal{E}_{B_{r+\rho}}^{K_s}(\tau G(\tilde{u}), \tau G(\tilde{u})) \\
 & \leq c [(\partial_t u(t), \tau^2 g(\tilde{u}(t))) + \mathcal{E}(u(t), \tau^2 g(\tilde{u}(t)))] \\
 & \quad + cq^\gamma \rho^{-\alpha} \|G(\tilde{u}(t))^2\|_{L^1(B_{r+\rho})} + c \|g(\tilde{u}(t))\|_{L^1(B_{r+\rho})} \widehat{\text{Tail}}_K(u(t), r + \frac{1}{2}\rho, r + \rho) \\
 & \leq c(f(t), \tau^2 g(\tilde{u}(t))) + cq^\gamma \rho^{-\alpha} \|G(\tilde{u}(t))^2\|_{L^1(B_{r+\rho})} + c \|g(\tilde{u}(t))\|_{L^1(B_{r+\rho})} \widehat{\text{Tail}}_K(u(t), r + \frac{1}{2}\rho, r + \rho) \\
 & \leq cq^\gamma \rho^{-\alpha} \|G(\tilde{u}(t))^2\|_{L^1(B_{r+\rho})} + c \|g(\tilde{u}(t))\|_{L^1(B_{r+\rho})} \widehat{\text{Tail}}_K(u(t), r + \frac{1}{2}\rho, r + \rho), \tag{4-16}
 \end{aligned}$$

where $c > 0$ is the constant from Lemma 4.7 and we tested the equation with $\tau^2 g(u)$, where $\tau = \tau_{r,\rho/2}$. Moreover, by the definition of \tilde{u} , we used

$$(f(t), \tau^2 g(\tilde{u}(t))) \leq c\rho^{-\alpha} \|G(\tilde{u}(t))^2\|_{L^1(B_{r+\rho})}.$$

We observe that

$$(\partial_t u)g(\tilde{u}) = \begin{cases} \frac{1}{2q} \partial_t(\tilde{u}^{2q}), & u \leq M, \\ \frac{1}{2} M^{2q-2} \partial_t(\tilde{u}^2), & u > M. \end{cases}$$

Next, we define $\chi \in C^1(\mathbb{R})$ to be a function satisfying

$$0 \leq \chi \leq 1, \quad \|\chi'\|_\infty \leq 16((r+\rho)^\alpha - r^\alpha)^{-1}, \quad \chi(t_0 - (\frac{1}{4}(r+\rho))^\alpha) = 0, \quad \chi \equiv 1 \quad \text{in } I_{r/4}^\ominus(t_0).$$

By multiplying (4-16) with χ^2 and integrating over $(t_0 - (\frac{1}{4}(r+\rho))^\alpha, t)$ for some arbitrary $t \in I_{r/4}^\ominus(t_0)$, we obtain

$$\begin{aligned} & \int_{B_{r+\rho}} \chi^2(t) \tau^2(x) H(\tilde{u}(t, x)) \, dx + \int_{t_0 - ((r+\rho)/4)^\alpha}^t \chi^2(s) \mathcal{E}_{B_{r+\rho}}^{K_s}(\tau G(\tilde{u}(s)), \tau G(\tilde{u}(s))) \, ds \\ & \leq c_2 q^\gamma \rho^{-\alpha} \int_{t_0 - ((r+\rho)/4)^\alpha}^t \chi^2(s) \|G(\tilde{u}(s))^2\|_{L^1(B_{r+\rho})} \, ds \\ & \quad + c_2 \int_{t_0 - ((r+\rho)/4)^\alpha}^t \chi(s) |\chi'(s)| \int_{B_{r+\rho}} \tau^2(x) H(\tilde{u}(s, x)) \, dx \, ds \\ & \quad + c_2 \int_{t_0 - ((r+\rho)/4)^\alpha}^t \chi^2(s) \|g(\tilde{u}(s))\|_{L^1(B_{r+\rho})} \widehat{\text{Tail}}_K(u(s), r + \frac{1}{2}\rho, r + \rho) \, ds \end{aligned}$$

for some $c_2 > 0$, where

$$H(t) = \begin{cases} \frac{1}{2q} t^{2q}, & t \leq M, \\ \frac{1}{2} M^{2q-2} t^2, & t > M. \end{cases}$$

Consequently,

$$\begin{aligned} & \sup_{t \in I_{r/4}^\ominus} \int_{B_r} H(\tilde{u}(t, x)) \, dx + \int_{I_{r/4}^\ominus} \mathcal{E}_{B_{r+\rho}}^{K_s}(\tau G(\tilde{u}(s)), \tau G(\tilde{u}(s))) \, ds \\ & \leq c_3 q^\gamma (\rho^{-\alpha} \vee ((r+\rho)^\alpha - r^\alpha)^{-1}) (\|H(\tilde{u})\|_{L^1(I_{(r+\rho)/4}^\ominus \times B_{r+\rho})} + \|G(\tilde{u})^2\|_{L^1(I_{(r+\rho)/4}^\ominus \times B_{r+\rho})}) \\ & \quad + c_3 \|g(\tilde{u})\|_{L^1(I_{(r+\rho)/4}^\ominus \times B_{r+\rho})} \sup_{t \in I_{(r+\rho)/4}^\ominus} \widehat{\text{Tail}}_K(u(t), r + \frac{1}{2}\rho, r + \rho) \end{aligned}$$

for some $c_3 > 0$. Now, we take the limit $M \nearrow \infty$. By monotone convergence, the definitions of g , G , and H , and Lemma 4.5,

$$\begin{aligned} & \sup_{t \in I_{r/4}^\ominus} \int_{B_r} \tilde{u}^{2q}(t, x) \, dx + \int_{I_{r/4}^\ominus} \mathcal{E}_{B_{r+\rho}}^{K_s}(\tau \tilde{u}^q(s), \tau \tilde{u}^q(s)) \, ds \\ & \leq c_4 q^\gamma (\rho^{-\alpha} \vee ((r+\rho)^\alpha - r^\alpha)^{-1}) \|\tilde{u}^{2q}\|_{L^1(I_{(r+\rho)/4}^\ominus \times B_{r+\rho})} \\ & \quad + c_4 q \|\tilde{u}^{2q-1}\|_{L^1(I_{(r+\rho)/4}^\ominus \times B_{r+\rho})} \sup_{t \in I_{(r+\rho)/4}^\ominus} \widehat{\text{Tail}}_K(u(t), r + \frac{1}{2}\rho, r + \rho) \end{aligned}$$

for some $c_4 > 0$. Recall $\kappa = 1 + \alpha/d > 1$. By Hölder interpolation and Sobolev inequality (Sob), we have

$$\begin{aligned} \|\tilde{u}^{2q}\|_{L^\kappa(I_r^\ominus \times B_r)} &\leq \left(\sup_{t \in I_r^\ominus} \|\tilde{u}^{2q}(t)\|_{L^1(B_r)}^{\kappa-1} \int_{I_r^\ominus} \|\tilde{u}^{2q}(s)\|_{L^{d/(d-\alpha)}(B_r)} ds \right)^{\frac{1}{\kappa}} \\ &\leq cq^\gamma (\rho^{-\alpha} \vee ((r+\rho)^\alpha - r^\alpha)^{-1}) \|\tilde{u}^{2q}\|_{L^1(I_{r+\rho}^\ominus \times B_{r+\rho})} \\ &\quad + cq \|\tilde{u}^{2q-1}\|_{L^1(I_{(r+\rho)/4}^\ominus \times B_{r+\rho})} \sup_{t \in I_{(r+\rho)/4}^\ominus} \widehat{\text{Tail}}_K(u(t), r + \frac{1}{2}\rho, r + \rho). \end{aligned} \tag{4-17}$$

We will now demonstrate how to perform the Moser iteration for positive exponents for nonlocal equations. Inequality (4-17) is the key estimate for the iteration scheme. The main difficulty compared to the classical local case is the treatment of the tail term.

Let us define $c_i = 2^{-(i+1)(d+\varepsilon)/\alpha} < 1$ for $\varepsilon > 0$ to be determined later and $i \in \mathbb{N}$. By Hölder’s and Young’s inequalities we have, for each $i \in \mathbb{N}$, the estimate

$$\begin{aligned} q \|\tilde{u}^{2q-1}\|_{L^1(I_{(r+\rho)/4}^\ominus \times B_{r+\rho})} \sup_{t \in I_{(r+\rho)/4}^\ominus} \widehat{\text{Tail}}_K(u(t), r + \frac{1}{2}\rho, r + \rho) \\ \leq (q(c_i\rho)^{-\alpha} \|\tilde{u}^{2q}\|_{L^1(I_{(r+\rho)/4}^\ominus \times B_{r+\rho})})^{\frac{2q-1}{2q}} \left(q^{\frac{1}{2q}} (c_i\rho)^{\alpha \frac{2q-1}{2q}} (r+\rho)^{\frac{d+\alpha}{2q}} \sup_{t \in I_{(r+\rho)/4}^\ominus} \widehat{\text{Tail}}_K(u(t), r + \frac{1}{2}\rho, r + \rho) \right) \\ \leq q(c_i\rho)^{-\alpha} \|\tilde{u}^{2q}\|_{L^1(I_{(r+\rho)/4}^\ominus \times B_{r+\rho})} + \left(q^{\frac{1}{2q}} (c_i\rho)^{\alpha \frac{2q-1}{2q}} (r+\rho)^{\frac{d+\alpha}{2q}} \sup_{t \in I_{(r+\rho)/4}^\ominus} \widehat{\text{Tail}}_K(u(t), r + \frac{1}{2}\rho, r + \rho) \right)^{2q}. \end{aligned}$$

Combining this estimate with (4-17) and taking both sides to the power $1/(2q)$ yields

$$\begin{aligned} \|\tilde{u}\|_{L^{2q\kappa}(I_r^\ominus \times B_r)} &\leq c^{\frac{1}{2q}} q^{\frac{\gamma}{2q}} c_i^{-\frac{\alpha}{2q}} (\rho^{-\frac{\alpha}{2q}} \vee ((r+\rho)^\alpha - r^\alpha)^{-\frac{1}{2q}}) \|\tilde{u}\|_{L^{2q}(I_{r+\rho}^\ominus \times B_{r+\rho})} \\ &\quad + c^{\frac{1}{2q}} q^{\frac{1}{2q}} (c_i\rho)^{\alpha \frac{2q-1}{2q}} (r+\rho)^{\frac{d+\alpha}{2q}} \sup_{t \in I_{(r+\rho)/4}^\ominus} \widehat{\text{Tail}}_K(u(t), r + \frac{1}{2}\rho, r + \rho) \\ &\leq c^{\frac{1}{2q}} q^{\frac{\gamma}{2q}} c_i^{-\frac{\alpha}{2q}} (\rho^{-\frac{\alpha}{2q}} \vee ((r+\rho)^\alpha - r^\alpha)^{-\frac{1}{2q}}) \\ &\quad \times \left(\|\tilde{u}\|_{L^{2q}(I_{r+\rho}^\ominus \times B_{r+\rho})} + (c_i\rho)^\alpha (r+\rho)^{\frac{d+\alpha}{2q}} \sup_{t \in I_{(r+\rho)/4}^\ominus} \widehat{\text{Tail}}_K(u(t), r + \frac{1}{2}\rho, r + \rho) \right). \end{aligned}$$

Recall that, by (2-11), we have the estimate

$$\widehat{\text{Tail}}_K(u(t), r + \frac{1}{2}\rho, r + \rho) \leq c\rho^{-\alpha} \left(\frac{r+\rho}{\rho} \right)^d \widehat{\text{Tail}}_{K,\alpha}(u(t), R).$$

Fix $q_0 \geq 1$ and $q_i = q_0\kappa^i$, and set $\rho_i = 2^{-i-1}R$ and $r_{i+1} = r_i - \rho_{i+1}$, $r_0 = R$. Note that $r_i \searrow \frac{1}{2}R$. For every $i \in \mathbb{N}$, using

$$(\rho_i^{-\frac{\alpha}{2q_{i-1}}} \vee ((r_i + \rho_i)^\alpha - r_i^\alpha)^{-\frac{1}{2q_{i-1}}}) \leq c^{\frac{1}{2q_{i-1}}} R^{-\frac{\alpha}{2q_{i-1}}} 2^{\frac{i+1}{q_{i-1}}},$$

we obtain

$$\begin{aligned} \|\tilde{u}\|_{L^{q_i}(I_{r_i}^\ominus \times B_{r_i})} &\leq c^{\frac{1}{2q_{i-1}}} q_{i-1}^{\frac{\gamma}{2q_{i-1}}} R^{-\frac{\alpha}{2q_{i-1}}} 2^{\frac{d+\varepsilon+2}{2q_{i-1}}(i+1)} \\ &\quad \times \left(\|\tilde{u}\|_{L^{2q_{i-1}}(I_{r_{i-1}}^\ominus \times B_{r_{i-1}})} + 2^{-(d+\varepsilon)(i+1)} \rho_i^\alpha R^{\frac{d+\alpha}{2q_{i-1}}} \sup_{t \in I_{(r_i+\rho_i)/4}^\ominus} \widehat{\text{Tail}}_K(u(t), r_i + \frac{1}{2}\rho_i, r_i + \rho_i) \right) \\ &\leq c^{\frac{1}{2q_{i-1}}} q_{i-1}^{\frac{\gamma}{2q_{i-1}}} R^{-\frac{\alpha}{2q_{i-1}}} 2^{\frac{d+\varepsilon+2}{2q_{i-1}}(i+1)} \\ &\quad \times \left(\|\tilde{u}\|_{L^{2q_{i-1}}(I_{r_{i-1}}^\ominus \times B_{r_{i-1}})} + R^{\frac{d+\alpha}{2q_{i-1}}} 2^{-(i+1)\varepsilon} \sup_{t \in I_{R/4}^\ominus} \widehat{\text{Tail}}_{K,\alpha}(u(t), R) \right). \end{aligned} \tag{4-18}$$

