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FEDERICO FRANCESCHINI AND WIKTORIA ZATOŃ

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We show that the singular set Σ in the classical obstacle problem can be locally covered by a C^∞ hypersurface, up to an "exceptional" set E, which has Hausdorff dimension at most $n-2$ (countable in the $n=2$ case). Outside this exceptional set, the solution admits a polynomial expansion of arbitrarily large order. We also prove that $\Sigma \setminus E$ is extremely unstable with respect to monotone perturbations of the boundary datum. We apply this result to the planar Hele-Shaw flow, showing that the free boundary can have singular points for at most countable many times.

1. Introduction

1.1. *The classical obstacle problem.* The classical obstacle problem consists in studying the solutions of the variational problem

$$\min\left\{\frac{1}{2}\int_{B_1} |\nabla w|^2 : w \geq \varphi \text{ in } B_1 \subseteq \mathbb{R}^n,\ w = g \text{ on } \partial B_1\right\},$$

where $g : \partial B_1 \to \mathbb{R}$ and $\varphi : B_1 \to \mathbb{R}$ are given, with $\varphi < g$ on ∂B_1. In two dimensions an intuitive interpretation of this problem is the following: The graph of the minimizer w represents the shape of a thin membrane stretched over $\bar{B}_1$ and fixed on ∂B_1 along the profile g. The hypograph of φ represents a solid "obstacle" above which the membrane must lie, possibly touching it. The Dirichlet energy, finally, corresponds to the linearization of the surface energy of the membrane, which is assumed proportional to the area of the graph of w.

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It is well known (see for example the exposition in [Fernández-Real and Ros-Oton 2022, Chapter 5]) that there exists a unique optimal shape v and that it enjoys $C^{1,1}_{\text{loc}}$ regularity, provided that φ is smooth enough. Furthermore, $u := v - \varphi$ solves the Euler–Lagrange equation

$$\Delta u = -\Delta\varphi \chi_{\{u>0\}} \quad \text{in } B_1.$$

One of the most challenging problems is to understand the a priori unknown interface $\partial\{u > 0\}$, called the "free boundary". Unless $\Delta\varphi < 0$, simple examples show that the free boundary can be any closed set; hence it is standard to assume that φ is superharmonic. Thus, we deal with solutions of

$$\begin{cases} \Delta u = f \chi_{\{u>0\}} & \text{in } B_1, \\ u \geq 0 & \text{in } B_1, \end{cases} \tag{1-1}$$

where $f \in C^\infty(B_1)$ is given and positive.

By classical works of Caffarelli [1977; 1998], the free boundary $\partial\{u > 0\}$ splits into a regular and a singular part:

$$\partial\{u > 0\} = \text{Reg}(u) \cup \Sigma(u).$$

Points in these sets can be characterized, for example, by density considerations:

$$\begin{aligned} x_\circ \in \text{Reg}(u) &\iff |\{u = 0\} \cap B_r(x_\circ)| = \tfrac{1}{2}|B_r| + o(r^n), \\ x_\circ \in \Sigma(u) &\iff |\{u = 0\} \cap B_r(x_\circ)| = o(r^n). \end{aligned}$$

Caffarelli showed that $\text{Reg}(u)$ is relatively open in the free boundary and, locally, is a C^1 hypersurface (smoothness and analyticity were proved later in [Kinderlehrer and Nirenberg 1977]). On the other hand, $\Sigma(u)$ can always be covered, locally, by a single C^1 hypersurface (see [Caffarelli 1998, Theorem 8]). Thus, when we speak about "regularity of $\Sigma(u)$" we are actually speaking about the regularity of the manifold covering it. In this paper we will improve the smoothness of this hypersurface. We remark that $\Sigma(u)$ can display a very wild structure, as long as it is contained inside a single hypersurface; see example (1-2).

1.2. *The singular set: important examples and known results.* The following simple example shows that $\Sigma(u)$ can be rather wild. Furthermore, it can have Hausdorff dimension equal to $n-1$, thus it can be as "large" as $\text{Reg}(u)$. Consider the function

$$u(x) := x_n^2 + h(x') \quad \text{for } (x', x_n) \in \mathbb{R}^{n-1} \times \mathbb{R}, \tag{1-2}$$

where $h \in C^\infty(\mathbb{R}^{n-1})$ is nonnegative. Possibly multiplying h by a small factor, u solves (1-1) (with some f depending on h) and

$$\Sigma(u) = \{x_n = 0\} \cap \{h = 0\}, \qquad \text{Reg}(u) = \varnothing.$$

Hence $\Sigma(u)$ can be any closed subset of $\{x_n = 0\}$. See the second point below for even wilder examples due to Schaeffer [1977], where the contact set has nonempty interior. In this paper we show that, locally, $\Sigma(u)$ is always contained in a C^∞ hypersurface (the hyperplane $\{x_n = 0\}$, in this example), except for an

$(n-2)$-dimensional piece (the empty set, in this example). Before turning to the statements, let us try to give an overview of what is known about $\Sigma(u)$.

- Concerning the pointwise structure, $\Sigma(u)$ consists precisely of those points $x_\circ \in \partial\{u > 0\}$ such that

$$r^{-2}u(x_\circ + rx) \to p_{2,x_\circ}(x) \quad \text{as } r \downarrow 0,$$

where $p_{2,x_\circ}$ is a convex and 2-homogeneous polynomial with $\Delta p_{2,x_\circ} = f(x_\circ)$. Thus, zooming in on $x_\circ$, one sees the contact set $\{u = 0\}$ clustering around the linear space $\{p_{2,x_\circ} = 0\}$. When $\dim\{p_{2,x_\circ} = 0\} = m$ for some integer $m \le n-1$, this suggests that $\Sigma(u)$ displays, qualitatively, an m-dimensional structure at $x_\circ$.

- Concerning the local structure, when $n = 2$ and f is real analytic, Sakai [1993] gave a complete characterization of the possible shapes of the free boundary around a singular point. In brief, $\Sigma(u)$ is locally either an analytic curve, or an isolated point, around which $\partial\{u > 0\}$ is the union of at most two analytic arcs. In particular, $\Sigma(u)$ has codimension 1 inside the free boundary. We remark that his approach relies on complex analysis techniques [Sakai 1991]. On the other hand, Schaeffer [1977] constructed examples of rather wild free boundaries in the case where f is "just" C^∞. He showed that

$$\operatorname{int}(\{u = 0\} \cap \{x_n = 0\}) \quad \text{and} \quad \{u = 0\} \cap \{x_n = 0\}$$

can be any nested couple of relatively open and closed subsets of $\{x_n = 0\}$ (the interior part is taken with respect to the relative topology). In particular, the contact set might form infinitely many cusps, which in turn produce an arbitrarily closed subset of $\{x_n = 0\}$ as the singular set. This shows that the sharpness of Sakai's results is due to analytic rigidity.

- Despite the counterexamples of Schaeffer, it is still possible to obtain refined statements about the shape of $\Sigma(u)$. Caffarelli [1977; 1998] showed that $\Sigma(u)$ is locally contained in a C^1 hypersurface. More precisely, if we partition $\Sigma(u)$ into the strata

$$\Sigma_m := \{x_\circ \in \Sigma(u) : \dim\ker A_{x_\circ} = m\} \quad \text{for } m = 0, \dots, n-1,$$

then each Σ_m is locally contained in an m-dimensional C^1 manifold. Building on Weiss [1999] and Monneau [2003], Figalli and Serra [2019] extended these results by showing that $\Sigma(u) \setminus E$ can be covered by $C^{1,1}$ manifolds, where the excluded set E has Hausdorff dimension at most $n-2$.

- By the implicit function theorem, this type of covering result is tightly linked with the fact that u admits a polynomial expansion around singular points. From this perspective, Caffarelli showed that, at each $x_\circ \in \Sigma(u)$,

$$u(x) = p_{2,x_\circ}(x - x_\circ) + \sigma(x - x_\circ)|x - x_\circ|^2,$$

with an abstract dimensional modulus of continuity σ. This was improved in [Colombo et al. 2018], showing that $\sigma(x - x_\circ) \le C\big|\log|x - x_\circ|\big|^{-\varepsilon_\circ}$ for some dimensional $C, \varepsilon_\circ > 0$. Figalli and Serra [2019] proved that $\sigma(x - x_\circ) \le C|x - x_\circ|^{\alpha_\circ}$ when $x_\circ \in \Sigma_{n-1}$, and also that $\sigma(x - x_\circ) \le C|x - x_\circ|$ provided $x_\circ \in \Sigma(u) \setminus E$, where $\dim_{\mathcal{H}} E \le n-2$. Similar results were recovered in [Savin and Yu 2023] with independent methods. This analysis was pushed further by Figalli, Ros-Oton and Serra [Figalli et al. 2020] in the framework of "generic" regularity. They showed that, for all $\varepsilon > 0$ small, there is a set $E \subseteq \Sigma(u)$ with Hausdorff dimension at most $n-2$ such that, if $x_\circ \in \Sigma(u) \setminus E$, then

$$u(x) = P_{4,x_\circ}(x) + O(|x - x_\circ|^{5-\varepsilon}) \tag{1-3}$$

for some polynomial $P_{4,x_\circ}$ with $\Delta P_{4,x_\circ} = 1$.

1.3. *Main results of this paper.* Concerning expansion (1-3), the approach in [Figalli et al. 2020] was blocked at order 5, and new ideas were needed to go further, as we explain in detail in Section 1.4.2 below. In fact, the natural question whether such an expansion could be pushed to some order $k \geq 5$ (at most points) was explicitly raised in [Figalli 2018, p. 22]. The main contribution of this paper is providing a positive answer to this question: we prove that u admits a C^∞ polynomial expansion at most points in Σ.

Theorem 1.1. *Let $n \geq 2$, $mu > 0$ and $f \in C^\infty(B_1)$ be given, with $f \geq \mu$. Let $u \in C^{1,1}_{\mathrm{loc}}(B_1)$ be a solution of the obstacle problem (1-1), and let Σ be its singular set. Then there exists a closed set $\Sigma^\infty \subseteq \Sigma$ such that*

(i) $\dim_{\mathcal{H}}(\Sigma \setminus \Sigma^\infty) \leq n-2$ *(countable if $n = 2$),*

(ii) *locally, Σ^∞ is contained in one $(n-1)$-dimensional C^∞ manifold.*

Moreover, at every point $x \in \Sigma^\infty$, the solution u has a polynomial expansion of arbitrarily large order. That is, for every $x \in \Sigma^\infty$ and $k \in \mathbb{N}$, there exists a unique polynomial $P_{k,x}$ with $\deg P_{k,x} \leq k$ such that the expansion

$$|u(x+h) - P_{k,x}(h)| \leq C|h|^{k+1} \quad \text{for all } |h| \leq \tfrac{1}{2}(1-|x|) \tag{1-4}$$

holds with a constant C depending only on n, k, μ, $\|f\|_{C^{k+1}}$, $1-|x|$. We further have that $\Delta P_{k+2,x} = f_{k,x}$, where $f_{k,x}$ is the k-th Taylor polynomial of f centered at x. Finally, the map $\Sigma^\infty \ni x \mapsto (P_{k,x})_{k\in\mathbb{N}}$ is smooth in the sense of Whitney.[1]

We remark that there are solutions in dimension 2 with $f \equiv 1$ for which $0 \in \Sigma(u)$, but where expansion (1-4) does not hold at 0 for $k \geq 3$ (e.g., see the cusp-type solutions in [Sakai 1993]). In this sense the dimensional bound in (i) is optimal. Furthermore, example (1-2) shows one cannot hope to show that Σ is a smooth manifold, at least when the right-hand sides f are not analytic — this motivates (ii).

Example (1-2) is also a model situation which our result describes effectively: in this case $\Sigma = \Sigma^\infty$.

In dimension 2, Theorem 1.1 can be also read as an extension of Sakai's result to nonanalytic right-hand sides f (see the proof of Corollary 1.4 for a detailed comparison of the two results).

Our analysis further shows that the set Σ^∞ is extremely unstable with respect to monotone perturbations of the boundary datum. Indeed, following [Figalli et al. 2020], we also prove:

Theorem 1.2. *Let $\{u^t\}_{t\in(-1,1)}$ be a family of solutions to (1-1), with f independent from t, which is "uniformly monotone" in the sense that, for every $t \in (-1,1)$ and any compact set $K \subseteq \partial B_1 \cap \{u^t > 0\}$, there exists $c = c(t, K) > 0$ such that*

$$\min_{x\in K}(u^{t+h}(x) - u^t(x)) \geq ch \quad \text{for all } -1 < t < t+h < 1. \tag{1-5}$$

[1] We denote by $\dim_{\mathcal{H}}$ the Hausdorff dimension, and we refer to Whitney's definition of smoothness on a closed set [Whitney 1934, Section 3].

With the notation of Theorem 1.1, define the singular sets

$$\mathbf{\Sigma} := \{(x_\circ, t_\circ) : x_\circ \in \Sigma(u^{t_\circ})\},$$
$$\mathbf{\Sigma}^\infty := \{(x_\circ, t_\circ) : x_\circ \in \Sigma^\infty(u^{t_\circ})\}.$$

Then, denoting the standard projections by $\pi_t : B_1 \times (-1, 1) \to (-1, 1)$ and $\pi_x : B_1 \times (-1, 1) \to B_1$, we have that $\mathbf{\Sigma}$ is a graph over $\pi_x(\mathbf{\Sigma})$, and

(i) $\dim_{\mathcal{H}}(\pi_t(\mathbf{\Sigma}^\infty)) = 0$ *in any dimension* $n \geq 2$,

(ii) $\dim_{\mathcal{H}}(\pi_t(\mathbf{\Sigma})) = 0$ *in dimension* $n = 2$,

(iii) $\dim_{\mathcal{H}}(\pi_x(\mathbf{\Sigma} \setminus \mathbf{\Sigma}^\infty)) \leq n - 2$ (*countable if* $n = 2$).

Remark 1.3. The Hausdorff dimension bound in (i) can actually be improved to zero Minkowski dimension (see [Mattila 1995, Chapter 5] for the definition).

Combining this result with Sakai's classification we also get an improvement of [Figalli et al. 2020, Theorem 1.2] concerning the generic regularity of the free boundary in the planar Hele-Shaw flow.

Corollary 1.4. *Let $O \subseteq \mathbb{R}^2$ be an open and bounded set with Lipschitz boundary, and let $\Omega := \mathbb{R}^2 \setminus \overline{O}$. For each $t > 0$, let u^t be a weak solution of*

$$\begin{cases} \Delta u^t = \chi_{\{u^t>0\}} & \text{in } \Omega, \\ u^t = t & \text{on } \partial\Omega, \\ u^t \geq 0 & \text{in } \Omega. \end{cases} \tag{1-6}$$

Then the set of $t \in (0, \infty)$ such that $\Sigma(u^t) \neq \varnothing$ is countable.

1.4. *On the proofs of the main results.* Let us now explain the main ideas in the proof of Theorem 1.1, the general outline of the argument being inspired by [Figalli et al. 2020]. As pointed out, the key feature is the Taylor expansion (1-4). Furthermore, we can work in the top-dimensional stratum Σ_{n-1}, as the lower strata Σ_m, $m \in \{1, \dots, n-2\}$, have Hausdorff dimension at most m thanks to Caffarelli's covering result.

We will perform a blow-up analysis on the functions $u - \mathcal{P}_k$, where the $\mathcal{P}_k$ are suitable polynomials. The core of this blow-up analysis is an Almgren-type monotonicity formula for $u - \mathcal{P}_k$.

1.4.1. *Polynomial ansatz.* Similarly to [Figalli et al. 2023], we construct $\mathcal{P}_k$, the prototypical k-th Taylor polynomial of u at $0 \in \Sigma_{n-1}$. These polynomials should be approximate solutions of (1-1), that is to say

$$\begin{cases} \Delta\mathcal{P}_k = f + O(|x|^{k-1}) & \text{in } B_1, \\ \mathcal{P}_k \geq -O(|x|^{k+2}) & \text{in } B_1. \end{cases}$$

Furthermore, they should have an $(n-1)$-dimensional zero set (we are in the top-dimensional stratum Σ_{n-1}). Together with nonnegativity, this suggests that $\mathcal{P}_k$ is almost a square:

$$\mathcal{P}_k = (\text{polynomial})^2 + O(|x|^{k+2}).$$

The coefficients of $\mathcal{P}_k$ can be chosen with some freedom, which can be used to modulate the shape of the hypersurface $\{\mathcal{P}_k = 0\}$ around 0. Notice that, by Caffarelli's analysis, $\mathcal{P}_0 = \mathcal{P}_1 = 0$ and (in suitable coordinates) $\mathcal{P}_2 = \frac{1}{2}x_n^2$. For example, we will see that $\mathcal{P}_3$ needs to have a more complicated form:

$$\mathcal{P}_3 = \left(x_n + \frac{p_3}{x_n} + x_n R_2\right)^2 + O(|x|^{k+2}),$$

where p_3 is any 3-homogeneous harmonic polynomial odd in x_n and $R_2 = R_2[p_3]$ is a 2-homogeneous polynomial determined uniquely by p_3.

1.4.2. *Almgren monotonicity formula.* In [Figalli and Serra 2019], it was first noticed that the Almgren frequency function of $u - p_2$, that is

$$r \mapsto \frac{\|r\nabla(u-p_2)(r\,\cdot)\|_{L^2(B_1)}}{\|(u-p_2)(r\,\cdot)\|_{L^2(\partial B_1)}} =: \phi(r, u-p_2), \tag{1-7}$$

is increasing. This property is known to be very powerful and paved the way for most of the results of [Figalli and Serra 2019]. Similarly, the key observation that allows us to prove expansion (1-4) is that, for all $k \geq 2$, (a version of) the Almgren frequency function is (almost) monotone on functions of the form $u - \mathcal{P}_k$. Proving this crucial fact for all k is one of the main contributions of this paper. The case $k = 3$ was already obtained in [Figalli et al. 2020],[2] but their approach was blocked and the general case cannot be obtained by tuning their argument; let us explain why.

By known computations, for every function w,

$$\frac{d}{dr}\phi(r, w) \geq \frac{2}{r}\frac{\int_{B_1}(\phi(r,w)w_r - x\cdot\nabla w_r)\Delta w_r}{\|w_r\|^2_{L^2(\partial B_1)}}. \tag{1-8}$$

In [Figalli and Serra 2019], it was shown that the right-hand side is nonnegative if $w = u - \mathcal{P}_2$, but in order to carry out our arguments it is enough to prove that the negative part of the right-hand side is integrable around $r = 0$ when $w = u - \mathcal{P}_k$. In trying to do so, the term $w_r \Delta w_r$ has (up to some errors) the right sign, while the main difficulties come from the term $(x\cdot\nabla w_r)\Delta w_r$.

In order to control this last term in the case $k = 3$, the following crucial Lipschitz estimate was proved in [Figalli et al. 2020, Lemma 4.7]:

$$r\|\nabla(u-\mathcal{P}_3)\|_{L^\infty(B_r)} \leq C(\|(u-\mathcal{P}_3)(r\,\cdot)\|_{L^2(B_2)} + r^5) \tag{1-9}$$

for some constant C independent of u, $\mathcal{P}_3$, and $r \in \left(0, \frac{1}{4}\right)$. In order to prove (1-9), one needs to take incremental quotients of $u - \mathcal{P}_3$ along the flow of some circular vector fields $\{X_j\}$, which have the property that $X_j X_j \mathcal{P}_3 = O(|x|^3)$ (see [Figalli et al. 2020, Lemma 4.6]). Now, exploiting that the Laplacian is invariant under rotations (that such vector fields generate), one can prove (1-9). There is little hope to make this ingenious argument work for a general $u - \mathcal{P}_k$, since the Laplacian does not commute with more general diffeomorphisms, which, on the other hand, would be needed to ensure $X_j X_j \mathcal{P}_k = O(|x|^k)$.

[2] Actually, as a consequence of the method of proof, the authors also immediately obtain the same monotonicity for $u - \mathcal{P}_3 - P$ for some homogeneous harmonic polynomial of order 4. This then leads the expansion up to order $5-\varepsilon$, but to continue on what is missing is Almgren's frequency monotonicity for $u - \mathcal{P}_4$, where $\mathcal{P}_4 \neq \mathcal{P}_3 + P$.

With a completely different proof, in Section 3.2, we will prove the following weaker Lipschitz estimate, which — nontrivially — is still enough to get the integrability of the right-hand side of (1-8). For all $k \geq 2$, we have

$$r\|\nabla(u-\mathcal{P}_k)\|_{L^\infty(B_r)} \leq C(\|(u-\mathcal{P}_k)(r\,\cdot\,)\|_{L^2(B_\theta)} + r^{k+2})^{1-\beta}, \tag{1-10}$$

where $\beta > 0$ can be chosen arbitrarily small, $C, \theta > 1$ are constants independent of u, and $r \in (0, 1/\theta)$. This allows us to prove the monotonicity of a suitable modification of Almgren's frequency function introduced in [Figalli et al. 2020]. More precisely, the function

$$r \mapsto \phi^\gamma(r, u-\mathcal{P}_k) + Cr^\varepsilon, \quad \text{where } \phi^\gamma(r, w) := \frac{\|\nabla w_r\|^2_{L^2(B_1)} + \gamma r^{2\gamma}}{\|w_r\|^2_{L^2(\partial B_1)} + r^{2\gamma}}, \tag{1-11}$$

is increasing. Here $w_r := w(r\,\cdot\,)$, C, ε are positive constants and $\gamma > k+1$ is the truncation parameter.

1.4.3. *Blow-up analysis.* The monotonicity formula allows us to pursue a blow-up analysis to every order. Similarly to [Figalli and Serra 2019], we classify the possible blowups, that is, we study the possible limits of the normalized sequence

$$\tilde{v}_{r_\ell} := \frac{v_{r_\ell}}{\|v_\ell\|^2_{L^2(\partial B_1)}} \quad \text{as } r_\ell \downarrow 0, \quad \text{where } v_r := (u-\mathcal{P}_k)(r\,\cdot\,).$$

We show that $\tilde{v}_{r_\ell} \to q$, where q is a nontrivial global solution of a certain PDE: the Signorini problem. Furthermore, q is homogeneous of degree $\phi^\gamma(0^+, v)$. The blowup can be performed at each point of Σ_{n-1} of course, but q could be a nonpolynomial function or even have nonintegral homogeneity. At the points where this happens, there cannot be a Taylor polynomial of order $k+1$, so the expansion (1-4) must stop. These points must be shown to be "rare".

1.4.4. *Dimension reduction.* We show that $\phi^\gamma(0^+, u-\mathcal{P}_k) = k+1$ outside of a set of dimension at most $n-2$ for a suitable choice of $\mathcal{P}_k$, and the blowup q is a harmonic polynomial vanishing on $\{p_2 = 0\}$ (a particular class of solutions of the Signorini problem). This allows us to determine the next ansatz $\mathcal{P}_{k+1}$ in terms of $\mathcal{P}_k$ and q and prove the Taylor expansion up to order $k+1$ with remainder $o(r^{k+1})$. The various dimension reduction techniques are inspired by [Figalli and Serra 2019], but new barrier arguments are introduced to deal with the points with even frequency.

1.5. ***Structure of the paper.*** In Section 2 we fix the notation and recall some basic results on the obstacle problem and the Signorini problem. In Section 3.1 we give the construction of the polynomial ansatz $\mathcal{P}_k$. In Section 3.2 we prove our Lipschitz estimate (1-10). In Section 4 we prove the almost-monotonicity of the truncated frequency $\phi^\gamma(\,\cdot\,, u-\mathcal{P}_k)$. In Section 5 we perform and classify the blowups. In Section 6 we perform the dimension reductions distinguishing various cases; the proof of Theorem 1.1 is given in Section 6.4. In Section 7 we give the proof of the instability result Theorem 1.2.

In Appendix A we reprove a result [Figalli and Serra 2019, Remark 2.14] for a general right-hand side. In Appendix B we explain, line by line, which modifications are needed for a smooth right-hand side in the previous proofs. Finally, Appendix C contains two technical lemmas.

2. Preliminaries

2.1. ***Notation.*** We work in $\mathbb{R}^n$ endowed with its Euclidean structure and assume $n \geq 2$. We will often perform blowups: given a function $v : B_1 \to \mathbb{R}$, we set $v_r := v(r\,\cdot)$ which is defined in $B_{1/r}$; the parameter $r > 0$ is thought to be small. We remark that $\nabla v_r = r(\nabla v)_r$. We will sometimes write $X \lesssim_{a,b} Y$, meaning that $X \leq CY$ for some constant $C > 0$ which depends only on a and b.

2.2. ***Known results.*** Fix $\mu > 0$ and a function $f \in C^\infty(B_1)$ such that $f \geq \mu$ in B_1. We will denote by u any solution of

$$\begin{cases} \Delta u = f\chi_{\{u>0\}} & \text{in } B_1, \\ u \geq 0 & \text{in } B_1, \\ 0 \in \partial\{u > 0\}. \end{cases} \tag{2-1}$$

The last condition is added for normalization purposes, as we want to stay away from ∂B_1. We recall some basic properties of the solution u, relying on the classical theory by Caffarelli [1977; 1998], see also [Figalli et al. 2020, Section 3] for a summary of the known results. There exists

$$C = C(n, \mu, \|f\|_{L^\infty(B_1)}) > 0$$

such that

$$\|u\|_{C^{1,1}(B_{1/2})} \leq C \quad \text{and} \quad \sup_{B_{1/2}} u \geq \frac{1}{C}. \tag{2-2}$$

Thus we will assume throughout the paper that $u \in C^{1,1}_{\text{loc}}(B_1)$. We remark that the problem has a natural scaling; in fact, for any $r > 0$, we have that $r^{-2}u_r$ solves

$$\begin{cases} \Delta(r^{-2}u_r) = f_r\chi_{\{u_r>0\}} & \text{in } B_{1/r}, \\ u_r \geq 0 & \text{in } B_{1/r}, \\ 0 \in \partial\{u_r > 0\}. \end{cases}$$

The free boundary $\partial\{u > 0\}$ consists of regular points (in the neighborhood of which $\partial\{u > 0\}$ is an analytic hypersurface) and singular points $\Sigma \subseteq \partial\{u > 0\}$ (at which the volume density of $\{u = 0\}$ is 0). It is well known that the singular points are characterized by the condition that the blowup

$$p_{2,x_\circ}(x) := \lim_{r\to 0} r^{-2}u(x_\circ + rx)$$

exists and is a convex 2-homogeneous polynomial with $\Delta p_{2,x_\circ} \equiv f(x_\circ)$. When $x_\circ = 0$, we denote the blowup simply by p_2. The singular set Σ stratifies according to $\dim\{p_{2,x_\circ} = 0\}$. The strata

$$\Sigma_m := \{x_\circ \in \Sigma : \dim\{p_{2,x_\circ} = 0\} = m\} \quad \text{for } m = 0, \ldots, n-1$$

are locally contained in m-dimensional C^1 manifolds. As we want to prove a statement "up to sets of codimension 2", we will be mostly interested in the top-dimensional stratum Σ_{n-1}.

The following lemma is crucial for our analysis. It shows that, in Σ_{n-1}, the rate of convergence of u to its blowup is more than quadratic. It was proved in [Figalli and Serra 2019, Remark 2.14] for $f \equiv 1$; for completeness we give the proof for a general $f \in C^\delta(B_1)$ in Appendix A.

Lemma 2.1. *Assume that* $0 \in \Sigma_{n-1}$ *and* $r^{-2}u_r \to p_2$. *Then there are* $C, \alpha_\circ > 0$ *such that*

$$\sup_{B_r} |u - p_2| \le C r^{2+2\alpha_\circ} \quad \textit{for all } r \in \left(0, \tfrac{1}{2}\right). \tag{2-3}$$

In particular, we have

$$\{u_r = 0\} \cap B_1 \subseteq \{x : \operatorname{dist}(x, \{p_2 = 0\}) \le C r^{\alpha_\circ}\} \quad \textit{for all } r \in (0, 1). \tag{2-4}$$

The constants C *and* $\alpha_\circ$ *depend only on* $n,\ \mu,\ \delta,\ \|f\|_{C^\delta(B_1)}$, *where* $0 < \delta \le 1$ *can be freely chosen.*

Notice that (2-4) immediately follows from (2-3) because $\operatorname{dist}(\,\cdot\,, \{p_2 = 0\})^2 = p_2$ since $0 \in \Sigma_{n-1}$.

2.3. *Truncated frequency function.* We will make extensive use of the following functionals. For $w \in C^{1,1}_{\mathrm{loc}}(B_1)$, $r \in (0, 1)$ and a parameter $\gamma \ge 0$, let us define the nondimensional quantities

$$D(r, w) := r^{2-n} \int_{B_r} |\nabla w|^2 = \int_{B_1} |\nabla w_r|^2, \quad H(r, w) := r^{1-n} \int_{\partial B_r} w^2 = \int_{\partial B_1} w_r^2 \tag{2-5}$$

and the truncated frequency function

$$\phi^\gamma(r, w) := \frac{D(r, w) + \gamma r^{2\gamma}}{H(r, w) + r^{2\gamma}}, \tag{2-6}$$

which has been introduced in [Figalli et al. 2020]. By [Figalli et al. 2020, Lemma 2.3], the following formula is valid for all $w \in C^{1,1}_{\mathrm{loc}}(B_1)$ and $r \in (0, 1)$:

$$\frac{d}{dr}\phi^\gamma(r, w) \ge \frac{2}{r} \frac{\left(r^{2-n} \int_{B_r} w \Delta w\right)^2 + E^\gamma(r, w)}{(H(r, w) + r^{2\gamma})^2},$$

where

$$E^\gamma(r, w) := \left(r^{2-n} \int_{B_r} w \Delta w\right)(D(r, w) + \gamma r^{2\gamma}) - \left(r^{2-n} \int_{B_r} (x \cdot \nabla w) \Delta w\right)(H(r, w) + r^{2\gamma}).$$

Thus we have

$$\frac{d}{dr}\phi^\gamma(r, w) \ge \frac{2}{r} \frac{\int_{B_1} (\phi^\gamma(r, w) w_r - x \cdot \nabla w_r) \Delta w_r}{H(r, w) + r^{2\gamma}}. \tag{2-7}$$

We recall from [Figalli et al. 2020] the following result which says, roughly speaking, that the value of $\phi^\gamma(\,\cdot\,, v)$ corresponds to the power at which $H(\,\cdot\,, v)$ grows. This lemma will be used extensively to pass from L^2 norms over spheres to L^2 norms over thick shells.

Lemma 2.2 [Figalli et al. 2020, Lemma 4.1, Remark 4.2]. *Let* $R \in (0, 1)$, *and let* $w : B_R \to [0, \infty)$ *be a* $C^{1,1}$ *function. Assume that, for some* $\varepsilon \in (0, 1)$ *and a constant* $C_\circ > 0$, *we have*

$$\frac{d}{dr}(\phi^\gamma(r, w) + C_\circ r^\varepsilon) \ge \frac{2}{r} \frac{\left(r^{2-n} \int_{B_r} w \Delta w\right)^2}{(H(r, w) + r^{2\gamma})^2} \quad \textit{for all } r \in (0, R).$$

Then the following hold:

(a) *Suppose that* $0 < \underline{\lambda} \le \phi^\gamma(r, w) \le \bar{\lambda}$ *for all* $r \in (0, R)$. *Then, for any given* $\delta > 0$, *we have*

$$\frac{1}{C_\delta}\left(\frac{R}{r}\right)^{2\underline{\lambda}-\delta} \le \frac{H(R, w) + R^{2\gamma}}{H(r, w) + r^{2\gamma}} \le C_\delta\left(\frac{R}{r}\right)^{2\bar{\lambda}+\delta} \quad \textit{for all } r \in (0, R),$$

where C_δ *depends on* $n,\ \gamma,\ \varepsilon,\ \bar{\lambda},\ C_\circ,\ \delta$.

(b) *Assume in addition that*

$$\frac{r^{2-n}\int_{B_r} w\Delta w}{H(r, w) + r^{2\gamma}} \ge -C_\circ r^\varepsilon \quad \textit{for all } r \in (0, R).$$

Then, for $\lambda_* := \phi^\gamma(0^+, w)$, *we have* $\lambda_* \le \gamma$ *and*

$$e^{-C_\circ/\varepsilon^2}\left(\frac{R}{r}\right)^{2\lambda_*} \le \frac{H(R, w) + R^{2\gamma}}{H(r, w) + r^{2\gamma}}.$$

2.4. *The Signorini problem.* The Signorini problem, called also the thin obstacle problem, consists of the following system of PDEs

$$\begin{cases} \Delta q \le 0 \text{ and } q\Delta q = 0 & \text{in } \mathbb{R}^n, \\ \Delta q = 0 & \text{in } \mathbb{R}^n \setminus L, \\ q \ge 0 & \text{on } L, \end{cases} \tag{2-8}$$

where $L \subseteq \mathbb{R}^n$ is a hyperplane and q is at least continuous. Recall that the following regularity is known (see [Athanasopoulos and Caffarelli 2004]) for weak solutions: if $L = \{x_n = 0\}$ then $q \in C^{1,1/2}_{\text{loc}}(\{x_n \ge 0\})$.

