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THE 3D STRICT SEPARATION PROPERTY FOR THE NONLOCAL CAHN–HILLIARD EQUATION WITH SINGULAR POTENTIAL

ANDREA POIATTI

We consider the nonlocal Cahn–Hilliard equation with singular (logarithmic) potential and constant mobility in three-dimensional bounded domains and we establish the validity of the instantaneous strict separation property. This means that any weak solution, which is not a pure phase initially, stays uniformly away from the pure phases ± 1 from any positive time on. This work extends the result in dimension two for the same equation and gives a positive answer to the long-standing open problem of the validity of the strict separation property in dimensions higher than 2. In conclusion, we show how this property plays an essential role to achieve higher-order regularity for the solutions and to prove that any weak solution converges to a single equilibrium.

1. Introduction

The diffuse interface theory, also called the phase field method, is one of the oldest and most efficient approaches to multiphase problems. This approach is characterized by the notion of diffuse interface, meaning that the transition layer between the two phases or components has a narrow finite size. The interface is not explicitly tracked as in boundary integral and front-tracking methods. On the other hand, the phase state is incorporated into the macroscopic equations and the internal microstructures arise from the competition between the diffusion and aggregation mechanisms included in the free energy. The fundamental advantage of this theory is the natural representation of singular interfacial behaviors, such as topological change, self-intersection, merger and pinch-off.

Consider a mixture of two incompatible substances A and B, which is homogeneously distributed and isothermal. Under certain circumstances, namely if the temperature is above a critical threshold θ_c, this configuration is stable; however, if suddenly cooled down and kept at $\bar{\theta} < \theta_c$, the initially (macroscopically) homogeneous alloy evolves in a way such that A-rich and B-rich regions appear and grow. The Cahn–Hilliard equation was introduced in [Allen and Cahn 1979; Cahn and Hilliard 1958] to model this phenomenon in iron alloys, and it has now become a widespread model, since phase separation has become a paradigm also in cell biology (see, e.g., [Dolgin 2018]). Let Ω be a bounded domain in $\mathbb{R}^d$, $d = 2, 3$, filled with a binary solution consisting of A and B atoms, and let us fix a time horizon $T > 0$.

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We define their relative mass fraction difference as ϕ, which is the phase-field variable, whose smooth but highly localized variation is associated with the (diffuse) interface. If the mixture is isothermal and the molar volume is uniform and independent on pressure, the system evolves in order to minimize the free energy functional

$$\mathcal{U}(\phi) := \int_\Omega \Big(\frac{\epsilon}{2}|\nabla\phi|^2 + \Psi(\phi)\Big)\,\mathrm{d}x, \tag{1-1}$$

where $\Psi(\phi)$ is the Helmholtz free energy density

$$\Psi(s) = \frac{\bar{\alpha}}{2}((1+s)\ln(1+s) + (1-s)\ln(1-s)) - \frac{\alpha_0}{2}s^2 = F(s) - \frac{\alpha_0}{2}s^2 \quad \text{for all } s \in [-1, 1], \tag{1-2}$$

with $\bar{\alpha}$ such that $0 < \bar{\alpha} < \alpha_0$, constants related to the temperature of the mixture. The term ϵ is called capillary coefficient, related to the thickness of interfaces. The potential defined in this way is called *singular*, whereas many authors (see, e.g., [Fife 2000]) considered a proper approximation, which avoids the fact that Ψ' is unbounded at the pure phases -1 and 1: namely, the significant potential is considered to be still a double-well, but with the two local minima coinciding with the pure phases. The most common choice is polynomial of even degree, like the case $\Psi(s) = \frac{1}{4}(s^2-1)^2$. However, in the case of polynomial potentials, it is worth recalling that it is not possible to guarantee the existence of physical solutions, that is, solutions for which $-1 \le \phi(x, t) \le 1$. Following, e.g., [Lowengrub and Truskinovsky 1998], we get a differential description of the phenomenon of the phase separation as

$$\partial_t\phi + \operatorname{div} \boldsymbol{J} = 0 \quad \text{in } \Omega \times (0, T), \tag{1-3}$$

where ϕ is the order parameter and $\boldsymbol{J}$ is the diffusional flux given by Fick's law,

$$\boldsymbol{J} = -M(\phi)\nabla\frac{\delta\mathcal{U}(\phi)}{\delta\phi} = -M(\phi)\nabla(-\epsilon\Delta\phi + \Psi'(\phi)),$$

where $\delta\mathcal{U}(\phi)/\delta\phi$ is the variational derivative of $\mathcal{U}(\phi)$. The function $M(\phi)$ is the mobility of the substances and in this work will be considered as a unitary constant (see, for instance, [Cherfils et al. 2011; Elliott and Garcke 1996] for an analysis of the case of nonconstant and degenerate mobility, i.e., vanishing at the pure phases). The Cahn–Hilliard equation with constant mobility then reads

$$\begin{cases} \partial_t\phi = \Delta\mu & \text{in } \Omega \times (0, T), \\ \mu = -\epsilon\Delta\phi + \Psi'(\phi) & \text{in } \Omega \times (0, T), \end{cases} \tag{1-4}$$

with the initial condition ϕ_0 and two boundary conditions which are generally the following:

$$\partial_{\boldsymbol{n}}\phi = 0, \quad \partial_{\boldsymbol{n}}\mu = 0, \quad \text{on } \partial\Omega \times (0, T), \tag{1-5}$$

with $\boldsymbol{n}$ as the outer normal vector. The former condition means that no mass flux occurs at the boundary, while the latter requires the interface to be orthogonal at the boundary.

It is worth noticing that the free energy $\mathcal{U}$ in (1-1) only focuses on short range interactions between particles. Indeed, the gradient square term accounts for the fact that the local interaction energy is spatially dependent and varies across the interfacial surface due to spatial inhomogeneities in the concentration. Going back to the general approach of statistical mechanics, the mutual short and long range interactions between particles is described through convolution integrals weighted by interactions kernels. Following this approach, Giacomin and Lebowitz [1996; 1997; 1998] observed that a physically more rigorous derivation leads to nonlocal dynamics, which is the nonlocal Cahn–Hilliard equation. In particular, this equation is rigorously justified as a macroscopic limit of microscopic phase segregation models with particles conserving dynamics. In this case, the gradient term is replaced by a nonlocal spatial interaction integral, namely, the energy is defined as

$$\mathcal{E}(\phi) := -\frac{1}{2}\int_\Omega\int_\Omega J(x-y)\phi(x)\phi(y)\,\mathrm{d}x\mathrm{d}y + \int_\Omega F(\phi(x))\,\mathrm{d}x, \tag{1-6}$$

where J is a sufficiently smooth symmetric interaction kernel. Note that this functional is characterized by a competition between the mixing entropy F and a nonlocal demixing term. As shown in [Giacomin and Lebowitz 1997] (see also [Gal et al. 2017; 2023a]), the energy $\mathcal{U}$ can be seen as an approximation of $\mathcal{E}$, as long as we suitably redefine F as $\widetilde{F}(x,s) = F(s) - \frac{1}{2}(J*1)(x)s^2$. In particular, we can rewrite $\mathcal{E}$ as

$$\begin{aligned}\mathcal{E}(\phi) &= \frac{1}{4}\int_\Omega\int_\Omega J(x-y)|\phi(y)-\phi(x)|^2\,\mathrm{d}x\mathrm{d}y + \int_\Omega\Big(F(\phi(x)) - \frac{a(x)}{2}\phi^2(x)\Big)\,\mathrm{d}x\\ &= \frac{1}{4}\int_\Omega\int_\Omega J(x-y)|\phi(y)-\phi(x)|^2\,\mathrm{d}x\mathrm{d}y + \int_\Omega \widetilde{F}(\phi(x))\,\mathrm{d}x,\end{aligned}$$

with $a(x) = (J*1)(x)$. If we formally interpret $\widetilde{F}$ as the potential Ψ of (1-1), we realize that the (formal) first approximation of the nonlocal interaction is $\frac{k}{2}|\nabla\phi|^2$, for some $k>0$, as long as J is sufficiently peaked around 0. In the case $\Omega = \mathbb{T}^3$ (see, e.g., [Giacomin and Lebowitz 1998]), the term $J*1$ is a constant: thus $\mathcal{E}$ and $\mathcal{U}$ appear to be very similar. In particular, in this case, corresponding to set $a(x) = \alpha_0$, nonlocal-to-local asymptotics results have been obtained in [Davoli et al. 2021a; 2021b] (see also [Gal and Shomberg 2022]) for the nonlocal equation (1-7) below: namely, the solution to the nonlocal equation converges, under suitable conditions on the data of the problem, to the weak solution of (1-4)–(1-5).

The resulting nonlocal Cahn–Hilliard equation then reads (see [Gal et al. 2017; 2023a])

$$\begin{cases}\partial_t\phi - \Delta\mu = 0 & \text{in } \Omega\times(0,T),\\ \mu = F'(\phi) - J*\phi & \text{in } \Omega\times(0,T),\\ \partial_{\boldsymbol{n}}\mu = 0 & \text{on } \partial\Omega\times(0,T),\\ \phi(\,\cdot\,,0) = \phi_0 & \text{in } \Omega.\end{cases} \tag{1-7}$$

From now on we will refer to problem (1-4)–(1-5) as the local Cahn–Hilliard equation, in order to distinguish it from the nonlocal one in (1-7).

The well-posedness theory of Cahn–Hilliard equations with logarithmic (or singular) potential has been widely studied. The local Cahn–Hilliard equation (1-4)–(1-5) has been studied in [Abels and Wilke 2007; Debussche and Dettori 1995; Elliott and Luckhaus 1991; Giorgini et al. 2017; Londen and Petzeltová 2018; Miranville and Zelik 2004] (see also [Cherfils et al. 2011; Gal et al. 2023a] for a review and an insight analysis about this topic). Concerning the nonlocal Cahn–Hilliard equation, the physical relevance of nonlocal interactions was already pointed out in the pioneering paper [van der Waals 1982] (see also [Emmerich 2003, 4.2]) and studied for different kind of evolution equations, mainly Cahn–Hilliard and phase-field systems (see, e.g., [Bertozzi et al. 2007; Colli et al. 2007; Gajewski and Zacharias 2003; Gal and Grasselli 2014; Krejčí et al. 2007]). In particular, regarding the nonlocal system (1-7), the existence of weak solutions and their uniqueness, and the existence of the connected global attractor were proven in [Frigeri et al. 2016; Frigeri and Grasselli 2012a; 2012b]. Moreover, well-posedness and regularity of weak solutions are studied in [Gal et al. 2017], namely, in this work the authors establish the validity of the strict separation property in dimension two for the nonlocal Cahn–Hilliard equation (1-7) with constant mobility and singular potential. This means that if the initial state is not a pure phase (i.e., $\phi_0 \equiv 1$ or $\phi_0 \equiv -1$), then the corresponding solution stays away from the pure states in finite time, uniformly with respect to the initial datum. Exploiting this crucial property in dimension two, the authors derive straightforward consequences, such as further regularity results as well as the existence of regular finite-dimensional attractors and the convergence of a weak solution to a single equilibrium point. In [Gal et al. 2023a], the same authors propose an alternative argument to prove the strict separation property in dimension two, relying on a De Giorgi's iteration scheme (see [Gal et al. 2023a, Theorem 4.1]).

In the present work we extend the results of [Gal et al. 2023a] to the case of three-dimensional bounded domains, namely we prove the validity of the instantaneous strict separation property in dimension three for the system (1-7) with singular potential F. Our main result is the following: given a weak solution to (1-7),

$$\text{for all } \tau > 0 \text{ there exists } \delta > 0 \text{ such that } |\phi(x,t)| \le 1-\delta \quad \text{for a.e. } (x,t) \in \Omega \times (\tau, +\infty), \tag{1-8}$$

where δ depends on the parameters of the problem, the initial datum ϕ_0 and τ. Furthermore, we show that, if the initial datum ϕ_0 is more regular and already strictly separated from the pure phases, then (1-8) also holds with $\tau = 0$, i.e., the solution is uniformly strictly separated at almost any time $t \ge 0$. To assess the importance of property (1-8), similarly to [Gal et al. 2017], we infer some additional regularization results for any weak solution and we prove that each weak solution converges to a single stationary state.

