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SMALL SCALE FORMATION FOR THE 2-DIMENSIONAL BOUSSINESQ EQUATION

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We study the 2-dimensional incompressible Boussinesq equations without thermal diffusion, and aim to construct rigorous examples of small scale formations as time goes to infinity. In the viscous case, we construct examples of global smooth solutions satisfying $\sup_{\tau\in[0,t]}\|\nabla\rho(\tau)\|_{L^2} \gtrsim t^\alpha$ for some $\alpha > 0$. For the inviscid equation in the strip, we construct examples satisfying $\|\omega(t)\|_{L^\infty} \gtrsim t^3$ and $\sup_{\tau\in[0,t]}\|\nabla\rho(\tau)\|_{L^\infty} \gtrsim t^2$ during the existence of a smooth solution. These growth results hold for a broad class of initial data, where we only require certain symmetry and sign conditions. As an application, we also construct solutions to the 3-dimensional axisymmetric Euler equation whose velocity has infinite-in-time growth.

1. Introduction

The incompressible Boussinesq equations describe the motion of incompressible fluid under the influence of gravitational forces [Gill and Adrian 1982; Majda 2003; Pedlosky 1979]. Let us denote by $\rho(x,t)$ the density of the fluid (it can also represent the temperature, depending on the physical context) and $u(x,t)$ the velocity field. Throughout this paper, we consider the 2-dimensional incompressible Boussinesq equations in the absence of density/thermal diffusivity:

$$\begin{aligned} &\rho_t + u\cdot\nabla\rho = 0,\\ &u_t + u\cdot\nabla u = -\nabla p - \rho e_2 + \nu\Delta u, \quad x\in\Omega,\ \ t>0,\\ &\qquad \nabla\cdot u = 0, \end{aligned} \tag{1-1}$$

where the initial condition is $u(\cdot,0)=u_0$ and $\rho(\cdot,0)=\rho_0$. Here $e_2 := (0,1)^T$, and $\nu \ge 0$ is the viscosity coefficient. We assume the spatial domain Ω is one of the following: the whole space $\mathbb{R}^2$, the torus $\mathbb{T}^2 := (-\pi,\pi]^2$, or the strip $\mathbb{T}\times[0,\pi]$ that is periodic in x_1. When Ω is the strip, we impose the no-slip boundary condition $u|_{\partial\Omega}=0$ if $\nu>0$, and the no-flow boundary condition $u\cdot n|_{\partial\Omega}=0$ if $\nu=0$.

In the past decade, much progress has been made on the analysis of (1-1) in both the viscous case $\nu>0$ and inviscid case $\nu=0$. Below we briefly review the relevant literature and state our main results in each case.

1.1. ***The viscous case $\nu > 0$.*** If the equation for ρ has an additional thermal diffusion term $\kappa\Delta\rho$, global regularity of solutions is well known (see, e.g., [Temam 1988]) and follows from the classical methods for Navier–Stokes equations. In the absence of thermal diffusion, the first global-in-time regularity results were obtained by Hou and Li [2005] in the space $(u,\rho)\in H^m(\mathbb{R}^2)\times H^{m-1}(\mathbb{R}^2)$ for $m\ge 3$, and by

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Chae [2006] in the space $H^m(\mathbb{R}^2) \times H^m(\mathbb{R}^2)$ for $m \geq 3$. When $\Omega \subset \mathbb{R}^2$ is a bounded domain, Lai, Pan, and Zhao [Lai et al. 2011] proved global well-posedness of solutions in $H^3(\Omega) \times H^3(\Omega)$ with the no-slip boundary condition, and showed that the kinetic energy is uniformly bounded in time. The function space was improved by Hu, Kukavica, and Ziane [Hu et al. 2013] to $(u, \rho) \in H^m(\Omega) \times H^{m-1}(\Omega)$ for $m \geq 2$, where Ω is either a bounded domain, $\mathbb{R}^2$, or $\mathbb{T}^2$. In spaces with lower regularity, global well-posedness of weak solutions was obtained in [Abidi and Hmidi 2007; Danchin and Paicu 2011; Hmidi and Keraani 2007; Larios et al. 2013]. For the temperature patch problem, Gancedo and García-Juárez [2017; 2020] proved global regularity in two dimensions and local regularity in three dimensions.

Regarding upper bounds of the global-in-time solutions, for a bounded domain, Ju [2017] obtained that $\|\rho\|_{H^1(\Omega)} \lesssim e^{Ct^2}$. The e^{Ct^2} bound was improved to an exponential bound e^{Ct} in [Kukavica and Wang 2020] for $\Omega = \mathbb{T}^2$ or a bounded domain, and a super-exponential bound $e^{Ct^{(1+\beta)}}$ for some constant $\beta \approx 0.29$ for $\Omega = \mathbb{R}^2$. When $\Omega = \mathbb{T}^2$, they also obtained the uniform-in-time bound $\|u\|_{W^{2,p}(\mathbb{T}^2)} \leq C(p)$ for all $p \in [2, \infty)$. In recent work by Kukavica, Massatt, and Ziane [Kukavica et al. 2023], when Ω is a bounded domain, the upper bound of the norm of ρ has been improved to $\|\rho\|_{H^2(\Omega)} \leq C_\epsilon e^{\epsilon t}$ for all $\epsilon > 0$, and they also showed $\|u\|_{H^3} \leq C_\epsilon e^{\epsilon t}$ for all $\epsilon > 0$.

We would like to point out that all these results deal with *upper bounds* of solutions, and it is a natural question whether certain norms of solutions *can* actually grow to infinity as $t \to \infty$. When $\nu > 0$ and $\Omega = \mathbb{R}^2$, Brandolese and Schonbek [2012] proved that when the initial data ρ_0 does not have mean zero, $\|u(t)\|_{L^2(\mathbb{R}^2)}$ must grow to infinity like $(1+t)^{1/4}$. Here the growth mechanism is due to potential energy converting into kinetic energy, and does not necessarily imply growth in higher derivatives of u or ρ. To the best of our knowledge, there has been no example in the literature showing that $\|\rho(t)\|_{\dot{H}^m}$ or $\|u(t)\|_{\dot{H}^m}$ can actually grow to infinity as $t \to \infty$ for some $m \geq 1$. The goal of this paper is exactly to construct such examples in $\mathbb{R}^2$ and $\mathbb{T}^2$, where $\|\rho(t)\|_{\dot{H}^m} \to \infty$ as $t \to \infty$ for all $m \geq 1$. Since $\|\rho(t)\|_{L^2}$ is preserved in time, growth of $\|\rho(t)\|_{\dot{H}^m}$ implies that ρ has some small scale formation as $t \to \infty$.

In the viscous case, we set the spatial domain to be either $\mathbb{R}^2$ or $\mathbb{T}^2$, and assume that the initial data (ρ_0, u_0) satisfies the following assumptions (here we write $u_0 = (u_{01}, u_{02})^T$). See Figure 1 for an illustration of the assumptions on ρ_0.

(A1) $\rho_0, u_0 \in C^\infty(\Omega)$. If $\Omega = \mathbb{R}^2$, assume in addition that $\rho_0, u_0 \in C_c^\infty(\mathbb{R}^2)$.

(A2) ρ_0 and u_{02} are odd in x_2, and u_{01} is even in x_2. If $\Omega = \mathbb{T}^2$, assume in addition that ρ_0 and u_{02} are even in x_1, u_{01} is odd in x_1, and $\rho_0 = 0$ on the x_2-axis.[1]

(A3) ρ_0 is not identically zero, and $\rho_0 \geq 0$ for $x_2 \geq 0$.

As we show in Section 2.1, under these assumptions, both the potential energy $E_P(t) := \int_\Omega \rho(x,t) x_2 \, dx$ and kinetic energy $E_K(t) = \frac{1}{2}\|u(t)\|_{L^2(\Omega)}^2$ of the solution remain bounded for all times, and the total energy is decreasing in time. We prove that, for all $s \geq 1$, the Sobolev norm $\|\rho(t)\|_{\dot{H}^s}$ grows to infinity at least algebraically in t.

[1]Note that if the $\rho_0 = 0$ on the x_2-axis assumption is removed, the initial data would include some steady states with horizontally stratified density, which clearly would not lead to any growth.

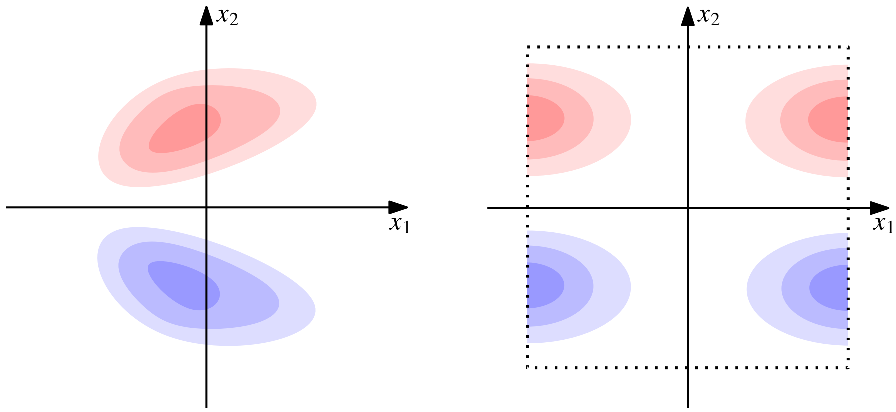

Figure 1. Illustration of the symmetry and sign assumptions on ρ_0 in the plane $\mathbb{R}^2$ (left) and torus $\mathbb{T}^2$ (right) for the viscous Boussinesq equations. Here red denotes positive ρ_0 and blue denotes negative ρ_0.

Theorem 1.1. *Assume* $\nu > 0$, *and let* $\Omega = \mathbb{R}^2$ *or* $\mathbb{T}^2$. *For any initial data* (ρ_0, u_0) *satisfying* (A1)–(A3), *the global-in-time smooth solution* (ρ, u) *to* (1-1) *satisfies the following*:

- *If* $\Omega = \mathbb{R}^2$, *we have*

$$\limsup_{t\to\infty} t^{-s/10} \|\rho(t)\|_{\dot{H}^s(\Omega)} = +\infty \quad \textit{for all } s \geq 1. \tag{1-2}$$

- *If* $\Omega = \mathbb{T}^2$, *we have*

$$\limsup_{t\to\infty} t^{-s(2s-1)/(8s-2)} \|\rho(t)\|_{\dot{H}^s(\Omega)} = +\infty \quad \textit{for all } s \geq 1. \tag{1-3}$$

Remark 1.2. It is a natural question whether these growth rates are sharp. While the powers are likely nonsharp, we point out that $\|\rho(t)\|_{H^1}$ cannot have exponential growth under the assumptions (A1)–(A3). Namely, following arguments similar to [Kukavica and Wang 2020], we show in Proposition 2.4 that, under the assumptions (A1)–(A3), $\|\rho(t)\|_{H^1}$ has a refined subexponential upper bound

$$\|\rho(t)\|_{H^1(\Omega)} \lesssim \exp(Ct^\alpha) \quad \text{for all } t > 0$$

for some constant $\alpha \in (0, 1)$. Therefore in this setting, the fastest possible growth rate of $\|\rho(t)\|_{H^1(\Omega)}$ is somewhere between algebraic and subexponential.

The proof of Theorem 1.1 is motivated by a recent result on small scale formation in solutions to incompressible porous media (IPM) equation by the first and third author [Kiselev and Yao 2023]. The main idea there was to use the monotonicity of the potential energy $E_P(t) = \int \rho(x, t) x_2 \, dx$: on the one hand, for solutions with certain symmetries, $E_P(t)$ is bounded below with $E_P'(t) = -\|\partial_1 \rho(t)\|_{\dot{H}^{-1}}^2$, thus the integral $\int_0^\infty \|\partial_1 \rho(t)\|_{\dot{H}^{-1}}^2 \, dt$ is finite; on the other hand, under certain symmetries, one can show that $\|\partial_1 \rho(t)\|_{\dot{H}^{-1}}^2$ can only be small if $\|\rho(t)\|_{H^s} \gg 1$ for some $s > 0$, leading to growth of ρ in Sobolev norms.

The IPM and Boussinesq equations are related in the sense that, in both equations, the density ρ is transported by an incompressible u, where $u = -\nabla p - \rho e_2$ in IPM, whereas $Du/Dt = -\nabla p - \rho e_2 + \nu \Delta u$ in Boussinesq equations. Since the velocity in Boussinesq equations has one more time derivative than

IPM, we formally expect that $E_P''(t)$ should be related to $-\|\partial_1\rho(t)\|_{\dot H^{-1}}^2$. While this turns out to be true, the situation is more delicate for the Boussinesq equations because $E_P''(t)$ also contains other terms coming from the pressure and viscosity terms. By carefully controlling these additional terms, we prove that if $\|\rho(t)\|_{H^s}$ grows too slowly for $s \geq 1$, $E_P'(t)$ would become unbounded below, contradicting the uniform-in-time bound of energy.