For each solution q we will consider its *singular set*,[3] defined by

$$\Sigma(q) := \{x \in L : q = |\nabla q| = 0\}. \tag{2-9}$$

We will be interested in homogeneous solutions, so, for every $\lambda \ge 0$ and every hyperplane $L \subseteq \mathbb{R}^n$, let us define

$$\mathcal{S}_\lambda(L) := \{q \in W^{1,2}_{\text{loc}} \cap C^0_{\text{loc}}(\mathbb{R}^n) : q \text{ is } \lambda\text{-homogeneous and solves (2-8)}\}. \tag{2-10}$$

We will use the following characterization of homogeneous solutions.

Lemma 2.3. *Let* $q \in W^{1,2}_{\text{loc}} \cap C^0_{\text{loc}}(\mathbb{R}^n)$ *be a weak solution of* (2-8), *and let* $\lambda \ge 0$. *Then* q *is* λ*-homogeneous if and only if*

$$\frac{D(r, q)}{H(r, q)} = \lambda \quad \textit{for all } r > 0.$$

Proof. Setting to zero the derivative with respect to r of the left-hand side, one formally gets

$$q(x) = \lambda x \cdot \nabla q(x).$$

One can make the computation rigorous using the $C^{1,\alpha}$ regularity of q; see [Fernández-Real 2022]. □

[3]We warn the reader that our singular set has nothing to do with the "singular points" of the free boundary of q. Our terminology is instead consistent with [Naber and Valtorta 2017].

Every $q \in \mathcal{S}(\nu^\perp)$, with ν a unit vector, can be split into its even and odd part with respect to L, namely

$$q^{\text{even}}(x) := \tfrac{1}{2}(q(x) + q(x - 2(x \cdot \nu)\nu)), \quad q^{\text{odd}}(x) := \tfrac{1}{2}(q(x) - q(x - 2(x \cdot \nu)\nu)), \tag{2-11}$$

so that $q = q^{\text{even}} + q^{\text{odd}}$. It is easy to show that q^{even} and q^{odd} solve (2-8) separately, thus it is natural to define

$$\mathcal{S}_\lambda^{\text{even}}(L) := \{q \in \mathcal{S}_\lambda(L) : q \text{ is even with respect to } L\},$$
$$\mathcal{S}_\lambda^{\text{odd}}(L) := \{q \in \mathcal{S}_\lambda(L) : q \text{ is odd with respect to } L\}.$$

When it is not relevant, we will drop the dependence on L. We gather information on these sets next.

Proposition 2.4. *For every $m \in \mathbb{N}$, the following hold*:

(i) *Every element of $\mathcal{S}_{2m+1}(L)$ vanishes on the obstacle L.*

(ii) *$\mathcal{S}_\lambda^{\text{odd}}(L)$ consists exactly of those λ-homogeneous harmonic polynomials that vanish on L, thus it's empty when $\lambda \notin \mathbb{N}$.*

(iii) *$\mathcal{S}_{2m}^{\text{even}}(L)$ consists exactly of those $2m$-homogeneous harmonic polynomials that are nonnegative on L.*

(iv) *If $q \in \mathcal{S}_{2m+1}^{\text{even}}(e_n^\perp)$, then $q(x) = -|x_n|(q_0(x') + x_n^2 q_1(x))$, where q_0 and q_1 are polynomials such that $q_0 \geq 0$ and $\Delta(-x_n q_0(x') + x_n^3 q_1(x)) = 0$.*

(v) *The* (*real*) *values of λ for which $\mathcal{S}_\lambda^{\text{even}}(L)$ is not empty are known only in dimension $n = 2$, in which case we have $\lambda \in \mathbb{N} \cup \{2m + \frac{3}{2} : m \in \mathbb{N}\}$.*

Proof. For (i) see [Figalli et al. 2020, Lemma 5.1]. To show (ii), notice that q is harmonic in a half-space and coincides with its odd reflection, thus it is harmonic everywhere. The third point is proven in [Garofalo and Petrosyan 2009, Lemma 1.3.4]. For (iv) see [Figalli et al. 2020, Appendix B]. The last point follows by separating variables and explicitly solving the resulting ODE. □

Remark 2.5. Using Proposition 2.4, it is easy to check that if $n = 2$ and $\lambda \in \mathbb{N}$, $\lambda \geq 2$, then $\Sigma(q^{\text{even}}) = \{0\}$.

3. Lipschitz estimates

For the sake of readability, we deal first with the case $f \equiv 1$ and $\mu = 1$. A list of notational changes needed to address a general f is given in Appendix B. This section is devoted to the derivation of the Lipschitz estimate (1-10), which contains the most original part of this work.

3.1. *Polynomial ansatz.* We denote by V_j the vector space of homogeneous polynomials of degree $j \geq 1$. We introduce the projection map $\pi_j : \mathbb{R}[x] \to V_j$, which sends a polynomial to its j-homogeneous part, and the map $\pi_{\leq j}$, which truncates it at degree j. We define for $k \geq 2$ the set $\boldsymbol{P}_k \subseteq V_2 \times \cdots \times V_k$ by saying that $(p_2, \ldots, p_k) \in \boldsymbol{P}_k$ if and only if

(i) p_j is a j-homogeneous polynomial for each $2 \leq j \leq k$,

(ii) $p_2 \geq 0$, $\Delta p_2 = 1$ and $\dim\{p_2 = 0\} = n - 1$,

(iii) $\Delta p_j = 0$ and p_j vanish on $\{p_2 = 0\}$ for each $3 \leq j \leq k$.

Notice that any p_2 satisfying (ii) must be of the form $p_2(x) = \frac{1}{2}(\nu \cdot x)^2$ for some unit vector ν; of course ν is not unique as we can always choose $-\nu$, but that is the only freedom we have. Furthermore, for every $(p_2, \dots, p_k) \in V_2 \times \cdots \times V_k$, we set

$$|(p_2, \dots, p_k)| := \sum_{2 \le j \le k} \|p_j\|_{L^2(\partial B_1)}.$$

Lemma 3.1. *Let $k \ge 2$ and $(p_2, \dots, p_k) \in \boldsymbol{P}_k$ be given. Then there exists a unique collection of polynomials*

$$(R_1, \dots, R_{k-1}) \in V_1 \times \cdots \times V_{k-1}$$

such that the following holds. If $p_2(x) = \frac{1}{2}(\nu \cdot x)^2$ for some unit vector ν and we define

$$\mathcal{A}_{k,\nu}(x) := (\nu \cdot x) + \sum_{j=1}^{k-1} (\nu \cdot x) R_j(x) + \sum_{j=3}^{k} \frac{p_j(x)}{(\nu \cdot x)},$$

then $\Delta\big(\frac{1}{2}\mathcal{A}_{k,\nu}^2\big) = 1 + O(|x|^k)$. Furthermore, each R_j is determined only by $(p_2, \dots, p_{j+1})$ and does not depend on which ν we choose, so that $\mathcal{A}_{k,-\nu} = -\mathcal{A}_{k,\nu}$. Finally, $\frac{1}{2}\mathcal{A}_{2,\nu}^2 = p_2$.

Proof. We prove the full statement by induction on k, beginning with $k = 2$. We compute

$$\Delta\big(\tfrac{1}{2}\mathcal{A}_{2,\nu}^2\big) = 1 + \Delta(2p_2R_1) + O(|x|^2),$$

thus R_1 must solve $\Delta(p_2R_1) = 0$; this is true if and only if $R_1 = 0$, as we can see with the following general argument. For $m \ge 1$, we consider the linear map $\delta_m : V_m \to V_m$ given by $\delta_m(q) := \Delta(p_2q)$. We claim that δ_m is a isomorphism. Indeed, if $\delta_m(q) = 0$, then the polynomial p_2q is a harmonic function that vanishes on the hyperplane $\{p_2 = 0\}$ along with its normal derivative, thus $p_2q \equiv 0$ (by reflection, p_2q is both even and odd with respect to $\{p_2 = 0\}$, thus $q \equiv 0$). As V_m has finite dimension, the map δ_m is invertible. In particular, $R_1 = \delta_1^{-1}(0) = 0$, regardless of p_2 and ν.

Assume that the full statement is proved up to some $k \ge 2$. Fix $(p_2, \dots, p_{k+1}) \in \boldsymbol{P}_{k+1}$ and ν, and for simplicity set $x_\nu := (\nu \cdot x)$. Notice that, for every $(R_1, \dots, R_k)$, we have $\mathcal{A}_{k+1,\nu} = \mathcal{A}_{k,\nu} + x_\nu R_k + p_{k+1}/x_\nu$. A direct computation again gives

$$\Delta\big(\tfrac{1}{2}\mathcal{A}_{k+1,\nu}^2\big) = \pi_{\le k-1}\big(\Delta\big(\tfrac{1}{2}\mathcal{A}_{k,\nu}^2\big)\big) + \pi_k\big(\Delta\big(\tfrac{1}{2}\mathcal{A}_{k,\nu}^2\big)\big) + \Delta p_{k+1} + \Delta\frac{p_3p_{k+1}}{2p_2} + \Delta(2p_2R_k) + O(|x|^{k+1}).$$

Hence we have $\Delta\frac{1}{2}\mathcal{A}_{k+1,\nu}^2 = 1 + O(|x|^{k+1})$ if and only if

$$\begin{cases} \Delta\frac{1}{2}\mathcal{A}_{k,\nu}^2 = 1 + O(|x|^k), \\ \pi_k\big(\Delta\frac{1}{2}\mathcal{A}_{k,\nu}^2\big) + \Delta p_3p_{k+1}/(2p_2) + \Delta 2p_2R_k = 0, \end{cases}$$

by inductive assumption the first equation determines uniquely $(R_1, \dots, R_{k-1})$ and thus $\mathcal{A}_{k,\nu}$. The second equation then determines uniquely R_k, indeed, as in the base step, we set

$$R_k := -\tfrac{1}{2}\delta_k^{-1}\bigg(\pi_k\Delta\big(\tfrac{1}{2}\mathcal{A}_{k,\nu}^2\big) + \Delta\frac{p_3p_{k+1}}{2p_2}\bigg).$$

Finally, by inductive assumption $\frac{1}{2}\mathcal{A}_{k,\nu}^2 = \frac{1}{2}\mathcal{A}_{k,-\nu}^2$, thus it is manifest that R_k does not depend on the choice of ν. ☐

This lemma shows that we can construct a function $\frac{1}{2}\mathcal{A}_k^2 : \boldsymbol{P}_k \to \mathbb{R}[x]$ defined by

$$\mathcal{A}_k^2 : (p_2, \dots, p_k) \mapsto \mathcal{A}_{k,\nu}^2, \quad \text{where } p_2(x) = (\nu \cdot x)^2. \tag{3-1}$$

Definition 3.2. Given $k \geq 2$, we define $\mathcal{P}_k : \boldsymbol{P}_k \to \mathbb{R}[x]$ by

$$\mathcal{P}_k(p_2, \dots, p_k) := \pi_{\leq k+1}\big(\tfrac{1}{2}\mathcal{A}_k^2\big), \tag{3-2}$$

where $\mathcal{A}_k^2 = \mathcal{A}_{k,\pm\nu}^2$ are constructed from $(p_2, \dots, p_k)$ as in Lemma 3.1.

We now give some simple properties of $\frac{1}{2}\mathcal{A}_k^2$ and $\mathcal{P}_k$; given a unit vector e, we write $\partial_e u = e \cdot \nabla u$.

Proposition 3.3. *Let $k \geq 2$, $(p_2, \dots, p_k) \in \boldsymbol{P}_k$ and $\tau > 0$ be such that $|(p_2, \dots, p_k)| \leq \tau$. Choose some unit vector ν for which $p_2(x) = \frac{1}{2}(\nu \cdot x)^2$. Then the polynomials $\frac{1}{2}\mathcal{A}_k^2(p_2, \dots, p_k)$ and $\mathcal{P}_k(p_2, \dots, p_k)$ satisfy*:

(i) *$\Delta \mathcal{P}_k = 1$ and $\partial_e\big(\frac{1}{2}\mathcal{A}_k^2\big) = \partial_e \mathcal{P}_k + O(|x|^{k+1})$, for any unit vector e.*

(ii) *We have $\mathcal{P}_k(p_2, \dots, p_k) = \mathcal{P}_{k-1}(p_2, \dots, p_{k-1}) + p_k + O(|x|^{k+1})$.*

(iii) *For all $|x| \leq r_0$, we have $\frac{1}{2} \leq |\partial_\nu \mathcal{A}_{k,\nu}(x)| \leq 2$, and thus*

$$\tfrac{1}{2}|\mathcal{A}_k(x)| \leq \big|\partial_\nu\big(\tfrac{1}{2}\mathcal{A}_k^2(x)\big)\big| \leq 2|\mathcal{A}_k(x)|,$$

where $r_0 = r_0(n, k, \tau) \in (0, 1)$.

(iv) *If u is a solution as in (2-1), $0 \in \Sigma_{n-1}$ and $r^{-2}u(r\,\cdot) \to p_2$, then by (2-3) we have, for all $0 < r < \frac{1}{2}$,*

$$\sup_{B_r \cap \{u=0\}} |\partial_\nu \mathcal{P}_k| \leq C r^{1+\alpha_\circ}$$

for some constant $C = C(n, k, \tau) > 0$.

Proof. Point (i) follows immediately from $\Delta\big(\frac{1}{2}\mathcal{A}_k^2\big) = 1 + O(|x|^k)$ and the fact that $\mathcal{P}_k - \frac{1}{2}\mathcal{A}_k^2 = O(|x|^{k+2})$. The second point can be easily checked by direct computation using the structure of the polynomial $\mathcal{A}_{k,\nu}$. For (iii), we compute $\partial_\nu \mathcal{A}_{k,\nu} = \partial_\nu(\nu \cdot x + O(|x|^2)) = 1 + O(|x|)$. Lastly, for the fourth point, note that by construction $\mathcal{P}_k = p_2 + O(|x|^3)$ and apply Lemma 2.1. □

3.2. ***Regularity estimates near the free boundary.*** We now turn to the Lipschitz estimate on functions of the form $u - \mathcal{P}_k$.

Proposition 3.4. *Let u be a solution of the obstacle problem (2-1) with $f \equiv 1$, $\mu = 1$, and suppose $0 \in \Sigma_{n-1}$. Let $k \geq 2$ be an integer, $\tau > 0$ and $\beta \in (0, \alpha_\circ^2/(k+2))$. Let $(p_2, \dots, p_k) \in \boldsymbol{P}_k$ be such that $|(p_2, \dots, p_k)| \leq \tau$. Suppose that*

$$r^{-2}u(r\,\cdot) \to p_2 = \tfrac{1}{2}x_n^2,$$

and set

$$v := u - \mathcal{P}_k,$$

where $\mathcal{P}_k = \mathcal{P}_k(p_2, \dots, p_k)$ is the polynomial ansatz from Definition 3.2. Then:

(i) *there are* $C > 0$ *and* $r_0 \in (0, 1)$, *depending on* n, k, τ, *such that, for each* $j = 1, \dots, n-1$ *and* $0 < r < r_0$, *we have*

$$\|\partial_j v_r\|_{L^\infty(B_1)} \le C(\|v_{\theta r}\|_{L^2(B_2 \setminus B_{1/2})} + r^{k+2}), \tag{3-3}$$

where $\theta = \theta(n, k) > 1$,

(ii) *there are* $C > 0$ *and* $r_0 \in (0, 1)$, *depending on* n, k, τ, β, *such that, for every* $0 < r < r_0$, *we have*

$$\|\partial_n v_r\|_{L^\infty(B_1)} \le C(\|v_{\theta r}\|_{L^2(B_2 \setminus B_{1/2})} + r^{k+2})^{1-\beta}, \tag{3-4}$$

where $\theta = \theta(n, k, \beta) > 1$.

As this result will be crucial let us explain briefly its proof. As $p_2 = \frac{1}{2}x_n^2$, we split $\mathbb{R}^n$ into the "tangential" directions $e_1, \dots, e_{n-1}$ and the "normal" direction e_n. The idea is to study the quantity

$$\sup_{B_r \cap \{u=0\}} |\partial_n \mathcal{P}_k|$$

for small r.

Geometrically this quantity tells us how much the zero set of $\mathcal{P}_k$ is a good approximation of the contact set $\{u = 0\}$ around the origin; see Proposition 3.3.

Analytically, $\sup_{B_r \cap \{u=0\}} |\partial_n \mathcal{P}_k|$ is crucial because it will be the "pivot" linking Lipschitz estimates along "tangential" directions with the one along the "normal" direction; let us see how.

First, taking difference quotients along tangential directions, we will prove that when $j \ne n$, we have,

$$r\|\partial_j v\|_{L^\infty(B_r)} \lesssim_{n,k,\tau} r^2 \cdot \sup_{\{u=0\} \cap B_r} |\partial_n \mathcal{P}_k| + \|v_r\|_{L^2(B_2 \setminus B_{1/2})} + r^{k+2}.$$

The fact that we have r^2 (and not r!) in front of $\sup_{\{u=0\} \cap B_r} |\partial_n \mathcal{P}_k|$ is the crucial gain, peculiar to the tangential derivatives.

Now we have to bound $\sup_{\{u=0\} \cap B_r} |\partial_n \mathcal{P}_k|$ from above. As this is much more complex, we just give the heuristics. First, notice that, as u is C^1, we have $\partial_n \mathcal{P}_k \equiv \partial_n v$ in $\{u = 0\}$. As v is harmonic in $\Omega := B_r \cap \{u > 0\}$, elliptic regularity suggests that we should be able to control $\|\partial_n v\|_{C^{0,\beta}(\Omega)}$ with $\|v\|_{C^{1,\beta}(\partial\Omega)} + \|v\|_{L^\infty(\Omega)}$. Furthermore, by Lemma 2.1, $\partial\Omega$ is close to the hyperplane $\{x_n = 0\}$, so we expect that the main contribution in $\|v\|_{C^{1,\beta}(\partial\Omega)}$ comes from the tangential derivatives $\|\partial_j v\|_{C^{0,\beta}(\partial\Omega)}$ for $j \ne n$. If we could take $\beta = 0$ and knew that $\partial\Omega$ was regular enough, this argument would give a bound on $\sup_{B_r} |\partial_n \mathcal{P}_k|$ in terms of $\|\partial_j v\|_{L^\infty(B_r)}$. But this is too much to ask: Schauder estimates break down at the Lipschitz scale and $\partial\Omega$ could be wild, this is why we lose a power β on the right-hand side of (3-4). The first issue will be fixed choosing β small and interpolating, the second will require us to construct a different set $\Omega \subseteq \{u > 0\}$ through a geometric barrier argument.

We start with a preliminary L^2-L^∞ estimate.

Lemma 3.5. *In the setting of Proposition 3.4, there exists a constant* $C = C(n, k, \tau)$ *such that*

$$\|v_r\|_{L^\infty(B_1)} \le C\|v_r\|_{L^2(B_2 \setminus B_{1/2})} + Cr^{k+2} \tag{3-5}$$

for all $0 < r < \frac{1}{2}$.

Proof. Recalling $\Delta\mathcal{P}_k = 1$, we have $\Delta v = \Delta(u - \mathcal{P}_k) = -\chi_{\{u=0\}} \le 0$. Using the mean-value inequality for superharmonic functions, for some $z \in \partial B_r$, we have

$$\min_{\bar{B}_r} v = v(z) \ge \fint_{B_{1/2}(z)} v_r \gtrsim_n -\|v_r\|_{L^2(B_2\setminus B_{1/2})}.$$

This provides the estimate from below. Inside $\{u = 0\} \cap B_r$, we have

$$v = -\tfrac{1}{2}\mathcal{A}_k^2 + O(|x|^{k+2}) \le Cr^{k+2}$$

for some $C = C(n, k, \tau)$. Then we "glue" the functions Cr^{k+2} and v, which is harmonic in $B_r \cap \{u > 0\}$, so that

$$V := \max\{Cr^{k+2}, v\} \quad \text{is subharmonic in } B_r.$$

Thus to estimate from above v on $B_r \cap \{u > 0\}$, we just use the mean-value property on V as above: this gives the upper bound up to corrections of size Cr^{k+2}. □

With a similar technique, we bound the first derivatives of v with $\sup_{\{u=0\}\cap B_r} |\partial_n \mathcal{P}_k|$.

Lemma 3.6. *In the setting of Proposition 3.4, there exists a constant $C = C(n, k, \tau)$, such that, for each $j = 1, \dots, n-1$ and $0 < r < \frac{1}{2}$, we have*

$$\|\partial_j v_r\|_{L^\infty(B_1)} \le Cr \cdot \sup_{\{u=0\}\cap B_r} |r\partial_n \mathcal{P}_k| + C\|v_r\|_{L^2(B_2\setminus B_{1/2})} + Cr^{k+2},$$
$$\|\partial_n v_r\|_{L^\infty(B_1)} \le C \sup_{\{u=0\}\cap B_r} |r\partial_n \mathcal{P}_k| + C\|v_r\|_{L^2(B_2\setminus B_{1/2})} + Cr^{k+2}.$$

We remark that $\|\partial_\ell v_r\|_{L^\infty(B_1)} = r\|\partial_\ell v\|_{L^\infty(B_r)}$ for all ℓ and $r > 0$.

Proof. We address first the case $j \ne n$. By construction, we have $\Delta\mathcal{P}_k = 1$, so $\Delta v = -\chi_{\{u=0\}}$. Hence, $\partial_j v$ is harmonic in $B_r \cap \{u > 0\}$. On the other hand, in $B_r \cap \{u = 0\}$, we have $\partial_j v = -\partial_j \mathcal{P}_k$, so by Proposition 3.3

$$|\partial_j \mathcal{P}_k| \le |\partial_j \mathcal{A}_k||\mathcal{A}_k| + O(|x|^{k+1}) \lesssim_{n,k,\tau} |x||\partial_n \mathcal{P}_k(x)| + |x|^{k+1}.$$

Here we crucially used that $|\partial_j \mathcal{A}_k| \lesssim_{n,k,\tau} |x|$ as $\mathcal{A}_k(x) = x_n + O(x^2)$. Hence,

$$\sup_{B_r\cap\{u=0\}} |\partial_j v| \le Cr^{k+1} + Cr \cdot \sup_{B_r\cap\{u=0\}} |\partial_n \mathcal{P}_k| =: K$$

for some $C = C(n, k, \tau)$. In order to estimate $\partial_j v$ on $B_r \cap \{u > 0\}$, we truncate it at levels K and $-K$ to obtain that

$$f := \max\{K, \partial_j v\} \quad \text{and} \quad g := \min\{-K, \partial_j v\}$$

are, respectively, subharmonic and superharmonic in B_r. Choose $x \in \partial B_r$ a maximum point of f in $\bar{B}_r$ and use the mean-value property:

$$\sup_{B_r\cap\{u>0\}} \partial_j v \le \sup_{B_r} f = f(x) \le \fint_{B_{1/2}(x)} f_r \lesssim_n K + \frac{1}{r}\|\partial_j v_r\|_{L^1(B_{3/2}\setminus B_{1/2})},$$

where we used that $B_{r/2}(x) \subseteq B_{3r/2} \setminus B_{r/2}$. By standard elliptic estimates we find

$$\|\nabla v_r\|_{L^1(B_{3/2}\setminus B_{1/2})} \lesssim_n \|\Delta v_r\|_{L^1(B_2\setminus B_{1/2})} + \|v_r\|_{L^2(B_2\setminus B_{1/2})}.$$

Recalling $\Delta v_r = -r^2\chi_{\{u_r=0\}} \le 0$, we integrate by parts with some smooth cut-off function

$$\chi_{B_2\setminus B_{1/2}} \le \psi \le \chi_{B_3\setminus B_{1/3}},$$

with $\|\psi\|_{C^2} \lesssim_n 1$, and we have

$$\int_{B_2\setminus B_{1/2}} |\Delta v_r| = -\int_{B_2\setminus B_{1/2}} \Delta v_r \le -\int_{B_3\setminus B_{1/3}} \Delta v_r \psi \lesssim_n \|v_r\|_{L^2(B_3\setminus B_{1/3})}.$$

The same computation with g instead of f provides an analogous estimate from below on $\partial_j v$. In conclusion we proved that, in every case, in B_r we have

$$|\partial_j v| \lesssim_n K + \frac{1}{r}\|v_r\|_{L^2(B_3\setminus B_{1/3})}.$$

Multiplying by r on both sides we find the first estimate.

The second estimate is proven with the same reasoning, just replacing K with

$$K' := Cr^{k+1} + C\cdot \sup_{B_r\cap\{u=0\}} |\partial_n \mathcal{P}_k|,$$

without the "extra r". □

We now use global Schauder estimates to bound from above the term $\sup_{\{u=0\}\cap B_r} |r\partial_n\mathcal{P}_k|$.

Lemma 3.7. *In the setting of Proposition 3.4, for any $\beta \in (0, \alpha_\circ^2/(k+2))$, there exists $C = C(n, k, \tau, \beta)$ such that, for all $0 < r < \frac{1}{2}$, we have*

$$\sup_{\{u=0\}\cap B_{r/4}} |r\partial_n\mathcal{P}_k| \le Cr^{\beta/(\alpha_\circ-\beta)}\|\partial_n v_r\|_{L^\infty(B_1)} + Cr^{\beta+\beta/\alpha_\circ}\|\nabla_{x'} v_r\|^{1-\beta/\alpha_\circ}_{L^\infty(B_1)} + C\|v_r\|_{L^\infty(B_1)} + Cr^{k+2}. \quad (3\text{-}6)$$

We recall that the dimensional constant $\alpha_\circ > 0$ has been defined in Lemma 2.1.

Proof. We will split the coordinates $x = (x', x_n)$ and denote by B_r' the intersection of B_r and $\{x_n = 0\}$. First of all, we recall from Lemma 2.1 that

$$\sup_{\{u=0\}\cap B_r} |\partial_n\mathcal{P}_k| = \sup_{\{u=0\}\cap B_r} |x_n + O(|x|^2)| \le C_\circ r^{1+\alpha_\circ} \quad \text{for all } 0 < r < 1 \quad (3\text{-}7)$$

for some $\alpha_\circ, C_\circ$ depending only on n and k. It is enough to prove the claim for $r \in (0, r_0)$, for some r_0 whose size will be constrained during the proof in terms of n, k, τ (recall that $\tau \ge |(p_2, \dots, p_3)|$). We will prove in detail only the upper bound, as the lower bound is derived in the same way, with a "symmetric" argument.

We choose a point $z \in \bar{B}_{r/4} \cap \{u = 0\}$ such that

$$\sup_{\{u=0\}\cap B_{r/4}} r\partial_n\mathcal{P}_k = r\partial_n\mathcal{P}_k(z).$$

Step 1. If r_0 is small enough, then, for all $r \in (0, r_0)$, we can find an open set Ω satisfying the following:

(i) Ω is a smooth domain inside B_{r_0}, that is $\Omega \cap B_{r_0} = \{x_n > \gamma(x')\} \cap B_{r_0}$ for some $\gamma \in C^\infty(B'_{r_0})$. Furthermore we have the following estimates on γ:

$$\|\gamma\|_{L^\infty(B'_r)} \le Cr^{1+\alpha_\circ}, \quad \|\nabla'\gamma\|_{L^\infty(B'_r)} \le Cr^{\alpha_\circ}, \quad [\nabla'\gamma]_{C^{\alpha_\circ}(B'_r)} \le C \tag{3-8}$$

for some $C = C(n, k, \tau)$.

(ii) $\Omega \subseteq \{u > 0\}$ and there exists $z^* \in \partial\Omega \cap \{u = 0\} \cap B_{r/2}$.

We emphasize that Ω, γ and z^* may depend on r, but the constant in (3-8) does not.

Proof of Step 1. We fix some $r \in (0, r_0)$. For all $b \in \mathbb{R}$ and $L = L(n, k, \tau)$ to be determined, we define the domains

$$\Omega(b) := \{x \in B_r : \partial_n \mathcal{P}_k(x) > \partial_n \mathcal{P}_k(z) + Lr^{\alpha_\circ - 1}|x' - z'|^2 + b\}.$$

Roughly speaking, $\Omega(b)$ looks like a perturbed paraboloid with vertex at $(z', z_n + b)$, provided $r_0 \lesssim_{n,k,\tau} 1$. Now, starting with b large, we decrease it until $\Omega(b)$ touches the contact set in B_r. That is, define

$$\Omega := \Omega(b^*), \quad \text{where } b^* := \inf\{b \in \mathbb{R} : \Omega(b) \cap B_r \subseteq \{u > 0\}\}.$$

We start checking that b^* is well defined. Thanks to (3-7), if there exists $x \in \{u = 0\} \cap B_r \cap \Omega(b)$ then

$$Lr^{\alpha_\circ - 1}|x' - z'|^2 + b < |\partial_n \mathcal{P}_k(x)| + |\partial_n \mathcal{P}_k(z)| \le 2C_\circ r^{1+\alpha_\circ}. \tag{3-9}$$

This shows that b^* is well defined and $b^* \le 2C_\circ r^{1+\alpha_\circ}$; in fact $\{u = 0\} \cap B_r \cap \Omega(b)$ must be empty for larger b. We also notice that $b \ge 0$, as $z \in \{u = 0\} \cap B_r \cap \Omega(b)$ for all $b < 0$.

We now prove (ii). Take $b < b^*$. By definition there exists $x_b \in \{u = 0\} \cap \Omega(b)$, and inequality (3-9) shows that $|x'_b - z'| \le \frac{1}{8}r$ for an appropriate choice of $L \gtrsim C_\circ$. Using the triangular inequality and (3-7) to estimate $|(x_b)_n - z_n|$, we find, for r_0 small, that

$$x_b \in \{u = 0\} \cap B_r \cap \Omega(b) \quad \Longrightarrow \quad x_b \in B_{r/2} \text{ and } b \le 2C_\circ r^{1+\alpha_\circ}. \tag{3-10}$$

We take $z^* \in \bar{B}_{r/2}$ to be any accumulation point of x_b as $b \uparrow b^*$. We clearly have $z^* \in \partial\Omega \cap \{u = 0\} \cap \bar{B}_{r/2}$.

Let us now prove (i). Consider the map $\Phi : x \mapsto (x', \partial_n \mathcal{P}_k(x))$. As $\partial_n \mathcal{P}_k(x) = x_n + O(x^2)$, we have that Φ is a diffeomorphism from B_{r_0}, provided r_0 is small. Let Ψ denote the n-th component of its inverse which is defined in some ball B_{R_0}. Clearly we have that $\Psi(y) = y_n + O(|y|^2)$, thus

$$|\nabla'\Psi(y)| \lesssim_{n,k,\tau} |y| \text{ and } \partial_n \Psi(y) \ge \tfrac{1}{2} \quad \text{for all } y \in B_{R_0}. \tag{3-11}$$

Therefore we conclude that $x \in \Omega$ if, and only if, $x \in B_r$ and

$$x_n = \Psi(x', \partial_n \mathcal{P}_k(x)) > \Psi(x', \partial_n \mathcal{P}_k(z) + Lr^{\alpha_\circ - 1}|x' - z'|^2 + b^*) := \gamma(x').$$

We have that $\gamma \in C^\infty(B'_{r_0})$ is well defined as, for r_0 small,

$$(x', \partial_n \mathcal{P}_k(z) + Lr^{\alpha_\circ - 1}|x' - z'|^2 + b^*) \in B_{R_0}.$$

This proves that Ω is a smooth domain. We remark that, while γ depends on r, Ψ only depends on r_0. This observation along with (3-11) easily gives the estimates on γ with constants independent of r. For example, we estimate, for all $x' \in B'_r$,

$$\begin{aligned}|\nabla'\gamma(x')| &\le |\nabla'\Psi(x', \partial_n\mathcal{P}_k(z) + Lr^{\alpha_\circ - 1}|x'-z'|^2 + b^*)| \\ &\qquad\qquad + 2Lr^{\alpha_\circ-1}|x'-z'|\,|\partial_n\Psi(x', \partial_n\mathcal{P}_k(z) + Lr^{\alpha_\circ-1}|x'-z'|^2 + b^*)| \\ &\lesssim_{n,\tau} |x'| + |\partial_n\mathcal{P}_k(z) + Lr^{\alpha_\circ-1}|x'-z'|^2 + b^*| + Lr^{\alpha_\circ-1}\|\partial_n\Psi\|_{L^\infty(B_{R_0})}|x'-z'|,\end{aligned}$$

and all the terms on the right-hand side are of order $r^{\alpha_\circ}$ or higher thanks to (3-9). The other estimates can be proven identically, using (3-9) and the fact that $\Psi(0,0) = |\nabla'\Psi(0,0)| = 0$.

Step 2. For each $\beta \in (0, \alpha_\circ]$, there is $C = C(n, k, \tau, \beta)$ such that

$$\frac{1}{C}\sup_{\{u=0\}\cap B_{r/4}} r\partial_n\mathcal{P}_k \le [\nabla'(v\circ\Gamma)_r]_{C^\beta(B'_{3/4})} + \|v_r\|_{L^\infty(B_1)},$$

where $\Gamma : B'_r \to \mathbb{R}^n$ is the graph of γ, that is, $\Gamma(x') = (x', \gamma(x'))$.

