ANALYSIS & PDE

Volume 18 No. 10 2025

FLORIAN GRUBE AND MORITZ KASSMANN

ROBUST NONLOCAL TRACE AND EXTENSION THEOREMS

msp

ANALYSIS AND PDE
Vol. 18 (2025), No. 10, pp. 2367–2414
DOI: 10.2140/apde.2025.18.2367

ROBUST NONLOCAL TRACE AND EXTENSION THEOREMS

FLORIAN GRUBE AND MORITZ KASSMANN

We prove trace and extension results for Sobolev-type function spaces that are well suited for nonlocal Dirichlet and Neumann problems including those for the fractional p-Laplacian. Our results are robust with respect to the order of differentiability. In this sense they align with the classical trace and extension theorems.

1. Introduction

We are concerned with well-posedness of nonlinear nonlocal equations in bounded domains, such as

$$\begin{aligned} (-\Delta)_p^s u &= f \quad \text{in } \Omega, \\ u &= g \quad \text{in } \mathbb{R}^d \setminus \Omega, \end{aligned} \tag{1-1}$$

where the fractional p-Laplacian is defined via

$$(-\Delta)_p^s u(x) = (1-s)\,\text{p.v.}\int_{\mathbb{R}^d} |u(x)-u(y)|^{p-2}(u(x)-u(y))\frac{\mathrm{d}y}{|x-y|^{d+sp}}.$$

A standard approach to problems like (1-1) is the variational approach, which is based on an energy functional and corresponding function spaces. Since the operator $(-\Delta)_p^s u$ is nonlocal, it is necessary to prescribe values $u(x)$ for $x \in \mathbb{R}^d \setminus \Omega$ in order for (1-1) to be well-posed. A possible yet restrictive option is to work in the Sobolev–Slobodeckij space $W^{s,p}(\mathbb{R}^d)$. Note that an assumption of the type $g \in W^{s,p}(\mathbb{R}^d)$ imposes unnatural restrictions since problem (1-1) does not involve any regularity of g in $\mathbb{R}^d \setminus \overline{\Omega}$ other than some weighted integrability. Popular workarounds include assumptions of the type $g \in W^{s,p}(\Omega_\varepsilon) \cap L^p(\mathbb{R}^d; (1+|x|)^{-d-sp}\,\mathrm{d}x)$ for some enlarged domain $\Omega_\varepsilon = \{x \in \mathbb{R}^d \mid \operatorname{dist}(x, \overline{\Omega}) < \varepsilon\}$.

We introduce and study trace spaces on $\mathbb{R}^d \setminus \Omega$ that allow for a natural variational approach to nonlocal nonlinear problems. An important feature of our approach is the robustness of our results as $s \to 1^-$. This allows for a theory of well-posedness for problems like (1-1) that is continuous in the parameter s at $s = 1$. In this case, problem (1-1) reduces to

$$\begin{aligned} -\operatorname{div}(|\nabla u|^{p-2}\nabla u) &= f \quad \text{in } \Omega, \\ u &= g \quad \text{on } \partial\Omega. \end{aligned}$$

Financial support by the German Research Foundation (GRK 2235 - 282638148, SFB 1283 - 317210226) is gratefully acknowledged.
MSC2020: primary 35J25, 35J60, 45G05, 46E35, 47G20; secondary 35A15.
Keywords: nonlocal Sobolev space, trace, extension, convergence of trace space.

In order to derive the setting of the variational approach, let us explain the definition of a weak solution to our model example (1-1). Given a sufficiently regular solution u to (1-1) and a regular test function $\varphi : \mathbb{R}^d \to \mathbb{R}$ with compact support in Ω, the following should hold:

$$\int_\Omega (-\Delta)_p^s u\, \varphi = \int_\Omega f\varphi,$$

which, after an application of Fubini's theorem, reads

$$\frac{1-s}{2} \iint_{(\Omega^c\times\Omega^c)^c} |u(x)-u(y)|^{p-2}(u(x)-u(y))(\varphi(x)-\varphi(y))\frac{\mathrm{d}y\,\mathrm{d}x}{|x-y|^{d+sp}} = \int_\Omega f\varphi. \tag{1-2}$$

This line motivates the following definition of an energy space. For a bounded open set $\Omega \subset \mathbb{R}^d$ and $1 \le p < \infty$, we consider the fractional Sobolev-type space

$$V^{s,p}(\Omega \,|\, \mathbb{R}^d) := \{u : \mathbb{R}^d \to \mathbb{R} \text{ measurable} \mid [u]_{V^{s,p}(\Omega \,|\, \mathbb{R}^d)} < \infty\}, \tag{1-3}$$

$$[u]^p_{V^{s,p}(A \,|\, B)} := (1-s)\int_A\int_B \frac{|u(x)-u(y)|^p}{|x-y|^{d+sp}}\,\mathrm{d}x\,\mathrm{d}y, \qquad A, B \in \mathcal{B}(\mathbb{R}^d). \tag{1-4}$$

We endow this space with the norm $\|u\|^p_{V^{s,p}(\Omega \,|\, \mathbb{R}^d)} := \|u\|^p_{L^p(\Omega)} + [u]^p_{V^{s,p}(\Omega \,|\, \mathbb{R}^d)}$. The space $V^{s,p}(\Omega \,|\, \mathbb{R}^d)$ is a separable Banach space and reflexive for $p > 1$; see, e.g., [Foghem Gounoue 2020, Chapter 3.4]. It is well known that this space converges to $W^{1,p}(\Omega)$ for $1 < p < \infty$ as $s \to 1^-$; see [Bourgain et al. 2001, Theorem 2] and [Foghem Gounoue 2023, Theorems 1.1, 1.3, 1.5]. In his famous work, Gagliardo [1957] proved that the classical trace $\gamma : W^{1,p}(\Omega) \to W^{1-1/p,p}(\partial\Omega)$ is linear and continuous and has a continuous right inverse. We are concerned with the search for a trace theorem and extension result for the fractional Sobolev spaces of type $V^{s,p}(\Omega \,|\, \mathbb{R}^d)$ onto the nonlocal boundary Ω^c such that the result is robust in the limit $s \to 1^-$.

Remark 1.1. In some more applied fields such as peridynamics, one studies nonlocal problems in bounded open sets Ω, where data are prescribed in a bounded open set $E \supset \Omega$; see [Mengesha and Du 2016]. Then, there is no need to discuss the decay at infinity, but the main challenge remains: quantify local behavior of functions across the boundary $\partial\Omega$. Our results apply to such problems directly as $\mathbb{R}^d$ can be replaced by a general set E.

Main results. We introduce a space of functions $\mathcal{T}^{s,p}(\Omega^c)$ defined on Ω^c, see (1-6), and prove trace and extension results which are robust in the parameter s; see Theorems 1.2 and 1.3. Lastly, we prove the asymptotic of the spaces $\mathcal{T}^{s,p}(\Omega^c)$ as well as some related weighted L^p spaces as $s \to 1^-$; see Theorem 1.4.

Due to the nonlocality of the operators under consideration, problems like (1-1) can be formulated in open sets, which are not necessarily connected. Since our main results do not require Ω to be connected, we define $\Omega \subset \mathbb{R}^d$ to be a Lipschitz domain if it is open and has a uniform Lipschitz boundary; see Section 2. We define measures

$$\mu_s(\mathrm{d}x) := \mathbb{1}_{\Omega^c}(x)(1-s)d_x^{-s}(1+d_x)^{-d-s(p-1)}\,\mathrm{d}x \tag{1-5}$$

on the Borel σ-algebra $\mathcal{B}(\mathbb{R}^d)$, $s \in (0,1)$, $1 \le p < \infty$, where $d_x := \operatorname{dist}(x, \partial\Omega)$ for $x \in \mathbb{R}^d$. We simply write $\mu_s(x)$ for the density of the measure μ_s with respect to the Lebesgue measure on $\mathbb{R}^d$. Given an open bounded set $A \subset \mathbb{R}^d$, note that $\mu_s(A) \asymp (1-s)\int_{A\cap\Omega^c} d_x^{-s}\,\mathrm{d}x$ and μ_s converges weakly for $s \to 1^-$ to the Hausdorff measure on $\partial\Omega \cap A$; see Lemma 5.1.

We introduce for $s \in (0,1)$, $1 \le p < \infty$, our trace spaces

$$\begin{aligned} \mathcal{T}^{s,p}(\Omega^c) &:= \{g : \Omega^c \to \mathbb{R} \text{ measurable} \mid \|g\|_{\mathcal{T}^{s,p}(\Omega^c)} < \infty\}, \\ \|g\|^p_{\mathcal{T}^{s,p}(\Omega^c)} &:= \|g\|^p_{L^p(\Omega^c;\mu_s)} + [g]^p_{\mathcal{T}^{s,p}(\Omega^c)}, \\ [f,g]^p_{\mathcal{T}^{s,p}(\Omega^c)} &:= \int_{\Omega^c}\int_{\Omega^c} \frac{|f(x)-f(y)|^{p-2}(f(x)-f(y))(g(x)-g(y))}{((|x-y|+d_x+d_y)\wedge 1)^{d+s(p-2)}}\mu_s(\mathrm{d}x)\mu_s(\mathrm{d}y). \end{aligned} \tag{1-6}$$

Here, we use the convention $[g]_{\mathcal{T}^{s,p}(\Omega^c)} = [g,g]_{\mathcal{T}^{s,p}(\Omega^c)}$. The space $\mathcal{T}^{s,p}(\Omega^c)$ is a separable Banach space (Hilbert space for $p = 2$) and reflexive for $p > 1$; see Lemma 2.2. Now we state the trace result and extension result for $p > 1$.

Theorem 1.2. *Let $\Omega \subset \mathbb{R}^d$ be a bounded Lipschitz domain, $s \in (0,1)$, $1 < p < \infty$. Then the trace operator*

$$\mathrm{Tr_s} : V^{s,p}(\Omega \mid \mathbb{R}^d) \to \mathcal{T}^{s,p}(\Omega^c), \quad u \mapsto u|_{\Omega^c},$$

is continuous and linear and there exists a continuous linear right inverse

$$\mathrm{Ext_s} : \mathcal{T}^{s,p}(\Omega^c) \to V^{s,p}(\Omega \mid \mathbb{R}^d), \quad g \mapsto \mathrm{Ext_s}(g),$$

which we call the nonlocal extension operator. Moreover, the continuity constants of the linear trace and extension operator only depend on Ω and a lower bound on s, as well as a lower and upper bound on p.

An extension of Theorem 1.2 to the case $p = 1$ requires a refined consideration. Analogously to the case $p > 1$, one might guess that the limit space of $V^{s,1}(\Omega \mid \mathbb{R}^d)$ as $s \to 1^-$ is $W^{1,1}(\Omega)$. But, in fact, the Sobolev space $W^{1,1}(\Omega)$ is too small to capture all functions such that $\liminf_{s\to 1^-}\|f\|_{V^{s,1}(\Omega\mid\mathbb{R}^d)}$ is finite. The limit space of $V^{s,1}(\Omega \mid \mathbb{R}^d)$ as $s \to 1^-$ turns out to be the space of functions of bounded variation $BV(\Omega)$; see [Dávila 2002, Theorem 1; Bourgain et al. 2001, Theorem 3′, Corollaries 2 and 5; Foghem Gounoue 2023, Theorems 1.3 and 1.4′]. It is well known that functions in $BV(\Omega)$ have a trace to the boundary $\partial\Omega$ that is integrable and the trace map to $L^1(\partial\Omega)$ is surjective; see [Gagliardo 1957], [Dávila 2002, Theorem 1] or [Leoni 2017, Theorem 18.13]. Theorem 1.2 may be extended to the case $p = 1$ as follows.

Theorem 1.3. *Let $\Omega \subset \mathbb{R}^d$ be a bounded Lipschitz domain, $s \in (0,1)$. Then the trace operator*

$$\mathrm{Tr_s} : V^{s,1}(\Omega \mid \mathbb{R}^d) \to L^1(\Omega^c; \mu_s(\mathrm{d}x)), \quad u \mapsto u|_{\Omega^c},$$

is continuous and linear. There exists a continuous linear right inverse

$$\mathrm{Ext_s} : \mathcal{T}^{s,1}(\Omega^c) \to V^{s,1}(\Omega \mid \mathbb{R}^d), \quad g \mapsto \mathrm{ext}(g).$$

The continuity constants of the linear trace and extension operator only depend on Ω and a lower bound on s. In addition, the norm of the extension operator in dimension $d = 1$ also depends on a lower bound on $1 - s$.

This result is analogous to the local setting where $L^1(\Omega^c;\mu_s)$ is a suitable replacement for $L^1(\partial\Omega)$. In particular, a direct analog of the trace result from Theorem 1.2 for $p=1$, i.e., $\|u\|_{\mathcal{T}^{s,1}(\Omega^c)} \lesssim \|u\|_{V^{s,1}(\Omega\,|\,\mathbb{R}^d)}$, cannot hold; see the counterexample in Remark 3.11. This is in alignment with the local setting. Recall that there exists a nonlinear bounded extension operator from $L^1(\partial\Omega)$ to $BV(\Omega)$; see, e.g., [Malý et al. 2018, Theorem 1.2]. It was shown in [Peetre 1979] that a continuous extension map of integrable functions on $\partial\Omega$ to a function of bounded variation in Ω cannot be linear. If we restrict ourselves to the Besov space $B^0_{1,1}(\partial\Omega)\subset L^1(\partial\Omega)$, then a continuous linear extension to functions $BV(\Omega)$ that is a right inverse to the trace map exists; see [Malý et al. 2018, Theorem 1.1]. A function $f\in L^1(\partial\Omega)$ is in the Besov space $B^0_{1,1}(\partial\Omega)$ whenever the seminorm $[f]_{B^0_{1,1}(\partial\Omega)}$ is finite, where

$$[f]_{B^0_{1,1}(\partial\Omega)} := \int_{\partial\Omega\times\partial\Omega} \frac{|f(x)-f(y)|}{|x-y|^{d-1}}(\sigma\otimes\sigma)(\mathrm{d}(x,y)).$$

Here, the measure σ is the surface measure on $\partial\Omega$. The Besov space $B^0_{1,1}(\partial\Omega)$ is a Banach space endowed with the norm $\|f\|_{B^0_{1,1}(\partial\Omega)} := \|f\|_{L^1(\partial\Omega)}+[f]_{B^0_{1,1}(\partial\Omega)}$. In Step 1 of the proof of Theorem 1.4, see Section 5, we show that our trace norm $\|f\|_{\mathcal{T}^{s,1}(\Omega^c)}$ converges to $\|f\|_{B^0_{1,1}(\partial\Omega)}$ as $s\to 1^-$ for any $f\in C^{0,1}_c(\mathbb{R}^d)$. In this regard, we recover the local extension theorem to $BV(\Omega)$ functions in the limit $s\to 1^-$ as the extension operator in Theorem 1.3 has a uniformly bounded norm in the same limit.

As mentioned above, the spaces $V^{s,p}(\Omega\,|\,\mathbb{R}^d)$, $1<p<\infty$, converge to the traditional Sobolev space $W^{1,p}(\Omega)$ as the order of differentiability s reaches 1. Having established the robust continuity of the trace and extension operators from Theorems 1.2 and 1.3, our second goal is to study the limiting properties of the spaces $\mathcal{T}^{s,p}(\Omega^c)$ for $s\to 1^-$ and to recover the classical trace and extension results for Sobolev spaces.

Theorem 1.4. *Let $\Omega\subset\mathbb{R}^d$ be a bounded Lipschitz domain, $s\in(0,1)$, $1<p<\infty$. Then*

$$\begin{aligned}
\|\mathrm{Tr_s}\,u\|_{L^p(\Omega^c;\mu_s)} &\to \|\gamma u\|_{L^p(\partial\Omega)}, && u\in W^{1,p}(\mathbb{R}^d),\\
[\mathrm{Tr_s}\,u]_{\mathcal{T}^{s,p}(\Omega^c)} &\to [\gamma u]_{W^{1-1/p,p}(\partial\Omega)}, && u\in W^{1,p}(\mathbb{R}^d),\\
\|\mathrm{Tr_s}\,u\|_{L^1(\Omega^c;\mu_s)} &\to \|\gamma u\|_{L^1(\partial\Omega)}, && u\in BV(\mathbb{R}^d),\\
[\mathrm{Tr_s}\,u]_{\mathcal{T}^{s,1}(\Omega^c)} &\to [\gamma u]_{B^0_{1,1}(\partial\Omega)}, && u\in C^{0,1}_c(\mathbb{R}^d),
\end{aligned}$$

as $s\to 1^-$. Here, γ denotes the classical trace operator and $B^0_{1,1}(\partial\Omega)$ is the Besov space defined above.

Remark 1.5. In the case of a bounded connected $C^{1,1}$-domain Ω and $p=2$, Theorems 1.2 and 1.4 have been established in [Grube and Hensiek 2024]; see the discussion of related literature below.

Applications to the Dirichlet problem. Let us present a well-posedness result for (1-1). We define the space of test functions for the Dirichlet problem as follows:

$$V_0^{s,p}(\Omega\,|\,\mathbb{R}^d) = \{v\in V^{s,p}(\Omega\,|\,\mathbb{R}^d) \mid v=0 \text{ a.e. on } \mathbb{R}^d\setminus\Omega\} \tag{1-7}$$

Definition 1.6. Let $\Omega\subset\mathbb{R}^d$ be a bounded Lipschitz domain, $s\in(0,1)$, $1<p<\infty$. Let $g\in\mathcal{T}^{s,p}(\Omega^c)$ and $f\in V^{s,p}(\Omega\,|\,\mathbb{R}^d)'\supset L^{p'}(\Omega)$. We say that $u\in V^{s,p}(\Omega\,|\,\mathbb{R}^d)$ is a weak solution of (1-1) if, for every $\varphi\in V_0^{s,p}(\Omega\,|\,\mathbb{R}^d)$, the equation (1-2) holds.

Here is our result on well-posedness of the Dirichlet problem.

Corollary 1.7. *Let $\Omega \subset \mathbb{R}^d$ be a bounded Lipschitz domain, $s_\star \le s < 1$, $1 < p < \infty$. Let $g \in \mathcal{T}^{s,p}(\Omega^c)$ and $f \in V^{s,p}(\Omega \,|\, \mathbb{R}^d)' \supset L^{p'}(\Omega)$. Then there exists a unique weak solution $u \in V^{s,p}(\Omega \,|\, \mathbb{R}^d)$ to problem (1-1). Moreover, there is a constant $c > 0$, depending only on p, Ω, $s_\star$, such that*

$$\|u\|_{V^{s,p}(\Omega \,|\, \mathbb{R}^d)} \le c(\|g\|_{\mathcal{T}^{s,p}(\Omega^c)} + \|f\|_{V^{s,p}(\Omega \,|\, \mathbb{R}^d)'}). \tag{1-8}$$

Proof. Let $V_g^{s,p}(\Omega \,|\, \mathbb{R}^d)$ be the set of all functions v of the form $v = \mathrm{Ext_s}(g) + v_0$ with $v_0 \in V_0^{s,p}(\Omega \,|\, \mathbb{R}^d)$ and $\mathrm{Ext_s}(g)$ as in Theorem 1.2. This set is a closed convex subset of $V^{s,p}(\Omega \,|\, \mathbb{R}^d)$. Let $I : V_g^{s,p}(\Omega \,|\, \mathbb{R}^d) \to \mathbb{R}$ be defined by

$$I(v) = \frac{1-s}{2p} \iint_{(\Omega^c \times \Omega^c)^c} \frac{|v(x) - v(y)|^p}{|x-y|^{d+sp}} \,\mathrm{d}y\,\mathrm{d}x - f(v).$$

We note that $f(v)$ is the duality pairing between the functional $f \in V^{s,p}(\Omega \,|\, \mathbb{R}^d)'$ and the function $v \in V_g^{s,p}(\Omega \,|\, \mathbb{R}^d)$. The functional I is strictly convex and weakly lower semicontinuous on the reflexive, separable Banach space $V_g^{s,p}(\Omega \,|\, \mathbb{R}^d)$. Since

$$|f(v)| \le \|f\|_{V^{s,p}(\Omega \,|\, \mathbb{R}^d)'} \|v\|_{V^{s,p}(\Omega \,|\, \mathbb{R}^d)} \le \delta \|v\|^p_{V^{s,p}(\Omega \,|\, \mathbb{R}^d)} + (p')^{-1}(\delta p)^{-1/(p-1)} \|f\|^{p'}_{V^{s,p}(\Omega \,|\, \mathbb{R}^d)'},$$

for every $\delta \in (0,1)$, we can apply the Poincaré inequality, see Proposition 2.1, to the function $v - \mathrm{Ext_s}(g)$ to obtain

$$I(v) \ge \frac{1}{4p} [v]^p_{V^{s,p}(\Omega \,|\, \mathbb{R}^d)} + c_1^{-1} \|v\|^p_{L^p(\Omega)} - c_1 \|f\|^{p'}_{V^{s,p}(\Omega \,|\, \mathbb{R}^d)'} - c_1 \|\mathrm{Ext_s}(g)\|^p_{V^{s,p}(\Omega \,|\, \mathbb{R}^d)}$$

for some constant c_1 depending on p and on the constant from Proposition 2.1. Thus, the functional I is coercive in the sense that $I(v) \to +\infty$ for $\|v\|_{V^{s,p}(\Omega \,|\, \mathbb{R}^d)} \to +\infty$. We have shown that I attains a unique minimizer u on the set $V_g^{s,p}(\Omega \,|\, \mathbb{R}^d)$. It is now straightforward to show that the function u solves problem (1-1). The claimed estimate follows from $I(u) \le I(\mathrm{Ext_s}(g))$, the above estimate and Theorem 1.2. □

Let us quickly review some related results on problems for nonlocal operators in bounded domains with given exterior data. Note that there are also approaches to nonlocal problems in bounded domains Ω with data given on $\partial\Omega$ such as [Grubb 2015], which we do not take into account here.

Some early well-posedness results for variational nonlocal problems of the type (1-1) can be found in [Servadei and Valdinoci 2012; 2013; Felsinger et al. 2015]. The case of homogeneous problems, i.e., when $g = 0$, is particularly simple and has been treated by several authors. Note that the vector space $\widetilde{D}^{s,p}(\Omega)$ in [Piersanti and Pucci 2017] equals the space $V_0^{s,p}(\Omega \,|\, \mathbb{R}^d)$. Existing results for nonzero data g often assume g to be regular in all of $\mathbb{R}^d$, e.g., [Di Castro et al. 2016, Theorem 2.3; Lindgren and Lindqvist 2017, Theorem 8; Acosta et al. 2019, Proposition 2.2]. As [Korvenpää et al. 2017, Example 1] shows, optimal results require extra care and more regularity than just suitable integrability of g in $\mathbb{R}^d$. Also, $g \in W^{s,p}(\Omega) \cap L^p(\mathbb{R}^d; (1+|x|)^{-d-sp}\,\mathrm{d}x)$ does not imply well-posedness as claimed in [Palatucci 2018], which is not essential at all for the main results of that work. Workarounds avoiding global $W^{s,p}(\mathbb{R}^d)$-regularity are used in [Korvenpää et al. 2016; 2017, Lemma 6; Brasco et al. 2018, Definition 2.10]. These approaches assume $g \in W^{s,p}(\Omega_\varepsilon) \cap L^p(\mathbb{R}^d; (1+|x|)^{-d-sp}\,\mathrm{d}x)$ for some enlarged domain $\Omega_\varepsilon = \{x \in \mathbb{R}^d \mid \mathrm{dist}(x, \overline{\Omega}) < \varepsilon\}$. Concerning the case $p = 1$, we refer to [Bucur et al. 2023] for results on existence and regularity of solutions to (1-1) with given exterior data.

Note that well-posedness and energy estimates similar to (1-8) are proved for $p = 2$ in [Foghem and Kassmann 2024] and for general p in [Foghem 2025]. The present work resolves the matter of optimal assumptions on exterior data g, which has been achieved for $p = 2$ and $C^{1,1}$-domains in [Grube and Hensiek 2024].

Remark 1.8. Note that the fractional p-Laplacian is well defined at a point $x \in \mathbb{R}^d$ if u is sufficiently regular in a neighborhood of x and $u \in L^{p-1}(\mathbb{R}^d; (1+|x|)^{-d-sp}\,\mathrm{d}x)$. For a variational approach, the tail space $L^p(\mathbb{R}^d; (1+|x|)^{-d-sp}\,\mathrm{d}x)$ is more natural, but modifications are possible.

Remark 1.9. For demonstration purposes, we have presented the well-posedness result for the fractional p-Laplacian. It is straightforward to extend to more general nonlinear operators of the form

$$\mathrm{p.v.}\int_{\mathbb{R}^d} \Phi(x, |u(x)-u(y)|)(u(x)-u(y))k_s(x,y)\,\mathrm{d}y$$

for appropriate functions Φ and kernels k_s, $s \in (0,1)$.

Related results. Let us discuss related results concerning function spaces, in particular trace theorems. As explained above, the main new feature of the energy space $V^{s,p}(\Omega \mid \mathbb{R}^d)$ is that functions in $V^{s,p}(\Omega \mid \mathbb{R}^d)$ satisfy some incremental regularity across the boundary plus some integrability at infinity. Dyda and Kassmann [2019] provide trace and extension results for $V^{s,p}(\Omega \mid \mathbb{R}^d)$ for rather general domains Ω.[1] The proof is based on a Whitney decomposition of Ω and Ω^c, which we employ here, too. However, the construction of the extension operator in [Dyda and Kassmann 2019] is much simpler and uses the Lebesgue measure. Thus, for $s \to 1^-$, one does not recover the classical extension result. In order to resolve this problem, we introduce the measure μ_s on Ω^c, which converges to the surface measure on $\partial\Omega$.

In [Bogdan et al. 2020], the authors prove a version of the Douglas identity and provide trace and extensions results for spaces like $V^{s,2}(\Omega \mid \mathbb{R}^d)$, where they allow for a large class of Lévy measures $\nu(\mathrm{d}h)$ instead of $|h|^{-d-2s}\,\mathrm{d}h$. The proof is based on a careful study of the Poisson kernel and provides a representation of the energy of the solution u to problems like (1-1) ($p = 2$) in terms of its trace on Ω^c. The article leaves open the question of robustness as $s \to 1^-$. Unlike [Bogdan et al. 2020], we define the trace space for general $p \geq 1$ with the help of explicitly given norms that allow for robustness and limit results as $s \to 1^-$. Extensions of the results in [Bogdan et al. 2020] to some nonlinear cases, still based on L^2-Lévy integrable kernels, can be found in [Bogdan et al. 2023].

A systematic study of generalizations of the energy space $V^{s,p}(\Omega \mid \mathbb{R}^d)$ in the case of $p = 2$ and a Lévy measure $\nu(\mathrm{d}h)$ can be found in [Foghem and Kassmann 2024], where functional inequalities, well-posedness results and some nonlocal-to-local convergence results are provided. The trace space is shown to contain a certain weighted L^2-space of functions on Ω^c. Foghem [2025] provided extensions to the general

[1] Note that in [Dyda and Kassmann 2019] the domain of integration in (1.6) and (1.7) has to be changed from $\Omega^c \times \Omega^c$ to $M \times \Omega^c$ with $M = \{x \in \Omega^c \mid \mathrm{dist}(x, \partial\Omega) < 1\}$.

case $p > 1$. Nonlocal energy spaces appear also in the context of Markov jump processes in [Vondraček 2021]. Here, the author considers the intersection with $L^2(\mathbb{R}^d; m)$, where $m(x) = \mathbb{1}_\Omega(x) + \mu(x)\mathbb{1}_{\Omega^c}(x)$ and $\mu(x)$ behaves like $\operatorname{dist}(x, \partial\Omega)^{-2s}$ for x close to $\partial\Omega$; see Remark 2.37 in [Foghem and Kassmann 2024] for detailed comments. This approach together with functional inequalities and questions of well-posedness has been studied for more general kernels in [Frerick et al. 2025].

The present work can be seen as an extension of results in [Grube and Hensiek 2024]. Here we treat general bounded Lipschitz domains and the full range $p \geq 1$ instead of bounded $C^{1,1}$-domains and $p = 2$ in the aforementioned work. Both works use the measure μ_s, but the construction of the extension operator is different. In the present work we employ the Whitney decomposition technique and not the Poisson extension. The study of nonlocal Neumann problems as in [Grube and Hensiek 2024] together with the asymptotic behavior for $s \to 1^-$ is possible in our framework, too. In order to keep the scope of this work reasonable, we defer this line of research until a later date.