1.2. *The inviscid case $\nu = 0$.* For the inviscid Boussinesq equations in two dimensions, it is well known that the system (1-1) can be rewritten into an equivalent system for the density ρ and the vorticity $\omega = \partial_1 u_2 - \partial_2 u_1$:

$$\begin{aligned} \rho_t + u\cdot\nabla\rho &= 0,\\ \omega_t + u\cdot\nabla\omega &= -\partial_1\rho, \end{aligned} \tag{1-4}$$

where the velocity u can be recovered from the vorticity ω from the Biot–Savart law $u = \nabla^\perp(-\Delta)^{-1}\omega$. While local well-posedness results are available in a variety of functional spaces for $\Omega = \mathbb{R}^2$, $\mathbb{T}^2$, or a bounded domain [Chae and Nam 1997; Chae et al. 1999; Danchin 2013], whether smooth initial data in $\mathbb{T}^2$ or $\mathbb{R}^2$ with finite energy can develop a finite-time singularity is an outstanding open question in fluid dynamics. Note that smooth, infinite-energy initial data can lead to a finite-time blowup, as shown in [Sarria and Wu 2015].

In the presence of boundary, there have been many exciting developments regarding finite-time singularity formation of solutions in the past few years. Luo and Hou [2014] provided numerical evidence for finite-time blowup in smooth solutions of the 3-dimensional axisymmetric Euler equation in a cylinder. When the domain has a corner, Elgindi and Jeong [2020] proved that blow-up can happen for inviscid Boussinesq equations with smooth initial data. When $\Omega = \mathbb{R}^2_+$ is the upper half-plane, Chen and Hou [2021] proved that solutions with $C^{1,\alpha}$ velocity and density can have a nearly self-similar finite-time blowup. Recently, for smooth initial data, Wang, Lai, Gómez-Serrano, and Buckmaster [Wang et al. 2023] used physics-informed neural networks to construct an approximate self-similar blow-up solution numerically. In a very recent preprint, Chen and Hou [2022] put forward an argument combining impressive analytical tools and computer assisted estimates to show that smooth initial data can lead to a stable nearly self-similar blowup.

Note that the inviscid Boussinesq equations (1-4) become the 2-dimensional Euler equation when $\rho \equiv 0$, where it is well known that $\|\nabla\omega(t)\|_{L^\infty}$ can have infinite-in-time growth [Denisov 2009; 2015; Kiselev and Šverák 2014; Nadirashvili 1991; Zlatoš 2015]. Therefore we will only focus on proving infinite-in-time growth of either $\nabla\rho$ (since ρ itself is preserved along the trajectory, one can at most obtain growth results for $\nabla\rho$) or L^p norms of ω itself not involving any derivatives (where such growth is not possible for the 2-dimensional Euler equation since $\|\omega\|_{L^p}$ is preserved in time).

Our first result is set up in the periodic domain $\Omega = \mathbb{T}^2$. We show that, for all smooth initial data (ρ_0, ω_0) in $\mathbb{T}^2$ under some symmetry assumptions, as long as ρ_0 takes values of different sign along the two line segments $\{0\}\times[0,\pi]$ and $\{\pi\}\times[0,\pi]$ (see the left figure of Figure 2 for an illustration), $\|\nabla\rho(t)\|_{L^\infty}$ must grow to infinity at least algebraically in time for all time during the existence of a smooth solution.

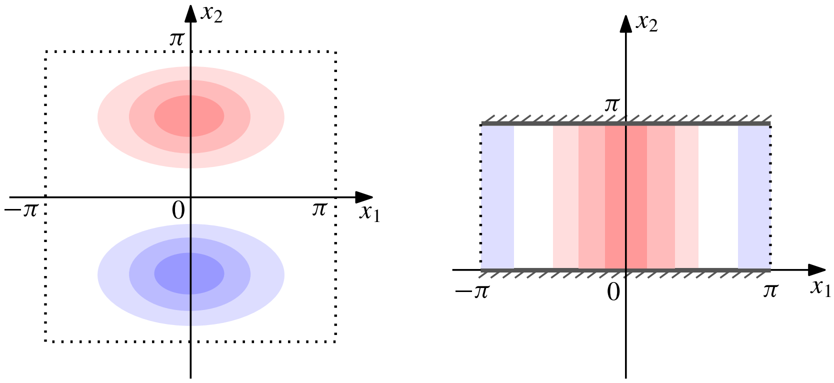

Figure 2. Illustration of the symmetry and sign assumptions on ρ_0 in the torus $\mathbb{T}^2$ (left) and the strip $\mathbb{T} \times [0, \pi]$ (right) for the inviscid Boussinesq equation. Here red denotes positive ρ_0 and blue denotes negative ρ_0.

Theorem 1.3. *Let $\rho_0 \in C^\infty(\mathbb{T}^2)$ be odd in x_2 and even in x_1, and $\omega_0 \in C^\infty(\mathbb{T}^2)$ be odd in both x_1 and x_2. Assume $\rho_0 \geq 0$ on $\{0\} \times [0, \pi]$ with $k_0 := \sup_{x_2 \in [0,\pi]} \rho_0(0, x_2) > 0$, and $\rho_0 \leq 0$ on $\{\pi\} \times [0, \pi]$. Then there exists some constant $c(\rho_0, \omega_0) > 0$ such that the corresponding solution (ρ, ω) to* (1-4) *satisfies*

$$\sup_{\tau \in [0,t]} \|\nabla \rho(\tau)\|_{L^\infty(\mathbb{T}^2)} > c(\rho_0, \omega_0) t^{1/2} \quad \text{for all } t \in [0, T), \tag{1-5}$$

where T is the lifespan of the smooth solution (ρ, ω).

Next we consider the inviscid Boussinesq equation in the strip $\mathbb{T} \times [0, \pi]$. Here the presence of boundary allows us to obtain a faster growth rate in $\|\nabla \rho(t)\|_{L^\infty}$: we prove that the growth is at least like t^2 in the strip (as compared to $t^{1/2}$ in Theorem 1.3). We are also able to obtain a superlinear lower bound for $\|\omega(t)\|_{L^p}$ (for $p = \infty$ it grows like t^3) and a linear lower bound for $\|u(t)\|_{L^\infty}$. Although these algebraic lower bounds are far from finite-time blowup, they hold for a broad class of initial data: no assumption on ω_0 is needed other than being odd in x_1, and ρ_0 only needs to be even in x_1 and satisfy some sign conditions along two line segments (see the right figure of Figure 2 for an illustration). The proofs are soft but might provide an insight into the behavior of smooth solutions during their lifespan.

Theorem 1.4. *Let $\Omega = \mathbb{T} \times [0, \pi]$. Let $\rho_0 \in C^\infty(\Omega)$ be even in x_1 and $\omega_0 \in C^\infty(\Omega)$ be odd in x_1. Assume that there exists $k_0 > 0$ such that $\rho_0 \geq k_0 > 0$ on $\{0\} \times [0, \pi]$ and $\rho_0 \leq 0$ on $\{\pi\} \times [0, \pi]$. Then there exist some constants $T_0(\rho_0, \omega_0) \geq 0$ and $c(\rho_0, \omega_0) > 0$ such that the corresponding solution (ρ, ω) to* (1-4) *satisfies*

$$\|\omega(t)\|_{L^p(\Omega)} \geq ct^{3-2/p} \quad \text{for all } p \in [1, \infty], \quad t \in [T_0, T), \tag{1-6}$$

$$\|u(t)\|_{L^\infty(\Omega)} \geq ct \qquad \text{for all } t \in [T_0, T), \tag{1-7}$$

and

$$\sup_{\tau \in [0,t]} \|\nabla \rho(\tau)\|_{L^\infty(\Omega)} > ct^2 \quad \text{for all } t \in [0, T), \tag{1-8}$$

where T is the lifespan of the smooth solution (ρ, ω). In particular, if $\int_{[0,\pi]\times[0,\pi]} \omega_0 \, dx \geq 0$, then $T_0 = 0$ in all the estimates above.

Remark 1.5. In the estimates for $\|\omega(t)\|_{L^p(\Omega)}$ and $\|u(t)\|_{L^\infty(\Omega)}$ above, it is necessary to have a "waiting time" T_0 depending on the initial data. This is because, for any $t_1 > 0$, there exists some initial data satisfying the assumption of Theorem 1.4 with $\omega(\cdot, t_1) \equiv 0$. (To see this, one can start with $\omega(\cdot, t_1) \equiv 0$ and go backwards in time.) That being said, it can be easily seen from the proof that, if $\int_{[0,\pi]\times[0,\pi]} \omega_0 \, dx \ge 0$, no waiting time is needed.

Remark 1.6. If the symmetry assumptions on ρ_0 and ω_0 are dropped, we still have $\|\omega(t)\|_{L^1(\Omega)} \gtrsim t$ for $t \gg 1$. This infinite-in-time growth implies that, given any steady state ω_s for the 2-dimensional Euler equation on the strip, we have $(0, \omega_s)$ is a nonlinearly unstable steady state for the inviscid Boussinesq equation. See Remark 3.3 for more discussions.

Remark 1.7. Note that the growth result in Theorem 1.4 also holds for the rectangular domain $[-\pi, \pi] \times [0, \pi]$, since the symmetries imposed on the initial data automatically implies $u \cdot n = 0$ on all boundaries of $[-\pi, \pi] \times [0, \pi]$ for all time. However, the proof of Theorem 1.4 does not apply to domains with smooth boundary. That being said, for any bounded domain that is symmetric about both the x_1 and x_2 axis and has a smooth boundary, one can proceed similarly as in Theorem 1.3 (and Lemma 3.1) to obtain the same growth of $\|\nabla\rho\|_{L^\infty}$ as in Theorem 1.3. We leave the details of the argument to interested readers.

For both Theorems 1.3 and 1.4, the proof is based on an interplay between various monotone and conservative quantities. Under the symmetry assumptions, one can easily check that the sign assumptions $\rho \ge 0$ on $\{0\} \times [0, \pi]$ and $\rho \le 0$ on $\{\pi\} \times [0, \pi]$ remain true for all times. This allows us to make the elementary but important observation that the vorticity integral $\int_{[0,\pi]\times[0,\pi]} \omega(x, t)\, dx$ is monotone increasing for all times. More precisely, for the strip, the growth is linear for all times during the existence of a smooth solution, whereas in $\mathbb{T}^2$ we relate the growth with $\|\nabla\rho(t)\|_{L^\infty}$. Another key ingredient is the relation between the vorticity integral and kinetic energy: since the kinetic energy has a uniform-in-time bound, we prove that if the vorticity integral is large, the L^p norm of vorticity must be much larger. For a strip, this allows us to upgrade the linear growth of $\|\omega(t)\|_{L^1}$ to superlinear growth for $\|\omega(t)\|_{L^p}$ for $p \in (1, \infty]$.

1.3. ***Infinite-in-time growth for the 3-dimensional axisymmetric Euler equation.*** The question whether the incompressible Euler equation in $\mathbb{R}^3$ can have a finite-time blowup from smooth initial data of finite energy is an outstanding open problem in nonlinear PDE and fluid dynamics. As we mentioned earlier, for the 3-dimensional axisymmetric Euler equation, when the equation is set up in a cylinder with boundary, Luo and Hou [2014] gave convincing numerical evidence that smooth initial data can lead to a finite-time singularity formation on the boundary. Recent numerical evidence by Hou and Huang [2022; 2023] and Hou [2022] suggests that the blowup can also happen in the interior of domain, but apparently not in self-similar fashion. The first rigorous blow-up result for finite-energy solutions was established in domains with corners by Elgindi and Jeong [2019]. For initial data in $C^{1,\alpha}$ in $\mathbb{R}^3$, Elgindi [2021] showed that such initial data can lead to a self-similar blowup. Very recently, using the connection between 3-dimensional axisymmetric Euler and Boussinesq equations, Chen and Hou [2022] set up a computer-assisted argument that smooth solutions to 3-dimensional axisymmetric Euler equation can form a stable nearly self-similar blowup. The singularity formation happens for initial data in a small neighborhood of a profile that is selected carefully with computer assistance.

In addition to the blow-up v.s. global-in-time regularity question, it is also interesting to investigate whether Sobolev norms of solutions to the 3-dimensional Euler equation can have infinite-in-time growth for broader classes of initial data. Choi and Jeong [2023] constructed smooth compactly supported initial data in $\mathbb{R}^3$ with $\|\nabla^2\omega(t)\|_{L^\infty}$ growing algebraically for all times, and $\|\omega(t)\|_{L^\infty}$ growing exponentially for finite (but arbitrarily long) time. It is also well known that the "two-and-a-half dimensional" solutions (i.e., where u only depends on $x,\ y$, not z) can lead to infinite-in-time linear growth of ω; see [Bardos and Titi 2007, Remark 3.1] for example. See the excellent survey [Drivas and Elgindi 2023] for more results on growth and singularity formation for 2-dimensional and 3-dimensional Euler equations.

