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We study the blow-up dynamics for the energy-critical 1-corotational wave map problem with target the 2-sphere. Raphaël and Rodnianski (*Publ. Math. Inst. Hautes Études Sci.* **115** (2012), 1–122) exhibited stable finite-time blow-up dynamics arising from smooth initial data. In this paper, we exhibit a sequence of new finite-time blow-up rates (quantized rates), which can still arise from well-localized smooth initial data. We closely follow the strategy of Raphaël and Schweyer (*Anal. PDE* **7**:8 (2014), 1713–1805), who exhibited a similar construction of the quantized blow-up rates for the harmonic map heat flow. The main difficulty in our wave map setting stems from the lack of dissipation and its critical nature, which we overcome by a systematic identification of correction terms in higher-order energy estimates.

1. Introduction

1.1. *Wave map problem.* For a map $\Phi : \mathbb{R}^{n+1} \to \mathbb{S}^n$, the wave map problem is given by

$$\partial_{tt}\Phi - \Delta\Phi = \Phi(|\nabla\Phi|^2 - |\partial_t\Phi|^2), \quad \vec{\Phi}(t) := (\Phi, \partial_t\Phi)(t) \in \mathbb{S}^n \times T_\Phi\mathbb{S}^n. \tag{1-1}$$

Problem (1-1) has an intrinsic derivation from the Lagrangian action

$$\frac{1}{2}\int_{\mathbb{R}^{n+1}} (|\nabla\Phi(x,t)|^2 - |\partial_t\Phi(x,t)|^2)\,dx\,dt, \tag{1-2}$$

which yields the energy conservation

$$E(\vec{\Phi}(t)) = \frac{1}{2}\int_{\mathbb{R}^n} |\nabla\Phi|^2 + |\partial_t\Phi|^2\,dx = E(\vec{\Phi}(0)). \tag{1-3}$$

In particular, for the case $n = 2$, (1-1) is called *energy-critical* since the conserved energy is invariant under the scaling symmetry: if $\vec{\Phi}(t, x)$ is a solution to (1-1), then $\vec{\Phi}_\lambda(t, x)$ is also a solution to (1-1):

$$\vec{\Phi}_\lambda(t, x) := \left(\Phi\left(\frac{t}{\lambda}, \frac{x}{\lambda}\right), \frac{1}{\lambda} \partial_t \Phi\left(\frac{t}{\lambda}, \frac{x}{\lambda}\right) \right),$$

and $\vec{\Phi}_\lambda(t, x)$ satisfies $E(\vec{\Phi}_\lambda) = E(\vec{\Phi})$.

When observing a complicated model, it makes sense from a physics perspective to extract the essential dynamics of the problem by reducing the degrees of freedom. Especially for field theories such as (1-1), the *geodesic approximation*—that is, a method of approximating the dynamics of the full problem as a geodesic motion over a space of static solutions—is prevalent (see [Manton and Sutcliffe 2004]).

To discuss static solutions in more detail, we focus on the solutions with finite energy. This assumption extends the spatial domain of Φ to $\mathbb{S}^2$ and allows the *topological degree* of Φ to be well-defined:

$$k = \frac{1}{|\mathbb{S}^2|} \int_{\mathbb{R}^2} \Phi^*(dw) = \frac{1}{4\pi} \int_{\mathbb{R}^2} \Phi \cdot (\partial_x \Phi \times \partial_y \Phi) \, dx \, dy.$$

Here, dw is the area form on $\mathbb{S}^2$ and k is given only as an integer. We also remark that k is conserved over time.

We now consider static solutions to (1-1),

$$\Delta \Phi + \Phi |\nabla \Phi|^2 = 0, \tag{1-4}$$

so-called *harmonic maps*. Recalling our Lagrangian action (1-2), harmonic maps are characterized as the (local) minimizer of the Dirichlet energy

$$\frac{1}{2} \int_{\mathbb{R}^2} |\nabla \Phi|^2 \, dx \, dy.$$

Assume the topological degree of a harmonic map Φ is $k \in \mathbb{Z}$. Then we have the inequality

$$
\begin{aligned}
\frac{1}{2} \int_{\mathbb{R}^2} |\nabla \Phi|^2 \, dx \, dy &= \frac{1}{2} \int_{\mathbb{R}^2} |\partial_x \Phi|^2 + |\partial_y \Phi|^2 \, dx \, dy \\
&= \frac{1}{2} \int_{\mathbb{R}^2} |\partial_x \Phi \pm \Phi \times \partial_y \Phi|^2 \, dx \, dy \mp \int_{\mathbb{R}^2} \partial_x \Phi \cdot (\Phi \times \partial_y \Phi) \, dx \, dy \\
&\geq \pm \int_{\mathbb{R}^2} \Phi \cdot (\partial_x \Phi \times \partial_y \Phi) \, dx \, dy = 4\pi |k|.
\end{aligned}
$$

Hence, in a given topological sector k, Φ satisfies the *Bogomolny equation* [1976]

$$\partial_x \Phi \pm \Phi \times \partial_y \Phi = 0 \quad \text{for } \pm k \geq 0. \tag{1-5}$$

That is, the field equation (1-4) can be reduced from a second-order PDE to a first-order PDE. From the stereographic projection, we can see that equation (1-5) is equivalent to the Cauchy–Riemann equation,[1] which clearly identifies the space of harmonic maps as the space of rational maps of degree k.

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[1]If k is negative, we adopt the *conjugate* Cauchy–Riemann equation instead of the Cauchy–Riemann equation. Thence, harmonic maps can be represented as rational maps with $\bar{z}$ as a complex variable.

Under the L^2 metric induced naturally from the kinetic energy formula, it is well known that the space of static solutions is *geodesically incomplete*, which leads us to expect a blow-up scenario of the low-energy problem.

1.2. *Corotational symmetry.*

We consider an ansatz of solutions to (1-1) with k-corotational symmetry:

$$\Phi(t, r, \theta) = \begin{pmatrix} \sin(u(t, r)) \cos k\theta \\ \sin(u(t, r)) \sin k\theta \\ \cos(u(t, r)) \end{pmatrix}, \tag{1-6}$$

where (r, θ) are polar coordinates on $\mathbb{R}^2$.

Under the k-corotational symmetry assumption, $u(t, r)$ satisfies

$$\begin{cases} \partial_{tt} u - \partial_{rr} u - (1/r) \partial_r u + k^2 f(u)/r^2 = 0, \\ u|_{t=0} = u_0, \quad \partial_t u|_{t=0} = \dot{u}_0, \end{cases} \quad f(u) = \frac{\sin 2u}{2}. \tag{1-7}$$

It is known that the flow (1-1) preserves such corotational symmetry (1-6) with smooth initial data at least locally in time; see [Raphaël and Rodnianski 2012].

Also, the energy functional (1-3) can be rewritten as

$$E(u, \dot{u}) := \pi \int_0^\infty \left(|\dot{u}|^2 + |\partial_r u|^2 + k^2 \frac{\sin^2 u}{r^2} \right) r \, dr = E(u_0, \dot{u}_0). \tag{1-8}$$

From the above expression, we can observe that a solution to (1-7) with finite energy must satisfy the boundary conditions

$$\lim_{r \to 0} u(r) = m\pi \quad \text{and} \quad \lim_{r \to \infty} u(r) = n\pi, \quad m, n \in \mathbb{Z}. \tag{1-9}$$

We have additional symmetries from the geometry of the target domain $\mathbb{S}^2$:

$$-u(t, r), \quad u(t, r) + \pi \tag{1-10}$$

are also solutions to (1-7). Thus, we restrict our solution space to a set of functions $(u, \dot{u})$ that have finite energy and satisfy the boundary conditions (1-9) with $m = 0$, $n = 1$, which provides the local well-posedness of (1-7) (see also [Klainerman and Machedon 1993; 1995; Krieger 2004; Tao 2001; Tataru 2005]).

1.3. *Harmonic map.*

With this restriction, the harmonic map is uniquely determined (up to scaling) and can be written explicitly as

$$Q(r) = 2 \tan^{-1} r^k. \tag{1-11}$$

Based on the geodesic approximation, it can be said that observing the vicinity of Q under the corotational symmetry assumption facilitates the analysis of blow-up dynamics. This has been proven as a rigorous statement in several past global regularity works (see [Christodoulou and Tahvildar-Zadeh 1993; Shatah and Tahvildar-Zadeh 1992; 1994; Struwe 2003]).

The above results proved that if a wave map blows up in finite time, such a singularity should be created by bubbling off of a nontrivial harmonic map (strictly) inside the backward light cone.

This statement has inspired other researches studying global behaviors of solutions, and many of the results have been developed based on the existence of nontrivial harmonic maps.

Firstly, there is global existence, which is a consequence of the preceding blow-up criteria. If the initial data cannot form a nontrivial harmonic map — that is, if the energy is less than the ground state energy — it can be naturally predicted that the solution exists globally in time, and mathematical proof is also contained in the previously mentioned global regularity results.

This study also allows us to consider the problems of energy threshold (see [Côte et al. 2008] for the symmetric case and [Krieger and Schlag 2012; Sterbenz and Tataru 2010a; 2010b; Tao 2008a; 2008b; 2008c; 2009a; 2009b] for the general case). In this case, it is also important to set an appropriate threshold value and the ground state energy is suitable for our problem setting. However, for other boundary conditions or other topological degrees, it is often given as an integer multiple of $E(Q, 0)$. The heuristic reason is that the degree condition cannot be satisfied with just one bubble (see [Côte et al. 2015a; Lawrie and Oh 2016]). This goes beyond suggesting the existence of a multibubble solution [Jendrej and Lawrie 2018; 2022a; 2022b; 2023; Rodriguez 2021] and serves as an opportunity to verify the soliton resolution conjecture [Duyckaerts et al. 2022; Jendrej and Lawrie 2025] (see also [Côte 2015; Côte et al. 2015a; 2015b; Jia and Kenig 2017]).

The most recent soliton resolution result [Jendrej and Lawrie 2025] fully characterizes the profile decomposition of the solution in all equivariant classes. Thus, our interest is to observe how the scale of the profile given by the harmonic map changes over time within the lifespan of the solution. In particular, for the case of low energy — that is, when the energy is slightly greater than the ground state energy — the geodesic approximation discussed earlier leads us to focus on the situation of having only one harmonic map as the blow-up profile.

1.4. *Blow-up near Q.* From a methodological perspective, studies investigating the blow-up of a single bubble can be broadly divided into the backward construction starting from Krieger, Schlag and Tataru [Krieger et al. 2008] and the forward construction inspired by Rodnianski and Sterbenz [2010] and Raphaël and Rodnianski [2012].

The former work obtained a continuum of blow-up rates for the case $k = 1$ via the iteration method and inspired other extended results such as stability under regular perturbations [Krieger and Miao 2020; Krieger et al. 2020] and the construction of more exotic solutions [Pillai 2023b; 2023a]. Beyond direct extensions of this approach, there is a classification result [Jendrej et al. 2022] via configuring radiations appropriately at the blow-up time. These constructions inevitably involve some constraints on regularity and degeneracy of the initial data.

The latter case adopts a method that accurately describes the initial data set that drives blowup. Although it is difficult (probably impossible) to form a family of blow-up rates as in the previous results, the emphasis is on being able to observe the construction of blow-up solutions with smooth initial data. Especially in [Raphaël and Rodnianski 2012], the authors explicitly describe an initial data set that is open under H^2 topology around Q and prove the so-called stable blowup, in which the solutions starting from that set blow up with a universal rate that slightly misses the self-similar one for all $k \geq 1$.

We note that the initial data set in the above result does not imply a universal blowup of all well-localized smooth data. Our main theorem says that there exist other smooth solutions that blow up in finite time with quantized rates corresponding to the excited regime.

1.5. *Main theorem.* We focus on the solution to (1-7) with 1-corotational initial data, i.e., $k = 1$. Let us restate the stable blowup result.

Theorem 1.1 (stable blowup for 1-corotational wave maps [Kim 2023; Raphaël and Rodnianski 2012]). *There exists a constant $\varepsilon_0 > 0$ such that, for all 1-corotational initial data $(u_0, \dot{u}_0)$ with*

$$\|u_0 - Q, \dot{u}_0\|_{\mathcal{H}^2} < \varepsilon_0, \tag{1-12}$$

the corresponding solutions to (1-7) blow up in finite time $0 < T = T(u_0, \dot{u}_0) < \infty$ as follows: for some $(u^*, \dot{u}^*) \in \mathcal{H}$,

$$\left\| u(t, r) - Q\left(\frac{r}{\lambda(t)}\right) - u^*, \partial_t u(t, r) - \dot{u}^* \right\|_{\mathcal{H}} \to 0 \quad as\ t \to T \tag{1-13}$$

with the universal blow-up speed

$$\lambda(t) = 2e^{-1}(1 + o_{t \to T}(1))(T - t)e^{-\sqrt{|\log(T-t)|}}. \tag{1-14}$$

Here, $\mathcal{H}$ and $\mathcal{H}^2$ are given by (1-24) and (1-25), respectively.

Remark (1-corotational symmetry). Raphaël and Rodnianski [2012] mentioned that the nature of the harmonic map, which varies depending on whether k is equal to 1 or not, leads to distinctive blow-up rates. As a result of the logarithmic calculation that occurs additionally only when $k = 1$, the universality of the blow-up rate in this case was unclear. The sharp constant $2e^{-1}$ in (1-14) was later obtained by Kim [2023] using a refined modulation analysis.

Nevertheless, the slow decaying nature of the harmonic map is rather an advantage in our analysis, which allows us to exhibit the following smooth blowup with the quantized blow-up rates corresponding to the excited regime.

Theorem 1.2 (quantized blowup for 1-corotational wave map). *For a natural number $\ell \geq 2$ and an arbitrarily small constant $\varepsilon_0 > 0$, there exists a smooth 1-corotational initial data $(u_0, \dot{u}_0)$ with*

$$\|u_0 - Q, \dot{u}_0\|_{\mathcal{H}} < \varepsilon_0 \tag{1-15}$$

such that the corresponding solution to (1-7) blows up in finite time $0 < T = T(u_0, \dot{u}_0) < \infty$ and satisfies (1-13), with the quantized blow-up speed

$$\lambda(t) = c(u_0, \dot{u}_0)(1 + o_{t \to T}(1)) \frac{(T - t)^{\ell}}{|\log(T - t)|^{\ell/(\ell-1)}}, \quad c(u_0, \dot{u}_0) > 0. \tag{1-16}$$

Remark (further regularity of asymptotic profile). The asymptotic profile $(u^*, \dot{u}^*)$ also has $\dot{H}^{\ell} \times \dot{H}^{\ell-1}$ regularity in the sense that certain ℓ-fold (resp. $(\ell-1)$-fold) derivatives of u^* (resp. $\dot{u}^*$) belong to L^2. This is a consequence of the fact that the ℓ-th-order energy of the radiative part of the solution satisfies the scaling invariance bound ($\mathcal{E}_{\ell} \leq C\lambda^{2(\ell-1)}$; see (4-13)) similar to [Raphaël and Schweyer 2014].

Remark (quantized blowup). The existence of (type-II) blow-up solutions with quantized blow-up rates has also been well studied in parabolic equations, especially for nonlinear heat equations. Starting with the discovery of formal mechanisms [Filippas et al. 2000; Herrero and Velázquez 1992; 1994], there

are classification works [Mizoguchi 2007; 2011] in the energy-supercritical regime. The proofs in this literature are based on the maximum principle (see [Matano and Merle 2004; 2009]).

Through modulation analysis, not relying on maximum principle, there have been some (type-II) quantized rate constructions in the critical parabolic equations such as [Hadžić and Raphaël 2019; Raphaël and Schweyer 2014] for the energy-critical case and [Collot et al. 2022] for the mass-critical case. See also [del Pino et al. 2020; Harada 2020], which rely on the inner-outer gluing method. Raphaël and Schweyer [2014] expected that their modulation technique could be robust enough to be propagated to dispersive models including the wave map problem, and the quantized rate constructions have been established in the energy-supercritical dispersive equations [Collot 2018; Ghoul et al. 2018; Merle et al. 2015]. To our knowledge, Theorem 1.2 provides the first rigorous quantized rate constructions for critical dispersive equations. We expect that our analysis can also be extended to other energy-critical dispersive equations such as the nonlinear wave equation.

Remark (instability of blowup). In contrast to Theorem 1.1, our initial data set is of codimension $(\ell - 1)$, similar to [Raphaël and Schweyer 2014], due to unstable directions inherent in the ODE system driving the blow-up dynamics. This similarity follows from the fact that the wave map problem and the harmonic map heat flow share the same ground states and linearized Hamiltonian under the 1-corotational symmetry. We also expect the stability formulated by constructing a smooth manifold of initial data leading to our quantized blow-up scenario.

1.6. *Notation.*

We introduce some notation needed for the proof before going into the strategy of the proof. We first use the bold notation for vectors in $\mathbb{R}^2$:

$$\boldsymbol{u} := \begin{pmatrix} u \\ \dot{u} \end{pmatrix}, \quad \boldsymbol{u}(r) := \begin{pmatrix} u(r) \\ \dot{u}(r) \end{pmatrix}. \tag{1-17}$$

For $\lambda > 0$, the $\dot{H}^1 \times L^2$ scaling is defined by

$$\boldsymbol{u}_\lambda(r) = \begin{pmatrix} u_\lambda(r) \\ \lambda^{-1} \dot{u}_\lambda(r) \end{pmatrix} := \begin{pmatrix} u(y) \\ \lambda^{-1} \dot{u}(y) \end{pmatrix}, \quad y := \frac{r}{\lambda}, \tag{1-18}$$

and the corresponding generator is denoted by

$$\boldsymbol{\Lambda u} := \begin{pmatrix} \Lambda u \\ \Lambda_0 \dot{u} \end{pmatrix} := -\frac{d\boldsymbol{u}_\lambda(r)}{d\lambda}\bigg|_{\lambda=1} = \begin{pmatrix} r\partial_r u(r) \\ (1 + r\partial_r)\dot{u}(r) \end{pmatrix}. \tag{1-19}$$

In general, we employ the $\dot{H}^k$ scaling generator

$$\Lambda_k u := -\frac{d}{d\lambda}(\lambda^{k-1} u_\lambda(r))\bigg|_{\lambda=1} = (-k + 1 + r\partial_r)u(r). \tag{1-20}$$

We now reformulate (1-7) using the vector-valued function $\boldsymbol{F} : \mathbb{R}^2 \to \mathbb{R}^2$:

$$\begin{cases} \partial_t \boldsymbol{u} = \boldsymbol{F}(\boldsymbol{u}), \\ \boldsymbol{u}|_{t=0} = \boldsymbol{u}_0, \end{cases} \quad \boldsymbol{u} = \boldsymbol{u}(t, r), \quad \boldsymbol{F}(\boldsymbol{u}) := \begin{pmatrix} \dot{u} \\ \Delta u - f(u)/r^2 \end{pmatrix}, \tag{1-21}$$

where $\Delta = \partial_{rr} + (1/r)\partial_r$.

We use two subsets of the real line:
$$\mathbb{R}_+ = \{r \in \mathbb{R} : x \geq 0\},$$
$$\mathbb{R}_+^* = \{r \in \mathbb{R} : x > 0\}.$$

We denote by χ a C^∞ radial cut-off function on $\mathbb{R}_+$:
$$\chi(r) = \begin{cases} 1 & \text{for } r \leq 1, \\ 0 & \text{for } r \geq 2. \end{cases}$$

We let $\chi_B(r) := \chi(r/B)$ for $B > 0$. Similarly, we denote by $\mathbf{1}_A(y)$ the indicator function on the set A. In particular, $\mathbf{1}_{B \leq y \leq 2B}$ will be rewritten as $\mathbf{1}_{y \sim B}$, or abusively as simply $\mathbf{1}_B$. The cut-off boundary B will often be chosen as the constant multiples of
$$B_0 := \frac{1}{b_1}, \quad B_1 := \frac{|\log b_1|^\gamma}{b_1}, \quad b_1 > 0. \tag{1-22}$$

Later, we will choose $\gamma = 1 + \bar{\ell}$, where ℓ appeared in Theorem 1.2. Here, we denote by $\bar{i}$ the remainder of dividing i by 2, i.e., $\bar{i} = i \mod 2$ for an integer i. We also write $L = \ell + \overline{\ell + 1}$, i.e., L is the smallest odd integer greater than or equal to ℓ. We also abuse the indicator notation $\mathbf{1}_{\{l \geq m\}}$:
$$\mathbf{1}_{\{l \geq m\}} = \begin{cases} 1 & \text{if } l \geq m, \\ 0 & \text{if } l < m, \end{cases} \quad l, m \in \mathbb{Z}.$$

We adopt the following $L^2(\mathbb{R}^2)$ inner product for radial functions u, v:
$$\langle u, v \rangle := \int_0^\infty u(r) v(r) r \, dr,$$

and the $L^2 \times L^2$ inner product for vector-valued functions $\boldsymbol{u}, \boldsymbol{v}$:
$$\langle \boldsymbol{u}, \boldsymbol{v} \rangle := \langle u, v \rangle + \langle \dot{u}, \dot{v} \rangle. \tag{1-23}$$

We introduce two Sobolev spaces $\mathcal{H}$ and $\mathcal{H}^2$ with the following norms:
$$\|u, \dot{u}\|_{\mathcal{H}}^2 := \int |\partial_y u|^2 + \frac{|u|^2}{y^2} + |\dot{u}|^2, \tag{1-24}$$

$$\|u, \dot{u}\|_{\mathcal{H}^2}^2 := \|u, \dot{u}\|_{\mathcal{H}}^2 + \int |\partial_y^2 u|^2 + |\partial_y \dot{u}|^2 + \frac{|\dot{u}|^2}{y^2} + \int_{|y| \leq 1} \frac{1}{y^2} \left(\partial_y u - \frac{u}{y} \right)^2, \tag{1-25}$$

where the above shorthand for integrals is given by $\int = \int_{\mathbb{R}^2}$.

For any $x := (x_1, \ldots, x_n) \in \mathbb{R}^n$, we set $|x|^2 = x_1^2 + \cdots + x_n^2$ and
$$\mathcal{B}^n := \{x \in \mathbb{R}^n, |x| \leq 1\},$$
$$\mathcal{S}^n := \partial \mathcal{B}^n = \{x \in \mathbb{R}^n, |x| = 1\}.$$

We use the Kronecker delta notation: $\delta_{ij} = 1$ for $i = j$ and $\delta_{ij} = 0$ for $i \neq j$.

1.7. *Strategy of the proof.* Our proof is based on the general modulation analysis scheme developed by Raphaël and Rodnianski [2012], Merle, Raphaël and Rodnianski [Merle et al. 2013] and Raphaël and Schweyer [2014], which also have difficulties arising from the energy-critical nature and the small equivariance index, including logarithmic computations. We closely follow the main strategy of [Raphaël and Schweyer 2014]. However, notable differences stem from the lack of dissipation in the higher-order (H^{L+1}, $L \gg 1$) energy estimates due to the dispersive nature of our problem. We overcome this difficulty by carefully correcting the higher-order energy functional to uncover the repulsive property (to identify terms with good sign), generalizing the computation in the H^2 energy estimates of [Raphaël and Rodnianski 2012].

Given an odd integer $L \geq 3$, we first construct the blow-up profile $\boldsymbol{Q}_b$ of the form

$$\boldsymbol{Q}_b := \boldsymbol{Q} + \boldsymbol{\alpha}_b := \begin{pmatrix} Q \\ 0 \end{pmatrix} + \sum_{i=1}^{L} b_i \boldsymbol{T}_i + \sum_{i=2}^{L+2} \boldsymbol{S}_i, \tag{1-26}$$

where $b = (b_1, \ldots, b_L)$ is a set of modulation parameters and $\boldsymbol{T}_i$ and $\boldsymbol{S}_i$ are deformation directions so that $(\boldsymbol{Q}_{b(t)})_{\lambda(t)}$ solves (1-21) approximately. Equivalently, $\boldsymbol{Q}_b$ satisfies

$$\partial_s \boldsymbol{Q}_b - \boldsymbol{F}(\boldsymbol{Q}_b) - \frac{\lambda_s}{\lambda} \boldsymbol{\Lambda} \boldsymbol{Q}_b \approx 0, \quad \frac{ds}{dt} = \frac{1}{\lambda(t)}. \tag{1-27}$$

From the imposed relations (1-27), the blow-up dynamics are determined by the evolution of the modulation parameters $b = (b_1, \ldots, b_L)$. The leading dynamics of b and $\boldsymbol{T}_i$ are determined by considering the linearized flow of (1-27) near $\boldsymbol{Q}$:

$$0 \approx \partial_s \boldsymbol{Q}_b - \boldsymbol{F}(\boldsymbol{Q}_b) - \frac{\lambda_s}{\lambda} \boldsymbol{\Lambda} \boldsymbol{Q}_b = \partial_s(\boldsymbol{Q}_b - \boldsymbol{Q}) - \boldsymbol{F}(\boldsymbol{Q}_b) + \boldsymbol{F}(\boldsymbol{Q}) - \frac{\lambda_s}{\lambda} \boldsymbol{\Lambda} \boldsymbol{Q}_b$$

$$\approx \partial_s \boldsymbol{\alpha}_b + \boldsymbol{H} \boldsymbol{\alpha}_b - \frac{\lambda_s}{\lambda} \boldsymbol{\Lambda}(\boldsymbol{Q} + \boldsymbol{\alpha}_b), \tag{1-28}$$

where $\boldsymbol{H}$ denotes the linearized Hamiltonian:

$$\boldsymbol{H} := \begin{pmatrix} 0 & -1 \\ H & 0 \end{pmatrix}, \quad H = -\Delta + \frac{f'(Q)}{y^2}. \tag{1-29}$$

After defining $\boldsymbol{T}_i$ inductively,

$$\boldsymbol{H} \boldsymbol{T}_{i+1} = -\boldsymbol{T}_i, \quad \boldsymbol{T}_0 := \boldsymbol{\Lambda} \boldsymbol{Q}, \tag{1-30}$$

equation (1-28) and the asymptotics $\boldsymbol{\Lambda} \boldsymbol{T}_i \sim (i-1) \boldsymbol{T}_i$ yield the leading dynamics of b:

$$-\frac{\lambda_s}{\lambda} = b_1, \quad (b_k)_s = b_{k+1} - (k-1)b_1 b_k, \quad b_{L+1} := 0, \quad 1 \leq k \leq L. \tag{1-31}$$

$\boldsymbol{S}_i$ appears to correct (1-28) to (1-27) containing some radiative terms from the difference $\boldsymbol{\Lambda} \boldsymbol{T}_i - (i-1) \boldsymbol{T}_i$ and the nonlinear effect from $\boldsymbol{F}(\boldsymbol{Q}_b) - \boldsymbol{F}(\boldsymbol{Q}) + \boldsymbol{H} \boldsymbol{\alpha}_b$. Then b drives the ODE system

$$(b_k)_s = b_{k+1} - \left(k - 1 + \frac{1}{(1 + \delta_{1k}) \log s}\right) b_1 b_k, \quad b_{L+1} := 0, \quad 1 \leq k \leq L. \tag{1-32}$$

We then choose a special solution of (1-32),

$$b_1(s) \sim \frac{\ell}{\ell-1}\left(\frac{1}{s} - \frac{(\ell-1)^{-1}}{s\log s}\right), \tag{1-33}$$

which leads to (1-16) from the relations

$$-\lambda_t = b_1 \quad \text{and} \quad \frac{ds}{dt} = \frac{1}{\lambda}.$$

Since the special solution we choose is formally codimension $(\ell-1)$ stable, we control the unstable directions in the vicinity of these special solutions to ODE system (1-32) by Brouwer's fixed point theorem.

Now, we decompose the solution $\boldsymbol{u} = \boldsymbol{u}(t,r)$ to (1-21) as

$$\boldsymbol{u} = (\boldsymbol{Q}_{b(t)} + \boldsymbol{\varepsilon})_{\lambda(t)} = (\boldsymbol{Q}_{b(t)})_{\lambda(t)} + \boldsymbol{w}, \quad \langle \boldsymbol{H}^i \boldsymbol{\varepsilon}, \boldsymbol{\Phi}_M \rangle = 0, \quad 0 \le i \le L, \tag{1-34}$$

where $\boldsymbol{\Phi}_M$ is defined in (3-1). The orthogonality conditions in (1-34) uniquely determine the decomposition by the implicit function theorem. Then we derive the evolution equation of $\boldsymbol{\varepsilon}$ from (1-21), which contains the formal modulation ODE (1-32) with some errors in terms of $\boldsymbol{\varepsilon}$.

To justify the formal modulation ODE (1-32), we need sufficient smallness of $\boldsymbol{\varepsilon}$ and we need to propagate it. For this purpose, we consider the higher-order energy associated to the linearized Hamiltonian H:

$$\mathcal{E}_{L+1} = \langle H^{(L+1)/2}\varepsilon, H^{(L+1)/2}\varepsilon \rangle + \langle H H^{(L-1)/2}\dot{\varepsilon}, H^{(L-1)/2}\dot{\varepsilon} \rangle. \tag{1-35}$$

This energy is coercive thanks to the orthogonality conditions in (1-34).

Thus, our analysis boils down to estimating the time derivative of $\mathcal{E}_{L+1}$. Unlike in [Raphaël and Schweyer 2014], we cannot employ dissipation to control the time derivative of $\mathcal{E}_{L+1}$ due to the dispersive nature of our problem. Instead, we use the repulsive property of the (supersymmetric) conjugated Hamiltonian $\widetilde{H}$ of H observed in [Raphaël and Rodnianski 2012; Rodnianski and Sterbenz 2010]. To illuminate the repulsive property in the energy estimate, we consider the linearized flow in terms of w from $\boldsymbol{w} = (w, \dot{w})$ and the well-known factorization:

$$w_{tt} + H_\lambda w = 0, \quad H_\lambda = A_\lambda^* A_\lambda, \quad A_\lambda = -\partial_r + \frac{\cos Q_\lambda}{r}.$$

Defining the higher-order derivatives adapted to H_λ and its corresponding operator

$$w_k := \mathcal{A}_\lambda^k w, \quad \mathcal{A}_\lambda = A_\lambda, \quad \mathcal{A}_\lambda^2 = A_\lambda^* A_\lambda, \quad \ldots, \quad \mathcal{A}_\lambda^k = \underbrace{\cdots A_\lambda^* A_\lambda A_\lambda^* A_\lambda}_{k \text{ times}},$$

the higher-order energy (1-35) can essentially be written as

$$\mathcal{E}_{L+1} \approx \lambda^{2L}(\langle w_{L+1}, w_{L+1} \rangle + \langle \partial_t w_L, \partial_t w_L \rangle)$$
$$= \lambda^{2L}(\langle \widetilde{H}_\lambda w_L, w_L \rangle + \langle \partial_t w_L, \partial_t w_L \rangle)$$

where $\widetilde{H}_\lambda = A_\lambda A_\lambda^*$ is the conjugated Hamiltonian of H_λ. As an advantage of the adoption of the Leibniz rule notation between an operator and a function,

$$\partial_t(Pf) = \partial_t(P)f + Pf_t, \quad \partial_t(P) := [\partial_t, P],$$

we can express the energy estimate for $\mathcal{E}_{L+1}$ succinctly:

$$\frac{d}{dt}\left\{\frac{\mathcal{E}_{L+1}}{2\lambda^{2L}}\right\} \approx \frac{1}{2}\langle\partial_t(\widetilde{H}_\lambda)w_L, w_L\rangle + \langle\widetilde{H}_\lambda w_L, \partial_t w_L\rangle + \langle\partial_{tt}w_L, \partial_t w_L\rangle$$

$$\approx \frac{1}{2}\langle\partial_t(\widetilde{H}_\lambda)w_L, w_L\rangle + 2\langle\partial_t w_L, \partial_t(\mathcal{A}_\lambda^L)w_t\rangle.$$

Integrating the second term by parts in time, we get

$$\frac{d}{dt}\left\{\frac{\mathcal{E}_{L+1}}{2\lambda^{2L}} - 2\langle w_L, \partial_t(\mathcal{A}_\lambda^L)w_t\rangle\right\} \approx \frac{1}{2}\langle\partial_t(\widetilde{H}_\lambda)w_L, w_L\rangle + 2\langle w_L, \partial_t(\mathcal{A}_\lambda^L)w_2\rangle.$$

Raphaël and Rodnianski [2012] exhibited the repulsive property by directly calculating the following identity with the advantage of $L = 1$:

$$\langle w_1, \partial_t(\mathcal{A}_\lambda)w_2\rangle = \tfrac{1}{2}\langle\partial_t(\widetilde{H}_\lambda)w_1, w_1\rangle \le 0.$$

However, this computation does not seem to directly extend to our case $L \ge 3$. We overcome this problem by first writing $\mathcal{A}_\lambda^L = \widetilde{H}_\lambda \mathcal{A}_\lambda^{L-2}$ and pulling out the repulsive term using the Leibniz rule:

$$\langle w_L, \partial_t(\mathcal{A}_\lambda^L)w_2\rangle = \langle w_L, \partial_t(\widetilde{H}_\lambda)w_L\rangle + \langle\widetilde{H}_\lambda w_L, \partial_t(\mathcal{A}_\lambda^{L-2})w_2\rangle$$

$$\approx \langle w_L, \partial_t(\widetilde{H}_\lambda)w_L\rangle - \langle\partial_{tt}w_L, \partial_t(\mathcal{A}_\lambda^{L-2})w_2\rangle.$$

Again integrating by parts in time, we obtain

$$\frac{d}{dt}\left\{\frac{\mathcal{E}_{L+1}}{2\lambda^{2L}} - 2\big(\langle w_L, \partial_t(\mathcal{A}_\lambda^L)w_t\rangle - \langle\partial_t w_L, \partial_t(\mathcal{A}_\lambda^{L-2})w_2\rangle + \langle w_L, \partial_t(\mathcal{A}_\lambda^{L-2})\partial_t w_2\rangle\big)\right\}$$

$$\approx \frac{5}{2}\langle\partial_t(\widetilde{H}_\lambda)w_L, w_L\rangle + 2\langle w_L, \partial_t(\mathcal{A}_\lambda^{L-2})w_4\rangle.$$

Repeating the above correction procedure, we arrive at the term with good sign:

$$\frac{d}{dt}\left\{\frac{\mathcal{E}_{L+1}}{2\lambda^{2L}} + \text{corrections}\right\} \approx \frac{2L-1}{2}\langle\partial_t(\widetilde{H}_\lambda)w_L, w_L\rangle + 2\langle w_L, \partial_t(\mathcal{A}_\lambda)w_{L+1}\rangle$$

$$\approx \frac{2L+1}{2}\langle\partial_t(\widetilde{H}_\lambda)w_L, w_L\rangle \le 0.$$

In the actual energy estimate, there are also error terms such as the profile equation error and nonlinear terms in ε. For these nonlinear terms, we also estimate the intermediate energies $\mathcal{E}_k$, which can be defined similarly to $\mathcal{E}_{L+1}$.

