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MARGULIS LEMMA ON RCD(K, N) SPACES

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We extend the Margulis lemma for manifolds with lower Ricci curvature bounds to the RCD(K, N) setting. As one of our main tools, we obtain improved regularity estimates for regular Lagrangian flows on these spaces.

1. Introduction

The main result of this paper extends the Margulis lemma to RCD(K, N) spaces. Recall that for a group G, we say an (ordered) generating set $\beta = \{\gamma_1, \dots, \gamma_n\} \subset G$ is a *nilpotent basis of length* n if for all $i, j \in \{1, \dots, n\}$ one has $[\gamma_i, \gamma_j] \in \langle\{\gamma_1, \dots, \gamma_{i-1}\}\rangle$.

Theorem 1.1. *For each $K \in \mathbb{R}$, $N \geq 1$, there exist $\varepsilon > 0$ and $C \in \mathbb{N}$ such that if $(X, d, \mathfrak{m}, p)$ is a pointed* RCD(K, N) *space of rectifiable dimension n, the image of the map*

$$\pi_1(B_\varepsilon(p), p) \to \pi_1(X, p)$$

induced by inclusion contains a subgroup of index $\leq C$ that admits a nilpotent basis of length $\leq n$.

From [Kapovitch and Wilking 2011], Theorem 1.1 is known to hold when X is a smooth Riemannian manifold. On the other hand, Breuillard, Green and Tao [Breuillard et al. 2012, Corollary 11.17] proved that, after quotienting by a finite normal subgroup, Theorem 1.1 holds in more general metric spaces with nice packing properties.

The proof strategy of Theorem 1.1 is similar to that of Kapovitch and Wilking, including a reverse induction argument (see Theorem 1.14). Nevertheless, there are quite a few technical challenges to generalizing their arguments to a nonsmooth framework. An important tool used in [Kapovitch and Wilking 2011] is the gradient flow of smooth functions with suitable integral Hessian bounds and their

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associated regularity estimates. In the nonsmooth framework, these gradient flows are by necessity replaced by the regular Lagrangian flows (RLFs) of Sobolev vector fields. Intuitively, RLFs are the appropriate notion of flows in a context where pointwise defined flows do not make sense and might not be unique. When restricted to smooth vector fields on Riemannian manifolds, RLFs coincide with the classical flows almost everywhere.

Several regularity results have been obtained for RLFs in [Brué and Semola 2020a; 2020b; Brué et al. 2022], but are not quite strong enough to give the necessary estimates; see the discussion at the end of Section 1.1 for more details. The main technical contribution of this paper, therefore, is to establish new regularity estimates for regular Lagrangian flows on the $\mathrm{RCD}(K, N)$ spaces. These estimates match the effective estimates known for smooth manifolds with a Ricci curvature lower bound. We mention that related estimates of this type have also been employed successfully in other works to study the structure of Ricci limit spaces (see [Cheeger and Colding 1996; Colding and Naber 2012; Kapovitch and Li 2018]) and $\mathrm{RCD}(K, N)$ spaces (see [Brué and Semola 2020b; Deng 2020]).

For the rest of the paper we shall assume some basic familiarity with the theory of $\mathrm{RCD}(K, N)$ spaces, and in particular that of its first and second order calculus framework. We refer to [Sturm 2006a; 2006b; Lott and Villani 2009; Ambrosio et al. 2014a; 2014b; 2015; Savaré 2014; Gigli 2015; Mondino and Naber 2019; Gigli 2018; Brué and Semola 2020b], among others, for a detailed treatment.

1.1. ***Main regularity estimates on RLFs.*** Let us first define regular Lagrangian flows [Ambrosio 2004; Ambrosio and Trevisan 2014] and maximal functions [Stein 1993].

Definition 1.2. Let $(X, d, \mathfrak{m})$ be an $\mathrm{RCD}(K, N)$ space, $T > 0$, and let $V : [0, T] \to L^2_{\mathrm{loc}}(TX)$ be a time-dependent vector field. A Borel map $\boldsymbol{X} : [0, T] \times X \to X$ is called a *regular Lagrangian flow* (*RLF*) to V if the following holds:

$R.1$ $\boldsymbol{X}_0(x) = x$ and $[0, T] \ni t \mapsto \boldsymbol{X}_t(x)$ is continuous for every $x \in X$.

$R.2$ For every $f \in \mathrm{TestF}(X)$ and $\mathfrak{m}$-a.e. $x \in X$, $t \mapsto f(\boldsymbol{X}_t(x))$ is in $W^{1,1}([0, T])$ and

$$\frac{d}{dt} f(\boldsymbol{X}_t(x)) = df(V(t))(\boldsymbol{X}_t(x)) \quad \text{for a.e. } t \in [0, T]. \tag{1.3}$$

$R.3$ There exists a constant $C(V)$ such that $(\boldsymbol{X}_t)_* m \le C\mathfrak{m}$ for all t in $[0, T]$.

Definition 1.4. Let $(X, d, \mathfrak{m})$ be an $\mathrm{RCD}(K, N)$ space, $R > 0$, and $h : X \to \mathbb{R}^+$ measurable. The *R-maximal function* $\mathrm{Mx}_R(h) : X \to \mathbb{R}$ is defined as

$$\mathrm{Mx}_R(h)(x) := \sup_{0 < r \le R} \fint_{B_r(x)} h \, d\mathfrak{m}.$$

For simplicity, we denote Mx_1 by Mx.

The following regularity result is our substitute for smoothness in the context of regular Lagrangian flows. Roughly speaking, it establishes that for a vector field V, if one has enough integral control on $\mathrm{Mx}(|\nabla V|)$ along most flow lines that start in a ball B, then the RLF of V maps most points of B to a ball of similar scale. Theorem 1.5 will be later used as a base of induction to produce stronger quantitative estimates along flow lines in Section 5.

Theorem 1.5. *Let* $\rho > 0$, $T > 0$, $L \geq 1$, $D \geq 0$, $(X, d, \mathfrak{m})$ *an* $\mathrm{RCD}(-(N-1), N)$ *space,* $V \in L^1([0,T]; H^{1,2}_{C,s}(TX))$ *a vector field with* $\|V(t)\|_\infty \leq L$ *and* $\|\mathrm{div}(V(t))\|_\infty \leq D$ *for all* $t \in [0, T]$, $\boldsymbol{X} : [0, T] \times X \to X$ *its RLF, and define* $H : X \to \mathbb{R}$ *as* $H(y) := \int_0^T Mx_\rho(|\nabla V(t)|)(\boldsymbol{X}_t(y))\,dt$. *Then there are* $\delta(D, T, N) > 0$, $M(D, T, N) > 0$, *such that if* $x \in X$ *satisfies*

$$\limsup_{r\to 0} \frac{\mathfrak{m}(\{y \in B_r(x) | H(y) > \delta\})}{\mathfrak{m}(B_r(x))} < \tfrac{1}{2}, \tag{1.6}$$

then there is $r_x \leq \rho/100$ *and a representative* $\widetilde{\boldsymbol{X}} : [0, T] \times X \to X$ *of the RLF to* V *such that for all* $r \leq r_x$ *the following holds*:

*S.*1 *There is* $A_r \subset B_r(x)$ *with* $\mathfrak{m}(A_r) \geq \frac{1}{M}\mathfrak{m}(B_r(x))$ *and*

$$\widetilde{\boldsymbol{X}}_t(A_r) \subset B_{2r}(\widetilde{\boldsymbol{X}}_t(x)) \quad \text{for all } t \in [0, T].$$

*S.*2 *For all* $t \in [0, T]$,

$$\frac{1}{M}\mathfrak{m}(B_r(x)) \leq \mathfrak{m}(B_r(\widetilde{\boldsymbol{X}}_t(x))) \leq M\mathfrak{m}(B_r(x)).$$

Moreover, $\widetilde{\boldsymbol{X}}$ *can be chosen so that any point* $x \in X$ *satisfying* (1.6) *also satisfies* $S.1$ *and* $S.2$ *for* r *sufficiently small* (*depending on* x).

For smooth vector fields on Riemannian manifolds, the previous result follows immediately from the infinitesimal to local property in differential calculus (see [Kapovitch and Wilking 2011, Lemma 3.7; Colding and Naber 2012, Proposition 3.6]). This issue is far more delicate in the nonsmooth setting since one cannot perform infinitesimal calculus pointwise. To overcome this, we directly obtain quantitative estimates on all scales using some new technical arguments developed in [Deng 2020], which builds on the ideas of [Kapovitch and Wilking 2011; Colding and Naber 2012].

The proof of Theorem 1.5 uses a similar technique as [Deng 2020, Lemma 5.1], generalizing it to a wider class of flows. However, in order to successfully use it to perform the topological arguments required for Theorem 1.1, we need to adjust the RLF to obtain the appropriate representative $\widetilde{\boldsymbol{X}}$ mentioned at the end of the theorem, while in [Deng 2020] the flow one initially works with is already good enough for the required application (roughly this is because, in [Deng 2020], one can show that the flows starting from close to a given point should always limit in some sense to a geodesic, which can be identified canonically and without ambiguity, whereas in this work, all objects considered are defined "almost-everywhere" and there was no natural canonical limit to begin with).

Definition 1.7. Let $M(1, T, N) > 0$ be given by Theorem 1.5, $(X, d, \mathfrak{m})$ an $\mathrm{RCD}(-(N-1), N)$ space, $V : [0, T] \to L^2_{\mathrm{loc}}(TX)$ a vector field, and $\boldsymbol{X} : [0, T] \times X \to X$ its RLF. We say that $x \in X$ is a *point of essential stability* of $\boldsymbol{X}$ if there is $r_x > 0$ such that $S.1$ and $S.2$ hold for all $r \leq r_x$.

Corollary 1.8. *For each* $N \geq 1$, $T \geq 0$, $D \geq 0$, $r \geq 0$, $L \geq 0$ *and* $\varepsilon > 0$, *there are* $R \geq 1$, $\eta > 0$, *such that the following holds. Let* $(X, d, \mathfrak{m}, p)$ *be an* $\mathrm{RCD}(-(N-1), N)$ *space,* $V \in H^{1,2}_{C,s}(TX)$ *a vector field with* $\|V\|_\infty \leq L$, $\|\mathrm{div}(V)\|_\infty \leq D$, *and* $\boldsymbol{X} : [0, T] \times X \to X$ *its RLF. Assume that for all* $s \in [1, R]$ *one has*

$$\fint_{B_s(p)} |\nabla V|^2\,d\mathfrak{m} \leq \eta.$$

Then if $G \subset X$ denotes the set of points of essential stability of $\mathbf{X}$, one has

$$(G \cap B_r(p)) \geq (1-\varepsilon)\mathfrak{m}(B_r(p)).$$

We remark that for noncollapsed RCD(K, N) spaces, a version of these regularity results were obtained in [Brué et al. 2022] using alternative methods relying on estimates of the Green's function, which cannot be readily applied in collapsed cases. Moreover, the use of the Green's function in [Brué et al. 2022] resulted in the dependence of various estimates on nonstructural information such as the space itself, somewhat inevitably since the Green's function naturally contains global information about X. This is undesirable for the application at present since we will need to consider sequences of RCD spaces and therefore cannot make use of estimates which depend on the space. Indeed this dependence can be avoided by adapting the scheme of [Deng 2020]. We point out that the advantage of using the Green's function in the noncollapsed setting is that one obtains optimal infinitesimal Lipschitz estimates [Brué et al. 2022, Theorem 1.6], which does not seem to be readily obtainable using the methods employed here.

1.2. *Induction theorem.* In this subsection we state Theorem 1.14; our main technical result from which Theorem 1.1 follows. Recall that for a semilocally simply connected space X, we can identify its fundamental group $\pi_1(X)$ with the group of deck transformations of its universal cover $\widetilde{X}$.

Definition 1.9. Let X be a semilocally simply connected geodesic space and $\widetilde{X}$ its universal cover. We say a function $f : \widetilde{X} \to \widetilde{X}$ is of *deck type* if there is an automorphism $f_* \in \mathrm{Aut}(\pi_1(X))$ such that for all $g \in \pi_1(X)$ and $x \in \widetilde{X}$, one has $f(g(x)) = f_*(g)(f(x))$.

Example 1.10. If $f \in \pi_1(X)$, then it is of deck type with $f_*(g) := f \circ g \circ f^{-1}$.

Definition 1.11. For metric spaces X, Y, a function $f : X \to Y$, and $r > 0$, the *distortion at scale r* is defined as the map $\mathrm{dt}_r(f) : X \times X \to [0, r]$ with

$$\mathrm{dt}_r(f)(x_1, x_2) := \min\{r, |d_X(x_1, x_2) - d_Y(f(x_1), f(x_2))|\}.$$

If X is equipped with a measure $\mathfrak{m}$, we say that $x \in X$ is a *point of essential continuity* of f if there exists $r_0 > 0$ such that for all $r \leq r_0$ there is a subset $A_r \subset B_r(x)$ with $\mathfrak{m}(A_r) \geq \frac{1}{2}\mathfrak{m}(B_r(x))$ and $f(A_r) \subset B_{2r}(f(x))$.

The next definition is a nonsmooth version of the *maps with zoom-in property* from [Kapovitch and Wilking 2011]. Although the notion is very technical, these are precisely the properties present in gradient flows of δ-splittings (and as we will show, also in the RLFs of δ-splittings in the RCD setting).

Definition 1.12. Let $(X_i^j, d_i^j, \mathfrak{m}_i^j, p_i^j)$, $j \in \{1, 2\}$ be two sequences of pointed RCD(K, N) spaces. We say that a sequence of measurable functions $f_i : [X_i^1, p_i^1] \to [X_i^2, p_i^2]$ is *good at all scales* (GS) if there is a sequence of measurable functions $f_i^{-1} : [X_i^2, p_i^2] \to [X_i^1, p_i^1]$ such that $f_i^{-1} \circ f_i = \mathrm{Id}_{X_i^1}$ almost everywhere and $f_i \circ f_i^{-1} = \mathrm{Id}_{X_i^2}$ almost everywhere, satisfying the following:

(1) $(f_i)_*(\mathfrak{m}_i^1) \ll \mathfrak{m}_i^2$ and $(f_i^{-1})_*(\mathfrak{m}_i^2) \ll \mathfrak{m}_i^1$ for all i.

(2) There is $R_0 > 0$ and sequences $S_i^j \subset B_1(p_i^j)$ for $j \in \{1, 2\}$ with $\mathfrak{m}_i^j(S_i^j) \geq \frac{1}{2}\mathfrak{m}_i^j(B_1(p_i^j))$ and $f_i(S_i^1) \subset B_{R_0}(p_i^2)$, $f_i^{-1}(S_i^2) \subset B_{R_0}(p_i^1)$.

(3) There is a sequence $\varepsilon_i \to 0$ and sequences of subsets $U_i^j \subset X_i^j$ for $j \in \{1, 2\}$ such that:

(a) The points of U_i^1 (resp. U_i^2) are of essential continuity of f_i (resp. f_i^{-1}).

(b) f_i (resp. f_i^{-1}) restricted to U_i^1 (resp. U_i^2) is measure preserving.

(c) For all $R > 0$ and $j \in \{1, 2\}$, one has

$$\lim_{i\to\infty} \frac{\mathfrak{m}_i^j(U_i^j \cap B_R(p_i^j))}{\mathfrak{m}_i^j(B_R(p_i^j))} = 1.$$

(d) For all $x_i^1 \in U_i^1$, $x_i^2 \in U_i^2$, $r \leq 1$, one has

$$\fint_{B_r(x_i^1)^{\times 2}} \mathrm{dt}_r(f_i)(a, b)\, d(\mathfrak{m}_i^1 \times \mathfrak{m}_i^1)(a, b) \leq \varepsilon_i r,$$

$$\fint_{B_r(x_i^2)^{\times 2}} \mathrm{dt}_r(f_i^{-1})(a, b)\, d(\mathfrak{m}_i^2 \times \mathfrak{m}_i^2)(a, b) \leq \varepsilon_i r.$$

Definition 1.13. Let Γ be a group, $G \leq \Gamma$ a subgroup admitting a nilpotent basis $\beta = \{\gamma_1, \dots, \gamma_n\}$, and $\varphi \in \mathrm{Aut}(\Gamma)$. We say that φ *respects* β if it preserves $\langle\{\gamma_1, \dots, \gamma_m\}\rangle$ for each m, and acts trivially on $\langle\{\gamma_1, \dots, \gamma_m\}\rangle / \langle\{\gamma_1, \dots, \gamma_{m-1}\}\rangle$ for each m.

Theorem 1.14. *Let $(X_i, d_i, \mathfrak{m}_i, p_i)$ be a sequence of pointed* RCD$\left(-\frac{1}{i}, N\right)$ *spaces of rectifiable dimension n and a pointed compact metric space (Y, y) of diameter D for which the sequence (X_i, p_i) converges in the pointed Gromov–Hausdorff sense to $(\mathbb{R}^k \times Y, (0, y))$. Let $\widetilde{X}_i$ be the sequence of universal covers, $\tilde{p}_i \in \widetilde{X}_i$ in the preimage of p_i, $\Gamma_i \leq \pi_1(X_i)$ be the group generated by the elements $g \in \pi_1(X_i)$ with $d(g\tilde{p}_i, \tilde{p}_i) \leq 2D + 1$, and for each $j \in \{1, \dots, \ell\}$, $f_{j,i} : [\widetilde{X}_i, \tilde{p}_i] \to [\widetilde{X}_i, \tilde{p}_i]$ a sequence of deck type maps with the GS property. Then for some $C > 0$ and i large enough, Γ_i contains a subgroup $G_i \leq \Gamma_i$ with the following properties*:

- $[\Gamma_i, G_i] \leq C$.
- *G_i admits a nilpotent basis β_i of length $\leq n - k$.*
- *$(f_{j,i})_*^{C!}$ respects β_i for each j.*

Naber and Zhang [2016, Appendix A] proved a blown-down version of Theorem 1.14 for Riemannian manifolds. The techniques they used to obtain this version from Theorem 1.14 (also present in [Kapovitch and Wilking 2011]) apply to RCD(K, N) spaces, giving the following result.

Corollary 1.15. *Let $(X, d, \mathfrak{m}, p)$ be a pointed* RCD(K, N) *space of rectifiable dimension k. Then there is $\varepsilon > 0$ such that if a pointed* RCD(K, N) *space $(X', d', \mathfrak{m}', p')$ of rectifiable dimension n satisfies*

$$d_{GH}((X', p'), (X, p)) < \varepsilon,$$

then the image of the map

$$\pi_1(B_\varepsilon(p'), p') \to \pi_1(X', p')$$

induced by inclusion contains a subgroup of index $\leq C(X, p)$ that admits a nilpotent basis of length $\leq n-k$.

1.3. ***Main ideas of the proof.*** A natural approach to prove a result like Theorem 1.1 is to consider a contradicting sequence of pointed $\mathrm{RCD}(K, N)$ spaces $(X_i, d_i, \mathfrak{m}_i, p_i)$ of rectifiable dimension n, and $\varepsilon_i \to 0$ for which the group $\Gamma_i := j_*(\pi_1(B_{\varepsilon_i}(p_i), p_i))$ does not contain a subgroup of index $\leq i$ admitting a nilpotent basis of length $\leq n$, where $j_* : \pi_1(B_{\varepsilon_i}(p_i), p_i) \to \pi_1(X_i, p_i)$ is the natural map induced by inclusion. After slowly blowing up and taking a subsequence, one can assume the universal covers $(\widetilde{X}_i, \tilde{d}_i, \widetilde{\mathfrak{m}}_i, \tilde{p}_i)$ converge in the pointed measured Gromov–Hausdorff sense to a pointed $\mathrm{RCD}(0, N)$ space $(X, d, \mathfrak{m}, p)$, and the actions of Γ_i on these spaces converge equivariantly to a Lie group $\Gamma \leq \mathrm{Iso}(X)$.

From here, it would be easy to obtain via well-established techniques that the identity connected component $\Gamma_0 \leq \Gamma$ is nilpotent and $[\Gamma : \Gamma_0] < \infty$. This nice behavior can be traced back to the groups Γ_i using Gromov–Hausdorff approximations $\psi_i : \Gamma_i \to \Gamma$. This would finish the proof, if not for the possibility that there may be subgroups $H_i \leq \Gamma_i$ too small for the Gromov–Hausdorff approximations to detect them. Recall that while a sequence of Gromov–Hausdorff approximations describes very well the geometry of a sequence of spaces at a certain scale, it fails to see

- features that are too small,
- features that are too far from the basepoints.

To remedy the issue of having subgroups $H_i \leq \Gamma_i$ which are too small, one could blow up the sequence by factors $\lambda_i \to \infty$ to a scale at which the groups H_i are visible, and again take a subsequence in such a way that the actions of Γ_i on the spaces $(\widetilde{X}_i, \lambda_i \tilde{d}_i, \widetilde{\mathfrak{m}}_i, \tilde{p}_i)$ converge equivariantly to a Lie group Γ' acting by isometries on a new limit space $(X', d', \mathfrak{m}', p')$. The problem with doing so is that relevant elements of the original group Γ_i may be sent too far for the new Gromov–Hausdorff approximations $\psi' : \Gamma_i \to \Gamma'$ to see them.

In order to understand how the elements of H_i interact with the elements lost due the blow-up, we need to bring these elements back by homotopy. For this purpose, the gradient flow of semiconcave (resp. harmonic) functions is used in [Kapovitch et al. 2010] (resp. [Kapovitch and Wilking 2011]). In the setting of $\mathrm{RCD}(K, N)$ spaces, the regular Lagrangian flows play the role of such tools. However, since this process has to be done multiple times, without prior knowledge about scale and location, one needs to control the regularity of such flows at all small scales. This is the reason for the technical nature of Theorems 1.5 and 1.14. The maps $f_{j,i}$ in Theorem 1.14 are precisely these isometries in Γ_i that were sent too far and then brought back by composing them with an appropriate regular Lagrangian flow.

1.4. ***Open problems.*** In the context of Theorem 1.1, it has been conjectured that the nilpotent group can be taken so that its torsion lies in its center. This is not known even for Riemannian manifolds of sectional curvature $\geq K/(N-1)$ [Kapovitch et al. 2010; 2018] (see also [Fukaya and Yamaguchi 1992, Conjecture 0.16]).

Conjecture 1.16. *For each $K \in \mathbb{R}$, $N \geq 1$, there exist $\varepsilon > 0$ and $C \in \mathbb{N}$ such that if $(X, d, \mathfrak{m}, p)$ is a pointed $\mathrm{RCD}(K, N)$ space of rectifiable dimension n, the image of the map*

$$\pi_1(B_\varepsilon(p), p) \to \pi_1(X, p)$$

induced by inclusion contains a subgroup of index $\leq C$ whose torsion elements are contained in its center.

On the other hand, it is also a very challenging problem to find an explicit expression for $C(K, N)$ in Theorem 1.1. Such an expression hasn't been found even for Riemannian manifolds of sectional curvature $\geq K/(N-1)$ (see [Kapovitch et al. 2010]).