As far as we are aware, this is the first time the instantaneous strict separation property is shown in three-dimensional bounded domains for the Cahn–Hilliard equation with constant mobility and singular (logarithmic) potential. Indeed, the only available result in dimension three regards the nonlocal Cahn–Hilliard equation with degenerate mobility and singular potential; see [Londen and Petzeltová 2011]. For the local Cahn–Hilliard equation the instantaneous separation property was first proven to hold in [Miranville and Zelik 2004], but only in dimension two. Concerning dimension three, only the asymptotic (i.e., from some positive time on, depending on the specific initial datum) separation property was proven in [Abels and Wilke 2007] for the local Cahn–Hilliard equation, but nothing is known about its instantaneous (i.e., from *any* positive time on) counterpart. The main issue which so far seemed to be hard to overcome in dimension three for both local and nonlocal cases is the use of the Trudinger–Moser inequality (see, e.g., [Nagai et al. 1997]), which, in dimension $d = 2, 3$, reads

$$\int_\Omega e^{|f(x)|}\,\mathrm{d}x \le C e^{C\|f\|^d_{W^{1,d}(\Omega)}} \quad \text{for all } f \in W^{1,d}(\Omega), \tag{1-9}$$

for some positive constant C independent of f, but depending on the dimension d and on the Lebesgue d-dimensional measure of Ω. In dimension two this inequality is easy to be handled, since it concerns only the $H^1(\Omega)$ norm of f. Indeed, if one assumes that

$$F''(s) \le Ce^{C|F'(s)|} \quad \text{for all } s \in (-1,1), \tag{1-10}$$

for some constant $C > 0$ (see, e.g., [Gal et al. 2023a, (E2)] or [Gal et al. 2017]), which is satisfied by the logarithmic potential

$$F(s) = \frac{\bar{\alpha}}{2}\big((1+s)\ln(1+s) + (1-s)\ln(1-s)\big) \quad \text{for all } s \in [-1,1], \tag{1-11}$$

then, exploiting (1-9) as done in [Gal et al. 2017] or adopting an argument as in [Gal et al. 2023a, Theorem 3.1], one can control the quantity $\|F''(\phi(t))\|_{L^p(\Omega)}$, for any $p \ge 2$, uniformly in time and this is the key tool to prove the validity of the separation property in two dimensions for example of the nonlocal Cahn–Hilliard equation with constant mobility and singular potential. In the case of three-dimensional bounded domains, (1-9) leads to the necessity of a control of the $W^{1,3}(\Omega)$ norm of f and this does not seem to be feasible in this context. Thus the proof proposed in [Gal et al. 2017] does not hold in dimension three. Moreover, also the alternative proof in [Gal et al. 2023a] to allow the control of $\|F''(\phi(t))\|_{L^p(\Omega)}$ is not viable in dimension three, due to the fact that the embedding $H^1(\Omega) \hookrightarrow L^q(\Omega)$ holds only for $q \in [2,6]$, so that a result like [Gal et al. 2023a, (3.3)–(3.6)] cannot be obtained.

Here we are able to establish the (strict) separation property in three dimensions by avoiding the control of the quantity $F''(\phi(t))$ in any $L^p(\Omega)$ space. We do not assume condition (1-10) on F any more (see assumptions (H_2)–(H_3) and Remark 4.2 below), but we only rely on some natural growth conditions of F' and F'' near the endpoints ± 1. The idea is to perform a De Giorgi's iteration scheme on each interval of the form $(T-\tilde{\tau}, T)$, with $T > 0$ arbitrary and $\tilde{\tau}$ suitably chosen, similarly to the proof of [Gal et al. 2023a, Theorem 4.1], but modifying the argument in order to fully exploit the property that $F''(1-2\delta)^{-4} = O(\delta^4)$, for $\delta > 0$ sufficiently small (see (4-32)). This is possible in the estimates by treating in a suitable way all the terms leading to the presence of a quantity of the kind $F''(1-2\delta)^{-\gamma}$, with $0 \le \gamma < 4$ (see, e.g., the term Z_2 in the proof of [Gal et al. 2023a, Theorem 4.1]). To this aim, we first show the validity of a novel Poincaré-type inequality (Lemma 3.1), which is applied to a particular family of truncated functions obtained from the weak solution ϕ (namely, a family $\phi_\rho = (\phi-\rho)^+$, for some suitable $\rho \in (0,1)$). This can be obtained heavily relying on the conservation of total mass (i.e.,

$$\int_\Omega \phi_0(x)\,\mathrm{d}x = \int_\Omega \phi(x,t)\,\mathrm{d}x$$

for any $t \ge 0$), that is one of the most important properties of the solution. By means of this Poincaré-type inequality, in the De Giorgi's scheme we get, at the end of the estimates, a term of the kind $F''(1-2\delta)^{-4}\delta^{-5} = O(\delta^{-1})$ and this, together with the use of the growth condition of F' near 1, permits to obtain the strict separation property by choosing a suitably small $\tilde{\tau}$ depending on δ. Since the size of δ and the related quantity $\tilde{\tau}$ do not depend on T, we repeat the same argument on each time interval $(T-\tilde{\tau}, T)$ for arbitrary $T > 0$, extending the result of the separation property on the entire interval $(\tau, +\infty)$, for $\tau > 0$ arbitrarily fixed at the beginning, completing in this way the proof of the validity of (1-8).

As future work, it is worth noticing that the strict separation property could pave the way for the study of other related problems with logarithmic potential in dimension three. For example, one could study the nonlocal Cahn–Hilliard–Oono equation (see, e.g., [Della Porta and Grasselli 2015]), the nonlocal Cahn–Hilliard–Hele–Shaw system (see, e.g., [Della Porta et al. 2018]) as well as other hydrodynamic phase-field models for binary fluid mixtures of incompressible viscous fluids (see also Remark 4.7).

The paper is organized as follows. In Section 2 we introduce the functional setting. Section 3 is devoted to the presentation some preliminaries, which are essential in the proofs, in particular the new Poincaré-type inequality. In the same section we also recall some already-known results concerning well-posedness of the nonlocal Cahn–Hilliard equation and we present a Lemma on geometric convergence of numerical sequences, which is a key tool for De Giorgi's type arguments. Section 4 contains the main result concerning the strict separation property in dimension three for the system (1-7), together with its proof. In conclusion, in Section 5 we present some consequences of the validity of the strict separation property, namely we show some regularization results and we prove that any weak solution to (1-7) converges to a single equilibrium.

2. Mathematical setting

Let Ω be a smooth bounded domain in $\mathbb{R}^3$. The Sobolev spaces are denoted as usual by $W^{k,p}(\Omega)$, where $k \in \mathbb{N}$ and $1 \le p \le \infty$, with norm $\|\cdot\|_{W^{k,p}(\Omega)}$. The Hilbert space $W^{k,2}(\Omega)$ is denoted by $H^k(\Omega)$ with norm $\|\cdot\|_{H^k(\Omega)}$. In particular, we will adopt the notation

$$H = L^2(\Omega), \quad V = H^1(\Omega), \quad V_2 = \{v \in H^2(\Omega) : \partial_{\boldsymbol{n}} v = 0 \text{ on } \partial\Omega\}.$$

Moreover, given a space X, we denote by $\boldsymbol{X}$ the space of vectors of three components, each one belonging to X. We then denote by $(\cdot\,,\cdot)$ the inner product in H and by $\|\cdot\|$ the induced norm. We indicate by $(\cdot\,,\cdot)_V$ and $\|\cdot\|_V$ the canonical inner product and its induced norm in V, respectively. We also define the integral mean of a function f as

$$\bar{f} := \frac{\int_\Omega f(x)\,\mathrm{d}x}{|\Omega|},$$

where $|\Omega|$ stands for the three-dimensional Lebesgue measure of the set Ω. We then introduce

$$H_0 = \{v \in H : \bar{f} = 0\}, \quad V_0 = \{v \in V : \bar{f} = 0\}, \quad V_0' = \left\{v \in V' : \frac{\langle f, 1\rangle}{|\Omega|} = 0\right\},$$

endowed with the norms of H, V and V'. Thanks to the Poincaré-Wirtinger inequality, it follows that $(\|\nabla u\|^2_{L^2(\Omega)} + |\bar{u}|^2)^{1/2}$ is a norm on V equivalent to $\|u\|_V$. The Laplace operator $A_0 : V_0 \to V_0'$ defined by $\langle A_0 u, v\rangle = (\nabla u, \nabla v)$ is an isomorphism. We denote by $\mathcal{N}$ its inverse map and we set $\|f\|_* := \|\nabla \mathcal{N} f\|$, which is a norm on V_0' equivalent to the canonical one. Moreover, we recall that

$$\|f - \bar{f}\|_*^2 + |\bar{f}|^2 \tag{2-1}$$

is a norm V' which is equivalent to the standard one. Next, we recall the following Gagliardo–Nirenberg inequality (see, e.g., [Brezis 2011, Chapter 9]):

$$\|u\|_{L^p(\Omega)} \le C(p)\|u\|^{\frac{6-p}{2p}} \|u\|_V^{\frac{3(p-2)}{2p}} \quad \text{for all } u \in V \text{ and } p \in [2,6], \tag{2-2}$$

where the constant $C(p)$ depends on Ω and p. From this inequality, in the case $p = \frac{10}{3}$ we get

$$\|u\|_{L^{10/3}(\Omega)} \le \widehat{C}\|u\|^{\frac{2}{5}} \|u\|_V^{\frac{3}{5}} \quad \text{for all } u \in V, \tag{2-3}$$

with $\widehat{C} > 0$ depending on Ω.

3. Preliminaries

Here we present some preliminary results, which are essential for the proof of our main theorem.

3.1. *A Poincaré-type inequality.* First we state the following generalized version of the well known Poincaré's inequality:

Lemma 3.1. *Let I be either a compact interval or an interval of the kind $[\tau, +\infty)$, with $\tau > 0$. Let $\mathcal{K} \subset \mathbb{R}$ be a set of indices and $\{f_\rho\}_{\rho\in\mathcal{K}} \subset L^\infty(I; V) \cap C(I; H)$. Assume also that, for any $\rho \in \mathcal{K}$ and for any $t \in I$, $f_\rho(t) \equiv 0$ on the set $E(t) := \{x \in \Omega : g(t,x) \le 1 - 2\delta\} \subset \Omega$, with $g \in C(I; L^q(\Omega))$, $q \ge 1$, and $\delta \in \left(0, \frac{1}{2}\right)$. Moreover, for a fixed $\varepsilon > 0$ sufficiently small, assume that for any $t \in I$ the set $\{x \in \Omega : g(t,x) \le 1 - 2\delta - \varepsilon\} \subset E(t)$ has strictly positive Lebesgue measure. In the case the interval I is $[\tau, +\infty)$, assume additionally that for any sequence $\{t_l\}_l$, such that $t_l \to \infty$ as $l \to \infty$, there exists a (nonrelabeled) subsequence $\{t_l\}_l$, a function $g^\star \in L^r(\Omega)$, $r \ge 1$, and $\tilde{\varepsilon} > 0$, such that $g(t_l) \to g^\star$ strongly in $L^r(\Omega)$ as $l \to \infty$ and the set $\{x \in \Omega : g^\star(x) \le 1 - 2\delta - \tilde{\varepsilon}\}$ has strictly positive Lebesgue measure.*

Then there exists a uniform (in ρ and t) constant $C_P > 0$ such that

$$\|f_\rho(t)\| \le C_P \|\nabla f_\rho(t)\| \quad \textit{for all } t \in I \textit{ and } \rho \in \mathcal{K}. \tag{3-1}$$

Remark 3.2. Since $\{f_\rho\}_\rho \subset C(I; H) \cap L^\infty(I; V) \hookrightarrow C_w(I; V)$, where $C_w(I; V)$ denotes the V-valued weakly continuous functions (see, e.g., [Boyer and Fabrie 2013, Lemma II.5.9]), it makes sense to ask for conditions at *any* time $t \in I$.

Proof. Due to $\{f_\rho\}_\rho \subset C_w(I; V)$, $f_\rho(t) \in V$ for any $\rho \in \mathcal{K}$ and any $t \in I$. Assume by contradiction that (3-1) is false. Then there exist a sequence $\{\rho_n\}_{n\in\mathbb{N}} \subset \mathcal{K}$ and a sequence $\{t_n\}_{n\in\mathbb{N}} \subset I$ such that

$$\|f_{\rho_n}(t_n)\| > n\|\nabla f_{\rho_n}(t_n)\| \quad \text{for all } n \in \mathbb{N}.$$

We then set

$$w_n := \frac{f_{\rho_n}(t_n)}{\|f_{\rho_n}(t_n)\|}, \qquad \text{with } \|w_n\| = 1.$$

We need to consider two cases:

(1) Either the interval I is compact or there exists a nonrelabeled subsequence of $\{t_n\}_n$ which is entirely contained in the set $[\tau, M] \subset I$, for some $M < +\infty$. In this case there exists another nonrelabeled subsequence of times and $t^\star \in I$, with $t^\star < +\infty$, such that $t_n \to t^\star$.

