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OPTIMAL REGULARITY AND THE LIOUVILLE PROPERTY FOR STABLE SOLUTIONS TO SEMILINEAR ELLIPTIC EQUATIONS IN $\mathbb{R}^n$ WITH $n \geq 10$

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Let $0 \leq f \in C^{0,1}(\mathbb{R})$. Given a domain $\Omega \subset \mathbb{R}^n$, we prove that any stable solution to the equation $-\Delta u = f(u)$ in Ω satisfies

- a BMO interior regularity, when $n = 10$,
- a Morrey $M^{p_n, 4+2/(p_n-2)}$ interior regularity, when $n \geq 11$, where

$$p_n = \frac{2(n-2\sqrt{n-1}-2)}{n-2\sqrt{n-1}-4}.$$

This result is optimal as hinted by, e.g., Brezis and Vázquez (1997), Cabré and Capella (2006), and Dupaigne (2011), and answers an open question raised by Cabré, Figalli, Ros-Oton and Serra (2020). As an application, we show a sharp Liouville property: any stable solution $u \in C^2(\mathbb{R}^n)$ to $-\Delta u = f(u)$ in $\mathbb{R}^n$ satisfying the growth condition

$$|u(x)| = \begin{cases} o(\log|x|) & \text{as } |x| \to +\infty, \quad \text{when } n = 10, \\ o(|x|^{-n/2+\sqrt{n-1}+2}) & \text{as } |x| \to +\infty, \quad \text{when } n \geq 11, \end{cases}$$

must be a constant. This extends the well-known Liouville property for radial stable solutions obtained by Villegas (2007).

1. Introduction

Let Ω be a bounded domain of $\mathbb{R}^n$ with $n \geq 2$. Given any local Lipschitz function $f : \mathbb{R} \to \mathbb{R}$ (for short $f \in C^{0,1}(\mathbb{R})$), we consider the semilinear elliptic equation

$$-\Delta u = f(u) \quad \text{in } \Omega, \tag{1-1}$$

which is the Euler–Lagrange equation for the energy functional

$$\mathcal{E}(u) := \int_\Omega \Big(\frac{1}{2}|Du|^2 - F(u)\Big)\, dx, \tag{1-2}$$

where $F(t) = \int_0^t f(s)\, ds$ for $t \in \mathbb{R}$. A function $u \in W^{1,2}(\Omega)$ is called a weak solution to (1-1) if $f(u) \in L^1_{\mathrm{loc}}(\Omega)$ and

$$\int_\Omega Du \cdot D\xi \, dx = \int_\Omega f(u)\xi \, dx \quad \text{for all } \xi \in C_c^\infty(\Omega),$$

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that is, u is a critical point of the energy functional $\mathcal{E}$. We say that a weak solution u is *stable* in Ω if $f'_-(u) \in L^1_{\mathrm{loc}}(\Omega)$ and

$$\int_\Omega f'_-(u)\xi^2\,dx \le \int_\Omega |D\xi|^2\,dx \quad \text{for all } \xi \in C_c^\infty(\Omega), \tag{1-3}$$

that is, the second variation of the energy functional $\mathcal{E}$ is nonnegative. Here and below,

$$f'_-(t) = \liminf_{h\to 0} \frac{f(t+h)-f(t)}{h} \quad \text{for all } t \in \mathbb{R},$$

and note that $f'_-(t) = f'(t)$ whenever $f \in C^1(\mathbb{R})$.

The study of stable solutions to semilinear elliptic equations can be traced to the seminal paper [Crandall and Rabinowitz 1975]. The regularity of stable solutions provides an important way to understand the regularity of the extremal solution $u^\star$ to the Gelfand-type problem

$$\begin{cases} -\Delta u = \lambda^\star f(u) & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega \end{cases} \tag{1-4}$$

for some positive constant $\lambda^\star > 0$. We refer to [Brezis 2003; Cabré 2017; Gelfand 1963] for a comprehensive analysis of (1-4) and related topics. Note that the extremal solution $u^\star$ can be approximated by stable solutions $\{u_\lambda\}_{\lambda<\lambda^\star}$; see, e.g., [Dupaigne 2011].

In dimension $n \le 9$, Brezis [2003] introduced an open problem: is the extremal solution $u^\star$ to (1-4) bounded for some f and Ω? Since $u^\star$ is approximated by stable solutions $\{u_\lambda\}_{\lambda<\lambda^\star}$, it suffices to establish some a priori bound for stable solutions. In recent years, there were several strong efforts to study regularity for stable solutions and hence for Brezis' open problem. In particular, a positive answer was given by Nedev [2000], when $n \le 3$, and by Cabré [2010], when $n = 4$ (see also [Cabré 2019] for an alternative proof).

Very recently, Cabré, Figalli, Ros-Oton and Serra [Cabré et al. 2020] provided a complete answer to Brezis' open problem when $f \ge 0$ based on certain Morrey-type estimates for $n \ge 3$. Throughout this paper, for $p \in [1,\infty)$ and $\beta \in (0,n)$, we define the Morrey norm as

$$\|w\|_{M^{p,\beta}(\Omega)} := \sup_{y\in\Omega,\, r>0} \left(r^{\beta-n} \int_{\Omega\cap B_r(y)} |w|^p\,dx \right)^{1/p} < \infty, \tag{1-5}$$

where $B_r(y)$ denotes the ball with center y and radius $r > 0$. We simply write B_r when the center of the ball is at the origin. In addition, following the convention, we denote by $C(a, b, \dots)$ a positive constant depending only on the parameters $a, b, \dots$.

In dimension $n \ge 10$, in particular, [Cabré et al. 2020, Theorem 1.9] established the following regularity of stable solutions to (1-1).

Theorem 1.1 [Cabré et al. 2020]. *Suppose that $f \in C^{0,1}(\mathbb{R})$ is nonnegative. If $u \in C^2(B_1)$ is a stable solution to (1-1) in B_1, then*

$$\|u\|_{M^{p,2+4/(p-2)}(B_{1/2})} \le C(n,p)\|u\|_{L^1(B_1)} \quad \textit{for every } p < p_n, \tag{1-6}$$

where

$$p_n := \begin{cases} \infty & \text{if } n = 10, \\ \dfrac{2(n-2\sqrt{n-1}-2)}{n-2\sqrt{n-1}-4} & \text{if } n \geq 11. \end{cases} \tag{1-7}$$

Moreover, suppose additionally that f is nondecreasing and Ω is a bounded domain of class C^3. If $u \in C^2(\Omega) \cap C^0(\overline{\Omega})$ is a stable solution to (1-1) in Ω with boundary $u = 0$ on $\partial\Omega$, then

$$\|u\|_{M^{p,2+4/(p-2)}(\Omega)} \leq C(n, p, \Omega)\|u\|_{L^1(\Omega)} \quad \text{for every } p < p_n. \tag{1-8}$$

We remark that the exponent $n - 2\sqrt{n-1} - 4$ changes sign when $n = 10$, which has already appeared in, e.g., [Gui et al. 1992].

However, for the endpoint case $p = p_n$, [Cabré et al. 2020, Section 1.3] pointed out that it is an open question whether (1-6) holds.

As hinted at by earlier results in the radial symmetric case [Cabré and Capella 2006], when $n = 10$, instead of $L^\infty = M^{\infty,2}$, a more suitable space to consider is a class of functions with bounded mean oscillations (BMO space), as remarked therein. Indeed, $u(x) = -2\log|x|$ is a stable solution to (1-1) in B_1, with $f(u) = 2(n-2)e^u$. Obviously, $u \in \mathrm{BMO}(B_1)$ but $u \notin L^\infty(B_1)$. Here and below, the BMO norm is defined as

$$\|u\|_{\mathrm{BMO}(\Omega)} := \sup_{y\in\Omega, r>0} \inf_{c\in\mathbb{R}} \fint_{\Omega\cap B_r(y)} |u(x) - c|\, dx,$$

where, $\fint_E v\, dx$ denotes the integral average of v on a measurable set E.