Last, let us mention recent trace and extension results for nonlocal function spaces, where problems similar to ours occur but the setup is conceptually different. In [Tian and Du 2017] the trace space $H^{1/2}(\partial\Omega)$ is recovered as the trace space of a certain $L^2(\Omega)$-space with a nonlocal interaction kernel that has a localizing property at the boundary $\partial\Omega$. The analogous result for $W^{s-1/p,p}(\partial\Omega)$ is proved in [Du et al. 2022a]. The result is extended further to domains with very rough boundaries including those with spatially varying dimension in [Foss 2021]. See [Scott and Du 2024] for applications to nonlocal equations with Dirichlet data given on $\partial\Omega$. Given a localization parameter $\delta > 0$ and a domain Ω, the authors of [Du et al. 2022b] study trace and extension operators between the domain and a layer $\{x \in \Omega^c \mid \operatorname{dist}(x, \partial\Omega) < \delta\}$. The operators are shown to be robust as $\delta \to 0$, which makes it possible to recover classical trace results. For more details we refer to the discussion in [Grube and Hensiek 2024, Section 1.2].

The development of nonlocal function spaces and related trace and extension results benefits greatly from classical results for Sobolev, Sobolev–Slobodeckij, or Besov spaces. Early results on trace spaces for $W^{1,p}(\Omega)$ can be found in [Aronszajn 1955; Slobodeckiĭ and Babič 1956; Prodi 1956; Gagliardo 1957; Slobodeckiĭ 1958] and the monograph [Nečas 1967]. See [Nečas 2012] for an English translation and, in particular, Chapter 2.5 therein. Lipschitz domains and fractional-order spaces are covered in [Grisvard 2011], e.g., in Theorems 1.5.1.3 and 1.5.2.1. For domains with corners see also [Yakovlev 1967]. The corresponding state-of-the-art around this time is summarized in Chapter 1, Sections 7–9 of [Lions and Magenes 1972]. Another standard reference focusing on contributions of researchers from the Soviet Union is [Besov et al. 1975, Chapter IV]. Another important monograph in this direction is [Triebel 1983], in particular Chapters 3.3.3 and 3.3.4. Trace and extensions results are provided in [Marschall 1987] under minimal regularity assumptions on the domains. A survey of results on boundary value problems for higher-order elliptic equations with degeneracies along the boundary is given in [Nikolskiĭ et al. 1988]. Kim [2007] extends well-known trace assertions for weighted Sobolev spaces. The aforementioned list is rather selective and not complete at all. Even some fundamental problems such as a trace result for $H^s(\Omega)$, $1 < s < \frac{3}{2}$, and Ω a bounded Lipschitz domain are not covered in the list above; see [Ding 1996].

Very useful references for our work are contributions of A. Jonsson and H. Wallin [Jonsson and Wallin 1978; 1984; Jonsson 1994]. The setting in the aforementioned references includes results for subsets of the Euclidean space endowed with general doubling measures. This is related to our framework because we consider measure spaces $(\Omega^c; \mu_s)$ with μ_s as in (1-5). Moreover, the construction used in the proof of the extension result Theorem 1.2 is inspired by the corresponding results Theorems 3.1 and 4.1 in [Jonsson and Wallin 1978].

Outline. In Section 2 we fix the notation and shortly introduce function spaces used throughout this work. The trace embeddings are studied in Section 3. We divide the proofs of the trace results from Theorems 1.2 and 1.3 into the L^p-embedding, see Proposition 3.9, and the seminorm inequality, see Proposition 3.10. We construct the extension operator in Section 4. The extension theorems are proven in Proposition 4.5 as well as Proposition 4.6 with precise dependencies of the operator norms. Lastly, the limiting properties of the spaces $\mathcal{T}^{s,p}(\Omega^c)$, see Theorem 1.4, are proven in Section 5.

2. Preliminaries

2.1. ***Notation.*** For two real numbers $a, b \in \mathbb{R}$, we write $a \wedge b = \min\{a, b\}$, $a \vee b := \max\{a, b\}$ and $\lfloor a \rfloor = \max(-\infty, a] \cap \mathbb{Z}$. The ball of radius $r > 0$ centered at $x \in \mathbb{R}^d$ in the d-dimensional Euclidean space is written as $B_r(x)$ or $B_r^{(d)}$ whenever we want to specify the dimension. For a set A, we denote by $\mathbb{1}_A$ the indicator function on A. An open set $\Omega \subset \mathbb{R}^d$ is said to have a uniform Lipschitz boundary if there exists a localization radius $r > 0$ and a constant $L > 0$ such that, for any boundary point $z \in \partial\Omega \neq \varnothing$, there exists a translation and rotation $T_z : \mathbb{R}^d \to \mathbb{R}^d$ satisfying $T_z(z) = 0$ as well as a Lipschitz continuous function $\phi_z : \mathbb{R}^{d-1} \to \mathbb{R}$ whose Lipschitz constant is bounded by L such that

$$T_z(\Omega \cap B_r(z)) = \{(x', x_d) \in B_r(0) \mid \phi_z(x') > x_d\};$$

see, e.g., [Leoni 2017, Definition 13.11] and the discussion in [Grisvard 2011, Chapter 1.2.1]. An open set $\varnothing \neq B \subset \mathbb{R}^d$ is said to satisfy the uniform exterior cone condition if we find an opening angle α and a height $h_0 > 0$ such that, for any $z \in \partial\Omega$, there exists an exterior cone $\mathcal{C}_z \subset \overline{\Omega}^c$ with apex at z and height h_0. The notion of the uniform interior cone condition is defined analogously. Note that an open set with a uniform Lipschitz boundary satisfies both uniform interior and exterior cone conditions. The interior cones (resp. the exterior cones) can simply be constructed via

$$\mathcal{C}_z := T_z^{-1}\big\{(x', x_d) \in B_r(0) \mid x_d < -\tfrac{1}{2}L|x'|\big\}$$

for $z \in \partial\Omega$. We say that $\Omega \subset \mathbb{R}^d$ is a Lipschitz domain if it is open and has a uniform Lipschitz boundary. Notice that we do not assume Ω to be connected. Nevertheless, a bounded Lipschitz domain has finitely many connected components since the uniform interior cone condition bounds the volume of each connected component from below by a uniform positive constant. We denote the distance of x to a closed set $A \subset \mathbb{R}^d$ by $\operatorname{dist}(x, A) = \inf\{|x - a| \mid a \in A\}$. When the dependencies are clear, we write for short $d_x := \operatorname{dist}(x, \partial\Omega)$ for any $x \in \mathbb{R}^d$. Furthermore, we use for $r > 0$ the notation

$$\Omega_r^{\text{ext}} := \{x \in \overline{\Omega}^c \mid d_x < r\}, \qquad \Omega_{\text{ext}}^r := \{x \in \overline{\Omega}^c \mid d_x \geq r\}. \tag{2-1}$$

We denote by $\mathcal{H}^{(d-l)}$ the normalized $(d-l)$-dimensional Hausdorff measure on $\mathbb{R}^d$. The surface measure of the $(d-1)$-dimensional unit sphere will be written for short as $\mathcal{H}^{(d-1)}(\partial B_1) = \omega_{d-1}$. To shorten the notation, we write σ for the surface measure on $\partial\Omega$. The inner radius of the domain Ω we denote by

$$\operatorname{inr}(\Omega) := \sup\{r \mid B_r \subset \Omega\}.$$

We will use lowercase letters $c_1, c_2, \ldots$ with running indices as constants in our proofs and reset them after every proof. When we introduce a new constant, we write $C = C(\ldots)$ to indicate what the constant depends on, i.e., $C = C(d, \Omega) > 0$ depends only on the dimension d and the set Ω.

2.2. ***Function spaces.*** The following function spaces will be used throughout this work. We denote by $W^{s,p}(\Omega)$ (resp. $W^{s,p}(\partial\Omega)$), $s \in (0, 1)$, $p \geq 1$, the Sobolev–Slobodeckij space of functions in $u \in L^p(\Omega)$ satisfying

$$[u]_{W^{s,p}(\Omega)} := [u]_{V^{s,p}(\Omega\,|\,\Omega)} < \infty$$

endowed with the norm $\|u\|^p_{W^{s,p}(\Omega)} := \|u\|^p_{L^p(\Omega)} + [u]^p_{W^{s,p}(\Omega)}$ (resp. $\partial\Omega$ with the surface measure). See (1-4) for the definition of the seminorm $[\,\cdot\,]_{V^{s,p}(A\,|\,B)}$. We write $BV(\Omega)$ for the space of functions $u \in L^1(\Omega)$ with bounded variation endowed with the norm $\|u\|_{BV(\Omega)} := \|u\|_{L^1(\Omega)} + |\nabla u|(\Omega)$. The Bessel potential spaces $H^{s,p}(\mathbb{R}^d)$ are defined in (3-5). As mentioned in the introduction, a variational approach to equations like (1-1) leads naturally to function spaces like $V^{s,p}(\Omega\,|\,\mathbb{R}^d)$ which we introduced in (1-3). These function spaces are the focus of our study. They were first introduced in [Servadei and Valdinoci 2012; 2014; Felsinger et al. 2015] for the case $p = 2$. We also refer to [Dipierro et al. 2017], in which the nonlocal normal operator was introduced, and [Foghem Gounoue 2020; Foghem and Kassmann 2024; Foghem 2025] for an intensive study of these spaces for general p. It is well known that $V^{s,p}(\Omega\,|\,\mathbb{R}^d)$ is a separable Banach space (Hilbert space for $p = 2$) which is reflexive for $1 < p < \infty$; see, e.g., [Foghem Gounoue 2020, Chapter 3.4].

The spaces $V^{s,p}(\Omega\,|\,\mathbb{R}^d)$ allow for a Poincaré inequality, which is an important ingredient for the proof of well-posedness for the Dirichlet problem (1-1) together with an energy estimate; see Corollary 1.7. We will need a version of the Poincaré inequality that is robust as s reaches 1.

Proposition 2.1 [Foghem 2025, Theorem 10.1]. *Let $p > 1$ and $s_\star \in (0, 1)$. Let $\Omega \subset \mathbb{R}^d$ be a bounded Lipschitz domain. Then there exists $c > 0$ such that, for all $s_\star \leq s < 1$ and $u \in V_0^{s,p}(\Omega\,|\,\mathbb{R}^d)$,*

$$\|u\|_{L^p(\Omega)} \leq c\|u\|_{V^{s,p}(\Omega\,|\,\mathbb{R}^d)}. \tag{2-2}$$

Let us recall our trace spaces $\mathcal{T}^{s,p}(\Omega^c)$, which are introduced in (1-6). For $s \in (0, 1)$, $1 \leq p < \infty$ and $A, B \in \mathcal{B}(\Omega^c)$, we define

$$[f, g]^p_{\mathcal{T}^{s,p}(A\,|\,B)} := \int_A\int_B \frac{|f(x) - f(y)|^{p-2}(f(x) - f(y))(g(x) - g(y))}{((|x - y| + d_x + d_y) \wedge 1)^{d+s(p-2)}}\mu_s(\mathrm{d}x)\mu_s(\mathrm{d}y) \tag{2-3}$$

with the convention $[g]_{\mathcal{T}^{s,p}(A\,|\,B)} = [g, g]_{\mathcal{T}^{s,p}(A\,|\,B)}$. Note that $[f, g]_{\mathcal{T}^{s,p}(\Omega^c)} = [f, g]_{\mathcal{T}^{s,p}(\Omega^c\,|\,\Omega^c)}$. We employ standard techniques to prove that these spaces are separable Banach spaces (resp. Hilbert spaces if $p = 2$).

Lemma 2.2. *Let Ω be an open set. The space $\mathcal{T}^{s,p}(\Omega^c)$ is a separable Banach space, reflexive for $1 < p < \infty$, and in the case $p = 2$, it is a separable Hilbert space with inner product*

$$(u, v)_{\mathcal{T}^{s,2}(\Omega^c)} = (u, v)_{L^2(\Omega^c,\mu_s)} + [u, v]^2_{\mathcal{T}^{s,2}(\Omega^c)}.$$

Proof. To prove completeness, we take a Cauchy sequence $\{u_n\}_n \subset \mathcal{T}^{s,p}(\Omega^c)$. Then $v_n(x) := u_n(x)\mu_s(x)^{1/p}$ is Cauchy in $L^p(\Omega^c)$ with limit $v \in L^p(\Omega^c)$. Define $u(x) := v(x)\mu_s(x)^{-1/p}$. Then u is the limit of u_n with respect to $\| \cdot \|_{L^p(\Omega^c;\mu_s)}$. Take a subsequence $\{u_{n_l}\}_l$ converging a.e. to u on $\mathbb{R}^d$. Then, by Fatou's lemma, we have

$$[u - u_{n_l}]^p_{\mathcal{T}^{s,p}(\Omega^c)} \le \liminf_{k\to\infty} [u_{n_k} - u_{n_l}]^p_{\mathcal{T}^{s,p}(\Omega^c)} \to 0 \quad \text{as } l \to \infty.$$

Separability follows from the fact that the map $\iota : \mathcal{T}^{s,p}(\Omega^c) \to L^p(\Omega^c) \times L^p(\Omega^c \times \Omega^c)$,

$$u \mapsto \left(x \mapsto u(x)\mu_s(x)^{1/p},\ (x, y) \mapsto \frac{u(x) - u(y)}{((|x - y| + d_x + d_y) \wedge 1)^{d/p+s(p-2)/p}} \mu_s(x)^{1/p}\mu_s(y)^{1/p} \right),$$

is an isometric injection. As $\iota(\mathcal{T}^{s,p}(\Omega^c))$ is closed and since $L^p(\Omega^c) \times L^p(\Omega^c \times \Omega^c)$ is separable, so is $\mathcal{T}^{s,p}(\Omega^c)$. In the same manner, as $L^p(\Omega^c) \times L^p(\Omega^c \times \Omega^c)$ is reflexive for $1 < p < \infty$, so is $\mathcal{T}^{s,p}(\Omega^c)$. □

The functions from $\mathcal{T}^{s,p}(\Omega^c)$ have some regularity at the boundary because the weight in the seminorm $[\,\cdot\,,\cdot\,]_{\mathcal{T}^{s,p}(\Omega^c)}$ becomes $((|x - y|) \wedge 1)^{-d-s(p-2)}$ as $x, y \to \partial\Omega$. Thereby, for sufficiently large s, the functions in the trace space $\mathcal{T}^{s,p}(\Omega^c)$ themselves have a trace onto the boundary $\partial\Omega$. This is a direct consequence of Theorem 1.2.

Corollary 2.3. *The space $\mathcal{T}^{s,p}(\Omega^c)$ is continuously embedded in $W^{s-1/p,p}(\partial\Omega)$ for any $s \in (1/p, 1)$ and $p \in (1, \infty)$. The embedding is surjective. The continuity constant depends only on Ω, p and a lower bound on s.*

Proof. By Theorem 1.2, the extension $\mathrm{Ext}_s : \mathcal{T}^{s,p}(\Omega^c) \to V^{s,p}(\Omega \,|\, \mathbb{R}^d)$ is continuous and the continuity constant $c_1 > 0$ depends only on Ω, p and a lower bound on s. The space $V^{s,p}(\Omega \,|\, \mathbb{R}^d)$ is embedded in $W^{s,p}(\Omega)$ with the embedding constant depending only on a lower bound on s. The result follows from the classical trace result $W^{s,p}(\Omega) \to W^{s-1/p,p}(\partial\Omega)$. The embedding is surjective since we can extend a function from $W^{s-1/p,p}(\partial\Omega)$ to an element from $W^{s,p}(\mathbb{R}^d) \hookrightarrow V^{s,p}(\Omega \,|\, \mathbb{R}^d) \hookrightarrow \mathcal{T}^{s,p}(\Omega^c)$. □

3. Trace theorem

Here we aim to prove the trace parts of Theorems 1.2 and 1.3. This proof is carried out in Propositions 3.9 and 3.10. Essential building blocks in the respective proofs are an approximation to the classical L^p-trace embedding in Theorem 3.5 and, for $p = 1$, a Hardy-type inequality provided in Theorem 3.6. On a more technical level, we use upper and lower bounds of the distance function; see Lemmas 3.7 and 3.8.

To prove Theorem 3.5 we apply techniques developed in [Jonsson and Wallin 1984]. In particular, we use the interpolation between Bessel potential spaces on $\mathbb{R}^d$. For this reason we need a Sobolev extension operator for fractional Sobolev spaces $W^{s,p}(\Omega)$ whose continuity constant is independent of s. The existence of such an extension is well known in the literature. We provide this result in the following theorem for the convenience of the reader.

Theorem 3.1 [Jonsson and Wallin 1984, Chapter VI.2, Theorem 3; Triebel 1995]. *Let $\Omega \subset \mathbb{R}^d$ be a connected Lipschitz domain. There exists a linear map E, which extends measurable functions $f : \Omega \to \mathbb{R}$ such that $E : L^p(\Omega) \to L^p(\mathbb{R}^d)$ for all $p \geq 1$ and, with some constant $C = C(d, \Omega, p) > 0$, for any $0 < s \leq 1$,*

$$\|Ef\|_{W^{s,p}(\mathbb{R}^d)} \leq C\|f\|_{W^{s,p}(\Omega)}.$$

Remark 3.2. The extension is constructed via a Whitney decomposition of $\overline{\Omega}^c$, a smooth partition of unity and copying mean values of f from inside to respective Whitney cubes. The construction of the extension Ef is independent of s and p and satisfies $E : W^{1,p}(\Omega) \to W^{1,p}(\mathbb{R}^d)$. Real interpolation allows us to choose the constant $C(d, \Omega, p)$ in the theorem independent of s.

Analogously to the measure μ_s from (1-5), we define for $s \in (0, 1)$ the measure

$$\tau_s(\mathrm{d}x) = \frac{1-s}{d_x^s} \mathbb{1}_\Omega(x)\,\mathrm{d}x \tag{3-1}$$

on the σ-algebra $\mathcal{B}(\mathbb{R}^d)$. Recall that $d_x = \operatorname{dist}(x, \partial\Omega)$. The measure $\tau_s(\mathrm{d}x)$ plays the same role as μ_s but is supported inside Ω. We use it in Theorem 3.5 for the proof of the trace part of our main theorems; see also Propositions 3.9 and 3.10. In contrast to μ_s, the measure τ_s does not need the additional term $(1 + d_x)^{-d-s(p-1)}$ for the decay at infinity since the open set Ω is assumed to be bounded throughout this work. The following lemma shows how balls scale under τ_s. This scaling plays a crucial role in Theorem 3.5.

Lemma 3.3. *Let $\Omega \subset \mathbb{R}^d$ be a bounded Lipschitz domain with a localization radius $r_0 > 0$. There exists a constant $C = C(d, \Omega) > 0$ such that, for any $s \in (0, 1)$, $0 < r \leq \frac{1}{2}r_0$ and $x \in \Omega$,*

$$\tau_s(B_r(x)) \leq C r^{d-s}. \tag{3-2}$$

Proof. Let $d \geq 2$. If $r \leq d_x$, i.e., $B_r(x) \Subset \Omega$, then $d_y \geq r - |x-y|$ for any $y \in B_r(x)$ and, thus,

$$\tau_s(B_r(x)) = \int_{B_r(x)} \frac{1-s}{d_y^s}\,\mathrm{d}y \leq \int_{B_r(x)} \frac{1-s}{(r-|x-y|)^s}\,\mathrm{d}y = \omega_{d-1}\int_0^r \frac{1-s}{(r-t)^s} t^{d-1}\,\mathrm{d}t \leq \omega_{d-1} r^{d-s}.$$

Now we consider the case $r > d_x$, i.e., $B_r(x) \cap \partial\Omega \neq \varnothing$. Without loss of generality we assume that $0 \in \partial\Omega$ is a minimizer of d_x. Since Ω is a Lipschitz domain, we find a Lipschitz map $\phi : \mathbb{R}^{d-1} \to \mathbb{R}$ such that $\Omega \cap B_{r_0} = \{(y', y_d) \in B_{r_0} \mid y_d < \phi(y')\}$. The Lipschitz constant of ϕ is bounded by $L > 0$ independent of x. A simple calculation yields, for any $y = (y', y_d) \in B_r(x) \cap \Omega$,

$$\begin{aligned} |y| &\leq |x| + |y - x| \leq 2r, \\ |y_d - \phi(y')| &\leq \inf_{(\tilde{y}', \phi(\tilde{y}')) \in B_{r_0}} |y_d - \phi(\tilde{y}')| + |\phi(y') - \phi(\tilde{y}')| \\ &\leq 2^{1/2}(1+L) \inf_{(\tilde{y}', \phi(\tilde{y}')) \in B_{r_0}} |y - (\tilde{y}', \phi(\tilde{y}'))| \\ &= 2^{1/2}(1+L) d_y. \end{aligned} \tag{3-3}$$

In the case that the minimizer of d_y is not in the graph of ϕ, we simply pick a smaller r_0 depending only on the constant L. Therefore,

$$\begin{aligned}\tau_s(B_r(x)) &\le 2^{s/2}(1+L)^s \int_{B_{2r}\cap\{y_d<\phi(y')\}} \frac{1-s}{|y_d-\phi(y')|^s}\,\mathrm{d}(y',y_d)\\ &\le 2(1+L)^s\omega_{d-2}(2r)^{d-1}\int_0^{(2+L)r}\frac{1-s}{y_d^s}\,\mathrm{d}y_d \le 2^{d+1}(2+L)\omega_{d-2}r^{d-s}. \end{aligned} \tag{3-4}$$

The proof in the case $d=1$ is straightforward. Note that similar arguments as in this proof are employed in the proof of Lemma 4.1. □

In the proof of Theorem 3.5, we use interpolation results, which we explain now. Let G_α, $\alpha>0$ be the Bessel potential kernel. We introduce the Bessel potential spaces

$$H^{\alpha,p}(\mathbb{R}^d) := \{g:\mathbb{R}^d\to\mathbb{R} \mid \exists f\in L^p(\mathbb{R}^d) : g = G_\alpha * f\} \tag{3-5}$$

with the canonical norm $\|g\|_{H^{\alpha,p}(\mathbb{R}^d)} := \|f\|_{L^p(\mathbb{R}^d)}$ if $g\in H^{\alpha,p}(\mathbb{R}^d)$ and $g=G_\alpha * f$. The convolution of the Bessel potential kernel with the function f can be written as $G_\alpha * f = \mathscr{F}^{-1}((1+|\cdot|^2)^{-\alpha/2}\mathscr{F}f)$, where $\mathscr{F}$ is the Fourier transformation; see [Bergh and Löfström 1976, p. 139, Definition 6.2.3]. We refer the reader to [Aronszajn and Smith 1961] for more details on the kernel G_α. We recall the real interpolation result

$$[H^{\alpha_0,p}(\mathbb{R}^d), H^{\alpha_1,p}(\mathbb{R}^d)]_{p\theta} = W^{s,p}(\mathbb{R}^d), \tag{3-6}$$

where $0<\alpha_0<\alpha_1$, $\theta\in(0,1)$, $s=(1-\theta)\alpha_0+\theta\alpha_1$ and $p\ge 1$; see, e.g., [Bergh and Löfström 1976, Theorem 6.2.4]. Analogously to [Jonsson and Wallin 1984, Chapter V], we calculate bounds on the Bessel potential kernel G_{α_i} for some $0<\alpha_0<s<\alpha_1$, see Lemma 3.4, and prove an approximate trace result inside Ω; see Theorem 3.5.

The following lemma is a slight modification of [Jonsson and Wallin 1984, Lemma C] that fits our setting. The well-known estimates of the Bessel potential kernel, its gradient and decay at infinity are crucial in the proof. For more details on the Bessel potential we refer to [Taibleson 1964, Chapter IV]. In particular, we need to pay attention to the constants and their dependencies.

Lemma 3.4 [Jonsson and Wallin 1984, Chapter V, Lemma C]. *Let $\Omega\subset\mathbb{R}^d$ be a bounded connected Lipschitz domain, $0<s_\star\le s<1$ and $1<p_\star\le p\le p^\star<\infty$. We set*

$$\alpha_0 := s\frac{1+p}{2p},\qquad \alpha_1 := 1+\frac{s}{2p}, \tag{3-7}$$

and $\beta_i := \alpha_i - s/p$ for $i\in\{0,1\}$. There exists a constant $C=C(d,\Omega,p_\star,p^\star,s_\star)>0$ such that, for all $0<r\le\frac{1}{2}r_0$ and $f\in L^p(\mathbb{R}^d)$, we have

$$\frac{1}{r^{d-s}}\iint_{\substack{\Omega\times\Omega\\ |x-y|<r}} |G_{\alpha_i}*f(x)-G_{\alpha_i}*f(y)|^p\tau_s(\mathrm{d}y)\tau_s(\mathrm{d}x) \le C r^{p\beta_i}\|f\|^p_{L^p(\mathbb{R}^d)}, \tag{3-8}$$

$$\int_\Omega |G_{\alpha_i}*f(x)|^p\tau_s(\mathrm{d}x) \le C\|f\|^p_{L^p(\mathbb{R}^d)}. \tag{3-9}$$

Proof. In [Jonsson and Wallin 1984, Chapter V, Lemma C], the statement is proven for doubling measures satisfying (3-2) under the assumptions $0<\beta_i<1$ and $0<\alpha_i\neq d$. The proof uses estimates of the Bessel potential kernel G_α; see [loc. cit., Chapter V, Lemmas 1, A, B]. Carefully inspecting the proof of [loc. cit., Chapter V, Lemma C], we find that the resulting constant depends on the constant C from (3-2), a lower bound $0<\beta_{i,\star}\le\beta_i$ and an upper bound $\beta_i\le\beta_i^\star<1$, as well as a lower bound on $|d-\alpha_i|$. We calculate

$$0<s_\star\frac{p_\star-1}{2p_\star}\le s\frac{p-1}{2p}=\beta_0\le\frac{1}{2}<1,$$
$$0<1-\frac{1}{p_\star}<1-\frac{s}{2p}=\beta_1\le 1-\frac{s_\star}{p^\star}<1.$$

Furthermore, we have

$$|d-\alpha_0|=d-s\frac{1+p}{2p}\ge(d-1)+\frac{p-1}{2p}\ge\frac{p_\star-1}{2p_\star}>0$$

and

$$|d-\alpha_1|=\begin{cases} d-1-\dfrac{s}{2p}\ge 1-\dfrac{1}{2p_\star}>0, & d\ge 2,\\ \dfrac{s}{2p}\ge\dfrac{s_\star}{2p^\star}>0, & d=1.\end{cases}$$

This yields the estimates with dependencies of the constants as claimed. □

Theorem 3.5 (approximate trace inequality). *Let $\Omega\subset\mathbb{R}^d$ be a bounded Lipschitz domain, $1<p_\star<p^\star<\infty$ and $s_\star\in(0,1)$. There exists a constant $C=C(d,\Omega,p_\star,p^\star,s_\star)>0$ such that, for every $s\in(s_\star,1)$, $p_\star\le p\le p^\star$ and $u\in W^{s,p}(\Omega)$,*

$$\int_\Omega|u(x)|^p\tau_s(\mathrm{d}x)+\int_\Omega\int_\Omega\frac{|u(x)-u(y)|^p}{((|x-y|+d_x+d_y)\wedge 1)^{d+s(p-2)}}\tau_s(\mathrm{d}y)\tau_s(\mathrm{d}x)\le C\|u\|^p_{W^{s,p}(\Omega)}. \tag{3-10}$$

Before we give the proof of this theorem we want to motivate it. In anticipation of Section 5, the left-hand side of (3-10) converges in the limit $s\to 1$ to

$$\int_{\partial\Omega}|u|^p\,\mathrm{d}\sigma+\int_{\partial\Omega\times\partial\Omega}\frac{|u(x)-u(y)|^p}{(|x-y|\wedge 1)^{d-1+p(1-1/p)}}(\sigma\otimes\sigma)(x,y)\asymp\|u\|^p_{W^{1-1/p,p}(\partial\Omega)}.$$

Thereby, we retrieve the classical trace result $W^{1,p}(\Omega)\to W^{1-1/p,p}(\partial\Omega)$ in the limit $s\to 1^-$.

Proof. Since Ω is a bounded open set with a uniform Lipschitz boundary, Ω decomposes into finitely many connected components Ω_i, $i\in\{1,\dots,I\}$, and each Ω_i is a connected bounded Lipschitz domain. First, we prove (3-10) for each Ω_i.