Proof of Step 2. We observe that, by definition of z^*,

$$\partial_n\mathcal{P}_k(z) \le \partial_n\mathcal{P}_k(z^*),$$

so we have

$$\sup_{\{u=0\}\cap B_{r/4}} r\partial_n\mathcal{P}_k = r\partial_n\mathcal{P}_k(z) \le r\partial_n\mathcal{P}_k(z^*) \le \|\partial_n v_r\|_{L^\infty(\partial\widetilde{\Omega}\cap B_{1/2})},$$

where we used the rescaled domain $\widetilde{\Omega} := \Omega/r$, whose boundary is the graph of $\tilde{\gamma} = \gamma(r\,\cdot)/r$. We now employ global Schauder estimates in $\widetilde{\Omega}$ (see, e.g., [Gilbarg and Trudinger 1983, Theorem 8.33]) to control the right-hand side:

$$\|\partial_n v_r\|_{L^\infty(\partial\widetilde{\Omega}\cap B_{1/2})} \lesssim_{n,\beta} [\nabla'(v\circ\Gamma)_r]_{C^\beta(B'_{3/4})} + \|\Delta v_r\|_{L^\infty(\widetilde{\Omega})} + \|v_r\|_{L^\infty(B_1)}. \tag{3-12}$$

Note that $\widetilde{\Omega}$ lies in $\{u_r > 0\}$, where $\Delta v_r \equiv 0$. We remark that the previous estimate holds with a constant which depends on $\|\nabla\tilde{\gamma}\|_{C^{\alpha_\circ}(B_1)}$, which is bounded independently by r from (3-8):

$$\|\nabla\tilde{\gamma}\|_{L^\infty(B'_1)} = \|\nabla\gamma\|_{L^\infty(B'_r)} \le Cr^{\alpha_\circ} \quad\text{and}\quad [\nabla\tilde{\gamma}]_{C^{\alpha_\circ}(B'_1)} = r^{\alpha_\circ}\|\nabla\gamma\|_{C^{\alpha_\circ}(B'_r)} \le Cr^{\alpha_\circ}.$$

Furthermore, we also used that, thanks to (3-8), the graph of $\tilde{\gamma}$ is contained in the strip $\left\{|y_n| \le \frac{1}{10}\right\} \cap B_1$, provided that r_0 is small.

Step 3. There is $C = C(n, k, \tau)$ such that

$$\begin{aligned}\frac{1}{C}\|\nabla'(v\circ\Gamma)_r\|_{L^\infty(B'_{3/4})} &\le \|\nabla' v_r\|_{L^\infty(B_1)} + r^{\alpha_\circ}\|\partial_n v_r\|_{L^\infty(B_1)}, \\ \frac{1}{C}[\nabla'(v\circ\Gamma)_r]_{C^{\alpha_\circ}(B'_{3/4})} &\le r^{1+\alpha_\circ}.\end{aligned}$$

Proof of Step 3. Let $\widetilde{\Gamma}$ denote the graph of $\tilde{\gamma}$, that is, $\widetilde{\Gamma}(y') = (y', \tilde{\gamma}(y'))$. We have

$$\nabla'(v\circ\Gamma)_r = \nabla'(v_r\circ\widetilde{\Gamma}) = \nabla' v_r + (\partial_n v_r\circ\widetilde{\Gamma})\nabla'\tilde{\gamma},$$

so the first bound follows taking the $L^\infty(B'_{3/4})$ norms, using $\widetilde{\Gamma}(B'_{3/4}) \subseteq B'_1$ and $\|\nabla\tilde{\gamma}\|_{L^\infty(B'_1)} \le Cr^{\alpha_\circ}$. For the second bound we use the optimal regularity of u (see (2-2)):

$$\begin{aligned}[\nabla'(v\circ\Gamma)_r]_{C^{\alpha_\circ}(B'_{3/4})} &= r^{1+\alpha_\circ}[(\nabla' v\circ\Gamma)(\nabla'\Gamma)]_{C^{\alpha_\circ}(B'_{3r/4})}\\ &\le r^{1+\alpha_\circ}(\|\nabla v\|_{L^\infty(B_r)}[\nabla'\gamma]_{C^{\alpha_\circ}(B'_r)} + [\nabla v]_{C^{0,1}(B'_r)}[\Gamma]_{C^{\alpha_\circ}(B'_r)}\|\nabla'\Gamma\|_{L^\infty(B'_r)})\\ &\le r^{1+\alpha_\circ}\|\gamma\|_{C^{\alpha_\circ}(B'_r)}\|v\|_{C^{1,1}(B_{1/2})}\\ &\lesssim_{n,k,\tau} r^{1+\alpha_\circ}.\end{aligned}$$

Step 4. There is $C = C(n,k,\tau,\beta)$ such that

$$\frac{1}{C}[\nabla'(v\circ\Gamma)_r]_{C^\beta(B'_{3/4})} \le r^{\alpha_\circ^2/\beta} + r^{\beta/(\alpha_\circ-\beta)}\|\partial_n v_r\|_{L^\infty(B_1)} + r^{\beta+\beta/\alpha_\circ}\|\nabla' v_r\|^{1-\beta/\alpha_\circ}_{L^\infty(B_1)}. \tag{3-13}$$

Proof of Step 4. The idea is to bound the C^β norm in (3-12) with an interpolation of the L^∞ and the $C^{\alpha_\circ}$ norms, which we bounded in Step 3. This gives

$$\begin{aligned}[\nabla'(v\circ\Gamma)_r]_{C^\beta(B'_{3/4})} &\le [\nabla'(v\circ\Gamma)_r]^{\beta/\alpha_\circ}_{C^{\alpha_\circ}(B'_1)}[\nabla'(v\circ\Gamma)_r]^{1-\beta/\alpha_\circ}_{L^\infty(B'_1)}\\ &\le Cr^{\beta+\beta/\alpha_\circ}(\|\nabla' v_r\|_{L^\infty(B_1)} + r^{\alpha_\circ}\|\partial_n v_r\|_{L^\infty(B_1)})^{1-\beta/\alpha_\circ}\\ &\le Cr^{\alpha_\circ}(r^{\beta/(\alpha_\circ-\beta)}\|\partial_n v_r\|_{L^\infty(B_1)})^{1-\beta/\alpha_\circ} + Cr^{\beta+\beta/\alpha_\circ}\|\nabla' v_r\|^{1-\beta/\alpha_\circ}_{L^\infty(B_1)},\end{aligned}$$

where in the last line we used the subadditivity of $t\mapsto t^{1-\beta/\alpha_\circ}$. Finally we obtain (3-13) using Young's inequality on the first term, with exponent $1/p = \beta/\alpha_\circ$:

$$r^{\alpha_\circ}(r^{\beta/(\alpha_\circ-\beta)}\|\partial_n v_r\|_{L^\infty(B_1)})^{1-\beta/\alpha_\circ} \lesssim_{\alpha_\circ,\beta} r^{\alpha_\circ^2/\beta} + r^{\beta/(\alpha_\circ-\beta)}\|\partial_n v_r\|_{L^\infty(B_1)}.$$

Combining Steps 2 and 4 and recalling that $\alpha_\circ^2/\beta > k+2$, we obtain (3-6) for all $r\in(0,r_0)$, as the constants do not depend on r. □

With the help of the two previous lemmas, we can now prove our main Lipschitz estimate by using $\sup_{B_r\cap\{u=0\}}|\partial_n\mathcal{P}_k|$ as "pivot".

Proof of Proposition 3.4. We first address (ii), the estimate in the normal direction. All constants will depend on n, k, τ, β. Linking Lemma 3.7 with the second estimate of Lemma 3.6, we get

$$\frac{1}{C}\|\partial_n v_r\|_{L^\infty(B_{1/4})} \le r^{\beta/(\alpha_\circ-\beta)}\|\partial_n v_r\|_{L^\infty(B_1)} + r^{\beta+\beta/\alpha_\circ}\|\nabla_{x'} v_r\|^{1-\beta/\alpha_\circ}_{L^\infty(B_1)} + \Lambda(r),$$

where we set for brevity

$$\Lambda(r) := \|v\|_{L^\infty(B_r)} + r^{k+2}.$$

Notice that Λ is increasing in r. Now we use trivial bounds and the tangential estimate of Lemma 3.6 to deal with the central terms on the right-hand side:

$$r^{\beta+\beta/\alpha_\circ}\|\nabla_{x'} v_r\|^{1-\beta/\alpha_\circ}_{L^\infty(B_1)} \le Cr^{\beta+\beta/\alpha_\circ}(r\|\partial_n v_r\|_{L^\infty(B_1)} + \Lambda(2r))^{1-\beta/\alpha_\circ}.$$

Then, using Young's inequality with $1/p = \beta/\alpha_\circ$ and $r^{\alpha_\circ/\beta} \le r^{k+2}$, we obtain

$$\begin{aligned} r^{\beta+\beta/\alpha_\circ}(r\|\partial_n v_r\|_{L^\infty(B_1)})^{1-\beta/\alpha_\circ} &= r(r^{\alpha_\circ\beta/(\alpha_\circ-\beta)}\|\partial_n v_r\|_{L^\infty(B_1)})^{1-\beta/\alpha_\circ} \\ &\lesssim_{\alpha_\circ,\beta} r^{\alpha_\circ\beta/(\alpha_\circ-\beta)}\|\partial_n v_r\|_{L^\infty(B_1)} + r^{k+2}. \end{aligned}$$

Thus, by subadditivity of $t \mapsto t^{1-\beta/\alpha_\circ}$ and enlarging the constants, we finally arrive at

$$\|\partial_n v_r\|_{L^\infty(B_{1/4})} \le C r^{\alpha_\circ\beta/(\alpha_\circ-\beta)}\|\partial_n v_r\|_{L^\infty(B_1)} + C\Lambda(2r)^{1-\beta/\alpha_\circ}. \tag{3-14}$$

We conclude by iteration of this inequality. Let us set

$$f(r) := \|\partial_n v_r\|_{L^\infty(B_1)} = r\|\partial_n v\|_{L^\infty(B_r)} \quad \text{and} \quad \delta := \frac{\alpha_\circ\beta}{\alpha_\circ-\beta}.$$

Then (3-14) reads as

$$f\big(\tfrac14 r\big) \le C r^\delta f(r) + C\Lambda(2r)^{1-\beta/\alpha_\circ} \quad \text{for all } 0 < r < \tfrac14.$$

Since f and Λ are increasing functions, we can iterate this inequality $N \sim k/\delta$ times (N does not depend on r), and it becomes

$$f(r) \le C_N\Lambda(4^N r)^{1-\beta/\alpha_\circ} + C_N f\big(\tfrac14\big) r^{k+2} \quad \text{for all } 0 < r < 4^{-N},$$

and $f\big(\tfrac14\big)$ is again bounded by a dimensional constant, by optimal regularity. Finally, we use Lemma 3.5 to replace $\|v_{4^N r}\|_{L^\infty(B_1)}$ with $\|v_{4^N r}\|_{L^2(B_2\setminus B_{1/2})} + r^{k+2}$. We have proved that

$$\|\partial_n v_r\|_{L^\infty(B_1)} \le C(\|v_{\theta r}\|_{L^2(B_2\setminus B_{1/2})} + r^{k+2})^{1-\beta/\alpha_\circ},$$

with $C = C(n, k, \tau, \beta)$ and $\theta(n, k, \beta) = 4^N$, for all $r \in (0, r_0)$. Since $\beta \in (0, \alpha_\circ/(k+2))$ we proved (3-4).

We turn to the proof of the tangential estimate (3-3). Combining the previous step with Lemma 3.6, we have, for $C > 0$ and $r \in (0, r_0)$,

$$\begin{aligned} \frac{1}{C}\|\partial_j v_r\|_{L^\infty(B_1)} &\le r(\|v_{\theta r}\|_{L^2(B_2\setminus B_{1/2})} + r^{k+2})^{1-\beta/\alpha_\circ} + \|v_r\|_{L^2(B_2\setminus B_{1/2})} + r^{k+2} \\ &\le (r^{1-\beta(k+2)/\alpha_\circ} + 1)(\|v_{\theta r}\|_{L^2(B_2\setminus B_{1/2})} + r^{k+2}). \end{aligned}$$

We used that θ is large and Lemma 3.5 to bound

$$\|v_r\|_{L^2(B_2\setminus B_{1/2})} \le \|v_r\|_{L^\infty(B_2)} \le \|v_{\theta r}\|_{L^2(B_1)} \lesssim_{n,k,\theta} \|v_{\theta r}\|_{L^2(B_2\setminus B_{1/2})}.$$

Note that with the choice $\beta := \alpha_\circ^2/(k+4)$, we get rid of the β-dependence in the constants and obtain the claimed estimate. □

4. Monotonicity of the truncated frequency

In this section we show that the truncated frequency function $\phi^\gamma(r, u - \mathcal{P}_k)$ from (2-6) is almost monotone for $\gamma < k+2$, regardless of the $p_3, \dots, p_k$. All the proofs in this section do not change for a generic smooth f, as we use the inequalities of the previous section as "black boxes".

The core of the monotonicity is the following computational lemma.

Lemma 4.1. *Let $k \geq 2$, $\tau > 0$, and let $u : B_1 \to \mathbb{R}$ be a solution of the obstacle problem (2-1), with $f \equiv 1$ and $\mu = 1$. Assume $r^{-2}u(r\,\cdot) \to p_2$, and take $(p_2, \dots, p_k) \in \boldsymbol{P}_k$ such that $|(p_2, \dots, p_k)| \leq \tau$. Consider $v := u - \mathcal{P}_k$, where $\mathcal{P}_k = \mathcal{P}_k(p_2, \dots, p_k)$ is constructed as in Definition 3.2. For each $\gamma \in [0, k+2)$ and $\beta \in (0, \alpha_\circ/(k+2))$, set*

$$\epsilon := \min\{\alpha_\circ - \beta(k+2), k+2-\gamma\} > 0,$$

where the dimensional constant $\alpha_\circ$ is the one of Lemma 2.1. Then there exists $r_0 = r_0(k, n, \tau, \beta) \in (0, 1)$ such that, for all $0 < r < r_0$,

$$\frac{d}{dr}\phi^\gamma(r, v) \geq -Cr^{\epsilon-1}(g^\gamma(r, v) + 1)(\phi^\gamma(r) + 1) \tag{4-1}$$

and

$$\frac{\int_{B_r} |v_r \Delta v_r|}{H(r, v) + r^{2\gamma}} \leq Cr^\epsilon (g^\gamma(r, v) + 1), \tag{4-2}$$

where we set

$$g^\gamma(r, v) := \frac{\|v_{\theta r}\|^2_{L^2(B_2 \setminus B_{1/2})}}{H(r, v) + r^{2\gamma}}.$$

Here $\theta = \theta(k, \beta)$ and $C = C(n, k, \tau, \beta)$ are constants.

Proof. Throughout the proof, C will be a constant depending on n, k, τ, β. Up to a rotation of the coordinate axes, we may assume $p_2 = \frac{1}{2}x_n^2$. We begin with recalling the estimate on the derivative of ϕ^γ, compare (2-7),

$$\frac{d}{dr}\phi^\gamma(r, v) \geq \frac{2}{r}\frac{\phi^\gamma(r, v)\int_{B_1} v_r \Delta v_r - \int_{B_1}(x \cdot \nabla v_r)\Delta v_r}{H(r, v) + r^{2\gamma}}.$$

Using $\operatorname{supp} \Delta v_r \subseteq \{u_r = 0\}$, we reduce (4-1) to a bound from below on the quantity

$$\frac{2}{r}\frac{\|\phi^\gamma(r)v_r - x \cdot \nabla v_r\|_{L^\infty(B_1 \cap \{u_r = 0\})}\int_{B_1 \cap \{u_r=0\}} |\Delta v_r|}{H(r, v) + r^{2\gamma}}.$$

For $x \in B_1 \cap \{u_r = 0\}$ and $r < r_0(n, k, \tau, \beta)$, we have

(i) $|x_n| \leq Cr^{\alpha_\circ}$, see Lemma 2.1;

(ii) $|v_r(x)| \leq Cr|\partial_n(\mathcal{P}_k)_r(x)| + Cr^{k+2}$, see Proposition 3.3 (ii);

(iii) $|\partial_j v_r(x)| \leq Cr|\partial_n(\mathcal{P}_k)_r(x)| + Cr^{k+2}$ for all $j \neq n$, see Lemma 3.6;

(iv) $r|\partial_n \mathcal{P}_k(x)| \leq C(\|v_{\theta r}\|_{L^2(B_2 \setminus B_{1/2})} + r^{k+2})^{1-\beta}$, see Proposition 3.4.

Putting together all these bounds we get, for all $x \in B_1 \cap \{u_r = 0\}$,

$$\begin{aligned}
|\phi^\gamma(r)v_r - x \cdot \nabla v_r| &\leq (\phi^\gamma(r) + 1)(|v_r| + |x_n|\,|\partial_n(\mathcal{P}_k)_r| + |\nabla_{x'} v_r|) \\
&\leq C(\phi^\gamma(r) + 1)(r^{\alpha_\circ}\|\partial_n(\mathcal{P}_k)_r\|_{L^\infty(B_1 \cap \{u_r=0\})} + r^{k+2}) \\
&\leq C(\phi^\gamma(r) + 1)r^{\alpha_\circ}(\|v_{\theta r}\|_{L^2(B_3 \setminus B_{1/2})} + r^{k+2})^{1-\beta} + C(\phi^\gamma(r) + 1)r^{k+2} \\
&\leq C(\phi^\gamma(r) + 1)r^\epsilon(\|v_{\theta r}\|_{L^2(B_3 \setminus B_{1/2})} + r^{k+2}) + C(\phi^\gamma(r) + 1)r^{k+2},
\end{aligned}$$

where in the last line we argued

$$(\|v_{\theta r}\|_{L^2(B_3\setminus B_{1/2})}+r^{k+2})^{1-\beta}\le r^{-\beta(k+2)}(\|v_{\theta r}\|_{L^2(B_3\setminus B_{1/2})}+r^{k+2}).$$

Thus, using $(H(r,v)+r^{2\gamma})^{-1/2}\le r^{-\gamma}$ and $r^{k+2-\gamma}\le r^{\epsilon}$, we get

$$\frac{\|\phi^\gamma(r)v_r-x\cdot\nabla v_r\|_{L^\infty(B_1\cap\{u_r=0\})}}{(H(r,v)+r^{2\gamma})^{1/2}}\le r^{\epsilon}(\phi^\gamma(r)+1)(g^\gamma(r)^{1/2}+1).$$

Now, using (ii) and (iv) to estimate $|\nabla v_r(x)|$ for $x\in B_1\cap\{u_r=0\}$ as above, we get

$$\frac{\|v_r\|_{L^\infty(B_1\cap\{u_r=0\})}}{(H(r,v)+r^{2\gamma})^{1/2}}\le r^{\epsilon}(g^\gamma(r)^{1/2}+1).$$

Thanks to this observation, to prove both (4-1) and (4-2) we only need to show

$$\frac{\int_{B_1\cap\{u_r=0\}}|\Delta v_r|}{(H(r,v)+r^{2\gamma})^{1/2}}\le Cg^\gamma(r)^{1/2}.$$

As $\Delta v_r=-r^2\chi_{\{u_r=0\}}$, we can integrate by parts with some regular cut-off $\chi_{B_1}\le\psi\le\chi_{B_2}$:

$$\int_{B_1\cap\{u_r=0\}}|\Delta v_r|=-\int_{B_1}\Delta v_r\le-\int_{B_2}\Delta v_r\psi\lesssim_n\|v_r\|_{L^1(B_2\setminus B_1)},$$

which concludes the proof. □

We now make a specific choice of β and derive from these preliminary bounds the monotonicity of the truncated frequency.

Proposition 4.2. *Let $k\ge 2$, $\tau>0$, and let u be a solution of the obstacle problem (2-1) with $f\equiv 1$ and $\mu=1$. Assume $r^{-2}u(r\,\cdot)\to p_2$, and take $(p_2,\dots,p_k)\in\boldsymbol{P}_k$ such that $|(p_2,\dots,p_k)|\le\tau$. Consider $v:=u-\mathcal{P}_k$, where $\mathcal{P}_k=\mathcal{P}_k(p_2,\dots,p_k)$ is constructed as in Definition 3.2, and, for each $\gamma\in(0,k+2)$, set*

$$\varepsilon(\gamma):=\min\bigl\{\tfrac{1}{2}\alpha_\circ;\,k+2-\gamma\bigr\},$$

where the dimensional constant $\alpha_\circ$ is the one of Lemma 2.1. Then, there exist $C(n,k,\tau,\gamma)>0$ and $r_0(n,k,\tau)\in(0,1)$ such that, for all $0<r<r_0$, we have

$$\frac{d}{dr}\phi^\gamma(r)\ge -Cr^{\varepsilon-1},\quad \phi^\gamma(r)\le C\quad\text{and}\quad \frac{\int_{B_r}|v_r\Delta v_r|}{H(r,v)+r^{2\gamma}}\le Cr^{\varepsilon}. \tag{4-3}$$

In particular, $\phi^\gamma(0^+,v)=\lim_{r\downarrow 0}\phi^\gamma(r,v)$ exists and $\phi^\gamma(0^+,v)\le\gamma$.

Proof. Fix $\gamma_\circ\in(0,k+2)$, and let $\varepsilon_\circ=\varepsilon(\gamma_\circ)$. Let $r_0=r_0(n,k,\tau,\beta=\alpha_\circ/(2(k+2)))$ be as in Lemma 4.1. By Lemma 4.1 and estimate (4-1), we only need to show that the functions $g^{\gamma_\circ}(\cdot)$ and $\phi^{\gamma_\circ}(\cdot)$ are uniformly bounded in the interval $(0,r_0]$. We are going to prove it by increasing iteratively the parameter γ, exactly as in the proof of [Figalli et al. 2020, Lemma 4.3]. Throughout the proof, C_γ will denote a general constant depending on $n,\ k,\ \tau,\ \gamma$, and similarly for $C_{\gamma,\delta}$.

First notice that by (2-2) the functions $\phi^0(\,\cdot\,, v)$ and $g^0(\,\cdot\,)$ are uniformly bounded in $[0, r_0]$. Fix any $\gamma \in [0, \gamma_\circ]$. The core of the proof is the observation

$$\begin{cases} \phi^\gamma + g^\gamma \le C_\gamma & \text{in } (0, r_0], \\ 0 < 5\delta \le \varepsilon_\circ, \end{cases} \implies \quad \phi^{\gamma+\delta} + g^{\gamma+\delta} \le C_{\gamma,\delta} \quad \text{in } (0, r_0]. \tag{4-4}$$

We iterate this observation to reach the conclusion. Define the sequence $\gamma_0 = 0,\ \gamma_{j+1} := \gamma_j + \frac{1}{5}\varepsilon_\circ$, where $j \ge 0$. With a finite number of iterations we get closer than $\frac{1}{5}\varepsilon_\circ$ to $\gamma_\circ$, and applying (4-4) once more with an appropriate δ we get to $\gamma_\circ$.

We prove (4-4) for a generic γ. Keeping in mind $r, \delta \le 1$, we estimate

$$\phi^{\gamma+\delta}(r) = \frac{D(r, v) + (\gamma+\delta) r^{2\gamma+2\delta}}{H(r, v) + r^{2\gamma+2\delta}} \le \frac{1}{r^{2\delta}} \frac{D(r, v) + \gamma r^{2\gamma}}{H(r, v) + r^{2\gamma}} + 1 \le \frac{C_\gamma}{r^{2\delta}}$$

and

$$g^{\gamma+\delta}(r) = \frac{\|v_{\theta r}\|^2_{L^2(B_3 \setminus B_{1/2})}}{H(r, v) + r^{2\gamma+2\delta}} \le \frac{1}{r^{2\delta}} \frac{\|v_{\theta r}\|^2_{L^2(B_2 \setminus B_{1/2})}}{H(r, v) + r^{2\gamma}} \le \frac{C_\gamma}{r^{2\delta}}.$$

Now we apply Lemma 4.1 with $\beta := \alpha_\circ/(2(k+2))$ and $\gamma \to \gamma + \delta$:

$$\frac{d}{dr}\phi^{\gamma+\delta}(r) \ge -C r^{\epsilon-1}(g^{\gamma+\delta}(r) + 1)(\phi^{\gamma+\delta}(r) + 1) \ge -C_\gamma r^{\varepsilon_\circ - 4\delta - 1} \ge -C_\gamma r^{\delta-1},$$

where we used $\epsilon(\beta, \gamma+\delta) \ge \varepsilon_\circ$ and the smallness of δ. Integrating this inequality and using $C^{1,1}$ estimates, we obtain

$$\phi^{\gamma+\delta}(r) \le \phi^{\gamma+\delta}(r_0) + C_\gamma r_0^\delta/\delta \lesssim_{n,k,\tau} C_{\gamma,\delta}$$

for all $r \in (0, r_0]$. The uniform boundedness of $g^{\gamma+\delta}$ is now a consequence of Lemma 2.2 (a), as we can take $\bar\lambda = C_{\gamma,\delta}$. Indeed, for any $0 < 2\theta r < r_0$ and any $R \in \left(\frac{1}{2} r\theta, 2\theta r\right)$, we have

$$\frac{H(R, v) + R^{2\gamma+2\delta}}{H(r, v) + r^{2\gamma+2\delta}} \lesssim_{n,k,\tau} \left(\frac{R}{r}\right)^{C_{\gamma,\delta}} \le C_{\gamma,\delta}.$$

Thus, for all $r \in (0, r_0/(2\theta)]$, we have

$$g^{\gamma+\delta}(r) = \frac{\|v_{\theta r}\|^2_{L^2(B_2 \setminus B_{1/2})}}{H(r, v) + r^{2\gamma+2\delta}} \lesssim_{n,k,\tau} \fint_{\theta r/2}^{2\theta r} \frac{H(R, v)\, dR}{H(r, v) + r^{2\gamma+2\delta}} \le C_{\gamma,\delta}.$$

Finally $g^{\gamma+\delta}$ is clearly bounded in $[r_0/(2\theta), r_0]$ by (2-2). This concludes the proof. □

The following is an immediate corollary of Proposition 4.2.

Corollary 4.3. *With the same notation from Proposition 4.2, we have that, for all $r \in (0, r_0)$,*

$$\frac{d}{dr}\log(r^{-2\lambda}(H(r, u - \mathcal{P}_k) + r^{2\gamma})) \ge -C r^{\varepsilon-1}, \tag{4-5}$$

provided $\lambda \le \phi^\gamma(0^+, v)$. Here $r_0,\ C,\ \varepsilon$ are the same positive numbers from Proposition 4.2.

Proof. Computing the derivative and using Proposition 4.2, we obtain

$$\frac{d}{dr}\log(r^{-2\lambda}(H(r,v)+r^{2\gamma})) = \frac{2}{r}(\phi^{\gamma}(r,v)-\lambda)+\frac{2}{r}\frac{\int_{B_1} v_r\Delta v_r}{H(r,v)+r^{2\gamma}}$$
$$\geq -C\int_0^r s^{\varepsilon-1}\,ds - Cr^{\varepsilon-1} \geq -Cr^{\varepsilon-1}. \qquad \square$$

5. The sets $\Sigma^{k\text{-th}}$ and higher-order blowups

The estimates obtained in the previous two sections did not assume any relationship between u and $\mathcal{P}_k$, beside the crucial fact that p_2 was the blowup of u at 0. In this section, instead, we define the set of points at which u admits a polynomial expansion of order k. This expansion will identify the polynomial $\mathcal{P}_k$ up to order k, leaving some freedom for the $k+1$ terms (compare Proposition 3.3 (ii)).

The results of this section never use explicitly the simplifying assumption $f \equiv 1$; they use it implicitly, though, employing the results of the previous two sections. Hence for a generic smooth f, all the results of this section apply without change.

Recall that $\boldsymbol{P}_k$ was defined at the beginning of Section 3.1.

Definition 5.1. Let $u : B_1 \to [0,\infty)$ solve (2-1) and $x_\circ \in \Sigma_{n-1}$. We say the following:

(i) $\Sigma^{2\text{nd}} := \Sigma_{n-1}$.

(ii) $x_\circ \in \Sigma^{3\text{rd}}$ if
$$r^{-3}(u(x_\circ + r\,\cdot\,) - p_{2,x_\circ}(r\,\cdot\,)) \to p_{3,x_\circ} \quad \text{in } W^{1,2}_{\text{loc}}(\mathbb{R}^n)\cap C^0_{\text{loc}}(\mathbb{R}^n)$$
as $r \downarrow 0$, where $p_{3,x_\circ}$ is some 3-homogeneous harmonic polynomial vanishing on $\{p_{2,x_\circ}=0\}$. Thus we have $(p_{2,x_\circ}, p_{3,x_\circ}) \in \boldsymbol{P}_3$.

(iii) $x_\circ \in \Sigma^{k\text{-th}}$ for some integer $k \geq 4$, if $x_\circ \in \Sigma^{(k-1)\text{-th}}$ and
$$r^{-k}(u(x_\circ + r\,\cdot\,) - \mathcal{P}_{k-1,x_\circ}(r\,\cdot\,)) \to p_{k,x_\circ} \quad \text{in } W^{1,2}_{\text{loc}}(\mathbb{R}^n)\cap C^0_{\text{loc}}(\mathbb{R}^n)$$
as $r \downarrow 0$, where $\mathcal{P}_{k-1,x_\circ} = \mathcal{P}_{k-1}(p_{2,x_\circ},\dots,p_{k-1,x_\circ})$ is the polynomial ansatz from Definition 3.2 and the limit $p_{k,x_\circ}$ is some k-homogeneous harmonic polynomial vanishing on $\{p_{2,x_\circ}=0\}$. Thus we have $(p_{2,x_\circ},\dots,p_{k,x_\circ}) \in \boldsymbol{P}_k$.

When $x_\circ = 0$, we simply drop $x_\circ$ from $p_{j,x_\circ}$.

Remark 5.2. Let $u : B_1 \to [0,\infty)$ solve (2-1) and suppose $0 \in \Sigma^{k\text{-th}}$ for some $k \geq 2$. Then

(i) $\phi^{\gamma}(0^+, u-\mathcal{P}_k)$ exists for all $\gamma \in [k, k+2)$, see Proposition 4.2;

(ii) for every $\delta > 0$, we have $r^{2\phi^{\gamma}(0^+,u-\mathcal{P}_k)+\delta} \ll H(r, u-\mathcal{P}_k) = o(r^{2k})$ as $r \downarrow 0$. The lower bound follows by Lemma 2.2, the upper bound by continuity of the embedding $W^{1,2}(B_1) \to L^2(\partial B_1)$.

With help of Corollary 4.3, we prove a Monneau-type monotonicity formula; see [Monneau 2003]. The argument is an adaptation of [Figalli and Serra 2019, Lemma 4.1].

Proposition 5.3 (Monneau-type monotonicity). *Let $u : B_1 \to [0,\infty)$ solve (2-1). Suppose $0 \in \Sigma^{k\text{-th}}$ for some $k \geq 3$. Let q be any polynomial such that $(p_{2,0}, \dots, p_{k-1,0}, q) \in \boldsymbol{P}_k$, with $|(p_{2,0}, \dots, p_{k-1,0}, q)| \leq \tau$ for some number $\tau > 0$. Set*

$$w := u - \mathcal{P}_k(p_{2,0}, \dots, p_{k-1,0}, q).$$

Then there exists $r_0 = r_0(n,k,\tau)$ such that, for all $r \in (0, r_0)$,

$$r^{-2k} H(r,w) \leq C \quad \text{and} \quad \frac{d}{dr}(r^{-2k} H(r,w)) \geq -C r^{\varepsilon-1}$$

for some constants $C = C(n,k,\tau)$ and $\varepsilon(n,k) > 0$.

Proof. For the sake of the proof, let us fix $\gamma = k + 1 + \frac{1}{2}$; all constants are allowed to depend on n, k, γ, even if not explicitly stated, and can change value from line to line. We begin by showing $\phi^\gamma(0^+, w) \geq k$. By construction of the $\mathcal{P}_k$ (see Proposition 3.3), we have

$$w = u - \mathcal{P}_{k,0} - p_{k,0} + q + O(x^{k+1}),$$

so using $0 \in \Sigma^{k\text{-th}}$ we find

$$H(r,w)^{1/2} \leq H(r, u - \mathcal{P}_k)^{1/2} + H(r, p_k)^{1/2} + H(r,q)^{1/2} + H(r, O(|x|^{k+1}))^{1/2} = o(r^k) + O(r^k).$$

If $\phi^\gamma(0^+, w) < k$ were true, then Lemma 2.2 would give, for $r \ll 1$,

$$r^{2\phi^\gamma(0^+,w)+\delta} \ll H(r,w) + r^{2\gamma} \lesssim r^{2k},$$

which would be a contradiction for $\delta > 0$ small. Hence, $\phi^\gamma(0^+, w) \geq k$, and we can apply Corollary 4.3 with $\gamma = k + 1 + \frac{1}{2}$ to find that the function

$$f(r) := \log(r^{-2k}(H(r,w) + r^{2\gamma})) + C r^\varepsilon$$

is increasing in $(0, r_0)$ for appropriate r_0, C, ε depending on n, k, τ. Using Equation (2-2) we have that $\|w\|_{L^\infty(B_1)} \leq C$, and so $f(r_0) \leq C$. Thus, by monotonicity of f,

$$r^{-2k}(H(r,w) + r^{2\gamma}) \leq C$$

for $r \in (0, r_0)$. Inserting again this estimate in Corollary 4.3, we get

$$\frac{d}{dr}(r^{-2k}(H(r,w) + r^{2\gamma})) \geq -C r^{\varepsilon-1}(r^{-2k}(H(r,w) + r^{2\gamma})) \geq -C r^{\varepsilon-1},$$

and as $(d/dr) r^{2(\gamma-k)} \ll r^{\varepsilon-1}$ we conclude. □

The following result proves the continuity of the map $x \mapsto \mathcal{P}_{k,x}$, defined on $\Sigma^{k\text{-th}}$, for $k \geq 2$. Our argument is a direct adaptation of [Figalli and Serra 2019, Proposition 4.5] for the case $k = 3$.