Organization of the paper. In Section 2, we construct the approximate blow-up profile with the description of the ODE dynamics of the modulation equations. Section 3 is devoted to the decomposition of the solution into the blow-up profile constructed in the previous section and the remaining error. We also introduce the bootstrap setting to control the error and establish a Lyapunov-type monotonicity for the higher-order energy with respect to such error. Section 4 provides the proof of Theorem 1.2 by closing the bootstrap with some standard topological arguments.

2. Construction of the approximate solution

In this section, we construct the approximate blow-up profile Q_b, represented by a deformation of the harmonic map Q through modulation parameters $b = (b_1, \ldots, b_L)$. We also derive formal dynamical laws of b, which leads to our desired blow-up rate.

2.1. *The linearized dynamics.* It is natural to look into the linearized dynamics of our system near the stationary solution Q. Let $\boldsymbol{u} = \boldsymbol{Q} + \boldsymbol{\varepsilon}$, where $\boldsymbol{Q} = (Q, 0)^t$ and $\boldsymbol{u}$ is the solution to (1-21). Then $\boldsymbol{\varepsilon}$ satisfies

$$\partial_t \boldsymbol{\varepsilon} = \boldsymbol{F}(\boldsymbol{Q} + \boldsymbol{\varepsilon}) - \boldsymbol{F}(\boldsymbol{Q}) = \begin{pmatrix} \dot{\varepsilon} \\ \Delta\varepsilon - (f(Q+\varepsilon) - f(Q))/r^2 \end{pmatrix}$$

$$= \begin{pmatrix} \dot{\varepsilon} \\ \Delta\varepsilon - r^{-2} f'(Q)\varepsilon \end{pmatrix} - \frac{1}{r^2} \begin{pmatrix} 0 \\ f(Q+\varepsilon) - f(Q) - f'(Q)\varepsilon \end{pmatrix}.$$

Ignoring higher-order terms for $\boldsymbol{\varepsilon}$ and setting $\lambda = 1$ (i.e., $r = y$), we roughly obtain the linearized system

$$\partial_t \boldsymbol{\varepsilon} + \boldsymbol{H}\boldsymbol{\varepsilon} = 0, \quad \boldsymbol{H}\boldsymbol{\varepsilon} = \begin{pmatrix} 0 & -1 \\ H & 0 \end{pmatrix} \begin{pmatrix} \varepsilon \\ \dot{\varepsilon} \end{pmatrix}, \tag{2-1}$$

where H is the Schrödinger operator with explicitly computable potential $f'(Q)$ from (1-7) and (1-11),

$$H := -\Delta + \frac{V}{y^2}, \quad V = f'(Q) = \frac{y^4 - 6y^2 + 1}{(y^2 + 1)^2}. \tag{2-2}$$

Due to the scaling invariance, we have $H \Lambda Q = 0$, where

$$\Lambda Q = \frac{2y}{1 + y^2}. \tag{2-3}$$

However, ΛQ slightly fails to belong to $L^2(\mathbb{R}^2)$, so we call ΛQ the *resonance* of H. The positivity of ΛQ on $\mathbb{R}_+^*$ allows us to factorize H:

$$H = A^*A, \quad A = -\partial_y + \frac{Z}{y}, \quad A^* = \partial_y + \frac{1+Z}{y}, \quad Z(y) = \cos Q = \frac{1 - y^2}{1 + y^2}. \tag{2-4}$$

The above factorization facilitates examining the formal kernel of H on $\mathbb{R}_+^*$, denoted by $\mathrm{Ker}(H)$. More precisely, the equivalent form

$$Au = -\partial_y u + \partial_y(\log \Lambda Q)u = -\Lambda Q \partial_y\left(\frac{u}{\Lambda Q}\right), \tag{2-5}$$

$$A^*u = \frac{1}{y}\partial_y(yu) + \partial_y(\log \Lambda Q)u = \frac{1}{y\Lambda Q}\partial_y(uy\Lambda Q) \tag{2-6}$$

yields, for $y > 0$, $\mathrm{Ker}(H) = \mathrm{Span}(\Lambda Q, \Gamma)$, where

$$\Gamma(y) = \Lambda Q \int_1^y \frac{dx}{x(\Lambda Q(x))^2} = \begin{cases} O(1/y) & \text{as } y \to 0, \\ y/4 + O(\log y/y) & \text{as } y \to \infty. \end{cases} \tag{2-7}$$

From variation of parameters, we obtain the formal inverse of H:

$$H^{-1}f = \Lambda Q \int_0^y f\Gamma x \, dx - \Gamma \int_0^y f\Lambda Q x \, dx, \tag{2-8}$$

so the inverse of $\boldsymbol{H}$ is given by

$$\boldsymbol{H}^{-1} := \begin{pmatrix} 0 & H^{-1} \\ -1 & 0 \end{pmatrix}.$$

We remark that the inverse formula (2-8) is uniquely determined by the boundary condition at the origin: for any smooth function f with $f = O(1)$, we have $H^{-1}f = O(y^2)$ near the origin.

On the other hand, the supersymmetric conjugate operator $\widetilde{H}$ is given by

$$\widetilde{H} := AA^* = -\Delta + \frac{\widetilde{V}}{y^2}, \quad \widetilde{V}(y) = (1+Z)^2 - \Lambda Z = \frac{4}{y^2 + 1}. \tag{2-9}$$

We note that $\widetilde{H}$ has a repulsive property represented by its potential

$$\widetilde{V} = \frac{4}{y^2 + 1} > 0, \quad \Lambda \widetilde{V} = -\frac{8y^2}{(y^2 + 1)^2} \leq 0. \tag{2-10}$$

Based on the commutation relation

$$AH = \widetilde{H}A,$$

we can naturally define higher-order derivatives adapted to the linearized Hamiltonian H inductively:

$$f_0 := f, \quad f_{k+1} := \begin{cases} Af_k & \text{for } k \text{ even,} \\ A^* f_k & \text{for } k \text{ odd.} \end{cases} \tag{2-11}$$

For the sake of simplicity, we denote the corresponding operator as follows:

$$\mathcal{A} := A, \quad \mathcal{A}^2 := A^*A, \quad \mathcal{A}^3 := AA^*A, \quad \ldots, \quad \mathcal{A}^k := \underbrace{\cdots A^*AA^*A}_{k \text{ times}}. \tag{2-12}$$

We observe that f needs an odd parity condition near the origin to define f_k. More precisely, for any smooth function f, (2-5) implies

$$f_1 = Af \sim -y\partial_y(y^{-1}f) \tag{2-13}$$

near $y = 0$. Thus, f must degenerate near the origin as $f = cy + O(y^2)$, and so $Af = c'y + O(y^2)$. Here, the leading term $c'y$ comes from the cancellation

$$Ay = O(y^2), \tag{2-14}$$

which is a direct consequence of (2-13). However, f_2 does not degenerate near the origin like f, since A^* does not have any cancellation like (2-14). Hence, f should be more degenerate near the origin as $f = cy + c'y^3 + O(y^4)$. Furthermore, if f_k is to be well-defined for all $k \in \mathbb{N}$, f must satisfy the following condition: for all $p \in \mathbb{N}$, f has a Taylor expansion near the origin:

$$f(y) = \sum_{k=0}^{p} c_k y^{2k+1} + O(y^{2p+3}). \tag{2-15}$$

In Appendix A of [Raphaël and Schweyer 2014], it is proved that, for a well-localized smooth 1-corotational map $\Phi(r, \theta)$, the corresponding u is a smooth function that satisfies (2-15).

2.2. *Admissible functions.* As mentioned earlier, the leading dynamics of the blow-up are determined by the leading growth of tails from the blow-up profile. In the same way as [Collot 2018; Raphaël and Schweyer 2014], we first define an "admissible" vector-valued function characterized by three different indices, which represent a certain behavior near the origin and infinity, and the position of a nonzero coordinate.

Definition 2.1 (admissible functions). We say that a smooth vector-valued function $\boldsymbol{f} : \mathbb{R}_+ \to \mathbb{R}^2$ is admissible of degree $(p_1, p_2, \iota) \in \mathbb{N} \times \mathbb{Z} \times \{0, 1\}$ if it satisfies the following:

(i) $\boldsymbol{f}$ is situated on the $(\iota+1)$-th coordinate, i.e.,

$$\boldsymbol{f} = \begin{pmatrix} f \\ 0 \end{pmatrix} \quad \text{if } \iota = 0 \quad \text{and} \quad \boldsymbol{f} = \begin{pmatrix} 0 \\ f \end{pmatrix} \quad \text{if } \iota = 1. \tag{2-16}$$

In such cases, we use f and $\boldsymbol{f}$ interchangeably.

(ii) We can expand f near $y = 0$: for all $2p \geq p_1$,

$$f(y) = \sum_{\substack{k=p_1-\iota \\ k \text{ is even}}}^{2p} c_k y^{k+1} + O(y^{2p+3}), \tag{2-17}$$

and similar expansions hold after taking derivatives.

(iii) The adapted derivatives f_k have the following bounds: for all $k \geq 0$ and $y \geq 1$,

$$|f_k(y)| \lesssim y^{p_2-1-\iota-k}(1 + |\log y| \mathbf{1}_{p_2-k-\iota \geq 1}). \tag{2-18}$$

Remark. The logarithmic term in (2-18) comes from integrating y^{-1}.

From (2-3), we can easily check that $\Lambda \boldsymbol{Q} = (\Lambda Q, 0)^t$ is admissible of degree $(0, 0, 0)$. The next lemma says that admissible functions are designed to be compatible with the linearized operator $\boldsymbol{H}$.

Lemma 2.2 (action of $\boldsymbol{H}$ and $\boldsymbol{H}^{-1}$ on admissible functions). *Let $\boldsymbol{f}$ be an admissible function of degree* (p_1, p_2, ι). *Recall $\bar{i} = i \mod 2$. Then,*

(i) *for all $k \in \mathbb{N}$, $\boldsymbol{H}^k \boldsymbol{f}$ is admissible of degree*

$$(\max(p_1 - k, \iota),\ p_2 - k,\ \overline{\iota+k}), \tag{2-19}$$

(ii) *for all $k \in \mathbb{N}$ and $p_2 \geq \iota$, $\boldsymbol{H}^{-k} \boldsymbol{f}$ is admissible of degree*

$$(p_1 + k,\ p_2 + k,\ \overline{\iota+k}). \tag{2-20}$$

Proof. (i) This claim directly comes from the facts

$$\boldsymbol{H} = \begin{pmatrix} 0 & -1 \\ H & 0 \end{pmatrix}, \quad \boldsymbol{H}^2 = \begin{pmatrix} -H & 0 \\ 0 & -H \end{pmatrix}.$$

More precisely, the maximum choice, $\max(p_1 - k, \iota)$, appears from the cancellation (2-14) near the origin. Near infinity, the degree condition $p_2 - k$ is a consequence of the simple relation $Hf = f_2$.

(ii) It suffices to calculate the case $k = 1$ by induction. For $\iota = 0$,

$$\boldsymbol{H}^{-1} f = \begin{pmatrix} 0 & H^{-1} \\ -1 & 0 \end{pmatrix} \begin{pmatrix} f \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ -f \end{pmatrix},$$

and $\boldsymbol{H}^{-1} f$ is admissible of degree $(p_1 + 1, p_2 + 1, 1)$. For $\iota = 1$, we have

$$\boldsymbol{H}^{-1} f = \begin{pmatrix} 0 & H^{-1} \\ -1 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ f \end{pmatrix} = \begin{pmatrix} H^{-1} f \\ 0 \end{pmatrix}.$$

Instead of using the formal inverse formula (2-8) directly, we utilize the relation (2-6) as

$$A H^{-1} f = \frac{1}{y \Lambda Q} \int_0^y f \Lambda Q x \, dx, \tag{2-21}$$

and the relation (2-5) as

$$H^{-1} f = -\Lambda Q \int_0^y \frac{A H^{-1} f}{\Lambda Q} \, dx. \tag{2-22}$$

Near the origin, (2-21) gives the following expansion for $A H^{-1} f$:

$$A H^{-1} f = \sum_{\substack{k=p_1-1 \\ k \text{ is even}}}^{2p} \tilde{c}_k y^{k+2} + O(y^{2p+4}), \tag{2-23}$$

and thus $H^{-1} f$ satisfies the Taylor expansion

$$H^{-1} f = \sum_{\substack{k=p_1-1 \\ k \text{ is even}}}^{2p} \tilde{c}_k y^{k+3} + O(y^{2p+5}) = \sum_{\substack{k=p_1+1-0 \\ k \text{ is even}}}^{2p} \tilde{c}_k y^{k+1} + O(y^{2p+3}). \tag{2-24}$$

For $y \geq 1$, (2-21) and (2-22) imply

$$|A H^{-1} f| \lesssim \int_0^y |f| \, dx \lesssim \int_1^y x^{p_2-2} (1 + |\log x| \mathbf{1}_{p_2 \geq 2}) \, dx$$
$$\lesssim y^{(p_2+1)-1-0-1} (1 + |\log y| \mathbf{1}_{p_2 \geq 1}), \tag{2-25}$$

$$|H^{-1} f| \lesssim \frac{1}{y} \int_0^y |x A H^{-1} f| \, dx \lesssim \frac{1}{y} \int_1^y x^{p_2} (1 + |\log x| \mathbf{1}_{p_2 \geq 1}) \, dx$$
$$\lesssim y^{(p_2+1)-0-1} (1 + |\log y| \mathbf{1}_{p_2 \geq 0}), \tag{2-26}$$

and we obtain (2-18) for f and f_1. The higher derivative results come from $H(H^{-1} f) = f$. Hence $\boldsymbol{H}^{-1} f$ is admissible of degree $(p_1 + 1, p_2 + 1, 0)$. $\square$

Lemma 2.2 yields the presence of the admissible functions which generate the generalized null space of $\boldsymbol{H}$, which we now define formally.

Definition 2.3 (generalized kernel of $\boldsymbol{H}$). For each $i \geq 0$, we define an admissible function $\boldsymbol{T}_i$ of degree $(i, i, \bar{i})$ as

$$\boldsymbol{T}_i := (-\boldsymbol{H})^{-i} \boldsymbol{\Lambda} \boldsymbol{Q}. \tag{2-27}$$

Remark. By the definition of the admissible functions, we will use the notation T_i as a scalar function.

2.3. b_1-admissible functions. We will keep track of the logarithmic weight $|\log b_1|$ from the blow-up profiles to be constructed later. In this sense, the logarithmic loss of T_i hinders our analysis, so we settle this problem by introducing a new class of functions.

Definition 2.4 (b_1-admissible functions). We say that a smooth vector-valued function $\boldsymbol{f} : \mathbb{R}_+^* \times \mathbb{R}_+ \to \mathbb{R}^2$ is b_1-admissible of degree $(p_1, p_2, \iota) \in \mathbb{N} \times \mathbb{Z} \times \{0, 1\}$ if it satisfies the following:

(i) $\boldsymbol{f}$ is situated on the $(\iota+1)$-th coordinate (so we use f and $\boldsymbol{f}$ interchangeably).

(ii) $f = f(b_1, y)$ can be expressed as a finite sum of smooth functions of the form $h(b_1) \tilde{f}(y)$, where $\tilde{f}(y)$ has a Taylor expansion (2-17) and $h(b_1)$ satisfies,

$$\text{for all } l \geq 0, \quad \left| \frac{\partial^l h_j}{\partial b_1^l} \right| \lesssim \frac{1}{b_1^l}, \quad b_1 > 0. \tag{2-28}$$

(iii) f and its adapted derivatives f_k given by (2-11) have the following bounds: there exists a constant $c_{p_2} > 0$ such that, for all $k \geq 0$ and $y \geq 1$,

$$|f_k(b_1, y)| \lesssim y^{p_2 - k - 1 - \iota} \left(g_{p_2 - k - \iota}(b_1, y) + \frac{|\log y|^{c_{p_2}}}{y^2} + \frac{\mathbf{1}_{\{p_2 \geq k + 3 + \iota, y \geq 3B_0\}}}{y^2 b_1^2 |\log b_1|} \right), \tag{2-29}$$

and, for all $l \geq 1$,

$$\left| \frac{\partial^l}{\partial b_1^l} f_k(b_1, y) \right| \lesssim \frac{y^{p_2 - k - 1 - \iota}}{b_1^l |\log b_1|} \left(\tilde{g}_{p_2 - k - \iota}(b_1, y) + \frac{|\log y|^{c_{p_2}}}{y^2} + \frac{\mathbf{1}_{\{p_2 \geq k + 3 + \iota, y \geq 3B_0\}}}{y^2 b_1^2} \right), \tag{2-30}$$

where B_0 is given by (1-22) and $g_l, \tilde{g}_l$ are defined as

$$g_l(b_1, y) = \frac{1 + |\log(b_1 y)| \mathbf{1}_{\{l \geq 1\}}}{|\log b_1|} \mathbf{1}_{y \leq 3B_0}, \quad \tilde{g}_l(b_1, y) = \frac{1 + |\log y| \mathbf{1}_{\{l \geq 1\}}}{|\log b_1|} \mathbf{1}_{y \leq 3B_0}. \tag{2-31}$$

Remark. One may think that the asymptotics (2-29) and (2-30) are quite artificial, however, the functions $g_\ell(b_1, y)$ and $\tilde{g}_\ell(b_1, y)$ will appear in the cancellation by the radiation in Lemma 2.6. Then the indicator part $\mathbf{1}_{p_2 \geq k + 3 + \iota, y \geq 3B_0}$ comes from integrating g_ℓ in the region $1 \leq y \leq 3B_0$ to take $\boldsymbol{H}^{-1}$, which can be seen in more detail in the proof of the following lemma.

Lemma 2.5 (action of $\boldsymbol{H}$ and $\boldsymbol{H}^{-1}$ on b_1-admissible functions). *Let $\boldsymbol{f}$ be a b_1-admissible function of degree (p_1, p_2, ι). Then,*

(i) *for all $k \in \mathbb{N}$, $\boldsymbol{H}^k \boldsymbol{f}$ is b_1-admissible of degree*

$$(\max(p_1 - k, \iota), p_2 - k, \overline{\iota + k}), \tag{2-32}$$

(ii) *for all $k \in \mathbb{N}$ and $p_2 \geq \iota$, $\boldsymbol{H}^{-k} \boldsymbol{f}$ is b_1-admissible of degree*

$$(p_1 + k, p_2 + k, \overline{\iota + k}), \tag{2-33}$$

(iii) *the operators*

$$\Lambda : f \mapsto \Lambda f \quad \text{and} \quad b_1 \frac{\partial}{\partial b_1} : f \mapsto b_1 \frac{\partial f}{\partial b_1}$$

preserve the degree.

Proof. (i) We can borrow the proof of Lemma 2.2 since b_1 is independent of H.

(ii) Similar to the proof of Lemma 2.2, it suffices to consider the case $\iota = 1$ and $k = 1$. Near the origin, we still use (2-23) and (2-24) for $\tilde{f}$ from $h(b_1)\tilde{f}(y)$ in Definition 2.4.

However, for $y \geq 1$, we need a subtle calculation to integrate the terms containing g_l and $\tilde{g}_l$, defined in (2-31). More precisely, (2-25) implies, for $1 \leq y \leq 3B_0$,

$$
\begin{aligned}
|AH^{-1}f| &\lesssim \int_1^y x^{p_2-2} g_{p_2-1}(b_1, x) + x^{p_2-4}|\log x|^{c_{p_2}}\, dx \\
&\lesssim \int_1^y x^{p_2-2} \frac{1 + |\log(b_1 x)|\mathbf{1}_{\{p_2 \geq 2\}}}{|\log b_1|}\, dx + y^{p_2-3}|\log y|^{1+c_{p_2}} \\
&\lesssim \frac{1}{b_1^{p_2-1}|\log b_1|} \int_0^{b_1 y} x^{p_2-2}(1 + |\log x|\mathbf{1}_{\{p_2 \geq 2\}})\, dx + y^{p_2-3}|\log y|^{1+c_{p_2}} \\
&\lesssim y^{p_2-1} \frac{1 + |\log(b_1 y)|\mathbf{1}_{\{p_2 \geq 1\}}}{|\log b_1|} + y^{p_2-3}|\log y|^{1+c_{p_2}} \\
&= y^{(p_2+1)-1-1-0}\left(g_{(p_2+1)-1}(b_1, y) + \frac{|\log y|^{1+c_{p_2}}}{y^2}\right),
\end{aligned}
\tag{2-34}
$$

and, for $y \geq 3B_0$,

$$
\begin{aligned}
|AH^{-1}f| &\lesssim \int_1^y x^{p_2-2} g_{p_2-1}(b_1, x) + x^{p_2-4}|\log x|^{c_{p_2}} + \frac{x^{p_2-4}\mathbf{1}_{\{p_2 \geq 4, x \geq 3B_0\}}}{b_1^2|\log b_1|}\, dx \\
&\lesssim \frac{1}{b_1^{p_2-1}|\log b_1|} + \frac{y^{p_2-3}\mathbf{1}_{\{p_2 \geq 4\}}}{b_1^2|\log b_1|} + y^{p_2-3}|\log y|^{1+c_{p_2}} \\
&\lesssim y^{(p_2+1)-1-1-0}\left(\frac{\mathbf{1}_{\{p_2 \geq 1+3, y \geq 3B_0\}}}{y^2 b_1^2|\log b_1|} + \frac{|\log y|^{1+c_{p_2}}}{y^2}\right).
\end{aligned}
\tag{2-35}
$$

Once again, (2-26) and (2-34) yield, for $1 \leq y \leq 3B_0$,

$$
\begin{aligned}
|H^{-1}f| &\lesssim \frac{1}{y} \int_1^y x^{p_2} g_{p_2}(b_1, x) + x^{p_2-3}|\log x|^{1+c_{p_2}}\, dx \\
&= y^{(p_2+1)-1-0}\left(g_{p_2+1}(b_1, y) + \frac{|\log y|^{2+c_{p_2}}}{y^2}\right),
\end{aligned}
$$

and (2-35) implies, for $y \geq 3B_0$,

$$
\begin{aligned}
|H^{-1}f| &\lesssim \frac{1}{y} \int_1^y x^{p_2-2}|\log x|^{1+c_{p_2}} + \frac{x^{p_2-2}\mathbf{1}_{\{p_2 \geq 4, x \geq 3B_0\}}}{b_1^2|\log b_1|}\, dx \\
&\lesssim y^{(p_2+1)-1-0}\left(\frac{\mathbf{1}_{\{p_2 \geq 3, y \geq 3B_0\}}}{y^2 b_1^2|\log b_1|} + \frac{|\log y|^{2+c_{p_2}}}{y^2}\right),
\end{aligned}
$$

and we obtain (2-29) for f and f_1. The higher derivative results come from $H(H^{-1}f) = f$. We can easily prove (2-30) by replacing g_l with $\tilde{g}_l$ and dividing by $b_1^l|\log b_1|$. Hence $\boldsymbol{H}^{-1}\boldsymbol{f}$ is b_1-admissible of degree $(p_1 + 1, p_2 + 1, 0)$.

(iii) Note that

$$\mathbf{\Lambda} f = \begin{cases} (\Lambda f, 0)^t & \text{if } \iota = 0, \\ (0, \Lambda_0 f)^t & \text{if } \iota = 1, \end{cases}$$

and $\Lambda_0 f = f + \Lambda f$; therefore we get the desired result since Λ preserves the parity of f and its adapted derivative satisfies the bound

$$|(\Lambda f)_k| \lesssim |y f_{k+1}| + |f_k| + y^{p_2 - k - 3 - \iota}, \quad y \geq 1,$$

which was established in [Raphaël and Schweyer 2014].

Near the origin, the property of the operator $b_1(\partial/\partial b_1)$ comes from the fact that $b_1(\partial/\partial b_1)$ preserves the parity of f. For $y \geq 1$, (2-30) multiplied by b_1 with $l = 1$ is bounded by (2-29) with the bound

$$\frac{\tilde{g}_l(b_1, y)}{|\log b_1|} \lesssim g_l(b_1, y). \qquad \square$$

2.4. Control of the extra growth. The elements of the null space of $\boldsymbol{H}$, which was defined in (2-27), serve as a kind of tail in our blow-up profile. Since we basically plan a bubbling off of the blowup by scaling, the situation where the scaling generator $\boldsymbol{\Lambda}$ is taken by the tails $\boldsymbol{T}_i$ naturally emerges. Especially for $i \geq 2$, the leading asymptotics of $\boldsymbol{\Lambda} \boldsymbol{T}_i$ matches that of $(i-1)\boldsymbol{T}_i$ and determines the leading dynamical laws. However, the extra growth of $\boldsymbol{\Lambda} \boldsymbol{T}_i - (i-1)\boldsymbol{T}_i$ is inadequate to close our analysis. We will eliminate it by adding some radiations, which were first introduced in [Merle et al. 2013].

We now define the radiation situated on the first coordinate as follows: for small $b_1 > 0$,

$$\boldsymbol{\Sigma}_{b_1} = \begin{pmatrix} \Sigma_{b_1} \\ 0 \end{pmatrix}, \quad \Sigma_{b_1} = H^{-1}\{-c_{b_1} \chi_{B_0/4} \Lambda Q + d_{b_1} H[(1 - \chi_{B_0}) \Lambda Q]\}, \tag{2-36}$$

where

$$c_{b_1} = \frac{4}{\int \chi_{B_0/4}(\Lambda Q)^2} = \frac{1}{|\log b_1|} + O\left(\frac{1}{|\log b_1|^2}\right), \tag{2-37}$$

$$d_{b_1} = c_{b_1} \int_0^{B_0} \chi_{B_0/4} \Lambda Q \Gamma y \, dy = O\left(\frac{1}{b_1^2 |\log b_1|}\right). \tag{2-38}$$

From the inverse formula (2-8), we obtain the asymptotics near the origin and infinity:

$$\Sigma_{b_1} = \begin{cases} c_{b_1} T_2 & \text{for } y \leq \frac{1}{4} B_0, \\ 4\Gamma & \text{for } y \geq 3B_0. \end{cases} \tag{2-39}$$

To deal with $\boldsymbol{T}_1$, which is radiative itself, we further define

$$\tilde{c}_{b_1} := \frac{\langle \Lambda_0 \Lambda Q, \Lambda Q \rangle}{\langle \chi_{B_0/4} \Lambda Q, \Lambda Q \rangle} = \frac{1}{2|\log b_1|} + O\left(\frac{1}{|\log b_1|^2}\right). \tag{2-40}$$

Lemma 2.6 (cancellation by the radiation). *For $i \geq 1$, let $\boldsymbol{\Theta}_i$ be defined as*

$$\boldsymbol{\Theta}_1 := \boldsymbol{\Lambda} \boldsymbol{T}_1 - \tilde{c}_{b_1} \chi_{B_0/4} \boldsymbol{T}_1, \tag{2-41}$$

$$\text{for } i \geq 2, \quad \boldsymbol{\Theta}_i := \boldsymbol{\Lambda} \boldsymbol{T}_i - (i-1)\boldsymbol{T}_i - (-\boldsymbol{H})^{-i+2}\boldsymbol{\Sigma}_{b_1}, \tag{2-42}$$

where $\boldsymbol{T}_i$ is given by (2-27). Then $\boldsymbol{\Theta}_i$ is b_1-admissible of degree $(i, i, \bar{i})$.

Remark. As mentioned earlier, our radiation Σ_{b_1} cancels the extra growth of $\Lambda T_2 - T_2 \sim y$ from the asymptotics

$$T_2 = y \log y + cy + O\left(\frac{|\log y|^2}{y}\right), \quad \Lambda T_2 = y \log y + (c+1)y + O\left(\frac{|\log y|^2}{y}\right)$$

by 4Γ in (2-39). Since T_2 and Γ are elements of the generalized null space of H, the above cancellation holds for all Θ_i, $i \geq 2$.

Proof. <u>Step 1</u>: $i = 1$. Note that $\Theta_1 = (0, \Theta_1)^t$ and

$$\Theta_1 = \Lambda_0 \Lambda Q - \tilde{c}_{b_1} \Lambda Q \chi_{B_0/4},$$

and we have that Θ_1 is b_1-admissible of degree $(1, 1, 1)$ from the explicit formulae

$$\Lambda Q(y) = \frac{2y}{1 + y^2}, \quad \Lambda_0 \Lambda Q(y) = \frac{4y}{(1 + y^2)^2}$$

and the bounds, for $l \geq 1$,

$$\left|\frac{\partial^l c_{b_1}}{\partial b_1^l}\right| + \left|\frac{\partial^l \tilde{c}_{b_1}}{\partial b_1^l}\right| \lesssim \frac{1}{b_1^l |\log b_1|^2}, \quad \left|\frac{\partial^l d_{b_1}}{\partial b_1^l}\right| \lesssim \frac{1}{b_1^{l+2} |\log b_1|}, \quad \left|\frac{\partial^l \chi_{B_0}}{\partial b_1^l}\right| \lesssim \frac{\mathbf{1}_{y \sim B_0}}{b_1^l}. \tag{2-43}$$

<u>Step 2</u>: $i = 2$. Now, we use induction on $i \geq 2$. For $i = 2$, (2-39) and the admissibility of T_2 imply that Θ_2 satisfies the desired condition near zero (2-17) since

$$\Theta_2 = \begin{pmatrix} \Theta_2 \\ 0 \end{pmatrix} = \begin{pmatrix} \Lambda T_2 - T_2 - \Sigma_{b_1} \\ 0 \end{pmatrix}. \tag{2-44}$$

To exhibit the behavior near infinity, we deal with the cases $1 \leq y \leq 3B_0$ and $y \geq 3B_0$ separately. The inverse formula (2-8) yields, for $1 \leq y \leq 3B_0$,

$$\begin{aligned}
\Sigma_{b_1}(y) &= \Gamma \int_0^y c_{b_1} \chi_{B_0/4} (\Lambda Q)^2 x \, dx - \Lambda Q \int_0^y c_{b_1} \chi_{B_0/4} \Lambda Q \Gamma x \, dx + d_{b_1}(1 - \chi_{B_0}) \Lambda Q \\
&= y \frac{\int_0^y \chi_{B_0/4} (\Lambda Q)^2 x}{\int \chi_{B_0/4} (\Lambda Q)^2 x} + O\left(\frac{1 + y}{|\log b_1|}\right), \tag{2-45}
\end{aligned}$$

$$\begin{aligned}
\Theta_2(y) &= y + O\left(\frac{|\log y|^2}{y}\right) - y \frac{\int_0^y \chi_{B_0/4} (\Lambda Q)^2 x}{\int \chi_{B_0/4} (\Lambda Q)^2} + O\left(\frac{1 + y}{|\log b_1|}\right) \\
&= y \frac{\int_y^{B_0} \chi_{B_0/4} (\Lambda Q)^2 x}{\int \chi_{B_0/4} (\Lambda Q)^2} + O\left(\frac{1 + y}{|\log b_1|}\right) + O\left(\frac{|\log y|^2}{y}\right) \\
&= O\left(\frac{1 + y}{|\log b_1|}(1 + |\log(b_1 y)|)\right). \tag{2-46}
\end{aligned}$$

For $y \geq 3B_0$, (2-7) implies

$$\Sigma_{b_1}(y) = \Gamma \int_0^y c_{b_1} \chi_{B_0/4} (\Lambda Q)^2 x \, dx = y + O\left(\frac{\log y}{y}\right). \tag{2-47}$$

Hence, for $y \geq 1$, Θ_2 satisfies (2-29) for the case $k = 0$ as

$$|\Theta_2(y)| \lesssim y^{2-0-1-0} g_2(b_1, y) + y^{2-0-3-0}(\log y)^2. \tag{2-48}$$

The higher derivatives, namely f_k and $\partial^l f_k / \partial b_1^l$, can also be estimated by using (2-21), the bounds of the coefficients (2-37), (2-38), (2-43) and the commutator relation

$$A(\Lambda f) = Af + \Lambda Af - \frac{\Lambda Z}{y} f, \quad H(\Lambda f) = 2Hf + \Lambda Hf - \frac{\Lambda V}{y^2} f,$$

where Z and V are given by (2-2) and (2-4), respectively. Here, we can easily check that $\Lambda Z / y$ is an odd function and $\Lambda V / y^2$ is an even function. Furthermore, for $y \geq 1$,

$$\left| \frac{\partial^k}{\partial y^k} \left(\frac{\Lambda Z}{y} \right) \right| \lesssim \frac{1}{1 + y^{k+3}}, \quad \left| \frac{\partial^k}{\partial y^k} \left(\frac{\Lambda V}{y} \right) \right| \lesssim \frac{1}{1 + y^{k+4}}. \tag{2-49}$$

Therefore, Θ_2 is b_1-admissible of degree $(2, 2, 0)$.