Consequently,

$$\begin{aligned} \sup_{I_{R/2}^\ominus \times B_{R/2}} \tilde{u} &\leq \left(\prod_{i=1}^\infty c^{\frac{1}{2q_{i-1}}} q_{i-1}^{\frac{\gamma}{2q_{i-1}}} R^{-\frac{\alpha}{2q_{i-1}}} 2^{\frac{d+\varepsilon+2}{2q_{i-1}}(i+1)} \right) \|\tilde{u}\|_{L^{2q_0}(I_R^\ominus \times B_R)} \\ &\quad + \left[\sum_{i=1}^\infty \left(\prod_{j=i}^\infty c^{\frac{1}{2q_{j-1}}} q_{j-1}^{\frac{\gamma}{2q_{j-1}}} R^{-\frac{\alpha}{2q_{j-1}}} 2^{\frac{d+\varepsilon+2}{2q_{j-1}}(j+1)} \right) R^{\frac{d+\alpha}{2q_{i-1}}} 2^{-(i+1)\varepsilon} \right] \sup_{t \in I_{R/4}^\ominus} \widehat{\text{Tail}}_{K,\alpha}(u(t), R). \end{aligned}$$

Note that $\sum_{i=0}^\infty \kappa^{-i} = (d + \alpha)/\alpha$ and also $\sum_{i=0}^\infty i/\kappa^i =: c_3 < \infty$. Therefore,

$$\begin{aligned} \prod_{i=1}^\infty (cq_{i-1})^{\frac{\gamma}{2q_{i-1}}} &\leq (cq_0)^{\frac{\gamma}{2q_0}} \sum_{i=0}^\infty \kappa^{-i} \kappa^{\frac{\gamma}{2q_0} \sum_{i=0}^\infty \frac{i}{\kappa^i}} \leq c(q_0, \kappa, \gamma) < \infty, \\ \prod_{i=1}^\infty 2^{\frac{d+\varepsilon+2}{2q_{i-1}}(i+1)} &\leq 2^{\frac{d+\varepsilon+2}{2q_0} \sum_{i=0}^\infty \frac{i+2}{\kappa^i}} \leq 2^{\frac{(d+\varepsilon+2)c_4}{2q_0}} < \infty, \\ \prod_{j=i}^\infty R^{-\frac{\alpha}{2q_{j-1}}} &= R^{-\frac{\alpha}{2q_{i-1}} \sum_{j=0}^\infty \kappa^{-j}} = R^{-\frac{d+\alpha}{2q_{i-1}}}. \end{aligned}$$

As a consequence,

$$\begin{aligned} \prod_{i=1}^\infty c^{\frac{1}{2q_{i-1}}} q_{i-1}^{\frac{\gamma}{2q_{i-1}}} R^{-\frac{\alpha}{2q_{i-1}}} 2^{\frac{d+\varepsilon+2}{2q_{i-1}}(i+1)} &\leq c(q_0, \kappa, d) R^{-\frac{d+\alpha}{2q_k}} 2^{\frac{d+\varepsilon+2}{2q_k} \sum_{i=0}^\infty \frac{i+k+2}{\kappa^i}} \\ &\leq c(q_0, \kappa, d) R^{-\frac{d+\alpha}{2q_0}} 2^{\frac{(d+\varepsilon+2)c_5}{2q_0}}, \\ \sum_{i=1}^\infty \left(\prod_{j=i}^\infty c^{\frac{1}{2q_{j-1}}} q_{j-1}^{\frac{\gamma}{2q_{j-1}}} R^{-\frac{\alpha}{2q_{j-1}}} 2^{\frac{d+\varepsilon+2}{2q_{j-1}}(j+1)} \right) R^{\frac{d+\alpha}{2q_{i-1}}} 2^{-(i+1)\varepsilon} &\leq c \sum_{i=1}^\infty 2^{\frac{(d+\varepsilon+2)c_5}{2q_{i-1}}(i+1)} 2^{-(i+1)\varepsilon} \\ &\leq c 2^{\frac{(d+\varepsilon+2)c_6}{2q_0}} \sum_{i=1}^\infty 2^{-(i+1)\varepsilon} \\ &\leq c(d, q_0, \kappa, \varepsilon), \end{aligned}$$

where we used that $(i + 1)/\kappa^{i-1}$ is bounded from above by some constant $c_6 = c_6(\kappa)$. Therefore, choosing $\varepsilon = 1$ and $q_0 = 1$, we deduce that, for some $c > 0$,

$$\sup_{I_{R/2}^\ominus \times B_{R/2}} \tilde{u} \leq c \left(\int_{I_R^\ominus} \int_{B_R} \tilde{u}^2(t, x) \, dx \, dt \right)^{\frac{1}{2}} + c \sup_{t \in I_{R/4}^\ominus} \widehat{\text{Tail}}_{K,\alpha}(u(t), R).$$

As a consequence, using the definition of \tilde{u} as well as the triangle inequality for the L^2 -norm, we deduce

$$\sup_{I_{R/2}^\ominus \times B_{R/2}} u \leq c \left(\int_{I_R^\ominus} \int_{B_R} u^2(t, x) \, dx \, dt \right)^{\frac{1}{2}} + c \sup_{t \in I_{R/4}^\ominus} \widehat{\text{Tail}}_{K,\alpha}(u(t), R) + cR^\alpha \|f\|_{L^\infty}.$$

This proves the desired result. □

5. Local tail estimate

In this section, local tail estimates for supersolutions to (PDE) and ($\widehat{\text{PDE}}$) (see Corollary 5.3) as well as the corresponding stationary equations (ell-PDE) and (ell- $\widehat{\text{PDE}}$) (see Corollary 5.4) are established. The main auxiliary results are Lemmas 5.1 and 5.2, whose proofs use similar ideas as in Lemmas 3.1 and 3.4. Central ingredients in the proof are the assumptions (UJS) and ($\widehat{\text{UJS}}$), which allow us to derive local tail estimates without having to assume a pointwise lower bound of the jumping kernel. They are applied in a similar way as in [Schulze 2019], where symmetric nonlocal operators are considered.

Lemma 5.1. *Assume that (K1_{loc}), (cutoff), and (UJS) hold for some $\theta \in [d/\alpha, \infty]$. Moreover, assume (Sob) if $\theta < \infty$. Then there exist $c_1, c_2 > 0$ such that, for every $0 < \rho \leq r \leq 1$, every nonnegative function $u \in V(B_{2r} | \mathbb{R}^d)$, and every $S > 0$ with $S \geq \sup_{B_{r+\rho}} u$, we have*

$$\text{Tail}_K(u, r, r + \rho) \leq c_1 \frac{1}{S \rho^d} \mathcal{E}(u, \tau^2(u - 2S)) + c_2 \left(\frac{r + \rho}{\rho} \right)^d \rho^{-\alpha} S,$$

where $B_{2r} \subset \Omega$ and $\tau = \tau_{r, \rho}$.

Proof. We define $w = u - 2S$. Note that, by definition, $w \in [-2S, -S]$ in $B_{r+\rho}$. We separate the proof into several steps.

Step 1: First, we claim that, for some $c > 0$, we have

$$\mathcal{E}_{B_{r+\rho}}^{K_s}(\tau w, \tau w) \leq \mathcal{E}_{B_{r+\rho}}^{K_s}(u, \tau^2 w) + c S^2 (r + \rho)^d \rho^{-\alpha}. \quad (5-1)$$

We compute

$$\begin{aligned} \mathcal{E}_{B_{r+\rho}}^{K_s}(u, \tau^2 w) &= \int_{B_{r+\rho}} \int_{B_{r+\rho}} (w(x) - w(y))(\tau^2 w(x) - \tau^2 w(y)) K_s(x, y) \, dy \, dx \\ &= \mathcal{E}_{B_{r+\rho}}^{K_s}(\tau w, \tau w) - \int_{B_{r+\rho}} \int_{B_{r+\rho}} w(x)w(y)(\tau(x) - \tau(y))^2 K_s(x, y) \, dy \, dx. \end{aligned}$$

We estimate, using (cutoff),

$$\int_{B_{r+\rho}} \int_{B_{r+\rho}} w(x)w(y)(\tau(x) - \tau(y))^2 K_s(x, y) \, dy \, dx \leq 4S^2 \mathcal{E}_{B_{r+\rho}}^{K_s}(\tau, \tau) \leq c_1 S^2 (r + \rho)^d \rho^{-\alpha}$$

for some $c_1 > 0$, which directly implies (5-1).

Step 2: Next, we claim that there exists $c > 0$ such that

$$-\mathcal{E}_{B_{r+\rho}}^{K_a}(u, \tau^2 w) \leq \frac{1}{2} \mathcal{E}_{B_{r+\rho}}^{K_s}(\tau w, \tau w) + c S^2 (r + \rho)^d \rho^{-\alpha}. \quad (5-2)$$

For the proof, we use the same arguments as in the proof of the Caccioppoli estimate:

$$\begin{aligned} -\mathcal{E}_{B_{r+\rho}}^{K_a}(u, \tau^2 w) &= \int_{B_{r+\rho}} \int_{B_{r+\rho}} (w(y) - w(x))(\tau^2 w(x) + \tau^2 w(y)) K_a(x, y) \, dy \, dx \\ &= \int_{B_{r+\rho}} \int_{B_{r+\rho}} (\tau w(y) - \tau w(x))(\tau w(y) + \tau w(x)) K_a(x, y) \, dy \, dx \\ &\quad + \int_{B_{r+\rho}} \int_{B_{r+\rho}} w(x)w(y)(\tau^2(x) - \tau^2(y)) K_a(x, y) \, dy \, dx \\ &=: J_1 + J_2. \end{aligned}$$

Using Hölder's and Young's inequalities as well as (K1_{loc}) and (2-8), we obtain, for every $\delta > 0$,

$$\begin{aligned} J_1 &\leq \delta \mathcal{E}_{B_{r+\rho}}^J(\tau w, \tau w) + c_2 \int_{B_{r+\rho}} \int_{B_{r+\rho}} (\tau w(y) + \tau w(x))^2 \frac{|K_a(x, y)|^2}{J(x, y)} dy dx \\ &\leq c\delta \mathcal{E}_{B_{r+\rho}}^{K_s}(\tau w, \tau w) + 2c_2 \int_{B_{r+\rho}} \tau^2 w^2(x) \left(\int_{B_{r+\rho}} \frac{|K_a(x, y)|^2}{J(x, y)} dy \right) dx \\ &\leq 2c\delta \mathcal{E}_{B_{r+\rho}}^{K_s}(\tau w, \tau w) + c_3 S^2 (r + \rho)^d \rho^{-\alpha} \end{aligned}$$

for $c_2, c_3 > 0$ depending on δ . Again, by Hölder's and Young's inequalities as well as (K1_{loc}), (cutoff), and (2-8), we estimate

$$\begin{aligned} J_2 &\leq \frac{1}{2} \int_{B_{r+\rho}} \int_{B_{r+\rho}} |w(x)| |w(y)| (\tau(y) - \tau(x))^2 J(x, y) dy dx \\ &\quad + \frac{1}{2} \int_{B_{r+\rho}} \int_{B_{r+\rho}} |w(x)| |w(y)| (\tau(y) + \tau(x))^2 \frac{|K_a(x, y)|^2}{J(x, y)} dy dx \\ &\leq 2S^2 \mathcal{E}_{B_{r+\rho}}^J(\tau, \tau) + 8S^2 \int_{B_{r+\rho}} \left(\tau^2(x) \int_{B_{r+\rho}} \frac{|K_a(x, y)|^2}{J(x, y)} dy \right) dx \\ &\leq c_4 S^2 \mathcal{E}_{B_{r+\rho}}^{K_s}(\tau, \tau) + c_4 S^2 (r + \rho)^d \rho^{-\alpha} \\ &\leq c_5 S^2 (r + \rho)^d \rho^{-\alpha} \end{aligned}$$

for $c_4, c_5 > 0$. From here, (5-2) directly follows.

Step 3: We claim that there exist constants $c, c' > 0$ such that

$$-\mathcal{E}_{(B_{r+\rho} \times B_{r+\rho})^c}(u, \tau^2 w) \leq c S^2 (r + \rho)^d \rho^{-\alpha} - c' S \rho^d \text{Tail}_K(u, r, r + \rho). \quad (5-3)$$

First, we rewrite the term on the left-hand side of the above line:

$$\begin{aligned} -\mathcal{E}_{(B_{r+\rho} \times B_{r+\rho})^c}(u, \tau^2 w) &= -2 \iint_{(B_{r+\rho} \times B_{r+\rho})^c} (u(x) - u(y)) \tau^2 w(x) K(x, y) dy dx \\ &= -2 \int_{B_{r+\rho}} \int_{B_{r+\rho}^c \cap \{u(y) \geq S\}} (u(y) - u(x)) \tau^2(x) (2S - u(x)) K(x, y) dy dx \\ &\quad + 2 \int_{B_{r+\rho}} \int_{B_{r+\rho}^c \cap \{u(y) \leq S\}} (u(x) - u(y)) \tau^2(x) (2S - u(x)) K(x, y) dy dx \\ &=: I_1 + I_2. \end{aligned} \quad (5-4)$$

For I_2 , we obtain

$$\begin{aligned} I_2 &\leq 4S \int_{B_{r+\rho}} \int_{B_{r+\rho}^c \cap \{u(y) \leq S\}} (u(x) - u(y))_+ \tau^2(x) K(x, y) dy dx \\ &\leq 8S^2 \int_{B_{r+\rho}} \int_{B_{r+\rho}^c} (\tau(x) - \tau(y))^2 K(x, y) dy dx \\ &\leq 8S^2 \int_{B_{r+\rho}} \Gamma^{K_s}(\tau, \tau)(x) dx \\ &\leq c_6 S^2 (r + \rho)^d \rho^{-\alpha} \end{aligned}$$

for some $c_6 > 0$, where we used (1-2), (cutoff), and that $K \geq 0$ and $u \geq 0$ globally. We treat I_1 in the following way (see [Schulze 2019]):

$$\begin{aligned} I_1 &\leq -2S \int_{B_{r+\rho}} \int_{B_{r+\rho}^c \cap \{u(y) \geq S\}} (u(y) - S)\tau^2(x)K(x, y) \, dy \, dx \\ &\leq -2S \int_{B_{r+\rho}} \int_{B_{r+\rho}^c} (u(y) - S)\tau^2(x)K(x, y) \, dy \, dx \\ &\leq -2S \int_{B_{r+\rho/4}} \int_{B_{r+\rho}^c} u(y)\tau^2(x)K(x, y) \, dy \, dx + 2S^2 \int_{B_{r+\rho}} \int_{B_{r+\rho}^c} (\tau(x) - \tau(y))^2 K(x, y) \, dy \, dx \\ &\leq -\frac{S}{8} \int_{B_{r+\rho/4}} \int_{B_{r+\rho}^c} u(y)K(x, y) \, dy \, dx + c_7 S^2 (r + \rho)^d \rho^{-\alpha} \end{aligned}$$

for some $c_7 > 0$, where we used that $u, K \geq 0$, $u \leq S$ in $B_{r+\rho}$, $\tau^2 \geq \frac{1}{16}$ in $B_{r+\rho/4}$, (1-2), and (cutoff). Finally, note that, due to (UJS),

$$\begin{aligned} \rho^d \text{Tail}_K(u, r, r + \rho) &= \rho^d \sup_{x \in B_r} \int_{B_{r+\rho}^c} u(y)K(x, y) \, dy \\ &\leq c_8 \sup_{x \in B_r} \int_{B_{r+\rho}^c} u(y) \left(\int_{B_{\rho/4}(x)} K(z, y) \, dz \right) dy \\ &\leq c_8 \int_{B_{r+\rho}^c} u(y) \left(\int_{B_{r+\rho/4}} K(x, y) \, dx \right) dy \\ &= c_8 \int_{B_{r+\rho/4}} \int_{B_{r+\rho}^c} u(y)K(x, y) \, dy \, dx \end{aligned} \tag{5-5}$$

for some $c_8 > 0$. Consequently,

$$I_1 \leq -c_9 S \rho^d \text{Tail}_K(u, r, r + \rho) + c_{10} S^2 (r + \rho)^d \rho^{-\alpha},$$

where $c_9, c_{10} > 0$ are constants.