Proposition 5.4. *Let $u : B_1 \to [0,\infty)$ solve (2-1) and $k \geq 2$. Then the map $\Sigma^{k\text{-th}} \ni x \mapsto \mathcal{P}_{k,x}$ is continuous. Furthermore, there exists a constant $\tau(n,k)$ such that*

$$\sup\{|(p_{2,x}, \dots, p_{k,x})| : x \in \Sigma^{k\text{-th}} \cap B_{1/2}\} \leq \tau(n,k). \tag{5-1}$$

Proof. We first prove the bound on $\tau(n,k)$. Since $x\in\Sigma^{k\text{-th}}$ implies $x\in\Sigma^{j\text{-th}}$ for $j\le k$, we proceed by induction on k. The inductive step follows from Proposition 5.3 applied to the functions $u(x+\cdot)$ and $q=0$, which allows us to deduce that $p_{k+1,x}$ is bounded in terms of $n,\ k$ and $\tau(n,k-1)$. We can take as the base step $k=2$, for which $|(p_{2,x})|\le\frac{1}{2}$.

Let us prove continuity at 0. Again, we proceed inductively and suppose that the statement is true for $k-1$ (see [Figalli and Serra 2019] for $k-1=2$). Let $(x_\ell)_{\ell\in\mathbb{N}}\subseteq\Sigma^{k\text{-th}}\cap B_{1/2}$ with $x_\ell\to 0$, and choose a sequence of rotations $(R_\ell)_{\ell\in\mathbb{N}}\subseteq \mathrm{SO}(n)$ mapping $\{p_{2,x_\ell}=0\}$ to $\{p_{2,0}=0\}$ for each ℓ and satisfying $R_\ell\to\mathrm{id}$. We apply Proposition 5.3 to the functions $u(x_\ell+\cdot)$ and the polynomials $q_\ell:=p_{k,0}\circ R_\ell$, and since

$$u(x_\ell+y)-\mathcal{P}_{k,x_\ell}(p_{2,x_\ell},\dots,p_{k-1,x_\ell},q)=u(x_\ell+y)-\mathcal{P}_{k-1,x_\ell}(y)+q_\ell(y)+O(|y|^{k+1}),$$

we find that the function

$$r\mapsto\int_{\partial B_1}\left|\frac{u(x_\ell+r\,\cdot)-\mathcal{P}_{k-1,x_\ell}(r\,\cdot)}{r^k}-p_{k,0}\circ R_\ell+\frac{O(|rx|^{k+1})}{r^k}\right|^2d\sigma+Cr^\varepsilon \tag{5-2}$$

is increasing in $(0,r_0)$ for all $\ell\in\mathbb{N}$, for some r_0 and C uniform in ℓ. Using this information for the constant sequence $x_\ell=0$, we find that, for any $\delta>0$, there is $r_\delta<\min\{r_0,\delta\}$ such that

$$\int_{\partial B_1}\left|\frac{u(r_\delta\,\cdot)-\mathcal{P}_{k-1,0}(r_\delta\,\cdot)}{r_\delta^k}-p_{k,0}+\frac{O(|r_\delta x|^{k+1})}{r_\delta^k}\right|^2d\sigma\le\delta. \tag{5-3}$$

Using (5-2) we estimate, for each ℓ,

$$\begin{aligned}\int_{\partial B_1}|p_{k,x_\ell}-p_{k,0}\circ R_\ell|^2&=\lim_{r\downarrow 0}\int_{\partial B_1}\left|\frac{u(x_\ell+r\,\cdot)-\mathcal{P}_{k-1,x_\ell}(r\,\cdot)}{r^k}-p_{k,0}\circ R_\ell\right|^2d\sigma\\&\le\int_{\partial B_1}\left|\frac{u(x_\ell+r_\delta\,\cdot)-\mathcal{P}_{k-1,x_\ell}(r_\delta\,\cdot)}{r_\delta^k}-p_{k,0}\circ R_\ell+O(r_\delta)\right|^2d\sigma+Cr_\delta^\varepsilon.\end{aligned}$$

As $\mathcal{P}_{k-1,x_\ell}\to\mathcal{P}_{k-1,0}$ by inductive assumption, taking the upper limit in ℓ on both sides and using (5-3), we find

$$\limsup_\ell\int_{\partial B_1}|p_{k,x_\ell}-p_{k,0}|^2\le\delta,$$

and letting $\delta\downarrow 0$ we conclude. □

The following definition is useful to quantify the rate at which $\mathcal{P}_k$ approximates u.

Definition 5.5 (frequency). For $u:B_1\to[0,\infty)$ a solution to (2-1) and $k\ge 2$, define the k-th frequency $\lambda_k:\Sigma^{k\text{-th}}\to[k,k+2]$ by

$$\lambda_k(x):=\sup\{\phi^\gamma(0^+,u(x+\cdot)-\mathcal{P}_{k,x}):\gamma\in[k,k+2)\}.$$

At $x=0$, we write $\lambda_k:=\lambda_k(0)$.

We comment that in the definition above we indeed have $\lambda_k(\Sigma^{k\text{-th}})\subseteq[k,k+2]$. First, the fact that $\phi^\gamma(0^+,u-\mathcal{P}_k)\ge k$ holds for every $\gamma\in[k,k+2)$ was observed in Proposition 5.3. Second, we always have $\phi^\gamma(0^+,u-\mathcal{P}_k)\le\gamma$, as observed in Proposition 4.2. The following lemma shows that indeed ϕ^γ is a truncation of the frequency, that is $\phi^\gamma(0^+,u-\mathcal{P}_k)=\min\{\lambda_k,\gamma\}$.

Lemma 5.6. *For $u : B_1 \to [0, \infty)$ a solution to (2-1) and $k \geq 2$, consider $x_\circ \in \Sigma^{k\text{-}th}$. Then, for all $\gamma \in (\lambda_k, k+2)$,*

$$\lambda_k(x_\circ) = \phi^\gamma(0^+, u(x_\circ + \cdot\,) - \mathcal{P}_{k,x_\circ}) = \lim_{r\downarrow 0} \phi(r, u(x_\circ + \cdot\,) - \mathcal{P}_{k,x_\circ}),$$

where $\phi(r, v) := D(r, v)/H(r, v)$ is the (nontruncated) Almgren frequency function.

Proof. For simplicity, let $x_\circ = 0$ and set $v := u - \mathcal{P}_k$. By Lemma 2.2, for each $\delta > 0$, there is a constant c_δ and a radius r_δ such that $C_\delta r^{2\lambda_k+\delta} \ll H(r, u - \mathcal{P}_k) + r^{2\gamma}$ for every $0 < r < r_\delta$. Hence, after picking $0 < \delta < \frac{1}{10}(\gamma - \lambda_k)$, we find

$$\phi^\gamma(0^+, v) = \lim_{r\downarrow 0} \frac{\phi(r, v) + o(1)}{1 + o(1)} = \lim_{r\downarrow 0} \phi(r, v) =: \tilde{\lambda},$$

where $\tilde{\lambda}$ does not depend on the choice of $\gamma \in (\lambda_k, k+2)$. On the one hand, $\tilde{\lambda} \leq \gamma$ holds for any such γ, implying $\tilde{\lambda} \leq \lambda_k$; see Proposition 4.2. On the other hand, $\lambda_k \geq \phi^\gamma(0^+, v) = \tilde{\lambda}$, by definition. □

We now give a more flexible characterization of $\Sigma^{k\text{-th}}$.

Lemma 5.7. *For every solution $u : B_1 \to [0, \infty)$ to (2-1) and $k \geq 2$, we have*

$$\Sigma^{k\text{-}th} \equiv \widetilde{\Sigma}^{k\text{-}th} := \Big\{x \in \Sigma^{(k-1)\text{-}th} : \exists (q_2, \dots, q_k) \in \boldsymbol{P}_k,\ \exists r_\ell \downarrow 0 \text{ such that } r_\ell^{-k}(u - \mathcal{P}_{k-1}(q_2, \dots, q_{k-1}))_{r_\ell} \rightharpoonup q_k \text{ in } W^{1,2}_{\mathrm{loc}}(\mathbb{R}^n)\Big\}.$$

Proof. We just need to show that $\widetilde{\Sigma}^{k\text{-th}} \subseteq \Sigma^{k\text{-th}}$, because the other inclusion follows by definition. Let $0 \in \widetilde{\Sigma}^{k\text{-th}}$. We know that

$$r_\ell^{-k}(u - \mathcal{P}_{k-1}(q_2, \dots, q_{k-1}))_{r_\ell} \rightharpoonup q_k \quad \text{in } W^{1,2}_{\mathrm{loc}}(\mathbb{R}^n)$$

for a certain $(q_2, \dots, q_k) \in \boldsymbol{P}_k$ and a certain sequence $r_\ell \downarrow 0$.

We first show that necessarily $q_j = p_{j,0}$ for all $2 \leq j \leq k-1$. To prove this we reason inductively and exploit the fact that $0 \in \Sigma^{j\text{-th}}$; in particular, we have the uniform convergence

$$\lim_{r\to 0} r^{-j}(u - \mathcal{P}_j(p_{2,0}, \dots, p_{j,0}))_r = 0.$$

Suppose $(p_{2,0}, \dots, p_{j-1,0}) = (q_2, \dots, q_{j-1})$ holds for some $j \geq 2$. By Proposition 3.3 (ii), we have

$$\begin{aligned}
0 &= \lim_\ell \frac{(u - \mathcal{P}_k(q_2, \dots, q_j))_{r_\ell}}{r_\ell^j} \\
&= \lim_\ell \frac{(u - \mathcal{P}_j(p_{2,0}, \dots, p_{j,0}))_{r_\ell}}{r_\ell^j} + \frac{(\mathcal{P}_j(p_{2,0}, \dots, p_{j,0}) - \mathcal{P}_k(q_2, \dots, q_k))_{r_\ell}}{r_\ell^j} \\
&= 0 + \lim_\ell \frac{(\mathcal{P}_{j-1}(p_{2,0}, \dots, p_{j-1,0}) + p_{j,0} - \mathcal{P}_{j-1}(q_2, \dots, q_{j-1}) - q_j + O(|x|^{j+1}))_{r_\ell}}{r_\ell^j} \\
&= p_{j,0} - q_j,
\end{aligned}$$

in $L^2(\partial B_1)$. This completes the inductive step. The same computation gives also the base step $q_2 = p_{2,0}$.

Now let $\mathcal{P}_k = \mathcal{P}_k(p_2, \dots, p_{k-1}, q_k)$. Then $\mathcal{P}_k = \mathcal{P}_{k-1} + q_k + P$ for some $(k+1)$-homogeneous harmonic polynomial P (see (ii) in Proposition 3.3). We set $v = u - \mathcal{P}_k$ and notice that by assumption $r_\ell^{-k} v_{r_\ell} \rightharpoonup 0$

in $W^{1,2}_{\mathrm{loc}}(\mathbb{R}^n)$. We will show that this convergence is strong, locally uniform and happens along the full range $r \downarrow 0$, which in turn implies that $0 \in \Sigma^{k\text{-th}}$.

Take r_0 as in Proposition 4.2 and fix some $\gamma \in (k+1, k+2)$. As in (ii) in Remark 5.2, we obtain $H(r_\ell, v) = o(r_\ell^{2k})$, which as in Proposition 5.3 implies $\lambda := \lim_{r\to 0} \phi^\gamma(r, v) \geq k$. Since by Proposition 4.2 $\phi^\gamma(\,\cdot\,, v)$ is bounded by C_γ in $(0, r_0)$, we have

$$r^{-2k} \int_{B_R} |\nabla v_r|^2 \leq r^{-2k} \phi^\gamma(Rr, v)(H(Rr, v) + (Rr)^{2\gamma}) \leq C_\gamma r^{-2k}(H(Rr, v) + (Rr)^{2\gamma}),$$

provided $Rr < r_0$. We now exploit that the logarithm of the right-hand side is almost monotone in r thanks to Corollary 4.3 and get

$$\begin{aligned}\limsup_{r\downarrow 0} \log\Bigl(r^{-2k} \int_{B_R} |\nabla v_r|^2\Bigr) &\leq \log C_\gamma + \lim_{s\downarrow 0} \log(s^{-k}(H(s, v) + s^{2\gamma})) \\ &= \log C_\gamma + \lim_{\ell\to\infty} \log(r_\ell^{-k}(H(r_\ell, v) + r_\ell^{2\gamma})) = -\infty,\end{aligned}$$

thus $\lim_{r\downarrow 0} \|r^{-k}\nabla v_k\|_{L^2(B_R)} = 0$ for all fixed $R > 0$. The proof of local uniform convergence is very similar: namely, using Lemma 3.5 and then Lemma 2.2, we have

$$\|v_r\|_{L^\infty(B_R)} \leq C\|v_{Rr}\|_{L^2(B_2\setminus B_{1/2})} + C(Rr)^{k+2} \leq CH(Rr, v)^{1/2} + C(Rr)^{k+2},$$

provided Rr is small, thus we can divide by r^k and argue as before exploiting the log-monotonicity. □

With the same kind of reasoning we can prove the following basic lemma.

Lemma 5.8. *Let $u : B_1 \to [0, \infty)$ be a solution to (2-1) and $k \geq 2$, and suppose $0 \in \Sigma^{k\text{-th}}$ with $\lambda_k > k+1$. Then $0 \in \Sigma^{(k+1)\text{-th}}$ and $p_{k+1} = 0$.*

Proof. Set $v := u - \mathcal{P}_k$, and pick any $\gamma \in (\lambda_k, k+2)$, so that $\phi^\gamma(0^+, v) > k+1$. Arguing as in the proof of Lemma 5.7, we find

$$r^{-2(k+1)} \int_{B_R} |\nabla v_r|^2 \lesssim r^{-2(k+1)}(H(Rr, v) + (Rr)^{2\gamma}),$$

provided $Rr < r_0 \ll 1$. On the other hand, as $\phi^\gamma(0^+, v) > k+1$, we have $\phi^\gamma(r, u - \mathcal{P}_k) > k+1$ for $r \ll 1$. Thus, with Lemma 2.2 we deduce $H(r, u - \mathcal{P}_k) = o(r^{2(k+1)})$. Taking the above estimate into account, we conclude $r^{-2(k+1)} v_r \to 0$ in $W^{1,2}_{\mathrm{loc}}(\mathbb{R}^n)$, thus by Lemma 5.7, we have $0 \in \Sigma^{(k+1)\text{-th}}$. □

Finally, we study the blowups of $u - \mathcal{P}_k$ when Lemma 5.8 does not apply. Our argument is an adaptation of [Figalli and Serra 2019, Proposition 2.10]. We will study the sequence of functions $\tilde{v}_r := H(r, u - \mathcal{P}_k)^{-1/2}(u - \mathcal{P}_k)(r\,\cdot\,)$ as $r \downarrow 0$. Any limit of $\tilde{v}_r$ will be a λ_k-homogeneous solution of a certain PDE (the Signorini problem (5-4)) but not necessarily a polynomial.

Proposition 5.9. *Let $0 \in \Sigma^{k\text{-th}}$ with $\lambda_k \leq k+1$. Let $(r_\ell)_{\ell\in\mathbb{N}}$ be an infinitesimal sequence, and let $x_\ell \in \Sigma^{k\text{-th}} \cap B_{r_\ell}$. For every ℓ, set $v_{x_\ell} := u(x_\ell + \cdot\,) - \mathcal{P}_{k,x_\ell}$, and suppose that $\lambda_k(x_\ell) \to \lambda_k$. Consider the sequence*

$$\tilde{v}_{r_\ell, x_\ell} := \frac{v_{x_\ell}(r_\ell\,\cdot\,)}{H(r_\ell, v_{x_\ell})^{1/2}}.$$

Then:

(i) $(\tilde{v}_{r_\ell,x_\ell})_{\ell\in\mathbb{N}}$ *is bounded in* $W^{1,2}_{\mathrm{loc}}(\mathbb{R}^n)$ *and* $C^{0,1/(n+1)}_{\mathrm{loc}}(\mathbb{R}^n)$.

(ii) *If* $\tilde{v}_{r_\ell,x_\ell} \rightharpoonup q \in W^{1,2}_{\mathrm{loc}}(\mathbb{R}^n)$, *then the convergence is in fact strong and* q *must be a nontrivial* λ_k-*homogeneous solution of the Signorini problem with obstacle* $\{p_2 = 0\}$, *that is*

$$\begin{cases} \Delta q \le 0 \text{ and } q\Delta q = 0 & \text{in } \mathbb{R}^n, \\ \Delta q = 0 & \text{in } \mathbb{R}^n \setminus \{p_2 = 0\}, \\ q \ge 0 & \text{on } \{p_2 = 0\}. \end{cases} \tag{5-4}$$

Finally, *if* $\lambda_k < k+1$, *then* q *is even with respect to the thin obstacle.*

Proof. For the sake of readability, we set $v_\ell := v_{x_\ell}$ and $\tilde{v}_\ell := \tilde{v}_{r_\ell,x_\ell}$. Furthermore, we will omit the dependence of the constants on n and k, and set $\delta := \frac{1}{100}\varepsilon$, where $\varepsilon(n,k)$ is the same as in Proposition 5.3.

Without loss of generality, assume ℓ is large enough that $x_\ell \in B_{1/2}$ and $\lambda_k(x_\ell) \le \lambda_k + \delta$. Within this proof we fix $\gamma := k+1+\frac{3}{4}$, so by Lemma 5.6 we have that $\lambda_k(x_\ell) = \phi^\gamma(0^+, v_\ell)$ for all ℓ.

Step 1. We claim that there are $\varepsilon, r_0 \in \left(0, \frac{1}{2}\right)$ and $C_0, c_0 > 0$, all independent of ℓ and ℓ_0, such that if we define

$$f_{x_\ell}(r) := \phi^\gamma(r, v_\ell) + C_0 r^\varepsilon \quad \text{and} \quad h_{x_\ell}(r) := r^{-2k} H(r, v_\ell) + C_0 r^\varepsilon,$$

then:

- f_{x_ℓ} and h_{x_ℓ} are continuous and increasing on $[0, r_0]$ and converge uniformly in this interval to f_0 and h_0, respectively. Furthermore, $h_{x_\ell}(0^+) = 0$ identically.
- We have $f_{x_\ell}(r) \le \lambda_k + 2\delta$ and $H(r, v_\ell) \ge c_0 r^{2\lambda_k + 5\delta}$ for all $r \in [0, r_0]$ and all $\ell > \ell_0$.

The fact that, for each ℓ, both f_{x_ℓ} and h_{x_ℓ} are increasing is a consequence of Proposition 4.2 and Proposition 5.3, respectively. By Proposition 5.4 we can choose $\tau := \tau(n,k)$ such that r_0 and C_0 can be taken uniform in ℓ. Since $x_\ell \in \Sigma^{k\text{-th}}$, we already observed in Remark 5.2 that $h_{x_\ell}(0^+) = 0$. Furthermore, by assumption, $f_{x_\ell}(0^+) = \lambda_k(x_\ell) \to \lambda_k = f_0(0^+)$, thus $f_{x_\ell} \to f_0$ and $h_{x_\ell} \to h_0$ pointwise. As they are monotone and the limit functions are continuous, the convergence must be uniform, and thus (a) is proved. We turn to (b): possibly taking a smaller r_0, we have that $f_0 \le \lambda_k + \delta$ in $[0, r_0]$, and thus by uniform convergence there is ℓ_0 such that $f_{x_\ell} \le \lambda_k + 2\delta$. Now the last statement follows if we apply Lemma 2.2 with $w = v_\ell,\ R = r_0,\ \bar{\lambda} = \lambda_k + \delta,\ \delta = \frac{1}{100}\varepsilon$.

We will use many times through the proof that

$$c_0 r^{2\gamma} \le H(r, v_\ell) r^{1/2} \quad \text{in } [0, r_0], \tag{5-5}$$

uniformly in $\ell > \ell_0$. This is a direct consequence of (b) and the fact that $2\gamma - \frac{1}{2} > 2\lambda_k + 5\delta$.

Step 2. We prove (i). Fix some $R > 1$ and for $Rr_\ell < r_0$ write

$$D(R, \tilde{v}_\ell) \le \phi^\gamma(Rr_\ell, v_\ell) \frac{H(Rr_\ell, v_\ell) + (Rr_\ell)^{2\gamma}}{H(r_\ell, v_\ell)} \le C_R \frac{H(Rr_\ell, v_\ell)}{H(r_\ell, v_\ell)} + C_R \frac{r_\ell^{2\gamma}}{H(r_\ell, v_\ell)} \le C_R + o(1),$$

where we used (5-5) and Lemma 2.2 to find the second and the first addendum, respectively. Thus $\|\nabla \tilde{v}_\ell\|_{L^2(B_R)}$ is bounded in ℓ.

Now we want to combine the uniform L^2 bound on $\nabla\tilde{v}_\ell$ and the Lipschitz estimate on $\nabla_{x'}\tilde{v}_r$ to produce uniform Hölder bounds. Fix some ℓ and choose coordinates so that $p_{2,x_\ell} = \frac{1}{2}x_n^2$. By Proposition 3.4, we have, for $2\theta R r_\ell < r_0$ and $j \neq n$,

$$\|\partial_j \tilde{v}_\ell\|_{L^\infty(B_R)} \leq C\|\tilde{v}_\ell\|_{L^2(B_{2\theta R}\setminus B_{\theta R/2})} + C\frac{r^{k+2}}{H(r_\ell, v_\ell)^{1/2}}.$$

The second term is $o(1)$ in ℓ again by (5-5), while for the first we employ again Lemma 2.2 to find

$$\|v_\ell(r_\ell\,\cdot\,)\|^2_{L^2(B_{2\theta R}\setminus B_{\theta R/2})} = \int_{\theta R/2}^{2\theta R} H(sr_\ell, v_\ell)s^{n-1}\,ds \leq C_R H(r_\ell, v_\ell). \tag{5-6}$$

Hence, $\|\partial_j \tilde{v}_\ell\|_{L^\infty(B_R)} \leq C_R$ for all $j \neq n$. By Lemma C.1, this gives the Hölder bound

$$[\tilde{v}_\ell]_{C^{0,1/(n+1)}(B_{R/2})} \leq C_R(\|\nabla_{x'}\tilde{v}_\ell\|_{L^\infty(B_R)} + \|\nabla\tilde{v}_\ell\|_{L^2(B_R)}) \leq C_R.$$

This concludes the proof of (i).

Step 3. We turn to (ii) and prove that q solves (5-4). Since $\Delta v_\ell = -\chi_{\{u(x_\ell+\,\cdot\,)=0\}} \leq 0$, we have that $\Delta q \leq 0$ weakly in $\mathbb{R}^n$. Furthermore, integrating by parts with some cut-off function $\chi_{B_R} \leq \psi \leq \chi_{B_{2R}}$ leads to

$$\int_{B_R} |\Delta\tilde{v}_\ell| \leq -\int_{\mathbb{R}^n} \Delta\tilde{v}_\ell\psi \leq C_R\|\tilde{v}_\ell\|_{L^2(B_{2R}\setminus B_R)} \leq C_R,$$

where in the last step we argued as in (5-6). Hence by compactness $\Delta\tilde{v}_\ell \overset{*}{\rightharpoonup} \Delta q$ in $C_c(\mathbb{R}^n)^*$. On the other hand, by (i), $\tilde{v}_\ell \to q$ locally uniformly, and so

$$\tilde{v}_\ell\Delta\tilde{v}_\ell \overset{*}{\rightharpoonup} q\Delta q \quad \text{in } C_c(\mathbb{R}^n)^*.$$

We now apply Proposition 4.2 to v_ℓ with our particular choice of γ and recall that by (4-3) we have, for $Rr_\ell < r_0$,

$$\int_{B_R} |\tilde{v}_\ell\Delta\tilde{v}_\ell| \frac{H(r_\ell, v_\ell)}{H(Rr_\ell, v_\ell) + (Rr_\ell)^{2\gamma}} \leq C_R r_\ell^\varepsilon = o(1).$$

Notice that the constants are independent of ℓ as, by Proposition 5.4, we can choose a uniform $\tau = \tau(n,k)$ for all $x_\ell \in \Sigma^{k\text{-th}} \cap B_{1/2}$. Sending $\ell \uparrow \infty$, we get $q\Delta q = 0$. In order to show $\Delta q = 0$ outside $\{p_{2,0} = 0\}$, we exploit once again Lemma 2.1 to find

$$B_{R/2} \cap \operatorname{supp}(\Delta\tilde{v}_\ell) \subseteq \{\operatorname{dist}(\{p_{2,x_\ell} = 0\}, \cdot\,) \leq C_R r_\ell^{\alpha_\circ}\},$$

and as $p_{2,x_\ell} \to p_{2,0}$ and R can be taken arbitrarily large, we deduce that $\operatorname{supp}\Delta q \subseteq \{p_{2,0} = 0\}$. It remains to show that q is nonnegative on the thin obstacle. Up to a rotation we can assume $p_{2,0} = \frac{1}{2}x_n^2$. Pick $x_* \in \{x_n = 0\}$ and consider some sequence $(y_\ell)_{\ell\in\mathbb{N}}$ such that

$$y_\ell \in \{\mathcal{A}_{k,x_\ell}(r_\ell\,\cdot\,) = 0\}, \qquad y_\ell \to x_*.$$

Thus, by locally uniform convergence and (5-5),

$$q(x_*) = \lim_\ell \tilde{v}_{r_\ell}(y_\ell) = \lim_\ell \frac{u(r_\ell y_\ell) - \frac{1}{2}\mathcal{A}^2_{k,x_\ell}(r_\ell y_\ell) + O(r_\ell^{k+2})}{H(r_\ell, v_\ell)^{1/2}} \geq 0.$$

To construct such a sequence set $y_\ell := \Phi_\ell(x_*)$, where $\Phi_\ell \in C^\infty(B_{R_\ell})$ are the inverse functions of $\Psi_\ell : x \mapsto (x', r_\ell^{-1}(\mathcal{A}_{k,x_\ell})_{r_\ell})$. Notice that $\Psi_\ell \to \mathrm{id}$ in C^1_{loc} and that $R_\ell \uparrow +\infty$ as $\ell \to \infty$. So $\Phi_\ell \to \mathrm{id}$ and $x_* \in B_{R_\ell}$ eventually, thus $y_\ell \to x_*$. Therefore

$$q \geq 0 \quad \text{on} \quad \{x_n = 0\}.$$

Hence we proved that q is a global solution of the Signorini problem (5-4).

Step 4. We show that $\tilde{v}_\ell \to q$ in $W^{1,2}_{\mathrm{loc}}(\mathbb{R}^n)$ and that q is λ_k-homogeneous. Fix any $\eta \in C^\infty_c(\mathbb{R}^n)$ and exploit as before that $\|\tilde{v}_{r_\ell} \Delta \tilde{v}_{r_\ell}\|_{L^1(B_R)} \to 0$ and integrate by parts in $\mathbb{R}^n$:

$$\begin{aligned} \int |\nabla(\eta\tilde{v}_\ell)|^2 &= -\int \eta\tilde{v}_\ell \Delta(\eta\tilde{v}_\ell) = -\int (\eta\tilde{v}_\ell^2 \Delta\eta + 2\eta\tilde{v}_\ell \nabla\eta \cdot \nabla\tilde{v}_\ell + \eta^2\tilde{v}_\ell\Delta\tilde{v}_\ell) \\ &\leq -\int (\eta\Delta\eta\tilde{v}_\ell^2 + 2\eta\tilde{v}_\ell\nabla\eta\cdot\nabla\tilde{v}_\ell) + C(\eta)\|\tilde{v}_\ell\Delta\tilde{v}_\ell\|_{L^1(B_R)}. \end{aligned}$$

Taking the upper limit and using $\nabla\tilde{v}_\ell \rightharpoonup \nabla q$ in $L^2_{\mathrm{loc}}(\mathbb{R}^n)$ and $\tilde{v}_\ell \to q$ in $C^0_{\mathrm{loc}}(\mathbb{R}^n)$, we get

$$\limsup_\ell \int |\nabla(\eta\tilde{v}_\ell)|^2 \leq -\int (\eta\Delta\eta q^2 + 2\eta q \nabla\eta\cdot\nabla q + \eta^2 q\Delta q) = \int |\nabla(\eta q)|^2,$$

where we used $q\Delta q = 0$. By weak lower semicontinuity we always have the converse inequality, thus $\nabla(\eta\tilde{v}_\ell) \to \nabla(\eta q)$ strongly in $L^2(\mathbb{R}^n)$. This in particular gives, for every $R > 0$,

$$\phi(R, q) = \lim_\ell \phi(R, \tilde{v}_\ell) = \lim_\ell \phi(Rr_\ell, v_\ell) = \lim_\ell \phi^\gamma(Rr_\ell, v_\ell),$$

where in the last line we used (5-5). On the other hand, (a) in Step 1 implies

$$\lim_\ell \phi^\gamma(Rr_\ell, v_\ell) = \lim_\ell f_{x_\ell}(Rr_\ell) = f_0(0^+) = \lambda_k,$$

thus $\phi(R, q) \equiv \lambda_k$ for all $R > 0$. As a standard consequence, we have that q is λ_k-homogeneous; see [Athanasopoulos et al. 2008].

Step 5. We finally prove that, for $\lambda_k < k+1$, we have $q^{\mathrm{odd}} = 0$. Notice that by Proposition 2.4 q^{odd} is harmonic and thus has integral homogeneity; hence the only nontrivial case is when $\lambda_k = k$. We need to show that q is orthogonal in $L^2(\partial B_1)$ to every k-homogeneous harmonic polynomial P vanishing on $\{p_{2,0} = 0\}$. Fix such a P and apply Proposition 5.3 with

$$w_\ell := u(x_\ell + \cdot\,) - \mathcal{P}_k(p_{2,x_\ell}, \dots, p_{k-1,x_\ell}, p_{k,x_\ell} - P \circ R_\ell),$$

where R_ℓ are rotations sending $\{p_{2,x_\ell=0}\}$ to $\{p_{2,0} = 0\}$ and $R_\ell \to \mathrm{id}$. Thus, with constants uniform in ℓ (P is fixed),

$$\begin{aligned} r^{-2k}H(r, w_\ell) + Cr^\varepsilon &= Cr^\varepsilon + \int_{\partial B_1} \left(\frac{v_\ell(r\,\cdot)}{r^k} + P\circ R_\ell + O\left(\frac{|x|^{k+1}}{r^k}\right)\right)^2 \\ &\geq \lim_{r\to 0} r^{-2k}H(r, w_\ell) + Cr^\varepsilon = \int_{\partial B_1} P^2. \end{aligned}$$

Now divide by the sequence $\varepsilon_\ell := (H(r_\ell, v_\ell) r_\ell^{-2k})^{1/2}$, which by Step 1 (b) satisfies (recall $\lambda_k = k$)

$$r_\ell^{5\delta} \le \varepsilon_\ell^2 \le h_{x_\ell}(r_\ell) = o(1).$$

We compute the squares and rearrange the terms to get

$$\int_{\partial B_1} \tilde{v}_\ell^2 \varepsilon_\ell + 2 \int_{\partial B_1} \tilde{v}_\ell P \circ R_\ell \ge -C \frac{r_\ell^\varepsilon}{\varepsilon_\ell} \ge -C r_\ell^{\varepsilon - 5\delta/2}.$$

Since $\delta \le \frac{1}{100}\varepsilon$, we can send $\ell \to \infty$ and get

$$\int_{\partial B_1} q P \ge 0.$$

The conclusion follows by linearity in P. □

Remark 5.10. An important application of Proposition 5.9 is when the sequence x_ℓ is identically equal to 0.

Remark 5.11. Step 1 of the proof shows that the function $\Sigma^{k\text{-th}} \ni x \mapsto \phi^\gamma(u(x + \cdot) - \mathcal{P}_{k,x}, 0^+)$ is upper semicontinuous. In fact, with the same notations we have, for each $r < r_0$,

$$\limsup_\ell \phi^\gamma(u(x_\ell + \cdot) - \mathcal{P}_{k,x_\ell}, 0^+) \le \limsup_\ell \phi^\gamma(u(x_\ell + \cdot) - \mathcal{P}_{k,x_\ell}, r) + C r^\varepsilon = \phi^\gamma(u - \mathcal{P}_k, r) + C r^\varepsilon,$$

and the conclusion follows letting $r \downarrow 0$.

Proposition 5.9 shows that in order to pursue our analysis further we need to have some basic knowledge about homogeneous solutions of the Signorini Problem (5-4). In the next chapter we will use extensively the results reported in Section 2.4.

6. Estimating the size of the sets $\Sigma^{k\text{-th}} \setminus \Sigma^{(k+1)\text{-th}}$

Throughout this section u will be a solution of (2-1) with $f \equiv 1$ and $\mu = 1$. We will show that, for all $k \ge 2$,

$$\dim_{\mathcal{H}}(\Sigma^{k\text{-th}} \setminus \Sigma^{(k+1)\text{-th}}) \text{ is less than or equal to } n-2 \text{ and is countable if } n = 2.$$

In the last subsection we will show how this constrains the geometry of Σ. We remark that, by Caffarelli's analysis, $\Sigma \setminus \Sigma^{2\text{nd}}$ has locally finite $\mathcal{H}^{n-2}$ measure (see, e.g., [Caffarelli 1998, Theorem 8 (c)]).

