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RELATIVE HEAT CONTENT ASYMPTOTICS FOR SUB-RIEMANNIAN MANIFOLDS

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The relative heat content associated with a subset $\Omega \subset M$ of a sub-Riemannian manifold is defined as the total amount of heat contained in Ω at time t, with uniform initial condition on Ω, allowing the heat to flow outside the domain. We obtain a fourth-order asymptotic expansion in the square root of t of the relative heat content associated with relatively compact noncharacteristic domains. Compared to the classical heat content that was studied by Rizzi and Rossi (*J. Math. Pures Appl.* (9) **148** (2021), 267–307), several difficulties emerge due to the absence of Dirichlet conditions at the boundary of the domain. To overcome this lack of information, we combine a rough asymptotics for the temperature function at the boundary, coupled with stochastic completeness of the heat semigroup. Our technique applies to any (possibly rank-varying) sub-Riemannian manifold that is globally doubling and satisfies a global weak Poincaré inequality, including in particular sub-Riemannian structures on compact manifolds and Carnot groups.

1. Introduction

We study the asymptotics of the relative heat content in sub-Riemannian geometry. The latter is a vast generalization of Riemannian geometry; indeed a sub-Riemannian manifold M is a smooth manifold where a metric is defined only on a subset of preferred directions $\mathcal{D}_x \subset T_xM$ at each point $x \in M$ (called horizontal directions). For example, $\mathcal{D}$ can be a sub-bundle of the tangent bundle, but we will consider the most general case of rank-varying distributions. Moreover, we assume that $\mathcal{D}$ satisfies the so-called Hörmander condition, which ensures that M is horizontally path connected, and that the usual length-minimization procedure yields a well-defined metric.

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Let M be a sub-Riemannian manifold, equipped with a smooth measure ω, let $\Omega \subset M$ be an open relatively compact subset of M, with smooth boundary, and consider the Cauchy problem for the heat equation in this setting

$$\begin{aligned} (\partial_t - \Delta) u(t,x) &= 0 \qquad \text{for all } (t,x) \in (0,\infty) \times M, \\ u(0,\cdot) &= \mathbb{1}_\Omega \quad \text{in } L^2(M,\omega), \end{aligned} \tag{1}$$

where $\mathbb{1}_\Omega$ is the indicator function of the set Ω, and Δ is the sub-Laplacian, defined with respect to ω. By classical spectral theory, there exists a unique solution to (1),

$$u(t,x) = e^{t\Delta}\mathbb{1}_\Omega(x) \quad \text{for all } x \in M,\ t > 0,$$

where $e^{t\Delta}$ denotes the heat semigroup in $L^2(M,\omega)$, associated with Δ. The *relative heat content* is the function

$$H_\Omega(t) = \int_\Omega u(t,x)\, d\omega(x) \quad \text{for all } t > 0.$$

This quantity has been studied in connection with geometric properties of subsets of $\mathbb{R}^n$, starting from the seminal work of De Giorgi [1954], where he introduced the notion of perimeter of a set in $\mathbb{R}^n$ and proved a characterization of sets of finite perimeter in terms of the heat kernel. His result was subsequently refined, using techniques of functions of bounded variation: it was proven in [Ledoux 1994] for balls in $\mathbb{R}^n$, and in [Miranda et al. 2007] for general subsets of $\mathbb{R}^n$, that a borel set $\Omega \subset \mathbb{R}^n$ with finite Lebesgue measure has finite perimeter à la De Giorgi if and only if

$$\text{there exists } \lim_{t\to 0} \frac{\sqrt{\pi}}{\sqrt{t}}(|\Omega| - H_\Omega(t)) = P(\Omega), \tag{2}$$

where $|\cdot|$ is the Lebesgue measure and P is the perimeter measure in $\mathbb{R}^n$. Notice that (2) is equivalent to a first-order[1] asymptotic expansion of $H_\Omega(t)$. A further development in this direction was then obtained in [Angiuli et al. 2013], where the authors extended (2) to an asymptotic expansion of order 3 in $\sqrt{t}$, assuming the boundary of $\Omega \subset \mathbb{R}^n$ to be a $C^{1,1}$ set. For simplicity, we state here the result of [Angiuli et al. 2013, Theorem 1.1] assuming $\partial\Omega$ is smooth:[2]

$$H_\Omega(t) = |\Omega| - \frac{1}{\sqrt{\pi}} P(\Omega) t^{1/2} + \frac{(n-1)^2}{12\sqrt{\pi}} \int_{\partial\Omega} \left(H^2_{\partial\Omega}(x) + \frac{2}{(n-1)^2} c_{\partial\Omega}(x) \right) d\mathcal{H}^{n-1}(x) t^{3/2} + o(t^{3/2}) \tag{3}$$

as $t \to 0$, where $\mathcal{H}^{n-1}$ is the Hausdorff measure and, denoting by $k_i^{\partial\Omega}(x)$ the principal curvatures of $\partial\Omega$ at the point x,

$$H_{\partial\Omega}(x) = \frac{1}{n-1}\sum_{i=1}^{n-1} k_i^{\partial\Omega}(x), \qquad c_{\partial\Omega}(x) = \sum_{i=1}^{n-1} k_i^{\partial\Omega}(x)^2.$$

In the Riemannian setting, van den Berg and Gilkey [2015] proved the existence of a complete asymptotic expansion for $H_\Omega(t)$, generalizing (3), when $\partial\Omega$ is smooth. Moreover, they were able to compute explicitly the coefficients of the expansion up to order 4 in $\sqrt{t}$. Their techniques are based

[1] Here and throughout the paper, the notion of order is computed with respect to $\sqrt{t}$.

[2] The statement of Theorem 1.1 in [Angiuli et al. 2013] differs from (3) by a sign in the third-order coefficient: the correct sign appears a few lines below the statement, in the expansion of the function $K_t(E, E^c)$.

on pseudodifferential calculus and cannot be immediately adapted to the sub-Riemannian setting. In particular, what is missing is a global parametrix estimate for the heat kernel $p_t(x, y)$, see [van den Berg and Gilkey 2015, Section 2.3]: for any $k \in \mathbb{N}$, there exist $J_k, C_k > 0$ such that

$$\left\| p_t(x, y) - \sum_{j=0}^{J_k} p_t^j(x, y) \right\|_{C^k(M\times M)} \le C_k t^k \quad \text{as } t \to 0, \tag{4}$$

where $p_t^j(x, y)$ are suitable smooth functions, given explicitly in terms of the Euclidean heat kernel and iterated convolutions. The closest estimate analogue to (4) in the sub-Riemannian setting is the one proved recently in [Colin de Verdière et al. 2021, Theorem A] (see Theorem 2.9 for the precise statement), where the authors show an asymptotic expansion of the heat kernel in an *asymptotic neighborhood* of the diagonal, which is not enough to reproduce (4) and thus the argument of van den Berg and Gilkey. Moreover, in this case, $p_t^j(x, y)$ is expressed in terms of the heat kernel of the nilpotent approximation and iterated convolutions, thus posing technical difficulties for the explicit computations of the coefficients (which would be no longer "simple" Gaussian-type integrals).

Under the assumption of not having characteristic points, we prove the existence of the asymptotic expansion of $H_\Omega(t)$, up to order 4 in $\sqrt{t}$, as $t \to 0$. We remark that we include also the rank-varying case. In order to state our main results, let us introduce the following operator, acting on smooth functions compactly supported close to $\partial\Omega$:

$$N\phi = 2g(\nabla\phi, \nabla\delta) + \phi\Delta\delta,$$

where $\delta : M \to \mathbb{R}$ denotes the sub-Riemannian signed distance function from $\partial\Omega$; see Section 4 for precise definitions.

Theorem 1.1. *Let M be a compact sub-Riemannian manifold, equipped with a smooth measure ω, and let $\Omega \subset M$ be an open subset whose boundary is smooth and has no characteristic points. Then, as $t \to 0$,*

$$H_\Omega(t) = \omega(\Omega) - \frac{1}{\sqrt{\pi}}\sigma(\partial\Omega)t^{1/2} - \frac{1}{12\sqrt{\pi}} \int_{\partial\Omega} (N(\Delta\delta) - 2(\Delta\delta)^2)\, d\sigma\, t^{3/2} + o(t^2), \tag{5}$$

where σ denotes the sub-Riemannian perimeter measure.

Remark 1.2. The compactness assumption in Theorem 1.1 is technical and can be relaxed by requiring, instead, global doubling of the measure and a global Poincaré inequality; see Section 7 and in particular Theorem 7.3. Some notable examples satisfying these assumptions are:

- M is a Lie group with polynomial volume growth, the distribution is generated by a family of left-invariant vector fields satisfying the Hörmander condition and ω is the Haar measure. This family includes also Carnot groups.
- $M = \mathbb{R}^n$, equipped with a sub-Riemannian structure induced by a family of vector fields $\{Y_1, \dots, Y_N\}$ with bounded coefficients together with their derivatives, and satisfying the Hörmander condition.
- M is a complete Riemannian manifold, equipped with the Riemannian measure, and with nonnegative Ricci curvature.

See Section 7.1 for further details. In all these examples, Theorem 1.1 holds.

The strategy of the proof of Theorem 1.1 follows a strategy similar to that of [Rizzi and Rossi 2021], inspired by the method introduced in [Savo 1998], used for the classical heat content (6). However, as we are going to explain in Section 1.1, new technical difficulties arise, the main one being related to the fact that now $u(t,\cdot)|_{\partial\Omega} \neq 0$. At order zero, we obtain the following result; see Section 2 for precise definitions.

Theorem 1.3. *Let M be a sub-Riemannian manifold, equipped with a smooth measure ω and let $\Omega \subset M$ be an open relatively compact subset, whose boundary is smooth and has no characteristic points. Let $x \in \partial\Omega$ and consider a chart of privileged coordinates $\psi : U \to V \subset \mathbb{R}^n$ centered at x such that $\psi(U \cap \Omega) = V \cap \{z_1 > 0\}$. Then,*

$$\lim_{t\to 0} u(t,x) = \int_{\{z_1>0\}} \hat{p}_1^x(0,z)\, d\hat{\omega}^x(z) = \frac{1}{2} \quad \textit{for all } x \in \partial\Omega,$$

where $\hat{\omega}^x$ denotes the nilpotentization of ω at x and $\hat{p}_t^x$ denotes the heat kernel associated with the nilpotent approximation of M at x and measure $\hat{\omega}^x$.

This result can be seen as a partial generalization of [Capogna et al. 2013, Proposition 3], where the authors proved an asymptotic expansion of $u(t,x)$ up to order 1 in $\sqrt{t}$ for $x \in \partial\Omega$ for a special class of noncharacteristic domains in Carnot groups.

Remark 1.4. Our proof of Theorem 1.3 does not yield an asymptotic series for $u(t,\cdot)|_{\partial\Omega}$ at order higher than 0. Indeed a complete asymptotic series of this quantity seems difficult to achieve; see Section 6.

Remark 1.5. When $\partial\Omega$ has no characteristic points, the conormal bundle

$$\mathcal{A}(\partial\Omega) := \{\lambda \in T^*M : \langle \lambda, T_{\pi(\lambda)}\partial\Omega\rangle = 0\}$$

does not intersect the characteristic set and, as a consequence, the principal symbol of the sub-Laplacian is elliptic near $\mathcal{A}(\partial\Omega)$. Thus, it is likely that microlocal analysis techniques in the spirit of [Colin de Verdière et al. 2018] could yield the existence of a complete asymptotic expansion of the relative heat content (but not an explicit expression and geometric interpretation of the coefficients). We thank Yves Colin de Verdière and the anonymous referee for pointing out this fact.

1.1. *Strategy of the proof of Theorem 1.1.*

To better understand the new technical difficulties in the study of the relative heat content $H_\Omega(t)$, let us compare it with the classical heat content $Q_\Omega(t)$ and illustrate the strategy of the proof of Theorem 1.1.

The classical heat content. We highlight the differences between the relative heat content $H_\Omega(t)$ and the classical one $Q_\Omega(t)$: Let $\Omega \subset M$ an open set in M. Then, for all $t > 0$, we have

$$H_\Omega(t) = \int_\Omega u(t,x)\, d\omega(x), \qquad Q_\Omega(t) = \int_\Omega u_0(t,x)\, d\omega(x), \tag{6}$$

where $u(t,x)$ is the solution to (1) and $u_0(t,x)$ is the solution to the Dirichlet problem for the heat equation, associated with Ω, i.e.,

$$\begin{aligned} (\partial_t - \Delta)u_0(t,x) &= 0 \quad \text{for all } (t,x) \in (0,\infty) \times \Omega, \\ u_0(t,x) &= 0 \quad \text{for all } (t,x) \in (0,\infty) \times \partial\Omega, \\ u_0(0,x) &= 1 \quad \text{for all } x \in \Omega. \end{aligned} \tag{7}$$

The crucial difference is that $u_0(t,\cdot)|_{\partial\Omega}=0$ for any $t>0$, whereas $u(t,\cdot)|_{\partial\Omega}\neq 0$ in general. Thus, there is no a priori relation between $H_\Omega(t)$ and $Q_\Omega(t)$: the only relevant information is given by domain monotonicity, which implies that

$$Q_\Omega(t)\le H_\Omega(t)\quad \text{for all } t>0,$$

and clearly this does not give the asymptotics of the latter. See also [van den Berg 2013] for other comparison results in the Euclidean setting.

Failure of Duhamel's principle. In [Rizzi and Rossi 2021], we established a complete asymptotic expansion of $Q_\Omega(t)$, as $t\to 0$, provided that $\partial\Omega$ has no characteristic points. The proof of this result relied on an iterated application of the Duhamel's principle and the fact that $u_0(t,x)|_{\partial\Omega}=0$. Following the same strategy, we apply Duhamel's principle to a localized version of $H_\Omega(t)$: Fix a function $\phi\in C_c^\infty(M)$, compactly supported in a tubular neighborhood around $\partial\Omega$ and such that $0\le\phi\le 1$ and ϕ is identically 1, close to $\partial\Omega$. Then, using off-diagonal estimates for the heat kernel, one can prove that

$$\omega(\Omega)-H_\Omega(t)=I\phi(t,0)+O(t^\infty)\quad \text{as } t\to 0, \tag{8}$$

where $I\phi(t,r)$ is defined for $t>0$ and $r\ge 0$ as

$$I\phi(t,r)=\int_{\Omega_r}(1-u(t,x))\phi(x)\,d\omega(x), \tag{9}$$

where $\Omega_r=\{x\in\Omega:\delta(x)>r\}$, with $\delta:\Omega\to\mathbb{R}$ denoting the distance function from the boundary. Hence, the small-time behavior of $H_\Omega(t)$ is captured by $I\phi(t,0)$. By Duhamel's principle and the sub-Riemannian mean value lemma, see Section 4 for details, we obtain

$$I\phi(t,0)=\frac{1}{\sqrt{\pi}}\int_0^t\int_{\partial\Omega}(1-u(\tau,y))\phi(y)\,d\sigma(y)\,(t-\tau)^{-1/2}\,d\tau+O(t)\quad \text{as } t\to 0. \tag{10}$$

For the classical heat content, u_0 satisfies Dirichlet boundary condition, see (7); hence (10) would give the first-order asymptotics (and then one could iterate). On the contrary, in this case, we do not have prior knowledge of $u(t,y)$ as $y\in\partial\Omega$ and $t\to 0$. Thus, already for the first-order asymptotics, Duhamel's principle alone is not enough, and we need some information on the asymptotic behavior of $u(t,\cdot)|_{\partial\Omega}$.

First-order asymptotics. We study the asymptotics of $u(t,\cdot)|_{\partial\Omega}$. Using the notion of nilpotent approximation of a sub-Riemannian manifold, see Section 2.3, we deduce the zero-order asymptotic expansion of $u(t,\cdot)|_{\partial\Omega}$ as $t\to 0$, proving Theorem 1.3. This is enough to infer the first-order expansion of $H_\Omega(t)$, by means of (10). At this point, we iterate the Duhamel's principle to obtain the higher-order terms of the expansion of $H_\Omega(t)$. However, already at the first iteration, we obtain the following formula for $I\phi$:

$$\begin{aligned}I\phi(t,0)=&\frac{1}{\sqrt{\pi}}\int_0^t\int_{\partial\Omega}(1-u(\tau,\cdot))\phi\,d\sigma\,(t-\tau)^{-1/2}\,d\tau\\&+\frac{1}{2\pi}\int_0^t\int_0^\tau\int_{\partial\Omega}(1-u(\hat\tau,\cdot))N\phi\,d\sigma\,((\tau-\hat\tau)(t-\tau))^{-1/2}\,d\hat\tau\,d\tau+O(t^{3/2})\end{aligned} \tag{11}$$

as $t\to 0$. Therefore, the zero-order asymptotic expansion of $u(t,\cdot)|_{\partial\Omega}$ no longer suffices for obtaining the second-order asymptotics of $H_\Omega(t)$.

The outside contribution $I^c\phi$. We mentioned that the crucial difference between $H_\Omega(t)$ and $Q_\Omega(t)$, defined in (6), is related to the fact that $u(t,\cdot)|_{\partial\Omega} \neq 0$, whereas $u_0(t,\cdot)|_{\partial\Omega} = 0$ for any $t > 0$. From a physical viewpoint, this distinction comes from the fact that, since the boundary $\partial\Omega$ is no longer insulated, the solution to (1) can flow also outside of Ω, whereas the solution to the Dirichlet problem (7) is confined in Ω, and the external temperature is 0. Hence, we can imagine that the asymptotic expansion of $H_\Omega(t)$ is affected by the boundary, both from the inside and from the outside of Ω.

Interpreting $I\phi$ as the *inside contribution* to the asymptotics of H_Ω, we are going to formalize the physical intuition of having heat flowing outside of Ω, defining an *outside contribution*, $I^c\phi$, to the asymptotics.[3] The starting observation is the following simple relation: Setting

$$K_\Omega(t) = \int_{M\setminus\Omega} u(t,x)\,d\omega(x) \quad \text{for all } t > 0,$$

we have, by the divergence theorem,

$$H_\Omega(t) + K_\Omega(t) = \omega(\Omega) \quad \text{for all } t > 0. \tag{12}$$

Similarly to (9), for a suitable smooth function ϕ, one may define a localized version of $K_\Omega(t)$, which we call $I^c\phi(t,r)$, so that

$$K_\Omega(t) = I^c\phi(t,0) + O(t^\infty) \quad \text{as } t \to 0; \tag{13}$$

see Section 5.1 for precise definitions. Using (8), (12) and (13), we show the relation

$$I\phi(t,0) - I^c\phi(t,0) = O(t^\infty) \quad \text{as } t \to 0,$$

for a suitable smooth function ϕ. On the other hand, for the localized quantity $I\phi(t,0) - I^c\phi(t,0)$ we have a Duhamel's principle, thanks to which we are able to study the asymptotic expansion, up to order 3, of the *integral* of $u(t,x)$ over $\partial\Omega$; see Theorem 5.4. The limitation to the order 3 of the asymptotics is technical and seems difficult to overcome; see Remark 5.5. Inserting this asymptotics in (11), we obtain the asymptotics *up to order* 3 of the expansion of $H_\Omega(t)$ as $t \to 0$.

Fourth-order asymptotics. Since we have at disposal only the asymptotics of the integral of $u(t,x)$ over $\partial\Omega$, up to order 3, we need a finer argument to obtain the fourth-order asymptotics of $H_\Omega(t)$. The simple but compelling relation is based once again on (8), (12) and (13), thanks to which we can write

$$\omega(\Omega) - H_\Omega(t) = \tfrac{1}{2}(I\phi(t,0) + I^c\phi(t,0)) + O(t^\infty) \quad \text{as } t \to 0.$$

Now for the sum of the contributions $I\phi(t,0) + I^c\phi(t,0)$, the Duhamel's principle implies

$$\begin{aligned} I\phi(t,0) &+ I^c\phi(t,0) \\ &= \frac{2}{\sqrt{\pi}}\sigma(\partial\Omega)t^{1/2} + \frac{1}{2\pi}\int_0^t\int_0^\tau\int_{\partial\Omega}(1-2u(\hat\tau,x))N\phi(y)\,d\sigma(y)((\tau-\hat\tau)(t-\tau))^{-1/2}\,d\hat\tau\,d\tau + o(t). \end{aligned}$$

[3]The notation "superscript c" stands for complement. Indeed the outside contribution is the inside contribution of the complement of Ω, see Section 5.1.

This time notice how the integral of $u(t,x)$ over $\partial\Omega$ appears in a first-order term (as opposed to what happened in (10) or (11)); thus its asymptotic expansion up to order 3 implies a fourth-order expansion for $H_\Omega(t)$, concluding the proof of Theorem 1.1.

1.2. ***From the heat kernel asymptotics to the relative heat content asymptotics.*** In [Colin de Verdière et al. 2021, Theorem A], the authors proved the existence of small-time asymptotics of the hypoelliptic heat kernel, $p_t(x,y)$; see Theorem 2.9 below for the precise statement. In Theorem 1.3 we are able to exploit this result to obtain the zero-order asymptotics of the function

$$u(t,x) = e^{t\Delta}\mathbb{1}_\Omega(x) = \int_\Omega p_t(x,y)\,d\omega(y) \quad \text{for all } t>0,\ x\in\partial\Omega.$$

However, we are not able to extend Theorem 1.3 to higher-order asymptotics since, roughly speaking, the remainder terms in Theorem 2.9 are not uniform as $t\to 0$. If we had a better control on the remainders, we could indeed integrate (in a suitable way) the small-time heat kernel asymptotics to obtain the corresponding expansion for $u(t,x)$. Finally, from such an expansion, the relative heat content asymptotics would follow from the localization principle (8) and the (iterated) Duhamel's principle (10). This is done in Section 6.

1.3. ***Characteristic points.*** In order to prove our main results, we need the noncharacteristic assumption on the domain Ω. We recall that for a subset $\Omega\subset M$ with smooth boundary, $x\in\partial\Omega$ is a characteristic point if $\mathcal{D}_x\subset T_x(\partial\Omega)$. As was the case for the classical heat content, see [Rizzi and Rossi 2021], the noncharacteristic assumption is crucial to follow our strategy, since it guarantees the smoothness of the signed distance function close to $\partial\Omega$; see Theorem 4.1. Nevertheless, one might ask whether Theorem 1.1 holds for domains with characteristic points, at least formally.

On the one hand, the coefficients, up to order 2, are well-defined even in presence of characteristic points; see [Balogh 2003]. While, for what concerns the integrand of the third-order coefficient, its integrability, with respect to the sub-Riemannian induced measure σ, is related to integrability of the sub-Riemannian mean curvature $\mathcal{H}$, with respect to the Riemannian induced measure. The latter is a nontrivial property, which has been studied in [Danielli et al. 2012], and holds in the Heisenberg group, for surfaces with mildly degenerate characteristic points in the sense of [Rossi 2023].

On the other hand, differently from what happens in the case of the Dirichlet problem, the heat kernel $p_t(x,y)$ associated with (1) is smooth at the boundary of Ω for positive times, even in presence of characteristic points. Thus, in principle, there is no obstacle in obtaining an asymptotic expansion of $H_\Omega(t)$ also in that case. Moreover, in Carnot groups of step 2, a result similar to (2) holds; see [Bramanti et al. 2012; Garofalo and Tralli 2023]. In particular, the characterization of sets of finite horizontal perimeter in Carnot groups of step 2 is independent of the presence of characteristic points, indicating that an asymptotic expansion such as (5) may still hold, dropping the noncharacteristic assumption.

1.4. ***Notation.*** Throughout the article, for a set $U\subset M$, we will use the notation $C_c^\infty(U)$, even in the compact case, so that all the statements need not be modified in the noncompact case, when the generalization is possible; see Theorem 7.3. Moreover, in the noncompact and complete case, the set $\Omega\subset M$ is assumed to be open and bounded.

2. Preliminaries

We recall some essential facts in sub-Riemannian geometry, following [Agrachev et al. 2020].

2.1. ***Sub-Riemannian geometry.*** Let M be a smooth, connected finite-dimensional manifold. A sub-Riemannian structure on M is defined by a set of N global smooth vector fields $X_1, \dots, X_N$, called a *generating frame*. The generating frame defines a *distribution* of subspaces of the tangent spaces at each point $x \in M$, given by

$$\mathcal{D}_x = \operatorname{span}\{X_1(x), \dots, X_N(x)\} \subseteq T_x M. \tag{14}$$

We assume that the distribution satisfies the *Hörmander condition*, i.e., the Lie algebra of smooth vector fields generated by $X_1, \dots, X_N$, evaluated at the point x, coincides with $T_x M$, for all $x \in M$. The generating frame induces a norm on the distribution at x, namely

$$\|v\|_g = \inf\left\{\sum_{i=1}^N u_i^2 : \sum_{i=1}^N u_i X_i(x) = v\right\} \quad \text{for all } v \in \mathcal{D}_x,$$

which, in turn, defines an inner product on $\mathcal{D}_x$ by polarization, which we denote by $g_x(v, v)$. Let $T > 0$. We say that $\gamma : [0, T] \to M$ is a *horizontal curve* if it is absolutely continuous and

$$\dot\gamma(t) \in \mathcal{D}_{\gamma(t)} \quad \text{for a.e. } t \in [0, T].$$

This implies that there exists $u : [0, T] \to \mathbb{R}^N$ such that

$$\dot\gamma(t) = \sum_{i=1}^N u_i(t) X_i(\gamma(t)) \quad \text{for a.e. } t \in [0, T].$$

Moreover, we require that $u \in L^2([0, T], \mathbb{R}^N)$. If γ is a horizontal curve, then the map $t \mapsto \|\dot\gamma(t)\|_g$ is integrable on $[0, T]$. We define the *length* of a horizontal curve as

$$\ell(\gamma) = \int_0^T \|\dot\gamma(t)\|_g \, dt.$$

The *sub-Riemannian distance* is defined, for any $x, y \in M$, by

$$d_{\mathrm{SR}}(x, y) = \inf\{\ell(\gamma) : \gamma \text{ horizontal curve between } x \text{ and } y\}.$$

By the Chow–Rashevsky theorem, the distance $d_{\mathrm{SR}} : M \times M \to \mathbb{R}$ is finite and continuous. Furthermore it induces the same topology as the manifold one.

Remark 2.1. The above definition includes all classical constant-rank sub-Riemannian structures as in [Montgomery 2002; Rifford 2014] (where $\mathcal{D}$ is a vector distribution and g a symmetric and positive tensor on $\mathcal{D}$), but also general rank-varying sub-Riemannian structures. Moreover, the same sub-Riemannian structure can arise from different generating families.

2.2. ***The relative heat content.*** Let M be a sub-Riemannian manifold. Let ω be a smooth measure on M, i.e., by a positive tensor density. The *divergence* of a smooth vector field is defined by

$$\operatorname{div}_\omega(X)\omega = \mathcal{L}_X \omega \quad \text{for all } X \in \Gamma(TM),$$

where $\mathcal{L}_X$ denotes the Lie derivative in the direction of X. The *horizontal gradient* of a function $f \in C^\infty(M)$, denoted by ∇f, is defined as the horizontal vector field (i.e., tangent to the distribution at each point) such that

$$g_x(\nabla f(x), v) = v(f)(x) \quad \text{for all } v \in \mathcal{D}_x,$$

where v acts as a derivation on f. In terms of a generating frame as in (14), one has

$$\nabla f = \sum_{i=1}^N X_i(f) X_i \quad \text{for all } f \in C^\infty(M).$$

We recall the divergence theorem (we stress that M is not required to be orientable): Let $\Omega \subset M$ be open with smooth boundary. Then

$$\int_\Omega (f \operatorname{div}_\omega X + g(\nabla f, X))\, d\omega = -\int_{\partial\Omega} f g(X, \nu)\, d\sigma \tag{15}$$

for any smooth function f and vector field X such that the vector field fX is compactly supported. In (15), ν is the inward-pointing normal vector field to Ω and σ is the induced sub-Riemannian measure on $\partial\Omega$ (i.e., the one whose density is $\sigma = |i_\nu \omega|_{\partial\Omega}$).

The *sub-Laplacian* is the operator $\Delta = \operatorname{div}_\omega \circ \nabla$, acting on $C^\infty(M)$. Again, we may write its expression with respect to a generating frame (14), obtaining

$$\Delta f = \sum_{i=1}^N \{X_i^2(f) + X_i(f) \operatorname{div}_\omega(X_i)\} \quad \text{for all } f \in C^\infty(M). \tag{16}$$

We denote by $L^2(M, \omega)$, or simply by L^2, the space of real functions on M which are square-integrable with respect to the measure ω. Let $\Omega \subset M$ be an open relatively compact set with smooth boundary. This means that the closure $\overline{\Omega}$ is a compact manifold with smooth boundary. We consider the *Cauchy problem for the heat equation* on Ω; that is, we look for functions u such that

$$\begin{aligned} (\partial_t - \Delta) u(t, x) &= 0 \qquad \text{for all } (t, x) \in (0, \infty) \times M, \\ u(0, \cdot) &= \mathbb{1}_\Omega \quad \text{in } L^2(M, \omega), \end{aligned} \tag{17}$$

where $u(0, \cdot)$ is a shorthand notation for the L^2-limit of $u(t, x)$ as $t \to 0$. Notice that Δ is symmetric with respect to the L^2-scalar product and negative; moreover, if (M, d_{SR}) is complete as a metric space, it is essentially self-adjoint; see [Strichartz 1986]. Thus, there exists a unique solution to (17), and it can be represented as

$$u(t, x) = e^{t\Delta} \mathbb{1}_\Omega(x) \quad \text{for all } x \in M,\ t > 0,$$

where $e^{t\Delta} : L^2 \to L^2$ denotes the heat semigroup, associated with Δ. We remark that, for all $\varphi \in L^2$, the function $e^{t\Delta}\varphi$ is smooth for all $(t, x) \in (0, \infty) \times M$, by the hypoellipticity of the heat operator, and there exists a heat kernel associated with (17), i.e., a positive function $p_t(x, y) \in C^\infty((0, +\infty) \times M \times M)$ such that

$$u(t, x) = \int_M p_t(x, y) \mathbb{1}_\Omega(y)\, d\omega(y) = \int_\Omega p_t(x, y)\, d\omega(y). \tag{18}$$

Definition 2.2 (relative heat content). Let $u(t,x)$ be the solution to (17). We define the *relative heat content*, associated with Ω, as

$$H_\Omega(t) = \int_\Omega u(t,x)\, d\omega(x) \quad \text{for all } t > 0.$$

Remark 2.3. If we consider, instead of Ω, a set which is the closure of an open set, then the Cauchy problem (17) has a unique solution and relative heat content is still well-defined.

We recall here a property of the solution to (17): it satisfies a weak maximum principle, meaning that

$$0 \le u(t,x) \le 1 \quad \text{for all } x \in \Omega, \text{ for all } t > 0. \tag{19}$$

This can be proven following the blueprint of the Riemannian proof (see [Grigoryan 2009, Theorem 5.11]).

Definition 2.4 (characteristic point). We say that $x \in \partial\Omega$ is a *characteristic point*, or tangency point, if the distribution is tangent to $\partial\Omega$ at x, that is,

$$\mathcal{D}_x \subseteq T_x(\partial\Omega).$$

We will assume that $\partial\Omega$ has no characteristic points. We say in this case that Ω is a *noncharacteristic domain*.

2.3. *Nilpotent approximation of M.*

We introduce the notion of nilpotent approximation of a sub-Riemannian manifold; see [Jean 2014; Bellaïche 1996] for details. This will be used only in Sections 3 and 6.

Sub-Riemannian flag. Let M be an n-dimensional sub-Riemannian manifold with distribution $\mathcal{D}$. We define the *flag* of $\mathcal{D}$ as the sequence of subsheaves $\mathcal{D}^k \subset TM$ such that

$$\mathcal{D}^1 = \mathcal{D}, \qquad \mathcal{D}^{k+1} = \mathcal{D}^k + [\mathcal{D}, \mathcal{D}^k] \quad \text{for all } k \ge 1,$$

with the convention that $\mathcal{D}^0 = \{0\}$. Under the Hörmander condition, the flag of the distribution defines an exhaustion of T_xM for any point $x \in M$; i.e., there exists $r(x) \in \mathbb{N}$ such that

$$\{0\} = \mathcal{D}^0_x \subset \mathcal{D}^1_x \subset \cdots \subset \mathcal{D}^{r(x)-1}_x \subsetneq \mathcal{D}^{r(x)}_x = T_xM. \tag{20}$$

The number $r(x)$ is called *degree of nonholonomy* at x. We set $n_k(x) = \dim \mathcal{D}^k_x$ for any $k \ge 0$. Then the collection of $r(x)$ integers

$$(n_1(x), \ldots, n_{r(x)}(x))$$

is called *growth vector* at x, and we have $n_{r(x)}(x) = n = \dim M$. Associated with the growth vector, we can define the *sub-Riemannian weights* $w_i(x)$ at x, setting for any $i \in \{1, \ldots, n\}$,

$$w_i(x) = j \quad \text{if and only if} \quad n_{j-1}(x) + 1 \le i \le n_j(x). \tag{21}$$

A point $x \in M$ is said to be *regular* if the growth vector is constant in a neighborhood of x, and *singular* otherwise. The sub-Riemannian structure on M is said to be *equiregular* if all points of M are regular. In this case, the weights are constant as well on M. Finally, given any $x \in M$, we define the *homogeneous dimension* of M at x as

$$\mathcal{Q}(x) = \sum_{i=1}^{r(x)} i(n_i(x) - n_{i-1}(x)) = \sum_{i=1}^{n} w_i(x).$$

We recall that, if x is regular, then $\mathcal{Q}(x)$ coincides with the Hausdorff dimension of (M, d_{SR}) at x; see [Mitchell 1985]. Moreover, $\mathcal{Q}(x) > n$ for any $x \in M$ such that $\mathcal{D}_x \subsetneq T_x M$.

Privileged coordinates. Let M be a sub-Riemannian manifold with generating frame (14) and f be the germ of a smooth function f at $x \in M$. We call *nonholonomic derivative* of order $k \in \mathbb{N}$ of f the quantity

$$X_{j_1} \cdots X_{j_k} f(x)$$

for any family of indices $\{j_1, \dots, j_k\} \subset \{1, \dots, N\}$. Then, the *nonholonomic order* of f at the point x is

$$\mathrm{ord}_x(f) = \min\{k \in \mathbb{N} : \text{there exists } \{j_1, \dots, j_k\} \subset \{1, \dots, N\} \text{ such that } X_{j_1} \cdots X_{j_k} f(x) \neq 0\}.$$

Definition 2.5 (privileged coordinates). Let M be a n-dimensional sub-Riemannian manifold and $x \in M$. A system of local coordinates $(z_1, \dots, z_n)$ centered at x is said to be *privileged* at x if

$$\mathrm{ord}_x(z_j) = w_j(x) \quad \text{for all } j = 1, \dots, n.$$

Notice that privileged coordinates $(z_1, \dots, z_n)$ at x satisfy the following property:

$$\partial_{z_i|x} \in \mathcal{D}_x^{w_i}, \quad \partial_{z_i|x} \notin \mathcal{D}_x^{w_i - 1} \quad \text{for all } i = 1, \dots, n. \tag{22}$$

A local frame of TM consisting of n vector fields $\{Z_1, \dots, Z_n\}$ and satisfying (22) is said to be *adapted* to the flag (20) at x. Thus, privileged coordinates are always adapted to the flag. In addition, given a local frame adapted to the sub-Riemannian flag at x, say $\{Z_1, \dots, Z_n\}$, we can define a set of privileged coordinates at x, starting from $\{Z_1, \dots, Z_n\}$, i.e.,

$$\mathbb{R}^n \ni (z_1, \dots, z_n) \mapsto e^{z_1 Z_1} \circ \cdots \circ e^{z_n Z_n}(x). \tag{23}$$

Moreover, in these coordinates, the vector field Z_1 is exactly ∂_{z_1}.

Nilpotent approximation. Let M be a sub-Riemannian manifold and let $x \in M$ with weights as in (21). Consider $\psi = (z_1, \dots, z_n) : U \to V$ a chart of privileged coordinates at x, where $U \subset M$ is a relatively compact neighborhood of x and $V \subset \mathbb{R}^n$ is a neighborhood of 0. Then, for any $\varepsilon \in \mathbb{R}$, we can define the *dilation* at x as

$$\delta_\varepsilon : \mathbb{R}^n \to \mathbb{R}^n, \quad \delta_\varepsilon(z) = (\varepsilon^{w_1(x)} z_1, \dots, \varepsilon^{w_n(x)} z_n). \tag{24}$$

Using such dilations, we obtain the nilpotent (or first-order) approximation of the generating frame (14); indeed setting $Y_i = \psi_* X_i$ for any $i = 1 \dots, N$, define

$$\widehat{X}_i^x = \lim_{\varepsilon \to 0} \varepsilon \delta_{(1/\varepsilon)*}(Y_i) \quad \text{for all } i = 1 \dots, N, \tag{25}$$

where the limit is taken in the C^∞-topology of $\mathbb{R}^n$. Notice that the vector field $\widehat{X}_i^x$ is defined on the whole $\mathbb{R}^n$, even though Y_i was defined only on $V \subset \mathbb{R}^n$.

Theorem 2.6. *Let M be a n-dimensional sub-Riemannian manifold with generating frame $\{X_1, \dots, X_N\}$ and consider its first-order approximation at x as in (25). Then, the frame $\{\widehat{X}_1^x, \dots, \widehat{X}_N^x\}$ of vector fields on $\mathbb{R}^n$ generates a nilpotent Lie algebra of step $r(x) = w_n(x)$ and satisfies the Hörmander condition.*

The proof of this theorem can be found in [Jean 2014]. Recall that a Lie algebra is said to be nilpotent of step s if s is the smallest integer such that all the brackets of length greater than s are zero.

Definition 2.7 (nilpotent approximation). Let M be a sub-Riemannian manifold and let $x \in M$. Then, Theorem 2.6 implies that the frame $\{\widehat{X}_1^x, \dots, \widehat{X}_N^x\}$ is a generating frame for a sub-Riemannian structure on $\mathbb{R}^n$: we denote the resulting sub-Riemannian manifold by $\widehat{M}^x$. This is the so-called *nilpotent approximation* of M at the point x.

Notice that the sub-Riemannian distance of $\widehat{M}^x$, denoted by $\hat{d}^x$, is 1-homogeneous with respect to the dilations (24).

Remark 2.8. Up to isometries, the nilpotent approximation of M at x coincides with the Gromov–Hausdorff metric tangent space of (M, d_{SR}) at x. Moreover, $\widehat{M}^x$ is isometric to a quotient of a Carnot group. See [Gromov 1996; Bellaïche 1996; Montgomery 2002] for further details.

Nilpotentized sub-Laplacian. Let M be a sub-Riemannian manifold, equipped with a smooth measure ω, and let $(z_1, \dots, z_n)$ be a set of privileged coordinates at $x \in M$. We will use the same symbol ω to denote measure in coordinates. The *nilpotentization* $\hat{\omega}^x$ of ω at x is defined as

$$\langle \hat{\omega}^x, f \rangle = \lim_{\varepsilon \to 0} \frac{1}{|\varepsilon|^{\mathcal{Q}(x)}} \langle \delta_\varepsilon^* \omega, f \rangle \quad \text{for all } f \in C_c^\infty(\mathbb{R}^n). \tag{26}$$

Notice that, denoting by $dz = dz_1 \cdots dz_n$ the Lebesgue measure on $\mathbb{R}^n$, we have

$$\delta_\varepsilon^*(dz) = |\varepsilon|^{\mathcal{Q}(x)} dz \quad \text{for all } \varepsilon \neq 0.$$

Thus, the limit in (26) exists. Finally, we can define the *nilpotentized sub-Laplacian* according to (16), acting on $C^\infty(\mathbb{R}^n)$, i.e.,

$$\hat{\Delta}^x = \operatorname{div}_{\hat{\omega}^x}(\hat{\nabla}^x) = \sum_{i=1}^N (\widehat{X}_i^x)^2. \tag{27}$$

We remark that in (27) there is no divergence term, since

$$\operatorname{div}_{\hat{\omega}^x}(\widehat{X}_i^x) = 0 \quad \text{for all } i \in \{1, \dots, N\}.$$

As in the general sub-Riemannian context, in the nilpotent approximation $\widehat{M}^x$, we may consider the Cauchy heat problem (17) in $L^2(\mathbb{R}^n, \hat{\omega}^x)$. We will the denote the associated heat kernel as

$$\hat{p}_t^x(z, z') \in C^\infty((0, +\infty) \times \mathbb{R}^n \times \mathbb{R}^n).$$

Heat kernel asymptotics. Let M be a sub-Riemannian manifold, equipped with a smooth measure ω and denote by $p_t(x, y)$ the heat kernel (18). We have the following result.

Theorem 2.9 [Colin de Verdière et al. 2021, Theorem A]. *Let M be a sub-Riemannian manifold and let $\psi : U \to V$ be a chart of privileged coordinates at $x \in M$. Then, for any $m \in \mathbb{N}$,*

$$|\varepsilon|^{\mathcal{Q}(x)} p_{\varepsilon^2\tau}(\delta_\varepsilon(z), \delta_\varepsilon(z')) = \hat{p}^x_\tau(z, z') + \sum_{i=1}^{m} \varepsilon^i f^x_i(\tau, z, z') + o(|\varepsilon|^m) \quad \text{as } \varepsilon \to 0, \tag{28}$$

in the C^∞-topology of $(0, \infty) \times V \times V$, where the f^x_i are smooth functions satisfying the following homogeneity property: for $i = 0, \dots, m$,

$$|\varepsilon|^{\mathcal{Q}(x)} \varepsilon^{-i} f^x_i(\varepsilon^2\tau, \delta_\varepsilon(z), \delta_\varepsilon(z')) = f^x_i(\tau, z, z') \quad \text{for all } (\tau, z, z') \in (0, \infty) \times \mathbb{R}^n \times \mathbb{R}^n, \tag{29}$$

where, for $i = 0$, we set $f^x_0(\tau, z, z') = \hat{p}^x_\tau(z, z')$. In (28), we are considering the heat kernel p_t in coordinates, with a little abuse of notation.

Remark 2.10. We will drop the dependence on the center of the privileged coordinates if there is no confusion.

3. Small-time asymptotics of $u(t, x)$ at the boundary

We prove here Theorem 1.3, regarding the zero-order asymptotics of $u(t, \cdot)|_{\partial\Omega}$ as $t \to 0$.

Theorem 3.1. *Let M be a compact sub-Riemannian manifold, equipped with a smooth measure ω and let $\Omega \subset M$ be an open subset, whose boundary is smooth and has no characteristic points. Let $x \in \partial\Omega$ and consider a chart of privileged coordinates $\psi : U \to V \subset \mathbb{R}^n$ centered at x such that $\psi(U \cap \Omega) = V \cap \{z_1 > 0\}$. Then,*

$$\lim_{t \to 0} u(t, x) = \int_{\{z_1 > 0\}} \hat{p}^x_1(0, z)\, d\hat{\omega}^x(z) = \frac{1}{2} \quad \text{for all } x \in \partial\Omega,$$

where $\hat{\omega}^x$ denotes the nilpotentization of ω at x and $\hat{p}^x_t$ denotes the heat kernel associated with the nilpotent approximation of M at x and measure $\hat{\omega}^x$.

Remark 3.2. A chart of privileged coordinates such that $\psi(U \cap \Omega) = V \cap \{z_1 > 0\}$ always exists, provided that $\partial\Omega$ has no characteristic points. Indeed, in this case, there exists a tubular neighborhood of the boundary, see Theorem 4.1, which is built through the flow of $\nabla\delta$, namely

$$G : (-r_0, r_0) \times \partial\Omega \to \Omega^{r_0}_{-r_0}, \quad G(t, q) = e^{t\nabla\delta}(q),$$

is a diffeomorphism such that $G_*\partial_t = \nabla\delta$ and $\delta(G(t, q)) = t$. Here $\delta : M \to \mathbb{R}$ is the signed distance function[4] from $\partial\Omega$ and $\Omega^{r_0}_{-r_0} = \{-r_0 < \delta < r_0\}$; see Section 4.1 for precise definitions. Therefore, choosing

[4]We warn the reader that δ without a subscript always denotes the signed distance function and should not be confused with dilations δ_ε.

an adapted frame for the distribution at x, say $\{Z_1, \dots, Z_n\}$, where $Z_1 = \nabla\delta$, we can define a set of privileged coordinates as in (23):

$$\mathbb{R}^n \ni (z_1, \dots, z_n) \mapsto e^{z_1 Z_1} \circ \underbrace{e^{z_2 Z_2} \circ \cdots \circ e^{z_n Z_n}(x)}_{\varphi(z_2,\dots,z_n)} = G(z_1, \varphi(z_2, \dots, z_n)). \tag{30}$$

The resulting set of coordinates ψ satisfies $\psi_*(\nabla\delta) = \partial_{z_1}$ and, denoting by V the neighborhood of 0 in $\mathbb{R}^n$ where ψ is invertible, $\psi(U \cap \Omega) = \{z_1 > 0\} \cap V$. Here, $e^{sX}(q)$ denotes the flow of the vector field X, starting at q, evaluated at time s.

Proof of Theorem 3.1. Let $p_t(x, y)$ be the heat kernel of M. Then we may write

$$u(t, x) = \int_\Omega p_t(x, y)\, d\omega(y) \quad \text{for all } x \in M.$$

For a fixed $x \in M$, denoting by U any relatively compact neighborhood of x, we have

$$\begin{aligned} u(t, x) &= \int_{U\cap\Omega} p_t(x, y)\, d\omega(y) + \int_{\Omega\setminus U} p_t(x, y)\, d\omega(y) \\ &= \int_{U\cap\Omega} p_t(x, y)\, d\omega(y) + O(t^\infty) \end{aligned}$$

as $t \to 0$. Indeed, since the heat kernel is exponentially decaying outside the diagonal, see [Jerison and Sánchez-Calle 1986, Proposition 3],

$$\int_{\Omega\setminus U} p_t(x, y)\, d\omega(y) \le \omega(\Omega \setminus U) C_U e^{-c_U/t} = O(t^\infty) \quad \text{as } t \to 0. \tag{31}$$

Now, for $x \in \partial\Omega$, fix the set of privileged coordinates $\psi : U \to V \subset \mathbb{R}^n$, defined as in the statement, and assume without loss of generality that $\delta_\varepsilon(V) \subset V$ for any $|\varepsilon| \le 1$, where δ_ε is the dilation (24) of the nilpotent approximation of M. Also set

$$V_\varepsilon = \delta_\varepsilon(V \cap \{z_1 > 0\}) \quad \text{for all } |\varepsilon| \le 1.$$

When the limits exist, we have

$$\lim_{t\to 0} u(t, x) = \lim_{t\to 0} \int_{U\cap\Omega} p_t(x, y)\, d\omega(y) = \lim_{t\to 0} \int_{V_1} p_t(0, z)\, d\omega(z), \tag{32}$$

where, in the last equation, we are considering the expression of the heat kernel and the measure in coordinates. We want to apply (28) at order 1 in ε, so let us rephrase the statement as follows: for any compact set $K \subset V$,

$$|\varepsilon|^{\mathcal{Q}} p_{\varepsilon^2\tau}(0, \delta_\varepsilon(z)) = \hat{p}_\tau(0, z) + \varepsilon R(\varepsilon, \tau, z) \quad \text{as } \varepsilon \to 0, \tag{33}$$

where R is a smooth function such that

$$\sup_{\varepsilon\in[-1,1],\ z\in K} |R(\varepsilon, \tau, z)| \le C(\tau, K), \tag{34}$$

with $C(\tau, K) > 0$. Notice that (34) is not uniform in τ, in the sense that $\tau \mapsto C(\tau, K)$ can explode as $\tau \to 0$, in general. Moreover, without loss of generality and, up to restrictions of U, we can assume that

(34) holds globally on $\overline{V}_1$. For a fixed parameter $L > 1$, we set $\tau = 1/L$ and $\varepsilon^2 = tL$ in (33), obtaining

$$|tL|^{\mathcal{Q}/2} p_t(0, \delta_{\sqrt{tL}}(z)) = \hat{p}_{1/L}(0, z) + \sqrt{tL} R(\sqrt{tL}, 1/L, z) \quad \text{as } t \to 0,$$

where the remainder R is bounded as $t \to 0$ on the compact sets of V, but with a constant depending on L. Inserting the above expansion in (32), and writing the measure in coordinates $d\omega(z) = \omega(z)\, dz$, with $\omega(\,\cdot\,) \in C^\infty(V_1)$, we have

$$\begin{aligned} u(t, x) &= \int_{V_1} p_t(0, z)\omega(z)\, dz + O(t^\infty) \\ &= \int_{V_{\sqrt{tL}}} p_t(0, z)\omega(z)\, dz + \int_{V_1 \setminus V_{\sqrt{tL}}} p_t(0, z)\, d\omega(z) + O(t^\infty) \\ &= \int_{V_1} |tL|^{\mathcal{Q}/2} p_t(0, \delta_{\sqrt{tL}}(z))\omega(\delta_{\sqrt{tL}}(z))\, dz + \int_{V_1 \setminus V_{\sqrt{tL}}} p_t(0, z)\, d\omega(z) + O(t^\infty) \\ &= \int_{V_1} \big(\hat{p}_{1/L}(0, z) + \sqrt{tL} R\big(\sqrt{tL}, 1/L, z\big)\big)\omega(\delta_{\sqrt{tL}}(z))\, dz \qquad (35) \\ &\qquad\qquad + \int_{V_1 \setminus V_{\sqrt{tL}}} p_t(0, z)\, d\omega(z) + O(t^\infty), \qquad (36) \end{aligned}$$

where in the third equality we performed the change of variable $z \mapsto \delta_{1/\sqrt{tL}}(z)$ in the first integral. Let us discuss the terms appearing in (35) and (36). First of all, for any $L > 1$, by definition of the nilpotentization of ω given in (26), we get

$$\lim_{t \to 0} \int_{V_1} \hat{p}_{1/L}(0, z)\omega(\delta_{\sqrt{tL}}(z))\, dz = \int_{V_1} \hat{p}_{1/L}(0, z)\, d\hat{\omega}(z).$$

Moreover, for a fixed $L > 1$, the integral of R is bounded as $t \to 0$; Therefore, using (34), we have

$$\left| \sqrt{tL} \int_{V_1} R(\sqrt{tL}, 1/L, z)\, d\omega(z) \right| \le C_L \sqrt{t} \quad \text{for all } t \le 1,$$

where $C_L > 0$ is a constant depending on the fixed L. Secondly, by an upper Gaussian bound for the heat kernel in compact sub-Riemannian manifold [Jerison and Sánchez-Calle 1986, Theorem 2], we obtain the following estimate for (36):

$$\int_{V_1 \setminus V_{\sqrt{tL}}} p_t(0, z)\, d\omega(z) \le \int_{V_1 \setminus V_{\sqrt{tL}}} \frac{C_1 e^{-\beta\, d_{\mathrm{SR}}^2(0,z)/t}}{t^{\mathcal{Q}/2}}\, d\omega(z), \qquad (37)$$

where $C_1, \beta > 0$ are positive constants. Now, by the ball-box theorem [Jean 2014, Theorem 2.1], the sub-Riemannian distance function at the origin is comparable with the sub-Riemannian distance of $\widehat{M}^x$, denoted by $\hat{d}$. In particular, there exists a constant $c > 0$ such that

$$d_{\mathrm{SR}}^2(0, z) \ge c\, \hat{d}^2(0, z) \quad \text{for all } z \in V. \qquad (38)$$

Since in (37) we are integrating over the set $V_1 \setminus V_{\sqrt{tL}}$ and $\hat{d}$ is 1-homogeneous with respect to δ_ε, we conclude that

$$d_{\mathrm{SR}}^2(0, z) \ge c\, tL \quad \text{for all } z \in V_1 \setminus V_{\sqrt{tL}}.$$

Therefore, using also (38), the term (37) can be estimated as follows:

$$\begin{aligned}\int_{V_1\setminus V_{\sqrt{tL}}} p_t(0,z)\,d\omega(z) &\leq C_1 e^{-c\beta L/2}\int_{V_1}\frac{e^{-\beta d^2_{\mathrm{SR}}(0,z)/(2t)}}{t^{\mathcal{Q}/2}}\,d\omega(z)\\ &\leq C_1 e^{-c\beta L/2}\int_{V_1}\frac{e^{-\beta c\,\hat{d}^2(0,z)/(2t)}}{t^{\mathcal{Q}/2}}\omega(z)\,dz \leq \widetilde{C}e^{-c\beta L/2},\end{aligned}\tag{39}$$

where $\widetilde{C}>0$ is independent of t and L. The last inequality in (39) follows from the fact that, after a change of variable $z\mapsto\delta_{1/\sqrt{t}}(z)$, the integral

$$\int_{V_1}\frac{e^{-\beta c\hat{d}^2(0,z)/(2t)}}{t^{\mathcal{Q}/2}}\omega(z)\,dz<+\infty$$

is uniformly bounded with respect to $t\in[0,\infty)$.

Therefore, for any $L>1$, we obtain the following estimates for the limit of u:

$$\begin{aligned}\limsup_{t\to 0} u(t,x) &\leq \int_{V_1}\hat{p}_{1/L}(0,z)\,d\hat{\omega}(z)+\widetilde{C}e^{-c\beta L/2},\\ \liminf_{t\to 0} u(t,x) &\geq \int_{V_1}\hat{p}_{1/L}(0,z)\,d\hat{\omega}(z)-\widetilde{C}e^{-c\beta L/2}.\end{aligned}\tag{40}$$

In order to evaluate the limits in (40), let us firstly notice that, since $\hat{p}$ enjoys upper and lower Gaussian bounds (see for example [Colin de Verdière et al. 2021, Appendix C]), reasoning as we did for (39), we can prove

$$\int_{V_1}\hat{p}_{1/L}(0,z)\,d\hat{\omega}(z)=\int_{\{z_1>0\}}\hat{p}_{1/L}(0,z)\,d\hat{\omega}(z)+O(e^{-\beta' L}).\tag{41}$$

Secondly, thanks to (29) for $\hat{p}$, we have the parity property

$$\hat{p}_t(0,z)=\hat{p}_t(0,\delta_{-1}(z))\quad\text{for all } t>0,\ z\in\mathbb{R}^n,$$

and, by the choice of privileged coordinates, $\delta_{-1}(\{z_1>0\})=\{z_1<0\}$. Thus, using also the stochastic completeness of the nilpotent approximation, we obtain, for any $t\geq 0$,

$$1=\int_{\mathbb{R}^n}\hat{p}_t(0,z)\,d\hat{\omega}(z)=\int_{\{z_1>0\}}\hat{p}_t(0,z)\,d\hat{\omega}(z)+\int_{\{z_1<0\}}\hat{p}_t(0,z)\,d\hat{\omega}(z)=2\int_{\{z_1>0\}}\hat{p}_t(0,z)\,d\hat{\omega}(z),$$

having performed the change of variables $z\mapsto\delta_{-1}(z)$ in the last equality. Hence, the integral in (41) is

$$\int_{V_1}\hat{p}_{1/L}(0,z)\,d\hat{\omega}(z)=\frac{1}{2}+O(e^{-\beta' L}).$$

Finally, we optimize the inequalities (40) with respect to L, taking $L\to\infty$ and concluding the proof. □

Remark 3.3. In the noncompact case, if M is globally doubling and supports a global Poincaré inequality, the proof above is still valid; see Theorem 7.3. Otherwise, a different proof is needed; see [Rossi 2021, Appendix D] for details.

4. First-order asymptotic expansion of $H_\Omega(t)$

In this section, we introduce the technical tools that allow us to prove the first-order asymptotic expansion of the relative heat content starting from Theorem 3.1. The new ingredient is a definition of an operator I_Ω, which depends on the base set Ω.

4.1. ***A mean value lemma.*** Define $\delta : M \to \mathbb{R}$ to be the signed distance function from $\partial\Omega$, i.e.,

$$\delta(x) = \begin{cases} d_{\mathrm{SR}}(x, \partial\Omega), & x \in \Omega, \\ -d_{\mathrm{SR}}(x, \partial\Omega), & x \in M \setminus \Omega, \end{cases}$$

where $d_{\mathrm{SR}}(\,\cdot\,, \partial\Omega) : M \to [0, +\infty)$ denotes the usual distance function from $\partial\Omega$. Let us introduce the following notation: for any $a, b \in \mathbb{R}$, with $a < b$, we set

$$\Omega_a^b = \{x \in M : a < \delta(x) < b\},$$

with the understanding that if b (or a) is omitted, it is assumed to be $+\infty$ (or $-\infty$), for example[5]

$$\Omega_r = \Omega_r^{+\infty} = \{x \in M : r < \delta(x)\}.$$

In the noncharacteristic case, [Franceschi et al. 2020, Proposition 3.1] can be extended without difficulties to the signed distance function.

Theorem 4.1 (double-sided tubular neighborhood). *Let M be a sub-Riemannian manifold and let $\Omega \subset M$ be an open relatively compact subset of M whose boundary is smooth and has no characteristic points. Let $\delta : M \to \mathbb{R}$ be the signed distance function from $\partial\Omega$. Then, we have:*

(i) *δ is Lipschitz with respect to the sub-Riemannian distance and $\|\nabla\delta\|_g \le 1$ a.e.*

(ii) *There exists $r_0 > 0$ such that $\delta : \Omega_{-r_0}^{r_0} \to \mathbb{R}$ is smooth.*

(iii) *There exists a smooth diffeomorphism $G : (-r_0, r_0) \times \partial\Omega \to \Omega_{-r_0}^{r_0}$ such that*

$$\delta(G(t, y)) = t \quad \text{and} \quad G_*\partial_t = \nabla\delta \quad \text{for all } (t, y) \in (-r_0, r_0) \times \partial\Omega.$$

Moreover, $\|\nabla\delta\|_g \equiv 1$ on $\Omega_{-r_0}^{r_0}$.

In particular, the following coarea formula for the signed distance function holds:

$$\int_{\Omega_0^r} v(x)\, d\omega(x) = \int_0^r \int_{\partial\Omega_s} v(s, y)\, d\sigma(y)\, ds \quad \text{for all } r \ge 0, \tag{42}$$

where σ is the induced measure on $\partial\Omega_s$, namely the positive measure with density $|i_{\nabla\delta}\omega|_{|\partial\Omega_s}$. From (42), we deduce the sub-Riemannian mean value lemma; see [Rizzi and Rossi 2021, Theorem 4.1] for a proof.

[5] Notice that the set $\Omega_{-\infty}^{+\infty}$ is equal to M; thus omitting both indices can create confusion. We will never do that and Ω will always denote the starting subset of M.

Proposition 4.2. *Let M be a compact sub-Riemannian manifold, equipped with a smooth measure ω, let $\Omega \subset M$ be an open subset of M with smooth boundary and no characteristic points and let $\delta : M \to \mathbb{R}$ be the signed distance function from $\partial\Omega$. Fix a smooth function $v \in C^\infty(M)$ and define*

$$F(r) = \int_{\Omega_r} v(x) d\omega(x) \quad \textit{for all } r \geq 0. \tag{43}$$

Then there exists $r_0 > 0$ such that the function F is smooth on $[0, r_0)$ and, for $0 \leq r < r_0$,

$$F''(r) = \int_{\Omega_r} \Delta v(x)\, d\omega(x) - \int_{\partial\Omega_r} v(y)\, \mathrm{div}_\omega(\nu(y))\, d\sigma(y),$$

where ν is the inward-pointing unit normal to Ω_r, and σ is the induced measure on $\partial\Omega_r$.

Remark 4.3. If $v \in C_c^\infty(M)$, then neither M nor Ω is required to be compact for Proposition 4.2 to be true; indeed its proof relies on (42), which continues to hold, and the divergence theorem (15), which applies if $\mathrm{supp}(v)$ is compact. Moreover, we remark that ν_r is equal to $\nabla\delta$ up to sign. We prefer to keep ν_r in (43), since we are going to apply it when the integral is performed over Ω_r or its complement.

If we choose the function v in the definition of F to be $1 - u(t, x)$, where $u(t, \cdot) = e^{t\Delta}\mathbb{1}_\Omega$, then F satisfies a nonhomogeneous one-dimensional heat equation.

Corollary 4.4. *Under the hypotheses of Proposition 4.2, the function*

$$F(t, r) = \int_{\Omega_r} (1 - u(t, x))\, d\omega(x) \quad \textit{for all } t > 0,\ r \geq 0, \tag{44}$$

where $u(t, x) = e^{t\Delta}\mathbb{1}_\Omega(x)$, satisfies the following nonhomogeneous one-dimensional heat equation:

$$(\partial_t - \partial_r^2)F(t, r) = \int_{\partial\Omega_r} (1 - u(t, \cdot))\, \mathrm{div}_\omega(\nu)\, d\sigma, \quad t > 0,\ r \in [0, r_0). \tag{45}$$

Here ν is the inward-pointing unit normal to Ω_r, and σ is the induced measure on $\partial\Omega_r$.

Corollary 4.4 holds only for $r \leq r_0$; however, we would like to extend it to the *whole* positive half-line, in order to apply a Duhamel's principle. This can be done up to an error which is exponentially small.

4.2. *Localization principle.*

Proposition 4.5. *Let M be a compact sub-Riemannian manifold, equipped with a smooth measure ω, and let $\Omega \subset M$ be an open subset of M, with smooth boundary. Moreover, let $K \subset M$ be a closed set such that $K \cap \partial\Omega = \emptyset$. Then*

$$\mathbb{1}_\Omega(x) - u(t, x) = O(t^\infty) \quad \textit{uniformly for } x \in K,$$

where $u(t, x) = e^{t\Delta}\mathbb{1}_\Omega(x)$.

Proof. The statement is a direct consequence of the off-diagonal estimate for the heat kernel in compact sub-Riemannian manifold (see [Jerison and Sánchez-Calle 1986, Proposition 3]):

$$p_t(x, y) \leq C_a e^{-c_a/t} \quad \text{for all } x, y \text{ with } d(x, y) \geq a,\ t < 1, \tag{46}$$

for suitable constants $C_a, c_a > 0$, depending only on a. Now, since $K \cap \partial\Omega = \varnothing$, we can write K as a disjoint union

$$K = K_1 \sqcup K_2, \quad \text{with } K_1 \subset \Omega,\ K_2 \subset M \setminus \Omega.$$

At this point, for $i = 1, 2$, set $a_i = d_{\mathrm{SR}}(K_i, \partial\Omega) > 0$ by hypothesis, and let $x \in K_1$. Then, using the stochastic completeness of M, we have

$$|\mathbb{1}_\Omega(x) - u(t,x)| = 1 - u(t,x) = \int_{M\setminus\Omega} p_t(x,y)\, d\omega(y) \le C_1 e^{-c_1/t}\omega(M\setminus\Omega), \tag{47}$$

which is exponentially decaying, uniformly in K_1. Analogously, if $x \in K_2$, we have

$$|\mathbb{1}_\Omega(x) - u(t,x)| = u(t,x) = \int_M p_t(x,y)\mathbb{1}_\Omega(y)\, d\omega(y) = \int_\Omega p_t(x,y)\, d\omega(y) \le C_2 e^{-c_2/t}\omega(\Omega),$$

uniformly in K_2. □

Remark 4.6. In the noncompact case, Proposition 4.5 may fail. Indeed, on the one hand the off-diagonal estimate (46) is not always available; on the other hand the measure of $M \setminus \Omega$ appearing in (47) is infinite. Under additional assumption on M, we are able to recover a localization principle; see Section 7.

Let M be compact. Thanks to Proposition 4.5, we can extend the function F defined in (44) to a solution to a nonhomogeneous heat equation such as (45) on the whole half-line. More precisely, let $\phi, \eta \in C_c^\infty(M)$ such that

$$\phi + \eta \equiv 1, \quad \mathrm{supp}(\phi) \subset \Omega_{-r_0}^{r_0}, \quad \mathrm{supp}(\eta) \subset \Omega^{-r_0/2} \cup \Omega_{r_0/2}, \tag{48}$$

where r_0 is defined in Proposition 4.2. We have then, for $r \in [0, r_0)$,

$$\begin{aligned} F(t,r) &= \int_{\Omega_r} (1-u(t,x))\phi(x)\, d\omega(x) + \int_{\Omega_r} (1-u(t,x))\eta(x)\, d\omega(x) \\ &= \int_{\Omega_r} (1-u(t,x))\phi(x)\, d\omega(x) + \int_{\mathrm{supp}(\eta)\cap\Omega_r} (1-u(t,x))\eta(x)\, d\omega(x) \\ &= \int_{\Omega_r} (1-u(t,x))\phi(x)\, d\omega(x) + O(t^\infty), \end{aligned} \tag{49}$$

where we used Proposition 4.5 to deal with the second term, having set $K = \mathrm{supp}(\eta) \cap \Omega_r$. For this reason, we may focus on the first term in (49).

Definition 4.7. For all $t > 0$ and $r \ge 0$, we define the operators $I_\Omega, \Lambda_\Omega : C_c^\infty(\Omega_{-r_0}^{r_0}) \to C^\infty((0,\infty)\times[0,\infty))$, associated with Ω, by

$$I_\Omega\phi(t,r) = \int_{\Omega_r} (1-u(t,x))\phi(x)\, d\omega(x),$$

$$\Lambda_\Omega\phi(t,r) = -\partial_r I_\Omega\phi(t,r) = -\int_{\partial\Omega_r} (1-u(t,y))\phi(y)\, d\sigma(y)$$

for any $\phi \in C_c^\infty(\Omega_{-r_0}^{r_0})$, and where σ denotes the induced measure on $\partial\Omega_r$ and $u(t,\cdot) = e^{t\Delta}\mathbb{1}_\Omega(\cdot)$.

Remark 4.8. We stress that, for every $\phi \in C_c^\infty(\Omega_{-r_0}^{r_0})$, $I_\Omega\phi$, $\Lambda_\Omega\phi$ are indeed smooth in both variables thanks to the choice of the parameter $r_0 > 0$ as in Proposition 4.2, together with the smoothness of the solution to the heat equation. Moreover, $\Lambda_\Omega\phi$ is compactly supported in the r-variable.

Thanks to the localization principle, we can improve Corollary 4.4, obtaining a better result for $I_\Omega\phi(t,r)$.

Lemma 4.9. *Let $L = \partial_t - \partial_r^2$ be the one-dimensional heat operator. Then, for any $\phi \in C_c^\infty(\Omega_{-r_0}^{r_0})$,*

$$L(I_\Omega\phi(t,r)) = I_\Omega\Delta\phi(t,r) + \Lambda_\Omega N_\Omega\phi(t,r) \quad \text{for all } t > 0,\ r \geq 0,$$

where N_Ω is the operator defined by

$$N_\Omega\phi = 2g(\nabla\phi, \nu) + \phi \operatorname{div}_\omega(\nu) \quad \text{for all } \phi \in C_c^\infty(\Omega_{-r_0}^{r_0}), \tag{50}$$

and ν is the inward-pointing unit normal to Ω.

4.3. *Duhamel's principle for $I_\Omega\phi$.* We recall for the convenience of the reader a one-dimensional version of the Duhamel's principle; see [Rizzi and Rossi 2021, Lemma 5.4].

Lemma 4.10 (Duhamel's principle). *Let $f \in C((0,\infty)\times[0,\infty))$, $v_0, v_1 \in C([0,\infty))$, such that $f(t,\cdot)$ and v_0 are compactly supported and assume that*

$$\text{there exists } \lim_{t\to 0} f(t,r) \quad \text{for all } r \geq 0.$$

Consider the nonhomogeneous heat equation on the half-line:

$$\begin{aligned} Lv(t,r) &= f(t,r) && \text{for } t > 0,\ r > 0, \\ v(0,r) &= v_0(r) && \text{for } r > 0, \\ \partial_r v(t,0) &= v_1(t) && \text{for } t > 0, \end{aligned} \tag{51}$$

where $L = \partial_t - \partial_r^2$. Then, for $t > 0$ and $r \geq 0$, the solution to (51) is given by

$$v(t,r) = \int_0^\infty e(t,r,s)v_0(s)\,ds + \int_0^t\int_0^\infty e(t-\tau,r,s)f(\tau,s)\,ds\,d\tau - \int_0^t e(t-\tau,r,0)v_1(\tau)\,d\tau, \tag{52}$$

where $e(t,r,s)$ is the Neumann heat kernel on the half-line, that is,

$$e(t,r,s) = \frac{1}{\sqrt{4\pi t}}\left(e^{-(r-s)^2/(4t)} + e^{-(r+s)^2/(4t)}\right). \tag{53}$$

Finally, we apply Lemma 4.10 to obtain an asymptotic equality for $I_\Omega\phi$. The main difference with the result of [Rizzi and Rossi 2021, Theorem 5.6] is that the former will not be a true first-order asymptotic expansion.

Corollary 4.11. *Let M be a compact sub-Riemannian manifold, equipped with a smooth measure ω, and let $\Omega \subset M$ be an open subset whose boundary is smooth and has no characteristic points. Then, for any* *function* $\phi \in C_c^\infty(\Omega_{-r_0}^{r_0})$,

$$I_\Omega\phi(t,0) = \frac{1}{\sqrt{\pi}} \int_0^t \int_{\partial\Omega} (1-u(\tau,y))\phi(y)\,d\sigma(y)(t-\tau)^{-1/2}\,d\tau + O(t)$$

as $t \to 0$, *where* $u(t,\cdot) = e^{t\Delta}\mathbb{1}_\Omega(\cdot)$.

Proof. By Lemma 4.9, the function $I_\Omega\phi(t,r)$ satisfies the following Neumann problem on the half-line:

$$\begin{aligned} LI_\Omega\phi(t,r) &= f(t,r) && \text{for } t>0,\ r>0,\\ I_\Omega\phi(0,r) &= 0 && \text{for } r>0,\\ \partial_r I_\Omega\phi(t,0) &= -\Lambda_\Omega\phi(t,0) && \text{for } t>0,\end{aligned}$$

where the source is given by $f(t,r) = I_\Omega\Delta\phi(t,r) + \Lambda_\Omega N_\Omega\phi(t,r)$. Thus, applying Duhamel's formula (52), we have

$$I_\Omega\phi(t,0) = \int_0^t \int_0^{+\infty} e(t-\tau,0,s) f(\tau,s)\,ds\,d\tau + \frac{1}{\sqrt{\pi}} \int_0^t \frac{1}{\sqrt{t-\tau}} \Lambda_\Omega\phi(t,0)\,d\tau.$$

Since the source is uniformly bounded by the weak maximum principle (19), the first integral is a remainder of order t as $t \to 0$, concluding the proof. $\square$

Remark 4.12. We mention that a relevant role in the sequel will be played by the operators I_Ω, see Definition 4.7, associated with either Ω or its complement Ω^c.

4.4. ***First-order asymptotics.*** In this section we prove the first-order asymptotic expansion of $H_\Omega(t)$; see Theorem 1.1 at order 1. We will use Corollary 4.11 for the *inside contribution*:

$$I\phi(t,r) = \int_{\Omega_r} (1-u(t,x))\phi(x)\,d\omega(x) \quad \text{for all } t>0,\ r\geq 0, \tag{54}$$

for any $\phi \in C_c^\infty(\Omega_{-r_0}^{r_0})$, and where σ denotes the induced measure on $\partial\Omega_r$ and $u(t,\cdot) = e^{t\Delta}\mathbb{1}_\Omega(\cdot)$ is the solution to (17). The quantity (54) is just Definition 4.7, applied with base set $\Omega \subset M$.

Theorem 4.13. *Let M be a compact sub-Riemannian manifold, equipped with a smooth measure ω, and let $\Omega \subset M$ be an open subset whose boundary is smooth and has no characteristic points. Then,*

$$H_\Omega(t) = \omega(\Omega) - \frac{1}{\sqrt{\pi}}\sigma(\partial\Omega)t^{1/2} + o(t^{1/2}) \quad \textit{as } t \to 0.$$

Proof. Let $\phi \in C_c^\infty(\Omega_{-r_0}^{r_0})$ be as in (48); namely

$$0 \leq \phi \leq 1 \qquad \text{and} \qquad \phi \equiv 1 \quad \text{in } \Omega_{-r_0/2}^{r_0/2}.$$

Then, by the localization principle, see (49), we have that

$$\omega(\Omega) - H_\Omega(t) = I\phi(t,0) + O(t^\infty) \quad \text{as } t \to 0. \tag{55}$$

Applying Corollary 4.11, we have

$$I\phi(t,0) = \frac{1}{\sqrt{\pi}} \int_0^t \int_{\partial\Omega} (1-u(\tau,y))\phi(y)\,d\sigma(y)(t-\tau)^{-1/2}\,d\tau + O(t) \quad \text{as } t \to 0. \tag{56}$$

Thus, to infer the first-order term of the asymptotic expansion we have to compute the following limit:

$$\lim_{t\to 0}\frac{I\phi(t,0)}{t^{1/2}}=\lim_{t\to 0}\frac{1}{t^{1/2}\sqrt{\pi}}\int_0^t\int_{\partial\Omega}(1-u(\tau,y))\phi(y)\,d\sigma(y)(t-\tau)^{-1/2}\,d\tau. \tag{57}$$

Firstly, by the change of variable in the integral $\tau\mapsto t\tau$, we rewrite the argument of the limit (57) as

$$\frac{1}{\sqrt{\pi}}\int_0^1\int_{\partial\Omega}(1-u(t\tau,y))\phi(y)\,d\sigma(y)(1-\tau)^{-1/2}\,d\tau.$$

Secondly, we apply the dominated convergence theorem. Indeed, on the one hand, by Theorem 3.1 we have pointwise convergence

$$(1-u(t\tau,y))\phi(y)\xrightarrow{t\to 0}\tfrac{1}{2}\phi(y)\quad\text{for all } y\in\partial\Omega,\ \tau\in(0,1),$$

and on the other hand, by the maximum principle

$$\left|\int_{\partial\Omega}(1-u(t\tau,y))\phi(y)\,d\sigma(y)(1-\tau)^{-1/2}\right|\le\int_{\partial\Omega}|\phi|\,d\sigma(1-\tau)^{-1/2}\in L^1(0,1)$$

for any $t>0$. Therefore, we finally obtain that

$$I\phi(t,0)=\sqrt{\frac{t}{\pi}}\int_{\partial\Omega}\phi(y)\,d\sigma(y)+o(t^{1/2})\quad\text{as } t\to 0.$$

Recalling that $\phi_{|\partial\Omega}\equiv 1$, we conclude the proof. □

Remark 4.14. The above technique used to evaluate the first-order coefficient causes a loss of precision in the remainder, with respect to the expression (56), where the remainder is $O(t)$. This loss comes from the application of Theorem 3.1, which does not contain any remainder estimate.

5. Higher-order asymptotic expansion of $H_\Omega(t)$

We iterate Duhamel's formula (52) for the inside contribution to study the higher-order asymptotics of $H_\Omega(t)$. We obtain the following expression for $I\phi$ at order 3:

$$\begin{aligned}I\phi(t,0)=\frac{1}{\sqrt{\pi}}&\int_0^t\int_{\partial\Omega}(1-u(\tau,\cdot))\phi\,d\sigma(t-\tau)^{-1/2}\,d\tau\\&+\frac{1}{2\pi}\int_0^t\int_0^\tau\int_{\partial\Omega}(1-u(\hat\tau,\cdot))N\phi\,d\sigma((\tau-\hat\tau)(t-\tau))^{-1/2}\,d\hat\tau\,d\tau+O(t^{3/2}),\end{aligned}\tag{58}$$

where $u(t,\cdot)=e^{t\Delta}\mathbb{1}_\Omega(\cdot)$ denotes the solution to (17) and N is the operator defined by

$$N\phi=2g(\nabla\phi,\nabla\delta)+\phi\Delta\delta\quad\text{for all }\phi\in C_c^\infty(\Omega_{-r_0}^{r_0}),\tag{59}$$

with $\delta:M\to\mathbb{R}$ the signed distance function from $\partial\Omega$. The computations for deriving (58) are technical. We refer to the Appendix for further details, and in particular to Lemma A.6. Motivated by (58), we introduce the following functional.

Definition 5.1. Let M be a sub-Riemannian manifold, equipped with a smooth measure ω, let $\Omega \subset M$ be a relatively compact subset with smooth boundary and let $v \in C^\infty((0,+\infty)\times M)$. Define the functional $\mathcal{G}_v$, for any $\phi \in C_c^\infty(\Omega_{-r_0}^{r_0})$ as

$$\mathcal{G}_v[\phi](t) = \frac{1}{2\sqrt{\pi}} \int_0^t \int_{\partial\Omega} v(\tau,\cdot)\phi \, d\sigma (t-\tau)^{-1/2} \, d\tau \quad \text{for all } t \geq 0, \tag{60}$$

where σ is the sub-Riemannian induced measure on $\partial\Omega$.

Notice that the functional $\mathcal{G}_v$ is linear with respect to the subscript function v, by linearity of the integral. Moreover, when the function v is chosen to be the solution to (17), we easily obtain the following corollary of Theorem 3.1, which is just a rewording of (57).

Corollary 5.2. *Let M be a compact sub-Riemannian manifold, equipped with a smooth measure ω, and let $\Omega \subset M$ be an open subset whose boundary is smooth and has no characteristic points. Let $\phi \in C_c^\infty(\Omega_{-r_0}^{r_0})$. Then,*

$$\mathcal{G}_u[\phi](t) = \frac{1}{2\sqrt{\pi}} \int_{\partial\Omega} \phi(y)\, d\sigma(y) t^{1/2} + o(t^{1/2}) \quad \text{as } t \to 0.$$

Then, we can rewrite (58) in a compact notation:

$$I\phi(t,0) = 2\mathcal{G}_{1-u}[\phi](t) + \frac{1}{\sqrt{\pi}} \int_0^t \mathcal{G}_{1-u}[N\phi](t)\, d\sigma (t-\tau)^{-1/2}\, d\tau + O(t^{3/2}). \tag{61}$$

However, on the one hand, the application of Corollary 5.2 to (58) does not give any new information on the asymptotics of $H_\Omega(t)$, as the first term produces an error of $o(t^{1/2})$. On the other hand, it is clear that an asymptotic series of $\mathcal{G}_u$ is enough to deduce the small-time expansion of $H_\Omega(t)$.

5.1. ***The outside contribution and an asymptotic series for $\mathcal{G}_u[\phi]$.*** In this section, we deduce an asymptotic series, at order 3, of $\mathcal{G}_u[\phi](t)$ as $t \to 0$. This is done by exploiting the fact that the diffusion of heat is not confined in Ω, and as a result we can define an *outside contribution*, namely the quantity obtained from Definition 4.7, applied with base set $\Omega^c \subset M$:

$$I^c\phi(t,r) = \int_{(\Omega^c)_r} (1-u^c(t,x))\phi(x)\, d\omega(x) \quad \text{for all } t>0,\ r \geq 0, \tag{62}$$

for any $\phi \in C_c^\infty(\Omega_{-r_0}^{r_0})$, and where σ denotes the induced measure on the boundary of $(\Omega^c)_r$ and $u^c(t,x) = e^{t\Delta}\mathbb{1}_{\Omega^c}(x)$. We remark that, since Ω and its complement share the boundary, then $(\Omega^c)_{-r_0}^{r_0} = \Omega_{-r_0}^{r_0}$. It is convenient to introduce (62), because it turns out that the quantity $I\phi - I^c\phi$, where $I\phi$ is the inside contribution (54), has an explicit asymptotic series in integer powers of t.

Proposition 5.3. *Let M be a compact sub-Riemannian manifold, equipped with a smooth measure ω, and let $\Omega \subset M$ be an open subset with smooth boundary. Let $\phi \in C_c^\infty(\Omega_{-r_0}^{r_0})$. Then, for any $k \in \mathbb{N}$,*

$$I\phi(t,0) - I^c\phi(t,0) = \sum_{i=1}^k a_i(\phi)t^i + O(t^{k+1}) \quad \text{as } t \to 0, \tag{63}$$

where

$$a_i(\phi) = \int_{\partial\Omega} g(\nabla(\Delta^{i-1}\phi), \nabla\delta)\, d\sigma \quad \text{for } i \geq 1.$$

Proof. Recall that in the definition of the outside contribution (62), the integrand function involves $u^c(t,x)=e^{t\Delta}\mathbb{1}_{\Omega^c}(x)$. Since M is compact, and hence stochastically complete, we have

$$1-u^c(t,x)=e^{t\Delta}1(x)-e^{t\Delta}\mathbb{1}_{\Omega^c}(x)=u(t,x) \quad \text{for all } t>0,\ x\in M,$$

having used the pointwise equality $1-\mathbb{1}_{\Omega^c}=\mathbb{1}_{\Omega}$ in $M\setminus\partial\Omega$. Therefore, we can write the difference $I\phi(t,0)-I^c\phi(t,0)$ as follows:

$$\begin{aligned} I\phi(t,0)-I^c\phi(t,0) &= \int_{\Omega}(1-u(t,\cdot))\phi\,d\omega-\int_{\Omega^c}(1-u^c(t,\cdot))\phi\,d\omega \\ &= \int_{\Omega}(1-u(t,\cdot))\phi\,d\omega-\int_{\Omega^c}u(t,\cdot)\phi\,d\omega \\ &= \int_{\Omega}\phi(x)\,d\omega(x)-\int_{M}u(t,x)\phi(x)\,d\omega(x). \end{aligned} \tag{64}$$

Since $u(t,x)$ is the solution to (17), the function (64) is smooth as $t\in[0,\infty)$. Indeed, the smoothness in the open interval is guaranteed by hypoellipticity of the sub-Laplacian. At $t=0$, the divergence theorem, together with the fact that ϕ has compact support in M, implies that

$$\begin{aligned} \partial_t^i\left(\int_M u(t,x)\phi(x)\,d\omega(x)\right) &= \int_M \partial_t^i(u(t,x)\phi(x))\,d\omega(x)=\int_M \Delta^i u(t,x)\phi(x)\,d\omega(x) \\ &= \int_M u(t,x)\Delta^i\phi(x)\,d\omega(x) \xrightarrow{t\to 0} \int_{\Omega}\Delta^i\phi(x)\,d\omega(x). \end{aligned}$$

The previous limit shows that (64) is smooth at $t=0$, and also that its asymptotic expansion at order k, as $t\to 0$, coincides with its k-th Taylor polynomial at $t=0$. Finally, we recover (63), applying once again the divergence theorem:

$$\int_{\Omega}\Delta^i\phi\,d\omega=-\int_{\partial\Omega}g(\nabla(\Delta^{i-1}\phi),\nu)\,d\sigma=-\int_{\partial\Omega}g(\nabla(\Delta^{i-1}\phi),\nabla\delta)\,d\sigma,$$

recalling that $\nu=\nabla\delta$ is the inward-pointing normal vector to Ω at its boundary. □

Applying the (iterated) Duhamel's principle (52) to the difference $I\phi-I^c\phi$, we are able to obtain relevant information on the functional $\mathcal{G}_u$.

Theorem 5.4. *Let M be a compact sub-Riemannian manifold, equipped with a smooth measure ω, and let $\Omega\subset M$ be an open subset whose boundary is smooth and has no characteristic points. Then, for any $\phi\in C_c^\infty(\Omega_{-r_0}^{r_0})$,*

$$\mathcal{G}_u[\phi](t)=\frac{1}{2\sqrt{\pi}}\int_{\partial\Omega}\phi\,d\sigma\,t^{1/2}+\frac{1}{8}\int_{\partial\Omega}\phi\Delta\delta\,d\sigma\,t+o(t^{3/2}) \quad \text{as } t\to 0. \tag{65}$$

Proof. Let us study the difference of the inside and outside contributions $I\phi(t,0)-I^c\phi(t,0)$. On the one hand, we have an iterated Duhamel's principle, see Lemma A.7, which we report here:

$$\begin{aligned} (I\phi-I^c\phi)(t,0) &= 2\mathcal{G}_{1-2u}[\phi](t)+\frac{1}{2}\int_{\partial\Omega}N\phi\,d\sigma\,t+\frac{1}{2\pi}\int_0^t\int_0^\tau \mathcal{G}_{1-2u}[N^2\phi](\hat\tau)((\tau-\hat\tau)(t-\tau))^{-1/2}\,d\hat\tau\,d\tau \\ &\quad +\frac{1}{4\sqrt{\pi}}\int_0^t\int_{\partial\Omega}(1-2u(\tau,\cdot))(4\Delta-N^2)\phi\,d\sigma(t-\tau)^{1/2}\,d\tau+O(t^2), \end{aligned} \tag{66}$$

where we recall that N is the operator acting on smooth functions compactly supported close to $\partial\Omega$ defined by

$$N\phi = 2g(\nabla\phi, \nabla\delta) + \phi\Delta\delta \quad \text{for all } \phi \in C_c^\infty(\Omega_{-r_0}^{r_0}).$$

Using Corollary 5.2 and the linearity of $\mathcal{G}_v$ with respect to v, we know that

$$\mathcal{G}_{1-2u}[\phi](t) = o(t^{1/2}) \quad \text{as } t \to 0, \text{ for all } \phi \in C_c^\infty(\Omega_{-r_0}^{r_0}). \tag{67}$$

Therefore, applying (67) to the function $N^2\phi \in C_c^\infty(\Omega_{-r_0}^{r_0})$, we obtain

$$\frac{1}{2\pi}\int_0^t\int_0^\tau \mathcal{G}_{1-2u}[N^2\phi](\hat{\tau})((\tau-\hat{\tau})(t-\tau))^{-1/2}\,d\hat{\tau}\,d\tau = o(t^{3/2}) \quad \text{as } t \to 0. \tag{68}$$

In addition, an application of Theorem 3.1 and the dominated convergence theorem implies that

$$\int_0^t\int_{\partial\Omega}(1-2u(\tau,\cdot))(4\Delta - N^2)\phi\,d\sigma\,(t-\tau)^{1/2}\,d\tau = o(t^{3/2}) \quad \text{as } t \to 0. \tag{69}$$

Thus, using (68) and (69), we can improve (66), obtaining

$$I\phi(t,0) - I^c\phi(t,0) = 2\mathcal{G}_{1-2u}[\phi](t) + \frac{1}{2}\int_{\partial\Omega} N\phi\,d\sigma\,t + o(t^{3/2}). \tag{70}$$

On the other hand, the quantity $I\phi(t,0) - I^c\phi(t,0)$ has a complete asymptotic series by Proposition 5.3, which at order 3 becomes

$$I\phi(t,0) - I^c\phi(t,0) = \int_{\partial\Omega} g(\nabla\phi, \nabla\delta)\,d\sigma\,t + o(t^{3/2}) \quad \text{as } t \to 0. \tag{71}$$

Comparing (70) and (71), we deduce that, as $t \to 0$,

$$\begin{aligned} 2\mathcal{G}_{1-2u}[\phi](t) &= -\frac{1}{2}\int_{\partial\Omega} N\phi\,d\sigma\,t + o(t^{3/2}) + \int_{\partial\Omega} g(\nabla\phi, \nabla\delta)\,d\sigma\,t + o(t^{3/2}) \\ &= -\frac{1}{2}\int_{\partial\Omega}\phi\Delta\delta\,d\sigma\,t + o(t^{3/2}). \end{aligned}$$

Finally, using the linearity of the functional $\mathcal{G}_v[\phi]$ with respect to v, we conclude the proof. □

Remark 5.5. The asymptotics (65) for the functional $\mathcal{G}_u[\phi](t)$ is the best result that we are able to achieve. In the expression (66), the problematic term is given by (69), i.e.,

$$\int_0^t\int_{\partial\Omega}(1-2u(\tau,\cdot))(4\Delta - N^2)\phi\,d\sigma\,(t-\tau)^{1/2}\,d\tau,$$

which cannot be expressed in terms of $\mathcal{G}_u$; hence the only relevant information is given by Theorem 3.1. In conclusion, we cannot repeat the strategy of the proof of Theorem 5.4, replacing the series of $\mathcal{G}_u$ at order 3 in (66) to deduce the higher-order terms.

5.2. *Fourth-order asymptotics.* In this section we prove Theorem 1.1. We recall here the statement.

Theorem 5.6. *Let M be a compact sub-Riemannian manifold, equipped with a smooth measure ω, and let $\Omega \subset M$ be an open subset whose boundary is smooth and has no characteristic points. Then, as $t \to 0$,*

$$H_\Omega(t) = \omega(\Omega) - \frac{1}{\sqrt{\pi}}\sigma(\partial\Omega)t^{1/2} - \frac{1}{12\sqrt{\pi}}\int_{\partial\Omega}(2g(\nabla\delta, \nabla(\Delta\delta)) - (\Delta\delta)^2)\, d\sigma\, t^{3/2} + o(t^2).$$

Before giving the proof of the theorem, let us comment on its strategy. Recall that, on the one hand, for a cutoff function $\phi \in C_c^\infty(\Omega_{-r_0}^{r_0})$ which is identically 1 close to $\partial\Omega$, see (48), the localization principle (55) holds, namely

$$\omega(\Omega) - H_\Omega(t) = I\phi(t,0) + O(t^\infty) \quad \text{as } t \to 0. \tag{72}$$

Moreover, by the iterated Duhamel's principle for $I\phi(t,0)$, see Lemma A.6, we can deduce expression (61), namely

$$I\phi(t,0) = 2\mathcal{G}_{1-u}[\phi](t) + \frac{1}{\sqrt{\pi}}\int_0^t \mathcal{G}_{1-u}[N\phi](t)\, d\sigma (t-\tau)^{-1/2}\, d\tau + O(t^{3/2}). \tag{73}$$

On the other hand, we have an asymptotic series of the functional $\mathcal{G}_u$ at order 3; see Theorem 3.1. Therefore, if we naively insert this series in (73), we can obtain, at most, a third-order asymptotic expansion of the relative heat content $H_\Omega(t)$, whereas we are interested in the fourth-order expansion.

Using the outside contribution, we are able to overcome this difficulty. In particular, applying Proposition 5.3, for a function $\phi \in C_c^\infty(\Omega_{-r_0}^{r_0})$ which is identically 1 close to $\partial\Omega$, we have the following asymptotic relation:

$$I\phi(t,0) = I^c\phi(t,0) + O(t^\infty) \quad \text{as } t \to 0. \tag{74}$$

Notice that (74) is a direct consequence of Proposition 5.3 since all the coefficients of the expansion vanish. Therefore, thanks to (74), we can rephrase (72) as

$$\omega(\Omega) - H_\Omega(t) = \tfrac{1}{2}(I\phi(t,0) + I^c\phi(t,0)) + O(t^\infty) \quad \text{as } t \to 0. \tag{75}$$

The advantage of (75) is that we can now apply the iterated Dirichlet principle for the sum $I\phi + I^c\phi$; see Lemma A.8. Already at order 3, we obtain

$$(I\phi + I^c\phi)(t,0) = \frac{2}{\sqrt{\pi}}\int_{\partial\Omega}\phi\, d\sigma\, t^{1/2} + \frac{1}{\sqrt{\pi}}\int_0^t \mathcal{G}_{1-2u}[N\phi](\tau)(t-\tau)^{-1/2}\, d\tau + O(t^{3/2}), \tag{76}$$

where N is the operator defined in (59). As we can see, in (76), the functional $\mathcal{G}_u$ occurs for the first time in the second iteration of the Duhamel's principle, as opposed to the expansion for $I\phi$, where it appeared already in the first application; see (73). Hence we gain an order with respect to the asymptotic series of $\mathcal{G}_u$. More generally, if we were able to develop the k-th order asymptotics for $\mathcal{G}_u$, this would imply the $(k+1)$-th order expansion for $H_\Omega(t)$.

Proof of Theorem 5.6. Following the discussion above, it is enough to expand the sum $I\phi + I^c\phi$, with $\phi \in C_c^\infty(\Omega_{-r_0}^{r_0})$. For this quantity, Lemma A.8 holds, namely we have the following iterated version of

Duhamel's principle:

$$\begin{aligned}(I\phi+I^c\phi)(t,0) &= \frac{2}{\sqrt{\pi}}\int_{\partial\Omega}\phi\,d\sigma\,t^{1/2}+\frac{1}{\sqrt{\pi}}\int_0^t \mathcal{G}_{1-2u}[N\phi](\tau)(t-\tau)^{-1/2}\,d\tau\\ &\quad+\frac{1}{6\sqrt{\pi}}\int_{\partial\Omega}(4\Delta+N^2)\phi\,d\sigma\,t^{3/2}\\ &\quad+\frac{1}{4\pi^{3/2}}\int_0^t\int_0^\tau\int_0^{\hat\tau}\mathcal{G}_{1-2u}[N^3\phi](s)((\hat\tau-s)(\tau-\hat\tau)(t-\tau))^{-1/2}\,ds\,d\hat\tau\,d\tau\\ &\quad+\frac{1}{4\sqrt{\pi}}\int_0^t\mathcal{G}_{1-2u}[(6N\Delta-N^3-2\Delta N)\phi](\tau)(t-\tau)^{1/2}\,d\tau+O(t^{5/2}),\end{aligned}\tag{77}$$

where N is defined in (59). Moreover, recall that by Theorem 5.4, the functional $\mathcal{G}_{1-2u}[\phi]$ has the following expansion for any $\phi\in C_c^\infty(\Omega_{-r_0}^{r_0})$:

$$\mathcal{G}_{1-2u}[\phi](t)=-\frac{1}{4}\int_{\partial\Omega}\phi\Delta\delta\,d\sigma\,t+o(t^{3/2})\quad\text{as } t\to 0.\tag{78}$$

Thus, replacing the term $\mathcal{G}_{1-2u}[N\phi]$ in (77) with the expansion (78) for $N\phi\in C_c^\infty(\Omega_{-r_0}^{r_0})$, we obtain the following asymptotic as $t\to 0$:

$$\begin{aligned}I\phi(t,0)+I^c\phi(t,0) &= \frac{2}{\sqrt{\pi}}\int_{\partial\Omega}\phi\,d\sigma\,t^{1/2}-\frac{1}{3\sqrt{\pi}}\left(\int_{\partial\Omega}N\phi\Delta\delta\,d\sigma\right)t^{3/2}\\ &\quad+\frac{1}{6\sqrt{\pi}}\int_{\partial\Omega}(4\Delta+N^2)\phi\,d\sigma\,t^{3/2}+o(t^2)\end{aligned}\tag{79}$$

for any $\phi\in C_c^\infty(\Omega_{-r_0}^{r_0})$. In particular, if we choose $\phi\in C_c^\infty(\Omega_{-r_0}^{r_0})$ such that $\phi\equiv 1$ close to $\partial\Omega$, then on the one hand, from (79), we obtain, as $t\to 0$,

$$I\phi(t,0)+I^c\phi(t,0)=\frac{2}{\sqrt{\pi}}\sigma(\partial\Omega)t^{1/2}+\frac{1}{6\sqrt{\pi}}\int_{\partial\Omega}(2g(\nabla\delta,\nabla(\Delta\delta))-(\Delta\delta)^2)\,d\sigma\,t^{3/2}+o(t^2).$$

On the other hand, the asymptotic relation (75) holds. □

Third-order vs. fourth-order asymptotics. We stress that we could have obtained the third-order asymptotic expansion of $H_\Omega(t)$ *without* introducing the sum of the inside and outside contributions $I\phi+I^c\phi$, and only using the Duhamel's principle for $I\phi$, see Lemma A.6, and the asymptotic series for $\mathcal{G}_u$, see Theorem 5.4. However, for the improvement to the fourth-order asymptotics, the argument of the sum of contributions seems necessary.

5.3. *The weighted relative heat content.* Adapting the proof of Theorem 5.6, one can prove a slightly more general result which we state here for completeness.

Proposition 5.7. *Let M be a compact sub-Riemannian manifold, equipped with a smooth measure ω, and let $\Omega\subset M$ be an open subset whose boundary is smooth and has no characteristic points. Let $\chi\in C_c^\infty(M)$ and define the* ***weighted relative heat content***

$$H_\Omega^\chi(t)=\int_\Omega u(t,x)\chi(x)\,d\omega(x)\quad\textit{for all } t>0.$$

Then, as $t \to 0$,

$$
\begin{aligned}
H_\Omega^\chi(t) = \int_\Omega \chi \, d\omega - \frac{1}{\sqrt{\pi}} \int_{\partial\Omega} \chi \, d\sigma \, t^{1/2} - \frac{1}{2} \int_{\partial\Omega} g(\nabla\chi, \nabla\delta) \, d\sigma \, t \\
- \left(\frac{1}{12\sqrt{\pi}} \int_{\partial\Omega} (4\Delta + N^2)\chi \, d\sigma - \frac{1}{6\sqrt{\pi}} \int_{\partial\Omega} (N\chi)\Delta\delta \, d\sigma \right) t^{3/2} \\
- \frac{1}{2} \int_{\partial\Omega} g(\nabla(\Delta\chi), \nabla\delta) \, d\sigma \, t^2 + o(t^2).
\end{aligned}
$$

Proof. Let us consider a cutoff function ϕ as in (48). Then, applying the usual localization argument, see (49), we have

$$
\int_\Omega \chi(x) \, d\omega(x) - H_\Omega^\chi(t) = I[\phi\chi](t, 0) + O(t^\infty) \quad \text{as } t \to 0,
$$

where now the function $\phi\chi \in C_c^\infty(\Omega_{-r_0}^{r_0})$ and $\phi\chi = \chi$ close to $\partial\Omega$.

As we did in the proof of Theorem 5.6, we relate $H_\Omega^\chi(t)$ with the sum of contributions. Applying Proposition 5.3, we have the following asymptotic relation at order 4:

$$
I[\phi\chi](t, 0) - I^c[\phi\chi](t, 0) = \int_{\partial\Omega} g(\nabla\chi, \nabla\delta) \, d\sigma \, t + \int_{\partial\Omega} g(\nabla(\Delta\chi), \nabla\delta) \, d\sigma \, t^2 + o(t^2),
$$

as $t \to 0$, having used the fact that $\phi\chi \equiv \chi$ close to $\partial\Omega$. Notice that this relation coincides with (74) when $\chi \equiv 1$ close to $\partial\Omega$. Thus, we obtain

$$
\begin{aligned}
\int_\Omega \chi(x) \, d\omega(x) - H_\Omega^\chi(t) = \frac{1}{2}(I[\phi\chi](t, 0) + I^c[\phi\chi](t, 0)) \\
+ \int_{\partial\Omega} g(\nabla\chi, \nabla\delta) \, d\sigma \, t + \int_{\partial\Omega} g(\nabla(\Delta\chi), \nabla\delta) \, d\sigma \, t^2 + o(t^2) \quad \text{as } t \to 0.
\end{aligned}
$$

Finally, applying (79) for $I[\phi\chi](t, 0) + I^c[\phi\chi](t, 0)$, we conclude. □

Remark 5.8. We compare the coefficients of the expansions of $H_\Omega(t)$ and $Q_\Omega(t)$, defined in (6), respectively. On the one hand, by [Rizzi and Rossi 2021, Theorem 5.8], the k-th coefficient of the expansion of $Q_\Omega(t)$ is of the form

$$
-\int_{\partial\Omega} D_k(\chi) \, d\sigma \quad \text{for all } \chi \in C_c^\infty(M),
$$

where D_k is a differential operator acting on $C_c^\infty(M)$ and belonging to $\operatorname{span}_{\mathbb{R}}\{\Delta, N\}$ as algebra of operators. On the other hand, Proposition 5.7 shows that this is no longer true for the third coefficient of the expansion of $H_\Omega(t)$, as we need to add the operator multiplication by $\Delta\delta$.

6. An alternative approach using the heat kernel asymptotics

As we can see by a first application of Duhamel's principle, see (10), and its iterations, the small-time asymptotics of $u(t, \cdot)|_{\partial\Omega}$, together with uniform estimates on the remainder with respect to $x \in \partial\Omega$, would be enough to determine the asymptotic expansion of the relative heat content, at any order.

In Theorem 3.1, we studied the zero-order asymptotics of $u(t, \cdot)|_{\partial\Omega}$. The technique used for its proof does not work at higher-order, since the exponential remainder term in (40) would be unbounded as $t \to 0$.

In this section, we comment how such a higher-order asymptotics of $u(t,\cdot)|_{\partial\Omega}$ can be obtained exploiting the asymptotic formula for the heat kernel proved in [Colin de Verdière et al. 2021, Theorem A].

Let M be a compact sub-Riemannian manifold and $\Omega \subset M$ an open subset with smooth boundary. For $x \in \partial\Omega$, let us consider $\psi = (z_1, \dots, z_n) : U \to V$ a chart of privileged coordinates centered at x, with U a relatively compact set. Since the heat kernel is exponentially decaying outside the diagonal, see (31),

$$\begin{aligned} u(t,x) = \int_\Omega p_t(x,y)\,d\omega(y) &= \int_{\Omega\cap U} p_t(x,y)\,d\omega(y) + O(t^\infty) \\ &= \int_{V_1} p_t(0,z)\,d\omega(z) + O(t^\infty), \end{aligned} \tag{80}$$

where $V_1 = \psi(U \cap \Omega)$, and we denote with the same symbols ω and $p_t(0,z)$ the coordinate expression of the measure and heat kernel, respectively. For example, if $x \in \partial\Omega$ is noncharacteristic, we may choose ψ as in (30), and then $V_1 = V \cap \{z_1 > 0\}$. Recall the asymptotic expansion of the heat kernel of Theorem 2.9, evaluated in $(0,z)$: for any $m \in \mathbb{N}$ and compact set $K \subset (0,\infty) \times V$,

$$|\varepsilon|^{\mathcal{Q}} p_{\varepsilon^2\tau}(0, \delta_\varepsilon(z)) = \hat{p}_\tau(0,z) + \sum_{i=0}^{m} \varepsilon^i f_i(\tau, 0, z) + o(|\varepsilon|^m) \quad \text{as } \varepsilon \to 0, \tag{81}$$

uniformly as $(\tau, z) \in K$, where $\mathcal{Q}$, $\hat{p}$ and the f_i are defined in Section 2. We will omit the dependence on the center of the privileged coordinates, x, it being fixed for the moment. At this point, we would like to integrate (81) to get information of $u(t,x)$ as $t \to 0$. Proceeding formally, let us choose the parameters ε, τ in (81) such that

$$\varepsilon^2\tau = t, \qquad \varepsilon = t^{\alpha/(2\alpha+1)}, \qquad \tau = t^{1/(2\alpha+1)}, \tag{82}$$

for some $\alpha > 0$ to be fixed. For convenience of notation, set

$$V_s = \delta_s(V_1) \quad \text{for all } s \in [-1, 1].$$

Then, split the integral over V_1 in (80) in two, so that the first one is computed on V_ε and the second one is computed on its complement in V_1, i.e., $V_1 \setminus V_\varepsilon$. Notice that, by usual off-diagonal estimates, see [Jerison and Sánchez-Calle 1986, Proposition 3] and our choice of the parameter ε as in (82), the following is a remainder term, independent of the value of α:

$$\int_{V_1\setminus V_\varepsilon} p_t(0,z)\,d\omega(z) = O(e^{-\beta\varepsilon^2/t}) = O(t^\infty) \quad \text{as } t \to 0.$$

Thus, writing the measure in coordinates $d\omega(z) = \omega(z)\,dz$ with $\omega(\cdot) \in C^\infty(V_1)$, we have, as $t \to 0$,

$$\begin{aligned} u(t,x) = \int_{V_\varepsilon} p_t(0,z)\omega(z)\,dz + O(t^\infty) &= \int_{V_1} \varepsilon^{\mathcal{Q}} p_{\varepsilon^2\tau}(0,\delta_\varepsilon(z))\omega(\delta_\varepsilon(z))\,dz + O(t^\infty) \\ &= \int_{V_1} \Big(\hat{p}_\tau(0,z) + \sum_{i=0}^{m-1} \varepsilon^i f_i(\tau,0,z) + \varepsilon^m R_m(\varepsilon,\tau,z) \Big) \omega(\delta_\varepsilon(z))\,dz + O(t^\infty), \end{aligned} \tag{83}$$

where R_m is a smooth function on $[-1,1]\times(0,\infty)\times\mathbb{R}^n$ such that

$$\sup_{\varepsilon\in[-1,1],\ z\in K}|R_m(\varepsilon,\tau,z)|\leq C_m(\tau,K) \tag{84}$$

for any compact set $K\subset\mathbb{R}^n$, according to (81). Up to restricting the domain of privileged coordinates U, we can assume that (84) holds on $\overline{V}$. By our choices (82), we would like the term

$$t^{m\alpha/(2\alpha+1)}\int_{V_1}|R_m(t^{\alpha/(2\alpha+1)},t^{1/(2\alpha+1)},z)|\omega(\delta_{t^{\alpha/(2\alpha+1)}}(z))\,dz \tag{85}$$

to be an error term of order greater than $(m-1)/2$ as $t\to 0$. Thus, assume for the moment that for all $K\subset V$ compact and, for all $m\in\mathbb{N}$, there exist $\ell=\ell(m,K)\in\mathbb{N}$ and $C_m(K)>0$ such that

$$\sup_{\substack{\varepsilon\in[-1,1]\\ z\in K}}|R_m(\varepsilon,\tau,z)|\leq\frac{C_m(K)}{\tau^\ell}\quad\text{for all }\tau\in(0,1). \tag{$\boldsymbol{H}$}$$

Thanks to assumption ($\boldsymbol{H}$), choosing α large enough, we see that (85) is a $o(t^{(m-1)/2})$. Performing the change of variable $z\mapsto\delta_{1/\sqrt{\tau}}(z)$ in (83), and exploiting the homogeneity properties of $\hat{p}$ and f_i, namely (29), we finally obtain the following expression for u as $t\to 0$:

$$u(t,x)=\int_{V_{t^{-1/(2(2\alpha+1))}}}\left(\hat{p}_1(0,z)+\sum_{i=0}^{m-1}t^{i/2}a_i(z)\right)\omega(\delta_{\sqrt{t}}(z))\,dz+o(t^{(m-1)/2}), \tag{86}$$

having set $a_i(z)=f_i(1,0,z)$, for all $i\in\mathbb{N}$. Therefore, we find an asymptotic expansion of $u(t,x)$ under assumption ($\boldsymbol{H}$), which is crucial to overcome the fact that (81) is formulated on an asymptotic neighborhood of the diagonal, and not uniformly as $\tau\to 0$. It is likely[6] that ($\boldsymbol{H}$) can be proven in the nilpotent case, and more generally when the ambient manifold is $M=\mathbb{R}^n$ and the generating family of the sub-Riemannian structure, $\{X_1,\dots,X_N\}$ satisfies the uniform Hörmander polynomial condition; see [Colin de Verdière et al. 2021, Appendix B] for details. Although this strategy could be used to prove the existence of an asymptotic expansion of $H_\Omega(t)$, we refrain to go in this direction since two technical difficulties would arise nonetheless:

- *Uniformity of the expansion of $u(t,x)$ with respect to $x\in\partial\Omega$.* In the nonequiregular case, see Section 2.3 for details, the expansion (81) is not uniform as x varies in compact subsets of M; hence the same would be true for the expansion (86).
- *Computations of the coefficients.* The coefficients appearing in (86) depend on the nilpotent approximation at $x\in\partial\Omega$ and are not clearly related to the invariants of $\partial\Omega$.

Our strategy avoids almost completely the knowledge of the small-time asymptotics of $u(t,\cdot)_{\partial\Omega}$, it being based on an asymptotic series of the auxiliary functional $\mathcal{G}_u$. Moreover, we stress that our method to prove the asymptotics of $H_\Omega(t)$ up to order 4, see Theorem 1.1, holds for any sub-Riemannian manifold, including also the nonequiregular ones.

[6] Personal communication by Yves Colin de Verdière, Luc Hillairet and Emmanuel Trélat.

Remark 6.1. In order to pass from (86) to the asymptotic expansion of $H_\Omega(t)$, we would use Duhamel's formula, which holds under the noncharacteristic assumption. This means that, even though (81) of course is true even in presence of characteristic points, we can't say much about the asymptotics of $H_\Omega(t)$ in the general case.

7. The noncompact case

In the noncompact case, we have the following difficulties:

- The localization principle, see Proposition 4.5, may fail.
- Set $u(t,x) = e^{t\Delta}\mathbb{1}_\Omega(x)$ and $u^c(t,x) = e^{t\Delta}\mathbb{1}_{\Omega^c}(x)$. If the manifold is not stochastically complete, the relation $u(t,x) + u^c(t,x) = 1$ does not hold.
- The Gaussian bounds for the heat kernel and its time-derivatives, à la [Jerison and Sánchez-Calle 1986, Theorem 3], may not hold; thus Lemma A.3 may not be true.

Definition 7.1. Let M be a sub-Riemannian manifold, equipped with a smooth measure ω. We say that (M,ω) is (*globally*) *doubling* if there exist constants $C_D > 0$ such that

$$V(x, 2\rho) \le C_D V(x,\rho) \quad \text{for all } \rho > 0,\ x \in M,$$

where $V(x,\rho) = \omega(B_\rho(x))$. We say that (M,ω) satisfies a (*global*) *weak Poincaré inequality*, if there exist constants $C_P > 0$ such that

$$\int_{B_\rho(x)} |f - f_{x,\rho}|^2\, d\omega \le C_P \rho^2 \int_{B_{2\rho}(x)} \|\nabla f\|^2\, d\omega, \qquad \rho > 0,\ x \in M,$$

for any smooth function $f \in C^\infty(M)$. Here

$$f_{x,\rho} = \frac{1}{V(x,\rho)} \int_{B_\rho(x)} f\, d\omega.$$

We refer to these properties as local whenever they hold for any $\rho < \rho_0$.

Remark 7.2. If M is a sub-Riemannian manifold, equipped with a smooth globally doubling measure ω, then it is stochastically complete, namely

$$\int_M p_t(x,y)\, d\omega(y) = 1 \quad \text{for all } t > 0,\ x \in M.$$

This is a straightforward consequence of the characterization given by [Sturm 1994, Theorem 4] on the volume growth of balls.

Theorem 7.3. *Let M be a complete sub-Riemannian manifold, equipped with a smooth measure ω. Assume that (M,ω) is globally doubling and satisfies a global weak Poincaré inequality. Then, there exist constants $C_k, c_k > 0$ for any integer $k \ge 0$, depending only on C_D, C_P, such that, for any $x, y \in M$ and $t > 0$,*

$$|\partial_t^k p_t(x,y)| \le \frac{C_k t^{-k}}{V(x,\sqrt{t})} \exp\left(-\frac{d_{\mathrm{SR}}^2(x,y)}{c_k t}\right), \tag{87}$$

where we recall $V(x,\sqrt{t}) = \omega(B_{\sqrt{t}}(x))$.

In addition, there exist constants $C_\ell, c_\ell > 0$, depending only on C_D, C_P, such that, for any $x, y \in M$ and $t > 0$,

$$p_t(x,y) \geq \frac{C_\ell}{V(x,\sqrt{t})} \exp\left(-\frac{d^2_{\mathrm{SR}}(x,y)}{c_\ell t}\right). \tag{88}$$

Proof. Define the sub-Riemannian Hamiltonian as the smooth function $H : T^*M \to \mathbb{R}$,

$$H(\lambda) = \frac{1}{2}\sum_{i=1}^{N} \langle \lambda, X_i\rangle^2, \quad \lambda \in T^*M,$$

where $\{X_1, \dots, X_N\}$ is a generating family for the sub-Riemannian structure. Then, following the notation of [Sturm 1996], one can easily verify that

$$\mathcal{E}(u,v) = \int_M 2H(du,dv)\,d\omega \quad \text{for all } u, v \in C_c^\infty(M),$$

where H is the sub-Riemannian Hamiltonian viewed as a bilinear form on fibers, defines a strongly local Dirichlet form with domain $\mathrm{dom}(\mathcal{E}) = C_c^\infty(M)$. Notice that the Friedrichs extension of $\mathcal{E}$ is exactly the sub-Laplacian. Moreover, the intrinsic metric

$$d_I(x,y) = \sup\{|u(x)-u(y)| : u \in C_c^\infty(M),\ |2H(du,du)| \leq 1\} \quad \text{for all } x, y \in M$$

coincides with the usual sub-Riemannian distance, as $|2H(du,du)| = \|\nabla u\|^2$; see [Barilari et al. 2016, Chapter 2, Proposition 12.4]. Thus, $\mathcal{E}$ is also strongly regular and, by our assumptions on (M,ω), [Saloff-Coste 1992, Theorem 4.3] holds true, proving (87). For the Gaussian lower bound (88), it is enough to apply [Sturm 1996, Corollary 4.10]; see also [Saloff-Coste 1992, Theorem 4.2]. □

Remark 7.4. Theorem 7.3 ensures that the time-derivatives of the heat kernel satisfy Gaussian bounds, which are sufficient to prove Lemma A.3 in the noncompact case. This lemma is crucial to obtain the asymptotics expansion of $H_\Omega(t)$ at order *strictly greater* than 1.

We prove now the noncompact analogue of Proposition 4.5.

Corollary 7.5. *Under the assumptions of Theorem 7.3, let $\Omega \subset M$ be an open subset with smooth boundary. Then, for any $K \subset M$ closed subset of M such that $K \cap \partial\Omega = \varnothing$, we have*

$$\mathbb{1}_\Omega(x) - u(t,x) = O(t^\infty) \quad \text{as } t \to 0, \text{ uniformly for } x \in K,$$

where $u(t,x) = e^{t\Delta}\mathbb{1}_\Omega(x)$ is the solution to (17).

Proof. Let us assume that $K \subset \Omega$ such that $K \cap \partial\Omega = \varnothing$. The other part of the statement can be done similarly.

Since M is stochastically complete, see Remark 7.2, for any $x \in K$, we can write

$$\mathbb{1}_\Omega(x) - u(t,x) = 1 - e^{t\Delta}\mathbb{1}_\Omega(x) = e^{t\Delta}1(x) - e^{t\Delta}\mathbb{1}_\Omega(x) = \int_{M\setminus\Omega} p_t(x,y)\,d\omega(y).$$

Thanks to Theorem 7.3, we can apply (87) for $k=0$, obtaining

$$\int_{M\setminus\Omega} p_t(x,y)\,d\omega(y) \le \int_{M\setminus\Omega} \frac{C_0}{V(x,\sqrt{t})}\exp\left(-\frac{d^2_{\mathrm{SR}}(x,y)}{c_0 t}\right)d\omega(y)$$

for suitable constants $C_0, c_0>0$ not depending on $x, y \in M,\ t>0$. Now, fix $L>1$; since $K\subset\Omega$ is closed with empty intersection with $\partial\Omega$, and thus well-separated from $\partial\Omega$, we deduce there exists $a=a(K)>0$ such that $d_{\mathrm{SR}}(x,y)>a$ for any $x\in K,\ y\in M\setminus\Omega$, and so

$$\begin{aligned}\int_{M\setminus\Omega} p_t(x,y)\,d\omega(y) &\le \int_{M\setminus\Omega} \frac{C_0}{V(x,\sqrt{t})}\exp\left(-\frac{d^2_{\mathrm{SR}}(x,y)}{c_0 t}\right)d\omega(y)\\ &\le \exp\left(-\frac{C(a,L)}{c_0 t}\right)\int_{M\setminus\Omega} \frac{C_0}{V(x,\sqrt{t})}\exp\left(-\frac{d^2_{\mathrm{SR}}(x,y)}{2^L c_0 t}\right)d\omega(y),\end{aligned} \tag{89}$$

where $C(a,L)=a^2(2^L-1)/2^L>0$. Thus, if we prove that the integral in (89) is finite, we conclude. Firstly, recall the Gaussian lower bound (88), which holds thanks to Theorem 7.3:

$$p_t(x,y) \ge \frac{C_\ell}{V(x,\sqrt{t})}\exp\left(-\frac{d^2_{\mathrm{SR}}(x,y)}{c_\ell t}\right) \tag{90}$$

for suitable constants $C_\ell, c_\ell>0$, not depending on $x,y\in M,\ t>0$. Secondly, by the doubling property of ω, it is well known that there exists $C'_D,\ s>0$ depending only on C_D such that

$$V(x,R)\le C'_D\left(\frac{R}{\rho}\right)^s V(x,\rho)\quad \text{for all } \rho\le R. \tag{91}$$

Therefore, choosing $L>1$ so big that $\tilde c^2=(2^L c_0)/c_\ell>1$ and applying (91) for $\rho=\sqrt{t}$ and $R=\tilde c\sqrt{t}$, we have $R>\rho$ and

$$V(x,\tilde c\sqrt{t})\le \widetilde{C}\,V(x,\sqrt{t})\quad \text{for all } t>0, \tag{92}$$

having let $\widetilde{C}=C'_D\tilde c^s>0$. Finally, using (92) and the Gaussian lower bound (90), we can estimate the integral in (89) as follows:

$$\begin{aligned}\int_{M\setminus\Omega}\frac{1}{V(x,\sqrt{t})}\exp\left(-\frac{d^2_{\mathrm{SR}}(x,y)}{2^L c_0 t}\right)d\omega(y) &\le \int_M \frac{\widetilde{C}}{V(x,\tilde c\sqrt{t})}\exp\left(-\frac{d^2_{\mathrm{SR}}(x,y)}{c_\ell \tilde c t}\right)d\omega(y)\\ &\le \frac{\widetilde{C}}{C_\ell}\int_M p_{\tilde t}(x,y)\,d\omega(y)\le\frac{\widetilde{C}}{C_\ell},\end{aligned}$$

where $\tilde t=\tilde c t$. Since the resulting constant does not depend on $x\in K$, we conclude the proof. □

Using Corollary 7.5 and adopting the same strategy of the compact case, one can finally prove the following result.

Theorem 7.6. *Let M be a complete sub-Riemannian manifold, equipped with a smooth measure ω. Assume that (M,ω) is globally doubling and satisfies a global weak Poincaré inequality. Let $\Omega\subset M$ be* *an open and bounded subset whose boundary is smooth and has no characteristic points. Then, as* $t \to 0$,

$$H_\Omega(t) = \omega(\Omega) - \frac{1}{\sqrt{\pi}}\sigma(\partial\Omega)t^{1/2} - \frac{1}{12\sqrt{\pi}}\int_{\partial\Omega}\left(2g(\nabla\delta, \nabla(\Delta\delta)) - (\Delta\delta)^2\right) d\sigma\, t^{3/2} + o(t^2).$$

Remark 7.7. Theorem 7.6 holds true also for the weighted relative heat content; see Section 5.3. In both cases, we do not know whether its assumptions are sharp in the noncompact case.

7.1. *Notable examples.* We list here some notable examples of sub-Riemannian manifolds satisfying the assumptions of Theorem 7.3. For these examples Theorem 7.6 is valid.

- M is a Lie group with polynomial volume growth, the distribution is generated by a family of left-invariant vector fields satisfying the Hörmander condition and ω is the Haar measure. This family includes also Carnot groups. See for example [Varopoulos 1996; Saloff-Coste 1992; Gallagher and Sire 2012].
- $M = \mathbb{R}^n$, equipped with a sub-Riemannian structure induced by a family of vector fields $\{Y_1, \dots, Y_N\}$ with bounded coefficients together with their derivatives, and satisfying the Hörmander condition. Under these assumptions, the Lebesgue measure is doubling, see [Nagel et al. 1985, Theorem 1], and the Poincaré inequality is verified in [Jerison 1986]. We remark that these works provide the local properties of Definition 7.1, with constants depending only on the C^k-norms of the vector fields Y_i for $i = 1, \dots, N$. Thus, if the C_k-norms are globally bounded, we obtain the corresponding global properties.
- M is a complete Riemannian manifold with metric g, equipped with the Riemannian measure, and with nonnegative Ricci curvature.

We mention that a Riemannian manifold M with Ricci curvature bounded below by a negative constant satisfies only locally Definition 7.1, i.e., for some $\rho_0 < \infty$, depending on the Ricci bound. Nevertheless, we can prove Corollary 7.5 in this case, as [Li and Yau 1986, Corollary 3.1] provides an upper Gaussian bound, and a lower bound as (88) holds; see [Bakry and Qian 1999, Corollary 2]. Thus, the first-order asymptotic expansion of $H_\Omega(t)$, see Theorem 4.13, is valid in this setting.

Appendix: Iterated Duhamel's principle for $I_\Omega\phi(t, 0)$

In this section, we study the iterated Duhamel's principle for the $I_\Omega\phi$; see Definition 4.7. The main result is Lemma A.6, which will imply formulas (58), (66) and (77).

The next proposition is a version of the iterated Duhamel's principle taken from [Rizzi and Rossi 2021, Proposition A.1], which we recall here.

Proposition A.1. *Let* $F \in C^\infty((0, \infty) \times [0, +\infty))$ *be a smooth function compactly supported in the second variable and let* $L = \partial_t - \partial_r^2$. *Assume that the following conditions hold*:

(i) $L^k F(0, r) = \lim_{t\to 0} L^k F(t, r)$ *exists in the sense of distributions*[7] *for any* $k \geq 0$.

(ii) $L^k F(t, 0)$ *and* $\partial_r L^k F(t, 0)$ *converge to a finite limit as* $t \to 0$ *for any* $k \geq 0$.

[7] Namely, for any $\psi \in C^\infty([0, \infty))$, there exists finite $\lim_{t\to 0}\int_0^\infty f(t, r)\psi(r)\, dr$. With a slight abuse of notation, we define $\int_0^\infty f(0, r)\psi(r)\, dr = \lim_{t\to 0}\int_0^\infty f(t, r)\psi(r)\, dr$.

Then, for all $m \in \mathbb{N}$ and $t > 0$, we have

$$F(t,0) = \sum_{k=0}^{m}\Big(\frac{t^k}{k!}\int_0^\infty e(t,r,0)L^kF(0,r)\,dr - \frac{1}{\sqrt{\pi}k!}\int_0^t \partial_r L^k F(\tau,0)(t-\tau)^{k-1/2}\,d\tau\Big) + \frac{1}{m!}\int_0^t\int_0^\infty e(t-\tau,r,0)L^{m+1}F(\tau,r)(t-\tau)^m\,dr\,d\tau, \tag{93}$$

where $e(t,r,s)$ is the Neumann heat kernel on the half-line; see (53).

We want to apply Proposition A.1 to the function $I_\Omega\phi(t,0)$; thus, we study in detail the operators L^kI_Ω for any $k \ge 1$. Define iteratively the family of matrices of operators, acting on smooth functions,

$$M_{kj} = \begin{pmatrix} Q_{kj} & S_{kj} \\ P_{kj} & R_{kj}\end{pmatrix},$$

as follows. Set

$$M_{10} = \begin{pmatrix} \Delta & -\Delta N_\Omega \\ N_\Omega & -N_\Omega^2+\Delta\end{pmatrix} \quad\text{and}\quad M_{11} = \begin{pmatrix} 0 & N_\Omega \\ 0 & 0\end{pmatrix},$$

and, for all $k \ge 1$ and $0 \le j \le k$, set

$$M_{kj} = M_{10}M_{k-1,j} + M_{11}M_{k-1,j-1}, \tag{94}$$

while $M_{kj} = 0$ for all other values of the indices, i.e., $k < 0$, $j < 0$ or $k < j$. Here N_Ω is the operator defined in (50), namely

$$N_\Omega\phi = 2g(\nabla\phi,\nu) + \phi\ \mathrm{div}_\omega(\nu) \quad \text{for all } \phi \in C_c^\infty(\Omega_{-r_0}^{r_0}), \tag{95}$$

where ν is the inward-pointing normal from Ω.

Recall the definition of I_Ω and Λ_Ω: for any $\phi \in C_c^\infty(\Omega_{-r_0}^{r_0})$ and for all $t > 0$, $r \ge 0$,

$$I_\Omega\phi(t,r) = \int_{\Omega_r}(1-u(t,x))\phi(x)\,d\omega(x),$$
$$\Lambda_\Omega\phi(t,r) = -\partial_r I_\Omega\phi(t,r) = -\int_{\partial\Omega_r}(1-u(t,y))\phi(y)\,d\sigma(y),$$

where $u(t,\cdot) = e^{t\Delta}\mathbb{1}_\Omega(\cdot)$. Iterations of $L^kI_\Omega\phi$ satisfy the following lemma.

Lemma A.2. *Let M be a sub-Riemannian manifold, equipped with a smooth measure ω, and let $\Omega \subset M$ be an open relatively compact subset whose boundary is smooth and has no characteristic points. Then, as operators on $C_c^\infty(\Omega_{-r_0}^{r_0})$, we have:*

(i) $LI_\Omega = I_\Omega\Delta + \Lambda_\Omega N_\Omega$.

(ii) $L\Lambda_\Omega = \Lambda_\Omega(-N_\Omega^2 + \Delta) + \partial_t I_\Omega N_\Omega - I_\Omega\Delta N_\Omega$.

(iii) *For any $k \in \mathbb{N}$,*

$$L^kI_\Omega = \sum_{j=0}^{k}\frac{\partial^j}{\partial t^j}(\Lambda_\Omega P_{kj} + I_\Omega Q_{kj}) \quad \textit{and} \quad L^k\Lambda_\Omega = \sum_{j=0}^{k}\frac{\partial^j}{\partial t^j}(\Lambda_\Omega R_{kj} + I_\Omega S_{kj}).$$

Here we mean that, for any $\phi \in C_c^\infty(\Omega_{-r_0}^{r_0})$, the operator L^k acts on the functions $I_\Omega\phi(t,r)$, $\Lambda_\Omega\phi(t,r)$. Analogously the right-hand side when evaluated in ϕ is a function of (t,r).

Proof. The proof of items (i) and (ii) follows from Proposition 4.2 and the divergence theorem; see [Rizzi and Rossi 2021, Lemma A.2]. We show how to recover the iterative law (94).

Consider the vector $V = (I_\Omega, \Lambda_\Omega)$. Then by items (i) and (ii), we have

$$LV = (LI_\Omega, L\Lambda_\Omega) = VM_{10} + \partial_t VM_{11}. \tag{96}$$

Notice that the operator L^k contains at most k derivatives with respect to t. Therefore we have

$$L^k V = \sum_{j=0}^{k} \partial_t^j (VM_{kj}) \quad \text{for all } k \geq 0.$$

On the other hand, we can evaluate $L^k V$, using (96),

$$\begin{aligned} L^k V = L(L^{k-1}V) &= \sum_{j=0}^{k-1} L\partial_t^j (VM_{k-1,j}) = \sum_{j=0}^{k-1} \partial_t^j (LVM_{k-1,j}) \\ &= \sum_{j=0}^{k-1} \partial_t^j VM_{10}M_{k-1,j} + \sum_{j=0}^{k-1} \partial_t^{j+1} VM_{11}M_{k-1,j}. \end{aligned}$$

Reorganizing the sum, we find (94), concluding the proof. □

We want to apply Proposition A.1 to $I_\Omega\phi(t,r)$ for $k \geq 2$, in order to obtain higher-order asymptotics. However, Lemma A.2 shows that $L^k I_\Omega$ for $k \geq 2$, involves time derivatives of $u(t,x)$ which are not well-defined at $\partial\Omega$ as $t \to 0$. Therefore, we consider the following approximation of $I_\Omega\phi$ and $\Lambda_\Omega\phi$, respectively: fix $\varepsilon > 0$ and define, for any $t > 0$, $r \geq 0$,

$$I_\varepsilon\phi(t,r) = = \int_{\Omega_r} (1 - u_\varepsilon(t,x))\phi(x)\, d\omega(x),$$

$$\Lambda_\varepsilon\phi(t,r) = -\partial_r I_\varepsilon\phi(t,r) = \int_{\partial\Omega_r} (1 - u_\varepsilon(t,x))\phi(y)\, d\sigma(y),$$

where $u_\varepsilon(t,x) = e^{t\Delta}\mathbb{1}_{\Omega_\varepsilon}(x)$. We recall that, for any $a \in \mathbb{R}$, $\Omega_a = \{x \in M : \delta(x) > a\}$. Notice that, by the dominated convergence theorem, we have

$$I_\varepsilon\phi(t,0) \xrightarrow{\varepsilon\to 0} I_\Omega\phi(t,0) \quad \text{uniformly on } [0,T],$$

and, in addition, Lemma A.2 holds unchanged also for I_ε and Λ_ε.

Lemma A.3. *Let M be a compact sub-Riemannian manifold, equipped with a smooth measure ω, and let $\Omega \subset M$ be an open subset whose boundary is smooth and has no characteristic points. Let $\psi \in C^\infty([0,\infty))$, $\varepsilon \in (0, r_0)$ and define*

$$\psi^{(-1)}(r) = \int_0^r \psi(s)\, ds \quad \textit{for all } r \geq 0.$$

Then, for any $\phi \in C_c^\infty(\Omega_{-r_0}^{r_0})$, the following identities hold:

(i) $\displaystyle\lim_{t\to 0}\int_0^\infty \frac{\partial^j}{\partial t^j}\Lambda_\varepsilon\phi(t,r)\psi(r)\,dr = \begin{cases} \int_{\Omega_0^\varepsilon}\phi(\psi\circ\delta)\,d\omega & \text{if } j=0,\\ -\int_{\Omega_\varepsilon}\Delta^j(\phi(\psi\circ\delta))\,d\omega & \text{if } j\geq 1,\end{cases}$

(ii) $\displaystyle\lim_{t\to 0}\int_0^\infty \frac{\partial^j}{\partial t^j}I_\varepsilon\phi(t,r)\psi(r)\,dr = \begin{cases} \int_{\Omega_0^\varepsilon}\phi(\psi^{(-1)}\circ\delta)\,d\omega & \text{if } j=0,\\ -\int_{\Omega_\varepsilon}\Delta^j(\phi(\psi^{(-1)}\circ\delta))\,d\omega & \text{if } j\geq 1,\end{cases}$

(iii) $\displaystyle\frac{\partial^j}{\partial t^j}\Lambda_\varepsilon\phi(0,0) = \begin{cases} \int_{\partial\Omega}\phi\,d\sigma & \text{if } j=0,\\ 0 & \text{if } j\geq 1,\end{cases}$

(iv) $\displaystyle\frac{\partial^j}{\partial t^j}I_\varepsilon\phi(0,0) = \begin{cases} d\omega & \text{if } j=0,\\ -\int_{\Omega_\varepsilon}\Delta^j\phi\,d\omega & \text{if } j\geq 1,\end{cases}$

where, we recall, $\Omega_\varepsilon=\{x\in M:\delta(x)>\varepsilon\}$ *and* $\Omega_0^\varepsilon=\Omega\setminus\Omega_\varepsilon$.

Remark A.4. The only difference with respect to [Rizzi and Rossi 2021, Lemma A.4] is item (iii), which now holds only as $t\to 0$ and not for all positive times.

Proof of Lemma A.3. We claim that, for any $j\geq 1$,

$$\lim_{t\to 0}\int_\Omega \phi(x)\Delta^j u_\varepsilon(t,x)\,d\omega(x)=\int_{\Omega_\varepsilon}\Delta^j\phi(x)\,d\omega(x). \tag{97}$$

Let us prove it by induction: For $j=1$, applying the divergence theorem, we have

$$\int_\Omega \phi\Delta u_\varepsilon\,d\omega = -\int_{\partial\Omega}\phi g(\nabla u_\varepsilon,\nabla\delta)\,d\sigma+\int_{\partial\Omega}u_\varepsilon g(\nabla\phi,\nabla\delta)\,d\sigma+\int_\Omega u_\varepsilon\Delta\phi\,d\omega. \tag{98}$$

Let us discuss the first term in (98): By the divergence theorem applied with respect to the set Ω^c, we have

$$\int_{\partial\Omega}\phi g(\nabla u_\varepsilon,\nabla\delta)\,d\sigma=\int_{\Omega^c}\phi\Delta u_\varepsilon\,d\omega+\int_{\partial\Omega}u_\varepsilon g(\nabla\phi,\nabla\delta)\,d\sigma-\int_{\Omega^c}u_\varepsilon\Delta\phi\,d\omega. \tag{99}$$

Then, using [Jerison and Sánchez-Calle 1986, Theorem 3] and noticing that $d_{\mathrm{SR}}(x,y)\geq\varepsilon$ for any $x\in\Omega_\varepsilon$ and $y\in\Omega^c$, we conclude that in the limit as $t\to 0$, (99) converges to 0. This proves (97) for $j=1$. For $j>1$, proceeding by induction, we conclude. Finally, using the coarea formula (42), we complete the proof of the statement as in the usual argument of [Savo 1998, Lemma 5.6]. □

Remark A.5. In the noncompact case, under the assumption of Theorem 7.3, the above lemma holds. In particular, on the one hand, the divergence theorem holds since ϕ has compact support. On the other hand, notice that the time derivative estimates (87) are enough to ensure that (99) converges to 0 as $t\to 0$, regardless of the compactness of the set of integration. The same is true for $j>1$, where higher-order time derivatives appear.

The next step is to apply the iterated Duhamel's principle (93) to I_ε, which now satisfies its assumptions, then, pass to the limit as $\varepsilon\to 0$. The computations are long but straightforward: we report here the result at order $t^{5/2}$.

Lemma A.6. *Under the same assumptions of Lemma A.3, let $\phi \in C_c^\infty(\Omega_{-r_0}^{r_0})$. Then, as $t \to 0$, we have*

$$\begin{aligned} I_\Omega\phi(t,0) = 2\mathcal{G}_{1-u}[\phi](t) &+ \frac{1}{\sqrt{\pi}}\int_0^t \mathcal{G}_{1-u}[N_\Omega\phi](\tau)(t-\tau)^{-1/2}\,d\tau \\ &+ \frac{1}{2\pi}\int_0^t\int_0^\tau \mathcal{G}_{1-u}[N_\Omega^2\phi](\hat\tau)((\tau-\hat\tau)(t-\tau))^{-1/2}\,d\hat\tau\,d\tau \\ &+ \frac{1}{4\pi^{3/2}}\int_0^t\int_0^\tau\int_0^{\hat\tau} \mathcal{G}_{1-u}[N_\Omega^3\phi](s)((\hat\tau-s)(\tau-\hat\tau)(t-\tau))^{-1/2}\,ds\,d\hat\tau\,d\tau \\ &+ \frac{1}{4\sqrt{\pi}}\int_0^t\int_{\partial\Omega}(1-u(\tau,\cdot))(4\Delta-N_\Omega^2)\phi\,d\sigma\,(t-\tau)^{1/2}\,d\tau \\ &+ \frac{1}{4\sqrt{\pi}}\int_0^t \mathcal{G}_{1-u}[(6N_\Omega\Delta-N_\Omega^3-2\Delta N_\Omega)\phi](\tau)(t-\tau)^{1/2}\,d\tau + O(t^{5/2}), \end{aligned} \tag{100}$$

where $u(t,\cdot) = e^{t\Delta}\mathbb{1}_\Omega$ and $\mathcal{G}_u[\phi]$ is the functional defined in (60). We recall that N_Ω is the operator defined in (95).

The expression (58) is a direct consequence of A.6. Moreover, we can apply it, when the base set is chosen to be Ω^c. Then, evaluating the difference between $I_\Omega\phi(t,0)$ and $I_{\Omega^c}\phi(t,0)$ we obtain the asymptotic equality (66), which is proved in the next lemma. We use the shorthands I, I^c for I_Ω and I_{Ω^c} respectively.

Lemma A.7. *Under the same assumptions of Lemma A.3, let $\phi \in C_c^\infty(\Omega_{-r_0}^{r_0})$. Then, as $t \to 0$, we have*

$$\begin{aligned} (I\phi - I^c\phi)(t,0) = 2\mathcal{G}_{1-2u}[\phi](t) &+ \frac{1}{2}\int_{\partial\Omega} N\phi\,d\sigma\,t \\ &+ \frac{1}{2\pi}\int_0^t\int_0^\tau \mathcal{G}_{1-2u}[N^2\phi](\hat\tau)((\tau-\hat\tau)(t-\tau))^{-1/2}\,d\hat\tau\,d\tau \\ &+ \frac{1}{4\sqrt{\pi}}\int_0^t\int_{\partial\Omega}(1-2u(\tau,\cdot))(4\Delta-N^2)\phi\,d\sigma\,(t-\tau)^{1/2}\,d\tau + O(t^2), \end{aligned}$$

where N is the operator given by

$$N\phi = 2g(\nabla\phi,\nabla\delta) + \phi\Delta\delta \quad \textit{for all } \phi \in C_c^\infty(\Omega_{-r_0}^{r_0}),$$

with $\delta : M \to \mathbb{R}$ the signed distance function from $\partial\Omega$.

Proof. Firstly, we apply Lemma A.6 to $I\phi$: we obtain exactly the expression (100), with the operator $N_\Omega = N$. Secondly, for the outside contribution, recall that we have the following equality of smooth functions:

$$1 - u^c(t,x) = 1 - e^{t\Delta}\mathbb{1}_{\Omega^c}(x) = e^{t\Delta}\mathbb{1}_\Omega(x) = u(t,x) \quad \text{for all } t>0,\ x \in M.$$

Therefore, when we apply Lemma A.6 to $I^c\phi$, we replace $1-u^c(t,\cdot) = 1 - e^{t\Delta}\mathbb{1}_{\Omega^c}$ with the function $u(t,\cdot) = e^{t\Delta}\mathbb{1}_\Omega(\cdot)$. Moreover, the operator N_{Ω^c} defined in (95), for the set Ω^c, is equal to $-N$, since the inward-pointing normal to Ω^c is $-\nabla\delta$. Therefore, writing the difference of the two contributions, and noticing that Ω and its complement share the boundary, we have

$$\begin{aligned}(I\phi - I^c\phi)(t,0) = 2\mathcal{G}_{1-2u}[\phi](t) + \frac{1}{\sqrt{\pi}}\int_0^t \mathcal{G}_1[N\phi](\tau)(t-\tau)^{-1/2}\,d\tau \\ + \frac{1}{2\pi}\int_0^t\int_0^\tau \mathcal{G}_{1-2u}[N^2\phi](\hat\tau)((\tau-\hat\tau)(t-\tau))^{-1/2}\,d\hat\tau\,d\tau \\ + \frac{1}{4\pi^{3/2}}\int_0^t\int_0^\tau\int_0^{\hat\tau} \mathcal{G}_1[N^3\phi](s)((\hat\tau-s)(\tau-\hat\tau)(t-\tau))^{-1/2}\,ds\,d\hat\tau\,d\tau \qquad (101) \\ + \frac{1}{4\sqrt{\pi}}\int_0^t\int_{\partial\Omega}(1-2u(\tau,\cdot))(4\Delta - N^2)\phi\,d\sigma(t-\tau)^{1/2}\,d\tau \\ + \frac{1}{4\sqrt{\pi}}\int_0^t \mathcal{G}_1[(6N\Delta - N^3 - 2\Delta N)\phi](\tau)(t-\tau)^{1/2}\,d\tau + O(t^{5/2}). \qquad (102)\end{aligned}$$

To conclude, it is enough to notice that the functional $\mathcal{G}_1$ can be explicitly computed:

$$\mathcal{G}_1[\phi](t) = \frac{1}{\sqrt{\pi}}\int_{\partial\Omega}\phi\,d\sigma\,t^{1/2} \quad \text{for all } \phi\in C_c^\infty(\Omega_{-r_0}^{r_0}).$$

Thus, the terms in (101) and (102) are remainder of order $O(t^2)$. □

Applying Lemma A.6 to the sum of $I_\Omega\phi(t,0)$ and $I_{\Omega^c}\phi(t,0)$ instead, we obtain (77). The proof of this result is not provided here, as it is similar to the proof of Lemma A.7.

Lemma A.8. *Under the same assumptions of Lemma A.3, let $\phi\in C_c^\infty(\Omega_{-r_0}^{r_0})$. Then, as $t\to 0$, we have*

$$\begin{aligned}(I\phi + I^c\phi)(t,0) = \frac{2}{\sqrt{\pi}}\int_{\partial\Omega}\phi\,d\sigma\,t^{1/2} + \frac{1}{\sqrt{\pi}}\int_0^t \mathcal{G}_{1-2u}[N\phi](\tau)(t-\tau)^{-1/2}\,d\tau \\ + \frac{1}{6\sqrt{\pi}}\int_{\partial\Omega}(4\Delta + N^2)\phi\,d\sigma\,t^{3/2} \\ + \frac{1}{4\pi^{3/2}}\int_0^t\int_0^\tau\int_0^{\hat\tau} \mathcal{G}_{1-2u}[N^3\phi](s)((\hat\tau-s)(\tau-\hat\tau)(t-\tau))^{-1/2}\,ds\,d\hat\tau\,d\tau \\ + \frac{1}{4\sqrt{\pi}}\int_0^t \mathcal{G}_{1-2u}[(6N\Delta - N^3 - 2\Delta N)\phi](\tau)(t-\tau)^{1/2}\,d\tau + O(t^{5/2}).\end{aligned}$$

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MINKOWSKI INEQUALITY ON COMPLETE RIEMANNIAN MANIFOLDS WITH NONNEGATIVE RICCI CURVATURE

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We consider Riemannian manifolds of dimension at least 3, with nonnegative Ricci curvature and Euclidean volume growth. For every open bounded subset with smooth boundary we establish the validity of an optimal Minkowski inequality. We also characterise the equality case, provided the domain is strictly outward minimising and strictly mean convex. Along with the proof, we establish in full generality sharp monotonicity formulas, holding along the level sets of p-capacitary potentials in p-nonparabolic manifolds with nonnegative Ricci curvature.

1. Introduction

1A. *Statements of the main results.* Given an open bounded convex domain with smooth boundary $\Omega \subseteq \mathbb{R}^n$, $n \geq 3$, the classical Minkowski inequality, originally proven in [Minkowski 1903], gives a sharp lower bound for the average of the mean curvature H of $\partial\Omega$ in terms of the inverse of its surface radius, that is,

$$\left(\frac{|\mathbb{S}^{n-1}|}{|\partial\Omega|}\right)^{\frac{1}{n-1}} \leq \fint_{\partial\Omega} \frac{\mathrm{H}}{n-1}\, \mathrm{d}\sigma,$$

with the equality satisfied if and only if Ω is a ball. It was clear to many authors that such inequality deserved to be further investigated. For example one would like to relax the convexity assumption on one hand, and to prove that the inequality holds on more general ambient manifolds on the other.

The first question has been positively answered using techniques based on geometric flows [Huisken 2009], optimal transport [Chang and Wang 2013; Castillon 2010], and recently also nonlinear potential theory [Fogagnolo et al. 2019; Agostiniani et al. 2022a]. The latter method actually provides the most general statement available so far, namely the extended Minkowski inequality

$$\left(\frac{|\partial\Omega^*|}{|\mathbb{S}^{n-1}|}\right)^{\frac{n-2}{n-1}} \leq \frac{1}{|\mathbb{S}^{n-1}|}\int_{\partial\Omega}\left|\frac{\mathrm{H}}{n-1}\right| \mathrm{d}\sigma \tag{1-1}$$

holding for every open bounded domain $\Omega \subseteq \mathbb{R}^n$ with smooth boundary. Here Ω^* denotes the *strictly outward minimising hull* of Ω. The precise definition of Ω^* is reported in (4-12) below and analysed in full detail in [Fogagnolo and Mazzieri 2022]. However, in this preliminary discussion, we just point out that Ω^* minimises the perimeter among bounded subsets containing Ω.

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Many improvements can be found in the literature also concerning the question of extending the Minkowski inequality to more general settings. Firstly Gallego and Solanes [2005] established quermassintegral inequalities for convex domains in the hyperbolic space. Using the inverse mean curvature flow (IMCF for short), de Lima and Girão [2016] extended the result to star-shaped and strictly mean-convex domains lying in the same ambient manifold. The IMCF has been also employed to establish a Minkowski-type inequality for outward minimising sets sitting in the Schwarzschild manifold by Wei [2018], in the anti-de Sitter–Schwarzschild manifold by Brendle, Hung and Wang [Brendle et al. 2016], and on asymptotically flat static manifolds by McCormick [2018].

A natural context in which to test the validity of a Minkowski inequality is provided by complete noncompact Riemannian manifolds with nonnegative Ricci curvature. A very recent work [Brendle 2023] actually points in this direction. Indeed, choosing $f = 1$ in Corollary 1.5 of that work a nonsharp Minkowski inequality can be deduced for complete Riemannian manifolds with nonnegative sectional curvature and Euclidean volume growth. In the present paper, we prove the following theorem.

Theorem 1.1 (extended Minkowski inequality). *Let (M, g) be a complete Riemannian manifold with* $\mathrm{Ric} \geq 0$ *and Euclidean volume growth. Let $\Omega \subseteq M$ be an open bounded set with smooth boundary. Then*

$$\left(\frac{|\partial\Omega^*|}{|\mathbb{S}^{n-1}|}\right)^{\frac{n-2}{n-1}} \mathrm{AVR}(g)^{\frac{1}{n-1}} \leq \frac{1}{|\mathbb{S}^{n-1}|} \int_{\partial\Omega} \left|\frac{\mathrm{H}}{n-1}\right| \mathrm{d}\sigma, \tag{1-2}$$

where $\mathrm{AVR}(g)$ is the asymptotic volume ratio of (M, g), H is the mean curvature of $\partial\Omega$ with respect to the outward normal unit vector and Ω^* is the strictly outward minimising hull of Ω.

In the case a strictly outward minimising $\Omega \subset M$ with strictly mean-convex boundary achieves the identity in (1-2), we show that $M \smallsetminus \Omega$ splits as a (truncated) cone.

Theorem 1.2 (rigidity for the Minkowski inequality). *A bounded strictly outward minimising subset $\Omega \subset M$ with smooth strictly mean-convex boundary satisfies*

$$\left(\frac{|\partial\Omega|}{|\mathbb{S}^{n-1}|}\right)^{\frac{n-2}{n-1}} \mathrm{AVR}(g)^{\frac{1}{n-1}} = \frac{1}{|\mathbb{S}^{n-1}|} \int_{\partial\Omega} \frac{\mathrm{H}}{n-1} \,\mathrm{d}\sigma$$

if and only if $(M \smallsetminus \Omega, g)$ is isometric to

$$\left([\rho_0, +\infty) \times \partial\Omega,\ \mathrm{d}\rho \otimes \mathrm{d}\rho + \left(\frac{\rho}{\rho_0}\right)^2 g_{\partial\Omega}\right), \quad \text{where } \rho_0 = \left(\frac{|\partial\Omega|}{\mathrm{AVR}(g)|\mathbb{S}^{n-1}|}\right)^{\frac{1}{n-1}}.$$

Some comments are in order about the above statements. First, we recall for the reader's convenience that the asymptotic volume ratio of (M, g) is given by

$$\mathrm{AVR}(g) = \lim_{r \to +\infty} \frac{|B(o, r)|}{r^n |\mathbb{B}^n|}$$

for some $o \in M$. The fact that, on complete manifolds with nonnegative Ricci curvature, the above limit is well-defined and does not depend on the base point o, is a consequence of the classical Bishop–Gromov volume comparison theorem. Moreover, one has that $0 \leq \mathrm{AVR}(g) \leq 1$, with $\mathrm{AVR}(g) = 1$ if and only if

(M, g) is the standard n-dimensional Euclidean space. Beside the intrinsic fundamental role played by manifolds with nonnegative Ricci curvature with Euclidean volume growth in geometric analysis, this class includes a diversity of explicit manifolds naturally arising from different fields, such as asymptotically locally Euclidean spaces (ALE for short) *gravitational instantons*. These are noncompact hyperkähler Ricci flat 4-dimensional manifolds playing a role in the study of Euclidean quantum gravity theory, gauge theory and string theory (see [Hawking 1977; Eguchi and Hanson 1979; Kronheimer 1989a; 1989b; Minerbe 2009; 2010; 2011]).

It is worth noticing that inequality (1-2) is sharp and it provides the optimal Minkowski inequality on manifolds with nonnegative Ricci curvature for *outward minimising subsets*, see Corollary 4.6. These subsets are mean-convex and satisfy $|\partial\Omega^*| = |\partial\Omega|$, so that the Minkowski inequality reads

$$\left(\frac{|\partial\Omega|}{|\mathbb{S}^{n-1}|}\right)^{\frac{n-2}{n-1}} \mathrm{AVR}(g)^{\frac{1}{n-1}} \le \frac{1}{|\mathbb{S}^{n-1}|}\int_{\partial\Omega} \frac{\mathrm{H}}{n-1}\, d\sigma,$$

in this case. In addition to the Euclidean spaces, where it is immediately seen that balls achieve the identity in (1-2), the sharpness of this inequality is checked in far greater generality, as specified in Remark 4.7 below.

Combining Theorem 1.1 with the sharp isoperimetric inequality for manifolds with nonnegative Ricci curvature, first proved in dimension 3 in [Agostiniani et al. 2020, Theorem 1.4] and recently extended to any dimension in [Brendle 2023] (see also [Fogagnolo and Mazzieri 2022; Johne 2021; Balogh and Kristály 2023]), reading

$$\frac{|\mathbb{S}^{n-1}|^n}{|\mathbb{B}^n|^{n-1}} \mathrm{AVR}(g) \le \frac{|\partial\Omega^*|^n}{|\Omega^*|^{n-1}},$$

we get the following sharp volumetric version of the Minkowski inequality.

Theorem 1.3 (volumetric Minkowski inequality). *Let (M, g) be a complete Riemannian manifold with* $\mathrm{Ric} \ge 0$ *and Euclidean volume growth. Let $\Omega \subseteq M$ be an open bounded set with smooth boundary. Then*

$$\left(\frac{|\Omega|}{|\mathbb{B}^n|}\right)^{\frac{n-2}{n}} \mathrm{AVR}(g)^{\frac{2}{n}} \le \frac{1}{|\mathbb{S}^{n-1}|}\int_{\partial\Omega} \left|\frac{\mathrm{H}}{n-1}\right| d\sigma, \tag{1-3}$$

where $\mathrm{AVR}(g)$ *is the asymptotic volume ratio of (M, g), H is the mean curvature of $\partial\Omega$ with respect to the outward normal unit vector. Moreover, the equality is satisfied if and only if (M, g) is isometric to the flat Euclidean space and Ω is a ball.*

As for the extended Minkowski inequality, (1-3) is easily recognised to be sharp, while the rigidity statement directly follows from the rigidity of the isoperimetric inequality. We finally point out that earlier contributions to the volumetric Minkowski inequality were given in [Chang and Wang 2011; Qiu 2015], holding in the flat Euclidean space and under stronger geometric assumptions on the boundary of Ω.

1B. ***Outline of the proof.*** We now describe the main features of our approach, which is in line with [Agostiniani and Mazzieri 2020; Agostiniani et al. 2020; 2022a; Fogagnolo et al. 2019]. Given (M, g) a Riemannian n-manifold, $n \ge 3$, with nonnegative Ricci curvature, and an open bounded subset $\Omega \subseteq M$

with smooth boundary we consider, for every $1 < p < n$, the p-capacitary potential associated to Ω. This is the solution u to the problem

$$\begin{cases} \Delta_g^{(p)} u = 0 & \text{on } M \smallsetminus \overline{\Omega}, \\ u = 1 & \text{on } \partial\Omega, \\ u(x) \to 0 & \text{as } d_g(x, o) \to +\infty, \end{cases} \tag{1-4}$$

where $\Delta_g^{(p)}$ is the p-Laplace operator associated with the metric g, and $d_g(\,\cdot\,, o)$ is the distance induced by g to some fixed reference point o. Provided the manifold (M, g) is p-nonparabolic (see Definition 2.5 below, as well as [Holopainen 1990; 1999]), the solution to problem (1-4) exists and is unique. Such a solution is commonly referred to as the *p-capacitary potential* associated with Ω. It is worth specifying that manifolds with Euclidean volume growth (i.e., $\mathrm{AVR}(g) > 0$) do satisfy the p-nonparabolicity assumption for $1 < p < n$ by the characterisation given in [Holopainen 1999, Proposition 5.10]. As a crucial step in our method, we will establish families of *monotonicity formulas*, holding along the level sets of the p-capacitary potentials associated with Ω. More precisely, for every $t \in [1, +\infty)$, we set

$$F_p^\beta(t) = t^{\beta \frac{(n-1)(p-1)}{(n-p)}} \int_{\{u=1/t\}} |\mathrm{D}u|^{(\beta+1)(p-1)} \, \mathrm{d}\sigma, \tag{1-5}$$

and we show that for

$$\beta > \frac{n-p}{(p-1)(n-1)}$$

the above quantity admits a nonincreasing $\mathscr{C}^1(1, +\infty)$ representative.

Some remarks are mandatory at this stage. First of all, let us point out that the monotonicity statement provided here for the functions F_p^β holds in full generality and with no restriction on the geometry of Ω. As such, it is also new for domains sitting in $\mathbb{R}^n$, where the same conclusions were provided in [Fogagnolo et al. 2019] only for convex domains, and in fact for smooth level sets flows. In the general case, it is well known that the level sets flow of p-harmonic functions might present a much less regular behaviour since no general bound is available for the Hausdorff dimension of the critical set. To overcome these difficulties, the authors in [Agostiniani et al. 2022a] settled for the *effective inequalities*

$$\lim_{t \to +\infty} F_p^\beta(t) \leq F_p^\beta(1) \quad \text{and} \quad (F_p^\beta)'(1) \leq 0. \tag{1-6}$$

The derivation of these two bounds, however, heavily relied on the compactness of the critical set of u, that is a particular feature of spaces with finite topology, and as such it is not directly viable in our setting (see [Menguy 2000]). In contrast with this, the present treatment provides the desired extension to the nonlinear setting and to the general framework of nonnegatively Ricci curved p-nonparabolic manifolds of the monotonicity formulas discovered in [Colding 2012; Colding and Minicozzi 2014b; Agostiniani and Mazzieri 2020; Agostiniani et al. 2020] for harmonic functions. As a second remark, to let the reader appreciate the $\mathscr{C}^1$-regularity result, we observe that in principle even the fact that formula (1-5) yields a well-posed definition is not granted for free. The most serious difficulty here is that the set of singular values cannot be controlled through Sard's theorem, since p-harmonic functions only enjoy a mild — though optimal — $\mathscr{C}^{1,\beta}$-regularity. We managed to solve these problems also taking advantage of recent insights given in [Gigli and Violo 2023]. The full statement of the monotonicity theorem is found in Theorem 3.1 below.

Through the monotonicity of F_p^β, with $\beta = (p-1)^{-1}$, we arrive at the following *L^p-Minkowski inequality*

$$\mathrm{C}_p(\Omega)^{\frac{n-p-1}{n-p}} \,\mathrm{AVR}(g)^{\frac{1}{n-p}} \leq \frac{1}{|\mathbb{S}^{n-1}|} \int_{\partial\Omega} \left| \frac{\mathrm{H}}{n-1} \right|^p \mathrm{d}\sigma, \tag{1-7}$$

where $\mathrm{C}_p(\Omega)$ is the normalised p-capacity of Ω defined in (2-5) below. A major advantage we draw out of the full monotonicity of F_p^β is the bypassing of the computation of its limit as $t \to +\infty$ when reaching for (1-7). Indeed, this step is now replaced by a suitable contradiction argument that combines the full monotonicity of our quantities with the sharp *iso-p-capacitary inequality* (see Theorem 4.1 below)

$$\frac{\mathrm{C}_p(\mathbb{B}^n)^n}{|\mathbb{B}^n|^{n-p}} \,\mathrm{AVR}(g)^p \leq \frac{\mathrm{C}_p(\Omega)^n}{|\Omega|^{n-p}}. \tag{1-8}$$

Such a statement is of independent interest in our opinion and can be achieved by taking advantage of the already-mentioned sharp isoperimetric inequality in manifolds with nonnegative Ricci curvature and Euclidean volume growth, following rather classical arguments (see, e.g., [Jauregui 2012]).

With the L^p-Minkowski inequality (1-7) at hand, the extended Minkowski inequality (1-7) simply follows by letting $p \to 1^+$ since

$$\lim_{p \to 1^+} \mathrm{C}_p(\Omega) = \frac{|\partial\Omega^*|}{|\mathbb{S}^{n-1}|},$$

as proven in [Fogagnolo and Mazzieri 2022, Theorem 1.2]. This particular feature of our approach, namely the fact that the Minkowski inequality is obtained as the limit of its L^p-versions, makes the rigidity statement a particularly nontrivial task, although we show that (1-7) holds with equality only on cones. This leads us to prove the rigidity statement, Theorem 1.2, through an argument involving the study of the IMCF starting at boundaries of domains that saturate the Minkowski inequality (1-1). More precisely, we first show that the flow is smooth and given by constantly mean-curved totally umbilical hypersurfaces for a short time. This crucially exploits the nonnegativity of the Ricci curvature (Lemma 4.8). Then, a splitting procedure along such flow, inspired by [Huisken and Ilmanen 2001], shows that an outer neighbourhood of $\partial\Omega$ is isometric to a truncated cone with the same volume ratio as $\mathrm{AVR}(g)$, and this allows us to conclude (Lemma 4.9).

1C. ***Further monotonicity-rigidity results.*** Beside the monotonicity-rigidity properties of F_p^β discussed above, we also establish analogous ones for the function

$$F_p^\infty(t) = t^{\frac{n-1}{n-p}} \sup_{\{u=1/t\}} |\mathrm{D}u|.$$

This is the content of Theorem 3.2, which is again proved in the general setting of p-nonparabolic manifolds with nonnegative Ricci curvature, extending [Fogagnolo et al. 2019, Theorem 1.3]. As geometric consequences of this statement, we provide a rigidity result under pinching conditions and a sphere theorem for smooth boundaries in manifolds with $\mathrm{Ric} \geq 0$ (see Theorems 4.11 and 4.12 below) and Euclidean volume growth. It is worth mentioning that the monotonicity of F_p^∞ also leads to a new insight on the critical set of the p-capacitary potential, which we believe deserves some further investigation. Namely, it turns out that every level set of u displays some nonempty relatively open region, where $\mathrm{D}u$ does not vanish, and where in particular u is smooth (see Corollary 3.3).

1D. *Summary.* In Section 2 we report, for the ease of the reader, some relevant facts from the theory of p-harmonic functions on Riemannian manifolds, focussing on the regularity theory as well as on the existence and uniqueness of solutions to (1-4). Some important — though already well known — estimates and identities are also recalled in this section. Section 3 is devoted to the proof of monotonicity-rigidity theorems (see Theorems 3.1 and 3.2). After having introduced a convenient conformally related setting, we restate them in this framework and we conclude the section with their proofs. In Section 4, after having provided (1-8), we make use of these tools to prove the L^p-Minkowski inequality (see Theorem 4.3), deduce the extended Minkowski inequality Theorem 1.1 and some rigidity results under pinching conditions as consequences of the monotonicity-rigidity theorems.

2. The p-capacitary potential in Riemannian manifolds

We have collected here, for the sake of future reference, some substantially well-known results that will be repeatedly applied in our arguments. Before considering the specific case of problem (1-4), we recall the definition of p-harmonic functions, as well as their regularity estimates. We then analyse the existence and uniqueness of the solution u_p to (1-4) on complete Riemannian manifolds. It turns out that these questions are intimately related to the notion of p-nonparabolicity, and p-nonparabolic manifolds will then constitute the natural setting for the monotonicity-rigidity theorems. We afterwards recall some global standard estimates on u_p and its gradient as well as a Kato-type identity for p-harmonic functions.

2A. *p-harmonic functions and regularity.* Given an open subset U of a complete Riemannian manifold (M, g), we say that $v \in W^{1,p}(U)$ is p-harmonic if

$$\int_U \big\langle |\mathrm{D}v|^{p-2}\mathrm{D}v \,\big|\, \mathrm{D}\psi \big\rangle \,\mathrm{d}\mu = 0 \tag{2-1}$$

for any test function $\psi \in \mathscr{C}_c^\infty(U)$. With $\langle \cdot \,|\, \cdot \rangle$ we denote as usual the scalar product induced by the underlying Riemannian metric g on the tangent space at each point. Regularity results for p-harmonic functions (see [Tolksdorf 1984; DiBenedetto 1983; Lieberman 1988]) ensure that v belongs to $\mathscr{C}^{1,\beta}_{\mathrm{loc}}(U)$ for some $\beta \in (0, 1)$ and is smooth around each point where $|\mathrm{D}v| > 0$.

Since the $\mathscr{C}^{1,\beta}$-regularity is not sufficient to employ Sard's theorem, we are going to heavily rely on the coarea formula. We report it here for ease of further references. The statement below follows from [Maggi 2012, Lemma 18.5 and Theorem 18.1] coupled with standard approximation results.

Proposition 2.1 (coarea formula). *Let (M, g) be a complete Riemannian manifold. Consider a locally Lipschitz function $v : U \to [0, +\infty)$ on some open subset $U \subseteq M$ such that $v^{-1}([a, b])$ is compact for every $[a, b] \subset (0, +\infty)$. Then the following hold*:

(1) $|\{v = t\} \cap \mathrm{Crit}(v)| = 0$ *for almost every* $t \in [0, +\infty)$.

(2) *For every measurable* f *such that* $f|\mathrm{D}v| \in L^1_{\mathrm{loc}}(U)$ *we have* $f \in L^1(\{v = t\})$ *for almost every* $t \in (0, +\infty)$ *and*

$$\int_U \psi(v) f |\mathrm{D}v| \, \mathrm{d}\mu = \int_0^{+\infty} \psi(t) \int_{\{v=t\}} f \, \mathrm{d}\sigma \, \mathrm{d}t \tag{2-2}$$

for every ψ *bounded measurable function compactly supported in* $(0, +\infty)$. *In particular,*

$$t \mapsto \int_{\{v=t\}} f \, \mathrm{d}\sigma \in L^1_{\mathrm{loc}}(0, +\infty),$$

and its equivalence class does not depend on the representative of f.

Remark 2.2. If $h \in L^1_{\mathrm{loc}}(U)$ and $h = 0$ almost everywhere on $\mathrm{Crit}(v)$, the function $f = h|\mathrm{D}v|^{-1}$, satisfies the assumptions of Proposition 2.1(2). Clearly, if $f \in L^1(U)$, then (2-2) holds for every ψ bounded measurable, even without compact support.

With the idea of applying the previous result for $f = |\mathrm{D}|\mathrm{D}v|^{p-1}|$, a higher integrability degree of p-harmonic functions is required. We refer the reader to [Lou 2008, Lemma 2.1] for a self-contained proof of the following lemma in the Euclidean case. The general case follows in the same way, as it is ultimately due to a careful integration of the Bochner identity. Indeed, computations are the same provided a lower bound on the Ricci tensor is in force, which is always true locally (see [Benatti 2022, Appendix C] for a complete proof).

Lemma 2.3. *Let* (M, g) *be a complete Riemannian manifold and* $U \subseteq M$ *be an open subset. Given* $v \in W^{1,p}(U)$ *a* p*-harmonic function, then* $|\mathrm{D}v|^{p-1} \in W^{1,2}_{\mathrm{loc}}(U)$.

Given $U \subseteq M$ with Lipschitz boundary, a p-harmonic function $u \in W^{1,p}(U)$ attains some Dirichlet data $g \in L^p(\partial U)$ if u coincides with g on ∂U in the sense of the trace operator.

2B. ***p-nonparabolic manifolds and the p-capacitary potential.*** Given a noncompact Riemannian manifold M, we consider the p-capacitary potential of a bounded set with smooth boundary $\Omega \subset M$, that is, a function $u \in W^{1,p}(M \smallsetminus \overline{\Omega})$ solving (1-4). The function u belongs to $\mathscr{C}^{1,\beta}(M \smallsetminus \Omega)$ (see [Lieberman 1988]) and it is smooth near the points where the gradient does not vanish. In particular, by the Hopf maximum principle in [Tolksdorf 1983, Proposition 3.2.1] the datum on $\partial\Omega$ is attained smoothly.

We now focus on some classical sufficient conditions to ensure the existence of the p-capacitary potential, which turns out to be related to the notion of p-Green's function we are going to recall.

Definition 2.4 (p-Green's function). Let (M, g) be a complete Riemannian manifold. Let $\mathrm{Diag}(M) = \{(x, x) \in M \times M \mid x \in M\}$. For $p \geq 1$, we say that $G_p : M \times M \smallsetminus \mathrm{Diag}(M) \to \mathbb{R}$ is a p-Green's function for M if it weakly satisfies $\Delta_p G(o, \cdot) = -\delta_o$ for any $o \in M$, where δ_o is the Dirac delta centred at o, that is, if it holds

$$\int_M \big\langle |\mathrm{D}G_p(o, \cdot)|^{p-2} \, \mathrm{D}G_p(o, \cdot) \,\big|\, \mathrm{D}\psi \big\rangle \, \mathrm{d}\mu = \psi(o)$$

for any $\psi \in \mathscr{C}^\infty_c(M)$.

The notion of p-Green's function calls for that of p-nonparabolic Riemannian manifold.

Definition 2.5 (p-nonparabolicity). We say that a complete noncompact Riemannian manifold (M, g) is p-nonparabolic if there exists a *positive* p-Green's function $G_p : M \times M \smallsetminus \mathrm{Diag}(M) \to \mathbb{R}$. With the expression *$p$-Green function* we are in fact referring to the positive minimal one.

The notion of p-nonparabolicity is intimately related to the existence of a solution to (1-4), in that if the positive p-Green's function of a p-nonparabolic Riemannian manifold vanishes at infinity, then such a solution exists for any open bounded subset $\Omega \subset M$ with smooth boundary. A complete and self-contained proof of this fact is provided in the Appendix of [Fogagnolo and Mazzieri 2022]. We report the statement of such basic thought fundamental result.

Theorem 2.6 (existence of the p-capacitary potential). *Let (M, g) be a complete noncompact p-nonparabolic Riemannian manifold. Let $\Omega \subset M$ be an open bounded subset with smooth boundary. Assume also that the p-Green's function G_p satisfies $G_p(o, x) \to 0$ as $d_g(o, x) \to +\infty$ for some $o \in M$. Then, there exists a unique solution u_p to (1-4).*

If (M, g) is a complete noncompact Riemannian manifold with $\mathrm{Ric} \geq 0$ and Euclidean volume growth, then it is in fact p-nonparabolic for every $1 < p < n$ and the p-Green's function satisfies

$$G_p(o, x) \leq \mathrm{C}\, d_g(o, x)^{-\frac{n-p}{p-1}} \tag{2-3}$$

for some constant C. This is a direct consequence of [Holopainen 1999, Proposition 5.10].

We find convenient to recall here the definition of p-capacity of an open bounded subset $\Omega \subset M$ together with a normalised version of it which turns out to be more advantageous for our computations.

Definition 2.7 (p-capacity and normalised p-capacity). Let (M, g) be a complete noncompact Riemannian manifold, and let Ω be an open bounded subset of M. For $1 < p < n$, the p-capacity of Ω is defined as

$$\mathrm{Cap}_p(\Omega) = \inf\left\{ \int_M |\mathrm{D}v|^p \, \mathrm{d}\mu \;\middle|\; v \in \mathscr{C}_c^\infty(M),\ v \geq 1 \text{ on } \Omega \right\}. \tag{2-4}$$

On the other hand, the normalised p-capacity of Ω is defined as

$$\mathrm{C}_p(\Omega) = \frac{1}{|\mathbb{S}^{n-1}|}\left(\frac{p-1}{n-p}\right)^{p-1} \mathrm{Cap}_p(\Omega). \tag{2-5}$$

A function u solving (1-4) realises the p-capacity of the initial set Ω, and actually one can also characterise such quantity with a suitable integral on $\partial\Omega$. We resume these facts in the following statement.

Proposition 2.8. *Let (M, g) be a complete noncompact p-nonparabolic Riemannian manifold for some $1 < p < n$. Let $\Omega \subset M$ be an open bounded subset with smooth boundary. Then the solution u_p to (1-4) realises*

$$\mathrm{C}_p(\Omega) = \frac{1}{|\mathbb{S}^{n-1}|}\left(\frac{p-1}{n-p}\right)^{p-1} \int_{M \smallsetminus \overline{\Omega}} |\mathrm{D}u_p|^p \, \mathrm{d}\mu. \tag{2-6}$$

Moreover, we have that

$$\mathrm{C}_p(\Omega) = \frac{1}{|\mathbb{S}^{n-1}|}\left(\frac{p-1}{n-p}\right)^{p-1} \int_{\{u_p = 1/t\}} |\mathrm{D}u_p|^{p-1} \, \mathrm{d}\sigma \tag{2-7}$$

holds for almost every $t \in [1, +\infty)$, including any $1/t$ that is a regular value for u_p.

Proof. The function u_p can be approximated in $W^{1,p}(M \smallsetminus \overline{\Omega})$ by functions φ in $\mathscr{C}_c^\infty(M)$ which satisfy $\varphi \geq 1$ on Ω. Then

$$\operatorname{Cap}_p(\Omega) \leq \int_{M \smallsetminus \overline{\Omega}} |\mathrm{D}u_p|^p \, \mathrm{d}\mu.$$

On the other hand, the weak formulation in (2-1) can be relaxed in duality with functions in $W_0^{1,p}(M \smallsetminus \overline{\Omega})$. Hence, taking any competitor $\psi \in \mathscr{C}_c^\infty(M)$ with $\psi \geq 1$ on Ω, $u_p - \psi \in W_0^{1,p}(M \smallsetminus \overline{\Omega})$, we get that

$$\int_{M \smallsetminus \overline{\Omega}} |\mathrm{D}u_p|^p \, \mathrm{d}\mu = \int_{M \smallsetminus \overline{\Omega}} \langle |\mathrm{D}u_p|^{p-2} \mathrm{D}u_p, \mathrm{D}u_p \rangle \, \mathrm{d}\mu = \int_{M \smallsetminus \overline{\Omega}} \langle |\mathrm{D}u_p|^{p-2} \mathrm{D}u_p, \mathrm{D}\psi \rangle \, \mathrm{d}\mu.$$

Applying the Hölder inequality to the right-hand side, we are left with

$$\int_{M \smallsetminus \overline{\Omega}} |\mathrm{D}v|^p \, \mathrm{d}\mu \leq \int_{M \smallsetminus \overline{\Omega}} |\mathrm{D}\psi|^p \, \mathrm{d}\mu$$

for every competitor ψ in (2-4), proving (2-6). Since $|\mathrm{D}u_p| \in L^p(M \smallsetminus \overline{\Omega})$, applying the coarea formula (2-2) with $f = |\mathrm{D}u_p|^{p-1}$ to (2-6) (see Remark 2.2) one can obtain that

$$\operatorname{Cap}_p(\Omega) = \int_0^1 \int_{\{u_p = \tau\}} |\mathrm{D}u_p|^{p-1} \, \mathrm{d}\sigma \, \mathrm{d}\tau. \tag{2-8}$$

Employing again the coarea formula (2-2) with $f = |\mathrm{D}u_p|^{p-1}$ and integration by parts we get

$$\begin{aligned} \int_0^1 \varphi'(\tau) \int_{\{u_p = \tau\}} |\mathrm{D}u_p|^{p-1} \, \mathrm{d}\sigma \, \mathrm{d}\tau &= \int_{M \smallsetminus \overline{\Omega}} \varphi'(u_p) |\mathrm{D}u_p|^p \, \mathrm{d}\mu = -\int_{M \smallsetminus \overline{\Omega}} |\mathrm{D}u_p|^{p-2} \langle \mathrm{D}u_p, \mathrm{D}(\varphi(u_p)) \rangle \, \mathrm{d}\mu \\ &= \int_{M \smallsetminus \overline{\Omega}} \varphi(u_p) \operatorname{div}(|\mathrm{D}u_p|^{p-2} \mathrm{D}u_p) \, \mathrm{d}\mu = 0 \end{aligned}$$

for every $\varphi \in \mathscr{C}_c^\infty(0, 1)$, which gives that

$$\tau \mapsto \int_{\{u_p = \tau\}} |\mathrm{D}u_p|^{p-1} \, \mathrm{d}\sigma$$

admits a constant representative; that coupled with (2-8) yields (2-7), letting $t = 1/\tau$. $\square$

In particular, evaluating (2-7) at $t = 1$, which is a regular value by the Hopf maximum principle [Tolksdorf 1983, Proposition 3.2.1], we have that

$$\mathrm{C}_p(\Omega) = \frac{1}{|\mathbb{S}^{n-1}|} \left(\frac{p-1}{n-p} \right)^{p-1} \int_{\partial\Omega} |\mathrm{D}u_p|^{p-1} \, \mathrm{d}\sigma.$$

Moreover, one can actually relate the capacity of $\Omega_t = \{u > 1/t\} \cup \Omega$ to the capacity of Ω. The proof of the following lemma is contained in [Holopainen 1990, Lemma 3.8].

Proposition 2.9. *Let (M, g) be a complete noncompact p-nonparabolic Riemannian manifold for some $1 < p < n$. Let $\Omega \subset M$ be an open bounded subset with smooth boundary. Then a solution u_p to (1-4) realises*

$$\mathrm{C}_p(\Omega_t) = t^{p-1} \mathrm{C}_p(\Omega) \tag{2-9}$$

for every $t \in [1, +\infty)$, where $\Omega_t = \{u > 1/t\} \cup \Omega$. In particular, the map $t \mapsto \mathrm{C}_p(\Omega_t)$ is smooth.

2C. ***Li–Yau-type estimates.*** We provide a sharp lower estimate for the p-Green's function, extending the well-known

$$d_g(o,x)^{2-n} \leq G_2(o,x) \tag{2-10}$$

holding true for any couple of points o, x belonging to a 2-nonparabolic Riemannian manifolds with nonnegative Ricci curvature. The proof of (2-10) builds on the Laplacian comparison, which applies to showing that

$$\Delta\, d_g(o,\,\cdot\,)^{2-n} \geq 0$$

in the sense of distributions. This amounts to saying that

$$-\int_M \langle \mathrm{D} d_g(o,\cdot)^{2-n} \,|\, \mathrm{D}\psi\rangle \,\mathrm{d}\mu = \int_M d_g(o,\cdot)^{2-n} \Delta\psi \,\mathrm{d}\mu \geq 0 \tag{2-11}$$

for any test function $\psi \in \mathscr{C}^\infty_c(M)$. This leads to (2-10) substantially through the maximum principle. We refer the reader to [Agostiniani et al. 2020, Lemma 2.12] for details. The nonlinear version of (2-10), that, to our knowledge, has not been explicitly pointed out in literature yet, actually relies on (2-11) too.

Proposition 2.10 (sharp lower bound for the p-Green's function). *Let (M, g) be a complete p-nonparabolic Riemannian manifold*, $1 < p < n$, *with* $\mathrm{Ric} \geq 0$. *Let $o \in M$. Then, we have*

$$d_g(o,x)^{-\frac{n-p}{p-1}} \leq G_p(o,x) \tag{2-12}$$

for any $x \in M \setminus \{o\}$.

Proof. Fix for simplicity $o \in M$, and let $r(x) = d_g(o,x)$. We first show that $\Delta_p r^{-(n-p)/(p-1)} \geq 0$ holds in the weak sense, that is,

$$\int_M \langle |\mathrm{D} r^{-\frac{n-p}{p-1}}|^{p-2} \mathrm{D} r^{-\frac{n-p}{p-1}}, \mathrm{D}\psi\rangle \,\mathrm{d}\mu \leq 0$$

for any $\psi \in \mathscr{C}^\infty_c(M)$. In fact, we have

$$\begin{aligned}\int_M \langle |\mathrm{D} r^{-\frac{n-p}{p-1}}|^{p-2} \mathrm{D} r^{-\frac{n-p}{p-1}}, \mathrm{D}\psi\rangle \,\mathrm{d}\mu &= -\left(\frac{n-p}{p-1}\right)^{p-1} \int_M r^{1-n} \langle \mathrm{D} r, \mathrm{D}\psi\rangle \,\mathrm{d}\mu \\ &= \frac{1}{n-2}\left(\frac{n-p}{p-1}\right)^{p-1} \int_M \langle \mathrm{D} r^{2-n}, \mathrm{D}\psi\rangle \,\mathrm{d}\mu \leq 0,\end{aligned}$$

where the last inequality is the Laplacian comparison theorem (2-11).

Let now be $\delta > 0$. Since both $r^{-(n-p)/(p-1)}$ and G_p vanish at infinity, we have $r^{-(n-p)/(p-1)} \leq G_p + \delta$ on $\partial B(o,R)$ for any $R > 0$ big enough. On the other hand, the general result [Serrin 1964, Theorem 12] ensures that $G_p(o,x)$ is asymptotic to $r(x)^{-(n-p)/(p-1)}$ as $d_g(o,x) \to 0^+$, and thus we also get $r^{-(n-p)/(p-1)} \leq G_p + \delta$ on $\partial B(o,\varepsilon)$ for any $\varepsilon > 0$ small enough. Thus, applying the comparison principle to the subsolution $r^{-(n-p)/(p-1)}$ and to the solution $G_p + \delta$ (with respect to the p-Laplacian), in the annulus $B(o,R) \setminus \overline{B(o,\varepsilon)}$, we get $r^{-(n-p)/(p-1)} \leq G_p + \delta$ on such an annulus. Letting $\varepsilon \to 0^+$ and $R \to +\infty$, we deduce that the same holds on the whole $M \setminus \{o\}$. Finally, letting $\delta \to 0^+$, we are left with (2-12). □

Coupling (2-3) and (2-12) with the comparison principle, we deduce the following important estimate for the p-capacitary potential.

Theorem 2.11. *Let (M, g) be a complete p-nonparabolic Riemannian manifold for some for some $1 < p < n$, with $\mathrm{Ric} \geq 0$. Let $\Omega \subset M$ be a bounded subset with smooth boundary, and let u_p be its p-capacitary potential. Then, there exists a positive constant C_1 such that*

$$\mathrm{C}_1\, d_g(o, x)^{-\frac{n-p}{p-1}} \leq u_p(x) \tag{2-13}$$

for any $x \in M \smallsetminus \Omega$. If in addition (M, g) has Euclidean volume growth, then there also exists another positive constant C_2 such that

$$u_p(x) \leq \mathrm{C}_2\, d_g(o, x)^{-\frac{n-p}{p-1}}. \tag{2-14}$$

Proof. In light of (2-12) and (2-3), this one holding true if (M, g) satisfies the additional Euclidean volume growth assumption, it suffices to show that there exist positive constants C_1 and C_2 such that $\mathrm{C}_1 G_p \leq u_p \leq \mathrm{C}_2 G_p$. Choose any $\mathrm{C}_1 < 1/\sup_{\partial\Omega} u_p$. Then, $\mathrm{C}_1 G_p < u_p$ on $\partial\Omega$. Moreover, since both u_p and G_p vanish at infinity, for any $\delta > 0$ we have $\mathrm{C}_1 G_p < u_p + \delta$ on $\partial B(o, R)$ for any R big enough. The comparison principle applied to the p-harmonic functions $u_p + \delta$ and G_p in $B(o, R) \smallsetminus \overline{\Omega}$ shows that $\mathrm{C}_1 G_p < u + \delta$ in the latter subset. The radius R being arbitrarily big, this implies that, by passing to the limit as $R \to +\infty$, that $\mathrm{C}_1 G_p < u_p + \delta$ in the whole $M \smallsetminus \Omega$. Letting $\delta \to 0^+$ leaves us with $\mathrm{C}_1 G_p \leq u_p$, and consequently with (2-13). The inequality $u_p \leq \mathrm{C}_2 G_p$, yielding (2-14), is shown the same way. □

We now couple (2-13) with the general Cheng–Yau-type inequality for p-harmonic functions on manifolds with nonnegative Ricci curvature provided in [Wang and Zhang 2011]. It asserts that a p-harmonic function v, with $1 < p < n$ defined in a ball $B(o, 2R) \subset M$, where M is endowed with a Riemannian metric such that $\mathrm{Ric} \geq 0$, satisfies the estimate

$$\sup_{B(o,R)} \frac{|\mathrm{D}v|}{v} \leq \frac{\mathrm{C}}{R} \tag{2-15}$$

for a constant C depending only on the dimension of the ambient manifold and p. With these tools we immediately obtain:

Proposition 2.12. *Let (M, g) be a p-nonparabolic Riemannian manifold for some $1 < p < n$, with $\mathrm{Ric} \geq 0$. Let $\Omega \subset M$ be a bounded subset with smooth boundary, and let u_p be its p-capacitary potential. Then, there exists a positive constant C such that*

$$|\mathrm{D}u_p| u_p^{-\frac{n-1}{n-p}} \leq \mathrm{C} \tag{2-16}$$

holds on the whole $M \smallsetminus \Omega$.

Proof. By the $\mathscr{C}^1$-regularity of u_p, it clearly suffices to show that (2-16) holds outside some compact set containing $\overline{\Omega}$. Let then $o \in \Omega$ and $R > 0$ be such that $\Omega \subset B(o, R)$, and let $x \in M \smallsetminus \overline{B(o, 4R)}$. With this choice, we have $B(x, 2d_g(o, x) - 2R) \subset M \smallsetminus \overline{B(o, 2R)}$. Thus, applying inequality (2-15) to the function u_p, on $B(x, d_g(o,x)-R)$, we get

$$\frac{|\mathrm{D}u_p|}{u_p^{\frac{n-1}{n-p}}}(x) \leq \mathrm{C}\frac{u_p(x)}{d_g(o,x)-R}u_p^{-\frac{n-1}{n-p}}(x) \leq 2\mathrm{C}\frac{u_p^{-\frac{p-1}{n-p}}(x)}{d_g(o,x)}$$

and the rightmost-hand side is bounded by means of (2-13). □

2D. *Kato-type identity and a warped product splitting theorem.* Finally, we give the statement of the refined Kato-type identity for p-harmonic functions obtained in [Fogagnolo et al. 2019, Proposition 4.4], which will be at the core of the monotonicity and rigidity of F_p^β.

Definition 2.13 (geometry of level sets and orthogonal decomposition). Let (M,g) be a Riemannian manifold and v be a smooth function on M. At any point where $|\mathrm{D}v| \neq 0$ we denote by h and H respectively the second fundamental form and the mean curvature of the level set of u with respect to the unit normal $\mathrm{D}v/|\mathrm{D}v|$ and $g^\top$ the metric induced by g on the level set of u. Finally, for a given differentiable function f, we denote by $\mathrm{D}^\top f$ the tangential part of the gradient, according to the orthogonal decomposition

$$\mathrm{D}^\perp f = \left\langle \mathrm{D}f, \frac{\mathrm{D}v}{|\mathrm{D}v|}\right\rangle \frac{\mathrm{D}v}{|\mathrm{D}v|} \quad \text{and} \quad \mathrm{D}^\top f = \mathrm{D}f - \mathrm{D}^\perp f.$$

In particular, the following formula holds:

$$|\mathrm{D}|\mathrm{D}f||^2 = |\mathrm{D}^\top|\mathrm{D}f||^2 + |\mathrm{D}^\perp|\mathrm{D}f||^2.$$

We are now ready to state the Kato-type identity for p-harmonic function.

Proposition 2.14 (Kato-type identity). *Let (M,g) be a Riemannian manifold and let v be a p-harmonic function on some subset of M, $p>1$. Then, in an open neighbourhood of a point where $|\mathrm{D}v| \neq 0$, the following identity holds*:

$$|\mathrm{D}\mathrm{D}v|^2 - \left(1+\frac{(p-1)^2}{n-1}\right)|\mathrm{D}|\mathrm{D}v||^2 = |\mathrm{D}v|^2\left|\mathrm{h} - \frac{\mathrm{H}}{n-1}g^\top\right|^2 + \left(1-\frac{(p-1)^2}{n-1}\right)|\mathrm{D}^\top|\mathrm{D}v||^2,$$

according to the notation in Definition 2.13. Moreover, if, for some $t_0 \in \mathbb{R}$, $|\mathrm{D}v|>0$ and

$$\left|\mathrm{h} - \frac{\mathrm{H}}{n-1}g^\top\right|^2 = 0, \quad |\mathrm{D}^\top|\mathrm{D}v||^2 = 0$$

hold at each point of $\{v \geq t_0\}$, then the Riemannian manifold $(\{v \geq t_0\}, g)$ is isometric to the warped product $([t_0,+\infty) \times \{v=t_0\}, \mathrm{d}t \otimes \mathrm{d}t + \eta^2(t) g_{\{v=t_0\}})$, where the relation between v, η and t is given by

$$\eta(t) = \left(\frac{v'(t_0)}{v'(t)}\right)^{\frac{p-1}{n-1}}. \tag{2-17}$$

3. Monotonicity-rigidity theorems

In this section we are going to prove our *monotonicity formulas* in the p-nonparabolic setting. The results we present here are the natural extensions of the ones shown in [Agostiniani and Mazzieri 2020;

Agostiniani et al. 2020], as well as of the ones obtained in [Fogagnolo et al. 2019; Agostiniani et al. 2022a]. In the first two mentioned papers the authors established the monotonicity in the case of the harmonic potential, respectively in $\mathbb{R}^n$ and in a general 2-nonparabolic manifold with nonnegative Ricci curvature, whereas an analogous theory has been developed in the case of the p-capacitary potential in the Euclidean setting in the second two papers. More precisely, in [Fogagnolo et al. 2019], the authors worked out the smooth computations and took advantage of the fact that the p-capacitary potential associated with a convex domain is smooth and has no critical points (see [Colesanti et al. 2015; Lewis 1977]), whereas the main technical achievement in [Agostiniani et al. 2022a] is the treatment of the general case, when the critical points are present and even possibly arranged in sets of full measure. On the other hand, the approach presented in that work only produces *effective inequalities* (1-6), that are anyway sufficient to prove Theorem 1.1 in the flat setting, as mentioned in the Introduction. Here, we extend these results to the setting of p-nonparabolic manifolds and we improve them, establishing the full monotonicity of the integral quantities defined in (3-1) along the p-capacitary level sets flow.

As usual, the main difficulty amounts to ensuring that the monotonicity survives the singular values of u, that, as far as we know, could even form a set of positive measure. Inspired by the analysis in [Gigli and Violo 2023], where the authors were forced to face severe technical problems caused by the typical low regularity of the nonsmooth setting, we compute the derivative of our integral quantities (3-1) in the distributional sense, appealing to the full strength of the coarea formula in Proposition 2.1, and exploiting the integrability properties of the p-harmonic functions in Lemma 2.3.

From now on, except where it is necessary, we fix $1 < p < n$ *and we drop the subscript* p *when we consider a solution* u_p *to the problem* (1-4).

3A. *Statement of the monotonicity-rigidity theorems.* Let $u : M \smallsetminus \Omega \to \mathbb{R}$ be a solution of (1-4). For $\beta \in [0, +\infty)$ we consider the function

$$F_p^\beta(t) = t^{\beta \frac{(n-1)(p-1)}{(n-p)}} \int_{\{u=1/t\}} |\mathrm{D}u|^{(\beta+1)(p-1)} \, \mathrm{d}\sigma \tag{3-1}$$

defined for every $t \geq 1$ such that $|\{u = 1/t\} \cap \mathrm{Crit}(u)| = 0$, which is fulfilled for almost every $t \in [1, +\infty)$ by Proposition 2.1. We also set

$$F_p^\infty(t) = t^{\frac{n-1}{n-p}} \sup_{\{u=1/t\}} |\mathrm{D}u|, \tag{3-2}$$

which is defined on the whole $[1, +\infty)$. If $1/t$ is a regular value for u, then F_p^β is differentiable at t for every $\beta \in [0, +\infty)$ and its derivative is

$$(F_p^\beta)'(t) = -\beta t^{\beta \frac{(n-1)(p-1)}{(n-p)} - 2} \int_{\{u=1/t\}} |\mathrm{D}u|^{(\beta+1)(p-1)-1} \left(\mathrm{H} - \frac{(n-1)(p-1)}{(n-p)} |\mathrm{D} \log u| \right) \mathrm{d}\sigma. \tag{3-3}$$

As said before, the aim of this section is to prove monotonicity-rigidity theorems for $t \mapsto F_p^\beta(t)$ and $t \mapsto F_p^\infty(t)$.

Theorem 3.1 (monotonicity-rigidity theorem for F_p^β). *Let* (M, g) *be a* p*-nonparabolic Riemannian manifold with* $\mathrm{Ric} \geq 0$. *Let* $\Omega \subseteq M$ *be a bounded open subset with smooth boundary. Let* F_p^β *be the* *function defined in (3-1) with*

$$\frac{n-p}{n-1)(p-1)} < \beta < +\infty.$$

Then F_p^β belongs to $W^{2,1}(1,+\infty)$ and the identity

$$\begin{aligned}(F_p^\beta)'(t) = -\beta\left(\frac{(n-2)(p-1)}{(n-p)}\right)^{(\beta+1)(p-1)} &\int_{\{u\le 1/t\}\smallsetminus \mathrm{Crit}(u)} u^{2-\beta\frac{(p-1)(n-1)}{(n-p)}} |\mathrm{D}u|^{(\beta+1)(p-1)-1} \\ &\times\left\{\left[\beta-\frac{(n-p)}{(n-1)(p-1)}\right]\left[\mathrm{H}-\left[\frac{(n-1)(p-1)}{(n-p)}\right]|\mathrm{D}\log u|\right]^2\right. \\ &\left.+\left|\mathrm{h}-\frac{\mathrm{H}}{n-1}g^\top\right|^2+(p-1)\left[\beta+\frac{p-2}{p-1}\right]\frac{|\mathrm{D}^\top|\mathrm{D}u||^2}{|\mathrm{D}u|^2}+\mathrm{Ric}\left(\frac{\mathrm{D}u}{|\mathrm{D}u|},\frac{\mathrm{D}u}{|\mathrm{D}u|}\right)\right\}\mathrm{d}\mu \end{aligned} \tag{3-4}$$

holds for every $t\in[1,+\infty)$ and

$$\begin{aligned}(F_p^\beta)''(t) = \beta\left(\frac{(n-2)(p-1)}{(n-p)}\right)^{(\beta+1)(p-1)} t^{\beta\frac{(n-1)(p-1)}{(n-p)}-2} &\int_{\{u=1/t\}} |\mathrm{D}u|^{(\beta+1)(p-1)-2} \\ &\times\left\{\left[\beta-\frac{(n-p)}{(n-1)(p-1)}\right]\left[\mathrm{H}-\left[\frac{(n-1)(p-1)}{(n-p)}\right]|\mathrm{D}\log u|\right]^2\right. \\ &\left.+\left|\mathrm{h}-\frac{\mathrm{H}}{n-1}g^\top\right|^2+(p-1)\left[\beta+\frac{p-2}{p-1}\right]\frac{|\mathrm{D}^\top|\mathrm{D}u||^2}{|\mathrm{D}u|^2}+\mathrm{Ric}\left(\frac{\mathrm{D}u}{|\mathrm{D}u|},\frac{\mathrm{D}u}{|\mathrm{D}u|}\right)\right\}\mathrm{d}\mu \end{aligned} \tag{3-5}$$

holds for almost every $t\in[1,+\infty)$. In particular, F_p^β admits a convex and monotone nonincreasing $\mathscr{C}^1$ representative. Moreover, $(F_p^\beta)'(t_0)=0$ at some $t_0\ge 1$ such that $1/t_0$ is a regular value for u if and only if $(\{u\le 1/t_0\},g)$ is isometric to

$$\left([\tau_0,+\infty)\times\{u=1/t_0\},\ \mathrm{d}\tau\otimes\mathrm{d}\tau+\left(\frac{\tau}{\tau_0}\right)^2 g_{\{u=1/t_0\}}\right),\quad \textit{where } \tau_0=\left(\frac{|\{u=1/t_0\}|}{\mathrm{AVR}(g)|\mathbb{S}^{n-1}|}\right)^{\frac{1}{n-1}}.$$

In this case $\{u=1/t_0\}$ is a connected totally umbilical hypersurface with constant mean curvature in $(M\smallsetminus\Omega,g)$.

We also highlight that the rigidity statement is expressed in terms of the derivative. However, if $F_p^\beta(t)=F_p^\beta(T)$ for $1\le t<T<+\infty$ such that $1/t$ and $1/T$ are regular values for u, the rigidity statement still triggers. Indeed, since the set of regular values is open, monotonicity ensures the existence of a decreasing sequence $(t_j)_{j\in\mathbb{N}}$ such that $t_j\to t$ as $j\to+\infty$, $1/t_j$ is regular for u and $(F_p^\beta)'(t_j)=0$. Since $t\mapsto F_p^\beta(t)$ is smooth in a neighbourhood of t, this implies that $(F_p^\beta)'(t)=0$; hence the splitting of $\{u\le 1/t\}$.

Theorem 3.2 (monotonicity-rigidity theorem for F_p^∞). *Let (M,g) be a p-nonparabolic Riemannian manifold with* $\mathrm{Ric}\ge 0$. *Let $\Omega\subseteq M$ be a bounded open subset with smooth boundary. Let F_p^∞ be the function defined in (3-2). Then F_p^∞ is a continuous monotone nonincreasing function. Furthermore, we* *have*

$$\left[\mathrm{H}_g - \frac{(n-1)(p-1)}{(n-p)}|\mathrm{D}\log u|_g\right](x_t) = -(p-1)\frac{\partial}{\partial \nu_t}\log\frac{|\mathrm{D}u|_g}{u^{\frac{n-1}{n-p}}}(x_t) \geq 0, \tag{3-6}$$

where $x_t \in \{u = 1/t\}$ *is the point where* $\sup_{\{u=1/t\}}|\mathrm{D}u|_g/u^{(n-1)/(n-p)}$ *is achieved and* $\nu_t = -\mathrm{D}u/|\mathrm{D}u|_g$ *is the unit normal to* $\{u = 1/t\}$. *Moreover,* $F_p^\infty(t_0) = F_p^\infty(T)$ *for some* $t_0 < T$ *or the equality holds in* (3-6) *for some* t_0 *such that* $1/T$ *and* $1/t_0$ *are regular for* u *if and only if* $(\{u \leq 1/t_0\}, g)$ *is isometric to*

$$\left([\tau_0, +\infty) \times \{u = 1/t_0\},\ \mathrm{d}\tau \otimes \mathrm{d}\tau + \left(\frac{\tau}{\tau_0}\right)^2 g_{\{u=1/t_0\}}\right), \quad \text{where } \tau_0 = \left(\frac{|\{u = 1/t_0\}|}{\mathrm{AVR}(g)|\mathbb{S}^{n-1}|}\right)^{\frac{1}{n-1}}.$$

In this case $\{u = 1/t_0\}$ *is a connected totally umbilical hypersurface with constant mean curvature in* $(M \smallsetminus \Omega, g)$.

A direct consequence of the monotonicity of F_p^∞ is the following regularity theorem for the p-capacitary potential.

Corollary 3.3. *The function* F_p^∞ *is strictly positive. In particular, every level of* u *has at least one regular point.*

We want also to emphasise that these theorems can be applied in particular in $\mathbb{R}^n$ for every Ω open bounded with smooth boundary, where they naturally extend the monotonicity-rigidity theorems in [Fogagnolo et al. 2019; Agostiniani et al. 2022a].

We conclude this introduction by rewriting the functions F_p^β and F_p^∞ defined in (3-1) and (3-2) in a different formulation. We make use of this tool only to simplify computations, but as shown in [Agostiniani and Mazzieri 2020; Fogagnolo et al. 2019; Agostiniani et al. 2022b] monotonicity-rigidity theorems have their counterpart in this framework. Let (M, g) be a complete p-nonparabolic Riemannian manifold with $\mathrm{Ric} \geq 0$ and $u : M \smallsetminus \Omega \to \mathbb{R}$ be the solution of the problem (1-4). We consider the conformally related Riemannian manifold $(M \smallsetminus \Omega, \tilde{g})$, where $\tilde{g}$ is given by

$$\tilde{g} = u^{2\left(\frac{p-1}{n-p}\right)} g. \tag{3-7}$$

It is also convenient to consider the new variable

$$\varphi = -\frac{(p-1)(n-2)}{(n-p)}\log u, \tag{3-8}$$

so that the metric $\tilde{g}$ can be equivalently rewritten as

$$\tilde{g} = \mathrm{e}^{-\frac{2\varphi}{n-2}} g.$$

With the same formal computation as in [Fogagnolo et al. 2019], one can prove that $\Delta_{\tilde{g}}^p \varphi = 0$ on $M \smallsetminus \overline{\Omega}$ where $\Delta_{\tilde{g}}^p$ is the p-Laplace operator with respect to the metric $\tilde{g}$.