Step 4: Now, we want to combine (5-1), (5-2), and (5-3). First, we observe that

$$\mathcal{E}_{B_{r+\rho}}^{K_s}(u, \tau^2 w) = \mathcal{E}(u, \tau^2 w) - \mathcal{E}_{(B_{r+\rho} \times B_{r+\rho})^c}(u, \tau^2 w) - \mathcal{E}_{B_{r+\rho}}^{K_a}(u, \tau^2 w).$$

Together, we obtain

$$\mathcal{E}_{B_{r+\rho}}^{K_s}(\tau w, \tau w) \leq \mathcal{E}(u, \tau^2 w) + c_{11} S^2 (r + \rho)^d \rho^{-\alpha} - c_{12} S \rho^d \text{Tail}_K(u, r, r + \rho) + \frac{1}{2} \mathcal{E}_{B_{r+\rho}}^{K_s}(\tau w, \tau w)$$

for $c_{11}, c_{12} > 0$. Since $L \geq 0$, we conclude

$$\text{Tail}_K(u, r, r + \rho) \leq c_{13} \frac{1}{S \rho^d} \mathcal{E}(u, \tau^2 w) + c_{14} S \left(\frac{r + \rho}{\rho} \right)^d \rho^{-\alpha},$$

where $c_{13}, c_{14} > 0$ are constants. This yields the desired result. □

Next, we prove a similar estimate for the dual form.

Lemma 5.2. *Assume that $(K1_{\text{glob}})$, (cutoff), and (\widehat{UJS}) hold for some $\theta \in [d/\alpha, \infty]$. Moreover, assume (Sob) if $\theta < \infty$. Then there exist $c_1, c_2 > 0$ such that, for every $0 < \rho \leq r \leq 1$, every nonnegative function $u \in V(B_{2r} | \mathbb{R}^d) \cap L^{2\theta'}(\mathbb{R}^d)$, and every $S \geq \sup_{B_{r+\rho}} u$, we have*

$$\widehat{\text{Tail}}_K(u, r, r + \rho) \leq c_1 \frac{1}{S\rho^d} \widehat{\mathcal{E}}(u, \tau^2(u - 2S)) + c_2 \left(\frac{r + \rho}{\rho}\right)^d \rho^{-\alpha} S,$$

where $B_{2r} \subset \Omega$, $\tau = \tau_{r,\rho}$.

Proof. As in the proof of Lemma 5.1, we define $w = u - 2S$ and observe that $w \in [-S, -2S]$ in $B_{r+\rho}$. The proof is separated into several steps.

Step 1: First, we recall from Step 1 in the proof of Lemma 5.1 that, for some $c > 0$, we have

$$\mathcal{E}_{B_{r+\rho}}^{K_s}(\tau w, \tau w) \leq \mathcal{E}_{B_{r+\rho}}^{K_s}(u, \tau^2 w) + cS^2(r + \rho)^d \rho^{-\alpha}. \tag{5-6}$$

Step 2: In analogy with Step 2 in the proof of Lemma 5.1, we claim that, for some $c > 0$,

$$-\widehat{\mathcal{E}}^{K_a}(u, \tau^2 w) \leq \frac{1}{2} \mathcal{E}_{B_{r+\rho}}^{K_s}(\tau w, \tau w) + cS^2(r + \rho)^d \rho^\alpha. \tag{5-7}$$

To see this, we estimate

$$\begin{aligned} -\widehat{\mathcal{E}}^{K_a}(u, \tau^2 w) &= \int_{B_{r+\rho}} \int_{B_{r+\rho}} (\tau^2 w(x) - \tau^2 w(y))(w(x) + w(y)) K_a(x, y) \, dy \, dx \\ &\quad + 4S \int_{B_{r+\rho}} \int_{B_{r+\rho}} (\tau^2 w(x) - \tau^2 w(y)) K_a(x, y) \, dy \, dx \\ &:= I_1 + I_2. \end{aligned}$$

For I_1 , we compute

$$\begin{aligned} I_1 &= \int_{B_{r+\rho}} \int_{B_{r+\rho}} (\tau^2 w^2(x) - \tau^2 w^2(y)) K_a(x, y) \, dy \, dx \\ &\quad + \int_{B_{r+\rho}} \int_{B_{r+\rho}} w(x)w(y)(\tau^2(x) - \tau^2(y)) K_a(x, y) \, dy \, dx, \end{aligned}$$

and from the same arguments as in the proof of Step 2 in the proof of Lemma 5.1, we conclude

$$I_1 \leq \frac{1}{4} \mathcal{E}_{B_{r+\rho}}^{K_s}(\tau w, \tau w) + cS^2(r + \rho)^d \rho^{-\alpha},$$

using $(K1_{\text{glob}})$ and (cutoff). For I_2 , we observe

$$\begin{aligned} I_2 &= 2S \int_{B_{r+\rho}} \int_{B_{r+\rho}} (\tau w(x) - \tau w(y))(\tau(x) + \tau(y)) K_a(x, y) \, dy \, dx \\ &\quad + 2S \int_{B_{r+\rho}} \int_{B_{r+\rho}} (\tau w(x) + \tau w(y))(\tau(x) - \tau(y)) K_a(x, y) \, dy \, dx \\ &=: I_{2,1} + I_{2,2}. \end{aligned}$$

Now, using $(K1_{\text{glob}})$, (2-8), and (cutoff),

$$\begin{aligned} I_{2,1} &\leq \frac{1}{8} \mathcal{E}_{B_{r+\rho}}^{K_s}(\tau w, \tau w) + cS^2 \int_{B_{r+\rho}} \tau^2(x) \left(\int_{B_{r+\rho}} \frac{|K_a(x, y)|^2}{J(x, y)} dy \right) dx \\ &\leq \frac{1}{8} \mathcal{E}_{B_{r+\rho}}^{K_s}(\tau w, \tau w) + cS^2 \mathcal{E}_{B_{r+\rho}}^{K_s}(\tau, \tau) + cS^2 \rho^{-\alpha} \int_{B_{r+\rho}} \tau^2(x) dx \\ &\leq \frac{1}{8} \mathcal{E}_{B_{r+\rho}}^{K_s}(\tau w, \tau w) + cS^2(r + \rho)^d \rho^{-\alpha}, \end{aligned}$$

and, again using $(K1_{\text{glob}})$, (2-8), and (cutoff),

$$\begin{aligned} I_{2,2} &\leq cS^2 \mathcal{E}_{B_{r+\rho}}^{K_s}(\tau, \tau) + \int_{B_{r+\rho}} \tau^2 w^2(x) \left(\int_{B_{r+\rho}} \frac{|K_a(x, y)|^2}{K_s(x, y)} dy \right) dx \\ &\leq cS^2(r + \rho)^d \rho^{-\alpha} + \frac{1}{8} \mathcal{E}_{B_{r+\rho}}^{K_s}(\tau w, \tau w) + c\rho^{-\alpha} \int_{B_{r+\rho}} \tau^2 w^2(x) dx \\ &\leq \frac{1}{8} \mathcal{E}_{B_{r+\rho}}^{K_s}(\tau w, \tau w) + cS^2(r + \rho)^d \rho^{-\alpha}. \end{aligned}$$

Altogether, we have proved (5-7).

Step 3: Moreover, we claim that, for some constants $c, c' > 0$,

$$-\widehat{\mathcal{E}}_{(B_{r+\rho} \times B_{r+\rho})^c}(u, \tau^2 w) \leq cS^2(r + \rho)^d \rho^{-\alpha} - c'S\rho^d \widehat{\text{Tail}}_K(u, r, r + \rho). \quad (5-8)$$

First, we write the decomposition

$$\begin{aligned} -\mathcal{E}_{(B_{r+\rho} \times B_{r+\rho})^c}(\tau^2 w, u) &= -2 \int_{B_{r+\rho}} \int_{B_{r+\rho}^c} \tau^2 w(x) u(x) K(x, y) dy dx + 2 \int_{B_{r+\rho}^c} \int_{B_{r+\rho}} \tau^2 w(y) u(x) K(x, y) dy dx \\ &=: J_1 + J_2. \end{aligned}$$

For J_1 , using the definition of w , nonnegativity of u , and (1-2), we compute

$$\begin{aligned} J_1 &= 2 \int_{B_{r+\rho}} \int_{B_{r+\rho}^c} \tau^2(x) (2S - u(x)) u(x) K(x, y) dy dx \\ &\leq 4S^2 \int_{B_{r+\rho}} \int_{B_{r+\rho}^c} (\tau(x) - \tau(y))^2 K_s(x, y) dy dx \\ &\leq cS^2(r + \rho)^2 \rho^{-\alpha}. \end{aligned}$$

For J_2 , using that $\tau^2 \geq \frac{1}{16}$ in $B_{r+\rho/4}$, we observe

$$\begin{aligned} J_2 &= 2 \int_{B_{r+\rho}^c} \int_{B_{r+\rho}} \tau^2(y) (u(y) - 2S) u(x) K(x, y) dy dx \\ &\leq -2S \int_{B_{r+\rho}} \int_{B_{r+\rho}^c} \tau^2(y) u(x) K(x, y) dx dy \\ &\leq -\frac{S}{8} \int_{B_{r+\rho/4}} \int_{B_{r+\rho}^c} u(x) K(x, y) dx dy. \end{aligned}$$

Finally, using $(\widehat{\text{UJS}})$ and the same argument as in (5-5), we can prove that

$$\rho^d \widehat{\text{Tail}}_K(u, r, r + \rho) \leq c \int_{B_{r+\rho/4}} \int_{B_{r+\rho}^c} u(x) K(x, y) dx dy.$$

Altogether, we have established (5-8), as desired.

Step 4: Combining (5-6), (5-7), and (5-8), we obtain

$$\begin{aligned} \mathcal{E}_{B_{r+\rho}}^{K_s}(\tau w, \tau w) &\leq \mathcal{E}_{B_{r+\rho}}^{K_s}(u, \tau^2 w) + cS^2(r + \rho)^d \rho^{-\alpha} \\ &= \widehat{\mathcal{E}}(u, \tau^2 w) - \widehat{\mathcal{E}}_{B_{r+\rho}}^{K_a}(u, \tau^2 w) - \widehat{\mathcal{E}}_{(B_{r+\rho} \times B_{r+\rho})^c}(u, \tau^2 w) + cS^2(r + \rho)^d \rho^{-\alpha} \\ &\leq \widehat{\mathcal{E}}(u, \tau^2 w) + cS^2(r + \rho)^d \rho^{-\alpha} + \frac{1}{2} \mathcal{E}_{B_{r+\rho}}^{K_s}(\tau w, \tau w) - cS\rho^d \widehat{\text{Tail}}_K(u, r, r + \rho). \end{aligned}$$

Consequently,

$$\widehat{\text{Tail}}_K(u, r, r + \rho) \leq c \frac{1}{S\rho^d} \widehat{\mathcal{E}}(u, \tau^2 w) + c \left(\frac{r + \rho}{\rho} \right)^d \rho^{-\alpha} S,$$

as desired. \square

Lemma 5.1 can be used to bound $\text{Tail}_K(u, r, r + \rho)$ from above by the supremum of u . First, we provide such an estimate for weak supersolutions to the stationary equations (ell-PDE) and (ell-PDE), which is a direct corollary of Lemma 5.1 applied with $S = \sup_{B_{r+\rho}} u$.

Corollary 5.3. *Assume that (cutoff) holds.*

- (i) *Assume $(K1_{\text{loc}})$ and (UJS) hold for some $\theta \in [d/\alpha, \infty]$. Moreover, assume (Sob) if $\theta < \infty$. Then there exists $c > 0$ such that, for every $0 < \rho \leq r \leq 1$ and every nonnegative, weak supersolution u to (ell-PDE) in B_{2r} , we have*

$$\text{Tail}_K(u, r, r + \rho) \leq c \left(\frac{r + \rho}{\rho} \right)^d \left(\rho^{-\alpha} \sup_{B_{r+\rho}} u + \|f\|_{L^\infty} \right),$$

where $B_{2r} \subset \Omega$.