In this chapter we repeatedly use the facts and notation concerning the Signorini problem recalled and/or established in Section 2.4. In particular, we will use $\mathcal{S}_k$, $\mathcal{S}^{\text{even}}$, q^{even}, q^{odd}, $\Sigma(q), \dots$.

We need to understand the nature of points in $\Sigma^{k\text{-th}} \setminus \Sigma^{(k+1)\text{-th}}$. Therefore, suppose $0 \in \Sigma^{k\text{-th}}$ and $0 \notin \Sigma^{(k+1)\text{-th}}$. We necessarily have $\lambda_k \le k+1$; see Lemma 5.8. Notice that, with the notation of Proposition 5.9 and Lemma 2.2, we have

$$\frac{(u - \mathcal{P}_k)(r\,\cdot)}{r^{k+1}} = \frac{H(r, u - \mathcal{P}_k)^{1/2}}{r^{k+1}} \frac{(u - \mathcal{P}_k)(r\,\cdot)}{H(r, u - \mathcal{P}_k)^{1/2}} = O(r^{\lambda_k - (k+1)}) \tilde{v}_{r,0}.$$

As every accumulation point of $\tilde{v}_{r,0}$ equals some nonzero $q \in \mathcal{S}_{k+1}(\{p_2 = 0\})$ (see Proposition 5.9), in order to conclude $0 \notin \Sigma^{(k+1)\text{-th}}$ (see the flexible definition of this set from Lemma 5.7), either we must have $\lambda_k < k+1$ or $\lambda_k = k+1$ and every accumulation point q satisfies $q^{\text{even}} \not\equiv 0$.

These observations inspire the following trichotomy. If $x \in \Sigma^{k\text{-th}} \setminus \Sigma^{(k+1)\text{-th}}$ then exactly one of the following happens:

(1) $\lambda_k(x) = k$,

(2) $\lambda_k(x) \in (k, k+1)$,

(3) $\lambda_k(x) = k+1$, but every accumulation point of $r^{-(k+1)}(u(x+\cdot) - \mathcal{P}_{k,x})(r\,\cdot)$ has a nonzero even part.

We rephrase these cases with a notation closer to that adopted in [Figalli and Serra 2019; Figalli et al. 2020]. Namely, for each $k \geq 2$, define

$$\Sigma^{>k} := \Sigma^{k\text{-th}} \cap \{\lambda_k > k\}, \qquad \Sigma^{\geq k+1} := \Sigma^{k\text{-th}} \cap \{\lambda_k \geq k+1\}.$$

So we have the descending chain of inclusions

$$\Sigma_{n-1} = \Sigma^{2\text{nd}} = \Sigma^{>2} \supseteq \cdots \supseteq \Sigma^{k\text{-th}} \supseteq \Sigma^{>k} \supseteq \Sigma^{\geq k+1} \supseteq \Sigma^{(k+1)\text{-th}} \supseteq \cdots \supseteq \bigcap_{j \geq 2} \Sigma^{j\text{-th}} =: \Sigma^{\infty}.$$

With this notation case (1) corresponds to the set $\Sigma^{k\text{-th}} \setminus \Sigma^{>k}$, case (2) to $\Sigma^{>k} \setminus \Sigma^{\geq k+1}$ and case (3) to $\Sigma^{\geq k+1} \setminus \Sigma^{(k+1)\text{-th}}$. In the next subsections we will address separately each case.

We point out that, in cases (1) and (3), the parity of k will play a role in our arguments. This is related to the different shape of the functions in $\mathcal{S}_k^{\text{even}}$ according to the parity of k (see Proposition 2.4).

6.1. ***The size of*** $\boldsymbol{\Sigma^{k\text{-th}} \setminus \Sigma^{>k}}$**.** We start by showing that when k is even, the set $\Sigma^{k\text{-th}} \setminus \Sigma^{>k}$ is in fact empty. This is a simple consequence of the following monotonicity formula, which is an extension of [Figalli et al. 2020, Lemma 4.14] to higher values of k.

Lemma 6.1. *Let u solve (2-1), and let $0 \in \Sigma^{k\text{-th}}$ for some $k \geq 2$. Let $v := u - \mathcal{P}_k$, and let P be any k-homogeneous harmonic polynomial such that $P \geq 0$ on $\{p_2 = 0\}$. Then there exists $\varepsilon, r_0 > 0$ depending on n and k such that, for all $r \in (0, r_0)$,*

$$\frac{d}{dr}\left(r^{-k} \int_{\partial B_1} v_r P\right) \leq C r^{\varepsilon - 1}$$

for some constant C depending only on n, k, $\|P\|_{L^2(\partial B_1)}$.

Proof. The proof is identical to in [Figalli et al. 2020, Lemma 4.14]; we give it nevertheless for the reader's convenience. Integration by parts and the fact that P is harmonic lead to

$$\frac{d}{dr}\int_{\partial B_1} v_r P = \frac{1}{r}\left(\int_{\partial B_1} v_r \partial_\nu P + \int_{B_1} \Delta v_r P\right) = \frac{1}{r}\left(k \int_{\partial B_1} v_r q + \int_{B_1} \Delta v_r P\right),$$

where we used the homogeneity of P to deduce that $\partial_\nu P = kP$ on ∂B_1. As $\Delta v_r = -r^2 \chi_{\{u_r = 0\}}$, we can rewrite this as

$$\frac{d}{dr}\left(r^{-k} \int_{\partial B_1} v_r P\right) = -\frac{1}{r^{k-1}} \int_{B_1 \cap \{u_r = 0\}} P.$$

We have

$$r^{-k}\|v_r\|_{L^2(B_1)} = \|\tilde{v}_r\|_{L^2(B_1)}(r^{-2k}H(r,v))^{1/2} \le C(n,k)$$

for $r \lesssim 1$ sufficiently small, thanks to Proposition 5.9 (i) and Proposition 5.3. Combining this estimate with the Lipschitz bounds from Proposition 3.4 (ii), with $\beta \in (0, 1/(k+2))$ to be chosen, we find

$$\{u_r = 0\} \cap B_1 \subseteq \{x \in B_1 : r|\partial_n \mathcal{P}_k|(rx) = r^2|x_n + O(|x|^2)| \le Cr^{k(1-\beta)}\},$$

with some constant C depending on n and k only (as we can choose $\tau = \tau(n,k)$ from Proposition 5.4). This shows that $|\{u_r = 0\} \cap B_1| \le Cr^{k(1-\beta)-2}$. On the other hand, by the maximum principle, we have $-P \le C|x_n|$ in B_1. Hence, using Lemma 2.1, we obtain

$$-\int_{B_1 \cap \{u_r=0\}} P \le Cr^{\alpha_\circ}|\{u_r = 0\} \cap B_1| \le Cr^{k+\alpha_\circ-k\beta-2},$$

and the lemma follows choosing $\beta = \alpha_\circ/(2k)$. □

As a simple corollary we get our claim.

Corollary 6.2. *For every even integer $k \ge 2$, we have $\Sigma^{k\text{-th}} \setminus \Sigma^{>k} = \varnothing$.*

Proof. Let us assume, on the contrary, that $0 \in \Sigma^{k\text{-th}} \setminus \Sigma^{>k}$, that is $\lambda_k = k$. Then, by Proposition 5.9, any accumulation point q of $\tilde{v}_r = (u - \mathcal{P}_k)_r / H(r, u - \mathcal{P}_k)^{1/2}$ lies in $\mathcal{S}_k^{\text{even}}(\{p_2 = 0\}) \setminus \{0\}$. Furthermore, by Proposition 2.4, any such q satisfies the assumptions of Lemma 6.1. As $0 \in \Sigma^{k\text{-th}}$, we moreover have $v_r/r^k \to 0$, thus after combining this with Lemma 6.1 with $P = q$, we find

$$r^{-k}\int_{\partial B_1} v_r q \le Cr^{\varepsilon}$$

for small $r \le 1$. Dividing by $H(r, u - \mathcal{P}_k)^{1/2}$ leads to

$$\int_{\partial B_1} \tilde{v}_{r_\ell} q \le C\frac{r_\ell^{k+\varepsilon}}{H(r_\ell, u - \mathcal{P}_k)^{1/2}}.$$

Thanks to (ii) in Remark 5.2, we deduce that the right-hand side vanishes as $\ell \uparrow \infty$, implying $\int_{\partial B_1} q^2 \le 0$ and contradicting $\|q\|_{L^2(\partial B_1)} = 1$. □

Let us now consider an odd k. We point out that, for $k = 3$, we still have $\Sigma^{3\text{rd}} \setminus \Sigma^{\ge 3} = \varnothing$, but the proof is more refined; see [Figalli et al. 2020, Proposition 5.8]. We will instead rely on a more robust argument which will be also employed later to deal with the case $\lambda_k = k+1$ (see Lemma 6.10). The main step is contained in the following lemma, based on a barrier argument.

Lemma 6.3. *Let $k \ge 3$ be odd. For all $x \in \Sigma^{k\text{-th}} \setminus \Sigma^{>k}$ and $\varepsilon > 0$, there is $\varrho = \varrho(\varepsilon, x) > 0$ such that, for each $0 < r < \varrho$, there exists $q \in \mathcal{S}_k^{\text{even}}(\{p_{2,x} = 0\})$ such that*

$$\Sigma(u) \cap B_r(x) \subseteq \Sigma(q) + B_{\varepsilon r}(x). \tag{6-1}$$

Recall that $\Sigma(q) := \{q = |\nabla q| = 0\} \cap \{p_{2,x} = 0\}$ was defined in (2-9).

Proof. Up to an isometry, suppose $x = 0$ and $p_2 = \frac{1}{2}x_n^2$. We argue by contradiction and (rescaling the space) suppose that there are $\varepsilon_\circ > 0$ and $r_\ell \downarrow 0$ such that

$$\Sigma(u_{r_\ell}) \cap \{y \in B_1 : \operatorname{dist}(y, \Sigma(q)) > \varepsilon_\circ\} \neq \varnothing \quad \text{for all } q \in \mathcal{S}_k^{\text{even}}.$$

Thanks to Proposition 5.9, we can extract a subsequence (that we do not rename) such that

$$\frac{(u - \mathcal{P}_k)_{r_\ell}}{H(r_\ell, u - \mathcal{P}_k)^{1/2}} \to \bar{q} \in \mathcal{S}_k^{\text{even}} \setminus \{0\} \quad \text{in } C^0_{\text{loc}}(\mathbb{R}^n). \tag{6-2}$$

Thus, there are $y_\ell \in B_1$ such that

$$y_\ell \in \Sigma(u_{r_\ell}), \quad \operatorname{dist}(y_\ell, \Sigma(\bar{q})) \geq \varepsilon_\circ, \quad y_\ell \to y_\infty \in \{x_n = 0\}.$$

By Proposition 2.4 we can write $\bar{q}(x) = -|x_n|(q_0(x') + x_n^2 q_1(x))$ for some polynomials q_0 and q_1, with $q_0 \geq 0$. For brevity, we set $h_\ell := H(r_\ell, u - \mathcal{P}_k)^{1/2}$ and remark that $r_\ell^{k+\delta} \leq C_0 h_\ell \ll r_\ell^k$ for some constants C_0 and δ (see Remark 5.2).

As $y_\ell \to y_\infty$, in order to reach a contradiction, it suffices to show that, for some small radius $R > 0$ and all ℓ large,

$$\Sigma(u_{r_\ell}) \cap B_R(y_\infty) = \varnothing. \tag{6-3}$$

The rest of the proof is devoted to showing (6-3) for a suitable R independent of ℓ.

We start by choosing a radius ρ as follows. As $y_\infty \in \{x_n = 0\} \setminus \Sigma(\bar{q}) \subseteq \{q_0 > 0\}$, we can find some small $\rho, m \in (0, 1)$ such that

$$q(x) \leq -m|x_n| \quad \text{for } x \in B_{4\rho}(y_\infty), \tag{6-4}$$

and we can also require that $\rho \leq \frac{1}{100}m$.

Let us introduce some notation. Define the set of points that admit a barrier as

$$Z_\ell := \{z \in B_\rho(y_\infty) : \exists\, \phi_{z,\ell} \text{ of class } C^2 \text{ in a neighborhood of } B_\rho(z) \text{ solving (6-5)}\},$$

where

$$\begin{cases} \phi_{z,\ell}(z) = 0, & \\ \phi_{z,\ell} \geq 0 & \text{in } B_\rho(z), \\ \Delta\phi_{z,\ell} < r_\ell^2 & \text{in } \overline{B_\rho(z)}, \\ u(r_\ell\,\cdot) < \phi_{z,\ell} & \text{on } \partial B_\rho(z). \end{cases} \tag{6-5}$$

We also set

$$\Gamma_\ell := \{\mathcal{A}_k(r_\ell\,\cdot) = 0\} \cap B_2.$$

For ℓ large in terms of n, k and τ, we have that Γ_ℓ is a smooth hypersurface inside B_2, which converges to the hyperplane $\{x_n = 0\}$, say, in the C^2 norm. Furthermore, combining Lemma 2.1 and the fact that $\mathcal{A}_k(x) = x_n + O(|x|^2)$, we get

$$(\{u_{r_\ell} = 0\} \cup \Gamma_\ell) \cap B_2 \subseteq \{|x_n| \leq C_1 r_\ell^{\alpha_\circ}\} \cap B_2 \tag{6-6}$$

for some constant $C_1 = C_1(n, k) > 0$.

Claim. There is ℓ_0 such that, for $\ell > \ell_0$, the following hold:

(i) All points in $B_\rho(y_\infty)$ belonging to the hypersurface Γ_ℓ admit a barrier, that is, $\Gamma_\ell \cap B_\rho(y_\infty) \subseteq Z_\ell$.

(ii) The function u vanishes on Z_ℓ. Moreover, Z_ℓ is open and contained in the interior of the contact set $\{u(r_\ell\,\cdot\,) = 0\}$.

(iii) There is a dimensional constant $N(n) > 0$ such that $(\Sigma(u_{r_\ell}) \cap B_{\rho/2N}(y_\infty)) \setminus \Gamma_\ell = \varnothing$.

The combination of these three claims will give (6-3) with $R = \rho/(2N)$. In fact, as there are no singular points in the interior of the singular set, (i) and (ii) give $\Sigma(u_{r_\ell}) \cap B_{\rho/2N}(y_\infty) \cap \Gamma_\ell = \varnothing$. We conclude with (iii). □

Proof of the Claim. We begin by proving (ii). First, for any $z \in Z_\ell$ and any ξ close to z, we can define

$$\phi_{\xi,\ell}(x) := \phi_{z,\ell}(x + (z - \xi)).$$

By continuity of translations, $\phi_{\xi,\ell}$ solves (6-5) on $B_\rho(\xi)$ for $|\xi - z|$ small enough; hence Z_ℓ is open. We now apply the comparison principle, using the last two properties of the barrier in (6-5) to find $u|_{Z_\ell} \equiv 0$. Notice that, for $z \in Z_\ell$, two cases arise: either for all $c > 0$ we have $u(r_\ell\,\cdot\,) < \phi_{z,\ell} + c$ on $B_\rho(z)$, or there exist the largest $c_* > 0$ such that $u(r_\ell x_*) = \phi_{z,\ell}(x_*) + c_*$ for some $x_* \in \overline{B_\rho(z)}$. In the first case, evaluating at z and sending $c \downarrow 0$, we get $u(r_\ell z) = 0$. In the second case, we notice that by (6-5) we must have $x_* \notin \partial B_\rho(z)$, thus we get

$$r_\ell^2 \chi_{\{u(r_\ell\cdot)>0\}}(x_*) = \Delta u_{r_\ell}(x_*) \le \Delta\phi_{z,\ell}(x_*) < r_\ell^2 \quad \Rightarrow \quad x_* \notin \{u(r_\ell\,\cdot\,) > 0\}.$$

Then $0 = u(r_\ell x_*) = \phi_{z,\ell}(x_*) + c_* \ge c_* > 0$, a contradiction.

Next, we turn to the proof of (iii). First recall that there exists a dimensional modulus of continuity (see [Caffarelli 1998, Theorem 8 and Corollary 11]) such that, for all $x \in \Sigma(u)$, we have

$$\|u(x + \cdot\,) - p_{2,x}\|_{L^\infty(B_r)} \le r^2\omega(r).$$

Suppose by contradiction that we can find

$$y_\ell^* \in (\Sigma(u_{r_\ell}) \cap B_{\rho/2N}(y_\infty)) \setminus \Gamma_\ell$$

for arbitrarily large ℓ. We set for brevity $p_\ell := p_{2,r_\ell y_\ell^*}$. As p_ℓ is convex and $\Delta p_\ell \ge 1$, for every such ℓ, we choose a unit vector e_ℓ such that

$$p_\ell(x) \ge \frac{1}{2n}(e_\ell \cdot x)^2.$$

Now by item (ii), we have, for all $z \in B_{\rho/N}(y_\infty) \cap \Gamma_\ell$,

$$\begin{aligned} 0 = u(r_\ell z) = u_{r_\ell}(y_\ell^* + (z - y_\ell^*)) &\ge p_\ell(z - y_\ell^*) - |z - y_\ell^*|^2\omega(|z - y_\ell^*|) \\ &\ge \frac{1}{2n}(e_\ell \cdot (z - y_\ell^*))^2 - |z - y_\ell^*|^2\omega(|z - y_\ell^*|). \end{aligned}$$

From this inequality we reach a contradiction: On one hand we have $|z - y_\ell^*| \le 1/N$, so we can require N to be large enough that $\omega(|z - y_\ell^*|) \le 1/(100n)$. On the other hand we can choose z in such a way that the nonzero vector $z - y_\ell^*$ is almost aligned with e_ℓ, thus we get a contradiction dividing by $|z - y_\ell^*|^2$.

In order to prove (i) we need to construct a barrier for all $z \in B_\rho(y_\infty) \cap \Gamma_\ell$. Set

$$\phi_{z,\ell}(x) := \left(1 - \frac{h_\ell}{r_\ell^2}\right)\frac{1}{2}\mathcal{A}_k^2(r_\ell x) + \frac{h_\ell}{4n}|x' - z'|^2.$$

We have to check that $\phi_{z,\ell}$ indeed satisfies (6-5) for ℓ large. The first two equations in (6-5) are clearly fulfilled for any ℓ. For the third condition, let us compute, for $x \in B_\rho(z)$,

$$\Delta\phi_{z,\ell} = r_\ell^2 - h_\ell + C_2 r_\ell^{k+2} + \frac{2(n-1)}{4n}h_\ell \le r_\ell^2 - \tfrac{1}{2}h_\ell + C_0 C_2 r_\ell h_\ell,$$

where we used, for some $C_2 = C_2(n,k)$, that $\Delta\frac{1}{2}\mathcal{A}_k^2 \le 1 + C_2|x|^k$ and $r_\ell^{k+1} \le C_0 h_\ell$. Hence, $\Delta\phi_{z,\ell} < r_\ell^2$ as soon as ℓ is large enough.

We turn to the last condition of (6-5). For any fixed $\eta \le \rho^2/(100n)$, we have, by uniform convergence of (6-2) for ℓ large,

$$u_{r_\ell} \le \tfrac{1}{2}\mathcal{A}_k^2(r_\ell\,\cdot\,) + h_\ell\bar{q} + h_\ell\eta$$

in B_2. As for some constant $C_3 = C_3(n,k) > 0$, we have, in B_2,

$$\tfrac{1}{2}\mathcal{A}_k^2(x) \le \tfrac{1}{2}x_n^2 + C_3|x|^3,$$

and recalling the choice of ρ from (6-4), we get, for $x \in \overline{B_\rho(z)}$,

$$\begin{aligned} u(r_\ell x) &\le \tfrac{1}{2}\mathcal{A}_k^2(r_\ell x) - h_\ell m|x_n| + h_\ell\eta \\ &\le \left(1 - \frac{h_\ell}{r_\ell^2}\right)\frac{1}{2}\mathcal{A}_k^2(r_\ell x) + \frac{h_\ell}{2}x_n^2 - h_\ell m|x_n| + C_3 h_\ell r_\ell|x|^3 + h_\ell\eta \\ &= \phi_{z,\ell}(x) + h_\ell\left(\frac{1}{2}x_n^2 - m|x_n| + C_3 r_\ell|x|^3 + \eta - \frac{1}{4n}|x' - z'|^2\right). \end{aligned}$$

We show that, whenever $x \in \partial B_\rho(z)$, the term between parentheses is negative. Using (6-6) and the fact that $|x| \le 2$, we get, for all $x \in \partial B_\rho(z)$,

$$\begin{aligned} \frac{1}{2}x_n^2 - m|x_n| + C_3 r_\ell|x|^3 + \eta - \frac{1}{4n}|x' - z'|^2 &= \frac{1}{4n}|x_n - z_n|^2 + \frac{1}{2}x_n^2 - m|x_n| + C_3 r_\ell|x|^3 + \eta - \frac{\rho^2}{4n} \\ &\le x_n^2 - m|x_n| + C_1^2 r_\ell^{2\alpha_\circ} + 8C_3 r_\ell + \eta - \frac{\rho^2}{4n}. \end{aligned}$$

We claim that this quantity is negative as soon as

$$r_\ell^{\alpha_\circ} \le \min\left\{\frac{\rho}{C_1}, \frac{\eta}{8C_3}, \frac{\eta}{C_1^2}, 1\right\}.$$

In fact by (6-6), we have, uniformly in z,

$$|x_n| \le |x_n - z_n| + |z_n| \le \rho + C_1 r_\ell^{\alpha_\circ} \le 2\rho \le \frac{2m}{100},$$

thus $x_n^2 - m|x_n| \le 0$ and

$$C_1 r_\ell^2 + 8C_3 r_\ell + \eta - \frac{\rho^2}{4n} \le 3\eta - \frac{\rho^2}{4n} \le \frac{3\rho^2}{100n} - \frac{\rho^2}{4n} < 0. \qquad \square$$

Exploiting some recent volume estimates for the tubular neighborhood of the critical set of harmonic functions from [Naber and Valtorta 2017], we can now deduce the following.

Lemma 6.4. *Given $\beta_1 > n-2$ and $k \geq 3$ odd, there exists an $\hat{\varepsilon} = \hat{\varepsilon}(n, \beta_1)$ small such that the following holds. Let $E \subseteq \mathbb{R}^n$ be any set satisfying*

$$E \subseteq B_r(x) \cap (\Sigma(q) + B_{\hat{\varepsilon} r}(x))$$

for some $r \in (0, 1)$, $x \in \mathbb{R}^n$ and $q \in \mathcal{S}_k^{\text{even}}(L) \setminus \{0\}$ for some hyperplane L. Then E can be covered with $\lfloor \gamma^{-\beta_1} \rfloor$ balls of radius γr centered at points of E for $\gamma = \frac{1}{5}\hat{\varepsilon}$.

Proof. By translation and scaling we can recover the general case from the case $r = 1$, $x = 0$. Let $\hat{\varepsilon} \in (0, 1)$ be a parameter to be fixed later, and take q as in the statement, recalling that q vanishes on L (see Proposition 2.4). For simplicity we assume $L = \{x_n = 0\}$ and consider Q, the odd (with respect to L) extension of $q|_{\{x_n>0\}}$ to $\mathbb{R}^n$. Q is harmonic, and it is easily checked that $\Sigma(q) \subseteq \{Q = |\nabla Q| = 0\} =: \Sigma(Q)$; hence a fortiori

$$E \subseteq B_1 \cap \{\text{dist}(\cdot\,, \Sigma(Q)) \leq \hat{\varepsilon}\}.$$

As Q is harmonic and nonzero, we can apply the volume estimates in [Naber and Valtorta 2017, Theorem 1.1] to find

$$\mathcal{H}^n(B_2 \cap \{\text{dist}(\cdot\,, \Sigma(Q)) \leq t\}) \leq C(n)t^2$$

for all $t \in (0, 1)$. Now, consider a covering of E of the form $\{B_{\hat{\varepsilon}}(x)\}_{x \in E}$. By Vitali's covering lemma, there exists a disjoint subcollection $\{B_{\hat{\varepsilon}}(x_i)\}_{i \in I}$ such that

$$E \subseteq \bigcup_{x \in E} \overline{B_{\hat{\varepsilon}}(x)} \subseteq \bigcup_{i \in I} B_{5\hat{\varepsilon}}(x_i).$$

We need to estimate the cardinality of I. Denoting by ω_n the volume of the unit ball in $\mathbb{R}^n$ and using that $B_{\hat{\varepsilon}}(x_i) \subseteq (E + B_{\hat{\varepsilon}} \subseteq \Sigma(Q) + B_{2\hat{\varepsilon}}) \cap B_2$, we have

$$\omega_n \hat{\varepsilon}^n \# I = \mathcal{H}^n\Bigl(\bigcup_{i \in I} \overline{B_{\hat{\varepsilon}}(x_i)}\Bigr) \leq \mathcal{H}^n(\{\text{dist}(\cdot\,, \Sigma(Q)) \leq 2\hat{\varepsilon}\}) \leq C(n)\hat{\varepsilon}^2,$$

and thus $\# I \leq C(n)\hat{\varepsilon}^{2-n}$. As $\beta_1 > n-2$, choosing $\hat{\varepsilon}(n, \beta_1)$ small enough, we find $\# I \leq \bigl(\frac{1}{5}\hat{\varepsilon}\bigr)^{-\beta_1}$, which finishes the proof. □

We employ Lemma 6.4 to get a Reifenberg-type result. We need to incorporate the lower-semicontinuous function τ into the statement, as we will use this result in the next section.

Proposition 6.5 [Figalli et al. 2020, Proposition 7.5]. *Let $\tau : E \to \mathbb{R}$ be a lower-semicontinuous function and $E \subseteq \mathbb{R}^n$ be a measurable set with the following property. For any $\varepsilon > 0$ and $x \in E$, there exists $\varrho = \varrho(x, \varepsilon) > 0$ such that, for all $r \in (0, \varrho)$, there exist a hyperplane L, an odd integer $k \geq 3$ and $q \in \mathcal{S}_k^{\text{even}}(L) \setminus \{0\}$ such that*

$$E \cap \overline{B_r(x)} \cap \tau^{-1}([\tau(x), +\infty)) \subseteq \Sigma(q) + \overline{B_{\varepsilon r}(x)}.$$

Then $\dim_{\mathcal{H}}(E) \leq n-2$.

Proof. The result follows by iterating Lemma 6.4, and we skip the details. See for example the proof of Propositions 7.3 or 7.5 in [Figalli et al. 2020]. □

This finally gives the desired dimensional estimate.

Corollary 6.6. *Let $k \geq 3$ be odd. Then $\dim_{\mathcal{H}}(\Sigma^{k\text{-}th} \setminus \Sigma^{>k}) \leq n-2$. Furthermore, if $n=2$, then $\Sigma^{k\text{-}th} \setminus \Sigma^{>k}$ is discrete in Σ.*

Proof. Recall that if $n=2$, then $\Sigma(q)=\{0\}$ for every $q \in \mathcal{S}_k^{\text{even}} \setminus \{0\}$. Pick any $x \in \Sigma^{k\text{-th}} \setminus \Sigma^{>k}$ and apply Lemma 6.3 with $\varepsilon := \frac{1}{2}$. Then, for all $r < \varrho\left(x, \frac{1}{2}\right)$, we have

$$\Sigma \cap (B_r(x) \setminus \overline{B_{r/2}(x)}) = \varnothing.$$

This clearly implies that $\Sigma \cap B_\varrho(x) = \{x\}$, thus x is isolated in Σ.

For the case $n \geq 3$ we apply Proposition 6.5 to $E := \Sigma^{k\text{-th}} \setminus \Sigma^{>k}$ and the constant function $\tau \equiv 1$. The hypothesis are satisfied thanks to Lemma 6.3. □

6.2. *The size of* $\Sigma^{>k} \setminus \Sigma^{\geq k+1}$. The key idea behind this dimensional reduction is that at an accumulation point of $\Sigma^{>k} \setminus \Sigma^{\geq k+1}$, the blowup gains a translation symmetry along the direction of the approaching sequence. This observation corresponds to Lemmas 6.8 or 6.9 in [Figalli et al. 2020].

Lemma 6.7. *Let $0 \in \Sigma^{>k} \setminus \Sigma^{\geq k+1}$ for some $k \geq 2$. Suppose there exists an infinitesimal sequence $r_\ell \downarrow 0$ and points $x_\ell \in \Sigma^{k\text{-}th} \cap B_{r_\ell}$, $x_\ell \neq 0$ such that $\lambda_k(x_\ell) \to \lambda_k$. Assume further that, as $\ell \to \infty$, we have*

(i) $x_\ell / r_\ell \to y_\infty \in \bar{B}_1$,

(ii) $\tilde{v}_{r_\ell} = (u - \mathcal{P}_k)_{r_\ell} / H(r_\ell, u - \mathcal{P}_k)^{1/2} \to q$ *in* $C^0_{\text{loc}}(\mathbb{R}^n)$ *for some* $q \in \mathcal{S}_{\lambda_k}(\{p_2 = 0\}) \setminus \{0\}$.

Then $y_\infty \in \{p_2 = 0\}$ and $q = q(y_\infty + \cdot\,)$.

Proof. Consider a sequence $(x_\ell)_{\ell \in \mathbb{N}} \subseteq \Sigma^{k\text{-th}} \cap B_{r_\ell}$ as in the statement of the lemma. We begin by recalling that $y_\infty \in \{p_2 = 0\}$ because $r_\ell^{-2} u(r_\ell \cdot) \to p_2$ uniformly in B_2. Then we apply Proposition 5.9 with varying centers $(x_\ell)_{\ell \in \mathbb{N}}$, and after passing to a subsequence (denoted again with r_ℓ) we have

$$\tilde{v}_{r_\ell, x_\ell} := \frac{u(x_\ell + r_\ell \cdot\,) - \mathcal{P}_{k, x_\ell}(r_\ell \cdot\,)}{\|(u(x_\ell + \cdot\,) - \mathcal{P}_{k, x_\ell})_{r_\ell}\|_{L^2(\partial B_1)}} \to Q$$

in $C^0_{\text{loc}}(\mathbb{R}^n)$ for some $Q \in \mathcal{S}_{\lambda_k}(\{p_2 = 0\}) \setminus \{0\}$. On the other hand, by uniform convergence,

$$q(y_\infty + \cdot\,) = \lim_\ell \tilde{v}_{r_\ell}\left(\frac{x_\ell}{r_\ell} + \cdot\right) = \lim_\ell \frac{u(x_\ell + r_\ell \cdot\,) - \mathcal{P}_k(x_\ell + r_\ell \cdot\,)}{H(r_\ell, u - \mathcal{P}_k)^{1/2}}.$$

So putting everything together we can write

$$\tilde{v}_{r_\ell}\left(\frac{x_\ell}{r_\ell} + \cdot\right) = \tilde{v}_{r_\ell, x_\ell} \cdot I_\ell + J_\ell, \tag{6-7}$$

where

$$I_\ell := \frac{H(r_\ell, u(x_\ell + \cdot\,) - \mathcal{P}_{k, x_\ell}(\cdot\,))^{1/2}}{H(r_\ell, u - \mathcal{P}_k)^{1/2}} = \frac{\|u(x_\ell + r_\ell \cdot\,) - \mathcal{P}_{k, x_\ell}(r_\ell \cdot\,)\|_{L^2(\partial B_1)}}{\|u(r_\ell \cdot\,) - \mathcal{P}_k(r_\ell \cdot\,)\|_{L^2(\partial B_1)}}$$

is a numerical sequence and

$$J_\ell := \frac{\mathcal{P}_{k,x_\ell}(r_\ell\,\cdot\,) - \mathcal{P}_k(x_\ell + r_\ell\,\cdot\,)}{H(r_\ell, u - \mathcal{P}_k)^{1/2}}$$

is a sequence of harmonic polynomials of degree at most $k+1$. Now two cases arise:

$$\text{either} \quad \sup_\ell I_\ell < \infty \quad \text{or} \quad I_{\ell_m} \uparrow \infty \quad \text{for some subsequence} \quad \ell_m \to \infty.$$

Let us begin with the first case. Up to a subsequence that we do not rename, we have $I_\ell \to \alpha$ for some $\alpha \geq 0$. Equation (6-7) then implies that $J_\ell \to J$ locally uniformly to some harmonic polynomial J of degree at most $k+1$. Thus, sending $\ell \uparrow \infty$ in (6-7), we obtain

$$q(y_\infty + \,\cdot\,) = \alpha Q + J. \tag{6-8}$$

We exploit homogeneity: for large $R > 0$, we have

$$R^{\lambda_k} q\Big(\frac{y_\infty}{R} + \cdot\Big) = R^{\lambda_k}\alpha Q + J(R\,\cdot\,),$$

so $\lim_{R\uparrow\infty} R^{-\lambda_k} J(Rx)$ exists for every $x \in \mathbb{R}^n$. As λ_k is not an integer and J is a polynomial, the only possibility is that $\lim_{R\uparrow\infty} R^{-\lambda_k} J(Rx) = 0$ for all x, and so $\deg J \leq k$. Hence the last identity reads

$$q = \alpha Q.$$

Inserting this back in (6-8), we find $q(y_\infty + \,\cdot\,) = q + J$. Now, using again that q is homogeneous, for any $R > 0$, we have

$$R\Big(q\Big(\frac{y_\infty}{R} + \cdot\Big) - q(\,\cdot\,)\Big) = R^{1-\lambda_k} J(R\,\cdot\,).$$

Sending $R \to \infty$, the left-hand side converges to $y_\infty \cdot \nabla q$, but as before the right-hand side can only converge to 0, so $y_\infty \cdot \nabla q = 0$.