We define α_0 and α_1 as in (3-7) depending on p and s. We set $\theta:=\frac{s(p-1)}{(2+s)p}\in(0,1)$ and notice that

$$0<s_\star\frac{p_\star-1}{(2+s_\star)p_\star}\le\theta\le\frac{p^\star-1}{3p^\star}<1.$$

Most importantly, the relation $s=(1-\theta)\alpha_0+\theta\alpha_1$ is true. By Theorem 3.1, it is sufficient to prove the existence of a constant $C>0$ such that

$$\int_{\Omega_i}|u(x)|^p\tau_s(\mathrm{d}x)+\int_{\Omega_i}\int_{\Omega_i}\frac{|u(x)-u(y)|^p}{((|x-y|+d_x+d_y)\wedge 1)^{d+s(p-2)}}\tau_s(\mathrm{d}y)\tau_s(\mathrm{d}x)\le C\|u\|^p_{W^{s,p}(\mathbb{R}^d)}$$

for any $u \in W^{s,p}(\mathbb{R}^d)$. Let $c_1 > 0$ be the constant from Lemma 3.4. The equality (3-9) proves the continuity of the restriction operator $Ru(x) = u|_{\Omega_i}$ as a map from $R : H^{\alpha_i,p}(\mathbb{R}^d) \to L^p(\Omega_i, \tau_s(\mathrm{d}x))$, $i = 0, 1$. Real interpolation yields the continuity of

$$R : [H^{\alpha_0,p}(\mathbb{R}^d), H^{\alpha_1,p}(\mathbb{R}^d)]_{p\theta} = W^{s,p}(\mathbb{R}^d) \to [L^p(\Omega_i, \tau_s), L^p(\Omega_i, \tau_s)]_{p\theta} = L^p(\Omega_i, \tau_s)$$

with the continuity constant c_1; see, e.g., [Bergh and Löfström 1976]. Now we consider the second term on the left-hand side of (3-10) with Ω_i in place of Ω. Let $u \in W^{s,p}(\mathbb{R}^d)$. We write

$$\begin{aligned}\int_{\Omega_i}\int_{\Omega_i} &\frac{|u(x)-u(y)|^p}{((|x-y|+d_x+d_y)\wedge 1)^{d+s(p-2)}}\tau_s(\mathrm{d}y)\tau_s(\mathrm{d}x) \\ &\le 2\sum_{n=0}^{\infty} 2^{ns(p-1)} \iint_{\substack{\Omega_i\times\Omega_i \\ 2^{-n-1}\le|x-y|<2^{-n}}} |u(x)-u(y)|^p \frac{(\tau_s\otimes\tau_s)(\mathrm{d}(y,x))}{|x-y|^{d-s}} \\ &\qquad + \iint_{\substack{\Omega_i\times\Omega_i \\ 1\le|x-y|}} |u(x)-u(y)|^p \tau_s(\mathrm{d}y)\tau_s(\mathrm{d}x) =: \mathrm{(I)} + \mathrm{(II)}.\end{aligned}$$

We estimate (II) using the continuity of R shown above. We have

$$\mathrm{(II)} \le 2^p \tau_s(\Omega_i) \int_{\Omega_i} |Ru(x)|^p \tau_s(\mathrm{d}x) \le c_1^p 2^p \tau_s(\Omega_i) \|u\|^p_{W^{s,p}(\mathbb{R}^d)}.$$

We set $H := L^p(\Omega_i \times \Omega_i, |x-y|^{-d+s}\tau_s(\mathrm{d}y)\tau_s(\mathrm{d}x))$ and define for any $1 < q \le \infty$, $\beta > 0$ the space of sequences

$$l^{\beta,q} := \{(h_n)_n \mid h_n \in H\}, \quad \|(h_n)\|_{l^{\beta,q}} := \big\|(2^{n\beta}\|h_n\|_H)_n\big\|_{l^q(\mathbb{N})}.$$

Notice that

$$\mathrm{(I)} = \|((u(x)-u(y))\mathbb{1}_{2^{-n-1}\le|x-y|<2^{-n}})_n\|^p_{l^{s-s/p,p}}. \tag{3-11}$$

We define the linear map

$$Tf(x,y) := ((f(x)-f(y))\mathbb{1}_{2^{-n-1}\le|x-y|<2^{-n}})_n, \quad f : \mathbb{R}^d \to \mathbb{R}.$$

Lemma 3.4, in particular (3-8), shows the continuity of $T : H^{\alpha_i,s} \to l^{\beta_i,\infty}$ with $\beta_i = \alpha_i - s/p$ and the continuity constant c_1, $i = 0, 1$. Real interpolation yields the continuity of $T : [H^{\alpha_0,p}(\mathbb{R}^d), H^{\alpha_1,p}(\mathbb{R}^d)]_{p\theta} = W^{s,p}(\mathbb{R}^d) \to [l^{\beta_0,\infty}, l^{\beta_1,\infty}]_{p\theta} = l^{(1-\theta)\beta_0+\theta\beta_1,p}$ with the continuity constant c_1; see, e.g., [Bergh and Löfström 1976]. This proves the claimed inequality for each connected component Ω_i by (3-11) and

$$(1-\theta)\beta_0 + \theta\beta_1 = (1-\theta)\alpha_0 + \theta\alpha_1 - \frac{s}{p} = s - \frac{s}{p}.$$

Simply summing over $i \in \{1, \dots, I\}$ yields a constant $c_2 = c_2(d, \Omega, p_\star, p^\star, s_\star) > 0$ such that

$$\|u\|^p_{L^p(\Omega;\tau_s)} \le c_2 \|u\|^p_{W^{s,p}(\Omega)}. \tag{3-12}$$

It remains to prove the existence of a constant $c_3 = c_3(d, \Omega, p_\star, p^\star, s_\star) > 0$ such that, for any $i \ne j$,

$$\int_{\Omega_i}\int_{\Omega_j} \frac{|u(x)-u(y)|^p}{((|x-y|+d_x+d_y)\wedge 1)^{d+s(p-2)}}\tau_s(\mathrm{d}y)\tau_s(\mathrm{d}x) \le c_3 \|u\|^p_{W^{s,p}(\Omega)}.$$

Since the distance between any two connected components is bounded from below by a uniform constant, this is an immediate consequence of the triangle inequality and Lemma 3.3, as well as (3-12). □

Theorem 3.5 is not true in the case $p = 1$; see Remark 3.11. If we only keep the first term on the left-hand side in the estimate (3-10), then it is a fractional Hardy inequality; see, e.g., [Dyda 2004]. In [Dyda and Kijaczko 2022, Theorem 4], such a Hardy inequality is proven with a constant whose dependency on the parameter s is not known. Since the dependency on s is crucial in our setup, we prove the following theorem based on a Hardy inequality on the half-space with the best constant; see Theorem B.1.

Theorem 3.6 (Hardy inequality). *Let $\Omega \subset \mathbb{R}^d$ be a bounded Lipschitz domain and $s \in (0, 1)$. There exists a constant $C = C(d, \Omega) > 0$ such that*

$$(1-s)\int_\Omega \frac{|u(x)|}{d_x^s}\,\mathrm{d}x \le C\left(\|u\|_{L^1(\Omega)} + s(1-s)\int_{\Omega\times\Omega}\frac{|u(x)-u(y)|}{|x-y|^{d+s}}\,\mathrm{d}(x,y)\right)$$

for any $u \in W^{s,1}(\Omega)$.

Before we state the proof of the theorem, let us remark that the previous inequality, in the limit $s \to 1^-$, yields the classical trace embedding $W^{1,1}(\partial\Omega) \to L^1(\partial\Omega)$ since the measure τ_s converges weakly to the surface measure on $\partial\Omega$.

Proof. It is sufficient to prove the statement for any connected component of Ω in place of Ω. Thus, we can assume without loss of generality that Ω is a connected bounded Lipschitz domain. Therefore, we can cover the boundary with finitely many neighborhoods U_i and bi-Lipschitz maps $\phi_i : U_i \to B_1(0)$ such that $\phi_i(U_i \cap \Omega) = B_1(0)_+ := \{(x', x_d) \in B_1(0) \mid x_d > 0\}$, $i \in \{1, \dots, N\}$; see, e.g., [Grisvard 2011, Chapter 1.2.1]. We denote the distance of $\Omega \cap \bigcap_{i=1}^N U_i^c$ to the boundary $\partial\Omega$ by $2r_0 > 0$. We fix

$$U_0 := \left\{x \in \mathbb{R}^d \;\middle|\; \operatorname{dist}\left(x, \Omega \cap \bigcap_{i=1}^N U_i^c\right) < r_0\right\} \subset \Omega.$$

Notice that $\{U_i \mid i = 0, \dots, N\}$ is an open cover of $\overline{\Omega}$ and $\operatorname{dist}(U_0, \partial\Omega) \ge r_0$. Next, we pick a partition of unity $\eta_i \in C_c^\infty(U_i)$ adapted to U_i, i.e., $\sum_{i=0}^N \eta_i = 1$ on $\overline{\Omega}$. We define $\tilde{\eta}_i := \eta_i \circ \phi_i^{-1} \in C_c^{0,1}(B_1(0))$. Let $c_1 = c_1(\tilde{\eta}_1, \dots, \tilde{\eta}_N) \ge 1$ such that $[\tilde{\eta}_i]_{C^{0,1}} \le c_1$ for all $i = 1, \dots, N$. Without loss of generality, we assume that $\tilde{\eta}_i = 1$ in $B_{1/2}(0)$ for all $i = 1, \dots, N$. Then

$$\begin{aligned}\int_\Omega \frac{|u(x)|}{d_x^s}\,\mathrm{d}x \\ = \sum_{i=1}^N \int_{B_1(0)_+} \frac{|u(\phi_i^{-1}(x))|}{d^s_{\phi_i^{-1}(x)}}\eta_i(\phi_i^{-1}(x))|\det(D\phi_i^{-1}(x))|\,\mathrm{d}x + \int_{U_0}\eta_0(x)\frac{|u(x)|}{d_x^s}\,\mathrm{d}x =: \sum_{i=1}^N (\mathrm{I}_i) + (\mathrm{II}).\end{aligned}$$

We define $u_i := u \circ \phi_i^{-1}$ for all $i = 1, \dots, N$. By the bi-Lipschitz continuity of the ϕ_i, we find a constant $c_2 = c_2(\phi_1, \dots, \phi_N) > 1$ such that $c_2^{-1} x_d \le d_{\phi_i^{-1}(x)} \le c_2 x_d$ for any $x \in B_1(0)_+$ and $i \in \{1, \dots, N\}$; see (3-3). Further, we find a constant $c_3 = c_3(\phi_1, \dots, \phi_N) \ge 1$ such that both $[\phi_i^{-1}]_{C^{0,1}}$ and $[\phi_i]_{C^{0,1}}$ are bounded from above by c_3 and from below by c_3^{-1} for all i. We apply Theorem B.1 to the function $\tilde{\eta}_i u_i$ to find

$$\begin{aligned}
(\mathrm{I}_i) &\le c_2 c_3^d \int_{\mathbb{R}^d_+} \frac{|\tilde{\eta}_i(x) u_i(x)|}{x_d^s} \,\mathrm{d}x \\
&\le c_2 c_3^d \mathcal{D}_{s,1}^{-1} \int_{\mathbb{R}^d_+ \times \mathbb{R}^d_+} \frac{|\tilde{\eta}_i(x) u_i(x) - \tilde{\eta}_i(y) u_i(y)|}{|x-y|^{d+s}} \,\mathrm{d}(x,y) \\
&\le c_2 c_3^d \mathcal{D}_{s,1}^{-1} \bigg(\int_{B_1(0)_+ \times B_1(0)_+} \tilde{\eta}_i(x) \frac{|u_i(x) - u_i(y)|}{|x-y|^{d+s}} \,\mathrm{d}(x,y) \\
&\qquad + \int_{B_1(0)_+ \times B_1(0)_+} |u_i(y)| \frac{|\tilde{\eta}_i(x) - \tilde{\eta}_i(y)|}{|x-y|^{d+s}} \,\mathrm{d}(x,y) \\
&\qquad + 2 \int_{B_1(0)_+} \tilde{\eta}_i(x) |u_i(x)| \int_{B_1(0)^c} |x-y|^{-d-s} \,\mathrm{d}y \,\mathrm{d}x \bigg) =: (\mathrm{III}_i) + (\mathrm{IV}_i) + (\mathrm{V}_i).
\end{aligned}$$

The first term in the previous estimate, i.e., (III_i), can be simply estimated using a change of variables and the bi-Lipschitz continuity of ϕ_i:

$$(\mathrm{III}_i) \le c_2 c_3^{4d+s} \mathcal{D}_{s,1}^{-1} \int_{(U_i \cap \Omega) \times (U_i \cap \Omega)} \frac{|u(x) - u(y)|}{|x-y|^{d+s}} \,\mathrm{d}(x,y).$$

To estimate (IV_i), we calculate

$$\int_{B_1(0)} \frac{|\tilde{\eta}_i(x) - \tilde{\eta}_i(y)|}{|x-y|^{d+s}} \,\mathrm{d}x \le c_1 \frac{\omega_{d-1} 2^{1-s}}{1-s}.$$

Using this, we find

$$(\mathrm{IV}_i) \le c_1 c_2 c_3^d \mathcal{D}_{s,1}^{-1} \frac{2\omega_{d-1}}{1-s} \int_{B_1(0)_+} |u_i(y)| \,\mathrm{d}y \le c_1 c_2 c_3^{2d} \mathcal{D}_{s,1}^{-1} \frac{2\omega_{d-1}}{1-s} \int_{U_i \cap \Omega} |u(x)| \,\mathrm{d}x.$$

Now, we estimate (V_i). Since $\tilde{\eta}_i \in C_c^{0,1}(B_1)$, we find a constant $c_4 \ge 1$ such that $\tilde{\eta}_i(x) \le c_4(1-|x|)$ for all $i = 1, \dots, N$. We notice for any $x \in B_1(0)$

$$\int_{B_1(0)^c} |x-y|^{-d-s} \,\mathrm{d}y \le \int_{B_{(1-|x|)}(x)^c} |x-y|^{-d-s} \,\mathrm{d}y = \omega_{d-1} \frac{(1-|x|)^{-s}}{s}$$

and, thus,

$$(\mathrm{V}_i) \le 2 c_2 c_3^d c_4 \frac{\mathcal{D}_{s,1}^{-1}}{s} \omega_{d-1} \int_{B_1(0)_+} |u_i(x)| \,\mathrm{d}x \le 2 c_2 c_3^{2d} c_4 \frac{\mathcal{D}_{s,1}^{-1}}{s} \omega_{d-1} \int_{U_i \cap \Omega} |u(x)| \,\mathrm{d}x.$$

To estimate (II), we simply notice that the distance function is bounded from below by r_0 on U_0. So, finally, we put everything together. This yields

$$\int_\Omega \frac{|u(x)|}{d_x^s} \,\mathrm{d}x \le \frac{c_6}{1-s} \int_\Omega |u(x)| \,\mathrm{d}x + c_7 s \int_{\Omega \times \Omega} \frac{|u(x) - u(y)|}{|x-y|^{d+s}} \,\mathrm{d}(x,y).$$

Here

$$c_6 := (r_0 \wedge 1)^{-1} + 2\omega_{d-1} N c_1 c_2 c_3^{2d} c_4 c_5, \quad c_7 := N c_2 c_3^{4d+1} c_5$$

and c_5 is the constant from Lemma B.2. □

The next two lemmas are technical tools which we employ in the proof of the trace result; see Propositions 3.9 and 3.10. They allow us to rewrite the distance functions appearing in the measure μ_s as an integral over Ω. This enables us to use the regularity of the functions from $V^{s,p}(\Omega\,|\,\mathbb{R}^d)$ in Ω when we prove the trace result.

Lemma 3.7. *Let $\varnothing \neq B \subset \mathbb{R}^d$ be an open set, $s > 0$ and $f : [0,\infty) \to [0,\infty)$ be a nonincreasing function. For any $x \in \overline{B}^c$, we have*

$$\int_B \frac{f(|x-z|)}{|x-z|^{d+s}}\,\mathrm{d}z \le \frac{\omega_{d-1}}{s}\frac{f(\operatorname{dist}(x,B))}{\operatorname{dist}(x,B)^s}.$$

If B is bounded, then there exists a constant $C = C(d,B)$ such that, for any $x \in \overline{B}^c$,

$$\int_B \frac{f(|x-z|)}{|x-z|^{d+s}}\,\mathrm{d}z \le \frac{C}{s}\frac{f(\operatorname{dist}(x,B))}{\operatorname{dist}(x,B)^s(1+\operatorname{dist}(x,B))^d}.$$

Proof. Fix $x \in \overline{B}^c$. We use $B \subset B_{\operatorname{dist}(x,B)}(x)^c$ and apply polar coordinates to get

$$\int_B \frac{f(|x-z|)}{|x-z|^{d+s}}\,\mathrm{d}z \le \int_{B_{\operatorname{dist}(x,B)}(x)^c} \frac{f(|x-z|)}{|x-z|^{d+s}}\,\mathrm{d}z = \omega_{d-1}\int_{\operatorname{dist}(x,B)}^{\infty} f(t)t^{-1-s}\,\mathrm{d}t \le \omega_{d-1}\frac{f(\operatorname{dist}(x,B))}{s\operatorname{dist}(x,B)^s}.$$

In the case that $\operatorname{dist}(x,B) < 1$, the second claim for bounded B is a direct consequence of the first statement. If B is bounded and $\operatorname{dist}(x,B) \ge 1$, then

$$\int_B \frac{f(|x-z|)}{|x-z|^{d+s}}\,\mathrm{d}z \le |B|\frac{f(\operatorname{dist}(x,B))}{\operatorname{dist}(x,B)^{d+s}} \le |B|2^d\frac{f(\operatorname{dist}(x,B))}{\operatorname{dist}(x,B)^s(1+\operatorname{dist}(x,B))^d}. \qquad \square$$

Lemma 3.8. *Let $\varnothing \neq B \subset \mathbb{R}^d$ be an open set satisfying the uniform interior cone condition with a compact boundary. Then there exists a constant $C = C(d,B) > 0$ such that, for any nonincreasing function $f : [0,\infty) \to [0,\infty)$ and any $s > 0$,*

$$\frac{f(2\operatorname{dist}(x,\overline{B}))}{\operatorname{dist}(x,\overline{B})^s(1+\operatorname{dist}(x,\overline{B}))^d} \le C\int_B \frac{f(|x-z|)}{|x-z|^{d+s}}\mathbb{1}_{B_{1+\operatorname{dist}(x,\overline{B})}(x)}(z)\,\mathrm{d}z$$

for all $x \in \overline{B}^c$.

Proof. Fix $x \in \overline{B}^c$ and a minimizer $x_0 \in \partial B$ of the distance $\operatorname{dist}(x,\overline{B})$. Since B satisfies the uniform interior cone condition we find an interior cone $\mathcal{C}$ with apex at x_0 whose height h_0 and open angle are independent of x_0. Without loss of generality, we assume $h_0 \le 1$. Let $\widetilde{\mathcal{C}} := \{z \in \mathcal{C} \mid |z-x_0| < \operatorname{dist}(x,\overline{B})\}$ be a subcone with a reduced height. Notice $\widetilde{\mathcal{C}} \subset B_{1+\operatorname{dist}(x,\overline{B})}(x) \cap B$. For any $z \in \widetilde{\mathcal{C}}$, we have

$$|x-z| \le |x-x_0| + |x-z| \le \operatorname{dist}(x,\overline{B}) + \min\{\operatorname{dist}(x,\overline{B}), h_0\} \le 2\operatorname{dist}(x,\overline{B}).$$

Thus, the claim simply follows from

$$\begin{aligned}\int_B \frac{f(|x-z|)}{|x-z|^{d+s}}\mathbb{1}_{B_{1+\operatorname{dist}(x,\overline{B})}(x)}(z)\,\mathrm{d}z &\ge \frac{f(2\operatorname{dist}(x,\overline{B}))}{(2\operatorname{dist}(x,\overline{B}))^{d+s}}|\widetilde{\mathcal{C}}| = f(2\operatorname{dist}(x,\overline{B}))\frac{c_1(\min\{\operatorname{dist}(x,\overline{B}),h_0\})^d}{(2\operatorname{dist}(x,\overline{B}))^{d+s}} \\ &\ge \frac{c_1h_0^d}{2^{d+1}}\frac{f(2\operatorname{dist}(x,\overline{B}))}{\operatorname{dist}(x,\overline{B})^s(1+\operatorname{dist}(x,\overline{B}))^d}.\end{aligned}$$

Here we used $|\widetilde{\mathcal{C}}| = c_1(\min\{\operatorname{dist}(x,\overline{B}),h_0\})^d$, where $c_1 > 0$ depends only on d and the opening angle of $\widetilde{\mathcal{C}}$, which is independent of x_0. $\square$

We are now in the position to prove the trace part in Theorems 1.2 and 1.3. We split the proof into two propositions. The following proposition contains the trace embedding $V^{s,p}(\Omega \,|\, \mathbb{R}^d) \to L^p(\Omega^c; \mu_s(\mathrm{d}x))$ for all $1 \le p < \infty$. The estimates of the seminorm $[\,\cdot\,]_{\mathcal{T}^{s,p}(\Omega^c)}$ for $1 < p < \infty$ are proven thereafter in Proposition 3.10. Recall the definition of the sets Ω_r^{ext} and Ω_{ext}^r in (2-1) for given $r > 0$.

Proposition 3.9. *Let $\Omega \subset \mathbb{R}^d$, $d \in \mathbb{N}$, be a bounded Lipschitz domain, $s_\star \in (0,1)$ and $1 < p^\star < \infty$. There exists a constant $C = C(\Omega, p^\star, s_\star) > 0$ such that*

$$\|\mathrm{Tr}_s\, u\|_{L^p(\Omega^c;\mu_s)} \le C \|u\|_{V^{s,p}(\Omega \,|\, \mathbb{R}^d)} \tag{3-13}$$

for any $s \in (s_\star, 1)$, $1 \le p \le p^\star$ and $u \in V^{s,p}(\Omega \,|\, \mathbb{R}^d)$

Proof. We split the integration domain of $\|\mathrm{Tr}_s\, u\|^p_{L^p(\Omega^c;\mu_s)}$ into Ω_1^{ext} and Ω_{ext}^1. Let $c_1 = c_1(d,\Omega) > 0$ be the constant from Lemma 3.8 when applied to $B = \Omega$ and $f = 1$. We have

$$\begin{aligned}\|\mathrm{Tr}_s\, u\|^p_{L^p(\Omega_1^{\mathrm{ext}};\mu_s)} &\le (1-s)\int_{\Omega_1^{\mathrm{ext}}} \frac{|u(x)|^p}{\mathrm{dist}(x,\Omega)^s}\,\mathrm{d}x \le 2^d(1-s)c_1 \int_{\Omega_1^{\mathrm{ext}}}\int_\Omega \frac{|u(x)|^p}{|x-z|^{d+s}}\,\mathrm{d}z\,\mathrm{d}x\\ &\le 2^{d+p}(1-s)c_1\left[\int_{\Omega_1^{\mathrm{ext}}}\int_\Omega \frac{|u(x)-u(z)|^p}{|x-z|^{d+s}}\,\mathrm{d}z\,\mathrm{d}x + \int_{\Omega_1^{\mathrm{ext}}}\int_\Omega \frac{|u(z)|^p}{|x-z|^{d+s}}\,\mathrm{d}z\,\mathrm{d}x\right]\\ &=: (\mathrm{I}) + (\mathrm{II}).\end{aligned}$$

The term (I) is estimated easily via

$$(\mathrm{I}) \le 2^{d+p^\star} c_1 (\mathrm{diam}\,\Omega + 1)^{p^\star - 1} [u]^p_{V^{s,p}(\Omega \,|\, \Omega_1^{\mathrm{ext}})}.$$

An application of Lemma 3.7 with $B = \Omega_1^{\mathrm{ext}}$ and Theorem 3.5 (resp. Theorem 3.6 in the case $p = 1$) yields the following bound on the term (II):

$$(\mathrm{II}) \le 2^{d+p}(1-s)c_1 \frac{\omega_{d-1}}{s}\int_\Omega \frac{|u(z)|^p}{d_z^s}\,\mathrm{d}z \le 2^{d+p^\star}\frac{\omega_{d-1}}{s_\star} c_1 c_2 ([u]^p_{W^{s,p}(\Omega)} + \|u\|^p_{L^p(\Omega)}).$$

Here $c_2 > 0$ is the constant from Theorem 3.5 in the case $p > 1$. In the case $p = 1$ let c_2 be the constant from Theorem 3.6. For the estimate of Ω_{ext}^1, we define $\mathrm{diam}\,\Omega + 1 =: c_3 \ge 1$ and notice that $d_x \ge |x-z|/c_3$ as well as $\Omega_{\mathrm{ext}}^1 \subset B_1(z)^c$ holds for any $z \in \Omega$ and $x \in \Omega_{\mathrm{ext}}^1$. Thus,

$$\begin{aligned}\|\mathrm{Tr}_s\, u\|^p_{L^p(\Omega_{\mathrm{ext}}^1;\mu_s)} &\le 2^{p^\star}(1-s)\int_{\Omega_{\mathrm{ext}}^1} \fint_\Omega \frac{|u(x)-u(z)|^p + |u(z)|^p}{d_x^s(1+d_x)^{d+s(p-1)}}\,\mathrm{d}z\,\mathrm{d}x\\ &\le 2^{p^\star} c_3^{d+sp^\star}(1-s)\int_{\Omega_{\mathrm{ext}}^1}\fint_\Omega \frac{|u(x)-u(z)|^p}{|x-z|^{d+sp}}\,\mathrm{d}z\,\mathrm{d}x + 2^{p^\star}(1-s)\int_{\Omega_{\mathrm{ext}}^1}\fint_\Omega \frac{|u(z)|^p}{d_x^{d+sp}}\,\mathrm{d}z\,\mathrm{d}x\\ &\le \frac{2^{p^\star}c_3^{d+sp^\star}}{|\Omega|}[u]^p_{V^{s,p}(\Omega \,|\, \Omega_{\mathrm{ext}}^1)} + \frac{\omega_{d-1}2^{p^\star}c_3^{d+sp^\star}(1-s)}{sp|\Omega|}\|u\|^p_{L^p(\Omega)}.\end{aligned} \tag{3-14}$$

In the last step we used

$$\int_{\Omega_{\mathrm{ext}}^1} \frac{1}{d_x^{d+sp}}\,\mathrm{d}x \le c_3^{d+sp}\int_{B_1(x_0)^c} \frac{1}{|x_0 - x|^{d+sp}}\,\mathrm{d}x = c_3^{d+sp}\frac{\omega_{d-1}}{sp}, \tag{3-15}$$

where $x_0 \in \Omega$ is a fixed point. Combining the estimates of (I) and (II), as well as (3-14) yields (3-13). □

Proposition 3.10. *Let $\Omega \subset \mathbb{R}^d$, $d \in \mathbb{N}$, be a bounded Lipschitz domain, $s_\star \in (0,1)$ and $1 < p_\star < p^\star < \infty$. There exists a constant $C = C(\Omega, p_\star, p^\star, s_\star) > 0$ such that, for any $s \in (s_\star, 1)$, $p_\star \le p \le p^\star$ and $u \in V^{s,p}(\Omega \,|\, \mathbb{R}^d)$,*

$$[\mathrm{Tr_s}\, u]_{\mathcal{T}^{s,p}(\Omega^c)} \le C \|u\|_{V^{s,p}(\Omega \,|\, \mathbb{R}^d)}. \tag{3-16}$$

Proof. We fix $\rho := \mathrm{inr}(\Omega) > 0$ and divide the integration domain of $[\mathrm{Tr_s}\, u]^p_{\mathcal{T}^{s,p}(\Omega^c)}$ into $\Omega^{\mathrm{ext}}_\rho \times \Omega^{\mathrm{ext}}_\rho$, $\Omega^c \times \Omega^\rho_{\mathrm{ext}}$ and $\Omega^\rho_{\mathrm{ext}} \times \Omega^c$. By symmetry the estimates for $\Omega^c \times \Omega^\rho_{\mathrm{ext}}$ and $\Omega^\rho_{\mathrm{ext}} \times \Omega^c$ are equivalent. Thus, we settle on $\Omega^\rho_{\mathrm{ext}} \times \Omega^c$. Since $|x-y| + d_x + d_y \ge \rho$ for any $x \in \Omega^c$ and $y \in \Omega^\rho_{\mathrm{ext}}$, we have

$$\begin{aligned}
[\mathrm{Tr_s}\, u]^p_{\mathcal{T}^{s,p}(\Omega^\rho_{\mathrm{ext}} \,|\, \Omega^c)} &\le \int_{\Omega^\rho_{\mathrm{ext}}} \int_{\Omega^c} \frac{|u(x)-u(y)|^p}{(1 \wedge \rho)^{d+p^\star}} \mu_s(\mathrm{d}x)\mu_s(\mathrm{d}y) \\
&\le \frac{2^p(1-s)^2}{(1 \wedge \rho)^{d+p^\star}} \biggl[\int_{\Omega^c} \frac{|u(x)|^p}{d_x^s(1+d_x)^{d+s(p-1)}} \int_{\Omega^\rho_{\mathrm{ext}}} \frac{1}{d_y^s(1+d_y)^{d+s(p-1)}} \,\mathrm{d}y\,\mathrm{d}x \\
&\qquad + \int_{\Omega^\rho_{\mathrm{ext}}} \frac{|u(y)|^p}{d_y^s(1+d_y)^{d+s(p-1)}} \int_{\Omega^c} \frac{1}{d_x^s(1+d_x)^{d+s(p-1)}} \,\mathrm{d}x\,\mathrm{d}y \biggr].
\end{aligned} \tag{3-17}$$

After covering Ω^{ext}_1 by finitely many balls, a calculation similar to (3-4) yields a constant $c_1 > 0$, independent of s, such that

$$\int_{\Omega^{\mathrm{ext}}_1} d_x^{-s}\,\mathrm{d}x \le c_1(1-s)^{-1}.$$

By possibly enlarging the constant, we assume $c_1 \ge \omega_{d-1}(1+\mathrm{diam}(\Omega))^{d+p^\star}$. With this observation and (3-15), we find

$$\int_{\Omega^c} \frac{1}{d_x^s(1+d_x)^{d+s(p-1)}}\,\mathrm{d}x \le \int_{\Omega^{\mathrm{ext}}_1} \frac{1}{d_x^s}\,\mathrm{d}x + \int_{\Omega^1_{\mathrm{ext}}} d_x^{-d-sp}\,\mathrm{d}x \le \frac{c_1}{s(1-s)}. \tag{3-18}$$

Now, we combine this estimate with Proposition 3.9:

$$\begin{aligned}
[\mathrm{Tr_s}\, u]^p_{\mathcal{T}^{s,p}(\Omega^\rho_{\mathrm{ext}} \,|\, \Omega^c)} &\le \frac{2^{p^\star+1}}{(1\wedge\rho)^{d+p^\star}} \frac{c_1}{s} \|\mathrm{Tr_s}\, u\|^p_{L^p(\Omega^c;\mu_s)} \\
&\le \frac{2^{p^\star+1}}{(1\wedge\rho)^{d+p^\star}} \frac{c_1 c_2}{s_\star} \|u\|^p_{V^{s,p}(\Omega \,|\, \mathbb{R}^d)}.
\end{aligned}$$

Here $c_2 = c_2(\Omega, p^\star, s_\star) > 0$ is the constant from Proposition 3.9.