Now notice that, since $g \in C(I; L^q(\Omega))$, $q \geq 1$, we get $g(t_n) \to g(t^\star)$ in $L^q(\Omega)$. Therefore, there exists a subsequence $\{g(t_{n_j})\}_j$ such that, as $j \to \infty$,

$$g(t_{n_j}) \to g(t^\star) \quad \text{a.e. in } \Omega.$$

Let us now set $D := \{x \in \Omega : g(t^\star, x) \leq 1 - 2\delta - \varepsilon\}$, and

$$\alpha = |D| > 0,$$

which is possible by assumption. Then by the Severini–Egorov theorem (notice that Ω has finite measure, so this theorem can be applied), there exists a measurable subset $B \subset \Omega$ such that $|B| < \frac{\alpha}{2}$ and such that, as $j \to \infty$,

$$g(t_{n_j}) \to g(t^\star) \quad \text{uniformly on } \Omega \setminus B.$$

Therefore, we also deduce that $|D \setminus B| > \frac{\alpha}{2} > 0$ and that also

$$g(t_{n_j}) \to g(t^\star) \quad \text{uniformly on } D \setminus B.$$

This means that there exists a $\bar{J} \in \mathbb{N}$ such that, for any $x \in D \setminus B$,

$$|g(t_{n_j}, x) - g(t^\star, x)| < \varepsilon \quad \text{for all } j \geq \bar{J},$$

implying that, for any $x \in D \setminus B$, by definition of the set D,

$$g(t_{n_j}, x) = g(t_{n_j}, x) - g(t^\star, x) + g(t^\star, x) \leq \varepsilon + 1 - 2\delta - \varepsilon = 1 - 2\delta \quad \text{for all } j \geq \bar{J}.$$

This means, by the assumptions, that

$$D \setminus B \subset E(t_{n_j}) \subset \{x \in \Omega : w_{n_j}(x) = 0\} \quad \text{for all } j \geq \bar{J},$$

implying

$$D \setminus B \subset \bigcap_{j \geq \bar{J}} \{x \in \Omega : w_{n_j}(x) = 0\}, \quad |D \setminus B| > \frac{\alpha}{2}.$$

(2) The interval I is of the form $[\tau, +\infty)$ and there are no bounded subsequences of $\{t_n\}_n$, i.e., $t_n \to +\infty$ as $n \to \infty$. In this case we have by assumption that, up to a nonrelabeled subsequence, there exists $g^\star \in L^r(\Omega)$, $r \geq 1$, such that $g(t_n) \to g^\star$ strongly in $L^r(\Omega)$. Thus there exists a subsequence $\{g(t_{n_j})\}_j$ such that

$$g(t_{n_j}) \to g^\star \quad \text{a.e. in } \Omega.$$

As in case (1), we set $D := \{x \in \Omega : g^\star(x) \leq 1 - 2\delta - \tilde{\varepsilon}\}$, and

$$\alpha = |D| > 0,$$

which is again possible by assumption. Then we can repeat exactly the same arguments as in case (1) to obtain again that

$$D \setminus B \subset E(t_{n_j}) \subset \{x \in \Omega : \; w_{n_j}(x) = 0\} \quad \text{for all } j \geq \bar{J},$$

implying

$$D \setminus B \subset \bigcap_{j \geq \bar{J}} \{x \in \Omega : \; w_{n_j}(x) = 0\}, \qquad |D \setminus B| > \frac{\alpha}{2}.$$

Clearly notice that in this case the set B will be such that there exists a $\bar{J} \in \mathbb{N}$ such that, for any $x \in D \setminus B$,

$$|g(t_{n_j}, x) - g^\star(x)| < \tilde{\varepsilon} \quad \text{for all } j \geq \bar{J}.$$

In both cases (1) and (2), since w_{n_j} is uniformly bounded in V, there exists $w \in V$ such that, by the Rellich–Kondrachov theorem, as $j \to \infty$,

$$w_{n_j} \rightharpoonup w \quad \text{in } V, \qquad w_{n_j} \to w \quad \text{in } H, \qquad \nabla w_{n_j} \rightharpoonup \nabla w \quad \text{in } H,$$

up to a nonrelabeled subsequence. Moreover, since $\|\nabla w_{n_j}\| < 1/n_j$, we deduce, by weak lower sequential semicontinuity of the L^2-norm, that $\nabla w \equiv 0$ almost everywhere in Ω and thus, being Ω connected, $w \equiv \kappa$ almost everywhere in Ω, with κ constant. Therefore, since also, up to another subsequence, $w_{n_j} \to w$ almost everywhere in Ω, we have $w \equiv 0$ on $D \setminus B$ (of positive Lebesgue measure) up to a zero measure set. But this clearly implies that $\kappa = 0$, which is a contradiction, since $\|w\| = 1$ (because $\|w_{n_j}\| = 1$ and $w_{n_j} \to w$ in H as $j \to \infty$). This concludes the proof. □

3.2. ***The state of the art for the three-dimensional nonlocal Cahn–Hilliard equation.*** For the sake of completeness we state here the already-known results concerning the nonlocal Cahn–Hilliard equation with constant mobility and singular potential in three-dimensional bounded domains. We first consider the following assumptions:

(H_1) $J \in W^{1,1}_{\text{loc}}(\mathbb{R}^3)$, with $J(x) = J(-x)$.

(H_2) $F \in C([-1, 1]) \cap C^2(-1, 1)$ fulfills

$$\lim_{s \to -1} F'(s) = -\infty, \qquad \lim_{s \to 1} F'(s) = +\infty, \qquad F''(s) \geq \alpha > 0 \qquad \text{for all } \; s \in (-1, 1).$$

We extend $F(s) = +\infty$ for any $s \notin [-1, 1]$. Without loss of generality, $F(0) = 0$ and $F'(0) = 0$. In particular, this entails that $F(s) \geq 0$ for any $s \in [-1, 1]$. Also, we assume that there exists $\gamma \in (0, 1)$ such that F'' is nondecreasing in $[1 - \gamma, 1)$ and nonincreasing in $(-1, -1 + \gamma]$.

Theorem 3.3. *Assume that (H_1)–(H_2) hold and also that $\phi_0 \in L^\infty(\Omega)$ such that $\|\phi_0\|_{L^\infty} \leq 1$ and $|\bar{\phi}_0| = m < 1$. Then there exists a unique weak solution to (1-7) such that, for any $T > 0$,*

$$\phi \in L^\infty(\Omega \times (0, T)) : \quad \textit{for all } t > 0, \;\; |\phi(t)| < 1, \quad \textit{a.e. in } \Omega,$$
$$\phi \in L^2(0, T; V) \cap H^1(0, T; H),$$
$$\mu \in L^2(0, T; V), \qquad F'(\phi) \in L^2(0, T; V),$$

such that

$$\langle \partial_t \phi, v\rangle + (\nabla\mu, \nabla v) = 0 \quad \textit{for all } v \in V, \quad \textit{a.e. in } (0,T), \tag{3-2}$$

$$\mu = F'(\phi) - J * \phi \quad \textit{a.e. in } \Omega \times (0,T), \tag{3-3}$$

and $\phi(\,\cdot\,,0) = \phi_0(\,\cdot\,)$ *in* Ω. *The weak solution also satisfies the energy identity* ($\mathcal{E}$ *is defined in* (1-6))

$$\mathcal{E}(\phi(t)) + \int_s^t \|\nabla\mu(\tau)\|^2 \,\mathrm{d}\tau = \mathcal{E}(\phi(s)) \quad \textit{for all } 0 \le s \le t < \infty. \tag{3-4}$$

Moreover, *for any* $\tau > 0$,

$$\sup_{t\ge\tau} \|\partial_t\phi(t)\|_{V'} + \sup_{t\ge\tau} \|\partial_t\phi\|_{L^2(t,t+1,H)} \le \frac{K_0}{\sqrt{\tau}}, \tag{3-5}$$

$$\sup_{t\ge\tau} \|\mu(t)\|_V + \sup_{t\ge\tau} \|\phi(t)\|_V \le \frac{K_0}{\sqrt{\tau}}, \tag{3-6}$$

$$\|F'(\phi)\|_{L^\infty(\tau,t;V)} + \|\mu\|_{L^2(t,t+1,V_2)} \le K_1 \quad \textit{for all } t \ge \tau, \tag{3-7}$$

$$\|\nabla\mu\|_{L^q(t,t+1;L^p(\Omega))} + \|\nabla\phi\|_{L^q(t,t+1;L^p(\Omega))} \le K_2 \;\; \textit{if} \;\; \frac{3p-6}{2p} = \frac{2}{q} \quad \textit{for all } p \in [2,6] \textit{ and } t \ge \tau, \tag{3-8}$$

where the positive constant K_0 *depends only on the initial datum energy* $\mathcal{E}(\phi_0)$, $\bar{\phi}_0$, Ω *and the parameters of the system*, *whereas* $K_1 = K_1(\tau)$ *and* $K_2 = K_2(\tau)$ *also depend on* τ. *Furthermore* K_2 *depends on also* q, p. *In conclusion*, *there holds the following continuous dependence estimate*: *for every two weak solutions* ϕ_1 *and* ϕ_2 *to* (1-7) *on* $[0,T]$, *with initial data* ϕ_{01} *and* ϕ_{02}, *respectively*, *we have*, *for all* $t \in [0,T]$,

$$\|\phi_1(t) - \phi_2(t)\|_{V'}^2 \le \|\phi_{01} - \phi_{02}\|_{V'}^2 + K|\bar{\phi}_{01} - \bar{\phi}_{02}|e^{CT},$$

where C *is a positive constant and*

$$K = C(\|F'(\phi_1)\|_{L^1(0,T;L^1(\Omega))} + \|F'(\phi_2)\|_{L^1(0,T;L^1(\Omega))}).$$

Remark 3.4. The proof of the above theorem can be found in [Gal et al. 2017, Theorems 3.4, 4.1, Proposition 4.2] and [Della Porta et al. 2018, Proposition 3.1]; see also [Gal et al. 2023b, Theorem 4.1] and [Poiatti and Signori 2024, Theorem 2.2] for a comprehensive result in the more general case of an advective nonlocal Cahn–Hilliard equation in two and three dimensions, respectively. In particular, we refer to [Gal et al. 2023b, Theorem 4.1, (4.4)] and [Della Porta et al. 2018, Proposition 3.1, (3.53)], which still hold in the nonadvective case $\boldsymbol{u} = \mathbf{0}$, for the validity of the energy identity (3-4), whereas (3-5) is shown in [Gal et al. 2017, Theorem 4.1, (4.2)]. Estimates (3-6)–(3-8) can be found in Theorem 4.1, (4.3), Proposition 4.2, (4.7), and Proposition 4.2, (4.9) of [Gal et al. 2017], respectively.

Remark 3.5. If we assume additionally that $\nabla F'(\phi_0) \in \boldsymbol{H}$ we can actually extend (3-5)–(3-8) to $\tau = 0$, since the initial datum is more regular and one can argue as in [Della Porta et al. 2018, Section 4] to obtain the desired regularity departing from the initial time. This means that the solution ϕ with initial datum ϕ_0 is indeed a strong solution to problem (1-7).

Remark 3.6. Notice that from condition (3-7) we can also deduce by Sobolev embeddings that

$$\|F'(\phi)\|_{L^\infty(\tau,\infty;L^p(\Omega))} \le K_3(\tau, p) \quad \text{for all } p \in [1,6], \tag{3-9}$$

where $K_3(\tau, p)$ depends on K_1, Ω and p.

Remark 3.7. We highlight that the previous theorem and our following main result concerning the strict separation property in dimension three heavily rely on the assumption $\bar{\phi}_0 \in (-1, 1)$ (see also [Kenmochi et al. 1995] for the local Cahn–Hilliard equation). This is physically reasonable since $\bar{\phi}_0 = 1$ (or $\bar{\phi}_0 = -1$) means that the initial condition is a pure phase, so that no phase separation takes place in Ω, unless we assume the existence of a source or reaction term (see, for instance [Grasselli et al. 2023]).

3.3. ***A lemma on geometric convergence of sequences.*** We present here one of the key tools for the application of De Giorgi's iteration argument. This lemma can be found, e.g., in [DiBenedetto 1993, Chapter I, Lemma 4.1], [Ladyženskaja et al. 1968, Chapter 2, Lemma 5.6], and it has also been proposed in [Gal et al. 2023a, Lemma 4.3].

Lemma 3.8. *Let $\{y_n\}_{n\in\mathbb{N}\cup\{0\}} \subset \mathbb{R}^+$ satisfy the recursive inequalities*

$$y_{n+1} \le C b^n y_n^{1+\varepsilon} \quad \textit{for all } n \ge 0, \tag{3-10}$$

for some $C > 0$, $b > 1$ and $\varepsilon > 0$. If

$$y_0 \le \theta := C^{-\frac{1}{\varepsilon}} b^{-\frac{1}{\varepsilon^2}}, \tag{3-11}$$

then

$$y_n \le \theta b^{-\frac{n}{\varepsilon}} \quad \textit{for all } n \ge 0, \tag{3-12}$$

and consequently $y_n \to 0$ for $n \to \infty$.

Proof. The proof can be easily carried out directly by induction. Indeed, the case $n = 0$ is trivial. Then assume that (3-12) holds for n. We prove that it also holds for $n+1$. In particular we have by (3-10) and recalling (3-11),

$$y_{n+1} \le C b^n y_n^{1+\varepsilon} \le C b^n \theta^{1+\varepsilon} b^{-\frac{n}{\varepsilon}(1+\varepsilon)} = C\theta^{1+\varepsilon} b^{-\frac{n}{\varepsilon}} = \theta b^{-\frac{n+1}{\varepsilon}} C\theta^\varepsilon b^{\frac{1}{\varepsilon}} \le \theta b^{-\frac{n+1}{\varepsilon}},$$

where we exploited the definition of θ in (3-11). This means that (3-12) also holds for $n+1$, concluding the proof by induction. □

We now present our main results, concerning the instantaneous strict separation property in three-dimensional bounded domains.

4. Main results

Let us assume, additionally to (H$_2$), the following hypotheses on the singular potential F:

(H$_3$) As $\delta \to 0^+$ we assume

$$\frac{1}{F'(1-2\delta)} = O\left(\frac{1}{|\ln(\delta)|}\right), \qquad \frac{1}{F''(1-2\delta)} = O(\delta), \tag{4-1}$$

and analogously

$$\frac{1}{|F'(-1+2\delta)|}=O\left(\frac{1}{|\ln(\delta)|}\right),\quad \frac{1}{F''(-1+2\delta)}=O(\delta). \tag{4-2}$$

Remark 4.1. Notice that these conditions are verified by the logarithmic potential (1-11). Indeed,

$$F'(s)=\frac{\bar{\alpha}}{2}\ln\left(\frac{1+s}{1-s}\right),\quad F''(s)=\frac{\bar{\alpha}}{1-s^2};$$

thus

$$F'(1-2\delta)=\frac{\bar{\alpha}}{2}\ln\left(\frac{1-\delta}{\delta}\right),\qquad F''(1-2\delta)=\frac{\bar{\alpha}}{4\delta(1-\delta)},$$
$$F'(-1+2\delta)=\frac{\bar{\alpha}}{2}\ln\left(\frac{\delta}{1-\delta}\right),\quad F''(-1+2\delta)=\frac{\bar{\alpha}}{4\delta(1-\delta)},$$

clearly implying assumption (H_3).