On the other hand, when $n \geq 11$, also hinted at by the results in [Cabré and Capella 2006], the range $p \leq p_n$ is the best possible in (1-6). Besides, it was proven in [Brezis and Vázquez 1997] that the function $u(x) = |x|^{-2/(q_n-1)} - 1$ is the extremal solution to

$$-\Delta u = \lambda^\star(1+u)^{q_n} \ \text{ in } B_1, \quad u = 0 \ \text{ on } \partial B_1, \tag{1-9}$$

with

$$\lambda^\star = \frac{2}{q_n} \quad \text{and} \quad q_n := \frac{n-2\sqrt{n-1}}{n-2\sqrt{n-1}-4}.$$

We note that q_n here is exactly the standard exponent in [Joseph and Lundgren 1973]. It is easy to see that $u \in M^{p,2+4/(p-2)}(B_{1/2})$ if and only if $p \leq p_n$. Recall that, by [Dupaigne 2011, Section 3.2.2], such an extremal solution can be approximated by stable solutions. We also refer to, e.g., [Farina 2007] for some earlier work on Lane–Emden equations, which also hints at the optimality of our results.

The first main purpose of this paper is to establish the following regularity at the endpoint p_n for stable solutions to (1-1), when $n \geq 10$, and then answer the above open question in [Cabré et al. 2020].

Theorem 1.2. *Suppose $f \in C^{0,1}(\mathbb{R})$ is nonnegative. For any stable solution $u \in C^2(B_1)$ to (1-1) in B_1, when $n = 10$, we have*

$$\|u\|_{\mathrm{BMO}(B_{1/2})} \leq C(n)\|u\|_{L^1(B_1)}, \tag{1-10}$$

and when $n \geq 11$, we have

$$\|u\|_{M^{p_n,2+4/(p_n-2)}(B_{1/2})} \leq C(n)\|u\|_{L^1(B_1)}. \tag{1-11}$$

Moreover, suppose additionally that f *is nondecreasing and* Ω *is a bounded smooth convex domain. For any positive stable solution* $u \in C^2(\overline{\Omega})$ *to* (1-1) *with boundary* $u = 0$ *on* $\partial\Omega$, *when* $n = 10$, *we have*

$$\|u\|_{\mathrm{BMO}(\Omega)} \le C(n, \Omega)\|u\|_{L^1(\Omega)}, \tag{1-12}$$

and when $n \ge 11$, *we have*

$$\|u\|_{M^{p_n,2+4/(p_n-2)}(\Omega)} \le C(n, \Omega)\|u\|_{L^1(\Omega)}. \tag{1-13}$$

As a direct consequence of the above a priori estimates, we have the following result for stable solutions in $W^{1,2}$.

Corollary 1.3. *Suppose that* $\Omega \subset \mathbb{R}^n$ *is a bounded smooth convex domain and that* $f \in C^{0,1}(\mathbb{R})$ *is nonnegative, nondecreasing, convex, and satisfies* $f(t)/t \to +\infty$ *as* $t \to +\infty$. *For any stable solution* $u \in W_0^{1,2}(\Omega)$ *to* (1-1) *with boundary* $u = 0$ *on* $\partial\Omega$, *we have* (1-12) *when* $n = 10$, *and* (1-13) *when* $n \ge 11$.

Remark 1.4. (i) While writing this paper, we learned via personal communication that Figalli and Mayboroda have independently proved (1-10) in Theorem 1.2 with $n = 10$ via a similar argument.

(ii) In Theorem 1.2 and Corollary 1.3 we only consider bounded smooth convex domains so as to avoid technical discussions on the boundary estimate. We believe that after suitable modifications, it is possible to relax this assumption to bounded domains of C^3 class, as in [Cabré et al. 2020].

As an application of Theorem 1.2, we prove the following Liouville property for stable solutions to the equation

$$-\Delta u = f(u) \quad \text{in } \mathbb{R}^n \tag{1-14}$$

for $f \in C^{0,1}(\mathbb{R}^n)$.

Theorem 1.5. *Let* $n \ge 10$ *and* $0 \le f \in C^{0,1}_{\mathrm{loc}}(\mathbb{R})$. *Suppose that* $u \in C^2(\mathbb{R}^n)$ *is a nonconstant stable solution to* (1-14) *in* $\mathbb{R}^n$.

If u *is nonconstant, then*

$$⨍_{B_{4R}\setminus B_R} |u(x)|\,dx \ge \begin{cases} c\log R & \text{for all } R \ge R_0, \quad \text{if } n = 10, \\ cR^{-n/2+2+\sqrt{n-1}} & \text{for all } R \ge R_0, \quad \text{if } n \ge 11, \end{cases} \tag{1-15}$$

for some $R_0 \ge 2$ *and* $c > 0$.

In particular, if u *satisfies the growth condition*

$$|u(x)| = \begin{cases} o(\log|x|) & \text{as } |x| \to +\infty, \quad \text{when } n = 10, \\ o(|x|^{-n/2+2+\sqrt{n-1}}) & \text{as } |x| \to +\infty, \quad \text{when } n \ge 11, \end{cases} \tag{1-16}$$

then u *must be a constant.*

This problem has attracted a lot of attention in the literature. First of all, for radial stable solutions, Villegas [2007] obtained the following sharp Liouville property based on the monotone property by Cabré and Capella [2004]; see also [Dupaigne 2011; Villegas 2007].

Theorem 1.6 [Villegas 2007]. *Let $n \geq 2$ and $f \in C^1(\mathbb{R})$. Suppose that $u \in C^2(\mathbb{R}^n)$ is a radial stable solution to (1-14).*

If u is not constant, then

$$|u(x)| \geq \begin{cases} M \log|x| & \text{whenever } |x| \geq r_0, \quad \text{when } n = 10, \\ M|x|^{-n/2+\sqrt{n-1}+2} & \text{whenever } |x| \geq r_0, \quad \text{when } n \neq 10, \end{cases} \tag{1-17}$$

for some $M > 0$ and $r_0 \geq 10$.

In particular, if u satisfies the growth condition (1-16), then u must be a constant.

Note that for radial stable solutions $u(x)$, the condition (1-15) is equivalent to (1-17). Indeed, by [Villegas 2007], $u(r) = u(re_1)$ is always monotone, and hence

$$\min\{|u(4r)|, |u(r)|\} \leq \fint_{B_{4r}\setminus B_r} |u(x)|\,dx \leq \max\{|u(4r)|, |u(r)|\} \quad \text{for all } r > 0,$$

which implies the equivalence between (1-15) and (1-17).

Let $\beta_n = -\frac{1}{2}n + 2 + \sqrt{n-1}$. Then $\beta_n < 0$ when $n \geq 11$, and $\beta_n > 0$ when $n \leq 9$. The sharpness of Theorem 1.6 (and also Theorem 1.5) is demonstrated in the following sense by Villegas [2007] (with a slight modification at $n = 10$).

(i) When $n \neq 10$, the radial smooth function $(1+|x|^2)^{\beta_n/2}$ is a stable solution to the equation $-\Delta u = f_{\beta_n}(u)$ in $\mathbb{R}^n$, where, when $n \geq 11$,

$$f_{\beta_n}(s) := \begin{cases} 0 & \text{if } s \leq 0, \\ \beta_n(\beta_n - 2)s^{1-4/\beta_n} - \beta_n(\beta_n + n - 2)s^{1-2/\beta_n} & \text{if } s > 0, \end{cases}$$

and, when $n \leq 9$,

$$f_{\beta_n}(s) := \begin{cases} \beta_n(\beta_n - 2)s^{1-4/\beta_n} - \beta_n(\beta_n + n - 2)s^{1-2/\beta_n} & \text{if } s \geq 1, \\ -(\beta_n - 2)(n+2)(s-1) - n\beta_n & \text{if } s < 1. \end{cases}$$

See [Villegas 2007, Example 3.1] for details. Note that, when $n \geq 11$, by $\beta_n < 0$ and $\beta_n + n - 2 > 0$, we have $f_{\beta_n} \geq 0$ in $\mathbb{R}$, while, when $n \leq 9$, we have that $f_{\beta_n} \leq 0$ in $\mathbb{R}$.