Lastly, we prove the inequality for the more delicate part of the seminorm, where higher-order singularities close to the boundary may occur. For $x, y \in \Omega^{\mathrm{ext}}_\rho$, we have

$$(|x-y| + d_x + d_y) \le 4\rho + \mathrm{diam}(\Omega) + 1 =: c_3$$

and, thus,

$$(|x-y| + d_x + d_y) \wedge 1 \ge c_3^{-1}(|x-y| + d_x + d_y).$$

We apply Lemma 3.8 twice with the monotone decreasing function $f(r) = \bigl(|x-y| + \tfrac{1}{2}r + d_y\bigr)^{-d-s(p-2)}$ and then again with the function $f(r) = \bigl(|x-y| + \tfrac{1}{2}|x-z| + \tfrac{1}{2}r\bigr)^{-d-s(p-2)}$. We now let $c_4 = c_4(\Omega, d) > 0$

and $r>0$ be the constants from Lemma 3.8. This yields

$$
\begin{aligned}
&[\operatorname{Tr}_s u]^p_{\mathcal{T}^{s,p}(\Omega^{\text{ext}}_\rho\,|\,\Omega^{\text{ext}}_\rho)}\\
&\quad\le (1-s)c_3^{d+p}c_4\int_{\Omega^{\text{ext}}_\rho}\int_{\Omega^{\text{ext}}_\rho}\int_\Omega \frac{|u(x)-u(y)|^p}{|x-z|^{d+s}(|x-y|+|x-z|/2+d_y)^{d+s(p-2)}}\,\mathrm{d}z\,\mathrm{d}x\,\mu_s(\mathrm{d}y)\\
&\quad\le (1-s)^2c_3^{d+p}c_4^2\int_{\Omega^{\text{ext}}_\rho}\int_{\Omega^{\text{ext}}_\rho}\int_\Omega\int_\Omega \frac{|u(x)-u(y)|^p|x-z|^{-d-s}|y-w|^{-d-s}}{(|x-y|+(|x-z|+|y-w|)/2)^{d+s(p-2)}}\,\mathrm{d}w\,\mathrm{d}z\,\mathrm{d}x\,\mathrm{d}y\\
&\quad\le 2^{p^\star}4^{d+p^\star}c_3^{d+p^\star}c_4^2((\mathrm{III})+2(\mathrm{IV})).
\end{aligned}
$$

Here we added $\pm u(z)\pm u(w)$ and used the triangle inequality. The terms are

$$
\begin{aligned}
(\mathrm{III}) &:= (1-s)^2\int_\Omega\int_\Omega |u(z)-u(w)|^p a(z,w)\,\mathrm{d}w\,\mathrm{d}z,\\
a(z,w) &:= \int_{\Omega^{\text{ext}}_\rho}\int_{\Omega^{\text{ext}}_\rho}\frac{1}{|x-z|^{d+s}|y-w|^{d+s}(|x-y|+2|x-z|+2|y-w|)^{d+s(p-2)}}\,\mathrm{d}x\,\mathrm{d}y,\\
(\mathrm{IV}) &:= (1-s)^2\int_{\Omega^{\text{ext}}_\rho}\int_\Omega |u(x)-u(w)|^p b(x,w)\,\mathrm{d}w\,\mathrm{d}x,\\
b(x,w) &:= \int_{\Omega^{\text{ext}}_\rho}\int_\Omega\frac{1}{|x-z|^{d+s}|y-w|^{d+s}(|x-y|+2|x-z|+2|y-w|)^{d+s(p-2)}}\,\mathrm{d}z\,\mathrm{d}y.
\end{aligned}
$$

Estimate of (III): Our goal is to find an appropriate estimate of kernel a to apply Theorem 3.5. Notice that

$$|x-y|+2|x-z|+2|y-w|\ge |z-w|+|x-z|+|y-w| \quad \text{for any } x,y,w,z\in\mathbb{R}^d.$$

Thus, for any $z,w\in\Omega$,

$$a(z,w)\le \int_{\Omega^{\text{ext}}_\rho}\int_{\Omega^{\text{ext}}_\rho}\frac{1}{|x-z|^{d+s}|y-w|^{d+s}(|z-w|+|x-z|+|y-w|)^{d+s(p-2)}}\,\mathrm{d}x\,\mathrm{d}y.$$

Now, we apply Lemma 3.7 twice with $B=\Omega^{\text{ext}}_\rho$. We use the function $f(t_1,t_2)=(|z-w|+t_1+t_2)^{-d-s(p-2)}$ which is decreasing in both t_1 and t_2. Thereby,

$$a(z,w)\le \frac{\omega_{d-1}^2}{s^2}\frac{1}{d_z^s d_w^s(|z-w|+d_z+d_w)^{d+s(p-2)}}. \tag{3-19}$$

This yields the desired estimate for (III) via Theorem 3.5:

$$
\begin{aligned}
(\mathrm{III}) &\le \frac{\omega_{d-1}^2}{s^2}(1-s)^2\int_\Omega\int_\Omega\frac{|u(z)-u(w)|^p}{d_z^s d_w^s(|z-w|+d_z+d_w)^{d+s(p-2)}}\,\mathrm{d}w\,\mathrm{d}z\\
&\le c_5\frac{\omega_{d-1}^2}{s^2}\|u\|^p_{W^{s,p}(\Omega)}\\
&\le c_5\frac{\omega_{d-1}^2}{s_\star^2}\|u\|^p_{V^{s,p}(\Omega\,|\,\mathbb{R}^d)}.
\end{aligned}
$$

Here $c_5=c_5(d,\Omega,p_\star,p^\star,s_\star)>0$ is the constant from Theorem 3.5.

Estimate of (IV): Our approach to estimate (IV) is similar to the proof of [Dyda and Kassmann 2019, Theorem 5]. Analogously to the estimate of a, see (3-19), we find

$$b(x, w) \le \frac{\omega_{d-1}^2}{s^2} \frac{1}{d_x^s d_w^s (|x-w| + d_x + d_w)^{d+s(p-2)}}$$

for any $x \in \Omega_\rho^{\text{ext}}$ and $w \in \Omega$. We define

$$\text{(V)} := (1-s)^2 \int_{\Omega_\rho^{\text{ext}}} \int_\Omega \frac{|u(x) - u(w)|^p}{d_x^s d_w^s (|x-w| + d_x + d_w)^{d+s(p-2)}} \, \mathrm{d}w \, \mathrm{d}x.$$

The previous estimate of b yields

$$\text{(IV)} \le \omega_{d-1}^2 s_\star^{-2} \text{(V)}.$$

Claim: We will show that there exists a constant $c_6 = c_6(d, \Omega, \rho, p^\star) > 0$ such that

$$\text{(V)} \le c_6 \left(\frac{1}{s^2(p_\star - 1)} [u]^p_{V^{s,p}(\Omega \,|\, \Omega_\rho^{\text{ext}})} + (1-s)^2 \int_\Omega \int_\Omega \frac{|u(z) - u(w)|^p}{d_z^s d_w^s (|z-w| + d_z + d_w)^{d+s(p-2)}} \, \mathrm{d}w \, \mathrm{d}z \right).$$

Let $\mathcal{W}(\mathbb{R}^d \setminus \overline{\Omega})$ be the Whitney decomposition from Appendix A and recall that $\rho = \operatorname{inr}(\Omega)$. We define the set of Whitney cubes $Q \in \mathcal{W}(\mathbb{R}^d \setminus \overline{\Omega})$ with $\operatorname{diam}(Q) \le \rho$ by $\mathcal{W}_\rho(\mathbb{R}^d \setminus \overline{\Omega})$. As in Appendix A and [Dyda and Kassmann 2019], we denote by $\widetilde{Q} \subset \Omega$ the reflected Whitney cube for any cube $Q \in \mathcal{W}_\rho(\mathbb{R}^d \setminus \overline{\Omega})$. The collection of reflected Whitney cubes satisfies a bounded overlap property; see (A-4). Let $N \in \mathbb{N}$ be the constant from the bounded overlap property. Furthermore, the distance to the boundary as well as the diameter of the reflected cubes are comparable to the original cubes with the constant $M > 0$ from (A-2). By the covering properties of the Whitney cubes (A-1) and the reflecting cubes (A-3), we find

$$\text{(V)} \le \sum_{Q_1, Q_2 \in \mathcal{W}_\rho(\mathbb{R}^d \setminus \overline{\Omega})} (1-s)^2 \int_{Q_1} \int_{\widetilde{Q}_2} \frac{|u(x) - u(w)|^p}{d_x^s d_w^s (|x-w| + d_x + d_w)^{d+s(p-2)}} \, \mathrm{d}w \, \mathrm{d}x. \tag{3-20}$$

For the moment we fix two cubes $Q_1, Q_2 \in \mathcal{W}(\mathbb{R}^d \setminus \overline{\Omega})$ satisfying $\operatorname{diam} Q_1, \operatorname{diam} Q_2 \le \rho$. For each $x \in Q_1$ and $w \in \widetilde{Q}_2$, we define

$$z(x, w) := q_{\widetilde{Q}_1} + \left(\frac{x - q_{Q_1}}{2l(Q_1)} + \frac{w - q_{\widetilde{Q}_2}}{2l(\widetilde{Q}_2)} \right) l(\widetilde{Q}_1) \in \widetilde{Q}_1.$$

The map z connects points in $Q_1, \widetilde{Q}_2$ with points in $\widetilde{Q}_1$ in a continuous way. We will use it for a change of variables in either x or w. Therefore,

$$\begin{aligned} &\int_{Q_1} \int_{\widetilde{Q}_2} \frac{|u(x)-u(w)|^p}{d_x^s d_w^s (|x-w|+d_x+d_w)^{d+s(p-2)}} \, \mathrm{d}w \, \mathrm{d}x \\ &\le 2^p \int_{Q_1} \int_{\widetilde{Q}_2} \frac{|u(z(x,w))-u(w)|^p}{d_x^s d_w^s (|x-w|+d_x+d_w)^{d+s(p-2)}} \, \mathrm{d}w \, \mathrm{d}x + 2^p \int_{Q_1} \int_{\widetilde{Q}_2} \frac{|u(x)-u(z(x,w))|^p}{d_x^s d_w^s (|x-w|+d_x+d_w)^{d+s(p-2)}} \, \mathrm{d}w \, \mathrm{d}x \\ &=: 2^p(\text{(VI)} + \text{(VII)}). \end{aligned} \tag{3-21}$$

We make a few observations before estimating (VI). For $x \in Q_1$ and $w \in \widetilde{Q}_2$, we have $d_x \geq M^{-1} d_{z(x,w)}$ as well as

$$\begin{aligned}
|x-w|+d_x+d_w &\geq M^{-1}(|x-w|+\operatorname{dist}(Q_1, \widetilde{Q}_1))+(1-M^{-1})|x-w|+d_w \\
&\geq M^{-1}(\operatorname{dist}(Q_1, \widetilde{Q}_2)+\operatorname{dist}(Q_1, \widetilde{Q}_1))+(1-M^{-1})(\operatorname{dist}(Q_1, \partial\Omega)+\operatorname{dist}(\widetilde{Q}_2, \partial\Omega))+d_w \\
&\geq M^{-1}\operatorname{dist}(\widetilde{Q}_1, \widetilde{Q}_2)+(1-M^{-1})M^{-1}\operatorname{dist}(\widetilde{Q}_1, \partial\Omega)+(2-M^{-1})\operatorname{dist}(\widetilde{Q}_2, \partial\Omega) \\
&\geq \frac{(1-M^{-1})M^{-1}\big(\operatorname{dist}(\widetilde{Q}_1, \widetilde{Q}_2)+\sum_{i=1}^2 \operatorname{dist}(\widetilde{Q}_i, \partial\Omega)\big)(|z(x,w)-w|+d_{z(x,w)}+d_w)}{(\operatorname{diam}\widetilde{Q}_1+\operatorname{dist}(\widetilde{Q}_1, \widetilde{Q}_2)+\operatorname{diam}\widetilde{Q}_2)+\sum_{i=1}^2(\operatorname{dist}(\widetilde{Q}_i, \partial\Omega)+\operatorname{diam}\widetilde{Q}_i)} \\
&\geq (1-M^{-1})M^{-1}\tfrac{2}{3}(|z(x,w)-w|+d_{z(x,w)}+d_w).
\end{aligned}$$

We define $c_7 := \frac{3}{2}M^2(M-1)^{-1} > 1$ and use the previous calculation to estimate (VI) by

$$\begin{aligned}
(\mathrm{VI}) &\leq M^s c_7^{d+s(p-2)} \int_{\widetilde{Q}_2}\int_{Q_1} \frac{|u(z(x,w))-u(w)|^p}{d_{z(x,w)}^s d_w^s(|z(x,w)-w|+d_{z(x,w)}+d_w)^{d+s(p-2)}}\,\mathrm{d}x\,\mathrm{d}w \\
&\leq M^s c_7^{d+s(p-2)}\left(\frac{2l(Q_1)}{l(\widetilde{Q}_1)}\right)^d \int_{\widetilde{Q}_2}\int_{\widetilde{Q}_1} \frac{|u(z)-u(w)|^p}{d_z^s d_w^s(|z-w|+d_z+d_w)^{d+s(p-2)}}\,\mathrm{d}z\,\mathrm{d}w \\
&\leq M c_7^{d+(0\vee(p^\star-2))}(2M)^d \int_{\widetilde{Q}_1}\int_{\widetilde{Q}_2} \frac{|u(z)-u(w)|^p}{d_z^s d_w^s(|z-w|+d_z+d_w)^{d+s(p-2)}}\,\mathrm{d}w\,\mathrm{d}z. \qquad (3\text{-}22)
\end{aligned}$$

Here we used the change of variables $z = z(x,w)$. We set $c_8 := 2^{d+p^\star} c_7^{d+(0\vee(p^\star-2))} M^{d+1}$. Now we sum (VI) over all Whitney cubes in $\mathcal{W}_\rho(\mathbb{R}^d \setminus \overline{\Omega})$. By the bounded overlap property of the Whitney decomposition, see (A-4), we have

$$\begin{aligned}
\sum_{Q_1, Q_2 \in \mathcal{W}_\rho(\mathbb{R}^d\setminus\overline{\Omega})} (1-s)^2 2^p (\mathrm{VI}) &\leq c_8(1-s)^2 \sum_{Q_1, Q_2 \in \mathcal{W}_\rho(\mathbb{R}^d\setminus\overline{\Omega})} \int_{\widetilde{Q}_1}\int_{\widetilde{Q}_2} \frac{|u(z)-u(w)|^p}{d_z^s d_w^s(|z-w|+d_z+d_w)^{d+s(p-2)}}\,\mathrm{d}w\,\mathrm{d}z \\
&\leq N^2 c_8(1-s)^2 \int_\Omega\int_\Omega \frac{|u(z)-u(w)|^p}{d_z^s d_w^s(|z-w|+d_z+d_w)^{d+s(p-2)}}\,\mathrm{d}w\,\mathrm{d}z. \qquad (3\text{-}23)
\end{aligned}$$

Now we estimate (VII). We make a few observations upon the choice of the Whitney decomposition and reflected cubes in Appendix A. For $x \in Q_1$ and $w \in \widetilde{Q}_2$, we have

$$\begin{aligned}
d_x &\geq (1+M^{-1})^{-1}(d_x+d_{z(x,w)}) \geq (1+M^{-1})^{-1}|x-z(x,w)|, \\
d_w &\geq \operatorname{dist}(\widetilde{Q}_2, \partial\Omega), \\
d_x+d_w+|x-w| &\geq \operatorname{dist}(Q_1, \partial\Omega)+\operatorname{dist}(\widetilde{Q}_2, \partial\Omega)+\operatorname{dist}(Q_1, \widetilde{Q}_2), \\
|x-z(x,w)| &\leq \operatorname{dist}(Q_1, \widetilde{Q}_1) \leq M\operatorname{dist}(Q_1, \partial\Omega).
\end{aligned}$$

We set

$$J(Q_1, Q_2) := \frac{\operatorname{dist}(Q_1, \partial\Omega)^{d+s(p-1)}}{\operatorname{dist}(\widetilde{Q}_2, \partial\Omega)^s(\operatorname{dist}(Q_1, \partial\Omega)+\operatorname{dist}(\widetilde{Q}_2, \partial\Omega)+\operatorname{dist}(Q_1, \widetilde{Q}_2))^{d+s(p-2)}}.$$

Therefore,

$$\begin{aligned}(\mathrm{VII}) &\leq (1+M^{-1})^s M^{d+s(p-1)} J(Q_1,\widetilde{Q}_2)\int_{Q_1}\int_{\widetilde{Q}_2}\frac{|u(x)-u(z(x,w))|^p}{|x-z(x,w)|^{d+sp}}\,\mathrm{d}w\,\mathrm{d}x\\ &\leq (1+M^{-1})M^{d+p-1}J(Q_1,\widetilde{Q}_2)\Big(\frac{2l(\widetilde{Q}_2)}{l(\widetilde{Q}_1)}\Big)^d\int_{Q_1}\int_{\widetilde{Q}_1}\frac{|u(x)-u(z)|^p}{|x-z|^{d+sp}}\,\mathrm{d}z\,\mathrm{d}x.\end{aligned}$$

Here we used the change of variables $z := z(x,w)$. Set $c_9 := (1+M^{-1})M^{d+p^\star-1}2^{d+p^\star}$. By (A-4),

$$\begin{aligned}(1-s)^2 &\sum_{Q_1,Q_2\in\mathcal{W}_\rho(\mathbb{R}^d\setminus\bar{\Omega})} 2^p(\mathrm{VII})\\ &\leq c_9(1-s)^2\sum_{Q_1,Q_2\in\mathcal{W}_\rho(\mathbb{R}^d\setminus\bar{\Omega})} J(Q_1,\widetilde{Q}_2)\Big(\frac{l(\widetilde{Q}_2)}{l(\widetilde{Q}_1)}\Big)^d\int_{Q_1}\int_{\widetilde{Q}_1}\frac{|u(x)-u(z)|^p}{|x-z|^{d+sp}}\,\mathrm{d}z\,\mathrm{d}x. \quad (3\text{-}24)\end{aligned}$$

Therefore, it is sufficient to prove that

$$I(Q_1) := \sum_{Q_2\in\mathcal{W}_\rho(\mathbb{R}^d\setminus\bar{\Omega})} J(Q_1,\widetilde{Q}_2)\Big(\frac{l(\widetilde{Q}_2)}{l(\widetilde{Q}_1)}\Big)^d$$

is bounded independent of Q_1. We fix $Q_1\in\mathcal{W}_\rho(\mathbb{R}^d\setminus\Omega)$ and set $a := \mathrm{dist}(Q_1,\partial\Omega)$. Let $\hat{q}_{Q_1}\in\partial\Omega$ be a minimizer of the distance of q_{Q_1} to $\partial\Omega$; i.e., $|\hat{q}_{Q_1}-q_{Q_1}| = \mathrm{dist}(q_{Q_1},\partial\Omega)$. By the properties of the Whitney cubes, we have for any $w\in\widetilde{Q}_2$

$$\mathrm{dist}(\widetilde{Q}_2,\partial\Omega)\geq \tfrac{1}{2}(\mathrm{dist}(\widetilde{Q}_2,\partial\Omega)+\mathrm{diam}\,\widetilde{Q}_2)\geq \tfrac{1}{2}d_w, \quad (3\text{-}25)$$

$$\begin{aligned}|w-b| &\leq |w-q_{Q_1}|+|q_{Q_1}-\hat{q}_{Q_1}| \leq \mathrm{diam}\,Q_1+\mathrm{dist}(\widetilde{Q}_2,Q_1)+\mathrm{dist}(q_{Q_1},\partial\Omega)\\ &\leq \mathrm{dist}(\widetilde{Q}_2,Q_1)+3\,\mathrm{dist}(Q_1,\partial\Omega)\leq 4\,\mathrm{dist}(\widetilde{Q}_2,Q_1). \quad (3\text{-}26)\end{aligned}$$

We estimate $I(Q_1)$ using the properties of the Whitney cubes, (3-25), (3-26) and (A-4):

$$\begin{aligned}I(Q_1) &\leq M^d4^d\sum_{Q_2\in\mathcal{W}_\rho(\mathbb{R}^d\setminus\bar{\Omega})}\frac{\mathrm{diam}(\widetilde{Q}_2)^d a^{s(p-1)}}{\mathrm{dist}(\widetilde{Q}_2,\partial\Omega)^s(a+\mathrm{dist}(\widetilde{Q}_2,\partial\Omega)+\mathrm{dist}(Q_1,\widetilde{Q}_2))^{d+s(p-2)}}\\ &= M^d4^d\sum_{Q_2\in\mathcal{W}_\rho(\mathbb{R}^d\setminus\bar{\Omega})}\int_{\widetilde{Q}_2}\frac{a^{s(p-1)}}{\mathrm{dist}(\widetilde{Q}_2,\partial\Omega)^s(a+\mathrm{dist}(\widetilde{Q}_2,\partial\Omega)+\mathrm{dist}(Q_1,\widetilde{Q}_2))^{d+s(p-2)}}\,\mathrm{d}w\\ &\leq 2^s4^{d+s(p-2)}M^d4^dN\int_{\Omega}\frac{a^{s(p-1)}}{d_w^s(a+d_w+|w-\hat{q}_{Q_1}|)^{d+s(p-2)}}\,\mathrm{d}w. \quad (3\text{-}27)\end{aligned}$$

We claim that the integral in the last line is bounded independent of a and $\hat{q}_{Q_1}$.

We localize the boundary in a neighborhood of $\hat{q}_{Q_1}$. Let $r_0>0$ be the localization radius, and let $\phi: B_{r_0}(\hat{q}_{Q_1})\to B_1(0)$ be a bi-Lipschitz flattening of the boundary since Ω has a uniform Lipschitz boundary. A change of variables and an estimate similar to (3-3) yields a constant $c_{10}=c_{10}(d,\Omega)\geq 1$ such that

$$\int_{\Omega\cap B_{r_0}(\hat{q}_{Q_1})}\frac{a^{s(p-1)}}{d_w^s(a+d_w+|w-\hat{q}_{Q_1}|)^{d+s(p-2)}}\,\mathrm{d}w\leq c_{10}\int_{B_1(0)_+}\frac{a^{s(p-1)}}{w_d^s(a+|w|)^{d+s(p-2)}}\,\mathrm{d}w.$$

To calculate this integral, we apply the coarea formula; see, e.g., [Federer 1969] with $(r,t)=(w_d,|w|)$ in the case $d\geq 2$:

$$\int_{B_1(0)_+}\frac{a^{s(p-1)}}{w_d^s(a+|w|)^{d+s(p-2)}}\,\mathrm{d}w\leq \omega_{d-2}\int_0^1\int_0^t\frac{a^{s(p-1)}t^{d-2}}{r^s(a+t)^{d+s(p-2)}}\,\mathrm{d}r\,\mathrm{d}t$$
$$\leq\frac{\omega_{d-2}}{1-s}\int_0^1\frac{a^{s(p-1)}t^{d-1-s}}{(a+t)^{d+s(p-2)}}\,\mathrm{d}t$$
$$\leq\frac{\omega_{d-2}}{1-s}\int_0^1\frac{a^{s(p-1)}}{(a+t)^{1+s(p-1)}}\,\mathrm{d}t\leq\frac{\omega_{d-2}}{(1-s)s(p-1)}. \tag{3-28}$$

Here we used

$$\mathcal{H}^{(d-2)}(\{w\in\Omega\mid w_d=r,\,|w|=t\})\leq\omega_{d-2}\mathbb{1}_{r\leq t}t^{d-2}.$$

A similar calculation shows the same estimate in the case $d=1$. Furthermore, the remainder of the integral on the right-hand side of (3-27), i.e.,

$$\int_{\Omega\cap B_{r_0}(\hat{q}_{Q_1})^c}\frac{a^{s(p-1)}}{d_w^s(a+d_w+|w-\hat{q}_{Q_1}|)^{d+s(p-2)}}\,\mathrm{d}w,$$

is easily bounded independent of a since $a+d_w+|w-\hat{q}_{Q_1}|\geq r_0$ for $w\in\Omega\cap B_{r_0}(\hat{q}_{Q_1})^c$ and $a\leq 4\rho$.

Therefore $(1-s)I(Q_1)$ is bounded independent of Q_1 by a constant $c_{11}=c_{11}(d,\Omega,p_\star,p^\star,s_\star)>0$. We combine (3-24), (3-27) and (3-28) as well as (A-4) to obtain

$$(1-s)^2\sum_{Q_1,Q_2\in\mathcal{W}_\rho(\mathbb{R}^d\setminus\overline{\Omega})}2^p(\mathrm{VII})\leq c_9c_{11}N(1-s)\int_{\Omega_\rho^{\mathrm{ext}}}\int_\Omega\frac{|u(x)-u(z)|^p}{|x-z|^{d+sp}}\,\mathrm{d}z\,\mathrm{d}x$$
$$=c_9c_{11}N[u]^p_{V^{s,p}(\Omega\,|\,\Omega_\rho^{\mathrm{ext}})}. \tag{3-29}$$

Finally, we combine (3-20), (3-21), (3-23) and (3-29) and the claim follows. The constant is given by $c_6:=N\max\{Nc_8,c_9c_{11}\}$.

We finish the estimate of (IV) using the previous claim and Theorem 3.5:

$$(\mathrm{V})\leq c_6\left(\frac{1}{s^2(p_\star-1)}[u]^p_{V^{s,p}(\Omega\,|\,\Omega_\rho^{\mathrm{ext}})}+c_5\|u\|^p_{W^{s,p}(\Omega)}\right).$$

Combining the estimates of (III) and (IV) yields (3-16). □

Remark 3.11. As mentioned in the introduction, the trace embedding $V^{s,1}(\Omega\,|\,\mathbb{R}^d)\to\mathcal{T}^{s,1}(\Omega^c)$ cannot be continuous. This may be seen as follows: Consider the sequence of functions

$$u_n(x):=\begin{cases}0, & x\in\Omega,\\ n^{1-s}, & x\in\Omega^{\mathrm{ext}}_{1/n},\\ 0, & x\in\Omega^{1/n}_{\mathrm{ext}},\end{cases}$$

for $n\in\mathbb{N}$. By Lemmas 3.7 and 3.8 one easily sees that $\|u_n\|_{V^{s,1}(\Omega\,|\,\mathbb{R}^d)}\asymp\|u_n\|_{L^1(\Omega^c;\mu_s)}\asymp 1$, but a simple calculation yields $[u_n]_{\mathcal{T}^{s,1}(\Omega^c)}\asymp\ln(n)\to\infty$ as $n\to\infty$. A similar sequence of functions proves Theorem 3.5 to be false for $p=1$.

4. Extension results

The aim of this section is to prove the extension parts of Theorems 1.2 and 1.3. This proof is carried out in Propositions 4.5 and 4.6. We are able to treat the cases $1 < p < \infty$ and $p = 1$ together.