Remark 4.2. As already pointed out in the Introduction, assumption (H_3) does not make any explicit reference to the typical extra condition (1-10). Indeed, as far as we know, this is the first proof of the instantaneous separation property concerning nonlocal Cahn–Hilliard equation with constant mobility and singular potential (problem (1-7)) in which it is not exploited any constraint on $\|F''(\phi(t))\|_{L^q(\Omega)}$, for some $q\geq 2$ and for almost any $t\geq\tau$, with $\tau>0$. Indeed, in our proof we simply rely on some natural growth conditions of F' and F'' near the endpoints ± 1. Note that assumptions (H_2)–(H_3) on the potential F are somehow minimal, in the sense that the proof of the separation property in dimension three works only in this case (or for more singular potentials than the logarithmic one). This seems to suggest that the use of the logarithmic potential when modeling phase separation phenomena with the help of the nonlocal Cahn–Hilliard equation with constant mobility could be a good choice, since it preserves all the basic physical properties expected from the solution.

We can now state our main theorem.

Theorem 4.3. *Let $\Omega\subset\mathbb{R}^3$ be a smooth bounded domain and let assumptions (H_1)–(H_3) hold. Assume that $\phi_0\in L^\infty(\Omega)$ such that $\|\phi_0\|_{L^\infty}\leq 1$ and $|\overline{\phi}_0|=m<1$. Then for any $\tau>0$ there exists $\delta\in(0,1)$, depending on τ, m and the initial datum, such that the unique weak solution to problem (1-7) given in Theorem 3.3 satisfies*

$$|\phi(x,t)|\leq 1-\delta\quad \text{for a.e. } (x,t)\in\Omega\times(\tau,+\infty),$$

i.e., the instantaneous strict separation property from the pure phases ± 1 holds.

Remark 4.4. Observe that the quantity δ given in the theorem strongly depends on the specific entire trajectory, therefore, by the uniqueness of the solution, on the initial datum ϕ_0. This means that we cannot have an explicit dependence of δ, e.g., on the initial datum energy.

As a byproduct of the main theorem, we also prove that, if the initial datum ϕ_0 is more regular and already separated from the pure phases, i.e., there exists $\delta_0 \in (0, 1]$ such that

$$\|\phi_0\|_{L^\infty(\Omega)} \le 1 - \delta_0,$$

then the unique solution ϕ departing from ϕ_0, which is now strong from the time $t = 0$ (see Remark 3.5), is strictly separated on $[0, +\infty)$, i.e., it remains separated from the pure phases *uniformly* for almost any time $t \ge 0$.

Corollary 4.5. *Under the same hypotheses of Theorem 4.3, if we assume additionally that $\nabla F'(\phi_0) \in \boldsymbol{H}$, and that ϕ_0 is strictly separated, i.e., there exists $\delta_0 \in (0, 1]$ such that*

$$\|\phi_0\|_{L^\infty(\Omega)} \le 1 - \delta_0,$$

then there exists $\delta \in (0, 1)$, depending on τ, m, δ_0 and the initial datum, such that the unique strong solution to problem (1-7) given in Remark 3.5 satisfies

$$|\phi(x, t)| \le 1 - \delta, \quad \text{for a.e. } (x, t) \in \Omega \times [0, +\infty),$$

i.e., the instantaneous strict separation property from the pure phases ± 1 holds for almost any time $t \ge 0$.

Remark 4.6. Observe that, since by Theorem 4.3 the solution ϕ in Corollary 4.5 is strictly separated on time sets of the kind $(\tau, +\infty)$, for any $\tau > 0$, it is enough to show that there exists an interval $[0, T_1]$ ($T_1 > 0$) on which the solution is separated to obtain the strict separation over $[0, +\infty)$, choosing $\tau = T_1$. As it will be clear from the proof of Corollary 4.5, T_1 can be explicitly computed as a function of the parameters of the problem and the initial datum.

4.1. ***Proof of Theorem 4.3.*** We divide the proof into two steps. In the first we show that we can apply Lemma 3.1 to a specific family of functions, which will be of essential importance in the second step, when we adopt a De Giorgi's iteration scheme (as in [Gal et al. 2023a, Theorem 4.1]) to obtain the desired result.

Step 1. Application of Lemma 3.1 to a family of truncated functions. Let us consider the unique solution ϕ departing from ϕ_0, whose existence and regularity is stated in Theorem 3.3. We make the following observations: first fix any $\tau > 0$.

- Since $|\overline{\phi}_0| \le m < 1$, there exists $\hat{\delta} > 0$ and an $\varepsilon > 0$ such that

$$m \le 1 - 2\hat{\delta} - \varepsilon. \tag{4-3}$$

In particular we may choose $\varepsilon := (1 - m)/2 > 0$ and $\hat{\delta} := (1 - m)/4 > 0$. Thanks to the conservation of total mass, we have that for any $\rho \in \mathbb{R}^+$, $\rho \ge 1 - 2\hat{\delta}$, and for any $t \in [0, +\infty)$, the function

$$\phi_\rho(x, t) := (\phi(x, t) - \rho)^+ \tag{4-4}$$

vanishes on the set (independent of ρ)

$$E(t) := \{x \in \Omega : \phi(x, t) \le 1 - 2\hat{\delta}\}, \tag{4-5}$$

which is such that

$$|\{x \in \Omega : \phi(x, t) \le 1 - 2\hat{\delta} - \varepsilon\}| > 0 \quad \text{for all } t \ge 0. \tag{4-6}$$

Proof. To prove this observation, let us assume by contradiction that, for some $\tilde{t} \geq 0$,

$$|\{x \in \Omega : \phi(x, \tilde{t}) \leq 1 - 2\hat{\delta} - \varepsilon\}| = 0.$$

By the conservation of total mass we get, for any $t \geq 0$,

$$(1 - 2\hat{\delta} - \varepsilon)|\Omega| \geq m|\Omega| \geq \int_\Omega \phi_0(x)\, \mathrm{d}x = \int_\Omega \phi(x, t)\, \mathrm{d}x,$$

but then we get a contradiction, since $|\Omega| = |\{x \in \Omega : \phi(x, \tilde{t}) > 1 - 2\hat{\delta} - \varepsilon\}|$ and

$$(1 - 2\hat{\delta} - \varepsilon)|\Omega| \geq \int_\Omega \phi(x, \tilde{t})\, \mathrm{d}x > (1 - 2\hat{\delta} - \varepsilon)|\{x \in \Omega : \phi(x, \tilde{t}) > 1 - 2\hat{\delta} - \varepsilon\}|. \qquad \square$$

- We aim to apply Lemma 3.1 with $\mathcal{K} = [1 - 2\hat{\delta}, 1]$, $\{f_\rho\}_{\rho\in\mathcal{K}} = \{\phi_\rho\}_{\rho\in\mathcal{K}}$, $I = [\tau, +\infty)$, $g = \phi$, $\delta = \hat{\delta}$, $\tilde{\varepsilon} = \varepsilon$. Indeed we verify all the assumptions:

- We have $\{\phi_\rho\}_\rho \subset L^\infty(I; V) \cap C(I; H)$, $\phi \in C(I; H)$, and (4-5) and (4-6) hold for any $t \in I$.

- Let $\{t_l\}_l$ be any sequence such that $t_l \to \infty$. By (3-6), there exists a constant $C(\tau) > 0$ such that

$$\sup_{t\geq\tau} \|\phi\|_V \leq C(\tau).$$

Therefore, since V is reflexive, there exist a (nonrelabeled) subsequence $\{t_l\}_l$ and a function $g^\star \in V$ (which could depend on the subsequence) such that, as $l \to \infty$,

$$\phi(t_l) \rightharpoonup g^\star \quad \text{in } V,$$

implying by compactness that

$$\phi(t_l) \to g^\star \quad \text{in } H. \tag{4-7}$$

Now notice that this strong convergence also implies, by the conservation of total mass, that

$$\int_\Omega \phi_0(x)\, \mathrm{d}x = \int_\Omega \phi(x, t_l)\, \mathrm{d}x \to \int_\Omega g^\star(x)\, \mathrm{d}x,$$

and thus also $g^\star$ enjoys the same total mass as the initial datum ϕ_0:

$$\int_\Omega g^\star(x)\, \mathrm{d}x = \int_\Omega \phi_0(x)\, \mathrm{d}x.$$

This means that we can repeat exactly the same argument as the one adopted to get (4-6) to infer

$$|\{x \in \Omega : g^\star(x) \leq 1 - 2\hat{\delta} - \varepsilon\}| > 0, \tag{4-8}$$

so that, having chosen $\tilde{\varepsilon} = \varepsilon$ and $g = \phi$, thanks to (4-7)–(4-8), we have completed the verification of the assumptions of Lemma 3.1.

In the end we can conclude that there exists a uniform (in ρ and t) constant $C_{P,+} > 0$ such that

$$\|\phi_\rho(t)\| \leq C_{P,+}\|\nabla\phi_\rho(t)\| \tag{4-9}$$

for any $t \in [\tau, +\infty)$ and any $\rho \in [1 - 2\hat{\delta}, 1]$.

- Since in the last part of the proof we need to reproduce all the arguments on the functions

$$\tilde{\phi}_\rho(x,t) := (\phi(x,t)+\rho)^- = (-\phi(x,t)-\rho)^+, \tag{4-10}$$

with $\rho \ge 1-2\hat{\delta}$, we observe that (4-5) and (4-6) still hold substituting ϕ with $-\phi$, simply because, again by the conservation of mass, $m|\Omega| \ge \int_\Omega -\phi(x,t)\,dx$ for any $t \ge \tau$. Therefore again the assumptions of Lemma 3.1 are satisfied (with $g = -\phi$), and thus that there exists a uniform (in ρ and t) constant $C_{P,-} > 0$ (which is possibly different from $C_{P,+}$) such that

$$\|\tilde{\phi}_\rho(t)\| \le C_{P,-}\|\nabla\tilde{\phi}_\rho(t)\| \tag{4-11}$$

for any $t \in [\tau,+\infty)$ and for any $\rho \in [1-2\hat{\delta}, 1]$. Thus we introduce the constant $C_P := \max\{C_{P,+}, C_{P,-}\}$ so that both (4-9) and (4-11) hold with the same constant C_P, i.e.,

$$\|\phi_\rho(t)\| \le C_P\|\nabla\phi_\rho(t)\|, \quad \|\tilde{\phi}_\rho(t)\| \le C_P\|\nabla\tilde{\phi}_\rho(t)\|, \tag{4-12}$$

for any $t \ge \tau$ and any $\rho \in [1-2\hat{\delta}, 1]$. Note that the constant C_P depends on the specific solution ϕ we used, thus, since ϕ is uniquely determined by ϕ_0, we have that C_P depends in a nontrivial way on the initial datum.

Step 2. De Giorgi's iteration scheme. We perform a De Giorgi's iteration scheme following the one presented in [Gal et al. 2023a, Lemma 4.1]. Let us fix δ sufficiently small such that $\delta \le \hat{\delta}$, so that (4-12) holds for any $\rho \in [1-2\delta, 1]$. Set then $\tilde{\tau} > 0$ such that

$$\tilde{\tau} = \frac{2^{-20}\delta^5(F''(1-2\delta))^4F'(1-2\delta)}{3C(\tau)\|\nabla J\|^5_{L^1(B_r)}\widehat{C}^{\frac{3}{2}}(1+C_P^2)^{\frac{3}{2}}}, \tag{4-13}$$

where C_P is given in (4-12), $\widehat{C}$ is defined in (2-3) and B_r is a ball centered at $\mathbf{0}$ of radius $r>0$ sufficiently large such that $x-\Omega \subset B_r$ for any $x \in \Omega$ (see also [Giorgini 2024] for this observation on B_r). Now observe that, since, by (4-1), there exists a positive constant $C_F > 0$ such that, for δ sufficiently small,

$$0 < \frac{1}{F''(1-2\delta)} \le C_F\delta \quad \text{and} \quad 0 < \frac{1}{F'(1-2\delta)} \le \frac{C_F}{|\ln(\delta)|},$$

we have

$$\begin{aligned}\frac{\frac{8\delta^2}{\tilde{\tau}}}{\frac{\|\nabla J\|^2_{L^1(B_r)}}{2F''(1-2\delta)}} &= \frac{16\delta^2F''(1-2\delta)}{\|\nabla J\|^2_{L^1(B_r)}}\frac{3C(\tau)\|\nabla J\|^5_{L^1(B_r)}\widehat{C}^{\frac{3}{2}}(1+C_P^2)^{\frac{3}{2}}}{2^{-20}\delta^5(F''(1-2\delta))^4F'(1-2\delta)}\\ &= \frac{3C(\tau)\|\nabla J\|^3_{L^1(B_r)}\widehat{C}^{\frac{3}{2}}(1+C_P^2)^{\frac{3}{2}}}{2^{-24}\delta^3(F''(1-2\delta))^3F'(1-2\delta)} \le \frac{\widetilde{C}}{|\ln(\delta)|} \to 0 \quad \text{as } \delta \to 0^+,\end{aligned}$$

where

$$\widetilde{C} := \frac{3C(\tau)\|\nabla J\|^3_{L^1(B_r)}\widehat{C}^{\frac{3}{2}}(1+C_P^2)^{\frac{3}{2}}C_F^4}{2^{-24}} > 0,$$

so that

$$\frac{\frac{8\delta^2}{\tilde{\tau}}}{\frac{\|\nabla J\|^2_{L^1(B_r)}}{2F''(1-2\delta)}} = O\left(\frac{1}{|\ln(\delta)|}\right).$$

This means that we can find a sufficiently small $\delta > 0$ so that

$$\max\left\{\frac{\|\nabla J\|^2_{L^1(B_r)}}{2F''(1-2\delta)}, \frac{8\delta^2}{\tilde{\tau}}\right\} = \frac{\|\nabla J\|^2_{L^1(B_r)}}{2F''(1-2\delta)}. \tag{4-14}$$

Choose now $T > 0$ such that $T - 3\tilde{\tau} \geq \frac{\tau}{2}$ $\left(\text{for example, one can start with } T = 3\tilde{\tau} + \frac{\tau}{2}\right)$. Up to reducing the size of δ, and thus of $\tilde{\tau}$, we can find $\tilde{\tau}$ such that

$$2\tilde{\tau} + \frac{\tau}{2} \leq \tau. \tag{4-15}$$

Let us then fix $\delta > 0$ (and thus $\tilde{\tau} > 0$) so that also (4-14) and (4-15) hold. Notice that the choice of δ and $\tilde{\tau}$ *does not* depend on the specific T, but clearly depends on τ.