- (ii) *Assume $(K1_{\text{glob}})$ and $(\widehat{\text{UJS}})$ holds for some $\theta \in [d/\alpha, \infty]$. Moreover, assume (Sob) if $\theta < \infty$. Then there exists $c > 0$ such that, for every $0 < \rho \leq r \leq 1$ and every nonnegative, weak subsolution u to (ell-PDE) in B_{2r} , we have*

$$\widehat{\text{Tail}}_K(u, r, r + \rho) \leq c \left(\frac{r + \rho}{\rho} \right)^d \left(\rho^{-\alpha} \sup_{B_{r+\rho}} u + \|f\|_{L^\infty} \right),$$

where $B_{2r} \subset \Omega$.

One can also deduce an estimate for the L^1 -parabolic tail,

$$\int_{I_{r/2}^\ominus} \text{Tail}_K(u(t), r, r + \rho) dt,$$

for supersolutions to (PDE) and (PDE) from Lemma 5.1.

Corollary 5.4. *Assume that (cutoff) holds.*

- (i) *Assume (K1_{loc}) and (UJS) hold for some $\theta \in [d/\alpha, \infty]$. Moreover, assume (Sob) if $\theta < \infty$. Then there exists $c > 0$ such that, for every $0 < \rho \leq r \leq 1$ and every nonnegative, weak supersolution u to (PDE) in $I_r^\ominus(t_0) \times B_{2r}$, we have*

$$\int_{I_{r/2}^\ominus} \text{Tail}_K(u(t), r, r + \rho) dt \leq c \left(\frac{r + \rho}{\rho}\right)^d \left(\left(\frac{r + \rho}{\rho}\right)^{\alpha \vee 1} \sup_{I_{(r+\rho)/2}^\ominus \times B_{r+\rho}} u + (r + \rho)^\alpha \|f\|_{L^\infty}\right),$$

where $B_{2r} \subset \Omega$.

- (ii) *Assume (K1_{glob}) and ($\widehat{\text{UJS}}$) hold for some $\theta \in [d/\alpha, \infty]$. Moreover, assume (Sob) if $\theta < \infty$. Then there exists $c > 0$ such that, for every $0 < \rho \leq r \leq 1$ and every nonnegative, weak supersolution u to ($\widehat{\text{PDE}}$) in $I_r^\ominus(t_0) \times B_{2r}$, we have*

$$\int_{I_{r/2}^\ominus} \widehat{\text{Tail}}_K(u(t), r, r + \rho) dt \leq c \left(\frac{r + \rho}{\rho}\right)^d \left(\left(\frac{r + \rho}{\rho}\right)^{\alpha \vee 1} \sup_{I_{(r+\rho)/2}^\ominus \times B_{r+\rho}} u + (r + \rho)^\alpha \|f\|_{L^\infty}\right),$$

where $B_{2r} \subset \Omega$.

Proof. We only explain the proof of (i). The proof of (ii) works in the same way, but relies on Lemma 5.2 instead of Lemma 5.1. We write $S = \sup_{I_{(r+\rho)/2}^\ominus \times B_{r+\rho}} u$ and define $w = u - 2S$. We also observe that $\partial_t(w^2) = 2w\partial_t u$. From Lemma 5.1 and the fact that u is a supersolution to (PDE), we deduce

$$\begin{aligned} \frac{c}{2S\rho^d} \int_{B_{r+\rho}} \tau^2(x)\partial_t(w^2)(t, x) dx + \text{Tail}_K(u(t), r, r + \rho) & \\ & \leq c \frac{1}{S\rho^d} [(\partial_t u(t), \tau^2 w(t)) + \mathcal{E}(u(t), \tau^2 w(t))] + cS \left(\frac{r + \rho}{\rho}\right)^d \rho^{-\alpha} \\ & \leq c \frac{1}{S\rho^d} (f(t), \tau^2 w(t)) + cS \left(\frac{r + \rho}{\rho}\right)^d \rho^{-\alpha} \\ & \leq c \left(\frac{r + \rho}{\rho}\right)^d (\|f\|_{L^\infty} + S\rho^{-\alpha}), \end{aligned}$$

where $c > 0$ is the constant from Lemma 5.1 and we tested the equation with $\tau^2 w$, where $\tau = \tau_{r,\rho}$. Let $\chi \in C^1(\mathbb{R})$ be a nonnegative function with

$$\chi(t_0 - (\frac{1}{2}(r + \rho))^\alpha) = 0, \quad \chi \equiv 1 \text{ in } I_{r/2}^\ominus, \quad \|\chi\|_\infty \leq 1, \quad \|\chi'\|_\infty \leq 8((r + \rho)^\alpha - r^\alpha)^{-1}.$$

Multiplying by χ^2 and integrating over $(t_0 - (\frac{1}{2}(r + \rho))^\alpha, t)$ for some arbitrary $t \in I_{r/2}^\ominus$, we obtain

$$\begin{aligned} \frac{c}{2S\rho^d} \int_{B_{r+\rho}} \chi^2(t)\tau^2(x)w^2(t, x) dx + \int_{t_0 - ((r+\rho)/2)^\alpha}^t \chi^2(s) \text{Tail}_K(u(s), r, r + \rho) ds & \\ \leq c_1 \int_{t_0 - ((r+\rho)/2)^\alpha}^t \chi^2(s) S \left(\frac{r + \rho}{\rho}\right)^d \rho^{-\alpha} ds + c_1 \left(\frac{r + \rho}{\rho}\right)^d (r + \rho)^\alpha \|f\|_{L^\infty} & \\ + c_1 \int_{t_0 - ((r+\rho)/2)^\alpha}^t \frac{1}{S\rho^d} \chi(s)|\chi'(s)| \int_{B_{r+\rho}} \tau^2(x)w^2(s, x) dx ds, & \end{aligned}$$

where $c_1 > 0$ is a constant. Consequently, using that $w^2 \leq 4S^2$,

$$\begin{aligned} \sup_{t \in I_{r/2}^\ominus} \frac{c}{2S\rho^d} \int_{B_r} w^2(t, x) \, dx + \int_{I_{r/2}^\ominus} \text{Tail}_K(u(s), r, r + \rho) \, ds \\ \leq c_2 \left(\frac{r + \rho}{\rho}\right)^{d+(\alpha \vee 1)} \sup_{I_{(r+\rho)/2}^\ominus \times B_{r+\rho}} u + c_2 \left(\frac{r + \rho}{\rho}\right)^d (r + \rho)^\alpha \|f\|_{L^\infty}, \end{aligned}$$

where $c_2 > 0$ and we used that, for some $c > 0$,

$$((r + \rho)^\alpha - r^\alpha)^{-1} \leq c\rho^{-(\alpha \vee 1)}(r + \rho)^{(\alpha \vee 1) - \alpha}.$$

This concludes the proof. □

6. Harnack inequalities

The goal of this section is to complete the proofs of our main results: Theorem 1.1 and Theorem 1.4. In Section 6.1, we give improved versions of the local boundedness estimates from Sections 3 and 4, which do not involve tail terms. These results make use of the tail estimates obtained in Corollary 5.3 and are the key ingredients in the proof of Theorem 1.4. In Section 6.2 we combine local boundedness estimates with the weak Harnack inequalities from [Kassmann and Weidner 2022] and obtain our main results.

We point out that the proof of Theorem 1.1 does not rely on the tail estimates from Section 5. It is an open question — even in the symmetric case — how to derive a parabolic Harnack inequality involving only local quantities from suitable tail estimates, as one does in the stationary case. Section 6.3 is dedicated to this issue.

6.1. Local boundedness without tail terms. We obtain local L^∞ - L^p -estimates for solutions to (ell-PDE) and (ell-PDE) (see Theorem 6.2). In comparison with Theorem 3.6, the estimates only contain purely local quantities. The underlying procedure works exactly as for symmetric forms. However, note that we need to redo the iteration in Theorem 3.6 in order to prove Theorem 6.1 since the quantities Tail_K and $\text{Tail}_{K,\alpha}$ are in general not comparable.

The following theorem is the key result on our path towards L^∞ - L^p -estimates for nonnegative solutions to (ell-PDE) and (ell-PDE) since it no longer involves nonlocal quantities.

Theorem 6.1. *Assume that (cutoff) and (Sob) hold.*

- (i) *Assume that $(K1)_{\text{loc}}$ and (UJS) hold for some $\theta \in [d/\alpha, \infty]$. Then, for every $\delta \in (0, 1]$, there exists $c > 0$ such that, for every $0 < R \leq 1$ and every nonnegative, weak solution u to (ell-PDE) in $B_{2R} \subset \Omega$, we have*

$$\sup_{B_{R/2}} u \leq c \left(\int_{B_R} u^2(x) \, dx \right)^{1/2} + \delta \sup_{B_R} u + cR^\alpha \|f\|_{L^\infty}. \tag{6-1}$$

- (ii) *Assume that $(K1)_{\text{glob}}$ and $(\widehat{\text{UJS}})$ hold for some $\theta \in (d/\alpha, \infty]$. Then, for every $\delta \in (0, 1]$, there exists $c > 0$ such that, for every $0 < R \leq 1$ and every nonnegative, weak solution u to (ell-PDE) in B_{2R} , estimate (6-1) holds.*

We present two proofs of this theorem based on the De Giorgi iteration and the Moser iteration. Both proofs rely on a combination of the iteration schemes established in Sections 3 and 4 and the tail estimate from Corollary 5.4.

Proof of Theorem 6.1 (based on De Giorgi iteration). The proof of (i) is analogous to the proof of Theorem 3.6 (i). We define $(l_i)_i, (\rho_i)_i, (r_i)_i, (w_i)_i$ in the same way. Moreover, we set $A_i = \|w_i\|_{L^1(B_{r_i})}$. Note that

$$\left(\frac{r_i + \rho_i}{\rho_i}\right)^d = \left(\frac{\left(1 + \left(\frac{1}{2}\right)^i\right) + \left(\frac{1}{2}\right)^i}{\left(\frac{1}{2}\right)^{i+1}}\right)^d \leq 2^{(i+2)d}.$$

Consequently, Corollary 5.3 (i) — applied with $r = r_i + \frac{1}{2}\rho_i$ and $\rho = \frac{1}{2}\rho_i$ — yields

$$\text{Tail}_K(u, r_i + \frac{1}{2}\rho_i, r_i + \rho_i) \leq c_1 2^{i(d+2)} R^{-\alpha} \left(\sup_{B_R} u + R^\alpha \|f\|_{L^\infty}\right) \tag{6-2}$$

for some $c_1 > 0$. Moreover, by following the arguments in the proof of Theorem 3.6 (i), we derive the following analog of (3-16):

$$A_i \leq c_2 \frac{1}{(l_i - l_{i-1})^{2/\kappa'}} \left(\sigma(r_i, \rho_i) + \frac{\text{Tail}_K(u, r_i + \frac{1}{2}\rho_i, r_i + \rho_i) + \|f\|_{L^\infty}}{l_i - l_{i-1}}\right) A_{i-1}^{1+1/\kappa'} \tag{6-3}$$

for some $c_2 > 0$, where we can choose $\kappa = d/(d - \alpha)$ using that u is a subsolution to the stationary equation (ell-PDE) in (3-13). We combine (6-2) and (6-3) and obtain

$$A_i \leq \frac{c_3}{R^\alpha M^{2/\kappa'}} 2^{\gamma i} \left(1 + \frac{\sup_{B_R} u + R^\alpha \|f\|_{L^\infty}}{M}\right) A_{i-1}^{1+1/\kappa'},$$

where $c_3 > 0$ and $\gamma > 1$ are constants. We proceed as in the proof of Theorem 3.6 (i) and choose

$$M := \delta \left(\sup_{B_R} u + \|f\|_{L^\infty}\right) + C^{\kappa^2/2} c_3^{\kappa'/2} \delta^{-\kappa'/2} R^{-\alpha\kappa'/2} A_0^{1/2},$$

where $C := 2^\gamma > 1$ and conclude

$$A_0 \leq \left(\frac{c_3}{\delta R^\alpha M^{2/\kappa'}}\right)^{-\kappa'} C^{-\kappa'^2},$$

and therefore we obtain from Lemma 7.1 in [Giusti 2003]

$$\sup_{B_{R/2}} u \leq M = \delta \left(\sup_{B_R} u + R^\alpha \|f\|_{L^\infty}\right) + c_3 \delta^{-\kappa'/2} \left(\int_{B_R} u^2(x) dx\right)^{1/2}$$

for some $c_3 > 0$, as desired.

In order to prove (ii), we follow the arguments in the proof of Theorem 3.6 (ii) and derive the following analog of (6-3):

$$A_i \leq c_4 \frac{R^{d(1/\kappa' - \mu)}}{(l_i - l_{i-1})^{2\mu}} \left(\sigma(r_i, \rho_i) \left(1 + \left(\frac{l_i}{l_i - l_{i-1}}\right)^2\right) + \frac{\text{Tail}_K(u, r_i + \frac{1}{2}\rho_i, r_i + \rho_i) + \|f\|_{L^\infty}}{l_i - l_{i-1}}\right) A_{i-1}^{1+\mu}$$

for some $c_4 > 0$, where $\mu = 1/\kappa' - 1/\theta$ and $\kappa = d/(d - \alpha)$. As before, by Corollary 5.3 (ii) — applied with $r = r_i + \frac{1}{2}\rho_i$ and $\rho = \frac{1}{2}\rho_i$ — we prove

$$\widehat{\text{Tail}}_K(u, r_i + \frac{1}{2}\rho_i, r_i + \rho_i) \leq c_1 2^{i(d+2)} R^{-\alpha} \left(\sup_{B_R} u + R^\alpha \|f\|_{L^\infty} \right). \tag{6-4}$$

By combining (6-4) with the previous estimate, we deduce

$$A_i \leq \frac{c_5}{\delta R^{\mu d} M^{2\mu}} C^{\gamma i} A_{i-1}^{1+\mu}$$

for some $c_5 > 0$ and $\gamma > 1$. From here, the desired result follows by the same arguments as in the proof of (i). \square

Proof of Theorem 6.1 (based on Moser iteration). We explain how to prove (ii). The proof of (i) follows exactly the same arguments. Our proof is based on the Moser iteration and works in a similar way to the proof of Theorem 4.8. Let us define $(\rho_i)_i$, $(r_i)_i$, and $(q_i)_i$ in the same way, but set $\kappa = d/(d - \alpha)$.