<u>Step 3</u>: *Induction on i.* Suppose that Θ_i is b_1-admissible of degree $(i, i, \bar{i})$. For even i, we have that Θ_{i+1} is b_1-admissible of degree $(i + 1, i + 1, \overline{i + 1})$ since

$$\Theta_{i+1} = \begin{pmatrix} 0 \\ \Lambda_0 T_{i+1} - i T_{i+1} - (-H)^{-i/2+1} \Sigma_{b_1} \end{pmatrix} = \begin{pmatrix} 0 \\ \Lambda T_i - (i-1) T_i - (-H)^{-i/2+1} \Sigma_{b_1} \end{pmatrix} = \begin{pmatrix} 0 \\ \Theta_i \end{pmatrix}.$$

For odd i, we have

$$\boldsymbol{H} \boldsymbol{\Theta}_{i+1} = \begin{pmatrix} 0 & 1 \\ H & 0 \end{pmatrix} \begin{pmatrix} \Theta_{i+1} \\ 0 \end{pmatrix}$$

$$= \begin{pmatrix} 0 \\ H \Lambda T_{i+1} - i H T_{i+1} - H(-H)^{-(i+1)/2+1} \Sigma_{b_1} \end{pmatrix}$$

$$= \begin{pmatrix} 0 \\ \Lambda H T_{i+1} - (i-2) H T_{i+1} - y^{-2} \Lambda V T_{i+1} + (-H)^{-(i-1)/2+1} \Sigma_{b_1} \end{pmatrix}$$

$$= - \begin{pmatrix} 0 \\ \Lambda T_i - (i-2) T_i - (-H)^{-(i-1)/2+1} \Sigma_{b_1} + y^{-2} \Lambda V T_{i+1} \end{pmatrix}$$

$$= - \begin{pmatrix} 0 \\ \Lambda_0 T_i - (i-1) T_i - (-H)^{-(i-1)/2+1} \Sigma_{b_1} \end{pmatrix} + \begin{pmatrix} 0 \\ y^{-2} \Lambda V T_{i+1} \end{pmatrix}$$

$$= -\boldsymbol{\Theta}_i + \begin{pmatrix} 0 \\ y^{-2} \Lambda V T_{i+1} \end{pmatrix}.$$

The Taylor expansion condition (2-17) of $(0, y^{-2} \Lambda V T_{i+1})^t$ comes from the definition of $\boldsymbol{T}_i$ and the cancellation $\Lambda V = O(y^2)$ near $y = 0$.

For $y \geq 1$, (2-49) implies

$$\mathcal{A}^k \left(\frac{\Lambda V}{y^2} T_{i+1} \right) \lesssim \sum_{j=0}^k \frac{1}{y^{j+4}} y^{i-(k-j)} |\log y|^{c_i} \lesssim y^{i-3-k-1} |\log y|^{c_i}.$$

Hence $(0, y^{-2} \Lambda V T_{i+1})^t$ is b_1-admissible of degree $(i, i, 1)$; the desired result comes from Lemma 2.5. $\square$

2.5. *Adapted norms of b_1 admissible functions.* The next lemma yields some suitable norms corresponding to the adapted derivatives of b_1-admissible functions.

Lemma 2.7 (adapted norms of b_1-admissible function). *For $i \geq 1$, a b_1-admissible function f of degree $(i, i, \bar{i})$ has the following bounds*:

(i) *Global bounds*:

$$\|f_{k-\bar{i}}\|_{L^2(|y|\leq 2B_1)} \lesssim \begin{cases} b_1^{k-i}|\log b_1|^{\gamma(i-k-2)-1} & \text{if } k \leq i-3, \\ b_1^{k-i}/|\log b_1| & \text{if } k = i-2,\, i-1, \\ 1 & \text{if } k \geq i. \end{cases} \tag{2-50}$$

(ii) *Logarithmic weighted bounds*:

$$\sum_{k=0}^{m} \left\| \frac{1+|\log y|}{1+y^{m-k}} f_{k-\bar{i}} \right\|_{L^2(|y|\leq 2B_1)} \lesssim \begin{cases} b_1^{m-i}|\log b_1|^C & \text{for } m \leq i-1, \\ |\log b_1|^C & \text{for } m \geq i. \end{cases} \tag{2-51}$$

(iii) *Improved global bounds*:

$$\sum_{j=0}^{k-\bar{i}} \|y^{-(k-\bar{i}-j)} f_j\|_{L^2(y\sim B_1)} \lesssim b_1^{k-i}|\log b_1|^{\gamma(i-k-2)-1}. \tag{2-52}$$

Here, $B_1 = |\log b_1|^\gamma/b_1$ and $\gamma = 1 + \bar{\ell}$.

Remark. Due to the growth in (2-29), it is indispensable to restrict the integration domain by taking the L^2 norm. Later, we will attach a cutoff function χ_{B_1} to the profile modifications. Considering Leibniz's rule, the adapted derivative $\mathcal{A}^k$ can be taken on such modifications or the cutoff function. Then the global bounds (2-50) yield some estimates for the former case and (2-52) gives those for the latter case. The choice of cutoff region B_1 will be determined by the localization of our blow-up profile, which can be seen in more detail in Proposition 2.10.

Proof. (i) From (2-29), $f_{k-\bar{i}}$ satisfies the following estimate for $y \geq 2$:

$$|f_{k-\bar{i}}| \lesssim y^{i-k-1}\left(g_{i-k}(b_1, y) + \frac{|\log y|^{c_{p_2}}}{y^2} + \frac{\mathbf{1}_{\{i\geq k+3,\, y\geq 3B_0\}}}{y^2 b_1^2 |\log b_1|}\right).$$

Therefore, we obtain (2-50) for $i \geq k+1$:

$$\|f_{k-\bar{i}}\|_{L^2(|y|\leq 2B_1)} \lesssim \|\mathbf{1}_{|y|\leq 2}\|_{L^2} + \left\| y^{i-k-1}\frac{1+|\log(b_1 y)|}{|\log b_1|}\right\|_{L^2(2\leq|y|\leq 3B_0)}$$

$$+ \|y^{i-k-3}|\log y|^{c_i}\|_{L^2(2\leq|y|\leq 2B_1)} + \left\| \frac{y^{i-k-3}\mathbf{1}_{\{i\geq k+3\}}}{b_1^2|\log b_1|}\right\|_{L^2(3B_0\leq|y|\leq 2B_1)}$$

$$\lesssim 1 + \frac{b_1^{k-i}}{|\log b_1|} + b_1^{(k-i+2)\mathbf{1}_{\{i\geq k+2\}}}|\log b_1|^C + \frac{B_1^{i-k-2}}{b_1^2|\log b_1|}\mathbf{1}_{\{i\geq k+3\}}$$

$$\lesssim \frac{b_1^{k-i}}{|\log b_1|}|\log b_1|^{\gamma(i-k-2)\mathbf{1}_{\{i\geq k+3\}}},$$

and the case $i \leq k$ also holds similarly.

(ii) The logarithmic weighted bounds (2-51) are nothing but (2-50) multiplied by the logarithmic loss $|\log b_1|^C$ and then using the fact that $|\log y|/|\log b_1| \lesssim 1$ on $2 \le |y| \le 3B_0$.

(iii) We can prove (2-52) from a pointwise estimate in the region $y \sim B_1$:

$$|y^{-(k-\bar{i}-j)} f_j| \lesssim y^{i-k-3} \left(|\log y|^C + \frac{\mathbf{1}_{\{i \ge \bar{i}+j+3\}}}{b_1^2 |\log b_1|} \right) \lesssim \frac{y^{i-k-1}}{|\log b_1|^{2\gamma+1}}, \tag{2-53}$$

and the proof is complete. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ □

2.6. *Approximate blow-up profiles.* From now on, we fix

$$\ell \ge 2 \quad \text{and} \quad L = \ell + \overline{\ell+1}.$$

We construct the blow-up profiles based on the generalized kernels T_i. To be more specific, our blow-up scenario is created by bubbling off Q via scaling and adding $b_i T_i$; the evolution of λ is determined by the system of dynamical laws for $b = (b_1, \ldots, b_L)$. Here, we are faced with unnecessary growth caused by linear and nonlinear terms. To minimize this growth, we define the homogeneous functions, which do not affect the evolution of b (i.e., $b_i T_i$). We note that this kind of construction was introduced in [Raphaël and Schweyer 2014].

Definition 2.8 (homogeneous functions). Write $J = (J_1, \ldots, J_L)$ and $|J|_2 = \sum_{k=1}^L k J_k$. We say that a smooth vector-valued function $S(b, y) = S(b_1, \ldots, b_L, y)$ is homogeneous of degree $(p_1, p_2, \iota, p_3) \in \mathbb{N} \times \mathbb{Z} \times \{0, 1\} \times \mathbb{N}$ if it can be expressed as a finite sum of smooth functions of the form $\left(\prod_{i=1}^L b_i^{J_i} \right) S_J(y)$, where $S_J(y)$ is a b_1-admissible function of degree (p_1, p_2, ι) with $|J|_2 = p_3$.

Proposition 2.9 (construction of the approximate profile). *Given a large constant $M > 0$, there exists a small constant $0 < b^*(M) \ll 1$ such that a C^1 map*

$$b : s \mapsto (b_1(s), \ldots, b_L(s)) \in \mathbb{R}_+^* \times \mathbb{R}^{L-1}$$

verifies the existence of a slowly modulated profile Q_b given by

$$Q_b := Q + \alpha_b, \ \alpha_b := \sum_{i=1}^L b_i T_i + \sum_{i=2}^{L+2} S_i, \tag{2-54}$$

which drives the equation

$$\partial_s Q_b - F(Q_b) + b_1 \Lambda Q_b = \mathbf{Mod}(t) + \psi_b, \tag{2-55}$$

where $\mathbf{Mod}(t) := (\mathrm{Mod}(t), \dot{\mathrm{Mod}}(t))^t$ establishes the dynamical law of b:

$$\mathbf{Mod}(t) = \sum_{i=1}^L ((b_i)_s + (i - 1 + c_{b_1,i}) b_1 b_i - b_{i+1}) \left(T_i + \sum_{j=i+1}^{L+2} \frac{\partial S_j}{\partial b_i} \right), \tag{2-56}$$

where we set $b_{L+1} = 0$ for convenience and $c_{b_1,i}$ is defined by

$$c_{b_1,i} = \begin{cases} \tilde{c}_{b_1} = \dfrac{\langle \Lambda_0 \Lambda Q, \Lambda Q \rangle}{\langle \chi_{B_0/4} \Lambda Q, \Lambda Q \rangle} & \text{for } i = 1, \\[2ex] c_{b_1} = \dfrac{4}{\int \chi_{B_0/4} (\Lambda Q)^2} & \text{for } i \ne 1. \end{cases} \tag{2-57}$$

Here, $\boldsymbol{T}_i$ is given by (2-27) *and $\boldsymbol{S}_i$ is a homogeneous function of degree $(i, i, \bar{i}, i)$ satisfying*

$$S_1 = 0, \quad \frac{\partial \boldsymbol{S}_i}{\partial b_j} = 0 \quad \text{for } 2 \le i \le j \le L. \tag{2-58}$$

Moreover, the restrictions $|b_k| \lesssim b_1^k$ and $0 < b_1 < b^*(M)$ yield the estimates below for $\boldsymbol{\psi}_b = (\psi_b, \dot{\psi}_b)^t$:

(i) *Global bound: for $2 \le k \le L - 1$,*

$$\|\mathcal{A}^k \psi_b\|_{L^2(|y| \le 2B_1)} + \|\mathcal{A}^{k-1} \dot{\psi}_b\|_{L^2(|y| \le 2B_1)} \lesssim b_1^{k+1} |\log b_1|^C, \tag{2-59}$$

$$\|\mathcal{A}^L \psi_b\|_{L^2(|y| \le 2B_1)} + \|\mathcal{A}^{L-1} \dot{\psi}_b\|_{L^2(|y| \le 2B_1)} \lesssim \frac{b_1^{L+1}}{|\log b_1|^{1/2}}, \tag{2-60}$$

$$\|\mathcal{A}^{L+1} \psi_b\|_{L^2(|y| \le 2B_1)} + \|\mathcal{A}^L \dot{\psi}_b\|_{L^2(|y| \le 2B_1)} \lesssim \frac{b_1^{L+2}}{|\log b_1|}. \tag{2-61}$$

(ii) *Logarithmic weighted bound: for $m \ge 1$ and $0 \le k \le m$,*

$$\left\| \frac{1 + |\log y|}{1 + y^{m-k}} \mathcal{A}^k \psi_b \right\|_{L^2(|y| \le 2B_1)} \lesssim b_1^{m+1} |\log b_1|^C, \quad m \le L + 1, \tag{2-62}$$

$$\left\| \frac{1 + |\log y|}{1 + y^{m-k}} \mathcal{A}^k \dot{\psi}_b \right\|_{L^2(|y| \le 2B_1)} \lesssim b_1^{m+2} |\log b_1|^C, \quad m \le L. \tag{2-63}$$

(iii) *Improved local bound: for all $2 \le k \le L + 1$,*

$$\|\mathcal{A}^k \psi_b\|_{L^2(|y| \le 2M)} + \|\mathcal{A}^{k-1} \dot{\psi}_b\|_{L^2(|y| \le 2M)} \lesssim C(M) b_1^{L+3}. \tag{2-64}$$

Here, $B_0 = 1/b_1$ and $B_1 = |\log b_1|^\gamma / b_1$.

Remark. As can be seen in the following proof, the homogeneous profile $\boldsymbol{S}_i$ is eventually derived from the b_1-admissible function $\boldsymbol{\Theta}_{i-1}$ with some nonlinear effects.

Proof. <u>Step 1</u>: Linearization. We pull out the modulation law of b from linearizing the renormalized equation. Recall

$$\boldsymbol{F}(\boldsymbol{u}) := \begin{pmatrix} \dot{u} \\ \Delta u - f(u)/r^2 \end{pmatrix}.$$

Since $\boldsymbol{F}(\boldsymbol{Q}) = 0$, we have

$$\partial_s \boldsymbol{Q}_b + b_1 \boldsymbol{\Lambda} \boldsymbol{Q}_b - \boldsymbol{F}(\boldsymbol{Q}_b) = \partial_s \boldsymbol{\alpha}_b + b_1 \boldsymbol{\Lambda}(\boldsymbol{Q} + \boldsymbol{\alpha}_b) - (\boldsymbol{F}(\boldsymbol{Q} + \boldsymbol{\alpha}_b) - \boldsymbol{F}(\boldsymbol{Q}))$$

$$=: b_1 \boldsymbol{\Lambda} \boldsymbol{Q} + (\partial_s + b_1 \boldsymbol{\Lambda}) \boldsymbol{\alpha}_b + \boldsymbol{H} \boldsymbol{\alpha}_b + \boldsymbol{N}(\boldsymbol{\alpha}_b),$$

where $\boldsymbol{N}$ denotes the higher-order terms:

$$\boldsymbol{N}(\boldsymbol{\alpha}_b) := \frac{1}{y^2} \begin{pmatrix} 0 \\ f(Q + \alpha_b) - f(Q) - f'(Q)\alpha_b \end{pmatrix}, \quad \boldsymbol{\alpha}_b = \begin{pmatrix} \alpha_b \\ \dot{\alpha}_b \end{pmatrix}. \tag{2-65}$$

Note that

$$\partial_s \boldsymbol{\alpha}_b = \sum_{i=1}^{L} \left[(b_i)_s \boldsymbol{T}_i + \sum_{j=i+1}^{L+2} (b_i)_s \frac{\partial \boldsymbol{S}_j}{\partial b_i} \right] = \sum_{i=1}^{L} \left[(b_i)_s \boldsymbol{T}_i + \sum_{j=1}^{i-1} (b_j)_s \frac{\partial \boldsymbol{S}_i}{\partial b_j} \right] + \sum_{i=1}^{L} (b_i)_s \frac{\partial \boldsymbol{S}_{L+1}}{\partial b_i} + \sum_{i=1}^{L} (b_i)_s \frac{\partial \boldsymbol{S}_{L+2}}{\partial b_i}.$$

Rearranging the linear terms to the degree with respect to b_1 and using the fact $\boldsymbol{H}\boldsymbol{T}_{i+1} = -\boldsymbol{T}_i$ for $1 \leq i \leq L-1$,

$$b_1 \boldsymbol{\Lambda} \boldsymbol{Q} + (\partial_s + b_1 \boldsymbol{\Lambda})\boldsymbol{\alpha}_b + \boldsymbol{H}\boldsymbol{\alpha}_b$$

$$= \sum_{i=1}^{L}[(b_i)_s \boldsymbol{T}_i + b_1 b_i \boldsymbol{\Lambda}\boldsymbol{T}_i - b_{i+1}\boldsymbol{T}_i] + \sum_{i=1}^{L}\left[\boldsymbol{H}\boldsymbol{S}_{i+1} + b_1 \boldsymbol{\Lambda}\boldsymbol{S}_i + \sum_{j=1}^{i-1}(b_j)_s \frac{\partial \boldsymbol{S}_i}{\partial b_j}\right]$$

$$+ b_1 \boldsymbol{\Lambda}\boldsymbol{S}_{L+1} + \boldsymbol{H}\boldsymbol{S}_{L+2} + \sum_{i=1}^{L}(b_i)_s \frac{\partial \boldsymbol{S}_{L+1}}{\partial b_i} + b_1 \boldsymbol{\Lambda}\boldsymbol{S}_{L+2} + \sum_{i=1}^{L}(b_i)_s \frac{\partial \boldsymbol{S}_{L+2}}{\partial b_i}. \quad (2\text{-}66)$$

From Lemma 2.6,

$$(b_1)_s \boldsymbol{T}_1 + b_1^2 \boldsymbol{\Lambda}\boldsymbol{T}_1 - b_2 \boldsymbol{T}_1 = ((b_1)_s + b_1^2 \tilde{c}_{b_1} - b_2)\boldsymbol{T}_1 - b_1^2 \tilde{c}_{b_1}(1 - \chi_{B_0/4})\boldsymbol{T}_1 + b_1^2 \boldsymbol{\Theta}_1,$$

and, for $2 \leq i \leq L$,

$$(b_i)_s \boldsymbol{T}_i + b_1 b_i \boldsymbol{\Lambda}\boldsymbol{T}_i - b_{i+1}\boldsymbol{T}_i$$

$$= ((b_i)_s + (i-1+c_{b_1})b_1 b_i - b_{i+1})\boldsymbol{T}_i + b_1 b_i(-\boldsymbol{H})^{-i+2}(\boldsymbol{\Sigma}_{b_1} - c_{b_1}\boldsymbol{T}_2) + b_1 b_i \boldsymbol{\Theta}_i. \quad (2\text{-}67)$$

Hence, we can separate $\mathbf{Mod}(t)$ from the right-hand side of (2-66) to get the expression

$$\mathbf{Mod}(t) - b_1^2 \tilde{c}_{b_1}(1 - \chi_{B_0/4})\boldsymbol{T}_1 + \sum_{i=2}^{L} b_1 b_i(-\boldsymbol{H})^{-i+2}(\boldsymbol{\Sigma}_{b_1} - c_{b_1}\boldsymbol{T}_2)$$

$$+ \sum_{i=1}^{L}\left[\boldsymbol{H}\boldsymbol{S}_{i+1} + b_1 b_i \boldsymbol{\Theta}_i + b_1 \boldsymbol{\Lambda}\boldsymbol{S}_i - \sum_{j=1}^{i-1}((j-1+c_{b_1,j})b_1 b_j - b_{j+1})\frac{\partial \boldsymbol{S}_i}{\partial b_j}\right]$$

$$+ \boldsymbol{H}\boldsymbol{S}_{L+2} + b_1 \boldsymbol{\Lambda}\boldsymbol{S}_{L+1} - \sum_{i=1}^{L}((i-1+c_{b_1,i})b_1 b_i - b_{i+1})\frac{\partial \boldsymbol{S}_{L+1}}{\partial b_i}$$

$$+ b_1 \boldsymbol{\Lambda}\boldsymbol{S}_{L+2} - \sum_{i=1}^{L}((i-1+c_{b_1,i})b_1 b_i - b_{i+1})\frac{\partial \boldsymbol{S}_{L+2}}{\partial b_i}. \quad (2\text{-}68)$$

<u>Step 2</u>: Construction of $\boldsymbol{S}_i$. One can observe that the second and third lines of (2-68) provide the definition of the homogeneous profiles $\boldsymbol{S}_i$ inductively. We need to pull out the additional homogeneous functions from $\boldsymbol{N}(\boldsymbol{\alpha}_b) = (0, N(\alpha_b))^t$ via Taylor's theorem:

$$N(\alpha_b) = \frac{1}{y^2}\left\{\sum_{j=2}^{(L+1)/2} \frac{f^{(j)}(Q)}{j!}\alpha_b^j + N_0(\alpha_b)\alpha_b^{(L+3)/2}\right\},$$

where $N_0(\alpha_b)$ is the coefficient of the remainder term:

$$N_0(\alpha_b) = \frac{1}{((L+1)/2)!}\int_0^1 (1-\tau)^{(L+1)/2} f^{((L+3)/2)}(Q + \tau\alpha_b)\, d\tau.$$

Roughly speaking, $N_0(\alpha_b) = O(b_1^{L+3})$. We also rewrite the Taylor polynomial part of $N(\alpha_b)$ in terms of the degree of b_1: for the L-tuple $J := (J_2, J_4, \ldots, J_{L-1}, \tilde{J}_2, \tilde{J}_4, \ldots, \tilde{J}_{L+1})$,

$$\sum_{j=2}^{(L+1)/2} \frac{f^{(j)}(Q)}{j!}\alpha_b^j = \sum_{i=1}^{(L+1)/2} P_{2i} + R',$$

where

$$P_i := \sum_{j=2}^{(L+1)/2} \sum_{\substack{|J|_1=j \\ |J|_2=i}} c_{j,J} \prod_{k=1}^{(L-1)/2} (b_{2k}T_{2k})^{J_{2k}} \prod_{k=1}^{(L+1)/2} S_{2k}^{\tilde{J}_{2k}},$$

$$R' := \sum_{j=2}^{(L+1)/2} \sum_{\substack{|J|_1=j \\ |J|_2 \geq L+3}} c_{j,J} \prod_{k=1}^{(L-1)/2} (b_{2k}T_{2k})^{J_{2k}} \prod_{k=1}^{(L+1)/2} S_{2k}^{\tilde{J}_{2k}},$$

$$c_{j,J} = \frac{f^{(j)}(Q)}{\prod_{k=1}^{(L-1)/2} J_{2k}! \prod_{k=1}^{(L+1)/2} \tilde{J}_{2k}!},$$

with two distinct counting notations

$$|J|_1 := \sum_{k=1}^{(L-1)/2} J_{2k} + \sum_{k=1}^{(L+1)/2} \tilde{J}_{2k}, \quad |J|_2 := \sum_{k=1}^{(L-1)/2} 2k J_{2k} + \sum_{k=1}^{(L+1)/2} 2k \tilde{J}_{2k}.$$

In short, $P_{2i} = O(b_1^{2i})$ and $R' = O(b_1^{L+3})$. We collect all $O(b_1^{L+3})$ terms as follows:

$$R := N_0(\alpha_b)\alpha_b^{(L+3)/2} + R'. \tag{2-69}$$

We claim that $\boldsymbol{P}_{2i}/y^2 = (0, P_{2i}/y^2)$ is homogeneous of degree $(2i-1, 2i-1, 1, 2i)$ for $1 \leq i \leq \frac{1}{2}(L+1)$. The case $i = 1$ is trivial since $P_2 = 0$. For $2 \leq i \leq \frac{1}{2}(L+1)$, we recall that P_{2i}/y^2 is a linear combination of the following monomials: for $|J|_1 = j$, $|J|_2 = 2i$ and $2 \leq j \leq i$,

$$\frac{f^{(j)}(Q)}{y^2} \prod_{k=1}^{i} (b_{2k}T_{2k})^{J_{2k}} \prod_{k=1}^{i} S_{2k}^{\tilde{J}_{2k}}.$$

Near the origin, we observe that T_{2k} and S_{2k} are odd functions, and the parity of a function $f^{(j)}(Q)$ is determined by the parity of j, so each monomial is either an odd or even function. Hence it suffices to calculate the leading power of the Taylor expansion of each function constituting the monomial:

$$T_{2k} \sim y^{2k+1}, \quad S_{2k} \sim O(b_1^{2k})y^{2k+1} \quad \text{and} \quad f^{(j)}(Q) \sim y^{\bar{j}+1},$$

and the leading power of each monomial is given by

$$b_1^{\sum_{k=1}^{i} 2k J_{2k}} \cdot b_1^{\sum_{k=1}^{i} 2k \tilde{J}_{2k}} = b_1^{2i}, \tag{2-70}$$

$$y^{-2} y^{\bar{j}+1} y^{\sum_{k=1}^{i} (2k+1)J_{2k}} y^{\sum_{k=1}^{i} (2k+1)\tilde{J}_{2k}} = y^{2i+j-1-\bar{j}}.$$

Therefore, the Taylor expansion condition (2-17) comes from the fact that $j - 1 - \bar{j}$ is a positive odd integer when $j \geq 2$.

Similarly, for $y \geq 1$, we have that $|T_{2k}| \lesssim y^{2k-1}\log y$, $|S_{2k}| \lesssim b_1^{2k} y^{2k-1}$ and $|f^{(j)}(Q)| \lesssim y^{-1+\bar{j}}$ imply

$$\left| \frac{f^{(j)}(Q)}{y^2} \prod_{k=1}^{i} b_{2k}^{J_{2k}} T_{2k}^{J_{2k}} \prod_{k=1}^{i} S_{2k}^{\tilde{J}_{2k}} \right| \lesssim b_1^{2i} |y^{-3+\bar{j}}| \prod_{k=1}^{i} |y^{2k-1}\log y|^{J_{2k}} \prod_{k=1}^{i} |y^{2k-1}|^{\tilde{J}_{2k}}$$

$$\lesssim b_1^{2i} y^{2i-j-3+\bar{j}} |\log y|^C$$

$$\lesssim b_1^{2i} y^{2i-5} |\log y|^C \tag{2-71}$$

with the fact that $j - \bar{j} \geq 2$. We can easily estimate the higher derivatives of each monomial.

Under the setting $P_{2k+1} := (0,0)^t$ for $k \in \mathbb{N}$, we obtain the final definition of S_i: $S_1 := 0$ and, for $i = 1, \ldots, L+1$,

$$S_{i+1} := (-H)^{-1}\left(b_1 b_i \Theta_i + b_1 \Lambda S_i + \frac{P_{i+1}}{y^2} - \sum_{j=1}^{i-1}((j-1+c_{b_1,j})b_1 b_j - b_{j+1})\frac{\partial S_i}{\partial b_j}\right). \quad (2\text{-}72)$$

From the homogeneity of P_i/y^2 established above and Lemmas 2.5 and 2.6, we can prove that S_i is homogeneous of degree $(i, i, \bar{i}, i)$ for $1 \le i \le L+2$ with (2-58) via induction. To sum up, we get (2-55) by collecting remaining errors into ψ_b:

$$\psi_b := -b_1^2 \tilde{c}_{b_1}(1 - \chi_{B_0/4})T_1 + \sum_{i=2}^{L} b_1 b_i (-H)^{-i+2}\widetilde{\Sigma}_{b_1}$$
$$+ b_1 \Lambda S_{L+2} - \sum_{i=1}^{L}((i-1+c_{b_1,i})b_1 b_i - b_{i+1})\frac{\partial S_{L+2}}{\partial b_i} + \frac{R}{y^2}, \quad (2\text{-}73)$$

where $\widetilde{\Sigma}_{b_1} := \Sigma_{b_1} - c_{b_1}T_2$ and $R = (0, R)^t$ from (2-69).

Step 3: Error bounds. Now, it remains to prove the Sobolev bounds (2-59)–(2-64). We can treat the errors involving S_{L+2} in (2-73) easily. Since S_{L+2} is homogeneous of degree $(L+2, L+2, 1, L+2)$, Lemma 2.5 ensures that the functions containing S_{L+2} are homogeneous of degree $(L+2, L+2, 1, L+3)$, and thus the desired bounds come from Lemma 2.7.

The other errors require separate integration to conclude. We start with the first line of (2-73). Noting that $T_1 = (0, T_1)^t$ and $\Lambda Q \sim 1/y$ on $y \ge 1$, we have, for $k \ge 0$,

$$|\mathcal{A}^k(1 - \chi_{B_0/4})T_1| \lesssim y^{-(k+1)}\mathbf{1}_{y \ge B_0/4}, \quad (2\text{-}74)$$

which imply (2-59), (2-60) and (2-61): for $2 \le k \le L+1$,

$$\|b_1^2 \tilde{c}_{b_1} \mathcal{A}^{k-1}(1 - \chi_{B_0/4})T_1\|_{L^2(|y| \le 2B_1)} \lesssim \frac{b_1^2}{|\log b_1|}\|y^{-k}\|_{L^2(B_0/4 \le |y| \le 2B_1)} \lesssim \frac{b_1^{k+1}}{|\log b_1|}. \quad (2\text{-}75)$$

For $2 \le i \le L$, we rewrite
$$(-H)^{i+2}\widetilde{\Sigma}_{b_1} = \begin{cases} ((-H)^{-i/2+1}\widetilde{\Sigma}_{b_1}, 0)^t & \text{for even } i, \\ (0, -(-H)^{-(i-1)/2+1}\widetilde{\Sigma}_{b_1})^t & \text{for odd } i \end{cases} \quad (2\text{-}76)$$

using the fact $H^{-2} = -H^{-1}$. Moreover, $\text{supp}(\widetilde{\Sigma}_{b_1}) \subset \{|y| \ge \frac{1}{4}B_0\}$ and, for $k \ge 0$, we have the following crude bound: for $\frac{1}{4}B_0 \le y \le 2B_1$,

$$|\mathcal{A}^{k-\bar{i}}H^{-(i-\bar{i})/2+1}\widetilde{\Sigma}_{b_1}| \lesssim y^{i-k-1}\frac{|\log y|}{|\log b_1|} \lesssim y^{i-k-1}. \quad (2\text{-}77)$$

Hence, for $1 \le k < i \le L$, we obtain (2-59) from the estimation

$$\|b_1 b_i \mathcal{A}^{k-\bar{i}}H^{-(i-\bar{i})/2+1}\widetilde{\Sigma}_{b_1}\|_{L^2(|y| \le 2B_1)} \lesssim b_1^{i+1}\|y^{i-k-1}\|_{L^2(B_0/4 \le |y| \le 2B_1)} \lesssim b_1^{k+1}|\log b_1|^{\gamma(i-k)}. \quad (2\text{-}78)$$

We also observe, for $k \ge i$,
$$\mathcal{A}^{k-\bar{i}}H^{-(i-\bar{i})/2+1}\widetilde{\Sigma}_{b_1} = \mathcal{A}^{k-i}H\widetilde{\Sigma}_{b_1}, \quad (2\text{-}79)$$

and together with the sharp bounds

$$|H\widetilde{\Sigma}_{b_1}| \lesssim \frac{\mathbf{1}_{y \geq B_0/4}}{|\log b_1|}\frac{1}{y}, \quad |\mathcal{A}^j H\widetilde{\Sigma}_{b_1}| \lesssim \frac{\mathbf{1}_{y \sim B_0}}{B_0^{j+1}|\log b_1|}, \quad j \geq 1, \tag{2-80}$$

this implies (2-59), (2-60) and (2-61):

$$\|b_1 b_i \mathcal{A}^{k-i} H\widetilde{\Sigma}_{b_1}\|_{L^2(|y| \leq 2B_1)} \lesssim \frac{b_1^{i+1}}{|\log b_1|}\|y^{i-k-1}\|_{L^2(B_0/4 \leq |y| \leq 2B_1)} \lesssim \frac{b_1^{k+1}}{|\log b_1|^{1/2}},$$

$$\|b_1 b_i \mathcal{A}^{L+1-i} H\widetilde{\Sigma}_{b_1}\|_{L^2(|y| \leq 2B_1)} \lesssim \frac{b_1^{i+1}}{B_0^{L+1-i}|\log b_1|} \lesssim \frac{b_1^{L+2}}{|\log b_1|}.$$

The logarithmic weighted bounds (2-62) and (2-63) come from the above estimation with the trivial bound $|\log y/\log b_1| \lesssim 1$ on $\frac{1}{4}B_0 \leq y \leq 2B_1$ and the fact that the errors in the first line of (2-73) are supported in $y \geq \frac{1}{4}B_0$. This support property also yields the improved local bound (2-64) by choosing $b^*(M)$ small enough.

Now, we move to the last error: $\mathbf{R}/y^2$. Recalling (2-69), we observe that $\mathbf{R}/y^2 = (0, R/y^2)$ has two parts: a sum of monomials like P_{2i}/y^2 and nonlinear terms

$$\frac{1}{y^2}N_0(\alpha_b)\alpha_b^{(L+3)/2}.$$

For the monomial part, we borrow the calculation of P_{2i}/y^2: (2-70), (2-71). Under the range $|J|_1 = j$, $|J|_2 \geq L+3$, $2 \leq j \leq \frac{1}{2}(L+1)$, those k-th suitable derivatives (i.e., $\mathcal{A}^k$) have the pointwise bounds

$$\begin{cases} b_1^{L+3} & \text{for } y \leq 1, \\ b_1^{|J|_2} y^{|J|_2-k-5}|\log y|^C & \text{for } 1 \leq y \leq 2B_1, \end{cases} \tag{2-81}$$

and we simply obtain the bounds (2-59)–(2-64) via integrating the above bound. It remains to estimate the nonlinear term. For $y \leq 1$, we utilize the parity of $f^{((L+3)/2)}(Q)$ and α_b. We already know that α_b is an odd function with the leading term $O(b_1^2)y^3$, the parity of $f^{((L+3)/2)}(Q)$ is opposite that of $\frac{1}{2}(L+3)$, and $N_0(\alpha_b)\alpha_b^{(L+3)/2}/y^2$ is an odd function with the leading term $O(b_1^{L+3})y^{3(L+3)/2-1-\overline{(L+3)/2}}$. Hence, for $1 \leq k \leq L$,

$$\left\|\mathcal{A}^k\left(\frac{N_0(\alpha_b)}{y^2}\alpha_b^{(L+3)/2}\right)\right\|_{L^\infty(y \leq 1)} \lesssim b_1^{L+3}.$$

For $1 \leq y \leq 2B_1$, the simple bound

$$|\partial_y^k(Q + \tau\alpha_b)| \lesssim \frac{|\log b_1|^C}{y^{k+1}}, \quad k \geq 1,$$

implies

$$|N_0(\alpha_b)| \lesssim 1, \quad |\partial_y^k N_0(\alpha_b)| \lesssim \frac{|\log b_1|^C}{y^{k+1}} \text{ for } k \geq 1.$$

From the Leibniz rule and the crude bound $|\partial_y^k \alpha_b| \lesssim b_1^2|\log b_1|y^{1-k}$, we have

$$\left|\mathcal{A}^k\left(\frac{N_0(\alpha_b)}{y^2}\alpha_b^{(L+3)/2}\right)\right| \lesssim \sum_{j=0}^k \frac{|\partial_y^j(N_0(\alpha_b)\alpha_b^{(L+3)/2})|}{y^{2+k-j}} \lesssim b_1^{L+3}|\log b_1|^C y^{(L+3)/2-2-k} \tag{2-82}$$

for $0 \leq k \leq L$, and the above pointwise bound yields (2-59)–(2-64) via integration. $\qquad\square$

2.7. *Localization of the approximate profile.* In the previous construction, we observe that the blow-up profile does not approximate the solution of (2-55) on the region $y \geq 2B_1$. Hence it is necessary to cut off the overgrowth of each tail.