(ii) When $n = 10$, the radial smooth function $-\frac{1}{2}\log(1 + |x|^2)$ is a stable solution to the equation $-\Delta u = f(u)$ in $\mathbb{R}^n$, where $f(s) = (n-2)e^{2s} + 2e^{4s} \geq 0$ in $\mathbb{R}$. This is a slight modification of [Villegas 2007, Example 3.1] with $n = 10$. See the Appendix for details.

For general (nonradial) stable solutions $u \in C^2(\mathbb{R}^n)$ to $-\Delta u = f(u)$ in $\mathbb{R}^n$, it is then natural to ask if certain Liouville properties similar to Theorem 1.6 hold. Namely, when f satisfies certain regularity assumptions,

- if u satisfies (1-16), then is it necessary that u is a constant?
- if u is nonconstant, is it possible to give some sharp lower bound for $|u|$ toward ∞?

Suppose that $0 \leq f \in C^1(\mathbb{R})$ and $u \in C^2(\mathbb{R}^n)$ is a stable solution to (1-14). When $n \leq 4$, Dupaigne and Farina [2023] proved that if $|u|$ is bounded, then u must be a constant. Recently, with the aid of [Cabré et al. 2020], Dupaigne and Farina [2022] showed that if $n \leq 9$ and $u(x) \geq -C[1 + \log|x|]^\gamma$ for some

$\gamma \geq 1$ and $C > 0$, or if $n = 10$ and $u \geq -C$ for some constant $C > 0$, then u must be a constant. When $n \geq 10$, our result Theorem 1.5 finally answers the two questions above.

Ideas of the proofs. We sketch the ideas to prove Theorems 1.2 and 1.5. All of them heavily rely on the following decay estimate on the Dirichlet energy.

Lemma 1.7. *Let $n \geq 10$ and $f \in C^{0,1}(\mathbb{R})$. For any $y \in \mathbb{R}^n$ and $t > 0$, if $u \in C^2(B_{2t}(y))$ is a stable solution to (1-1) in $B_{2t}(y)$, one has*

$$\left(\frac{r}{t}\right)^{-2(1+\sqrt{n-1})} \int_{B_r(y)} |Du|^2\,dx \leq C(n) \int_{B_t(y)\setminus B_{t/2}(y)} |Du|^2\,dx \quad \text{for all } r \leq \frac{t}{2}. \tag{1-18}$$

See Section 2 for the proof of Lemma 1.7; the key point is that we take a suitable test function in a celebrated lemma of [Cabré et al. 2020] (see Lemma 2.1 below). One may compare it with [Cabré et al. 2020, Lemma 2.1] in the case where $3 \leq n \leq 9$.

We also recall the following lemma, which was essentially established in [Cabré et al. 2020, Lemma A.2 and Proposition 2.5] together with the proofs therein. For the convenience of the reader, we give a sketch of the proof at the beginning of Section 3.

Lemma 1.8. *Let $0 \leq f \in C^{0,1}(\mathbb{R})$. For any stable solution $u \in C^2(B_{2t}(y))$ to (1-1) in $B_{2t}(y)$, one has*

$$\left(\int_{B_{t/2}(y)} |Du|^2\,dx\right)^{1/2} \leq C(n)t^{-n/2} \int_{B_t(y)} |Du|\,dx \tag{1-19}$$

and

$$\int_{B_{t/2}(y)} |Du|\,dx \leq C(n)t^{-1} \int_{B_t(y)} |u|\,dx. \tag{1-20}$$

Applying Lemma 1.7, Lemma 1.8 and some known boundary estimate, we are able to prove Theorem 1.2 and Corollary 1.3. This is clarified in Section 3.

In order to prove Theorem 1.5, an auxiliary and crucial proposition is shown in Section 4, which is specifically applied in the case $n = 10$.

Proposition 1.9. *Let $n \geq 3$. Suppose that $u \in W^{1,1}_{\mathrm{loc}}(\mathbb{R}^n)$ is superharmonic, that is, $-\Delta u \geq 0$ in $\mathbb{R}^n$ in the distributional sense. For any $0 < r < R < \infty$, we have*

$$\int_{B_R\setminus B_r} |Du||x|^{-n+1}\,dx \leq C(n) \fint_{B_{r/2}\setminus B_{r/4}} |u|\,dz + C(n) \fint_{B_{4R}\setminus B_{2R}} |u|\,dz. \tag{1-21}$$

The main idea of showing Proposition 1.9 goes as follows. First, it is known that

$$Du_\delta(x) = D\Delta^{-1}[\Delta(u_\delta\eta)](x) \quad \text{for } x \in B_R \setminus B_r,$$

where u_δ is a standard smooth mollification of u and η is a suitable cut-off function. Next, thanks to the key fact $-\Delta u_\delta \geq 0$, via some subtle kernel estimates and integration by parts, we are able to prove (1-21) for u_δ, and then a standard approximation gives (1-21) as desired.

Theorem 1.5 is eventually proved in the last section. The case $n \geq 11$ is relatively simple. In fact, by Lemmas 1.7 and 1.8, one can build up the following:

$$r^{-(1+\sqrt{n-1})}\left(\int_{B_r} |Du|^2\,dx\right)^{1/2} \leq C(n) R^{n/2-2-\sqrt{n-1}} \fint_{B_{3R}\setminus B_{3R/4}} |u|\,dx \quad \text{for all } 0 < r < R < \infty$$

for stable solutions, which allows us to conclude Theorem 1.5 for $n \geq 11$.

As for the case when $n = 10$, we first employ Lemma 1.7 and repeat Lemma 1.8 to get

$$r^{-(1+\sqrt{n-1})}\left(\int_{B_r} |Du|^2\,dx\right)^{1/2} \leq C(n) \frac{1}{\log R} \int_{B_{R^2}\setminus B_4} |Du||x|^{-n+1}\,dx \quad \text{for all } 0 < r < R < \infty,$$

which, when $R > 2^5 + r > 4$ and thanks to Proposition 1.9 with r and R therein replaced by 4 and R^2, is then bounded from above by

$$C(n)\frac{1}{\log R}\left(\fint_{B_2\setminus B_1} |u(z)|\,dz + \fint_{B_{4R^2}\setminus B_{2R^2}} |u(z)|\,dz\right).$$

From this we conclude Theorem 1.5 when $n = 10$.

2. Proof of Lemma 1.7

Towards Lemma 1.7 we recall the following a priori bound by [Cabré et al. 2020, Lemma 2.1], which is obtained by taking the test function $(x \cdot Du)\eta$ in the stability condition (1-3).

Lemma 2.1. *Let $u \in C^2(B_1)$ be a stable solution to (1-1) in B_1, with $f \in C^{0,1}(\mathbb{R})$. Then, for all cut-off functions $\eta \in C_c^{0,1}(B_1)$,*

$$\int_{B_1} |x \cdot Du|^2 |D\eta|^2\,dx \geq (n-2)\int_{B_1} |Du|^2\eta^2\,dx + 2\int_{B_1} |Du|^2 (x\cdot D\eta)\eta\,dx - 4\int_{B_1} (x\cdot Du)(Du\cdot D\eta)\eta\,dx. \tag{2-1}$$

For convenience, for any $0 < r < t < \infty$ and $y \in \mathbb{R}^n$, we define the annulus $A_{r,t}(y) := B_t(y) \setminus \overline{B_r(y)}$; for simplicity, we write $A_{r,t} = A_{r,t}(0)$.