We now define the sequence

$$k_n = 1 - \delta - \frac{\delta}{2^n} \quad \text{for all } n \geq 0, \tag{4-16}$$

where

$$1 - 2\delta < k_n < k_{n+1} < 1 - \delta \quad \text{for all } n \geq 1, \quad k_n \to 1 - \delta \quad \text{as } n \to \infty, \tag{4-17}$$

and the sequence of times

$$t_n = \begin{cases} T - 3\tilde{\tau} & \text{if } n = -1, \\ t_{n-1} + \frac{\tilde{\tau}}{2^n} & \text{if } n \geq 0, \end{cases} \tag{4-18}$$

which satisfies

$$t_{-1} < t_n < t_{n+1} < T - \tilde{\tau} \quad \text{for all } n \geq 0.$$

We now introduce a cutoff function $\eta_n \in C^1(\mathbb{R})$ by setting

$$\eta_n(t) := \begin{cases} 0 & \text{if } t \leq t_{n-1}, \\ 1 & \text{if } t \geq t_n \end{cases} \quad \text{and} \quad |\eta_n'(t)| \leq \frac{2^{n+1}}{\tilde{\tau}}, \tag{4-19}$$

on account of the above definition of the sequence $\{t_n\}_n$. Recalling (4-4), we then set $\rho = k_n$,

$$\phi_n(x,t) := (\phi - k_n)^+, \tag{4-20}$$

and, for any $n \geq 0$, we introduce the interval $I_n = [t_{n-1}, T]$ and the set

$$A_n(t) := \{x \in \Omega : \phi(x,t) - k_n \geq 0\} \quad \text{for all } t \in I_n.$$

Clearly, we have

$$\begin{aligned} I_{n+1} &\subseteq I_n && \text{for all } n \geq 0, \\ A_{n+1}(t) &\subseteq A_n(t) && \text{for all } n \geq 0 \text{ and } t \in I_{n+1}. \end{aligned}$$

In conclusion, we set

$$y_n = \int_{I_n} \int_{A_n(s)} 1 \, \mathrm{d}x \, \mathrm{d}s \quad \text{for all } n \geq 0.$$

Now, for any $n \geq 0$, we consider the test function $v = \phi_n \eta_n^2$, and integrate over $[t_{n-1}, t]$, $t_n \leq t \leq T$. Then

$$\int_{t_{n-1}}^t \langle \partial_t \phi, \phi_n \eta_n^2 \rangle \mathrm{d}s + \int_{t_{n-1}}^t \int_{A_n(s)} F''(\phi) \nabla \phi \cdot \nabla \phi_n \eta_n^2 \, \mathrm{d}x \, \mathrm{d}s = \int_{t_{n-1}}^t \int_{A_n(s)} \eta_n^2 (\nabla J * \phi) \cdot \nabla \phi_n \, \mathrm{d}x \, \mathrm{d}s, \tag{4-21}$$

since $\nabla F'(\phi(t)) = F''(\phi) \nabla \phi(t)$, for almost every $x \in \Omega$ and for any $t \geq \frac{\tau}{2}$, which can be proven, e.g., by a truncation argument as in [He and Wu 2021, Lemma 3.2], applied for any $t \geq \frac{\tau}{2}$. Indeed, as in [He and Wu 2021, (3.5)], we obtain $\nabla F'(\phi(t)) = F''(\phi) \nabla \phi(t)$ in the sense of distribution and thus, since $\nabla F'(\phi) \in L^\infty\big(\frac{\tau}{2}, \infty; \boldsymbol{H}\big)$, we immediately infer that the equality holds also almost everywhere in Ω, for any $t \geq \frac{\tau}{2}$. Now, as in [Gal et al. 2023a], for δ sufficiently small we obtain

$$\int_{t_{n-1}}^t \eta_n^2 \int_{A_n(s)} F''(\phi) \nabla \phi \cdot \nabla \phi_n \, \mathrm{d}x \, \mathrm{d}s \geq F''(1 - 2\delta) \int_{t_{n-1}}^t \eta_n^2 \|\nabla \phi_n\|^2 \, \mathrm{d}s, \tag{4-22}$$

and, for the right-hand side of (4-21), recalling that $|\phi| < 1$ a.e. in $\Omega \times (0, +\infty)$, we find

$$\begin{aligned}
\int_{t_{n-1}}^t \int_{A_n(s)} &(\nabla J * \phi) \cdot \nabla \phi_n \eta_n^2 \, \mathrm{d}x \, \mathrm{d}s \\
&\leq \frac{1}{2} F''(1 - 2\delta) \int_{t_{n-1}}^t \eta_n^2 \|\nabla \phi_n\|^2 \, \mathrm{d}s + \frac{1}{2F''(1 - 2\delta)} \int_{t_{n-1}}^t \int_{A_n(s)} \eta_n^2 |\nabla J * \phi|^2 \, \mathrm{d}x \, \mathrm{d}s \\
&\leq \frac{1}{2} F''(1 - 2\delta) \int_{t_{n-1}}^t \eta_n^2 \|\nabla \phi_n\|^2 \, \mathrm{d}s + \frac{1}{2F''(1 - 2\delta)} \int_{t_{n-1}}^t \|\nabla J * \phi\|_{L^\infty(\Omega)}^2 \int_{A_n(s)} \mathrm{d}x \, \mathrm{d}s \\
&\leq \frac{1}{2} F''(1 - 2\delta) \int_{t_{n-1}}^t \eta_n^2 \|\nabla \phi_n\|^2 \, \mathrm{d}s + \frac{\|\nabla J\|_{L^1(B_r)}^2}{2F''(1 - 2\delta)} \int_{t_{n-1}}^t \int_{A_n(s)} \mathrm{d}x \, \mathrm{d}s \\
&\leq \frac{1}{2} F''(1 - 2\delta) \int_{t_{n-1}}^t \eta_n^2 \|\nabla \phi_n\|^2 \, \mathrm{d}s + \frac{\|\nabla J\|_{L^1(B_r)}^2}{2F''(1 - 2\delta)} y_n,
\end{aligned} \tag{4-23}$$

where we have applied (see, e.g., [Brezis 2011, Theorem 4.33])

$$\|\nabla J * \phi\|_{L^\infty(\Omega)} \leq \|\nabla J\|_{L^1(B_r)} \|\phi\|_{L^\infty(\Omega)} \leq \|\nabla J\|_{L^1(B_r)}. \tag{4-24}$$

Moreover, we have

$$\int_{t_{n-1}}^t \langle \partial_t \phi, \phi_n \eta_n^2 \rangle \, \mathrm{d}s = \frac{1}{2} \|\phi_n(t)\|^2 - \int_{t_{n-1}}^t \|\phi_n(s)\|^2 \eta_n \partial_t \eta_n \, \mathrm{d}s. \tag{4-25}$$

Note that, since $|\phi| < 1$ a.e. in Ω, for any $t \geq \frac{\tau}{2}$,

$$0 \leq \phi_n \leq 2\delta \quad \text{a.e. in } \Omega \text{ for all } t \geq \frac{\tau}{2}. \tag{4-26}$$

Then, by the above inequality,

$$\int_{t_{n-1}}^{t}\|\phi_n(s)\|^2\eta_n\partial_t\eta_n\,\mathrm{d}s=\int_{t_{n-1}}^{t}\int_{\Omega}\phi_n^2(s)\eta_n\partial_t\eta_n\,\mathrm{d}x\,\mathrm{d}s=\int_{t_{n-1}}^{t}\int_{A_n(s)}\phi_n^2(s)\eta_n\partial_t\eta_n\,\mathrm{d}x\,\mathrm{d}s$$
$$\leq\int_{t_{n-1}}^{t}\int_{A_n(s)}(2\delta)^2\frac{2^{n+1}}{\tilde{\tau}}\,\mathrm{d}x\,\mathrm{d}s\leq\frac{2^{n+3}\delta^2}{\tilde{\tau}}y_n. \tag{4-27}$$

Plugging (4-22), (4-23), (4-25) and (4-27) into (4-21), we find

$$\frac{1}{2}\|\phi_n(t)\|^2+\frac{1}{2}F''(1-2\delta)\int_{t_{n-1}}^{t}\eta_n^2\|\nabla\phi_n(s)\|^2\,\mathrm{d}s\leq 2^{n+1}\max\left\{\frac{\|\nabla J\|^2_{L^1(B_r)}}{2F''(1-2\delta)},\frac{8\delta^2}{\tilde{\tau}}\right\}y_n$$

for any $t\in[t_n,T]$. Thanks to the choice of δ and $\tilde{\tau}$, we recall (4-14), implying

$$\max_{t\in I_{n+1}}\|\phi_n(t)\|^2\leq X_n,\qquad F''(1-2\delta)\int_{I_{n+1}}\|\nabla\phi_n\|^2\,\mathrm{d}s\leq X_n, \tag{4-28}$$

where

$$X_n:=2^{n+1}\frac{\|\nabla J\|^2_{L^1(B_r)}}{F''(1-2\delta)}y_n.$$

On the other hand, for any $t\in I_{n+1}$ and for almost any $x\in A_{n+1}(t)$, we get

$$\phi_n(x,t)=\phi(x,t)-\left[1-\delta-\frac{\delta}{2^n}\right]$$
$$=\underbrace{\phi(x,t)-\left[1-\delta-\frac{\delta}{2^{n+1}}\right]}_{\phi_{n+1}(x,t)\geq 0}+\delta\left[\frac{1}{2^n}-\frac{1}{2^{n+1}}\right]\geq\frac{\delta}{2^{n+1}},$$

which implies

$$\int_{I_{n+1}}\int_{\Omega}|\phi_n|^3\,\mathrm{d}x\,\mathrm{d}s\geq\int_{I_{n+1}}\int_{A_{n+1}(s)}|\phi_n|^3\,\mathrm{d}x\,\mathrm{d}s\geq\left(\frac{\delta}{2^{n+1}}\right)^3\int_{I_{n+1}}\int_{A_{n+1}(s)}\mathrm{d}x\,\mathrm{d}s=\left(\frac{\delta}{2^{n+1}}\right)^3y_{n+1}.$$

Then we have

$$\left(\frac{\delta}{2^{n+1}}\right)^3y_{n+1}\leq\int_{I_{n+1}}\int_{\Omega}|\phi_n|^3\,\mathrm{d}x\,\mathrm{d}s=\int_{I_{n+1}}\int_{A_n(s)}|\phi_n|^3\,\mathrm{d}x\,\mathrm{d}s$$
$$\leq\left(\int_{I_{n+1}}\int_{\Omega}|\phi_n|^{\frac{10}{3}}\,\mathrm{d}x\,\mathrm{d}s\right)^{\frac{9}{10}}\left(\int_{I_{n+1}}\int_{A_n(s)}\mathrm{d}x\,\mathrm{d}s\right)^{\frac{1}{10}}. \tag{4-29}$$

Notice that, thanks to (2-3) and (4-12) (which holds thanks to (4-17)), we get

$$\int_{I_{n+1}}\int_{\Omega}|\phi_n|^{\frac{10}{3}}\,\mathrm{d}x\,\mathrm{d}s\leq\widehat{C}\int_{I_{n+1}}\|\phi_n\|_V^2\|\phi_n\|^{\frac{4}{3}}\,\mathrm{d}s\leq\widehat{C}\int_{I_{n+1}}\left(\|\phi_n\|^2+\|\nabla\phi_n\|^2\right)\|\phi_n\|^{\frac{4}{3}}\,\mathrm{d}s$$
$$\leq\widehat{C}(1+C_P^2)\int_{I_{n+1}}\|\nabla\phi_n\|^2\|\phi_n\|^{\frac{4}{3}}\,\mathrm{d}s,$$

where we have chosen an equivalent norm on V. Observe now that, by (4-28),

$$\begin{aligned}\int_{I_{n+1}}\int_\Omega |\phi_n|^{\frac{10}{3}}\,\mathrm{d}x\,\mathrm{d}s &\le \widehat{C}(1+C_P^2)\int_{I_{n+1}}\|\nabla\phi_n\|^2\|\phi_n\|^{\frac{4}{3}}\,\mathrm{d}s\\ &\le \widehat{C}(1+C_P^2)\max_{t\in I_{n+1}}\|\phi_n(t)\|^{\frac{4}{3}}\int_{I_{n+1}}\|\nabla\phi_n\|^2\,\mathrm{d}s\\ &\le \frac{\widehat{C}(1+C_P^2)}{F''(1-2\delta)}X_n^{\frac{2}{3}}F''(1-2\delta)\int_{I_{n+1}}\|\nabla\phi_n\|^2\,\mathrm{d}s\\ &\le \frac{\widehat{C}(1+C_P^2)}{F''(1-2\delta)}X_n^{\frac{5}{3}} \le \frac{2^{\frac{5n}{3}+\frac{5}{3}}\|\nabla J\|_{L^1(B_r)}^{\frac{10}{3}}\widehat{C}(1+C_P^2)}{(F''(1-2\delta))^{\frac{8}{3}}}y_n^{\frac{5}{3}}.\end{aligned}$$