Proposition 2.10 (localization of the approximate profile). *Assume the hypotheses of Proposition 2.9, and assume moreover the a priori bounds*

$$|(b_1)_s| \lesssim b_1^2, \quad |b_L| \lesssim \frac{b_1^L}{|\log b_1|} \quad \text{when } \ell = L - 1. \tag{2-83}$$

Then the localized profile $\widetilde{Q}_b$ given by

$$\widetilde{Q}_b = Q + \chi_{B_1} \alpha_b \tag{2-84}$$

drives the equation

$$\partial_s \widetilde{Q}_b - F(\widetilde{Q}_b) + b_1 \Lambda \widetilde{Q}_b = \chi_{B_1} \mathbf{Mod}(t) + \tilde{\psi}_b, \tag{2-85}$$

where $\mathbf{Mod}(t)$ was defined in (2-56) and $\tilde{\psi}_b = (\tilde{\psi}_b, \dot{\tilde{\psi}}_b)^t$ satisfies the following bounds:

(i) *Global bound: for all $2 \leq k \leq L - 1$,*

$$\|\mathcal{A}^k \tilde{\psi}_b\|_{L^2} + \|\mathcal{A}^{k-1} \dot{\tilde{\psi}}_b\|_{L^2} \lesssim b_1^{k+1} |\log b_1|^C, \tag{2-86}$$

$$\|\mathcal{A}^L \tilde{\psi}_b\|_{L^2} + \|\mathcal{A}^{L-1} \dot{\tilde{\psi}}_b\|_{L^2} \lesssim b_1^{L+1} |\log b_1|, \tag{2-87}$$

$$\|\mathcal{A}^{L+1} \tilde{\psi}_b\|_{L^2} + \|\mathcal{A}^L \dot{\tilde{\psi}}_b\|_{L^2} \lesssim \frac{b_1^{L+2}}{|\log b_1|}. \tag{2-88}$$

(ii) *Logarithmic weighted bound: for $m \geq 1$ and $0 \leq k \leq m$,*

$$\left\| \frac{1 + |\log y|}{1 + y^{m-k}} \mathcal{A}^k \tilde{\psi}_b \right\|_{L^2} \lesssim b_1^{m+1} |\log b_1|^C, \quad m \leq L + 1, \tag{2-89}$$

$$\left\| \frac{1 + |\log y|}{1 + y^{m-k}} \mathcal{A}^k \dot{\tilde{\psi}}_b \right\|_{L^2} \lesssim b_1^{m+2} |\log b_1|^C, \quad m \leq L. \tag{2-90}$$

(iii) *Improved local bound: for all $2 \leq k \leq L + 1$,*

$$\|\mathcal{A}^k \tilde{\psi}_b\|_{L^2(|y| \leq 2M)} + \|\mathcal{A}^{k-1} \dot{\tilde{\psi}}_b\|_{L^2(|y| \leq 2M)} \lesssim C(M) b_1^{L+3}. \tag{2-91}$$

Remark. This proposition says that our cutoff function χ_{B_1} does not affect the estimates (2-59)–(2-64) in Proposition 2.9. Although such bounds came from integrating over the region $|y| \leq 2B_1$, there are two main reasons why this is possible. First, we do not need to keep track of the logarithmic weight $|\log b_1|$ except for (2-61) corresponding to the highest-order derivative. Second, (2-61) was derived from the sharp pointwise bound (2-80), which only depends on B_0. Thus, $B_1 = |\log b_1|^\gamma / b_1$ just needs to be large enough to obtain (2-88) by increasing γ.

Proof. Noting that $\tilde{\psi}_b = \psi_b$ on $|y| \leq B_1$, we see that (2-64) directly implies the local bound (2-91). For the other estimates, we will prove the global bounds (2-86) and (2-88) first, and the less demanding

logarithmic weighted bounds (2-89) and (2-90) later. By a straightforward calculation, $\tilde{\boldsymbol{\psi}}_b$ is given by

$$\tilde{\boldsymbol{\psi}}_b = \chi_{B_1}\boldsymbol{\psi}_b + (\partial_s(\chi_{B_1}) + b_1(y\chi')_{B_1})\boldsymbol{\alpha}_b + b_1(1 - \chi_{B_1})\boldsymbol{\Lambda}\boldsymbol{Q}$$
$$- \begin{pmatrix} 0 \\ \Delta(\chi_{B_1}\alpha_b) - \chi_{B_1}\Delta(\alpha_b) \end{pmatrix} - \frac{1}{y^2}\begin{pmatrix} 0 \\ f(\tilde{Q}_b) - f(Q) - \chi_{B_1}(f(Q_b) - f(Q)) \end{pmatrix}. \quad (2\text{-}92)$$

Before we estimate $\chi_{B_1}\boldsymbol{\psi}_b$ in (2-92), we introduce a useful asymptotics of cutoff:

$$\mathcal{A}^k(\chi_{B_1}f) = \chi_{B_1}\mathcal{A}^k f + \mathbf{1}_{y\sim B_1}\sum_{j=0}^{k-1}O(y^{-(k-j)})\mathcal{A}^j f. \quad (2\text{-}93)$$

Applying the above asymptotics to $\chi_{B_1}\boldsymbol{\psi}_b$, we get from Proposition 2.9 that we only need to estimate the errors localized in $y \sim B_1$. From (2-53), (2-74), (2-77), (2-81) and (2-82), we obtain the following pointwise bounds: for $y \sim B_1$ and $0 \le j \le k$,

$$|y^{-(k-j)}\mathcal{A}^j\psi_{b_1}| \lesssim \sum_{i=1}^{(L-1)/2}b_1^{2i+1}y^{2i-k-1} \lesssim b_1^{k+1}|\log b_1|^{\gamma(L-1-k)}B_1^{-1} \quad (2\text{-}94)$$

and

$$|y^{-(k-1-j)}\mathcal{A}^j\dot{\psi}_{b_1}| \lesssim \sum_{i=1}^{(L+1)/2}b_1^{2i}y^{2i-k-2} + \frac{b_1^{L+3}y^{L+1-k}}{|\log b_1|^{2\gamma+1}} + (b_1^{k+4} + b_1^{(L+3)/2+k+1})|\log b_1|^C$$
$$\lesssim b_1^{k+1}|\log b_1|^{\gamma(L-k)}B_1^{-1}.$$

These pointwise bounds directly imply the global bounds (2-86), (2-87) and (2-88) if we choose $\gamma \ge 1$.

For the second term in the right-hand side of (2-92), we recall

$$\boldsymbol{\alpha}_b = \begin{pmatrix} \alpha_b \\ \dot{\alpha}_b \end{pmatrix} = \begin{pmatrix} \sum_{i=1,\text{even}}^{L}b_i T_i + \sum_{i=2,\text{even}}^{L+2}S_i \\ \sum_{i=1,\text{odd}}^{L}b_i T_i + \sum_{i=2,\text{odd}}^{L+2}S_i \end{pmatrix}.$$

From the a priori bound $|b_{1,s}| \lesssim b_1^2$,

$$|\partial_s(\chi_{B_1}) + b_1(y\chi')_{B_1}| \lesssim \left(\frac{|b_{1,s}|}{b_1} + b_1\right)|(y\chi')_{B_1}| \lesssim b_1\mathbf{1}_{y\sim B_1}. \quad (2\text{-}95)$$

One can easily check that (2-93) still holds even if we replace the cutoff function χ_{B_1} with other cutoff functions supported in $y \sim B_1$. Hence the cutoff asymptotics (2-93) and the admissibility of T_i imply, for $1 \le i \le L$,

$$\|b_i\mathcal{A}^{k-\bar{i}}(\partial_s(\chi_{B_1}) + b_1(y\chi')_{B_1})T_i\|_{L^2} \lesssim \sum_{j=0}^{k-\bar{i}}b_1|b_i|\|y^{-(k-j-\bar{i})}\mathcal{A}^j T_i\|_{L^2(y\sim B_1)} \lesssim b_1|b_i|\|y^{i-k-1}|\log y|\|_{L^2(y\sim B_1)}$$
$$\lesssim b_1^{k+1-i}|b_i||\log b_1|^{\gamma(i-k)+1}, \quad (2\text{-}96)$$

and, for $2 \le i \le L+2$, Lemma 2.7 implies

$$\|\mathcal{A}^{k-\bar{i}}(\partial_s(\chi_{B_1}) + b_1(y\chi')_{B_1})S_i\|_{L^2} \lesssim b_1\sum_{j=0}^{k-\bar{i}}\|y^{-(k-j-\bar{i})}\mathcal{A}^j S_i\|_{L^2(y\sim B_1)} \lesssim b_1^{k+1}|\log b_1|^{\gamma(i-k-2)-1}, \quad (2\text{-}97)$$

so we obtain the global bounds (2-86) and (2-87). In (2-96), we cannot cancel $\log y$ from T_i: the additional $|\log b_1|$ appears. Thus, we need to choose $\gamma = 1 + \bar{\ell}$ for the case $(k, i) = (L + 1, L)$, which corresponds to (2-88). We note that $\gamma = 1$ when $\ell = L - 1$ since we have the additional $|\log b_1|$ gain of b_L from (2-83).

The third term in the right-hand side of (2-92) can be estimated as

$$\|b_1 \mathcal{A}^k (1 - \chi_{B_1}) \Lambda Q\|_{L^2} \lesssim b_1 \|y^{-k-1}\|_{L^2(y \geq B_1)} \lesssim \frac{b_1^{k+1}}{|\log b_1|^{\gamma k}}.$$

Finally, we compute (2-92):

$$\Delta(\chi_{B_1} \alpha_b) - \chi_{B_1} \Delta(\alpha_b) = (\Delta \chi_{B_1}) \alpha_b + 2 \partial_y (\chi_{B_1}) \partial_y (\alpha_b),$$

$$f(\widetilde{Q}_b) - f(Q) - \chi_{B_1}(f(Q_b) - f(Q)) = \chi_{B_1} \alpha_b \int_0^1 [f'(Q + \tau \chi_{B_1} \alpha_b) - f'(Q + \tau \alpha_b)] \, d\tau,$$

and we can easily check that each term is localized in $y \sim B_1$. In this region, the rough bounds

$$|f^{(k)}| \lesssim 1 \quad \text{and} \quad |\partial_y^k Q| + |\partial_y^k \chi_{B_1}| \lesssim y^{-k}$$

yield

$$\left| \frac{\partial^k}{\partial y^k} \left(\Delta(\chi_{B_1} \alpha_b) - \chi_{B_1} \Delta(\alpha_b) + \frac{f(\widetilde{Q}_b) - f(Q) - \chi_{B_1}(f(Q_b) - f(Q))}{y^2} \right) \right| \lesssim \frac{|\alpha_b|}{y^{k+2}},$$

and we can borrow the estimation of $\partial_s (\chi_{B_1}) \alpha_b$, namely (2-96) and (2-97).

The logarithmic weighted bounds (2-89) and (2-90) basically come from the fact that $|\log y| \sim |\log b_1|$ on $y \sim B_1$. We further use the decay property $|\log y|^C / y \to 0$ as $y \to \infty$ for the third term in the right-hand side of (2-92). $\qquad \square$

We also introduce another localization that depends on ℓ to verify the further regularity found in the remark after Theorem 1.2 on page 2419.

Proposition 2.11 (localization for the case when $\ell = L$). *Assume the hypotheses of Proposition 2.10. Then the localized profile $\widehat{Q}_b$ given by*

$$\widehat{Q}_b = \widetilde{Q}_b + \zeta_b := \widetilde{Q}_b + (\chi_{B_0} - \chi_{B_1}) b_L T_L \tag{2-98}$$

drives the equation

$$\partial_s \widehat{Q}_b - F(\widehat{Q}_b) + b_1 \Lambda \widehat{Q}_b = \widehat{\mathrm{Mod}}(t) + \hat{\psi}_b, \tag{2-99}$$

where $\widehat{\mathrm{Mod}}(t)$ is given by

$$\widehat{\mathrm{Mod}}(t) = \chi_{B_1} \mathrm{Mod}(t) + (\chi_{B_0} - \chi_{B_1})((b_L)_s + (L - 1 + c_{b,L}) b_1 b_L) T_L \tag{2-100}$$

and $\hat{\psi}_b = (\hat{\psi}_b, \dot{\hat{\psi}}_b)^t$ satisfies the bounds

$$\|\mathcal{A}^L (\hat{\psi}_b - (\chi_{B_1} - \chi_{B_0}) b_L T_{L-1})\|_{L^2} \lesssim b_1^{L+1}, \tag{2-101}$$

$$\|\mathcal{A}^{L-1}(\dot{\hat{\psi}}_b - (\partial_s \chi_{B_0} + b_1 (y \chi')_{B_0}) b_L T_L)\|_{L^2} \lesssim b_1^{L+1}. \tag{2-102}$$

Proof. Note that $F(\widetilde{Q}_b + \zeta_b) - F(\widetilde{Q}_b) = (\chi_{B_0} - \chi_{B_1})b_L T_{L-1}$. From (2-67) and (2-56), we have

$$\partial_s \widehat{Q}_b - F(\widehat{Q}_b) + b_1 \Lambda \widehat{Q}_b = \chi_{B_1} \mathbf{Mod}(t) + \tilde{\psi}_b + \partial_s \zeta_b - (F(\widetilde{Q}_b + \zeta_b) - F(\widetilde{Q}_b)) + b_1 \Lambda \zeta_b$$
$$= \widehat{\mathbf{Mod}}(t) + b_1 b_L (\chi_{B_0} - \chi_{B_1})\{(-H)^{L+2}\widetilde{\Sigma}_{b_1} + \theta_L\}$$
$$+ \tilde{\psi}_b - (\partial_s(\chi_{B_1}) + b_1(y\chi')_{B_1})b_L T_L$$
$$+ (\partial_s(\chi_{B_0}) + b_1(y\chi')_{B_0})b_L T_L + (\chi_{B_1} - \chi_{B_0})b_L T_{L-1}. \quad (2\text{-}103)$$

From the above identity, we can see that the last line of (2-103) is exactly subtracted from $\hat{\psi}_b$ in (2-101) and (2-102). Hence we need to estimate the second term and second line of the right-hand side of (2-103). We point out that the logarithm weight $|\log b_1|$ in (2-87) comes from the estimate (2-96) when $i = L$, which is eliminated in the second line of the right-hand side of (2-103). For the second term of the right-hand side of (2-103), we can borrow the bound (2-80) and use Lemma 2.7. $\qquad\square$

Proposition 2.12 (localization for the case when $\ell = L - 1$). *Assume the hypotheses of Proposition 2.10. Then the localized profile $\widehat{Q}_b$ given by*

$$\widehat{Q}_b = \widetilde{Q}_b + \zeta_b := \widetilde{Q}_b + (\chi_{B_0} - \chi_{B_1})(b_{L-1} T_{L-1} + b_L T_L) \quad (2\text{-}104)$$

drives the equation

$$\partial_s \widehat{Q}_b - F(\widehat{Q}_b) + b_1 \Lambda \widehat{Q}_b = \widehat{\mathbf{Mod}}(t) + \hat{\psi}_b, \quad (2\text{-}105)$$

where $\widehat{\mathbf{Mod}}(t)$ is given by

$$\widehat{\mathbf{Mod}}(t) = \chi_{B_1}\mathbf{Mod}(t) + (\chi_{B_0} - \chi_{B_1})((b_{L-1})_s + (L - 2 + c_{b,L-1})b_1 b_{L-1})T_{L-1}$$
$$+ (\chi_{B_0} - \chi_{B_1})((b_L)_s + (L - 1 + c_{b,L})b_1 b_L)T_L \quad (2\text{-}106)$$

and $\hat{\psi}_b = (\hat{\psi}_b, \dot{\hat{\psi}}_b)^t$ satisfies the bounds

$$\|\mathcal{A}^{L-1}(\hat{\psi}_b - (\partial_s \chi_{B_0} + b_1(y\chi')_{B_0})b_{L-1} T_{L-1} - (\chi_{B_1} - \chi_{B_0})b_L T_{L-1})\|_{L^2} \lesssim b_1^L, \quad (2\text{-}107)$$

$$\|\mathcal{A}^{L-2}(\dot{\hat{\psi}}_b - (\partial_s \chi_{B_0} + b_1(y\chi')_{B_0})b_L T_L + b_{L-1}H(\chi_{B_1} - \chi_{B_0})T_L)\|_{L^2} \lesssim b_1^L. \quad (2\text{-}108)$$

Remark. We point out that Propositions 2.11 and 2.12 provide improved bounds (2-101), (2-102), (2-107) and (2-108) compared to (2-86) and (2-87) in Proposition 2.10. These improved bounds will be essential to prove the monotonicity formula (4-12) later.

Proof. Note that

$$F(\widetilde{Q}_b + \zeta_b) - F(\widetilde{Q}_b) = -H\zeta_b - NL(\zeta_b) - L(\zeta_b), \quad (2\text{-}109)$$

where

$$NL(\zeta_b) = \begin{pmatrix} 0 \\ NL(\zeta_b) \end{pmatrix} := \frac{1}{y^2} \begin{pmatrix} 0 \\ f(\widetilde{Q}_b + \zeta_b) - f(\widetilde{Q}_b) - f'(\widetilde{Q}_b)\zeta_b \end{pmatrix}, \quad (2\text{-}110)$$

$$L(\zeta_b) = \begin{pmatrix} 0 \\ L(\zeta_b) \end{pmatrix} := \frac{1}{y^2} \begin{pmatrix} 0 \\ (f'(\widetilde{Q}_b) - f'(Q))\zeta_b \end{pmatrix}. \quad (2\text{-}111)$$

From (2-67) and (2-56), we have

$$
\begin{aligned}
\partial_s \widehat{\boldsymbol{Q}}_b - F(\widehat{\boldsymbol{Q}}_b) + b_1 \Lambda \widehat{\boldsymbol{Q}}_b &= \chi_{B_1} \mathbf{Mod}(t) + \tilde{\boldsymbol{\psi}}_b + \partial_s \boldsymbol{\zeta}_b - (F(\widetilde{\boldsymbol{Q}}_b + \boldsymbol{\zeta}_b) - F(\widetilde{\boldsymbol{Q}}_b)) + b_1 \Lambda \boldsymbol{\zeta}_b \\
&= \widehat{\mathbf{Mod}}(t) + b_1 b_{L-1}(\chi_{B_0} - \chi_{B_1})\{(-\boldsymbol{H})^{L+1}\widetilde{\Sigma}_{b_1} + \boldsymbol{\theta}_{L-1}\} \\
&\quad + b_1 b_L(\chi_{B_0} - \chi_{B_1})\{(-\boldsymbol{H})^{L+2}\widetilde{\Sigma}_{b_1} + \boldsymbol{\theta}_L\} + \boldsymbol{NL}(\boldsymbol{\zeta}_b) + \boldsymbol{L}(\boldsymbol{\zeta}_b) \\
&\quad + \tilde{\boldsymbol{\psi}}_b - (\partial_s(\chi_{B_1}) + b_1(y\chi')_{B_1})(b_{L-1}\boldsymbol{T}_{L-1} + b_L \boldsymbol{T}_L) \\
&\quad + (\partial_s(\chi_{B_0}) + b_1(y\chi')_{B_0})b_L \boldsymbol{T}_L + (\chi_{B_1} - \chi_{B_0})b_L \boldsymbol{T}_{L-1} + \boldsymbol{H}\boldsymbol{\zeta}_b. \quad (2\text{-}112)
\end{aligned}
$$

Based on the proof of the previous proposition, it suffices to show that

$$
\|\mathcal{A}^{L-2}NL(\zeta_b)\|_{L^2} + \|\mathcal{A}^{L-2}L(\zeta_b)\|_{L^2} \lesssim b_1^L,
$$

which comes from the following crude pointwise bounds on $B_0 \le y \le 2B_1$: for $k \ge 0$,

$$
|\mathcal{A}^k NL(\zeta_b)| \lesssim b_1^{2L-2} y^{2L-6-k} |\log b_1|^C, \qquad |\mathcal{A}^k L(\zeta_b)| \lesssim b_1^L y^{L-4-k} |\log b_1|^C. \qquad \square
$$

2.8. *Dynamical laws of $b = (b_1, \ldots, b_L)$.* As previously mentioned, the blow-up rate is determined by the evolution of the vector b, so we will figure out its dynamical laws from (2-56): for $1 \le k \le L$,

$$
(b_k)_s = b_{k+1} - \left(k - 1 + \frac{1}{(1 + \delta_{1k})\log s}\right) b_1 b_k, \quad b_{L+1} = 0. \qquad (2\text{-}113)
$$

One can check that the above system has L independent solutions characterized by the number of nonzero coordinates: for $1 \le k \le L$, we have $b = (b_1, \ldots, b_k, 0, \ldots, 0)$. Here, we adopt two special solutions (recall that there are two ℓs that can achieve the same L) among them.

Lemma 2.13 (special solutions for the b system). *For all $\ell \ge 2$, the vector of functions*

$$
b_k^e(s) = \frac{c_k}{s^k} + \frac{d_k}{s^k \log s} \quad \text{for } 1 \le k \le \ell, \quad b_k^e \equiv 0 \quad \text{for } k > \ell \qquad (2\text{-}114)
$$

solves (2-113) *approximately: for $1 \le k \le L$,*

$$
(b_k^e)_s + \left(k - 1 + \frac{1}{(1 + \delta_{1k})\log s}\right) b_1^e b_k^e - b_{k+1}^e = O\left(\frac{1}{s^{k+1}(\log s)^2}\right) \quad \text{as } s \to +\infty, \qquad (2\text{-}115)
$$

where the sequence $(c_k, d_k)_{k=1,\ldots,\ell}$ is given by

$$
c_1 = \frac{\ell}{\ell - 1}, \quad c_{k+1} = -\frac{\ell - k}{\ell - 1} c_k, \quad 1 \le k \le \ell, \qquad (2\text{-}116)
$$

and, for $2 \le k \le \ell - 1$,

$$
d_1 = -\frac{\ell}{(\ell - 1)^2}, \quad d_2 = -d_1 + \frac{1}{2}c_1^2, \quad d_{k+1} = -\frac{\ell - k}{\ell - 1} d_k + \frac{\ell(\ell - k)}{(\ell - 1)^2} c_k. \qquad (2\text{-}117)
$$

Remark. The recurrence relations (2-116) and (2-117) are obtained by substituting (2-114) into (2-115) and comparing the coefficients of s^{-k} and $(s^k \log s)^{-1}$, yielding the proof.

For our solution b to drive the system like the special solution b^e, we should control the fluctuation

$$\frac{U_k(s)}{s^k(\log s)^\beta} := b_k(s) - b_k^e(s) \quad \text{for } 1 \le k \le \ell. \tag{2-118}$$

Here, (2-114) and (2-115) restrict the range of β to $1 < \beta < 2$; we will choose $\beta = \frac{5}{4}$ later. The next lemma provides the evolution of $U = (U_1, \ldots, U_\ell)$ from (2-113).

Lemma 2.14 (evolution of U). *Let $b_k(s)$ be a solution to* (2-113) *and U be defined by* (2-118). *Then U solves*

$$s(U)_s = A_\ell U + O\left(\frac{1}{(\log s)^{2-\beta}} + \frac{|U| + |U|^2}{\log s}\right), \tag{2-119}$$

where the $\ell \times \ell$ matrix A_ℓ has the form

$$A_\ell = \begin{pmatrix} 1 & 1 & & & & \\ -c_2 & \frac{\ell-2}{\ell-1} & 1 & & (0) & \\ -2c_3 & & \frac{\ell-3}{\ell-1} & 1 & & \\ \vdots & & & \ddots & \ddots & \\ -(\ell-2)c_{\ell-1} & & (0) & & \frac{1}{\ell-1} & 1 \\ -(\ell-1)c_\ell & & & & & 0 \end{pmatrix}. \tag{2-120}$$

Moreover, there exists an invertible matrix P_ℓ such that $A_\ell = P_\ell^{-1} D_\ell P_\ell$ with

$$D_\ell = \begin{pmatrix} -1 & & & & \\ & \frac{2}{\ell-1} & & (0) & \\ & & \frac{3}{\ell-1} & & \\ & & & \ddots & \\ & (0) & & 1 & \\ & & & & \frac{\ell}{\ell-1} \end{pmatrix}. \tag{2-121}$$

Proof. Observing the relation

$$(k-1)c_1 - k = \frac{(k-1)\ell}{\ell-1} - k = -\frac{\ell-k}{\ell-1},$$

we obtain (2-119) and (2-120) since

$$(b_k)_s + \left(k - 1 + \frac{1}{(1+\delta_{1k})\log s}\right)b_1 b_k - b_{k+1}$$

$$= \frac{1}{s^{k+1}(\log s)^\beta}\left[s(U_k)_s - kU_k + O\left(\frac{|U|}{\log s}\right)\right] + O\left(\frac{1}{s^{k+1}(\log s)^2}\right)$$

$$\qquad\qquad + \frac{1}{s^{k+1}(\log s)^\beta}\left[(k-1)c_k U_1 + (k-1)c_1 U_k - U_{k+1} + O\left(\frac{|U| + |U|^2}{\log s}\right)\right]$$

$$= \frac{1}{s^{k+1}(\log s)^\beta}\left[s(U_k)_s + (k-1)c_k U_1 - \frac{\ell-k}{\ell-1}U_k - U_{k+1}\right]$$

$$\qquad\qquad\qquad\qquad\qquad\qquad + O\left(\frac{1}{s^{k+1}(\log s)^2} + \frac{|U| + |U^2|}{s^{k+1}(\log s)^{1+\beta}}\right). \tag{2-122}$$

Equation (2-121) is obtained by substituting $\alpha = 1$ in [Collot 2018, Lemma 2.17]. $\qquad\square$

Remark. Since the above process can be seen as linearizing (2-113) around our special solution b^e, the appearance of the matrix A_ℓ is quite natural. We also note that $\ell - 1$ unstable directions corresponding to $\ell - 1$ positive eigenvalues yield the (formal) codimension $(\ell - 1)$ restriction of our initial data.

3. The trapped solutions

Our goal in this section is to implement the blow-up dynamics constructed in the previous section into the real solution u. To do this, we first decompose the solution u as the blow-up profile and the error, i.e., $u = (\widetilde{Q}_b + \varepsilon)_\lambda = \widetilde{Q}_{b,\lambda} + w$. For the term "error" to be meaningful, we need to control the "direction" and "size" of $w = \varepsilon_\lambda$.

Here, ε must be orthogonal to the directions that provoke blowup from $\widetilde{Q}_{b,\lambda}$. Such orthogonal conditions determine the system modulation equations of the dynamical parameters b as designed in Section 2.8.

In this process, ε appears as an error that is small in some suitable norms. The smallness is required in order to keep the leading-order evolution laws unchanged (2-113). We describe the set of initial data and the trapped conditions represented by some bootstrap bounds for such suitable norms, i.e, the higher-order energies.

After establishing estimates of modulation parameters, we also establish a Lyapunov-type monotonicity of the higher-order energies to close our bootstrap assumptions.

3.1. *Decomposition of the flow.* We recall the approximate direction $\mathbf{\Phi}_M$ which was defined in [Collot 2018]. For a large constant $M > 0$, we define

$$\mathbf{\Phi}_M = \sum_{p=0}^{L} c_{p,M} H^{*p}(\chi_M \mathbf{\Lambda} Q), \quad H^* = \begin{pmatrix} 0 & H \\ -1 & 0 \end{pmatrix}, \tag{3-1}$$

where $c_{p,M}$ is given by

$$c_{0,M} = 1, \quad c_{k,M} = (-1)^{k+1} \frac{\sum_{p=0}^{k-1} c_{p,M} \langle H^{*p}(\chi_M \mathbf{\Lambda} Q), T_k \rangle}{\langle \chi_M \mathbf{\Lambda} Q, \mathbf{\Lambda} Q \rangle}, \quad 1 \le k \le L. \tag{3-2}$$

One can easily verify (see [Collot 2018, Section 3.1.1]) that H^* is an adjoint operator of H in the sense that

$$\langle Hu, v \rangle = \langle u, H^* v \rangle,$$

and $\mathbf{\Phi}_M = (\Phi_M, 0)$ satisfies

$$\langle \mathbf{\Phi}_M, \mathbf{\Lambda} Q \rangle = \langle \chi_M \mathbf{\Lambda} Q, \mathbf{\Lambda} Q \rangle \sim 4 \log M, \quad |c_{p,M}| \lesssim M^p, \quad \|\Phi_M\|_{L^2}^2 \sim c \log M. \tag{3-3}$$

We then obtain our desired decomposition by imposing a collection of orthogonal directions, which approximates the generalized kernel defined in Definition 2.3.

Lemma 3.1 (decomposition). *Let $u(t)$ be a solution to (1-21) starting close enough to Q in $\mathcal{H}$. Then there exist C^1 functions $\lambda(t)$ and $b(t) = (b_1, \dots, b_L)$ such that u can be decomposed as*

$$u = (\widetilde{Q}_{b(t)} + \varepsilon)_{\lambda(t)}, \tag{3-4}$$

where $\widetilde{Q}_b$ is given in Proposition 2.10 and ε satisfies the orthogonality conditions

$$\langle \varepsilon, H^{*i}\Phi_M \rangle = 0, \quad \text{for } 0 \leq i \leq L. \tag{3-5}$$

and an orbital stability estimate

$$|b(t)| + \|\varepsilon\|_{\mathcal{H}} \ll 1. \tag{3-6}$$

Remark. Equation (3-7) says that the elements of $\{\langle \cdot, H^{*i}\Phi_M \rangle\}_{i \geq 0}$ serve as coordinate functions on the space $\text{Span}\{T_i\}_{i \geq 0}$.