The method used in this section is essentially inspired by [Jonsson and Wallin 1984, Chapter V]. We decompose the domain Ω into Whitney cubes and consider neighborhoods of each cube intersected with Ω^c. The extension is constructed by copying weighted mean values of the exterior data g from this intersection into the respective cube; see (4-11). The weights are taken with respect to a measure that behaves like μ_s close to the boundary $\partial\Omega$.

Throughout this section we fix an open nonempty proper subset $\Omega \subset \mathbb{R}^d$. We will introduce additional assumptions on Ω when needed. Further, we fix a dyadic Whitney-decomposition $\mathcal{W}(\Omega)$ of Ω consisting of cubes with parallel sides to the axes of $\mathbb{R}^d$ such that

(a) $\Omega = \bigcup_{Q\in\mathcal{W}(\Omega)} Q$,

(b) the interior of the cubes are mutually disjoint,

(c) for all cubes $Q \in \mathcal{W}(\Omega)$, the diameter is comparable to the distance to the boundary of Ω; i.e.,

$$\operatorname{diam} Q \le d(Q, \partial\Omega) \le 4 \operatorname{diam} Q. \tag{4-1}$$

We set $s_Q \in 2^{\mathbb{Z}}$ to be the side length of a cube Q and let $q_Q \in Q$ be the center of the cube Q. We denote the diameter of the cube Q by $l_Q = \operatorname{diam}(Q) = \sqrt{d} s_Q$. This decomposition satisfies the following: Suppose $Q_1, Q_2 \in \mathcal{W}(\Omega)$ touch each other. Then

$$\tfrac{1}{4} \operatorname{diam} Q_1 \le \operatorname{diam} Q_2 \le 4 \operatorname{diam} Q_1. \tag{4-2}$$

Additionally, we denote by $Q^\star$ a cube having the same center as $Q \in \mathcal{W}(\Omega)$ but side length $\big(1 + \frac{1}{8}\big)s_Q$. We denote the collection of these scaled cubes by

$$\mathcal{W}^\star(\Omega) := \{Q^\star \mid Q \in \mathcal{W}(\Omega)\}.$$

These scaled cubes satisfy a finite overlap property; i.e., $\sum_{Q^\star\in\mathcal{W}^\star(\Omega)} \mathbb{1}_{Q^\star} \le N$, where $N \in \mathbb{N}$ is a fixed number for the remainder of this section. Additionally, two cubes $Q_1^\star, Q_2^\star \in \mathcal{W}^\star(\Omega)$ have nonempty intersection if and only if Q_1 and Q_2 touch. We define $J_i \subset \mathcal{W}(\Omega)$ to be the set of all cubes with side lengths 2^{-i}, and set

$$D_i := \bigcup_{Q\in J_i} Q, \qquad D_{\ge i} := \bigcup_{j\ge i} D_j. \tag{4-3}$$

Analogously to [Jonsson and Wallin 1984, Section 1.2], we introduce a specific partition of unity on Ω which we will use to construct an extension operator $\mathcal{T}^{s,p}(\Omega^c) \to V^{s,p}(\Omega \,|\, \mathbb{R}^d)$. We emphasize that the construction of the extension is independent of p. Let $\psi \in C_c^\infty(\mathbb{R}^d)$ be a bump function such that $\psi = 1$ on $\big[-\frac{1}{2}, \frac{1}{2}\big]^d$ and $\psi = 0$ on $\big(\big[-\frac{1}{2}\big(1+\frac{1}{8}\big), \frac{1}{2}\big(1+\frac{1}{8}\big)\big]^d\big)^c$, $0 \le \psi \le 1$. Then we define for any $Q \in \mathcal{W}(\Omega)$ a translated and rescaled version of ψ,

$$\psi_Q(x) := \psi\left(\frac{x - q_Q}{s_Q}\right),$$

and our partition functions

$$\phi_Q(x) := \frac{\psi_Q(x)}{\sum_{\widetilde{Q}\in\mathcal{W}(\Omega)} \psi_{\widetilde{Q}(x)}}. \tag{4-4}$$

Then obviously $\sum_Q \phi_Q = \mathbb{1}_\Omega$ holds. Furthermore, we set

$$\begin{aligned}\rho &:= \tfrac{1}{2}\operatorname{inr}(\Omega) \wedge \tfrac{1}{2} > 0,\\ \kappa &:= \left\lfloor \log_2 \frac{\rho}{\sqrt{d}} \right\rfloor,\\ \mathcal{W}_{\le\kappa}(\Omega) &:= \{Q \in \mathcal{W}(\Omega) \mid s_Q \le 2^\kappa\}.\end{aligned} \tag{4-5}$$

Since $l_Q = \sqrt{d} s_Q$ for any $Q \in \mathcal{W}$ and the cubes have dyadic side lengths, $l_Q \le \rho$ for any $Q \in \mathcal{W}_{\le\kappa}$. For any $x \in Q \in \mathcal{W}$, we know

$$\begin{aligned} d_x &\le l_Q + \operatorname{dist}(Q, \partial\Omega) \le 5l_Q,\\ d_x &\ge \operatorname{dist}(Q, \partial\Omega) \ge \tfrac{1}{4} l_Q.\end{aligned}$$

Therefore,

$$\left\{x \in \Omega \mid d_x < \tfrac{1}{4}\rho\right\} \subset \bigcup_{Q\in\mathcal{W}_{\le\kappa}(\Omega)} Q \subset \{x \in \Omega \mid d_x < 5\rho\}. \tag{4-6}$$

For any cube $Q \in \mathcal{W}_{\le\kappa}$ such that all neighboring cubes are in $\mathcal{W}_{\le\kappa}$, we have

$$\sum_{Q'\in\mathcal{W}_{\le\kappa}} \phi_{Q'}(x) = 1, \quad x \in Q.$$

By (4-2), all cubes Q with a side length that is at most $2^{\kappa-2}$ only have neighboring cubes in $\mathcal{W}_{\le\kappa}$. Therefore,

$$\sum_{Q\in\mathcal{W}_{\le\kappa}} \phi_Q(x) = 1, \quad x \in D_{\ge -\kappa+2}. \tag{4-7}$$

We define for $s \in (0, 1)$ the measure on $\mathcal{B}(\mathbb{R}^d)$

$$\tilde{\mu}_s(\mathrm{d}z) = \mathbb{1}_{\Omega^c}(z)\frac{1-s}{d_z^s}\,\mathrm{d}z. \tag{4-8}$$

This measure behaves like μ_s, see (1-5), near the boundary $\partial\Omega$. We will use $\tilde{\mu}_s$ to construct the extension of a function $g : \Omega^c \to \mathbb{R}$; see (4-11). In particular, the value of the extension $\mathrm{Ext_s}(g)$ in a cube $Q \in \mathcal{W}_{\le\kappa}$ will depend on a $\tilde{\mu}_s$-mean of g in a neighborhood of Q. Since $\tilde{\mu}_s$ converges weakly to the surface measure on $\partial\Omega$, we recover a classical Whitney extension of functions in $W^{1-1/p,p}(\partial\Omega)$ in the limit $s \to 1^-$. We set for $Q \in \mathcal{W}(\Omega)$

$$a_{Q,s} := (\tilde{\mu}_s(B_{6l_Q}(q_Q)))^{-1}. \tag{4-9}$$

Since $\operatorname{dist}(q_Q, \partial\Omega) \le 5l_Q$, the intersection $B_{6l_Q}(q_Q) \cap \Omega^c$ has nonempty interior. The following lemma shows the order of $a_{Q,s}$ in terms of l_Q and s for Lipschitz domains. The estimate (4-10) is essential in Propositions 4.5 and 4.6.

Lemma 4.1. *Let $\varnothing \neq \Omega \subset \mathbb{R}^d$ be a Lipschitz domain. There exists a constant $C = C(d, \Omega) > 1$ such that, for any $s \in (0, 1)$ and $Q \in \mathcal{W}_{\leq \kappa+6}(\Omega)$,*

$$C^{-1} l_Q^{s-d} \leq a_{Q,s} \leq C l_Q^{s-d}. \tag{4-10}$$

Proof. Let $z_Q \in \partial\Omega$ be a minimizer of the distance of q_Q to the boundary $\partial\Omega$. Since the distance $\operatorname{dist}(q_Q, \partial\Omega)$ is bounded from above by $5l_Q$, we have $B_{l_Q}(z_Q) \subset B_{6l_Q}(q_Q) \subset B_{11l_Q}(z_Q)$. Since Ω has a uniform Lipschitz boundary, we find a radius $r = r(\Omega) > 0$ and a constant $L = L(\Omega) > 0$, both independent of z_Q, and a rotation and translation $T_{z_Q} : \mathbb{R}^d \to \mathbb{R}^d$ as well as a Lipschitz continuous function $\phi : \mathbb{R}^{d-1} \to \mathbb{R}$ such that

$$T_{z_Q}(\Omega \cap B_r(z_q)) = \{(x', x_d) \in B_r(z_Q) \mid x_d > \phi(x')\} \quad \text{and} \quad [\phi]_{C^{0,1}} \leq L.$$

Without loss of generality, we assume T_{z_Q} to be the identity map, in particular, $z_Q = 0$. By arguments similar to (3-3), we find $2(1+L)d_x \geq |x_d - \phi(x')|$ for any $(x', x_d) \in B_r(0)$ such that $x_d > \phi(x')$. First we assume that $11l_Q \leq r$. Then proceeding as in (3-4) yields

$$a_{Q,s}^{-1} \leq (2(1+L))^s \int_{B_{11l_Q} \cap \{x_d \geq \phi(x')\}} \frac{1-s}{(x_d - \phi(x'))^s} \,\mathrm{d}(x', x_d) \leq 2(11+L)(11)^d l_Q^{d-s}.$$

If $11l_Q > r$, then we simply cover $B_{11l_Q}(z_Q) \cap \partial\Omega$ by finitely many balls $B_1, \dots, B_N$. Since l_Q is bounded from above by $2^6\rho$, the number of balls of radius r which are needed can be picked uniformly. Set $A := \Omega^c \cap B_{11lQ}(z_Q) \cap \bigcap_j B_j^c$ and $r_1 := \operatorname{dist}(\partial\Omega, A)$. We pick the balls $B_1, \dots, B_N$ such that $r_1 > \frac{1}{2}r$. Then, by a similar calculation as above,

$$\begin{aligned} a_{Q,s}^{-1} \leq \sum_{j=1}^{N} \tilde{\mu}_s(B_j) + \tilde{\mu}_s(A) &\leq N2(11+L)(11)^d r^{d-s} + 2^7 11^d \omega_{d-1}(r \wedge 1)^{-1}(\rho \vee 1) l_Q^{d-s} \\ &\leq (N2(11+L)(11)^{2d} + 2^7 11^d \omega_{d-1}(r \wedge 1)^{-1}(\rho \vee 1)) l_Q^{d-s}. \end{aligned}$$

For the upper bound in (4-10), we simply notice that

$$a_{Q,s}^{-1} \geq \int_{B_{l_Q \wedge r} \cap \{x_d \geq \phi(x')\}} \frac{1-s}{(x_d - \phi(x'))^s} \,\mathrm{d}(x', x_d)$$

and proceed in a similar fashion. □

For $g \in L^p_{\text{loc}}(\mathbb{R}^d)$, we define the extension $\operatorname{Ext_s}(g)$ as

$$\operatorname{Ext_s}(g)(x) := \begin{cases} \sum_{Q \in \mathcal{W}_{\leq\kappa}(\Omega)} \phi_Q(x) a_{Q,s} \int_{\Omega^c \cap B_{6l_Q}(q_Q)} g(z) \tilde{\mu}_s(\mathrm{d}z) & \text{for } x \in \Omega, \\ g(x) & \text{for } x \in \Omega^c. \end{cases} \tag{4-11}$$

For any $Q \in \mathcal{W}_{\leq\kappa}(\Omega)$, we have

$$\sup\{\operatorname{dist}(x, \partial\Omega) \mid x \in Q^\star\} \leq \operatorname{dist}(Q^\star, \partial\Omega) + \operatorname{diam}(Q^\star) \leq 4\rho + \tfrac{9}{8}\rho \leq 6\rho.$$

Therefore, $\operatorname{Ext_s}(g)(x) = 0$ for $x \in \Omega$ such that $d_x > 6\rho$. Additionally, the definition of $\operatorname{Ext_s}(g)$ inside Ω depends only on the values of g on $\Omega_{6\rho}^{\text{ext}} \subset \Omega_{3\,\text{inr}(\Omega)}^{\text{ext}}$. We could use the measure μ_s introduced in (1-5) instead of $\tilde{\mu}_s$ in the definition of the extension because $\operatorname{Ext_s}(g)|_\Omega$ does not depend on the values of g far away from the boundary. But the benefit of $\tilde{\mu}_s$ is that the extension is independent of the parameter p.

We begin by proving some properties of $\mathrm{Ext_s}$ analogous to [Jonsson 1994, Lemma 1]. The proof follows the same lines as [Jonsson and Wallin 1984, Chapter V, Lemma D]. We define for cubes $Q_1, Q_2 \in \mathcal{W}_{\le\kappa}(\Omega)$ and $g \in L^p_{\mathrm{loc}}(\mathbb{R}^d)$

$$J_p(q_{Q_1}, q_{Q_2}) := \left(a_{Q_1,s} a_{Q_2,s} \int_{B_{30l_{Q_1}}(q_{Q_1})} \int_{B_{30l_{Q_2}}(q_{Q_2})} |g(z_1) - g(z_2)|^p \tilde{\mu}_s(\mathrm{d}z_2) \tilde{\mu}_s(\mathrm{d}z_1) \right)^{1/p}. \tag{4-12}$$

Lemma 4.2. *Let $s \in (0, 1)$ and $1 \le p < \infty$, and assume that Ω satisfies (4-10). Further, let $Q_1, Q_2 \in \mathcal{W}_{\le\kappa-2}(\Omega)$. There exists a constant $C = C(d, \Omega, \psi, \mathcal{W}(\Omega)) > 0$ such that, for any $g \in L^p(\Omega^c)$ and $x \in Q_1$, $y \in Q_2$ as well as $b \in \mathbb{R}$:*

(a) $|\mathrm{Ext_s}(g)(x) - \mathrm{Ext_s}(g)(y)| \le C J_p(q_{Q_1}, q_{Q_2})$,

(b) $|\nabla \mathrm{Ext_s}(g)(x)| \le C l_{Q_1}^{-1} J_p(q_{Q_1}, q_{Q_2})$,

(c) $|\mathrm{Ext_s}(g)(x) - b| \le C \big(a_{Q_1,s} \int_{B_{30l_{Q_1}}(q_{Q_1})} |g(z_1) - b|^p \tilde{\mu}_s(\mathrm{d}z_1) \big)^{1/p}$,

(d) $|\nabla \mathrm{Ext_s}(g)(w)| \le C l_Q^{-1} \big(a_{Q,s} \int_{B_{30l_Q}(q_Q)} |g(z)|^p \tilde{\mu}_s(\mathrm{d}z) \big)^{1/p}$ *for any $w \in Q \in \mathcal{W}(\Omega)$.*

Proof. (a) By (4-7),

$$\sum_{Q \in \mathcal{W}_{\le\kappa}} \phi_Q(x) = 1 = \sum_{Q \in \mathcal{W}_{\le\kappa}} \phi_Q(y).$$

By the choice of $a_{Q,s}$, we find

$$\begin{aligned} &\mathrm{Ext_s}(g)(x) - \mathrm{Ext_s}(g)(y) \\ &\quad= \sum_{Q \in \mathcal{W}_{\le\kappa}(\Omega)} \phi_Q(x) a_{Q,s} \int_{B_{6l_Q}(q_Q)} (g(z_1) - \mathrm{Ext_s}(g)(y)) \tilde{\mu}_s(\mathrm{d}z_1) \\ &\quad= \sum_{Q, \widetilde{Q} \in \mathcal{W}_{\le\kappa}(\Omega)} \phi_Q(x) \phi_{\widetilde{Q}}(y) a_{Q,s} a_{\widetilde{Q},s} \int_{B_{6l_Q}(q_Q)} \int_{B_{6l_{\widetilde{Q}}}(q_{\widetilde{Q}})} (g(z_1) - g(z_2)) \tilde{\mu}_s(\mathrm{d}z_2) \tilde{\mu}_s(\mathrm{d}z_1). \end{aligned}$$

We apply Hölder's inequality to find

$$\begin{aligned} &|\mathrm{Ext_s}(g)(x) - \mathrm{Ext_s}(g)(y)| \\ &\le \sum_{Q, \widetilde{Q} \in \mathcal{W}_{\le\kappa}(\Omega)} \phi_Q(x) \phi_{\widetilde{Q}}(y) a_{Q,s} a_{\widetilde{Q},s} \big(\tilde{\mu}_s(B_{6l_Q}(q_Q)) \tilde{\mu}_s(B_{6l_{\widetilde{Q}}}(q_{\widetilde{Q}})) \big)^{1-1/p} \\ &\qquad\qquad \times \left(\int_{B_{6l_Q}(q_Q)} \int_{B_{6l_{\widetilde{Q}}}(q_{\widetilde{Q}})} |g(z_1) - g(z_2)|^p \tilde{\mu}_s(\mathrm{d}z_2) \tilde{\mu}_s(\mathrm{d}z_1) \right)^{1/p} \\ &= \sum_{Q, \widetilde{Q} \in \mathcal{W}_{\le\kappa}(\Omega)} \phi_Q(x) \phi_{\widetilde{Q}}(y) \left(a_{Q,s} a_{\widetilde{Q},s} \int_{B_{6l_Q}(q_Q)} \int_{B_{6l_{\widetilde{Q}}}(q_{\widetilde{Q}})} |g(z_1) - g(z_2)|^p \tilde{\mu}_s(\mathrm{d}z_2) \tilde{\mu}_s(\mathrm{d}z_1) \right)^{1/p}. \end{aligned}$$

Let $Q \in \mathcal{W}_{\le\kappa}$ be a cube such that $\phi_Q(x) \ne 0$. Then Q touches Q_1 by the definition of ϕ_Q. By (4-2), we find

$$|t - q_{Q_1}| \le |t - q_Q| + |q_Q - q_{Q_1}| \le 6 l_Q + (l_Q + l_{Q_1}) \le (6 \cdot 4 + 4 + 1) l_{Q_1} \le 30 l_{Q_1}$$

for any $t \in B_{6l_Q}(q_Q)$. Let $c_1 = c_1(d, \Omega, p) > 1$ be the constant from (4-10); then

$$a_{Q,s} \le c_1 l_Q^{s-d} \le c_1 4^{d-s} l_{Q_1}^{s-d} \le c_1^2 4^d a_{Q_1,s}.$$

By the finite overlap property, there exist at most $N-1$ cubes touching Q_1. The same holds for Q_1 replaced by Q_2. Therefore,

$$|\mathrm{Ext_s}(g)(x)-\mathrm{Ext_s}(g)(y)| \\ \le (N-1)^2(c_1^2 4^d)^{2/p}\Big(a_{Q_1,s}a_{Q_2,s}\int_{B_{30l_{Q_1}}(q_{Q_1})}\int_{B_{30l_{Q_2}}(q_{Q_2})}|g(z_1)-g(z_2)|^p\tilde{\mu}_s(\mathrm{d}z_2)\tilde{\mu}_s(\mathrm{d}z_1)\Big)^{1/p}.$$

(b) By (4-7), $\sum_{Q\in\mathcal{W}_{\le\kappa}}\phi_Q=1$ on $D_{\ge 2-\kappa}$, and thus $\sum_{Q\in\mathcal{W}_{\le\kappa}}\nabla\phi_Q=0$ on $D_{\ge 2-\kappa}$. We write

$$\nabla\,\mathrm{Ext_s}(g)(x)=\sum_{Q\in\mathcal{W}_{\le\kappa}(\Omega)}\nabla\phi_Q(x)\Big(a_{Q,s}\int_{B_{6l_Q}(q_Q)}g(z_1)\tilde{\mu}_s(\mathrm{d}z_1)-\mathrm{Ext_s}(g)(y)\Big) \\ =\sum_{Q,\widetilde{Q}\in\mathcal{W}_{\le\kappa}(\Omega)}\nabla\phi_Q(x)\phi_{\widetilde{Q}}(y)a_{Q,s}a_{\widetilde{Q},s}\int_{B_{6l_Q}(q_Q)}\int_{B_{6l_{\widetilde{Q}}}(q_{\widetilde{Q}})}(g(z_1)-g(z_2))\tilde{\mu}_s(\mathrm{d}z_2)\tilde{\mu}_s(\mathrm{d}z_1).$$

By definition of the partition of unity $\phi_{(\cdot)}$, see (4-4), there exists a positive constant $c_2=c_2(\mathcal{W}(\Omega),d,\psi)$ such that, for any $Q\in\mathcal{W}_{\le\kappa}$ touching Q_1 or Q_1 itself, we have

$$|\nabla\phi_Q(x)|\le c_2 s_Q^{-1}\le 4\sqrt{d}c_2 l_{Q_1}^{-1}.$$

We calculate using the same arguments as in the proof of (a) and get

$$|\nabla\,\mathrm{Ext_s}(g)(x)|\le l_{Q_1}^{-1}4\sqrt{d}c_2(N-1)^2(c_1^2 4^d)^{2/p}J_p(q_{Q_1},q_{Q_2}).$$

The proofs of (c) and (d) follow the same lines as the proofs of (a) and (b). Therefore, we omit them. □

Lemma 4.3 [Jonsson 1994, Lemma 2; Jonsson and Wallin 1984, Chapter V, Lemma 2]. *Let $s\in(0,1)$, $a>0$, $h:\Omega^c\to\mathbb{R}$. Set, for $x\in\Omega$,*

$$f(x)=\int_{B_{al_Q}(q_Q)}h(t)\tilde{\mu}_s(\mathrm{d}t)$$

if $x\in\mathring{Q}$ for $Q\in J_i$, $i\in\mathbb{N}$, and $f(x)=0$ otherwise. There exist constants $C=C(d,a)>0$ and $a_0=a_0(d,a)>0$ such that, for any $x_0\in\mathbb{R}^d$ and $0<r\le\infty$,

$$\int_{D_i\cap B_r(x_0)}f(x)\,\mathrm{d}x\le C2^{-id}\int_{\Omega^{\mathrm{ext}}_{\sqrt{d}a2^{-i}}\cap B_{r+a_0 2^{-i}}(x_0)}h(t)\tilde{\mu}_s(\mathrm{d}t). \tag{4-13}$$

Lemma 4.4. *Assume $a,b>0$ and $p^\star\ge 1$. There exists a constant $C=C(a,b,d,p_\star)>0$ such that, for $s\in(0,1)$ and $1\le p\le p^\star$,*

$$\sum_{j=0}^{\infty}2^{j(d+s(p-2))}\iint_{\substack{|x-y|\le a2^{-j}\\ d_x\le b2^{-j}}}|g(x)-g(y)|^p\tilde{\mu}_s(\mathrm{d}y)\tilde{\mu}_s(\mathrm{d}x) \\ \le\frac{C}{d+s(p-2)}\iint_{\substack{|x-y|\le a\\ d_x\le b}}\frac{|g(x)-g(y)|^p}{((|x-y|+d_x+d_y)\wedge 1)^{d+s(p-2)}}\mu_s(\mathrm{d}y)\mu_s(\mathrm{d}x).$$

It is only due to this lemma that the norm of the extension operator in Theorem 1.3 in the case $d=1$ depends on a lower bound of $(1-s)$.

Proof. The left-hand side of the inequality is equal to

$$\sum_{j=0}^{\infty}\sum_{k,n\geq j}2^{j(d+s(p-2))}\iint_{\substack{a2^{-n-1}\leq|x-y|\leq a2^{-n}\\ b2^{-k-1}\leq d_x\leq b2^{-k}}}|g(x)-g(y)|^p\tilde{\mu}_s(\mathrm{d}y)\tilde{\mu}_s(\mathrm{d}x)$$

$$=\bigg(\sum_{n\geq k\geq 0}\sum_{j=0}^{k}+\sum_{k>n\geq 0}\sum_{j=0}^{n}\bigg)2^{j(d+s(p-2))}\iint_{\substack{a2^{-n-1}\leq|x-y|\leq a2^{-n}\\ b2^{-k-1}\leq d_x\leq b2^{-k}}}|g(x)-g(y)|^p\tilde{\mu}_s(\mathrm{d}y)\tilde{\mu}_s(\mathrm{d}x)$$

$$\leq 2^{d+p+1}\sum_{n,k\geq 0}\frac{2^{(k\wedge n)(d+s(p-2))}}{d+s(p-2)}(1+a+b)^{2d+2s(p-1)}\iint_{\substack{a2^{-n-1}\leq|x-y|\leq a2^{-n}\\ b2^{-k-1}\leq d_x\leq b2^{-k}}}|g(x)-g(y)|^p\mu_s(\mathrm{d}y)\mu_s(\mathrm{d}x)$$

$$=:\text{(I)}.$$

For $x,y\in\Omega^c$ satisfying $|x-y|\leq a2^{-n}$ and $d_x\leq b2^{-k}$, we have $|x-y|+d_x+d_y\leq 2(a+b)2^{-(k\wedge n)}$. Therefore,

$$\text{(I)}\leq\frac{(2(a+b)+1)^{4d+4p+1}}{d+s(p-2)}\iint_{\substack{|x-y|\leq a\\ d_x\leq b}}\frac{|g(x)-g(y)|^p}{((|x-y|+d_x+d_y)\wedge 1)^{d+s(p-2)}}\mu_s(\mathrm{d}y)\mu_s(\mathrm{d}x). \qquad\square$$

Now we are in the position to prove the continuity of the L^p part. Recall the definition of the sets Ω_r^{ext} and Ω_{ext}^r in (2-1) for given $r>0$.

Proposition 4.5. *Let $s\in(0,1)$ and $1\leq p\leq p^\star<\infty$, and assume that Ω satisfies (4-10). Then, for every measurable $g:\Omega^c\to\mathbb{R}$,*

$$\|\mathrm{Ext}_s(g)\|_{L^p(\Omega)}\leq\frac{C}{s^{1/p}}\|g\|_{L^p(\Omega^{\text{ext}}_{3\,\text{inr}(\Omega)};\mu_s)}$$

with a constant $C=C(d,\Omega,p^\star)>0$ which is independent of s and p.

Proof. Firstly, $\int_\Omega\phi_Q(x)\,\mathrm{d}x\leq|Q^\star|\leq\left(1+\frac{1}{8}\right)^d s_Q^d$ for any $Q\in\mathcal{W}(\Omega)$. Let $c_1=c_1(d,\Omega)>1$ be the constant from (4-10). Recall the definitions of ρ and κ in (4-5). For any $Q\in\mathcal{W}_{\leq\kappa}$ and $z\in\Omega^c\cap B_{6l_Q}(q_Q)$, we know $d_z\leq 6l_Q=6\sqrt{d}s_Q\leq 6\sqrt{d}2^\kappa\leq 6\rho$. We use the finite overlap property of the Whitney decomposition to estimate

$$\|\mathrm{Ext}_s(g)\|^p_{L^p(\Omega)}\leq\left(1+\tfrac{1}{8}\right)^d N^p\sum_{Q\in\mathcal{W}_{\leq\kappa}(\Omega)}s_Q^d\bigg(\fint_{B_{6l_Q}(q_Q)}|g(z)|\tilde{\mu}_s(\mathrm{d}z)\bigg)^p$$

$$\leq 2^d c_1 N^p(6\rho+1)^{d+p-1}\sqrt{d}^{\,s-d}\sum_{Q\in\mathcal{W}_{\leq\kappa}(\Omega)}s_Q^s\int_{B_{6l_Q}(q_Q)}|g(z)|^p\mu_s(\mathrm{d}z)=:(\star).$$

Now, we consider $i\in\mathbb{N}$ and two cubes $Q_1,Q_2\in\mathcal{W}_{\leq\kappa}(\Omega)$ such that $s_{Q_1}=s_{Q_2}=2^{-i}$. If

$$|q_{Q_1}-q_{Q_2}|\geq 6(l_{Q_1}+l_{Q_2})=12\sqrt{d}2^{-i},$$

then $B_{6l_{Q_1}(q_{Q_1})} \cap B_{6l_{Q_2}(q_{Q_2})} = \varnothing$. The number of cubes $Q \in \mathcal{W}_{\le\kappa}$ with side length 2^{-i} that fit in the ball $B_{12\sqrt{d}2^{-i}}(q_{Q_1})$ is bounded from above by $\lceil 12\sqrt{d}2^{-i}/s_Q\rceil^d = \lceil 12\sqrt{d}\rceil^d$. Therefore,

$$\#\{Q \in \mathcal{W}_{\le\kappa} \mid s_Q = 2^{-i},\, B_{6l_{Q_1}(q_{Q_1})} \cap B_{6l_Q(q_Q)} \neq \varnothing\} \le \lceil 12\sqrt{d}\rceil^d. \tag{4-14}$$

We set $c_2 := 2^d c_1 N^p (6\rho+1)^{d+p-1}$. For any $z \in \Omega^c$ such that there exists a cube $Q \in \mathcal{W}_{\le\kappa}$ which satisfies $z \in B_{6l_Q}(q_Q)$, we have

$$d_z \le |z - q_Q| \le 6l_Q \le 6\rho.$$

We change the order of summation and use the above consideration to estimate

$$\begin{aligned}(\star) &\le c_2 \sum_{i=0}^{\infty} \sum_{Q\in\mathcal{W}_{\le\kappa}(\Omega)\cap J_i} 2^{-is} \int_{B_{6\cdot 2^{-i}}(q_Q)} |g(z)|^p \mu_s(\mathrm{d}z) \\ &\le c_2 \lceil 12\sqrt{d}\rceil^d \frac{1}{1-2^{-s}} \int_{\Omega^{\mathrm{ext}}_{6\rho}} |g(z)|^p \mu_s(\mathrm{d}z) \le c_2 \lceil 12\sqrt{d}\rceil^d \frac{2}{s} \|g\|^p_{L^p(\Omega^{\mathrm{ext}}_{3\,\mathrm{inr}(\Omega)};\mu_s)}.\end{aligned}$$

Thus, the proposition is proven with the constant

$$C := 2^{d+1} c_1 N^{p^\star} (6\rho+1)^{d+p^\star-1} \lceil 12\sqrt{d}\rceil^d. \qquad \square$$

Proposition 4.6. *Let $s \in (0, 1)$ and $1 \le p \le p^\star < \infty$. We assume that Ω is bounded and satisfies (4-10). Then, for every $g \in \mathcal{T}^{s,p}(\Omega^c)$,*

$$[\mathrm{Ext_s}(g)]_{V^{s,p}(\Omega\,|\,\mathbb{R}^d)} \le \frac{C}{(d+s(p-2))^{1/p} s^{2/p}} \|g\|_{\mathcal{T}^{s,p}(\Omega^c)}$$

with a constant $C = C(d, \Omega, p^\star) > 0$ which is independent of s and p.