The second case is simpler. We divide (6-7) by I_{ℓ_m} and find, after passing to a subsequence of ℓ_m, that

$$0 = Q + \tilde{J}$$

for some harmonic polynomial $\tilde{J}$ of degree at most $k+1$. This is a contradiction, since $Q \neq 0$ is a $\lambda_k \in (k, k+1)$ homogeneous function and hence not a polynomial. This finishes the proof. □

Lemma 6.7 triggers a Federer-type dimension reduction, exactly as in [Figalli and Serra 2019].

Proposition 6.8 [Figalli et al. 2020, Proposition 7.3]. *Let $E \subseteq \mathbb{R}^n$, $f : E \to \mathbb{R}$ and $m \in \{1, \dots, n\}$. Assume that, for any $\varepsilon > 0$ and $x \in E$, there exists $\varrho = \varrho(x, \varepsilon) > 0$ such that, for all $r \in (0, \varrho)$, we have*

$$E \cap \overline{B_r(x)} \cap f^{-1}([f(x) - \varrho, f(x) + \varrho]) \subseteq \Pi_{x,r} + B_{\varepsilon r}$$

for some m-dimensional plane $\Pi_{x,r}$ passing through x (possibly depending on r). Then $\dim_{\mathcal{H}}(E) \leq m$.

We combine Lemma 6.7 and Proposition 6.8 to prove the dimensional estimate.

Proposition 6.9. *For every $k \geq 2$, we have $\dim_{\mathcal{H}}(\Sigma^{>k} \setminus \Sigma^{\geq k+1}) \leq n-2$. Moreover, if $n = 2$, then $\Sigma^{>k} \setminus \Sigma^{\geq k+1}$ consists of isolated points if k is odd and is empty if k is even.*

Proof. We want to apply Proposition 6.8 with $E := \Sigma^{>k} \setminus \Sigma^{\geq k+1}$, $m = n-2$ and the function f given on E by $x \mapsto \lambda_k(x) \in (k, k+1)$. It suffices to show that, for all $x \in E$ and for all $\varepsilon > 0$, there exist $\varrho = \varrho(x, \varepsilon) > 0$ and an $(n-2)$-dimensional plane $\Pi_{x,r}$ passing through x such that

$$E \cap B_r(x_\circ) \cap \lambda_k^{-1}([\lambda_k(x) - \varrho, \lambda_k(x) + \varrho]) \subseteq \{x : \operatorname{dist}(x, \Pi_{x,r}) \leq \varepsilon r\} \quad \text{for all } r \in (0, \varrho).$$

We argue by contradiction. Assume that, for $x = 0$ and some $\varepsilon_\circ > 0$, the above does not hold. Then we make the following simple geometric claim. For each ℓ there exists $r_\ell \in (0, 2^{-\ell})$ and $n-1$ points $x_\ell^{(1)}, \dots, x_\ell^{(n-1)}$ in $E \cap B_{r_\ell}$ such that

$$|x_\ell^{(1)} \wedge \cdots \wedge x_\ell^{(n-1)}| \geq \delta r_\ell^{n-1}, \quad |\lambda_k(x_\ell^{(j)}) - \lambda_k| \leq 2^{-\ell}$$

for all $j \in \{1, \dots, n-1\}$ and for some $\delta = \delta(n, \varepsilon_\circ) \in (0, 1)$. In particular, $\{x_\ell^{(1)}, \dots, x_\ell^{(n-1)}\}$ span a hyperplane and, for each fixed j, the sequence $(x_\ell^{(j)})_{\ell \in \mathbb{N}}$ lies in $E \subseteq \Sigma^{k\text{-th}}$, with

$$\lambda_k(x_\ell^{(j)}) \to \lambda_k.$$

We extract a finite number of subsequences to ensure $x_\ell^{(j)}/r_\ell \to y_\infty^{(j)}$ for each j. Exploiting the lower bound on the exterior product, we again have that

$$\dim \operatorname{span}\{y_\infty^{(1)}, \dots, y_\infty^{(n-1)}\} = n-1.$$

Now we apply Proposition 5.9 to each $(x_\ell^{(j)})_{\ell \in \mathbb{N}}$ and get

$$\tilde{v}_{r_\ell} = \frac{(u - \mathcal{P}_k)_{r_\ell}}{H(r_\ell, u - \mathcal{P}_k)^{1/2}} \to q \quad \text{in } C^0_{\text{loc}}$$

for some $q \in \mathcal{S}_{\lambda_k}$. Notice that, taking at each time a subsequence, q can be taken the same for all j's. By Lemma 6.7, we conclude that q is translation-invariant in the directions $y_\infty^{(j)}$ for all $1 \leq j \leq n-1$; hence q is a 1-dimensional homogeneous solution to the obstacle problem vanishing at the origin. Thus, after a rotation of coordinates, we must have $q(x) = -A|x_n| + Bx_n$ for some constants $A \geq 0$ and $B \in \mathbb{R}$, which contradicts $\lambda_k > 1$.

Let us sketch the geometric argument needed to construct such $\{x_\ell^{(1)}, \dots, x_\ell^{(n-1)}\}$. Fixing ℓ, we pick any $(n-2)$-plane Π_0 and any $x^{(1)} \in (E \cap B_{r_\ell}) \setminus (\Pi_0 + B_{\varepsilon_\circ})$. Then we choose any plane Π_1 containing $x^{(1)}$ and any $x^{(2)}$ in $(E \cap B_{r_\ell}) \setminus (\Pi_1 + B_{\varepsilon_\circ})$. We can go on in this way and construct the whole set $\{x^{(1)}, \dots, x^{(n-1)}\}$. Finally, we notice that, by compactness,

$$\delta := \min\{|z_1 \wedge \cdots \wedge z_{n-1}| : z_j \in \mathbb{R}^n, \forall j \ \operatorname{dist}(z_j, \operatorname{span}\{z_i : i \neq j, 1 \leq i \leq n-1\}) \geq \varepsilon_\circ\} > 0.$$

We are left with the case $n = 2$. Recall that if q is a λ-homogeneous solution to the thin obstacle problem, then $\lambda \in \mathbb{N}_+ \cup \left\{2m - \frac{1}{2} : m \in \mathbb{N}_+\right\}$ (see Proposition 2.4). Thus, having in mind Proposition 5.9, we find that $\Sigma^{>k} \setminus \Sigma^{\geq k+1}$ is empty for k even. If k is odd, we find $\lambda_k(x_\circ) = k + \frac{1}{2}$ for every $x_\circ \in \Sigma^{>k} \setminus \Sigma^{\geq k+1}$. If this set was not discrete, we could apply Lemma 6.7 and reach a contradiction, obtaining a one-dimensional and $\left(k + \frac{1}{2}\right)$-homogeneous solution of the thin obstacle problem. $\square$

6.3. *The size of* $\Sigma^{\geq k+1} \setminus \Sigma^{(k+1)\text{-}th}$.

Lemma 6.10. *Let* $k \geq 2$, $x \in \Sigma^{\geq k+1} \setminus \Sigma^{(k+1)\text{-}th}$ *and* $\varepsilon > 0$. *Then there exists* $\varrho = \varrho(\varepsilon, x) > 0$ *such that, for each* $r \in (0, \varrho)$, *there is* $q \in \mathcal{S}^{\text{even}}_{k+1}(\{p_{2,x} = 0\}) \setminus \{0\}$ *such that*

$$\Sigma(u) \cap B_r(x) \subseteq \Sigma(q) + B_{\varepsilon r}(x). \tag{6-9}$$

Recall that $\Sigma(q) = \{q = |\nabla q| = 0\} \cap \{p_{2,x} = 0\}$ *was defined in* (2-9).

Proof of the case $(k+1)$ *even.* Up to an isometry, we can assume $x = 0$ and $p_2 = \frac{1}{2}x_n^2$. We argue by contradiction and rescale everything: we find $\varepsilon_\circ > 0$ and a sequence $r_\ell \downarrow 0$ such that

$$y_\ell \in \Sigma(u(r_\ell \,\cdot)) \cap B_1 \quad \text{and} \quad \operatorname{dist}(y_\ell, \Sigma(q')) \geq \varepsilon_\circ$$

for all $q' \in \mathcal{S}^{\text{even}}_{k+1}(\{x_n = 0\})$. Up to taking subsequences, we can assume $y_\ell \to y_\infty \in \{x_n = 0\}$, and by Proposition 5.9,

$$\frac{(u - \mathcal{P}_k)(r_\ell \,\cdot)}{r_\ell^{k+1}} \to \bar{q} \in \mathcal{S}_{k+1}(\{x_n = 0\})$$

in $C^0_{\text{loc}}(\mathbb{R}^n)$. Since $0 \notin \Sigma^{(k+1)\text{-th}}$, we have $\bar{q}^{\text{even}} \neq 0$. Rearranging the terms we can equivalently write

$$w_\ell := \frac{\big(u - \frac{1}{2}\mathcal{A}^2_{k+1}(p_{2,0}, \ldots, p_{k,0}, \bar{q}^{\text{odd}})\big)(r_\ell \,\cdot)}{r_\ell^{k+1}} \to \bar{q}^{\text{even}} \quad \text{in } C^0_{\text{loc}}(\mathbb{R}^n).$$

Now recall that, $k+1$ being even, we have $\Sigma(\bar{q}^{\text{even}}) = \{\bar{q}^{\text{even}} = 0\}$, thus $\eta := \bar{q}^{\text{even}}(y_\infty) > 0$, as y_∞ lies on the thin obstacle. So we can find a small radius $\delta > 0$ and a large ℓ_0 such that, for all $\ell > \ell_0$, we have

$$\inf_{B_\delta(y_\infty)} w_\ell \geq \inf_{B_\delta(y_\infty)} \bar{q}^{\text{even}} - \|w_\ell - \bar{q}^{\text{even}}\|_{L^\infty(B_2)} \geq \tfrac{1}{2}\eta - \tfrac{1}{4}\eta = \tfrac{1}{4}\eta.$$

However, eventually we will have $y_\ell \in B_\delta(y_\infty)$, and this is a contradiction as

$$0 \geq -\frac{\mathcal{A}^2_{k+1}(r_\ell y_\ell)}{2r_\ell^{k+1}} = w_\ell(y_\ell) \geq \inf_{B_\delta(y_\infty)} w_\ell \geq \tfrac{1}{4}\eta.$$

We remark that we only used that $y_\ell \in \{u_{r_\ell} = 0\}$, not that the y_ℓ were singular points. □

Proof of the case $(k+1)$ *odd.*. Arguing by contradiction as in the even case, we find $\varepsilon_\circ > 0$ and a sequence $r_\ell \downarrow 0$ such that, for each ℓ,

$$y_\ell \in \Sigma(u(r_\ell \,\cdot)) \cap B_1 \quad \text{and} \quad \operatorname{dist}(y_\ell, \Sigma(q)) \geq \varepsilon_\circ \quad \text{for all } q \in \mathcal{S}^{\text{even}}_{k+1}(\{x_n = 0\}) \setminus \{0\}.$$

We can also assume that $y_\ell \to y_\infty \in \{x_n = 0\}$ and

$$w_\ell := \frac{\big(u - \frac{1}{2}\mathcal{A}^2_{k}(p_{2,0}, \ldots, p_{k,0}, \bar{q}^{\text{odd}})\big)(r_\ell \,\cdot)}{r_\ell^{k+1}} \to \bar{q}^{\text{even}} \quad \text{in } C^0_{\text{loc}}(\mathbb{R}^n).$$

From this point the proof is conducted analogously to the proof of Lemma 6.3; it suffices to replace k with $k+1$ and h_ℓ with r_ℓ^{k+1}. □

Remark 6.11. In the case $k+1$ even the proof actually gives a stronger result: as we only used that $y_\ell \in \{u=0\}$, we can replace $\Sigma(u)$ with the full contact set. In other words we can replace (6-9) with

$$\{u=0\} \cap B_r(x) \subseteq \Sigma(q) + B_{\varepsilon r}(x). \tag{6-10}$$

We can conclude now exactly as in the case $\lambda_k = k$ for k odd.

Corollary 6.12. *Suppose $k \geq 2$. Then $\dim_{\mathcal{H}}(\Sigma^{\geq k+1} \setminus \Sigma^{(k+1)\text{-}th}) \leq n-2$. Furthermore, if $n=2$, then $\Sigma^{\geq k+1} \setminus \Sigma^{(k+1)\text{-}th}$ is discrete in the full Σ.*

Proof. Recall that if $n=2$ then $\Sigma(q) = \{0\}$ for every $q \in \mathcal{S}_{k+1}^{\text{even}}$. Pick $x \in \Sigma^{\geq k+1} \setminus \Sigma^{(k+1)\text{-th}}$ and apply Lemma 6.10 with $\varepsilon := \frac{1}{2}$. This gives that, for all $r < \varrho\big(x, \frac{1}{2}\big)$, we have

$$\Sigma(u) \cap (B_r(x) \setminus \overline{B_{r/2}(x)}) = \varnothing.$$

This clearly gives $\Sigma(u) \cap B_\varrho(x) = \{x\}$, thus x is isolated in $\Sigma(u)$.

For $n \geq 3$, we argue as in Corollary 6.6; namely, we apply Proposition 6.5 to $E := \Sigma^{\geq k} \setminus \Sigma^{(k+1)\text{-th}}$. The assumptions are satisfied thanks to Lemma 6.10. □

In particular we notice that in dimension $n=2$ this forces the sets $\Sigma^{k\text{-th}}$ to be closed.

Corollary 6.13. *If $n=2$, then the sets $\Sigma^{k\text{-}th}$ are closed for all $k \geq 2$.*

Proof. We prove the assertion by induction on k. Let $(x_\ell)_{\ell \in \mathbb{N}} \subseteq \Sigma^{(k+1)\text{-th}} \setminus \{0\}$ be a sequence with $x_\ell \to 0$. In particular, we have $x_\ell \in \Sigma^{\geq k+1}$. By inductive assumption we can assume $0 \in \Sigma^{k\text{-th}}$, and by the upper semicontinuity of the truncated frequency we get $\lambda_k(0) \geq k+1$ (see Remark 5.11). But by Corollary 6.12 the origin cannot lie in $\Sigma^{\geq k+1} \setminus \Sigma^{(k+1)\text{-th}}$, because it is an accumulation point of the sequence of singular points x_ℓ. Since $\Sigma^{\text{2nd}} = \Sigma_{n-1}$ is closed, by lower semicontinuity of the rank, the proof is finished. □

6.4. ***The geometry of Σ^∞.*** Let us put together the results obtained so far. By definition,

$$\Sigma^\infty := \bigcap_{k \geq 2} \Sigma^{k\text{-th}}.$$

In the last three subsections we proved the following.

Proposition 6.14. *We have $\dim_{\mathcal{H}}(\Sigma \setminus \Sigma^\infty) \leq n-2$. If $n=2$, then $\Sigma \setminus \Sigma^\infty$ is countable.*

Proof. By definition we have that

$$\Sigma \setminus \Sigma^\infty = (\Sigma \setminus \Sigma_{n-1}) \cup \bigcup_{j \geq 2} (\Sigma^{j\text{-th}} \setminus \Sigma^{>j}) \cup \bigcup_{j \geq 2} (\Sigma^{>j} \setminus \Sigma^{\geq j+1}) \cup \bigcup_{j \geq 3} (\Sigma^{\geq j} \setminus \Sigma^{j\text{-th}}).$$

But now

- $\dim_{\mathcal{H}}(\Sigma \setminus \Sigma_{n-1}) \leq n-2$ (discrete if $n=2$), by [Caffarelli 1998, Theorem 8];
- $\dim_{\mathcal{H}}(\Sigma^{j\text{-th}} \setminus \Sigma^{>j}) \leq n-2$ (discrete if $n=2$), by Corollaries 6.2 and 6.6;
- $\dim_{\mathcal{H}}(\Sigma^{j\text{-th}} \setminus \Sigma^{>j}) \leq n-2$ (discrete if $n=2$), by Proposition 6.9;
- $\dim_{\mathcal{H}}(\Sigma^{\geq j} \setminus \Sigma^{j\text{-th}}) \leq n-2$ (discrete if $n=2$), by Corollary 6.12. □

At each point of Σ^∞ we have Taylor polynomials of every order, and they vary smoothly in the sense of Whitney. This also gives that Σ^∞ locally is contained in a smooth hypersurface. Let us first phrase a suitable statement.

Theorem 6.15. *Let $E \subseteq \mathbb{R}^n$ be any set, and, for each $k \in \mathbb{N}$, consider a collection of polynomials $\{P_{k,x}\}_{x\in E}$ of degree at most k. Suppose that these polynomials satisfy*

(i) *$P_{k,x} = \pi_{\le k}(P_{k+\ell,x})$ for all $k, \ell \in \mathbb{N}$ and $x \in E$,*

(ii) *for each $k \in \mathbb{N}$, there is a constant $C(k)$ such that, for each multi-index α, $|\alpha| \le k$, we have*

$$|\partial^\alpha P_{k,x}(0) - \partial^\alpha P_{k,y}(x-y)| \le C(k)|x-y|^{k-|\alpha|+1} \quad \text{for all } x, y \in E.$$

Then there exists a function $F \in C^\infty(\mathbb{R}^n)$ such that, for each $x \in E$ and $k \in \mathbb{N}$, we have

$$F(x+h) = P_{k,x}(h) + O(|h|^{k+1}) \quad \text{as } |h| \to 0.$$

Proof. This is just a restatement of Whitney's extension theorem for smooth functions. The interested reader can find in Appendix C how to derive this formulation from the original, namely [Whitney 1934, Theorem I]. □

Lemma 6.16. *Let u be a solution to the obstacle problem (2-1). Then Σ^∞ is closed and locally covered by one smooth manifold of dimension $n-1$.*

Proof. The main idea is to combine the implicit function theorem and Whitney's extension theorem (Theorem 6.15). We will first prove the covering and then the closeness.

As the statement is local we can assume that $0 \in \Sigma^\infty$ and that u solves (2-1) in $B_2(0) \subseteq \mathbb{R}^n$. We want to apply Whitney's extension theorem (Theorem 6.15) with $E := \Sigma^\infty \cap B_1$ and the polynomials

$$P_{k,x} := \pi_{\le k}(\mathcal{P}_{k,x}) \quad \text{for all } x \in \Sigma^\infty \cap B_1, \quad k \ge 0.$$

Assumption (i) holds because $\mathcal{P}_{k+\ell}$ and $\mathcal{P}_k$ agree up to order k (see also Lemma 5.7). We need to show that (ii) holds. It is not restrictive to do it only for some fixed $k \ge 3$. To do so we exploit our previous analysis on $\Sigma^{(k+1)\text{-th}}$. More precisely, combining Lemma 3.5 with the uniform estimate in Proposition 5.4 and growth estimates from Proposition 4.2 and Lemma 2.2, we find $R = R(n,k)$ and $C = C(n,k)$ such that, for all $x \in \Sigma^{(k+1)\text{-th}} \cap B_1$ and $0 < r < R < \frac{1}{2}$, we have

$$\|u(x+\cdot) - P_{k,x}\|_{L^\infty(B_r(0))} \le Cr^{k+1}. \tag{6-11}$$

Thus this must hold, a fortiori, for all $x \in \Sigma^\infty \cap B_1$. Let now $x_1, x_2 \in \Sigma^\infty \cap B_1$ such that $|x_1 - x_2| \le \frac{1}{10}R$. Then, since $B_{2|x_1-x_2|}(x_1) \subseteq B_{4|x_1-x_2|}(x_2)$, by (6-11) applied at x_1 with $r_1 = 2|x_1-x_2|$ and at x_2 with $r_2 = 4|x_1-x_2|$, together with the triangle inequality, we find

$$\|P_{k,x_1}(\cdot - x_1) - P_{k,x_2}(\cdot - x_2)\|_{L^\infty(B_{2|x_1-x_2|}(x_1))} \le C|x_1-x_2|^{k+1}. \tag{6-12}$$

If we consider the polynomial $Q = P_{k,x_1}(\cdot - x_1) - P_{k,x_2}(\cdot - x_2)$, equation (6-12) reads

$$\|Q(x_1 + 2|x_1-x_2|\cdot)\|_{L^\infty(B_1)} \le C|x_1-x_2|^{k+1},$$

hence by the equivalence of norms on the space of polynomials of degree bounded by k, we conclude

$$\|(\partial^\alpha Q)(x_1+2|x_1-x_2|\,\cdot\,)\|_{L^\infty(B_1)} \le C|x_1-x_2|^{k+1-|\alpha|}$$

for all multi-index $|\alpha|\le k$, with some $C=C(n,k)$. In particular, looking at the center of B_1, we get

$$|\partial^\alpha P_{k,x_1}(x_2-x_1)-\partial^\alpha P_{k,x_2}(0)| \le C(n,k)|x_1-x_2|^{k+1-l}, \tag{6-13}$$

and this proves that assumption (ii) holds. By the Whitney extension theorem, there exists a C^∞ function $F:\mathbb{R}^n\to\mathbb{R}$ for which

$$F(x)=P_{k,x_\circ}(x-x_\circ)+O(|x-x_\circ|^{k+1})$$

at every $x_\circ\in\Sigma^\infty\cap\overline{B}_1$.

We now conclude as in [Figalli et al. 2020, Proposition 8.1c)], using the implicit function theorem. As $\Sigma^\infty\subseteq\{\nabla F=0\}$ and $\nabla^2F(x_1)=\nabla^2 p_{2,x_1}(0)$ has rank 1, we conclude with the implicit function theorem that $\{\nabla F=0\}$ is a smooth hypersurface in some neighborhood of x_1.

Let us now prove that Σ^∞ is closed. Suppose $0\in\overline{\Sigma^\infty}$, so there exists $x_\ell\in\Sigma^\infty$ such that $x_\ell\to 0$. First observe that $0\in\Sigma$ since the full singular set Σ is closed, and hence p_2 exists. We define by continuity $\mathcal{P}_k:=\lim_\ell \mathcal{P}_{k,x_\ell}$ for all $k\ge 3$; this is a well-posed definition as by Lemma 6.16 the map $x\mapsto\mathcal{P}_{k,x}$ is Lipschitz on Σ^∞ and hence admits a unique extension to the closure. Now, by Proposition 5.3, for some constants $C,\varepsilon>0$ and a radius r_0, both independent of ℓ, we have, for all $r\in(0,r_0)$,

$$\frac{d}{dr}(r^{-2k}H(r,u(x_\ell+\cdot\,)-\mathcal{P}_{k,x_\ell}))\ge -Cr^{\varepsilon-1} \quad\text{and}\quad r^{-2k}H(r,u(x_\ell+\cdot\,)-\mathcal{P}_{k,x_\ell})\le C.$$

We can pass both these inequalities to the limit $\ell\to\infty$ and apply the same reasoning to Proposition 4.2 to show that the sequence $r^{-k}(u-\mathcal{P})_r$ is uniformly bounded in $W^{1,2}_{\mathrm{loc}}(\mathbb{R}^n)$. Hence it is immediate that $0\in\widetilde{\Sigma}^{(k-1)\text{-th}}$ (see Lemma 5.7); as k was arbitrary we conclude $0\in\Sigma^\infty$. □

We conclude this section proving Theorem 1.1.

Proof of Theorem 1.1. The bound on the dimension of $\Sigma\setminus\Sigma^\infty$ is given by Proposition 6.14 and the covering by Lemma 6.16. The polynomial expansion has been shown in (6-11) above, we simply take $P_{k,x_\circ}:=\pi_{\le k}(\mathcal{P}_{k,x_\circ})$. Although we often assumed $f\equiv 1$ and $\mu=1$ to simplify the notation, essentially no modifications are needed for a general f. The reader can find a complete account of the modifications needed in the statements and in the proofs in Appendix B. □

In the following section we aim to explain in which sense the set Σ^∞ is unstable and disappears after a slight perturbation of the boundary data in the obstacle problem.

7. Extension to a monotone family of solutions

In this section we aim to prove Theorem 1.2 and Corollary 1.4. For simplicity we take $f\equiv 1$. This allows us to use verbatim some lemmas from [Figalli et al. 2020] and shortens the notation, without affecting the proofs. We list the changes needed in Appendix B.

We remark that in Sections 7.1 and 7.2 we only assume to have a monotone family of solutions, while in Section 7.3 we work under the "uniform monotonicity" assumption (7-8).

7.1. ***Setup and strategy.*** For the rest of the section, we let $u : \bar{B}_1 \times [-1, 1] \to \mathbb{R}$, $u \geq 0$, be a monotone 1-parameter family of solutions of the obstacle problem, namely

$$\begin{cases} \Delta u(\,\cdot\,, t) = \chi_{\{u(\,\cdot\,,t)>0\}}, \\ 0 \leq u(\,\cdot\,, s) \leq u(\,\cdot\,, t) \qquad \text{in } B_1 \text{ for } -1 \leq s \leq t \leq 1. \end{cases} \tag{7-1}$$

We will also use the notation $u^t := u(\,\cdot\,, t)$. We will assume in addition that $u \in C^0(\bar{B}_1 \times [-1, 1])$. We remark that this continuity property in t follows by the maximum principle whenever $u|_{\partial B_1 \times [-1,1]}$ is continuous.

We will often think of t as the time parameter, as intuitively we imagine lifting the boundary datum of a solution of (2-1). However, no equation in t is given.

For each fixed t, we can apply the results of the previous sections, so we introduce further notation for the following subsets of $\bar{B}_1 \times [-1, 1]$:

$$\begin{aligned} \mathbf{\Sigma} &:= \{(x_\circ, t_\circ) : x_\circ \in \Sigma(u(\,\cdot\,, t_\circ))\}, \\ \mathbf{\Sigma}_{n-1} &:= \{(x_\circ, t_\circ) : x_\circ \in \Sigma_{n-1}(u(\,\cdot\,, t_\circ))\}, \\ \mathbf{\Sigma}^{k\text{-th}} &:= \{(x_\circ, t_\circ) : x_\circ \in \Sigma^{k\text{-th}}(u(\,\cdot\,, t_\circ))\}, \qquad k \geq 2, \\ \mathbf{\Sigma}^{>k} &:= \{(x_\circ, t_\circ) : x_\circ \in \Sigma^{>k}(u(\,\cdot\,, t_\circ))\}, \qquad k \geq 2, \\ \mathbf{\Sigma}^{\geq k+1} &:= \{(x_\circ, t_\circ) : x_\circ \in \Sigma^{\geq k+1}(u(\,\cdot\,, t_\circ))\}, \quad k \geq 2, \\ \mathbf{\Sigma}^{\infty} &:= \{(x_\circ, t_\circ) : x_\circ \in \Sigma^{\infty}(u(\,\cdot\,, t_\circ))\}. \end{aligned} \tag{7-2}$$

This setup (up to $k = 4$) has already been considered in [Figalli et al. 2020]. As we use the same notation, we begin recalling two important lemmas from [Figalli et al. 2020] about the set $\mathbf{\Sigma}$.

Lemma 7.1 [Figalli et al. 2020, Lemma 6.2]. *Let $u \in C^0(\bar{B}_1 \times [-1, 1])$ solve (7-1). Then*

(i) $\mathbf{\Sigma} \cap \bar{B}_\varrho \times [-1, 1]$ *is closed for any $\varrho < 1$, and*

$$\mathbf{\Sigma} \cap \bar{B}_\varrho \times [-1, 1] \ni (x_k, t_k) \to (x_\infty, t_\infty) \quad \Rightarrow \quad p_{2,x_k,t_k} \to p_{2,x_\infty,t_\infty}.$$

(ii) *If $(x_\circ, t_1)$ and $(x_\circ, t_2)$ both belong to $\mathbf{\Sigma}$ and $t_1 < t_2$, then there exists $r > 0$ such that $u(x, t)$ is independent of t for all $(x, t) \in B_r(x_\circ) \times [t_1, t_2]$.*

The next result concerns the quantitative behavior of the first blowup $p_{2,k} := p_{2,x_k,t_k}$ with respect to the convergence $x_k \to 0$ (here it is assumed $(x_k, t_k) \in \mathbf{\Sigma}$ for some sequence of times).

Lemma 7.2 [Figalli et al. 2020, Lemma 6.3]. *Let $u \in C^0(\bar{B}_1 \times [-1, 1])$ solve (7-1), let $(x_k, t_k) \in \mathbf{\Sigma}$, $(0, 0) \in \mathbf{\Sigma}$ and assume that $x_k \to 0$. If we set $p_2 := p_{2,0,0}$, then we have*

$$\left\| p_{2,k} - p_2\left(\frac{x_k}{|x_k|} + \cdot\right)\right\|_{L^\infty(B_1)} \leq C\omega(2|x_k|) \quad \textit{and} \quad \|p_{2,k} - p_2\|_{L^\infty(B_1)} \leq C\omega(2|x_k|)$$

for some dimensional modulus of continuity ω. In addition,

$$\operatorname{dist}\left(\frac{x_k}{|x_k|}, \{p_2 = 0\}\right) \to 0 \quad \textit{as } k \to \infty.$$

Our strategy follows the one exhibited in [Figalli et al. 2020]. For each $(x_\circ, t_\circ) \in \mathbf{\Sigma}^{\geq k+1}$, we first prove the approximation

$$\|u(x_\circ + \cdot\,, t_\circ) - P_{k,x_\circ,t_\circ}\|_{L^\infty(B_r)} \leq C r^{k+1}, \tag{7-3}$$

where the polynomial $P_{k,x_\circ,t_\circ}$ of degree at most k is unique and $\Delta P_{k,x_\circ,t_\circ} = 1$. Using the fact that the polynomials above are almost positive together with barrier-type arguments (see Lemma 7.11), we are able to conclude a "cleaning property" in space-time in the following sense. For each $(x_\circ, t_\circ) \in \mathbf{\Sigma}^{\geq k+1}$, there exist $\varrho > 0$ and $C > 0$ (depending on $n,\ k,\ x_\circ$) such that

$$\{(x,t) \in B_\varrho(x_\circ) \times (t_\circ, 1) : t - t_\circ > C|x - x_\circ|^k\} \cap \{u = 0\} = \varnothing. \tag{7-4}$$

This property expresses the instability of $\Sigma^{\geq k+1}(u^t)$ with respect to increments of the t parameter. From here, we will conclude with the next geometric measure theory result, that the set $\pi_t(\mathbf{\Sigma}^\infty)$ has zero Hausdorff dimension.

Proposition 7.3 [Figalli et al. 2020, Corollary 7.8]. *Let $E \subseteq \mathbb{R}^n \times [-1,1]$, let (x,t) denote a point in $\mathbb{R}^n \times [-1,1]$, and let $\pi_x : (x,t) \mapsto x$ and $\pi_t : (x,t) \mapsto t$ be the standard projections. Assume that, for some $\beta \in (0,n]$ and $s > \beta$, we have*:

- $\dim_{\mathcal{H}}(\pi_x(E)) \leq \beta$.
- *For all $(x_\circ, t_\circ) \in E$ and $\varepsilon > 0$, there exists $\varrho = \varrho_{x_\circ,t_\circ,\varepsilon} > 0$ such that*

$$\{(x,t) \in B_\varrho(x_\circ) \times [-1,1] : t - t_\circ > |x - x_\circ|^{s-\varepsilon}\} \cap E = \varnothing.$$

Then $\dim_{\mathcal{H}}(\pi_t(E)) \leq \beta/s$.

To obtain Corollary 1.4 we also need to take care of the points where the expansion (7-3) fails for some k. To this end we need to generalize to one-parameter solutions some of the previous results.

7.2. ***Adaptation of previous sections to family of solutions.*** In this section we establish an analog of Theorem 1.1 for monotone families of solutions. The generalization of the polynomial expansion is obvious in the set $\pi_x(\mathbf{\Sigma}^\infty)$, so the only nontrivial task is to show that $\pi_x(\mathbf{\Sigma} \setminus \mathbf{\Sigma}^\infty)$ has again Hausdorff dimension at most $n-2$. We will show this by repeating the arguments of Section 6. While the arguments of Sections 6.1 and 6.3 adapt immediately by exploiting monotonicity, the arguments of Section 6.2 require a bit more care. Specifically, we have to check Lemma 6.7 for varying times.

We start observing that in Proposition 5.9 one can also consider the varying time parameter.

Proposition 7.4. *Let $u \in C^0(\bar{B}_1 \times [-1,1])$ solve (7-1) and $(0,0) \in \mathbf{\Sigma}^{k\text{-}th}$, with $\lambda_k = \lambda_{k,0}(0) \leq k+1$. Let $(r_\ell)_{\ell \in \mathbb{N}}$ be an infinitesimal sequence, and let $x_\ell \in \Sigma^{k\text{-}th}(u^{t_\ell}) \cap B_{r_\ell}$. For every ℓ, set*

$$v_\ell := u(x_\ell + \cdot\,, t_\ell) - \mathcal{P}_{k,x_\ell,t_\ell},$$

and suppose that $\lambda_{k,t_\ell}(x_\ell) \to \lambda_k$. Consider the sequence

$$\tilde{v}_\ell := \frac{v_\ell(r_\ell\,\cdot)}{H(r_\ell, v_\ell)^{1/2}}.$$

Then:

(i) $(\tilde{v}_\ell)_{\ell\in\mathbb{N}}$ *is bounded in* $W^{1,2}_{\mathrm{loc}}(\mathbb{R}^n)$ *and* $C^{0,1/(n+1)}_{\mathrm{loc}}(\mathbb{R}^n)$.