1.5. ***Structure of the paper.*** In Section 2, we cover the background material we will need. In Section 3, we prove Theorem 3.1, which provides us with subgroups $\Upsilon_i \triangleleft \Gamma_i$ that play the role of identity connected components in the discrete groups Γ_i.

In Section 4 we prove Theorem 1.5 and Corollary 1.8, allowing us to find points of essential stability, and in Section 5 we study how essential stability allows one to obtain stronger estimates. In Section 6 we prove properties of GS maps, and in Section 7 we give two ways to construct GS maps (cf. [Kapovitch and Wilking 2011, Section 3]).

In Section 8 we show Theorem 8.1, reducing Theorem 1.14 to the case $Y \neq \{*\}$ (cf. [Kapovitch and Wilking 2011, Section 5]). In Section 9 we prove Theorem 1.14 and with it Theorem 1.1 and Corollary 1.15.

2. Preamble

2.1. ***Notation.*** For a set A, we denote by $A^{\times 2}$ the set $A \times A$. If $A \subset X$, we denote by $\chi_A : X \to [0, 1]$ the characteristic function of A. For a group G and $g \in G$, we denote by $g_* \in \operatorname{Aut}(G)$ the map $h \mapsto ghg^{-1}$. For metric spaces (X, d_X) and (Y, d_Y), we denote by $X \times Y$ the L^2 product. That is, for $x_1, x_2 \in X$, $y_1, y_2 \in Y$,

$$d_{X\times Y}((x_1, y_1), (x_2, y_2)) := \sqrt{d_X(x_1, x_2)^2 + d_Y(y_1, y_2)^2}.$$

We say a pointed metric measure space $(X, d, \mathfrak{m}, p)$ is *normalized* if

$$\int_{B_1(p)} (1 - d(p, \cdot))\, d\mathfrak{m} = 1.$$

For $m \in \mathbb{N}$, we denote by $\mathbb{R}^m$ the m-dimensional Euclidean space equipped with its usual metric, and by $\mathcal{H}^m$ the m-dimensional Hausdorff measure for which the metric measure space $(\mathbb{R}^m, \mathcal{H}^m, 0)$ is normalized.

To a metric space (X, d), we can adjoin a point $*$ at infinite distance from any point of X to get a new space we denote by $X \cup \{*\}$. Similarly, to any group G we can adjoin an element $*$ whose product with any element of G is defined as $*$, obtaining a binary operation on $G \cup \{*\}$.

We write $C(\alpha, \beta, \gamma)$ to denote a constant C that depends only on the quantities α, β, γ.

2.2. **RCD(K, N)** ***spaces; doubling, isometries, covers, and geodesics.*** The main objects of this text are RCD(K, N) spaces. We note that a large number of papers in the literature work with a condition known as RCD*(K, N), originally introduced in [Bacher and Sturm 2010]. Since it is now known that this condition is equivalent to the RCD(K, N) condition [Cavalletti and Milman 2021; Li 2024], we will make no distinction between them.

One of the most powerful tools in the study of RCD(K, N) spaces is the Bishop–Gromov inequality [Sturm 2006b].

Theorem 2.1 (Sturm). *For each $K \in \mathbb{R}$, $N \geq 1$, $R > 0$, $\lambda > 1$ there is $C(K, N, R, \lambda) > 0$ such that for any pointed* $\mathrm{RCD}(K, N)$ *space* $(X, d, \mathfrak{m}, p)$, *and any* $r \leq R$, *one has*

$$\mathfrak{m}(B_{\lambda r}(p)) \leq C \cdot \mathfrak{m}(B_r(p)).$$

Moreover, for fixed K, N, R, if $\lambda \to 1$ then $C \to 1$.

Corollary 2.2. *Let $(X_i, d_i, \mathfrak{m}_i, p_i)$ be a sequence of pointed* $\mathrm{RCD}(K, N)$ *spaces and consider a sequence of subsets $U_i \subset X_i$. Then the following are equivalent*:

- *For all $R > 0$, there is a sequence $\eta_i \to 0$ such that*

$$\mathfrak{m}_i(U_i \cap B_R(p_i)) \geq (1 - \eta_i)\mathfrak{m}_i(B_R(p_i)).$$

- *For all $R \geq \delta > 0$, there is a sequence $\varepsilon_i \to 0$ such that if $x \in B_R(p_i)$, one has*

$$\mathfrak{m}_i(U_i \cap B_\delta(x)) \geq (1 - \varepsilon_i)\mathfrak{m}_i(B_\delta(x)).$$

In either case, we say that the sequence U_i has asymptotically full measure.

Proof. Assume the first condition holds. If the second condition fails for some $R \geq \delta > 0$, then after passing to a subsequence, there would be $\varepsilon > 0$ and $x_i \in B_R(p_i)$ with

$$\mathfrak{m}_i(B_\delta(x_i) \backslash U_i) \geq \varepsilon \cdot \mathfrak{m}_i(B_\delta(x_i)). \tag{2.3}$$

By the triangle inequality and Theorem 2.1, there is $C(K, N, R, \delta) > 0$ with

$$\mathfrak{m}_i(B_{R+\delta}(p_i)) \leq \mathfrak{m}_i(B_{2R+\delta}(x_i)) \leq C \cdot \mathfrak{m}_i(B_\delta(x_i)). \tag{2.4}$$

Since $B_\delta(x_i) \subset B_{R+\delta}(p_i)$, combining (2.3) and (2.4) we get

$$\mathfrak{m}_i(B_{R+\delta}(p_i) \backslash U_i) \geq \varepsilon \cdot \mathfrak{m}_i(B_{R+\delta}(p_i))/C,$$

contradicting our hypothesis.

The other implication is evident by taking $\delta = R$ and $x = p_i$. □

The following well-known facts follow from Theorem 2.1 (see [Stein 1993, p. 12; Kapovitch and Wilking 2011, p. 6]). For definition of maximal function, see for example Definition 1.4.

Proposition 2.5. *Let $(X, d, \mathfrak{m})$ be an* $\mathrm{RCD}(K, N)$ *space, $h : X \to \mathbb{R}^+$ measurable, and $R > 0$.*

(1) *For all $\delta > 0$,*

$$\mathfrak{m}(\{x \in X \mid Mx_R(h)(x) \geq \delta\}) \leq \frac{C(K, N, R)}{\delta} \int_X h \, d\mathfrak{m}.$$

(2) *For all $\alpha > 1$,*

$$\|Mx_R(h)\|_\alpha \leq C(K, N, R, \alpha)\|h\|_\alpha.$$

(3) *For all $\alpha > 1$, $s \leq R/2$,*

$$Mx_s(Mx_s(h)^\alpha) \leq C(K, N, R, \alpha)Mx_R(h^\alpha).$$

For a proper metric space X, the topology that we use on its group of isometries $\mathrm{Iso}(X)$ is the compact-open topology, which in this setting coincides with both the topology of pointwise convergence and the topology of uniform convergence on compact sets. This topology makes $\mathrm{Iso}(X)$ a locally compact second countable metric group. In the case $(X, d, \mathfrak{m})$ is an RCD(K, N) space, $\mathrm{Iso}(X)$ is a Lie group [Guijarro and Santos-Rodríguez 2019; Sosa 2018].

Theorem 2.6 (Sosa, Guijarro and Santos-Rodríguez). *Let $(X, d, \mathfrak{m})$ be an* RCD(K, N) *space. Then* $\mathrm{Iso}(X)$ *is a Lie group.*

The RCD(K, N) condition can be checked locally (see [Erbar et al. 2015, Section 3]). Hence if $(X, d, \mathfrak{m})$ is an RCD(K, N) space and $\rho : \widetilde{X} \to X$ is a covering space, $\widetilde{X}$ admits a unique measure making it an RCD(K, N) space, and for which ρ is a local isomorphism of metric measure spaces (see [Mondino and Wei 2019, Section 2.3]). Whenever we have a covering space of an RCD(K, N) space, we assume it is equipped with such measure. This allows one to lift estimates on maximal functions [Kapovitch and Wilking 2011, Lemma 1.6].

Proposition 2.7. *Let $(X, d, \mathfrak{m})$ be an* RCD(K, N), *$\rho : (\widetilde{X}, \tilde{d}, \tilde{\mathfrak{m}}) \to (X, d, \mathfrak{m})$ a covering space, $x \in X$, $\tilde{x} \in \rho^{-1}(x)$, $f : X \to \mathbb{R}^+$ measurable. Then for all $r \leq R$, one has*

$$\fint_{B_r(\tilde{x})} (f \circ \rho)\, d\tilde{\mathfrak{m}} \leq C(K, N, R) \fint_{B_r(x)} f\, d\mathfrak{m}.$$

In particular,

$$Mx_R(f \circ \rho) \leq C(K, N, R) \cdot Mx_R(f) \circ \rho.$$

An important topological property of RCD(K, N) spaces is that they are semilocally simply connected [Wang 2024].

Theorem 2.8 (Wang). *Let $(X, d, \mathfrak{m})$ be an* RCD(K, N) *space. Then X is semilocally simply connected, so its universal cover $\widetilde{X}$ is simply connected and we can identify $\pi_1(X)$ with the group of deck transformations $\widetilde{X} \to \widetilde{X}$.*

The following is a well-known equivalence of semilocal simple-connectedness (see for example [Calcut and McCarthy 2009]). We include its proof for completeness.

Proposition 2.9. *Let X be a semilocally simply connected geodesic space. Then for each compact set $K \subset X$ there is $\delta > 0$ with the property that any two curves $\alpha, \beta : [0, 1] \to K$ sharing endpoints and at uniform distance $\leq \delta$ are homotopic relative to their endpoints.*

Proof. By hypothesis, X admits an open cover $\mathcal{U}$ with the property that each loop contained in an element of $\mathcal{U}$ is contractible in X. It is an easy exercise to check that if $U \in \mathcal{U}$, and $\sigma_1, \sigma_2 : [0, 1] \to U$ are two paths with the same endpoints, then σ_1 and σ_2 are homotopic relative to their endpoints as curves in X.

Consider a compact set $K \subset X$. Then there is $\delta > 0$ such that 2δ is a Lebesgue number of $\mathcal{U}$ as a cover of K. That is, for any $x \in K$ there is an element of $\mathcal{U}$ that contains $B_{2\delta}(x)$. Then take two curves $\alpha, \beta : [0, 1] \to K$ with the same endpoints and at uniform distance $\leq \delta$. We claim that they are homotopic relative to their endpoints as curves in X.

To see this, take a partition $0 = t_0 < t_1 < \cdots < t_k = 1$ with the property that $\alpha(t) \in B_\delta(\alpha(t_j))$ for all $t \in [t_{j-1}, t_j]$, $j \in \{1, \ldots, k\}$. For $j \in \{1, \ldots, k\}$ define $\gamma_j : [0, 1] \to X$ to be a curve that agrees with β along $[0, t_j]$, with α along $[t_{j+1}, 1]$, and along $[t_j, t_{j+1}]$ is a minimizing curve connecting $\beta(t_j)$ with $\alpha(t_{j+1})$. Notice that $\gamma_k = \beta$.

It is then easy to see by induction that α and γ_j are homotopic relative to their endpoints. Indeed, if we set $\gamma_0 := \alpha$, then for each $j \in \{1, \ldots, k\}$ the curves γ_{j-1} and γ_j are identical except along an interval where their images are contained in $B_{2\delta}(\alpha(t_j))$, and hence in an element of $\mathcal{U}$ and consequently homotopic relative to their endpoints. □

To conclude this subsection, we note that by the Kuratowski–Ryll-Nardzewski measurable selection theorem, for any $\mathrm{RCD}(K, N)$ space $(X, d, \mathfrak{m})$, there is a measurable map

$$\gamma_{\cdot,\cdot}(\cdot) : X \times X \times [0, 1] \to X$$

such that, for all $x, y \in X$, the map $[0, 1] \ni s \mapsto \gamma_{x,y}(s)$ is a constant speed geodesic from x to y. For the rest of this paper, for each $(X, d, \mathfrak{m})$ we fix such a choice of γ. This allows us to state the segment inequality for $\mathrm{RCD}(K, N)$ spaces [Deng 2020, Theorem 3.22].

Theorem 2.10. *Let $(X, d, \mathfrak{m})$ be an $\mathrm{RCD}(K, N)$ space, $h : X \to \mathbb{R}^+$ measurable, $p \in X$, and $r \leq R$. Then*

$$\fint_{B_r(p)^{\times 2}} d(x, y)\left[\int_0^1 h(\gamma_{x,y}(s))\, ds\right] d(\mathfrak{m} \times \mathfrak{m})(x, y) \leq r \cdot C(K, N, R) \fint_{B_{2r}(p)} h\, d\mathfrak{m}.$$

We will also need the following variation of the Lebesgue differentiation theorem (see [Stein and Shakarchi 2005; Heinonen et al. 2015]).

Definition 2.11. Let $(X, d, \mathfrak{m})$ be a metric measure space. We say that a family of measures $\mathcal{V}$ on X has *bounded eccentricity* if there are $M \geq 1 \geq \eta > 0$ such that $\nu \leq M\mathfrak{m}$ for all $\nu \in \mathcal{V}$, and a map $\theta : \mathcal{V} \to X$ such that for all $\nu \in \mathcal{V}$ there is $r(\nu) > 0$ with $\operatorname{supp}(\nu) \subset B_r(\theta(\nu))$ and $\nu(B_r(\theta(\nu))) \geq \eta\, \mathfrak{m}(B_r(\theta(\nu)))$. We then say that a net $\nu_i \in \mathcal{V}$ converges to $x \in X$ if $\theta(\nu_i) = x$ for all large i and $r(\nu_i) \to 0$.

Lemma 2.12. *Let $(X, d, \mathfrak{m})$ be an $\mathrm{RCD}(K, N)$ space, $f \in L^1(\mathfrak{m})$, and $\mathcal{V}$ a family of measures of bounded eccentricity. Then for $\mathfrak{m}$-almost every $x \in X$ we have*

$$f(x) = \lim_{\nu \to x} \frac{1}{\nu(X)} \int_X f\, d\nu.$$

Proof. For $\alpha > 0$, define

$$E_\alpha := \left\{x \in X : \limsup_{\nu \to x} \frac{1}{\nu(X)} \left| \int_X f(y) - f(x)\, d\nu \right| > 2\alpha \right\}.$$

Given $\varepsilon > 0$, pick a continuous function $g \in L^1(\mathfrak{m})$ with

$$\|f - g\|_{L^1(\mathfrak{m})} \leq \varepsilon. \tag{2.13}$$

For $\nu \in \mathcal{V}$ with $r(\nu) \le 1$ and $\theta(\nu) = x$ we have

$$\frac{1}{\nu(X)}\left|\int_X (f(y) - f(x))\, d\nu(y)\right| \le \frac{1}{\nu(X)}\left|\int_X (f(y) - g(y))\, d\nu(y)\right| + \frac{1}{\nu(X)}\left|\int_X (g(y) - g(x))\, d\nu(y)\right| + |g(x) - f(x)|. \quad (2.14)$$

Since g is continuous, for all $x \in X$ we have

$$\lim_{\nu \to x} \frac{1}{\nu(X)}\left|\int_X (g(y) - g(x))\, d\nu(y)\right| = 0. \quad (2.15)$$

To deal with the first summand, we compute

$$\frac{1}{\nu(X)}\left|\int_X (f(y) - g(y))\, d\nu(y)\right| \le \frac{M}{\eta\, \mathfrak{m}(B_r(x))}\int_{B_r(x)} |f(y) - g(y)|\, d\mathfrak{m} \le \frac{M}{\eta}\mathrm{Mx}(|f - g|)(x). \quad (2.16)$$

Combining (2.14), (2.15), and (2.16), we get

$$E_\alpha \subset \left\{\mathrm{Mx}(|f - g|) \ge \frac{\eta\,\alpha}{M}\right\} \cup \{|f - g| \ge \alpha\}.$$

Then from (2.13) and Proposition 2.5(1) we obtain

$$\mathfrak{m}(E_\alpha) \le \frac{C(K, N)M}{\eta\alpha}\varepsilon.$$

Since ε was arbitrary we get $\mathfrak{m}(E_\alpha) = 0$, and hence the result. ☐

2.3. *Gromov–Hausdorff topology.*

Definition 2.17. Let (X_i, p_i) be a sequence of pointed proper metric spaces. We say that it *converges in the pointed Gromov–Hausdorff sense* to a proper pointed metric space (X, p) if there is a sequence of functions $\varphi_i : X_i \to X \cup \{*\}$ with $\varphi_i(p_i) \to p$ such that, for each $R > 0$,

$$\varphi_i^{-1}(B_R(p)) \subset B_{2R}(p_i) \text{ for } i \text{ large enough},$$

$$\lim_{i\to\infty} \sup_{x_1, x_2 \in B_{2R}(p_i)} |d(\varphi_i(x_1), \varphi_i(x_2)) - d(x_1, x_2)| = 0,$$

$$\lim_{i\to\infty} \sup_{y \in B_R(p)} \inf_{x \in B_{2R}(p_i)} d(\varphi_i(x), y) = 0.$$

If, in addition, $(X_i, d_i, \mathfrak{m}_i)$, $(X, d, \mathfrak{m})$ are metric measure spaces, the maps φ_i are Borel measurable, and

$$\int_X f \cdot d((\varphi_i)_*\mathfrak{m}_i) \to \int_X f \cdot d\mathfrak{m}$$

for all $f : X \to \mathbb{R}$ bounded continuous with compact support, then we say that $(X_i, d_i, \mathfrak{m}_i, p_i)$ converges to $(X, d, \mathfrak{m}, p)$ in the *pointed measured Gromov–Hausdorff sense*.

Remark 2.18. Whenever a sequence of pointed spaces (X_i, p_i) converges in the pointed Gromov–Hausdorff sense to some pointed space (X, p), we implicitly assume the existence of the maps φ_i, called *Gromov–Hausdorff approximations* satisfying the above conditions, and if a sequence $x_i \in X_i$ is such that $\varphi_i(x_i) \to x \in X$, by an abuse of notation we say that x_i *converges* to x.

The topology induced by this convergence is also given by a metric [Gromov 2007].

Theorem 2.19 (Gromov). *There is a metric d_{GH} in the class of pointed proper metric spaces modulo pointed isometry with the property that a sequence (X_i, p_i) converges to a space (X, p) in the pointed Gromov–Hausdorff sense if and only if $d_{GH}((X_i, p_i), (X, p)) \to 0$.*

Remark 2.20. The only property we will need about this metric is that if (Y, y) is a pointed compact geodesic space for which

$$d_{GH}((\mathbb{R}^k \times Y, (0, y)), (\mathbb{R}^k, 0)) \leq \tfrac{1}{100} \quad \text{for some } k \in \mathbb{N},$$

then $\operatorname{diam}(Y) \leq \frac{1}{10}$.

One of the main features of the class of $\mathrm{RCD}(K, N)$ spaces is the compactness property. Theorem 2.21 follows immediately from Gromov's compactness criterion [2007, Proposition 5.2], and Theorem 2.22 was proven in [Gigli et al. 2015] building upon [Lott and Villani 2009; Sturm 2006a; 2006b; Ambrosio et al. 2014b].

Theorem 2.21. *If $(X_i, d_i, \mathfrak{m}_i, p_i)$ is a sequence of pointed $\mathrm{RCD}(K, N)$ spaces, then one can find a subsequence for which (X_i, p_i) converges in the pointed Gromov–Hausdorff sense to some pointed proper geodesic space (X, p).*

Notice that for any pointed $\mathrm{RCD}(K,N)$ space $(X, d, \mathfrak{m}, p)$, there is a unique $c > 0$ for which $(X, d, c\mathfrak{m}, p)$ is normalized.

Theorem 2.22. *The class of pointed normalized $\mathrm{RCD}(K, N)$ spaces is closed under pointed measured Gromov–Hausdorff convergence. Moreover, if $(X_i, d_i, \mathfrak{m}_i, p_i)$ is a sequence of $\mathrm{RCD}(K - \varepsilon_i, N)$ spaces such that $\varepsilon_i \to 0$ and (X_i, p_i) converges in the pointed Gromov–Hausdorff sense to a pointed proper metric space (X, p), then X admits a measure $\mathfrak{m}$ that makes it a normalized $\mathrm{RCD}(K, N)$ space, and after passing to a subsequence, there are $c_i > 0$ for which $(X_i, d_i, c_i\mathfrak{m}_i, p_i)$ converges in the pointed measured Gromov–Hausdorff sense to $(X, d, \mathfrak{m}, p)$.*

Definition 2.23. Let $(X, d, \mathfrak{m})$ be an $\mathrm{RCD}(K, N)$ space and $m \in \mathbb{N}$. We say that $p \in X$ is an *m-regular* point if for each $\lambda_i \to \infty$, the sequence $(\lambda_i X, p)$ converges in the pointed Gromov–Hausdorff sense to $(\mathbb{R}^m, 0)$.

Mondino and Naber [2019] showed that the set of regular points in an $\mathrm{RCD}(K, N)$ space has full measure. This result was refined by Brué and Semola [2020b] who showed that most points have the same local dimension.

Theorem 2.24 (Brué and Semola). *Let $(X, d, \mathfrak{m})$ be an $\mathrm{RCD}(K, N)$ space. Then there is a unique $m \in \mathbb{N} \cap [0, N]$ such that the set of m-regular points in X has full measure. This number m is called the rectifiable dimension of X.*

The Cheeger–Gromoll splitting theorem was extended by Gigli [2014] to this setting.

Theorem 2.25 (Gigli). *Let $(X, d, \mathfrak{m})$ be an $\mathrm{RCD}(0, N)$ space of rectifiable dimension n and assume the metric space (X, d) contains an isometric copy of $\mathbb{R}^m$, then there is $c > 0$ and an $\mathrm{RCD}(0, N - m)$ space (Y, d^Y, ν) of rectifiable dimension $n - m$ such that $(X, d, c\mathfrak{m})$ is isomorphic to the product $(\mathbb{R}^m \times Y, d^{\mathbb{R}^m} \times d^Y, \mathcal{H}^m \otimes \nu)$. In particular $m \leq n$, and if $m = n$ then Y is a point.*

Corollary 2.26. *Let $(X_i, d_i, \mathfrak{m}_i, p_i)$ be a sequence of pointed normalized* RCD$(-\delta_i, N)$ *spaces with $\delta_i \to 0$. If (X_i, p_i) converges in the pointed Gromov–Hausdorff sense to $(\mathbb{R}^k, 0)$, then $(X_i, d_i, \mathfrak{m}_i, p_i)$ converges to $(\mathbb{R}^k, d^{\mathbb{R}^k}, \mathcal{H}^k, 0)$ in the pointed measured Gromov–Hausdorff sense as well.*

Corollary 2.27 below follows from Theorem 2.25 the same way [Cheeger and Gromoll 1971/72, Theorem 3] follows from the splitting theorem for smooth manifolds.