Proof. It is clear that $H^i T_j = 0$ for $i > j$. For $0 \leq i \leq j$,

$$\langle \Phi_M, H^i T_j \rangle = (-1)^i \langle \Phi_M, T_{j-i} \rangle$$

$$= (-1)^i \sum_{p=0}^{j-i-1} c_{p,M} \langle H^{*p}(\chi_M \Lambda Q), T_{j-i} \rangle + (-1)^j c_{j-i,M} \langle \chi_M \Lambda Q, \Lambda Q \rangle$$

$$= (-1)^j \langle \chi_M \Lambda Q, \Lambda Q \rangle \delta_{i,j}. \tag{3-7}$$

Now, we consider $\varepsilon := u_{1/\lambda} - \widetilde{Q}_b$ as a map in the (λ, b, u) basis. By the implicit function theorem, (3-4) is deduced from the nondegeneracy of the Jacobian

$$\left| \left(\frac{\partial}{\partial(\lambda, b)} \langle \varepsilon, H^{*i}\Phi_M \rangle \right)_{0 \leq i \leq L} \right|_{(\lambda, b, u) = (1, 0, Q)} = (-1)^{L+1} |(\langle T_j, H^{*i}\Phi_M \rangle)_{0 \leq i, j \leq L}|$$

$$= |(\langle \Phi_M, H^i T_j \rangle)_{0 \leq i, j \leq L}|$$

$$= |((-1)^j \langle \chi_M \Lambda Q, \Lambda Q \rangle \delta_{i,j})_{0 \leq i, j \leq L}|$$

$$= (-1)^{(L+1)/2} \langle \chi_M \Lambda Q, \Lambda Q \rangle^{L+1} \neq 0. \qquad \square$$

3.2. *Equation for the error.* Based on the previously established decomposition

$$u = \widetilde{Q}_{b(t), \lambda(t)} + w = (\widetilde{Q}_{b(s)} + \varepsilon(s))_{\lambda(s)},$$

(1-21) turns into the following evolution equation of ε:

$$\partial_s \varepsilon - \frac{\lambda_s}{\lambda} \Lambda \varepsilon + H\varepsilon = -\left(\partial_s \widetilde{Q}_b - \frac{\lambda_s}{\lambda} \Lambda \widetilde{Q}_b \right) + F(\widetilde{Q}_b + \varepsilon) + H\varepsilon$$

$$= -(\partial_s \widetilde{Q}_b - F(\widetilde{Q}_b) + b_1 \Lambda \widetilde{Q}_b) + \left(\frac{\lambda_s}{\lambda} + b_1 \right) \Lambda \widetilde{Q}_b + F(\widetilde{Q}_b + \varepsilon) - F(\widetilde{Q}_b) + H\varepsilon$$

$$= -\widetilde{\text{Mod}}(t) - \tilde{\psi}_b - NL(\varepsilon) - L(\varepsilon), \tag{3-8}$$

where

$$\widetilde{\text{Mod}}(t) := \chi_{B_1} \text{Mod}(t) - \left(\frac{\lambda_s}{\lambda} + b_1 \right) \Lambda \widetilde{Q}_b, \quad \widetilde{\text{Mod}}(t) := \begin{pmatrix} \widetilde{\text{Mod}}(t) \\ \dot{\widetilde{\text{Mod}}}(t) \end{pmatrix} \tag{3-9}$$

$$NL(\varepsilon) := \frac{1}{y^2} \begin{pmatrix} 0 \\ f(\widetilde{Q}_b + \varepsilon) - f(\widetilde{Q}_b) - f'(\widetilde{Q}_b)\varepsilon \end{pmatrix}, \quad L(\varepsilon) := \frac{1}{y^2} \begin{pmatrix} 0 \\ (f'(\widetilde{Q}_b) - f'(Q))\varepsilon \end{pmatrix}. \tag{3-10}$$

For later analysis, we also employ the following evolution equation of w:

$$\partial_t w + H_\lambda w = \frac{1}{\lambda} \mathcal{F}_\lambda, \quad \mathcal{F} = -\widetilde{\text{Mod}}(t) - \tilde{\psi}_b - NL(\varepsilon) - L(\varepsilon), \tag{3-11}$$

where

$$\boldsymbol{H}_\lambda = \begin{pmatrix} 0 & -1 \\ H_\lambda & +0 \end{pmatrix} := \begin{pmatrix} 0 & -1 \\ -\Delta + r^{-2} f'(Q_\lambda) & +0 \end{pmatrix}, \tag{3-12}$$

We notice that the $\boldsymbol{NL}$ and $\boldsymbol{L}$ terms are situated on the second coordinate:

$$\boldsymbol{NL}(\boldsymbol{\varepsilon}) = \begin{pmatrix} 0 \\ NL(\varepsilon) \end{pmatrix}, \quad \boldsymbol{L}(\boldsymbol{\varepsilon}) = \begin{pmatrix} 0 \\ L(\varepsilon) \end{pmatrix}. \tag{3-13}$$

We also introduce another decomposition

$$\boldsymbol{u} = \widehat{\boldsymbol{Q}}_{b(t),\lambda(t)} + \hat{\boldsymbol{w}} = (\widehat{\boldsymbol{Q}}_{b(s)} + \hat{\boldsymbol{\varepsilon}}(s))_{\lambda(s)},$$

which depends on whether $\ell = L$ (Proposition 2.11) or $\ell = L - 1$ (Proposition 2.12). The evolution equation of $\hat{\boldsymbol{\varepsilon}}$ is given by

$$\partial_s \hat{\boldsymbol{\varepsilon}} - \frac{\lambda_s}{\lambda} \boldsymbol{\Lambda} \hat{\boldsymbol{\varepsilon}} + \boldsymbol{H} \hat{\boldsymbol{\varepsilon}} = -\widehat{\boldsymbol{Mod}}'(t) - \hat{\boldsymbol{\psi}}_b - \widehat{\boldsymbol{NL}}(\hat{\boldsymbol{\varepsilon}}) - \widehat{\boldsymbol{L}}(\hat{\boldsymbol{\varepsilon}}), \tag{3-14}$$

where

$$\widehat{\boldsymbol{Mod}}'(t) := \widehat{\boldsymbol{Mod}}(t) - \left(\frac{\lambda_s}{\lambda} + b_1 \right) \boldsymbol{\Lambda} \widehat{\boldsymbol{Q}}_b, \tag{3-15}$$

$$\widehat{\boldsymbol{NL}}(\hat{\boldsymbol{\varepsilon}}) := \frac{1}{y^2} \begin{pmatrix} 0 \\ f(\widehat{Q}_b + \hat{\varepsilon}) - f(\widehat{Q}_b) - f'(\widehat{Q}_b)\hat{\varepsilon} \end{pmatrix}, \quad \widehat{\boldsymbol{L}}(\hat{\boldsymbol{\varepsilon}}) := \frac{1}{y^2} \begin{pmatrix} 0 \\ (f'(\widehat{Q}_b) - f'(Q))\hat{\varepsilon} \end{pmatrix}. \tag{3-16}$$

We also employ the evolution equation of $\hat{\boldsymbol{w}}$:

$$\partial_t \hat{\boldsymbol{w}} + \boldsymbol{H}_\lambda \hat{\boldsymbol{w}} = \frac{1}{\lambda} \widehat{\boldsymbol{\mathcal{F}}}_\lambda, \quad \widehat{\boldsymbol{\mathcal{F}}} = -\widehat{\boldsymbol{Mod}}'(t) - \hat{\boldsymbol{\psi}}_b - \widehat{\boldsymbol{NL}}(\hat{\boldsymbol{\varepsilon}}) - \widehat{\boldsymbol{L}}(\hat{\boldsymbol{\varepsilon}}). \tag{3-17}$$

3.3. *Initial data setting for the bootstrap.* In this subsection, we describe our initial data and the bootstrap assumption. To do this, we recall the fluctuation (2-118), i.e., $U = (U_1, \ldots, U_\ell)$,

$$U_k(s) = s^k (\log s)^\beta (b_k(s) - b_k^e(s)).$$

We also define the adapted higher-order energies given by

$$\mathcal{E}_k := \langle \varepsilon_k, \varepsilon_k \rangle + \langle \dot{\varepsilon}_{k-1}, \dot{\varepsilon}_{k-1} \rangle, \quad 2 \le k \le L + 1. \tag{3-18}$$

We set our renormalized space-time variables (s, y) as follows: for a large enough $s_0 \gg 1$,

$$y = \frac{r}{\lambda(t)}, \quad s(t) = s_0 + \int_0^t \frac{d\tau}{\lambda(\tau)}.$$

For the sake of simplicity, we use a transformed fluctuation $V = (V_1(s), \ldots, V_\ell(s))$,

$$V = P_\ell U, \tag{3-19}$$

where P_ℓ yields the diagonalization (2-121). Then we illustrate the modulation parameters b as a sum of the exact solutions $b^e(s)$ and $V(s)$: for $\ell = L - 1$ or L,

$$b(s) = b^e(s) + \left(\frac{(P_\ell^{-1} V(s))_1}{s(\log s)^\beta}, \ldots, \frac{(P_\ell^{-1} V(s))_\ell}{s^\ell (\log s)^\beta}, b_{\ell+1}(s), \ldots, b_L(s) \right).$$

Now, we assume some smallness conditions for our initial data $\boldsymbol{u}_0(s_0) = (u_0, \dot{u}_0)$ as follows: for large constants $M = M(L)$, $K = K(L, M)$, $s_0 = s_0(L, M, K)$, we set the initial data $\boldsymbol{u}_0 = \boldsymbol{u}(s_0)$ as

$$\boldsymbol{u}_0 = (\widetilde{\boldsymbol{Q}}_{b(s_0)} + \boldsymbol{\varepsilon}(s_0))_{\lambda(s_0)}, \tag{3-20}$$

where $\boldsymbol{\varepsilon}(s_0)$ satisfies the orthogonality conditions (3-5), we have the smallness of higher-order energies

$$\mathcal{E}_k(s_0) \leq b_1^{2L+4}(s_0), \tag{3-21}$$

and $b(s_0)$ satisfies the smallness of the stable modes:

$$|V_1(s_0)| \leq \frac{1}{4},$$
$$|b_L(s_0)| \leq \frac{1}{s_0^{(L-1)c_1}(\log s_0)^{3/2}} \quad \text{for } \ell = L - 1, \tag{3-22}$$

where $c_1 = \ell/(\ell - 1)$. Furthermore, we may assume

$$\lambda(s_0) = 1 \tag{3-23}$$

up to rescaling.

Proposition 3.2 (existence of trapped solutions). *Given $\boldsymbol{u}(s_0)$ of the form* (3-20) *satisfying* (3-5), (3-21) *and* (3-22), *there exists an initial direction of the unstable modes*

$$(V_2(s_0), \ldots, V_\ell(s_0)) \in \mathcal{B}^{\ell-1} \tag{3-24}$$

such that the corresponding solution to (1-21) *becomes **trapped**; namely, it satisfies the following bounds for all $s \geq s_0$:*

- *Control of the higher-order energies: for $2 \leq k \leq \ell - 1$,*

$$\mathcal{E}_k(s) \leq b_1^{2(k-1)c_1}|\log b_1|^K,$$
$$\mathcal{E}_{L+1}(s) \leq K\frac{b_1^{2L+2}}{|\log b_1|^2}, \tag{3-25}$$

$$\mathcal{E}_L(s) \leq \begin{cases} K\lambda^{2(L-1)} & \text{when } \ell = L, \\ b_1^{2L}|\log b_1|^K & \text{when } \ell = L - 1, \end{cases} \tag{3-26}$$

$$\mathcal{E}_{L-1}(s) \leq K\lambda^{2(L-2)} \qquad \text{when } \ell = L - 1. \tag{3-27}$$

- *Control of the stable modes:*

$$|V_1(s)| \leq 1,$$
$$|b_L(s)| \leq \frac{1}{s^L(\log s)^\beta} \quad \text{when } \ell = L - 1. \tag{3-28}$$

- *Control of the unstable modes:*

$$(V_2(s), \ldots, V_\ell(s)) \in \mathcal{B}^{\ell-1}. \tag{3-29}$$

Under the initial setting of $(\boldsymbol{\varepsilon}(s_0), V(s_0), b_{\ell+1}(s_0), \ldots, b_L(s_0))$ (see (3-20)–(3-22) and (3-24)), we define an exit time

$$s^* = \sup\{s \geq s_0 : \text{(3-25)–(3-29) hold on } [s_0, s]\}. \tag{3-30}$$

From (3-20)–(3-22) and (3-24), it is clear that (3-25)–(3-29) hold at $s = s_0$. We will prove Proposition 3.2 in Section 4 by contradiction, assume that

$$s^* < \infty \quad \text{for all } (V_2(s_0), \ldots, V_\ell(s_0)) \in \mathcal{B}^{\ell-1}. \tag{3-31}$$

At the exit time s^*, we claim that only (3-29) fails among the bootstrap bounds in Proposition 3.2 through establishing estimates of modulation parameters and some monotonicity formulae of the higher-order energies. Then, the codimension $(\ell - 1)$ stability (2-121) leads to a contradiction by Brouwer's fixed point theorem.

3.4. *Modulation equations.* Now we provide the evolution of the modulation parameters from the orthogonality conditions (3-5).

Lemma 3.3 (modulation equations). *The modulation parameters* $(\lambda, b_1, \ldots, b_L)$ *satisfy the bounds*

$$\left| \frac{\lambda_s}{\lambda} + b_1 \right| + \sum_{i=1}^{L-1} |(b_i)_s + (i - 1 + c_{b_1,i})b_1 b_i - b_{i+1}| \lesssim C(M) b_1 (\sqrt{\mathcal{E}_{L+1}} + b_1^{L+2}), \tag{3-32}$$

$$|(b_L)_s + (L - 1 + c_{b_1,L})b_1 b_L| \lesssim \frac{\sqrt{\mathcal{E}_{L+1}}}{\sqrt{\log M}} + C(M) b_1^{L+3}. \tag{3-33}$$

Remark. The bounds (3-32) and (3-25) allow us to obtain the a priori assumption (2-83).

Proof. <u>Step 1</u>: *Modulation identity.* Write $D(t) = (D_0(t), \ldots, D_L(t))$, where $D_i(t)$ is given by

$$D_0(t) := -\left(\frac{\lambda_s}{\lambda} + b_1 \right), \quad D_i(t) := (b_i)_s + (i - 1 + c_{b_1,i})b_1 b_i - b_{i+1}, \quad b_{L+1} = 0.$$

We take the vector-valued inner product (1-23) of (3-8) with $\boldsymbol{H}^{*k}\boldsymbol{\Phi}_M$ for $0 \leq k \leq L$. Then we have the identity

$$\langle \widetilde{\mathbf{Mod}}(t), \boldsymbol{H}^{*k}\boldsymbol{\Phi}_M \rangle + \langle \boldsymbol{H}\boldsymbol{\varepsilon}, \boldsymbol{H}^{*k}\boldsymbol{\Phi}_M \rangle$$
$$= \frac{\lambda_s}{\lambda} \langle \boldsymbol{\Lambda}\boldsymbol{\varepsilon}, \boldsymbol{H}^{*k}\boldsymbol{\Phi}_M \rangle - \langle \tilde{\boldsymbol{\psi}}_b, \boldsymbol{H}^{*k}\boldsymbol{\Phi}_M \rangle - \langle \boldsymbol{NL}(\boldsymbol{\varepsilon}) + \boldsymbol{L}(\boldsymbol{\varepsilon}), \boldsymbol{H}^{*k}\boldsymbol{\Phi}_M \rangle. \tag{3-34}$$

<u>Step 2</u>: *Estimates for each term in* (3-34). We claim that the left-hand side of (3-34) gives the main contribution needed to prove (3-32) and (3-33).

(i) $\widetilde{\mathbf{Mod}}(t)$ *terms.* First, $\chi_{B_1}\boldsymbol{\alpha}_b = \boldsymbol{\alpha}_b$ holds on $|y| \leq 2M$ for small enough b_1. We also have the pointwise bound

$$|\boldsymbol{\Lambda}\boldsymbol{\alpha}_b| + \sum_{i=1}^{L} \sum_{j=i+1}^{L+2} \left| \frac{\partial S_j}{\partial b_i} \right| \lesssim b_1 C(M) \quad \text{for } |y| \leq 2M$$

from our blow-up profile construction. Hence we estimate the $\widetilde{\mathbf{Mod}}(t)$ term in (3-34) by the transversality (3-7) and the compact support property of $\mathbf{\Phi}_M$:

$$
\begin{aligned}
\langle \widetilde{\mathbf{Mod}}(t), \boldsymbol{H}^{*k}\boldsymbol{\Phi}_M \rangle &= D_0(t)\langle \boldsymbol{\Lambda}\boldsymbol{Q}_b, \boldsymbol{H}^{*k}\boldsymbol{\Phi}_M \rangle + \sum_{i=1}^{L} D_i(t)\Big\langle \boldsymbol{T}_i + \sum_{j=i+1}^{L+2} \frac{\partial \boldsymbol{S}_j}{\partial b_i}, \boldsymbol{H}^{*k}\boldsymbol{\Phi}_M \Big\rangle \\
&= \sum_{i=0}^{L} D_i(t)\langle \boldsymbol{T}_i, \boldsymbol{H}^{*k}\boldsymbol{\Phi}_M \rangle + \Big\langle D_0(t)\boldsymbol{\Lambda}\boldsymbol{\alpha}_b + \sum_{i=1}^{L}\sum_{j=i+1}^{L+2} D_i(t)\frac{\partial \boldsymbol{S}_j}{\partial b_i}, \boldsymbol{H}^{*k}\boldsymbol{\Phi}_M \Big\rangle \\
&= (-1)^k D_k(t)\langle \boldsymbol{\Lambda}\boldsymbol{Q}, \boldsymbol{\Phi}_M \rangle + O(C(M)b_1|D(t)|). \tag{3-35}
\end{aligned}
$$

(ii) *Linear terms.* For $0 \le k \le L-1$, we have

$$
\langle \boldsymbol{H}\boldsymbol{\varepsilon}, \boldsymbol{H}^{*k}\boldsymbol{\Phi}_M \rangle = \langle \boldsymbol{\varepsilon}, \boldsymbol{H}^{*(k+1)}\boldsymbol{\Phi}_M \rangle = 0
$$

from the orthogonal conditions (3-5). For $k = L$, the Cauchy–Schwarz inequality implies

$$
|\langle \boldsymbol{\varepsilon}, \boldsymbol{H}^{*(L+1)}\boldsymbol{\Phi}_M \rangle| = |\langle \boldsymbol{H}^{L+1}\boldsymbol{\varepsilon}, \boldsymbol{\Phi}_M \rangle| \lesssim \sqrt{\log M}\sqrt{\mathcal{E}_{L+1}}. \tag{3-36}
$$

(iii) *Scaling terms.* We can estimate the scaling term in (3-34) from the compact support property of $\boldsymbol{\Phi}_M$ and the coercivity bound (A-15):

$$
\begin{aligned}
\Big| \frac{\lambda_s}{\lambda}\langle \boldsymbol{\Lambda}\boldsymbol{\varepsilon}, \boldsymbol{H}^{*k}\boldsymbol{\Phi}_M \rangle \Big| &\le (b_1 + |D_0(t)|)|\langle \boldsymbol{\Lambda}\boldsymbol{\varepsilon}, \boldsymbol{H}^{*k}\boldsymbol{\Phi}_M \rangle| \\
&\lesssim (b_1 + |D_0(t)|)C(M)\sqrt{\mathcal{E}_{L+1}}. \tag{3-37}
\end{aligned}
$$

(iv) $\tilde{\boldsymbol{\psi}}_b$ *terms.* Here, the improved local bound (2-91) implies

$$
|\langle \tilde{\boldsymbol{\psi}}_b, \boldsymbol{H}^{*k}\boldsymbol{\Phi}_M \rangle| \lesssim C(M)b_1^{L+3}. \tag{3-38}
$$

(v) $\boldsymbol{NL}(\boldsymbol{\varepsilon})$ and $\boldsymbol{L}(\boldsymbol{\varepsilon})$ *terms.* Using the coercivity bound (A-15) with the crude bound $|NL(\varepsilon)| \lesssim |\varepsilon|^2/y^2$ and $|L(\varepsilon)| \lesssim b_1^2|\varepsilon|/y$,

$$
|\langle \boldsymbol{NL}(\boldsymbol{\varepsilon}), \boldsymbol{H}^{*i}\boldsymbol{\Phi}_M \rangle| \lesssim C(M)\mathcal{E}_{L+1}, \quad |\langle \boldsymbol{L}(\boldsymbol{\varepsilon}), \boldsymbol{H}^{*i}\boldsymbol{\Phi}_M \rangle| \lesssim C(M)b_1^2\sqrt{\mathcal{E}_{L+1}}. \tag{3-39}
$$

Step 3: *Conclusion.* Injecting the estimates from (3-35)–(3-39) into (3-34), we obtain

$$
\begin{aligned}
(-1)^k D_k(t)\langle \boldsymbol{\Lambda}\boldsymbol{Q}, \boldsymbol{\Phi}_M \rangle &+ O(C(M)b_1|D(t)|) \\
&= O(\sqrt{\log M}\sqrt{\mathcal{E}_{L+1}})\delta_{kL} + O(C(M)b_1(\sqrt{\mathcal{E}_{L+1}} + b_1^{L+2})) \tag{3-40}
\end{aligned}
$$

for $0 \le k \le L$. Dividing the above equation by $\langle \boldsymbol{\Lambda}\boldsymbol{Q}, \boldsymbol{\Phi}_M \rangle$, (3-3) implies

$$
D_k(t) + O(C(M)b_1|D(t)|) = O\left(\frac{\sqrt{\mathcal{E}_{L+1}}}{\sqrt{\log M}}\right)\delta_{kL} + O(C(M)b_1(\sqrt{\mathcal{E}_{L+1}} + b_1^{L+2})),
$$

which yields (3-32) and (3-33). $\qquad\square$

3.5. Improved modulation equation of b_L. At first glance, (3-33) seems sufficient to close the modulation equation for b_L because of the presence of $\sqrt{\log M}$. However, our desired blow-up scenario comes from the exact solution b_L^e, and (3-33) is inadequate to close the bootstrap bounds for stable/unstable modes $V(s)$. Thus, we need to obtain further logarithmic room by adding some correction to b_L.

Lemma 3.4 (improved modulation equation of b_L). *Let $B_\delta = B_0^\delta$ and*

$$\tilde{b}_L = b_L + (-1)^L \frac{\langle H^L \varepsilon, \chi_{B_\delta} \Lambda Q \rangle}{4\delta |\log b_1|} \tag{3-41}$$

for some small enough universal constant $0 < \delta \ll 1$. Then $\tilde{b}_L$ satisfies

$$|\tilde{b}_L - b_L| \lesssim b_1^{L+1-C\delta} \tag{3-42}$$

and

$$|(\tilde{b}_L)_s + (L - 1 + c_{b,L}) b_1 \tilde{b}_L| \lesssim \frac{\sqrt{\mathcal{E}_{L+1}}}{\sqrt{|\log b_1|}}. \tag{3-43}$$

Remark. We point out that $\tilde{b}_L$ is well-defined at time $s = s_0$ since $\tilde{b}_L - b_L$ only depends on b_1 and ε.

Proof. We obtain (3-42) from the coercivity bound (A-15) and (3-32):

$$|\langle H^L \varepsilon, \chi_{B_\delta} \Lambda Q \rangle| \lesssim |\langle H^{(L-1)/2} \dot{\varepsilon}, \chi_{B_\delta} \Lambda Q \rangle| \lesssim C(M) \delta b_1^{-C\delta} \sqrt{\mathcal{E}_{L+1}} \lesssim b_1^{L+1-C\delta}, \tag{3-44}$$

We also know

$$\frac{d}{ds} \langle H^L \varepsilon, \chi_{B_\delta} \Lambda Q \rangle = \langle H^L \varepsilon_s, \chi_{B_\delta} \Lambda Q \rangle + \langle H^L \varepsilon, (\chi_{B_\delta})_s \Lambda Q \rangle. \tag{3-45}$$

We compute the last inner product in (3-45) similar to (3-44):

$$|\langle H^L \varepsilon, (\chi_{B_\delta})_s \Lambda Q \rangle| = |\delta(b_1)_s b_1^{-1}| |\langle H^{(L-1)/2} \dot{\varepsilon}, (y \partial_y \chi)_{B_\delta} \Lambda Q \rangle| \lesssim C(M) \delta b_1^{1-\delta} \sqrt{\mathcal{E}_{L+1}}. \tag{3-46}$$

Using (3-8), we obtain an identity similar to (3-34):

$$\langle H^L \varepsilon_s, \chi_{B_\delta} \Lambda Q \rangle = -\langle H^L \widetilde{\text{Mod}}(t), \chi_{B_\delta} \Lambda Q \rangle - \langle H^{L+1} \varepsilon, \chi_{B_\delta} \Lambda Q \rangle + \frac{\lambda_s}{\lambda} \langle H^L \Lambda \varepsilon, \chi_{B_\delta} \Lambda Q \rangle$$
$$- \langle H^L \tilde{\psi}_b, \chi_{B_\delta} \Lambda Q \rangle - \langle H^L NL(\varepsilon), \chi_{B_\delta} \Lambda Q \rangle - \langle H^L L(\varepsilon), \chi_{B_\delta} \Lambda Q \rangle.$$

Considering the support of $\chi_{B_\delta} \Lambda Q$, we can borrow all the estimates in Step 2 of the proof of Lemma 3.3 by replacing the weight $\log M$ and $C(M)$ with $|\log b_1|$ and $b_1^{-C\delta}$, respectively. Hence Lemma 3.3 and (3-46) give a "B_δ version" of (3-40):

$$\frac{d}{ds} \langle H^L \varepsilon, \chi_{B_\delta} \Lambda Q \rangle = (-1)^{L+1} D_L(t) \langle \Lambda Q, \chi_{B_\delta} \Lambda Q \rangle + O(b_1^{1-C\delta} |D(t)|)$$
$$+ O(\sqrt{|\log b_1|} \sqrt{\mathcal{E}_{L+1}}) + O(b_1^{1-C\delta} (\sqrt{\mathcal{E}_{L+1}} + b_1^{L+2}))$$
$$= (-1)^{L+1} 4\delta |\log b_1| D_L(t) + O(\sqrt{|\log b_1|} \sqrt{\mathcal{E}_{L+1}}).$$

Hence we obtain (3-43) as follows:

$$|(\tilde{b}_L)_s + (L - 1 + c_{b,L}) b_1 \tilde{b}_L| \lesssim |\langle H^L \varepsilon, \chi_{B_\delta} \Lambda Q \rangle| \left| b_1 + \frac{d}{ds} \left\{ \frac{1}{4\delta \log b_1} \right\} \right| + \frac{\sqrt{\mathcal{E}_{L+1}}}{\sqrt{|\log b_1|}}$$
$$\lesssim \frac{\sqrt{\mathcal{E}_{L+1}}}{\sqrt{|\log b_1|}} + b_1^{L+2-C\delta}. \qquad \square$$

3.6. *Lyapunov monotonicity for $\mathcal{E}_{L+1}$.* A simple way to control the adapted higher-order energy $\mathcal{E}_{L+1}$ is to estimate its time derivative. However, we cannot obtain enough estimates to close the bootstrap bound (3-25) with $\mathcal{E}_{L+1}$ by itself, i.e., with $b_1 \sim -\lambda_t$:

$$\frac{d}{dt}\left\{\frac{\mathcal{E}_{L+1}}{\lambda^{2L}}\right\} \le Cb_1 \frac{\mathcal{E}_{L+1}}{\lambda^{2L+1}}, \quad \frac{\mathcal{E}_{L+1}(t)}{\lambda^{2L}(t)} \le \frac{\mathcal{E}_{L+1}(0)}{\lambda^{2L}(0)} + C\int_0^t b_1(\tau)\frac{\mathcal{E}_{L+1}(\tau)}{\lambda^{2L+1}(\tau)}\,d\tau$$

$$\le K\int_0^t \frac{b_1(\tau)}{\lambda^{2L+1}(\tau)}\frac{b_1^{2(L+1)}(\tau)}{|\log b_1(\tau)|^2}\,d\tau$$

$$\lesssim \frac{K}{\lambda^{2L}(t)}\frac{b_1^{2(L+1)}(t)}{|\log b_1(t)|^2}.$$

Thus, we use the repulsive property of the conjugated Hamiltonian $\widetilde{H}$ of H observed in [Raphaël and Rodnianski 2012; Rodnianski and Sterbenz 2010] with some additional integration by parts to pull out the accurate corrections.

Proposition 3.5 (Lyapunov monotonicity for $\mathcal{E}_{L+1}$). *We have the bound*

$$\frac{d}{dt}\left\{\frac{\mathcal{E}_{L+1}}{\lambda^{2L}} + O\left(\frac{b_1 C(M)\mathcal{E}_{L+1}}{\lambda^{2L}}\right)\right\} \le C\frac{b_1}{\lambda^{2L+1}}\left[\frac{b_1^{L+1}}{|\log b_1|}\sqrt{\mathcal{E}_{L+1}} + \frac{\mathcal{E}_{L+1}}{\sqrt{\log M}}\right]. \tag{3-47}$$

Proof. <u>Step 1</u>: *Evolution of adapted derivatives.* We start by introducing the rescaled version of the operators A and A^*:

$$A_\lambda := -\partial_r + \frac{Z_\lambda}{r}, \quad A_\lambda^* := \partial_r + \frac{1 + Z_\lambda}{r}, \quad Z_\lambda(r) = Z\left(\frac{r}{\lambda}\right) = \frac{1 - (r/\lambda)^2}{1 + (r/\lambda)^2}.$$

We also recall H_λ in (3-12) and define its conjugate operator $\widetilde{H}_\lambda$ as the rescaled version of the linearized operator H and its conjugate $\widetilde{H}$:

$$H_\lambda := A_\lambda^* A_\lambda = -\Delta + \frac{V_\lambda}{r^2}, \quad V(y) = \frac{y^4 - 6y^2 + 1}{(y^2 + 1)^2},$$

$$\widetilde{H}_\lambda := A_\lambda A_\lambda^* = -\Delta + \frac{\widetilde{V}_\lambda}{r^2}, \quad \widetilde{V}(y) = \frac{4}{y^2 + 1}.$$

In the same manner as (2-12), we define the rescaled version of the adapted derivative operator

$$\mathcal{A}_\lambda := A_\lambda, \quad \mathcal{A}_\lambda^2 := A_\lambda^* A_\lambda, \quad \mathcal{A}_\lambda^3 := A_\lambda A_\lambda^* A_\lambda, \quad \dots, \quad \mathcal{A}_\lambda^k := \underbrace{\cdots A_\lambda^* A_\lambda A_\lambda^* A_\lambda}_{k \text{ times}}, \tag{3-48}$$

so the higher-order derivatives of $\boldsymbol{w} = (w, \dot{w})^t$ adapted to the Hamiltonian H_λ are given by

$$w_k := \mathcal{A}_\lambda^k w, \quad \dot{w}_k := \mathcal{A}_\lambda^k \dot{w}.$$

One can easily check that $w_k = (\varepsilon_k)_\lambda / \lambda^k$ and $\dot{w}_k = (\dot{\varepsilon}_k)_\lambda / \lambda^{k+1}$, and our target energy can be written as

$$\frac{\mathcal{E}_{L+1}}{\lambda^{2L}} = \langle w_{L+1}, w_{L+1}\rangle + \langle \dot{w}_L, \dot{w}_L\rangle = \langle \widetilde{H}_\lambda w_L, w_L\rangle + \langle \dot{w}_L, \dot{w}_L\rangle. \tag{3-49}$$

To describe the evolution of w_k and $\dot{w}_k$, we first rewrite the flow (3-11) of $\boldsymbol{w} = (w, \dot{w})$ componentwise:

$$\begin{cases} w_t - \dot{w} = \mathcal{F}_1, \\ \dot{w}_t + H_\lambda w = \mathcal{F}_2, \end{cases} \qquad \begin{pmatrix} \mathcal{F}_1 \\ \mathcal{F}_2 \end{pmatrix} := \frac{1}{\lambda} \mathcal{F}_\lambda = \frac{1}{\lambda} \begin{pmatrix} \mathcal{F} \\ \dot{\mathcal{F}} \end{pmatrix}_\lambda. \tag{3-50}$$

Substituting $\mathcal{A}_\lambda^k$ given by (3-48) into (3-50), we obtain the evolution equation of w_k

$$\begin{cases} \partial_t w_k - \dot{w}_k = [\partial_t, \mathcal{A}_\lambda^k] w + \mathcal{A}_\lambda^k \mathcal{F}_1, \\ \partial_t \dot{w}_k + w_{k+2} = [\partial_t, \mathcal{A}_\lambda^k] \dot{w} + \mathcal{A}_\lambda^k \mathcal{F}_2. \end{cases} \tag{3-51}$$

Lastly, we employ the following notation: for any time-dependent operator P,

$$\partial_t(P) := [\partial_t, P],$$

which yields the Leibniz rule between the operator and function:

$$\partial_t(Pf) = \partial_t(P)f + Pf_t. \tag{3-52}$$

Step 2: *First energy identity.* Recalling (3-49), we compute the energy identity

$$\begin{aligned} \partial_t \left(\frac{\mathcal{E}_{L+1}}{2\lambda^{2L}} \right) &= \frac{1}{2} \langle \partial_t(\widetilde{H}_\lambda) w_L, w_L \rangle + \langle \widetilde{H}_\lambda w_L, \partial_t w_L \rangle + \langle \dot{w}_L, \partial_t \dot{w}_L \rangle \\ &= \frac{1}{2} \langle \partial_t(\widetilde{H}_\lambda) w_L, w_L \rangle + \langle \widetilde{H}_\lambda w_L, \partial_t(\mathcal{A}_\lambda^L) w \rangle + \langle \dot{w}_L, \partial_t(\mathcal{A}_\lambda^L) \dot{w} \rangle \\ &\qquad\qquad\qquad\qquad + \langle \widetilde{H}_\lambda w_L, \mathcal{A}_\lambda^L \mathcal{F}_1 \rangle + \langle \dot{w}_L, \mathcal{A}_\lambda^L \mathcal{F}_2 \rangle. \end{aligned} \tag{3-53}$$

We will check that the last two terms of (3-53) satisfy the desired bound (3-47) later. Unlike the last two terms of (3-53), when the first three terms of (3-53) are estimated using coercivity (A-15) directly, we obtain the insufficient bound

$$\frac{b_1}{\lambda^{2L+1}} C(M) \mathcal{E}_{L+1}. \tag{3-54}$$

One can employ repulsive property (2-10) for the first term of (3-53) with the modulation equation (3-32):

$$\partial_t(\widetilde{H}_\lambda) = -\frac{\lambda_t}{\lambda} \frac{(\Lambda \widetilde{V})_\lambda}{r^2} = -\frac{b_1 + O(b_1^{L+2})}{\lambda^3} \frac{8}{(1+y^2)^2} \quad \Longrightarrow \quad \langle \partial_t(\widetilde{H}_\lambda) w_L, w_L \rangle < 0. \tag{3-55}$$

We claim that the sum $\langle \widetilde{H}_\lambda w_L, \partial_t(\mathcal{A}_\lambda^L) w \rangle + \langle \dot{w}_L, \partial_t(\mathcal{A}_\lambda^L) \dot{w} \rangle$ in (3-53) is eventually negative like (3-55) by adding some corrections. For this, we start by employing (3-51) to exchange $\widetilde{H}_\lambda w_L$ for $-\partial_t \dot{w}_L$:

$$\langle \widetilde{H}_\lambda w_L, \partial_t(\mathcal{A}_\lambda^L) w \rangle = -\langle \partial_t \dot{w}_L, \partial_t(\mathcal{A}_\lambda^L) w \rangle + \langle \partial_t(\mathcal{A}_\lambda^L) \dot{w}, \partial_t(\mathcal{A}_\lambda^L) w \rangle + \langle \mathcal{A}_\lambda^L \mathcal{F}_2, \partial_t(\mathcal{A}_\lambda^L) w \rangle, \tag{3-56}$$

and we can treat the first term on the right-hand side of (3-56) via integration by parts in time with (3-50):

$$\begin{aligned} -\langle \partial_t \dot{w}_L, \partial_t(\mathcal{A}_\lambda^L) w \rangle + \partial_t \langle \dot{w}_L, \partial_t(\mathcal{A}_\lambda^L) w \rangle &= \langle \dot{w}_L, \partial_{tt}(\mathcal{A}_\lambda^L) w \rangle + \langle \dot{w}_L, \partial_t(\mathcal{A}_\lambda^L) w_t \rangle \\ &= \langle \dot{w}_L, \partial_t(\mathcal{A}_\lambda^L) \dot{w} \rangle + \langle \dot{w}_L, \partial_{tt}(\mathcal{A}_\lambda^L) w \rangle + \langle \dot{w}_L, \partial_t(\mathcal{A}_\lambda^L) \mathcal{F}_1 \rangle. \end{aligned} \tag{3-57}$$

In short, we add a correction to the energy identity to transform the inner product $\langle \widetilde{H}_\lambda w_L, \partial_t(\mathcal{A}_\lambda^L) w \rangle$ to the inner product $\langle \dot{w}_L, \partial_t(\mathcal{A}_\lambda^L) \dot{w} \rangle$ in (3-53) up to some errors from (3-56) and (3-57):

$$\langle \widetilde{H}_\lambda w_L, \partial_t(\mathcal{A}_\lambda^L) w \rangle + \partial_t D_{0,1,1} = \langle \dot{w}_L, \partial_t(\mathcal{A}_\lambda^L) \dot{w} \rangle + E_{0,1,1} + E_{0,1,2} + F_{0,1,1} + F_{0,1,2}, \tag{3-58}$$

where

$$D_{0,1,1} = \langle \dot{w}_L, \partial_t(\mathcal{A}_\lambda^L)w \rangle,$$
$$E_{0,1,1} = \langle \dot{w}_L, \partial_{tt}(\mathcal{A}_\lambda^L)w \rangle,$$
$$E_{0,1,2} = \langle \partial_t(\mathcal{A}_\lambda^L)\dot{w}, \partial_t(\mathcal{A}_\lambda^L)w \rangle,$$
$$F_{0,1,1} = \langle \dot{w}_L, \partial_t(\mathcal{A}_\lambda^L)\mathcal{F}_1 \rangle,$$
$$F_{0,1,2} = \langle \mathcal{A}_\lambda^L \mathcal{F}_2, \partial_t(\mathcal{A}_\lambda^L)w \rangle.$$

However, the inner product $\langle \dot{w}_L, \partial_t(\mathcal{A}_\lambda^L)\dot{w} \rangle$ in (3-53) is also not small enough to close our bootstrap by itself. Thus, we use (3-51) again to exchange $\dot{w}_L$ for $\partial_t w_L$:

$$\langle \dot{w}_L, \partial_t(\mathcal{A}_\lambda^L)\dot{w} \rangle = \langle \partial_t w_L, \partial_t(\mathcal{A}_\lambda^L)\dot{w} \rangle - \langle \partial_t(\mathcal{A}_\lambda^L)w, \partial_t(\mathcal{A}_\lambda^L)\dot{w} \rangle - \langle \mathcal{A}_\lambda^L \mathcal{F}_1, \partial_t(\mathcal{A}_\lambda^L)\dot{w} \rangle. \qquad (3\text{-}59)$$