From now on, given (M, g) *a* p*-nonparabolic manifold with* $\mathrm{Ric} \geq 0$ *and* u *a solution to* (1-4), φ *will be the function obtained by* u *through* (3-8), *whereas* $\tilde{g}$ *will indicate the metric on* $M \smallsetminus \Omega$ *obtained from* u *and* g *through* (3-7).

The gradient of φ is related to the one of u by

$$|\nabla\varphi|_{\tilde{g}} = \frac{(n-2)(p-1)}{(n-p)} \frac{|\mathrm{D}u|_g}{u^{\frac{n-1}{n-p}}}, \tag{3-9}$$

where ∇ is the Levi-Civita connection associated to the metric $\tilde{g}$. We can observe that if t is a regular value for u then $s = -[(p-1)(n-2)/(n-p)]\log t$ is a regular value for φ, thanks to (3-8) and the previous relation. Moreover, we recognise from the above expression and the estimate (2-16) that the fundamental property of $|\nabla\varphi|_{\tilde{g}}$ is uniformly bounded, that is, there exists a constant C such that

$$|\nabla\varphi|_{\tilde{g}} \leq \mathrm{C} \tag{3-10}$$

on the whole $M \smallsetminus \Omega$.

Using (3-9), the family of functions $t \mapsto F_p^\beta(t)$ for $\beta \in [0, +\infty]$ defined in (3-1) and (3-2) can be rewritten in terms of $\tilde{g}$ and φ obtained through (3-8) and (3-7). For any $\beta \in [0, +\infty)$ we can now consider the function

$$\Phi_p^\beta(s) = \int_{\{\varphi=s\}} |\nabla\varphi|_{\tilde{g}}^{(\beta+1)(p-1)} \, \mathrm{d}\sigma_{\tilde{g}}, \tag{3-11}$$

whenever $s \geq 0$ is such that $|\{\varphi = s\} \cap \mathrm{Crit}(\varphi)|$. Correspondingly we set

$$\Phi_p^\infty(s) = \sup_{\{\varphi=s\}} |\nabla\varphi|_{\tilde{g}}, \tag{3-12}$$

which is defined on the whole $[0, +\infty)$. The function Φ_p^β can be obtained from F_p^β through a change of variable, that is,

$$\Phi_p^\beta(s) = F_p^\beta(\mathrm{e}^{\frac{(n-p)}{(p-1)(n-2)}s}).$$

For $\beta < +\infty$ it thus holds that

$$(\Phi_p^\beta)'(s) = \frac{(n-p)}{(p-1)(n-2)} \mathrm{e}^{\frac{(n-p)}{(p-1)(n-2)}s} (F_p^\beta)'(\mathrm{e}^{\frac{(n-p)}{(p-1)(n-2)}s})$$

for almost every $s \in [0, +\infty)$. The previous relations reveal how proving the monotonicity results for F_p^β and F_p^∞, stated in Theorems 3.1 and 3.2, are equivalent to show the same one for Φ_p^β and Φ_p^∞. The same argument applies for the regularity of F_p^β.

3B. ***Proof of monotonicity-rigidity theorems.*** A basic property we will need is the essential uniform boundedness of Φ_p^β of Φ_p^∞ defined in (3-11) and (3-12).

Lemma 3.4. *Let be $1 < p < n$, and (M, g) be a p-nonparabolic Riemannian manifold. Let $\Omega \subset M$ be a open bounded subset with smooth boundary. For every $\beta \in [0, +\infty)$, Φ_p^β is essentially uniformly bounded, namely $\Phi_p^\beta(s) \leq C$ for almost every $s \in [0, +\infty)$, including any s that is regular for φ. Moreover, the function Φ_p^∞ is uniformly bounded.*

Proof. It suffices to write Φ_p^β as

$$\Phi_p^\beta(s) = \int_{\{\varphi=s\}} |\nabla\varphi|_{\tilde g}^{(\beta+1)(p-1)}\, d\sigma_{\tilde g} \le C^{\beta(p-1)} \int_{\{\varphi=s\}} |\nabla\varphi|_{\tilde g}^{p-1}\, d\sigma_{\tilde g}$$
$$= C^{\beta(p-1)} \left[\frac{(n-2)(p-1)}{(n-p)}\right]^{p-1} \int_{\{u=1/t\}} |Du|^{p-1}\, d\sigma,$$

where C is the constant in (3-10), the last identity is due to (3-9) and (3-8) taking

$$s = -\left[\frac{(p-1)(n-2)}{n-p}\right] \log t.$$

By (2-7) we have that the integral on the rightmost-hand side coincides with $\mathrm{Cap}_p(\Omega)$ for almost any t, including any of those such that $1/t$ is a regular value for u. This settles the boundedness of Φ_p^β for finite β. On the other hand the uniform boundedness of Φ_p^∞ is a direct consequence of (3-10) alone. □

From now on, we will drop the subscript $\tilde g$ whenever it is clear to which metric we are referring.

Suppose by now that $\beta \in [0, +\infty)$ and consider the vector field

$$X = e^{-\frac{(n-p)}{(n-2)(p-1)}\varphi} |\nabla\varphi|^{p-2}\big(\nabla|\nabla\varphi|^{\beta(p-1)} + (p-2)\nabla^\perp|\nabla\varphi|^{\beta(p-1)}\big), \tag{3-13}$$

defined in a neighbourhood of each point such that $|\nabla\varphi| > 0$, where the function φ is actually smooth, being p-harmonic with respect to the metric $\tilde g$. This vector field is related to the derivative of Φ_p^β through the following identity.

Proposition 3.5. *Let (M, g) be a p-nonparabolic Riemannian manifold with* Ric ≥ 0. *For every $\beta \in [0, +\infty)$, the function $s \mapsto \Phi_p^\beta(s)$ defined in (3-11) belongs to $W^{1,1}_{\mathrm{loc}}(0, +\infty)$ and its derivative is given by*

$$e^{-\frac{(n-p)}{(n-2)(p-1)}s} (\Phi_p^\beta)'(s) = \frac{1}{p-1} \int_{\{\varphi=s\}} \left\langle X, \frac{\nabla\varphi}{|\nabla\varphi|}\right\rangle d\sigma \tag{3-14}$$

for almost every $s \in [0, +\infty)$, where X is the vector field defined in (3-13).

Before starting the proof, observe that the quantity appearing in the left-hand side of (3-14) is actually well-defined for almost every $s \in (0, +\infty)$ even if X is a priori defined only where $|\nabla\varphi| > 0$. Indeed, by Proposition 2.1 $|\mathrm{Crit}\,\varphi \cap \{\varphi = s\}| = 0$ for almost every $s \in (0, +\infty)$.

Proof. By the definition of X, it is easy to check that

$$e^{-\frac{(n-p)}{(n-2)(p-1)}\varphi} \left\langle |\nabla\varphi|^{p-2}\nabla|\nabla\varphi|^{\beta(p-1)}, \frac{\nabla\varphi}{|\nabla\varphi|}\right\rangle = \frac{1}{p-1}\left\langle X, \frac{\nabla\varphi}{|\nabla\varphi|}\right\rangle$$

holds around each point such that $|\nabla\varphi| \ne 0$. Hence, it remains only to prove that $\Phi_p^\beta(s) \in W^{1,1}_{\mathrm{loc}}(0 + \infty)$ and that

$$(\Phi_p^\beta)'(s) = \int_{\{\varphi=s\}} \left\langle |\nabla\varphi|^{p-2}\nabla|\nabla\varphi|^{\beta(p-1)}, \frac{\nabla\varphi}{|\nabla\varphi|}\right\rangle d\sigma$$

holds for almost any $s \in (0, \infty)$. Let $\eta \in \mathscr{C}_c^\infty(0, +\infty)$. Since $|\nabla\varphi| \leq C$ by (3-10), applying the coarea formula (2-2) with $f = |\nabla\varphi|^{(\beta+1)(p-1)}$ and the chain rule we obtain that

$$\begin{aligned}\int_0^{+\infty} \eta'(s)\Phi_p^\beta(s)\,\mathrm{d}s &= \int_0^{+\infty} \eta'(s) \int_{\{\varphi=s\}} |\nabla\varphi|^{(\beta+1)(p-1)}\,\mathrm{d}\sigma\,\mathrm{d}s \\ &= \int_{M\smallsetminus\bar{\Omega}} \eta'(s)\langle\nabla\varphi, \nabla\varphi\rangle|\nabla\varphi|^{(\beta+1)(p-1)-1}\,\mathrm{d}\mu \\ &= \int_{M\smallsetminus\bar{\Omega}} \langle\nabla(\eta(\varphi)), \nabla\varphi\rangle|\nabla\varphi|^{(\beta+1)(p-1)-1}\,\mathrm{d}\mu.\end{aligned}$$

Integrating by parts the right-hand side, $\Delta^{(p)}\varphi = 0$ yields

$$\int_0^{+\infty} \eta'(s)\Phi_p^\beta(s)\,\mathrm{d}s = -\int_{M\smallsetminus\bar{\Omega}} \eta(\varphi)\langle|\nabla\varphi|^{p-2}\nabla|\nabla\varphi|^{\beta(p-1)}, \nabla\varphi\rangle\,\mathrm{d}\mu.$$

Thanks to (3-10) and Lemma 2.3, we are in position to apply the coarea formula in Proposition 2.1 with $f = \langle|\nabla\varphi|^{p-2}\nabla|\nabla\varphi|^{\beta(p-1)}, \nabla\varphi\rangle/|\nabla\varphi|$ (see Remark 2.2) to get

$$\int_0^{+\infty} \eta'(s)\Phi_p^\beta(s)\,\mathrm{d}s = -\int_0^1 \eta(s) \int_{\{\varphi=s\}} \left\langle|\nabla\varphi|^{p-2}\nabla|\nabla\varphi|^{\beta(p-1)}, \frac{\nabla\varphi}{|\nabla\varphi|}\right\rangle \mathrm{d}\sigma\,\mathrm{d}s,$$

which ensures both that $\Phi_p^\beta \in W^{1,1}_{\mathrm{loc}}(0, +\infty)$ and (3-14). □

The nonnegative divergence of X is what substantially rules the monotonicity of Φ_p^β, and this is true when β ranges in a suitable set of parameters.

Lemma 3.6 (divergence of X). *Let (M, g) be a p-nonparabolic manifold and X be the vector field defined in (3-13). Then*

$$\operatorname{div} X = \mathrm{e}^{-\frac{(n-p)}{(n-2)(p-1)}\varphi} Q$$

holds at any point such that $|\nabla\varphi| > 0$, with

$$\begin{aligned}Q = \beta(p-1)|\nabla\varphi|^{\beta(p-1)+p-2}\Bigg\{\left|\mathrm{h} - \frac{\mathrm{H}}{n-1}\tilde{g}^\top\right|^2 + (p-1)\left[\beta + \frac{p-2}{p-1}\right]\frac{|\nabla^\top|\nabla\varphi||^2}{|\nabla\varphi|^2} \\ + (p-1)^2\left[\beta - \frac{(n-p)}{(p-1)(n-1)}\right]\frac{|\nabla^\perp|\nabla\varphi||^2}{|\nabla\varphi|^2} + \mathrm{Ric}_g\left(\frac{\nabla\varphi}{|\nabla\varphi|^2}, \frac{\nabla\varphi}{|\nabla\varphi|^2}\right)\Bigg\},\end{aligned} \tag{3-15}$$

where h *and* H *are respectively the second fundamental form and the mean curvature of the level sets of φ with respect to the unit normal $\nabla\varphi/|\nabla\varphi|$, $\nabla^\top$ is defined in Definition 2.13 and* Ric_g *denotes the Ricci tensor of the background metric. In particular,*

$$\operatorname{div}(X) \geq 0 \quad \textit{for } \frac{n-p}{(n-1)(p-1)} \leq \beta < +\infty.$$

Proof. The proof follows the same lines of [Agostiniani et al. 2022a, Lemma 4.1], replacing accordingly the vector fields $W = |\nabla\varphi|^{p-2}\nabla|\nabla\varphi|^{\beta(p-1)}$ and $Z = (p-2)|\nabla\varphi|^{p-2}\nabla^\perp|\nabla\varphi|^{\beta(p-1)}$. The Ricci curvature term appears computing the divergence of W thanks to the Bochner identity for p-harmonic functions, as the reader can see following [Fogagnolo et al. 2019, Proposition 4.3]. □

Suppose that $|\nabla\varphi| \neq 0$ everywhere. We can apply the divergence theorem in the domain $\{s < \varphi < S\}$ to obtain

$$\int_{\{\varphi=S\}} \left\langle X, \frac{\nabla\varphi}{|\nabla\varphi|}\right\rangle \mathrm{d}\sigma - \int_{\{\varphi=s\}} \left\langle X, \frac{\nabla\varphi}{|\nabla\varphi|}\right\rangle \mathrm{d}\sigma = \int_{\{s<\varphi<S\}} \operatorname{div} X \,\mathrm{d}\mu \geq 0. \tag{3-16}$$

Using (3-14) we deduce that

$$\mathrm{e}^{-\frac{(n-p)}{(n-2)(p-1)}s}(\Phi_p^\beta)'(s) \leq \mathrm{e}^{-\frac{(n-p)}{(n-2)(p-1)}S}(\Phi_p^\beta)'(S).$$

This almost concludes the proof of the monotonicity theorem for Φ_p^β with

$$\frac{n-p}{(n-1)(p-1)} < \beta < +\infty,$$

assuming the absence of critical points. Indeed, by integrating it, monotonicity will follow as in [Fogagnolo et al. 2019, Theorem 3.4]. This case lies in the same trail blazed in [Agostiniani and Mazzieri 2020] since if $|\nabla\varphi| \neq 0$, the p-Laplace operator is elliptic nondegenerate, and thus the techniques used for harmonic functions fit perfectly.

If we want to pursue the previous path, even when the critical set of φ is not empty, we are first committed to providing a version of (3-16) that holds even in presence of critical values. The main issue is that $\operatorname{div}(X)$ does not belong to L^1_{loc} a priori. Following the same lines of [Gigli and Violo 2023, Proposition 4.6], testing $s \mapsto \mathrm{e}^{-s(n-p)/((n-2)(p-1))}(\Phi_p^\beta)'(s)$ against nonnegative functions $\eta \in \mathscr{C}_c^\infty(0, +\infty)$ and using the coarea formula Proposition 2.1 for $f = \langle X, \nabla\varphi/|\nabla\varphi|\rangle(1 - \chi_{\mathrm{Crit}\,\varphi})$, one gets

$$(p-1)\int_0^{+\infty} \eta'(s)\mathrm{e}^{-\frac{(n-p)}{(n-2)(p-1)}s}(\Phi_p^\beta)'(s)\,\mathrm{d}s = \int_{M\smallsetminus\mathrm{Crit}(\varphi)} \langle X, \nabla[\eta(\varphi)]\rangle\,\mathrm{d}\mu.$$

We now would like to integrate by parts and use the nonnegativity of $\operatorname{div}(X)$ outside the critical set of φ. In doing this, we are hampered by the fact that $\operatorname{div}(\chi_{M\smallsetminus\mathrm{Crit}\,\varphi}X)$ is actually a measure that is possibly not absolutely continuous. Hence we can aim to prove that $s \mapsto \mathrm{e}^{-s(n-p)/((n-2)(p-1))}(\Phi_p^\beta)'(s)$ belongs to $\mathrm{BV}_{\mathrm{loc}}(0, +\infty)$, but not the absolute continuity. Differently from the nonsmooth case, we can here employ the higher regularity of φ outside its critical set to refine the result.

Proposition 3.7. *Let (M, g) be a p-nonparabolic Riemannian manifold with* $\mathrm{Ric} \geq 0$. *Let $\Omega \subseteq M$ be an open bounded subset with smooth boundary. For every*

$$\frac{n-p}{(n-1)(p-1)} < \beta < +\infty,$$

the function $s \mapsto \mathrm{e}^{-s(n-p)/((n-2)(p-1))}(\Phi_p^\beta)'(s)$ defined in (3-14) belongs to $W^{1,1}_{\mathrm{loc}}(0, +\infty)$ and its derivative is given by

$$\left(\mathrm{e}^{-\frac{(n-p)}{(n-2)(p-1)}s}(\Phi_p^\beta)'(s)\right)' = \frac{1}{p-1}\int_{\{\varphi=s\}} \frac{\operatorname{div} X}{|\nabla\varphi|}\,\mathrm{d}\sigma \tag{3-17}$$

for almost every $s \in [0, +\infty)$, where X is the vector field defined in (3-13).

We remark again that the quantity appearing in the left-hand side of (3-17) is actually well-defined for almost every $s \in (0, +\infty)$ even if X is a priori defined only where $|\nabla\varphi| > 0$. Indeed, by Proposition 2.1

$|\mathrm{Crit}\,\varphi \cap \{\varphi = s\}| = 0$ for almost every $s \in (0, +\infty)$. Moreover, since $\varphi \in \mathscr{C}^\infty$ around each point where $|\nabla\varphi| > 0$, the field X is smooth around such points; thus its divergence can be classically computed.

Proof. Proposition 3.7 follows if we prove that $\operatorname{div}(X)(1 - \chi_{\mathrm{Crit}(\varphi)})$ belongs to $L^1_{\mathrm{loc}}(M \smallsetminus \overline{\Omega})$ and

$$(p-1)\int_0^{+\infty} \eta'(s) \mathrm{e}^{-\frac{(n-p)}{(n-2)(p-1)}s} (\Phi_p^\beta)'(s)\,\mathrm{d}s = -\int_{M \smallsetminus \mathrm{Crit}\,\varphi} \eta(\varphi) \operatorname{div} X \,\mathrm{d}\mu \tag{3-18}$$

holds for every $\eta \in \mathscr{C}^\infty_c(0, +\infty)$. By the coarea formula in Proposition 2.1, with $f = \operatorname{div}(X)(1 - \chi_{\mathrm{Crit}(\varphi)})$, we would get

$$\int_{M \smallsetminus \mathrm{Crit}\,\varphi} \eta(\varphi) \operatorname{div} X \,\mathrm{d}\mu = \int_0^{+\infty} \eta(s) \int_{\{\varphi = s\}} \frac{\operatorname{div} X}{|\nabla\varphi|} \,\mathrm{d}\sigma\,\mathrm{d}t,$$

which implies both that $\mathrm{e}^{-(n-p)s/(n-2)(p-1)}(\Phi_p^\beta)' \in W^{1,1}_{\mathrm{loc}}(0, +\infty)$ and (3-17).

Step 1: proof for nonnegative η. Let $\eta \in \mathscr{C}^\infty_c(0, +\infty)$ be nonnegative. For any given $\varepsilon > 0$ consider the smooth nonnegative cut-off function $\chi_\varepsilon : [0, +\infty) \to \mathbb{R}$ defined as

$$\begin{cases} \chi_\varepsilon(t) = 0 & \text{in } t < \frac{1}{2}\varepsilon, \\ 0 < \chi_\varepsilon'(t) \leq 2\varepsilon^{-1} & \text{in } \frac{1}{2}\varepsilon \leq t \leq \frac{3}{2}\varepsilon, \\ \chi_\varepsilon(t) = 1 & \text{in } t > \frac{3}{2}\varepsilon. \end{cases}$$

Consider accordingly the vector field $X_\varepsilon = \chi_\varepsilon(|\nabla\varphi|^{\beta(p-1)})X$, where X is the vector field given in (3-13). Let $\eta \in \mathscr{C}^\infty_c(0, +\infty)$ be nonnegative. We notice that $|\langle X_\varepsilon, \nabla\varphi\rangle| \leq |\langle X, \nabla\varphi\rangle|$ which is in $L^2_{\mathrm{loc}}(M \smallsetminus \overline{\Omega})$ by (3-10) and Lemma 2.3. Hence (3-14), the coarea formula with $f = \eta'(\varphi)\langle X, \nabla\varphi/|\nabla\varphi|\rangle$ and the dominated convergence theorem imply

$$\int_0^{+\infty} \eta'(s) \mathrm{e}^{-\frac{(n-p)}{(n-2)(p-1)}s} (\Phi_p^\beta)'(s)\,\mathrm{d}s = \lim_{\varepsilon \to 0^+} \frac{1}{p-1} \int_M \eta'(\varphi)\langle X_\varepsilon, \nabla\varphi\rangle \,\mathrm{d}\mu.$$

Employing the coarea formula in (2-2) with $f = \langle X_\varepsilon, \nabla\varphi/|\nabla\varphi|\rangle$ and integration by parts, we obtain that

$$\begin{aligned} \int_0^{+\infty} &\eta'(s) \int_{\{\varphi = s\}} \left\langle X_\varepsilon, \frac{\nabla\varphi}{|\nabla\varphi|}\right\rangle \mathrm{d}\sigma\,\mathrm{d}s \\ &= \int_M \eta'(\varphi)\langle X_\varepsilon, \nabla\varphi\rangle \,\mathrm{d}\mu = -\int_M \operatorname{div}(X_\varepsilon)\eta(\varphi)\,\mathrm{d}\mu \\ &= -\int_{M \smallsetminus N_{\varepsilon/2}} \eta(\varphi)\chi_\varepsilon(|\nabla\varphi|^{\beta(p-1)}) \operatorname{div} X \,\mathrm{d}\mu - \int_{N_{3\varepsilon/2} \smallsetminus N_{\varepsilon/2}} \eta(\varphi)\chi_\varepsilon'(|\nabla\varphi|^{\beta(p-1)})\langle X, \nabla|\nabla\varphi|^{\beta(p-1)}\rangle \,\mathrm{d}\mu, \end{aligned}$$

where $N_\delta = \{|\nabla\varphi|^{\beta(p-1)} < \delta\}$ for every $\delta > 0$. By the monotone convergence theorem, the first integral in the rightmost-hand side gives

$$\lim_{\varepsilon \to 0^+} \int_{M \smallsetminus N_{\varepsilon/2}} \eta(\varphi)\chi_\varepsilon(|\nabla\varphi|^{\beta(p-1)}) \operatorname{div} X \,\mathrm{d}\mu = \int_{M \smallsetminus \mathrm{Crit}(\varphi)} \eta(\varphi) \operatorname{div} X \,\mathrm{d}\mu \geq 0.$$

To achieve Step 1, it thus remains to prove that the second integral vanishes as $\varepsilon \to 0^+$. Observe that the integral in question is always nonnegative, as $\langle X, \nabla|\nabla\varphi|^{\beta(p-1)}\rangle \geq 0$, $\eta \geq 0$ and $\chi_\varepsilon' \geq 0$. Hence, we only need to estimate it from above with a quantity that vanishes as $\varepsilon \to 0^+$. Since $|\nabla\varphi|^{\beta(p-1)} \geq \varepsilon/2$ on $N_{3\varepsilon/2} \setminus N_{\varepsilon/2}$, φ is smooth in such a region. The coarea formula in Proposition 2.1 and $\chi_\varepsilon' \leq 2/\varepsilon$ would give

$$\int_{N_{3\varepsilon/2}\setminus N_{\varepsilon/2}} \eta(\varphi)\chi_\varepsilon'(|\nabla\varphi|^{\beta(p-1)})\langle X, \nabla|\nabla\varphi|^{\beta(p-1)}\rangle \,\mathrm{d}\mu \leq \frac{2}{\varepsilon}\int_{\varepsilon/2}^{3\varepsilon/2}\int_{\partial N_s} \frac{\langle \eta(\varphi)X, \nabla|\nabla\varphi|^{\beta(p-1)}\rangle}{|\nabla|\nabla\varphi|^{\beta(p-1)}|}\,\mathrm{d}\sigma\,\mathrm{d}s. \tag{3-19}$$

However, to apply Proposition 2.1 without further specifications, the set $N_{3\varepsilon/2} \setminus N_{\varepsilon/2}$ should be compactly contained in $M \setminus \overline{\Omega}$ for every $\varepsilon > 0$ small enough. Since $|\nabla\varphi| > 0$ on $\partial\Omega$ and $\varphi \in \mathscr{C}^{1,\beta}_{\mathrm{loc}}(M \setminus \Omega)$, it is clear that the set $N_{3\varepsilon/2} \setminus N_{\varepsilon/2}$ does not touch $\partial\Omega$. Nonetheless, it could be unbounded. This is not a real issue since we are integrating $\eta(\varphi)$, which has compact support. More rigorously, choose $S > 0$ such that $\eta(s) = 0$ for every $s \geq S$. Let $\xi : \mathbb{R} \to [0, 1]$ be a smooth cut-off function such that $\xi = 1$ on $[0, S]$ and $\xi = 0$ on $[2S, +\infty)$. Observe that the function $\xi(\varphi)|\nabla\varphi|^{\beta(p-1)} + (1-\xi(\varphi))$ is smooth outside $\operatorname{Crit}\varphi$, its sublevels $\widetilde{N}_\delta$ are compact for $\delta < 1$ and its gradient coincides with $\nabla|\nabla\varphi|^{\beta(p-1)}$ on the support of $\eta(\varphi)$. Moreover, one can replace N_δ with $\widetilde{N}_\delta$ in both sides of (3-19) without changing the value of the integrals. Indeed, such sets coincide on the support of $\eta(\varphi)$, where integrations are actually performed. Hence, (3-19) holds. Up to the end of this step, we will implicitly use this truncation argument when the coarea formula is applied.

Let $0 < R < 1$ be a regular value for $|\nabla\varphi|$. Define $\mathcal{H}$ as

$$\mathcal{H}(r) = \int_{\partial N_r} \frac{\langle \eta(\varphi)X, \nabla|\nabla\varphi|^{\beta(p-1)}\rangle}{|\nabla|\nabla\varphi|^{\beta(p-1)}|}\,\mathrm{d}\sigma \geq 0$$

for every $r \in (0, R)$ that is a regular value of $|\nabla\varphi|$, hence for almost every $r \in (0, R)$ thanks to Sard's theorem applied to the smooth function $|\nabla\varphi|$. We claim that $\mathcal{H}(r)$ vanishes as $r \to 0^+$. This is enough for Step 1, since it would give

$$\frac{2}{\varepsilon}\int_{\varepsilon/2}^{3\varepsilon/2}\int_{\partial N_s} \frac{\langle \eta(\varphi)X, \nabla|\nabla\varphi|^{\beta(p-1)}\rangle}{|\nabla|\nabla\varphi|^{\beta(p-1)}|}\,\mathrm{d}\sigma\,\mathrm{d}s \leq 2 \sup_{r\in[\frac{\varepsilon}{2},\frac{3\varepsilon}{2}]} \mathcal{H}(r) \to 0$$

as $\varepsilon \to 0^+$.

Let $0 < t < r < R$ be two regular values for $|\nabla\varphi|$, applying the divergence theorem to the smooth vector field X on $N_r \setminus N_t$ we get

$$\begin{aligned}\mathcal{H}(r) - \mathcal{H}(t) &= \int_{\partial N_r} \frac{\langle \eta(\varphi)X, \nabla|\nabla\varphi|^{\beta(p-1)}\rangle}{|\nabla|\nabla\varphi|^{\beta(p-1)}|}\,\mathrm{d}\sigma - \int_{\partial N_t} \frac{\langle \eta(\varphi)X, \nabla|\nabla\varphi|^{\beta(p-1)}\rangle}{|\nabla|\nabla\varphi|^{\beta(p-1)}|}\,\mathrm{d}\sigma \\ &= \int_{N_r\setminus N_t} \operatorname{div}(\eta(\varphi)X)\,\mathrm{d}\mu = \int_{N_r\setminus N_t} \eta(\varphi)\operatorname{div}(X)\,\mathrm{d}\mu + \int_{N_r\setminus N_t} \langle X, \nabla\varphi\rangle\eta'(\varphi)\,\mathrm{d}\mu.\end{aligned} \tag{3-20}$$

Since $\operatorname{Ric} \geq 0$ and

$$|\nabla\varphi|^2\left|\mathrm{h} - \frac{\mathrm{H}}{n-1}g^\top\right|^2 \geq 0,$$

by (3-15) we have that

$$
\begin{aligned}
\operatorname{div} X \geq{}& \beta(p-1)^2 \mathrm{e}^{-\frac{(n-p)}{(p-1)(n-2)}\varphi} |\nabla\varphi|^{\beta(p-1)+p-4} \\
&\times\left(\left[\beta+\frac{p-2}{p-1}\right]|\nabla^{\top}|\nabla\varphi||^2+(p-1)\left[\beta-\frac{(n-p)}{(p-1)(n-1)}\right]|\nabla^{\perp}|\nabla\varphi||^2\right) \\
\geq{}& \mathrm{C}\,\beta^2 p(p-1)^2 \mathrm{e}^{-\frac{(n-p)}{(p-1)(n-2)}\varphi}|\nabla\varphi|^{\beta(p-1)+p-4}(|\nabla^{\perp}|\nabla\varphi||^2+|\nabla^{\top}|\nabla\varphi||^2) \\
\geq{}& \mathrm{C}\,p\,\mathrm{e}^{-\frac{(n-p)}{(p-1)(n-2)}\varphi}\frac{|\nabla\varphi|^{p-2}|\nabla|\nabla\varphi|^{\beta(p-1)}|^2}{|\nabla\varphi|^{\beta(p-1)}} \geq \mathrm{C}\,\frac{\langle X, \nabla|\nabla\varphi|^{\beta(p-1)}\rangle}{|\nabla\varphi|^{\beta(p-1)}},
\end{aligned}
$$

where

$$
\mathrm{C}=\frac{1}{p\beta}\min\left\{\left[\beta+\frac{p-2}{p-1}\right],\,(p-1)\left[\beta-\frac{(n-p)}{(p-1)(n-1)}\right]\right\}>0.
$$

If we plug the above estimate into (3-20) and use the coarea formula in Proposition 2.1 with $f=|\nabla\varphi|^{-\beta(p-1)+p-2}\big|\nabla|\nabla\varphi|^{\beta(p-1)}\big|$, we get

$$
\mathcal{H}(r)-\mathcal{H}(t)-\int_{N_r\smallsetminus N_t}\langle X,\nabla\varphi\rangle\eta'(\varphi)\,\mathrm{d}\mu \geq \mathrm{C}\int_t^r\frac{\mathcal{H}(s)}{s}\,\mathrm{d}s. \tag{3-21}
$$

On the other hand, the map

$$
t\mapsto\mathcal{G}(t)=\int_{N_t\smallsetminus\operatorname{Crit}\varphi}\langle X,\nabla\varphi\rangle\eta'(\varphi)\,\mathrm{d}\mu
$$

is a well-defined bounded function in $\mathscr{C}^0([0,R])$. Indeed, $\eta'(\varphi)$ has compact support and

$$
|\langle X,\nabla\varphi\rangle|\leq\beta(p-1)|\nabla\varphi|^{\beta(p-1)}|\nabla|\nabla\varphi|^{p-1}|\in L^2_{\mathrm{loc}}(N_R\smallsetminus\operatorname{Crit}\varphi)
$$

by Lemma 2.3. Equation (3-21) states that $t\mapsto\mathcal{H}(t)-\mathcal{G}(t)$ is monotonically increasing, whereas $\mathcal{H}(s)\geq 0$ for almost every $s\in(0,r)$. Thus, $t\mapsto\mathcal{H}(t)=\mathcal{H}(t)-\mathcal{G}(t)+\mathcal{G}(t)$ admits a limit as $t\to 0^+$, being the sum of a monotone and a continuous function. Denote by $\mathcal{H}(0)$ such a limit. Since $\mathcal{G}(t)\to 0$ as $t\to 0^+$, by dominated convergence theorem and $\mathcal{H}(0)\geq 0$, we have

$$
\mathcal{H}(R)-\mathcal{G}(R)\geq[\mathcal{H}(R)-\mathcal{G}(R)]-[\mathcal{H}(0)-\mathcal{G}(0)]\geq\mathrm{C}\int_0^R\frac{\mathcal{H}(s)}{s}\,\mathrm{d}s.
$$

Hence $\mathcal{H}(s)\to 0$ as $s\to 0^+$; otherwise $\mathcal{H}(s)/s$ would not belong to $L^1(0,R)$, contradicting the boundedness of $\mathcal{H}(R)-\mathcal{G}(R)$.

Step 2: conclusions. In the previous step we proved (3-18) for every nonnegative function $\eta\in\mathscr{C}^\infty_c(0,+\infty)$. Let be $K\subset M\smallsetminus\overline{\Omega}$. Then, there exists an $\eta_K\in\mathscr{C}^\infty_c(0,+\infty)$, $\eta_K\geq 0$, such that $\eta_K(\varphi)\geq 1$ on K. In particular, since $\operatorname{div}(X)\geq 0$ outside $\operatorname{Crit}(\varphi)$ we have

$$
\begin{aligned}
\int_K\operatorname{div}(X)(1-\chi_{\operatorname{Crit}(\varphi)})\,\mathrm{d}\mu &\leq \int_{M\smallsetminus\operatorname{Crit}(\varphi)}\eta_K(\varphi)\operatorname{div}(X)\,\mathrm{d}\mu \\
&= -(p-1)\int_0^{+\infty}\eta_K'(s)\mathrm{e}^{-\frac{(n-p)}{(n-2)(p-1)}s}(\Phi_p^\beta)'(s)\,\mathrm{d}s,
\end{aligned}
$$

which is finite thanks to Proposition 3.5. This ensures that $\operatorname{div}(X)(1-\chi_{\operatorname{Crit}(\varphi)})$ belongs to $L^1_{\mathrm{loc}}(M \smallsetminus \overline{\Omega})$. Approximating the positive and the negative part of a general $\eta \in \mathscr{C}^\infty_c(0,+\infty)$, that are nonnegative Lipschitz with compact support, we can conclude. □

Proof of Theorem 3.1. We use an argument due to [Colding and Minicozzi 2014a]. By Propositions 3.7 and 3.5, Φ^β_p is $W^{2,1}_{\mathrm{loc}}(0,+\infty)$. By (3-17), $s \mapsto \mathrm{e}^{-s(n-p)/((n-2)(p-1))}(\Phi^\beta_p)'(s)$ is nondecreasing. Then for every $0 \le s < S < +\infty$ we have

$$\mathrm{e}^{\frac{(n-p)}{(n-2)(p-1)}(S-s)}(\Phi^\beta_p)'(s) \le (\Phi^\beta_p)'(S).$$

Integrating the above inequality, we get

$$\frac{(n-2)(p-1)}{(n-p)}\big(\mathrm{e}^{\frac{(n-p)}{(n-2)(p-1)}(S-s)}-1\big)(\Phi^\beta_p)'(s) \le \Phi^\beta_p(S)-\Phi^\beta_p(s) \tag{3-22}$$

for every $0 \le s < S < +\infty$. Suppose, by contradiction, that $(\Phi^\beta_p)'(s) > 0$ for some $s \in [0,+\infty)$. Passing to the limit as $S \to +\infty$ in (3-22) we would get that $\Phi^\beta_p(S) \to +\infty$ against the boundedness property ensured by Lemma 3.4. Hence, $(\Phi^\beta_p)'(s) \le 0$ and in particular $s \mapsto \Phi^\beta_p(s)$ is nonincreasing. Notice that Φ^β_p is a bounded, nonincreasing $\mathscr{C}^1(0,+\infty)$ function. Then $(\Phi^\beta_p)'(s) \to 0$ as $s \to +\infty$. Coupling this with the coarea formula in Proposition 2.1 for $f = \operatorname{div}(X)(1-\chi_{\operatorname{Crit}(\varphi)})/|\nabla\varphi|$ one gets that

$$\begin{aligned}\mathrm{e}^{-\frac{(n-p)}{(n-2)(p-1)}s}(\Phi^\beta_p)'(s) &= \lim_{S\to+\infty} \mathrm{e}^{-\frac{(n-p)}{(n-2)(p-1)}s}(\Phi^\beta_p)'(s) - \mathrm{e}^{-\frac{(n-p)}{(n-2)(p-1)}S}(\Phi^\beta_p)'(S) \\ &= \lim_{S\to+\infty} -\int_{\{s\le\varphi\le S\}\smallsetminus\operatorname{Crit}(\varphi)} \operatorname{div} X \,\mathrm{d}\mu = -\int_{\{\varphi\ge s\}\smallsetminus\operatorname{Crit}(\varphi)} \operatorname{div} X\,\mathrm{d}\mu \end{aligned} \tag{3-23}$$

for almost every $s \in [0,+\infty)$, which also ensures that $\operatorname{div} X \in L^1(M \smallsetminus (\overline{\Omega} \cup \operatorname{Crit}(\varphi)))$. We also observe that (3-23) holds actually for every $s \in [0,+\infty)$ and this is why (3-4) is in turn true for every $t \in [1,\infty)$. Indeed, the left-hand side is continuous by the statement. By the locality of the gradient, $\{\varphi = s\} \cap \operatorname{Crit}\varphi$ is negligible with respect to the volume measure μ, since φ is a $\mathscr{C}^{1,\beta}$ function. The integration in (3-23) can be thus performed on $\{\varphi > s\} \cap \operatorname{Crit}\varphi$. This shows that the right-hand side is right-continuous (hence continuous) by the monotone convergence theorem.

One can now obtain (3-4) rewriting (3-23) in terms of u. The proof proceeds through direct computations. The main ones are contained in [Fogagnolo et al. 2019, Section 3.3], the only difference is the Ricci term that can be computed as

$$\operatorname{Ric}(\nabla\varphi,\nabla\varphi) = \left[\frac{(p-1)(n-2)}{(n-p)}\right]^2 u^{-2\frac{n+p-2}{n-p}} \operatorname{Ric}(\mathrm{D}u,\mathrm{D}u).$$

Consequently, (3-5) follows by (3-4) and coarea formula.

For the rigidity statement, suppose that $(F^\beta_p)'(t_0) = 0$ for some $t_0 \in [1,+\infty)$ regular for u. Then by (3-4)

$$\left|\mathrm{h} - \frac{\mathrm{H}}{n-1} g^\top\right|_g = 0 \quad\text{and}\quad |\mathrm{D}^\top|\mathrm{D}u|_g|_g = 0$$

hold on $\{u \le 1/t_0\} \smallsetminus \operatorname{Crit}(u)$. By Proposition 2.14, $(\{u \le 1/t_0\}, g)$ splits to a warped product near the level set $\{u = 1/t_0\}$. In particular, the mean curvature H depends only on u. By (3-3) also $|\mathrm{D}u|$ depends only on u and

$$\frac{\partial}{\partial u}|\mathrm{D}u|_g = \frac{\mathrm{H}}{p-1} = \frac{n-1}{n-p}\frac{|\mathrm{D}u|_g}{u}.$$

Integrating it, we get that for some $A(t_0) > 0$ the identity

$$|\mathrm{D}u|_g = u^{\frac{n-1}{p-1}} A(t_0)$$

holds, which gives that $|\mathrm{D}u|_g$ never vanishes on $\{u \leq 1/t_0\}$ by the continuity of gradient. Recalling the relation between u, η and t in (2-17), we obtain that $\eta(t) = B(t_0)t_0 t + (1 - B(t_0))$ for some $B(t_0) > 0$. If we define the new coordinate as

$$\tau = t + \frac{1 - B(t_0)}{B(t_0)t_0} \quad \text{and} \quad \tau_0 = \frac{1}{t_0(B(t_0) - 1)},$$

we have that $\{\tau \geq \tau_0\} = \{u \leq 1/t_0\}$, $\eta(t) = \tau/\tau_0$ and $\mathrm{d}\tau = -\mathrm{d}t$. To sum up, we have proven that $(\{u \leq 1/t_0\}, g)$ is isometric to

$$\left([\tau_0, +\infty) \times \{u = 1/t_0\},\ \mathrm{d}\tau \otimes \mathrm{d}\tau + \left(\frac{\tau}{\tau_0}\right)^2 g_{\{u=1/t_0\}}\right),$$

leaving us only to characterise τ_0. Observe that, by the conical splitting, the measures of the level sets of τ satisfy

$$|\{\tau = R\}| = \left(\frac{R}{\tau_0}\right)^{n-1} |\{u = 1/t_0\}|.$$

One can easily prove that on a cone

$$1 = \lim_{R \to +\infty} \frac{|\{\tau \leq R\}|}{|B(o, R)|} = \lim_{R \to +\infty} \frac{|\{\tau = R\}|}{|\partial B(o, R)|},$$

which can be used to compute the claimed value of τ_0,

$$\mathrm{AVR}(g) = \lim_{R \to +\infty} \frac{|\{\tau = R\}|}{R^{n-1}|\mathbb{S}^{n-1}|} = \frac{|\{u = 1/t_0\}|}{\tau_0^{n-1}|\mathbb{S}^{n-1}|}.$$ □

We conclude this section by sketching the proof of the monotonicity-rigidity theorem for Φ_p^∞, which does not require much more effort than in $\mathbb{R}^n$ [Fogagnolo et al. 2019].

Proof of Theorem 3.2. Lemma 5.1 in [Fogagnolo et al. 2019] holds also in this setting. The only difference in proving that $|\nabla\varphi|^p$ is a subsolution of the nondegenerate uniformly elliptic operator

$$\mathscr{L}(f) = \Delta f + (p-2)\nabla\nabla f\left(\frac{\nabla\varphi}{|\nabla\varphi|}, \frac{\nabla\varphi}{|\nabla\varphi|}\right) - \frac{n-p}{n-2}\langle\nabla f, \nabla\varphi\rangle,$$

acting on smooth f in a neighbourhood of points such that $|\nabla\varphi| > 0$, is that the curvature term that appears when the Bochner identity for p-harmonic functions is applied can be controlled by $\mathrm{Ric} \geq 0$. We claim that

$$|\nabla\varphi|(x) \leq \sup_{\{\varphi = s\}} |\nabla\varphi| \tag{3-24}$$

for every $s \in [0, +\infty)$ and $x \in \{\varphi \geq s\}$, which is the main ingredient in the proof of [Fogagnolo et al. 2019, Theorem 3.5]. Firstly suppose that $\Phi_p^\infty(s) > 0$ and let be $0 < \delta < \Phi_p^\infty(s)$. By (3-10), $|\nabla\varphi| \leq \mathrm{C}$ uniformly in $M \smallsetminus \Omega$. For some $S > s$ let

$$w = |\nabla\varphi|^p - \sup_{\{\varphi=s\}} |\nabla\varphi|^p - \mathrm{C}^p \mathrm{e}^{\frac{n-p}{(n-2)(p-1)}(\varphi-S)}$$

be defined on $\{s \leq \varphi \leq S\} \smallsetminus N_\delta$, where $N_\delta = \{|\nabla\varphi| < \delta\}$. Since $w \leq 0$ on the boundary of $\{s \leq \varphi \leq S\} \smallsetminus N_\delta$ and $\mathscr{L}(w) \geq 0$ in its interior, by the maximum principle we have that

$$|\nabla\varphi|^p \leq \sup_{\{\varphi=s\}} |\nabla\varphi|^p + \mathrm{C}^p \mathrm{e}^{\frac{n-p}{(n-2)(p-1)}(\varphi-S)} \tag{3-25}$$

on $\{s \leq \varphi \leq S\} \smallsetminus N_\delta$. Moreover, since $|\nabla\varphi| < \delta$ on N_δ, (3-25) is thus satisfied in the whole $\{s \leq \varphi \leq S\}$. Passing to the limit as $S \to +\infty$, (3-24) is proven for $s \in [0, +\infty)$ such that $\Phi_p^\infty(s) > 0$.

We now prove Corollary 3.3, namely that $\Phi_p^\beta(s) > 0$ for every $s \in [0, +\infty)$, which in particular yields (3-24) proving the monotonicity. Suppose by contradiction that $\Phi_p^\infty(s) = 0$ for some $s \in [0, +\infty)$. By Proposition 2.1 there exists a sequence of $(s_j)_{j\in\mathbb{N}}$, $s_j \to s$ as $j \to +\infty$ and $\Phi_p^\infty(s_j) > 0$. If, up to a subsequence, we can assume that $\Phi_p^\infty(s_j) \to 0$, then we can conclude. Indeed, $\Phi_p^\infty(s_j) \geq |\nabla\varphi|(x)$ for every $x \in \{\varphi \geq s\}$ and $\Phi_p^\infty(s_j) \to 0$ as $j \to +\infty$; hence $|\nabla\varphi| = 0$ on $\{\varphi \geq s\}$, contradicting the unboundedness of φ. Suppose now that every subsequence of $\Phi_p^\infty(s_j)$ does not vanish. Then there would be a $\delta > 0$ and $J \in \mathbb{N}$ such that $\Phi_p^\infty(s_j) > \delta$ for every $j \geq J$. Since level sets of φ are compact, $\Phi_p^\beta(s_j)$ is actually achieved at some point $x_{s_j} \in \{\varphi = s_j\}$. Moreover, $(x_{s_j})_{j\in\mathbb{N}}$ is bounded, since it is contained in $\{\varphi \leq s\}$. Hence, we can assume that there exists $x \in \{\varphi \leq s\}$ such that $x_{s_j} \to x$ as $j \to +\infty$. Since φ is $\mathscr{C}^1$, we obtain that $\varphi(x) = s$ and $|\nabla\varphi|(x) \geq \delta$, contradicting the fact that $\Phi_p^\infty(s) = 0$.

Using a similar argument we can infer that $s \mapsto \Phi_p^\beta(s)$ is left continuous. Indeed, by contradiction there would be a $\delta > 0$ such that $\Phi_p^\infty(s) \geq \Phi_p^\infty(s_0) + \delta$ for any $s < s_0$. Let $x_s \in \{\varphi = s\}$ such that $\Phi_p^\infty(s) = |\nabla\varphi|(x_s)$. By the compactness of $\{\varphi \leq s_0\}$, there exists a sequence $(s_j)_{j\in\mathbb{N}}$ and a point $x \in \{\varphi \leq s_0\}$ such that $s_j < s_0$, $s_j \to s_0$ and $x_{s_j} \to x$. Since $\varphi \in \mathscr{C}^1$, we have $\varphi(x) = s_0$ and $|\nabla\varphi|(x) \geq \Phi_p^\infty(s_0) + \delta$, contradicting the definition of Φ_p^∞. To prove the right continuity it is the enough to prove that $s \mapsto \Phi_p^\infty(s)$ is lower semicontinuous. Since $\Phi_p^\infty > 0$, the maximum of $|\nabla\varphi|$ on $\{\varphi = s\}$ is achieved at a regular point x. Let $(s_j)_{j\in\mathbb{N}}$ be a sequence such that $s_j \to s$ as $j \to +\infty$. Seeing as $|\nabla\varphi|$ is continuous, there exists a sequence of points $(x_{s_j})_{j\in\mathbb{N}}$ such that $x_{s_j} \in \{\varphi = s_j\}$ and $x_{s_j} \to x$ as $j \to +\infty$. Since $|\nabla\varphi|(x_{s_j}) \leq \Phi_p^\infty(s_j)$ for every $j \in \mathbb{N}$, we complete the proof.

We turn to prove the second part of Theorem 3.2. Since x_t is a point of maximum for the function $|\mathrm{D}u|_g / u^{(n-1)/(n-p)}$ on $\{u \leq 1/t\}$, its derivative with respect to the normal unit vector $\nu_t = -\mathrm{D}u/|\mathrm{D}u|_g$ is nonpositive. Hence (3-6) follows by direct computations. To conclude, both rigidity statements follow in the same way as in [Fogagnolo et al. 2019, Theorem 3.5], since $|\mathrm{D}u|_g^p / u^{p(n-1)/(n-p)}$ is also a subsolution of $\mathscr{L} f = 0$, thanks to (3-9). $\square$

4. Geometric consequences of the monotonicity theorems

In this section, we prove the geometric implications of the monotonicity-rigidity theorems, which are the Minkowski inequalities, a rigidity result under a pinching condition and a sphere theorem. The proof of these theorems follows, along with the monotonicity already mentioned, by a contradiction argument that involves the iso-p-capacitary inequality, which we are going to state and prove immediately since we believe it to be of independent interest.

4A. ***Iso-p-capacitary inequality.*** We provide the sharp iso-p-capacitary inequality in complete noncompact Riemannian manifolds with nonnegative Ricci curvature and Euclidean volume growth. As for the standard iso-p-capacitary inequality in the Euclidean setting, the proof fully relies on the isoperimetric inequality combined with a Pólya–Szegő principle. In particular, the sharpness of the inequality that follows is a direct consequence of the sharp isoperimetric constant in this setting, which has been found first in dimension 3 in [Agostiniani et al. 2020] and later extended to all dimensions in [Brendle 2023]. See also [Fogagnolo and Mazzieri 2022; Balogh and Kristály 2023; Johne 2021] for related results. The proof below is classical, and it is inspired by [Jauregui 2012], where it is illustrated for the 2-capacity in $\mathbb{R}^n$.

Theorem 4.1 (iso-p-capacitary inequality). *Let (M, g) be a complete, noncompact Riemannian manifold with nonnegative Ricci curvature and Euclidean volume growth. Let be $\Omega \subseteq M$ open bounded subset with smooth boundary. Then*

$$\frac{\mathrm{Cap}_p(\mathbb{B}^n)^n}{|\mathbb{B}^n|^{n-p}} \mathrm{AVR}(g)^p \leq \frac{\mathrm{Cap}_p(\Omega)^n}{|\Omega|^{n-p}}. \tag{4-1}$$

Moreover, if the equality holds then (M, g) is isometric to the Euclidean space and Ω is a ball.

Proof. By (2-6) and the coarea formula in Proposition 2.1 we have that

$$\mathrm{Cap}_p(\Omega) = \int_{M \smallsetminus \overline{\Omega}} |\mathrm{D}u|^p \,\mathrm{d}\mu = \int_0^1 \int_{\{u=\tau\}} |\mathrm{D}u|^{p-1} \,\mathrm{d}\sigma \,\mathrm{d}\tau. \tag{4-2}$$

The Hölder inequality with exponents $a = p$ and $b = p/(p-1)$ gives

$$|\{u=\tau\}|^p \leq \left(\int_{\{u=\tau\}} |\mathrm{D}u|^{p-1} \,\mathrm{d}\sigma \right) \left(\int_{\{u=\tau\}} \frac{1}{|\mathrm{D}u|} \,\mathrm{d}\sigma \right)^{p-1} \tag{4-3}$$

for almost every $\tau \in (0, 1]$. Let $V' : (0, 1] \to \mathbb{R}$ be defined as

$$V'(\tau) = -\int_{\{u=\tau\}} \frac{1}{|\mathrm{D}u|} \,\mathrm{d}\sigma. \tag{4-4}$$

Moreover, let $V : (0, 1] \to \mathbb{R}$ be the primitive of $V'(\tau)$ chosen as

$$V(\tau) = |\Omega| - \int_\tau^1 V'(s) \,\mathrm{d}s = |\Omega_\tau \smallsetminus \mathrm{Crit}(u)|, \tag{4-5}$$

where the second identity is obtained coupling (4-4) with the coarea formula (2-2) applied with $f = (1 - \chi_{\mathrm{Crit}(u)})|\mathrm{D}u|^{-1}$ (see Remark 2.2).

By the isoperimetric inequality in [Brendle 2023, Corollary 1.3] we have that

$$|\{u=\tau\}| \geq |\partial\Omega_\tau| \geq |\Omega_\tau|^{\frac{n-1}{n}} \mathrm{AVR}(g)^{\frac{1}{n}} n |\mathbb{B}^n|^{\frac{1}{n}} \geq V(\tau)^{\frac{n-1}{n}} \mathrm{AVR}(g)^{\frac{1}{n}} n |\mathbb{B}^n|^{\frac{1}{n}}. \tag{4-6}$$

Let $R(\tau)$ be the radius of the ball in $\mathbb{R}^n$ which has volume $V(\tau)$. Then $V(\tau) = |\mathbb{B}^n| R(\tau)^n$ and $V'(\tau) = |\mathbb{S}^{n-1}| R(\tau)^{n-1} R'(\tau)$. Coupling (4-6) with (4-2), (4-3) and (4-4) we obtain

$$\begin{aligned}\operatorname{Cap}_p(\Omega) &\geq \int_0^1 \frac{|\{u=\tau\}|^p}{[-V'(\tau)]^{p-1}}\,\mathrm{d}\tau \geq n^p (|\mathbb{B}^n| \operatorname{AVR}(g))^{\frac{p}{n}} \int_0^1 \frac{V(\tau)^{\frac{p(n-1)}{n}}}{[-V'(\tau)]^{p-1}}\,\mathrm{d}\tau \\ &= |\mathbb{S}^{n-1}| \operatorname{AVR}(g)^{\frac{p}{n}} \int_0^1 \frac{R(\tau)^{n-1}}{[-R'(\tau)]^{p-1}}\,\mathrm{d}\tau.\end{aligned}$$

Let now $v : \{|x| \geq R(1)\} \subset \mathbb{R}^n \to (0,1]$ be the function which is τ on $\{|x| = R(\tau)\}$. By (4-6) and (2-16) there exists a positive constant $\mathrm{C} = \mathrm{C}(p,n)$ such that

$$-V'(\tau) = \int_{\{u=\tau\}} \frac{1}{|\mathrm{D}u|}\,\mathrm{d}\sigma \geq \mathrm{C} |\Omega|^{\frac{n-1}{n}} \tau^{\frac{n-p}{p-1}}.$$

Seeing as

$$|\mathrm{D}v| = -\frac{1}{R'(\tau)} = -|\mathbb{S}^{n-1}| \frac{R^{n-1}(\tau)}{V'(\tau)},$$

the function v is locally Lipschitz. Since $|\mathbb{S}^{n-1}| R(\tau)^{n-1} = |\{|x| = R(\tau)\}| = |\{v = \tau\}|$, by the coarea formula (2-2) applied with $f = |\mathrm{D}v|^{p-1}$(see Remark 2.2) we have

$$\begin{aligned}|\mathbb{S}^{n-1}| \operatorname{AVR}(g)^{\frac{p}{n}} \int_0^1 \frac{R(\tau)^{n-1}}{[-R'(\tau)]^{p-1}}\,\mathrm{d}\tau &= \operatorname{AVR}(g)^{\frac{p}{n}} \int_0^1 \int_{\{v=\tau\}} |\mathrm{D}v|^{p-1}\,\mathrm{d}\sigma\,\mathrm{d}\tau \\ &= \operatorname{AVR}(g)^{\frac{p}{n}} \int_{\{|x| \geq R(1)\}} |\mathrm{D}v|^p\,\mathrm{d}x \geq \operatorname{AVR}(g)^{\frac{p}{n}} \operatorname{Cap}_p(\{|x| < R(1)\}),\end{aligned}$$

where the last one is by the definition of the p-capacity (2-4) in flat $\mathbb{R}^n$. Using (2-9) and the fact that $|\{|x| \leq R(1)\}| = V(1) = |\Omega|$, we finally obtain

$$\operatorname{AVR}(g)^{\frac{p}{n}} \operatorname{Cap}_p(\{|x| < R(1)\}) = \operatorname{AVR}(g)^{\frac{p}{n}} \operatorname{Cap}_p(\mathbb{B}^n) R(1)^{n-p} = \operatorname{AVR}(g)^{\frac{p}{n}} \frac{\operatorname{Cap}_p(\mathbb{B}^n)}{|\mathbb{B}^n|^{\frac{n-p}{n}}} |\Omega|^{\frac{n-p}{n}},$$

and consequently (4-1).

Clearly, if the equality holds in (4-1) then also the equality holds in the use of the isoperimetric inequality, and [Brendle 2023, Theorem 1.2] forces the rigidity both of the ambient manifold and Ω. ☐

We conclude this subsection with the following remark, whose importance will be clarified in the very proof of the L^p-Minkowski inequality (Theorem 4.3 below), where a sharp lower bound for the p-capacity of the superlevel sets of the p-capacitary potential of Ω will be needed.

Remark 4.2. We observe that, replacing Ω and u with $\Omega_t = \{u > 1/t\} \cup \Omega$ and $u_t = tu$ respectively and defining $V : (0,1] \to \mathbb{R}$ in (4-5) as

$$V(\tau) = |\Omega_t \cup \{u_t = 1\}| + \int_\tau^1 \int_{\{u_t = s\}} \frac{1}{|\mathrm{D}u_t|}\,\mathrm{d}\sigma\,\mathrm{d}s = |\Omega_{\tau/t} \smallsetminus (\operatorname{Crit}(u) \cap \{\tau < u_t < 1\})|,$$

we obtain that

$$\frac{\operatorname{Cap}_p(\mathbb{B}^n)^n}{|\mathbb{B}^n|^{n-p}} \operatorname{AVR}(g)^p \leq \frac{\operatorname{Cap}_p(\Omega_t)^n}{|\Omega_t|^{n-p}}$$

holds for every $t \in [1, +\infty)$.

4B. ***Minkowski inequality.*** We are now ready to prove the L^p-Minkowski inequality in our setting. Let (M, g) be a noncompact, complete Riemannian manifold with $\mathrm{Ric} \geq 0$ and Euclidean volume growth. Consider the function $t \mapsto F_p(t)$ defined in (3-1) as F_p^β with $\beta = 1/(p-1)$. By (2-9) we can rewrite F_p in a more geometric fashion as

$$F_p(t) = t^{\frac{n-1}{n-p}} \int_{\{u=1/t\}} |\mathrm{D}u|^p \, \mathrm{d}\sigma = \left(\frac{\mathrm{C}_p(\Omega_t)}{\mathrm{C}_p(\Omega)}\right)^{-\frac{n-p-1}{n-p}} \int_{\{u_t=1\}} |\mathrm{D}u_t|^p \, \mathrm{d}\sigma, \tag{4-7}$$

where $u_t = tu$ and $\Omega_t = \{u > 1/t\} \cup \Omega$.

Theorem 4.3 (L^p-Minkowski inequality). *Let (M, g) be complete Riemannian manifold with* $\mathrm{Ric} \geq 0$ *and Euclidean volume growth. Let $\Omega \subseteq M$ be a open bounded subset with smooth boundary. Then, for every $1 < p < n$, the following inequality holds*:

$$\mathrm{C}_p(\Omega)^{\frac{n-p-1}{n-p}} \, \mathrm{AVR}(g)^{\frac{1}{n-p}} \leq \frac{1}{|\mathbb{S}^{n-1}|} \int_{\partial\Omega} \left|\frac{\mathrm{H}}{n-1}\right|^p \mathrm{d}\sigma. \tag{4-8}$$

Moreover, the equality holds in (4-8) if and only if $(M \smallsetminus \Omega, g)$ is isometric to

$$\left([\rho_0, +\infty) \times \partial\Omega, \; \mathrm{d}\rho \otimes \mathrm{d}\rho + \left(\frac{\rho}{\rho_0}\right)^2 g_{\partial\Omega}\right), \quad \textit{where } \rho_0 = \left(\frac{|\partial\Omega|}{\mathrm{AVR}(g)|\mathbb{S}^{n-1}|}\right)^{\frac{1}{n-1}}.$$

Proof. We first show that

$$\mathrm{C}_p(\Omega)^{\frac{n-p-1}{n-p}} \, \mathrm{AVR}(g)^{\frac{1}{n-p}} \leq \frac{1}{|\mathbb{S}^{n-1}|} \left(\frac{p-1}{n-p}\right)^p \int_{\partial\Omega} |\mathrm{D}u|^p \, \mathrm{d}\sigma \tag{4-9}$$

holds for any open subset $\Omega \subseteq M$ with smooth boundary.

Let then $\theta < \mathrm{AVR}(g)$ and suppose by contradiction that there exists an open subset $\Omega \subseteq M$ with smooth boundary such that

$$\mathrm{C}_p(\Omega)^{\frac{n-p-1}{n-p}} \, \theta^{\frac{1}{n-p}} \geq \frac{1}{|\mathbb{S}^{n-1}|} \left(\frac{p-1}{n-p}\right)^p \int_{\partial\Omega} |\mathrm{D}u|^p \, \mathrm{d}\sigma.$$

Define $\tau = 1/t \in (0, 1]$. By Theorem 3.1, the function $\tau \mapsto F_p(\tau)$ is nondecreasing for $\tau \in (0, 1]$. Exploiting this monotonicity as in (4-7) we have

$$\left(\frac{n-p}{p-1}\right)^p |\mathbb{S}^{n-1}| \theta^{\frac{1}{n-p}} \geq \mathrm{C}_p(\Omega)^{-\frac{n-p-1}{n-p}} \int_{\partial\Omega} |\mathrm{D}u|^p \, \mathrm{d}\sigma \geq \mathrm{C}_p(\Omega_\tau)^{-\frac{n-p-1}{n-p}} \int_{\{u=\tau\}} |\mathrm{D}u_\tau|^p \, \mathrm{d}\sigma, \tag{4-10}$$

where $u_\tau = u/\tau$. The Hölder inequality with conjugate exponents $a = (p+1)/p$ and $b = p+1$ yields

$$\mathrm{Cap}_p(\Omega_\tau)^{\frac{p+1}{p}} \leq \left(\int_{\{u=\tau\}} |\mathrm{D}u_\tau|^p \, \mathrm{d}\sigma\right) \left(\int_{\{u=\tau\}} \frac{1}{|\mathrm{D}u_\tau|} \, \mathrm{d}\sigma\right)^{\frac{1}{p}}.$$

Therefore, plugging it into (4-10), we get

$$|\mathbb{S}^{n-1}| \mathrm{C}_p(\Omega_\tau)^{\frac{n}{n-p}} \leq \left(\frac{n-p}{p-1}\right) \theta^{\frac{p}{n-p}} \int_{\{u=\tau\}} \frac{1}{|\mathrm{D}u_\tau|} \, \mathrm{d}\sigma.$$

Using (2-9) and integrating both sides we obtain

$$|\mathbb{S}^{n-1}|\mathrm{C}_p(\Omega)^{\frac{n}{n-p}}\int_\tau^1 s^{-\frac{n(p-1)}{n-p}-1}\,\mathrm{d}s \le \left(\frac{n-p}{p-1}\right)\theta^{\frac{p}{n-p}}\int_\tau^1\int_{\{u=s\}}\frac{1}{|\mathrm{D}u|}\,\mathrm{d}\sigma\,\mathrm{d}s,$$

which, together with the coarea formula (2-2) with $f=(1-\chi_{\mathrm{Crit}(u)})|\mathrm{D}u|^{-1}$ (see Remark 2.2), leaves us with

$$\frac{|\mathbb{S}^{n-1}|}{n}(\mathrm{C}_p(\Omega_\tau)^{\frac{n}{n-p}}-\mathrm{C}_p(\Omega)^{\frac{n}{n-p}})\le\theta^{\frac{p}{n-p}}|\Omega_\tau\setminus(\Omega\cup\mathrm{Crit}(u))|$$

for every $\tau\in[0,1)$. Applying the sharp iso-p-capacitary inequality (4-1) to the left-hand side we obtain

$$\mathrm{AVR}(g)^{\frac{p}{n-p}}(|\Omega_\tau|-\mathrm{C}_p(\Omega)^{\frac{n}{n-p}})\le\theta^{\frac{p}{n-p}}|\Omega_\tau|.$$

Dividing both sides by $|\Omega_\tau|$ and passing to the limit as $\tau\to 0$, we get a contradiction with $\theta<\mathrm{AVR}(g)$, proving that for any $\theta<\mathrm{AVR}(g)$

$$\mathrm{C}_p(\Omega)^{\frac{n-p-1}{n-p}}\theta^{\frac{1}{n-p}} < \frac{1}{|\mathbb{S}^{n-1}|}\left(\frac{p-1}{n-p}\right)^p\int_{\partial\Omega}|\mathrm{D}u|^p\,\mathrm{d}\sigma$$

holds for every any bounded open $\Omega\subset M$ with smooth boundary. Letting $\theta\to\mathrm{AVR}(g)^-$ yields (4-9).

To conclude observe that Theorem 3.1 implies $(F_p)'(1)\le 0$ and thus, thanks to (3-3), we have

$$\int_{\partial\Omega}\left(\frac{p-1}{n-p}\right)|\mathrm{D}u|^p\,\mathrm{d}\sigma\le\int_{\partial\Omega}|\mathrm{D}u|^{p-1}\frac{\mathrm{H}}{n-1}\,\mathrm{d}\sigma.$$

By the Hölder inequality with conjugate exponents $a=p/(p-1)$ and $b=p$, we get

$$\int_{\partial\Omega}|\mathrm{D}u|^p\,\mathrm{d}\sigma\le\left(\frac{n-p}{p-1}\right)^p\int_{\partial\Omega}\left|\frac{\mathrm{H}}{n-1}\right|^p\mathrm{d}\sigma, \tag{4-11}$$

which coupled with (4-9) concludes the proof of (4-8).

If we now assume that the equality holds in (4-8), then the two sides of (4-11) are identical too. In particular, by (3-3), $F_p'(1)=0$ and the rigidity statement in Theorem 3.1 applies. □

Remark 4.4 (a sharp bound on F_p^β and other geometric inequalities). The previous proof combines a lower bound on $F_p(+\infty)$ with $F_p'(1)\le 0$. Such an argument can be generalised for every

$$\beta\ge\frac{n-p}{(n-1)(p-1)}.$$

In fact, with a similar reasoning one can get

$$\lim_{t\to+\infty}F_p^\beta(t)\ge\left(\frac{n-p}{p-1}\right)^{\beta(p-1)}\mathrm{C}_p(\partial\Omega)^{1-\beta\frac{p-1}{n-p}}\,\mathrm{AVR}(g)^{\beta\frac{p-1}{n-p}},$$

and couple it with $(F_p^\beta)'(1)\le 0$ to obtain the family of inequalities

$$\mathrm{C}_p(\partial\Omega)^{1-\beta\frac{p-1}{n-p}}\,\mathrm{AVR}(g)^{\beta\frac{p-1}{n-p}}\le\frac{1}{|\mathbb{S}^{n-1}|}\int_{\partial\Omega}\left|\frac{\mathrm{H}}{n-1}\right|^{(\beta+1)(p-1)}\mathrm{d}\sigma$$

depending on parameters

$$\beta \geq \frac{n-p}{(n-1)(p-1)} \quad \text{and} \quad 1 < p < n$$

(see [Benatti 2022, Theorem 4.2.1] and its proof for the details). Among them, we have the above-mentioned L^p-Minkowski inequality for $\beta = 1/(p-1)$ and the Willmore-type inequality proved in [Agostiniani et al. 2020, Theorem 1.1] for $\beta = (n-p)/(p-1)$.

In order to derive the extended Minkowski inequality we want to briefly recall the definition of outward minimising sets and the notion of strictly outward minimising hull in accordance to [Huisken and Ilmanen 2001] and some related properties that the interested reader can find in [Fogagnolo and Mazzieri 2022]. We are denoting with $\partial^* E$ the reduced boundary of a finite perimeter set E.

Definition 4.5 (outward minimising and strictly outward minimising sets). Let (M, g) be a complete Riemannian manifold. Let $E \subset M$ be a bounded measurable set with finite perimeter. E is *outward minimising* if for any $F \supseteq E$ we have $|\partial^* E| \leq |\partial^* F|$, where by $\partial^* F$ we denote the reduced boundary of a set F. We say E is *strictly outward minimising* if it is outward minimising and whenever $|\partial^* E| = |\partial^* F|$ for some $F \supseteq E$ we have that $|F \smallsetminus E| = 0$.

We can define the *strictly outward minimising hull* Ω^* of an open bounded subset Ω with smooth boundary as

$$\Omega^* = \operatorname{Int} E \quad \text{for some bounded } E \text{ containing } \Omega \text{ such that } |E| = \inf_{F \in \mathrm{SOMBE}(\Omega)} |F|, \tag{4-12}$$

where by $\mathrm{SOMBE}(\Omega)$ we denote the family of all bounded strictly outward minimising sets containing Ω and $\operatorname{Int} E$ is the measure-theoretic interior of E. As a consequence of [Fogagnolo and Mazzieri 2022, Theorem 1.1], if (M, g) is a manifold with nonnegative Ricci curvature and Euclidean volume growth, then Ω^* as defined above is unique and it is a maximal volume solution to the problem of area minimisation with obstacle Ω, that is,

$$|\partial^* \Omega^*| = \inf\{|\partial^* F| \mid F \text{ is bounded and } \Omega \subseteq F\}.$$

Outward minimising sets can be characterised as those satisfying

$$|\partial \Omega| = |\partial \Omega^*|. \tag{4-13}$$

The relation between the strictly outward minimising hull of a bounded set with smooth boundary Ω and its p-capacity in the family of manifolds we are working on is resumed in the limit

$$\lim_{p \to 1^+} \mathrm{C}_p(\Omega) = \frac{|\partial \Omega^*|}{|\mathbb{S}^{n-1}|}.$$

Such a result is contained in the far more general [Fogagnolo and Mazzieri 2022, Theorem 1.2], having in mind the relation between the p-capacity and the normalised p-capacity given in Definition 2.7. Letting $p \to 1^+$ in the L^p-Minkowski inequality (4-8) and employing the dominated convergence theorem complete the proof of the extended Minkowski inequality of Theorem 1.1,

$$\left(\frac{|\partial \Omega^*|}{|\mathbb{S}^{n-1}|} \right)^{\frac{n-2}{n-1}} \mathrm{AVR}(g)^{\frac{1}{n-1}} \leq \frac{1}{|\mathbb{S}^{n-1}|} \int_{\partial \Omega} \left| \frac{\mathrm{H}}{n-1} \right| \mathrm{d}\sigma. \tag{4-14}$$

Outward minimising sets are mean-convex, as a simple variational argument immediately shows, and satisfy (4-13). As a corollary, the Minkowski inequality can be simplified for this particular class of subsets as in the following statement.

Corollary 4.6 (Minkowski inequality for outward minimising sets). *Let (M, g) be complete Riemannian manifold with* $\mathrm{Ric} \geq 0$ *and Euclidean volume growth. Let $\Omega \subseteq M$ be a bounded outward minimising subset with smooth boundary, then*

$$\left(\frac{|\partial\Omega|}{|\mathbb{S}^{n-1}|}\right)^{\frac{n-2}{n-1}} \mathrm{AVR}(g)^{\frac{1}{n-1}} \leq \frac{1}{|\mathbb{S}^{n-1}|}\int_{\partial\Omega} \frac{\mathrm{H}}{n-1}\,\mathrm{d}\sigma. \tag{4-15}$$

Remark 4.7 (sharpness of the Minkowski inequality for outward minimising sets). The sharpness of the Minkowski inequality for outward minimising sets (4-15) is not difficult to check even in nonflat spaces. In fact, in a $\mathscr{C}^1$-asymptotically conical manifold, where the metric g approaches the cone metric $\mathrm{d}\rho \otimes \mathrm{d}\rho + \rho^2 g_L$ in the $\mathscr{C}^1$-topology, big level sets of ρ are outward minimising (see, e.g., [Benatti et al. 2024, Lemma 4.3]) and is straightforward to check that $\{\rho = R\}$ saturates (4-15) in the limit as $R \to +\infty$.

Going beyond asymptotically conical spaces, one can infer the sharpness of the Minkowski inequality for outward minimising sets in manifolds of nonnegative Ricci curvature and Euclidean volume growth of dimension $n \leq 7$. Indeed, the proof of [Fogagnolo and Mazzieri 2022, Theorem 1.3] can be readapted by exploiting (4-15) in place of the Willmore-type inequality [Agostiniani et al. 2022b, Theorem 1.1]. This would allow showing that the infimum among all outward minimising smooth sets of $|\partial\Omega|^{-(n-2)/(n-1)} \int_{\partial\Omega} \mathrm{H}$ is the lower bound given by (4-15), exactly in the same way [Fogagnolo and Mazzieri 2022, Theorem 1.3] provides the sharpness of the Willmore-type inequality.

4C. ***Rigidity statement.*** We finally characterise the subsets Ω that saturate the inequality (4-14). We are getting this rigidity result evolving $\partial\Omega$ by smooth IMCF, proving that, in an outer neighbourhood of $\partial\Omega$, the manifold is a truncated cone with the same volume ratio of (M, g). The conclusion then follows from a generalisation of the Bishop–Gromov theorem.

Going into more detail, since $\partial\Omega$ is strictly mean-convex, we can consider a sequence of sets Ω_t with $t \in [0, T)$ such that $\partial\Omega_t = F_t(\partial\Omega)$, where $F_t : \partial\Omega \to M$ satisfies

$$\frac{\mathrm{d}}{\mathrm{d}t} F_t(\partial\Omega) = \frac{1}{\mathrm{H}_t}\nu_t, \tag{4-16}$$

where ν_t and H_t are respectively the outer unit normal and the mean curvature of $\partial\Omega_t$. The conical splitting we aim for is inspired by an argument contained in [Huisken and Ilmanen 2001, Section 8]. The first step consists in the following fundamental lemma.

Lemma 4.8. *Let (M, g) be a complete Riemannian manifold with* $\mathrm{Ric} \geq 0$ *and $\Sigma \subseteq M$ a totally umbilical closed hypersurface such that* $\mathrm{Ric}(\nu, \nu) = 0$ *where ν is the normal unit vector field to Σ. Then Σ has* ***constant*** *mean curvature.*

Proof. The (traced) Codazzi–Mainardi equations and the totally umbilicity yield

$$\mathrm{Ric}_{j\nu} = \mathrm{D}_i \mathrm{h}_{ij} - \mathrm{D}_j \mathrm{H} = -\frac{n-2}{n-1}\mathrm{D}_j \mathrm{H}$$

for any $j = 1, \dots, n-1$. Consider, at a fixed point on Σ, the vector $\eta_\lambda = \lambda\, \mathrm{D}^\top \mathrm{H} + \nu$, with $\lambda \in \mathbb{R}$. Since $\mathrm{Ric}(\nu, \nu) = 0$, we have

$$0 \leq \mathrm{Ric}(\eta_\lambda, \eta_\lambda) = 2\,\mathrm{Ric}_{j\nu}\, \eta_\lambda^j \eta_\lambda^\nu + \mathrm{Ric}_{ij}\, \eta_\lambda^i \eta_\lambda^j = -2\lambda \frac{n-2}{n-1} |\mathrm{D}^\top \mathrm{H}|^2 + \lambda^2\, \mathrm{Ric}_{ij}\, \mathrm{D}^i \mathrm{H}\, \mathrm{D}^j \mathrm{H}$$

for every $\lambda \in \mathbb{R}$. This can happen only if $|\mathrm{D}^\top \mathrm{H}| = 0$, so that H is constant on Σ. □

The following straightforward but very important consequence of the Bishop–Gromov monotonicity ensures in particular that if an outer neighbourhood of a bounded open set with smooth boundary $\Omega \subset M$ is isometric to a truncated cone, then the whole complement of Ω is isometric to a truncated cone based at $\partial\Omega$.

Lemma 4.9. *Let (M, g) be a complete noncompact Riemannian manifold with $\mathrm{Ric} \geq 0$. Let $K \subset M$ be a bounded open set. Suppose there exists an outer neighbourhood $A \subset M \setminus K$ of K such that (A, g) is isometric to*

$$\left([\rho_0, \rho_1] \times \partial K,\ \mathrm{d}\rho \otimes \mathrm{d}\rho + \left(\frac{\rho}{\rho_0}\right)^2 g_{\partial K} \right)$$

for $0 < \rho_0 < \rho_1$. Then

$$|\partial K| \geq \rho_0^{n-1} |\mathbb{S}^{n-1}|\, \mathrm{AVR}(g), \tag{4-17}$$

and the equality holds if and only if $(M \setminus K, g)$ is isometric to

$$\left([\rho_0, +\infty) \times \partial K,\ \mathrm{d}\rho \otimes \mathrm{d}\rho + \left(\frac{\rho}{\rho_0}\right)^2 g_{\partial K} \right).$$

Proof. Consider the cone $(C, \hat{g})$ given by

$$\left((0, \rho_1) \times \partial K,\ \mathrm{d}\rho \otimes \mathrm{d}\rho + \left(\frac{\rho}{\rho_0}\right)^2 g_{\partial K} \right),$$

and the Riemannian manifold, with a conical singularity, obtained by gluing $(C, \hat{g})$ with $(M \setminus (K \cup A), g)$ along $\{\rho = \rho_1\}$. By our assumptions, such a manifold is well-defined with nonnegative Ricci curvature outside of the tip o of C, and coincides with (M, g) in the complement of K. In C, the geodesic distance from o is given by ρ, and in particular, by Bishop–Gromov monotonicity,

$$\frac{|\{\rho = r\}|}{r^{n-1} |\mathbb{S}^{n-1}|} \geq \mathrm{AVR}(g)$$

for any $r \in (0, \rho_1)$. Since $|\{\rho = \rho_0\}| = |\partial K|$, setting $r = \rho_0$ proves (4-17). If equality holds, then, by the rigidity statement in the Bishop–Gromov theorem for manifolds with a conical singularity, the whole manifold we constructed is isometric to a cone, and in particular, $(M \setminus K, g)$ splits as claimed. This well-known, slightly enhanced version of the Bishop–Gromov rigidity statement can be readily deduced from its classic proof, or seen as a very special case of its version for nonsmooth metric spaces [De Philippis and Gigli 2016]. □

We finally have at our disposal all the tools we need to work out the splitting argument leading to Theorem 1.2.

Proof of Theorem 1.2. Suppose that some strictly outward minimising $\Omega \subset M$ with strictly mean-convex boundary satisfies

$$\left(\frac{|\partial\Omega|}{|\mathbb{S}^{n-1}|}\right)^{\frac{n-2}{n-1}} \mathrm{AVR}(g)^{\frac{1}{n-1}} = \frac{1}{|\mathbb{S}^{n-1}|}\int_{\partial\Omega} \frac{\mathrm{H}}{n-1}\, \mathrm{d}\sigma. \tag{4-18}$$

Since $\partial\Omega$ is by assumption strictly mean-convex, we can evolve it by (smooth) IMCF $\partial\Omega_t$ defined in (4-16) for $t \in [0, T)$. By the [Huisken and Ilmanen 2001, Smooth Start Lemma 2.4], up to shortening the time interval, we can assume that Ω_t is strictly outward minimising for any $t \in [0, T)$. Indeed, since Ω is strictly outward minimising, the flow coincides for a short time with the weak notion of IMCF, which exists in our setting by [Mari et al. 2022, Theorem 1.7]. The sublevel sets of the weak IMCF being strictly outward minimising is a basic and fundamental property illustrated in [Huisken and Ilmanen 2001, Minimizing Hull Property 1.4]. Consider then the function $\mathcal{Q} : [0, T) \to \mathbb{R}$ defined by

$$\mathcal{Q}(t) = |\partial\Omega_t|^{-\frac{n-2}{n-1}} \int_{\partial\Omega_t} \mathrm{H}_t\, \mathrm{d}\sigma.$$

By evolution equations for curvature flows derived for example in [Huisken and Polden 1999, Theorem 3.2], a straightforward computation shows that

$$\mathcal{Q}'(t) = -|\partial\Omega_t|^{-\frac{n-2}{n-1}} \int_{\partial\Omega_t} \frac{|\mathring{\mathrm{h}}_t|^2 + \mathrm{Ric}(\nu_t, \nu_t)}{\mathrm{H}_t}\, \mathrm{d}\sigma \leq 0,$$

where by $\mathring{\mathrm{h}}_t$ we denote the trace-free part of the second fundamental form h_t of $\partial\Omega_t$. On the other hand, the strict inequality for some $t \in [0, T)$ would result in a contradiction to the Minkowski inequality. Thus $\mathcal{Q}'(t)$ vanishes for any $t \in [0, T)$ and, in particular $\partial\Omega_t$ satisfies (4-18) for any $t \in [0, T)$. Hence, $\partial\Omega_t$ is totally umbilical and satisfies $\mathrm{Ric}(\nu_t, \nu_t) = 0$ for every $t \in [0, T)$. By Lemma 4.8 $\partial\Omega_t$ has constant mean curvature for every $t \in [0, T)$.

On $\{0 \leq t < T\}$, the solution to the weak level set formulation of the IMCF w, which in our smooth case just means $\{w = t\} = \partial\Omega_t$, satisfies the relation

$$\mathrm{H}_t = \mathrm{div}\left(\frac{\mathrm{D}w}{|\mathrm{D}w|}\right)(x_t) = |\mathrm{D}w|(x_t)$$

at any $x_t \in \partial\Omega_t$. Hence, since $\mathrm{H}_t > 0$, a well-known extension of the Gauss' lemma yields

$$g = \frac{\mathrm{d}w \otimes \mathrm{d}w}{|\mathrm{D}w|^2} + g_{\partial\Omega_t} = \frac{\mathrm{d}t \otimes \mathrm{d}t}{\mathrm{H}_t^2} + g_{\partial\Omega_t}. \tag{4-19}$$

The evolution equation (see [Huisken and Polden 1999, Theorem 3.2(i)]) satisfied by $g_{\partial\Omega_t}$ is

$$\frac{\partial}{\partial t} g_{\partial\Omega_t} = 2\frac{\mathrm{h}_t}{\mathrm{H}_t} g_{\partial\Omega_t} = \frac{2}{n-1} g_{\partial\Omega_t},$$

where the last identity is due to the total umbilicity of $\partial\Omega_t$. Integrating such equation we deduce

$$g_{\partial\Omega_t} = \mathrm{e}^{\frac{2t}{(n-1)}} g_{\partial\Omega}. \tag{4-20}$$

On the other hand, the evolution equation for the mean curvature along the IMCF (see [Huisken and Polden 1999, Theorem 3.2(v)]) gives

$$\frac{\partial}{\partial t}\mathrm{H}_t = -\Delta_{\partial\Omega_t}\left(\frac{1}{\mathrm{H}_t}\right) - \frac{1}{\mathrm{H}_t}[|\mathrm{h}_t|^2 + \mathrm{Ric}(\nu_t, \nu_t)] = -\frac{\mathrm{H}_t}{n-1},$$

where the last identity is due to the fact that $\partial\Omega_t$ is totally umbilical, $\mathrm{Ric}(\nu_t, \nu_t) = 0$ and the mean curvature H_t of $\partial\Omega_t$ depends only on t. Integrating it we obtain that

$$\mathrm{H}_t = \mathrm{e}^{-\frac{t}{n-1}}\mathrm{H}_0, \tag{4-21}$$

where H_0 is the mean curvature of $\partial\Omega$.

Plugging (4-20) and (4-21) into (4-19), we deduce that $(\{0 \le t < T\}, g)$ is isometric to

$$\left([0, T) \times \partial\Omega,\ \mathrm{e}^{\frac{2t}{n-1}}\frac{\mathrm{d}t \otimes \mathrm{d}t}{\mathrm{H}_0^2} + \mathrm{e}^{\frac{2t}{n-1}} g_{\partial\Omega}\right).$$

Performing the change of variables

$$\rho = \frac{(n-1)}{\mathrm{H}_0}\mathrm{e}^{\frac{t}{(n-1)}},$$

the metric can be written as

$$\left([\rho_0, \rho(T)) \times \partial\Omega,\ \mathrm{d}\rho \otimes \mathrm{d}\rho + \left(\frac{\rho}{\rho_0}\right)^2 g_{\partial\Omega}\right), \quad \text{where } \rho_0 = \frac{(n-1)}{\mathrm{H}_0}.$$

On the other hand, since by assumption $\partial\Omega$ saturates the Minkowski inequality, that is, (4-18) holds, we immediately get

$$\rho_0 = \left(\frac{|\partial\Omega|}{\mathrm{AVR}(g)|\mathbb{S}^{n-1}|}\right)^{\frac{1}{n-1}},$$

and we conclude by the rigidity statement in Lemma 4.9 that the whole $M \smallsetminus \Omega$ is isometric to a truncated cone. □

In the following remark, we briefly discuss how the assumptions for the rigidity can be relaxed in small dimensions.

Remark 4.10. In dimension $3 \le n \le 7$, an open bounded subset Ω with smooth strictly mean-convex boundary satisfying

$$\left(\frac{|\partial\Omega^*|}{|\mathbb{S}^{n-1}|}\right)^{\frac{n-2}{n-1}} \mathrm{AVR}(g)^{\frac{1}{n-1}} = \frac{1}{|\mathbb{S}^{n-1}|}\int_{\partial\Omega}\frac{\mathrm{H}}{n-1}\,\mathrm{d}\sigma$$

is a priori strictly outward minimising, and thus, in this case, such an assumption can be dropped. Indeed, by approximating Ω via mean curvature flow with smooth strictly outward minimising domains, as described in [Huisken and Ilmanen 2001, Lemma 5.6], we deduce that (4-14) holds also for $\mathscr{C}^{1,1}$-hypersurfaces. In particular, the Minkowski inequality holds also for the strictly outward minimising hull of Ω (see the regularity results recalled in [Huisken and Ilmanen 2001, Regularity Theorem 1.3] and [Fogagnolo and Mazzieri 2022, Theorem 2.18]) for every Ω with smooth boundary, provided the dimensional bound holds. We can then argue by contradiction. Suppose that Ω^* does not coincide with Ω. Then

$$\left(\frac{|\partial\Omega^*|}{|\mathbb{S}^{n-1}|}\right)^{\frac{n-2}{n-1}} \mathrm{AVR}(g)^{\frac{1}{n-1}} = \frac{1}{|\mathbb{S}^{n-1}|}\int_{\partial\Omega}\frac{\mathrm{H}}{n-1}\,\mathrm{d}\sigma > \frac{1}{|\mathbb{S}^{n-1}|}\int_{\partial\Omega^*}\frac{\mathrm{H}}{n-1}\,\mathrm{d}\sigma,$$

where the last inequality is due to the fact that $\mathrm{H}=0$ on $\partial\Omega^*\smallsetminus\partial\Omega$. But this contradicts the Minkowski inequality for Ω^*; hence $\Omega=\Omega^*$.

4D. ***A pinching condition and a sphere theorem.*** In this subsection, we exploit the monotonicity of the function $t\mapsto F_p^\infty(t)$ defined in (3-2) to prove a couple of rigidity statements involving a pinching condition on the mean curvature of $\partial\Omega$ and an a priori bound on the gradient of the p-capacitary potential associated to Ω. These results without any convexity assumption are also new in $\mathbb{R}^n$, and they constitute the complete nonlinear generalisation of [Borghini et al. 2019, Corollary 1.4 and 1.9]. For convex subsets of the Euclidean space they are the content of [Fogagnolo et al. 2019, Corollary 2.16 and 2.17].

Theorem 4.11. *Let (M,g) be a complete Riemannian manifold with $\mathrm{Ric}\geq 0$ and Euclidean volume growth. If there exists an open bounded subset $\Omega\subseteq M$ with smooth boundary satisfying*

$$-\left[\frac{\mathrm{AVR}(g)}{\mathrm{C}_p(\Omega)}\right]^{\frac{1}{n-p}} \leq \frac{\mathrm{H}}{n-1} \leq \left[\frac{\mathrm{AVR}(g)}{\mathrm{C}_p(\Omega)}\right]^{\frac{1}{n-p}} \tag{4-22}$$

on every point of $\partial\Omega$, then $(M\smallsetminus\Omega,g)$ is isometric to

$$\left([\rho_0,+\infty)\times\partial\Omega,\ \mathrm{d}\rho\otimes\mathrm{d}\rho+\left(\frac{\rho}{\rho_0}\right)^2 g_{\partial\Omega}\right),\quad \textit{where } \rho_0=\left(\frac{|\partial\Omega|}{\mathrm{AVR}(g)|\mathbb{S}^{n-1}|}\right)^{\frac{1}{n-1}}.$$

In this case $\partial\Omega$ is a connected totally umbilical hypersurface with constant mean curvature in $(M\smallsetminus\Omega,g)$.

Proof. We can argue by contradiction as in Theorem 4.3 to prove that

$$\left(\frac{n-p}{p-1}\right)\left[\frac{\mathrm{AVR}(g)}{\mathrm{C}_p(\Omega)}\right]^{\frac{1}{n-p}} \leq \sup_{\partial\Omega}|\mathrm{D}u|.$$

Indeed, we can follow the same lines replacing the consequence of the monotonicity of F_p with the corresponding of F_p^∞, which thanks to (2-9) can be rewritten as

$$F_p^\infty(t)=t^{\frac{n-1}{n-p}}\sup_{\{u=1/t\}}|\mathrm{D}u| = \left(\frac{\mathrm{C}_p(\Omega_t)}{\mathrm{C}_p(\Omega)}\right)^{\frac{1}{n-p}}\sup_{\{u_t=1\}}|\mathrm{D}u_t|,$$

where $u_t=tu$ and $\Omega_t=\{u>1/t\}\cup\Omega$. Accordingly, we employ the Hölder inequality with conjugate exponents $a=+\infty$ and $b=1$, that is,

$$\mathrm{Cap}_p(\Omega_t)^{\frac{1}{p}} \leq \sup_{\{u=1/t\}}|\mathrm{D}u_t|\left(\int_{\{u=1/t\}}\frac{1}{|\mathrm{D}u_t|}\,\mathrm{d}\sigma\right)^{\frac{1}{p}}.$$

In the end, by Theorem 3.2 we get

$$\sup_{\partial\Omega}|\mathrm{D}u| \leq \frac{(n-p)}{(p-1)(n-1)}\sup_{\partial\Omega}|\mathrm{H}|$$

and the equality holds if and only if $(M\smallsetminus\Omega,g)$ splits as in the statement. Condition (4-22) easily implies the equality. □

The above result is a rigidity theorem under a pinching condition on the mean curvature of $\partial\Omega$ with respect to its p-capacity. From the proof above we can also get that

$$\frac{1}{p-1}\left[\frac{\mathrm{AVR}(g)}{\mathrm{C}_p(\Omega)}\right]^{\frac{1}{n-p}} \leq \sup_{\partial\Omega}\left|\frac{\mathrm{D}u}{n-p}\right| \tag{4-23}$$

and the equality is satisfied only on metric cones. The previous inequality gives a lower bound on the gradient of u on $\partial\Omega$ in terms of the p-capacity of Ω that, when attained, forces (M, g) to be (isometric to) $\mathbb{R}^n$ with Ω a Euclidean ball.

Theorem 4.12. *Let (M, g) be a complete Riemannian manifold with* $\mathrm{Ric} \geq 0$ *curvature and Euclidean volume growth. Let $\Omega \subseteq M$ be an open bounded subset with smooth boundary, u the p-capacitary potential associated to Ω and assume that*

$$\sup_{\partial\Omega}\left|\frac{\mathrm{D}u}{n-p}\right| \leq \frac{1}{p-1}\,\mathrm{AVR}(g)^{\frac{1}{p-1}}\left(\frac{|\mathbb{S}^{n-1}|}{|\partial\Omega|}\right)^{\frac{1}{n-1}}. \tag{4-24}$$

Then (M, g) is isometric to $\mathbb{R}^n$ with the Euclidean metric and Ω is a ball.

Proof. Under the assumption (4-24), we get

$$\mathrm{C}_p(\Omega) = \left(\frac{p-1}{n-p}\right)^{p-1}\frac{1}{|\mathbb{S}^{n-1}|}\int_{\partial\Omega}|\mathrm{D}u|^{p-1}\,\mathrm{d}\sigma \leq \mathrm{AVR}(g)\left(\frac{|\mathbb{S}^{n-1}|}{|\partial\Omega|}\right)^{-\frac{n-p}{n-1}},$$

which yields

$$\left(\frac{|\mathbb{S}^{n-1}|}{|\partial\Omega|}\right)^{\frac{n-p}{n-1}} \leq \frac{\mathrm{AVR}(g)}{\mathrm{C}_p(\Omega)} \leq (p-1)^{n-p}\sup_{\partial\Omega}\left|\frac{\mathrm{D}u}{n-p}\right|^{n-p} \leq \mathrm{AVR}(g)^{\frac{n-p}{p-1}}\left(\frac{|\mathbb{S}^{n-1}|}{|\partial\Omega|}\right)^{\frac{n-p}{n-1}}, \tag{4-25}$$

where we used (4-23) together with the condition (4-24). Thus, we obtain that $\mathrm{AVR}(g) = 1$, and hence, by the Bishop–Gromov theorem, that (M, g) is isometric to $\mathbb{R}^n$ with the Euclidean metric. Since all inequalities in (4-25) become equalities, by the second one we can apply the rigidity statement in Theorem 3.2 which ensures that $\partial\Omega$ is a compact connected and totally umbilical hypersurface of $\mathbb{R}^n$, that is, Ω is a ball. □

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THE WILLMORE FLOW OF TORI OF REVOLUTION

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We study long-time existence and asymptotic behavior for the L^2-gradient flow of the Willmore energy, under the condition that the initial datum is a torus of revolution. We show that if an initial datum has Willmore energy below 8π then the solution of the Willmore flow converges for $t \to \infty$ to the Clifford torus, possibly rescaled and translated. The energy threshold of 8π turns out to be optimal for such a convergence result. We give an application to the conformally constrained Willmore minimization problem.

1. Introduction

Let $f : \Sigma \to \mathbb{R}^3$ be a smooth immersion of a two-dimensional manifold without boundary. Its *Willmore energy* is

$$\mathcal{W}(f) = \frac{1}{4} \int_\Sigma |\vec{H}|^2 \, \mathrm{d}\mu, \tag{1-1}$$

where $\vec{H}$ denotes the mean curvature vector and $\mathrm{d}\mu$ the induced Riemannian measure. Its critical points are called *Willmore immersions* and satisfy

$$\Delta \vec{H} + Q(\mathring{A})\vec{H} = 0, \tag{1-2}$$

where Δ denotes the Laplace–Beltrami operator, $\mathring{A}$ is the trace-free second fundamental form and Q is quadratic in $\mathring{A}$ (see (2-3)). If $f(\Sigma)$ is orientable (or two-sided, which is equivalent in $\mathbb{R}^3$) then $\vec{H} = (\kappa_1 + \kappa_2)\vec{N}$, with κ_1, κ_2 the principal curvatures of $f(\Sigma)$ and $\vec{N}$ a smooth normal vector field. The L^2-gradient flow of the Willmore functional with given initial datum f_0, a smooth immersion, is

$$\partial_t f = -(\Delta \vec{H} + Q(\mathring{A})\vec{H}), \tag{1-3}$$

with $f(t = 0) = f_0$. This fourth-order quasilinear geometric evolution equation has been extensively studied in [Kuwert and Schätzle 2001; 2002], where a blow-up criterion is formulated. With the aid of this criterion the same authors proved in [Kuwert and Schätzle 2004] long-time existence and convergence for the *flow of spherical immersions* under the assumption that the initial immersion $f_0 : \mathbb{S}^2 \to \mathbb{R}^3$ satisfies $\mathcal{W}(f_0) < 8\pi$. The energy threshold of 8π is shown to be sharp in [Blatt 2009] for the convergence of spherical immersions.

In the classical work [Mayer and Simonett 2002] the Willmore flow is studied numerically, not only for spheres but also for surfaces of different genus, such as tori. See also [Barrett et al. 2019] for other numerical examples. In [Mayer and Simonett 2002, Section 8.1] it is stated that the flow converges for

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all tori that the authors looked at, which was astounding as this behavior differs fundamentally from the surface diffusion flow, where the hole of all initial tori seems to close and the curvature blows up; see [Mayer 2001; Barrett et al. 2019]. Our goal is to understand analytically what happens to tori along the Willmore flow. In this article we only look at the special case of tori of revolution.

Definition 1.1. In the sequel we identify $\mathbb{S}^1 = \mathbb{R}/\mathbb{Z}$ and set $\mathbb{H}^2 := \mathbb{R} \times (0, \infty)$. We call an immersion $f : \mathbb{S}^1 \times \mathbb{S}^1 \to \mathbb{R}^3$ a *torus of revolution* if there exists an immersed curve $\gamma \in C^\infty(\mathbb{S}^1, \mathbb{H}^2)$, $\gamma = (\gamma^{(1)}, \gamma^{(2)})$, such that

$$f(u, v) = \begin{pmatrix} \gamma^{(1)}(u) \\ \gamma^{(2)}(u)\cos(2\pi v) \\ \gamma^{(2)}(u)\sin(2\pi v) \end{pmatrix}. \tag{1-4}$$

We call γ *profile curve* and we will frequently denote f as in (1-4) by F_γ.

An essential element in our argument is that the property of being a torus of revolution is preserved along the Willmore flow. Hence the evolution by Willmore flow can also be regarded as a time evolution of the profile curves. In the arguments to come we will take advantage of an interplay between the revolution symmetry and the blow-up-criterion developed in [Kuwert and Schätzle 2001; 2002]. With this technique we have identified a geometric quantity whose boundedness ensures convergence. This quantity is the *hyperbolic length* of the profile curves given by

$$\mathcal{L}_{\mathbb{H}^2}(\gamma) := \int_{\mathbb{S}^1} \frac{|\gamma'(x)|}{\gamma^{(2)}(x)}\,\mathrm{d}x, \qquad \gamma \in C^\infty(\mathbb{S}^1, \mathbb{R} \times (0, \infty)).$$

Strikingly, the *hyperbolic geometry* of the curve evolution is decisive for the convergence behavior. We recall that the hyperbolic plane $\mathbb{H}^2 = \mathbb{R} \times (0, \infty)$ is endowed with the metric $g_{(x,y)} = y^{-2}\,\mathrm{d}x\,\mathrm{d}y$.

Now we can state our main convergence criterion:

Theorem 1.2. *Let $f : [0, T) \times \mathbb{S}^1 \times \mathbb{S}^1 \to \mathbb{R}^3$ be a maximal evolution by Willmore flow such that $f(0)$ is a torus of revolution. Then $f(t)$ is a torus of revolution for all $t \in [0, T)$. Suppose that $(\gamma(t))_{t\in[0,T)}$ is a collection of profile curves of $f(t)$. If*

$$\liminf_{t\to T} \mathcal{L}_{\mathbb{H}^2}(\gamma(t)) < \infty, \tag{1-5}$$

then $T = \infty$ and the Willmore flow converges (*up to reparametrizations*) *in C^k for all k to a Willmore torus of revolution f_∞.*

We remark that the concept of C^k-convergence that we impose is a *geometric* one; see Appendix C (Definition C.7) for details. From now on, the term C^k-convergence is understood up to reparametrizations as in Definition C.7.

That the *hyperbolic geometry* of the profile curve plays a role is not surprising — there is an interesting correspondence between the Willmore energy of tori of revolution and the hyperbolic elastic energy of curves, observed in [Langer and Singer 1984a]. With this correspondence one can for example show the Willmore conjecture for tori of revolution; see [Langer and Singer 1984b]. Other applications of this relationship include [Dall'Acqua et al. 2008; Mandel 2018]. To the authors' knowledge, this is the first time that this correspondence is used in a problem depending on time.

The main question now is to identify which initial data generate evolutions with bounded hyperbolic length. It turns out that the same energy threshold of 8π needed for spherical immersions (see [Kuwert and Schätzle 2004]) is needed in the case of tori of revolution.

Theorem 1.3. *Let $f_0 : \mathbb{S}^1 \times \mathbb{S}^1 \to \mathbb{R}^3$ be a torus of revolution satisfying $\mathcal{W}(f_0) \leq 8\pi$. Let $f : [0, T) \times (\mathbb{S}^1 \times \mathbb{S}^1) \to \mathbb{R}^3$ evolve by the Willmore flow with initial datum f_0. Then $T = \infty$ and f converges in C^k for all $k \in \mathbb{N}$ to the Clifford torus, possibly rescaled and translated in the direction* $(1, 0, 0)$.

Here the Clifford torus is the surface of revolution given by

$$(u, v) \mapsto \Big(\tfrac{1}{\sqrt{2}} \sin(2\pi u), \big(1 + \tfrac{1}{\sqrt{2}} \cos(2\pi u)\big) \cos(2\pi v), \big(1 + \tfrac{1}{\sqrt{2}} \cos(2\pi u)\big) \sin(2\pi v)\Big). \tag{1-6}$$

Notice that it is not important which parametrization we choose since C^k-convergence is a geometric concept. The Clifford torus arises from stereographic projection of the minimal surface $\frac{1}{\sqrt{2}}(\mathbb{S}^1 \times \mathbb{S}^1) \subset \mathbb{S}^3$. From the solution [Marques and Neves 2014] of the famous Willmore conjecture we know that the Clifford torus is the global minimum of the Willmore energy among tori in $\mathbb{R}^3$ and the unique minimum modulo smooth conformal transformations (of $\mathbb{R}^3$) and reparametrizations. Our method relies on a *gap theorem* for Willmore tori of revolution, which is a consequence of [Müller and Spener 2020]; see Proposition 2.4. This relates to the findings in [Mondino and Nguyen 2014].

The convergence result in Theorem 1.3 holds up to surprisingly little invariances. It is often expected that such convergence results can only be shown up to invariances of the Willmore energy, i.e., reparametrizations and conformal transformations. The fact that we do not have to apply conformal transformations along the flow to achieve convergence is explained by the use of a Łojasiewicz–Simon gradient inequality. This inequality is a purely analytical tool, so the invariances will not play a role. For the limit immersion, we can rule out all conformal transformations that break the rotational symmetry and even more — symmetry-preserving Möbius inversions can also be ruled out due to the fact that they are not invariances of the Willmore flow equation. What remains is just scaling and translation in the direction $(1, 0, 0)$. This is not surprising since both transformations preserve the symmetry we consider and also preserve solutions of the Willmore flow equation, possibly rescaling appropriately in time.

We also prove that the energy threshold of 8π is sharp by constructing explicit nonconvergent evolutions with initial data f_0 satisfying $\mathcal{W}(f_0) > 8\pi$. There are multiple reasons why this number could be a universal threshold for any genus. The most striking is the inequality of Li and Yau that shows that immersions of Willmore energy below 8π are embeddings; see [Li and Yau 1982, Theorem 6]. Another property is that the metric of tori of energy $\leq 8\pi - \delta$, $\delta > 0$, is uniformly controlled up to Möbius transformations and reparametrizations; see [Schätzle 2013, Theorem 1.1] for details. As pointed out in [Simon 1993, p. 282; Kuwert et al. 2010], there exist surfaces of arbitrary genus with energy below 8π.

As already announced, we also show optimality of the energy bound of 8π.

Theorem 1.4. *For any $\varepsilon > 0$ there exists a torus of revolution $f_0 : \mathbb{S}^1 \times \mathbb{S}^1 \to \mathbb{R}^3$ such that $\mathcal{W}(f_0) < 8\pi + \varepsilon$ and the maximal Willmore flow $(f(t))_{t\in[0,T)}$ develops a singularity (in finite or infinite time). More precisely, one of the following phenomena occurs:*

(1) (*Concentration of curvature*) *The second fundamental form* $(\|A(t)\|_{L^\infty(\Sigma)})_{t\in[0,T)}$ *is unbounded. This singularity can occur in finite or infinite time.*

(2) (*Diameter blow-up in infinite time*) $T=\infty$ *and* $\lim_{t\to\infty}\operatorname{diam}(f(t))(\mathbb{S}^1\times\mathbb{S}^1)=\infty$.

In both cases the Willmore flow cannot converge in C^2.

The singular behavior as described in Theorem 1.4 will actually occur for each initial immersion F_γ, as in Definition 1.1, with γ a curve of *vanishing total curvature*; see (3-20). This gives a class of singular examples for the Willmore flow. The total curvature also plays a significant role in earlier constructions of singular examples; see [Blatt 2009] for $\Sigma=\mathbb{S}^2$.

As a consequence of our main result we are able to show that each rectangular conformal class contains a torus of revolution of energy below 8π. This result has far-reaching consequences for the minimization of the Willmore energy with fixed conformal class, studied for example in [Kuwert and Schätzle 2013]. In this article the authors show that minimizers in a given conformal class exist under the condition that the class contains an element of Willmore energy below 8π. By our result this condition is satisfied for every *rectangular* conformal class.

This paper is organized as follows. In Section 2 we fix the notation and collect some useful facts on elastic curves in the hyperbolic plane and on tori of revolution. Section 3 exploits the consequences of the initial datum being a torus of revolution for the symmetry properties of the evolution, for the possible singularities and the limit. It also contains the proofs of the main results and of the optimality results. In the last section we give the application on existence of tori of revolution with energy below 8π in each conformal class. Some useful results on smooth convergence (see Definition 2.1 below) and the Willmore flow are collected in the Appendix.

2. Geometric preliminaries

2.1. ***Notation.*** We first recall some basic definitions from differential geometry. Let Σ be a two-dimensional smooth manifold and $f:\Sigma\to\mathbb{R}^n$ be a smooth immersion. In this paper all manifolds are assumed to have no boundary. If we talk about tori of revolution, we need to impose the restriction that $n=3$, but we will also discuss some results on the Willmore flow that remain valid in any codimension, i.e., for all $n\geq 3$. Let g be the induced Riemannian metric and ∇ the Levi-Civita connection on Σ, and denote the set of smooth vector fields on Σ by $\mathcal{V}(\Sigma)$. For $X\in\mathcal{V}(\Sigma)$ and $h\in\mathcal{C}^\infty(\Sigma,\mathbb{R}^n)$ we define $D_Xh\in\mathcal{C}^\infty(\Sigma,\mathbb{R}^n)$ as

$$D_Xh:=\sum_{i=1}^n X(h_i)\vec{e}_i,\quad \text{whenever } h=\sum_{i=1}^n h_i\vec{e}_i\in C^\infty(M;\mathbb{R}^n),$$

and $\{\vec{e}_1,\vec{e}_2,\vec{e}_3,\dots,\vec{e}_n\}$ is the canonical basis of $\mathbb{R}^n$ (see also Appendix B). The second fundamental form of Σ is $A:\mathcal{V}(\Sigma)\times\mathcal{V}(\Sigma)\to\mathcal{C}^\infty(\Sigma,\mathbb{R}^n)$, given by

$$A(X,Y):=D_X(D_Yf)-D_{\nabla_XY}f. \tag{2-1}$$

We remark that for all $p \in \Sigma$ one has $A_p(X, Y) \in df_p(T_p\Sigma)^\perp$; we say it takes values in the *normal bundle*. Moreover $A_p(X, Y)$ only depends on $X(p)$,$Y(p)$. Its trace-free part $\mathring{A}$ is given by

$$\mathring{A}(X, Y) := A(X, Y) - \tfrac{1}{2} g(X, Y)\vec{H},$$

where the mean curvature vector $\vec{H}$ is the trace of the bilinear form (2-1) and can be computed by

$$\vec{H}(p) = A(e_1, e_1) + A(e_2, e_2),$$

with $\{e_1, e_2\}$ being an orthonormal basis of $T_p\Sigma$. Similarly (see Appendix A for details) we have

$$|A|^2 = \sum_{i,j=1}^{2} \langle A(e_i, e_j), A(e_i, e_j)\rangle_{\mathbb{R}^n}.$$

With these definitions we may introduce the Willmore flow of a smooth immersion $f_0 : \Sigma \to \mathbb{R}^n$. We say that a smooth family of smooth immersions $f : [0, T) \times \Sigma \to \mathbb{R}^n$, where $T > 0$, evolves by the *Willmore flow* with initial datum f_0 if f satisfies

$$\partial_t f = -(\Delta\vec{H} + Q(\mathring{A})\vec{H}) \quad \text{in } (0, T) \times \Sigma, \tag{2-2}$$

with $f(t = 0) = f_0$. Here, Δ denotes the *normal Laplacian*, i.e., for an orthonormal basis $\{e_1, e_2\}$ that is a basis of $T_p\Sigma$ with respect to $f(t, \cdot)^* g_{\mathbb{R}^n}$ one has

$$\Delta\vec{H} = \sum_{i=1}^{2} (\nabla^\perp)^2 \vec{H}(e_i, e_i),$$

where $\nabla^\perp_X Y = (D_X Y)^\perp$ (see (B-2), (B-3) for details). With the same notation as above, the quadratic operator Q is given by

$$(Q(\mathring{A})\vec{H})(t, p) = \sum_{i,j=1}^{2} \langle \mathring{A}(e_i, e_j), \vec{H}\rangle_{\mathbb{R}^n} \mathring{A}(e_i, e_j). \tag{2-3}$$

Since (2-2) is well-posed for smooth initial immersions f_0 (see [Kuwert and Schätzle 2002, Proposition 1.1]) we will always assume that the evolution is maximal, i.e., nonextendable in the class of smooth immersions.

To study the behavior of $f(t)$ as $t \to T$ we use the following notion of smooth convergence on compact sets from [Kuwert and Schätzle 2001, Theorem 4.2]; see also [Breuning 2015] and Appendix C.

Definition 2.1 (Smooth convergence of immersions). Let Σ and $\widehat{\Sigma}$ be smooth two-dimensional manifolds and $(f_j)_{j=1}^\infty : \Sigma \to \mathbb{R}^n$ and $\hat{f} : \widehat{\Sigma} \to \mathbb{R}^n$ be smooth immersions. Define

$$\widehat{\Sigma}(m) := \{p \in \widehat{\Sigma} : |\hat{f}(p)| < m\}, \quad m \in \mathbb{N}. \tag{2-4}$$

We say that f_j *converges to* $\hat{f}$ *smoothly on compact subsets of* $\mathbb{R}^n$ if for each $j \in \mathbb{N}$ there exists a diffeomorphism $\phi_j : \widehat{\Sigma}(j) \to U_j$ for some open $U_j \subset \Sigma$, and a normal vector field $u_j \in C^\infty(\widehat{\Sigma}(j), \mathbb{R}^n)$ satisfying

$$f_j \circ \phi_j = \hat{f} + u_j \quad \text{on } \widehat{\Sigma}(j), \tag{2-5}$$

as well as $\|(\hat{\nabla}^{\perp})^k u_j\|_{L^\infty(\widehat{\Sigma}(j))} \to 0$ as $j \to \infty$ for all $k \in \mathbb{N}_0$. Here $\hat{\nabla}$ is the Levi-Civita connection on $(\widehat{\Sigma}, g_{\hat{f}})$ and $(\hat{\nabla}^{\perp})^k u_j$ is defined as in Appendix B. Additionally, we require that for each $R > 0$ there exists $j(R) \in \mathbb{N}$ such that $j \geq j(R)$ implies that $f_j^{-1}(B_R(0)) \subset U_j$.

We exploit a fundamental correspondence between the Willmore energy of tori and the elastic energy of curves in the hyperbolic plane already used in several works since its observation in [Langer and Singer 1984a].

2.2. ***Curves in the hyperbolic plane.*** We consider the hyperbolic half-plane $\mathbb{H}^2 = \{(x^{(1)}, x^{(2)}) \in \mathbb{R} \times (0, \infty)\}$ endowed with the metric

$$g_{\mathbb{H}^2}(v, w) = \frac{1}{z^2} \langle v, w\rangle_{\mathbb{R}^2}, \qquad v, w \in T_z\mathbb{H}^2,$$

and define $|v|_{\mathbb{H}^2} = \sqrt{g_{\mathbb{H}^2}(v, v)},\ v \in T_z\mathbb{H}^2$. For a smooth immersed curve $\gamma = (\gamma^{(1)}, \gamma^{(2)})$ in $\mathbb{H}^2$, $\gamma \in C^\infty(\mathbb{S}^1, \mathbb{H}^2)$, the length is as in the Introduction given by

$$\mathcal{L}_{\mathbb{H}^2}(\gamma) := \int_0^1 \frac{|\gamma'(x)|_{\mathbb{R}^2}}{\gamma^{(2)}(x)}\,\mathrm{d}x = \int_0^1 \mathrm{d}s, \tag{2-6}$$

where $\mathrm{d}s = |\partial_x \gamma|_{\mathbb{H}^2}\,\mathrm{d}x$ denotes the arc length parameter, and the derivative with respect to x is abbreviated with the prime. As usual, $\partial_s = \partial_x / |\partial_x\gamma|_{\mathbb{H}^2}$ denotes the arc length derivative. The curvature vector field of γ is given by

$$\kappa[\gamma] = \nabla_s \partial_s \gamma = \begin{pmatrix} \partial_s^2\gamma^{(1)} - (2/\gamma^{(2)})\partial_s\gamma^{(1)}\partial_s\gamma^{(2)} \\ \partial_s^2\gamma^{(2)} + (1/\gamma^{(2)})((\partial_s\gamma^{(1)})^2 - (\partial_s\gamma^{(2)})^2) \end{pmatrix} \tag{2-7}$$

as an element of $T_z\mathbb{H}^2$ [Dall'Acqua and Spener 2017, (12)]. Here ∇_s denotes the covariant derivative along γ with respect to the Levi-Civita connection on $\mathbb{H}^2$. We write $\kappa = \kappa[\gamma]$ if the curve is clear from the context. The *elastic energy* $\mathcal{E}$ of γ is then defined to be

$$\mathcal{E}(\gamma) := \int_\gamma |\kappa|^2_{\mathbb{H}^2}\,\mathrm{d}s.$$

Its critical points are called *free hyperbolic elastica* and satisfy

$$(\nabla_s^{\perp})^2\kappa + \tfrac{1}{2}|\kappa|^2_{\mathbb{H}^2}\kappa - \kappa = 0,$$

where $\nabla_s^{\perp}\eta = \nabla_s\eta - \langle\nabla_s\eta, \partial_s\gamma\rangle_{\mathbb{H}^2}\partial_s\gamma$ is the covariant derivative on the normal bundle of γ.

We collect some results connecting the length and the elastic energy of smooth closed curves in the hyperbolic plane.

Theorem 2.2 [Müller and Spener 2020, Theorem 5.3]. *For each $\varepsilon > 0$ there exists $c(\varepsilon) > 0$ such that*

$$\frac{\mathcal{E}(\gamma)}{\mathcal{L}_{\mathbb{H}^2}(\gamma)} \geq c(\varepsilon)$$

for all immersed and closed curves $\gamma \in C^\infty(\mathbb{S}^1, \mathbb{H}^2)$ such that $\mathcal{E}(\gamma) \leq 16 - \varepsilon$.

Note that the energy threshold of 16 is sharp for this result; see [Müller and Spener 2020].

We also fix the notion of the Euclidean length of the curve $\gamma : \mathbb{S}^1 \to \mathbb{H}^2 \subset \mathbb{R}^2$, which is given by $\mathcal{L}_{\mathbb{R}^2}(\gamma)$. We also consider the *Euclidean curvature* of $\gamma : \mathbb{S}^1 \to \mathbb{R}^2$, which we will denote by

$$\vec{\kappa}_{\mathrm{euc}}[\gamma] := \frac{1}{|\gamma'|}\frac{d}{dt}\frac{\gamma'}{|\gamma'|},$$

and the *Euclidean scalar curvature* $\kappa_{\mathrm{euc}}[\gamma] := (1/|\gamma'|^2)\langle \gamma'', n\rangle_{\mathbb{R}^2}$. To finish this section we discuss some relations between Euclidean and hyperbolic length.

Lemma 2.3. *Let* $\gamma \in C^\infty(\mathbb{S}^1, \mathbb{H}^2)$ *and* $a, b \in [0, 1]$. *Then*

$$\gamma^{(2)}(b)e^{-\mathcal{L}_{\mathbb{H}^2}(\gamma)} \le \gamma^{(2)}(a) \le \gamma^{(2)}(b)e^{\mathcal{L}_{\mathbb{H}^2}(\gamma)} \tag{2-8}$$

and

$$\mathcal{L}_{\mathbb{H}^2}(\gamma) \ge \frac{\mathcal{L}_{\mathbb{R}^2}(\gamma)}{\sup_{\mathbb{S}^1}\gamma^{(2)}}. \tag{2-9}$$

Proof. For γ, a, b as in the statement, we find by (2-6)

$$\mathcal{L}_{\mathbb{H}^2}(\gamma) \ge \int_a^b \frac{|(\gamma^{(2)})'|}{\gamma^{(2)}}\,\mathrm{d}x \ge |\log\gamma^{(2)}(b) - \log\gamma^{(2)}(a)|,$$

and therefore $\log\gamma^{(2)}(b) - \mathcal{L}_{\mathbb{H}^2}(\gamma) \le \log\gamma^{(2)}(a) \le \log\gamma^{(2)}(b) + \mathcal{L}_{\mathbb{H}^2}(\gamma)$. Taking exponentials (2-8) follows. For (2-9) we simply estimate

$$\mathcal{L}_{\mathbb{H}^2}(\gamma) = \int_{\mathbb{S}^1}\frac{|\gamma'(u)|}{\gamma^2(u)}\,\mathrm{d}u \ge \frac{1}{\sup_{\mathbb{S}^1}\gamma^{(2)}}\int_{\mathbb{S}^1}|\gamma'(u)|\,\mathrm{d}u = \frac{\mathcal{L}_{\mathbb{R}^2}(\gamma)}{\sup_{\mathbb{S}^1}\gamma^{(2)}}. \qquad \square$$

2.3. ***Tori of revolution in*** $\mathbb{R}^3$**.** Here we collect some basic facts about tori of revolution. More precisely we express some geometric quantities associated to tori of revolution using only their profile curves. If $F_\gamma : \mathbb{S}^1 \times \mathbb{S}^1 \to \mathbb{R}^3$ is chosen as in Definition 1.1 we can compute the first fundamental form with respect to the local coordinates (u, v) of $\mathbb{S}^1 \times \mathbb{S}^1$. This yields the associated surface measure on the Riemannian manifold $(\mathbb{S}^1 \times \mathbb{S}^1, g = F_\gamma^* g_{\mathbb{R}^3})$ given by

$$\mathrm{d}\mu_g = 2\pi\gamma^{(2)}(u)|\gamma'(u)|_{\mathbb{R}^2}\,\mathrm{d}u\,\mathrm{d}v. \tag{2-10}$$

As we have already announced, the Willmore energy of F_γ can also be expressed only in terms of γ using the fundamental relationship

$$\mathcal{W}(F_\gamma) = \tfrac{\pi}{2}\mathcal{E}(\gamma); \tag{2-11}$$

see [Langer and Singer 1984a; Dall'Acqua and Spener 2018, Theorem 4.1]. Moreover, let κ be the hyperbolic curvature vector field of γ in $\mathbb{H}^2$. Then

$$-\langle(\nabla_s^\perp)^2\kappa + \tfrac{1}{2}|\kappa|^2_{\mathbb{H}^2}\kappa - \kappa, n\rangle_{\mathbb{H}^2} = 2(\gamma^{(2)})^4\big(\Delta H + 2H\big(\tfrac{1}{4}H^2 - K\big)\big), \tag{2-12}$$

where $n = (-\partial_s\gamma^{(2)}, \partial_s\gamma^{(1)})$ is the normal vector field along γ (see [Dall'Acqua and Spener 2018, Theorem 4.1]). In particular, F_γ is a Willmore torus of revolution if and only if γ is a hyperbolic elastica. In Appendix A we discuss the relationship between (2-12) and (1-2).

An immediate consequence of [Müller and Spener 2020, Proposition 6.5] (that builds on findings in [Langer and Singer 1984b]) is the following.

Proposition 2.4 (A gap theorem for Willmore tori of revolution). *Let $f : \mathbb{S}^1 \times \mathbb{S}^1 \to \mathbb{R}^3$ be a Willmore torus of revolution that satisfies $\mathcal{W}(f) \le 8\pi$. Then f is, up to reparametrization, the Clifford torus possibly rescaled and translated in the direction* $(1, 0, 0)^T$.

Proof. Let $f = F_\gamma$ be as in the statement with profile curve $\gamma \in C^\infty(\mathbb{S}^1, \mathbb{H}^2)$. From (2-12) we know that γ is a hyperbolic elastica. From (2-11) we can conclude that $\mathcal{E}(\gamma) \le 16$. By [Müller and Spener 2020, Proposition 6.5] we obtain that γ has to coincide (up to reparametrization) with the profile curve of the Clifford torus up to isometries of $\mathbb{H}^2$. This however implies that f is, up to reparametrization, the Clifford torus possibly rescaled and translated in the direction $(1, 0, 0)$. □

Another important quantity for our discussion is the second fundamental form $A[F_\gamma]$, which we will also express in terms of γ. A property which we will later make extensive use of is the fact that for a torus of revolution $f = F_\gamma$, $|A[F_\gamma]|^2 \in C^\infty(\mathbb{S}^1 \times \mathbb{S}^1)$ is a function that depends only on u (a parameter that describes the profile curve) and not on v (a parameter that describes the revolution). This is the reason why curvature concentration is "passed along" the revolution. We will describe this more precisely in Section 3.4. For this section it is enough to observe by a direct computation (see [Dall'Acqua and Spener 2018, p. 118]) that with respect to the normal $N_{F_\gamma} = (\partial_u F_\gamma \times \partial_v F_\gamma)/|\partial_u F_\gamma \times \partial_v F_\gamma|$ the principal curvatures are given by

$$\kappa_1[F_\gamma](u, v) = -\kappa_{\mathrm{euc}}[\gamma](u) \quad \text{and} \quad \kappa_2[F_\gamma](u, v) = \frac{(\gamma^{(1)})'(u)}{|\gamma'(u)|\gamma^{(2)}(u)}.$$

With this at hand, one can derive a useful bound for the length of the profile curve in terms of surface quantities.

Lemma 2.5. *Suppose that $f = F_\gamma : \mathbb{S}^1 \times \mathbb{S}^1 \to \mathbb{R}^3$ is a torus of revolution with profile curve γ. Then*

$$\mathcal{L}_{\mathbb{R}^2}(\gamma) \le \mu_{g_f}(\mathbb{S}^1 \times \mathbb{S}^1)^{1/2} \mathcal{W}(f)^{1/2}.$$

Proof. We may without loss of generality assume that γ is parametrized with constant velocity, i.e., $|\gamma'| = \mathcal{L}_{\mathbb{R}^2}(\gamma) =: L$. Recall from Appendix A that $\vec{H}(u, v) = (\kappa_1(u, v) + \kappa_2(u, v))N_f(u, v)$, where

$$N_f(u, v) = \frac{1}{\mathcal{L}_{\mathbb{R}^2}(\gamma)} \begin{pmatrix} (\gamma^{(2)})'(u) \\ -(\gamma^{(1)})'(u)\cos(2\pi v) \\ -(\gamma^{(1)})'(u)\sin(2\pi v) \end{pmatrix} \quad \text{with } u, v \in \mathbb{S}^1.$$

We show next that

$$-2L = \int_{\mathbb{S}^1 \times [0,1/2]} \vec{H} \cdot e_3 \, \mathrm{d}\mu_{g_f}. \tag{2-13}$$

Plugging in the quantities characterized in this section and using

$$(\gamma^{(1)})'^2 + (\gamma^{(2)})'^2 = L^2 \quad \text{and} \quad (\gamma^{(1)})''(\gamma^{(1)})' + (\gamma^{(2)})''(\gamma^{(2)})' = 0$$

we obtain

$$\begin{aligned}
\int_{\mathbb{S}^1\times[0,1/2]} \vec{H}\cdot e_3 \,\mathrm{d}\mu_{g_f} &= 2\pi \int_0^1\int_0^{1/2} (\kappa_1(u,v)+\kappa_2(u,v))(N_f(u,v)\cdot e_3)|\gamma'(u)|\gamma^{(2)}(u)\,\mathrm{d}v\,\mathrm{d}u \\
&= 2\pi \int_0^1\int_0^{1/2} \Big(-\kappa_{\mathrm{euc}}[\gamma](u)+\frac{(\gamma^{(1)})'(u)}{L\gamma^{(2)}(u)}\Big)[-(\gamma^{(1)})'(u)\sin(2\pi v)]\gamma^{(2)}(u)\,\mathrm{d}v\,\mathrm{d}u \\
&= -[-\cos(2\pi v)]_0^{1/2} \int_0^1 \Big(\frac{(\gamma^{(1)})''(\gamma^{(2)})'-(\gamma^{(2)})''(\gamma^{(1)})'}{L^3}+\frac{(\gamma^{(1)})'}{L\gamma^{(2)}}\Big)(\gamma^{(1)})'\gamma^{(2)}\,\mathrm{d}u \\
&= -2\frac{1}{L^3}\int_0^1 ((\gamma^{(1)})''(\gamma^{(1)})'(\gamma^{(2)})'-(\gamma^{(2)})''(\gamma^{(1)})'^2)\gamma^{(2)}\,\mathrm{d}u - \frac{2}{L}\int_0^1 (\gamma^{(1)})'^2\,\mathrm{d}u \\
&= -2\frac{1}{L^3}\int_0^1 (-(\gamma^{(2)})''(\gamma^{(2)})'^2-(\gamma^{(2)})''(\gamma^{(1)})'^2)\gamma^{(2)}\,\mathrm{d}u - \frac{2}{L}\int_0^1 (\gamma^{(1)})'^2\,\mathrm{d}u \\
&= \frac{2}{L^3}\int_0^1 (\gamma^{(2)})''L^2\gamma^{(2)}\,\mathrm{d}u - \frac{2}{L}\int_0^1 (\gamma^{(1)})'^2\,\mathrm{d}u \\
&= -\frac{2}{L}\int_0^1 (\gamma^{(2)})'^2\,\mathrm{d}u - \frac{2}{L}\int_0^1 (\gamma^{(1)})'^2\,\mathrm{d}u,
\end{aligned}$$

where we have used integration by parts in the last step. Adding up the integrands and once again using $(\gamma^{(1)})'^2 + (\gamma^{(2)})'^2 = L^2$, we obtain (2-13). From (2-13) and the Cauchy–Schwarz inequality we also conclude

$$2L \le \int_{\mathbb{S}^1\times\mathbb{S}^1} |\vec{H}|\,\mathrm{d}\mu_{g_f} \le 2\mathcal{W}(f)^{1/2}\mu_{g_f}(\mathbb{S}^1\times\mathbb{S}^1)^{1/2}. \qquad \square$$

A quantity which we will also study is the diameter.

Lemma 2.6. *Let $f = F_\gamma : \mathbb{S}^1\times\mathbb{S}^1 \to \mathbb{R}^3$ be a torus of revolution with profile curve γ. Then,*

$$\operatorname{diam}(F_\gamma(\mathbb{S}^1\times\mathbb{S}^1)) \le \tfrac{1}{2}\mathcal{L}_{\mathbb{R}^2}(\gamma) + 2\|\gamma^{(2)}\|_{L^\infty}.$$

Proof. Let $(u,v), (u',v') \in \mathbb{S}^1\times\mathbb{S}^1$ and $f = F_\gamma$ be as in the statement. Without loss of generality we can assume that $\gamma^{(2)}(u) \le \gamma^{(2)}(u')$. We start proving

$$|f(u,v) - f(u',v')| \le |\gamma(u)-\gamma(u')| + \sqrt{2}\gamma^{(2)}(u)\sqrt{1-\cos(2\pi(v-v'))}.$$

First observe that $|f(u,v)-f(u',v')| \le |f(u',v')-f(u,v')|+|f(u,v')-f(u,v)|$. Using the definition of the Euclidean distance we find $|f(u',v')-f(u,v')| = |\gamma(u)-\gamma(u')|$. Similarly,

$$\begin{aligned}
|f(u,v')-f(u,v)| &= \gamma^{(2)}(u)\sqrt{(\cos(2\pi v)-\cos(2\pi v'))^2+(\sin(2\pi v)-\sin(2\pi v'))^2} \\
&= \gamma^{(2)}(u)\sqrt{2-2\cos(2\pi(v-v'))}.
\end{aligned}$$

Both computations imply the desired estimate, and the asserted diameter bound follows immediately. $\square$

3. The Willmore flow of tori of revolution

In this section we understand the interplay between the rotational symmetry and the *curvature concentration criterion*, which is able to detect *singularities* of the Willmore flow. This gives us a better understanding of the singularities that can arise in our symmetric setting. We will then prove the main theorems by excluding those singularities in certain circumstances.

3.1. ***Singularities of the Willmore flow.*** In this section we summarize what singularities of the Willmore flow look like. The following result summarizes a list of results that have been obtained previously in other articles on the Willmore flow. It exposes the diameter of appropriate parabolic rescalings as a quantity whose control is sufficient for convergence. The appropriate rescaling is given by a *concentration property* of the Willmore flow; see Appendix D. In the following discussion we will use the two parameters ε_0 and c_0 which have been introduced in Theorem D.1.

Theorem 3.1 (Convergence criterion of the Willmore flow; proof in Appendix D). *Let Σ be a compact two-dimensional manifold without boundary and let $f : [0, T) \times \Sigma \to \mathbb{R}^n$ be a maximal evolution by the Willmore flow with initial datum f_0. Consider an arbitrary sequence $(t_j)_{j\in\mathbb{N}} \subset (0, T)$ with $t_j \to T$. Then, the* ***concentration radii***

$$r_j := \sup\left\{ r > 0 : \text{ for all } x \in \mathbb{R}^n \text{ one has } \int_{f(t_j)^{-1}(B_r(x))} |A(t_j)|^2 \,\mathrm{d}\mu_{g_{f(t_j)}} \le \varepsilon_0 \right\}, \tag{3-1}$$

$j \in \mathbb{N}$, satisfy $t_j + c_0 r_j^4 < T$ for all $j \in \mathbb{N}$. Further, the maps

$$\tilde{f}_{j,c_0} : \Sigma \to \mathbb{R}^3, \quad \tilde{f}_{j,c_0} := \frac{f(t_j + c_0 r_j^4)}{r_j},$$

are called ***concentration rescalings*** *and one of the following alternatives occurs*

Case 1: *convergent evolution. There exists $\delta > 0$ such that $\delta < r_j < 1/\delta$. Then $T = \infty$. If additionally $(\operatorname{diam}(\tilde{f}_{j,c_0}))_{j\in\mathbb{N}}$ is uniformly bounded then the Willmore flow converges to a Willmore immersion. More precisely there exists a Willmore immersion $f_\infty : \Sigma \to \mathbb{R}^n$ such that $f(t) \to f_\infty$ in C^k for all $k \in \mathbb{N}$ as $t \to \infty$.*

Case 2: *blow-up or blow-down. A subsequence of $(r_j)_{j\in\mathbb{N}}$ goes either to zero or to infinity. In this case one has $\operatorname{diam}(\tilde{f}_{j,c_0}) \to \infty$ as $j \to \infty$.*

In particular, if $(\operatorname{diam}(\tilde{f}_{j,c_0}))_{j\in\mathbb{N}}$ is uniformly bounded, then $T = \infty$ and the Willmore flow converges to a Willmore immersion $f_\infty : \Sigma \to \mathbb{R}^n$ in C^k for all $k \in \mathbb{N}$.

In the coming sections we will study the relation between the diameter of the concentration rescalings and the hyperbolic length of the profile curves. Having understood this we will finally be able to obtain Theorems 1.2 and 1.3.

3.2. ***Dimension reduction.*** We have already announced that the rotational symmetry is preserved along the flow. This section is devoted to the proof of this fact, see Lemma 3.3. In the proof of Lemma 3.3 we will make use of an alternative characterization of tori of revolution, see Definition 1.1, which we state next.

Proposition 3.2. *Let $f : \mathbb{S}^1 \times \mathbb{S}^1 \to \mathbb{R}^3$ be a smooth immersion. Then, f is a torus of revolution if and only if*

$$\text{for all } \phi \in \mathbb{S}^1 \quad f(u, v+\phi) = R_{2\pi\phi} f(u, v), \quad \text{where } R_z = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos z & -\sin z \\ 0 & \sin z & \cos z \end{pmatrix}, \tag{3-2}$$

$$\text{for all } u \in \mathbb{S}^1 \quad f^{(3)}(u, 0) = 0 \quad \text{and} \quad f^{(2)}(u_0, 0) \geq 0 \ \text{ for one value } u_0 \in \mathbb{S}^1. \tag{3-3}$$

Proof. If f is a torus of revolution then (3-2) and (3-3) can be checked by direct computation. If (3-2) and (3-3) hold for some immersion $f : \mathbb{S}^1 \times \mathbb{S}^1 \to \mathbb{R}^3$ then one can define a smooth curve $\gamma : \mathbb{S}^1 \to \mathbb{R}^2$ by $\gamma(u) := (f^{(1)}(u, 0), f^{(2)}(u, 0))$. Equation (1-4) is then easy to check, but it also needs to be shown that $\gamma(u) \in \mathbb{H}^2$ for all $u \in \mathbb{S}^1$. So far we have

$$f(u, v) = \big(\gamma^{(1)}(u), \gamma^{(2)}(u)\cos(2\pi v), \gamma^{(2)}(u)\sin(2\pi v)\big) \quad \text{for all } (u, v) \in \mathbb{S}^1 \times \mathbb{S}^1.$$

If now there exists a point $u_0 \in \mathbb{S}^1 \times \mathbb{S}^1$ such that $\gamma^{(2)}(u_0) = 0$ then one can compute

$$\partial_v f(u_0, v) = (0, 0, 0)^T \quad \text{for all } v \in \mathbb{S}^1,$$

which is a contradiction to the fact that f is an immersion. Hence $\gamma^{(2)}$ may not change sign or attain the value zero. As a consequence, $\gamma^{(2)} > 0$ and the claim follows. □

In particular, given a torus of revolution its profile curve is given by $\gamma(u) := (f^{(1)}(u, 0), f^{(2)}(u, 0))$. Note that — by inspection of the previous proof — each immersion $f : \mathbb{S}^1 \times \mathbb{S}^1 \to \mathbb{R}^3$ that fulfills (3-2), as well as $f^{(3)}(u, 0) = 0$ for all $u \in \mathbb{S}^1$, must satisfy $f^{(2)}(\,\cdot\,, 0) \neq 0$. In particular it cannot change sign. Thus, either $f^{(2)}(\,\cdot\,, 0) > 0$ or $f^{(2)}(\,\cdot\,, 0) < 0$. In the latter case $f(\,\cdot\,, \cdot + 1)$ defines a torus of revolution. This shows also consistency of our definition with [Blatt 2009, Definition 2.2], whose results we will need later.

When it comes to evolutions $(f(t))_{t\geq 0}$, we however want to work without reparametrizations of $f(t)$ along the flow and hence we specify $\gamma^{(2)} = f^{(2)}(\,\cdot\,, 0) > 0$ (and we check that this remains satisfied along the flow).

Lemma 3.3. *Let $f_0 : \mathbb{S}^1 \times \mathbb{S}^1 \to \mathbb{R}^3$ be a torus of revolution and let $(f(t))_{t\in[0,T)} : \mathbb{S}^1 \times \mathbb{S}^1 \to \mathbb{R}^3$ evolve by the Willmore flow with initial datum f_0. Then $(f(t))_{t\in[0,T)}$ is a torus of revolution for all $t \in [0, T)$.*

Proof. We prove that $(f(t))_{t\in[0,T)}$ satisfies (3-2) and (3-3) for all $t \in [0, T)$ so that the claim follows from Proposition 3.2.

Let $\phi \in \mathbb{S}^1$. We observe that $R_{2\pi\phi}$ is an isometry in $\mathbb{R}^3$ and $(u, v) \mapsto (u, v+\phi)$ is a diffeomorphism. Hence $(R_{2\pi\phi}^{-1} f(t)(\,\cdot\,, \cdot + \phi))_{t\in[0,T)} : \mathbb{S}^1 \times \mathbb{S}^1 \to \mathbb{R}^3$ is an evolution by Willmore flow with initial value $R_{2\pi\phi}^{-1} f_0(\,\cdot\,, \cdot + \phi)$. Recall now that f_0 satisfies (3-2), i.e., $R_{2\pi\phi}^{-1} f_0(\,\cdot\,, \cdot + \phi) = f_0$. By the uniqueness result for the Willmore flow, see [Kuwert and Schätzle 2002, Proposition 1.1], we obtain that

$$R_{2\pi\phi}^{-1} f(t)(u, v+\phi) = f(t)(u, v) \quad \text{for all } (u, v) \in \mathbb{S}^1 \times \mathbb{S}^1,$$

that is, (3-2). In particular, there exist smooth functions $x, y, z : [0, T) \times \mathbb{S}^1$ such that

$$f(t)(u, v) = R_{2\pi v}(f(t)(u, 0)) = R_{2\pi v} \begin{pmatrix} x(t, u) \\ y(t, u) \\ z(t, u) \end{pmatrix}. \tag{3-4}$$

As an intermediate step for (3-3) we show that $f(t)^{(3)}(u, 0) = 0$ for all $t > 0$ and $u \in \mathbb{S}^1$, i.e., $z \equiv 0$ on $[0, T) \times \mathbb{S}^1$. Set

$$S := \sup\{s \in [0, T) : f(t) \text{ is a torus of revolution for all } t \in [0, s]\}.$$

We show that $S = T$. If $S < T$ then observe that $z(S, u) = 0$ for all $u \in \mathbb{S}^1$ by smoothness of $(f(t))_{t\in[0,T)}$ and the fact that $f(t)^{(3)}(u, 0) = 0$ for all $t \in [0, S)$ and $u \in \mathbb{S}^1$. As additionally $y(S, \cdot)$ is nonnegative and $f(S)$ is an immersion, $f(S)$ is a torus of revolution by Proposition 3.2.

Restart the flow with $\tilde{f}_0 := f(S)$ (if $S = 0$ there is no need to restart). Choose now c_0, ρ for $\tilde{f}_0$ to be as in Theorem D.1 and consider the time interval $I := [S, S + (1/c_0)\rho^4]$. The Willmore flow equation in the local coordinates (u, v) of $\mathbb{S}^1 \times \mathbb{S}^1$ reads

$$\partial_t f(t) = P(A(t), \nabla^\perp A(t), (\nabla^\perp)^2 A(t)) \vec{N}_{f(t)},$$

where

$$\vec{N}_{f(t)} := \frac{\partial_u f(t) \times \partial_v f}{|\partial_u f(t) \times \partial_v f(t)|}$$

and $P(A, \nabla^\perp A, (\nabla^\perp)^2 A)$ is a scalar quantity that can be bounded in terms of $\|g\|_{L^\infty(\mathbb{S}^1\times\mathbb{S}^1)}$ and $\|(\nabla^\perp)^k A\|_{L^\infty(\mathbb{S}^1\times\mathbb{S}^1)}$ $(k = 0, 1, 2)$. All of those remain bounded in I by (D-1) and the explanation afterwards. The idea now is to consider the evolution equation satisfied by $z(t, u)^2$. Since

$$\vec{N}_{f(t)}(u, v) = \frac{1}{\sqrt{\det(g(t))}} R_{2\pi v} \begin{pmatrix} y(t, u)\partial_u y(t, u) + \partial_u z(t, u) z(t, u) \\ -y(t, u)\partial_u x(t, u) \\ -z(t, u)\partial_u x(t, u) \end{pmatrix},$$

we find

$$\begin{aligned} \partial_t(z(t, u)^2) &= 2z(t, u)\partial_t z(t, u) = 2z(t, u) P(A(t), \nabla^\perp A(t), (\nabla^\perp)^2 A(t)) \vec{N}^{(3)}_{f(t)}(u, 0) \\ &= -2\frac{1}{\sqrt{\det(g(t))}} P(A(t), \nabla^\perp A(t), (\nabla^\perp)^2 A(t)) \partial_u x(t, u) z(t, u)^2. \end{aligned}$$

By Theorem D.1 for fixed $u \in \mathbb{S}^1$ we have obtained

$$\begin{cases} \partial_t(z(t, u)^2) \le C z(t, u)^2, & t \in I, \\ z(S, u)^2 = 0, \end{cases}$$

and hence $z(t, u) = 0$ for all $t \in I$ and all $u \in \mathbb{S}^1$, as u was chosen arbitrarily. Similar to before, again by Proposition 3.2 and the discussion afterwards it can be shown that $y(t, \cdot) > 0$ for all $t \in I$. This is finally a contradiction to the choice of S and thus $S = T$. The claim follows. □

The previous lemma implies that for each Willmore evolution $(f(t))_{t\geq 0}$ starting at a torus of revolution $f_0 : \mathbb{S}^1 \times \mathbb{S}^1 \to \mathbb{R}^3$ there exists a unique smooth evolution of curves $(\gamma(t))_{t\in[0,T)} \subset C^\infty(\mathbb{S}^1, \mathbb{H}^2)$, $\gamma(t)(u) = f(t)(u, 0)$ such that

$$f(t)(u, v) = \begin{pmatrix} \gamma^{(1)}(t)(u) \\ \gamma^{(2)}(t)(u)\cos(2\pi v) \\ \gamma^{(2)}(t)(u)\sin(2\pi v) \end{pmatrix}, \tag{3-5}$$

whereupon the flow can also be seen as an evolution of $(\gamma(t))_{t\in[0,T)}$.

3.3. *Symmetry of the limit immersion.* Theorem 3.1 provides us with a general convergence criterion for the Willmore flow and yields a smooth limit immersion f_∞, which is a Willmore immersion. In this section we need to check that the revolution symmetry is passed along to the limit; i.e., we will prove that under certain conditions the limit immersion f_∞ is a (Willmore) torus of revolution. Let us stress that this not trivial because the notion of convergence is *geometric*, i.e., invariant with respect to reparametrization. Hence classical results about pointwise convergence cannot be applied.

The arguments in this section make frequent use of the fact that to each torus of revolution $f = F_\gamma : \mathbb{S}^1 \times \mathbb{S}^1 \to \mathbb{R}^3$ one can easily associate a smooth orthonormal frame with respect to g_f, given by

$$E_1(u, v) := \frac{1}{|\gamma'(u)|}\frac{\partial}{\partial u}, \quad E_2(u, v) := \frac{1}{2\pi\gamma^{(2)}(u)}\frac{\partial}{\partial v}. \tag{3-6}$$

This orthonormal frame also has some further interesting properties, for example that it diagonalizes the second fundamental form $A[f]$, and hence yields the principal curvatures of f. The first principal curvature

$$\kappa_1[f] = \langle A[f]_{(u,v)}(E_1, E_1), N_f\rangle_{\mathbb{R}^3} = -\kappa_{\mathrm{euc}}[\gamma](u)$$

coincides up to a sign with the Euclidean scalar curvature of the profile curve, while the second principal curvature

$$\kappa_2[f] = \langle A[f]_{(u,v)}(E_2, E_2), N_f\rangle_{\mathbb{R}^3} = \frac{(\gamma^{(1)})'(u)}{|\gamma'(u)|\gamma^{(2)}(u)}$$

depends heavily on the distance of the profile curve to the revolution axis. This will be of great use when it comes to explicit estimates involving the second fundamental form.

Lemma 3.4 (Revolution symmetry of the limit). *Suppose that $f : [0, \infty) \times (\mathbb{S}^1 \times \mathbb{S}^1) \to \mathbb{R}^3$ is a global evolution by Willmore flow, convergent to some Willmore immersion $f_\infty : \mathbb{S}^1 \times \mathbb{S}^1 \to \mathbb{R}^3$ in C^k for all $k \in \mathbb{N}$. Suppose further that $f(0)$ is a torus of revolution and $(\gamma(t))_{t\in[0,\infty)} \subset C^\infty(\mathbb{S}^1, \mathbb{R}^2)$ is as in (3-5). Then f_∞ is (up to reparametrization) a Willmore torus of revolution. A profile curve γ_∞ of f_∞ can be obtained by a $C^m(\mathbb{S}^1, \mathbb{R}^2)$-limit of appropriate reparametrizations of a sequence $(\gamma(t_j))_{j\in\mathbb{N}}$, $t_j \to \infty$. Here $m \in \mathbb{N}$ is arbitrary. In particular $\gamma_\infty \in C^\infty(\mathbb{S}^1, \mathbb{H}^2)$ is a hyperbolic elastica.*

Proof. Let $(t_j)_{j\in\mathbb{N}} \subset [0, \infty)$ be an arbitrary sequence such that $t_j \to \infty$.

<u>Step 1</u>: bounds for the profile curves. After reparametrization we may assume without loss of generality that $(\gamma(t_j))_{j\in\mathbb{N}}$ is parametrized with constant Euclidean speed.

Now fix $m \in \mathbb{N}$ arbitrary. To bound the $W^{m,2}$-norm of $(\gamma(t_j))_{j\in\mathbb{N}}$ we first bound $\|\gamma(t_j)\|_{L^\infty(\mathbb{S}^1,\mathbb{R}^2)}$. To this end we observe by (3-5) that

$$\|\gamma(t_j)\|_{L^\infty(\mathbb{S}^1,\mathbb{R}^2)} = \|f(t_j)\|_{L^\infty(\mathbb{S}^1\times\mathbb{S}^1,\mathbb{R}^3)}.$$

Now $\|f(t_j)\|_{L^\infty}$ is uniformly bounded because it converges in C^k for all $k \in \mathbb{N}$ to f_∞, whose image is a compact subset of $\mathbb{R}^3$. Note that we have used here that the L^∞-norm is not affected by reparametrization. Next we bound $\mathcal{L}_{\mathbb{R}^2}(\gamma(t_j)) = \|\partial_u\gamma(t_j)\|_{L^\infty}$. We use Lemmas 2.5 and D.7 to compute

$$\mathcal{L}_{\mathbb{R}^2}(\gamma(t_j)) \le \mathcal{W}(f(t_j))^{1/2}\mu_{g_{f(t_j)}}(\mathbb{S}^1\times\mathbb{S}^1)^{1/2} \le \operatorname{diam}(f(t_j)(\mathbb{S}^1\times\mathbb{S}^1))\mathcal{W}(f(t_j)).$$

Notice that $\operatorname{diam}(f(t_j)(\mathbb{S}^1\times\mathbb{S}^1)) \le 2\|f(t_j)\|_{L^\infty(\mathbb{S}^1\times\mathbb{S}^1,\mathbb{R}^3)}$, which is uniformly bounded in j. By Lemma C.5 and the fact that $\mathbb{S}^1\times\mathbb{S}^1$ is compact we infer that $\mathcal{W}(f(t_j)) \to \mathcal{W}(f_\infty)$ and hence $(\mathcal{W}(f(t_j)))_{j\in\mathbb{N}}$ is also uniformly bounded. We conclude the boundedness of $(\mathcal{L}_{\mathbb{R}^2}(\gamma(t_j)))_{j\in\mathbb{N}}$.

Further, we bound second derivatives uniformly in j. To this end we introduce the following notation. For a torus of revolution $f : \mathbb{S}^1\times\mathbb{S}^1 \to \mathbb{R}^3$ with profile curve $\gamma \in C^\infty(\mathbb{S}^1,\mathbb{H}^2)$ we introduce the vector field on $\mathbb{S}^1\times\mathbb{S}^1$

$$\partial_s|_{(u,v)} = \frac{1}{|\partial_u\gamma(u)|_{\text{euc}}}\,\frac{\partial}{\partial u}\Big|_{(u,v)}.$$

One easily checks that $g_f(\partial_s,\partial_s) = 1$ and

$$\begin{pmatrix} -\vec{\kappa}_{\text{euc}}[\gamma](u) \\ 0 \end{pmatrix} = A_{(u,0)}[f](\partial_s,\partial_s) \quad \text{for all } u \in \mathbb{S}^1.$$

By Remark D.4, $\|A[f(t_j)]\|_{L^\infty}$ is uniformly bounded in j. This is why

$$\|\vec{\kappa}_{\text{euc}}[\gamma(t_j)]\|_{L^\infty} \le \|A[f(t_j)]\|_{L^\infty}\|g_{f(t_j)}(\partial_s,\partial_s)\|^2_{L^\infty}$$

is also uniformly bounded in j. We next control all higher-order arclength derivatives of the curvature of $\gamma(t_j)$ uniformly in j. Easy tensor calculus and $\partial_s = \partial_u/|\partial_u\gamma(t_j)(u)|$ implies with (B-4)

$$\begin{aligned}\frac{1}{|\partial_u\gamma(t_j)(u)|}\begin{pmatrix} -\partial_u\vec{\kappa}_{\text{euc}}[\gamma(t_j)](u) \\ 0 \end{pmatrix} &= -D_{\partial_s}\begin{pmatrix} \vec{\kappa}_{\text{euc}}[\gamma(t_j)](u) \\ 0 \end{pmatrix} = D_{\partial_s}A[f(t_j)](\partial_s,\partial_s) \\ &= \nabla^\perp_{\partial_s}A(\partial_s,\partial_s) - \sum_{i=1}^{2}\langle A(\partial_s,\partial_s), A(\partial_s,E_i)\rangle_{\mathbb{R}^3} D_{E_i}[f(t_j)], \end{aligned} \tag{3-7}$$

where $\{E_1,E_2\}$ is an arbitrary orthonormal basis of $T_{(u,0)}(\mathbb{S}^1\times\mathbb{S}^1)$ with respect to $g_{f(t_j)}$ and we have used the (slightly ambiguous) shorthand notation A for $A[f(t_j)]$. Choosing $E_1 = \partial_s$ and

$$E_2(u,v) = \frac{1}{\gamma(t_j)^{(2)}(u)}\,\frac{\partial}{\partial v}\Big|_{(u,v)},$$

we obtain with (B-3)

$$\begin{aligned}\frac{1}{|\partial_u\gamma(t_j)(u)|}\begin{pmatrix} -\partial_u\vec{\kappa}_{\text{euc}}[\gamma(t_j)](u) \\ 0 \end{pmatrix} &= \nabla^\perp_{\partial_s}A(\partial_s,\partial_s) - |A(\partial_s,\partial_s)|^2 D_{\partial_s}f(t_j) \\ &= \nabla^\perp A(\partial_s,\partial_s,\partial_s) + A(\nabla_{\partial_s}\partial_s,\partial_s) + A(\partial_s,\nabla_{\partial_s}\partial_s) - |A(\partial_s,\partial_s)|^2 D_{\partial_s}f(t_j) \\ &= \nabla^\perp A(\partial_s,\partial_s,\partial_s) - |A(\partial_s,\partial_s)|^2 D_{\partial_s}f(t_j), \end{aligned} \tag{3-8}$$

where we have used in the last step that $\nabla_{\partial_s}\partial_s = 0$, which is an immediate consequence of the formula $df_p(\nabla_X Y) = \nabla^{\mathbb{R}^3}_{df_p(X)}(df_{(\,\cdot\,)}(Y))$ applied with $f = f(t_j)$. Note that

$$D_{\partial_s} f(u,0) = \frac{1}{|\gamma'(u)|} D_{\partial_u} f = \frac{1}{|\gamma'(u)|}(\partial_u f)$$

has Euclidean norm equal to 1. We obtain, since $g_{f(t_j)}(\partial_s,\partial_s) \le 1$, that

$$\frac{|\partial_u \vec{\kappa}_{\mathrm{euc}}[\gamma(t_j)](u)|}{|\partial_u \gamma(t_j)(u)|} \le \|\nabla^\perp A[f(t_j)]\|_{L^\infty} + \|A\|^2_{L^\infty}.$$

If we introduce the differential operator $\partial^{\mathrm{arc}} := \partial_u/|\partial_u\gamma(t_j)|$ on $\mathbb{S}^1$, we have obtained

$$\|\partial^{\mathrm{arc}}\vec{\kappa}_{\mathrm{euc}}[\gamma(t_j)]\|_{L^\infty} \le \|\nabla^\perp A[f(t_j)]\|_{L^\infty} + \|A\|^2_{L^\infty}. \tag{3-9}$$

Next we obtain by differentiating (3-8) and using the shorthand notation $f = f(t_j)$, as well as $\nabla_{\partial_s}\partial_s = 0$, again proceeding as in (3-7) and (3-8)

$$\begin{aligned}
&\begin{pmatrix} -(\partial^{\mathrm{arc}})^2\vec{\kappa}_{\mathrm{euc}}[\gamma(t_j)](u) \\ 0 \end{pmatrix} \\
&= D_{\partial_s}[\nabla^\perp A(\partial_s,\partial_s,\partial_s) - |A(\partial_s,\partial_s)|^2 D_{\partial_s} f] \\
&= D_{\partial_s}\nabla^\perp A(\partial_s,\partial_s,\partial_s) - \partial_s(|A(\partial_s,\partial_s)|^2) D_{\partial_s} f - |A(\partial_s,\partial_s)|^2 D_{\partial_s} D_{\partial_s} f \\
&= \nabla^\perp_{\partial_s}\nabla^\perp A(\partial_s,\partial_s,\partial_s) - (\nabla^\perp A(\partial_s,\partial_s,\partial_s), A(\partial_s,\partial_s)) D_{\partial_s} f - \partial_s(|A(\partial_s,\partial_s)|^2) D_{\partial_s} f - |A(\partial_s,\partial_s)|^2 D_{\partial_s} D_{\partial_s} f \\
&= (\nabla^\perp)^2 A(\partial_s,\partial_s,\partial_s,\partial_s) - (\nabla^\perp A(\partial_s,\partial_s,\partial_s), A(\partial_s,\partial_s)) D_{\partial_s} f - \partial_s(|A(\partial_s,\partial_s)|^2) D_{\partial_s} f - |A(\partial_s,\partial_s)|^2 D_{\partial_s} D_{\partial_s} f.
\end{aligned}$$

Note that since A is normal and $\nabla_{\partial_s}\partial_s = 0$ we have

$$\begin{aligned}
\partial_s|A(\partial_s,\partial_s)|^2 &= 2(D_{\partial_s} A(\partial_s,\partial_s), A(\partial_s,\partial_s)) \\
&= 2(\nabla^\perp_{\partial_s} A(\partial_s,\partial_s), A(\partial_s,\partial_s)) = 2(\nabla^\perp A(\partial_s,\partial_s,\partial_s), A(\partial_s,\partial_s)).
\end{aligned}$$

Moreover we have

$$D_{\partial_s} D_{\partial_s} f = (D_{\partial_s} D_{\partial_s} f)^T + A(\partial_s,\partial_s).$$

An easy computation[1] now reveals that $(D_{\partial_s} D_{\partial_s} f)^T = 0$ and we obtain

$$\begin{aligned}
&\begin{pmatrix} -(\partial^{\mathrm{arc}})^2\vec{\kappa}_{\mathrm{euc}}[\gamma(t_j)](u) \\ 0 \end{pmatrix} \\
&\qquad = (\nabla^\perp)^2 A(\partial_s,\partial_s,\partial_s,\partial_s) - 3(\nabla^\perp A(\partial_s,\partial_s,\partial_s), A(\partial_s,\partial_s)) D_{\partial_s} f - |A(\partial_s,\partial_s)|^2 A(\partial_s,\partial_s).
\end{aligned}$$

For short we write

$$\begin{pmatrix} -(\partial^{\mathrm{arc}})^2\vec{\kappa}_{\mathrm{euc}}[\gamma(t_j)](u) \\ 0 \end{pmatrix} = (\nabla^\perp)^2 A + \nabla^\perp A * A * D_{\partial_s} f + A * A * A,$$

which implies

$$\|(\partial^{\mathrm{arc}})^2\vec{\kappa}_{\mathrm{euc}}\|_{L^\infty} \le C[\|(\nabla^\perp)^2 A\|_{L^\infty} + \|\nabla A\|_{L^\infty}\,\|A\|_{L^\infty} + \|A\|^3_{L^\infty}].$$

[1] Recall that the normal to the curve γ coincides up to a sign with the normal to $f(\Sigma)$.

Inductively one shows that for all $m \in \mathbb{N}$

$$\begin{pmatrix} -(\partial^{\mathrm{arc}})^m \vec{\kappa}_{\mathrm{euc}}[\gamma(t_j)](u) \\ 0 \end{pmatrix} = (\nabla^\perp)^m A + P_1(A, \nabla^\perp A, \dots, (\nabla^\perp)^{m-1} A) * D_{\partial_s} f + P_2(A, \nabla^\perp A, \dots, (\nabla^\perp)^{m-2} A), \quad (3\text{-}10)$$

where P_1 is a real-valued polynomial of degree ≤ 2 and P_2 is an $\mathbb{R}^3$-valued polynomial of degree ≤ 3.

We conclude from (3-10) that for all $m \in \mathbb{N}$

$$\|\partial_u^m \gamma(t_j)\|_{L^\infty} \leq C(m) \mathcal{L}_{\mathbb{R}^2}(\gamma(t_j))^m \left[\|(\nabla^\perp)^m A\|_{L^\infty} + \sum_{i=0}^{m-1} \|(\nabla^\perp)^i A\|_{L^\infty}^3 \right]. \quad (3\text{-}11)$$

Hence for each fixed $m \in \mathbb{N}$ we can bound $(\gamma(t_j))_{j \in \mathbb{N}}$ uniformly in $W^{m+1,\infty}(\mathbb{S}^1, \mathbb{R}^2)$ and hence obtain a convergent subsequence in $C^m(\mathbb{S}^1, \mathbb{R}^2)$ for any m.

Step 2: the limit curve is a profile curve. By a diagonal argument we can also obtain a sequence $t_j \to \infty$ (no relabeling) and $\gamma_\infty \in C^\infty(\mathbb{S}^1, \mathbb{R}^2)$ such that $\gamma(t_j)$ converges to γ_∞ in $C^m(\mathbb{S}^1, \mathbb{R}^2)$ for all $m \in \mathbb{N}$ (classical convergence). Note also that γ_∞ is parametrized with constant Euclidean speed and $\gamma_\infty^{(2)} \geq 0$ on $\mathbb{S}^1$. We next show that $\gamma_\infty \in C^\infty(\mathbb{S}^1, \mathbb{H}^2)$, i.e., $\inf_{\mathbb{S}^1} \gamma_\infty^{(2)} > 0$. Indeed, assume the opposite, i.e., there exists $u_0 \in \mathbb{S}^1$ such that $\gamma_\infty^{(2)}(u_0) = 0$. Notice that this and $\gamma_\infty^{(2)} \geq 0$ also yield $(\gamma_\infty^{(2)})'(u_0) = 0$. As a consequence, we infer that there exist $C > 0$ and $\delta_0 > 0$ such that $0 \leq \gamma_\infty^{(2)}(u) \leq C|u - u_0|^2$ for all $u \in (u_0 - \delta_0, u_0 + \delta_0)$. The fact that γ_∞ is parametrized with constant Euclidean velocity also yields that $|(\gamma_\infty^{(1)})'(u_0)| = \mathcal{L}_{\mathbb{R}^2}(\gamma_\infty) > 0$. With this information we now estimate the following quantity for arbitrary $\delta \in (0, \delta_0)$:

$$Q := \int_0^1 \frac{|(\gamma_\infty^{(1)})'(u)|^2}{\gamma_\infty^{(2)}(u)} \, \mathrm{d}u \geq \int_{u_0-\delta}^{u_0+\delta} \frac{|(\gamma_\infty^{(1)})'(u)|^2}{C|u-u_0|^2} \, \mathrm{d}u \geq \frac{1}{C\delta^2} \int_{u_0-\delta}^{u_0+\delta} |(\gamma_\infty^{(1)})'(u)|^2 \, \mathrm{d}u.$$

Taking the limit $\delta \to 0+$ yields infinity on the right-hand side, since

$$\frac{1}{2\delta} \int_{u_0-\delta}^{u_0+\delta} |(\gamma_\infty^{(1)})'(u)|^2 \, \mathrm{d}u \to |(\gamma_\infty^{(1)})'(u_0)|^2 = \mathcal{L}_{\mathbb{R}^2}(\gamma_\infty)^2 > 0.$$

We infer that $Q = \infty$. On the other hand, Fatou's lemma and the explicit formula for the second principal curvature κ_2 of a surface imply that

$$\begin{aligned} Q &\leq \liminf_{j\to\infty} \int_0^1 \frac{|(\gamma(t_j)^{(1)})'|^2}{\gamma(t_j)^{(2)}} \, \mathrm{d}u = \liminf_{j\to\infty} \frac{\mathcal{L}_{\mathbb{R}^2}(\gamma(t_j))}{2\pi} \int_0^1 \int_0^1 \frac{2\pi |(\gamma(t_j)^{(1)})'|^2}{\gamma(t_j)^{(2)} \mathcal{L}_{\mathbb{R}^2}(\gamma(t_j))} \, \mathrm{d}u \, \mathrm{d}v \\ &\leq \liminf_{j\to\infty} \frac{\mathcal{L}_{\mathbb{R}^2}(\gamma(t_j))}{2\pi} \int_0^1 \int_0^1 \kappa_2[F_{\gamma(t_j)}]^2 \{2\pi \gamma(t_j)^{(2)} \mathcal{L}_{\mathbb{R}^2}(\gamma(t_j))\} \, \mathrm{d}u \, \mathrm{d}v \\ &\leq \liminf_{j\to\infty} \frac{\mathcal{L}_{\mathbb{R}^2}(\gamma(t_j))}{2\pi} \int_{\mathbb{S}^1 \times \mathbb{S}^1} |A[F_{\gamma(t_j)}]|^2 \, \mathrm{d}\mu_{F_{\gamma(t_j)}} \\ &= \liminf_{j\to\infty} \frac{\mathcal{L}_{\mathbb{R}^2}(\gamma(t_j))}{2\pi} \int_{\mathbb{S}^1 \times \mathbb{S}^1} |A[f(t_j)]|^2 \, \mathrm{d}\mu_{f(t_j)} = \liminf_{j\to\infty} \frac{2\mathcal{L}_{\mathbb{R}^2}(\gamma(t_j))}{\pi} \mathcal{W}(f(t_j)), \end{aligned}$$

where the last identity is due to the Gauss–Bonnet theorem; see (A-4). Recall from estimates in Step 1 that $\mathcal{L}_{\mathbb{R}^2}(\gamma(t_j))$ is uniformly bounded. As a consequence of this one infers that $Q < \infty$, a contradiction. We obtain therefore that $\gamma_\infty \in C^\infty(\mathbb{S}^1, \mathbb{H}^2)$.

Step 3: convergence of the associated surfaces. By the following proposition (Proposition 3.5), the tori of revolution $F_{\gamma(t_j)}$ converge to F_{γ_∞} classically in C^k for all k. Since $F_{\gamma(t_j)}$ is a reparametrization of $f(t_j)$ for all $j \in \mathbb{N}$, also $f(t_j)$ converges to F_{γ_∞} in C^k for all k. By assumption however, $f(t_j)$ also converges to f_∞ in C^k for all k (in general not anymore classically, but in the sense of Definition C.7). Applying Corollary C.12 we infer that f_∞ coincides up to reparametrization with F_{γ_∞}. In particular f_∞ is (up to reparametrization) a torus of revolution. Since f_∞ is also a Willmore immersion it must (up to reparametrization) be a Willmore torus of revolution. By (2-12) we infer also that γ_∞ is a hyperbolic elastica. □

The following proposition is needed to complete the proof of the previous lemma.

Proposition 3.5. *Let $m \geq 1$ and suppose that $(\gamma_j)_{j\in\mathbb{N}} \subset C^\infty(\mathbb{S}^1, \mathbb{H}^2)$ converges in $C^m(\mathbb{S}^1, \mathbb{R}^2)$ (classically) to some immersed curve $\gamma \in C^m(\mathbb{S}^1, \mathbb{H}^2)$. Then F_{γ_j} converges classically to F_γ in $C^m(\mathbb{S}^1 \times \mathbb{S}^1)$.*

Proof. We will use without further notice the characterization of C^m-convergence in Proposition C.9. We show the claim only for $m = 1$, the other cases follow by induction. We define $w_j : \mathbb{S}^1 \times \mathbb{S}^1 \to \mathbb{R}^3$ via

$$w_j(u, v) := F_{\gamma_j}(u, v) - F_\gamma(u, v) = \begin{pmatrix} \gamma_j^{(1)}(u) - \gamma^{(1)}(u) \\ (\gamma_j^{(2)}(u) - \gamma^{(2)}(u)) \cos(2\pi v) \\ (\gamma_j^{(2)}(u) - \gamma^{(2)}(u)) \sin(2\pi v) \end{pmatrix} \tag{3-12}$$

and we show that $\|w_j\|_{L^\infty(\mathbb{S}^1\times\mathbb{S}^1,\hat{g})}, \|Dw_j\|_{L^\infty(\mathbb{S}^1\times\mathbb{S}^1,\hat{g})} \to 0$ as $j \to \infty$. Here $\hat{g} = F_\gamma^* g_{\mathbb{R}^3}$ is the metric induced by F_γ. The fact that $\|w_j\|_{L^\infty} \to 0$ follows directly from (3-12) by the estimate

$$\|w_j\|_{L^\infty} \leq \|\gamma_j - \gamma\|_{L^\infty} \to 0.$$

Let E_1, E_2 be the orthonormal frame as in (3-6). Then

$$\|Dw_j\|_{L^\infty} = \sup_{\mathbb{S}^1\times\mathbb{S}^1} \sup_{g(X,X)\leq 1} |Dw_j(X)| = \sup_{\mathbb{S}^1\times\mathbb{S}^1} \sup_{\theta_1^2+\theta_2^2\leq 1} |Dw_j(\theta_1 E_1 + \theta_2 E_2)|, \tag{3-13}$$

and

$$\begin{aligned} |Dw_j(E_1)| &= \frac{1}{|\gamma'(u)|}\left|\frac{\partial w_j}{\partial u}\right| \leq \frac{1}{|\gamma'(u)|}\|\gamma_j' - \gamma'\|_\infty \leq \frac{1}{\inf_{\mathbb{S}^1}|\gamma'|}\|\gamma_j' - \gamma'\|_\infty, \\ |Dw_j(E_2)| &= \frac{1}{2\pi\gamma^{(2)}(u)}\left|\frac{\partial w_j}{\partial v}\right| \leq \frac{1}{\inf_{\mathbb{S}^1}\gamma^{(2)}}\|\gamma_j - \gamma\|_{L^\infty}. \end{aligned}$$

Note that $\inf_{\mathbb{S}^1}|\gamma'| > 0$ as γ is immersed and $\inf_{\mathbb{S}^1}\gamma^{(2)} > 0$ since $\gamma \in C^\infty(\mathbb{S}^1, \mathbb{H}^2)$ and $\mathbb{S}^1$ is compact. The claim follows from (3-13) since $\gamma_j \to \gamma$ in C^1. □

3.4. ***Rotational symmetry and concentration.*** In this section we will prove a lemma that controls the distance of the concentration points to the axis of revolution. Here the revolution symmetry will play an important role. The following lemma is the main observation that rules out Case 2 in Theorem 3.1.

Lemma 3.6 (Distance control for concentration points). *Let $f : [0, T) \times (\mathbb{S}^1 \times \mathbb{S}^1) \to \mathbb{R}^3$ be a maximal evolution by Willmore flow such that $f(0)$ is a torus of revolution. Suppose that $t_j \to T$. Let $(r_j)_{j\in\mathbb{N}}$ be as in Theorem 3.1 and let $x_j \in \mathbb{R}^3$ be such that*

$$\int_{f(t_j)^{-1}(\overline{B_{r_j}(x_j)})} |A[f(t_j)]|^2 \, \mathrm{d}\mu_{g_{f(t_j)}} \geq \varepsilon_0. \tag{3-14}$$

Let $h_j \in \mathbb{R}$, $\rho_j > 0$ and $\sigma_j \in \mathbb{S}^1$ such that x_j/r_j is expressed in cylindrical coordinates by[2] *$x_j/r_j = (h_j, \rho_j\sigma_j)$. Then $(\rho_j)_{j\in\mathbb{N}}$ is bounded.*

Proof. We first use scaling properties to obtain that

$$\int_{(f(t_j)/r_j)^{-1}(\overline{B_1(x_j/r_j)})} \left| A\left[\frac{f(t_j)}{r_j}\right]\right|^2 \mathrm{d}\mu_{g_{f(t_j)/r_j}} \geq \varepsilon_0. \tag{3-15}$$

Now write $x_j/r_j = (h_j, \rho_j\sigma_j)$ as in the statement. Since $f(t_j)/r_j$ has a revolution symmetry (see Lemma 3.3), we conclude from (3-15) that the curvature concentration does not only happen at points but actually on circles. More precisely,

$$\int_{(f(t_j)/r_j)^{-1}(\overline{B_1(h_j,\rho_j\sigma)})} \left| A\left[\frac{f(t_j)}{r_j}\right]\right|^2 \mathrm{d}\mu_{g_{f(t_j)/r_j}} \geq \varepsilon_0 \quad \text{for all } \sigma \in \mathbb{S}^1. \tag{3-16}$$

Next, we define for each $\rho > 0$ the maximal number of disjoint closed balls of radius 1 needed to cover the circle $(0, \rho\mathbb{S}^1) \subset \mathbb{R}^3$

$$N(\rho) := \max\Big\{l \in \mathbb{N} : \text{there exist } \omega_1, \dots, \omega_l \in \mathbb{S}^1 \\ \text{such that } \overline{B_1((0,\rho\omega_1))}, \dots, \overline{B_1((0,\rho\omega_l))} \text{ are pairwise disjoint}\Big\}.$$

This number depends only on the radius of the circle and not on its position in $\mathbb{R}^3$. By compactness of $\mathbb{S}^1$, $N(\rho)$ is well-defined and finite. Moreover, using (3-16) on $N(\rho_j)$ disjoint balls that cover $(h_j, \rho_j\, \mathbb{S}^1)$ and that preimages of disjoint sets are always disjoint, we infer

$$\int_{\mathbb{S}^1\times\mathbb{S}^1} \left| A\left[\frac{f(t_j)}{r_j}\right]\right|^2 \mathrm{d}\mu_{g_{f(t_j)/r_j}} \geq N(\rho_j)\varepsilon_0.$$

Note that this implies by scaling properties and the Gauss–Bonnet theorem that

$$N(\rho_j) \leq \frac{1}{\varepsilon_0}\int_{\mathbb{S}^1\times\mathbb{S}^1} |A[f(t_j)]|^2 \, \mathrm{d}\mu_{g_{f(t_j)}} = \frac{1}{\varepsilon_0}\mathcal{W}(f(t_j)) \leq \frac{\mathcal{W}(f_0)}{\varepsilon_0}.$$

To infer that ρ_j is bounded it suffices now to show that $N(\rho) \to \infty$ as $\rho \to \infty$. To this end we prove that

$$N(\rho) \geq \frac{\pi}{4\arccos(1 - 8/\rho^2)} \quad \text{for } \rho \geq 4. \tag{3-17}$$

[2] That is, $h_j = x_j^{(1)}/r_j \in \mathbb{R}$, $\rho_j = \sqrt{(x_j^{(2)})^2 + (x_j^{(3)})^2}/r_j \geq 0$ and $\sigma_j = (x_j^{(2)}, x_j^{(3)})/(\rho_j r_j) \in \mathbb{S}^1$. We consider a cylinder with axis in the direction $(1, 0, 0)$.

Let us first fix $\rho \geq 4$. Note first that the squared Euclidean distance in $\mathbb{R}^3$ between $(0, \rho\cos(\alpha), \rho\sin(\alpha))$ and $(0, \rho\cos(\beta), \rho\sin(\beta))$ is given by

$$d^2_{\alpha,\beta} := 2\rho^2(1-\cos(\alpha-\beta)).$$

Also observe that the balls $\overline{B_1((0, \rho\cos(\alpha), \rho\sin(\alpha))}$, $\overline{B_1((0, \rho\cos(\beta), \rho\sin(\beta))}$ are disjoint if and only if $d^2_{\alpha,\beta} > 4$. Hence it suffices to find distinct values $\alpha_1, \dots, \alpha_{\widetilde{N}} \in [0, 2\pi)$ such that for all $i, j \in \{1, \dots, \widetilde{N}\}$ one has

$$d^2_{\alpha_i,\alpha_j} \geq 16 \quad \text{for all } i, j \in \{1, \dots, \widetilde{N}\}.$$

We claim that the choice of $\alpha_j := j \arccos(1 - 8/\rho^2)$, $j = 1, \dots, \widetilde{N}$, with

$$\widetilde{N} = \left\lfloor \frac{\pi}{4\arccos(1-8/\rho^2)} \right\rfloor$$

has the desired properties. Indeed, note that $\alpha_1, \dots, \alpha_{\widetilde{N}} \in \left[0, \frac{\pi}{4}\right]$ which implies that $|\alpha_i - \alpha_j| \in \left[0, \frac{\pi}{2}\right]$ for all i, j. Using evenness of cos and monotonicity of cos in $\left[0, \frac{\pi}{2}\right]$ we obtain for all $i, j \in \{1, \dots, \widetilde{N}\}$

$$\begin{aligned} d^2_{\alpha_i,\alpha_j} &= 2\rho^2(1-\cos(\alpha_i-\alpha_j)) = 2\rho^2\Big(1-\cos\Big(|i-j|\arccos\Big(1-\frac{8}{\rho^2}\Big)\Big)\Big) \\ &\geq 2\rho^2\Big(1-\cos\Big(1\cdot\arccos\Big(1-\frac{8}{\rho^2}\Big)\Big)\Big) = 16. \end{aligned}$$

We have thus shown (3-17) and thus the claim follows. □

Remark 3.7. The lemma reveals an interesting property of the Willmore flow of tori of revolution. Suppose that $T < \infty$. Then by Theorem 3.1 and in particular the property $t_j + c_0 r_j^4 < T$, necessarily $r_j \to 0$. Now let $(x_j)_{j\in\mathbb{N}}$ be a collection of *points of concentration*, i.e., points where (3-14) holds true. From the previous lemma we know that the distance of x_j/r_j to the x-axis is bounded. Hence the distance $(x_j)_{j\in\mathbb{N}}$ to the x-axis tends to zero. In other words, finite-time-concentration may only happen close to the x-axis.

3.5. *Proof of Theorems 1.2 and 1.3.*

Proof of Theorem 1.2. Let $f : [0, T) \times \mathbb{S}^1 \times \mathbb{S}^1 \to \mathbb{R}^3$ be as in the statement. That $f(t)$ is a torus of revolution for all $t \in [0, T)$ follows from Lemma 3.3. Thus we can actually choose $(\gamma(t))_{t\in[0,T)}$ as in the statement; see also the discussion after Lemma 3.3. Let $t_j \to T$ be such that $\mathcal{L}_{\mathbb{H}^2}(\gamma(t_j)) \leq M$ for some $M > 0$ and let $r_j > 0$ and $\tilde{f}_{j,c_0}$ be as in Theorem 3.1. By Theorem 3.1 it is sufficient for the convergence of the Willmore flow that $(\operatorname{diam}(\tilde{f}_{j,c_0}))_{j\in\mathbb{N}}$ is bounded. Notice that we assume a bound on $\mathcal{L}_{\mathbb{H}^2}$ at t_j and we want a bound on the diameter at $t_j + c_0 r_j^4$. To this end we define $\tilde{f}_{j,0} := f(t_j)/r_j$ and choose for all $j \in \mathbb{N}$, x_j as in (3-14). Such a choice of x_j exists due to the definition of r_j in Theorem 3.1. We write $x_j/r_j = (h_j, \rho_j\sigma_j)$, $\rho_j > 0$ and $\sigma_j \in \mathbb{S}^1$ as in Lemma 3.6 and infer from Lemma 3.6 that $(\rho_j)_{j\in\mathbb{N}}$ is bounded, say $\rho_j \leq C$ for all $j \in \mathbb{N}$. Note that by the choice of x_j, in particular (3-15), for all $j \in \mathbb{N}$ one has $\operatorname{dist}(x_j/r_j, \tilde{f}_{j,0}(\mathbb{S}^1 \times \mathbb{S}^1)) \leq 1$. Now we look at $\tilde{\gamma}_j = \gamma(t_j)/r_j$, which is clearly a profile curve of $\tilde{f}_{j,0}$ and satisfies also $\mathcal{L}_{\mathbb{H}^2}(\tilde{\gamma}_j) \leq M$ by scaling invariance of the hyperbolic length. By the distance estimate we can find $u_j, v_j \in \mathbb{S}^1$ such that

$$\left|\frac{1}{r_j}[(x_j^{(2)}, x_j^{(3)}) - \gamma_j^{(2)}(u_j)(\cos(2\pi v_j), \sin(2\pi v_j))]\right| \leq 1.$$

Hence we infer that

$$\tilde{\gamma}_j^{(2)}(u_j) \leq 1 + \left| \frac{1}{r_j}(x_j^{(2)}, x_j^{(3)}) \right| \leq 1 + \rho_j \leq 1 + C.$$

From the bounded hyperbolic length and (2-8) we infer that

$$\sup_{\mathbb{S}^1} \tilde{\gamma}_j^{(2)} \leq \tilde{\gamma}_j^{(2)}(u_j) e^{\mathcal{L}_{\mathbb{H}^2}(\tilde{\gamma}_j)} \leq (1+C)e^M.$$

This implies also by (2-9) that

$$\mathcal{L}_{\mathbb{R}^2}(\tilde{\gamma}_j) \leq \sup_{\mathbb{S}^1} \tilde{\gamma}_j^{(2)} \mathcal{L}_{\mathbb{H}^2}(\tilde{\gamma}_j) \leq M(1+C)e^M,$$

and from Lemma 2.6 we now infer

$$\operatorname{diam}(\tilde{f}_{j,0}) \leq \tfrac{1}{2}\mathcal{L}_{\mathbb{R}^2}(\tilde{\gamma}_j) + 2 \sup_{\mathbb{S}^1} \tilde{\gamma}_j^{(2)} \leq D \tag{3-18}$$

for some constant $D \geq 0$. We now define $\tilde{f}_j(s) := f(t_j + s r_j^4)/r_j$, $s \in [0, c_0]$, taking into account the parabolic scaling. It is easy to see that then $\tilde{f}_j$ is a solution of the Willmore flow equation and $\tilde{f}_j(0) = \tilde{f}_{j,0}$ and $\tilde{f}_j(c_0) = \tilde{f}_{j,c_0}$. Hence we can estimate by Lemma D.6

$$\operatorname{diam}(f_{j,c_0}) \leq C(\mathcal{W}(\tilde{f}_{j,0}))(\operatorname{diam}(\tilde{f}_{j,0}) + c_0^{1/4}).$$

Using that by scaling invariance $\mathcal{W}(\tilde{f}_{j,0}) = \mathcal{W}(f(t_j)) \leq \mathcal{W}(f_0)$ and (3-18) we obtain

$$\operatorname{diam}(f_{j,c_0}) \leq C(\mathcal{W}(f_0))(D + c_0^{1/4}). \tag{3-19}$$

By Theorem 3.1 this implies that $T = \infty$ and $(f(t))_{t \in [0,\infty)}$ is a convergent evolution. It only remains to show that the limit is a torus of revolution. This is however a direct consequence of Lemma 3.4. □

Proof of Theorem 1.3. Let $(f(t))_{t\in[0,T)}$ and $(\gamma(t))_{t\in[0,T)}$ be as in the statement. We distinguish two cases.

Case 1: $\mathcal{W}(f_0) < 8\pi$. To show long-time existence and convergence of the evolution we apply Theorem 1.2. To this end we need to show that

$$\liminf_{t \to T} \mathcal{L}_{\mathbb{H}^2}(\gamma(t)) < \infty.$$

First we observe that $(\gamma(t))_{t\in[0,T)}$ satisfies

$$\mathcal{E}(\gamma(t)) = \frac{2}{\pi}\mathcal{W}(F_{\gamma(t)}) = \frac{2}{\pi}\mathcal{W}(f(t)) \leq \frac{2}{\pi}\mathcal{W}(f_0) < 16.$$

We apply Theorem 2.2 with $\varepsilon := 16 - \frac{2}{\pi}\mathcal{W}(f_0)$ to find that for each $t \in [0, T)$ one has

$$\mathcal{L}_{\mathbb{H}^2}(\gamma(t)) \leq \frac{1}{c(\varepsilon)}\mathcal{E}(\gamma(t)) = \frac{2}{\pi c(\varepsilon)}\mathcal{W}(f(t)) \leq \frac{2}{\pi c(\varepsilon)}\mathcal{W}(f_0),$$

and hence the hyperbolic length is uniformly bounded for $t \in [0, T)$. By Theorem 1.2 the evolution converges in C^k for all k and the limit, say $f_\infty : \mathbb{S}^1 \times \mathbb{S}^1 \to \mathbb{R}^3$, is a Willmore torus of revolution. By the gradient flow properties of the Willmore flow and Lemma C.3 we obtain that $\mathcal{W}(f_\infty) \leq \mathcal{W}(f_0) < 8\pi$.

We obtain from Proposition 2.4 that f_∞ is, up to reparametrization, a Clifford torus, possibly rescaled and translated in the direction $(1, 0, 0)^T$. The claim follows.

Case 2: $\mathcal{W}(f_0) = 8\pi$. We first claim that f_0 is not a Willmore surface. Indeed, if it were then it would by Proposition 2.4 be a rescaled and translated reparametrization of a Clifford torus. But the Willmore energy of the Clifford torus is $2\pi^2$, contradicting $\mathcal{W}(f_0) = 8\pi$. Hence

$$\frac{d}{dt}\mathcal{W}(f(t))\Big|_{t=0} = -\|\nabla_{L^2}\mathcal{W}(f_0)\|^2_{L^2(\Sigma)} < 0,$$

which implies that there exists $t_0 > 0$ such that $\mathcal{W}(f(t_0)) < 8\pi$. We restart the Willmore flow with $f(t_0)$ which satisfies the assumptions of Case 1 and hence converges to a reparametrization of the Clifford torus, possibly rescaled and translated in the direction $(1, 0, 0)^T$. The claim follows. □

3.6. *Optimality.* We show that the upper bound of 8π on the Willmore energy of the initial datum in Theorem 1.3 is sharp by proving Theorem 1.4. In the statement of this theorem, the geometric quantities that may possibly degenerate along the flow are the second fundamental form or the diameter. On contrary, the statement of Theorem 1.2 suggests another quantity which must degenerate — the hyperbolic length. In the following we will construct the nonconvergent evolutions and study the relation between the degenerating quantities.

Lemma 3.8 (The singular evolutions). *For any $\varepsilon > 0$ there exists a torus of revolution $f_0 : \mathbb{S}^1 \times \mathbb{S}^1 \to \mathbb{R}^3$ such that $\mathcal{W}(f_0) < 8\pi + \varepsilon$, and the maximal Willmore flow $(f(t))_{t\in[0,T)}$ starting at f_0 satisfies*

$$\lim_{t\to T} \mathcal{L}_{\mathbb{H}^2}(\gamma(t)) = \infty.$$

The main idea is to start the flow with an immersed curve that has *total curvature*

$$T[\gamma] := \frac{1}{2\pi}\int_\gamma \kappa_{\text{euc}}[\gamma]\,\mathrm{d}s \tag{3-20}$$

equal to zero. This quantity $T[\,\cdot\,]$ turns out to be a flow invariant and can hence be helpful to classify possible limits of convergent evolution. This in turn can also be used to show that some evolutions cannot be convergent.

Lemma 3.9. *The total curvature T, defined on curves in $W^{2,2}(\mathbb{S}^1, \mathbb{R}^2)_{\text{imm}} := \{\gamma \in W^{2,2}(\mathbb{S}^1, \mathbb{R}^2) : \gamma \text{ immersed}\}$ is integer-valued and weakly continuous in the relative topology of $W^{2,2}(\mathbb{S}^1, \mathbb{R}^2)_{\text{imm}}$. Moreover it is a flow invariant for the Willmore flow of tori of revolution; i.e., if $(f(t))_{t\in[0,T)}$ is an evolution by the Willmore flow with profile curve $(\gamma(t))_{t\in[0,T)}$ then $T[\gamma(t)] = T[\gamma(0)]$ for all $t \in [0, T)$.*

Proof. The fact that $T[\,\cdot\,]$ is integer-valued and an invariant with respect to regular homotopies is very classical and follows from the Whitney–Graustein theorem. Since $\gamma(t) = f(t)(u, 0)$ (see (3-5)) and $t \mapsto f(t)$ is a regular homotopy, so is $t \mapsto \gamma(t)$. Hence we can also conclude that it is a Willmore flow invariant. The weak $W^{2,2}$-continuity follows immediately from the formula

$$T[\gamma] := \frac{1}{2\pi}\int_0^1 \frac{1}{|\gamma'|}((\gamma^{(2)})''(\gamma^{(1)})' - (\gamma^{(1)})''(\gamma^{(2)})')\,\mathrm{d}x$$

and the compact embedding $W^{2,2} \hookrightarrow C^1$. □

Proof of Lemma 3.8. Fix $\varepsilon > 0$. By [Müller and Spener 2020, Corollary 6.4] there exists a curve γ_ε such that $16 \le \mathcal{E}(\gamma_\varepsilon) < 16+\varepsilon$ and $T[\gamma_\varepsilon] = 0$, where $T[\,\cdot\,]$ is given as in (3-20). Now start the flow with $f_0 = F_{\gamma_\varepsilon} : \mathbb{S}^1 \times \mathbb{S}^1 \to \mathbb{R}^3$ defined as in (1-4) with profile curve γ_ε and let $(f(t))_{t\in[0,T)}$ be the corresponding evolution by the Willmore flow. Assume that for $(\gamma(t))_{t\in[0,\infty)}$ as in (3-5) one has

$$\liminf_{t\to T} \mathcal{L}_{\mathbb{H}^2}(\gamma(t)) < \infty.$$

By Theorem 1.2 we obtain that then $T = \infty$ and $(f(t))_{t\in[0,\infty)}$ is convergent to a Willmore torus of revolution f_∞. Let now $t_j \to \infty$ be a sequence such that $\mathcal{L}_{\mathbb{H}^2}(\gamma(t_j)) \le M < \infty$ for all $j \in \mathbb{N}$. By Lemma 3.4 we obtain that an appropriate reparametrization of $\gamma(t_j)$ converges in $C^k(\mathbb{S}^1, \mathbb{R}^2)$ to some $\gamma_\infty \in C^\infty(\mathbb{S}^1, \mathbb{H}^2)$, which is a profile curve of f_∞, i.e., up to a reparametrization one has $f_\infty = F_{\gamma_\infty}$. By (2-12) we infer that γ_∞ is a hyperbolic elastica.

Now we choose $\phi_j \in C^4(\mathbb{S}^1, \mathbb{S}^1)$ such that $\gamma(t_j)\circ\phi_j$ converges to γ_∞ classically in $C^4(\mathbb{S}^1, \mathbb{R}^2)$. Then, by the previous lemma

$$T[\gamma_\infty] = \lim_{j\to\infty} T[\gamma(t_j)] = T[\gamma(0)] = 0.$$

Hence γ_∞ is a hyperbolic elastica with vanishing Euclidean total curvature. By [Müller and Spener 2020, Corollary 5.8] there exist no hyperbolic elastica of vanishing total curvature. We obtain a contradiction and the claim follows. □

As an important ingredient for case (2) in Theorem 1.4, we need to show that global evolutions under the Willmore flow of tori of revolution with unbounded hyperbolic length and no curvature concentration must have unbounded diameter.

Lemma 3.10 (Diameter blow-up). *Let $f_0 : \mathbb{S}^1 \times \mathbb{S}^1 \to \mathbb{R}^3$ be a torus of revolution and let $(f(t))_{t\in[0,\infty)}$ evolve by the Willmore flow with initial datum f_0. Let $\gamma(t) = f(t)(\,\cdot\,,0)$ be the profile curve of $f(t)$ for all $t \ge 0$. Assume that $(A(t))_{t\in[0,\infty)}$ is bounded in $L^\infty(\Sigma)$ and $\lim_{t\to\infty} \mathcal{L}_{\mathbb{H}^2}(\gamma(t)) = \infty$. Then*

$$\lim_{t\to\infty} \operatorname{diam}(f(t)(\mathbb{S}^1 \times \mathbb{S}^1)) = \infty.$$

Proof. We first introduce the constant $D := \sup_{t\in[0,\infty)} \|A(t)\|_{L^\infty} < \infty$. Next we assume for a contradiction that there exists some $t_j \to T = \infty$ such that $\operatorname{diam}(f(t_j)(\mathbb{S}^1 \times \mathbb{S}^1)) \le M < \infty$ for all $j \in \mathbb{N}$. Let $(r_j)_{j\in\mathbb{N}} \subset (0,\infty)$ be as in Theorem 3.1. Note that there exists $x_j \in \mathbb{R}^3$

$$\varepsilon_0 \le \int_{f(t_j)^{-1}(\overline{B_{r_j}(x_j)})} |A[f(t_j)]|^2 \,\mathrm{d}\mu_{g_{f(t_j)}} \le D^2 \mu_{g_{f(t_j)}}(f(t_j)^{-1}(\overline{B_{r_j}(x_j)})).$$

By (D-2) we have that

$$\mu_{g_{f(t_j)}}(f(t_j)^{-1}(\overline{B_{r_j}(x_j)})) \le C\mathcal{W}(f(t_j))r_j^2 \le C\mathcal{W}(f_0)r_j^2.$$

In particular we find by the previous two equations

$$r_j^2 \ge \frac{\varepsilon_0}{D^2 C\mathcal{W}(f_0)}; \tag{3-21}$$

i.e., there exists $\delta > 0$ such that $r_j \geq \delta$ for all $j \in \mathbb{N}$. Since we have assumed that $\operatorname{diam}(f(t_j)(\mathbb{S}^1 \times \mathbb{S}^1)) \leq M$ we obtain that

$$\operatorname{diam}\left(\frac{f(t_j)}{r_j}(\mathbb{S}^1 \times \mathbb{S}^1)\right) \leq \frac{1}{r_j} \operatorname{diam}(f(t_j)(\mathbb{S}^1 \times \mathbb{S}^1)) \leq \frac{1}{\delta} M.$$

Now recall that $\tilde{f}_j(s) := f(t_j + s r_j^4)/r_j$, $s \in [0, c_0]$, defines a solution of the Willmore flow, with $\tilde{f}_j(0) = f(t_j)/r_j$ and $\tilde{f}_j(c_0) = \tilde{f}_{j,c_0}$, defined as in Theorem 3.1. With Lemma D.6 we obtain thus that

$$\operatorname{diam}(\tilde{f}_{j,c_0}) \leq C\left(\mathcal{W}\left(\frac{f(t_j)}{r_j}\right)\right)\left(\operatorname{diam}\left(\frac{f(t_j)}{r_j}\right) + c_0^{1/4}\right) \leq C(\mathcal{W}(f_0))\left(\frac{M}{\delta} + c_0^{1/4}\right),$$

which is uniformly bounded in j. This implies by Theorem 3.1 that there exists a Willmore immersion $f_\infty : \mathbb{S}^1 \times \mathbb{S}^1 \to \mathbb{R}^3$ such that $f(t) \to f_\infty$ in C^k for all $k \in \mathbb{N}$. By Lemma 3.4, f_∞ is a Willmore torus of revolution. In particular, up to reparametrization one has $f_\infty = F_{\gamma_\infty}$ for some $\gamma_\infty \in C^\infty(\mathbb{S}^1, \mathbb{H}^2)$. We next claim that there exists $\delta > 0$ such that $\inf_{\mathbb{S}^1} \gamma(t)^{(2)} > \delta$ for all $t \in [0, \infty)$. To this end observe

$$\begin{aligned}\lim_{t\to\infty} \inf_{\mathbb{S}^1} \gamma(t)^{(2)} &= \lim_{t\to\infty} \inf_{\mathbb{S}^1\times\mathbb{S}^1} \sqrt{(f(t)^{(2)})^2 + (f(t)^{(3)})^2} \\ &= \inf_{\mathbb{S}^1\times\mathbb{S}^1} \sqrt{(f_\infty^{(2)})^2 + (f_\infty^{(3)})^2} = \inf_{\mathbb{S}^1} \gamma_\infty^{(2)} > 0,\end{aligned}$$

since $\gamma_\infty^{(2)}(u) > 0$ for all $u \in \mathbb{S}^1$ and $\mathbb{S}^1$ is compact. Note that we have used here that the infimum expression is independent of the parametrization of $f(t)$. This and the fact that $(f(t))_{t\in[0,\infty)}$ is a smoothly evolving family of tori of revolution implies $\inf_{\mathbb{S}^1} \gamma(t)^{(2)} > \delta$ for all $t \in [0, \infty)$. Next we look at the surface area of $f(t)$, i.e.,

$$\mu_{g_{f(t)}}(\mathbb{S}^1 \times \mathbb{S}^1) = 2\pi \int_0^1 |\gamma(t)'(u)| \gamma^{(2)}(t)(u)\, du,$$

and infer

$$\mu_{g_{f(t_j)}}(\mathbb{S}^1 \times \mathbb{S}^1) \geq 2\pi\delta^2 \mathcal{L}_{\mathbb{H}^2}(\gamma(t_j)) \to \infty.$$

With Lemma D.7 it follows

$$M \geq \operatorname{diam}(f(t_j)(\mathbb{S}^1 \times \mathbb{S}^1)) \geq \sqrt{\frac{\mu_{g_{f(t_j)}}(\mathbb{S}^1 \times \mathbb{S}^1)}{\mathcal{W}(f(t_j))}} \geq \sqrt{\frac{\mu_{g_{f(t_j)}}(\mathbb{S}^1 \times \mathbb{S}^1)}{\mathcal{W}(f_0)}} \to \infty.$$

A contradiction. We infer that $\lim_{t\to\infty} \operatorname{diam}(f(t)(\mathbb{S}^1 \times \mathbb{S}^1)) = \infty$. □

In the proof we have used without further notice that the concept of tori of revolution in [Blatt 2009, Definition 2.2] coincides with our definition in Definition 1.1, at least up to reparametrization. For details recall Proposition 3.2 and the discussion afterwards.

Proof of Theorem 1.4. Let $\varepsilon > 0$ be as in the statement and f_0 be as in Lemma 3.8. Then the evolution $(f(t))_{t\in[0,T)}$ satisfies $\lim_{t\to T} \mathcal{L}_{\mathbb{H}^2}(\gamma(t)) = \infty$. Next let $t_j \uparrow T$ be a sequence. Let $\varepsilon_0 > 0$, $c_0 > 0$ and $(r_j)_{j\in\mathbb{N}}$ be as in Theorem 3.1. We distinguish now two cases.

Case 1: there exists a subsequence of r_j that converges to zero. We claim that then condition (1) in the statement occurs. To this end assume that $(\|A(t)\|_{L^\infty})_{t\in[0,T)}$ is bounded, say $D := \sup_{t\in[0,T)} \|A(t)\|_{L^\infty} < \infty$.

Then one has by (3-1) that for all $j \in \mathbb{N}$ there exists $x_j \in \mathbb{R}^3$ such that

$$\varepsilon_0 \leq \int_{f(t_j)^{-1}(\overline{B_{r_j}(x_j)})} |A(t_j)|^2 \, \mathrm{d}\mu_{g_{f(t_j)}} \leq D^2 \mu_{g_{f(t_j)}}(f(t_j)^{-1}(\overline{B_{r_j}(x_j)})).$$

Using (D-2) we find that $\varepsilon_0 \leq c\mathcal{W}(f_0)D^2 r_j^2$. This is a contradiction to the condition that up to a subsequence $r_j \to 0$. Hence we have shown that $(\|A(t)\|_{L^\infty(\Sigma)})_{t\in[0,T)}$ is unbounded.

Case 2: there exists $\delta > 0$ such that $r_j \geq \delta$ for all $j \in \mathbb{N}$. First observe that in this case $T = \infty$ since $t_j + c_0 r_j^4 < T$ by Theorem 3.1. If condition (1) in the statement holds true, i.e., $(\|A(t)\|_{L^\infty(\Sigma)})_{t\geq 0}$ is unbounded, there is nothing to prove. Hence we may assume that $(\|A(t)\|_{L^\infty(\Sigma)})_{t\geq 0}$ is bounded. Since $\lim_{t\to\infty} \mathcal{L}_{\mathbb{H}^2}(\gamma(t)) = \infty$, by Lemma 3.10 we find that $\lim_{t\to\infty} \operatorname{diam}(f(t))(\mathbb{S}^1 \times \mathbb{S}^1) = \infty$ and hence condition (2) occurs. This proves the claim. □

4. An application: energy minimization among conformal constraints

A very vivid field of research is the minimization of the Willmore energy among all tori that are *conformally equivalent* to a reference torus. Being conformally equivalent means that the surface can be parametrized with a *conformal* immersion of the reference torus. Taking a reference torus of the form $\mathbb{C}/(\mathbb{Z}+\omega\mathbb{Z})$ one can also associate to every torus its *conformal class*, defined as follows.

Definition 4.1 (Conformal class; see [Ndiaye and Schätzle 2015, p. 293]). Let $S \subset \mathbb{R}^3$ be a smooth torus. Then there exists a unique $\omega \in \mathbb{C}$ satisfying $|\omega| \geq 1$, $\operatorname{Im}(\omega) > 0$ and $\operatorname{Re}(\omega) \in \left[0, \frac{1}{2}\right]$ such that there exists a conformal smooth immersion

$$F : \frac{\mathbb{C}}{\mathbb{Z}+\omega\mathbb{Z}} \to S,$$

i.e.,

$$g^F_{i,j} = e^{2u}\delta_{i,j} \quad \text{for some } u \in C^\infty\left(\frac{\mathbb{C}}{\mathbb{Z}+\omega\mathbb{Z}}\right). \tag{4-1}$$

The value $\omega = \omega(S) \in \mathbb{C}$ is then called the *conformal class of S*. If ω is purely imaginary, we call the torus *rectangular*.