Corollary 2.27. *Let $(\widetilde{Y}, d, \mathfrak{m})$ be an* RCD$(0, N)$ *space of rectifiable dimension n for which $\widetilde{Y}/\operatorname{Iso}(\widetilde{Y})$ is compact. Then there are $m \leq n$ and a compact metric space Z for which $\widetilde{Y}$ is isometric to the product $\mathbb{R}^m \times Z$.*

The rectifiable dimension is lower semicontinuous [Kitabeppu 2019].

Theorem 2.28 (Kitabeppu). *Let $(X_i, d_i, \mathfrak{m}_i, p_i)$ be a sequence of pointed* RCD(K, N) *spaces of rectifiable dimension m. Assume (X_i, p_i) converges in the pointed Gromov–Hausdorff sense to (X, p). If $\mathfrak{m}$ is a measure on X that makes it an* RCD(K, N) *space, then $(X, d, \mathfrak{m})$ has rectifiable dimension at most m.*

2.4. *Equivariant Gromov–Hausdorff convergence.* In the setting of Gromov–Hausdorff convergence, there is a notion of convergence of group actions [Fukaya and Yamaguchi 1992, Section 3]. For a pointed proper metric space (X, p), we equip its isometry group $\operatorname{Iso}(X)$ with the metric d_0^p given by

$$d_0^p(h_1, h_2) := \inf_{r>0} \left\{ \frac{1}{r} + \sup_{x \in B_r(p)} d(h_1 x, h_2 x) \right\} \tag{2.29}$$

for $h_1, h_2 \in \operatorname{Iso}(X)$. It is easy to see that this metric is left invariant, induces the compact-open topology, and makes $\operatorname{Iso}(X)$ a proper metric space.

Recall that if a sequence of pointed proper metric spaces (X_i, p_i) converges in the pointed Gromov–Hausdorff sense to the pointed proper metric space (X, p), one has Gromov–Hausdorff approximations $\varphi_i : X_i \to X \cup \{*\}$.

Definition 2.30. Consider a sequence of pointed proper metric spaces (X_i, p_i) that converges in the pointed Gromov–Hausdorff sense to a pointed proper metric space (X, p), a sequence of closed groups of isometries $\Gamma_i \leq \operatorname{Iso}(X_i)$, and a closed group $\Gamma \leq \operatorname{Iso}(X)$. Equip Γ_i with the metric $d_0^{p_i}$ and Γ with the metric d_0^p. We say that the sequence Γ_i *converges equivariantly* to Γ if there is a sequence of Gromov–Hausdorff approximations $\psi_i : \Gamma_i \to \Gamma \cup \{*\}$ such that for each $R > 0$ one has

$$\lim_{i \to \infty} \sup_{g \in B_R(\mathrm{Id}_{X_i})} \sup_{x \in B_R(p_i)} d(\varphi_i(gx), \psi_i(g)(\varphi_i x)) = 0.$$

Isometry groups of proper spaces satisfy a compactness property [Fukaya and Yamaguchi 1992, Proposition 3.6].

Theorem 2.31 (Fukaya and Yamaguchi). *Let (Y_i, q_i) be a sequence of proper metric spaces that converges in the pointed Gromov–Hausdorff sense to a proper space (Y, q), and take a sequence $\Gamma_i \leq \operatorname{Iso}(Y_i)$ of closed groups of isometries. Then, after taking a subsequence, Γ_i converges equivariantly to a closed group $\Gamma \leq \operatorname{Iso}(Y)$, and the sequence $(Y_i/\Gamma_i, [q_i])$ converges in the pointed Gromov–Hausdorff sense* *to* $(Y/\Gamma, [q])$. *Moreover, if* $\rho_i : Y_i \to Y_i/\Gamma_i$, $\rho : Y \to Y/\Gamma$ *are the projections, there are* $\delta_i \to 0$, $R_i \to \infty$, *and Gromov–Hausdorff approximations* $\tilde{\varphi}_i : Y_i \to Y \cup \{*\}$, $\varphi_i : Y_i/\Gamma_i \to Y/\Gamma \cup \{*\}$ *such that for all* $x \in B_{R_i}(q_i)$ *one has*

$$d(\varphi_i(\rho_i(x)), \rho(\tilde{\varphi}_i(x))) \le \delta_i. \tag{2.32}$$

As a consequence of Theorems 2.25 and 2.31, one gets the following well-known result.

Proposition 2.33. *For each* $i \in \mathbb{N}$, *let* $(X_i, d_i, \mathfrak{m}_i, p_i)$ *be a pointed* $\mathrm{RCD}\big(-\frac{1}{i}, N\big)$ *space of rectifiable dimension n. Assume* (X_i, p_i) *converges in the pointed Gromov–Hausdorff sense to* (X, p), *there is a sequence of closed groups of isometries* $\Gamma_i \le \mathrm{Iso}(X_i)$ *that converges equivariantly to* $\Gamma \le \mathrm{Iso}(X)$, *and the sequence of pointed metric spaces* $(X_i/\Gamma_i, [p_i])$ *converges in the pointed Gromov–Hausdorff sense to* $(\mathbb{R}^k \times Y, (0, q))$ *for some pointed proper metric space* (Y, q).

Then there is a pointed metric space $(\widetilde{Y}, \tilde{q})$ *for which X is isomorphic to the product* $\mathbb{R}^k \times \widetilde{Y}$, *the* Γ*-action respects the splitting* $\mathbb{R}^k \times \widetilde{Y}$, *and acts trivially on the first factor. In particular, if* $k = n$, *then* $\widetilde{Y}$ *is a point.*

Proof. By Theorem 2.31, $X/\Gamma = \mathbb{R}^k \times Y$, and one can use the submetry $\rho : X \to X/\Gamma$ to lift the lines of $\mathbb{R}^k$ to lines in X passing through p. By Theorem 2.22, X admits a measure that makes it an $\mathrm{RCD}(0, N)$ space, so by Theorems 2.25 and 2.28, we get the desired splitting $X = \mathbb{R}^k \times \widetilde{Y}$ with the property that $\rho(x, \tilde{q}) = (x, q)$ for all $x \in \mathbb{R}^k$.

Now we show that the action of Γ respects the $\widetilde{Y}$-fibers. Let $g \in \Gamma$ and assume $g(x_1, \tilde{q}) = (x_2, y)$ for some $x_1, x_2 \in \mathbb{R}^k$, $y \in \widetilde{Y}$. Then for all $t \ge 1$, one has

$$\begin{aligned} t|x_1 - x_2| &= d(\rho(x_1 + t(x_2 - x_1), \tilde{q}), \rho(x_1, \tilde{q})) \\ &= d(\rho(x_1 + t(x_2 - x_1), \tilde{q}), \rho(x_2, y)) \\ &\le d^X((x_1 + t(x_2 - x_1), \tilde{q}), (x_2, y)) \\ &= \sqrt{|(t-1)(x_2 - x_1)|^2 + d^{\widetilde{Y}}(\tilde{q}, y)^2}. \end{aligned}$$

As $t \to \infty$, this is only possible if $x_1 = x_2$. This shows that $g(x, \tilde{q}) = (x, y)$ for some $y \in \widetilde{Y}$ independent of $x \in \mathbb{R}^k$. Now assume $g(x_1, z) = (x_2, z')$ for some $z, z' \in \widetilde{Y}$. Then

$$\begin{aligned} t^2|x_1 - x_2|^2 + d^{\widetilde{Y}}(\tilde{q}, z)^2 &= d^X((x_1 + t(x_2 - x_1), \tilde{q}), (x_1, z))^2 \\ &= d^X((x_1 + t(x_2 - x_1), y), (x_2, z'))^2 \\ &= |(t-1)(x_2 - x_1)|^2 + d^{\widetilde{Y}}(y, z')^2. \end{aligned}$$

This is only possible if $x_1 = x_2$, showing that Γ acts trivially on the $\mathbb{R}^k$-factor. ☐

2.5. *δ-splittings.* Let us recall some results on δ-splittings. For proof and detailed discussions see for example [Bruè et al. 2023, Section 3.1].

Lemma 2.34. *Let* $(X_i, d_i, \mathfrak{m}_i, p_i)$ *be a sequence of* $\mathrm{RCD}\big(-\frac{1}{i}, N\big)$ *spaces for which* (X_i, p_i) *converges in the pointed Gromov–Hausdorff sense to* $(\mathbb{R}^k \times Y, (0, y))$ *for some metric space* (Y, y). *Then for any* *sequence of Gromov–Hausdorff approximations* $\varphi_i : X_i \to \mathbb{R}^k \times Y \cup \{*\}$, *there are sequences* $\delta_i \to 0$, $R_i \to \infty$, *and a sequence of* $L(N)$*-Lipschitz functions* $h^i \in H^{1,2}(X_i; \mathbb{R}^k)$ *such that*

- h^i *is harmonic* (*equivalently*, ∇h^i *is divergence free*) *in* $B_{R_i}(p_i)$,
- *for all* $r \in [1, R_i]$, *one has*

$$\fint_{B_r(p_i)} \Bigg[\sum_{j_1, j_2 = 1}^{k} |\langle \nabla h^i_{j_1}, \nabla h^i_{j_2} \rangle - \delta_{j_1, j_2}| + \sum_{j=1}^{k} |\nabla \nabla h^i_j|^2 \Bigg] d\mathfrak{m}_i \le \delta_i^2,$$

- *for all* $x \in B_{R_i}(p_i)$ *one has*

$$|h^i(x) - \pi(\varphi_i x)| \le \delta_i, \tag{2.35}$$

where $\pi : \mathbb{R}^k \times Y \to \mathbb{R}^k$ *is the projection.*

Lemma 2.36. *Let* $(X_i, d_i, \mathfrak{m}_i, p_i)$ *be a sequence of* RCD$\big(-\frac{1}{i}, N\big)$. *Assume there are sequences* $\delta_i \to 0$, $R_i \to \infty$, *and a sequence of* L*-Lipschitz functions* $h^i \in H^{1,2}(X_i; \mathbb{R}^k)$ *with* $h^i(p_i) = 0$ *for all* i *and such that*

- h^i *is harmonic* (*equivalently*, ∇h^i *is divergence free*) *in* $B_{R_i}(p_i)$,
- *for all* $r \in [1, R_i]$, *one has*

$$\fint_{B_r(p_i)} \Bigg[\sum_{j_1, j_2 = 1}^{k} |\langle \nabla h^i_{j_1}, \nabla h^i_{j_2} \rangle - \delta_{j_1, j_2}| + \sum_{j=1}^{k} |\nabla \nabla h^i_j|^2 \Bigg] d\mathfrak{m}_i \le \delta_i^2.$$

Then, *after taking a subsequence*, *there is a metric space* (Y, y) *and a sequence of Gromov–Hausdorff approximations* $\varphi_i : X_i \to \mathbb{R}^k \times Y \cup \{*\}$ *for which*

$$\sup_{x \in B_{R_i}(p_i)} |h^i(x) - \pi(\varphi_i x)| \to 0 \quad \text{as } i \to \infty, \tag{2.37}$$

where $\pi : \mathbb{R}^k \times Y \to \mathbb{R}^k$ *is the projection.*

Remark 2.38. In the literature, Lemmas 2.34 and 2.36 are often stated without equations (2.35) and (2.37). However, these equations follow from how the functions h_i (resp. φ_i) are constructed in the proof of Lemma 2.34 (resp. Lemma 2.36). Similarly, the maps h_i are usually only defined on balls around p_i with radii going to infinity, but thanks to the existence of good cut-off functions [Mondino and Naber 2019, Lemma 3.1], we can assume they are fully defined on the spaces X_i.

2.6. ***Regular Lagrangian flows.*** In RCD(K, N) spaces, there exist flows of certain Sobolev vector fields. For the definition of RLFs, see for example Definition 1.2. For sufficiently regular vector fields, RLFs satisfy an existence and uniqueness property [Ambrosio and Trevisan 2014].

Theorem 2.39. *Let* $(X, d, \mathfrak{m})$ *be an* RCD(K, N) *space*, *and assume* $V \in L^1([0, T], L^2(TX))$ *satisfies* $V(t) \in D(\mathrm{div})$ *for a.e.* $t \in [0, T]$ *with*

$$\mathrm{div}(V(\cdot)) \in L^1([0, T], L^2(\mathfrak{m})), \quad (\mathrm{div}(V(\cdot)))^- \in L^1([0, T], L^\infty(\mathfrak{m})), \quad \nabla V(\cdot) \in L^1([0, T], L^2(T^{\otimes 2} X)).$$

Then there exists a unique (*up to* $\mathfrak{m}$*-a.e. equality*) *RLF* $\boldsymbol{X} : [0, T] \times X \to X$ *for* V *satisfying*

$$(\boldsymbol{X}_t)_*(\mathfrak{m}) \le \exp\biggl(\int_0^t \|\mathrm{div}(V(s))^-\|_{L^\infty(\mathfrak{m})}\, ds\biggr)\mathfrak{m} \tag{2.40}$$

for every $t \in [0, T]$.

The estimate (2.40) can be localized [Gigli and Violo 2023, Proposition 5.3].

Proposition 2.41. *Let* $(X, d, \mathfrak{m})$, T, V, *and* $\boldsymbol{X}$ *be as in Theorem 2.39. Then for any* $S \in \mathcal{B}(X)$ *and* $t \in [0, T]$ *one has*

$$(\boldsymbol{X}_t)_*(\mathfrak{m}|_S) \le \exp\biggl(\int_0^t \|\mathrm{div}(V(s))^-\|_{L^\infty((\boldsymbol{X}_s)_*(\mathfrak{m}|_S))}\, ds\biggr)\mathfrak{m}.$$

Remark 2.42. From $R.2$, we get that if $\|V(t)\|_\infty \le L$ for all $t \in [0, T]$ and some $L > 0$, then for $\mathfrak{m}$-a.e. $x \in X$, the map

$$[0, T] \ni t \mapsto \boldsymbol{X}_t(x) \quad \text{is } L\text{-Lipschitz.} \tag{2.43}$$

Thus, after modifying $\boldsymbol{X}$ on a set of measure zero, we can always assume (2.43) holds for all $x \in X$ (see [Gigli and Tamanini 2021, Theorem A.4]).

For nice vector fields, there is a reverse flow [Deng 2020, Proposition 3.12].

Proposition 2.44. *Let* $(X, d, \mathfrak{m})$, V, $\boldsymbol{X}$, *be as in Theorem 2.39, and define* $\overline{V} : [0, T] \to L^2(TX)$ *as*

$$\overline{V}(t)(x) := -V(T - t)(x)$$

for each $t \in [0, T]$, $x \in X$. *Then there is a map* $\overline{\boldsymbol{X}} : [0, T] \times X \to X$ *which is an RLF for* $\overline{V}$ *and for* $\mathfrak{m}$*-a.e.* $x \in X$ *one has*

$$\overline{\boldsymbol{X}}_t(\boldsymbol{X}_T(x)) = \boldsymbol{X}_{T-t}(x) \quad \textit{for all } t \in [0, T].$$

Remark 2.45. If $\|\mathrm{div}(V(t))\|_\infty \le D$ for all $t \in [0, T]$ and some $D > 0$, (2.40) implies

$$e^{-DT}\mathfrak{m} \le (\boldsymbol{X}_t(\,\cdot\,))_*(\mathfrak{m}) \le e^{DT}\mathfrak{m} \quad \text{for all } t \in [0, T]. \tag{2.46}$$

The integral first variation formula extends to $\mathrm{RCD}(K, N)$ spaces [Brué et al. 2022, Corollary 4.2].

Theorem 2.47. *Let* $r > 0$, $(X, d, \mathfrak{m})$ *an* $\mathrm{RCD}(K, N)$ *space, and* V *a time-dependent vector field satisfying the conditions of Theorem 2.39. Set*

$$\begin{gathered} \mathrm{dt}_r : [0, T] \times X \times X \to [0, r], \\ \mathrm{dt}_r(t)(a, b) := \sup_{s \in [0,t]} \mathrm{dt}_r(\boldsymbol{X}_s)(a, b). \end{gathered} \tag{2.48}$$

Let S_1, S_2 *be Borel subsets of* X *with finite positive measure, and define*

$$\Gamma(t) := \{(a, b) \in S_1 \times S_2 \mid \mathrm{dt}_r(t)(a, b) < r\}. \tag{2.49}$$

Then the map $t \mapsto \int_{S_1 \times S_2} \mathrm{dt}_r(t)(a, b)\, d(\mathfrak{m} \times \mathfrak{m})(a, b)$ *is Lipschitz on* $[0, T]$ *and for a.e.* $t \in [0, T]$ *one has*

$$\frac{d}{dt}\int_{S_1 \times S_2} \mathrm{dt}_r(t)(a, b)\, d(\mathfrak{m}\times\mathfrak{m})(a, b) \le \int_0^1\int_{\Gamma(t)} d(\boldsymbol{X}_t(a), \boldsymbol{X}_t(b))|\nabla V(t)|(\gamma_{\boldsymbol{X}_t(a),\boldsymbol{X}_t(b)}(s))\, d(\mathfrak{m}\times\mathfrak{m})(a, b)\, ds.$$

Remark 2.50. Although [Brué et al. 2022, Corollary 4.2] was stated only for the noncollapsed case (i.e., $\mathfrak{m} = \mathscr{H}^N$), its proof follows that of [Deng 2020, Proposition 3.27] (see also [Brué et al. 2022, Proposition 4.1] for additional comments) and in particular works without the noncollapsed assumption.

2.7. *Group norms.* Let (X, p) be a pointed proper geodesic space and $\Gamma \leq \mathrm{Iso}(X)$ a closed group of isometries. The *norm* $\|\cdot\|_p : \Gamma \to \mathbb{R}$ associated to p is defined as $\|g\|_p := d(gp, p)$. We denote as $\mathcal{G}(\Gamma, X, p, r)$ the subgroup of Γ generated by the elements of norm $\|\cdot\|_p \leq r$. The *norm spectrum* $\sigma(\Gamma)$ is defined as the set of $r \geq 0$ for which $\mathcal{G}(\Gamma, X, p, r) \neq \mathcal{G}(\Gamma, X, p, r - \varepsilon)$ for all $\varepsilon > 0$. Notice that we always have $0 \in \sigma(\Gamma)$. If we want redundancy we sometimes write $\sigma(\Gamma, X, p)$ to denote the spectrum of the action of Γ on the pointed space (X, p). See also [Sormani and Wei 2004; 2015; Plaut 2021] for similar notions of group spectra and their relationship.

Proposition 2.51. *If Γ is equipped with the metric d_0^p from* (2.29), *and $\Gamma = \mathcal{G}(\Gamma, X, p, D)$ for some $D > 0$, then $\Gamma = \langle B_{D+2\sqrt{2}+\varepsilon}(\mathrm{Id}_X)\rangle$ for all $\varepsilon > 0$.*

Proof. From (2.29) with $r = 1/\sqrt{2}$, for all $g \in \Gamma$ one gets

$$\|g\|_p \leq d_0^p(g, I_X) \leq \|g\|_p + 2\sqrt{2}.$$

Then $\{g \in \Gamma \mid \|g\|_p \leq D\} \subset B_{D+2\sqrt{2}+\varepsilon}(\mathrm{Id}_X)$ for all $\varepsilon > 0$. □

It also satisfies a continuity property [Santos-Rodríguez and Zamora 2023, Proposition 47].

Proposition 2.52. *Let (X_i, p_i) be a sequence of pointed proper metric spaces that converges in the pointed Gromov–Hausdorff sense to (X, p) and consider a sequence of closed isometry groups $\Gamma_i \leq \mathrm{Iso}(X_i)$ that converges equivariantly to a closed group $\Gamma \leq \mathrm{Iso}(X)$. Then for any convergent sequence of real numbers $r_i \in \sigma(\Gamma_i)$, the limit $\lim_{i\to\infty} r_i$ lies in $\sigma(\Gamma)$.*

Remark 2.53. It is possible that an element in $\sigma(\Gamma)$ is not a limit of elements in $\sigma(\Gamma_i)$, so this spectrum is not necessarily continuous with respect to equivariant convergence (see [Kapovitch and Wilking 2011, Example 1]).

Proposition 2.54. *For any $a > 0$, one has $\mathcal{G}(\Gamma, X, p, a) = \mathcal{G}(\Gamma, X, p, a + \varepsilon)$ for $\varepsilon > 0$ small enough.*

Proof. Assuming the proposition fails, there is a sequence of elements g_i not in $\mathcal{G}(\Gamma, X, p, a)$ with $\|g_i\|_p \to a$. As the sequence $\|g_i\|_p$ is bounded, after taking a subsequence we can assume $g_i \to g$ for some $g \in \Gamma$ with $\|g\|_p = a$. Then for large enough i, $\|g^{-1}g_i\|_p < a$, so $g_i = (g)(g^{-1}g_i) \in \mathcal{G}(\Gamma, X, p, a)$, which is a contradiction. □

Corollary 2.55. *For any $[a, b] \subset (0, \infty)$, the following are equivalent*:

- $\sigma(\Gamma) \cap \big(a, b] = \varnothing$.
- $\mathcal{G}(\Gamma, X, p, a) = \mathcal{G}(\Gamma, X, p, b)$.

It is well known that when a group action is co-compact, the spectrum is bounded [Gromov 2007, Proposition 5.28].

Lemma 2.56. *Let (X, p) be a pointed proper geodesic space and $\Gamma \leq \mathrm{Iso}(X)$ a closed group of isometries. Then $r \leq 2 \cdot \mathrm{diam}(X/\Gamma)$ for all $r \in \sigma(\Gamma, X, p)$.*

To prove Theorem 1.14, one needs to control the number of generators of the groups Γ_i. This was done in [Santos-Rodríguez and Zamora 2023, Theorem 80] after [Kapovitch and Wilking 2011, Theorem 2.5].

Lemma 2.57. *Let $(X, d, \mathfrak{m}, p)$ be a pointed $\mathrm{RCD}(K, N)$ space, and $\Gamma \leq \mathrm{Iso}(X)$ a discrete group of measure preserving isometries with $\Gamma = \mathcal{G}(\Gamma, X, p, D)$. Then Γ can be generated by at most $C(K, N, D)$ elements.*

2.8. *Group theory.* In this section we cover basic group theory results needed later. Proofs of Propositions 2.58 and 2.60 below can be found in [Fukaya and Yamaguchi 1992, Section 4].

Proposition 2.58. *Let G be a group generated by k elements and $H \leq G$ a subgroup of index $[G : H] \leq M$. Then there is a characteristic subgroup $H' \triangleleft G$ with $H' \leq H$ and $[G : H'] \leq C(M, k)$.*

Remark 2.59. By Lemma 2.57 and Proposition 2.58, whenever Theorem 1.14 holds, we may assume the subgroups $G_i \triangleleft \Gamma_i$ are characteristic.