Coming back to (4-29), we immediately infer

$$\begin{aligned}\left(\frac{\delta}{2^{n+1}}\right)^3 y_{n+1} &\le \left(\int_{I_{n+1}}\int_\Omega|\phi_n|^{\frac{10}{3}}\,\mathrm{d}x\,\mathrm{d}s\right)^{\frac{9}{10}}\left(\int_{I_{n+1}}\int_{A_n(s)}\mathrm{d}x\,\mathrm{d}s\right)^{\frac{1}{10}}\\ &\le \frac{2^{\frac{3}{2}n+\frac{3}{2}}\|\nabla J\|_{L^1(B_r)}^3\widehat{C}^{\frac{9}{10}}(1+C_P^2)^{\frac{9}{10}}}{(F''(1-2\delta))^{\frac{12}{5}}}y_n^{\frac{3}{2}}y_n^{\frac{1}{10}}\\ &= \frac{2^{\frac{3}{2}n+\frac{3}{2}}\|\nabla J\|_{L^1(B_r)}^3\widehat{C}^{\frac{9}{10}}(1+C_P^2)^{\frac{9}{10}}}{(F''(1-2\delta))^{\frac{12}{5}}}y_n^{\frac{8}{5}}. \qquad (4\text{-}30)\end{aligned}$$

In conclusion, we end up with

$$y_{n+1} \le \frac{2^{\frac{9}{2}n+\frac{9}{2}}\|\nabla J\|_{L^1(B_r)}^3\widehat{C}^{\frac{9}{10}}(1+C_P^2)^{\frac{9}{10}}}{\delta^3(F''(1-2\delta))^{\frac{12}{5}}}y_n^{\frac{8}{5}} \quad \text{for all } n\ge 0. \qquad (4\text{-}31)$$

Thus we can apply Lemma 3.8. In particular, we have

$$b = 2^{\frac{9}{2}} > 1,\qquad C = \frac{2^{\frac{9}{2}}\|\nabla J\|_{L^1(B_r)}^3\widehat{C}^{\frac{9}{10}}(1+C_P^2)^{\frac{9}{10}}}{\delta^3(F''(1-2\delta))^{\frac{12}{5}}} > 0,\qquad \varepsilon = \frac{3}{5},$$

to get that $y_n \to 0$, as long as

$$y_0 \le C^{-\frac{5}{3}}b^{-\frac{25}{9}},$$

that is,

$$y_0 \le \frac{2^{-20}\delta^5(F''(1-2\delta))^4}{\|\nabla J\|_{L^1(B_r)}^5\widehat{C}^{\frac{3}{2}}(1+C_P^2)^{\frac{3}{2}}}. \qquad (4\text{-}32)$$

We are left with a last estimate: thanks to (3-7), we know that $\|F'(\phi)\|_{L^\infty(\tau/2,\infty;L^1(\Omega))} \le C(\tau)$ and F' is monotone in a neighborhood of $+1$, so that we infer

$$y_0 = \int_{I_0}\int_{A_0(s)} 1\,\mathrm{d}x\,\mathrm{d}s \le \int_{I_0}\int_{\{x\in\Omega:\ \phi(x,s)\ge 1-2\delta\}} 1\,\mathrm{d}x\,\mathrm{d}s \le \int_{I_0}\int_{A_0(s)}\frac{|F'(\phi)|}{F'(1-2\delta)}\,\mathrm{d}x\,\mathrm{d}s \le \frac{3C(\tau)\tilde{\tau}}{F'(1-2\delta)}.$$

Therefore, if we ensure that

$$\frac{3C(\tau)\tilde{\tau}}{F'(1-2\delta)} \leq \frac{2^{-20}\delta^5(F''(1-2\delta))^4}{\|\nabla J\|_{L^1(B_r)}^5 \widehat{C}^{\frac{3}{2}}(1+C_P^2)^{\frac{3}{2}}},$$

then (4-32) holds. Having fixed $\tilde{\tau}$ in (4-13) such that

$$\tilde{\tau} = \frac{2^{-20}\delta^5(F''(1-2\delta))^4 F'(1-2\delta)}{3C(\tau)\|\nabla J\|_{L^1(B_r)}^5 \widehat{C}^{\frac{3}{2}}(1+C_P^2)^{\frac{3}{2}}}, \tag{4-33}$$

we obtain the result. Notice that δ is fixed, so $\tilde{\tau} > 0$ is not infinitesimal, but it depends on ϕ_0 in a nontrivial way (thus not only on the initial energy) through C_P.

In the end, passing to the limit in y_n as $n \to \infty$, we have obtained that

$$\|(\phi - (1-\delta))^+\|_{L^\infty(\Omega\times(T-\tilde{\tau},T))} = 0,$$

since, as $n \to \infty$,

$$y_n \to \left|\left\{(x,t) \in \Omega \times [T-\tilde{\tau}, T] : \phi(x,t) \geq 1-\delta\right\}\right| = 0.$$

We now repeat exactly the same argument for the case $(\phi - (-1+\delta))^-$ (using $\phi_n(t) = (\phi(t)+k_n)^-$). Notice that also for this second case we have the same constant C_P (see (4-12)). Moreover, the argument is exactly the same due to assumption (4-2), which implies that

$$\frac{1}{F''(-1+2\delta)} = O(\delta) \quad \text{and} \quad \frac{1}{|F'(-1+2\delta)|} = O\left(\frac{1}{|\ln(\delta)|}\right),$$

for δ sufficiently small. We can then choose the minima between the δ and $\tilde{\tau}$ obtained in the two cases, to get in the end that there exists a couple $\delta > 0$, $\tilde{\tau} > 0$ such that

$$-1+\delta \leq \phi(x,t) \leq 1-\delta, \quad \text{a.e. in } \Omega \times (T-\tilde{\tau}, T). \tag{4-34}$$

Finally, notice that, due to the choice of T, we have $T - \tilde{\tau} = 2\tilde{\tau} + \frac{\tau}{2} \leq \tau$; therefore we can repeat the same procedure on the interval $(T, T+\tilde{\tau})$ (this means that the new starting time will be $t_{-1} = T - 2\tilde{\tau} \geq \frac{\tau}{2}$) and so on, reaching eventually the entire interval $[\tau, +\infty)$. Clearly δ and $\tilde{\tau}$ are always the same, since the constant C_P is uniform over the entire interval $[\tau, +\infty)$ and the time horizon T does not enter in any of the estimates. The proof is thus concluded.

Remark 4.7. We point out that the same proof holds for the case of convective nonlocal Cahn–Hilliard equation

$$\begin{cases} \partial_t\phi + \boldsymbol{u}\cdot\nabla\phi - \Delta\mu = 0 & \text{in } \Omega \times (0,T), \\ \mu = F'(\phi) - J*\phi & \text{in } \Omega \times (0,T), \\ \partial_{\boldsymbol{n}}\mu = 0 & \text{on } \partial\Omega \times (0,T), \\ \phi(\cdot,0) = \phi_0 & \text{in } \Omega, \end{cases} \tag{4-35}$$

where $\boldsymbol{u}$ is a sufficiently regular divergence free vector field, such that $\boldsymbol{u}\cdot\boldsymbol{n} = 0$ on $\partial\Omega \times (0,T)$. Indeed, Theorem 3.3 can be mostly extended also to this case (see, e.g., [Gal et al. 2017, Section 6], in which a related system, the nonlocal Cahn–Hilliard–Navier–Stokes system, is analyzed). Moreover, in the proof of

Theorem 4.3 the term $\boldsymbol{u}\cdot\nabla\phi$ does not appear, since in (4-21) we should get an additional $(\boldsymbol{u}\cdot\nabla\phi, \phi_n\eta_n^2)$, which is zero thanks to the assumptions on $\boldsymbol{u}$. Therefore, the separation property could a priori be obtained also in the couplings of the nonlocal Cahn–Hilliard equation with some hydrodynamic models, like Navier–Stokes equations (see, e.g., [Abels and Terasawa 2020] or [Gal et al. 2017, Section 6] for some examples of such models).

Remark 4.8. One might think that the proof of Theorem 4.3 could be adapted also to the conserved Allen–Cahn equation

$$\begin{cases} \partial_t\phi+\mu-\bar{\mu}=0 & \text{in } \Omega\times(0,T),\\ \mu=\Psi'(\phi)-\Delta\phi & \text{in } \Omega\times(0,T),\\ \partial_{\boldsymbol{n}}\phi=0 & \text{on } \partial\Omega\times(0,T),\\ \phi(\cdot\,,0)=\phi_0 & \text{in } \Omega, \end{cases} \tag{4-36}$$

where Ψ is defined in (1-2). Indeed, this has been obtained in [Grasselli and Poiatti 2024] in the case of multicomponent conserved Allen–Cahn equation in two and three dimensions, and it is valid also for (4-36). In the proof one loses the term

$$\int_{t_{n-1}}^{t}\int_{A_n(s)} F''(\phi)\nabla\phi\cdot\nabla\phi_n\eta_n^2\,\mathrm{d}x\,\mathrm{d}s,$$

which is substituted by $\int_{t_{n-1}}^{t}\int_{A_n(s)} F'(\phi)\phi_n\eta_n^2\,\mathrm{d}x\,\mathrm{d}s \geq F'(1-2\delta)\int_{t_{n-1}}^{t}\int_{A_n(s)}\phi_n\eta_n^2\,\mathrm{d}x\,\mathrm{d}s$: the presence of the first derivative of F instead of the second derivative, since $F'(1-2\delta)\to+\infty$ as $\delta\to 0^+$, is still enough to carry out the De Giorgi's iteration scheme, by heavily exploiting estimate (4-26). We also mention the fact that in two-dimensional bounded domains the instantaneous strict separation property for (4-36) was proven before in [Giorgini et al. 2022], by a completely different argument.

Remark 4.9. Assumption (H_3) shows that the strict separation property also holds for more general and singular potentials F than the logarithmic one (1-2). Furthermore, by slightly adapting the proof of Theorem 4.3, one can show that the same property also holds for more general double well potentials than F. For instance, one could deal with a chemical potential $\mu=\Psi'(\phi)+(J*1)\phi-J*\phi$, with Ψ defined in (1-2) and obtain an analogous result. Notice that in this new setting the nonlocal term $J*\phi$ is related to diffusion effects (see [Gal et al. 2023a]). Also, in the case of nonconstant mobility $M(\phi)$, the proof should work well as long as it is nondegenerate (i.e., bounded below by a strictly positive constant) and the existence of strong solutions is given. In conclusion, another possible extension could be in the case of dynamic boundary conditions (see, e.g., [Knopf and Signori 2021]): first one needs to assess the existence of strong solutions and the instantaneous regularization of weak solutions, and then apply a De Giorgi's iteration scheme, which seems harder due to the presence of boundary terms which have to be carefully handled.

4.2. ***Proof of Corollary 4.5.*** Observe that, due to Remark 4.6, we only need to prove that the unique solution ϕ departing from ϕ_0 is strictly separated from the pure phases in a neighborhood of the initial time. To this aim we perform again a De Giorgi's iteration scheme, in this case without the use of a cutoff function of the form (4-19). Indeed, the necessity of the cutoff function is merely to eliminate the presence of the initial datum in the estimates, but in our case, up to choosing $\delta \le \delta_0/2$, this problem does not appear any more, as we shall see. Again the Step 1 of the proof of Theorem 4.3 is still valid, and we adopt the same notation. Clearly, thanks to Remark 3.5, we can choose $\tau = 0$, so that again

$$\|\phi_\rho(t)\| \le C_P \|\nabla \phi_\rho(t)\|, \quad \|\tilde{\phi}_\rho(t)\| \le C_P \|\nabla \tilde{\phi}_\rho(t)\|, \quad \text{for almost any } t \ge 0 \text{ and for any } \rho \in [1-2\hat{\delta}, 1]. \tag{4-37}$$

We then start from Step 2. Let us fix δ sufficiently small such that $\delta \le \min\{\hat{\delta}, \delta_0/2\}$, so that (4-37) holds for any $\rho \in [1-2\delta, 1]$. Set then $\tilde{\tau} > 0$ such that (4-43) below holds. As in Theorem 4.3, we define the same sequence (4-16), but we do not need to consider any sequence of times, since we will always use the same, fixed, interval $I := [0, \tilde{\tau}]$. Then we define again

$$\phi_n(x, t) := (\phi - k_n)^+, \tag{4-38}$$

and, for any $n \ge 0$, we introduce the set

$$A_n(t) := \{x \in \Omega : \phi(x, t) - k_n \ge 0\} \quad \text{for all } t \in I,$$

so that

$$A_{n+1}(t) \subseteq A_n(t) \quad \text{for all } n \ge 0 \text{ and } t \in I.$$

We thus set

$$y_n = \int_I \int_{A_n(s)} 1 \, \mathrm{d}x \, \mathrm{d}s \quad \text{for all } n \ge 0.$$

Now, for any $n \ge 0$ we consider the test function $w = \phi_n$, and integrate over $[0, t]$, $t \le \tilde{\tau}$. Then we have, as in Theorem 4.3,

$$\frac{1}{2}\|\phi_n(t)\|^2 + \int_0^t \int_{A_n(s)} F''(\phi)\nabla\phi \cdot \nabla\phi_n \, \mathrm{d}x \, \mathrm{d}s = \int_0^t \int_{A_n(s)} (\nabla J * \phi) \cdot \nabla\phi_n \, \mathrm{d}x \, \mathrm{d}s + \frac{1}{2}\|\phi_n(0)\|^2.$$