Proof of Lemma 1.7. It suffices to prove

$$\left(\frac{r}{t}\right)^{-2(1+\sqrt{n-1})} \int_{B_r(y)} |Du|^2\,dx \leq C(n) \int_{A_{r,t}(y)} |Du|^2\,dx \quad \text{for all } r \leq \frac{t}{2}. \tag{2-2}$$

Indeed, applying (2-2) to $\frac{1}{2}t$ and t, one has

$$\left(\frac{1}{2}\right)^{-2(1+\sqrt{n-1})} \int_{B_{t/2}(y)} |Du|^2\,dx \leq C(n) \int_{A_{t/2,t}(y)} |Du|^2\,dx. \tag{2-3}$$

If $\frac{1}{4}t \le r < \frac{1}{2}t$, by $B_r(y) \subset B_{t/2}(y)$ and $\frac{1}{4} \le r/t \le 1$, inequality (2-3) gives

$$\left(\frac{r}{t}\right)^{-2(1+\sqrt{n-1})} \int_{B_r(y)} |Du|^2\,dx \le C(n) \int_{A_{t/2,t}(y)} |Du|^2\,dx. \tag{2-4}$$

If $0 < r < \frac{1}{4}t$, applying (2-2) to r and $\frac{1}{2}t$, and noting $A_{r,t/2} \subset B_{t/2}$, one gets

$$\left(\frac{r}{t/2}\right)^{-2(1+\sqrt{n-1})} \int_{B_r(y)} |Du|^2\,dx \le C(n) \int_{A_{r,t/2}(y)} |Du|^2\,dx \le C(n) \int_{B_{t/2}(y)} |Du|^2\,dx,$$

which together with (2-3) yields

$$\left(\frac{r}{t}\right)^{-2(1+\sqrt{n-1})} \int_{B_r(y)} |Du|^2\,dx \le C(n) \int_{A_{t/2,t}(y)} |Du|^2\,dx.$$

From this and (2-4) we conclude (1-18).

To prove (2-2), without loss of generality we may assume that $t = 1$ and $y = 0$. Indeed, if $u(x)$ is a stable solution to $-\Delta u = f(u)$ in $B_{2t}(y)$, then $v(x) = u(tx + y)$ is the stable solution to $-\Delta v = t^2 f(v)$ in B_2. Note that, up to a change of variable, u satisfies (2-2) if and only if v satisfies (2-2) with $t = 1$ and $y = 0$.

Write $a = 2(1 + \sqrt{n-1})$. Let $r \in \left(0, \frac{1}{2}\right]$ be fixed and set

$$\eta = \begin{cases} r^{-a/2} & \text{if } 0 \le |x| \le r, \\ |x|^{-a/2}\phi & \text{if } r < |x| \le 1, \end{cases} \tag{2-5}$$

where $\phi \in C_c^\infty(B_1)$ satisfies

$$\phi = 1 \quad \text{in } B_{3/4} \qquad \text{and} \qquad |D\phi| \le 5\chi_{B_1 \setminus B_{3/4}}. \tag{2-6}$$

Clearly, $\eta \in C_c^{0,1}(B_1)$. Since $\eta = r^{-a/2}$ in B_r and hence $D\eta = 0$ in B_r, substituting η in inequality (2-1) one has

$$\begin{aligned} \int_{A_{r,1}} |x \cdot Du|^2 |D\eta|^2\,dx \ge (n-2) r^{-a} \int_{B_r} |Du|^2\,dx + (n-2) \int_{A_{r,1}} |Du|^2 \eta^2\,dx \\ + 2\int_{A_{r,1}} |Du|^2 (x \cdot D\eta)\eta\,dx - 4\int_{A_{r,1}} (x \cdot Du)(Du \cdot D\eta)\eta\,dx. \end{aligned} \tag{2-7}$$

Noting that

$$D\eta = -\tfrac{1}{2} a |x|^{-a/2-2} x\phi + |x|^{-a/2} D\phi \quad \text{in } A_{r,1},$$

one has

$$\begin{aligned} &2\int_{A_{r,1}} |Du|^2 (x \cdot D\eta)\eta\,dx - 4\int_{A_{r,1}} (x \cdot Du)(Du \cdot D\eta)\eta\,dx \\ &= -a \int_{A_{r,1}} |Du|^2 |x|^{-a} \phi^2\,dx + 2\int_{A_{r,1}} |Du|^2 (x \cdot D\phi)\phi |x|^{-a}\,dx + 2a \int_{a_{r,1}} (x \cdot du)^2 |x|^{-a-2} \phi^2\,dx \\ &\qquad - 4\int_{A_{r,1}} (x \cdot Du)(Du \cdot D\phi)\phi |x|^{-a}\,dx. \end{aligned} \tag{2-8}$$

Moreover, by

$$|D\eta|^2 = \tfrac{1}{4}a^2|x|^{-a-2}\phi^2 - 2a|x|^{-a-2}(x\cdot D\phi)\phi + |x|^{-a}|D\phi|^2,$$

one can write

$$\int_{A_{r,1}} (Du\cdot x)^2|D\eta|^2\,dx = \frac{a^2}{4}\int_{A_{r,1}} (Du\cdot x)^2|x|^{-a-2}\phi^2\,dx + \int_{A_{r,1}} (Du\cdot x)^2|x|^{-a}|D\phi|^2\,dx \\ - a\int_{A_{r,1}} (Du\cdot x)^2|x|^{-a-2}(x\cdot D\phi)\phi\,dx. \quad (2\text{-}9)$$

Using (2-8) for the left-hand side of (2-7), and (2-9) for the last two terms in the right-hand side of (2-7), and then moving all terms including $D\phi$ to the left-hand side and all other terms to the right-hand side, we have

$$\begin{aligned}
&\int_{A_{r,1}} |x\cdot Du|^2|D\phi|^2|x|^{-a}\,dx - 2\int_{A_{r,1}} |Du|^2(x\cdot D\phi)\phi|x|^{-a}\,dx \\
&\quad + 4\int_{A_{r,1}} (x\cdot Du)(Du\cdot D\phi)\phi|x|^{-a}\,dx - a\int_{A_{r,1}} |x|^{-a-2}(x\cdot Du)^2\phi(x\cdot D\phi)\,dx \\
&\quad \ge (n-2)r^{-a}\int_{B_r} |Du|^2\,dx + (n-2)\int_{A_{r,1}} |Du|^2|x|^{-a}\phi^2\,dx \\
&\qquad - a\int_{A_{r,1}} |Du|^2|x|^{-a}\phi^2\,dx + 2a\int_{A_{r,1}} (x\cdot Du)^2|x|^{-a-2}\phi^2\,dx - \frac{a^2}{4}\int_{A_{r,1}} (Du\cdot x)^2|x|^{-a-2}\phi^2\,dx \\
&\quad = (n-2)r^{-a}\int_{B_r} |Du|^2\,dx + \int_{A_{r,1}} \bigl\{(n-2-a)|Du|^2 + \bigl(2a - \tfrac{1}{4}a^2\bigr)(Du\cdot x)^2|x|^{-2}\bigr\}|x|^{-a}\phi^2\,dx. \quad (2\text{-}10)
\end{aligned}$$

Note that, by $|D\phi| = 0$ in $B_{3/4}$ and $|D\phi| \le 5$ in B_1 as in (2-6) and $a > 2$,

$$\begin{aligned}
&\int_{A_{r,1}} |x\cdot Du|^2|D\phi|^2|x|^{-a}\,dx - 2\int_{A_{r,1}} |Du|^2(x\cdot D\phi)\phi|x|^{-a}\,dx \\
&\quad + 4\int_{A_{r,1}} (x\cdot Du)(Du\cdot D\phi)\phi|x|^{-a}\,dx - a\int_{A_{r,1}} |x|^{-a-2}(x\cdot Du)^2\phi(x\cdot D\phi)\,dx \\
&\qquad \le C(n)\int_{A_{3/4,1}} |Du|^2\,dx. \quad (2\text{-}11)
\end{aligned}$$

Additionally, note that $n \ge 10$ implies $a = 2(1+\sqrt{n-1}) \ge 8$, and hence

$$2a - \tfrac{1}{4}a^2 = \tfrac{1}{4}a(8-a) \le 0.$$

By $|x|^{-1}|x\cdot Du| \le |Du|$ in $A_{r,1}$, we have

$$(n-2-a)|Du|^2 + \bigl(2a - \tfrac{1}{4}a^2\bigr)(Du\cdot x)^2|x|^{-2} \ge \bigl(n-2+a-\tfrac{1}{4}a^2\bigr)|Du|^2.$$

Since

$$n-2+a-\tfrac{1}{4}a^2 = -\bigl(\tfrac{1}{2}a - [1-\sqrt{n-1}]\bigr)\bigl(\tfrac{1}{2}a - [1+\sqrt{n-1}]\bigr) = 0,$$

we have

$$(n-2-a)|Du|^2 + \bigl(2a - \tfrac{1}{4}a^2\bigr)(Du\cdot x)^2|x|^{-2} \ge 0 \quad \text{in } A_{r,1}, \quad (2\text{-}12)$$

which means that the last term in the right-hand side of (2-10) is nonnegative. From this, together with (2-10) and (2-11), we conclude (2-2). The proof is complete. □

Remark 2.2. Recall that in [Cabré et al. 2020], the authors used the test function $\eta = |x|^{-a/2}\xi$ with $\xi \in C_c^\infty(B_1)$, which was not enough to get (2-2).