(ii) *If* $\tilde{v}_\ell \rightharpoonup q \in W^{1,2}_{\mathrm{loc}}(\mathbb{R}^n)$, *then the convergence is strong and* q *must be a nontrivial* λ_k*-homogeneous solution of the thin obstacle problem* (5-4) *with obstacle* $\{p_2 = 0\}$, *that is*

$$\begin{cases} \Delta q \le 0 \ \text{and} \ q\Delta q = 0 & \text{in } \mathbb{R}^n, \\ \Delta q = 0 & \text{in } \mathbb{R}^n \setminus \{p_2 = 0\}, \\ q \ge 0 & \text{on } \{p_2 = 0\}. \end{cases}$$

Finally, if $\lambda_k < k+1$ *then* q *is even with respect to the thin obstacle.*

Proof. Given the convergence assumption $\lambda_{k,t_\ell}(x_\ell) \to \lambda_k$ and Lemma 7.2, the proof is almost identical to Proposition 5.9. □

We now turn to the time-dependent version of Lemma 6.7.

Lemma 7.5. *Let* $u \in C^0(\bar{B}_1 \times [-1,1])$ *solve* (7-1), *let* $k \ge 2$ *and suppose* $(0,0) \in \mathbf{\Sigma}^{>k} \setminus \mathbf{\Sigma}^{\ge k+1}$, *that is* $\lambda_k = \lambda_{k,0}(0) \in (k, k+1)$. *Suppose there exists an infinitesimal sequence* $r_\ell \downarrow 0$ *and* $(x_\ell, t_\ell) \in \mathbf{\Sigma}^{k\text{-th}} \cap B_{r_\ell}$ *such that* $\lambda_{k,t_\ell}(x_\ell) \to \lambda_k$. *Assume further that, as* $\ell \uparrow \infty$, *we have*

(i) $x_\ell / r_\ell \to y_\infty \in \bar{B}_1$,

(ii) $\tilde{v}_{r_\ell} = (u_0 - \mathcal{P}_{k,0,0})_{r_\ell} / H(r_\ell, u_0 - \mathcal{P}_{k,0,0})^{1/2} \to q$ *in* $C^0_{\mathrm{loc}}(\mathbb{R}^n)$ *for some* $q \in \mathcal{S}_{\lambda_k}(\{p_{2,0,0} = 0\}) \setminus \{0\}$.

Then $y_\infty \in \{p_{2,0,0} = 0\}$ *and* $q = q(y_\infty + \cdot\,)$.

Proof. Whenever $x = t = 0$, we simplify the notation by dropping the indices, e.g., $p_{2,0,0} = p_2$. Consider a sequence $(x_\ell, t_\ell)_{\ell\in\mathbb{N}} \subseteq \mathbf{\Sigma}^{k\text{-th}} \cap B_{r_\ell}$ as in the statement of the lemma. Note that $y_\infty \in \{p_2 = 0\}$ due to Lemma 7.2. Applying Proposition 7.4 with varying centers $(x_\ell)_{\ell\in\mathbb{N}}$ and respective sequence of times $(t_\ell)_{\ell\in\mathbb{N}}$, we find (after passing to a subsequence)

$$\tilde{v}_\ell := \frac{u(x_\ell + r_\ell\,\cdot\,, t_\ell) - \mathcal{P}_{k,x_\ell,t_\ell}(r_\ell\,\cdot\,)}{\|(u(x_\ell + \cdot\,, t_\ell) - \mathcal{P}_{k,x_\ell,t_\ell})_{r_\ell}\|_{L^2(\partial B_1)}} \to Q$$

in $C^0_{\mathrm{loc}}(\mathbb{R}^n)$ for some $Q \in \mathcal{S}_{\lambda_k}(\{p_2 = 0\}) \setminus \{0\}$. On the other hand, by uniform convergence,

$$q(y_\infty + \cdot\,) = \lim_\ell \frac{u(x_\ell + r_\ell\,\cdot\,) - \mathcal{P}_k(x_\ell + r_\ell\,\cdot\,)}{H(r_\ell, u - \mathcal{P}_k)^{1/2}}.$$

We write

$$\frac{u(x_\ell + r_\ell\,\cdot\,) - \mathcal{P}_k(x_\ell + r_\ell\,\cdot\,)}{H(r_\ell, u - \mathcal{P}_k)^{1/2}} = \frac{u(x_\ell + r_\ell\,\cdot\,) - u(x_\ell + r_\ell\,\cdot\,, t_\ell)}{\|(u(x_\ell + \cdot\,, t_\ell) - \mathcal{P}_{k,x_\ell,t_\ell})_{r_\ell}\|_{L^2(B_1)}} \cdot a_\ell I_\ell + \tilde{v}_\ell \cdot b_\ell I_\ell + J_\ell,$$

where

$$I_\ell := \frac{\|(u(x_\ell + \cdot\,, t_\ell) - \mathcal{P}_{k,x_\ell,t_\ell})_{r_\ell}\|_{L^2(B_1)} + H(r_\ell, u(x_\ell + \cdot\,, t_\ell) - \mathcal{P}_{k,x_\ell,t_\ell})^{1/2}}{H(r_\ell, u - \mathcal{P}_k)^{1/2}}$$

is a numerical sequence and

$$J_\ell := \frac{\mathcal{P}_{k,x_\ell}(r_\ell\,\cdot\,) - \mathcal{P}_k(x_\ell + r_\ell\,\cdot\,, t_\ell)}{H(r_\ell, u - \mathcal{P}_k)^{1/2}}$$

is a sequence of harmonic polynomials of degree at most $k+1$. The numerical sequences

$$a_\ell := \frac{\|(u(x_\ell+\cdot\,,t_\ell)-\mathcal{P}_{k,x_\ell,t_\ell})_{r_\ell}\|_{L^2(B_1)}}{\|(u(x_\ell+\cdot\,,t_\ell)-\mathcal{P}_{k,x_\ell,t_\ell})_{r_\ell}\|_{L^2(B_1)}+H(r_\ell,u(x_\ell+\cdot\,,t_\ell)-\mathcal{P}_{k,x_\ell,t_\ell})^{1/2}}$$

and

$$b_\ell := \frac{H(r_\ell,u(x_\ell+\cdot\,,t_\ell)-\mathcal{P}_{k,x_\ell,t_\ell})^{1/2}}{\|(u(x_\ell+\cdot\,,t_\ell)-\mathcal{P}_{k,x_\ell,t_\ell})_{r_\ell}\|_{L^2(B_1)}+H(r_\ell,u(x_\ell+\cdot\,,t_\ell)-\mathcal{P}_{k,x_\ell,t_\ell})^{1/2}}$$

are both bounded by 1 and hence, up to a subsequence, converge to some a and b, respectively, in $[0,1]$. Now two cases arise:

(i) $\sup_\ell I_\ell < \infty$.

(ii) $I_{\ell_m} \uparrow \infty$ for some subsequence $\ell_m \to \infty$.

Let us begin with the first case. Up to a subsequence that we do not rename, we have $I_\ell \to \alpha$, and passing to the limit as $\ell \to \infty$ in (7-5) in $L^2(B_1)$ implies that J_ℓ converges to some harmonic polynomial J of degree at most $k+1$. We find that

$$q(y_\infty+\cdot\,) = a\alpha w + b\alpha Q + J \tag{7-5}$$

in L^2 for some function w having a constant sign. Combining this fact and exploiting the homogeneity of q and Q, as in Proposition 5.9, we find

$$q(\,\cdot\,) \le a\alpha Q(\,\cdot\,) + R^{-\lambda_k} J(R\,\cdot\,) \quad \text{or} \quad q \ge a\alpha Q + R^{-\lambda_k} J(R\,\cdot\,). \tag{7-6}$$

Next, we remark that J does not have a constant sign, as it is harmonic and vanishes somewhere on the line segment $\overline{-y_\infty 0}$. The last property is seen as follows. First, note that, for large ℓ, we have $H(r_\ell,u-\mathcal{P}_k)^{1/2} \gg r_\ell^{\lambda_k+\delta} \gg r_\ell^{k+1}$. On the other hand, we calculate

$$H(r_\ell,u-\mathcal{P}_k)^{1/2} J_\ell(0) \le C r_\ell^{k+2} \quad \text{and} \quad H(r_\ell,u-\mathcal{P}_k)^{1/2} J_\ell\Big(-\frac{x_\ell}{r_\ell}\Big) \ge -C r_\ell^{k+2}.$$

Hence, there exists a sequence of points $\bar y_\ell \in \overline{-(x_\ell/r_\ell)0}$ with $|J_\ell(\bar y_\ell)| \le C r_\ell$, and so J vanishes at some point in the line segment $\overline{-y_\infty 0}$.

As J does not have a constant sign, there are directions $x_\pm \in \mathbb{S}^{n-1}$ with $J(Rx_\pm) \to \pm\infty$ as $R \to \infty$. Combining this with (7-6), we thus find $\deg J \le k$ and

$$q \le a\alpha Q \quad \text{or} \quad q \ge a\alpha Q.$$

Thus, in any of the cases, we have found two ordered λ_k-homogeneous solutions to the thin obstacle problem, and they must be equal, see [Figalli et al. 2020, Lemma A.4]. Inserting this back in (7-5), we find $q(y_\infty+\cdot\,) = q + J$. Therefore, for any $R > 0$, we have

$$R\Big(q\Big(\frac{y_\infty}{R}+\cdot\Big)-q(\,\cdot\,)\Big) \le R^{1-\lambda_k} J(R\,\cdot\,) \quad \text{or} \quad R\Big(q\Big(\frac{y_\infty}{R}+\cdot\Big)-q(\,\cdot\,)\Big) \ge R^{1-\lambda_k} J(R\,\cdot\,).$$

As the left-hand side is bounded (and converges to $y_\infty \cdot \nabla q$) as $R \to \infty$, we exploit the fact that J does not have a constant sign to find that the k-th coefficients must vanish. And so

$$y_\infty \cdot \nabla q \le 0 \quad \text{or} \quad y_\infty \cdot \nabla q \ge 0.$$

Reasoning as in Step 3 in the proof of [Figalli et al. 2020, Lemma 6.5], we find that in any of the cases we must have $y_\infty \cdot \nabla q \equiv 0$, as otherwise $y_\infty \cdot \nabla q$ would be a multiple of an eigenfunction to some elliptic problem on a subset on the sphere, which contradicts the high homogeneity of $y_\infty \cdot \nabla q$.

The second case is simpler. We divide (7-5) by I_{ℓ_m} and find, after passing to a subsequence of ℓ_m,

$$0 = aw + bQ + \tilde{J}$$

for some harmonic polynomial $\tilde{J}$ of degree at most $k+1$. This is a contradiction, as the three functions $\tilde{J}$, w and Q are not linearly dependent. Indeed, w has a sign, $Q \neq 0$ is a $\lambda_k \in (k, k+1)$ homogeneous function and $\tilde{J}$ a harmonic polynomial with no constant sign. The fact that $\tilde{J}$ vanishes somewhere can be checked as we checked that J vanishes somewhere, using that

$$I_\ell \ge \frac{H(r_\ell, u(x_\ell + \cdot\,, t_\ell) - \mathcal{P}_{k,x_\ell,t_\ell})^{1/2}}{H(r_\ell, u - \mathcal{P}_k)^{1/2}} \gg \frac{r_\ell^{k+1}}{H(r_\ell, u - \mathcal{P}_k)^{1/2}}. \qquad \square$$

In order to perform the necessary dimension reductions we need some adaptations of Section 6. We start with the following variation of Proposition 6.5, taken from [Figalli et al. 2023].

Proposition 7.6 [Figalli et al. 2023, Proposition 7.6]. *Let $k \ge 2$ and $E \subseteq \mathbb{R}^n \times \mathbb{R}$. Suppose that*

$$\forall (x,t) \in E, \quad \forall \varepsilon > 0, \quad \exists \varrho > 0, \quad \forall r \in (0, \varrho), \quad \exists L \textit{ hyperplane}, \quad \exists q \in \mathcal{S}_k^{\text{even}}(L)$$

such that

$$\pi_x(E \cap (\overline{B_r(x)} \times (-\infty, t])) \subseteq \Sigma(q) + \overline{B_{\varepsilon r}(x)}.$$

Then $\dim_{\mathcal{H}}(E) \le n-2$.

We want to apply this proposition to $E = \mathbf{\Sigma}^{k\text{-th}} \setminus \mathbf{\Sigma}^{(k+1)\text{-th}}$. We check in the next two lemmas that this is possible.

Lemma 7.7. *Let $k \ge 2$ and $(0,0) \in \mathbf{\Sigma}^{k\text{-th}} \setminus \mathbf{\Sigma}^{(k+1)\text{-th}}$, and suppose $\lambda := \lambda_k(0,0)$ is an even integer. Then,*

$$\forall \varepsilon > 0, \quad \exists \varrho > 0, \quad \forall r \in (0, \varrho), \quad \exists q \in \mathcal{S}_\lambda^{\text{even}}(\{p_{2,0,0} = 0\})$$

such that

$$\pi_x(\{u = 0\} \cap (\overline{B_r(x)} \times [0,1])) \subseteq \Sigma(q) + \overline{B_{\varepsilon r}(x)}.$$

Proof. Notice that, by Lemma 6.1, $\lambda = k+1$. Further we have by monotonicity

$$\pi_x(\{u = 0\} \cap (\overline{B_r(x)} \times [0,1])) \subseteq \{u(\cdot\,, 0) = 0\} \cap \overline{B_r(x)}.$$

Now by Lemma 6.10, and taking Remark 6.11 into account, the set $\{u(\cdot\,,0) = 0\} \cap \overline{B_r(x)}$ can be covered with a tubular neighborhood of the singular set of a Signorini solution, provided r is small enough. $\square$

For odd frequencies we use a control "in the past".

Lemma 7.8. *Let $k \geq 2$ and $(0,0) \in \boldsymbol{\Sigma}^{k\text{-}th} \setminus \boldsymbol{\Sigma}^{(k+1)\text{-}th}$, and suppose $\lambda := \lambda_k(0,0)$ is an odd integer. Then,*

$$\forall \varepsilon > 0, \quad \exists \varrho > 0, \quad \forall r \in (0, \varrho), \quad \exists q \in \mathcal{S}_\lambda^{\text{even}}(\{p_{2,0,0} = 0\})$$

such that

$$\pi_x(\boldsymbol{\Sigma} \cap (\overline{B_r(x)} \times [-1, 0])) \subseteq \Sigma(q) + \overline{B_{\varepsilon r}(x)}.$$

Proof. Notice that either $\lambda = k$ or $\lambda = k+1$. In the first case, we reproduce the proof of Lemma 6.3. In the second case, we reproduce the proof of Lemma 6.10 for the odd case. In both cases, it suffices to replace u with $u(\cdot\,, 0)$, and the argument for a single solution can be applied. The key point is that, by monotonicity, the barriers $\{\phi_{z,\ell}\}$ will work for all $u(\cdot\,, t)$ for $t \leq 0$. Indeed, following the proof with the same notation, one arrives at

$$\{\mathcal{A}_{k,0,0}(r_\ell\,\cdot) = 0\} \cap B_\rho(y_\infty) \subseteq Z_\ell \subseteq \operatorname{int}\{u(r_\ell\,\cdot\,, 0) = 0\} \subseteq \operatorname{int}\{u(r_\ell\,\cdot\,, t) = 0\}.$$

Hence the contact set of $u(\cdot\,, t)$ is fat around y_∞. This gives $\Sigma(u(r_\ell\,\cdot\,, t)) \cap B_{\rho/N}(y_\infty) = \varnothing$ for some dimensional constant N and for all $t \leq 0$ (see [Caffarelli 1998, Theorem 7] or the proof of [Figalli et al. 2020, Lemma 9.4]). This is the desired contradiction as $y_\ell \to y_\infty$, where $y_\ell \in \Sigma(u(r_\ell\,\cdot\,, t_\ell))$. □

Putting all these results together, we can prove the main theorem of this section. It is an extension of the fifth-order approximation result [Figalli et al. 2020, Theorem 8.7] to every order. For a fixed solution, this is just the content of our main Theorem 1.1.

Theorem 7.9. *Let $u \in C^0(\bar{B}_1 \times [-1, 1])$ solve (7-1). Then $\dim_{\mathcal{H}}(\pi_x(\boldsymbol{\Sigma} \setminus \boldsymbol{\Sigma}^\infty)) \leq n-2$, and the set is countable if $n = 2$. Moreover, for every $k \geq 2$, there exist constants $C = C(n,k)$ and $\rho = \rho(n,k)$ such that*

$$\|u(x_\circ + \cdot\,, t_\circ) - P_{k,x_\circ,t_\circ}\|_{L^\infty(B_r)} \leq C r^{k+1} \tag{7-7}$$

holds with a unique polynomial $P_{k,x_\circ,t_\circ}$ of degree at most k and $\Delta P_{k,x_\circ,t_\circ} = 1$, for all $0 < r < \rho$ and $(x_\circ, t_\circ) \in \boldsymbol{\Sigma}^\infty \cap B_{1/2} \times (-1, 1)$.

Proof. We recall from [Figalli et al. 2020, Proposition 8.1] that $\dim_{\mathcal{H}}(\pi_x(\boldsymbol{\Sigma} \setminus \boldsymbol{\Sigma}_{n-1})) \leq n-2$ and that $\dim_{\mathcal{H}}(\pi_x(\boldsymbol{\Sigma} \setminus \boldsymbol{\Sigma}_{n-1}))$ is countable if $n = 2$. Thus we need to show that, for all $k \geq 2$,

(i) $\dim_{\mathcal{H}}(\pi_x(\boldsymbol{\Sigma}^{\geq k} \setminus \boldsymbol{\Sigma}^{k\text{-th}})) \leq n-2$ (countable if $n = 2$),

(ii) $\dim_{\mathcal{H}}(\pi_x(\boldsymbol{\Sigma}^{k\text{-th}} \setminus \boldsymbol{\Sigma}^{>k})) \leq n-2$ (countable if $n = 2$),

(iii) $\dim_{\mathcal{H}}(\pi_x(\boldsymbol{\Sigma}^{>k} \setminus \boldsymbol{\Sigma}^{\geq k+1})) \leq n-2$ (countable if $n = 2$).

By Lemmas 7.8 and 7.7, to prove (i) and (ii) we can use Proposition 7.6 (or an obvious version of it for future times) with $E = \boldsymbol{\Sigma}^{\geq k} \setminus \boldsymbol{\Sigma}^{k\text{-th}}$ and $E = \boldsymbol{\Sigma}^{k\text{-th}} \setminus \boldsymbol{\Sigma}^{>k}$, respectively.

We turn to the proof of (iii). We can apply Proposition 6.8 to the set $E = \pi_x(\boldsymbol{\Sigma}^{>k} \setminus \boldsymbol{\Sigma}^{\geq k+1})$ using the function $f(x_\circ) := \lambda_{k,\tau(x_\circ)}(x_\circ)$, where $\tau : \pi_x(\boldsymbol{\Sigma}) \to [-1, 1]$ is defined by

$$\tau(x_\circ) := \min\{t \in [-1, 1] : (x_\circ, t) \in \boldsymbol{\Sigma}\}.$$

The assumptions of Proposition 6.8 hold for such E: if not we could argue by contradiction and blow up exactly as in Proposition 6.9. The only difference is that we have to use Lemma 7.5 above, instead of Lemma 6.7. □

7.3. *Cleaning lemmas in the time variable.* Following [Figalli et al. 2020], in this section we consider any monotone family of solutions $\{u^t\}_{t\in(-1,1)}$ of (2-1) in B_1, which additionally satisfy the following "uniform monotonicity" condition:

$$\begin{aligned}&\text{For every } t\in(-1,1) \text{ and any compact set } K_t\subseteq \partial B_1\cap\{u^t>0\},\\ &\qquad \text{there exists } c_{K_t}>0 \text{ such that } \inf_{x\in K_t}(u^{t'}(x)-u^t(x))\ge c_{K_t}(t'-t) \text{ for all } -1<t<t'<1.\end{aligned} \tag{7-8}$$

This condition rules out the existence of regions that remain stationary as we increase the parameter t. Combining this observation with (iii) in Lemma 7.1, one gets that $\mathbf{\Sigma}$ is a graph above B_1 in the sense that

$$x\in\Sigma(u^t)\cap\Sigma(u^s)\quad\Rightarrow\quad s=t.$$

We now turn to the "cleaning lemmas", namely Lemmas 7.10 and 7.11. Using a barrier argument, we show that if u^0 is $O(r^\kappa)$-close to a polynomial ansatz in B_r, then u^t is positive in B_r as soon as $t\sim r^\kappa$: thus the contact set was "cleaned" from B_r. The larger the κ, the faster this cleaning takes place. Then we combine this reasoning with the polynomial expansions given by the $\mathcal{P}_k$.

Lemma 7.10. *Let $u\in C^0(\bar B_1\times[-1,1])$ solve (7-1) and satisfy the uniform monotonicity condition (7-8). Assume $(0,0)\in\mathbf{\Sigma}$, and let $\mathcal{P}$ be a solution of $\Delta\mathcal{P}=1$ such that*

$$|u(\cdot,0)-\mathcal{P}|\le Cr^\kappa\quad \text{in } B_r \text{ for all } r\in\left(0,\tfrac12\right)$$

for some $C,\kappa>0$. Then there exist $r_\circ, c>0$ such that

$$u(\cdot,t)\ge\mathcal{P}+crt-Cr^\kappa\quad\text{in } B_{r/4} \text{ for all } r\in(0,r_\circ).$$

Proof. This is a combination of Lemmas 9.1 and 9.2 in [Figalli et al. 2020]. □

The next result shows that, if $(x_\circ,t_\circ)\in\mathbf{\Sigma}^{\ge k+1}$, then the contact set surely disappears from $B_r(x_\circ)$ after $t-t_\circ\sim r^k$ units of time.

Lemma 7.11. *Let $u\in C^0(\bar B_1\times[-1,1])$ solve (7-1) and satisfy the uniform monotonicity condition (7-8). Suppose $(0,0)\in\mathbf{\Sigma}^{\ge k+1}$ for some $k\ge2$. Then there exists $r, C_0>0$ depending on n and k such that*

$$\{(x,t)\in B_r\times(0,1): t>C_0|x|^k\}\cap\{u=0\}=\varnothing.$$

Proof. Since $0\in\Sigma^{\ge k+1}(u^t)$, there exists $C(n,k)>0$ such that, for every $r\in\left(0,\tfrac12\right)$,

$$|u(\cdot,0)-\mathcal{P}_k|\le Cr^{k+1}\quad\text{in } B_r.$$

Moreover, recall from Proposition 3.3 that $\mathcal{P}_k$ is almost positive, in the sense of

$$\mathcal{P}_k\ge -C(n,k)|x|^{k+2}\quad\text{in } B_1.$$

Combining this with Lemma 7.10 with $\mathcal{P}=\mathcal{P}_k$ and $\kappa=k+1$, we get

$$u(\cdot,t)\ge\mathcal{P}_k+crt-Cr^{k+1}\ge -Cr^{k+1}+crt\quad\text{in } B_{r/4} \text{ for all } r\in(0,r_\circ) \text{ for all } t\ge0$$

for some $r_\circ, c>0$. Now evaluating this at $(x,t)\in\partial B_r\times(0,1)$, with $t>C_0r^k$, we get $u(x,t)>0$ as soon as r is small enough and C_0 is large enough in terms of c and C. □

We finally prove Theorem 1.2, combining Lemma 7.11 with Proposition 7.3.

Proof of Theorem 1.2. For any $k \geq 2$, we can apply Proposition 7.3 to the set $E = \mathbf{\Sigma}^{\geq k+1}$ with $\beta = n$ and $s = k+1$, as the assumptions are satisfied thanks to Lemma 7.11. Hence, we get

$$\dim_{\mathcal{H}}(\pi_t(\mathbf{\Sigma}^\infty)) \leq \dim_{\mathcal{H}}(\pi_t(\mathbf{\Sigma}^{\geq k+1})) \leq \frac{n}{k+1},$$

and (i) follows letting $k \uparrow \infty$. As noted in Remark 1.3, this bound can be improved to a Minkowski dimension bound by directly applying Lemma 4.2 in [Fernández-Real and Ros-Oton 2021], which is a refinement of Proposition 7.3.

For (ii) it suffices to show that $\pi_t(\mathbf{\Sigma} \setminus \mathbf{\Sigma}^\infty)$ has zero Hausdorff dimension. By Proposition 6.14, the set $\pi_x(\mathbf{\Sigma} \setminus \mathbf{\Sigma}^\infty)$ is countable, provided $n = 2$. On the other hand, by the strict monotonicity condition (7-8), $\mathbf{\Sigma}$ is a graph above the space variables and hence $\mathbf{\Sigma} \setminus \mathbf{\Sigma}^\infty$ is also countable; this finishes the proof. Finally, (iii) is contained in Theorem 7.9. □

We turn to the proof of Corollary 1.4. We remark that, for analytic f, we have at most countable many singular times (combining Theorem 1.2 with [Sakai 1993, Theorem 1.1]). For smooth f, Theorem 1.2 gives that singular times have zero Hausdorff dimension.

Proof of Corollary 1.4. We divide the proof into two steps.

Step 1. The set $\Sigma(u^t) \setminus \Sigma^\infty(u^t)$ is not empty at most for countably many times.

The result follows directly from Theorem 1.2 (iii) provided we show that $\{u^t\}$ satisfies the uniform monotonicity condition (7-8). For completeness we give the argument: fix $t, h > 0$ and $K \Subset \{u^t > 0\}$. For brevity, we work with the assumption that Ω is connected and thus unbounded. Notice that $w := u^{t+h} - u^t$ is harmonic in $\{u^t > 0\}$, which is connected. By Schauder estimates and Lipschitz regularity of $\partial\Omega$, we have that $\operatorname{dist}(\{u^t = 0\}, O) > \delta$ for some $\delta = \delta(n, \partial\Omega, t) > 0$. Hence we can build an open and connected set V with Lipschitz boundary such that

$$\overline{O} \cup K \subseteq V \Subset \{u^t > 0\}.$$

By comparison we have $w \geq h \cdot \phi$, where ϕ solves

$$\begin{cases} \Delta\phi = 0 & \text{in } V \setminus \overline{O}, \\ \phi = 1 & \text{on } \partial O = \partial\Omega, \\ \phi = 0 & \text{in } \partial V. \end{cases}$$

As $\phi > 0$ in $V \setminus \overline{O}$, we have $c := \min_K \phi > 0$, so, for all $h > 0$ and $x \in K$, we have

$$u^{t+h}(x) - u^t(x) \geq h \min_K \phi = ch.$$

We used that V, and hence ϕ, did not depend on h.

Step 2. The set $\Sigma^\infty(u^t)$ is not empty for at most countably many times.

Assume $0 \in \Sigma^\infty(u^0)$. Then we will show that we have an instantaneous cleaning of the zero set, that is: there exists a universal $\delta > 0$ such that $B_\delta \cap \{u^t = 0\} = \varnothing$ for all $t > 0$. In fact, referring to the classification provided in [Sakai 1993, Theorem 1.1], we have that 0 must be a "degenerate" point (case 2a), that is:

$\{u^0 = 0\} \cap B_\delta$ must be an analytic arc (it cannot be an isolated point). In particular, $\Delta u^0 = 1$ in B_δ and $u^t - u^0$ is harmonic and nonnegative in B_δ, thus it is strictly positive in $B_{\delta/2}$ since, by assumption (7-8), it cannot be the zero function.

We explain how to prove that 0 is not a "double point" (case 2b) nor a "cusp" (case 2c). If 0 was a double point, it would be the tangency point of two distinct analytic arcs, but since the expansion of u holds at any order these two arcs should have the same Taylor expansion; hence they are the same arc (so we are in case 2a). If 0 was a cusp point, the cusp should be of the form given in [Sakai 1993, Proposition 4.1]; in particular, up to a rotation, we would have two different functions $\alpha, \beta : [0, \delta) \to \mathbb{R}$ such that

$$\{u^0 = 0\} \cap B_\delta = \{(x, y) : \alpha(x) \le y \le \beta(x),\ x \ge 0\} \cap B_\delta.$$

But by the Lipschitz estimate (1-10), we get, for all $k \ge 2$,

$$|\mathcal{A}_k(x, \alpha(x))| + |\mathcal{A}_k(x, \beta(x)| \lesssim \sup_{\{u^0=0\}\cap B_r} |\partial_n \mathcal{P}_k| \lesssim r^k, \quad x \in [0, \delta).$$

This shows that the graphs of α and β are both tangent to the manifold $\{\mathcal{A}_k = 0\}$ up to order $k-1$. As k was arbitrary, this forces α and β to have the same polynomial expansions. By Proposition 4.1 in [Sakai 1993], this requires that $\alpha \equiv \beta$, a contradiction. □

Appendix A: Proof of Lemma 2.1

We quickly prove Lemma 2.1 for a solution of (2-1) with $f \in C^\delta(B_1)$ for some $\delta \in (0, 1]$. This is just an adaptation of the argument given in [Figalli and Serra 2019].

In this section we will call "universal" any constant depending on n, μ, δ, $\|f\|_{C^\delta(B_1)}$. We also assume that $0 \in \partial\{u > 0\}$ and $0 \in \Sigma(u)$, meaning that there exists a sequence $r_k \downarrow 0$ such that

$$\frac{|\{u = 0\} \cap B_{r_k}|}{|B_{r_k}|} \to 0 \quad \text{as } k \to \infty.$$

Lemma A.1. *There is a universal constant C such that, for all $r \in \left(0, \tfrac{1}{2}\right)$,*

$$r^2 \le C \sup_{\partial B_r} u, \quad \|u\|_{L^\infty(B_r)} \le Cr^2, \quad \|Du\|_{L^\infty(B_r)} \le Cr, \quad \|D^2 u\|_{L^\infty(B_r)} \le C. \tag{A-1}$$

Proof. See [Caffarelli 1998, Theorem 2 and Lemma 5]. □

From this we classify all possible blowups.

Lemma A.2. *Up to subsequences, we have that*

$$r_k^{-2} u(r_k \,\cdot\,) \rightharpoonup f(0)\, p_2 \quad \text{in } C^{1,1}_{\mathrm{loc}}(\mathbb{R}^n), \tag{A-2}$$

where p is a 2-homogeneous nonnegative polynomial with $\Delta p_2 = 1$. We denote with $\boldsymbol{P}$ the set of such polynomials.

Proof. Set $v_k := r_k^{-2} u(r_k\,\cdot) \in C^{1,1}(B_{1/r_k})$. By weak* compactness, v_k has a limit point $v \in C^{1,1}_{\mathrm{loc}}(\mathbb{R}^n)$ with $v \geq 0,\ v(0) = 0$ and

$$\|\nabla^2 v\|_{L^\infty(\mathbb{R}^n)} \leq \liminf_k \|\nabla^2 v_k\|_{L^\infty(B_{1/2r_k})} \leq \|\nabla^2 u\|_{L^\infty(B_{1/2})} \leq C.$$

Since $0 \in \Sigma(u)$, we also have that $f(r_k\,\cdot)\chi_{\{v_k=0\}} \to f(0)$ in $L^1_{\mathrm{loc}}(\mathbb{R}^n)$. A nonnegative entire function with Laplacian $f(0)$ and bounded Hessian must be in $\boldsymbol{P}$. □

Now we show that the blowups are unique using the Weiss monotonicity formula for the adjusted energy; see [Weiss 1999]. We set

$$W_\lambda(r, v) := r^{-2\lambda}\{D(r, v) - \lambda H(r, v)\}.$$

Lemma A.3. *There is a universal constant C such that, for all $p \in \boldsymbol{P}$ and $r \in (0, 1)$, we have*

$$\frac{d}{dr} W_2(r, u - f(0)p) \geq -C r^{\delta-1}. \tag{A-3}$$

Proof. Set $v := u - f(0)p$, and directly compute

$$\frac{d}{dr} W_2(r, u - f(0)p) \geq \frac{2}{r^5} \int_{B_1} (2v_r - x \cdot \nabla v_r)\Delta v_r.$$

Notice that $|\Delta v_r + r^2 f_r \chi_{\{u_r=0\}}| \leq r^2 \sup_{B_r} |f - f(0)|$. And thus

$$\begin{aligned}\int_{B_1} (2v_r - x \cdot \nabla v_r)\Delta v_r &\geq -r^2 \int_{B_1\cap\{u=0\}} (2v_r - x \cdot \nabla v_r) f_r - C \int_{B_1} |2v_r - x \cdot \nabla v_r| r^{2+\delta} \\ &\geq r^2 \int_{B_1\cap\{u=0\}} \underbrace{(2p_r - x \cdot \nabla p_r)}_{=0} f_r - C r^{4+\delta} \geq -C r^{4+\delta}.\end{aligned}$$ □

We deduce uniqueness of blowups and Monneau's almost-monotonicity formula.