As it turns out, all tori of revolution are rectangular (see also [Langer and Singer 1984a, Proposition 7]).

Proposition 4.2. *Suppose that $\gamma \in C^\infty(\mathbb{S}^1, \mathbb{H}^2)$. Then $F_\gamma(\mathbb{S}^1 \times \mathbb{S}^1)$, the torus with profile curve γ, has conformal class*

$$\omega(F_\gamma(\mathbb{S}^1 \times \mathbb{S}^1)) = \begin{cases} i\mathcal{L}_{\mathbb{H}^2}(\gamma)/(2\pi), & \mathcal{L}_{\mathbb{H}^2}(\gamma) \geq 2\pi, \\ i2\pi/\mathcal{L}_{\mathbb{H}^2}(\gamma), & \mathcal{L}_{\mathbb{H}^2}(\gamma) < 2\pi. \end{cases}$$

In particular, each torus of revolution is rectangular and $\omega(F_\gamma(\mathbb{S}^1 \times \mathbb{S}^1))$ is a continuous function of $\mathcal{L}_{\mathbb{H}^2}(\gamma)$.

Proof. Let $\bar{\gamma} : \mathbb{R} \to \mathbb{R}$ be the $\frac{1}{2\pi}\mathcal{L}_{\mathbb{H}^2}(\gamma)$-periodic reparametrization of γ with constant hyperbolic velocity 2π. If $\mathcal{L}_{\mathbb{H}^2}(\gamma) \geq 2\pi$ we choose the smooth immersion

$$F : \frac{\mathbb{C}}{\mathbb{Z} + (i\mathcal{L}_{\mathbb{H}^2}(\gamma)/(2\pi))\mathbb{Z}} \to F_\gamma(\mathbb{S}^1 \times \mathbb{S}^1)$$

by

$$F(s+it) = \begin{pmatrix} \bar{\gamma}^1(t) \\ \bar{\gamma}^2(t)\cos(2\pi s) \\ \bar{\gamma}^2(t)\sin(2\pi s) \end{pmatrix}. \tag{4-2}$$

An easy computation shows $g^F_{1,2} = g^F_{2,1} = 0$ and

$$g^F_{1,1} = (\bar{\gamma}^1)'^2 + (\bar{\gamma}^2)'^2, \quad g^F_{2,2} = 4\pi^2(\bar{\gamma}^2)^2.$$

Therefore by our choice of parametrization

$$\frac{g^F_{1,1}}{g^F_{2,2}} = \frac{(\bar{\gamma}^1)'^2 + (\bar{\gamma}^2)'^2}{4\pi^2(\bar{\gamma}^2(t))^2} = 1.$$

Hence (4-1) is satisfied and F is a conformal immersion. Moreover one readily checks that $\omega = i\mathcal{L}_{\mathbb{H}^2}(\gamma)/(2\pi)$ meets the requirements of Definition 4.1.

If $\mathcal{L}_{\mathbb{H}^2}(\gamma) < 2\pi$ we choose

$$\widetilde{F} : \frac{\mathbb{C}}{\mathbb{Z} + i(2\pi/\mathcal{L}_{\mathbb{H}^2}(\gamma))\mathbb{Z}} \to F_\gamma(\mathbb{S}^1 \times \mathbb{S}^1)$$

to be given by

$$\widetilde{F}(s+it) := F\left(\frac{\mathcal{L}_{\mathbb{H}^2}(\gamma)}{2\pi}t + i\frac{\mathcal{L}_{\mathbb{H}^2}(\gamma)}{2\pi}s\right),$$

where F is as in (4-2) and the claim follows also in this case arguing as before. □

Remark 4.3. The conformal class of the Clifford torus is $\omega = i$. Indeed, its defining curve is

$$\gamma(t) = \begin{pmatrix} 0 \\ 1 \end{pmatrix} + \frac{1}{\sqrt{2}}\begin{pmatrix} \cos(t) \\ \sin(t) \end{pmatrix}, \quad t \in [-\pi, \pi).$$

From this we conclude with the residue theorem (more precisely [Freitag and Busam 2005, Proposition III.7.10]) that

$$\begin{aligned}
\mathcal{L}_{\mathbb{H}^2}(\gamma) &= \int_{-\pi}^{\pi} \frac{1}{\sqrt{2}+\sin(t)}\,\mathrm{d}t = \int_{-\pi}^{\pi} \frac{1}{\sqrt{2}+2\cos(t/2)\sin(t/2)}\,\mathrm{d}t \\
&= \int_{-\pi}^{\pi} \frac{1}{\sqrt{2}+2\frac{\tan(t/2)}{1+\tan^2(t/2)}}\,\mathrm{d}t = \int_{-\pi}^{\pi} \frac{1+\tan^2(t/2)}{\sqrt{2}(1+\tan^2(t/2))+2\tan(t/2)}\,\mathrm{d}t \\
&= 2\int_{-\infty}^{\infty} \frac{1}{\sqrt{2}(1+z^2)+2z}\,\mathrm{d}z = 2\int_{-\infty}^{\infty} \frac{1}{\sqrt{2}\big(z-\frac{-1+i}{\sqrt{2}}\big)\big(z-\frac{-1-i}{\sqrt{2}}\big)} \\
&= 2(2\pi i)\sum_{a:\mathrm{Im}(a)>0} \mathrm{Res}\left(\frac{-1}{\sqrt{2}\big(z-\frac{-1+i}{\sqrt{2}}\big)\big(z-\frac{1-i}{\sqrt{2}}\big)}, a\right) = 4\pi i\frac{1}{\sqrt{2}\big(\frac{-1+i}{\sqrt{2}}-\frac{-1-i}{\sqrt{2}}\big)} = 2\pi.
\end{aligned}$$

An interesting problem is the minimization of the Willmore functional in each conformal class.

Definition 4.4 (Conformally constrained Willmore minimization). For ω as in Definition 4.1 we set

$$M_{3,1}(\omega) := \inf\left\{\mathcal{W}(f) : f : \frac{\mathbb{C}}{\mathbb{Z}+\omega\mathbb{Z}} \to \mathbb{R}^3 \text{ conformal immersion}\right\}.$$

In [Ndiaye and Schätzle 2015, Proposition D.1] the authors show that there exists some $b_0 \geq 1$ such that $b \geq b_0$ implies $M_{3,1}(ib) < 8\pi$. Our first contribution in this context is the new insight that $b_0 = 1$. We prove the existence of tori of revolution with Willmore energy smaller than 8π in each conformal class $\omega = ib$, $b \geq 1$, via the Willmore flow studied in Theorem 1.3. Note that $\mathbb{C}/(\mathbb{Z}+ib\mathbb{Z})$ and $\mathbb{S}^1 \times \mathbb{S}^1$ are diffeomorphic with diffeomorphism $\phi : \mathbb{S}^1 \times \mathbb{S}^1 \to \mathbb{C}/(\mathbb{Z}+ib\mathbb{Z})$ being given by $\phi(u, v) = u + ibv$. Hence the results about the Willmore flow in Theorem 1.3 apply also for surfaces defined on $\mathbb{C}/(\mathbb{Z}+ib\mathbb{Z})$.

Theorem 4.5. *For each $b \geq 1$ there exists a torus of revolution T_b such that $\omega(T_b) = ib$ and $\mathcal{W}(T_b) < 8\pi$.*

Proof. From the construction in the proof of [Ndiaye and Schätzle 2015, Proposition D.1] follows that there exists $b_0 > 1$ such that for all $b \geq b_0$ there exists a torus T_b as in the statement. Note that actually the authors construct only a $C^{1,1}$-torus of revolution T_b, but by mollification of the profile curve one can easily obtain a smooth torus of revolution that satisfies the same requirements and differs not too much in the conformal class as the hyperbolic length depends continuously on γ.

It remains to prove the claim for $b \in [1, b_0)$. For this choose $f_0 : \mathbb{C}/(\mathbb{Z}+ib_0\mathbb{Z}) \to \mathbb{R}^3$ to be a smooth conformal parametrization of T_{b_0} and let $(f(t))_{t\in(0,\infty)}$ be the evolution of f_0 by the Willmore flow, which is global and smoothly convergent to the Clifford torus (possibly rescaled and translated in the direction $(1, 0, 0)$) by Theorem 1.3. Moreover, $f(t)$ is a torus of revolution for all $t \geq 0$. Let $\gamma(t) = f(t)(\,\cdot\,, 0)$ be the profile curve of $f(t)$ for all $t \geq 0$, i.e., $f(t) = F_{\gamma(t)}$. By (3-5), $t \mapsto \gamma(t)$ is a smooth family of curves for $t \geq 0$ and in particular $\mathcal{L}_{\mathbb{H}^2}(\gamma(t))$ depends smoothly on t. By Proposition 4.2 one obtains that $t \mapsto (1/i)\omega(F_{\gamma(t)})$ is real-valued and depends continuously on t. We show next that along a subsequence $t \mapsto (1/i)\omega(F_{\gamma(t)})$ tends to 1 as $t \to \infty$. By Lemma 3.4 we obtain that there exists some $t_j \to \infty$ such that an appropriate reparametrization of $\gamma(t_j)$ converges in $C^2(\mathbb{S}^1, \mathbb{R}^2)$ to $\gamma_\infty \in C^\infty(\mathbb{S}^1, \mathbb{H}^2)$, a profile curve of the Clifford torus (possibly rescaled and translated in the direction $(1, 0, 0)$). Thus we have

$$2\pi = \mathcal{L}_{\mathbb{H}^2}(\gamma_\infty) = \lim_{j\to\infty} \mathcal{L}_{\mathbb{H}^2}(t_j), \tag{4-3}$$

i.e., $(1/i)\omega(F_{\gamma(t_j)}) \to 1$ as $j \to \infty$. Since $(1/i)\omega(F_{\gamma(0)}) = b_0$, each value between 1 and b_0 is attained by the intermediate value theorem. From this the existence of a torus of revolution T_b for each $b \in [1, b_0)$ follows. □

Remark 4.6. Theorem 4.5 can also be proven using the results in [Müller and Spener 2020] concerning the elastic flow in $\mathbb{H}^2$ (which also dissipates the Willmore energy).

In [Kuwert and Schätzle 2013] the authors prove that the infimum in a conformal class ω is attained once one can find a competitor with energy below 8π. For $\omega = ib$ our small energy tori serve as such competitors and show that the infimum is attained.

Corollary 4.7. *For each $b \geq 1$ the infimum $M_{3,1}(ib)$ is attained and the map $b \to M_{3,1}(ib)$ is continuous on $[1, \infty)$.*

Proof. Theorem 7.3 and Proposition 5.1 in [Kuwert and Schätzle 2013] show that each $b \geq 1$ where $M_{3,1}(ib) < 8\pi$ is a point of continuity of $b \mapsto M_{3,1}(ib)$ and a point where the infimum in the definition of $M_{3,1}$ is attained. The claim then follows directly from this result and Theorem 4.5. □

The symmetries of the Willmore energy might suggest that the infimum of the Willmore energy in each class of rectangular tori (i.e., $\omega = ib$) is attained at a torus of revolution. This is in general still open. Far reaching results are obtained using a formulation of the Willmore energy in $\mathbb{S}^3$ by means of the *stereographic projection*. Since the stereographic projection is conformal it does also not change the conformal class. Looking at the Willmore energy in $\mathbb{S}^3$ one can find tori with a lot of symmetries: For $\alpha \in (0, 1)$ one can look at $\alpha\mathbb{S}^1 + \sqrt{1-\alpha^2}\mathbb{S}^1$. The stereographic projections of all of these are tori of revolution. In particular, these are good candidates for minimizers in their conformal classes $\omega = i\sqrt{1-\alpha^2}/\alpha$. For $\alpha = \frac{1}{\sqrt{2}}$ we obtain the Clifford torus which is the global minimizer and hence surely the minimizer in its conformal class. In [Ndiaye and Schätzle 2014; 2015] the authors show that for conformal classes close to the Clifford torus one still gets minimizers of the form $\alpha\mathbb{S}^1 \times \sqrt{1-\alpha^2}\mathbb{S}^1$. More precisely, the result [Ndiaye and Schätzle 2015, Theorem 3.1] shows that there exists $b_1 > 1$ such that for all $b \leq b_1$ one has that $M_{3,1}(b)$ is attained by

$$\Sigma_b := P\left(\frac{1}{\sqrt{1+b^2}}\mathbb{S}^1 \times \frac{b}{\sqrt{1+b^2}}\mathbb{S}^1\right),$$

where $P : \mathbb{S}^3 \to \mathbb{R}^3$ denotes the stereographic projection. The authors also obtain that $b_1 < \infty$. The critical value b_1 can be understood as a point where a symmetry of the minimizers breaks down. They also note that this property has to break down for large conformal classes; see [Ndiaye and Schätzle 2015, p. 293–294]. In the following we will be able to find an explicit upper bound on the symmetry-breaking value b_1. This result is now obtained by energy comparison. There are other (sharper) results using a stability discussion of Σ_b in $\mathbb{S}^3$; see [Kuwert and Lorenz 2013].

Corollary 4.8. *Let $b_1 \geq 1$ be such that for $b \leq b_1$ the minimizer for $M_{3,1}(b)$ is attained by*

$$\Sigma_b := P\left(\frac{1}{\sqrt{1+b^2}}\mathbb{S}^1 \times \frac{b}{\sqrt{1+b^2}}\mathbb{S}^1\right),$$

where $P : \mathbb{S}^3 \to \mathbb{R}^3$ denotes the stereographic projection. Then

$$b_1 < \frac{4}{\pi} + \sqrt{\frac{16}{\pi^2} - 1} \simeq 2.06136. \tag{4-4}$$

Proof. Let $b > 1$ be such that Σ_b is a minimizer and let T_b be the torus constructed in Theorem 4.5. Then, necessarily, $\mathcal{W}(\Sigma_b) \leq \mathcal{W}(T_b) < 8\pi$. This inequality implies the claim once we have shown that $\mathcal{W}(\Sigma_b) = \pi^2(b + 1/b)$.

For this according to [Topping 2000, equation (9)] for all $f : \Sigma \to \mathbb{R}^3$

$$\mathcal{W}(f) = \int_\Sigma \left(\frac{1}{4}|\widetilde{H}|^2_{P^{-1}(f)} + 1\right) \mathrm{d}\mu_f,$$

where $\widetilde{H}$ denotes the mean curvature of $P^{-1}(f)$ in $\mathbb{S}^3$ and μ_f denotes the surface measure in $\mathbb{S}^3$. By [Ndiaye and Schätzle 2014, equation (2.3)] we have $|\vec{H}_{\mathbb{R}^4}|^2 = |\widetilde{H}|^2 + 4$ and hence we obtain

$$\mathcal{W}(f) = \frac{1}{4}\int_\Sigma |\vec{H}_{\mathbb{R}^4}(P^{-1}(f))|^2 \, \mathrm{d}\mu_f.$$

Having now arrived in $\mathbb{R}^4$ and using that

$$P^{-1}(\Sigma_b) = \frac{1}{\sqrt{1+b^2}}\mathbb{S}^1 \times \frac{b}{\sqrt{1+b^2}}\mathbb{S}^1,$$

we can define $r := 1/\sqrt{1+b^2}$ and use the parametrization

$$F : \mathbb{S}^1 \times \mathbb{S}^1 \ni (\phi, \theta) \mapsto \begin{pmatrix} r\cos(2\pi\phi) \\ r\sin(2\pi\phi) \\ \sqrt{1-r^2}\cos(2\pi\theta) \\ \sqrt{1-r^2}\sin(2\pi\theta) \end{pmatrix} \in \mathbb{R}^4.$$

A computation reveals that

$$g = 4\pi^2 \begin{pmatrix} r^2 & 0 \\ 0 & 1-r^2 \end{pmatrix}.$$

We obtain that

$$\left\{ \frac{1}{2\pi r}\frac{\partial}{\partial\phi}, \frac{1}{2\pi\sqrt{1-r^2}}\frac{\partial}{\partial\theta} \right\}$$

is an orthonormal basis of $T_{(\phi,\theta)}(\mathbb{S}^1, \mathbb{S}^1)$ and hence

$$\vec{H}_{\mathbb{R}^4}(F) = \frac{1}{4\pi^2 r^2}\frac{\partial^2 F}{\partial\phi^2} + \frac{1}{4\pi^2(1-r^2)}\frac{\partial^2 F}{\partial\theta^2},$$

which implies that

$$|\vec{H}_{\mathbb{R}^4}(F)|^2 = \frac{1}{r^2} + \frac{1}{1-r^2}.$$

Also note that $\sqrt{\det(g)} = 4\pi^2 r\sqrt{1-r^2}$. The Willmore energy then reads

$$\mathcal{W}(\Sigma_b) = \frac{1}{4}\left(\frac{1}{r^2} + \frac{1}{1-r^2}\right)4\pi^2 r\sqrt{1-r^2} = \pi^2\left(\frac{\sqrt{1-r^2}}{r} + \frac{r}{\sqrt{1-r^2}}\right)$$

and the claim follows using that by definition of r one has $r = 1/\sqrt{1+b^2}$. □

Appendix A: Consistency between extrinsic and intrinsic view

In literature there are multiple ways to define geometric quantities like curvature. This also leads to different notions of the Willmore energy and its gradient flow. Here we want to convince the reader that all those notions are consistent with the one we chose. For this we first have to do some computations in local coordinates. Let M be a smooth two-dimensional manifold, $f : M \to \mathbb{R}^3$ be an immersion and $\psi : M \to \mathbb{R}^2$ be a chart for M with coordinates (u^1, u^2). Given vector field $X = x^i(\partial/\partial u^i)$ and $Y = y^j(\partial/\partial u^j)$ then

$$A(X, Y) = x^i y^j \left(\frac{\partial^2 f}{\partial u^i \partial u^j} - \Gamma^k_{i,j} \frac{\partial f}{\partial u^k} \right), \tag{A-1}$$

where $\Gamma^k_{i,j}$ are the Christoffel symbols defined using the metric $g_{ij} = \langle (\partial/\partial u^i) f, (\partial/\partial u^j) f \rangle$. In particular, we see that the second fundamental form is symmetric.

If $f : M \to \mathbb{R}^3$ is an isometric immersion then for each local chart (u^1, u^2) of M one can define a unit normal field

$$\vec{N} = \frac{\partial_{u^1} f \times \partial_{u^2} f}{|\partial_{u^1} f \times \partial_{u^2} f|}$$

for (u^1, u^2) and rewrite

$$A(X, Y) = x^i y^j \left(\frac{\partial^2 f}{\partial u^i \partial u^j} - g^{kl} \left\langle \frac{\partial^2 f}{\partial u^i \partial u^j}, \frac{\partial f}{\partial u^l} \right\rangle_{\mathbb{R}^3} \frac{\partial f}{\partial u^k} \right) = x^i y^j \left\langle \frac{\partial^2 f}{\partial u^i \partial u^j}, \vec{N} \right\rangle_{\mathbb{R}^3} \vec{N}. \tag{A-2}$$

If $f : M \to f(M) \subset \mathbb{R}^3$ is now an isometric embedding and $f(M)$ is orientable, $\vec{N}$ is independent of the chosen chart and (A-2) coincides with the usual definition of the second fundamental form.

Let us now choose normal coordinates (u^1, u^2) and fix $e_1 = \partial f/\partial u^1$ and $e_2 = \partial f/\partial u^2$. Then by (A-2) we find

$$A(e_i, e_j) = h^j_i \vec{N},$$

where h^j_i denote the usual coefficients of the Weingarten map. Then, the mean curvature (vector) and Gauss curvature are given by

$$\begin{aligned} \vec{H} &= A(e_1, e_1) + A(e_2, e_2) = (h^1_1 + h^2_2)\vec{N} = H\vec{N}, \\ K &:= \langle A(e_1, e_1), A(e_2, e_2)\rangle_{\mathbb{R}^3} - \langle A(e_1, e_2), A(e_2, e_1)\rangle_{\mathbb{R}^3} = h^1_1 h^2_2 - (h^1_2)^2, \end{aligned} \tag{A-3}$$

where H denotes the *scalar* mean curvature. For $Q(\mathring{A})H$, the "cubic"-term in the Willmore equation, one easily derives

$$Q(\mathring{A})\vec{H} = \tfrac{1}{2} H(H^2 - 4K).$$

With similar computations,

$$|A|^2 = |H|^2 - 2K = \sum_{i,j=1}^{2} \langle A(e_i, e_j), A(e_i, e_j)\rangle_{\mathbb{R}^3},$$

and hence for each toroidal immersion $f : \mathbb{S}^1 \times \mathbb{S}^1 \to \mathbb{R}^3$ one has by the Gauss–Bonnet theorem

$$\int_\Sigma |A|^2 \, \mathrm{d}\mu_f = 4\mathcal{W}(f). \tag{A-4}$$

Similarly, again in the case of tori, an easy computation shows that $|\mathring{A}|^2 = \frac{1}{2} H^2 - 2K$ and

$$\int_\Sigma |\mathring{A}|^2 \, \mathrm{d}\mu_f = 2\mathcal{W}(f).$$

Also, note that $|\mathring{A}|^2 \le |A|^2$.

Appendix B: Tensor calculus

Throughout the article, we use a nonstandard notation for some differential geometric concepts involving connections, derivatives and tensors. We discuss here that our notation is consistent with that used in [Kuwert and Schätzle 2001; 2002; 2004], since many results cited there are used. Here we shall briefly introduce these concepts and clarify their meaning. Let M be a smooth two-dimensional manifold and $f \in C^\infty(M; \mathbb{R}^n)$ be an immersion. Moreover, let ∇ be the Levi-Civita connection on M. For a vector field $X \in \mathcal{V}(M)$ we define the *full derivative* $D_X : C^\infty(M; \mathbb{R}^n) \to C^\infty(M; \mathbb{R}^n)$ via

$$D_X G := \sum_{i=1}^n X(G_i)\vec{e}_i, \quad \text{whenever } G = \sum_{i=1}^n G_i \vec{e}_i \in C^\infty(M; \mathbb{R}^n), \tag{B-1}$$

and $\{\vec{e}_1, \vec{e}_2, \dots, \vec{e}_n\}$ is the canonical basis of $\mathbb{R}^n$. We say that $G \in C^\infty(M; \mathbb{R}^n)$ is a *normal vector field* if $G(p) \perp df_p(T_pM)$ for all $p \in M$. We define for short $N_pM := df_p(T_pM)^\perp$ and $NM := \bigsqcup_{p\in M} N_pM$ the *normal bundle*. For such a normal vector field $G \in C^\infty(M, NM)$ we define the *normal connection* of G to be

$$\nabla^\perp_X G|_p := \pi_{N_pM}(D_X G|_p) = D_X G^\perp, \tag{B-2}$$

where π_U denotes the orthogonal projection on U. A normal vector field that will be used very frequently is $Y = A(Z, W)$ for some $Z, W \in \mathcal{V}(M)$. This is however not just a normal vector field but each of its components is also a (2, 0)-tensor — we may think of $p \to A_p(Z, W)$ as a (2, 0)-tensor on M with values in the normal bundle NM, i.e., a for each $p \in M$ it is a multilinear map from T_pM^2 to N_pM. If we do so, the standard concept of *tensorial connections* (see [Lee 2018, Lemma 4.6]) is not applicable, since it is needed that the tensor takes values in $\mathbb{R}$. One can however overcome this by using two *different connections*, namely ∇ and $\nabla^\perp$. More precisely, for a $(k, 0)$-tensor $F : p \mapsto (F_p : T_pM^k \to N_pM)$ on M with values in the normal bundle NM we can define a $(k+1, 0)$-tensor $\nabla^\perp F$ via

$$\nabla^\perp F(X_1, \dots, X_{k+1}) := \nabla^\perp_{X_1} F(X_2, \dots, X_{k+1}) - \sum_{j=2}^{k+1} F(X_2, \dots, \nabla_{X_1} X_j, \dots, X_{k+1}) \tag{B-3}$$

for $X_1, \dots, X_{k+1} \in \mathcal{V}(M)$. It can easily be checked that $\nabla^\perp F$ is indeed a $(k+1)$-tensor, i.e., $\nabla^\perp F_p$ depends only on $X_1(p), \dots, X_{k+1}(p)$. Moreover, if F is a (0, 0)-tensor on M with values in NM, i.e., $F \in C^\infty(M; NM)$, then the notation of $\nabla^\perp F$ coincides with the previous definition in (B-2). We remark that in [Kuwert and Schätzle 2001; 2002; 2004], $\nabla^\perp$ and ∇ are both denoted by ∇. The $L^\infty(M)$-norm of a $(k, 0)$-tensor F on M with values in NM is defined to be

$$\|F\|_{L^\infty(M)} := \sup_{p\in M} \sup_{\{E_1,E_2\} \text{ orthonormal basis of } T_pM} \sum_{i_1,\dots,i_k=1}^{2} |F(E_{i_1}, \dots, E_{i_k})|,$$

where $|\cdot|$ denotes the norm in $\mathbb{R}^n$. We will also use very frequently [Kuwert and Schätzle 2002, equation (2.7)], which we state here for the reader's convenience. Let $f \in C^\infty(M; \mathbb{R}^n)$ be an immersion with second fundamental form A and normal bundle NM. Then for each $G \in C^\infty(M; NM)$ and $X \in \mathcal{V}(M)$

one has

$$D_X G = \nabla_X^{\perp} G - \sum_{i=1}^{2} \langle G, A(X, E_i) \rangle_{\mathbb{R}^n} D_{E_i} f, \tag{B-4}$$

where $\{E_1, E_2\}$ is an arbitrary orthonormal basis of $T_p M$ with respect to $g_f := f^* g_{\mathbb{R}^n}$. We also remark that we can define a tensorial version of D, treated as a tensor on M with values in $\mathbb{R}^n$. The transformation law we prescribe here is analogous to (B-3), namely if F is a $(k, 0)$-tensor on M with values in $\mathbb{R}^n$, we define for $X_1, \dots, X_{k+1} \in \mathcal{V}(M)$

$$DF(X_1, \dots, X_{k+1}) := D_{X_1} F(X_2, \dots, X_{k+1}) - \sum_{j=2}^{k+1} F(X_2, \dots, \nabla_{X_1} X_j, \dots, X_{k+1}).$$

As an important special case we obtain for $f \in C^\infty(M, \mathbb{R}^n)$

$$D^2 f(X, Y) = D_X D_Y f - D_{\nabla_X Y} f.$$

If f is additionally an immersion, this formula yields exactly the second fundamental form (see (2-1)). Hence one could also write $A[f] = D^2 f$.

Appendix C: On the smooth convergence of surfaces

Here we present some useful results concerning smooth convergence on compact subsets of $\mathbb{R}^n$, which we will simply call smooth convergence.

We remark that smooth convergence, see Definition 2.1, actually takes place in the equivalence class of surfaces that coincide up to reparametrization, more precisely

Remark C.1. Consider a sequence of immersions $(f_j)_{j \in \mathbb{N}}$, $f_j : \Sigma \to \mathbb{R}^n$, that converges to $\hat{f}$ smoothly on compact subsets of $\mathbb{R}^n$ and a sequence of diffeomorphisms $(\Psi_j)_{j \in \mathbb{N}}$, $\Psi_j : \Sigma_j \to \Sigma$, with Σ_j a smooth manifold without boundary. Then it follows from the definition of smooth convergence that $f_j \circ \Psi_j$ converges to $\hat{f}$ smoothly on compact subsets of $\mathbb{R}^n$. Moreover if $\Psi : \widetilde{\Sigma} \to \widehat{\Sigma}$ is yet another diffeomorphism then f_j also converges to $f \circ \Psi$ smoothly on compact subsets of $\mathbb{R}^n$.

Remark C.2. In general, smooth convergence is not *topology-preserving*, i.e., the topologies of $\widehat{\Sigma}$ and Σ need not coincide; see [Breuning 2015, Figure 6]. The situation is better if Σ is connected and $\widehat{\Sigma}$ has a compact component C. Lemma 4.3 in [Kuwert and Schätzle 2001] gives that Σ, $\widehat{\Sigma}$ are diffeomorphic. By the previous remark they can then also chosen to be equal.

Next we examine how relevant geometric quantities behave with respect to smooth convergence, for instance the diameter.

Lemma C.3. *Suppose that $(f_j)_{j=1}^{\infty} : \Sigma \to \mathbb{R}^n$ is a sequence that converges smoothly on compact subsets of $\mathbb{R}^n$ to $\hat{f} : \widehat{\Sigma} \to \mathbb{R}^n$. Then*

$$\operatorname{diam} \hat{f}(\widehat{\Sigma}) \le \liminf_{j \to \infty} \operatorname{diam} f_j(\Sigma).$$

Proof. Suppose that $(\hat{f}(p_k))_{k=1}^{\infty}$, $(\hat{f}(q_k))_{k=1}^{\infty} \subset \hat{f}(\widehat{\Sigma})$ are sequences such that

$$|\hat{f}(p_k) - \hat{f}(q_k)| \to \operatorname{diam} \hat{f}(\widehat{\Sigma}).$$

Then, by Definition 2.1 for each $k \in \mathbb{N}$ there exists $j(k) \in \mathbb{N}$ such that $p_k, q_k \in \widehat{\Sigma}(j)$ for all $j \geq j(k)$. Now (2-5) implies that for all $j \geq j(k)$

$$\begin{aligned}|\hat{f}(p_k) - \hat{f}(q_k)| &\leq |f_j \circ \phi_j(p_k) - f_j \circ \phi_j(q_k)| + |u_j(p_k) - u_j(q_k)| \\ &\leq \operatorname{diam} f_j(\Sigma) + 2\|u_j\|_{L^\infty(\widehat{\Sigma}(j))}.\end{aligned}$$

Letting first $j \to \infty$ and then $k \to \infty$ we obtain the claim. □

Now we study the lower semicontinuity with respect to smooth convergence of the Willmore energy. As a first step we prove the following result.

Lemma C.4. *Let $(f_j)_{j\in\mathbb{N}}$, $f_j : \Sigma \to \mathbb{R}^n$ be a sequence of immersions that converges smoothly on compact subsets of $\mathbb{R}^n$ to an immersion $\hat{f} : \widehat{\Sigma} \to \mathbb{R}^n$. Let (U, ψ) be a chart for $\widehat{\Sigma}$ such that $U \subset \widehat{\Sigma}(J)$ for some $J \in \mathbb{N}$ and $\hat{g}_{i\tau} \circ \psi^{-1} \in C^1(\overline{\psi(U)})$ $\widehat{\Gamma}^{\alpha}_{i\tau} \circ \psi^{-1} \in C^0(\overline{\psi(U)})$ for all i, τ, α, and $\det(\hat{g})$, $\hat{g}_{11}$ are bounded from below by some positive $\delta > 0$, where $\hat{g}_{i\tau}$ and $\widehat{\Gamma}^{\alpha}_{i\tau}$ denote the metric and Christoffel's symbols induced by $\hat{f}$ on $\widehat{\Sigma}$. Moreover we require that $\|D^2\hat{f}\|_{L^\infty(U,g_{\hat{f}})}$, $\|A[\hat{f}]\|_{L^\infty(U,g_{\hat{f}})}$, $\|DA[\hat{f}]\|_{L^\infty(U,g_{\hat{f}})} < \infty$. Let $(\phi_j)_{j=1}^{\infty}$, $\phi_j : \widehat{\Sigma} \to \Sigma$, be a sequence of diffeomorphisms as in Definition 2.1. Let $\hat{g}(m)$ be the first fundamental form induced by $f_m \circ \phi_m$ on U with respect to the chart (U, ψ) and $H(m) := H_{f_m\circ\phi_m}$ be the mean curvature of $f_m \circ \phi_m$.*

Then, $\hat{g}(m) \circ \psi^{-1}$ converges to $\hat{g} \circ \psi^{-1}$ uniformly in $\psi(U)$ and $H(m) \circ \psi^{-1}$ converges to $H_{\hat{f}} \circ \psi^{-1}$ uniformly in $\psi(U)$.

Proof. For $m > J$ let u_m be as in Definition 2.1 such that on $\widehat{\Sigma}(m)$ one has

$$f_m \circ \phi_m + u_m = \hat{f} \qquad \text{and} \qquad \|(\hat{\nabla}^\perp)^k u_m\|_{L^\infty(\widehat{\Sigma}(m))} \to 0, \quad m \to \infty. \tag{C-1}$$

Let (y^1, y^2) be the local coordinates induced by (U, ψ); in particular for all $h \in C^\infty(\Sigma; \mathbb{R}^d)$, $d \in \mathbb{N}$, in particular observe that $\partial h/\partial y^i = (\partial(h \circ \psi^{-1})/\partial e_i) \circ \psi$ for all $h \in C^\infty(\Sigma; \mathbb{R}^d)$, $d \in \mathbb{N}$. Our first intermediate claim is that $\partial u_m/\partial y^i$ and $\partial^2 u_m/(\partial y^i\, \partial y^\tau)$ converge to zero uniformly in U for all i, τ.

In the following we let $E_1, E_2 \in \mathcal{V}(U)$ be the smooth orthonormal frame on $(U, g_{\hat{f}})$ which we obtain by applying the Gram-Schmidt procedure on $\{\partial/\partial y^1, \partial/\partial y^2\}$, i.e.,

$$E_1 = \frac{1}{\sqrt{\hat{g}_{1,1}}} \frac{\partial}{\partial y^1} \quad \text{and} \quad E_2 = \frac{1}{\sqrt{\hat{g}_{1,1}}\sqrt{\det(\hat{g})}} \Big(\hat{g}_{11} \frac{\partial}{\partial y^2} - \hat{g}_{12} \frac{\partial}{\partial y^1} \Big).$$

Note that by (B-4)

$$\frac{\partial u_m}{\partial y^i} = D_{(\partial/\partial y^i)} u_m = \hat{\nabla}^\perp_{(\partial/\partial y^i)} u_m - \sum_{j=1}^{2} \Big\langle u_m, A[\hat{f}]\Big(\frac{\partial}{\partial y^i}, E_j\Big)\Big\rangle_{\mathbb{R}^3} D_{E_j} \hat{f}$$

and hence on U we have

$$\left|\frac{\partial u_m}{\partial y^i}\right| \leq \|\hat{\nabla}^\perp u_m\|_{L^\infty(U)} + 2\|A[\hat{f}]\|_{L^\infty(U)} |\hat{g}_{i,i}|^{1/2} \|u_m\|_{L^\infty(U)}. \tag{C-2}$$

Estimating

$$\|\hat{\nabla}^{\perp} u_m\|_{L^\infty(U)} \le \|\hat{\nabla}^{\perp} u_m\|_{L^\infty(\widehat{\Sigma}(m))} \to 0, \quad \|u_m\|_{L^\infty(U)} \le \|u_m\|_{L^\infty(\widehat{\Sigma}(m))} \to 0, \quad |\hat{g}_{i,i}| \le \|\hat{g}_{i,i} \circ \psi^{-1}\|_{L^\infty(\psi(U))},$$

we infer that $\partial u_m / \partial y^i$ converges to zero uniformly on U. Next we compute for all i, τ, writing for short $A = A[\hat{f}]$,

$$\begin{aligned}
\frac{\partial^2 u_m}{\partial y^\tau \partial y^i} &= D_{\partial/\partial y^\tau} D_{\partial/\partial y^i} u_m = D_{\partial/\partial y^\tau}\Big(\hat{\nabla}^{\perp}_{\partial/\partial y^i} u_m - \sum_{j=1}^{2} \Big\langle u_m, A\Big(\frac{\partial}{\partial y^i}, E_j\Big)\Big\rangle_{\mathbb{R}^n} D_{E_j} f\Big) \\
&= D_{\partial/\partial y^\tau} \hat{\nabla}^{\perp}_{\partial/\partial y^i} u_m - \sum_{j=1}^{2} D_{\partial/\partial y^\tau}\Big[\Big\langle u_m, A\Big(\frac{\partial}{\partial y^i}, E_j\Big)\Big\rangle_{\mathbb{R}^n} D_{E_j} f\Big] \\
&= \hat{\nabla}^{\perp}_{\partial/\partial y^\tau} \hat{\nabla}^{\perp}_{\partial/\partial y^i} u_m - \sum_{l=1}^{2} \Big\langle \hat{\nabla}^{\perp}_{\partial/\partial y^i} u_m, A\Big(\frac{\partial}{\partial y^\tau}, E_l\Big)\Big\rangle_{\mathbb{R}^n} D_{E_l} f \\
&\quad - \sum_{j=1}^{2} \Big\langle \frac{\partial u_m}{\partial y^\tau}, A\Big(\frac{\partial}{\partial y^i}, E_j\Big)\Big\rangle_{\mathbb{R}^n} D_{E_j} \hat{f} - \sum_{j=1}^{2} \Big\langle u_m, D_{\partial/\partial y^\tau} A\Big(\frac{\partial}{\partial y^i}, E_j\Big)\Big\rangle_{\mathbb{R}^n} D_{E_j} f \\
&\quad - \sum_{j=1}^{2} \Big\langle u_m, A\Big(\frac{\partial}{\partial y^i}, E_j\Big)\Big\rangle_{\mathbb{R}^n} D_{\partial/\partial y^\tau} D_{E_j} \hat{f} \\
&= (\hat{\nabla}^{\perp})^2 u_m\Big(\frac{\partial}{\partial y^\tau}, \frac{\partial}{\partial y^i}\Big) + \hat{\nabla}^{\perp} u_m\Big(\hat{\nabla}_{\partial/\partial y^\tau} \frac{\partial}{\partial y^i}\Big) \\
&\quad - \sum_{l=1}^{2} \Big\langle \hat{\nabla}^{\perp} u_m\Big(\frac{\partial}{\partial y^i}\Big), A\Big(\frac{\partial}{\partial y^\tau}, E_l\Big)\Big\rangle_{\mathbb{R}^n} D_{E_l} f - \sum_{j=1}^{2} \Big\langle \frac{\partial u_m}{\partial y^\tau}, A\Big(\frac{\partial}{\partial y^i}, E_j\Big)\Big\rangle_{\mathbb{R}^n} D_{E_j} \hat{f} \\
&\quad - \sum_{j=1}^{2} \Big\langle u_m, DA\Big(\frac{\partial}{\partial y^\tau}, \frac{\partial}{\partial y^i}, E_j\Big) + A\Big(\hat{\nabla}_{\partial/\partial y^\tau} \frac{\partial}{\partial y^i}, E_j\Big) + A\Big(\frac{\partial}{\partial y^i}, \hat{\nabla}_{\partial/\partial y^\tau} E_j\Big)\Big\rangle_{\mathbb{R}^n} D_{E_j} \hat{f} \\
&\quad - \sum_{j=1}^{2} \Big\langle u_m, A\Big(\frac{\partial}{\partial y^i}, E_j\Big)\Big\rangle_{\mathbb{R}^n} \Big[D^2 \hat{f}\Big(\frac{\partial}{\partial y^\tau}, E_j\Big) + D\hat{f}(\hat{\nabla}_{\partial/\partial y^\tau} E_j)\Big].
\end{aligned}$$

All terms that appear here as arguments of tensors can be bounded in L^∞-norm with quantities that we assumed to be bounded. Notice that a bound on $\hat{\nabla}_{\partial/\partial y^\tau} \partial/\partial y^i$ needs the fact that the Christoffel symbols lie in $C^0(\overline{\psi(U)})$. Bounding $\hat{\nabla}_{\partial/\partial y^\tau} E_j$ in terms of the given quantities needs the explicit representation of E_j that we discussed above. Here we also need that $\det(\hat{g})$, $\hat{g}_{11}$ are bounded from below uniformly in U. We obtain with a straightforward computation that $\partial^2 u_m / (\partial y^\tau \partial y^i)$ converges to zero uniformly in U.

We now show that $\hat{g}(m)$ converges to $\hat{g}$ uniformly on U, which implies the convergence claimed in the statement. First note that by (C-1) and (C-2)

$$\frac{\partial (f_m \circ \phi_m)}{\partial y^\tau} = \frac{\partial \hat{f}}{\partial y^\tau} + o(1),$$

where $\partial \hat{f} / \partial y^\tau$ are bounded by assumption. Hence, $\partial(f_m \circ \phi_m)/\partial y^\tau$ and $\hat{g}(m)$ are uniformly bounded. Now we can compute using (C-1)

$$\hat{g}_{i\tau} = \Big\langle \frac{\partial \hat{f}}{\partial y^i}, \frac{\partial \hat{f}}{\partial y^\tau} \Big\rangle_{\mathbb{R}^n} = \hat{g}_{i\tau}(m) + \Big\langle \frac{\partial (f_m \circ \phi_m)}{\partial y^i}, \frac{\partial u_m}{\partial y^\tau} \Big\rangle_{\mathbb{R}^n} + \Big\langle \frac{\partial (f_m \circ \phi_m)}{\partial y^\tau}, \frac{\partial u_m}{\partial y^i} \Big\rangle_{\mathbb{R}^n} + \Big\langle \frac{\partial u_m}{\partial y^\tau}, \frac{\partial u_m}{\partial y^i} \Big\rangle_{\mathbb{R}^n}.$$

By the arguments above, the last three terms are uniformly convergent to zero and so convergence of the first fundamental form is shown. Note in particular that also $\hat{g}^{-1}(m)$ converges to $\hat{g}^{-1}$ since we assumed that $\det(\hat{g})$ is strictly bounded from below.

Observe now that by (A-1) and (A-3)

$$\vec{H}_{\hat{f}} = \hat{g}^{i\tau}\left(\frac{\partial^2 \hat{f}}{\partial y^i \partial y^\tau} - \widehat{\Gamma}^\alpha_{i\tau}\frac{\partial \hat{f}}{\partial y^\alpha}\right),$$
$$\vec{H}(m) = \hat{g}^{i\tau}(m)\left(\frac{\partial^2 (f_m \circ \phi_m)}{\partial y^i \partial y^\tau} - \widehat{\Gamma}^\alpha_{i\tau}(m)\frac{\partial (f_m \circ \phi_m)}{\partial y^\alpha}\right),$$

where $\widehat{\Gamma}^\alpha_{i\tau}(m)$ denotes the Christoffel symbols of the immersion $f_m \circ \phi_m$ with respect to the chart (U, ψ). We have already discussed the uniform convergence of all terms that $H(m)$ consists of except for the Christoffel symbols. The convergence of those however follows analogously to the convergence of $\hat{g}(m)$ from the classical formula

$$\widehat{\Gamma}^\alpha_{i\tau}(m) = g^{\alpha\beta}(m)\left\langle \frac{\partial^2 (f_m \circ \phi_m)}{\partial y^i \partial y^\tau}, \frac{\partial (f_m \circ \phi_m)}{\partial y^\beta}\right\rangle_{\mathbb{R}^n}. \qquad \square$$

Lemma C.5. *Suppose that $(f_j)_{j=1}^\infty : \Sigma \to \mathbb{R}^n$ is a sequence of immersions that converges smoothly on compact subsets of $\mathbb{R}^n$ to an immersion $\hat{f} : \widehat{\Sigma} \to \mathbb{R}^n$. Then*

$$\mathcal{W}(\hat{f}) \le \liminf_{j\to\infty} \mathcal{W}(f_j).$$

Additionally, if $\widehat{\Sigma}$ is compact then $\mathcal{W}(\hat{f}) = \lim_{j\to\infty} \mathcal{W}(f_j)$.

Proof. We start choosing a cover $\{(U_p, \psi_p)\}_{p\in\widehat{\Sigma}}$ of $\widehat{\Sigma}$ such that U_p is an open neighborhood of p. Since each p is contained in some $\Sigma(m_p)$ for some $m_p \in \mathbb{N}$ and $\Sigma(m_p)$ is open, we may assume that $U_p \subset \Sigma(m_p)$ by possibly shrinking U_p. Let V_p be a neighborhood of p compactly contained in U_p. Then in each chart (V_p, ψ_p), $\hat{g}_{it}$ and Γ^α_{it} are bounded and $\det(\hat{g})$ is uniformly bounded from below by some $\delta = \delta(p) > 0$. By second countability there exist countably many points $\{p_\nu\}_{\nu=1}^\infty$ such that $\{(V_{p_\nu}, \psi_{p_\nu})\}_{\nu=1}^\infty$ is a cover of $\widehat{\Sigma}$ and there exists a locally finite partition of unity $(\eta_\nu)_{\nu=1}^\infty$ of smooth and compactly supported functions that satisfy $\operatorname{supp}(\eta_\nu) \subset V_{p_\nu}$. Now we infer by Lemma C.4 (taking diffeomorphisms ϕ_m as in (C-1)) and Fatou's lemma

$$\begin{aligned}
\int_{\widehat{\Sigma}} H_{\hat{f}}^2 \,\mathrm{d}\mu_{\hat{f}} &= \sum_{\nu=1}^\infty \int_{V_{p_\nu}} \eta_\nu H_{\hat{f}}^2 \,\mathrm{d}\mu_{\hat{f}} = \sum_{\nu=1}^\infty \int_{\psi_{p_\nu}(V_{p_\nu})} (\eta_\nu \circ \psi_{p_\nu}^{-1})(H_{\hat{f}} \circ \psi_{p_\nu}^{-1})^2 \sqrt{\det \hat{g}} \circ \psi_{p_\nu}^{-1} \,\mathrm{d}x \\
&= \sum_{\nu=1}^\infty \lim_{m\to\infty} \int_{\psi_{p_\nu}(V_{p_\nu})} (\eta_\nu \circ \psi_{p_\nu}^{-1})(H_{f_m\circ\phi_m} \circ \psi_{p_\nu}^{-1})^2 \sqrt{\det \hat{g}(m)} \circ \psi_{p_\nu}^{-1} \,\mathrm{d}x \\
&\le \liminf_{m\to\infty} \sum_{\nu=1}^\infty \int_{\psi_{p_\nu}(V_{p_\nu})} (\eta_\nu \circ \psi_{p_\nu}^{-1})(H_{f_m\circ\phi_m} \circ \psi_{p_\nu}^{-1})^2 \sqrt{\det \hat{g}(m)} \circ \psi_{p_\nu}^{-1} \,\mathrm{d}x \\
&= \liminf_{m\to\infty} \sum_{\nu=1}^\infty \int_{V_{p_\nu}} \eta_\nu H_{f_m\circ\phi_m}^2 \,\mathrm{d}\mu_{f_m\circ\phi_m} = \liminf_{m\to\infty} \int_{\widehat{\Sigma}} H_{f_m\circ\phi_m}^2 \,\mathrm{d}\mu_{f_m\circ\phi_m}.
\end{aligned}$$

All in all we obtain $\mathcal{W}(\hat{f}) \le \liminf_{m\to\infty} \mathcal{W}(f_m \circ \phi_m) = \liminf_{m\to\infty} \mathcal{W}(f_m)$ as the Willmore energy does not depend on the reparametrization. If $\widehat{\Sigma}$ is compact then the partition of unity can be chosen to be finite and the last claim follows then with the same techniques. $\square$

Lemma C.6 [Breuning 2015, Corollary 1.4]. *Suppose that $f_j : \Sigma \to \mathbb{R}^n$ and $\hat{f} : \widehat{\Sigma} \to \mathbb{R}^n$ are such that f_j converges to $\hat{f}$ smoothly on compact subsets of $\mathbb{R}^n$. Then the surface measures $f_j^*\mu_{g_j}$ converge in $C_0(\mathbb{R}^n)'$ to $\hat{f}^*\mu_{\hat{f}}$.*

A second concept of convergence that is related to smooth convergence is the C^l-convergence which we also use throughout the article.

Definition C.7. We say that a sequence of immersions $(f_j)_{j\in\mathbb{N}}$, $f_j : \Sigma \to \mathbb{R}^n$, defined on a two-dimensional manifold Σ without boundary converges to $\hat{f} : \Sigma \to \mathbb{R}^n$ in $C^l(\Sigma)$, $l \in \mathbb{N}$, if there exist diffeomorphisms $\phi_j : \Sigma \to \Sigma$ for all $j \in \mathbb{N}$ and $u_j : \Sigma \to N\Sigma$ such that $f_j \circ \phi_j + u_j = \hat{f}$ on Σ and $\|(\hat{\nabla}^\perp)^k u_j\|_{L^\infty(\Sigma)} \to 0$ as $j \to \infty$ for all $k \in \{0, \dots, l\}$.

Remark C.8. The two concepts of convergence we discussed are obviously related. Indeed, if $f_j : \widetilde{\Sigma} \to \mathbb{R}^n$ is a sequence that converges smoothly on compact subsets to some $\hat{f} : \widetilde{\Sigma} \to \mathbb{R}^n$ and $\widetilde{\Sigma}$ is compact, then f_j converges to $\hat{f}$ in C^l for all $l \in \mathbb{N}$. We further say that a family $(f(t))_{t\in[0,\infty)}$ converges to $\hat{f}$ in C^l for all l if for each sequence $t_j \to \infty$ one has $f(t_j) \to \hat{f}$ as $j \to \infty$.

We will now present an alternative characterization of C^l convergence in which we do not need to require that u_j are orthogonal. However we have to pay a price — in this case one needs to control the full derivative. Even though we expect this result to be true even in higher codimension, we formulate it only in the case of $n = 3$ for the sake of simplicity. This will be sufficient for our purposes.

Proposition C.9. *Let Σ be a compact orientable two-dimensional manifold without boundary and $f_j : \Sigma \to \mathbb{R}^3$ be a sequence of immersions and $k \ge 2$. Then f_j converges to a limit immersion $\hat{f} : \Sigma \to \mathbb{R}^3$ in C^k if and only if there exist $w_j \in C^k(\Sigma, \mathbb{R}^3)$ and C^k-smooth diffeomorphisms $\psi_j : \Sigma \to \Sigma$ such that for j large enough*

$$f_j \circ \psi_j = \hat{f} + w_j \quad \text{on } \Sigma$$

and for all $k \in \mathbb{N}$ one has $\|D^k w_j\|_{L^\infty(\Sigma, g_{\hat{f}})} \to 0$ as $j \to \infty$.

Proof. First assume that $f_j : \Sigma \to \mathbb{R}^3$ converges to $\hat{f} : \Sigma \to \mathbb{R}^3$ in $C^k(\Sigma)$. Then, for j large enough one can find $u_j \in C^k(\Sigma, N\Sigma)$ and C^k-diffeomorphisms $\phi_j : \Sigma \to \Sigma$ such that

$$f_j \circ \phi_j = \hat{f} + u_j \quad \text{on } \Sigma$$

and for all $k \in \mathbb{N}$ one has $\|(\hat{\nabla}^\perp)^k u_j\|_{L^\infty} \to 0$ as $j \to \infty$. Now we choose $\psi_j := \phi_j$ and $w_j := u_j$. It only remains to show that $\|D^k w_j\|_{L^\infty} \to 0$ as $k \to \infty$. For $k = 1$ we observe that for each $X \in \mathcal{V}(M)$ one has by (B-4)

$$D_X w_j = \hat{\nabla}^\perp_X w_j - \sum_{i=1}^{2} \langle w_j, A[\hat{f}](X, E_i)\rangle_{\mathbb{R}^3} D_{E_i} f = \hat{\nabla}^\perp_X u_j - \sum_{i=1}^{2} \langle u_j, A[\hat{f}](X, E_i)\rangle_{\mathbb{R}^3} D_{E_i} f. \tag{C-3}$$

We obtain that

$$\|Dw_j\|_{L^\infty} \leq \|\hat{\nabla}^\perp u_j\|_{L^\infty} + C\|u_j\|_{L^\infty}\|A[\hat{f}]\|_{L^\infty}\|D\hat{f}\|_{L^\infty}.$$

Since Σ is compact, $\|A[\hat{f}]\|_{L^\infty}$ and $\|D\hat{f}\|_{L^\infty}$ are finite and thus $\|Dw_j\|_{L^\infty} \to 0$ as $j \to \infty$. The estimates for $k \geq 2$ follow easily by using iterated versions of (C-3).

For the converse, suppose we have diffeomorphisms ψ_j and w_j as in the statement. We denote by $C^k(\Sigma; \mathbb{R})$ the set of all C^k-smooth real-valued maps from Σ of $\mathbb{R}$ equipped with the norm $\|f\|_{C^k(\Sigma;\mathbb{R})} := \sum_{l=1}^k \|\hat{\nabla}^l f\|_{L^\infty}$, where $\hat{\nabla}$ here denotes the tensorial connection with respect to the Levi-Civita connection on $(\Sigma, g_{\hat{f}})$; see [Lee 2018, Lemma 4.6]. We also endow $C^k(\Sigma; \mathbb{R}^3)$ with the norm $\|f\|_{C^k(\Sigma;\mathbb{R}^3)} = \sum_{l=1}^k \|D^l f\|_{L^\infty}$. Moreover we define $\mathrm{Diffeo}^k(\Sigma, \Sigma)$ to be the set of all C^k smooth diffeomorphisms of Σ. Note that $\mathrm{Diffeo}^k(\Sigma, \Sigma)$ is a smooth Banach manifold with the compact-open topology and for all $\phi \in \mathrm{Diffeo}^k(\Sigma)$ the tangent space $T_\phi \mathrm{Diffeo}^k(\Sigma, \Sigma)$ can be identified with $\mathcal{V}(\Sigma)$. This fact follows from [Wittmann 2019; Hirsch 1976, Chapter 2, Theorem 1.7]. Let now $N_{\hat{f}}$ be a smooth unit normal field along $\hat{f}$. (Here orientability of Σ is needed). We now define for all $k \in \mathbb{N}$ the map

$$F : \mathrm{Diffeo}^k(\Sigma, \Sigma) \times C^k(\Sigma; \mathbb{R}) \to C^k(\Sigma; \mathbb{R}^3), \quad F(\eta, \beta) := (\hat{f} + \beta N_{\hat{f}}) \circ \eta. \tag{C-4}$$

It is easy to show that for all $X \in \mathcal{V}(\Sigma)$ and $\alpha \in C^k(\Sigma; \mathbb{R})$ one has $d_{(\mathrm{id},0)}F(X, \alpha) = D_X \hat{f} + \alpha N_{\hat{f}}$. Having this formula, one checks that $d_{(\mathrm{id},0)}F : T_{(\mathrm{id},0)}(\mathrm{Diffeo}^k(\Sigma, \Sigma) \times C^k(\Sigma; \mathbb{R})) \to T_{\hat{f}}(C^k(\Sigma; \mathbb{R}^3)) \simeq C^k(\Sigma; \mathbb{R}^3)$ is an isomorphism. As a consequence one can find a small neighborhood V of $(\mathrm{id}, 0)$ such that $F|_V$ is a diffeomorphism. We conclude that for all $k \in \mathbb{N}$ there exists $\varepsilon > 0$ such that $\|g - \hat{f}\|_{C^k(\Sigma;\mathbb{R}^3)} < \varepsilon$ implies that there exists $\eta \in \mathrm{Diffeo}^k$ and $\beta \in C^k$ such that $g = (\hat{f} + \beta N_{\hat{f}}) \circ \eta$. Next we look at $g = \hat{f} + w_j$. For j large enough one has that there exists $\eta_j \in \mathrm{Diffeo}^k$ and $\beta_j \in C^k$ such that

$$\hat{f} + w_j = (\hat{f} + \beta_j N_{\hat{f}}) \circ \eta_j$$

and thus we infer that

$$f_j \circ \phi_j = (\hat{f} + \beta_j N_{\hat{f}}) \circ \eta_j.$$

We compose with η_j^{-1} to obtain

$$f_j \circ \phi_j \circ \eta_j^{-1} = \hat{f} + \beta_j N_{\hat{f}}.$$

Defining $\psi_j := \phi_j \circ \eta_j^{-1}$ and $u_j := \beta_j N_{\hat{f}}$ we obtain that $f_j \circ \psi_j = \hat{f} + u_j$ and $u_j \in C^k(\Sigma, N\Sigma)$. It remains to show that $\|(\hat{\nabla}^\perp)^l u_j\| \to 0$ for all $l = 1, \ldots, k$. To do so we compute for any $X \in \mathcal{V}(\Sigma)$

$$\hat{\nabla}_X^\perp u_j = \hat{\nabla}_X^\perp(\beta_j N_f) = X(\beta_j) N_{\hat{f}} + \beta_j \hat{\nabla}_X^\perp N_{\hat{f}}.$$

Note that $X(\beta_j) = \hat{\nabla}_X \beta_j$ and thus

$$\|\hat{\nabla}^\perp u_j\|_{L^\infty} \leq \|\beta_j\|_{C^1(\Sigma,\mathbb{R})}(1 + \|\hat{\nabla}^\perp N_{\hat{f}}\|_{L^\infty}).$$

Observe that $\|\hat{\nabla}^\perp N_{\hat{f}}\|_{L^\infty}$ is finite by the compactness of Σ. Similarly one can show that

$$\|(\hat{\nabla}^\perp)^j u_j\|_{L^\infty} \leq C(k, \Sigma, \hat{f})\|\beta_j\|_{C^j(\Sigma,\mathbb{R})} \quad \text{for all } j = 1, \ldots, k. \tag{C-5}$$

Note that $\hat{f} + w_j \to \hat{f}$ in $C^k(\Sigma; \mathbb{R}^3)$ and the fact that F, defined in (C-4), is a local diffeomorphism implies that (η_j, β_j) converges to $(\mathrm{id}, 0)$ in $\mathrm{Diffeo}^k(\Sigma, \Sigma) \times C^k(\Sigma)$. Thus β_j converges to 0 in $C^k(\widehat{\Sigma})$. This and (C-5) verify Definition C.7 for $l = k$. The claim is shown. $\square$

Also C^l-convergence is not affected by reparametrizations and Remark C.1 can be formulated also for the C^l-convergence. This implies in particular that limits with respect to C^l-convergence are not unique. In the rest of this section we will however show that, in our setting, C^l-limits are unique up to reparametrizations. Let us first fix what we mean by classical C^l convergence.

Definition C.10. We say that a sequence of immersions $(h_j)_{j=1}^{\infty}$, $h_j : \Sigma \to \mathbb{R}^n$, converges classically in C^l to some immersion $h : \Sigma \to \mathbb{R}^n$ if $u_j := h - h_j : \Sigma \to \mathbb{R}^n$ satisfies $\|D^k u_j\|_{L^\infty(\Sigma)} \to 0$ for all $k = 0, \dots, l$.

Proposition C.11. *Let $(f_j)_{j=1}^{\infty} : \mathbb{S}^1 \times \mathbb{S}^1 \to \mathbb{R}^3$ be a sequence of smooth immersions and $l \geq 2$. Let $f, h : \mathbb{S}^1 \times \mathbb{S}^1 \to \mathbb{R}^3$ be such that f_j converges to f in C^l and f_j converges to h classically in C^l. Then f and h coincide up to reparametrization, i.e., there exists a C^l-diffeomorphism $\phi : \mathbb{S}^1 \times \mathbb{S}^1 \to \mathbb{S}^1 \times \mathbb{S}^1$ such that $h = f \circ \phi$.*

Proof. Since f_j converges to f in C^l there exist diffeomorphisms ϕ_j of $\mathbb{S}^1 \times \mathbb{S}^1$ and maps $u_j : \mathbb{S}^1 \times \mathbb{S}^1 \to \mathbb{R}^3$ such that

$$f_j \circ \phi_j + u_j = f \quad \text{on } \mathbb{S}^1 \times \mathbb{S}^1, \tag{C-6}$$

and $\|u_j\|_{L^\infty}$, $\|Du_j\|_{L^\infty}$ converge to zero. Moreover there exist v_j such that

$$f_j + v_j = h \quad \text{on } \mathbb{S}^1 \times \mathbb{S}^1, \tag{C-7}$$

and $\|v_j\|_{L^\infty}$, $\|Dv_j\|_{L^\infty}$ converge to zero.

<u>Step 1</u>: $(\phi_j)_{j=1}^{\infty}$ converges uniformly to some $\phi \in C^0(\mathbb{S}^1 \times \mathbb{S}^1)$ that satisfies $h = f \circ \phi$. First note that functions on $\mathbb{S}^1 \times \mathbb{S}^1$ can be periodically extended on $\mathbb{R}^2$. Doing so and tacitly identifying all the functions we defined above with their unique periodic extensions we infer that (C-6) and (C-7) hold on the whole of $\mathbb{R}^2$. From both equations we infer that

$$h \circ \phi_j - v_j \circ \phi_j + u_j = f \quad \text{on } \mathbb{R}^2. \tag{C-8}$$

Since we deal now with functions in $C^1(\mathbb{R}^2; \mathbb{R}^3)$, we can compute derivatives simply using the Jacobi matrix. By the chain rule

$$(Dh(\phi_j) - Dv_j(\phi_j))D\phi_j + Du_j = Df \quad \text{in } \mathbb{R}^2. \tag{C-9}$$

We claim that $\|D\phi_j\|_{L^\infty(\mathbb{R}^{2\times 2})}$ is bounded. For this assume that a subsequence (which we do not relabel) satisfies $\|D\phi_j\|_{L^\infty} \to \infty$ and let $p_j \in \mathbb{S}^1 \times \mathbb{S}^1$ be such that $|D\phi_j(p_j)| = \|D\phi_j\|_{L^\infty}$, where $|\cdot|$ is a suitable matrix norm. Evaluating (C-9) at p_j and dividing by $\|D\phi_j\|_{L^\infty}$ we obtain

$$\big(Dh(\phi_j(p_j)) - Dv_j(\phi_j(p_j))\big)\frac{D\phi_j(p_j)}{\|D\phi_j\|_{L^\infty}} + \frac{1}{\|D\phi_j\|_{L^\infty}}Du_j(p_j) = \frac{1}{\|D\phi_j\|_{L^\infty}}Df(p_j). \tag{C-10}$$

By the boundedness of $\phi_j : \mathbb{R}^2 \to \mathbb{S}^1 \times \mathbb{S}^1$ and the choice of p_j one can choose a subsequence such that $(\phi_j(p_j))_{j=1}^{\infty}$ converges to some $q \in \mathbb{S}^1 \times \mathbb{S}^1$ and $D\phi_j(p_j)/\|D\phi_j\|_{L^\infty}$ converges to some $B \in \mathbb{R}^{2\times 2}$ that satisfies $|B|=1$. Note that by the requirements on u_j, v_j and the fact that the first fundamental forms of f, h with respect to the local coordinates (u, v) are bounded one has $\|Du_j\|_{L^\infty(\mathbb{R}^2,\mathbb{R}^{2\times 3})}, \|Dv_j\|_{L^\infty(\mathbb{R}^2,\mathbb{R}^{2\times 3})} \to 0$ as $j \to \infty$. Passing to the limit in (C-10) we obtain

$$Dh(q)B = 0.$$

This is a contradiction to h being an immersion and $|B| = 1$. Hence $\|D\phi_j\|_{L^\infty(\mathbb{R}^2,\mathbb{R}^{2,2})}$ is bounded. Note also that $\phi_j : \mathbb{R}^2 \to \mathbb{S}^1 \times \mathbb{S}^1$ is uniformly bounded as it takes values only in $\mathbb{S}^1 \times \mathbb{S}^1$. By the Arzelà–Ascoli theorem there exists a subsequence (which we do not relabel) and $\phi \in C^0(\mathbb{S}^1 \times \mathbb{S}^1)$ such that $\phi_j \to \phi$ on $\mathbb{S}^1 \times \mathbb{S}^1$. We can now go back to (C-8) and pass to the limit there to obtain

$$h \circ \phi = f \quad \text{on } \mathbb{S}^1 \times \mathbb{S}^1. \tag{C-11}$$

Step 2: ϕ is a local C^l diffeomorphism; i.e., ϕ is C^l smooth and for all $p \in \mathbb{S}^1 \times \mathbb{S}^1$ there exists an open neighborhood U containing p such that $\phi_{|U}$ is a diffeomorphism onto its image. To this end fix $p \in \mathbb{S}^1 \times \mathbb{S}^1$ and recall that, being h an immersion, there exists an open neighborhood W of $\phi(p)$ such that $h_{|W}$ is a diffeomorphism onto its image $V := h(W)$. We denote by $\tilde{h} : V \to W$ the inverse of $h_{|W}$. By (C-11) we obtain

$$\phi = \tilde{h} \circ f \quad \text{on } f^{-1}(V). \tag{C-12}$$

Notice that since $\phi(p) \in W$ it follows that $f(p) = h(\phi(p)) \in V$ and hence $p \in f^{-1}(V)$ so that $f^{-1}(V)$ is an open neighborhood of p. Now there exists another open neighborhood G of p such that $f_{|G}$ is a C^l-diffeomorphism onto its image. Defining $U = G \cap f^{-1}(V)$ we obtain that $\phi_{|U}$ is a C^l-diffeomorphism as a composition of two diffeomorphisms. Note in particular that $D\phi(p)$ is invertible at each point $p \in \mathbb{S}^1 \times \mathbb{S}^1$. This implies in particular, as $\mathbb{S}^1 \times \mathbb{S}^1$ is connected and $\phi \in C^l$ that $\operatorname{sgn}(\det(D\phi))$ is constant.

Step 3: $\deg(\phi) = \pm 1$. Recall that the *mapping degree* of ϕ is given by

$$\deg(\phi) := \sum_{x \in \phi^{-1}(\{y\})} \operatorname{sgn}(\det(D\phi(x))) \tag{C-13}$$

for any choice of $y \in \mathbb{S}^1 \times \mathbb{S}^1$. See [Outerelo and Ruiz 2009, Chapter 3] or [Guillemin and Pollack 1974, Chapter 3, Section 3] for the well-definedness of deg, e.g., the independence of the definition of the chosen y and finiteness of the sum in the definition. We make use of the degree-integration formula (see [Guillemin and Pollack 1974, p. 188]) to compute $\deg(\phi)$. Since $\phi : \mathbb{S}^1 \times \mathbb{S}^1 \to \mathbb{S}^1 \times \mathbb{S}^1$ is sufficiently smooth, one has for all differential forms ω on $\mathbb{S}^1 \times \mathbb{S}^1$ that

$$\int_{\mathbb{S}^1 \times \mathbb{S}^1} \phi^* \omega = \deg(\phi) \int_{\mathbb{S}^1 \times \mathbb{S}^1} \omega,$$

where $\phi^*\omega$ is defined as in [Guillemin and Pollack 1974, p. 166]. Let $\eta \in C_0^\infty(\mathbb{R}^3)$ be arbitrary. Take $\omega_\eta(u, v) := \eta(h(u, v))\sqrt{\det Dh^T Dh}\, du \wedge dv$. Then

$$\int_{\mathbb{S}^1 \times \mathbb{S}^1} \omega_\eta = \int_0^1 \int_0^{2\pi} \eta(h(u, v))\sqrt{\det(Dh^T Dh)}\, \mathrm{d}u\, \mathrm{d}v = \int \eta \, \mathrm{d}h^* \mu_h, \tag{C-14}$$

since $Dh^T Dh$ is the first fundamental form of $(\mathbb{S}^1 \times \mathbb{S}^1, g_h)$. Note that by Lemma C.6 $f^*\mu_f$ coincides with $h^*\mu_h$ as both measures are $C_0(\mathbb{R}^n)'$-limits of $f_j^*\mu_{f_j}$. Hence by (C-14)

$$\int \eta \,\mathrm{d} f^*\mu_f = \int \eta \,\mathrm{d} h^*\mu_h = \int_{\mathbb{S}^1\times\mathbb{S}^1} \omega_\eta. \tag{C-15}$$

Using now that $f = h \circ \phi$ we can also compute $\int \eta \,\mathrm{d} f^*\mu_f$ in another way. Since $s := \operatorname{sgn}\det D\phi$ is constant, by definition of $\phi^*\omega_\eta$ we obtain

$$\begin{aligned}\int \eta \,\mathrm{d} f^*\mu_f &= \int_0^1\int_0^1 \eta(f(u,v))\sqrt{\det(Df^T Df)}\,\mathrm{d}u\,\mathrm{d}v\\ &= \int_0^1\int_0^1 \eta(h(\phi(u,v)))\sqrt{\det(Dh^T Dh)}|\det(D\phi)|\,\mathrm{d}u\,\mathrm{d}v\\ &= s\int_0^1\int_0^1 \eta(h(\phi(u,v)))\sqrt{\det(Dh^T Dh)}\det(D\phi)\,\mathrm{d}u\,\mathrm{d}v\\ &= s\int_{\mathbb{S}^1\times\mathbb{S}^1} \phi^*\omega_\eta = s\cdot\deg(\phi)\int_{\mathbb{S}^1\times\mathbb{S}^1}\omega_\eta.\end{aligned}$$

This and (C-15) yields that $\deg(\phi) = 1/s = \pm 1$.

Conclusion: The fact that $\deg(\phi) = \pm 1$, $\operatorname{sgn}(\det(D\phi))$ is constant together with (C-13) imply that $\phi^{-1}(\{y\})$ must be a singleton for any choice of $y \in \mathbb{S}^1 \times \mathbb{S}^1$. This proves the injectivity of ϕ. Surjectivity follows directly from [Outerelo and Ruiz 2009, Chapter 3, Remark 1.5(2)]. We finally end up with a surjective and injective local diffeomorphism. By this inverse function theorem, this is also a global diffeomorphism. □

Corollary C.12. *If $(f_j)_{j=1}^\infty : \mathbb{S}^1 \times \mathbb{S}^1 \to \mathbb{R}^3$ converges in C^l to some $f : \mathbb{S}^1 \times \mathbb{S}^1 \to \mathbb{R}^3$ and also to some $h : \mathbb{S}^1 \times \mathbb{S}^1 \to \mathbb{R}^3$. Then there exists a C^l diffeomorphism $\phi : \mathbb{S}^1 \times \mathbb{S}^1 \to \mathbb{S}^1 \times \mathbb{S}^1$ such that $f = h \circ \phi$.*

Proof. If f_j converges to h in C^l then by Proposition C.9 there exists a sequence of diffeomorphisms $(\psi_j)_{j=1}^\infty$ of $\mathbb{S}^1 \times \mathbb{S}^1$ such that $f_j \circ \psi_j$ converges to h classically in C^l. Since (nonclassical) C^l convergence is not affected by reparametrizations, we infer that also $f_j \circ \psi_j$ converges to f in C^l. By Proposition C.11 applied to $f_j \circ \psi_j$ we infer that $f = h \circ \phi$ for a C^l-diffeomorphism ϕ of $\mathbb{S}^1 \times \mathbb{S}^1$. □

Appendix D: On the Willmore flow

Here we mention some previous results on the Willmore flow, which we will use. Since we need the precise formulations and constants we state them here for the readers convenience. We start with a short time existence and uniqueness result. We remark that this result is not the only short time existence result in the literature (see, e.g., [Simonett 2001]), but it is the most useful for the formulation we use.

Theorem D.1 [Kuwert and Schätzle 2002, Theorem 1.2]. *Suppose that $f_0 : \Sigma \to \mathbb{R}^n$ is a smooth immersion. Then there exist constants $\varepsilon_0 > 0$, $c_0 < \infty$ that depend only on n such that for all $\rho > 0$ that satisfy*

$$\sup_{x\in\mathbb{R}^n} \int_{f_0^{-1}(B_\rho(x))} |A[f_0]|^2 \,\mathrm{d}\mu_{f_0} \le \varepsilon_0$$

there exists a unique maximal smooth Willmore flow $(f(t))_{t\in[0,T)}$ *starting at* f_0 *that satisfies* $T \geq c_0\rho^4$. *Moreover, for all* $m \geq 0$ *there exists* $C = C(n, m, f_0)$ *such that*

$$\|(\nabla^{\perp})^m A[f(t)]\|_{L^\infty(\Sigma)} \leq C \quad \textit{for all } t \in [0, c_0\rho^4]. \tag{D-1}$$

Note that (D-1) is not in the statement of [Kuwert and Schätzle 2002, Theorem 1.2] but in its proof; see [Kuwert and Schätzle 2002, equation (4.27)]. In fact the bound of the derivatives of the curvature are crucial in the proof of the short time existence theorem. In addition to bounds on the curvature one also needs a bound on the metric. Let us emphasize that this bound is (in finite time) implied by the curvature bounds as part of a more general result; see [Hamilton 1982, Lemma 14.2]. Once short time existence is shown one can look at long time existence. The most important blow up criterion obtained so far is the one discussed in Theorem D.5 below. It says that if $T < \infty$ then the curvature has to concentrate. One can ask what happens to other quantities once the curvature degenerates. By Simon's monotonicity formula, the "density" will not degenerate. Indeed, in [Simon 1993, equation (1.3)], a local bound for the surface measure is shown. A useful implication stated in [Kuwert and Schätzle 2001, Lemma 4.1] is that there exists $c > 0$ such that for all proper immersions $f : \Sigma \to \mathbb{R}^n$ (Σ compact and without boundary) one has

$$\frac{\mu_f(f^{-1}(B_\rho(x_0)))}{\rho^2} \leq c\mathcal{W}(f) \quad \text{for all } \rho > 0, \tag{D-2}$$

where we further assume that Σ is a torus so that its Euler characteristic vanishes.

Up to this point, no examples of evolutions where the curvature degenerates are known, even though there exists one candidate for this phenomenon; see [Mayer and Simonett 2002].

Close to local minimizers curvature concentration cannot occur and one deduces convergence with the aid of a Łojasiewicz–Simon gradient inequality.

Theorem D.2 [Chill et al. 2009, Lemma 4.1]. *Let* $f_W : \Sigma \to \mathbb{R}^n$ *be a Willmore immersion of a compact manifold* Σ *without boundary, and let* $k \in \mathbb{N}$, $\delta > 0$. *Then there exists* $\varepsilon = \varepsilon(f_W) > 0$ *such that the following is true: suppose that* $(f(t))_{t\in[0,T)}$ *is a Willmore flow of* Σ *satisfying*

$$\|f_0 - f_W\|_{W^{2,2}\cap C^1} < \varepsilon$$

and

$$\mathcal{W}(f(t)) \geq \mathcal{W}(f_W) \quad \textit{whenever } \|f(t)\circ\Phi(t) - f_W\|_{C^k} \leq \delta, \tag{D-3}$$

for some appropriate diffeomorphisms $\Phi(t) : \Sigma \to \Sigma$.

Then this Willmore flow exists globally, that is, $T = \infty$, *and converges, after reparametrization by appropriate diffeomorphisms* $\widetilde{\Phi}(t) : \Sigma \to \Sigma$, *smoothly to a Willmore immersion* f_∞. *That is,*

$$f(t)\circ\widetilde{\Phi}(t) \to f_\infty \quad \textit{as } t \to \infty.$$

Moreover, $\mathcal{W}(f_\infty) = \mathcal{W}(f_W)$ *and* $\|f_0 - f_W\|_{C^k} < \delta$.

Remark D.3. Notice that ε in the statement does not change if instead of f_W one considers the translated Willmore surface $f_W + \bar{x}$ for $\bar{x} \in \mathbb{R}^n$. Indeed, if f_0 satisfies

$$\|f_0 - (f_W + \bar{x})\|_{W^{2,2}\cap C^1} < \varepsilon = \varepsilon(f_W),$$

then clearly $f_0 - \bar{x}$ satisfies the assumptions on the initial datum stated in Theorem D.2 so that the corresponding Willmore flow $\tilde{f}(t)$ converges. Due to the uniqueness of the solution for the Willmore flow, $\tilde{f}(t) = f(t) - \bar{x}$ with $f(t)$ the solution of the Willmore flow which starts in f_0. Hence, also $f(t)$ converges.

Remark D.4. We also remark that in case that the Willmore flow converges in C^k for all k one obtains uniform bounds on all derivatives of the second fundamental form, i.e., for all $m \in \mathbb{N}_0$ there exists $C = C(m, f_0)$ such that

$$\|(\nabla^{\perp})^m A[f(t)]\|_{L^\infty} \leq C \quad \text{for all } t \in [0, \infty).$$

Not every evolution of the Willmore flow is convergent. What one can however always obtain is a *Willmore concentration limit* of appropriate parabolic rescalings. Below we will introduce the Willmore concentration limit rigorously since we need to examine it for the proof of Theorem 3.1.

Theorem D.5 (Willmore concentration limit [Kuwert and Schätzle 2001, Section 4]). *Let Σ be a compact two-dimensional manifold without boundary and let $f : [0, T) \times \Sigma \to \mathbb{R}^n$ be immersions evolving by the Willmore flow with initial datum f_0. Let $\varepsilon_0 > 0$ and c_0 be defined as in Theorem D.1.*

Then for each sequence $(t_j)_{j=1}^{\infty} \nearrow T$ there exist $(x_j)_{j=1}^{\infty} \subset \mathbb{R}^n$, $(r_j)_{j=1}^{\infty} \subset (0, \infty)$ (defined as in (3-1)) and $c_0 > 0$ such that

$$t_j + c_0 r_j^4 < T \quad \text{for all } j \in \mathbb{N} \tag{D-4}$$

and

$$\tilde{f}_j := \frac{1}{r_j}(f(t_j + c_0 r_j^4, \cdot) - x_j) : \Sigma \to \mathbb{R}^n \tag{D-5}$$

converges smoothly on compact subsets of $\mathbb{R}^n$ to a proper Willmore immersion $\hat{f} : \widehat{\Sigma} \to \mathbb{R}^n$, where $\widehat{\Sigma} \neq \varnothing$ is a smooth two-dimensional manifold without boundary. Moreover

$$\liminf_{j\to\infty} \int_{B_j} |A(t_j + c_0 r_j^4)|^2 \, \mathrm{d}\mu_{g(t_j + c_0 r_j^4)} > 0, \tag{D-6}$$

where $B_j = (f(t_j + c_0 r_j^4))^{-1}(\overline{B_{r_j}(x_j)})$.

Now we are finally ready to prove Theorem 3.1.

Proof of Theorem 3.1. The first part of the statement follows from (D-4). From Theorem D.5 it follows that there exists a sequence $(x_j)_{j\in\mathbb{N}} \subset \mathbb{R}^n$ and a proper Willmore immersion $\hat{f} : \widehat{\Sigma} \to \mathbb{R}^n$ such that

$$\tilde{f}_{j,c_0} - \frac{x_j}{r_j} \to \hat{f}, \tag{D-7}$$

smoothly as $j \to \infty$. Now we examine the asymptotics of $(r_j)_{j\in\mathbb{N}}$.

If there exists a subsequence of the radii r_j that tends to zero or infinity. By [Chill et al. 2009, Theorem 1.1], $\widehat{\Sigma}$ is not compact. In particular $\operatorname{diam}(\hat{f}(\widehat{\Sigma})) = \infty$ since otherwise $\hat{f}(\widehat{\Sigma})$ lies in a compact set of $\mathbb{R}^n$ which is a contradiction to the properness of $\hat{f}$. By lower semicontinuity of the diameter, see

Lemma C.3, we infer

$$\infty = \operatorname{diam}(\hat{f}(\widehat{\Sigma})) \leq \liminf_{j\to\infty} \operatorname{diam}\left(\tilde{f}_{j,c_0} - \frac{x_j}{r_j}\right) = \liminf_{j\to\infty} \operatorname{diam}(\tilde{f}_{j,c_0}).$$

Hence we have shown that (2) occurs.

Suppose on contrary that $(r_j)_{j\in\mathbb{N}}$ has no subsequence that tends to zero or infinity. Then there exists $\delta > 0$ such that $\delta < r_j < 1/\delta$ for all $j \in \mathbb{N}$ and Case 1 occurs. Necessarily from (D-4) we see that $T = \infty$.

It remains to show that a bound on the diameter ensures full convergence to a Willmore immersion. Suppose therefore that $\operatorname{diam}(\tilde{f}_{j,c_0}) \leq M$ for all $j \in \mathbb{N}$. Note that then - once again by lower semicontinuity, see Lemma C.3,

$$\operatorname{diam}(\hat{f}(\widehat{\Sigma})) \leq \liminf_{j\to\infty} \operatorname{diam}\left(f_{j,c_0} - \frac{x_j}{r_j}\right) = \liminf_{j\to\infty} \operatorname{diam}(f_{j,c_0}) \leq M.$$

Since $\hat{f}$ is proper this ensures that $\widehat{\Sigma}$ is compact. By [Kuwert and Schätzle 2001, Lemma 4.3] we infer that $\widehat{\Sigma} = \mathbb{S}^1 \times \mathbb{S}^1$ and the convergence in (D-7) is actually convergence in C^k for all $k \in \mathbb{N}$. Now we define

$$\tilde{f}_j : [0, c_0] \times \mathbb{S}^1 \times \mathbb{S}^1 \to \mathbb{R}^n, \qquad \tilde{f}_j(s) := \frac{f(t_j + s r_j^4)}{r_j}.$$

Note that by scaling properties of the Willmore gradient $\tilde{f}_j$ solves the Willmore flow equation. By (D-7) we can now fix $j_0 \in \mathbb{N}$ and a smooth diffeomorphism $\Phi : \mathbb{S}^1 \times \mathbb{S}^1 \to \mathbb{S}^1 \times \mathbb{S}^1$ such that

$$\left\| \tilde{f}_{j_0,c_0} \circ \Phi - \frac{x_{j_0}}{r_{j_0}} - \hat{f} \right\|_{C^2} < \varepsilon = \varepsilon(\hat{f}), \tag{D-8}$$

where $\varepsilon(\hat{f})$ is chosen as in Theorem D.2. By Remark D.3 we also have $\varepsilon(\hat{f}) = \varepsilon(\hat{f} + x_{j_0}/r_{j_0})$. We infer by Theorem D.2 that the Willmore flow starting at $\tilde{f}_{j_0,c_0} \circ \Phi$ exists globally and converges (up to reparametrization) to a Willmore immersion $f_\infty : \mathbb{S}^1 \times \mathbb{S}^1 \to \mathbb{R}^n$. By geometric uniqueness of Willmore evolutions we infer that $\tilde{f}_{j_0} \circ \Phi$, first defined on $[0, c_0]$, extends to a global evolution, i.e., defined on $[0, \infty)$, and converges (up to reparametrization) to f_∞. Again by geometric uniqueness we infer that $\tilde{f}_{j_0}$ extends to a global evolution converging (up to reparametrization) to $f_\infty \circ \Phi^{-1}$. Using scaling properties of the Willmore flow we infer that f extends to a global evolution by Willmore flow that converges to $r_{j_0} f_\infty$, which is again a Willmore immersion.

To show the last sentence of the claim we first observe that a uniform bound on the diameter implies that Case 2 may not occur, in particular $r_j \in (\delta, 1/\delta)$ for some $\delta > 0$. Then the fact that $t_j + c_0 r_j^4 < T$ for all j and $t_j \to T$ implies that $T = \infty$. Convergence follows then according to case (1) with the diameter bound. □

With this theorem we have proved that boundedness of $\operatorname{diam}(\tilde{f}_{j,c_0})$ implies convergence. The fact that the $\tilde{f}_{j,c_0}$ need information about $f(t_j + c_0 r_j^4)$ and not just about $f(t_j)$ adds a technical difficulty — the time shift might cause geometric quantities to degenerate. Luckily, the diameter is not so much affected by (bounded) time shifts, as we shall see in the following:

Lemma D.6 (Evolution of diameter and area). *Suppose that $f : [0, T) \times \Sigma \to \mathbb{R}^n$ is a maximal evolution by Willmore flow. Then there exist constants $C_1 = C_1(\mathcal{W}(f(0)))$, $C_2 = C_2(\mathcal{W}(f(0)))$ depending monotonically on $\mathcal{W}(f(0))$ such that*

$$\mu_{g_{f(t)}}(\Sigma) \le \mu_{g_{f(0)}}(\Sigma) + C_1(\mathcal{W}(f(0)))t^{1/2} \tag{D-9}$$

and

$$\operatorname{diam}(f(t)(\Sigma)) \le C_2(\mathcal{W}(f(0)))(\operatorname{diam}(f(0)(\Sigma)) + t^{1/4}).$$

Proof. First we remark that, since the Willmore flow is a gradient flow, for all $s \ge 0$

$$\int_0^s \int_\Sigma |\partial_t f(t)|^2 \, \mathrm{d}\mu_{g_{f(t_j)}} = \mathcal{W}(f(0)) - \mathcal{W}(f(s)) \le \mathcal{W}(f(0)). \tag{D-10}$$

By [Kuwert and Schätzle 2002, equation (2.16)] we have

$$\begin{aligned}\left|\frac{d}{dt}\mu_{g_{f(t)}}(\Sigma)\right| &= \left|\int_\Sigma \langle \vec{H}[f(t)], \partial_t f(t)\rangle \, \mathrm{d}\mu_{g_{f(t)}}\right| \\ &\le \left(\int_\Sigma |\vec{H}[f(t)]|^2 \, \mathrm{d}\mu_{g_{f(t)}}\right)^{1/2} \left(\int_\Sigma |\partial_t f(t)|^2 \, \mathrm{d}\mu_{g_{f(t)}}\right)^{1/2} \\ &\le 2\sqrt{\mathcal{W}(f(t))}\left(\int_\Sigma |\partial_t f(t)|^2 \, \mathrm{d}\mu_{g_{f(t)}}\right)^{1/2}.\end{aligned}$$

Integrating with respect to t and since $t \mapsto \mathcal{W}(f(t))$ is decreasing we obtain

$$\begin{aligned}|\mu_{g_{f(s)}}(\Sigma) - \mu_{g_{f(0)}}(\Sigma)| &\le 2\sqrt{\mathcal{W}(f(0))} \int_0^s \left(\int_\Sigma |\partial_t f(t)|^2 \, \mathrm{d}\mu_{g_{f(t)}}\right)^{1/2} \mathrm{d}t \\ &\le 2\sqrt{\mathcal{W}(f(0))}s^{1/2}\left(\int_0^s \int_\Sigma |\partial_t f(t)|^2 \, \mathrm{d}\mu_{g_{f(t)}} \, \mathrm{d}t\right)^{1/2} \le 2\mathcal{W}(f(0))s^{1/2},\end{aligned}$$

using (D-10) in the last step. The estimate in (D-9) follows if we choose $C_1(W) = 2\mathcal{W}(f(0))$. Next we use a generalization of [Simon 1993, Lemma 1.1] (see the following lemma) for immersed surfaces to obtain that there exists $C_S > 0$ such that $\operatorname{diam}(f(\Sigma))^2 \le C_S \mathcal{W}(f)\mu_{g_f}(\Sigma)$. Using this, (D-9) and Lemma D.7 we obtain

$$\begin{aligned}\operatorname{diam}(f(t)(\Sigma))^2 &\le C_S\mathcal{W}(f(t))\mu_{g_{f(t)}}(\Sigma) \le C_S\mathcal{W}(f(0))(\mu_{g_{f(0)}}(\Sigma) + 2\mathcal{W}(f(0))t^{1/2}) \\ &\le C_S\mathcal{W}(f(0))\big(\mathcal{W}(f(0)) \operatorname{diam}(f(0))^2 + 2W(f(0))t^{1/2}\big). \\ &\le C_S\mathcal{W}(f(0))^2(\operatorname{diam}(f(0))^2 + 2t^{1/2}) \\ &\le 2C_S\mathcal{W}(f(0))^2(\operatorname{diam}(f(0)) + t^{1/4})^2.\end{aligned}$$

The choice of $C_2(W) := 2C_S W^2$ does the job. □

In this proof we have used the following lemma, which generalizes [Simon 1993, Lemma 1.1].

Lemma D.7 (cf. [Simon 1993, Lemma 1.1]). *There exists $C_S = C_S(n) > 0$ such that for all immersions $f : \Sigma \to \mathbb{R}^n$ of a compact connected two-dimensional manifold without boundary Σ one has*

$$\frac{\mu_{g_f}(\Sigma)}{\mathcal{W}(f)} \le \operatorname{diam}(f(\Sigma))^2 \le C_S(n)\mu_{g_f}(\Sigma)\mathcal{W}(f).$$

Proof. Let Σ be as in the statement. By [Simon 1993, Lemma 1.1] we infer that for all $n \in \mathbb{N}$ there exists $c(n) > 0$ such that for all embeddings $f : \Sigma \to \mathbb{R}^n$ one has

$$\frac{\mu_{g_f}(\Sigma)}{\mathcal{W}(f)} \leq \operatorname{diam}(f(\Sigma))^2 \leq c(n)\mu_{g_f}(\Sigma)\mathcal{W}(f). \tag{D-11}$$

We need to generalize this result to immersions. Let $N \in \mathbb{N}$ be such that each smooth two-dimensional manifold can be smoothly embedded into $\mathbb{R}^N$. Such a constant N exists due to Nash's embedding theorem (or alternatively one can derive $N = 4$ explicitly using a handle decomposition). We will show that the desired estimate is satisfied with the constant $C_S(n) := c(n + N)$. To this end let $f : \Sigma \to \mathbb{R}^n$ be an immersion and $\iota : \Sigma \to \mathbb{R}^N$ be an embedding. For fixed $\varepsilon > 0$ define $f_\varepsilon : \Sigma \to \mathbb{R}^{n+N}$ via $f_\varepsilon(p) := (f(p), \varepsilon\iota(p))^T$. It is easy to check that f_ε is an embedding. We infer by (D-11) that

$$\frac{\mu_{g_{f_\varepsilon}}(\Sigma)}{\mathcal{W}(f_\varepsilon)} \leq \operatorname{diam}(f_\varepsilon(\Sigma))^2 \leq c(n+N)\mu_{g_{f_\varepsilon}}(\Sigma)\mathcal{W}(f_\varepsilon). \tag{D-12}$$

Next we pass to the limit as $\varepsilon \to 0$. First we examine the diameter. Note that for all $x, y \in \Sigma$ one has

$$|f_\varepsilon(x) - f_\varepsilon(y)|^2 = |f(x) - f(y)|^2 + \varepsilon^2|\iota(x) - \iota(y)|^2.$$

From this one easily infers

$$\operatorname{diam}(f(\Sigma))^2 \leq \operatorname{diam}(f_\varepsilon(\Sigma))^2 \leq \operatorname{diam}(f(\Sigma))^2 + \varepsilon^2 \operatorname{diam}(\iota(\Sigma)).$$

Since Σ is compact we find that $\operatorname{diam}(\iota(\Sigma)) < \infty$. Hence

$$\lim_{\varepsilon\to 0} \operatorname{diam}(f_\varepsilon(\Sigma)) = \operatorname{diam}(f(\Sigma)).$$

One readily checks that $f_\varepsilon \to (f, 0)$ in C^k for all k. From Lemma C.5 one infers then that $\lim_{\varepsilon\to 0} \mathcal{W}(f_\varepsilon) = \mathcal{W}((f, 0)) = \mathcal{W}(f)$. That $\mathcal{W}((f, 0)) = \mathcal{W}(f)$ can easily be checked since

$$A[(f, 0))](X, Y) = D^2(f, 0)(X, Y) = (D^2 f(X, Y), 0),$$

where the last identity is due to the fact that D is defined componentwise; see (B-1). Using methods similar to Lemma C.5 one can also check $\lim_{\varepsilon\to 0} \mu_{g_{f_\varepsilon}}(\Sigma) = \mu_{g_{(f,0)}}(\Sigma) = \mu_{g_f}(\Sigma)$. This being shown, the claim follows from (D-12) letting $\varepsilon \to 0$. □

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OPTIMAL PRANDTL EXPANSION AROUND A CONCAVE BOUNDARY LAYER

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We show an optimal stability result for boundary layer solutions of the Navier–Stokes equation in a half-plane, under a mild concavity condition on the boundary layer profile. The key point is the derivation of sharp Gevrey estimates for the linearized Navier–Stokes equation in vorticity form, on a time interval uniform in ν. As the nonlocal boundary condition on the vorticity prevents us from deriving direct estimates, we use a novel iteration scheme, similar to a splitting method in numerical analysis. Our result is a big step forward compared to our previous work (*Duke Math. J.* **167** (2018), 2531–2631), where we proved stability of boundary layer expansions of shear flow type. Indeed, the approach of the present paper is much more robust than the one in that previous work, which was based on the Fourier transform and hence only adapted to expansions independent of the tangential variable. Moreover, we are now able to relax the assumption of strict concavity made in our previous work to obtain the optimal Gevrey $\frac{3}{2}$ stability, which was not satisfied by generic boundary layer expansions. We provide in this way the first justification of unsteady boundary layer theory outside the analytic setting.

1. Introduction

We are interested in the high Reynolds number dynamics of the Navier–Stokes equation in a half-plane:

$$\begin{aligned} \partial_t u^\nu - \nu\Delta u^\nu + \nabla p^\nu + u^\nu\cdot\nabla u^\nu = 0, \quad t>0,\ x\in\mathbb{T},\ y>0, \\ \nabla\cdot u^\nu = 0, \quad t\ge 0,\ x\in\mathbb{T},\ y>0, \\ u^\nu|_{y=0}=0, \quad u^\nu|_{t=0}=u_0, \end{aligned} \tag{1-1}$$

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where ν stands for the inverse Reynolds number. Note that we consider periodic boundary conditions in x, but could consider decay conditions as well. As is well known, the Navier–Stokes solution u^ν exhibits a boundary layer near $y=0$, that is a region of high velocity gradients generated by the no-slip condition. A famous modeling of this boundary layer was provided by Prandtl. In modern language, he provided approximate solutions of Navier–Stokes equations in the form of multiscale asymptotic expansions:

$$v=\sum_{i=0}^{N}\sqrt{\nu}^{i}U^{E,i}(t,x,y)+\sum_{i=0}^{N}\sqrt{\nu}^{i}(V_1^{\mathrm{bl},i}(t,x,y/\sqrt{\nu}),\sqrt{\nu}V_2^{\mathrm{bl},i}(t,x,y/\sqrt{\nu})),\tag{1-2}$$

where the profiles $U^{E,i}=U^{E,i}(t,x,y)$ describe the flow away from the boundary, and the profiles $V^{\mathrm{bl},i}=V^{\mathrm{bl},i}(t,x,Y)$ are boundary layer correctors that go to zero exponentially fast in variable $Y=y/\sqrt{\nu}$. We stress that there is a factor $\sqrt{\nu}$ between the amplitudes of the horizontal and vertical components of the boundary layer profiles: this is consistent with the divergence-free condition. In particular, the leading order term $U^E:=U^{E,0}$ solves the Euler equation, while the leading order boundary corrector $V^{\mathrm{bl}}:=V^{\mathrm{bl},0}$ solves the modified Prandtl equation

$$\begin{aligned}\partial_t V_1^{\mathrm{bl}}+(U_1^E|_{y=0}+V_1^{\mathrm{bl}})\partial_x V^{\mathrm{bl},1}+V_1^{\mathrm{bl}}\partial_x U_1^E|_{y=0}+(Y\partial_y U_2^E|_{y=0}+V_2^{\mathrm{bl}})\partial_Y V_1^{\mathrm{bl}}-\partial_Y^2 V_1^{\mathrm{bl}}&=0,\\ \partial_x V^{\mathrm{bl},1}+\partial_Y V_2^{\mathrm{bl}}&=0,\\ V_1^{\mathrm{bl}}|_{Y=0}=-U_1^E|_{y=0},\quad V^{\mathrm{bl}}\to 0,\quad Y\to+\infty.&\end{aligned}$$

Prandtl boundary layer theory has revealed much about the mechanism of vorticity generation in fluids and has contributed to the quantitative understanding of some model problems, notably the description of the Blasius flow near a flat plate. It can moreover be rigorously justified under strong symmetry conditions on the flow and its perturbations; see for instance [Lopes Filho et al. 2008; Mazzucato and Taylor 2008]. Still, under generic perturbations, Navier–Stokes flows of type (1-2) are known to experience instabilities, due to two main mechanisms:

- Boundary layer separation, which corresponds to a loss of monotonicity and concavity of the boundary layer profile V_1^{bl}, under an adverse pressure gradient. Mathematically, it corresponds to some ill-posedness or blow-up of the Prandtl model.
- Hydrodynamic instabilities of Tollmien–Schlichting-type, experienced by concave boundary layer flows.

These phenomena have crucial consequences in hydrodynamics and aerodynamics. From the mathematical point of view, describing the stability/instability properties of flows v of type (1-2) is a difficult topic. The evolution of the perturbation $w=u^\nu-v$ obeys the perturbed Navier–Stokes system

$$\begin{aligned}\partial_t w-\nu\Delta w+\nabla q+v\cdot\nabla w+w\cdot\nabla v&=-w\cdot\nabla w+r, && t>0,\ x\in\mathbb{T},\ y>0,\\ \nabla\cdot w&=0, && t\ge 0,\ x\in\mathbb{T},\ y>0,\\ w|_{y=0}=0,\quad w|_{t=0}&=w_0.\end{aligned}\tag{1-3}$$

Here, r represents a remainder term due to the approximation v, while w_0 is a given initial perturbation of the velocity. We will assume that r and w_0 are of the order $O(\nu^n)$ in some norm with $n\gg 1$. In the case of r, this is realized by taking N large enough in (1-2). More precisely, one has to consider functional frameworks such that the equations of both Prandtl-type and Euler-type are uniquely solvable at least locally in time. Then, the point is to understand under which conditions one can obtain uniform (in ν) estimates of w in a suitable norm, that is justification of the Prandtl theory.

An important result in this direction is due to Sammartino and Caflisch [1998a; 1998b], who proved local well-posedness of Euler and Prandtl equations, as well as stability results for (1-3) in the case of analytic data. This stability result is then extended by [Fei et al. 2018; Kukavica et al. 2020; 2022; Maekawa 2014; Wang and Wang 2020; Wang et al. 2017], all of which require the analyticity near the boundary. This general analytic stability result is somehow optimal, in view of [Grenier 2000a]; see also [Grenier and Nguyen 2019]. Grenier studied the case where the Prandtl expansion v in (1-2) is a shear flow: this means that

$$v = (V_1^{\mathrm{bl}}(t, x, y/\sqrt{\nu}), 0), \tag{1-4}$$

where V_1^{bl} solves the heat equation

$$\partial_t V_1^{\mathrm{bl}} - \partial_Y^2 V_1^{\mathrm{bl}} = 0, \qquad V_1^{\mathrm{bl}}|_{Y=0} = 0. \tag{1-5}$$

He proved that for some profiles V_1^{bl} that have initially inflection points, the linearized version of (1-3) admits growing perturbations of the form

$$w^\nu(t, x, y) \approx e^{\alpha t/\nu^{1/2}} e^{ix/\nu^{1/2}} \widetilde{w}^\nu(y),$$

with fixed $\alpha > 0$. This shows that high frequencies $k \approx 1/\nu^{1/2}$ in variable x may be amplified by $e^{\alpha k t}$. In other words, to obtain a bound independent of ν over a time $T = O(1)$ will only be possible if those modes k have amplitude less than $e^{-\delta k}$, with $\delta \leq \alpha T$. This necessary exponential decay of the frequency spectrum corresponds to analytic perturbations. Let us note that the result of Grenier relies on the so-called Rayleigh instability, which is an inviscid instability mechanism for shear flows with inflection points. In terms of hydrodynamics of the boundary layer, the appearance of inflection points corresponds to the separation phenomenon. Hence, it is a framework in which various negative results exist for the Prandtl equation itself [E and Engquist 1997; Gérard-Varet and Dormy 2010; Gérard-Varet and Nguyen 2012; Kukavica et al. 2017].

The case without inflection points, corresponding to the nicer situation where the boundary layer profile V_1^{bl} is concave in variable Y, is much more involved. Again, the natural first step is to consider the shear flow situation (1-4). The stability of shear flows within the Navier–Stokes equation is an old topic of hydrodynamics, notably studied by Tollmien and Schlichting. See [Drazin and Reid 2004] for a detailed account. They showed that generic concave shear flows, although stable in the Euler evolution, exhibit instability in the Navier–Stokes one (albeit with a growth rate vanishing with viscosity). This is the so-called Tollmien–Schlichting instability, revisited on a rigorous basis by Grenier, Guo and Nguyen [Grenier et al. 2016]. Roughly, by using a proper rescaling of these unstable eigenmodes, one can construct for the linearization of (1-3) solutions of the type

$$w^\nu(t, x, y) \approx e^{\alpha t/\nu^{1/4}} e^{ix/\nu^{3/8}} \widetilde{w}^\nu(y).$$

This time, high frequencies $k \approx 1/\nu^{3/8}$ may be amplified by $e^{\alpha k^{2/3} t}$. This is still not compatible with Sobolev uniform bounds. More precisely, under the assumption that the spectral radius of the linearized Navier–Stokes operator is given by the growth rate of the Tollmien–Schlichting instability, one can obtain exponential bounds on the semigroup and from there show nonlinear Sobolev instability of Prandtl expansions of shear flow type; see [Grenier and Nguyen 2017; 2024].

Nevertheless, in the setting of concave boundary layer flows, the class of data w_0 for which one can hope to have uniform (in ν) local (in time) control of w is larger than analytic: namely, one may expect control for data whose Fourier spectrum in x decays like $O(e^{-k^{2/3}})$. This corresponds to the so-called Gevrey class of exponent $\frac{3}{2}$.

To show such optimal stability result for general "concave" Prandtl expansions is the main goal of the present paper. It goes much beyond our result [Gérard-Varet et al. 2018], limited to the case when the boundary layer is of shear type (1-4). See also the recent development [Chen et al. 2022], still on shear flow expansions. Precise statements will be given in Section 2. Three preliminary remarks are in order:

- The approach in [Gérard-Varet et al. 2018] was very much based on the Fourier transform in x, made easy because (1-4) is independent of x. It does not adapt to general Prandtl expansions. The approach in the present paper relies on very different ideas.

- The main step in our approach is the derivation of stability estimates for the linearized equations

$$\begin{aligned} \partial_t w - \nu \Delta w + \nabla q + v \cdot \nabla w + w \cdot \nabla v = f, \quad t > 0, \quad x \in \mathbb{T}, \quad y > 0, \\ \nabla \cdot w = 0, \quad t \geq 0, \quad x \in \mathbb{T}, \quad y > 0, \\ w|_{y=0} = 0, \quad w|_{t=0} = w_0. \end{aligned} \tag{1-6}$$

But to derive such bounds, we do not make any assumption on the spectral radius of the linearized operator, in contrast with the works [Grenier and Nguyen 2017; 2024].

- A strong point of our analysis is that it applies to boundary layer profiles V_1^{bl} that are concave in Y but not necessarily strictly concave. See Section 2 for detailed hypotheses. This is important for applications, as can be seen from (1-5): there, $\partial_Y^2 V_1^{\mathrm{bl}}$ vanishes at the boundary for $Y = 0$ at positive times. Despite such possible degeneracies, we are able to reach Gevrey $\frac{3}{2}$ stability: this was not the case in our previous paper [Gérard-Varet et al. 2018], where our Gevrey exponent for stability was less than $\frac{3}{2}$ for nonstrictly concave flows.

The outline of the paper is as follows. Section 2 contains our main assumptions and stability results. We notably explain how our assumptions are adapted to generic boundary layer expansions. Section 3 gives an overview of our proof. The key point is the analysis of system (1-6), expressed in vorticity form. While this form allows to get rid of the pressure term, we face the difficulty that the vorticity $\omega = \operatorname{curl} v$ satisfies an intricate nonlocal condition, which forbids good direct stability estimates. To overcome this issue, we construct (and estimate) ω through an iteration scheme, where each step of the iteration can be split in two:

- In a first substep, we solve the linearized equation but with an artificial Neumann boundary condition on ω. This change in boundary condition allows to obtain stability estimates through the use of weighted norms inspired by the analysis of the hydrostatic Euler equations and subsequent works [Brenier 1999; Grenier 2000b]. This is where concavity is involved. These estimates, which would be wrong with the Dirichlet conditions for the velocity, remain valid under this modified boundary condition.

- In the next substep, we solve the linearized equation with zero source term or initial data but with a inhomogeneous Dirichlet condition on the velocity, correcting the error of the previous substep. This time, as the forcing is only through the boundary, the corresponding solution is more localized and of a parabolic nature. This allows for stabilizing effects.

More elements of the strategy are provided in Section 3. Afterwards, Section 4 details the estimates useful for the first substep of the iteration scheme, and Section 5 details the construction of the boundary corrector of the second substep. Eventually, Sections 6 and 7 provide the final linear and nonlinear estimates respectively.

2. Statements of the results

To state our stability result, we first introduce our functional framework. Let $p \in [1, \infty]$, $K \geq 1$, and $\nu \in (0, 1]$. For simplicity we assume $\nu^{-1/2} \in \mathbb{N}$, but it is not at all essential to our argument. We set

$$\|f\|_{G^p_{3/2}} = \sum_{j=0}^{\nu^{-1/2}} \frac{1}{(j!)^{3/2}} \sup_{j_2=0,\dots,j} \|e^{-Kt(j+1)}\beta_{j_2}\partial_x^{j-j_2} f\|_{L^p_t(0,1/K;L^2_{x,y})}, \tag{2-1}$$

where

$$\beta_{j_2} = \chi^{j_2}\partial_y^{j_2}, \quad \chi(y) = 1 - e^{-\kappa y}. \tag{2-2}$$

Here $\kappa \in (0, 1]$ is a fixed number, which will be taken small enough. We note that $\|f\|_{G^p_{3/2}}$ depends on $\nu, \kappa \in (0, 1]$ and $K \geq 1$, though we drop this dependence to simplify the notation. Note that for each fixed ν the norm $\|f\|_{G^p_{3/2}}$ is of Sobolev-type, but if $\|f\|_{G^p_{3/2}}$ is uniformly bounded in ν, it implies a usual Gevrey $\frac{3}{2}$ regularity for the C^∞ function f. The reason we can restrict to $j \leq \nu^{-1/2}$ in the sum above is that, in (1-3), the stretching term $\nabla v = O(\nu^{-1/2})$ creates at most an amplification $O(e^{C\nu^{-1/2}t})$. For $j \sim \nu^{-1/2}$, it is therefore balanced by the factor $e^{-Kt(j+1)}$ for large enough K. This means that we will be able to close an estimate considering only derivatives up to order $\nu^{-1/2}$.

Our main theorem is the following. Let us set $H^1_{0,\sigma}(\mathbb{T}\times\mathbb{R}_+) = \{f \in H^1_0(\mathbb{T}\times\mathbb{R}_+)^2 \mid \operatorname{div} f = 0 \text{ in } \mathbb{T}\times\mathbb{R}_+\}$, the space of all H^1 solenoidal vector fields satisfying the no-slip boundary condition at $Y = 0$.

Theorem 2.1 (nonlinear stability of concave Prandtl expansions). *Let $v = v(t, x, y)$ be a divergence-free vector field that fulfills the regularity and concavity conditions gathered in the Assumptions below but is not necessarily of type (1-2). There exists $\kappa_0 > 0$ such that the following statement holds for any $\kappa \in (0, \kappa_0]$: there exist $C > 0$, $K > 0$, $\delta_0 > 0$ such that, for all $\nu \leq K^{-2}$, if $r \in L^2(0, 1/K; L^2(\mathbb{T}\times\mathbb{R}_+)^2)$ and $w_0 \in H^1_{0,\sigma}(\mathbb{T}\times\mathbb{R}_+)$ satisfy*

$$[|w_0|]_{G_{3/2}} + [|\operatorname{rot} w_0|]_{G_{3/2}} \leq \delta_0\nu^{\frac{9}{4}}, \quad \|r\|_{G^2_{3/2}} \leq \delta_0\nu^{\frac{11}{4}}, \tag{2-3}$$

then the system (1-3) *has a unique solution* $w \in C([0, 1/K], H^1_{0,\sigma}(\mathbb{T}\times\mathbb{R}_+))$ *satisfying*

$$\|w\|_{G^\infty_{3/2}} + \nu^{\frac{1}{2}}\|\mathrm{rot}\, w\|_{G^\infty_{3/2}} \le C\nu^{-\frac{1}{2}}([|w_0|]_{G_{3/2}} + [|\mathrm{rot}\, w_0|]_{G_{3/2}} + \nu^{-\frac{1}{2}}\|r\|_{G^2_{3/2}}). \tag{2-4}$$

Here $\mathrm{rot}\, w = \partial_x w_2 - \partial_y w_1$ *and*

$$[|w_0|]_{G_{3/2}} = \sum_{j=0}^{\nu^{-1/2}} \frac{1}{(j!)^{3/2}} \sup_{j_2=0,\dots,j} \|\beta_{j_2}\partial_x^{j-j_2} w_0\|_{L^2_{x,y}}.$$

To complete the statement of our theorem, it remains to describe the set of assumptions on v that yield Theorem 2.1. Of course, these assumptions are designed to be satisfied by Prandtl expansions of type (1-2), when V_1^{bl} has some mild concavity. Due to the boundary layer variable Y, it is more convenient to work with rescaled variables $(\tau, X, Y) := \nu^{-1/2}(t, x, y)$. Accordingly, we shall express our assumptions directly on

$$V(\tau, X, Y) := v(t, x, y), \quad \tau > 0, \quad X \in \mathbb{T}_\nu, \quad Y > 0.$$

Here, $\mathbb{T}_\nu := \nu^{-1/2}\mathbb{T}$. We set

$$\Omega = \partial_X V_2 - \partial_Y V_1, \tag{2-5}$$

which describes the vorticity field of the approximation in the rescaled variables. We also set

$$\chi_\nu = \chi(\nu^{\frac{1}{2}} Y) = 1 - e^{-\kappa\nu^{1/2} Y}. \tag{2-6}$$

Note that $\kappa \in (0, 1]$ is fixed but taken small enough. Also, in the rescaled variables, our almost Gevrey norm $\|\cdot\|_{G^p_{3/2}}$ becomes

$$|||F|||_p = \sum_{j=0}^{\nu^{-1/2}} \frac{1}{(j!)^{3/2}\nu^{j/2}} \sup_{j_2=0,\dots,j} \|e^{-K\tau\nu^{1/2}(j+1)} B_{j_2}\partial_X^{j-j_2} F\|_{L^p_\tau(0,1/(K\nu^{1/2}); L^2_{X,Y})}, \qquad B_{j_2} = \chi_\nu^{j_2}\partial_Y^{j_2}. \tag{2-7}$$

We state our key assumptions in terms of V and Ω.

Assumptions. (i) *Divergence-free and Dirichlet condition on* V:

$$\partial_X V_1 + \partial_Y V_2 = 0, \qquad V|_{Y=0} = 0. \tag{2-8}$$

Moreover, there exist constants $C_* \ge 1$ *and* $C_0^*, C_1^*, C_2^* > 0$ *such that the following statements hold for any* $\nu \in (0, 1]$ *and* $K \ge 1$:

(ii) *Almost Gevrey* L^∞ *bounds for* V *and* $\nabla\Omega$: *For any* $\kappa \in (0, 1]$, *we have*

$$\begin{aligned}\sum_{j=0}^{\nu^{-1/2}} \frac{1}{(j!)^{3/2}\nu^{j/2}} \sup_{j_2=0,\dots,j} \Bigg(&\|e^{-K\tau\nu^{1/2} j} B_{j_2}\partial_X^{j-j_2} V_1\|_{L^\infty_{\tau,X,Y}} + \kappa \left\| e^{-K\tau\nu^{1/2} j} \frac{\partial_X^j V_2}{\chi_\nu} \right\|_{L^\infty_{\tau,X,Y}} \\ &+ \nu^{-\frac{1}{2}}(j+1)^{\frac{1}{2}} \|e^{-K\tau\nu^{1/2} j} B_{j_2}\partial_X^{j-j_2}\partial_X V_1\|_{L^\infty_{\tau,X,Y}} + (j+1)^{\frac{1}{2}}\|e^{-K\tau\nu^{1/2} j} B_{j_2}\partial_X^{j-j_2}\partial_Y V_1\|_{L^\infty_{\tau,X,Y}} \\ &+ \nu^{-\frac{1}{2}} \left\| \frac{1+Y}{1+\nu^{1/2}Y} e^{-K\tau\nu^{1/2} j} B_{j_2}\partial_X^{j-j_2}\partial_X\Omega \right\|_{L^\infty_{\tau,X,Y}} + \left\| \left(\frac{1+Y}{1+\nu^{1/2}Y}\right)^2 e^{-K\tau\nu^{1/2} j} B_{j_2}\partial_X^{j-j_2}\partial_Y\Omega \right\|_{L^\infty_{\tau,X,Y}} \Bigg) \le C_0^*.\end{aligned}$$

Here $L^\infty_{\tau,X,Y} = L^\infty_\tau(0, 1/(K\nu^{1/2}); L^\infty_{X,Y})$.

(iii) *Derivative bounds for V and Ω*: *We have*

$$\|V\|_{L^\infty_{\tau,X,Y}} + \nu^{-\frac{1}{2}}\|\partial_X V\|_{L^\infty_{\tau,X,Y}} + \left\|\frac{1+Y}{1+\nu^{1/2}Y}\partial_Y V_1\right\|_{L^\infty_{\tau,X,Y}} + \nu^{-\frac{1}{2}}\left\|\frac{1+Y}{1+\nu^{1/2}Y}\partial_X\Omega\right\|_{L^\infty_{\tau,X,Y}}$$
$$+\left\|\left(\frac{1+Y}{1+\nu^{1/2}Y}\right)^2\partial_Y\Omega\right\|_{L^\infty_{\tau,X,Y}} + \nu^{-\frac{1}{2}}\left\|\left(\frac{Y}{1+\nu^{1/2}Y}\right)^2\partial_\tau\partial_Y\Omega\right\|_{L^\infty_{\tau,X,Y}}$$
$$+\nu^{-\frac{1}{2}}\left\|\frac{Y(1+Y)}{(1+\nu^{1/2}Y)^2}\partial^2_{XY}\Omega\right\|_{L^\infty_{\tau,X,Y}} + \left\|\frac{Y(1+Y)^2}{(1+\nu^{1/2}Y)^3}\partial^2_Y\Omega\right\|_{L^\infty_{\tau,X,Y}} \le C_1^*. \tag{2-9}$$

(iv) *Monotonicity of Ω*: *Set $\rho(Y) = C_*((1+Y/\nu^{1/4})^{-2} + \nu^{1/2}(1+Y)^{-2} + \nu)$. Then we have*

$$\partial_Y\Omega + \rho \ge 0 \tag{2-10}$$

and

$$\nu^{-\frac{1}{2}}\left\|\frac{Y}{1+\nu^{1/2}Y}\frac{\partial^2_{XY}\Omega}{\sqrt{\partial_Y\Omega+2\rho}}\right\|_{L^\infty_{\tau,X,Y}} + \left\|\frac{Y(1+Y)}{(1+\nu^{1/2}Y)^2}\frac{\partial^2_Y\Omega}{\sqrt{\partial_Y\Omega+2\rho}}\right\|_{L^\infty_{\tau,X,Y}} \le C_2^*. \tag{2-11}$$

Remark 2.2 (link between the Prandtl expansions and the Assumptions). Let us explain how the set of assumptions above relates to the Prandtl expansions as given in (1-2).

(i) The divergence-free and Dirichlet conditions are satisfied by Prandtl expansions of type (1-2). Fields $U^{E,i}$ solve Euler or linearized Euler equations, while fields $V^{\mathrm{bl},i}$ solve Prandtl or linearized Prandtl equations: in both cases, they are divergence-free. Moreover, they are constructed alternatively in order to satisfy the Dirichlet boundary condition: once $U^{E,i}$ is constructed, $V^{\mathrm{bl},i}$ is constructed so that

$$U_1^{E,i}|_{y=0} + V_1^{\mathrm{bl},i}|_{Y=0} = 0.$$

Then, $U^{E,i+1}$ is constructed by solving an Euler-type equation with the nonpenetration condition

$$U_2^{E,i+1}|_{y=0} + V_2^{\mathrm{bl},i}|_{Y=0} = 0.$$

More precisely, one can construct $(U^{E,i}, V^{\mathrm{bl},i})$ in this way for $i \le N-1$ and conclude with

$$U^{E,N}(t,x,y) := (0, -V_2^{\mathrm{bl},N-1}(t,x,0)), \quad V^{\mathrm{bl},N} := 0.$$

(ii) Assumption (ii) amounts essentially to a Gevrey $\frac{3}{2}$ bound on solutions $U^{E,i}$ and $V^{\mathrm{bl},i}$ of Euler-like and Prandtl-like equations, respectively. Such solutions exist locally in time. For the Euler equations, we refer to [Kukavica and Vicol 2011]. For the Prandl equations, as mentioned before, the works [Kukavica and Vicol 2013; Sammartino and Caflisch 1998a] provide local-in-time solutions for analytic data. These local solutions being analytic, they belong to the Gevrey class $\frac{3}{2}$. More recently, Gevrey local-in-time well-posedness of the Prandtl equation has been established in [Dietert and Gérard-Varet 2019] (see [Gerard-Varet and Masmoudi 2015; Li and Yang 2020] for preliminary partial results). Also, if v is given by (1-2), as $V_2(\tau, X, Y) = v_2(t,x,y)$ is zero at the boundary $Y=0$, we can write

$$V_2 = \int_0^Y \partial_Y V_2 \approx \int_0^Y (\nu^{\frac{1}{2}}(\partial_y V_2^{E,0} + \partial_Y V_2^{\mathrm{bl},0}) + \cdots) = O(\nu^{\frac{1}{2}}Y) = O\Big(\frac{1}{\kappa}\chi_\nu(Y)\Big) \quad \text{at } Y=0,$$

so that $(1/\kappa)(V_2/\chi_\nu)$ is under control as required in (ii).

(iii) Again, Assumption (iii) is satisfied by classical Prandtl expansions of type (1-2). To check that, one has to keep in mind that $\partial_\tau \sim \nu^{1/2}\partial_t$, $\partial_X \sim \nu^{1/2}\partial_x$, so that for Prandtl expansions, which depend smoothly on t and x, any τ- or X-derivative allows to gain $\nu^{1/2}$. This explains for instance the factor $\nu^{-1/2}$ in front of the second and fourth terms of (2-9), related to $\partial_X V$ and $\partial_X \Omega$. In the same spirit, as $\partial_Y \sim \nu^{1/2}\partial_y$, for the Euler part of the Prandtl expansion (which depends smoothly on y), any Y-derivative allows to gain $\nu^{1/2}$. This remark does not apply to the boundary layer part of the expansion, as it depends genuinely on Y. Still, this part has good decay in Y (typically like e^{-Y} or $(1+Y)^{-N}$ for large Y). This is coherent with the weights $(1+Y)/(1+\nu^{1/2}Y)$ or $Y/(1+\nu^{1/2}Y)$ that can be found in (2-9) in front of terms with Y derivatives: outside the boundary layer ($Y \gg 1$), it yields a gain of $\nu^{1/2}$, but in the boundary layer ($Y \sim 1$), it yields some decay information on the boundary layer terms.

(iv) In the case when v is given by Prandtl expansions of type (1-2),

$$\partial_Y \Omega = \partial^2_{XY} V_2 - \partial^2_Y V_1 = -\partial^2_Y V_1^{\mathrm{bl}} + O(\nu) + O(\sqrt{\nu}(1+Y)^{-2})$$

Here, the $O(\nu)$ comes from the Euler part of the Prandtl expansion. The $O(\sqrt{\nu}(1+Y)^{-2})$ corresponds to the boundary layer profiles $V^{\mathrm{bl},i}$, $i \geq 1$. The last two terms in the definition of the weight ρ allow to control them for C_* large enough. Hence, condition (2-10) is essentially a (nonstrict) concavity condition on the leading term of the Prandtl boundary layer, $V^{\mathrm{bl}} := V^{\mathrm{bl},0}$. Moreover, by the addition of the sublayer term $(1+(Y/\nu^{1/4}))^{-2}$ in the definition of ρ, we allow any sign for $\partial^2_Y V^P_{0,1}$ in the sublayer $0 \leq Y \leq O(\nu^{1/4})$, and the concavity is only needed for $Y \geq O(\nu^{1/4})$. In the original variables this sublayer is of the order $O(\nu^{3/4})$, which is typical order of Kolmogorov dissipation length in the theory of turbulence.

As regards (2-11), we notice that for Prandtl expansions:

$$\partial^2_{XY}\Omega = -\partial_X \partial^2_Y V_1^{\mathrm{bl}} + O(\nu^{\frac{3}{2}}) + O(\nu(1+Y)^{-2}) \quad \text{and} \quad \partial^2_Y \Omega = -\partial^3_Y V_1^{\mathrm{bl}} + +O(\nu^{\frac{3}{2}}) + O(\nu^{\frac{1}{2}}(1+Y)^{-2}).$$

Hence, by taking into account the bound $1/\sqrt{\partial_Y \Omega + 2\rho} \leq 1/(C_* \nu^{1/2})$, the condition (2-11) is essentially verified if V_1^{bl} satisfies

$$\nu^{-\frac{1}{2}} \left\| \frac{Y \partial_X \partial^2_Y V_1^{\mathrm{bl}}}{\sqrt{-\partial^2_Y V_1^{\mathrm{bl}} + 2C_*(1+Y/\nu^{1/4})^{-2}}} \right\|_{L^\infty_{\tau,X,Y}} + \left\| \frac{Y(1+Y)\partial^3_Y V_1^{\mathrm{bl}}}{\sqrt{-\partial^2_Y V_1^{\mathrm{bl}} + 2C_*(1+Y/\nu^{1/4})^{-2}}} \right\|_{L^\infty_{\tau,X,Y}} \leq C < \infty.$$

In the next section, we will explain the general strategy for the proof of our main stability theorem. More precisely, we will briefly describe our stability analysis of the linearized equation (1-6) for f a given force. This is the core of our paper: the transition from linear to nonlinear stability is more standard. As explained before, we shall work with the rescaled variables (τ, X, Y). We set

$$W(\tau, X, Y) := w(t, x, y), \qquad F(\tau, X, Y) := \sqrt{\nu} f(t, x, y), \qquad W_0(X, Y) := w_0(x, y)$$

(and still $V(\tau, X, Y) = v(t, x, y)$). System (1-6) becomes

$$\begin{aligned} \partial_\tau W - \nu^{\frac{1}{2}} \Delta W + \nabla Q + V \cdot \nabla W + W \cdot \nabla V = F, \quad & \tau > 0, \quad X \in \mathbb{T}_\nu, \quad Y > 0, \\ \nabla \cdot W = 0, \quad & \tau \geq 0, \quad X \in \mathbb{T}_\nu, \quad Y > 0, \\ W|_{Y=0} = 0, \quad W|_{\tau=0} = W_0. & \end{aligned} \tag{2-12}$$

The main result on this linear system is:

Theorem 2.3. *Suppose that the Assumptions hold. Then there exists $\kappa_0 \in (0, 1]$ such that the following statement holds for any $\kappa \in (0, \kappa_0]$. There exists $K_0 = K_0(\kappa, C_*, C_j^*) \geq 1$ such that if $K \geq K_0$ then the system (2-12) admits a unique solution $W \in C([0, 1/(K\nu^{1/2})]; H^1_{0,\sigma}(\mathbb{T}_\nu \times \mathbb{R}_+))$ satisfying*

$$\|\|W\|\|_\infty + \|\|\mathrm{rot}\, W\|\|_\infty \leq C((\nu^{-\frac{1}{2}} + K^{\frac{1}{2}}\nu^{-\frac{1}{4}})[\|W_0\|] + \nu^{-1}[\|\mathrm{rot}\, W_0\|] + \nu^{-\frac{5}{4}}\|\|F\|\|_2). \tag{2-13}$$

Here $\mathrm{rot}\, W = \partial_X W_2 - \partial_Y W_1$ *and*

$$[\|W_0\|] = \sum_{j=0}^{\nu^{-1/2}} \frac{1}{(j!)^{3/2}\nu^{j/2}} \sup_{j_2=0,\ldots,j} \|B_{j_2}\partial_X^{j-j_2} W_0\|_{L^2_{X,Y}},$$

and C is a universal constant.

As a consequence, we have the following result in the original variables. Note that, from $F(\tau, X, Y) = \nu^{1/2} f(t, x, y)$, we have $\nu^{-5/4}\|\|F\|\|_2 = \nu^{-3/2}\|f\|_{G^2_{3/2}}$.

Theorem 2.4. *Suppose that the Assumptions hold. Then there exists $\kappa_0 \in (0, 1]$ such that the following statement holds for any $\kappa \in (0, \kappa_0]$. There exists $K_0 = K_0(\kappa, C_*, C_j^*) \geq 1$ such that if $K \geq K_0$ then the system (1-6) admits a unique solution $w \in C([0, 1/K]; H^1_{0,\sigma}(\mathbb{T} \times \mathbb{R}_+))$ satisfying*

$$\|w\|_{G^\infty_{3/2}} + \nu^{\frac{1}{2}}\|\mathrm{rot}\, w\|_{G^\infty_{3/2}} \leq C\nu^{-\frac{1}{2}}((1 + K^{\frac{1}{2}}\nu^{\frac{1}{4}})[|w_0|]_{G_{3/2}} + [|\mathrm{rot}\, w_0|]_{G_{3/2}} + \nu^{-\frac{1}{2}}\|f\|_{G^2_{3/2}}). \tag{2-14}$$

Here $\mathrm{rot}\, w = \partial_x w_2 - \partial_y w_1$ *and*

$$[|w_0|]_{G_{3/2}} = \sum_{j=0}^{\nu^{-1/2}} \frac{1}{(j!)^{3/2}} \sup_{j_2=0,\ldots,j} \|\beta_{j_2}\partial_x^{j-j_2} w_0\|_{L^2_{x,y}},$$

and C is a universal constant.

3. General strategy

Estimates on system (2-12) will be performed at the level of the vorticity field $\omega = \mathrm{rot}\, W := \partial_X W_2 - \partial_Y W_1$:

$$\begin{gathered}(\partial_\tau + V \cdot \nabla - \nu^{\frac{1}{2}}\Delta)\omega + W \cdot \nabla\Omega = \mathrm{rot}\, F, \quad \nabla \cdot W = 0, \quad \tau > 0, \quad X \in \mathbb{T}_\nu, \quad Y > 0, \\ W|_{Y=0} = 0.\end{gathered} \tag{3-1}$$

We recall that $\tau = \nu^{-1/2}t$: the point is to get estimates that are valid over time intervals of size $\nu^{-1/2}$, which is difficult due to the stretching term $W \cdot \nabla\Omega$. Classical estimates and Gronwall's lemma would only yield a control on time intervals $O(1)$. We have to use both our Gevrey functional framework and concavity condition.

Actually, several difficulties are already captured by the toy model

$$\begin{gathered}(\partial_\tau - \nu^{\frac{1}{2}}\Delta)\omega + W_2\partial_Y\Omega = 0, \quad \omega = \mathrm{rot}\, W, \quad \nabla \cdot W = 0, \quad \tau > 0, \quad X \in \mathbb{T}_\nu, \quad Y > 0, \\ W|_{Y=0} = 0,\end{gathered} \tag{3-2}$$

where $\Omega = \Omega(Y)$ (for simplicity, we assume no dependence on τ and X). We shall stick to this model for what follows.

In the case of the inviscid equation

$$\partial_\tau \omega + W_2 \partial_Y \Omega = 0, \quad \omega = \operatorname{rot} W, \quad \nabla \cdot W = 0, \quad W_2|_{Y=0} = 0$$

under the strict sign condition $\partial_Y \Omega \geq C > 0$, a trick that goes back to [Grenier 2000b] is to test the equation against $\omega/(\partial_Y \Omega)$. By the cancellation

$$\int W_2 \partial_Y \Omega \frac{\omega}{\partial_Y \Omega} = \int W_2 \operatorname{rot} W = -\frac{1}{2} \int \partial_X |W|^2 = 0,$$

one can obtain a uniform-in-time control on the weighted quantity $\|\omega/\sqrt{\partial_Y \Omega}\|_{L^2} \sim \|\omega\|_{L^2}$. However, back to the model (3-2), we are facing two difficulties:

(1) Inspired by the case of Prandtl layers, we must consider situations where $\partial_Y \Omega$ vanishes or even becomes slightly negative; see Assumption (iv).

(2) Even in the simpler case $\partial_Y \Omega \geq C > 0$, the weighted estimate above is not compatible with the introduction of viscosity and no-slip conditions.

We recall that these difficulties are not purely technical, as no uniform-in-ν stability estimate is expected below Gevrey $\frac{3}{2}$ regularity. To overcome these issues, we shall proceed in two steps.

3A. ***First step*****:** ***Gevrey estimates for artificial boundary conditions.*** The first step consists in deriving Gevrey bounds for the same equation, but with pure slip instead of no-slip conditions. For the real vorticity equation, this will be performed in Section 4. For our toy model, this means that we consider

$$\begin{gathered}(\partial_\tau - \nu^{\frac{1}{2}} \Delta)\omega + W_2 \partial_Y \Omega = 0, \quad \omega = \operatorname{rot} W, \quad \nabla \cdot W = 0, \quad \tau > 0, \quad X \in \mathbb{T}_\nu, \quad Y > 0, \\ W_2|_{Y=0} = \omega|_{Y=0} = 0.\end{gathered} \tag{3-3}$$

The main point in this change of boundary conditions is that difficulty (2) mentioned above disappears: the Dirichlet condition on ω goes well with integration by parts, and in the case $\partial_Y \Omega \geq C > 0$, one can achieve again some good control on $\|\omega/\sqrt{\partial_Y \Omega}\|_{L^2}$. Still, we have to explain how to obtain stability under the less stringent condition in Assumption (iv). Here, we need Gevrey regularity. Let us for simplicity forget about Y-derivatives, which are not important for the toy model, and set

$$\omega^j := e^{-K\tau\nu^{1/2}(j+1)} \partial_X^j \omega, \quad W^j := e^{-K\tau\nu^{1/2}(j+1)} \partial_X^j W.$$

The point is to obtain a bound on

$$\sum_{j \leq \nu^{-1/2}} \frac{1}{(j!)^{3/2} \nu^{j/2}} \|\omega^j\|_{L^2_{X,Y}}.$$

As $\Omega = \Omega(Y)$, the equation satisfied by ω^j is

$$(K\nu^{1/2}(j+1) + \partial_\tau - \nu^{\frac{1}{2}} \Delta)\omega^j + W_2^j \partial_Y \Omega = 0. \tag{3-4}$$

Roughly, the idea is to control a weighted Gevrey norm of the form

$$\sum_{j \leq \nu^{-1/2}} \frac{1}{(j!)^{3/2} \nu^{j/2}} \left\| \frac{\omega^j}{\sqrt{\partial_Y \Omega + 2\rho_j}} \right\|_{L^2_{X,Y}},$$

where ρ_j is added to compensate for possible degeneracies of $\partial_Y \Omega$. Testing (3-4) against $\omega_j/(\partial_Y \Omega + 2\rho_j)$, we find

$$K\nu^{\frac{1}{2}}(j+1)\left\|\frac{\omega^j}{\sqrt{\partial_Y\Omega+2\rho_j}}\right\|_{L^2}^2+\frac{1}{2}\frac{d}{d\tau}\left\|\frac{\omega^j}{\sqrt{\partial_Y\Omega+2\rho_j}}\right\|_{L^2}^2+\nu^{\frac{1}{2}}\left\|\frac{\nabla\omega^j}{\sqrt{\partial_Y\Omega+2\rho_j}}\right\|_{L^2}^2$$
$$=-\nu^{\frac{1}{2}}\int\nabla\frac{1}{\partial_Y\Omega+2\rho_j}\cdot\nabla\omega^j\omega^j-\int W_2^j\partial_Y\Omega\frac{\omega^j}{\partial_Y\Omega+2\rho_j}$$
$$=\nu^{\frac{1}{2}}\int\frac{\nabla\partial_Y\Omega}{(\partial_Y\Omega+2\rho_j)^2}\cdot\nabla\omega^j\omega^j+\nu^{\frac{1}{2}}\int\frac{\nabla\rho_j}{(\partial_Y\Omega+2\rho_j)^2}\cdot\nabla\omega^j\omega^j+\int W_2^j\frac{2\rho_j}{\partial_Y\Omega+2\rho_j}\omega^j, \tag{3-5}$$

where we used again the cancellation property $\int W_2^j\omega^j=0$. One must then choose ρ_j so that the three terms at the right are controlled by the left-hand side for K large enough. Roughly, this can be achieved by taking ρ_j in the form $\rho_j(Y)\approx\rho+(1+\lambda_jY)^{-2},\ \lambda_j:=(j+1)^{1/2}$. To give an idea of why it works, let us consider the first and last terms. As regards the first one, we write

$$\nu^{\frac{1}{2}}\int\frac{\nabla\partial_Y\Omega}{(\partial_Y\Omega+2\rho_j)^2}\cdot\nabla\omega^j\omega^j=\nu^{\frac{1}{2}}\int_{\{Y\geq 1/\lambda_j\}}\frac{1}{Y\sqrt{\partial_Y\Omega+2\rho_j}}\frac{Y\nabla\partial_Y\Omega}{\sqrt{\partial_Y\Omega+2\rho_j}}\cdot\frac{\nabla\omega^j}{\sqrt{\partial_Y\Omega+2\rho_j}}\frac{\omega^j}{\sqrt{\partial_Y\Omega+2\rho_j}}$$
$$+\nu^{\frac{1}{2}}O\left(\left\|\frac{\nabla\omega^j}{\sqrt{\partial_Y\Omega+2\rho_j}}\right\|_{L^2}\left\|\frac{\omega^j}{\sqrt{\partial_Y\Omega+2\rho_j}}\right\|_{L^2}\right).$$

The second term on the right side corresponds to the contribution of the region $Y\leq 1/\lambda_j$, for which the weight $\partial_Y\Omega+2\rho_j$ is bounded from below and raises no issue (we further assumed here that $\partial_Y\nabla\Omega$ for the sake of brevity). As regards the first term on the right side, for all $Y\geq 1/\lambda_j$, we use the bounds

$$\frac{1}{Y\sqrt{\partial_Y\Omega+2\rho_j}}\leq\frac{1}{Y\sqrt{2\rho_j}}\leq C\lambda_j\quad\text{and}\quad\frac{|Y\nabla\partial_Y\Omega|}{\sqrt{\partial_Y\Omega+2\rho_j}}\leq\frac{|Y\nabla\partial_Y\Omega|}{\sqrt{\partial_Y\Omega+2\rho}}\leq C,$$

where we used Assumption (iv). We end up with

$$\nu^{\frac{1}{2}}\int\frac{\nabla\partial_Y\Omega}{(\partial_Y\Omega+2\rho_j)^2}\cdot\nabla\omega^j\omega^j\leq C\nu^{\frac{1}{2}}\lambda_j\left\|\frac{\nabla\omega^j}{\sqrt{\partial_Y\Omega+2\rho_j}}\right\|_{L^2}\left\|\frac{\omega^j}{\sqrt{\partial_Y\Omega+2\rho_j}}\right\|_{L^2},$$

which is absorbed by the left-hand side under the constraint $\lambda_j\lesssim(j+1)^{1/2}$. As regards the third term on the right side of (3-5), we use the inequality

$$\frac{\rho_j}{\sqrt{\partial_Y\Omega+2\rho_j}}\leq\frac{\sqrt{\rho_j}}{\sqrt{2}}\leq C\left(\sqrt{\nu}+\frac{1}{\lambda_jY}\right)$$

to obtain

$$\int W_2^j\frac{2\rho_j}{\partial_Y\Omega+2\rho_j}\omega^j\leq C\sqrt{\nu}\|W_2^j\|_{L^2}\left\|\frac{\omega^j}{\sqrt{\partial_Y\Omega+2\rho_j}}\right\|_{L^2}+\frac{C}{\lambda_j}\left\|\frac{W_2^j}{Y}\right\|_{L^2}\left\|\frac{\omega^j}{\sqrt{\partial_Y\Omega+2\rho_j}}\right\|_{L^2}$$
$$\leq C\left(\sqrt{\nu}\|W_2^j\|_{L^2}+\frac{1}{\lambda_j}\|\partial_YW_2^j\|_{L^2}\right)\left\|\frac{\omega^j}{\sqrt{\partial_Y\Omega+2\rho_j}}\right\|_{L^2},$$

where the second line comes from Hardy's inequality. Using that $\|\partial_Y W_2^j\|_{L^2} = \|\partial_X W_1^j\|_{L^2} \approx \|W_1^{j+1}\|_{L^2}$, we have, for any sequence (a_j),

$$\sum_j \frac{1}{(j!)^{3/2}\nu^{j/2}} a_j \|\partial_Y W_2^j\|_{L^2} \approx \sum_j \frac{1}{(j!)^{3/2}\nu^{j/2}} \nu^{\frac{1}{2}}(j+1)^{\frac{3}{2}} a_{j-1}\|W_1^j\|_{L^2}.$$

In other words, at Gevrey $\frac{3}{2}$ regularity, $a_j\|\partial_Y W_2^j\|_{L^2}$ behaves like $\nu^{1/2}(j+1)^{3/2}a_{j-1}\|W_1^j\|_{L^2}$. Combining this with a control of $\|W^j\|_{L^2}$ by $\|\omega_j/\sqrt{\partial_Y\Omega+2\rho_j}\|_{L^2}$ and with a precise statement to be given in Section 4, the previous bound is in the same spirit as

$$\int W_2^j \frac{2\rho_j}{\partial_Y\Omega+2\rho_j}\omega^j \le C\frac{\nu^{1/2}(j+1)^{3/2}}{\lambda_{j-1}} \left\| \frac{\omega^j}{\sqrt{\partial_Y\Omega+2\rho_j}} \right\|_{L^2}^2,$$

which allows a control by the left-hand side of (3-5) as soon as $(j+1)^{1/2} \lesssim \lambda_j$. Hence the choice $\lambda_j = (j+1)^{1/2}$.

Of course, the elements above provide only glimpses of the approach carried out in the first step of our stability study. The full study of the vorticity equation with artificial boundary conditions is given in Section 4.

3B. ***Recovery of the right boundary conditions.*** We give again a few elements on the toy model (3-2). The analysis of the complete model is carried in Section 5. After the first step, one has a solution of system (3-3), with the same initial condition and same boundary condition $W_2|_{Y=0} = 0$ as in (3-2) but not the same boundary condition on the tangential velocity: $h := W_1|_{Y=0} \neq 0$. Note that by the first step and the trace theorem, one is able to get a Gevrey bound for h: as shown rigorously in the next sections, one may get an estimate of the form

$$|||h|||_{\mathrm{bc}} := \sum_{j\le\nu^{-1/2}} \frac{1}{(j!)^{3/2}\nu^{j/2}} \|h^j\|_{L^2((0,1/(K\nu^{1/2}));L^2_X)} \le \frac{C}{K^{1/4}}\left(\|W_0\|_{L^2} + C\sum_{j\le\nu^{-1/2}} \frac{1}{(j!)^{3/2}\nu^{j/2}}\|\omega_0^j\|_{L^2_{X,Y}}\right),$$

where W_0 and $\omega_0 := \operatorname{rot} W_0$ are the initial data for the velocity and vorticity, respectively.

Working in Gevrey regularity, the point is then to solve

$$\begin{gathered}(\partial_\tau - \nu^{\frac{1}{2}}\Delta)\omega + W_2\partial_Y\Omega = 0, \quad \omega = \operatorname{rot} W, \quad \nabla\cdot W = 0, \quad \tau > 0, \quad X \in \mathbb{T}_\nu, \quad Y > 0, \\ W_2|_{Y=0} = 0, \quad W_1|_{Y=0} = h, \quad W|_{t=0} = 0.\end{gathered} \tag{3-6}$$

The main idea is to use the following scheme:

Step (a): We solve the approximate Stokes equation

$$\begin{gathered}(\partial_\tau - \nu^{\frac{1}{2}}\Delta)\omega = 0, \quad \omega = \operatorname{rot} W, \quad \nabla\cdot W = 0, \\ W_2|_{Y=0} = 0, \quad W_1|_{Y=0} = h, \quad W|_{t=0} = 0\end{gathered} \tag{3-7}$$

and obtain in this way a solution $W_a = (W_{a,1}, W_{a,2}) = W_a[h]$.

Step (b): We correct the stretching term created by the previous approximation by considering the full equation with artificial boundary condition:

$$\begin{gathered}(\partial_\tau - \nu^{\frac{1}{2}}\Delta)\omega + W_2\partial_Y\Omega = -W_{a,2}\partial_Y\Omega, \quad \omega = \operatorname{rot} W, \quad \nabla\cdot W = 0,\\ W_2|_{Y=0} = 0, \quad \omega|_{Y=0} = 0, \quad W|_{t=0} = 0.\end{gathered} \tag{3-8}$$

We denote by $W_b = W_b[h]$ the solution of such a system. It can be seen as a functional of h through W_a.

Step (c): At the end of the Steps (a) and (b), the function $W - W_a - W_b$ solves formally the same system as W, replacing h by $R_{\mathrm{bc}}[h] := -W_{b,1}[h]|_{Y=0}$. The point is to show that, for K large enough,

$$|||R_{\mathrm{bc}}[h]|||_{\mathrm{bc}} \le \tfrac{1}{2}|||h|||_{\mathrm{bc}}, \tag{3-9}$$

which allows us to solve (3-6) by iteration.

Obviously, to establish (3-9), one must have careful Gevrey stability estimates for systems (3-7) and (3-8). The estimates for (3-8) follow from the same ideas as those described in Section 3A to treat (3-3) (the initial condition is just replaced by a source term). As regards (3-7), the initial data being zero, one can take the Laplace transform in τ and the Fourier transform in X and solve explicitly the resulting ordinary differential equation in Y. It leads to sharp L^2 estimates on W and its derivatives on the Fourier–Laplace side, which transfer to L^2 estimates in the physical space by the Plancherel theorem.

All the analysis in the framework of the vorticity equation is provided in Section 5. In this setting, the iteration scheme mentioned above has to be modified, because the advection term creates extra difficulties. Namely, one has to add an intermediate step between Steps (a) and (b) above; see Section 5 for details.

Of course, we have indicated here key ideas for the stability analysis of the linearized system (1-6). One has then to go from these estimates to the nonlinear Theorem 2.1. This will be achieved in Section 7. Finally we introduce the simplified notation

$$\|f\| = \|f\|_{L^2_{X,Y}}, \quad \langle f, g\rangle = \langle f, g\rangle_{L^2_{X,Y}}$$

for convenience.

4. Vorticity estimate under artificial boundary condition

In accordance with the strategy described in the previous section, we consider here the solution to the system

$$\begin{gathered}-\nu^{\frac{1}{2}}\Delta\omega + \partial_\tau\omega + V\cdot\nabla\omega + W\cdot\nabla\Omega = \operatorname{rot} F + G, \quad \omega = \operatorname{rot} W, \quad \nabla\cdot W = 0,\\ \tau > 0, \quad X \in \mathbb{T}_\nu, \quad Y > 0,\\ W_2|_{Y=0} = \omega|_{Y=0} = 0, \quad W|_{\tau=0} = W_0.\end{gathered} \tag{4-1}$$

Here a given force term $G \in L^2(0, 1/(K\nu^{1/2}); L^2\cap\dot{H}^{-1})$, where $\dot{H}^{-1}$ is the dual space of the homogeneous Sobolev space $\dot{H}^1_0(\mathbb{T}_\nu\times\mathbb{R}_+)$ (the subscript 0 means the zero boundary trace), is also introduced for later use. As usual, the velocity W is given in terms of the stream function ϕ, i.e.,

$$W = \nabla^{\perp}\phi = \begin{pmatrix} \partial_Y \phi \\ -\partial_X \phi \end{pmatrix}, \tag{4-2}$$

and $\phi \in \dot{H}^1_0(\mathbb{T} \times \mathbb{R}_+)$ is the unique solution to the Poisson equation $-\Delta\phi = \omega$ with the zero Dirichlet boundary condition $\phi|_{Y=0} = 0$. This formulation is well defined, and the unique solvability of (4-1) in the class $\omega \in C([0, 1/(K\nu^{1/2})]; L^2 \cap \dot{H}^{-1}) \cap L^2(0, 1/(K\nu^{1/2}); H^1_0)$ is shown without difficulty (under the regularity condition we impose on V, Ω, and the forces). The reason why the regularity $\omega \in C([0, 1/(K\nu^{1/2})]; \dot{H}^{-1})$ is preserved is that the term $-V \cdot \nabla\omega - W \cdot \nabla\Omega + \operatorname{rot} F + G$ has a bound in $\dot{H}^{-1}$ (in space) such as

$$\|-V \cdot \nabla\omega - W \cdot \nabla\Omega + \operatorname{rot} F + G\|_{L^2\dot{H}^{-1}} \le \|V\|_{L^\infty}\|\omega\|_{L^2L^2} + \|\Omega\|_{L^\infty}\|W\|_{L^2L^2} + \|F\|_{L^2L^2} + \|G\|_{L^2\dot{H}^{-1}}$$

and also $\|\operatorname{rot} W_0\|_{\dot{H}^{-1}} \le \|W_0\|_{L^2}$ for the initial vorticity. Hence the space $C([0, 1/(K\nu^{1/2})]; \dot{H}^{-1})$ for the vorticity field and the regularity $\phi(\tau, \cdot) \in \dot{H}^1_0(\mathbb{T} \times \mathbb{R}_+)$ for the stream function are compatible in our setting. By the parabolic regularity of the system, the ν-dependent estimates for the higher-order derivatives are easily obtained, and thus, our main interest here is the *uniform* estimate in time and ν. To this end, for $\boldsymbol{j} = (j_1, j_2)$ with $j_1 + j_2 = j$, we set

$$\omega^{\boldsymbol{j}} = e^{-K\tau\nu^{1/2}(j+1)} B_{j_2}\partial_X^{j_1}\omega, \quad (\nabla\phi)^{\boldsymbol{j}} = e^{-K\tau\nu^{1/2}(j+1)} B_{j_2}\partial_X^{j_1}\nabla\phi, \tag{4-3}$$

and similarly, $(\Delta\omega)^{\boldsymbol{j}} = e^{-K\tau\nu^{1/2}(j+1)} B_{j_2}\partial_X^{j_1}\Delta\omega$. We also set

$$V^{\boldsymbol{j}} = e^{-K\tau\nu^{1/2} j} B_{j_2}\partial_X^{j_1} V, \quad (\nabla\Omega)^{\boldsymbol{j}} = e^{-K\tau\nu^{1/2} j} B_{j_2}\partial_X^{j_1}\nabla\Omega. \tag{4-4}$$

From the first equation of (4-1), we observe that $\omega^{\boldsymbol{j}}$ satisfies, by setting $\boldsymbol{l} = (l - l_2, l_2)$,

$$\begin{aligned}
-\nu^{\frac12}(\Delta\omega)^{\boldsymbol{j}} &+ (\partial_\tau + K\nu^{\frac12}(j+1) + V\cdot\nabla)\omega^{\boldsymbol{j}} + (\nabla^\perp\phi)^{\boldsymbol{j}} \cdot \nabla\Omega \\
&= -V_2[B_{j_2}, \partial_Y] e^{-K\tau\nu^{1/2}(j+1)}\partial_X^{j_1}\omega \\
&\quad - \sum_{l=0}^{j-1} \sum_{\max\{0, l+j_2-j\} \le l_2 \le \min\{l, j_2\}} \binom{j_2}{l_2}\binom{j-j_2}{l-l_2} V^{\boldsymbol{j}-\boldsymbol{l}} \cdot (\nabla\omega)^{\boldsymbol{l}} \\
&\quad - \sum_{l=0}^{j-1} \sum_{\max\{0, l+j_2-j\} \le l_2 \le \min\{l, j_2\}} \binom{j_2}{l_2}\binom{j-j_2}{l-l_2} (\nabla^\perp\phi)^{\boldsymbol{l}} \cdot (\nabla\Omega)^{\boldsymbol{j}-\boldsymbol{l}} \\
&\quad + \operatorname{rot} F^{\boldsymbol{j}} - [B_{j_2}, \partial_Y]\partial_X^{j_1} e^{-K\tau\nu^{1/2}(j+1)} F_1 + G^{\boldsymbol{j}}.
\end{aligned} \tag{4-5}$$

Here the sum $\sum_{l=0}^{j-1}$ is defined to be 0 for $j = 0$, and the definitions of $F^{\boldsymbol{j}}$ and $G^{\boldsymbol{j}}$ are straightforward.

To simplify notations let us introduce weighted seminorms; for a given nonnegative smooth function $\xi_j = \xi_j(\tau, X, Y)$, we set

$$M_{p,j,\xi_j}[\omega] = \sup_{j_2 = 0,\dots,j} \|\xi_j \omega^{(j-j_2, j_2)}\|_{L^p_\tau(0,1/(K\nu^{1/2}); L^2_{X,Y})} \tag{4-6}$$

and also set, with the definition $\boldsymbol{\xi} = (\xi_j)_{j=0}^{\infty}$,

$$|||F|||'_{p,\boldsymbol{\xi}} = \sum_{j=0}^{\nu^{-1/2}} \frac{\nu^{1/(2p)}(j+1)^{1/p}}{(j!)^{3/2}\nu^{j/2}} M_{p,j,\xi_j}[F]. \tag{4-7}$$

Note that

$$|||F|||'_{\infty,\mathbf{1}} = |||F|||_{\infty}, \quad \mathbf{1} = (1, 1, \dots). \tag{4-8}$$

The choice of ξ_j is essential in the stability estimate for ω^j. We will take

$$\xi_j = \frac{1}{\sqrt{\partial_Y \Omega + 2\rho_j}}, \tag{4-9}$$

where

$$\rho_j = K^{\frac{1}{4}} C_* (1 + (j+1)^{\frac{1}{2}} Y)^{-2} + C_* \left(\left(1 + \frac{Y}{\nu^{1/4}}\right)^{-2} + \nu^{\frac{1}{2}} (1+Y)^{-2} + \nu \right). \tag{4-10}$$

See Section 3 for more on the origin of this weight. We also introduce the norm of the boundary trace as

$$|||\partial_Y \phi|_{Y=0}|||_{\mathrm{bc}} = \sum_{j=0}^{\nu^{-1/2}} \frac{\nu^{1/4}(j+1)^{1/2}}{(j!)^{3/2}\nu^{j/2}} \|e^{-K\tau\nu^{1/2}(j+1)} \partial_X^j \partial_Y \phi|_{Y=0}\|_{L^2(0,1/(K\nu^{1/2}); L^2_X)}. \tag{4-11}$$

The main result of this section is:

Proposition 4.1. *There exists $\kappa_1 \in (0, 1]$ such that the following statement holds for any $\kappa \in (0, \kappa_1]$. There exists $K_1 = K_1(\kappa, C_*, C_j^*) \geq 1$ such that if $K \geq K_1$ then the system (4-1) admits a unique solution $\omega \in C([0, 1/(K\nu^{1/2})]; L^2 \cap \dot{H}^{-1}) \cap L^2(0, 1/(K\nu^{1/2}); H_0^1)$ satisfying*

$$\begin{aligned} &|||\omega|||'_{\infty,\boldsymbol{\xi}} + K^{\frac{1}{2}} |||\omega|||'_{2,\boldsymbol{\xi}} + K^{\frac{1}{4}} |||\nabla\phi|||'_{2,\mathbf{1}} + K^{\frac{1}{4}} |||\partial_Y \phi|_{Y=0}|||_{\mathrm{bc}} \\ &\quad \leq C \biggl(\|W_0\|_{L^2_{X,Y}} + \nu^{-}[\|\mathrm{rot}\, W_0\|] + (C_2^* + 1)\nu^{-\frac{1}{2}} |||F|||'_{2,\tilde{\xi}^{(1)}} \\ &\qquad\qquad + \frac{1}{K^{1/2}\nu^{1/2}} |||G|||'_{2,\tilde{\xi}^{(2)}} + \frac{1}{K^{1/2}\nu^{1/4}} \|G\|_{L^2(0,1/(K\nu^{1/2}); \dot{H}^{-1})} \biggr). \end{aligned} \tag{4-12}$$

Here $C > 0$ is a universal constant, while the weight $\tilde{\xi}^{(k)}$ is defined as

$$\tilde{\xi}^{(k)} = \left(\frac{\xi_j}{(j+1)^{k/2}} \right)_{j=0}^{\infty}.$$

Remark 4.2. (1) From the bound $1/\xi_j \leq (C_1^* + 8K^{1/4}C_*)^{1/2}$ in (4-18) below, we have

$$K^{\frac{3}{16}} |||\omega|||'_{2,\mathbf{1}} \leq K^{\frac{3}{16}} (C_1^* + 8K^{\frac{1}{4}} C_*)^{\frac{1}{2}} |||\omega|||'_{2,\boldsymbol{\xi}} \leq K^{\frac{1}{2}} |||\omega|||'_{2,\boldsymbol{\xi}} \tag{4-13}$$

if K is large enough further depending only on C_1^* and C_*. Estimates (4-13) and (4-12) gives the estimate of $K^{3/16} |||\omega|||'_{2,\mathbf{1}}$.

(2) By the definition of (4-7), we have

$$\nu^{-\frac{1}{2}}|||F|||'_{2,\tilde{\xi}^{(1)}} = \nu^{-\frac{1}{4}}\sum_{j=0}^{\nu^{-1/2}}\frac{M_{2,j,\xi_j}[F]}{(j!)^{3/2}\nu^{j/2}}, \quad \nu^{-\frac{1}{2}}|||G|||'_{2,\tilde{\xi}^{(2)}} = \nu^{-\frac{1}{4}}\sum_{j=0}^{\nu^{-1/2}}\frac{M_{2,j,\xi_j}[G]}{(j!)^{3/2}\nu^{j/2}(j+1)^{1/2}}.$$

Since $\xi_j \le 1/\sqrt{\rho_j} \le 1/(C_*\nu^{1/2})$ by the definitions (4-9)–(4-10) with the monotonicity condition (2-10), we have

$$\nu^{-\frac{1}{2}}|||F|||'_{2,\tilde{\xi}^{(1)}} \le \frac{|||F|||_2}{C_*\nu^{3/4}}. \tag{4-14}$$

Before going into the details of the proof of Proposition 4.1, let us give a lemma for the weight ξ_j and ρ_j, which will be used frequently. By the concavity condition on $\partial_Y\Omega$ in Assumption (iv) and the definition of ρ_j we have:

Lemma 4.3. *There exists $C>0$ such that the following estimates hold for any $j\ge 0$:*

$$\begin{aligned} \xi_j^2 \le \frac{1}{\rho_j} &\le \frac{1}{C_*\max\{K^{1/4}(1+(j+1)^{1/2}Y)^{-2},\nu\}} && \text{for } Y\ge 0,\\ \frac{1}{\rho_j} &\le \frac{4}{K^{1/4}C_*} && \text{for } 0\le Y\le (j+1)^{-\frac{1}{2}}. \end{aligned} \tag{4-15}$$

In particular,

$$\left\|\frac{1+\nu^{1/2}Y}{1+Y}\xi_j\right\|_{L^\infty} + \left\|\frac{1+\nu^{1/2}Y}{Y}\xi_j\right\|_{L^\infty(\{Y\ge (j+1)^{-1/2}\})} \le C(j+1)^{\frac{1}{2}}. \tag{4-16}$$

Moreover,

$$\|\rho_j\|_{L^\infty} \le 4K^{\frac{1}{4}}C_*, \quad \left\|\frac{Y\partial_Y\rho_j}{\rho_j}\right\|_{L^\infty} \le 2 \tag{4-17}$$

and

$$\left\|\frac{1}{\xi_j}\right\|_{L^\infty} \le (C_1^*+8K^{\frac{1}{4}}C_*)^{\frac{1}{2}}, \quad \sup_{j\ge 1}\left\|\frac{\xi_j}{\xi_{j-1}}\right\|_{L^\infty} \le C. \tag{4-18}$$

The proof of Lemma 4.3 is a straightforward consequence of the definitions of ξ_j and ρ_j, so we omit the details.

4A. ***Vorticity estimate for the modified system.*** In this subsection we collect lemmas for the solution to (4-5) and give the estimate for the vorticity. The main result of this subsection is as follows.

Proposition 4.4. *There exists $\kappa_1'\in(0,1]$ such that the following statement holds for any $\kappa\in(0,\kappa_1']$. There exists $K_1' = K_1'(\kappa, C_*, C_j^*)\ge 1$ such that if $K\ge K_1'$ then the system (4-1) admits a unique solution $\omega\in C([0,1/(K\nu^{1/2})];L^2\cap\dot{H}^{-1})\cap L^2(0,1/(K\nu^{1/2});H_0^1)$ satisfying*

$$|||\nabla\omega|||'_{2,\tilde{\xi}^{(1)}} + |||\omega|||'_{2,\boldsymbol{\xi}} + K^{\frac{1}{2}}|||\omega|||'_{2,\boldsymbol{\xi}} \\ \le C\left(\nu^{-\frac{1}{2}}[\|\mathrm{rot}\, W_0\|] + \frac{C_2^*+1}{\nu^{1/2}}|||F|||'_{2,\tilde{\xi}^{(1)}} + \frac{1}{K^{1/2}\nu^{1/2}}|||G|||'_{2,\tilde{\xi}^{(2)}} + |||W|||'_{2,\mathbf{1}}\right). \tag{4-19}$$

Here $C>0$ is a universal constant.

Since the unique solvability of the linear system (4-1) itself follows from the standard theory of parabolic equations, we focus on establishing the estimate (4-19). Then the core part of the proof of Proposition 4.4 consists of the calculation of the inner product for each term in (4-5) with $\xi_j^2\omega^{\boldsymbol{j}}$, where $\boldsymbol{j}=(j_1,j_2)$ with $j_1+j_2=j$ and the weight ξ_j is defined as in (4-9). Let us start from the following lemma. The number $\tau_0\in(0,1/(K\nu^{1/2})]$ is taken arbitrarily below.

Lemma 4.5. *There exists $K_{1,1}=K_{1,1}(C_1^*,C_*)\ge 1$ such that if $K\ge K_{1,1}$ then we have*

$$\int_0^{\tau_0}\langle -\nu^{\frac12}(\Delta\omega)^{\boldsymbol{j}},\xi_j^2\omega^{\boldsymbol{j}}\rangle\,d\tau$$
$$\ge \tfrac12\nu^{\frac12}\|\xi_j(\nabla\omega)^{\boldsymbol{j}}\|^2_{L^2(0,\tau_0;L^2_{X,Y})}-C\nu^{\frac12}(\kappa\nu^{\frac12}j_2)^2M_{2,j-1,\xi_{j-1}}[\partial_Y\omega]^2-C(C_2^*+1)\nu^{\frac12}(j+1)\|\xi_j\omega^{\boldsymbol{j}}\|^2_{L^2(0,\tau_0;L^2_{X,Y})}.$$

Here $C>0$ is a universal constant.

Proof. Let us write $\chi_\nu'=(\chi')(\nu^{1/2}Y)=\kappa e^{-\kappa\nu^{1/2}Y}$. We will frequently use the identity

$$[B_{j_2},\partial_Y]=-\nu^{\frac12}j_2\chi_\nu' B_{j_2-1}\partial_Y=-\frac{\nu^{1/2}j_2\chi_\nu'}{\chi_\nu}B_{j_2}. \tag{4-20}$$

Then we observe that

$$(\Delta\omega)^{\boldsymbol{j}}=e^{-K\tau\nu^{1/2}(j+1)}B_{j_2}\partial_X^{j_1}\Delta\omega=\nabla\cdot(\nabla\omega)^{\boldsymbol{j}}-\frac{\nu^{1/2}j_2\chi_\nu'}{\chi_\nu}(\partial_Y\omega)^{\boldsymbol{j}} \tag{4-21}$$

and

$$\nabla\omega^{\boldsymbol{j}}=(\nabla\omega)^{\boldsymbol{j}}+\nu^{\frac12}j_2\chi_\nu' e^{-K\tau\nu^{1/2}}(\partial_Y\omega)^{(j_1,j_2-1)}\boldsymbol{e}_2,\quad \omega^{\boldsymbol{j}}=\chi_\nu e^{-K\tau\nu^{1/2}}(\partial_Y\omega)^{(j_1,j_2-1)}. \tag{4-22}$$

Here $\boldsymbol{e}_2=(0,1)$. Hence integration by parts gives

$$\begin{aligned}&\int_0^{\tau_0}-\nu^{\frac12}\langle(\Delta\omega)^{\boldsymbol{j}},\xi_j^2\omega^{\boldsymbol{j}}\rangle\,d\tau\\
&=\nu^{\frac12}\int_0^{\tau_0}\Big(\|\xi_j(\nabla\omega)^{\boldsymbol{j}}\|^2+2\nu^{\frac12}j_2e^{-K\tau\nu^{1/2}}\langle\xi_j(\nabla\omega)^{\boldsymbol{j}},\chi_\nu'\xi_j(\partial_Y\omega)^{(j_1,j_2-1)}\rangle+\langle(\nabla\omega)^{\boldsymbol{j}}\cdot\nabla(\xi_j^2),\omega^{\boldsymbol{j}}\rangle\Big)\,d\tau\\
&\ge\tfrac34\nu^{\frac12}\|\xi_j(\nabla\omega)^{\boldsymbol{j}}\|^2_{L^2(0,\tau_0;L^2)}-C\nu^{\frac12}(\kappa\nu^{\frac12}j_2)^2\|\xi_{j-1}(\partial_Y\omega)^{(j_1,j_2-1)}\|^2_{L^2(0,\tau_0;L^2)}\\
&\qquad-\nu^{\frac12}\int_0^{\tau_0}|\langle(\nabla\omega)^{\boldsymbol{j}}\cdot\nabla(\xi_j^2),\omega^{\boldsymbol{j}}\rangle|\,d\tau.\end{aligned}$$

Here we have used $\|\xi_j/\xi_{j-1}\|_{L^\infty}\le C$ in the last line as stated in Lemma 4.3. When $j_2=0$, the term $(\partial_Y\omega)^{(j_1,j_2-1)}$ is defined as 0 for convenience. It suffices to estimate $\langle(\nabla\omega)^{\boldsymbol{j}}\cdot\nabla(\xi_j^2),\omega^{\boldsymbol{j}}\rangle$. We have

$$\nabla(\xi_j^2)=-\frac{\nabla\partial_Y\Omega+2\nabla\rho_j}{\sqrt{\partial_Y\Omega+2\rho_j}}\xi_j^3, \tag{4-23}$$

which yields

$$|\langle(\nabla\omega)^{\boldsymbol{j}}\cdot\nabla(\xi_j^2),\omega^{\boldsymbol{j}}\rangle|\le\|\xi_j(\nabla\omega)^{\boldsymbol{j}}\|\bigg(\bigg\|\frac{\nabla\partial_Y\Omega}{\sqrt{\partial_Y\Omega+2\rho_j}}\xi_j^2\omega^{\boldsymbol{j}}\bigg\|+\bigg\|\frac{2\partial_Y\rho_j}{\sqrt{\partial_Y\Omega+2\rho_j}}\xi_j^2\omega^{\boldsymbol{j}}\bigg\|\bigg).$$

To estimate $\|(\nabla\partial_Y\Omega/\sqrt{\partial_Y\Omega+2\rho_j})\xi_j^2\omega^{\boldsymbol{j}}\|$, we decompose the integral about Y into $0\le Y\le (j+1)^{-1/2}$ and $Y\ge (j+1)^{-1/2}$. Then we see from Lemma 4.3 with $\xi_j^2/\sqrt{\partial_Y\Omega+2\rho_j}=\xi_j^3\le 1/\rho_j^{3/2}$,

$$\begin{aligned}
\Big\|\frac{\nabla\partial_Y\Omega}{\sqrt{\partial_Y\Omega+2\rho_j}}\xi_j^2\omega^{\boldsymbol{j}}\Big\|_{L^2(\{0<Y<(j+1)^{-1/2}\})} &\le \Big\|\frac{1}{\rho_j^{3/2}}\Big\|_{L^\infty(\{0<Y<(j+1)^{-1/2}\})}\|\nabla\partial_Y\Omega\omega^{\boldsymbol{j}}\|_{L^2(\{0<Y<(j+1)^{-1/2}\})}\\
&\le \frac{2}{(K^{1/4}C_*)^{3/2}}\Big\|\frac{Y}{1+\nu^{1/2}Y}\nabla\partial_Y\Omega\Big\|_{L^\infty}\Big\|\frac{\omega^{\boldsymbol{j}}}{Y}\Big\|\\
&\le \frac{CC_1^*}{(K^{1/4}C_*)^{3/2}}\|\partial_Y\omega^{\boldsymbol{j}}\|.
\end{aligned}$$

Here we have used Assumption (iii) and the Hardy inequality $\|\omega^{\boldsymbol{j}}/Y\|\le 4\|\partial_Y\omega^{\boldsymbol{j}}\|$. Then by using (4-22) for $\partial_Y\omega^{\boldsymbol{j}}$ and (4-18) we have

$$\begin{aligned}
\|\partial_Y\omega^{\boldsymbol{j}}\| &\le \|(\partial_Y\omega)^{\boldsymbol{j}}\|+\kappa\nu^{\frac12}j_2\|(\partial_Y\omega)^{(j_1,j_2-1)}\|\\
&\le \Big\|\frac{1}{\xi_j}\Big\|_{L^\infty}\|\xi_j(\partial_Y\omega)^{\boldsymbol{j}}\|+\kappa\nu^{\frac12}j_2\Big\|\frac{1}{\xi_{j-1}}\Big\|_{L^\infty}\|\xi_{j-1}(\partial_Y\omega)^{(j_1,j_2-1)}\|\\
&\le C(C_1^*+K^{\frac14}C_*)^{\frac12}(\|\xi_j(\partial_Y\omega)^{\boldsymbol{j}}\|+\kappa\nu^{\frac12}j_2\|\xi_{j-1}(\partial_Y\omega)^{(j_1,j_2-1)}\|).
\end{aligned}\tag{4-24}$$

On the other hand, we have from Assumption (iv) and (4-16) in Lemma 4.3,

$$\begin{aligned}
\Big\|\frac{\nabla\partial_Y\Omega}{\sqrt{\partial_Y\Omega+2\rho_j}}\xi_j^2\omega^{\boldsymbol{j}}\Big\|_{L^2(\{Y\ge(j+1)^{-1/2}\})} &\le \Big\|\frac{Y\nabla\partial_Y\Omega}{(1+\nu^{1/2}Y)\sqrt{\partial_Y\Omega+2\rho_j}}\Big\|\Big\|\frac{1+\nu^{1/2}Y}{Y}\xi_j\Big\|_{L^\infty(\{Y\ge(j+1)^{-1/2}\})}\|\xi_j\omega^{\boldsymbol{j}}\|\\
&\le CC_2^*(j+1)^{\frac12}\|\xi_j\omega^{\boldsymbol{j}}\|.
\end{aligned}$$

Next we estimate the term $\|(2\partial_Y\rho_j/\sqrt{\partial_Y\Omega+2\rho_j})\xi_j^2\omega^{\boldsymbol{j}}\|$. To this end we observe that

$$\begin{aligned}
|\partial_Y\rho_j| &\le 2(j+1)^{\frac12}K^{\frac14}C_*(1+(j+1)^{\frac12}Y)^{-3}+2C_*\nu^{\frac12}(1+Y)^{-3}+2C_*\nu^{-\frac14}\Big(1+\frac{Y}{\nu^{1/4}}\Big)^{-3}\\
&\le \begin{cases} 2(j+1)^{\frac12}\rho_j+2C_*/Y, & 0<Y<(j+1)^{-\frac12},\\ 2(j+1)^{\frac12}\rho_j+2\rho_j/Y, & Y\ge(j+1)^{-\frac12},\end{cases}
\end{aligned}$$

which gives, from Lemma 4.3,

$$\begin{aligned}
\Big\|\frac{2\partial_Y\rho_j}{\sqrt{\partial_Y\Omega+2\rho_j}}\xi_j^2\omega^{\boldsymbol{j}}\Big\|_{L^2} &\le 4(j+1)^{\frac12}\|\xi_j\omega^{\boldsymbol{j}}\|+2C_*\Big\|\frac{\xi_j^3\omega^{\boldsymbol{j}}}{Y}\Big\|_{L^2(\{0<Y<(j+1)^{-1/2}\})}+2\Big\|\frac{\rho_j\xi_j^3\omega^{\boldsymbol{j}}}{Y}\Big\|_{L^2(\{Y\ge(j+1)^{-1/2}\})}\\
&\le 4(j+1)^{\frac12}\|\xi_j\omega^{\boldsymbol{j}}\|+\frac{2C_*}{(K^{1/4}C_*)^{3/2}}\Big\|\frac{\omega^{\boldsymbol{j}}}{Y}\Big\|+2(j+1)^{\frac12}\|\xi_j\omega^{\boldsymbol{j}}\|.
\end{aligned}$$

Then we apply the Hardy inequality $\|\omega^{\boldsymbol{j}}/Y\|\le 4\|\partial_Y\omega^{\boldsymbol{j}}\|$ and then use (4-24). Collecting these, we obtain

$$\begin{aligned}
&|\langle(\nabla\omega)^{\boldsymbol{j}}\cdot\nabla(\xi_j^2),\omega^{\boldsymbol{j}}\rangle|\\
&\quad\le \|\xi_j(\nabla\omega)^{\boldsymbol{j}}\|\bigg(\frac{C(C_1^*+1)(C_1^*+K^{1/4}C_*)^{1/2}}{(K^{1/4}C_*)^{3/2}}(\|\xi_j(\partial_Y\omega)^{\boldsymbol{j}}\|+\kappa\nu^{\frac12}j_2\|\xi_{j-1}(\partial_Y\omega)^{(j_1,j_2-1)}\|)\\
&\qquad\qquad+C(C_2^*+1)(j+1)^{\frac12}\|\xi_j\omega^{\boldsymbol{j}}\|\bigg).
\end{aligned}$$

Thus, by taking K large enough depending only on C_1^* and C_*, we obtain the desired estimate as stated in Lemma 4.6. □

Lemma 4.6. *There exists $K_{1,2} = K_{1,2}(C_1^*, C_*) \geq 1$ such that if $K \geq K_{1,2}$ then we have*

$$\begin{aligned}\int_0^{\tau_0} \langle (\partial_\tau + K\nu^{\frac{1}{2}}(j+1) + V\cdot\nabla)\omega^{\boldsymbol{j}}, \xi_j^2\omega^{\boldsymbol{j}}\rangle \, d\tau &\geq \tfrac{1}{2}\|\xi_j\omega^{\boldsymbol{j}}(\tau_0)\|^2_{L^2_{X,Y}} - \tfrac{1}{2}\|\xi_j\omega^{\boldsymbol{j}}(0)\|^2_{L^2_{X,Y}} + \tfrac{1}{2}K\nu^{\frac{1}{2}}(j+1)\|\xi_j\omega^{\boldsymbol{j}}\|^2_{L^2(0,\tau_0;L^2_{X,Y})} \\ &\quad - \frac{CC_1^*\nu^{1/2}}{K^{1/4}C_*}(\|\xi_j(\partial_Y\omega)^{\boldsymbol{j}}\|^2_{L^2(0,\tau_0;L^2_{X,Y})} + (\kappa\nu^{\frac{1}{2}}j)^2 M_{2,j-1,\xi_{j-1}}[\partial_Y\omega]^2).\end{aligned}$$

Here $C > 0$ is a universal constant.

Proof. Integration by parts yields

$$\begin{aligned}\int_0^{\tau_0} \langle (\partial_\tau + K\nu^{\frac{1}{2}}(j+1) + V\cdot\nabla)\omega^{\boldsymbol{j}}, \xi_j^2\omega^{\boldsymbol{j}}\rangle \, d\tau &= \tfrac{1}{2}\|\xi_j\omega^{\boldsymbol{j}}(\tau_0)\|^2_{L^2_{X,Y}} - \tfrac{1}{2}\|\xi_j\omega^{\boldsymbol{j}}(0)\|^2_{L^2_{X,Y}} + K\nu^{\frac{1}{2}}(j+1)\|\xi_j\omega^{\boldsymbol{j}}\|^2_{L^2(0,\tau_0;L^2_{X,Y})} \\ &\quad - \frac{1}{2}\int_0^{\tau_0} \langle \partial_\tau(\xi_j^2) + V\cdot\nabla(\xi_j^2), (\omega^{\boldsymbol{j}})^2\rangle \, d\tau.\end{aligned}$$

As for the term $\langle \partial_\tau(\xi_j^2), (\omega^{\boldsymbol{j}})^2\rangle$, we decompose the integral about Y into $\{0 < Y < (j+1)^{-1/2}\}$ and $\{Y \geq (j+1)^{-1/2}\}$ and compute as follows:

$$\begin{aligned}&|\langle \partial_\tau(\xi_j^2), (\omega^{\boldsymbol{j}})^2\rangle| \\ &\leq \left\| \left(\frac{Y}{1+\nu^{1/2}Y}\right)^2 \partial_\tau\partial_Y\Omega \right\|_{L^\infty} \left\| \left(\frac{1+\nu^{1/2}Y}{Y}\right)\xi_j^2\omega^{\boldsymbol{j}} \right\|^2 \\ &\leq C_1^*\nu^{\frac{1}{2}}\left(\|(1+\nu^{\frac{1}{2}}Y)\xi_j^2\|^2_{L^\infty(\{0<Y<(j+1)^{-1/2}\})} \left\|\frac{\omega^{\boldsymbol{j}}}{Y}\right\|^2 + \left\|\left(\frac{1+\nu^{1/2}Y}{Y}\right)\xi_j\right\|^2_{L^\infty(\{Y\geq(j+1)^{-1/2}\})} \|\xi_j\omega^{\boldsymbol{j}}\|^2\right) \\ &\leq C_1^*\nu^{\frac{1}{2}}\left(\frac{C}{(K^{1/4}C_*)^2}\|\partial_Y\omega^{\boldsymbol{j}}\|^2 + C(j+1)\|\xi_j\omega^{\boldsymbol{j}}\|^2\right) \quad \text{(by the Hardy inequality and Lemma 4.3).} \qquad (4\text{-}25)\end{aligned}$$

Next we have

$$|\langle V\cdot\nabla(\xi_j^2), (\omega^{\boldsymbol{j}})^2\rangle| \leq \left\| \frac{V\cdot\nabla(\partial_Y\Omega + 2\rho_j)}{\partial_Y\Omega + 2\rho_j} \right\|_{L^\infty} \|\xi_j\omega^{\boldsymbol{j}}\|^2.$$

Then we have from Assumption (iii) and Lemma 4.3,

$$\begin{aligned}\left\|\frac{V_1\partial_Y\partial_X\Omega}{\partial_Y\Omega + 2\rho_j}\right\|_{L^\infty} &\leq \left\|\frac{Y(1+Y)}{(1+\nu^{1/2}Y)^2}\partial_X\partial_Y\Omega\right\|_{L^\infty} \left\|\frac{V_1(1+\nu^{1/2}Y)^2}{Y(1+Y)\rho_j}\right\|_{L^\infty} \\ &\leq C_1^*\nu^{\frac{1}{2}}\left(2\left\|\frac{V_1}{Y(1+Y)\rho_j}\right\|_{L^\infty} + 2\|V_1\|_{L^\infty}\left\|\frac{(\nu^{1/2}Y)^2}{Y(1+Y)\rho_j}\right\|_{L^\infty}\right) \leq C(C_1^*)^2\nu^{\frac{1}{2}}(j+1).\end{aligned}$$

Here we have computed, using $V_1|_{Y=0}=0$,

$$\begin{aligned}\left\|\frac{V_1}{Y(1+Y)\rho_j}\right\|_{L^\infty} &\le \|\partial_Y V_1\|_{L^\infty}\left\|\frac{1}{\rho_j}\right\|_{L^\infty(\{0<Y<(j+1)^{-1/2}\})}+\|V_1\|_{L^\infty}\left\|\frac{1}{Y(1+Y)\rho_j}\right\|_{L^\infty(\{Y\ge(j+1)^{-1/2}\})}\\ &\le C_1^*(j+1).\end{aligned}$$

Similarly,

$$\begin{aligned}\left\|\frac{V_2(\partial_Y^2\Omega+2\partial_Y\rho_j)}{\partial_Y\Omega+2\rho_j}\right\|_{L^\infty} &\le \left\|\frac{Y(1+Y)^2}{(1+\nu^{1/2}Y)^3}\partial_Y^2\Omega\right\|_{L^\infty}\left\|\frac{V_2(1+\nu^{1/2}Y)^3}{Y(1+Y)^2\rho_j}\right\|_{L^\infty}+\left\|\frac{V_2}{Y}\right\|_{L^\infty}\left\|\frac{Y\partial_Y\rho_j}{\rho_j}\right\|_{L^\infty}\\ &\le CC_1^*\left(\left\|\frac{V_2}{Y(1+Y)^2\rho_j}\right\|_{L^\infty}+\|V_2\|_{L^\infty}\left\|\frac{(\nu^{1/2}Y)^3}{Y(1+Y)^2\rho_j}\right\|_{L^\infty}\right)+2C_1^*\nu^{\frac12}\\ &\le CC_1^*\left(\|\partial_Y V_2\|_{L^\infty}\left\|\frac{1}{(1+Y)^2\rho_j}\right\|_{L^\infty}+\|V_2\|_{L^\infty}\nu^{\frac32}\left\|\frac{1}{\rho_j}\right\|_{L^\infty}\right)+2C_1^*\nu^{\frac12}\\ &\le CC_1^*(C_1^*+1)\nu^{\frac12}(j+1). \qquad \text{(by Lemma 4.3)}.\end{aligned}$$

Note that we have also used $\|\partial_Y V_2\|_{L^\infty}=\|\partial_X V_1\|_{L^\infty}\le C_1^*\nu^{1/2}$. Collecting these and applying the identity (4-22) for $\partial_Y\omega^j$ in (4-25) (that is, we use (4-24)), we obtain the desired estimate by taking K large enough depending only on C_1^* and C_*. □

Lemma 4.7. *It follows that*

$$\int_0^{\tau_0}|\langle(\nabla^\perp\phi)^j\cdot\nabla\Omega,\xi_j^2\omega^j\rangle|\,d\tau\le\frac{C(R_{j,\text{Lemma 4.7}}[\nabla\phi])^2}{\nu^{1/2}(j+1)}+\tfrac18 K\nu^{\frac12}(j+1)\|\xi_j\omega^j\|^2_{L^2(0,\tau_0;L^2_{X,Y})}, \tag{4-26}$$

where

$$R_{j,\text{Lemma 4.7}}[\nabla\phi] := \left(\frac{C_1^*}{K^{1/2}}+\frac{(K^{1/4}C_*)^{1/2}}{K^{1/2}}+\kappa^{\frac12}\right)\nu^{\frac12}(j+1)M_{2,j}[\nabla\phi]+\frac{(K^{1/2}C_*)^{1/2}}{K^{1/2}}\delta_{j\le\nu^{-1/2}-1}\frac{M_{2,j+1}[\partial_Y\phi]}{(j+1)^{1/2}}.$$

Here

$$\delta_{j\le\nu^{-1/2}-1}=\begin{cases}1 & \text{for } 0\le j\le\nu^{-1/2}-1,\\ 0 & \text{for } j=\nu^{-1/2}.\end{cases}$$

Moreover, there exists $K_{1,3}=K_{1,3}(C_1^*,C_*)\ge1$ such that if $K\ge K_{1,3}$ then

$$\sum_{j=0}^{\nu^{-1/2}}\frac{R_{j,\text{Lemma 4.7}}[\nabla\phi]}{(j!)^{3/2}\nu^{j/2}\nu^{1/4}(j+1)^{1/2}}\le C|||\nabla\phi|||'_{2,\mathbf{1}}. \tag{4-27}$$

Here $C>0$ is a universal constant.

Proof. It suffices to show

$$\int_0^{\tau_0}|\langle(\partial_X\phi)^j,\omega^j\rangle|\,d\tau\le 2\kappa\nu^{\frac12}j_2(M_{2,j}[\nabla\phi])^2, \tag{4-28}$$

$$\int_0^{\tau_0} |\langle \rho_j (\partial_X \phi)^{\boldsymbol{j}}, \xi_j^2 \omega^{\boldsymbol{j}} \rangle| \, d\tau$$
$$\leq \begin{cases} C(K^{\frac{1}{4}} C_*)^{\frac{1}{2}} \Big(\frac{M_{2,j+1}[\partial_Y \phi]}{(j+1)^{1/2}} + \kappa \nu^{\frac{1}{2}} (j+1)^{\frac{1}{2}} M_{2,j}[\nabla \phi] \Big) \|\xi_j \omega^{\boldsymbol{j}}\|_{L^2(0,\tau_0; L^2_{X,Y})}, & 0 \leq j \leq \nu^{-\frac{1}{2}} - 1, \\ C(K^{\frac{1}{4}} C_*)^{\frac{1}{2}} M_{2,j}[\partial_X \phi] \|\xi_j \omega^{\boldsymbol{j}}\|_{L^2(0,\tau_0; L^2_{X,Y})}, & j = \nu^{-\frac{1}{2}}, \end{cases} \tag{4-29}$$

and
$$\int_0^{\tau_0} |\langle (\partial_Y \phi)^{\boldsymbol{j}} \partial_X \Omega, \xi_j^2 \omega^{\boldsymbol{j}} \rangle| \, d\tau \leq C C_1^* \nu^{\frac{1}{2}} (j+1)^{\frac{1}{2}} M_{2,j}[\partial_Y \phi] \|\xi_j \omega^{\boldsymbol{j}}\|_{L^2(0,\tau_0; L^2_{X,Y})}. \tag{4-30}$$

Let us start from (4-28). To compute $\langle (\partial_X \phi)^{\boldsymbol{j}}, \omega^{\boldsymbol{j}} \rangle$, we first observe that
$$\omega^{\boldsymbol{j}} = \nabla \cdot (\nabla \phi)^{\boldsymbol{j}} - \frac{\nu^{1/2} j_2 \chi_\nu'}{\chi_\nu} (\partial_Y \phi)^{\boldsymbol{j}}. \tag{4-31}$$

Then we have, from integration by parts and $[B_{j_2}, \partial_Y] = -((\nu^{1/2} j_2 \chi_\nu')/\chi_\nu) B_{j_2}$,
$$\begin{aligned} \langle (\partial_X \phi)^{\boldsymbol{j}}, \omega^{\boldsymbol{j}} \rangle &= -\langle \nabla (\partial_X \phi)^{\boldsymbol{j}}, (\nabla \phi)^{\boldsymbol{j}} \rangle - \nu^{\frac{1}{2}} j_2 \langle (\partial_Y \phi)^{(j_1+1, j_2-1)}, \chi_\nu' (\partial_Y \phi)^{\boldsymbol{j}} \rangle \\ &= -\langle \partial_X (\nabla \phi)^{\boldsymbol{j}}, (\nabla \phi)^{\boldsymbol{j}} \rangle - 2\nu^{\frac{1}{2}} j_2 \langle \chi_\nu' (\partial_Y \phi)^{(j_1+1, j_2-1)}, (\partial_Y \phi)^{\boldsymbol{j}} \rangle \\ &= -2\nu^{\frac{1}{2}} j_2 \langle \chi_\nu' (\partial_Y \phi)^{(j_1+1, j_2-1)}, (\partial_Y \phi)^{\boldsymbol{j}} \rangle. \end{aligned}$$

Hence we have, from $\|\chi_\nu'\|_{L^\infty} = \kappa$,
$$\int_0^{\tau_0} |\langle (\partial_X \phi)^{\boldsymbol{j}}, \omega^{\boldsymbol{j}} \rangle| \, d\tau \leq 2\kappa \nu^{\frac{1}{2}} j_2 M_{2,j}[\partial_Y \phi]^2. \tag{4-32}$$

To estimate $\int_0^{\tau_0} |\langle \rho_j (\partial_X \phi)^{\boldsymbol{j}}, \xi_j^2 \omega^{\boldsymbol{j}} \rangle| \, d\tau$, the key inequality from the definition (4-10) is
$$\xi_j \rho_j \leq \sqrt{\rho_j} \leq C(K^{\frac{1}{4}} C_*)^{\frac{1}{2}} (1 + (j+1)^{\frac{1}{2}} Y)^{-1} + C\nu^{\frac{1}{2}}, \tag{4-33}$$

where $\nu^{1/2}(j+1) \leq 2$ is used. Thus we have from the Hardy inequality
$$\begin{aligned} &\int_0^{\tau_0} |\langle \rho_j (\partial_X \phi)^{\boldsymbol{j}}, \xi_j^2 \omega^{\boldsymbol{j}} \rangle| \, d\tau \\ &\quad \leq \int_0^{\tau_0} \|\xi_j \rho_j (\partial_X \phi)^{\boldsymbol{j}}\| \|\xi_j \omega^{\boldsymbol{j}}\| \, d\tau \\ &\quad \leq \frac{C(K^{1/4} C_*)^{1/2}}{(j+1)^{1/2}} \int_0^{\tau_0} \left\| \frac{(\partial_X \phi)^{\boldsymbol{j}}}{Y} \right\| \|\xi_j \omega^{\boldsymbol{j}}\| \, d\tau + C\nu^{\frac{1}{2}} \|(\partial_X \phi)^{\boldsymbol{j}}\|_{L^2(0,\tau_0;L^2)} \|\xi_j \omega^{\boldsymbol{j}}\|_{L^2(0,\tau_0;L^2)} \\ &\quad \leq \frac{C(K^{1/4} C_*)^{1/2}}{(j+1)^{1/2}} \|\partial_Y (\partial_X \phi)^{\boldsymbol{j}}\|_{L^2(0,\tau_0;L^2)} \|\xi_j \omega^{\boldsymbol{j}}\|_{L^2(0,\tau_0;L^2)} + C\nu^{\frac{1}{2}} \|(\partial_X \phi)^{\boldsymbol{j}}\|_{L^2(0,\tau_0;L^2)} \|\xi_j \omega^{\boldsymbol{j}}\|_{L^2(0,\tau_0;L^2)}. \end{aligned}$$

Then the desired estimate for $0 \leq j \leq \nu^{-1/2} - 1$ follows from $K\tau\nu^{1/2} \leq 1$ and
$$\partial_Y (\partial_X \phi)^{\boldsymbol{j}} = e^{K\tau\nu^{1/2}} (\partial_Y \phi)^{(j_1+1, j_2)} + \nu^{\frac{1}{2}} j_2 \chi_\nu' (\partial_Y \phi)^{(j_1+1, j_2-1)}. \tag{4-34}$$

On the other hand, the estimate for $j = \nu^{-1/2}$ easily follows from
$$\|\xi_j \rho_j (\partial_X \phi)^{\boldsymbol{j}}\| \leq \|\sqrt{\rho_j}\|_{L^\infty} \|(\partial_X \phi)^{\boldsymbol{j}}\| \leq C(K^{\frac{1}{4}} C_*)^{\frac{1}{2}} \|(\partial_X \phi)^{\boldsymbol{j}}\|. \tag{4-35}$$

Finally we have, from Assumption (iii) and Lemma 4.3,

$$\|\xi_j(\partial_Y\phi)^j\partial_X\Omega\| \le \left\|\frac{1+Y}{1+\nu^{1/2}Y}\partial_X\Omega\right\|_{L^\infty}\left\|\frac{1+\nu^{1/2}Y}{1+Y}\xi_j\right\|_{L^\infty}\|(\partial_Y\phi)^j\| \le CC_1^*\nu^{\frac{1}{2}}(j+1)^{\frac{1}{2}}\|(\partial_Y\phi)^j\|,$$

which gives

$$\int_0^{\tau_0}|\langle(\partial_Y\phi)^j\partial_X\Omega,\xi_j^2\omega^j\rangle|\,d\tau \le CC_1^*\nu^{\frac{1}{2}}(j+1)^{\frac{1}{2}}M_{2,j}[\partial_Y\phi]\|\xi_j\omega^j\|_{L^2(0,\tau_0;L^2_{X,Y})}.$$

Collecting these, we obtain (4-26), for the identity

$$\partial_Y\Omega\xi_j^2 = \frac{\partial_Y\Omega}{\partial_Y\Omega+2\rho_j} = 1-2\rho_j\xi_j^2$$

holds. The estimate (4-27) is verified from the definition

$$|||\nabla\phi|||'_{2,\mathbf{1}} = \sum_{j=0}^{\nu^{-1/2}}\frac{\nu^{1/4}(j+1)^{1/2}}{(j!)^{3/2}\nu^{j/2}}M_{2,j}[\nabla\phi]$$

and

$$\begin{aligned}\sum_{j=0}^{\nu^{-1/2}-1}\frac{M_{2,j+1}[\partial_Y\phi]}{(j!)^{3/2}\nu^{j/2}\nu^{1/4}(j+1)} &= \sum_{j=0}^{\nu^{-1/2}-1}\frac{\nu^{1/2}(j+1)^{3/2}M_{2,j+1}[\nabla\phi]}{((j+1)!)^{3/2}\nu^{(j+1)/2}\nu^{1/4}(j+1)}\\ &\le \sum_{j=1}^{\nu^{-1/2}}\frac{\nu^{1/4}j^{1/2}M_{2,j}[\nabla\phi]}{(j!)^{3/2}\nu^{j/2}} \le |||\nabla\phi|||'_{2,\mathbf{1}}.\end{aligned}$$

□

Lemma 4.8. *Let $j_2\ge 1$. Then it follows that*

$$\int_0^{\tau_0}|\langle V_2[B_{j_2},\partial_Y]e^{-K\tau\nu^{1/2}(j+1)}\partial_X^{j_1}\omega,\xi_j^2\omega^j\rangle|\,d\tau \le CC_1^*\nu^{\frac{1}{2}}j_2\|\xi_j\omega^j\|^2_{L^2(0,\tau_0;L^2_{X,Y})}. \tag{4-36}$$

Here $C>0$ is a universal constant.

Proof. The estimate directly follows from (4-20) and

$$|V_2\chi'_\nu| \le \left\|\frac{V_2}{Y}\right\|_{L^\infty}|Y\chi'_\nu| \le \|\partial_X V_1\|_{L^\infty}|Y\chi'_\nu| \le C_1^*\nu^{\frac{1}{2}}|Y\chi'_\nu| \le CC_1^*\chi_\nu$$

by Assumption (iii) and $\kappa\nu^{1/2}Ye^{-\kappa\nu^{1/2}Y} \le C\chi_\nu$ for a universal constant $C>0$. □

Lemma 4.9. *Let $j\ge 1$. It follows that*

$$\int_0^{\tau_0}\left|\left\langle\sum_{l=0}^{j-1}\sum_{\max\{0,l+j_2-j\}\le l_2\le\min\{l,j_2\}}\binom{j_2}{l_2}\binom{j-j_2}{l-l_2}V^{j-l}\cdot(\nabla\omega)^l,\xi_j^2\omega^j\right\rangle\right|d\tau$$
$$\le \frac{C}{\kappa}R_{j,\text{Lemma 4.9}}[\omega]\|\xi_j\omega^j\|_{L^2(0,\tau_0;L^2_{X,Y})},$$

where

$$R_{j,\text{Lemma 4.9}}[\omega] := \sum_{l=0}^{j-1}(j-l+1)^{\frac{1}{2}}\min\{l+1,j-l+1\}\binom{j}{l}N_{\infty,j-l}[V]M_{2,l+1,\xi_l}[\omega],$$

and

$$N_{\infty,j}[V] := \sup_{j_2=0,\dots,j}\left(\|B_{j_2}\partial_X^{j-j_2}V_1\|_{L^\infty(0,1/(K\nu^{1/2});L^\infty_{X,Y})} + \kappa\left\|\frac{\partial_X^j V_2}{\chi_\nu}\right\|_{L^\infty(0,1/(K\nu^{1/2});L^\infty_{X,Y})}\right).$$

Moreover,

$$\sum_{j=0}^{\nu^{-1/2}} \frac{R_{j,\text{Lemma 4.9}}[\omega]}{(j!)^{3/2}\nu^{j/2}\nu^{1/4}(j+1)^{1/2}} \le CC_0^*|||\omega|||'_{2,\xi}. \tag{4-37}$$

Here $C>0$ is a universal constant.