Proof. We set $c_1 := (\sqrt{d}2^{\kappa+1}) \wedge \frac{1}{2^4} < 2\rho \wedge \frac{1}{8}$ and $j_0 := -\kappa - 6$, where κ and ρ are as in (4-5) and split the integration domain of the seminorm into

$$\begin{aligned}[\mathrm{Ext_s}(g)]^p_{V^{s,p}(\Omega\,|\,\mathbb{R}^d)} &= (1-s) \int_\Omega \int_{\{|h|\ge c_1\}} \frac{|\mathrm{Ext_s}(g)(x) - \mathrm{Ext_s}(g)(x+h)|^p}{|h|^{d+sp}} \,\mathrm{d}h\,\mathrm{d}x \\ &\quad + \left(\sum_{j=j_0}^{\infty} (1-s) \int_{D_j} \int_{\{|h|<c_1 2^{-j}\}} \frac{|\mathrm{Ext_s}(g)(x) - \mathrm{Ext_s}(g)(x+h)|^p}{|h|^{d+sp}} \,\mathrm{d}h\,\mathrm{d}x \right) \\ &\quad + \left(\sum_{j=j_0}^{\infty} (1-s) \int_{D_j} \int_{\{c_1 2^{-j}\le|h|<c_1\}} \frac{|\mathrm{Ext_s}(g)(x) - \mathrm{Ext_s}(g)(x+h)|^p}{|h|^{d+sp}} \,\mathrm{d}h\,\mathrm{d}x \right) \\ &\quad + \left(\sum_{j<j_0} (1-s) \int_{D_j} \int_{\{|h|<c_1\}} \frac{|\mathrm{Ext_s}(g)(x) - \mathrm{Ext_s}(g)(x+h)|^p}{|h|^{d+sp}} \,\mathrm{d}h\,\mathrm{d}x \right) \\ &=: \mathrm{(I)} + \mathrm{(II)} + \mathrm{(III)} + \mathrm{(IV)}.\end{aligned}$$

Recall that D_j is the collection of Whitney cubes with side length 2^{-j}; see (4-3). We handle these four terms separately.

Estimate of (IV): For any $x \in D_j$, $j < j_0$ and $|h| < c_1$, the distance of x as well as $x+h$ to the boundary $\partial\Omega$ is bigger or equal to

$$\begin{aligned} d_x - |h| &\geq \operatorname{dist}(D_j, \partial\Omega) - c_1 \\ &\geq \sqrt{d}2^{-j-2} - c_1 \\ &\geq \sqrt{d}(2^{\kappa+4} - 2^{\kappa+1}) \\ &> 6\rho. \end{aligned}$$

Therefore,

$$\mathrm{Ext_s}(g)(x) = 0 = \mathrm{Ext_s}(g)(x+h) \quad \text{and} \quad (\mathrm{IV}) = 0.$$

Estimate of (I): Using the triangle inequality and the definition of the extension $\mathrm{Ext_s}(g)$, we estimate (I) as follows:

$$\begin{aligned} (\mathrm{I}) &\leq 2^p(1-s)\bigg(\int_\Omega \int_{|h|\geq c_1} \frac{|\mathrm{Ext_s}(g)(x)|^p}{|h|^{d+sp}}\,\mathrm{d}h\,\mathrm{d}x + \int_\Omega \int_{|h|\geq c_1} \frac{|\mathrm{Ext_s}(g)(x+h)|^p}{|h|^{d+sp}}\,\mathrm{d}h\,\mathrm{d}x\bigg) \\ &\leq 2\frac{2^p(1-s)}{sp} c_1^{-sp}\omega_{d-1}\|\mathrm{Ext_s}(g)\|^p_{L^p(\Omega)} + 2^p 2^{-d-s(p-1)}|\Omega| \int_{\Omega^c} |g(y)|^p \mu_s(\mathrm{d}y). \end{aligned}$$

By Proposition 4.5 and the previous inequality, we find a constant $c_2 = c_2(d, \Omega, p^\star) > 0$ such that

$$(\mathrm{I}) \leq \frac{1-s}{s^2} \frac{2^{p+1}\omega_{d-1}c_2^p}{pc_1^p} \|g\|^p_{L^p(\Omega^c;\mu_s)} + 2^p|\Omega| \|g\|^p_{L^p(\Omega^c;\mu_s)}.$$

This is the desired estimate for (I).

Estimate of (II): For the moment we fix $j \in \mathbb{N}$, $j \geq j_0$, $Q_j \in J_j$, $x \in Q_j$ and $|h| < c_1 2^{-j} \leq \frac{1}{2^4} s_{Q_j}$. Under these assumptions, $x + h \in Q_j^\star$, where $Q_j^\star$ is the cube with the same center as Q_j but side length $\big(1+\frac{1}{8}\big)s_{Q_j}$. Thus, for any $t \in [0, 1]$, the vector $x + th$ is either in Q_j or in a neighboring cube, say Q, touching Q_j. By (4-2),

$$\tfrac{1}{4} l_{Q_j} \leq l_Q \leq 4 l_{Q_j} \quad \text{and} \quad |q_{Q_j} - q_Q| \leq \operatorname{diam}(Q_j) + \operatorname{diam}(Q) \leq (1+4) l_{Q_j}.$$

Further, for $z \in B_{30 l_Q}(q_Q)$, we find

$$|z - q_{Q_j}| \leq (30 \cdot 4 + 5) l_{Q_j}.$$

Set $c_3 := 30 \cdot 4 + 5$, and let $c_4 > 0$ be the constant from Lemma 4.2 and $c_5 > 0$ be the constant from (4-10). We want to apply Lemma 4.2(b) and (d) to estimate (II). We set $j_1 := j_0 + 8 = -\kappa + 2$ and write

$$\begin{aligned} (\mathrm{II}) = \sum_{j=j_1}^{\infty} (1-s) \int_{D_j} \int_{\{|h|<c_1 2^{-j}\}} \frac{|\mathrm{Ext_s}(g)(x) - \mathrm{Ext_s}(g)(x+h)|^p}{|h|^{d+sp}}\,\mathrm{d}h\,\mathrm{d}x \\ + \sum_{j=j_0}^{j_1-1} (1-s) \int_{D_j} \int_{\{|h|<c_1 2^{-j}\}} \frac{|\mathrm{Ext_s}(g)(x) - \mathrm{Ext_s}(g)(x+h)|^p}{|h|^{d+sp}}\,\mathrm{d}h\,\mathrm{d}x \\ =: (\mathrm{II}_1) + (\mathrm{II}_2). \end{aligned}$$

For all $j \geq j_1$, all neighboring cubes of Q_j have side lengths at most 2^κ, and thus $B_{c_1 2^{-j}}(x) \subset D_{\geq -\kappa}$. Since $\mathrm{Ext_s}(g)$ is smooth in Ω, the fundamental theorem of calculus and Lemma 4.2(b) yield

$$\begin{aligned}
&|\mathrm{Ext_s}(g)(x) - \mathrm{Ext_s}(g)(x+h)| \\
&\quad = \left| \int_0^1 \nabla \mathrm{Ext_s}(g)(x+th) \cdot h \,\mathrm{d}t \right| \\
&\quad \leq |h| \sup_{B_{c_1 2^{-j}}(x)} |\nabla \mathrm{Ext_s}(g)| \\
&\quad \leq c_4 |h| \frac{2^{j+2}}{\sqrt{d}} \left(c_5^2 \sqrt{d}^{2(s-d)} 2^{2(j+2)(d-s)} \int_{(B_{c_3 l_{Q_j}}(q_{Q_j}))^2} |g(z_1) - g(z_2)|^p (\tilde{\mu}_s \otimes \tilde{\mu}_s)(\mathrm{d}(z_1, z_2)) \right)^{1/p}.
\end{aligned}$$

Using this, we estimate

$$\begin{aligned}
(\mathrm{II}_1) &\leq (1-s) c_4^p 4^{2(d-s)+p} c_5^2 \sum_{j=j_0}^\infty 2^{2j(d-s)+jp} \sum_{Q \in J_j} \int_{|h|<c_1 2^{-j}} |h|^{-d+p(1-s)} \,\mathrm{d}h \\
&\qquad \times \int_Q \int_{(B_{c_3 l_Q}(q_Q))^2} |g(z_1) - g(z_2)|^p (\tilde{\mu}_s \otimes \tilde{\mu}_s)(\mathrm{d}(z_1, z_2)) \,\mathrm{d}x \\
&\leq \frac{\omega_{d-1} c_1^{p(1-s)} c_4^p 4^{2(d-s)+p} c_5^2}{p} \sum_{j=j_0}^\infty 2^{2j(d-s)+jp-jp(1-s)} \sum_{Q \in J_j} \\
&\qquad \times \int_Q \int_{\substack{(B_{c_3 l_Q}(q_Q))^2 \\ |z_1 - z_2| \leq 2\sqrt{d} c_3 2^{-j}}} |g(z_1) - g(z_2)|^p (\tilde{\mu}_s \otimes \tilde{\mu}_s)(\mathrm{d}(z_1, z_2)) \,\mathrm{d}x.
\end{aligned}$$

We define the functions

$$f_j : \Omega \to \mathbb{R} \quad \text{and} \quad h_j : \Omega^c \to \mathbb{R}$$

via

$$\begin{aligned}
h_j(z_1) &:= \int_{|z_1 - z_2| \leq 2\sqrt{d} c_3 2^{-j}} |g(z_1) - g(z_2)|^p \tilde{\mu}_s(\mathrm{d}z_2), \qquad z_1 \in \Omega^c, \\
f_j(x) &:= \int_{B_{c_3 l_Q}(q_Q)} h_j(z_1) \tilde{\mu}_s(\mathrm{d}z_1), \qquad x \in \Omega,
\end{aligned}$$

whenever there exists $Q \in J_j$ such that $x \in \mathring{Q}$, otherwise we set $f_j = 0$. With this notation we estimate (II_1) using Lemma 4.3 and $d_z \leq c_3 \sqrt{d} 2^\kappa$ for $z \in B_{c_3 l_Q}(q_Q) \cap \Omega^c$ and $Q \in \mathcal{W}_{\leq \kappa}$:

$$\begin{aligned}
(\mathrm{II}_1) &\leq \frac{\omega_{d-1} c_1^{p(1-s)} c_4^p 4^{2(d-s)+p} c_5^2}{p} \sum_{j=j_0}^\infty 2^{2j(d-s)+jps} \int_{D_j} f_j(x) \,\mathrm{d}x \\
&\leq \frac{\omega_{d-1} (c_1 \vee 1)^p c_4^p 4^{2d+p} c_5^2}{p} c_6 \sum_{j=j_0}^\infty 2^{j(d+s(p-2))} \int_{\Omega^{\mathrm{ext}}_{\sqrt{d} c_3 2^{-j}}} h_j(z_1) \tilde{\mu}_s(\mathrm{d}z_1) \\
&\leq \frac{c_8}{d+s(p-2)} \int_{\substack{\Omega^{\mathrm{ext}}_{\sqrt{d} c_3} \times \Omega^c \\ |z_1 - z_2| \leq 2\sqrt{d} c_3}} \frac{|g(z_1) - g(z_2)|^p}{((|z_1 - z_2| + d_{z_1} + d_{z_2}) \wedge 1)^{d+s(p-2)}} (\tilde{\mu}_s \otimes \tilde{\mu}_s)(\mathrm{d}(z_1, z_2)).
\end{aligned}$$

In the last inequality we used Lemma 4.4. Here $c_6 = c_6(d, c_3) > 0$ is the constant from Lemma 4.3, $c_7 = c_7(d, p, c_3) > 0$ is the constant from Lemma 4.4 and

$$c_8 = c_8(d, p, c_1, c_3, c_5, c_6, c_7) := \frac{\omega_{d-1}(c_1 \vee 1)^p c_4^p 4^{2d+p} c_5^2}{p} (c_3\sqrt{d}2^{\kappa} + 1)^{2d+2(p-1)} c_6 c_7.$$

Just as in the proof of the estimate of (II_1), we apply the fundamental theorem of calculus and Lemma 4.2(d), (4-10), and Lemma 4.3 to estimate (II_2):

$$\begin{aligned}
(\mathrm{II}_2) &\le \frac{\omega_{d-1}(c_1 2^{-j_0})^{p(1-s)}}{p} \sum_{j=j_0}^{j_1-1} \sum_{Q\in J_j} \int_Q \max_{y\in Q^\star} |\nabla \operatorname{Ext_s}(g)(y)|^p \, \mathrm{d}x \\
&\le \frac{\omega_{d-1}(c_1 2^{-j_0})^{p(1-s)}}{p} \left(c_4 \frac{2^{(j_1+1)}}{\sqrt{d}}\right)^p c_5 (\sqrt{d}2^{-j_1+1})^{s-d} \sum_{j=j_0}^{j_1-1} \sum_{Q\in J_j} \int_Q \int_{B_{c_3 l_Q}(q_Q)} |g(z)|^p \tilde{\mu}_s(\mathrm{d}z)\, \mathrm{d}x \\
&\le \frac{\omega_{d-1}(c_1 2^{-j_0})^{p(1-s)}}{p} \left(c_4 \frac{2^{(j_1+1)}}{\sqrt{d}}\right)^p c_5 (\sqrt{d}2^{-j_1+1})^{s-d} \sum_{j=j_0}^{j_1-1} c_6 \int_{\Omega^{\mathrm{ext}}_{\sqrt{d}c_3 2^{-j}}} |g(z)|^p \tilde{\mu}_s(\mathrm{d}z) \\
&\le \frac{\omega_{d-1}(c_1 2^{-j_0} \vee 1)^p}{p} \left(c_4 \frac{2^{(j_1+1)}}{\sqrt{d}}\right)^p c_5 \frac{(1+\sqrt{d}c_3 2^{-j_0})^{d+p-1}}{(\sqrt{d}2^{-j_1+1} \wedge 1)^d} c_6 (j_1 - j_0) \|g\|^p_{L^p(\Omega^{\mathrm{ext}}_{\sqrt{d}c_3 2^{-j_0}};\mu_s)}.
\end{aligned}$$

Estimate of (III): We put $h_m := c_1 2^{-m}$ and write

$$\begin{aligned}
(\mathrm{III}) &= (1-s) \sum_{j=j_0}^{\infty} \sum_{m=0}^{j-1} \int_{D_j} \int_{\{h_{m+1}\le |h| < h_m\}} \frac{|\operatorname{Ext_s}(g)(x) - \operatorname{Ext_s}(g)(x+h)|^p}{|h|^{d+sp}} \, \mathrm{d}h\, \mathrm{d}x \\
&\le (1-s) \sum_{m=0}^{\infty} \sum_{j=m+j_0}^{\infty} \int_{D_j} \int_{\{h_{m+1}\le |h| < h_m\}} \frac{|\operatorname{Ext_s}(g)(x) - \operatorname{Ext_s}(g)(x+h)|^p}{|h|^{d+sp}} \, \mathrm{d}h\, \mathrm{d}x \\
&= (1-s) \sum_{m=0}^{\infty} \int_{D_{\ge m+j_0}} \int_{\{h_{m+1}\le |h| < h_m\}} \frac{|\operatorname{Ext_s}(g)(x) - \operatorname{Ext_s}(g)(x+h)|^p}{|h|^{d+sp}} \, \mathrm{d}h\, \mathrm{d}x \\
&\le (1-s) \sum_{m=0}^{\infty} h_{m+1}^{-d-sp} \bigg(\int_{D_{\ge m+j_0}} \int_{\Omega} |\operatorname{Ext_s}(g)(x) - \operatorname{Ext_s}(g)(y)|^p \mathbb{1}_{|x-y|\le h_m} \, \mathrm{d}y\, \mathrm{d}x \\
&\qquad\qquad + \int_{D_{\ge m+j_0}} \int_{\Omega^c} |\operatorname{Ext_s}(g)(x) - g(y)|^p \mathbb{1}_{|x-y|\le h_m} \, \mathrm{d}y\, \mathrm{d}x \bigg) \\
&=: (1-s)((\mathrm{III}_1) + (\mathrm{III}_2)).
\end{aligned}$$

Estimate of (III_1): For $x \in Q \subset D_{\ge m+j_0}$ and $y \in \Omega$ such that $|x-y| \le h_m$, we have

$$\begin{aligned}
d_y \le d_x + |x-y| &\le \operatorname{dist}(Q, \partial\Omega) + \operatorname{diam}(Q) + h_m \\
&\le (5\sqrt{d} + c_1) 2^{-m-j_0} \\
&\le \sqrt{d} 2^{3-j_0-m}
\end{aligned}$$

and, therefore, $y \in D_{\geq m+j_0-3}$ by (4-1). We calculate

$$\begin{aligned}(\mathrm{III}_1) &\leq \sum_{m=0}^{\infty}\sum_{k=m+j_0}^{\infty}\sum_{n=m+j_0-3}^{\infty} h_{m+1}^{-d-sp}\int_{D_n}\int_{D_k\cap B_{h_m}(y)} |\mathrm{Ext_s}(g)(x)-\mathrm{Ext_s}(g)(y)|^p\,\mathrm{d}x\,\mathrm{d}y\\ &= \sum_{m=j_1-j_0+3}^{\infty}\sum_{k=m+j_0}^{\infty}\sum_{n=m+j_0-3}^{\infty} h_{m+1}^{-d-sp}\int_{D_n}\int_{D_k\cap B_{h_m}(y)} |\mathrm{Ext_s}(g)(x)-\mathrm{Ext_s}(g)(y)|^p\,\mathrm{d}x\,\mathrm{d}y\\ &\quad+\sum_{m=0}^{j_1-j_0+2}\sum_{k=m+j_0}^{\infty}\sum_{n=m+j_0-3}^{\infty} h_{m+1}^{-d-sp}\int_{D_n}\int_{D_k\cap B_{h_m}(y)} |\mathrm{Ext_s}(g)(x)-\mathrm{Ext_s}(g)(y)|^p\,\mathrm{d}x\,\mathrm{d}y\\ &=: (\mathrm{III}_{1,1})+(\mathrm{III}_{1,2}).\end{aligned}\tag{4-15}$$

We estimate $(\mathrm{III}_{1,1})$ first. For

$$m \geq j_1-j_0+3=11,\quad x\in Q_k\in J_k \text{ and } y\in Q_n\in J_n,\quad n\geq m+j_0-3,\quad k\geq m+j_0$$

such that $|x-y|\leq h_m$ and $z_1\in B_{30l_{Q_k}}(q_{Q_k})$, $z_2\in B_{30l_{Q_n}}(q_{Q_n})$, we have $n,k\geq j_1$ and

$$\begin{aligned}|z_1-z_2| &\leq |z_1-q_{Q_k}|+|q_{Q_k}-x|+|x-y|+|y-q_{Q_n}|+|q_{Q_n}-z_2|\\ &\leq 31\sqrt{d}2^{-k}+c_1 2^{-m}+31\sqrt{d}2^{-n}\leq c_9 2^{-m},\end{aligned}$$

where $c_9 := 31\sqrt{d}2^{-j_0}+c_1+31\sqrt{d}2^{3-j_0}$. Note that $s_{Q_k}, s_{Q_n}\leq 2^{\kappa-2}$. Lemma 4.2(a) yields

$$\begin{aligned}&|\mathrm{Ext_s}(g)(x)-\mathrm{Ext_s}(g)(y)|^p\\ &\qquad\leq c_4^p a_{Q_k,s}a_{Q_n,s}\int_{B_{30l_{Q_k}}(q_{Q_k})}\int_{B_{30l_{Q_n}}(q_{Q_n})}|g(z_1)-g(z_2)|^p\mathbb{1}_{|z_1-z_2|\leq c_9 2^{-m}}\tilde{\mu}_s(\mathrm{d}z_2)\tilde{\mu}_s(\mathrm{d}z_1)\\ &\qquad= c_4^p a_{Q_k,s}a_{Q_n,s}\tilde{f}_m(x),\end{aligned}\tag{4-16}$$

where $\tilde{f}_m:\mathbb{R}^d\to\mathbb{R}$ is defined by

$$\tilde{f}_m(x) := \int_{B_{30l_Q}(q_Q)}\tilde{h}_m(z_1)\tilde{\mu}_s(\mathrm{d}z_1)$$

for $x\in\mathbb{R}^d$ whenever there exists $Q\in J_k$ such that $x\in\mathring{Q}$, otherwise we set $\tilde{f}_m=0$. Here $\tilde{h}_m:\Omega^c\to\mathbb{R}$ is defined by

$$\tilde{h}_m(z_1) := \int_{B_{30l_{Q_n}}(q_{Q_n})}|g(z_1)-g(z_2)|^p\mathbb{1}_{|z_1-z_2|\leq c_9 2^{-m}}\tilde{\mu}_s(\mathrm{d}z_2),\qquad z_1\in\Omega^c.$$

Thus, Lemma 4.3 together with (4-16) yields

$$\begin{aligned}\int_{D_n}\int_{D_k\cap B_{h_m}(y)}|\mathrm{Ext_s}(g)(x)-\mathrm{Ext_s}(g)(y)|^p\,\mathrm{d}y\,\mathrm{d}x &\leq c_4^p a_{Q_k,s}a_{Q_n,s}\int_{D_n}\int_{D_k\cap B_{h_m}(y)}\tilde{f}_m(x)\,\mathrm{d}x\,\mathrm{d}y\\ &\leq c_4^p c_{10}2^{-kd}a_{Q_k,s}a_{Q_n,s}\int_{D_n}\int_{B_{h_m+a_1 2^{-k}}(y)}\tilde{h}_m(z_1)\tilde{\mu}_s(\mathrm{d}z_1)\,\mathrm{d}y\\ &=: (\widetilde{\mathrm{III}}_{1,1}).\end{aligned}\tag{4-17}$$

Here $c_{10} = c_{10}(d) > 0$ and $a_1 = a_1(d) > 0$ are the constants from Lemma 4.3. For $y \in D_n$ and $z_1 \in B_{h_m + a_1 2^{-k}}(y) \cap \Omega^c$, we find

$$d_{z_1} \le |y - z_1| \le (c_1 + a_1 2^{-j_0}) 2^{-m}$$

and set $c_{11} := (c_1 + a_1 2^{-j_0})$. Then, $(\widetilde{\mathrm{III}}_{1,1})$ becomes, after applying Lemma 4.3 again,

$$\begin{aligned}(\widetilde{\mathrm{III}}_{1,1}) &\le \frac{c_4^p c_{10}}{2^{kd}} a_{Q_k,s} a_{Q_n,s} \int_{D_n} \int_{B_{30 l_{Q_n}}(q_{Q_n})} \int_{\Omega^{\mathrm{ext}}_{c_{11} 2^{-m}}} |g(z_1) - g(z_2)|^p \mathbb{1}_{|z_1 - z_2| \le c_9 2^{-m}} \tilde{\mu}_s(\mathrm{d}z_1) \tilde{\mu}_s(\mathrm{d}z_2) \, \mathrm{d}y \\ &\le \frac{c_4^p c_{12} c_{10}}{2^{(k+n)d}} a_{Q_k,s} a_{Q_n,s} \int_{\Omega^c} \int_{\Omega^{\mathrm{ext}}_{c_{11} 2^{-m}}} |g(z_1) - g(z_2)|^p \mathbb{1}_{|z_1 - z_2| \le c_9 2^{-m}} \tilde{\mu}_s(\mathrm{d}z_1) \tilde{\mu}_s(\mathrm{d}z_2) \\ &\le \frac{c_4^p c_5 c_{12} c_{10}}{2^{s(k+n)}} \int_{\Omega^c} \int_{\Omega^{\mathrm{ext}}_{c_{11} 2^{-m}}} |g(z_1) - g(z_2)|^p \mathbb{1}_{|z_1 - z_2| \le c_9 2^{-m}} \tilde{\mu}_s(\mathrm{d}z_1) \tilde{\mu}_s(\mathrm{d}z_2). \end{aligned} \tag{4-18}$$

In the last inequality, we used (4-10) and $c_{12} = c_{12}(d) > 0$ is the constant from Lemma 4.3. We set $c_{13} := c_4^p c_5 c_{10} c_{12} 2^{d+p} (c_0 \wedge 1)^{-d-p}$. Recall that $j_1 - j_0 = 8$. The estimates (4-17) and (4-18) yield

$$\begin{aligned}(\mathrm{III}_{1,1}) &\le c_{13} \sum_{m = j_1 - j_0 + 3}^{\infty} \sum_{k = m + j_0}^{\infty} \sum_{n = m + j_0 - 3}^{\infty} 2^{m(d+sp)} 2^{-s(k+n)} \\ &\qquad\qquad \times \int_{\Omega^c} \int_{\Omega^{\mathrm{ext}}_{c_{11} 2^{-m}}} |g(z_1) - g(z_2)|^p \mathbb{1}_{|z_1 - z_2| \le c_9 2^{-m}} \tilde{\mu}_s(\mathrm{d}z_1) \tilde{\mu}_s(\mathrm{d}z_2) \\ &= \frac{c_{13}}{2^{2sj_0 - 3s}} \left(\frac{2^s}{2^s - 1} \right)^2 \sum_{m=11}^{\infty} 2^{m(d + s(p-2))} \int_{\Omega^c} \int_{\Omega^{\mathrm{ext}}_{c_{11} 2^{-m}}} |g(z_1) - g(z_2)|^p \mathbb{1}_{|z_1 - z_2| \le c_9 2^{-m}} \tilde{\mu}_s(\mathrm{d}z_1) \tilde{\mu}_s(\mathrm{d}z_2) \\ &\le c_{14} \frac{(1 + c_9 + c_{11})^{2d + 2(p-1)}}{s^2 (d + s(p-2))} \int_{\Omega^{\mathrm{ext}}_{c_{11}}} \int_{\Omega^c \cap B_{c_9}(z_2)} \frac{|g(z_1) - g(z_2)|^p}{((|z_1 - z_2| + d_{z_1} + d_{z_2}) \wedge 1)^{d + s(p-2)}} \mu_s(\mathrm{d}z_1) \mu_s(\mathrm{d}z_2). \end{aligned}$$

In the last estimate we used Lemma 4.4. Here $c_{14} := 2^{5 - 2(j_0 \wedge 0)} c_{13} c_{15}$ and $c_{15} = c_{15}(d, p, c_9, c_{11}) > 0$ is the constant from Lemma 4.4. This is the desired estimate for $(\mathrm{III}_{1,1})$. To estimate $(\mathrm{III}_{1,2})$, we calculate

$$\begin{aligned}(\mathrm{III}_{1,2}) &\le \frac{j_1 - j_0 + 2}{(c_1 2^{-(j_1 - j_0 - 1)})^{d+sp}} \sum_{k=j_0}^{\infty} \sum_{n=j_0-3}^{\infty} \int_{D_k} \int_{D_n \cap B_{c_1}(y)} |\mathrm{Ext_s}(g)(x) - \mathrm{Ext_s}(g)(y)|^p \, \mathrm{d}x \, \mathrm{d}y \\ &\le 2^{p+1} \frac{10}{(c_1 2^{-7})^{d+sp}} \sum_{k,n=j_0-3}^{\infty} \int_{D_k} \int_{D_n \cap B_{c_1}(y)} |\mathrm{Ext_s}(g)(x)|^p \, \mathrm{d}x \, \mathrm{d}y \\ &\le \frac{2^{p+5}}{(c_1 2^{-7})^{d+sp}} \omega_{d-1} c_1^d \|\mathrm{Ext_s}(g)\|^p_{L^p(\Omega)} \\ &\le \frac{2^{p+5} \omega_{d-1} c_1^d}{(c_1 2^{-7} \wedge 1)^{d+p}} \frac{c_2^p}{s} \|g\|^p_{L^p(\Omega^{\mathrm{ext}}_{3\,\mathrm{inr}(\Omega)}; \mu_s)}. \end{aligned}$$

Here we used Proposition 4.5. Combining the estimates of $(\mathrm{III}_{1,1})$ and $(\mathrm{III}_{1,2})$ yields the desired estimate of (III_1).