Note that, due to the choice of $\delta \le \delta_0/2$, thanks to the strict separation of the initial datum, we immediately infer that $\|\phi_n(0)\| = 0$ for any $n \ge 0$. Following the same arguments as in the proof of Theorem 4.3, we obtain

$$\frac{1}{2}\|\phi_n(t)\|^2 + \frac{1}{2}F''(1-2\delta)\int_0^t \|\nabla\phi_n(s)\|^2 \, \mathrm{d}s \le \frac{\|\nabla J\|^2_{L^1(B_r)}}{2F''(1-2\delta)} y_n$$

for any $t \in [0, \tilde{\tau}]$. Observe that we do not see the presence of the term related to $1/\tilde{\tau}$ (estimated in (4-27)), since it is a consequence of the use of the cutoff function (4-19). This implies

$$\max_{t \in I} \|\phi_n(t)\|^2 \le Z_n, \qquad F''(1-2\delta)\int_I \|\nabla\phi_n\|^2 \, \mathrm{d}s \le Z_n, \tag{4-39}$$

where

$$Z_n := \frac{\|\nabla J\|^2_{L^1(B_r)}}{F''(1-2\delta)} y_n.$$

Observe that, for any $t \in I$ and for almost any $x \in A_{n+1}(t)$, we get

$$\phi_n(x, t) = \underbrace{\phi(x, t) - \left[1 - \delta - \frac{\delta}{2^{n+1}}\right]}_{\phi_{n+1}(x,t) \ge 0} + \delta\left[\frac{1}{2^n} - \frac{1}{2^{n+1}}\right] \ge \frac{\delta}{2^{n+1}},$$

which implies

$$\int_I \int_\Omega |\phi_n|^3 \, dx \, ds \geq \int_I \int_{A_{n+1}(s)} |\phi_n|^3 \, dx \, ds \geq \left(\frac{\delta}{2^{n+1}}\right)^3 \int_I \int_{A_{n+1}(s)} dx \, ds = \left(\frac{\delta}{2^{n+1}}\right)^3 y_{n+1}.$$

Then we have, as in (4-29),

$$\left(\frac{\delta}{2^{n+1}}\right)^3 y_{n+1} \leq \left(\int_I \int_\Omega |\phi_n|^{\frac{10}{3}} \, dx \, ds\right)^{\frac{9}{10}} \left(\int_I \int_{A_n(s)} dx \, ds\right)^{\frac{1}{10}}. \tag{4-40}$$

Again thanks to (2-3) and (4-37), we have

$$\int_I \int_\Omega |\phi_n|^{\frac{10}{3}} \, dx \, ds \leq \widehat{C}(1 + C_P^2) \int_I \|\nabla \phi_n\|^2 \|\phi_n\|^{\frac{4}{3}} \, ds,$$

so that, by (4-39),

$$\begin{aligned}
&\int_I \int_\Omega |\phi_n|^{\frac{10}{3}} \, dx \, ds \\
&\quad \leq \widehat{C}(1 + C_P^2) \int_I \|\nabla \phi_n\|^2 \|\phi_n\|^{\frac{4}{3}} \, ds \leq \widehat{C}(1 + C_P^2) \max_{t \in I} \|\phi_n\|^{\frac{4}{3}} \int_I \|\nabla \phi_n\|^2 \, ds \\
&\quad \leq \frac{\widehat{C}(1 + C_P^2)}{F''(1 - 2\delta)} Z_n^{\frac{2}{3}} F''(1 - 2\delta) \int_I \|\nabla \phi_n\|^2 \, ds \leq \frac{\widehat{C}(1 + C_P^2)}{F''(1 - 2\delta)} Z_n^{\frac{5}{3}} \leq \frac{\|\nabla J\|_{L^1(B_r)}^{\frac{10}{3}} \widehat{C}(1 + C_P^2)}{(F''(1 - 2\delta))^{\frac{8}{3}}} y_n^{\frac{5}{3}}.
\end{aligned}$$

Therefore, we immediately infer from (4-40) that

$$\begin{aligned}
\left(\frac{\delta}{2^{n+1}}\right)^3 y_{n+1} &\leq \left(\int_I \int_\Omega |\phi_n|^{\frac{10}{3}} \, dx \, ds\right)^{\frac{9}{10}} \left(\int_I \int_{A_n(s)} dx \, ds\right)^{\frac{1}{10}} \\
&\leq \frac{\|\nabla J\|_{L^1(B_r)}^3 \widehat{C}^{\frac{9}{10}} (1 + C_P^2)^{\frac{9}{10}}}{(F''(1 - 2\delta))^{\frac{12}{5}}} y_n^{\frac{3}{2}} y_n^{\frac{1}{10}} = \frac{\|\nabla J\|_{L^1(B_r)}^3 \widehat{C}^{\frac{9}{10}} (1 + C_P^2)^{\frac{9}{10}}}{(F''(1 - 2\delta))^{\frac{12}{5}}} y_n^{\frac{8}{5}}.
\end{aligned} \tag{4-41}$$

In conclusion, we end up with

$$y_{n+1} \leq \frac{2^{3n+3} \|\nabla J\|_{L^1(B_r)}^3 \widehat{C}^{\frac{9}{10}} (1 + C_P^2)^{\frac{9}{10}}}{\delta^3 (F''(1 - 2\delta))^{\frac{12}{5}}} y_n^{\frac{8}{5}} \quad \text{for all } n \geq 0,$$

and we can apply Lemma 3.8. In particular, we have

$$b = 2^3 > 1, \quad C = \frac{2^3 \|\nabla J\|_{L^1(B_r)}^3 \widehat{C}^{\frac{9}{10}} (1 + C_P^2)^{\frac{9}{10}}}{\delta^3 (F''(1 - 2\delta))^{\frac{12}{5}}} > 0, \quad \varepsilon = \frac{3}{5},$$

to get that $y_n \to 0$, as long as

$$y_0 \leq C^{-\frac{5}{3}} b^{-\frac{25}{9}},$$

i.e.,

$$y_0 \le \frac{2^{-\frac{40}{3}}\delta^5(F''(1-2\delta))^4}{\|\nabla J\|^5_{L^1(B_r)}\widehat{C}^{\frac{3}{2}}(1+C_P^2)^{\frac{3}{2}}}. \tag{4-42}$$

In conclusion, since we know by (3-7) and Remark 3.5 that $\|F'(\phi)\|_{L^\infty(0,\infty;L^1(\Omega))} \le C$, we infer

$$y_0 = \int_I\int_{A_0(s)} 1\,\mathrm{d}x\,\mathrm{d}s \le \int_I\int_{A_0(s)} \frac{|F'(\phi)|}{F'(1-2\delta)}\,\mathrm{d}x\,\mathrm{d}s \le \frac{C\tilde{\tau}}{F'(1-2\delta)}.$$

Having fixed $\tilde{\tau}$ so that

$$\tilde{\tau} = \frac{2^{-\frac{40}{3}}\delta^5(F''(1-2\delta))^4F'(1-2\delta)}{C\|\nabla J\|^5_{L^1(B_r)}\widehat{C}^{\frac{3}{2}}(1+C_P^2)^{\frac{3}{2}}}, \tag{4-43}$$

we have

$$\frac{C\tilde{\tau}}{F'(1-2\delta)} \le \frac{2^{-\frac{40}{3}}\delta^5(F''(1-2\delta))^4}{\|\nabla J\|^5_{L^1(B_r)}\widehat{C}^{\frac{3}{2}}(1+C_P^2)^{\frac{3}{2}}},$$

so that (4-42) holds. In the end, passing to the limit in y_n as $n \to \infty$, we have obtained that

$$\|(\phi-(1-\delta))^+\|_{L^\infty(\Omega\times(0,\tilde{\tau}))} = 0.$$

We now repeat exactly the same argument for the case $(\phi-(-1+\delta))^-$ (using $\phi_n(t) = (\phi(t)+k_n)^-$), to get in the end that there exist $\delta > 0$, $\tilde{\tau} > 0$ such that

$$-1+\delta \le \phi(x,t) \le 1-\delta, \quad \text{a.e. in } \Omega\times(0,\tilde{\tau}). \tag{4-44}$$

Notice that $\tilde{\tau}$ can be explicitly computed as a function of the parameters of the problem and the initial datum (see (4-43)). The proof is then concluded, recalling Remark 4.6 with $T_1 = \tilde{\tau}$.

5. Some consequences of the strict separation property

In this section we collect some results which are straightforward consequences of the strict separation property proven in Theorem 4.3.

5.1. *Regularization in finite time.* First we show that the weak solution given by Theorem 3.3 actually regularizes more. Indeed, we have a first immediate consequence:

Corollary 5.1. *Under the same assumptions of Theorem 4.3, for any $\tau > 0$, there exists a constant $C = C(\tau) > 0$ such that*

$$\|F'(\phi(t))\|_{L^\infty(\Omega)} + \|\mu(t)\|_{L^\infty(\Omega)} \le C \quad \textit{for all } t \ge \tau.$$

Proof. The proof is immediate, since by the strict separation property we deduce $\|F'(\phi(t))\|_{L^\infty(\Omega)} \le C$ for any $t \ge \tau$ and then by comparison we get the L^∞- control on μ. ☐

Furthermore, we can also obtain the Hölder regularity of the weak solutions:

Corollary 5.2. *Under the same assumptions of Theorem 4.3, for any $\tau > 0$, there exists $C = C(\tau) > 0$ and $\kappa = \kappa(\tau, \delta) \in (0, 1)$ such that*

$$|\phi(x_1, t_1) - \phi(x_2, t_2)| \le C\big(|x_1 - x_2|^\kappa + |t_1 - t_2|^{\frac{\kappa}{2}}\big), \tag{5-1}$$

$$|\mu(x_1, t_1) - \mu(x_2, t_2)| \le C\big(|x_1 - x_2|^\kappa + |t_1 - t_2|^{\frac{\kappa}{2}}\big) \tag{5-2}$$

for all $(x_1, t_1), (x_2, t_2) \in \Omega_t$, where $\Omega_t = [t, t+1] \times \overline{\Omega}$ and $t \ge \tau$.

Proof. We can argue as in [Gal and Grasselli 2014, Lemma 2.11]. In particular, we rewrite the system (1-7) in the form

$$\partial_t \phi = \operatorname{div}(a(x, \phi, \nabla\phi)), \quad (a(x, \phi, \nabla\phi) \cdot \boldsymbol{n})_{|\partial\Omega} = 0,$$

with

$$a(x, \phi, \nabla\phi) := F''(\phi)\nabla\phi - \nabla J * \phi.$$

Since by (H_1) we have $J \in W^{1,1}_{\text{loc}}(\mathbb{R}^3)$, $F''(s) \ge \alpha$ for any $s \in (-1, 1)$ by (H_2) and $\|\nabla J * \phi\|_{L^\infty(\Omega)} \le \|\nabla J\|_{L^1(B_r)}$ by (4-24), by Young's inequality we get

$$\begin{aligned} a(x, \phi, \nabla\phi) \cdot \nabla\phi &= F''(\phi)|\nabla\phi|^2 - (\nabla J * \phi) \cdot \nabla\phi \\ &\ge \alpha|\nabla\phi|^2 - \|\nabla J\|_{L^1(B_r)}|\nabla\phi| \\ &\ge \frac{\alpha}{2}|\nabla\phi|^2 - \frac{1}{2\alpha}\|\nabla J\|^2_{L^1(B_r)}, \end{aligned}$$

and, similarly, by Corollary 5.1 and (4-24),

$$|a(x, \phi, \nabla\phi)| \le \|F''(\phi)\|_{L^\infty(\Omega)}|\nabla\phi| + \|\nabla J * \phi\|_{L^\infty(\Omega)} \le C_1|\nabla\phi| + \|\nabla J\|_{L^1(B_r)}$$

for some positive constant C_1 depending on τ, δ. Therefore we infer the desired estimate (5-1) applying [Dung 2000, Corollary 4.2]. Then, by the regularity of F, we immediately deduce the same result for μ, concluding the proof. □

In order to obtain higher-order spatial regularity for the phase variable ϕ, we need to strengthen the assumptions on the interaction kernel J. In particular, we assume

(H_4) Either $J \in W^{2,1}(\mathcal{B}_R)$, where $\mathcal{B}_R := \{x \in \mathbb{R}^3 : |x| < R\}$, with $R \sim \operatorname{diam}(\Omega)$ such that $\overline{\Omega} \subset \mathcal{B}_R$ and $x - \Omega \subset B_R$ for any $x \in \Omega$, or J is admissible in the sense of [Bedrossian et al. 2011, Definition 1].

Remark 5.3. As noticed in [Gal et al. 2017, Remark 5.9], we observe that Newtonian and second-order Bessel potentials satisfy assumption (H_4), namely they are admissible in the sense of [Bedrossian et al. 2011, Definition 1].