3. Proofs of Equation (1-1) and Corollary 1.3

In this section we prove Theorem 1.2 and Corollary 1.3. First, we sketch a proof of Lemma 1.8.

Proof of Lemma 1.8. Up to considering $v(x) = u(tx + y)$, we may assume that $t = 1$ and $y = 0$. Inequality (1-20) is given by [Cabré et al. 2020, Lemma A.2]. Inequality (1-19) reads as $\|Du\|_{L^2(B_{1/2})} \le C(n)\|Du\|_{L^1(B_1)}$ and will follow from the proof of [Cabré et al. 2020, Proposition 2.5], where the authors proved that

$$\|Du\|_{L^2(B_{1/2})} \le C(n)\|u\|_{L^1(B_1)}. \tag{3-1}$$

In their proof, first they obtained a bound of $\|Du\|_{L^2(B_{1/2})}$ via $\|Du\|_{L^1(B_{1/2})}$ and some other small terms. Next, they used $\|Du\|_{L^1(B_{1/2})} \le C(n)\|u\|_{L^1(B_1)}$. Finally, via an iteration argument, they got (3-1). If we directly apply the iteration argument without using $\|Du\|_{L^1(B_{1/2})} \le C(n)\|u\|_{L^1(B_1)}$, we get $\|Du\|_{L^2(B_{1/2})} \le C(n)\|Du\|_{L^1(B_1)}$. □

Recall that $u_E = ⨍_E u\,dx$ denotes the integral average of u on a measurable set E. The interior regularity (1-10) and (1-11) in Theorem 1.2 is a consequence of Lemma 1.7 and (1-19), together with a standard embedding argument.

Proofs of (1-10) and (1-11) in Theorem 1.2. Let $u \in C^2(B_2)$ be a stable solution to (1-1). Write $\beta = n - 2 - 2\sqrt{n-1}$. For any $y \in B_{1/2}$, if $r > \frac{1}{8}$, by Lemma 1.8 we have

$$r^{\beta-n}\int_{B_r(y)\cap B_{1/2}} |Du|^2\,dx \le C(n) ⨍_{B_{1/2}} |Du|^2\,dx \le C(n)\|u\|^2_{L^1(B_1)},$$

and if $0 < r < \frac{1}{8}$, by Lemmas 1.7 and 1.8 again we have

$$r^{\beta-n}\int_{B_r(y)\cap B_{1/2}} |Du|^2\,dx \le r^\beta ⨍_{B_r(y)} |Du|^2\,dx \le C(n) ⨍_{B_{1/4}(y)} |Du|^2\,dx \le C(n)\|u\|^2_{L^1(B_1)}.$$

This means that $Du \in M^{2,\beta}(B_{1/2})$ with $\|Du\|_{M^{2,\beta}(B_{1/2})} \le C(n)\|u\|_{L^1(B_1)}$.

If $n = 10$, then $\beta = 2$ and $2\beta/(\beta-2) = \infty$. Thanks to the Sobolev–Poincaré inequality, one can easily check that $Du \in M^{2,\beta}(B_{1/2})$ implies $u \in \mathrm{BMO}(B_{1/2})$, with a norm bound

$$\|u\|_{\mathrm{BMO}(B_{1/2})} \le C(n)\|Du\|_{M^{2,\beta}(B_{1/2})}.$$

If $n \ge 11$, then $p_n = 2\beta/(\beta-2) < \infty$ and $\beta = 2 + 4/(p_n - 2)$. By the embedding result in [Adams 1975] and also [Cabré and Charro 2021, Section 4], $Du \in M^{2,\beta}(B_{1/2})$ implies $u \in M^{2\beta/(\beta-2),\beta}(B_{1/2})$, with its norm bound

$$\|u\|_{M^{p_n,\beta}(B_{1/2})} \le C(n)\|Du\|_{M^{2,\beta}(B_{1/2})}.$$

This proves (1-10) and (1-11). □

To prove the global regularity (1-12) and (1-13) in Theorem 1.2, we need the following a priori L^∞-bound in a neighborhood of $\partial\Omega$ for a C^2 solution when Ω is a bounded smooth convex domain; see [Cabré 2010, Proposition 3.2] and [Chen and Li 1993; de Figueiredo et al. 1982; Gidas et al. 1979]. For $\rho > 0$, we write

$$\Omega_\rho := \{x \in \Omega : \operatorname{dist}(x, \partial\Omega) < \rho\}.$$

Lemma 3.1. *Suppose that $f \in C^{0,1}(\mathbb{R})$ is nonnegative and Ω is a smooth convex domain in $\mathbb{R}^n$. There exist positive constants ρ and γ depending only on the domain Ω such that, for any positive solution $u \in C^2(\Omega) \cap C^0(\overline{\Omega})$ to (1-1), one has*

$$\|u\|_{L^\infty(\Omega_\rho)} \le \frac{1}{\gamma}\|u\|_{L^1(\Omega)}. \tag{3-2}$$

Note that, as $f \ge 0$, the maximum principle shows that any solution $u \in C^2(\Omega) \cap C^0(\overline{\Omega})$ to (1-1) with zero boundary is always nonnegative, and the strong maximum principle further shows that u is always positive in the domain Ω.

Proofs of (1-12) and (1-13) in Theorem 1.1. Let $\beta = n - 2 - 2\sqrt{n-1}$, and let ρ, γ be as in Lemma 3.1. We first consider the case $n \ge 11$. For any $y \in \overline{\Omega}$ and $r > 0$, write

$$\begin{aligned} r^{\beta-n}\int_{\Omega\cap B_r(y)} |u|^{p_n}\,dx &= r^{\beta-n}\int_{\Omega_\rho\cap B_r(y)} |u|^{p_n}\,dx + r^{\beta-n}\int_{(\Omega\setminus\Omega_\rho)\cap B_r(y)} |u|^{p_n}\,dx \\ &:= \Phi_1(y,r) + \Phi_2(y,r). \end{aligned}$$

To see (1-12), we only need to prove $\Phi_1(y,r) \le C(n,\Omega)\|u\|^{p_n}_{L^1(\Omega)}$ and $\Phi_2(y,r) \le C(n,\rho,\Omega)\|u\|^{p_n}_{L^1(\Omega)}$ for any $y \in \Omega$ and $r > 0$.

Note that

$$r^{\beta-n}|\Omega_\rho \cap B_r(y)| \le \begin{cases} C(n) & \text{when } r < 1, \\ |\Omega_\rho| & \text{when } r > 1, \end{cases}$$

so by $2 < \beta < n$ and Lemma 3.1, we have

$$\Phi_1(y,r) \le r^{\beta-n}|\Omega_\rho \cap B_r(y)|\,\|u\|^{p_n}_{L^\infty(\Omega_\rho)} \le C(n,\Omega)\|u\|^{p_n}_{L^1(\Omega)}.$$

Next, to get $\Phi_2(y,r) \le C(n,\rho,\Omega)\|u\|^{p_n}_{L^1(\Omega)}$ for any $y \in \Omega$ and $r > 0$, we only need to consider $y \in \Omega\setminus\Omega_\rho$ and $0 < r < \frac{1}{8}\rho$. Indeed, for $y \in \Omega_\rho$, if $r < \operatorname{dist}(y, \Omega\setminus\Omega_\rho)$, then $\Phi_2(y,r) = 0$, and if $r \ge \operatorname{dist}(y, \Omega\setminus\Omega_\rho)$, then $\Phi_2(y,r) \le C(n)\Phi_2(\bar{y}, 2r)$, where $\bar{y}$ is the closest point in $\Omega\setminus\Omega_\rho$ and $B(y,r) \subset B(\bar{y}, 2r)$. Moreover, for any $y \in \Omega\setminus\Omega_\rho$ and $r \ge \frac{1}{8}\rho$,

$$\Phi_2(y,r) \le \rho^{\beta-n}\int_{\Omega\setminus\Omega_\rho} |u|^{p_n}\,dx \le \sum_{i=1}^N \rho^{\beta-n}\int_{\Omega\setminus\Omega_\rho\cap B_{\rho/9}(x_i)} |u|^{p_n}\,dx = \sum_{i=1}^N \Phi\big(x_i, \tfrac{1}{9}\rho\big),$$

where $\big\{B\big(x_i, \frac{1}{9}\rho\big)\big\}_{i=1}^N$ is a cover of the compact set $\Omega\setminus\Omega_\rho$, $\{x_i\}_{i=1}^N \subset \Omega\setminus\Omega_\rho$ and N depends only on Ω and ρ.