Note that by following the arguments of the proof of Theorem 4.8, but using that u is a subsolution to the stationary equation in (4-17), we can derive the following analog of (4-18):

$$\begin{aligned} \|\tilde{u}\|_{L^{q_i}(B_{r_i})} &\leq c \frac{1}{2^{2q_i-1}} q_{i-1}^{\frac{\gamma}{2q_i-1}} R^{-\frac{\alpha}{2q_i-1}} 2^{\frac{d+\varepsilon+2}{2q_i-1}(i+1)} \\ &\quad \times \left(\|\tilde{u}\|_{L^{2q_{i-1}}(B_{r_{i-1}})} + 2^{-(d+\varepsilon+\alpha)(i+1)} R^{\alpha + \frac{d}{2q_{i-1}}} \widehat{\text{Tail}}_K(u, r_i + \frac{1}{2}\rho_i, r_i + \rho_i) \right). \end{aligned}$$

By combining this estimate with (6-4), we obtain

$$\begin{aligned} \|\tilde{u}\|_{L^{q_i}(B_{r_i})} &\leq c \frac{1}{2^{2q_i-1}} q_{i-1}^{\frac{\gamma}{2q_i-1}} R^{-\frac{\alpha}{2q_i-1}} 2^{\frac{d+\varepsilon+2}{2q_i-1}(i+1)} \\ &\quad \times \left(\|\tilde{u}\|_{L^{2q_{i-1}}(B_{r_{i-1}})} + R^{\frac{d}{2q_{i-1}}} 2^{-(i+1)(\varepsilon+\alpha-2)} \left(\sup_{B_R} u + R^\alpha \|f\|_{L^\infty} \right) \right). \end{aligned}$$

From here, the proof follows in a similar manner to the proof of Theorem 4.8. First, we observe that

$$\begin{aligned} \sup_{B_{R/2}} \tilde{u} &\leq \left(\prod_{i=1}^{\infty} c \frac{1}{2^{2q_i-1}} q_{i-1}^{\frac{\gamma}{2q_i-1}} R^{-\frac{\alpha}{2q_i-1}} 2^{\frac{d+\varepsilon+2}{2q_i-1}(i+1)} \right) \|\tilde{u}\|_{L^{2q_0}(B_R)} \\ &\quad + \left[\sum_{i=1}^{\infty} \left(\prod_{j=i}^{\infty} c \frac{1}{2^{2q_j-1}} q_{j-1}^{\frac{\gamma}{2q_j-1}} R^{-\frac{\alpha}{2q_j-1}} 2^{\frac{d+\varepsilon+2}{2q_j-1}(j+1)} \right) R^{\frac{d}{2q_{i-1}}} 2^{-(i+1)(\varepsilon+\alpha-2)} \right] \left(\sup_{B_R} u + R^\alpha \|f\|_{L^\infty} \right). \end{aligned}$$

Moreover, by similar arguments as in the proof of Theorem 4.8,

$$\sum_{i=1}^{\infty} \left(\prod_{j=i}^{\infty} c \frac{1}{2^{2q_j-1}} q_{j-1}^{\frac{\gamma}{2q_j-1}} R^{-\frac{\alpha}{2q_j-1}} 2^{\frac{d+\varepsilon+2}{2q_j-1}(j+1)} \right) R^{\frac{d}{2q_{i-1}}} 2^{-(i+1)(\varepsilon+\alpha-2)} \leq c \sum_{i=1}^{\infty} \frac{2^{\frac{(d+\varepsilon+2)c_5}{2q_0}(i+1)}}{2^{(i+1)(\varepsilon+\alpha-2)}}$$

using that

$$\sum_{i=0}^{\infty} \kappa^{-i} = \frac{d}{\alpha} \quad \text{and} \quad \sum_{i=0}^{\infty} \frac{i}{\kappa^i} < \infty,$$

where $\kappa = d/(d - \alpha)$.

Now, choose $\varepsilon \geq 1$ large enough that

$$\sum_{i=1}^{\infty} 2^{-(i+1)\frac{\varepsilon+\alpha-2}{2}} \leq \frac{\delta}{2c}.$$

Then, let us choose $q_0 \geq 1$ large enough that

$$\frac{(d + \varepsilon + 2)c_5}{2q_0} \leq \frac{\varepsilon + \alpha - 2}{2}.$$

In that case,

$$c \sum_{i=1}^{\infty} 2^{\frac{(d+\varepsilon+2)c_5}{2q_0}(i+1)} 2^{-(i+1)(\varepsilon+\alpha-2)} \leq c \sum_{i=k+1}^{\infty} 2^{-(i+1)\frac{(\varepsilon+\alpha-2)}{2}} \leq \frac{1}{2}\delta.$$

Therefore,

$$\sup_{B_{R/2}} \tilde{u} \leq c \left(\int_{B_R} \tilde{u}^{2q_0}(x) \, dx \right)^{\frac{1}{2q_0}} + \frac{1}{2}\delta \left(\sup_{B_R} u + R^\alpha \|f\|_{L^\infty} \right).$$

As a consequence, using the definition of \tilde{u} as well as the triangle inequality for the L^{2q_0} -norm, we deduce

$$\sup_{B_{R/2}} u \leq c \left(\int_{B_R} u^{2q_0}(x) \, dx \right)^{\frac{1}{2q_0}} + \frac{1}{2}\delta \sup_{B_R} u + cR^\alpha \|f\|_{L^\infty}.$$

It remains to prove the desired estimate (6-1) in the case $q_0 > 1$. This follows from Young’s inequality:

$$\left(\int_{B_R} u^{2q_0}(x) \, dx \right)^{\frac{1}{2q_0}} \leq \sup_{B_R} u^{\frac{2q_0-2}{2q_0}} \left(\int_{B_R} u^2(x) \, dx \right)^{\frac{1}{2q_0}} \leq \frac{\delta}{2c} \sup_{B_R} u + c \left(\int_{B_R} u^2(x) \, dx \right)^{\frac{1}{2}}. \quad \square$$

By a standard iteration argument one can deduce local boundedness of nonnegative solutions to (ell-PDE) and (ell-PDE) from Theorem 6.1.

Theorem 6.2. *Assume that (cutoff) and (\mathcal{E}_\geq) hold.*

- (i) *Assume that $(K1_{loc})$ and (UJS) hold for some $\theta \in [d/\alpha, \infty]$. Then there exists $c > 0$ such that, for every $0 < R \leq 1$, every $p \in (0, 2]$, and every nonnegative, weak solution u to (ell-PDE) in B_{2R} , we have*

$$\sup_{B_{R/4}} u \leq c \left(\int_{B_{R/2}} u^p(x) \, dx \right)^{\frac{1}{p}} + cR^\alpha \|f\|_{L^\infty}, \tag{6-5}$$

where $B_{2R} \subset \Omega$.

- (ii) *Assume that $(K1_{glob})$ and (\widehat{UJS}) hold for some $\theta \in (d/\alpha, \infty]$. Then there exists $c > 0$ such that, for every $0 < R \leq 1$, every $p \in (0, 2]$, and every nonnegative, weak solution u to (ell-PDE) in B_{2R} , estimate (6-5) holds.*

Proof. We restrict ourselves to proving (i). The proof of (ii) follows in the same way. The proof works as in [Di Castro et al. 2014, pp. 1828-1829]. Let us point out that this proof crucially relies on local boundedness of u , i.e.,

$$\sup_{B_{R/2}} u < \infty, \tag{6-6}$$

which follows from Theorem 3.6 and Theorem 4.8, since $\text{Tail}_{K,\alpha}(u, R)$ and $\widehat{\text{Tail}}_{K,\alpha}(u, R)$ are finite under the assumptions of this theorem due to Lemma 2.13 (i) and Lemma 2.13 (ii), respectively. Let $\frac{1}{4} \leq t < s \leq \frac{1}{2}$. We conclude from Theorem 6.1 and a classical covering argument

$$\sup_{B_{tR}} u \leq c_1(s-t)^{-d/2} \left(\int_{B_{sR}} u^2(x) dx \right)^{1/2} + c_2 R^\alpha \|f\|_{L^\infty} + c_2 \delta \sup_{B_{sR}} u,$$

where $c_1, c_2 > 0$ are constants. By Young’s inequality (applied with $2/p, 2/(2-p) \geq 1$),

$$\begin{aligned} \sup_{B_{tR}} u &\leq c_1(s-t)^{-d/2} \sup_{B_{sR}} u^{(2-p)/2} \left(\int_{B_{sR}} u^p(x) dx \right)^{1/2} + c_2 \delta \sup_{B_{sR}} u + c_2 R^\alpha \|f\|_{L^\infty} \\ &\leq (c_2 \delta + \frac{1}{4}) \sup_{B_{sR}} u + c_3(s-t)^{-d/p} \left(\int_{B_{sR}} u^p(x) dx \right)^{1/p} + c_2 R^\alpha \|f\|_{L^\infty} \end{aligned}$$

for some $c_3 > 0$. By choosing $\delta = 1/(4c_2)$, we obtain

$$\sup_{B_{tR}} u \leq \frac{1}{2} \sup_{B_{sR}} u + c_4(s-t)^{-d/p} \left(\int_{B_{R/2}} u^p(x) dx \right)^{1/p} + c_4 R^\alpha \|f\|_{L^\infty}$$

for $c_4 > 0$, and the result follows from the application of Lemma 1.1 in [Giaquinta and Giusti 1982] using (6-6). □

6.2. Proofs of main results. In this section we provide the proofs of our main results: Theorem 1.1 and Theorem 1.4. Let us recall the following theorem from [Kassmann and Weidner 2022].

Theorem 6.3 (weak Harnack inequality). *Assume (K2), (cutoff), (Poinc), and (Sob).*

- (i) *Assume that (K1_{loc}) holds for some $\theta \in [d/\alpha, \infty]$. Then there is $c > 0$ such that, for every $0 < R \leq 1$ and every nonnegative, weak supersolution u to (PDE) in $I_R(t_0) \times B_{2R}$, we have*

$$\inf_{(t_0+R^\alpha-(R/2)^\alpha, t_0+R^\alpha) \times B_{R/2}} u \geq c \left(\int_{(t_0-R^\alpha, t_0-R^\alpha+(R/2)^\alpha) \times B_{R/2}} u(t, x) dx dt - R^\alpha \|f\|_{L^\infty} \right), \tag{6-7}$$

where $B_{2R} \subset \Omega$.

- (ii) *Assume that (K1_{glob}) holds for some $\theta \in (d/\alpha, \infty]$. Then there is $c > 0$ such that, for every $0 < R \leq 1$ and every nonnegative, weak supersolution u to $(\widehat{\text{PDE}})$ in $I_R(t_0) \times B_{2R}$, estimate (6-7) holds.*

Now we prove Theorem 1.1 and Theorem 1.4. Both results require the weak Harnack inequality Theorem 6.3.

Proof of Theorem 1.1. We only prove (i) since the proof of (ii) follows the same line of arguments. Part (i) follows from a combination of Theorem 6.3 and Theorem 3.6 (or Theorem 4.8). First, we deduce from Theorem 3.6 (or Theorem 4.8) and a classical covering argument that, for every $\frac{1}{4} \leq t < s \leq \frac{1}{2}$,

$$\sup_{I_{tR/2}^\ominus \times B_{tR}} u \leq c_1(s-t)^{-(d+\alpha)/2} \left(\int_{I_{sR/2}^\ominus} \int_{B_{sR}} u^2(t, x) dx dt \right)^{1/2} + \sup_{t \in I_{R/4}^\ominus} \text{Tail}_{K,\alpha}(u(t), R) + c_2 R^\alpha \|f\|_{L^\infty}.$$

By a similar iteration argument as in the proof of Theorem 6.2, we deduce

$$\sup_{I_{R/8}^\ominus \times B_{R/4}} u \leq c_2 \left(\int_{I_{R/4}^\ominus \times B_{R/2}} u(t, x) \, dx \, dt \right) + c_2 \sup_{t \in I_{R/4}^\ominus} \text{Tail}_{K,\alpha}(u(t), R) + c_2 R^\alpha \|f\|_{L^\infty}. \tag{6-8}$$

Next, Theorem 6.3 yields

$$\inf_{(t_0+(1-2^{-\alpha})R^\alpha, t_0+R^\alpha) \times B_{R/2}} u \geq c_1 \left(\int_{(t_0-R^\alpha, t_0-(1-2^{-\alpha})R^\alpha) \times B_{R/2}} u(t, x) \, dx \, dt - R^\alpha \|f\|_{L^\infty} \right) \tag{6-9}$$

for some $c_1 > 0$. Note that

$$(t_0 - R^\alpha, t_0 - (1 - 2^{-\alpha})R^\alpha) = I_{R/2}^\ominus(t_0 - (1 - 2^{-\alpha})R^\alpha).$$

Consequently, by (6-8),

$$\begin{aligned} & \sup_{I_{R/8}^\ominus(t_0-(1-2^{-\alpha})R^\alpha) \times B_{R/4}} u \\ & \leq c_2 \left(\int_{I_{R/4}^\ominus(t_0-(1-2^{-\alpha})R^\alpha) \times B_{R/2}} u(t, x) \, dx \, dt \right) + c_2 \sup_{t \in I_{R/4}^\ominus(t_0-(1-2^{-\alpha})R^\alpha)} \text{Tail}_{K,\alpha}(u(t), R) + c_2 R^\alpha \|f\|_{L^\infty} \\ & \leq c_3 \left(\int_{(t_0-R^\alpha, t_0-(1-2^{-\alpha})R^\alpha) \times B_{R/2}} u(t, x) \, dx \, dt \right) + c_3 \sup_{t \in I_{R/4}^\ominus(t_0-(1-2^{-\alpha})R^\alpha)} \text{Tail}_{K,\alpha}(u(t), R) + c_3 R^\alpha \|f\|_{L^\infty} \\ & \leq c_4 \inf_{(t_0+(1-2^{-\alpha})R^\alpha, t_0+R^\alpha) \times B_{R/2}} u + c_4 \sup_{t \in (t_0-(1-2^{-\alpha}+4^{-\alpha})R^\alpha, t_0-(1-2^{-\alpha})R^\alpha)} \text{Tail}_{K,\alpha}(u(t), R) + c_4 R^\alpha \|f\|_{L^\infty} \end{aligned}$$

for some $c_2, c_3, c_4 > 0$. The proof is finished upon noticing that

$$I_{R/8}^\ominus(t_0 - (1 - 2^{-\alpha})R^\alpha) = (t_0 - (1 - 2^{-\alpha} + 8^{-\alpha})R^\alpha, t_0 - (1 - 2^{-\alpha})R^\alpha). \quad \square$$

Proof of Theorem 1.4. This result follows directly by combining Theorems 6.2 and 6.3, where we apply Theorem 6.2 with $p = 1$. □

6.3. Challenges in the parabolic case. Let us assume that (cutoff), (\mathcal{E}_\geq) , $(K1_{\text{loc}})$, $(K2)$, and (UJS) hold for some $\theta \in [d/\alpha, \infty]$. The goal of this section is to discuss the validity of a parabolic version of Theorem 1.4, i.e., to investigate the estimate

$$\sup_{(t_0-c_1R^\alpha, t_0-c_2R^\alpha) \times B_{R/4}} u \leq C \left(\inf_{(t_0+c_2R^\alpha, t_0+R^\alpha) \times B_{R/2}} u + R^\alpha \|f\|_{L^\infty} \right) \tag{6-10}$$

for some $C > 0$ and $0 < c_2 < c_1 < 1$ for nonnegative, weak solutions u to (PDE) in $I_R^\ominus \times B_{2R}$, where $B_{2R} \subset \Omega$. In order to keep the presentation short, we will not discuss weak solutions to $(\widehat{\text{PDE}})$ here.