It is well known that, away from the axis of symmetry, the 3-dimensional axisymmetric Euler equation is closely related to the inviscid 2-dimensional Boussinesq equations; see [Majda and Bertozzi 2002, Section 5.4.1]. To see this connection, recall that the 3-dimensional axisymmetric Euler equation can be reduced to the system

$$D_t(ru^\theta)=0,\qquad D_t\left(\frac{\omega^\theta}{r}\right)=\frac{\partial_z(ru^\theta)^2}{r^4},\tag{1-9}$$

where u^θ and ω^θ only depend on $r,\ z,\ t$, and $D_t:=\partial_t+u^r\partial_r+u^z\partial_z$ is the material derivative. Heuristically speaking, ru^θ plays the role of ρ in the Boussinesq equation, whereas ω^θ/r plays the role of ω in the Boussinesq equation. Here (u^r,u^z) can be recovered from ω^θ/r by the Biot–Savart law

$$(u^r,u^z)=\frac{1}{r}(-\partial_z\psi,\,\partial_r\psi),\quad \text{where } -\frac{1}{r}\partial_r\left(\frac{1}{r}\partial_r\psi\right)-\frac{1}{r^2}\partial_z^2\psi=\frac{\omega^\theta}{r}.\tag{1-10}$$

We note that the analog of Theorem 1.4 holds for the 3-dimensional axisymmetric Euler equation. We set the spatial domain to be a (not rotating) Taylor–Couette tank

$$\Omega=\{(r,\theta,z): r\in[\pi,2\pi],\ \theta\in\mathbb{T},\ z\in\mathbb{T}\},\tag{1-11}$$

with no-penetration boundary condition at $r=\pi,2\pi$ and periodic boundary conditions in z. Our assumptions and results are as follows.

Theorem 1.8. *Consider the* 3*-dimensional axisymmetric Euler equation* (1-9)–(1-10) *set on the domain* Ω *in* (1-11). *Let $u_0^\theta\in C^\infty(\Omega)$ be even in z and $\omega_0^\theta\in C^\infty(\Omega)$ be odd in z. Assume that there exists $k_0>0$ such that $u_0^\theta\ge k_0>0$ on $z=\pi$ and $|u_0^\theta|\le\frac{1}{8}k_0$ on $z=0$. Then there exist some constants $T_0(u_0)\ge 0$ and $c(u_0)>0$ such that the corresponding solution satisfies*

$$\|\omega^\theta(t)\|_{L^p(\Omega)}\ge ct^{3-2/p}\quad \text{for all } p\in[1,\infty],\quad t\in[T_0,T)\tag{1-12}$$

and

$$\|u(t)\|_{L^\infty(\Omega)}\ge ct\quad \text{for all } t\in[T_0,T),\tag{1-13}$$

where T is the lifespan of the smooth solution. In particular, if $\int_0^\pi\int_\pi^{2\pi}\omega_0^\theta\,dr\,dz\ge 0$, then $T_0=0$ in both estimates above.

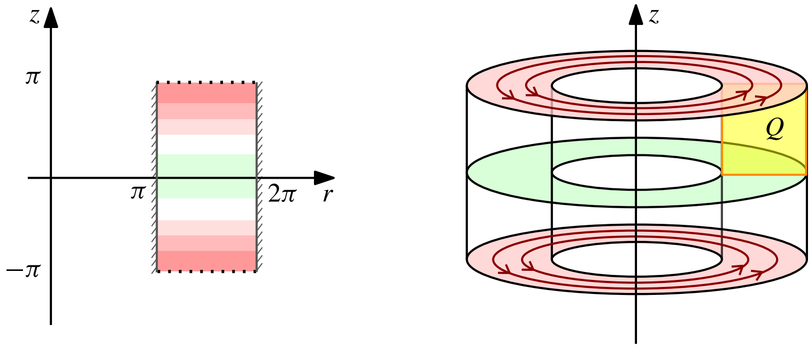

Figure 3. Illustration of the domain and assumptions on u_0^θ for the 3-dimensional axisymmetric Euler equation. The left figure illustrates u_0^θ on the rz plane, and the right figure shows the 3-dimensional setting. Here red denotes positive u_0^θ (and deeper color means larger magnitude), and green denotes u_0^θ with a smaller magnitude (whose sign can be positive or negative). With such initial data, we will show that the "secondary flow" within the yellow square Q grows to infinity as $t \to \infty$.

See Figure 3 for an illustration of the domain and initial data. Note that our setting is almost the same as the Hou–Luo scenario [Luo and Hou 2014], except that we replace the cylinder by an annular cylinder. While our growth estimates are far from a finite-time blowup, they hold for a broad class of initial data: in addition to some symmetry assumptions on u_0^θ and ω_0^θ, all we need is u_0^θ being uniformly positive on $z = \pi$ and having small magnitude on $z = 0$. The proof is a simple argument analogous to Theorem 1.4 for Boussinesq equations, where the key idea is the interplay between the monotonicity of a vorticity integral and the boundedness of kinetic energy.

After the completion of this manuscript, we became aware of work by Serre [1991; 1999], where he studied the 3-dimensional axisymmetric Euler equation in the same domain as in our setting and obtained linear growth of vorticity.

2. Small scale formation for viscous Boussinesq equation

In this section, we aim to prove Theorem 1.1. To begin with, we discuss some properties on the solution (ρ, u) when the initial data satisfies (A1)–(A3). Under the assumption (A1), it is well known that $\rho(\cdot, t)$ and $u(\cdot, t)$ remain in $C^\infty(\Omega)$. And if $\Omega = \mathbb{R}^2$, we have $\rho(\cdot, t) \in C_c^\infty(\mathbb{R}^2)$ and $u(\cdot, t) \in H^k(\mathbb{R}^2)$ for all $k \in \mathbb{N}$ and $t \geq 0$; see, e.g., [Chae 2006; Hou and Li 2005].

Note that the symmetry in (A2) holds true for all times thanks to the uniqueness of solutions. If $\Omega = \mathbb{T}^2$, the additional symmetry in x_1 leads to $u_1(\cdot, t) = 0$ on the x_2-axis for all times, thus $\rho(0, x_2, t) = 0$ for all $x_2 \in \mathbb{T}$ and $t \geq 0$.

The symmetry in x_2 in (A2) also gives $u_2(\cdot, t) = 0$ on the x_1-axis for all times, and combining it with (A3) gives $\rho(x, t) \geq 0$ for $x_2 \geq 0$ and all $t \geq 0$.

We also note that, due to the incompressibility of u, all L^p norms of ρ are conserved in time; that is,

$$\|\rho(t, \cdot)\|_{L^p(\Omega)} = \|\rho_0\|_{L^p(\Omega)} \quad \text{for all } t \geq 0, \quad p \in [1, \infty]. \tag{2-1}$$

2.1. ***Evolution of the potential and kinetic energy.*** Let us define the *potential energy* and *kinetic energy* of the solution as, respectively,

$$E_P(t) := \int_\Omega \rho(x,t)x_2\,dx \quad \text{and} \quad E_K(t) := \frac{1}{2}\int_\Omega |u(x,t)|^2\,dx. \tag{2-2}$$

As we will see, the evolution of these energies plays a crucial role in the proof of Theorem 1.1. The rate of change of E_P can be easily computed as

$$E_P'(t) = \int_\Omega \rho_t x_2\,dx = \int_\Omega -u\cdot(\nabla\rho)x_2\,dx = \int_\Omega \rho u_2\,dx, \tag{2-3}$$

where the last equality follows from the divergence theorem and $\nabla\cdot u = 0$, and note that the boundary integral in the divergence theorem is zero: in $\mathbb{R}^2$ it follows from $\rho(\,\cdot\,,t)$ having compact support, and in $\mathbb{T}^2$ it follows from the symmetries in (A2).

Similarly, one can compute the rate of change of the kinetic energy E_K as

$$E_K'(t) = -\int_\Omega \rho u_2\,dx - \nu\int_\Omega |\nabla u|^2\,dx.$$

Combining the two equations, the total energy $E_P(t)+E_K(t)$ is nonincreasing in time, and more precisely we have

$$E_P(t)+E_K(t)+\nu\int_0^t\int_\Omega |\nabla u(x,s)|^2\,dx\,ds = E_K(0)+E_P(0) \quad \text{for all } t\geq 0. \tag{2-4}$$

From our discussion above, $\rho(\,\cdot\,,t)$ remains odd in x_2 for all $t\geq 0$, and the property (A3) holds for all $t\geq 0$. Thus $E_P(t)$ is positive for all times. Combining this with (2-4) gives

$$0\leq E_P(t)\leq E_P(0)+E_K(0) \quad \text{and} \quad 0\leq E_K(t)\leq E_P(0)+E_K(0) \qquad \text{for all } t\geq 0. \tag{2-5}$$

In addition, using that $E_P(t)\geq 0$ and $E_K(t)\geq 0$ for all $t\geq 0$, we can send $t\to\infty$ in (2-4) to obtain

$$\nu\int_0^\infty \|\nabla u(t)\|_{L^2(\Omega)}^2\,dt \leq E_P(0)+E_K(0). \tag{2-6}$$

In the next lemma we compute the second derivative of E_P, which will be used later.

Lemma 2.1. *Let* (ρ,u) *be a solution to* (1-1) *with initial data* (ρ_0,u_0) *satisfying* (A1)–(A3). *Then the potential energy* E_P *defined in* (2-2) *satisfies*

$$E_P''(t) = A(t)+B(t)-\delta(t) \quad \textit{for all } t\geq 0, \tag{2-7}$$

where

$$A(t) := \sum_{i,j=1}^{2}\int_\Omega ((-\Delta)^{-1}\partial_2\rho)\partial_i u_j\partial_j u_i\,dx, \quad B(t) := \nu\int_\Omega \rho\Delta u_2\,dx, \quad \textit{and} \quad \delta(t) := \|\partial_1\rho\|_{\dot H^{-1}(\Omega)}^2. \tag{2-8}$$

Proof. Differentiating (2-3) in time, we get

$$E_P''(t) = \int_\Omega -u\cdot\nabla(\rho u_2)+\rho(-\partial_2 p-\rho+\nu\Delta u_2)\,dx = \int_\Omega \rho(-\partial_2 p-\rho+\nu\Delta u_2)\,dx, \tag{2-9}$$

where the second equality follows from the incompressibility of u and the fact that the boundary integral is zero as we apply the divergence theorem: for $\Omega = \mathbb{R}^2$ it follows from $\rho(\cdot, t)$ having compact support, whereas for $\Omega = \mathbb{T}^2$ we are using $u \cdot n = 0$ on the boundary of $[-\pi, \pi]^2$ due to our symmetry assumptions in (A2). Comparing (2-9) with our goal (2-7), it suffices to show that

$$\int_\Omega \rho(-\partial_2 p - \rho)\,dx = A(t) - \delta(t). \tag{2-10}$$

To do so, we take divergence in the equation for u in (1-1). Using the incompressibility of u, we get $\nabla \cdot (u \cdot \nabla u) = -\Delta p - \partial_2 \rho$, and hence

$$p = (-\Delta)^{-1}\nabla \cdot (u \cdot \nabla u) + (-\Delta)^{-1}\partial_2 \rho,$$

where $(-\Delta)^{-1}$ is the inverse Laplacian in Ω (which is either $\mathbb{R}^2$ or $\mathbb{T}^2$) defined in the standard way using Fourier transform (for $\Omega = \mathbb{R}^2$) or Fourier series (for $\Omega = \mathbb{T}^2$). Therefore it follows that

$$\begin{aligned} -\partial_2 p - \rho &= -\partial_2(-\Delta)^{-1}\nabla \cdot (u \cdot \nabla u) - (-\Delta)^{-1}\partial_{22}\rho - \rho \\ &= -\sum_{i,j=1}^{2} \partial_2(-\Delta)^{-1}(\partial_i u_j \partial_j u_i) + (-\Delta)^{-1}\partial_{11}\rho. \end{aligned}$$

This immediately yields that

$$\begin{aligned} \int_\Omega \rho(-\partial_2 p - \rho)\,dx &= -\sum_{i,j=1}^{2} \int_\Omega \rho \partial_2(-\Delta)^{-1}(\partial_i u_j \partial_j u_i)\,dx + \int_\Omega \rho(-\Delta)^{-1}\partial_{11}\rho\,dx \\ &= A(t) - \delta(t), \end{aligned}$$

where the second equality follows from integration by parts. This finishes the proof. □

The relation between $\delta(t)$ and $\|\rho(t)\|_{\dot{H}^s(\Omega)}$ has been investigated in [Kiselev and Yao 2023]. Below we state the results from that paper and give a slightly improved estimate for the $\Omega = \mathbb{R}^2$ case.[2] For the sake of completeness, we give a proof in the Appendix. In the statement of the lemma we replace $\rho(t)$ by μ to emphasize that the estimate does not depend on the equation that $\rho(t)$ satisfies.