On the other hand, for any $y\in\Omega\backslash\Omega_\rho$ and $0<r<\frac{1}{8}\rho$, since u is a stable solution in $B_\rho(y)\subset\Omega$, by (1-11) with a scaling argument, we have $u\in M^{p_n,\beta}(B_{\rho/8}(y))$ with $\|u\|_{M^{p_n,\beta}(B_{\rho/8}(y))}\le C(n,\rho)\|u\|_{L^1(B_{\rho/2}(y))}$, in particular

$$\Phi_2(y,r)\le r^\beta \fint_{B_r(y)} |u|^{p_n}\,dx\le C(n,\rho)\|u\|^{p_n}_{L^1(\Omega)}$$

as desired. This proves (1-13).

In the case $n=10$, for any $y\in\Omega$, if $r>\frac{1}{9}\rho$, we have

$$r^{-n}\int_{\Omega\cap B_r(y)}|u|\,dx\le C(n,\rho)\|u\|_{L^1(\Omega)}.$$

Below we assume that $0<r<\frac{1}{9}\rho$. If $y\in\Omega\backslash\Omega_{8\rho/9}$, we have $\rho<\frac{9}{8}\operatorname{dist}(y,\partial\Omega)$. Since $0<r<\frac{1}{8}\operatorname{dist}(y,\partial\Omega)$ and u is a stable solution in $B_{\operatorname{dist}(y,\partial\Omega)}(y)\subset\Omega$, by (1-10) with a scaling we have

$$\fint_{B_r(y)}|u-u_{B_r(y)}|\,dx\le C(n,\rho)\|u\|_{L^1(B_{\operatorname{dist}(y,\partial\Omega)}(y))}\le C(n,\rho)\|u\|_{L^1(\Omega)}.$$

For $y\in\Omega_{8\rho/9}$, noting $0<r<\frac{1}{9}\rho\le\operatorname{dist}(y,\partial\Omega_\rho)$, one has $\Omega\cap B_r(y)\subset\Omega\setminus\Omega_\rho$. Thus

$$r^{-n}\int_{\Omega\cap B_r(y)}|u|\,dx=r^{-n}\int_{\Omega_\rho\cap B_r(y)}|u|\,dx\le C(n,\rho)\|u\|_{L^1(\Omega)}.$$

Combining these estimates, we obtain (1-12). □

We finally prove Corollary 1.3.

Proof of Corollary 1.3. Let $u\in W^{1,2}_0(\Omega)$ be a stable solution to (1-1) with zero boundary. By [Dupaigne 2011, Corollary 3.2.1] (see also the proof in [Cabré et al. 2020, Theorem 4.1] and [Dupaigne and Farina 2023, Theorem 5]), there is a nonnegative, nondecreasing sequence (f_k) of convex functions in $C^1(\mathbb{R})$ such that $f_k\to f$ pointwise in $[0,\infty)$ and a nondecreasing sequence (u_k) in $C^2(\overline{\Omega})\cap W^{1,2}_0(\Omega)$ such that u_k is a weak stable solution to

$$-\Delta u_k=f_k(u_k)\quad\text{in }\Omega,\qquad u_k=0\quad\text{on }\partial\Omega \tag{3-3}$$

and

$$u_k\to u\quad\text{in }W^{1,2}(\Omega)\qquad\text{as }k\to+\infty.$$

If $n=10$, applying (1-12) to u_k, one has

$$\fint_{\Omega\cap B_r(y)}\left|u_k(x)-\fint_{\Omega\cap B_r(y)}u_k\,dz\right|dx\le\|u_k\|_{\mathrm{BMO}(\Omega)}\le C(n,\Omega)\int_\Omega|u_k|\,dx\quad\text{for all }r>0\text{ for all }y\in\overline{\Omega}.$$

Since $u_k\to u$ in $W^{1,2}(\Omega)$ as $k\to+\infty$, we conclude that $\|u\|_{\mathrm{BMO}(\Omega)}\le C(n)\|u\|_{L^1(\Omega)}$ as desired.

If $n\ge 11$, applying (1-13) to u_k, we have

$$r^{\beta-n}\int_{\Omega\cap B_r(y)}|u_k|^{p_n}\,dx\le C(n,\Omega,\rho)(\|u_k\|_{L^1(\Omega)})^{p_n}\quad\text{for all }y\in\overline{\Omega}\text{ for all }r>0, \tag{3-4}$$

where $\beta = 2p_n/(p_n-2) \in (0,n)$. Since $u_k \to u$ in $W^{1,2}(\Omega)$ as $k \to +\infty$, we deduce that $u_k \in L^{p_n}(\Omega)$ uniformly in $k \geq 0$, and hence $u_k \to u$ weakly in $L^{p_n}(\Omega)$. Thus, letting $k \to +\infty$ in (3-4), we conclude $\|u\|_{M^{p_n,\beta}(\Omega)} \leq C(n)\|u\|_{L^1(\Omega)}$ as desired. □

4. Proof of Proposition 1.9

Let $0 < r < R < \infty$. Let $\eta \in C_c^\infty(A_{r/4,4R})$ satisfy

$$0 \leq \eta \leq 1 \quad \text{in } A_{r/4,4R} \qquad \text{and} \qquad \eta = 1 \quad \text{in } A_{r/2,2R}, \tag{4-1}$$

$$|D\eta|^2 + |D^2\eta| \leq \frac{C}{r^2} \quad \text{in } A_{r/4,r/2} \qquad \text{and} \qquad |D\eta|^2 + |D^2\eta| \leq \frac{C}{R^2} \quad \text{in } A_{2R,4R}, \tag{4-2}$$

where $C > 0$ is a universal constant.

Let $u_\delta = u * \phi_\delta$ for $\delta > 0$, where ϕ_δ is the standard smooth mollifier and is supported in $B(0,\delta)$. Recall that $u \in W^{1,1}_{\mathrm{loc}}(\mathbb{R}^n)$ and $u_\delta \to u$ in $W^{1,1}_{\mathrm{loc}}(\mathbb{R}^n)$. Since $-\Delta u \geq 0$ is a locally finite measure, we have $-\Delta u_\delta = (-\Delta u) * \phi_\delta \geq 0$ everywhere. By $u_\delta \eta \in C_c^\infty(\mathbb{R}^n)$, one has

$$u_\delta\eta(x) = \Delta^{-1}[\Delta(u_\delta\eta)](x) = c(n)\int_{\mathbb{R}^n} \frac{1}{|x-y|^{n-2}}\Delta(u_\delta\eta)(y)\,dy \quad \text{for all } x \in \mathbb{R}^n,$$

and hence

$$D(u_\delta\eta)(x) = D\Delta^{-1}[\Delta(u_\delta\eta)](x) = c(n)(2-n)\int_{\mathbb{R}^n} \frac{x-y}{|x-y|^{n}}\Delta(u_\delta\eta)(y)\,dy \quad \text{for all } x \in \mathbb{R}^n.$$