As in the elliptic case, the general strategy to establish (6-10) would be to first prove an L^∞ - L^p -estimate of the form (given any $p \in (0, 2]$)

$$\sup_{I_{R/8}^\ominus \times B_{R/4}} u \leq c \left(\int_{I_{R/4}^\ominus} \int_{B_{R/2}} u^p(t, x) \, dx \, dt \right)^{1/p} + c R^\alpha \|f\|_{L^\infty} \tag{6-11}$$

and to deduce (6-10) after combination with the weak parabolic Harnack inequality of Theorem 6.3 as in the proof of Theorem 1.1.

A natural approach in order to show (6-11) would be to proceed as in the proof of Theorem 3.6 but to apply Corollary 5.4 in order to estimate the nonlocal tail by a local quantity. However, as Corollary 5.4 only provides an estimate for $\int_{I_{r/2}^\ominus} \text{Tail}_K(u(t), r, r + \rho) dt$ but not for $\sup_{I_{r/2}^\ominus} \text{Tail}_K(u(t), r, r + \rho)$, one needs to come up with a new idea to bridge the gap between

$$\sup_{I_{r/2}^\ominus} \text{Tail}_K(u(t), r, r + \rho) \quad \text{and} \quad \int_{I_{r/2}^\ominus} \text{Tail}_K(u(t), r, r + \rho) dt.$$

Note that the same issue appears in the symmetric case and has not been solved so far. There seems to be no proof of a parabolic Harnack inequality (6-10) for jumping kernels $K(x, y) \asymp |x - y|^{-d-\alpha}$ that uses only analytic arguments. Note that via probabilistic methods, an estimate of the form (6-10) has been proved in the symmetric case in [Bass and Levin 2002; Chen and Kumagai 2003].

Let us explain how to deduce (6-11) under the condition that u satisfies the following two additional assumptions:

- (a) There exists $c_0 > 0$ such that, for every $\frac{1}{2}R \leq r \leq R$ and $0 < \rho \leq r \leq r + \rho \leq R$,

$$\sup_{t \in I_{(r+\rho)/4}^\ominus} \text{Tail}_K(u(t), r + \frac{1}{2}\rho, r + \rho) \leq c_0 \sup_{I_{(r+\rho/2)/2}^\ominus \times B_{r+\rho}} u. \tag{6-12}$$

- (b) We have $\sup_{I_{R/4}^\ominus(t_0)} \text{Tail}_{K,\alpha}(u(t), R) < \infty$.

Remark 6.4. (i) Naturally, the constant c in (6-11) will depend on c_0 .

- (ii) (6-12) holds for global solutions to (PDE) in the symmetric case (see [Strömquist 2019b]).
- (iii) It has been proposed in [Kim 2019] to establish (6-12) for every weak solution u to (PDE) in $I \times B_{2R}$ with prescribed nonlocal parabolic boundary data $g \in L^\infty(I \times \mathbb{R}^d) \cap C(\bar{I} \times \mathbb{R}^d)$, with c_0 depending only on g . The proof of [Kim 2019, Lemma 5.3] is not complete.
- (iv) Note that (b) is an additional restriction and does not naturally follow from our weak solution concept. We refer to Section 2.3 for a more detailed discussion of finiteness of tail terms.

In order to establish (6-11), we need to prove an analog of (6-1). As in the proof of Theorem 3.6, we derive (3-16), and by combining it with (6-12), we deduce, for every $\delta > 0$,

$$A_i \leq \frac{c_1}{R^\alpha M^{2/\kappa'}} 2^{\gamma i} \left(1 + \frac{\sup_{I_{R/2}^\ominus \times B_R} u + R^\alpha \|f\|_{L^\infty}}{M} \right) A_{i-1}^{1+1/\kappa'}$$

for some $c_1 > 0$ and $\gamma > 1$. Here, $\kappa = 1 + \alpha/d$. By choosing

$$M := \delta \left(\sup_{I_{R/2}^\ominus \times B_R} u + R^\alpha \|f\|_{L^\infty} \right) + C^{\kappa^2/2} c_1^{\kappa'/2} \delta^{-\kappa'/2} R^{-\alpha\kappa'/2} A_0^{1/2},$$

where $C := 2^\gamma > 1$, we can deduce

$$\sup_{I_{R/8}^\ominus \times B_{R/2}} u \leq \delta \left(\sup_{I_{R/2}^\ominus \times B_R} u + R^\alpha \|f\|_{L^\infty} \right) + c_2 \delta^{-\kappa'/2} \left(\int_{I_{R/4}^\ominus} \int_{B_R} u^2(t, x) dx dt \right)^{1/2} \tag{6-13}$$

for some $c_2 > 0$. This estimate is a parabolic analog of (6-1). Note that (6-13) can also be established via the arguments from the proof of Theorem 4.8 using the Moser iteration.

Next, we intend to prove (6-11) by adapting the arguments in the proof of Theorem 6.2 to the parabolic setting.

As in the elliptic case, a standard covering argument yields, for every $\frac{1}{4} \leq t < s \leq \frac{1}{2}$,

$$\sup_{I_{tR/2}^\ominus \times B_{tR}} u \leq c_3(s-t)^{-(d+\alpha)/2} \left(\int_{I_{sR/2}^\ominus} \int_{B_{sR}} u^2(t, x) \, dx \, dt \right)^{1/2} + c_4 R^\alpha \|f\|_{L^\infty} + c_4 \delta \sup_{I_{sR/2}^\ominus \times B_{sR}} u,$$

where $c_3, c_4 > 0$ are constants. By Young’s inequality and choosing $\delta = 1/c_4$, we arrive at

$$\sup_{I_{tR/2}^\ominus \times B_{tR}} u \leq \frac{1}{2} \sup_{I_{sR/2}^\ominus \times B_{sR}} u + c_4(s-t)^{-(d+\alpha)/p} \left(\int_{I_{sR/2}^\ominus} \int_{B_{sR}} u^p(t, x) \, dx \, dt \right)^{1/p} + c_4 R^\alpha \|f\|_{L^\infty},$$

where $p \in (0, 2]$ can be chosen arbitrarily.

Now, (6-11) follows from [Giaquinta and Giusti 1982, Lemma 1.1], but this only applies if

$$\sup_{I_{R/4}^\ominus \times B_{R/2}} u < \infty. \tag{6-14}$$

In order to obtain (6-14), we apply Theorem 3.6 (or Theorem 4.8) and use condition (b) on u . This concludes the proof of (6-11) under the additional assumptions (a) and (b).

Appendix

The following lemma justifies the way we deal with the weak formulation of (PDE), or $(\widehat{\text{PDE}})$, in the proof of Theorem 3.6 after testing with $\phi(t, x) = \tau^2(x)(u(t, x) - k)_+$ for some $k \geq 0$, where u is a subsolution to the respective equation. In fact, ϕ is a priori not differentiable in t , which prevents us from integrating by parts. The idea of the proof is to test the equation with an auxiliary function having the required smoothness properties in t . This can be achieved with the help of Steklov averages. For symmetric nonlocal equations, such lemmas are well known (see [Felsinger and Kassmann 2013; Strömqvist 2019a]). We adapt the idea of the proof of [Felsinger and Kassmann 2013] to the nonsymmetric case. Note that Lemma A.2 in [Felsinger and Kassmann 2013] is not sufficient for the proof of (A.4) in [Felsinger and Kassmann 2013]. Our proof fixes the gap in their argument.

Lemma A.1. *Assume (cutoff).*

- (i) *Assume that $(K1_{\text{loc}})$ holds for some $\theta \in [d/\alpha, \infty]$. Moreover, assume (Sob) if $\theta < \infty$. Let $u \in V(B_{r+\rho} | \mathbb{R}^d)$ be a weak subsolution to (PDE). Then, for every $[t_1, t_2] \subset I$, every $0 < \rho \leq r \leq 1$ with $B_{r+\rho} \subset \Omega$, every $k \geq 0$, and every $\chi \in C_c^1(\mathbb{R})$,*

$$\begin{aligned} & \chi^2(t_2) \int_{B_{r+\rho}} (u(t_2) - k)_+^2 \tau^2 \, dx - \chi^2(t_1) \int_{B_{r+\rho}} (u(t_1) - k)_+^2 \tau^2 \, dx \\ & \quad - \int_{t_1}^{t_2} \partial_t(\chi^2(t)) \int_{B_{r+\rho}} (u(t) - k)_+^2 \tau^2 \, dx \, dt + \int_{t_1}^{t_2} \chi^2(t) \mathcal{E}(u(t), \tau^2(u(t) - k)_+) \, dt \\ & \leq \int_{t_1}^{t_2} \chi^2(t) \int_{B_{r+\rho}} f(t, x) \tau^2(x) (u(t, x) - k)_+ \, dx \, dt, \end{aligned}$$

where $\tau = \tau_{r,\rho/2}$.

(ii) Assume that $(\text{K1}_{\text{glob}})$ holds for some $\theta \in [d/\alpha, \infty]$. Moreover, assume (Sob) if $\theta < \infty$. Let $u \in V(B_{r+\rho} | \mathbb{R}^d)$ be a weak subsolution to $(\widehat{\text{PDE}})$. Then, for every $[t_1, t_2] \subset I$, every $0 < \rho \leq r \leq 1$ with $B_{r+\rho} \subset \Omega$, every $k \geq 0$, and every $\chi \in C_c^1(\mathbb{R})$,

$$\begin{aligned} & \chi^2(t_2) \int_{B_{r+\rho}} (u(t_2) - k)_+^2 \tau^2 \, dx - \chi^2(t_1) \int_{B_{r+\rho}} (u(t_1) - k)_+^2 \tau^2 \, dx \\ & \quad - \int_{t_1}^{t_2} \partial_t (\chi^2(t)) \int_{B_{r+\rho}} (u(t) - k)_+^2 \tau^2 \, dx \, dt + \int_{t_1}^{t_2} \chi^2(t) \widehat{\mathcal{E}}(u(t), \tau^2(u(t) - k)_+) \, dt \\ & \leq \int_{t_1}^{t_2} \chi^2(t) \int_{B_{r+\rho}} f(t, x) \tau^2(x) (u(t, x) - k)_+ \, dx \, dt. \end{aligned}$$

Proof. Given $v \in L^1((0, T); X)$ for some Banach space X , we define its Steklov average $v_h(t, x) = \int_t^{t+h} v(s, \cdot) \, ds$ if $t+h \in I$ and $v_h(t, x) = 0$ otherwise. Observe that

$$\partial_t u_h(t, x) = \frac{1}{h} (u(t+h, x) - u(t, x)) = \int_t^{t+h} \partial_s u(s, x) \, ds.$$

According to Lemma A.1 in [Felsinger and Kassmann 2013], we have

$$\|v_h(t) - v(t)\|_{L^2} \rightarrow 0 \quad \text{as } h \searrow 0 \quad \text{if } v \in C((0, T); L^2(B_{r+\rho})), \quad (\text{A-1})$$

$$\|v_h - v\|_{L^2([t_1, t_2]; X)} \rightarrow 0 \quad \text{as } h \searrow 0, \quad (\text{A-2})$$

$$\|v_h\|_{L^2([t_1, t_2]; X)} \leq \|v\|_{L^2([t_1, t_2]; X)}. \quad (\text{A-3})$$

We first explain how to prove (i). Let $t \in I$. We use the test function $\phi = \tau^2(u_h(t) - k)_+$, and after integrating over $(t, t+h)$ for some $h > 0$ such that $t+h \in I$ and dividing by h , we obtain

$$\int_{B_{r+\rho}} \partial_t u_h(t, x) \phi(t, x) \, dx + \mathcal{E}(u_h(t), \phi(t)) \leq (f(t), \phi(t)).$$

Note that $t \mapsto u_h(t, x)$ is differentiable for a.e. $x \in B_{r+\rho}$, and therefore

$$\partial_t u_h(t, x) \phi(t, x) = \frac{1}{2} \partial_t [(u_h(t, x) - k)_+^2] \tau^2(x).$$