Proof. We first observe that

$$\binom{j_2}{l_2}\binom{j-j_2}{l-l_2} \le \binom{j}{l}, \quad 0\le j_2\le l_2\le l\le j, \tag{4-38}$$

and

$$\#\{l_2\in\mathbb{N}\cup\{0\} \mid \max\{0, l+j_2-j\}\le l_2\le \min\{l, j_2\}\} \le \min\{l+1, j-l+1\}. \tag{4-39}$$

Hence we have

$$\begin{aligned}\int_0^{\tau_0}\left|\left\langle\sum_{l=0}^{j-1}\sum_{\max\{0,l+j_2-j\}\le l_2\le\min\{l,j_2\}}\binom{j_2}{l_2}\binom{j-j_2}{l-l_2}\boldsymbol{V}^{\boldsymbol{j-l}}\cdot(\nabla\omega)^{\boldsymbol{l}}, \xi_j^2\omega^{\boldsymbol{j}}\right\rangle\right| d\tau \\ \le \sum_{l=0}^{j-1}\binom{j}{l}\min\{l+1, j-l+1\}\|\xi_j\boldsymbol{V}^{\boldsymbol{j-l}}\cdot(\nabla\omega)^{\boldsymbol{l}}\|_{L^2(0,\tau_0;L^2)}\|\xi_j\omega^{\boldsymbol{j}}\|_{L^2(0,\tau_0;L^2)}.\end{aligned}$$

From the definition of ξ_j, we see, for $0\le l\le j-1$,

$$\frac{\xi_j}{\xi_l} \le \sqrt{1+\frac{(1+(j+1)^{1/2}Y)^{-2}}{(1+(l+1)^{1/2}Y)^{-2}}} \le C(j+l-1)^{\frac12},$$

where $C>0$ is a universal constant, and thus,

$$\|\xi_j\boldsymbol{V}^{\boldsymbol{j-l}}\cdot(\nabla\omega)^{\boldsymbol{l}}\|_{L^2(0,\tau_0;L^2)} \le C(j+l-1)^{\frac12}\|\xi_l\boldsymbol{V}^{\boldsymbol{j-l}}\cdot(\nabla\omega)^{\boldsymbol{l}}\|_{L^2(0,\tau_0;L^2)}.$$

Next we have

$$\begin{aligned}\|\xi_l V_1^{\boldsymbol{j-l}}(\partial_X\omega)^{\boldsymbol{l}}\|_{L^2(0,\tau_0;L^2)} &\le \left\|\frac{\xi_l}{\xi_{l+1}}\right\|_{L^\infty}\|V_1^{\boldsymbol{j-l}}\|_{L^\infty}\|\xi_{l+1}\omega^{(l_1+1,l_2)}\|_{L^2(0,\tau_0;L^2)} \\ &\le CN_{\infty,j-l}[V]M_{2,l+1,\xi_{l+1}}[\omega],\end{aligned}$$

and similarly,

$$\begin{aligned}\|\xi_l V_2^{\boldsymbol{j-l}}(\partial_Y\omega)^{\boldsymbol{l}}\|_{L^2(0,\tau_0;L^2)} &\le \left\|\frac{\xi_l}{\xi_{l+1}}\right\|_{L^\infty}\left\|\frac{V_2^{\boldsymbol{j-l}}}{\chi_\nu}\right\|_{L^\infty}\|\xi_{l+1}\omega^{(l_1,l_2+1)}\|_{L^2(0,\tau_0;L^2)} \\ &\le \frac{C}{\kappa}N_{\infty,j-l}[V]M_{2,l+1,\xi_{l+1}}[\omega],\end{aligned}$$

Here we have used from $\partial_X V_1 + \partial_Y V_2 = 0$ that $V_2^{j-l}/\chi_\nu = (\partial_Y V_2)^{(j_1-l_1, j_2-l_2-1)} = -V_1^{(j_1-l_1+1, j_2-l_2-1)}$ for $j_2 - l_2 \geq 1$, which satisfies $\|V_2^{j-l}/\chi_\nu\|_{L^\infty} \leq C N_{\infty, j-l}[V]$. The estimate (4-37) follows from

$$\begin{aligned}
&\sum_{j=0}^{\nu^{-1/2}} \frac{1}{(j!)^{3/2}\nu^{j/2}\nu^{1/4}(j+1)^{1/2}} \sum_{l=0}^{j-1} (j-l+1)^{\frac12} \min\{l+1,\, j-l+1\} \binom{j}{l} \{(j-l)!\,(l+1)!\}^{\frac32} \nu^{(j+1)/2} \\
&\qquad\qquad \times \frac{N_{\infty,j-l}[V]}{((j-l)!)^{3/2}\nu^{(j-l)/2}} \frac{M_{2,l+1,\xi_l}[\omega]}{((l+1)!)^{3/2}\nu^{(l+1)/2}} \\
&\leq \sum_{j=0}^{\nu^{-1/2}} \sum_{l=0}^{j-1} (j-l+1)^{\frac12} \min\{l+1,\, j-l+1\} \frac{(l+1)^{3/2}}{(j+1)^{1/2}(l+2)^{1/2}} \left(\frac{(j-l)!\,l!}{j!}\right)^{\frac12} \\
&\qquad\qquad \times \frac{N_{\infty,j-l}[V]}{((j-l)!)^{3/2}\nu^{(j-l)/2}} \frac{\nu^{1/4}(l+2)^{1/2} M_{2,l+1,\xi_l}[\omega]}{((l+1)!)^{3/2}\nu^{(l+1)/2}} \\
&\leq C \sum_{j=0}^{\nu^{-1/2}} \sum_{l=0}^{j-1} \frac{N_{\infty,j-l}[V]}{((j-l)!)^{3/2}\nu^{(j-l)/2}} \frac{\nu^{1/4}(l+2)^{1/2} M_{2,l+1,\xi_l}[\omega]}{((l+1)!)^{3/2}\nu^{(l+1)/2}}.
\end{aligned}$$

Here we have used, for $j \geq 1$,

$$(j-l+1)^{\frac12} \min\{l+1,\, j-l+1\} \frac{(l+1)^{3/2}}{(j+1)^{1/2}(l+2)^{1/2}} \left(\frac{(j-l)!\,l!}{j!}\right)^{\frac12} \leq C, \quad 0 \leq l \leq j-1, \tag{4-40}$$

with a universal constant $C > 0$. Here the key is the following estimate for each $k = 0, 1, 2, 3$:

$$\frac{(j-l)!\,l!}{j!} \leq \frac{C}{(j+1)^{1+k}} \quad \text{for } 1+k \leq l \leq j-1-k. \tag{4-41}$$

Then we obtain (4-37) from the Young inequality by convolution in the l^1 space. □

Lemma 4.10. *Let $j \geq 1$. It follows that*

$$\begin{aligned}
\int_0^{\tau_0} \left| \left\langle \sum_{l=0}^{j-1} \sum_{\max\{0, l+j_2-j\} \leq l_2 \leq \min\{l, j_2\}} \binom{j_2}{l_2}\binom{j-j_2}{l-l_2} (\nabla^\perp \phi)^l \cdot (\nabla\Omega)^{j-l}, \xi_j^2 \omega^j \right\rangle \right| d\tau& \\
\leq C R_{j,\text{Lemma 4.10}}[\nabla\phi] \|\xi_j \omega^j\|_{L^2(0,\tau_0; L^2_{X,Y})}&,
\end{aligned}$$

where

$$\begin{aligned}
&R_{j,\text{Lemma 4.10}}[\nabla\phi] \\
&\quad := C_2^* \nu^{\frac12} j (M_{2,j}[\nabla\phi] + \nu^{\frac12} j M_{2,j-1}[\nabla\phi]) \\
&\qquad\qquad + (j+1)^{\frac12} \sum_{l=0}^{j-2} \min\{l+1,\, j-l+1\} \binom{j}{l} N_{\infty,j-l}[\nabla\Omega] (M_{2,l+1}[\partial_Y\phi] + \nu^{\frac12}(l+1) M_{2,l}[\nabla\phi]) \\
&\qquad\qquad\qquad + \nu^{\frac12}(j+1)^{\frac32} N_{\infty,1}[\nabla\Omega] M_{2,j-1}[\partial_Y\phi]
\end{aligned}$$

and

$$\begin{aligned}
&N_{\infty,j-l}[\nabla\Omega] \\
&\quad := \sup_{j_2=0,\dots,j} \left(\left\| \left(\frac{1+Y}{1+\nu^{1/2}Y}\right)^2 (\partial_Y\Omega)^j \right\|_{L^2(0,1/(K\nu^{1/2}); L^2_{X,Y})} + \nu^{-\frac12} \left\| \frac{1+Y}{1+\nu^{1/2}Y} (\partial_X\Omega)^j \right\|_{L^2(0,1/(K\nu^{1/2}); L^2_{X,Y})} \right).
\end{aligned}$$

Here the second term on the right-hand side is defined as zero when $j=1$. Moreover,

$$\sum_{j=0}^{\nu^{-1/2}} \frac{R_{j,\text{Lemma 4.10}}[\nabla\phi]}{(j!)^{3/2}\nu^{j/2}\nu^{1/4}(j+1)^{1/2}} \leq C(C_0^*+C_2^*)|||\nabla\phi|||'_{2,\mathbf{1}}. \tag{4-42}$$

Proof. As in the proof of Lemma 4.9, we have, from (4-38) and (4-39),

$$\begin{aligned}\int_0^{\tau_0}\biggl|\biggl\langle \sum_{l=0}^{j-1}\sum_{\max\{0,l+j_2-j\}\leq l_2\leq\min\{l,j_2\}} \binom{j_2}{l_2}\binom{j-j_2}{l-l_2}(\nabla^\perp\phi)^{\boldsymbol{l}}\cdot(\nabla\Omega)^{\boldsymbol{j}-\boldsymbol{l}}, \xi_j^2\omega^{\boldsymbol{j}}\biggr\rangle\biggr|\,d\tau \\ \leq \sum_{l=0}^{j-1}\binom{j}{l}\min\{l+1,\,j-l+1\}\|\xi_j(\nabla^\perp\phi)^{\boldsymbol{l}}\cdot(\nabla\Omega)^{\boldsymbol{j}-\boldsymbol{l}}\|_{L^2(0,\tau_0;L^2)}\|\xi_j\omega^{\boldsymbol{j}}\|_{L^2(0,\tau_0;L^2)}.\end{aligned}$$

Then we have, from Lemma 4.3,

$$\begin{aligned}\|\xi_j(\partial_Y\phi)^{\boldsymbol{l}}(\partial_X\Omega)^{\boldsymbol{j}-\boldsymbol{l}}\|_{L^2(0,\tau_0;L^2)} &\leq \left\|\frac{(1+\nu^{1/2}Y)\xi_j}{1+Y}\right\|_{L^\infty}\left\|\frac{1+Y}{1+\nu^{1/2}Y}(\partial_X\Omega)^{\boldsymbol{j}-\boldsymbol{l}}\right\|_{L^\infty}\|(\partial_Y\phi)^{\boldsymbol{l}}\|_{L^2(0,\tau_0;L^2)} \\ &\leq C\nu^{\frac12}(j+1)^{\frac12}N_{\infty,j-l}[\nabla\Omega]M_{2,l}[\partial_Y\phi].\end{aligned}$$

Let $j\geq 2$ and $0\leq l\leq j-2$. Then,

$$\begin{aligned}&\|\xi_j(\partial_X\phi)^{\boldsymbol{l}}(\partial_Y\Omega)^{\boldsymbol{j}-\boldsymbol{l}}\|_{L^2(0,\tau_0;L^2)} \\ &\quad\leq \left\|\frac{(1+\nu^{1/2}Y)\xi_j}{1+Y}\right\|_{L^\infty}\left\|\left(\frac{1+Y}{1+\nu^{1/2}Y}\right)^2(\partial_Y\Omega)^{\boldsymbol{j}-\boldsymbol{l}}\right\|_{L^\infty}\left\|\frac{1+\nu^{1/2}Y}{1+Y}(\partial_X\phi)^{\boldsymbol{l}}\right\|_{L^2(0,\tau_0;L^2)} \\ &\quad\leq C(j+1)^{\frac12}N_{\infty,j-l}[\nabla\Omega](\|\partial_Y(\partial_X\phi)^{\boldsymbol{l}}\|_{L^2(0,\tau_0;L^2)}+\nu^{\frac12}\|(\partial_X\phi)^{\boldsymbol{l}}\|_{L^2(0,\tau_0;L^2)}),\end{aligned}$$

where the Hardy inequality is applied in the last line. Then (4-34) gives

$$\begin{aligned}&\|\xi_j(\partial_X\phi)^{\boldsymbol{l}}(\partial_Y\Omega)^{\boldsymbol{j}-\boldsymbol{l}}\|_{L^2(0,\tau_0;L^2)} \\ &\quad\leq C(j+1)^{\frac12}N_{\infty,j-l}[\nabla\Omega](M_{2,l+1}[\partial_Y\phi]+\kappa\nu^{\frac12}(l+1)M_{2,l}[\nabla\phi]),\quad 0\leq l\leq j-2.\end{aligned}$$

As for the case $l=j-1$, by recalling $\xi_j\leq 1/\sqrt{\partial_Y\Omega+2\rho}$, we compute

$$\begin{aligned}\|\xi_j(\partial_X\phi)^{\boldsymbol{l}}(\partial_Y\Omega)^{\boldsymbol{j}-\boldsymbol{l}}\|_{L^2(0,\tau_0;L^2)} &\leq \left\|\frac{Y}{1+\nu^{1/2}Y}\xi_j(\partial_Y\Omega)^{\boldsymbol{j}-\boldsymbol{l}}\right\|_{L^\infty}\left\|\frac{1+\nu^{1/2}Y}{Y}(\partial_X\phi)^{\boldsymbol{l}}\right\|_{L^2(0,\tau_0;L^2)} \\ &\leq C\left(\left\|\frac{Y}{1+\nu^{1/2}Y}\frac{\partial_{XY}^2\Omega}{\sqrt{\partial_Y\Omega+2\rho}}\right\|_{L^\infty}+\left\|\frac{Y}{1+\nu^{1/2}Y}\frac{\chi_\nu\partial_Y^2\Omega}{\sqrt{\partial_Y\Omega+2\rho}}\right\|_{L^\infty}\right) \\ &\qquad\times(\|\partial_Y(\partial_X\phi)^{\boldsymbol{l}}\|_{L^2(0,\tau_0;L^2)}+\nu^{\frac12}\|(\partial_X\phi)^{\boldsymbol{l}}\|_{L^2(0,\tau_0;L^2)}).\end{aligned}$$

Here we have used the Hardy inequality and that, when $l=j-1$, either $(\partial_Y\Omega)^{\boldsymbol{j}-\boldsymbol{l}}=\partial_{XY}^2\Omega$ or $\chi_\nu\partial_Y^2\Omega$. Then, by using $\|((1+\nu^{1/2}Y)/Y)\chi_\nu\|_{L^\infty}\leq C\nu^{1/2}$, Assumption (iii), and (4-34), we have

$$\|\xi_j(\partial_X\phi)^{\boldsymbol{l}}(\partial_Y\Omega)^{\boldsymbol{j}-\boldsymbol{l}}\|_{L^2(0,\tau_0;L^2)}\leq CC_2^*\nu^{\frac12}(M_{2,l+1}[\partial_Y\phi]+\kappa\nu^{\frac12}(l+1)M_{2,l}[\nabla\phi]),\quad l=j-1.$$

Collecting these, we obtain the term $R_{j,\text{Lemma 4.10}}[\nabla\phi]$ by noticing ${}_jC_l = j$ for $l = j-1$, as desired. The estimate (4-42) is proved as in (4-37) but by also using the Young inequality for convolution in the l^1 space together with the following estimates for $j \geq 2$:

$$(j+1)^{\frac{1}{2}} \min\{l+1,\, j-l+1\} \frac{(l+1)^{3/2}}{(j+1)^{1/2}(l+2)^{1/2}} \left(\frac{(j-l)!\,l!}{j!}\right)^{\frac{1}{2}} \leq C, \quad 0 \leq l \leq j-2,$$

$$(j+1)^{\frac{1}{2}} \min\{l+1,\, j-l+1\} \frac{l+1}{(j+1)^{1/2}(l+1)^{1/2}} \left(\frac{(j-l)!\,l!}{j!}\right)^{\frac{1}{2}} \leq C, \quad 0 \leq l \leq j-2.$$

Note that the condition $l \leq j-2$ is crucial here, for we apply (4-41). We omit the details. $\square$

Lemma 4.11. *There exists $K_{1,4} = K_{1,4}(C_1^*, C_*) \geq 1$ such that, for $K \geq K_{1,4}$,*

$$\int_0^{\tau_0} \langle \operatorname{rot} F^j - [B_{j_2}, \partial_Y]\partial_X^{j_1} e^{-K\tau\nu^{1/2}(j+1)} F_1, \xi_j^2 \omega^j \rangle \, d\tau$$
$$\leq C(C_2^*+1) M_{2,j,\xi_j}[F] (\|\xi_j (\nabla\omega)^j\|_{L^2(0,\tau_0;L^2_{X,Y})} + \kappa\nu^{\frac{1}{2}} j M_{2,j-1,\xi_{j-1}}[\partial_Y\omega] + (j+1)^{\frac{1}{2}} \|\xi_j\omega^j\|_{L^2(0,\tau_0;L^2_{X,Y})})$$

and

$$\int_0^{\tau_0} \langle G^j, \xi_j^2 \omega^j \rangle \, d\tau \leq M_{2,j,\xi_j}[G] \|\xi_j \omega^j\|_{L^2(0,\tau_0;L^2_{X,Y})}.$$

Here $C > 0$ is a universal constant.

Proof. The estimate about G^j is straightforward and we focus on the estimate about F^j. Integration by parts and also (4-20) yield

$$\int_0^{\tau_0} \langle \operatorname{rot} F^j - [B_{j_2}, \partial_Y]\partial_X^{j_1} e^{-K\tau\nu^{1/2}(j+1)} F_1, \xi_j^2 \omega^j \rangle \, d\tau$$
$$= \int_0^{\tau_0} \langle F^j, \nabla^\perp(\xi_j^2 \omega^j) \rangle + \nu^{\frac{1}{2}} j_2 \langle \chi'_\nu F_1^j, \xi_j^2 e^{-K\tau\nu^{1/2}} (\partial_Y\omega)^{(j_1,j_2-1)} \rangle \, d\tau.$$

The second term is bounded from above by $C\kappa\nu^{1/2} j_2 \|\xi_j F_1^j\|_{L^2(0,\tau_0;L^2)} M_{2,j-1,\xi_{j-1}}[\partial_Y\omega]$, and thus we focus on the first term:

$$\int_0^{\tau_0} \langle F^j, \nabla^\perp(\xi_j^2 \omega^j) \rangle \, d\tau$$
$$= \int_0^{\tau_0} \langle F^j \cdot \nabla^\perp(\xi_j^2), \omega^j \rangle + \langle F^j, \xi_j^2 (\nabla^\perp\omega)^j \rangle + \nu^{\frac{1}{2}} j_2 \langle F_1^j, \xi_j^2 \chi'_\nu e^{-K\tau\nu^{1/2}} (\partial_Y\omega)^{(j_1,j_2-1)} \rangle \, d\tau$$
$$\leq \int_0^{\tau_0} \langle F^j \cdot \nabla^\perp(\xi_j^2), \omega^j \rangle \, d\tau + M_{2,j,\xi_j}[F] \|\xi_j (\nabla\omega)^j\|_{L^2(0,\tau_0;L^2)} + C\kappa\nu^{\frac{1}{2}} j_2 M_{2,j,\xi_j}[F] M_{2,j-1,\xi_{j-1}}[\partial_Y\omega].$$

Then, from Assumption (iv) and Lemma 4.3, and by recalling

$$\nabla^\perp(\xi_j^2) = -\frac{\nabla^\perp(\partial_Y\Omega + 2\rho_j)}{\sqrt{\partial_Y\Omega + 2\rho_j}} \xi_j^3 = -\frac{\nabla^\perp\partial_Y\Omega}{\sqrt{\partial_Y\Omega + 2\rho_j}} \xi_j^3 - 2(\nabla^\perp\rho_j)\xi_j^4,$$

we have

$$
\begin{aligned}
&\langle F^{j}\cdot\nabla^{\perp}(\xi_j^2),\omega^{j}\rangle\\
&\le \|\xi_j F^{j}\|\Biggl(\biggl\|\frac{Y\nabla(\partial_Y\Omega+2\rho_j)}{(1+\nu^{1/2}Y)\sqrt{\partial_Y\Omega+2\rho_j}}\xi_j^2\biggr\|_{L^\infty(\{0<Y<(j+1)^{-1/2}\})}\biggl\|\frac{1+\nu^{1/2}Y}{Y}\omega^{j}\biggr\|_{L^2(\{0<Y<(j+1)^{-1/2}\})}\\
&\qquad+\biggl\|\frac{Y\nabla\partial_Y\Omega}{(1+\nu^{1/2}Y)\sqrt{\partial_Y\Omega+2\rho_j}}\biggr\|_{L^\infty(\{Y\ge(j+1)^{-1/2}\})}\biggl\|\frac{1+\nu^{1/2}Y}{Y}\xi_j\biggr\|_{L^\infty(\{Y\ge(j+1)^{-1/2}\})}\|\xi_j\omega^{j}\|\\
&\qquad+\|Y\partial_Y\rho_j\xi_j^2\|_{L^\infty}\biggl\|\frac{1}{Y}\biggr\|_{L^\infty(\{Y\ge(j+1)^{-1/2}\})}\|\xi_j\omega^{j}\|\Biggr)\\
&\le C\|\xi_j F^{j}\|\Biggl(\Bigl(C_2^*\|\xi_j^2\|_{L^\infty(\{0<Y<(j+1)^{-1/2}\})}+\|Y\partial_Y\rho_j\xi_j^2\|_{L^\infty}\|\xi_j\|_{L^\infty(\{0<Y<(j+1)^{-1/2}\})}\Bigr)\biggl\|\frac{\omega^{j}}{Y}\biggr\|\\
&\qquad+(C_2^*+1)(j+1)^{\frac12}\|\xi_j\omega^{j}\|\Biggr)\\
&\le C\|\xi_j F^{j}\|\biggl(\Bigl(\frac{C_2^*}{K^{1/4}C_*}+\frac{1}{(K^{1/4}C_*)^{1/2}}\Bigr)\|\partial_Y\omega^{j}\|+(C_2^*+1)(j+1)^{\frac12}\|\xi_j\omega^{j}\|\biggr).
\end{aligned}
$$

Thus, the estimate (4-24) for $\partial_Y\omega^{j}$ yields the desired estimate by taking K large enough depending only on C_1^* and C_*. □

Proof of Proposition 4.4. We are now in position to prove Proposition 4.4. Lemmas 4.5–4.11 imply that, by taking the supremum over $j_2=0,\dots,j$,

$$
\begin{aligned}
&\nu^{\frac14}M_{2,j,\xi_j}[\nabla\omega]+M_{\infty,j,\xi_j}[\omega]+(K\nu^{\frac12}(j+1))^{\frac12}M_{2,j,\xi_j}[\omega]\\
&\le C\Biggl(\sup_{j_2=0,\dots,j}\|\xi_j\omega^{j}(0)\|+\kappa\nu^{\frac14}\nu^{\frac12}jM_{2,j-1,\xi_{j-1}}[\nabla\omega]\\
&\quad+\frac{R_{j,\text{Lemma 4.7}}[\nabla\phi]}{\nu^{1/4}(j+1)^{1/2}}+\frac{\kappa^{-1}R_{j,\text{Lemma 4.9}}[\omega]+R_{j,\text{Lemma 4.10}}[\nabla\phi]+M_{2,j,\xi_j}[G]}{(K\nu^{1/2}(j+1))^{1/2}}+(C_2^*+1)\nu^{-\frac14}M_{2,j,\xi_j}[F]\Biggr)
\end{aligned}
$$

for $j=0,1,\dots,\nu^{-1/2}$. Here $K\ge1$ is taken large enough depending only on C_* and C_j^*, while $C>0$ is a universal constant. Hence, by taking the sum $\sum_{j=0}^{\nu^{-1/2}}$ with the factor $1/((j!)^{3/2}\nu^{j/2})$, we obtain

$$
\begin{aligned}
&|||\nabla\omega|||'_{2,\tilde\xi^{(1)}}+|||\omega|||'_{\infty,\xi}+K^{\frac12}|||\omega|||'_{2,\xi}\\
&\quad\le C\Biggl(\sum_{j=0}^{\nu^{-1/2}}\frac{1}{(j!)^{3/2}\nu^{j/2}}\sup_{j_2=0,\dots,j}\|\xi_j\omega^{j}(0)\|+\kappa|||\nabla\omega|||'_{2,\tilde\xi^{(1)}}+\frac{C_0^*}{K^{1/2}\kappa}|||\omega|||'_{2,\xi}\\
&\qquad+\Bigl(1+\frac{C_0^*+C_2^*}{K^{1/2}}\Bigr)|||\nabla\phi|||'_{2,\mathbf{1}}+\frac{1}{K^{1/2}\nu^{1/2}}|||G|||'_{2,\tilde\xi^{(2)}}+\frac{C_2^*+1}{\nu^{1/2}}|||F|||'_{2,\tilde\xi^{(1)}}\Biggr).
\end{aligned}
$$

Thus we obtain (4-19) by first taking $\kappa>0$ small enough and then by taking K large enough, and also by using $\xi_j\le1/(C_*\nu^{1/2})\le1/\nu^{1/2}$ to bound $\|\xi_j\omega^{j}(0)\|$. Note that the required smallness on κ is independent of ν, K, C_*, and C_j^*, while the required largeness of K depends only on κ, C_*, and C_j^*. The proof of Proposition 4.4 is complete. □

4B. ***Estimate for the velocity in terms of the vorticity.*** In this subsection we give the estimate of the stream function ϕ in terms of the vorticity ω. We remind the reader that $\omega = -\Delta\phi$ with the boundary condition $\phi|_{Y=0} = 0$.

Proposition 4.12. *There exists $\kappa_2 \in (0,1]$ such that, for any $K \geq 1$, $\kappa \in (0,\kappa_2]$, and $p \in [1,\infty]$,*

$$|||\nabla\phi|||'_{p,\mathbf{1}} \leq C(K^{\frac14}C_* + C_1^*)^{\frac12}|||\omega|||'_{p,\xi} + C\nu^{1/(2p)}\|\nabla\phi^{(0,0)}\|_{L^p(0,1/(K\nu^{1/2});L^2_{X,Y})}.$$

Here $C > 0$ is a universal constant.

Proof. It suffices to show

$$\sum_{j=1}^{\nu^{-1/2}} \frac{\nu^{1/(2p)}(j+1)^{1/p}}{(j!)^{3/2}\nu^{j/2}} M_{p,j,1}[\nabla\phi] \\ \leq C(K^{\frac14}C_* + C_1^*)^{\frac12}|||\omega|||'_{p,\xi} + C\nu^{1/(2p)}\|\nabla\phi^{(0,0)}\|_{L^p(0,1/(K\nu^{1/2});L^2_{X,Y})}. \tag{4-43}$$

Let $j \geq 1$, and let us recall that $\omega^{\boldsymbol{j}} = e^{-K\tau\nu^{1/2}(j+1)}B_{j_2}\partial_X^{j-j_2}\omega$ with $\omega = -\Delta\phi$. Computations similar to those in (4-21) imply

$$\omega^{\boldsymbol{j}} = -\nabla\cdot(\nabla\phi)^{\boldsymbol{j}} + \frac{\nu^{1/2}j_2\chi'_\nu}{\chi_\nu}(\partial_Y\phi)^{\boldsymbol{j}}.$$

Then integration by parts together with the identity

$$\nabla\phi^{\boldsymbol{j}} = (\nabla\phi)^{\boldsymbol{j}} + \nu^{\frac12}j_2\chi'_\nu e^{-K\tau\nu^{1/2}}(\partial_Y\phi)^{(j-j_2,j_2-1)}\boldsymbol{e}_2$$

yields

$$\langle\omega^{\boldsymbol{j}},\phi^{\boldsymbol{j}}\rangle = \|(\nabla\phi)^{\boldsymbol{j}}\|^2 + 2\nu^{\frac12}j_2e^{-K\tau\nu^{1/2}}\langle\chi'_\nu(\partial_Y\phi)^{\boldsymbol{j}},(\partial_Y\phi)^{(j-j_2,j_2-1)}\rangle. \tag{4-44}$$

Then $\langle\omega^{\boldsymbol{j}},\phi^{\boldsymbol{j}}\rangle \leq \|\xi_j\omega^{\boldsymbol{j}}\|\|\phi^{\boldsymbol{j}}/\xi_j\|$, and the definition of ξ_j in (4-9) gives

$$\left\|\frac{\phi^{\boldsymbol{j}}}{\xi_j}\right\| = \|\sqrt{\partial_Y\Omega + 2\rho_j}\phi^{\boldsymbol{j}}\| \leq \left\|\left(\frac{1+Y}{1+\nu^{1/2}Y}\right)^2\partial_Y\Omega\right\|_{L^\infty}^{1/2}\left\|\frac{1+\nu^{1/2}Y}{1+Y}\phi^{\boldsymbol{j}}\right\| + \sqrt2\|\sqrt{\rho_j}\phi^{\boldsymbol{j}}\| \\ \leq (C_1^*)^{\frac12}(C\|\partial_Y\phi^{\boldsymbol{j}}\| + \nu^{\frac12}\|\phi^{\boldsymbol{j}}\|) + \sqrt2\|\sqrt{\rho_j}\phi^{\boldsymbol{j}}\|.$$

Here we have used Assumption (iii) and the Hardy inequality. Next the definition of ρ_j in (4-10) implies

$$\sqrt{\rho_j} \leq K^{\frac18}C_*^{\frac12}(1+(j+1)^{\frac12}Y)^{-1} + C_*^{\frac12}\left(\left(1+\frac{Y}{\nu^{1/4}}\right)^{-1} + \nu^{\frac14}(1+Y)^{-1} + \nu^{\frac12}\right),$$

which gives, from the Hardy inequality, $\nu^{1/2}(j+1) \leq 2$, and $K \geq 1$,

$$\|\sqrt{\rho_j}\phi^{\boldsymbol{j}}\| \leq CK^{\frac18}C_*^{\frac12}(j+1)^{-\frac12}\|\partial_Y\phi^{\boldsymbol{j}}\| + C_*^{\frac12}\nu^{\frac12}\|\phi^{\boldsymbol{j}}\|.$$

Thus we have

$$\left\|\frac{\phi^{\boldsymbol{j}}}{\xi_j}\right\| \leq C(C_1^* + K^{\frac14}C_*)^{\frac12}\|\partial_Y\phi^{\boldsymbol{j}}\| + C(C_1^* + C_*)^{\frac12}\nu^{\frac12}\|\phi^{\boldsymbol{j}}\|.$$

Thus (4-44) and the identity $\partial_Y \phi^{j} = (\partial_Y \phi)^{j} + \nu^{1/2} j_2 \chi_\nu' e^{-K\tau\nu^{1/2}} (\partial_Y \phi)^{(j-j_2, j_2-1)}$ finally give

$$\|(\nabla\phi)^{j}\| \le C(C_1^* + K^{\frac14} C_*)^{\frac12} \|\xi_j \omega^{j}\| + C\kappa \nu^{\frac12} j_2 \|(\partial_Y\phi)^{(j-j_2,j_2-1)}\| + \tfrac{1}{16}\nu^{\frac12}\|\phi^{j}\|.$$

Here $C > 0$ is a universal constant. Taking the supremum over $j_2 = 0, \dots, j$ yields

$$M_{p,j,1}[\nabla\phi] \le C(C_1^* + K^{\frac14} C_*)^{\frac12} M_{p,j,\xi_j}[\omega] + C\kappa\nu^{\frac12} j M_{p,j-1,1}[\nabla\phi] + \tfrac{1}{16}\nu^{\frac12} M_{p,j,1}[\phi].$$

Thus we have, from $M_{p,j,1}[\phi] \le M_{p,j-1,1}[\nabla\phi]$ and $(j+1)/j \le 2$ for $j \ge 1$,

$$\sum_{j=1}^{\nu^{-1/2}} \frac{\nu^{1/(2p)}(j+1)^{1/p}}{(j!)^{3/2}\nu^{j/2}} M_{p,j,1}[\nabla\phi] \\ \le C(K^{\frac14}C_* + C_1^*)^{\frac12} |||\omega|||'_{p,\xi} + \Big(C\kappa + \frac18\Big) \sum_{j=0}^{\nu^{-1/2}} \frac{\nu^{1/(2p)}(j+1)^{1/p}}{(j!)^{3/2}\nu^{j/2}} M_{p,j,1}[\nabla\phi].$$

Here $C > 0$ is a universal constant. By taking κ small enough we obtain (4-43). □

In view of the estimate in Proposition 4.12, our next task is to show the estimate of the zeroth-order term $\nabla\phi^{(0,0)}$.

Proposition 4.13. *Let $\kappa_2 \in (0, 1]$ be the number in Proposition 4.12. There exists $K_2 = K_2(C_*, C_1^*) \ge 1$ such that, for any $K \ge K_2$ and $\kappa \in (0, \kappa_2]$,*

$$\nu^{\frac14}\|\omega^{(0,0)}\|_{L^2(0,1/(K\nu^{1/2});L^2_{X,Y})} + \|\nabla\phi^{(0,0)}\|_{L^\infty(0,1/(K\nu^{1/2});L^2_{X,Y})} + K^{\frac12}\nu^{\frac14}\|\nabla\phi^{(0,0)}\|_{L^2(0,1/(K\nu^{1/2});L^2_{X,Y})} \\ \le C\bigg(\|W_0\|_{L^2_{X,Y}} + \frac{1}{K^{1/2}\nu^{1/4}}\|F\|_{L^2(0,1/(K\nu^{1/2});L^2_{X,Y})} + \frac{1}{K^{1/2}\nu^{1/4}}\|G\|_{L^2(0,1/(K\nu^{1/2});\dot H^{-1})} + |||\omega|||'_{2,\xi}\bigg). \tag{4-45}$$

Here $C > 0$ is a universal constant.

Proof. It suffices to show

$$\nu^{\frac14}\|\omega^{(0,0)}\|_{L^2(0,1/(K\nu^{1/2});L^2_{X,Y})} + \|\nabla\phi^{(0,0)}\|_{L^\infty(0,1/(K\nu^{1/2});L^2_{X,Y})} + K^{\frac12}\nu^{\frac14}\|\nabla\phi^{(0,0)}\|_{L^2(0,1/(K\nu^{1/2});L^2_{X,Y})} \\ \le C\bigg(\|W_0\|_{L^2_{X,Y}} + \frac{1}{K^{1/2}\nu^{1/4}}\|F\|_{L^2(0,1/(K\nu^{1/2});L^2_{X,Y})} + \frac{1}{K^{1/2}\nu^{1/4}}\|G\|_{L^2(0,1/(K\nu^{1/2});\dot H^{-1})} \\ + \frac{C_1^*}{K^{1/2}}|||\partial_Y\phi|||'_{2,\mathbf{1}}\bigg). \tag{4-46}$$

Indeed, estimate (4-45) is a direct consequence of (4-46) and Proposition 4.12 by taking K large enough depending only on C_1^* and C_*. To prove (4-45), let us go back to (4-1) and take the inner product with $\eta_R\phi$ for (4-1), where $\eta_R = \eta(Y/R)$ with a smooth cut-off η such that $\eta = 1$ for $0 \le Y \le 1$ and $\eta = 0$ for $Y \ge 1$. Then, taking the limit $R \to \infty$ after integration by parts verifies the identity

$$\nu^{\frac12}\|\omega^{(0,0)}\|^2 + \frac12\frac{d}{d\tau}\|\nabla\phi^{(0,0)}\|^2 + K\nu^{\frac12}\|\nabla\phi^{(0,0)}\|^2 \\ = -\langle\Delta\phi^{(0,0)}, V\cdot\nabla\phi^{(0,0)}\rangle + \langle F^{(0,0)}, \nabla^\perp\phi^{(0,0)}\rangle + \langle G^{(0,0)}, \phi^{(0,0)}\rangle, \quad \tau > 0. \tag{4-47}$$

Note that $|\langle F^{(0,0)},\nabla^\perp\phi^{(0,0)}\rangle|\le\|F\|\|\nabla\phi^{(0,0)}\|$ and $|\langle G^{(0,0)},\phi^{(0,0)}\rangle|\le\|G\|_{\dot H^{-1}}\|\nabla\phi^{(0,0)}\|$. Thus it suffices to focus on the term $-\langle\Delta\phi^{(0,0)},V\cdot\nabla\phi^{(0,0)}\rangle$. Integration by parts and $\nabla\cdot V=0$ imply

$$\begin{aligned}
-\langle\Delta\phi^{(0,0)},V\cdot\nabla\phi^{(0,0)}\rangle
&=\langle\partial_X\phi^{(0,0)},(\partial_X V)\cdot\nabla\phi^{(0,0)}\rangle+\langle\partial_Y\phi^{(0,0)},(\partial_Y V)\cdot\nabla\phi^{(0,0)}\rangle\\
&=\langle\partial_X\phi^{(0,0)},(\partial_X V)\cdot\nabla\phi^{(0,0)}\rangle-\langle\partial_Y\phi^{(0,0)},(\partial_X V_2)\partial_Y\phi^{(0,0)}\rangle+\langle\partial_Y\phi^{(0,0)},(\partial_Y V_1)\partial_X\phi^{(0,0)}\rangle\\
&\le 2C_1^*\nu^{\frac12}\|\nabla\phi^{(0,0)}\|^2+\langle\partial_Y\phi^{(0,0)},(\partial_Y V_1)\partial_X\phi^{(0,0)}\rangle.
\end{aligned}$$

Here we have used Assumption (ii). Then the last term is estimated as

$$\begin{aligned}
\langle\partial_Y\phi^{(0,0)},(\partial_Y V_1)\partial_X\phi^{(0,0)}\rangle&\le\left\|\frac{Y}{1+\nu^{1/2}Y}\partial_Y V_1\right\|_{L^\infty}\|\partial_Y\phi^{(0,0)}\|\left\|\frac{1+\nu^{1/2}Y}{Y}\partial_X\phi^{(0,0)}\right\|\\
&\le C_1^*\|\partial_Y\phi^{(0,0)}\|(C\|\partial_{XY}^2\phi^{(0,0)}\|+\nu^{\frac12}\|\partial_X\phi^{(0,0)}\|).
\end{aligned}$$

Here we have used Assumption (ii) and the Hardy inequality. Hence, by taking K large enough depending only on C_1^*, we obtain

$$\nu^{\frac12}\|\omega^{(0,0)}\|^2+\frac12\frac{d}{d\tau}\|\nabla\phi^{(0,0)}\|^2+K\nu^{\frac12}\|\nabla\phi^{(0,0)}\|^2\le\frac{C(C_1^*)^2}{K\nu^{1/2}}\|\partial_X\partial_Y\phi^{(0,0)}\|^2+C(\|F\|^2+\|G\|_{\dot H^{-1}}^2).$$

Integrating about τ shows (4-46), for $\nu^{-1/2}\|\partial_X\partial_Y\phi^{(0,0)}\|^2_{L^2(0,1/(K\nu^{1/2});L^2_{X,Y})}\le(|||\partial_Y\phi^{(0,0)}|||'_{2,\mathbf{1}})^2$ holds. □

4C. *Proof of Proposition 4.1*. Propositions 4.12 and 4.13 yield

$$\begin{aligned}
K^{\frac14}|||\nabla\phi|||'_{2,\mathbf{1}}\le C\biggl(K^{\frac14}(K^{\frac14}C_*+C_1^*)^{\frac12}|||\omega|||'_{2,\boldsymbol{\xi}}+\|W_0\|_{L^2_{X,Y}}&\\
+\frac{1}{K^{1/2}\nu^{1/4}}\|F\|_{L^2(0,1/(K\nu^{1/2});L^2_{X,Y})}+\frac{1}{K^{1/2}\nu^{1/4}}\|G\|_{L^2(0,1/(K\nu^{1/2});\dot H^{-1})}\biggr).&\qquad(4\text{-}48)
\end{aligned}$$

Then (4-48) and Proposition 4.4 give

$$\begin{aligned}
&|||\omega|||'_{\infty,\xi}+K^{\frac12}|||\omega|||'_{2,\xi}+K^{\frac14}|||\nabla\phi|||'_{2,\mathbf{1}}\\
&\quad\le C\biggl(\|W_0\|_{L^2_{X,Y}}+\nu^{-}[\|\mathrm{rot}\,W_0\|]+(C_2^*+1)\nu^{-\frac12}|||F|||'_{2,\tilde\xi^{(1)}}\\
&\qquad\qquad+\frac{1}{K^{1/2}\nu^{1/2}}|||G|||'_{2,\tilde\xi^{(2)}}+\frac{1}{K^{1/2}\nu^{1/4}}\|G\|_{L^2(0,1/(K\nu^{1/2});\dot H^{-1})}\biggr).\qquad(4\text{-}49)
\end{aligned}$$

It remains to estimate the boundary trace $|||\partial_Y\phi|_{Y=0}|||_{\mathrm{bc}}$. By the interpolation inequality we have

$$|\partial_X^j\partial_Y\phi(\tau,X,0)|\le C\|\partial_X^j\partial_Y^2\phi(\tau,X,\cdot)\|_{L^2_Y}^{1/2}\|\partial_X^j\partial_Y\phi(\tau,X,\cdot)\|_{L^2_Y}^{1/2},$$

which implies

$$\begin{aligned}
&K^{\frac14}\|\partial_Y\phi^{(j,0)}|_{Y=0}\|_{L^2(0,1/(K\nu^{1/2});L^2_X)}\\
&\quad\le CK^{\frac14}\|\partial_Y^2\phi^{(j,0)}\|^{\frac12}_{L^2(0,1/(K\nu^{1/2});L^2_{X,Y})}\|\partial_Y\phi^{(j,0)}\|^{\frac12}_{L^2(0,1/(K\nu^{1/2});L^2_{X,Y})}\\
&\quad\le C(K^{\frac14}\|\omega^{(j,0)}\|_{L^2(0,1/(K\nu^{1/2});L^2_{X,Y})})^{\frac12}(K^{\frac14}\|\partial_Y\phi^{(j,0)}\|_{L^2(0,1/(K\nu^{1/2});L^2_{X,Y})})^{\frac12}.\qquad(4\text{-}50)
\end{aligned}$$

Here we used the Calderón–Zygmund inequality. Since (4-18) yields

$$\|\omega^{(j,0)}\|_{L^2(0,1/(K\nu^{1/2});L^2_{X,Y})} \le (C_1^* + 8K^{\frac14}C_*)^{\frac12} M_{2,j,.\xi_j}[\omega],$$

we have from (4-49) that, by taking K further large enough if necessary,

$$\begin{aligned} K^{\frac14}|||\partial_Y\phi|_{Y=0}|||_{\mathrm{bc}} &\le C(K^{\frac12}|||\omega|||'_{2,\xi})^{\frac12}(K^{\frac14}|||\nabla\phi|||'_{2,\mathbf{1}})^{\frac12} \\ &\le C\biggl(\|W_0\|_{L^2_{X,Y}} + \nu^{-}[\|\mathrm{rot}\, W_0\|] + (C_2^*+1)\nu^{-\frac12}|||F|||'_{2,\tilde\xi^{(1)}} \\ &\qquad + \frac{1}{K^{1/2}\nu^{1/2}}|||G|||'_{2,\tilde\xi^{(2)}} + \frac{1}{K^{1/2}\nu^{1/4}}\|G\|_{L^2(0,1/(K\nu^{1/2});\dot H^{-1})}\biggr). \end{aligned}$$

The proof of Proposition 4.1 is complete. □

5. Construction of the boundary corrector

In the previous section, we constructed a solution to the vorticity equation with arbitrary initial data but artificial boundary conditions: we replaced condition $W_1|_{Y=0} = 0$ by $\omega|_{Y=0} = 0$. Hence, to prove Theorem 2.3, we still need to understand how to correct the Neumann condition, that is how to construct solutions for systems of the following type:

$$\begin{aligned} -\nu^{\frac12}\Delta\omega + \partial_\tau\omega + V\cdot\nabla\omega + \nabla^\perp\phi\cdot\nabla\Omega = 0, &\quad \tau>0,\quad X\in\mathbb{T}_\nu,\quad Y>0, \\ \phi|_{Y=0}=0,\quad \partial_Y\phi|_{Y=0}=h,\quad \phi|_{\tau=0}=0.& \end{aligned} \tag{5-1}$$

Here $\phi(\tau,\cdot)$ is the stream function associated with the vorticity $\omega(\tau,\cdot)$, i.e., $\phi\in\dot H^1_0(\mathbb{T}_\nu\times\mathbb{R}_+)$ is the unique solution to $-\Delta\phi=\omega$ subject to the zero Dirichlet boundary condition. Such a construction will be performed through an iteration, with first approximation given by the Stokes equation.

5A. ***Stokes estimate.*** In this subsection we consider the solution to the Stokes equations (in terms of the stream function):

$$\begin{aligned} -\nu^{\frac12}\Delta\omega + \partial_\tau\omega = 0, &\quad \tau>0,\quad X\in\mathbb{T}_\nu,\quad Y>0, \\ \phi|_{Y=0}=0,\quad \partial_Y\phi|_{Y=0}=h,\quad \phi|_{\tau=0}=0.& \end{aligned} \tag{5-2}$$

Here $\phi\in\dot H^1_0(\mathbb{T}_\nu\times\mathbb{R}_+)$ is the stream function associated with ω, and h is a given boundary data satisfying $h(\tau)=0$ for $\tau=0$ and $\tau\ge 1/(K\nu^{1/2})$, and the norm $|||h|||_{\mathrm{bc}}$ is defined as

$$|||h|||_{\mathrm{bc}} = \sum_{j=0}^{\nu^{-1/2}} \frac{\nu^{1/4}(j+1)^{1/2}}{(j!)^{3/2}\nu^{j/2}}\|e^{-K\tau\nu^{1/2}(j+1)}\partial_X^j h\|_{L^2(0,1/(K\nu^{1/2});L^2_X)} < \infty. \tag{5-3}$$

Set $\psi = e^{-K\tau\nu^{1/2}(j+1)}\partial_X^{j_1}\phi$, $0\le j_1\le j$, with the zero extension for $\tau\le 0$, and let $\hat\psi=\hat\psi(\lambda,\alpha,Y)$ be the Fourier (in X and τ) transform of ψ. Then, since $-\Delta\phi=\omega$, the function $\hat\psi$ obeys the ODE

$$\begin{aligned} \nu^{\frac12}(\partial_Y^2-\alpha^2)^2\hat\psi - (i\lambda + K\nu^{\frac12}(j+1))(\partial_Y^2-\alpha^2)\hat\psi = 0, &\quad Y>0, \\ \hat\psi|_{Y=0}=0,\quad \partial_Y\hat\psi|_{Y=0}=\hat g^{(j_1)},& \end{aligned} \tag{5-4}$$

where $\lambda \in \mathbb{R}$ and $\hat{g}^{(j_1)}$ is the Fourier transform of $g^{(j_1)} := e^{-K\tau\nu^{1/2}(j+1)}\partial_X^{j_1} h$. We note that

$$\alpha = \nu^{\frac{1}{2}} n, \tag{5-5}$$

where n is the n-th Fourier mode in the original variable $x \in \mathbb{T}$. Assuming the decay of $(|\alpha|\hat{\psi}, \partial_Y \hat{\psi})$ and the boundedness of $\hat{\psi}$, we obtain the formula

$$\begin{aligned} \hat{\psi}(\lambda, \alpha, Y) &= -\frac{e^{-\gamma Y} - e^{-|\alpha| Y}}{\gamma - |\alpha|} \hat{g}^{(j_1)}(\lambda, \alpha), \\ \gamma = \gamma_j(\lambda, \alpha, \nu, K) &= \sqrt{\alpha^2 + K(j+1) + \frac{i\lambda}{\nu^{1/2}}}, \end{aligned} \tag{5-6}$$

where the square root is taken so that the real part is positive, and it follows that

$$|\alpha| \le \sqrt{\alpha^2 + K(j+1)} \le \operatorname{Re}(\gamma) \le |\gamma| \le \sqrt{2}\operatorname{Re}(\gamma). \tag{5-7}$$

This inequality will be freely used. We can also check the identity

$$\partial_Y \hat{\psi}(\lambda, \alpha, Y) = -e^{-\gamma Y} \hat{g}^{(j_1)}(\lambda, \alpha) + \operatorname{sgn}(\alpha)\alpha \hat{\psi}(\lambda, \alpha, Y). \tag{5-8}$$

We also have, from (5-6),

$$-(\partial_Y^2 - \alpha^2)\hat{\psi} = (\gamma + |\alpha|) e^{-\gamma Y} \hat{g}^{(j_1)}. \tag{5-9}$$

This formula will be used in estimating the vorticity field.

Lemma 5.1. *There exists $\kappa' \in (0, 1]$ such that the following statement holds for any $\kappa \in (0, \kappa']$. Let $j_1 = 0, \dots, j$ and $j_2 = j - j_1$. Then*

$$|B_{j_2} i\alpha \hat{\psi}(\lambda, \alpha, Y)| \le \frac{C\nu^{j_2/2} j_2! |\alpha \hat{g}^{(j_1)}|}{j_2 + 1} \left(Y e^{-\operatorname{Re}(\gamma) Y/2} + e^{-|\alpha| Y/2} \left| \frac{1 - e^{-(\gamma - |\alpha|)Y}}{\gamma - |\alpha|} \right| \right), \tag{5-10}$$

$$|B_{j_2} \partial_Y \hat{\psi}(\lambda, \alpha, Y)| \le \frac{C\nu^{j_2/2} j_2! |\hat{g}^{(j_1)}|}{j_2 + 1} e^{-\operatorname{Re}(\gamma) Y/2}. \tag{5-11}$$

As a consequence,

$$\begin{aligned} &\left(\sum_{\alpha \in \nu^{1/2}\mathbb{Z}} \|B_{j_2} i\alpha \hat{\psi}(\cdot, \alpha, \cdot)\|_{L^2_{\lambda, Y}}^2 + \|B_{j_2} \partial_Y \hat{\psi}(\cdot, \alpha, \cdot)\|_{L^2_{\lambda, Y}}^2 \right)^{\frac{1}{2}} \\ &\qquad \le \frac{C\nu^{j_2/2} j_2!}{K^{1/4}(j+1)^{1/4}(j_2+1)} \left(\sum_{\alpha \in \nu^{1/2}\mathbb{Z}} \|\hat{g}^{(j_1)}(\cdot, \alpha)\|_{L^2_\lambda}^2 \right)^{\frac{1}{2}}. \end{aligned} \tag{5-12}$$

We also have

$$\left(\sum_{\alpha \in \nu^{1/2}\mathbb{Z}} \left\| \frac{1}{1+Y} B_{j_2} i\alpha \hat{\psi}(\cdot, \alpha, \cdot) \right\|_{L^2_{\lambda, Y}}^2 \right)^{\frac{1}{2}} \le \frac{C\nu^{j_2/2} j_2!}{K^{1/2}(j+1)^{1/2}(j_2+1)} \left(\sum_{\alpha \in \nu^{1/2}\mathbb{Z}} \|\alpha \hat{g}^{(j_1)}(\cdot, \alpha)\|_{L^2_\lambda}^2 \right)^{\frac{1}{2}}. \tag{5-13}$$

Here $C > 0$ is a universal constant.

Proof. We first show (5-10) for $B_{j_2} i\alpha\hat{\psi}$. It suffices to consider the case $j_2 \geq 1$, for the case $j_2 = 0$ is trivial from (5-6). We observe from (5-6) that

$$B_{j_2}\hat{\psi} = -\frac{\hat{g}^{(j_1)}\chi_\nu^{j_2}}{\gamma - |\alpha|}((-\gamma)^{j_2}e^{-\gamma Y} - (-|\alpha|)^{j_2}e^{-|\alpha|Y})$$
$$= -\frac{(-\gamma)^{j_2} - (-|\alpha|)^{j_2}}{\gamma - |\alpha|}\chi_\nu^{j_2}e^{-\gamma Y}\hat{g}^{(j_1)} + (-|\alpha|)^{j_2}\chi_\nu^{j_2}e^{-|\alpha|Y}\hat{g}^{(j_1)}\frac{1 - e^{-(\gamma-|\alpha|)Y}}{\gamma - |\alpha|}. \tag{5-14}$$

Since

$$(-\gamma)^{j_2} - (-|\alpha|)^{j_2} = (-1)^{j_2}\sum_{l_2=0}^{j_2-1}\binom{j_2}{l_2}(\gamma - |\alpha|)^{j_2-l_2}|\alpha|^{l_2},$$

we have, from $\binom{j_2}{l_2} \leq j_2\binom{j_2-1}{l_2}$ for $0 \leq l_2 \leq j_2 - 1$,

$$\left|\frac{(-\gamma)^{j_2} - (-|\alpha|)^{j_2}}{\gamma - |\alpha|}\right| \leq \sum_{l_2=0}^{j_2-1}\binom{j_2}{l_2}\big|\gamma - |\alpha|\big|^{j_2-l_2-1}|\alpha|^{l_2} \leq j_2\sum_{l_2=0}^{j_2-1}\binom{j_2-1}{l_2}\big|\gamma - |\alpha|\big|^{j_2-l_2-1}|\alpha|^{l_2}$$
$$= j_2\big(\big|\gamma - |\alpha|\big| + |\alpha|\big)^{j_2-1} \leq j_2(3|\gamma|)^{j_2-1}.$$

Here we have used $|\alpha| \leq |\gamma|$ by (5-7). Then the inequality $\chi_\nu = 1 - e^{-\kappa\nu^{1/2}Y} \leq \kappa\nu^{1/2}Y$ implies

$$\left|\frac{(-\gamma)^{j_2} - (-|\alpha|)^{j_2}}{\gamma - |\alpha|}\chi_\nu^{j_2}e^{-\gamma Y}\right| \leq j_2\kappa\nu^{\frac{1}{2}}Y(3\kappa\nu^{\frac{1}{2}}|\gamma|Y)^{j_2-1}e^{-\operatorname{Re}(\gamma)Y}$$
$$\leq j_2\kappa\nu^{j_2/2}Y(3\sqrt{2}\kappa\operatorname{Re}(\gamma)Y)^{j_2-1}e^{-\operatorname{Re}(\gamma)Y} \quad \text{(by (5-7))}.$$

From the bound $r^k e^{-r} \leq (k/e)^k$ and the Stirling bound $(k/e)^k \leq (2\pi)^{-1/2}k^{-1/2}k!$ for $k \in \mathbb{N}$, we have

$$\big(\tfrac{1}{2}\operatorname{Re}(\gamma)Y\big)^{j_2-1}e^{-\operatorname{Re}(\gamma)Y/2} \leq \frac{(j_2-1)!}{\sqrt{2\pi}(j_2-1)^{1/2}}, \qquad j_2 \geq 2.$$

This gives, when $6\sqrt{2}\kappa \leq \frac{1}{2}$,

$$\left|\frac{(-\gamma)^{j_2} - (-|\alpha|)^{j_2}}{\gamma - |\alpha|}\chi_\nu^{j_2}e^{-\gamma Y}\right| \leq \frac{\nu^{j_2/2}j_2!}{(j_2+1)}Ye^{-\operatorname{Re}(\gamma)Y/2}, \qquad j_2 \geq 1.$$

Similarly, we have, for $j_2 \geq 1$,

$$|(-|\alpha|)^{j_2}\chi_\nu^{j_2}e^{-|\alpha|Y}| \leq \frac{\nu^{j_2/2}j_2!}{j_2+1}e^{-|\alpha|Y/2}.$$

Hence (5-10) for $B_{j_2} i\alpha\hat{\psi}$ follows by collecting these with (5-14). The estimate for $B_{j_2}\partial_Y\hat{\psi}$ is proved in the same manner in view of (5-8), and we omit the details. Estimate (5-12) follows from (5-10) and the Plancherel theorem, by observing the estimates for the multipliers

$$\|\alpha Ye^{-\operatorname{Re}(\gamma)Y/2}\|_{L^2_Y} \leq \frac{C}{K^{1/4}(j+1)^{1/4}}, \tag{5-15}$$

$$\left\|\alpha e^{-|\alpha|Y/2}\left|\frac{1 - e^{-(\gamma-|\alpha|)Y}}{\gamma - |\alpha|}\right|\right\|_{L^2_Y} \leq \frac{C}{K^{1/4}(j+1)^{1/4}}. \tag{5-16}$$

Here $C > 0$ is a universal constant. Estimate (5-15) is a consequence of (5-7). As for (5-16), we divide into two cases. (i) The case $|\alpha| \le \frac{1}{2}K^{1/2}(j+1)^{1/2}$: in this case we have, from (5-7),

$$|\gamma - |\alpha|| \ge |\gamma| - |\alpha| \ge \frac{|\alpha| + K^{1/2}(j+1)^{1/2}}{C}$$

with a universal constant $C > 0$, which gives

$$\begin{aligned}\left\| \alpha e^{-|\alpha|Y/2} \left| \frac{1 - e^{-(\gamma - |\alpha|)Y}}{\gamma - |\alpha|} \right| \right\|_{L^2_Y} &\le \frac{C}{|\alpha| + K^{1/2}(j+1)^{1/2}} \|\alpha e^{-|\alpha|Y/2}\|_{L^2_Y} \\ &\le \frac{C|\alpha|^{1/2}}{|\alpha| + K^{1/2}(j+1)^{1/2}} \le \frac{C}{K^{1/4}(j+1)^{1/4}}.\end{aligned}$$

(ii) The case $|\alpha| \ge \frac{1}{2}K^{1/2}(j+1)^{1/2}$: in this case we used the bound

$$\sup_{\mathrm{Re}(z)>0} \left| \frac{1-e^{-z}}{z} \right| \le C,$$

which gives

$$\left\| \alpha e^{-|\alpha|Y/2} \left| \frac{1 - e^{-(\gamma - |\alpha|)Y}}{\gamma - |\alpha|} \right| \right\|_{L^2_Y} \le C \|\alpha Y e^{-|\alpha|Y/2}\|_{L^2_Y} \le \frac{C}{|\alpha|^{1/2}} \le \frac{C}{K^{1/4}(j+1)^{1/4}}.$$

The proof of (5-16) is complete, and (5-12) is proved. Estimate (5-13) is proved similarly by using (5-10), the Plancherel theorem, and

$$\left\| \frac{Y}{1+Y} e^{-\mathrm{Re}(\gamma)Y/2} \right\|_{L^2_Y} \le \frac{C}{K^{3/4}(j+1)^{3/4}}, \tag{5-17}$$

$$\left\| \frac{1}{1+Y} e^{-|\alpha|Y/2} \left| \frac{1 - e^{-(\gamma - |\alpha|)Y}}{\gamma - |\alpha|} \right| \right\|_{L^2_Y} \le \frac{C}{K^{1/2}(j+1)^{1/2}}. \tag{5-18}$$

Here $C > 0$ is a universal constant. Indeed, (5-17) is straightforward, while in (5-18), the estimate becomes worse due to the case $|\alpha| \le \frac{1}{2}K^{1/2}(j+1)^{1/2}$ with $|\alpha| \ll 1$, where we compute

$$\left\| \frac{1}{1+Y} e^{-|\alpha|Y/2} \left| \frac{1 - e^{-(\gamma - |\alpha|)Y}}{\gamma - |\alpha|} \right| \right\|_{L^2_Y} \le \frac{C}{|\alpha| + K^{1/2}(j+1)^{1/2}} \left\| \frac{1}{1+Y} e^{-|\alpha|Y/2} \right\|_{L^2_Y} \le \frac{C}{K^{1/2}(j+1)^{1/2}}.$$

Here we essentially use the factor $1/(1+Y)$ to obtain the uniform estimate in α. □

In Propositions 5.2 and 5.4 below we give estimates for the solution to (5-1) given by the formula as above in terms of the Fourier transform. We always take κ small enough such that $\kappa \in (0, \kappa']$ as in Lemma 5.1.

Proposition 5.2 (estimate for velocity). *It follows that*

$$\sum_{j=0}^{\nu^{-1/2}} \frac{\nu^{1/4}(j+1)^{3/4}}{(j!)^{3/2}\nu^{j/2}} M_{2,j,1}[\nabla\phi] + \sum_{j=0}^{\nu^{-1/2}} \frac{1}{(j!)^{3/2}\nu^{j/2+1/4}(j+1)^{1/2}} M_{2,j,1/(1+Y)}[\partial_X \phi] \le \frac{C}{K^{1/4}} |||h|||_{\mathrm{bc}}. \tag{5-19}$$

Here $C > 0$ is a universal constant.

Proof. Assume that $M_{2,j,1}[\nabla\phi] = \|(\nabla\phi)^{\boldsymbol{j}}\|_{L^2(0,1/(K\nu^{1/2});L^2_{X,Y})}$ for some $\boldsymbol{j} = (j_1, j_2)$ with $j_1 + j_2 = j$. Note that this j_1 depends on j, and we write $j_1[j]$ if necessary. By the Plancherel theorem the estimate (5-12) implies

$$\|(\nabla\phi)^{\boldsymbol{j}}\|_{L^2(0,1/(K\nu^{1/2});L^2_{X,Y})} \le \frac{C\nu^{j-j_1[j]/2}(j-j_1[j])!}{K^{1/4}(j+1)^{1/4}(j-j_1[j]+1)}\|h^{(j_1)}\|_{L^2(0,1/(K\nu^{1/2});L^2_X)},$$
$$h^{(j_1)} = e^{-K\tau\nu^{1/2}(j_1+1)}\partial_X^{j_1}h.$$

Thus we have

$$\begin{aligned}
&\sum_{j=0}^{\nu^{-1/2}} \frac{\nu^{1/4}(j+1)^{3/4}}{(j!)^{3/2}\nu^{j/2}} M_{2,j,1}[\nabla\phi] \\
&\quad\le \frac{C}{K^{1/4}} \sum_{j=0}^{\nu^{-1/2}} \binom{j}{j-j_1[j]}^{-1} \frac{1}{(j-j_1[j]+1)} \left(\frac{j_1[j]!}{j!}\right)^{\frac{1}{2}} \left(\frac{j+1}{j_1[j]+1}\right)^{\frac{1}{2}} \\
&\qquad\qquad \times \left(\frac{\nu^{1/4}(j_1[j]+1)^{1/2}}{(j_1[j]!)^{3/2}\nu^{j_1[j]/2}}\|h^{(j_1[j])}\|_{L^2(0,1/(K\nu^{1/2});L^2_X)}\right).
\end{aligned}$$

We decompose the summation in the right-hand side as $\sum_{j_1[j]=j}$ (i.e., j's such that $0 \le j \le \nu^{-1/2}$ and $j_1[j] = j$) and $\sum_{j_1[j]\le j-1}$ (i.e., j's such that $0 \le j \le \nu^{-1/2}$ and $j_1[j] \le j-1$). Then the sum of $\sum_{j_1[j]=j}$ is bounded from above by $|||h|||_{\mathrm{bc}}$, while the sum of $\sum_{j_1[j]\le j-1}$ is bounded as

$$\begin{aligned}
&\sum_{j_1[j]\le j-1} \binom{j}{j-j_1[j]}^{-1} \frac{1}{(j-j_1[j]+1)} \left(\frac{j_1[j]!}{j!}\right)^{\frac{1}{2}} \left(\frac{j+1}{j_1[j]+1}\right)^{\frac{1}{2}} \\
&\qquad\qquad \times \left(\frac{\nu^{1/4}(j_1[j]+1)^{1/2}}{(j_1[j]!)^{3/2}\nu^{j_1[j]/2}}\|h^{(j_1[j])}\|_{L^2(0,1/(K\nu^{1/2});L^2_X)}\right) \\
&\quad\le \sum_{j_1[j]\le j-1} \binom{j}{j-j_1[j]}^{-1} \frac{1}{(j-j_1[j]+1)} \left(\frac{j_1[j]!}{j!}\right)^{\frac{1}{2}} \left(\frac{j+1}{j_1[j]+1}\right)^{\frac{1}{2}} \sup_{0\le k\le \nu^{-1/2}} \\
&\qquad\qquad \times \left(\frac{\nu^{1/4}(k+1)^{1/2}}{(k!)^{3/2}\nu^{k/2}}\|h^{(k)}\|_{L^2(0,1/(K\nu^{1/2});L^2_X)}\right) \\
&\quad\le C|||h|||_{\mathrm{bc}}.
\end{aligned}$$

Indeed, it suffices to use

$$\sum_{j_1[j]\le j-1} \binom{j}{j-j_1[j]}^{-1} \frac{1}{(j-j_1[j]+1)} \left(\frac{j_1[j]!}{j!}\right)^{\frac{1}{2}} \left(\frac{j+1}{j_1[j]+1}\right)^{\frac{1}{2}} \le C \sum_{j_1[j]\le j-1} (j+1)^{-\frac{3}{2}} \le C. \tag{5-20}$$

Next we prove the estimate about $M_{2,j,1/(1+Y)}[\partial_X\phi]$. Arguing as above, we have from (5-13) that, for $0 \le j \le \nu^{-1/2} - 1$,

$$M_{2,j,1/(1+Y)}[\partial_X\phi] \le \frac{C\nu^{(j-j_1[j])/2}(j-j_1[j])!}{K^{1/2}(j+1)^{1/2}(j-j_1[j]+1)}\|\partial_X h^{(j_1[j])}\|_{L^2(0,1/(K\nu^{1/2});L^2_X)},$$

where $j_1[j]$ is taken similarly as in the above argument. Thus we have

$$\begin{aligned}
&\sum_{j=0}^{\nu^{-1/2}} \frac{1}{(j!)^{3/2}\nu^{j/2+1/4}(j+1)^{1/2}} M_{2,j,1/(1+Y)}[\partial_X\phi] \\
&\quad\le \frac{C}{K^{1/2}} \sum_{j=0}^{\nu^{-1/2}-1} \frac{(j-j_1[j])!}{(j!)^{3/2}\nu^{j_1/2+1/4}(j+1)(j-j_1[j]+1)} \|\partial_X h^{(j_1)}\|_{L^2(0,1/(K\nu^{1/2});L^2_X)} \\
&\qquad\qquad + \frac{M_{2,j,1/(1+Y)}[\partial_X\phi]}{(j!)^{3/2}\nu^{j/2+1/4}(j+1)^{1/2}}\bigg|_{j=\nu^{-1/2}} \\
&\quad\le \frac{C}{K^{1/2}} \sum_{j=0}^{\nu^{-1/2}-1} \frac{(j-j_1[j])!}{(j!)^{3/2}\nu^{j_1/2+1/4}(j+1)(j-j_1[j]+1)} \|h^{(j_1+1)}\|_{L^2(0,1/(K\nu^{1/2});L^2_X)} \\
&\qquad\qquad + \frac{\nu^{1/4}(j+1)^{3/4}M_{2,j,1}[\partial_X\phi]}{(j!)^{3/2}\nu^{j/2}}\bigg|_{j=\nu^{-1/2}}.
\end{aligned}$$

The second term is bounded from above by $(C/K^{1/4})|||h|||_{\mathrm{bc}}$, as we have shown above. As for the first term, we again decompose the summation $\sum_{j=0}^{\nu^{-1/2}-1}$ into $\sum_{j_1[j]=j}$ and $\sum_{j_1[j]\le j-1}$, as we have done previously. Then the sum of $\sum_{j_1[j]=j}$ is bounded from above by $C|||h|||_{\mathrm{bc}}$, while the sum of $\sum_{j_1[j]\le j-1}$ is estimated as

$$\begin{aligned}
&\sum_{j_1[j]\le j-1} \frac{(j-j_1[j])!}{(j!)^{3/2}\nu^{j_1[j]/2+1/4}(j+1)(j-j_1[j]+1)} \|h^{(j_1[j]+1)}\|_{L^2(0,1/(K\nu^{1/2});L^2_X)} \\
&\quad\le \sum_{j_1[j]\le j-1} \frac{(j-j_1[j])!\,(j_1[j]+1)!}{j!}\left(\frac{(j_1[j]+1)!}{j!}\right)^{\frac{1}{2}} \frac{1}{(j+1)(j_1[j]+1)^{1/2}(j-j_1[j]+1)} \\
&\qquad\qquad \times \sup_{0\le k\le \nu^{-1/2}} \left(\frac{\nu^{1/4}(k+1)^{1/2}}{(k!)^{3/2}\nu^{k/2}}\|h^{(k)}\|_{L^2(0,1/(K\nu^{1/2});L^2_X)}\right) \\
&\quad\le C\sum_{j=0}^{\nu^{-1/2}} \frac{1}{(j+1)^{3/2}} \sup_{0\le k\le \nu^{-1/2}} \left(\frac{\nu^{1/4}(k+1)^{1/2}}{(k!)^{3/2}\nu^{k/2}}\|h^{(k)}\|_{L^2(0,1/(K\nu^{1/2});L^2_X)}\right) \le C|||h|||_{\mathrm{bc}}. \qquad \square
\end{aligned}$$

Next we show the estimate for the vorticity field. The argument is similar to the one for the velocity.

Lemma 5.3. *There exists $\kappa'' \in (0,1]$ such that the following statement holds for any $\kappa \in (0,\kappa'']$. Let $j_1 = 0,\dots,j$ and $j_2 = j - j_1$. Then*

$$|B_{j_2}(\partial_Y^2-\alpha^2)\hat{\psi}(\lambda,\alpha,Y)| + |YB_{j_2}\partial_Y(\partial_Y^2-\alpha^2)\hat{\psi}(\lambda,\alpha,Y)| \le \frac{C\nu^{j_2/2}j_2!}{j_2+1}|\gamma|e^{-\mathrm{Re}(\gamma)Y/2}|\hat{g}^{(j_1)}|. \tag{5-21}$$

As a consequence, for $\theta' \in \left[-\frac{1}{2}, 2\right]$,

$$\begin{aligned}
&\bigg(\sum_{\alpha\in\nu^{1/2}\mathbb{Z}} \|Y^{1+\theta'}B_{j_2}(\partial_Y^2-\alpha^2)\hat{\psi}(\cdot,\alpha,\cdot)\|^2_{L^2_{\lambda,Y}} + \|Y^{2+\theta'}B_{j_2}\alpha(\partial_Y^2-\alpha^2)\hat{\psi}(\cdot,\alpha,\cdot)\|^2_{L^2_{\lambda,Y}} \\
&\qquad\qquad + \|Y^{2+\theta'}B_{j_2}\partial_Y(\partial_Y^2-\alpha^2)\hat{\psi}(\cdot,\alpha,\cdot)\|^2_{L^2_{\lambda,Y}}\bigg)^{\frac{1}{2}} \\
&\quad\le \frac{C\nu^{j_2/2}j_2!}{K^{\theta'/2+1/4}(j+1)^{\theta'/2+1/4}(j_2+1)}\bigg(\sum_{\alpha\in\nu^{1/2}\mathbb{Z}} \|\hat{g}^{(j_1)}(\cdot,\alpha)\|^2_{L^2_\lambda}\bigg)^{\frac{1}{2}}.
\end{aligned} \tag{5-22}$$

Here $C > 0$ is a universal constant.

Proof. Estimate (5-21) follows from (5-9) by arguing as in the proof of (5-10). Estimate (5-22) then follows from (5-21), the Plancherel theorem, and

$$\|Y^{1+m}|\gamma|e^{-\operatorname{Re}(\gamma)Y/2}\|_{L^2_Y} \le \frac{C}{(\operatorname{Re}(\gamma))^{m+1/2}} \le \frac{C}{(|\alpha|+K^{1/2}(j+1)^{1/2})^{m+1/2}} \quad \text{(by (5-7))}$$

for $m \in \left[-\frac{1}{2}, 3\right]$. The details are omitted here. □

Proposition 5.4 (estimate for vorticity). *Let* $\theta \in [0, 2]$. *It follows that*

$$\sum_{j=0}^{\nu^{-1/2}} \frac{(j+1)^{1/4}}{(j!)^{3/2}\nu^{j/2}} \nu^{\frac{1}{4}}(j+1)^{\frac{1}{2}}(M_{2,j,Y}[\omega] + M_{2,j,Y^2}[\nabla\omega]) \le \frac{C}{K^{1/4}}|||h|||_{\mathrm{bc}} \tag{5-23}$$

and

$$\sum_{j=0}^{\nu^{-1/2}} \frac{(j+1)^{(\theta-1)/2}}{(j!)^{3/2}\nu^{j/2+1/4}}(M_{2,j,Y^{3/2+\theta}}[\partial_X\omega] + \nu^{\frac{1}{2}}M_{2,j,Y^{3/2+\theta}}[\partial_Y\omega]) \le \frac{C}{K^{\theta/2}}|||h|||_{\mathrm{bc}}. \tag{5-24}$$

Here $C > 0$ *is a universal constant.*

Proof. Estimate (5-23) is a consequence of (5-22) with $\theta' = 0$, by introducing $j_1[j]$ as in the proof of Proposition 5.2. As for (5-24), we have from (5-22) with $\theta' = \theta - \frac{1}{2}$ that

$$\sum_{j=0}^{\nu^{-1/2}} \frac{\nu^{1/4}(j+1)^{(\theta-1)/2}}{(j!)^{3/2}\nu^{j/2}} M_{2,j,Y^{3/2+\theta}}[\partial_Y\Delta\phi] \le \frac{C}{K^{\theta/2}}|||h|||_{\mathrm{bc}}.$$

Next we have from $M_{2,j,Y^{3/2+\theta}}[\partial_X\Delta\phi] \le CM_{2,j+1,Y^{3/2+\theta}}[\Delta\phi]$ that

$$\begin{aligned}
\sum_{j=0}^{\nu^{-1/2}-1} \frac{(j+1)^{(\theta-1)/2}}{(j!)^{3/2}\nu^{j/2+1/4}} M_{2,j,Y^{3/2+\theta}}[\partial_X\Delta\phi] &\le C\sum_{j=0}^{\nu^{-1/2}-1} \frac{(j+1)^{(\theta-1)/2}}{(j!)^{3/2}\nu^{j/2+1/4}} M_{2,j+1,Y^{3/2+\theta}}[\Delta\phi] \\
&= C\sum_{j=0}^{\nu^{-1/2}-1} \frac{\nu^{1/4}(j+1)^{3/2+(\theta-1)/2}}{((j+1)!)^{3/2}\nu^{(j+1)/2}} M_{2,j+1,Y^{3/2+\theta}}[\Delta\phi] \\
&= C\sum_{j=1}^{\nu^{-1/2}} \frac{\nu^{1/4}j^{\theta/2+1}}{(j!)^{3/2}\nu^{j/2}} M_{2,j,Y^{3/2+\theta}}[\Delta\phi].
\end{aligned}$$

By arguing as in the proof of Proposition 5.2, the application of (5-22) gives

$$C\sum_{j=1}^{\nu^{-1/2}} \frac{\nu^{1/4}j^{\theta/2+1}}{(j!)^{3/2}\nu^{j/2}} M_{2,j,Y^{3/2+\theta}}[\Delta\phi] \le \frac{C}{K^{\theta+1/2}} \sum_{j=1}^{\nu^{-1/2}} \frac{\nu^{1/4}(j+1)^{1/2}}{(j!)^{3/2}\nu^{j/2}} \|e^{-K\tau\nu^{1/2}(j+1)}\partial_X^j h\|_{L^2(0,1/(K\nu^{1/2});L^2_X)},$$

where the smoothing factor $(j+1)^{-\theta'/2-1/4}$ with $\theta'=\theta+\frac{1}{2}$ in (5-22) plays a key role. When $j=\nu^{-1/2}$, we have

$$
\begin{aligned}
&\frac{(j+1)^{(\theta-1)/2}}{(j!)^{3/2}\nu^{j/2+1/4}} M_{2,j,Y^{3/2+\theta}}[\partial_X \Delta\phi]\bigg|_{j=\nu^{-1/2}} \\
&\quad\le \frac{\nu^{1/4}(j+1)^{(\theta+1)/2}}{(j!)^{3/2}\nu^{j/2}} M_{2,j,Y^{3/2+\theta}}[\partial_X \Delta\phi]\bigg|_{j=\nu^{-1/2}} \\
&\quad\le \frac{C}{K^{\theta/2}} \sum_{j=0}^{\nu^{-1/2}} \frac{\nu^{1/4}(j+1)^{1/2}}{(j!)^{3/2}\nu^{j/2}} \|e^{-K\tau\nu^{1/2}(j+1)}\partial_X^j h\|_{L^2(0,1/(K\nu^{1/2});L^2_X)} \quad \bigl(\text{by (5-22) with } \theta'=\theta-\tfrac{1}{2}\bigr) \\
&\quad\le \frac{C}{K^{\theta/2}} |||h|||_{\mathrm{bc}}. \qquad \square
\end{aligned}
$$

5B. *Vorticity transport estimate.* Propositions 5.2 and 5.4 of the previous paragraph reflect a strong difference between the weighted fields $(\nabla\phi)^j$ and $(\Delta\phi)^j$ associated to the Stokes solution ϕ of (5-1): the former is not localized near the boundary, while the latter is, at scale $(K(j+1))^{-1/2}$. This is due to a harmonic nonlocalized part in ϕ, see expression (5-6). As a consequence, as shown in Proposition 5.4, for the vorticity field the weight Y^θ gives a gain of $(j+1)^{-\theta/2}$. In particular, the transport term $V\cdot\nabla\Delta\phi$ shares similar properties. When working in the Gevrey class $\frac{3}{2}$, this term can be seen to be formally of the same size as the Stokes term $\nu^{1/2}\Delta^2\phi-\partial_\tau\Delta\phi$. Hence, we need to add one step to our iteration in which we solve the heat-transport equations

$$
\begin{aligned}
-\nu^{\frac{1}{2}}\Delta\omega+\partial_\tau\omega+V\cdot\nabla\omega=H, \qquad \tau>0,\ \ X\in\mathbb{T}_\nu,\ \ Y>0,\\
\phi|_{Y=0}=\omega|_{Y=0}=0, \qquad \phi|_{\tau=0}=0.
\end{aligned}
\tag{5-25}
$$

Here $\phi\in\dot H^1_0(\mathbb{T}_\nu\times\mathbb{R}_+)$ is the stream function associated with ω, and the source term $H\in L^2\dot H^{-1}$ will be the transport term created by the Stokes approximation. A key point in dealing with this equation rather than with the full vorticity equation is that we will be able to propagate weighted estimates with weight Y^θ, which is crucial to have sharp bounds. In the last step of our iteration, we will correct nonlocal stretching terms using the vorticity equation with artificial boundary conditions, using the bounds of Section 4. The main result of this paragraph is:

Proposition 5.5. *There exists $K_3=K_3(C_1^*)\ge 1$ such that if $K\ge K_3$ then the system (5-25) admits a unique solution $\omega\in C([0,1/(K\nu^{1/2})];L^2\cap\dot H^{-1})\cap L^2(0,1/(K\nu^{1/2});H^1_0)$ satisfying, for $0\le j\le\nu^{-1/2}$, $\kappa\in(0,1]$, and $\theta=0,1,2$,*

$$
\begin{aligned}
&\nu^{\frac{1}{4}}M_{2,j,Y^\theta}[\nabla\omega]+M_{\infty,j,Y^\theta}[\omega]+K^{\frac{1}{2}}\nu^{\frac{1}{4}}(j+1)^{\frac{1}{2}}M_{2,j,Y^\theta}[\omega] \\
&\quad\le C\biggl(\kappa\nu^{\frac{3}{4}}jM_{2,j-1,Y^\theta}[\nabla\omega]+\nu^{\frac{1}{4}}\theta M_{2,j,Y^{\theta-1}}[\omega]+\frac{1}{K^{1/4}\nu^{1/4}(j+1)^{1/4}}M_{2,j,Y^{\theta+1/2}}[H] \\
&\qquad\qquad+\frac{1}{\kappa K^{1/2}\nu^{1/4}(j+1)^{1/2}}\sum_{l=0}^{j-1}\min\{l+1,\,j-l+1\}\binom{j}{l}N_{\infty,j-l}[V]M_{2,l+1,Y^\theta}[\omega]\biggr).
\end{aligned}
\tag{5-26}
$$

Here $C>0$ is a universal constant.

Remark 5.6. The solution ω to (5-25) in Proposition 5.5 has the regularity

$$(\partial_\tau - \nu^{\frac{1}{2}}\Delta)Y^\theta\omega \in L^2_{\mathrm{loc}}([0,\infty); L^2(\mathbb{T}_\nu \times \mathbb{R}_+)), \quad \theta = 0, 1, 2,$$

with the Dirichlet boundary condition. Hence, the maximal regularity for the heat equation implies

$$\partial_\tau Y^\theta\omega,\ \Delta(Y^\theta\omega) \in L^2_{\mathrm{loc}}([0,\infty); L^2(\mathbb{T}_\nu \times \mathbb{R}_+)).$$

To prove Proposition 5.5 let us recall that $\omega^j = e^{-K\tau\nu^{1/2}(j+1)} B_{j_2}\partial_X^{j-j_2}\omega$ satisfies

$$\begin{aligned}
&-\nu^{\frac{1}{2}}(\Delta\omega)^j + \partial_\tau\omega^j + K\nu^{\frac{1}{2}}(j+1)\omega^j + V\cdot\nabla\omega^j \\
&= -V_2[B_{j_2},\partial_Y]e^{-K\tau\nu^{1/2}(j+1)}\partial_X^{j_1}\omega - \sum_{l=0}^{j-1}\ \sum_{\max\{0,l+j_2-j\}\le l_2\le\min\{l,j_2\}} \binom{j_2}{l_2}\binom{j-j_2}{l-l_2} V^{j-l}\cdot(\nabla\omega)^l + H^j. \qquad (5\text{-}27)
\end{aligned}$$

Then (5-26) is proved by taking the inner product in (5-27) with $Y^{2\theta}\omega^j$ for each $\theta = 0, 1, 2$, and then by taking the supremum over $j_2 = 0, \dots, j$ and about $\tau_0 \in (0, 1/(K\nu^{1/2})]$. Hence the proof proceeds as in the proof of Proposition 4.4.

Lemma 5.7. *There exists $C > 0$ such that, for any $K \ge 1$ and $\kappa \in (0,1]$,*

$$\int_0^{\tau_0} \langle -\nu^{\frac{1}{2}}(\Delta\omega)^j, Y^{2\theta}\omega^j\rangle\,d\tau \ge \tfrac{3}{4}\nu^{\frac{1}{2}}\|Y^\theta(\nabla\omega)^j\|^2_{L^2(0,\tau_0;L^2_{X,Y})} - C\nu^{\frac{1}{2}}(\kappa\nu_2^j)^2 M_{2,j-1,Y^\theta}[\partial_Y\omega]^2 - C\theta^2\nu^{\frac{1}{2}} M_{2,j,Y^{\theta-1}}[\omega]^2.$$

Proof. The proof is similar to (and much simpler than) the one of Lemma 4.5. Indeed, the only difference is the presence of the weight $Y^{2\theta}$ with $\theta = 0, 1, 2$, which creates the term

$$2\theta\nu^{\frac{1}{2}}\int_0^{\tau_0}\langle Y^\theta(\partial_Y\omega)^j, Y^{\theta-1}\omega^j\rangle\,d\tau$$

after integration by parts. This is responsible for the last term in the estimate of this lemma. The details are omitted. □

Lemma 5.8. *There exists $K_{3,2} = K_{3,2}(C_1^*) \ge 1$ such that if $K \ge K_{3,2}$ then*

$$\int_0^{\tau_0}\langle \partial_\tau\omega^j + K\nu^{\frac{1}{2}}(j+1)\omega^j + V\cdot\nabla\omega^j, Y^{2\theta}\omega^j\rangle\,d\tau \ge \tfrac{1}{2}\|Y^\theta\omega^j(\tau_0)\|^2 + \tfrac{3}{4}K\nu^{\frac{1}{2}}(j+1)\|Y^\theta\omega^j\|^2_{L^2(0,\tau_0;L^2_{X,Y})}.$$

Proof. The proof is a simple modification of the one of Lemma 4.6. We note that the initial data is taken as zero, and integration by parts gives

$$\int_0^{\tau_0}\langle V\cdot\nabla\omega^j, Y^{2\theta}\omega^j\rangle\,d\tau \le \theta\left\|\frac{V_2}{Y}\right\|_{L^\infty}\|Y^\theta\omega^j\|^2_{L^2(0,\tau_0;L^2)}.$$

Then the desired estimate follows by taking K large enough depending only on C_1^* for $\|V_2/Y\|_{L^\infty} \le \|\partial_Y V_2\|_{L^\infty} = \|\partial_X V_1\|_{L^\infty} \le C_1^*\nu^{1/2}$. The details are omitted. □

Lemma 5.9. *Let $j_2 \geq 1$. It follows that*

$$\int_0^{\tau_0} \langle -V_2[B_{j_2}, \partial_Y] e^{-K\tau\nu^{1/2}(j+1)} \partial_X^{j_1}\omega, Y^{2\theta}\omega^{\boldsymbol{j}}\rangle \, d\tau \leq C C_1^* \nu^{\frac{1}{2}} j_2 \|Y^\theta \omega^{\boldsymbol{j}}\|^2_{L^2(0,\tau_0; L^2_{X,Y})}.$$

Here $C > 0$ is a universal constant.

Proof. The proof is similar to the one of Lemma 4.7. The details are omitted here. □

Lemma 5.10. *Let $j \geq 1$. It follows that*

$$\int_0^{\tau_0} \left\langle -\sum_{l=0}^{j-1} \sum_{\max\{0, l+j_2-j\} \leq l_2 \leq \min\{l, j_2\}} \binom{j_2}{l_2}\binom{j-j_2}{l-l_2} V^{\boldsymbol{j}-\boldsymbol{l}} \cdot (\nabla\omega)^{\boldsymbol{l}}, Y^{2\theta}\omega^{\boldsymbol{j}} \right\rangle d\tau \leq \frac{C}{\kappa} R_{j,\text{Lemma 5.10}}[\omega] M_{2,j,Y^\theta}[\omega],$$

where

$$R_{j,\text{Lemma 5.10}}[\omega] = \sum_{l=0}^{j-1} \min\{l+1, j-l+1\} \binom{j}{l} N_{\infty, j-l}[V] M_{2,l+1,Y^\theta}[\omega].$$

Here $C > 0$ is a universal constant, and $N_{\infty, j-l}[V]$ is defined as in Lemma 4.9.

Proof. The proof is similar to the one of Lemma 4.9. The details are omitted here. □

Lemma 5.11. *It follows that*

$$\int_0^{\tau_0} \langle H^{\boldsymbol{j}}, Y^{2\theta}\omega^{\boldsymbol{j}}\rangle \, d\tau \leq \begin{cases} C M_{2,j,Y^{\theta+1/2}}[H] (M_{2,j,Y^\theta}[\partial_Y\omega] + \kappa\nu^{\frac{1}{2}} j M_{2,j-1,Y^\theta}[\nabla\omega])^{\frac{1}{2}} (M_{2,j,Y^\theta}[\omega])^{\frac{1}{2}}, & \theta = 0, \\ C M_{2,j,Y^{\theta+1/2}}[H] (M_{2,j,Y^{\theta-1}}[\omega])^{\frac{1}{2}} (M_{2,j,Y^\theta}[\omega])^{\frac{1}{2}}, & \theta = 1, 2. \end{cases}$$

Here $C > 0$ is a universal constant.

Proof. The estimate follows from the inequality

$$\langle H^{\boldsymbol{j}}, Y^{2\theta}\omega^{\boldsymbol{j}}\rangle \leq \|Y^{\theta+\frac{1}{2}} H^{\boldsymbol{j}}\| \|Y^{\theta-\frac{1}{2}}\omega^{\boldsymbol{j}}\| \leq \|Y^{\theta+\frac{1}{2}} H^{\boldsymbol{j}}\| \|Y^{\theta-1}\omega^{\boldsymbol{j}}\|^{\frac{1}{2}} \|Y^\theta\omega^{\boldsymbol{j}}\|^{\frac{1}{2}}$$

and the Hardy inequality for $\theta = 0$:

$$\|Y^{-1}\omega^{\boldsymbol{j}}\| \leq C\|\partial_Y \omega^{\boldsymbol{j}}\| \leq C(\|(\partial_Y\omega)^{\boldsymbol{j}}\| + \kappa\nu^{\frac{1}{2}} j_2 \|(\partial_Y\omega)^{(j_1, j_2-1)}\|).$$ □

Proof of Proposition 5.5. It suffices to show the estimate (5-26), but it follows from Lemmas 5.7–5.11 by considering the cases $\theta = 0$ and $\theta = 1, 2$ separately. The details are omitted here. □

Corollary 5.12. *There exists $\kappa_3 \in (0, 1]$ such that the following statement holds for any $\kappa \in (0, \kappa_3]$. There exists $K_3' = K_3'(\kappa, C_0^*, C_1^*) \geq 1$ such that if $K \geq K_3'$ then the system (5-25) admits a unique solution $\omega \in C([0, 1/(K\nu^{1/2})]; L^2 \cap \dot{H}^{-1}) \cap L^2(0, 1/(K\nu^{1/2}); H_0^1)$ satisfying, for $\theta = 0, 1, 2$,*

$$\sum_{j=0}^{\nu^{-1/2}} \frac{(j+1)^{\theta/2-1/4}}{(j!)^{3/2}\nu^{j/2}} (\nu^{\frac{1}{4}} M_{2,j,Y^\theta}[\nabla\omega] + M_{\infty,j,Y^\theta}[\omega] + K^{\frac{1}{2}}\nu^{\frac{1}{4}}(j+1)^{\frac{1}{2}} M_{2,j,Y^\theta}[\omega]) \leq \frac{C}{K^{1/4}} \sum_{\theta'=0}^{\theta} \sum_{j=0}^{\nu^{-1/2}} \frac{1}{(j!)^{3/2}\nu^{j/2+1/4}(j+1)^{(1-\theta')/2}} M_{2,j,Y^{\theta'+1/2}}[H], \quad (5\text{-}28)$$

and

$$|||\nabla\phi|||'_{2,\mathbf{1}} + |||\partial_Y\phi|_{Y=0}|||_{\mathrm{bc}} \le \frac{C}{K^{3/4}} \sum_{\theta'=0}^{1} \sum_{j=0}^{\nu^{-1/2}} \frac{1}{(j!)^{3/2}\nu^{j/2+1/4}(j+1)^{(1-\theta')/2}} M_{2,j,Y^{\theta'+1/2}}[H]. \tag{5-29}$$

Here $C > 0$ is a universal constant.

Proof. Let us first show (5-28). By virtue of Proposition 5.5 we have, for $\theta = 0, 1, 2$,

$$\begin{aligned}
&\sum_{\theta'=0}^{\theta} \sum_{j=0}^{\nu^{-1/2}} \frac{(j+1)^{\theta'/2-1/4}}{(j!)^{3/2}\nu^{j/2}} \big(\nu^{\frac14} M_{2,j,Y^{\theta'}}[\nabla\omega] + M_{\infty,j,Y^{\theta'}}[\omega] + K^{\frac12}\nu^{\frac14}(j+1)^{\frac12} M_{2,j,Y^{\theta'}}[\omega]\big)\\
&\quad\le C \sum_{\theta'=0}^{\theta} \sum_{j=0}^{\nu^{-1/2}} \frac{(j+1)^{\theta'/2-1/4}}{(j!)^{3/2}\nu^{j/2}}\\
&\qquad\qquad \times \Big(\kappa\nu^{\frac34} j M_{2,j-1,Y^{\theta'}}[\nabla\omega] + \nu^{\frac14}\theta' M_{2,j,Y^{\theta'-1}}[\omega] + \frac{1}{K^{1/4}\nu^{1/4}(j+1)^{1/4}} M_{2,j,Y^{\theta'+1/2}}[H]\\
&\qquad\qquad\qquad + \frac{1}{\kappa K^{1/2}\nu^{1/4}(j+1)^{1/2}} \sum_{l=0}^{j-1} \min\{l+1,\, j-l+1\} \binom{j}{l} N_{\infty,j-l}[V] M_{2,l+1,Y^{\theta'}}[\omega]\Big)\\
&\quad\le C\kappa \sum_{\theta'=0}^{\theta} \sum_{j=0}^{\nu^{-1/2}-1} \frac{(j+1)^{\theta'/2-1/4}}{(j!)^{3/2}\nu^{j/2}} \nu^{\frac14} M_{2,j,Y^{\theta'}}[\nabla\omega]\\
&\qquad + C \sum_{\theta'=0}^{\theta} \theta' \sum_{j=0}^{\nu^{-1/2}} \frac{(j+1)^{\theta'-1/2-1/4}}{(j!)^{3/2}\nu^{j/2}} \nu^{\frac14}(j+1)^{\frac12} M_{2,j,Y^{\theta'-1}}[\omega]\\
&\qquad + C \sum_{\theta'=0}^{\theta} \sum_{j=0}^{\nu^{-1/2}} \frac{(j+1)^{\theta'/2-1/4}}{(j!)^{3/2}\nu^{j/2}} \frac{1}{K^{1/4}\nu^{1/4}(j+1)^{1/4}} M_{2,j,Y^{\theta'+1/2}}[H]\\
&\qquad + C \sum_{\theta'=0}^{\theta} \sum_{j=0}^{\nu^{-1/2}} \frac{(j+1)^{\theta'/2-1/4}}{(j!)^{3/2}\nu^{j/2}} \frac{1}{\kappa K^{1/2}\nu^{1/4}(j+1)^{1/2}}\\
&\qquad\qquad \times \sum_{l=0}^{j-1} \min\{l+1,\, j-l+1\} \binom{j}{l} N_{\infty,j-l}[V] M_{2,l+1,Y^{\theta'}}[\omega].
\end{aligned} \tag{5-30}$$

Here $C > 0$ is a universal constant. As for the last term in (5-30), arguing as at the end of the proof of Lemma 4.9, we find that

$$\begin{aligned}
&\sum_{j=0}^{\nu^{-1/2}} \frac{(j+1)^{\theta'/2-1/4}}{(j!)^{3/2}\nu^{j/2}} \frac{1}{K^{1/2}\nu^{1/4}(j+1)^{1/2}} \sum_{l=0}^{j-1} \min\{l+1,\, j-l+1\} \binom{j}{l} N_{\infty,j-l}[V] M_{2,l+1,Y^{\theta'}}[\omega]\\
&\qquad\qquad \le \frac{CC_0^*}{\kappa K^{1/2}} \sum_{j=0}^{\nu^{-1/2}} \frac{(j+1)^{\theta'/2-1/4}}{(j!)^{3/2}\nu^{j/2}} \nu^{\frac14}(j+1)^{\frac12} M_{2,j,Y^{\theta'}}[\omega].
\end{aligned}$$

Hence (5-28) follows by taking κ small enough that $C\kappa \le \frac12$, and then by taking K large enough that $CC_0^*/(\kappa K) \le \frac12$.

To show (5-29) let ϕ be the stream function associated to ω, and it suffices to prove the embedding inequality

$$\begin{aligned}\|\!|\nabla\phi|\!\|'_{2,\mathbf{1}} &\le \sum_{j=0}^{\nu^{-1/2}} \frac{(j+1)^{1/4}}{(j!)^{3/2}\nu^{j/2}} \nu^{\frac14}(j+1)^{\frac12} M_{2,j,1}[\nabla\phi] \\ &\le C\sum_{j=0}^{\nu^{-1/2}} \frac{(j+1)^{1/4}}{(j!)^{3/2}\nu^{j/2}} \nu^{\frac14}(j+1)^{\frac12} M_{2,j,Y}[\omega] \end{aligned} \tag{5-31}$$

and the interpolation inequality

$$\begin{aligned}&\|\!|\partial_Y\phi|_{Y=0}|\!\|_{\mathrm{bc}} \\ &:= \sum_{j=0}^{\nu^{-1/2}} \frac{\nu^{1/4}(j+1)^{1/2}}{(j!)^{3/2}\nu^{j/2}} \|e^{-K\tau\nu^{1/2}(j+1)}\partial_X^j\partial_Y\phi|_{Y=0}\|_{L^2(0,1/(K\nu^{1/2});L^2_X)} \\ &\le C\biggl(\sum_{j=0}^{\nu^{-1/2}} \frac{(j+1)^{-1/4}}{(j!)^{3/2}\nu^{j/2}} \nu^{\frac14}(j+1)^{\frac12} M_{2,j,1}[\omega]\biggr)^{\frac12} \biggl(\sum_{j=0}^{\nu^{-1/2}} \frac{(j+1)^{1/4}}{(j!)^{3/2}\nu^{j/2}} \nu^{\frac14}(j+1)^{\frac12} M_{2,j,Y}[\omega]\biggr)^{\frac12}. \end{aligned} \tag{5-32}$$

Then (5-29) follows from (5-28) with (5-31) and (5-32). The proof of (5-31) proceeds as in the proof of Proposition 4.12. Indeed, from

$$\omega^{\boldsymbol{j}} = -\nabla\cdot(\nabla\phi)^{\boldsymbol{j}} + \frac{\nu^{1/2} j_2\chi'_\nu}{\chi_\nu}(\partial_Y\phi)^{\boldsymbol{j}}$$

and integration by parts, we have

$$\begin{aligned}\|(\nabla\phi)^{\boldsymbol{j}}\|^2 &= \langle\omega^{\boldsymbol{j}},\phi^{\boldsymbol{j}}\rangle - 2\nu^{\frac12} j_2 e^{-K\tau\nu^{1/2}}\langle\chi'_\nu(\partial_Y\phi)^{\boldsymbol{j}},(\partial_Y\phi)^{(j-j_2,j_2-1)}\rangle \\ &\le \|Y\omega^{\boldsymbol{j}}\|\left\|\frac{\phi^{\boldsymbol{j}}}{Y}\right\| + 2\nu^{\frac12} j_2\kappa\|(\partial_Y\phi)^{\boldsymbol{j}}\|\|(\partial_Y\phi)^{(j-j_2,j_2-1)}\| \\ &\le C\|Y\omega^{\boldsymbol{j}}\|\|\partial_Y\phi^{\boldsymbol{j}}\| + 2\nu^{\frac12} j_2\kappa\|(\partial_Y\phi)^{\boldsymbol{j}}\|\|(\partial_Y\phi)^{(j-j_2,j_2-1)}\|.\end{aligned}$$

Here the Hardy inequality is used in the last line. Then the identity

$$\partial_Y\phi^{\boldsymbol{j}} = (\partial_Y\phi)^{\boldsymbol{j}} + \nu^{\frac12} j_2\chi'_\nu e^{-K\tau\nu^{1/2}}(\partial_Y\phi)^{(j-j_2,j_2-1)}$$

yields

$$\|(\nabla\phi)^{\boldsymbol{j}}\| \le C(\|Y\omega^{\boldsymbol{j}}\| + \nu^{\frac12} j_2\kappa\|(\partial_Y\phi)^{(j-j_2,j_2-1)}\|).$$

This estimate gives

$$\begin{aligned}&\sum_{j=0}^{\nu^{-1/2}} \frac{(j+1)^{1/4}}{(j!)^{3/2}\nu^{j/2}} \nu^{\frac14}(j+1)^{\frac12} M_{2,j,1}[\nabla\phi] \\ &\quad\le C\sum_{j=0}^{\nu^{-1/2}} \frac{(j+1)^{1/4}}{(j!)^{3/2}\nu^{j/2}} \nu^{\frac14}(j+1)^{\frac12}(M_{2,j,Y}[\omega] + \nu^{\frac12} j\kappa M_{2,j-1,1}[\nabla\phi]) \\ &\quad\le C\sum_{j=0}^{\nu^{-1/2}} \frac{(j+1)^{1/4}}{(j!)^{3/2}\nu^{j/2}} \nu^{\frac14}(j+1)^{\frac12} M_{2,j,Y}[\omega] + C\kappa\sum_{j=0}^{\nu^{-1/2}} \frac{(j+1)^{1/4}}{(j!)^{\gamma}\nu^{j/2}} \nu^{\frac14}(j+1)^{\frac12} M_{2,j,1}[\nabla\phi],\end{aligned}$$

where $C>0$ is a universal constant. This proves (5-31) if κ is small enough that $C\kappa \le \frac{1}{2}$. As for (5-32), we observe from (4-50) that

$$\|e^{-K\tau\nu^{1/2}(j+1)}\partial_X^j\partial_Y\phi|_{Y=0}\|_{L^2(0,1/(K\nu^{1/2});L^2_X)} \le C((j+1)^{-\frac{1}{4}}\|\omega^{(j,0)}\|_{L^2(0,1/(K\nu^{1/2});L^2_X)})^{\frac{1}{2}}((j+1)^{\frac{1}{4}}\|\partial_Y\phi^{(j,0)}\|_{L^2(0,1/(K\nu^{1/2});L^2_X)})^{\frac{1}{2}},$$

which implies, from the Schwarz inequality,

$$|||\partial_Y\phi|_{Y=0}|||_{\mathrm{bc}} \le C\biggl(\sum_{j=0}^{\nu^{-1/2}}\frac{(j+1)^{-1/4}}{(j!)^{\gamma}\nu^{j/2}}\nu^{\frac{1}{4}}(j+1)^{\frac{1}{2}}M_{2,j,1}[\omega]\biggr)^{\frac{1}{2}}\biggl(\sum_{j=0}^{\nu^{-1/2}}\frac{(j+1)^{1/4}}{(j!)^{\gamma}\nu^{j/2}}\nu^{\frac{1}{4}}(j+1)^{\frac{1}{2}}M_{2,j,1}[\nabla\phi]\biggr)^{\frac{1}{2}}.$$

Then (5-31) shows (5-32). □

Corollary 5.13. *In Corollary 5.12, let $H=-V\cdot\nabla\omega_{1,1}[h]$, where $\omega_{1,1}[h]$ is the solution to (5-2) in Propositions 5.2 and 5.4. Then*

$$\sum_{j=0}^{\nu^{-1/2}}\frac{(j+1)^{\theta/2-1/4}}{(j!)^{3/2}\nu^{j/2}}(\nu^{\frac{1}{4}}M_{2,j,Y^\theta}[\nabla\omega]+M_{\infty,j,Y^\theta}[\omega]+K^{\frac{1}{2}}\nu^{\frac{1}{4}}(j+1)^{\frac{1}{2}}M_{2,j,Y^\theta}[\omega]) \le \frac{CC_0^*}{K^{1/4}}|||h|||_{\mathrm{bc}} \tag{5-33}$$

and

$$|||\nabla\phi|||'_{2,\mathbf{1}}+|||\partial_Y\phi|_{Y=0}|||_{\mathrm{bc}} \le \frac{CC_0^*}{K^{3/4}}|||h|||_{\mathrm{bc}}. \tag{5-34}$$

Moreover, we have

$$\sum_{j=0}^{\nu^{-1/2}}\frac{1}{(j!)^{3/2}\nu^{j/2+1/4}(j+1)^{1/2}}M_{2,j,1/(1+Y)}[\partial_X\phi] \le \frac{CC_0^*}{K^{3/4}}|||h|||_{\mathrm{bc}}. \tag{5-35}$$

Here $C>0$ is a universal constant.

Proof. To show (5-33) and (5-34), it suffices to prove, for $\theta'=0,1,2$,

$$\sum_{j=0}^{\nu^{-1/2}}\frac{1}{(j!)^{3/2}\nu^{j/2}(j+1)^{(1-\theta')/2}}M_{2,j,Y^{\theta'+1/2}}[H] \le CC_0^*\sum_{j=0}^{\nu^{-1/2}}\frac{1}{(j!)^{3/2}\nu^{j/2}(j+1)^{(1-\theta')/2}}(M_{2,j,Y^{3/2+\theta'}}[\partial_X\omega_{1,1}]+\nu^{\frac{1}{2}}M_{2,j,Y^{3/2+\theta'}}[\partial_Y\omega_{1,1}]). \tag{5-36}$$

Then (5-33) and (5-34) follow from (5-28), (5-29), (5-24) and (5-36). To show (5-36), we observe that

$$H^j=-\sum_{l=0}^{j}\sum_{\max\{0,l+j_2-j\}\le l_2\le\min\{l,j_2\}}\binom{j_2}{l_2}\binom{j-j_2}{l-l_2}V^{j-l}\cdot(\nabla\omega_{1,1})^l.$$

Thus we have

$$\|Y^{\theta'+\frac{1}{2}}H^j\| \le \sum_{l=0}^{j}\binom{j}{l}\sum_{\max\{0,l+j_2-j\}\le l_2\le\min\{l,j_2\}}(\|\partial_Y V_1^{j-l}\|_{L^\infty}\|Y^{\frac{3}{2}+\theta'}(\partial_X\omega_{1,1})^l\|+\|\partial_Y V_2^{j-l}\|_{L^\infty}\|Y^{\frac{3}{2}+\theta'}(\partial_Y\omega_{1,1})^l\|).$$

Set

$$N_{\infty,j}[\nabla V_1] = (j+1)^{\frac{1}{2}} \sup_{j_2=0,\dots,j} \left(\nu^{-\frac{1}{2}} \|(\partial_X V_1)^{\boldsymbol{j}}\|_{L^\infty_{\tau,X,Y}} + \|(\partial_Y V_1)^{\boldsymbol{j}}\|_{L^\infty_{\tau,X,Y}}\right). \tag{5-37}$$

Since

$$\begin{aligned}\|\partial_Y V_1^{\boldsymbol{j}-\boldsymbol{l}}\|_{L^\infty} &\le \|(\partial_Y V_1)^{\boldsymbol{j}-\boldsymbol{l}}\|_{L^\infty} + \kappa\nu^{\frac{1}{2}}(j_2-l_2)\|(\partial_Y V_1)^{(j_1-l_1,j_2-l_2-1)}\|_{L^\infty}\\ &\le (j-l+1)^{-\frac{1}{2}} N_{\infty,j-l}[\nabla V_1] + \kappa\nu^{\frac{1}{2}}(j-l)^{\frac{1}{2}} N_{\infty,j-l-1}[\nabla V_1]\end{aligned}$$

and similarly

$$\begin{aligned}\|\partial_Y V_2^{\boldsymbol{j}-\boldsymbol{l}}\|_{L^\infty} &\le \|(\partial_Y V_2)^{\boldsymbol{j}-\boldsymbol{l}}\|_{L^\infty} + \kappa\nu^{\frac{1}{2}}(j_2-l_2)\|(\partial_Y V_2)^{(j_1-l_1,j_2-l_2-1)}\|_{L^\infty}\\ &= \|(\partial_X V_1)^{\boldsymbol{j}-\boldsymbol{l}}\|_{L^\infty} + \kappa\nu^{\frac{1}{2}}(j_2-l_2)\|(\partial_X V_1)^{(j_1-l_1,j_2-l_2-1)}\|_{L^\infty}\\ &\le \nu^{\frac{1}{2}}\left((j-l+1)^{-\frac{1}{2}} N_{\infty,j-l}[\nabla V_1] + \kappa\nu^{\frac{1}{2}}(j-l)^{\frac{1}{2}} N_{\infty,j-l-1}[\nabla V_1]\right),\end{aligned}$$

we obtain

$$\begin{aligned}M_{2,j,Y^{\theta'+1/2}}[H] \le \sum_{l=0}^{j}\binom{j}{l}\min\{l+1,\, j-l+1\}\left((j-l+1)^{-\frac{1}{2}} N_{\infty,j-l}[\nabla V_1] + \kappa\nu^{\frac{1}{2}}(j-l)^{\frac{1}{2}} N_{\infty,j-l-1}[\nabla V_1]\right)\\ \times \left(M_{2,l,Y^{3/2+\theta'}}[\partial_X\omega_{1,1}] + \nu^{\frac{1}{2}} M_{2,l,Y^{3/2+\theta'}}[\partial_Y\omega_{1,1}]\right).\end{aligned}$$

Then (5-36) follows from the Young inequality for convolution in the l^1 space. For example, using

$$\frac{(l+1)^{(1-\theta')/2}}{(j+1)^{(1-\theta')/2}(j-l+1)^{1/2}} \le C \quad \text{for } \theta' = 0,\, 1,\, 2$$

and

$$\left(\frac{(j-l)!\,l!}{j!}\right)^{\frac{1}{2}} \min\{l+1,\, j-l+1\} \le C,$$

we have

$$\begin{aligned}&\sum_{j=0}^{\nu^{-1/2}}\sum_{l=0}^{j}\frac{1}{(j+1)^{1-\theta'/2}}\left(\frac{(j-l)!\,l!}{j!}\right)^{\frac{1}{2}}\min\{l+1,\, j-l+1\}(j-l+1)^{-\frac{1}{2}}(l+1)^{(1-\theta')/2}\\ &\qquad\qquad \times\left(\frac{1}{((j-l)!)^{3/2}\nu^{(j-l)/2}} N_{\infty,j-l}[\nabla V_1]\right)\left(\frac{1}{(l!)^{3/2}\nu^{l/2}(l+1)^{(1-\theta')/2}} M_{2,l,Y^{3/2+\theta'}}[\partial_X\omega_{1,1}]\right)\\ &\quad\le C\sum_{j=0}^{\nu^{-1/2}}\sum_{l=0}^{j}\left(\frac{1}{((j-l)!)^{3/2}\nu^{(j-l)/2}} N_{\infty,j-l}[\nabla V_1]\right)\left(\frac{1}{(l!)^{3/2}\nu^{l/2}(l+1)^{(1-\theta')/2}} M_{2,l,Y^{3/2+\theta'}}[\partial_X\omega_{1,1}]\right)\\ &\quad\le CC_0^*\sum_{j=0}^{\nu^{-1/2}}\frac{1}{(j!)^{3/2}\nu^{j/2}(j+1)^{(1-\theta')/2}} M_{2,j,Y^{3/2+\theta'}}[\partial_X\omega_{1,1}].\end{aligned}$$

The other terms are handled in the same manner and we omit the details. The proof of (5-33)–(5-34) is complete. Finally let us prove (5-35). The key is to apply the interpolation-type inequality proved in

Proposition A.2. Indeed, Proposition A.2 implies, for the stream function ϕ associated with ω,

$$
\begin{aligned}
&\sum_{j=0}^{\nu^{-1/2}} \frac{1}{(j!)^{3/2}\nu^{j/2+1/4}(j+1)^{1/2}} M_{2,j,1/(1+Y)}[\partial_X \phi] \\
&\le C\sum_{\theta=0}^{1}\sum_{j=0}^{\nu^{-1/2}-1} \frac{1}{(j!)^{3/2}\nu^{j/2+1/4}(j+1)^{1/2}}(j+1)^{\theta/2-\frac14} M_{2,j+1,Y^{1+\theta}}[\omega] \\
&\qquad + C\sum_{j=0}^{\nu^{-1/2}-1} \frac{1}{(j!)^{3/2}\nu^{j/2+1/4}(j+1)^{1/2}} \kappa\nu^{\frac12} j\,(M_{2,j-1,Y}[\omega]+M_{2,j-1,1}[\nabla\phi]) \\
&\qquad\qquad + \frac{1}{(j!)^{3/2}\nu^{j/2+1/4}(j+1)^{1/2}} M_{2,j,1/(1+Y)}[\partial_X\phi]\Big|_{j=\nu^{-1/2}} \\
&\le C\sum_{\theta=0}^{1}\sum_{j=0}^{\nu^{-1/2}} \frac{\nu^{1/4}(j+1)^{\theta/2+3/4}}{(j!)^{3/2}\nu^{j/2}} M_{2,j,Y^{1+\theta}}[\omega] + C\sum_{j=0}^{\nu^{-1/2}} \frac{\nu^{1/4}(j+1)^{1/2}}{(j!)^{3/2}\nu^{j/2}} M_{2,j,Y}[\omega] + C|||\nabla\phi|||'_{2,\mathbf{1}} \\
&\le \frac{CC_\kappa}{K^{3/4}} |||h|||_{\mathrm{bc}}.
\end{aligned}
$$

Here we have used (5-33) and (5-34) in the last line. □

5C. *Full construction of boundary corrector.* We set

$$\omega_{\mathrm{app},1} = \omega_{\mathrm{app},1}[h] = \omega_{1,1}[h] + \omega_{1,2}[h],$$

where $\omega_{1,1}[h]$ is the solution to (5-2) in Propositions 5.2–5.4, and $\omega_{1,2}[h]$ is the solution to (5-25) with $H = -V\cdot\nabla\omega_{1,1}[h]$ as in Corollary 5.13. Then the approximate solution ω_{app} to the full system (5-1) is constructed in the form

$$\omega_{\mathrm{app}} = \omega_{\mathrm{app},1} + \tilde\omega_1,$$

which leads to the equations for $\tilde\omega_1 = \tilde\omega_1[h]$, as

$$
\begin{aligned}
-\nu^{\frac12}\Delta\tilde\omega_1 + \partial_\tau\tilde\omega_1 + V\cdot\nabla\tilde\omega_1 + \nabla^\perp\tilde\phi_1\cdot\nabla\Omega = -\nabla^\perp\phi_{\mathrm{app},1}\cdot\nabla\Omega, \quad \tau>0,\quad X\in\mathbb{T}_\nu,\quad Y>0,\\
\tilde\phi_1|_{Y=0} = \tilde\omega_1|_{Y=0} = 0, \quad \tilde\omega_1|_{\tau=0}=0.
\end{aligned}
\tag{5-38}
$$

Here $\tilde\phi_1$ and $\phi_{\mathrm{app},1}$ are the stream functions associated with $\tilde\omega_1$ and $\omega_{\mathrm{app},1}$, respectively. Let us first give the estimate for the force term $-\nabla^\perp\phi_{\mathrm{app},1}\cdot\nabla\Omega$.

Proposition 5.14. *Let $\kappa_3\in(0,1]$ be the number in Corollary 5.12. For any $\kappa\in(0,\kappa_3]$ there exists $K_3' = K_3'(\kappa, C_*, C_j^*)\ge 1$ such that, for any $K\ge K_3'$,*

$$
\begin{aligned}
&\frac{1}{K^{1/2}\nu^{1/2}} |||\nabla^\perp\phi_{\mathrm{app},1}\cdot\nabla\Omega|||'_{2,\tilde\xi^{(2)}} + \frac{1}{K^{1/2}\nu^{1/4}} \|\nabla^\perp\phi_{\mathrm{app},1}\cdot\nabla\Omega\|_{L^2(0,1/(K\nu^{1/2});\dot H^{-1})} \\
&\qquad \le \frac{1}{K^{1/4}}\Bigg(\sum_{j=0}^{\nu^{-1/2}} \frac{1}{(j!)^{3/2}\nu^{j/2+1/4}(j+1)^{1/2}} M_{2,j,1/(1+Y)}[\partial_X\phi_{\mathrm{app},1}] + 2|||\nabla\phi_{\mathrm{app},1}|||'_{2,\mathbf{1}}\Bigg).
\end{aligned}
\tag{5-39}
$$

Proof. Let us recall that

$$\frac{1}{\nu^{1/2}}|||\nabla^{\perp}\phi_{\mathrm{app},1}\cdot\nabla\Omega|||'_{2,\tilde{\xi}^{(2)}} = \sum_{j=0}^{\nu^{-1/2}} \frac{1}{(j!)^{3/2}\nu^{j/2}\nu^{1/4}(j+1)^{1/2}} \sup_{j_2=0,\dots,j} \|\xi_j e^{-K\tau\nu^{1/2}(j+1)} B_{j_2}\partial_X^{j-j_2}(\nabla^{\perp}\phi_{\mathrm{app},1}\cdot\nabla\Omega)\|_{L^2(0,1/(K\nu^{1/2});L^2_{X,Y})}.$$

Thus we consider the estimate of

$$e^{-K\tau\nu^{1/2}(j+1)} B_{j_2}\partial_X^{j-j_2}(\nabla^{\perp}\phi_{\mathrm{app},1}\cdot\nabla\Omega) = (\nabla^{\perp}\phi_{\mathrm{app},1})^{\boldsymbol{j}}\cdot\nabla\Omega + \sum_{l=0}^{j-1}\sum_{\max\{0,l+j_2-j\}\le l_2\le\min\{l,j_2\}} \binom{j_2}{l_2}\binom{j-j_2}{l-l_2}(\nabla^{\perp}\phi_{\mathrm{app},1})^{\boldsymbol{l}}\cdot(\nabla\Omega)^{\boldsymbol{j}-\boldsymbol{l}},$$

where $\boldsymbol{j}=(j-j_2,j_2)$ and $\boldsymbol{l}=(l-l_2,l_2)$. We observe that, from the definition of ρ_j in (4-10), Assumption (iii), and $K\ge 1$,

$$\begin{aligned}
\|\xi_j\partial_X\phi^{\boldsymbol{j}}_{\mathrm{app},1}\partial_Y\Omega\| &= \left\|\frac{\partial_Y\Omega}{\sqrt{\partial_Y\Omega+2\rho_j}}\partial_X\phi^{\boldsymbol{j}}_{\mathrm{app},1}\right\| \\
&\le C\|(|\partial_Y\Omega|^{\frac12}+\sqrt{\rho_j})\partial_X\phi^{\boldsymbol{j}}_{\mathrm{app},1}\| \\
&\le C\left\|\left(\frac{1+Y}{1+\nu^{1/2}Y}\right)^2\partial_Y\Omega\right\|^{\frac12}_{L^\infty}\left\|\frac{1+\nu^{1/2}Y}{1+Y}\partial_X\phi^{\boldsymbol{j}}_{\mathrm{app},1}\right\| \\
&\qquad + C(K^{1/4}C_*)^{\frac12}\left\|\frac{1}{1+Y}\partial_X\phi^{\boldsymbol{j}}_{\mathrm{app},1}\right\| + CC_*^{\frac12}\nu^{\frac12}\|\partial_X\phi^{\boldsymbol{j}}_{\mathrm{app},1}\| \\
&\le C(C_1^*+K^{1/4}C_*)^{\frac12}\left\|\frac{1}{1+Y}\partial_X\phi^{\boldsymbol{j}}_{\mathrm{app},1}\right\| + C(C_1^*+C_*)^{\frac12}\nu^{\frac12}\|\partial_X\phi^{\boldsymbol{j}}_{\mathrm{app},1}\|.
\end{aligned}$$

On the other hand, we have

$$\begin{aligned}
\|\xi_j(\partial_Y\phi_{\mathrm{app},1})^{\boldsymbol{j}}\partial_X\Omega\| &\le \left\|\frac{1+Y}{1+\nu^{1/2}Y}\partial_X\Omega\right\|_{L^\infty}\left\|\frac{1+\nu^{1/2}Y}{1+Y}\xi_j\right\|_{L^\infty}\|(\partial_Y\phi_{\mathrm{app},1})^{\boldsymbol{j}}\| \\
&\le CC_1^*\nu^{\frac12}(j+1)^{\frac12}\|(\partial_Y\phi_{\mathrm{app},1})^{\boldsymbol{j}}\|.
\end{aligned}$$

Here we have used (4-16) and Assumption (iii). Thus we have, from $C_*\ge 1$,

$$\|\xi_j(\nabla^{\perp}\phi_{\mathrm{app},1})^{\boldsymbol{j}}\cdot\nabla\Omega\|_{L^2(0,1/(K\nu^{1/2});L^2_{X,Y})} \le C(C_1^*+K^{\frac14}C_*)^{\frac12}M_{2,j,1/(1+Y)}[\partial_X\phi_{\mathrm{app},1}] + C(C_1^*+C_*)\nu^{\frac12}(j+1)^{\frac12}M_{2,j,1}[\partial_Y\phi_{\mathrm{app},1}]. \tag{5-40}$$

Next we see

$$\left\|\xi_j\sum_{l=0}^{j-1}\sum_{\max\{0,l+j_2-j\}\le l_2\le\min\{l,j_2\}}\binom{j_2}{l_2}\binom{j-j_2}{l-l_2}(\nabla^{\perp}\phi_{\mathrm{app},1})^{\boldsymbol{l}}\cdot(\nabla\Omega)^{\boldsymbol{j}-\boldsymbol{l}}\right\| \le \sum_{l=0}^{j-1}\binom{j}{l}\sum_{\max\{0,l+j_2-j\}\le l_2\le\min\{l,j_2\}}\|\xi_j(\nabla^{\perp}\phi_{\mathrm{app},1})^{\boldsymbol{l}}\cdot(\nabla\Omega)^{\boldsymbol{j}-\boldsymbol{l}}\|$$

and

$$
\begin{aligned}
&\|\xi_j (\nabla^\perp \phi_{\mathrm{app},1})^{\boldsymbol{l}} \cdot (\nabla \Omega)^{\boldsymbol{j}-\boldsymbol{l}}\| \\
&\quad \le \left\| \left(\frac{1+Y}{1+\nu^{1/2}Y} \right)^2 (\partial_Y \Omega)^{j-l} \right\|_{L^\infty} \left\| \frac{1+\nu^{1/2}Y}{1+Y} \xi_j \right\|_{L^\infty} \left\| \frac{1+\nu^{1/2}Y}{1+Y} \partial_X \phi^{\boldsymbol{l}}_{\mathrm{app},1} \right\| \\
&\qquad\qquad + \left\| \frac{1+Y}{1+\nu^{1/2}Y} (\partial_X \Omega)^{j-l} \right\|_{L^\infty} \left\| \frac{1+\nu^{1/2}Y}{1+Y} \xi_j \right\|_{L^\infty} \|(\partial_Y \phi_{\mathrm{app},1})^{\boldsymbol{l}}\| \\
&\quad \le C(j+1)^{\frac12} N_{\infty, j-l, ((1+Y)/(1+\nu^{1/2}Y))^2} [\partial_Y \Omega] \left\| \frac{1}{1+Y} \partial_X \phi^{\boldsymbol{l}}_{\mathrm{app},1} \right\| + C\nu^{\frac12} (j+1)^{\frac12} N_{\infty, j-l} [\nabla \Omega] \|(\nabla \phi_{\mathrm{app},1})^{\boldsymbol{l}}\|.
\end{aligned}
$$

Thus we have

$$
\begin{aligned}
&\left\| \xi_j \sum_{l=0}^{j-1} \sum_{\max\{0, l+j_2-j\} \le l_2 \le \min\{l, j_2\}} \binom{j_2}{l_2} \binom{j-j_2}{l-l_2} (\nabla^\perp \phi_{\mathrm{app},1})^{\boldsymbol{l}} \cdot (\nabla \Omega)^{\boldsymbol{j}-\boldsymbol{l}} \right\|_{L^2(0,1/(K\nu^{1/2}); L^2_{X,Y})} \\
&\quad \le C(j+1)^{\frac12} \sum_{l=0}^{j-1} \min\{l+1,\, j-l+1\} \binom{j}{l} N_{\infty, j-l} [\nabla \Omega] \\
&\qquad\qquad \times (M_{2,l,1/(1+Y)} [\partial_X \phi_{\mathrm{app},1}] + \nu^{\frac12} M_{2,l,1} [\nabla \phi_{\mathrm{app},1}]). \qquad (5\text{-}41)
\end{aligned}
$$

We note that

$$
(j+1)^{\frac12} \min\{l+1,\, j-l+1\} \left(\frac{(j-l)!\, l!}{j!} \right)^{\frac12} \le C, \qquad 1 \le l \le j-1.
$$

Taking into account this uniform bound — by decomposing the sum $\sum_{l=0}^{j-1}$ into the "$l=0$" term and the sum $\sum_{l=1}^{j-1}$ — and collecting (5-40) and (5-41), we obtain, from the Young inequality for convolution in the l^1 space,

$$
\begin{aligned}
&\frac{1}{K^{1/2}\nu^{1/2}} |||\nabla^\perp \phi_{\mathrm{app},1} \cdot \nabla \Omega|||'_{2,\tilde{\xi}^{(2)}} \\
&\qquad \le \frac{1}{K^{1/4}} \left(\sum_{j=0}^{\nu^{-1/2}} \frac{1}{(j!)^{3/2} \nu^{j/2+1/4} (j+1)^{1/2}} M_{2,j,1/(1+Y)} [\partial_X \phi_{\mathrm{app},1}] + |||\nabla \phi_{\mathrm{app},1}|||'_{2,\mathbf{1}} \right), \qquad (5\text{-}42)
\end{aligned}
$$

where K has been taken large enough depending on C_*, C_1^*, and C_κ. As for the estimate of

$$
\|\nabla^\perp \phi_{\mathrm{app},1} \cdot \nabla \Omega\|_{L^2(0,1/(K\nu^{1/2}); \dot{H}^{-1})},
$$

let us take any $\eta \in \dot{H}^1_0(\mathbb{T} \times \mathbb{R}_+)$. Then we have

$$
\begin{aligned}
\langle \nabla^\perp \phi_{\mathrm{app},1} \cdot \nabla \Omega, \eta \rangle &= \left\langle \frac{1+Y}{1+\nu^{1/2}Y} \nabla^\perp \phi_{\mathrm{app},1} \cdot \nabla \Omega, \frac{\eta}{1+Y} \right\rangle + \left\langle \nabla^\perp \phi_{\mathrm{app},1} \cdot \nabla \Omega, \frac{\nu^{1/2} Y \eta}{1+\nu^{1/2}Y} \right\rangle \\
&= \left\langle \frac{1+Y}{1+\nu^{1/2}Y} \nabla^\perp \phi_{\mathrm{app},1} \cdot \nabla \Omega, \frac{\eta}{1+Y} \right\rangle - \left\langle \Omega, \nabla^\perp \phi_{\mathrm{app},1} \cdot \nabla \left(\frac{\nu^{1/2} Y \eta}{1+\nu^{1/2}Y} \right) \right\rangle.
\end{aligned}
$$

This implies

$$
\begin{aligned}
&|\langle \nabla^\perp \phi_{\mathrm{app},1}\cdot\nabla\Omega,\eta\rangle| \\
&\qquad\le \left\|\frac{1+Y}{1+\nu^{1/2}Y}\nabla^\perp\phi_{\mathrm{app},1}\cdot\nabla\Omega\right\|\left\|\frac{\eta}{1+Y}\right\| + \left\|\frac{1+Y}{1+\nu^{1/2}Y}\Omega\nabla^\perp\phi_{\mathrm{app},1}\right\|\left\|\frac{1+\nu^{1/2}Y}{1+Y}\nabla\left(\frac{\nu^{1/2}Y\eta}{1+\nu^{1/2}Y}\right)\right\| \\
&\qquad\le C\left\|\frac{1+Y}{1+\nu^{1/2}Y}\nabla^\perp\phi_{\mathrm{app},1}\cdot\nabla\Omega\right\|\|\partial_Y\eta\| + C\nu^{\frac12}\left\|\frac{1+Y}{1+\nu^{1/2}Y}\Omega\nabla^\perp\phi_{\mathrm{app},1}\right\|\|\nabla\eta\|,
\end{aligned}
$$

where the Hardy inequality was used several times. Hence we obtain

$$
\begin{aligned}
&\|\nabla^\perp\phi_{\mathrm{app},1}\cdot\nabla\Omega\|_{\dot H^{-1}} \\
&\quad\le C\left\|\frac{1+Y}{1+\nu^{1/2}Y}\nabla^\perp\phi_{\mathrm{app},1}\cdot\nabla\Omega\right\| + C\nu^{\frac12}\left\|\frac{1+Y}{1+\nu^{1/2}Y}\Omega\nabla^\perp\phi_{\mathrm{app},1}\right\| \\
&\quad\le C\left\|\frac{1+Y}{1+\nu^{1/2}Y}\partial_X\Omega\right\|_{L^\infty}\|\partial_Y\phi_{\mathrm{app},1}\| + C\left\|\left(\frac{1+Y}{1+\nu^{1/2}Y}\right)^2\partial_Y\Omega\right\|_{L^\infty}\left\|\frac{1+\nu^{1/2}Y}{1+Y}\partial_X\phi_{\mathrm{app},1}\right\| \\
&\qquad\qquad + C\nu^{\frac12}\left\|\frac{1+Y}{1+\nu^{1/2}Y}\Omega\right\|_{L^\infty}\|\nabla\phi_{\mathrm{app},1}\| \\
&\quad\le CC_1^*(\nu^{\frac12}\|\nabla\phi_{\mathrm{app},1}\| + \|\partial_Y\partial_X\phi_{\mathrm{app},1}\|).
\end{aligned}
$$

Then

$$
\begin{aligned}
&\frac{1}{K^{1/2}\nu^{1/4}}\|\nabla^\perp\phi_{\mathrm{app},1}\cdot\nabla\Omega\|_{L^2(0,1/(K\nu^{1/2});\dot H^{-1})} \\
&\le \frac{CC_1^*}{K^{1/2}\nu^{1/4}}(\nu^{\frac12}\|\nabla\phi_{\mathrm{app},1}\|_{L^2(0,1/(K\nu^{1/2});L^2_{X,Y})} + \|\partial_X\partial_Y\phi_{\mathrm{app},1}\|_{L^2(0,1/(K\nu^{1/2});L^2_{X,Y})}) \le \frac{1}{K^{1/4}}|||\nabla\phi_{\mathrm{app},1}|||'_{2,\mathbf{1}}. \qquad \square
\end{aligned}
$$

Propositions 4.1 and 5.14 yield:

Corollary 5.15. *There exists $\kappa_4\in(0,1]$ such that the following statement holds for any $\kappa\in(0,\kappa_4]$. There exists $K_4 = K_4(\kappa, C_*, C_j^*)\ge 1$ such that if $K\ge K_4$ then the system (5-38) admits a unique solution $\tilde\omega_1\in C([0,1/(K\nu^{1/2})]; L^2\cap\dot H^{-1})\cap L^2(0,1/(K\nu^{1/2}); H_0^1)$ satisfying*

$$
|||\tilde\omega_1|||'_{\infty,\xi} + K^{\frac12}|||\tilde\omega_1|||'_{2,\xi} + K^{\frac14}|||\nabla\tilde\phi_1|||'_{2,\mathbf{1}} + K^{\frac14}|||\partial_Y\tilde\phi_1|_{Y=0}|||_{\mathrm{bc}} \le \frac{1}{K^{1/2}}|||h|||_{\mathrm{bc}}. \tag{5-43}
$$

Proof. Propositions 4.1 and 5.14 give

$$
\begin{aligned}
&|||\tilde\omega_1|||'_{\infty,\xi} + K^{\frac12}|||\tilde\omega_1|||'_{2,\xi} + K^{\frac14}|||\nabla\tilde\phi_1|||'_{2,\mathbf{1}} + K^{\frac14}|||\partial_Y\tilde\phi_1|_{Y=0}|||_{\mathrm{bc}} \\
&\qquad\qquad\le \frac{C}{K^{1/4}}\left(\sum_{j=0}^{\nu^{-1/2}}\frac{1}{(j!)^{3/2}\nu^{j/2+1/4}(j+1)^{1/2}}M_{2,j,1/(1+Y)}[\partial_X\phi_{\mathrm{app},1}] + |||\nabla\phi_{\mathrm{app},1}|||'_{2,\mathbf{1}}\right).
\end{aligned}
$$

Here $C>0$ is a universal constant. Recall that $\phi_{\mathrm{app},1}[h] = \phi_{1,1}[h]+\phi_{1,2}[h]$, where $\phi_{1,j}[h]$ is the stream function associated with $\omega_{1,j}[h]$. Then the assertion follows from Proposition 5.2 for $\phi_{1,1}[h]$ and Corollary 5.13 for $\phi_{1,2}[h]$. $\square$

From the construction, the vorticity $\omega_{\mathrm{app}} = \omega_{\mathrm{app}}[h] = \omega_{\mathrm{app},1}[h] + \tilde{\omega}_1[h]$ satisfies

$$\begin{aligned} -\nu^{\frac{1}{2}}\Delta\omega_{\mathrm{app}} + \partial_\tau\omega_{\mathrm{app}} + V\cdot\nabla\omega_{\mathrm{app}} + \nabla^\perp\phi_{\mathrm{app}}\cdot\nabla\Omega = 0, \quad & \tau > 0, \quad X\in\mathbb{T}_\nu, \quad Y>0, \\ \phi_{\mathrm{app}}|_{Y=0} = 0, \quad \partial_Y\phi_{\mathrm{app}}|_{Y=0} = h + R_{\mathrm{bc}}[h], \quad & \phi_{\mathrm{app}}|_{\tau=0} = 0. \end{aligned} \tag{5-44}$$

Here ϕ_{app} is the stream function associated with ω_{app}, and $R_{\mathrm{bc}}[h]$ is the linear operator defined as

$$R_{\mathrm{bc}}[h] = \partial_Y\phi_{1,2}[h]|_{Y=0} + \partial_Y\tilde{\phi}_1[h]|_{Y=0}. \tag{5-45}$$

We note that the operator R_{bc} is well defined on the Banach space

$$Z_{\mathrm{bc}} = \{h \in L^2(0, 1/(K\nu^{\frac{1}{2}}); L^2_X) \mid \|h\|_{Z_{\mathrm{bc}}} := |||h|||_{\mathrm{bc}} < \infty\}. \tag{5-46}$$

Proposition 5.16. *There exists $\kappa_5 \in (0,1]$ such that the following statement holds for any $\kappa \in (0,\kappa_5]$. There exists $K_5 = K_5(\kappa, C_*, C_j^*) \ge 1$ such that if $K \ge K_5$ then the map $R_{\mathrm{bc}} : Z_{\mathrm{bc}} \to Z_{\mathrm{bc}}$ defined by (5-45) satisfies*

$$|||R_{\mathrm{bc}}[h]|||_{\mathrm{bc}} \le \tfrac{1}{2}|||h|||_{\mathrm{bc}}. \tag{5-47}$$

Hence, the operator $I + R_{\mathrm{bc}}$ is invertible in Z_{bc}, and the map

$$\Phi_{\mathrm{bc}}[h] := \phi_{\mathrm{app}}[(I+R_{\mathrm{bc}})^{-1}h], \quad h \in Z_{\mathrm{bc}}, \tag{5-48}$$

gives the solution to (5-1) and satisfies

$$|||\nabla\Phi_{\mathrm{bc}}[h]|||'_{2,\mathbf{1}} \le C|||h|||_{\mathrm{bc}}. \tag{5-49}$$

Here $C > 0$ is a universal constant.

Proof. By the definition of R_{bc} in (5-45), estimate (5-47) is a consequence of Corollaries 5.13 and 5.15, by taking κ small first and then K large enough depending only on C_*, C_j^*, and C_κ. In particular, we have

$$|||(I+R_{\mathrm{bc}})^{-1}h|||_{\mathrm{bc}} \le 2|||h|||_{\mathrm{bc}}, \quad h\in Z_{\mathrm{bc}}. \tag{5-50}$$

Then Proposition 5.2 and Corollaries 5.13–5.15 give (5-49). ☐

6. Full estimate for linearization

We have constructed the solution to (2-12) of the form

$$W = \nabla^\perp\phi = \nabla^\perp\Phi_{\mathrm{slip}} + \nabla^\perp\Phi_{\mathrm{bc}}[h], \quad h = -\partial_Y\Phi_{\mathrm{slip}}|_{Y=0} \in Z_{\mathrm{bc}}, \tag{6-1}$$

where $\nabla^\perp\Phi_{\mathrm{slip}}$ is the velocity field associated with the solution to (4-1) and

$$\Phi_{\mathrm{bc}}[h] = \phi_{\mathrm{app},1}[(I+R_{\mathrm{bc}})^{-1}h] + \tilde{\phi}_1[(I+R_{\mathrm{bc}})^{-1}h], \quad \phi_{\mathrm{app},1} = \phi_{1,1} + \phi_{1,2}.$$

To simplify the notation we will write $\phi_{\mathrm{app},1}$ for $\phi_{\mathrm{app},1}[(I+R_{\mathrm{bc}})^{-1}h]$ below. So far we have the bound of $\nabla^\perp\phi_{1,1}$ only in the norm $|||\cdot|||'_{2,\mathbf{1}}$. To obtain the estimates of $|||\nabla\phi|||_\infty$ and $|||\omega|||_\infty$ we need the extra work.

Proposition 6.1. *There exists $\kappa_6 \in (0,1]$ such that the following statement holds for any $\kappa \in (0,\kappa_6]$. There exists $K_6 = K_6(C_0^*, C_1^*) \ge 1$ such that if $K \ge K_6$ then the solution to (2-12) constructed as in (6-1) satisfies*

$$\nu^{\frac14}|||\omega|||_\infty + K^{\frac12}\nu^{\frac14}|||\nabla\phi|||_\infty \le \frac{C(C_0^*+C_1^*)}{\nu^{1/4}}\big(|||\nabla\phi|||'_{2,\mathbf{1}} + |||\Delta(\phi-\phi_{\mathrm{app},1})|||'_{2,\mathbf{1}} + |||\Delta\phi_{\mathrm{app},1}|||'_{2,Y}\big) + C\big(K^{\frac12}[\|W_0\|] + \nu^{\frac14}[\|\mathrm{rot}\, W_0\|] + |||F|||_2\big).$$

Here $C>0$ is a universal constant.

The proof of Proposition 6.1 is similar to the one of Proposition 4.4, and we postpone it to Appendix B. Admitting Proposition 6.1, we will now complete the proof of Theorem 2.3. Let us recall (6-1). We first observe from Proposition 4.1 and Remark 4.2 that

$$|||\Delta\Phi_{\mathrm{slip}}|||'_{2,\mathbf{1}} + |||\nabla\Phi_{\mathrm{slip}}|||'_{2,\mathbf{1}} + |||\partial_Y\Phi_{\mathrm{slip}}|_{Y=0}|||_{\mathrm{bc}} \le \frac{1}{K^{1/8}}\big(\|W_0\|_{L^2_{X,Y}} + \nu^{-\frac12}[\|\mathrm{rot}\, W_0\|] + \nu^{-\frac34}|||F|||_2\big) \tag{6-2}$$

by taking K large enough. On the other hand, Proposition 5.16 (for $\nabla\Phi_{\mathrm{bc}}$), Corollary 5.15 and Remark 4.2(1) (for $\Delta(\Phi_{\mathrm{bc}}-\phi_{\mathrm{app},1}) = \Delta\tilde\phi_1$), Proposition 5.4 and Corollary 5.13 (for $\Delta\phi_{\mathrm{app},1} = \Delta\phi_{1,1}+\Delta\phi_{1,2}$), and (6-2) give

$$\begin{aligned}|||\nabla\Phi_{\mathrm{bc}}|||'_{2,1} + |||\Delta(\Phi_{\mathrm{bc}}-\phi_{\mathrm{app},1})|||'_{2,\mathbf{1}} + |||\Delta\phi_{\mathrm{app},1}|||'_{2,Y} &\le C|||\partial_Y\Phi_{\mathrm{slip}}|_{Y=0}|||_{\mathrm{bc}} \\ &\le \frac{C}{K^{1/8}}\big(\|W_0\|_{L^2_{X,Y}} + \nu^{-\frac12}[\|\mathrm{rot}\, W_0\|] + \nu^{-\frac34}|||F|||_2\big).\end{aligned} \tag{6-3}$$

Here $C>0$ is a universal constant. By applying the estimate in Proposition 6.1 and by taking K large enough, the proof of Theorem 2.3 is complete. ☐

7. Nonlinear stability: proof of Theorem 2.1

Let us recall the nonlinear system (1-3). Theorem 2.1 is a consequence of Theorem 2.4 for the linear system (1-6) and the bilinear estimate in Lemma 7.1 stated below. We observe that

$$-w\cdot\nabla w = w\,\mathrm{rot}\, w + \nabla\tilde q$$

for any solenoidal vector field w, so the bilinear term we consider here is of the form $f\,\mathrm{rot}\, g$. To this end we fix $K \ge 1$ and $\nu \in (0,1]$, and let X be the Banach space of solenoidal vector fields $f = (f_1, f_2)$ on $[0, 1/K] \times \mathbb{R}^2_+$ defined as

$$X = \Big\{ f \in C\Big(\Big[0, \frac1K\Big]; H^1_{0,\sigma}(\mathbb{T}\times\mathbb{R}_+)\Big) \;\Big|\; \|f\|_X = \|f\|_{G^\infty_{3/2}} + \nu^{\frac12}\|\mathrm{rot}\, f\|_{G^\infty_{3/2}} < \infty \Big\},$$

where $\|\cdot\|_{G^\infty_{3/2}}$ is defined in (2-1) with $p=\infty$.

Lemma 7.1. *There exists a universal constant $C>0$ such that, for any $f, g \in X$,*

$$\|f\,\mathrm{rot}\, g\|_{G^2_{3/2}} \le \frac{C}{K^{1/2}}\nu^{-\frac34}\|f\|_X\|g\|_X. \tag{7-1}$$

Proof. We compute

$$\begin{aligned}\|f\,\mathrm{rot}\,g\|_{G^2_{3/2}} &\le C\sum_{j=0}^{\nu^{-1/2}}\frac{1}{j!^{3/2}}\sup_{|\boldsymbol{j}|=j}\sum_{\boldsymbol{l}\le\boldsymbol{j}}\binom{\boldsymbol{j}}{\boldsymbol{l}}\|f^{\boldsymbol{l}}(\mathrm{rot}\,g)^{\boldsymbol{j}-\boldsymbol{l}}\|_{L^2(0,1/K;L^2_{x,y})}\\ &\le \frac{C}{K^{1/2}}\sum_{j=0}^{\nu^{-1/2}}\frac{1}{j!^{3/2}}\sup_{|\boldsymbol{j}|=j}\sum_{\boldsymbol{l}\le\boldsymbol{j}}\binom{\boldsymbol{j}}{\boldsymbol{l}}\|f^{\boldsymbol{l}}(\mathrm{rot}\,g)^{\boldsymbol{j}-\boldsymbol{l}}\|_{L^\infty(0,1/K;L^2_{x,y})}.\end{aligned}$$

As $\binom{\boldsymbol{j}}{\boldsymbol{l}}\le\binom{|\boldsymbol{j}|}{|\boldsymbol{l}|}$ and as, for all $l\in\mathbb{N}_0$,

$$\sharp\{\boldsymbol{l},\ |\boldsymbol{l}|=l,\ \boldsymbol{l}\le\boldsymbol{j}\}=\sharp\{l_2,\ \max(0,\ l-j+j_2)\le l_2\le\min(j_2,\ l)\}\le\min(l+1,\ j-l+1),$$

we end up with

$$\begin{aligned}&\|f\,\mathrm{rot}\,g\|_{G^2_{3/2}}\\ &\quad\le\frac{C}{K^{1/2}}\sum_{j=0}^{\nu^{-1/2}}\frac{1}{j!^{3/2}}\sum_{l=0}^{j}\min(l+1,\ j-l+1)\binom{j}{l}\sup_{|\boldsymbol{l}|=l}\sup_{|\boldsymbol{k}|=j-l}\|f^{\boldsymbol{l}}(\mathrm{rot}\,g)^{\boldsymbol{k}}\|_{L^\infty_tL^2_{x,y}}\\ &\quad\le\frac{C}{K^{1/2}}\sum_{j=0}^{\nu^{-1/2}}\sum_{0\le l\le j/2}(l+1)\binom{j}{l}^{-\frac12}\frac{1}{l!^{3/2}}\sup_{|\boldsymbol{l}|=l}\|f^{\boldsymbol{l}}\|_{L^\infty_{t,x,y}}\frac{1}{(j-l)!^{3/2}}\sup_{|\boldsymbol{k}|=j-l}\|(\mathrm{rot}\,g)^{\boldsymbol{k}}\|_{L^\infty_tL^2_{x,y}}\\ &\qquad+\frac{C}{K^{1/2}}\sum_{j=0}^{\nu^{-1/2}}\sum_{j/2<l\le j}(j-l+1)\binom{j}{l}^{-\frac12}\frac{1}{l!^{3/2}}\sup_{|\boldsymbol{l}|=l}\|f^{\boldsymbol{l}}\|_{L^\infty_tL^2_xL^\infty_y}\frac{1}{(j-l)!^{3/2}}\sup_{|\boldsymbol{k}|=j-l}\|(\mathrm{rot}\,g)^{\boldsymbol{k}}\|_{L^\infty_tL^\infty_xL^2_y}\\ &\quad\le\frac{C}{K^{1/2}}\sum_{j=0}^{\nu^{-1/2}}\sum_{0\le l\le j/2}(l+1)^{\frac52}\binom{j}{l}^{-\frac12}\frac{1}{(l+1)!^{3/2}}\\ &\qquad\qquad\times\sup_{|\boldsymbol{l}|=l}(\|\partial_xf^{\boldsymbol{l}}\|_{L^\infty_tL^2_xL^2_y}+\|f^{\boldsymbol{l}}\|_{L^\infty_tL^2_xL^2_y})^{\frac12}(\|\partial_x\partial_yf^{\boldsymbol{l}}\|_{L^\infty_tL^2_xL^2_y}+\|\partial_yf^{\boldsymbol{l}}\|_{L^\infty_tL^2_xL^2_y})^{\frac12}\\ &\qquad\qquad\times\frac{1}{(j-l)!^{3/2}}\sup_{|\boldsymbol{k}|=j-l}\|(\mathrm{rot}\,g)^{\boldsymbol{k}}\|_{L^\infty_tL^2_{x,y}}\\ &\qquad+\frac{C}{K^{1/2}}\sum_{j=0}^{\nu^{-1/2}}\sum_{j/2<l\le j}(j-l+1)^{\frac52}\binom{j}{l}^{-\frac12}\frac{1}{l!^{3/2}}\sup_{|\boldsymbol{l}|=l}\|f^{\boldsymbol{l}}\|^{\frac12}_{L^\infty_tL^2_xL^2_y}\|\partial_yf^{\boldsymbol{l}}\|^{\frac12}_{L^\infty_tL^2_xL^2_y}\\ &\qquad\qquad\times\frac{1}{(j-l+1)!^{3/2}}\sup_{|\boldsymbol{k}|=j-l}(\|\partial_x(\mathrm{rot}\,g)^{\boldsymbol{k}}\|_{L^\infty_tL^\infty_xL^2_y}+\|(\mathrm{rot}\,g)^{\boldsymbol{k}}\|_{L^\infty_tL^\infty_xL^2_y}).\end{aligned}$$

Here we have used the Sobolev embedding type inequality. By using the bound

$$\begin{aligned}&\sup_{|\boldsymbol{l}|=l}(\|\partial_xf^{\boldsymbol{l}}\|_{L^\infty_tL^2_xL^2_y}+\|f^{\boldsymbol{l}}\|_{L^\infty_tL^2_xL^2_y})^{\frac12}(\|\partial_x\partial_yf^{\boldsymbol{l}}\|_{L^\infty_tL^2_xL^2_y}+\|\partial_yf^{\boldsymbol{l}}\|_{L^\infty_tL^2_xL^2_y})^{\frac12}\\ &\qquad\le\nu^{-\frac14}\sup_{l\le|\boldsymbol{l}|\le l+1}\|f^{\boldsymbol{l}}\|_{L^\infty_tL^2_{x,y}}+\nu^{\frac14}\sup_{l\le|\boldsymbol{l}|\le l+1}\|\partial_yf^{\boldsymbol{l}}\|_{L^\infty_tL^2_{x,y}}\end{aligned}$$

and by observing that there exists $C > 0$ such that, for $\binom{j}{l}^{-1/2}(l+1)^{5/2} \le C$ for $0 \le l \le \frac{1}{2}j$, we have

$$\frac{C}{K^{1/2}} \sum_{j=0}^{\nu^{-1/2}} \sum_{0 \le l \le j/2} (l+1)^{\frac{5}{2}} \binom{j}{l}^{-\frac{1}{2}} \frac{1}{(l+1)!^{3/2}} \times \sup_{|\boldsymbol{l}|=l} (\|\partial_x f^{\boldsymbol{l}}\|_{L^\infty_t L^2_x L^2_y} + \|f^{\boldsymbol{l}}\|_{L^\infty_t L^2_x L^2_y})^{\frac{1}{2}} (\|\partial_x \partial_y f^{\boldsymbol{l}}\|_{L^\infty_t L^2_x L^2_y} + \|\partial_y f^{\boldsymbol{l}}\|_{L^\infty_t L^2_x L^2_y})^{\frac{1}{2}} \times \frac{1}{(j-l)!^{3/2}} \sup_{|\boldsymbol{k}|=j-l} \|(\mathrm{rot}\, g)^{\boldsymbol{k}}\|_{L^\infty_t L^2_{x,y}}$$

$$\le \frac{C}{K^{1/2}\nu^{1/4}} \sum_{j=0}^{\nu^{-1/2}} \sum_{0 \le l \le j/2} \frac{1}{(l+1)!^{3/2}} \sup_{l \le |\boldsymbol{l}| \le l+1} \|f^{\boldsymbol{l}}\|_{L^\infty_t L^2_{x,y}} \frac{1}{(j-l)!^{3/2}} \sup_{|\boldsymbol{k}|=j-l} \|(\mathrm{rot}\, g)^{\boldsymbol{k}}\|_{L^\infty_t L^2_{x,y}}$$
$$+ \frac{C\nu^{1/4}}{K^{1/2}} \sum_{j=0}^{\nu^{-1/2}} \sum_{0 \le l \le j/2} \frac{1}{(l+1)!^{3/2}} \sup_{l \le |\boldsymbol{l}| \le l+1} \|\partial_y f^{\boldsymbol{l}}\|_{L^\infty_t L^2_{x,y}} \frac{1}{(j-l)!^{3/2}} \sup_{|\boldsymbol{k}|=j-l} \|(\mathrm{rot}\, g)^{\boldsymbol{k}}\|_{L^\infty_t L^2_{x,y}}$$
$$\le \frac{C}{K^{1/2}\nu^{1/4}} \|f\|_{G^\infty_{3/2}} \|\mathrm{rot}\, g\|_{G^\infty_{3/2}} + \frac{C\nu^{1/4}}{K^{1/2}} \|\partial_y f\|_{G^\infty_{3/2}} \|\mathrm{rot}\, g\|_{G^\infty_{3/2}},$$

where the discrete Young's convolution inequality is applied in the last line together with the estimate

$$\sum_{j=0}^{\nu^{-1/2}} \frac{1}{j!^{3/2}} \sup_{|\boldsymbol{j}|=j} \|\partial_y f^{\boldsymbol{j}}\|_{L^\infty_t L^2_{x,y}} \le C \|\partial_y f\|_{G^\infty_{3/2}}.$$

Similarly, since $(j-l+1)^{5/2}\binom{j}{l}^{-1/2} \le C$ for $\frac{1}{2}j \le l \le j$, we have

$$\frac{C}{K^{1/2}} \sum_{j=0}^{\nu^{-1/2}} \sum_{j/2 < l \le j} (j-l+1)^{\frac{5}{2}} \binom{j}{l}^{-\frac{1}{2}} \frac{1}{l!^{3/2}} \sup_{|\boldsymbol{l}|=l} \|f^{\boldsymbol{l}}\|^{\frac{1}{2}}_{L^\infty_t L^2_x L^2_y} \|\partial_y f^{\boldsymbol{l}}\|^{\frac{1}{2}}_{L^\infty_t L^2_x L^2_y} \times \frac{1}{(j-l+1)!^{3/2}} \sup_{|\boldsymbol{k}|=j-l} (\|\partial_x (\mathrm{rot}\, g)^{\boldsymbol{k}}\|_{L^\infty_t L^\infty_x L^2_y} + \|(\mathrm{rot}\, g)^{\boldsymbol{k}}\|_{L^\infty_t L^\infty_x L^2_y})$$

$$\le \frac{C}{K^{1/2}} \sum_{j=0}^{\nu^{-1/2}} \sum_{j/2 < l \le j} \frac{1}{l!^{3/2}} \sup_{|\boldsymbol{l}|=l} (\nu^{-\frac{1}{4}} \|f^{\boldsymbol{l}}\|_{L^\infty_t L^2_x L^2_y} + \nu^{\frac{1}{4}} \|\partial_y f^{\boldsymbol{l}}\|_{L^\infty_t L^2_x L^2_y}) \times \frac{1}{(j-l+1)!^{3/2}} \sup_{j-l \le |\boldsymbol{k}| \le j-l+1} \|(\mathrm{rot}\, g)^{\boldsymbol{k}}\|_{L^\infty_t L^\infty_x L^2_y}$$
$$\le \frac{C}{K^{1/2}\nu^{1/4}} \|f\|_{G^\infty_{3/2}} \|\mathrm{rot}\, g\|_{G^\infty_{3/2}} + \frac{C\nu^{1/4}}{K^{1/2}} \|\partial_y f\|_{G^\infty_{3/2}} \|\mathrm{rot}\, g\|_{G^\infty_{3/2}}.$$

Hence the result follows from Lemma C.1. □

Proof of Theorem 2.1. Let C be the universal constant in Theorem 2.4. Then the standard fixed-point theorem in the closed convex set

$$X_R = \left\{ f \in C\left(\left[0, \frac{1}{K}\right]; H^1_{0,\sigma}(\mathbb{T} \times \mathbb{R}_+)\right) \,\middle|\, \|f\|_X \le R \right\}, \quad R = 4C\delta_0 \nu^{\frac{7}{4}},$$

is applied by using Theorem 2.4 and Lemma 7.1, if $\nu \le K^{-2}$ holds and if δ_0 is sufficiently small. We note that the smallness condition $[|w_0|] + [|\mathrm{rot}\, w_0|] \le \delta_0 \nu^{9/4}$, $\|r\|_{G^2_{3/2}} \le \delta_0 \nu^{11/4}$, is needed to close the estimate. Since the argument is standard we omit the details. □

Appendix A: Interpolation estimate for solutions to the Poisson equation

Lemma A.1. *Assume that $Y^k\omega \in L^2(\mathbb{T}_\nu \times \mathbb{R}_+)$ for $k = 0, 1, 2$. Let $\phi \in \dot{H}^1_0(\mathbb{T}_\nu \times \mathbb{R}_+)$ be the solution to the Poisson equation $-\Delta\phi = \omega$ in $\mathbb{T}_\nu \times \mathbb{R}_+$ with $\phi|_{Y=0} = 0$. Then there exists $C > 0$ such that, for any $j \ge 0$, we have*

$$\sup_{Y>0} \|\phi(\,\cdot\,, Y)\|_{L^2(\mathbb{T}_\nu)} \le C((j+1)^{-\frac{1}{4}} \|Y\omega\|_{L^2(\mathbb{T}_\nu\times\mathbb{R}_+)} + (j+1)^{\frac{1}{4}} \|Y^2\omega\|_{L^2(\mathbb{T}_\nu\times\mathbb{R}_+)}). \tag{A-1}$$

Proof. The solution is given by the formula

$$\phi(X, Y) = \int_0^Y e^{-(Y-Y')(-\partial_X^2)^{1/2}} \int_{Y'}^\infty e^{-(Y''-Y')(-\partial_X^2)^{1/2}} \omega(\,\cdot\,, Y'')\, dY''\, dY'.$$

Here $e^{-Y(-\partial_X^2)^{1/2}}$ is the Poisson semigroup. Then we have

$$\|\phi(\,\cdot\,, Y)\|_{L^2(\mathbb{T}_\nu)} \le \int_0^Y \int_{Y'}^\infty \|\omega(Y'')\|_{L^2(\mathbb{T}_\nu)}\, dY''\, dY'.$$

By decomposing the integral $\int_0^Y$ into $\int_0^{\min\{Y,(j+1)^{-1/2}\}}$ and $\int_{\min\{Y,(j+1)^{-1/2}\}}^Y$, we have, from the Hölder inequality,

$$\sup_{Y>0} \|\phi(\,\cdot\,, Y)\|_{L^2(\mathbb{T}_\nu)} \le C(j+1)^{-\frac{1}{4}} \|Y\omega\|_{L^2(\mathbb{T}_\nu\times\mathbb{R})} + C(j+1)^{\frac{1}{4}} \|Y^2\omega\|_{L^2(\mathbb{T}_\nu\times\mathbb{R}_+)}. \qquad \square$$

Lemma A.1 yields the following:

Proposition A.2. *Let $\phi \in \dot{H}^1_0(\mathbb{T}_\nu \times \mathbb{R}_+)$ be the solution to the Poisson equation $-\Delta\phi = \omega$ in $\mathbb{T}_\nu \times \mathbb{R}_+$ with $\phi|_{Y=0} = 0$. Then, for any $j \ge 0$, we have*

$$\begin{aligned} &M_{2,j,1/(1+Y)}[\partial_X\phi] \\ &\le C(j+1)^{-\frac{1}{4}} M_{2,j+1,Y}[\omega] + C(j+1)^{\frac{1}{4}} M_{2,j+1,Y^2}[\omega] + C\kappa\nu^{\frac{1}{2}} j (M_{2,j-1,Y}[\omega] + M_{2,j-1,1}[\nabla\phi]). \end{aligned} \tag{A-2}$$

Here $C > 0$ is a universal constant.

Proof. Since $-\Delta\partial_X\phi = \partial_X\omega$, we have $-(\Delta\partial_X\phi)^j = \partial_X\omega^j$. Then we use the commutator relation

$$-(\Delta\phi)^j = -\nabla\cdot(\nabla\phi)^j + \nu^{\frac{1}{2}} j 2 \frac{\chi_\nu'}{\chi_\nu} (\partial_Y\phi)^j = -\Delta\phi^j + \partial_Y\left(\nu^{\frac{1}{2}} j 2 \frac{\chi_\nu'}{\chi_\nu} \phi^j\right) + \nu^{\frac{1}{2}} j 2 \frac{\chi_\nu'}{\chi_\nu} (\partial_Y\phi)^j.$$

Thus we have the following Poisson equation for ϕ^j:

$$-\Delta\phi^j = \omega^j - \partial_Y\left(\nu^{\frac{1}{2}} j 2 \frac{\chi_\nu'}{\chi_\nu} \phi^j\right) - \nu^{\frac{1}{2}} j 2 \frac{\chi_\nu'}{\chi_\nu} (\partial_Y\phi)^j.$$

Then we decompose $\phi^{\boldsymbol{j}}$ into $\phi_1+\phi_{2,1}+\phi_{2,2}$, so that

$$-\Delta\phi_1=\omega^{\boldsymbol{j}},\quad -\Delta\phi_{2,1}=-\partial_Y\left(\nu^{\frac12}j_2\frac{\chi_\nu'}{\chi_\nu}\phi^{\boldsymbol{j}}\right),\quad -\Delta\phi_{2,2}=-\nu^{\frac12}j_2\frac{\chi_\nu'}{\chi_\nu}(\partial_Y\phi)^{\boldsymbol{j}},$$

subject to the Dirichlet boundary condition. Then Lemma A.1 implies, for $\partial_X\phi_1$,

$$\sup_{Y>0}\|\partial_X\phi_1(\,\cdot\,,Y)\|_{L^2(\mathbb{T}_\nu)}\le C((j+1)^{-\frac14}\|Y\partial_X\omega^{\boldsymbol{j}}\|_{L^2(\mathbb{T}_\nu\times\mathbb{R}_+)}+(j+1)^{\frac14}\|Y^2\partial_X\omega^{\boldsymbol{j}}\|_{L^2(\mathbb{T}_\nu\times\mathbb{R}_+)}).\tag{A-3}$$

On the other hand, the simple energy estimate gives

$$\|\nabla\phi_{2,1}\|\le\nu^{\frac12}j_2\left\|\frac{\chi_\nu'}{\chi_\nu}\phi^{\boldsymbol{j}}\right\|\le\kappa\nu^{\frac12}j_2\|(\partial_Y\phi)^{(j_1,j_2-1)}\|.$$

As for $\phi_{2,2}$, from

$$\frac{1}{\chi_\nu}(\partial_Y\phi)^{\boldsymbol{j}}=e^{-K\tau\nu^{1/2}}(\partial_Y^2\phi)^{(j_1,j_2-1)}=e^{-K\tau\nu^{1/2}}(-\omega^{(j_1,j_2-1)}-\partial_X^2\phi^{(j_1,j_2-1)}),$$

the Hardy inequality, and integration by parts, we have

$$\|\nabla\phi_{2,2}\|\le C\kappa\nu^{\frac12}j_2(\|Y\omega^{(j_1,j_2-1)}\|+\|\partial_X\phi^{(j_1,j_2-1)}\|).$$

Hence we obtain the desired estimate by taking the L^2 norm in time and by taking the supremum over $\boldsymbol{j}$ such that $|\boldsymbol{j}|=j$. □

Appendix B: Proof of Proposition 6.1

Let us go back to (4-1) with $G=0$, but now we impose the no-slip boundary condition $\phi|_{Y=0}=\partial_Y\phi|_{Y=0}=0$ in this appendix. Then we have

$$-\nu^{\frac12}(\Delta\omega)^{\boldsymbol{j}}+(\partial_\tau+K\nu^{\frac12}(j+1))\omega^{\boldsymbol{j}}=-(V\cdot\nabla\omega)^{\boldsymbol{j}}-(\nabla^\perp\phi\cdot\nabla\Omega)^{\boldsymbol{j}}+(\operatorname{rot}F)^{\boldsymbol{j}}=(\operatorname{div}H)^{\boldsymbol{j}},\tag{B-1}$$

where

$$H=-V\omega-\Omega\nabla^\perp\phi+(F_2,-F_1).$$

The idea is to take the L^2 inner product with $\partial_\tau\phi^{\boldsymbol{j}}$, which gives the estimates of $|||\nabla\phi|||_\infty$ and $|||\Delta\phi|||_\infty$ in terms of $|||\nabla\phi|||_{2,\mathbf{1}}'$. The most technical part is the computation of the viscous term $\langle(\Delta\omega)^{\boldsymbol{j}},\partial_\tau\phi^{\boldsymbol{j}}\rangle$ when $j_2\neq0$, for which one needs to convert the vertical derivative $\partial_Y^2\omega$ into the tangential ones by using equation (B-1).

Lemma B.1. *For any $\kappa\in(0,1]$ and $K\ge1$, we have*

$$\begin{aligned}\int_0^{\tau_0}&\langle(\partial_\tau+K\nu^{\frac12}(j+1))\omega^{\boldsymbol{j}},\partial_\tau\phi^{\boldsymbol{j}}\rangle\,d\tau\\&\ge\tfrac12\|\partial_\tau(\nabla\phi)^{\boldsymbol{j}}\|^2_{L^2(0,\tau_0;L^2_{X,Y})}+\tfrac12K\nu^{\frac12}(j+1)(\|(\nabla\phi)^{\boldsymbol{j}}(\tau_0)\|^2-\|(\nabla\phi)^{\boldsymbol{j}}(0)\|^2)\\&\qquad-C\kappa^2K\nu^{\frac12}j(\nu^{\frac12}j^{\frac32})^2M_{\infty,j-1,1}[\nabla\phi]^2-C(\kappa\nu^{\frac12}j)^2M_{2,j-1,1}[\partial_\tau\nabla\phi]^2.\end{aligned}$$

Here C is a universal constant.

Proof. Let us recall the identity

$$\omega^{\boldsymbol{j}} = -(\Delta\phi)^{\boldsymbol{j}} = -\nabla\cdot(\nabla\phi)^{\boldsymbol{j}} + \nu^{\frac{1}{2}} j_2 \frac{\chi_\nu'}{\chi_\nu}(\partial_Y\phi)^{\boldsymbol{j}}, \tag{B-2}$$

which implies

$$\langle(\partial_\tau + K\nu^{\frac{1}{2}}(j+1))\omega^{\boldsymbol{j}}, \partial_\tau\phi^{\boldsymbol{j}}\rangle = \|\partial_\tau(\nabla\phi)^{\boldsymbol{j}}\|^2 + 2\nu^{\frac{1}{2}} j_2 \left\langle \frac{\chi_\nu'}{\chi_\nu}\partial_\tau(\partial_Y\phi)^{\boldsymbol{j}}, \partial_\tau\phi^{\boldsymbol{j}}\right\rangle + \tfrac{1}{2}K\nu^{\frac{1}{2}}(j+1)\partial_\tau\|(\nabla\phi)^{\boldsymbol{j}}\|^2 + 2\nu^{\frac{1}{2}} j_2 K\nu^{\frac{1}{2}}(j+1)\left\langle\frac{\chi_\nu'}{\chi_\nu}(\partial_Y\phi)^{\boldsymbol{j}}, \partial_\tau\phi^{\boldsymbol{j}}\right\rangle.$$

Then, from $\partial_\tau\phi^{\boldsymbol{j}} = \chi_\nu\partial_\tau(e^{-K\tau\nu^{1/2}}(\partial_Y\phi)^{(j_1,j_2-1)})$ for $j_2\ge 1$, we have

$$\int_0^{\tau_0} 2\nu^{\frac{1}{2}} j_2 \left\langle\frac{\chi_\nu'}{\chi_\nu}\partial_\tau(\partial_Y\phi)^{\boldsymbol{j}}, \partial_\tau\phi^{\boldsymbol{j}}\right\rangle d\tau \ge -\tfrac{1}{4}\|\partial_\tau(\nabla\phi)^{\boldsymbol{j}}\|^2_{L^2(0,\tau_0;L^2)} - C(\kappa\nu^{\frac{1}{2}} j)^2(M_{2,j-1,1}[\partial_\tau\nabla\phi]^2 + (K\nu^{\frac{1}{2}})^2 M_{2,j-1,1}[\nabla\phi]^2),$$

while we have, from integration by parts in time,

$$\begin{aligned}
&\int_0^{\tau_0} 2\nu^{\frac{1}{2}} j_2 K\nu^{\frac{1}{2}}(j+1)\left\langle\frac{\chi_\nu'}{\chi_\nu}(\partial_Y\phi)^{\boldsymbol{j}}, \partial_\tau\phi^{\boldsymbol{j}}\right\rangle d\tau \\
&\quad= 2\nu^{\frac{1}{2}} j_2 K\nu^{\frac{1}{2}}(j+1)(e^{-K\tau_0\nu^{1/2}}\langle\chi_\nu'(\partial_Y\phi)^{\boldsymbol{j}}, (\partial_Y\phi)^{(j_1,j_2-1)}\rangle(\tau_0) - \langle\chi_\nu'(\partial_Y\phi)^{\boldsymbol{j}}, (\partial_Y\phi)^{(j_1,j_2-1)}\rangle(0)) \\
&\qquad - 2\nu^{\frac{1}{2}} j_2 K\nu^{\frac{1}{2}}(j+1)\int_0^{\tau_0} e^{-K\tau\nu^{1/2}}\langle\partial_\tau(\partial_Y\phi)^{\boldsymbol{j}}, \chi_\nu'(\partial_Y\phi)^{(j_1,j_2-1)}\rangle\, d\tau \\
&\quad\ge 2\nu^{\frac{1}{2}} j_2 K\nu^{\frac{1}{2}}(j+1)(e^{-K\tau_0\nu^{1/2}}\langle\chi_\nu'(\partial_Y\phi)^{\boldsymbol{j}}(\tau_0), (\partial_Y\phi)^{(j_1,j_2-1)}(\tau_0)\rangle - \langle\chi_\nu'(\partial_Y\phi)^{\boldsymbol{j}}(0), (\partial_Y\phi)^{(j_1,j_2-1)}(0)\rangle) \\
&\qquad - \tfrac{1}{4}\|\partial_\tau(\partial_Y\phi)^{\boldsymbol{j}}\|^2_{L^2(0,\tau_0;L^2)} - C(K\kappa\nu j^2)^2 M_{2,j-1,1}[\nabla\phi]^2.
\end{aligned}$$

We also observe that, for $j_2\ge 1$,

$$\begin{aligned}
&\langle\chi_\nu'(\partial_Y\phi)^{\boldsymbol{j}}, (\partial_Y\phi)^{(j_1,j_2-1)}\rangle \\
&\quad= e^{-K\tau\nu^{1/2}}\langle\chi_\nu'\chi_\nu(\partial_Y\partial_Y\phi)^{(j_1,j_2-1)}, (\partial_Y\phi)^{(j_1,j_2-1)}\rangle \\
&\quad= -\tfrac{1}{2}e^{-K\tau\nu^{1/2}}\langle\partial_Y(\chi_\nu'\chi_\nu)(\partial_Y\phi)^{(j_1,j_2-1)}, (\partial_Y\phi)^{(j_1,j_2-1)}\rangle - e^{-K\tau\nu^{1/2}}\nu^{\frac{1}{2}}(j_2-1)\|\chi_\nu'(\partial_Y\phi)^{(j_1,j_2-1)}\|^2.
\end{aligned}$$

Thus we conclude also from $K\tau\nu^{1/2}\le 1$ that

$$\begin{aligned}
&\int_0^{\tau_0} 2\nu^{\frac{1}{2}} j_2 K\nu^{\frac{1}{2}}(j+1)\left\langle\frac{\chi_\nu'}{\chi_\nu}(\partial_Y\phi)^{\boldsymbol{j}}, \partial_\tau\phi^{\boldsymbol{j}}\right\rangle d\tau \\
&\quad\ge -CK\nu^{\frac{1}{2}}(\kappa\nu^{\frac{1}{2}} j)^2(j\|(\partial_Y\phi)^{(j_1,j_2-1)}(\tau_0)\|^2 + \|(\partial_Y\phi)^{(j_1,j_2-1)}(0)\|^2) \\
&\qquad - \tfrac{1}{4}\|\partial_\tau(\partial_Y\phi)^{\boldsymbol{j}}\|^2_{L^2(0,\tau_0;L^2)} - C(K\kappa\nu j^2)^2 M_{2,j-1,1}[\nabla\phi]^2.
\end{aligned}$$

Combining the above and $M_{2,j-1,1}[\nabla\phi]^2 \le (K\nu^{1/2})^{-1}M_{\infty,j-1,1}[\nabla\phi]^2$, we obtain the desired estimate. □

Lemma B.2. *For any $\kappa \in (0,1]$ and $K \geq 1$, we have*

$$\begin{aligned}\int_0^{\tau_0} \langle -\nu^{\frac12}(\Delta\omega)^{\boldsymbol{j}}, \partial_\tau \phi^{\boldsymbol{j}}\rangle \, d\tau \geq \tfrac12 \nu^{\frac12}(\|\omega^{\boldsymbol{j}}(\tau_0)\|^2 - \|\omega^{\boldsymbol{j}}(0)\|^2) - \tfrac14 M_{2,j,1}[\partial_\tau \nabla\phi]^2 \\ - C(\kappa\nu^{\frac12} j)^2 (M_{2,j-1,1}[\partial_\tau\nabla\phi]^2 + (\nu^{\frac12}(j-1))^2 M_{2,j-2,1}[\partial_\tau\nabla\phi]^2) \\ - C\kappa^2 \nu^{\frac12}(M_{\infty,j,1}[\omega]^2 + (\nu^{\frac12} j)^2 M_{\infty,j-1,1}[\omega]) \\ - CK\nu^{\frac12} j(\kappa\nu^{\frac12} j^{\frac32})^2 (M_{\infty,j-1,1}[\nabla\phi]^2 + (\nu^{\frac12}(j-1))^2 M_{\infty,j-2,1}[\nabla\phi]^2) \\ - C(M_{2,j,1}[H]^2 + (\nu^{\frac12} j)^2 M_{2,j-1,1}[H]^2).\end{aligned}$$

Here C is a universal constant.

Proof. We observe from

$$\begin{aligned}(\Delta\omega)^{\boldsymbol{j}} &= \nabla\cdot(\nabla\omega)^{\boldsymbol{j}} - \nu^{\frac12} j_2 \frac{\chi_\nu'}{\chi_\nu}(\partial_Y\omega)^{\boldsymbol{j}}, \qquad \chi_\nu' = \kappa e^{-\kappa\nu^{1/2}Y}, \\ \nabla\partial_\tau\phi^{\boldsymbol{j}} &= \partial_\tau(\nabla\phi)^{\boldsymbol{j}} + \nu^{\frac12} j_2 \frac{\chi_\nu'}{\chi_\nu}\partial_\tau\phi^{\boldsymbol{j}} \boldsymbol{e}_2,\end{aligned} \tag{B-3}$$

and integration by parts that

$$\langle -\nu^{\frac12}(\Delta\omega)^{\boldsymbol{j}}, \partial_\tau\phi^{\boldsymbol{j}}\rangle = \nu^{\frac12}\langle(\nabla\omega)^{\boldsymbol{j}}, \partial_\tau(\nabla\phi)^{\boldsymbol{j}}\rangle + 2\nu j_2 \left\langle \frac{\chi_\nu'}{\chi_\nu}(\partial_Y\omega)^{\boldsymbol{j}}, \partial_\tau\phi^{\boldsymbol{j}}\right\rangle.$$

Then the similar identities

$$\begin{aligned}(\nabla\omega)^{\boldsymbol{j}} &= \nabla\omega^{\boldsymbol{j}} - \nu^{\frac12} j_2\frac{\chi_\nu'}{\chi_\nu}\omega^{\boldsymbol{j}}\boldsymbol{e}_2, \\ \nabla\cdot\partial_\tau(\nabla\phi)^{\boldsymbol{j}} &= \partial_\tau(\Delta\phi)^{\boldsymbol{j}} + \nu^{\frac12} j_2 \frac{\chi_\nu'}{\chi_\nu}\partial_\tau(\partial_Y\phi)^{\boldsymbol{j}},\end{aligned} \tag{B-4}$$

together with integration by parts, yield

$$\langle -\nu^{\frac12}(\Delta\omega)^{\boldsymbol{j}}, \partial_\tau\phi^{\boldsymbol{j}}\rangle = \nu^{\frac12}\langle\omega^{\boldsymbol{j}}, \partial_\tau\omega^{\boldsymbol{j}}\rangle - 2\nu j_2\left\langle\frac{\chi_\nu'}{\chi_\nu}\omega^{\boldsymbol{j}}, \partial_\tau(\partial_Y\phi)^{\boldsymbol{j}}\right\rangle + 2\nu j_2\left\langle\frac{\chi_\nu'}{\chi_\nu}(\partial_Y\omega)^{\boldsymbol{j}}, \partial_\tau\phi^{\boldsymbol{j}}\right\rangle. \tag{B-5}$$

Again from the above identities about the commutators we have, for $j_2 \geq 1$,

$$\left\langle\frac{\chi_\nu'}{\chi_\nu}\omega^{\boldsymbol{j}}, \partial_\tau(\partial_Y\phi)^{\boldsymbol{j}}\right\rangle = -\left\langle\frac{\chi_\nu'}{\chi_\nu}(\partial_Y\omega)^{\boldsymbol{j}}, \partial_\tau\phi^{\boldsymbol{j}}\right\rangle - \nu^{\frac12}\left\langle\frac{\chi_\nu''}{\chi_\nu}\omega^{\boldsymbol{j}}, \partial_\tau\phi^{\boldsymbol{j}}\right\rangle - \nu^{\frac12}(2j_2-1)\left\langle\left(\frac{\chi_\nu'}{\chi_\nu}\right)^2\omega^{\boldsymbol{j}}, \partial_\tau\phi^{\boldsymbol{j}}\right\rangle.$$

Here $\chi_\nu'' = -\kappa^2 e^{-\kappa\nu^{1/2}Y}$. Thus (B-5) is written as

$$\begin{aligned}\langle -\nu^{\frac12}(\Delta\omega)^{\boldsymbol{j}}, \partial_\tau\phi^{\boldsymbol{j}}\rangle = \nu^{\frac12}\langle\omega^{\boldsymbol{j}}, \partial_\tau\omega^{\boldsymbol{j}}\rangle + 4\nu j_2\left\langle\frac{\chi_\nu'}{\chi_\nu}(\partial_Y\omega)^{\boldsymbol{j}}, \partial_\tau\phi^{\boldsymbol{j}}\right\rangle \\ + 2\nu^{\frac32} j_2\left\langle\frac{\chi_\nu''}{\chi_\nu}\omega^{\boldsymbol{j}}, \partial_\tau\phi^{\boldsymbol{j}}\right\rangle + 2\nu^{\frac32} j_2(2j_2-1)\left\langle\left(\frac{\chi_\nu'}{\chi_\nu}\right)^2\omega^{\boldsymbol{j}}, \partial_\tau\phi^{\boldsymbol{j}}\right\rangle.\end{aligned} \tag{B-6}$$

Let us compute the term $\langle(\chi_\nu'/\chi_\nu)(\partial_Y\omega)^{\boldsymbol{j}}, \partial_\tau\phi^{\boldsymbol{j}}\rangle$. From the identity

$$\frac{1}{\chi_\nu}(\partial_Y\omega)^{\boldsymbol{j}} = e^{-K\tau\nu^{1/2}}(\partial_Y^2\omega)^{(j_1,j_2-1)} = e^{-K\tau\nu^{1/2}}((\Delta\omega)^{(j_1,j_2-1)} - \partial_X^2\omega^{(j_1,j_2-1)}),$$

we have

$$\left\langle \frac{\chi_\nu'}{\chi_\nu}(\partial_Y\omega)^{\boldsymbol{j}},\, \partial_\tau\phi^{\boldsymbol{j}}\right\rangle = e^{-K\tau\nu^{1/2}}\langle \chi_\nu'(\Delta\omega)^{(j_1,j_2-1)},\, \partial_\tau\phi^{\boldsymbol{j}}\rangle + \langle \chi_\nu'\omega^{(j_1+1,j_2-1)},\, \partial_\tau\partial_X\phi^{\boldsymbol{j}}\rangle.$$

Since $\nu^{1/2}(\Delta\omega)^{(j_1,j_2-1)} = (\partial_\tau + K\nu^{1/2}j)\omega^{(j_1,j_2-1)} - (\operatorname{div} H)^{(j_1,j_2-1)}$, the identity (B-6) is written as

$$\begin{aligned}\langle -\nu^{\frac12}(\Delta\omega)^{\boldsymbol{j}},\, \partial_\tau\phi^{\boldsymbol{j}}\rangle &= \nu^{\frac12}\langle\omega^{\boldsymbol{j}},\, \partial_\tau\omega^{\boldsymbol{j}}\rangle + 4\nu^{\frac12}j_2e^{-K\tau\nu^{1/2}}\langle\chi_\nu'(\partial_\tau + K\nu^{\frac12}j)\omega^{(j_1,j_2-1)},\, \partial_\tau\phi^{\boldsymbol{j}}\rangle\\ &\quad - 4\nu^{\frac12}j_2e^{-K\tau\nu^{1/2}}\langle\chi_\nu'(\operatorname{div} H)^{(j_1,j_2-1)},\, \partial_\tau\phi^{\boldsymbol{j}}\rangle + 4\nu j_2\langle\chi_\nu'\omega^{(j_1+1,j_2-1)},\, \partial_\tau\partial_X\phi^{\boldsymbol{j}}\rangle\\ &\quad + 2\nu^{\frac32}j_2\left\langle\frac{\chi_\nu''}{\chi_\nu}\omega^{\boldsymbol{j}},\, \partial_\tau\phi^{\boldsymbol{j}}\right\rangle + 2\nu^{\frac32}j_2(2j_2-1)\left\langle\left(\frac{\chi_\nu'}{\chi_\nu}\right)^2\omega^{\boldsymbol{j}},\, \partial_\tau\phi^{\boldsymbol{j}}\right\rangle. \end{aligned}\tag{B-7}$$

Next we compute the term $\nu^{1/2}j_2e^{-K\tau\nu^{1/2}}\langle\chi_\nu'(\partial_\tau + K\nu^{1/2}j)\omega^{(j_1,j_2-1)},\, \partial_\tau\phi^{\boldsymbol{j}}\rangle$ in (B-7): from the identities in (B-4), we have

$$\begin{aligned}e^{-K\tau\nu^{1/2}}\langle\chi_\nu'(\partial_\tau + K\nu^{\frac12}j)\omega^{(j_1,j_2-1)},\, \partial_\tau\phi^{\boldsymbol{j}}\rangle &= e^{-K\tau\nu^{1/2}}\langle\chi_\nu'(\partial_\tau + K\nu^{\frac12}j)(\nabla\phi)^{(j_1,j_2-1)},\, \partial_\tau(\nabla\phi)^{\boldsymbol{j}}\rangle\\ &\quad + 2\nu^{\frac12}j_2e^{-K\tau\nu^{1/2}}\left\langle\frac{(\chi_\nu')^2}{\chi_\nu}(\partial_\tau + K\nu^{\frac12}j)(\partial_Y\phi)^{(j_1,j_2-1)},\, \partial_\tau\phi^{\boldsymbol{j}}\right\rangle\\ &\quad + \nu^{\frac12}e^{-K\tau\nu^{1/2}}\langle\chi_\nu''(\partial_\tau + K\nu^{\frac12}j)(\partial_Y\phi)^{(j_1,j_2-1)},\, \partial_\tau\phi^{\boldsymbol{j}}\rangle.\end{aligned}$$

By setting $(\nabla\phi)^{\tilde{\boldsymbol{j}}-1} = e^{-K\tau\nu^{1/2}}(\nabla\phi)^{(j_1,j_2-1)}$ for simplicity, we have

$$\begin{aligned}&e^{-K\tau\nu^{1/2}}\langle\chi_\nu'(\partial_\tau + K\nu^{\frac12}j)\omega^{(j_1,j_2-1)},\, \partial_\tau\phi^{\boldsymbol{j}}\rangle\\ &\quad= \langle\chi_\nu'\partial_\tau(\nabla\phi)^{\tilde{\boldsymbol{j}}-1},\, \partial_\tau(\nabla\phi)^{\boldsymbol{j}}\rangle + 2\nu^{\frac12}j_2\left\langle\frac{(\chi_\nu')^2}{\chi_\nu}\partial_\tau(\partial_Y\phi)^{\tilde{\boldsymbol{j}}-1},\, \partial_\tau\phi^{\boldsymbol{j}}\right\rangle + \nu^{\frac12}\langle\chi_\nu''\partial_\tau(\partial_Y\phi)^{\tilde{\boldsymbol{j}}-1},\, \partial_\tau\phi^{\boldsymbol{j}}\rangle\\ &\qquad + K\nu^{\frac12}j\left(\langle\chi_\nu'(\nabla\phi)^{\tilde{\boldsymbol{j}}-1},\, \partial_\tau(\nabla\phi)^{\boldsymbol{j}}\rangle + 2\nu^{\frac12}j_2\left\langle\frac{(\chi_\nu')^2}{\chi_\nu}(\partial_Y\phi)^{\tilde{\boldsymbol{j}}-1},\, \partial_\tau\phi^{\boldsymbol{j}}\right\rangle + \nu^{\frac12}\langle\chi_\nu''(\partial_Y\phi)^{\tilde{\boldsymbol{j}}-1},\, \partial_\tau\phi^{\boldsymbol{j}}\rangle\right).\end{aligned}$$

Since

$$\begin{aligned}\partial_\tau(\nabla\phi)^{\boldsymbol{j}} &= \chi_\nu\partial_\tau(\partial_Y\nabla\phi)^{\tilde{\boldsymbol{j}}-1} = \chi_\nu\partial_Y\partial_\tau(\nabla\phi)^{\tilde{\boldsymbol{j}}-1} - \nu^{\frac12}(j_2-1)\chi_\nu'\partial_\tau(\nabla\phi)^{\tilde{\boldsymbol{j}}-1},\\ \partial_\tau\phi^{\boldsymbol{j}} &= \chi_\nu\partial_\tau(\partial_Y\phi)^{\tilde{\boldsymbol{j}}-1},\end{aligned}$$

we then arrive at

$$\begin{aligned}&\nu^{\frac12}j_2e^{-K\tau\nu^{1/2}}\langle\chi_\nu'(\partial_\tau + K\nu^{\frac12}j)\omega^{(j_1,j_2-1)},\, \partial_\tau\phi^{\boldsymbol{j}}\rangle\\ &= \nu^{\frac12}j_2\Big\{-\tfrac12\langle\partial_Y(\chi_\nu'\chi_\nu)\partial_\tau(\nabla\phi)^{\tilde{\boldsymbol{j}}-1},\, \partial_\tau(\nabla\phi)^{\tilde{\boldsymbol{j}}-1}\rangle - \nu^{\frac12}(j_2-1)\langle(\chi_\nu')^2\partial_\tau(\nabla\phi)^{\tilde{\boldsymbol{j}}-1},\, \partial_\tau(\nabla\phi)^{\tilde{\boldsymbol{j}}-1}\rangle\\ &\qquad + 2\nu^{\frac12}j_2\langle(\chi_\nu')^2\partial_\tau(\partial_Y\phi)^{\tilde{\boldsymbol{j}}-1},\, \partial_\tau(\partial_Y\phi)^{\tilde{\boldsymbol{j}}-1}\rangle + \nu^{\frac12}\langle\chi_\nu''\partial_\tau(\partial_Y\phi)^{\tilde{\boldsymbol{j}}-1},\, \chi_\nu\partial_\tau(\partial_Y\phi)^{\tilde{\boldsymbol{j}}-1}\rangle\\ &\qquad + K\nu^{\frac12}j\big(\langle\chi_\nu'(\nabla\phi)^{\tilde{\boldsymbol{j}}-1},\, \partial_\tau(\nabla\phi)^{\boldsymbol{j}}\rangle + 2\nu^{\frac12}j_2\langle(\chi_\nu')^2(\partial_Y\phi)^{\tilde{\boldsymbol{j}}-1},\, \partial_\tau(\partial_Y\phi)^{\tilde{\boldsymbol{j}}-1}\rangle\\ &\qquad + \nu^{\frac12}\langle\chi_\nu''(\partial_Y\phi)^{\tilde{\boldsymbol{j}}-1},\, \chi_\nu\partial_\tau(\partial_Y\phi)^{\tilde{\boldsymbol{j}}-1}\rangle\big)\Big\}\\ &\geq -C(\kappa\nu^{\frac12}j_2)^2\|\partial_\tau(\nabla\phi)^{\tilde{\boldsymbol{j}}-1}\|^2\\ &\quad + K\nu j_2j\big(\langle\chi_\nu'(\nabla\phi)^{\tilde{\boldsymbol{j}}-1},\, \partial_\tau(\nabla\phi)^{\boldsymbol{j}}\rangle + \nu^{\frac12}j_2\partial_\tau\|\chi_\nu'(\partial_Y\phi)^{\tilde{\boldsymbol{j}}-1}\|^2 + \tfrac12\nu^{\frac12}\partial_\tau\langle\chi_\nu''(\partial_Y\phi)^{\tilde{\boldsymbol{j}}-1},\, \chi_\nu(\partial_Y\phi)^{\tilde{\boldsymbol{j}}-1}\rangle\big).\end{aligned}\tag{B-8}$$

Here we have used the fact that it suffices to consider the case $j_2 \geq 1$, and C is a universal constant. Hence, by going back to (B-7), we have

$$
\begin{aligned}
&\langle -\nu^{\frac{1}{2}}(\Delta\omega)^{\boldsymbol{j}}, \partial_\tau \phi^{\boldsymbol{j}} \rangle \\
&\quad \geq \nu^{\frac{1}{2}} \langle \omega^{\boldsymbol{j}}, \partial_\tau \omega^{\boldsymbol{j}} \rangle - C(\kappa \nu^{\frac{1}{2}} j_2)^2 \| \partial_\tau (\nabla\phi)^{\tilde{\boldsymbol{j}}-1} \|^2 \\
&\qquad + K\nu j_2 j \big(\langle \chi_\nu' (\nabla\phi)^{\tilde{\boldsymbol{j}}-1}, \partial_\tau (\nabla\phi)^{\boldsymbol{j}} \rangle + \nu^{\frac{1}{2}} j_2 \partial_\tau \| \chi_\nu' (\partial_Y \phi)^{\tilde{\boldsymbol{j}}-1} \|^2 + \tfrac{1}{2} \nu^{\frac{1}{2}} \partial_\tau \langle \chi_\nu'' (\partial_Y\phi)^{\tilde{\boldsymbol{j}}-1}, \chi_\nu (\partial_Y \phi)^{\tilde{\boldsymbol{j}}-1} \rangle \big) \\
&\qquad - 4\nu^{\frac{1}{2}} j_2 e^{-K\tau\nu^{1/2}} \langle \chi_\nu' (\operatorname{div} H)^{(j_1, j_2-1)}, \partial_\tau \phi^{\boldsymbol{j}} \rangle + 4\nu j_2 \langle \chi_\nu' \omega^{(j_1+1, j_2-1)}, \partial_\tau \partial_X \phi^{\boldsymbol{j}} \rangle \\
&\qquad + 2\nu^{\frac{3}{2}} j_2 \left\langle \frac{\chi_\nu''}{\chi_\nu} \omega^{\boldsymbol{j}}, \partial_\tau \phi^{\boldsymbol{j}} \right\rangle + 2\nu^{\frac{3}{2}} j_2 (2j_2 - 1) \left\langle \left(\frac{\chi_\nu'}{\chi_\nu} \right)^2 \omega^{\boldsymbol{j}}, \partial_\tau \phi^{\boldsymbol{j}} \right\rangle. \qquad \text{(B-9)}
\end{aligned}
$$

Here C is a universal constant. Next we observe from $\partial_\tau \phi^{\boldsymbol{j}} = \chi_\nu \partial_\tau (\partial_Y \phi)^{\tilde{\boldsymbol{j}}-1}$ that

$$
-4\nu^{\frac{1}{2}} j_2 e^{-K\tau\nu^{1/2}} \langle \chi_\nu' (\operatorname{div} H)^{(j_1, j_2-1)}, \partial_\tau \phi^{\boldsymbol{j}} \rangle \geq -C\kappa \nu^{\frac{1}{2}} j_2 (\| H_1^{(j_1+1, j_2-1)} \| + \| H_2^{\boldsymbol{j}} \|) \| \partial_\tau (\partial_Y \phi)^{\tilde{\boldsymbol{j}}-1} \| \qquad \text{(B-10)}
$$

and also

$$
4\nu j_2 \langle \chi_\nu' \omega^{(j_1+1, j_2-1)}, \partial_\tau \partial_X \phi^{\boldsymbol{j}} \rangle \geq -C\kappa \nu j_2 \| \omega^{(j_1+1, j_2-1)} \| \| \partial_\tau \partial_X \phi^{\boldsymbol{j}} \|, \qquad \text{(B-11)}
$$

$$
2\nu^{\frac{3}{2}} j_2 \left\langle \frac{\chi_\nu''}{\chi_\nu} \omega^{\boldsymbol{j}}, \partial_\tau \phi^{\boldsymbol{j}} \right\rangle \geq -C\kappa^2 \nu^{\frac{3}{2}} j_2 \| \omega^{\boldsymbol{j}} \| \| \partial_\tau (\partial_Y \phi)^{\tilde{\boldsymbol{j}}-1} \|. \qquad \text{(B-12)}
$$

Finally let us compute the term $\nu^{1/2} \langle (\chi_\nu'/\chi_\nu)^2 \omega^{\boldsymbol{j}}, \partial_\tau \phi^{\boldsymbol{j}} \rangle$ when $j_2 \geq 1$. If $j_2 = 1$ then

$$
\begin{aligned}
&\nu^{\frac{1}{2}} \left\langle \left(\frac{\chi_\nu'}{\chi_\nu} \right)^2 \omega^{\boldsymbol{j}}, \partial_\tau \phi^{\boldsymbol{j}} \right\rangle \\
&\quad = \nu^{\frac{1}{2}} \langle (\chi_\nu')^2 e^{-K\tau\nu^{1/2}} (\partial_Y \omega)^{(j_1, 0)}, \partial_\tau (e^{-K\tau\nu^{1/2}} (\partial_Y \phi)^{(j_1, 0)}) \rangle \\
&\quad = \nu^{\frac{1}{2}} \langle e^{-K\tau\nu^{1/2}} \nabla \partial_Y \phi^{(j_1, 0)}, \nabla ((\chi_\nu')^2 \partial_\tau (e^{-K\tau\nu^{1/2}} (\partial_Y \phi)^{(j_1, 0)})) \rangle \\
&\quad = \tfrac{1}{2} \nu^{\frac{1}{2}} \partial_\tau \| \chi_\nu' e^{-K\tau\nu^{1/2}} \nabla \partial_Y \phi^{(j_1, 0)} \|^2 + 2\nu \langle \chi_\nu'' \chi_\nu' e^{-K\tau\nu^{1/2}} \partial_Y^2 \phi^{(j_1, 0)}, \partial_\tau (e^{-K\tau\nu^{1/2}} (\partial_Y \phi)^{(j_1, 0)}) \rangle \\
&\quad \geq \tfrac{1}{2} \nu^{\frac{1}{2}} \partial_\tau \| \chi_\nu' e^{-K\tau\nu^{1/2}} \nabla \partial_Y \phi^{(j_1, 0)} \|^2 - C\kappa^3 \nu \| \omega^{(j_1, 0)} \| \| \partial_\tau (\partial_Y \phi)^{\tilde{\boldsymbol{j}}-1} \|. \qquad \text{(B-13)}
\end{aligned}
$$

If $j_2 \geq 2$ then

$$
\nu^{\frac{1}{2}} \left\langle \left(\frac{\chi_\nu'}{\chi_\nu} \right)^2 \omega^{\boldsymbol{j}}, \partial_\tau \phi^{\boldsymbol{j}} \right\rangle = e^{-2K\tau\nu^{1/2}} \nu^{\frac{1}{2}} \langle (\chi_\nu')^2 (\partial_Y^2 \omega)^{(j_1, j_2-2)}, \partial_\tau \phi^{\boldsymbol{j}} \rangle, \qquad \text{(B-14)}
$$

and then by using the identity

$$
\nu^{\frac{1}{2}} (\Delta\omega)^{(j_1, j_2-2)} = (\partial_\tau + K\nu^{\frac{1}{2}} (j-1)) \omega^{(j_1, j_2-2)} - (\operatorname{div} H)^{(j_1, j_2-2)},
$$

we have

$$
\begin{aligned}
\nu^{\frac{1}{2}} \left\langle \left(\frac{\chi_\nu'}{\chi_\nu} \right)^2 \omega^{\boldsymbol{j}}, \partial_\tau \phi^{\boldsymbol{j}} \right\rangle &= -\nu^{\frac{1}{2}} \langle (\chi_\nu')^2 \omega^{(j_1+2, j_2-2)}, \partial_\tau \phi^{\boldsymbol{j}} \rangle \\
&\quad + e^{-2K\tau\nu^{1/2}} \langle (\chi_\nu')^2 (\partial_\tau + K\nu^{\frac{1}{2}} (j-1)) \omega^{(j_1, j_2-2)}, \partial_\tau \phi^{\boldsymbol{j}} \rangle \\
&\quad - e^{-2K\tau\nu^{1/2}} \langle (\chi_\nu')^2 (\operatorname{div} H)^{(j_1, j_2-2)}, \partial_\tau \phi^{\boldsymbol{j}} \rangle. \qquad \text{(B-15)}
\end{aligned}
$$

As for the second term on the right-hand side of (B-15), we have, for $j \geq j_2 \geq 2$,

$$
\begin{aligned}
&e^{-2K\tau\nu^{1/2}}\langle(\chi_\nu')^2(\partial_\tau+K\nu^{\frac12}(j-1))\omega^{(j_1,j_2-2)},\partial_\tau\phi^{\boldsymbol{j}}\rangle\\
&\quad= e^{-2K\tau\nu^{1/2}}\langle(\chi_\nu')^2(\partial_\tau+K\nu^{\frac12}(j-1))\partial_X\phi^{(j_1,j_2-2)},\partial_\tau\partial_X\phi^{\boldsymbol{j}}\rangle\\
&\qquad\qquad - e^{-2K\tau\nu^{1/2}}\langle(\chi_\nu')^2(\partial_\tau+K\nu^{\frac12}(j-1))(e^{2K\tau\nu^{1/2}}(\partial_Y\phi)^{\tilde{\boldsymbol{j}}-1}),\partial_\tau(\partial_Y\phi)^{\tilde{\boldsymbol{j}}-1}\rangle\\
&\quad\geq -\kappa^2(\|\partial_\tau\partial_X\phi^{(j_1,j_2-2)}\|+K\nu^{\frac12}j\|\partial_X\phi^{(j_1,j_2-2)}\|)\|\partial_\tau\partial_X\phi^{\boldsymbol{j}}\|\\
&\qquad\qquad -\kappa^2(\|\partial_\tau(\partial_Y\phi)^{\tilde{\boldsymbol{j}}-1}\|+K\nu^{\frac12}j\|(\partial_Y\phi)^{\tilde{\boldsymbol{j}}-1}\|)\|\partial_\tau(\partial_Y\phi)^{\tilde{\boldsymbol{j}}-1}\|.
\end{aligned}
$$

Since it is straightforward to see that

$$
\begin{aligned}
-\nu^{\frac12}\langle(\chi_\nu')^2\omega^{(j_1+2,j_2-2)},\partial_\tau\phi^{\boldsymbol{j}}\rangle &\geq -\kappa^2\nu^{\frac12}\|\omega^{(j_1+1,j_2-2)}\|\|\partial_\tau(\partial_X\phi)^{\boldsymbol{j}}\|,\\
-e^{-2K\tau\nu^{1/2}}\langle(\chi_\nu')^2(\operatorname{div}H)^{(j_1,j_2-2)},\partial_\tau\phi^{\boldsymbol{j}}\rangle &\geq -\kappa^2(\|H_1^{(j_1+1,j_2-2)}\|+\|H_2^{(j_1,j_2-1)}\|)\|\partial_\tau(\partial_Y\phi)^{\tilde{\boldsymbol{j}}-1}\|,
\end{aligned}
$$

we obtain, for $j_2 \geq 2$,

$$
\begin{aligned}
&\nu^{\frac12}\left\langle\left(\frac{\chi_\nu'}{\chi_\nu}\right)^2\omega^{\boldsymbol{j}},\partial_\tau\phi^{\boldsymbol{j}}\right\rangle\\
&\quad\geq -\kappa^2(\|\partial_\tau\partial_X\phi^{(j_1,j_2-2)}\|+K\nu^{\frac12}j\|\partial_X\phi^{(j_1,j_2-2)}\|+\nu^{\frac12}\|\omega^{(j_1+1,j_2-2)}\|)\|\partial_\tau(\partial_X\phi)^{\boldsymbol{j}}\|\\
&\qquad -\kappa^2(\|\partial_\tau(\partial_Y\phi)^{\tilde{\boldsymbol{j}}-1}\|+K\nu^{\frac12}j\|(\partial_Y\phi)^{\tilde{\boldsymbol{j}}-1}\|+\|H_1^{(j_1+1,j_2-2)}\|+\|H_2^{(j_1,j_2-1)}\|)\|\partial_\tau(\partial_Y\phi)^{\tilde{\boldsymbol{j}}-1}\|.
\end{aligned}
\tag{B-16}
$$

Collecting (B-9)–(B-12) with (B-13) (for $j_2 = 1$) and (B-16) (for $j_2 \geq 2$), we conclude the desired estimate by using the bound

$$
\begin{aligned}
M_{2,j,1}[f]^2 &= \sup_{|\boldsymbol{j}|=j}\|f^{\boldsymbol{j}}\|^2_{L^2(0,1/(K\nu^{1/2});L^2_{X,Y})}\\
&\leq \frac{1}{K\nu^{1/2}}\sup_{|\boldsymbol{j}|=j}\|f^{\boldsymbol{j}}\|^2_{L^\infty(0,1/(K\nu^{1/2});L^2_{X,Y})}\\
&= \frac{1}{K\nu^{1/2}}M_{\infty,j,1}[f]^2.
\end{aligned}
$$

□

As a consequence of Lemmas B.1 and B.2, we obtain:

Corollary B.3. *There exists $\kappa_B \in (0,1]$ such that, for any $\kappa \in (0,\kappa_B]$ and $K \geq 1$,*

$$
\begin{aligned}
&\nu^{\frac14}\sum_{j=0}^{\nu^{-1/2}}\frac{1}{(j!)^{3/2}\nu^{j/2}}M_{\infty,j,1}[\omega]+K^{\frac12}\nu^{\frac14}\sum_{j=0}^{\nu^{-1/2}}\frac{(j+1)^{1/2}}{(j!)^{3/2}\nu^{j/2}}M_{\infty,j,1}[\nabla\phi]+\sum_{j=0}^{\nu^{-1/2}}\frac{1}{(j!)^{3/2}\nu^{j/2}}M_{2,j,1}[\partial_\tau\nabla\phi],\\
&\leq C\left(\nu^{\frac14}\sum_{j=0}^{\nu^{-1/2}}\frac{1}{(j!)^{3/2}\nu^{j/2}}\|\omega^{\boldsymbol{j}}|_{\tau=0}\|+K^{\frac12}\sum_{j=0}^{\nu^{-1/2}}\frac{\nu^{1/4}(j+1)^{1/2}}{(j!)^{3/2}\nu^{j/2}}\|(\nabla\phi)^{\boldsymbol{j}}|_{\tau=0}\|+\sum_{j=0}^{\nu^{-1/2}}\frac{1}{(j!)^{3/2}\nu^{j/2}}M_{2,j,1}[H]\right).
\end{aligned}
$$

Here C is a universal constant.