Noting

$$\Delta(u_\delta\eta)(y) = \Delta u_\delta(y)\eta(y) + \Delta\eta(y)u_\delta(y) + 2Du_\delta(y)\cdot D\eta(y),$$

for $0 < \delta \ll \frac{1}{8}r$, we write

$$\begin{aligned}
\int_{A_{r,R}} |Du_\delta||x|^{-n+1}\,dx &= \int_{A(r,R)} |D(u_\delta\eta)||x|^{-n+1}\,dx \\
&= \int_{A_{r,R}} \left|\int_{\mathbb{R}^n} \frac{x-y}{|x-y|^n}\Delta(u_\delta\eta)(y)\,dy\right| |x|^{-n+1}\,dx \\
&\leq C(n)\int_{A_{r,R}} \left|\int_{\mathbb{R}^n} \frac{x-y}{|x-y|^n}\Delta u_\delta(y)\eta(y)\,dy\right| |x|^{-n+1}\,dx \\
&\quad + C(n)\int_{A_{r,R}} \left|\int_{\mathbb{R}^n} \frac{x-y}{|x-y|^n}u_\delta(y)\Delta\eta(y)\,dy\right| |x|^{-n+1}\,dx \\
&\quad + C(n)\int_{A_{r,R}} \left|\int_{\mathbb{R}^n} \frac{x-y}{|x-y|^n}Du_\delta(y)\cdot D\eta(y)\,dy\right| |x|^{-n+1}\,dx \\
&=: I_1 + I_2 + I_3.
\end{aligned}$$

In order to control I_1 from above, first by $-\Delta u_\delta \geq 0$ and (4-1), one has

$$I_1 \leq \int_{\mathbb{R}^n}\left(\int_{\mathbb{R}^n} |x-y|^{-n+1}|x|^{-n+1}\,dx\right)(-\Delta u_\delta)(y)\eta(y)\,dy.$$

Employing the triangle inequality, for $y \in \mathbb{R}^n$, we further get

$$\begin{aligned}\int_{\mathbb{R}^n} |x-y|^{-n+1}|x|^{-n+1}\,dx &\le 2^{n-1}\int_{\{|x|>2|y|\}} |x|^{-2n+2}\,dx + 2^{n-1}\int_{\{|x|<|y|/2\}} |x|^{-n+1}|y|^{-n+1}\,dx \\ &\qquad + \int_{\{|y|/2\le |x|\le 2|y|\}} |x-y|^{-n+1}|y|^{-n+1}\,dx \\ &\le C(n)|y|^{-n+2} + C(n)|y|^{-n+2} + \int_{\{|y-x|\le 3|y|\}} |x-y|^{-n+1}|y|^{-n+1}\,dx \\ &\le C(n)|y|^{-n+2}. \end{aligned} \tag{4-3}$$

This together with $-\Delta u_\delta \ge 0$ again gives

$$I_1 \le C(n)\int_{\mathbb{R}^n} (-\Delta u_\delta)|y|^{-n+2}\eta(y)\,dy.$$

Via integration by parts and using $\eta \in C_c^\infty(A_{r/4,4R})$, we have

$$\int_{\mathbb{R}^n} (-\Delta u_\delta)|y|^{-n+2}\eta(y)\,dy = \int_{A_{r/4,4R}} u_\delta[-\Delta|y|^{-n+2}\eta(y) + D|y|^{-n+2}\cdot D\eta(y) - |y|^{-n+2}\Delta\eta(y)]\,dy.$$

Observing that $\Delta|y|^{n-2} = 0$ in $A_{r/4,4R}$ and using (4-1) and (4-2), we arrive at

$$\begin{aligned} I_1 &\le C(n)\int_{A_{r/4,4R}} u_\delta(y)[(2-n)|y|^{-n}y\cdot D\eta(y) - |y|^{-n+2}\Delta\eta(y)]\,dy \\ &\le C(n)\int_{A_{r/4,4R}} |u_\delta(y)|[r^{-n}\chi_{A_{r/4,r/2}} + R^{-n}\chi_{A_{2R,4R}}]\,dy \\ &\le C(n)⨍_{A_{r/4,r/2}} |u_\delta|\,dz + ⨍_{A_{2R,4R}} |u_\delta|\,dz. \end{aligned}$$

For I_2, by (4-3) and (4-1),

$$\begin{aligned} I_2 &\le \int_{\mathbb{R}^n}\biggl(\int_{A(r,R)} |x-y|^{-n+1}|x|^{-n+1}\,dx\biggr)|u_\delta|(y)|\Delta\eta(y)|\,dy \\ &\le C(n)\int_{\mathbb{R}^n} |y|^{-n+2}|u_\delta|(y)|\Delta\eta(y)|\,dy \\ &\le C(n)\int_{\mathbb{R}^n} |u_\delta(y)|[r^{-n}\chi_{A_{r/4,r/2}} + R^{-n}\chi_{A_{2R,4R}}]\,dy \\ &\le C(n)⨍_{A_{r/4,r/2}} |u_\delta|\,dz + C(n)⨍_{A_{2R,4R}} |u_\delta|\,dz. \end{aligned}$$

Now let us estimate I_3. First via integration by parts one gets

$$\begin{aligned}&\int_{\mathbb{R}^n} |x-y|^{-n}(x-y)Du_\delta(y)\cdot D_\delta\eta(y)\,dy \\ &\qquad = \int_{\mathbb{R}^n} |x-y|^{-n}(x-y)u_\delta(y)\Delta\eta(y)\,dy + \int_{\mathbb{R}^n} u_\delta(y)D[|x-y|^{-n}(x-y)]D\eta(y)\,dy.\end{aligned}$$

Since

$$|D[|x-y|^{-n}(x-y)]| \le C(n)|x-y|^{-n},$$

we obtain

$$\left|\int_{\mathbb{R}^n} |x-y|^{-n}(x-y) Du_\delta(y)\cdot D\eta(y)\,dy\right| \\ \le C(n)\left|\int_{\mathbb{R}^n} |x-y|^{-n+1}u_\delta(y)\Delta\eta(y)\,dy\right| + C(n)\int_{\mathbb{R}^n} |x-y|^{-n}|u_\delta(y)||D\eta(y)|\,dy.$$

As a consequence,

$$I_3 \le C(n)I_2 + C(n)\int_{\mathbb{R}^n}\left(\int_{A_{r,R}} |x-y|^{-n}|x|^{-n+1}\,dx\right)|u_\delta(y)||D\eta(y)|\,dy =: C(n)I_2 + C(n)\tilde{I}_3.$$

In order to estimate $\tilde{I}_3$, first we note that (4-1) gives

$$\tilde{I}_3 \le C(n)\int_{\mathbb{R}^n}\left(\int_{A_{r,R}} |x-y|^{-n}|x|^{-n+1}\,dx\right)|u_\delta(y)|[r^{-1}\chi_{A_{r/4,r/2}} + R^{-1}\chi_{A_{2R,4R}}]\,dy.$$

For any $x\in A_{r,R}$, if $y\in A_{r/4,r/2}$, we have $|x-y|\ge \frac{1}{2}|x|$, and hence

$$\int_{A_{r,R}} |x-y|^{-n}|x|^{-n+1}\,dx \le C(n)\int_{A_{r,R}} |x|^{-2n+1}\,dx \le C(n)r^{-n+1};$$

if $y\in A_{2R,4R}$, then $|x-y|\ge R$, and hence

$$\int_{A_{r,R}} |x-y|^{-n}|x|^{-n+1}\,dx \le C(n)R^{-n}\int_{A_{r,R}} |x|^{-n+1}\,dx \le C(n)R^{-n+1}.$$

Thus it follows that

$$\tilde{I}_3 \le C(n)\int_{\mathbb{R}^n} |u_\delta(y)|[r^{-n}\chi_{A_{r/4,r/2}} + R^{-n}\chi_{A_{2R,4R}}]\,dy \le C(n)\fint_{A_{r/4,r/2}} |u_\delta|\,dz + C(n)\fint_{A_{2R,4R}} |u_\delta|\,dz.$$

To conclude,

$$\int_{A_{r,R}} |Du_\delta||x|^{-n+1}\,dx \le C(n)\fint_{A_{r/4,r/2}} |u_\delta|\,dz + C(n)\fint_{A_{2R,4R}} |u_\delta|\,dz.$$

By letting $\delta\to 0$ and noting $u_\delta\to u$ in $W^{1,1}_{\mathrm{loc}}$, we conclude (1-21). □