Lemma 2.2. (a) *Assume $\Omega = \mathbb{R}^2$. Consider all $\mu \in C_c^\infty(\mathbb{R}^2)$ that are odd in x_2 and not identically zero. For all such μ, there exists $c_1(s, \|\mu\|_{L^1}, \|\mu\|_{L^2}) > 0$ such that*

$$\|\mu\|_{\dot{H}^s(\mathbb{R}^2)} \geq c_1(\|\partial_1 \mu\|^2_{\dot{H}^{-1}(\mathbb{R}^2)})^{-s/4} \quad \text{for all } s > 0. \tag{2-11}$$

(b) *Assume $\Omega = \mathbb{T}^2$. Consider all $\mu \in C^\infty(\mathbb{T}^2)$ that are not identically zero, odd in x_2, even in x_1, with $\mu = 0$ on the x_2-axis, and $\mu \geq 0$ in $\mathbb{T} \times [0, \pi]$. For all such μ, there exists $c_2\big(s, \int_{\mathbb{T}\times[0,\pi]} \mu^{1/3}\,dx\big) > 0$ such that*

$$\|\mu\|_{\dot{H}^s(\mathbb{T}^2)} \geq c_2(\|\partial_1 \mu\|^2_{\dot{H}^{-1}(\mathbb{T}^2)})^{-s+1/2} \quad \text{for all } s > \tfrac{1}{2}. \tag{2-12}$$

[2] In [Kiselev and Yao 2023], the estimate corresponding to (2-11) is [Kiselev and Yao 2023, (3.4)], where an extra condition $\|\partial_1 \mu\|^2_{\dot{H}^{-1}} < \frac{1}{4}\|\mu\|^2_{L^2}$ was imposed. In this lemma we give a slightly improved estimate where this assumption is dropped.

2.2. *Infinite-in-time growth of Sobolev norms.* Using Lemma 2.1, for any $t_2 > t_1 \geq 0$, integrating E''_P from t_1 to t_2 we get

$$E'_P(t_2) - E'_P(t_1) = \int_{t_1}^{t_2} A(t)\,dt + \int_{t_1}^{t_2} B(t)\,dt - \int_{t_1}^{t_2} \delta(t)\,dt. \tag{2-13}$$

In the next lemma we estimate the two integrals $\int_{t_1}^{t_2} A(t)\,dt$ and $\int_{t_1}^{t_2} B(t)\,dt$ on the right-hand side.

Lemma 2.3. *Assume* $\nu > 0$. *Let* (ρ, u) *be a solution to* (1-1) *with initial data* (ρ_0, u_0) *satisfying* (A1)–(A3). *Then*, *for all* $t_2 > t_1 \geq 0$, $A(t)$ *defined in* (2-8) *satisfies*

$$\int_{t_1}^{t_2} |A(t)|\,dt \leq C(\rho_0) \int_{t_1}^{t_2} \|\nabla u(t)\|^2_{L^2(\Omega)}\,dt. \tag{2-14}$$

Furthermore, *for all* $s \geq 1$ *and* $t_2 > t_1 \geq 0$, $B(t)$ *defined in* (2-8) *satisfies*

$$\int_{t_1}^{t_2} |B(t)|\,dt \leq C(s, \rho_0)\nu \left(\int_{t_1}^{t_2} \|\nabla u(t)\|^2_{L^2(\Omega)}\,dt \right)^{1/2} \left(\int_{t_1}^{t_2} \|\rho(t)\|^{2/s}_{\dot{H}^s(\Omega)}\,dt \right)^{1/2}. \tag{2-15}$$

Proof. Let us show (2-14) first. Let $f := (-\Delta)^{-1}\partial_2\rho$; we claim that

$$\|f(\cdot, t)\|_{L^\infty(\Omega)} \leq C(\rho_0) \quad \text{for all } t \geq 0. \tag{2-16}$$

Once this is proved, it follows that

$$\int_{t_1}^{t_2} |A(t)|\,dt \leq \int_{t_1}^{t_2} \|f\|_{L^\infty(\Omega)} \|\nabla u\|^2_{L^2(\Omega)}\,dt \leq C(\rho_0) \int_{t_1}^{t_2} \|\nabla u\|^2_{L^2(\Omega)}\,dt.$$

To estimate $\|f\|_{L^\infty(\Omega)}$, we recall the following Hardy–Littlewood–Sobolev inequality for $\Omega = \mathbb{R}^2$ or $\mathbb{T}^2$: (when $\Omega = \mathbb{T}^2$, the function g needs to satisfy an additional assumption $\int_\Omega g(x)\,dx = 0$)

$$\|(-\Delta)^{-\alpha/2} g\|_{L^q(\Omega)} \leq C(\alpha, p, q)\|g\|_{L^p(\Omega)} \quad \text{for } 0 < \alpha < 2,\ \ 1 < p < q < \infty,\ \text{and } \frac{1}{q} = \frac{1}{p} - \frac{\alpha}{2}.$$

We choose $\alpha = 1$, $q = 4$, $p = \frac{4}{3}$, and $g = (-\Delta)^{1/2} f(\cdot, t)$ (note that $g = (-\Delta)^{-1/2}\partial_2\rho$ indeed has mean zero when $\Omega = \mathbb{T}^2$). Then the above inequality becomes

$$\|f(\cdot, t)\|_{L^4(\Omega)} \leq C\|(-\Delta)^{1/2} f\|_{L^{4/3}(\Omega)} = C\|(-\Delta)^{-1/2}\partial_2\rho\|_{L^{4/3}(\Omega)} \leq C\|\rho\|_{L^{4/3}(\Omega)} \leq C(\rho_0),$$

and we also have

$$\|(-\Delta)^{1/2} f(\cdot, t)\|_{L^4(\Omega)} = \|(-\Delta)^{-1/2}\partial_2\rho\|_{L^4(\Omega)} \leq C\|\rho\|_{L^4(\Omega)} \leq C(\rho_0).$$

In the above two estimates, the second-to-last inequality in both equations is due to the Riesz transform being bounded in $L^p(\Omega)$ for $1 < p < \infty$, and the last inequality in both equations comes from (2-1). Combining these estimates together, we have

$$\|f(\cdot, t)\|_{W^{1,4}(\Omega)} \leq C(\rho_0) \quad \text{for all } t \geq 0.$$

Then the boundedness of f follows immediately from Morrey's inequality $W^{1,4}(\Omega) \subset C^{0,1/2}(\Omega)$ for both $\Omega = \mathbb{R}^2$ and $\mathbb{T}^2$. This leads to $\|f(\cdot, t)\|_{L^\infty(\Omega)} \leq C\|f(\cdot, t)\|_{W^{1,4}(\Omega)} \leq C(\rho_0)$ for all $t \geq 0$, which proves (2-16).

Now we turn to the estimate for $B(t)$. Applying the divergence theorem to the definition of $B(t)$ from (2-8), we see that

$$\int_{t_1}^{t_2} |B(t)|\,dt = \nu \int_{t_1}^{t_2} \left| \int_\Omega \nabla\rho \cdot \nabla u_2\,dx \right| dt \le \nu \left(\int_{t_1}^{t_2} \|u(t)\|^2_{\dot H^1(\Omega)}\,dt \right)^{1/2} \left(\int_{t_1}^{t_2} \|\rho(t)\|^2_{\dot H^1(\Omega)}\,dt \right)^{1/2}, \tag{2-17}$$

where we used the Cauchy–Schwarz inequality in the last step. Using the Gagliardo–Nirenberg interpolation inequality, we obtain

$$\begin{aligned}\int_{t_1}^{t_2} |B(t)|\,dt &\le \nu \left(\int_{t_1}^{t_2} \|\nabla u(t)\|^2_{L^2(\Omega)}\,dt \right)^{1/2} \left(\int_{t_1}^{t_2} C(s) \|\rho(t)\|^{2(1-1/s)}_{L^2} \|\rho(t)\|^{2/s}_{\dot H^s(\Omega)}\,dt \right)^{1/2} \\ &\le C(s,\rho_0)\nu \left(\int_{t_1}^{t_2} \|\nabla u(t)\|^2_{L^2(\Omega)}\,dt \right)^{1/2} \left(\int_{t_1}^{t_2} \|\rho(t)\|^{2/s}_{\dot H^s(\Omega)}\,dt \right)^{1/2},\end{aligned}$$

where the last inequality follows from (2-1). This finishes the proof of (2-15). □

Now we are ready to prove Theorem 1.1.

Proof of Theorem 1.1. The main idea of the proof is to estimate all terms in (2-13) for $t_1 = T$ and $t_2 = 2T$ for $T \gg 1$, and obtain a contradiction if $\sup_{t\in[T,2T]} \|\rho(t)\|_{\dot H^s}$ grows slower than a certain power of T.

First, to bound the left-hand side of (2-13), note that (2-3) and the Cauchy–Schwarz inequality yield

$$|E_P'(t)| \le \|\rho(t)\|_{L^2} \|u(t)\|_{L^2} \le \|\rho_0\|_{L^2} \sqrt{2E_K(t)} \le C(\rho_0,u_0) < \infty \quad \text{for all } t \ge 0, \tag{2-18}$$

where the second inequality follows from (2-1) and the definition of E_K in (2-2), and the third inequality follows from (2-5). Thus

$$|E_P'(2T) - E_P'(T)| \le C_0(\rho_0,u_0) < \infty \quad \text{for all } T > 0. \tag{2-19}$$

Plugging the estimates (2-19) and (2-14) into the identity (2-13), we have

$$\int_T^{2T} \delta(t)\,dt \le C_0(\rho_0,u_0) + C_1(\rho_0) \int_T^{2T} \|\nabla u(t)\|^2_{L^2(\Omega)}\,dt + \int_T^{2T} |B(t)|\,dt \quad \text{for all } T > 0. \tag{2-20}$$

Next we will bound the two integrals on the right-hand side from above, and $\int_T^{2T} \delta(t)\,dt$ from below. Let us define

$$\eta(T) := \int_T^{2T} \|\nabla u(t)\|^2_{L^2(\Omega)}\,dt \quad \text{and} \quad M_s(T) := \sup_{t\in[T,2T]} \|\rho(t)\|_{\dot H^s(\Omega)}.$$

Combining (2-4) and (2-5) yields

$$\int_0^\infty \|\nabla u(t)\|^2_{L^2(\Omega)}\,dt \le \nu^{-1} C(\rho_0,u_0) < \infty,$$

where we also used the assumption $\nu > 0$. This implies

$$\lim_{T\to\infty} \eta(T) = 0. \tag{2-21}$$

To bound $\int_T^{2T} |B(t)|\,dt$, we use (2-15) and the definitions of $\eta(T)$ and $M_s(T)$ to get

$$\begin{aligned}\int_T^{2T} |B(t)|\,dt &\le C(s,\rho_0)\nu\biggl(\int_T^{2T} \|\nabla u\|_{L^2(\Omega)}^2\,dt\biggr)^{1/2}\biggl(\int_T^{2T} \|\rho\|_{\dot H^s(\Omega)}^{2/s}\,dt\biggr)^{1/2}\\ &\le C_2(s,\rho_0,\nu)\eta(T)^{1/2}M_s(T)^{1/s}T^{1/2} \qquad \text{for all } s\ge 1,\quad T>0. \end{aligned}\tag{2-22}$$

Next we will bound the integral $\int_T^{2T}\delta(t)\,dt$ from below. If $\Omega=\mathbb{R}^2$, assumption (A2) allows us to apply Lemma 2.2 (a) to $\rho(\,\cdot\,,t)$ (and note that its L^1 and L^2 norms are preserved in time), so there exists $c_3(s,\rho_0)>0$ such that

$$\|\rho(t)\|_{\dot H^s(\mathbb{R}^2)} \ge c_3(s,\rho_0)\delta(t)^{-s/4} \quad \text{for all } s>0,\quad t>0. \tag{2-23}$$

And if $\Omega=\mathbb{T}^2$, using assumptions (A2) and (A3) (note that these imply that $\int_{\mathbb{T}\times[0,\pi]}\rho(x,t)^{1/3}\,dx$ is preserved in time), by Lemma 2.2 (b), there exists $c_4(s,\rho_0)>0$ such that

$$\|\rho(t)\|_{\dot H^s(\mathbb{T}^2)} \ge c_4(s,\rho_0)\delta(t)^{-(s-1/2)} \quad \text{for all } s>\tfrac12,\quad t>0. \tag{2-24}$$