Proposition 2.60. *Let A be an abelian group generated by m elements, and $\varphi : G \to A$ a surjective morphism with finite kernel. Then G contains a finite index abelian subgroup generated by m elements.*

Proposition 2.61. *Let G be a group, $H \triangleleft G$ a normal subgroup, $a, b \in G$ such that $[a, b] \in H$, and $H_0 \triangleleft H$ a characteristic subgroup of H with $[H : H_0] \leq M$. Then for all $C \geq 2M$ one has $[a^{C!}, b] \in H_0$.*

Proof. In the group G/H_0, set $\alpha := aH_0$ and $\beta := bH_0$. Then $\alpha\beta\alpha^{-1} = \beta h$ for some $h \in H/H_0$. A direct computation shows that $\alpha^k\beta\alpha^{-k} = \beta(h)(\alpha h\alpha^{-1}) \cdots (\alpha^{k-1}h\alpha^{-k+1})$. As H/H_0 is normal in G/H_0 and $|H/H_0| \leq M$, one gets that $\alpha^{M!}h\alpha^{-M!} = h$, so

$$\alpha^{M!M}\beta\alpha^{-M!M} = \beta \prod_{j=0}^{M!M-1} (\alpha^j h\alpha^{-j}) = \beta\left(\prod_{j=0}^{M!-1} (\alpha^j h\alpha^{-j})\right)^M = \beta.$$

If $C \geq 2M$, then $C!$ is a multiple of $M!M$, and

$$[\alpha^{C!}, \beta] = \alpha^{C!}\beta\alpha^{-C!}\beta^{-1} = \alpha^{M!M}(\cdots(\alpha^{M!M}\beta\alpha^{-M!M})\cdots\alpha^{-M!M})\beta^{-1} = \beta\beta^{-1} = e_{G/H_0}$$

This shows that $[a^{C!}, b] \in H_0$. □

Proposition 2.62. *Let Γ be a group, $G \triangleleft \Gamma$ a characteristic subgroup admitting a nilpotent basis, $g \in \Gamma$, $\varphi \in \mathrm{Aut}(\Gamma)$, and $C \in 2\mathbb{Z}$. If $[\Gamma : G] \leq C/2$, then the nilpotent basis in G is preserved by $\varphi^{C!}$ if and only if it is preserved by $(\varphi \circ g_*)^{C!}$.*

Proof. First we observe that for any $k \in \mathbb{N}$ we have

$$\begin{aligned}(\varphi \circ g_*)^k &= (\varphi \circ g_* \circ \varphi^{-1})(\varphi^2 \circ g_* \circ \varphi^{-2}) \cdots (\varphi^k \circ g_* \circ \varphi^{-k})\varphi^k \\ &= (\varphi(g))_*(\varphi^2(g))_* \cdots (\varphi^k(g))_*\varphi^k \\ &= (\varphi(g)\varphi^2(g) \cdots \varphi^k(g))_*\varphi^k. \end{aligned} \tag{2.63}$$

On the other hand, as G is characteristic in Γ, the group $G_* := \{x_* : \Gamma \to \Gamma \mid x \in G\}$ is normal in $\mathrm{Aut}(\Gamma)$, so one has $y_*^C G_* = G_*$ for all $y \in \Gamma$. Also, notice that $\varphi^{(C/2)!}(g)G = gG$ in Γ/G, so $(\varphi^{(C/2)!}(g))_* G_* = g_* G_*$ in $\mathrm{Aut}(\Gamma)/G_*$. Thus if $\ell = (C-1)!/(C/2)!$, using (2.63) we have

$$(\varphi \circ g_*)^{C!} G_* = ((\varphi(g)\varphi^2(g)\cdots\varphi^{(C/2)!}(g))_*)^{C\ell}\varphi^{C!}G_* = \varphi^{C!}G_*.$$

This implies that $(\varphi \circ g_*)^{C!}$ and $\varphi^{C!}$ differ only by an element of G_*, which clearly respects the nilpotent basis in G. □

We will also need the following version of the Bieberbach theorem [Fukaya and Yamaguchi 1992, Section 4].

Theorem 2.64 (Fukaya and Yamaguchi). *Let $G \leq \mathrm{Iso}(\mathbb{R}^m)$ be a closed group of isometries and $G_0 \leq G$ its identity connected component. Then G/G_0 contains a finite index abelian subgroup generated by at most m elements.*

Corollary 2.65. *Let Z be a compact metric space, $\Gamma \leq \mathrm{Iso}(\mathbb{R}^m \times Z)$ a closed group of isometries and $\Gamma_0 \leq \Gamma$ its identity connected component. If $\mathrm{Iso}(Z)$ is a Lie group, then Γ/Γ_0 contains a finite index abelian subgroup generated by at most m elements.*

Proof. Notice that for each $(x, z) \in \mathbb{R}^m \times Z$, the $\mathbb{R}^m$-fiber passing through (x, z) can be characterized as the union of the images of all infinite geodesics passing through (x, z). This implies that $\mathrm{Iso}(\mathbb{R}^m \times Z)$ respects the splitting $\mathbb{R}^m \times Z$ and decomposes as $\mathrm{Iso}(\mathbb{R}^m \times Z) = \mathrm{Iso}(\mathbb{R}^m) \times \mathrm{Iso}(Z)$. Let $G \leq \mathrm{Iso}(\mathbb{R}^m)$ be the image of Γ under the projection $\pi : \mathrm{Iso}(\mathbb{R}^m \times Z) \to \mathrm{Iso}(\mathbb{R}^m)$. As Γ is closed and $\mathrm{Iso}(Z) = \mathrm{Ker}(\pi)$ is compact, G is closed in $\mathrm{Iso}(\mathbb{R}^m)$.

We claim that $\pi(\Gamma_0) = G_0$. Assuming the contrary, as G_0 is connected, there would be a sequence $x_i \in G_0 \backslash \pi(\Gamma_0)$ with $x_i \to e_{G_0}$. Pick $g_i \in \Gamma$ with $\pi(g_i) = x_i$. Since $\mathrm{Iso}(Z)$ is compact, after passing to a subsequence we can assume $g_i \to g_\infty$ for some $g_\infty \in \mathrm{Ker}(\pi)$. Then $g_\infty^{-1} g_i \to e_\Gamma$, so for i large enough one has $g_\infty^{-1} g_i \in \Gamma_0$. This would mean that $\pi(g_\infty^{-1} g_i) = \pi(g_i) = x_i \in \pi(\Gamma_0)$, which is a contradiction.

Let $H := \pi^{-1}(G_0) \cap \Gamma$. We claim that $[H : \Gamma_0] < \infty$. Otherwise, there would be a sequence $h_i \in H \cap \mathrm{Ker}(\pi)$ with $h_i^{-1} h_j \in H \backslash \Gamma_0$ for all $i \neq j$. As $\mathrm{Ker}(\pi)$ is compact, after taking a subsequence we can assume $h_i \to h_\infty$ for some $h_\infty \in \mathrm{Ker}(\pi)$. This would mean that for i, j large enough, one has $h_i^{-1} h_j \in \Gamma_0$, which is a contradiction.

The above implies that Γ/Γ_0 is a finite extension of $(\Gamma/\Gamma_0)/(H/\Gamma_0) = \Gamma/H \cong G/G_0$, so the result follows from Theorem 2.64 and Proposition 2.60. □

3. Groups of connected components

The goal of this section is to prove the following result (cf. [Fukaya and Yamaguchi 1992, Theorem 3.10] and [Santos-Rodríguez and Zamora 2023, Lemma 58]). The groups Υ_i play the role of "connected component of the identity" in the groups Γ_i.

Theorem 3.1. *Let (X_i, p_i) be a sequence of proper geodesic spaces that converges in the pointed Gromov–Hausdorff sense to a space (X, p), $\Gamma_i \leq \mathrm{Iso}(X_i)$ a sequence of closed groups of isometries that converges* *equivariantly to a closed group* $\Gamma \leq \mathrm{Iso}(X)$, *and* $\psi_i : \Gamma_i \to \Gamma \cup \{*\}$ *the Gromov Hausdorff approximations given by Definition 2.30. Assume*

- $\Gamma_i = \mathcal{G}(\Gamma_i, X_i, p_i, D)$ *for some* $D > \sqrt{2}$,
- Γ_0, *the connected component of the identity of* Γ, *is open*,
- Γ/Γ_0 *is finitely presented.*

Then there are subgroups $\Upsilon_i \triangleleft \Gamma_i$ *such that*

- Υ_i *is normal in* Γ_i *for* i *large enough*,
- *for any* $R > 0$, $\Upsilon_i = \langle \psi_i^{-1}(B_R(\mathrm{Id}_X) \cap \Gamma_0)\rangle$ *for* i *large enough*,
- *for* i *large enough, there are surjective morphisms* $\Gamma/\Gamma_0 \to \Gamma_i/\Upsilon_i$.

Proof. Let $r > 0$ be such that $B_{2r}(\mathrm{Id}_X) \subset \Gamma_0$. First we show that for any fixed $R \geq r$ and $\delta \in (0, r]$, the subgroup of Γ_i generated by $\psi_i^{-1}(B_\delta(\mathrm{Id}_X))$ in Γ_i coincides with $\langle \psi_i^{-1}(B_R(\mathrm{Id}_X) \cap \Gamma_0)\rangle$ for large enough i. To see this, first take a collection $y_1, \dots, y_n \in B_R(\mathrm{Id}_X) \cap \Gamma_0$ with

$$B_R(\mathrm{Id}_X) \cap \Gamma_0 \subset \bigcup_{j=1}^{n} B_{\delta/10}(y_j).$$

By connectedness, for each $j \in \{1, \dots, n\}$ we can construct a sequence $e = z_{j,0}, \dots, z_{j,k_j} = y_j$ in Γ with $d(z_{j,\ell-1}, z_{j,\ell}) \leq \delta/10$ for each $\ell \in \{1, \dots, k_j\}$. Since all $z_{j,\ell}$ are contained in a compact subset of Γ_0, if i is large enough, for any element $x \in \psi_i^{-1}(B_R(\mathrm{Id}_X) \cap \Gamma_0)$ we can find y_j with $d(y_j, \psi_i(x)) \leq \delta/10$, and $e = x_0, \dots, x_{k_j} = x$ in Γ_i with $d(z_{j,\ell}, \psi_i(x_\ell)) \leq \delta/10$ for each ℓ. This allows us to write $x = (x_1)(x_1^{-1}x_2)\cdots(x_{k_j-1}^{-1}x_{k_j})$ as a product of k_j elements in $\psi_i^{-1}(B_\delta(\mathrm{Id}_X))$, proving our claim. Set Υ_i to be the subgroup of Γ_i generated by $\psi_i^{-1}(B_r(\mathrm{Id}_X))$.

Choose $\delta > 0$ small enough so that for all $g \in B_{3D}(\mathrm{Id}_X)$, $h \in B_\delta(\mathrm{Id}_X)$ one has $ghg^{-1} \in B_{r/2}(\mathrm{Id}_X)$. Then for large enough i, the conjugate of an element in $\psi_i^{-1}(B_\delta(\mathrm{Id}_X))$ by an element in $\psi_i^{-1}(B_{3D}(\mathrm{Id}_X))$ lies in $\psi_i^{-1}(B_r(\mathrm{Id}_X))$. By Proposition 2.51, $\psi_i^{-1}(B_{3D}(\mathrm{Id}_X))$ generates Γ_i and $\psi_i^{-1}(B_\delta(\mathrm{Id}_X))$ generates Υ_i for large enough i, implying that Υ_i is normal in Γ_i.

Let $S_0 = \{\bar{s}_1, \dots, \bar{s}_k\} \subset \Gamma/\Gamma_0$ be a finite symmetric generating set containing all connected components intersecting $B_{4D}(\mathrm{Id}_X)$, $S = \{s_1, \dots, s_k\} \subset \Gamma$ a set of representatives, and for each $j \in \{1, \dots, k\}$, pick a sequence $g_i^j \in \Gamma_i$ with $\psi_i(g_i^j) \to s_j$. Then define $h_i' : S_0 \to \Gamma_i/\Upsilon_i$ as $h_i'(\bar{s}_j) := g_i^j \Upsilon_i \in \Gamma_i/\Upsilon_i$. It is easy to check that $h_i'(s_j)$ does not depend on the choices of the representatives s_j nor the sequences g_i^j for i large enough.

By hypothesis, Γ/Γ_0 admits a presentation $\langle S_0, W\rangle$ with W a finite set of words. For $\bar{s}_{i_1} \dots \bar{s}_{i_\ell} \in W$, one has $d_0^p(\psi_i(g_i^{i_1}) \cdots \psi_i(g_i^{i_\ell}), \psi_i(g_i^{i_1} \cdots g_i^{i_\ell})) < r$ for i large enough, and hence $\psi_i(g_i^{i_1} \cdots g_i^{i_\ell}) \in \Gamma_0$. This means, again for i large enough, that $g_i^{i_1} \cdots g_i^{i_\ell} \in \Upsilon_i$, and $h_i'(s_{i_1}) \cdots h_i'(s_{i_\ell}) = g_i^{i_1} \cdots g_i^{i_\ell} \Upsilon_i = \Upsilon_i \in \Gamma_i/\Upsilon_i$. As there are only finitely many words in W, the functions $h_i' : S_0 \to \Gamma_i/\Upsilon_i$ extend to group morphisms $h_i : \Gamma/\Gamma_0 \to \Gamma_i/\Upsilon_i$. As S_0 intersects each connected component in $B_{4D}(\mathrm{Id}_X)$ and Γ_i is generated by $B_{3D}(\mathrm{Id}_{X_i})$, the maps h_i are surjective. □

The following result deals with the base of induction in the proof of Theorem 1.14.

Lemma 3.2. *Let $(X_i, d_i, \mathfrak{m}_i, p_i)$ be a sequence of* RCD(K, N) *spaces of rectifiable dimension n, and $\Gamma_i \leq \mathrm{Iso}(X_i)$ a sequence of closed groups of isometries. Assume the sequence $(X_i, d_i, \mathfrak{m}_i, p_i)$ converges in the pointed measured Gromov–Hausdorff sense to a pointed* RCD(K, N) *space $(X, d, \mathfrak{m}, p)$ of rectifiable dimension n. If there is $D > 0$ such that $\Gamma_i = \mathcal{G}(\Gamma_i, X_i, p_i, D)$ for all i, and Γ_i converges equivariantly to the trivial group, then the groups Γ_i are trivial for i large enough.*

Proof. Clearly, we can assume $D > \sqrt{2}$. Let $\Upsilon_i \leq \Gamma_i$ be the subgroups given by Theorem 3.1. Then $\Gamma_i = \Upsilon_i = \langle \psi_i^{-1}(\mathrm{Id}_X)\rangle$ for i large enough. From the definition of equivariant convergence, it is easy to see that the ψ_i-preimage of an open compact subgroup of Γ is a subgroup in Γ_i for i large enough; hence $\Gamma_i = \psi_i^{-1}(\mathrm{Id}_X)$. This means that Γ_i are small subgroups in the sense of [Santos-Rodríguez and Zamora 2023, Definition 66 and Remark 75], so by [loc. cit., Theorem 93] the result follows. □

4. Proof of main regularity estimates on RLFs

In this section we prove Theorem 1.5 and Corollary 1.8, the key step being Lemma 4.1.

Proof of Theorem 1.5. Let $r_x \leq \rho/100$ be such that for all $r \leq r_x$ one has

$$\mathfrak{m}(\{y \in B_r(x) \mid H(y) \leq \delta\}) \geq \tfrac{1}{2}\mathfrak{m}(B_r(x)).$$

Lemma 4.1. *Fix $r \leq r_x$. If δ is small enough, there is $x_r \in B_r(x) \cap \{H \leq \delta\}$ and a constant $C_0(N) > 1$ which is independent of r such that*:

S_r.1 *There is $B'_r(x_r) \subseteq B_r(x_r)$ such that $\mathfrak{m}(B'_r(x_r)) \geq (1 - \sqrt{\delta})\mathfrak{m}(B_r(x_r))$ and*

$$\boldsymbol{X}_t(B'_r(x_r)) \subseteq B_{2r}(\boldsymbol{X}_t(x_r)) \quad \text{for all } t \in [0, T].$$

S_r.2 *For all $t \in [0, T]$,*

$$\frac{1}{C_0}\mathfrak{m}(B_r(x_r)) \leq \mathfrak{m}(B_r(\boldsymbol{X}_t(x_r))) \leq C_0\mathfrak{m}(B_r(x_r)).$$

Lemma 4.1 is proven by an induction on time following the scheme of [Deng 2020, Section 5]. We now give an outline of this proof.

First choose $x_{r,0}$ so that $H(x_{r,0}) \leq \delta$. Then by the Bishop–Gromov inequality, the estimates S_r.1 and S_r.2 trivially hold for $x_r = x_{r,0}$ up to time $r/(10L)$. This serves as the base of induction. We then assume there is $x_{r,k}$ with $H(x_{r,k}) \leq \delta$ and such that S_r.1 and S_r.2 hold for $x_r = x_{r,k}$ along the interval $[0, kr/(10L)]$. The goal is then to show there is $x_{r,k+1}$ with $H(x_{r,k+1}) \leq \delta$ and such that S_r.1 and S_r.2 hold for $x_r = x_{r,k+1}$ along the interval $[0, t_k]$, where $t_k := \min\{(k+1)r/(10L), T\}$.

In order to achieve this, we first combine the fact that flow lines are L-Lipschitz with the inductive hypothesis to obtain integral estimates on $\mathrm{dt}_r(t_k)$ over carefully chosen sets (see (4.5) and (4.7)), from which we deduce that a significant portion of $B_{2r}(x)$ stays within $7r$ of $\boldsymbol{X}_t(x_{r,k})$ up to time t_k. This allows us to choose $x_{r,k+1} \in B_r(x) \cap \{H \leq \delta\}$ so that along the interval $[0, t_k]$, most of the flow lines starting at $B_r(x_{r,k+1})$ stay within $2r$ of $\boldsymbol{X}_t(x_{r,k+1})$. This is enough to guarantee that both S_r.1 and the first inequality of S_r.2 hold up to time t_k.

By the Bishop–Gromov inequality, we also have the other inequality, but with a worse constant. In order to improve this constant back to the original one, we perform the analysis of the previous paragraph but in the reverse direction using the flow $\overline{\boldsymbol{X}}$ given by Proposition 2.44. We show that for each $t \in [0, t_k]$, under the reverse flow $\overline{\boldsymbol{X}}_t$ a significant portion of $B_r(\boldsymbol{X}_t(x_{r,k+1}))$ returns to $B_{2r}(x_{r,k+1})$, and hence close to x. This is enough to improve the volume ratio to the desired constant.

We now turn to the actual proof, where for the sake of detail, we also present the case $k = 0$.

Proof of Lemma 4.1. Let $I_0 = [0, r/(10L)]$ and fix some $x_{r,0} \in B_r(x)$ with $H(x_{r,0}) \le \delta$. By Remark 2.42, for any $y \in B_r(x)$ and any $t \in I_0$, we have

$$d(\boldsymbol{X}_t(y), \boldsymbol{X}_t(x_{r,0})) \le d(\boldsymbol{X}_t(y), y) + d(y, x) + d(x, x_{r,0}) + d(x_{r,0}, \boldsymbol{X}_t(x_{r,0})) \le 2L\frac{r}{10L} + 2r < 3r. \tag{4.2}$$

Define $\mathrm{dt}_r(t)$ as in (2.48), and set $S_1 = B_r(x) \cap \{H \le \delta\}$, $S_2 = B_{2r}(x)$, and $\Gamma(t)$ as in (2.49). By Theorem 2.47, we obtain

$$\begin{aligned}\int_{S_1\times S_2} \mathrm{dt}_r\left(\frac{r}{10L}\right)&(y,z)\, d(\mathfrak{m}\times\mathfrak{m})(y,z)\\ &= \int_{I_0}\frac{d}{dt}\int_{S_1\times S_2}\mathrm{dt}_r(t)(y,z)\, d(\mathfrak{m}\times\mathfrak{m})(y,z)\, dt\\ &\le \int_{I_0}\int_0^1\int_{\Gamma(t)} d(\boldsymbol{X}_t(y), \boldsymbol{X}_t(z))|\nabla V(t)|(\gamma_{\boldsymbol{X}_t(y),\boldsymbol{X}_t(z)}(s))\, d(\mathfrak{m}\times\mathfrak{m})(y,z)\, ds\, dt. \end{aligned}\tag{4.3}$$

Using (2.46) and a change of variables, for any $t \in I_0$ we have

$$\begin{aligned}\int_0^1\int_{\Gamma(t)} d(\boldsymbol{X}_t(y), \boldsymbol{X}_t(z))|\nabla V(t)|(\gamma_{\boldsymbol{X}_t(y),\boldsymbol{X}_t(z)}(s))\, d(\mathfrak{m}\times\mathfrak{m})(y,z)\, ds&\\ \le e^{DT}\int_0^1\int_{\boldsymbol{X}_t(\Gamma(t))} d(y,z)|\nabla V(t)|(\gamma_{y,z}(s))\, d(\mathfrak{m}\times\mathfrak{m})(y,z)\, ds&.\end{aligned}$$

Furthermore,

$$\begin{aligned}e^{DT}\int_0^1\int_{\boldsymbol{X}_t(\Gamma(t))} d(y,z)|\nabla V(t)|(\gamma_{y,z}(s))\, &d(\mathfrak{m}\times\mathfrak{m})(y,z)\, ds\\ &\le e^{DT}\int_0^1\int_{B_{6r}(\boldsymbol{X}_t(x_{r,0}))^{\times 2}} d(y,z)|\nabla V(t)|(\gamma_{y,z}(s))\, d(\mathfrak{m}\times\mathfrak{m})(y,z)\, ds\\ &\le e^{DT}C(N)r\mathfrak{m}(B_{6r}(\boldsymbol{X}_t(x_{r,0})))^2 \fint_{B_{12r}(\boldsymbol{X}_t(x_{r,0}))}|\nabla V(t)|\, d\mathfrak{m}\\ &\le e^{DT}C(N)r\mathfrak{m}(B_r(x))^2 \fint_{B_{12r}(\boldsymbol{X}_t(x_{r,0}))}|\nabla V(t)|\, d\mathfrak{m},\end{aligned}$$

where we used (4.2) for the second line, Theorem 2.10 for the third line, and Theorem 2.1 for the fourth line. Combining the above estimates starting from (4.3), we obtain

$$\begin{aligned}\int_{S_1\times S_2}\mathrm{dt}_r\left(\frac{r}{10L}\right)(y,z)\, d(\mathfrak{m}\times\mathfrak{m})(y,z) &\le e^{DT}C(N)r\mathfrak{m}(B_r(x))^2\int_{I_0}\fint_{B_{12r}(\boldsymbol{X}_t(x_{r,0}))}|\nabla V(t)|\, d\mathfrak{m}\, dt\\ &\le e^{DT}C(N)r\mathfrak{m}(B_r(x))^2\int_{I_0}\mathrm{Mx}_\rho(|\nabla V(t)|)(\boldsymbol{X}_t(x_{r,0}))\, dt\\ &\le e^{DT}C(N)r\mathfrak{m}(B_r(x))^2 H(x_{r,0}) \le e^{DT}C(N)r\mathfrak{m}(B_r(x))^2\delta.\end{aligned}$$

By Chebyshev's inequality, there is $x_{r,1} \in B_r(x) \cap \{H \le \delta\}$ with

$$\int_{B_{2r}(x)} \mathrm{dt}_r\left(\frac{r}{10L}\right)(x_{r,1}, y)\, d\mathfrak{m}(y) \le e^{DT} C(N) r \mathfrak{m}(B_r(x))\delta.$$

By Theorem 2.1, we have

$$\mathfrak{m}(B_r(x_{r,1})) \ge \frac{1}{C(N)} \mathfrak{m}(B_r(x)),$$

thus another instance of Chebyshev's inequality implies there is $B_{r,1}(x_{r,1}) \subseteq B_r(x_{r,1})$ with $\mathfrak{m}(B_{r,1}(x_{r,1})) \ge (1-\sqrt{\delta})\mathfrak{m}(B_r(x_{r,1}))$ and

$$\mathrm{dt}_r\left(\frac{r}{10L}\right)(x_{r,1}, z) \le e^{DT} C(N) r\sqrt{\delta} \quad \text{for all } z \in B_{r,1}(x_{r,1}).$$

Hence if $e^{DT}C(N)\sqrt{\delta} < 1$, then by the definition of $\mathrm{dt}_r(t)$, for all $t \in I_0$ and $z \in B_{r,1}(x_{r,1})$ we have

$$d(\boldsymbol{X}_t(x_{r,1}), \boldsymbol{X}_t(z)) < d(x, z) + e^{DT} C(N) r\sqrt{\delta} < 2r,$$

so $\boldsymbol{X}_t(B_{r,1}(x_{r,1})) \subset B_{2r}(\boldsymbol{X}_t(x_{r,1}))$ for all $t \in I_0$. As

$$B_{r/2}(x_{r,1}) \subset B_r(\boldsymbol{X}_t(x_{r,1})) \subset B_{3r/2}(x_{r,1}) \quad \text{for all } t \in I_0,$$

from Theorem 2.1 we have, for all $t \in I_0$,

$$\frac{1}{C}\mathfrak{m}(B_r(x_{r,1})) \le \mathfrak{m}(B_r(\boldsymbol{X}_t(x_{r,1}))) \le C\mathfrak{m}(B_r(x_{r,1})).$$

The argument above establishes $S_r.1$ and $S_r.2$ up to time $r/(10L)$. Now we show we can establish the same estimate up to time T provided δ is small enough.