We note that

$$\sum_{j=0}^{\nu^{-1/2}} \frac{\nu^{1/4}(j+1)^{1/2}}{(j!)^{3/2}\nu^{j/2}} \|(\nabla\phi)^{\boldsymbol{j}}|_{\tau=0}\| \le C \sum_{j=0}^{\nu^{-1/2}} \frac{1}{(j!)^{3/2}\nu^{j/2}} \|(\nabla\phi)^{\boldsymbol{j}}|_{\tau=0}\| = C[\|\nabla\phi|_{\tau=0}\|]$$

since $j \le \nu^{-1/2}$. By virtue of Corollary B.3, it remains to estimate

$$\sum_{j=1}^{\nu^{-1/2}} \frac{1}{(j!)^{3/2}\nu^{j/2}} M_{2,j,1}[H].$$

Recall that $H = -V\omega - \Omega\nabla^{\perp}\phi + (F_2, -F_1)$. Hence it suffices to show:

Lemma B.4. *For any $\kappa \in (0, 1]$ and $K \ge 1$, we have*

$$\sum_{j=0}^{\nu^{-1/2}} \frac{1}{(j!)^{3/2}\nu^{j/2}} M_{2,j,1}[\Omega\nabla\phi] \le \frac{C(C_0^* + C_1^*)}{\nu^{1/4}} |||\nabla\phi|||'_{2,\mathbf{1}}, \tag{B-17}$$

$$\sum_{j=0}^{\nu^{-1/2}} \frac{1}{(j!)^{3/2}\nu^{j/2}} M_{2,j,1}[V\omega] \le \frac{C(C_0^* + C_1^*)}{\nu^{1/4}} (|||\Delta(\phi - \phi_{\mathrm{app},1})|||'_{2,\mathbf{1}} + |||\Delta\phi_{\mathrm{app},1}|||'_{2,Y}). \tag{B-18}$$

Here $\phi_{\mathrm{app},1} = (\phi_{1,1} + \phi_{1,2})[(I + R_{\mathrm{bc}})^{-1}h]$ with $h = -\partial_Y \Phi_{\mathrm{slip}}|_{Y=0}$, and C is a universal constant.

Proof. We give a sketch of the proof only for (B-18), for (B-17) is proved in a similar manner. Let $|\boldsymbol{j}| = j$. Then

$$\sum_{j=1}^{\nu^{-1/2}} \frac{1}{j!^{3/2}\nu^{j/2}} M_{2,j,1}[V\omega] \le \sum_{j=0}^{\nu^{-1/2}} \frac{1}{j!^{3/2}\nu^{j/2}} \max_{|\boldsymbol{j}|=j} \sum_{\boldsymbol{l}\le\boldsymbol{j}} \binom{\boldsymbol{j}}{\boldsymbol{l}} \|V^{\boldsymbol{l}}\omega^{\boldsymbol{j}-\boldsymbol{l}}\|_{L^2(0,1/(K\nu^{1/2});L^2)}.$$

Here $V^{\boldsymbol{j}} = e^{-K\tau\nu^{1/2}j} B_{j_2}\partial_X^{j_1} V$, while $\omega^{\boldsymbol{j}} = e^{-K\tau\nu^{1/2}(j+1)} B_{j_2}\partial_X^{j_1}\omega$. Since $\omega = -\Delta(\phi - \phi_{\mathrm{app},1}) - \Delta\phi_{\mathrm{app},1}$ by virtue of the construction, we have

$$\begin{aligned}
&\|V^{\boldsymbol{l}}\omega^{\boldsymbol{j}-\boldsymbol{l}}\|_{L^2(0,1/(K\nu^{1/2});L^2)} \\
&\quad \le \|V^{\boldsymbol{l}}\|_{L^\infty} \|(\Delta(\phi - \phi_{\mathrm{app},1}))^{\boldsymbol{j}-\boldsymbol{l}}\|_{L^2(0,1/(K\nu^{1/2});L^2)} + \|\partial_Y V^{\boldsymbol{l}}\|_{L^\infty} \|Y(\Delta\phi_{\mathrm{app},1})^{\boldsymbol{j}-\boldsymbol{l}}\|_{L^2(0,1/(K\nu^{1/2});L^2)}.
\end{aligned}$$

By using $\binom{\boldsymbol{j}}{\boldsymbol{l}} \le \binom{j}{l}$ with $l = |\boldsymbol{l}|$, we have

$$\begin{aligned}
&\frac{1}{j!^{3/2}\nu^{j/2}} \sum_{\boldsymbol{l}\le\boldsymbol{j}} \binom{\boldsymbol{j}}{\boldsymbol{l}} \|V^{\boldsymbol{l}}\omega^{\boldsymbol{j}-\boldsymbol{l}}\|_{L^2(0,1/(K\nu^{1/2});L^2)} \\
&\le \sum_{\boldsymbol{l}\le\boldsymbol{j}} \left(\frac{l!\,(j-l)!}{j!}\right)^{\frac{1}{2}} \frac{M_{2,j-l,1}[\Delta(\phi - \phi_{\mathrm{app},1})] + M_{2,j-l,Y}[\Delta\phi_{\mathrm{app},1}]}{(j-l)!^{3/2}\nu^{(j-l)/2}} \frac{1}{l!^{3/2}\nu^{l/2}} \max_{|\boldsymbol{l}|=l} (\|V^{\boldsymbol{l}}\|_{L^\infty} + \|\partial_Y V^{\boldsymbol{l}}\|_{L^\infty}).
\end{aligned}$$

Next we observe that, for all $l \in \mathbb{N} \cup \{0\}$,

$$\#\{\boldsymbol{l} \mid |\boldsymbol{l}| = l,\ \boldsymbol{l} \le \boldsymbol{j}\} = \#\{l_2,\ \max(0,\ l - j + j_2) \le l_2 \le \min(j_2,\ l)\} \le \min(l+1,\ j - l + 1),$$

which gives the bound of the form $\sum_{l\le j} \le \sum_{l=0}^{j} \min(l+1,\, j-l+1)$. Hence we have

$$\begin{aligned}
&\frac{1}{j!^{3/2}\nu^{j/2}} \sum_{\boldsymbol{l}\le \boldsymbol{j}} \binom{\boldsymbol{j}}{\boldsymbol{l}} \|V^{\boldsymbol{l}}\omega^{\boldsymbol{j}-\boldsymbol{l}}\|_{L^2(0,1/(K\nu^{1/2});L^2)} \\
&\quad \le \sum_{l=0}^{j} \min(l+1,\, j-l+1)\Big(\frac{l!\,(j-l)!}{j!}\Big)^{\frac12} \\
&\qquad\qquad \times \frac{M_{2,j-l,1}[\Delta(\phi-\phi_{\mathrm{app},1})]+M_{2,j-l,Y}[\Delta\phi_{\mathrm{app},1}]}{(j-l)!^{3/2}\nu^{(j-l)/2}} \frac{1}{l!^{3/2}\nu^{l/2}} \max_{|\boldsymbol{l}|=l}(\|V^{\boldsymbol{l}}\|_{L^\infty}+\|\partial_Y V^{\boldsymbol{l}}\|_{L^\infty}).
\end{aligned}$$

Since $\min(l+1,\, j-l+1)(l!\,(j-l)!/j!)^{1/2}$ is uniformly bounded about $0\le l\le j$, the Young inequality for l^1 convolution gives the inequality

$$\begin{aligned}
&\sum_{j=0}^{\nu^{-1/2}} \frac{1}{j!^{3/2}\nu^{j/2}} \max_{|\boldsymbol{j}|=j} \sum_{\boldsymbol{l}\le \boldsymbol{j}} \binom{\boldsymbol{j}}{\boldsymbol{l}} \|V^{\boldsymbol{l}}\omega^{\boldsymbol{j}-\boldsymbol{l}}\|_{L^2(0,1/(K\nu^{1/2});L^2)} \\
&\quad \le C \sum_{j=0}^{\nu^{-1/2}} \frac{1}{j!^{3/2}\nu^{j/2}} \max_{|\boldsymbol{j}|=j}(\|V^{\boldsymbol{j}}\|_{L^\infty}+\|\partial_Y V^{\boldsymbol{j}}\|_{L^\infty}) \\
&\qquad\qquad \times \sum_{j=0}^{\nu^{-1/2}} \frac{1}{j!^{3/2}\nu^{j/2}} \max_{|\boldsymbol{j}|=j}(M_{2,\boldsymbol{j},1}[\Delta(\phi-\phi_{\mathrm{app},1})]+M_{2,\boldsymbol{j},Y}[\Delta\phi_{\mathrm{app},1}]).
\end{aligned}$$

Then the desired estimate follows by noticing $\partial_Y V^{\boldsymbol{j}} = (\partial_Y V)^{\boldsymbol{j}} + \nu^{1/2} j_2 \chi_\nu'(\partial_Y V)^{(j_1,j_2-1)}$ and the bound of the form $|||f|||_2 \le \nu^{-1/4}|||f|||'_{2,\mathbf{1}}$. $\square$

Proposition 6.1 follows from Corollary B.3 and Lemma B.4.

Appendix C: Estimate of the Biot–Savart law

Lemma C.1. *The following statement holds if κ is sufficiently small. Assume that*

$$f \in C([0,1/K];\, H^1(\mathbb{T}\times\mathbb{R}_+)^2)$$

satisfies $\operatorname{div} f = 0$ *for* $y>0$ *and* $f_2|_{y=0}=0$. *Then*

$$\|\nabla f\|_{G^p_{3/2}} \le C\|\operatorname{rot} f\|_{G^p_{3/2}}, \qquad p\in[1,\infty].$$

Here C is a universal constant.

Proof. We observe that $\partial_y f_1 = \operatorname{rot} f + \partial_x f_2$ and $\partial_y f_2 = -\partial_x f_1$. Hence it suffices to show

$$\|\partial_x f\|_{G^p_{3/2}} \le C\|\operatorname{rot} f\|_{G^p_{3/2}}.$$

Since $f=\nabla^\perp\phi$ with the stream function ϕ and $-\Delta\phi=\omega$ with $\omega=\operatorname{rot} g$ and $\phi|_{y=0}=0$, we have

$$-(\Delta\partial_x\phi)^{\boldsymbol{j}} = \partial_x\omega^{\boldsymbol{j}}, \quad \omega^{\boldsymbol{j}} = e^{-Kt(j+1)}\chi^{j_2}\partial_y^{j_2}\partial_x^{j_1}\omega, \quad j_1+j_2=j.$$

By virtue of the identity $-(\Delta\partial_x\phi)^{j} = -\nabla\cdot(\partial_x\nabla\phi)^{j} + j_2(\chi'/\chi)(\partial_y\partial_x\phi)^{j}$, integration by parts gives

$$\|(\nabla\partial_x\phi)^{j}\|^2 + 2j_2\left\langle\frac{\chi'}{\chi}(\partial_y\partial_x\phi)^{j},\,\partial_x\phi^{j}\right\rangle = -\langle\omega^{j},\,\partial_x^2\phi^{j}\rangle.$$

Since $\partial_x\phi^{j} = e^{-Kt}\chi(\partial_y\partial_x\phi)^{(j_1,j_2-1)}$, we thus have

$$\|(\nabla\partial_x\phi)^{j}\| \le C(\|\omega^{j}\| + \kappa j\|(\partial_y\partial_x\phi)^{(j_1,j_2-1)}\|),$$

where C is a universal constant. This estimate implies $\|\partial_x\nabla\phi\|_{G^p_{3/2}} \le C(\|\omega\|_{G^p_{3/2}} + \kappa\|\partial_x\partial_y\phi\|_{G^p_{3/2}})$, and thus, by taking κ small enough, we obtain $\|\partial_x\nabla\phi\|_{G^p_{3/2}} \le C\|\omega\|_{G^p_{3/2}}$. $\square$

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NONLOCAL OPERATORS RELATED TO NONSYMMETRIC FORMS II: HARNACK INEQUALITIES

MORITZ KASSMANN AND MARVIN WEIDNER

Local boundedness and Harnack inequalities are studied for solutions to parabolic and elliptic integrodifferential equations whose governing nonlocal operators are associated with nonsymmetric forms. We present two independent proofs, one based on the De Giorgi iteration and the other on the Moser iteration technique. This article is a continuation of work of Kassmann and Weidner (2022), where Hölder regularity and a weak Harnack inequality are proved in a similar setup.

1. Introduction

The aim of this work is to prove local boundedness estimates and a Harnack inequality for weak solutions to parabolic equations of type

$$\partial_t u - Lu = f \quad \text{in } I_R(t_0) \times B_{2R} \subset \mathbb{R}^{d+1}, \tag{PDE}$$

where $B_{2R} \subset \Omega$ is some ball, $I_R(t_0) := (t_0 - R^\alpha, t_0 + R^\alpha) \subset \mathbb{R}$, and $f \in L^\infty(I_R(t_0) \times B_{2R})$. Equation (PDE) is governed by a linear nonlocal operator of the form

$$-Lu(x) = 2\,\text{p.v.} \int_{\mathbb{R}^d} (u(x) - u(y))K(x, y)\,dy. \tag{1-1}$$

Such operators are determined by jumping kernels $K : \mathbb{R}^d \times \mathbb{R}^d \to [0, \infty]$, which are allowed to be nonsymmetric. We also investigate solutions to the equation

$$\partial_t u - \widehat{L}u = f \quad \text{in } I_R(t_0) \times B_{2R} \subset \mathbb{R}^{d+1}, \tag{$\widehat{\text{PDE}}$}$$

which is driven by the dual operator $\widehat{L}$ associated with L.

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In this work, we prove local boundedness of weak solutions to (PDE) and $(\widehat{\text{PDE}})$ via an adaptation of the De Giorgi method to nonlocal operators with nonsymmetric jumping kernels. We also provide an alternative proof of local boundedness via the Moser iteration. Finally, combined with the weak Harnack inequality from [Kassmann and Weidner 2022], we obtain a full Harnack inequality.

The novelty of our result consists in the lack of symmetry of the underlying operator. Let us write the decomposition $K = K_s + K_a$, where the symmetric part K_s and antisymmetric part K_a are given by

$$K_s(x, y) = \tfrac{1}{2}K(x, y) + K(y, x), \quad K_a(x, y) = \tfrac{1}{2}K(x, y) - K(y, x), \quad x, y \in \mathbb{R}^d.$$

Note that the nonnegativity of K implies

$$|K_a(x, y)| \le K_s(x, y). \tag{1-2}$$

We can write for the nonsymmetric bilinear form associated with L

$$\mathcal{E}(u, v) := 2\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}(u(x) - u(y))v(x)K(x, y)\,\mathrm{d}y\,\mathrm{d}x =: \mathcal{E}^{K_s}(u, v) + \mathcal{E}^{K_a}(u, v),$$

where

$$\mathcal{E}^{K_s}(u, v) = \int_{\mathbb{R}^d}\int_{\mathbb{R}^d}(u(x) - u(y))(v(x) - v(y))K_s(x, y)\,\mathrm{d}y\,\mathrm{d}x,$$

$$\mathcal{E}^{K_a}(u, v) = \int_{\mathbb{R}^d}\int_{\mathbb{R}^d}(u(x) - u(y))(v(x) + v(y))K_a(x, y)\,\mathrm{d}y\,\mathrm{d}x.$$

In order to treat the antisymmetric part of the bilinear form, a refinement of the existing techniques for symmetric operators is required.

We have in mind the following three prototypes of kernels K for $\alpha \in (0, 2)$:

$$K_1(x, y) = g(x, y)|x - y|^{-d-\alpha}, \tag{1-3}$$

where $g : \mathbb{R}^d \times \mathbb{R}^d \to [\lambda, \Lambda]$ is a suitable nonsymmetric function for $0 < \lambda \le \Lambda < \infty$,

$$K_2(x, y) = |x - y|^{-d-\alpha} + (V(x) - V(y))\mathbb{1}_{\{|x-y|\le L\}}(x, y)|x - y|^{-d-\alpha}, \tag{1-4}$$

where $L \in (0, \infty]$ and $V : \mathbb{R}^d \to \mathbb{R}^d$ is a suitable function, and

$$K_3(x, y) = |x - y|^{-d-\alpha}\mathbb{1}_D(x - y) + |x - y|^{-d-\beta}\mathbb{1}_C(x - y), \tag{1-5}$$

where $C \subset \mathbb{R}^d$ is a cone, $D \subset \mathbb{R}^d$ is a double-cone such that $C \cap D = \varnothing$, and $0 < \beta < \frac{1}{2}\alpha$.

1.1. *Main results.*

Our first main result is the following Harnack inequality for weak solutions to (PDE). We state and discuss our assumptions in Section 2.

Theorem 1.1. *Assume (K2), (cutoff), $(K^{\le}_{\mathrm{loc}})$, (Sob), and (Poinc) for some $\alpha \in (0, 2)$. Let $f \in L^\infty(I \times \Omega)$.*

(i) *Assume that $(\mathrm{K1}_{\mathrm{loc}})$ holds for some $\theta \in [d/\alpha, \infty]$. Then there exist $c > 0$ and $0 < c_1 < c_2 < c_3 < c_4 \le 1$ such that, for every $0 < R \le 1$ and every nonnegative, weak solution u to (PDE) in $I_R(t_0) \times B_{2R}$,*

$$\sup_{(t_0 - c_2R^\alpha, t_0 - c_1R^\alpha)\times B_{R/4}} u \le c \inf_{(t_0 + c_1R^\alpha, t_0 + c_4R^\alpha)\times B_{R/2}} u + c \sup_{(t_0 - c_3R^\alpha, t_0 - c_1R^\alpha)} \mathrm{Tail}_{K,\alpha}(u, R) + cR^\alpha\|f\|_{L^\infty}, \tag{1-6}$$

where $B_{2R} \subset \Omega \subset \mathbb{R}^d$.

(ii) *Assume that* (K1$_{\text{glob}}$) *holds for some* $\theta \in (d/\alpha, \infty]$. *Then there exist* $c > 0$ *and* $0 < c_1 < c_2 < c_3 < c_4 \le 1$ *such that, for every* $0 < R \le 1$ *and every nonnegative, weak solution* u *to* ($\widehat{\text{PDE}}$) *in* $I_R(t_0) \times B_{2R}$,

$$\sup_{(t_0-c_2R^\alpha, t_0-c_1R^\alpha)\times B_{R/4}} u \le c \inf_{(t_0+c_1R^\alpha, t_0+c_4R^\alpha)\times B_{R/2}} u + c \sup_{(t_0-c_3R^\alpha, t_0-c_1R^\alpha)} \widehat{\text{Tail}}_{K,\alpha}(u, R) + cR^\alpha \|f\|_{L^\infty}. \tag{1-7}$$

The aforementioned Harnack inequality for nonnegative weak solutions u to (PDE) is a direct consequence of a weak Harnack inequality as it was proved in [Kassmann and Weidner 2022] (see Theorem 6.3) and an L^∞-L^1-estimate of the form (see Theorem 3.6 or Theorem 4.8)

$$\sup_{(t_0-(R/8)^\alpha, t_0)\times B_{R/2}} u \le c\left(\fint_{(t_0-(R/4)^\alpha, t_0)\times B_R} u + \sup_{(t_0-(R/4)^\alpha, t_0)} \text{Tail}_{K,\alpha}(u, R) + R^\alpha \|f\|_{L^\infty}\right). \tag{1-8}$$

Therefore large parts of this paper are dedicated to proving (1-8). Given $0 < R \le 1$, the nonlocal tail term is defined as

$$\text{Tail}_{K,\alpha}(v, R, x_0) := R^\alpha \int_{B_{2R}(x_0)\setminus B_{R/2}(x_0)} \frac{|v(y)|}{|x_0 - y|^{d+\alpha}}\, \mathrm{d}y + \sup_{x\in B_{3R/2}(x_0)} \int_{B_{2R}(x_0)^c} |v(y)| K(x, y)\, \mathrm{d}y.$$

For a detailed discussion of nonlocal tail terms, we refer the reader to Section 2.3.

Remark 1.2 (time-inhomogeneous kernels). It is possible to extend Theorem 1.1 to time-inhomogeneous jumping kernels $k : I \times \mathbb{R}^d \times \mathbb{R}^d \to [0, \infty]$ by following an approach similar to that in [Kassmann and Weidner 2022]. For k_s, we may assume pointwise comparability with a time-homogeneous jumping kernel satisfying (cutoff), ($\mathcal{E}_\ge$), and ($K^{\le}_{\text{loc}}$). In place of the first estimate in (K1$_{\text{loc}}$), we need

$$\left\| \int_{B_{2r}} \frac{|k_a(\cdot\,; \cdot\,, y)|^2}{J(\cdot\,, y)}\, \mathrm{d}y \right\|_{L^{\mu,\theta}_{t,x}(I_r\times B_{2r})} \le C$$

for a suitable symmetric jumping kernel $J : \mathbb{R}^d \times \mathbb{R}^d \to [0, \infty]$. The parameters (μ, θ) have to satisfy the compatibility condition

$$\frac{d}{\alpha\theta} + \frac{1}{\mu} < 1. \tag{CP}$$

Then, if suitable time-inhomogeneous analogs to (K2) and (UJS), or ($\widehat{\text{UJS}}$), hold, we can prove a Harnack inequality of the form (1-6) and (1-7) for nonnegative, weak solutions to the corresponding parabolic equations (PDE) and ($\widehat{\text{PDE}}$), respectively. For solutions to (PDE) we can also allow for equality in (CP) if $\theta > d/\alpha$. The range of exponents prescribed by (CP) align with the important classical results from the local theory; see [Aronson and Serrin 1967; Ivanov et al. 1966; Ladyzhenskaya et al. 1968]. Note that, by scaling arguments, one can see $d/(\alpha\theta) + 1/\mu = 1$ is the limit case for regularity results in Hölder spaces.

Remark 1.3. We observe that there is a positive distance of size $2(1 - 2^{-\alpha})R^\alpha$ between the two time intervals in the estimates (1-6) and (1-7). The existence of such time delay in the parabolic Harnack inequality comes from the method of proof we employ; see [Moser 1964]. For nonlocal equations, as for example the fractional heat equation, it can be neglected; see [Bonforte et al. 2017; Dier et al. 2020].

The second main result of this article concerns the corresponding stationary problems

$$-Lu = f \quad \text{in } B_{2R}, \tag{ell-PDE}$$

$$-\widehat{L}u = f \quad \text{in } B_{2R}, \tag{ell-$\widehat{\text{PDE}}$}$$

where $f \in L^\infty(B_{2R})$. We obtain the following elliptic Harnack inequality for weak solutions.

Theorem 1.4. *Assume* (K2), (cutoff), *and* $(\mathcal{E}_\geq)$ *for some* $\alpha \in (0, 2)$. *Let* $f \in L^\infty(\Omega)$.

(i) *Assume that* $(\mathrm{K1}_{\mathrm{loc}})$ *and* (UJS) *hold for some* $\theta \in [d/\alpha, \infty]$. *Then there exists* $c > 0$ *such that, for every* $0 < R \leq 1$ *and every nonnegative, weak solution* u *to* (ell-PDE) *in* B_{2R},

$$\sup_{B_{R/4}} u \leq c\Big(\inf_{B_{R/2}} u + R^\alpha \|f\|_{L^\infty}\Big), \tag{1-9}$$

where $B_{2R} \subset \Omega \subset \mathbb{R}^d$.

(ii) *Assume that* $(\mathrm{K1}_{\mathrm{glob}})$ *and* $(\widehat{\mathrm{UJS}})$ *hold for some* $\theta \in (d/\alpha, \infty]$. *Then there exists* $c > 0$ *such that, for every* $0 < R \leq 1$ *and every nonnegative, weak solution* u *to* (ell-$\widehat{\text{PDE}}$) *in* B_{2R}, *estimate* (1-9) *holds.*

As in (1-9), for elliptic equations, we are able to estimate the supremum of u by local quantities only. To this end, we prove a suitable estimate of the nonlocal tail term (see Corollary 5.3).

In the parabolic case, the situation is more complicated since we require the tail estimate to be uniform in t. The same difficulty occurs in the symmetric case. We comment on possible corresponding extensions of Theorem 1.1 in Section 6.3.

Remark 1.5. All constants in Theorems 1.1 and 1.4 depend only on d, α, θ and the constants in $(\mathrm{K1}_{\mathrm{loc}})$, (K2), (cutoff), (Poinc), (Sob), (UJS), $(K^{\leq}_{\mathrm{loc}})$, $(\mathcal{E}_\geq)$.

Remark 1.6. Theorems 1.1 and 1.4 remain valid for solutions u to (PDE), ($\widehat{\text{PDE}}$), and (ell-PDE), (ell-$\widehat{\text{PDE}}$) if $f \in L^\infty(I_R(t_0); L^\Theta(B_{2R}))$ and $f \in L^\Theta(B_{2R})$, respectively, for some $\theta \in (d/\alpha, \infty)$ with only marginal manipulations in the proofs. We exclude more general source terms in this work.

The contributions of this work can be summarized as follows:

(i) The main accomplishment is the extension of elliptic and parabolic regularity results — including full Harnack inequalities — for nonlocal problems to operators with *nonsymmetric* jumping kernels. In light of example (1-4), the operators under consideration include nonlocal counterparts of second-order differential operators in divergence form with a drift term

$$-\mathcal{L}u = -\partial_i(a_{i,j}\partial_j u) + b_i\partial_i u \quad \text{and} \quad -\widehat{\mathcal{L}}u = -\partial_i(a_{i,j}\partial_j u + b_i u),$$

respectively. Our results align with the corresponding theory for local operators; see [Aronson and Serrin 1967; Gilbarg and Trudinger 1983; Ladyzhenskaya et al. 1968; Stampacchia 1965].

(ii) As nonsymmetric kernels require a careful treatment, several parts of the energy methods for nonlocal operators are refined in this work. For instance, we give a new proof of local boundedness using the Moser iteration for positive exponents (see Section 4).

Moreover, as illustrated in example (1-5), nonsymmetric jumping kernels might naturally involve terms of lower-order, causing a difference between the growth behavior at zero and infinity. We introduce tail terms which take into account this phenomenon (see Section 2.3).

(iii) Technical issues of minor importance in other works are clarified, e.g., the treatment of Steklov averages (see the Appendix).

1.2. *Related literature.* The study of Harnack inequalities for symmetric nonlocal operators has become an active field of research in the past 20 years. It has been observed that a classical elliptic Harnack inequality of the form

$$\sup_{B_r} u \le c \inf_{B_r} u \tag{1-10}$$

fails even for harmonic functions u with respect to the fractional Laplacian $(-\Delta)^{\alpha/2}$ in B_{2r} if one merely assumes u to be nonnegative in the solution domain B_{2r}; see [Kassmann 2007]. Indeed, due to the nonlocality it is necessary either to assume u to be globally nonnegative — as in [Riesz 1938] and in this article — or to add the nonlocal tail of u_- to the right-hand side of (1-10). Such an estimate was proposed in [Kassmann 2011]. We refer to both estimates as a *Harnack inequality* in the context of this article.

A lot of research activity has centered around the challenge to establish a Harnack inequality for a larger class of nonlocal operators. First, we comment on corresponding elliptic regularity results for symmetric nonlocal operators related to energy forms. A Harnack inequality and Hölder estimates were proved in [Di Castro et al. 2014; 2016] for operators with a jumping kernel that is pointwise comparable to the kernel of the fractional p-Laplacian by a nonlocal De Giorgi-type iteration. This method was refined in [Cozzi 2017] to allow for more general nonlinearities. [Schulze 2019] considers a class of linear integrodifferential operators governed by jumping kernels satisfying an average integral bound instead of a pointwise lower bound.

However, it is well known that for the deduction of interior Hölder regularity estimates a weak Harnack inequality (see Theorem 6.3) is sufficient. Such inequalities hold for a much larger class of operators. In fact, only comparability of the energy forms to the $H^{\alpha/2}$-seminorm on small scales and a suitable upper bound for the probability of large jumps are required; see [Dyda and Kassmann 2020]. That is why operators with singular jumping measures that may be anisotropic (see [Chaker and Kassmann 2020; Chaker et al. 2019]) also satisfy Hölder regularity estimates. However, the Harnack inequality may fail for singular operators as was already observed in [Bogdan and Sztonyk 2005]. Hence it is an exciting (and still open) question to find equivalent conditions on the jumping kernel for which a (weak) elliptic Harnack inequality will hold. For α-stable translation-invariant operators, conditions on the jumping kernel are established in [Bogdan and Sztonyk 2005] that are equivalent to a Harnack inequality.

Second, we comment on parabolic Harnack inequalities of the form

$$\sup_{I_r^{\ominus}\times B_r} u \le c \inf_{I_r^{\oplus}\times B_r} u \tag{1-11}$$

for globally nonnegative solutions u to (PDE). Note that such results imply corresponding estimates for weak solutions to the stationary equation (ell-PDE). So far, parabolic Harnack inequalities have not been obtained via purely analytic methods, not even in the symmetric case. A major challenge in the parabolic case seems to be the correct treatment of the time-dependence in the nonlocal tail terms. For a discussion of this issue, we refer the reader to Section 2 and Section 6.3.

Parabolic Hölder estimates and local boundedness have been obtained via an adaptation of the nonlocal De Giorgi method in [Ding et al. 2021; Kim 2019; 2020; Strömqvist 2019b]. A proof of Hölder estimates based on Moser's technique can be found in [Felsinger and Kassmann 2013].

Using the corresponding Hunt process and its heat kernel, parabolic Harnack inequalities of the form (1-11) were first proved for symmetric Dirichlet forms with jumping measures pointwise comparable to the α-stable kernel in [Bass and Levin 2002; Chen and Kumagai 2003]. The authors also obtain two-sided heat kernel bounds. Numerous articles have analyzed the exact relationship between parabolic and elliptic Harnack inequalities, heat kernel bounds, and Hölder regularity estimates for nonlocal operators in connection to the geometry of the underlying metric measure space. Such a program was carried out in a series of papers [Chen et al. 2019; 2020; Grigor'yan et al. 2014; 2015; 2018]. On $\mathbb{R}^d$ it turns out that (1-11) is equivalent to a Poincaré inequality (see (Poinc)), a pointwise upper bound of the jumping kernel, and (UJS).

In contrast to the symmetric case, for nonlocal operators associated with nonsymmetric forms, pointwise estimates have not yet been studied systematically. Some results have been obtained making use of a sector-type condition. Well-posedness of the Dirichlet problem is proved in [Felsinger et al. 2015]. In the present article and in [Kassmann and Weidner 2022], we provide Harnack inequalities and interior Hölder regularity estimates for nonlocal operators that contain a nonlocal drift term of lower-order. These results can be regarded as nonlocal counterparts of the famous regularity results for local equations by Aronson and Serrin [1967] and Ladyzhenskaya, Solonnikov, and Ural'tceva [Ladyzhenskaya et al. 1968] in the linear case. Hölder estimates for kinetic integrodifferential equations including certain nonlocal operators with nonsymmetric jumping kernels are established in [Imbert and Silvestre 2020] using an adaptation of the De Giorgi iteration. The class of nonsymmetric kernels in their work does not contain the class of kernels in our work, and vice versa.

Note that, as an application of the regularity estimates in [Kassmann and Weidner 2022], it is possible to establish Markov chain approximation results not only for diffusion processes with drift terms, but also for certain nonsymmetric jump processes; see [Weidner 2023]. In light of [Chen et al. 2020] and [Grigor'yan et al. 2018], we consider it an interesting problem to establish heat kernel estimates for nonlocal operators associated with nonsymmetric forms, and to investigate their stability on general doubling metric measure spaces, as well as their connection to Harnack inequalities.

1.3. *Outline.* This article is structured as follows: In Section 2 we state and discuss our assumptions and the notion of a weak solution to (PDE) and ($\widehat{\text{PDE}}$). A Caccioppoli-type estimate for nonsymmetric forms and an a priori L^∞-L^2-estimate involving the nonlocal tail is proved in Section 3 using a nonsymmetric version of the De Giorgi iteration. An analogous result is established in Section 4 using a nonlocal adaptation of the Moser iteration technique for large positive exponents. Note that Sections 3 and 4 are fully independent of one another. In Section 5 we establish an upper bound for the nonlocal tails of supersolutions to (PDE) and ($\widehat{\text{PDE}}$). Our two main results, Theorems 1.1 and 1.4, are proved in Section 6.

2. Preliminaries

In this section we state and discuss the assumptions in our main results (see Section 2.1). Moreover, we provide the notion of a super- or subsolution to (PDE) and $(\widehat{\text{PDE}})$, as well as the corresponding stationary equations (ell-PDE) and (ell-$\widehat{\text{PDE}}$) (see Section 2.2). Another goal of this section is to introduce nonlocal tail terms which suit the class of nonsymmetric operators under consideration and are designed in such a way that they are compatible with the iteration techniques carried out in the remainder of this article (see Section 2.3).

We introduce the following notation: First of all, given $a, b \in \mathbb{R}$, we write $a \wedge b = \min\{a, b\}$ and $a \vee b := \max\{a, b\}$. Moreover, given a set $M \subset \mathbb{R}^d \times \mathbb{R}^d$, we write

$$\mathcal{E}_M(u, v) := \iint_M (u(x) - u(y))v(x)K(x, y)\,dx\,dy.$$

Analogously, we define $\mathcal{E}_M^{K_s}$ and $\mathcal{E}_M^{K_a}$. If $M := B_r \times B_r$ for a ball $B_r \subset \mathbb{R}^d$, we write $\mathcal{E}_{B_r} = \mathcal{E}_{B_r \times B_r}$.

2.1. ***Discussion of main assumptions.*** In this section, we list and discuss the assumptions which are imposed on the jumping kernels K in the course of this article. Except for (UJS), all other assumptions have already been discussed in detail in [Kassmann and Weidner 2022].

First, we assume throughout this article that K_s satisfies the Lévy-integrability condition

$$\left(x \mapsto \int_{\mathbb{R}^d} (|x - y|^2 \wedge 1) K_s(x, y)\,dy \right) \in L^1_{\text{loc}}(\mathbb{R}^d). \tag{2-1}$$

In the following, let $\Omega \subset \mathbb{R}^d$ be an open set. Let us now fix $\alpha \in (0, 2)$ and $\theta \in [d/\alpha, \infty]$. The first two assumptions were introduced and discussed in [Kassmann and Weidner 2022].

Assumption (K1). Let $J : \mathbb{R}^d \times \mathbb{R}^d \to [0, \infty]$ be a symmetric jumping kernel satisfying (cutoff) and let $\theta \in [d/\alpha, \infty]$.

- K satisfies (K1_{loc}) if there is $C > 0$ such that, for every ball $B_{2r} \subset \Omega$ with $r \le 1$,

$$\left\| \int_{B_{2r}} \frac{|K_a(\cdot, y)|^2}{J(\cdot, y)}\,dy \right\|_{L^\theta(B_{2r})} \le C, \qquad \mathcal{E}^J_{B_{2r}}(v, v) \le C\mathcal{E}^{K_s}_{B_{2r}}(v, v) \quad \text{for all } v \in L^2(B_{2r}). \tag{K1$_{\text{loc}}$}$$

- K satisfies $(\text{K1}_{\text{glob}})$ if there is $C > 0$ such that, for every ball $B_{2r} \subset \Omega$ with $r \le 1$,

$$\left\| \int_{\mathbb{R}^d} \frac{|K_a(\cdot, y)|^2}{J(\cdot, y)}\,dy \right\|_{L^\theta(\mathbb{R}^d)} \le C, \qquad \mathcal{E}^J_{B_{2r}}(v, v) \le C\mathcal{E}^{K_s}_{B_{2r}}(v, v) \quad \text{for all } v \in L^2(B_{2r}). \tag{K1$_{\text{glob}}$}$$

Assumption (K2). There exist $C > 0$, $D < 1$, and a symmetric jumping kernel j such that, for every ball $B_{2r} \subset \Omega$ with $r \le 1$ and every $v \in L^2(B_{2r})$ with $\mathcal{E}^{K_s}_{B_{2r}}(v, v) < \infty$,

$$K(x, y) \ge (1 - D)j(x, y) \quad \text{for all } x, y \in B_{2r}, \qquad \mathcal{E}^{K_s}_{B_{2r}}(v, v) \le C\mathcal{E}^{j}_{B_{2r}}(v, v). \tag{K2}$$

Remark 2.1. (i) (K1_{loc}) ensures that the quantities in (2-6) and (PDE) are well defined (see Lemma 2.9) and simultaneously requires that $\mathcal{E}^{K_a}$ is a term of lower-order. It gives rise to a nonlocal drift, analogous to $(b, \nabla u)$, where $b \in L^{2\theta}(\mathbb{R}^d)$ with $\theta \in \left[\frac{1}{2}d, \infty\right]$.

(ii) (K2) is only needed in the proof of the weak Harnack inequality (see Theorem 6.3). It ensures that the symmetric kernel $K_s - |K_a|$ is locally coercive with respect to $\mathcal{E}^{K_s}$.

(iii) For a detailed discussion of $(\mathrm{K1}_{\mathrm{loc}})$ and (K2) including their redundancy, we refer the reader to [Kassmann and Weidner 2022]. Equations $(\mathrm{K1}_{\mathrm{loc}})$ and (K2) are verified for the examples K_1, K_2, and K_3 from above in Section 8 of that paper.

(iv) In the simplest case, $(\mathrm{K1}_{\mathrm{loc}})$ (and $(\mathrm{K1}_{\mathrm{glob}})$) and (K2) hold with $J = j = K_s$. However, allowing for general symmetric kernels J and j significantly increases the class of admissible operators.

The following two assumptions on K only depend on the symmetric part. They are standard in the regularity for nonlocal operators associated with symmetric forms.

Assumption (cutoff). There is $c > 0$ such that, for every $0 < \rho \le r \le 1$ and $z \in \Omega$ such that $B_{r+\rho}(z) \subset \Omega$, there is a radially decreasing function $\tau = \tau_{z,r,\rho}$ centered at $z \in \mathbb{R}^d$ with $\operatorname{supp}(\tau) \subset \overline{B_{r+\rho}(z)}$, $0 \le \tau \le 1$, $\tau \equiv 1$ on $B_r(z)$, $|\nabla \tau| \le \frac{3}{2}\rho^{-1}$, and

$$\sup_{x \in B_{r+\rho}(z)} \Gamma^{K_s}(\tau, \tau)(x) \le c\rho^{-\alpha}, \tag{cutoff}$$

where

$$\Gamma^{K_s}(\tau, \tau)(x) := \int_{\mathbb{R}^d} (\tau(x) - \tau(y))^2 K_s(x, y)\,\mathrm{d}y$$

is the carré du champ associated with $\mathcal{E}^{K_s}$.

Note that $\Gamma^{K_s}(\tau, \tau)$ can be interpreted as the density of the energy $\mathcal{E}^{K_s}(\tau, \tau)$. Such an object is often called "carré du champ" in the literature.

Assumption $(\mathcal{E}_{\ge})$. There exists $c > 0$ such that, for every ball $B_{2r} \subset \Omega$ and every $v \in L^2(B_{2r})$,

$$\mathcal{E}^{K_s}_{B_{2r}}(v, v) \ge c[v]^2_{H^{\alpha/2}(B_{2r})}. \tag{$\mathcal{E}_{\ge}$}$$

Remark 2.2. (i) A sufficient condition for (cutoff) to hold for every $\tau_{z,r,\rho}$ is (see [Kassmann and Weidner 2022]): there is $c > 0$ such that, for every $0 < \zeta \le \rho \le r \le 1$ and $z \in \mathbb{R}^d$ with $B_{r+\rho}(z) \subset \Omega$,

$$\sup_{x \in B_{r+\rho}(z)} \left(\int_{\mathbb{R}^d \setminus B_\zeta(x)} K_s(x, y)\,\mathrm{d}y \right) \le c\zeta^{-\alpha}. \tag{2-2}$$

(ii) $(\mathcal{E}_{\ge})$ is a classical coercivity condition on K_s. It is significantly weaker than a pointwise lower bound of the form $K_s(x, y) \ge c|x - y|^{-d-\alpha}$ since it allows for non-fully-supported kernels such as K_3 (see (1-5)).

(iii) Under $(\mathcal{E}_{\ge})$, we have the following Poincaré and Sobolev inequalities: there is $c > 0$ such that, for every ball $B_{r+\rho} \subset \Omega$ with $0 < \rho \le r \le 1$ and $v \in L^2(B_{r+\rho})$,

$$\|v^2\|_{L^{d/(d-\alpha)}(B_r)} \le c\mathcal{E}^{K_s}_{B_{r+\rho}}(v, v) + c\rho^{-\alpha}\|v^2\|_{L^1(B_{r+\rho})}, \tag{Sob}$$

$$\int_{B_r} (v(x) - [v]_{B_r})^2\,\mathrm{d}x \le cr^\alpha \mathcal{E}^{K_s}_{B_r}(v, v), \tag{Poinc}$$

where $[v]_{B_r} = \fint_{B_r} v(x)\,\mathrm{d}x$. Equation (Poinc) is not explicitly needed in any of the proofs of this article. Nevertheless it is required for Theorem 6.3 to hold and therefore appears in the assumptions of our main result Theorem 1.1.

The following assumption did not appear in [Kassmann and Weidner 2022] and is designed to estimate nonlocal tails of supersolutions to (PDE) from above. It is required for the proof of the Harnack inequality.

Assumption (UJS).

- K satisfies (UJS) if there exists $c > 0$ such that, for every $x, y \in \mathbb{R}^d$ and every $r \le \left(\frac{1}{4} \wedge \frac{1}{4}|x-y|\right)$ with $B_r(x) \subset \Omega$,
$$K(x, y) \le c \fint_{B_r(x)} K(z, y)\,\mathrm{d}z. \tag{UJS}$$
- K satisfies $(\widehat{\text{UJS}})$ if there exists $c > 0$ such that, for every $x, y \in \mathbb{R}^d$ and every $r \le \left(\frac{1}{4} \wedge \frac{1}{4}|x-y|\right)$ with $B_r(x) \subset \Omega$,
$$K(y, x) \le c \fint_{B_r(x)} K(y, z)\,\mathrm{d}z. \tag{$\widehat{\text{UJS}}$}$$

Remark 2.3. (i) If K satisfies both conditions $(\widehat{\text{UJS}})$ and (UJS), then K_s satisfies (UJS).

(ii) Also for symmetric kernels the conditions (cutoff), (Poinc), and (Sob) are known to be insufficient for a Harnack inequality to hold; see [Bogdan and Sztonyk 2005].

(iii) Analogs to (UJS) for symmetric jumping kernels appeared in [Chen et al. 2020; Schulze 2019]. A pointwise version of (UJS) was considered in [Bass and Kassmann 2005].

Remark 2.4. (i) (UJS) clearly holds if $K(x, y)$ is pointwise comparable to $|x-y|^{-d-\alpha}$ for every $x, y \in \mathbb{R}^d$. However, (UJS) neither implies nor is implied by $(\mathcal{E}_{\ge})$.

(ii) Assume a global version of (K2), namely
$$|K_a(x, y)| \le D K_s(x, y) \quad \text{for all } x \in \Omega, \quad y \in \mathbb{R}^d. \tag{2-3}$$
Then $(1-D)K_s \le K \le 2K_s$, and therefore (UJS) is equivalent to
$$K_s(x, y) \le c \fint_{B_r(x)} K_s(z, y)\,\mathrm{d}z$$
for $x, y \in \mathbb{R}^d$ and $r \le \left(\frac{1}{4} \wedge \frac{1}{4}|x-y|\right)$ with $B_r(x) \subset \Omega$, i.e., it remains to verify (UJS) for K_s.

(iii) In [Schulze 2019] it was proved that kernels of the form
$$K_s(x, y) = \mathbb{1}_S(x-y)|x-y|^{-d-\alpha}$$
satisfy (UJS) if $S = -S$, and there exists $c > 0$ such that, for every $x \in S$ and $r \le \left(\frac{1}{4}|x| \wedge \frac{1}{4}\right)$, we have that $|B_r(x)| \le c|B_r(x) \cap S|$.

We provide sufficient conditions for (UJS) to hold for the examples K_1, K_2, K_3 in (1-3)–(1-5).

Example 2.5. (i) Let $K_1(x,y) = g(x,y)|x-y|^{-d-\alpha}$ be as in (1-3). It was shown in [Kassmann and Weidner 2022] that (2-3) holds for K_1 with $D = (\Lambda-\lambda)/(\Lambda+\lambda) < 1$. As

$$2K_s(x,y) = (g(x,y)+g(y,x))|x-y|^{-d-\alpha},$$

it follows that (UJS) holds for K.

(ii) Let K_2 be as in (1-4). Then, the antisymmetric part of K_2 is given by

$$K_a(x,y) = (V(x)-V(y))\mathbb{1}_{\{|x-y|\le L\}}(x,y)|x-y|^{-d-\alpha} \le K_s(x,y) = |x-y|^{-d-\alpha}.$$

Therefore, (UJS) holds if there exists $c>0$ such that, for every $x, y \in \mathbb{R}^d$ and $r \le \left(\frac{1}{4}|x-y| \wedge \frac{1}{4}\right)$ with $B_r(x) \subset \Omega$,

$$1 + (V(x)-V(y))\mathbb{1}_{\{|x-y|\le L\}} \le c \fint_{B_r(x)} 1 + (V(z)-V(y))\mathbb{1}_{\{|z-y|\le L\}} \,\mathrm{d}z. \tag{2-4}$$

(iii) We claim that (UJS) holds for K_3. Let us prove the following more general statement: Let $S \subset \mathbb{R}^d$ with $0 \in S$ and $c > 0$ such that, for every $x \in S$ and $r < \frac{1}{4}$, we have $|S \cap B_r(x)|/r^d \ge c$. Then,

$$K(x,y) = \mathbb{1}_S(x-y)|x-y|^{-d-\alpha}$$

satisfies (UJS) and ($\widehat{\text{UJS}}$).

In fact it suffices to prove that

$$\mathbb{1}_S(x-y) \le c \fint_{B_r(x)} \mathbb{1}_S(z-y)\,\mathrm{d}z \tag{2-5}$$

in order to deduce (UJS). Note that ($\widehat{\text{UJS}}$) follows by consideration of $-S$. We compute

$$\mathbb{1}_S(x-y) \le c\frac{|B_r(x-y)\cap S|}{r^d} = c\frac{|B_r(x)\cap(y+S)|}{r^d} = c\fint_{B_r(x)} \mathbb{1}_S(z-y)\,\mathrm{d}z.$$

Finally, we introduce the assumption of an upper bound of the jumping kernel which will be used only to prove an L^∞-L^2 + Tail estimate (see Theorem 3.6) and is not required for the proof of the main theorems. However it follows from (UJS) and (cutoff).

Assumption ($K_{\text{loc}}^{\le}$). There exists $c > 0$ such that, for every ball $B_{2r} \subset \Omega$ with $r \le 1$ and every $x, y \in B_{2r}$,

$$K(x,y) \le c|x-y|^{-d-\alpha}. \tag{$K_{\text{loc}}^{\le}$}$$

Remark 2.6. Note that ($K_{\text{loc}}^{\le}$) follows from (UJS) and (cutoff). Indeed, for any $x, y \in \mathbb{R}^d$ with $|x-y| \le 4$ and $r = \frac{1}{16}|x-y| \le \left(\frac{1}{4} \wedge \frac{1}{4}|x-y|\right)$, we have $B_r(x) \subset B_r(y)^c$, and therefore

$$K(x,y) \le c_1 \fint_{B_r(x)} K(z,y)\,\mathrm{d}z \le c_2 r^{-d} \int_{B_r(y)^c} K(z,y)\,\mathrm{d}z \le c_3 r^{-d-\alpha} \le c_4|x-y|^{-d-\alpha}$$

for some constants $c_1, c_2, c_3, c_4 > 0$.

2.2. ***Weak solution concept.*** We introduce the following function spaces for $\Omega \subset \mathbb{R}^d$:

$$V(\Omega\,|\,\mathbb{R}^d) = \{v : \mathbb{R}^d \to \mathbb{R} \text{ s.t. } v|_\Omega \in L^2(\Omega) \text{ and } (v(x)-v(y))K_s^{1/2}(x,y) \in L^2(\Omega\times\mathbb{R}^d)\},$$
$$H_\Omega(\mathbb{R}^d) = \{v \in V(\mathbb{R}^d\,|\,\mathbb{R}^d) \text{ s.t. } v = 0 \text{ on } \mathbb{R}^d \setminus \Omega\}$$

equipped with

$$\|v\|^2_{V(\Omega|\mathbb{R}^d)} = \|v\|^2_{L^2(\Omega)} + \int_\Omega\int_{\mathbb{R}^d} (v(x)-v(y))^2 K_s(x,y)\,\mathrm{d}y\,\mathrm{d}x,$$
$$\|v\|^2_{H_\Omega(\mathbb{R}^d)} = \|v\|^2_{L^2(\mathbb{R}^d)} + \mathcal{E}^{K_s}(v,v).$$

We emphasize that both spaces are completely determined by the symmetric part of the jumping kernel K_s. Moreover, for $\alpha \in (0,2)$, we define $V^\alpha(\Omega\,|\,\mathbb{R}^d)$ and $H^\alpha_\Omega(\mathbb{R}^d)$ as the corresponding function spaces associated with $K_s(x,y) = |x-y|^{-d-\alpha}$.

We are ready to define the notion of a weak solution to (PDE) and ($\widehat{\text{PDE}}$). Let us define $\theta' := \theta/(\theta-1)$ as the Hölder conjugate exponent of θ.

Definition 2.7. Let $\Omega \subset \mathbb{R}^d$ be a bounded domain, $I \subset \mathbb{R}$ a finite interval, and $f \in L^\infty(I\times\Omega)$.

(i) We say that $u \in L^2_{\text{loc}}(I; V(\Omega\,|\,\mathbb{R}^d))$ is a weak supersolution to (PDE) in $I\times\Omega$ if the weak $L^2(\Omega)$-derivative $\partial_t u$ exists, $\partial_t u \in L^1_{\text{loc}}(I; L^2(\Omega))$, and

$$(\partial_t u(t), \phi) + \mathcal{E}(u(t), \phi) \le (f(t), \phi) \quad \text{for all } t \in I \text{ and for all } \phi \in H_\Omega(\mathbb{R}^d) \text{ with } \phi \le 0. \tag{2-6}$$

We call u a weak subsolution if (2-6) holds for every $\phi \ge 0$. We call u a weak solution, if it is a supersolution and a subsolution.

(ii) We say that $u \in L^2_{\text{loc}}(I; V(\Omega\,|\,\mathbb{R}^d) \cap L^{2\theta'}(\mathbb{R}^d))$ is a weak supersolution to ($\widehat{\text{PDE}}$) in $I\times\Omega$ if the weak $L^2(\Omega)$-derivative $\partial_t u$ satisfies the same properties as before and

$$(\partial_t u(t), \phi) + \widehat{\mathcal{E}}(u(t), \phi) \le (f(t), \phi) \quad \text{for all } t \in I \text{ and for all } \phi \in H_\Omega(\mathbb{R}^d) \text{ with } \phi \le 0.$$

Weak (sub-)solutions to ($\widehat{\text{PDE}}$) are defined in analogy with (i).

Next, we introduce the solution concept for stationary equations.

Definition 2.8. Let $\Omega \subset \mathbb{R}^d$ be a bounded domain and $f \in L^\infty(\Omega)$.

(i) We say that $u \in V(\Omega\,|\,\mathbb{R}^d)$ is a weak supersolution to (ell-PDE) in Ω if

$$\mathcal{E}(u, \phi) \le (f, \phi) \quad \text{for all } \phi \in H_\Omega(\mathbb{R}^d) \text{ with } \phi \le 0. \tag{2-7}$$

We call u a weak subsolution if (2-7) holds for every $\phi \ge 0$. We call u a weak solution if it is a supersolution and a subsolution.

(ii) We say that $u \in V(\Omega\,|\,\mathbb{R}^d) \cap L^{2\theta'}(\mathbb{R}^d)$ is a weak supersolution to (ell-$\widehat{\text{PDE}}$) in Ω if

$$\widehat{\mathcal{E}}(u, \phi) \le (f, \phi) \quad \text{for all } \phi \in H_\Omega(\mathbb{R}^d) \text{ with } \phi \le 0.$$

(Sub)solutions to (ell-$\widehat{\text{PDE}}$) are defined in analogy with (i).

Let us point out that the solution concept also makes sense under much weaker assumptions on u without any change in the proofs being needed; see [Felsinger and Kassmann 2013]. In particular, one can drop the condition that the weak time derivative $\partial_t u$ exists.

We will only consider solutions on special time-space cylinders $I_R(t_0) \times B_{2R}$, where $B_{2R} \subset \Omega$ is a ball, $I_R(t_0) = (t_0 - R^\alpha, t_0 + R^\alpha)$, $0 < R \le 1$, and $t_0 \in \mathbb{R}$. Moreover,

$$I_R^{\ominus}(t_0) := (t_0 - R^\alpha, t_0), \quad I_R^{\oplus}(t_0) := (t_0, t_0 + R^\alpha).$$

Recall the following lemma, which was proved in [Kassmann and Weidner 2022]. It ensures that the expressions in Definitions 2.7 and 2.8 are well defined.

Lemma 2.9 [Kassmann and Weidner 2022, Lemma 2.2]. *Let $0 < \rho \le r \le 1$ and $B_{2r} \subset \Omega$.*

(i) *Assume that one of the following is true:*

- (K1$_{\mathrm{loc}}$) *holds with $\theta = \infty$,*
- (K1$_{\mathrm{loc}}$) *holds with $\theta \in [d/\alpha, \infty)$ and* (Sob) *holds.*

Then $\mathcal{E}(u, \phi)$ is well defined for $u \in V(B_{r+\rho} | \mathbb{R}^d)$ and $\phi \in H_{B_{r+\rho}}(\mathbb{R}^d)$.

(ii) *Assume that* (K1$_{\mathrm{glob}}$) *holds with $\theta \in [d/\alpha, \infty]$. Then $\widehat{\mathcal{E}}(u, \phi)$ is well defined for $\phi \in H_{B_{r+\rho/2}}(\mathbb{R}^d)$ and $u \in V(B_{r+\rho/2} | \mathbb{R}^d) \cap L^{2\theta'}(\mathbb{R}^d)$.*

The following lemma is of central importance in the proofs of the Caccioppoli estimates for nonsymmetric nonlocal operators. Note that the proof in the special case $\theta = \infty$ is trivial.

Lemma 2.10 [Kassmann and Weidner 2022, Lemma 2.4]. (i) *Assume that* (K1$_{\mathrm{loc}}$) *holds for some $\theta \in [d/\alpha, \infty]$. Moreover, assume* (Sob) *if $\theta < \infty$. Then, there exists $c_1 > 0$ such that, for every $\delta > 0$, there is $C(\delta) > 0$ such that, for every $v \in L^2(B_{r+\rho})$ with $\operatorname{supp}(v) \subset B_{r+\rho/2}$ and every ball $B_{2r} \subset \Omega$ with $0 < \rho \le r \le 1$, we have*

$$\int_{B_{r+\rho}} v^2(x) \left(\int_{B_{r+\rho}} \frac{|K_a(x,y)|^2}{J(x,y)} \, \mathrm{d}y \right) \mathrm{d}x \le \delta \mathcal{E}^{K_s}_{B_{r+\rho}}(v, v) + c_1 (C(\delta) + \delta \rho^{-\alpha}) \|v^2\|_{L^1(B_{r+\rho})}. \tag{2-8}$$

Moreover, if $\theta \in (d/\alpha, \infty]$, the constant $C(\delta)$ has the following form:

$$C(\delta) = \begin{cases} \|W\|_{L^\infty(B_{r+\rho})}, & \theta = \infty, \\ \delta^{d/(d-\theta\alpha)} \|W\|^{\theta\alpha/(\theta\alpha - d)}_{L^\theta(B_{r+\rho})}, & \theta \in (d/\alpha, \infty), \end{cases} \quad \text{where } W(x) := \int_{B_{r+\rho}} \frac{|K_a(x,y)|^2}{J(x,y)} \, \mathrm{d}y. \tag{2-9}$$

(ii) *Assume that* (K1$_{\mathrm{glob}}$) *holds for some $\theta \in [d/\alpha, \infty]$. Moreover, assume* (Sob) *if $\theta < \infty$. Then* (2-8) *and* (2-9) *hold with*

$$\int_{\mathbb{R}^d} \frac{|K_a(x,y)|^2}{J(x,y)} \, \mathrm{d}y \quad \textit{instead of} \quad \int_{B_{r+\rho}} \frac{|K_a(x,y)|^2}{J(x,y)} \, \mathrm{d}y.$$

2.3. ***Nonlocal tail terms.*** Due to the nonlocality of the problems under consideration, certain nonlocal tail terms naturally enter the picture. For references concerning the treatment of tail terms in the study of symmetric nonlocal operators, we refer the reader to [Chen et al. 2020; Di Castro et al. 2014; 2016]. It is crucial for our analysis to make sure that the respective tail terms are finite for any weak solution under reasonable assumptions on K and that the tail terms are compatible with the iteration techniques carried out in the remainder of this article.

Given any ball $B_{2r}(x_0) \subset \Omega$, a function $v \in V(B_{2r}(x_0) \,|\, \mathbb{R}^d)$, and $0 < r_1 < r_2 \le 2r$, we define

$$\mathrm{Tail}_K(v, r_1, r_2, x_0) := \sup_{x \in B_{r_1}(x_0)} \int_{B_{r_2}(x_0)^c} |v(y)| K(x, y)\, dy,$$

$$\widehat{\mathrm{Tail}}_K(v, r_1, r_2, x_0) := \sup_{x \in B_{r_1}(x_0)} \int_{B_{r_2}(x_0)^c} |v(y)| K(y, x)\, dy.$$

Remark 2.11. (i) For $0 < \rho_1 \le r_1$ and $0 < \rho_2 \le r_2$, we have $\mathrm{Tail}_K(v, \rho_1, r_2) \le \mathrm{Tail}_K(v, r_1, \rho_2)$.

(ii) Note that Tail_K has been introduced in [Schulze 2019] for symmetric kernels.

We would like to point out that Tail_K will naturally appear in the proofs of the Caccioppoli estimates in Sections 3 and 4. However, it is not suitable for De Giorgi-type and Moser-type iteration arguments. Therefore, we introduce another nonlocal tail term defined as follows:

$$\mathrm{Tail}_{K,\alpha}(u, R, x_0) := R^\alpha \int_{B_{2R}(x_0)\setminus B_{R/2}(x_0)} \frac{|u(y)|}{|x_0 - y|^{d+\alpha}}\, dy + \sup_{x \in B_{3R/2}(x_0)} \int_{B_{2R}(x_0)^c} |u(y)| K(x, y)\, dy,$$

$$\widehat{\mathrm{Tail}}_{K,\alpha}(u, R, x_0) := R^\alpha \int_{B_{2R}(x_0)\setminus B_{R/2}(x_0)} \frac{|u(y)|}{|x_0 - y|^{d+\alpha}}\, dy + \sup_{x \in B_{3R/2}(x_0)} \int_{B_{2R}(x_0)^c} |u(y)| K(y, x)\, dy.$$

$\mathrm{Tail}_{K,\alpha}$ can be regarded as a hybrid between a tail term for general kernels introduced in [Schulze 2019] and a tail term for rotationally symmetric kernels as in [Chen et al. 2020; Di Castro et al. 2016].

The advantage of $\mathrm{Tail}_{K,\alpha}$ is that it fits the iteration schemes, since, for short connections, the weight is a radial function. Moreover, it still takes into account the correct decay of the jumping kernel K for long jumps, which might be of lower-order due to the presence of a nonlocal drift term (see K_3 in (1-5)). Since we do not want to impose any pointwise upper bound on K for long jumps, the second summand contains the supremum in x.

We have the following connection between Tail_K and $\mathrm{Tail}_{K,\alpha}$.

Lemma 2.12. *Assume* $(K^{\le}_{\mathrm{loc}})$. *Let* $0 < \rho \le r \le r + \rho \le R \le 1$, $x_0 \in \mathbb{R}^d$, *and* $v \in V(B_R(x_0) \,|\, \mathbb{R}^d)$. *Then we have*

$$\mathrm{Tail}_K(v, r, r+\rho, x_0) \le c\rho^{-\alpha} \left(\frac{r+\rho}{\rho}\right)^d \mathrm{Tail}_{K,\alpha}(u, R, x_0), \tag{2-10}$$

$$\widehat{\mathrm{Tail}}_K(v, r, r+\rho, x_0) \le c\rho^{-\alpha} \left(\frac{r+\rho}{\rho}\right)^d \widehat{\mathrm{Tail}}_{K,\alpha}(u, R, x_0). \tag{2-11}$$

Proof. We use that, for

$$x \in B_r(x_0), \quad y \in B_{r+\rho}(x_0)^c \cap B_{2R}(x_0), \quad \text{and} \quad z \in B_{r+\rho}(x_0)^c \cap B_{2R}(x_0)^c = B_{2R}(x_0)^c,$$

we have

$$|y - x_0| \le \frac{r+\rho}{\rho} |y - x|, \qquad |z - x_0| \le 2|z - x|,$$

which implies upon $(K^{\le}_{\mathrm{loc}})$ that, for every $x \in B_r(x_0)$,

$$\begin{aligned}
\int_{B_{r+\rho}(x_0)^c} v(y)K(x,y)\,\mathrm{d}y \\
&\le \int_{B_{r+\rho}(x_0)^c \cap B_{2R}(x_0)} v(y)K(x,y)\,\mathrm{d}y + \int_{B_{2R}(x_0)^c} v(z)K(x,z)\,\mathrm{d}z \\
&\le c_1 \left(\frac{r+\rho}{\rho}\right)^{d+\alpha} \int_{B_{r+\rho}(x_0)^c \cap B_{2R}(x_0)} v(y)|x_0-y|^{-d-\alpha}\,\mathrm{d}y + c_1 \int_{B_{2R}(x_0)^c} v(z)K(x,z)\,\mathrm{d}z \\
&\le c_2 \rho^{-\alpha} \left(\frac{r+\rho}{\rho}\right)^{d} \mathrm{Tail}_{K,\alpha}(v,R),
\end{aligned}$$

where $c_1, c_2 > 0$. This proves (2-10), as desired. The proof of (2-11) works in the same way. □

Moreover, $\mathrm{Tail}_{K,\alpha}(u,R,x_0)$ and $\widehat{\mathrm{Tail}}_{K,\alpha}(u,R,x_0)$ are finite for any $u \in V(B_{2R}(x_0)\,|\,\mathbb{R}^d)$ under natural and nonrestrictive assumptions on K. This property is of some importance to us since it allows us to work with the natural function space $V(B_{2R}(x_0)\,|\,\mathbb{R}^d)$ associated with K.

Lemma 2.13. *Assume* (cutoff) *and* $(\mathcal{E}_{\ge})$.

(i) *If* (UJS) *holds, then* $\mathrm{Tail}_{K,\alpha}(u,R,x_0) < \infty$ *for every* $u \in V(B_{2R}(x_0)\,|\,\mathbb{R}^d)$,

(ii) *If* $(\widehat{\mathrm{UJS}})$ *holds, then* $\widehat{\mathrm{Tail}}_{K,\alpha}(u,R,x_0) < \infty$ *for every* $u \in V(B_{2R}(x_0)\,|\,\mathbb{R}^d)$.

Proof. We restrict ourselves to proving (i). The proof of (ii) follows via analogous arguments. By (cutoff), it clearly suffices to prove that

$$\int_{B_{2R}(x_0)\setminus B_{R/2}(x_0)} |u(y)|^2|x_0-y|^{-d-\alpha}\,\mathrm{d}y + \sup_{x \in B_{3R/2}(x_0)} \int_{B_{2R}(x_0)^c} |u(y)|^2 K(x,y)\,\mathrm{d}y < \infty. \tag{2-12}$$

We start by proving finiteness of the first summand. This can be achieved by the same argument as in the proof of Proposition 12 in [Dyda and Kassmann 2019]. Since $|x-y| \le 3|x_0-y|$ for every $x \in B_{R/4}(x_0)$ and $y \in \mathbb{R}^d \setminus B_{R/4}(x_0)$, we compute

$$\begin{aligned}
\int_{B_{2R}(x_0)\setminus B_{R/2}(x_0)} |u(y)|^2|x_0-y|^{-d-\alpha}\,\mathrm{d}y \\
&\le 3 \int_{B_{2R}(x_0)\setminus B_{R/2}(x_0)} \fint_{B_{R/4}(x_0)} |u(y)|^2|x-y|^{-d-\alpha}\,\mathrm{d}x\,\mathrm{d}y \\
&\le c \int_{B_{2R}(x_0)} \fint_{B_{R/4}(x_0)} |u(y)-u(x)|^2|x-y|^{-d-\alpha}\,\mathrm{d}x\,\mathrm{d}y \\
&\qquad + c \fint_{B_{R/4}(x_0)} |u(x)|^2 \left(\int_{B_{2R}(x_0)\setminus B_{R/2}(x_0)} |x-y|^{-d-\alpha}\,\mathrm{d}y \right) \mathrm{d}x \\
&\le c R^{-d} \mathcal{E}^{\alpha}_{B_{2R}(x_0)}(u,u) + c \fint_{B_{R/4}(x_0)} |u(x)|^2 \left(\int_{B_{R/4}(x)^c} |x-y|^{-d-\alpha}\,\mathrm{d}y \right) \mathrm{d}x \\
&\le c R^{-d} \mathcal{E}^{\alpha}_{B_{2R}(x_0)}(u,u) + c R^{-d-\alpha} \|u\|^2_{L^2(B_{R/4}(x_0))} < \infty.
\end{aligned}$$

Finiteness of the quantity on the right follows from $(\mathcal{E}_{\ge})$ and since $u \in V(B_{2R}(x_0)\,|\,\mathbb{R}^d)$.

For the second summand in (2-12), we estimate using (UJS) and (cutoff) that, for every $x \in B_{3R/2}(x_0)$,

$$\begin{aligned}
&\int_{B_{2R}(x_0)^c} |u(y)|^2 K(x,y)\,\mathrm{d}y \\
&\quad\le \int_{B_{2R}(x_0)^c} \fint_{B_{R/4}(x)} |u(y)|^2 K(z,y)\,\mathrm{d}z\,\mathrm{d}y \\
&\quad\le cR^{-d} \int_{B_{2R}(x_0)^c} \fint_{B_{2R}(x_0)} |u(y)-u(z)|^2 K_s(z,y)\,\mathrm{d}z\,\mathrm{d}y + 2\fint_{B_{R/4}(x)} |u(z)|^2 \Big(\int_{B_{2R}(x_0)^c} K_s(z,y)\,\mathrm{d}y \Big) \mathrm{d}z \\
&\quad\le cR^{-d}[u]^2_{V(B_{2R}(x_0)\,|\,\mathbb{R}^d)} + c\fint_{B_{R/4}(x)} |u(z)|^2 \Big(\int_{B_{R/4}(z)^c} K_s(z,y)\,\mathrm{d}y \Big) \mathrm{d}z \\
&\quad\le cR^{-d}[u]^2_{V(B_{2R}(x_0)\,|\,\mathbb{R}^d)} + cR^{-d-\alpha}\|u\|^2_{L^2(B_{2R}(x_0))} < \infty.
\end{aligned}$$

Here we used that $B_{R/4}(x) \subset B_{2R}(x_0)$ for every $x \in B_{3R/2}(x_0)$. □

Remark 2.14. Note that (UJS) and ($\widehat{\text{UJS}}$) are not necessary for $\mathrm{Tail}_{K,\alpha}(u,R,x_0)$ and $\widehat{\mathrm{Tail}}_{K,\alpha}(u,R,x_0)$ to be finite, respectively. Consider for example a jumping kernel K whose symmetric part satisfies global versions of ($\mathcal{E}_{\ge}$) and ($K^{\le}_{\mathrm{loc}}$), namely;

$$\mathcal{E}^{K_s}(u,u) \ge c[u]^2_{H^{\alpha/2}(B_r)} \quad \text{for all } v \in L^2(B_r),\ r>0, \qquad K(x,y) \le c|x-y|^{-d-\alpha} \quad \text{for all } x,y \in \mathbb{R}^d,$$

then we have that $V(B_{2R}\,|\,\mathbb{R}^d) = V^\alpha(B_{2R}\,|\,\mathbb{R}^d)$. Therefore,

$$\mathrm{Tail}_{K,\alpha}(u,R,x_0) \le c\,\mathrm{Tail}_\alpha(u,R,x_0) = R^\alpha \int_{B_{R/2}(x_0)} |u(y)|\,|x_0-y|^{-d-\alpha}\,\mathrm{d}y < \infty \quad \text{for all } u \in V(B_{2R}\,|\,\mathbb{R}^d).$$

Remark 2.15. (i) Later, we will require finiteness of $\mathrm{Tail}_{K,\alpha}(u,R,x_0)$ and $\widehat{\mathrm{Tail}}_{K,\alpha}(u,R,x_0)$ in order to deduce local boundedness of weak solutions to (ell-PDE) and (ell-$\widehat{\text{PDE}}$) from Theorem 3.6 and Theorem 4.8, respectively. The above lemma shows that under the natural assumptions (cutoff), ($\mathcal{E}_{\ge}$), and (UJS) or ($\widehat{\text{UJS}}$), finiteness of the tail terms for weak solutions follows already from the solution concept.

(ii) For parabolic equations, the aforementioned assumptions merely imply finiteness of

$$\mathrm{Tail}_{K,\alpha}(u(t),R,x_0) \quad \text{and} \quad \widehat{\mathrm{Tail}}_{K,\alpha}(u(t),R,x_0)$$

for a.e. t, but do not yield a uniform upper bound in t.

(iii) Since parabolic tails of the form $\sup_{t\in I}\mathrm{Tail}_K(u(t),r,r+\rho,x_0)$ and $\sup_{t\in I}\widehat{\mathrm{Tail}}_K(u(t),r,r+\rho,x_0)$ naturally appear in the analysis of solutions to (PDE) and ($\widehat{\text{PDE}}$), respectively, it is an important research question to investigate these quantities and to derive suitable estimates. First results have been obtained in [Strömqvist 2019b], where an estimate for $\sup_{t\in I}\mathrm{Tail}_K(u(t),r,r+\rho,x_0)$ is derived for global solutions u to (PDE) in the symmetric case under pointwise bounds for K. Another attempt has been made in [Kim 2019] for solutions to a parabolic boundary value problem with given continuous, bounded data. However, the proof of [Kim 2019, Lemma 5.3] is not complete.

3. Local boundedness via De Giorgi iteration

The goal of this section is to prove that the supremum of a weak subsolution u to (PDE), or to $(\widehat{\text{PDE}})$, can locally be estimated from above by the L^2-norm of u and a nonlocal tail term (see Theorem 3.6). Under the assumption that the tail term is finite, this result is the key to proving the Harnack inequality. The strategy of proof is based on the De Giorgi iteration for nonlocal operators, as adopted in [Cozzi 2017; Di Castro et al. 2014; 2016].

3.1. *Caccioppoli estimates.* In this section nonlocal Caccioppoli estimates are established. They are derived by testing the weak formulation of (PDE), or of $(\widehat{\text{PDE}})$, with a test function of the form $\tau^2(u-k)_+$. The lack of symmetry of the jumping kernel K calls for a refinement of the existing proofs for symmetric operators. The main technical ingredient is Lemma 2.10. Such estimates will be used in Section 3.2 to set up a De Giorgi-type iteration scheme which allows us to prove Theorem 3.6.

The following lemma can be regarded as a generalization of Proposition 8.5 in [Cozzi 2017] to nonsymmetric jumping kernels.

Lemma 3.1. *Assume that* ($K1_{\text{loc}}$) *and* (cutoff) *hold for some* $\theta \in [d/\alpha, \infty]$. *Moreover, assume* (Sob) *if* $\theta < \infty$. *Then there exist* $c_1, c_2 > 1$ *such that, for every* $0 < \rho \le r \le 1$, *every* $l \in \mathbb{R}$, *and every function* $u \in V(B_{r+\rho} \,|\, \mathbb{R}^d)$, *we have*

$$\mathcal{E}^{K_s}_{B_{r+\rho}}(\tau w_+, \tau w_+) - \mathcal{E}_{B_{r+\rho}}(w_-, \tau w_+) \\ \le c_1 \mathcal{E}(u, \tau^2 w_+) + c_2 \rho^{-\alpha} \|w_+^2\|_{L^1(B_{r+\rho})} + c_2 \|w_+\|_{L^1(B_{r+\rho})} \operatorname{Tail}_K\big(w_+, r + \tfrac{1}{2}\rho, r+\rho\big), \quad (3\text{-}1)$$

where $B_{2r} \subset \Omega$, $w = u - l$, *and* $\tau = \tau_{r,\rho/2}$.

Proof. Step 1: We claim that there exists a constant $c > 0$ such that

$$\mathcal{E}^{K_s}_{B_{r+\rho}}(\tau w_+, \tau w_+) - \mathcal{E}^{K_s}_{B_{r+\rho}}(w_-, \tau w_+) \le \mathcal{E}^{K_s}_{B_{r+\rho}}(u, \tau^2 w_+) + c\rho^{-\alpha} \|w_+^2\|_{L^1(B_{r+\rho})}. \quad (3\text{-}2)$$

Observe that by the algebraic identities

$$a - b = ((a-l)_+ - (b-l)_+) - ((a-l)_- - (b-l)_-),$$
$$(w_1 - w_2)(\tau_1^2 w_1 - \tau_2^2 w_2) = (\tau_1 w_1 - \tau_2 w_2)^2 - w_1 w_2 (\tau_1 - \tau_2)^2,$$

we have that

$$\mathcal{E}^{K_s}_{B_{r+\rho}}(\tau w_+, \tau w_+) - \mathcal{E}^{K_s}_{B_{r+\rho}}(w_-, \tau w_+) \\ = \mathcal{E}^{K_s}_{B_{r+\rho}}(u, \tau^2 w_+) + \int_{B_{r+\rho}} \int_{B_{r+\rho}} w_+(x) w_+(y) (\tau(x) - \tau(y))^2 K_s(x,y) \, dy \, dx.$$

Thus, (3-2) follows immediately from (cutoff).

Step 2: For every $\delta > 0$, there exists $c > 0$ such that

$$\mathcal{E}^{K_a}_{B_{r+\rho}}(u, \tau^2 w_+) \ge -\mathcal{E}^{K_a}_{B_{r+\rho}}(w_-, \tau^2 w_+) - \delta \mathcal{E}^{K_s}_{B_{r+\rho}}(\tau w_+, \tau w_+) - c\rho^{-\alpha} \|w_+^2\|_{L^1(B_{r+\rho})}. \quad (3\text{-}3)$$

For the proof, we first observe the algebraic identity

$$(w_1 - w_2)(\tau_1^2 w_1 + \tau_2^2 w_2) = (\tau_1^2 w_1^2 - \tau_2^2 w_2^2) + w_1 w_2 (\tau_2^2 - \tau_1^2).$$

Thus, we obtain

$$\mathcal{E}^{K_a}_{B_{r+\rho}}(u,\tau^2 w_+) = -\mathcal{E}^{K_a}_{B_{r+\rho}}(w_-,\tau^2 w_+) + \int_{B_{r+\rho}}\int_{B_{r+\rho}} (\tau^2 w_+^2(x) - \tau^2 w_+^2(y)) K_a(x,y)\,\mathrm{d}y\,\mathrm{d}x \\ + \int_{B_{r+\rho}}\int_{B_{r+\rho}} w_+(x)w_+(y)(\tau^2(y)-\tau^2(x))K_a(x,y)\,\mathrm{d}y\,\mathrm{d}x =: I_1 + I_2 + I_3.$$

For I_2, we estimate, using (K1$_{\text{loc}}$) and (2-8),

$$I_2 \ge -\tfrac{1}{2}\delta\mathcal{E}^{K_s}_{B_{r+\rho}}(\tau w_+,\tau w_+) - c\int_{B_{r+\rho}} \tau^2(x) w_+^2(x)\left(\int_{B_{r+\rho}} \frac{|K_a(x,y)|^2}{J(x,y)}\right)\mathrm{d}x \\ \ge -\delta\mathcal{E}^{K_s}_{B_{r+\rho}}(\tau w_+,\tau w_+) - c\rho^{-\alpha}\|w_+^2\|_{L^1(B_{r+\rho})}.$$

For I_3, using the standard estimate

$$(\tau^2(x)-\tau^2(y)) \le 2(\tau(x)-\tau(y))^2 + 2(\tau(x)-\tau(y))(\tau(x)\wedge\tau(y)), \tag{3-4}$$

estimate (1-2), (cutoff), and (K1$_{\text{loc}}$), we get

$$I_3 \ge -2\int_{B_{r+\rho}}\int_{B_{r+\rho}} (w_+^2(x)\vee w_+^2(y))(\tau(x)-\tau(y))^2 K_s(x,y)\,\mathrm{d}y\,\mathrm{d}x \\ - 2\int_{B_{r+\rho}}\int_{B_{r+\rho}} (w_+^2(x)\vee w_+^2(y))(\tau(x)\wedge\tau(y))|\tau(x)-\tau(y)||K_a(x,y)|\,\mathrm{d}y\,\mathrm{d}x \\ \ge -c\rho^{-\alpha}\|w_+^2\|_{L^1(B_{r+\rho})} - \int_{B_{r+\rho}} \tau^2(x)w_+^2(x)\left(\int_{B_{r+\rho}} \frac{|K_a(x,y)|^2}{J(x,y)}\,\mathrm{d}y\right)\mathrm{d}x \\ \ge -c\rho^{-\alpha}\|w_+^2\|_{L^1(B_{r+\rho})} - \delta\mathcal{E}^{K_s}_{B_{r+\rho}}(\tau w_+,\tau w_+).$$

This proves (3-3).

Step 3: Next, let us show how to prove

$$-\mathcal{E}_{(B_{r+\rho}\times B_{r+\rho})^c}(u,\tau^2 w_+) \le 2\left(\int_{B_{r+\rho}} w_+(x)\,\mathrm{d}x\right)\operatorname{Tail}_K\left(w_+, r+\tfrac{1}{2}\rho, r+\rho\right). \tag{3-5}$$

We estimate

$$-\mathcal{E}_{(B_{r+\rho}\times B_{r+\rho})^c}(u,\tau^2 w_+) = 2\int_{B_{r+\rho/2}}\int_{B^c_{r+\rho}} (u(y)-u(x))\tau^2 w_+(x)K(x,y)\,\mathrm{d}y\,\mathrm{d}x \\ \le 2\int_{B_{r+\rho/2}}\int_{B^c_{r+\rho}} (u(y)-u(x))_+\tau^2 w_+(x)K(x,y)\,\mathrm{d}y\,\mathrm{d}x \\ \le 2\int_{B_{r+\rho/2}}\int_{B^c_{r+\rho}} (u(y)-l)_+\tau^2 w_+(x)K(x,y)\,\mathrm{d}y\,\mathrm{d}x \\ \le 2\int_{B_{r+\rho/2}} w_+(x)\sup_{z\in B_{r+\rho/2}}\left(\int_{B^c_{r+\rho}} w_+(y)K(z,y)\,\mathrm{d}y\right)\mathrm{d}x,$$

where we used that K is nonnegative and $\tau\equiv 0$ in $B^c_{r+\rho/2}$.

Step 4: We will now combine (3-2), (3-3), and (3-5). Observe

$$\mathcal{E}^{K_s}_{B_{r+\rho}}(u,\tau^2 w_+) = \mathcal{E}(u,\tau^2 w_+) - \mathcal{E}^{K_a}_{B_{r+\rho}}(u,\tau^2 w_+) - \mathcal{E}_{(B_{r+\rho}\times B_{r+\rho})^c}(u,\tau^2 w_+).$$

Altogether, we immediately obtain the desired result by choosing $\delta > 0$ from Step 2 small enough. □

Note that $-\mathcal{E}_{B_{r+\rho}}(w_-,\tau^2 w_+) \geq 0$ since $K \geq 0$. Thus, we have the following corollary of Lemma 3.1.

Corollary 3.2. *Assume that* (K1$_{\text{loc}}$) *and* (cutoff) *hold for some* $\theta \in [d/\alpha,\infty]$. *Moreover, assume* (Sob) *if* $\theta < \infty$. *Then there exist* $c_1, c_2 > 0$ *such that, for every* $0 < \rho \leq r \leq 1$, *every* $l \in \mathbb{R}$, *and every function* $u \in V(B_{r+\rho}\,|\,\mathbb{R}^d)$, *we have*

$$\begin{aligned}\mathcal{E}^{K_s}_{B_{r+\rho}}&(\tau w_+,\tau w_+)\\ &\leq c_1\mathcal{E}(u,\tau^2 w_+) + c_2\rho^{-\alpha}\|w_+^2\|_{L^1(B_{r+\rho})} + c_2\|w_+\|_{L^1(B_{r+\rho})}\operatorname{Tail}_K\big(w_+, r+\tfrac{1}{2}\rho, r+\rho\big),\end{aligned}\tag{3-6}$$

where $B_{2r} \subset \Omega$, $w = u - l$, *and* $\tau = \tau_{r,\rho/2}$.

Remark 3.3. Let us point out that both Caccioppoli-type inequalities (3-1) and (3-6) appear in the literature for symmetric jumping kernels. Inequality (3-1) was introduced in [Cozzi 2017] (see also [Caffarelli et al. 2011; Cozzi 2019]) and is used to prove Hölder estimates for small α. For our purposes, inequality (3-6) is sufficient.

Next, we present a Caccioppoli inequality that is tailored to subsolutions to ($\widehat{\text{PDE}}$). Due to the different shape of the bilinear form, we obtain an additional summand on the right-hand side of the estimate.

Lemma 3.4. *Assume that* (K1$_{\text{loc}}$) *and* (cutoff) *hold for some* $\theta \in [d/\alpha,\infty]$. *Moreover, assume* (Sob) *if* $\theta < \infty$. *Then there exist* $c_1, c_2 > 1$ *such that, for every* $0 < \rho \leq r \leq 1$, *every* $l \in \mathbb{R}$, *and every function* $u \in V(B_{r+\rho}\,|\,\mathbb{R}^d)$, *we have*

$$\begin{aligned}\mathcal{E}^{K_s}_{B_{r+\rho}}&(\tau w_+,\tau w_+) - \mathcal{E}_{B_{r+\rho}}(w_-,\tau w_+)\\ &\leq c_1\widehat{\mathcal{E}}(u,\tau^2 w_+) + c\rho^{-\alpha}\|w_+^2\|_{L^1(B_{r+\rho})} + c_2 l^2\rho^{-\alpha}\left[|A(l,r+\rho)| + |B_{r+\rho}|\left(\frac{|A(l,r+\rho)|}{|B_{r+\rho}|}\right)^{1/\theta'}\right]\\ &\qquad + c_2\|w_+\|_{L^1(B_{r+\rho})}\widehat{\operatorname{Tail}}_K\big(u, r+\tfrac{1}{2}\rho, r+\rho\big),\end{aligned}\tag{3-7}$$

where $B_{2r} \subset \Omega$, $w = u - l$, $\tau = \tau_{r,\rho/2}$, *and* $A(l,r+\rho) = \{x \in B_{r+\rho} : w_+ > 0\}$.

Proof. The proof follows the structure of the proof of Lemma 3.1.

Step 1: As before, there exists a constant $c > 0$ such that

$$\mathcal{E}^{K_s}_{B_{r+\rho}}(\tau w_+,\tau w_+) - \mathcal{E}^{K_s}_{B_{r+\rho}}(w_-,\tau w_+) \leq \mathcal{E}^{K_s}_{B_{r+\rho}}(u,\tau^2 w_+) + c\rho^{-\alpha}\|w_+^2\|_{L^1(B_{r+\rho})}.\tag{3-8}$$

Step 2: We claim that, for every $\delta > 0$, there exists $c > 0$ such that

$$\begin{aligned}\widehat{\mathcal{E}}^{K_a}_{B_{r+\rho}}(u,\tau^2 w_+) \geq -\widehat{\mathcal{E}}^{K_a}_{B_{r+\rho}}(w_-,\tau^2 w_+) - \delta\mathcal{E}^{K_s}_{B_{r+\rho}}(\tau w_+,\tau w_+) - c\rho^{-\alpha}\|w_+^2\|_{L^1(B_{r+\rho})}&\\ - cl^2\rho^{-\alpha}\left[|A(l,r+\rho)| + |B_{r+\rho}|\left(\frac{|A(l,r+\rho)|}{|B_{r+\rho}|}\right)^{1/\theta'}\right]&.\end{aligned}\tag{3-9}$$

This is the main part of the proof, and it differs from Step 2 in Lemma 3.1. First, we observe

$$\widehat{\mathcal{E}}^{K_a}_{B_{r+\rho}}(u,\tau^2 w_+) = \mathcal{E}^{K_a}_{B_{r+\rho}}(\tau^2 w_+,u) = -\mathcal{E}^{K_a}_{B_{r+\rho}}(\tau^2 w_+,w_-) + \mathcal{E}^{K_a}_{B_{r+\rho}}(\tau^2 w_+,w_+) + \mathcal{E}^{K_a}_{B_{r+\rho}}(\tau^2 w_+,l).$$

To estimate the second term, observe

$$(\tau_1^2 w_1 - \tau_2^2 w_2)(w_1 + w_2) = (\tau_1^2 w_1^2 - \tau_2^2 w_2^2) + w_1 w_2(\tau_1^2 - \tau_2^2).$$

Thus, we note that, for every $\delta > 0$, there exists $c > 0$ such that

$$\begin{aligned}\mathcal{E}^{K_a}_{B_{r+\rho}}(\tau^2 w_+, w_+) &= \int_{B_{r+\rho}}\int_{B_{r+\rho}} (\tau^2 w_+^2(x) - \tau^2 w_+^2(y)) K_a(x,y)\,\mathrm{d}y\,\mathrm{d}x \\ &\qquad + \int_{B_{r+\rho}}\int_{B_{r+\rho}} w_+(x) w_+(y)(\tau^2(x) - \tau^2(y)) K_a(x,y)\,\mathrm{d}y\,\mathrm{d}x \\ &\geq -\delta \mathcal{E}^{K_s}_{B_{r+\rho}}(\tau w_+, \tau w_+) - c\rho^{-\alpha}\|w_+^2\|_{L^1(B_{r+\rho})}.\end{aligned}$$

The estimate in the last step works exactly as in the estimation of I_2 and I_3 in the proof of Lemma 3.1.

The estimate of the remaining term $\mathcal{E}^{K_a}_{B_{r+\rho}}(\tau^2 w_+, l)$ goes as follows:

$$\begin{aligned}\mathcal{E}^{K_a}_{B_{r+\rho}}(\tau^2 w_+, l) &= 2l \int_{B_{r+\rho}}\int_{B_{r+\rho}} (\tau^2 w_+(x) - \tau^2 w_+(y)) K_a(x,y)\,\mathrm{d}y\,\mathrm{d}x \\ &= 2l \int_{B_{r+\rho}}\int_{B_{r+\rho}} (\tau(x) - \tau(y))(\tau w_+(x) + \tau w_+(y)) K_a(x,y)\,\mathrm{d}y\,\mathrm{d}x \\ &\qquad + 2l \int_{B_{r+\rho}}\int_{B_{r+\rho}} (\tau(x) + \tau(y))(\tau w_+(x) - \tau w_+(y)) K_a(x,y)\,\mathrm{d}y\,\mathrm{d}x \\ &=: J_1 + J_2.\end{aligned}$$

To estimate J_1, we apply (cutoff) and (2-8):

$$\begin{aligned}J_1 &\geq -4l \int_{A(l,r+\rho)}\int_{B_{r+\rho}} |\tau(x) - \tau(y)|\tau w_+(x)|K_a(x,y)|\,\mathrm{d}y\,\mathrm{d}x \\ &\geq -cl^2 \int_{A(l,r+\rho)} \Gamma^J(\tau,\tau)(x)\,\mathrm{d}x - c\int_{A(l,r+\rho)}\int_{B_{r+\rho}} \tau^2 w_+^2(x)\frac{|K_a(x,y)|^2}{J(x,y)}\,\mathrm{d}y\,\mathrm{d}x \\ &\geq -c\rho^{-\alpha} l^2 |A(l, r+\rho)| - \delta\mathcal{E}^{K_s}_{B_{r+\rho}}(\tau w_+, \tau w_+) - c\rho^{-\alpha}\|w_+^2\|_{L^1(B_{r+\rho})}.\end{aligned}$$

J_2 can also be estimated with the help of (cutoff) and ($K1_{\mathrm{loc}}$):

$$\begin{aligned}J_2 &\geq -4l \int_{A(l,r+\rho)}\int_{B_{r+\rho}} (\tau(x) + \tau(y))|\tau w_+(x) - \tau w_+(y)||K_a(x,y)|\,\mathrm{d}y\,\mathrm{d}x \\ &\geq -8l \int_{A(l,r+\rho)}\int_{B_{r+\rho}} |\tau(x) - \tau(y)||\tau w_+(x) - \tau w_+(y)||K_s(x,y)|\,\mathrm{d}y\,\mathrm{d}x \\ &\qquad - 8l \int_{A(l,r+\rho)}\int_{B_{r+\rho}} (\tau(x) \wedge \tau(y))|\tau w_+(x) - \tau w_+(y)||K_a(x,y)|\,\mathrm{d}y\,\mathrm{d}x \\ &\geq -cl^2 \int_{A(l,r+\rho)} \Gamma^{K_s}(\tau,\tau)(x)\,\mathrm{d}x - \delta\mathcal{E}^{K_s}_{B_{r+\rho}}(\tau w_+, \tau w_+) \\ &\qquad - \delta\mathcal{E}^{J}_{B_{r+\rho}}(\tau w_+, \tau w_+) - cl^2 \int_{A(l,r+\rho)}\int_{B_{r+\rho}} (\tau^2(x) \wedge \tau^2(y))\frac{|K_a(x,y)|^2}{J(x,y)}\,\mathrm{d}y\,\mathrm{d}x \\ &\geq -c\delta\mathcal{E}^{K_s}_{B_{r+\rho}}(\tau w_+, \tau w_+) - cl^2\rho^{-\alpha}|A(l,r+\rho)| - cl^2\rho^{-\alpha}|B_{r+\rho}|\left(\frac{|A(l,r+\rho)|}{|B_{r+\rho}|}\right)^{1/\theta'}.\end{aligned}$$

Here, we used that, by (K1$_{\mathrm{loc}}$) and Hölder's inequality,

$$l^2 \int_{A(l,r+\rho)} \int_{B_{r+\rho}} (\tau^2(x) \wedge \tau^2(y)) \frac{|K_a(x,y)|^2}{J(x,y)} \,\mathrm{d}y\,\mathrm{d}x \le l^2 \int_{A(l,r+\rho)} \tau^2(x) \biggl(\int_{B_{r+\rho}} \frac{|K_a(x,y)|^2}{J(x,y)} \,\mathrm{d}y \biggr) \mathrm{d}x$$
$$\le c l^2 \|\tau^2\|_{L^{\theta'}(A(l,r+\rho))}$$
$$\le c l^2 \rho^{-\alpha} |B_{r+\rho}| \biggl(\frac{|A(l,r+\rho)|}{|B_{r+\rho}|} \biggr)^{1/\theta'},$$

since

$$1 \le c|B_{r+\rho}|^{-\alpha/d+1-1/\theta'} \le c\rho^{-\alpha}|B_{r+\rho}|^{1-1/\theta'}$$

for some constant $c > 0$ because $\theta \ge d/\alpha$, which implies that

$$-\frac{\alpha}{d} + 1 - \frac{1}{\theta'} \in \Bigl[-\frac{\alpha}{d}, 0 \Bigr) \quad \text{and} \quad \rho \le r \le 1.$$

Step 3: Next, let us demonstrate how to prove

$$-\widehat{\mathcal{E}}_{(B_{r+\rho} \times B_{r+\rho})^c}(u, \tau^2 w_+) \le 2 \biggl(\int_{B_{r+\rho}} w_+(x) \,\mathrm{d}x \biggr) \widehat{\mathrm{Tail}}_K\bigl(u, r + \tfrac{1}{2}\rho, r+\rho\bigr). \tag{3-10}$$

We estimate

$$-\widehat{\mathcal{E}}_{(B_{r+\rho} \times B_{r+\rho})^c}(u, \tau^2 w_+)$$
$$= 2 \int_{B^c_{r+\rho}} \int_{B_{r+\rho/2}} \tau^2 w_+(y) u(x) K(x,y) \,\mathrm{d}y\,\mathrm{d}x - 2 \int_{B_{r+\rho/2}} \int_{B^c_{r+\rho}} \tau^2 w_+(x) u(x) K(x,y) \,\mathrm{d}y\,\mathrm{d}x$$
$$\le 2 \int_{B_{r+\rho/2}} w_+(y) \biggl(\int_{\mathbb{R}^d \setminus B_{r+\rho}} u(x) K(x,y) \,\mathrm{d}x \biggr) \mathrm{d}y,$$

where we used that K is nonnegative and $\tau \equiv 0$ in $B^c_{r+\rho/2}$. Note that the second summand in the first step is negative since $w_+(x)u(x) \ge 0$, and can therefore be neglected.

Step 4: We will now combine (3-8), (3-9), and (3-10). Observe that

$$\mathcal{E}^{K_s}_{B_{r+\rho}}(u, \tau^2 w_+) = \widehat{\mathcal{E}}(u, \tau^2 w_+) - \widehat{\mathcal{E}}^{K_a}_{B_{r+\rho}}(u, \tau^2 w_+) - \widehat{\mathcal{E}}_{(B_{r+\rho} \times B_{r+\rho})^c}(u, \tau^2 w_+).$$

Altogether, we immediately obtain the desired result by choosing $\delta > 0$ from Step 2 small enough. □

Corollary 3.5. *Assume that (K1$_{\mathrm{glob}}$) and (cutoff) hold for some $\theta \in [d/\alpha, \infty]$. Moreover, assume (Sob) if $\theta < \infty$. Then there exist $c_1, c_2 > 0$ such that, for every $0 < \rho \le r \le 1$, every $l \in \mathbb{R}$, and every function $u \in V(B_{r+\rho} | \mathbb{R}^d)$, we have*

$$\mathcal{E}^{K_s}_{B_{r+\rho}}(\tau w_+, \tau w_+)$$
$$\le c_1 \widehat{\mathcal{E}}(u, \tau^2 w_+) + c_2 \rho^{-\alpha} \|w_+^2\|_{L^1(B_{r+\rho})} + c_2 l^2 \rho^{-\alpha} \biggl[|A(l,r+\rho)| + |B_{r+\rho}| \biggl(\frac{|A(l,r+\rho)|}{|B_{r+\rho}|} \biggr)^{1/\theta'} \biggr]$$
$$+ c_2 \|w_+\|_{L^1(B_{r+\rho})} \widehat{\mathrm{Tail}}_K\bigl(w_+, r + \tfrac{1}{2}\rho, r+\rho\bigr), \tag{3-11}$$

where $B_{2r} \subset \Omega$, $w = u - l$, and $\tau = \tau_{r,\rho/2}$.

3.2. *Local boundedness.* The following theorem is the main result of this section. It yields a priori local boundedness of subsolutions to (PDE), or to ($\widehat{\text{PDE}}$), if the nonlocal tail is finite.

Theorem 3.6. *Assume that* ($K_{\text{loc}}^{\leq}$), (cutoff), *and* (Sob) *hold.*

(i) *Assume that* (K1_{loc}) *holds for some $\theta \in [d/\alpha, \infty]$. Then there exists $c > 0$ such that, for every $0 < R \leq 1$, every $\delta \in (0, 1]$, and every nonnegative, weak subsolution u to* (PDE) *in $I_R^{\ominus}(t_0) \times B_{2R}$,*

$$\sup_{I_{R/8}^{\ominus} \times B_{R/2}} u \leq c\delta^{-(d+\alpha)/(2\alpha)} \left(\fint_{I_{R/4}^{\ominus}} \fint_{B_R} u^2(t,x)\,\mathrm{d}x\,\mathrm{d}t \right)^{1/2} + \delta \sup_{t \in I_{R/4}^{\ominus}} \mathrm{Tail}_{K,\alpha}(u(t), R) + \delta R^{\alpha} \|f\|_{L^\infty},$$

where $B_{2R} \subset \Omega$.

(ii) *Assume that* (K1_{glob}) *holds for some $\theta \in (d/\alpha, \infty]$. Then there exists $c > 0$ such that, for every $0 < R \leq 1$, every $\delta \in (0, 1]$, and every nonnegative, weak subsolution u to* ($\widehat{\text{PDE}}$) *in $I_R^{\ominus}(t_0) \times B_{2R}$,*

$$\sup_{I_{R/8}^{\ominus} \times B_{R/2}} u \leq c\delta^{-\tilde{\kappa}'/2} \left(\fint_{I_{R/4}^{\ominus}} \left(\fint_{B_R} u^{2\theta'}(t,x)\,\mathrm{d}x \right)^{1/\theta'} \mathrm{d}t \right)^{1/2} + \delta \sup_{t \in I_{R/4}^{\ominus}} \widehat{\mathrm{Tail}}_{K,\alpha}(u(t), R) + \delta R^{\alpha} \|f\|_{L^\infty},$$

where $B_{2R} \subset \Omega$ and $\tilde{\kappa} = 1 + \alpha/d - 1/\theta > 1$.

Proof. We first explain how to prove (i). Let $l > 0$, and define $w_l := (u-l)_+$. Let $r, \rho > 0$ such that $\frac{1}{2}R \leq r \leq R$ and $\rho \leq r \leq r + \rho \leq R$. Let $\tau = \tau_{r,\rho/2}$. Moreover, we define $\chi \in C^1(\mathbb{R})$ to be a function satisfying

$$0 \leq \chi \leq 1, \quad \|\chi'\|_\infty \leq 16((r+\rho)^\alpha - r^\alpha)^{-1}, \quad \chi(t_0 - ((r+\rho)/4)^\alpha) = 0, \quad \chi \equiv 1 \quad \text{in } I_{r/4}^{\ominus}(t_0).$$

Since u is a weak subsolution to (PDE), Lemma A.1 yields, for any $t \in I_{r/4}^{\ominus}(t_0)$,

$$\begin{aligned}
\int_{B_{r+\rho}} \chi^2(t)\tau^2(x) w_l^2(t,x)\,\mathrm{d}x &+ \int_{t_0-((r+\rho)/4)^\alpha}^{t} \chi^2(s)\mathcal{E}(u(s), \tau^2 w_l(s))\,\mathrm{d}s \\
&\leq \int_{t_0-((r+\rho)/4)^\alpha}^{t} \chi^2(s)(f(s), \tau^2 w_l(s))\,\mathrm{d}s + 2\int_{t_0-((r+\rho)/4)^\alpha}^{t} \chi(s)|\chi'(s)| \int_{B_{r+\rho}} \tau^2(x) w_l^2(s,x)\,\mathrm{d}x\,\mathrm{d}s \\
&\leq \|f\|_{L^\infty} \int_{I_{(r+\rho)/4}^{\ominus}} \|w_l(s)\|_{L^1(B_{r+\rho})}\,\mathrm{d}s + c_1((r+\rho)^\alpha - r^\alpha)^{-1} \int_{I_{(r+\rho)/4}^{\ominus}} \|w_l^2(s)\|_{L^1(B_{r+\rho})}\,\mathrm{d}s
\end{aligned}$$

for some constant $c_1 > 0$. Applying Corollary 3.2, we obtain

$$\begin{aligned}
\sup_{t \in I_{r/4}^{\ominus}} \int_{B_r} w_l^2(t,x)\,\mathrm{d}x &+ \int_{I_{r/4}^{\ominus}} \mathcal{E}_{B_{r+\rho}}^{K_s}(\tau w_l(s), \tau w_l(s))\,\mathrm{d}s \\
&\leq c_2(\rho^{-\alpha} \vee ((r+\rho)^\alpha - r^\alpha)^{-1}) \int_{I_{(r+\rho)/4}^{\ominus}} \|w_l^2(s)\|_{L^1(B_{r+\rho})}\,\mathrm{d}s \\
&\quad + c_2 \|w_l\|_{L^1(I_{(r+\rho)/4}^{\ominus} \times B_{r+\rho})} \Big(\sup_{t \in I_{(r+\rho)/4}^{\ominus}} \mathrm{Tail}_K\big(u(t), r + \tfrac{1}{2}\rho, r+\rho\big) + \|f\|_{L^\infty} \Big)
\end{aligned} \tag{3-12}$$

for some $c_2 > 0$. Recall $\kappa = 1+\alpha/d > 1$. Hölder interpolation and the Sobolev inequality (Sob) yield

$$\begin{aligned}\|w_l^2\|_{L^\kappa(I^\ominus_{r/4}\times B_r)} &\le \left(\sup_{t\in I^\ominus_{r/4}} \|w_l^2(t)\|^{\kappa-1}_{L^1(B_r)} \int_{I^\ominus_{r/4}} \|w_l^2(s)\|_{L^{d/(d-\alpha)}(B_r)}\,\mathrm{d}s\right)^{1/\kappa}\\ &\le c_3\sigma(r,\rho)\|w_l^2\|_{L^1(I^\ominus_{(r+\rho)/4}\times B_{r+\rho})}\\ &\qquad + c_3\|w_l\|_{L^1(I^\ominus_{(r+\rho)/4}\times B_{r+\rho})}\Big(\sup_{t\in I^\ominus_{(r+\rho)/4}} \mathrm{Tail}_K\big(u(t), r+\tfrac12\rho, r+\rho\big) + \|f\|_{L^\infty}\Big),\end{aligned} \tag{3-13}$$

where $c_3 > 0$ and we used that there is $c > 0$ such that

$$(\rho^{-\alpha}\vee((r+\rho)^\alpha - r^\alpha)^{-1}) \le c\rho^{-(\alpha\vee 1)}(r+\rho)^{(\alpha\vee 1)-\alpha} =: \sigma(r,\rho).$$

Furthermore, set

$$|A(l,r)| := \int_{I^\ominus_{r/4}} |\{x\in B_r : u(s,x) > l\}|\,\mathrm{d}s.$$

Then, by application of Hölder's inequality, with κ and $\kappa/(\kappa-1)$ both in time and in space, and (3-13),

$$\begin{aligned}&\|w_l^2\|_{L^1(I^\ominus_{r/4}\times B_r)}\\ &\quad\le |A(l,r)|^{1/\kappa'}\|w_l^2\|_{L^\kappa(I^\ominus_{r/4}\times B_r)}\\ &\quad\le c_4|A(l,r)|^{1/\kappa'}\Big[\sigma(r,\rho)\|w_l^2\|_{L^1(I^\ominus_{(r+\rho)/4}\times B_{r+\rho})}\\ &\qquad\qquad + \|w_l\|_{L^1(I^\ominus_{(r+\rho)/4}\times B_{r+\rho})}\Big(\sup_{t\in I^\ominus_{(r+\rho)/4}} \mathrm{Tail}_K\big(u(t), r+\tfrac12\rho, r+\rho\big) + \|f\|_{L^\infty}\Big)\Big],\end{aligned} \tag{3-14}$$

where $c_4 > 0$ is a constant. Let now $0 < k < l$ be arbitrary. Then the following hold:

$$\begin{aligned}\|w_l^2\|_{L^1(I^\ominus_{(r+\rho)/4}\times B_{r+\rho})} &\le \|w_k^2\|_{L^1(I^\ominus_{(r+\rho)/4}\times B_{r+\rho})},\\ \|w_l\|_{L^1(I^\ominus_{(r+\rho)/4}\times B_{r+\rho})} &\le \frac{\|w_k^2\|_{L^1(I^\ominus_{(r+\rho)/4}\times B_{r+\rho})}}{l-k},\\ |A(l,r)| &\le \frac{\|w_k^2\|_{L^1(I^\ominus_{(r+\rho)/4}\times B_{r+\rho})}}{(l-k)^2}.\end{aligned} \tag{3-15}$$

By combining (3-14) and (3-15), we obtain

$$\begin{aligned}&\|w_l^2\|_{L^1(I^\ominus_{r/4}\times B_r)}\\ &\quad\le c_5|A(l,r)|^{1/\kappa'}\left(\sigma(r,\rho) + \frac{\sup_{t\in I^\ominus_{(r+\rho)/4}} \mathrm{Tail}_K\big(u(t), r+\frac12\rho, r+\rho\big) + \|f\|_{L^\infty}}{l-k}\right)\|w_k^2\|_{L^1(I^\ominus_{(r+\rho)/4}\times B_{r+\rho})}\\ &\quad\le c_6(l-k)^{-2/\kappa'}\left(\sigma(r,\rho) + \frac{\sup_{t\in I^\ominus_{(r+\rho)/4}} \mathrm{Tail}_K\big(u(t), r+\frac12\rho, r+\rho\big) + \|f\|_{L^\infty}}{l-k}\right)\|w_k^2\|^{1+1/\kappa'}_{L^1(I^\ominus_{(r+\rho)/4}\times B_{r+\rho})}\end{aligned}$$

for some $c_5, c_6 > 0$. The plan for the remainder of the proof is to iterate the above estimate. Recall (2-10), which we will apply in the sequel. Let us now set up the iteration scheme. For this purpose, we define two sequences $l_i = M(1-2^{-i})$ and $\rho_i = 2^{-i-1}R,\ i\in\mathbb{N}$, where $M>0$ is to be determined later. We also set

$$r_0 = R,\quad r_{i+1} = r_i - \rho_{i+1} = \tfrac{1}{2}R\bigl(1+\bigl(\tfrac{1}{2}\bigr)^{i+1}\bigr),\quad \text{and}\quad l_0 = 0.$$

Then $r_i \searrow \frac{1}{2}R$ and $l_i \nearrow M$ as $i\to\infty$.

Note that $\sigma(r_i,\rho_i)\le c_7 R^{-\alpha}2^{2i}$ for some $c_7>0$. Define $A_i = \|w_{l_i}^2\|_{L^1(I^{\ominus}_{r_i/4}\times B_{r_i})}$. Then

$$\begin{aligned}
A_i &\le c_8\frac{1}{(l_i-l_{i-1})^{2/\kappa'}}\left(\sigma(r_i,\rho_i)+\frac{\sup_{t\in I^{\ominus}_{r_i/4}}\mathrm{Tail}_K\bigl(u(t),r_i+\frac{1}{2}\rho_i,r_i+\rho_i\bigr)+\|f\|_{L^\infty}}{l_i-l_{i-1}}\right)A_{i-1}^{1+1/\kappa'}\\
&\le c_9\frac{1}{(l_i-l_{i-1})^{2/\kappa'}}\left(\sigma(r_i,\rho_i)+\rho_i^{-\alpha}\left(\frac{r_i}{\rho_i}\right)^d\frac{\sup_{t\in I^{\ominus}_{R/4}}\mathrm{Tail}_{K,\alpha}(u(t),R)+R^\alpha\|f\|_{L^\infty}}{l_i-l_{i-1}}\right)A_{i-1}^{1+1/\kappa'}\\
&\le c_{10}\frac{2^{2i/\kappa'}}{M^{2/\kappa'}}\left(\frac{2^{2i}}{R^\alpha}+\frac{2^{(1+\alpha+d)i}}{R^\alpha}\frac{\sup_{t\in I^{\ominus}_{R/4}}\mathrm{Tail}_{K,\alpha}(u(t),R)+R^\alpha\|f\|_{L^\infty}}{M}\right)A_{i-1}^{1+1/\kappa'}\\
&\le\frac{c_{11}}{R^\alpha M^{2/\kappa'}}2^{\gamma i}\left(1+\frac{\sup_{t\in I^{\ominus}_{R/4}}\mathrm{Tail}_{K,\alpha}(u(t),R)+R^\alpha\|f\|_{L^\infty}}{M}\right)A_{i-1}^{1+1/\kappa'}
\end{aligned}\tag{3-16}$$

for $c_8,c_9,c_{10},c_{11}>0,\ \gamma>1$. Note that here we also applied (2-10). If, given $\delta\in(0,1]$, we choose

$$M\ge\delta\Bigl(\sup_{t\in I^{\ominus}_{R/4}}\mathrm{Tail}_{K,\alpha}(u(t),R)+R^\alpha\|f\|_{L^\infty}\Bigr)$$

then,

$$A_i\le\frac{c_{12}}{\delta R^\alpha M^{2/\kappa'}}C^iA_{i-1}^{1+1/\kappa'},$$

where $C:=2^{2/\kappa'+2}>1$ and $c_{12}>0$. We choose

$$M:=\delta\Bigl(\sup_{t\in I^{\ominus}_{R/4}}\mathrm{Tail}_{K,\alpha}(u(t),R)+R^\alpha\|f\|_{L^\infty}\Bigr)+C^{\kappa'^2/2}c_{12}^{\kappa'/2}\delta^{-\kappa'/2}R^{-\alpha\kappa'/2}A_0^{1/2}.$$

It follows that

$$A_0\le c_{12}^{-\kappa'}\delta^{\kappa'}R^{\alpha\kappa'}M^2C^{-\kappa'^2}=\left(\frac{c_{12}}{\delta R^\alpha M^{2/\kappa'}}\right)^{-\kappa'}C^{-\kappa'^2},$$

and therefore we know from Lemma 7.1 in [Giusti 2003] that $A_i\searrow 0$ as $i\to\infty$, i.e.,

$$\begin{aligned}
\sup_{I^{\ominus}_{R/8}\times B_{R/2}}u\le M&=\delta\Bigl(\sup_{t\in I^{\ominus}_{R/4}}\mathrm{Tail}_{K,\alpha}(u(t),R)+R^\alpha\|f\|_{L^\infty}\Bigr)+C^{\kappa'^2/2}c_{12}^{\kappa'/2}\delta^{-\kappa'/2}R^{-\alpha\kappa'/2}A_0^{1/2}\\
&=\delta\Bigl(\sup_{t\in I^{\ominus}_{R/4}}\mathrm{Tail}_{K,\alpha}(u(t),R)+R^\alpha\|f\|_{L^\infty}\Bigr)+c_{13}\delta^{-\kappa'/2}\left(R^{-\alpha\kappa'}\int_{I^{\ominus}_{R/4}}\int_{B_R}u^2(t,x)\,\mathrm{d}x\,\mathrm{d}t\right)^{1/2}
\end{aligned}$$

for $c_{13}>0$. Note that, by the definition of κ, we have $\alpha\kappa'=\alpha+d$. Therefore,

$$\sup_{I^{\ominus}_{R/8}\times B_{R/2}}u\le\delta\sup_{t\in I^{\ominus}_{R/4}}\mathrm{Tail}_{K,\alpha}(u(t),R)+\delta R^\alpha\|f\|_{L^\infty}+c_{14}\delta^{-\kappa'/2}\left(\fint_{I^{\ominus}_{R/4}}\fint_{B_R}u^2(t,x)\,\mathrm{d}x\,\mathrm{d}t\right)^{1/2}$$

for some $c_{14}>0$. This proves (i).

To prove (ii), observe that, instead of (3-12), applying Corollary 3.5 to a weak subsolution u to $(\widehat{\text{PDE}})$ yields

$$\sup_{t\in I^{\ominus}_{r/4}}\int_{B_r} w_l^2(t,x)\,\mathrm{d}x+\int_{I^{\ominus}_{r/4}}\mathcal{E}^{K_s}_{B_{r+\rho}}(\tau w_l(t),\tau w_l(t))\,\mathrm{d}t$$
$$\leq c_1\sigma(r,\rho)\|w_l^2(t)\|_{L^1(I^{\ominus}_{(r+\rho)/4}\times B_{r+\rho})}+c_1l^2\rho^{-\alpha}\left[|A(l,r+\rho)|+|B_{r+\rho}|^{1/\theta}\int_{I^{\ominus}_{(r+\rho)/4}}|B_{r+\rho}\cap\{u(t,x)>l\}|^{1/\theta'}\,\mathrm{d}t\right]$$
$$+c_1\|w_l(t)\|_{L^1(I^{\ominus}_{(r+\rho)/4}\times B_{r+\rho})}\Big(\sup_{t\in I^{\ominus}_{(r+\rho)/4}}\widehat{\operatorname{Tail}}_K\big(u(t),r+\tfrac{1}{2}\rho,r+\rho\big)+\|f\|_{L^\infty}\Big)$$

for some $c_1>0$. Proceeding as in the proof of (i), we derive the following estimate as a replacement of (3-14), where $\tilde{\kappa}:=\kappa-1/\theta>1$:

$$\int_{I^{\ominus}_{r/4}}\|w_l^2(t)\|_{L^{\theta'}(B_r)}\,\mathrm{d}t$$
$$\leq|A(l,r)|^{1/\tilde{\kappa}'}\bigg(\int_{I^{\ominus}_{r/4}}\|w_l^2(t)\|^{\tilde{\kappa}}_{L^{\tilde{\kappa}\theta'}(B_r)}\,\mathrm{d}t\bigg)^{1/\tilde{\kappa}}$$
$$\leq|A(l,r)|^{1/\tilde{\kappa}'}\bigg(\sup_{t\in I^{\ominus}_{r/4}}\|w_l^2(t)\|^{\tilde{\kappa}-1}_{L^1(B_r)}\int_{I^{\ominus}_{r/4}}\|w_l^2(s)\|_{L^{d/(d-\alpha)}(B_r)}\,\mathrm{d}s\bigg)^{1/\tilde{\kappa}}$$
$$\leq c_2|A(l,r+\rho)|^{1/\tilde{\kappa}'}\bigg[\sigma(r,\rho)\|w_l^2\|_{L^1(I^{\ominus}_{(r+\rho)/4}\times B_{r+\rho})}$$
$$+l^2\rho^{-\alpha}\left[|A(l,r+\rho)|+|B_{r+\rho}|^{1/\theta}\int_{I^{\ominus}_{(r+\rho)/4}}|B_{r+\rho}\cap\{u(t,x)>l\}|^{1/\theta'}\,\mathrm{d}t\right]$$
$$+\|w_l\|_{L^1(I^{\ominus}_{(r+\rho)/4}\times B_{r+\rho})}\Big(\sup_{t\in I^{\ominus}_{(r+\rho)/4}}\operatorname{Tail}_K\big(u(t),r+\tfrac{1}{2}\rho,r+\rho\big)+\|f\|_{L^\infty}\Big)\bigg]$$
$$\leq c_3|B_{r+\rho}|^{1/\theta}|A(l,r+\rho)|^{1/\tilde{\kappa}'}\left[\sigma(r,\rho)\bigg(1+\Big(\frac{l}{l-k}\Big)^2\bigg)+\frac{\sup_{t\in I^{\ominus}_{(r+\rho)/4}}\operatorname{Tail}_K\big(u(t),r+\frac{1}{2}\rho,r+\rho\big)+\|f\|_{L^\infty}}{l-k}\right]$$
$$\times\int_{I^{\ominus}_{(r+\rho)/4}}\|w_k^2(t)\|_{L^{\theta'}(B_{r+\rho})}\,\mathrm{d}t$$
$$\leq c_4\frac{|B_{r+\rho}|^{1/\theta}}{(l-k)^{2/\tilde{\kappa}'}}\left[\sigma(r,\rho)\bigg(1+\Big(\frac{l}{l-k}\Big)^2\bigg)+\frac{\sup_{t\in I^{\ominus}_{(r+\rho)/4}}\operatorname{Tail}_K\big(u(t),r+\frac{1}{2}\rho,r+\rho\big)+\|f\|_{L^\infty}}{l-k}\right]$$
$$\times\bigg(\int_{I^{\ominus}_{(r+\rho)/4}}\|w_k^2(t)\|_{L^{\theta'}(B_{r+\rho})}\,\mathrm{d}t\bigg)^{1+1/\tilde{\kappa}'}$$

for some $c_2,c_3,c_4>0$, and we used

$$\int_{I^{\ominus}_{(r+\rho)/4}}|B_{r+\rho}\cap\{u(t,x)>l\}|^{1/\theta'}\,\mathrm{d}t\leq(l-k)^{-2}\int_{I^{\ominus}_{(r+\rho)/4}}\|w_k^2(t)\|_{L^{\theta'}(B_{r+\rho})}\,\mathrm{d}t \tag{3-17}$$

and applied (3-15). From here, the proof basically proceeds as before. We define sequences (l_i), (ρ_i), and (r_i) as before, write $A_i=\int_{I^{\ominus}_{r_i/4}}\|w_{l_i}(t)\|_{L^{\theta'}(B_{r_i})}\,\mathrm{d}t$, and deduce that, for any $\delta\in(0,1]$, by choosing

$$M\geq\delta\Big(\sup_{t\in I^{\ominus}_{R/4}}\operatorname{Tail}_{K,\alpha}(u(t),R)+R^\alpha\|f\|_{L^\infty}\Big),$$

we deduce that

$$A_i \leq \frac{c_5}{\delta R^{\alpha-d/\theta} M^{2/\tilde{\kappa}'}} C^i A_{i-1}^{1+1/\tilde{\kappa}'},$$

where $C > 1$ and $c_5 > 0$ are constants. We choose

$$M := \delta\Big(\sup_{t\in I^{\ominus}_{R/4}} \operatorname{Tail}_{K,\alpha}(u(t), R) + R^{\alpha}\|f\|_{L^\infty}\Big) + C^{\tilde{\kappa}'^2/2} c_5^{\tilde{\kappa}'/2} \delta^{-\tilde{\kappa}'/2} R^{-(\alpha-d/\theta)\tilde{\kappa}'/2} A_0^{1/2}.$$

It follows that

$$A_0 \leq c_5^{-\tilde{\kappa}'} \delta^{\tilde{\kappa}'} R^{(\alpha-d/\theta)\tilde{\kappa}'} M^2 C^{-\tilde{\kappa}'^2} = \left(\frac{c_5}{\delta R^{\alpha-d/\theta} M^{d/\tilde{\kappa}'}}\right)^{-\tilde{\kappa}'} C^{-\tilde{\kappa}'^2},$$

and therefore we know from Lemma 7.1 in [Giusti 2003] that $A_i \searrow 0$ as $i \to \infty$, i.e.,

$$\begin{aligned}\sup_{I^{\ominus}_{R/8}\times B_{R/2}} u \leq M &= \delta\Big(\sup_{t\in I^{\ominus}_{R/4}} \operatorname{Tail}_{K,\alpha}(u(t), R) + R^{\alpha}\|f\|_{L^\infty}\Big) + C^{\tilde{\kappa}'^2/2} c_5^{\tilde{\kappa}'/2} \delta^{-\tilde{\kappa}'/2} R^{-(\alpha-d/\theta)\tilde{\kappa}'/2} A_0^{1/2} \\ &= \delta\Big(\sup_{t\in I^{\ominus}_{R/4}} \operatorname{Tail}_{K,\alpha}(u(t), R) + R^{\alpha}\|f\|_{L^\infty}\Big) + c_6\delta^{-\tilde{\kappa}'/2}\left(\fint_{I^{\ominus}_{R/4}}\left(\fint_{B_R} u^{2\theta'}(t,x)\,\mathrm{d}x\right)^{1/\theta'} \mathrm{d}t\right)^{1/2}\end{aligned}$$

for $c_6 > 0$, where we used $(\alpha - d/\theta)\tilde{\kappa}' = \alpha + d/\theta'$. ☐

Remark 3.7. Let us comment on the appearance of the $L^{2,2\theta'}_{t,x}$-norm of u in the estimate (ii) for subsolutions to ($\widehat{\text{PDE}}$). In fact, this term appears since we iterate the $L^{2,2\theta'}_{t,x}$-norms of w_{l_i} in the proof of (ii). In fact, upon estimating

$$|B_{r+\rho}|^{1/\theta} \int_{I^{\ominus}_{(r+\rho)/4}} |B_{r+\rho} \cap \{u(t,x) > l\}|^{1/\theta'}\,\mathrm{d}t \leq c|I^{\ominus}_{(r+\rho)/4} \times B_{r+\rho}|^{1/\theta} |A(l, r+\rho)|^{1/\theta'},$$

instead of (3-17), we could iterate the $L^{2,2}$-norms of w_{l_i} as in the proof of (i), however, only as long as

$$\mu := \frac{1}{\kappa'} - \frac{1}{\theta} = \frac{\alpha}{d+\alpha} - \frac{1}{\theta} > 0.$$

This means that we would have to restrict ourselves to the suboptimal range $\theta \in ((d+\alpha)/\alpha, \infty]$. In the local case, an analogous phenomenon appears in Chapter VI.13 in [Lieberman 1996].

Note that, for subsolutions (ell-$\widehat{\text{PDE}}$), the analogous condition reads $\mu := \alpha/d - 1/\theta > 0$, which allows us to estimate the supremum of u by the L^2-norm, as expected for the full range $\theta \in (d/\alpha, \infty]$.

We now state the analog to Theorem 3.6 for stationary solutions.

Theorem 3.8. *Assume that* ($K^{\leq}_{\text{loc}}$), (cutoff), *and* (Sob) *hold.*

(i) *Assume that* (K1_{loc}) *holds for some* $\theta \in [d/\alpha, \infty]$. *Then there exists* $c > 0$ *such that, for every* $0 < R \leq 1$, *every* $\delta \in (0, 1]$, *and every nonnegative, weak subsolution* u *to* (ell-PDE) *in* $B_{2R} \subset \Omega$,

$$\sup_{B_{R/2}} u \leq c\delta^{-d/(2\alpha)}\left(\fint_{B_R} u^2(x)\,\mathrm{d}x\right)^{1/2} + \delta \operatorname{Tail}_{K,\alpha}(u, R) + R^{\alpha}\|f\|_{L^\infty}. \tag{3-18}$$

(ii) *Assume that* (K1$_{\text{glob}}$) *holds for some* $\theta \in (d/\alpha, \infty]$. *Then there exists* $c > 0$ *such that, for every* $0 < R \le 1$, *every* $\delta \in (0, 1]$, *and every nonnegative, weak subsolution* u *to* (ell-$\widehat{\text{PDE}}$) *in* $B_{2R} \subset \Omega$,

$$\sup_{B_{R/2}} u \le c\delta^{-1/(2\mu)} \left(\fint_{B_R} u^2(x)\,\mathrm{d}x \right)^{1/2} + \delta\, \widehat{\mathrm{Tail}}_{K,\alpha}(u, R) + R^{\alpha} \|f\|_{L^\infty},$$

where $\mu := \alpha/d - 1/\theta \in (0, \alpha/d]$.

The first estimate can be read off from Theorem 3.6 (i). The proof of (ii) works similar to the proof of Theorem 3.6 (ii) up to small modifications in the sense of the aforementioned remark. The factor $\delta^{-d/(2\alpha)}$ in (3-18) stems from defining $\kappa = d/(d-\alpha)$ and $\kappa' = d/\alpha$ in the stationary case.

4. Local boundedness via Moser iteration

The goal of this section is to give another proof of Theorem 3.6 via the Moser iteration for positive exponents (see Theorem 4.8). For our main result there is no need of a second proof. However, we consider this independent approach interesting due to the wide range of applicability of the Moser iteration. While local boundedness for symmetric nonlocal operators has been established in numerous works by the De Giorgi iteration technique, the following proof of local boundedness (see Theorem 4.8) using a Moser iteration scheme seems to be new.

The Moser iteration for positive exponents is arguably more complicated than for negative exponents for the following two reasons: Roughly speaking, one would like to use test-functions of the form $\phi = \tau^2 u^{2q-1}$ for $q > 1$. Unfortunately, ϕ a priori does not belong to the correct function space unless u is bounded. Since boundedness of u is one of the main goals of this section, such an assumption is illegal. Instead, we truncate the monomial u^{2q-1} in an adequate way, similar to [Aronson and Serrin 1967]. The second reason concerns the appearance of nonlocal tail terms (see Section 3) due to the nonlocality of the equation. These quantities require special treatment in order to make the iteration work.

Note that Sections 3 and 4 are fully independent of each other.

4.1. ***Algebraic estimates.*** The first step is to establish suitable algebraic estimates, which can be seen and will be used as nonlocal analogs to the chain rule. Note that an estimate similar to (4-1) was established in [Brasco and Parini 2016]. We also refer to [Kassmann and Weidner 2022], where the Moser iteration schemes were established for negative and small positive exponents for the same class of nonsymmetric nonlocal operators.

Lemma 4.1. *Let* $g : [0, \infty) \to [0, \infty)$ *be continuously differentiable. Assume that* g *is increasing and that* $g(0) = 0$. *Set* $G(t) := \int_0^t g'(\tau)^{1/2}\,\mathrm{d}\tau$. *Then, for every* $s, t \ge 0$,

$$(t-s)(g(t)-g(s)) \ge (G(t)-G(s))^2, \tag{4-1}$$

$$\frac{(g(t) \wedge g(s))|t-s|}{|G(t)-G(s)|} \le G(t) \wedge G(s), \tag{4-2}$$

$$\frac{|g(t)-g(s)|}{|G(t)-G(s)|} \le g'(t \vee s)^{1/2}. \tag{4-3}$$

Proof. Note that, by assumption, $t \mapsto G'(t) = g'(t)^{1/2}$ is nonnegative. Let us assume without loss of generality that $s \le t$. First, we compute, with the help of Jensen's inequality,

$$(t-s)(g(t)-g(s)) = (t-s)\int_s^t g'(\tau)\,\mathrm{d}\tau = (t-s)\int_s^t G'(\tau)^2\,\mathrm{d}\tau \ge \left(\int_s^t G'(\tau)\,\mathrm{d}\tau\right)^2 = (G(t)-G(s))^2,$$

which proves (4-1). Next,

$$\frac{|G(t)-G(s)|}{|t-s|} = \fint_s^t G'(\tau)\,\mathrm{d}\tau \ge G'(s).$$

Moreover, we compute

$$g(s) = \int_0^s g'(\tau)\,\mathrm{d}\tau \le g'(s)^{1/2}\int_0^s g'(\tau)^{1/2}\,\mathrm{d}\tau = G'(s)G(s).$$

This implies

$$\frac{|G(t)-G(s)|}{|t-s|} \ge \frac{g(s)}{G(s)},$$

which proves (4-2). For (4-3), we compute, using the chain rule and again that $G'(t) = g'(t)^{1/2}$ is nondecreasing,

$$\frac{|g(t)-g(s)|}{|G(t)-G(s)|} = \left|\fint_{G(s)}^{G(t)} [g\circ G^{-1}]'(\tau)\,\mathrm{d}\tau\right| = \fint_{G(s)}^{G(t)} g'(G^{-1}(\tau))^{1/2}\,\mathrm{d}\tau \le g'(t)^{1/2}. \qquad \square$$

The following lemma has already been established and applied in [Kassmann and Weidner 2022] (see Lemma 3.2 therein).

Lemma 4.2. *Let $G : [0,\infty) \to \mathbb{R}$. Then, for any $\tau_1, \tau_2 \ge 0$ and $t, s > 0$,*

$$(\tau_1^2 \wedge \tau_2^2)|G(t)-G(s)|^2 \ge \tfrac{1}{2}|\tau_1 G(t)-\tau_2 G(s)|^2 - (\tau_1-\tau_2)^2(G^2(t)\vee G^2(s)), \tag{4-4}$$

$$(\tau_1^2 \vee \tau_2^2)|G(t)-G(s)|^2 \le 2|\tau_1 G(t)-\tau_2 G(s)|^2 + 2(\tau_1-\tau_2)^2(G^2(t)\vee G^2(s)). \tag{4-5}$$

From now on, let us define the functions $g : [0,\infty) \to [0,\infty)$ and $G(t) = \int_0^t g'(s)^{1/2}\,\mathrm{d}s$ for $M > 0$ and $q \ge 1$ via

$$g(t) = \begin{cases} t^{2q-1}, & t \le M, \\ M^{2q-1} + (2q-1)M^{2q-2}(t-M), & t > M, \end{cases}$$

$$G(t) = \begin{cases} \dfrac{\sqrt{2q-1}}{q}t^q, & t \le M, \\ \dfrac{\sqrt{2q-1}}{q}M^q + \sqrt{2q-1}(t-M)M^{q-1}, & t > M. \end{cases}$$

One easily checks that g is continuously differentiable, increasing, and satisfies $g(0) = 0$. Therefore g satisfies the assumptions of Lemma 4.1. Moreover, note that g is convex.

The following lemma is a direct consequence of the definition of g.

Lemma 4.3. *For every $t \geq 0$,*

$$G'(t) = g'(t)^{1/2} \leq q\frac{G(t)}{t}, \tag{4-6}$$

$$g(t)t \leq \frac{q^2}{2q-1}G^2(t). \tag{4-7}$$

Proof. Let us start by proving the first estimate. In the case $t \leq M$, a direct computation shows,

$$g'(t)^{1/2} = \sqrt{2q-1}t^{q-1} = q\frac{\sqrt{2q-1}}{q}t^{q-1} = q\frac{G(t)}{t}.$$

For $t > M$, we use $\sqrt{2q-1} \leq q$ to compute

$$\begin{aligned} g'(t)^{1/2} = \sqrt{2q-1}M^{q-1} &= q\frac{\frac{\sqrt{2q-1}}{q}(M^q + (t-M)M^{q-1})}{t} \\ &\leq q\frac{\frac{\sqrt{2q-1}}{q}M^q + \sqrt{2q-1}(t-M)M^{q-1}}{t} = q\frac{G(t)}{t}. \end{aligned}$$

This proves (4-6). For (4-7), in the case $t \leq M$, we compute

$$g(t)t = t^{2q} = \frac{q^2}{2q-1}G^2(t).$$

In the case $t > M$, we use $\sqrt{2q-1} \leq q$ to compute

$$g(t)t = t^2M^{2q-2} = \frac{q^2}{2q-1}\left(\frac{\sqrt{2q-1}}{q}(M^q + (t-M)M^{q-1})\right)^2 \leq \frac{q^2}{2q-1}G^2(t). \qquad \square$$

Remark 4.4. Note that (4-6) already implies a slightly weaker version of the estimate in (4-7). Indeed, by (4-6),

$$q^2G^2(t) \geq (G'(t)t)^2 = g'(t)t^2 \geq g(t)t,$$

where we used convexity and $g(0) = 0$ in the last estimate.

Lemma 4.5. *Let $q \geq 1$. Then, for every $s, t \geq 0$, we have*

$$(G(t) - G(s))^2 \nearrow \frac{2q-1}{q^2}(t^q - s^q)^2 \quad \text{as } M \nearrow \infty.$$

Proof. Clearly,

$$(G(t) - G(s))^2 \to \frac{2q-1}{q^2}(t^q - s^q)^2$$

as $M \to \infty$, since, for $t, s < M$, we already have

$$(G(t) - G(s))^2 = \frac{2q-1}{q^2}(t^q - s^q)^2.$$

It remains to prove that the convergence is monotone. Let us fix $t > s > 0$. First, we observe that $M \mapsto (G(t) - G(s))^2$ is continuous. Now, clearly, for $M < t < s$, we have

$$(G(t) - G(s))^2 = (2q-1)M^{2q-2}(t-s)^2,$$

which is increasing in M. In the case $s < M < t$,

$$(G(t)-G(s))^2 = \left(\frac{\sqrt{2q-1}}{q}M^q + (t-M)\sqrt{2q-1}M^{q-1} - \frac{\sqrt{2q-1}}{q}s^q\right)^2.$$

This expression is clearly monotone in M as long as $t > M$, since

$$\frac{\mathrm{d}}{\mathrm{d}M}\frac{\sqrt{2q-1}}{q}M^q + (t-M)\sqrt{2q-1}M^{q-1} = (q-1)\sqrt{2q-1}(t-M)M^{q-2} \geq 0.$$

This proves the desired result. □

4.2. ***Caccioppoli estimates.*** Now, we are in the position to prove the following Caccioppoli-type estimate. We emphasize that $\tau^2 g(\tilde{u}) \in H_{B_{r+\rho}}(\mathbb{R}^d)$ in the lemma below, where $\tilde{u} = u + R^\alpha \|f\|_{L^\infty}$, whenever $u \in V(B_{r+\rho}\,|\,\mathbb{R}^d)$. This is a direct consequence of the definition of g.

Lemma 4.6. *Assume that* (K1$_{\text{loc}}$) *and* (cutoff) *hold for some $\theta \in [d/\alpha, \infty]$. Moreover, assume* (Sob) *if $\theta < \infty$. Then there exist $c_1, c_2 > 0$ such that, for every $0 < \rho \leq r \leq 1$, every nonnegative function $u \in V(B_{r+\rho}\,|\,\mathbb{R}^d)$, and every $q \geq 1$, we have*

$$\begin{aligned}\mathcal{E}^{K_s}_{B_{r+\rho}}&(\tau G(\tilde{u}), \tau G(\tilde{u}))\\ &\leq c_1\mathcal{E}(u, \tau^2 g(\tilde{u})) + c_2\rho^{-\alpha}\|G(\tilde{u})^2\|_{L^1(B_{r+\rho})} + c_2\|g(\tilde{u})\|_{L^1(B_{r+\rho})}\operatorname{Tail}_K\big(u, r+\tfrac{1}{2}\rho, r+\rho\big),\end{aligned}$$

where $B_{2r} \subset \Omega$, $\tau = \tau_{r,\rho/2}$, and $\tilde{u} = u + R^\alpha\|f\|_{L^\infty}$.

Proof. We define

$$M := \{(x, y) \in B_{r+\rho} \times B_{r+\rho} : u(x) > u(y)\}.$$

Note that, for $(x, y) \in M$, we have $g(u(x)) \geq g(u(y))$ and $G(u(x)) \geq G(u(y))$. The proof is divided into several steps.

Step 1: First, we claim that, for some $c_1, c_2 > 0$,

$$\mathcal{E}_{B_{r+\rho}}(u, \tau^2 g(\tilde{u})) \geq c_1\mathcal{E}^{K_s}_{B_{r+\rho}}(\tau G(\tilde{u}), \tau G(\tilde{u})) - c_2\rho^{-\alpha}\|G(\tilde{u})^2\|_{L^1(B_{r+\rho})}. \tag{4-8}$$

For the symmetric part, we compute the following using the symmetry of K_s (see also Lemma 2.3 in [Kassmann and Weidner 2022]):

$$\begin{aligned}\mathcal{E}^{K_s}_{B_{r+\rho}}(u, \tau^2 g(\tilde{u})) &= 2\iint_M (\tilde{u}(x)-\tilde{u}(y))(\tau^2(x)g(\tilde{u}(x)) - \tau^2(y)g(\tilde{u}(y)))K_s(x,y)\,\mathrm{d}y\,\mathrm{d}x\\ &= 2\iint_M (\tilde{u}(x)-\tilde{u}(y))(g(\tilde{u}(x)) - g(\tilde{u}(y)))\tau^2(x)K_s(x,y)\,\mathrm{d}y\,\mathrm{d}x\\ &\qquad\qquad + 2\iint_M (\tilde{u}(x)-\tilde{u}(y))g(\tilde{u}(y))(\tau^2(x)-\tau^2(y))K_s(x,y)\,\mathrm{d}y\,\mathrm{d}x\\ &= I_s + J_s.\end{aligned}$$

For the nonsymmetric part, we compute, using the antisymmetry of K_a and with the help of Lemma 2.3 in [Kassmann and Weidner 2022],

$$\begin{aligned}\mathcal{E}^{K_a}_{B_{r+\rho}}(u,\tau^2 g(\tilde{u})) &= 2\iint_M (\tilde{u}(x)-\tilde{u}(y))(\tau^2(x)g(\tilde{u}(x))+\tau^2(y)g(\tilde{u}(y)))K_a(x,y)\,\mathrm{d}y\,\mathrm{d}x\\ &= 2\iint_M (\tilde{u}(x)-\tilde{u}(y))(g(\tilde{u}(x))-g(\tilde{u}(y)))\tau^2(x)K_a(x,y)\,\mathrm{d}y\,\mathrm{d}x\\ &\qquad\qquad + 2\iint_M (\tilde{u}(x)-\tilde{u}(y))g(\tilde{u}(y))(\tau^2(x)+\tau^2(y))K_a(x,y)\,\mathrm{d}y\,\mathrm{d}x\\ &= I_a + J_a.\end{aligned}$$

By adding I_s+I_a and using (4-1), (4-4), as well as (cutoff), we obtain

$$\begin{aligned}I_s+I_a &= 2\iint_M (\tilde{u}(x)-\tilde{u}(y))(g(\tilde{u}(x))-g(\tilde{u}(y)))\tau^2(x)K(x,y)\,\mathrm{d}y\,\mathrm{d}x\\ &\geq \iint_M (G(\tilde{u}(x))-G(\tilde{u}(y)))^2(\tau^2(x)\wedge\tau^2(y))K(x,y)\,\mathrm{d}y\,\mathrm{d}x\\ &\geq \tfrac{1}{2}\mathcal{E}^{K_s}_{B_{r+\rho}}(\tau G(\tilde{u}),\tau G(\tilde{u})) - c\rho^{-\alpha}\|G(\tilde{u})^2\|_{L^1(B_{r+\rho})}\\ &\qquad\qquad - \frac{1}{2}\int_{B_{r+\rho}}\int_{B_{r+\rho}} (\tau G(\tilde{u}(x))-\tau G(\tilde{u}(y)))^2|K_a(x,y)|\,\mathrm{d}y\,\mathrm{d}x.\end{aligned}$$

For the nonsymmetric part, using $(\mathrm{K1}_{\mathrm{loc}})$ and (2-8), we find that, for every $\varepsilon>0$, there is $c>0$ such that

$$\begin{aligned}&\int_{B_{r+\rho}}\int_{B_{r+\rho}} (\tau G(\tilde{u}(x))-\tau G(\tilde{u}(y)))^2|K_a(x,y)|\,\mathrm{d}y\,\mathrm{d}x\\ &\qquad \leq \varepsilon\mathcal{E}^{K_s}_{B_{r+\rho}}(\tau G(\tilde{u}),\tau G(\tilde{u})) + c\int_{B_{r+\rho}} \tau^2(x)G^2(\tilde{u}(x))\bigg(\int_{B_{r+\rho}}\frac{|K_a(x,y)|^2}{J(x,y)}\,\mathrm{d}y\bigg)\mathrm{d}x\\ &\qquad \leq 2\varepsilon\mathcal{E}^{K_s}_{B_{r+\rho}}(\tau G(\tilde{u}),\tau G(\tilde{u})) + c\rho^{-\alpha}\|G(\tilde{u})^2\|_{L^1(B_{r+\rho})}.\end{aligned}$$

Consequently,

$$I_s+I_a \geq \tfrac{1}{4}\mathcal{E}^{K_s}_{B_{r+\rho}}(\tau G(\tilde{u}),\tau G(\tilde{u})) - c\rho^{-\alpha}\|G(\tilde{u})^2\|_{L^1(B_{r+\rho})}.$$

For J_s, we use (4-2), (4-5), and (cutoff) to prove that, for every $\varepsilon>0$, there exists $c>0$ such that

$$\begin{aligned}J_s &\geq -\iint_M |G(\tilde{u}(x))-G(\tilde{u}(y))|G(\tilde{u}(y))|\tau(x)-\tau(y)|(\tau(x)\vee\tau(y))K_s(x,y)\,\mathrm{d}y\,\mathrm{d}x\\ &\geq -\varepsilon\iint_M (G(\tilde{u}(x))-G(\tilde{u}(y)))^2(\tau^2(x)\vee\tau^2(y))K_s(x,y)\,\mathrm{d}y\,\mathrm{d}x - c\rho^{-\alpha}\|G(\tilde{u})^2\|_{L^1(B_{r+\rho})}\\ &\geq -\varepsilon\mathcal{E}^{K_s}_{B_{r+\rho}}(\tau G(\tilde{u}),\tau G(\tilde{u})) - c\rho^{-\alpha}\|G(\tilde{u})^2\|_{L^1(B_{r+\rho})}.\end{aligned}$$

Next, we estimate J_a and prove, using (3-4), (1-2), (4-2), (cutoff), and (4-5), that, for every $\varepsilon>0$, there is $c>0$ such that

$$\begin{aligned}J_a &\geq -8\iint_M |\tilde{u}(x)-\tilde{u}(y)|g(\tilde{u}(y))(\tau^2(x)\wedge\tau^2(y))|K_a(x,y)|\,\mathrm{d}y\,\mathrm{d}x\\ &\qquad\qquad - 8\iint_M |\tilde{u}(x)-\tilde{u}(y)|g(\tilde{u}(y))(\tau(x)-\tau(y))^2K_s(x,y)\,\mathrm{d}y\,\mathrm{d}x\end{aligned}$$

$$
\begin{aligned}
&\geq -\varepsilon \iint_M (G(\tilde u(x)) - G(\tilde u(y)))^2 (\tau^2(x)\wedge\tau^2(y)) J(x,y)\,\mathrm{d}y\,\mathrm{d}x \\
&\qquad\qquad - c\iint_M G(\tilde u(y))(\tau^2(x)\wedge\tau^2(y)) \frac{|K_a(x,y)|^2}{J(x,y)}\,\mathrm{d}y\,\mathrm{d}x - c\rho^{-\alpha}\|G(\tilde u)^2\|_{L^1(B_{r+\rho})} \\
&\geq -2\varepsilon \mathcal{E}^{K_s}_{B_{r+\rho}}(\tau G(\tilde u), \tau G(\tilde u)) - c\rho^{-\alpha}\|G(\tilde u)^2\|_{L^1(B_{r+\rho})},
\end{aligned}
$$

where we used (K1$_{\text{loc}}$) and (2-8) in the last step to estimate

$$
c\iint_M G^2(\tilde u(y))(\tau^2(x)\wedge\tau^2(y)) \frac{|K_a(x,y)|^2}{J(x,y)}\,\mathrm{d}y\,\mathrm{d}x \leq 2\varepsilon \mathcal{E}^{K_s}_{B_{r+\rho}}(\tau G(\tilde u), \tau G(\tilde u)) + c\rho^{-\alpha}\|G(\tilde u)^2\|_{L^1(B_{r+\rho})}
$$

and used Lemma 2.6 in [Kassmann and Weidner 2022], (K1$_{\text{loc}}$), (4-4), and (cutoff) to estimate

$$
\begin{aligned}
&\iint_M (G(\tilde u(x)) - G(\tilde u(y)))^2 (\tau^2(x)\wedge\tau^2(y)) J(x,y)\,\mathrm{d}y\,\mathrm{d}x \\
&\qquad\qquad \leq c\int_{B_{r+\rho}}\int_{B_{r+\rho}} (G(\tilde u(x)) - G(\tilde u(y)))^2 (\tau^2(x)\wedge\tau^2(y)) K_s(x,y)\,\mathrm{d}y\,\mathrm{d}x \\
&\qquad\qquad \leq c\mathcal{E}^{K_s}_{B_{r+\rho}}(\tau G(\tilde u), \tau G(\tilde u)) + c\rho^{-\alpha}\|G(\tilde u)^2\|_{L^1(B_{r+\rho})}.
\end{aligned}
\tag{4-9}
$$

Altogether, we obtain

$$
\mathcal{E}_{B_{r+\rho}}(u, \tau^2 g(\tilde u)) \geq \left[\tfrac14 - 2\varepsilon\right]\mathcal{E}^{K_s}_{B_{r+\rho}}(\tau G(\tilde u), \tau G(\tilde u)) - c\rho^{-\alpha}\|G(\tilde u)^2\|_{L^1(B_{r+\rho})}.
$$

The desired estimate (4-8) now follows by choosing $\varepsilon > 0$ small enough.

<u>Step 2</u>: In addition, we claim

$$
-\mathcal{E}_{(B_{r+\rho}\times B_{r+\rho})^c}(u, \tau^2 g(\tilde u)) \leq 2\|g(\tilde u)\|_{L^1(B_{r+\rho})} \sup_{z\in B_{r+\rho/2}} \left(\int_{B^c_{r+\rho}} u(y)K(z,y)\,\mathrm{d}y\right).
\tag{4-10}
$$

To see this, we compute

$$
\begin{aligned}
&-\mathcal{E}_{(B_{r+\rho}\times B_{r+\rho})^c}(u, \tau^2 g(\tilde u)) \\
&= -2\int_{(B_{r+\rho}\times B_{r+\rho})^c} (u(x)-u(y))\tau^2(x) g(\tilde u(x)) K(x,y)\,\mathrm{d}y\,\mathrm{d}x \\
&= -2\int_{B_{r+\rho}} \tau^2(x)u(x)g(\tilde u(x))\left(\int_{B^c_{r+\rho}} K(x,y)\,\mathrm{d}y\right)\mathrm{d}x + 2\int_{B_{r+\rho}} \tau^2(x)g(\tilde u(x))\left(\int_{B^c_{r+\rho}} u(y)K(x,y)\,\mathrm{d}y\right)\mathrm{d}x \\
&\leq 2\|g(\tilde u)\|_{L^1(B_{r+\rho})} \sup_{z\in B_{r+\rho/2}}\left(\int_{B^c_{r+\rho}} u(y)K(z,y)\,\mathrm{d}y\right)
\end{aligned}
$$

using $u, K \geq 0$ and $\operatorname{supp}(\tau) \subset B_{r+\rho/2}$.

<u>Step 3</u>: Observe that

$$
\mathcal{E}^{K_s}_{B_{r+\rho}}(u, \tau^2 g(\tilde u)) = \mathcal{E}(u, \tau^2 g(\tilde u)) - \mathcal{E}_{(B_{r+\rho}\times B_{r+\rho})^c}(u, \tau^2 g(\tilde u)).
$$

Therefore, combining (4-8) and (4-10) yields the desired result. ☐

The following Caccioppoli-type estimate is designed for the dual equation.

Lemma 4.7. *Assume that* ($K1_{\mathrm{glob}}$) *and* (cutoff) *hold for some* $\theta \in (d/\alpha, \infty]$. *Moreover, assume* (Sob) *if* $\theta < \infty$. *Then there exist* $c_1, c_2, \gamma > 0$ *such that, for every* $0 < \rho \le r \le 1$, *every nonnegative function* $u \in V(B_{r+\rho} | \mathbb{R}^d) \cap L^{2\theta'}(\mathbb{R}^d)$, *and every* $q \ge 1$, *we have*

$$\mathcal{E}^{K_s}_{B_{r+\rho}}(\tau G(\tilde{u}), \tau G(\tilde{u})) \le c_1 \widehat{\mathcal{E}}(u, \tau^2 g(\tilde{u})) + c_2 q^{\gamma} \rho^{-\alpha} \|G(\tilde{u})^2\|_{L^1(B_{r+\rho})} + c_2 \|g(\tilde{u})\|_{L^1(B_{r+\rho})} \widehat{\mathrm{Tail}}_K\big(u, r + \tfrac{1}{2}\rho, r+\rho\big),$$

where $B_{2r} \subset \Omega$, $\tau = \tau_{r,\rho/2}$, *and* $\tilde{u} = u + R^{\alpha} \|f\|_{L^\infty}$.

Proof. Step 1: We claim that there exists $c > 0$ such that, for some $\gamma \ge 1$,

$$\widehat{\mathcal{E}}_{B_{r+\rho}}(u, \tau^2 g(\tilde{u})) \ge c_1 \mathcal{E}^{K_s}_{B_{r+\rho}}(\tau G(\tilde{u}), \tau G(\tilde{u})) - c_2 q^{\gamma} \rho^{-\alpha} \|G(\tilde{u})^2\|_{L^1(B_{r+\rho})}. \tag{4-11}$$

Let M be as in the proof of Lemma 4.6. Moreover, we observe the algebraic identity

$$(a+b)(\tau_1^2 g(\tilde{a}) - \tau_2^2 g(\tilde{b})) = (\tilde{a} - \tilde{b})(g(\tilde{a}) - g(\tilde{b}))\tau_1^2 + 2b(g(\tilde{a}) - g(\tilde{b}))\tau_1^2 + (a+b) g(\tilde{b})(\tau_1^2 - \tau_2^2).$$

We use again Lemma 2.3 in [Kassmann and Weidner 2022] to estimate

$$\begin{aligned}
\widehat{\mathcal{E}}^{K_a}_M(u, \tau^2 g(\tilde{u})) &= 2 \iint_M (\tilde{u}(x) - \tilde{u}(y))(g(\tilde{u}(x)) - g(\tilde{u}(y)))\tau^2(x) K_a(x,y)\,\mathrm{d}y\,\mathrm{d}x \\
&\qquad + 4 \iint_M u(y)(g(\tilde{u}(x)) - g(\tilde{u}(y)))\tau^2(x) K_a(x,y)\,\mathrm{d}y\,\mathrm{d}x \\
&\qquad + 4 \iint_M (u(x) + u(y)) g(\tilde{u}(y))(\tau^2(x) - \tau^2(y)) K_a(x,y)\,\mathrm{d}y\,\mathrm{d}x \\
&\ge 2 \iint_M (\tilde{u}(x) - \tilde{u}(y))(g(\tilde{u}(x)) - g(\tilde{u}(y)))\tau^2(x) K_a(y,x)\,\mathrm{d}y\,\mathrm{d}x \\
&\qquad - 4 \iint_M \tilde{u}(x) |g(\tilde{u}(x)) - g(\tilde{u}(y))| \tau^2(x) |K_a(x,y)|\,\mathrm{d}y\,\mathrm{d}x \\
&\qquad - 8 \iint_M g(\tilde{u}(x)) \tilde{u}(x) |\tau^2(x) - \tau^2(y)| |K_a(x,y)|\,\mathrm{d}y\,\mathrm{d}x \\
&= I_a + M_a + N_a,
\end{aligned}$$

where we used that $u(x) \ge u(y)$ and $g(u(x)) \ge g(u(y))$ on M, as well as $u \le \tilde{u}$. As in the proof of Lemma 4.6, we can write the decomposition

$$\mathcal{E}^{K_s}_{B_{r+\rho}}(u, \tau^2 g(\tilde{u})) = I_s + J_s.$$

Then, using (4-1), (2-8), and (cutoff), we estimate

$$\begin{aligned}
I_s + I_a &= 2 \iint_M (\tilde{u}(x) - \tilde{u}(y))(g(\tilde{u}(x)) - g(\tilde{u}(y)))\tau^2(x) K(x,y)\,\mathrm{d}y\,\mathrm{d}x \\
&\ge \tfrac{1}{4} \mathcal{E}^{K_s}_{B_{r+\rho}}(\tau G(\tilde{u}), \tau G(\tilde{u})) - c\rho^{-\alpha} \|G(\tilde{u})^2\|_{L^1(B_{r+\rho})}.
\end{aligned}$$

For M_a, we use (4-3), (4-6), (4-5), and (cutoff) to obtain

$$
\begin{aligned}
M_a &\geq -4q \iint_M |G(\tilde{u}(x)) - G(\tilde{u}(y))| G(\tilde{u}(x)) \tau^2(x) |K_a(x,y)| \,\mathrm{d}y\,\mathrm{d}x \\
&\geq -\varepsilon \iint_M (G(\tilde{u}(x)) - G(\tilde{u}(y)))^2 (\tau^2(y) \vee \tau^2(x)) J(x,y) \,\mathrm{d}y\,\mathrm{d}x \\
&\qquad\qquad - cq^2 \iint_M \tau^2(x) G^2(\tilde{u}(x)) \frac{|K_a(x,y)|^2}{J(x,y)} \,\mathrm{d}y\,\mathrm{d}x \\
&\geq -c\varepsilon \mathcal{E}^{K_s}_{B_{r+\rho}}(\tau G(\tilde{u}), \tau G(\tilde{u})) - cq^{\gamma_1} \rho^{-\alpha} \|G(\tilde{u})^2\|_{L^1(B_{r+\rho})}
\end{aligned}
$$

for some $\gamma_1 > 0$, where we used that, by ($\mathrm{K1}_{\mathrm{glob}}$) and (2-9) applied with some $\delta \leq \varepsilon/(cq^2)$,

$$
\begin{aligned}
cq^2 \iint_M \tau^2(x) G^2(\tilde{u}(x)) \frac{|K_a(x,y)|^2}{J(x,y)} \,\mathrm{d}y\,\mathrm{d}x \qquad\qquad& \\
\leq \varepsilon \mathcal{E}^{K_s}_{B_{r+\rho}}(\tau G(\tilde{u}), \tau G(\tilde{u})) + cq^2(\delta^{-\gamma_2} + \delta) \rho^{-\alpha} \|G(\tilde{u})^2\|_{L^1(B_{r+\rho})}&
\end{aligned}
\tag{4-12}
$$

for some $\gamma_2 > 0$, and moreover, by (3-4), (cutoff), and using the same argument as in (4-9),

$$
\begin{aligned}
&\varepsilon \iint_M (G(\tilde{u}(x)) - G(\tilde{u}(y)))^2 (\tau^2(y) \vee \tau^2(x)) J(x,y) \,\mathrm{d}y\,\mathrm{d}x \\
&\qquad\leq 2\varepsilon \int_{B_{r+\rho}} \int_{B_{r+\rho}} (G(\tilde{u}(x)) - G(\tilde{u}(y)))^2 (\tau^2(y) \wedge \tau^2(x)) J(x,y) \,\mathrm{d}y\,\mathrm{d}x + c\rho^{-\alpha} \|G(\tilde{u})^2\|_{L^1(B_{r+\rho})} \\
&\qquad\leq c\varepsilon \mathcal{E}^{K_s}_{B_{r+\rho}}(\tau G(\tilde{u}), \tau G(\tilde{u})) + c\rho^{-\alpha} \|G(\tilde{u})^2\|_{L^1(B_{r+\rho})}.
\end{aligned}
$$

For N_a, we compute, using (4-7), (3-4), and (1-2),

$$
\begin{aligned}
N_a &\geq -cq \iint_M G^2(\tilde{u}(x)) (\tau(x) - \tau(y))^2 K_s(x,y) \,\mathrm{d}y\,\mathrm{d}x \\
&\qquad\qquad - cq \iint_M G^2(\tilde{u}(x)) (\tau(x) \wedge \tau(y)) |\tau(x) - \tau(y)| |K_a(x,y)| \,\mathrm{d}y\,\mathrm{d}x \\
&\geq -cq\rho^{-\alpha} \|G(\tilde{u})^2\|_{L^1(B_{r+\rho})} - cq^2 \int_{B_{r+\rho}} \int_{B_{r+\rho}} \tau^2(x) G^2(\tilde{u}(x)) \frac{|K_a(x,y)|^2}{J(x,y)} \,\mathrm{d}y\,\mathrm{d}x \\
&\geq -cq^{\gamma_3} \rho^{-\alpha} \|G(\tilde{u})^2\|_{L^1(B_{r+\rho})} - \varepsilon \mathcal{E}^{K_s}_{B_{r+\rho}}(\tau G(\tilde{u}), \tau G(\tilde{u}))
\end{aligned}
$$

for some $\gamma_3 > 0$, where we applied (cutoff) and used the same argument as in (4-12) to estimate the second summand in the last step. Altogether, we have shown

$$
\widehat{\mathcal{E}}_{B_{r+\rho}}(u, \tau^2 g(\tilde{u})) \geq \left[\tfrac{1}{4} - 3\varepsilon\right] \mathcal{E}^{K_s}_{B_{r+\rho}}(\tau G(\tilde{u}), \tau G(\tilde{u})) - cq^{\gamma} \rho^{-\alpha} \|G(\tilde{u})^2\|_{L^1(B_{r+\rho})}.
$$

Thus, by choosing $\varepsilon > 0$ small enough, we obtain (4-11), as desired.

<u>Step 2</u>: Moreover, we have

$$
-\widehat{\mathcal{E}}_{(B_{r+\rho} \times B_{r+\rho})^c}(u, \tau^2 g(\tilde{u})) \leq c \|g(\tilde{u})\|_{L^1(B_{r+\rho})} \sup_{z \in B_{r+\rho/2}} \left(\int_{B^c_{r+\rho}} u(y) K(y,z) \,\mathrm{d}y \right).
\tag{4-13}
$$

The proof works similar to the proof of Step 2 in Lemma 4.6:

$$
\begin{aligned}
&-\widehat{\mathcal{E}}_{(B_{r+\rho}\times B_{r+\rho})^c}(u,\tau^2 g(\tilde{u}))\\
&\quad= -2\int_{(B_{r+\rho}\times B_{r+\rho})^c} (\tau^2 g(\tilde{u}(x)) - \tau^2 g(\tilde{u}(y)))u(x)K(x,y)\,\mathrm{d}y\,\mathrm{d}x\\
&\quad= -2\int_{B_{r+\rho}} \tau^2(x)u(x)g(\tilde{u}(x))\bigg(\int_{B^c_{r+\rho}} K(x,y)\,\mathrm{d}y\bigg)\mathrm{d}x + 2\int_{B_{r+\rho}} \tau^2(y)g(\tilde{u}(y))\bigg(\int_{B^c_{r+\rho}} u(x)K(x,y)\,\mathrm{d}x\bigg)\mathrm{d}y\\
&\quad\le 2\|g(\tilde{u})\|_{L^1(B_{r+\rho})} \sup_{z\in B_{r+\rho/2}} \bigg(\int_{B^c_{r+\rho}} u(y)K(y,z)\,\mathrm{d}y\bigg)
\end{aligned}
$$

using $u, K \ge 0$ and $\operatorname{supp}(\tau) \subset B_{r+\rho/2}$. □

4.3. *Local boundedness.* Now, we will show how to prove Theorem 3.6 via the Moser iteration. Note that we get a slightly better bound for subsolutions to $(\widehat{\text{PDE}})$ compared to Theorem 3.6 (ii).

Theorem 4.8. *Assume that* $(K^{\le}_{\text{loc}})$, (cutoff), *and* (Sob) *hold.*

(i) *Assume* (K1_{loc}) *holds for some* $\theta \in [d/\alpha, \infty]$. *Then there exists* $c > 0$ *such that, for every* $0 < R \le 1$ *and every nonnegative, weak subsolution* u *to* (PDE) *in* $I^{\ominus}_R(t_0) \times B_{2R}$,

$$
\sup_{I^{\ominus}_{R/8}\times B_{R/2}} u \le c\bigg(\fint_{I^{\ominus}_{R/4}}\fint_{B_R} u^2(t,x)\,\mathrm{d}x\,\mathrm{d}t\bigg)^{1/2} + c\sup_{t\in I^{\ominus}_{R/4}} \operatorname{Tail}_{K,\alpha}(u(t),R) + cR^{\alpha}\|f\|_{L^\infty}, \tag{4-14}
$$

where $B_{2R} \subset \Omega$.

(ii) *Assume* $(\text{K1}_{\text{glob}})$ *holds for some* $\theta \in (d/\alpha, \infty]$. *Then there exists* $c > 0$ *such that, for every* $0 < R \le 1$ *and every nonnegative, weak subsolution* u *to* $(\widehat{\text{PDE}})$ *in* $I^{\ominus}_R(t_0) \times B_{2R}$,

$$
\sup_{I^{\ominus}_{R/8}\times B_{R/2}} u \le c\bigg(\fint_{I^{\ominus}_{R/4}}\fint_{B_R} u^2(t,x)\,\mathrm{d}x\,\mathrm{d}t\bigg)^{1/2} + c\sup_{t\in I^{\ominus}_{R/4}} \widehat{\operatorname{Tail}}_{K,\alpha}(u(t),R) + cR^{\alpha}\|f\|_{L^\infty}, \tag{4-15}
$$

where $B_{2R} \subset \Omega$.

Proof. We will only demonstrate the proof of (ii). The proof of (i) follows via the same arguments, but uses Lemma 4.6 instead of Lemma 4.7. Let $0 < \rho \le r \le r+\rho \le R$ and $q \ge 1$. By applying Lemma 4.7, we obtain

$$
\begin{aligned}
&c\int_{B_{r+\rho}} \tau^2(x)\partial_t u(t,x)g(\tilde{u}(t,x))\,\mathrm{d}x + \mathcal{E}^{K_s}_{B_{r+\rho}}(\tau G(\tilde{u}),\tau G(\tilde{u}))\\
&\qquad\le c[(\partial_t u(t),\tau^2 g(\tilde{u}(t))) + \mathcal{E}(u(t),\tau^2 g(\tilde{u}(t)))]\\
&\qquad\qquad + cq^{\gamma}\rho^{-\alpha}\|G(\tilde{u}(t))^2\|_{L^1(B_{r+\rho})} + c\|g(\tilde{u}(t))\|_{L^1(B_{r+\rho})}\widehat{\operatorname{Tail}}_K\big(u(t), r+\tfrac{1}{2}\rho, r+\rho\big)\\
&\qquad\le c(f(t),\tau^2 g(\tilde{u}(t))) + cq^{\gamma}\rho^{-\alpha}\|G(\tilde{u}(t))^2\|_{L^1(B_{r+\rho})} + c\|g(\tilde{u}(t))\|_{L^1(B_{r+\rho})}\widehat{\operatorname{Tail}}_K\big(u(t), r+\tfrac{1}{2}\rho, r+\rho\big)\\
&\qquad\le cq^{\gamma}\rho^{-\alpha}\|G(\tilde{u}(t))^2\|_{L^1(B_{r+\rho})} + c\|g(\tilde{u}(t))\|_{L^1(B_{r+\rho})}\widehat{\operatorname{Tail}}_K\big(u(t), r+\tfrac{1}{2}\rho, r+\rho\big),
\end{aligned} \tag{4-16}
$$

where $c > 0$ is the constant from Lemma 4.7 and we tested the equation with $\tau^2 g(u)$, where $\tau = \tau_{r,\rho/2}$. Moreover, by the definition of $\tilde{u}$, we used

$$(f(t), \tau^2 g(\tilde{u}(t))) \le c\rho^{-\alpha} \|G(\tilde{u}(t))^2\|_{L^1(B_{r+\rho})}.$$

We observe that

$$(\partial_t u) g(\tilde{u}) = \begin{cases} \frac{1}{2q}\partial_t(\tilde{u}^{2q}), & u \le M, \\ \frac{1}{2} M^{2q-2}\partial_t(\tilde{u}^2), & u > M. \end{cases}$$

Next, we define $\chi \in C^1(\mathbb{R})$ to be a function satisfying

$$0 \le \chi \le 1, \quad \|\chi'\|_\infty \le 16((r+\rho)^\alpha - r^\alpha)^{-1}, \quad \chi\big(t_0 - \big(\tfrac{1}{4}(r+\rho)\big)^\alpha\big) = 0, \quad \chi \equiv 1 \quad \text{ in } I^\ominus_{r/4}(t_0).$$

By multiplying (4-16) with χ^2 and integrating over $\big(t_0 - \big(\tfrac{1}{4}(r+\rho)\big)^\alpha, t\big)$ for some arbitrary $t \in I^\ominus_{r/4}(t_0)$, we obtain

$$\begin{aligned}
&\int_{B_{r+\rho}} \chi^2(t)\tau^2(x) H(\tilde{u}(t,x))\,\mathrm{d}x + \int_{t_0-((r+\rho)/4)^\alpha}^t \chi^2(s)\mathcal{E}^{K_s}_{B_{r+\rho}}(\tau G(\tilde{u}(s)), \tau G(\tilde{u}(s)))\,\mathrm{d}s \\
&\quad \le c_2 q^\gamma \rho^{-\alpha} \int_{t_0-((r+\rho)/4)^\alpha}^t \chi^2(s)\|G(\tilde{u}(s))^2\|_{L^1(B_{r+\rho})}\,\mathrm{d}s \\
&\qquad\qquad + c_2 \int_{t_0-((r+\rho)/4)^\alpha}^t \chi(s)|\chi'(s)| \int_{B_{r+\rho}} \tau^2(x) H(\tilde{u}(s,x))\,\mathrm{d}x\,\mathrm{d}s \\
&\qquad\qquad\qquad + c_2 \int_{t_0-((r+\rho)/4)^\alpha}^t \chi^2(s)\|g(\tilde{u}(s))\|_{L^1(B_{r+\rho})}\, \widehat{\mathrm{Tail}}_K\big(u(s), r+\tfrac{1}{2}\rho, r+\rho\big)\,\mathrm{d}s
\end{aligned}$$

for some $c_2 > 0$, where

$$H(t) = \begin{cases} \frac{1}{2q} t^{2q}, & t \le M, \\ \frac{1}{2} M^{2q-2} t^2, & t > M. \end{cases}$$

Consequently,

$$\begin{aligned}
&\sup_{t \in I^\ominus_{r/4}} \int_{B_r} H(\tilde{u}(t,x))\,\mathrm{d}x + \int_{I^\ominus_{r/4}} \mathcal{E}^{K_s}_{B_{r+\rho}}(\tau G(\tilde{u}(s)), \tau G(\tilde{u}(s)))\,\mathrm{d}s \\
&\quad \le c_3 q^\gamma (\rho^{-\alpha} \vee ((r+\rho)^\alpha - r^\alpha)^{-1})(\|H(\tilde{u})\|_{L^1(I^\ominus_{(r+\rho)/4} \times B_{r+\rho})} + \|G(\tilde{u})^2\|_{L^1(I^\ominus_{(r+\rho)/4} \times B_{r+\rho})}) \\
&\qquad\qquad + c_3 \|g(\tilde{u})\|_{L^1(I^\ominus_{(r+\rho)/4} \times B_{r+\rho})} \sup_{t \in I^\ominus_{(r+\rho)/4}} \widehat{\mathrm{Tail}}_K\big(u(t), r+\tfrac{1}{2}\rho, r+\rho\big)
\end{aligned}$$

for some $c_3 > 0$. Now, we take the limit $M \nearrow \infty$. By monotone convergence, the definitions of g, G, and H, and Lemma 4.5,

$$\begin{aligned}
&\sup_{t \in I^\ominus_{r/4}} \int_{B_r} \tilde{u}^{2q}(t,x)\,\mathrm{d}x + \int_{I^\ominus_{r/4}} \mathcal{E}^{K_s}_{B_{r+\rho}}(\tau \tilde{u}^q(s), \tau \tilde{u}^q(s))\,\mathrm{d}s \\
&\quad \le c_4 q^\gamma (\rho^{-\alpha} \vee ((r+\rho)^\alpha - r^\alpha)^{-1}) \|\tilde{u}^{2q}\|_{L^1(I^\ominus_{(r+\rho)/4} \times B_{r+\rho})} \\
&\qquad\qquad + c_4 q \|\tilde{u}^{2q-1}\|_{L^1(I^\ominus_{(r+\rho)/4} \times B_{r+\rho})} \sup_{t \in I^\ominus_{(r+\rho)/4}} \widehat{\mathrm{Tail}}_K\big(u(t), r+\tfrac{1}{2}\rho, r+\rho\big)
\end{aligned}$$

for some $c_4 > 0$. Recall $\kappa = 1+\alpha/d > 1$. By Hölder interpolation and Sobolev inequality (Sob), we have

$$\|\tilde{u}^{2q}\|_{L^\kappa(I_r^\ominus \times B_r)} \le \left(\sup_{t\in I_r^\ominus} \|\tilde{u}^{2q}(t)\|_{L^1(B_r)}^{\kappa-1} \int_{I_r^\ominus} \|\tilde{u}^{2q}(s)\|_{L^{d/(d-\alpha)}(B_r)}\,\mathrm{d}s\right)^{\frac{1}{\kappa}}$$
$$\le cq^\gamma(\rho^{-\alpha} \vee ((r+\rho)^\alpha - r^\alpha)^{-1})\|\tilde{u}^{2q}\|_{L^1(I_{r+\rho}^\ominus \times B_{r+\rho})}$$
$$+ cq\|\tilde{u}^{2q-1}\|_{L^1(I_{(r+\rho)/4}^\ominus \times B_{r+\rho})} \sup_{t\in I_{(r+\rho)/4}^\ominus} \widehat{\operatorname{Tail}}_K\big(u(t), r+\tfrac{1}{2}\rho, r+\rho\big). \tag{4-17}$$

We will now demonstrate how to perform the Moser iteration for positive exponents for nonlocal equations. Inequality (4-17) is the key estimate for the iteration scheme. The main difficulty compared to the classical local case is the treatment of the tail term.

Let us define $c_i = 2^{-(i+1)(d+\varepsilon)/\alpha} < 1$ for $\varepsilon > 0$ to be determined later and $i \in \mathbb{N}$. By Hölder's and Young's inequalities we have, for each $i \in \mathbb{N}$, the estimate

$$q\|\tilde{u}^{2q-1}\|_{L^1(I_{(r+\rho)/4}^\ominus \times B_{r+\rho})} \sup_{t\in I_{(r+\rho)/4}^\ominus} \widehat{\operatorname{Tail}}_K\big(u(t), r+\tfrac{1}{2}\rho, r+\rho\big)$$
$$\le \big(q(c_i\rho)^{-\alpha}\|\tilde{u}^{2q}\|_{L^1(I_{(r+\rho)/4}^\ominus \times B_{r+\rho})}\big)^{\frac{2q-1}{2q}} \Big(q^{\frac{1}{2q}}(c_i\rho)^{\alpha\frac{2q-1}{2q}}(r+\rho)^{\frac{d+\alpha}{2q}} \sup_{t\in I_{(r+\rho)/4}^\ominus} \widehat{\operatorname{Tail}}_K\big(u(t), r+\tfrac{1}{2}\rho, r+\rho\big)\Big)$$
$$\le q(c_i\rho)^{-\alpha}\|\tilde{u}^{2q}\|_{L^1(I_{(r+\rho)/4}^\ominus \times B_{r+\rho})} + \Big(q^{\frac{1}{2q}}(c_i\rho)^{\alpha\frac{2q-1}{2q}}(r+\rho)^{\frac{d+\alpha}{2q}} \sup_{t\in I_{(r+\rho)/4}^\ominus} \widehat{\operatorname{Tail}}_K\big(u(t), r+\tfrac{1}{2}\rho, r+\rho\big)\Big)^{2q}.$$

Combining this estimate with (4-17) and taking both sides to the power $1/(2q)$ yields

$$\|\tilde{u}\|_{L^{2q\kappa}(I_r^\ominus \times B_r)} \le c^{\frac{1}{2q}} q^{\frac{\gamma}{2q}} c_i^{-\frac{\alpha}{2q}}(\rho^{-\frac{\alpha}{2q}} \vee ((r+\rho)^\alpha - r^\alpha)^{-\frac{1}{2q}})\|\tilde{u}\|_{L^{2q}(I_{r+\rho}^\ominus \times B_{r+\rho})}$$
$$+ c^{\frac{1}{2q}} q^{\frac{1}{2q}}(c_i\rho)^{\alpha\frac{2q-1}{2q}}(r+\rho)^{\frac{d+\alpha}{2q}} \sup_{t\in I_{(r+\rho)/4}^\ominus} \widehat{\operatorname{Tail}}_K\big(u(t), r+\tfrac{1}{2}\rho, r+\rho\big)$$
$$\le c^{\frac{1}{2q}} q^{\frac{\gamma}{2q}} c_i^{-\frac{\alpha}{2q}}(\rho^{-\frac{\alpha}{2q}} \vee ((r+\rho)^\alpha - r^\alpha)^{-\frac{1}{2q}})$$
$$\times \Big(\|\tilde{u}\|_{L^{2q}(I_{r+\rho}^\ominus \times B_{r+\rho})} + (c_i\rho)^\alpha (r+\rho)^{\frac{d+\alpha}{2q}} \sup_{t\in I_{(r+\rho)/4}^\ominus} \widehat{\operatorname{Tail}}_K\big(u(t), r+\tfrac{1}{2}\rho, r+\rho\big)\Big).$$

Recall that, by (2-11), we have the estimate

$$\widehat{\operatorname{Tail}}_K\big(u(t), r+\tfrac{1}{2}\rho, r+\rho\big) \le c\rho^{-\alpha}\left(\frac{r+\rho}{\rho}\right)^d \widehat{\operatorname{Tail}}_{K,\alpha}(u(t), R).$$

Fix $q_0 \ge 1$ and $q_i = q_0\kappa^i$, and set $\rho_i = 2^{-i-1}R$ and $r_{i+1} = r_i - \rho_{i+1}$, $r_0 = R$. Note that $r_i \searrow \frac{1}{2}R$. For every $i \in \mathbb{N}$, using

$$(\rho_i^{-\frac{\alpha}{2q_{i-1}}} \vee ((r_i+\rho_i)^\alpha - r_i^\alpha)^{-\frac{1}{2q_{i-1}}}) \le c^{\frac{1}{2q_{i-1}}} R^{-\frac{\alpha}{2q_{i-1}}} 2^{\frac{i+1}{q_{i-1}}},$$

we obtain

$$\|\tilde{u}\|_{L^{q_i}(I_{r_i}^\ominus \times B_{r_i})} \le c^{\frac{1}{2q_{i-1}}} q_{i-1}^{\frac{\gamma}{2q_{i-1}}} R^{-\frac{\alpha}{2q_{i-1}}} 2^{\frac{d+\varepsilon+2}{2q_{i-1}}(i+1)}$$
$$\times \Big(\|\tilde{u}\|_{L^{2q_{i-1}}(I_{r_{i-1}}^\ominus \times B_{r_{i-1}})} + 2^{-(d+\varepsilon)(i+1)}\rho_i^\alpha R^{\frac{d+\alpha}{2q_{i-1}}} \sup_{t\in I_{(r_i+\rho_i)/4}^\ominus} \widehat{\operatorname{Tail}}_K\big(u(t), r_i+\tfrac{1}{2}\rho_i, r_i+\rho_i\big)\Big)$$
$$\le c^{\frac{1}{2q_{i-1}}} q_{i-1}^{\frac{\gamma}{2q_{i-1}}} R^{-\frac{\alpha}{2q_{i-1}}} 2^{\frac{d+\varepsilon+2}{2q_{i-1}}(i+1)}$$
$$\times \Big(\|\tilde{u}\|_{L^{2q_{i-1}}(I_{r_{i-1}}^\ominus \times B_{r_{i-1}})} + R^{\frac{d+\alpha}{2q_{i-1}}} 2^{-(i+1)\varepsilon} \sup_{t\in I_{R/4}^\ominus} \widehat{\operatorname{Tail}}_{K,\alpha}(u(t), R)\Big). \tag{4-18}$$

Consequently,

$$\sup_{I^{\ominus}_{R/2}\times B_{R/2}} \tilde u \le \left(\prod_{i=1}^{\infty} c^{\frac{1}{2q_{i-1}}} q_{i-1}^{\frac{\gamma}{2q_{i-1}}} R^{-\frac{\alpha}{2q_{i-1}}} 2^{\frac{d+\varepsilon+2}{2q_{i-1}}(i+1)}\right)\|\tilde u\|_{L^{2q_0}(I^{\ominus}_R\times B_R)} + \left[\sum_{i=1}^{\infty}\left(\prod_{j=i}^{\infty} c^{\frac{1}{2q_{j-1}}} q_{j-1}^{\frac{\gamma}{2q_{j-1}}} R^{-\frac{\alpha}{2q_{j-1}}} 2^{\frac{d+\varepsilon+2}{2q_{j-1}}(j+1)}\right) R^{\frac{d+\alpha}{2q_{i-1}}} 2^{-(i+1)\varepsilon}\right] \sup_{t\in I^{\ominus}_{R/4}} \widehat{\mathrm{Tail}}_{K,\alpha}(u(t),R).$$

Note that $\sum_{i=0}^{\infty}\kappa^{-i}=(d+\alpha)/\alpha$ and also $\sum_{i=0}^{\infty} i/\kappa^i =: c_3<\infty$. Therefore,

$$\begin{aligned}\prod_{i=1}^{\infty}(cq_{i-1})^{\frac{\gamma}{2q_{i-1}}} &\le (cq_0)^{\frac{\gamma}{2q_0}\sum_{i=0}^{\infty}\kappa^{-i}} \kappa^{\frac{\gamma}{2q_0}\sum_{i=0}^{\infty}\frac{i}{\kappa^i}} \le c(q_0,\kappa,\gamma)<\infty,\\ \prod_{i=1}^{\infty} 2^{\frac{d+\varepsilon+2}{2q_{i-1}}(i+1)} &\le 2^{\frac{d+\varepsilon+2}{2q_0}\sum_{i=0}^{\infty}\frac{i+2}{\kappa^i}} \le 2^{\frac{(d+\varepsilon+2)c_4}{2q_0}}<\infty,\\ \prod_{j=i}^{\infty} R^{-\frac{\alpha}{2q_{j-1}}} &= R^{-\frac{\alpha}{2q_{i-1}}\sum_{j=0}^{\infty}\kappa^{-j}} = R^{-\frac{d+\alpha}{2q_{i-1}}}.\end{aligned}$$

As a consequence,

$$\begin{aligned}\prod_{i=1}^{\infty} c^{\frac{1}{2q_{i-1}}} q_{i-1}^{\frac{\gamma}{2q_{i-1}}} R^{-\frac{\alpha}{2q_{i-1}}} 2^{\frac{d+\varepsilon+2}{2q_{i-1}}(i+1)} &\le c(q_0,\kappa,d) R^{-\frac{d+\alpha}{2q_k}} 2^{\frac{d+\varepsilon+2}{2q_k}\sum_{i=0}^{\infty}\frac{i+k+2}{\kappa^i}}\\ &\le c(q_0,\kappa,d) R^{-\frac{d+\alpha}{2q_0}} 2^{\frac{(d+\varepsilon+2)c_5}{2q_0}},\\ \sum_{i=1}^{\infty}\left(\prod_{j=i}^{\infty} c^{\frac{1}{2q_{j-1}}} q_{j-1}^{\frac{\gamma}{2q_{j-1}}} R^{-\frac{\alpha}{2q_{j-1}}} 2^{\frac{d+\varepsilon+2}{2q_{j-1}}(j+1)}\right) R^{\frac{d+\alpha}{2q_{i-1}}} 2^{-(i+1)\varepsilon} &\le c\sum_{i=1}^{\infty} 2^{\frac{(d+\varepsilon+2)c_5}{2q_{i-1}}(i+1)} 2^{-(i+1)\varepsilon}\\ &\le c 2^{\frac{(d+\varepsilon+2)c_6}{2q_0}} \sum_{i=1}^{\infty} 2^{-(i+1)\varepsilon}\\ &\le c(d,q_0,\kappa,\varepsilon),\end{aligned}$$

where we used that $(i+1)/\kappa^{i-1}$ is bounded from above by some constant $c_6=c_6(\kappa)$. Therefore, choosing $\varepsilon=1$ and $q_0=1$, we deduce that, for some $c>0$,

$$\sup_{I^{\ominus}_{R/2}\times B_{R/2}} \tilde u \le c\left(\fint_{I^{\ominus}_R}\fint_{B_R} \tilde u^2(t,x)\,\mathrm{d}x\,\mathrm{d}t\right)^{\frac12} + c\sup_{t\in I^{\ominus}_{R/4}} \widehat{\mathrm{Tail}}_{K,\alpha}(u(t),R).$$

As a consequence, using the definition of $\tilde u$ as well as the triangle inequality for the L^2-norm, we deduce

$$\sup_{I^{\ominus}_{R/2}\times B_{R/2}} u \le c\left(\fint_{I^{\ominus}_R}\fint_{B_R} u^2(t,x)\,\mathrm{d}x\,\mathrm{d}t\right)^{\frac12} + c\sup_{t\in I^{\ominus}_{R/4}} \widehat{\mathrm{Tail}}_{K,\alpha}(u(t),R) + cR^{\alpha}\|f\|_{L^\infty}.$$

This proves the desired result. □

5. Local tail estimate

In this section, local tail estimates for supersolutions to (PDE) and ($\widehat{\text{PDE}}$) (see Corollary 5.3) as well as the corresponding stationary equations (ell-PDE) and (ell-$\widehat{\text{PDE}}$) (see Corollary 5.4) are established. The main auxiliary results are Lemmas 5.1 and 5.2, whose proofs use similar ideas as in Lemmas 3.1 and 3.4. Central ingredients in the proof are the assumptions (UJS) and ($\widehat{\text{UJS}}$), which allow us to derive local tail estimates without having to assume a pointwise lower bound of the jumping kernel. They are applied in a similar way as in [Schulze 2019], where symmetric nonlocal operators are considered.

Lemma 5.1. *Assume that* ($K1_{\text{loc}}$), (cutoff), *and* (UJS) *hold for some* $\theta \in [d/\alpha, \infty]$. *Moreover, assume* (Sob) *if* $\theta < \infty$. *Then there exist* $c_1, c_2 > 0$ *such that, for every* $0 < \rho \le r \le 1$, *every nonnegative function* $u \in V(B_{2r} \,|\, \mathbb{R}^d)$, *and every* $S > 0$ *with* $S \ge \sup_{B_{r+\rho}} u$, *we have*

$$\operatorname{Tail}_K(u, r, r+\rho) \le c_1 \frac{1}{S\rho^d} \mathcal{E}(u, \tau^2(u - 2S)) + c_2 \left(\frac{r+\rho}{\rho}\right)^d \rho^{-\alpha} S,$$

where $B_{2r} \subset \Omega$ *and* $\tau = \tau_{r,\rho}$.

Proof. We define $w = u - 2S$. Note that, by definition, $w \in [-2S, -S]$ in $B_{r+\rho}$. We separate the proof into several steps.

Step 1: First, we claim that, for some $c > 0$, we have

$$\mathcal{E}^{K_s}_{B_{r+\rho}}(\tau w, \tau w) \le \mathcal{E}^{K_s}_{B_{r+\rho}}(u, \tau^2 w) + cS^2(r+\rho)^d \rho^{-\alpha}. \tag{5-1}$$

We compute

$$\begin{aligned}\mathcal{E}^{K_s}_{B_{r+\rho}}(u, \tau^2 w) &= \int_{B_{r+\rho}} \int_{B_{r+\rho}} (w(x) - w(y))(\tau^2 w(x) - \tau^2 w(y)) K_s(x, y) \,\mathrm{d}y \,\mathrm{d}x \\ &= \mathcal{E}^{K_s}_{B_{r+\rho}}(\tau w, \tau w) - \int_{B_{r+\rho}} \int_{B_{r+\rho}} w(x) w(y) (\tau(x) - \tau(y))^2 K_s(x, y) \,\mathrm{d}y \,\mathrm{d}x.\end{aligned}$$

We estimate, using (cutoff),

$$\int_{B_{r+\rho}} \int_{B_{r+\rho}} w(x) w(y) (\tau(x) - \tau(y))^2 K_s(x, y) \,\mathrm{d}y \,\mathrm{d}x \le 4S^2 \mathcal{E}^{K_s}_{B_{r+\rho}}(\tau, \tau) \le c_1 S^2 (r+\rho)^d \rho^{-\alpha}$$

for some $c_1 > 0$, which directly implies (5-1).

Step 2: Next, we claim that there exists $c > 0$ such that

$$-\mathcal{E}^{K_a}_{B_{r+\rho}}(u, \tau^2 w) \le \tfrac{1}{2} \mathcal{E}^{K_s}_{B_{r+\rho}}(\tau w, \tau w) + cS^2(r+\rho)^d \rho^{-\alpha}. \tag{5-2}$$

For the proof, we use the same arguments as in the proof of the Caccioppoli estimate:

$$\begin{aligned}-\mathcal{E}^{K_a}_{B_{r+\rho}}(u, \tau^2 w) &= \int_{B_{r+\rho}} \int_{B_{r+\rho}} (w(y) - w(x))(\tau^2 w(x) + \tau^2 w(y)) K_a(x, y) \,\mathrm{d}y \,\mathrm{d}x \\ &= \int_{B_{r+\rho}} \int_{B_{r+\rho}} (\tau w(y) - \tau w(x))(\tau w(y) + \tau w(x)) K_a(x, y) \,\mathrm{d}y \,\mathrm{d}x \\ &\qquad + \int_{B_{r+\rho}} \int_{B_{r+\rho}} w(x) w(y) (\tau^2(x) - \tau^2(y)) K_a(x, y) \,\mathrm{d}y \,\mathrm{d}x \\ &=: J_1 + J_2.\end{aligned}$$

Using Hölder's and Young's inequalities as well as (K1$_{\text{loc}}$) and (2-8), we obtain, for every $\delta > 0$,

$$
\begin{aligned}
J_1 &\le \delta \mathcal{E}^J_{B_{r+\rho}}(\tau w, \tau w) + c_2 \int_{B_{r+\rho}} \int_{B_{r+\rho}} (\tau w(y) + \tau w(x))^2 \frac{|K_a(x,y)|^2}{J(x,y)} \,\mathrm{d}y \,\mathrm{d}x \\
&\le c\delta \mathcal{E}^{K_s}_{B_{r+\rho}}(\tau w, \tau w) + 2c_2 \int_{B_{r+\rho}} \tau^2 w^2(x) \biggl(\int_{B_{r+\rho}} \frac{|K_a(x,y)|^2}{J(x,y)} \,\mathrm{d}y \biggr) \mathrm{d}x \\
&\le 2c\delta \mathcal{E}^{K_s}_{B_{r+\rho}}(\tau w, \tau w) + c_3 S^2 (r+\rho)^d \rho^{-\alpha}
\end{aligned}
$$

for $c_2, c_3 > 0$ depending on δ. Again, by Hölder's and Young's inequalities as well as (K1$_{\text{loc}}$), (cutoff), and (2-8), we estimate

$$
\begin{aligned}
J_2 &\le \frac{1}{2} \int_{B_{r+\rho}} \int_{B_{r+\rho}} |w(x)||w(y)|(\tau(y) - \tau(x))^2 J(x,y) \,\mathrm{d}y \,\mathrm{d}x \\
&\qquad\qquad + \frac{1}{2} \int_{B_{r+\rho}} \int_{B_{r+\rho}} |w(x)||w(y)|(\tau(y) + \tau(x))^2 \frac{|K_a(x,y)|^2}{J(x,y)} \,\mathrm{d}y \,\mathrm{d}x \\
&\le 2S^2 \mathcal{E}^J_{B_{r+\rho}}(\tau, \tau) + 8S^2 \int_{B_{r+\rho}} \biggl(\tau^2(x) \int_{B_{r+\rho}} \frac{|K_a(x,y)|^2}{J(x,y)} \,\mathrm{d}y \biggr) \mathrm{d}x \\
&\le c_4 S^2 \mathcal{E}^{K_s}_{B_{r+\rho}}(\tau, \tau) + c_4 S^2 (r+\rho)^d \rho^{-\alpha} \\
&\le c_5 S^2 (r+\rho)^d \rho^{-\alpha}
\end{aligned}
$$

for $c_4, c_5 > 0$. From here, (5-2) directly follows.

<u>Step 3</u>: We claim that there exist constants $c, c' > 0$ such that

$$
-\mathcal{E}_{(B_{r+\rho} \times B_{r+\rho})^c}(u, \tau^2 w) \le c S^2 (r+\rho)^d \rho^{-\alpha} - c' S \rho^d \operatorname{Tail}_K(u, r, r+\rho). \tag{5-3}
$$

First, we rewrite the term on the left-hand side of the above line:

$$
\begin{aligned}
-\mathcal{E}_{(B_{r+\rho} \times B_{r+\rho})^c}(u, \tau^2 w) &= -2 \iint_{(B_{r+\rho} \times B_{r+\rho})^c} (u(x) - u(y)) \tau^2 w(x) K(x,y) \,\mathrm{d}y \,\mathrm{d}x \\
&= -2 \int_{B_{r+\rho}} \int_{B^c_{r+\rho} \cap \{u(y) \ge S\}} (u(y) - u(x)) \tau^2(x)(2S - u(x)) K(x,y) \,\mathrm{d}y \,\mathrm{d}x \\
&\qquad + 2 \int_{B_{r+\rho}} \int_{B^c_{r+\rho} \cap \{u(y) \le S\}} (u(x) - u(y)) \tau^2(x)(2S - u(x)) K(x,y) \,\mathrm{d}y \,\mathrm{d}x \\
&=: I_1 + I_2.
\end{aligned} \tag{5-4}
$$

For I_2, we obtain

$$
\begin{aligned}
I_2 &\le 4S \int_{B_{r+\rho}} \int_{B^c_{r+\rho} \cap \{u(y) \le S\}} (u(x) - u(y))_+ \tau^2(x) K(x,y) \,\mathrm{d}y \,\mathrm{d}x \\
&\le 8S^2 \int_{B_{r+\rho}} \int_{B^c_{r+\rho}} (\tau(x) - \tau(y))^2 K(x,y) \,\mathrm{d}y \,\mathrm{d}x \\
&\le 8S^2 \int_{B_{r+\rho}} \Gamma^{K_s}(\tau, \tau)(x) \,\mathrm{d}x \\
&\le c_6 S^2 (r+\rho)^d \rho^{-\alpha}
\end{aligned}
$$

for some $c_6 > 0$, where we used (1-2), (cutoff), and that $K \geq 0$ and $u \geq 0$ globally. We treat I_1 in the following way (see [Schulze 2019]):

$$\begin{aligned} I_1 &\leq -2S \int_{B_{r+\rho}} \int_{B^c_{r+\rho} \cap \{u(y) \geq S\}} (u(y) - S)\tau^2(x) K(x, y) \, dy \, dx \\ &\leq -2S \int_{B_{r+\rho}} \int_{B^c_{r+\rho}} (u(y) - S)\tau^2(x) K(x, y) \, dy \, dx \\ &\leq -2S \int_{B_{r+\rho/4}} \int_{B^c_{r+\rho}} u(y)\tau^2(x) K(x, y) \, dy \, dx + 2S^2 \int_{B_{r+\rho}} \int_{B^c_{r+\rho}} (\tau(x) - \tau(y))^2 K(x, y) \, dy \, dx \\ &\leq -\frac{S}{8} \int_{B_{r+\rho/4}} \int_{B^c_{r+\rho}} u(y) K(x, y) \, dy \, dx + c_7 S^2 (r+\rho)^d \rho^{-\alpha} \end{aligned}$$

for some $c_7 > 0$, where we used that $u, K \geq 0$, $u \leq S$ in $B_{r+\rho}$, $\tau^2 \geq \frac{1}{16}$ in $B_{r+\rho/4}$, (1-2), and (cutoff). Finally, note that, due to (UJS),

$$\begin{aligned} \rho^d \operatorname{Tail}_K(u, r, r+\rho) &= \rho^d \sup_{x \in B_r} \int_{B^c_{r+\rho}} u(y) K(x, y) \, dy \\ &\leq c_8 \sup_{x \in B_r} \int_{B^c_{r+\rho}} u(y) \left(\int_{B_{\rho/4}(x)} K(z, y) \, dz \right) dy \\ &\leq c_8 \int_{B^c_{r+\rho}} u(y) \left(\int_{B_{r+\rho/4}} K(x, y) \, dx \right) dy \\ &= c_8 \int_{B_{r+\rho/4}} \int_{B^c_{r+\rho}} u(y) K(x, y) \, dy \, dx \end{aligned} \tag{5-5}$$

for some $c_8 > 0$. Consequently,

$$I_1 \leq -c_9 S \rho^d \operatorname{Tail}_K(u, r, r+\rho) + c_{10} S^2 (r+\rho)^d \rho^{-\alpha},$$

where $c_9, c_{10} > 0$ are constants.

Step 4: Now, we want to combine (5-1), (5-2), and (5-3). First, we observe that

$$\mathcal{E}^{K_s}_{B_{r+\rho}}(u, \tau^2 w) = \mathcal{E}(u, \tau^2 w) - \mathcal{E}_{(B_{r+\rho} \times B_{r+\rho})^c}(u, \tau^2 w) - \mathcal{E}^{K_a}_{B_{r+\rho}}(u, \tau^2 w).$$

Together, we obtain

$$\mathcal{E}^{K_s}_{B_{r+\rho}}(\tau w, \tau w) \leq \mathcal{E}(u, \tau^2 w) + c_{11} S^2 (r+\rho)^d \rho^{-\alpha} - c_{12} S \rho^d \operatorname{Tail}_K(u, r, r+\rho) + \tfrac{1}{2} \mathcal{E}^{K_s}_{B_{r+\rho}}(\tau w, \tau w)$$

for $c_{11}, c_{12} > 0$. Since $L \geq 0$, we conclude

$$\operatorname{Tail}_K(u, r, r+\rho) \leq c_{13} \frac{1}{S\rho^d} \mathcal{E}(u, \tau^2 w) + c_{14} S \left(\frac{r+\rho}{\rho} \right)^d \rho^{-\alpha},$$

where $c_{13}, c_{14} > 0$ are constants. This yields the desired result. □

Next, we prove a similar estimate for the dual form.

Lemma 5.2. *Assume that* (K1$_{\text{glob}}$), (cutoff), *and* ($\widehat{\text{UJS}}$) *hold for some* $\theta \in [d/\alpha, \infty]$. *Moreover, assume* (Sob) *if* $\theta < \infty$. *Then there exist* $c_1, c_2 > 0$ *such that, for every* $0 < \rho \le r \le 1$, *every nonnegative function* $u \in V(B_{2r} | \mathbb{R}^d) \cap L^{2\theta'}(\mathbb{R}^d)$, *and every* $S \ge \sup_{B_{r+\rho}} u$, *we have*

$$\widehat{\text{Tail}}_K(u, r, r+\rho) \le c_1 \frac{1}{S\rho^d} \widehat{\mathcal{E}}(u, \tau^2(u-2S)) + c_2 \left(\frac{r+\rho}{\rho}\right)^d \rho^{-\alpha} S,$$

where $B_{2r} \subset \Omega$, $\tau = \tau_{r,\rho}$.

Proof. As in the proof of Lemma 5.1, we define $w = u - 2S$ and observe that $w \in [-S, -2S]$ in $B_{r+\rho}$. The proof is separated into several steps.

Step 1: First, we recall from Step 1 in the proof of Lemma 5.1 that, for some $c > 0$, we have

$$\mathcal{E}^{K_s}_{B_{r+\rho}}(\tau w, \tau w) \le \mathcal{E}^{K_s}_{B_{r+\rho}}(u, \tau^2 w) + cS^2 (r+\rho)^d \rho^{-\alpha}. \tag{5-6}$$

Step 2: In analogy with Step 2 in the proof of Lemma 5.1, we claim that, for some $c > 0$,

$$-\widehat{\mathcal{E}}^{K_a}(u, \tau^2 w) \le \tfrac{1}{2} \mathcal{E}^{K_s}_{B_{r+\rho}}(\tau w, \tau w) + cS^2 (r+\rho)^d \rho^{\alpha}. \tag{5-7}$$

To see this, we estimate

$$\begin{aligned} -\widehat{\mathcal{E}}^{K_a}(u, \tau^2 w) &= \int_{B_{r+\rho}} \int_{B_{r+\rho}} (\tau^2 w(x) - \tau^2 w(y))(w(x) + w(y)) K_a(x,y) \, dy \, dx \\ &\qquad + 4S \int_{B_{r+\rho}} \int_{B_{r+\rho}} (\tau^2 w(x) - \tau^2 w(y)) K_a(x,y) \, dy \, dx \\ &:= I_1 + I_2. \end{aligned}$$

For I_1, we compute

$$\begin{aligned} I_1 &= \int_{B_{r+\rho}} \int_{B_{r+\rho}} (\tau^2 w^2(x) - \tau^2 w^2(y)) K_a(x,y) \, dy \, dx \\ &\qquad + \int_{B_{r+\rho}} \int_{B_{r+\rho}} w(x) w(y) (\tau^2(x) - \tau^2(y)) K_a(x,y) \, dy \, dx, \end{aligned}$$

and from the same arguments as in the proof of Step 2 in the proof of Lemma 5.1, we conclude

$$I_1 \le \tfrac{1}{4} \mathcal{E}^{K_s}_{B_{r+\rho}}(\tau w, \tau w) + cS^2 (r+\rho)^d \rho^{-\alpha},$$

using (K1$_{\text{glob}}$) and (cutoff). For I_2, we observe

$$\begin{aligned} I_2 &= 2S \int_{B_{r+\rho}} \int_{B_{r+\rho}} (\tau w(x) - \tau w(y))(\tau(x) + \tau(y)) K_a(x,y) \, dy \, dx \\ &\qquad + 2S \int_{B_{r+\rho}} \int_{B_{r+\rho}} (\tau w(x) + \tau w(y))(\tau(x) - \tau(y)) K_a(x,y) \, dy \, dx \\ &=: I_{2,1} + I_{2,2}. \end{aligned}$$

Now, using (K1$_{\text{glob}}$), (2-8), and (cutoff),

$$\begin{aligned} I_{2,1} &\le \tfrac{1}{8}\mathcal{E}^{K_s}_{B_{r+\rho}}(\tau w, \tau w) + cS^2 \int_{B_{r+\rho}} \tau^2(x)\biggl(\int_{B_{r+\rho}} \frac{|K_a(x,y)|^2}{J(x,y)}\,\mathrm{d}y\biggr)\mathrm{d}x \\ &\le \tfrac{1}{8}\mathcal{E}^{K_s}_{B_{r+\rho}}(\tau w, \tau w) + cS^2\mathcal{E}^{K_s}_{B_{r+\rho}}(\tau, \tau) + cS^2\rho^{-\alpha}\int_{B_{r+\rho}} \tau^2(x)\,\mathrm{d}x \\ &\le \tfrac{1}{8}\mathcal{E}^{K_s}_{B_{r+\rho}}(\tau w, \tau w) + cS^2(r+\rho)^d\rho^{-\alpha}, \end{aligned}$$

and, again using (K1$_{\text{glob}}$), (2-8), and (cutoff),

$$\begin{aligned} I_{2,2} &\le cS^2\mathcal{E}^{K_s}_{B_{r+\rho}}(\tau, \tau) + \int_{B_{r+\rho}} \tau^2 w^2(x)\biggl(\int_{B_{r+\rho}} \frac{|K_a(x,y)|^2}{K_s(x,y)}\,\mathrm{d}y\biggr)\mathrm{d}x \\ &\le cS^2(r+\rho)^d\rho^{-\alpha} + \tfrac{1}{8}\mathcal{E}^{K_s}_{B_{r+\rho}}(\tau w, \tau w) + c\rho^{-\alpha}\int_{B_{r+\rho}} \tau^2 w^2(x)\,\mathrm{d}x \\ &\le \tfrac{1}{8}\mathcal{E}^{K_s}_{B_{r+\rho}}(\tau w, \tau w) + cS^2(r+\rho)^d\rho^{-\alpha}. \end{aligned}$$

Altogether, we have proved (5-7).