Let us rewrite (2-23) and (2-24) above in a unified manner for the two cases $\Omega=\mathbb{R}^2$ and $\mathbb{T}^2$, so we do not need to repeat similar proofs twice. For Ω either being $\mathbb{R}^2$ or $\mathbb{T}^2$, let us define

$$\alpha_\Omega := \begin{cases} \frac14 s & \Omega=\mathbb{R}^2,\\ s-\frac12 & \Omega=\mathbb{T}^2,\end{cases} \qquad \underline{s}_\Omega := \begin{cases} 0 & \Omega=\mathbb{R}^2,\\ \frac12 & \Omega=\mathbb{T}^2,\end{cases} \qquad c_\Omega(s,\rho_0) := \begin{cases} c_3(s,\rho_0) & \Omega=\mathbb{R}^2,\\ c_4(s,\rho_0) & \Omega=\mathbb{T}^2.\end{cases} \tag{2-25}$$

With this notation, (2-23) and (2-24) become

$$\|\rho(t)\|_{\dot H^s(\Omega)} \ge c_\Omega(s,\rho_0)\delta(t)^{-\alpha_\Omega} \quad \text{for all } s>\underline{s}_\Omega,\quad t>0. \tag{2-26}$$

Combining (2-26) with the definition of M_s gives

$$\int_T^{2T}\delta(t)\,dt \ge \int_T^{2T} c_\Omega^{1/\alpha_\Omega}\|\rho(t)\|_{\dot H^s(\Omega)}^{-1/\alpha_\Omega}\,dt \ge c_\Omega^{1/\alpha_\Omega}M_s(T)^{-1/\alpha_\Omega}T \quad \text{for all } s>\underline{s}_\Omega,\quad T>0. \tag{2-27}$$

Applying the bounds (2-22) and (2-27) and the definition of $\eta(T)$ to the inequality (2-20) (and noting that $\underline{s}_\Omega<1$), we have

$$c_5 M_s(T)^{-1/\alpha_\Omega}T \le C_0 + C_1\eta(T) + C_2\eta(T)^{1/2}M_s(T)^{1/s}T^{1/2} \quad \text{for all } s\ge 1,\quad T>0,$$

where $c_5 := c_\Omega(s,\rho_0)^{1/\alpha_\Omega}$, $C_0 := C_0(\rho_0,u_0)$, $C_1 := C_1(\rho_0)$, and $C_2 := C_2(s,\rho_0,\nu)$ — note that they are all strictly positive and do not depend on T. Rearranging the terms, the inequality is equivalent to

$$(c_5 - C_2\eta(T)^{1/2}T^{-1/2}M_s(T)^{1/s+1/\alpha_\Omega})M_s(T)^{-1/\alpha_\Omega}T \le C_0 + C_1\eta(T) \quad \text{for all } s\ge 1,\quad T>0. \tag{2-28}$$

We claim that this implies

$$\limsup_{T\to\infty} T^{-1/2}M_s(T)^{1/s+1/\alpha_\Omega} = +\infty \quad \text{for all } s\ge 1. \tag{2-29}$$

Towards a contradiction, assume

$$A := \limsup_{T\to\infty} T^{-1/2}M_s(T)^{1/s+1/\alpha_\Omega} < \infty \quad \text{for some } s\ge 1.$$

Combining this assumption with (2-21) gives

$$\limsup_{T\to\infty} \eta(T)^{1/2} T^{-1/2} M_s(T)^{1/s+1/\alpha_\Omega} = \big(\limsup_{T\to\infty} T^{-1/2} M_s(T)^{1/s+1/\alpha_\Omega}\big)\big(\lim_{T\to\infty} \eta(T)^{1/2}\big) = 0,$$

so the parenthesis in (2-28) converges to c_5 as $T \to \infty$. For the remaining term on the left-hand of (2-28), we have

$$\begin{aligned}\liminf_{T\to\infty} M_s(T)^{-1/\alpha_\Omega} T &= \liminf_{T\to\infty} (T^{-1/2} M_s(T)^{1/s+1/\alpha_\Omega})^{-s/(s+\alpha_\Omega)} T^{(s+2\alpha_\Omega)/(2(s+\alpha_\Omega))} \\ &= \liminf_{T\to\infty} A^{-s/(s+\alpha_\Omega)} T^{(s+2\alpha_\Omega)/(2(s+\alpha_\Omega))} = +\infty. \end{aligned} \tag{2-30}$$

The above discussion yields that the liminf of the left-hand side of (2-28) is $+\infty$. This contradicts (2-21), which says the right-hand side of (2-28) goes to $C_0 < \infty$ as $T \to \infty$. This finishes the proof of the claim (2-29).

Finally, using the definition of M_s, we have that (2-29) is equivalent to

$$\limsup_{t\to\infty} t^{-1/2} \|\rho(t)\|_{\dot H^s}^{1/s+1/\alpha_\Omega} = +\infty.$$

Recalling the definition of α_Ω from (2-25), we see that the desired estimates (1-2) and (1-3) follow immediately. □

Although it is unclear whether the algebraic rates are sharp, in the next proposition we show that, under the assumptions (A1)–(A3), $\|\rho(t)\|_{H^1(\Omega)}$ can at most have subexponential growth.

Proposition 2.4. *Let $\Omega = \mathbb{R}^2$ or $\mathbb{T}^2$. For any initial data (ρ_0, u_0) satisfying* (A1)–(A3), $\|\rho(t)\|_{H^1(\Omega)}$ *satisfies the subexponential bound*

$$\|\rho(t)\|_{H^1(\Omega)} \lesssim \exp(Ct^\alpha) \quad \text{for all } t > 0,$$

for some constant $\alpha \in (0, 1)$.

Proof. The proposition can be proved by making a slight modification to [Kukavica and Wang 2020, Theorem 3.1]. For the sake of completeness, we will provide a sketch of the proof. For both $\Omega = \mathbb{T}^2$ and $\mathbb{R}^2$, standard energy estimates give that $\|\nabla\rho(t)\|_{L^2(\Omega)}$ satisfies the estimate

$$\frac{d}{dt}\|\nabla\rho(t)\|_{L^2(\Omega)} \le \|\nabla u(t)\|_{L^\infty} \|\nabla\rho(t)\|_{L^2(\Omega)},$$

which leads to

$$\|\nabla\rho(t)\|_{L^2(\Omega)} \lesssim \exp\left(\int_0^t \|\nabla u(s)\|_{L^\infty(\Omega)}\, ds\right) \|\nabla\rho_0\|_{L^2(\Omega)}. \tag{2-31}$$

Recall that (2-6) gives

$$\int_0^\infty \|\nabla u(t)\|_{L^2}^2\, dt \le C(\nu, \rho_0, u_0). \tag{2-32}$$

Here the time integrability of $\|\nabla u(t)\|_{L^2}^2$ follows from the symmetry assumptions in our setting, and it allows us to obtain a refined upper bound compared to [Kukavica and Wang 2020, Theorem 3.1]. Namely, combining (2-32) with the Gagliardo–Nirenberg inequality

$$\|\nabla u\|_{L^\infty(\Omega)} \le \|\nabla u\|_{L^2(\Omega)}^{(p-2)/(2p-2)} \|\nabla^2 u\|_{L^p(\Omega)}^{p/(2p-2)} \quad \text{for } p > 2 \tag{2-33}$$

and Hölder's inequality, the exponent in (2-31) can be bounded above by

$$\int_0^t \|\nabla u(s)\|_{L^\infty(\Omega)}\,ds \le C(p,\nu,\rho_0,u_0)\bigg(\int_0^t \|\nabla^2 u\|_{L^p(\Omega)}^{(2p)/(3p-2)}\,ds\bigg)^{(3p-2)/(4p-4)}. \tag{2-34}$$

When $\Omega=\mathbb{T}^2$, by [Kukavica and Wang 2020, Theorem 2.1], $\|u(t)\|_{W^{2,p}} < C(p,\nu,\rho_0,u_0)$ for all $p<\infty$. So one can choose $p\gg 1$ to obtain the subexponential upper bound

$$\|\nabla\rho(t)\|_{L^2} \le C(\rho_0)\exp(C(\epsilon,\nu,\rho_0,u_0)t^{3/4+\epsilon}) \quad \text{for any } \epsilon>0,\ \ t>0. \tag{2-35}$$

Next we move on to the $\Omega=\mathbb{R}^2$ case. In this case, it suffices to prove $\|\nabla^2 u(t)\|_{L^p} < C(p,\nu,\rho_0,u_0)$ for all $p<\infty$ under our symmetry setting. Once this is shown, an identical argument as (2-31)–(2-35) again leads to the subexponential growth, since all these estimates also hold for $\mathbb{R}^2$.

To begin with, we show that $\|\omega(t)\|_{L^2}$ is uniformly bounded in time under our symmetry assumptions. Noting from (1-1) that ω satisfies $\omega_t + u\cdot\nabla\omega = \nu\Delta\omega - \partial_1\rho$, we can obtain a standard energy inequality

$$\begin{aligned}\frac{d}{dt}\|\omega(t)\|_{L^2(\mathbb{R}^2)}^2 + \nu\|\nabla\omega(t)\|_{L^2(\mathbb{R}^2)}^2 &= 2\int_{\mathbb{R}^2}\rho(t,x)\partial_1\omega(t,x)\,dx\\ &\le \tfrac12\nu\|\nabla\omega(t)\|_{L^2(\mathbb{R}^2)}^2 + C(\nu)\|\rho(t)\|_{L^2(\mathbb{R}^2)}.\end{aligned}$$

Since $\|\rho(t)\|_{L^2}$ is conserved, the above estimate leads to $(d/dt)\|\omega(t)\|_{L^2(\mathbb{R}^2)}^2 \le C(\nu,\rho_0)$. Combining this with (2-32) (and recall $\|\omega\|_{L^2}=\|\nabla u\|_{L^2}$), we have

$$\|\omega(t)\|_{L^2(\mathbb{R}^2)} < C(\nu,\rho_0,u_0) \quad \text{for all } t\ge 0. \tag{2-36}$$

Following the notation from [Kukavica and Wang 2020], let us define $\zeta=\omega-\partial_1(I-\Delta)^{-1}\rho$ to be the modified vorticity. Since one has $\|\partial_1(I-\Delta)^{-1}\rho\|_{W^{1,p}} \le C(p,\rho_0)$ for all $1<p<\infty$, it implies

$$\|\zeta-\omega\|_{L^p} \le C(p,\rho_0) \quad\text{and}\quad \|\nabla\zeta-\nabla\omega\|_{L^p} \le C(p,\rho_0). \tag{2-37}$$

Combining (2-36) and (2-37) gives a uniform-in-time bound $\|\zeta(t)\|_{L^2} < C(\nu,\rho_0,u_0)$. Now, let us define $\psi_p(t) := \int_{\mathbb{R}^2}|\nabla\zeta(t)|^p$ for $p\ge 2$. Using (1-1), one can express the equation for ζ as [Kukavica and Wang 2020, (2.21)]

$$\zeta_t + u\cdot\nabla\zeta = \nu\Delta\zeta + F, \qquad F := [\partial_1(I-\Delta)^{-1}, u\cdot\nabla]\rho - ((I-\Delta)^{-1}\Delta - I)\partial_1\rho.$$

A straightforward calculation yields that $\psi_2'(t) = 2\int_{\mathbb{R}^2}\nabla\zeta\cdot\nabla F\,dx - 2\nu\int_{\mathbb{R}^2}|\nabla^2\zeta|^2\,dx$. Using the interpolation inequality

$$\|\nabla^2\zeta\|_{L^2} \ge \frac{\|\nabla\zeta\|_{L^2}^2}{C\|\zeta\|_{L^2}},$$

we obtain

$$\psi_2'(t) + \frac{\psi_2^2}{C\|\zeta\|_{L^2}^2} \le 2\int_{\mathbb{R}^2}\nabla\zeta\cdot\nabla F\,dx.$$

To obtain an estimate of the right-hand side, a more careful analysis is required, and the same argument as in [Kukavica and Wang 2020, (3.2)] gives that

$$\psi_2'(t) + \frac{\psi_2^2}{C\|\zeta\|_{L^2}^2} \le C\psi_2 + C.$$

Thus the above uniform-in-time bound for $\|\zeta\|_{L^2}^2$ gives a uniform-in-time bound for $\psi_2(t)$. For any $2 \le p < \infty$, [Kukavica and Wang 2020, (3.3)] gives

$$\psi_{2p}'(t) + \frac{\psi_{2p}^2}{C\psi_p^2} \le Cp^2\psi_{2p} + Cp^5\psi_{2p}^{(p-1)/p}.$$

One can use induction (for $p = 2, 4, 8, \ldots$) to obtain a uniform-in-time bound $\psi_p(t) \le C(p, \nu, \rho_0, u_0)$, and combining this bound with (2-37) gives

$$\|\nabla^2 u(t)\|_{L^p} \le C(p)\|\nabla\omega(t)\|_{L^p} \le C(p)(\|\nabla\zeta(t)\|_{L^p} + C(p, \rho_0)) \le C(p, \nu, \rho_0, u_0).$$

Finally, choosing an arbitrarily large $p \gg 1$ and plugging the above uniform-in-time estimate into (2-34), we again have the subexponential upper bound (2-35) for $\Omega = \mathbb{R}^2$. □

3. Infinite-in-time growth for inviscid Boussinesq and 3-dimensional Euler

3.1. ***Vorticity lemma for flows with fixed kinetic energy.*** Before proving the main theorems, let us start with a simple observation: it says that for any vector field u in a square $Q = [0, \pi]^2$ with a fixed kinetic energy, if its vorticity integral $A := \int_Q \omega\, dx$ is big, then, for $1 < p \le \infty$, $\|\omega\|_{L^p}$ must be even bigger, at least of order $A^{3-2/p}$.