Let $k \in \mathbb{N}$ with $k < \lceil 10TL/r \rceil$, and assume there is $x_{r,k} \in B_r(x)$ such that

$S_{r,k}.1$ There exists $B'_r(x_{r,k}) \subseteq B_r(x_{r,k})$ with $\mathfrak{m}(B'_r(x_{r,k})) \ge (1-\sqrt{\delta})\mathfrak{m}(B_r(x_{r,k}))$ and

$$\boldsymbol{X}(B'_r(x_{r,k})) \subset B_{2r}(x_{r,k}) \quad \text{for all } t \in \left[0, \frac{kr}{10L}\right].$$

$S_{r,k}.2$ For all $t \in [0, kr/(10L)]$,

$$\frac{1}{C_0}\mathfrak{m}(B_r(x_{r,k})) \le \mathfrak{m}(B_r(\boldsymbol{X}_t(x_{r,k}))) \le C_0\mathfrak{m}(B_r(x_{r,k})).$$

Set

$$t_k := \min\left\{\frac{(k+1)r}{10L}, T\right\} \quad \text{and} \quad I_k := [0, t_k].$$

From $S_{r,k}.1$ and Remark 2.42, for all $t \in I_k$ we have

$$\boldsymbol{X}_t(B'_r(x_{r,k})) \subset B_{2r+r/2}(\boldsymbol{X}_t(x_{r,k})). \tag{4.4}$$

Let $S_1 := B'_r(x_{r,k})$, $S_2 := B_{2r}(x)$, and $\Gamma(t)$ be given by (2.49). By Theorem 2.47,

$$\begin{aligned}
&\int_{S_1\times S_2} \mathrm{dt}_r(t_k)(y,z)\,d(\mathfrak{m}\times\mathfrak{m})(y,z)\\
&\qquad= \int_{I_k} \frac{d}{dt}\int_{S_1\times S_2} \mathrm{dt}_r(t)(y,z)\,d(\mathfrak{m}\times\mathfrak{m})(y,z)\,dt\\
&\qquad\le \int_{I_k}\int_0^1\int_{\Gamma(t)} d(\boldsymbol{X}_t(y),\boldsymbol{X}_t(z))|\nabla V(t)|(\gamma_{\boldsymbol{X}_t(y),\boldsymbol{X}_t(z)}(s))\,d(\mathfrak{m}\times\mathfrak{m})(y,z)\,ds\,dt\\
&\qquad\le \int_{I_k} e^{DT}\int_0^1\int_{\boldsymbol{X}_t(\Gamma(t))} d(y,z)|\nabla V(t)|(\gamma_{y,z}(s))\,d(\mathfrak{m}\times\mathfrak{m})(y,z)\,ds\,dt\\
&\qquad\le \int_{I_k} e^{DT}\int_0^1\int_{B_{6r}(\boldsymbol{X}_t(x_{r,k}))^{\times 2}} d(y,z)|\nabla V(t)|(\gamma_{y,z}(s))\,d(\mathfrak{m}\times\mathfrak{m})(y,z)\,ds\,dt\\
&\qquad\le \int_{I_k} e^{DT}C(N)r\mathfrak{m}(B_{12r}(\boldsymbol{X}_t(x_{r,k})))^2 ⨍_{B_{6r}(\boldsymbol{X}_t(x_{r,k}))} |\nabla V(t)|\,d\mathfrak{m}\,dt\\
&\qquad\le \int_{I_k} e^{DT}C_0^2C(N)r\mathfrak{m}(B_r(x))^2 ⨍_{B_{12r}(\boldsymbol{X}_t(x_{r,k}))} |\nabla V(t)|\,d\mathfrak{m}\,dt. \qquad (4.5)
\end{aligned}$$

where we used (2.46), (4.4), Theorem 2.10, and Theorem 2.1 with (4.4). From the above estimates we get

$$\begin{aligned}
\int_{S_1\times S_2} \mathrm{dt}_r(t_k)(y,z)\,d(\mathfrak{m}\times\mathfrak{m})(y,z) &\le e^{DT}C_0^2C(N)r\mathfrak{m}(B_r(x))^2\int_{I_k} ⨍_{B_{12r}(\boldsymbol{X}_t(x_{r,k}))} |\nabla V(t)|\,d\mathfrak{m}\,dt\\
&\le e^{DT}C_0^2C(N)r\mathfrak{m}(B_r(x))^2\int_{I_k} \mathrm{Mx}_\rho(|\nabla V(t)|)(\boldsymbol{X}_t(x_{r,k}))\,dt\\
&\le e^{DT}C_0^2C(N)r\mathfrak{m}(B_r(x))^2 H(x_{r,k})\\
&\le e^{DT}C_0^2C(N)r\mathfrak{m}(B_r(x))^2\delta.
\end{aligned}$$

By Chebyshev's inequality, there is some $x' \in B'_r(x_{r,k})$ with

$$\int_{B_{2r}(x)} \mathrm{dt}_r(t_k)(x',y)\,d\mathfrak{m}(y) \le e^{DT}C_0^2C(N)r\mathfrak{m}(B_r(x))\delta.$$

Thus there are $C(D,T,N) > 1$ and $B_k \subset B_{2r}(x)$ with

$$\begin{aligned}
\mathrm{dt}_r(t_k)(x',y) &\le Cr\sqrt{\delta} \quad \text{for all } y \in B_k,\\
\mathfrak{m}(B_k) &\ge (1-\sqrt{\delta}/2)\mathfrak{m}(B_{2r}(x)). \qquad (4.6)
\end{aligned}$$

As $x' \in B'_r(x_{r,k})$, from (4.4) we have $\boldsymbol{X}_t(x') \in B_{2r+r/2}(\boldsymbol{X}_t(x_{r,k}))$ for all $t \in I_k$, provided $C\sqrt{\delta} < 1$. Thus for all $t \in I_k$, we also have

$$\boldsymbol{X}_t(B_k) \subset B_{7r}(\boldsymbol{X}_t(x_{r,k})).$$

Define $S_1 = B_k$, $S_2 = B_r(x) \cap \{H \le \delta\}$, and $\Gamma(t)$ as in (2.49). Similar to before, we have

$$\begin{aligned}\int_{S_1\times S_2} \mathrm{dt}_r(t_k)(y,z)\, d(\mathfrak{m}\times\mathfrak{m})(y,z) &\le e^{DT} C_0^2 C(N) r\mathfrak{m}(B_r(x))^2 \int_{I_k} \fint_{B_{20r}(\boldsymbol{X}_t(x_{r,k}))} |\nabla V(t)|\, d\mathfrak{m}\, dt \\ &\le e^{DT} C_0^2 C(N) r\mathfrak{m}(B_r(x))^2 \int_{I_k} \mathrm{Mx}_\rho(|\nabla V|)(\boldsymbol{X}_t(x_{r,k}))\, dt \\ &\le e^{DT} C_0^2 C(N) r\mathfrak{m}(B_r(x))^2 \delta. \end{aligned} \tag{4.7}$$

Thus there is $x_{r,k+1} \in B_r(x) \cap \{H \le \delta\}$ such that

$$\int_{B_k} \mathrm{dt}_r(t_k)(x_{r,k+1}, y)\, d\mathfrak{m}(y) \le e^{DT} C_0^2 C(N) r\mathfrak{m}(B_r(x))\delta.$$

From (4.6) and Theorem 2.1, there are $C(D,T,N) > 1$ and $B_r'(x_{r,k+1}) \subset B_r(x_{r,k+1})$ with

$$\mathrm{dt}_r(t_k)(x_{r,k+1}, y) \le C\sqrt{\delta} r \quad \text{for all } y \in B_r'(x_{r,k+1}),$$
$$\mathfrak{m}(B_r'(x_{r,k+1})) \ge (1-\sqrt{\delta})\mathfrak{m}(B_r(x_{r,k+1})).$$

Thus if $C\sqrt{\delta} < 1$, for all $t \in I_k$ we have

$$\boldsymbol{X}_t(B_r'(x_{r,k+1})) \subseteq B_{2r}(\boldsymbol{X}_t(x_{r,k+1})).$$

Also, from Theorem 2.1 and (2.46) we have, for some $C(D,T,N) > 1$,

$$\begin{aligned}\mathfrak{m}(B_r(\boldsymbol{X}_t(x_{r,k+1}))) &\ge \frac{1}{C}\mathfrak{m}(B_{2r}(\boldsymbol{X}_t(x_{r,k+1}))) \ge \frac{1}{C}\mathfrak{m}(\boldsymbol{X}_t(B_r'(x_{r,k+1}))) \\ &\ge \frac{1}{C}\mathfrak{m}(B_r(x_{r,k+1})). \end{aligned} \tag{4.8}$$

To obtain the other direction of the volume estimate corresponding to $S_r.2$, we consider the reversal of the flow. Fix $t \in I_k$, define $\overline{V} \in L^1([0,t]; H^{1,2}_{C,s}(TX))$ as

$$\overline{V}(s) := -V(t-s) \quad \text{for all } s \in [0,t],$$

and let $\overline{\boldsymbol{X}} : [0,t]\times X \to X$ be its RLF. Define $\mathrm{dt}_r' : [0,t]\times X\times X \to [0,r]$ as

$$\mathrm{dt}_r'(s)(y,z) := \sup_{0\le u\le s} \mathrm{dt}_r(\overline{\boldsymbol{X}}_u)(y,z),$$

$S_1 = \boldsymbol{X}_t(B_r'(x_{r,k+1}))$, and $S_2 = B_r(\boldsymbol{X}_t(x_{r,k+1}))$. Similar to before we have

$$\begin{aligned}\int_{S_1\times S_2} \mathrm{dt}_r'(t)(y,z)\, d(\mathfrak{m}\times\mathfrak{m})(y,z) &\le e^{DT} C_0^2 C(N) r\mathfrak{m}(B_r(x_{r,k+1}))^2 \int_0^t \fint_{B_{20r}(\boldsymbol{X}_{t-s}(x_{r,k+1}))} |\nabla \overline{V}(s)|\, d\mathfrak{m}\, ds \\ &\le e^{DT} C_0^2 C(N) r\mathfrak{m}(B_r(x))^2 \int_0^t \mathrm{Mx}_\rho(|\nabla V(s)|)(\boldsymbol{X}_{t-s}(x_{r,k+1}))\, ds \\ &\le e^{DT} C_0^2 C(N) r\mathfrak{m}(B_r(x))^2 \delta. \end{aligned}$$

Thus we have $x'' \in \boldsymbol{X}_t(B'_r(x_{r,k+1}))$ with

$$\int_{B_r(\boldsymbol{X}_t(x_{r,k+1}))} \mathrm{dt}'_r(t)(x'', y)\, d\mathfrak{m}(y) \le e^{DT} C_0^2 C(N) r \mathfrak{m}(B_r(x))\delta.$$

Hence there are $C(D, T, N) > 0$ and $A' \subset B_r(\boldsymbol{X}_t(x_{r,k+1}))$ such that

$$\begin{aligned} \mathrm{dt}'_r(t)(x'', y) &\le Cr\sqrt{\delta} \quad \text{for all } y \in A', \\ \mathfrak{m}(A') &\ge (1-\sqrt{\delta})\mathfrak{m}(B_r(\boldsymbol{X}_t(x_{r,k+1}))). \end{aligned}$$

Thus we have, for some $C(D, T, N) > 1$,

$$\begin{aligned} \mathfrak{m}(B_r(x_{r,k+1})) &\ge \frac{1}{C}\mathfrak{m}(B_{2r}(x_{r,k+1})) \ge \frac{1}{C}\mathfrak{m}(\overline{\boldsymbol{X}}_t(A')) \\ &\ge \frac{1}{C}\mathfrak{m}(B_r(\boldsymbol{X}_t(x_{r,k+1}))). \end{aligned}$$

Combining with (4.8) we have the desired volume bound, concluding the induction step. The result follows by taking $x_r := x_{r,k}$ with $k = \lfloor 10TL/r \rfloor$. □

Take

$$\begin{aligned} x_r &\in B_r(x) \cap \{H \le \delta\}, & B'_r(x_r) &\subset B_r(x_r), \\ x_{r/2} &\in B_{r/2}(x) \cap \{H \le \delta\}, & B'_{r/2}(x_{r/2}) &\subset B_{r/2}(x_{r/2}) \end{aligned}$$

given by Lemma 4.1. That is, they satisfy $S_r.1$ and $S_r.2$ with r and $r/2$ respectively. We claim that

$$d(\boldsymbol{X}_t(x_r), \boldsymbol{X}_t(x_{r/2})) \le 20r \quad \text{for all } t \in [0, T]. \tag{4.9}$$

Take $S_1 = B_r(x)$ and $S_2 = B'_{r/2}(x_{r/2})$. Arguing as before, we can find $x' \in B'_{r/2}(x_{r/2})$ with

$$\int_{B_r(x)} \mathrm{dt}_r(T)(x', y)\, d\mathfrak{m}(y) \le e^{DT} C(N) r \mathfrak{m}(B_r(x))\delta,$$

and $B''_r(x) \subset B_r(x)$ such that

$$\mathrm{dt}_r(T)(x', y) < r \quad \text{for all } y \in B''_r(x), \tag{4.10}$$

$$\mathfrak{m}(B''_r(x)) \ge (1-\sqrt{\delta})\mathfrak{m}(B_r(x)). \tag{4.11}$$

Hence for all $t \in [0, T]$, $y \in B'_r(x)$, using (4.10) we have

$$\begin{aligned} d(\boldsymbol{X}_t(x_{r/2}), \boldsymbol{X}_t(y)) &\le d(\boldsymbol{X}_t(x_{r/2}), \boldsymbol{X}_t(x')) + d(\boldsymbol{X}_t(x'), \boldsymbol{X}_t(y)) \\ &\le r + r + d(x', y) \le 4r. \end{aligned} \tag{4.12}$$

In a similar fashion, one can find a subset $B'''_r(x) \subset B_r(x)$ with

$$\mathbf{X}_t(B'''_r(x)) \subset B_{10r}(\boldsymbol{X}_t(x_r)), \tag{4.13}$$

$$\mathfrak{m}(B'''_r(x)) \ge (1-\sqrt{\delta})\mathfrak{m}(B_r(x)). \tag{4.14}$$

From (4.11) and (4.14), one can find $z \in B''_r(x) \cap B'''_r(x)$. Then from (4.12) and (4.13) applied to z, we conclude (4.9).

Notice that by iterated applications of (4.9), for $r_1, r_2 \le r_x$ and $t \in [0, T]$, one gets

$$d(\boldsymbol{X}_t(x_{r_1}), \boldsymbol{X}_t(x_{r_2})) \le 100 \max\{r_1, r_2\}. \tag{4.15}$$

Now we will use what we have proven so far to construct an adjusted representative $\widetilde{\boldsymbol{X}}$ of the RLF to V with the property that any $x \in X$ satisfying (1.6) also satisfies $S.1$ and $S.2$ for r sufficiently small. Let $S \subset X$ denote the set of x satisfying (1.6) and for each $x \in S$ define $r_x \le \rho/100$ such that for all $r \le r_x$ one has

$$\frac{\mathfrak{m}(\{H > \delta\} \cap B_r(x))}{\mathfrak{m}(B_r(x))} \le \tfrac{1}{2}.$$

As the construction of r_x only uses measurable functions, guaranteed from Kuratowski–Ryll-Nardzewski measurable selection theorem (see for example [Deng 2020, Remark 2.26]) we can take a measurable choice of $r_x : S \to \mathbb{R}$. Moreover, the same is true for x_r given by Lemma 4.1, allowing us to define a measurable map $\tilde{x} : \mathbb{R}^+ \times X \to X$ as

$$\tilde{x}(r, x) := \begin{cases} x_r & \text{if } x \in S,\ r \le r_x, \\ x & \text{otherwise.} \end{cases}$$

Then let us define the adjusted flow $\widetilde{\boldsymbol{X}} : [0, T] \times X \to X$ as

$$\widetilde{\boldsymbol{X}}(x, t) = \lim_{r \to 0} \boldsymbol{X}(\tilde{x}(r, x), t).$$

By (4.15), the limit exists and satisfies $S.1$ and $S.2$ for all $x \in S$ and $r \le r_x$. Now we need to verify that $\widetilde{\boldsymbol{X}}$ is also a regular Lagrangian flow.

$R.1$ holds as (2.43) passes to the limit trajectories. Given $x \in S$, $r \le r_x$, choose a set $A_r(x) \subset B_r(x)$ satisfying $S.1$, and consider the probability measures

$$\mu_{r,x}(t) = (\boldsymbol{X}_t)_* \left(\frac{\chi_{A_r(x)}}{\mathfrak{m}(A_r(x))} \mathfrak{m} \right).$$

From the definition of $\boldsymbol{X}$, for all $f \in \mathrm{TestF}(X)$ we have

$$\frac{d}{dt} \int_X f \, d\mu_{r,x}(t) = \int_X df(V(t)) \, d\mu_{r,x}(t). \tag{4.16}$$

Also notice that, by $S.1$, for all $t \in [0, T]$ we have

$$\mathrm{supp}(\mu_{r,x}(t)) \subset B_{2r}(\widetilde{\boldsymbol{X}}_t(x)), \tag{4.17}$$

and, by $S.2$, the map $\mu_{r,x}(t) \overset{\theta}{\longmapsto} \widetilde{\boldsymbol{X}}_t(x)$ makes

$$\{\mu_{r,x}(t) \mid x \in S, r \le r_x, t \in [0, T]\}$$

a family of bounded eccentricity. Let $f \in \mathrm{TestF}(X)$ and $t \in [0, T]$. From Lemma 2.12 we have, for $\mathfrak{m}$-a.e. $x \in S$,

$$df(V(t))(\widetilde{\boldsymbol{X}}_t(x)) = \lim_{r \to 0} \int_X df(V(t)) \, d\mu_{r,x}(t).$$

Hence, for all $t_0, t_1 \in [0, T]$ and $\mathfrak{m}$-a.e. $x \in S$,

$$\begin{aligned}\int_{t_0}^{t_1} df(V(t))(\widetilde{\boldsymbol{X}}_t(x))dt &= \lim_{r\to 0}\int_{t_0}^{t_1}\int_X df(V(t))\,d\mu_{r,x}(t)\\ &= \lim_{r\to 0}\int_X f\,d\mu_{r,x}(t_1) - \int_X f\,d\mu_{r,x}(t_0)\\ &= f(\widetilde{\boldsymbol{X}}_{t_1}(x)) - f(\widetilde{\boldsymbol{X}}_{t_0}(x)),\end{aligned}$$

where we used dominated convergence on the first two lines, (4.16) on the second, and (4.17) on the third. This implies $\widetilde{\boldsymbol{X}}$ satisfies $R.2$ for $\mathfrak{m}$-almost all $x \in S$. Since $\widetilde{\boldsymbol{X}}$ and $\boldsymbol{X}$ coincide on $X\backslash S$, and $\boldsymbol{X}$ satisfies $R.2$, we deduce $\widetilde{\boldsymbol{X}}$ satisfies $R.2$ as well.