Integrating by parts in time once more,

$$\langle \partial_t w_L, \partial_t(\mathcal{A}_\lambda^L)\dot{w} \rangle - \partial_t \langle w_L, \partial_t(\mathcal{A}_\lambda^L)\dot{w} \rangle = -\langle w_L, \partial_{tt}(\mathcal{A}_\lambda^L)\dot{w} \rangle - \langle w_L, \partial_t(\mathcal{A}_\lambda^L)\dot{w}_t \rangle$$
$$= \langle w_L, \partial_t(\mathcal{A}_\lambda^L)w_2 \rangle - \langle w_L, \partial_{tt}(\mathcal{A}_\lambda^L)\dot{w} \rangle - \langle w_L, \partial_t(\mathcal{A}_\lambda^L)\mathcal{F}_2 \rangle. \quad (3\text{-}60)$$

To sum it up, we obtain a relation similar to (3-58):

$$\langle \dot{w}_L, \partial_t(\mathcal{A}_\lambda^L)\dot{w} \rangle + \partial_t D_{0,2,1} = \langle w_L, \partial_t(\mathcal{A}_\lambda^L)w_2 \rangle + E_{0,2,1} + E_{0,2,2} + F_{0,2,1} + F_{0,2,2}, \qquad (3\text{-}61)$$

where

$$D_{0,2,1} = -\langle w_L, \partial_t(\mathcal{A}_\lambda^L)\dot{w} \rangle,$$
$$E_{0,2,1} = -\langle w_L, \partial_{tt}(\mathcal{A}_\lambda^L)\dot{w} \rangle,$$
$$E_{0,2,2} = -\langle \partial_t(\mathcal{A}_\lambda^L)w, \partial_t(\mathcal{A}_\lambda^L)\dot{w} \rangle,$$
$$F_{0,2,1} = -\langle \mathcal{A}_\lambda^L \mathcal{F}_1, \partial_t(\mathcal{A}_\lambda^L)\dot{w} \rangle,$$
$$F_{0,2,2} = -\langle w_L, \partial_t(\mathcal{A}_\lambda^L)\mathcal{F}_2 \rangle.$$

Raphaël and Rodnianski [2012] directly checked that $\langle w_1, \partial_t(\mathcal{A}_\lambda^L)w_2 \rangle < 0$ in the case $L = 1$. In contrast, when $L \geq 3$, we cannot obtain similar information from $\langle w_L, \partial_t(\mathcal{A}_\lambda^L)w_2 \rangle$ by itself. We pull out the repulsive terms using the Leibniz rule:

$$\langle w_L, \partial_t(\mathcal{A}_\lambda^L)w_2 \rangle = \langle w_L, \partial_t(\widetilde{H}_\lambda)w_L \rangle + \langle w_L, \widetilde{H}_\lambda \partial_t(\mathcal{A}_\lambda^{L-2})w_2 \rangle$$
$$= \langle w_L, \partial_t(\widetilde{H}_\lambda)w_L \rangle + \langle \widetilde{H}_\lambda w_L, \partial_t(\mathcal{A}_\lambda^{L-2})w_2 \rangle. \qquad (3\text{-}62)$$

We observe that the second inner product in (3-62) has the same form as the first inner product in (3-58); we can iterate integration by parts, which leads to the following recurrence equations: for $0 \leq k \leq \frac{1}{2}(L-1)$,

$$\langle \widetilde{H}_\lambda w_L, \partial_t(\mathcal{A}_\lambda^{L-2k})w_{2k} \rangle + \partial_t D_{k,1,1} = \langle \dot{w}_L, \partial_t(\mathcal{A}_\lambda^{L-2k})\dot{w}_{2k} \rangle + E_{k,1,1} + E_{k,1,2} + F_{k,1,1} + F_{k,1,2}, \quad (3\text{-}63)$$

where

$$D_{k,1,1} = \langle \dot{w}_L, \partial_t(\mathcal{A}_\lambda^{L-2k})w_{2k} \rangle,$$
$$E_{k,1,1} = \langle \dot{w}_L, \partial_{tt}(\mathcal{A}_\lambda^{L-2k})w_{2k} \rangle,$$
$$E_{k,1,2} = \langle \partial_t(\mathcal{A}_\lambda^L)\dot{w}, \partial_t(\mathcal{A}_\lambda^{L-2k})w_{2k} \rangle + \langle \dot{w}_L, \partial_t(\mathcal{A}_\lambda^{L-2k})\partial_t(H_\lambda^k)w \rangle,$$
$$F_{k,1,1} = \langle \dot{w}_L, \partial_t(\mathcal{A}_\lambda^{L-2k})H_\lambda^k \mathcal{F}_1 \rangle,$$
$$F_{k,1,2} = \langle \mathcal{A}_\lambda^L \mathcal{F}_2, \partial_t(\mathcal{A}_\lambda^{L-2k})w_{2k} \rangle$$

and

$$\langle \dot{w}_L, \partial_t(\mathcal{A}_\lambda^{L-2k})\dot{w}_{2k}\rangle + \partial_t D_{k,2,1} = \langle w_L, \partial_t(\mathcal{A}_\lambda^{L-2k})w_{2k+2}\rangle + E_{k,2,1} + E_{k,2,2} + F_{k,2,1} + F_{k,2,2}, \quad (3\text{-}64)$$

where

$$D_{k,2,1} = -\langle w_L, \partial_t(\mathcal{A}_\lambda^{L-2k})\dot{w}_{2k}\rangle,$$

$$E_{k,2,1} = -\langle w_L, \partial_{tt}(\mathcal{A}_\lambda^{L-2k})\dot{w}_{2k}\rangle,$$

$$E_{k,2,2} = -\langle \partial_t(\mathcal{A}_\lambda^L)w, \partial_t(\mathcal{A}_\lambda^{L-2k})\dot{w}_{2k}\rangle - \langle w_L, \partial_t(\mathcal{A}_\lambda^{L-2k})\partial_t(H_\lambda^k)\dot{w}\rangle,$$

$$F_{k,2,1} = -\langle \mathcal{A}_\lambda^L \mathcal{F}_1, \partial_t(\mathcal{A}_\lambda^{L-2k})\dot{w}_{2k}\rangle,$$

$$F_{k,2,2} = -\langle w_L, \partial_t(\mathcal{A}_\lambda^{L-2k})\mathcal{F}_2\rangle.$$

We can also pull out the repulsive term like (3-62) from (3-64): for $0 \le k \le \frac{1}{2}(L-3)$,

$$\langle w_L, \partial_t(\mathcal{A}_\lambda^{L-2k})w_{2k+2}\rangle = \langle w_L, \partial_t(\widetilde{H}_\lambda)w_L\rangle + \langle \widetilde{H}_\lambda w_L, \partial_t(\mathcal{A}_\lambda^{L-2k-2})w_{2k+2}\rangle. \quad (3\text{-}65)$$

The displays (3-63), (3-64) and (3-65) allow us to iterate our recurrence relations. For $k = \frac{1}{2}(L-1)$, we can verify that (3-64) is negative from the facts

$$\partial_t(A_\lambda) = \partial_t(A_\lambda^*) = \frac{-\lambda_t}{\lambda}\frac{(\Lambda Z)_\lambda}{r},$$

$$\langle \partial_t(\widetilde{H}_\lambda)w_L, w_L\rangle = \langle \partial_t(A_\lambda A_\lambda^*)w_L, w_L\rangle$$

$$= \langle \partial_t(A_\lambda)A_\lambda^* w_L, w_L\rangle + \langle A_\lambda \partial_t(A_\lambda^*)w_L, w_L\rangle$$

$$= 2\langle \partial_t(A_\lambda)w_{L+1}, w_L\rangle.$$

Hence we write the following decomposition of the term $\langle \widetilde{H}_\lambda w_L, \partial_t(\mathcal{A}_\lambda^L)w\rangle$ of (3-53):

$$\langle \widetilde{H}_\lambda w_L, \partial_t(\mathcal{A}_\lambda^L)w\rangle + \sum_{k=0}^{(L-1)/2}\sum_{i=1}^{2}\partial_t D_{k,i,1} = \frac{L}{2}\langle \partial_t(\widetilde{H}_\lambda)w_L, w_L\rangle + \sum_{k=0}^{(L-1)/2}\sum_{i,j=1}^{2}(E_{k,i,j} + F_{k,i,j}).$$

Similarly, we write the following decomposition the term $\langle \dot{w}_L, \partial_t(\mathcal{A}_\lambda^L)\dot{w}\rangle$ of (3-53):

$$\langle \dot{w}_L, \partial_t(\mathcal{A}_\lambda^L)\dot{w}\rangle + \sum_{k=0}^{(L-1)/2}\sum_{i=1}^{2}\partial_t(1 - \delta_{k,0}\delta_{i,1})D_{k,i,1}$$

$$= \frac{L}{2}\langle \partial_t(\widetilde{H}_\lambda)w_L, w_L\rangle + \sum_{k=0}^{(L-1)/2}\sum_{i,j=1}^{2}(1 - \delta_{k,0}\delta_{i,1})(E_{k,i,j} + F_{k,i,j}).$$

Together with the first term and the last two terms of (3-53), we obtain the following initial identity of $\mathcal{E}_{L+1}$:

$$\partial_t\left\{\frac{\mathcal{E}_{L+1}}{2\lambda^{2L}} + \sum_{k=0}^{(L-1)/2}\sum_{i=1}^{2}(2 - \delta_{k,0}\delta_{i,1})D_{k,i,1}\right\} = \frac{2L+1}{2}\langle \partial_t(\widetilde{H}_\lambda)w_L, w_L\rangle + \langle \widetilde{H}_\lambda w_L, \mathcal{A}_\lambda^L \mathcal{F}_1\rangle + \langle \dot{w}_L, \mathcal{A}_\lambda^L \mathcal{F}_2\rangle$$

$$+ \sum_{k=0}^{(L-1)/2}\sum_{i,j=1}^{2}(2 - \delta_{k,0}\delta_{i,1})(E_{k,i,j} + F_{k,i,j}). \quad (3\text{-}66)$$

Step 3: *Second energy identity.* We find additional corrections from $E_{k,i,1}$, which contain $\partial_{tt}(\mathcal{A}_\lambda^{L-2k})$. More precisely, from Lemma C.1,

$$E_{k,1,1} = \langle \dot{w}_L, \partial_{tt}(\mathcal{A}_\lambda^{L-2k})w_{2k}\rangle$$

$$= \sum_{m=2k}^{L-1} \frac{\lambda_{tt}}{\lambda^{L+1-m}} \langle (\Phi_{m,L,k}^{(1)})_\lambda w_m, \dot{w}_L\rangle + \sum_{m=2k}^{L-1} \frac{O(b_1^2)}{\lambda^{L+2-m}} \langle (\Phi_{m,L,k}^{(2)})_\lambda w_m, \dot{w}_L\rangle$$

and

$$E_{k,2,1} = -\langle w_L, \partial_{tt}(\mathcal{A}_\lambda^{L-2k})\dot{w}_{2k}\rangle$$

$$= -\sum_{m=2k}^{L-1} \frac{\lambda_{tt}}{\lambda^{L+1-m}} \langle (\Phi_{m,L,k}^{(1)})_\lambda \dot{w}_m, w_L\rangle - \sum_{m=2k}^{L-1} \frac{O(b_1^2)}{\lambda^{L+2-m}} \langle (\Phi_{m,L,k}^{(2)})_\lambda \dot{w}_m, w_L\rangle,$$

where $\Phi_{m,L,k}^{(j_1)}(y) := \Phi_{m-2k,L-2k}^{(j_1)}(y)$ with $j_1 = 1, 2$, so that

$$|\Phi_{m,L,k}^{(j_1)}(y)| \lesssim \frac{1}{1+y^{L+2-m}}.$$

Here, we cannot treat λ_{tt} directly because we do not have estimates on second derivatives of the modulation parameters (and we did not set $\lambda_t = -b_1$). Thus, we add $(b_1)_t$ to λ_{tt} and use (3-32):

$$\frac{\lambda_{tt}}{\lambda^{L+1-m}} \langle (\Phi_{m,L,k}^{(1)})_\lambda w_m, \dot{w}_L\rangle = \frac{(\lambda_t + b_1)_t}{\lambda^{L+1-m}} \langle (\Phi_{m,L,k}^{(1)})_\lambda w_m, \dot{w}_L\rangle + \frac{O(b_1^2)}{\lambda^{L+2-m}} \langle (\Phi_{m,L,k}^{(1)})_\lambda w_m, \dot{w}_L\rangle. \quad (3\text{-}67)$$

We then correct (3-67) via integration by parts in time with (3-51):

$$\frac{(\lambda_t + b_1)_t}{\lambda^{L+1-m}} \langle (\Phi_{m,L,k}^{(1)})_\lambda w_m, \dot{w}_L\rangle - \partial_t \left(\frac{\lambda_t + b_1}{\lambda^{L+1-m}} \langle (\Phi_{m,L,k}^{(1)})_\lambda w_m, \dot{w}_L\rangle \right)$$

$$= (\lambda_t + b_1) \left\langle \partial_t \left(\frac{1}{\lambda^{L+1-m}} (\Phi_{m,L,k}^{(1)})_\lambda \right) w_m, \dot{w}_L\right\rangle + \frac{\lambda_t + b_1}{\lambda^{L+1-m}} [\langle (\Phi_{m,L,k}^{(1)})_\lambda \partial_t w_m, \dot{w}_L\rangle + \langle (\Phi_{m,L,k}^{(1)})_\lambda w_m, \partial_t \dot{w}_L\rangle]$$

$$= -\frac{\lambda_t(\lambda_t + b_1)}{\lambda^{L+2-m}} \langle (\Lambda_{m-L}\Phi_{m,L,k}^{(1)})_\lambda w_m, \dot{w}_L\rangle - \frac{\lambda_t + b_1}{\lambda^{L+1-m}} \langle (\Phi_{m,L,k}^{(1)})_\lambda (\dot{w}_m + \partial_t(\mathcal{A}_\lambda^m)w + \mathcal{A}_\lambda^m \mathcal{F}_1), \dot{w}_L\rangle$$

$$+ \frac{\lambda_t + b_1}{\lambda^{L+1-m}} \langle (\Phi_{m,L,k}^{(1)})_\lambda w_m, w_{L+2} - \partial_t(\mathcal{A}_\lambda^L)\dot{w} - \mathcal{A}_\lambda^L \mathcal{F}_2\rangle.$$

We can also obtain the same correction for $E_{k,2,1}$:

$$\frac{(\lambda_t + b_1)_t}{\lambda^{L+1-m}} \langle (\Phi_{m,L,k}^{(1)})_\lambda \dot{w}_m, w_L\rangle - \partial_t \left(\frac{\lambda_t + b_1}{\lambda^{L+1-m}} \langle (\Phi_{m,L,k}^{(1)})_\lambda \dot{w}_m, w_L\rangle \right)$$

$$= -\frac{\lambda_t(\lambda_t + b_1)}{\lambda^{L+2-m}} \langle (\Lambda_{m-L}\Phi_{m,L,k}^{(1)})_\lambda \dot{w}_m, w_L\rangle - \frac{\lambda_t + b_1}{\lambda^{L+1-m}} \langle (\Phi_{m,L,k}^{(1)})_\lambda (w_{m+2} - \partial_t(\mathcal{A}_\lambda^m)\dot{w} - \mathcal{A}_\lambda^m \mathcal{F}_2), w_L\rangle$$

$$+ \frac{\lambda_t + b_1}{\lambda^{L+1-m}} \langle (\Phi_{m,L,k}^{(1)})_\lambda \dot{w}_m, \dot{w}_L + \partial_t(\mathcal{A}_\lambda^L)w + \mathcal{A}_\lambda^L \mathcal{F}_1\rangle.$$

Rearranging the existing errors $E_{k,i,j}$, $F_{k,i,j}$, while introducing a new correction notation $D_{k,i,2}$ and new error notation $E_{k,i,j}^*$, $F_{k,i,j}^*$ for $0 \leq k \leq \frac{1}{2}(L-1)$ and $i = 1, 2$, we have

$$E_{k,i,1} - \partial_t D_{k,i,2} + E_{k,i,2} + F_{k,i,1} + F_{k,i,2} = E_{k,i,1}^* + E_{k,i,2}^* + F_{k,i,1}^* + F_{k,i,2}^*, \quad (3\text{-}68)$$

where

$$D_{k,1,2} = \sum_{m=2k}^{L-1} \frac{\lambda_t + b_1}{\lambda^{L+1-m}} \langle (\Phi_{m,L,k}^{(1)})_\lambda w_m, \dot{w}_L \rangle,$$

$$E_{k,1,1}^* = -\sum_{m=2k}^{L-1} \frac{\lambda_t(\lambda_t + b_1)}{\lambda^{L+2-m}} \langle (\Lambda_{m-L}\Phi_{m,L,k}^{(1)})_\lambda w_m, \dot{w}_L \rangle - \sum_{m=2k}^{L-1} \frac{\lambda_t + b_1}{\lambda^{L+1-m}} \langle (\Phi_{m,L,k}^{(1)})_\lambda (\dot{w}_m + \partial_t(\mathcal{A}_\lambda^m)w), \dot{w}_L \rangle$$

$$+ \sum_{m=2k}^{L-1} \frac{\lambda_t + b_1}{\lambda^{L+1-m}} \langle (\Phi_{m,L,k}^{(1)})_\lambda w_m, w_{L+2} - \partial_t(\mathcal{A}_\lambda^L)\dot{w} \rangle,$$

$$E_{k,1,2}^* = E_{k,1,2} + \sum_{m=2k}^{L-1} \frac{O(b_1^2)}{\lambda^{L+2-m}} \langle (\Phi_{m,L,k}^{(2)})_\lambda w_m, \dot{w}_L \rangle,$$

$$F_{k,1,1}^* = F_{k,1,1} - \sum_{m=2k}^{L-1} \frac{\lambda_t + b_1}{\lambda^{L+1-m}} \langle (\Phi_{m,L,k}^{(1)})_\lambda \mathcal{A}_\lambda^m \mathcal{F}_1, \dot{w}_L \rangle,$$

$$F_{k,1,2}^* = F_{k,1,2} - \sum_{m=2k}^{L-1} \frac{\lambda_t + b_1}{\lambda^{L+1-m}} \langle (\Phi_{m,L,k}^{(1)})_\lambda w_m, \mathcal{A}_\lambda^L \mathcal{F}_2 \rangle,$$

and

$$D_{k,2,2} = -\sum_{m=2k}^{L-1} \frac{\lambda_t + b_1}{\lambda^{L+1-m}} \langle (\Phi_{m,L,k}^{(1)})_\lambda \dot{w}_m, w_L \rangle,$$

$$E_{k,2,1}^* = \sum_{m=2k}^{L-1} \frac{\lambda_t(\lambda_t + b_1)}{\lambda^{L+2-m}} \langle (\Lambda_{m-L}\Phi_{m,L,k}^{(1)})_\lambda \dot{w}_m, w_L \rangle + \sum_{k=2m}^{L-1} \frac{\lambda_t + b_1}{\lambda^{L+1-m}} \langle (\Phi_{m,L,k}^{(1)})_\lambda (w_{m+2} - \partial_t(\mathcal{A}_\lambda^m)\dot{w}), w_L \rangle$$

$$- \sum_{m=2k}^{L-1} \frac{\lambda_t + b_1}{\lambda^{L+1-m}} \langle (\Phi_{m,L,k}^{(1)})_\lambda \dot{w}_m, \dot{w}_L + \partial_t(\mathcal{A}_\lambda^L)w \rangle,$$

$$E_{k,2,2}^* = E_{k,2,2} - \sum_{m=2k}^{L-1} \frac{O(b_1^2)}{\lambda^{L+2-m}} \langle (\Phi_{m,L,k}^{(2)})_\lambda \dot{w}_m, w_L \rangle,$$

$$F_{k,2,1}^* = F_{k,2,1} - \sum_{m=2k}^{L-1} \frac{\lambda_t + b_1}{\lambda^{L+1-m}} \langle (\Phi_{m,L,k}^{(1)})_\lambda \dot{w}_m, \mathcal{A}_\lambda^L \mathcal{F}_1 \rangle,$$

$$F_{k,2,2}^* = F_{k,2,2} - \sum_{m=2k}^{L-1} \frac{\lambda_t + b_1}{\lambda^{L+1-m}} \langle (\Phi_{m,L,k}^{(1)})_\lambda \mathcal{A}_\lambda^m \mathcal{F}_2, w_L \rangle.$$

Hence we obtain the modified energy identity

$$\partial_t \left\{ \frac{\mathcal{E}_{L+1}}{2\lambda^{2L}} + \sum_{k=0}^{(L-1)/2} \sum_{i,j=1}^{2} (2 - \delta_{k,0}\delta_{i,1}) D_{k,i,j} \right\} = \frac{2L+1}{2} \langle \partial_t(\widetilde{H}_\lambda)w_L, w_L \rangle + \langle \widetilde{H}_\lambda w_L, \mathcal{A}_\lambda^L \mathcal{F}_1 \rangle + \langle \dot{w}_L, \mathcal{A}_\lambda^L \mathcal{F}_2 \rangle$$

$$+ \sum_{k=0}^{(L-1)/2} \sum_{i,j=1}^{2} (2 - \delta_{k,0}\delta_{i,1})(E_{k,i,j}^* + F_{k,i,j}^*). \quad (3\text{-}69)$$

Step 4: *Error estimation.* All we need is to estimate all inner products except the repulsive one $\langle \partial_t (\widetilde{H}_\lambda) w_L, w_L \rangle$. We can classify such inner products into two main categories: quadratic terms with respect to w (i.e., $D_{k,i,j}$ and $E^*_{k,i,j}$), or those involving $\mathcal{F}_i$, $i = 1, 2$ (i.e., $F^*_{k,i,j}$ and the last terms of (3-53)).

(i) $D_{k,i,j}$ *terms.* From (C-1) and Lemma C.1, all inner products of $D_{k,i,j}$ can be written as sums of terms of the form, for $0 \le m \le L - 1$,

$$\frac{O(b_1)}{\lambda^{2L}} \langle \Phi_{m,L} \varepsilon_m, \dot{\varepsilon}_L \rangle, \quad \frac{O(b_1)}{\lambda^{2L}} \langle \Phi_{m,L} \dot{\varepsilon}_m, \varepsilon_L \rangle, \quad |\Phi_{m,L}(y)| \lesssim \frac{1}{1 + y^{L+2-m}}.$$

Indeed, the $\Phi_{m,L}$ included in each of the above inner products are different functions (e.g., $\Phi^{(j_1)}_{m-2k,L-2k}$, $\Phi^{(j_2)}_{m,L,k}$, $\Lambda_{m-L} \Phi^{(1)}_{m,L,k}$, ...), but we abuse the notation because they are all rational functions with the same asymptotics. From the coercive property (A-15), we obtain the desired bound for the correction in (3-47):

$$|\langle \Phi_{m,L} \varepsilon_m, \dot{\varepsilon}_L \rangle| \lesssim \left\| \frac{\varepsilon_m}{1 + y^{L+2-m}} \right\|_{L^2} \sqrt{\mathcal{E}_{L+1}} \lesssim C(M) \mathcal{E}_{L+1},$$

$$|\langle \Phi_{m,L} \dot{\varepsilon}_m, \varepsilon_L \rangle| \lesssim \left\| \frac{1 + |\log y|}{1 + y^{L+1-m}} \dot{\varepsilon}_m \right\|_{L^2} \sqrt{\mathcal{E}_{L+1}} \lesssim C(M) \mathcal{E}_{L+1}.$$

(ii) $E^*_{k,i,j}$ *terms.* Similarly, all inner products of $E^*_{k,i,j}$ can be written as sums of terms of the form, for $0 \le m, n \le L - 1$,

$$\frac{O(b_1^2)}{\lambda^{2L+1}} \langle \Phi_{m,L} \varepsilon_m, \dot{\varepsilon}_L \rangle, \quad \frac{O(b_1^2)}{\lambda^{2L+1}} \langle \Phi_{m,L} \dot{\varepsilon}_m, \varepsilon_L \rangle, \quad \frac{O(b_1^2)}{\lambda^{2L+1}} \langle \Phi_{m,L} \dot{\varepsilon}_m, \Phi_{n,L} \varepsilon_n \rangle,$$

$$\frac{O(b_1^2)}{\lambda^{2L+1}} \langle \Phi_{m,L} \dot{\varepsilon}_m, \dot{\varepsilon}_L \rangle, \quad \frac{O(b_1^2)}{\lambda^{2L+1}} \langle \Phi_{m,L} \varepsilon_m, \varepsilon_{L+2} \rangle, \quad \frac{O(b_1^2)}{\lambda^{2L+1}} \langle \Phi_{m,L} \varepsilon_{m+2}, \varepsilon_L \rangle,$$

which are bounded by

$$\frac{b_1^2}{\lambda^{2L+1}} C(M) \mathcal{E}_{L+1}.$$

(iii) $F^*_{k,i,j}$ *and the last two terms of* (3-53). Recalling $\mathcal{F}_1 = \lambda^{-1} \mathcal{F}_\lambda$ and $\mathcal{F}_2 = \lambda^{-2} \dot{\mathcal{F}}_\lambda$, all inner products of $F^*_{k,i,j}$ can be written as sums of terms of the form, for $0 \le m \le L - 1$,

$$\frac{O(b_1)}{\lambda^{2L+1}} \langle \Phi_{m,L} \mathcal{A}^m \mathcal{F}, \dot{\varepsilon}_L \rangle, \quad \frac{O(b_1)}{\lambda^{2L+1}} \langle \Phi_{m,L} \dot{\varepsilon}_m, \mathcal{A}^L \mathcal{F} \rangle, \quad \frac{O(b_1)}{\lambda^{2L+1}} \langle \Phi_{m,L} \varepsilon_m, \mathcal{A}^L \dot{\mathcal{F}} \rangle, \tag{3-70}$$

$$\frac{O(b_1)}{\lambda^{2L+1}} \langle \Phi_{m,L} \mathcal{A}^m \dot{\mathcal{F}}, \varepsilon_L \rangle, \quad \frac{1}{\lambda^{2L+1}} \langle \varepsilon_{L+1}, \mathcal{A}^{L+1} \mathcal{F} \rangle, \quad \frac{1}{\lambda^{2L+1}} \langle \dot{\varepsilon}_L, \mathcal{A}^L \dot{\mathcal{F}} \rangle. \tag{3-71}$$

We claim that $\mathcal{F}$ and $\dot{\mathcal{F}}$ satisfy the following estimates: for $0 \le k \le L - 1$,

$$\|\mathcal{A}^{L+1} \mathcal{F}\|_{L^2} + \|\mathcal{A}^L \dot{\mathcal{F}}\|_{L^2} \lesssim b_1 \left[\frac{b_1^{L+1}}{|\log b_1|} + \sqrt{\frac{\mathcal{E}_{L+1}}{\log M}} \right], \tag{3-72}$$

$$\left\| \frac{1 + |\log y|}{1 + y^{L+1-k}} \mathcal{A}^k \mathcal{F} \right\|_{L^2} \lesssim b_1^{L+2} |\log b_1|^C, \tag{3-73}$$

$$\left\| \frac{1 + |\log y|}{1 + y^{L+1-k}} \mathcal{A}^k \dot{\mathcal{F}} \right\|_{L^2} \lesssim \frac{b_1^{L+1}}{|\log b_1|} + \sqrt{\frac{\mathcal{E}_{L+1}}{\log M}}. \tag{3-74}$$

Assuming the claims (3-72)–(3-74) with the coercivity (A-15), we can estimate $F_{k,i,j}^*$ terms as follows: the three inner products in (3-70) are bounded by

$$\frac{b_1}{\lambda^{2L+1}} C(M) b_1^{L+2} |\log b_1|^C \sqrt{\mathcal{E}_{L+1}}.$$

For the three inner products in (3-71), we obtain the sharp bound

$$\frac{b_1}{\lambda^{2L+1}} \left(\frac{b_1^{L+1}}{|\log b_1|} + \sqrt{\frac{\mathcal{E}_{L+1}}{\log M}} \right) \sqrt{\mathcal{E}_{L+1}}$$

from (3-72), (3-74) and the sharp coercivity bound

$$\left\| \frac{\varepsilon_L}{y(1 + |\log y|)} \right\|_{L^2}^2 \leq C \langle \widetilde{H} \varepsilon_L, \varepsilon_L \rangle \leq C \mathcal{E}_{L+1}.$$

Hence it remains to prove (3-72)–(3-74).

<u>Step 5</u>: Proof of (3-72), (3-73) and (3-74). Recalling (3-11), we have $\boldsymbol{\mathcal{F}} = (\mathcal{F}, \dot{\mathcal{F}})^t$ and

$$\begin{pmatrix} \mathcal{F} \\ \dot{\mathcal{F}} \end{pmatrix} = -\widetilde{\mathbf{Mod}}(t) - \tilde{\boldsymbol{\psi}}_b - \boldsymbol{NL}(\boldsymbol{\varepsilon}) - \boldsymbol{L}(\boldsymbol{\varepsilon}), \quad \boldsymbol{NL}(\boldsymbol{\varepsilon}) = \begin{pmatrix} 0 \\ NL(\varepsilon) \end{pmatrix}, \quad \boldsymbol{L}(\boldsymbol{\varepsilon}) = \begin{pmatrix} 0 \\ L(\varepsilon) \end{pmatrix}.$$

Thus, we will estimate each of the above four errors.

(i) $\tilde{\boldsymbol{\psi}}_b$ *term*. It directly follows from the global and logarithmic weighted bounds of Proposition 2.10.

(ii) $\widetilde{\mathbf{Mod}}(t)$ *term*. Recall (3-9): we have

$$\widetilde{\mathbf{Mod}}(t) = -\left(\frac{\lambda_s}{\lambda} + b_1 \right) \left(\boldsymbol{\Lambda} \boldsymbol{Q} + \sum_{i=1}^{L} b_i \boldsymbol{\Lambda}(\chi_{B_1} T_i) + \sum_{i=2}^{L+2} \boldsymbol{\Lambda}(\chi_{B_1} S_i) \right)$$
$$+ \sum_{i=1}^{L} ((b_i)_s + (i - 1 + c_{b,i}) b_1 b_i - b_{i+1}) \chi_{B_1} \left(T_i + \sum_{j=i+1}^{L+2} \frac{\partial S_j}{\partial b_i} \right). \quad (3\text{-}75)$$

Due to Lemma 3.3, the logarithmic weighted bounds (3-73) and (3-74) are derived from the finiteness of the integrals

$$\int \left| \frac{1 + |\log y|}{1 + y^{L+1-k}} \mathcal{A}^k \left[\Lambda Q + \sum_{i=1}^{L} b_i \Lambda_{1-\bar{i}}(\chi_{B_1} T_i) + \sum_{i=2}^{L+2} \Lambda_{1-\bar{i}}(\chi_{B_1} S_i) \right] \right|^2 \lesssim 1,$$

$$\sum_{i=1}^{L} \int \left| \frac{1 + |\log y|}{1 + y^{L+1-k}} \mathcal{A}^k \left[\chi_{B_1} T_i + \chi_{B_1} \sum_{j=i+1}^{L+2} \frac{\partial S_j}{\partial b_i} \right] \right|^2 \lesssim 1,$$

which comes from the admissibility of T_i and Lemma 2.7. For the global bounds (3-72), we need to gain one extra b_1 as follows: since $A \Lambda Q = 0$, the admissibility of T_i and Lemma 2.7 imply

$$\int \left| \mathcal{A}^{L+1} \Lambda Q + \sum_{i=1}^{L} b_i \mathcal{A}^{L+1-\bar{i}} [\Lambda_{1-\bar{i}}(\chi_{B_1} T_i)] + \sum_{i=2}^{L+2} \mathcal{A}^{L+1-\bar{i}} [\Lambda_{1-\bar{i}}(\chi_{B_1} S_i)] \right|^2$$
$$\lesssim \sum_{i=1}^{L} \int_{y \leq 2B_1} b_1^i \left| \frac{(1 + |\log y|) y^{i-2}}{1 + y^L} \right|^2 + \sum_{i=2}^{L+1} b_1^{2i} + \frac{b_1^{2(L+1)}}{|\log b_1|^2} \lesssim b_1^2.$$

For (3-75), we additionally use the cancellation $\mathcal{A}^L T_i = 0$ for $1 \leq i \leq L$ to estimate

$$\sum_{i=1}^{L} \int |\mathcal{A}^{L+1-\bar{i}}(\chi_{B_1} T_i)|^2 \lesssim \sum_{i=1}^{L} \int_{y \sim B_1} \left| \frac{y^{i-2} \log y}{y^L} \right|^2 \lesssim b_1^2,$$

$$\sum_{j=i+1}^{L+2} \int \left| \mathcal{A}^{L+1-\bar{i}} \left[\chi_{B_1} \frac{\partial S_j}{\partial b_i} \right] \right|^2 \lesssim \sum_{j=i+1}^{L+2} b_1^{2(j-i)} + \frac{b_1^{2(L+1-i)}}{|\log b_1|^2} \lesssim b_1^2.$$

Hence (3-72) comes from Lemma 3.3:

$$\|\mathcal{A}^{L+1}\widetilde{\mathrm{Mod}}(t)\|_{L^2} + \|\mathcal{A}^{L}\dot{\widetilde{\mathrm{Mod}}}(t)\|_{L^2} \lesssim b_1 \left[\frac{b_1^{L+1}}{|\log b_1|} + \sqrt{\frac{\mathcal{E}_{L+1}}{\log M}} \right].$$

For the remaining two terms, $NL(\varepsilon)$ and $L(\varepsilon)$, we follow the approach developed in [Raphaël and Schweyer 2014]. We deal with the cases $y \leq 1$ and $y \geq 1$ separately.