Lemma 3.1. *Let $Q := [0, \pi]^2$. For any vector field $u \in C^\infty(Q)$, let $\omega := \partial_1 u_2 - \partial_2 u_1$. Let us define*

$$E_0 := \int_Q |u|^2\, dx \quad \text{and} \quad A := \int_Q \omega(x)\, dx.$$

Then we have the following lower bound for $\|\omega\|_{L^p(Q)}$:

$$\|\omega\|_{L^p(Q)} \ge c_0 \max\{E_0^{-1+1/p}|A|^{3-2/p}, |A|\} \quad \text{for all } p \in [1, \infty], \tag{3-1}$$

where $c_0 = (128\pi^2)^{-1} > 0$ is a universal constant.

Proof. Without loss of generality, assume $A > 0$. (If $A < 0$, we can prove the estimate for $-u$, whose vorticity integral would be positive.) By Green's theorem, we have

$$\int_{\partial Q} |u(x)|\, ds \ge \int_{\partial Q} u(x) \cdot dl = \int_Q \omega(x)\, dx = A,$$

where the integral in ds denotes the (scalar) line integral with respect to arclength, and the integral in dl denotes the (vector) line integral counterclockwise along ∂Q.

For any $r \in \left[0, \frac{\pi}{2}\right)$, let us define

$$Q_r := [r, \pi - r] \times [r, \pi - r].$$

Note that $Q_0 = Q$ and Q_r shrinks to a point as $r \nearrow \frac{\pi}{2}$. Let us define

$$r_0 := \inf\left\{ r \in \left[0, \tfrac{\pi}{2}\right) : \int_{\partial Q_r} |u(x)|\, ds = \tfrac{1}{2} A \right\}.$$

Since

$$\int_{\partial Q_0} |u(x)|\, ds > A \qquad \text{and} \qquad \int_{\partial Q_r} |u(x)|\, ds \to 0 \quad \text{as } r \nearrow \tfrac{\pi}{2},$$

the above definition leads to a well-defined $r_0 \in \left(0, \frac{\pi}{2}\right)$, and in addition we have

$$\int_{\partial Q_r} |u(x)|\, ds > \tfrac{1}{2} A \quad \text{for all } r \in [0, r_0).$$

Next we claim that

$$r_0 < 16\pi E_0 A^{-2}. \tag{3-2}$$

To show this, note that, for all $0 < r < r_0$, we can apply the Cauchy–Schwarz inequality on ∂Q_r (and use $|\partial Q_r| < 4\pi$) to obtain

$$\int_{\partial Q_r} |u|^2\, ds \geq \frac{1}{4\pi} \left(\int_{\partial Q_r} |u|\, ds \right)^2 > \frac{A^2}{16\pi}.$$

Integrating the above inequality for $r \in (0, r_0)$ over the direction transversal to ∂Q_r $\left(\text{and noting that } \bigcup_{r \in (0, r_0)} \partial Q_r = Q \setminus Q_{r_0}\right)$, we obtain

$$E_0 \geq \int_{Q \setminus Q_{r_0}} |u|^2\, dx = \int_0^{r_0} \int_{\partial Q_r} |u|^2\, ds\, dr > \frac{A^2 r_0}{16\pi},$$

which yields the claim (3-2). Note that (3-2) implies

$$|Q \setminus Q_{r_0}| = \int_0^{r_0} |\partial Q_r|\, dr \leq \min\{4\pi r_0, \pi^2\} \leq \min\{64\pi^2 E_0 A^{-2}, \pi^2\}. \tag{3-3}$$

By Green's theorem and the definition of r_0,

$$\int_{Q \setminus Q_{r_0}} \omega\, dx = \int_{\partial Q} u \cdot dl - \int_{\partial Q_{r_0}} u \cdot dl \geq A - \tfrac{1}{2} A = \tfrac{1}{2} A. \tag{3-4}$$

Finally, we apply Hölder's inequality to bound $\|\omega\|_{L^p(Q)}$ from below for $p \in [1, \infty]$:

$$\|\omega\|_{L^p(Q)} \geq \|\omega\|_{L^p(Q \setminus Q_{r_0})} \geq \left(\int_{Q \setminus Q_{r_0}} \omega\, dx \right) |Q \setminus Q_{r_0}|^{-1+1/p} \quad \text{for all } p \in [1, \infty].$$

Applying the estimates (3-4) and (3-3) in the above inequality finishes the proof of (3-1) with a universal constant $c_0 = (128\pi^2)^{-1}$. □

3.2. ***Infinite-in-time growth for inviscid Boussinesq equations.*** Now we are ready to prove the infinite-in-time growth results. Let us start with Theorem 1.3 for $\Omega = \mathbb{T}^2$.

Proof of Theorem 1.3. Using the Biot–Savart law $u = \nabla^\perp(-\Delta)^{-1}\omega$, one can easily check that, in $\mathbb{T}^2 = (-\pi, \pi]^2$, the even-odd symmetry of ρ and odd-odd symmetry of ω is preserved for all times. This implies the odd-even symmetry of u_1 and even-odd symmetry of u_2 hold for all times. In particular, defining

$$Q := [0, \pi] \times [0, \pi],$$

we have $u \cdot n = 0$ on ∂Q for all times.

For any $x \in \mathbb{T}^2$ and $t \geq 0$, let $\Phi_t(x)$ be the flow map defined by

$$\partial_t \Phi_t(x) = u(\Phi_t(x), t), \quad \Phi_0(x) = x.$$

Using $u \cdot n = 0$ on ∂Q for all times (and $u = 0$ at the four corners of ∂Q), for any $x \in \partial Q$, $\Phi_t(x)$ remains on the same side of ∂Q for all times during the existence of a smooth solution. Combining this with the fact that ρ is preserved along the flow map, the assumptions on ρ_0 implies

$$\rho(0, x_2, t) \geq 0 \quad \text{and} \quad \rho(\pi, x_2, t) \leq 0 \qquad \text{for all } x_2 \in [0, \pi], \ \ t \geq 0. \tag{3-5}$$

Note that the odd-in-x_2 symmetry of ρ_0 yields $\rho_0(0, 0) = \rho_0(0, \pi) = 0$, so the supremum in $k_0 := \sup_{x_2 \in [0,\pi]} \rho_0(0, x_2) > 0$ is achieved at some $\rho(0, a)$ for $a \in (0, \pi)$. In addition, by continuity of ρ_0, there exists some $b \in (0, a)$ such that $\rho_0(0, b) = \frac{1}{2}k_0$ and $\rho_0 \geq \frac{1}{2}k_0$ on $\{0\} \times [b, a]$.

Since $u \cdot n = 0$ on ∂Q for all times, $\Phi_t(0, a)$ and $\Phi_t(0, b)$ remain on the line segment $\{0\} \times (0, \pi)$ for all times. Define

$$h(t) := |\Phi_t(0, b) - \Phi_t(0, a)|, \tag{3-6}$$

which is strictly positive as long as u remains smooth. Note $\rho(\Phi_t(0, a), t) = k_0$ and $\rho(\Phi_t(0, b), t) = \frac{1}{2}k_0$ for all times. This implies

$$\|\nabla\rho(t)\|_{L^\infty(Q)} \geq \frac{|\rho(\Phi_t(0, b), t) - \rho(\Phi_t(0, a), t)|}{|\Phi_t(0, b) - \Phi_t(0, a)|} \geq \tfrac{1}{2}k_0 h(t)^{-1} \tag{3-7}$$

for all times during the existence of a smooth solution.

Next let us define

$$A(t) := \int_Q \omega(x, t)\, dx;$$

we make a simple but useful observation about the monotonicity of $A(t)$. Using the symmetries and the facts $\nabla \cdot u = 0$ in Q and $u \cdot n = 0$ on ∂Q, we find

$$\begin{aligned} A'(t) &= -\int_Q u(x, t) \cdot \nabla\omega(x, t)\, dx - \int_Q \partial_{x_1}\rho(x, t)\, dx \\ &= \int_0^\pi \rho(0, x_2, t)\, dx_2 - \int_0^\pi \rho(\pi, x_2, t)\, dx_2 \geq \tfrac{1}{2}k_0 h(t), \end{aligned} \tag{3-8}$$

where the inequality follows from (3-5), the definition of $h(t)$, and the fact that $\rho(\cdot, t) \geq \frac{1}{2}k_0$ on the line segment connecting $\Phi_t(0, a)$ and $\Phi_t(0, b)$. We now integrate (3-8) in $[0, t]$ and apply (3-7). This yields

$$A(t) \geq \tfrac{1}{4}k_0^2 \int_0^t \|\nabla\rho(\tau)\|_{L^\infty(Q)}^{-1}\, d\tau + A(0). \tag{3-9}$$

In order to apply Lemma 3.1, we need to bound $\|u(t)\|_{L^2(Q)}^2$ from above. From the same calculation in Section 2.1, the sum of the kinetic and potential energies is conserved in $\mathbb{T}^2$, and hence it is also conserved in Q due to the symmetries

$$\frac{1}{2}\int_Q |u(x,t)|^2\, dx + \int_Q x_2\rho(x,t)\, dx = \frac{1}{2}\int_Q |u_0(x)|^2\, dx + \int_Q x_2\rho_0(x)\, dx.$$

Since ρ is advected by the flow, $\|\rho(t)\|_{L^1(Q)}$ is conserved in time, so $\left|\int_Q x_2\rho(x,t)\, dx\right| \leq \pi\|\rho_0\|_{L^1(Q)}$ for all times. This implies

$$\int_Q |u(x,t)|^2\, dx \leq \int_Q |u_0(x)|^2\, dx + 4\pi\|\rho_0\|_{L^1(Q)} =: E_0(\rho_0, u_0)$$

for all times. Now we can apply Lemma 3.1 with $p = +\infty$ to conclude

$$\|\omega(t)\|_{L^\infty} \geq c_0 E_0^{-1} A(t)^3 \geq c_0 E_0^{-1}\left(\tfrac{1}{4}k_0^2 \int_0^t \|\nabla\rho(\tau)\|_{L^\infty}^{-1}\, d\tau + A(0)\right)^3, \tag{3-10}$$

where we used (3-9) in the last step. Note that $A(0)$ may be positive or negative.

On the other hand, the Lagrangian form of the evolution equation for vorticity

$$\frac{d}{dt}\omega(\Phi_t(x), t) = -\partial_{x_1}\rho(\Phi_t(x), t)$$

implies that

$$\|\omega(t)\|_{L^\infty} \leq \int_0^t \|\nabla\rho(\tau)\|_{L^\infty}\, d\tau + \|\omega_0\|_{L^\infty}. \tag{3-11}$$

Combining (3-10) and (3-11), we arrive at

$$\int_0^t \|\nabla\rho(\tau)\|_{L^\infty}\, d\tau + \|\omega_0\|_{L^\infty} \geq c_0 E_0^{-1}\left(\tfrac{1}{4}k_0^2 \int_0^t \|\nabla\rho(\tau)\|_{L^\infty}^{-1}\, d\tau + A_0\right)^3. \tag{3-12}$$

Let us define

$$F(t) := \int_0^t \|\nabla\rho(\tau)\|_{L^\infty}\, d\tau.$$

Since the Cauchy–Schwarz inequality yields

$$\int_0^t \|\nabla\rho(\tau)\|_{L^\infty}^{-1}\, d\tau \geq t^2\left(\int_0^t \|\nabla\rho(\tau)\|_{L^\infty}\, d\tau\right)^{-1} \geq t^2 F(t)^{-1} \quad \text{for all } t > 0,$$

plugging it into (3-12) gives an inequality relating $F(t)$ to itself:

$$F(t) \geq c_0 E_0^{-1}\left(\tfrac{1}{4}k_0^2 t^2 F(t)^{-1} + A_0\right)^3 - \|\omega_0\|_{L^\infty}. \tag{3-13}$$

Our goal is to show that there exists some $c_1(\rho_0, \omega_0) > 0$ such that

$$F(t) \geq c_1(\rho_0, \omega_0) t^{3/2} \quad \text{for all } t \geq 1. \tag{3-14}$$

Towards a contradiction, suppose (3-14) does not hold at some $t_1 \geq 1$, so $t_1^2 F(t_1)^{-1} \geq c_1^{-1} t_1^{1/2}$. Since $t_1 \geq 1$, one can choose c_1 sufficiently small (only depending on initial data) such that the right-hand side of (3-13) is bounded below by $4^{-4} c_0 E_0^{-1} k_0^6 c_1^{-3} t_1^{3/2}$. On the other hand, the left-hand side is bounded above by $c_1 t_1^{3/2}$. Thus we obtain a contradiction if we further require $c_1 < 4^{-1}(c_0 E_0^{-1} k_0^6)^{1/4}$.