Let $S_k := \left\{x \in S \mid r_x \geq \frac{1}{k}\right\}$. For all $r \leq \frac{1}{k}$, $y \in X$, we have

$$\begin{aligned}\int_{S_k}\frac{\chi_{\boldsymbol{X}_t(A_r(x))}(y)}{\mathfrak{m}(A_r(x))}\,d\mathfrak{m}(x) &\leq M^2\int_{S_k}\frac{\chi_{B_{2r}(y)}(\boldsymbol{X}_t(x))}{\mathfrak{m}(B_r(\boldsymbol{X}_t(x)))}\,d\mathfrak{m}(x)\\ &\leq M^2\int_{B_{2r}(y)}\frac{1}{\mathfrak{m}(B_r(z))}\,d((\boldsymbol{X}_t)_*(\mathfrak{m}))(z)\\ &\leq M^2e^{DT}\int_{B_{2r}(y)}\frac{1}{\mathfrak{m}(B_r(z))}\,d\mathfrak{m}(z)\\ &\leq M^2e^{DT}C(N), \end{aligned}\tag{4.18}$$

where the first line follows from $S.1$ and $S.2$, the second from a change of variables, the third from (2.46), and the fourth from Theorem 2.1. Then, given any $0 \leq f \in \mathrm{TestF}(X)$, we compute

$$\begin{aligned}\int_X f\,d((\tilde{\boldsymbol{X}}_t)_*(\mathfrak{m}|_{S_k})) &= \int_{S_k}(f\circ\widetilde{\boldsymbol{X}}_t)\,d\mathfrak{m}\\ &\leq \lim_{r\to 0}\int_{S_k}\int_X f(y)\,d\mu_{r,x}(t)(y)\,d\mathfrak{m}(x)\\ &= \lim_{r\to 0}\int_X f(y)\int_{S_k}\frac{\chi_{\boldsymbol{X}_t(A_r(x))}(y)}{\mathfrak{m}(A_r(x))}\,d\mathfrak{m}(x)\,d((\boldsymbol{X}_t)_*\mathfrak{m})(y)\\ &\leq M^2e^{DT}C(N)\int_X f(y)\,d((\boldsymbol{X}_t)_*\mathfrak{m})(y)\\ &\leq M^2e^{2DT}C(N)\int_X f\,d\mathfrak{m}, \end{aligned}\tag{4.19}$$

where we used (4.17) on the second line, Tonelli's theorem on the third, (4.18) on the fourth, and (2.46) on the fifth. Since f was arbitrary, (4.19) implies that $(\widetilde{\boldsymbol{X}}_t)_*\mathfrak{m}|_{S_k} \leq \mathfrak{C}\mathfrak{m}$ for some $\mathfrak{C}(D, T, N) > 0$. Hence

$$\begin{aligned}(\widetilde{\boldsymbol{X}}_t)_*\mathfrak{m} &= (\widetilde{\boldsymbol{X}}_t)_*(\mathfrak{m}|_{X\backslash S}) + (\widetilde{\boldsymbol{X}}_t)_*(\mathfrak{m}|_S)\\ &= (\boldsymbol{X}_t)_*(\mathfrak{m}|_{X\backslash S}) + \lim_{k\to\infty}(\widetilde{\boldsymbol{X}}_t)_*(\mathfrak{m}|_{S_k})\\ &\leq (e^{DT}+\mathfrak{C})\mathfrak{m},\end{aligned}$$

establishing $R.3$ for $\widetilde{\boldsymbol{X}}$, so we conclude $\widetilde{\boldsymbol{X}}$ is an RLF for b. □

Proof of Corollary 1.8. By Proposition 2.5(2), for all $s \in [1, R-1]$, one has

$$\fint_{B_s(p)} \mathrm{Mx}(|\nabla V|)^2 \, d\mathfrak{m} \le C(N)\eta.$$

Define $H : X \to \mathbb{R}$ as $H(x) := \int_0^T \mathrm{Mx}(|\nabla V|)(\boldsymbol{X}_t(x)) \, dt$. Then by the Cauchy-Schwarz inequality and (2.46) one gets

$$\begin{aligned}
\left[\fint_{B_r(p)} H(x) \, d\mathfrak{m}(x)\right]^2 &\le \fint_{B_r(p)} \left[\int_0^T \mathrm{Mx}(|\nabla V|)(\boldsymbol{X}_t(x)) \, dt\right]^2 d\mathfrak{m}(x) \\
&\le T \fint_{B_r(p)} \int_0^T \mathrm{Mx}(|\nabla V|)^2(\boldsymbol{X}_t(x)) \, dt \, d\mathfrak{m}(x) \\
&\le T e^{2DT} \int_0^T \fint_{\boldsymbol{X}_t(B_r(x))} \mathrm{Mx}(|\nabla V|)^2(\boldsymbol{X}_t(x)) \, d\mathfrak{m}(x) \, dt \\
&\le C(D, T, N) \int_0^T \fint_{B_{r+LT}(x)} \mathrm{Mx}(|\nabla V|)^2(\boldsymbol{X}_t(x)) \, d\mathfrak{m}(x) \, dt \\
&\le C(D, T, N)\eta.
\end{aligned} \tag{4.20}$$

Let $\delta(D, T, N) > 0$ be given by Theorem 1.5. From (4.20), there is $C(D, T, N) > 1$ such that

$$\mathfrak{m}(\{H \le \delta\} \cap B_r(p)) \ge (1 - C\sqrt{\eta}) \, \mathfrak{m}(B_r(p)). \tag{4.21}$$

By Theorem 1.5, G contains the density points of $\{H \le \delta\}$, so the result follows from (4.21) provided $\eta \le \varepsilon^2/C^2$. □

5. Self-improving stability

In this section, we show that combining essential stability with integral control on the covariant derivative of the corresponding vector field, one can improve the conditions of essential stability to much better estimates. For example, one could compare Corollary 5.13 with $S.2$ and Proposition 5.14 with $S.1$.

This improvement is attained by induction on the radius. Roughly speaking, if for some small $r > 0$ one has $S.1$, $S.2$, and enough control on the covariant derivative of the vector field, then at a scale slightly larger than r, one can obtain conditions similar to $S.1$ and $S.2$ but with better constants. This is the content of Lemma 5.1 (cf. [Kapovitch and Wilking 2011, Lemma 3.7; Colding and Naber 2012, Proposition 3.6]).

Lemma 5.1. *For each $N \ge 1$, $M \ge 1$, there are $\lambda(N) \ge 4$, $\varepsilon(N, M) > 0$, such that the following holds. Let $(X, d, \mathfrak{m})$ be an* RCD$(-(N-1), N)$ *space, $x \in X$, $V \in L^1([0, T]; H^{1,2}_{C,s}(TX))$ a divergence-free vector field, and $\boldsymbol{X} : [0, T] \times X \to X$ the RLF of V. Assume*

$$\int_0^T Mx_4(|\nabla V(t)|)(\boldsymbol{X}_t(x)) \, dt \le \varepsilon.$$

For some $r \le 1/\lambda$ one has

$$\frac{1}{M}\mathfrak{m}(B_r(x)) \le \mathfrak{m}(B_r(\boldsymbol{X}_t(x))) \le M\mathfrak{m}(B_r(x)) \quad \textit{for all } t \in [0, T], \tag{5.2}$$

and there is $S_r \subset B_r(x)$ *with*

$$\mathfrak{m}(S_r) \geq \frac{1}{M}\mathfrak{m}(B_r(x)),$$
$$\boldsymbol{X}_t(S_r) \subset B_{2r}(\boldsymbol{X}_t(x)) \quad \textit{for all } t \in [0, T].$$

Then

$$\tfrac{1}{2}\mathfrak{m}(B_{\lambda r}(x)) \leq \mathfrak{m}(B_{\lambda r}(\boldsymbol{X}_t(x))) \leq 2\mathfrak{m}(B_{\lambda r}(x)) \quad \textit{for all } t \in [0, T], \tag{5.3}$$

and there is $A_{\lambda r} \subset B_{\lambda r}(x)$ *with*

$$\boldsymbol{X}_t(A_{\lambda r}) \subset B_{(\lambda+4)r}(\boldsymbol{X}_t(x)) \qquad \textit{for all } t \in [0, T], \tag{5.4}$$

$$\mathfrak{m}(\boldsymbol{X}_t(A_{\lambda r}) \cap B_{\lambda r}(\boldsymbol{X}_t(x))) \geq \tfrac{9}{10}\mathfrak{m}(B_{\lambda r}(\boldsymbol{X}_t(x))) \quad \textit{for all } t \in [0, T]. \tag{5.5}$$

Proof. Pick $\lambda(N) \geq 5$ such that

$$\mathfrak{m}(B_{(\lambda+5)s}(y)) \leq \tfrac{101}{100}\mathfrak{m}(B_{\lambda s}(y)) \tag{5.6}$$

for all $y \in X$, $s \leq 10/\lambda$. With this choice of λ, (5.3) will follow from (5.4) and (5.5). Let $\mathrm{dt}_r(t)$ be given by (2.48) and

$$\Gamma(t) := \{(a, b) \in S_r \times B_{\lambda r}(x) \mid \mathrm{dt}_r(t)(a, b) < r\}.$$

Notice that for each $t \in [0, T]$, $(a, b) \in \Gamma(t)$, one has $d(\boldsymbol{X}_t(a), \boldsymbol{X}_t(x)) \leq 2r$, and

$$\begin{aligned} d(\boldsymbol{X}_t(b), \boldsymbol{X}_t(x)) &\leq d(\boldsymbol{X}_t(b), \boldsymbol{X}_t(a)) + d(\boldsymbol{X}_t(a), \boldsymbol{X}_t(x)) \\ &\leq d(a, b) + r + 2r \\ &\leq \lambda r + 4r. \end{aligned} \tag{5.7}$$

Then

$$\begin{aligned} &\fint_{S_r \times B_{\lambda r}(x)} \mathrm{dt}_r(T)\, d(\mathfrak{m} \times \mathfrak{m}) \\ &\quad \leq \frac{M}{\mathfrak{m}(B_r(x))^2} \int_0^T \int_{\Gamma(t)} d(\boldsymbol{X}_t(y), \boldsymbol{X}_t(z)) \left[\int_0^1 |\nabla V(t)|(\gamma_{\boldsymbol{X}_t(y), \boldsymbol{X}_t(z)}(s))\, ds\right] d(\mathfrak{m} \times \mathfrak{m})(y, z)\, dt \\ &\quad \leq \frac{M}{\mathfrak{m}(B_r(x))^2} \int_0^T \int_{\boldsymbol{X}_t(\Gamma(t))} d(y, z) \left[\int_0^1 |\nabla V(t)|(\gamma_{y,z}(s))\, ds\right] d(\mathfrak{m} \times \mathfrak{m})(y, z)\, dt \\ &\quad \leq \frac{M}{\mathfrak{m}(B_r(x))^2} \int_0^T \int_{B_{(\lambda+4)r}(\boldsymbol{X}_t(x))^{\times 2}} d(y, z) \left[\int_0^1 |\nabla V(t)|(\gamma_{y,z}(s))\, ds\right] d(\mathfrak{m} \times \mathfrak{m})(y, z)\, dt \\ &\quad \leq C(N) \cdot M^3 \cdot r \cdot \int_0^T \fint_{B_{(2\lambda+8)r}(\boldsymbol{X}_t(x))} |\nabla V(t)|(y)\, d\mathfrak{m}(y)\, dt \\ &\quad \leq C(N) \cdot M^3 \cdot r \cdot \int_0^T \mathrm{Mx}_4(|\nabla V(t)|)(\boldsymbol{X}_t(x))\, dt \leq C(N) M^3 \varepsilon r, \end{aligned} \tag{5.8}$$

where the first inequality follows from Theorem 2.47, the second from Tonelli's theorem, the third from (5.7), and the fourth from Theorem 2.10 and (5.2). Hence there is $y \in S_r$ with

$$\fint_{B_{\lambda r}(x)} \mathrm{dt}_r(T)(y,z)\, d\mathfrak{m}(z) \le C(N) M^3 \varepsilon r. \tag{5.9}$$

We can then define $A_{\lambda r} := \{z \in B_{\lambda r}(x) \mid \mathrm{dt}_r(T)(y,z) < r\}$, which by (5.7) satisfies (5.4), and by (5.9) satisfies

$$\mathfrak{m}(A_{\lambda r}) \ge (1 - C(N) M^3 \varepsilon)\mathfrak{m}(B_{\lambda r}(x)) \ge \tfrac{99}{100}\mathfrak{m}(B_{\lambda r}(x)), \tag{5.10}$$

provided ε is small enough. To verify (5.5), fix $t \in [0, T]$, consider the vector field $\overline{V} \in L^1([0,t], H^{1,2}_{C,s}(TX))$ given by $\overline{V}(s) := -V(t-s)$, and $\overline{\boldsymbol{X}} : [0,t] \times X \to X$ its RLF. Also set

$$\begin{aligned} &\overline{\mathrm{dt}}_r(\cdot)(\cdot,\cdot) : [0,t] \times X \times X \to \mathbb{R}, \\ &\overline{\mathrm{dt}}_r(s)(y,z) := \sup_{u \in [0,s]} \mathrm{dt}_r(\overline{X}_u)(y,z), \end{aligned}$$

and define $\overline{\Gamma}(s) := \{(a,b) \in \boldsymbol{X}_t(S_r) \times B_{\lambda r}(\boldsymbol{X}_t(x)) \mid \overline{\mathrm{dt}}_r(s)(a,b) < r\}$. Then for all $s \in [0,t]$ and $(a,b) \in \overline{\Gamma}(s)$, one has $d(\overline{\boldsymbol{X}}_s(a), \boldsymbol{X}_{t-s}(x)) \le 2r$, and

$$\begin{aligned} d(\overline{\boldsymbol{X}}_s(b), \boldsymbol{X}_{t-s}(x)) &\le d(\overline{\boldsymbol{X}}_s(b), \overline{\boldsymbol{X}}_s(a)) + d(\overline{\boldsymbol{X}}_s(a), \boldsymbol{X}_{t-s}(x)) \\ &\le d(a,b) + r + 2r \\ &\le \lambda r + 5r. \end{aligned} \tag{5.11}$$

As in (5.8), using (5.11) instead of (5.7) we get

$$\begin{aligned} &\fint_{\boldsymbol{X}_t(S_r) \times B_{\lambda r}(\boldsymbol{X}_t(x))} \overline{\mathrm{dt}}_r(t)\, d(\mathfrak{m} \times \mathfrak{m}) \\ &\quad\le \frac{2M}{\mathfrak{m}(B_r(x))^2} \int_0^t \int_{\overline{\Gamma}(s)} d(\overline{\boldsymbol{X}}_s(y), \overline{\boldsymbol{X}}_s(z)) \left[\int_0^1 |\nabla \overline{V}(s)|(\gamma_{\overline{\boldsymbol{X}}_s(y), \overline{\boldsymbol{X}}_s(z)}(u))\, du\right] d(\mathfrak{m} \times \mathfrak{m})(y,z)\, ds \\ &\quad\le \frac{2M}{\mathfrak{m}(B_r(x))^2} \int_0^t \int_{\overline{\boldsymbol{X}}_s(\overline{\Gamma}(s))} d(y,z) \left[\int_0^1 |\nabla \overline{V}(s)|(\gamma_{y,z}(u))\, du\right] d(\mathfrak{m} \times \mathfrak{m})(y,z)\, ds \\ &\quad\le \frac{2M}{\mathfrak{m}(B_r(x))^2} \int_0^t \int_{B_{(\lambda+5)r}(\boldsymbol{X}_{t-s}(x))^{\times 2}} d(y,z) \left[\int_0^1 |\nabla \overline{V}(s)|(\gamma_{y,z}(u))\, du\right] d(\mathfrak{m} \times \mathfrak{m})(y,z)\, ds \\ &\quad\le C(N) M^3 r \int_0^t \fint_{B_{(2\lambda+10)r}(\boldsymbol{X}_{t-s}(x))} |\nabla \overline{V}(s)|(y)\, d\mathfrak{m}(y)\, ds \\ &\quad\le C(N) M^3 r \int_0^t \mathrm{Mx}_4(|\nabla V(t-s)|)(\boldsymbol{X}_{t-s}(x))\, ds \le C(N) M^3 \varepsilon r. \end{aligned}$$

Pick $y_t \in \boldsymbol{X}_t(S_r)$ such that

$$\fint_{B_{\lambda r}(\boldsymbol{X}_t(x))} \overline{\mathrm{dt}}_r(t)(y_t, z)\, d\mathfrak{m}(z) \le C(N) M^3 \varepsilon r.$$

Then the set $A^t_{\lambda r} := \{z \in B_{\lambda r}(\boldsymbol{X}_t(x)) \mid \overline{\mathrm{dt}}_r(t)(y_t, z) < r\}$ satisfies

$$\mathfrak{m}(A^t_{\lambda r}) \geq (1 - C(N)M^3\varepsilon)\mathfrak{m}(B_{\lambda r}(\boldsymbol{X}_t(x))).$$

Since $\overline{X}_t(A^t_{\lambda r}) \subset B_{(\lambda+5)r}(x)$ and $\overline{X}_t$ is measure preserving, using (5.6) we have

$$\mathfrak{m}(B_{\lambda r}(x)) \geq \tfrac{98}{100}\mathfrak{m}(B_{\lambda r}(\boldsymbol{X}_t(x))) \tag{5.12}$$

provided ε is small enough. We conclude

$$\begin{aligned}\mathfrak{m}(\boldsymbol{X}_t(A_{\lambda r}) \cap B_{\lambda r}(\boldsymbol{X}_t(x))) &\geq \mathfrak{m}(\boldsymbol{X}_t(A_{\lambda r})) - \mathfrak{m}(B_{(\lambda+4)r}(\boldsymbol{X}_t(x)) \backslash B_{\lambda r}(\boldsymbol{X}_t(x)))\\ &\geq \mathfrak{m}(A_{\lambda r}) - \tfrac{1}{100}\mathfrak{m}(B_{\lambda r}(\boldsymbol{X}_t(x)))\\ &\geq \tfrac{99}{100}\mathfrak{m}(B_{\lambda r}(x)) - \tfrac{1}{100}\mathfrak{m}(B_{\lambda r}(\boldsymbol{X}_t(x)))\\ &\geq \tfrac{9}{10}\mathfrak{m}(B_{\lambda r}(\boldsymbol{X}_t(x))),\end{aligned}$$

where we used (5.4) on the first inequality, (5.6) on the second, (5.10) on the third, and (5.12) on the fourth. □

Corollary 5.13. *Let* $(X, d, \mathfrak{m})$ *be an* $\mathrm{RCD}(-(N-1), N)$ *space*, $x \in X$, $V \in L^1([0, T]; H^{1,2}_{C,s}(TX))$ *a divergence-free vector field*, *and* $\boldsymbol{X} : [0, T] \times X \to X$ *the RLF of* V. *Assume* x *is a point of essential stability of* $\boldsymbol{X}$ *and*

$$\int_0^T Mx_4(|\nabla V(t)|)(\boldsymbol{X}_t(x))\,dt \leq \varepsilon.$$

If ε *is small enough*, *depending only on* N, *then for all* $r \leq 1$, $t \in [0, T]$, *one has*

$$\tfrac{1}{2}\mathfrak{m}(B_r(x)) \leq \mathfrak{m}(B_r(\boldsymbol{X}_t(x))) \leq 2\mathfrak{m}(B_r(x)).$$

Proof. By the definition of essential stability, there is $M(N) > 0$ for which the hypotheses of Lemma 5.1 hold for small enough $r \leq 1$. By Lemma 5.1, if they hold for a certain r, then they hold for λr so we can apply Lemma 5.1 repeatedly, and (5.3) is valid for all $r \leq 1/\lambda$. □

Proposition 5.14. *There is* $C_0(N) > 0$ *such that*, *under the conditions of Corollary 5.13*, *for all* $r \leq 1$ *there is* $A_r \subset B_r(x)$ *such that*

$$\mathfrak{m}(A_r) \geq (1 - C_0\varepsilon)\mathfrak{m}(B_r(x)), \tag{5.15}$$

$$\boldsymbol{X}_t(A_r) \subset B_{2r}(\boldsymbol{X}_t(x)) \quad \textit{for all } t \in [0, T]. \tag{5.16}$$

Proof. By the definition of essential stability, for $r \leq 1$ sufficiently small, there is $S_{r/10} \subset B_{r/10}(x)$ with $\mathfrak{m}(S_{r/10}) \geq \frac{1}{M(N)}\mathfrak{m}(B_{r/10}(x))$ and $\boldsymbol{X}_t(S_{r/10}) \subset B_{r/5}(\boldsymbol{X}_t(x))$ for all $t \in [0, T]$. Set

$$\begin{aligned}&\mathrm{dist}_{r/10}(\,\cdot\,)(\,\cdot\,,\cdot\,) : [0, T] \times X \times X \to [0, r/10],\\ &\mathrm{dist}_{r/10}(t)(y, z) := \sup_{s\in[0,t]} \mathrm{dist}_{r/10}(\boldsymbol{X}_s)(y, z),\\ &\qquad \Gamma(t) := \{(a, b) \in S_{r/10} \times B_r(x) \mid \mathrm{dist}_{r/10}(t)(a, b) < r/10\}.\end{aligned}$$

Then for $(y, z) \in \Gamma(t)$, for each $t \in [0, T]$ we have

$$\begin{aligned} d(\boldsymbol{X}_t(b), \boldsymbol{X}_t(x)) &\le d(\boldsymbol{X}_t(b), \boldsymbol{X}_t(a)) + d(\boldsymbol{X}_t(a), \boldsymbol{X}_t(x)) \\ &\le d(a, b) + r/10 + r/5 \\ &\le r/2 + r/10 + r/10 + r/5 < r, \end{aligned} \tag{5.17}$$

so as in (5.8), using Corollary 5.13 one gets

$$\fint_{S_{r/10} \times B_r(x)} \mathrm{dt}_{r/10}(T)\, d(\mathfrak{m} \times \mathfrak{m}) \le C(N) \cdot r \cdot \int_0^T \mathrm{Mx}_4(|\nabla V(t)|)(\boldsymbol{X}_t(x))\, dt \le C(N)\varepsilon r.$$

Then there is $y \in S_{r/10}$ with

$$\fint_{B_r(x)} \mathrm{dt}_{r/10}(T)(y, z)\, d\mathfrak{m}(z) \le C(N)\varepsilon r. \tag{5.18}$$

We can then define $A_r := \{z \in B_r(x) \mid \mathrm{dt}_{r/10}(T)(y, z) < r/10\}$, which by (5.18) satisfies (5.15) and by (5.17) also (5.16). The above analysis shows that (5.15) and (5.16) hold for all r small enough. An identical argument (using $A_{r/10}$ instead of $S_{r/10}$) shows that if (5.15) and (5.16) hold for some $r/10 \le 1/10$, then they hold for r. □

Proposition 5.19. *There is $C_0(N) > 0$ such that under the conditions of Corollary 5.13, for all $r \le 1$, one has*

$$\fint_{B_r(x)^{\times 2}} \mathrm{dt}_r(\boldsymbol{X}_T)\, d(\mathfrak{m} \times \mathfrak{m}) \le C_0 \varepsilon r.$$

Proof. For $A_r \subset B_r(x)$ given by Proposition 5.14, a computation analogous to (5.8) yields

$$\fint_{A_r^{\times 2}} \mathrm{dt}_r(\boldsymbol{X}_T)\, d(\mathfrak{m} \times \mathfrak{m}) \le C(N)\varepsilon r.$$

Combining this with (5.15), we get the result. □

Definition 5.20. Let $(X, d, \mathfrak{m})$ be an RCD$(-(N-1), N)$ space, $V_1, \dots, V_k \in L^1([0, 1]; H^{1,2}_{C,s}(TX))$ divergence-free vector fields, $\boldsymbol{X}^j : [0, 1] \times X \to X$ their RLFs, and $x_1, \dots, x_k \in X$ such that

- x_j is a point of essential stability of $\boldsymbol{X}^j$ for each $j \in \{1, \dots, k\}$.
- $\boldsymbol{X}^j_1(x_j) = x_{j+1}$ for each $j \in \{1, \dots, k-1\}$.

If $V \in L^1([0, 1]; H^{1,2}_{C,s}(TX))$ is given by

$$V(t) := k \cdot V_j(kt - j + 1) \quad \text{for } t \in \left[\frac{j-1}{k}, \frac{j}{k}\right],$$

and $\boldsymbol{X} : [0, 1] \times X \to X$ is its RLF, then we say x_1 is a point of *weak essential stability* of $\boldsymbol{X}$.

The following proposition shows that, under suitable conditions, weak essential stability can be upgraded to essential stability, allowing one to concatenate well behaved flows; a crucial step in the proof of the rescaling theorem.