Lemma A.4. *For all $p \in \boldsymbol{P}$, we have $W_2(0^+, u - f(0)p) = 0$ and*

$$\frac{d}{dr}(r^{-4} H(r, u - f(0)p)) \geq -C r^{\delta-1} \quad \textit{for all } r \in (0, 1), \tag{A-4}$$

with C universal. In particular, the blowup is unique at singular points and there exists a universal modulus of continuity $\omega : (0, 1) \to \mathbb{R},\ \omega(0^+) = 0$, such that

$$r^{-4} H(r, u - f(0)p_2) \leq \omega(r) \quad \textit{for all } r \in (0, 1),$$

provided p_2 is the blowup.

Proof. Choose some subsequence $r_k \downarrow 0$ and $p \in \boldsymbol{P}$ such that $r_k^{-2} u_{r_k} \to p$. Then, by Lemma A.3 and (A-2), we have

$$\begin{aligned} W_2(0^+, u - f(0)p) &= \lim_k W_2(r_k, u - f(0)q) = \lim_k D(1, r_k^{-2}u_{r_k} - f(0)q) - 2H(1, r_k^{-2}u_{r_k} - f(0)q) \\ &= \int_{B_1} |\nabla(p - q)|^2 - 2\int_{\partial B_1} (p - q)^2 = 0,\end{aligned}$$

where in the last step we used that p and q are 2-homogeneous and $\Delta p = \Delta q$.

Integrating (A-3) we get $W_2(r, v) \geq -Cr^\delta$, so by direct computation

$$\begin{aligned}\frac{d}{dr}(r^{-4}H(r, u - f(0)p)) &= \frac{2}{r}\left\{W_2(r, v) + \frac{1}{r^4}\int_{B_1} v_r \Delta v_r\right\}\\ &\geq \frac{2}{r}\left\{-Cr^\delta + \int_{B_1\cap\{u_r=0\}} \underbrace{f(0)pf(r\,\cdot)}_{\geq 0} - Cr^\delta\right\}\\ &\geq -Cr^{\delta-1}.\end{aligned}$$

This immediately gives uniqueness of the blowups, let us prove the existence of a universal rate of convergence of u to such blowups. Arguing by contradiction, one finds $\epsilon > 0$ and u_k such that

$$r_k^{-4}H(r_k, u_k - f(0)p_{2,k}) \geq \epsilon.$$

Setting $v_k := r_k^{-2}u_k(r_k\,\cdot)$ and arguing as in Lemma A.2, one finds $q \in \boldsymbol{P}$ such that $v_k \to f(0)q$ in $C^1_{\mathrm{loc}}(\mathbb{R}^n)$. Now we get a contradiction using Monneau's monotonicity on u_k and q:

$$\begin{aligned}\epsilon &\leq H(1, v_k - f(0)p_{2,k}) \lesssim H(1, v_k - f(0)q) + H(1, f(0)q - f(0)p_{2,k})\\ &\leq H(1, v_k - f(0)q) + r_k^{-4}H(r_k, u_k - f(0)q) + Cr_k^\delta\\ &\leq 2H(1, v_k - f(0)q) + Cr_k^\delta \to 0\end{aligned}$$

as $k \to \infty$. □

From now on we will denote with $f(0)p_2$ the unique blowup. Let us give several preliminary estimates on the function $v := u - f(0)p_2$.

Lemma A.5. *Take any $p \in \boldsymbol{P}$ and set $v_r := (u - f(0)p)_r$. Then the following estimates hold with universal constants for all $r \in \left(0, \tfrac{1}{2}\right)$:*

$$\begin{aligned}\Delta v_r &= -r^2 f_r \chi_{\{u_r=0\}} + O(r^{2+\delta}),\\ \|v_r\|_{L^\infty(B_1)} &\lesssim \|v_r\|_{L^2(B_2\setminus B_{1/2})} + r^{2+\delta},\\ r^2|\{u_r = 0\}\cap B_1| &\lesssim \|v_r\|_{L^2(B_2\setminus B_{1/2})} + r^{2+\delta},\\ v_r\Delta v_r &= r^4 f(0) f_r p_2 \chi_{\{u_r=0\}} + v_r O(r^{2+\delta}),\\ \|\nabla v_r\|_{L^2(B_1)} &\lesssim \|v_r\|_{L^2(B_2\setminus B_{1/2})} + r^{2+\delta},\\ [v_r]_{C^{0,\delta/(2\delta+n-1)}} &\lesssim \|v_r\|_{L^2(B_2\setminus B_{1/2})} + r^{2+\delta}, \quad \textit{provided } \dim\{p = 0\} = n - 1.\end{aligned}$$

Proof. The first is a direct computation exploiting Hölder continuity of f.

For the second we notice that

- $\Delta v \leq Cr^\delta$ in B_r and $\Delta v \geq -Cr^\delta$ in $B_r \cap \{u > 0\}$.
- $v \leq 0$ in $B_r \cap \{u = 0\}$.

Hence sub- and superharmonic comparisons give the result as in Lemma 3.5.

For the third, we choose $\chi_{B_1} \le \eta \le \chi_{B_2}$ and compute

$$\begin{aligned}\mu r^2|\{u_r=0\}\cap B_1| &\le \int_{B_1} r^2 f_r \chi_{\{u=0\}} \le C r^{2+\delta} - \int_{B_1} \Delta v_r \\ &= \int_{B_1} \underbrace{\Delta\Big(\frac{Cr^{2+\delta}|x|^2}{2n} - v_r\Big)}_{\ge 0} \le \int_{B_2} \Delta\Big(\frac{Cr^{2+\delta}|x|^2}{2n} - v_r\Big)\eta \\ &\le C_\eta(\|v_r\|_{L^2(B_2\setminus B_1)} + r^{2+\delta}).\end{aligned}$$

The fourth is a direct computation.

Since $v_r\Delta v_r \ge -Cr^{2+\delta}|v_r|$, for the fifth we can use the Caccioppoli inequality:

$$\begin{aligned}\int_{B_2} \eta^2|\nabla v_r|^2 &= -2\int_{B_2} \eta v_r \nabla v_r\cdot\nabla\eta - \int_{B_2}\eta^2 v_r\Delta v_r \\ &\le 4\|\eta\nabla v_r\|_{L^2(B_2)}\|v_r\|_{L^2(B_2\setminus B_1)} + Cr^{2+\delta}\int_{B_2}|v_r| \\ &\le \tfrac{1}{2}\|\eta\nabla v_r\|^2_{L^2(B_2)} + C(\|v_r\|^2_{L^2(B_2\setminus B_1)} + r^{2(2+\delta)}),\end{aligned}$$

where η is as above.

For the sixth, assume $p_2 = \frac{1}{2}x_n^2$ and consider, for $0<t<1$, $j\ne n$, the function

$$w_\pm(x) := \frac{v_r(x\pm te_j)-v_r(x)}{t^\delta} = \frac{u_r(x\pm te_j)-u_r(x)}{t^\delta}.$$

Notice that, with constants uniform in t, we have

$$\|w_\pm\|_{L^2(B_2\setminus B_{1/2})} \lesssim t^{1-\delta}\|\nabla v_r\|_{L^2(B_4\setminus B_{1/4})} \lesssim \|v_r\|_{L^2(B_8\setminus B_{1/8})} + r^{2+\delta}.$$

On the other hand, in $\{u_r>0\}\cap B_1$ we have $\Delta w_\pm \lesssim r^{2+\delta}$, and in $\{u_r=0\}\cap B_1$ we have $w_\pm \ge 0$. Thus the function

$$\min\Big\{w_\pm + Cr^{2+\delta}\frac{1-|x|^2}{2n}; 0\Big\}$$

is superharmonic in B_1. Using the minimum principle and the previous estimate, we get

$$\min_{\bar B_1} w_\pm \ge -C(\|v_r\|_{L^2(B_4\setminus B_{1/4})} + r^{2+\delta}).$$

By the symmetry $w_\pm(x\mp te_j) = -w_\mp(x)$, we also have the upper bound on a smaller ball. Since all the constants are uniform in t, we conclude, using the following estimate (see Lemma C.1),

$$[f]_{C^{0,\delta/(2\delta+n-1)}(B_1)} \lesssim_{n,\delta} \sum_{j=1}^{n-1} \sup_{x\in B_1,|t|\le 1} \frac{|f(x+te_j)-f(x)|}{|t|^\delta} + \|\partial_n f\|_{L^2(B_2)},$$

valid for every $f\in \mathrm{Lip}(B_2)$. □

Since the blowup is well defined, we can from now on assume to be in the top-dimensional stratum, that is $p_2 = \frac{1}{2}x_n^2$. Arguing as is Section 4, we exploit the truncated frequency ϕ^γ with some $\gamma(\delta) > 2$.

Lemma A.6. *Let $p \in \boldsymbol{P}$ and $\gamma = 2 + \frac{1}{8}\delta$, and set $v := u - f(0)p$. Then there is $\varepsilon = \varepsilon(\delta) > 0$ such that the following inequalities hold for all $r \in (0, 1)$:*

$$\phi^\gamma(r, v) \geq 2 - Cr^\varepsilon, \quad \phi^\gamma(r, v) \leq C, \quad \frac{d}{dr}\phi^\gamma(r, v) \geq -Cr^{\varepsilon-1}, \tag{A-5}$$

with C universal. Furthermore, we also have

$$\frac{\int_{B_1} v_r \Delta v_r}{H(r, v) + r^{2\gamma}} \geq -Cr^\varepsilon. \tag{A-6}$$

Proof. For the first inequality in (A-5), we employ Lemma A.3 and get

$$\begin{aligned}\phi^\gamma(r, v) - 2 &= \frac{D(r) - 2H(r) + (\gamma - 2)r^{2\gamma}}{H(r) + r^{2\gamma}} \\ &\geq \frac{W_2(r)}{r^{-4}H(r) + r^{2\gamma-4}} \geq -Cr^{\delta-2(\gamma-2)},\end{aligned}$$

so we can set $\varepsilon := \frac{3}{4}\delta$. For the second, we need to estimate from below with $-Cr^{\varepsilon-1}$ the term

$$\frac{2}{r(H(r) + r^{2\gamma})} \int_{B_1} (\lambda_r v_r - x \cdot \nabla v_r)\Delta v_r \, dx,$$

where for brevity $\lambda_r := \phi^\gamma(r, v)$ (see Proposition 4.2). Recall that

$$|\Delta v_r + r^2 f_r \chi_{\{u_r=0\}}| \leq Cr^{2+\delta},$$

and estimate each term recalling that p_2 is 2-homogeneous:

$$\begin{aligned}\int_{B_1} (\lambda_r v_r - x \cdot \nabla v_r)\Delta v_r &= -r^2 \int_{B_1 \cap \{u_r=0\}} (\lambda_r v_r - x \cdot \nabla v_r) f_r - Cr^{2+\delta} \int_{B_1} |\lambda_r v_r - x \cdot \nabla v_r| \\ &\geq r^2 \int_{B_1 \cap \{u_r=0\}} (\lambda_r p_r - 2p_r) f(0) f_r - Cr^{4+\delta}(\lambda_r + 1) \\ &\geq r^4 \underbrace{(\lambda_r - 2)}_{\geq -Cr^\varepsilon} \underbrace{\int_{B_1 \cap \{u_r=0\}} pf(0) f_r}_{\geq 0} - Cr^{4+\delta}(\lambda_r + 1) \\ &\geq -Cr^4(r^\varepsilon + r^\delta(\lambda_r + 1)),\end{aligned}$$

so with crude bounds the frequency solves the ODI

$$\lambda_r' \geq -Cr^{3-2\gamma}(r^\varepsilon + r^\delta(\lambda_r + 1)) \geq -Cr^{3+\varepsilon-2\gamma}(\lambda_r + 1). \tag{A-7}$$

From here we see that $\log(1 + \lambda_r)$ is almost monotone and bounded above by some constant, provided $\gamma < 2 + \frac{1}{2}\varepsilon$. Thus plugging this back into (A-7), we get

$$\lambda_r' \geq -Cr^{3+\varepsilon-2\gamma},$$

which was the claim up to redefining ε. Equation (A-6) follows as in the proof of Lemma A.4 above. □

Hence $\phi^\gamma(0^+, v) \geq 2$ exists for all p, and we want to show that there is a universal number $\alpha_\circ > 0$ such that $\phi^\gamma(0^+, u - f(0)p_2) \geq 2 + 2\alpha_\circ$, provided p_2 is indeed the blowup at 0. Let us show how to conclude from here. Up to universal constants we have the following: By Lemma A.5, we have

$$\|v\|_{L^\infty(B_1)} \lesssim \|v_r\|_{L^2(B_2 \setminus B_{1/2})} + r^{2+\delta}.$$

But $\phi^\gamma \leq C$ in (0, 1), so by Lemma 2.2, we have in turn

$$\|v_r\|^2_{L^2(B_2 \setminus B_{1/2})} \lesssim H\big(\tfrac{1}{2}r\big) + r^{2\gamma}$$

and $\gamma > 2$. Now, since $\phi^\gamma(0^+, v) = 2 + 2\alpha_\circ$, we have, again by Lemma 2.2, that

$$H(r) + r^{2\gamma} \lesssim r^{2(2+2\alpha_\circ)},$$

hence putting everything together we obtain Lemma 2.1:

$$\|v\|_{L^\infty(B_1)} \lesssim r^{2+2\alpha_\circ}.$$

So we are left to show that

$$\lambda_2(0) := \phi^\gamma(0^+, u - f(0)p_2) \geq 2 + 2\alpha_\circ, \tag{A-8}$$

and it is also clear that we can work under the assumption that $\lambda_2(0) \leq 2 + \frac{1}{16}\delta$, otherwise (A-8) holds with $\alpha_\circ = \frac{1}{64}\delta$. The following proposition is crucial and the proof follows the same line of Proposition 5.9 (or also of [Figalli and Serra 2019, Proposition 2.12]). As the only technical complications are settled by the bounds gathered in Lemma A.5, we omit the proof.

Lemma A.7. *Assume $0 \in \Sigma_{n-1}$ and $\lambda_2 \leq 2 + \frac{1}{16}\alpha$. Then the sequence*

$$\tilde{v}_r := \frac{v_r}{\|v_r\|_{L^2(\partial B_1)}}$$

is bounded in $W^{1,2}_{\mathrm{loc}}(\mathbb{R}^n) \cap C^{\delta/(2\delta+n-1)}_{\mathrm{loc}}(\mathbb{R}^n)$. Furthermore, every accumulation point of $\{v_r\}_{r>0}$ solves the Signorini problem (5-4) and is λ_2-homogeneous.

The following combination of Monneau monotonicity and the characterization of blowups will prove (A-8). The proof is in fact very similar to Step 5 in the proof of Proposition 5.9.

Lemma A.8. *There cannot be sequences u_k, f_k, μ_k, δ_k, with $0 \in \partial\{u_\ell > 0\}$ and*

$$\sup_\ell \left(\|f_\ell\|_{C^{\delta_\ell}(B_1)} + \frac{1}{\delta_\ell} + \frac{1}{\mu_\ell} \right) < +\infty,$$

such that $\lambda_2^{(k)} \downarrow 2$, where

$$\lambda_2^{(k)} := \phi^{2+\delta_k/8}(0^+, u_k - f_k(0)p_2^{(k)}).$$

In particular, (A-8) holds for some $\alpha_\circ = \alpha_\circ(n, k, \delta, \|f\|_{C^\delta(B_1)}) \in (0, 1)$.

Proof. **Step 1**. If $\tilde{v}_{r_k} \rightharpoonup q$ in $W^{1,2}_{\mathrm{loc}}$ then, for all $p \in \boldsymbol{P}$, we have

$$\int_{\partial B_1} q(p_2 - p) \geq 0. \tag{A-9}$$

Proof of Step 1. Define $\varepsilon_k^2 := H(r_k, v)$ and notice that, by the growth Lemma 2.2 and the compactness of the trace operator, we have, for k large,

$$r_k^\delta \ll \varepsilon_k \to 0,$$

where we used that $\phi^\gamma(r_k) \le 2 + \frac{1}{100}\delta$ for all k large enough. By Monneau monotonicity (Lemma A.4) applied to p instead of p_2, we have

$$\int_{\partial B_1} (\varepsilon_k \tilde{v}_{r_k} + p_2 - p)^2 + C r_k^\delta \ge \int_{\partial B_1} (p_2 - p)^2,$$

computing the squares and dividing by ε_k we get

$$\varepsilon_k \int_{\partial B_1} \tilde{v}_{r_k}^2 + 2\int_{\partial B_1} \tilde{v}_{r_k}(p_2 - p) + C\frac{r_k^\delta}{\varepsilon_k} \ge 0,$$

and sending $k \uparrow \infty$ we obtain (A-9). We remark that all the constants in these computations are universal.

Step 2. If q is a 2-homogeneous harmonic polynomial such that (A-9) holds for all $p \in \boldsymbol{P}$, then $q \le 0$ on the hyperplane $\{p_2 = 0\}$.

Proof of Step 2. This is exactly [Figalli and Serra 2019, Lemma 2.12].

Step 3. For each $u_k,\ f_k,\ \mu_k,\ \delta_k$ as in the assumptions, Lemma A.6 gives q_k, a $\lambda_2^{(k)}$-homogeneous solution of the Signorini problem with $\|q_k\|_{L^2(\partial B_1)} = 1$. It is easy to see that, by compactness, $q_k \to q$, where q is a 2-homogeneous solution of Signorini with $\|q\|_{L^2(\partial B_1)} = 1$. Thus, q is a harmonic polynomial, nonnegative on the thin obstacle (see Proposition 2.4). But this contradicts Step 1, up to taking a diagonal subsequence. A careful verification that all the bounds are uniform is the same as Step 1 in the proof of Proposition 5.9, and it is not repeated here. □

Appendix B: Adaptations for general right-hand sides

In this section we collect the modification needed to work with a general f and μ.

The main difference is that $u - \mathcal{P}_k$ will not be harmonic in $\{u > 0\} \cap B_r$, but its Laplace operator will be of size $O(r^k)$. This is the size of the error we would have in every estimate. Keeping this in mind, it is clear that all the arguments go through with the same proof, provided we can indeed construct $\mathcal{P}_k$ with the same properties as before. This is not a hard task. We will, for completeness, list also the other modifications needed. Let us remark that all constants that in the case $f \equiv 1$ depend on n and k will now also depend on μ and $\|f\|_{C^k}$.

In the following, we provide a generalization of Section 3.1. We begin with the respective polynomial ansatz, which will additionally depend on the Taylor expansion of f and on the center of expansion. We will denote by $F_{k,x}$ the k-th Taylor polynomial of f at x, that is

$$F_{k,x}(h) := \sum_{|\alpha| \le k} \frac{\partial^\alpha f(x)}{\alpha!} h^\alpha.$$

The sets $\boldsymbol{P}_k$ and V_j are the same as in Section 3.1.

Lemma B.1. *Let $k \ge 2$, $f \in C^{k-1}(B_1)$, $x \in B_1$ and $(p_2, \dots, p_k) \in \boldsymbol{P}_k$ be given. Let ν be any unit vector such that $p_2(h) = \frac{1}{2}(h \cdot \nu)^2$. There exists a unique collection of polynomials*

$$(R_1, \dots, R_{k-1}) \in V_1 \times \cdots \times V_{k-1}$$

such that if we define the polynomial

$$\mathcal{A}_{x,k,\nu}(h) := (\nu \cdot h) + \sum_{j=1}^{k-1} (\nu \cdot h) R_j(y) + \sum_{j=3}^{k} \frac{p_j(h)}{(\nu \cdot h)},$$

then

$$\Delta\big(\tfrac{1}{2} f(x) \mathcal{A}^2_{x,k,\nu}\big)(h) = F_{k-1,x}(h) + O(|h|^k).$$

Furthermore, each R_j is determined (analytically) only by $(p_2, \dots, p_{j+1})$ and the coefficients of $F_{k,x}$. In particular, each R_j does not depend on ν, so $\mathcal{A}_{x,k,-\nu} = -\mathcal{A}_{x,k,\nu}$.

Proof. The proof is almost identical to the proof of Lemma 3.1, the only difference being that we have to take into account the Taylor expansion of f. Let us work out explicitly the case $k = 2$. By a direct computation we find

$$\Delta\big(\tfrac{1}{2} f(x) \mathcal{A}^2_{x,k,\nu}\big)(h) = f(x) + \Delta(2 f(x) p_2 R_1)(h) + O(|h|^2).$$

Thus, the right (and unique) choice for R_1 is

$$R_1 := \frac{1}{2 f(x)} \delta_1^{-1}(F_{1,x}),$$

where the linear isomorphisms $\delta_m : V_m \to V_m$ were introduced in the proof of Lemma 3.1. ☐

Using Lemma B.1, we can define the polynomial ansatz functions $\mathcal{A}^2_k, \mathcal{P}_k : B_1 \times \boldsymbol{P}_k \to \mathbb{R}[h]$, which now depend explicitly also on the center of expansion x. We set

$$\mathcal{A}_k(x; p_2, \dots, p_{k-1}) := \mathcal{A}^2_{x,k,\nu}, \quad \mathcal{P}_k(x; p_2, \dots, p_{k-1}) := \pi_{\le k+1}\big(\tfrac{1}{2} f(x) \mathcal{A}^2_{x,k,\nu}\big),$$

and notice that the dependence on f is hidden in the dependence on x. Once again any norm of $\mathcal{P}_k$ is bounded by constants depending on n, k, $\|f\|_{C^{k-1}(B_1)}$ and $|(p_2, \dots, p_k)|$. Furthermore, the function $\mathcal{P}_k(x; \cdot)$ is injective.

With this construction we obtain

$$\Delta \mathcal{P}_k(x; p_2, \dots, p_k)(h) = f(x+h) + O(|h|^k) \quad \text{and} \quad \mathcal{P}_k(x; p_2, \dots, p_k)(h) \ge -C|h|^{k+2}, \tag{B-1}$$

where the big O is a C^k function of x and h. Here comes the only difference with the case in which $f \equiv 1$. When we apply the Laplace operator to the function $v := u(x_\circ + \cdot\,) - \mathcal{P}_k(x_\circ; p_2, \dots, p_k)$, we get, in $B_r(x_\circ)$,

$$\Delta v = -f(x_\circ + \cdot\,)\chi_{\{u(x_\circ + \cdot\,) = 0\}} + O(r^k), \tag{B-2}$$

while in the case $f \equiv 1$ we had $\Delta v = -\chi_{\{u(x_\circ + \cdot\,)=0\}}$ exactly.

We now state an analog of Proposition 3.3, which contained all crucial properties of the ansatz.

Proposition B.2. *Let $k \geq 2$, $(p_2, \dots, p_k) \in \boldsymbol{P}_k$ and $\tau > 0$ be such that $|(p_2, \dots, p_k)| \leq \tau$. Choose some unit vector ν for which $p_2(x) = \frac{1}{2}(\nu \cdot x)^2$. Let $f \in C^{k-1,1}(B_1)$, and let $|x_\circ| < \frac{1}{2}$. Then the polynomials $\mathcal{A}_k^2(x_\circ; p_2, \dots, p_k)$ and $\mathcal{P}_k(x_\circ; p_2, \dots, p_k)$ satisfy:*

(i) $\Delta \mathcal{P}_k = f(x_\circ + \cdot\,) + O(|\cdot|^k)$ *and* $\partial_e\big(\frac{1}{2}\mathcal{A}_k^2\big) = \partial_e \mathcal{P}_k + O(|\cdot|^{k+1})$ *for any unit vector e.*

(ii) *We have* $\mathcal{P}_k(x_\circ; p_2, \dots, p_k) = \mathcal{P}_{k-1}(x_\circ, p_2, \dots, p_{k-1}) + p_k + O(|\cdot|^{k+1})$.

(iii) *For all $|h| \leq r_0$, we have $\frac{1}{2} \leq |\partial_\nu \mathcal{A}_{x_\circ,k,\nu}(h)| \leq 2$, and thus*

$$\tfrac{1}{2}|\mathcal{A}_k(h)| \leq \big|\partial_\nu\big(\tfrac{1}{2}\mathcal{A}_k^2(h)\big)\big| \leq 2|\mathcal{A}_k(h)|,$$

where $r_0 = r_0(n, k, \tau, \|f\|_{C^k(B_1)}) \in (0, 1)$.

(iv) *If u is a solution as in* (2-1), $0 \in \Sigma_{n-1}$ *and* $r^{-2}u(r\,\cdot\,) \to p_2$, *then by* (2-3) *we have, for all* $0 < r < \frac{1}{2}$,

$$\sup_{B_r(x_\circ)\cap\{u=0\}} |\partial_\nu \mathcal{P}_k| \leq C r^{1+\alpha_\circ}$$

for some constant $C = C(n, k, \tau, \|f\|_{C^k})$.

Now that the polynomial ansatz has the right formal properties (i.e., (B-1), (B-2) and those collected in Proposition B.2), it is simple to check that the rest of the arguments go through. The rest of this section is a list of the modifications needed to obtain Theorem 1.1 in its full generality.

- Lemmas 3.5 and 3.6 are the same: even if, in $B_r \cap \{u > 0\}$, the function $u - \mathcal{P}_k$ is not harmonic, pointwise we have $\Delta(u - \mathcal{P}_k) = O(r^k)$. The comparison principle we use remains valid in this case.

- The proof of Lemma 3.7 is identical, except for the fact that in Ω our function is not harmonic. This is used only in (3-12), where we have the term $\|\Delta v_r\|_{L^\infty(\tilde{\Omega}\cap B_1)}$, but it can be absorbed into the term Cr^{k+2}.

- In Lemma 4.1, we also pick up an extra term, which, however, is much smaller than the one we are estimating. Indeed we have

$$\int_{B_1} |v_r \Delta v_r| \leq M \int_{B_1\cap\{u=0\}} |v_r| + Cr^k \int_{B_1} |v_r|.$$

The first integral is treated as in Lemma 4.1. Since we have $|v_r| \lesssim r^2$ around a contact point, for the second term we estimate

$$\frac{1}{r}\frac{Cr^{k+2}}{H(r, v) + r^{2\gamma}} \lesssim r^{k+1-\gamma} = r^{\epsilon-1}.$$

As $|x \cdot \nabla v_r| \lesssim r^2$ in B_r, the same reasoning applies to the term $\int_{B_1} |(x \cdot \nabla v_r)\Delta v_r|$.

The rest of Section 4 goes on with exactly the same proofs.

- Section 5 essentially uses the statements of two previous sections as black boxes. The only modification is in the very definition of the sets $\Sigma^{k\text{-th}}$, namely, for $x_\circ \in \Sigma^{k\text{-th}}$, we use the ansatz

$$\mathcal{P}_{k,x_\circ} := \mathcal{P}_k(x_\circ; p_{2,x_\circ}, \dots, p_{k,x_\circ}),$$

which is again continuous in the $x_\circ$ variable. Notice that, to use our argument, we need it to make sense to construct $\mathcal{P}_k$ in $\Sigma^{k\text{-th}}$; hence we require that, at a minimum, $f \in C^k(B_1)$.

• Concerning Section 6, note that we can no longer use that $\Delta\mathcal{P}_k = 0$ in $\{u > 0\}$. Thus in Lemma 6.1 we introduce a small modification, namely in the term

$$\int_{B_1} P\Delta v_r = -\int_{B_1\cap\{u_r=0\}} fP + O(r^{k+2})\int_{B_1} P,$$

but we underline that the extra factor $O(r^{k+2})$ does not affect any subsequent computation.

• In Lemma 6.3, the definition of the barrier function needs to be adapted. Namely (6-5) must be replaced with

$$\begin{cases} \phi_{z,\ell}(z) = 0, & \\ \phi_{z,\ell} \geq 0 & \text{in } B_\rho(z), \\ \Delta\phi_{z,\ell} < r_\ell^2 f(r_\ell\,\cdot\,) & \text{in } \overline{B_\rho(z)}, \\ u(r_\ell\,\cdot\,) < \phi_{z,\ell} & \text{on } \partial B_\rho(z), \end{cases}$$

so that the proofs of Claim (ii) and Claim (iii) are the same. For Claim (i) we must use the following barrier:

$$\phi_{z,\ell}(x) := \left(1 - \frac{h_\ell}{r_\ell^2}\right)\frac{f(0)}{2}\mathcal{A}_k^2(r_\ell x) + \frac{h_\ell}{4nM}|x' - z'|^2.$$

• Sections 6.1, 6.2, 6.3 do not require further modifications.

• In the proof of Lemma 6.16, the constant of (6-11) depends on $\|\nabla^k f\|_{L^\infty}$.

• As they are based on Section 6, Sections 7.1 and 7.2 do not require any modification.

• In Section 7.3, it is easily checked that Lemma 7.11 works if we assume that $\Delta\mathcal{P} = f + O(r^\kappa)$ instead of $\Delta\mathcal{P} = 1$, and the cleaning works just as before.

Appendix C: Auxiliary lemmas

Lemma C.1. *For every $u \in \mathrm{Lip}(B_2)$, $1 \leq j \leq n$ and $\beta \in (0, 1]$, define*

$$[\delta_j u]_\beta := \sup_{x\in B_1, |t|\leq 1} \frac{|u(x + te_j) - u(x)|}{|t|^\beta}.$$

Then, for all $p > 1$,

$$[u]_{C^{0,\sigma}(B_1)} \leq C\left(\sum_{1\leq j<n} [\delta_j u]_\beta + \|\partial_n u\|_{L^p(B_2)}\right),$$

where $C = C(n, \beta, p)$ and $\sigma = \beta(p-1)/(\beta p + n - 1)$.

Proof. By homogeneity we can assume that the right-hand side is 1. Set $h = (0, \dots, 0, r)$, and consider $A_r := B'_{r^\alpha} \times [0, r]$, where $\theta > 0$ is small. By Fubini's theorem, we can find some $z' \in B'_{r^\theta}$ such that

$$\int_0^r |\partial_n u(z', s)|^p\, ds \leq r^{\theta(1-n)}\|\partial_n u\|^p_{L^p(A_r)} \leq r^{\theta(1-n)}.$$

The fundamental theorem of calculus and Hölder's inequality give

$$\begin{aligned}|u(0)-u(h)| &\le |u(0)-u(z',0)|+|u(z',0)-u(z',r)|+|u(z',r)-u(0,r)|\\ &\le 2n|z'|^{\beta}+\int_0^r |\partial_n u(z',s)|\,ds\\ &\lesssim_n r^{\theta\beta}+r^{\theta(1-n)/p}r^{1/p'}.\end{aligned}$$

Since $|h|=r$, u is σ-Hölder continuous for every $\sigma\le\min\{\theta\beta,\theta(1-n)/p+1/p'\}$, and maximizing with respect to the interpolation parameter $\theta>0$ we get the optimal value for σ. □

Let us finally give, for completeness, the proof of our statement of Whitney's theorem for C^∞ functions.

Proof of Theorem 6.15. Let us define the functions $f_\alpha: E\to\mathbb{R}$ for each multi-index $\alpha\in\mathbb{N}^n$ by

$$f_\alpha(x):=\partial^\alpha P_{|\alpha|,x}(\,\cdot\,)\quad (=\partial^\alpha P_{|\alpha|+\ell,x}(0)\text{ for all }\ell\in\mathbb{N}).$$

Assumption (ii) with $k=|\alpha|$ immediately gives

$$|f_\alpha(x)-f_\alpha(y)|=|\partial^\alpha P_{|\alpha|,x}(0)-\partial^\alpha P_{|\alpha|,y}(x-y)|\le C(|\alpha|)|x-y|.$$

Thus each f_α admits a canonical Lipschitz extension to $\bar{E}$, which we don't rename.

For each $x, y, \in\bar{E}$, $m\in\mathbb{N}$ and $|\alpha|\le m$, define the remainder

$$R_{m,\alpha}(x,y):=f_\alpha(x)-\sum_{|\beta|\le m-|\alpha|}\frac{f_{\alpha+\beta}(y)}{\beta!}(x-y)^\beta. \tag{C-1}$$

If we show that each remainder satisfies $|R_{\alpha,m}(x,y)|\le C(m)|x-y|^{|\alpha|-m+1}$ for all $x,y\in E$, then we can extend it by continuity, so that it holds in the full $\overline{E\times E}$, and conclude by applying [Whitney 1934, Theorem I] verbatim. To check this, notice that the left-hand side of assumption (ii) is just (C-1) in disguise:

$$\begin{aligned}R_{m,\alpha}(x,y)&=\partial^\alpha P_{m,x}(0)-\sum_{|\beta|\le m-|\alpha|}\frac{\partial^\beta(\partial^\alpha P_{m,y})(0)}{\beta!}(x-y)^\beta\\ &=\partial^\alpha P_{m,x}(0)-\partial^\alpha P_{m,y}(x-y)=O(|x-y|^{m-|\alpha|+1}),\end{aligned}$$

where we used that polynomials equal their Taylor expansion of sufficiently high degree (and here $\deg\partial^\alpha P_{m,y}\le m-|\alpha|$). This concludes the proof. □

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FEDERICO FRANCESCHINI: ffederico@ias.edu
Department of Mathematics, ETH Zürich, Zürich, Switzerland
Current address: *School of Mathematics, Institute for Advanced Study, Princeton, NJ, United Stated*

WIKTORIA ZATOŃ: wiktoria.beckmann@gmail.com
Institut für Mathematik, Universität Zürich, Zürich, Switzerland

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