(iii) $NL(\varepsilon)$ *term*: (a) $y \leq 1$. From a Taylor Lagrange formula in Lemma B.1, $NL(\varepsilon)$ also satisfies a Taylor Lagrange formula

$$NL(\varepsilon) = \sum_{i=0}^{(L-1)/2} c_i y^{2i+1} + r_\varepsilon, \tag{3-76}$$

where

$$|c_i| \lesssim C(M)\mathcal{E}_{L+1}, \quad |\mathcal{A}^k r_\varepsilon| \lesssim y^{L-k}|\log y|C(M)\mathcal{E}_{L+1}, \quad 0 \leq k \leq L. \tag{3-77}$$

Since the expansion part of $NL(\varepsilon)$ is an odd function, that of $\mathcal{A}^k NL(\varepsilon)$ also has a single parity from the cancellation $A(y) = O(y^2)$. Using (3-77), we obtain

$$|\mathcal{A}^k NL(\varepsilon)(y)| \lesssim C(M)|\log y|\mathcal{E}_{L+1}, \quad 0 \leq k \leq L, \tag{3-78}$$

and thus we conclude

$$\|\mathcal{A}^L NL(\varepsilon)\|_{L^2(y \leq 1)} + \left\| \frac{1 + |\log y|^C}{1 + y^{L+1-k}} \mathcal{A}^k NL(\varepsilon) \right\|_{L^2(y \leq 1)} \lesssim C(M)\mathcal{E}_{L+1} \lesssim b_1^{2L+1}.$$

(b) $y \geq 1$. Let

$$NL(\varepsilon) = \zeta^2 N_1(\varepsilon), \quad \zeta = \frac{\varepsilon}{y}, \quad N_1(\varepsilon) = \int_0^1 (1-\tau) f''(\widetilde{Q}_b + \tau\varepsilon) \, d\tau. \tag{3-79}$$

We have the following bounds for $i \geq 0$, $j \geq 1$ and $1 \leq i + j \leq L$:

$$\left\| \frac{\partial_y^i \zeta}{y^{j-1}} \right\|_{L^\infty(y \geq 1)} + \left\| \frac{\partial_y^i \zeta}{y^j} \right\|_{L^2(y \geq 1)} \lesssim |\log b_1|^{C(K)} b_1^{m_{i+j+1}}, \quad \|\zeta\|_{L^2(y \geq 1)} \lesssim 1, \tag{3-80}$$

$$|N_1(\varepsilon)| \lesssim 1, \quad |\partial_y^k N_1(\varepsilon)| \lesssim |\log b_1|^{C(K)} \left[\frac{1}{y^{k+1}} + b_1^{m_{k+1}} \right], \quad 1 \leq k \leq L, \tag{3-81}$$

where

$$m_{k+1} = \begin{cases} kc_1 & \text{if } 1 \leq k \leq L-2, \\ L & \text{if } k = L-1, \\ L+1 & \text{if } k = L. \end{cases} \tag{3-82}$$

The estimates (3-80) are consequences of Lemma B.1 and the orbital stability (3-6). We can prove the estimates (3-81) by borrowing the proof of (3-77) in [Raphaël and Schweyer 2014] (see page 1768 line 1 in that work), since we can obtain the crude bound

$$|\partial_y^k \widetilde{Q}_b| \lesssim |\log b_1|^C \left[\frac{1}{y^{k+1}} + \sum_{i=1}^{(L+1)/2} b_1^{2i} y^{2i-1-k} 1_{y \leq 2B_1} \right]$$

$$\lesssim \frac{|\log b_1|^C}{y^{k+1}}.$$

Returning to the estimates for $NL(\varepsilon)$, we have the trivial bound,

$$\text{for } 0 \leq k \leq L, \quad \left| \frac{1 + |\log y|^C}{y^{L+1-k}} \mathcal{A}^k NL(\varepsilon) \right| \lesssim \left| \frac{\mathcal{A}^k NL(\varepsilon)}{y^{L-k}} \right|,$$

and (3-79) and (3-81) imply

$$\left| \frac{\mathcal{A}^k NL(\varepsilon)}{y^{L-k}} \right| \lesssim \sum_{k=0}^{L} \frac{|\partial_y^k NL(\varepsilon)|}{y^{L-k}}$$

$$\lesssim \sum_{k=0}^{L} \frac{1}{y^{L-k}} \sum_{i=0}^{k} |\partial_y^i \zeta^2| |\partial_y^{k-i} N_1(\varepsilon)|$$

$$\lesssim \sum_{k=0}^{L} \frac{|\log b_1|^{C(K)}}{y^{L-k}} \left[|\partial_y^k \zeta^2| + \sum_{i=0}^{k-1} b_1^{m_{k-i+1}} |\partial_y^i \zeta^2| \right]$$

$$\lesssim \sum_{k=0}^{L} \frac{|\log b_1|^{C(K)}}{y^{L-k}} \left[\sum_{i=0}^{k} |\partial_y^i \zeta| |\partial_y^{k-i} \zeta| + \sum_{i=0}^{k-1} \sum_{j=0}^{i} b_1^{m_{k-i+1}} |\partial_y^j \zeta| |\partial_y^{i-j} \zeta| \right].$$

Writing $I_1 = k - i$, $I_2 = i$, there exists $J_2 \in \mathbb{N}$ such that

$$\max(0, 1-i) \leq J_2 \leq \min(L+1-k, L-i), \quad J_1 = L+1-k-J_2,$$

and we have

$$1 \leq I_1 + J_1 \leq L, \quad 1 \leq I_2 + J_2 \leq L, \quad I_1 + I_2 + J_1 + J_2 = L + 1.$$

Thus,

$$\left\| \frac{\partial_y^i \zeta \cdot \partial_y^{k-i} \zeta}{y^{L-k}} \right\|_{L^2(y \geq 1)} \leq \left\| \frac{\partial_y^{I_1} \zeta}{y^{J_1-1}} \right\|_{L^\infty(y \geq 1)} \left\| \frac{\partial_y^{I_2} \zeta}{y^{J_2}} \right\|_{L^2(y \geq 1)}$$

$$\lesssim |\log b_1|^{C(K)} b_1^{m_{I_1+J_1+1}} b_1^{m_{I_2+J_2+1}} \lesssim b_1^{\delta(L)} b_1^{L+2}$$

since

$$m_{I_1+J_1+1} + m_{I_2+J_2+1} = \begin{cases} (L+1)c_1 & \text{if } I_1 + J_1 < L - 1 \text{ and } I_2 + J_2 < L - 1, \\ L + 2c_1 & \text{if } I_1 + J_1 = L - 1 \text{ or } I_2 + J_2 = L - 1, \\ L + 1 + c_1 & \text{if } I_1 + J_1 = L \text{ or } I_2 + J_2 = L, \end{cases}$$

$$> L + 2.$$

We calculate the latter term similarly except for the cases $k = L$ and $0 \le i = j \le k - 1$. Here, we use the energy bound $\|\zeta\|_{L^2(y \ge 1)} \lesssim 1$ and obtain

$$|\log b_1|^{C(K)} b_1^{m_{L-i+1}} \|\partial_y^i \zeta \cdot \zeta\|_{L^2(y \ge 1)} \lesssim |\log b_1|^{C(K)} b_1^{m_{L-i+1}} \|\partial_y^i \zeta\|_{L^\infty(y \ge 1)}$$

$$\lesssim \begin{cases} |\log b_1|^{C(K)} b_1^{(L+1)c_1} & \text{if } 0 < i < L - 1, \\ |\log b_1|^{C(K)} b_1^{L+2c_1} & \text{if } i = 1, \, L - 2, \\ |\log b_1|^{C(K)} b_1^{L+1+c_1} & \text{if } i = 0, \, L - 1, \end{cases}$$

$$\lesssim b_1^{\delta(L)} b_1^{L+2}.$$

The remaining case can be estimated by the following inequalities: since $k - i \ge 1$, $I_1 + J_1 \ge 1$, $I_2 + J_2 \ge 1$ and $I_1 + I_2 + J_1 + J_2 = L + 1 - (k - i)$,

$$|\log b_1|^{C(K)} b_1^{m_{k-i+1} + m_{I_1+J_1+1} + m_{I_2+J_2+1}} \lesssim \begin{cases} |\log b_1|^{C(K)} b_1^{(L+1)c_1} & \text{if } k - i < L - 1, \\ |\log b_1|^{C(K)} b_1^{L+2c_1} & \text{if } k - i = L - 1, \end{cases}$$

$$\lesssim b_1^{\delta(L)} b_1^{L+2}.$$

(iv) $L(\varepsilon)$ *term*: (a) $y \le 1$. Similar to the case $NL(\varepsilon)$, we obtain a Taylor Lagrange formula for $L(\varepsilon)$:

$$L(\varepsilon) = b_1^2 \left[\sum_{i=0}^{(L-1)/2} \tilde{c}_i y^{2i+1} + \tilde{r}_\varepsilon \right], \tag{3-83}$$

where

$$|\tilde{c}_i| \lesssim C(M)\sqrt{\mathcal{E}_{L+1}}, \quad |\mathcal{A}^k \tilde{r}_\varepsilon| \lesssim y^{L-k} |\log y| C(M)\sqrt{\mathcal{E}_{L+1}}, \quad 0 \le k \le L. \tag{3-84}$$

Using the cancellation $A(y) = O(y^2)$ and (3-84), we obtain

$$|\mathcal{A}^k L(\varepsilon)(y)| \lesssim C(M) b_1^2 |\log y| \sqrt{\mathcal{E}_{L+1}}, \quad 0 \le k \le L, \tag{3-85}$$

and thus we conclude

$$\|\mathcal{A}^L L(\varepsilon)\|_{L^2(y \le 1)} + \left\| \frac{1 + |\log y|^C}{1 + y^{L+1-k}} \mathcal{A}^k L(\varepsilon) \right\|_{L^2(y \le 1)} \lesssim C(M) b_1^2 \sqrt{\mathcal{E}_{L+1}}.$$

(b) $y \ge 1$. Let

$$L(\varepsilon) = \varepsilon N_2(\alpha_b), \quad N_2(\alpha_b) = \frac{f'(\widetilde{Q}_b) - f'(Q)}{y^2} = \frac{\chi_{B_1} \alpha_b}{y^2} \int_0^1 f''(Q + \tau \chi_{B_1} \alpha_b) \, d\tau.$$

Similar to (3-81), we have the bound

$$|\partial_y^k N_2| \lesssim \frac{b_1^2 |\log b_1|^C}{y^{k+1}}, \quad 0 \le k \le L. \tag{3-86}$$

Since $L(\varepsilon)$ satisfies the pointwise bound

$$\left| \frac{\mathcal{A}^k L(\varepsilon)}{y^{L-k}} \right| \lesssim \sum_{i=0}^k \frac{|\partial_y^i \varepsilon| \, |\partial_y^{k-i} N_2|}{y^{L-k}} \lesssim b_1^2 |\log b_1|^C \sum_{i=0}^k \frac{|\partial_y^i \varepsilon|}{y^{L+1-i}}, \tag{3-87}$$

this yields the desired result. □

4. Proof of the main theorem

4.1. *Proof of Proposition 3.2.* Step 1: *Control of the scaling law.* We have the bound

$$-\frac{\lambda_s}{\lambda} = \frac{c_1}{s} + \frac{d_1}{s \log s} + O\left(\frac{1}{s(\log s)^\beta}\right).$$

We rewrite this as

$$\left|\frac{d}{ds}(\log(s^{c_1}(\log s)^{d_1}\lambda(s)))\right| \lesssim \frac{1}{s(\log s)^\beta};$$

integration and (3-23) give

$$\lambda(s) = \frac{s_0^{c_1}(\log s_0)^{d_1}}{s^{c_1}(\log s)^{d_1}}\left(1 + O\left(\frac{1}{(\log s_0)^{\beta-1}}\right)\right). \tag{4-1}$$

Note that

$$\frac{d}{ds}\left(\frac{b_1^{2n}(\log b_1)^{2m}}{\lambda^{2k-2}}\right) = 2\frac{b_1^{2n-1}(\log b_1)^{2m}}{\lambda^{2k-2}}\left[(k-1)b_1^2 + b_{1s}\left(n + \frac{m}{\log b_1}\right) + O(b_1^{L+2})\right]. \tag{4-2}$$

From Lemma 3.3 with (2-118), (2-115) and (3-28),

$$(k-1)b_1^2 + b_{1s}\left(n + \frac{m}{\log b_1}\right) = (k-1)b_1^2 + \left(b_2 - c_{b_1,1}b_1^2\right)\left(n + \frac{m}{\log b_1}\right) + O(b_1^{L+2})$$

$$= (k-1)b_1^2 + nb_2 + \frac{2mb_2 - nb_1^2}{2\log b_1} + O\left(\frac{b_1^2}{(\log b_1)^2}\right)$$

$$= \frac{(k-1)c_1^2 + nc_2}{s^2} + \frac{2(k-1)c_1d_1 - nd_2 - mc_2 + \frac{1}{2}nc_1^2}{s^2 \log s} + O\left(\frac{1}{s^2(\log s)^\beta}\right).$$

The recurrence relations (2-116) and (2-117) imply

$$(k-1)c_1^2 + nc_2 = c_1\left((k-1)\frac{\ell}{\ell-1} - n\right)$$

and

$$2(k-1)c_1d_1 - nd_2 + \frac{1}{2}nc_1^2 = d_1(2(k-1)c_1 + n) < 0.$$

Hence, if we set $n = L+1$ and $m = -1$ for $k = L+1$, $c_1 \geq L/(L-1)$ implies

$$(k-1)b_1^2 + b_{1s}\left(n + \frac{m}{\log b_1}\right) \geq \frac{1}{s^2}\left(\frac{c_1}{L-1} + O\left(\frac{1}{\log s}\right)\right) > 0,$$

and, if we set $n = (k-1)c_1$ and $m = m(k, L)$ large enough for $k \leq L$,

$$(k-1)b_1^2 + b_{1s}\left(n + \frac{m}{\log b_1}\right) \geq \frac{c_1}{s^2 \log s}\left(\frac{m}{2} + O\left(\frac{1}{(\log s)^{\beta-1}}\right)\right) > 0$$

for all $s \in [s_0, s^*)$ with sufficiently large s_0. Thus,

$$\frac{b_1^{2(L+1)}(0)}{(\log b_1(0))^2 \lambda^{2L}(0)} \leq \frac{b_1^{2(L+1)}(t)}{(\log b_1(t))^2 \lambda^{2L}(t)} \tag{4-3}$$

and

$$\frac{b_1^{2(k-1)c_1}(0)|\log b_1(0)|^m}{\lambda^{2(k-1)}(0)} \leq \frac{b_1^{2(k-1)c_1}(t)|\log b_1(t)|^m}{\lambda^{2(k-1)}(t)}. \tag{4-4}$$

<u>Step 2</u>: *Improved bound on $\mathcal{E}_{L+1}$.* We integrate the Lyapunov monotonicity (3-47) and inject the bootstrap bounds (3-21) and (3-25):

$$
\mathcal{E}_{L+1}(t) \lesssim \frac{\lambda^{2L}(t)}{\lambda^{2L}(0)}(1 + b_1 C(M))\mathcal{E}_{L+1}(0) + b_1 C(M)\mathcal{E}_{L+1}(t)
$$
$$
+ \left[\frac{K}{\sqrt{\log M}} + \sqrt{K}\right]\lambda^{2L}(t) \int_0^t \frac{b_1}{\lambda^{2L+1}} \frac{b_1^{2(L+1)}}{|\log b_1|^2}\, d\tau
$$
$$
\lesssim \frac{b_1^{2(L+1)}(t)}{|\log b_1(t)|^2} + \left[\frac{K}{\sqrt{\log M}} + \sqrt{K}\right]\lambda^{2L}(t) \int_0^t \frac{b_1}{\lambda^{2L+1}} \frac{b_1^{2(L+1)}}{|\log b_1|^2}. \tag{4-5}
$$

To deal with the integral in (4-5), one can directly replace λ and b_1 with functions of s using (4-1) and (2-118). However, the fact that s_0 in (4-1) depends on the bootstrap constant K requires (more) care in direct substitution. On behalf of this approach, we integrate by parts using (4-2), (4-3) and the fact $c_1 \geq L/(L-1)$:

$$
\int_0^t \frac{b_1}{\lambda^{2L+1}} \frac{b_1^{2(L+1)}}{|\log b_1|^2} = -\int_0^t \frac{\lambda_t}{\lambda^{2L+1}} \frac{b_1^{2(L+1)}}{|\log b_1|^2} + \int_0^t O(b_1^{L+2}) \frac{b_1^{2(L+1)}}{\lambda^{2L+1}|\log b_1|^2}
$$
$$
= \frac{1}{2L}\left[\frac{b_1^{2(L+1)}(t)}{\lambda^{2L}(t)|\log b_1(t)|^2} - \frac{b_1^{2(L+1)}(0)}{\lambda^{2L}(0)|\log b_1(0)|^2}\right]
$$
$$
- \frac{1}{2L}\int_0^t \frac{1}{\lambda^{2L}}\left(\frac{b_1^{2(L+1)}}{|\log b_1|^2}\right)_t + \int_0^t O(b_1^{L+2})\frac{b_1^{2(L+1)}}{\lambda^{2L+1}|\log b_1|^2}
$$
$$
\leq \frac{b_1^{2(L+1)}(t)}{\lambda^{2L}(t)|\log b_1(t)|^2} + \int_0^t \frac{b_1}{\lambda^{2L+1}}\left(\frac{L^2-1}{L^2} + \frac{C}{|\log b_1|}\right)\frac{b_1^{2(L+1)}}{|\log b_1|^2},
$$

and we obtain the bound

$$
\int_0^t \frac{b_1}{\lambda^{2L+1}} \frac{b_1^{2(L+1)}}{|\log b_1|^2} \lesssim \frac{b_1^{2(L+1)}(t)}{\lambda^{2L}(t)|\log b_1(t)|^2},
$$

and therefore,

$$
\mathcal{E}_{L+1}(t) \lesssim \left[1 + \frac{K}{\sqrt{\log M}} + \sqrt{K}\right]\frac{b_1^{2(L+1)}(t)}{|\log b_1(t)|^2} \leq \frac{K}{2}\frac{b_1^{2(L+1)}(t)}{|\log b_1(t)|^2}. \tag{4-6}
$$

<u>Step 3</u>: *Improved bound on $\mathcal{E}_k$.* We now claim the improved bound on the intermediate energies: for $2 \leq k \leq L$,

$$
\mathcal{E}_k \leq b_1^{2(k-1)c_1}|\log b_1|^{C+K/2}. \tag{4-7}
$$

This follows from the monotonicity formula, for $2 \leq k \leq L$,

$$
\frac{d}{dt}\left\{\frac{\mathcal{E}_k}{\lambda^{2k-2}}\right\} \leq C\frac{b_1|\log b_1|^C}{\lambda^{2k-1}}(\sqrt{\mathcal{E}_{k+1}} + b_1^k + b_1^{\delta(k)+(k-1)c_1})\sqrt{\mathcal{E}_k} \tag{4-8}
$$

for some universal constants C, $\delta > 0$ independent of the bootstrap constant K. Estimate (4-7) will be proved in Appendix D. We integrate the above monotonicity formula ($K/2$ comes from $\sqrt{\mathcal{E}_k}$) and obtain

$$
\mathcal{E}_k \lesssim b_1^{2(k-1)c_1}|\log b_1|^{C+K/2} + \lambda^{2k-2}(t)\int_0^t \frac{b_1^{1+2(k-1)c_1}}{\lambda^{2k-1}}|\log b_1|^{C+K/2}. \tag{4-9}
$$

In this case, we directly substitute λ and b_1 with functions of s since the possible large coefficient can be absorbed by $|\log b_1|^C$. From (4-1), (2-114) and (2-118),

$$\lambda^{2k-2}(t) \int_0^t \frac{b_1^{1+2(k-1)c_1}}{\lambda^{2k-1}} |\log b_1|^{C+K/2} \, d\tau = \lambda^{2k-2}(s) \int_{s_0}^s \frac{b_1^{1+2(k-1)c_1}}{\lambda^{2k-2}} |\log b_1|^{C+K/2} \, d\sigma$$

$$\lesssim \frac{(\log s)^{C+K/2}}{s^{2(k-1)c_1}} \int_{s_0}^s \frac{1}{\sigma} \, d\sigma$$

$$\lesssim b_1^{2(k-1)c_1} |\log b_1|^{C+K/2}. \tag{4-10}$$

However, these improved bounds (4-7) are inadequate to close the bootstrap bounds when $\ell = L$ (3-26) and when $\ell = L - 1$ (3-27) due to the logarithm factor. In these cases, we employ alternative energies defined by

$$\widehat{\mathcal{E}}_\ell := \langle \hat{\varepsilon}_\ell, \hat{\varepsilon}_\ell \rangle + \langle \dot{\hat{\varepsilon}}_{\ell-1}, \dot{\hat{\varepsilon}}_{\ell-1} \rangle. \tag{4-11}$$

We can easily check that

$$\widehat{\mathcal{E}}_\ell = \mathcal{E}_\ell + O(b_1^{2\ell} |\log b_1|^2).$$

Then we have the monotonicity formulae

$$\frac{d}{dt} \left\{ \frac{\widehat{\mathcal{E}}_\ell}{\lambda^{2\ell-2}} + O\left(\frac{b_1^{2\ell} |\log b_1|^2}{\lambda^{2\ell-2}} \right) \right\} \leq \frac{b_1^{\ell+1} |\log b_1|^\delta}{\lambda^{2\ell-1}} (b_1^\ell |\log b_1| + \sqrt{\mathcal{E}_\ell}). \tag{4-12}$$

Integrating (4-12), the initial bounds (3-21) and the bootstrap bounds (3-26), (3-27) imply

$$\frac{\widehat{\mathcal{E}}_\ell(t)}{\lambda^{2(\ell-1)}(t)} \lesssim \frac{b_1^{2\ell} |\log b_1|^2(t)}{\lambda^{2\ell-2}(t)} + \frac{\widehat{\mathcal{E}}_\ell(0) + b_1^{2\ell}(0) |\log b_1(0)|^2}{\lambda^{2(\ell-1)}(0)} + \int_0^t \frac{b_1^{\ell+1} |\log b_1|^\delta}{\lambda^{2\ell-1}} (b_1^\ell |\log b_1| + \sqrt{\mathcal{E}_\ell}) \, d\tau$$

$$\lesssim 1 + \int_0^t \frac{b_1^{\ell+1} |\log b_1|^{\delta'}}{\lambda^\ell} \, d\tau \lesssim 1 + \int_{s_0}^s \frac{1}{\sigma (\log \sigma)^{\ell/(\ell-1) - \delta'}} \, d\sigma \lesssim \frac{K}{2}.$$

The monotonicity formulae (4-8), (4-12) are proved in Appendix D.

Remark. We remark that the exponent $1 + 2(k-1)c_1$ of b_1 in (4-9) can be replaced by $1 + \delta + 2(k-1)c_1$ for some small $\delta > 0$ when $2 \leq k \leq \ell - 1$, so we can improve the bound (4-10) to $b_1^{2(k-1)c_1 + \delta} |\log b_1|^C$. Hence, for $2 \leq k \leq \ell$, we get the uniform bounds

$$\mathcal{E}_k \lesssim \lambda^{2k-2}. \tag{4-13}$$

Step 4: *Control of stable/unstable parameters.* We make use of the modified modulation parameters $\tilde{b} = (b_1, \ldots, b_{L-1}, \tilde{b}_L)$ with $\tilde{b}_L$ given by (3-41) and the corresponding fluctuation $\tilde{V} = P_\ell \tilde{U}$, where $\tilde{U} = (\tilde{U}_1, \ldots, \tilde{U}_\ell)$ is defined by

$$\frac{\tilde{U}_k}{s^k (\log s)^\beta} = \tilde{b}_k - b_k^e, \quad 1 \leq k \leq \ell.$$

We note that the existence of $V(s_0)$ in Proposition 3.2 is equivalent to the existence of $\tilde{V}(s_0)$ from the remark on page 2453 and (3-42) in view of

$$|V - \tilde{V}| \lesssim s^L |\log s|^\beta |b_L - \tilde{b}_L| \lesssim s^L |\log s|^\beta b_1^{L+1-C\delta} \lesssim \frac{1}{s^{1/2}}. \tag{4-14}$$

Hence we can replace $\widetilde{V}$ for all the V of the initial assumptions (3-22), (3-24) and bootstrap bounds (3-28), (3-29) in Section 3.3. In particular, we replace the assumption (3-31) with

$$\tilde{s}^* < \infty \quad \text{for all } (V_2(s_0), \ldots, V_\ell(s_0)) \in \mathcal{B}^{\ell-1}, \tag{4-15}$$

where $\tilde{s}^*$ denotes the modified exit time to indicate that V has been changed to $\widetilde{V}$.

We start by closing the bootstrap bounds for the stable parameters b_L (for the case $\ell = L - 1$) and $\widetilde{V}_1$, then we rule out the assumption of the unstable parameters $(\widetilde{V}_2(s), \ldots, \widetilde{V}_\ell(s))$ via showing a contradiction by Brouwer's fixed point theorem.

(i) *Stable parameter b_L when $\ell = L - 1$:* Recalling Lemma 3.4, we have

$$|(\tilde{b}_L)_s + (L - 1 + c_{b,L})b_1\tilde{b}_L| \lesssim \frac{\sqrt{\mathcal{E}_{L+1}}}{\sqrt{|\log b_1|}}. \tag{4-16}$$

Note that $c_1 = (L - 1)/(L - 2)$ and $b_1 \sim c_1/s + d_1/(s \log s)$. Then, from (3-28) and (4-16),

$$\begin{aligned}
\frac{d}{ds}(s^{(L-1)c_1}(\log s)^{3/2}\tilde{b}_L) &= s^{(L-1)c_1-1}(\log s)^{3/2}\left((L-1)c_1 + \frac{3/2}{\log s}\right)\tilde{b}_L \\
&\quad - s^{(L-1)c_1}(\log s)^{3/2}\left((L-1+c_{b,L})b_1\tilde{b}_L + O\left(\frac{\sqrt{\mathcal{E}_{L+1}}}{\sqrt{|\log b_1|}}\right)\right) \\
&= s^{(L-1)c_1-1}(\log s)^{3/2}O\left(\frac{1}{s^L(\log s)^{1+\beta}} + \frac{1}{s^L(\log s)^{3/2}}\right) \\
&= O(s^{(L-1)c_1-L-1}).
\end{aligned}$$

We integrate the above equation and estimate using the initial condition (3-22)

$$|b_L(s)| \lesssim b_1^{L+1-C\delta} + \frac{s_0^{(L-1)c_1}(\log s_0)^{3/2}|\tilde{b}_L(s_0)|}{s^{(L-1)c_1}(\log s)^{3/2}} + \frac{1 + (s_0/s)^{(L-1)c_1-L}}{s^L(\log s)^{3/2}} \leq \frac{1/2}{s^L(\log s)^{\beta}}$$

with the fact $(L-1)c_1 > L$. Here, we choose $\beta = \frac{5}{4}$.

To control the modes $\widetilde{V}$, we rewrite (2-119) for our $\tilde{b}$ as follows:

$$s(\widetilde{U})_s - A_\ell\widetilde{U} = O\left(\frac{1}{(\log s)^{3/2-\beta}}\right) \tag{4-17}$$

using (2-122) and Lemmas 3.3 and 3.4. Here, the reduced exponent $\frac{3}{2}$ comes from (4-16). By the definition of $\widetilde{V}$, (4-17) is equivalent to

$$s(\widetilde{V})_s - D_\ell\widetilde{V} = O\left(\frac{1}{(\log s)^{3/2-\beta}}\right), \tag{4-18}$$

where D_ℓ is given by (2-121).

(ii) *Stable mode $\widetilde{V}_1$:* The first coordinate of (4-18) can be written as

$$s(\widetilde{V}_1)_s + \widetilde{V}_1 = (s\widetilde{V}_1)_s = O\left(\frac{1}{(\log s)^{3/2-\beta}}\right).$$

Hence we improve the bound for $\widetilde{V}_1(s)$ from the initial assumption (3-22):

$$|\widetilde{V}_1(s)| \lesssim \frac{s_0}{s}|\widetilde{V}_1(s_0)| + \frac{C}{s}\int_{s_0}^{s}\frac{d\tau}{(\log \tau)^{3/2-\beta}} \leq \frac{1}{2}.$$

(iii) *Unstable mode $\widetilde{V}_k$, $2 \leq k \leq \ell$*: Our goal is to construct a continuous map $f : \mathcal{B}^{\ell-1} \to \mathcal{S}^{\ell-1}$ defined as

$$f(\widetilde{V}_2(s_0), \ldots, \widetilde{V}_\ell(s_0)) = (\widetilde{V}_2(\tilde{s}^*), \ldots, \widetilde{V}_\ell(\tilde{s}^*)).$$

The assumption (4-15) yields that f can be well-defined on $\mathcal{B}^{\ell-1}$ and the improved bootstrap bounds give the exit condition $(\widetilde{V}_2(\tilde{s}^*), \ldots, \widetilde{V}_\ell(\tilde{s}^*)) \in \mathcal{S}^{\ell-1}$.

We obtain the outgoing behavior of the flow map $s \mapsto (\widetilde{V}_2, \ldots, \widetilde{V}_\ell)$ from (4-18): for all time $s \in [s_0, \tilde{s}^*]$ such that $\sum_{i=2}^{\ell} \widetilde{V}_i^2 \geq \frac{1}{2}$,

$$\frac{d}{ds}\left(\sum_{i=2}^{\ell}\widetilde{V}_i^2\right) = 2\sum_{i=2}^{\ell}(\widetilde{V}_i)_s\widetilde{V}_i = \frac{2}{s}\sum_{i=2}^{\ell}\left[\frac{i}{\ell-1}\widetilde{V}_i^2 + O\left(\frac{1}{(\log s)^{3/2-\beta}}\right)\right] > 0. \tag{4-19}$$

We note that (4-19) implies two key results. First, (4-19) allows us to prove the continuity of f by showing the continuity of the map $(\widetilde{V}_2(s_0), \ldots, \widetilde{V}_\ell(s_0)) \mapsto \tilde{s}^*$ with some standard arguments (see [Côte et al. 2011, Lemma 6]).

Second, if we choose $s = s_0$ and $(\widetilde{V}_2(s_0), \ldots, \widetilde{V}_\ell(s_0)) \in \mathcal{S}^{\ell-1}$, we have $\sum_{i=2}^{\ell}\widetilde{V}_i^2(s) > 1$ for any $s > s_0$, and so $\tilde{s}^* = s_0$. Hence f is an identity map on $\mathcal{S}^{\ell-1}$ itself, which contradicts to Brouwer's fixed point theorem. $\qquad\square$

4.2. *Proof of Theorem 1.2.* Recall that there exists $c(u_0, \dot{u}_0) > 0$ such that

$$\lambda(s) = \frac{c(u_0, \dot{u}_0)}{s^{c_1}(\log s)^{d_1}}\left[1 + O\left(\frac{1}{(\log s_0)^{\beta-1}}\right)\right].$$

Using $T - t = \int_s^{\infty}\lambda(s)\,ds < \infty$, we have $T < \infty$ and

$$(T-t)^{\ell-1} = c'(u_0, \dot{u}_0)s^{-1}(\log s)^{\ell/(\ell-1)}[1 + o_{t\to T}(1)] = c''(u_0, \dot{u}_0)\lambda(s)^{(\ell-1)/\ell}(\log s)[1 + o_{t\to T}(1)].$$

Therefore, we obtain

$$\lambda(t) = c'''(u_0, \dot{u}_0)\frac{(T-t)^\ell}{|\log(T-t)|^{\ell/(\ell-1)}}[1 + o_{t\to T}(1)].$$

The strong convergence (1-13) follows as in [Raphaël and Rodnianski 2012].

Appendix A: Coercive properties

We recall that $\mathbf{\Phi}_M = (\Phi_M, 0)^t$, and hence the orthogonality conditions (3-5) are equivalent to

$$\langle \varepsilon, H^i\Phi_M\rangle = \langle \dot{\varepsilon}, H^i\Phi_M\rangle = 0, \quad 0 \leq i \leq \tfrac{1}{2}(L-1). \tag{A-1}$$

In this section, we claim that the above equivalent orthogonality conditions yield the coercive property of the higher-order energy $\mathcal{E}_{k+1}$:

$$\mathcal{E}_{k+1} = \langle \varepsilon_{k+1}, \varepsilon_{k+1}\rangle + \langle \dot{\varepsilon}_k, \dot{\varepsilon}_k\rangle, \quad 1 \leq k \leq L. \tag{A-2}$$

Our desired result is deduced from the coercivity of $\{\|v_m\|_{L^2}^2\}_{m=1}^{L+1}$ under the orthogonality conditions

$$\langle v, H^i \Phi_M \rangle = 0, \quad 0 \leq i \leq \left\lfloor \tfrac{1}{2}(m-1) \right\rfloor. \tag{A-3}$$

First, we restate Lemma B.5 of [Raphaël and Schweyer 2014], which established the coercivity of $\|v_m\|_{L^2}^2$ when m is even. For the rest of the paper, we use [RS14] to abbreviate this work.

Lemma A.1 (coercivity of $\|v_{2k+2}\|_{L^2}^2$). *Let $0 \leq k \leq \frac{1}{2}(L-1)$ and $M = M(L) > 0$ be a large constant. Then there exists $C(M) > 0$ such that the following holds. For all radially symmetric v with*

$$\int |v_{2k+2}|^2 + \int \frac{|v_{2k+1}|^2}{y^2(1+y^2)}$$
$$+ \sum_{i=0}^{k} \int \frac{|v_{2i-1}|^2}{y^6(1+|\log y|^2)(1+y^{4(k-i)})} + \frac{|v_{2i}|^2}{y^4(1+|\log y|^2)(1+y^{4(k-i)})} < \infty \tag{A-4}$$

(we write $v_{-1} = 0$) and (A-3) *for $m = 2k+2$, we have*

$$\int |v_{2k+2}|^2 \geq C(M) \left\{ \int \frac{|v_{2k+1}|^2}{y^2(1+|\log y|^2)} \right.$$
$$\left. + \sum_{i=0}^{k} \int \left[\frac{|v_{2i-1}|^2}{y^6(1+|\log y|^2)(1+y^{4(k-i)})} + \frac{|v_{2i}|^2}{y^4(1+|\log y|^2)(1+y^{4(k-i)})} \right] \right\}. \tag{A-5}$$

We additionally prove the coercivity of $\|v_m\|_{L^2}^2$ when m is odd, which is an unnecessary step in [RS14].