Estimate of (III$_2$): In contrast to (III$_1$), we integrate over Ω^c where the extension is just g. For $x \in Q \in J_k$, $k \geq m + j_0$ and $y \in \Omega^c$ such that $|x-y| \leq h_m = c_1 2^{-m}$, the distance of y to the center of the cube q_Q is smaller than

$$d_y \leq |y - q_Q| \leq |y-x| + |x - q_Q| \leq c_1 2^{-m} + \sqrt{d} 2^{-k} \leq \sqrt{d}(c_1 + 2^{-j_0}) 2^{-m}.$$

Set $c_{16} := 2c_1^{-1}\sqrt{d}(c_1 + 2^{-j_0} + 1)$. Thus $(1-s)c_{16}^{-s} h_{m+1}^{-s} \leq (1-s) d_y^{-s}$. With this we split $(1-s)(\mathrm{III}_2)$ into

$$\begin{aligned}
&(1-s)(\mathrm{III}_2)\\
&\quad \leq \sum_{m=0}^{j_1-j_0-1} \frac{(1-s)}{h_{m+1}^{d+sp}} \sum_{k=m+j_0}^{\infty} \sum_{Q \in J_k} \int_Q \int_{B_{c_{16}2^{-m}}(q_Q)} |\mathrm{Ext_s}(g)(x) - g(y)|^p \mathbb{1}_{|x-y|\leq h_m}\, \mathrm{d}y\, \mathrm{d}x\\
&\qquad + c_{16}^s \sum_{m=j_1-j_0}^{\infty} h_{m+1}^{-d-s(p-1)} \sum_{k=m+j_0}^{\infty} \sum_{Q\in J_k} \int_Q \int_{B_{c_{16}2^{-m}}(q_Q)} |\mathrm{Ext_s}(g)(x) - g(y)|^p \mathbb{1}_{|x-y|\leq h_m} \tilde{\mu}_s(\mathrm{d}y)\, \mathrm{d}x\\
&\quad =: (\mathrm{III}_{2,1}) + (\mathrm{III}_{2,2}).
\end{aligned} \tag{4-19}$$

We estimate (III$_{2,1}$) by

$$\begin{aligned}
(\mathrm{III}_{2,1}) &\leq 2^p (j_1 - j_0)\Bigg((1-s) \sum_{k=j_0}^{\infty} \sum_{Q \in J_k} \int_Q \int_{B_{c_{16}}(q_Q)} |\mathrm{Ext_s}(g)(x)|^p \mathbb{1}_{|x-y|\leq c_1}\, \mathrm{d}y\, \mathrm{d}x\\
&\qquad\qquad + c_{16}^{d+sp} \sum_{k=j_0}^{\infty} \sum_{Q\in J_k} \int_Q \int_{B_{c_{16}}(q_Q)} |g(y)|^p \mathbb{1}_{|x-y|\leq c_1} \mu_s(\mathrm{d}y)\, \mathrm{d}x\Bigg)\\
&\leq 2^p 8\big((1-s)\omega_{d-1} c_1^d \|\mathrm{Ext_s}(g)\|^p_{L^p(\Omega)} + c_{16}^{d+sp} \omega_{d-1} c_1^d \|g\|^p_{L^p(\Omega^{\mathrm{ext}}_{c_{16}};\mu_s)}\big)\\
&\leq 2^p 8\bigg((1-s)\omega_{d-1} c_1^d \frac{c_2}{s} \|g\|^p_{L^p(\Omega^{\mathrm{ext}}_{3\,\mathrm{inr}(\Omega)};\mu_s)} + c_{16}^{d+sp} \omega_{d-1} c_1^d \|g\|^p_{L^p(\Omega^{\mathrm{ext}}_{c_{16}};\mu_s)}\bigg).
\end{aligned}$$

Here, we used Proposition 4.5. We apply Lemma 4.2(c) and (4-10) to estimate (III$_{2,2}$):

$$\begin{aligned}
&(\mathrm{III}_{2,2})\\
&\leq \sum_{m=j_1-j_0}^{\infty} \frac{c_4^p c_5 c_{16}^s}{h_{m+1}^{d+s(p-1)}} \sum_{k=m+j_0}^{\infty} 2^{k(d-s)} \sum_{Q\in J_k} \int_Q \int_{B_{30 l_Q}(q_Q)} \int_{B_{c_{16}/2^m}(q_Q)} |g(z) - g(y)|^p \tilde{\mu}_s(\mathrm{d}y) \tilde{\mu}_s(\mathrm{d}z)\, \mathrm{d}x.
\end{aligned} \tag{4-20}$$

For $z \in B_{30\sqrt{d}2^{-k}}(q_Q)$ and $y \in \Omega^c$ with $|q_Q - y| \leq c_{16} 2^{-m}$, we have

$$|y - z| \leq (c_{16} + \sqrt{d}30) 2^{-m}.$$

We write $c_{17} := c_{16} + \sqrt{d}30$. Now, we apply Lemma 4.3 with $r = +\infty$ and conclude, with a positive constant $c_{18} = c_{18}(d)$,

$$\begin{aligned}
&\sum_{Q\in J_k} \int_Q \int_{B_{30 l_Q}(q_Q)} \int_{B_{c_{16}2^{-m}}(q_Q)} |g(z) - g(y)|^p \tilde{\mu}_s(\mathrm{d}y) \tilde{\mu}_s(\mathrm{d}z)\, \mathrm{d}x\\
&\qquad\qquad \leq c_{18} 2^{-kd} \int_{\Omega^{\mathrm{ext}}_{30\sqrt{d}2^{-m}}} \int_{B_{c_{17}2^{-m}}(z)} |g(z) - g(y)|^p \tilde{\mu}_s(\mathrm{d}y) \tilde{\mu}_s(\mathrm{d}z).
\end{aligned} \tag{4-21}$$

By (4-20), (4-21) and Lemma 4.4,

$$(\mathrm{III}_{2,2}) \le \frac{c_4^p c_5 c_{16}^s}{c_1^{d+s(p-1)}} \frac{2^{-s(j_0\wedge 0)}}{1-2^{-s}} c_{18} \sum_{m=0}^{\infty} 2^{m(d+s(p-2))} \int_{\Omega^{\mathrm{ext}}_{30\sqrt{d}2^{-m}}} \int_{B_{c_{17}2^{-m}}(z)} |g(z)-g(y)|^p \tilde{\mu}_s(\mathrm{d}y)\tilde{\mu}_s(\mathrm{d}z)$$
$$\le \frac{c_{20}}{s(d+s(p-2))} \int_{\Omega^{\mathrm{ext}}_{30\sqrt{d}}} \int_{B_{c_{17}}(z)} \frac{|g(z)-g(y)|^p}{((|y-z|+d_z+d_y)\wedge 1)^{d+s(p-2)}} \mu_s(\mathrm{d}y)\mu_s(\mathrm{d}z).$$

Here the constant is

$$c_{20} := 2^{2-(j_0\wedge 0)}(c_1\wedge 1)^{-d-(p-1)} c_4^p c_5 c_{16} (60\sqrt{d}+2c_{17})^{d+|p-2|} c_{19},$$

where $c_{19} = c_{19}(d, p, c_{17}) > 0$ is the constant from Lemma 4.4. Combining $(\mathrm{III}_{2,1})$ and $(\mathrm{III}_{2,2})$ yields the desired estimate of (III_2). Further, the previous estimates of (III_1) and (III_2) finish the proof of the bound on (III). □

Proof of Theorems 1.2 and 1.3. The proof of Theorem 1.2 follows from Propositions 3.9 and 3.10, Lemma 4.1, and Propositions 4.5 and 4.6. The proof of Theorem 1.2 does not require Proposition 3.10. □

5. Nonlocal to local

In this section, we prove the convergence of the trace spaces $\mathcal{T}^{s,p}(\Omega^c) \to W^{1-1/p,p}(\partial\Omega)$ as $s \to 1^-$ as claimed in Theorem 1.4.

The following lemma is a minor extension of [Grube and Hensiek 2024, Lemma 4.1] to Lipschitz domains. Note that the term $(1-s)/d_x^s$ in the measure μ_s is responsible for the reduction of Ω^c to $\partial\Omega$. Recall the definition of the sets Ω_r^{ext} and Ω^r_{ext} in (2-1) for given $r > 0$.

Lemma 5.1. *Let $\Omega \subset \mathbb{R}^d$ be a Lipschitz domain, $d \ge 2$, and $0 < r \le \infty$. We define a family of measures $\bar{\mu}_s$ on $(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))$ via*

$$\bar{\mu}_s(x) := \frac{1-s}{d_x^s} \mathbb{1}_{\Omega_r^{\mathrm{ext}}}(x).$$

Let σ be the surface measure on the Lipschitz submanifold $\partial\Omega$, and set $\sigma(D) := \sigma(\partial\Omega \cap D)$ for all sets $D \in \mathcal{B}(\mathbb{R}^d)$. Then $\{\bar{\mu}_s\}_s$ converges weakly to σ as $s \to 1^-$, i.e., when integrated against test functions in $C_c(\mathbb{R}^d)$.

Remark. (1) In dimension $d = 1$, the previous convergence result reads

$$\bar{\mu}_s \to \sum_{x_0 \in \partial\Omega} \delta_{x_0} \text{ weakly,}$$

i.e., when tested against $C_c(\mathbb{R})$ functions. Here δ_{x_0} is the Dirac measure in the boundary point $x_0 \in \partial\Omega$.

(2) In [Grube and Hensiek 2024, Lemma 4.1], the first author and his coauthor proved Lemma 5.1 for bounded C^1-domains. Below, we adopt this proof and explain necessary differences to that result.

Proof. Let $f \in C_c(\mathbb{R}^d)$. We have shown in [Grube and Hensiek 2024, Lemma 4.1] without using that the boundary $\partial\Omega$ is C^1 that $\int_{\Omega^{\varepsilon}_{\text{ext}}} |f|\bar{\mu}_s \to 0$ as $s \to 1^-$ for any $\varepsilon > 0$. Thus, the problem localizes. Without loss of generality, we find a cube $Q = (-\rho, \rho)^d$ and a Lipschitz continuous map $\phi : \mathbb{R}^{d-1} \to \mathbb{R}$ such that $\Omega \cap Q = \{(x', x_d) \mid x_d < \phi(x')\} \cap Q$. Furthermore, on this cube we have

$$\int_{Q\cap\Omega^c} f \, \mathrm{d}\bar{\mu}_s = \int_{(-\rho,\rho)^{d-1}} \int_{\phi(x')}^{\rho} f(x', x_d) \frac{1-s}{d^s_{(x',x_d)}} \, \mathrm{d}x_d \, \mathrm{d}x'.$$

Since $\frac{1-s}{x_d - \phi(x')}$ is an approximate identity evaluating at $x_d = \phi(x')$ as $s \to 1^-$, it remains to show that

$$\frac{x_d - \phi(x')}{d_{(x',x_d)}} \to \sqrt{1 + |\nabla\phi(x')|^2} \quad \text{as } x_d \to \phi(x') \tag{5-1}$$

for almost every $x' \in (-\rho, \rho)^{d-1}$. Since ϕ is Lipschitz continuous, it is differentiable at almost every point $x' \in (-\rho, \rho)^{d-1}$ by Rademacher's theorem. In [Grube and Hensiek 2024, Lemma 4.1], we used continuous differentiability of ϕ to show (5-1). Here we only assume ϕ to be Lipschitz continuous. We fix $x' \in (-\rho, \rho)^{d-1}$ such that ϕ is differentiable at x'. For $x_d > \phi(x')$ such that $x_d < \rho$, we pick a minimizer of the distance of $x = (x', x_d)$ to the surface $\partial\Omega$ and call it $y = y(x_d) = (y', \phi(y'))$. Analogously to the estimate (3-3), we have

$$|x_d - \phi(x')| \le 2(1 + [\phi]_{C^{0,1}}) d_x. \tag{5-2}$$

Furthermore, we define the hyperplane

$$P := \{(z', \phi(x') + (z' - x') \cdot \nabla\phi(x')) \mid z' \in (-\rho, \rho)^{d-1}\}$$

which is tangential to the surface at x. Let $z = z(x_d) = (z', z_d)$ be the minimizer of $x = x(x', x_d)$ to the plane P. A small calculation yields

$$\begin{aligned}
z' &= x' + \frac{x_d - \phi(x')}{1 + |\phi(x')|^2} \nabla\phi(x'), \\
z_d &= \phi(x') + \frac{|\nabla\phi(x')|^2}{1 + |\nabla\phi(x')|^2}(x_d - \phi(x')), \\
\operatorname{dist}(x, P) = |x - z| &= \frac{x_d - \phi(x')}{\sqrt{1 + |\nabla\phi(x')|^2}}.
\end{aligned}$$

Since ϕ is differentiable, the error function $r : (-\rho, \rho)^{d-1} \to \mathbb{R}$ given by

$$r(w') := \phi(w') - \phi(x') - (w' - x') \cdot \nabla\phi(x')$$

satisfies

$$\frac{r(w')}{|w' - x'|} \to 0 \quad \text{as } w' \to x'.$$

Since $|y' - x'| \le d_x \le |x_d - \phi(x')|$, we know

$$\frac{r(y')}{|x_d - \phi(x')|} \to 0 \quad \text{as } x_d \to \phi(x'). \tag{5-3}$$

Now, we will estimate $\operatorname{dist}(x, P)$ by d_x and an error and vice versa. Let $y_d \in \mathbb{R}$ such that $(y', y_d) \in P$. By the triangle inequality, we have

$$|(y', y_d) - (x', x_d)| \le |(y', \phi(y')) - (x', x_d)| + |(y', y_d) - (y', \phi(y'))| = d_x + r(y').$$

Since $(y', y_d) \in P$ and z is the minimizer of the distance of x to P, we have $\operatorname{dist}(x, P) \le d_x + r(y')$. Again by the triangle inequality,

$$d_x \le |(z', \phi(z')) - (x', x_d)| \le |(z', z_d) - (x', x_d)| + |(z', \phi(z')) - (z', z_d)| = \operatorname{dist}(x, P) + r(z').$$

Therefore,

$$\begin{aligned}\left|1 - \frac{d_x}{\operatorname{dist}(x, P)}\right| &= \frac{|\operatorname{dist}(x, P) - d_x|}{\operatorname{dist}(x, P)} \le \frac{\max\{|r(y')|, |r(z')|\}}{\operatorname{dist}(x, P)} \\ &= \sqrt{1 + |\nabla\phi(x')|^2}\frac{\max\{|r(y')|, |r(z')|\}}{|x_d - \phi(x')|} \to 0\end{aligned}$$

as $x_d \to \phi(x')$ by (5-3), the choice of z' and the properties of the error function r. By the previous calculation of $\operatorname{dist}(x, P)$ and this convergence, (5-1) follows. Since $(1-s)/|x_d - \phi(x')|^s$ is an approximate identity, we have for any $x' \in (-\rho, \rho)^{d-1}$ such that ϕ is differentiable at x'

$$\int_{\phi(x')}^{\rho} \frac{1-s}{d^s_{(x', x_d)}} f(x', x_d)\,\mathrm{d}x_d \to f(x', 0)\sqrt{1 + |\nabla\phi(x')|^2} \quad \text{as } s \to 1^-.$$

Since f has compact support, there exists $R > 0$ such that $\operatorname{supp}(f) \subset B_R(0)$. By (5-2), we have

$$\begin{aligned}\left|\int_{\phi(x')}^{\rho} \frac{1-s}{d^s_{(x', x_d)}} f(x', x_d)\,\mathrm{d}x_d\right| &\le \|f\|_{L^\infty}\mathbb{1}_{B_R(0)}(x') \int_{\phi(x')}^{\rho} \frac{1-s}{d^s_{(x', x_d)}}\,\mathrm{d}x_d \\ &\le 2(1 + [\phi]_{C^{0,1}})\|f\|_{L^\infty}\mathbb{1}_{B_R(0)}(x') \int_{\phi(x')}^{\rho} \frac{1-s}{|x_d - \phi(x')|^s}\,\mathrm{d}x_d \\ &\le 2(1 + [\phi]_{C^{0,1}})(\rho + 1)\|f\|_{L^\infty}\mathbb{1}_{B_R(0)}(x').\end{aligned}$$

By dominated convergence,

$$\int_{Q \cap \Omega^c} f\,\mathrm{d}\bar{\mu}_s \to \int_{\partial\Omega \cap Q} f\,\mathrm{d}\sigma \quad \text{as } s \to 1^-.$$

Following the proof of [Grube and Hensiek 2024, Lemma 4.1], the result follows. □

We are now in the position to prove the convergence theorem. We use similar arguments as in the proof of [Grube and Hensiek 2024, Proposition 4.3, Theorem 1.4].

Proof of Theorem 1.4. First, we prove the convergence result for $u \in C_c^{0,1}(\mathbb{R}^d)$. Then we apply a density argument.

Step 1: Let us assume $u \in C_c^{0,1}(\mathbb{R}^d)$.

L^p part: We split Ω^c into the union $\Omega_1^{\text{ext}} \cup \Omega^1_{\text{ext}}$. On Ω^1_{ext} we apply the estimates from the proof of the trace continuity. By (3-14), there exists a constant $c_1 = c_1(d, \Omega, p) > 0$ such that

$$\lim_{s \to 1^-} \|\operatorname{Tr}_s u\|^p_{L^p(\Omega^1_{\text{ext}}; \mu_s)} \le c_1 \lim_{s \to 1^-} [u]^p_{V^{s,p}(\Omega \,|\, \Omega^1_{\text{ext}})}.$$

Notice that

$$[u]^p_{V^{s,p}(\Omega\,|\,\Omega^1_{\text{ext}})} \le (1-s)\int_{B_1(0)^c}\int_\Omega \frac{|u(y)-u(y+h)|^p}{|h|^{d+sp}}\,\mathrm{d}y\,\mathrm{d}h$$
$$\le 2^p(1-s)\|u\|^p_{L^p(\mathbb{R}^d)}\int_{B_1(0)^c}\frac{1}{|h|^{d+sp}}\,\mathrm{d}h = \frac{2^p\omega_{d-1}}{p}(1-s)\|u\|_{L^p(\mathbb{R}^d)} \tag{5-4}$$

and, thus, $\lim_{s\to 1^-}\|\mathrm{Tr_s}\,u\|^p_{L^p(\Omega^1_{\text{ext}};\mu_s)}=0$. On Ω_1^{ext} we consider the family of measures $\{\bar\mu_s\}$ from Lemma 5.1 with $r=1$. This family converges weakly to the surface measure on $\partial\Omega$, which we denote by σ. Thus, we conclude the claim via

$$\|\mathrm{Tr_s}\,u\|^p_{L^p(\Omega_1^{\text{ext}};\mu_s)} = \int_{\mathbb{R}^d}\frac{|u(x)|^p}{(1+d_x)^{d+s(p-1)}}\bar\mu_s(\mathrm{d}x) \le \int_{\mathbb{R}^d}|u(x)|^p\bar\mu_s(\mathrm{d}x) \to \int_{\partial\Omega}|u(x)|^p\,\mathrm{d}\sigma(x)$$

and similarly

$$\|\mathrm{Tr_s}\,u\|^p_{L^p(\Omega_1^{\text{ext}};\mu_s)} \ge \int_{\mathbb{R}^d}\frac{|u(x)|^p}{(1+d_x)^{d+p-1}}\bar\mu_s(\mathrm{d}x) \to \int_{\partial\Omega}|u(x)|^p\,\mathrm{d}\sigma(x) \quad \text{as } s\to 1^-$$

because $x\mapsto(1+d_x)^{-d-p+1}$ is continuous.

Seminorm part: The main task is to show

$$\iint_{\Omega^c\times\Omega^c}\frac{|u(x)-u(y)|^p(1+d_x)^{-d-s(p-1)}(1+d_y)^{-d-s(p-1)}}{((|x-y|+d_x+d_y)\wedge 1)^{d+s(p-2)}}\,\mathrm{d}(\bar\mu_s\otimes\bar\mu_s)((x,y))$$
$$\to \iint_{\partial\Omega\times\partial\Omega}\frac{|u(x)-u(y)|^p}{|x-y|^{d+p-2}}\,\mathrm{d}(\sigma\otimes\sigma)(x,y) \quad \text{as } s\to 1^-$$

for some $r>0$ in the definition of $\bar\mu_s$. The choice of r is arbitrary and will be made later. Let $\bar\mu_s$ be the measure from Lemma 5.1 to the parameter r. As in the proof of Proposition 3.10, we split $\Omega^c\times\Omega^c$ into the union $\Omega_r^{\text{ext}}\times\Omega_r^{\text{ext}}\cup\Omega^c\times\Omega^r_{\text{ext}}\cup\Omega^r_{\text{ext}}\times\Omega^c$. The proof in the case $\Omega^c\times\Omega^r_{\text{ext}}$ is the same as the one in the case $\Omega^r_{\text{ext}}\times\Omega^c$ and shows that $[\mathrm{Tr_s}\,u]^p_{\mathcal{T}^{s,p}(\Omega^r_{\text{ext}}\,|\,\Omega^c)}$ converges to zero. By (3-17), (5-4) and a calculation similar to (3-18), we find a constant $c_2=c_2(\Omega,p)>0$ such that

$$[\mathrm{Tr_s}\,u]^p_{\mathcal{T}^{s,p}(\Omega^r_{\text{ext}}\,|\,\Omega^c)} \le 2^p\biggl((1-s)\|\mathrm{Tr_s}\,u\|^p_{L^p(\Omega^c;\mu_s)}\int_{\Omega^r_{\text{ext}}} d_x^{-d-sp}\,\mathrm{d}x + \|\mathrm{Tr_s}\,u\|^p_{L^p(\Omega^r_{\text{ext}};\mu_s)}\mu_s(\Omega^c)\biggr)$$
$$\le 2^p\biggl(\frac{(1-s)c_2r^{-sp}}{s}\|\mathrm{Tr_s}\,u\|^p_{L^p(\Omega^c;\mu_s)} + \frac{c_2}{s}\|\mathrm{Tr_s}\,u\|^p_{L^p(\Omega^r_{\text{ext}};\mu_s)}\biggr) \to 0$$

as $s\to 1-$. Now, we consider the interesting part $\Omega_r^{\text{ext}}\times\Omega_r^{\text{ext}}$. We would like to apply Lemma 5.1 to the function

$$h_s(x,y) := \frac{|u(x)-u(y)|^p(1+d_x)^{-d-s(p-1)}(1+d_y)^{-d-s(p-1)}}{((|x-y|+d_x+d_y)\wedge 1)^{d+s(p-2)}}$$

since

$$[\mathrm{Tr_s}\,u]^p_{\mathcal{T}^{s,p}(\Omega_r^{\text{ext}}\,|\,\Omega_r^{\text{ext}})} = \iint_{\Omega_r^{\text{ext}}\times\Omega_r^{\text{ext}}} h_s(x,y)\,\mathrm{d}(\bar\mu_s\otimes\bar\mu_s)((x,y))$$

and

$$h_s(x,y) \to \frac{|u(w)-u(z)|^p}{|w-z|^{d+p-2}} \quad \text{for } x \to w \in \partial\Omega,\ y \to z \in \partial\Omega,\ s \to 1^- \quad \text{for } w \neq z.$$

Lemma 5.1 cannot be applied directly because h_s is neither continuous on $\Omega^c \times \Omega^c$ nor independent of s. We resolve this issue by arguments similar to the ones used in [Grube and Hensiek 2024, Proposition 4.3]. Let us fix a radial bump function $\varphi \in C_c^\infty(\overline{B_2(0)})$, $0 \le \varphi \le 1$, $\varphi = 1$ in B_1, and define $\varphi_\varepsilon(x,y) := \varphi(|x-y|/\varepsilon)$ for $\varepsilon \in (0,1)$. We set

$$a_s^\star := \begin{cases} 1, & p \ge 2, \\ \varepsilon^{-(1-s)(2-p)}, & 1 \le p < 2, \end{cases} \qquad a_{s,\star} := \begin{cases} \varepsilon^{(1-s)(p-2)}, & p \ge 2, \\ 1, & 1 \le p < 2, \end{cases}$$

$$h_\varepsilon^\star(x,y) := \frac{|u(x)-u(y)|^p}{(|x-y| \wedge 1)^{d+p-2}}(1-\varphi_\varepsilon(x,y)),$$

$$h_{\varepsilon,\star}(x,y) := \frac{|u(x)-u(y)|^p((1+d_x)(1+d_y))^{-d-p+1}}{((|x-y|+d_x+d_y) \wedge 1)^{d+p-2}}(1-\varphi_\varepsilon(x,y)),$$

$$g_{s,\varepsilon}(x,y) := h_s(x,y)\varphi_\varepsilon(x,y).$$

For every $s \in (0,1)$ and every $\varepsilon > 0$, we have $a_{s,\star}h_{\varepsilon,\star} + g_{s,\varepsilon} \le h_s \le a_s^\star h_\varepsilon^\star + g_{s,\varepsilon}$. We will now consider the limit $s \to 1^-$ and subsequently $\varepsilon \to 0^+$ of the upper and lower bound in this inequality integrated against $\bar\mu_s \otimes \bar\mu_s$. Both $h_{\varepsilon,\star}$ and $h_\varepsilon^\star$ are continuous and bounded on $\mathbb{R}^d \times \mathbb{R}^d$. By Lemma 5.1, $\{\mu_s\}_s$ converges weakly to σ, and thus the sequence of product measures $\{\mu_s \otimes \mu_s\}$ converges weakly to $\sigma \otimes \sigma$. Therefore,

$$\begin{aligned} &\iint h_{\varepsilon,\star}(x,y)\,\mathrm{d}(\bar\mu_s \otimes \bar\mu_s)((x,y)) \\ &\quad \to \iint_{\partial\Omega\times\partial\Omega} \frac{|u(x)-u(y)|^p}{|x-y|^{d+p-2}}(1-\varphi_\varepsilon(x,y))\,\mathrm{d}(\sigma\otimes\sigma)(x,y) \quad \text{as } s \to 1^-, \\ &\quad \to \iint_{\partial\Omega\times\partial\Omega} \frac{|u(x)-u(y)|^p}{|x-y|^{d+p-2}}\,\mathrm{d}(\sigma\otimes\sigma)(x,y) \quad \text{as } \varepsilon \to 0+. \end{aligned}$$

The same is true for $h_\varepsilon^\star$. Furthermore, $a_s^\star, a_{s,\star} \to 1^-$. Now, we show that

$$\lim_{s\to 1^-} \iint g_{s,\varepsilon}\,\mathrm{d}(\bar\mu_s \otimes \bar\mu_s)(x,y) \to 0 \quad \text{as } \varepsilon \to 0+.$$

Note

$$\iint g_{s,\varepsilon}(x,y)\,\mathrm{d}(\bar\mu_s \otimes \bar\mu_s)(x,y) \le [u]_{C^{0,1}} \iint_{\substack{\Omega_r^{\text{ext}}\times\Omega_r^{\text{ext}} \\ |x-y|<2\varepsilon}} |x-y|^{-d-s(p-2)+2}\,\mathrm{d}(\bar\mu_s \otimes \bar\mu_s)(x,y).$$

The problem localizes. Since Ω is a bounded Lipschitz domain, we find a uniform localization radius $r_0 > 0$. Let $\mathcal{Q}$ be a cover of $\partial\Omega$ with balls $B \in \mathcal{Q}$ with radius $r_0 > 0$ such that the union of these balls with half their radii still contains $\partial\Omega$. Now, we fix $r > 0$ from the beginning of the proof such that $\bigcup_{B\in\mathcal{Q}} \frac{1}{2}B \supset \Omega_r^{\text{ext}}$, where $\frac{1}{2}B$ is the ball with half the radius. We assume $\varepsilon < \frac{1}{4}r_0$ such that, for any $x, y \in \Omega_r^{\text{ext}}$ satisfying $x \in \frac{1}{2}B$, $B \in \mathcal{Q}$, $|x-y| < 2\varepsilon$, we have $y \in B$. Thus, we only need to consider one ball $B \in \mathcal{Q}$. After translation we assume, with loss of generality, $B = B_{r_0}(0)$. We flatten the boundary $\partial\Omega$ that lies in B with the function $\phi \in C^{0,1}(\mathbb{R}^{d-1})$ such that $\Omega \cap B = \{(x', x_d) \in B \mid x_d < \phi(x')\}$. Since

Ω has a uniform Lipschitz boundary, the Lipschitz constant of ϕ does not depend on the position of B. For any $x \in B \cap \Omega^c$, we have $d_x \geq (1 + \|\phi\|_{C^{0,1}})^{-1}|x_d - \phi(x')|$. Therefore,

$$\iint_{(\Omega^c \cap Q)\times(\Omega^c \cap Q)} \mathbb{1}_{B_{2\varepsilon}(x)}(y)|x-y|^{-d-s(p-2)+2p}\,\mathrm{d}(\bar{\mu}_s \otimes \bar{\mu}_s)(x,y)$$
$$\leq (1+\|\phi\|_{C^{0,1}})^2 \int_{B^{(d-1)}_{r_0}(0)} \int_{\phi(x')}^{2r_0} \int_{B^{(d-1)}_{2\varepsilon}(x')} \int_{\phi(y')}^{2r_0} \frac{(1-s)^2|x'-y'|^{-d-s(p-2)+2p}}{(x_d-\phi(x'))^s(y_d-\phi(y'))^s}\,\mathrm{d}y_d\,\mathrm{d}y'\,\mathrm{d}x_d\,\mathrm{d}x'$$
$$\leq r_0^{2-2s}(1+\|\phi\|_{C^{0,1}})^2 \frac{\omega_{d-2}^2}{d-1} r_0^{d-1} \int_0^{2\varepsilon} t^{(1-s)(p-2)}\,\mathrm{d}t$$
$$= r_0^{2-2s}(1+\|\phi\|_{C^{0,1}})^2 \frac{\omega_{d-2}^2}{d-1} r_0^{d-1} \frac{(2\varepsilon)^{(1-s)(p-2)+1}}{(1-s)(p-2)+1} \to (1+\|\phi\|_{C^{0,1}})^2 \frac{\omega_{d-2}^2}{d-1} r_0^{d-1} 2\varepsilon \to 0.$$

In the last line, we first consider the limit $s \to 1^-$ and then the limit $\varepsilon \to 0+$. Thus,

$$\iint g_{s,\varepsilon}(x,y)\,\mathrm{d}(\bar{\mu}_s \otimes \bar{\mu}_s)(x,y) \leq \sum_{B \in \mathcal{Q}} \iint_{Q\times Q} \mathbb{1}_{B_{2\varepsilon}(x)}(y)|x-y|^{-d-s(p-2)+2p}\,\mathrm{d}(\bar{\mu}_s \otimes \bar{\mu}_s)(x,y) \to 0.$$

The result for $C_c^{0,1}(\mathbb{R}^d)$ functions follows from

$$a_{s,\star} h_{\varepsilon,\star} + g_{s,\varepsilon} \leq h_s \leq a_s^{\star} h_\varepsilon^{\star} + g_{s,\varepsilon}.$$

The proof of $[u]_{\mathcal{T}^{s,1}(\Omega^c)} \to [u]_{B^0_{1,1}(\partial\Omega)}$ follows analogously.