We multiply by $\chi^2(t)$ and integrate over (t_1, t_2) . Integration by parts yields

$$\begin{aligned} & \int_{B_{r+\rho}} \chi^2(t_2) (u_h(t_2) - k)_+^2 \tau^2 \, dx - \int_{B_{r+\rho}} \chi^2(t_1) (u_h(t_1) - k)_+^2 \tau^2 \, dx \\ & \quad - \int_{t_1}^{t_2} \int_{B_{r+\rho}} \partial_t (\chi^2(t)) (u_h(t) - k)_+^2 \tau^2 \, dx \, dt + \int_{t_1}^{t_2} \chi^2(t) \mathcal{E}(u_h(t), \tau^2(u_h(t) - k)_+) \, dt \\ & \leq \int_{t_1}^{t_2} \chi^2(t) \int_{B_{r+\rho}} f(t, x) \tau^2(x) (u_h(t, x) - k)_+ \, dx \, dt. \end{aligned}$$

Since $\| (u_h(t) - k)_+ - (u(t) - k)_+ \|_{L^2(B_{r+\rho})} \leq \|u_h(t) - u(t)\|_{L^2(B_{r+\rho})}$, it follows by (A-1) that

$$\int_{B_{r+\rho}} (u_h(t) - k)_+^2 \tau^2 \, dx \rightarrow \int_{B_{r+\rho}} (u(t) - k)_+^2 \tau^2 \, dx \quad \text{for } t \in [t_1, t_2].$$

Moreover, (A-2) implies

$$\begin{aligned} \int_{t_1}^{t_2} \partial_t(\chi^2(t)) \int_{B_{r+\rho}} (u_h(t) - k)_+^2 \tau^2 \, dx \, dt &\rightarrow \int_{t_1}^{t_2} \int_{B_{r+\rho}} \partial_t(\chi^2(t))(u(t) - k)_+^2 \tau^2 \, dx \, dt, \\ \int_{t_1}^{t_2} \chi^2(t) \int_{B_{r+\rho}} f(t, x) \tau^2(x) (u_h(t, x) - k)_+ \, dx \, dt &\rightarrow \int_{t_1}^{t_2} \chi^2(t) \int_{B_{r+\rho}} f(t, x) \tau^2(x) (u(t, x) - k)_+ \, dx \, dt \end{aligned}$$

as $h \searrow 0$. It remains to prove that

$$\int_{t_1}^{t_2} \chi^2(t) \mathcal{E}(u_h(t), \tau^2(u_h(t) - k)_+) \, dt \rightarrow \int_{t_1}^{t_2} \chi^2(t) \mathcal{E}(u(t), \tau^2(u(t) - k)_+) \, dt. \quad (\text{A-4})$$

In Lemma A.2 in [Felsinger and Kassmann 2013], the authors established a related convergence property for symmetric energy forms. However, their proof has a gap, since Lemma A.2 does not suffice to deduce the desired result (even in the symmetric case), since, if $\Phi = f(u)$, it does not hold in general that $\Phi_h = f(u_h)$.

We define

$$\begin{aligned} V(t, x, y) &= u(t, x) - u(t, y), \\ W(t, x, y) &= \tau^2(x)(u(t, x) - k)_+ - \tau^2(y)(u(t, y) - k)_+, \\ \tilde{W}(t, x, y) &= \tau^2(x)(u_h(t, x) - k)_+ - \tau^2(y)(u_h(t, y) - k)_+. \end{aligned}$$

Our goal is to show that

$$\int_{t_1}^{t_2} |\mathcal{E}(u_h(t) - u(t), \tau^2(u_h(t) - k)_+)| \, dt \rightarrow 0, \quad (\text{A-5})$$

$$\int_{t_1}^{t_2} |\mathcal{E}(u(t), \tau^2(u_h - k)_+ - \tau^2(u(t) - k)_+)| \, dt \rightarrow 0. \quad (\text{A-6})$$

To establish (A-5), we split

$$\begin{aligned} \int_{t_1}^{t_2} |\mathcal{E}(u_h(t) - u(t), \tau^2(u_h(t) - k)_+)| \, dt \\ \leq \|\mathcal{E}_{B_{r+\rho}}^{K_s}(u_h - u, \tau^2(u_h - k)_+)\|_{L^1([t_1, t_2])} + \|\mathcal{E}_{B_{r+\rho}}^{K_a}(u_h - u, \tau^2(u_h - k)_+)\|_{L^1([t_1, t_2])} \\ + \|\mathcal{E}_{(B_{r+\rho} \times B_{r+\rho})^c}(u_h - u, \tau^2(u_h - k)_+)\|_{L^1([t_1, t_2])} \\ =: I_1 + I_2 + I_3, \end{aligned}$$

and establish the convergence of each term separately. For I_1 , we estimate

$$\begin{aligned} I_1 &\leq \int_{t_1}^{t_2} \int_{B_{r+\rho}} \int_{B_{r+\rho}} |V_h(t, x, y) - V(t, x, y)| |\tilde{W}(t, x, y)| K_s(x, y) \, dy \, dx \, dt \\ &\leq \|(V_h - V) K_s^{1/2}\|_{L^2([t_1, t_2] \times B_{r+\rho} \times B_{r+\rho})} \|\tilde{W} K_s^{1/2}\|_{L^2([t_1, t_2] \times B_{r+\rho} \times B_{r+\rho})} \\ &\leq \|u_h - u\|_{L^2([t_1, t_2]; V(B_{r+\rho} | \mathbb{R}^d))} \|\mathcal{E}_{B_{r+\rho}}^{K_s}(\tau^2(u_h - k)_+, \tau^2(u_h - k)_+)\|_{L^1([t_1, t_2])}^{1/2} \\ &\leq \|u_h - u\|_{L^2([t_1, t_2]; V(B_{r+\rho} | \mathbb{R}^d))} \|\tau^2 u\|_{L^2([t_1, t_2]; V(B_{r+\rho} | \mathbb{R}^d))} \\ &\rightarrow 0, \end{aligned}$$

where we used (A-2) and (A-3), that

$$u, \phi \in L^2([t_1, t_2]; V(B_{r+\rho} | \mathbb{R}^d)),$$

and the fact that, due to the Markov property of \mathcal{E}^{K_s} and (A-3),

$$\begin{aligned} \mathcal{E}_{B_{r+\rho}}^{K_s} (\tau^2(u_h - k)_+, \tau^2(u_h - k)_+) &\leq \mathcal{E}_{B_{r+\rho}}^{K_s} (\tau^2 u_h, \tau^2 u_h) \\ &= \mathcal{E}_{B_{r+\rho}}^{K_s} ([\tau^2 u]_h, [\tau^2 u]_h) \leq \|\tau^2 u\|_{V(B_{r+\rho} | \mathbb{R}^d)}^2. \end{aligned} \tag{A-7}$$

For I_2 ,

$$\begin{aligned} I_2 &\leq \int_{t_1}^{t_2} \int_{B_{r+\rho}} \int_{B_{r+\rho}} |V_h(t, x, y) - V(t, x, y)| \tau^2(x) (u_h(t, x) - k)_+ |K_a(x, y)| \, dy \, dx \, dt \\ &\leq \|(V_h - V) J^{1/2}\|_{L^2([t_1, t_2] \times B_{r+\rho} \times B_{r+\rho})} \left\| \int_{B_{r+\rho/2}} (u_h(\cdot, x) - k)_+^2 \left(\int_{B_{r+\rho}} \frac{|K_a(x, y)|^2}{J(x, y)} \, dy \right) dx \right\|_{L^1([t_1, t_2])}^{1/2} \\ &\leq c \|u_h - u\|_{L^2([t_1, t_2]; V(B_{r+\rho} | \mathbb{R}^d))} \|u\|_{L^2([t_1, t_2]; L^{2\theta'}(B_{r+\rho/2}))} \\ &\leq c \|u_h - u\|_{L^2([t_1, t_2]; V(B_{r+\rho} | \mathbb{R}^d))} \|u\|_{L^2([t_1, t_2]; V(B_{r+\rho} | \mathbb{R}^d))} \\ &\rightarrow 0, \end{aligned}$$

where $c > 0$ might depend on ρ , and we used (K1_{loc}), (A-2), (A-3), that

$$u \in L^2([t_1, t_2]; V(B_{r+\rho} | \mathbb{R}^d)),$$

and (Sob). For I_3 , we obtain

$$\begin{aligned} I_3 &\leq 2 \int_{t_1}^{t_2} \int_{B_{r+\rho}} \int_{B_{r+\rho}^c} |V_h(t, x, y) - V(t, x, y)| \tau^2(x) (u_h(t, x) - k)_+ K_s(x, y) \, dy \, dx \, dt \\ &\leq 2 \|(V_h - V) K_s^{1/2}\|_{L^2([t_1, t_2] \times B_{r+\rho} \times B_{r+\rho}^c)} \left\| \int_{B_{r+\rho/2}} (u_h(\cdot, x) - k)_+^2 \Gamma^{K_s}(\tau, \tau)(x) \, dx \right\|_{L^1([t_1, t_2])}^{1/2} \\ &\leq c \rho^{-\alpha/2} \|u_h - u\|_{L^2([t_1, t_2]; V(B_{r+\rho} | \mathbb{R}^d))} \|u\|_{L^2([t_1, t_2] \times B_{r+\rho})} \\ &\rightarrow 0, \end{aligned}$$

where we used (1-2), (cutoff), (A-2), (A-3) and that

$$u \in L^2([t_1, t_2]; V(B_{r+\rho} | \mathbb{R}^d)).$$

It remains to prove (A-6). Again, we split

$$\begin{aligned} &\int_{t_1}^{t_2} |\mathcal{E}(u(t), \tau^2(u_h - k)_+ - \tau^2(u - k)_+)| \, dt \\ &\leq \|\mathcal{E}_{B_{r+\rho}}^{K_s}(u, \tau^2(u_h - k)_+ - \tau^2(u - k)_+)\|_{L^1([t_1, t_2])} \\ &\quad + \|\mathcal{E}_{B_{r+\rho}}^{K_a}(u, \tau^2(u_h - k)_+ - \tau^2(u - k)_+)\|_{L^1([t_1, t_2])} \\ &\quad + \|\mathcal{E}_{(B_{r+\rho} \times B_{r+\rho})^c}(u, \tau^2(u_h - k)_+ - \tau^2(u - k)_+)\|_{L^1([t_1, t_2])} \\ &=: J_1 + J_2 + J_3, \end{aligned}$$

Convergence of J_1 can be proved as follows. First, by Hölder’s inequality,

$$J_1 \leq \|u\|_{L^2([t_1, t_2]; V(B_{r+\rho} | \mathbb{R}^d))} \left\| \int_{B_{r+\rho}} \int_{B_{r+\rho}} |W(\cdot, x, y) - \tilde{W}(\cdot, x, y)|^2 K_s(x, y) \, dy \, dx \right\|_{L^1([t_1, t_2])}^{1/2}.$$

Since $u \in L^2([t_1, t_2]; V(B_{r+\rho} | \mathbb{R}^d))$, it suffices to prove that the second factor converges to zero in order to conclude that $J_1 \rightarrow 0$. For this, we claim that there exist $\xi(t, x, y), \tilde{\xi}(t, x, y) \in [0, 1]$ such that

$$\begin{aligned} \tau^2(x)(u(t, x) - k)_+ - \tau^2(y)(u(t, y) - k)_+ &= \xi(t, x, y)[f(t, x) - f(t, y)], \\ \tau^2(x)(u_h(t, x) - k)_+ - \tau^2(y)(u_h(t, y) - k)_+ &= \tilde{\xi}(t, x, y)[f_h(t, x) - f_h(t, y)], \end{aligned}$$

where we define $f(t, x) = \tau^2(x)(u(t, x) - k)$. In fact, it is easy to see that

$$\xi(t, x, y) = \begin{cases} 1, & u(t, x), u(t, y) > k, \\ 0, & u(t, x), u(t, y) \leq k, \\ \frac{f(t, x)}{f(t, x) - f(t, y)}, & u(t, x) > k \geq u(t, y), \\ \frac{f(t, y)}{f(t, y) - f(t, x)}, & u(t, y) > k \geq u(t, x), \end{cases}$$

$$\tilde{\xi}(t, x, y) = \begin{cases} 1, & u_h(t, x), u_h(t, y) > k, \\ 0, & u_h(t, x), u_h(t, y) \leq k, \\ \frac{f_h(t, x)}{f_h(t, x) - f_h(t, y)}, & u_h(t, x) > k \geq u_h(t, y), \\ \frac{f_h(t, y)}{f_h(t, y) - f_h(t, x)}, & u_h(t, y) > k \geq u_h(t, x) \end{cases}$$

have the desired properties. We estimate

$$\begin{aligned} & \left\| \int_{B_{r+\rho}} \int_{B_{r+\rho}} |W(\cdot, x, y) - \tilde{W}(\cdot, x, y)|^2 K_s(x, y) \, dy \, dx \right\|_{L^1([t_1, t_2])}^{1/2} \\ & \leq 2 \left\| \int_{B_{r+\rho}} \int_{B_{r+\rho}} |\tilde{\xi}(\cdot, x, y)|^2 [(f_h(t, x) - f(t, x)) - (f_h(t, y) - f(t, y))]^2 K_s(x, y) \, dy \, dx \right\|_{L^1([t_1, t_2])}^{1/2} \\ & \quad + 2 \left\| \int_{B_{r+\rho}} \int_{B_{r+\rho}} |\tilde{\xi}(\cdot, x, y) - \xi(\cdot, x, y)|^2 [f(t, x) - f(t, y)]^2 K_s(x, y) \, dy \, dx \right\|_{L^1([t_1, t_2])}^{1/2} \\ & \leq J_{1,1} + J_{1,2}. \end{aligned}$$

For $J_{1,1}$, note that

$$J_{1,1} \leq 2 \|f_h - f\|_{L^2([t_1, t_2]; V(B_{r+\rho} | \mathbb{R}^d))} \rightarrow 0,$$

where we used that $|\tilde{\xi}| \leq 1$, $f \in L^2([t_1, t_2]; V(B_{r+\rho} | \mathbb{R}^d))$, and (A-2). For $J_{1,2}$, we observe that

$$|\tilde{\xi}(t, x, y) - \xi(t, x, y)| \rightarrow 0 \quad \text{as } h \searrow 0 \quad \text{for a.e. } t, x, y.$$

Since $f \in L^2([t_1, t_2]; V(B_{r+\rho} | \mathbb{R}^d))$, it follows from dominated convergence that $J_{1,2}$ also goes to 0.