Step 3: Moreover, we claim that, for some constants $c, c' > 0$,

$$-\widehat{\mathcal{E}}_{(B_{r+\rho}\times B_{r+\rho})^c}(u, \tau^2 w) \le cS^2(r+\rho)^d\rho^{-\alpha} - c'S\rho^d\,\widehat{\mathrm{Tail}}_K(u, r, r+\rho). \tag{5-8}$$

First, we write the decomposition

$$\begin{aligned} &-\mathcal{E}_{(B_{r+\rho}\times B_{r+\rho})^c}(\tau^2 w, u) \\ &\qquad = -2\int_{B_{r+\rho}}\int_{B^c_{r+\rho}} \tau^2 w(x)u(x)K(x,y)\,\mathrm{d}y\,\mathrm{d}x + 2\int_{B^c_{r+\rho}}\int_{B_{r+\rho}} \tau^2 w(y)u(x)K(x,y)\,\mathrm{d}y\,\mathrm{d}x \\ &\qquad =: J_1 + J_2. \end{aligned}$$

For J_1, using the definition of w, nonnegativity of u, and (1-2), we compute

$$\begin{aligned} J_1 &= 2\int_{B_{r+\rho}}\int_{B^c_{r+\rho}} \tau^2(x)(2S - u(x))u(x)K(x,y)\,\mathrm{d}y\,\mathrm{d}x \\ &\le 4S^2\int_{B_{r+\rho}}\int_{B^c_{r+\rho}} (\tau(x)-\tau(y))^2 K_s(x,y)\,\mathrm{d}y\,\mathrm{d}x \\ &\le cS^2(r+\rho)^2\rho^{-\alpha}. \end{aligned}$$

For J_2, using that $\tau^2 \ge \frac{1}{16}$ in $B_{r+\rho/4}$, we observe

$$\begin{aligned} J_2 &= 2\int_{B^c_{r+\rho}}\int_{B_{r+\rho}} \tau^2(y)(u(y) - 2S)u(x)K(x,y)\,\mathrm{d}y\,\mathrm{d}x \\ &\le -2S\int_{B_{r+\rho}}\int_{B^c_{r+\rho}} \tau^2(y)u(x)K(x,y)\,\mathrm{d}x\,\mathrm{d}y \\ &\le -\frac{S}{8}\int_{B_{r+\rho/4}}\int_{B^c_{r+\rho}} u(x)K(x,y)\,\mathrm{d}x\,\mathrm{d}y. \end{aligned}$$

Finally, using $(\widehat{\text{UJS}})$ and the same argument as in (5-5), we can prove that

$$\rho^d \, \widehat{\text{Tail}}_K(u, r, r+\rho) \le c \int_{B_{r+\rho/4}} \int_{B_{r+\rho}^c} u(x) K(x, y) \, dx \, dy.$$

Altogether, we have established (5-8), as desired.

Step 4: Combining (5-6), (5-7), and (5-8), we obtain

$$\begin{aligned}
\mathcal{E}^{K_s}_{B_{r+\rho}}(\tau w, \tau w) &\le \mathcal{E}^{K_s}_{B_{r+\rho}}(u, \tau^2 w) + cS^2 (r+\rho)^d \rho^{-\alpha} \\
&= \widehat{\mathcal{E}}(u, \tau^2 w) - \widehat{\mathcal{E}}^{K_a}_{B_{r+\rho}}(u, \tau^2 w) - \widehat{\mathcal{E}}_{(B_{r+\rho} \times B_{r+\rho})^c}(u, \tau^2 w) + cS^2 (r+\rho)^d \rho^{-\alpha} \\
&\le \widehat{\mathcal{E}}(u, \tau^2 w) + cS^2 (r+\rho)^d \rho^{-\alpha} + \tfrac{1}{2} \mathcal{E}^{K_s}_{B_{r+\rho}}(\tau w, \tau w) - cS\rho^d \, \widehat{\text{Tail}}_K(u, r, r+\rho).
\end{aligned}$$

Consequently,

$$\widehat{\text{Tail}}_K(u, r, r+\rho) \le c \frac{1}{S\rho^d} \widehat{\mathcal{E}}(u, \tau^2 w) + c \left(\frac{r+\rho}{\rho} \right)^d \rho^{-\alpha} S,$$

as desired. □

Lemma 5.1 can be used to bound $\text{Tail}_K(u, r, r+\rho)$ from above by the supremum of u. First, we provide such an estimate for weak supersolutions to the stationary equations (ell-PDE) and $(\text{ell-}\widehat{\text{PDE}})$, which is a direct corollary of Lemma 5.1 applied with $S = \sup_{B_{r+\rho}} u$.

Corollary 5.3. *Assume that (cutoff) holds.*

(i) *Assume* (K1_{loc}) *and* (UJS) *hold for some* $\theta \in [d/\alpha, \infty]$. *Moreover, assume* (Sob) *if* $\theta < \infty$. *Then there exists* $c > 0$ *such that, for every* $0 < \rho \le r \le 1$ *and every nonnegative, weak supersolution* u *to* (ell-PDE) *in* B_{2r}, *we have*

$$\text{Tail}_K(u, r, r+\rho) \le c \left(\frac{r+\rho}{\rho} \right)^d \left(\rho^{-\alpha} \sup_{B_{r+\rho}} u + \|f\|_{L^\infty} \right),$$

where $B_{2r} \subset \Omega$.

(ii) *Assume* (K1_{glob}) *and* $(\widehat{\text{UJS}})$ *holds for some* $\theta \in [d/\alpha, \infty]$. *Moreover, assume* (Sob) *if* $\theta < \infty$. *Then there exists* $c > 0$ *such that, for every* $0 < \rho \le r \le 1$ *and every nonnegative, weak subsolution* u *to* $(\text{ell-}\widehat{\text{PDE}})$ *in* B_{2r}, *we have*

$$\widehat{\text{Tail}}_K(u, r, r+\rho) \le c \left(\frac{r+\rho}{\rho} \right)^d \left(\rho^{-\alpha} \sup_{B_{r+\rho}} u + \|f\|_{L^\infty} \right),$$

where $B_{2r} \subset \Omega$.

One can also deduce an estimate for the L^1-parabolic tail,

$$\int_{I^{\ominus}_{r/2}} \text{Tail}_K(u(t), r, r+\rho) \, dt,$$

for supersolutions to (PDE) and $(\widehat{\text{PDE}})$ from Lemma 5.1.

Corollary 5.4. *Assume that (cutoff) holds.*

(i) *Assume* ($\mathrm{K1_{loc}}$) *and* (UJS) *hold for some* $\theta \in [d/\alpha, \infty]$. *Moreover, assume* (Sob) *if* $\theta < \infty$. *Then there exists* $c > 0$ *such that, for every* $0 < \rho \le r \le 1$ *and every nonnegative, weak supersolution* u *to* (PDE) *in* $I_r^{\ominus}(t_0) \times B_{2r}$, *we have*

$$\int_{I^{\ominus}_{r/2}} \mathrm{Tail}_K(u(t), r, r+\rho)\,\mathrm{d}t \le c\Big(\frac{r+\rho}{\rho}\Big)^d \bigg(\Big(\frac{r+\rho}{\rho}\Big)^{\alpha\vee 1} \sup_{I^{\ominus}_{(r+\rho)/2}\times B_{r+\rho}} u + (r+\rho)^{\alpha}\|f\|_{L^\infty}\bigg),$$

where $B_{2r} \subset \Omega$.

(ii) *Assume* ($\mathrm{K1_{glob}}$) *and* ($\widehat{\mathrm{UJS}}$) *hold for some* $\theta \in [d/\alpha, \infty]$. *Moreover, assume* (Sob) *if* $\theta < \infty$. *Then there exists* $c > 0$ *such that, for every* $0 < \rho \le r \le 1$ *and every nonnegative, weak supersolution* u *to* ($\widehat{\mathrm{PDE}}$) *in* $I_r^{\ominus}(t_0) \times B_{2r}$, *we have*

$$\int_{I^{\ominus}_{r/2}} \widehat{\mathrm{Tail}}_K(u(t), r, r+\rho)\,\mathrm{d}t \le c\Big(\frac{r+\rho}{\rho}\Big)^d \bigg(\Big(\frac{r+\rho}{\rho}\Big)^{\alpha\vee 1} \sup_{I^{\ominus}_{(r+\rho)/2}\times B_{r+\rho}} u + (r+\rho)^{\alpha}\|f\|_{L^\infty}\bigg),$$

where $B_{2r} \subset \Omega$.

Proof. We only explain the proof of (i). The proof of (ii) works in the same way, but relies on Lemma 5.2 instead of Lemma 5.1. We write $S = \sup_{I^{\ominus}_{(r+\rho)/2}\times B_{r+\rho}} u$ and define $w = u - 2S$. We also observe that $\partial_t(w^2) = 2w\partial_t u$. From Lemma 5.1 and the fact that u is a supersolution to (PDE), we deduce

$$\begin{aligned}\frac{c}{2S\rho^d}\int_{B_{r+\rho}} &\tau^2(x)\partial_t(w^2)(t,x)\,\mathrm{d}x + \mathrm{Tail}_K(u(t), r, r+\rho)\\
&\le c\frac{1}{S\rho^d}[(\partial_t u(t), \tau^2 w(t)) + \mathcal{E}(u(t), \tau^2 w(t))] + cS\Big(\frac{r+\rho}{\rho}\Big)^d \rho^{-\alpha}\\
&\le c\frac{1}{S\rho^d}(f(t), \tau^2 w(t)) + cS\Big(\frac{r+\rho}{\rho}\Big)^d \rho^{-\alpha}\\
&\le c\Big(\frac{r+\rho}{\rho}\Big)^d (\|f\|_{L^\infty} + S\rho^{-\alpha}),\end{aligned}$$

where $c > 0$ is the constant from Lemma 5.1 and we tested the equation with $\tau^2 w$, where $\tau = \tau_{r,\rho}$. Let $\chi \in C^1(\mathbb{R})$ be a nonnegative function with

$$\chi\big(t_0 - \big(\tfrac12(r+\rho)\big)^{\alpha}\big) = 0, \quad \chi \equiv 1 \text{ in } I^{\ominus}_{r/2}, \quad \|\chi\|_\infty \le 1, \quad \|\chi'\|_\infty \le 8((r+\rho)^{\alpha} - r^{\alpha})^{-1}.$$

Multiplying by χ^2 and integrating over $\big(t_0 - \big(\tfrac12(r+\rho)\big)^{\alpha}, t\big)$ for some arbitrary $t \in I^{\ominus}_{r/2}$, we obtain

$$\begin{aligned}&\frac{c}{2S\rho^d}\int_{B_{r+\rho}} \chi^2(t)\tau^2(x)w^2(t,x)\,\mathrm{d}x + \int_{t_0-((r+\rho)/2)^{\alpha}}^{t} \chi^2(s)\,\mathrm{Tail}_K(u(s), r, r+\rho)\,\mathrm{d}s\\
&\le c_1\int_{t_0-((r+\rho)/2)^{\alpha}}^{t} \chi^2(s)S\Big(\frac{r+\rho}{\rho}\Big)^d \rho^{-\alpha}\,\mathrm{d}s + c_1\Big(\frac{r+\rho}{\rho}\Big)^d (r+\rho)^{\alpha}\|f\|_{L^\infty}\\
&\qquad + c_1\int_{t_0-((r+\rho)/2)^{\alpha}}^{t} \frac{1}{S\rho^d}\chi(s)|\chi'(s)|\int_{B_{r+\rho}} \tau^2(x)w^2(s,x)\,\mathrm{d}x\,\mathrm{d}s,\end{aligned}$$

where $c_1 > 0$ is a constant. Consequently, using that $w^2 \le 4S^2$,

$$\sup_{t\in I^{\ominus}_{r/2}} \frac{c}{2S\rho^d}\int_{B_r} w^2(t,x)\,\mathrm{d}x + \int_{I^{\ominus}_{r/2}} \mathrm{Tail}_K(u(s),r,r+\rho)\,\mathrm{d}s \\ \le c_2\Big(\frac{r+\rho}{\rho}\Big)^{d+(\alpha\vee 1)} \sup_{I^{\ominus}_{(r+\rho)/2}\times B_{r+\rho}} u + c_2\Big(\frac{r+\rho}{\rho}\Big)^{d}(r+\rho)^{\alpha}\|f\|_{L^\infty},$$

where $c_2 > 0$ and we used that, for some $c > 0$,

$$((r+\rho)^{\alpha} - r^{\alpha})^{-1} \le c\rho^{-(\alpha\vee 1)}(r+\rho)^{(\alpha\vee 1)-\alpha}.$$

This concludes the proof. □

6. Harnack inequalities

The goal of this section is to complete the proofs of our main results: Theorem 1.1 and Theorem 1.4. In Section 6.1, we give improved versions of the local boundedness estimates from Sections 3 and 4, which do not involve tail terms. These results make use of the tail estimates obtained in Corollary 5.3 and are the key ingredients in the proof of Theorem 1.4. In Section 6.2 we combine local boundedness estimates with the weak Harnack inequalities from [Kassmann and Weidner 2022] and obtain our main results.

We point out that the proof of Theorem 1.1 does not rely on the tail estimates from Section 5. It is an open question — even in the symmetric case — how to derive a parabolic Harnack inequality involving only local quantities from suitable tail estimates, as one does in the stationary case. Section 6.3 is dedicated to this issue.

6.1. ***Local boundedness without tail terms.*** We obtain local L^∞-L^p-estimates for solutions to (ell-PDE) and (ell-$\widehat{\text{PDE}}$) (see Theorem 6.2). In comparison with Theorem 3.6, the estimates only contain purely local quantities. The underlying procedure works exactly as for symmetric forms. However, note that we need to redo the iteration in Theorem 3.6 in order to prove Theorem 6.1 since the quantities Tail_K and $\mathrm{Tail}_{K,\alpha}$ are in general not comparable.

The following theorem is the key result on our path towards L^∞-L^p-estimates for nonnegative solutions to (ell-PDE) and (ell-$\widehat{\text{PDE}}$) since it no longer involves nonlocal quantities.

Theorem 6.1. *Assume that (cutoff) and (Sob) hold.*

(i) *Assume that ($\mathrm{K1}_{\mathrm{loc}}$) and (UJS) hold for some $\theta \in [d/\alpha, \infty]$. Then, for every $\delta \in (0,1]$, there exists $c > 0$ such that, for every $0 < R \le 1$ and every nonnegative, weak solution u to (ell-PDE) in $B_{2R} \subset \Omega$, we have*

$$\sup_{B_{R/2}} u \le c\Big(\fint_{B_R} u^2(x)\,\mathrm{d}x\Big)^{1/2} + \delta\sup_{B_R} u + cR^{\alpha}\|f\|_{L^\infty}. \tag{6-1}$$

(ii) *Assume that ($\mathrm{K1}_{\mathrm{glob}}$) and ($\widehat{\text{UJS}}$) hold for some $\theta \in (d/\alpha, \infty]$. Then, for every $\delta \in (0,1]$, there exists $c > 0$ such that, for every $0 < R \le 1$ and every nonnegative, weak solution u to (ell-$\widehat{\text{PDE}}$) in B_{2R}, estimate (6-1) holds.*

We present two proofs of this theorem based on the De Giorgi iteration and the Moser iteration. Both proofs rely on a combination of the iteration schemes established in Sections 3 and 4 and the tail estimate from Corollary 5.4.

Proof of Theorem 6.1 (*based on De Giorgi iteration*). The proof of (i) is analogous to the proof of Theorem 3.6 (i). We define $(l_i)_i$, $(\rho_i)_i$, $(r_i)_i$, $(w_i)_i$ in the same way. Moreover, we set $A_i = \|w_i\|_{L^1(B_{r_i})}$. Note that

$$\left(\frac{r_i+\rho_i}{\rho_i}\right)^d = \left(\frac{(1+(\frac{1}{2})^i)+(\frac{1}{2})^i}{(\frac{1}{2})^{i+1}}\right)^d \le 2^{(i+2)d}.$$

Consequently, Corollary 5.3 (i) — applied with $r = r_i + \frac{1}{2}\rho_i$ and $\rho = \frac{1}{2}\rho_i$ — yields

$$\mathrm{Tail}_K\big(u, r_i + \tfrac{1}{2}\rho_i, r_i+\rho_i\big) \le c_1 2^{i(d+2)} R^{-\alpha}\Big(\sup_{B_R} u + R^\alpha \|f\|_{L^\infty}\Big) \tag{6-2}$$

for some $c_1 > 0$. Moreover, by following the arguments in the proof of Theorem 3.6 (i), we derive the following analog of (3-16):

$$A_i \le c_2 \frac{1}{(l_i - l_{i-1})^{2/\kappa'}}\left(\sigma(r_i,\rho_i) + \frac{\mathrm{Tail}_K\big(u, r_i+\frac{1}{2}\rho_i, r_i+\rho_i\big) + \|f\|_{L^\infty}}{l_i - l_{i-1}}\right) A_{i-1}^{1+1/\kappa'} \tag{6-3}$$

for some $c_2 > 0$, where we can choose $\kappa = d/(d-\alpha)$ using that u is a subsolution to the stationary equation (ell-PDE) in (3-13). We combine (6-2) and (6-3) and obtain

$$A_i \le \frac{c_3}{R^\alpha M^{2/\kappa'}} 2^{\gamma i}\left(1 + \frac{\sup_{B_R} u + R^\alpha \|f\|_{L^\infty}}{M}\right) A_{i-1}^{1+1/\kappa'},$$

where $c_3 > 0$ and $\gamma > 1$ are constants. We proceed as in the proof of Theorem 3.6 (i) and choose

$$M := \delta\Big(\sup_{B_R} u + \|f\|_{L^\infty}\Big) + C^{\kappa'^2/2} c_3^{\kappa'/2} \delta^{-\kappa'/2} R^{-\alpha\kappa'/2} A_0^{1/2},$$

where $C := 2^\gamma > 1$ and conclude

$$A_0 \le \left(\frac{c_3}{\delta R^\alpha M^{2/\kappa'}}\right)^{-\kappa'} C^{-\kappa'^2},$$

and therefore we obtain from Lemma 7.1 in [Giusti 2003]

$$\sup_{B_{R/2}} u \le M = \delta\Big(\sup_{B_R} u + R^\alpha \|f\|_{L^\infty}\Big) + c_3 \delta^{-\kappa'/2}\left(\fint_{B_R} u^2(x)\,\mathrm{d}x\right)^{1/2}$$

for some $c_3 > 0$, as desired.

In order to prove (ii), we follow the arguments in the proof of Theorem 3.6 (ii) and derive the following analog of (6-3):

$$A_i \le c_4 \frac{R^{d(1/\kappa'-\mu)}}{(l_i-l_{i-1})^{2\mu}}\left(\sigma(r_i,\rho_i)\left(1+\left(\frac{l_i}{l_i-l_{i-1}}\right)^2\right) + \frac{\mathrm{Tail}_K\big(u, r_i+\frac{1}{2}\rho_i, r_i+\rho_i\big)+\|f\|_{L^\infty}}{l_i-l_{i-1}}\right) A_{i-1}^{1+\mu}$$

for some $c_4 > 0$, where $\mu = 1/\kappa' - 1/\theta$ and $\kappa = d/(d-\alpha)$. As before, by Corollary 5.3 (ii) — applied with $r = r_i + \frac{1}{2}\rho_i$ and $\rho = \frac{1}{2}\rho_i$ — we prove

$$\widehat{\mathrm{Tail}}_K\big(u, r_i + \tfrac{1}{2}\rho_i, r_i + \rho_i\big) \le c_1 2^{i(d+2)} R^{-\alpha}\Big(\sup_{B_R} u + R^{\alpha}\|f\|_{L^\infty}\Big). \tag{6-4}$$

By combining (6-4) with the previous estimate, we deduce

$$A_i \le \frac{c_5}{\delta R^{\mu d} M^{2\mu}} C^{\gamma i} A_{i-1}^{1+\mu}$$

for some $c_5 > 0$ and $\gamma > 1$. From here, the desired result follows by the same arguments as in the proof of (i). □

Proof of Theorem 6.1 (*based on Moser iteration*). We explain how to prove (ii). The proof of (i) follows exactly the same arguments. Our proof is based on the Moser iteration and works in a similar way to the proof of Theorem 4.8. Let us define $(\rho_i)_i$, $(r_i)_i$, and $(q_i)_i$ in the same way, but set $\kappa = d/(d-\alpha)$.

Note that by following the arguments of the proof of Theorem 4.8, but using that u is a subsolution to the stationary equation in (4-17), we can derive the following analog of (4-18):

$$\|\tilde{u}\|_{L^{q_i}(B_{r_i})} \le c^{\frac{1}{2q_{i-1}}} q_{i-1}^{\frac{\gamma}{2q_{i-1}}} R^{-\frac{\alpha}{2q_{i-1}}} 2^{\frac{d+\varepsilon+2}{2q_{i-1}}(i+1)} \times \Big(\|\tilde{u}\|_{L^{2q_{i-1}}(B_{r_{i-1}})} + 2^{-(d+\varepsilon+\alpha)(i+1)} R^{\alpha+\frac{d}{2q_{i-1}}}\, \widehat{\mathrm{Tail}}_K\big(u, r_i + \tfrac{1}{2}\rho_i, r_i + \rho_i\big)\Big).$$

By combining this estimate with (6-4), we obtain

$$\|\tilde{u}\|_{L^{q_i}(B_{r_i})} \le c^{\frac{1}{2q_{i-1}}} q_{i-1}^{\frac{\gamma}{2q_{i-1}}} R^{-\frac{\alpha}{2q_{i-1}}} 2^{\frac{d+\varepsilon+2}{2q_{i-1}}(i+1)} \times \Big(\|\tilde{u}\|_{L^{2q_{i-1}}(B_{r_{i-1}})} + R^{\frac{d}{2q_{i-1}}} 2^{-(i+1)(\varepsilon+\alpha-2)}\Big(\sup_{B_R} u + R^{\alpha}\|f\|_{L^\infty}\Big)\Big).$$

From here, the proof follows in a similar manner to the proof of Theorem 4.8. First, we observe that

$$\sup_{B_{R/2}} \tilde{u} \le \Big(\prod_{i=1}^{\infty} c^{\frac{1}{2q_{i-1}}} q_{i-1}^{\frac{\gamma}{2q_{i-1}}} R^{-\frac{\alpha}{2q_{i-1}}} 2^{\frac{d+\varepsilon+2}{2q_{i-1}}(i+1)}\Big)\|\tilde{u}\|_{L^{2q_0}(B_R)} + \Big[\sum_{i=1}^{\infty}\Big(\prod_{j=i}^{\infty} c^{\frac{1}{2q_{j-1}}} q_{j-1}^{\frac{\gamma}{2q_{j-1}}} R^{-\frac{\alpha}{2q_{j-1}}} 2^{\frac{d+\varepsilon+2}{2q_{j-1}}(j+1)}\Big) R^{\frac{d}{2q_{i-1}}} 2^{-(i+1)(\varepsilon+\alpha-2)}\Big]\Big(\sup_{B_R} u + R^{\alpha}\|f\|_{L^\infty}\Big).$$

Moreover, by similar arguments as in the proof of Theorem 4.8,

$$\sum_{i=1}^{\infty}\Big(\prod_{j=i}^{\infty} c^{\frac{1}{2q_{j-1}}} q_{j-1}^{\frac{\gamma}{2q_{j-1}}} R^{-\frac{\alpha}{2q_{j-1}}} 2^{\frac{d+\varepsilon+2}{2q_{j-1}}(j+1)}\Big) R^{\frac{d}{2q_{i-1}}} 2^{-(i+1)(\varepsilon+\alpha-2)} \le c\sum_{i=1}^{\infty} \frac{2^{\frac{(d+\varepsilon+2)c_5}{2q_0}(i+1)}}{2^{(i+1)(\varepsilon+\alpha-2)}}$$

using that

$$\sum_{i=0}^{\infty} \kappa^{-i} = \frac{d}{\alpha} \quad \text{and} \quad \sum_{i=0}^{\infty} \frac{i}{\kappa^i} < \infty,$$

where $\kappa = d/(d-\alpha)$.

Now, choose $\varepsilon \geq 1$ large enough that

$$\sum_{i=1}^{\infty} 2^{-(i+1)\frac{\varepsilon+\alpha-2}{2}} \leq \frac{\delta}{2c}.$$

Then, let us choose $q_0 \geq 1$ large enough that

$$\frac{(d+\varepsilon+2)c_5}{2q_0} \leq \frac{\varepsilon+\alpha-2}{2}.$$

In that case,

$$c\sum_{i=1}^{\infty} 2^{\frac{(d+\varepsilon+2)c_5}{2q_0}(i+1)} 2^{-(i+1)(\varepsilon+\alpha-2)} \leq c \sum_{i=k+1}^{\infty} 2^{-(i+1)\frac{(\varepsilon+\alpha-2)}{2}} \leq \tfrac{1}{2}\delta.$$

Therefore,

$$\sup_{B_{R/2}} \tilde{u} \leq c\left(\fint_{B_R} \tilde{u}^{2q_0}(x)\,\mathrm{d}x \right)^{\frac{1}{2q_0}} + \tfrac{1}{2}\delta\Big(\sup_{B_R} u + R^{\alpha}\|f\|_{L^\infty}\Big).$$

As a consequence, using the definition of $\tilde{u}$ as well as the triangle inequality for the L^{2q_0}-norm, we deduce

$$\sup_{B_{R/2}} u \leq c\left(\fint_{B_R} u^{2q_0}(x)\,\mathrm{d}x \right)^{\frac{1}{2q_0}} + \tfrac{1}{2}\delta \sup_{B_R} u + cR^{\alpha}\|f\|_{L^\infty}.$$

It remains to prove the desired estimate (6-1) in the case $q_0 > 1$. This follows from Young's inequality:

$$\left(\fint_{B_R} u^{2q_0}(x)\,\mathrm{d}x \right)^{\frac{1}{2q_0}} \leq \sup_{B_R} u^{\frac{2q_0-2}{2q_0}} \left(\fint_{B_R} u^{2}(x)\,\mathrm{d}x \right)^{\frac{1}{2q_0}} \leq \frac{\delta}{2c}\sup_{B_R} u + c\left(\fint_{B_R} u^{2}(x)\,\mathrm{d}x \right)^{\frac{1}{2}}. \qquad \square$$

By a standard iteration argument one can deduce local boundedness of nonnegative solutions to (ell-PDE) and (ell-$\widehat{\text{PDE}}$) from Theorem 6.1.

Theorem 6.2. *Assume that (cutoff) and $(\mathcal{E}_{\geq})$ hold.*

(i) *Assume that $(\mathrm{K1}_{\mathrm{loc}})$ and (UJS) hold for some $\theta \in [d/\alpha, \infty]$. Then there exists $c > 0$ such that, for every $0 < R \leq 1$, every $p \in (0, 2]$, and every nonnegative, weak solution u to (ell-PDE) in B_{2R}, we have*

$$\sup_{B_{R/4}} u \leq c\left(\fint_{B_{R/2}} u^{p}(x)\,\mathrm{d}x \right)^{\frac{1}{p}} + cR^{\alpha}\|f\|_{L^\infty}, \tag{6-5}$$

where $B_{2R} \subset \Omega$.

(ii) *Assume that $(\mathrm{K1}_{\mathrm{glob}})$ and ($\widehat{\text{UJS}}$) hold for some $\theta \in (d/\alpha, \infty]$. Then there exists $c > 0$ such that, for every $0 < R \leq 1$, every $p \in (0, 2]$, and every nonnegative, weak solution u to (ell-$\widehat{\text{PDE}}$) in B_{2R}, estimate (6-5) holds.*

Proof. We restrict ourselves to proving (i). The proof of (ii) follows in the same way. The proof works as in [Di Castro et al. 2014, pp. 1828-1829]. Let us point out that this proof crucially relies on local boundedness of u, i.e.,

$$\sup_{B_{R/2}} u < \infty, \tag{6-6}$$

which follows from Theorem 3.6 and Theorem 4.8, since $\mathrm{Tail}_{K,\alpha}(u,R)$ and $\widehat{\mathrm{Tail}}_{K,\alpha}(u,R)$ are finite under the assumptions of this theorem due to Lemma 2.13 (i) and Lemma 2.13 (ii), respectively. Let $\frac{1}{4}\le t<s\le\frac{1}{2}$. We conclude from Theorem 6.1 and a classical covering argument

$$\sup_{B_{tR}} u \le c_1(s-t)^{-d/2}\biggl(\fint_{B_{sR}} u^2(x)\,\mathrm{d}x\biggr)^{1/2} + c_2R^\alpha\|f\|_{L^\infty} + c_2\delta\sup_{B_{sR}} u,$$

where $c_1, c_2>0$ are constants. By Young's inequality (applied with $2/p, 2/(2-p)\ge 1$),

$$\begin{aligned}\sup_{B_{tR}} u &\le c_1(s-t)^{-d/2}\sup_{B_{sR}} u^{(2-p)/2}\biggl(\fint_{B_{sR}} u^p(x)\,\mathrm{d}x\biggr)^{1/2} + c_2\delta\sup_{B_{sR}} u + c_2R^\alpha\|f\|_{L^\infty}\\ &\le \bigl(c_2\delta+\tfrac{1}{4}\bigr)\sup_{B_{sR}} u + c_3(s-t)^{-d/p}\biggl(\fint_{B_{sR}} u^p(x)\,\mathrm{d}x\biggr)^{1/p} + c_2R^\alpha\|f\|_{L^\infty}\end{aligned}$$

for some $c_3>0$. By choosing $\delta=1/(4c_2)$, we obtain

$$\sup_{B_{tR}} u \le \tfrac{1}{2}\sup_{B_{sR}} u + c_4(s-t)^{-d/p}\biggl(\fint_{B_{R/2}} u^p(x)\,\mathrm{d}x\biggr)^{1/p} + c_4R^\alpha\|f\|_{L^\infty}$$

for $c_4>0$, and the result follows from the application of Lemma 1.1 in [Giaquinta and Giusti 1982] using (6-6). □

6.2. *Proofs of main results.* In this section we provide the proofs of our main results: Theorem 1.1 and Theorem 1.4. Let us recall the following theorem from [Kassmann and Weidner 2022].

Theorem 6.3 (weak Harnack inequality). *Assume (K2), (cutoff), (Poinc), and (Sob).*

(i) *Assume that $(\mathrm{K1}_{\mathrm{loc}})$ holds for some $\theta\in[d/\alpha,\infty]$. Then there is $c>0$ such that, for every $0<R\le 1$ and every nonnegative, weak supersolution u to (PDE) in $I_R(t_0)\times B_{2R}$, we have*

$$\inf_{(t_0+R^\alpha-(R/2)^\alpha,\,t_0+R^\alpha)\times B_{R/2}} u \ge c\biggl(\fint_{(t_0-R^\alpha,\,t_0-R^\alpha+(R/2)^\alpha)\times B_{R/2}} u(t,x)\,\mathrm{d}x\,\mathrm{d}t - R^\alpha\|f\|_{L^\infty}\biggr), \tag{6-7}$$

where $B_{2R}\subset\Omega$.

(ii) *Assume that $(\mathrm{K1}_{\mathrm{glob}})$ holds for some $\theta\in(d/\alpha,\infty]$. Then there is $c>0$ such that, for every $0<R\le 1$ and every nonnegative, weak supersolution u to $(\widehat{\mathrm{PDE}})$ in $I_R(t_0)\times B_{2R}$, estimate (6-7) holds.*

Now we prove Theorem 1.1 and Theorem 1.4. Both results require the weak Harnack inequality Theorem 6.3.

Proof of Theorem 1.1. We only prove (i) since the proof of (ii) follows the same line of arguments. Part (i) follows from a combination of Theorem 6.3 and Theorem 3.6 (or Theorem 4.8). First, we deduce from Theorem 3.6 (or Theorem 4.8) and a classical covering argument that, for every $\frac{1}{4}\le t<s\le\frac{1}{2}$,

$$\sup_{I^\ominus_{tR/2}\times B_{tR}} u \le c_1(s-t)^{-(d+\alpha)/2}\biggl(\fint_{I^\ominus_{sR/2}}\fint_{B_{sR}} u^2(t,x)\,\mathrm{d}x\,\mathrm{d}t\biggr)^{1/2} + \sup_{t\in I^\ominus_{R/4}} \mathrm{Tail}_{K,\alpha}(u(t),R) + c_2R^\alpha\|f\|_{L^\infty}.$$

By a similar iteration argument as in the proof of Theorem 6.2, we deduce

$$\sup_{I^{\ominus}_{R/8}\times B_{R/4}} u \le c_2\left(\fint_{I^{\ominus}_{R/4}\times B_{R/2}} u(t,x)\,\mathrm{d}x\,\mathrm{d}t\right) + c_2 \sup_{t\in I^{\ominus}_{R/4}} \mathrm{Tail}_{K,\alpha}(u(t),R) + c_2 R^{\alpha}\|f\|_{L^\infty}. \tag{6-8}$$

Next, Theorem 6.3 yields

$$\inf_{(t_0+(1-2^{-\alpha})R^{\alpha},t_0+R^{\alpha})\times B_{R/2}} u \ge c_1\left(\fint_{(t_0-R^{\alpha},t_0-(1-2^{-\alpha})R^{\alpha})\times B_{R/2}} u(t,x)\,\mathrm{d}x\,\mathrm{d}t - R^{\alpha}\|f\|_{L^\infty}\right) \tag{6-9}$$

for some $c_1>0$. Note that

$$(t_0-R^{\alpha},t_0-(1-2^{-\alpha})R^{\alpha}) = I^{\ominus}_{R/2}(t_0-(1-2^{-\alpha})R^{\alpha}).$$

Consequently, by (6-8),

$$\begin{aligned}
&\sup_{I^{\ominus}_{R/8}(t_0-(1-2^{-\alpha})R^{\alpha})\times B_{R/4}} u\\
&\quad\le c_2\left(\fint_{I^{\ominus}_{R/4}(t_0-(1-2^{-\alpha})R^{\alpha})\times B_{R/2}} u(t,x)\,\mathrm{d}x\,\mathrm{d}t\right) + c_2 \sup_{t\in I^{\ominus}_{R/4}(t_0-(1-2^{-\alpha})R^{\alpha})} \mathrm{Tail}_{K,\alpha}(u(t),R) + c_2R^{\alpha}\|f\|_{L^\infty}\\
&\quad\le c_3\left(\fint_{(t_0-R^{\alpha},t_0-(1-2^{-\alpha})R^{\alpha})\times B_{R/2}} u(t,x)\,\mathrm{d}x\,\mathrm{d}t\right) + c_3 \sup_{t\in I^{\ominus}_{R/4}(t_0-(1-2^{-\alpha})R^{\alpha})} \mathrm{Tail}_{K,\alpha}(u(t),R) + c_3R^{\alpha}\|f\|_{L^\infty}\\
&\quad\le c_4 \inf_{(t_0+(1-2^{-\alpha})R^{\alpha},t_0+R^{\alpha})\times B_{R/2}} + c_4 \sup_{t\in(t_0-(1-2^{-\alpha}+4^{-\alpha})R^{\alpha},t_0-(1-2^{-\alpha})R^{\alpha})} \mathrm{Tail}_{K,\alpha}(u(t),R) + c_4R^{\alpha}\|f\|_{L^\infty}
\end{aligned}$$

for some $c_2,c_3,c_4>0$. The proof is finished upon noticing that

$$I^{\ominus}_{R/8}(t_0-(1-2^{-\alpha})R^{\alpha}) = (t_0-(1-2^{-\alpha}+8^{-\alpha})R^{\alpha},t_0-(1-2^{-\alpha})R^{\alpha}). \qquad \square$$

Proof of Theorem 1.4. This result follows directly by combining Theorems 6.2 and 6.3, where we apply Theorem 6.2 with $p=1$. $\square$

6.3. ***Challenges in the parabolic case.*** Let us assume that (cutoff), ($\mathcal{E}_{\ge}$), ($\mathrm{K1_{loc}}$), (K2), and (UJS) hold for some $\theta\in[d/\alpha,\infty]$. The goal of this section is to discuss the validity of a parabolic version of Theorem 1.4, i.e., to investigate the estimate

$$\sup_{(t_0-c_1R^{\alpha},t_0-c_2R^{\alpha})\times B_{R/4}} u \le C\left(\inf_{(t_0+c_2R^{\alpha},t_0+R^{\alpha})\times B_{R/2}} u + R^{\alpha}\|f\|_{L^\infty}\right) \tag{6-10}$$

for some $C>0$ and $0<c_2<c_1<1$ for nonnegative, weak solutions u to (PDE) in $I^{\ominus}_R\times B_{2R}$, where $B_{2R}\subset\Omega$. In order to keep the presentation short, we will not discuss weak solutions to ($\widehat{\mathrm{PDE}}$) here.

As in the elliptic case, the general strategy to establish (6-10) would be to first prove an L^∞-L^p-estimate of the form (given any $p\in(0,2]$)

$$\sup_{I^{\ominus}_{R/8}\times B_{R/4}} u \le c\left(\fint_{I^{\ominus}_{R/4}}\fint_{B_{R/2}} u^p(t,x)\,\mathrm{d}x\,\mathrm{d}t\right)^{1/p} + cR^{\alpha}\|f\|_{L^\infty} \tag{6-11}$$

and to deduce (6-10) after combination with the weak parabolic Harnack inequality of Theorem 6.3 as in the proof of Theorem 1.1.

A natural approach in order to show (6-11) would be to proceed as in the proof of Theorem 3.6 but to apply Corollary 5.4 in order to estimate the nonlocal tail by a local quantity. However, as Corollary 5.4 only provides an estimate for $\int_{I^{\ominus}_{r/2}} \mathrm{Tail}_K(u(t), r, r+\rho)\,\mathrm{d}t$ but not for $\sup_{I^{\ominus}_{r/2}} \mathrm{Tail}_K(u(t), r, r+\rho)$, one needs to come up with a new idea to bridge the gap between

$$\sup_{I^{\ominus}_{r/2}} \mathrm{Tail}_K(u(t), r, r+\rho) \quad \text{and} \quad \int_{I^{\ominus}_{r/2}} \mathrm{Tail}_K(u(t), r, r+\rho)\,\mathrm{d}t.$$

Note that the same issue appears in the symmetric case and has not been solved so far. There seems to be no proof of a parabolic Harnack inequality (6-10) for jumping kernels $K(x, y) \asymp |x-y|^{-d-\alpha}$ that uses only analytic arguments. Note that via probabilistic methods, an estimate of the form (6-10) has been proved in the symmetric case in [Bass and Levin 2002; Chen and Kumagai 2003].

Let us explain how to deduce (6-11) under the condition that u satisfies the following two additional assumptions:

(a) There exists $c_0 > 0$ such that, for every $\frac{1}{2}R \le r \le R$ and $0 < \rho \le r \le r+\rho \le R$,

$$\sup_{t \in I^{\ominus}_{(r+\rho)/4}} \mathrm{Tail}_K\big(u(t), r+\tfrac{1}{2}\rho, r+\rho\big) \le c_0 \sup_{I^{\ominus}_{(r+\rho/2)/2} \times B_{r+\rho}} u. \tag{6-12}$$

(b) We have $\sup_{I^{\ominus}_{R/4}(t_0)} \mathrm{Tail}_{K,\alpha}(u(t), R) < \infty$.

Remark 6.4. (i) Naturally, the constant c in (6-11) will depend on c_0.

(ii) (6-12) holds for global solutions to (PDE) in the symmetric case (see [Strömqvist 2019b]).

(iii) It has been proposed in [Kim 2019] to establish (6-12) for every weak solution u to (PDE) in $I \times B_{2R}$ with prescribed nonlocal parabolic boundary data $g \in L^{\infty}(I \times \mathbb{R}^d) \cap C(\bar{I} \times \mathbb{R}^d)$, with c_0 depending only on g. The proof of [Kim 2019, Lemma 5.3] is not complete.

(iv) Note that (b) is an additional restriction and does not naturally follow from our weak solution concept. We refer to Section 2.3 for a more detailed discussion of finiteness of tail terms.

In order to establish (6-11), we need to prove an analog of (6-1). As in the proof of Theorem 3.6, we derive (3-16), and by combining it with (6-12), we deduce, for every $\delta > 0$,

$$A_i \le \frac{c_1}{R^{\alpha} M^{2/\kappa'}} 2^{\gamma i} \left(1 + \frac{\sup_{I^{\ominus}_{R/2} \times B_R} u + R^{\alpha} \|f\|_{L^{\infty}}}{M}\right) A_{i-1}^{1+1/\kappa'}$$

for some $c_1 > 0$ and $\gamma > 1$. Here, $\kappa = 1 + \alpha/d$. By choosing

$$M := \delta\Big(\sup_{I^{\ominus}_{R/2} \times B_R} u + R^{\alpha} \|f\|_{L^{\infty}}\Big) + C^{\kappa'^2/2} c_1^{\kappa'/2} \delta^{-\kappa'/2} R^{-\alpha\kappa'/2} A_0^{1/2},$$

where $C := 2^{\gamma} > 1$, we can deduce

$$\sup_{I^{\ominus}_{R/8} \times B_{R/2}} u \le \delta\Big(\sup_{I^{\ominus}_{R/2} \times B_R} u + R^{\alpha} \|f\|_{L^{\infty}}\Big) + c_2 \delta^{-\kappa'/2} \bigg(\fint_{I^{\ominus}_{R/4}} \fint_{B_R} u^2(t, x)\,\mathrm{d}x\,\mathrm{d}t\bigg)^{1/2} \tag{6-13}$$

for some $c_2 > 0$. This estimate is a parabolic analog of (6-1). Note that (6-13) can also be established via the arguments from the proof of Theorem 4.8 using the Moser iteration.

Next, we intend to prove (6-11) by adapting the arguments in the proof of Theorem 6.2 to the parabolic setting.

As in the elliptic case, a standard covering argument yields, for every $\frac{1}{4} \leq t < s \leq \frac{1}{2}$,

$$\sup_{I^{\ominus}_{tR/2}\times B_{tR}} u \leq c_3(s-t)^{-(d+\alpha)/2}\left(\fint_{I^{\ominus}_{sR/2}}\fint_{B_{sR}} u^2(t,x)\,\mathrm{d}x\,\mathrm{d}t\right)^{1/2} + c_4R^{\alpha}\|f\|_{L^\infty} + c_4\delta \sup_{I^{\ominus}_{sR/2}\times B_{sR}} u,$$

where $c_3, c_4 > 0$ are constants. By Young's inequality and choosing $\delta = 1/c_4$, we arrive at

$$\sup_{I^{\ominus}_{tR/2}\times B_{tR}} u \leq \tfrac{1}{2}\sup_{I^{\ominus}_{sR/2}\times B_{sR}} u + c_4(s-t)^{-(d+\alpha)/p}\left(\fint_{I^{\ominus}_{R/4}}\fint_{B_{R/2}} u^p(t,x)\,\mathrm{d}x\,\mathrm{d}t\right)^{1/p} + c_4R^{\alpha}\|f\|_{L^\infty},$$

where $p \in (0,2]$ can be chosen arbitrarily.

Now, (6-11) follows from [Giaquinta and Giusti 1982, Lemma 1.1], but this only applies if

$$\sup_{I^{\ominus}_{R/4}\times B_{R/2}} u < \infty. \tag{6-14}$$

In order to obtain (6-14), we apply Theorem 3.6 (or Theorem 4.8) and use condition (b) on u. This concludes the proof of (6-11) under the additional assumptions (a) and (b).

Appendix

The following lemma justifies the way we deal with the weak formulation of (PDE), or ($\widehat{\text{PDE}}$), in the proof of Theorem 3.6 after testing with $\phi(t,x) = \tau^2(x)(u(t,x)-k)_+$ for some $k \geq 0$, where u is a subsolution to the respective equation. In fact, ϕ is a priori not differentiable in t, which prevents us from integrating by parts. The idea of the proof is to test the equation with an auxiliary function having the required smoothness properties in t. This can be achieved with the help of Steklov averages. For symmetric nonlocal equations, such lemmas are well known (see [Felsinger and Kassmann 2013; Strömqvist 2019a]). We adapt the idea of the proof of [Felsinger and Kassmann 2013] to the nonsymmetric case. Note that Lemma A.2 in [Felsinger and Kassmann 2013] is not sufficient for the proof of (A.4) in [Felsinger and Kassmann 2013]. Our proof fixes the gap in their argument.

Lemma A.1. *Assume (cutoff).*

(i) *Assume that $(K1_{\mathrm{loc}})$ holds for some $\theta \in [d/\alpha, \infty]$. Moreover, assume (Sob) if $\theta < \infty$. Let $u \in V(B_{r+\rho}\,|\,\mathbb{R}^d)$ be a weak subsolution to (PDE). Then, for every $[t_1,t_2] \subset I$, every $0 < \rho \leq r \leq 1$ with $B_{r+\rho} \subset \Omega$, every $k \geq 0$, and every $\chi \in C^1_c(\mathbb{R})$,*

$$\begin{aligned}\chi^2(t_2)\int_{B_{r+\rho}} (u(t_2)-k)_+^2\tau^2\,\mathrm{d}x - \chi^2(t_1)\int_{B_{r+\rho}} (u(t_1)-k)_+^2\tau^2\,\mathrm{d}x\\ - \int_{t_1}^{t_2}\partial_t(\chi^2(t))\int_{B_{r+\rho}} (u(t)-k)_+^2\tau^2\,\mathrm{d}x\,\mathrm{d}t + \int_{t_1}^{t_2}\chi^2(t)\mathcal{E}(u(t),\tau^2(u(t)-k)_+)\,\mathrm{d}t\\ \leq \int_{t_1}^{t_2}\chi^2(t)\int_{B_{r+\rho}} f(t,x)\tau^2(x)(u(t,x)-k)_+\,\mathrm{d}x\,\mathrm{d}t,\end{aligned}$$

where $\tau = \tau_{r,\rho/2}$.

(ii) *Assume that* (K1$_{\text{glob}}$) *holds for some* $\theta \in [d/\alpha, \infty]$. *Moreover, assume* (Sob) *if* $\theta < \infty$. *Let* $u \in V(B_{r+\rho} \,|\, \mathbb{R}^d)$ *be a weak subsolution to* ($\widehat{\text{PDE}}$). *Then, for every* $[t_1, t_2] \subset I$, *every* $0 < \rho \le r \le 1$ *with* $B_{r+\rho} \subset \Omega$, *every* $k \ge 0$, *and every* $\chi \in C_c^1(\mathbb{R})$,

$$
\begin{aligned}
\chi^2(t_2) \int_{B_{r+\rho}} (u(t_2)-k)_+^2 \tau^2 \,\mathrm{d}x - \chi^2(t_1) \int_{B_{r+\rho}} (u(t_1)-k)_+^2 \tau^2 \,\mathrm{d}x \\
- \int_{t_1}^{t_2} \partial_t(\chi^2(t)) \int_{B_{r+\rho}} (u(t)-k)_+^2 \tau^2 \,\mathrm{d}x\,\mathrm{d}t + \int_{t_1}^{t_2} \chi^2(t) \widehat{\mathcal{E}}(u(t), \tau^2(u(t)-k)_+) \,\mathrm{d}t \\
\le \int_{t_1}^{t_2} \chi^2(t) \int_{B_{r+\rho}} f(t,x) \tau^2(x) (u(t,x)-k)_+ \,\mathrm{d}x\,\mathrm{d}t.
\end{aligned}
$$

Proof. Given $v \in L^1((0,T); X)$ for some Banach space X, we define its Steklov average $v_h(t,x) = ⨍_t^{t+h} v(s,\cdot)\,\mathrm{d}s$ if $t+h \in I$ and $v_h(t,x) = 0$ otherwise. Observe that

$$
\partial_t u_h(t,x) = \frac{1}{h}(u(t+h,x) - u(t,x)) = ⨍_t^{t+h} \partial_s u(s,x)\,\mathrm{d}s.
$$

According to Lemma A.1 in [Felsinger and Kassmann 2013], we have

$$
\|v_h(t) - v(t)\|_{L^2} \to 0 \quad \text{as } h \searrow 0 \ \text{ if } v \in C((0,T); L^2(B_{r+\rho})), \tag{A-1}
$$

$$
\|v_h - v\|_{L^2([t_1,t_2];X)} \to 0 \quad \text{as } h \searrow 0, \tag{A-2}
$$

$$
\|v_h\|_{L^2([t_1,t_2];X)} \le \|v\|_{L^2([t_1,t_2];X)}. \tag{A-3}
$$

We first explain how to prove (i). Let $t \in I$. We use the test function $\phi = \tau^2(u_h(t)-k)_+$, and after integrating over $(t, t+h)$ for some $h > 0$ such that $t+h \in I$ and dividing by h, we obtain

$$
\int_{B_{r+\rho}} \partial_t u_h(t,x)\phi(t,x)\,\mathrm{d}x + \mathcal{E}(u_h(t), \phi(t)) \le (f(t), \phi(t)).
$$

Note that $t \mapsto u_h(t,x)$ is differentiable for a.e. $x \in B_{r+\rho}$, and therefore

$$
\partial_t u_h(t,x)\phi(t,x) = \tfrac{1}{2}\partial_t[(u_h(t,x)-k)_+^2]\tau^2(x).
$$

We multiply by $\chi^2(t)$ and integrate over (t_1,t_2). Integration by parts yields

$$
\begin{aligned}
\int_{B_{r+\rho}} \chi^2(t_2)(u_h(t_2)-k)_+^2 \tau^2 \,\mathrm{d}x - \int_{B_{r+\rho}} \chi^2(t_1)(u_h(t_1)-k)_+^2 \tau^2 \,\mathrm{d}x \\
- \int_{t_1}^{t_2}\int_{B_{r+\rho}} \partial_t(\chi^2(t))(u_h(t)-k)_+^2 \tau^2 \,\mathrm{d}x\,\mathrm{d}t + \int_{t_1}^{t_2} \chi^2(t) \mathcal{E}(u_h(t), \tau^2(u_h(t)-k)_+)\,\mathrm{d}t \\
\le \int_{t_1}^{t_2} \chi^2(t) \int_{B_{r+\rho}} f(t,x)\tau^2(x)(u_h(t,x)-k)_+ \,\mathrm{d}x\,\mathrm{d}t.
\end{aligned}
$$

Since $\||(u_h(t)-k)_+ - (u(t)-k)_+|\tau^2\|_{L^2(B_{r+\rho})} \le \|u_h(t) - u(t)\|_{L^2(B_{r+\rho})}$, it follows by (A-1) that

$$
\int_{B_{r+\rho}} (u_h(t)-k)_+^2 \tau^2 \,\mathrm{d}x \to \int_{B_{r+\rho}} (u(t)-k)_+^2 \tau^2 \,\mathrm{d}x \quad \text{for } t \in [t_1, t_2].
$$

Moreover, (A-2) implies

$$\int_{t_1}^{t_2} \partial_t(\chi^2(t)) \int_{B_{r+\rho}} (u_h(t)-k)_+^2 \tau^2 \,\mathrm{d}x\,\mathrm{d}t \to \int_{t_1}^{t_2}\int_{B_{r+\rho}} \partial_t(\chi^2(t))(u(t)-k)_+^2\tau^2 \,\mathrm{d}x\,\mathrm{d}t,$$

$$\int_{t_1}^{t_2} \chi^2(t) \int_{B_{r+\rho}} f(t,x)\tau^2(x)(u_h(t,x)-k)_+ \,\mathrm{d}x\,\mathrm{d}t \to \int_{t_1}^{t_2} \chi^2(t) \int_{B_{r+\rho}} f(t,x)\tau^2(x)(u(t,x)-k)_+ \,\mathrm{d}x\,\mathrm{d}t$$

as $h \searrow 0$. It remains to prove that

$$\int_{t_1}^{t_2} \chi^2(t)\mathcal{E}(u_h(t), \tau^2(u_h(t)-k)_+)\,\mathrm{d}t \to \int_{t_1}^{t_2} \chi^2(t)\mathcal{E}(u(t), \tau^2(u(t)-k)_+)\,\mathrm{d}t. \tag{A-4}$$

In Lemma A.2 in [Felsinger and Kassmann 2013], the authors established a related convergence property for symmetric energy forms. However, their proof has a gap, since Lemma A.2 does not suffice to deduce the desired result (even in the symmetric case), since, if $\Phi = f(u)$, it does not hold in general that $\Phi_h = f(u_h)$.

We define

$$\begin{aligned}
V(t,x,y) &= u(t,x)-u(t,y),\\
W(t,x,y) &= \tau^2(x)(u(t,x)-k)_+ - \tau^2(y)(u(t,y)-k)_+,\\
\widetilde{W}(t,x,y) &= \tau^2(x)(u_h(t,x)-k)_+ - \tau^2(y)(u_h(t,y)-k)_+.
\end{aligned}$$

Our goal is to show that

$$\int_{t_1}^{t_2} |\mathcal{E}(u_h(t)-u(t), \tau^2(u_h(t)-k)_+)|\,\mathrm{d}t \to 0, \tag{A-5}$$

$$\int_{t_1}^{t_2} |\mathcal{E}(u(t), \tau^2(u_h-k)_+ - \tau^2(u(t)-k)_+)|\,\mathrm{d}t \to 0. \tag{A-6}$$

To establish (A-5), we split

$$\begin{aligned}
&\int_{t_1}^{t_2} |\mathcal{E}(u_h(t)-u(t), \tau^2(u_h(t)-k)_+)|\,\mathrm{d}t\\
&\qquad\le \|\mathcal{E}^{K_s}_{B_{r+\rho}}(u_h-u, \tau^2(u_h-k)_+)\|_{L^1([t_1,t_2])} + \|\mathcal{E}^{K_a}_{B_{r+\rho}}(u_h-u, \tau^2(u_h-k)_+)\|_{L^1([t_1,t_2])}\\
&\qquad\qquad + \|\mathcal{E}_{(B_{r+\rho}\times B_{r+\rho})^c}(u_h-u, \tau^2(u_h-k)_+)\|_{L^1([t_1,t_2])}\\
&\qquad =: I_1 + I_2 + I_3,
\end{aligned}$$

and establish the convergence of each term separately. For I_1, we estimate

$$\begin{aligned}
I_1 &\le \int_{t_1}^{t_2}\int_{B_{r+\rho}}\int_{B_{r+\rho}} |V_h(t,x,y)-V(t,x,y)||\widetilde{W}(t,x,y)|K_s(x,y)\,\mathrm{d}y\,\mathrm{d}x\,\mathrm{d}t\\
&\le \|(V_h-V)K_s^{1/2}\|_{L^2([t_1,t_2]\times B_{r+\rho}\times B_{r+\rho})} \|\widetilde{W}K_s^{1/2}\|_{L^2([t_1,t_2]\times B_{r+\rho}\times B_{r+\rho})}\\
&\le \|u_h-u\|_{L^2([t_1,t_2];V(B_{r+\rho}|\mathbb{R}^d))} \|\mathcal{E}^{K_s}_{B_{r+\rho}}(\tau^2(u_h-k)_+, \tau^2(u_h-k)_+)\|^{1/2}_{L^1([t_1,t_2])}\\
&\le \|u_h-u\|_{L^2([t_1,t_2];V(B_{r+\rho}|\mathbb{R}^d))} \|\tau^2 u\|_{L^2([t_1,t_2];V(B_{r+\rho}|\mathbb{R}^d))}\\
&\to 0,
\end{aligned}$$

where we used (A-2) and (A-3), that

$$u, \phi \in L^2([t_1, t_2]; V(B_{r+\rho} | \mathbb{R}^d)),$$

and the fact that, due to the Markov property of $\mathcal{E}^{K_s}$ and (A-3),

$$\begin{aligned}\mathcal{E}^{K_s}_{B_{r+\rho}}(\tau^2(u_h - k)_+, \tau^2(u_h - k)_+) &\le \mathcal{E}^{K_s}_{B_{r+\rho}}(\tau^2 u_h, \tau^2 u_h)\\ &= \mathcal{E}^{K_s}_{B_{r+\rho}}([\tau^2 u]_h, [\tau^2 u]_h) \le \|\tau^2 u\|^2_{V(B_{r+\rho} | \mathbb{R}^d)}.\end{aligned} \tag{A-7}$$

For I_2,

$$\begin{aligned}I_2 &\le \int_{t_1}^{t_2}\int_{B_{r+\rho}}\int_{B_{r+\rho}} |V_h(t,x,y) - V(t,x,y)|\tau^2(x)(u_h(t,x)-k)_+|K_a(x,y)|\,\mathrm{d}y\,\mathrm{d}x\,\mathrm{d}t\\ &\le \|(V_h - V)J^{1/2}\|_{L^2([t_1,t_2]\times B_{r+\rho}\times B_{r+\rho})}\left\|\int_{B_{r+\rho/2}}(u_h(\,\cdot\,,x)-k)_+^2\left(\int_{B_{r+\rho}}\frac{|K_a(x,y)|^2}{J(x,y)}\,\mathrm{d}y\right)\mathrm{d}x\right\|^{1/2}_{L^1([t_1,t_2])}\\ &\le c\|u_h - u\|_{L^2([t_1,t_2];V(B_{r+\rho}|\mathbb{R}^d))}\|u\|_{L^2([t_1,t_2];L^{2\theta'}(B_{r+\rho/2}))}\\ &\le c\|u_h - u\|_{L^2([t_1,t_2];V(B_{r+\rho}|\mathbb{R}^d))}\|u\|_{L^2([t_1,t_2];V(B_{r+\rho}|\mathbb{R}^d))}\\ &\to 0,\end{aligned}$$

where $c > 0$ might depend on ρ, and we used (K1$_{\text{loc}}$), (A-2), (A-3), that

$$u \in L^2([t_1, t_2]; V(B_{r+\rho} | \mathbb{R}^d)),$$

and (Sob). For I_3, we obtain

$$\begin{aligned}I_3 &\le 2\int_{t_1}^{t_2}\int_{B_{r+\rho}}\int_{B^c_{r+\rho}} |V_h(t,x,y) - V(t,x,y)|\tau^2(x)(u_h(t,x)-k)_+ K_s(x,y)\,\mathrm{d}y\,\mathrm{d}x\,\mathrm{d}t\\ &\le 2\|(V_h - V)K_s^{1/2}\|_{L^2([t_1,t_2]\times B_{r+\rho}\times B^c_{r+\rho})}\left\|\int_{B_{r+\rho/2}}(u_h(\,\cdot\,,x)-k)_+^2\Gamma^{K_s}(\tau,\tau)(x)\,\mathrm{d}x\right\|^{1/2}_{L^1([t_1,t_2])}\\ &\le c\rho^{-\alpha/2}\|u_h - u\|_{L^2([t_1,t_2];V(B_{r+\rho}|\mathbb{R}^d))}\|u\|_{L^2([t_1,t_2]\times B_{r+\rho})}\\ &\to 0,\end{aligned}$$

where we used (1-2), (cutoff), (A-2), (A-3) and that

$$u \in L^2([t_1, t_2]; V(B_{r+\rho} | \mathbb{R}^d)).$$

It remains to prove (A-6). Again, we split

$$\begin{aligned}&\int_{t_1}^{t_2} |\mathcal{E}(u(t), \tau^2(u_h - k)_+ - \tau^2(u(t)-k)_+)|\,\mathrm{d}t\\ &\le \|\mathcal{E}^{K_s}_{B_{r+\rho}}(u, \tau^2(u_h-k)_+ - \tau^2(u-k)_+)\|_{L^1([t_1,t_2])}\\ &\qquad + \|\mathcal{E}^{K_a}_{B_{r+\rho}}(u, \tau^2(u_h-k)_+ - \tau^2(u-k)_+)\|_{L^1([t_1,t_2])}\\ &\qquad\qquad + \|\mathcal{E}_{(B_{r+\rho}\times B_{r+\rho})^c}(u, \tau^2(u_h-k)_+ - \tau^2(u-k)_+)\|_{L^1([t_1,t_2])}\\ &=: J_1 + J_2 + J_3,\end{aligned}$$

Convergence of J_1 can be proved as follows. First, by Hölder's inequality,

$$J_1 \le \|u\|_{L^2([t_1,t_2];V(B_{r+\rho}\,|\,\mathbb{R}^d))} \left\| \int_{B_{r+\rho}} \int_{B_{r+\rho}} |W(\,\cdot\,,x,y) - \widetilde{W}(\,\cdot\,,x,y)|^2 K_s(x,y)\,\mathrm{d}y\,\mathrm{d}x \right\|_{L^1([t_1,t_2])}^{1/2}.$$

Since $u \in L^2([t_1,t_2]; V(B_{r+\rho}\,|\,\mathbb{R}^d))$, it suffices to prove that the second factor converges to zero in order to conclude that $J_1 \to 0$. For this, we claim that there exist $\xi(t,x,y),\ \tilde{\xi}(t,x,y) \in [0,1]$ such that

$$\begin{aligned} \tau^2(x)(u(t,x)-k)_+ - \tau^2(y)(u(t,y)-k)_+ &= \xi(t,x,y)[f(t,x)-f(t,y)],\\ \tau^2(x)(u_h(t,x)-k)_+ - \tau^2(y)(u_h(t,y)-k)_+ &= \tilde{\xi}(t,x,y)[f_h(t,x)-f_h(t,y)], \end{aligned}$$

where we define $f(t,x) = \tau^2(x)(u(t,x)-k)$. In fact, it is easy to see that

$$\xi(t,x,y) = \begin{cases} 1, & u(t,x), u(t,y) > k,\\ 0, & u(t,x), u(t,y) \le k,\\ \dfrac{f(t,x)}{f(t,x)-f(t,y)}, & u(t,x) > k \ge u(t,y),\\ \dfrac{f(t,y)}{f(t,y)-f(t,x)}, & u(t,y) > k \ge u(t,x), \end{cases}$$

$$\tilde{\xi}(t,x,y) = \begin{cases} 1, & u_h(t,x), u_h(t,y) > k,\\ 0, & u_h(t,x), u_h(t,y) \le k,\\ \dfrac{f_h(t,x)}{f_h(t,x)-f_h(t,y)}, & u_h(t,x) > k \ge u_h(t,y),\\ \dfrac{f_h(t,y)}{f_h(t,y)-f_h(t,x)}, & u_h(t,y) > k \ge u_h(t,x) \end{cases}$$

have the desired properties. We estimate

$$\begin{aligned} &\left\| \int_{B_{r+\rho}} \int_{B_{r+\rho}} |W(\,\cdot\,,x,y) - \widetilde{W}(\,\cdot\,,x,y)|^2 K_s(x,y)\,\mathrm{d}y\,\mathrm{d}x \right\|_{L^1([t_1,t_2])}^{1/2}\\ &\le 2\left\| \int_{B_{r+\rho}} \int_{B_{r+\rho}} |\tilde{\xi}(\,\cdot\,,x,y)|^2[(f_h(t,x)-f(t,x)) - (f_h(t,y)-f(t,y))]^2 K_s(x,y)\,\mathrm{d}y\,\mathrm{d}x \right\|_{L^1([t_1,t_2])}^{1/2}\\ &\qquad + 2\left\| \int_{B_{r+\rho}} \int_{B_{r+\rho}} |\tilde{\xi}(\,\cdot\,,x,y) - \xi(\,\cdot\,,x,y)|^2 [f(t,x)-f(t,y)]^2 K_s(x,y)\,\mathrm{d}y\,\mathrm{d}x \right\|_{L^1([t_1,t_2])}^{1/2}\\ &\le J_{1,1} + J_{1,2}. \end{aligned}$$

For $J_{1,1}$, note that

$$J_{1,1} \le 2\|f_h - f\|_{L^2([t_1,t_2];V(B_{r+\rho}\,|\,\mathbb{R}^d))} \to 0,$$

where we used that $|\tilde{\xi}| \le 1,\ f \in L^2([t_1,t_2]; V(B_{r+\rho}\,|\,\mathbb{R}^d))$, and (A-2). For $J_{1,2}$. we observe that

$$|\tilde{\xi}(t,x,y) - \xi(t,x,y)| \to 0 \quad \text{as } h \searrow 0 \quad \text{for a.e. } t,x,y.$$

Since $f \in L^2([t_1,t_2]; V(B_{r+\rho}\,|\,\mathbb{R}^d))$, it follows from dominated convergence that $J_{1,2}$ also goes to 0.

For J_2, we estimate

$$
\begin{aligned}
J_2 &\le \int_{t_1}^{t_2}\int_{B_{r+\rho}}\int_{B_{r+\rho}} |V(t,x,y)|\tau^2(x)|(u_h(t,x)-k)_+ - (u(t,x)-k)_+||K_a(x,y)|\,\mathrm{d}y\,\mathrm{d}x\,\mathrm{d}t \\
&\le \|u\|_{L^2([t_1,t_2];V(B_{r+\rho}|\mathbb{R}^d))} \left\| \int_{B_{r+\rho}} |(u_h(\cdot\,,x)-k)_+ - (u(\cdot\,,x)-k)_+|^2 \left(\int_{B_{r+\rho}} \frac{|K_a(x,y)|^2}{J(x,y)}\,\mathrm{d}y\right)\mathrm{d}x \right\|_{L^1([t_1,t_2])}^{1/2} \\
&\le c\|u\|_{L^2([t_1,t_2];V(B_{r+\rho}|\mathbb{R}^d))}\|u_h-u\|_{L^2([t_1,t_2];L^{2\theta'}(B_{r+\rho}))} \\
&\to 0,
\end{aligned}
$$

where we used (K1$_{\mathrm{loc}}$),

$$
|(u_h(t,x)-k)_+ - (u(t,x)-k)_+| \le |u_h(t,x)-u(t,x)|, \quad u \in L^2([t_1,t_2];V(B_{r+\rho}|\mathbb{R}^d)),
$$

(Sob), and (A-2). To prove convergence of J_3, we proceed as follows:

$$
\begin{aligned}
J_3 &\le 2\int_{t_1}^{t_2}\int_{B_{r+\rho}}\int_{B^c_{r+\rho}} |V(t,x,y)|\tau^2(x)|(u_h(t,x)-k)_+ - (u(t,x)-k)_+|K_s(x,y)\,\mathrm{d}y\,\mathrm{d}x\,\mathrm{d}t \\
&\le 2\|u\|_{L^2([t_1,t_2];V(B_{r+\rho}|\mathbb{R}^d))} \left\| \int_{B_{r+\rho/2}} |(u_h(\cdot\,,x)-k)_+ - u(\cdot\,,x)-k)|^2\Gamma^{K_s}(\tau,\tau)(x)\,\mathrm{d}x \right\|_{L^1([t_1,t_2])}^{1/2} \\
&\le c\rho^{-\alpha/2}\|u\|_{L^2([t_1,t_2];V(B_{r+\rho}|\mathbb{R}^d))}\|u_h-u\|_{L^2([t_1,t_2]\times B_{r+\rho})} \\
&\to 0,
\end{aligned}
$$

where we used (cutoff), (A-2), and

$$
u \in L^2([t_1,t_2];V(B_{r+\rho}|\mathbb{R}^d)).
$$

Altogether, this proves (A-4), and we deduce the desired result. Let us now prove (ii). In analogy to the proof of (i), it is only left to show

$$
\int_{t_1}^{t_2} \widehat{\mathcal{E}}(u_h(t),\tau^2(u_h(t)-k)_+)\,\mathrm{d}t \to \int_{t_1}^{t_2} \widehat{\mathcal{E}}(u(t),\tau^2(u(t)-k)_+)\,\mathrm{d}t. \tag{A-8}
$$

We will establish (A-8) by proving the following two properties:

$$
\int_{t_1}^{t_2} |\mathcal{E}(u_h(t)-u(t),\tau^2(u_h(t)-k)_+)|\,\mathrm{d}t \to 0, \tag{A-9}
$$

$$
\int_{t_1}^{t_2} |\mathcal{E}(u(t),\tau^2(u_h-k)_+ - \tau^2(u(t)-k)_+)|\,\mathrm{d}t \to 0. \tag{A-10}
$$

Let us first prove (A-9). In analogy to the proof of (A-5), we split

$$
\begin{aligned}
&\int_{t_1}^{t_2} |\widehat{\mathcal{E}}(u_h(t)-u(t),\tau^2(u_h(t)-k)_+)|\,\mathrm{d}t \\
&\quad\le \|\mathcal{E}^{K_s}_{B_{r+\rho}}(u_h-u,\tau^2(u_h-k)_+)\|_{L^1([t_1,t_2])} + \|\widehat{\mathcal{E}}^{K_a}_{B_{r+\rho}}(u_h-u,\tau^2(u_h-k)_+)\|_{L^1([t_1,t_2])} \\
&\qquad\qquad + \|\widehat{\mathcal{E}}_{(B_{r+\rho}\times B_{r+\rho})^c}(u_h-u,\tau^2(u_h-k)_+)\|_{L^1([t_1,t_2])} \\
&\quad=: \widehat{I_1}+\widehat{I_2}+\widehat{I_3}.
\end{aligned}
$$

In (i), we already showed that $\widehat{I_1} \to 0$. Let us estimate $\widehat{I_2}$ as follows:

$$\begin{aligned}
\widehat{I_2} &\le \|(u_h - u)\widetilde{W}|K_a|\|_{L^1([t_1,t_2])} \\
&\le \|\mathcal{E}^{K_s}_{B_{r+\rho}}(\tau^2(u_h-k)_+, \tau^2(u_h-k)_+)\|^{1/2}_{L^1([t_1,t_2])} \|u_h - u\|_{L^2([t_1,t_2];L^{2\theta'}(B_{r+\frac{\rho}{2}}))} \\
&\le c\|\tau^2 u\|_{L^2([t_1,t_2];V(B_{r+\rho}|\mathbb{R}^d))} \|u_h - u\|_{L^2([t_1,t_2];V(B_{r+\rho}|\mathbb{R}^d))} \\
&\to 0,
\end{aligned}$$

where we used (K1$_{\text{glob}}$), (A-2), (A-7), and (Sob). Moreover, $\widehat{I_3}$ can be treated as follows:

$$\begin{aligned}
\widehat{I_3} &\le \|\mathcal{E}^{K_s}_{(B_{r+\rho}\times B_{r+\rho})^c}(u_h - u, \tau^2(u_h-k)_+)\|_{L^1([t_1,t_2])} \\
&\quad + \left\| \int_{B_{r+\rho/2}} \int_{B^c_{r+\rho}} (u_h(\cdot,x) - u(\cdot,x))\tau^2(x)(u_h(\cdot,x)-k)_+ |K_a(x,y)| \,\mathrm{d}y\,\mathrm{d}x \right\|_{L^1([t_1,t_2])} \\
&\quad + \left\| \int_{B^c_{r+\rho}} \int_{B_{r+\rho/2}} (u_h(\cdot,x) - u(\cdot,x))\tau^2(y)(u_h(\cdot,y)-k)_+ |K_a(x,y)| \,\mathrm{d}y\,\mathrm{d}x \right\|_{L^1([t_1,t_2])} \\
&=: \widehat{I_{3,1}} + \widehat{I_{3,2}} + \widehat{I_{3,3}}.
\end{aligned}$$

The proof of convergence for $\widehat{I_{3,1}}$ goes exactly like for I_3. For $\widehat{I_{3,2}}$, we estimate using the assumptions (K1$_{\text{glob}}$) and (cutoff):

$$\begin{aligned}
\widehat{I_{3,2}} + \widehat{I_{3,3}} &\le \left\| \int_{\mathbb{R}^d} |u_h(\cdot,x) - u(\cdot,x)|^2 \left(\int_{\mathbb{R}^d} \frac{|K_a(x,y)|^2}{J(x,y)} \,\mathrm{d}y \right) \mathrm{d}x \right\|^{1/2}_{L^1([t_1,t_2])} \\
&\qquad \times \left\| \int_{B_{r+\rho/2}} (u_h(\cdot,x)-k)^2_+ \Gamma^J(\tau,\tau)(x)\,\mathrm{d}x \right\|^{1/2}_{L^1([t_1,t_2])} \\
&\le c\rho^{-\alpha/2} \|u_h - u\|_{L^2([t_1,t_2];L^{2\theta'}(\mathbb{R}^d))} \|(u_h - k)_+\|_{L^2([t_1,t_2]\times B_{r+\rho})} \\
&\le c\rho^{-\alpha/2} \|u_h - u\|_{L^2([t_1,t_2];L^{2\theta'}(\mathbb{R}^d))} \|u\|_{L^2([t_1,t_2]\times B_{r+\rho})} \\
&\to 0,
\end{aligned}$$

where we used (A-2), (A-3), and

$$u \in L^2([t_1,t_2]; L^{2\theta'}(\mathbb{R}^d)).$$

We have established (A-9). To prove (A-10), let us again split

$$\begin{aligned}
&\int_{t_1}^{t_2} |\widehat{\mathcal{E}}(u(t), \tau^2(u_h-k)_+ - \tau^2(u(t)-k)_+)| \,\mathrm{d}t \\
&\le \|\mathcal{E}^{K_s}_{B_{r+\rho}}(u, \tau^2(u_h-k)_+ - \tau^2(u-k)_+)\|_{L^1([t_1,t_2])} \\
&\qquad + \|\widehat{\mathcal{E}}^{K_a}_{B_{r+\rho}}(u, \tau^2(u_h-k)_+ - \tau^2(u-k)_+)\|_{L^1([t_1,t_2])} \\
&\qquad\qquad + \|\widehat{\mathcal{E}}_{(B_{r+\rho}\times B_{r+\rho})^c}(u, \tau^2(u_h-k)_+ - \tau^2(u-k)_+)\|_{L^1([t_1,t_2])} \\
&=: \widehat{J_1} + \widehat{J_2} + \widehat{J_3}.
\end{aligned}$$

Note that $\widehat{J_1} = J_1 \to 0$. For $\widehat{J_2}$, we estimate

$$\widehat{J_2} \le \|u\|_{L^2([t_1,t_2];L^{2\theta'}(B_{r+\rho}))} \left\| \int_{B_{r+\rho}} \int_{B_{r+\rho}} |W(\,\cdot\,,x,y) - \widetilde{W}(\,\cdot\,,x,y)|^2 K_s(x,y)\,\mathrm{d}y\,\mathrm{d}x \right\|_{L^1([t_1,t_2])}^{1/2},$$

where we used (K1$_{\text{glob}}$) and that $u \in L^2([t_1,t_2]; L^{2\theta'}(B_{r+\rho}))$. We conclude that $\widehat{J_2} \to 0$ since the second factor converges to zero, as we proved already when dealing with J_1.

To estimate $\widehat{J_3}$, we proceed as follows:

$$\begin{aligned}
\widehat{J_3} &\le \|\mathcal{E}^{K_s}_{(B_{r+\rho}\times B_{r+\rho})^c}(u, \tau^2(u_h-k)_+ - \tau^2(u-k)_+)\|_{L^1([t_1,t_2])} \\
&\qquad + \left\| \int_{B_{r+\rho/2}} \int_{B^c_{r+\rho}} \tau^2(x)|(u_h-k)_+(x) - (u-k)_+(x)|u(x)|K_a(x,y)|\,\mathrm{d}y\,\mathrm{d}x \right\|_{L^1([t_1,t_2])} \\
&\qquad\qquad + \left\| \int_{B^c_{r+\rho}} \int_{B_{r+\rho/2}} \tau^2(y)|(u_h-k)_+(y) - (u-k)_+(y)|u(x)|K_a(x,y)|\,\mathrm{d}y\,\mathrm{d}x \right\|_{L^1([t_1,t_2])} \\
&= \widehat{J}_{3,1} + \widehat{J}_{3,2} + \widehat{J}_{3,3}.
\end{aligned}$$

Note that $\widehat{J}_{3,1} \to 0$ follows similarly to the proof of $J_3 \to 0$. $\widehat{J}_{3,2}$ and $\widehat{J}_{3,3}$ are estimated as follows, using similar arguments as in the estimates of $\widehat{I}_{3,2}$ and $\widehat{I}_{3,3}$:

$$\begin{aligned}
\widehat{J}_{3,2} + \widehat{J}_{3,3} &\le \|(u_h-k)_+ - (u-k)_+\|_{L^2([t_1,t_2];L^{2\theta'}(\mathbb{R}^d))} \left\| \int_{B_{r+\rho/2}} u^2(x)\Gamma^{K_s}(\tau,\tau)(x)\,\mathrm{d}x \right\|_{L^1([t_1,t_2])}^{1/2} \\
&\le c\rho^{-\alpha/2}\|u_h - u\|_{L^2([t_1,t_2];L^{2\theta'}(\mathbb{R}^d))}\|u\|_{L^2([t_1,t_2]\times B_{r+\rho})} \\
&\to 0,
\end{aligned}$$

where we used (cutoff) and (K1$_{\text{glob}}$), as well as (A-2) and $u \in L^2([t_1,t_2]; L^{2\theta'}(B_{r+\rho}))$. This proves (ii). □

Remark A.2. We point out that the above proof can be extended to more general test functions ϕ of the form $\phi = \pm\tau^2 g(u)$, where $g : [0,\infty) \to [0,\infty)$. This way, it would be possible to generalize the notion of a weak solution to (PDE), or to ($\widehat{\text{PDE}}$), in $I \times \Omega$, in the sense that the assumption $\partial_t u \in L^1_{\text{loc}}(I, L^2(\Omega))$, where $\partial_t u$ is the weak $L^2(\Omega)$-derivative of u, can be replaced by $u \in C(I; L^2(\Omega))$.

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TRANSFERENCE OF SCALE-INVARIANT ESTIMATES FROM LIPSCHITZ TO NONTANGENTIALLY ACCESSIBLE TO UNIFORMLY RECTIFIABLE DOMAINS

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In relatively nice geometric settings, in particular, on Lipschitz domains, absolute continuity of elliptic measure with respect to the surface measure is equivalent to Carleson measure estimates, to square function estimates, and to ε-approximability, for solutions to the second-order divergence-form elliptic partial differential equations $Lu = -\operatorname{div}(A\nabla u) = 0$. In more general situations, notably, in an open set Ω with a uniformly rectifiable boundary, absolute continuity of elliptic measure with respect to the surface measure may fail, already for the Laplacian. In the present paper, extending and clarifying our previous work (*Duke Math J.* **165**:12 (2016), 2331–2389), we demonstrate that nonetheless, Carleson measure estimates, square function estimates, and ε-approximability remain valid in such Ω, for solutions of $Lu = 0$, provided that such solutions enjoy these properties in Lipschitz subdomains of Ω.

Moreover, we establish a general real-variable transference principle, from Lipschitz to chord-arc domains, and from chord-arc to open sets with uniformly rectifiable boundary, that is not restricted to harmonic functions or even to solutions of elliptic equations. In particular, this allows one to deduce the first Carleson measure estimates and square function bounds for higher-order systems on open sets with uniformly rectifiable boundaries and to treat subsolutions and subharmonic functions.

1. Introduction

In the setting of a Lipschitz domain $\Omega \subset \mathbb{R}^{n+1}$, $n \geq 1$, for any divergence-form elliptic operator $L = -\operatorname{div}(A\nabla)$ with bounded measurable coefficients, the following are equivalent:

(i) Every bounded solution u, of the equation $Lu = 0$ in Ω, satisfies the *Carleson measure estimate* (see Definition 1.9 with $F = |\nabla u| / \|u\|_{L^\infty(\Omega)}$).

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(ii) Every bounded solution u, of the equation $Lu = 0$ in Ω, is *ε-approximable* for every $\varepsilon > 0$ (see Definition 1.11).

(iii) The elliptic measure associated to L, ω_L, is (quantitatively) absolutely continuous with respect to the Lebesgue measure, $\omega_L \in A_\infty(\sigma)$ on $\partial\Omega$.

(iv) Uniform *square function/nontangential maximal function* ("S/N") estimates hold locally in "sawtooth" subdomains of Ω (see Definition 1.15).

Historically, Dahlberg [1980a] obtained an extension of Garnett's ε-approximability result, observing that (iv) implies (ii) in the harmonic case.[1] The explicit connection of ε-approximability with the A_∞ property of harmonic measure, i.e., that (ii) implies (iii), appears in [Kenig et al. 2000] (where this implication is established not only for the Laplacian, but for general divergence-form elliptic operators). That (iii) implies (iv) is proved for harmonic functions in [Dahlberg 1980b],[2] and, for null solutions of general divergence-form elliptic operators, in [Dahlberg et al. 1984]. Finally, Kenig, Kirchheim, Pipher and Toro [Kenig et al. 2016] have recently shown that (i) implies (iii), whereas, on the other hand, (i) may be seen, via good-lambda and John–Nirenberg arguments, to be equivalent to the local version of one direction of (iv) (the "$S < N$" direction).[3]

The main goal of the present paper is to show that while (iii) may fail on general uniformly rectifiable domains even for harmonic functions [Bishop and Jones 1990], or might be not applicable in the absence of a suitable concept of elliptic measure (e.g., for systems), (i), (ii) and (iv) carry over from Lipschitz domains to the complement of a uniformly rectifiable set. The novelty of the present work lies in the fact that we develop a general transference principle, from Lipschitz domains to chord-arc domains and thence to domains with uniformly rectifiable boundaries, that will allow us to carry out this program by a purely real-variable mechanism. In particular, this both extends and clarifies our previous work [Hofmann et al. 2016]. But let us start with more historical context.

In the past several decades, uniformly rectifiable sets have been identified as the most general geometric setting in which many standard harmonic-analytic properties continue to hold. In particular, it was shown in the early 90's that uniform rectifiability of a set E is equivalent to boundedness of all sufficiently nice singular integral operators with odd kernels in $L^2(E)$ [David and Semmes 1991], and, much more recently, that uniform rectifiability is equivalent to boundedness of the Riesz transform in $L^2(E)$ (see [Mattila et al. 1996] for the case $n = 1$ and [Nazarov et al. 2014] in general).

However, it seemed to be vital for many standard boundary estimates for solutions of elliptic PDEs in a domain Ω that, in addition to uniform rectifiability of its boundary, Ω should possess some additional topological features, ensuring a reasonably nice approach to the boundary. In some respects, this is indeed true. In particular, it has been known that (i)–(iv) hold for harmonic functions on chord-arc domains, that is, nontangentially accessible domains with Ahlfors–David regular boundaries (see Definitions 1.1 and 1.6 below, and [Jerison and Kenig 1982; Dahlberg et al. 1984; David and Jerison 1990]). Such domains

[1] This implication holds more generally for null solutions of divergence-form elliptic equations; see [Kenig et al. 2000; Hofmann et al. 2015].

[2] And thus all four properties hold for harmonic functions in Lipschitz domains, by the result of [Dahlberg 1977].

[3] We will prove this fact in much greater generality in this paper.

satisfy an interior and exterior corkscrew condition (quantitative openness) and a Harnack chain condition (quantitative connectedness). At the same time, the counterexample of [Bishop and Jones 1990] showed that absolute continuity of harmonic measure with respect to the Lebesgue measure (iii) may fail on a general set with a uniformly rectifiable boundary: they construct a one-dimensional (uniformly) rectifiable set E in the complex plane, for which harmonic measure with respect to $\Omega = \mathbb{C} \setminus E$, is singular with respect to Hausdorff H^1 measure on E. Much more recently, under the natural and rather minimal background assumptions that Ω satisfies an interior corkscrew condition, and has an Ahlfors–David regular boundary, quantitative absolute continuity of harmonic measure with respect to surface measure (either property (iii) above, or the weak-A_∞ property, i.e., property (iii) in the absence of doubling), has now been characterized in the harmonic case, thus establishing the necessity of some connectivity assumption in this context: property (iii) (respectively, its weaker nondoubling version) is equivalent to uniform rectifiability of $\partial\Omega$, along with some version of accessibility to the boundary, either the semiuniformity condition of [Aikawa and Hirata 2008] in the doubling case [Azzam 2021], or respectively, the "weak local John condition", which gives access to an ample portion of the boundary, locally, from each interior point of Ω [Azzam et al. 2020]. Thus, while some connectivity is indeed required to obtain property (iii), in [Hofmann et al. 2016] the authors proved that, nonetheless, Carleson measure estimates (i) and ε-approximability (ii) for harmonic functions (and implicitly, for solutions of a certain more general class of elliptic equations) remain valid on all domains with a uniformly rectifiable boundary, in the absence of any connectivity assumption. Shortly thereafter, it was shown that, at least in the presence of interior corkscrew points, each of the necessary properties (i) and (ii) is also *sufficient* for uniform rectifiability [Garnett et al. 2018].

The present paper introduces a new transference mechanism, which illustrates that for certain classes of scale-invariant estimates (e.g., Carleson measure bounds, or square function/nontangential maximal function estimates) the passage from such estimates on Lipschitz domains to analogous results on chord-arc domains and further to the same bounds on all open sets with uniformly rectifiable boundaries is, in fact, a real-variable phenomenon. That is, for a given function F defined in the complement of a codimension 1, uniformly rectifiable set $E \subset \mathbb{R}^{n+1}$, if one has suitable bounds for F on Lipschitz domains, then these automatically carry over to $\mathbb{R}^{n+1} \setminus E$. This immediately gives a series of new results in very general PDE settings (for solutions of second-order elliptic PDEs with coefficients satisfying a Carleson measure condition, for solutions of higher-order systems, for nonnegative subsolutions), but clearly the power of having a general, purely real-variable scheme, goes beyond these applications. Let us now discuss the details. We begin by defining several basic concepts.

Definition 1.1 (ADR). We say that a set $E \subset \mathbb{R}^{n+1}$ is n-dimensional *Ahlfors–David regular* (or simply *ADR*) if it is closed, and if there is some uniform constant $C \geq 1$ such that

$$C^{-1} r^n \leq \sigma(\Delta(x,r)) \leq C r^n \quad \text{for all } r \in (0, \operatorname{diam}(E)),\ x \in E, \tag{1.2}$$

where $\operatorname{diam}(E)$ may be infinite. Here, $\Delta(x,r) := E \cap B(x,r)$ is the surface ball of radius r, and $\sigma := H^n|_E$ is the surface measure on E, where H^n denotes n-dimensional Hausdorff measure.

Definition 1.3 (UR and UR character). An n-dimensional ADR (hence closed) set $E \subset \mathbb{R}^{n+1}$ is n-dimensional *uniformly rectifiable* (or simply *UR*) if and only if it contains *big pieces of Lipschitz images* of $\mathbb{R}^n$ (*BPLI*). This means that there are positive constants θ, $M_0 > 1$ such that for each $x \in E$ and each $r \in (0, \operatorname{diam}(E))$ there is a Lipschitz mapping $\rho = \rho_{x,r} : \mathbb{R}^n \to \mathbb{R}^{n+1}$, with Lipschitz constant no larger than M_0, such that

$$H^n\big(E \cap B(x,r) \cap \rho(\{z \in \mathbb{R}^n : |z| < r\})\big) \geq \theta^{-1} r^n.$$

Additionally, the *UR character* of E is just the triple of constants (θ, M_0, C), where C is the ADR constant; or equivalently, the quantitative bounds involved in any particular characterization of uniform rectifiability.

Note that, in particular, a UR set is closed by definition, so that $\mathbb{R}^{n+1} \setminus E$ is open, but need not be connected.

We recall that n-dimensional rectifiable sets are characterized by the property that they can be covered, up to a set of H^n-measure 0, by a countable union of Lipschitz images of $\mathbb{R}^n$; we observe that BPLI is a quantitative version of this fact.

It is worth mentioning that there exist sets that are ADR (and that even form the boundary of an open set satisfying interior corkscrew and Harnack chain conditions), but that are totally nonrectifiable (e.g., see the construction of Garnett's "4-corners Cantor set" in [David and Semmes 1993, Chapter 1]).

Definition 1.4 (corkscrew condition). Following [Jerison and Kenig 1982], we say that an open set $\Omega \subset \mathbb{R}^{n+1}$ satisfies the *corkscrew condition* if for some uniform constant $C > 1$ and for every surface ball $\Delta := \Delta(x,r) = B(x,r) \cap \partial\Omega$, with $x \in \partial\Omega$ and $0 < r < \operatorname{diam}(\partial\Omega)$, there is a ball $B(X_\Delta, C^{-1}r) \subset B(x,r) \cap \Omega$. The point $X_\Delta \subset \Omega$ is called a *corkscrew point* relative to Δ. We note that we may allow $r < C' \operatorname{diam}(\partial\Omega)$ for any fixed C' simply by adjusting the constant C.

Definition 1.5 (Harnack chain condition). Again following [Jerison and Kenig 1982], we say that an open set Ω satisfies the *Harnack chain condition* if there is a uniform constant $C \geq 1$ such that for every pair of points $X, X' \in \Omega$ there is a chain of balls $B_1, B_2, \dots, B_N \subset \Omega$ with

$$N \leq C\left(2 + \log_2^+ \frac{|X - X'|}{\min\{\operatorname{dist}(X, \partial\Omega), \operatorname{dist}(X', \partial\Omega)\}}\right),$$

$X \in B_1$, $X' \in B_N$, $B_k \cap B_{k+1} \neq \varnothing$ for every $1 \leq k \leq N-1$, and $C^{-1} \operatorname{diam}(B_k) \leq \operatorname{dist}(B_k, \partial\Omega) \leq C \operatorname{diam}(B_k)$ for every $1 \leq k \leq N$. The chain of balls is called a *Harnack chain*. We remark that in general, the estimate for N can be worse than logarithmic, but as is well known, in the presence of an interior corkscrew condition, it is necessarily logarithmic if it holds at all.

Definition 1.6 (NTA, 1-sided NTA, CAD, and 1-sided CAD). We say that an open set $\Omega \subset \mathbb{R}^{n+1}$ is 1-*sided nontangentially accessible* (or simply 1-*sided NTA*) if it satisfies the Harnack chain condition, and Ω satisfies the (interior) corkscrew condition. Additionally, the 1-*sided NTA character* of Ω is just the collection of constants involved in the fact that Ω is 1-sided NTA, that is, the (interior) corkscrew constant, as well as the constant from the Harnack chain condition.

As in [Jerison and Kenig 1982], we say that an $\Omega \subset \mathbb{R}^{n+1}$ is *nontangentially accessible* (or simply *NTA*) if it satisfies the Harnack chain condition, and if both Ω and $\Omega_{\text{ext}} := \mathbb{R}^{n+1} \setminus \overline{\Omega}$ satisfy the corkscrew condition. The *NTA character* of Ω is the collection of constants involved in the fact that Ω is NTA, that is, the interior and exterior corkscrew constants, as well as the constant from the Harnack chain condition.

We say that an open set $\Omega \subset \mathbb{R}^{n+1}$ is a 1-*sided chord-arc domain*, or simply 1-*sided CAD*, (resp. *chord-arc domain*, or simply *CAD*) if it is 1-sided NTA (resp. NTA) and has ADR boundary. The 1-*sided CAD character* (resp. *CAD character*) is the 1-sided NTA character (resp. NTA character) together with the ADR constant.

Definition 1.7 (Lipschitz graph domain). We say that $\Omega \subset \mathbb{R}^{n+1}$ is a *Lipschitz graph domain* if there is some Lipschitz function $\psi : \mathbb{R}^n \to \mathbb{R}$ and some coordinate system such that

$$\Omega = \{(x', t) : \ x' \in \mathbb{R}^n, \ t > \psi(x')\}.$$

We refer to $M = \|\nabla\psi\|_{L^\infty(\mathbb{R}^n)}$ as the *Lipschitz constant* of Ω.

Definition 1.8 (bounded Lipschitz domain). We say that an open connected set $\Omega \subset \mathbb{R}^{n+1}$ is a *bounded Lipschitz domain* if there exist $r_\Omega > 0, \ M, C_0, m \geq 1, \ \{x_j\}_{j=1}^m \subset \partial\Omega, \ \{r_j\}_{j=1}^m$, with $C_0^{-1} r_\Omega < r_j < C_0 r_\Omega$ for every $1 \leq j \leq m$, such that the following conditions hold. First, $\partial\Omega \subset \bigcup_{j=1}^m B(x_j, r_j)$. Second, for each $1 \leq j \leq m$ there is some Lipschitz graph domain V_j, with $x_j \in \partial V_j$ and with Lipschitz constant at most M, such that $U_j \cap \Omega = U_j \cap V_j$, where U_j is a cylinder of height $8(M+1)r_j$, radius $2r_j$, and with axis parallel to the t-axis (in the coordinates associated with V_j). We refer to the triple (M, m, C_0) as the *Lipschitz character* of Ω.

As we pointed out above and as can be seen from the definitions, nontangentially accessible domains possess certain quantitative topological features. One can show that a CAD satisfies a property analogous to Definition 1.3, but using big pieces of Lipschitz subdomains, rather than big pieces of Lipschitz images (see Proposition 3.20), the crucial difference being that in some sense a nice access to the boundary of a Lipschitz domain is retained, contrary to the general UR case.

Finally, let us define the scale-invariant estimates at the center of this paper.

Definition 1.9 CME. Let $\Omega \subset \mathbb{R}^{n+1}$ be an open set and let $F \in L^2_{\mathrm{loc}}(\Omega)$. We say that F satisfies the *Carleson measure estimate* (or simply *CME*) on Ω if

$$\|F\|_{\mathrm{CME}(\Omega)} := \sup_{x \in \partial\Omega, \, 0<r<\infty} \frac{1}{r^n} \iint_{B(x,r)\cap\Omega} |F(Y)|^2 \operatorname{dist}(Y, \partial\Omega) \, dY < \infty. \tag{1.10}$$

Definition 1.11 (ε-approximable). Let $\Omega \subset \mathbb{R}^{n+1}$ be an open set. Let $u \in L^\infty(\Omega)$, with $\|u\|_{L^\infty(\Omega)} \leq 1$, and let $\varepsilon \in (0, 1)$. We say that u is ε-*approximable* on Ω if there is a constant C_ε and a function $\varphi = \varphi^\varepsilon \in W^{1,1}_{\mathrm{loc}}(\Omega)$ satisfying

$$\|u - \varphi\|_{L^\infty(\Omega)} < \varepsilon \tag{1.12}$$

and

$$\sup_{x \in \partial\Omega, \, 0<r<\infty} \frac{1}{r^n} \iint_{B(x,r)\cap\Omega} |\nabla\varphi(Y)| \, dY \leq C_\varepsilon. \tag{1.13}$$

Let Ω be an open set. The cone with vertex at $x \in \partial\Omega$ and aperture $\kappa > 0$ is defined as

$$\Gamma_\Omega(x) := \Gamma_{\Omega,\kappa}(x) := \{Y \in \Omega \cap B(x, r) : |Y - x| \leq (1+\kappa) \operatorname{dist}(Y, \partial\Omega)\}, \qquad x \in \partial\Omega. \tag{1.14}$$

Given $r > 0$, we write $\Gamma^r_\Omega(x) := \Gamma_\Omega(x) \cap B(x, r)$ for the truncated cone. With a slight abuse of notation if Ω is unbounded and $\partial\Omega$ bounded, our cones will be truncated. More precisely, in that scenario, we will write $\Gamma_\Omega(\,\cdot\,)$ to denote $\Gamma_\Omega^{C\,\mathrm{diam}(\partial\Omega)}(\,\cdot\,)$, where $C \geq 2$ is a fixed harmless constant. In this way, when $\partial\Omega$ is bounded, so are the cones, all being contained in a $C'\,\mathrm{diam}(\partial\Omega)$-neighborhood of $\partial\Omega$. We will sometimes refer to these cones as "traditional" to distinguish them from some dyadic cones which will be introduced later; see (2.23).

Definition 1.15 (nontangential maximal function, area integral, and square function). Let Ω be an open set. For $H \in C(\Omega)$ (i.e., H is a continuous function in Ω) we define the *nontangential maximal function* as

$$N_{*,\Omega}H(x) := N_{*,\Omega,\kappa}H(x) := \sup_{Y\in\Gamma_{\Omega,\kappa}(x)} |H(Y)|, \qquad x \in \partial\Omega; \tag{1.16}$$

for $G \in L^2_{\mathrm{loc}}(\Omega)$, we define the *area integral* as

$$\mathcal{A}_\Omega G(x) := \mathcal{A}_{\Omega,\kappa}G(x) := \left(\iint_{\Gamma_{\Omega,\kappa}(x)} |G(Y)|^2\,\mathrm{dist}(Y,\partial\Omega)^{1-n}\,dY\right)^{\frac12}, \qquad x \in \partial\Omega; \tag{1.17}$$

and, for $u \in W^{1,2}_{\mathrm{loc}}(\Omega)$, we define the *square function* as

$$S_\Omega u(x) := S_{\Omega,\kappa}u(x) := \left(\iint_{\Gamma_{\Omega,\kappa}(x)} |\nabla u(Y)|^2\,\mathrm{dist}(Y,\partial\Omega)^{1-n}\,dY\right)^{\frac12}, \qquad x \in \partial\Omega. \tag{1.18}$$

For any $r > 0$, we write $N^r_{*,\Omega}$, $\mathcal{A}^r_\Omega$, and S^r_Ω to denote the *truncated* nontangential maximal function, area integral, and square function respectively, where $\Gamma_\Omega(\,\cdot\,)$ is replaced by the truncated cone $\Gamma^r_\Omega(\,\cdot\,)$.

Let us now list some highlights of the main results of this paper (see Corollary 3.1, Theorem 3.31 and Theorem 3.6 for the precise statements in the body of the paper and also Notation 2.56). First, we have that Carleson measure estimates on Lipschitz domains imply Carleson measure estimates in CAD, which, in turn, imply Carleson measure estimates on the sets with UR boundaries, via the following formalism.

Theorem 1.19 (transference of Carleson measure estimates[4]). (i) *Let $D \subset \mathbb{R}^{n+1}$ be a chord-arc domain and $F \in L^2_{\mathrm{loc}}(D)$. If F satisfies the Carleson measure estimate on all bounded Lipschitz subdomains of D then F satisfies the Carleson measure estimate on D as well.*

(ii) *Let $E \subset \mathbb{R}^{n+1}$ be an n-dimensional uniformly rectifiable set and let $F \in L^2_{\mathrm{loc}}(\mathbb{R}^{n+1}\setminus E)$. If F satisfies the Carleson measure estimate on all bounded chord-arc subdomains of $\mathbb{R}^{n+1}\setminus E$, then F satisfies the Carleson measure estimate on $\mathbb{R}^{n+1}\setminus E$ as well.*

(iii) *Let $E \subset \mathbb{R}^{n+1}$ be an n-dimensional uniformly rectifiable set and let $F \in L^2_{\mathrm{loc}}(\mathbb{R}^{n+1}\setminus E)$. If F satisfies the Carleson measure estimate on all bounded Lipschitz subdomains of $\mathbb{R}^{n+1}\setminus E$, then F satisfies the Carleson measure estimate on $\mathbb{R}^{n+1}\setminus E$ as well.*

[4] In the statement we have omitted the dependence in the Carleson estimates on the various geometric parameters. The precise statements (see Theorems 3.31 and 3.6) given in the body of the paper impose that the Carleson measure estimates hold for any bounded Lipschitz (resp. chord-arc) subdomain with a bound depending on the Lipschitz (resp. CAD) character. The latter means that for all subdomains with Lipschitz (resp. CAD) character controlled by some uniform quantity, say M, the corresponding Carleson measure estimates hold with an associated uniform constant depending on M. The conclusions should also include that the resulting Carleson estimates depend on the CAD character of D (resp. UR character of E), as well as on the Carleson estimates of F in the subdomains.

We remark that Theorem 1.19(ii) was already implicit in our previous work [Hofmann et al. 2016], although in the present paper we give a proof of this result that is simpler than the corresponding argument there. The main new ingredient in Theorem 1.19 is part (i); part (iii) is an immediate corollary of parts (i) and (ii).

Next, in the class of open sets with UR or ADR boundary, or in the class of chord-arc domains or 1-sided chord-arc domains, the Carleson measure estimates are equivalent to local and global area integral bounds (aka square function estimates).

Theorem 1.20. *Let $\Omega \subset \mathbb{R}^{n+1}$ be an open set with ADR boundary and suppose that we have a collection $\{\Omega'\}_{\Omega' \in \Sigma}$ such that each $\Omega' \in \Sigma$ is an open subset of Ω, $\partial\Omega'$ is ADR boundary, and also that all of its local sawtooth subdomains (see Section 2) belong to Σ. Let $G \in L^2_{\mathrm{loc}}(\Omega)$ and $H \in C(\Omega)$ and assume that*

$$\left(\frac{1}{r^n}\iint_{B(X,r)} |G(Y)|^2 \delta(Y)\,dY\right)^{\frac{1}{2}} \leq C\|H\|_{L^\infty(B(X,2r))} \quad \text{for all } B(X,2r) \subset \Omega.$$

The following statements are equivalent:

(i) $\|G\|_{\mathrm{CME}(\Omega')} \lesssim \|H\|^2_{L^\infty(\Omega')}$ *for all* $\Omega' \in \Sigma$.

(ii) $\|\mathcal{A}_{\Omega'}G\|_{L^q(\partial\Omega')} \leq C\|N_{*,\Omega'}H\|_{L^q(\partial\Omega')}$ *for all* $\Omega' \in \Sigma$ *and for some* $0 < q < \infty$.

(iii) $\|\mathcal{A}_{\Omega'}G\|_{L^q(\partial\Omega')} \leq C\|N_{*,\Omega'}H\|_{L^q(\partial\Omega')}$ *for all* $\Omega' \in \Sigma$ *and for all* $0 < q < \infty$.

This result is a particular case of Theorem 4.8 (and Remarks 4.20, 2.37, and 2.38), which actually contains considerably more detailed statements, as well as equivalence to local area integral bounds.

Finally, we discuss transference for the converse bounds on nontangential maximal function in terms of the square function and their connection with ε-approximability. In this context, one has to tie up explicitly the arguments of $\mathcal{A}$ and N_*. Our first result is a reduced version of the combination of Theorems 5.1 and 5.24 stated in Corollary 5.50. We do not explain in detail conditions (5.2) and (5.25) now, but let us mention that, generally, they are harmless bounds on interior cubes, which, in the context of solutions of elliptic PDE follow from well-known interior estimates.

Theorem 1.21. *Let $D \subset \mathbb{R}^{n+1}$ be a chord-arc domain. Let $u \in W^{1,2}_{\mathrm{loc}}(D) \cap C(D)$ so that (5.2) and (5.25) hold for some $p > 2$. Assume that for every bounded Lipschitz subdomain $\Omega \subset \mathbb{R}^{n+1} \setminus E$*

$$\|N_{*,\Omega}(u - u(X^+_\Omega))\|_{L^2(\partial\Omega)} \leq C\|S_\Omega u\|_{L^2(\partial\Omega)} \tag{1.22}$$

holds with a constant depending on n and the Lipschitz character of Ω, and where X^+_Ω is any interior corkscrew point of Ω at the scale of $\mathrm{diam}(\partial\Omega)$. *Then, for every $\kappa > 0$, if ∂D is bounded*

$$\|N_{*,D,\kappa}(u - u(X^+_D))\|_{L^q(\partial D)} \leq C'\|S_{D,\kappa}u\|_{L^q(\partial D)} \quad \text{for all } 0 < q < \infty,$$

and if ∂D is unbounded and $u(X) \to 0$ as $|X| \to \infty$ then

$$\|N_{*,D,\kappa}u\|_{L^q(\partial D)} \leq C'\|S_{D,\kappa}u\|_{L^q(\partial D)} \quad \text{for all } 0 < q < \infty,$$

where C' depends on q, n, the CAD character of D, the implicit constants in (5.2) and (5.25), the constant C in (1.22), and κ; and where X^+_D is any interior corkscrew point of D at the scale of $\mathrm{diam}(\partial D)$.

We mention one further result that is stated in full detail below as Theorem 6.1. The interior bound (6.2) is, again, a fairly harmless prerequisite which follows from known interior estimates in the context of solutions of elliptic PDEs. We remark that the estimate (1.24) itself (see below) would not make much sense for general uniformly rectifiable sets because of topological obstructions (there is no preferred component for a corkscrew point in such a general context), and for that reason we pass directly to ε-approximability.

Theorem 1.23. *Let $E \subset \mathbb{R}^{n+1}$ be an n-dimensional uniformly rectifiable, and suppose that*

$$u \in W^{1,2}_{\mathrm{loc}}(\mathbb{R}^{n+1} \setminus E) \cap C(\mathbb{R}^{n+1} \setminus E) \cap L^\infty(\mathbb{R}^{n+1} \setminus E)$$

satisfies (6.2). Assume, in addition, that

$$\|\nabla u\|_{\mathrm{CME}(\mathbb{R}^{n+1}\setminus E)} \le C_0' \|u\|_{L^\infty(\mathbb{R}^{n+1}\setminus E)}$$

and that for every bounded chord-arc subdomain $\Omega \subset \mathbb{R}^{n+1} \setminus E$

$$\|N_{*,\Omega}(u - u(X_\Omega^+))\|_{L^2(\partial\Omega)} \le C \|S_\Omega u\|_{L^2(\partial\Omega)} \tag{1.24}$$

holds with a constant depending on n and the CAD character of Ω, and where X_Ω^+ is any interior corkscrew point of Ω at the scale of $\mathrm{diam}(\partial\Omega)$. *Then u is ε-approximable on $\mathbb{R}^{n+1} \setminus E$, with the implicit constants depending on n, the UR character of E, the constant in (6.2) and in C_0'.*

Theorem 1.23/Theorem 6.1 is simply a formalization of results that were implicit in [Hofmann et al. 2016], and we state it here, without proof, for the record.

Let us reiterate that the fact that our results provide a "black box" real-variable transference principle allows one to use them considerably beyond the traditional scope. We can treat, for instance, subsolutions and supersolutions of elliptic equations. Another example is higher-order elliptic systems. The best available results to date in this context are restricted to Lipschitz domains [Dahlberg et al. 1997]. Here we establish, for instance, the following estimates.

Let $K, m \in \mathbb{N}$. Let E be an n-dimensional uniformly rectifiable set and u be a weak solution to the system

$$Lu = \sum_{k=1}^{K} \sum_{|\alpha|=|\beta|=m} a^{jk}_{\alpha\beta} \partial^\alpha \partial^\beta u^k = 0, \qquad j = 1, \dots, K,$$

on $\mathbb{R}^{n+1} \setminus E$. Here, $a^{jk}_{\alpha\beta}$, $1 \le \alpha, \beta \le n+1$, $1 \le j, k \le K$, $\alpha = (\alpha_1, \dots, \alpha_{n+1}) \in \mathbb{N}_0^{n+1}$ are real constant symmetric coefficients satisfying the Legendre–Hadamard ellipticity condition (see (7.18)). Then u satisfies the $S < N$ estimates in $\mathbb{R}^{n+1} \setminus E$, that is,

$$\|S_{\mathbb{R}^{n+1}\setminus E}(\nabla^{m-1} u)\|_{L^p(E)} \le C \|N_{*,\mathbb{R}^{n+1}\setminus E}(|\nabla^{m-1} u|)\|_{L^p(E)}, \qquad 0 < p < \infty.$$

Furthermore, if $D \subset \mathbb{R}^{n+1}$ is a chord-arc domain with an unbounded boundary and $\nabla^{m-1} u$ vanishes at infinity, we also have the converse estimate

$$\|N_{*,D}(\nabla^{m-1} u)\|_{L^q(\partial D)} \le C \|S_D(\nabla^{m-1} u)\|_{L^q(\partial D)} \quad \text{for all } 0 < q < \infty.$$

Similar results are valid locally and on bounded domains. We also obtain a version of ε-approximability and Carleson measure estimates in this general context. The reader can consult Section 7 for a detailed discussion of these results and other applications.

Let us conclude this introduction with an outline of the organization of the paper. In Section 2, we develop some preliminary material, including notation and definitions, and we state some known results that will be useful in the sequel.

In Section 3, we give the proof of Theorem 1.19, showing first that Carleson measure estimates ("CME") may be transferred from Lipschitz subdomains to chord-arc domains (part (i)), and then from chord-arc subdomains to the complement of a uniformly rectifiable set (part (ii)). As noted above, the first step is new, while the second step is a very general version of a result whose proof was implicit in [Hofmann et al. 2016], established here by a simpler argument than in that work. These results, along with those in Section 5, comprise the main new contributions of the paper (although some of our applications in Section 7 are also novel).

In Section 4, we prove Theorem 1.20/Theorem 4.8, in which, using the well-known technique of good-λ inequalities, we show that abstract versions of CME are equivalent to abstract versions of so-called "$S < N$" bounds (in the generality that we consider here, the notation $\mathcal{A} < N$ seems more appropriate), which express the control of a square function by a nontangential maximal function, in L^p norm.

In Section 5, we consider the reverse "$N < S$" bounds (see Theorem 1.21 above), and show that these may be transferred from Lipschitz subdomains to chord-arc domains.

In Section 6, we state a detailed version of Theorem 1.23.

We note that the results in Sections 3–6 are of a purely real-variable nature, and we do not assume, per se, that we are dealing with solutions (or sub/supersolutions) of a PDE, although at certain points we do impose abstract versions of Caccioppoli's inequality and/or Moser's local boundedness.

Finally, in Section 7, we present several PDE applications of our abstract results.

2. Preliminaries

We start with some further notation and definitions.

- We use the letters c, C to denote harmless positive constants, not necessarily the same at each occurrence, which depend only on dimension and the constants appearing in the hypotheses of the theorems (which we refer to as the "allowable parameters"). We shall also sometimes write $a \lesssim b$ and $a \approx b$ to mean, respectively, that $a \leq Cb$ and $0 < c \leq a/b \leq C$, where the constants c and C are as above, unless explicitly noted to the contrary. At times, we shall designate by M a particular constant whose value will remain unchanged throughout the proof of a given lemma or proposition, but which may have a different value during the proof of a different lemma or proposition.

- Given a closed set $E \subset \mathbb{R}^{n+1}$, we shall use lower case letters x, y, z, etc., to denote points on E, and capital letters X, Y, Z, etc., to denote generic points in $\mathbb{R}^{n+1}$ (especially those in $\mathbb{R}^{n+1} \setminus E$).

- The open $(n+1)$-dimensional Euclidean ball of radius r will be denoted by $B(x, r)$ when the center x lies on E, or $B(X, r)$ when the center X lies in $\mathbb{R}^{n+1} \setminus E$. A surface ball is denoted by $\Delta(x, r) := B(x, r) \cap E$ where unless otherwise specified we implicitly assume that $x \in E$.

- Given a Euclidean ball B or surface ball Δ, its radius will be denoted r_B or r_Δ, respectively.

• Given a Euclidean or surface ball $B = B(X, r)$ or $\Delta = \Delta(x, r)$, its concentric dilate by a factor of $\kappa > 0$ will be denoted by $\kappa B := B(X, \kappa r)$ or $\kappa\Delta := \Delta(x, \kappa r)$.

• Given a (fixed) closed set $E \subset \mathbb{R}^{n+1}$, for $X \in \mathbb{R}^{n+1}$, we set $\delta(X) := \operatorname{dist}(X, E)$.

• We let H^n denote n-dimensional Hausdorff measure, and let $\sigma := H^n|_E$ denote the "surface measure" on E.

• We will also work with open sets $\Omega \subset \mathbb{R}^{n+1}$ in which case the previous notation and definitions easily adapt by letting $E := \partial\Omega$.

• For a Borel set $A \subset \mathbb{R}^{n+1}$, we let 1_A denote the usual indicator function of A, i.e., $1_A(x) = 1$ if $x \in A$, and $1_A(x) = 0$ if $x \notin A$.

• For a Borel set $A \subset \mathbb{R}^{n+1}$, we let $\operatorname{int}(A)$ denote the interior of A.

• Given a Borel measure μ, and a Borel set A, with positive and finite μ measure, we set $⨍_A f\, d\mu := \mu(A)^{-1}\int_A f\, d\mu$.

• We shall use the letter I (and sometimes J) to denote a closed $(n+1)$-dimensional Euclidean dyadic cube with sides parallel to the coordinate axes, and we let $\ell(I)$ denote the side length of I. If $\ell(I) = 2^{-k}$, then we set $k_I := k$. Given an ADR set $E \subset \mathbb{R}^{n+1}$, we use Q to denote a dyadic "cube" on E. The latter exist (see [David and Semmes 1991; Christ 1990]) and enjoy certain properties which we enumerate in Lemma 2.1 below.

Lemma 2.1 (existence and properties of the "dyadic grid" [David and Semmes 1991; 1993; Christ 1990]). *Suppose that $E \subset \mathbb{R}^{n+1}$ is an n-dimensional ADR set. Then there exist constants $a_0 > 0$, $\gamma > 0$ and $C_1 < \infty$, depending only on dimension and the ADR constant, such that for each $k \in \mathbb{Z}$ there is a collection of Borel sets ("cubes")*

$$\mathbb{D}_k := \{Q_j^k \subset E : j \in \mathfrak{I}_k\},$$

where $\mathfrak{I}_k$ denotes some (possibly finite) index set depending on k, satisfying:

(i) $E = \bigcup_j Q_j^k$ *for each* $k \in \mathbb{Z}$.

(ii) *If* $m \geq k$ *then either* $Q_i^m \subset Q_j^k$ *or* $Q_i^m \cap Q_j^k = \varnothing$.

(iii) *For each* (j, k) *and each* $m < k$, *there is a unique* i *such that* $Q_j^k \subset Q_i^m$.

(iv) $\operatorname{diam}(Q_j^k) \leq C_1 2^{-k}$.

(v) *Each* Q_j^k *contains some "surface ball"* $\Delta(x_j^k, a_0 2^{-k}) := B(x_j^k, a_0 2^{-k}) \cap E$.

(vi) $H^n(\{x \in Q_j^k : \operatorname{dist}(x, E \setminus Q_j^k) \leq \varrho 2^{-k}\}) \leq C_1 \varrho^\gamma H^n(Q_j^k)$ *for all* k, j *and for all* $\varrho \in (0, a_0)$.

A few remarks are in order concerning this lemma.

• In the setting of a general space of homogeneous type, this lemma has been proved in [Christ 1990], with the dyadic parameter $\frac{1}{2}$ replaced by some constant $\delta \in (0, 1)$. In fact, one may always take $\delta = \frac{1}{2}$ (see [Hofmann et al. 2017b, proof of Proposition 2.12]). In the presence of the Ahlfors–David property (1.2), the result already appears in [David and Semmes 1991; 1993].

• For our purposes, we may ignore those $k \in \mathbb{Z}$ such that $2^{-k} \gtrsim \operatorname{diam}(E)$, in the case that the latter is finite.

• We shall denote by $\mathbb{D} = \mathbb{D}(E)$ the collection of all relevant Q_j^k, i.e.,

$$\mathbb{D} := \bigcup_k \mathbb{D}_k,$$

where, if $\operatorname{diam}(E)$ is finite, the union runs over those k such that $2^{-k} \lesssim \operatorname{diam}(E)$. When E is bounded, there exists a cube $Q_0 \in \mathbb{D}(\partial\Omega)$ such that $Q_0 = \partial\Omega$ and $Q \in \mathbb{D}_{Q_0}$ for any $Q \in \mathbb{D}(\partial\Omega)$.

• For a dyadic cube $Q \in \mathbb{D}_k$, we shall set $\ell(Q) = 2^{-k}$, and we shall refer to this quantity as the "length" of Q. Evidently, $\ell(Q) \approx \operatorname{diam}(Q)$.

• For a dyadic cube $Q \in \mathbb{D}$, we let $k(Q)$ denote the "dyadic generation" to which Q belongs, i.e., we set $k = k(Q)$ if $Q \in \mathbb{D}_k$; thus, $\ell(Q) = 2^{-k(Q)}$.

• Given $Q \in \mathbb{D}$ we write $\widetilde{Q}$ to denote the dyadic parent of Q, that is, the unique dyadic cube $\widetilde{Q}$ with $Q \subset \widetilde{Q}$ and $\ell(\widetilde{Q}) = 2\ell(Q)$. Also, the children of Q are the dyadic cubes $Q' \subset Q$ with $\ell(Q') = \ell(Q)/2$.

• Properties (iv) and (v) imply that, for each cube $Q \in \mathbb{D}$, there is a point $x_Q \in E$, a Euclidean ball $B(x_Q, r)$ and a surface ball $\Delta(x_Q, r) := B(x_Q, r) \cap E$ such that $c\ell(Q) \le r \le \ell(Q)$ for some uniform constant $0 < c < 1$ and

$$\Delta(x_Q, 2r) \subset Q \subset \Delta(x_Q, Cr) \tag{2.2}$$

for some uniform constant C. We shall denote this ball and surface ball by

$$B_Q := B(x_Q, r), \quad \Delta_Q := \Delta(x_Q, r), \tag{2.3}$$

and we shall refer to the point x_Q as the "center" of Q.

Definition 2.4. Let $E \subset \mathbb{R}^{n+1}$ be an n-dimensional ADR set. By $M^{\mathbb{D}} = M^{\mathbb{D}(E)}$ we denote the dyadic Hardy–Littlewood maximal function on E, that is, for $f \in L^1_{\mathrm{loc}}(E)$

$$M^{\mathbb{D}} f(x) = \sup_{x \in Q \in \mathbb{D}(E)} \fint_Q |f(y)| \, d\sigma(y),$$

and, for $0 < p < \infty$, we also write $M_p^{\mathbb{D}} f = M^{\mathbb{D}}(|f|^p)^{1/p}$. Analogously, if $Q_0 \in \mathbb{D}(E)$, we write $M_{Q_0}^{\mathbb{D}}$ for the dyadic Hardy–Littlewood maximal function localized to Q_0,

$$M_{Q_0}^{\mathbb{D}} f(x) = \sup_{x \in Q \in \mathbb{D}_{Q_0}} \fint_Q |f(y)| \, d\sigma(y),$$

where $\mathbb{D}_{Q_0}(E) = \{Q \in \mathbb{D}(E) : Q \subset Q_0\}$, and, for $0 < p < \infty$, we also write $M_{Q_0,p}^{\mathbb{D}} f = M_{Q_0}^{\mathbb{D}}(|f|^p)^{1/p}$.

Let $\Omega \subset \mathbb{R}^{n+1}$ be an open set so that $\partial\Omega$ is ADR. Let $\mathcal{W} = \mathcal{W}(\Omega)$ denote a collection of (closed) dyadic Whitney cubes of Ω, so that the cubes in $\mathcal{W}$ form a pairwise nonoverlapping covering of Ω, which satisfy

$$4 \operatorname{diam}(I) \le \operatorname{dist}(4I, E) \le \operatorname{dist}(I, \partial\Omega) \le 40 \operatorname{diam}(I) \quad \text{for all } I \in \mathcal{W} \tag{2.5}$$

(just dyadically divide the standard Whitney cubes, as constructed in [Stein 1970, Chapter VI], into cubes with side length $\frac{1}{8}$ as large) and also

$$\tfrac{1}{4}\operatorname{diam}(I_1) \le \operatorname{diam}(I_2) \le 4\operatorname{diam}(I_1),$$

whenever I_1 and I_2 touch.

Next, we choose a small parameter $0 < \tau_0 < 2^{-4}$ (depending only on dimension), so that for any $I \in \mathcal{W}$, and any $\tau \in (0, \tau_0]$, the concentric dilate $I^*(\tau) := (1+\tau)I$ still satisfies the Whitney property

$$\operatorname{diam} I \approx \operatorname{diam} I^*(\tau) \approx \operatorname{dist}(I^*(\tau), \partial\Omega) \approx \operatorname{dist}(I, \partial\Omega), \qquad 0 < \tau \le \tau_0. \tag{2.6}$$

Moreover, for $\tau \le \tau_0$ small enough, and for any $I, J \in \mathcal{W}$, we have that $I^*(\tau)$ meets $J^*(\tau)$ if and only if I and J have a boundary point in common, and that, if $I \ne J$, then $I^*(\tau)$ misses $3J/4$.

Definition 2.7 (Whitney-dyadic structure). Let $\Omega \subset \mathbb{R}^{n+1}$ be an open set so that $\partial\Omega$ is ADR. Let $\mathcal{W} = \mathcal{W}(\Omega)$ denote a collection of (closed) dyadic Whitney cubes of Ω as in (2.5). Let $\mathbb{D} = \mathbb{D}(\partial\Omega)$ be the collection of dyadic cubes from Lemma 2.1 and given the parameters $\eta < 1$ and $K > 1$, set

$$\mathcal{W}_Q^0 := \{I \in \mathcal{W} : \eta^{\frac{1}{4}}\ell(Q) \le \ell(I) \le K^{\frac{1}{2}}\ell(Q),\ \operatorname{dist}(I, Q) \le K^{\frac{1}{2}}\ell(Q)\}, \tag{2.8}$$

A *Whitney-dyadic structure* for Ω with parameters η and K is a family $\{\mathcal{W}_Q\}_{Q\in\mathbb{D}} \subset \mathcal{W}$ satisfying the following conditions:

(i) $\mathcal{W}_Q^0 \ne \varnothing$ for every $Q \in \mathbb{D}$.

(ii) $\mathcal{W}_Q^0 \subset \mathcal{W}_Q$ for every $Q \in \mathbb{D}$.

(iii) There exists $C \ge 1$ such that, for every $Q \in \mathbb{D}$,

$$\begin{aligned} C^{-1}\eta^{\frac{1}{2}}\ell(Q) \le \ell(I) \le CK^{\frac{1}{2}}\ell(Q) \quad &\text{for all } I \in \mathcal{W}_Q, \\ \operatorname{dist}(I, Q) \le CK^{\frac{1}{2}}\ell(Q) \quad &\text{for all } I \in \mathcal{W}_Q. \end{aligned} \tag{2.9}$$

In principle, for the previous definition, η and K are arbitrary, but we will typically need to assume that η is sufficiently small and K is sufficiently large. We will do so and as a consequence the constant C will be independent of η and K and will depend on dimension, ADR, and some other intrinsic constants depending on the different scenarios on which we work. In particular, it is convenient to assume, and we will do so, that $K \ge 40^2 n$ so that given any $I \in \mathcal{W}$ such that $\ell(I) \lesssim \operatorname{diam}(E)$, if we write Q_I^* for (one) nearest dyadic cube to I with $\ell(I) = \ell(Q_I^*)$ then $I \in \mathcal{W}_{Q_I^*}^0 \subset \mathcal{W}_{Q_I^*}$. Note that there can be more than one choice of Q_I^*, but at this point we fix one so that in what follows Q_I^* is unambiguously defined.

Below we will discuss a few special cases depending on whether we have some extra information about Ω or $\partial\Omega$. The main idea consists in constructing some kind of "Whitney regions" which will allow us to introduce some "Carleson boxes" and "sawtooth subdomains". The construction of the Whitney regions depends very much on the background assumptions, having extra information about Ω or $\partial\Omega$ will allow us to augment the collections $\mathcal{W}_Q^0$ to define $\mathcal{W}_Q$ so that we gain some connectivity on the corresponding Whitney regions and hence the resulting subdomains would have better properties. We consider four cases. In the first one, treated in Section 2.1, we assume only that $\Omega = \mathbb{R}^{n+1} \setminus E$, where

E is ADR (but is not necessarily UR) and we set $\mathcal{W}_Q = \mathcal{W}_Q^0$ (here we do not gain any connectivity). The second case is considered in Section 2.2 and deals with $\Omega = \mathbb{R}^{n+1} \setminus E$, where E is UR, in which case we can invoke Lemma 2.42 below and use the Lipschitz graphs associated to the good regimes so that the augmented collection $\mathcal{W}_Q$ creates two nice Whitney regions, one each lying respectively above and below the Lipschitz graph. Third, when Ω is a 1-sided CAD we can augment $\mathcal{W}_Q^0$ using that D is Harnack chain connected so that the resulting collections $\mathcal{W}_Q$ give some Whitney regions which produce Carleson boxes and sawtooth subdomains which are 1-sided CAD; see Section 2.3. We repeat the same construction in our last case in Section 2.4, where Ω is a CAD. The fact that Ω satisfies the exterior corkscrew condition allows us to conclude that Carleson boxes and sawtooth subdomains are as well.

To continue with our discussion let $\Omega \subset \mathbb{R}^{n+1}$ be an open set so that $\partial\Omega$ is ADR. Let $\mathcal{W} = \mathcal{W}(\Omega)$ and $\mathbb{D} = \mathbb{D}(\partial\Omega)$ be as above and let $\{\mathcal{W}_Q\}_{Q\in\mathbb{D}}$ be a Whitney-dyadic structure for Ω with some parameters η and K (we will assume that η is sufficiently small and K is sufficiently large). Fix $0 < \tau \le \tau_0/4$ as above. Given an arbitrary $Q \in \mathbb{D}$, we may define an associated *Whitney region* U_Q (not necessarily connected), as

$$U_Q = U_{Q,\tau} := \bigcup_{I\in\mathcal{W}_Q} I^*(\tau). \tag{2.10}$$

For later use, it is also convenient to introduce some fattened version of U_Q

$$\widehat{U}_Q = U_{Q,2\tau} := \bigcup_{I\in\mathcal{W}_Q} I^*(2\tau). \tag{2.11}$$

When the particular choice of $\tau \in (0, \tau_0]$ is not important, for the sake of notational convenience, we may simply write I^* and U_Q in place of $I^*(\tau)$ and $U_{Q,\tau}$.

We may also define the *Carleson box* relative to $Q \in \mathbb{D}$, by

$$T_Q = T_{Q,\tau} := \operatorname{int}\Bigg(\bigcup_{Q'\in\mathbb{D}_Q} U_{Q,\tau}\Bigg), \tag{2.12}$$

where

$$\mathbb{D}_Q := \{Q' \in \mathbb{D} : Q' \subset Q\}. \tag{2.13}$$

Let us note that we may choose K large enough so that, for every Q,

$$T_{Q,\tau} \subset T_{Q,\tau_0} \subset B_Q^* := B(x_Q, K\ell(Q)). \tag{2.14}$$

We also observe that for any $N \ge 1$ we have

$$B_Q \cap \Omega \subset T_{Q,\tau/N}. \tag{2.15}$$

To see this, let $Y \in B_Q \cap \Omega = B(x_Q, r) \cap \Omega$ (see (2.2), (2.3)) and pick $I \in \mathcal{W}$ with $I \ni Y$. Note that $\ell(I) \le \operatorname{dist}(I, \partial\Omega)/4 \le |Y - x_Q|/4 < r/4 \le \ell(Q)/4$. Take $\hat{y} \in Q$ so that $\operatorname{dist}(Y, Q) = |Y - \hat{y}|$ and select $Q_Y \ni \hat{y}$ with $\ell(Q_Y) = \ell(I) \le \ell(Q)/4$. Thus, $Q_Y \in \mathbb{D}_Q$ and

$$\operatorname{dist}(I, Q_Y) \le |Y - \hat{y}| = \operatorname{dist}(Y, Q) \le |Y - x_Q| < r \le \ell(Q).$$

All these show that $I \in \mathcal{W}_Q^0 \subset \mathcal{W}_Q$ and consequently $Y \in \operatorname{int}(I^*(\tau/N)) \subset T_{Q,\tau/N}$ as desired.

It is convenient to introduce the *Carleson box* T_Δ relative to $\Delta = \Delta(x, r)$, with $x \in \partial\Omega$ and $0 < r < \operatorname{diam}(\partial\Omega)$. Let $k(\Delta)$ denote the unique $k \in \mathbb{Z}$ such that $2^{-k-1} < 200r \le 2^{-k}$ and set

$$\mathbb{D}^\Delta := \{Q \in \mathbb{D}_{k(\Delta)} : Q \cap 2\Delta \neq \varnothing\}.$$

We then define

$$T_\Delta = T_{\Delta,\tau} := \operatorname{int}\Big(\bigcup_{Q\in\mathbb{D}^\Delta} \overline{T}_Q\Big). \tag{2.16}$$

Much as in [Hofmann and Martell 2014, (3.60)] if we write $B_\Delta = B(x, r)$ so that $\Delta = B_\Delta \cap E$, we have by taking K possibly larger

$$\tfrac{5}{4}B_\Delta \cap \Omega \subset T_\Delta \subset B(x, Kr) \cap \Omega. \tag{2.17}$$

For future reference, we also introduce dyadic sawtooth regions as follows. Given a family $\mathcal{F}$ of disjoint cubes $\{Q_j\} \subset \mathbb{D}$, we define the *global discretized sawtooth* relative to $\mathcal{F}$ by

$$\mathbb{D}_\mathcal{F} := \mathbb{D} \setminus \bigcup_{\mathcal{F}} \mathbb{D}_{Q_j}, \tag{2.18}$$

i.e., $\mathbb{D}_\mathcal{F}$ is the collection of all $Q \in \mathbb{D}$ that are not contained in any $Q_j \in \mathcal{F}$. Given some fixed cube Q, the *local discretized sawtooth* relative to $\mathcal{F}$ by

$$\mathbb{D}_{\mathcal{F},Q} := \mathbb{D}_Q \setminus \bigcup_{\mathcal{F}} \mathbb{D}_{Q_j} = \mathbb{D}_\mathcal{F} \cap \mathbb{D}_Q. \tag{2.19}$$

Note that we can also allow $\mathcal{F}$ to be empty in which case $\mathbb{D}_\varnothing = \mathbb{D}$ and $\mathbb{D}_{\varnothing,Q} = \mathbb{D}_Q$.

Similarly, we may define geometric sawtooth regions as follows. Given a family $\mathcal{F} \subset \mathbb{D}$ of disjoint cubes as before, we define the *global sawtooth* and the *local sawtooth* relative to $\mathcal{F}$ by respectively

$$\Omega_\mathcal{F} := \operatorname{int}\Big(\bigcup_{Q'\in\mathbb{D}_\mathcal{F}} U_{Q'}\Big), \quad \Omega_{\mathcal{F},Q} := \operatorname{int}\Big(\bigcup_{Q'\in\mathbb{D}_{\mathcal{F},Q}} U_{Q'}\Big). \tag{2.20}$$

Note that $\Omega_{\varnothing,Q} = T_Q$. For the sake of notational convenience, we set

$$\mathcal{W}_\mathcal{F} := \bigcup_{Q'\in\mathbb{D}_\mathcal{F}} \mathcal{W}_{Q'}, \quad \mathcal{W}_{\mathcal{F},Q} := \bigcup_{Q'\in\mathbb{D}_{\mathcal{F},Q}} \mathcal{W}_{Q'}, \tag{2.21}$$

so that in particular, we may write

$$\Omega_{\mathcal{F},Q} = \operatorname{int}\Big(\bigcup_{I\in\mathcal{W}_{\mathcal{F},Q}} I^*\Big). \tag{2.22}$$

Finally, for every $x \in \partial\Omega$, we define nontangential approach regions, *dyadic cones*, as

$$\Gamma(x) = \bigcup_{Q\in\mathbb{D}:Q\ni x} U_Q. \tag{2.23}$$

Their local (or truncated) versions are given by

$$\Gamma^Q(x) = \bigcup_{Q'\in\mathbb{D}_Q:Q'\ni x} U_{Q'}, \quad x \in Q. \tag{2.24}$$

When $\partial\Omega$ is bounded, there exists a cube $Q_0 \in \mathbb{D}(\partial\Omega)$ such that $Q_0 = \partial\Omega$ and $Q \in \mathbb{D}_{Q_0}$ for any $Q \in \mathbb{D}(\partial\Omega)$. In particular, $\Gamma_Q(\cdot) \subset \Gamma_{Q_0}(\cdot) \subset \{X \in \Omega : \operatorname{dist}(X, \partial\Omega) \lesssim \operatorname{diam}(\partial\Omega)\}$ and all the cones are bounded.

Note that all the previous objects have been defined using the Whitney regions U_Q (made out of dilated Whitney cubes $I^*(\tau)$). One can analogously use the fattened Whitney regions $\widehat{U}_Q$ (composed of the union of dilated Whitney cubes $I^*(2\tau)$). In that case we will use the notation $\widehat{T}_Q$, $\widehat{T}_\Delta$, $\widehat{\Omega}_{\mathcal{F}}$, $\widehat{\Omega}_{\mathcal{F},Q}$, $\widehat{\Gamma}(\cdot)$, $\widehat{\Gamma}^Q(\cdot)$.

We will always assume that K is large enough (say $K \geq 10^4 n$) so that $\widehat{\Gamma}_{\Omega,1}(x) \subset \Gamma(x)$ (see (1.14)) for every $x \in \partial\Omega$. Indeed, let $Y \in \Gamma_{\Omega,1}(x)$ and pick $I \in \mathcal{W}$ with $Y \in I$. Take $Q \in \mathbb{D}$ with $Q \ni x$ and $\ell(Q) = \ell(I)$. Then,

$$\operatorname{dist}(I, Q) \leq |Y - x| \leq 2\operatorname{dist}(Y, \partial\Omega) \leq 2(\operatorname{diam}(I) + \operatorname{dist}(I, \partial\Omega)) \leq 82\operatorname{diam}(I) < 100\sqrt{n}\,\ell(Q).$$

Hence, $I \in \mathcal{W}_Q^0 \subset \mathcal{W}_Q$ provided $100\sqrt{n} \leq \sqrt{K}$ and thus $I \subset U_Q \subset \Gamma(x)$ as desired.

Remark 2.25. It is convenient to introduce a condition on interior Whitney balls that is much weaker than CME itself. Let $\Omega \subset \mathbb{R}^{n+1}$ be an open set. For every $F \in L^2_{\mathrm{loc}}(\Omega)$ we set

$$\|F\|_{\mathbb{C}_0(\Omega)} := \sup_{X \in \Omega} \frac{1}{\delta(X)^{n-1}} \iint_{B(X,\delta(X)/2)} |F(Y)|^2\, dY, \tag{2.26}$$

where $\delta(\cdot) = \operatorname{dist}(\cdot, \partial\Omega)$.

Note that for any $X \in \Omega$ we have that $B(X, \delta(X)/2) \subset B(\hat{x}, 3\delta(X)/2) \cap \Omega$ with $\hat{x} \in \partial\Omega$ so that $\delta(X) = |X - \hat{x}|$, and $\delta(Y) \geq \delta(X)/2$ for every $Y \in B(X, \delta(X)/2)$. Hence,

$$\|F\|_{\mathbb{C}_0(\Omega)} \leq 2\left(\tfrac{3}{2}\right)^n \|F\|_{\mathrm{CME}(\Omega)}, \tag{2.27}$$

and $\|F\|_{\mathbb{C}_0(\Omega)} < \infty$ is necessary for (1.10) to hold.

We note that in all applications to the CME for solutions of elliptic PDEs, $\|F\|_{\mathbb{C}_0(\Omega)}$ will be bounded automatically, by Caccioppoli's inequality (since F will be of the form ∇u or $\nabla^m u$ with u being a bounded solution). We shall discuss this in more detail together with the corresponding applications.

We introduce a dyadic version of Definition 1.9. Given $\Omega \subset \mathbb{R}^{n+1}$, an open set with $\partial\Omega$ being ADR, let $\{\mathcal{W}_Q\}_{Q \in \mathbb{D}(\partial\Omega)}$ be a Whitney-dyadic structure for Ω with some parameters η and K. We define, for every $F \in L^2_{\mathrm{loc}}(\Omega)$,

$$\|F\|_{\mathrm{CME}^{\mathrm{dyad}}(\Omega)} := \sup_{Q \in \mathbb{D}(\partial\Omega)} \frac{1}{\sigma(Q)} \iint_{T_Q} |F(X)|^2 \operatorname{dist}(X, \partial\Omega)\, dX. \tag{2.28}$$

We are going to show that

$$\|F\|_{\mathrm{CME}(\Omega)} \lesssim \|F\|_{\mathrm{CME}^{\mathrm{dyad}}(\Omega)} + \|F\|_{\mathbb{C}_0(\Omega)}. \tag{2.29}$$

To obtain this, fix $x \in \partial\Omega$ and $0 < r < \infty$. Set $\mathcal{W}_{x,r} = \{I \in \mathcal{W}(\Omega) : I \cap B(x, r) \neq \varnothing\}$ and note that given $I \in \mathcal{W}_{x,r}$, if we pick $Z_I \in I \cap B(x, r)$, then (2.5) implies

$$\operatorname{diam}(I) \leq \operatorname{dist}(I, \partial\Omega) \leq |Z_I - x| < r. \tag{2.30}$$

Set

$$\mathcal{W}_{x,r}^{\mathrm{small}} = \{I \in \mathcal{W}_{x,r} : \ell(I) < \operatorname{diam}(\partial\Omega)/4\}, \quad \mathcal{W}_{x,r}^{\mathrm{big}} = \{I \in \mathcal{W}_{x,r} : \ell(I) \geq \operatorname{diam}(\partial\Omega)/4\},$$

with the understanding that $\mathcal{W}_{x,r}^{\text{big}} = \varnothing$ if $\operatorname{diam}(\partial\Omega) = \infty$. Using this notation and writing $\delta = \operatorname{dist}(\,\cdot\,, \partial\Omega)$ we have

$$\iint_{B(x,r)\cap\Omega} |F|^2\delta\,dX \le \sum_{I\in\mathcal{W}_{x,r}^{\text{small}}} \iint_I |F|^2\delta\,dX + \sum_{I\in\mathcal{W}_{x,r}^{\text{big}}} \iint_I |F|^2\delta\,dX = \mathrm{I} + \mathrm{II}, \tag{2.31}$$

where we understand that $\mathrm{II} = 0$ if $\mathcal{W}_{x,r}^{\text{big}} = \varnothing$.

To estimate I we set $r_0 = \min\{r, \operatorname{diam}(\partial\Omega)/4\}$ and pick $k_2 \in \mathbb{Z}$ so that $2^{k_2-1} \le r_0 < 2^{k_2}$. Set

$$\mathcal{D}_1 = \{Q \in \mathbb{D}(\partial\Omega) : \ell(Q) = 2^{k_2},\ Q \cap B(x, 3r) \ne \varnothing\}.$$

Given $I \in \mathcal{W}_{x,r}^{\text{small}}$ we pick $y \in \partial\Omega$ so that $\operatorname{dist}(I, \partial\Omega) = \operatorname{dist}(I, y)$. Hence there exists a unique $Q_I \in \mathbb{D}(\partial\Omega)$ so that $y \in Q_I$ and $\ell(Q_I) = \ell(I) < r_0 \le \operatorname{diam}(\partial\Omega)/4$ by (2.30). Also,

$$\operatorname{dist}(I, Q_I) \le \operatorname{dist}(I, y) = \operatorname{dist}(I, \partial\Omega) \le 40\operatorname{diam}(I) = 40\sqrt{n}\,\ell(Q).$$

This implies that $I \in \mathcal{W}_{Q_I}^0 \subset \mathcal{W}_{Q_I}$, provided $0 < \eta \le 1$ and $K \ge 40\sqrt{n}$. On the other hand, by (2.30)

$$|y - x| \le \operatorname{dist}(y, I) + \operatorname{diam}(I) + |Z_I - x| < 3r;$$

hence there exists a unique $Q \in \mathcal{D}_1$ so that $y \in Q$. Since $\ell(Q_I) < r_0 < 2^{k_2} = \ell(Q)$, we conclude that $Q_I \subset Q$ and consequently $I \subset \operatorname{int}(U_{Q_I}) \subset T_Q$. In short we have shown that if $I \in \mathcal{W}_{x,r}^{\text{small}}$, there exists $Q \in \mathcal{D}_1$ so that $I \subset T_Q$. Thus,

$$\mathrm{I} \lesssim \sum_{Q\in\mathcal{D}_1} \iint_{T_Q} |F|^2\delta\,dX \le \|F\|_{\mathrm{CME}^{\text{dyad}}(\Omega)} \sum_{Q\in\mathcal{D}_1} \sigma(Q) \lesssim \|F\|_{\mathrm{CME}^{\text{dyad}}(\Omega)} r^n,$$

where we have used the fact that $\mathcal{D}_1$ is a pairwise disjoint family, that $\bigcup_{Q\in\mathcal{D}_1} Q \subset B(x, Cr) \cap \partial\Omega$ (with C depending on dimension and ADR), and that $\partial\Omega$ is ADR.

We now estimate II when nonempty, in which case $\operatorname{diam}(\partial\Omega) < \infty$. Using the properties of the Whitney cubes and recalling (2.26) we arrive at

$$\mathrm{II} \lesssim \sum_{I\in\mathcal{W}_{x,r}^{\text{big}}} \ell(I) \iint_I |F|^2\,dX \lesssim \|F\|_{\mathcal{C}_0(\Omega)} \sum_{I\in\mathcal{W}_{x,r}^{\text{big}}} \ell(I)^n \le \|F\|_{\mathcal{C}_0(\Omega)} \sum_{\operatorname{diam}(\partial\Omega)/4\le 2^k < r} 2^{kn}\#\{I \in \mathcal{W}_{x,r}^{\text{big}} : \ell(I) = 2^k\}.$$

To estimate the last term we observe that if $Y \in I \in \mathcal{W}_{x,r}^{\text{big}}$ we have by (2.5)

$$|Y - x| \le \operatorname{diam}(I) + \operatorname{dist}(I, \partial\Omega) + \operatorname{diam}(\partial\Omega) \lesssim \ell(I).$$

This and the fact that Whitney cubes have nonoverlapping interiors imply

$$\begin{aligned}\#\{I \in \mathcal{W}_{x,r}^{\text{big}} : \ell(I) = 2^k\} &= 2^{-k(n+1)} \sum_{I\in\mathcal{W}_{x,r}^{\text{big}}:\ell(I)=2^k} |I| \\ &= 2^{-k(n+1)} \Bigg| \bigcup_{I\in\mathcal{W}_{x,r}^{\text{big}}:\ell(I)=2^k} I \Bigg| \le 2^{-k(n+1)} |B(x, C2^k)| \lesssim 1. \end{aligned} \tag{2.32}$$

Therefore,

$$\mathrm{II} \lesssim \|F\|_{\mathbb{C}_0(\Omega)} \sum_{\operatorname{diam}(\partial\Omega)/4 \le 2^k < r} 2^{kn} \lesssim \|F\|_{\mathbb{C}_0(\Omega)} r^n.$$

Collecting the estimates for I and II we obtain (2.29).

Definition 2.33 (dyadic nontangential maximal function, area integral, and square function). Let $\Omega \subset \mathbb{R}^{n+1}$ be an open set with $\partial\Omega$ being ADR and let $\{\mathcal{W}_Q\}_{Q\in\mathbb{D}(\partial\Omega)}$ be a Whitney-dyadic structure for Ω with some parameters η and K. For $H \in C(\Omega)$ (i.e., H is continuous function in Ω), we define the *dyadic nontangential maximal function* as

$$N_* H(x) := \sup_{Y\in\Gamma(x)} |H(Y)|, \qquad x \in \partial\Omega; \tag{2.34}$$

for $G \in L^2_{\mathrm{loc}}(\Omega)$, we define the *dyadic area integral* as

$$\mathcal{A}G(x) := \left(\iint_{\Gamma(x)} |G(Y)|^2 \operatorname{dist}(Y, E)^{1-n}\, dY\right)^{\frac{1}{2}}, \qquad x \in \partial\Omega; \tag{2.35}$$

and, for $u \in W^{1,2}_{\mathrm{loc}}(\Omega)$, we define the *dyadic square function* as

$$Su(x) := \left(\iint_{\Gamma(x)} |\nabla u(Y)|^2 \operatorname{dist}(Y, \partial\Omega)^{1-n}\, dY\right)^{\frac{1}{2}}, \qquad x \in \partial\Omega. \tag{2.36}$$

For any $Q \in \mathbb{D}(\partial\Omega)$, we write N_*^Q, $\mathcal{A}^Q$, and S^Q to denote the *local* (or *truncated*) dyadic nontangential maximal function, area integral, and square function respectively, where $\Gamma(\cdot)$ is replaced by the local cone $\Gamma^Q(\cdot)$. Finally, $\widehat{N}_*$, $\widehat{\mathcal{A}}$, $\widehat{S}$ or $\widehat{N}_*^Q$, $\widehat{\mathcal{A}}^Q$, $\widehat{S}^Q$ stand for the corresponding objects associated to the fattened cones $\widehat{\Gamma}(\cdot)$ or their local versions $\widehat{\Gamma}^Q(\cdot)$.

Remark 2.37. It is convenient to compare the two types of cones, the "traditional" and the dyadic (see (1.14) and (2.23)). Fix a Whitney-dyadic structure $\{\mathcal{W}_Q\}_{Q\in\mathbb{D}(\partial\Omega)}$ for Ω with parameters η and K. It is straightforward to see that there exists κ such that the dyadic cones $\Gamma(x)$ are contained in $\Gamma_\Omega(x)$ for all $x \in \partial\Omega$. Indeed, if $Y \in I^*(2\tau)$ with $I \in \mathcal{W}_Q$ and $Q \ni x$ then by (2.9)

$$\begin{aligned}|Y-x| &\le \operatorname{diam}(I^*(2\tau)) + \operatorname{dist}(I, Q) + \operatorname{diam}(Q) \lesssim K^{\frac{1}{2}}\ell(Q) \lesssim K^{\frac{1}{2}}\eta^{-\frac{1}{2}}\ell(I)\\ &\lesssim K^{\frac{1}{2}}\eta^{-\frac{1}{2}} \operatorname{dist}(I, \partial\Omega) \le K^{\frac{1}{2}}\eta^{-\frac{1}{2}} \operatorname{dist}(Y, \partial\Omega);\end{aligned}$$

hence $Y \in \Gamma_{\Omega, K^{1/2}\eta^{-1/2}}(x)$. And we have shown that $\Gamma(x) \subset \widehat{\Gamma}(x) \subset \Gamma_{\Omega, K^{1/2}\eta^{-1/2}}$. Conversely, given $\kappa > 0$, there exist η and K (depending on κ) such that if $\{\mathcal{W}_Q\}_{Q\in\mathbb{D}(\partial\Omega)}$ is a Whitney-dyadic structure for Ω with parameters η and K then $\Gamma_{\Omega,\kappa}(x) \subset \Gamma(x)$ for all $x \in \partial\Omega$. As a matter of fact, given $Y \in \Gamma_{\Omega,\kappa}(x)$, let $I \in \mathcal{W}$ with $I \ni Y$ and pick $Q \in \mathbb{D}(\partial\Omega)$ with $Q \ni x$ and $\ell(I) = \ell(Q)$ (recall that if $\partial\Omega$ is bounded we have assumed that $\delta(Y) \lesssim \operatorname{diam}(\partial\Omega)$, hence such a cube Q always exists). Then,

$$\operatorname{dist}(I, Q) \le |Y-x| \le (1+\kappa)\operatorname{dist}(Y, \partial\Omega) \le (1+\kappa)(\operatorname{diam}(I) + \operatorname{dist}(I, \partial\Omega)) \lesssim (1+\kappa)\ell(I) = (1+\kappa)\ell(Q).$$

Thus, if $K^{1/2} \gg 1+\kappa$, then $I \in \mathcal{W}_Q^0 \subset \mathcal{W}_Q$ and $Y \in I \subset U_Q \subset \Gamma(x)$ as desired.

Remark 2.38. In the previous remark we were able to compare the dyadic and the traditional cones and this gives comparisons between the associated nontangential maximal functions, area integrals, or square functions by adjusting the different parameters. It is also convenient to see how to incorporate the "change on the aperture" on the traditional cones via or on the dyadic cones. In the case of traditional cones this amounts to considering different values of the aperture κ. For the dyadic cones one can "change the aperture" using $U_Q = U_{Q,\tau}$ versus $\widehat{U}_Q = U_{Q,2\tau}$, or even by considering Whitney-dyadic structures with different parameters.

In the case of the traditional cones, one has, for every $0 < p < \infty$ and κ, κ' and for every $F \in C(\Omega)$ and $G \in W^{1,2}_{\mathrm{loc}}(\Omega)$,

$$\|N_{*,\Omega,\kappa} F\|_{L^p(\partial\Omega)} \approx_{\kappa,\kappa'} \|N_{*,\Omega,\kappa'} F\|_{L^p(\partial\Omega)}, \qquad \|\mathcal{A}_{\Omega,\kappa} G\|_{L^p(\partial\Omega)} \lesssim_{\kappa,\kappa'} \|\mathcal{A}_{\Omega,\kappa'} G\|_{L^p(\partial\Omega)}. \tag{2.39}$$

The first estimate can be found in [Hofmann et al. 2010, Proposition 2.2]. For the second estimate we refer to [Milakis et al. 2013, Proposition 4.5] in the case of Ω being a CAD, a simpler argument (valid also in the former case) can be carried out by adapting [Martell and Prisuelos-Arribas 2017, Proposition 3.2(i)]. Further details are left to the interested reader.

For the dyadic cones, Remark 2.37 says that if $\{\mathcal{W}_Q\}_{Q\in\mathbb{D}(\partial\Omega)}$ is a Whitney-dyadic structure for Ω with parameters $\eta \ll 1$ and $K \gg 1$ then $\Gamma(x) \subset \widehat{\Gamma}(x) \subset \Gamma_{\Omega,\kappa}(x)$ for some large $\kappa > 0$ and for every $x \in \partial\Omega$. On the other hand, let $\{\mathcal{W}'_Q\}_{Q\in\mathbb{D}(\partial\Omega)}$ be a Whitney-dyadic structure for Ω with parameters $\eta' \ll 1$ and $K' \gg 1$ and we write $\Gamma'(x)$ for the associated dyadic cone. As observed before we have that $\Gamma_{\Omega,1}(x) \subset \Gamma'(x)$. Write N_* and $\mathcal{A}$ (resp. N'_* and $\mathcal{A}'$) as in (2.34) and (2.35) for the cones Γ (resp. Γ'). These and (2.39) allow us to obtain that for every $0 < p < \infty$ and for every $F \in C(\Omega)$

$$\|N_* F\|_{L^p(\partial\Omega)} \le \|\widehat{N}_* F\|_{L^p(\partial\Omega)} \le \|N_{*,\Omega,\kappa} F\|_{L^p(\partial\Omega)} \lesssim_\kappa \|N_{*,\Omega,1} F\|_{L^p(\partial\Omega)} \le \|N'_* F\|_{L^p(\partial\Omega)}$$

and, for every $G \in W^{1,2}_{\mathrm{loc}}(\Omega)$,

$$\|\mathcal{A} G\|_{L^p(\partial\Omega)} \le \|\widehat{\mathcal{A}} G\|_{L^p(\partial\Omega)} \le \|\mathcal{A}_{\Omega,\kappa} G\|_{L^p(\partial\Omega)} \lesssim_\kappa \|\mathcal{A}_{\Omega,1} G\|_{L^p(\partial\Omega)} \le \|\mathcal{A}' G\|_{L^p(\partial\Omega)}.$$

2.1. *Case ADR.* Here we assume that $\Omega = \mathbb{R}^{n+1} \setminus E$, where E is merely ADR, but possibly not UR. Let us set $\mathcal{W}_Q = \mathcal{W}^0_Q$ (see (2.8)) and we clearly have (ii) and (iii) with $C = 1$ in Definition 2.7. For (i), we see that $\mathcal{W}^0_Q$ is nonempty, provided that we choose η small enough, and K large enough, depending only on dimension and the ADR constant of E. Indeed, given $Q \in \mathbb{D}(E)$, consider the ball $B_Q = B(x_Q, r)$, as defined in (2.2), (2.3), with $r \approx \ell(Q)$, so that $\Delta_Q = B_Q \cap E \subset Q$. By [Hofmann and Martell 2014, Lemma 5.3], we have that, for some $C = C(n, ADR)$,

$$|\{Y \in \mathbb{R}^{n+1} \setminus E : \operatorname{dist}(Y, E) < \varepsilon r\} \cap B_Q| \le C\varepsilon r^{n+1}$$

for every $0 < \varepsilon < 1$. Consequently, fixing $0 < \varepsilon_0 < 1$ small enough, there exists $X_Q \in B_Q/2$, with $\operatorname{dist}(X_Q, E) \ge \varepsilon_0 r$. Thus, $B(X_Q, \varepsilon_0 r/2) \subset B_Q \setminus E$. We shall refer to this point X_Q as a "corkscrew point" relative to Q, that is, relative to the surface ball Δ_Q (see (2.2) and (2.3)). Now observe that X_Q belongs to some Whitney cube $I \in \mathcal{W}$, which will belong to $\mathcal{W}^0_Q$ for η small enough and K large enough. Hence, $\{\mathcal{W}_Q\}_{Q\in\mathbb{D}(E)}$ is a Whitney-dyadic structure for $\mathbb{R}^{n+1} \setminus E$.

In [Hofmann et al. 2016] it was shown that the ADR property is inherited by all dyadic local sawtooths and all Carleson boxes:

Proposition 2.40 [Hofmann et al. 2016, Proposition A.2]. *Let $E \subset \mathbb{R}^{n+1}$ be a closed n-dimensional ADR set and let $\{\mathcal{W}_Q\}_{Q\in\mathbb{D}(E)}$ be a Whitney-dyadic structure for $\mathbb{R}^{n+1}\setminus E$ with parameters $\eta \ll 1$ and $K \gg 1$. Then all dyadic local sawtooths $\Omega_{\mathcal{F},Q}$ and all Carleson boxes T_Q have n-dimensional ADR boundaries. In all cases, the implicit constants are uniform and depend only on dimension, the ADR constant of E and the parameters η, K, and τ.*

2.2. *Case UR.* Here we assume that $\Omega = \mathbb{R}^{n+1}\setminus E$, where we further assume that E is UR. Much as before, since E is in particular ADR, if we take $\eta \ll 1$ and $K \gg 1$ (depending on n and the ADR constant of E), we can guarantee that $\mathcal{W}_Q^0 \neq \varnothing$. In this case we will exploit the additional fact that E is UR to construct some Whitney-dyadic structure with better properties. To do so, we would like to recall some results from [Hofmann et al. 2016] but we first give a definition to then continue with the main geometric lemma there.

Definition 2.41 [David and Semmes 1993]. . Let $\boldsymbol{S} \subset \mathbb{D}(E)$. We say that $\boldsymbol{S}$ is "coherent" if the following conditions hold:

(a) $\boldsymbol{S}$ contains a unique maximal element denoted by $Q(\boldsymbol{S})$ which contains all other elements of $\boldsymbol{S}$ as subsets.

(b) If Q belongs to $\boldsymbol{S}$, and if $Q \subset \widetilde{Q} \subset Q(\boldsymbol{S})$, then $\widetilde{Q} \in \boldsymbol{S}$.

(c) Given a cube $Q \in \boldsymbol{S}$, either all of its children belong to $\boldsymbol{S}$, or none of them do.

We say that $\boldsymbol{S}$ is "semicoherent" if only conditions (a) and (b) hold.

Lemma 2.42 (the bilateral corona decomposition [Hofmann et al. 2016, Lemma 2.2]). *Suppose that $E \subset \mathbb{R}^{n+1}$ is n-dimensional UR. Then given any positive constants $\eta \ll 1$ and $K \gg 1$, there is a disjoint decomposition $\mathbb{D}(E) = \mathcal{G}\cup\mathcal{B}$, satisfying the following properties.*

(i) *The "good" collection $\mathcal{G}$ is further subdivided into disjoint stopping time regimes such that each such regime $\boldsymbol{S}$ is coherent (see Definition 2.41).*

(ii) *The "bad" cubes, as well as the maximal cubes $Q(\boldsymbol{S})$ satisfy a Carleson packing condition:*

$$\sum_{Q'\subset Q,\, Q'\in\mathcal{B}} \sigma(Q') + \sum_{\boldsymbol{S}:Q(\boldsymbol{S})\subset Q} \sigma(Q(\boldsymbol{S})) \le C_{\eta,K}\sigma(Q) \quad \text{for all } Q \in \mathbb{D}(E).$$

(iii) *For each $\boldsymbol{S}$, there is a Lipschitz graph $\Gamma_{\boldsymbol{S}}$, with Lipschitz constant at most η, such that, for every $Q \in \boldsymbol{S}$,*

$$\sup_{x\in\Delta_Q^*} \operatorname{dist}(x,\Gamma_{\boldsymbol{S}}) + \sup_{y\in B_Q^*\cap\Gamma_{\boldsymbol{S}}} \operatorname{dist}(y,E) < \eta\ell(Q), \tag{2.43}$$

where $B_Q^* := B(x_Q, K\ell(Q))$ and $\Delta_Q^* := B_Q^*\cap E$.

As we have assumed that E is UR we make the corresponding bilateral corona decomposition of Lemma 2.42 with $\eta \ll 1$ and $K \gg 1$. Our goal is to construct, for each stopping time regime $\boldsymbol{S}$ in

Lemma 2.42, a pair of CAD domains $\Omega_{\boldsymbol{S}}^{\pm}$, which provide a good approximation to E, at the scales within $\boldsymbol{S}$, in some appropriate sense. To be a bit more precise, $\Omega_{\boldsymbol{S}} := \Omega_{\boldsymbol{S}}^{+} \cup \Omega_{\boldsymbol{S}}^{-}$ will be constructed as a sawtooth region relative to some family of dyadic cubes, and the nature of this construction will be essential to the dyadic analysis that we will use below.

Given $Q \in \mathbb{D}(E)$, for this choice of η and K, we set as above $B_Q^* := B(x_Q, K\ell(Q))$, where we recall that x_Q is the center of Q (see (2.2), (2.3)). For a fixed stopping time regime $\boldsymbol{S}$, we choose a coordinate system so that $\Gamma_{\boldsymbol{S}} = \{(z, \varphi_{\boldsymbol{S}}(z)) : z \in \mathbb{R}^n\}$, where $\varphi_{\boldsymbol{S}} : \mathbb{R}^n \to \mathbb{R}$ is a Lipschitz function with $\|\varphi\|_{\mathrm{Lip}} \le \eta$.

Claim 2.44 [Hofmann et al. 2016, Claim 3.4]. *If $Q \in \boldsymbol{S}$, and $I \in \mathcal{W}_Q^0$, then I lies either above or below $\Gamma_{\boldsymbol{S}}$. Moreover*, $\operatorname{dist}(I, \Gamma_{\boldsymbol{S}}) \ge \eta^{1/2}\ell(Q)$ (*and therefore, by* (2.43), $\operatorname{dist}(I, \Gamma_{\boldsymbol{S}}) \approx \operatorname{dist}(I, E)$, *with implicit constants that may depend on η and K*).

Next, given $Q \in \boldsymbol{S}$, we augment $\mathcal{W}_Q^0$. We split $\mathcal{W}_Q^0 = \mathcal{W}_Q^{0,+} \cup \mathcal{W}_Q^{0,-}$, where $I \in \mathcal{W}_Q^{0,+}$ if I lies above $\Gamma_{\boldsymbol{S}}$, and $I \in \mathcal{W}_Q^{0,-}$ if I lies below $\Gamma_{\boldsymbol{S}}$. Choosing K large and η small enough, by (2.43), we may assume that both $\mathcal{W}_Q^{0,\pm}$ are nonempty. We focus on $\mathcal{W}_Q^{0,+}$, as the construction for $\mathcal{W}_Q^{0,-}$ is the same. For each $I \in \mathcal{W}_Q^{0,+}$, let X_I denote the center of I. Fix one particular $I_0 \in \mathcal{W}_Q^{0,+}$, with center $X_Q^+ := X_{I_0}$. Let $\widetilde{Q}$ denote the dyadic parent of Q (that is, the unique dyadic cube $\widetilde{Q}$ with $Q \subset \widetilde{Q}$ and $\ell(\widetilde{Q}) = 2\ell(Q)$), unless $Q = Q(\boldsymbol{S})$; in the latter case we simply set $\widetilde{Q} = Q$. Note that $\widetilde{Q} \in \boldsymbol{S}$, by the coherency of $\boldsymbol{S}$. By Claim 2.44, for each I in $\mathcal{W}_Q^{0,+}$, or in $\mathcal{W}_{\widetilde{Q}}^{0,+}$, we have

$$\operatorname{dist}(I, E) \approx \operatorname{dist}(I, Q) \approx \operatorname{dist}(I, \Gamma_{\boldsymbol{S}}),$$

where the implicit constants may depend on η and K. Thus, for each such I, we may fix a Harnack chain, call it $\mathcal{H}_I$, relative to the Lipschitz domain

$$\Omega_{\Gamma_{\boldsymbol{S}}}^{+} := \{(x, t) \in \mathbb{R}^{n+1} : t > \varphi_{\boldsymbol{S}}(x)\},$$

connecting X_I to X_Q^+. By the bilateral approximation condition (2.43), the definition of $\mathcal{W}_Q^0$, and the fact that $K^{1/2} \ll K$, we may construct this Harnack chain so that it consists of a bounded number of balls (depending on η and K), and stays a distance at least $c\eta^{1/2}\ell(Q)$ away from $\Gamma_{\boldsymbol{S}}$ and from E. We let $\mathcal{W}_Q^{*,+}$ denote the set of all $J \in \mathcal{W}$ which meet at least one of the Harnack chains $\mathcal{H}_I$, with $I \in \mathcal{W}_Q^{0,+} \cup \mathcal{W}_{\widetilde{Q}}^{0,+}$ (or simply $I \in \mathcal{W}_Q^{0,+}$, if $Q = Q(\boldsymbol{S})$), i.e.,

$$\mathcal{W}_Q^{*,+} := \{J \in \mathcal{W} : \text{there exists } I \in \mathcal{W}_Q^{0,+} \cup \mathcal{W}_{\widetilde{Q}}^{0,+} \text{ for which } \mathcal{H}_I \cap J \neq \varnothing\},$$

where as above, $\widetilde{Q}$ is the dyadic parent of Q, unless $Q = Q(\boldsymbol{S})$, in which case we simply set $\widetilde{Q} = Q$ (so the union is redundant). We observe that, in particular, each $I \in \mathcal{W}_Q^{0,+} \cup \mathcal{W}_{\widetilde{Q}}^{0,+}$ meets $\mathcal{H}_I$, by definition, and therefore

$$\mathcal{W}_Q^{0,+} \cup \mathcal{W}_{\widetilde{Q}}^{0,+} \subset \mathcal{W}_Q^{*,+}. \tag{2.45}$$

Of course, we may construct $\mathcal{W}_Q^{*,-}$ analogously. We then set

$$\mathcal{W}_Q^* := \mathcal{W}_Q^{*,+} \cup \mathcal{W}_Q^{*,-}.$$

It follows from the construction of the augmented collections $\mathcal{W}_Q^{*,\pm}$ that there are uniform constants c and C such that

$$\begin{aligned} c\eta^{\frac{1}{2}}\ell(Q) \le \ell(I) \le CK^{\frac{1}{2}}\ell(Q) &\quad \text{for all } I \in \mathcal{W}_Q^*, \\ \operatorname{dist}(I, Q) \le CK^{\frac{1}{2}}\ell(Q) &\quad \text{for all } I \in \mathcal{W}_Q^*. \end{aligned} \tag{2.46}$$

It is convenient at this point to introduce some additional terminology.

Definition 2.47. Given $Q \in \mathcal{G}$, and hence in some $\boldsymbol{S}$, we shall refer to the point X_Q^+ specified above, as the "center" of U_Q^+ (similarly, the analogous point X_Q^-, lying below $\Gamma_{\boldsymbol{S}}$, is the "center" of U_Q^-). We also set $Y_Q^\pm := X_{\widetilde{Q}}^\pm$, and we call this point the "modified center" of $U_Q^\pm$, where as above $\widetilde{Q}$ is the dyadic parent of Q, unless $Q = Q(\boldsymbol{S})$, in which case $Q = \widetilde{Q}$, and $Y_Q^\pm = X_Q^\pm$.

Observe that $\mathcal{W}_Q^{*,\pm}$ and hence also $\mathcal{W}_Q^*$ have been defined for any Q that belongs to some stopping time regime $\boldsymbol{S}$, that is, for any Q belonging to the "good" collection $\mathcal{G}$ of Lemma 2.42. We now set

$$\mathcal{W}_Q := \begin{cases} \mathcal{W}_Q^*, & Q \in \mathcal{G}, \\ \mathcal{W}_Q^0, & Q \in \mathcal{B}, \end{cases} \tag{2.48}$$

and for $Q \in \mathcal{G}$ we shall henceforth simply write $\mathcal{W}_Q^\pm$ in place of $\mathcal{W}_Q^{*,\pm}$. Note that by (2.8) when $Q \in \mathcal{B}$ and by (2.46) when $Q \in \mathcal{G}$ we clearly obtain (2.9) with C depending on n and the UR character of E. By construction $\mathcal{W}_Q^0 \subset \mathcal{W}_Q$. All these show that, provided $\eta \ll 1$ and $K \gg 1$ (depending on n and the UR character of E), $\{\mathcal{W}_Q\}_{Q\in\mathbb{D}(E)}$ is a Whitney-dyadic structure for $\mathbb{R}^{n+1} \setminus E$ with parameter η and K and with C depending on n and the UR character of E.

Given an arbitrary $Q \in \mathbb{D}(E)$ and $0 < \tau \le \tau_0/4$, we may define an associated Whitney region U_Q (not necessarily connected) as in (2.10) or the fattened version of $\widehat{U}_Q$ as in (2.11). In the present situation, if $Q \in \mathcal{G}$, then U_Q splits into exactly two connected components

$$U_Q^\pm = U_{Q,\tau}^\pm := \bigcup_{I\in\mathcal{W}_Q^\pm} I^*(\tau). \tag{2.49}$$

We note that for $Q \in \mathcal{G}$, each $U_Q^\pm$ is Harnack chain connected, by construction (with constants depending on the implicit parameters τ, η and K); moreover, for a fixed stopping time regime $\boldsymbol{S}$, if Q' is a child of Q, with both $Q', Q \in \boldsymbol{S}$, then $U_{Q'}^+ \cup U_Q^+$ is Harnack chain connected, and similarly for $U_{Q'}^- \cup U_Q^-$.

We may also define the Carleson boxes T_Q, global and local sawtooth regions $\Omega_{\mathcal{F}}$, $\Omega_{\mathcal{F},Q}$, cones Γ, and local cones Γ^Q as in (2.12), (2.20), (2.23), and (2.24).

Remark 2.50. We recall that, by construction (see (2.45), (2.48)), given $Q \in \mathcal{G}$, one has $\mathcal{W}_{\widetilde{Q}}^{0,\pm} \subset \mathcal{W}_Q$, where $\widetilde{Q}$ is the dyadic parent of Q. Therefore, $Y_Q^\pm \in U_Q^\pm \cap U_{\widetilde{Q}}^\pm$. Moreover, since $Y_Q^\pm$ is the center of some $I \in \mathcal{W}_{\widetilde{Q}}^{0,\pm}$, we have that $\operatorname{dist}(Y_Q^\pm, \partial U_Q^\pm) \approx \operatorname{dist}(Y_Q^\pm, \partial U_{\widetilde{Q}}^\pm) \approx \ell(Q)$ (with implicit constants possibly depending on η and/or K)

Remark 2.51. Given a stopping time regime $\boldsymbol{S}$ as in Lemma 2.42, for any semicoherent subregime (see Definition 2.41) $\boldsymbol{S}' \subset \boldsymbol{S}$ (including, of course, $\boldsymbol{S}$ itself), we now set

$$\Omega_{\boldsymbol{S}'}^\pm = \operatorname{int}\biggl(\bigcup_{Q\in\boldsymbol{S}'} U_Q^\pm\biggr), \tag{2.52}$$

and let $\Omega_{S'} := \Omega_{S'}^{+} \cup \Omega_{S'}^{-}$. Note that implicitly, $\Omega_{S'}$ depends upon τ (since $U_Q^{\pm}$ has such dependence). When it is necessary to consider the value of τ explicitly, we shall write $\Omega_{S'}(\tau)$.

The main geometric lemma for the associated sawtooth regions is the following.

Lemma 2.53 [Hofmann et al. 2016, Lemma 3.24]. *Let $\mathbf{S}$ be a given stopping time regime as in Lemma 2.42, and let $\mathbf{S}'$ be any nonempty, semicoherent subregime of $\mathbf{S}$. Then, for $0 < \tau \le \tau_0$, with τ_0 small enough, each of $\Omega_{S'}^{\pm}$ is a CAD with character depending only on n, τ, η, K, and the UR character of E.*

2.3. ***Case 1-sided CAD.*** Here we assume that Ω is a 1-sided CAD. In this case, we are basically in the situation which is similar to being within one regimen $\mathbf{S}$, at least as far as the construction of $\mathcal{W}_Q$ is concerned.

With $\mathcal{W} = \mathcal{W}(\Omega)$ and $\mathbb{D} = \mathbb{D}(\partial\Omega)$ as above, and for some give parameters $\eta < 1$, $K > 1$, we consider $\mathcal{W}_Q^0$ (see (2.8)). For any $Q \in \mathbb{D}$ we let X_Q be a corkscrew point relative to Q, more specifically, relative to Δ_Q (see (2.2), (2.3)). We note that in this scenario the existence of such point comes from the fact that Ω satisfies the (interior) corkscrew condition). For $\eta \ll 1$ and $K \gg 1$ depending on the CAD character of Ω we can guarantee that for every $Q \in \mathbb{D}$, if $I \in \mathcal{W}$ is so that $I \ni X_Q$ then $I \in \mathcal{W}_Q^0$. We then augment $\mathcal{W}_Q^0$ to $\mathcal{W}_Q^*$ as done in [Hofmann and Martell 2014, Section 3]. More precisely, use the fact that one can construct a Harnack chain to connect X_Q with any of the centers of the Whitney cubes in $\mathcal{W}_Q^0 \cup \mathcal{W}_{\widetilde{Q}}^0$, where $\widetilde{Q}$ is the dyadic parent of Q. Then $\mathcal{W}_Q^*$ is the family of all Whitney cubes which meet at least one ball in all those Harnack chains. Note that in the case when E is UR and $Q \in \mathbf{S}$ we have used a similar idea; the main difference is that the Harnack chain in that case comes from the fact that $\Omega_{\Gamma_S}^{+}$ is a Lipschitz domain, whereas here such property comes from the assumption that Ω is a 1-sided CAD and hence the Harnack chain condition holds. Set then $\mathcal{W}_Q = \mathcal{W}_Q^*$ and one can see that (with the appropriate choice of a sufficiently small η and a sufficiently large K depending on n and the CAD character of D) (2.9) holds. Moreover, the construction guarantees that $\mathcal{W}_Q^0 \cup \mathcal{W}_{\widetilde{Q}}^0 \subset \mathcal{W}_Q$, that we can cover with the Whitney cubes in $\mathcal{W}_Q$ all the Harnack chains connecting X_Q with any center of $I \in \mathcal{W}_Q^0 \cup \mathcal{W}_{\widetilde{Q}}^0 \subset \mathcal{W}_Q$, and also that if I, J are such that $I \ni X_Q$ and $J \ni X_{\widetilde{Q}}$ then $I, J \in \mathcal{W}_Q$. We note that by construction the Harnack chain condition holds in each Whitney region U_Q and so it does in $U_Q \cup U_{\widetilde{Q}}$. In either case the corresponding constant depends on the CAD character of D and the parameters η, K, τ.

In the present situation we have the following geometric result:

Lemma 2.54 [Hofmann and Martell 2014, Lemma 3.61]. *Let $\Omega \subset \mathbb{R}^{n+1}$ be a 1-sided CAD and let $\{\mathcal{W}_Q\}_{Q \in \mathbb{D}(\partial\Omega)}$ be a Whitney-dyadic structure for Ω with parameters $\eta \ll 1$ and $K \gg 1$ as just constructed. Then all of its dyadic sawtooths regions $\Omega_{\mathcal{F}}$ and $\Omega_{\mathcal{F},Q}$ and all Carleson boxes T_Q and T_Δ are also 1-sided CAD with character depending only on dimension, the 1-sided CAD character of Ω, and the parameters η, K, and τ.*

2.4. ***Case CAD.*** Here we assume that Ω is a CAD. This is, strictly speaking, a subcase of the case of 1-sided CAD above, but the extra assumption that Ω has exterior corkscrews can be inferred to the associated sawtooth regions and Carleson boxes.

With $\mathcal{W} = \mathcal{W}(\Omega)$ and $\mathbb{D} = \mathbb{D}(\partial\Omega)$ as above, and for some give parameters $\eta < 1$, $K > 1$, we consider $\mathcal{W}_Q^0$ (see (2.8)) and construct $\mathcal{W}_Q$ exactly as in the 1-sided CAD case since a CAD is in particular a 1-sided CAD. Hence, we have the very same properties, in particular, Lemma 2.54 applies. But we can additionally obtain the exterior corkscrew condition:

Lemma 2.55. *Let $\Omega \subset \mathbb{R}^{n+1}$ be a CAD and let $\{\mathcal{W}_Q\}_{Q\in\mathbb{D}(\partial\Omega)}$ be a Whitney-dyadic structure for Ω with parameters $\eta \ll 1$ and $K \gg 1$ as just constructed. Then all of its dyadic sawtooths regions $\Omega_{\mathcal{F}}$ and $\Omega_{\mathcal{F},Q}$ and all Carleson boxes T_Q and T_Δ are also CAD with character depending only on dimension, the CAD character of Ω, and the parameters η, K, and τ.*

Proof. As mentioned above we can apply Lemma 2.54; hence all the $\Omega_{\mathcal{F}}$, $\Omega_{\mathcal{F},Q}$, T_Q, and T_Δ are 1-sided CAD domains. It remains to see that any of them satisfy the exterior corkscrew condition. Let $\Omega_\star$ be one of these subdomains and take $x_\star \in \partial\Omega_\star$ and $0 < r < \operatorname{diam}(\partial\Omega_\star)$. By construction $\partial\Omega_\star \subset \overline{\Omega}$ and we consider two cases $0 \le \operatorname{dist}(x_\star, \partial\Omega) \le r/2$ and $\operatorname{dist}(x_\star, \partial\Omega) > r/2$. In the first scenario we pick $x \in \partial\Omega$ so that $|x_\star - x| = \operatorname{dist}(x_\star, \partial\Omega) \le r/2$ (notice that $x = x_\star$ if $x_\star \in \partial\Omega \cap \partial\Omega_\star$). Since Ω is a CAD, it satisfies the exterior corkscrew condition; hence we can find $X \in \Omega_{\rm ext} = \mathbb{R}^{n+1} \setminus \overline{\Omega}$ so that $B(X, c_0 r/2) \subset B(x, r/2) \cap \Omega_{\rm ext}$ where c_0 is the exterior corkscrew constant. Note that $\Omega_\star \subset \Omega$; hence $B(X, c_0 r/2) \subset (\Omega_\star)_{\rm ext}$. Also, $B(X, c_0 r/2) \subset B(x, r/2) \subset B(x_\star, r)$. This shows that X is an exterior corkscrew point relative to the surface ball $B(x_\star, r) \cap \partial\Omega_\star$ for the domain $\Omega_\star$ with constant $c_0/2$. Consider next the case on which $\operatorname{dist}(x_\star, \partial\Omega) > r/2$. Note that in particular $x_\star \in \Omega$ and therefore we can find two Whitney cubes $I, J \in \mathcal{W}$ so that $x \in \partial I^* \cap J$, $\partial I \cap \partial J \neq \varnothing$, $\operatorname{int}(I^*) \subset \Omega_\star$ and J is a Whitney cube which does not belong to any of the $\mathcal{W}_Q$ that define $\Omega_\star$. Note that $\ell(J) \ge \operatorname{dist}(x_\star, \partial\Omega)/C > r/(2C)$ for some uniform constant $C \ge 1$, that I^* misses $3J/4$ as observed before and that the center of J satisfies $X(J) \in (\Omega_\star)_{\rm ext}$. It is then clear that the open segment joining $x_\star$ with $X(J)$ is contained in $(\Omega_\star)_{\rm ext}$ and we pick X in that segment so that $|X - x_\star| = r/(8C)$ and hence $B(X, r/(16C)) \subset B(x_\star, r) \cap \Omega_\star$. This shows that X is an exterior corkscrew point relative to the surface ball $B(x_\star, r) \cap \partial\Omega_\star$ for the domain $\Omega_\star$ with constant $1/(16C)$. Therefore, we have shown that Ω satisfies the exterior corkscrew condition with implicit constant uniformly controlled by the CAD character of Ω. □

2.5. *Some important notation.* To complete this section we introduce the following notation which will be used in our main statements:

Notation 2.56. In the statements of our main results, we will assume that some estimates (e.g., Carleson estimates, "$\mathcal{A} < N$", "$N < S$", etc.) hold for a given family of subsets with constants depending on the character of those subsets and our goal is to transfer those estimates to the original set. It is crucial to explain how this dependence on the character is understood. To set the stage suppose that we are given some set $\mathbb{X} \subset \mathbb{R}^{n+1}$ and a family $\mathbb{S}_{\mathbb{X}} := \{\mathbb{Y}\}_{\mathbb{Y}\in\mathbb{S}_{\mathbb{X}}}$, $\mathbb{Y} \subset \mathbb{X}$. We assume that associated with $\mathbb{X}$ there is some collection of nonnegative parameters $M_{\mathbb{X}} \in [1, \infty)^{N_1}$ called its character and also that each $\mathbb{Y} \in \mathbb{S}_{\mathbb{X}}$ has some associated character $M_{\mathbb{Y}} \in [1, \infty)^{N_2}$, a collection of nonnegative parameters. Using this notation when we say that certain estimate holds for all $\mathbb{Y} \in \mathbb{S}_{\mathbb{X}}$ with constant $C_{\mathbb{Y}}$ depending on the character of $\mathbb{Y}$, we mean that $C_{\mathbb{Y}} = \Theta(M_{\mathbb{Y}})$ with $\Theta : [1, \infty)^{N_2} \to (0, \infty)$ being a nondecreasing function in each variable. Implicit in the arguments to transfer the desired estimate to $\mathbb{X}$, we will use only those sets $\mathbb{Y} \in \mathbb{S}_{\mathbb{X}}$ whose parameters in the character are all uniformly controlled by some constant M_0 (which will depend on the character of $\mathbb{X}$), and then all the corresponding constants in the assumed estimates for those sets will be controlled by $\Theta(M_0, \dots, M_0) < \infty$, and eventually the desired estimate on $\mathbb{X}$ will depend on $\Theta(M_0, \dots, M_0)$.

It is illustrative to present some examples explaining the previous abstract notation in some particular cases. Suppose that the goal is to show that some function F satisfies the Carleson measure estimate (1.10) in $\mathbb{X} = \mathbb{R}^{n+1} \setminus E$, with E being UR (see the second part of Theorem 3.31). In this case $M_{\mathbb{X}} \in [1, \infty)^3$ is the UR character of E, and we let $\mathbb{S}_{\mathbb{X}}$ be the collection of bounded chord-arc subdomains of $\mathbb{X}$, in which case $M_{\mathbb{Y}} \in [1, \infty)^4$ is the CAD character of $\mathbb{Y}$. With this in hand we show that there is a constant M_0 (depending only on $M_{\mathbb{X}}$, dimension, and the harmless discretionary parameters τ, η and K, and thus independent of F; see Lemma 2.53) so that the resulting estimate can be transferred from the collection of CAD with parameters in the character at most M_0, and hence the Carleson estimate (1.10) holds with a constant depending only on $\Theta(M_0, M_0, M_0, M_0)$, and other harmless parameters. Similarly, another example is the case that $\mathbb{X} = D$ is a CAD, hence $M_{\mathbb{X}} \in [1, \infty)^4$ is its CAD character, and $\mathbb{S}_{\mathbb{X}}$ is some collection of bounded Lipschitz chord-subdomains of $\mathbb{X}$; then $M_{\mathbb{Y}} \in [1, \infty)^3$ is the Lipschitz CAD character of $\mathbb{Y}$.

3. Transference of Carleson measure estimates

In this section we show how to transfer CME estimates from Lipschitz to CAD (see Theorem 3.6) and then from CAD to the complement of a UR set (see Theorem 3.31). These two independent results, each interesting in its own right, can be combined to give immediately the following:

Corollary 3.1. *Let $E \subset \mathbb{R}^{n+1}$ be an n-dimensional UR set and let $F \in L^2_{\mathrm{loc}}(\mathbb{R}^{n+1} \setminus E)$. If F satisfies the Carleson measure estimate (1.10) for every bounded Lipschitz subdomain of $\mathbb{R}^{n+1} \setminus E$ with constant depending on the Lipschitz character (see Notation 2.56), then F satisfies the Carleson measure estimate (1.10) in $\mathbb{R}^{n+1} \setminus E$ as well. More precisely, there exists a large constant M_0 (depending only n and the UR character of E[5]) so that using the notation in (1.10) there holds*

$$\|F\|_{\mathrm{CME}(\mathbb{R}^{n+1}\setminus E)} \le C \sup_{\Omega \subset \mathbb{R}^{n+1}\setminus E} \|F\|_{\mathrm{CME}(\Omega)}, \tag{3.2}$$

where the sup runs over all bounded Lipschitz subdomains $\Omega \subset \mathbb{R}^{n+1} \setminus E$ with parameters in the Lipschitz character at most M_0, and C depends as before only on n, and the UR character of E.

Remark 3.3. The previous result (and also Theorem 3.31) easily yields a version of itself where everything is localized to some open subset with UR boundary. More precisely, let $\Omega \subset \mathbb{R}^{n+1}$ be an open set with $\partial\Omega$ being UR and let $F \in L^2_{\mathrm{loc}}(\Omega)$. Then

$$\|F\|_{\mathrm{CME}(\Omega)} \le C \sup_{D \subset \Omega} \|F\|_{\mathrm{CME}(D)}, \tag{3.4}$$

where the sup runs over all bounded Lipschitz subdomains $D \subset \Omega$ with parameters in the Lipschitz character at most M_0, and C depends only on n and the UR character of $\partial\Omega$.

[5]Our estimates depend also on the discretionary parameters τ, η and K introduced above, but in turn each of these may be chosen to depend at most on n and the UR character of E.

To see this, write $F_\Omega := F$ in Ω and $F_\Omega = 0$ in $\mathbb{R}^{n+1} \setminus \overline{\Omega}$ so that $F \in L^2_{\mathrm{loc}}(\mathbb{R}^{n+1} \setminus \partial\Omega)$. Since $\partial\Omega$ is UR we can apply Corollary 3.1 to $E = \partial\Omega$ and (3.2) easily yields

$$\|F\|_{\mathrm{CME}(\Omega)} = \|F_\Omega\|_{\mathrm{CME}(\mathbb{R}^{n+1}\setminus\partial\Omega)} \le C \sup_{D\subset\mathbb{R}^{n+1}\setminus\partial\Omega} \|F_\Omega\|_{\mathrm{CME}(D)} = C \sup_{D\subset\Omega} \|F_\Omega\|_{\mathrm{CME}(D)}.$$

3.1. ***Transference of Carleson measure estimates: from Lipschitz to chord-arc domains.*** In this section we present a method to transfer the CME estimates from Lipschitz domains to CAD. Our main result is as follows:

Theorem 3.5. *Let $D \subset \mathbb{R}^{n+1}$ be a given CAD and assume that $F \in L^2_{\mathrm{loc}}(D)$ satisfies (2.26). If F satisfies the Carleson measure estimate (1.10) on all bounded Lipschitz subdomains of D with the constant $C = C_0$ depending on the Lipschitz constants of the underlying domains only, then F satisfies the Carleson measure estimate (1.10) in D as well, with the bound depending on C_0, the constant in (2.26), the NTA constants of D and the ADR constants of ∂D only.*

Theorem 3.6. *Let $D \subset \mathbb{R}^{n+1}$ be a given CAD and let $F \in L^2_{\mathrm{loc}}(D)$. If F satisfies the Carleson measure estimate (1.10) for every bounded Lipschitz subdomain of D with constant depending on the Lipschitz character (see Notation 2.56), then F satisfies the Carleson measure estimate (1.10) in D as well. More precisely, there exists a large constant M_0 (depending only n and the CAD character of D) so that using the notation in (1.10) there holds*

$$\|F\|_{\mathrm{CME}(D)} \le C \sup_{\Omega\subset D} \|F\|_{\mathrm{CME}(\Omega)}, \tag{3.7}$$

where the sup runs over all bounded Lipschitz subdomains $\Omega \subset D$ with parameters in the Lipschitz character at most M_0, and C depends as before only on n, and the CAD character of D.

Let us remark that in the course of the proof we ensure a suitable choice of a (sufficiently small) η and a (sufficiently large) K is (2.8) which strictly speaking affect the constant in (3.7). However, as all choices depend on dimension and the CAD character only, this does not affect the result as stated above.

In preparation to prove the previous result we start with the following version of the John–Nirenberg inequality. It is a suitable modification of [Hofmann and Mayboroda 2009, Lemma 10.1] which, in turn, was inspired by [Auscher et al. 2001, Lemma 2.14]. Here we present an alternative proof along the lines in [Marín et al. 2020, Lemma A.1]. Given Ω an open set with an ADR boundary, let Q_0 be either $\partial\Omega$, in which case $\mathbb{D}_{Q_0} = \mathbb{D}(\partial\Omega)$, or some fixed dyadic cube in $\mathbb{D}(\partial\Omega)$, in which case $\mathbb{D}_{Q_0}$ is defined in (2.13).

Lemma 3.8. *Let Ω be an open set with an ADR boundary, let Q_0 be either $\partial\Omega$ or a fixed cube in $\mathbb{D}(\partial\Omega)$, and for some given $\eta \ll 1$ and $K \gg 1$, consider a Whitney-dyadic structure $\{\mathcal{W}_Q\}_{Q\in\mathbb{D}(\partial\Omega)}$ for Ω with parameters η and K as in Definition 2.7. Let $F \in L^2_{\mathrm{loc}}(\Omega)$ and suppose that there exist $0 < \alpha < 1$ and $0 < N < \infty$ such that*

$$\sigma\{x \in Q : \mathcal{A}^Q F(x) > N\} \le \alpha\sigma(Q) \quad \textit{for all } Q \in \mathbb{D}_{Q_0}. \tag{3.9}$$

Then, for every $0 < p < \infty$ there exists $C_{\alpha,p}$ depending only on p and α such that

$$\sup_{Q\in\mathbb{D}_{Q_0}} \fint_Q \mathcal{A}^Q F(x)^p \, d\sigma(x) \le C_{\alpha,p} N^p. \tag{3.10}$$

Proof. We first claim that for all $Q \in \mathbb{D}_{Q_0}$

$$\mathcal{A}^Q F(x) \le \mathcal{A}^{Q'} F(x) + \inf_{y\in\widetilde{Q}'} \mathcal{A}^Q F(y) \quad \text{for all } x \in Q' \in \mathbb{D}_Q \setminus \{Q\}, \tag{3.11}$$

where $\widetilde{Q}'$ is the dyadic parent of Q'. This follows easily from the fact that if $x \in Q' \in \mathbb{D}_Q \setminus \{Q\}$ and $y \in \widetilde{Q}'$ then

$$\Gamma^Q(x) \setminus \Gamma^{Q'}(x) \subset \bigcup_{x\in P\in\mathbb{D}_Q\setminus\mathbb{D}_{Q'}} U_P = \bigcup_{\widetilde{Q}'\subset P\subset Q} U_P \subset \Gamma^Q(y).$$

Next, let us set

$$\Xi(t) := \sup_{Q\in\mathbb{D}_{Q_0}} \frac{\sigma(E_Q(t))}{\sigma(Q)} := \sup_{Q\in\mathbb{D}_{Q_0}} \frac{\sigma\{x \in Q : \mathcal{A}^Q F(x) > t\}}{\sigma(Q)}, \qquad 0 < t < \infty. \tag{3.12}$$

From (3.9) it follows that

$$\sigma(E_Q(N)) := \sigma\{x \in Q : \mathcal{A}^Q F(x) > N\} \le \alpha\sigma(Q) \quad \text{for all } Q \in \mathbb{D}_{Q_0}. \tag{3.13}$$

Fix now $Q \in \mathbb{D}_{Q_0}$, $\beta \in (\alpha, 1)$ (we will eventually let $\beta \to 1^+$) and, recalling the notation introduced in Definition 2.4 with $E = \partial\Omega$, set

$$F_Q(N) := \{x \in Q : M_Q^{\mathbb{D}}(1_{E_Q(N)})(x) > \beta\}. \tag{3.14}$$

Note that (3.13) ensures that

$$\fint_Q 1_{E_Q(N)}(y)\, d\sigma(y) = \frac{\sigma(E_Q(N))}{\sigma(Q)} \le \alpha < \beta; \tag{3.15}$$

hence we can extract a family of pairwise disjoint stopping-time cubes $\{Q_j\}_j \subset \mathbb{D}_Q \setminus \{Q\}$ so that $F_Q(N) = \bigcup_j Q_j$ and for every j

$$\frac{\sigma(E_Q(N) \cap Q_j)}{\sigma(Q_j)} > \beta; \qquad \frac{\sigma(E_Q(N) \cap Q')}{\sigma(Q')} \le \beta, \quad Q_j \subsetneq Q' \in \mathbb{D}_Q. \tag{3.16}$$

Fix $t > N$. Observe that $E_Q(t) \subset E_Q(N)$ and

$$\beta < 1 = 1_{E_Q(N)}(x) \le M_Q^{\mathbb{D}}(1_{E_Q(N)})(x) \quad \text{for } \sigma\text{-a.e. } x \in E_Q(t). \tag{3.17}$$

Hence,

$$\sigma(E_Q(t)) = \sigma(E_Q(t) \cap F_Q(N)) = \sum_j \sigma(E_Q(t) \cap Q_j).$$

For every j, by the second estimate in (3.16) applied to $\widetilde{Q}_j$, the dyadic parent of Q_j, we have

$$\sigma(E_Q(N) \cap \widetilde{Q}_j)/\sigma(\widetilde{Q}_j) \le \beta < 1;$$

therefore $\sigma(\widetilde{Q}_j \setminus E_Q(N))/\sigma(\widetilde{Q}_j) \ge 1 - \beta > 0$. In particular, we can pick $x_j \in \widetilde{Q}_j \setminus E_Q(N)$. This and (3.11) imply that for all $x \in Q_j$

$$\mathcal{A}^Q F(x) \le \mathcal{A}^{Q_j} F(x) + \inf_{y\in\widetilde{Q}_j} \mathcal{A}^Q F(y) \le \mathcal{A}^{Q_j} F(x) + \mathcal{A}^{Q_j} F(x_j) \le \mathcal{A}^{Q_j} F(x) + N.$$

Consequently, $\mathcal{A}^{Q_j} F(x) > t - N$ for every $x \in E_Q(t) \cap Q_j$, which further implies

$$\sigma(E_Q(t) \cap Q_j) \le \sigma\{x \in Q_j : \mathcal{A}^{Q_j} F(x) > t - N\} \le \Xi(t - N)\sigma(Q_j).$$

All these give

$$\begin{aligned}\sigma(E_Q(t)) &= \sum_j \sigma(E_Q(t)\cap Q_j) \le \Xi(t-N)\sum_j \sigma(Q_j)\\ &\le \Xi(t-N)\frac{1}{\beta}\sum_j \sigma(E_Q(N)\cap Q_j) \le \Xi(t-N)\frac{1}{\beta}\sigma(E_Q(N)) \le \Xi(t-N)\frac{\alpha}{\beta}\sigma(Q), \end{aligned} \tag{3.18}$$

where we have used the first estimate in (3.16), that the cubes $\{Q_j\}_j$ are pairwise disjoint and, finally, (3.13). Dividing by $\sigma(Q)$ and taking the supremum over all $Q \in \mathbb{D}_{Q_0}$ we obtain

$$\Xi(t) \le \frac{\alpha}{\beta}\Xi(t-N), \qquad t > N. \tag{3.19}$$

Since this estimate is valid for all $\beta \in (\alpha, 1)$, we can now let $\beta \to 1^-$, iterate the previous expression, and use the fact that $\Xi(t) \le 1$ to conclude that

$$\Xi(t) \le \alpha^{-1} e^{-(\log(\alpha^{-1})/N)t}, \qquad t > 0.$$

We finally see how the just-obtained estimate implies (3.10): for any $0 < p < \infty$,

$$\begin{aligned}\fint_Q \mathcal{A}^Q F(x)^p \, d\sigma(x) &= p\int_0^\infty \frac{\sigma\{x\in Q : \mathcal{A}^Q F(x) > t\}}{\sigma(Q)} t^p \frac{dt}{t}\\ &\le p\int_0^\infty \Xi(t)t^p \frac{dt}{t} \le p\alpha^{-1}\int_0^\infty e^{-(\log(\alpha^{-1})/N)t} t^p \frac{dt}{t}\\ &= p\alpha^{-1}\left(\frac{N}{\log(\alpha^{-1})}\right)^p \int_0^\infty e^{-t}t^p\frac{dt}{t} = C_{\alpha,p}N^p. \qquad \square\end{aligned}$$

To address the transference of the Carleson measure condition from Lipschitz to chord-arc domains we shall use the fact that chord-arc domains contain interior big pieces of Lipschitz subdomains.

Proposition 3.20 [David and Jerison 1990]. *Given $\Omega \subset \mathbb{R}^{n+1}$, a CAD, there exist constants $C \ge 2$ and $0 < \theta < 1$ such that for every surface ball $\Delta(x, r) = B(x, r) \cap \partial\Omega$, $x \in \partial\Omega$, $0 < r < \operatorname{diam}(\partial\Omega)$, there exists a bounded Lipschitz domain Ω' for which we have the following conditions*:

(i) *$H^n(\partial\Omega \cap \partial\Omega' \cap B(x, r)) \ge \theta H^n(\Delta(x, r)) \approx \theta r^n$.*

(ii) *There exists X_Δ so that $B(X_\Delta, r/C) \subset B(x, r) \cap \Omega \cap \Omega'$.*

(iii) *$\Omega' \subset \Omega \cap B(x, r)$.*

The Lipschitz character of Ω' as well as $0 < \theta < 1$ and $C \ge 2$ depend on n, the CAD character of D only (and are independent of x, r).

We remark that in [David and Jerison 1990], Proposition 3.20 is proved under weaker assumptions, namely, ADR and an interior corkscrew condition, and a "weak exterior corkscrew condition" which gives exterior disks rather than exterior balls, and with no hypothesis of Harnack chains — but if the Harnack chain condition is assumed, [Azzam et al. 2017] yields the exterior corkscrew condition, hence exterior disks implies exterior balls. Later on, in [Badger 2012], existence of big pieces of Lipschitz subdomains was also proved for usual NTA domains, with no upper ADR assumption on $\partial\Omega$ (the lower

ADR bound holds automatically in the presence of a two-sided corkscrew condition, by virtue of the relative isoperimetric inequality). For the applications that we have in mind here, neither amelioration is significant, and we will simply work with CAD domains in the sense of Definition 1.6.

For future reference we also would like to provide the following corollary.

Corollary 3.21. *Let $\Omega\subset\mathbb{R}^{n+1}$ be a CAD. There exist constants $C\geq 2$ and $0<\theta<1$ such that, for every $Q\in\mathbb{D}(\partial\Omega)$, there exists a bounded Lipschitz domain $\Omega_Q\subset\Omega$ for which, using the notation $B_Q=B(x_Q,r)$, $\Delta_Q=B_Q\cap\partial\Omega$, with $c\ell(Q)\leq r\leq\ell(Q)$ in (2.2), (2.3), we have the following*:

(i) $\sigma(\partial\Omega_Q\cap Q)\geq\theta\sigma(Q)\approx\theta\ell(Q)^n$.