Proposition 5.21. *Under the conditions of Definition 5.20, there is* $\eta(N) > 0$ *such that if*

$$\int_0^1 Mx(|\nabla V(t)|)(\boldsymbol{X}_t(x_1))\,dt = \sum_{j=1}^{k} \int_0^1 Mx(|\nabla V_j(t)|)(\boldsymbol{X}_t^j(x_j))\,dt \leq \eta,$$

then x_1 *is a point of essential stability of* $\boldsymbol{X}$.

Proof. By induction we can assume $k = 2$. By Corollary 5.13, and Proposition 5.14, we can apply Lemma 5.1 to both X^1 and X^2, provided η is small enough. By (5.3), for each $r \leq 1$, $t \in [0, 1]$, we get

$$\tfrac{1}{4}\mathfrak{m}(B_r(x_1)) \leq \mathfrak{m}(B_r(\boldsymbol{X}_t(x_1))) \leq 4\mathfrak{m}(B_r(x_1)).$$

Let $\overline{V}_1 \in L^1([0, 1]; H^{1,2}_{C,s}(TX))$ be given by $\overline{V}_1(t) := -V_1(1-t)$ and let $\overline{\boldsymbol{X}}^1 : [0, 1] \times X \to X$ be its RLF. Again by Lemma 5.1, for r small enough there are sets $A_r^1, A_r^2 \subset B_r(x_2)$ such that

$$\begin{gathered}\overline{\boldsymbol{X}}_t^1(A_r^1) \subset B_{2r}(\overline{\boldsymbol{X}}_t^1(x_2)), \quad \boldsymbol{X}_t^2(A_r^2) \subset B_{2r}(\boldsymbol{X}_t^2(x_2)),\\ \mathfrak{m}(\overline{\boldsymbol{X}}_t^1(A_r^1) \cap B_r(\overline{\boldsymbol{X}}_t^1(x_2))) \geq \tfrac{9}{10}\mathfrak{m}(B_r(\overline{\boldsymbol{X}}_t^1(x_2))),\\ \mathfrak{m}(\boldsymbol{X}_t^2(A_r^2) \cap B_r(\boldsymbol{X}_t^2(x_2))) \geq \tfrac{9}{10}\mathfrak{m}(B_r(\boldsymbol{X}_t^2(x_2)))\end{gathered}$$

for all $t \in [0, 1]$. Then $A_r := B_r(x_1) \cap \overline{\boldsymbol{X}}_1^1(A_r^1 \cap A_r^2)$ satisfies

$$\boldsymbol{X}_t(A_r) \subset B_{2r}(\boldsymbol{X}_t(x)) \quad \text{for all } t \in [0, 1],$$

and using (5.3) we conclude

$$\begin{aligned}\mathfrak{m}(A_r) &\geq \mathfrak{m}(B_r(x_1) \cap \overline{\boldsymbol{X}}_1^1(A_r^1)) - \mathfrak{m}(A_r^1 \backslash A_r^2)\\ &\geq \tfrac{9}{10}\mathfrak{m}(B_r(x_1)) - \tfrac{1}{10}\mathfrak{m}(B_r(x_2))\\ &\geq \left[\tfrac{9}{10} - \tfrac{1}{5}\right]\mathfrak{m}(B_r(x_1))\\ &\geq \tfrac{1}{2}\mathfrak{m}(B_r(x_1)).\end{aligned}$$

□

6. Properties of GS maps

In this section we prove the main properties of GS maps; they converge weakly to an isometry (Lemma 6.4), have the zoom-in property (Proposition 6.7), and can be concatenated (Proposition 6.8).

Remark 6.1. From condition (1) of Definition 1.12, we can assume that for all $x \in U_i^1$, $y \in U_i^2$, we have $(f_i^{-1})^{-1}(x) = \{f_i(x)\}$ and $(f_i)^{-1}(y) = \{f_i^{-1}(y)\}$.

Definition 6.2. For $j \in \{1, 2\}$, let $(X_i^j, d_i^j, \mathfrak{m}_i^j, p_i^j)$, be a sequence of pointed $\mathrm{RCD}(K, N)$ spaces for which (X_i^j, p_i^j) converges to (X^j, p^j) in the pointed Gromov–Hausdorff sense. We say a sequence of maps $f_i : X_i^1 \to X_i^2$ converges weakly to $f_\infty : X^1 \to X^2$ if there is a sequence of subsets $U_i \subset X_i^1$ with asymptotically full measure such that

$$\lim_{i\to\infty} \sup_{x \in U_i} d(\varphi_i^2 f_i(x), f_\infty \varphi_i^1(x)) = 0, \tag{6.3}$$

where $\varphi_i^j : X_i^j \to X^j \cup \{*\}$ are Gromov–Hausdorff approximations for $j \in \{1, 2\}$.

Lemma 6.4. *For $j \in \{1, 2\}$, let $(X_i^j, d_i^j, \mathfrak{m}_i^j, p_i^j)$, be a sequence of pointed* RCD(K, N) *spaces for which (X_i^j, p_i^j) converges to (X^j, p^j) in the pointed Gromov–Hausdorff sense. If $f_i : [X_i^1, p_i^1] \to [X_i^2, p_i^2]$ is a sequence of GS maps, then, after taking a subsequence, f_i converges weakly to an isometry $f_\infty : X^1 \to X^2$.*

Proof. Let $R_0 > 0$, $\varepsilon_i \to 0$, and $S_i^j, U_i^j \subset X_i^j$ be given by Definition 1.12 (see Remark 6.1).

Step 1: For i large enough, $x \in U_i^1$, $r \leq 1$, there is $A \subset B_r(x)$ such that

$$f_i(A) \subset B_{2r}(f_i(x)) \quad \text{and} \quad \mathfrak{m}_i^1(A) \geq \tfrac{1}{2}\mathfrak{m}_i^1(B_r(x)).$$

From the definition of essential continuity, the statement holds for r small enough (depending on x). We now see that if i is large enough, and there is $A_0 \subset B_{r/10}(x)$ such that $f_i(A_0) \subset B_{r/5}(f_i(x))$, and $\mathfrak{m}_i^1(A_0) \geq \frac{1}{2}\mathfrak{m}_i^1(B_{r/5}(x))$, then there is $A \subset B_r(x)$ such that $f_i(A) \subset B_{2r}(f_i(x))$, and $\mathfrak{m}_i^1(A) \geq \frac{1}{2}\mathfrak{m}_i^1(B_r(x))$. Since there is $C(K, N) > 0$ such that $\mathfrak{m}_i^1(B_r(x)) \leq C\mathfrak{m}_i^1(A_0)$, we have

$$\fint_{B_r(x_i^1)\times A_0} \mathrm{dt}_r(f_i)\, d(\mathfrak{m}_i^1 \times \mathfrak{m}_i^1) \leq Cr\varepsilon_i.$$

Hence if $A := \{y \in B_r(x) \mid d(fy, fx) < 2r\}$, one gets

$$r \cdot \frac{\mathfrak{m}_i^1(B_r(x)\backslash A)}{\mathfrak{m}_i^1(B_r(x))} = \frac{\int_{(B_r(x)\backslash A)\times A_0} \mathrm{dt}_r(f_i)\, d(\mathfrak{m}_i^1 \times \mathfrak{m}_i^1)}{\mathfrak{m}_i^1(B_r(x)) \cdot \mathfrak{m}_i^1(A_0)} \leq Cr\varepsilon_i,$$

implying that $\mathfrak{m}_i^1(A) \geq \frac{1}{2}\mathfrak{m}_i^1(B_r(x))$ provided $\varepsilon_i \leq \frac{1}{2C}$.

Step 2: For all distinct $x_i, y_i \in U_i^1$ with $d(x_i, y_i) \leq \frac{1}{2}$, one has

$$\limsup_{i\to\infty} \frac{d(f_i x_i, f_i y_i)}{d(x_i, y_i)} \leq 1.$$

Set $r_i := d(x_i, y_i)$ and assume, after taking a subsequence, that $d(f_i x_i, f_i y_i) \geq (1+\delta)r_i$ for some $\delta > 0$ and all i. By Step 1, there are subsets $A_i \subset B_{\delta r_i/10}(x_i)$, $B_i \subset B_{\delta r_i/10}(y_i)$ with $f_i(A_i) \subset B_{\delta r_i/5}(f_i x_i)$, $f_i(B_i) \subset B_{\delta r_i/5}(f_i y_i)$, $\mathfrak{m}_i^1(A_i) \geq \frac{1}{2}\mathfrak{m}_i^1(B_{\delta r_i/10}(x_i))$, and $\mathfrak{m}_i^1(B_i) \geq \frac{1}{2}\mathfrak{m}_i^1(B_{\delta r_i/10}(y_i))$. Since there is $C(K, N) > 0$ such that

$$\mathfrak{m}_i^1(B_{2r_i}(x_i)) \leq C \cdot \min\{\mathfrak{m}_i^1(A_i), \mathfrak{m}_i^1(B_i)\},$$

one has

$$\frac{\delta r_i}{10 \cdot C^2} \leq \frac{\int_{A_i\times B_i} \mathrm{dt}_r(f_i)\, d(\mathfrak{m}_i^1 \times \mathfrak{m}_i^1)}{\mathfrak{m}_i^1(B_{2r_i}(x_i))^2} \leq \fint_{B_{2r_i}(x_i)^{\times 2}} \mathrm{dt}_r(f)\, d(\mathfrak{m}_i^1 \times \mathfrak{m}_i^1) \leq 2r_i\varepsilon_i,$$

which is impossible as $\varepsilon_i \to 0$.

Step 3: For $R > 0$ and distinct $x_i, y_i \in U_i^1$ with $d(x_i, p_i^1), d(y_i, p_i^1) \leq R$, one has

$$\limsup_{i\to\infty} \frac{d(f_i x_i, f_i y_i)}{d(x_i, y_i)} \leq 1.$$

By Step 2, we can assume $d(x_i, y_i) \geq \frac{1}{2}$ for all i. For each i, choose a sequence $x_i = z_i^0, \dots, z_i^k = y_i \in X_i^1$ with $d(z_i^{j-1}, z_i^j) \leq \frac{1}{3}$ for each $j \in \{1, \dots, k\}$, $d(x_i, y_i) = \sum_{j=1}^k d(z_i^{j-1}, z_i^j)$, and $k = \lfloor 10R \rfloor$. For each $i \in \mathbb{N}$ and $j \in \{1, \dots, k\}$, let $w_i^j \in U_i^1$ be such that

$$d(w_i^j, z_i^j) \leq 2 \cdot \inf\{d(w, z_i^j) \mid w \in U_i^1\}.$$

As the sets U_i^1 have asymptotically full measure, $\sup_j d(w_i^j, z_i^j) \to 0$ as $i \to \infty$, and the claim follows from Step 2 applied to pairs (w_i^{j-1}, w_i^j).

Step 4: For $R > 0$ and $x_i \in U_i^1$ with $d(x_i, p_i^1) \leq R$, one has

$$\limsup_{i \to \infty} d(f_i x_i, p_i^2) \leq R_0 + R + 1.$$

As the sets U_i^1 have asymptotically full measure, for i large enough one can pick $y_i \in U_i^1 \cap S_i^1$. Then the result follows from Step 3 and the fact that $d(x_i, y_i) \leq R + 1$ for all i.

Step 5: For $R > 0$, $x_i \in U_i^1$ with $d(x_i, p_i^1) \leq R$, and $\delta > 0$, for large enough i there is

$$y_i \in B_\delta(x_i) \cap U_i^1 \cap f_i^{-1}(U_i^2).$$

Without loss of generality assume $\delta < \frac{1}{2}$. As the sets U_i^1 have asymptotically full measure, the sets $A_i := B_\delta(x_i) \cap U_i^1$ satisfy $\mathfrak{m}_i^1(A_i) \geq \frac{1}{2}\mathfrak{m}_i^1(B_\delta(x_i))$ for i large enough. Assuming the claim fails, one has from Step 2, after taking a subsequence, that $f_i(A_i) \subset B_{2\delta}(f_i x_i) \backslash U_i^2$ for all i. As f_i restricted to A_i is measure preserving, and the sets U_i^2 have asymptotically full measure, this means that

$$\frac{\mathfrak{m}_i^1(B_\delta(x_i))}{\mathfrak{m}_i^2(B_{2\delta}(f_i x_i))} \to 0 \quad \text{as } i \to \infty. \tag{6.5}$$

From Step 4 we know that $B_{2\delta}(f_i x_i) \subset B_{R_0+R+2}(p_i)$ for large i, and from the Bishop–Gromov inequality, there is $C(K, N, R_0, R, \delta) > 0$ such that

- $\mathfrak{m}_i^2(B_{R_0+R+2}(p_i^2)) \leq C \cdot \mathfrak{m}_i^2(S_i^2 \cap U_i^2)$ for large enough i,
- $\mathfrak{m}_i^1(B_{R_0}(p_i^1)) \leq C \cdot \mathfrak{m}_i^1(B_\delta(x_i))$.

Combining this with the fact that $f_i^{-1}(S_i^2 \cap U_i^2) \subset B_{R_0}(p_i^1)$, we get that

$$\frac{\mathfrak{m}_i^2(B_{2\delta}(f_i x_i))}{\mathfrak{m}_i^1(B_\delta(x_i))} \leq C^2$$

for i large enough, contradicting (6.5).

Step 6: For $R > 0$ and distinct $x_i, y_i \in U_i^1$ with $d(x_i, p_i^1), d(y_i, p_i^1) \leq R$, one has

$$\lim_{i \to \infty} |d(f_i x_i, f_i y_i) - d(x_i, y_i)| = 0.$$

From Step 3, one gets

$$\limsup_{i \to \infty} (d(f_i x_i, f_i y_i) - d(x_i, y_i)) \leq 0.$$

By Step 5, there are sequences $w_i, z_i \in U_i^1 \cap h_i(U_i^2)$ with $d(w_i, x_i), d(z_i, y_i) \to 0$. By Step 2, we have $d(f_i w_i, f_i x_i), d(f_i z_i, f_i y_i) \to 0$, and by Step 3 applied to h_i, one gets

$$\limsup_{i\to\infty}(d(w_i, z_i) - d(f_i w_i, f_i z_i)) \leq 0.$$

Hence

$$\limsup_{i\to\infty}(d(x_i, y_i) - d(f_i x_i, f_i y_i)) \leq 0.$$

Step 7: Lemma 6.4 holds.

Let $\varphi_i^j : X_i^j \to X^j \cup \{*\}$ be Gromov–Hausdorff approximations and fix $\mathcal{D} \subset X^1$ a countable dense set. For $x \in \mathcal{D}$, choose $x_i \in U_i^1$ converging to x. By Step 4 we can define (after taking a subsequence) $f'_\infty(x) \in X^2$ as

$$f'_\infty(x) := \lim_{i\to\infty} \varphi_i^2 f_i(x_i).$$

By a diagonal argument, this can be done simultaneously for all $x \in \mathcal{D}$. It is easy to see from Step 6 that $f'_\infty : \mathcal{D} \to X^2$ extends to an isometry $f_\infty : X^1 \to X^2$ and satisfies (6.3). □

Proposition 6.6. *For $j \in \{1, 2\}$, let $(X_i^j, d_i^j, \mathfrak{m}_i^j, p_i^j)$, be a sequence of pointed* RCD(K, N) *spaces that converges in the pointed measured Gromov–Hausdorff sense to a pointed* RCD(K, N) *space $(X^j, d^j, \mathfrak{m}^j, p^j)$, and assume there is a sequence $f_i : [X_i^1, p_i^1] \to [X_i^2, p_i^2]$ of GS maps. If sequences of sets $V_i^1 \subset X_i^1$ and $V_i^2 \subset X_i^2$ have asymptotically full measure, then the sequences $f_i(V_i^1) \subset X_i^2$ and $f_i^{-1}(V_i^2) \subset X_i^1$ have asymptotically full measure as well.*

Proof. Let $U_i^j \subset X_i^j$ be sets given by condition (3) of Definition 1.12. By replacing U_i^j and V_i^j by $U_i^j \cap V_i^j$, we can assume $U_i^j = V_i^j$ for all $j \in \{1, 2\}$, $i \in \mathbb{N}$. Fix $R > \delta > 0$, and consider a sequence $x_i \in B_R(p_i^1)$. As the sets U_i^1 have asymptotically full measure, by Step 5 above, there is a sequence $y_i^1 \in U_i^1 \cap f_i^{-1}(V_i^2)$ with $d(x_i, y_i^1) \to 0$. Define $y_i^2 := f_i y_i^1$, $A_i^j := U_i^j \cap B_\delta(y_i^j)$ for $j \in \{1, 2\}$. By Step 6 above, there is a sequence $\varepsilon_i \to 0$ such that

$$f_i(A_i^1) \subset B_{\delta+\varepsilon_i}(y_i^2),\ f_i^{-1}(A_i^2) \subset B_{\delta+\varepsilon_i}(y_i^1).$$

Then

$$\begin{aligned}\lim_{i\to\infty} \frac{\mathfrak{m}_i^1(f_i^{-1}(A_i^2) \cap B_\delta(x_i^1))}{\mathfrak{m}_i^1(B_\delta(x_i^1))} &\geq \lim_{i\to\infty} \frac{\mathfrak{m}_i^1(f_i^{-1}(A_i^2))}{\mathfrak{m}_i^1(A_i^1)} \geq \lim_{i\to\infty} \frac{\mathfrak{m}_i^2(A_i^2)}{\mathfrak{m}_i^2(f_i(A_i^1))} \\ &\geq \lim_{i\to\infty} \frac{\mathfrak{m}_i^2(B_\delta(y_i^2))}{\mathfrak{m}_i^2(B_\delta(y_i^2))} = 1.\end{aligned}$$

This shows that $f_i^{-1}(U_i^2)$ has asymptotically full measure. The result for $f_i(U_i^1)$ is analogous. □

Proposition 6.7. *Let $(X_i^j, d_i^j, \mathfrak{m}_i^j, p_i^j)$, $j \in \{1, 2\}$ be a pair of sequences of pointed* RCD(K, N) *spaces and $f_i : [X_i^1, p_i^1] \to [X_i^2, p_i^2]$ is a sequence of GS maps. Then there is a sequence of subsets $W_i^1 \subset X_i^1$ of asymptotically full measure with the property that for all $w_i \in W_i^1$ and $\lambda_i \to \infty$, the sequence $f_i : [\lambda_i X_i^1, w_i] \to [\lambda_i X_i^2, f_i(w_i)]$ is GS.*

Proof. Let $U_i^j \subset X_i^j$ be given by Definition 1.12 and consider a sequence $\delta_i \to 0$. Set

$$\chi_i^j := 1 - \chi_{U_i^j} : X_i^j \to \mathbb{R},$$

and $V_i^j := \{x \in U_i^j \mid \mathrm{Mx}(\chi_i^j)(x) \leq \delta_i\}$. Then by Proposition 2.5(1) and Proposition 6.6, if $\delta_i \to 0$ slowly enough, the sets

$$W_i^1 := V_i^1 \cap f_i^{-1}(V_i^2),\;\; W_i^2 := V_i^2 \cap f_i(V_i^1),$$

have asymptotically full measure. Moreover, by construction, for any sequences $\lambda_i \to \infty$ and $w_i \in W_i^1$, the sets W_i^1 and W_i^2 also have asymptotically full measure when regarded as subsets of the spaces $(X_i^1, \lambda_i d_i^1, \mathfrak{m}_i^1, w_i)$ and $(X_i^2, \lambda_i d_i^2, \mathfrak{m}_i^2, f_i(w_i))$, respectively.

Using the sets W_i^j as a replacement for U_i^j, all the properties of Definition 1.12 for $f_i : [\lambda_i X_i^1, w_i] \to [\lambda_i X_i^2, f_i(w_i)]$ follow from the ones of the original sequence, except for condition (2), which follows from Step 1 in the proof of Lemma 6.4. □

Proposition 6.8. *Let* $(X_i^1 j, d_i^j, \mathfrak{m}_i^j, p_i^j)$, $j \in \{1, 2, 3\}$ *be sequences of pointed* $\mathrm{RCD}(K, N)$ *spaces, and* $f_i : [X_i^1, p_i^1] \to [X_i^2, p_i^2]$, $h_i : [X_i^2, p_i^2] \to [X_i^3, p_i^3]$ *be sequences of GS maps. Then* $h_i \circ f_i : [X_i^1, p_i^1] \to [X_i^3, p_i^3]$ *is GS. Moreover, if* f_i *converges weakly to* f *and* h_i *converges weakly to* h, *then* $h_i f_i$ *converges weakly to* hf.

Proof. Let $U_i^1 \subset X_i^1$, $U_i^2 \subset X_i^2$, $V_i^2 \subset X_i^2$, $V_i^3 \subset X_i^3$ be given by Definition 1.12 applied to f_i and h_i respectively. Set $W_i' := U_i^1 \cap f_i^{-1}(U_i^2 \cap V_i^2 \cap h_i^{-1}(V_i^3))$, $\chi_i := 1 - \chi_{W_i'}$, and for a sequence $\delta_i \to 0$, define

$$W_i := \{x \in U_i^1 \mid \mathrm{Mx}(\chi_i)(x) \leq \delta_i\}.$$

By Proposition 2.5(1), if $\delta_i \to 0$ slowly enough, the sets $W_i \subset X_i^1$ have asymptotically full measure, so from Lemma 6.4, $S_i^1 := W_i \cap B_1(p_i^1)$ and $h_i f_i$ satisfy condition (2) from Definition 1.12. By Step 2 in the proof of Lemma 6.4 (applied to both f_i and f_i^{-1}), there is $\eta_i \to 0$ such that for all $r < 10$, $a, b \in W_i'$ with $d(a, b) \leq 2r$, one has

$$\mathrm{dt}_r(f_i)(a, b), \mathrm{dt}_r(h_i)(f_i a, f_i b) \leq \eta_i \cdot d(a, b).$$

This implies that W_i consists of essential continuity points of $h_i f_i$ provided $\delta_i, \eta_i \leq \frac{1}{2}$. Also, for $x \in W_i$, $r \leq \frac{1}{10}$, set $Z = B_r(x) \cap W_i'$. Then

$$\begin{aligned}
\frac{1}{r} \fint_{B_r(x)^{\times 2}} \mathrm{dt}_r(h_i f_i)\, d(\mathfrak{m}_i^1 \times \mathfrak{m}_i^1) &\leq 2 \frac{\mathfrak{m}_i^1(B_r(x) \backslash Z)}{\mathfrak{m}_i^1(B_r(x))} + \frac{1}{\mathfrak{m}_i^1(B_r(x))^2} \int_{Z^{\times 2}} \frac{\mathrm{dt}_r(h_i f_i)}{r}\, d(\mathfrak{m}_i^1 \times \mathfrak{m}_i^1) \\
&\leq 2\delta_i + \fint_{Z^{\times 2}} \frac{\mathrm{dt}_r(h_i)(f_i \cdot, f_i \cdot) + \mathrm{dt}_r(f_i)(\cdot, \cdot)}{r}\, d(\mathfrak{m}_i^1 \times \mathfrak{m}_i^1) \\
&\leq 2\delta_i + 4\eta_i.
\end{aligned}$$

This shows that $h_i f_i$ satisfies condition (3)(d) from Definition 1.12, with $r \leq \frac{1}{10}$ instead of $r \leq 1$. Identical arguments show that $f_i^{-1} h_i^{-1}$ also satisfy the corresponding properties in Definition 1.12. Conditions (1) and (3)(b) for $h_i f_i$ follow from the corresponding conditions for h_i and f_i.