Step 2: Let $1 < p < \infty$. We generalize the result from Step 1 to all functions in $W^{1,p}(\mathbb{R}^d)$ via a density argument. Firstly, there exists a constant $c_3 = c_3(d,p) > 0$ such that $\|u\|_{V^{s,p}(\Omega\,|\,\mathbb{R}^d)} \leq c_3\|u\|_{W^{s,p}(\mathbb{R}^d)}$. Combining this estimate with (3-13) and (3-16) yields

$$\|\mathrm{Tr_s}\, u\|_{\mathcal{T}^{s,p}(\Omega^c)} \leq c_4\|u\|_{V^{s,p}(\Omega\,|\,\mathbb{R}^d)} \leq c_3 c_4 \|u\|_{W^{1,p}(\mathbb{R}^d)}.$$

Here $c_4 > 0$ is the sum of the constants from Propositions 3.9 and 3.10. Now take any $u \in W^{1,p}(\mathbb{R}^d)$. Since $C_c^{0,1}(\mathbb{R}^d)$ is dense in $W^{1,p}(\mathbb{R}^d)$, we find a sequence $u_n \in C_c^{0,1}(\mathbb{R}^d)$ such that $\|u-u_n\|_{W^{1,p}(\mathbb{R}^d)} \to 0$ as $n \to \infty$. Since the classical trace is continuous, the mapping

$$\gamma_0 : W^{1,p}(\mathbb{R}^d) \xhookrightarrow{\text{cts.}} W^{1,p}(\Omega) \xrightarrow{\gamma} W^{1-1/p,p}(\partial\Omega)$$

is linear and continuous. Thus, uniformly in $s \in (0,1)$,

$$\|\mathrm{Tr_s}\, u - \mathrm{Tr_s}\, u_n\|_{\mathcal{T}^{s,p}(\Omega^c)} \leq c_1\|u-u_n\|_{W^{1,p}(\mathbb{R}^d)} \to 0,$$
$$\|\gamma_0 u - \gamma_0 u_n\|_{W^{1-1/p,p}(\partial\Omega)} \leq C_3\|u-u_n\|_{W^{1,p}(\mathbb{R}^d)} \to 0 \quad \text{as } n \to \infty.$$

Finally, Step 1 yields $\|u_n\|_{\mathcal{T}^{s,p}(\Omega^c)} \to \|u_n\|_{W^{1-1/p,p}(\partial\Omega)}$ as $s \to 1^-$. The proof of the statement for $d = 1$ follows with minor changes and we omit it. Notice in this case that functions in $W^{s,p}(\Omega)$ for $s > 1/p$ are continuous by Morrey's inequality. Lastly, the proof of $\|\mathrm{Tr_s}\, u\|_{L^1(\Omega;\mu_s)} \to \|\gamma u\|_{L^1(\partial\Omega)}$ follows analogously to the proof in Step 2 using the uniform trace embedding Proposition 3.9. ☐

Appendix A: Reflected Whitney cubes

The following results are taken from [Dyda and Kassmann 2019, Section 3.2]. Throughout this section, we fix a Lipschitz domain $\Omega \subset \mathbb{R}^d$. We fix a Whitney decomposition $\mathcal{W}(\mathbb{R}^d \setminus \overline{\Omega})$ of the open set with Lipschitz boundary $\mathbb{R}^d \setminus \overline{\Omega}$; i.e., each cube $Q \in \mathcal{W}(\mathbb{R}^d \setminus \overline{\Omega})$ satisfies $\operatorname{diam} Q \le \operatorname{dist}(Q, \partial\Omega) \le 4 \operatorname{diam} Q$. We denote the center of a Whitney cube $Q \in \mathcal{W}(\mathbb{R}^d \setminus \overline{\Omega})$ by $q_Q \in Q$. These cubes satisfy

$$\sum_{Q \in \mathcal{W}(\mathbb{R}^d \setminus \overline{\Omega})} \mathbb{1}_Q = \mathbb{1}_{\mathbb{R}^d \setminus \overline{\Omega}}. \tag{A-1}$$

Bounded Lipschitz domains are both interior and exterior thick; see [loc. cit., Definition 14 and 15]. Thereby, we find a constant $M > 1$ and a reflected Whitney cube $\widetilde{Q} \subset \Omega$ for any $Q \in \mathcal{W}(\mathbb{R}^d \setminus \overline{\Omega})$ such that $\operatorname{diam} Q \le \operatorname{inr}(\Omega) = \sup\{r \mid B_r \subset \Omega\}$ satisfying

$$\begin{gathered} \operatorname{diam} \widetilde{Q} \le \operatorname{dist}(\widetilde{Q}, \partial\Omega) \le 4 \operatorname{diam} \widetilde{Q}, \\ M^{-1} \operatorname{diam} Q \le \operatorname{diam} \widetilde{Q} \le M \operatorname{diam} Q, \\ \operatorname{dist}(Q, \widetilde{Q}) \le M \operatorname{dist}(Q, \partial\Omega). \end{gathered} \tag{A-2}$$

Again we denote the centers of the reflected cubes by $q_{\widetilde{Q}} \in \widetilde{Q}$. The collection of these reflected cubes cover the domain Ω; i.e.,

$$\bigcup_{\substack{Q \in \mathcal{W}(\mathbb{R}^d \setminus \overline{\Omega}) \\ \operatorname{diam} Q \le \operatorname{inr}(\Omega)}} \widetilde{Q} = \Omega. \tag{A-3}$$

Additionally, the reflected cubes satisfy the bounded overlap property; i.e., there exists a constant $N \ge 1$ such that

$$\sum_{\substack{Q \in \mathcal{W}(\mathbb{R}^d \setminus \overline{\Omega}) \\ \operatorname{diam} Q \le \operatorname{inr}(\Omega)}} \mathbb{1}_{\widetilde{Q}} \le N \mathbb{1}_{\Omega}; \tag{A-4}$$

see [loc. cit., Remark 19]. We define

$$\mathcal{W}_{\operatorname{inr}(\Omega)}(\mathbb{R}^d \setminus \Omega) := \{Q \in \mathcal{W}(\mathbb{R}^d \setminus \Omega) \mid \operatorname{diam} Q \le \operatorname{inr}(\Omega)\}.$$

Appendix B: Hardy inequality for the half-space

The following Hardy inequality for the half-space is proven in [Frank and Seiringer 2010; Bogdan and Dyda 2011] in the case $p = 2$. See [Dyda and Kijaczko 2024] for a corresponding weighted Hardy inequality. Note that the constant $\mathcal{D}_{s,p}$ is optimal.

Theorem B.1 [Frank and Seiringer 2010, Theorem 1.1; Dyda and Kijaczko 2024, Theorem 1]. *Let* $0 < s < 1$, $d \in \mathbb{N}$, $p \in [1, \infty)$ *with* $ps \neq 1$. *Then*

$$\mathcal{D}_{s,p} \int_{\mathbb{R}^d_+} \frac{|u(x)|^p}{x_d^{sp}} \, \mathrm{d}x \le \int_{\mathbb{R}^d_+ \times \mathbb{R}^d_+} \frac{|u(x) - u(y)|^p}{|x - y|^{d+sp}} \, \mathrm{d}(x, y) \tag{B-1}$$

for any $u \in W_0^{s,p}(\mathbb{R}^d_+) = \overline{C_c^\infty(\mathbb{R}^d_+)}^{\|\cdot\|_{W^{s,p}}}$. *The constant is given by*

$$\mathcal{D}_{s,p} := 2\pi^{(d-1)/2}\frac{\Gamma\left(\frac{1+sp}{2}\right)}{\Gamma\left(\frac{d+sp}{2}\right)}\int_0^1 \frac{|1-t^{(ps-1)/p}|^p}{(1-t)^{1+ps}}\,\mathrm{d}t. \tag{B-2}$$

Furthermore, in the case $p = 1$ *and* $d = 1$, *the inequality only holds for functions that are proportional to a nonincreasing function.*

Lemma B.2. *There exists a constant* $C = C(d) \geq 1$ *such that, for all* $0 < s < 1$,

$$C^{-1} \leq s\mathcal{D}_{s,1} \leq C,$$

where $\mathcal{D}_{s,1}$ *is the constant defined in* (B-2).

Proof. We split the integral in (B-2) into two parts. First,

$$\int_0^{1/2} \frac{|1-t^{s-1}|}{(1-t)^{1+s}}\,\mathrm{d}t \leq 2^{1+s}\int_0^{1/2} t^{s-1}\,\mathrm{d}t \leq 4\frac{\left(\frac{1}{2}\right)^s}{s} \leq \frac{4}{s}.$$

A lower bound in the same term is calculated similarly:

$$\int_0^{1/2} \frac{|1-t^{s-1}|}{(1-t)^{1+s}}\,\mathrm{d}t \geq \left(1-\left(\frac{1}{2}\right)^{1-s}\right)\int_0^{1/2} t^{s-1}\,\mathrm{d}t = \frac{2^{1-s}-1}{2s} \geq \frac{1-s}{4s}.$$

We move to the remaining part of the integral:

$$\int_{1/2}^1 \frac{|1-t^{s-1}|}{(1-t)^{1+s}}\,\mathrm{d}t \leq 2^{1-s}\int_{1/2}^1 \frac{1}{(1-t)^{1+s}}\left((1-s)\int_t^1 r^{-s}\,\mathrm{d}r\right)\mathrm{d}t \leq 2\int_{1/2}^1 \frac{1-s}{(1-t)^s}\,\mathrm{d}t \leq 2.$$

And a lower bound is calculated in a similar fashion:

$$\int_{1/2}^1 \frac{|1-t^{s-1}|}{(1-t)^{1+s}}\,\mathrm{d}t \geq \int_{1/2}^1 \frac{1}{(1-t)^{1+s}}\left((1-s)\int_t^1 r^{-s}\,\mathrm{d}r\right)\mathrm{d}t \geq \int_{1/2}^1 \frac{1-s}{(1-t)^s}\,\mathrm{d}t = \left(\frac{1}{2}\right)^{1-s} \geq \frac{1}{2}.$$

Therefore,

$$2\pi^{(d-1)/2}\frac{1}{\Gamma\left(\frac{d}{2}\right)\vee\Gamma\left(\frac{(d+1)}{2}\right)}\frac{1}{4s} \leq \mathcal{D}_{s,1} \leq 2\pi^{(d-1)/2}\frac{\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{d}{2}\right)\wedge\Gamma\left(\frac{(d+1)}{2}\right)}\frac{6}{s}. \qquad \square$$

Acknowledgements

The authors thank Juan Pablo Borthagaray for helpful discussions and Solveig Hepp and Thorben Hensiek for comments that made the arguments more perspicuous.

References

[Acosta et al. 2019] G. Acosta, J. P. Borthagaray, and N. Heuer, "Finite element approximations of the nonhomogeneous fractional Dirichlet problem", *IMA J. Numer. Anal.* **39**:3 (2019), 1471–1501. MR

[Aronszajn 1955] N. Aronszajn, "Boundary values of functions with finite Dirichlet integral", pp. 77–93 in *Conference on partial differential equations* (University of Kansas, Summer, 1954), Univ. of Kansas Dept. of Math., 1955.

[Aronszajn and Smith 1961] N. Aronszajn and K. T. Smith, "Theory of Bessel potentials, I", *Ann. Inst. Fourier* (*Grenoble*) **11** (1961), 385–475. MR

[Bergh and Löfström 1976] J. Bergh and J. Löfström, *Interpolation spaces: an introduction*, Grundl. Math. Wissen. **223**, Springer, 1976. MR

[Besov et al. 1975] O. V. Besov, V. P. Ilin, and S. M. Nikolskiĭ, Интегралные представления функциӣ и теоремы вложения, Izdat. "Nauka", Moscow, 1975. MR

[Bogdan and Dyda 2011] K. Bogdan and B. Dyda, "The best constant in a fractional Hardy inequality", *Math. Nachr.* **284**:5-6 (2011), 629–638. MR

[Bogdan et al. 2020] K. Bogdan, T. Grzywny, K. Pietruska-Pałuba, and A. Rutkowski, "Extension and trace for nonlocal operators", *J. Math. Pures Appl.* (9) **137** (2020), 33–69. MR

[Bogdan et al. 2023] K. Bogdan, T. Grzywny, K. Pietruska-Pałuba, and A. Rutkowski, "Nonlinear nonlocal Douglas identity", *Calc. Var. Partial Differential Equations* **62**:5 (2023), art. id. 151. MR

[Bourgain et al. 2001] J. Bourgain, H. Brezis, and P. Mironescu, "Another look at Sobolev spaces", pp. 439–455 in *Optimal control and partial differential equations*, edited by J. L. Menaldi et al., IOS, Amsterdam, 2001. MR

[Brasco et al. 2018] L. Brasco, E. Lindgren, and A. Schikorra, "Higher Hölder regularity for the fractional p-Laplacian in the superquadratic case", *Adv. Math.* **338** (2018), 782–846. MR

[Bucur et al. 2023] C. Bucur, S. Dipierro, L. Lombardini, J. M. Mazón, and E. Valdinoci, "(s, p)-harmonic approximation of functions of least $W^{s,1}$-seminorm", *Int. Math. Res. Not.* **2023**:2 (2023), 1173–1235. MR

[Dávila 2002] J. Dávila, "On an open question about functions of bounded variation", *Calc. Var. Partial Differential Equations* **15**:4 (2002), 519–527. MR

[Di Castro et al. 2016] A. Di Castro, T. Kuusi, and G. Palatucci, "Local behavior of fractional p-minimizers", *Ann. Inst. H. Poincaré C Anal. Non Linéaire* **33**:5 (2016), 1279–1299. MR

[Ding 1996] Z. Ding, "A proof of the trace theorem of Sobolev spaces on Lipschitz domains", *Proc. Amer. Math. Soc.* **124**:2 (1996), 591–600. MR

[Dipierro et al. 2017] S. Dipierro, X. Ros-Oton, and E. Valdinoci, "Nonlocal problems with Neumann boundary conditions", *Rev. Mat. Iberoam.* **33**:2 (2017), 377–416. MR

[Du et al. 2022a] Q. Du, T. Mengesha, and X. Tian, "Fractional Hardy-type and trace theorems for nonlocal function spaces with heterogeneous localization", *Anal. Appl.* (*Singap.*) **20**:3 (2022), 579–614. MR

[Du et al. 2022b] Q. Du, X. Tian, C. Wright, and Y. Yu, "Nonlocal trace spaces and extension results for nonlocal calculus", *J. Funct. Anal.* **282**:12 (2022), art. id. 109453. MR

[Dyda 2004] B. Dyda, "A fractional order Hardy inequality", *Illinois J. Math.* **48**:2 (2004), 575–588. MR

[Dyda and Kassmann 2019] B. Dyda and M. Kassmann, "Function spaces and extension results for nonlocal Dirichlet problems", *J. Funct. Anal.* **277**:11 (2019), art. id. 108134. MR

[Dyda and Kijaczko 2022] B. Dyda and M. Kijaczko, "On density of compactly supported smooth functions in fractional Sobolev spaces", *Ann. Mat. Pura Appl.* (4) **201**:4 (2022), 1855–1867. MR

[Dyda and Kijaczko 2024] B. Dyda and M. Kijaczko, "Sharp weighted fractional Hardy inequalities", *Studia Math.* **274**:2 (2024), 153–171. MR

[Federer 1969] H. Federer, *Geometric measure theory*, Grundl. Math. Wissen. **153**, Springer, 1969. MR

[Felsinger et al. 2015] M. Felsinger, M. Kassmann, and P. Voigt, "The Dirichlet problem for nonlocal operators", *Math. Z.* **279**:3-4 (2015), 779–809. MR

[Foghem 2025] G. Foghem, "Stability of complement value problems for p-Lévy operators", *NoDEA Nonlinear Differential Equations Appl.* **32**:1 (2025), art. id. 1. MR

[Foghem and Kassmann 2024] G. Foghem and M. Kassmann, "A general framework for nonlocal Neumann problems", *Commun. Math. Sci.* **22**:1 (2024), 15–66. MR

[Foghem Gounoue 2020] G. F. Foghem Gounoue, *L^2-theory for nonlocal operators on domains*, dissertation, Universität Bielefeld, 2020.

[Foghem Gounoue 2023] G. F. Foghem Gounoue, "A remake of Bourgain–Brezis–Mironescu characterization of Sobolev spaces", *Partial Differ. Equ. Appl.* **4**:2 (2023), art. id. 16. MR

[Foss 2021] M. Foss, "Traces on general sets in $\mathbb{R}^n$ for functions with no differentiability requirements", *SIAM J. Math. Anal.* **53**:4 (2021), 4212–4251. MR

[Frank and Seiringer 2010] R. L. Frank and R. Seiringer, "Sharp fractional Hardy inequalities in half-spaces", pp. 161–167 in *Around the research of Vladimir Maz'ya*, *I*, edited by A. Laptev, Int. Math. Ser. (N. Y.) **11**, Springer, 2010. MR

[Frerick et al. 2025] L. Frerick, C. Vollmann, and M. Vu, "The nonlocal Neumann problem", *J. Differential Equations* **443** (2025), art. id. 113553. MR

[Gagliardo 1957] E. Gagliardo, "Caratterizzazioni delle tracce sulla frontiera relative ad alcune classi di funzioni in n variabili", *Rend. Sem. Mat. Univ. Padova* **27** (1957), 284–305. MR

[Grisvard 2011] P. Grisvard, *Elliptic problems in nonsmooth domains*, Classics in Applied Mathematics **69**, Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA, 2011. MR

[Grubb 2015] G. Grubb, "Fractional Laplacians on domains, a development of Hörmander's theory of μ-transmission pseudodifferential operators", *Adv. Math.* **268** (2015), 478–528. MR

[Grube and Hensiek 2024] F. Grube and T. Hensiek, "Robust nonlocal trace spaces and Neumann problems", *Nonlinear Anal.* **241** (2024), art. id. 113481. MR

[Jonsson 1994] A. Jonsson, "Besov spaces on closed subsets of $\mathbb{R}^n$", *Trans. Amer. Math. Soc.* **341**:1 (1994), 355–370. MR

[Jonsson and Wallin 1978] A. Jonsson and H. Wallin, "A Whitney extension theorem in L_p and Besov spaces", *Ann. Inst. Fourier* (*Grenoble*) **28**:1 (1978), vi, 139–192. MR

[Jonsson and Wallin 1984] A. Jonsson and H. Wallin, "Function spaces on subsets of $\mathbb{R}^n$", *Math. Rep.* **2**:1 (1984), xiv+221. MR

[Kim 2007] D. Kim, "Trace theorems for Sobolev–Slobodeckij spaces with or without weights", *J. Funct. Spaces Appl.* **5**:3 (2007), 243–268. MR

[Korvenpää et al. 2016] J. Korvenpää, T. Kuusi, and G. Palatucci, "The obstacle problem for nonlinear integro-differential operators", *Calc. Var. Partial Differential Equations* **55**:3 (2016), art. id. 63. MR

[Korvenpää et al. 2017] J. Korvenpää, T. Kuusi, and G. Palatucci, "Fractional superharmonic functions and the Perron method for nonlinear integro-differential equations", *Math. Ann.* **369**:3-4 (2017), 1443–1489. MR

[Leoni 2017] G. Leoni, *A first course in Sobolev spaces*, 2nd ed., Graduate Studies in Mathematics **181**, Amer. Math. Soc., Providence, RI, 2017. MR

[Lindgren and Lindqvist 2017] E. Lindgren and P. Lindqvist, "Perron's method and Wiener's theorem for a nonlocal equation", *Potential Anal.* **46**:4 (2017), 705–737. MR

[Lions and Magenes 1972] J.-L. Lions and E. Magenes, *Non-homogeneous boundary value problems and applications, I*, Grundl. Math. Wissen. **181**, Springer, 1972. MR

[Malý et al. 2018] L. Malý, N. Shanmugalingam, and M. Snipes, "Trace and extension theorems for functions of bounded variation", *Ann. Sc. Norm. Super. Pisa Cl. Sci.* (5) **18**:1 (2018), 313–341. MR

[Marschall 1987] J. Marschall, "The trace of Sobolev–Slobodeckij spaces on Lipschitz domains", *Manuscripta Math.* **58**:1-2 (1987), 47–65. MR

[Mengesha and Du 2016] T. Mengesha and Q. Du, "Characterization of function spaces of vector fields and an application in nonlinear peridynamics", *Nonlinear Anal.* **140** (2016), 82–111. MR

[Nečas 1967] J. Nečas, *Les méthodes directes en théorie des équations elliptiques*, Masson et Cie, Paris, 1967. MR

[Nečas 2012] J. Nečas, *Direct methods in the theory of elliptic equations*, Springer, 2012. MR

[Nikolskiĭ et al. 1988] S. M. Nikolskiĭ, P. I. Lizorkin, and N. V. Miroshin, "Weighted function spaces and their applications to the investigation of boundary value problems for degenerate elliptic equations", *Izv. Vyssh. Uchebn. Zaved. Mat.* 8 (1988), 4–30. MR

[Palatucci 2018] G. Palatucci, "The Dirichlet problem for the p-fractional Laplace equation", *Nonlinear Anal.* **177** (2018), 699–732. MR

[Peetre 1979] J. Peetre, "A counterexample connected with Gagliardo's trace theorem", *Comment. Math. Spec. Issue* **2** (1979), 277–282. MR

[Piersanti and Pucci 2017] P. Piersanti and P. Pucci, "Existence theorems for fractional p-Laplacian problems", *Anal. Appl.* (*Singap.*) **15**:5 (2017), 607–640. MR

[Prodi 1956] G. Prodi, "Tracce sulla frontiera delle funzioni di Beppo Levi", *Rend. Sem. Mat. Univ. Padova* **26** (1956), 36–60. MR

[Scott and Du 2024] J. M. Scott and Q. Du, "Nonlocal problems with local boundary conditions, I: Function spaces and variational principles", *SIAM J. Math. Anal.* **56**:3 (2024), 4185–4222. MR

[Servadei and Valdinoci 2012] R. Servadei and E. Valdinoci, "Mountain pass solutions for non-local elliptic operators", *J. Math. Anal. Appl.* **389**:2 (2012), 887–898. MR

[Servadei and Valdinoci 2013] R. Servadei and E. Valdinoci, "Variational methods for non-local operators of elliptic type", *Discrete Contin. Dyn. Syst.* **33**:5 (2013), 2105–2137. MR

[Servadei and Valdinoci 2014] R. Servadei and E. Valdinoci, "Weak and viscosity solutions of the fractional Laplace equation", *Publ. Mat.* **58**:1 (2014), 133–154. MR

[Slobodeckiĭ 1958] L. N. Slobodeckiĭ, "S. L. Sobolev's spaces of fractional order and their application to boundary problems for partial differential equations", *Dokl. Akad. Nauk SSSR* (*N.S.*) **118**:2 (1958), 243–246. In Russian. MR

[Slobodeckiĭ and Babič 1956] L. N. Slobodeckiĭ and V. M. Babič, "On boundedness of the Dirichlet integrals", *Dokl. Akad. Nauk SSSR* (*N.S.*) **106** (1956), 604–606. In Russian. MR

[Taibleson 1964] M. H. Taibleson, "On the theory of Lipschitz spaces of distributions on Euclidean n-space, I: Principal properties", *J. Math. Mech.* **13** (1964), 407–479. MR

[Tian and Du 2017] X. Tian and Q. Du, "Trace theorems for some nonlocal function spaces with heterogeneous localization", *SIAM J. Math. Anal.* **49**:2 (2017), 1621–1644. MR

[Triebel 1983] H. Triebel, *Theory of function spaces*, Monogr. Math. **78**, Birkhäuser, Basel, 1983. MR

[Triebel 1995] H. Triebel, *Interpolation theory*, *function spaces*, *differential operators*, 2nd ed., Johann Ambrosius Barth, 1995. MR

[Vondraček 2021] Z. Vondraček, "A probabilistic approach to a non-local quadratic form and its connection to the Neumann boundary condition problem", *Math. Nachr.* **294**:1 (2021), 177–194. MR

[Yakovlev 1967] G. N. Yakovlev, "Traces of functions from the space W_p^l onto piecewise smooth surfaces", *Mat. Sb.* (*N.S.*) **74(116)** (1967), 526–543. In Russian; translated in *Mathematics of the USSR-Sbornik* **3**:4 (1967), 481–497. MR

Received 22 Dec 2023. Revised 2 Oct 2024. Accepted 8 Jan 2025.

FLORIAN GRUBE: fgrube@math.uni-bielefeld.de
Fakultät für Mathematik, Universität Bielefeld, Bielefeld, Germany

MORITZ KASSMANN: moritz.kassmann@uni-bielefeld.de
Fakultät für Mathematik, Universität Bielefeld, Bielefeld, Germany

Analysis & PDE

msp.org/apde

Cover image: Eric J. Heller: “Linear Ramp”

See inside back cover or msp.org/apde for submission instructions.

The subscription price for 2025 is US $475/year for the electronic version, and $735/year (+$70, if shipping outside the US) for print and electronic. Subscriptions, requests for back issues from the last three years and changes of subscriber address should be sent to MSP.

Analysis & PDE (ISSN 1948-206X electronic, 2157-5045 printed) at Mathematical Sciences Publishers, 2000 Allston Way # 59, Berkeley, CA 94701-4004, is published continuously online.

APDE peer review and production are managed by EditFlow® from MSP.

PUBLISHED BY

mathematical sciences publishers

nonprofit scientific publishing

http://msp.org/

Analysis & PDE

Volume 18 No. 10 2025