(ii) *For every $Q'\in\mathbb{D}(Q)$ such that there exists a point $y_{Q'}\in Q'\cap\partial\Omega_Q$, there exists $Y_{Q'}$ so that $B(Y_{Q'},\ell(Q')/C)\subset B(y_{Q'},\ell(Q'))\cap\Omega\cap\Omega_Q$, that is, $Y_{Q'}$ is a corkscrew relative to $B(y_{Q'},\ell(Q'))\cap\Omega$ and $\partial\Omega$, and $B(y_{Q'},\ell(Q'))\cap\partial\Omega_Q$ and Ω_Q. Furthermore, with the appropriate choice of η and K in (2.8), we have $B(Y_{Q'},\ell(Q')/C)\subset U_{Q'}$.*

(iii) $\Omega_Q\subset\Omega\cap B_Q$.

The Lipschitz character of Ω_Q as well as $0<\theta<1$, $C\geq 2$, depend on n, and the CAD character of Ω only (and are uniform in Q, Q').

Proof. The corollary follows directly from Proposition 3.20. Indeed, for any $Q\in\mathbb{D}(\partial\Omega)$ there exists $\Delta_Q\subset Q$ as in (2.2), (2.3). One can then build a Lipschitz domain Ω_Q from Proposition 3.20 corresponding to Δ_Q, and then the conditions (i) and (iii) in Proposition 3.20 give (i) and (iii) in Corollary 3.21, respectively. Condition (ii) in Corollary 3.21 follows from the fact that a Lipschitz domain Ω_Q is, in particular, a CAD, and hence, it has a corkscrew point relative to $B(y_{Q'},r')\cap\partial\Omega_Q$ since $r'\leq\ell(Q')\leq\ell(Q)\approx\operatorname{diam}(\partial\Omega_Q)$ (the $\approx$ follows from (ii) and (iii) in Proposition 3.20). Using the fact that $\Omega_Q\subset\Omega$, one can easily see that Y_Q is also a corkscrew point in Ω relative to $B(y_{Q'},r')\cap\partial\Omega$. It remains to observe that a suitable choice of η and K (uniform in Q') ensures that such a corkscrew point always belongs to $U_{Q'}$ and moreover, $B(Y_{Q'},C^{-1}\ell(Q'))\subset U_{Q'}$. □

We are now ready to prove Theorem 3.6:

Proof of Theorem 3.6. By (2.29) and Remark 3.34 we can reduce matters to estimate $\|F\|_{\mathrm{CME}^{\mathrm{dyad}}(D)}$. Fix some $Q\in\mathbb{D}$. According to Corollary 3.21 (along with the inner regularity property of the measure) there exists a bounded Lipschitz domain Ω_Q such that $\sigma(\partial\Omega_Q\cap Q)\geq\theta\sigma(Q)$, and the Lipschitz character of Ω_Q as well as $0<\theta<1$ depend only on n and the CAD character of D (and are uniformly in Q). The domain Ω_Q further satisfies properties (i)–(iii) in Corollary 3.21. Given $x\in Q\setminus\partial\Omega_Q$, since $\partial\Omega_Q$ is closed, there exists $r_x>0$ such that $B(x,r_x)\cap\partial\Omega_Q=\varnothing$. Pick then $Q_x\in\mathbb{D}$ with $\ell(Q_x)\ll\min\{\ell(Q),r_x\}$ so that $x\in Q_x$. Then, $x\in Q\cap Q_x$ and necessarily $Q_x\subset Q$. Also, $Q_x\subset B(x,r_x)$ since $x\in Q_x$ and $\operatorname{diam}(Q_x)\approx\ell(Q_x)\ll r_x$. Thus, $Q_x\subset Q\setminus\partial\Omega_Q$ and there exists a cube with maximal size $Q_x^{\max}\in\mathbb{D}_Q$ so that $Q_x^{\max}\subset Q\setminus\partial\Omega_Q$. Note that $Q_x^{\max}\subsetneq Q$ since $\sigma(\partial\Omega_Q\cap Q)>0$. Thus, by maximality, $\partial\Omega_Q\cap Q'\neq\varnothing$ for every Q' with $Q_x^{\max}\subsetneq Q'\subset Q$. Consider then $\mathcal{F}=\{Q_j\}_j\subset\mathbb{D}_Q\setminus\{Q\}$ the collection of such maximal cubes. By construction, the cubes in $\mathcal{F}$ are pairwise disjoint and also $Q\setminus\partial\Omega_Q=\bigcup_j Q_j$. Associated with $\mathcal{F}$ we build the corresponding local sawtooth $\Omega_{\mathcal{F},Q}$ (see (2.20)).

Note that if $Q' \subset Q_j \in \mathcal{F}$, then $Q' \subset Q_j \subset Q \setminus \partial\Omega_Q$; hence $\partial\Omega_Q \cap Q' = \varnothing$. Conversely, if $Q' \in \mathbb{D}_Q$ is such that $\partial\Omega_Q \cap Q' = \varnothing$, then $Q' \subset Q \setminus \partial\Omega_Q = \bigcup_j Q_j$ and there is $Q_j \in \mathcal{F}$ such that $Q' \cap Q_j \neq \varnothing$. If $Q_j \subsetneq Q'$ then by the maximality of Q_j we have $\partial\Omega_Q \cap Q' \neq \varnothing$, which is a contradiction. As a result, necessarily $Q' \subset Q_j$. All in one, for every $Q' \in \mathbb{D}_Q$, we have that $Q' \subset Q_j \in \mathcal{F}$ if and only if $\partial\Omega_Q \cap Q' = \varnothing$. Equivalently, given $Q' \in \mathbb{D}_Q$, one has that $Q' \in \mathbb{D}_{\mathcal{F},Q}$ if and only if $Q' \cap \partial\Omega_Q \neq \varnothing$.

Let $N \geq 1$ to be chosen and by Chebyshev's inequality

$$\begin{aligned}\sigma\{x \in \partial\Omega_Q \cap Q : \mathcal{A}^Q F(x) > N\} &\leq \frac{1}{N^2}\int_{\partial\Omega_Q \cap Q}\iint_{\Gamma^Q(x)} |F(Y)|^2\delta(Y)^{1-n}\,dY \\ &\leq \frac{1}{N^2}\sum_{Q' \in \mathbb{D}_Q} \sigma(\partial\Omega_Q \cap Q')\iint_{U_{Q'}} |F(Y)|^2\delta(Y)^{1-n}\,dY \\ &\approx \frac{1}{N^2}\sum_{Q' \in \mathbb{D}_{\mathcal{F},Q}} \frac{\sigma(\partial\Omega_Q \cap Q')}{\sigma(Q')}\iint_{U_{Q'}} |F(Y)|^2\delta(Y)\,dY \\ &\lesssim \frac{1}{N^2}\iint_{\Omega_{\mathcal{F},Q}} |F(Y)|^2\delta(Y)\,dY,\end{aligned}$$

where we have used that $\delta(Y) \approx \ell(Q')$ for every $Y \in U_Q'$ and also that the family $\{U_Q'\}_{Q' \in \mathbb{D}}$ has bounded overlap. We claim that

$$\frac{1}{\sigma(Q)}\iint_{\Omega_{\mathcal{F},Q}} |F(X)|^2\delta(X)\,dX \leq C\Big(\sup_{\Omega \subset D} \|F\|_{\mathrm{CME}(\Omega)} + \|F\|_{\mathbb{C}_0(D)}\Big), \tag{3.22}$$

where the sup runs over all bounded Lipschitz subdomains $\Omega \subset D$ with parameters in the Lipschitz character at most M_0, and C depends as before only on n, and the CAD character of D. Assuming this momentarily, and invoking (3.36), we conclude that

$$\begin{aligned}\sigma\{x \in Q : \mathcal{A}^Q F(x) > N\} &\leq \sigma(Q \setminus \partial\Omega_Q) + \frac{C}{N^2}\iint_{\Omega_{\mathcal{F},Q}} |F(Y)|^2\delta(Y)\,dY \\ &\leq (1-\theta)\sigma(Q) + \frac{C}{N^2}\sup_{\Omega \subset D}\|F\|_{\mathrm{CME}(\Omega)}\sigma(Q) \leq \Big(1 - \frac{\theta}{2}\Big)\sigma(Q),\end{aligned}$$

provided $N^2 = (2C/\theta)\sup_{\Omega \subset D}\|F\|_{\mathrm{CME}(\Omega)}$. Applying then the John–Nirenberg inequality, Lemma 3.8 with $Q_0 = E = \partial D$, which is ADR by assumption, extending F as 0 in $\mathbb{R}^{n+1} \setminus \overline{D}$, and with $p = 2$ we then conclude that

$$\sup_{Q \in \mathbb{D}_{Q_0}} \fint_Q \mathcal{A}^Q F(x)^2\sigma(x) \lesssim \sup_{\Omega \subset D}\|F\|_{\mathrm{CME}(\Omega)}.$$

In turn, this yields

$$\begin{aligned}\iint_{T_Q} |F|^2\delta\,dX &\leq \sum_{Q' \in \mathbb{D}_Q}\iint_{U_{Q'}} |F|^2\delta\,dX \approx \sum_{Q' \in \mathbb{D}_Q}\sigma(Q')\iint_{U_{Q'}} |F|^2\delta^{1-n}\,dX \\ &= \sum_{Q' \in \mathbb{D}_Q}\int_{Q'}\Big(\iint_{U_{Q'}} |F|^2\delta^{1-n}\,dX\Big)\,d\sigma \lesssim \int_Q\Big(\iint_{\Gamma^Q(x)} |F|^2\delta^{1-n}\,dY\Big)\,d\sigma(x) \\ &= \int_Q \mathcal{A}^Q F(x)^2\sigma(x) \lesssim \sigma(Q)\sup_{\Omega \subset D}\|F\|_{\mathrm{CME}(\Omega)}.\end{aligned}$$

Here we have used that $\delta(\cdot) \approx \ell(Q')$ in $U_{Q'}$ and the fact that the family $\{U_Q\}_{Q \in \mathbb{D}}$ has bounded overlap.

We are then left with showing (3.22). To this end, let us write

$$\iint_{\Omega_{\mathcal{F},Q}} |F(X)|^2\delta(X)\,dX \le \sum_{Q'\in\mathbb{D}_{\mathcal{F},Q}} \iint_{U_{Q'}} |F(Y)|^2\delta(Y)\,dY = \sum_{Q'\in\mathbb{D}^1_{\mathcal{F},Q}} \cdots + \sum_{Q'\in\mathbb{D}^2_{\mathcal{F},Q}} \cdots, \tag{3.23}$$

where, for some $\varepsilon > 0$ to be chosen,

$$\mathbb{D}^1_{\mathcal{F},Q} := \left\{ Q' \in \mathbb{D}_{\mathcal{F},Q} : \operatorname{dist}(U_{Q'}, \partial D) \le \frac{1}{\varepsilon}\operatorname{dist}(U_{Q'}, \partial\Omega_Q) \right\}, \quad \mathbb{D}^2_{\mathcal{F},Q} := \mathbb{D}_{\mathcal{F},Q} \setminus \mathbb{D}^1_{\mathcal{F},Q}.$$

Note that, in principle, $U_{Q'}$ can intersect $\partial\Omega_Q$. For later use it is convenient to record that $\ell(Q') \approx \operatorname{dist}(U_{Q'}, \partial D) \approx \operatorname{dist}(U_{Q'}, Q')$ by (2.8), (2.6), (2.9), (2.10).

Let $Q' \in \mathbb{D}^1_{\mathcal{F},Q}$, the fact that $Q' \in \mathbb{D}_{\mathcal{F},Q}$ implies that there exists $y \in Q' \cap \partial\Omega_Q$; hence

$$\begin{aligned} \varepsilon\ell(Q') &\approx \varepsilon \operatorname{dist}(U_{Q'}, \partial D) \le \operatorname{dist}(U_{Q'}, \partial\Omega_Q) \\ &\le \operatorname{dist}(U_{Q'}, y) \le \operatorname{dist}(U_{Q'}, Q') + \operatorname{diam}(Q') \lesssim \ell(Q'). \end{aligned} \tag{3.24}$$

In particular, for every $Y \in U_{Q'}$ with $Q' \in \mathbb{D}^1_{\mathcal{F},Q}$ we have

$$\delta(Y) = \operatorname{dist}(Y, \partial D) \lesssim \ell(Q') + \operatorname{dist}(U_{Q'}, \partial D) \lesssim \varepsilon^{-1}\operatorname{dist}(U_{Q'}, \partial\Omega_Q) \lesssim \varepsilon^{-1}\operatorname{dist}(Y, \partial\Omega_Q). \tag{3.25}$$

Note also that since $y' \in Q' \cap \partial\Omega_Q \neq \varnothing$, according to Corollary 3.21 part (ii), we can find $Y_{Q'}$ so that $B(Y_{Q'}, \ell(Q')/C) \subset B(y_{Q'}, \ell(Q')) \cap \Omega \cap \Omega_Q \cap U_{Q'}$. Hence, $\Omega_Q \cap U_{Q'} \neq \varnothing$, and then due to (3.24) and the fact that $U_{Q'}$ is connected by construction, we conclude that $U_{Q'} \subset \Omega_Q$. As a result,

$$\begin{aligned} \sum_{Q'\in\mathbb{D}^1_{\mathcal{F},Q}} \iint_{U_{Q'}} |F(Y)|^2\delta(Y)\,dY &\lesssim \varepsilon^{-1} \sum_{Q'\in\mathbb{D}^1_{\mathcal{F},Q}} \iint_{U_{Q'}} |F(Y)|^2 \operatorname{dist}(Y, \partial\Omega_Q)\,dY \\ &\lesssim \iint_{\Omega_Q} |F(Y)|^2 \operatorname{dist}(Y, \partial\Omega_Q)\,dY \le \sigma(Q) \sup_{\Omega\subset D} \|F\|_{\mathrm{CME}(\Omega)}, \end{aligned} \tag{3.26}$$

where we used (3.25), the finite overlap property of the family $\{U_{Q'}\}_{Q'\in\mathbb{D}}$, and the fact that Ω_Q is a bounded Lipschitz subdomain of D with character controlled by the CAD parameters in the last one. Note that $\Omega_Q \subset B(x_Q, C\ell(Q))$ for some uniform constant C, which justifies the bound by $\sigma(Q)$.

Consider next the family $\mathbb{D}^2_{\mathcal{F},Q}$ and we shall demonstrate that they satisfy a packing condition. Indeed, recall from above that $\ell(Q') \approx \operatorname{dist}(U_{Q'}, \partial D)$, so that in particular, if $Q' \in \mathbb{D}^2_{\mathcal{F},Q}$, then

$$\operatorname{dist}(U_{Q'}, \partial\Omega_Q) \lesssim \varepsilon\ell(Q'). \tag{3.27}$$

It follows that for a suitably small ε depending on the implicit constant in (3.27) and τ, we can ensure that fattened regions $\widehat{U}_{Q'}$ corresponding to $U_{Q'}$ (see (2.11)) necessarily intersect $\partial\Omega_Q$ and, moreover, $H^n(\widehat{U}_{Q'} \cap \partial\Omega_Q) \approx \ell(Q')^n$, while the family $\{\widehat{U}_{Q'}\}_{Q'}$ still has finite overlap. Since the Lipschitz character of $\partial\Omega_Q$ depends on the CAD character of D, we have that $H^n(\partial\Omega_Q) \approx \operatorname{diam}(\partial\Omega_Q)^n \approx \operatorname{diam}(\Omega_Q)^n \approx \ell(Q) \approx \sigma(Q)$, with implicit constants which are uniform in Q. Thus, all in all,

$$\sum_{Q'\in\mathbb{D}^2_{\mathcal{F},Q}} \sigma(Q') \approx \sum_{Q'\in\mathbb{D}^2_{\mathcal{F},Q}} \ell(Q')^n \approx \sum_{Q'\in\mathbb{D}^2_{\mathcal{F},Q}} H^n(\widehat{U}_{Q'} \cap \partial\Omega_Q) \lesssim H^n(\partial\Omega_Q) \approx \sigma(Q). \tag{3.28}$$

Consequently, using that one can cover $U_{Q'}$ by a uniform number of balls of the form $B(X, \delta(X)/2)$ with $X \in U_{Q'}$ (and hence $\delta(X) \approx \ell(Q')$) we arrive at

$$\sum_{Q' \in \mathbb{D}^2_{\mathcal{F},Q}} \iint_{U_{Q'}} |F(Y)|^2 \delta(Y)\, dY \lesssim \|F\|_{\mathbb{C}_0(D)} \sum_{Q' \in \mathbb{D}^2_{\mathcal{F},Q}} \sigma(Q') \lesssim \sigma(Q)\|F\|_{\mathbb{C}_0(D)}, \tag{3.29}$$

simply recalling the notation introduced in (2.26).

Collecting (3.23), (3.26), and (3.29) we conclude as desired (3.22), completing the proof. □

3.2. ***Transference of Carleson measure estimates*: *from chord-arc domains to the complement of a UR set.*** Let us now discuss the "transference" mechanism allowing one to pass from the Carleson measure estimates on CAD to those open sets with UR boundaries. The main idea consists in showing that if for some given F one can prove (1.10) on $D \subset \mathbb{R}^{n+1} \setminus E$, any bounded CAD, then (1.10) holds for $\mathbb{R}^{n+1} \setminus E$. This was proved in [Hofmann et al. 2016, Theorem 1.1] for $F = |\nabla u|/\|u\|_{L^\infty(\mathbb{R}^{n+1}\setminus E)}$ with u being a bounded harmonic function in $\mathbb{R}^{n+1} \setminus E$. On the other hand, it was already observed in Remark 4.28 of that work that harmonicity is not really needed and that one could take for instance $F = |\nabla u|/\|u\|_{L^\infty(\mathbb{R}^{n+1}\setminus E)}$ with u being a bounded solution of a second-order elliptic PDE or, more generally, $F = |\nabla^m u|/\|\nabla^{m-1} u\|_{L^\infty(\mathbb{R}^{n+1}\setminus E)}$ with u being a bounded solution of a $2m$-th order elliptic PDE, $m \in \mathbb{N}$. We shall come back to this point with more details in Section 7, and for now try to keep the discussion general for as long as possible.

Remark 3.30. There is a slightly glitchy point of notation point. For reasons of homogeneity, one might prefer to normalize so that $F = \operatorname{dist}(\cdot\,, E)|\nabla u|/\|u\|_{L^\infty(\mathbb{R}^{n+1}\setminus E)}$. However, making the function F and later on G and H in Section 4 depend on the open set (via its distance to the boundary) has its own dangers and kills the beauty of the generality here.

The following result is stated in [Hofmann et al. 2016, Theorem 1.1] exclusively for harmonic functions, but as noted in Remark 4.28 of that work, the same proof applies verbatim to any bounded function satisfying Caccioppoli's inequality along with CME in chord-arc subdomains. The argument further extends to the following formulation with a few changes. For the sake of self-containment we present below a somewhat different and more direct argument.

Theorem 3.31. *Let $E \subset \mathbb{R}^{n+1}$ be an n-dimensional UR set and let $F \in L^2_{\mathrm{loc}}(\mathbb{R}^{n+1} \setminus E)$. Given $\eta \ll 1$ and $K \gg 1$, consider the decomposition $\mathbb{D}(E) = \mathcal{G} \cup \mathcal{B}$ from Lemma 2.42, as well as a Whitney-dyadic structure $\{\mathcal{W}_Q\}_{Q \in \mathbb{D}(E)}$ for $\mathbb{R}^{n+1} \setminus E$ with parameters η and K; see Section 2.2. Then using the notation in (1.10) and (2.26) there holds*

$$\|F\|_{\mathrm{CME}(\mathbb{R}^{n+1}\setminus E)} \le C \max\Big\{\|F\|_{\mathbb{C}_0(\mathbb{R}^{n+1}\setminus E)}, \sup_{\boldsymbol{S} \subset \mathcal{G}} \|F\|_{\mathrm{CME}(\Omega^{\pm}_{\boldsymbol{S}})}\Big\}, \tag{3.32}$$

where $\Omega^{\pm}_{\boldsymbol{S}}$ is defined by (2.52) (with $\boldsymbol{S}' = \boldsymbol{S}$) and where C depends only on n, the UR character of E, and the choice of η, K, τ.

In particular, if F satisfies the Carleson measure estimate (1.10) for every bounded chord-arc subdomain $D \subset \mathbb{R}^{n+1} \setminus E$ with constants depending on the CAD character (see Notation 2.56) then F satisfies *the Carleson measure estimate (1.10) on $\mathbb{R}^{n+1}\setminus E$. More precisely, there exists a large constant M_0 (depending only n and the UR character of E) so that using the notation in (1.10) there holds*

$$\|F\|_{\mathrm{CME}(\mathbb{R}^{n+1}\setminus E)} \leq C \sup_{D\subset\mathbb{R}^{n+1}\setminus E} \|F\|_{\mathrm{CME}(D)}, \tag{3.33}$$

where the sup runs over all bounded chord-arc subdomains $D\subset\mathbb{R}^{n+1}\setminus E$ with parameters in the CAD character at most M_0, and C depends as before only on n and the UR character of E.

We note that much as in Remark 3.3 one can easily get a version of this result valid where everything is localized to some open subset with UR boundary. The precise statement and the details are left to the interested reader.

Remark 3.34. As already mentioned in Remark 2.25 and for PDE applications, the quantities $\|F\|_{\mathbb{C}_0(\mathbb{R}^{n+1}\setminus E)}$ or $\|F\|_{\mathbb{C}_0(D)}$ are harmless terms since they are typically finite, whether or not F satisfies Carleson measure estimates on some family of nice subdomains. However, one can also see that these terms are under-controlled when one imposes Carleson measure estimates on bounded Lipschitz subdomains. Let $E\subset\mathbb{R}^{n+1}$ be an n-dimensional ADR set, write $\delta(\cdot)=\operatorname{dist}(\cdot,E)$, and let $F\in L^2_{\mathrm{loc}}(\mathbb{R}^{n+1}\setminus E)$. Note that $\Omega_X=B(X,\delta(X))$ is a bounded Lipschitz subdomain of $\mathbb{R}^{n+1}\setminus E$ with all the parameters in the Lipschitz character bounded by $M_n\geq 1$ which depends just on n. Also if $Y\in B(X,\delta(X)/2)$ then $\operatorname{dist}(Y,\partial\Omega_X)\geq\delta(X)/2$ and $Y\in B(z,2\delta(X))$ for any $z\in\partial\Omega_X$. Thus, for any $z\in\partial\Omega_X$

$$\frac{1}{\delta(X)^{n-1}}\iint_{B(X,\delta(X)/2)}|F(Y)|^2\,dY \leq \frac{2}{\delta(X)^n}\iint_{B(z,2\delta(X))}|F(Y)|^2\operatorname{dist}(Y,\partial\Omega_X)\,dY$$

and, consequently,

$$\|F\|_{\mathbb{C}_0(\mathbb{R}^{n+1}\setminus E)} \leq 2^{n+1}\sup_{D\subset\mathbb{R}^{n+1}\setminus E}\|F\|_{\mathrm{CME}(D)}, \tag{3.35}$$

where the sup runs over all bounded Lipschitz subdomains of $\mathbb{R}^{n+1}\setminus E$ with all the parameters in the Lipschitz character at most $M_n\geq 1$. Analogously, if $F\in L^2_{\mathrm{loc}}(\Omega)$, where $\Omega\subset\mathbb{R}^{n+1}$ is an open set with $\partial\Omega$ being n-dimensional ADR, then

$$\|F\|_{\mathbb{C}_0(\Omega)} \leq 2^{n+1}\sup_{D\subset\Omega}\|F\|_{\mathrm{CME}(D)}, \tag{3.36}$$

where the sup runs over all bounded Lipschitz subdomains of Ω with all the parameters in the Lipschitz character at most $M_n\geq 1$.

Proof. We write $\delta(\cdot)=\operatorname{dist}(\cdot,E)$ and Define $\beta_Q=\iint_{U_{Q,\tau/2}}|F|^2\delta\,dX$ for every $Q\in\mathbb{D}=\mathbb{D}(E)$. Fix $Q_0\in\mathbb{D}$. Using the decomposition $\mathbb{D}(E)=\mathcal{G}\cup\mathcal{B}$ from Lemma 2.42

$$\begin{aligned}\iint_{T_{Q_0,\tau/2}}|F(X)|^2\delta(X)\,dX &\leq \sum_{Q\in\mathbb{D}_{Q_0}}\beta_Q = \sum_{Q\in\mathbb{D}_{Q_0}\cap\mathcal{B}}\beta_Q + \sum_{Q\in\mathbb{D}_{Q_0}\cap\mathcal{G}}\beta_Q \\ &= \sum_{Q\in\mathbb{D}_{Q_0}\cap\mathcal{B}}\beta_Q + \sum_{\boldsymbol{S}:\mathbb{D}_{Q_0}\cap\boldsymbol{S}\neq\varnothing}\ \sum_{Q\in\mathbb{D}_{Q_0}\cap\boldsymbol{S}}\beta_Q =: \Sigma_1+\Sigma_2,\end{aligned}$$

and we estimate each term in turn. For Σ_1 we observe that by construction the $U_{Q,\tau/2}$'s are uniformly bounded unions of Whitney cubes of size of the order of $\ell(Q)$ and with distance to E of the order of $\ell(Q)$ and it follows easily that $\beta_Q \lesssim C_0\sigma(Q)$, where the implicit constants depend only on n, the UR character of E, and the choice of η, K, τ. Hence,

$$\Sigma_1 \lesssim \|F\|_{\mathbb{C}_0(\mathbb{R}^{n+1}\setminus E)} \sum_{Q\in\mathbb{D}_{Q_0}\cap\mathcal{B}} \sigma(Q) \lesssim \|F\|_{\mathbb{C}_0(\mathbb{R}^{n+1}\setminus E)}\sigma(Q_0), \tag{3.37}$$

where in the last estimate we have used Lemma 2.42(ii).

Let us estimate Σ_2. Fix $\boldsymbol{S}$ so that $\mathbb{D}_{Q_0}\cap\boldsymbol{S}\neq\varnothing$ and write $Q_1 = Q_1(\boldsymbol{S}) = Q_0\cap Q(\boldsymbol{S})$. Note that if $Q\in\mathbb{D}_{Q_0}\cap\boldsymbol{S}$ then $Q\subset Q_1\subset Q(\boldsymbol{S})$ and by the coherency of $\boldsymbol{S}$ we conclude that $Q_1\in\boldsymbol{S}$. Set $\delta_{\boldsymbol{S}}^{\pm}(\,\cdot\,) = \operatorname{dist}(\,\cdot\,,\partial\Omega_{\boldsymbol{S}}^{\pm})$ (see (2.52) with $\boldsymbol{S}'=\boldsymbol{S}$). Note that $\Omega_{\boldsymbol{S}}^{\pm}$ is comprised of Whitney regions of the form $U_Q^{\pm} = U_{Q,\tau}^{\pm}$. Thus for $X\in U_{Q,\tau/2}^{\pm}$ with $Q\in\boldsymbol{S}$, we have that $\delta(X)\approx\delta_{\boldsymbol{S}}^{\pm}(X)$, where the implicit constants depend on τ. This, the fact that the family $\{U_Q^{\pm}\}_{Q\in\mathbb{D}}$ has bounded overlap and (2.14) easily give

$$\sum_{Q\in\mathbb{D}_{Q_0}\cap\boldsymbol{S}} \beta_Q = \sum_{Q\in\mathbb{D}_{Q_1}\cap\boldsymbol{S}} \beta_Q \approx \sum_{Q\in\mathbb{D}_{Q_1}\cap\boldsymbol{S}} \iint_{U_Q^{\pm}} |F|^2\delta_{\boldsymbol{S}}^{\pm}\,dX \lesssim \iint_{B_{Q_1}^*\cap\Omega_{\boldsymbol{S}}^{\pm}} |F|^2\delta_{\boldsymbol{S}}^{\pm}\,dX,$$

where $B_{Q_1}^* := B(x_{Q_1}, K\ell(Q_1))$. Pick now $X_1^{\pm}\in U_{Q_1,\tau/2}^{\pm}$ and choose $x_1^{\pm}\in\partial\Omega_{\boldsymbol{S}}^{\pm}$ so that $|X_1^{\pm}-x_1^{\pm}| = \delta_{\boldsymbol{S}}^{\pm}(X_1^{\pm})\approx\delta(X_1^{\pm})\approx\ell(Q_1)$. Therefore, $B_{Q_1}^*\subset B_{Q_1}^{**} = B_{Q_1}(x_1^{\pm}, C\ell(Q_1))$, where C depends on n, the UR character of E and η, K and τ. Thus,

$$\sum_{Q\in\mathbb{D}_{Q_0}\cap\boldsymbol{S}} \beta_Q \lesssim \iint_{B_{Q_1}^{**}\cap\Omega_{\boldsymbol{S}}^{\pm}} |F|^2\delta_{\boldsymbol{S}}^{\pm}\,dX \lesssim \|F\|_{\mathrm{CME}(\Omega_{\boldsymbol{S}}^{\pm})}\ell(Q_1)^n \approx \|F\|_{\mathrm{CME}(\Omega_{\boldsymbol{S}}^{\pm})}\sigma(Q_1). \tag{3.38}$$

Using this and recalling that $Q_1 = Q_1(\boldsymbol{S}) = Q_0\cap Q(\boldsymbol{S})$, we can bound Σ_2 as follows:

$$\begin{aligned}\Sigma_2 &= \sum_{\boldsymbol{S}:\mathbb{D}_{Q_0}\cap\boldsymbol{S}\neq\varnothing}\ \sum_{Q\in\mathbb{D}_{Q_0}\cap\boldsymbol{S}} \beta_Q \lesssim \sup_{\boldsymbol{S}\subset\mathcal{G}} \|F\|_{\mathrm{CME}(\Omega_{\boldsymbol{S}}^{\pm})} \sum_{\boldsymbol{S}:\mathbb{D}_{Q_0}\cap\boldsymbol{S}\neq\varnothing} \sigma(Q_0\cap Q(\boldsymbol{S}))\\ &= \sup_{\boldsymbol{S}\subset\mathcal{G}} \|F\|_{\mathrm{CME}(\Omega_{\boldsymbol{S}}^{\pm})}\Bigg(\sum_{\boldsymbol{S}:Q(\boldsymbol{S})\subset Q_0} \sigma(Q(\boldsymbol{S})) + \sum_{\substack{\boldsymbol{S}:\mathbb{D}_{Q_0}\cap\boldsymbol{S}\neq\varnothing\\ Q_0\subsetneq Q(\boldsymbol{S})}} \sigma(Q_0)\Bigg) =: \Sigma_{21}+\Sigma_{22}.\end{aligned}$$

Using Lemma 2.42(ii) we easily obtain

$$\sum_{\boldsymbol{S}:Q(\boldsymbol{S})\subset Q_0} \sigma(Q(\boldsymbol{S})) \lesssim \sigma(Q_0),$$

where the implicit constant depends only on n, the UR character of E, and the choice of η, K, τ. For the other term we note that the facts $\mathbb{D}_{Q_0}\cap\boldsymbol{S}\neq\varnothing$ and $Q_0\subsetneq Q(\boldsymbol{S})$ imply that $Q_0\in\boldsymbol{S}$ by the coherency of $\boldsymbol{S}$; hence $\Sigma_{22}=0$ if $Q_0\in\mathcal{B}$. On the other hand, if $Q_0\in\mathcal{G}$ there is a unique $\boldsymbol{S}_0\subset\mathcal{G}$ so that $Q_0\in\boldsymbol{S}_0$ and $\mathbb{D}_{Q_0}\cap\boldsymbol{S}=\varnothing$ for every $\boldsymbol{S}\neq\boldsymbol{S}_0$ with $Q_0\subsetneq Q(\boldsymbol{S})$. This clearly implies that in this case

$$\sum_{\substack{\boldsymbol{S}:\mathbb{D}_{Q_0}\cap\boldsymbol{S}\neq\varnothing\\ Q_0\subsetneq Q(\boldsymbol{S})}} \sigma(Q_0) = \sigma(Q_0).$$

If we finally collect all the obtained estimates we conclude that

$$\|F\|_{\mathrm{CME}^{\mathrm{dyad}}(\mathbb{R}^{n+1}\setminus E)} = \sup_{Q\in\mathbb{D}(E)} \frac{1}{\sigma(Q)} \iint_{T_{Q,\tau/2}} |F(X)|^2\delta(X)\,dX \\ \leq C\max\Big\{\|F\|_{\mathbb{C}_0(\mathbb{R}^{n+1}\setminus E)}, \sup_{S\subset\mathcal{G}} \|F\|_{\mathrm{CME}(\Omega_S^\pm)}\Big\}, \tag{3.39}$$

where C depends only on n, the UR character of E, and the choice of η, K, τ. Thus, the desired estimates follows from (2.29).

To complete the proof we look at the second part of the statement. By (3.35) and the fact that bounded Lipschitz domains are CAD with all the parameters in the CAD character by the Lipschitz character we have $\|F\|_{\mathbb{C}_0(\mathbb{R}^{n+1}\setminus E)} \lesssim \sup_{D\subset\mathbb{R}^{n+1}\setminus E}\|F\|_{\mathrm{CME}(D)}$, where the sup runs over all bounded CAD subdomains with character at most M_n. On the other hand, Lemma 2.53 establishes that all the $\Omega_{S}^{\pm}$'s are CAD with parameters in the CAD character all controlled by $M_0'\geq 1$ (depending on the allowable parameters). They are also bounded since every S has a maximal cube $Q(S)$ and hence $\Omega_S^\pm\subset B^*_{Q(S)}$ (see (2.14)). Consequently,

$$\sup_{S\subset\mathcal{G}}\|F\|_{\mathrm{CME}(\Omega_S^\pm)} \leq \sup_D \|F\|_{\mathrm{CME}(D)},$$

where the second sup runs over all bounded CAD with character at most M_0'. Taking $M_0=\max\{M_n, M_0'\}$, we easily see that (3.32) along with the above observations readily yield (3.33). □

4. Carleson estimates, $\mathcal{A} < N$ estimates and good-λ arguments

Given an open set $\Omega\subset\mathbb{R}^{n+1}$ with ADR boundary we recall the definitions of the area integral $\mathcal{A}$ and the nontangential maximal function N_* from Definition 2.33 or the corresponding fattened versions $\widehat{\mathcal{A}}$ and $\widehat{N}_*$ or the corresponding local versions. These are defined with respect to a $\{\mathcal{W}_Q\}_{Q\in\mathbb{D}}$, some Whitney-dyadic structure for Ω with some implicit parameters η and K. Note that according to these definitions, the cones are unbounded when $\partial\Omega$ is unbounded. On the other hand, when $\partial\Omega$ is bounded, so are the cones, all being contained in a $C\,\mathrm{diam}(\partial\Omega)$-neighborhood of $\partial\Omega$. We note also that when $\partial\Omega$ is bounded, there exists a cube $Q_0\in\mathbb{D}(\partial\Omega)$ such that $Q_0=\partial\Omega$ and for any $Q\in\mathbb{D}(\partial\Omega)$ we have $Q\in\mathbb{D}_{Q_0}$. It is, however, particularly useful to work with local versions $\mathcal{A}^Q$ and $\widehat{N}_*^Q$ or $\widehat{\mathcal{A}}^Q$ and $\widehat{N}_*^Q$.

Definition 4.1 ($\mathcal{A}<N$ estimates). Let $\Omega\subset\mathbb{R}^{n+1}$ be an open set with $\partial\Omega$ being ADR and let $\{\mathcal{W}_Q\}_{Q\in\mathbb{D}(\partial\Omega)}$ be a Whitney-dyadic structure for Ω with some parameters η and K. Consider also $G\in L^2_{\mathrm{loc}}(\Omega)$, $H\in C(\Omega)$, and $0<q<\infty$. We say that "$\mathcal{A}<N$" estimates hold for G, H on $L^q(\partial\Omega)$ if

$$\|\mathcal{A}G\|_{L^q(\partial\Omega)}\leq C\|\widehat{N}_*H\|_{L^q(\partial\Omega)}, \tag{4.2}$$

where the L^q norms are taken with respect to surface measure $\sigma:=H^n|_{\partial\Omega}$. Similarly, we will say that "$\mathcal{A}^{\mathbb{D}}<N^{\mathbb{D}}$" estimates hold for G, H on $L^q(\partial\Omega)$ if

$$\|\mathcal{A}^QG\|_{L^q(Q)}\leq C\|\widehat{N}_*^QH\|_{L^q(Q)} \quad \text{for all } Q\in\mathbb{D}(\partial\Omega), \tag{4.3}$$

with C independent of Q.

Remark 4.4. We observe that by Remarks 2.37 and 2.38, $\mathcal{A} < N$ estimates imply an analogous estimate for traditional cones, that is, for every $\kappa > 0$

$$\|\mathcal{A}_{\Omega,\kappa} G\|_{L^q(\partial\Omega)} \le C \|N_{*,\Omega,\kappa} H\|_{L^q(\partial\Omega)},$$

and the implicit constant depends on q, n, the ADR constant of $\partial\Omega$, the choice of η, K, τ, the constant in $\mathcal{A} < N$, and κ. On the other hand $\mathcal{A}^{\mathbb{D}} < N^{\mathbb{D}}$ estimates imply also some local $\mathcal{A} < N$ estimates with traditional cones. More precisely, for any $x \in \partial\Omega$ and $0 < r < 2\operatorname{diam}(\partial\Omega)$, using the notation in Definition 1.15, there exists K' depending on n, the ADR constant of $\partial\Omega$, the choice of η, K, τ, and the constant in Definition 2.7(iii) such that for every $\kappa > 0$

$$\|\mathcal{A}^r_{\Omega,\kappa} G\|_{L^q(\Delta(x,r))} \lesssim \|N^{K'r}_{*,\Omega,\kappa} H\|_{L^q(\Delta(x,K'r))}, \tag{4.5}$$

where $\Delta(x,r) = B(x,r) \cap \partial\Omega$, and the implicit constant depends on q, n, the ADR constant of $\partial\Omega$, the choice of η, K, τ, the constant in $\mathcal{A}^{\mathbb{D}} < N^{\mathbb{D}}$, and κ.

Fix then $\{\mathcal{W}_Q\}_{Q\in\mathbb{D}(\partial\Omega)}$ a Whitney-dyadic structure for Ω with some parameters η and K. Given $x \in \partial\Omega$ and $0 < r < 2\operatorname{diam}(\partial\Omega)$, write $\Delta = \Delta(x,r)$ and $B = B(x,r)$. We first consider the case $r \ll \operatorname{diam}(\partial\Omega)$. Note that for every $y \in \Delta$ we have $\Gamma^r(y) \subset 2B$. Also, if $\Gamma_{\Omega,1}(z) \cap 2B \neq \varnothing$ then $z \in 6\Delta$. Recall that we have always assumed that K is large enough (say $K \ge 10^4 n$) so that $\Gamma_{\Omega,1}(y) \subset \Gamma(y)$ for every $y \in \partial\Omega$. All these, together with Remark 2.38, give

$$\|\mathcal{A}^r_{\Omega,\kappa} G\|_{L^q(\Delta)} \le \|\mathcal{A}_{\Omega,\kappa}(G 1_{2B})\|_{L^q(\partial\Omega)} \lesssim \|\mathcal{A}_{\Omega,1}(G 1_{2B})\|_{L^q(\partial\Omega)} \le \|\mathcal{A}(G 1_{2B})\|_{L^q(6\Delta)}.$$

Let

$$\mathcal{D}_\Delta = \{Q \in \mathbb{D}(\partial\Omega) : Q \cap 6\Delta \neq \varnothing,\ C(\eta n)^{-1/2} r/4 \le \ell(Q) < C(\eta n)^{-1/2} r/2\}, \tag{4.6}$$

where C is the constant in (2.9) (it is here we use that $r \ll \operatorname{diam}(\partial\Omega)$ so that $C(\eta n)^{-1/2} r/2 < \operatorname{diam}(\partial\Omega)$, thus $\mathcal{D}_\Delta \neq \varnothing$). Suppose that $Q \subsetneq Q'$ with $Q \in \mathcal{D}_\Delta$ and let $Y \in U_{Q'}$. Then there is $I' \in \mathcal{W}_{Q'}$ with $Y \in \partial I^*(\tau)$ and by (2.5)

$$C(\eta n)^{-\frac{1}{2}} 2^{-1} r \le 2\ell(Q) \le \ell(Q') \le C\eta^{-\frac{1}{2}} \ell(I') \le C(\eta n) 4^{-1} \operatorname{dist}(4I', \partial\Omega) \le C(\eta n) 4^{-1} \operatorname{dist}(Y, \partial\Omega).$$

Hence, $\operatorname{dist}(Y, \partial\Omega) \ge 2r$ and $\Gamma(y) \cap 2B \subset \Gamma^Q(y)$ for every $y \in Q \in \mathcal{D}_\Delta$. Thus the $\mathcal{A}^{\mathbb{D}} < N^{\mathbb{D}}$ estimates give

$$\|\mathcal{A}^r_{\Omega,\kappa} G\|^q_{L^q(\Delta)} \lesssim \sum_{Q\in\mathcal{D}_\Delta} \|\mathcal{A}(G 1_{2B})\|^q_{L^q(Q)} \le \sum_{Q\in\mathcal{D}_\Delta} \|\mathcal{A}^Q G\|^q_{L^q(Q)} \lesssim \sum_{Q\in\mathcal{D}_\Delta} \|N^Q_* H\|^q_{L^q(Q)}.$$

Note next that for every $y \in Q \in \mathcal{D}_\Delta$ we have by (2.14) that $\Gamma^Q(y) \subset B(x_Q, K\ell(Q)) \cap \Omega \subset K'B \cap \Omega$. Hence, using again Remark 2.38 we have

$$\|\mathcal{A}^r_{\Omega,\kappa} G\|_{L^q(\Delta)} \lesssim \|N_*(H 1_{K'B})\|_{L^q(\partial\Omega)} \lesssim \|N_{*,\Omega,\min\{1,\kappa\}}(H 1_{K'B})\|_{L^q(\partial\Omega)} \le \|N^{3K'r}_{*,\Omega,\kappa} H\|_{L^q(3K'\Delta)},$$

where we have used that $\Gamma_{\Omega,1}(z) \cap K'B \neq \varnothing$ then $z \in 3K'\Delta$.

To conclude we consider the case $r \approx \operatorname{diam}(\partial\Omega)$. Hence $\partial\Omega$ is bounded and $\partial\Omega$ is itself a dyadic cube Q_0 and $\mathbb{D}(\partial\Omega) = \mathbb{D}_{Q_0}$. Then we easily obtain using some of the previous observations

$$\begin{aligned} \|\mathcal{A}^r_{\Omega,\kappa} G\|_{L^q(\Delta)} &\le \|\mathcal{A}_{\Omega,\kappa} G)\|_{L^q(\partial\Omega)} \lesssim \|\mathcal{A}_{\Omega,1} G\|_{L^q(\partial\Omega)} \\ &\le \|\mathcal{A}^{Q_0} G\|_{L^q(\partial\Omega)} \lesssim \|N^{Q_0}_* H\|_{L^q(\partial\Omega)} \\ &\lesssim \|N_{*,\Omega,\kappa} H\|_{L^q(\partial\Omega)} = \|N^{K'r}_{*,\Omega,\kappa} H\|_{L^q(\Delta(x,K'r))}, \end{aligned} \tag{4.7}$$

where the last estimate uses our convention that in the case Ω unbounded and $\partial\Omega$ bounded $\Gamma_\Omega(\cdot)$ is indeed $\Gamma_\Omega^{C \operatorname{diam}(\partial\Omega)}(\cdot)$.

Theorem 4.8. *Let $\Omega \subset \mathbb{R}^{n+1}$ be an open set with $\partial\Omega$ being ADR and let $\{\mathcal{W}_Q\}_{Q\in\mathbb{D}(\partial\Omega)}$ be a Whitney-dyadic structure for Ω with some parameters η and K. Given $G \in L^2_{\mathrm{loc}}(\Omega)$, $H \in C(\Omega)$, and $0 < q < \infty$, consider the following statements*:

- $(\boldsymbol{A})$ *Carleson measure estimate holds for $F = G/\|H\|_{L^\infty(\Omega)}$ on Ω, that is, $\|G\|_{\mathrm{CME}(\Omega)} \lesssim \|H\|^2_{L^\infty(\Omega)}$ (see (1.10)).*
- $(\boldsymbol{A})^{\mathbb{D}}$ *Dyadic Carleson measure estimate holds for $F = G/\|H\|_{L^\infty(\Omega)}$ on Ω, that is, $\|G\|_{\mathrm{CME}^{\mathrm{dyad}}(\Omega)} \lesssim \|H\|^2_{L^\infty(\Omega)}$ (see (2.28)).*
- $(\boldsymbol{A}_{\mathrm{loc}})$ *Carleson measure estimate holds on any (bounded) local sawtooth subdomain of Ω, in the sense that for any $Q \in \mathbb{D}(\partial\Omega)$ and any pairwise disjoint family of cubes $\mathcal{F} \subset \mathbb{D}_Q$, one has that $F = G/\|H\|_{L^\infty(\widehat{\Omega}_{\mathcal{F},Q})}$ satisfies the Carleson measure estimate on $\widehat{\Omega}_{\mathcal{F},Q}$, that is,*

$$\sup_{Q,\mathcal{F}} \|G\|_{\mathrm{CME}(\widehat{\Omega}_{\mathcal{F},Q})} / \|H\|^2_{L^\infty(\widehat{\Omega}_{\mathcal{F},Q})} < \infty,$$

where the sup runs over all $Q \in \mathbb{D}(\partial\Omega)$ and all pairwise disjoint family of cubes $\mathcal{F} \subset \mathbb{D}_Q$.

- $(\boldsymbol{B})_{\boldsymbol{q}}$ *$\mathcal{A} < N$ on $L^q(\partial\Omega)$ holds for G and H, in the sense of Definition 4.1, i.e., (4.2) is valid.*
- $(\boldsymbol{B}_{\mathrm{loc}})_{\boldsymbol{q}}$ *$\mathcal{A} < N$ on $L^q(\partial\widehat{\Omega}_{\mathcal{F},Q})$ holds for G and H in the sense of Definition 4.1 for any $Q \in \mathbb{D}(\partial\Omega)$ and any pairwise disjoint family of cubes $\mathcal{F} \subset \mathbb{D}_Q$, i.e., (4.2) is valid in $\widehat{\Omega}_{\mathcal{F},Q}$.*
- $(\boldsymbol{B})^{\mathbb{D}}_{\boldsymbol{q}}$ *$\mathcal{A}^{\mathbb{D}} < N^{\mathbb{D}}$ on $L^q(\partial\Omega)$ holds for G and H, in the sense of Definition 4.1, i.e., (4.3) is valid.*
- $(\boldsymbol{G\lambda})$ *There exists $\theta > 0$ such that for every $\varepsilon, \gamma > 0$ and for all $\alpha > 0$*

$$\sigma\{x \in \partial\Omega : \mathcal{A}G(x) > (1+\varepsilon)\alpha,\ \widehat{N}_* H(x) \le \gamma\alpha\} \le C(\gamma/\varepsilon)^\theta \sigma\{x \in \partial\Omega : \mathcal{A}G(x) > \alpha\}. \tag{4.9}$$

- $(\boldsymbol{G\lambda})^{\mathbb{D}}$ *There exists $\theta > 0$ such that for every $\varepsilon, \gamma > 0$ and for all $\alpha > 0$*

$$\begin{aligned} &\sigma\{x \in Q : \mathcal{A}^Q G(x) > (1+\varepsilon)\alpha,\ \widehat{N}^Q_* H(x) \le \gamma\alpha\} \\ &\qquad \le C(\gamma/\varepsilon)^\theta \sigma\{x \in Q : \mathcal{A}^Q G(x) > \alpha\} \quad \text{for any } Q \in \mathbb{D}(\partial\Omega). \end{aligned} \tag{4.10}$$

Consider, in addition, the condition

$$\left(\frac{1}{\delta(X)^n} \iint_{B(X,\delta(X)/2)} |G(Y)|^2 \delta(Y)\, dY\right)^{1/2} \le C\|H\|_{L^\infty(B(X,3\delta(X)/4))} \quad \text{for all } X \in \Omega. \tag{4.11}$$

Then the following implications hold:

$$(A_{\rm loc}) \quad \Longrightarrow \quad (G\lambda)^{\mathbb{D}} \quad \Longrightarrow \quad (G\lambda), \tag{4.12}$$

$$(A_{\rm loc}) \quad \Longrightarrow \quad (B)_q^{\mathbb{D}} \ \textit{for all}\ 0<q<\infty, \tag{4.13}$$

$$(B)_q^{\mathbb{D}} \ \textit{for some}\ 0<q<\infty \quad \Longrightarrow \quad (B)_q, \tag{4.14}$$

$$(B)_q^{\mathbb{D}} \ \textit{for some}\ 0<q<\infty \quad \Longrightarrow \quad (A)^{\mathbb{D}}, \tag{4.15}$$

$$(A)^{\mathbb{D}} \ \textit{and}\ (4.11) \quad \Longrightarrow \quad (A), \tag{4.16}$$

$$(B_{\rm loc})_q \ \textit{for some}\ 0<q<\infty \quad \Longrightarrow \quad (A)^{\mathbb{D}}. \tag{4.17}$$

In the previous implications the implicit constants of each of the conclusions depend on n, q, *the ADR character of* $\partial\Omega$, *the choice of* η, K, τ, *the constant in Definition 2.7*(iii), *as well as the implicit constants in the corresponding hypotheses.*

Remark 4.18. In the previous result it is understood that (A) and $(A)^{\mathbb{D}}$ are vacuous, unless $H \in L^\infty(\Omega)$. Regarding $(A_{\rm loc})$, if $H \notin L^\infty(\widehat{\Omega}_{\mathcal{F},Q})$, for some $Q \in \mathbb{D}(\partial\Omega)$ and for some pairwise disjoint family of cubes $\mathcal{F} \subset \mathbb{D}_Q$, then it is understood that $F = G/\|H\|_{L^\infty(\widehat{\Omega}_{\mathcal{F},Q})} = 0$ and $\|G\|_{\mathrm{CME}(\widehat{\Omega}_{\mathcal{F},Q})}/\|H\|_{L^\infty(\widehat{\Omega}_{\mathcal{F},Q})} = 0$. Hence, in the sup the only relevant sawtooths $\widehat{\Omega}_{\mathcal{F},Q}$ are those on which H is essentially bounded.

Remark 4.19. We note that the assumption (4.11) in (4.16) is only needed when Ω is unbounded and $\partial\Omega$ is bounded because all dyadic cones are contained in a $C\,\mathrm{diam}(\partial\Omega)$-neighborhood of E. Hence from $(A)^{\mathbb{D}}$ we only get information for F in that region. However, in all practical applications to solutions of elliptic PDEs (4.11) is easily justified by Caccioppoli's inequality.

Remark 4.20. It is possible to show the equivalence of previous conditions upon assuming that they hold in some class of sets. To be more precise, let $\Omega \subset \mathbb{R}^{n+1}$ be an open set with ADR boundary and suppose that we have a collection $\{\Omega'\}_{\Omega' \in \Sigma}$ such that each $\Omega' \in \Sigma$ is an open subset of Ω, $\partial\Omega'$ is ADR boundary, and also that $\widehat{\Omega}_{\mathcal{F},Q} \in \Sigma$ for every $Q \in \mathbb{D}(\partial\Omega')$ and any pairwise disjoint family of cubes $\mathcal{F} \subset \mathbb{D}_Q$. Assume further that

$$\left(\frac{1}{r^n}\iint_{B(X,r)} |G(Y)|^2 \delta(Y)\,dY\right)^{\frac12} \le C\|H\|_{L^\infty(B(X,2r))} \quad \text{for all } B(X,2r) \subset \Omega. \tag{4.21}$$

Then, (A) holds on every $\Omega' \in \Sigma$ if and only if $(B)_q^{\mathbb{D}}$ holds for every $\Omega' \in \Sigma$ and for all (some) $0<q<\infty$ if and only if $(B)_q$ holds for every $\Omega' \in \Sigma$ and for all (some) $0<q<\infty$; with the understanding that all implicit constants in the statements above are uniform within Σ. We have several examples of classes Σ. Suppose first that $\Omega = \mathbb{R}^{n+1} \setminus E$, with E being UR (resp. ADR). In that case Σ is the class of open sets $\Omega' \subset \mathbb{R}^{n+1} \setminus E$ with $\partial\Omega$ being UR (resp. ADR) and the implicit constant in each condition should depend on the UR (resp. ADR) character of each Ω'. Another interesting example is that when Ω is some given CAD (resp. 1-sided CAD) and Σ is the collection of chord-arc subdomains (resp. 1-sided chord-arc subdomains) $\Omega' \subset \Omega$, in that case the implicit constant in each condition should depend on the CAD (1-sided CAD) character of each Ω'.

Lemma 4.22. *Let $\Omega \subset \mathbb{R}^{n+1}$ be an open set with $\partial\Omega$ being ADR and let $\{\mathcal{W}_Q\}_{Q\in\mathbb{D}(\partial\Omega)}$ be a Whitney-dyadic structure for Ω with some parameters η and K. If (A_{loc}) holds for $G \in L^2_{\mathrm{loc}}(\Omega)$ and $H \in C(\Omega)$, then*

$$\|\mathcal{A}^{Q_0}G\|_{L^2(F)} \le C\sigma(Q_0)^{\frac12}\Big(\sup_{\mathcal{F}} \|G\|_{\mathrm{CME}(\widehat{\Omega}_{\mathcal{F},Q_0})}/\|H\|^2_{L^\infty(\widehat{\Omega}_{\mathcal{F},Q_0})}\Big)^{\frac12}\|\widehat{N}_*^{Q_0}H\|_{L^\infty(F)} \tag{4.23}$$

for every $Q_0 \in \mathbb{D}(\partial\Omega)$ and every Borel set $F \subset Q_0$, and where the sup is taken over all families $\mathcal{F} \in \mathbb{D}_{Q_0}$ which are pairwise disjoint. The constant C depends on n, the ADR character of $\partial\Omega$, the choice of η, K, τ, and the constant in Definition 2.7(iii).

Proof. We may assume without lost of generality that $\sigma(F) > 0$ and also that $\|\widehat{N}_*^{Q_0}H\|_{L^\infty(F)} < \infty$. Subdivide $Q_0 \in \mathbb{D}(\partial\Omega)$ dyadically and stop the first time that $Q \cap F = \varnothing$. This generates a possibly empty maximal (hence pairwise disjoint) family $\mathcal{F} = \{Q_j\}_j \subset \mathbb{D}_{Q_0} \setminus \{Q_0\}$, so that $Q_j \cap F = \varnothing$ for every $Q_j \in \mathcal{F}$, and $Q \cap F \neq \varnothing$ for every $Q \in \mathbb{D}_{\mathcal{F},Q_0}$.

Let us observe that if $Q \cap F \neq \varnothing$ then necessarily $Q \in \mathbb{D}_{\mathcal{F},Q_0}$; otherwise $Q \subset Q_j \in \mathcal{F}$ and hence $Q \cap F = \varnothing$, which is a contradiction. Recall that by construction for every $Y \in U_Q$ we have $\delta(Y) \approx \ell(Q) \approx \operatorname{dist}(Y, \partial\widehat{\Omega}_{\mathcal{F},Q_0})$ since, as explained above, $\widehat{\Omega}_{\mathcal{F},Q_0}$ is composed of fattened Whitney regions $\widehat{U}_Q$, which, in turn, have bounded overlap. Writing $\delta(\,\cdot\,) = \operatorname{dist}(\,\cdot\,, \partial\Omega)$, all these yield

$$\begin{aligned}
\int_F \mathcal{A}^{Q_0}G(x)^2\,d\sigma(x) &\le \int_F \sum_{x\in Q\in\mathbb{D}_{Q_0}} \iint_{U_Q} G(Y)^2\delta(Y)^{1-n}\,dY\,d\sigma(x)\\
&= \sum_{Q\in\mathbb{D}_{Q_0}} \sigma(F\cap Q)\iint_{U_Q} G(Y)^2\delta(Y)^{1-n}\,dY\\
&\lesssim \sum_{Q\in\mathbb{D}_{\mathcal{F},Q_0}} \iint_{\widehat{U}_Q} G(Y)^2\operatorname{dist}(Y,\partial\widehat{\Omega}_{\mathcal{F},Q_0})\,dY\\
&\lesssim \iint_{\widehat{\Omega}_{\mathcal{F},Q_0}} G(Y)^2\operatorname{dist}(Y,\partial\widehat{\Omega}_{\mathcal{F},Q_0})\,dY.
\end{aligned}$$

Pick then $y \in \partial\widehat{\Omega}_{\mathcal{F},Q_0}$ and use (A_{loc}) in the sawtooth domain $\widehat{\Omega}_{\mathcal{F},Q_0}$ to conclude

$$\begin{aligned}
\int_F \mathcal{A}^{Q_0}G(x)^2\,d\sigma(x) &\lesssim \iint_{B(y,2\operatorname{diam}(\widehat{\Omega}_{\mathcal{F},Q}))\cap\widehat{\Omega}_{\mathcal{F},Q_0}} G(Y)^2\operatorname{dist}(Y,\partial\widehat{\Omega}_{\mathcal{F},Q_0})\,dY\\
&\lesssim \|G\|_{\mathrm{CME}(\widehat{\Omega}_{\mathcal{F},Q_0})}\operatorname{diam}(\widehat{\Omega}_{\mathcal{F},Q_0})^n \le C_0\|H\|^2_{L^\infty(\widehat{\Omega}_{\mathcal{F},Q})}\operatorname{diam}(\widehat{\Omega}_{\mathcal{F},Q_0})^n\\
&\approx C_0\|G\|_{\mathrm{CME}(\widehat{\Omega}_{\mathcal{F},Q_0})}\|H\|^2_{L^\infty(\widehat{\Omega}_{\mathcal{F},Q_0})}\sigma(Q_0),
\end{aligned}$$

where

$$C_0 = \sup_{\mathcal{F}} \|G\|_{\mathrm{CME}(\widehat{\Omega}_{\mathcal{F},Q_0})}/\|H\|^2_{L^\infty(\widehat{\Omega}_{\mathcal{F},Q_0})}.$$

To conclude we observe that if $Y \in \widehat{\Omega}_{\mathcal{F},Q_0}$, then $Y \in \widehat{U}_Q$ for some $Q \in \mathbb{D}_{\mathcal{F},Q_0}$. The latter implies that we can find $z \in Q \cap F \neq \varnothing$. Hence $Y \in \Gamma^{Q_0}(z)$ and $|H(Y)| \le \widehat{N}_*^{Q_0}H(z) \le \|\widehat{N}_*^{Q_0}H\|_{L^\infty(F)}$. As a result,

$$\int_F \mathcal{A}^{Q_0}G(x)^2\,d\sigma(x) \lesssim C_0\|\widehat{N}_*^{Q_0}H\|^2_{L^\infty(F)}\sigma(Q_0). \qquad \square$$

4.1. ***Proof of Theorem 4.8*: $(A_{\mathrm{loc}}) \Longrightarrow (G\lambda)^{\mathbb{D}}$.** Fix $Q_0 \in \mathbb{D} = \mathbb{D}(\partial\Omega)$ and for any $\alpha > 0$, set

$$E_\alpha = \{x \in Q_0 : \mathcal{A}^{Q_0} G(x) > \alpha\}, \qquad F_\alpha = \{x \in Q_0 : \widehat{N}_*^{Q_0} H(x) \le \alpha\}.$$

Note that if $E_\alpha = \varnothing$ then (4.10) (with $Q = Q_0$) is trivial and there is nothing to prove. Assume then that $E_\alpha \neq \varnothing$.

We momentarily suppose that $E_\alpha \subsetneq Q_0$. Given $x \in E_\alpha$, the monotone convergence theorem guarantees that there exists $k_x \ge 0$ such that

$$\iint_{\bigcup_{\substack{x \in Q \in \mathbb{D}_{Q_0} \\ \ell(Q) \ge 2^{-k_x}}} U_Q} |G(Y)|^2 \delta(Y)^{1-n} > \alpha^2, \tag{4.24}$$

where $\delta(\cdot) = \operatorname{dist}(\cdot, \partial\Omega)$.

Let $Q_x \in \mathbb{D}_{Q_0}$ be the unique cube with $Q_x \ni x$ and $\ell(Q_x) = 2^{-k_x}$ and note that for every $y \in Q_x$

$$\Gamma^{Q_0}(y) = \bigcup_{y \in Q \in \mathbb{D}_{Q_0}} U_Q \supset \bigcup_{Q_x \subset Q \in \mathbb{D}_{Q_0}} U_Q = \bigcup_{\substack{x \in Q \in \mathbb{D}_{Q_0} \\ \ell(Q) \ge 2^{-k_x}}} U_Q.$$

This and (4.24) imply that $\mathcal{A}^{Q_0} G(y) > \alpha$. We have then show that for every $x \in E_\alpha$ there exists $Q_x \in \mathbb{D}_{Q_0}$ such that $Q_x \subset E_\alpha$. We can then take $Q_x^{\max}$, with $Q_x \subset Q_x^{\max} \subset Q_0$, the maximal cube so that $Q_x^{\max} \subset E_\alpha$. Note that $Q_x \subsetneq Q_0$ since $E_\alpha \subsetneq Q_0$. Write then $\mathcal{F} = \{Q_j\}_j \subset \mathbb{D}_{Q_0} \setminus \{Q_0\}$ for the collection of maximal (hence pairwise disjoint) cubes $Q_x^{\max}$ with $x \in E_\alpha$. By construction, $E_\alpha = \bigcup_{Q_j \in \mathcal{F}} Q_j$ and for every $Q_j \in \mathcal{F}$, by maximality, we can find $x_j \in \widetilde{Q}_j \setminus E_\alpha$, where $\widetilde{Q}_j$ is the dyadic parent of Q_j. In the latter scenario, if $x \in Q_j$

$$\Gamma^{Q_0}(x) = \bigcup_{x \in Q \in \mathbb{D}_{Q_0}} U_Q = \Big(\bigcup_{x \in Q \in \mathbb{D}_{Q_j}} U_Q\Big) \cup \Big(\bigcup_{Q_j \subsetneq Q \subset Q_0} U_Q\Big) \subset \Gamma^{Q_j}(x) \cup \Gamma^{Q_0}(x_j)$$

and, consequently,

$$\mathcal{A}^{Q_0} G(x) \le \mathcal{A}^{Q_j} G(x) + \mathcal{A}^{Q_0} G(x_j) \le \mathcal{A}^{Q_j} G(x) + \alpha, \qquad x \in Q_j.$$

Using this, for every $\varepsilon > 0$ we have

$$E_{(1+\varepsilon)\alpha} = E_{(1+\varepsilon)\alpha} \cap E_\alpha = \bigcup_{Q_j \in \mathcal{F}} E_{(1+\varepsilon)\alpha} \cap Q_j \subset \bigcup_{Q_j \in \mathcal{F}} \{x \in Q_j : \mathcal{A}^{Q_j} G(x) > \varepsilon\alpha\}.$$

This holds under the assumption $E_\alpha \subsetneq Q_0$ but it clearly extends to the case $E_\alpha \subsetneq Q_0$ by setting $\mathcal{F} = \{Q_0\}$. Hence, invoking Chebyshev's inequality and Lemma 4.22 in every Q_j, we arrive at

$$\begin{aligned}
\sigma(E_{(1+\varepsilon)\alpha} \cap F_{\gamma\alpha}) &\le \sum_{Q_j \in \mathcal{F}} \sigma(\{x \in Q_j : \mathcal{A}^{Q_j} G(x) > \varepsilon\alpha\} \cap F_{\gamma\alpha}) \\
&\le \frac{1}{(\varepsilon\alpha)^2} \sum_{Q_j \in \mathcal{F}} \int_{F_{\gamma\alpha} \cap Q_j} \mathcal{A}^{Q_j} G(x)^2 \, d\sigma(x) \\
&\lesssim \frac{1}{(\varepsilon\alpha)^2} \Big(\sup_{Q_0, \mathcal{F}} \|G\|_{\mathrm{CME}(\widehat{\Omega}_{\mathcal{F}, Q_0})} / \|H\|^2_{L^\infty(\widehat{\Omega}_{\mathcal{F}, Q_0})}\Big) \sum_{Q_j \in \mathcal{F}} \|\widehat{N}_*^{Q_0} H\|^2_{L^\infty(F_{\gamma\alpha} \cap Q_j)} \sigma(Q_j)
\end{aligned}$$

$$
\begin{aligned}
&\leq \left(\frac{\gamma}{\varepsilon}\right)^2 \Big(\sup_{Q_0,\mathcal{F}} \|G\|_{\mathrm{CME}(\widehat{\Omega}_{\mathcal{F},Q_0})}/\|H\|^2_{L^\infty(\widehat{\Omega}_{\mathcal{F},Q_0})}\Big) \sum_{Q_j\in\mathcal{F}} \sigma(Q_j)\\
&= \left(\frac{\gamma}{\varepsilon}\right)^2 \Big(\sup_{Q_0,\mathcal{F}} \|G\|_{\mathrm{CME}(\widehat{\Omega}_{\mathcal{F},Q_0})}/\|H\|^2_{L^\infty(\widehat{\Omega}_{\mathcal{F},Q_0})}\Big)\sigma\Big(\bigcup_{Q_j\in\mathcal{F}} Q_j\Big)\\
&= \left(\frac{\gamma}{\varepsilon}\right)^2 \Big(\sup_{Q_0,\mathcal{F}} \|G\|_{\mathrm{CME}(\widehat{\Omega}_{\mathcal{F},Q_0})}/\|H\|^2_{L^\infty(\widehat{\Omega}_{\mathcal{F},Q_0})}\Big)\sigma(E_\alpha),
\end{aligned}
$$

where the sup is taken over all $Q_0 \in \mathbb{D}$ and over all families $\mathcal{F} \in \mathbb{D}_{Q_0}$ which are pairwise disjoint. □

4.2. ***Proof of Theorem 4.8*: $(A_{\mathrm{loc}}) \Longrightarrow (B)_q^{\mathbb{D}}$ *for all* $0 < q < \infty$.** We start by observing that if $G \in L^2_{\mathrm{loc}}(\Omega)$ then for every $\Omega' \subset \Omega$ one has $\|G 1_{\Omega'}\|_{\mathrm{CME}(\widehat{\Omega}_{\mathcal{F},Q})} \leq \|G\|_{\mathrm{CME}(\widehat{\Omega}_{\mathcal{F},Q})}$ for every $Q \in \mathbb{D} = \mathbb{D}(\partial\Omega)$ and for every family of pairwise disjoint cubes $\mathcal{F} \in \mathbb{D}_Q$. This means that if (A_{loc}) holds for G and H then it also does for $G 1_{\Omega'}$ and H uniformly in Ω'. Therefore, from what we have proved so far, $(G\lambda)^{\mathbb{D}}$ holds for $G 1_{\Omega'}$ and H uniformly in Ω'.

Fix $x_0 \in \partial\Omega$ and given $k \in \mathbb{N}$ set

$$
\Omega_k = \{X \in B(x_0, k) \cap \Omega : |G(X)| \leq k,\ \delta(X) \geq k^{-1}\}
$$

and note that for every $0 < q < \infty$ and for every $x \in \partial\Omega$

$$
\mathcal{A}(G 1_{\Omega_k})(x)^2 = \iint_{\Gamma(x)\cap\Omega_k} |G(Y)|^2 \delta(Y)^{1-n}\, dY \leq k^{n+1} |B(x_0, k)| \approx k^{2(n+1)}.
$$

On the other hand, suppose that $x \in \partial\Omega$ is so that $\Gamma(x) \cap \Omega_k \neq \varnothing$. Pick $Z \in \Gamma(x) \cap \Omega_k \neq \varnothing$, then $Z \in I^*$ with $I \in \mathcal{W}_Q$ and $x \in Q \in \mathbb{D}$. Using (2.9) it follows that

$$
\begin{aligned}
|x - x_0| &\leq |x - x_Q| + \operatorname{diam}(Q) + \operatorname{dist}(I, Q) + \operatorname{diam}(I^*) + |Z - x_0| \lesssim \ell(I) + k\\
&\approx \delta(Z) + k \lesssim |X - z_0| + k \leq 2k.
\end{aligned}
$$

As a consequence, $\operatorname{supp} \mathcal{A}(G 1_{\Omega_k}) \subset B(x_0, CK)$. These, together with the fact that $\mathcal{A}^Q(G 1_{\Omega_k})(x) \leq \mathcal{A}(G 1_{\Omega_k})(x)$ for every $x \in \partial\Omega$, allow us to conclude that $\mathcal{A}(G 1_{\Omega_k}), \mathcal{A}^Q(G 1_{\Omega_k}) \in L^\infty_c(\partial\Omega) \subset L^q(\partial\Omega)$ for every $Q \in \mathbb{D}$, albeit with bounds that depend on k.

Using the previous observations and invoking $(G\lambda)^{\mathbb{D}}$ with $G 1_{\Omega_k}$ and H (with constant that is independent of k) we have for every $Q \in \mathbb{D}$

$$
\begin{aligned}
&\|\mathcal{A}^Q(G 1_{\Omega_k})\|^q_{L^q(Q)}\\
&\quad= (1+\varepsilon)^q \int_0^\infty q\alpha^q \sigma\{x \in Q : \mathcal{A}^Q(G 1_{\Omega_k})(x) > (1+\varepsilon)\alpha\} \frac{d\alpha}{\alpha}\\
&\quad\leq (1+\varepsilon)^q \int_0^\infty q\alpha^q \sigma\{x \in Q : \mathcal{A}^Q(G 1_{\Omega_k})(x) > (1+\varepsilon)\alpha,\ \widehat{N}^Q_* H(x) \leq \gamma\alpha\} \frac{d\alpha}{\alpha}\\
&\qquad\qquad + (1+\varepsilon)^q \int_0^\infty q\alpha^q \sigma\{x \in Q : \widehat{N}^Q_* H(x) > \gamma\alpha\} \frac{d\alpha}{\alpha}\\
&\quad\leq C\left(\frac{\gamma}{\varepsilon}\right)^\theta (1+\varepsilon)^q \int_0^\infty q\alpha^q \sigma\{x \in Q : \mathcal{A}^Q(G 1_{\Omega_k})(x) > \alpha\} \frac{d\alpha}{\alpha} + \left(\frac{1+\varepsilon}{\gamma}\right)^q \|\widehat{N}^Q_* H\|^q_{L^q(Q)}\\
&\quad= C\left(\frac{\gamma}{\varepsilon}\right)^\theta (1+\varepsilon)^q \|\mathcal{A}^Q(G 1_{\Omega_k})\|^q_{L^q(Q)} + \left(\frac{1+\varepsilon}{\gamma}\right)^q \|\widehat{N}_* H\|^q_{L^q(Q)}.
\end{aligned}
\tag{4.25}
$$

Pick $\varepsilon = 1$ and choose γ sufficiently small to ensure that $C\gamma^\theta 2^q < \frac{1}{2}$. Using that $\|\mathcal{A}^Q(G1_{\Omega_k})\|^q_{L^q(Q)} < \infty$ we can hide this term on the left-hand side of (4.25) and conclude that

$$\|\mathcal{A}^Q(G1_{\Omega_k})\|^q_{L^q(Q)} \lesssim \|\widehat{N}^Q_* H\|^q_{L^q(Q)}, \tag{4.26}$$

with an implicit constant depending on n, the ADR character of $\partial\Omega$, the choice of η, K, τ, the constant in Definition 2.7(iii), q, and the implicit constant in $(G\lambda)$, but nonetheless independent of k. By the monotone convergence theorem and the fact that $|G(X)| < \infty$ for a.e. $X \in \Omega$, since $G \in L^2_{\mathrm{loc}}(\Omega)$, it follows that $\mathcal{A}^Q(G1_{\Omega_k})(x) \nearrow \mathcal{A}^Q G(x)$. Then we can use the monotone convergence theorem to obtain from (4.26)

$$\|\mathcal{A}^Q G\|^q_{L^q(Q)} = \lim_{k\to\infty} \|\mathcal{A}^Q(G1_{\Omega_k})\|^q_{L^q(Q)} \lesssim \|\widehat{N}^Q_* H\|^q_{L^q(Q)},$$

completing the proof. □

Remark 4.27. The previous arguments easily yield that for any $0 < q < \infty$, one has that $(G\lambda)^{\mathbb{D}} \Rightarrow (B_{\mathrm{loc}})_q$ provided $\|\mathcal{A}^Q G\|_{L^q(Q)} < \infty$. A very similar argument gives that $(G\lambda) \Rightarrow (B)_q$ provided $\|\mathcal{A}G\|_{L^q(\partial\Omega)} < \infty$. Details are left to the interested reader.

4.3. ***Proof of Theorem 4.8*: $(G\lambda)^{\mathbb{D}} \Longrightarrow (G\lambda)$.** We note that if $\partial\Omega$ is bounded, then $\partial\Omega$ itself is the largest cube in $\mathbb{D} = \mathbb{D}(\partial\Omega)$, say $\partial\Omega = Q_0$; hence $(G\lambda)$ is a particular case of $(G\lambda)^{\mathbb{D}}$. Consider next the case $\partial\Omega$ unbounded and for every $k \in \mathbb{N}$ write

$$\Gamma^k(x) = \bigcup_{\substack{x\in Q\in\mathbb{D}\\ \ell(Q)\le 2^k}} U_Q, \qquad x \in \partial\Omega,$$

and associated with these cones define $\mathcal{A}^k$ and $\widehat{N}^k_*$. Given $Q \in \mathbb{D}_{-k}$, i.e., $\ell(Q) = 2^{-k}$, one easily sees that $\Gamma^{Q_0}(x) = \Gamma^k(x)$ for every $x \in Q_0$. Hence, for every $k \in \mathbb{N}$, using $(G\lambda)^{\mathbb{D}}$ we obtain

$$\begin{aligned}
&\sigma\{x \in \partial\Omega : \mathcal{A}^k G(x) > (1+\varepsilon)\alpha,\ \widehat{N}_* H(x) \le \gamma\alpha\}\\
&\qquad\le \sigma\{x \in \partial\Omega : \mathcal{A}^k G(x) > (1+\varepsilon)\alpha,\ \widehat{N}^k_* H(x) \le \gamma\alpha\}\\
&\qquad= \sum_{Q\in\mathbb{D}_{-k}} \sigma\{x \in Q : \mathcal{A}^k G(x) > (1+\varepsilon)\alpha,\ \widehat{N}^k_* H(x) \le \gamma\alpha\}\\
&\qquad= \sum_{Q\in\mathbb{D}_{-k}} \sigma\{x \in Q : \mathcal{A}^Q G(x) > (1+\varepsilon)\alpha,\ \widehat{N}^Q_* H(x) \le \gamma\alpha\}\\
&\qquad\lesssim \left(\frac{\gamma}{\varepsilon}\right)^\theta \sum_{Q\in\mathbb{D}_{-k}} \sigma\{x \in Q : \mathcal{A}^Q G(x) > \alpha\} = \left(\frac{\gamma}{\varepsilon}\right)^\theta \sum_{Q\in\mathbb{D}_{-k}} \sigma\{x \in Q : \mathcal{A}^k G(x) > \alpha\}\\
&\qquad= \left(\frac{\gamma}{\varepsilon}\right)^\theta \sigma\{x \in \partial\Omega : \mathcal{A}^k G(x) > \alpha\} \le \left(\frac{\gamma}{\varepsilon}\right)^\theta \sigma\{x \in \partial\Omega : \mathcal{A}G(x) > \alpha\}.
\end{aligned} \tag{4.28}$$

On the other hand, the monotone convergence theorem gives that $\mathcal{A}^k G(x) \nearrow \mathcal{A}G(x)$ as $k \to \infty$ and for every $x \in \partial\Omega$. Hence, another use of the monotone convergence theorem and (4.28) yield

$$\begin{aligned}
\sigma\{x \in \partial\Omega : \mathcal{A}G(x) > (1+\varepsilon)\alpha,\ \widehat{N}_* H(x) \le \gamma\alpha\} &= \lim_{k\to\infty} \sigma\{x \in \partial\Omega : \mathcal{A}^k G(x) > (1+\varepsilon)\alpha,\ \widehat{N}_* H(x) \le \gamma\alpha\}\\
&\lesssim (\gamma/\varepsilon)^\theta \sigma\{x \in \partial\Omega : \mathcal{A}G(x) > \alpha\},
\end{aligned}$$

□

4.4. ***Proof of Theorem 4.8*: $(B)_q^{\mathbb{D}}$ *for some* $0 < q < \infty \Longrightarrow (B)_q$.** We note that if $\partial\Omega$ is bounded, then $\partial\Omega$ itself is the largest cube in $\mathbb{D} = \mathbb{D}(\partial\Omega)$, say $\partial\Omega = Q_0$; hence $(B)_q$ is a particular case of $(B_{\rm loc})_q$. If $\partial\Omega$ is unbounded we use the same argument as in the previous proof:

$$\begin{aligned}\int_{\partial\Omega} \mathcal{A}^k G(x)^q \, d\sigma(x) &= \sum_{Q\in\mathbb{D}_{-k}} \int_Q \mathcal{A}^k G(x)^q \, d\sigma(x) = \sum_{Q\in\mathbb{D}_{-k}} \int_Q \mathcal{A}^Q G(x)^q \, d\sigma(x) \\ &\lesssim \sum_{Q\in\mathbb{D}_{-k}} \int_Q \widehat{N}_*^Q H(x)^q \, d\sigma(x) = \sum_{Q\in\mathbb{D}_{-k}} \int_Q \widehat{N}_*^k H(x)^q \, d\sigma(x) \\ &= \int_{\partial\Omega} \widehat{N}_*^k H(x)^q \, d\sigma(x) \le \int_{\partial\Omega} \widehat{N}_* H(x)^q \, d\sigma(x).\end{aligned}$$

From here, we obtain the desired estimate from the monotone convergence theorem and the fact that $\mathcal{A}^k G(x) \nearrow \mathcal{A}G(x)$ for every $x \in \partial\Omega$, as $k \to \infty$. □

4.5. ***Proof of Theorem 4.8*: $(B)_q^{\mathbb{D}}$ *for some* $0 < q < \infty \Longrightarrow (A)^{\mathbb{D}}$.** Assume that $(B_{\rm loc})_q$ for some $0 < q < \infty$ holds. We may assume that $H \in L^\infty(\Omega)$. Hence, for every $Q \in \mathbb{D}(\partial\Omega)$,

$$\int_Q \mathcal{A}^Q G(x)^q \, d\sigma(x) \le C_q^q \int_Q \widetilde{N}_*^Q H(x)^q \, d\sigma(x) \le C_q^q \|H\|_{L^\infty(\Omega)}^q \, \sigma(Q).$$

Writing $F := G(2^{1/q} C_q \|H\|_{L^\infty(\Omega)})^{-1}$, we have by Chebyshev's

$$\sigma\{x \in Q : \mathcal{A}^Q F(x) > 1\} \le \int_Q \mathcal{A}^Q F(x)^q \, d\sigma(x) \le \tfrac{1}{2}\sigma(Q).$$

We then invoke Lemma 3.8 with $p = 2$ and obtain

$$\sup_{Q\in\mathbb{D}_{Q_0}} \fint_Q \mathcal{A}^Q F(x)^2 \, d\sigma(x) \lesssim 1.$$

On the other hand, writing $\delta(\,\cdot\,) = \operatorname{dist}(\,\cdot\,, \partial\Omega)$, and recalling that the family $\{U_{Q'}\}_{Q'\in\mathbb{D}(\partial\Omega)}$ has bounded overlap, we see that

$$\begin{aligned}\iint_{T_Q} F^2 \delta \, dY &\approx \sum_{Q'\in\mathbb{D}_Q} \iint_{U_{Q'}} F^2 \delta \, dY \approx \sum_{Q'\in\mathbb{D}_Q} \sigma(Q') \iint_{U_{Q'}} F^2 \delta^{1-n} \, dY \\ &= \int_Q \sum_{x\in Q'\in\mathbb{D}_Q} \iint_{U_{Q'}} F^2 \delta^{1-n} \, dY \, d\sigma(x) \\ &\approx \int_Q \iint_{\Gamma^Q(x)} F^2 \delta^{1-n} \, dY \, d\sigma(x) = \int_Q \mathcal{A}^Q F(x)^2 \, d\sigma(x),\end{aligned} \tag{4.29}$$

Thus,

$$\|F\|_{\mathrm{CME}^{\rm dyad}(\Omega)} = \sup_{Q\in\mathbb{D}_{Q_0}} \frac{1}{\sigma(Q)} \iint_{T_Q} F(Y)^2 \delta(X) \, dY \lesssim 1.$$ □

4.6. ***Proof of Theorem 4.8*: $(A)^{\mathbb{D}}$ *and* (4.11) $\Longrightarrow (A)$.** This follows trivially from (2.29):

$$\|G\|_{\mathrm{CME}(\Omega)} \lesssim \|G\|_{\mathrm{CME}^{\rm dyad}(\Omega)} + \|G\|_{\mathbb{C}_0(\Omega)} \lesssim \|H\|_{L^\infty(\Omega)}^2,$$

which is the desired estimate. □

4.7. ***Proof of Theorem 4.8*: $(B_{\mathrm{loc}})_q$ *for some* $0 < q < \infty \implies (A)^{\mathbb{D}}$.** Write $\mathbb{D} = \mathbb{D}(\partial\Omega)$ and $\delta(\cdot) = \operatorname{dist}(\cdot, \partial\Omega)$. Assume (B_{loc}) and fix $Q_0 \in \mathbb{D}$. We may suppose that $H \in L^\infty(\Omega)$; otherwise there is nothing to prove. Recall that $\widehat{T}_{Q_0} = \widehat{\Omega}_{\varnothing, Q_0}$; hence (B_{loc}) implies that $A < N$ on $L^q(\partial T_{Q_0})$. Thus, Remark 4.4 yields for every $\kappa > 0$

$$\begin{aligned}\|\mathcal{A}_{\widehat{T}_{Q_0},\kappa} G\|^q_{L^q(\partial \widehat{T}_{Q_0})} &\lesssim \|N_{*,\widehat{T}_{Q_0},\kappa} H\|^q_{L^q(\partial \widehat{T}_{Q_0})} \\ &\le \|H\|^q_{L^\infty(\Omega)} H^n(\partial T_{Q_0}) \lesssim \|H\|^q_{L^\infty(\Omega)} \operatorname{diam}(\partial T_{Q_0})^n \\ &\lesssim \|H\|^q_{L^\infty(\Omega)} \ell(Q_0)^n \approx \|H\|^q_{L^\infty(\Omega)} \sigma(Q_0)^n, \end{aligned} \tag{4.30}$$

where we have used that ∂T_{Q_0} is upper ADR (see Remark A.2), (2.14), and that $\partial\Omega$ is ADR.

Let $x \in Q_0$ and $Y \in \Gamma^{Q_0}(x)$. Then $Y \in I^*$ with $I \in \mathcal{W}_Q$ with $x \in Q \in \mathbb{D}_{Q_0}$. Recalling that $I^* = I^*(\tau)$ and that $\widehat{T}_{Q_0}$ is defined using fattened Whitney cubes of the form $J^*(2\tau)$ we clearly see that $Y \in \mathcal{T}_{Q_0} \subset \widehat{T}_{Q_0}$ with $\delta(Y) \approx \operatorname{dist}(Y, \partial\widehat{T}_{Q_0})$. Consequently,

$$|Y - x| \le \operatorname{diam}(I) + \operatorname{dist}(I, Q) + \operatorname{diam}(Q) \lesssim \ell(I) \approx \delta(Y) \approx \operatorname{dist}(Y, \partial\widehat{T}_{Q_0}).$$

Then we can find κ depending on n, the ADR constants of $\partial\Omega$, η, K, and the constant in Definition 2.7(iii) such that $Y \in \Gamma_{\widehat{T}_{Q_0},\kappa}(x)$. Since $Q_0 \subset \partial\widehat{T}_{Q_0}$ (see [Hofmann and Martell 2014, Proposition 6.1]), we then obtain

$$\begin{aligned}\mathcal{A}^{Q_0} G(x) &= \left(\iint_{\Gamma^{Q_0}(x)} |G(Y)|^2 \delta(Y)^{1-n}\, dY\right)^{\frac12} \approx \left(\iint_{\Gamma^{Q_0}(x)} |G(Y)|^2 \operatorname{dist}(Y, \partial T_{Q_0})^{1-n}\, dY\right)^{\frac12} \\ &\le \left(\iint_{\Gamma_{\widehat{T}_{Q_0},\kappa}(x)} |G(Y)|^2 \operatorname{dist}(Y, \partial T_{Q_0})^{1-n}\, dY\right)^{\frac12} = \mathcal{A}_{\widehat{T}_{Q_0},\kappa} G(x).\end{aligned}$$

This and (4.30) imply

$$\fint_{Q_0} \mathcal{A}^{Q_0} G(x)^q \, d\sigma(x) \lesssim \fint_{Q_0} \mathcal{A}_{\widehat{T}_{Q_0},\kappa} G(x)^q \, d\sigma(x) \le C \|H\|^q_{L^\infty(\Omega)}.$$

Writing $F = G(C\|H\|_{L^\infty(\Omega)})^{-1}$, for N large enough, we obtain from Chebyshev's inequality

$$\sigma\{x \in Q_0 : \mathcal{A}^{Q_0} F(x) > N\} \le N^{-q} \int_{Q_0} \mathcal{A}^{Q_0} F(x)^q \, d\sigma(x) \le \tfrac12 \sigma(Q).$$

Since $Q_0 \in \mathbb{D}$ is arbitrary we can apply Lemma 3.8 with $p = 2$ and obtain

$$\sup_{Q \in \mathbb{D}_{Q_0}} \fint_Q \mathcal{A}^Q F(x)^2 \, d\sigma(x) \lesssim 1.$$

This and (4.29) give

$$\|F\|_{\mathrm{CME}^{\mathrm{dyad}}(\Omega)} = \sup_{Q \in \mathbb{D}_{Q_0}} \frac{1}{\sigma(Q)} \iint_{T_Q} F(Y)^2 \delta(X)\, dY \lesssim 1,$$

which is the desired estimate. □

5. Transference of $N < S$ estimates: from Lipschitz to chord-arc domains

Before starting, we introduce some notation. Let $D \subset \mathbb{R}^{n+1}$ be a bounded CAD. Given $Q \in \mathbb{D}(\partial D)$ or $\Delta = \Delta(x, r)$, with $x \in \partial D$ and $0 < r \lesssim \operatorname{diam}(\partial D)$, we will write X_Q^+ and X_Δ^+ to denote respectively some interior corkscrew points relative to Q (that is, relative to Δ_Q, see (2.2)) and Δ. When ∂D is bounded, we write X_D^+ to denote a corkscrew point relative to a surface ball $\Delta(x, 3\,\mathrm{diam}(\partial D)/2) = \partial D$ for some $x \in \partial D$.

Also, recall the dyadic Hardy–Littlewood maximal function from Definition 2.4. In addition, we will be using its continuous analogue. Let $E \subset \mathbb{R}^{n+1}$ be an n-dimensional ADR set. By $M = M_E$ we denote the continuous (noncentered) Hardy–Littlewood maximal function on E, that is, for $f \in L^1_{\mathrm{loc}}(E)$

$$Mf(x) = \sup_{\Delta \ni x} \fint_{\Delta} |f(y)|\, d\sigma(y),$$

where the sup is taken over all Δ, surface balls on E containing x. For $0 < p < \infty$, we also write $M_p f = M(|f|^p)^{1/p}$. It is clear from (2.2) that $M^{\mathbb{D}} f(x) \lesssim Mf(x)$ for every $x \in E$. The converse might fail pointwise, but both maximal functions are bounded in $L^p(E)$, $p > 1$.

We are now ready to state the main result of this section:

Theorem 5.1. *Let $D \subset \mathbb{R}^{n+1}$ be a CAD. Let $u \in W^{1,2}_{\mathrm{loc}}(D) \cap C(D)$ and assume that there exists $C_0 > 0$ such that for any $c \in \mathbb{R}$ and for any cube I with $2I \subset D$*

$$\sup_{X \in I} |u(X) - c| \le C_0 \left(\ell(I)^{-n-1} \iint_{2I} |u - c|^2 \, dX \right)^{\frac{1}{2}}. \tag{5.2}$$

Suppose that the $N < S$ estimates are valid on L^2 on all bounded Lipschitz subdomains $\Omega \subset D$, that is, for any bounded Lipschitz subdomain $\Omega \subset D$ there holds

$$\|N_{*,\Omega}(u - u(X_\Omega^+))\|_{L^2(\partial\Omega)} \le C_\Omega \|S_\Omega u\|_{L^2(\partial\Omega)}. \tag{5.3}$$

Here X_Ω^+ is any interior corkscrew point of Ω at the scale of $\mathrm{diam}(\Omega)$, *and the constant C_Ω in (5.3) depends on the Lipschitz character of Ω, the dimension n, the implicit choice of κ (the aperture of the cones in $N_{*,\Omega}$ and S_Ω), and the implicit corkscrew constant for the point X_Ω^+.*

Given $\eta \ll 1$ and $K \gg 1$, consider $\{\mathcal{W}_Q\}_{Q \in \mathbb{D}(\partial D)}$ a Whitney-dyadic structure for D with parameters η and K; see Section 2.4. Then there exist $0 < c_0 \ll 1$ and $C > 0$, depending on n, the CAD character of D, the choice of η, K, τ, such that for every $\varepsilon > 0$, for every $0 < \gamma < c_0\varepsilon/C_0$, for all $\alpha > 0$, and for all $Q \in \mathbb{D}(\partial D)$

$$\sigma\{x \in Q : N_*^Q(u - u(X_Q^+))(x) > (1+\varepsilon)\alpha,\ M^{\mathbb{D}}_{Q_0,2}(\widehat{S}^Q u)(x) \le \gamma\alpha\} \\ \le C^*_{\gamma,\varepsilon}\, \sigma\{x \in Q : N_*^Q(u - u(X_Q^+))(x) > \alpha\}, \tag{5.4}$$

where $C^*_{\gamma,\varepsilon} = (1 - \theta + C(\gamma/\varepsilon)^2)$ and $\theta \in (0,1)$ is from Corollary 3.21 (hence depends on n and the CAD character of D). Therefore

$$\|N_*^Q(u - u(X_Q^+))\|_{L^q(Q)} \le C' \|\widehat{S}^Q u\|_{L^q(Q)} \quad \text{for all } q > 2, \tag{5.5}$$

where C' depends on n, the CAD character of D, C_0, the choice of η, K, τ, and q.

As a consequence, for any $x \in \partial D$ and $0 < r < 2\,\mathrm{diam}(\partial D)$ there exists K' depending on n, the CAD character of D such that for every $\kappa > 0$

$$\|N^r_{*,D,\kappa}(u - u(X^+_{\Delta(x,r)}))\|_{L^q(\Delta(x,r))} \le C'' \|S^{K'r}_{D,\kappa} u\|_{L^q(\Delta(x,K'r))} \quad \text{for all } q > 2, \tag{5.6}$$

where $\Delta(x,r) = B(x,r) \cap \partial\Omega$, *and where* C'' *depends on* q, n, *the CAD character of* D, C_0, *and* κ. *In particular*, *if* ∂D *is bounded*

$$\|N_{*,D,\kappa}(u - u(X_D^+))\|_{L^q(\partial D)} \le C'' \|S_{D,\kappa} u\|_{L^q(\partial D)} \quad \textit{for all } q > 2 \tag{5.7}$$

and if ∂D *is unbounded and* $u(X) \to 0$ *as* $|X| \to \infty$ *then*

$$\|N_{*,D,\kappa} u\|_{L^q(\partial D)} \le C'' \|S_{D,\kappa} u\|_{L^q(\partial D)} \quad \textit{for all } q > 2. \tag{5.8}$$

We remark that contrary to the previous sections, we do not consider general $\mathcal{A}G$ and $N_* H$ anymore. This is a necessity, as the argument of the area integral has to be the gradient of the argument of the nontangential maximal function. The assumption (5.2) is a standard interior regularity estimate for solutions of elliptic equations (also known as Moser's local boundedness estimate). In principle, we need a somewhat different version. Recall that $\widehat{U}_Q$ is a fattened version of the Whitney region U_Q. We have

$$|u(Y_Q) - c| \le C_0 \left(\ell(Q)^{-n-1} \iint_{\widehat{U}_Q} |u - c|^2 \, dX \right)^{\frac{1}{2}}, \tag{5.9}$$

where Y_Q is any point lying in U_Q, so that there is a ball centered at Y_Q, of radius proportional to $\ell(Q)$, which lies inside $\widehat{U}_Q$. We note that if we assumed (5.2) or (5.9) without enlarging the integrals on the respective right-hand sides, we could obtain a version of (5.4)–(5.5) without enlarging the "aperture of cones" on the right-hand side (that is, with S^Q in place of $\widehat{S}^Q$). But that is minor and (5.2) looks a bit more familiar and more in line with (6.2) below.

Proof. To start, write $\mathbb{D} = \mathbb{D}(\partial D)$ and $\delta(\cdot) = \operatorname{dist}(\cdot, \partial D)$. Fix $\eta \ll 1$ and $K \gg 1$ and consider $\{\mathcal{W}_Q\}_{Q \in \mathbb{D}(\partial D)}$ a Whitney-dyadic structure for D with parameters η and K from Section 2.4. We claim that for every $Q \in \mathbb{D}$

$$\sup_{X,Y \in U_Q} |u(X) - u(Y)| \le C C_0 \inf_{z \in Q} \widehat{S}^Q u(z) \le C C_0 \fint_Q \widehat{S}^Q u \, d\sigma, \tag{5.10}$$

where C depends on n, η, K, τ, and the CAD character of D, and C_0 is the constant in (5.2). To see this observe that for every $Q \in \mathbb{D}$ and $X \in U_Q$ we have that $X \in I^*(\tau)$ for some $I \in \mathcal{W}_Q$. Let $I_X \subset D$ be the cube centered at X with side length $\tau \ell(I)$ so that $2I_X \subset I^*(2\tau) \subset \widehat{U}_Q$. Note that $\ell(I_X) \approx \ell(I) \approx \ell(Q)$. Then, (5.2) yields, for every $c \in \mathbb{R}$,

$$|u(X) - c| \le C_0 \left(\ell(I_X)^{-n-1} \iint_{2I_X} |u - c|^2 \, dX \right)^{\frac{1}{2}} \lesssim C_0 \left(\ell(Q)^{-n-1} \iint_{\widehat{U}_Q} |u - c|^2 \, dX \right)^{\frac{1}{2}}. \tag{5.11}$$

With this at hand, let $Q \in \mathbb{D}$ and $X, Y \in U_Q$ and $z \in Q$. Setting

$$c_Q := \frac{1}{|\widehat{U}_Q|} \iint_{\widehat{U}_Q} v \, dZ,$$

we obtain

$$\begin{aligned} |u(X) - u(Y)| &\le |u(X) - c_Q| + |u(Y) - c_Q| \lesssim C_0 \left(\ell(Q)^{-n-1} \iint_{\widehat{U}_Q} |u - c_Q|^2 \, dZ \right)^{\frac{1}{2}} \\ &\lesssim C_0 \left(\ell(Q)^{-n+1} \iint_{\widehat{U}_Q} |\nabla u|^2 \, dX \right)^{\frac{1}{2}} \approx C_0 \left(\iint_{\widehat{U}_Q} |\nabla u|^2 \delta^{1-n} \, dX \right)^{\frac{1}{2}} \le C_0 \widehat{S}^Q u(z), \end{aligned}$$

where the second inequality follows from (5.11), the third from Poincaré's inequality in the context of Whitney regions (see the argument in [Hofmann et al. 2017a, Proof of Lemma 3.1]), and the last from the fact that $\delta(\,\cdot\,) \approx \ell(Q)$ in $\widehat{U}_Q$. This proves our claim.

Let us fix $Q_0 \in \mathbb{D}$ and write $v := u - u(X^+_{Q_0})$, with $X^+_{Q_0}$ beginning the corkscrew relative to Q_0, that is, relative to the surface ball Δ_{Q_0} (see (2.2) and (2.3)). For every $\alpha > 0$ we set

$$E_\alpha := \{x \in Q_0 : N^{Q_0}_* v(x) > \alpha\}, \qquad F_\alpha := \{x \in Q_0 : M^{\mathbb{D}}_{Q_0,2}(\widehat{S}^{Q_0} v)(x) \le \alpha\},$$

where $M^{\mathbb{D}}_{Q,2}$ was defined in Definition 2.4. Our goal is to obtain for every $\alpha, \gamma, \varepsilon > 0$ with $0 < \gamma \ll \varepsilon / C_0$ there holds

$$\sigma(E_{(1+\varepsilon)\alpha} \cap F_{\gamma\alpha}) \le C^*_{\gamma,\varepsilon}\sigma(E_\alpha), \tag{5.12}$$

and we will me more specific about the constant $C^*_{\gamma,\varepsilon}$ momentarily. With this goal in mind we fix $\alpha, \gamma, \varepsilon > 0$. We may assume that $E_\alpha \neq \varnothing$; otherwise (5.4) is trivial.

Let $x \in E_\alpha$; then there exist $Q_x \in \mathbb{D}_{Q_0}$ with $x \in Q_x$ and $Y \in U_{Q_x}$ such that $|v(Y)| > \alpha$. Note that $U_{Q_x} \subset \Gamma^{Q_0}(y)$ for every $y \in Q_x$; hence $N^{Q_0}_* v(y) \ge |v(Y)| > \alpha$ and $Q_x \subset E_\alpha$. We can then take $Q^{\max}_x$, with $Q_x \subset Q^{\max}_x \subset Q_0$, the maximal cube so that $Q^{\max}_x \subset E_\alpha$. Write then $\mathcal{F} = \{Q_j\}_j \subset \mathbb{D}_{Q_0}$ for the collection of maximal (hence pairwise disjoint) cubes $Q^{\max}_x$ with $x \in E_\alpha$. By construction, $E_\alpha = \bigcup_{Q_j \in \mathcal{F}} Q_j$.

Given $Q \in \mathcal{F}$, invoke Corollary 3.21 and take a bounded Lipschitz domain $\Omega_Q \subset D$ satisfying properties (i)–(iii) in the statement. In particular, we set $F_Q := \partial\Omega_Q \cap Q \subset Q$ such that $\sigma(F_Q) \ge \theta\sigma(Q)$. Our goal is to show that

$$\sigma(E_{(1+\varepsilon)\alpha} \cap F_{\gamma\alpha} \cap F_Q) \le C\Big(\frac{\gamma}{\varepsilon}\Big)^2 C_{\Omega_Q}\sigma(Q), \tag{5.13}$$

where C_{Ω_Q} is the constant from (5.3); hence it depends on the Lipschitz character of Ω_Q, which in turn depends only on the CAD character of D, and C depends as well on the CAD character of D. Assuming this momentarily, we obtain (5.4):

$$\begin{aligned}
\sigma(E_{(1+\varepsilon)\alpha} \cap F_{\gamma\alpha}) = \sigma(E_{(1+\varepsilon)\alpha} \cap E_\alpha \cap F_{\gamma\alpha}) &= \sum_{Q \in \mathcal{F}} \sigma(E_{(1+\varepsilon)\alpha} \cap E_\alpha \cap Q) \\
&\le \sum_{Q \in \mathcal{F}} \big(\sigma(Q \setminus F_Q) + \sigma(E_{(1+\varepsilon)\alpha} \cap E_\alpha \cap F_Q)\big) \\
&\le \bigg(1 - \theta + C\Big(\frac{\gamma}{\varepsilon}\Big)^2 \sup_{Q \in \mathbb{D}} C_{\Omega_Q}\bigg) \sum_{Q \in \mathcal{F}} \sigma(Q) = C^*_{\gamma,\varepsilon}\sigma(E_\alpha),
\end{aligned}$$

where $C^*_{\gamma,\varepsilon} = (1 - \theta + C(\gamma/\varepsilon)^2 \sup_{Q \in \mathbb{D}} C_{\Omega_Q})$. Note that $\sup_{Q \in \mathbb{D}} C_{\Omega_Q} < \infty$ and ultimately depends on the CAD character of D, since all the Lipschitz characters of the Ω_Q are uniformly bounded depending on the CAD character of D (see Corollary 3.21) and our assumption states that C_{Ω_Q} depends on the Lipschitz character of Ω_Q, the dimension n, and the choice of κ (the aperture of the cones).

Let us then obtain (5.13). We may assume that the left-hand side is nonzero; hence we can pick $z_Q \in E_{(1+\varepsilon)\alpha} \cap F_{\gamma\alpha} \cap F_Q$. Let Y_Q be from Corollary 3.21(ii) whose existence is guaranteed by part (i) and note that $Y_Q \in U_Q$.

We need to consider two separate cases. First assume that $Q \subsetneq Q_0$. By the maximality of $Q \in \mathcal{F}$, we can find $\tilde{x} \in \widetilde{Q} \setminus E_\alpha$, where $\widetilde{Q}$ is the dyadic parent of Q. That is, $N_*^{Q_0} v(\tilde{x}) \le \alpha$ and, in particular, $|v(X)| \le \alpha$ for every $X \in U_{\widetilde{Q}}$ since $U_{\widetilde{Q}} \subset \Gamma^{Q_0}(\tilde{x})$. Note then that if $x \in Q$, then

$$N_*^{Q_0} v(x) = \sup_{Y \in \Gamma^{Q_0}(x)} |v(Y)| = \max\Big\{ \sup_{Y \in \Gamma^{Q}(x)} |v(Y)|, \max_{Y \in U_Q, \widetilde{Q} \subset Q \subset Q_0} |v(Y)| \Big\}$$
$$\le \max\{N_*^Q v(x), N_*^{Q_0} v(\tilde{x})\} \le \max\{N_*^Q v(x), \alpha\}. \tag{5.14}$$

Since $|v(X)| \le \alpha$ for every $X \in U_{\widetilde{Q}}$, we have that $|v(X^+_{\widetilde{Q}})| \le \alpha$, where $X^+_{\widetilde{Q}}$ is the interior corkscrew point relative to $\widetilde{Q}$ (with respect to D which is a CAD). Then, recalling that the construction of $\mathcal{W}_Q$ guarantees that $X^+_{\widetilde{Q}} \in U_Q$, and that $Y_Q \in U_Q$, we have, by (5.10),

$$|v(X^+_{\widetilde{Q}}) - v(Y_Q)| = |u(X^+_{\widetilde{Q}}) - u(Y_Q)| \le C C_0 \fint_Q \widehat{S}^Q u \, d\sigma \le C C_0 \fint_Q \widehat{S}^{Q_0} u \, d\sigma$$
$$= C C_0 \fint_Q \widehat{S}^{Q_0} v \, d\sigma \le M^{\mathbb{D}}_{Q_0,2}(\widehat{S}^{Q_0} v)(z_Q) \le C C_0 \gamma \alpha, \tag{5.15}$$

where we have used that $z_Q \in Q \cap F_{\gamma\alpha}$. As a consequence,

$$|v(Y_Q)| \le |v(Y_Q) - v(X^+_{\widetilde{Q}})| + |v(X^+_{\widetilde{Q}})| \le (1 + C C_0 \gamma)\alpha \le (1 + \varepsilon/2)\alpha, \tag{5.16}$$

where C depends on the CAD character of D, and provided $\gamma < (2CC_0)^{-1}\varepsilon =: 2c_0\varepsilon$. As a result, using (5.14), for every $x \in E_{(1+\varepsilon)\alpha}$ we arrive at

$$(1+\varepsilon)\alpha < N_*^{Q_0} v(x) = N_*^Q v(x) \le N_*^Q (v - v(Y_Q))(x) + |v(Y_Q)| \le N_*^Q (v - v(Y_Q))(x) + (1 + \varepsilon/2)\alpha,$$

and, consequently,

$$E_{(1+\varepsilon)\alpha} \cap F_{\gamma\alpha} \cap F_Q \subset \{x \in F_{\gamma\alpha} \cap F_Q : N_*^Q (v - v(Y_Q))(x) > \varepsilon\alpha/2\}, \tag{5.17}$$

where we recall that we are currently considering the case $E_\alpha \subsetneq Q$.

In the second case $Q = Q_0$; hence $\mathcal{F} = \{Q\}$ and $E_\alpha = Q$. Since $Y_Q, X^+_{Q_0} \in U_{Q_0}$ we can invoke (5.10) to obtain

$$|v(Y_Q)| = |u(Y_Q) - u(X_{Q_0})^+| \le C C_0 \fint_{Q_0} \widehat{S}^{Q_0} u \, d\sigma = C C_0 \fint_{Q_0} \widehat{S}^{Q_0} v \, d\sigma$$
$$\le M^{\mathbb{D}}_{Q_0,2}(\widehat{S}^{Q_0} v)(z_Q) \le C C_0 \gamma \alpha \le (1 + C C_0 \gamma)\alpha \le (1 + \varepsilon/2)\alpha, \tag{5.18}$$

where C depends on the CAD character of D, and provided $\gamma < (2CC_0)^{-1}\varepsilon =: 2c_0\varepsilon$. Consequently, for every $x \in E_{(1+\varepsilon)\alpha}$ we arrive at

$$(1+\varepsilon)\alpha < N_*^{Q_0} v(x) \le N_*^{Q_0} (v - v(Y_Q))(x) + |v(Y_Q)| \le N_*^Q (v - v(Y_Q))(x) + (1 + \varepsilon/2)\alpha.$$

Thus, $N_*^Q (v - v(Y_Q))(x) = N_*^{Q_0} (v - v(Y_Q))(x) > \varepsilon\alpha/2$ and (5.17) holds also in this case.

We can now merge the two cases. Pick $x \in F_{\gamma\alpha} \cap F_Q = \partial\Omega_Q \cap Q$ such that $N_*^Q (v - v(Y_Q))(x) > \varepsilon\alpha/2$. Then, there exist $Q' \in \mathbb{D}_Q$ with $Q' \ni x$ and $Y \in U_{Q'}$ such that $|v(Y) - v(Y_Q)| > \varepsilon\alpha/2$. Thus, $y_{Q'} := x \in Q' \cap \partial\Omega_Q = F_Q \cap Q'$ and applying once again condition (ii) of Corollary 3.21 we can find the corresponding $Y_{Q'} \in U_{Q'}$ so that

$$|Y_{Q'} - y_{Q'}| < \ell(Q') \le C \operatorname{dist}(Y_{Q'}, \partial\Omega_Q) := (1+\kappa) \operatorname{dist}(Y_{Q'}, \partial\Omega_Q),$$

where $C \ge 2$ is the constant from Corollary 3.21. This means that $Y_{Q'} \in \Gamma_{\Omega_Q}(y_{Q'}) = \Gamma_{\Omega_Q}(x)$ (see (1.14)). On the other hand, since $Y, Y_{Q'} \in U_{Q'}$ and $x \in F_{\gamma\alpha} \cap Q'$, one can see that (5.10) yields

$$\begin{aligned}|v(Y) - v(Y_{Q'})| = |u(Y) - u(Y_{Q'})| &\le C C_0 \fint_{Q'} \widehat{S}^{Q'} u \, d\sigma \le C C_0 \fint_{Q'} \widehat{S}^{Q_0} u \, d\sigma \\ &= C C_0 \fint_{Q'} \widehat{S}^{Q_0} v \, d\sigma \le C C_0 M^{\mathbb{D}}_{Q_0,2}(\widehat{S}^{Q_0} v)(x) \le C C_0 \gamma\alpha \le \varepsilon\alpha/4, \end{aligned} \tag{5.19}$$

provided $\gamma < c_0\varepsilon = (4CC_0)^{-1}\varepsilon$. Hence,

$$\varepsilon\alpha/2 < |v(Y) - v(Y_Q)| \le |v(Y) - v(Y_{Q'})| + |v(Y'_Q) - v(Y_Q)| \le \varepsilon\alpha/4 + |v(Y'_Q) - v(Y_Q)|$$

and

$$N_{*,\Omega_Q}(v - v(Y_Q))(x) = \sup_{Z \in \Gamma_Q(x)} |v(Y) - v(Y_Q)| \ge |v(Y_{Q'}) - v(Y_Q)| \ge \varepsilon\alpha/4.$$

All these yield

$$E_{(1+\varepsilon)\alpha} \cap F_{\gamma\alpha} \cap F_Q \subset \{x \in \partial\Omega_Q : N_{*,\Omega_Q}(v - v(Y_Q))(x) > \varepsilon\alpha/4\}.$$

Use Chebyshev's inequality and the assumption (5.3) we write

$$\begin{aligned}\sigma(E_{(1+\varepsilon)\alpha} \cap F_{\gamma\alpha} \cap F_Q) &\le \sigma\{x \in \partial\Omega_Q : N_{*,\Omega_Q}(v - v(Y_Q))(x) > \varepsilon\alpha/4\} \\ &\le \left(\frac{4}{\varepsilon\alpha}\right)^2 \int_{\partial\Omega_Q} N_{*,\Omega_Q}(v - v(Y_Q))(x)^2 \, dH^n(x) \\ &\lesssim C_{\Omega_Q} \frac{16}{(\varepsilon\alpha)^2} \int_{\partial\Omega_Q} (S_{\Omega_Q} v(x))^2 \, dH^n(x), \end{aligned} \tag{5.20}$$

where C_{Ω_Q} depends on n and the CAD of D, and so do all the implicit constants. Note that

$$\begin{aligned}\int_{\partial\Omega_Q} (S_{\Omega_Q} v(x))^2 \, dH^n(x) &= \int_{\partial\Omega_Q} \iint_{|Y-x| \le (1+\kappa)\operatorname{dist}(Y,\partial\Omega_Q)} |\nabla v(Y)|^2 \operatorname{dist}(Y, \partial\Omega_Q)^{1-n} \, dY \, dH^n(x) \\ &\le \iint_{\Omega_Q} |\nabla v(Y)|^2 \operatorname{dist}(Y, \partial\Omega_Q)^{1-n} H^n(B(Y, (2+\kappa)\operatorname{dist}(Y, \partial\Omega_Q)) \cap \partial\Omega_Q) \, dY \\ &\lesssim \iint_{\Omega_Q} |\nabla v(Y)|^2 \operatorname{dist}(Y, \partial\Omega_Q) \, dY \\ &\le \iint_{T_Q} |\nabla v(Y)|^2 \delta(Y) \, dY, \end{aligned} \tag{5.21}$$

where we have used that $\partial\Omega_Q$ is ADR with constant depending on the CAD character of Ω_G; hence ultimately on the CAD character of D, and the last inequality follows from the fact that $\Omega_Q \subset D \cap B_Q \subset T_Q$

(see (iii) in Corollary 3.21 and (2.15)) and, in particular, $\operatorname{dist}(Y, \partial\Omega_Q) \le \operatorname{dist}(Y, \partial D) = \delta(Y)$ for every $Y \in \Omega_Q$. Note that (4.29) with $\widetilde{G} = |\nabla v|$ implies

$$\begin{aligned}\iint_{T_Q} |\nabla v|^2 \delta \, dY &\approx \int_Q \iint_{\Gamma^Q(x)} |\nabla v|^2 \delta^{1-n} \, dY \, d\sigma(x) \\ &= \int_Q \widehat{S}^Q v^2 \, d\sigma \le M^{\mathbb{D}}_{Q_0,2}(\widehat{S}^{Q_0} v)(z_Q)^2 \sigma(Q) \le (\gamma\alpha)^2 \sigma(Q), \end{aligned} \tag{5.22}$$

where we have used that $z_Q \in F_{\gamma\alpha}$. Thus, (5.20), (5.21), and (5.22) imply

$$\sigma(E_{(1+\varepsilon)\alpha} \cap F_{\gamma\alpha} \cap F_Q) \lesssim C_{\Omega_Q} (\gamma/\varepsilon)^2 \sigma(Q),$$

which is (5.13).

To continue the proof, having at hand (5.4), an argument analogous to (4.25) yields (5.5). To be specific, we show that taking $\varepsilon > 0$ small enough depending on n and the CAD character of D and then taking $\gamma > 0$ small enough depending on the same parameters and ε, the estimate (5.4) yields (5.5). It is here that we use a possibility to pick $\varepsilon > 0$ sufficiently small. Indeed, fix any $q > 2$, $Q_0 \in \mathbb{D}$ and write $v := u - u(X_{Q_0})$. Then, much as in (4.25), for every $N > 1$

$$\begin{aligned}\mathrm{I}_N &:= \int_0^N q\alpha^q \sigma\{x \in Q_0 : N_*^{Q_0} v(x) > \alpha\} \, \frac{d\alpha}{\alpha} \\ &= (1+\varepsilon)^q \int_0^{N/(1+\varepsilon)} q\alpha^q \sigma\{x \in Q_0 : N_*^{Q_0} v(x) > (1+\varepsilon)\alpha\} \, \frac{d\alpha}{\alpha} \\ &\le (1+\varepsilon)^q \int_0^N q\alpha^q \sigma\{x \in Q_0 : N_*^{Q_0} v(x) > (1+\varepsilon)\alpha,\ M^{\mathbb{D}}_{Q_0,2}(\widehat{S}^{Q_0} v)(x) \le \gamma\alpha\} \frac{d\alpha}{\alpha} \\ &\qquad + \left(\frac{1+\varepsilon}{\gamma}\right)^q \| M^{\mathbb{D}}_{Q_0,2}(\widehat{S}^{Q_0} v)\|^q_{L^q(Q_0)} \\ &\le C^*_{\gamma,\varepsilon} (1+\varepsilon)^q \int_0^N q\alpha^q \sigma\{x \in Q_0 : N_*^{Q_0} v(x) > \alpha\} \, \frac{d\alpha}{\alpha} + \frac{(1+\varepsilon)^q}{\gamma^q} \| M^{\mathbb{D}}_{Q_0,2}(\widehat{S}^{Q_0} v)\|^q_{L^q(Q_0)} \\ &= \left(1 - \theta + C\left(\frac{\gamma}{\varepsilon}\right)^2 \sup_{Q\in\mathbb{D}} C_{\Omega_Q}\right)(1+\varepsilon)^q \mathrm{I}_N + \frac{(1+\varepsilon)^q}{\gamma^q} \| M^{\mathbb{D}}_{Q_0,2}(\widehat{S}^{Q_0} v)\|^q_{L^q(Q_0)}. \end{aligned} \tag{5.23}$$

At this point we first choose $\varepsilon > 0$ small enough so that $(1-\theta)(1+\varepsilon)^q < \frac{1}{4}$, and once ε is fixed we take $0 < \gamma < c_0\varepsilon/C_0$ small enough so that $C(\gamma/\varepsilon)^2 \sup_{Q\in\mathbb{D}} C_{\Omega_Q} (1+\varepsilon)^q < \frac{1}{4}$. With these choices and using that $I_N \le N^q \sigma(Q_0) < \infty$, we can hide this term with $I_N$on the left-hand side of (5.23) to obtain

$$\mathrm{I}_N \le 2(1+\varepsilon)^q/\gamma^q \| M^{\mathbb{D}}_{Q_0,2}(\widehat{S}^{Q_0} v)\|^q_{L^q(Q_0)}.$$

Noting that $\mathrm{I}_N \nearrow \|N_*^{Q_0} v\|^q_{L^q(Q_0)}$ as $N \to \infty$, and using that $M^{\mathbb{D}}_{Q_0,2}$ is bounded on $L^q(Q_0)$ since $q > 2$, we obtain as desired (5.5).

We next see how to obtain (5.6) using the ideas in Remark 4.4. Proceeding as there, once we have fixed $\{\mathcal{W}_Q\}_{Q\in\mathbb{D}(\partial D)}$ a Whitney-dyadic structure for D with some parameters η and K. Given $x \in \partial D$ and $0 < r < 2\operatorname{diam}(\partial D)$, write $\Delta = \Delta(x, r)$ and $B = B(x, r)$ and consider the case $r \ll \operatorname{diam}(\partial D)$. Then

$\Gamma^r(y) \subset 2B$ for every $y \in \Delta$, if $z \notin 6\Delta$ then $\Gamma_{\Omega,1}(z) \cap 2B = \varnothing$, and $\Gamma_{\Omega,1}(y) \subset \Gamma(y)$ for every $y \in \partial D$. All these and with Remark 2.38 imply

$$\begin{aligned}\|N^r_{*,D,\kappa}(u - u(X^+_\Delta))\|_{L^q(\Delta)} &\le \|N_{*,D,\kappa}((u - u(X^+_\Delta))1_{2B})\|_{L^q(\partial D)} \\ &\lesssim \|N_{*,D,1}((u - u(X^+_\Delta))1_{2B})\|_{L^q(\partial D)} \le \|N_*((u - u(X^+_\Delta))1_{2B})\|_{L^q(6\Delta)}.\end{aligned}$$

We introduce $\mathcal{D}_\Delta$ as in (4.6). Let $Q \in \mathcal{D}_\Delta$ and note that $\delta(X^+_Q) \approx \ell(Q) \approx r \approx \delta(X^+_\Delta)$ and also $|X^+_Q - X^+_\Delta| \lesssim r$. Hence we can use the Harnack chain condition to find a collection of cubes $I_1, \dots, I_N$ with $N \lesssim 1$ so that $X^+_Q \in I_0$, $X^+_\Delta \in I_N$, $\operatorname{dist}(4I_j, \partial D) \approx \ell(I_j) \approx r \approx \ell(Q)$ for $1 \le j \le N$, and there exists $X_j \in I_j \cap I_{j+1} \ne \varnothing$ for each $1 \le j \le N-1$. Write $X_0 = X_{Q^+}$, $X_N = X^+_\Delta$, and note that for every $1 \le j \le N$

$$\operatorname{dist}(I_j, Q) \le |X_j - x_Q| \le |X_j - X^+_Q| + |X^+_Q - x_Q| \lesssim \sum_{k=0}^{j-1} |X_k - X_{k+1}| + \ell(Q) \le \sum_{k=0}^{j-1} \operatorname{diam}(I_{k+1}) + \ell(Q) \approx \ell(Q).$$

Thus, there exist η' and K' depending on n, the CAD character of D, and fixed parameters η and K such that if $\{\mathcal{W}'_Q\}_{Q \in \mathbb{D}(\partial D)}$ is a Whitney-dyadic structure for D with parameters η' and K', and if $I \in \mathcal{W}$ with $I \cap 2I_j \ne \varnothing$, then $I \in (\mathcal{W}'_Q)^0 \subset \mathcal{W}'_Q$. Consequently, $2I \subset U'_Q$ (the Whitney region corresponding to Q with the Whitney-dyadic structure $\{\mathcal{W}'_Q\}_{Q \in \mathbb{D}(\partial D)}$). All these and (5.2) yield

$$\begin{aligned}|u(X^+_Q) - u(X^+_\Delta)| &= |u(X_0) - u(X_N)| \le \sum_{j=0}^{N-1} |u(X_j) - u(X_{j+1})| \\ &\le \sum_{j=0}^{N-1} \left| u(X_j) - \ell(2I_{j+1})^{-n-1} \iint_{2I_{j+1}} u\, dY \right| + \left| u(X_{j+1}) - \ell(2I_{j+1})^{-n-1} \iint_{2I_{j+1}} u\, dY \right| \\ &\lesssim \sup_{1 \le j \le N} \sup_{X \in I_j} \left| u(X) - \ell(2I_j)^{-n-1} \iint_{2I_j} u\, dY \right| \\ &\lesssim C_0 \sup_{1 \le j \le N} \left(\ell(2I_j)^{-n-1} \iint_{2I_j} \left| u - \ell(2I_j)^{-n-1} \iint_{2I_j} u \right|^2 dY \right)^{\frac12} \\ &\lesssim C_0 \sup_{1 \le j \le N} \ell(I_j) \left(\ell(2I_j)^{-n-1} \iint_{2I_j} |\nabla u|^2\, dY \right)^{\frac12} \\ &\approx C_0 \sup_{1 \le j \le N} \left(\iint_{2I_j} |\nabla u|^2 \delta^{1-n}\, dY \right)^{\frac12} \\ &\le C_0 \sup_{1 \le j \le N} \left(\iint_{U'_Q} |\nabla u|^2 \delta^{1-n}\, dY \right)^{\frac12} \le C_0 \inf_{y \in Q} \mathcal{A}'^{,Q}(\nabla u)(y),\end{aligned}$$

where $\mathcal{A}'^{,Q}$ is the local area integral to the cones $\Gamma'(\cdot)$ made up with the Whitney regions $U'_{Q'}$.

On the other hand for each $Q \in \mathcal{D}_\Delta$ we have much as before that

$$\Gamma(y) \cap 2B \subset \Gamma^Q(y) \subset \widehat{\Gamma}^Q(y) \subset B(x_Q, K\ell(Q)) \cap D \subset K'B \cap D$$

for every $y \in Q \in \mathcal{D}_\Delta$. Taking K' even larger we also have that $\Gamma^{\prime,Q}(y) \subset D \subset K'B \cap D$ for every $y \in Q \in \mathcal{D}_\Delta$. Thus all the previous considerations, (5.5) for $q > 2$, and Remark 2.38 give

$$\begin{aligned}
\|N^r_{*,D,\kappa}(u - u(X^+_\Delta))\|^q_{L^q(\Delta)} &\lesssim \sum_{Q\in\mathcal{D}_\Delta} \|N_*((u - u(X^+_\Delta))1_{2B})\|^q_{L^q(Q)} \\
&\le \sum_{Q\in\mathcal{D}_\Delta} \big(\|N^Q_*((u - u(X^+_Q))1_{2B})\|^q_{L^q(Q)} + |u(X^+_Q) - u(X^+_\Delta)|^q\sigma(Q)\big) \\
&\lesssim \sum_{Q\in\mathcal{D}_\Delta} \big(\|\widehat{S}^Q u\|^q_{L^q(Q)} + C_0 \inf_{y\in Q} \mathcal{A}^{\prime,Q}(\nabla u)(y)^q\sigma(Q)\big) \\
&\le \|\widehat{\mathcal{A}}(|\nabla u|1_{k'B})\|^q_{L^q(\partial D)} + C_0\|\mathcal{A}'(|\nabla u|1_{k'B})\|^q_{L^q(\partial D)} \\
&\lesssim (1 + C_0)\|\mathcal{A}_{D,\min\{1,\kappa\}}(|\nabla u|1_{k'B})\|^q_{L^q(\partial D)} \\
&\lesssim (1 + C_0)\|\mathcal{A}^{3K'r}_{D,\kappa}(|\nabla u|)\|^q_{L^q(3K'\Delta)} \\
&= (1 + C_0)\|S^{3K'r}_{D,\kappa} u\|^q_{L^q(3K'\Delta)},
\end{aligned}$$

where we have used that $\Gamma_{\Omega,1}(z) \cap K'B \ne \varnothing$ then $z \in 3K'\Delta$. This proves (5.6).

To complete the proof we observe that if ∂D is bounded then for any $x \in \partial D$ we have that $\partial D = \Delta(x, 3\,\mathrm{diam}(\partial D)/2)$. Thus (5.7) readily follows from (5.6). On the other hand, to obtain for (5.8) fix $x_0 =\in \partial D$ and write $\Delta_R = \Delta(x_0, R)$. Given $\varepsilon > 0$, there exist R_ε such that $|u(X)| < \varepsilon$ for every $|X - x_0| \ge R_\varepsilon$, with $X \in D$. By the corkscrew condition $B(X^+_{\Delta_R}, R/C) \subset B(x_0, R)$ for some $C > 1$ and then $|X^+_{\Delta_R} - x_0| \ge R/C$.

Fix $y \in \partial D$ and let $R > 2\max\{CR_\varepsilon, |y - x_0|\}$ so that $B(x_0, R_\varepsilon) \subset B(y, R)$ and $|X^+_{\Delta_R} - x_0| \ge R/C > R_\varepsilon$. Hence, $|u(X^+_{\Delta_R})| < \varepsilon$, $|u(Z)| < \varepsilon$ for every $D \setminus B(y, R)$, and

$$\begin{aligned}
|N_{*,\kappa}u(y) - N^R_{*,k}(u - u(X^+_{\Delta_R})(y)1_{\Delta_R}(y)| &= |N_{*,\kappa}u(y) - N_{*,k}((u - u(X^+_{\Delta_R})1_{B(y,R)}))(y)| \\
&\le |N_{*,\kappa}(u - (u - u(X^+_{\Delta_R})1_{B(y,R)}))(y)| \\
&\le |N_{*,\kappa}(u1_{D\setminus B(y,R)})(y)| + |u(X^+_{\Delta_R})| < 2\varepsilon.
\end{aligned}$$

This shows that for every $y \in \partial D$

$$\lim_{R\to\infty} N^R_{*,k}(u - u(X^+_{\Delta_R})(y)1_{\Delta_R}(y) = N_{*,\kappa}u(y).$$

Thus Fatou's lemma and (5.6) imply for every $q > 2$

$$\begin{aligned}
\int_{\partial D} N_{*,\kappa}u(y)^q\, d\sigma(y) &\le \liminf_{R\to\infty} \int_{\Delta_R} N^R_{*,k}(u - u(X^+_{\Delta_R})(y)^q\, d\sigma(y) \\
&\le C'' \liminf_{R\to\infty} \int_{K'\Delta_R} S^{K'R}u(y)^q\, d\sigma(y) \le C'' \int_{\partial D} Su(y)^q\, d\sigma(y). \qquad \square
\end{aligned}$$

Our next goal is to extend the previous result so that we have the $N < S$ estimates in all L^q. We need to introduce some notation. Recall that if D is a CAD, we have constructed a Whitney-dyadic structures $\{\mathcal{W}_Q\}_{Q\in\mathbb{D}(\partial D)}$ for D with parameters η and K; see Section 2.4. In the following result we will need to work with two different Whitney-dyadic structures associated with different parameters and we need to introduce some notation to distinguish between the associated objects. More specifically, let

$\{\mathcal{W}_Q\}_{Q\in\mathbb{D}(\partial D)}$ (resp. $\{\mathcal{W}'_Q\}_{Q\in\mathbb{D}(\partial D)}$) be a Whitney-dyadic structure for D with parameters $\eta \ll 1$ and $K \gg 1$ (resp. $\eta' \ll 1$ and $K' \gg 1$). Associated with $\{\mathcal{W}_Q\}_{Q\in\mathbb{D}(\partial D}$ (resp. $\{\mathcal{W}'_Q\}_{Q\in\mathbb{D}(\partial D)}$) we define the Whitney regions U_Q, the dyadic cones Γ and the local dyadic cones Γ^Q (resp. U'_Q, Γ', $\Gamma'^{,Q}$) as in (2.10), (2.23), or (2.24). With this we define N_*, N_*^Q, S, S^Q (resp. N'_*, $N_*'^{,Q}$, S', $S'^{,Q}$) as in Definition 2.33.

Theorem 5.24. *Let $D \subset \mathbb{R}^{n+1}$ be a CAD. Let $u \in W^{1,2}_{\mathrm{loc}}(D) \cap C(D)$ be so that (5.2) holds and assume that there exists $C'_0 > 0$ and $p > 2$ such that for any cube I with $2I \subset D$,*

$$\left(\ell(I)^{-n-1} \iint_I |\nabla u|^p \, dX\right)^{\frac{1}{p}} \le C'_0 \left(\ell(I)^{-n-1} \iint_{2I} |\nabla u|^2 \, dX\right)^{\frac{1}{2}}. \tag{5.25}$$

Suppose that the $N < S$ estimates are valid on L^p on all bounded chord-arc subdomains $\Omega \subset D$, that is, for any bounded chord-arc subdomain $\Omega \subset D$, there holds

$$\|N_{*,\Omega}(u - u(X^+_\Omega))\|_{L^p(\partial\Omega)} \le C_\Omega \|S_\Omega u\|_{L^p(\partial\Omega)}. \tag{5.26}$$

Here X^+_Ω is any interior corkscrew point of Ω at the scale of $\operatorname{diam}(\Omega)$, *and the constant C_Ω depends on the CAD character of Ω, the dimension n, p, the implicit choice of κ (the aperture of the cones in $N_{*,\Omega}$ and S_Ω), and the implicit corkscrew constant for the point X^+_Ω.*

Given $\eta \ll 1$ and $K \gg 1$, consider $\{\mathcal{W}_Q\}_{Q\in\mathbb{D}(\partial D)}$ a Whitney-dyadic structure for D with parameters η and K; see Section 2.4. Then, there exist $\eta' \ll \eta$ and $K' \gg K$ (depending on n, the CAD character of D, and the choice of η, K, τ) so that if $\{\mathcal{W}'_Q\}_{Q\in\mathbb{D}(\partial D)}$ is a Whitney-dyadic structure for D with parameters η and K, for every $Q \in \mathbb{D}(\partial D)$,

$$\|N_*^Q(u - u(X^+_Q))\|_{L^q(Q)} \le C' \|S'^{,Q} u\|_{L^q(Q)} \quad \textit{for all } 0 < q < \infty, \tag{5.27}$$

where C' depends on n, the CAD character of D, C_0, C'_0, q, and the choice of η, K, τ. Here N_*^Q is the nontangential maximal function associated with the Whitney-dyadic structure $\{\mathcal{W}_Q\}_{\mathbb{D}(\partial D)}$, while $S'^{,Q}$ is the square function with the associated with the Whitney-dyadic structure $\{\mathcal{W}'_Q\}_{\mathbb{D}(\partial D)}$.

As a consequence, for any $x \in \partial D$ and $0 < r < 2\operatorname{diam}(\partial D)$ there exists K' depending on n, the CAD character of D such that for every $\kappa > 0$

$$\|N^r_{*,D,\kappa}(u - u(X^+_{\Delta(x,r)})\|_{L^q(\Delta(x,r))} \le C'' \|S^{K'r}_{D,\kappa} u\|_{L^q(\Delta(x,K'r))} \quad \textit{for all } 0 < q < \infty, \tag{5.28}$$

where $\Delta(x,r) = B(x,r) \cap \partial\Omega$, and where C'' depends on q, n, the CAD character of D, C_0, C'_0, and κ. In particular, if ∂D is bounded then

$$\|N_{*,D,\kappa}(u - u(X^+_D))\|_{L^q(\partial D)} \le C'' \|S_{D,\kappa} u\|_{L^q(\partial D)} \quad \textit{for all } 0 < q < \infty, \tag{5.29}$$

and if ∂D is unbounded and $u(X) \to 0$ as $|X| \to \infty$ then

$$\|N_{*,D,\kappa} u\|_{L^q(\partial D)} \le C'' \|S_{D,\kappa} u\|_{L^q(\partial D)} \quad \textit{for all } 0 < q < \infty. \tag{5.30}$$

We note that (5.25) can be relaxed so that it suffices to assume that it holds for $I = 2J$ with $J \in \mathcal{W}(D)$. We also note that the same proof allows us to work with 1-sided CAD. That is, if D is a 1-sided CAD and (5.26) holds for all bounded 1-sided chord-arc subdomains then (5.27) and (5.29) hold for D. Further details are left to the interested reader.

Proof. For starters we fix $\eta \ll 1$ and $K \gg 1$ and let $\{\mathcal{W}_Q\}_{Q\in D}$ be Whitney-dyadic structure for D with parameters η and K. Let $\eta' \ll \eta$ be small enough and $K' \gg K$ large enough to be chosen and let $\{\mathcal{W}'_Q\}_{Q\in D}$ be Whitney-dyadic structure for D with parameters η' and K'. Taking into account (2.9) if $(\eta')^{1/4} \le C^{-1}\eta^{1/2}$ and $(K')^{1/2} \ge CK^{1/2}$, then $\mathcal{W}_Q \subset (\mathcal{W}'_Q)^0 \subset \mathcal{W}'_Q$ for every $Q \in \mathbb{D} = \mathbb{D}(\partial D)$. Consequently, $\widehat{\Gamma}^Q(x) \subset \Gamma'_Q(x)$ and $\widehat{S}^Q v(x) \le S'^{,Q} v(x)$ for every $x \in \partial D$, $Q \in \mathbb{D}$, and $v \in W^{1,2}_{\rm loc}(D)$.

Much as in the proof of Theorem 5.1, matters can be reduced to showing that for every $\alpha, \gamma, \varepsilon > 0$ with $0 < \gamma \ll \varepsilon/C_0$ and for any given $Q_0 \in \mathbb{D}$

$$\sigma\{x \in Q_0 : N_*^{Q_0}(u - u(X^+_{Q_0}))(x) > (1+\varepsilon)\alpha,\ S'^{,Q_0}u(x) \le \gamma\alpha\} \\ \le C^*_{\gamma,\varepsilon}\sigma\{x \in Q_0 : N_*^{Q_0}(u - u(X^+_{Q_0}))(x) > \alpha\}, \quad (5.31)$$

and we will me more specific about the constant $C^*_{\gamma,\varepsilon}$ momentarily.

Let us fix $Q_0 \in \mathbb{D}$ and write $v := u - u(X^+_{Q_0})$, with $X^+_{Q_0}$ begin the corkscrew relative to Q_0, that is, relative to the surface ball Δ_{Q_0} (see (2.2) and (2.3)). For every $\alpha > 0$ we set

$$E_\alpha := \{x \in Q_0 : N_*^{Q_0} v(x) > \alpha\}, \qquad \widetilde{F}_\alpha := \{x \in Q_0 : S'^{,Q_0} v(x) \le \alpha\}.$$

Our goal is to obtain

$$\sigma(E_{(1+\varepsilon)\alpha} \cap \widetilde{F}_{\gamma\alpha}) \le C^*_{\gamma,\varepsilon}\sigma(E_\alpha), \quad (5.32)$$

where $C^*_{\gamma,\varepsilon} = C(\gamma/\varepsilon)^p(1 + C'_0)\sup_{Q\in\mathbb{D},\widetilde{\mathcal{F}}} C_{\Omega_{\widetilde{\mathcal{F}},Q}}$, where the sup runs over all $Q \in \mathbb{D}$ and all pairwise disjoint families $\widetilde{F} \subset \mathbb{D}_Q \setminus \{Q\}$. Note that $\sup_{Q\in\mathbb{D},\widetilde{\mathcal{F}}} C_{\Omega_{\widetilde{\mathcal{F}},Q}} < \infty$ and ultimately depends on the CAD character of D, since all the sawtooth subdomains $\Omega_{\widetilde{\mathcal{F}},Q}$ are CAD with uniform constants (see Lemma 2.55) and our assumption states that $C_{\Omega_{\widetilde{\mathcal{F}},Q}}$ depends on the CAD character of $\Omega_{\widetilde{\mathcal{F}},Q}$.

With this goal in mind we fix $\alpha, \gamma, \varepsilon > 0$. We may assume that $E_\alpha \ne \varnothing$; otherwise (5.31) is trivial. As in the proof of Theorem 5.1we can find $\mathcal{F} = \{Q_j\}_j \subset \mathbb{D}_{Q_0}$, a family of maximal (hence pairwise disjoint) cubes with respect to the property $Q \subset E_\alpha$, so that $E_\alpha = \bigcup_{Q_j\in\mathcal{F}} Q_j$. We then fix $Q \in \mathcal{F}$ and we just need to see that

$$\sigma(E_{(1+\varepsilon)\alpha} \cap \widetilde{F}_{\gamma\alpha} \cap Q) \le C^*_{\gamma,\varepsilon}\sigma(Q), \quad (5.33)$$

assuming that $\gamma < c_0\varepsilon$ with a suitably small c_0 depending on n, the CAD character of D. We may assume that $\sigma(E_{(1+\varepsilon)\alpha} \cap \widetilde{F}_{\gamma\alpha} \cap Q) > 0$ and pick $z_Q \in E_{(1+\varepsilon)\alpha} \cap \widetilde{F}_{\gamma\alpha} \cap Q$. We follow the same argument of the proof of Theorem 5.1taking into account that the set $F_{\gamma\alpha}$ needs to be replaced by $\widetilde{F}_{\gamma\alpha}$. Here we do not invoke Corollary 3.21 and we formally take $F_Q = Q$. Also we take $Y_Q = X^+_Q$, the corkscrew relative to Q. We replace (5.15) by

$$|v(X^+_{\widetilde{Q}}) - v(Y_Q)| = |u(X^+_{\widetilde{Q}}) - u(Y_Q)| \le CC_0 \inf_{z\in Q} \widehat{S}^Q u(z) \le \widehat{S}^{Q_0} v(z_Q) \le S'^{,Q_0} v(z_Q) \le CC_0\gamma\alpha,$$

where we have used (5.10) and the fact that $z_Q \in Q \cap \widetilde{F}_{\gamma\alpha}$. Thus, assuming that $\gamma < (2CC_0)^{-1}\varepsilon =: 2c_0\varepsilon$, one arrives at (5.17) with $\widetilde{F}_{\gamma\alpha}$ in place of $F_{\gamma\alpha}$ and $F_Q = Q$ in the case $E_\alpha \subsetneq Q$. On the other hand, the same estimate holds in the case $Q = Q_0$ since $Y_Q = X_Q^+ = X_{Q_0}^+$; hence (5.18) becomes trivial. Thus we have obtained that in either case

$$E_{(1+\varepsilon)\alpha} \cap \widetilde{F}_{\gamma\alpha} \cap Q \subset \{x \in \widetilde{F}_{\gamma\alpha} \cap F_Q : N_*^Q(v - v(Y_Q))(x) > \varepsilon\alpha/2\} =: E_Q. \tag{5.34}$$

Let E'_Q be an arbitrary closed subset of E_Q with $\sigma(E'_Q) > 0$. Let $x \in Q \setminus E'_Q$. Since E'_Q is closed there exists $r_x > 0$ such that $B(x, r_x) \cap E'_Q = \varnothing$. Pick any $Q_x \in \mathbb{D}$ with $Q_x \ni x$ and $\ell(Q_x) \ll \min\{\ell(Q), r_x\}$. Then, $x \in Q_x \cap Q$ and necessarily $Q_x \subset Q$. Also $Q_x \subset B(x, r_x)$ since $x \in Q_x$ and $\operatorname{diam}(Q_x) \approx \ell(Q_x) \ll r_x$. All in one, $Q_x \subset Q \setminus E'_Q$ and there exists a maximal cube $Q_x^{\max} \in \mathbb{D}_Q$ so that $Q_x^{\max} \subset Q \setminus E'_Q$. Note that $Q_x^{\max} \subsetneq Q$; otherwise $E'_Q = \varnothing$ which contradicts the fact that $\sigma(E_{(1+\varepsilon)\alpha} \cap \widetilde{F}_{\gamma\alpha} \cap Q) > 0$. Let $\widetilde{\mathcal{F}}$ be the family of maximal (hence pairwise disjoint) cubes $Q_x^{\max}$ with $x \in E'_Q$. Note that $\widetilde{\mathcal{F}} \subset \mathbb{D}_Q \setminus \{Q\}$ and $Q \setminus E'_Q = \bigcup_{Q' \in \widetilde{\mathcal{F}}} Q'$.

Let $\Omega_\star = \widehat{\Omega}_{\widetilde{\mathcal{F}},Q}$. Let us write $\delta_\star(\cdot) = \operatorname{dist}(\cdot, \partial\Omega_\star)$ and $\sigma_\star = H^n|_{\partial\Omega_\star}$. We start with Chebyshev's inequality and the fact that $E'_Q \subset E_Q$

$$\sigma(E'_Q) \le \left(\frac{2}{\varepsilon\alpha}\right)^p \int_{E'_Q} N_*^Q(v - v(Y_Q))(x)^p \, d\sigma(x), \tag{5.35}$$

and now change the cones from those used in N_*^Q (dyadic, with respect to D) to the traditional ones (1.14) with respect to $\Omega_\star$. More precisely, let $x \in E'_Q = Q \setminus \bigcup_{Q' \in \widetilde{\mathcal{F}}} Q' \subset \partial\Omega_\star \cap \partial D$ (see [Hofmann and Martell 2014, Proposition 6.1]) and $Y \in \Gamma^Q(x)$. Then $Y \in I^*(\tau)$ with $I \in \mathcal{W}_{Q'}$ with $x \in Q' \in \mathbb{D}_Q$ and

$$|Y - x| \le \operatorname{diam}(I) + \operatorname{dist}(I, Q') + \operatorname{diam}(Q') \lesssim \ell(I).$$

Note that $Q' \in \mathbb{D}_{\widetilde{\mathcal{F}},Q}$; otherwise $Q' \subset Q'' \in \widetilde{\mathcal{F}}$ and hence $x \in \bigcup_{Q'' \in \widetilde{\mathcal{F}}} Q'' = Q \setminus E'_Q$. As a consequence, $\operatorname{int}(I^*(2\tau)) \subset \operatorname{int}(U_{Q',2\tau}) = \operatorname{int}(\widehat{U}_{Q'}) \subset \Omega_\star$ and $\delta_\star(Y) \gtrsim \ell(I)$. All this shows that $|Y - x| \lesssim \delta_\star(Y)$ and this means for some choice of κ (depending on the CAD character, and η and K), $Y \in \Gamma_{\Omega_\star,\kappa}(x)$ (see (1.14)). Thus, with the notation in (1.16),

$$N_*^Q(v - v(Y_Q))(x) = \sup_{Y \in \Gamma^Q(x)} |v(Y) - v(Y_Q)| \le \sup_{Y \in \Gamma_{\Omega_\star,\kappa}(x)} |v(Y) - v(Y_Q)| =: N_{*,\Omega_\star,\kappa}(v - v(Y_Q))(x).$$

and (5.35) leads to

$$\begin{aligned}
\sigma(E'_Q) &\le \left(\frac{2}{\varepsilon\alpha}\right)^p \int_{E'_Q} N_{*,\Omega_\star,\kappa}(v - v(Y_Q))(x)^p \, d\sigma_\star(x) \\
&\le \left(\frac{2}{\varepsilon\alpha}\right)^p \int_{\partial\Omega_\star} N_{*,\Omega_\star,\kappa}(v - v(Y_Q))(x)^p \, d\sigma_\star(x) \\
&\lesssim \frac{1}{(\varepsilon\alpha)^p} \int_{\partial\Omega_\star} N_{*,\Omega_\star,\kappa_0}(v - v(Y_Q))(x)^p \, d\sigma_\star(x),
\end{aligned} \tag{5.36}$$

where the last estimate follows from a change of aperture in the cones (see Remark 2.38). We remark that $Y_Q = X_Q^+$, which is a corkscrew point for Q with respect to D. By construction, if we take $I \in \mathcal{W}$ so that $X_Q^+ \in I$ then $I \in \mathcal{W}_Q$. Hence, much as before

$$\delta(Y_Q) \approx \ell(Q) \approx \ell(I) \lesssim \delta_\star(Y_Q) \le \delta(Y_Q).$$

Hence Y_Q is an interior corkscrew of $\Omega_\star$ at the scale $\operatorname{diam}(\Omega_\star)\approx\ell(Q)$ (see (2.14)). Note $v(\cdot)-v(Y_Q)=u(\cdot)-u(Y_Q)$ and $\nabla v=\nabla u$ in D. This and the fact that $\Omega_\star$ is a CAD (see Lemma 2.55) allow us to invoke (5.26), which together with (5.36), yields

$$\begin{aligned}\sigma(E'_Q)&\lesssim\frac{1}{(\varepsilon\alpha)^p}\int_{\partial\Omega_\star}N_{*,\Omega_\star,\kappa_0}(v-v(Y_Q))(x)^p\,d\sigma_\star(x)\\&\le C_{\Omega_\star}\frac{1}{(\varepsilon\alpha)^p}\int_{\partial\Omega_\star}S_{\Omega_\star,\kappa_0}v(x)^p\,d\sigma_\star(x)\\&\lesssim C_{\Omega_\star}\frac{1}{(\varepsilon\alpha)^p}\int_{\partial\Omega_\star}S_{\Omega_\star,1}v(x)^p\,d\sigma_\star(x)\\&=C_{\Omega_\star}\frac{1}{(\varepsilon\alpha)^p}\int_{E'_Q}S_{\Omega_\star,1}v(x)^p\,d\sigma_\star(x)+C_{\Omega_\star}\frac{1}{(\varepsilon\alpha)^p}\int_{\partial\Omega_\star\cap D}S_{\Omega_\star,1}v(x)^p\,d\sigma_\star(x)=:C_{\Omega_\star}(\mathrm{I}+\mathrm{II}),\end{aligned}\tag{5.37}$$

where the third estimate follows from a change of aperture in the cones (see Remark 2.38)) and the first equality from [Hofmann and Martell 2014, Propositions 6.1 and 6.3].

To estimate the previous terms we first need to introduce some notation. Given $x\in\partial\Omega_\star$ and for some parameter $N\ge1$ (depending on the CAD character of D) to be chosen later we write

$$\Gamma^1_{\Omega_\star,1}:=\Gamma_{\Omega_\star,1}\cap\{Y\in\Omega_\star:\delta(Y)\le\ell(Q)\},\qquad\Gamma^2_{\Omega_\star,1}:=\Gamma_{\Omega_\star,1}\setminus\Gamma^1_{\Omega_\star,1}.$$

To proceed let us observe that if $Q'\in\mathbb{D}_{\widetilde{\mathcal{F}},Q}$, then one can find $y_{Q'}\in Q'\cap E'_Q$; otherwise, $Q'\cap E'_Q=\varnothing$ and by construction there exists $Q''\in\widetilde{F}$ with $Q'\subset Q''$, contradicting the fact that $Q'\in\mathbb{D}_{\widetilde{\mathcal{F}},Q}$.

Given $x\in\partial\Omega_\star$, let $Y\in\Gamma^2_{\Omega,1}(x)\subset\Omega_\star=\widehat{\Omega}_{\widetilde{\mathcal{F}},Q}$. Then $Y\in\widehat{U}_{Q'}$ for some $Q'\in\mathbb{D}_{\widetilde{\mathcal{F}},Q}$. In particular, $Y\in\widehat{\Gamma}^{Q'}(y_{Q'})\subset\widehat{\Gamma}^{Q_0}(y_{Q'})$. Also, $\ell(Q)<\delta(Y)\approx\ell(Q')\le\ell(Q)$. This means that

$$\begin{aligned}\iint_{\Gamma^2_{\Omega_\star,1}(x)}|\nabla v|^2\delta^{1-n}\,dY&\le\sum_{\substack{Q'\in\mathbb{D}_{\widetilde{\mathcal{F}},Q}\\ \ell(Q')\approx\ell(Q)}}\iint_{\widehat{\Gamma}^{Q_0}(y_{Q'})}|\nabla v|^2\delta^{1-n}\,dY=\sum_{\substack{Q'\in\mathbb{D}_{\widetilde{\mathcal{F}},Q}\\ \ell(Q')\approx\ell(Q)}}\widehat{S}^{Q_0}v(y_{Q'})^2\\&\le\sum_{\substack{Q'\in\mathbb{D}_{\widetilde{\mathcal{F}},Q}\\ \ell(Q')\approx\ell(Q)}}S'^{,Q_0}v(y_{Q'})^2\le(\gamma\alpha)^2\#\{Q'\in\mathbb{D}_Q:\ell(Q')\approx\ell(Q)\}\lesssim(\gamma\alpha)^2.\end{aligned}\tag{5.38}$$

We next turn to estimate I. Let $x\in E'_Q\subset\partial\Omega_\star\cap\partial D$ (see [Hofmann and Martell 2014, Proposition 6.1]). Note first that if $Y\in\Gamma_{\Omega,1}(x)$, then $\delta(Y)\le|Y-x|\le2\delta_\star(Y)$ and thus (5.38) gives

$$\begin{aligned}\widehat{S}_{\Omega_\star,1}v(x)^2=\iint_{\Gamma_{\Omega_\star,1}(x)}|\nabla v|^2\delta_\star^{1-n}\,dY&\lesssim\iint_{\Gamma^1_{\Omega_\star,1}(x)}|\nabla v|^2\delta^{1-n}\,dY+\iint_{\Gamma^2_{\Omega_\star,1}(x)}|\nabla v|^2\delta^{1-n}\,dY\\&\lesssim\iint_{\Gamma^1_{\Omega_\star,1}(x)}|\nabla v|^2\delta^{1-n}\,dY+(\gamma\alpha)^2.\end{aligned}\tag{5.39}$$

Given $Y\in\Gamma^1_{\Omega,1}(x)\subset\Omega_\star\subset D$, one has $Y\in I_Y$ for some $I_Y\in\mathcal{W}$. Pick then $Q_Y\ni x$ with $\ell(I_Y)=\ell(Q_Y)$ and note that by (2.5), and since K is large enough,

$$\operatorname{dist}(Q_Y,I_Y)\le|x-Y|\le2\operatorname{dist}(Y,\partial\Omega_\star)\le2\delta(Y)\le82\operatorname{diam}(I_Y)=82\sqrt{n}\,\ell(Q_Y)\le K^{\frac12}\ell(Q_Y).$$

This means that $I_Y \in \mathcal{W}^0_{Q_Y} \subset \mathcal{W}_{Q_Y}$. Additionally,

$$\ell(Q_Y) = \ell(I_Y) \le \operatorname{dist}(I, \partial D) \le \delta(Y) \le \ell(Q)/N \le \ell(Q);$$

this together with the fact that $x \in Q_Y \cap Q$ gives that $Q_Y \in \mathbb{D}_Q$. Hence, $Y \in I_Y \subset U_{Q_Y} \subset \Gamma^Q(x) \subset \widehat{\Gamma}^{Q_0}(x)$ and eventually

$$\iint_{\Gamma^1_{\Omega_\star,1}(x)} |\nabla v|^2 \delta^{1-n}\, dY \le \widehat{S}^{Q_0} v(x)^2 \le S'^{,Q_0} v(x)^2 \le (\gamma\alpha)^2, \tag{5.40}$$

since $x \in E'_Q \subset E_Q$. This and (5.39) imply that

$$\mathrm{I} \lesssim \left(\frac{\gamma\alpha}{\varepsilon\alpha}\right)^p \sigma(E'_Q) \le \left(\frac{\gamma}{\varepsilon}\right)^p \sigma(Q). \tag{5.41}$$

Turning to II, we start with the following:

Claim 5.42. *We can take choose η' small enough and K' large enough (depending on n, the CAD character of D, and the choice of η, K, τ) such that for any $x \in \partial\Omega_\star \cap D$ there exists $y_x \in E'_Q$ such that if $J \in \mathcal{W}$ satisfies $4J \cap \Gamma^1_{\Omega_\star,1}(x) \ne \varnothing$, then $4J \subset \Gamma'^{,Q}(y_x)$ and, in particular, $\Gamma^1_{\Omega_\star,1}(x) \subset \Gamma'^{,Q}(y_x)$.*

Proof. Fix $x \in \partial\Omega_\star \cap D$. Then $x \in \partial\hat{I}$, where $\hat{I} := I^*(2\tau) = (1+2\tau)I$ with $I \in \mathcal{W}_{Q'}$, $Q' \in \mathbb{D}_{\widetilde{\mathcal{F}},Q}$. In this scenario we observed before that we can find pick $y_x = y_{Q'} \cap E_{Q'} \cap Q'$.

Let $Y \in 4J \cap \Gamma^1_{\Omega_\star,1}(x)$ and assume first that $|Y - x| \ge \ell(I)\tau/(2\sqrt{n})$. Pick $Q'' \in \mathbb{D}$ with $y_{Q'} \in Q''$ and $\delta_\star(Y)/2 < \ell(Q'') \le \delta_\star(Y)$. Note that $\ell(Q'') \le \delta_\star(Y) \le \delta(Y) \le \ell(Q)$ since $Y \in \Gamma^1_{\Omega_\star,1}(x)$ and hence $Q'' \subset Q$. Then, choosing N large enough, depending on n and the CAD character of D (recall that η, K have been already fixed depending also on the CAD character of D),

$$\begin{aligned}
\operatorname{dist}(4J, Q'') &\le |Y - y_{Q'}| \le |Y - x| + \operatorname{diam}(\hat{I}) + \operatorname{dist}(I, Q') + \operatorname{diam}(Q') \\
&\le |Y - x| + C K^{\frac12} \eta^{-\frac12} \ell(I) \le (1 + C K^{\frac12} \eta^{-\frac12} \tau^{-1}) |Y - x| \\
&\le \tfrac12 N |Y - x| \le N \delta_\star(Y) \le N \ell(Q''),
\end{aligned}$$

where we have used (2.9). Note also that by (2.5)

$$\ell(Q'') \le \delta_\star(Y) \le \delta(Y) \le \operatorname{diam}(4J) + \operatorname{dist}(4J, \partial D) \le 41 \operatorname{diam}(J) = 41\sqrt{n}\,\ell(J)$$

and

$$\ell(J) \le \operatorname{dist}(4J, \partial D)/\sqrt{n} \le \operatorname{dist}(4J, Q'')/\sqrt{n} \le N \ell(Q'').$$

All in one we have obtained that

$$N^{-1}\ell(J) \le \ell(Q'') \le 41\sqrt{n}\,\ell(J), \qquad \operatorname{dist}(4J, Q'') \le N \ell(Q'').$$

If we now take $J' \in \mathcal{W}$ with $J' \cap 4J \ne \varnothing$, then the properties of the Whitney cubes guarantee that $\ell(J') \approx \ell(J)$ and hence the previous estimates easily extend to J'. This means that choosing η' smaller and K' larger (depending on the CAD character of D), we have that $J' \in (\mathcal{W}'_{Q''})^0 \subset \mathcal{W}'_{Q''}$. Since $y_{Q'} \in Q''$, we then have that

$$4J \subset \bigcup_{J'\in\mathcal{W};\, J'\cap 4J\neq\varnothing} J' \subset \bigcup_{y_{Q'}\in Q'''\in\mathbb{D}_Q} \Big(\bigcup_{J'\in\mathcal{W}'_{Q'''}} I^*(\tau)\Big) = \bigcup_{y_{Q'}\in Q'''\in\mathbb{D}_Q} U'_{Q'''} = \Gamma^{\prime,Q}(y_{Q'}).$$

Consider finally the case on which $Y \in 4J \cap \Gamma^1_{\Omega_\star,1}(x)$ satisfies $|Y-x| < \ell(I)\tau/(2\sqrt{n})$ so that $Y \in (1+2\tau I) = I^*(2\tau) =: \hat{I}$ and $\ell(I) \approx \delta(Y) \approx \ell(J)$. Note then that if $J' \cap 4J \neq \varnothing$ we have $\ell(J') \approx \ell(J) \approx \ell(I)$. Since $I \in \mathcal{W}_{Q'}$, $Q' \in \mathbb{D}_{\widetilde{\mathcal{F}},Q}$ we have by (2.9) that

$$\eta^{\frac12}\ell(Q') \lesssim \ell(I) \approx \ell(J') \lesssim K^{\frac12}\ell(Q),$$

and

$$\operatorname{dist}(J',Q) \le \operatorname{diam}(J') + \operatorname{diam}(4J) + |Y-x| + \operatorname{diam}(\hat{I}) + \operatorname{dist}(I,Q) \lesssim \ell(I) + \operatorname{dist}(I,Q) \lesssim K^{\frac12}\ell(Q).$$

Thus, by taking η' smaller and K' bigger, if needed, we obtain that $J' \in (\mathcal{W}'_{Q'})^0$. Much as before the fact that $y_{Q'} \in Q'$ yields

$$4J \subset \bigcup_{J'\in\mathcal{W};\, J'\cap 4J\neq\varnothing} J' \subset \bigcup_{y_{Q'}\in Q'''\in\mathbb{D}_Q} \Big(\bigcup_{J'\in\mathcal{W}'_{Q'''}} I^*(\tau)\Big) = \bigcup_{y_{Q'}\in Q'''\in\mathbb{D}_Q} U'_{Q'''} = \Gamma^{\prime,Q}(y_{Q'}). \qquad \square$$

Let us now get back to the proof, specifically, to the estimate for II in (5.37). Let $\varpi > 0$ be small enough to be chosen and set for every $x \in \partial\Omega_\star \cap D$

$$\Gamma^3_{\Omega_\star,1}(x) = \{Y \in \Gamma^1_{\Omega_\star,1}(x) : \delta_\star(Y) \ge \varpi\delta(Y)\}, \quad \Gamma^4_{\Omega_\star,1}(x) = \{Y \in \Gamma^2_{\Omega_\star,1}(x) : \delta_\star(Y) \ge \varpi\delta(Y)\},$$

and

$$\Gamma^5_{\Omega_\star,1}(x) = \{Y \in \Gamma_{\Omega_\star,1}(x) : \delta_\star(Y) < \varpi\delta(Y)\}.$$

Thus

$$S_{\Omega_\star,1}v(x)^2 = \sum_{k=3}^{5} \iint_{\Gamma^k_{\Omega_\star,1}(x)} |\nabla v|^2 \delta_\star^{1-n}\, dY =: \sum_{k=3}^{5} g_k(x)^2. \tag{5.43}$$

Note that for $x \in \partial\Omega_\star \cap D$ invoking Claim 5.42 we obtain

$$g_3(x)^2 \le \varpi^{1-n} \iint_{\Gamma^1_{\Omega_\star,1}(x)} |\nabla v|^2 \delta^{1-n}\, dY \le \varpi^{1-n} \iint_{\Gamma^{\prime,Q}(y_x)} |\nabla v|^2 \delta^{1-n}\, dY = \varpi^{1-n} S^{\prime,Q}u(y_x)^2 \le (\gamma\alpha^2).$$

Analogously, by (5.38)

$$g_4(x)^2 \le \varpi^{1-n} \iint_{\Gamma^1_{\Omega_\star,2}(x)} |\nabla v|^2 \delta^{1-n}\, dY \le \varpi^{1-n} \iint_{\Gamma^{\prime,Q}(x)} |\nabla v|^2 \delta^{1-n}\, dY \lesssim (\gamma\alpha)^2.$$

As a result,

$$\begin{aligned}\int_{\partial\Omega_\star\cap D} (g_3^p + g_4^p)\, d\sigma_\star &\lesssim \varpi^{\frac12(1-n)p}(\gamma\alpha)^p \sigma_\star(\partial\Omega_\star) \\ &\lesssim \varpi^{\frac12(1-n)p}(\gamma\alpha)^p \ell(Q)^n \approx \varpi^{\frac12(1-n)p}(\gamma\alpha)^p \sigma(Q),\end{aligned} \tag{5.44}$$

where we have used that $\partial\Omega_\star$ is ADR with $\operatorname{diam}(\partial\Omega_\star) \lesssim \ell(Q)$ (see (2.14)).

We next consider g_5. Set $\mathcal{W}_\star = \{I \in \mathcal{W} : I \cap \partial\Omega_\star \neq \varnothing\}$ and note that $\partial\Omega_\star \cap D \subset \bigcup_{I\in\mathcal{W}_\star} I$. For every $x \in \partial\Omega_\star \cap D$ we then have that $x \in I_x \in \mathcal{W}_\star$ and also that $x \in \partial\hat{J}_x$ with $J_x \in \mathcal{W}_{Q_x}$, $Q_x \in \mathbb{D}_{\widetilde{\mathcal{F}},Q}$. If $Y \in \Gamma^5_{\Omega_\star,1}(x)$ and $\varpi < \frac{1}{4}$, then

$$\delta(Y) \leq |Y-x| + \delta(x) \leq 2\delta_\star(Y) + \delta(x) < 2\varpi\,\delta(Y) + \delta(x) < \tfrac{1}{2}\delta(Y) + \delta(x).$$

This and (2.5) yield

$$\delta(Y) \leq 2\delta(x) \leq 2(\operatorname{diam}(4J_x) + \operatorname{dist}(4J_x, \partial D)) < 100\operatorname{diam}(J_x)$$

and, for ϖ small enough,

$$|Y-x| \leq 2\delta_\star(Y) \leq 2\varpi\,\delta(Y) < 200\varpi\operatorname{diam}(J_x) < \tfrac{1}{8}\tau\ell(J_x).$$

Recalling that $\hat{J}_x := J_x^*(2\tau)$ with $\tau \leq \tau_0 \leq 2^{-4}$ it follows that $Y \in J_x^*(7\tau/4) \subset 2J_x$ and also $Y \in B(x, \ell(J_x))$. Hence, easy calculations lead to

$$\iint_{\Gamma^5_{\Omega_\star,1}(x)} \delta_\star^{\frac{p}{p-2}-n}\,dY \leq \max\{2^{\frac{p}{p-2}-n}, 1\} \iint_{B(x,\ell(J_x))} |x-Y|^{\frac{p}{p-2}-n}\,dY \lesssim \ell(J_x)^{2\frac{p-1}{p-2}} \approx \ell(I_x)^{2\frac{p-1}{p-2}}.$$

Using Hölder's inequality with $p/2$ we arrive at

$$\begin{aligned} g_5(x) &= \left(\iint_{\Gamma^5_{\Omega_\star,1}(x)} |\nabla v|^2 \delta_\star^{1-n}\,dY\right)^{\frac{1}{2}} \leq \left(\iint_{\Gamma^5_{\Omega_\star,1}(x)} \delta_\star^{\frac{p}{p-2}-n}\,dY\right)^{\frac{p-2}{2p}} \left(\iint_{\Gamma^5_{\Omega_\star,1}(x)} |\nabla v|^p \delta_\star^{-n}\,dY\right)^{\frac{1}{p}} \\ &\lesssim \ell(I_x)^{\frac{p-1}{p}} \left(\iint_{2J_x\cap B(x,2\delta_\star(x))\cap\Omega_\star} |\nabla v|^p \delta_\star^{-n}\,dY\right)^{\frac{1}{p}}. \end{aligned}$$

Next, for every $I \in \mathcal{W}_\star$ we set

$$\mathcal{W}_\star^I := \{J \in \mathcal{W} : J = J_x \text{ for some } x \in \partial\Omega_\star \cap I,\ 2J_x \cap \Gamma_{\Omega_\star,1}(x) \neq \varnothing\}$$

and obtain

$$\begin{aligned} \int_{\partial\Omega_\star\cap D} g_5^p\,d\sigma_\star &\leq \sum_{I\in\mathcal{W}_\star} \int_{\partial\Omega_\star\cap I} g_5^p\,d\sigma_\star \\ &\leq \sum_{I\in\mathcal{W}_\star} \ell(I)^{p-1} \int_{\partial\Omega_\star\cap I} \iint_{2J_x\cap B(x,2\delta_\star(x))\cap\Omega_\star} |\nabla v(Y)|^p \delta_\star(Y)^{-n}\,dY\,d\sigma_\star(x) \\ &\leq \sum_{I\in\mathcal{W}_\star} \ell(I)^{p-1} \sum_{J\in\mathcal{W}_\star^I} \iint_{2J\cap\Omega_\star} |\nabla v(Y)|^p \delta_\star(Y)^{-n} \sigma_\star(\partial\Omega_\star \cap B(Y, 2\delta_\star(x)))\,dY \\ &\lesssim \sum_{I\in\mathcal{W}_\star} \ell(I)^{p-1} \sum_{J\in\mathcal{W}_\star^I} \iint_{2J} |\nabla v(Y)|^p\,dY \\ &\lesssim C_0' \sum_{I\in\mathcal{W}_\star} \ell(I)^{p-1} \sum_{J\in\mathcal{W}_\star^I} \ell(J)^{(n+1)\frac{2-p}{2}} \left(\iint_{4J} |\nabla v(Y)|^2\,dY\right)^{\frac{p}{2}} \\ &\approx C_0' \sum_{I\in\mathcal{W}_\star} \ell(I)^n \sum_{J\in\mathcal{W}_\star^I} \left(\iint_{4J} |\nabla v(Y)|^2 \delta(Y)^{1-n} dY\right)^{\frac{p}{2}}, \end{aligned} \tag{5.45}$$

where we have used that $\partial\Omega_\star$ is ADR (see [Hofmann and Martell 2014, Lemma 3.61]), (5.25) (since $4J \subset D$ by (2.5)), that $\ell(J_x) \approx \ell(I)$ since $x \in I \cap \partial\hat{J}_x$ (hence $I \cap J \neq \varnothing$), and finally that $\delta(\cdot) \approx \ell(J)$ in $4J$ by (2.5).

Suppose next that $I \in \mathcal{W}_\star$ with $\ell(I) \ll \ell(Q)$. Note that if $J = J_x$ with $x \in \partial\Omega_\star \cap I$ then $x \in \partial\hat{J}_x \cap I$; hence $\ell(J_x) \approx \ell(I) \ll \operatorname{diam}(I)$ and $4J_x \subset \{Y \in D : \delta(Y) < \ell(Q)\}$. Thus, if $2J_x \cap \Gamma_{\Omega_\star,1}(x) \neq \varnothing$, necessarily $2J_x \cap \Gamma^1_{\Omega_\star,1}(x) \neq \varnothing$. We can then invoke Claim 5.42 with $J = J_x$ to find $y_x \in E'_Q$ so that

$$\begin{aligned}\sum_{J \in \mathcal{W}^I_\star} \left(\iint_{4J} |\nabla v|^2 \delta^{1-n} dY \right)^{\frac{p}{2}} &\le \left(\iint_{\Gamma^{\prime,Q}(y_x)} |\nabla v|^2 \delta^{1-n} dY \right)^{\frac{p}{2}} \#\{J \in \mathcal{W} : \partial\hat{J} \cap I \neq \varnothing\} \\ &\lesssim S^{\prime,Q}(y_x)^p \le S^{\prime,Q_0}(y_x)^p \le (\gamma\alpha)^p. \end{aligned} \tag{5.46}$$

Consider next the case $I \in \mathcal{W}_\star$ with $\ell(I) \gtrsim \ell(Q)$. For every $J \in \mathcal{W}^I_\star$ we have that $J = J_x$ for some $x \in \partial\Omega_\star \cap I$ and there exists $Z \in 2J \cap \Omega_\star$. As such $J \in \mathcal{W}_{Q_x}$ for some $Q_x \in \mathbb{D}_{\widetilde{\mathcal{F}},Q}$. In particular, $\ell(Q) \lesssim \ell(I) \approx \ell(J) \approx \ell(Q_x) \le \ell(Q)$. Take then an arbitrary $Y \in 4J \cap \Omega_\star$. Since $Z \in 2J$, one has $\delta(Y) \approx \ell(J) \approx \ell(Q)$. Also, $Z \in \Omega_\star = \widehat{\Omega}_{\widetilde{\mathcal{F}},Q}$; then $Z \in \widehat{U}_{Q'}$ for some $Q' \in \mathbb{D}_{\widetilde{\mathcal{F}},Q}$ and, as observed above, the latter implies that one can find $y_{Q'} \in Q' \cap E'_Q$. We claim that $4J \subset \Gamma^{\prime,Q}(y_x)$. To see this let $Y \in 4J \subset D$ and take $I_Y \in \mathcal{W}$ with $Y \in I_Y$. Note that by (2.5) and (2.9), $\ell(I_Y) \approx \delta(Y) \approx \ell(J) \approx \ell(Q)$ and

$$\operatorname{dist}(I_Y, Q) \le \operatorname{dist}(Y, Q) \le \operatorname{diam}(4J) + \operatorname{dist}(J, Q_x) \lesssim \ell(Q) + \ell(Q_x) \approx \ell(Q).$$

Thus taking η' smaller and K' larger if needed ((depending on n, the CAD character of D, and the choice of η, K, τ) we can ensure that $I_Y \in (\mathcal{W}'_Q)^0 \subset \mathcal{W}'_Q$ and since $y_{Q'} \in Q' \subset Q$ we conclude that $Z \in \Gamma^{\prime,Q}(y_x)$ as desired. All these give an estimate similar to (5.38):

$$\begin{aligned}\sum_{J \in \mathcal{W}^I_\star} \left(\iint_{4J} |\nabla v(Y)|^2 \delta(Y)^{1-n} dY \right)^{\frac{p}{2}} &\le \#\{J \in \mathcal{W} : \partial\hat{J} \cap I \neq \varnothing\} \left(\iint_{\Gamma^{\prime,Q}(y_x)} |\nabla v|^2 \delta^{1-n}\, dY \right)^{\frac{p}{2}} \\ &\lesssim S^{\prime,Q} v(y_{Q'}) \le S^{\prime,Q_0} v(y_{Q'})^p \le (\gamma\alpha)^p. \end{aligned} \tag{5.47}$$

We finally combine (5.45), (5.46), and (5.47) to obtain

$$\int_{\partial\Omega_\star \cap D} g_5^p \, d\sigma_\star \lesssim C'_0 (\gamma\alpha)^p \sum_{I \in \mathcal{W}_\star} \ell(I)^n. \tag{5.48}$$

To complete the proof we estimate the sum in the right-hand side. For every $I \in \mathcal{W}_\star$ pick $Z_I \in \partial\Omega_\star \cap I$ so that $\ell(I) \approx \delta(Z_I)$ and let $\Delta^I_\star := B(Z_I, \delta(Z_I)/2) \cap \partial\Omega_\star$, which is a surface ball with respect to $\Omega_\star$. The fact that $Z_I \in \partial\Omega_\star \subset \cap D$ implies that there exists $I' \in \mathcal{W}_{Q'}$ with $Q' \in \mathbb{D}_{\widetilde{\mathcal{F}},Q}$ and $Z_I \in \partial\hat{I}$. Then, $\ell(I) \approx \delta(Z_I) \approx \ell(I') \approx \ell(Q') \le \ell(Q)$ by (2.5) and (2.9)). Note that $Q \in \mathbb{D}_{\widetilde{\mathcal{F}},Q}$; hence $U_Q \subset \Omega_\star$. Pick $I_Q \in \mathcal{W}_Q$ (which is nonempty by construction) and note that $\ell(I_Q) \approx \ell(Q)$ by (2.9) and $I_Q \subset U_Q \subset \Omega_\star$. Hence $\ell(Q) \approx \operatorname{diam}(I_Q) \le \operatorname{diam}(\Omega_\star) \lesssim \ell(Q)$ by (2.14). All these show that $\delta(Z_I) \lesssim \operatorname{diam}(\partial\Omega_\star)$. Suppose next that $\Delta^I_\star \cap \Delta^J_\star \neq \varnothing$ for some $I, J \in \mathcal{W}_\star$ and let Y belong to that intersection. Assume for instance that $\ell(I) \le \ell(J)$ and note that

$$\delta(Z_J) \le |Z_J - Y| + |Y - Z_I| + \delta(Z_I) \le \tfrac{1}{2}\delta(Z_J) + \tfrac{3}{2}\delta(Z_I).$$

Hence, $\ell(J) \approx \delta(Z_J) \le 3\delta(Z_I) \approx \ell(I) \le \ell(J)$ and

$$\operatorname{dist}(I, J) \le |Z_I - Z_J| \le |Z_J - Y| + |Y - Z_I| \le \tfrac{1}{2}\delta(Z_J) + \tfrac{1}{2}\delta(Z_I) \approx \ell(I) + \ell(J) \approx \ell(I) \approx \ell(J).$$

As a consequence, the family $\{\Delta_\star^I\}_{I\in\mathcal{W}_\star}$ has bounded overlap and therefore

$$\sum_{I\in\mathcal{W}_\star} \ell(I)^n \approx \sum_{I\in\mathcal{W}_\star} \sigma_\star(\Delta_\star^I) \lesssim \sigma_\star\Big(\bigcup_{I\in\mathcal{W}_\star} \Delta_\star^I\Big) \le \sigma_\star(\partial\Omega_\star) \lesssim \operatorname{diam}(\partial\Omega_\star)^n \approx \ell(Q)^n \approx \sigma(Q),$$

where we have used that $\partial\Omega_\star$ is ADR (see [Hofmann and Martell 2014, Lemma 3.61]). This and (5.48) eventually yield

$$\int_{\partial\Omega_\star\cap D} g_5^p\, d\sigma_\star \lesssim C_0'(\gamma\alpha)^p \sigma(Q).$$

This, (5.37), (5.43), and (5.44) give

$$\mathrm{II} = \frac{1}{(\varepsilon\alpha)^p} \int_{\partial\Omega_\star\cap D} S_{\Omega_\star,1} v^p\, d\sigma_\star \lesssim \frac{1}{(\varepsilon\alpha)^p} \int_{\partial\Omega_\star\cap D} (g_3^p + g_4^p + g_5^p)\, d\sigma_\star \lesssim (1 + C_0')\Big(\frac{\gamma}{\varepsilon}\Big)^p \sigma(Q).$$

We next combine this with (5.37) and (5.41) to arrive at

$$\sigma(E_Q') \lesssim C_{\Omega_\star}(1 + C_0')\Big(\frac{\gamma}{\varepsilon}\Big)^p \sigma(Q).$$

Recalling that Let E_Q' be an arbitrary closed subset of E_Q with $\sigma(E_Q') > 0$, by inner regularity of the Hausdorff measure, we therefore obtain that

$$\sigma(E_{(1+\varepsilon)\alpha} \cap \widetilde{F}_{\gamma\alpha} \cap Q) \le \sigma(E_Q) \lesssim C_{\Omega_\star}(1 + C_0')\Big(\frac{\gamma}{\varepsilon}\Big)^p \sigma(Q).$$

We have then show (5.32) which in turn implies (5.31). With the latter estimate in hand and for any $0 < q < \infty$, we proceed as in (5.23):

$$\begin{aligned}
\mathrm{I}_N &:= \int_0^N q\alpha^q \sigma\{x\in Q_0 : N_*^{Q_0} v(x) > \alpha\}\, \frac{d\alpha}{\alpha} \\
&= (1+\varepsilon)^q \int_0^{N/(1+\varepsilon)} q\alpha^q \sigma\{x\in Q_0 : N_*^{Q_0} v(x) > (1+\varepsilon)\alpha\}\, \frac{d\alpha}{\alpha} \\
&\le (1+\varepsilon)^q \int_0^N q\alpha^q \sigma\{x\in Q_0 : N_*^{Q_0} v(x) > (1+\varepsilon)\alpha,\, S'^{,Q_0} v(x) \le \gamma\alpha\}\, \frac{d\alpha}{\alpha} + \Big(\frac{1+\varepsilon}{\gamma}\Big)^q \|S'^{,Q_0} v\|_{L^q(Q_0)}^q \\
&\le C_{\gamma,\varepsilon}^*(1+\varepsilon)^q \int_0^N q\alpha^q \sigma\{x\in Q_0 : N_*^{Q_0} v(x) > \alpha\}\, \frac{d\alpha}{\alpha} + \frac{(1+\varepsilon)^q}{\gamma^q} \|S'^{,Q_0} v\|_{L^q(Q_0)}^q \\
&= C\Big(\frac{\gamma}{\varepsilon}\Big)^p (1 + C_0')\Big(\sup_{Q\in\mathbb{D},\widetilde{\mathcal{F}}} C_{\Omega_{\widetilde{\mathcal{F}},Q}}\Big)(1+\varepsilon)^q \mathrm{I}_N + \frac{(1+\varepsilon)^q}{\gamma^q} \|S'^{,Q_0} v)\|_{L^q(Q_0)}^q.
\end{aligned} \tag{5.49}$$

At this point we first choose $\varepsilon = 1$ and next take $0 < \gamma < c_0\varepsilon/C_0$ small enough so that

$$C\gamma^p(1 + C_0') \sup_{Q\in\mathbb{D}} C_{\Omega_Q} 2^q < 1/1.$$

With these choices and using that $I_N \leq N^q \sigma(Q_0) < \infty$, we can hide this term on the left-hand side of (5.49) to obtain

$$\mathrm{I}_N \leq 2(1+\varepsilon)^q/\gamma^q \|S'^{,Q_0} v\|_{L^q(Q_0)}^q.$$

Noting that $\mathrm{I}_N \nearrow \|N_*^{Q_0} v\|_{L^q(Q_0)}^q$ as $N \to \infty$ we obtain as desired (5.27).

From (5.27) one can obtain (5.28), and hence (5.29) and (5.30) much as in the proof of Theorem 5.1 and we omit details. □

Combining Theorems 5.1 and 5.24 we can obtain the following:

Corollary 5.50. *Let $D \subset \mathbb{R}^{n+1}$ be a CAD. Let $u \in W^{1,2}_{\mathrm{loc}}(D) \cap C(D)$ so that (5.2) and (5.25) hold for some $p > 2$. Suppose that the $N < S$ estimates are valid on L^2 on all bounded Lipschitz subdomains $\Omega \subset D$ (see (5.3) in Theorem 5.1). Then (5.27)–(5.30) hold.*

Proof. Let $\Omega \subset D$ be an arbitrary bounded CAD. Since any bounded Lipschitz subdomain of Ω is also a subdomain of D we can apply Theorem 5.1 to obtain (5.7) for Ω and for every $q > 2$. That is, we have the $N < S$ estimates are valid on all bounded chord-arc subdomains $\Omega \subset D$ for $q = p > 2$. Hence, Theorem 5.24 applies to obtain the desired conclusions. □

6. From $N<S$ bounds on chord-arc domains to ε-approximability in the complement of a UR set

Recall the definition of ε-approximability (Definition 1.11). The second main result in [Hofmann et al. 2016], stated there for harmonic functions but proved in full generality, can be formulated as follows.

Theorem 6.1. *Let $E \subset \mathbb{R}^{n+1}$ be an n-dimensional UR set, $\mathbb{R}^{n+1} \setminus E$, and suppose that*

$$u \in W^{1,2}_{\mathrm{loc}}(\mathbb{R}^{n+1} \setminus E) \cap C(\mathbb{R}^{n+1} \setminus E) \cap L^\infty(\mathbb{R}^{n+1} \setminus E)$$

is such that for any cube I with $2I \subset \mathbb{R}^{n+1} \setminus E$

$$\sup_{X,Y \in I} |u(X) - u(Y)| \leq C_0 \left(\ell(I)^{1-n} \iint_{2I} |\nabla u|^2 \, dX \right)^{\frac{1}{2}} \tag{6.2}$$

and

$$\|\nabla u\|_{\mathrm{CME}(\mathbb{R}^{n+1}\setminus E)} \leq C_0' \|u\|_{L^\infty(\mathbb{R}^{n+1}\setminus E)}.$$

Assume, in addition, that $N < S$ estimates are valid on L^2 on all bounded chord-arc subdomains $\Omega \subset \mathbb{R}^{n+1} \setminus E$; that is, for any bounded chord-arc subdomain $\Omega \subset \mathbb{R}^{n+1} \setminus E$, there holds

$$\|N_{*,\Omega}(u - u(X_\Omega^+))\|_{L^2(\partial\Omega)} \leq C_\Omega \|S_\Omega u\|_{L^2(\partial\Omega)}. \tag{6.3}$$

Here X_Ω^+ is any interior corkscrew point of Ω at the scale of $\mathrm{diam}(\Omega)$, *and the constant C_Ω depends on the CAD character of Ω, the dimension n, p, the implicit choice of κ (the aperture of the cones in $N_{*,\Omega}$ and S_Ω), and the implicit corkscrew constant for the point X_Ω^+. Then u is ε-approximable on $\mathbb{R}^{n+1} \setminus E$, with the implicit constants depending only on n, the UR character of E, C_0, and C_0'.*

Strictly speaking, the previous result was proved in [Hofmann et al. 2016, Section 5] for harmonic functions but it was observed in Remark 5.29 of that work that the same argument can be carried out under the current assumptions.[6] Let us note that one can weaken (6.2) by just assuming that for any $Q \in \mathbb{D}(E)$ and for any connected component of U_Q^i there holds

$$\sup_{X,Y \in U_Q^i} |u(X) - u(Y)| \le C_0 \left(\ell(Q)^{-n-1} \iint_{\widehat{U}_Q^i} |u|^2 \, dX \right)^{\frac{1}{2}}. \tag{6.4}$$

Also, in the course of the proof one uses (6.3) for the bounded chord-arc subdomains of the form $\Omega = \Omega_{\boldsymbol{S}}^{\pm}$ defined by (2.52) (with $\boldsymbol{S}' = \boldsymbol{S}$). Further details are left to the interested reader.

7. Applications: solutions, subsolutions, and supersolutions of divergence-form elliptic equations with bounded measurable coefficients

7.1. ***Estimates for solutions of second-order divergence-form elliptic operators with coefficients satisfying a Carleson measure condition.*** Given an open set $\Omega \subset \mathbb{R}^{n+1}$, consider a divergence-form elliptic operator $L := -\operatorname{div}(A(\cdot)\nabla)$, defined in Ω, where A is an $(n+1) \times (n+1)$ matrix with real bounded measurable coefficients, possibly nonsymmetric, satisfying the ellipticity condition

$$\lambda^{-1}|\xi|^2 \le A(X)\xi \cdot, \xi := \sum_{i,j=1}^{n+1} A_{ij}(X)\xi_j\xi_i, \qquad |A(X)\xi \cdot \zeta| \le \lambda|\xi||\zeta|, \tag{7.1}$$

for some $\lambda \ge 1$, and for all $\xi, \zeta \in \mathbb{R}^{n+1}$, and for a.e. $X \in \Omega$. As usual, the divergence-form equation is interpreted in the weak sense; i.e., we say that $Lu = 0$ in Ω if $u \in W^{1,2}_{\mathrm{loc}}(\Omega)$ and

$$\iint_{\Omega} A(X)\nabla u(X) \cdot \nabla \Psi(X) \, dX = 0 \tag{7.2}$$

for all $\Psi \in C_0^\infty(\Omega)$.

Let us introduce some notation. Given an open set $\Omega \subset \mathbb{R}^{n+1}$ and A, an $(n+1) \times (n+1)$ matrix defined on $\mathbb{R}^{n+1} \setminus E$ with real bounded measurable coefficients, possibly nonsymmetric, satisfying the ellipticity condition (7.1), we say that $A \in \mathrm{KP}(\Omega)$ (the Kenig–Pipher class) if $|\nabla A(\cdot)| \operatorname{dist}(\cdot, \partial\Omega) \in L^\infty(\Omega)$ and $\|\nabla A\|_{\mathrm{CME}(\Omega)} < \infty$. It has been demonstrated in [Kenig and Pipher 2001] that if Ω is a Lipschitz domain and $A \in \mathrm{KP}(\Omega)$ then weak solutions to Lu satisfy square function/nontangential maximal function estimates and Carleson measure estimates on Ω. Strictly speaking, the class of matrices is slightly smaller and the details of the proof are only provided there for $N < S$ direction (and only for $p > 2$), but all ingredients are laid out for a reader to reconstruct a complete proof. One can also consult [David et al. 2019] for complete details presented in this and more general, higher codimensional, case. For the precise case we are considering here, the following result can be found in [Hofmann et al. 2017a, Appendix A]:[7]

[6] In [Hofmann et al. 2016, Remark 5.29], we inadvertently neglected to mention that our proof utilized estimate (6.3); in fact, it is utilized in an essential way. One should bear this in mind when comparing the statement of Theorem 6.1 with that Remark 5.29. The former is correct.

[7] The argument in [Hofmann et al. 2017a, Appendix A] follows that of [Kenig and Pipher 2001] very closely.

Let Ω be a Lipschitz domain and let $A \in \mathrm{KP}(\Omega)$. Then, any weak solution $u \in W^{1,2}_{\mathrm{loc}}(\Omega) \cap L^\infty(\Omega)$ to $Lu = 0$ in Ω satisfies $\|\nabla u\|_{\mathrm{CME}(\Omega)} \lesssim \|u\|^2_{L^\infty(\Omega)}$ with implicit constant depending on n, the Lipschitz character of Ω, ellipticity, and the implicit constants in $A \in \mathrm{KP}(\Omega)$. (7.3)

We also need the following auxiliary result (see [Kenig and Pipher 2001, Lemma 3.1]):

Lemma 7.4. *Let $E \subset \mathbb{R}^{n+1}$ be a closed set and let A be an $(n+1)\times(n+1)$ matrix defined on $\mathbb{R}^{n+1}\setminus E$ with real bounded measurable coefficients, possibly nonsymmetric, satisfying the ellipticity condition (7.1). If $A \in \mathrm{KP}(\mathbb{R}^{n+1}\setminus E)$ then $A \in \mathrm{KP}(D)$ for any subset $D \subset \mathbb{R}^{n+1}\setminus E$. Moreover, $\|\nabla A(\cdot)\,\mathrm{dist}(\cdot\,, \partial D)\|_{L^\infty(D)} \le \|\nabla A(\cdot)\,\mathrm{dist}(\cdot\,, E)\|_{L^\infty(\mathbb{R}^{n+1}\setminus E)}$ and*

$$\|\nabla A\|_{\mathrm{CME}(D)} \le C(\|\nabla A\|_{\mathrm{CME}(\mathbb{R}^{n+1}\setminus E)} + \|\nabla A(\cdot)\,\mathrm{dist}(\cdot\,, E)\|^2_{L^\infty(\mathbb{R}^{n+1}\setminus E)}),$$

where C depends only on dimension.

Proof. Note first that since $D \subset \mathbb{R}^{n+1}\setminus E$ then $\mathrm{dist}(X, \partial D) \le \mathrm{dist}(X, E)$ for every $X \in D$. In particular, one has $\|\nabla A(\cdot)|\,\mathrm{dist}(\cdot\,, \partial D)\|_{L^\infty(D)} \le \|\nabla A(\cdot)\,\mathrm{dist}(\cdot\,, E)\|_{L^\infty(\mathbb{R}^{n+1}\setminus E)}$.

Next, we fix $B(x, r)$ with $x \in \partial D$ and $0 < r < \infty$. We shall consider two cases. First, if $\mathrm{dist}(x, E) \le 2r$ we pick $z \in E$ with $\mathrm{dist}(x, E) = |x - z|$ and observe that $B(x, r) \subset B(z, 3r)$. Then,

$$\iint_{B(x,r)\cap D} |\nabla A(Y)|^2\,\mathrm{dist}(Y, \partial D)\,dY \le \iint_{B(z,3r)\cap D} |\nabla A(Y)|^2\,\mathrm{dist}(Y, E)\,dY \le (3r)^n \|\nabla A\|_{\mathrm{CME}(\mathbb{R}^{n+1}\setminus E)}.$$

In the second case, $\mathrm{dist}(X, E) > 2r$, we have $\mathrm{dist}(Y, E) > r$ and $\mathrm{dist}(Y, \partial D) \le |Y - x| < r$ for every $Y \in B(x, r) \cap D$. Hence,

$$\begin{aligned}\iint_{B(x,r)\cap D} |\nabla A(Y)|^2\,\mathrm{dist}(Y, \partial D)\,dY &\le \|\nabla A(\cdot)\,\mathrm{dist}(\cdot\,, E)\|^2_{L^\infty(\mathbb{R}^{n+1}\setminus E)} \iint_{B(z,r)\cap D} \frac{\mathrm{dist}(Y, \partial D)}{\mathrm{dist}(Y, E)^2}\,dY \\ &\le \|\nabla A(\cdot)\,\mathrm{dist}(\cdot\,, E)\|^2_{L^\infty(\mathbb{R}^{n+1}\setminus E)} r^{-1}|B(x, r)| \\ &= c_n \|\nabla A(\cdot)\,\mathrm{dist}(\cdot\,, E)\|^2_{L^\infty(\mathbb{R}^{n+1}\setminus E)} r^n.\end{aligned}$$

All these readily give the desired estimate. □

Theorem 7.5. *Let $E \subset \mathbb{R}^{n+1}$ be an n-dimensional UR set. Let A be an $(n+1)\times(n+1)$ matrix defined on $\mathbb{R}^{n+1}\setminus E$ with real bounded measurable coefficients, possibly nonsymmetric, satisfying the ellipticity condition (7.1) and so that $A \in \mathrm{KP}(\mathbb{R}^{n+1}\setminus E)$. Then any weak solution $u \in W^{1,2}_{\mathrm{loc}}(\mathbb{R}^{n+1}\setminus E)$ to $Lu = 0$ in $\mathbb{R}^{n+1}\setminus E$ satisfies the $S < N$ estimates*

$$\|S_{\mathbb{R}^{n+1}\setminus E} u\|_{L^p(E)} \le C \|N_{*,\mathbb{R}^{n+1}\setminus E} u\|_{L^p(E)}, \qquad 0 < p < \infty, \tag{7.6}$$

$$\|S^r_{\mathbb{R}^{n+1}\setminus E} u\|_{L^p(\Delta(x,r))} \lesssim \|N^{K'r}_{*,\mathbb{R}^{n+1}\setminus E} u\|_{L^p(\Delta(x,K'r))}, \quad 0 < p < \infty, \tag{7.7}$$

for any $x \in E$ and $0 < r < 2\,\mathrm{diam}(E)$, where $\Delta(x, r) = B(x, r) \cap E$, and where K' depends on n and the UR character of E; as well as its local dyadic analogue, for any Whitney-dyadic structure $\{\mathcal{W}_Q\}_{Q\in\mathbb{D}(E)}$, for $\mathbb{R}^{n+1}\setminus E$ with parameters η and K,

$$\|S^Q u\|_{L^p(Q)} \le C \|\widehat{N}^Q_* u\|_{L^p(Q)}, \quad Q \in \mathbb{D}(E), \quad 0 < p < \infty. \tag{7.8}$$

If, in addition, bounded, $u \in L^\infty(\mathbb{R}^{n+1}\setminus E)$ then the Carleson measure estimate

$$\|\nabla u\|_{\mathrm{CME}(\mathbb{R}^{n+1}\setminus E)} \le C\|u\|^2_{L^\infty(\mathbb{R}^{n+1}\setminus E)} \tag{7.9}$$

holds and u is ε-approximable on $\mathbb{R}^{n+1}\setminus E$, in the sense of Definition 1.11. All constants depend on n, the UR character of E, the ellipticity of A, $\|\nabla A(\cdot)\operatorname{dist}(\cdot,E)\|_{L^\infty(\mathbb{R}^{n+1}\setminus E)}$, $\|\nabla A\|_{\mathrm{CME}(\mathbb{R}^{n+1}\setminus E)}$, the aperture of the cone κ implicit in (7.6), and the implicit parameters η, K, τ implicit in (7.8).

Proof. Fix $A \in \mathrm{KP}(\mathbb{R}^{n+1}\setminus E)$ with ellipticity constant λ and take any weak solution $u \in W^{1,2}_{\mathrm{loc}}(\mathbb{R}^{n+1}\setminus E)$ to $Lu = 0$ in $\mathbb{R}^{n+1}\setminus E$.

Claim 7.10. *For any $\Omega \subset \mathbb{R}^{n+1}\setminus E$ with $\partial\Omega$ being UR there holds*

$$\|\nabla u\|_{\mathrm{CME}(\Omega)} \lesssim \|u\|^2_{L^\infty(\Omega)},$$

with an implicit constants on n, the UR character of E, λ, and the implicit constants in $A \in \mathrm{KP}(\mathbb{R}^{n+1}\setminus E)$.

Assuming this momentarily, and taking $\Omega = \mathbb{R}^{n+1}\setminus E$ we readily obtain (7.9). On the other hand, given an arbitrary $Q \in \mathbb{D}(E)$ and arbitrary pairwise disjoint family $\mathcal{F} \subset \mathbb{D}_Q$, let $G = \nabla u \in L^2_{\mathrm{loc}}(\mathbb{R}^{n+1}\setminus E)$ and $H = u \in C(\mathbb{R}^{n+1}\setminus E)$. Note that Proposition A.11 says that $\widehat{\Omega}_{\mathcal{F},Q}$ is an open set with UR boundary and with UR character depending on n and the UR character of E. Hence, Claim 7.10 says that

$$\|G\|_{\mathrm{CME}(\widehat{\Omega}_{\mathcal{F},Q})} = \|\nabla u\|_{\mathrm{CME}(\widehat{\Omega}_{\mathcal{F},Q})} \lesssim \|u\|^2_{L^\infty(\widehat{\Omega}_{\mathcal{F},Q})} = \|H\|^2_{L^\infty(\widehat{\Omega}_{\mathcal{F},Q})},$$

with a constant which is independent of u, Q and $\mathcal{F}$, and depends on n, the UR character of E, the ellipticity of A, and the implicit constants in $A \in \mathrm{KP}(\mathbb{R}^{n+1}\setminus E)$. This means that (A_{loc}) in Theorem 4.8 holds for the open set $\mathbb{R}^{n+1}\setminus E$. As such (4.13), (4.14), and Remark 4.4 imply (7.6)–(7.8).

Proof of Claim 7.10. Take an arbitrary any open subset $\Omega \subset \mathbb{R}^{n+1}\setminus E$ with $\partial\Omega$ being UR. We may assume that $0 < \|u\|_{L^\infty(\Omega)} < \infty$; otherwise the desired estimate is trivial. Set $A_\Omega := A$ in Ω and $A_\Omega := \mathrm{Id}$ (the identity matrix) in $\mathbb{R}^{n+1}\setminus\Omega$ which is an elliptic matrix with ellipticity constant at most λ. Note that Lemma 7.4 gives

$$\begin{aligned}
\|\nabla A_\Omega\|_{\mathrm{CME}(\mathbb{R}^{n+1}\setminus\partial\Omega)} &= \sup_{x\in\partial\Omega,0<r<\infty} \frac{1}{r^n}\iint_{B(x,r)\setminus\partial\Omega} |\nabla A_\Omega(Y)|^2 \operatorname{dist}(Y,\partial\Omega)\,dY \\
&= \sup_{x\in\partial\Omega,0<r<\infty} \frac{1}{r^n}\iint_{B(x,r)\cap\Omega} |\nabla A(Y)|^2 \operatorname{dist}(Y,\partial\Omega)\,dY \\
&\le \|\nabla A\|_{\mathrm{CME}(\Omega)} \le C_n(\|\nabla A\|_{\mathrm{CME}(\mathbb{R}^{n+1}\setminus E)} + \|\nabla A(\cdot)\operatorname{dist}(\cdot,E)\|^2_{L^\infty(\mathbb{R}^{n+1}\setminus E)})
\end{aligned}$$

and

$$\|\nabla A_\Omega \operatorname{dist}(\cdot,\partial\Omega)\|_{L^\infty(\mathbb{R}^{n+1}\setminus\partial\Omega)} = \|\nabla A \operatorname{dist}(\cdot,\partial\Omega)\|_{L^\infty(\Omega)} \le \|\nabla A \operatorname{dist}(\cdot,E)\|_{L^\infty(\mathbb{R}^{n+1}\setminus E)}.$$

Write also $u_\Omega = u$ in Ω and $u_\Omega := 0$ in $\mathbb{R}^{n+1}\setminus\Omega$. Note that $u_\Omega \in W^{1,2}_{\mathrm{loc}}(\mathbb{R}^{n+1}\setminus\partial\Omega)$ satisfies, in the weak sense, $-\operatorname{div}(A_\Omega\nabla u_\Omega) = Lu = 0$ in Ω and $-\operatorname{div}(A_\Omega\nabla u_\Omega) = 0$ and $\mathbb{R}^{n+1}\setminus\overline{\Omega} = 0$. This and the fact that Ω is open imply that $-\operatorname{div}(A_\Omega\nabla u_\Omega) = 0$ in $\mathbb{R}^{n+1}\setminus\partial\Omega$ in the weak sense. Note also that $u_\Omega \in L^\infty(\mathbb{R}^{n+1}\setminus\partial\Omega)$ implies $\|u_\Omega\|_{L^\infty(\mathbb{R}^{n+1}\setminus\partial\Omega)} = \|u\|_{L^\infty(\Omega)} < \infty$.