Lemma 5.4. *Under the same assumptions of Theorem 4.3, assuming also that J satisfies (H_4) and $F \in C^3(-1, 1)$, for any $\tau > 0$ there exists $C = C(\tau) > 0$ such that*

$$\|\phi\|_{L^{4/3}(t,t+1;H^2(\Omega))} \le C \quad \textit{for all } t \ge \tau. \tag{5-3}$$

Proof. We first observe that, since we can apply Theorem 3.3, by (3-7)–(3-8) we deduce that

$$\|\nabla\phi\|_{L^{8/3}(t,t+1;L^4(\Omega))}+\|\mu\|_{L^2(t,t+1;V_2)}\leq C(\tau)\quad \text{for all } t\geq\tau, \tag{5-4}$$

for some positive constant $C(\tau)$. Then, as in [Frigeri et al. 2016, Theorem 5], we proceed formally (these computations could be justified in a suitable approximating scheme, see, e.g., [Frigeri et al. 2016, Theorem 5, Step 3]) defining

$$\partial^2_{ij}\phi:=\frac{\partial^2\phi}{\partial x_i\partial x_j}\quad \text{for } i,\, j=1,2,3.$$

We now apply ∂^2_{ij} to the equation for the chemical potential μ and integrate on Ω, to infer

$$\int_\Omega \partial^2_{ij}\mu\partial^2_{ij}\phi\,\mathrm{d}x=\int_\Omega F''(\phi)(\partial^2_{ij}\phi)^2\,\mathrm{d}x-\int_\Omega \partial_i(\partial_j J*\phi)\partial^2_{ij}\phi\,\mathrm{d}x+\int_\Omega F'''(\phi)\partial_i\phi\partial_j\phi\partial^2_{ij}\phi,\quad i,j=1,2,3.$$

We now recall assumption (H$_4$), so that, by [Bedrossian et al. 2011, Lemma 2],

$$\|\partial_i(\partial_j J*\phi)\|_{L^2(\Omega)}\leq C\|\phi\|_{L^2(\Omega)}\leq C(\tau).$$

Therefore, by Cauchy–Schwartz and Young's inequalities, we infer, recalling that $F''(s)\geq\alpha$ for any $s\in(-1,1)$, and exploiting the separation property of Theorem 4.3, for any $t\geq\tau$,

$$\frac{\alpha}{2}\|\partial^2_{ij}\phi\|^2\leq C(1+\|\partial^2_{ij}\mu\|^2+\int_\Omega|\partial_i\phi|^2|\partial_j\phi|^2\,\mathrm{d}x)\leq C\big(1+\|\mu\|^2_{H^2(\Omega)}+\|\nabla\phi\|^4_{L^4(\Omega)}\big),\quad i,j=1,2,3,$$

which implies (5-3), thanks to (5-4). □

5.2. *Convergence to equilibrium.* We conclude the results of our paper by showing that the strict separation property is essential to study the longtime behavior of the single trajectory. In particular, we can follow [Della Porta et al. 2018, Section 6.2]: for the sake of completeness we give here a sketch of the proofs. We employ the typical strategy based on the Lyapunov property of the associated system (see (3-4)) and the well known Łojasiewicz–Simon inequality. Let us consider the set of admissible initial data

$$\mathcal{H}_m:=\{\phi\in L^\infty(\Omega):\|\phi\|_{L^\infty(\Omega)}\leq 1,\quad |\bar\phi|=m\},$$

with $m\in[0,1)$, and fix an initial datum $\phi_0\in\mathcal{H}_m$. Let then ϕ be the unique weak global-in-time solution departing from ϕ_0, whose existence and uniqueness is ensured by Theorem 3.3. We introduce the ω-limit set associated to ϕ_0, i.e.,

$$\omega(\phi_0)=\{\tilde\phi\in\mathcal{H}_m:\exists t_n\to\infty \text{ such that } \phi(t_n)\to\tilde\phi \text{ in } H\}.$$

By (3-6), ϕ is uniformly bounded in V, which is compactly embedded in H. Therefore, by standard results related to the intersection of nonempty, compact (in H), connected and nested sets, we infer that $\omega(\phi_0)$ is nonempty, compact and connected in $\mathcal{H}_m$. We now characterize the set $\omega(\phi_0)$, showing that it is composed by equilibrium points (i.e., stationary solutions) associated to (1-7), which are defined as:

Definition 5.5. ϕ_∞ is an equilibrium point to problem (1-7) if $\phi_\infty \in \mathcal{H}_m \cap V$ satisfies the stationary nonlocal Cahn–Hilliard equation

$$F'(\phi_\infty) - J * \phi_\infty = \mu_\infty \quad \text{in } \Omega, \tag{5-5}$$

where $\mu_\infty \in \mathbb{R}$ is a real constant.

As noticed also in [Della Porta et al. 2018], the existence of a (not necessarily unique, see, e.g., [Bates and Chmaj 1999]) solution to (5-5) can be proven by means of a fixed point argument. Moreover, as shown in [Della Porta et al. 2018, Lemma 6.1], any $\phi_\infty \in V \cap \mathcal{H}_m$ satisfying (5-5) is strictly separated from the pure phases, i.e., there exists $\delta > 0$ such that

$$\|\phi_\infty\|_{L^\infty(\Omega)} \le 1 - \delta.$$

If we now introduce the set of all the stationary points of the nonlocal Cahn–Hilliard equation,

$$\mathcal{S} := \{\phi_\infty \in \mathcal{H}_m \cap V : \phi_\infty \text{ satisfies (5-5)}\},$$

we can easily prove that $\omega(\phi_0) \subset \mathcal{S}$. Indeed, let us consider a sequence $t_n \to \infty$ such that $\phi(t_n) \to \tilde{\phi}$ in H, $\tilde{\phi} \in \omega(\phi_0)$. We then define the sequence of trajectories $\phi_n(t) := \phi(t + t_n)$ and $\mu_n(t) := \mu(t + t_n)$. Thanks to (3-6), we get, up to a nonrelabeled subsequence, that $\phi_n \overset{*}{\rightharpoonup} \phi^*$ in $L^\infty(0, \infty; V)$. Passing to the limit in the equations for ϕ_n, exploiting the results of Theorem 3.3, we infer that also ϕ^* satisfies (3-2)–(3-3) (we denote the corresponding chemical potential by μ^*), with initial datum $\phi^*(0) = \tilde{\phi}$. This last consideration follows from the fact that $\phi_n(0) = \phi(t_n) \to \tilde{\phi}$ strongly in H. Moreover, we clearly have $\lim_{n\to\infty} \mathcal{E}(\phi_n(t)) = \mathcal{E}(\phi^*(t))$ for all $t \ge 0$. By the energy identity (3-4), we infer that the energy $\mathcal{E}(\phi(\,\cdot\,))$ is nonincreasing in time, thus there exists E_∞ such that $\lim_{t\to\infty} \mathcal{E}(\phi(t)) = E_\infty$. This means that this convergence also holds for the subsequence $\{t + t_n\}_n$, thus

$$\mathcal{E}(\phi^*(t)) = \lim_{n\to\infty} \mathcal{E}(\phi_n(t)) = \lim_{n\to\infty} \mathcal{E}(\phi(t + t_n)) = E_\infty,$$

entailing that $\mathcal{E}(\phi^*(\,\cdot\,))$ is constant in time. Passing then to the limit in (3-4), which is valid for each ϕ_n, we obtain

$$E_\infty + \int_s^t \|\nabla \mu^*(\tau)\|^2 \, d\tau \le E_\infty \quad \text{for all } 0 \le s \le t < \infty,$$

implying $\nabla \mu^* = 0$ almost everywhere in Ω, and thus, by comparison in (3-2), also $\partial_t \phi^* = 0$ almost everywhere in Ω, for almost every $t \ge 0$. Therefore, we infer that ϕ^* is constant in time, namely $\phi^*(t) = \tilde{\phi}$ for all $t \ge 0$. Thus $\tilde{\phi}$ satisfies (5-5) with some constant $\mu_\infty \in \mathbb{R}$, and then $\tilde{\phi} \in \mathcal{S}$, implying, being $\tilde{\phi} \in \omega(\phi_0)$ arbitrary, $\omega(\phi_0) \subset \mathcal{S}$. Notice that in this way we have shown that, for any $\phi_\infty \in \omega(\phi_0)$,

$$\mathcal{E}(\phi_\infty) = E_\infty = \lim_{s\to\infty} \mathcal{E}(\phi(s)) = \inf_{s\ge 0} \mathcal{E}(\phi(s)) \le \mathcal{E}(\phi(t)) \quad \text{for all } t \ge 0. \tag{5-6}$$

We can then conclude by showing that $\omega(\phi_0)$ is a singleton. For the sake of clarity we present here the main tool, which is the Łojasiewicz–Simon inequality (see, e.g., [Della Porta et al. 2018, Proposition 6.2] or [Gajewski and Griepentrog 2006]):

Proposition 5.6. *Let $P_0 : H \to H_0$ be the projector operator. Assume that F satisfies (H$_2$) and is real analytic in $(-1, 1)$, $\phi \in V \cap L^\infty(\Omega)$ is such that $-1+\gamma \le \phi(x) \le 1-\gamma$, for any $x \in \overline{\Omega}$, for some $\gamma \in (0, 1)$ and $\phi_\infty \in \mathcal{S}$. Then there exists $\theta \in \left(0, \frac{1}{2}\right)$, $\eta > 0$ and a positive constant C such that*

$$|\mathcal{E}(\phi) - \mathcal{E}(\phi_\infty)|^{1-\theta} \le C\|P_0(F'(\phi) - J * \phi)\|_*, \tag{5-7}$$

whenever $\|\phi - \phi_\infty\| \le \eta$.

Remark 5.7. We observe that the logarithmic potential F is indeed real analytic in $(-1, 1)$, thus the assumption of the foregoing proposition is satisfied.

Theorem 5.8. *Under the same assumptions as in Theorem 4.3, suppose additionally that F is real analytic in $(-1, 1)$. Then the weak solution ϕ, departing from the initial datum $\phi_0 \in \mathcal{H}_m$ converges to a single equilibrium point ϕ_∞ (depending on ϕ_0) and $\omega(\phi_0) = \{\phi_\infty\}$. In particular we have*

$$\lim_{t\to\infty} \|\phi(t) - \phi_\infty\| = 0. \tag{5-8}$$

Proof. Thanks to (5-6), we infer that $\mathcal{E}(\phi(t)) \ge \mathcal{E}(\phi_\infty)$, $\mathcal{E}(\phi(t)) \to \mathcal{E}(\phi_\infty)$, as $t \to \infty$, for any $\phi_\infty \in \omega(\phi_0)$. Without loss of generality we can assume $\mathcal{E}(\phi(t)) > \mathcal{E}(\phi_\infty)$ for all $t \ge 0$. Indeed, if there exists $\bar{t} > 0$ such that $\mathcal{E}(\phi(\bar{t})) = \mathcal{E}(\phi_\infty)$, then clearly $\phi(t) = \phi(\bar{t})$ for any $t \ge \bar{t}$ and the claim follows, since then $\phi(t) = \phi_\infty$ for any $t \ge \bar{t}$. Let us now fix $\theta \in \left(0, \frac{1}{2}\right)$ and $\eta > 0$ given in Proposition 5.6, where we have chosen γ equal to the value of δ given in Theorem 4.3. By a contradiction argument as in [Frigeri et al. 2013, Theorem 4] it is possible to show that there exists $t^* > 0$ such that $\|\phi(t) - \phi_\infty\| \le \eta$, for all $t \ge t^*$. Therefore, since the solution ϕ enjoys the separation property (by Theorem 4.3) and thanks to the choice of γ, by Proposition 5.6 we get, for any $t \ge t^*$,

$$(\mathcal{E}(\phi) - \mathcal{E}(\phi_\infty))^{1-\theta} \le \|P_0(F'(\phi) - J * \phi)\|_* \le C\|P_0\mu\| \le \widehat{C}\|\nabla\mu\|,$$

where $\widehat{C} > 0$ depends on C and on the Poincaré–Wirtinger constant. Therefore, by means of the energy identity (3-4), we deduce, for any $t \ge t^*$,

$$-\frac{d}{dt}(\mathcal{E}(\phi) - \mathcal{E}(\phi_\infty))^\theta = -\theta(\mathcal{E}(\phi) - \mathcal{E}(\phi_\infty))^{\theta-1}\frac{d}{dt}\mathcal{E}(\phi) \ge \frac{\theta\|\nabla\mu\|^2}{\widehat{C}\|\nabla\mu\|} \ge \widetilde{C}\|\nabla\mu\|,$$

where $\widetilde{C} > 0$ is a positive constant independent of t. An integration over $(t^*, +\infty)$, for t^* sufficiently large, implies that $\nabla\mu \in L^1(t^*, \infty; \boldsymbol{H})$. By comparison, we deduce that also $\partial_t\phi \in L^1(t^*, \infty; V')$, so that

$$\phi(t) = \phi(t^*) + \int_{t^*}^{t} \partial_t\phi(\tau)\, d\tau \xrightarrow{t\to+\infty} \tilde{\phi} \quad \text{in } V',$$

for some $\tilde{\phi} \in V'$. Then we infer that $\phi(t)$ converges in V' as $t \to \infty$. By uniqueness of the limit in V', we can then conclude that $\omega(\phi_0)$ is a singleton, i.e., $\omega(\phi_0) = \{\tilde{\phi}\}$. From now on we will denote $\tilde{\phi}$ by ϕ_∞, since any $\phi_\infty \in \omega(\phi_0)$ coincides with $\tilde{\phi}$. Thanks to (3-6), we then get (5-8) by interpolation:

$$\|\phi(t) - \phi_\infty\| \le C\|\phi(t) - \phi_\infty\|_V^{1/2}\|\phi(t) - \phi_\infty\|_{V'}^{1/2} \le C\|\phi(t) - \phi_\infty\|_{V'}^{1/2} \xrightarrow{t\to+\infty} 0,$$

concluding the proof. □

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ANDREA POIATTI: andrea.poiatti@polimi.it
Dipartimento di Matematica, Politecnico di Milano, Milano, Italy

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