For J_2 , we estimate

$$\begin{aligned} J_2 &\leq \int_{t_1}^{t_2} \int_{B_{r+\rho}} \int_{B_{r+\rho}} |V(t, x, y)| \tau^2(x) |(u_h(t, x) - k)_+ - (u(t, x) - k)_+| |K_a(x, y)| \, dy \, dx \, dt \\ &\leq \|u\|_{L^2([t_1, t_2]; V(B_{r+\rho} | \mathbb{R}^d))} \left\| \int_{B_{r+\rho}} |(u_h(\cdot, x) - k)_+ - (u(\cdot, x) - k)_+|^2 \left(\int_{B_{r+\rho}} \frac{|K_a(x, y)|^2}{J(x, y)} \, dy \right) dx \right\|_{L^1([t_1, t_2])}^{1/2} \\ &\leq c \|u\|_{L^2([t_1, t_2]; V(B_{r+\rho} | \mathbb{R}^d))} \|u_h - u\|_{L^2([t_1, t_2]; L^{2\theta'}(B_{r+\rho}))} \\ &\rightarrow 0, \end{aligned}$$

where we used $(K1_{loc})$,

$$|(u_h(t, x) - k)_+ - (u(t, x) - k)_+| \leq |u_h(t, x) - u(t, x)|, \quad u \in L^2([t_1, t_2]; V(B_{r+\rho} | \mathbb{R}^d)),$$

(Sob), and (A-2). To prove convergence of J_3 , we proceed as follows:

$$\begin{aligned} J_3 &\leq 2 \int_{t_1}^{t_2} \int_{B_{r+\rho}} \int_{B_{r+\rho}^c} |V(t, x, y)| \tau^2(x) |(u_h(t, x) - k)_+ - (u(t, x) - k)_+| |K_s(x, y)| \, dy \, dx \, dt \\ &\leq 2 \|u\|_{L^2([t_1, t_2]; V(B_{r+\rho} | \mathbb{R}^d))} \left\| \int_{B_{r+\rho/2}} |(u_h(\cdot, x) - k)_+ - u(\cdot, x) - k|^2 \Gamma^{K_s}(\tau, \tau)(x) \, dx \right\|_{L^1([t_1, t_2])}^{1/2} \\ &\leq c \rho^{-\alpha/2} \|u\|_{L^2([t_1, t_2]; V(B_{r+\rho} | \mathbb{R}^d))} \|u_h - u\|_{L^2([t_1, t_2] \times B_{r+\rho})} \\ &\rightarrow 0, \end{aligned}$$

where we used (cutoff), (A-2), and

$$u \in L^2([t_1, t_2]; V(B_{r+\rho} | \mathbb{R}^d)).$$

Altogether, this proves (A-4), and we deduce the desired result. Let us now prove (ii). In analogy to the proof of (i), it is only left to show

$$\int_{t_1}^{t_2} \widehat{\mathcal{E}}(u_h(t), \tau^2(u_h(t) - k)_+) \, dt \rightarrow \int_{t_1}^{t_2} \widehat{\mathcal{E}}(u(t), \tau^2(u(t) - k)_+) \, dt. \tag{A-8}$$

We will establish (A-8) by proving the following two properties:

$$\int_{t_1}^{t_2} |\mathcal{E}(u_h(t) - u(t), \tau^2(u_h(t) - k)_+)| \, dt \rightarrow 0, \tag{A-9}$$

$$\int_{t_1}^{t_2} |\mathcal{E}(u(t), \tau^2(u_h - k)_+ - \tau^2(u(t) - k)_+)| \, dt \rightarrow 0. \tag{A-10}$$

Let us first prove (A-9). In analogy to the proof of (A-5), we split

$$\begin{aligned} &\int_{t_1}^{t_2} |\widehat{\mathcal{E}}(u_h(t) - u(t), \tau^2(u_h(t) - k)_+)| \, dt \\ &\leq \|\mathcal{E}_{B_{r+\rho}}^{K_s}(u_h - u, \tau^2(u_h - k)_+)\|_{L^1([t_1, t_2])} + \|\widehat{\mathcal{E}}_{B_{r+\rho}}^{K_a}(u_h - u, \tau^2(u_h - k)_+)\|_{L^1([t_1, t_2])} \\ &\quad + \|\widehat{\mathcal{E}}_{(B_{r+\rho} \times B_{r+\rho})^c}(u_h - u, \tau^2(u_h - k)_+)\|_{L^1([t_1, t_2])} \\ &=: \widehat{I}_1 + \widehat{I}_2 + \widehat{I}_3. \end{aligned}$$

In (i), we already showed that $\widehat{I}_1 \rightarrow 0$. Let us estimate \widehat{I}_2 as follows:

$$\begin{aligned} \widehat{I}_2 &\leq \|(u_h - u)\widetilde{W}|K_a|\|_{L^1([t_1, t_2])} \\ &\leq \|\mathcal{E}_{B_{r+\rho}}^{K_s}(\tau^2(u_h - k)_+, \tau^2(u_h - k)_+)\|_{L^1([t_1, t_2])}^{1/2} \|u_h - u\|_{L^2([t_1, t_2]; L^{2\theta'}(B_{r+\frac{\rho}{2}}))} \\ &\leq c\|\tau^2 u\|_{L^2([t_1, t_2]; V(B_{r+\rho}|\mathbb{R}^d))} \|u_h - u\|_{L^2([t_1, t_2]; V(B_{r+\rho}|\mathbb{R}^d))} \\ &\rightarrow 0, \end{aligned}$$

where we used (K1_{glob}), (A-2), (A-7), and (Sob). Moreover, \widehat{I}_3 can be treated as follows:

$$\begin{aligned} \widehat{I}_3 &\leq \|\mathcal{E}_{(B_{r+\rho} \times B_{r+\rho})^c}^{K_s}(u_h - u, \tau^2(u_h - k)_+)\|_{L^1([t_1, t_2])} \\ &\quad + \left\| \int_{B_{r+\rho/2}} \int_{B_{r+\rho}^c} (u_h(\cdot, x) - u(\cdot, x))\tau^2(x)(u_h(\cdot, x) - k)_+ |K_a(x, y)| \, dy \, dx \right\|_{L^1([t_1, t_2])} \\ &\quad + \left\| \int_{B_{r+\rho}^c} \int_{B_{r+\rho/2}} (u_h(\cdot, x) - u(\cdot, x))\tau^2(y)(u_h(\cdot, y) - k)_+ |K_a(x, y)| \, dy \, dx \right\|_{L^1([t_1, t_2])} \\ &=: \widehat{I}_{3,1} + \widehat{I}_{3,2} + \widehat{I}_{3,3}. \end{aligned}$$

The proof of convergence for $\widehat{I}_{3,1}$ goes exactly like for I_3 . For $\widehat{I}_{3,2}$, we estimate using the assumptions (K1_{glob}) and (cutoff):

$$\begin{aligned} \widehat{I}_{3,2} + \widehat{I}_{3,3} &\leq \left\| \int_{\mathbb{R}^d} |u_h(\cdot, x) - u(\cdot, x)|^2 \left(\int_{\mathbb{R}^d} \frac{|K_a(x, y)|^2}{J(x, y)} \, dy \right) dx \right\|_{L^1([t_1, t_2])}^{1/2} \\ &\quad \times \left\| \int_{B_{r+\rho/2}} (u_h(\cdot, x) - k)_+^2 \Gamma^J(\tau, \tau)(x) \, dx \right\|_{L^1([t_1, t_2])}^{1/2} \\ &\leq c\rho^{-\alpha/2} \|u_h - u\|_{L^2([t_1, t_2]; L^{2\theta'}(\mathbb{R}^d))} \| (u_h - k)_+ \|_{L^2([t_1, t_2] \times B_{r+\rho})} \\ &\leq c\rho^{-\alpha/2} \|u_h - u\|_{L^2([t_1, t_2]; L^{2\theta'}(\mathbb{R}^d))} \|u\|_{L^2([t_1, t_2] \times B_{r+\rho})} \\ &\rightarrow 0, \end{aligned}$$

where we used (A-2), (A-3), and

$$u \in L^2([t_1, t_2]; L^{2\theta'}(\mathbb{R}^d)).$$

We have established (A-9). To prove (A-10), let us again split

$$\begin{aligned} &\int_{t_1}^{t_2} |\widehat{\mathcal{E}}(u(t), \tau^2(u_h - k)_+ - \tau^2(u - k)_+)| \, dt \\ &\leq \|\mathcal{E}_{B_{r+\rho}}^{K_s}(u, \tau^2(u_h - k)_+ - \tau^2(u - k)_+)\|_{L^1([t_1, t_2])} \\ &\quad + \|\widehat{\mathcal{E}}_{B_{r+\rho}}^{K_a}(u, \tau^2(u_h - k)_+ - \tau^2(u - k)_+)\|_{L^1([t_1, t_2])} \\ &\quad + \|\widehat{\mathcal{E}}_{(B_{r+\rho} \times B_{r+\rho})^c}(u, \tau^2(u_h - k)_+ - \tau^2(u - k)_+)\|_{L^1([t_1, t_2])} \\ &=: \widehat{J}_1 + \widehat{J}_2 + \widehat{J}_3. \end{aligned}$$

Note that $\widehat{J}_1 = J_1 \rightarrow 0$. For \widehat{J}_2 , we estimate

$$\widehat{J}_2 \leq \|u\|_{L^2([t_1, t_2]; L^{2\theta'}(B_{r+\rho}))} \left\| \int_{B_{r+\rho}} \int_{B_{r+\rho}} |W(\cdot, x, y) - \widetilde{W}(\cdot, x, y)|^2 K_s(x, y) \, dy \, dx \right\|_{L^1([t_1, t_2])}^{1/2},$$

where we used (K1_{glob}) and that $u \in L^2([t_1, t_2]; L^{2\theta'}(B_{r+\rho}))$. We conclude that $\widehat{J}_2 \rightarrow 0$ since the second factor converges to zero, as we proved already when dealing with J_1 .

To estimate \widehat{J}_3 , we proceed as follows:

$$\begin{aligned} \widehat{J}_3 &\leq \|\mathcal{E}_{(B_{r+\rho} \times B_{r+\rho})^c}^{K_s}(u, \tau^2(u_h - k)_+ - \tau^2(u - k)_+)\|_{L^1([t_1, t_2])} \\ &\quad + \left\| \int_{B_{r+\rho/2}} \int_{B_{r+\rho}^c} \tau^2(x) |(u_h - k)_+(x) - (u - k)_+(x)| u(x) |K_a(x, y)| \, dy \, dx \right\|_{L^1([t_1, t_2])} \\ &\quad + \left\| \int_{B_{r+\rho}^c} \int_{B_{r+\rho/2}} \tau^2(y) |(u_h - k)_+(y) - (u - k)_+(y)| u(x) |K_a(x, y)| \, dy \, dx \right\|_{L^1([t_1, t_2])} \\ &= \widehat{J}_{3,1} + \widehat{J}_{3,2} + \widehat{J}_{3,3}. \end{aligned}$$

Note that $\widehat{J}_{3,1} \rightarrow 0$ follows similarly to the proof of $J_3 \rightarrow 0$. $\widehat{J}_{3,2}$ and $\widehat{J}_{3,3}$ are estimated as follows, using similar arguments as in the estimates of $\widehat{I}_{3,2}$ and $\widehat{I}_{3,3}$:

$$\begin{aligned} \widehat{J}_{3,2} + \widehat{J}_{3,3} &\leq \|(u_h - k)_+ - (u - k)_+\|_{L^2([t_1, t_2]; L^{2\theta'}(\mathbb{R}^d))} \left\| \int_{B_{r+\rho/2}} u^2(x) \Gamma^{K_s}(\tau, \tau)(x) \, dx \right\|_{L^1([t_1, t_2])}^{1/2} \\ &\leq c\rho^{-\alpha/2} \|u_h - u\|_{L^2([t_1, t_2]; L^{2\theta'}(\mathbb{R}^d))} \|u\|_{L^2([t_1, t_2] \times B_{r+\rho})} \\ &\rightarrow 0, \end{aligned}$$

where we used (cutoff) and (K1_{glob}), as well as (A-2) and $u \in L^2([t_1, t_2]; L^{2\theta'}(B_{r+\rho}))$. This proves (ii). \square

Remark A.2. We point out that the above proof can be extended to more general test functions ϕ of the form $\phi = \pm \tau^2 g(u)$, where $g : [0, \infty) \rightarrow [0, \infty)$. This way, it would be possible to generalize the notion of a weak solution to (PDE), or to ($\widehat{\text{PDE}}$), in $I \times \Omega$, in the sense that the assumption $\partial_t u \in L^1_{\text{loc}}(I, L^2(\Omega))$, where $\partial_t u$ is the weak $L^2(\Omega)$ -derivative of u , can be replaced by $u \in C(I; L^2(\Omega))$.

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
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Relative heat content asymptotics for sub-Riemannian manifolds ANDREI AGRACHEV, LUCA RIZZI and TOMMASO ROSSI	2997
Minkowski inequality on complete Riemannian manifolds with nonnegative Ricci curvature LUCA BENATTI, MATTIA FOGAGNOLO and LORENZO MAZZIERI	3039
The Willmore flow of tori of revolution ANNA DALL'ACQUA, MARIUS MÜLLER, REINER SCHÄTZLE and ADRIAN SPENER	3079
Optimal Prandtl expansion around a concave boundary layer DAVID GÉRARD-VARET, YASUNORI MAEKAWA and NADER MASMOUDI	3125
Nonlocal operators related to nonsymmetric forms, II: Harnack inequalities MORITZ KASSMANN and MARVIN WEIDNER	3189
Transference of scale-invariant estimates from Lipschitz to nontangentially accessible to uniformly rectifiable domains STEVE HOFMANN, JOSÉ MARÍA MARTELL and SVITLANA MAYBORODA	3251
Optimal regularity and the Liouville property for stable solutions to semilinear elliptic equations in \mathbb{R}^n with $n \geq 10$ FA PENG, YI RU-YA ZHANG and YUAN ZHOU	3335
A generalization of the Beurling–Malliavin majorant theorem IOANN VASILYEV	3355