Lemma A.2 (coercivity of $\|v_{2k+1}\|_{L^2}^2$). *Let $1 \leq k \leq \frac{1}{2}(L-1)$ and $M = M(L) > 0$ be a large constant. Then there exists $C(M) > 0$ such that the following holds. For all radially symmetric v with*

$$\int |v_{2k+1}|^2 + \int \frac{|v_{2k}|^2}{y^2} + \int \frac{|v_{2k-1}|^2}{y^4(1+|\log y|^2)}$$
$$+ \sum_{i=0}^{k-1} \int \frac{|v_{2i-1}|^2}{y^6(1+|\log y|^2)(1+y^{4(k-i)-2})} + \frac{|v_{2i}|^2}{y^4(1+|\log y|^2)(1+y^{4(k-i)-2})} < \infty \tag{A-6}$$

(we write $v_{-1} = 0$) and (A-3) *for $m = 2k+1$, we have*

$$\int |v_{2k+1}|^2 \geq C(M) \left\{ \int \frac{|v_{2k}|^2}{y^2} + \frac{|v_{2k-1}|^2}{y^4(1+|\log y|^2)} \right.$$
$$\left. + \sum_{i=0}^{k-1} \int \left[\frac{|v_{2i-1}|^2}{y^6(1+|\log y|^2)(1+y^{4(k-i)-2})} + \frac{|v_{2i}|^2}{y^4(1+|\log y|^2)(1+y^{4(k-i)-2})} \right] \right\}. \tag{A-7}$$

Remark. The case $k = 0$ is nothing but the coercivity of H described in Lemma B.1 of [RS14].

Based on the induction on k introduced in the proof of Lemma B.5 of [RS14], Lemma A.2 can be deduced from the following two lemmas, corresponding to the cases $k = 1$ and $k \to k+1$.

Lemma A.3 (coercivity of $\|v_3\|^2_{L^2}$). *Let $M = M(L) > 0$ be a large constant. Then there exists $C(M) > 0$ such that the following holds*: *for all radially symmetric v with*

$$\int |v_3|^2 + \int \frac{|v_2|^2}{y^2} + \int \frac{|v_1|^2}{y^4(1 + |\log y|^2)} + \int \frac{|v|^2}{y^4(1 + |\log y|^2)(1 + y^2)} < \infty$$

(we write $v_{-1} = 0$) and (A-3) *for $m = 3$, we have*

$$\int |v_3|^2 \geq C(M)\left\{\int \frac{|v_2|^2}{y^2} + \frac{|v_1|^2}{y^4(1 + |\log y|^2)} + \int \frac{|v|^2}{y^4(1 + |\log y|^2)(1 + y^2)}\right\}. \tag{A-8}$$

Proof. From the coercivity of H, we have

$$\int |v_3|^2 = \langle Hv_2, v_2 \rangle \geq C(M) \int \frac{|v_2|^2}{y^2}. \tag{A-9}$$

To prove the rest of (A-8), we claim the following weighted coercive bound:

$$\int \frac{|Hv|^2}{y^2(1 + |\log y|^2)} \geq C(M)\left\{\int \frac{|v|^2}{y^4(1 + |\log y|^2)(1 + y^2)} + \frac{|Av|^2}{y^4(1 + |\log y|^2)}\right\}. \tag{A-10}$$

By proving Lemma B.4 in [RS14], it is sufficient for (A-10) to prove only the subcoercivity estimate

$$\int \frac{|Hv|^2}{y^2(1 + |\log y|^2)} \gtrsim \int \frac{|\partial_y^2 v|^2}{y^2(1 + |\log y|^2)} + \int \frac{|\partial_y v|^2}{y^2(1 + |\log y|^2)(1 + y^2)}$$
$$+ \int \frac{|v|^2}{y^4(1 + |\log y|^2)(1 + y^2)} - C\left[\int \frac{|\partial_y v|^2}{1 + y^6} + \int \frac{|v|^2}{1 + y^8}\right]. \tag{A-11}$$

Unlike the region $y \leq 1$, which can be directly proved by borrowing the proof of Lemma B.4 in [RS14], we remark that (A-11) required some cautious estimates in the region $y \geq 1$: we have

$$\int_{y \geq 1} \frac{|Hv|^2}{y^2(1 + |\log y|^2)} \geq \int_{y \geq 1} \frac{|\partial_y(y\partial_y v)|^2}{y^4(1 + |\log y|^2)} - \int_{y \geq 1} |v|^2 \Delta\left(\frac{V}{y^4(1 + |\log y|^2)}\right)$$
$$+ \int_{y \geq 1} \frac{V^2|v|^2}{y^6(1 + |\log y|^2)} - C \int_{1 \leq y \leq 2} [|\partial_y v|^2 + |v|^2], \tag{A-12}$$

where $V(y) = 1 - 8y^2/(1 + y^2)^2$ is the potential part of H. Using the sharp logarithmic Hardy inequality employed in the proof of Lemma B.4 of [RS14], we obtain

$$\int_{y \geq 1} \frac{|\partial_y(y\partial_y v)|^2}{y^4(1 + |\log y|^2)} - \int_{y \geq 1} |v|^2 \Delta\left(\frac{1}{y^4(1 + |\log y|^2)}\right) \geq -C \int_{1 \leq y \leq 2} [|\partial_y v|^2 + |v|^2].$$

Now we employ the additional positive term in (A-12) with the asymptotics of the potential $V(y) = 1 + O(y^{-2})$ for $y \geq 1$,

$$\int_{y \geq 1} \frac{V^2|v|^2}{y^6(1 + |\log y|^2)} \geq (1 - \delta) \int_{y \geq 1} \frac{|v|^2}{y^6(1 + |\log y|^2)} - C \int \frac{|v|^2}{1 + y^8}. \qquad \square$$

Lemma A.4 (weighted coercivity bound). *For $k \geq 1$ and radially symmetric v with*

$$\int \frac{|v|^2}{y^4(1+|\log y|^2)(1+y^{4k+2})} + \frac{|Av|^2}{y^6(1+|\log y|^2)(1+y^{4k-2})} < \infty \qquad \text{(A-13)}$$

and

$$\langle v, \Phi_M \rangle = 0,$$

we have

$$\int \frac{|Hv|^2}{y^4(1+|\log y|^2)(1+y^{4k-2})}$$
$$\geq C(M) \left\{ \int \frac{|v|^2}{y^4(1+|\log y|^2)(1+y^{4k+2})} + \frac{|Av|^2}{y^6(1+|\log y|^2)(1+y^{4k-2})} \right\}. \quad \text{(A-14)}$$

Proof. We can prove (A-14) easily by replacing all $4k$ in the proof of Lemma B.4 of [RS14] with $4k - 2$ since the range of our k is $k \geq 1$. $\qquad \square$

From the previous lemmas, we obtain the coercivity of $\mathcal{E}_{k+1}$.

Lemma A.5 (coercivity of $\mathcal{E}_{k+1}$). *Let $1 \leq k \leq L$ and $M = M(L) > 0$ be a large constant. Then there exists $C(M) > 0$ such that*

$$\mathcal{E}_{k+1} = \langle \varepsilon_{k+1}, \varepsilon_{k+1} \rangle + \langle \dot{\varepsilon}_k, \dot{\varepsilon}_k \rangle$$
$$\geq C(M) \left[\sum_{i=0}^{k} \int \frac{|\varepsilon_i|^2}{y^2(1+y^{2(k-i)})(1+|\log y|^2)} + \sum_{i=0}^{k-1} \int \frac{|\dot{\varepsilon}_i|^2}{y^2(1+y^{2(k-1-i)})(1+|\log y|^2)} \right]. \quad \text{(A-15)}$$

Remark. The finiteness assumptions (A-4), (A-6) and (A-13) for (A-15) are satisfied from the well-localized smoothness of the 1-corotational map $(\Phi, \partial_t \Phi)$ (see Lemma A.1 in [RS14]).

Appendix B: Interpolation estimates

In this section, we provide some interpolation estimates for ε, i.e., the first coordinate part of $\boldsymbol{\varepsilon}$. We will employ these bounds to deal with $\boldsymbol{NL}(\boldsymbol{\varepsilon})$ and $\boldsymbol{L}(\boldsymbol{\varepsilon})$ terms in the evolution equation of $\boldsymbol{\varepsilon}$ (3-8).

Lemma B.1 (interpolation estimates). (i) *For $y \leq 1$, ε has a Taylor–Lagrange expansion*

$$\varepsilon = \sum_{i=1}^{(L+1)/2} c_i T_{L+1-2i} + r_\varepsilon,$$

where T_{2i} is the first coordinate part of $\boldsymbol{T}_{2i}$ and

$$|c_i| \lesssim C(M)\sqrt{\mathcal{E}_{L+1}}, \quad |\partial_y^k r_\varepsilon| \lesssim C(M) y^{L-k} |\log y| \sqrt{\mathcal{E}_{L+1}}, \quad 0 \leq k \leq L.$$

(ii) *For $y \leq 1$, ε satisfies the pointwise bounds*

$$|\varepsilon_k| \lesssim C(M) y^{1+\bar{k}} |\log y| \sqrt{\mathcal{E}_{L+1}}, \quad 0 \leq k \leq L-1,$$
$$|\varepsilon_L| \lesssim C(M) \sqrt{\mathcal{E}_{L+1}},$$
$$|\partial_y^k \varepsilon| \lesssim C(M) y^{\overline{k+1}} |\log y| \sqrt{\mathcal{E}_{L+1}}, \quad 0 \leq k \leq L.$$

(iii) *For $1 \leq k \leq L$ and $0 \leq i \leq k$,*

$$\int \frac{1 + |\log y|^C}{1 + y^{2(k-i+1)}} (|\varepsilon_i|^2 + |\partial_y^i \varepsilon|^2) + \left\| \frac{\partial_y^i \varepsilon}{y^{k-i}} \right\|_{L^\infty(y \geq 1)}^2 \lesssim |\log b_1|^C b_1^{2m_{k+1}},$$

where

$$m_{k+1} = \begin{cases} kc_1 & \text{if } 1 \leq k \leq L-2, \\ L & \text{if } k = L-1, \\ L+1 & \text{if } k = L. \end{cases}$$

Proof. It is provided by the proof of Lemma C.1 in [RS14]. $\qquad\square$

Appendix C: Leibniz rule for $\mathcal{A}^k$

Unlike [RS14], we encounter some terms in which ∂_t is applied more than once to $\mathcal{A}_\lambda^k$, such as $\partial_{tt}(\mathcal{A}_\lambda^k)$, $\partial_t(\mathcal{A}_\lambda^i)\partial_t(H_\lambda^j)$, etc. To control those terms, we recall the asymptotics

$$\partial_t(\mathcal{A}_\lambda^k) f_\lambda(r) = \frac{\lambda_t}{\lambda^{k+1}} \sum_{i=0}^{k-1} \Phi_{i,k}^{(1)}(y) f_i(y), \quad |\Phi_{i,k}^{(1)}(y)| \lesssim \frac{1}{1 + y^{k+2-i}}, \tag{C-1}$$

which were introduced in Appendices D and E of [RS14]. We note that near, the origin, $\Phi_{i,k}^{(1)}$ satisfies

$$\Phi_{i,k}^{(1)}(y) = \begin{cases} \sum_{p=0}^N c_{i,k,p} y^{2p} + O(y^{2N+2}), & k-i \text{ is even,} \\ \sum_{p=0}^N c_{i,k,p} y^{2p+1} + O(y^{2N+3}), & k-i \text{ is odd.} \end{cases} \tag{C-2}$$

Based on the above facts, we can obtain the following lemma.

Lemma C.1. *Let $1 \leq k \leq \frac{1}{2}(L-1)$. Then*

$$\partial_{tt}(\mathcal{A}_\lambda^k) f_\lambda(r) = \frac{\lambda_{tt}}{\lambda^{k+1}} \sum_{i=0}^{k-1} \Phi_{i,k}^{(1)}(y) f_i(y) + \frac{O(b_1^2)}{\lambda^{k+2}} \sum_{i=0}^{k-1} \Phi_{i,k}^{(2)}(y) f_i(y), \tag{C-3}$$

$$\partial_t(\mathcal{A}_\lambda^{L-2k}) \partial_t(H_\lambda^k) f_\lambda(r) = \frac{O(b_1^2)}{\lambda^{L+2}} \sum_{i=0}^{L-1} \Phi_{i,L}^{(3)}(y) f_i(y), \tag{C-4}$$

where

$$|\Phi_{i,k}^{(2)}(y)| \lesssim \frac{1}{1 + y^{k+2-i}}, \quad |\Phi_{i,L}^{(3)}(y)| \lesssim \frac{1}{1 + y^{L+3-i}}.$$

Proof. Recalling that $\partial_{tt}(\mathcal{A}_\lambda^k) f_\lambda = [\partial_t, \partial_t(\mathcal{A}_\lambda^k)] f_\lambda$ and

$$\frac{\lambda_t}{\lambda^{k+1}} \Phi_{i,k}^{(1)}(y) f_i(y) = \frac{\lambda_t}{\lambda^{k+1-i}} (\Phi_{i,k}^{(1)})_\lambda(r) \mathcal{A}_\lambda^i f_\lambda(r), \quad \partial_t \Phi_\lambda = -\frac{\lambda_t}{\lambda}(\Lambda\Phi)_\lambda,$$

we get (C-3) since

$$\left[\partial_t, \frac{\lambda_t}{\lambda^{k+1-i}}(\Phi_{i,k}^{(1)})_\lambda \mathcal{A}_\lambda^i \right] f_\lambda = \frac{\lambda_{tt}}{\lambda^{k+1-i}}(\Phi_{i,k}^{(1)})_\lambda \mathcal{A}_\lambda^i f_\lambda - \frac{(\lambda_t)^2}{\lambda^{k+2-i}}(\Lambda_{i-k}\Phi_{i,k}^{(1)})_\lambda \mathcal{A}_\lambda^i f_\lambda + \frac{\lambda_t}{\lambda^{k+1-i}}(\Phi_{i,k}^{(1)})_\lambda \partial_t(\mathcal{A}_\lambda^i) f_\lambda$$

$$= \frac{\lambda_{tt}}{\lambda^{k+1}} \Phi_{i,k}^{(1)}(y) f_i(y) + \frac{O(b_1^2)}{\lambda^{k+2}} \sum_{j=0}^i \Phi_{i,j,k}(y) f_j(y),$$

where

$$|\Phi_{i,j,k}(y)| \lesssim \frac{1}{1 + y^{k+2-j}}.$$

Moreover, we can easily check that $\Phi_{i,k}^{(2)}$ satisfies (C-2) because the scaling generator Λ preserves the asymptotics near the origin as well as at infinity.

To prove (C-4), we need to justify the terms of the form $\mathcal{A}^i \circ \Phi \mathcal{A}^j$. When j is an even number, we can use the Leibniz rule from Appendix D of [RS14]. However, when j is odd, terms such as $A \circ \Phi A$ appear, making the problem a bit more tricky.

Fortunately, our Φ from the terms of the form $\mathcal{A}^i \circ \Phi \mathcal{A}^{2j+1}$ have an expansion

$$\Phi(y) = \sum_{p=0}^{N} c_p y^{2p+1} + O(y^{2N+3})$$

near the origin since each $\Phi \mathcal{A}^{2j+1}$ comes from $\partial_t(H_\lambda^k)$ or $\partial_{tt}(H_\lambda^k)$, satisfying (C-2). Hence

$$(A \circ \Phi \mathcal{A}^{2j+1})f = (A\Phi)f_{2j+1} - \Phi \partial_y f_{2j+1} = \left(-\partial_y + \frac{1+2Z}{y}\right)\Phi \cdot f_{2j+1} - \Phi f_{2j+2} =: \Phi_1 f_{2j+1} - \Phi f_{2j+2},$$

where Φ_1 satisfies

$$\Phi_1(y) = \sum_{p=0}^{N} c_p y^{2p} + O(y^{2N+2})$$

near the origin. If we take A^* here,

$$\begin{aligned}
(H \circ \Phi \mathcal{A}^{2j+1})f &= A^*(\Phi_1 f_{2j+1} - \Phi f_{2j+2}) \\
&= (\partial_y \Phi_1)f_{2j+1} + (\Phi_1 - A^*\Phi)f_{2j+2} - \Phi \partial_y f_{2j+2} \\
&= (\partial_y \Phi_1)f_{2j+1} + \left(\Phi_1 - \partial_y \Phi - \frac{1+2Z}{y}\Phi\right)f_{2j+2} + \Phi f_{2j+3},
\end{aligned}$$

we can justify $\mathcal{A}^i \circ \Phi \mathcal{A}^{2j+1}$ by iterating the above calculation. $\qquad \square$

Appendix D: Monotonicity for the intermediate energy

Proposition D.1 (Lyapunov monotonicity for $\mathcal{E}_k$). *Let $2 \leq k \leq L$. We have*

$$\frac{d}{dt}\left\{\frac{\mathcal{E}_k}{\lambda^{2k-2}}\right\} \leq \frac{b_1 |\log b_1|^{C(k)}}{\lambda^{2k-1}}(\sqrt{\mathcal{E}_{k+1}} + b_1^k + b_1^{\delta(k)+(k-1)c_1})\sqrt{\mathcal{E}_k}, \tag{D-1}$$

where $C(k)$, $\delta(k) > 0$ are constants that depend only on k and L.

Proof. We compute the energy identity

$$\begin{aligned}
\partial_t\left(\frac{\mathcal{E}_k}{2\lambda^{2(k-1)}}\right) &= \langle \partial_t w_k, w_k \rangle + \langle \partial_t \dot{w}_{k-1}, \dot{w}_{k-1} \rangle \\
&= \langle \partial_t(\mathcal{A}_\lambda^k)w, w_k \rangle + \langle \partial_t(\mathcal{A}_\lambda^{k-1})\dot{w}, \dot{w}_{k-1} \rangle + \langle \mathcal{A}_\lambda^k \mathcal{F}_1, w_k \rangle + \langle \mathcal{A}_\lambda^{k-1} \mathcal{F}_2, \dot{w}_{k-1} \rangle. \tag{D-2}
\end{aligned}$$

We can directly estimate the first two terms of the right-hand side of (D-2) by Lemma C.1:

$$|\langle \partial_t(\mathcal{A}_\lambda^k)w, w_k\rangle| \lesssim \frac{b_1}{\lambda^{2k-1}} \sum_{m=0}^{k-1} |\langle \Phi_{m,k}^{(1)}\varepsilon_m, \varepsilon_k\rangle|$$

$$\lesssim \frac{b_1}{\lambda^{2k-1}} \sum_{m=0}^{k-1} \left\| \frac{\varepsilon_m}{1+y^{k+2-m}} \right\|_{L^2} \sqrt{\mathcal{E}_k} \lesssim \frac{b_1 C(M)}{\lambda^{2k-1}} \sqrt{\mathcal{E}_{k+1}\mathcal{E}_k}, \tag{D-3}$$

$$|\langle \partial_t(\mathcal{A}_\lambda^{k-1})\dot{w}, \dot{w}_{k-1}\rangle| \lesssim \frac{b_1}{\lambda^{2k-1}} \sum_{m=0}^{k-2} |\langle \Phi_{m,k-1}^{(1)}\dot{\varepsilon}_m, \dot{\varepsilon}_{k-1}\rangle| \lesssim \frac{b_1 C(M)}{\lambda^{2k-1}} \sqrt{\mathcal{E}_{k+1}\mathcal{E}_k}. \tag{D-4}$$

Then we conclude (D-1) from the bounds

$$\|\mathcal{A}^k\mathcal{F}\|_{L^2} + \|\mathcal{A}^{k-1}\dot{\mathcal{F}}\|_{L^2} \lesssim b_1|\log b_1|^C[b_1^k + b_1^{\delta(k)+(k-1)c_1}]. \tag{D-5}$$

The last two terms of the right-hand side of (D-2) is bounded by

$$\frac{b_1|\log b_1|^C}{\lambda^{2k-1}}(b_1^k + b_1^{\delta(k)+(k-1)c_1})\sqrt{\mathcal{E}_k}. \tag{D-6}$$

Now, it remains to prove (D-5), and we address it by separating $\mathcal{F} = (\mathcal{F}, \dot{\mathcal{F}})^t$ into four types, as we did for Step 5 in the proof of Proposition 3.5.

(i) $\tilde{\psi}_b$ *terms.* The contribution of $\tilde{\psi}_b$ terms to the above inequalities is estimated from the global weighted bounds of Proposition 2.10.

(ii) $\widetilde{\mathbf{Mod}}(t)$ *terms.* Similar to (ii) of Step 5 in the proof of Proposition 3.5 with the cancellation $\mathcal{A}^k T_i = 0$ for $1 \le i \le k$ and Lemma 2.7, we obtain

$$\int \left| \sum_{i=1}^L b_i \mathcal{A}^{k-\bar{i}}[\Lambda_{1-\bar{i}}(\chi_{B_1} T_i)] + \sum_{i=2}^{L+2} \mathcal{A}^{k-\bar{i}}[\Lambda_{1-\bar{i}}(\chi_{B_1} S_i)] \right|^2 \lesssim b_1^2,$$

$$\sum_{i=1}^L \int \left| \mathcal{A}^{k-\bar{i}}\left[\chi_{B_1} T_i + \chi_{B_1} \sum_{j=i+1}^{L+2} \frac{\partial S_j}{\partial b_i} \right] \right|^2 \lesssim b_1^{2(k-L)}|\log b_1|^{2\gamma(L-k)+2}.$$

Hence Lemma 3.3 and the bootstrap bound (3-25) imply

$$\|\mathcal{A}^k\widetilde{\mathbf{Mod}}(t)\|_{L^2} + \|\mathcal{A}^{k-1}\dot{\widetilde{\mathbf{Mod}}}(t)\|_{L^2} \lesssim b_1^{k-L}|\log b_1|^{\gamma(L-k)+1}\frac{b_1^{L+1}}{|\log b_1|} \lesssim b_1^{k+1}|\log b_1|^{\gamma(L-k)}.$$

(iii) $NL(\varepsilon)$ *term*: We can utilize the bound (3-78) near the origin. For $y \ge 1$, we recall the calculation and estimates from (iii) of Step 5 in the proof of Proposition 3.5: $\|\mathcal{A}^{k-1}NL(\varepsilon)\|_{L^2(y\ge 1)}$ is bounded by

$$|\log b_1|^C b_1^{m_I+1} b_1^{m_J+1} + |\log b_1|^C b_1^{m_X+1} b_1^{m_Y+1} b_1^{m_J+1},$$

where $I, J, X, Y, Z \ge 1$, $I+J = k$ and $X+Y+Z = k$. From the bootstrap bounds (3-25), (3-27) and the fact that $c_1 > 1$, we obtain

$$\|\mathcal{A}^{k-1}NL(\varepsilon)\|_{L^2(y\ge 1)} \lesssim |\log b_1|^{C(K)} b_1^{kc_1} \lesssim b_1^{1+\delta(k)+(k-1)c_1}.$$

(iv) $L(\varepsilon)$ *term*: With some modifications (replacing L by $k-1$, for instance), it is proved by (3-85) and (3-87). □

Remark. In step (iii) when $k = L$, we can avoid the case that either $I = L - 1$ or $J = L - 1$ by estimating $\|\partial_y^{L-1} N_1(\varepsilon)\|_{L^2(y \geq 1)}$ instead of $\|\partial_y^{L-1} N_1(\varepsilon)\|_{L^\infty(y \geq 1)}$.

Recall the modified higher-order energies

$$\widehat{\mathcal{E}}_\ell := \langle \hat{\varepsilon}_\ell, \hat{\varepsilon}_\ell \rangle + \langle \dot{\hat{\varepsilon}}_{\ell-1}, \dot{\hat{\varepsilon}}_{\ell-1} \rangle.$$

We rewrite the flow (3-17) componentwise: for $1 \leq k \leq \ell$,

$$\begin{cases} \partial_t \hat{w}_k - \dot{\hat{w}}_k = \partial_t(\mathcal{A}_\lambda^k)\hat{w} + \mathcal{A}_\lambda^k \widehat{\mathcal{F}}_1, \\ \partial_t \dot{\hat{w}}_k + \hat{w}_{k+2} = \partial_t(\mathcal{A}_\lambda^k)\dot{w} + \mathcal{A}_\lambda^k \widehat{\mathcal{F}}_2, \end{cases} \quad \begin{pmatrix} \widehat{\mathcal{F}}_1 \\ \widehat{\mathcal{F}}_2 \end{pmatrix} := \frac{1}{\lambda}\widehat{\mathcal{F}}_\lambda = \frac{1}{\lambda}\begin{pmatrix} \widehat{\mathcal{F}} \\ \dot{\widehat{\mathcal{F}}} \end{pmatrix}_\lambda. \tag{D-7}$$

Proposition D.2 (Lyapunov monotonicity for $\mathcal{E}_L$). *Let $\ell = L$. Then we have*

$$\frac{d}{dt}\left\{ \frac{\widehat{\mathcal{E}}_L}{\lambda^{2L-2}} + O\left(\frac{b_1^{2L}|\log b_1|^2}{\lambda^{2L-2}} \right) \right\} \leq \frac{b_1^{L+1}|\log b_1|^\delta}{\lambda^{2L-1}}(b_1^L|\log b_1| + \sqrt{\mathcal{E}_L}), \tag{D-8}$$

where $0 < \delta \ll 1$ is a sufficient small constant that depend only on L.

Proof. We compute the energy identity

$$\partial_t\left(\frac{\widehat{\mathcal{E}}_L}{2\lambda^{2(L-1)}} \right) = \langle \partial_t(\mathcal{A}_\lambda^L)\hat{w}, \hat{w}_L \rangle + \langle \partial_t(\mathcal{A}_\lambda^{L-1})\dot{w}, \dot{w}_{L-1} \rangle + \langle \mathcal{A}_\lambda^L \widehat{\mathcal{F}}_1, \hat{w}_L \rangle + \langle \mathcal{A}_\lambda^{L-1} \widehat{\mathcal{F}}_2, \dot{\hat{w}}_{L-1} \rangle. \tag{D-9}$$

We can directly estimate the first two terms of the right-hand side of (D-9) from the bounds (D-3), (D-4) and the fact $\varepsilon - \hat{\varepsilon} = \zeta_b$: we obtain the upper bound

$$\frac{b_1 C(M)}{\lambda^{2L-1}}\sqrt{\mathcal{E}_{L+1}\mathcal{E}_L} + \frac{b_1^{L+3}|\log b_1|^C}{\lambda^{2L-1}}\sqrt{\mathcal{E}_L} + \frac{b_1^{2L+3}|\log b_1|^C}{\lambda^{2L-1}}. \tag{D-10}$$

We can borrow steps (ii), (iii) and (iv) in the proof of Proposition D.1 to estimate the last two terms of the right-hand side of (D-9) except for the $\hat{\psi}_b$ terms. Also, by Proposition 2.11, all the inner products we have to deal with are

$$b_L \langle \mathcal{A}^L(\chi_{B_1} - \chi_{B_0})T_{L-1}, \hat{\varepsilon}_L \rangle, \quad b_L \langle \mathcal{A}^{L-1}(\partial_s \chi_{B_0} + b_1(y\chi')_{B_0})T_L, \dot{\hat{\varepsilon}}_{L-1} \rangle. \tag{D-11}$$

From the fact $\hat{\varepsilon} = \varepsilon$ and $\mathcal{A}^{L-1}T_{L-1} = (-1)^{(L-1)/2}\Lambda Q$, we obtain

$$\mathcal{A}^{L-1}(\chi_{B_1} - \chi_{B_0})T_{L-1} = (-1)^{(L-1)/2}(\chi_{B_1} - \chi_{B_0})\Lambda Q + (\mathbf{1}_{y \sim B_1} + \mathbf{1}_{y \sim B_0})O(y^{-1}|\log y|).$$

Hence the bootstrap bound (3-25) yields

$$\begin{aligned} |\langle \mathcal{A}^L(\chi_{B_1} - \chi_{B_0})T_{L-1}, \hat{\varepsilon}_L \rangle| &= |\langle \mathcal{A}^{L-1}(\chi_{B_1} - \chi_{B_0})T_{L-1}, \hat{\varepsilon}_{L+1} \rangle| \\ &\leq |\langle y^{-1}\mathbf{1}_{B_0 \leq y \leq 2B_1} + (\mathbf{1}_{y \sim B_1} + \mathbf{1}_{y \sim B_0})y^{-1}|\log y|, \varepsilon_{L+1} \rangle| \\ &\leq (|\log b_1|^{1/2} + |\log b_1|)\sqrt{\mathcal{E}_{L+1}} \leq b_1^{L+1}|\log b_1|^\delta. \end{aligned}$$

Note that $\dot{\hat{\varepsilon}} = \dot{\varepsilon} + b_L(\chi_{B_1} - \chi_{B_0})T_L$. The asymptotics (2-95) imply

$$\begin{aligned} |\langle \mathcal{A}^{L-1}(\partial_s \chi_{B_0} + b_1(y\chi')_{B_0})T_L, \dot{\varepsilon}_{L-1} \rangle| &\leq b_1|\langle \mathcal{A}^{L-2}(\mathbf{1}_{y \sim B_0}y^{L-2}|\log y|), \dot{\varepsilon}_L \rangle| \\ &\leq |\log b_1|\sqrt{\mathcal{E}_{L+1}} \leq b_1^{L+1}|\log b_1|^\delta. \end{aligned}$$

To estimate the last inner product, we employ the sharp asymptotics

$$b_1(y\chi')_{B_0} = -c_1\partial_s\chi_{B_0} + O\left(\frac{b_1\mathbf{1}_{y\sim B_0}}{|\log b_1|}\right)$$

from the fact $(b_1)_s = b_2 + O(b_1^2/|\log b_1|)$. Using the cancellation $\mathcal{A}^L T_L = 0$ and $\chi_{B_1} = 1$ on $y \sim B_0$, the remaining inner product can be written as

$$\frac{1}{L-1}b_L^2\langle\mathcal{A}^{L-1}\partial_s(\chi_{B_0}T_L), \mathcal{A}^{L-1}(\chi_{B_0}T_L)\rangle + O\left(\frac{b_1^{2L+1}}{|\log b_1|}\|\mathcal{A}^{L-1}(\mathbf{1}_{y\sim B_0}T_L)\|_{L^2}^2\right). \tag{D-12}$$

We can easily check that the second term in (D-12) is bounded by $b_1^{2L+1}|\log b_1|$. For the first term in (D-12), we use integration by parts in time to find out the correction for $\widehat{\mathcal{E}}_L$:

$$\frac{b_L^2}{\lambda^{2L-1}}\langle\mathcal{A}^{L-1}\partial_s(\chi_{B_0}T_L), \mathcal{A}^{L-1}(\chi_{B_0}T_L)\rangle = \frac{b_L^2}{2\lambda^{2L-1}}\partial_s\langle\mathcal{A}^{L-1}(\chi_{B_0}T_L), \mathcal{A}^{L-1}(\chi_{B_0}T_L)\rangle$$

$$= \frac{b_L^2}{2\lambda^{2L-2}}\partial_t\|\mathcal{A}^{L-1}(\chi_{B_0}T_L)\|_{L^2}^2.$$

By Lemma 3.3, we conclude (D-8):

$$\frac{b_L^2}{2\lambda^{2L-2}}\partial_t\|\mathcal{A}^{L-1}(\chi_{B_0}T_L)\|_{L^2}^2 - \partial_t\left(\frac{b_L^2}{2\lambda^{2L-2}}\|\mathcal{A}^{L-1}(\chi_{B_0}T_L)\|_{L^2}^2\right)$$

$$= -\partial_t\left(\frac{b_L^2}{2\lambda^{2L-2}}\right)\|\mathcal{A}^{L-1}(\chi_{B_0}T_L)\|_{L^2}^2 = \left(\frac{(L-1)b_L^2\lambda_t}{\lambda^{2L-1}} - \frac{b_L(b_L)_t}{\lambda^{2L-2}}\right)\|\mathcal{A}^{L-1}(\chi_{B_0}T_L)\|_{L^2}^2$$

$$= -\frac{b_L}{\lambda^{2L-1}}((b_L)_s + (L-1)b_1 b_L)O(|\log b_1|^2) = O\left(\frac{b_1^{2L+1}}{\lambda^{2L-1}}|\log b_1|\right). \qquad \square$$

Proposition D.3 (Lyapunov monotonicity for $\mathcal{E}_{L-1}$). *Let $\ell = L - 1$. Then we have*

$$\frac{d}{dt}\left\{\frac{\widehat{\mathcal{E}}_{L-1}}{\lambda^{2L-4}} + O\left(\frac{b_1^{2L-2}|\log b_1|^2}{\lambda^{2L-4}}\right)\right\} \le \frac{b_1^L|\log b_1|^\delta}{\lambda^{2L-3}}(b_1^{L-1}|\log b_1| + \sqrt{\mathcal{E}_{L-1}}), \tag{D-13}$$

where $0 < \delta \ll 1$ is a sufficient small constant that depends only on L.

Proof. Based on the proof of Proposition D.2 with Proposition 2.12, all the inner products we have to deal with are

$$b_L\langle\mathcal{A}^{L-1}(\chi_{B_1} - \chi_{B_0})T_{L-1}, \hat{\varepsilon}_{L-1}\rangle, \quad b_{L-1}\langle\mathcal{A}^{L-1}(\partial_s\chi_{B_0} + b_1(y\chi')_{B_0})T_{L-1}, \hat{\varepsilon}_{L-1}\rangle$$

$$b_{L-1}\langle\mathcal{A}^{L-2}H(\chi_{B_1} - \chi_{B_0})T_L, \dot{\hat{\varepsilon}}_{L-2}\rangle, \quad b_L\langle\mathcal{A}^{L-2}(\partial_s\chi_{B_0} + b_1(y\chi')_{B_0})T_L, \dot{\hat{\varepsilon}}_{L-2}\rangle.$$

By additionally considering $\hat{\varepsilon} = \varepsilon + b_{L-1}(\chi_{B_1} - \chi_{B_0})T_{L-1}$, we can estimate the above inner products similarly to (D-12) due to the derivative gain $\mathcal{A}^{L-2}H = \mathcal{A}^L$ and the logarithmic gain $|\log b_1|^{-\beta}$ from the bootstrap bound (3-28) for b_L when $\ell = L - 1$. The exact correction term is given by

$$-\partial_t\left(\frac{b_{L-1}^2}{2(L-2)\lambda^{2L-4}}\|\mathcal{A}^{L-1}(\chi_{B_0}T_{L-1})\|_{L^2}^2\right). \qquad \square$$

Acknowledgements

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