Finally, note that (3-14) directly implies $\sup_{\tau \in [0,t]} \|\nabla \rho(\tau)\|_{L^\infty} \geq c_1(\rho_0, \omega_0) t^{1/2}$ for all $t \geq 1$. For $t \in (0, 1)$, recall that the definition of k_0 and the fact $\rho(0, 0, t) = 0$ yield

$$\|\nabla \rho(t)\|_{L^\infty} \geq \tfrac{1}{\pi} k_0 \geq \big(\tfrac{1}{\pi} k_0\big) t^{1/2} \quad \text{for } t \in (0, 1).$$

Combining these two estimates finishes the proof. □

Remark 3.2. Theorem 1.3 does not give us an infinite-in-time growth result for $\omega(\cdot, t)$. All we have is the following conditional growth estimate coming from (3-10): if $\limsup_{t\to\infty} t^{-1} \|\nabla \rho(t)\|_{L^\infty} < \infty$, this must imply $\lim_{t\to\infty} \|\omega(t)\|_{L^\infty} = \infty$.

Proof of Theorem 1.4. The proof is similar to the previous one, and in fact it is easier due to the uniform positivity of ρ_0 on $\{0\} \times [0, \pi]$. Using the Biot–Savart law, one can check that the even-in-x_1 symmetry of ρ and odd-in-x_1 symmetry of ω is preserved for all times. Defining $Q := [0, \pi] \times [0, \pi]$, the symmetries and the boundary condition yield that $u \cdot n = 0$ on ∂Q for all times. In particular, this implies

$$\rho(0, x_2, t) \geq k_0 > 0 \quad \text{and} \quad \rho(\pi, x_2, t) \leq 0 \qquad \text{for all } x_2 \in [0, \pi], \quad t \geq 0, \tag{3-15}$$

during the existence of a smooth solution.

Again, let us define $A(t) := \int_Q \omega(x, t)\, dx$. A calculation similar to the previous proof shows that in this case

$$A'(t) \geq \int_0^\pi \rho(0, x_2, t)\, dx_2 - \int_0^\pi \rho(\pi, x_2, t)\, dx_2 \geq k_0 \pi,$$

where the last inequality follows from (3-15). This gives us a lower bound

$$A(t) \geq k_0 \pi t + A(0) \quad \text{for all } t \geq 0. \tag{3-16}$$

An identical argument as in the proof of Theorem 1.3 gives

$$\int_\Omega |u(x, t)|^2\, dx \leq E_0(\rho_0, u_0)$$

uniformly in time; thus we can apply Lemma 3.1 to obtain

$$\|\omega\|_{L^p(Q)} \geq c_0 E_0^{-1+1/p} |A(t)|^{3-2/p} \quad \text{for all } p \in [1, \infty]. \tag{3-17}$$

Also, note that Green's theorem yields

$$A(t) = \int_{\partial Q} u \cdot dl \leq 4\pi \|u(t)\|_{L^\infty}. \tag{3-18}$$

Regarding the growth of $\nabla \rho$, note that (3-11) still holds in a strip, so

$$\sup_{\tau \in [0,t]} \|\nabla \rho(\tau)\|_{L^\infty} \geq t^{-1}(\|\omega(t)\|_{L^\infty} - \|\omega_0\|_{L^\infty}) \quad \text{for all } t > 0. \tag{3-19}$$

Below we discuss two cases.

Case 1: $A(0) \ge 0$. In this case (3-16) gives

$$A(t) \ge k_0 \pi t \quad \text{for all } t > 0.$$

We then apply (3-17) and (3-18) to obtain lower bounds for $\|\omega(t)\|_{L^p(Q)}$ and $\|u(t)\|_{L^\infty}$:

$$\|\omega(t)\|_{L^p(Q)} \ge c_1(\rho_0, \omega_0) t^{3-2/p} \quad \text{for all } p \in [1, +\infty], \quad t \ge 0, \tag{3-20}$$

$$\|u(t)\|_{L^\infty(Q)} \ge \tfrac{1}{4} k_0 t \quad \text{for all } t \ge 0. \tag{3-21}$$

Regarding the growth of $\nabla\rho$, we apply (3-20) with $p = +\infty$ and combine it with (3-19) to obtain

$$\sup_{\tau\in[0,t]} \|\nabla\rho(\tau)\|_{L^\infty} \ge t^{-1}(c_1(\rho_0, \omega_0)t^3 - \|\omega_0\|_{L^\infty}),$$

which implies

$$\sup_{\tau\in[0,t]} \|\nabla\rho(\tau)\|_{L^\infty} \ge c_1(\rho_0, \omega_0)t^2 \quad \text{for all } t \ge \left(\frac{\|\omega_0\|_{L^\infty}}{c_1(\rho_0, \omega_0)}\right)^{1/3}.$$

Combining this large time estimate with the trivial lower bound $\|\nabla\rho(t)\|_{L^\infty} \ge \frac{1}{\pi} k_0$ for all times, there exists some $c_2(\rho_0, \omega_0) > 0$ such that

$$\sup_{\tau\in[0,t]} \|\nabla\rho(\tau)\|_{L^\infty} \ge c_2(\rho_0, \omega_0)t^2 \quad \text{for all } t \ge 0. \tag{3-22}$$

Case 2: $A_0 < 0$. In this case the right-hand side of (3-16) becomes positive for $t > |A_0|/(k_0\pi)$. In addition, we have

$$A(t) \ge \tfrac{1}{2} k_0 \pi t \quad \text{for all } t \ge T_0 =: \frac{2|A_0|}{k_0\pi}.$$

Once we obtain this (positive) linear lower bound for $t \ge T_0$, we can argue as in Case 1 to obtain lower bounds for $\|\omega(t)\|_{L^p(Q)}$, $\|u(t)\|_{L^\infty}$, and $\sup_{\tau\in[0,t]} \|\nabla\rho(\tau)\|_{L^\infty}$ for all $t \ge T_0$. In addition, combining the lower bound for $\|\nabla\rho(t)\|_{L^\infty}$ for $t \ge T_0$ with the trivial lower bound $\|\nabla\rho(t)\|_{L^\infty} \ge k_0/\pi$ for all times, we again have (3-22) with a smaller coefficient $c(\rho_0, \omega_0) > 0$ that only depends on the initial data. □

Remark 3.3. If the assumptions on symmetries of ρ_0 and ω_0 are dropped, the following simple argument still gives $\|\omega(t)\|_{L^1} \gtrsim t$ for $t \gg 1$. Let $Q_t := \{\Phi_t(x) : x \in [0, \pi] \times [0, \pi]\}$, and denote by

$$\Gamma_t^1 := \{\Phi_t(x) : x \in \{0\} \times [0, \pi]\} \quad \text{and} \quad \Gamma_t^2 := \{\Phi_t(x) : x \in \{\pi\} \times [0, \pi]\}$$

the left and right boundary of Q_t. (Since $u \cdot n = 0$ on $\partial\Omega$, the top and bottom boundaries of Q_t remain on $\partial\Omega$ for all times.) In addition, since ρ is preserved along the flow, at each t we have $\rho(\cdot, t)|_{\Gamma_t^1} \ge k_0 > 0$ and $\rho(\cdot, t)|_{\Gamma_t^2} \le 0$. Thus a computation similar to (3-8) in the moving domain Q_t gives

$$\frac{d}{dt}\int_{Q_t} \omega(x, t)\, dx = \int_{Q_t} -\partial_{x_1}\rho(t)\, dx \ge k_0\pi \quad \text{for all } t \ge 0.$$

Therefore, as long as the solution (ρ, ω) remains smooth, we have

$$\|\omega(t)\|_{L^1} \ge \int_{Q_t} \omega(x, t)\, dx \ge k_0\pi t - \|\omega_0\|_{L^1} \quad \text{for all } t \ge 0. \tag{3-23}$$

However, since Q_t is in general largely deformed from a square for $t \gg 1$, we are not able to apply Lemma 3.1 to obtain faster growth rate for higher L^p norms.

Note that given any steady state ω_s of the 2-dimensional Euler equation on the strip Ω, $(0, \omega_s)$ is automatically a steady state of the inviscid Boussinesq equations (1-4). Thus the infinite-in-time growth estimate (3-23) directly implies that any such steady state (with zero density) is nonlinearly unstable, in the sense that, for any $0 < k_0 \ll 1$, an arbitrarily small perturbation $\rho_0 = k_0 \cos(x_1)$, $\omega_0 = \omega_s$ leads to $\lim_{t\to\infty} \|\omega(t)\|_{L^1} = \infty$. See [Bedrossian et al. 2023; Castro et al. 2019; Deng et al. 2021; Doering et al. 2018; Masmoudi et al. 2022; Tao et al. 2020; Zillinger 2023] for more results on stability/instability of steady states of the inviscid or viscous Boussinesq equations.

3.3. ***Application to 3-dimensional axisymmetric Euler equation.*** In this subsection we will prove Theorem 1.8, whose proof is a close analog of Theorem 1.4.

Proof of Theorem 1.8. Using the Biot–Savart law, one can easily check that ω^θ remains odd in z and u^θ remains even in z for all times while the solution stays smooth. Combining these symmetries with the Biot–Savart law (1-10) gives $u^z = 0$ for $z = 0$ and $z = \pi$ for all times. For a point x on the rz-plane, let us define the flow-map $\Phi_t(x) : [\pi, 2\pi] \times \mathbb{T} \to [\pi, 2\pi] \times \mathbb{T}$, given by

$$\frac{d}{dt}\Phi_t(x) = (u_r(\Phi_t(x), t), u_z(\Phi_t(x), t)).$$

Since $u_z = 0$ on $z = \pi$, for any $x \in [\pi, 2\pi] \times \{\pi\}$, we have $\Phi_t(x)$ remains on $[\pi, 2\pi] \times \{\pi\}$. From the first equation in (1-9), we have ru^θ is conserved along the trajectory. Thus, for any point (r, π) with $r \in [\pi, 2\pi]$, we have

$$ru^\theta(r, \pi, t) \geq \pi u_0^\theta(\Phi_t^{-1}(r, \pi), 0) \geq \pi k_0,$$

where the last inequality follows from the assumption $u_0^\theta \geq k_0 > 0$ on $z = \pi$ and the fact that $\Phi_t^{-1}(r, \pi) \in [\pi, 2\pi] \times \{\pi\}$. This implies

$$u^\theta(r, \pi, t) \geq \tfrac{1}{2}k_0 > 0 \quad \text{for all } r \in [\pi, 2\pi], \quad t \geq 0. \tag{3-24}$$

Applying a similar argument for $z = 0$, the assumption $|u_0^\theta| < \frac{1}{8}k_0$ on $z = 0$ leads to

$$|u^\theta(r, 0, t)| \leq \tfrac{1}{4}k_0 \quad \text{for all } r \in [\pi, 2\pi], \quad t \geq 0. \tag{3-25}$$

Defining $Q := [\pi, 2\pi] \times [0, \pi]$ to be a square on the rz-plane, the above symmetry results give $(u^r, u^z) \cdot n = 0$ on ∂Q for all times. Using this boundary condition as well as the divergence-free property of (ru^r, ru^z) in (r, z) (which follows from (1-10)), we apply the divergence theorem to obtain

$$\begin{aligned}
\frac{d}{dt}\int_Q \omega^\theta(r, z, t)\, dr\, dz &= \int_Q (ru_r, ru_z) \cdot \nabla_{r,z}\left(\frac{\omega^\theta}{r}\right) + \frac{\partial_z (u^\theta)^2}{r}\, dr\, dz \\
&= \int_Q \frac{\partial_z (u^\theta)^2}{r}\, dr\, dz \\
&= \int_\pi^{2\pi} \frac{1}{r}(u^\theta(r, \pi, t)^2 - u^\theta(r, 0, t)^2)\, dr \\
&\geq \ln 2 \tfrac{3}{16}k_0^2 \geq \tfrac{1}{10}k_0^2
\end{aligned}$$

for all times during the existence of a smooth solution, where the last inequality follows from (3-24) and (3-25). This directly implies

$$A(t) := \int_Q \omega^\theta(r,z,t)\,dr\,dz \geq \tfrac{1}{10}k_0^2 t + \int_Q \omega_0^\theta\,dr\,dz.$$

In particular, if $\int_Q \omega_0^\theta\,dr\,dz \geq 0$, this implies

$$A(t) \geq \tfrac{1}{10}k_0^2 t \quad \text{for all } t \geq 0, \tag{3-26}$$

and if $\int_Q \omega_0^\theta\,dr\,dz < 0$, we have

$$A(t) \geq \tfrac{1}{20}k_0^2 t \quad \text{for all } t \geq T_0 =: 20k_0^{-2}\left|\int_Q \omega_0^\theta\,dr\,dz\right|. \tag{3-27}$$

Another ingredient we need is the energy conservation. It is well known that the kinetic energy is conserved for the 3-dimensional Euler equation, i.e., $\int_\Omega |u(x,t)|^2\,dx = \int_\Omega |u_0|^2\,dx$. Since Ω has an inner boundary with positive radius π, this implies, in the domain Q in the rz plane, we also have

$$\int_Q (u^r(r,z,t)^2 + u^z(r,z,t)^2)\,dr\,dz \leq E_0(u_0).$$

Recall that ω^θ and (u^r, u^z) are related by $\omega^\theta = \partial_r u^z - \partial_z u^r$. Thus we can apply Lemma 3.1 to conclude that

$$\|\omega^\theta(t)\|_{L^p(Q)} \geq c_0 E_0^{-1+1/p}|A(t)|^{3-2/p} \quad \text{for all } p \in [1,\infty],\ \ t \geq 0,$$

which directly leads to (1-12) once we plug estimates (3-26) and (3-27) of $A(t)$ into the above equation.