Fix $D \subset \Omega$ an arbitrary bounded Lipschitz subdomain and $F = \nabla u_\Omega / \|u_\Omega\|^2_{L^\infty(\Omega)}$. By Lemma 7.4, we have that $A_\Omega \in \mathrm{KP}(\Omega) \subset \mathrm{KP}(D)$ (with uniform bounds controlled by those of $A_\Omega \in \mathrm{KP}(\Omega)$, and hence ultimately on those of $A \in \mathrm{KP}(\mathbb{R}^{n+1} \setminus E)$). By (7.3) applied to u_Ω for the operator L_Ω in D we obtain

$$\|F\|_{\mathrm{CME}(D)} = \frac{\|\nabla u_\Omega\|_{\mathrm{CME}(D)}}{\|u_\Omega\|^2_{L^\infty(\Omega)}} \lesssim \frac{\|u_\Omega\|^2_{L^\infty(D)}}{\|u_\Omega\|^2_{L^\infty(\Omega)}} \le 1,$$

with implicit constant depending on n, the Lipschitz character of D', λ and the implicit constants of $A \in \mathrm{KP}(\mathbb{R}^{n+1} \setminus E)$. This and Corollary 3.1 (or Remark 3.3 for a more direct argument) to the UR set $\partial\Omega$ yield

$$\frac{\|\nabla u\|_{\mathrm{CME}(\Omega)}}{\|u\|^2_{L^\infty(\Omega)}} = \frac{\|\nabla u_\Omega\|_{\mathrm{CME}(\mathbb{R}^{n+1}\setminus\partial\Omega)}}{\|u_\Omega\|^2_{L^\infty(\Omega)}} = \|F\|_{\mathrm{CME}(\Omega)} \lesssim \sup_{D \subset \mathbb{R}^{n+1}\setminus\partial\Omega} \|F\|_{\mathrm{CME}(D)} = \sup_{D\subset\Omega} \|F\|_{\mathrm{CME}(D)} \lesssim 1,$$

with implicit constants depending only on n, the UR character of $\partial\Omega$, λ, and the implicit constants in $A \in \mathrm{KP}(\mathbb{R}^{n+1} \setminus E)$. This completes the proof of (7.9). □

To continue with the proof of Theorem 7.5 we are left with showing that if we further assume that $u \in L^\infty(\mathbb{R}^{n+1} \setminus E)$ then u is ε-approximable on $\mathbb{R}^{n+1} \setminus E$. Firstly, all auxiliary estimates (5.2), (5.25), and (6.2) hold for u in the open set $\mathbb{R}^{n+1} \setminus E$, and hence in any open subset $\Omega \subset \mathbb{R}^{n+1} \setminus E$, by the usual interior estimates for solutions of elliptic PDEs (see, e.g., [Kenig 1994]). We point out again that $N < S$ estimates (5.3) on all bounded Lipschitz subdomains of Ω hold essentially by [Kenig and Pipher 2001]. More precisely, let $D \subset \mathbb{R}^{n+1} \setminus E$ be an arbitrary chord-arc subdomain. For every a bounded Lipschitz subdomain $\Omega \subset D$, by Lemma 7.4 it follows that $A \in \mathrm{KP}(\Omega)$ with bounds that depend on the implicit constants in $A \in \mathrm{KP}(\mathbb{R}^{n+1} \setminus E)$. In turn (7.3) and [Kenig et al. 2016] yield that the associated elliptic measure belongs to the class $A_\infty(\partial\Omega)$ with respect to surface measure. Thus, [Dahlberg et al. 1984] allows us to obtain $N < S$ estimates are valid on L^q, $0 < q < \infty$, on Ω. Corollary 5.50 readily gives $N < S$ on L^q, $0 < q < \infty$. This together with the fact that we have already shown (7.9) allows us to invoke Theorem 6.1 to conclude as desired that u is ε-approximable with constants depending only on n, the UR character of E, λ, and the implicit constants in $A \in \mathrm{KP}(\mathbb{R}^{n+1} \setminus E)$. □

7.2. ***Estimates for subsolutions and supersolutions of second-order divergence-form elliptic operators with coefficients satisfying a Carleson measure condition.*** Our methods allow us to deal not only with solutions but also with subsolutions (thus, also with supersolutions) of the operators considered in the previous section. Before, stating the result let us recall that given an open set $\Omega \subset \mathbb{R}^{n+1}$ and a second-order divergence-form elliptic operators $L := -\operatorname{div}(A(\cdot)\nabla)$, defined in Ω, where A is an $(n+1)\times(n+1)$ matrix with real bounded measurable coefficients, possibly nonsymmetric, satisfying the ellipticity condition (7.1), we say that $u \in W^{1,2}_{\mathrm{loc}}(\Omega)$ is a weak L-subsolution (or, $Lu \le 0$) in Ω if

$$\iint_\Omega A(X)\nabla u(X) \cdot \nabla\Psi(X)\, dX \le 0 \tag{7.11}$$

for all $0 \le \Psi \in C_0^\infty(\Omega)$. Analogously, $u \in W^{1,2}_{\mathrm{loc}}(\Omega)$ is a weak L-supersolution (or, $Lu \ge 0$) if $-u$ is a subsolution.

We are now ready to state our main result in this section. We note that it applies in particular to the Laplace operator; hence the obtained estimates are valid for any subharmonic or superharmonic functions.

Theorem 7.12. *Let $E \subset \mathbb{R}^{n+1}$ be an n-dimensional UR set. Let A be an $(n+1)\times(n+1)$ matrix defined on $\mathbb{R}^{n+1}\setminus E$ with real bounded measurable coefficients, possibly nonsymmetric, satisfying the ellipticity condition (7.1) and so that $A \in \mathrm{KP}(\mathbb{R}^{n+1}\setminus E)$. Then any weak L-subsolution or L-supersolution $u \in W^{1,2}_{\mathrm{loc}}(\mathbb{R}^{n+1}\setminus E)$ in $\mathbb{R}^{n+1}\setminus E$ satisfies the $S < N$ estimates*

$$\|S_{\mathbb{R}^{n+1}\setminus E}u\|_{L^p(E)} \le C\|N_{*,\mathbb{R}^{n+1}\setminus E}u\|_{L^p(E)}, \qquad 0<p<\infty, \tag{7.13}$$

$$\|S^r_{\mathbb{R}^{n+1}\setminus E}u\|_{L^p(\Delta(x,r))} \lesssim \|N^{K'r}_{*,\mathbb{R}^{n+1}\setminus E}u\|_{L^p(\Delta(x,K'r))}, \quad 0<p<\infty, \tag{7.14}$$

for any $x\in E$ and $0<r<2\,\mathrm{diam}(E)$, where $\Delta(x,r)=B(x,r)\cap E$, and where K' depends on n and the UR character of E; as well as its local dyadic analogue, for any Whitney-dyadic structure $\{\mathcal{W}_Q\}_{Q\in\mathbb{D}(E)}$ for $\mathbb{R}^{n+1}\setminus E$ with parameters η and K,

$$\|S^Q u\|_{L^p(Q)} \le C\|\widehat{N}^Q_* u\|_{L^p(Q)}, \quad Q\in\mathbb{D}(E), \quad 0<p<\infty. \tag{7.15}$$

If, in addition, bounded, $u\in L^\infty(\mathbb{R}^{n+1}\setminus E)$ then the following Carleson measure estimate holds:

$$\|\nabla u\|_{\mathrm{CME}(\mathbb{R}^{n+1}\setminus E)} \le C\|u\|^2_{L^\infty(\mathbb{R}^{n+1}\setminus E)}. \tag{7.16}$$

All constants depend on n, the UR character of E, the ellipticity of A, $\|\nabla A(\cdot)\,\mathrm{dist}(\cdot,E)\|_{L^\infty(\mathbb{R}^{n+1}\setminus E)}$, $\|\nabla A\|_{\mathrm{CME}(\mathbb{R}^{n+1}\setminus E)}$, the aperture of the cone κ implicit in (7.6), and the parameters η, K, τ implicit in (7.8).

Proof. We start observing that we just need to consider the case where u is a weak L-subsolution (because if u is a weak L-supersolution then $-u$ is a weak L-subsolution). We proceed much in the proof of Theorem 7.12 and a careful reading shows that we just need a version of (7.3) valid for weak L-subsolutions. That is, we need to obtain the following:

> Let Ω be a Lipschitz domain and let $A\in\mathrm{KP}(\Omega)$. Then, any weak L-subsolution $u\in W^{1,2}_{\mathrm{loc}}(\Omega)\cap L^\infty(\Omega)$ in Ω satisfies $\|\nabla u\|_{\mathrm{CME}(\Omega)} \lesssim \|u\|^2_{L^\infty(\Omega)}$ with implicit constant depending on n, the Lipschitz character of Ω, ellipticity, and the implicit constants in $A\in\mathrm{KP}(\Omega)$. (7.17)

With this goal in mind, fix then an arbitrary weak L-subsolution $u\in W^{1,2}_{\mathrm{loc}}(\Omega)\cap L^\infty(\Omega)$ in Ω. We may suppose that u is a.e. nonnegative. Indeed, assume for the moment that we have proved (7.17) for a.e. nonnegative weak L-subsolutions, and let $u\in W^{1,2}_{\mathrm{loc}}(\Omega)$ be an arbitrary bounded weak L-subsolution, so that $\tilde u := u+\|u\|_{L^\infty(\Omega)} \in W^{1,2}_{\mathrm{loc}}(\Omega)\cap L^\infty(\Omega)$ is an a.e. nonnegative weak L-subsolution in Ω. We then observe that our assumption for a.e. nonnegative weak L-subsolutions yields the desired estimate for u:

$$\|\nabla u\|_{\mathrm{CME}(\Omega)} = \|\nabla\tilde u\|_{\mathrm{CME}(\Omega)} \lesssim \|\tilde u\|^2_{L^\infty(\Omega)} \le 2\|u\|^2_{L^\infty(\Omega)}.$$

Let us then verify (7.17) for an a.e. nonnegative weak L-subsolution $u\in W^{1,2}_{\mathrm{loc}}(\Omega)\cap L^\infty(\Omega)$. We observe that since $A\in\mathrm{KP}(\Omega)$, by (7.3) and [Kenig et al. 2016], it follows that the elliptic measure ω_L belongs to $A_\infty(\sigma)$ with $\sigma = H^n|_{\partial\Omega}$. With this in hand, we carefully follow the argument in [Cavero et al. 2020, proof of Theorem 1.1: (b) $\Rightarrow$ (a)] with u being the fixed a.e. nonnegative weak L-subsolution in Ω

in place of a solution and observing that Lipschitz domains are clearly 1-sided CAD. To justify that the argument can be adapted to the present situation we just need two observations. First, that u satisfies Caccioppoli's estimate (the proof is a straightforward modification of the standard argument using that u is a nonnegative a.e. weak L-subsolution). Second, in [Cavero et al. 2020, (3.64)] one has to replace "$= 0$" by "≤ 0" because in the present scenario u is a nonnegative a.e. weak L-subsolution (in place of a solution). With these two observations an interested reader could easily see that the argument goes through and eventually show that $\|\nabla u\|_{\mathrm{CME}(\Omega)} \lesssim \|u\|^2_{L^\infty(\Omega)}$. Hence, (7.17) holds and this completes the proof. □

7.3. *Higher-order elliptic equations and systems with constant coefficients.* In [Dahlberg et al. 1997] the authors obtained square function/nontangential maximal function estimates for higher-order elliptic equations and systems on bounded Lipschitz domains. These results have never been extended, even to CAD domains, and here we present a generalization of Carleson measure estimates to the complements of UR sets.

For any multiindex $\alpha = (\alpha_1, \dots, \alpha_{n+1}) \in \mathbb{N}_0^{n+1}$, we write $|\alpha| = \alpha_1 + \dots + \alpha_{n+1}$ and $\alpha! = \alpha_1! \cdots \alpha_{n+1}!$, where $0! = 1$. Also $\partial^\alpha = \partial^{\alpha_1} \dots \partial^{\alpha_{n+1}}$ and for every $Y \in \mathbb{R}^{n+1}$ we write $Y^\alpha = Y_1^{\alpha_1} \cdots Y_{n+1}^{\alpha_{n+1}}$, where $a^0 = 1$ for every $a \in \mathbb{R}$. Finally, ∇^k, $k \in \mathbb{N}$, stands for the vector of all partial derivatives of order k. For $k = 0$, ∇^0 is just the identity operator.

Let $K, m \in \mathbb{N}$. For every $1 \leq j, k \leq K$, let $L^{jk} = \sum_{|\alpha|=2m} a^{jk}_{\alpha\beta} \partial^\alpha$, where $\alpha = (\alpha_1, \dots, \alpha_{n+1}) \in \mathbb{N}_0^{n+1}$. The coefficients $a^{jk}_{\alpha\beta}$, $1 \leq \alpha, \beta \leq n+1$, $1 \leq j, k \leq K$ are real constants. Given an open set Ω and $u = (u_1, \dots, u_K)$, with $u_j \in W^{m,2}_{\mathrm{loc}}(\Omega)$, $1 \leq j \leq K$, we say that $Lu = 0$, if

$$\sum_{k=1}^K L^{jk} u^k = \sum_{k=1}^K \sum_{|\alpha|=|\beta|=m} a^{jk}_{\alpha\beta} \partial^\alpha \partial^\beta u^k = 0, \qquad j = 1, \dots, K,$$

as usual, in the weak sense, similarly to (7.2). Here, $W^{m,2}(\Omega)$ is the space of functions with all derivatives of orders $0, \dots, m$ in $L^2(\Omega)$ and $W^{m,2}_{\mathrm{loc}}(\Omega)$ is the space of functions locally in $W^{m,2}(\Omega)$. We assume, in addition, that L is symmetric: $L^{jk} = L^{kj}$ for $1 \leq j, k \leq K$, and that the Legendre–Hadamard ellipticity condition holds: there exists $\lambda > 0$ such that

$$\sum_{j,k=1}^K \sum_{|\alpha|=|\beta|=m} a^{jk}_{\alpha\beta} \xi^\alpha \xi^\beta \zeta_j \zeta_k \geq \lambda |\xi|^{2m} |\zeta|^2 \quad \text{for all } \zeta = (\zeta_1, \dots, \zeta_K) \in \mathbb{R}^K,\ \xi \in \mathbb{R}^{n+1}. \tag{7.18}$$

Theorem 7.19. *Let $E \subset \mathbb{R}^{n+1}$ be an n-dimensional UR set. Given $K, m \in \mathbb{N}$, let L be a symmetric constant-coefficient $2m$-order $K \times K$ system satisfying the Legendre–Hadamard ellipticity condition, as above. Then any weak solution $u \in [W^{m,2}_{\mathrm{loc}}(\mathbb{R}^{n+1} \setminus E) \cap C^{m-1}(\mathbb{R}^{n+1} \setminus E)]^K$ to $Lu = 0$ in $\mathbb{R}^{n+1} \setminus E$ satisfies the $S < N$ estimates*

$$\|S_{\mathbb{R}^{n+1}\setminus E}(\nabla^{m-1} u)\|_{L^p(E)} \leq C \|N_{*,\mathbb{R}^{n+1}\setminus E}(|\nabla^{m-1} u|)\|_{L^p(E)}, \qquad 0 < p < \infty, \tag{7.20}$$

$$\|S^r_{\mathbb{R}^{n+1}\setminus E}(\nabla^{m-1} u)\|_{L^p(\Delta(x,r))} \lesssim \|N^{K'r}_{*,\mathbb{R}^{n+1}\setminus E}(|\nabla^{m-1} u|)\|_{L^p(\Delta(x,K'r))}, \quad 0 < p < \infty, \tag{7.21}$$

for any $x \in E$ and $0 < r < 2\,\mathrm{diam}(E)$, where $\Delta(x,r) = B(x,r) \cap E$, and where K' depends on n and the UR character of E, as well as its local dyadic analogue, for any Whitney-dyadic structure $\{\mathcal{W}_Q\}_{Q \in \mathbb{D}(E)}$

for $\mathbb{R}^{n+1} \setminus E$ *with parameters* η *and* K,

$$\|S^Q(\nabla^{m-1}u)\|_{L^p(Q)} \leq C\|\widehat{N}_*^Q(|\nabla^{m-1}u|)\|_{L^p(Q)}, \quad Q \in \mathbb{D}(E), \quad 0 < p < \infty. \tag{7.22}$$

If u *is, in addition, such that* $\nabla^{m-1}u \in L^\infty(\Omega)$, *then the Carleson measure estimate*

$$\|\nabla^m u\|_{\mathrm{CME}(\mathbb{R}^{n+1}\setminus E)} \leq C\|\nabla^{m-1}u\|^2_{L^\infty(\mathbb{R}^{n+1}\setminus E)} \tag{7.23}$$

holds. All constants depend on n, *the UR character of* E, *the Legendre–Hadamard ellipticity constant*, $\sup_{j,k,\alpha,\beta}|a^{jk}_{\alpha\beta}|$, *the aperture of the cone* κ *implicit in* (7.20), *and the implicit parameters* η, K, τ *implicit in* (7.22).

Remark 7.24. It is easy to see that from the previous result, one can also obtain analogous estimates in any chord-arc domain $D \subset \mathbb{R}^{n+1}$. To see this let us consider any weak solution $u \in [W^{m,2}_{\mathrm{loc}}(D)]^K$ to $Lu = 0$ in D. Let $\tilde{u} := u$ in D and $\tilde{u} = 0 \in \mathbb{R}^{n+1} \setminus \bar{D}$. Then $\tilde{u} \in [W^{m,2}_{\mathrm{loc}}(\mathbb{R}^{n+1} \setminus \partial D)]^K$ satisfies $L\tilde{u} = 0$ in $\mathbb{R}^{n+1} \setminus \partial D$ in the weak sense. As such, and using the fact that since D is a CAD then ∂D is UR, we obtain (7.20) for $\tilde{u}$ in $\mathbb{R}^{n+1} \setminus \partial D$, which immediately gives the corresponding estimate for u in D. The same occurs with (7.23). Further details are left to the interested reader.

Proof. The proof runs much as that of Theorem 7.5. One replaces (7.3) with the fact that for any bounded Lipschitz domain $\Omega \subset \mathbb{R}^{n+1}$; it was shown in [Dahlberg et al. 1997, Theorem 2, p. 1455] that any weak solution $u \in [W^{m,2}_{\mathrm{loc}}(\Omega)]^K$ to $Lu = 0$ in Ω with $\nabla^{m-1}u \in L^\infty(\Omega)$ satisfies $\|\nabla^m u\|_{\mathrm{CME}(\Omega)} \lesssim \|\nabla^{m-1}u\|^2_{L^\infty(\Omega)}$. With this at hand the proof can be carried out mutatis mutandis. Further details are left to the interested reader. □

We can now state a higher-order version of Theorems 5.1 and 5.24:

Theorem 7.25. *Let* $D \subset \mathbb{R}^{n+1}$ *be a CAD, let* $K, m \in \mathbb{N}$ *and let* $u = (u_1, \dots, u_K) \in [W^{m,2}_{\mathrm{loc}}(D) \cap C^{m-1}(D)]^K$.

(1) *Assume that* (5.2) *holds with* $\nabla^{m-1}u$ *in place of* u. *Suppose that the* $(m-1)$*-th order* $N < S$ *estimates are valid on* L^2 *on all bounded Lipschitz subdomains* $\Omega \subset D$, *that is*, (5.3) *holds for any bounded Lipschitz subdomain* $\Omega \subset D$ *with* $\nabla^{m-1}u$ *in place of* u, *and where the constant may also depend on* m *and* K. *Then* (5.4)–(5.8) *hold replacing* u *by* $\nabla^{m-1}u$, *and where all the constants may also depend on* m *and* K.

(2) *Assume that* (5.2) *holds with* $\nabla^{m-1}u$ *in place of* u *and that* (5.25) *hold with* $\nabla^m u$ *in place of* ∇u *for some* $p > 2$. *Suppose that the* $(m-1)$*-th order* $N < S$ *estimates are valid on* L^p *on all bounded chord-arc* $\Omega \subset D$, *that is*, (5.26) *holds for any bounded chord-arc subdomain* $\Omega \subset D$ *with* $\nabla^{m-1}u$ *in place of* u, *and where the constant may also depend on* m *and* K. *Then* (5.27)–(5.30) *hold with* $\nabla^{m-1}u$ *in place of* u, *and where all the constants may also depend on* m *and* K.

(3) *Assume that* (5.2) *holds with* $\nabla^{m-1}u$ *in place of* u *and that* (5.25) *hold with* $\nabla^m u$ *in place of* ∇u *for some* $p > 2$. *Suppose that the* $(m-1)$*-th order* $N < S$ *estimates are valid on* L^2 *on all bounded Lipschitz subdomains* $\Omega \subset D$, *that is*, (5.3) *holds for any bounded Lipschitz subdomain* $\Omega \subset D$ *with* $\nabla^{m-1}u$ *in place of* u, *and where the constant may also depend on* m *and* K. *Then* (5.27)–(5.30) *hold replacing* u *by* $\nabla^{m-1}u$, *and where all the constants may also depend on* m *and* K.

Proof. The proof is fairly easy. Consider the vector $v = \nabla^{m-1}u \in [W^{1,2}_{\mathrm{loc}}(D) \cap C(D)]^{K(n-1)^{m-1}}$. Note that our current assumptions in (i)–(iii) imply that v satisfies (5.2). Also, in items (ii), (iii) we will have that v

satisfies (5.25). Note that (5.3) is satisfied by v in parts (i) and (iii), and (5.26) holds for v in part (ii). We also know that Theorems 5.1 and 5.24, and Corollary 5.50 can be easily extended to vector-valued functions u. With all these at hand, we readily obtain the corresponding estimates for v which translated into those stated for u. Further details are left to the interested reader. □

One can also obtain a higher-order version of Theorem 6.1 using the same ideas:

Theorem 7.26. *Let $E \subset \mathbb{R}^{n+1}$ be an n-dimensional UR set, $\mathbb{R}^{n+1} \setminus E$, and let $m, K \in N$. Suppose that $u \in [W^{m,2}_{\mathrm{loc}}(\mathbb{R}^{n+1} \setminus E) \cap C^{m-1}(\mathbb{R}^{n+1} \setminus E) \cap L^\infty(\mathbb{R}^{n+1} \setminus E)]^K$ is such that for any cube I with $2I \subset \mathbb{R}^{n+1} \setminus E$*

$$\sup_{X,Y \in I} |\nabla^{m-1} u(X) - \nabla^{m-1} u(Y)| \le C_0 \Big(\ell(I)^{1-n} \iint_{2I} |\nabla^m u|^2 \, dX \Big)^{\frac{1}{2}} \tag{7.27}$$

and

$$\|\nabla^m u\|_{\mathrm{CME}(\mathbb{R}^{n+1} \setminus E)} \le C_0' \|\nabla^{m-1} u\|_{L^\infty(\mathbb{R}^{n+1} \setminus E)}$$

Assume, in addition, $(m-1)$-th order that $N < S$ estimates are valid on L^2 on all bounded chord-arc subdomains $\Omega \subset \mathbb{R}^{n+1} \setminus E$, that is, for any bounded chord-arc subdomain $\Omega \subset \mathbb{R}^{n+1} \setminus E$, there holds

$$\|N_{*,\Omega}(\nabla^{m-1} u - \nabla^{m-1} u(X_\Omega^+))\|_{L^2(\partial\Omega)} \le C_\Omega \|S_\Omega(\nabla^{m-1} u)\|_{L^2(\partial\Omega)}. \tag{7.28}$$

Here X_Ω^+ is any interior corkscrew point of Ω at the scale of $\operatorname{diam}(\Omega)$, *and the constant C_Ω depends on the CAD character of Ω, the dimension n, m, K, p, the implicit choice of κ (the aperture of the cones in $N_{*,\Omega}$ and S_Ω), and the implicit corkscrew constant for the point X_Ω^+. Then $\nabla^{m-1} u$ is ε-approximable on $\mathbb{R}^{n+1} \setminus E$, with the implicit constants depending only on n, m, K, the UR character of E, C_0, and C_0'.*

As a corollary of all these we can obtain $N < S$ estimates and ε-approximability for solutions of a symmetric constant-coefficient $2m$-order $K \times K$ systems.

Theorem 7.29. *Given $K, m \in \mathbb{N}$, let L be a symmetric constant-coefficient $2m$-order $K \times K$ system, satisfying the Legendre–Hadamard ellipticity condition, as above.*

(i) *If $D \subset \mathbb{R}^{n+1}$ is a CAD, then any weak solution $u \in [W^{m,2}_{\mathrm{loc}}(D) \cap C^{m-1}(D)]^K$ to $Lu = 0$ in D satisfies for any $x \in \partial D$ and $0 < r < 2\operatorname{diam}(\partial D)$ and for every $\kappa > 0$*

$$\begin{aligned} &\|N^r_{*,D,\kappa}(\nabla^{m-1} u - \nabla^{m-1} u(X^+_{\Delta(x,r)}))\|_{L^q(\Delta(x,r))} \\ &\qquad \le C \|S^{C'r}_{D,\kappa}(\nabla^{m-1} u)\|_{L^q(\Delta(x,C'r))}, \quad \text{for all } 0 < q < \infty, \end{aligned} \tag{7.30}$$

where $\Delta(x,r) = B(x,r) \cap \partial\Omega$. Here C depends on n, q, K, m, the CAD character of D, the Legendre–Hadamard ellipticity constant, $\sup_{j,k,\alpha,\beta} |a^{jk}_{\alpha\beta}|$, *and the aperture of the cone κ, and C' depends on n and the CAD character of D. In particular, if ∂D is bounded then*

$$\|N_{*,D,\kappa}(\nabla^{m-1} u - \nabla^{m-1} u(X_D^+))\|_{L^q(\partial D)} \le C'' \|S_{D,\kappa}(\nabla^{m-1} u)\|_{L^q(\partial D)} \quad \text{for all } 0 < q < \infty, \tag{7.31}$$

and if ∂D is unbounded and $\nabla^{m-1} u(X) \to 0$ as $|X| \to \infty$ then

$$\|N_{*,D,\kappa}(\nabla^{m-1} u)\|_{L^q(\partial D)} \le C'' \|S_{D,\kappa}(\nabla^{m-1} u)\|_{L^q(\partial D)} \quad \text{for all } 0 < q < \infty. \tag{7.32}$$

(ii) *Let $E \subset \mathbb{R}^{n+1}$ be an n-dimensional UR set. Then any weak solution*

$$u \in [W^{m,2}_{\mathrm{loc}}(\mathbb{R}^{n+1} \setminus E) \cap C^{m-1}(\mathbb{R}^{n+1} \setminus E) \cap L^\infty(\mathbb{R}^{n+1} \setminus E)]^K$$

to $Lu=0$ in $\mathbb{R}^{n+1} \setminus E$ satisfies that $\nabla^{m-1}u$ is ε-approximable in $\mathbb{R}^{n+1} \setminus E$ with implicit constants depending on n, K, m, the UR character of E, the Legendre–Hadamard ellipticity constant, $\sup_{j,k,\alpha,\beta} |a^{jk}_{\alpha\beta}|$.

Proof. We aim to use Theorem 7.25(iii) and 7.26. To this end, we need to verify the interior estimates: (5.2) with $\nabla^{m-1}u$ in place of u, (5.25) with $\nabla^m u$ in place of ∇u for some $p > 2$, and (7.27), and to obtain $(m-1)$-th order $N < S$ estimates on L^2 on all bounded Lipschitz subdomains Ω and for any weak solution $u \in [W^{m,2}_{\mathrm{loc}}(\Omega) \cap C^{m-1}(\Omega)]^K$ to $Lu = 0$ in Ω. That is, we need to show that (7.28) holds on all bounded Lipschitz subdomains Ω. Let us start with the latter. To see this we introduce

$$P_{m-1,X^+_\Omega}u(X) = \sum_{|\alpha|\le m-1} \frac{\partial^\alpha u(X^+_\Omega)}{\alpha!}(X - X^+_\Omega)^\alpha, \qquad X \in \Omega.$$

and observe that $\nabla^k P_{m-1,X^+_\Omega}u(X^+_\Omega) = \nabla^k u(X^+_\Omega)$ for $0 \le k \le m-2$, $\nabla^{m-1}P_{m-1,X^+_\Omega}u(\cdot) \equiv \nabla^{m-1}u(X^+_\Omega)$, and $\nabla^m P_{m-1,X^+_\Omega}u \equiv 0$. Thus if we write $v = u - P_{m-1,X^+_\Omega}u(\cdot)$, we have that $v \in [W^{m,2}_{\mathrm{loc}}(\Omega) \cap C^{m-1}(\Omega)]^K$ is a weak solution to $Lv = 0$ in Ω satisfying $\nabla^k v(X^+_\Omega) = 0$ for all $0 \le k \le m-1$, $\nabla^{m-1}v = \nabla^{m-1}u - \nabla^{m-1}u(X^+_\Omega)$, and $\nabla^m v = \nabla^m u$. As such we can invoke [Dahlberg et al. 1997, Theorem 3, p. 1456] to obtain that

$$\begin{aligned}\|N_{*,\Omega}(\nabla^{m-1}u - \nabla^{m-1}u(X^+_\Omega))\|_{L^2(\partial\Omega)} &= \|N_{*,\Omega}(\nabla^{m-1}v)\|_{L^2(\partial\Omega)} \\ &\lesssim \|S_\Omega(\nabla^{m-1}v)\|_{L^2(\partial\Omega)} = \|S_\Omega(\nabla^{m-1}u)\|_{L^2(\partial\Omega)}.\end{aligned}$$

Turning to interior estimates, we recall from [Barton 2016, Corollary 22, p. 384] that for all solutions to $Lu = 0$ in $2I$ we have

$$\iint_I |\nabla^j u|^2\, dX \le C\ell(I)^{-2j} \iint_{2I} |u|^2\, dX, \qquad j = 0, \dots, m. \tag{7.33}$$

In fact, [Barton 2016] pertains to much more general elliptic systems with bounded measurable coefficients. It uses the weak Gårding inequality [Barton 2016, (10), p. 380]. To obtain the latter (with $\delta = 0$) we can see that Plancherel's theorem, the fact that we are currently considering the case with real constant coefficients, and the Legendre–Hadamard condition (7.18) easily yield, for every smooth compactly supported function φ,

$$\begin{aligned}\mathrm{Re}\langle \nabla^m\varphi, A\nabla^m\varphi\rangle_{\mathbb{R}^{n+1}} &= \mathrm{Re} \iint_{\mathbb{R}^{n+1}} \sum_{j,k=1}^K \sum_{|\alpha|=|\beta|=m} \overline{\partial^\alpha \varphi_j(X)}\, a^{jk}_{\alpha\beta} \partial^\beta \varphi_k(X)\, dX \\ &= \sum_{j,k=1}^K \sum_{|\alpha|=|\beta|=m} a^{jk}_{\alpha\beta}\, \mathrm{Re} \iint_{\mathbb{R}^{n+1}} (-2\pi i\xi)^\alpha (2\pi i\xi)^\beta\, \overline{\tilde\varphi_j(\xi)}\, \tilde\varphi_k(\xi)\, d\xi \\ &= \iint_{\mathbb{R}^{n+1}} \sum_{j,k=1}^K \sum_{|\alpha|=|\beta|=m} a^{jk}_{\alpha\beta} (2\pi\xi)^\alpha (2\pi\xi)^\beta\, \mathrm{Re}(\overline{\tilde\varphi_j(\xi)}\, \tilde\varphi_k(\xi))\, d\xi \\ &\ge \lambda \iint_{\mathbb{R}^{n+1}} (2\pi|\xi|)^{2m} |\tilde\varphi_j(X)|^2\, dX = \lambda \iint_{\mathbb{R}^{n+1}} |\nabla^m\varphi_j(\xi)|^2\, d\xi,\end{aligned}$$

and so [Barton 2016] applies to our setting.

Now, for constant-coefficient operators any derivative of a solution is still a solution, and, in fact, we will use $v := u - P_{m-1,X_I}u(\cdot)$ built similarly to above, only using X_I being the center of I in place of X_Ω^+. Clearly, $\nabla^m v = \nabla^m u$ is a solution too, and so a repeated application of (7.33) yields

$$\iint_I |\nabla^k v|^2 \, dX \le C\ell(I)^{-2(k-m)} \iint_{2I} |\nabla^m v|^2 \, dX, \quad k \ge m. \tag{7.34}$$

Taking $k > m-1$ large enough, depending on the dimension only, so that the Sobolev space $W^{k,2}(I)$ embeds into the Hölder space $C^{m-1,\alpha}(I)$, $\alpha > 0$, we can show that

$$\begin{aligned} \sup_{X,Y\in I} |\nabla^{m-1}u(X) - \nabla^{m-1}u(Y)| &= \sup_{X,Y\in I} |\nabla^{m-1}v(X) - \nabla^{m-1}v(Y)| \\ &\le C \sum_{j=0}^{k} \left(\ell(I)^{-1-n+2(j-m+1)} \iint_I |\nabla^j v|^2 \, dX \right)^{\frac{1}{2}}. \end{aligned} \tag{7.35}$$

For $j > m$ we use (7.34) to descend to $j = m$. For $j < m$, we use the Poincaré inequality to ascend to $j = m$, and all in all, the expression above is bounded by

$$C\left(\ell(I)^{1+n} \iint_{2I} |\nabla^m v|^2 \, dX \right)^{\frac{1}{2}} = C\left(\ell(I)^{1+n} \iint_{2I} |\nabla^m u|^2 \, dX \right)^{\frac{1}{2}},$$

as desired. This yields (7.27).

In order to obtain (5.2) with $\nabla^{m-1}u$ in place of u, we apply the same argument as above to $v := \nabla^{m-1}u - \vec{c}$ for some constant vector $\vec{c}$. The function v is also a solution of the initial system, and so (7.34) still holds. Much as above, by the Morrey inequality (or generalized Sobolev embeddings), for k large enough, depending on dimension only, we arrive at

$$\sup_I |v| \le C \sum_{j=0}^{k} \left(\ell(I)^{-1-n+2j} \iint_I |\nabla^j v|^2 \, dX \right)^{\frac{1}{2}} \le C\left(\ell(I)^{-1-n} \iint_{2I} |v|^2 \, dX \right)^{\frac{1}{2}}, \tag{7.36}$$

where we have used (7.33) and (7.34) for the second inequality.

Finally, the reverse Hölder inequality (5.25) with $\nabla^m u$ in place of ∇u was also proved in [Barton 2016, Theorem 24].

With all the previous ingredients we are ready to invoke Theorem 7.25(iii) and then Theorem 7.26 to obtain the desired estimates. □

Appendix: Sawtooths have UR boundaries

To start, recall from [Hofmann et al. 2016, Appendix A] the fact that the sawtooth regions and Carleson boxes inherit the ADR property. In that Appendix we treated simultaneously the case that the set E is ADR, but not necessarily UR, and also the case that E is UR. The point was that the Whitney regions in the two cases (and thus also the corresponding sawtooth regions and Carleson boxes) were somewhat different. In any case, the reader can easily see that, with the notation introduced in Definition 2.7, the arguments in [Hofmann et al. 2016, Appendix A] can be carried out for any ADR set E and with $\{\mathcal{W}_Q\}_{Q\in\mathbb{D}(E)}$ any Whitney-dyadic structure for $\mathbb{R}^{n+1} \setminus E$ with some parameters η and K. In turn, both if

E happens to be merely an ADR set as in Section 2.1, or a UR set as in Section 2.2, the corresponding constructions of Whitney-dyadic structure fit within the previous framework. Nonetheless, the same applies to any other Whitney-dyadic structure (constructed in a different way) but retaining the same properties.

Let us now recall some results from [Hofmann et al. 2016] that we shall use in the sequel.

Proposition A.1 [Hofmann et al. 2016, Proposition A.2]. *Let $E \subset \mathbb{R}^{n+1}$ be an n-dimensional ADR set and let $\{\mathcal{W}_Q\}_{Q\in\mathbb{D}(E)}$ be a Whitney-dyadic structure for $\mathbb{R}^{n+1}\setminus E$ with some parameters $\eta \ll 1$ and $K \gg 1$. Then all dyadic local sawtooths $\Omega_{\mathcal{F},Q}$ and all Carleson boxes T_Q have n-dimensional ADR boundaries. In all cases, the implicit constants are uniform and depend only on dimension, the ADR constant of E, parameters η, K, and the constant C in Definition 2.7*(iii).

Remark A.2. Let $\Omega \subset \mathbb{R}^{n+1}$ be an open set with ADR boundary and let $\{\mathcal{W}_Q\}_{Q\in\mathbb{D}(\partial\Omega)}$ be a Whitney-dyadic structure for Ω with parameters η and K. One can easily construct a Whitney-dyadic structure $\{\mathcal{W}'_Q\}_{Q\in\mathbb{D}(\partial\Omega)}$ for $\mathbb{R}^{n+1}\setminus\partial\Omega$ so that for every $I\in\mathcal{W}(\Omega)$ one has that $I\in\mathcal{W}_Q$ if and only if $I\in\mathcal{W}'_Q$, that is, the new Whitney-dyadic structure remains the same for the Whitney cubes contained in Ω. To construct such a Whitney-dyadic structure we define $(\mathcal{W}'_Q)^0$ as in (2.8) with the same parameters η and K but for all the Whitney cubes $I\in\mathcal{W}(\mathbb{R}^{n+1}\setminus\partial\Omega)$. For every $Q\in\mathbb{D}(\partial\Omega)$ we the set $\mathcal{W}'_Q := \mathcal{W}_Q \cup ((\mathcal{W}'_Q)^0 \cap \mathcal{W}(\mathbb{R}^{n+1}\setminus\bar{\Omega}))$. It is straightforward to see that $\{\mathcal{W}'_Q\}_{Q\in\mathbb{D}(\partial\Omega)}$ is a Whitney-dyadic structure for $\mathbb{R}^{n+1}\setminus\partial\Omega$ with parameters η and K and agreeing with $\{\mathcal{W}_Q\}_{Q\in\mathbb{D}(\partial\Omega)}$ when restricted to the Whitney cubes contained in Ω. Note also that the constants in Definition 2.7(iii) are the same for both.

We then note by Proposition A.1 all the associated dyadic local sawtooths $\Omega'_{\mathcal{F},Q}$ and all Carleson boxes T'_Q (contained in $\mathbb{R}^{n+1}\setminus\partial\Omega$) have n-dimensional ADR boundaries. In turn the agreement of $\{\mathcal{W}_Q\}_{Q\in\mathbb{D}(\partial\Omega)}$ with $\{\mathcal{W}'_Q\}_{Q\in\mathbb{D}(\partial\Omega)}$ inside Ω implies at the very least that all the associated dyadic local sawtooths $\Omega_{\mathcal{F},Q}$ and all Carleson boxes T_Q (contained now in Ω) have a boundary satisfying the upper ADR condition (that is the upper estimate in (1.2)) with constant depending on the ADR constant of $\partial\Omega$, η, K and the constant in Definition 2.7(iii).

In what follows we assume that E is an ADR set and fix $\{\mathcal{W}_Q\}_{Q\in\mathbb{D}(E)}$ a Whitney-dyadic structure for $\mathbb{R}^{n+1}\setminus E$ with some parameters η and K. As mentioned in Section 2, we always assume that if $\{\mathcal{W}_Q\}_{Q\in\mathbb{D}(E)}$ is a Whitney-dyadic structure for $\mathbb{R}^{n+1}\setminus E$ with some parameters η and K, then K is large enough (say $K \ge 40^2 n$) so that for any $\ell(I) \lesssim \operatorname{diam}(E)$ we have $I \in \mathcal{W}^0_{Q^*_I} \subset \mathcal{W}_{Q^*_I}$, where Q^*_I is some fixed nearest dyadic cube to I with $\ell(I) = \ell(Q^*_I)$. To simplify the notation, it is convenient to find $m_0 \in \mathbb{Z}_+$, $C_0 \in \mathbb{R}_+$ (say $2^{m_0} \approx C\max\{K, \eta^{-1}\}^{1/2}$, $C_0 = CK^{1/2}$, hence depending on η, K and the constant C in Definition 2.7(iii)) such that

$$2^{-m_0}\ell(Q) \le \ell(I) \le 2^{m_0}\ell(Q) \quad \text{and} \quad \operatorname{dist}(I,Q) \le C_0\ell(Q) \quad \text{for all } I\in\mathcal{W}_Q. \tag{A.3}$$

From now, we will use these parameters m_0 and C_0, rather than η, K and the constant C in Definition 2.7(iii).

Let us recall some notation from [Hofmann et al. 2016, Appendix A]. Given a cube $Q_0\in\mathbb{D}$ and a family $\mathcal{F}$ of disjoint cubes $\mathcal{F} = \{Q_j\} \subset \mathbb{D}_{Q_0}$ (for the case $\mathcal{F} = \varnothing$ the changes are straightforward and we leave them to the reader, also the case $\mathcal{F} = \{Q_0\}$ is disregarded since in that case $\Omega_{\mathcal{F},Q_0}$ is the null set).

We write $\Omega_\star = \Omega_{\mathcal{F},Q_0}$ and $\Sigma = \partial\Omega_\star \setminus E$. Given $Q \in \mathbb{D}$ we set

$$\mathbb{R}_Q := \bigcup_{Q' \in \mathbb{D}_Q} \mathcal{W}_{Q'}, \quad \text{and} \quad \Sigma_Q = \Sigma \bigcap \Big(\bigcup_{I \in \mathbb{R}_Q} I \Big).$$

Let C_1 be a sufficiently large constant, to be chosen below, depending on n, the ADR constant of E, m_0 and C_0. Let us introduce some new collections:

$$\begin{aligned}
\mathcal{F}_{||} &:= \{Q \in \mathbb{D} \setminus \{Q_0\} : \ell(Q) = \ell(Q_0),\ \operatorname{dist}(Q, Q_0) \le C_1 \ell(Q_0)\},\\
\mathcal{F}_{\top} &:= \{Q' \in \mathbb{D} : \operatorname{dist}(Q', Q_0) \le C_1 \ell(Q_0),\ \ell(Q_0) < \ell(Q') \le C_1 \ell(Q_0)\},\\
\mathcal{F}_{||}^* &:= \{Q \in \mathcal{F}_{||} : \Sigma_Q \ne \varnothing\} = \{Q \in \mathcal{F}_{||} : \text{there exists } I \in \mathbb{R}_Q \text{ such that } \Sigma \cap I \ne \varnothing\},\\
\mathcal{F}^* &:= \{Q \in \mathcal{F} : \Sigma_Q \ne \varnothing\} = \{Q \in \mathcal{F} : \text{there exists } I \in \mathbb{R}_Q \text{ such that } \Sigma \cap I \ne \varnothing\}.
\end{aligned}$$

We also set

$$\mathbb{R}_\perp = \bigcup_{Q \in \mathcal{F}^*} \mathbb{R}_Q, \quad \mathbb{R}_{||} = \bigcup_{Q \in \mathcal{F}_{||}^*} \mathbb{R}_Q, \quad \mathbb{R}_\top = \bigcup_{Q \in \mathcal{F}_\top} \mathcal{W}_Q.$$

Lemma A.4 [Hofmann et al. 2016, Lemma A.3]. *Set $\mathcal{W}_\Sigma = \{I \in \mathcal{W} : I \cap \Sigma \ne \varnothing\}$ and define*

$$\mathcal{W}_\Sigma^\perp = \bigcup_{Q \in \mathcal{F}^*} \mathcal{W}_{\Sigma,Q}, \quad \mathcal{W}_\Sigma^{||} = \bigcup_{Q \in \mathcal{F}_{||}^*} \mathcal{W}_{\Sigma,Q}, \quad \mathcal{W}_\Sigma^\top = \{I \in \mathcal{W}_\Sigma : Q_I^* \in \mathcal{F}_\top\},$$

where for every $Q \in \mathcal{F}^* \cup \mathcal{F}_{||}^*$ we set

$$\mathcal{W}_{\Sigma,Q} = \{I \in \mathcal{W}_\Sigma : Q_I^* \in \mathbb{D}_Q\},$$

and where we recall that Q_I^* is the nearest dyadic cube to I with $\ell(I) = \ell(Q_I^*)$ as defined above. Then

$$\mathcal{W}_\Sigma = \mathcal{W}_\Sigma^\perp \cup \mathcal{W}_\Sigma^{||} \cup \mathcal{W}_\Sigma^\top, \tag{A.5}$$

where

$$\mathcal{W}_\Sigma^\perp \subset \mathbb{R}_\perp, \quad \mathcal{W}_\Sigma^{||} \subset \mathbb{R}_{||}, \quad \mathcal{W}_\Sigma^\top \subset \mathbb{R}_\top. \tag{A.6}$$

As a consequence,

$$\Sigma = \Sigma_\perp \cup \Sigma_{||} \cup \Sigma_\top := \Big(\bigcup_{I \in \mathcal{W}_\Sigma^\perp} \Sigma \cap I \Big) \bigcup \Big(\bigcup_{I \in \mathcal{W}_\Sigma^{||}} \Sigma \cap I \Big) \bigcup \Big(\bigcup_{I \in \mathcal{W}_\Sigma^\top} \Sigma \cap I \Big). \tag{A.7}$$

Lemma A.8 [Hofmann et al. 2016, Lemma A.7]. *Given $I \in \mathcal{W}_\Sigma$, we can find $Q_I \in \mathbb{D}$, with $Q_I \subset Q_I^*$, such that $\ell(I) \approx \ell(Q_I)$, $\operatorname{dist}(Q_I, I) \approx \ell(I)$, and in addition,*

$$\sum_{I \in \mathcal{W}_{\Sigma,Q}} 1_{Q_I} \lesssim 1_Q \quad \text{for any } Q \in \mathcal{F}^* \cup \mathcal{F}_{||}^*, \tag{A.9}$$

$$\sum_{I \in \mathcal{W}_\Sigma^\top} 1_{Q_I} \lesssim 1_{B_{Q_0}^* \cap E}, \tag{A.10}$$

where the implicit constants depend on n, the ADR constant of E, m_0 and C_0, and where $B_{Q_0}^* = B(x_{Q_0}, C\ell(Q))$ with C large enough depending on the same parameters.

With the preceding results in hand, we turn to the main purpose of this appendix: to prove that uniform rectifiability is also inherited by the sawtooth domains and Carleson boxes.

Proposition A.11. *Let $E \subset \mathbb{R}^{n+1}$ be an n-dimensional UR set and let $\{\mathcal{W}_Q\}_{Q\in\mathbb{D}(E)}$ be a Whitney-dyadic structure for $\mathbb{R}^{n+1}\setminus E$ with some parameters $\eta \ll 1$ and $K \gg 1$. Then all dyadic local sawtooths $\Omega_{\mathcal{F},Q}$ and all Carleson boxes T_Q have n-dimensional UR boundaries. In all cases, the implicit constants are uniform and depend only on dimension, the UR character of E, and the parameters m_0 and C_0 (hence on the parameters η, K, and the constant C in Definition 2.7*(iii)).

The proof of this result follows the ideas from [Hofmann and Martell 2014, Appendix C], which in turn uses some ideas from Guy David, and uses the following singular integral characterization of UR sets, established in [David and Semmes 1991]. Suppose that $E \subset \mathbb{R}^{n+1}$ is n-dimensional ADR. The singular integral operators that we shall consider are those of the form

$$T_{E,\varepsilon} f(x) = T_\varepsilon f(x) := \int_E \mathcal{K}_\varepsilon(x-y) f(y)\, dH^n(y),$$

where $\mathcal{K}_\varepsilon(x) := \mathcal{K}(x)\Phi(|x|/\varepsilon)$, with $0 \le \Phi \le 1$, $\Phi(\rho) \equiv 1$ if $\rho \ge 2$, $\Phi(\rho) \equiv 0$ if $\rho \le 1$, and $\Phi \in C^\infty(\mathbb{R})$, and where the singular kernel $\mathcal{K}$ is an odd function, smooth on $\mathbb{R}^{n+1}\setminus\{0\}$, and satisfying

$$|\mathcal{K}(x)| \le C_0 |x|^{-n}, \tag{A.12}$$

$$|\nabla^m \mathcal{K}(x)| \le C_m |x|^{-n-m} \quad \text{for all } m \ge 1. \tag{A.13}$$

Then E is UR if and only if for every such kernel $\mathcal{K}$, we have that

$$\sup_{\varepsilon>0} \int_E |T_{E,\varepsilon} f|^2\, dH^n \le C_{\mathcal{K}} \int_E |f|^2\, dH^n. \tag{A.14}$$

We refer the reader to [David and Semmes 1991] for the proof. For $\mathcal{K}$ as above, set

$$\mathcal{T}_E f(X) := \int_E \mathcal{K}(X-y) f(y)\, dH^n(y), \qquad X \in \mathbb{R}^{n+1}\setminus E. \tag{A.15}$$

We define (possibly disconnected) nontangential approach regions $\Upsilon_\alpha(x)$ as follows. Set $\mathcal{W}_\alpha(x) := \{I \in \mathcal{W} : \operatorname{dist}(I, x) < \alpha\ell(I)\}$. Then we define

$$\Upsilon_\alpha(x) := \bigcup_{I\in\mathcal{W}_\alpha(x)} I^*$$

(thus, roughly speaking, α is the "aperture" of $\Upsilon_\alpha(x)$). Here $I^* = I^*(\tau)$ as in Section 2 with $0 < \tau \le \tau_0/4$, which is fixed. Note that these nontangential approach regions are slightly different that the ones introduced in (2.23) since they do not use the Whitney regions U_Q. For $F \in C(\mathbb{R}^{n+1}\setminus E)$ we may then also define a new nontangential maximal function (which is different than the one (2.34) although somehow comparable much as in Remark 2.37)

$$\mathcal{N}_{*,\alpha} F(x) := \sup_{Y\in\Upsilon_\alpha(x)} |F(Y)|.$$

We shall sometimes write simply $\mathcal{N}_*$ when there is no chance of confusion in leaving implicit the dependence on the aperture α. The following lemma is a standard consequence of the usual Cotlar inequality for maximal singular integrals, and we omit the proof.

Lemma A.16. *Suppose that $E \subset \mathbb{R}^{n+1}$ is an n-dimensional UR, and let $\mathcal{T}_E$ be defined as in (A.15). Then, for each $1 < p < \infty$ and $\alpha \in (0, \infty)$, there is a constant $C_{p,\alpha,\mathcal{K}}$ depending only on $p, n, \alpha, \mathcal{K}$ and the UR character of E such that*

$$\int_E (\mathcal{N}_{*,\alpha}(\mathcal{T}_E f))^p \, dH^n \le C_{\alpha,\mathcal{K}} \int_E |f|^p \, dH^n. \tag{A.17}$$

Proof of Proposition A.11. Write $\sigma = H^n|_E$. We fix $Q_0 \in \mathbb{D} = \mathbb{D}(E)$ and a family $\mathcal{F}$ of disjoint cubes $\mathcal{F} = \{Q_j\} \subset \mathbb{D}_{Q_0}$ (for the case $\mathcal{F} = \varnothing$ the changes are straightforward and we leave them to the reader, also the case $\mathcal{F} = \{Q_0\}$ is disregarded since $\Omega_{\mathcal{F},Q_0}$ is the null set). We write $\Omega_\star = \Omega_{\mathcal{F},Q_0}$, $E_\star = \partial\Omega_\star$, and $\sigma_\star = H^n|_{E_\star}$. We fix $0 \le \Phi \le 1$, $\Phi(\rho) \equiv 1$ if $\rho \ge 2$, $\Phi(\rho) \equiv 0$ if $\rho \le 1$, and $\Phi \in C^\infty(\mathbb{R})$. According to the previous considerations we fix $\varepsilon_0 > 0$ and our goal is to show that $T_{E_\star,\varepsilon_0}$ is bounded on $L^2(E_\star)$ with bounds that are independent of ε_0. To simplify the notation we write $\mathcal{K}_0 = \mathcal{K}_{\varepsilon_0}$ and set, for every $X \in \mathbb{R}^{n+1}$,

$$\mathcal{T}_{E,0} f(X) = \int_E \mathcal{K}_0(X-y) f(y) \, d\sigma(y), \quad \mathcal{T}_{E_\star,0} g(X) = \int_{E_\star} \mathcal{K}_0(X-y) g(y) \, d\sigma_\star(y).$$

We first observe that $\mathcal{K}_0$ is not singular and therefore, for any p, $1 < p < \infty$, and for every $f \in L^p(E)$, respectively $g \in L^p(E_\star)$, the previous operators are well-defined (by means of an absolutely convergent integral) for every $X \in \mathbb{R}^{n+1}$. Also for such functions it is easy to see that the dominated convergence theorem implies that $\mathcal{T}_{E,0} f, \mathcal{T}_{E_\star,0} g \in C(\mathbb{R}^{n+1})$.

Remark A.18. We notice that $\mathcal{K}_0$ is an odd smooth function which satisfies (A.12) and (A.13) with uniform constants (i.e., with no dependence on ε_0) and therefore the fact that E is UR implies that (A.14) and (A.17) hold with constants that do not depend on ε_0.

We are going to see that $\mathcal{T}_{E,0} : L^p(E) \to L^p(E_\star)$ for every $1 < p < \infty$. To do that we take $f \in L^p(E)$ and write

$$\int_{E_\star} |\mathcal{T}_{E,0} f(x)|^p \, d\sigma_\star(x) = \int_{E_\star \cap E} |\mathcal{T}_{E,0} f(x)|^p \, d\sigma_\star(x) + \int_{E_\star \setminus E} |\mathcal{T}_{E,0} f(x)|^p \, d\sigma_\star(x) =: \mathrm{I} + \mathrm{II}.$$

The estimate for I follows from the fact that E is UR

$$\mathrm{I} \le \int_E |\mathcal{T}_{E,0} f(x)|^p \, d\sigma(x) = \int_E |T_{E,\varepsilon_0} f(x)|^p \, d\sigma(x) \le C_{\mathcal{K}} \int_E |f(x)|^p \, d\sigma(x),$$

where we have used (A.14) and the standard Calderón–Zygmund theory (taking place in the ADR set E) and $C_{\mathcal{K}}$ does not depend on ε_0. For II we use that $\Sigma = E_\star \setminus E = \partial\Omega_\star \setminus E$ and invoke Lemmas A.4 and A.8; let Q_I be the cube constructed in the latter, so that

$$\mathrm{II} = \sum_{I \in \mathcal{W}_\Sigma} \int_{I \cap \Sigma} |\mathcal{T}_{E,0} f(x)|^p \, d\sigma_\star(x) = \sum_{I \in \mathcal{W}_\Sigma} \fint_{Q_I} \int_{I \cap \Sigma} |\mathcal{T}_{E,0} f(x)|^p \, d\sigma_\star(x) \, d\sigma(y).$$

Note that if $y \in Q_I$ and $x \in \Sigma \cap I$ then $\operatorname{dist}(I, y) \lesssim \ell(Q_I) \approx \ell(I)$. Then taking $\alpha > 0$ large enough we obtain that $I \subset \mathcal{W}_\alpha(y)$. Write $\widetilde{\mathcal{F}} = \mathcal{F}^* \cup \mathcal{F}^*_{||}$, and observe that by construction the cubes in $\widetilde{\mathcal{F}}$ are pairwise disjoint. Then by the ADR property of $E_\star$, along with Lemmas A.4 and A.8,

$$\begin{aligned}
\mathrm{II} &\le \sum_{I\in\mathcal{W}_\Sigma} \sigma_\star(\Sigma\cap I) \fint_{Q_I} |\mathcal{N}_{*,\alpha}(\mathcal{T}_{E,0}f)(y)|^p\,d\sigma(y)\\
&\lesssim \sum_{Q\in\widetilde{\mathcal{F}}}\sum_{I\in\mathcal{W}_{\Sigma,Q}} \int_{Q_I} |\mathcal{N}_{*,\alpha}(\mathcal{T}_{E,0}f)(y)|^p\,d\sigma(y) + \sum_{I\in\mathcal{W}_\Sigma^\top}\int_{Q_I} |\mathcal{N}_{*,\alpha}(\mathcal{T}_{E,0}f)(y)|^p\,d\sigma(y)\\
&\lesssim \sum_{Q\in\widetilde{\mathcal{F}}}\int_{Q} |\mathcal{N}_{*,\alpha}(\mathcal{T}_{E,0}f)(y)|^p\,d\sigma(y) + \int_{B^*_{Q_0}\cap E} |\mathcal{N}_{*,\alpha}(\mathcal{T}_{E,0}f)(y)|^p\,d\sigma(y)\\
&\lesssim \int_E |\mathcal{N}_{*,\alpha}(\mathcal{T}_{E,0}f)(y)|^p\,d\sigma(y) \lesssim \int_E |f(y)|^p\,d\sigma(y),
\end{aligned}$$

where in the last estimate we have employed Lemma A.16 and Remark A.18, and the implicit constants do not depend on ε_0.

We have thus established that $\mathcal{T}_{E,0} : L^p(E) \to L^p(E_\star)$ for every $1 < p < \infty$. Since $\mathcal{K}$ is odd, so is $\mathcal{K}_0$, and by duality we therefore obtain that

$$\mathcal{T}_{E_\star,0} : L^p(E_\star) \to L^p(E), \quad 1 < p < \infty. \tag{A.19}$$

Our goal is to show that $\mathcal{T}_{E_\star,0} : L^2(E_\star) \to L^2(E_\star)$ with bounds that do not depend on ε_0. Note that $\mathcal{T}_{E_\star,0}f$ is a continuous function for every $f \in L^2(E_\star)$ and therefore $\mathcal{T}_{E_\star,0}f|_{E_\star} = T_{E_\star,\varepsilon_0}f$ everywhere on $E_\star$.

We take $f \in L^2(E_\star)$ and write as before

$$\begin{aligned}
\int_{E_\star} |\mathcal{T}_{E_\star,0}f(x)|^2\,d\sigma_\star(x) &= \int_{E_\star\cap E} |\mathcal{T}_{E_\star,0}f(x)|^2\,d\sigma_\star(x) + \sum_{I\in\mathcal{W}_\Sigma}\int_{I\cap\Sigma} |\mathcal{T}_{E_\star,0}f(x)|^2\,d\sigma_\star(x)\\
&=: \mathrm{I} + \sum_{I\in\mathcal{W}_\Sigma} \mathrm{II}_I = \mathrm{I} + \mathrm{II}.
\end{aligned}\tag{A.20}$$

For I we use (A.19) with $p = 2$ and conclude the desired estimate

$$\mathrm{I} \le \int_{E_\star\cap E} |\mathcal{T}_{E_\star,0}f(x)|^2\,d\sigma_\star(x) \le \int_E |\mathcal{T}_{E_\star,0}f(x)|^2\,d\sigma(x) \le \int_{E_\star} |f(x)|^2\,d\sigma_\star(x). \tag{A.21}$$

We next fix $I \in \mathcal{W}_\Sigma$ and estimate each II_I. Let $M > 2$ be large parameter to be chosen below and set $\zeta_I = \ell(I)/M$, $\xi_I = M\ell(I)$. Write

$$\begin{aligned}
\mathcal{K}_0(x) &= \mathcal{K}_0(x)\Phi\Big(\frac{|x|}{\xi_I}\Big) + \mathcal{K}_0(x)\Big(\Phi\Big(\frac{|x|}{\zeta_I}\Big) - \Phi\Big(\frac{|x|}{\xi_I}\Big)\Big) + \mathcal{K}_0(x)\Big(1 - \Phi\Big(\frac{|x|}{\zeta_I}\Big)\Big)\\
&=: \mathcal{K}_{0,\xi_I}(x) + \mathcal{K}_{0,\zeta_I,\xi_I}(x) + \mathcal{K}_0^{\zeta_I}(x).
\end{aligned}\tag{A.22}$$

Corresponding to any of these kernels we respectively set the operators $\mathcal{T}_{E_\star,0,\xi_I}$, $\mathcal{T}_{E_\star,0,\zeta_I,\xi_I}$ and $\mathcal{T}^{\zeta_I}_{E_\star,0}$.

We start with $\mathcal{T}_{E_\star,0,\xi_I}$. Fix $x \in \Sigma\cap I$. Write $\Delta_{\star,I} = B(x,\xi_I)\cap E_\star$ and split $f = f_1 + f_2 := f 1_{\Delta_{\star,I}} + f 1_{E_\star\setminus\Delta_{\star,I}}$. Then we use Remark A.18, the fact $\operatorname{supp}\Phi \subset [1,\infty)$ and that $E_\star$ is ADR to easily obtain that for every $y \in Q_I$, with Q_I as in Lemma A.8,

$$\begin{aligned}
&|\mathcal{T}_{E_\star,0,\xi_I} f_1(x)| + |\mathcal{T}_{E_\star,0,\xi_I} f_1(y)| \\
&\qquad \le \int_{\Delta_{\star,I}} \left(|\mathcal{K}_0(x-z)| \Phi\left(\frac{|x-z|}{\xi_I}\right) + |\mathcal{K}_0(y-z)| \Phi\left(\frac{|y-z|}{\xi_I}\right)\right) |f(z)|\, d\sigma_\star(z) \\
&\qquad \lesssim \frac{1}{\xi_I^n} \int_{\Delta_{\star,I}} |f(y)|\, d\sigma_\star(z) \approx \fint_{\Delta_{\star,I}} |f(y)|\, d\sigma_\star(z) \le M_{E_\star} f(x),
\end{aligned} \tag{A.23}$$

where $M_{E_\star}$ is the Hardy–Littlewood maximal function on $E_\star$, and the constants are independent of ε_0 and I.

On the other hand, much as before we have that $\mathcal{K}_{0,\xi_I}$ is a Calderón–Zygmund kernel with constants that are uniform in ε_0 and ξ_I. Also, if M is taken large enough we have that $2|x-y| < M\ell(I) \le |x-z|$ for every $z \in E_\star \setminus \Delta_{\star,I}$, $x \in \Sigma \cap I$ and $y \in Q_I$. Therefore using standard Calderón–Zygmund estimates and the fact that $E_\star$ is ADR we obtain that for every and $y \in Q_I$

$$\begin{aligned}
|\mathcal{T}_{E_\star,0,\xi_I} f_2(x) - \mathcal{T}_{E_\star,0,\xi_I} f_2(y)| &\le \int_{E_\star \setminus \Delta_{\star,I}} |\mathcal{K}_{0,\xi_I}(x-z) - \mathcal{K}_{0,\xi_I}(y-z)|\, |f(z)|\, d\sigma_\star(z) \\
&\lesssim \int_{E_\star \setminus \Delta_{\star,I}} \frac{|x-y|}{|x-z|^{n+1}} |f(z)|\, d\sigma_\star(z) \lesssim_M M_{E_\star} f(x).
\end{aligned} \tag{A.24}$$

We next use (A.23) and (A.24) to conclude that

$$\begin{aligned}
\left| \mathcal{T}_{E_\star,0,\xi_I} f(x) - \fint_{Q_I} \mathcal{T}_{E_\star,0,\xi_I} f(y)\, d\sigma(y) \right| &\lesssim |\mathcal{T}_{E_\star,0,\xi_I} f_1(x)| + \fint_{Q_I} |\mathcal{T}_{E_\star,0,\xi_I} f_1(y)|\, d\sigma(y) \\
&\quad + \fint_{Q_I} |\mathcal{T}_{E_\star,0,\xi_I} f_2(x) - \mathcal{T}_{E_\star,0,\xi_I} f_2(y)|\, d\sigma(y) \lesssim M_{E_\star} f(x),
\end{aligned}$$

which in turn yields

$$\int_{\Sigma\cap I} \left| \mathcal{T}_{E_\star,0,\xi_I} f(x) - \fint_{Q_I} \mathcal{T}_{E_\star,0,\xi_I} f(y)\, d\sigma(y) \right|^2 d\sigma_\star(x) \lesssim \int_{\Sigma \cap I} M_{E_\star} f(x)^2\, d\sigma_\star(x). \tag{A.25}$$

We next introduce another operator

$$T_{E_\star,0,\xi_I} f(y) = \int_{z\in E_\star : |y-z| \ge \xi_I} \mathcal{K}_0(y-z) f(z)\, d\sigma_\star(z), \qquad y \in E.$$

We fix $x \in \Sigma \cap I$ and $y \in Q_I$. We first observe that, for M large enough, Remark A.18 and the ADR property for $E_\star$ imply that

$$\begin{aligned}
|\mathcal{T}_{E_\star,0,\xi_I} f(y) - T_{E_\star,0,\xi_I} f(y)| &\le \int_{E_\star} |\mathcal{K}_0(y-z)| \left| \Phi\left(\frac{|y-z|}{\xi_I}\right) - 1_{[1,\infty)}\left(\frac{|y-z|}{\xi_I}\right)\right| |f(z)|\, d\sigma_\star(z) \\
&\lesssim \frac{1}{\xi_I^n} \int_{z\in E_\star : |y-z| \le 2\xi_I} |f(z)|\, d\sigma_\star(z) \\
&\lesssim \frac{1}{\xi_I^n} \int_{z\in E_\star : |x-z| \le 3\xi_I} |f(z)|\, d\sigma_\star(z) \lesssim M_{E_\star} f(x).
\end{aligned}$$

On the other hand, we can introduce another decomposition

$$f = f_3 + f_4 := f 1_{B(y,\xi_I)\cap E_\star} + f 1_{E_\star \setminus B(y,\xi_I)},$$

and then for every $\bar{y} \in Q_I$

$$|T_{E_\star,0,\xi_I} f(y)| = |\mathcal{T}_{E_\star,0} f_4(y)| \le |\mathcal{T}_{E_\star,0} f_4(y) - \mathcal{T}_{E_\star,0} f_4(\bar{y})| + |\mathcal{T}_{E_\star,0} f_4(\bar{y})|$$
$$\le |\mathcal{T}_{E_\star,0} f_4(y) - \mathcal{T}_{E_\star,0} f_4(\bar{y})| + |\mathcal{T}_{E_\star,0} f(\bar{y})| + |\mathcal{T}_{E_\star,0} f_3(\bar{y})|. \tag{A.26}$$

We estimate each term in turn. We first observe that, for M large enough, $2|y - \bar{y}| < M\ell(I) \le |y - z|$ for every $z \in E_\star \setminus B(y, \xi_I)$ and $\bar{y} \in Q_I$. Therefore, using standard Calderón–Zygmund estimates and the fact that $E_\star$ is ADR, we obtain that for every and $\bar{y} \in Q_I$

$$|\mathcal{T}_{E_\star,0} f_4(y) - \mathcal{T}_{E_\star,0} f_4(\bar{y})| \le \int_{E_\star \setminus B(y,\xi_I)} |\mathcal{K}_0(y - z) - \mathcal{K}_0(\bar{y} - z)| \, |f(z)| \, d\sigma_\star(z)$$
$$\lesssim \int_{E_\star \setminus B(y,\xi_I)} \frac{|y - \bar{y}|}{|y - z|^{n+1}} |f(z)| \, d\sigma_\star(z) \lesssim M_{E_\star} f(x), \tag{A.27}$$

where we have used that, for M large enough, $x \in B(y, \xi_I/2)$. Fix $1 < p < 2$. We next average (A.26) on $\bar{y} \in Q_I$ and use (A.27) and (A.19) to obtain

$$\begin{aligned}
|T_{E_\star,0,\xi_I} f(y)| &\le \fint_{Q_I} \big(|\mathcal{T}_{E_\star,0} f_4(y) - \mathcal{T}_{E_\star,0} f_4(\bar{y})| + |\mathcal{T}_{E_\star,0} f(\bar{y})| + |\mathcal{T}_{E_\star,0} f_3(\bar{y})|\big) \, d\sigma(\bar{y}) \\
&\lesssim M_{E_\star} f(x) + M_E(\mathcal{T}_{E_\star,0} f)(y) + \sigma(Q_I)^{-\frac{1}{p}} \|\mathcal{T}_{E_\star,0} f_3\|_{L^p(E)} \\
&\lesssim M_{E_\star} f(x) + M_E(\mathcal{T}_{E_\star,0} f)(y) + \sigma(Q_I)^{-\frac{1}{p}} \|f_3\|_{L^p(E_\star)} \\
&\lesssim M_{E_\star} f(x) + M_E(\mathcal{T}_{E_\star,0} f)(y) + \left(\frac{1}{\ell(I)^n} \int_{B(y,\xi_I)\cap E_\star} |f(z)|^p \, d\sigma_\star(z)\right)^{\frac{1}{p}} \\
&\lesssim M_{E_\star,p} f(x) + M_E(\mathcal{T}_{E_\star,0} f)(y),
\end{aligned} \tag{A.28}$$

where M_E is the Hardy–Littlewood maximal function on E and we also write $M_{E_\star,p} f = M_{E_\star}(|f|^p)^{1/p}$. Note that this estimate holds for every $x \in \Sigma \cap I$ and for every $y \in Q_I$. Hence,

$$\int_{\Sigma\cap I} \left| \fint_{Q_I} \mathcal{T}_{E_\star,0,\xi_I} f(y) \, d\sigma(y) \right|^2 d\sigma_\star(x) \lesssim \int_{\Sigma\cap I} M_{E_\star,p} f(x)^2 \, d\sigma_\star(x) + \int_{Q_I} M_E(\mathcal{T}_{E_\star,0} f)(y)^2 \, d\sigma(y), \tag{A.29}$$

where we have used that $\sigma_\star(\Sigma \cap I) \lesssim \ell(I)^n$. We now gather (A.25) and (A.29) to obtain that for every $I \in \mathcal{W}_\Sigma$

$$\begin{aligned}
&\int_{\Sigma\cap I} |\mathcal{T}_{E_\star,0,\xi_I} f(x)|^2 \, d\sigma_\star(x) \\
&\quad\lesssim \int_{\Sigma\cap I} \left| \mathcal{T}_{E_\star,0,\xi_I} f(x) - \fint_{Q_I} \mathcal{T}_{E_\star,0,\xi_I} f(y) \, d\sigma(y) \right|^2 d\sigma_\star(x) + \int_{\Sigma\cap I} \left| \fint_{Q_I} \mathcal{T}_{E_\star,0,\xi_I} f(y) \, d\sigma(y) \right|^2 d\sigma_\star(x) \\
&\quad\lesssim \int_{\Sigma\cap I} M_{E_\star,p} f(x)^2 \, d\sigma_\star(x) + \int_{Q_I} M_E(\mathcal{T}_{E_\star,0} f)(y)^2 \, d\sigma(y).
\end{aligned} \tag{A.30}$$

We next consider $\mathcal{T}_{E_\star,0,\zeta_I,\xi_I}$. Note that for every $x \in \Sigma \cap I$ and $z \in E_\star$ we have

$$|\mathcal{K}_{0,\zeta_I,\xi_I}(z - x)| = |\mathcal{K}_0(z - x)| \left| \Phi\left(\frac{|z - x|}{\zeta_I}\right) - \Phi\left(\frac{|z - x|}{\xi_I}\right) \right| \lesssim \frac{1}{|z - x|^n} 1_{\zeta_I \le |z-x| \le 2\xi_I} \lesssim \frac{1}{\zeta_I^n} 1_{|z-x| \le 2\xi_I},$$

and therefore

$$\int_{\Sigma\cap I} |T_{E_\star,0,\zeta_I,\xi_I} f(x)|^2\, d\sigma_\star(x) \lesssim \int_{\Sigma\cap I}\bigg(\frac{1}{\zeta_I^n}\int_{B(x,2\xi_I)\cap E_\star} |f(z)|\, d\sigma_\star(z)\bigg)^2 d\sigma_\star(x)$$
$$\lesssim_M \int_{\Sigma\cap I} M_{E_\star} f(x)^2\, d\sigma_\star(x). \tag{A.31}$$

Let us finally address $\mathcal{T}^{\zeta_I}_{E_\star,0}$. Observe first that

$$\mathcal{K}_0^{\zeta_I}(\,\cdot\,) = \mathcal{K}(\,\cdot\,)\Phi\bigg(\frac{|\cdot|}{\varepsilon_0}\bigg)\bigg(1-\Phi\bigg(\frac{|\cdot|}{\zeta_I}\bigg)\bigg).$$

We consider three different cases.

<u>Case 1</u>: $\zeta_I \le \varepsilon_0/2$. We have that $\mathcal{K}_0^{\zeta_I} \equiv 0$ and thus $\mathcal{T}^{\zeta_I}_{E_\star,0} \equiv 0$.

<u>Case 2</u>: $\varepsilon_0/2 < \zeta_I \le 2\varepsilon_0$. In this case for every $x \in \Sigma\cap I$ and $z \in E_\star$

$$|\mathcal{K}_0^{\zeta_I}(x-z)| \lesssim \frac{1}{|x-z|^n} 1_{\varepsilon_0 \le |z-x| \le 2\zeta_I} \lesssim \frac{1}{\varepsilon_0^n} 1_{|z-x|\le 4\varepsilon_0},$$

and therefore

$$\int_{\Sigma\cap I} |\mathcal{T}^{\zeta_I}_{|!E_\star,0} f(x)|^2\, d\sigma_\star(x) \lesssim \int_{\Sigma\cap I}\bigg(\frac{1}{\varepsilon_0^n}\int_{B(x,4\varepsilon_0)\cap E_\star} |f(z)|\, d\sigma_\star(z)\bigg)^2 d\sigma_\star(x)$$
$$\lesssim \int_{\Sigma\cap I} M_{E_\star} f(x)^2\, d\sigma_\star(x), \tag{A.32}$$

where the implicit constants are independent of ε_0 and ζ_I.

<u>Case 3</u>: $\zeta_I > 2\varepsilon_0$. In this case $\mathcal{T}^{\zeta_I}_{E_\star,0} f$ is a double truncated integral whose smooth Calderón–Zygmund kernel $\mathcal{K}_0^{\zeta_I}$ is odd, smooth in $\mathbb{R}^{n+1}$ and satisfies the estimates (A.12), (A.13). with uniform bounds (i.e., independent of ε_0 and ζ_I). Fix $z_I \in \Sigma\cap I$ and notice that if $x \in \Sigma\cap I$ and $z \in B(x, 2\zeta_I)\cap E_\star$ then, taking M large enough, we have

$$|z-z_I| \le |z-x| + |x-z_I| \le 2\zeta_I + \operatorname{diam}(I) = \frac{\ell(I)}{2M} + \operatorname{diam}(I) < \frac{3}{2}\operatorname{diam}(I)$$

and therefore the fact that $\operatorname{supp}\mathcal{K}_0^{\zeta_I} \subset B(0, 2\zeta_I)$ immediately gives $\mathcal{T}^{\zeta_I}_{E_\star,0} f(x) = \mathcal{T}^{\zeta_I}_{E_\star,0}(f 1_{\tilde{\Delta}_{\star,I}})(x)$, where $\tilde{\Delta}_{\star,I} := \widetilde{B}_{\star,I}\cap E_\star := B(z_I, 2\operatorname{diam}(I))\cap E_\star$. Note that (2.5) yields

$$4\operatorname{diam}(I) \le \operatorname{dist}(4I, E) \le \operatorname{dist}(z_I, E) \le \operatorname{dist}(\widetilde{B}_{\star,I}, E) + 2\operatorname{diam}(I)$$

and therefore $\operatorname{dist}(\widetilde{B}_{\star,I}, E) \ge 2\operatorname{diam}(I)$. This implies that $3\widetilde{B}_{\star,I}/2 \subset \mathbb{R}^{n+1}\setminus E$. Also if $J \in \mathcal{W}$ satisfies that $J^*\cap\widetilde{B}_{\star,I} \ne \varnothing$ we can easily check that $\ell(I) \approx \ell(J)$ and $\operatorname{dist}(I, J) \lesssim \ell(I)$. This implies that only a bounded number of J's have the property that J^* intersects $\widetilde{B}_{\star,I}$. We recall that $\Sigma = E_\star\setminus E$ is a union of portion of faces of fattened Whitney cubes J^*. Thus we have

$$\tilde{\Delta}_{\star,I} \subset \bigcup_{m=1}^{M_0} F_{m,I},$$

where M_0 is a uniform constant and each $F_{m,I}$ is either a portion of a face of some J^*, or else $F_{m,I} = \varnothing$ (since M_0 is not necessarily equal to the number of faces, but is rather an upper bound for the number of faces.) Note also that $I \subset \widetilde{B}_{\star,I}$ and therefore we also have that

$$\Sigma \cap I \subset \bigcup_{m=1}^{M_0} F_{m,I}.$$

Thus

$$\begin{aligned}\int_{\Sigma\cap I} |\mathcal{T}^{\zeta_I}_{E_\star,0} f(x)|^2 \, d\sigma_\star(x) &= \int_{\Sigma\cap I} |\mathcal{T}^{\zeta_I}_{E_\star,0}(f 1_{\tilde{\Delta}_{\star,I}})(x)|^2 \, d\sigma_\star(x) \\ &\lesssim \sum_{1\le m,m'\le M_0} \int_{F_{m,I}} |\mathcal{T}^{\zeta_I}_{E_\star,0}(f 1_{F_{m',I}})(x)|^2 \, d\sigma_\star(x).\end{aligned}$$

In the case $m = m'$, we take the hyperplane $\mathcal{H}_{m,I}$ with $F_{m,I} \subset \mathcal{H}_{m,I}$ and then

$$\begin{aligned}\int_{F_{m,I}} |\mathcal{T}^{\zeta_I}_{E_\star,0}(f 1_{F_{m,I}})(x)|^2 \, d\sigma_\star(x) &\le \int_{\mathcal{H}_{m,I}} |\mathcal{T}^{\zeta_I}_{\mathcal{H}_{m,I},0}(f 1_{F_{m,I}})(x)|^2 \, dH^n(x) \\ &\lesssim \int_{F_{m,I}} |f(x)|^2 \, dH^n(x) = \int_{F_{m,I}} |f(x)|^2 \, d\sigma_\star(x),\end{aligned}$$

where, after a rotation, we have used the L^2 bounds of Calderón–Zygmund operators with nice kernels on $\mathbb{R}^n$. For $m \ne m'$ we consider two cases: either $\operatorname{dist}(F_{m,I}, F_{m',I}) \approx \ell(I)$ or $\operatorname{dist}(F_{m,I}, F_{m',I}) \ll \ell(I)$. In the first scenario, using that $\mathcal{K}_0^{\zeta_I}$ satisfies (A.12) uniformly we obtain that

$$\begin{aligned}\int_{F_{m,I}} |\mathcal{T}^{\zeta_I}_{E_\star,0}(f 1_{F_{m',I}})(x)|^2 \, d\sigma_\star(x) &\lesssim \int_{F_{m,I}} \left(\int_{F_{m',I}} \frac{1}{|x-z|^n} |f(z)| \, d\sigma_\star(z) \right)^2 d\sigma_\star(x) \\ &\lesssim \int_{F_{m,I}} \left(\frac{1}{\ell(I)^n} \int_{B(x,C\ell(I))\cap E_\star} |f(z)| \, d\sigma_\star(z) \right)^2 d\sigma_\star(x) \\ &\lesssim \int_{F_{m,I}} M_{E_\star} f(x)^2 \, d\sigma_\star(x).\end{aligned}$$

Finally if $\operatorname{dist}(F_{m,I}, F_{m',I}) \ll \ell(I)$, we have that $F_{m,I}$ and $F_{m',I}$ are contained in respective faces which either lie in the same hyperplane, or else meet at an angle of $\frac{\pi}{2}$. In the first case we may proceed as in the case $m = m'$. In the second case, after a possible rotation of coordinates, we may view $F^j_m \cup F^j_{m'}$ as lying in a Lipschitz graph with Lipschitz constant 1, so that we may estimate $\mathcal{T}^{\zeta_I}_{E_\star,0}$ using an extension of the Coifman–McIntosh–Meyer theorem:

$$\int_{F_{m,I}} |\mathcal{T}^{\zeta_I}_{E_\star,0}(f 1_{F_{m',I}})(x)|^2 \, d\sigma_\star(x) \lesssim \int_{F_{m',I}} |f(x)|^2 \, d\sigma_\star(x).$$

Gathering all the possible cases we may conclude that

$$\begin{aligned}\int_{\Sigma\cap I} |\mathcal{T}^{\zeta_I}_{E_\star,0} f(x)|^2 \, d\sigma_\star(x) &\lesssim \sum_{1\le m\le M_0} \int_{F_{m,I}} M_{E_\star} f(x)^2 \, d\sigma_\star(x) \\ &\lesssim \sum_{I'\in\mathcal{W}_\Sigma : I'\cap\tilde{\Delta}_{\star,I}\ne\varnothing} \int_{I'\cap\Sigma} M_{E_\star} f(x)^2 \, d\sigma_\star(x). \qquad \text{(A.33)}\end{aligned}$$

We now gather (A.30), (A.31) and (A.33) to get the following estimate for S_I after using (A.22):

$$\begin{aligned}
\mathrm{II}_I &= \int_{\Sigma\cap I} |\mathcal{T}_{E_\star,0} f(x)|^2\, d\sigma_\star(x) \\
&\lesssim \int_{\Sigma\cap I} |\mathcal{T}_{E_\star,0,\xi_I} f(x)|^2\, d\sigma_\star(x) + \int_{\Sigma\cap I} |\mathcal{T}_{E_\star,0,\zeta_I,\xi_I} f(x)|^2\, d\sigma_\star(x) + \int_{\Sigma\cap I} |\mathcal{T}^{\zeta_I}_{E_\star,0}|^2\, d\sigma_\star(x) \\
&\lesssim \int_{\Sigma\cap I} M_{E_\star,p} f(x)^2\, d\sigma_\star(x) + \int_{Q_I} M_E(\mathcal{T}_{E_\star,0} f)(y)^2\, d\sigma(y) \\
&\qquad\qquad + \sum_{I'\in\mathcal{W}_\Sigma : I'\cap\tilde{\Delta}_{\star,I}\neq\varnothing} \int_{I'\cap\Sigma} M_{E_\star} f(x)^2\, d\sigma_\star(x).
\end{aligned} \tag{A.34}$$

Note that since $1 < p < 2$ we have

$$\sum_{I\in\mathcal{W}_\Sigma} \int_{\Sigma\cap I} M_{E_\star,p} f(x)^2\, d\sigma_\star(x) \le \int_{E_\star} M_{E_\star,p} f(x)^2\, d\sigma_\star(x) \lesssim \int_{E_\star} |f(x)|^2\, d\sigma_\star(x). \tag{A.35}$$

On the other hand, recalling that $\widetilde{\mathcal{F}} = \mathcal{F}^* \cup \mathcal{F}^*_{||}$ is comprised of pairwise disjoint cubes, Lemmas A.4 and A.8 then imply that

$$\begin{aligned}
&\sum_{I\in\mathcal{W}_\Sigma} \int_{Q_I} M_E(\mathcal{T}_{E_\star,0} f)(y)^2\, d\sigma(y) \\
&\qquad = \sum_{Q\in\widetilde{\mathcal{F}}} \sum_{I\in\mathcal{W}_{\Sigma,Q}} \int_{Q_I} M_E(\mathcal{T}_{E_\star,0} f)(y)^2\, d\sigma(y) + \sum_{I\in\mathcal{W}_\Sigma^{\top}} \int_{Q_I} M_E(\mathcal{T}_{E_\star,0} f)(y)^2\, d\sigma(y) \\
&\qquad \lesssim \sum_{Q\in\widetilde{\mathcal{F}}} \int_{Q} M_E(\mathcal{T}_{E_\star,0} f)(y)^2\, d\sigma(y) + \int_{B^*_{Q_0}\cap E} M_E(\mathcal{T}_{E_\star,0} f)(y)^2\, d\sigma(y) \\
&\qquad \lesssim \int_{E} M_E(\mathcal{T}_{E_\star,0} f)(y)^2\, d\sigma(y) \lesssim \int_{E} |\mathcal{T}_{E_\star,0} f(y)|^2\, d\sigma(y) \lesssim \int_{E_\star} |f(y)|^2\, d\sigma_\star(y),
\end{aligned} \tag{A.36}$$

where in the last estimate we have used (A.19) with $p = 2$.

Finally, by the nature of the Whitney boxes (see (2.5)), we have that the family $\{2I\}_{I\in\mathcal{W}}$ has the bounded overlap property and therefore

$$\sum_{I\in\mathcal{W}_\Sigma} \sum_{I'\in\mathcal{W}_\Sigma : I'\cap\tilde{\Delta}_{\star,I}\neq\varnothing} 1_{\Sigma\cap I'} \lesssim \sup_{I'\in\mathcal{W}_\Sigma} \#\{I\in\mathcal{W}_\Sigma : I'\cap\Delta_{\star,I}\neq\varnothing\},$$

which we claim that is uniformly bounded. Indeed, fix $I'\in\mathcal{W}_\Sigma$ and let $I_1, I_2\in\mathcal{W}_\Sigma$ with $I'\cap\tilde{\Delta}_{\star,I_1}\neq\varnothing$ and $I'\cap\tilde{\Delta}_{\star,I_2}\neq\varnothing$. Recall that $\operatorname{dist}(\widetilde{B}_{\star,I}, E) \ge 2\operatorname{diam}(I)$ with $\widetilde{B}_{\star,I} = B(z_I, 2\operatorname{diam}(I))$ and $z_I\in I\cap\Sigma$. This implies that $\ell(I_1)\approx\ell(I')\approx\ell(I_2)$ and also $\operatorname{dist}(I_1, I_2)\lesssim\ell(I_1)$. This easily gives our claim. Using this we conclude that

$$\sum_{I\in\mathcal{W}_\Sigma} \sum_{I'\in\mathcal{W}_\Sigma : I'\cap\tilde{\Delta}_{\star,I}\neq\varnothing} \int_{I'\cap\Sigma} M_{E_\star} f(x)^2\, d\sigma_\star(x) \lesssim \int_{E_\star} M_{E_\star} f(x)^2\, d\sigma_\star(x) \lesssim \int_{E_\star} |f(x)|^2\, d\sigma_\star(x). \tag{A.37}$$

We now combine (A.34), (A.35), (A.36) and (A.37) to obtain that

$$\mathrm{II} = \sum_{I\in\mathcal{W}_\Sigma} \mathrm{II}_I \lesssim \int_{E_\star} |f(x)|^2\, d\sigma_\star(x).$$

This, (A.20), and (A.21) give as desired that

$$\int_{E_\star} |\mathcal{T}_{E_\star,0} f(x)|^2 \, d\sigma_\star(x) \lesssim \int_{E_\star} |f(x)|^2 \, d\sigma_\star(x),$$

and the implicit constant does not depend on ε_0. Hence, $\mathcal{T}_{E_\star,0} : L^2(E_\star) \to L^2(E_\star)$ with bounds that do not depend on ε_0. Since $\mathcal{T}_{E_\star,0} f$ is a continuous function for every $f \in L^2(E_\star)$, we have that $\mathcal{T}_{E_\star,0} f|_{E_\star} = T_{E_\star,\varepsilon_0} f$ everywhere on $E_\star$. Thus, all these show that $T_{E_\star,0} : L^2(E_\star) \to L^2(E_\star)$ uniformly in ε. This in turn gives, by the aforementioned result of [David and Semmes 1991], that $E_\star$ is UR as desired, and the proof is complete. □

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OPTIMAL REGULARITY AND THE LIOUVILLE PROPERTY FOR STABLE SOLUTIONS TO SEMILINEAR ELLIPTIC EQUATIONS IN $\mathbb{R}^n$ WITH $n \geq 10$

FA PENG, YI RU-YA ZHANG AND YUAN ZHOU

Let $0 \leq f \in C^{0,1}(\mathbb{R})$. Given a domain $\Omega \subset \mathbb{R}^n$, we prove that any stable solution to the equation $-\Delta u = f(u)$ in Ω satisfies

- a BMO interior regularity, when $n = 10$,
- a Morrey $M^{p_n, 4+2/(p_n-2)}$ interior regularity, when $n \geq 11$, where

$$p_n = \frac{2(n - 2\sqrt{n-1} - 2)}{n - 2\sqrt{n-1} - 4}.$$

This result is optimal as hinted by, e.g., Brezis and Vázquez (1997), Cabré and Capella (2006), and Dupaigne (2011), and answers an open question raised by Cabré, Figalli, Ros-Oton and Serra (2020). As an application, we show a sharp Liouville property: any stable solution $u \in C^2(\mathbb{R}^n)$ to $-\Delta u = f(u)$ in $\mathbb{R}^n$ satisfying the growth condition

$$|u(x)| = \begin{cases} o(\log|x|) & \text{as } |x| \to +\infty, \quad \text{when } n = 10, \\ o(|x|^{-n/2+\sqrt{n-1}+2}) & \text{as } |x| \to +\infty, \quad \text{when } n \geq 11, \end{cases}$$

must be a constant. This extends the well-known Liouville property for radial stable solutions obtained by Villegas (2007).

1. Introduction

Let Ω be a bounded domain of $\mathbb{R}^n$ with $n \geq 2$. Given any local Lipschitz function $f : \mathbb{R} \to \mathbb{R}$ (for short $f \in C^{0,1}(\mathbb{R})$), we consider the semilinear elliptic equation

$$-\Delta u = f(u) \quad \text{in } \Omega, \tag{1-1}$$

which is the Euler–Lagrange equation for the energy functional

$$\mathcal{E}(u) := \int_\Omega \Big(\frac{1}{2}|Du|^2 - F(u)\Big)\, dx, \tag{1-2}$$

where $F(t) = \int_0^t f(s)\, ds$ for $t \in \mathbb{R}$. A function $u \in W^{1,2}(\Omega)$ is called a weak solution to (1-1) if $f(u) \in L^1_{\mathrm{loc}}(\Omega)$ and

$$\int_\Omega Du \cdot D\xi \, dx = \int_\Omega f(u)\xi \, dx \quad \text{for all } \xi \in C_c^\infty(\Omega),$$

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that is, u is a critical point of the energy functional $\mathcal{E}$. We say that a weak solution u is *stable* in Ω if $f'_-(u) \in L^1_{\mathrm{loc}}(\Omega)$ and

$$\int_\Omega f'_-(u)\xi^2\,dx \le \int_\Omega |D\xi|^2\,dx \quad \text{for all } \xi \in C_c^\infty(\Omega), \tag{1-3}$$

that is, the second variation of the energy functional $\mathcal{E}$ is nonnegative. Here and below,

$$f'_-(t) = \liminf_{h\to 0} \frac{f(t+h)-f(t)}{h} \quad \text{for all } t \in \mathbb{R},$$

and note that $f'_-(t) = f'(t)$ whenever $f \in C^1(\mathbb{R})$.

The study of stable solutions to semilinear elliptic equations can be traced to the seminal paper [Crandall and Rabinowitz 1975]. The regularity of stable solutions provides an important way to understand the regularity of the extremal solution $u^\star$ to the Gelfand-type problem

$$\begin{cases} -\Delta u = \lambda^\star f(u) & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega \end{cases} \tag{1-4}$$

for some positive constant $\lambda^\star > 0$. We refer to [Brezis 2003; Cabré 2017; Gelfand 1963] for a comprehensive analysis of (1-4) and related topics. Note that the extremal solution $u^\star$ can be approximated by stable solutions $\{u_\lambda\}_{\lambda<\lambda^\star}$; see, e.g., [Dupaigne 2011].

In dimension $n \le 9$, Brezis [2003] introduced an open problem: is the extremal solution $u^\star$ to (1-4) bounded for some f and Ω? Since $u^\star$ is approximated by stable solutions $\{u_\lambda\}_{\lambda<\lambda^\star}$, it suffices to establish some a priori bound for stable solutions. In recent years, there were several strong efforts to study regularity for stable solutions and hence for Brezis' open problem. In particular, a positive answer was given by Nedev [2000], when $n \le 3$, and by Cabré [2010], when $n = 4$ (see also [Cabré 2019] for an alternative proof).

Very recently, Cabré, Figalli, Ros-Oton and Serra [Cabré et al. 2020] provided a complete answer to Brezis' open problem when $f \ge 0$ based on certain Morrey-type estimates for $n \ge 3$. Throughout this paper, for $p \in [1,\infty)$ and $\beta \in (0,n)$, we define the Morrey norm as

$$\|w\|_{M^{p,\beta}(\Omega)} := \sup_{y\in\Omega,\, r>0} \left(r^{\beta-n} \int_{\Omega\cap B_r(y)} |w|^p\,dx \right)^{1/p} < \infty, \tag{1-5}$$

where $B_r(y)$ denotes the ball with center y and radius $r > 0$. We simply write B_r when the center of the ball is at the origin. In addition, following the convention, we denote by $C(a,b,\dots)$ a positive constant depending only on the parameters $a, b, \dots$.

In dimension $n \ge 10$, in particular, [Cabré et al. 2020, Theorem 1.9] established the following regularity of stable solutions to (1-1).

Theorem 1.1 [Cabré et al. 2020]. *Suppose that $f \in C^{0,1}(\mathbb{R})$ is nonnegative. If $u \in C^2(B_1)$ is a stable solution to (1-1) in B_1, then*

$$\|u\|_{M^{p,2+4/(p-2)}(B_{1/2})} \le C(n,p)\|u\|_{L^1(B_1)} \quad \textit{for every } p < p_n, \tag{1-6}$$

where

$$p_n := \begin{cases} \infty & \textit{if } n = 10, \\ \dfrac{2(n-2\sqrt{n-1}-2)}{n-2\sqrt{n-1}-4} & \textit{if } n \geq 11. \end{cases} \tag{1-7}$$

Moreover, suppose additionally that f is nondecreasing and Ω is a bounded domain of class C^3. If $u \in C^2(\Omega) \cap C^0(\overline{\Omega})$ is a stable solution to (1-1) in Ω with boundary $u = 0$ on $\partial\Omega$, then

$$\|u\|_{M^{p,2+4/(p-2)}(\Omega)} \leq C(n, p, \Omega)\|u\|_{L^1(\Omega)} \quad \textit{for every } p < p_n. \tag{1-8}$$

We remark that the exponent $n - 2\sqrt{n-1} - 4$ changes sign when $n = 10$, which has already appeared in, e.g., [Gui et al. 1992].

However, for the endpoint case $p = p_n$, [Cabré et al. 2020, Section 1.3] pointed out that it is an open question whether (1-6) holds.

As hinted at by earlier results in the radial symmetric case [Cabré and Capella 2006], when $n = 10$, instead of $L^\infty = M^{\infty,2}$, a more suitable space to consider is a class of functions with bounded mean oscillations (BMO space), as remarked therein. Indeed, $u(x) = -2\log|x|$ is a stable solution to (1-1) in B_1, with $f(u) = 2(n-2)e^u$. Obviously, $u \in \mathrm{BMO}(B_1)$ but $u \notin L^\infty(B_1)$. Here and below, the BMO norm is defined as

$$\|u\|_{\mathrm{BMO}(\Omega)} := \sup_{y\in\Omega, r>0} \inf_{c\in\mathbb{R}} \fint_{\Omega\cap B_r(y)} |u(x) - c|\, dx,$$

where, $\fint_E v\, dx$ denotes the integral average of v on a measurable set E.

On the other hand, when $n \geq 11$, also hinted at by the results in [Cabré and Capella 2006], the range $p \leq p_n$ is the best possible in (1-6). Besides, it was proven in [Brezis and Vázquez 1997] that the function $u(x) = |x|^{-2/(q_n-1)} - 1$ is the extremal solution to

$$-\Delta u = \lambda^\star(1+u)^{q_n} \text{ in } B_1, \quad u = 0 \text{ on } \partial B_1, \tag{1-9}$$

with

$$\lambda^\star = \frac{2}{q_n} \quad \text{and} \quad q_n := \frac{n-2\sqrt{n-1}}{n-2\sqrt{n-1}-4}.$$

We note that q_n here is exactly the standard exponent in [Joseph and Lundgren 1973]. It is easy to see that $u \in M^{p,2+4/(p-2)}(B_{1/2})$ if and only if $p \leq p_n$. Recall that, by [Dupaigne 2011, Section 3.2.2], such an extremal solution can be approximated by stable solutions. We also refer to, e.g., [Farina 2007] for some earlier work on Lane–Emden equations, which also hints at the optimality of our results.

The first main purpose of this paper is to establish the following regularity at the endpoint p_n for stable solutions to (1-1), when $n \geq 10$, and then answer the above open question in [Cabré et al. 2020].

Theorem 1.2. *Suppose $f \in C^{0,1}(\mathbb{R})$ is nonnegative. For any stable solution $u \in C^2(B_1)$ to (1-1) in B_1, when $n = 10$, we have*

$$\|u\|_{\mathrm{BMO}(B_{1/2})} \leq C(n)\|u\|_{L^1(B_1)}, \tag{1-10}$$

and when $n \geq 11$, we have

$$\|u\|_{M^{p_n,2+4/(p_n-2)}(B_{1/2})} \leq C(n)\|u\|_{L^1(B_1)}. \tag{1-11}$$

Moreover, suppose additionally that f is nondecreasing and Ω is a bounded smooth convex domain. For any positive stable solution $u \in C^2(\overline{\Omega})$ to (1-1) with boundary $u = 0$ on $\partial\Omega$, when $n = 10$, we have

$$\|u\|_{\mathrm{BMO}(\Omega)} \le C(n, \Omega)\|u\|_{L^1(\Omega)}, \tag{1-12}$$

and when $n \ge 11$, we have

$$\|u\|_{M^{p_n, 2+4/(p_n-2)}(\Omega)} \le C(n, \Omega)\|u\|_{L^1(\Omega)}. \tag{1-13}$$

As a direct consequence of the above a priori estimates, we have the following result for stable solutions in $W^{1,2}$.

Corollary 1.3. *Suppose that $\Omega \subset \mathbb{R}^n$ is a bounded smooth convex domain and that $f \in C^{0,1}(\mathbb{R})$ is nonnegative, nondecreasing, convex, and satisfies $f(t)/t \to +\infty$ as $t \to +\infty$. For any stable solution $u \in W_0^{1,2}(\Omega)$ to (1-1) with boundary $u = 0$ on $\partial\Omega$, we have (1-12) when $n = 10$, and (1-13) when $n \ge 11$.*

Remark 1.4. (i) While writing this paper, we learned via personal communication that Figalli and Mayboroda have independently proved (1-10) in Theorem 1.2 with $n = 10$ via a similar argument.

(ii) In Theorem 1.2 and Corollary 1.3 we only consider bounded smooth convex domains so as to avoid technical discussions on the boundary estimate. We believe that after suitable modifications, it is possible to relax this assumption to bounded domains of C^3 class, as in [Cabré et al. 2020].

As an application of Theorem 1.2, we prove the following Liouville property for stable solutions to the equation

$$-\Delta u = f(u) \quad \text{in } \mathbb{R}^n \tag{1-14}$$

for $f \in C^{0,1}(\mathbb{R}^n)$.

Theorem 1.5. *Let $n \ge 10$ and $0 \le f \in C^{0,1}_{\mathrm{loc}}(\mathbb{R})$. Suppose that $u \in C^2(\mathbb{R}^n)$ is a nonconstant stable solution to (1-14) in $\mathbb{R}^n$.*

If u is nonconstant, then

$$\fint_{B_{4R}\setminus B_R} |u(x)|\,dx \ge \begin{cases} c\log R & \text{for all } R \ge R_0, \quad \text{if } n = 10, \\ cR^{-n/2+2+\sqrt{n-1}} & \text{for all } R \ge R_0, \quad \text{if } n \ge 11, \end{cases} \tag{1-15}$$

for some $R_0 \ge 2$ and $c > 0$.

In particular, if u satisfies the growth condition

$$|u(x)| = \begin{cases} o(\log|x|) & \text{as } |x| \to +\infty, \quad \text{when } n = 10, \\ o(|x|^{-n/2+2+\sqrt{n-1}}) & \text{as } |x| \to +\infty, \quad \text{when } n \ge 11, \end{cases} \tag{1-16}$$

then u must be a constant.

This problem has attracted a lot of attention in the literature. First of all, for radial stable solutions, Villegas [2007] obtained the following sharp Liouville property based on the monotone property by Cabré and Capella [2004]; see also [Dupaigne 2011; Villegas 2007].

Theorem 1.6 [Villegas 2007]. *Let $n \geq 2$ and $f \in C^1(\mathbb{R})$. Suppose that $u \in C^2(\mathbb{R}^n)$ is a radial stable solution to (1-14).*

If u is not constant, then

$$|u(x)| \geq \begin{cases} M \log|x| & \text{whenever } |x| \geq r_0, \quad \text{when } n = 10, \\ M|x|^{-n/2+\sqrt{n-1}+2} & \text{whenever } |x| \geq r_0, \quad \text{when } n \neq 10, \end{cases} \tag{1-17}$$

for some $M > 0$ and $r_0 \geq 10$.

In particular, if u satisfies the growth condition (1-16), then u must be a constant.

Note that for radial stable solutions $u(x)$, the condition (1-15) is equivalent to (1-17). Indeed, by [Villegas 2007], $u(r) = u(re_1)$ is always monotone, and hence

$$\min\{|u(4r)|, |u(r)|\} \leq \fint_{B_{4r}\setminus B_r} |u(x)|\,dx \leq \max\{|u(4r)|, |u(r)|\} \quad \text{for all } r > 0,$$

which implies the equivalence between (1-15) and (1-17).

Let $\beta_n = -\frac{1}{2}n + 2 + \sqrt{n-1}$. Then $\beta_n < 0$ when $n \geq 11$, and $\beta_n > 0$ when $n \leq 9$. The sharpness of Theorem 1.6 (and also Theorem 1.5) is demonstrated in the following sense by Villegas [2007] (with a slight modification at $n = 10$).

(i) When $n \neq 10$, the radial smooth function $(1+|x|^2)^{\beta_n/2}$ is a stable solution to the equation $-\Delta u = f_{\beta_n}(u)$ in $\mathbb{R}^n$, where, when $n \geq 11$,

$$f_{\beta_n}(s) := \begin{cases} 0 & \text{if } s \leq 0, \\ \beta_n(\beta_n - 2)s^{1-4/\beta_n} - \beta_n(\beta_n + n - 2)s^{1-2/\beta_n} & \text{if } s > 0, \end{cases}$$

and, when $n \leq 9$,

$$f_{\beta_n}(s) := \begin{cases} \beta_n(\beta_n - 2)s^{1-4/\beta_n} - \beta_n(\beta_n + n - 2)s^{1-2/\beta_n} & \text{if } s \geq 1, \\ -(\beta_n - 2)(n+2)(s-1) - n\beta_n & \text{if } s < 1. \end{cases}$$

See [Villegas 2007, Example 3.1] for details. Note that, when $n \geq 11$, by $\beta_n < 0$ and $\beta_n + n - 2 > 0$, we have $f_{\beta_n} \geq 0$ in $\mathbb{R}$, while, when $n \leq 9$, we have that $f_{\beta_n} \leq 0$ in $\mathbb{R}$.

(ii) When $n = 10$, the radial smooth function $-\frac{1}{2}\log(1+|x|^2)$ is a stable solution to the equation $-\Delta u = f(u)$ in $\mathbb{R}^n$, where $f(s) = (n-2)e^{2s} + 2e^{4s} \geq 0$ in $\mathbb{R}$. This is a slight modification of [Villegas 2007, Example 3.1] with $n = 10$. See the Appendix for details.

For general (nonradial) stable solutions $u \in C^2(\mathbb{R}^n)$ to $-\Delta u = f(u)$ in $\mathbb{R}^n$, it is then natural to ask if certain Liouville properties similar to Theorem 1.6 hold. Namely, when f satisfies certain regularity assumptions,

- if u satisfies (1-16), then is it necessary that u is a constant?
- if u is nonconstant, is it possible to give some sharp lower bound for $|u|$ toward ∞?

Suppose that $0 \leq f \in C^1(\mathbb{R})$ and $u \in C^2(\mathbb{R}^n)$ is a stable solution to (1-14). When $n \leq 4$, Dupaigne and Farina [2023] proved that if $|u|$ is bounded, then u must be a constant. Recently, with the aid of [Cabré et al. 2020], Dupaigne and Farina [2022] showed that if $n \leq 9$ and $u(x) \geq -C[1 + \log|x|]^\gamma$ for some

$\gamma \geq 1$ and $C > 0$, or if $n = 10$ and $u \geq -C$ for some constant $C > 0$, then u must be a constant. When $n \geq 10$, our result Theorem 1.5 finally answers the two questions above.

Ideas of the proofs. We sketch the ideas to prove Theorems 1.2 and 1.5. All of them heavily rely on the following decay estimate on the Dirichlet energy.

Lemma 1.7. *Let $n \geq 10$ and $f \in C^{0,1}(\mathbb{R})$. For any $y \in \mathbb{R}^n$ and $t > 0$, if $u \in C^2(B_{2t}(y))$ is a stable solution to (1-1) in $B_{2t}(y)$, one has*

$$\left(\frac{r}{t}\right)^{-2(1+\sqrt{n-1})} \int_{B_r(y)} |Du|^2\, dx \leq C(n) \int_{B_t(y)\setminus B_{t/2}(y)} |Du|^2\, dx \quad \textit{for all } r \leq \frac{t}{2}. \tag{1-18}$$

See Section 2 for the proof of Lemma 1.7; the key point is that we take a suitable test function in a celebrated lemma of [Cabré et al. 2020] (see Lemma 2.1 below). One may compare it with [Cabré et al. 2020, Lemma 2.1] in the case where $3 \leq n \leq 9$.

We also recall the following lemma, which was essentially established in [Cabré et al. 2020, Lemma A.2 and Proposition 2.5] together with the proofs therein. For the convenience of the reader, we give a sketch of the proof at the beginning of Section 3.

Lemma 1.8. *Let $0 \leq f \in C^{0,1}(\mathbb{R})$. For any stable solution $u \in C^2(B_{2t}(y))$ to (1-1) in $B_{2t}(y)$, one has*

$$\left(\int_{B_{t/2}(y)} |Du|^2\, dx\right)^{1/2} \leq C(n) t^{-n/2} \int_{B_t(y)} |Du|\, dx \tag{1-19}$$

and

$$\int_{B_{t/2}(y)} |Du|\, dx \leq C(n) t^{-1} \int_{B_t(y)} |u|\, dx. \tag{1-20}$$

Applying Lemma 1.7, Lemma 1.8 and some known boundary estimate, we are able to prove Theorem 1.2 and Corollary 1.3. This is clarified in Section 3.

In order to prove Theorem 1.5, an auxiliary and crucial proposition is shown in Section 4, which is specifically applied in the case $n = 10$.

Proposition 1.9. *Let $n \geq 3$. Suppose that $u \in W^{1,1}_{\mathrm{loc}}(\mathbb{R}^n)$ is superharmonic, that is, $-\Delta u \geq 0$ in $\mathbb{R}^n$ in the distributional sense. For any $0 < r < R < \infty$, we have*

$$\int_{B_R\setminus B_r} |Du||x|^{-n+1}\, dx \leq C(n) \fint_{B_{r/2}\setminus B_{r/4}} |u|\, dz + C(n) \fint_{B_{4R}\setminus B_{2R}} |u|\, dz. \tag{1-21}$$

The main idea of showing Proposition 1.9 goes as follows. First, it is known that

$$Du_\delta(x) = D\Delta^{-1}[\Delta(u_\delta \eta)](x) \quad \text{for } x \in B_R \setminus B_r,$$

where u_δ is a standard smooth mollification of u and η is a suitable cut-off function. Next, thanks to the key fact $-\Delta u_\delta \geq 0$, via some subtle kernel estimates and integration by parts, we are able to prove (1-21) for u_δ, and then a standard approximation gives (1-21) as desired.

Theorem 1.5 is eventually proved in the last section. The case $n \geq 11$ is relatively simple. In fact, by Lemmas 1.7 and 1.8, one can build up the following:

$$r^{-(1+\sqrt{n-1})}\left(\int_{B_r} |Du|^2\,dx\right)^{1/2} \leq C(n) R^{n/2-2-\sqrt{n-1}} \fint_{B_{3R}\setminus B_{3R/4}} |u|\,dx \quad \text{for all } 0<r<R<\infty$$

for stable solutions, which allows us to conclude Theorem 1.5 for $n \geq 11$.

As for the case when $n = 10$, we first employ Lemma 1.7 and repeat Lemma 1.8 to get

$$r^{-(1+\sqrt{n-1})}\left(\int_{B_r} |Du|^2\,dx\right)^{1/2} \leq C(n)\frac{1}{\log R} \int_{B_{R^2}\setminus B_4} |Du||x|^{-n+1}\,dx \quad \text{for all } 0<r<R<\infty,$$

which, when $R > 2^5 + r > 4$ and thanks to Proposition 1.9 with r and R therein replaced by 4 and R^2, is then bounded from above by

$$C(n)\frac{1}{\log R}\left(\fint_{B_2\setminus B_1} |u(z)|\,dz + \fint_{B_{4R^2}\setminus B_{2R^2}} |u(z)|\,dz\right).$$

From this we conclude Theorem 1.5 when $n = 10$.

2. Proof of Lemma 1.7

Towards Lemma 1.7 we recall the following a priori bound by [Cabré et al. 2020, Lemma 2.1], which is obtained by taking the test function $(x \cdot Du)\eta$ in the stability condition (1-3).

Lemma 2.1. *Let $u \in C^2(B_1)$ be a stable solution to (1-1) in B_1, with $f \in C^{0,1}(\mathbb{R})$. Then, for all cut-off functions $\eta \in C_c^{0,1}(B_1)$,*

$$\begin{aligned}\int_{B_1} |x\cdot Du|^2 |D\eta|^2\,dx \\ \geq (n-2)\int_{B_1} |Du|^2\eta^2\,dx + 2\int_{B_1} |Du|^2 (x\cdot D\eta)\eta\,dx - 4\int_{B_1} (x\cdot Du)(Du\cdot D\eta)\eta\,dx. \quad (2\text{-}1)\end{aligned}$$

For convenience, for any $0<r<t<\infty$ and $y\in\mathbb{R}^n$, we define the annulus $A_{r,t}(y) := B_t(y)\setminus \overline{B_r(y)}$; for simplicity, we write $A_{r,t} = A_{r,t}(0)$.

Proof of Lemma 1.7. It suffices to prove

$$\left(\frac{r}{t}\right)^{-2(1+\sqrt{n-1})} \int_{B_r(y)} |Du|^2\,dx \leq C(n)\int_{A_{r,t}(y)} |Du|^2\,dx \quad \text{for all } r \leq \frac{t}{2}. \quad (2\text{-}2)$$

Indeed, applying (2-2) to $\frac{1}{2}t$ and t, one has

$$\left(\frac{1}{2}\right)^{-2(1+\sqrt{n-1})} \int_{B_{t/2}(y)} |Du|^2\,dx \leq C(n)\int_{A_{t/2,t}(y)} |Du|^2\,dx. \quad (2\text{-}3)$$

If $\frac{1}{4}t \le r < \frac{1}{2}t$, by $B_r(y) \subset B_{t/2}(y)$ and $\frac{1}{4} \le r/t \le 1$, inequality (2-3) gives

$$\left(\frac{r}{t}\right)^{-2(1+\sqrt{n-1})} \int_{B_r(y)} |Du|^2\,dx \le C(n) \int_{A_{t/2,t}(y)} |Du|^2\,dx. \tag{2-4}$$

If $0 < r < \frac{1}{4}t$, applying (2-2) to r and $\frac{1}{2}t$, and noting $A_{r,t/2} \subset B_{t/2}$, one gets

$$\left(\frac{r}{t/2}\right)^{-2(1+\sqrt{n-1})} \int_{B_r(y)} |Du|^2\,dx \le C(n) \int_{A_{r,t/2}(y)} |Du|^2\,dx \le C(n) \int_{B_{t/2}(y)} |Du|^2\,dx,$$

which together with (2-3) yields

$$\left(\frac{r}{t}\right)^{-2(1+\sqrt{n-1})} \int_{B_r(y)} |Du|^2\,dx \le C(n) \int_{A_{t/2,t}(y)} |Du|^2\,dx.$$

From this and (2-4) we conclude (1-18).

To prove (2-2), without loss of generality we may assume that $t = 1$ and $y = 0$. Indeed, if $u(x)$ is a stable solution to $-\Delta u = f(u)$ in $B_{2t}(y)$, then $v(x) = u(tx + y)$ is the stable solution to $-\Delta v = t^2 f(v)$ in B_2. Note that, up to a change of variable, u satisfies (2-2) if and only if v satisfies (2-2) with $t = 1$ and $y = 0$.

Write $a = 2(1+\sqrt{n-1})$. Let $r \in \left(0, \frac{1}{2}\right]$ be fixed and set

$$\eta = \begin{cases} r^{-a/2} & \text{if } 0 \le |x| \le r, \\ |x|^{-a/2}\phi & \text{if } r < |x| \le 1, \end{cases} \tag{2-5}$$

where $\phi \in C_c^\infty(B_1)$ satisfies

$$\phi = 1 \quad \text{in } B_{3/4} \qquad \text{and} \qquad |D\phi| \le 5\chi_{B_1 \setminus B_{3/4}}. \tag{2-6}$$

Clearly, $\eta \in C_c^{0,1}(B_1)$. Since $\eta = r^{-a/2}$ in B_r and hence $D\eta = 0$ in B_r, substituting η in inequality (2-1) one has

$$\begin{aligned} \int_{A_{r,1}} |x \cdot Du|^2 |D\eta|^2\,dx \ge (n-2) r^{-a} \int_{B_r} |Du|^2\,dx + (n-2) \int_{A_{r,1}} |Du|^2 \eta^2\,dx \\ + 2\int_{A_{r,1}} |Du|^2 (x \cdot D\eta)\eta\,dx - 4\int_{A_{r,1}} (x \cdot Du)(Du \cdot D\eta)\eta\,dx. \end{aligned} \tag{2-7}$$

Noting that

$$D\eta = -\tfrac{1}{2} a |x|^{-a/2-2} x\phi + |x|^{-a/2} D\phi \quad \text{in } A_{r,1},$$

one has

$$\begin{aligned} &2\int_{A_{r,1}} |Du|^2 (x \cdot D\eta)\eta\,dx - 4\int_{A_{r,1}} (x \cdot Du)(Du \cdot D\eta)\eta\,dx \\ &= -a \int_{A_{r,1}} |Du|^2 |x|^{-a} \phi^2\,dx + 2\int_{A_{r,1}} |Du|^2 (x \cdot D\phi)\phi |x|^{-a}\,dx + 2a \int_{a_{r,1}} (x \cdot du)^2 |x|^{-a-2} \phi^2\,dx \\ &\qquad - 4\int_{A_{r,1}} (x \cdot Du)(Du \cdot D\phi)\phi |x|^{-a}\,dx. \end{aligned} \tag{2-8}$$

Moreover, by

$$|D\eta|^2 = \tfrac{1}{4}a^2|x|^{-a-2}\phi^2 - 2a|x|^{-a-2}(x\cdot D\phi)\phi + |x|^{-a}|D\phi|^2,$$

one can write

$$\int_{A_{r,1}} (Du\cdot x)^2|D\eta|^2\,dx = \frac{a^2}{4}\int_{A_{r,1}} (Du\cdot x)^2|x|^{-a-2}\phi^2\,dx + \int_{A_{r,1}} (Du\cdot x)^2|x|^{-a}|D\phi|^2\,dx \\ - a\int_{A_{r,1}} (Du\cdot x)^2|x|^{-a-2}(x\cdot D\phi)\phi\,dx. \tag{2-9}$$

Using (2-8) for the left-hand side of (2-7), and (2-9) for the last two terms in the right-hand side of (2-7), and then moving all terms including $D\phi$ to the left-hand side and all other terms to the right-hand side, we have

$$\begin{aligned}\int_{A_{r,1}} &|x\cdot Du|^2|D\phi|^2|x|^{-a}\,dx - 2\int_{A_{r,1}} |Du|^2(x\cdot D\phi)\phi|x|^{-a}\,dx \\ &+4\int_{A_{r,1}} (x\cdot Du)(Du\cdot D\phi)\phi|x|^{-a}\,dx - a\int_{A_{r,1}} |x|^{-a-2}(x\cdot Du)^2\phi(x\cdot D\phi)\,dx \\ &\geq (n-2)r^{-a}\int_{B_r}|Du|^2\,dx + (n-2)\int_{A_{r,1}}|Du|^2|x|^{-a}\phi^2\,dx \\ &\qquad - a\int_{A_{r,1}}|Du|^2|x|^{-a}\phi^2\,dx + 2a\int_{A_{r,1}}(x\cdot Du)^2|x|^{-a-2}\phi^2\,dx - \frac{a^2}{4}\int_{A_{r,1}}(Du\cdot x)^2|x|^{-a-2}\phi^2\,dx \\ &= (n-2)r^{-a}\int_{B_r}|Du|^2\,dx + \int_{A_{r,1}}\bigl\{(n-2-a)|Du|^2 + \bigl(2a-\tfrac{1}{4}a^2\bigr)(Du\cdot x)^2|x|^{-2}\bigr\}|x|^{-a}\phi^2\,dx.\end{aligned} \tag{2-10}$$

Note that, by $|D\phi| = 0$ in $B_{3/4}$ and $|D\phi|\leq 5$ in B_1 as in (2-6) and $a > 2$,

$$\begin{aligned}\int_{A_{r,1}} &|x\cdot Du|^2|D\phi|^2|x|^{-a}\,dx - 2\int_{A_{r,1}} |Du|^2(x\cdot D\phi)\phi|x|^{-a}\,dx \\ &+4\int_{A_{r,1}} (x\cdot Du)(Du\cdot D\phi)\phi|x|^{-a}\,dx - a\int_{A_{r,1}} |x|^{-a-2}(x\cdot Du)^2\phi(x\cdot D\phi)\,dx \\ &\leq C(n)\int_{A_{3/4,1}}|Du|^2\,dx.\end{aligned} \tag{2-11}$$

Additionally, note that $n \geq 10$ implies $a = 2(1+\sqrt{n-1}) \geq 8$, and hence

$$2a - \tfrac{1}{4}a^2 = \tfrac{1}{4}a(8-a) \leq 0.$$

By $|x|^{-1}|x\cdot Du| \leq |Du|$ in $A_{r,1}$, we have

$$(n-2-a)|Du|^2 + \bigl(2a-\tfrac{1}{4}a^2\bigr)(Du\cdot x)^2|x|^{-2} \geq \bigl(n-2+a-\tfrac{1}{4}a^2\bigr)|Du|^2.$$

Since

$$n-2+a-\tfrac{1}{4}a^2 = -\bigl(\tfrac{1}{2}a - [1-\sqrt{n-1}]\bigr)\bigl(\tfrac{1}{2}a - [1+\sqrt{n-1}]\bigr) = 0,$$

we have

$$(n-2-a)|Du|^2 + \bigl(2a-\tfrac{1}{4}a^2\bigr)(Du\cdot x)^2|x|^{-2} \geq 0 \quad \text{in } A_{r,1}, \tag{2-12}$$

which means that the last term in the right-hand side of (2-10) is nonnegative. From this, together with (2-10) and (2-11), we conclude (2-2). The proof is complete. □

Remark 2.2. Recall that in [Cabré et al. 2020], the authors used the test function $\eta = |x|^{-a/2}\xi$ with $\xi \in C_c^\infty(B_1)$, which was not enough to get (2-2).

3. Proofs of Equation (1-1) and Corollary 1.3

In this section we prove Theorem 1.2 and Corollary 1.3. First, we sketch a proof of Lemma 1.8.

Proof of Lemma 1.8. Up to considering $v(x) = u(tx + y)$, we may assume that $t = 1$ and $y = 0$. Inequality (1-20) is given by [Cabré et al. 2020, Lemma A.2]. Inequality (1-19) reads as $\|Du\|_{L^2(B_{1/2})} \le C(n)\|Du\|_{L^1(B_1)}$ and will follow from the proof of [Cabré et al. 2020, Proposition 2.5], where the authors proved that

$$\|Du\|_{L^2(B_{1/2})} \le C(n)\|u\|_{L^1(B_1)}. \tag{3-1}$$

In their proof, first they obtained a bound of $\|Du\|_{L^2(B_{1/2})}$ via $\|Du\|_{L^1(B_{1/2})}$ and some other small terms. Next, they used $\|Du\|_{L^1(B_{1/2})} \le C(n)\|u\|_{L^1(B_1)}$. Finally, via an iteration argument, they got (3-1). If we directly apply the iteration argument without using $\|Du\|_{L^1(B_{1/2})} \le C(n)\|u\|_{L^1(B_1)}$, we get $\|Du\|_{L^2(B_{1/2})} \le C(n)\|Du\|_{L^1(B_1)}$. □

Recall that $u_E = \fint_E u\,dx$ denotes the integral average of u on a measurable set E. The interior regularity (1-10) and (1-11) in Theorem 1.2 is a consequence of Lemma 1.7 and (1-19), together with a standard embedding argument.

Proofs of (1-10) and (1-11) in Theorem 1.2. Let $u \in C^2(B_2)$ be a stable solution to (1-1). Write $\beta = n - 2 - 2\sqrt{n-1}$. For any $y \in B_{1/2}$, if $r > \frac{1}{8}$, by Lemma 1.8 we have

$$r^{\beta-n}\int_{B_r(y)\cap B_{1/2}} |Du|^2\,dx \le C(n)\fint_{B_{1/2}} |Du|^2\,dx \le C(n)\|u\|^2_{L^1(B_1)},$$

and if $0 < r < \frac{1}{8}$, by Lemmas 1.7 and 1.8 again we have

$$r^{\beta-n}\int_{B_r(y)\cap B_{1/2}} |Du|^2\,dx \le r^\beta \fint_{B_r(y)} |Du|^2\,dx \le C(n)\fint_{B_{1/4}(y)} |Du|^2\,dx \le C(n)\|u\|^2_{L^1(B_1)}.$$

This means that $Du \in M^{2,\beta}(B_{1/2})$ with $\|Du\|_{M^{2,\beta}(B_{1/2})} \le C(n)\|u\|_{L^1(B_1)}$.

If $n = 10$, then $\beta = 2$ and $2\beta/(\beta-2) = \infty$. Thanks to the Sobolev–Poincaré inequality, one can easily check that $Du \in M^{2,\beta}(B_{1/2})$ implies $u \in \mathrm{BMO}(B_{1/2})$, with a norm bound

$$\|u\|_{\mathrm{BMO}(B_{1/2})} \le C(n)\|Du\|_{M^{2,\beta}(B_{1/2})}.$$

If $n \ge 11$, then $p_n = 2\beta/(\beta-2) < \infty$ and $\beta = 2 + 4/(p_n - 2)$. By the embedding result in [Adams 1975] and also [Cabré and Charro 2021, Section 4], $Du \in M^{2,\beta}(B_{1/2})$ implies $u \in M^{2\beta/(\beta-2),\beta}(B_{1/2})$, with its norm bound

$$\|u\|_{M^{p_n,\beta}(B_{1/2})} \le C(n)\|Du\|_{M^{2,\beta}(B_{1/2})}.$$

This proves (1-10) and (1-11). □

To prove the global regularity (1-12) and (1-13) in Theorem 1.2, we need the following a priori L^∞-bound in a neighborhood of $\partial\Omega$ for a C^2 solution when Ω is a bounded smooth convex domain; see [Cabré 2010, Proposition 3.2] and [Chen and Li 1993; de Figueiredo et al. 1982; Gidas et al. 1979]. For $\rho > 0$, we write

$$\Omega_\rho := \{x \in \Omega : \operatorname{dist}(x, \partial\Omega) < \rho\}.$$

Lemma 3.1. *Suppose that $f \in C^{0,1}(\mathbb{R})$ is nonnegative and Ω is a smooth convex domain in $\mathbb{R}^n$. There exist positive constants ρ and γ depending only on the domain Ω such that, for any positive solution $u \in C^2(\Omega) \cap C^0(\overline{\Omega})$ to (1-1), one has*

$$\|u\|_{L^\infty(\Omega_\rho)} \le \frac{1}{\gamma}\|u\|_{L^1(\Omega)}. \tag{3-2}$$

Note that, as $f \ge 0$, the maximum principle shows that any solution $u \in C^2(\Omega) \cap C^0(\overline{\Omega})$ to (1-1) with zero boundary is always nonnegative, and the strong maximum principle further shows that u is always positive in the domain Ω.

Proofs of (1-12) and (1-13) in Theorem 1.1. Let $\beta = n - 2 - 2\sqrt{n-1}$, and let ρ, γ be as in Lemma 3.1. We first consider the case $n \ge 11$. For any $y \in \overline{\Omega}$ and $r > 0$, write

$$\begin{aligned} r^{\beta-n}\int_{\Omega\cap B_r(y)} |u|^{p_n}\,dx &= r^{\beta-n}\int_{\Omega_\rho\cap B_r(y)} |u|^{p_n}\,dx + r^{\beta-n}\int_{(\Omega\setminus\Omega_\rho)\cap B_r(y)} |u|^{p_n}\,dx \\ &:= \Phi_1(y,r) + \Phi_2(y,r). \end{aligned}$$

To see (1-12), we only need to prove $\Phi_1(y,r) \le C(n,\Omega)\|u\|^{p_n}_{L^1(\Omega)}$ and $\Phi_2(y,r) \le C(n,\rho,\Omega)\|u\|^{p_n}_{L^1(\Omega)}$ for any $y \in \Omega$ and $r > 0$.

Note that

$$r^{\beta-n}|\Omega_\rho \cap B_r(y)| \le \begin{cases} C(n) & \text{when } r < 1, \\ |\Omega_\rho| & \text{when } r > 1, \end{cases}$$

so by $2 < \beta < n$ and Lemma 3.1, we have

$$\Phi_1(y,r) \le r^{\beta-n}|\Omega_\rho \cap B_r(y)|\|u\|^{p_n}_{L^\infty(\Omega_\rho)} \le C(n,\Omega)\|u\|^{p_n}_{L^1(\Omega)}.$$

Next, to get $\Phi_2(y,r) \le C(n,\rho,\Omega)\|u\|^{p_n}_{L^1(\Omega)}$ for any $y \in \Omega$ and $r > 0$, we only need to consider $y \in \Omega\setminus\Omega_\rho$ and $0 < r < \frac{1}{8}\rho$. Indeed, for $y \in \Omega_\rho$, if $r < \operatorname{dist}(y, \Omega\setminus\Omega_\rho)$, then $\Phi_2(y,r) = 0$, and if $r \ge \operatorname{dist}(y, \Omega\setminus\Omega_\rho)$, then $\Phi_2(y,r) \le C(n)\Phi_2(\bar{y}, 2r)$, where $\bar{y}$ is the closest point in $\Omega\setminus\Omega_\rho$ and $B(y,r) \subset B(\bar{y}, 2r)$. Moreover, for any $y \in \Omega\setminus\Omega_\rho$ and $r \ge \frac{1}{8}\rho$,

$$\Phi_2(y,r) \le \rho^{\beta-n}\int_{\Omega\setminus\Omega_\rho} |u|^{p_n}\,dx \le \sum_{i=1}^N \rho^{\beta-n}\int_{\Omega\setminus\Omega_\rho\cap B_{\rho/9}(x_i)} |u|^{p_n}\,dx = \sum_{i=1}^N \Phi\big(x_i, \tfrac{1}{9}\rho\big),$$

where $\big\{B\big(x_i, \frac{1}{9}\rho\big)\big\}_{i=1}^N$ is a cover of the compact set $\Omega\setminus\Omega_\rho$, $\{x_i\}_{i=1}^N \subset \Omega\setminus\Omega_\rho$ and N depends only on Ω and ρ.

On the other hand, for any $y\in\Omega\backslash\Omega_\rho$ and $0<r<\frac{1}{8}\rho$, since u is a stable solution in $B_\rho(y)\subset\Omega$, by (1-11) with a scaling argument, we have $u\in M^{p_n,\beta}(B_{\rho/8}(y))$ with $\|u\|_{M^{p_n,\beta}(B_{\rho/8}(y))}\le C(n,\rho)\|u\|_{L^1(B_{\rho/2}(y))}$, in particular

$$\Phi_2(y,r)\le r^\beta \fint_{B_r(y)}|u|^{p_n}\,dx\le C(n,\rho)\|u\|^{p_n}_{L^1(\Omega)}$$

as desired. This proves (1-13).

In the case $n=10$, for any $y\in\Omega$, if $r>\frac{1}{9}\rho$, we have

$$r^{-n}\int_{\Omega\cap B_r(y)}|u|\,dx\le C(n,\rho)\|u\|_{L^1(\Omega)}.$$

Below we assume that $0<r<\frac{1}{9}\rho$. If $y\in\Omega\backslash\Omega_{8\rho/9}$, we have $\rho<\frac{9}{8}\operatorname{dist}(y,\partial\Omega)$. Since $0<r<\frac{1}{8}\operatorname{dist}(y,\partial\Omega)$ and u is a stable solution in $B_{\operatorname{dist}(y,\partial\Omega)}(y)\subset\Omega$, by (1-10) with a scaling we have

$$\fint_{B_r(y)}|u-u_{B_r(y)}|\,dx\le C(n,\rho)\|u\|_{L^1(B_{\operatorname{dist}(y,\partial\Omega)}(y))}\le C(n,\rho)\|u\|_{L^1(\Omega)}.$$

For $y\in\Omega_{8\rho/9}$, noting $0<r<\frac{1}{9}\rho\le\operatorname{dist}(y,\partial\Omega_\rho)$, one has $\Omega\cap B_r(y)\subset\Omega\setminus\Omega_\rho$. Thus

$$r^{-n}\int_{\Omega\cap B_r(y)}|u|\,dx=r^{-n}\int_{\Omega_\rho\cap B_r(y)}|u|\,dx\le C(n,\rho)\|u\|_{L^1(\Omega)}.$$

Combining these estimates, we obtain (1-12). □

We finally prove Corollary 1.3.

Proof of Corollary 1.3. Let $u\in W^{1,2}_0(\Omega)$ be a stable solution to (1-1) with zero boundary. By [Dupaigne 2011, Corollary 3.2.1] (see also the proof in [Cabré et al. 2020, Theorem 4.1] and [Dupaigne and Farina 2023, Theorem 5]), there is a nonnegative, nondecreasing sequence (f_k) of convex functions in $C^1(\mathbb{R})$ such that $f_k\to f$ pointwise in $[0,\infty)$ and a nondecreasing sequence (u_k) in $C^2(\overline{\Omega})\cap W^{1,2}_0(\Omega)$ such that u_k is a weak stable solution to

$$-\Delta u_k=f_k(u_k)\quad\text{in }\Omega,\qquad u_k=0\quad\text{on }\partial\Omega \tag{3-3}$$

and

$$u_k\to u\quad\text{in }W^{1,2}(\Omega)\qquad\text{as }k\to+\infty.$$

If $n=10$, applying (1-12) to u_k, one has

$$\fint_{\Omega\cap B_r(y)}\left|u_k(x)-\fint_{\Omega\cap B_r(y)}u_k\,dz\right|dx\le\|u_k\|_{\mathrm{BMO}(\Omega)}\le C(n,\Omega)\int_\Omega|u_k|\,dx\quad\text{for all }r>0\text{ for all }y\in\overline{\Omega}.$$

Since $u_k\to u$ in $W^{1,2}(\Omega)$ as $k\to+\infty$, we conclude that $\|u\|_{\mathrm{BMO}(\Omega)}\le C(n)\|u\|_{L^1(\Omega)}$ as desired.

If $n\ge 11$, applying (1-13) to u_k, we have

$$r^{\beta-n}\int_{\Omega\cap B_r(y)}|u_k|^{p_n}\,dx\le C(n,\Omega,\rho)(\|u_k\|_{L^1(\Omega)})^{p_n}\quad\text{for all }y\in\overline{\Omega}\text{ for all }r>0, \tag{3-4}$$

where $\beta = 2p_n/(p_n-2) \in (0,n)$. Since $u_k \to u$ in $W^{1,2}(\Omega)$ as $k \to +\infty$, we deduce that $u_k \in L^{p_n}(\Omega)$ uniformly in $k \geq 0$, and hence $u_k \to u$ weakly in $L^{p_n}(\Omega)$. Thus, letting $k \to +\infty$ in (3-4), we conclude $\|u\|_{M^{p_n,\beta}(\Omega)} \leq C(n)\|u\|_{L^1(\Omega)}$ as desired. □

4. Proof of Proposition 1.9

Let $0 < r < R < \infty$. Let $\eta \in C_c^\infty(A_{r/4,4R})$ satisfy

$$0 \leq \eta \leq 1 \quad \text{in } A_{r/4,4R} \quad \text{and} \quad \eta = 1 \quad \text{in } A_{r/2,2R}, \tag{4-1}$$

$$|D\eta|^2 + |D^2\eta| \leq \frac{C}{r^2} \quad \text{in } A_{r/4,r/2} \quad \text{and} \quad |D\eta|^2 + |D^2\eta| \leq \frac{C}{R^2} \quad \text{in } A_{2R,4R}, \tag{4-2}$$

where $C > 0$ is a universal constant.

Let $u_\delta = u * \phi_\delta$ for $\delta > 0$, where ϕ_δ is the standard smooth mollifier and is supported in $B(0,\delta)$. Recall that $u \in W^{1,1}_{\mathrm{loc}}(\mathbb{R}^n)$ and $u_\delta \to u$ in $W^{1,1}_{\mathrm{loc}}(\mathbb{R}^n)$. Since $-\Delta u \geq 0$ is a locally finite measure, we have $-\Delta u_\delta = (-\Delta u) * \phi_\delta \geq 0$ everywhere. By $u_\delta \eta \in C_c^\infty(\mathbb{R}^n)$, one has

$$u_\delta \eta(x) = \Delta^{-1}[\Delta(u_\delta\eta)](x) = c(n) \int_{\mathbb{R}^n} \frac{1}{|x-y|^{n-2}} \Delta(u_\delta\eta)(y)\,dy \quad \text{for all } x \in \mathbb{R}^n,$$

and hence

$$D(u_\delta\eta)(x) = D\Delta^{-1}[\Delta(u_\delta\eta)](x) = c(n)(2-n) \int_{\mathbb{R}^n} \frac{x-y}{|x-y|^n} \Delta(u_\delta\eta)(y)\,dy \quad \text{for all } x \in \mathbb{R}^n.$$

Noting

$$\Delta(u_\delta\eta)(y) = \Delta u_\delta(y)\eta(y) + \Delta\eta(y)u_\delta(y) + 2Du_\delta(y)\cdot D\eta(y),$$

for $0 < \delta \ll \frac{1}{8}r$, we write

$$\begin{aligned}
\int_{A_{r,R}} |Du_\delta||x|^{-n+1}\,dx &= \int_{A(r,R)} |D(u_\delta\eta)||x|^{-n+1}\,dx \\
&= \int_{A_{r,R}} \left| \int_{\mathbb{R}^n} \frac{x-y}{|x-y|^n} \Delta(u_\delta\eta)(y)\,dy \right| |x|^{-n+1}\,dx \\
&\leq C(n) \int_{A_{r,R}} \left| \int_{\mathbb{R}^n} \frac{x-y}{|x-y|^n} \Delta u_\delta(y)\eta(y)\,dy \right| |x|^{-n+1}\,dx \\
&\quad + C(n) \int_{A_{r,R}} \left| \int_{\mathbb{R}^n} \frac{x-y}{|x-y|^n} u_\delta(y)\Delta\eta(y)\,dy \right| |x|^{-n+1}\,dx \\
&\quad + C(n) \int_{A_{r,R}} \left| \int_{\mathbb{R}^n} \frac{x-y}{|x-y|^n} Du_\delta(y)\cdot D\eta(y)\,dy \right| |x|^{-n+1}\,dx \\
&=: I_1 + I_2 + I_3.
\end{aligned}$$

In order to control I_1 from above, first by $-\Delta u_\delta \geq 0$ and (4-1), one has

$$I_1 \leq \int_{\mathbb{R}^n} \left(\int_{\mathbb{R}^n} |x-y|^{-n+1}|x|^{-n+1}\,dx \right) (-\Delta u_\delta)(y)\eta(y)\,dy.$$

Employing the triangle inequality, for $y \in \mathbb{R}^n$, we further get

$$\begin{aligned}\int_{\mathbb{R}^n} |x-y|^{-n+1}|x|^{-n+1}\,dx &\le 2^{n-1}\int_{\{|x|>2|y|\}} |x|^{-2n+2}\,dx + 2^{n-1}\int_{\{|x|<|y|/2\}} |x|^{-n+1}|y|^{-n+1}\,dx \\ &\qquad + \int_{\{|y|/2\le|x|\le 2|y|\}} |x-y|^{-n+1}|y|^{-n+1}\,dx \\ &\le C(n)|y|^{-n+2} + C(n)|y|^{-n+2} + \int_{\{|y-x|\le 3|y|\}} |x-y|^{-n+1}|y|^{-n+1}\,dx \\ &\le C(n)|y|^{-n+2}. \end{aligned} \tag{4-3}$$

This together with $-\Delta u_\delta \ge 0$ again gives

$$I_1 \le C(n)\int_{\mathbb{R}^n} (-\Delta u_\delta)|y|^{-n+2}\eta(y)\,dy.$$

Via integration by parts and using $\eta \in C_c^\infty(A_{r/4,4R})$, we have

$$\int_{\mathbb{R}^n} (-\Delta u_\delta)|y|^{-n+2}\eta(y)\,dy = \int_{A_{r/4,4R}} u_\delta[-\Delta|y|^{-n+2}\eta(y) + D|y|^{-n+2}\cdot D\eta(y) - |y|^{-n+2}\Delta\eta(y)]\,dy.$$

Observing that $\Delta|y|^{n-2} = 0$ in $A_{r/4,4R}$ and using (4-1) and (4-2), we arrive at

$$\begin{aligned} I_1 &\le C(n)\int_{A_{r/4,4R}} u_\delta(y)[(2-n)|y|^{-n}y\cdot D\eta(y) - |y|^{-n+2}\Delta\eta(y)]\,dy \\ &\le C(n)\int_{A_{r/4,4R}} |u_\delta(y)|[r^{-n}\chi_{A_{r/4,r/2}} + R^{-n}\chi_{A_{2R,4R}}]\,dy \\ &\le C(n)\fint_{A_{r/4,r/2}} |u_\delta|\,dz + \fint_{A_{2R,4R}} |u_\delta|\,dz. \end{aligned}$$

For I_2, by (4-3) and (4-1),

$$\begin{aligned} I_2 &\le \int_{\mathbb{R}^n}\left(\int_{A(r,R)} |x-y|^{-n+1}|x|^{-n+1}\,dx\right)|u_\delta|(y)|\Delta\eta(y)|\,dy \\ &\le C(n)\int_{\mathbb{R}^n} |y|^{-n+2}|u_\delta|(y)|\Delta\eta(y)|\,dy \\ &\le C(n)\int_{\mathbb{R}^n} |u_\delta(y)|[r^{-n}\chi_{A_{r/4,r/2}} + R^{-n}\chi_{A_{2R,4R}}]\,dy \\ &\le C(n)\fint_{A_{r/4,r/2}} |u_\delta|\,dz + C(n)\fint_{A_{2R,4R}} |u_\delta|\,dz. \end{aligned}$$

Now let us estimate I_3. First via integration by parts one gets

$$\begin{aligned} &\int_{\mathbb{R}^n} |x-y|^{-n}(x-y)Du_\delta(y)\cdot D_\delta\eta(y)\,dy \\ &\qquad = \int_{\mathbb{R}^n} |x-y|^{-n}(x-y)u_\delta(y)\Delta\eta(y)\,dy + \int_{\mathbb{R}^n} u_\delta(y)D[|x-y|^{-n}(x-y)]D\eta(y)\,dy. \end{aligned}$$

Since
$$|D[|x-y|^{-n}(x-y)]| \le C(n)|x-y|^{-n},$$
we obtain
$$\left|\int_{\mathbb{R}^n} |x-y|^{-n}(x-y) Du_\delta(y)\cdot D\eta(y)\,dy\right|$$
$$\le C(n)\left|\int_{\mathbb{R}^n} |x-y|^{-n+1} u_\delta(y)\Delta\eta(y)\,dy\right| + C(n)\int_{\mathbb{R}^n} |x-y|^{-n}|u_\delta(y)||D\eta(y)|\,dy.$$
As a consequence,
$$I_3 \le C(n) I_2 + C(n)\int_{\mathbb{R}^n}\left(\int_{A_{r,R}} |x-y|^{-n}|x|^{-n+1}\,dx\right)|u_\delta(y)||D\eta(y)|\,dy =: C(n)I_2 + C(n)\tilde{I}_3.$$

In order to estimate $\tilde{I}_3$, first we note that (4-1) gives
$$\tilde{I}_3 \le C(n)\int_{\mathbb{R}^n}\left(\int_{A_{r,R}} |x-y|^{-n}|x|^{-n+1}\,dx\right)|u_\delta(y)|[r^{-1}\chi_{A_{r/4,r/2}} + R^{-1}\chi_{A_{2R,4R}}]\,dy.$$
For any $x\in A_{r,R}$, if $y\in A_{r/4,r/2}$, we have $|x-y|\ge \frac{1}{2}|x|$, and hence
$$\int_{A_{r,R}} |x-y|^{-n}|x|^{-n+1}\,dx \le C(n)\int_{A_{r,R}} |x|^{-2n+1}\,dx \le C(n) r^{-n+1};$$
if $y\in A_{2R,4R}$, then $|x-y|\ge R$, and hence
$$\int_{A_{r,R}} |x-y|^{-n}|x|^{-n+1}\,dx \le C(n)R^{-n}\int_{A_{r,R}} |x|^{-n+1}\,dx \le C(n)R^{-n+1}.$$
Thus it follows that
$$\tilde{I}_3 \le C(n)\int_{\mathbb{R}^n} |u_\delta(y)|[r^{-n}\chi_{A_{r/4,r/2}} + R^{-n}\chi_{A_{2R,4R}}]\,dy \le C(n)\fint_{A_{r/4,r/2}} |u_\delta|\,dz + C(n)\fint_{A_{2R,4R}} |u_\delta|\,dz.$$

To conclude,
$$\int_{A_{r,R}} |Du_\delta||x|^{-n+1}\,dx \le C(n)\fint_{A_{r/4,r/2}} |u_\delta|\,dz + C(n)\fint_{A_{2R,4R}} |u_\delta|\,dz.$$
By letting $\delta\to 0$ and noting $u_\delta\to u$ in $W^{1,1}_{\mathrm{loc}}$, we conclude (1-21). □

5. Proof of Theorem 1.5

Since u satisfies (1-16), we know that u does not satisfy (1-15). We only need to show that if u is nonconstant, then (1-15) holds. Equivalently, it suffices to show that if u does not satisfy (1-15), then u is a constant. Namely, there exists a sequence $\{R_j\}_{j\in\mathbb{N}}$ tending toward ∞ such that
$$\frac{1}{\log R_j}\fint_{A_{R_j,4R_j}} |u(z)|\,dz \to 0 \quad \text{as } j\to\infty, \quad \text{when } n = 10, \tag{5-1}$$
and
$$R_j^{n/2-2-\sqrt{n-1}}\fint_{A_{R_j,4R_j}} |u(x)|\,dx \to 0 \quad \text{as } j\to\infty, \quad \text{when } n\ge 11. \tag{5-2}$$

On the other hand, given any $0 < r < \infty$, applying (1-18) for any $R > 4r$, we have

$$r^{-(1+\sqrt{n-1})}\left(\int_{B_r} |Du|^2\,dx\right)^{1/2} \le C(n)R^{-(1+\sqrt{n-1})}\left(\int_{A_{R,2R}} |Du|^2\,dx\right)^{1/2}. \tag{5-3}$$

Observe that the annulus $A_{1,2}$ can be covered by $\{B_{1/8}(y_i)\}_{i=1}^N$ with $y_1, \dots, y_N \in A_{1,2}$ and $N \le C(n)$:

$$\chi_{A_{1,2}} \le \sum_{i=1}^N \chi_{B_{1/8}(y_i)} \le \sum_{i=1}^N \chi_{B_{1/4}(y_i)} \le C(n)\chi_{A_{3/4,3}}.$$

Below we consider the case $n \ge 11$ and the case $n = 10$ separately.

Case $n \ge 11$. For each i, applying (1-19) and (1-20), one attains

$$\left(\int_{B_{R/8}(Ry_i)} |Du|^2\,dx\right)^{1/2} \le C(n)R^{-(n+2)/2}\int_{B_{R/4}(Ry_i)} |u|\,dx \le C(n)R^{(n-2)/2} ⨍_{A_{3R/4,3R}} |u|\,dx.$$

Thus by summing over all these balls,

$$\int_{A_{R,2R}} |Du|^2\,dx \le C(n)R^{n-2}\left(⨍_{A_{3R/4,3R}} |u|\,dx\right)^2,$$

and we eventually obtain from (5-3) that

$$r^{-(1+\sqrt{n-1})}\left(\int_{B_r} |Du|^2\,dx\right)^{1/2} \le C(n)R^{n/2-2-\sqrt{n-1}} ⨍_{A_{3R/4,3R}} |u|\,dx.$$

Taking $R = \frac{4}{3}R_j$, applying (5-2) and letting $j \to \infty$, one concludes

$$\int_{B_r} |Du|^2\,dx = 0.$$

By the arbitrariness of $r > 0$, we obtain $\|Du\|_{L^2(\mathbb{R}^n)} = 0$, which implies that u is a constant.

Case $n = 10$. For each i, applying (1-19), one attains

$$\left(\int_{B_{R/8}(Ry_i)} |Du|^2\,dx\right)^{1/2} \le C(n)R^{-n/2}\int_{B_{R/4}(Ry_i)} |Du|\,dx \le C(n)R^{(n-2)/2}\int_{A_{R/2,4R}} |Du||x|^{-n+1}\,dx.$$

Thus

$$\int_{A_{R,2R}} |Du|^2\,dx \le C(n)R^{n-2}\left(\int_{A_{R/2,4R}} |Du||x|^{-n+1}\,dx\right)^2. \tag{5-4}$$

We therefore obtain from (5-3) that

$$\begin{aligned} r^{-(1+\sqrt{n-1})}\left(\int_{B_r} |Du|^2\,dx\right)^{1/2} &\le C(n)R^{n/2-2-\sqrt{n-1}}\int_{A_{R/2,4R}} |Du||x|^{-n+1}\,dx \\ &= C(n)\int_{A_{R/2,4R}} |Du||x|^{-n+1}\,dx, \end{aligned}$$

where in the last identity we use $\frac{1}{2}n - 2 - \sqrt{n-1} = 5 - 2 - 3 = 0$.

For $R > 2^5 + r > 4$, let m be the largest integer such that $m \le \log_2 R - 3$. Applying (5-4) to $2^j R$ with $j = 1, \dots, m$, one has

$$r^{-(1+\sqrt{n-1})}\left(\int_{B_r} |Du|^2\,dx\right)^{1/2} \le C(n)\frac{1}{m}\sum_{j=1}^{m}\int_{A_{2^j R/2,4(2^j R)}} |Du||x|^{-n+1}\,dx$$
$$\le C(n)\frac{1}{m}\int_{A_{R,2^{m+2}R}} |Du||x|^{-n+1}\,dx \le C(n)\frac{1}{\log R}\int_{A_{4,R^2/2}} |Du||x|^{-n+1}\,dx.$$

By (1-21), one has

$$r^{-(1+\sqrt{n-1})}\left(\int_{B_r} |Du|^2\,dx\right)^{1/2} \le C(n)\frac{1}{\log R}\fint_{A_{1,2}} |u(z)|\,dz + C(n)\frac{1}{\log R^2}\fint_{A_{2R^2,4R^2}} |u(z)|\,dz.$$

Taking $R = \sqrt{R_j}$ and letting $j \to \infty$, by (5-1) one concludes

$$\int_{B_r} |Du|^2\,dx = 0.$$

Then the arbitrariness of $r > 0$ implies $\|Du\|_{L^2(\mathbb{R}^n)} = 0$, which further implies that u is a constant. $\square$

Appendix: A radial stable solution when $n = 10$

Suppose $n = 10$ in this appendix. Villegas [2007] proved that $\frac{1}{2}\log(1+|x|^2)$ is a stable solution to the equation $-\Delta u = -(n-2)e^{-2u} - 2e^{-4u}$ in $\mathbb{R}^n$. Note that $-(n-2)e^{-2s} - 2e^{-4s} \le 0$ in $\mathbb{R}$.

Below, we show that $u = -\frac{1}{2}\log(1+|x|^2)$ is a stable solution to the equation

$$-\Delta u = f(u) \quad \text{in } \mathbb{R}^n,$$

where $f(s) = (n-2)e^{2s} + 2e^{4s} \ge 0$ in $\mathbb{R}$.

First we show that u is a solution. Indeed, for any $x \in \mathbb{R}^n$, a direct calculation gives

$$-\Delta u(x) = ((1+|x|^2)^{-1}x_i)_{x_i} = \frac{n}{1+|x|^2} + 2\frac{|x|^2}{(1+|x|^2)^2} = (n-2)\frac{1}{1+|x|^2} + 2\frac{1}{(1+|x|^2)^2}.$$

Since $e^{2u(x)} = (1+|x|^2)^{-1}$, we have

$$-\Delta u(x) = (n-2)e^{2u(x)} + 2e^{4u(x)} = f(u(x)).$$

Next, we show that u is stable. Note that $f'(s) = 2(n-2)e^{2s} + 8e^{4s}$ for $s \in \mathbb{R}$. Given any $x \ne 0$, writing $r = |x|$ and noting $e^{2u(x)} = (1+|x|^2)^{-1}$, we have

$$f'(u(x)) = 2(n-2)e^{2u(x)} + 8e^{4u(x)} = \frac{2(n-2)}{1+r^2} + \frac{8}{(1+r^2)^2}.$$

Since $n = 10$, we have

$$f'(u(x)) = \frac{16r^2(1+r^2)+8r^2}{r^2(1+r^2)^2} = \frac{16r^4+24r^2}{r^2(1+r^2)^2} < \frac{16(1+r^2)^2}{r^2(1+r^2)^2} = \frac{(n-2)^2}{4|x|^2}.$$

By this and the Hardy inequality, we have

$$\int_{\mathbb{R}^n} f'(u)\xi^2\,dx \le \frac{(n-2)^2}{4}\int_{\mathbb{R}^n} \frac{\xi^2}{|x|^2}\,dx \le \int_{\mathbb{R}^n} |D\xi|^2\,dx \quad \text{for all } \xi \in C_c^\infty(\mathbb{R}^n).$$

Thus u is a stable solution to $-\Delta u = f(u)$ in $\mathbb{R}^n$.

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A GENERALIZATION OF THE BEURLING–MALLIAVIN MAJORANT THEOREM

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We prove a generalization of the Beurling–Malliavin majorant theorem. In more detail, we establish a new sufficient condition for a function to be a Beurling–Malliavin majorant. Our result is strictly more general than that of the Beurling–Malliavin majorant theorem. We also show that our result is sharp in a number of senses.

1. Introduction

Let $\mathrm{Lip}(\mathbb{R})$ denote the space of Lipschitz functions in $\mathbb{R}$ (i.e., functions f satisfying for all $x, y \in \mathbb{R}$ the inequality $|f(x) - f(y)| \le C|x - y|$ with $C > 0$ independent of x, y). By $\mathrm{Lip}(\xi, \mathbb{R})$ we shall denote all Lipschitz functions in $\mathbb{R}$ with the Lipschitz constant ξ.

The following theorem was first proved by A. Beurling and P. Malliavin.

Theorem A [Beurling and Malliavin 1962]. *Let $\omega : \mathbb{R} \to (0, 1]$ be a function such that $\log(1/\omega) \in L^1(\mathbb{R}, dx/(1+x^2))$, and $\log(1/\omega)$ is a Lipschitz function. Then for each $\delta > 0$ there exists a function $f \in L^2(\mathbb{R})$, which is not identically zero and which satisfies $\mathrm{spec}(f) \subset [0, \delta]$ and $|f(x)| \le \omega(x)$ for all $x \in \mathbb{R}$.*

For a function $f \in L^2(\mathbb{R})$, by $\mathrm{spec}(f)$ we mean the spectrum of f, i.e., the support of its Fourier transformation. Note that the spectrum is defined up to a set of the Lebesgue measure zero. Let us also remark that here the term "not identically zero" means "not zero almost everywhere". We shall further sometimes write just "nonzero" for brevity.

The Beurling–Malliavin theorems are considered by many experts to be among the most deep and important results of the 20th century harmonic analysis. Theorem A above is called the Beurling–Malliavin majorant theorem (or the first Beurling–Malliavin theorem). This result gives conditions for the majorant ω ensuring existence of a nonzero function whose spectrum lies in an arbitrary small interval and whose modulus is majorized by ω. This theorem is a crucial tool in the proof of the second Beurling–Malliavin theorem about the radius of completeness of an exponential system. Moreover, Theorem A was recently used by J. Bourgain and S. Dyatlov [2018] in the theory of resonances for hyperbolic surfaces. Deep connections of the first Beurling–Malliavin theorem with nowadays popular gap and type problems are discussed in [Borichev and Sodin 2011; Poltoratski 2012; Makarov and Poltoratski 2010].

Note that Theorem A is in a certain sense a contradiction to the following postulate, called the uncertainty principle: "It is impossible for a nonzero function and its Fourier transform to be simultaneously very small, unless the function is zero". Indeed, Theorem A shows that there exist nonzero functions that are "small" and whose Fourier transforms are also "small". Of course these smallnesses are different from each

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other and from the smallness in $L^2(\mathbb{R})$. So in fact Theorem A does not contradict the most well-known variant of the uncertainty principle, the Heisenberg inequality. For recent violations of the uncertainty principle of a completely different nature, see [Kislyakov and Perstneva 2021; Nazarov and Olevskii 2018].

In addition to the original proof of A. Beurling and P. Malliavin, there are many approaches to the proof of Beurling–Malliavin theorems due to H. Redheffer [1977], L. De Branges [1968], P. Kargaev [Koosis], N. Makarov and A. Poltoratski [Makarov and Poltoratski 2010], to name just some of them. V. Havin, J. Mashreghi and F. Nazarov [Mashregi et al. 2005] suggested a new proof of the first Beurling–Malliavin theorem. An essential novelty of their proof was that it was done by (almost) purely real methods and did not use complex analysis except at one place; see [Mashregi et al. 2005] and the remark right after the formulation of Theorem B below.

Among the goals of the present paper is to give a proof of a new nontrivial generalization of Theorem A. Before stating our main results, we recall some classical definitions and fix some notation.

One of the principle objects of this paper is the class of BM majorants.

Definition 1. Let ω be a bounded nonnegative function on $\mathbb{R}$. This function is called a Beurling–Malliavin majorant (we shall further write "BM majorant" to save space) if for any $\sigma > 0$ there exists a nonzero function $f \in L^2(\mathbb{R})$ such that

(a) $|f| \le \omega$,

(b) $\operatorname{spec}(f) \subset [0, \sigma]$.

The set of all BM majorants will be further referred to as the BM class. If the conditions (a) and (b) just above are satisfied for a function ω with some fixed $\sigma > 0$, then we call such function ω a σ-admissible majorant. If we replace the condition (a) with a stronger two-sided condition $C\omega \le |f| \le \omega$ for some constant $C > 0$, then what we get is the definition of a strictly admissible majorant.

Recall that the *Poisson measure* dP on $\mathbb{R}$ is defined by the formula

$$dP(x) := \frac{dx}{1+x^2}.$$

The corresponding weighted Lebesgue space $L^1(dP)$ is the space of all functions f that satisfy $\int_{\mathbb{R}} |f|\,dP < \infty$. The expression $\int_{\mathbb{R}} \log(1/\omega)\,dP$ will be sometimes further referred to as the logarithmic integral of ω.

Note that the condition $\log(1/\omega) \in L^1(dP)$ is necessary for ω to be a BM majorant, but not sufficient; see [Mashregi et al. 2005]. What Theorem A establishes is that some additional regularity suffices for admissibility.

We remind the reader of how one should modify the Cauchy kernel in order to extend the definition of the Hilbert transformation up to the space $L^1(dP)$.

Definition 2. The *Hilbert transformation* of a function $f \in L^1(dP)$ is defined as the principal value integral

$$\mathcal{H}f(x) := \fint_{\mathbb{R}} \left(\frac{1}{x-t} + \frac{t}{t^2+1} \right) f(t)\,dt.$$

It is worth noting that the integral above converges for almost all $x \in \mathbb{R}$.

To avoid ambiguity, we stress that this definition coincides, up to an additive constant, with the classical one for functions in $L^1(\mathbb{R})$.

Let us now introduce function classes that will play an important role in what follows. To this end, we first define an auxiliary system of intervals: $J_0 = [-2, 2)$, and for $j \in \mathbb{N}$,

$$J_j = [2^j, 2^{j+1}), \quad J_{-j} = [-2^{j+1}, -2^j).$$

Definition 3. Let $\beta \in (0, 1]$. If $\beta < 1$, then we shall say that an absolutely continuous function φ belongs to the class V_β if φ is a β-Hölder function on the interval J_j with the constant κ_j and moreover these constants satisfy

$$\left(\sum_{n\in\mathbb{Z}} 2^{-|j|}\kappa_j^{1/(1-\beta)}\right)^{1-\beta} < \infty. \tag{1}$$

In the case when $\beta = 1$, we use the convention $V_\beta = \mathrm{Lip}(\mathbb{R})$.

Note that these classes resemble homogeneous weighted Sobolev spaces. We are going to work with functions that belong to intersections $L^1(dP) \bigcap V_\beta$. From the functional-analytic point of view, these intersections are Banach spaces with respect to the norms $\|\cdot\|_{L^1(dP)} + \|\cdot\|_{V_\beta}$.

We are now in position to formulate the first main result of this paper to be proved in the next section.

Theorem 1. *Let* $\omega : \mathbb{R} \to (0, 1]$ *be a function such that* $\log(1/\omega) \in L^1(dP)$, *with* $\log(1/\omega)$ *absolutely continuous and satisfying* $\log(1/\omega) \in V_\beta$ *for some* $\beta \in (0, 1]$. *Then for each* $\delta > 0$ *there exists a function* $f \in L^2(\mathbb{R})$, *not identically zero, such that* $\mathrm{spec}(f) \subset [0, \delta]$ *and* $|f(x)| \le \omega(x)$ *for all* $x \in \mathbb{R}$.

Remark. We would like to stress that one can replace the intervals J_j in the definition of the spaces V_β with any system of intervals $[\lambda_j, \lambda_{j+1})$, where $\{\lambda_j\}$ is any sequence of reals satisfying $\lambda \le \lambda_{j+1}/\lambda_j \le \Lambda$ with $1 < \lambda < \Lambda < \infty$, in a way that the corresponding version of Theorem 1 holds true.

Remark. Throughout this paper, Ω will mean $\log(1/w)$ for a function $\omega : \mathbb{R} \to (0, 1]$.

In order to get some intuition of what a "typical" function satisfying $\Omega \in L^1(dP)$ and $\Omega \in V_\beta$ looks like, the reader is welcome to think of a function whose graph consists of an infinite number of "pits" and "hills"; see the pictures of Section 1.5 in [Mashregi et al. 2005] and Figure 1 below. Of course, the same intuition applies to the functions with Lipschitz logarithm and finite logarithmic integral (i.e., those satisfying the conditions of the first Beurling–Malliavin theorem). However, we shall shortly see that there are drastic differences between these classes of functions.

Indeed, let us compare our sufficient condition of Theorem 1 with these already known. First, it is obvious that our theorem is a generalization of the first Beurling–Malliavin theorem, since it is a particular case of our result that corresponds to $\beta = 1$.

A "typical" function in classes V_β is visualized at Figure 1.

There are many other sufficient "regularity" conditions for the admissibility; see for instance those contained in [Koosis 1988; 1992; Belov and Havin 2015]. However, all these conditions are either imposed on the Hilbert transform of Ω, or they claim that only some regularization or some minorant of ω is admissible. For instance, if the condition $(\log \omega(\cdot))/(1 + (\cdot)^2)^{1/2} \in \dot{W}^{1/2,2}(\mathbb{R})$ is fulfilled for

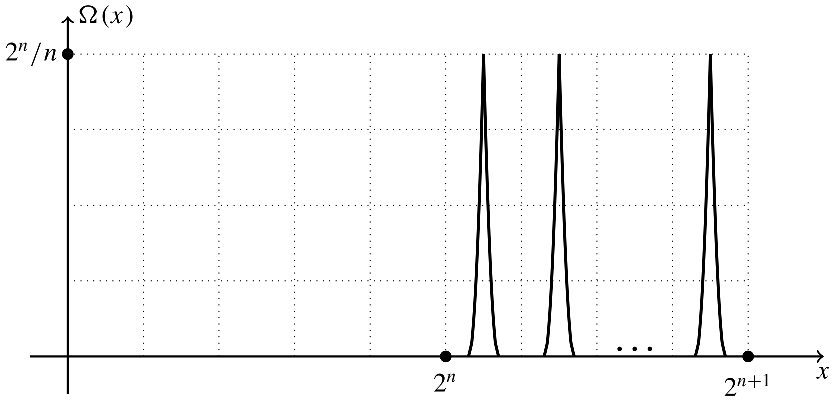

Figure 1. A "typical" function in V_β.

a function ω that has convergent logarithmic integral, then some regularization of this function is an admissible majorant; see [Beurling and Malliavin 1962].

Note that the following approximation property of the spaces V_β with $\beta \in (0, 1]$ is a direct consequence of our Theorem 1.

Corollary. *By a theorem of A. Baranov and V. Havin* [2006, Section 6], *we get that, for any* $\beta \in (0, 1]$, $\sigma > 0$, *and any* $\omega \in V_\beta$, *the space of all functions in* $L^2(\mathbb{R})$ *with the spectrum in* $\mathbb{R}\backslash[0, \sigma]$ *is not dense in the weighted Lebesgue space* $L^1(\omega)$.

We hope that our main results will find other applications in harmonic and complex analysis, in particular for the uncertainty principle and for exponential systems.

The main step of the proof of Theorem 1 is the following lemma.

Lemma 1 (a new variant of the global Nazarov lemma). *Let* $0 < \beta \leq 1$. *Suppose that* $\Omega \in L^1(dP) \cap V_\beta$ *is positive. Then, for each* $\varepsilon > 0$, *there exists a function* Ω_1, *satisfying*

(A) $\Omega(x) \leq \Omega_1(x)$ *for all* $x \in \mathbb{R}$,

(B) $\Omega_1 \in L^1(\mathbb{R}, dx/(1+x^2))$,

(C) $\mathcal{H}\Omega_1 \in \mathrm{Lip}(\varepsilon, \mathbb{R})$, *where* $\mathcal{H}$ *is the Hilbert transform on the real line.*

Indeed, Theorem 1 follows from Lemma 1, thanks to the following sufficient condition for a function to be a BM majorant, which is a consequence of a more general result, proved by Mashreghi and Havin.

Theorem B. *If* $\omega : \mathbb{R} \to (0, 1]$, $\log(1/\omega) \in L^1(dP)$ *and* $\|(\mathcal{H}\log(1/\omega))'\|_\infty < \pi\sigma$, *then* ω *is a* σ*-admissible majorant.*

Remark. The proof of Theorem B uses a one-dimensional construction coming from the classical (complex) theory of Hardy spaces on the unit circle. Namely, given a nonnegative function on the unit circle with convergent logarithmic integral there exists an analytic function whose modulus coincides with the former function. Such functions are called outer; see [Nikolski 2012] for details.

For necessary conditions for σ-admissible majorants, see [Belov 2007; 2008b; Baranov and Khavin 2006].

We briefly discuss main ideas lying behind our proof of Lemma 1. Our proof is inspired by that of the Nazarov lemma from [Mashregi et al. 2005]. Indeed, we use the beautiful idea of a so-called regularized system of intervals, which was first introduced by F. Nazarov and fruitfully used in [Mashregi et al. 2005]. Another important feature of the proof of the Nazarov lemma in [Mashregi et al. 2005] is a version of the Hadamard–Landau inequality. We have had to modify this result drastically in order for it to fit the conditions of our Lemma 1. This culminated in Lemma 4 of the present paper. On top of that, most estimates from the proof in [Mashregi et al. 2005] become considerably harder under our assumptions, in comparison to the Lipschitz condition of that work.

Note that Nazarov's lemma is by itself a highly nontrivial and very interesting result in harmonic analysis. To illustrate this, we mention [Stolyarov and Zatitskiy 2021], where the authors have utilized the main object of the Nazarov lemma, the regularized system of intervals, in some special form. For a multidimensional version of the classical Nazarov lemma, see our paper [Vasilyev 2022].

Let us now discuss the second main result of this article. Our Theorem 2, gives an answer to the following question: "How sharp is the result of Theorem 1 ?" The answer to this question is given in the following result.

Theorem 2. *For any $\beta \in (0, 1)$, there are functions $\omega : \mathbb{R} \to (0, 1]$ satisfying $\log(1/\omega) \in L^1(dP)$ and $2^{-|j|}\kappa_j^{1/(1-\beta)} \asymp 1$ in the notation of Theorem 1, that are not BM majorants.*

We remark that our Theorem 2 shows that the condition $\log(1/\omega) \in V_\beta$ in our Theorem 1 is sharp in a number of senses.

The proof of Theorem 2 builds upon one construction from [Belov and Havin 2015]. This construction says that smallness of a bandlimited function is "contagious": if such a function is small on an interval, it is also small on a much larger concentric interval. This construction is due to A. Borichev and it works only for majorants that have a growth strictly greater than linear at a sequence tending to infinity. Majorants that appear in the formulation of Theorem 2 have at most linear growth at infinity. Nevertheless, for some of these majorants, we were able to use a combination of Borichev's construction with an iteration method to prove Theorem 2.

The paper is organized as follows. Theorem 1 is proved in Sections 2 and 3. Section 4 is devoted to the proof of Theorem 2.

We finally mention some open questions concerning Theorems 1 and 2. The first question consists of determining whether the condition $\log(1/\omega) \in V_\beta$ in Theorem 1 can be weakened down to, roughly speaking, a condition of the kind "ω belongs to some Orlicz-type class, defined in the spirit of V_β classes". The second question concerns the system of intervals that are used in the definition of the spaces V_β. Namely, we would like to find a necessary and sufficient condition on the system of intervals instead of the dyadic system in Definition 3, for which the first theorem still holds. Yet another question is to find a multidimensional version of Theorem 1 which seems unavailable at the present time, according to [Han and Schlag 2020]. The fourth and the final question reads as follows. It would be also interesting to find counterparts of the main results of this paper in the context of the so-called model spaces, in spirit of Yu. S. Belov's early papers. The author plans to attack the aforementioned questions in the nearest future.

2. A new local Nazarov lemma

We accumulate here the list of the frequently used technical abbreviations and notation. For an interval $a \subset \mathbb{R}$ its length is denoted by $l(a)$, c_a will stand for the center of a and λa with λ positive will be the interval centered at c_a and whose edge length equals $\lambda l(a)$. Let I be an interval on the real line. We will denote by $T_I(x)$ the distance from $x \in \mathbb{R}$ to $\mathbb{R}\backslash I$. For a dyadic interval b, we will denote by $b^\sharp$ the dyadic parent of b. Throughout this paper, I^* will denote the unit interval $\left[-\frac{1}{2}, \frac{1}{2}\right]$. For $\beta \in (0, 1)$, we denote $\mathrm{Hol}_\beta(\kappa, I)$ the class of β-Hölder functions on the interval I, with the constant κ, i.e., all f defined on I such that for all $x \in I$ and $y \in I$ holds $|f(x) - f(y)| \le \kappa |x-y|^\beta$.

The main step of the proof of our new global Nazarov lemma is its following local variant.

Lemma 2 (a new local Nazarov lemma). *Let $I \subset \mathbb{R}$ be an interval and let $\beta \in (0, 1]$. Suppose that f is a nonnegative absolutely continuous function such that holds $f \in \mathrm{Hol}_\beta(\kappa, I)$ and $\|f\|_{L^\infty(I)} \le \delta l(I)$ for some $0 < \delta \le 1$ and $1 \le \kappa$. Then there exists a nonnegative function $F \in \mathrm{C}^\infty(\mathbb{R})$ such that*

(i) $F = 0$ *outside* $1.5I$,

(ii) $f(x) \le F(x)$ *for all* $x \in I$,

(iii) $\|(\mathcal{H}F)'\|_{L^\infty(\mathbb{R})} \lesssim \delta$,

(iv) $\int_{\mathbb{R}} F(x)\,dx \lesssim \int_I f + \kappa \delta^{-\beta} l(I)^{1-\beta} \left(\int_I f\right)^\beta$.

In the case when $\beta = 1$ in Lemma 2, the corresponding result coincides with Lemma 2.6 from [Mashregi et al. 2005].

In the formulation of Lemma 2 and until the end of the third section, the signs $\lesssim$ and $\gtrsim$ indicate that the left-hand (right-hand) part of an inequality is less than the right-hand (left-hand) part multiplied by a constant independent of δ, f, κ and I.

The rest of this section is entirely devoted to the proof of Lemma 2.

Proof of the new local Nazarov lemma. The following definition is very important.

Definition 4. We say that a dyadic interval $a \subset I$ is essential if $\|f\|_{L^\infty(a)} \ge \delta l(a)/2$. Denote by A the set of essential intervals.

It is straightforward to see that we have

$$\{x \in I : f(x) > 0\} \subseteq \bigcup_{a \in A} a.$$

However, we will not use this fact later on in our estimates.

Consider A^M, the set of maximal by inclusion elements of A. To each interval $a \in A^M$ we associate its tail $t(a)$. Informally, the tail $t(a)$ is a family of dyadic intervals that is composed of a countable number of finite series $t_p(a)$, $p = 0, 1, 2, \ldots,$ of dyadic intervals. For $p = 0$ we define $t_0(a) := a$ and for a fixed $p \geq 1$, the intervals of the family $t_p(a)$ all have length equal to $l(a)/2^p$ and their unions form the sets

$$a \cup \bigcup_{1\leq q\leq p} t_q(a) = \left\{ x \in \mathbb{R} : \frac{l(a)}{2} + l(a) \sum_{q=1}^{p-1} \frac{3^q}{2^q} \leq |x - c_a| < \frac{l(a)}{2} + l(a) \sum_{q=1}^{p} \frac{3^q}{2^q} \right\}.$$

For a detailed discussion of tails, see [Mashregi et al. 2005, Section 2.6.5]. In fact, after we have added these tails, we will get a regularized system of intervals; see [Mashregi et al. 2005, Sections 2.6 and 2.7]. Next, we define $B := \bigcup_{a\in A^M} t(a)$, and then pose $\tau := \{c \in B^M : c \subseteq I\}$. Here, B^M stands for the set of maximal by inclusion elements of B. Note that the system τ covers I, consists of dyadic intervals and any $c \in \tau$ satisfies $\delta l(c) \geq \|f\|_{L^\infty(c)}$; see [Mashregi et al. 2005].

Define for an interval $a \in \tau$ its neighborhood $N(a)$ by

$$N(a) := \left\{ b \in \tau : d(a, b) \leq 2l(a),\ \frac{1}{2} \leq \frac{l(a)}{l(b)} \leq 2 \right\}.$$

Note that $\#N(a) \leq 9$. We shall need the following property of the system τ.

Lemma 3. *Suppose that $a \in \tau$ and $b \in \tau\backslash N(a)$. If $l(b) \leq 2l(a)$ then $d(2a, 2b) \geq l(a)/2$, and if $l(b) = 2^k l(a)$ for some natural $k \geq 2$, then $d(2a, 2b) \geq 2 \cdot 3^{k-2} l(a)$.*

Proof. The proof of this lemma is not detailed here, since it can be found in [Mashregi et al. 2005, Section 2.6.6]. □

Define $2\tau := \{2c : c \in \tau\}$. As a direct consequence of the lemma, we deduce that the multiplicity $\#\{b \in 2\tau : x \in b\}$ is uniformly bounded in $x \in \mathbb{R}$. Indeed, if $b \in \tau\backslash N(a)$, then $d(2a, 2b) > 0$ and

$$\sup_{x\in\mathbb{R}} \#\{b \in 2\tau : x \in b\} \leq \sup_{a\in\tau} \#N(a) \lesssim 1.$$

Fix a bump function ϕ, i.e., $\phi \in C^\infty(\mathbb{R})$ satisfying $0 \leq \phi(x) \leq 1$ for all $x \in \mathbb{R}$, $\phi \equiv 0$ outside $1.5I^*$ and $\phi \equiv 1$ on I^*. Second, for an interval $a \in \tau$ define

$$\phi_a(\cdot) := \delta l(a) \phi\left(\frac{(\cdot) - c_a}{l(a)} \right).$$

Simple calculation shows that

$$\mathcal{H}\phi_b(\cdot) = \delta l(b) \mathcal{H}\phi\left(\frac{(\cdot) - c_b}{l(b)} \right).$$

Hence we infer the inequality $\|(\mathcal{H}\phi_b)'\|_{L^\infty(\mathbb{R})} \lesssim \delta$. We finally define F by

$$F := \sum_{a\in\tau} \phi_a.$$

Now we have to check the required properties of the majorant F. The first one follows readily from the definition of F. To prove the second one, note that for all $a \in \tau$ we have $\|f\|_{L^\infty(a)} \leq \delta l(a)$. Indeed, suppose the contrary, i.e., that $\|f\|_{L^\infty(a_0)} > \delta l(a_0)$ for some $a_0 \in \tau$. This means that

$$\|f\|_{L^\infty(a_0^\sharp)} \geq \|f\|_{L^\infty(a_0)} > \delta l(a_0) = \delta \left(\frac{l(a_0^\sharp)}{2} \right),$$

which in turn signifies that $a_0^\sharp$ is an essential interval and hence $a_0^\sharp \in \tau$. This contradicts the definition of τ. From here we deduce that if $x \in a \in \tau$, then

$$F(x) \geq \delta l(a) \geq \|f\|_{L^\infty(a)} \geq f(x).$$

Next we estimate the integral of the function F. To this end, we prove a variant of the Hadamard–Landau inequality which is appropriate for our goals.

Lemma 4. *Let a be an interval such that $a \in A^M$. Then we have*

$$\|f\|^2_{L^\infty(a)} \lesssim \left(\int_a f\right)\delta + \kappa\left(\int_a f\right)^\beta (\delta l(a))^{1-\beta},$$

where $C(r)$ is a positive constant, depending on r only.

Proof. Let $x_0 \in a$ be a point such that $\|f\|_{L^\infty(a)} = f(x_0)$. Suppose with no loss of generality that $a_+ - x_0 \geq l(a)/2$, where a_+ is the right end of the interval a. Consider a point $x \in (x_0, a_+)$. Since f is Hölder continuous, we hence infer the estimate

$$f(x) \geq f(x_0) - \kappa(x - x_0)^\beta.$$

Let $\upsilon := (f(x_0)/\kappa)^{1/\beta}$. We shall treat two cases separately, according to the value of υ. First, we suppose that $\upsilon < l(a)/2$. Observe that in this case the point $x_0 + \upsilon$ belongs to the interval a. We integrate the estimate just above using this observation and deduce that

$$\begin{aligned}\int_a f &\geq \int_{x_0}^{x_0+\upsilon/2} f(x)\,dx \geq \int_{x_0}^{x_0+\upsilon/2} f(x_0) - \kappa(x-x_0)^\beta\,dx \\ &= \frac{f(x_0)}{2}\left(\frac{f(x_0)}{\kappa}\right)^{1/\beta} - \frac{\kappa}{2^{1+\beta}(\beta+1)}\left(\frac{f(x_0)}{\kappa}\right)^{(\beta+1)/\beta} \\ &\gtrsim \frac{\|f\|^{(\beta+1)/\beta}_{L^\infty(a)}}{\kappa^{1/\beta}} = \frac{\|f\|^{2/\beta}_{L^\infty(a)}\|f\|^{(\beta-1)/\beta}_{L^\infty(a)}}{\kappa^{1/\beta}} \gtrsim \frac{\|f\|^{2/\beta}_{L^\infty(a)}(\delta l(a))^{(\beta-1)/\beta}}{\kappa^{1/\beta}},\end{aligned} \tag{2}$$

where the last bound above follows from the fact that $a \in A^M$. Hence we have that

$$\|f\|^2_{L^\infty(a)} \lesssim \kappa\left(\int_a f\right)^\beta (\delta l(a))^{1-\beta}.$$

Consider now the second case, where $\upsilon \geq l(a)/2$. In this case we shall use the fact that the point $x_0 + l(a)/2$ belongs to the interval a. Integrating the same inequality as in the first case yields

$$\int_a f \geq \int_{x_0}^{x_0+l(a)/2} f(x)\,dx \geq \frac{l(a)f(x_0)}{2} - \frac{\kappa l(a)^{\beta+1}}{2^{\beta+1}(\beta+1)} = \frac{l(a)}{2}\left(f(x_0) - \frac{\kappa}{\beta+1}\cdot\left(\frac{l(a)}{2}\right)^\beta\right). \tag{3}$$

Note that since $\upsilon \geq l(a)/2$, we have also that $f(x_0)/\kappa_a \geq (l(a)/2)^\beta$. Let us use this in the following way:

$$\int_a f \geq \frac{l(a)}{2}\left(f(x_0) - \frac{f(x_0)}{\beta+1}\right) \gtrsim \delta^{-1}\|f\|^2_{L^\infty(a)},$$

thanks to the fact that $a \in A^M$. Hence Lemma 4 is proved. ∎

So, let us start the estimates of the integral of the function F:

$$\begin{aligned}\int_{\mathbb{R}} F &\le \sum_{b\in A^M}\int_{\mathbb{R}}\phi_b+\sum_{c\in A^M}\sum_{b\in t(c)\setminus c}\int_{\mathbb{R}}\phi_b \le \delta\sum_{b\in A^M} l(b)^2+\delta\sum_{c\in A^M}\sum_{b\in t(c)\setminus c} l(b)^2\\ &\lesssim \delta\sum_{b\in A^M} l(b)^2+\delta\sum_{c\in A^M}\sum_{p=1}^{\infty}\sum_{b\in t_p(c)} l(b)^2 \lesssim \delta\sum_{b\in A^M} l(b)^2+\delta\sum_{c\in A^M}\sum_{p=1}^{\infty}3^p\Big(\frac{l(c)}{2^p}\Big)^2\\ &\lesssim \delta\sum_{c\in A^M} l(c)^2 \lesssim \delta^{-1}\sum_{c\in A^M}\|f\|^2_{L^\infty(c)}.\end{aligned}\tag{4}$$

We further use the result of Lemma 4 to continue the estimates of the integral of the function F:

$$\begin{aligned}\int_{\mathbb{R}} F &\lesssim \sum_{c\in A^M}\int_c f+\delta^{-1}\sum_{c\in A^M}\kappa\Big(\int_c f\Big)^{\beta}(\delta l(c))^{1-\beta}\\ &\le \int_I f+\delta^{-\beta}\kappa\Big(\sum_{c\in A^M}\int_c f\Big)^{\beta}\cdot\Big(\sum_{c\in A^M} l(c)\Big)^{1-\beta}\\ &\lesssim \int_I f+\delta^{-\beta}\kappa l(I)^{1-\beta}\Big(\int_I f\Big)^{\beta}.\end{aligned}\tag{5}$$

The last and second-to-last inequalities just above are in need of explanation. The last estimate uses the fact that intervals of A^M are nonoverlapping, whereas the penultimate bound follows from the Hölder inequality.

It remains to derive the inequality on the derivative of the Hilbert transformation of the function F. First, we shall obtain this estimate for $x\in\bigcup_{b\in\tau}2b$. Let $a(=a(x))$ denote the interval from τ such that $x\in 2a$. We isolate the neighborhood $N(a)$ from its complement in τ and infer the inequality

$$|(\mathcal{H}F)'(x)|\le\sum_{b\in N(a)}|(\mathcal{H}\phi_b)'(x)|+\sum_{\substack{b\in\tau\setminus N(a)\\ l(b)\le 2l(a)}}|(\mathcal{H}\phi_b)'(x)|+\sum_{k=2}^{\infty}\sum_{\substack{b\in\tau\setminus N(a)\\ l(b)=2^k l(a)}}|(\mathcal{H}\phi_b)'(x)|=:S_1+S_2+S_3.$$

We shall estimate the terms S_1, S_2 and S_3 separately. We start with the sum S_1, whose estimate turns out to be easy:

$$S_1\le \#N(a)\sup_{b\in\tau}\|(\mathcal{H}\phi_b)'\|_{L^\infty(\mathbb{R})}\lesssim\delta.$$

We further proceed to the second term. We use a simple estimate on the kernel of the Hilbert transformation, the fact that the system of intervals $\{2b\}_{b\in\tau}$ (by Lemma 3) has finite multiplicity and Lemma 3 to get

$$\begin{aligned}S_2 &\lesssim \sum_{\substack{b\in\tau\setminus N(a)\\ l(b)\le 2l(a)}}\int_{\mathbb{R}}\phi_b(t)\frac{\partial}{\partial x}\Big(\frac{1}{t-x}\Big)dt\lesssim\sum_{\substack{b\in\tau\setminus N(a)\\ l(b)\le 2l(a)}}\int_{\mathbb{R}}\frac{\phi_b(t)}{(t-x)^2}dt\\ &\lesssim\sum_{\substack{b\in\tau\setminus N(a)\\ l(b)\le 2l(a)}}\int_{1.5b}\frac{\delta l(a)\,dt}{(t-x)^2}\lesssim\delta l(a)\int_{\{|u|\ge l(a)/2\}}\frac{du}{|u|^2}\lesssim\delta.\end{aligned}$$

The third term can be estimated as well using Lemma 3:

$$S_3 \lesssim \sum_{k=2}^{\infty} \sum_{\substack{b\in\tau\setminus N(a)\\ l(b)=2^k l(a)}} \int_{2b} \frac{\phi_b(t)\,dt}{(t-x)^2} \leq \sum_{k=2}^{\infty} 2^k \delta l(a) \sum_{\substack{b\in\tau\setminus N(a)\\ l(b)=2^k l(a)}} \int_{2b} \frac{dt}{(t-x)^2} \lesssim \sum_{k=2}^{\infty} 2^k \delta l(a) \int_{\{|u|\geq 2\cdot 3^{k-2} l(a)\}} \frac{du}{u^2} \lesssim \delta,$$

and the lemma for $x \in \bigcup_{b\in\tau} 2b$ follows.

Next, if a point $z \in \mathbb{R}$ is situated at a positive distance from the set $\bigcup_{b\in\tau} 2b$, then denote by x the point of this set closest to z, and let $a(= a(x))$ be an interval as above. We infer the estimates

$$|(\mathcal{H}F)'(z)| \leq \sum_{b\in\tau\setminus N(a)} |(\mathcal{H}\phi_b)'(z)| + \sum_{b\in N(a)} |(\mathcal{H}\phi_b)'(z)| \lesssim \sum_{b\in\tau\setminus N(a)} \int_{\mathbb{R}} \frac{\phi_b(t)\,dt}{(t-x)^2} + \#N(a) \sup_{b\in\tau} \|(\mathcal{H}\phi_b)'\|_{L^\infty(\mathbb{R})}.$$

Thanks to the estimates of the terms S_1, S_2 and S_3, we conclude that the needed variant of the local Nazarov lemma is proved. □

3. Proof of a new global Nazarov lemma

In this section, we shall derive the global Nazarov lemma from the local one.

Proof. Until the end of the third section, the signs $\lesssim$ and $\gtrsim$ indicate that the left-hand (right-hand) part of an inequality is less than the right-hand (left-hand) part multiplied by a “harmless” positive constant.

Note that we may assume in the global Nazarov lemma that $\Omega(x) = 0$ for $|x| \leq R$, with R being an arbitrary large positive number. Indeed, if it is not the case, then consider the function $\Omega(\,\cdot\,) = \max(0, \Omega - \mathcal{M})(\,\cdot\,)$, where $\mathcal{M} := \max_{x\in B(0,R)} \Omega(x)$. If Ω_I is a majorant of the function Ω satisfying properties (B) and (C) then the function $\Omega_I + \mathcal{M}$ will be the desired majorant of the function Ω.

Fix $0 < \varepsilon \leq 1$ and choose $1 < R_1$ so big that

$$\int_{\mathbb{R}\setminus(-R_1,R_1)} \Omega\, dP \leq \varepsilon.$$

Since the series (1) converges, there exists a natural N_1 so big that for all $j > N_1$ it holds that $\kappa_j^{1/(1-\beta)} 2^{-j} \leq \varepsilon^{1/(1-\beta)}$. As a consequence, we infer for all such j the bound

$$\kappa_j 2^{j(\beta+1)} \leq \varepsilon 2^{2j}. \tag{6}$$

By the previous paragraph, we may assume Ω is equal to zero on the interval $(-\max(R_1, 2^{N_1}), \max(R_1, 2^{N_1}))$.

Recall the above-defined system of intervals $J_0 = [-2, 2)$, and, for $j \in \mathbb{N}$,

$$J_j = [2^j, 2^{j+1}), \qquad J_{-j} = [-2^{j+1}, -2^j).$$

Next, we shall prove for $x \in \mathbb{R}$ the inequality

$$\Omega(x) \lesssim \varepsilon|x|. \tag{7}$$

With no loss of generality, we suppose that $x > 0$ and we let $n \in \mathbb{N}$ be such that $2^n \leq |x| < 2^{n+1}$. First, note that according to the previous paragraph, the bound (7) is obvious once $|x| \lesssim 1$. Second, for $1 \lesssim |x|$

we will argue as in Lemma 4. We thus find a point $x_0 \in [2^n, 2^{n+1})$ such that $\Omega(x_0) = \|\Omega\|_{L^\infty(J_n)}$. Then, for any $y \in J_n$ we have

$$\Omega(y) \geq \Omega(x_0) - \kappa_n |y - x_0|^\beta.$$

Once again, without loss of generality we suppose that the point $x_0 + 2^n/2$ belongs to the interval J_n. We finally infer the chain of inequalities

$$\begin{aligned} \varepsilon &\geq \int \Omega\, dP \geq |x|^{-2} \int_{J_n} \Omega(y)\, dy \geq |x|^{-2} \int_{x_0}^{x_0+2^n/2} (\Omega(x_0) - \kappa_n (y - x_0)^\beta)\, dy \\ &\geq |x|^{-2}(2^{n-1}\Omega(x_0) - C(\beta)\kappa_n 2^{n(\beta+1)}). \end{aligned} \tag{8}$$

Hence, the bound (7) is proved, by virtue of (6).

Apply the local lemma to each interval J_j and the corresponding restriction $f_j = \Omega \llcorner J_j$. Indeed, Lemma 2 can be applied since these functions satisfy

$$\|f_j\|_\infty \leq \varepsilon 2^j \leq \varepsilon l(J_j)$$

by (7). Thus we obtain functions F_j for $j \in \mathbb{Z}$. The needed majorant Ω_1 is defined by

$$\Omega_1 = \sum_{j \in \mathbb{Z}} F_j.$$

Now, we shall check the required properties of Ω_1. The first property follows obviously from the local lemma. We proceed to the second one:

$$\begin{aligned} \int_{\mathbb{R}} \Omega_1(t)\, dP(t) &= \sum_{j \in \mathbb{Z}} \int_{\mathbb{R}} F_j(t)\, dP(t) \lesssim \sum_{j \in \mathbb{Z}} \int_{1.5 J_j} F_j(t) \frac{dt}{2^{2|j|}} \\ &\lesssim \varepsilon^{-\beta} \sum_{j \in \mathbb{Z}} 2^{|j|(\beta-1)} \kappa_j \left(\int_{1.5 J_j} \Omega(t) \frac{dt}{2^{2|j|}} \right)^\beta + \sum_{j \in \mathbb{Z}} \int_{1.5 J_j} \Omega(t) \frac{dt}{2^{2|j|}} \\ &\lesssim \varepsilon^{-\beta} \left(\sum_{j \in \mathbb{Z}} 2^{-|j|} \kappa_j^{1/(1-\beta)} \right)^{1-\beta} \cdot \left(\sum_{j \in \mathbb{Z}} \int_{1.5 J_j} \Omega(t) \frac{dt}{2^{2|j|}} \right)^\beta + \varepsilon \lesssim \varepsilon, \end{aligned} \tag{9}$$

where in the third inequality above we have used the local lemma and in the penultimate bound we have used the Hölder inequality.

So, it remains to check that the third conclusion holds. First, fix a point $x \in \mathbb{R}$. Second, denote by $S(x)$ the interval from the system $\mathcal{F} = \{J_j\}_{j \in \mathbb{Z}}$ such that $x \in S(x)$. Next, denote by $U(x)$ the subset of $\mathcal{F}$ consisting of $S(x)$ and its two neighbor intervals and by $W(x)$ its complement: $W(x) = \mathcal{F} \setminus U(x)$. Finally, write the function Ω_1 as a sum of two functions as follows:

$$\Omega_1 = \sum_{j \in W(x)} F_j + \sum_{j \in U(x)} F_j =: \omega_1 + \omega_2.$$

Since there is only a finite number of intervals in the family $U(x)$, we see that

$$|(\mathcal{H}\omega_2)'(x)| \lesssim \#U(x) \sup_{j\in U(x)} \|(\mathcal{H}F_j)'\|_\infty \lesssim \varepsilon,$$

where we have just used condition (iii) of Lemma 2 in the last estimate. On the other hand, since $\operatorname{supp}(\omega_1) \subseteq \bigcup_{j\in W(x)} 1.5J_j$ we deduce that

$$\operatorname{supp}(\omega_1) \subseteq \left\{t \in \mathbb{R} : |t-x| \geq \frac{l(S(x))}{4}\right\} \subseteq \left\{t \in \mathbb{R} : |t-x| \geq \frac{|x|}{16}\right\}.$$

Therefore, we arrive at the chain of inequalities

$$\begin{aligned}|(\mathcal{H}\omega_1)'(x)| &= \left|\left(\int_{\mathbb{R}} \omega_1(t)\frac{1}{t-x}\,dt\right)'\right| = \left|\int_{\mathbb{R}} \omega_1(t)\frac{\partial}{\partial x}\left(\frac{1}{t-x}\right)dt\right| \\ &= \int_{\mathbb{R}} \omega_1(t)\frac{1}{(t-x)^2}\,dt \lesssim \int_{\mathbb{R}} \Omega_1(t)\,dP(t) \lesssim \varepsilon,\end{aligned}$$

thanks to the bound (9).

Hence, the needed variant of the Nazarov lemma is proved. □

Thus, Theorem 1 is also proved, via Theorem B.

4. Sharpness of Lemma 1

Note that the proof of Theorem 2 is a direct consequence of the following proposition.

Proposition 1. *Let $\gamma > \frac{1}{2}$ and define $I_n := [2^n - 2^n/n^\gamma, 2^n + 2^n/n^\gamma]$ for $n \geq 3$. Consider for $x \in \mathbb{R}$ the function*

$$\omega(x) := \begin{cases}\exp(-n^{\gamma-1/2}T_{I_n}(x)) & \text{if } x \in I_n \text{ with } n \geq 3,\\ 1 & \text{otherwise.}\end{cases} \tag{10}$$

We claim that $\log(1/\omega) \in L^1(dP)$ and that $\log(1/\omega)$ satisfies the regularity assumption of Theorem 2, though ω is not a BM majorant.

The graph of the function $\Omega = \log(1/\omega)$ and the main idea of the proof below (i.e., the iteration) is illustrated at Figure 2.

Proof. The first two claims are easy to verify, so we omit their proofs.

Let σ be a positive constant and consider the Bernstein space $\mathcal{E}_{\sigma,1}$, i.e., the space of all entire functions f such that

$$|f(z)| \leq e^{\sigma|z|} \text{ for any } z \in \mathbb{C} \quad \text{and} \quad |f| \leq 1 \text{ on } \mathbb{R}.$$

Recall Lemma 1 from [Belov and Havin 2015].

Lemma A. *For any $\sigma > 0$ there exist a (small) $\alpha(\sigma) \in \left(0, \frac{1}{2}\right)$ and a (big) $h(\sigma) > 2$ such that for any $h \geq h(\sigma)$, any $f \in \mathcal{E}_{\sigma,1}$ and any compact interval $I \subset \mathbb{R}$*

$$|f| \leq e^{-hT_I} \text{ on } \mathbb{R} \quad \Longrightarrow \quad |f| \leq e^{-Ch|I|} \text{ on } \tilde{I},$$

where $C > 0$ is an absolute constant and $\tilde{I}$ is the interval centered at $c(I)$ with $|\tilde{I}| = h^{\alpha(\sigma)}|I|$.

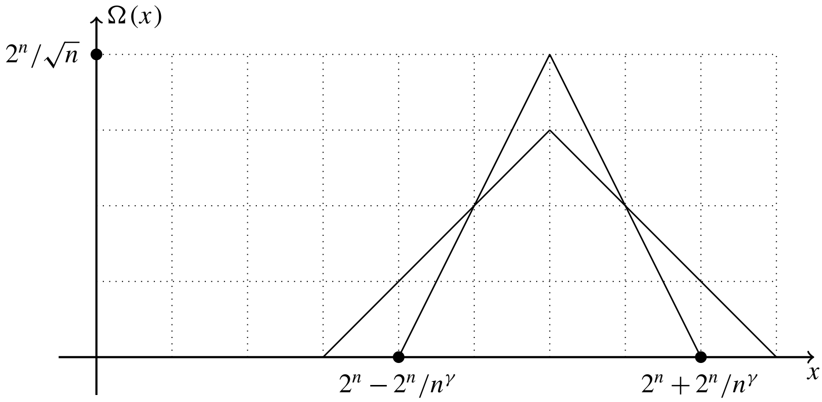

Figure 2. The main idea of the proof of Theorem 2.

Let us prove that ω is not in the BM class. Suppose the contrary. Hence, for a fixed $\sigma > 0$ there exists a function, not identically zero, satisfying $f \in L^2(\mathbb{R})$, $\operatorname{spec}(f) \subset [0, \sigma]$ and $|f(x)| \le \omega(x)$ for all real x. We shall now use Lemma A. Notice that the function f satisfies the conditions of this lemma with $h := n^\vartheta$, where $\vartheta := \gamma - \frac{1}{2} > 0$ and $I := I_n$ for $n \ge n(\gamma)$. We deduce from this lemma that there exists a universal constant C and a power $\alpha \in \left(0, \frac{1}{2}\right)$, depending only on σ such that

$$|f(x)| \le \exp(-C2^n)$$

on the interval $I_{n,1} := (n^{\vartheta\alpha}/2)I_n$.

Remark. From now until the end of the present article, the sign $X \asymp Y$ means that $\mathcal{C}_1 Y \le X \le \mathcal{C}_2 Y$ for some constants $\mathcal{C}_1$ and $\mathcal{C}_2$ depending only on γ, α, σ and C. In this case, we shall say that X is of order Y.

Note that the length of this interval satisfies the bound $|I_{n,1}| \asymp n^{\vartheta(\alpha-1)}2^n$. As a consequence, we infer that the inequality

$$|f(x)| \le \exp(-Cn^{\vartheta(\alpha-1)}T_{I_{n,1}}(x))$$

is valid for $x \in I_{n,1}$. This means that we can apply Lemma A once again, now for $h := Cn^{\vartheta(1-\alpha)}$ and $I := I_{n,1}$. This yields the bound

$$|f(x)| \le e^{-Ch|I_{n,1}|} = e^{-C^2 2^n},$$

which is true for $x \in I_{n,2} := ((Cn^{\vartheta(1-\alpha)})^\alpha/2)I_{n,1}$. It is not difficult to see that the corresponding interval $I_{n,2}$ has length of order

$$C^\alpha n^{\vartheta(1-\alpha)\alpha+\vartheta(\alpha-1)}2^n = C^\alpha n^{-\vartheta(1-\alpha)^2}2^n.$$

Acting inductively, after m steps, we arrive at the estimate $|f(x)| \le \exp(-C^m 2^n)$, verified by f for $x \in I_{n,m}$ with

$$|I_{n,m}| \asymp n^{-\vartheta(1-\alpha)^m}2^n.$$

Maybe, it is worth noting that $I_{k,m} \cap I_{n,m} = \varnothing$ for any natural m, once $n \ne k$. This results from the fact that we assume, as we can, that $C < 1$.

We are now in position to prove that $f = 0$ identically, which will lead to a contradiction. To this end, we estimate the logarithmic integral of f. For each natural number m it holds that

$$\int_{\mathbb{R}} \log|f(x)|\,dP(x) \le -\sum_{n\ge 3}\int_{I_{n,m}} 2^{-2n}C^m 2^n\,dx \asymp -\sum_{n\ge 3} n^{-\vartheta(1-\alpha)^m}.$$

Choosing m sufficiently large and recalling that $(1-\alpha)\in(0,1)$, we arrive at the formula

$$\int_{\mathbb{R}} \log|f(x)|\,dP(x) = -\infty.$$

Since $f \in L^2(\mathbb{R})$ has the spectrum in the interval $[0,\sigma]$, it hence belongs to the Hardy class $H^2(\mathbb{R})$. From the Jensen inequality, see [Havin and Jöricke 1994], we deduce that $f = 0$ identically, which contradicts our assumption. Hence, the second theorem is proved. □

Remark. Alas, our proof above does not work if one replaces in (10) and in the definition of intervals I_n the powers n^γ by θ^n with $\theta \in (1,2)$.

Remark. It can be seen exactly as above that the function ω_* is not a strictly admissible majorant; recall Definition 1. For a detailed discussion of strictly admissible majorants, see [Belov 2008a].

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