5. Proof of Theorem 1.5

Since u satisfies (1-16), we know that u does not satisfy (1-15). We only need to show that if u is nonconstant, then (1-15) holds. Equivalently, it suffices to show that if u does not satisfy (1-15), then u is a constant. Namely, there exists a sequence $\{R_j\}_{j\in\mathbb{N}}$ tending toward ∞ such that

$$\frac{1}{\log R_j}\fint_{A_{R_j,4R_j}} |u(z)|\,dz \to 0 \quad \text{as } j\to\infty, \quad \text{when } n=10, \tag{5-1}$$

and

$$R_j^{n/2-2-\sqrt{n-1}}\fint_{A_{R_j,4R_j}} |u(x)|\,dx \to 0 \quad \text{as } j\to\infty, \quad \text{when } n\ge 11. \tag{5-2}$$

On the other hand, given any $0 < r < \infty$, applying (1-18) for any $R > 4r$, we have

$$r^{-(1+\sqrt{n-1})}\left(\int_{B_r}|Du|^2\,dx\right)^{1/2} \le C(n)R^{-(1+\sqrt{n-1})}\left(\int_{A_{R,2R}}|Du|^2\,dx\right)^{1/2}. \tag{5-3}$$

Observe that the annulus $A_{1,2}$ can be covered by $\{B_{1/8}(y_i)\}_{i=1}^N$ with $y_1,\dots,y_N \in A_{1,2}$ and $N \le C(n)$:

$$\chi_{A_{1,2}} \le \sum_{i=1}^N \chi_{B_{1/8}(y_i)} \le \sum_{i=1}^N \chi_{B_{1/4}(y_i)} \le C(n)\chi_{A_{3/4,3}}.$$

Below we consider the case $n \ge 11$ and the case $n = 10$ separately.

Case $n \ge 11$. For each i, applying (1-19) and (1-20), one attains

$$\left(\int_{B_{R/8}(Ry_i)}|Du|^2\,dx\right)^{1/2} \le C(n)R^{-(n+2)/2}\int_{B_{R/4}(Ry_i)}|u|\,dx \le C(n)R^{(n-2)/2}\fint_{A_{3R/4,3R}}|u|\,dx.$$

Thus by summing over all these balls,

$$\int_{A_{R,2R}}|Du|^2\,dx \le C(n)R^{n-2}\left(\fint_{A_{3R/4,3R}}|u|\,dx\right)^2,$$

and we eventually obtain from (5-3) that

$$r^{-(1+\sqrt{n-1})}\left(\int_{B_r}|Du|^2\,dx\right)^{1/2} \le C(n)R^{n/2-2-\sqrt{n-1}}\fint_{A_{3R/4,3R}}|u|\,dx.$$

Taking $R = \frac{4}{3}R_j$, applying (5-2) and letting $j \to \infty$, one concludes

$$\int_{B_r}|Du|^2\,dx = 0.$$

By the arbitrariness of $r > 0$, we obtain $\|Du\|_{L^2(\mathbb{R}^n)} = 0$, which implies that u is a constant.

Case $n = 10$. For each i, applying (1-19), one attains

$$\left(\int_{B_{R/8}(Ry_i)}|Du|^2\,dx\right)^{1/2} \le C(n)R^{-n/2}\int_{B_{R/4}(Ry_i)}|Du|\,dx \le C(n)R^{(n-2)/2}\int_{A_{R/2,4R}}|Du||x|^{-n+1}\,dx.$$

Thus

$$\int_{A_{R,2R}}|Du|^2\,dx \le C(n)R^{n-2}\left(\int_{A_{R/2,4R}}|Du||x|^{-n+1}\,dx\right)^2. \tag{5-4}$$

We therefore obtain from (5-3) that

$$\begin{aligned} r^{-(1+\sqrt{n-1})}\left(\int_{B_r}|Du|^2\,dx\right)^{1/2} &\le C(n)R^{n/2-2-\sqrt{n-1}}\int_{A_{R/2,4R}}|Du||x|^{-n+1}\,dx \\ &= C(n)\int_{A_{R/2,4R}}|Du||x|^{-n+1}\,dx, \end{aligned}$$

where in the last identity we use $\frac{1}{2}n - 2 - \sqrt{n-1} = 5 - 2 - 3 = 0$.

For $R > 2^5 + r > 4$, let m be the largest integer such that $m \le \log_2 R - 3$. Applying (5-4) to $2^j R$ with $j = 1, \dots, m$, one has

$$r^{-(1+\sqrt{n-1})}\left(\int_{B_r} |Du|^2\,dx\right)^{1/2} \le C(n)\frac{1}{m}\sum_{j=1}^{m}\int_{A_{2^jR/2,4(2^jR)}} |Du||x|^{-n+1}\,dx$$
$$\le C(n)\frac{1}{m}\int_{A_{R,2^{m+2}R}} |Du||x|^{-n+1}\,dx \le C(n)\frac{1}{\log R}\int_{A_{4,R^2/2}} |Du||x|^{-n+1}\,dx.$$

By (1-21), one has

$$r^{-(1+\sqrt{n-1})}\left(\int_{B_r} |Du|^2\,dx\right)^{1/2} \le C(n)\frac{1}{\log R}\fint_{A_{1,2}} |u(z)|\,dz + C(n)\frac{1}{\log R^2}\fint_{A_{2R^2,4R^2}} |u(z)|\,dz.$$

Taking $R = \sqrt{R_j}$ and letting $j \to \infty$, by (5-1) one concludes

$$\int_{B_r} |Du|^2\,dx = 0.$$

Then the arbitrariness of $r > 0$ implies $\|Du\|_{L^2(\mathbb{R}^n)} = 0$, which further implies that u is a constant. □

Appendix: A radial stable solution when $n = 10$

Suppose $n = 10$ in this appendix. Villegas [2007] proved that $\frac{1}{2}\log(1+|x|^2)$ is a stable solution to the equation $-\Delta u = -(n-2)e^{-2u} - 2e^{-4u}$ in $\mathbb{R}^n$. Note that $-(n-2)e^{-2s} - 2e^{-4s} \le 0$ in $\mathbb{R}$.

Below, we show that $u = -\frac{1}{2}\log(1+|x|^2)$ is a stable solution to the equation

$$-\Delta u = f(u) \quad \text{in } \mathbb{R}^n,$$

where $f(s) = (n-2)e^{2s} + 2e^{4s} \ge 0$ in $\mathbb{R}$.

First we show that u is a solution. Indeed, for any $x \in \mathbb{R}^n$, a direct calculation gives

$$-\Delta u(x) = ((1+|x|^2)^{-1}x_i)_{x_i} = \frac{n}{1+|x|^2} + 2\frac{|x|^2}{(1+|x|^2)^2} = (n-2)\frac{1}{1+|x|^2} + 2\frac{1}{(1+|x|^2)^2}.$$

Since $e^{2u(x)} = (1+|x|^2)^{-1}$, we have

$$-\Delta u(x) = (n-2)e^{2u(x)} + 2e^{4u(x)} = f(u(x)).$$

Next, we show that u is stable. Note that $f'(s) = 2(n-2)e^{2s} + 8e^{4s}$ for $s \in \mathbb{R}$. Given any $x \ne 0$, writing $r = |x|$ and noting $e^{2u(x)} = (1+|x|^2)^{-1}$, we have

$$f'(u(x)) = 2(n-2)e^{2u(x)} + 8e^{4u(x)} = \frac{2(n-2)}{1+r^2} + \frac{8}{(1+r^2)^2}.$$

Since $n = 10$, we have

$$f'(u(x)) = \frac{16r^2(1+r^2) + 8r^2}{r^2(1+r^2)^2} = \frac{16r^4 + 24r^2}{r^2(1+r^2)^2} < \frac{16(1+r^2)^2}{r^2(1+r^2)^2} = \frac{(n-2)^2}{4|x|^2}.$$

By this and the Hardy inequality, we have

$$\int_{\mathbb{R}^n} f'(u)\xi^2\,dx \le \frac{(n-2)^2}{4}\int_{\mathbb{R}^n}\frac{\xi^2}{|x|^2}\,dx \le \int_{\mathbb{R}^n}|D\xi|^2\,dx \quad \text{for all } \xi \in C_c^\infty(\mathbb{R}^n).$$

Thus u is a stable solution to $-\Delta u = f(u)$ in $\mathbb{R}^n$.

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