Finally, applying Green's theorem in Q, we have

$$A(t) = \int_Q \omega^\theta drdz = \int_Q (\partial_r u^z - \partial_z u^r)\,dr\,dz = \int_{\partial Q} u\cdot dl \leq 4\pi\|u(t)\|_{L^\infty}.$$

Combining this with the estimates (3-26) and (3-27) directly gives (1-13), finishing the proof. □

Appendix: Proof of Lemma 2.2

In the appendix we prove Lemma 2.2. The proof is almost the same as in [Kiselev and Yao 2023] other than a small improvement in part (a). We sketch a proof for both parts below for the sake of completeness.

Proof of Lemma 2.2 (*a*). Here the proof mostly follows [Kiselev and Yao 2023, (3.4)], except that we make a small improvement dropping the assumption $\|\partial_1\mu\|_{\dot H^{-1}}^2 < \frac14\|\mu\|_{L^2}^2$ in that paper. Let us define

$$\delta := \|\partial_1\mu\|_{\dot H^{-1}(\mathbb{R}^2)}^2, \quad A := \|\mu\|_{L^2(\mathbb{R}^2)}^2.$$

Clearly,

$$\delta = \int_{\mathbb{R}^2} \frac{\xi_1^2}{|\xi|^2}|\hat\mu|^2\,d\xi \leq A.$$

Let us discuss the following two cases.

Case 1: $\delta < \frac{1}{4}A$. In this case let us define

$$D_\delta := \left\{(\xi_1, \xi_2) : \frac{|\xi_1|}{|\xi|} \ge \sqrt{\frac{2\delta}{A}}\right\}.$$

By definition of D_δ, we have

$$\delta \ge \int_{D_\delta} \frac{\xi_1^2}{|\xi|^2} |\hat{\mu}(\xi)|^2 \, d\xi \ge \frac{2\delta}{A} \int_{D_\delta} |\hat{\mu}|^2 \, d\xi.$$

This gives $\int_{D_\delta} |\hat{\mu}|^2 \, d\xi \le \frac{1}{2}A$, and thus $\int_{D_\delta^c} |\hat{\mu}|^2 \, d\xi \ge \frac{1}{2}A$. Note that D_δ^c can be expressed in polar coordinates as

$$D_\delta^c = \left\{(r\cos\theta, r\sin\theta) : r \ge 0, |\cos\theta| < \sqrt{\frac{2\delta}{A}}\right\}.$$

Since $\mu \in C_c^\infty(\mathbb{R}^2)$, we have

$$\|\hat{\mu}\|_{L^\infty(\mathbb{R}^2)} \le (2\pi)^{-1} \|\mu\|_{L^1(\mathbb{R}^2)} =: B.$$

Let $h_\delta > 0$ be such that $|D_\delta^c \cap \{|\xi_2| < h_\delta\}| = (4B^2)^{-1}A$, which we will estimate later. Such a definition gives

$$\int_{D_\delta^c \cap \{|\xi_2| \ge h_\delta\}} |\hat{\mu}|^2 \, d\xi = \int_{D_\delta^c} |\hat{\mu}|^2 \, d\xi - \int_{D_\delta^c \cap \{|\xi_2| < h_\delta\}} |\hat{\mu}|^2 \, d\xi \ge \tfrac{1}{2}A - (4B^2)^{-1}AB^2 = \tfrac{1}{4}A,$$

which implies

$$\|\mu\|_{\dot{H}^s(\mathbb{R}^2)}^2 \ge \int_{\mathbb{R}^2} |\xi_2|^{2s} |\hat{\mu}|^2 \, d\xi \ge h_\delta^{2s} \int_{D_\delta^c \cap \{|\xi_2| \ge h_\delta\}} |\hat{\mu}|^2 \, d\xi \ge \tfrac{1}{4}A h_\delta^{2s}. \tag{A-1}$$

To estimate h_δ, let us define $\theta_0 := \cos^{-1}(\sqrt{2\delta/A})$. Since $D_\delta^c \cap \{|\xi_2| < h_\delta\}$ consists of two identical triangles with height h_δ and base $2h_\delta \cot\theta_0$, we have

$$(4B^2)^{-1}A = |D_\delta^c \cap \{|\xi_2| < h_\delta\}| = 2h_\delta^2 \cot\theta_0 \le 4\sqrt{\delta} A^{-1/2} h_\delta^2,$$

where the inequality follows from $\cos\theta_0 = \sqrt{2\delta/A}$ and $\sin\theta_0 = \sqrt{1 - 2\delta/A} \ge 1/\sqrt{2}$, due the assumption $\delta < \frac{1}{4}A$ in Case 1. Therefore $h_\delta \ge (4B)^{-1} A^{3/4} \delta^{-1/4}$. Plugging it into (A-1) yields

$$\|\mu\|_{\dot{H}^s(\mathbb{R}^2)} \ge \tfrac{1}{2}\sqrt{A} h_\delta^s \ge c(s, A, B)\delta^{-s/4},$$

finishing the proof of Lemma 2.2 in Case 1.

Case 2: $\delta \ge \frac{1}{4}A$. As in Case 1, let us define $\|\hat{\mu}\|_{L^\infty(\mathbb{R}^2)} \le (2\pi)^{-1}\|\mu\|_{L^1(\mathbb{R}^2)} =: B$. Let $r_0 := (A/(2\pi B^2))^{1/2}$. Such a definition leads to

$$\int_{B(0,r_0)} |\hat{\mu}|^2 \, d\xi \le \pi r_0^2 \|\hat{\mu}\|_{L^\infty(\mathbb{R}^2)}^2 \le \tfrac{1}{2}A,$$

and thus

$$\|\mu\|_{\dot{H}^s(\mathbb{R}^2)}^2 \ge \int_{B(0,r_0)^c} |\xi|^{2s} |\hat{\mu}|^2 \, d\xi \ge r_0^{2s} \tfrac{1}{2}A \ge c(s, A, B)\delta^{-s/4},$$

where the last inequality follows from the assumption $\delta \ge \frac{1}{4}A$ in Case 2. This finishes the proof of part (a). ☐

Proof of Lemma 2.2 (*b*). This part is equivalent to the last (unnumbered) equation in the proof of Theorem 1.2 in [Kiselev and Yao 2023]. We sketch a proof below for completeness, and also to clarify the dependence of $c_2\big(s, \int_{\mathbb{T}\times[0,\pi]} \mu^{1/3}\,dx\big)$ in (2-12).

For any $k = (k_1, k_2) \in \mathbb{Z}^2$, the Fourier coefficient $\hat{\mu}(k_1, k_2)$ can be written as

$$\begin{aligned}\hat{\mu}(k_1,k_2) &= \frac{1}{(2\pi)^2}\int_{\mathbb{T}} e^{-ik_1x_1}\int_{\mathbb{T}} e^{-ik_2x_2}\mu(x_1,x_2)\,dx_2\,dx_1\\ &= \frac{1}{(2\pi)^2}\int_{\mathbb{T}} e^{-ik_1x_1}(-2i)\underbrace{\int_0^{\pi}\sin(k_2x_2)\mu(x_1,x_2)\,dx_2}_{=:g(x_1,k_2)}\,dx_1,\end{aligned} \tag{A-2}$$

where the last identity is due to μ being odd in x_2. With $g(x_1, k_2)$ defined in the last line of (A-2), when setting $k_2 = 1$, we claim that $g(x_1, 1)$ satisfies the following properties:

(a) $g(x_1, 1)$ is even in x_1 and nonnegative for all $x_1 \in \mathbb{T}$.

(b) $g(0, 1) = 0$.

(c) $\int_{\mathbb{T}} g(x_1, 1)\,dx_1 \geq c\big(\int_D \mu(x)^{1/3}\,dx\big)^3$ for some universal constant $c > 0$.

Here property (a) follows from the facts that μ is even in x_1 and nonnegative on $D := [0, \pi]^2$. Property (b) follows from $\mu(0, \cdot\,) \equiv 0$. For property (c), note that

$$\int_{\mathbb{T}} g(x_1,1)\,dx_1 = 2\int_0^{\pi} g(x_1,1)\,dx_1 = 2\int_D \sin(x_2)\mu(x)\,dx.$$

Combining Hölder's inequality with the fact that $\sin(x_2)\mu(x) \geq 0$ in D, we have

$$\int_D \sin(x_2)\mu(x)\,dx \geq \bigg(\int_D \sin(x_2)^{-1/2}\,dx\bigg)^{-2}\bigg(\int_D \mu(x)^{1/3}\,dx\bigg)^3 \geq c_0\bigg(\int_D \mu(x)^{1/3}\,dx\bigg)^3$$

for some universal constant $c_0 > 0$. This proves property (c).

For any $k_1 \in \mathbb{Z}$, let $\hat{g}(k_1)$ be the Fourier coefficient of $g(\cdot\,, 1)$; that is,

$$\hat{g}(k_1) := \frac{1}{2\pi}\int_{\mathbb{T}} e^{-ik_1x_1} g(x_1,1)\,dx. \tag{A-3}$$

Denote by $\bar{g} := \frac{1}{2\pi}\int_{\mathbb{T}} g(x_1, 1)\,dx_1$ the average of $g(\cdot\,, 1)$. Applying the definition of $\hat{g}$ to (A-2) gives

$$\hat{\mu}(k_1,1) = \frac{-2i}{2\pi}\hat{g}(k_1) \quad \text{for any } k_1 \in \mathbb{Z}. \tag{A-4}$$

This allows us to bound $\delta := \|\partial_1\mu\|^2_{\dot{H}^{-1}(\mathbb{T}^2)}$ from below as

$$\begin{aligned}\delta &\geq (2\pi)^2\sum_{k_1\in\mathbb{Z}\setminus\{0\}}\frac{k_1^2}{k_1^2+1}|\hat{\mu}(k_1,1)|^2\\ &\geq 2\sum_{k_1\in\mathbb{Z}\setminus\{0\}}|\hat{g}(k_1)|^2 = \frac{1}{\pi}\int_{\mathbb{T}}|g(x_1,1)-\bar{g}|^2\,dx_1.\end{aligned} \tag{A-5}$$

By property (c), $\bar{g} \ge c\big(\int_D \mu(x)^{1/3}\,dx\big)^3 > 0$. Applying [Kiselev and Yao 2023, Lemma 3.3] to $g(x_1, 1)$ yields

$$\|g(\cdot, 1)\|_{\dot{H}^s(\mathbb{T})} \ge c\left(s, \int_D \mu(x)^{1/3}\,dx\right)\delta^{-s+1/2} \quad \text{for all } s > \tfrac{1}{2}. \tag{A-6}$$

Note that

$$\|g(\cdot, 1)\|^2_{\dot{H}^s(\mathbb{T})} = 2\pi^3 \sum_{k_1 \ne 0} |k_1|^{2s} |\hat{\mu}(k_1, 1)|^2 \le \frac{\pi}{\sqrt{2}} \|\partial_1 \mu\|^2_{\dot{H}^{s-1}(\mathbb{T}^2)} \le \frac{\pi}{\sqrt{2}} \|\mu\|^2_{\dot{H}^s(\mathbb{T}^2)}, \tag{A-7}$$

where the first inequality follows by the assumption $s > \frac{1}{2}$. Finally, combining inequalities (A-6) and (A-7) gives (2-12). □

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