Notice that the proof of Lemma 6.4 still goes through if we replace 1 by $\frac{1}{10}$ in (3)(d). In particular, by Step 6, if we replace W_i by $W_i \cap B_{R_i}(p_i^1)$ for some sequence $R_i \to \infty$ diverging slowly enough, then (3)(d) holds for $r \in \left[\frac{1}{10}, 1\right]$ as well, and the sequence $h_i f_i$ is GS.

To verify the last claim, let $(X^j, d_j, \mathfrak{m}^j, p^j)$ be pointed RCD(K, N) spaces and $\varphi_i^j : X_i^j \to X^j \cup \{*\}$ Gromov–Hausdorff approximations for $j \in \{1, 2, 3\}$. Then by hypothesis, there are $\varepsilon_i \to 0$ and sets $A_i^j \subset X_i^j$ with $j \in \{1, 2\}$ having asymptotically full measure such that for $x \in A_i^1$, $y \in A_i^2$ one has

$$d(\varphi_i^2 f_i(x), f_\infty \varphi_i^1(x)), d(\varphi_i^3 h_i(y), h_\infty \varphi_i^2(y)) \leq \varepsilon_i.$$

Then by Proposition 6.6, the sets $A_i := A_i^1 \cap f_i^{-1}(A_i^2)$ have asymptotically full measure, and for all $x \in A_i$, one has

$$d(\varphi_i^3 h_i f_i(x), h_\infty f_\infty \varphi_i^1(x)) \leq d(\varphi_i^3 h_i f_i(x), h_\infty \varphi_i^2 f_i(x)) + d(h_\infty \varphi_i^2 f_i(x), h_\infty f_\infty \varphi_i^1(x)) \leq 2\varepsilon_i,$$

so $h_i f_i$ converges weakly to $h_\infty f_\infty$. □

7. Construction of GS maps

In this section we follow [Kapovitch and Wilking 2011] closely in order to construct GS maps out of RLFs of suitable functions. Specifically, Lemmas 7.1 and 7.7 are adaptations of Lemmas 3.8 and 3.6 of [Kapovitch and Wilking 2011], respectively. Both proofs make heavy use of the estimates obtained in Section 5.

Roughly speaking, Lemma 7.1 establishes that if a sequence of RCD spaces X_i converges to a space of the form $\mathbb{R}^k \times Y$, and the universal covers $\widetilde{X}_i$ converge to $\mathbb{R}^k \times \widetilde{Y}$, then any translation on the first factor of $\mathbb{R}^k \times \widetilde{Y}$ is a limit of GS maps $f_i : \widetilde{X}_i \to \widetilde{X}_i$ which are lifts of maps $X_i \to X_i$ homotopic to the identity Id_{X_i}. These maps are constructed via RLFs of the gradient vector fields of the δ-splittings given by Lemma 2.34.

Lemma 7.1. *Let $(X_i, d_i, \mathfrak{m}_i, p_i)$ be a sequence of* $\mathrm{RCD}\left(-\frac{1}{i}, N\right)$ *spaces,*

$$\rho_i : (\widetilde{X}_i, \tilde{d}_i, \tilde{\mathfrak{m}}_i, \tilde{p}_i) \to (X_i, d_i, \mathfrak{m}_i, p_i)$$

their universal covers, (Y, y), $(\widetilde{Y}, \tilde{y})$ a pair of pointed metric spaces, and a closed group $\Gamma \leq \mathrm{Iso}(\mathbb{R}^k \times \widetilde{Y})$ that acts trivially on the $\mathbb{R}^k$ factor with $\widetilde{Y}/\Gamma = Y$. Assume the sequences (X_i, p_i) and $(\widetilde{X}_i, \tilde{p}_i)$ converge in the pointed Gromov–Hausdorff sense to $(\mathbb{R}^k \times Y, (0, y))$ and $(\mathbb{R}^k \times \widetilde{Y}, (0, \tilde{y}))$, respectively, and the sequence of groups $\pi_1(X_i)$ converges to Γ. Let $\tilde{\varphi}_i : \widetilde{X}_i \to \mathbb{R}^k \times \widetilde{Y} \cup \{*\}$, $\varphi_i : X_i \to \mathbb{R}^k \times Y \cup \{*\}$ be the Gromov–Hausdorff approximations given by Theorem 2.31. Then for all $s \in \mathbb{R}^k$, there is a sequence $f_i : [\widetilde{X}_i, \tilde{p}_i] \to [\widetilde{X}_i, \tilde{p}_i]$ of deck type GS maps with $(f_i)_* = \mathrm{Id}_{\pi_1(X_i)}$, and such that f_i converges weakly to the map $\bar{s} : \mathbb{R}^k \times \widetilde{Y} \to \mathbb{R}^k \times \widetilde{Y}$, where $\bar{s}(x, y) := (x + s, y)$.

Proof. Notice that by replacing Y by $s^\perp \times Y$, where $s^\perp \leq \mathbb{R}^k$ denotes the orthogonal complement of s, we can assume $k = 1$ and $s > 0$. By Lemma 2.34, there are $\delta_i \to 0$, $R_i \to \infty$, and a sequence of $L(N)$-Lipschitz functions $h^i \in H^{1,2}(X_i)$ such that

- ∇h^i is divergence free in $B_{R_i}(p_i)$,

- for all $r \in [1, R_i]$, one has

$$\fint_{B_r(p_i)} \big[\big||\nabla h^i|^2 - 1\big| + |\nabla\nabla h^i|^2\big]\, d\mathfrak{m}_i \le \delta_i^2,$$

- for all $x \in B_{R_i}(p_i)$, one has

$$d(h^i(x), \pi(\varphi_i(x))) \le \delta_i,$$

where $\pi : \mathbb{R} \times Y \to \mathbb{R}$ is the projection.

Set $\tilde{h}^i : \widetilde{X}_i \to \mathbb{R}$ as $\tilde{h}^i := h^i \circ \rho_i$, and $\tilde{\pi} : \mathbb{R} \times \widetilde{Y} \to \mathbb{R}$ as $\tilde{\pi} := \pi \circ \rho$. By (2.32) one gets for $x \in B_{R_i}(\tilde{p}_i)$, after possibly updating δ_i and R_i, that

$$\begin{aligned} d(\tilde{h}^i(x), \tilde{\pi}(\tilde{\varphi}_i(x))) &\le d\big(h^i(\rho_i(x)), \pi(\varphi_i(\rho_i(x)))\big) + d\big(\pi(\varphi_i(\rho_i(x))), \pi(\rho(\tilde{\varphi}_i(x)))\big) \\ &\le \delta_i + \delta_i. \end{aligned}$$

Then by Proposition 2.7 one gets, after possibly updating δ_i and R_i, that

- $\nabla\tilde{h}^i$ is divergence free in $B_{R_i}(\tilde{p}_i)$,
- for all $r \in [1, R_i]$, one has

$$\fint_{B_r(\tilde{p}_i)} \big[\big||\nabla \tilde{h}^i|^2 - 1\big| + |\nabla\nabla \tilde{h}^i|^2\big]\, d\tilde{\mathfrak{m}}_i \le \delta_i^2,$$

- for all $x \in B_{R_i}(\tilde{p}_i)$, one has

$$d(\tilde{h}^i(x), \tilde{\pi}(\tilde{\varphi}_i(x))) \le \delta_i. \tag{7.2}$$

Set $V_i := s\nabla\tilde{h}^i$, $\boldsymbol{X}^i : [0, 1] \times \widetilde{X}_i \to \widetilde{X}_i$ the corresponding RLF, and $f_i := \boldsymbol{X}^i_1$. For $r \ge 1$, and i large enough, using the Cauchy-Schwarz inequality and Proposition 2.5(2), we have

$$\begin{aligned} \fint_{B_r(\tilde{p}_i)} \left[\int_0^1 \mathrm{Mx}_4(|\nabla V_i|)(\boldsymbol{X}^i_t(x))\, dt\right] d\widetilde{\mathfrak{m}}_i(x) &= \int_0^1 \left[\fint_{\boldsymbol{X}^i_t(B_r(\tilde{p}_i))} \mathrm{Mx}_4(|\nabla V_i|)\, d\widetilde{\mathfrak{m}}_i\right] dt \\ &\le C(N, s, r) \fint_{B_{r+sL}(\tilde{p}_i)} \mathrm{Mx}_4(|\nabla V_i|)\, d\widetilde{\mathfrak{m}}_i \\ &\le C(N, s, r) \sqrt{\fint_{B_{r+sL}(\tilde{p}_i)} \mathrm{Mx}_4(|\nabla V_i|)^2\, d\widetilde{\mathfrak{m}}_i} \\ &\le C(N, s, r)\delta_i. \end{aligned} \tag{7.3}$$

Set

$$U_i' := \left\{ x \in \widetilde{X}_i \;\middle|\; \int_0^1 \mathrm{Mx}_4(|\nabla V_i|)(\boldsymbol{X}^i_t(x))\, dt \le \sqrt{\delta_i} \right\},$$

and let $U_i \subset U_i'$ be the density points of U_i'. From (7.3), the sets U_i have asymptotically full measure. By Theorem 1.5, for i large enough, U_i consists of points of essential stability of $\boldsymbol{X}^i$, and hence of essential continuity of f_i. To verify part (2) of Definition 1.12, we notice that for all i we have

$$f_i(B_1(\tilde{p}_i)) \subset B_{1+sL}(\tilde{p}_i).$$

Applying Proposition 5.19 to the points in U_i, we see that part (3)(d) of Definition 1.12 holds. The corresponding properties for f_i^{-1} follow by identical arguments applied to the reverse flow, so we get that the maps f_i are good at all scales and converge, by Lemma 6.4, to a measure preserving isometry $f_\infty : \mathbb{R}\times\widetilde{Y} \to \mathbb{R}\times\widetilde{Y}$. It remains to show that f_∞ coincides with the translation $\bar{s}$.

For $q \in \mathbb{R}\times\widetilde{Y}$ with $d(q,(0,y)) < R$, choose $q_i \in U_i \cap f_i^{-1}(U_i)$ converging to q, and $\eta < \frac{1}{4}$. Then, for i large enough we have

$$\begin{aligned}\fint_{B_\eta(q_i)}\left|\int_0^1 |V_i|(\boldsymbol{X}_t^i(x))\,dt - s\right| d\widetilde{\mathfrak{m}}_i(x) &\le s\fint_{B_\eta(q_i)}\left[\int_0^1 \big||\nabla h_i|-1\big|(\boldsymbol{X}_t^i(x))\,dt\right] d\widetilde{\mathfrak{m}}_i(x)\\ &\le s\int_0^1\left[\fint_{\boldsymbol{X}_t^i(B_\eta(q_i))}\big||\nabla h_i|-1\big|\,d\widetilde{\mathfrak{m}}_i\right]dt\\ &\le C(N,R,s,\eta)\fint_{B_{\eta+R+sL}(p_i)}\big||\nabla h_i|-1\big|\,d\widetilde{\mathfrak{m}}_i\\ &\le C(N,R,s,\eta)\sqrt{\fint_{B_{\eta+R+sL}(p_i)}\big||\nabla h_i|^2-1\big|d\widetilde{\mathfrak{m}}_i}\\ &\le C(N,R,s,\eta)\delta_i.\end{aligned}$$

Hence, from the derivative formula [Deng 2020, Proposition 3.6], and using the fact that U_i have asymptotically full measure, we have

$$\fint_{B_\eta(q_i)\cap U_i}\max\{d(f_i(x),x)-s,0\}\,d\widetilde{\mathfrak{m}}_i(x) \le C(N,R,s,\eta)\delta_i. \tag{7.4}$$

From Step 2 of Lemma 6.4, we know that for i large enough, $d(f_i(x),x)$ varies by at most 5η for $x \in B_\eta(q_i)\cap U_i$. Since η was arbitrary, (7.4) implies that

$$d(f_\infty(q),q) = \lim_{i\to\infty} d(f_i(q_i),q_i) \le s. \tag{7.5}$$

Similarly, by the definition of RLF, if $\eta < \frac{1}{4}$,

$$\begin{aligned}\fint_{B_\eta(q_i)}\big|(\tilde{h}_i(f_i(x))-\tilde{h}_i(x))-s\big|\,d\widetilde{\mathfrak{m}}_i(x) &\le s\fint_{B_\eta(q_i)}\left[\int_0^1\big||\nabla\tilde{h}_i|^2-1\big|(\boldsymbol{X}_t^i(x))\,dt\right]d\widetilde{\mathfrak{m}}_i(x)\\ &\le C(N,s,R,\eta)\cdot\delta_i^2.\end{aligned}$$

Then, as η was arbitrary, from (7.2) we get

$$\tilde{\pi}(f_\infty q) - \tilde{\pi}(q) = \lim_{i\to\infty}[\tilde{h}_i(f_i(q_i)) - \tilde{h}_i(q_i)] = s. \tag{7.6}$$

Since $\bar{s}$ is the only map $\mathbb{R}\times\widetilde{Y} \to \mathbb{R}\times\widetilde{Y}$ satisfying (7.5) and (7.6), we get $f_\infty = \bar{s}$. □

Lemma 7.7 gives another way of constructing GS maps. One needs a sequence of vector fields V_i and for each i a point x_i of essential stability of the flow of V_i. If one has enough control on the covariant derivative ∇V_i along the trajectory of x_i, then after blowing up around x_i, one obtains GS maps as the endpoint maps of the flows of the vector fields V_i.

Lemma 7.7. *Let* $(X_i, d_i, \mathfrak{m}_i)$ *be a sequence of* $\mathrm{RCD}(-(N-1), N)$ *spaces*, $V_i \in L^1([0,1]; H^{1,2}_{C,s}(TX_i))$ *a sequence of piecewise constant on time, divergence-free vector fields*, $\boldsymbol{X}^i : [0,1] \times X_i \to X_i$ *their RLFs, and* $x_i \in X_i$ *a sequence such that* x_i *is a point of essential stability of* $\boldsymbol{X}^i$, *and*

$$\int_0^1 Mx(|\nabla V_i(t)|^{3/2})^{2/3}(\boldsymbol{X}^i_t(x_i))\,dt = \varepsilon_i.$$

If $\varepsilon_i \to 0$, *then for all* $\lambda_i \to \infty$, *the sequence of maps*

$$\boldsymbol{X}^i_1 : [\lambda_i X_i, x_i] \to [\lambda_i X_i, \boldsymbol{X}^i_1(x_i)]$$

has the GS property.

Proof. For $r \le \frac{1}{4}$, let

$$A^i_r := \{y \in B_r(x_i) \mid \boldsymbol{X}^i_t(y) \in B_{2r}(\boldsymbol{X}^i_t(x_i)) \text{ for all } t \in [0,1]\}.$$

By Corollary 5.13 and Proposition 5.14, we have, for i large enough,

$$\mathfrak{m}_i(B_r(\boldsymbol{X}_t(x_i))) \le 2\mathfrak{m}_i(B_r(x_i)). \tag{7.8}$$

$$\mathfrak{m}_i(A^i_r) \ge (1 - C(N)\varepsilon_i)\mathfrak{m}_i(B_r(x_i)), \tag{7.9}$$

Also, using (7.8) and Proposition 2.5(3),

$$\begin{aligned}\fint_{A^i_r}\int_0^1 \mathrm{Mx}_{1/2}(|\nabla V_i(t)|)(\boldsymbol{X}^i_t(y))\,dt\,d\mathfrak{m}_i(y) &= \int_0^1 \fint_{\boldsymbol{X}^i_t(A^i_r)} \mathrm{Mx}_{1/2}(|\nabla V_i(t)|)(y)\,d\mathfrak{m}_i(y)\,dt\\ &\le C(N)\int_0^1 \fint_{B_{2r}(\boldsymbol{X}^i_t(x_i))} \mathrm{Mx}_{1/2}(|\nabla V_i(t)|)(y)\,d\mathfrak{m}_i(y)\,dt\\ &\le C(N)\int_0^1 \mathrm{Mx}_{1/2}(\mathrm{Mx}_{1/2}(|\nabla V_i(t)|))(\boldsymbol{X}^i_t(x_i))\,dt\\ &\le C(N)\int_0^1 \mathrm{Mx}(|\nabla V_i(t)|^{3/2})^{2/3}(\boldsymbol{X}^i_t(x_i))\,dt\\ &\le C(N)\varepsilon_i.\end{aligned} \tag{7.10}$$

Let $U_i(r)$ be the density points of the set

$$\left\{ y \in A^i_r \;\middle|\; \int_0^1 \mathrm{Mx}_{1/2}(|\nabla V_i(t)|)(\boldsymbol{X}^i_t(y))\,dt \le \sqrt{\varepsilon_i} \right\}.$$

By Theorem 1.5, the set $U_i(r)$ consists of points of essential stability of $\boldsymbol{X}^i$ for i large enough. From (7.9) and (7.10), we have

$$\mathfrak{m}_i(U_i(r)) \ge (1 - C(N)\sqrt{\varepsilon_i})\mathfrak{m}_i(B_r(x_i)). \tag{7.11}$$

Given $\lambda_i \to \infty$ and $r_i \to 0$, by (7.11) and Theorem 2.1, if $\lambda_i r_i \to \infty$ slowly enough, the sets $U_i(r_i)$ have asymptotically full measure in the spaces $(X_i, \lambda_i d_i, \mathfrak{m}_i, x_i)$. By Proposition 5.19, for $y \in U_i(r_i)$, $r < 1/\lambda_i$, we get

$$\fint_{B_r(y)^{\times 2}} \mathrm{dt}_r(\boldsymbol{X}^i_1)\,d(\mathfrak{m}_i \times \mathfrak{m}_i) \le C(N)\sqrt{\varepsilon_i}\,r,$$

verifying part (3)(d) of Definition 1.12. The analogue properties for $\boldsymbol{X}^i_{-1} : [\lambda_i X_i, \boldsymbol{X}^i_1(x_i)] \to [\lambda_i X_i, x_i]$ follow from an identical argument. Property (2) of Definition 1.12 follows from the definition of essential stability. □

Definition 7.12. Let X be a geodesic space, $\rho : Y \to X$ a covering map, and $\varphi : [0, T] \times X \to X$ be a function such that for each $x \in X$, the map $t \mapsto \varphi(t, x)$ is continuous, and $\varphi(0, x) = x$. The *lift* of φ is defined to be the unique map $\psi : [0, T] \times Y \to Y$ such that for each $y \in Y$, the map $t \mapsto \psi(t, y)$ is continuous, $\psi(0, y) = y$, and $\rho(\psi(t, y)) = \varphi(t, y)$ for all $t \in [0, T]$.

Notice that if Y is the universal cover of X, then ψ is a deck type map with $\psi_* = \mathrm{Id}_{\pi_1(X)}$.

Proposition 7.13. *Let $(X, d, \mathfrak{m}, p)$ be a pointed* RCD$(-(N-1), N)$ *space, $(\widetilde{X}, \tilde{d}, \widetilde{\mathfrak{m}}, \tilde{p})$ its universal cover, $V \in L^1([0, T]; L^2(TX))$ a vector field satisfying the conditions of Theorem 2.39, $\boldsymbol{X} : [0, T] \times X \to X$ its RLF, and $\widetilde{V} : [0, T] \to L^2_{\mathrm{loc}}(T\widetilde{X})$ its lift. Then $\widetilde{\boldsymbol{X}} : [0, T] \times \widetilde{X} \to \widetilde{X}$, the lift of $\boldsymbol{X}$, is the RLF of $\widetilde{V}$. Moreover, if p is a point of essential stability of $\boldsymbol{X}$, then $\tilde{p}$ is a point of essential stability of $\widetilde{\boldsymbol{X}}$.*

Proof. Let $\rho : \widetilde{X} \to X$ be the projection. $R.1$ holds by construction. To verify $R.2$, notice that by linearity, it is enough to check it for $\tilde{f} \in \mathrm{TestF}(\widetilde{X})$ supported in a ball $\tilde{B} \subset \widetilde{X}$ sent isomorphically as a metric measure space to a ball $B = \rho(\tilde{B})$. For such $\tilde{f}$, it induces a function $f \in \mathrm{TestF}(X)$ supported in B with $\tilde{f}|_{\tilde{B}} = f \circ \rho|_{\tilde{B}}$. Then $R.2$ holds for $\widetilde{\boldsymbol{X}}$ and $\tilde{f}$ since it holds for $\boldsymbol{X}$ and f by locality of (1.3).

To verify $R.3$, consider a Borel partition $\{E_k\}_{k\in\mathbb{N}}$ of $\widetilde{X}$ consisting of subsets sent isomorphically by ρ as metric measure spaces to subsets of X. For a Borel set $A \subset \widetilde{X}$, and $t \in [0, T]$, setting

$$A_{k,\ell} := A \cap E_k \cap \widetilde{\boldsymbol{X}}_t^{-1}(E_\ell),$$

and using that $\boldsymbol{X}$ satisfies $R.3$, we get

$$\begin{aligned}\widetilde{\mathfrak{m}}(\widetilde{\boldsymbol{X}}_t(A)) &= \sum_{k,\ell\in\mathbb{N}} \widetilde{\mathfrak{m}}(\widetilde{\boldsymbol{X}}_t(A_{k,\ell})) = \sum_{k,\ell\in\mathbb{N}} \mathfrak{m}(\boldsymbol{X}_t(\rho(A_{k,\ell}))) \\ &\le \sum_{k,\ell\in\mathbb{N}} C\mathfrak{m}(\rho(A_{k,\ell})) = \sum_{k,\ell} C\widetilde{\mathfrak{m}}(A_{k,\ell}) = C\widetilde{\mathfrak{m}}(A),\end{aligned}$$

and hence $\widetilde{X}$ is the RLF of $\widetilde{V}$.

Now assume p is a point of essential stability. Let $R \ge 1$ be such that $\boldsymbol{X}([0, T] \times \{p\}) \subset B_R(p)$. By Proposition 2.9, there is $r_0 \le \frac{1}{10}$ such that any two curves $\alpha, \beta : [a, b] \to B_{2R}(p)$ sharing endpoints and at uniform distance $\le 10r_0$, are homotopic relative to their endpoints. Then for each $t \in [0, T]$, the ball $B_{2r_0}(\widetilde{\boldsymbol{X}}_t(\tilde{p}))$ is isomorphic as a metric measure space to $B_{2r_0}(\boldsymbol{X}_t(p))$, so for $r \le r_0$ small enough one has

$$\frac{1}{M}\widetilde{\mathfrak{m}}(B_r(\tilde{p})) \le \widetilde{\mathfrak{m}}(B_r(\widetilde{\boldsymbol{X}}_t(\tilde{p}))) \le M\widetilde{\mathfrak{m}}(B_r(\tilde{p})) \quad \text{for all } t \in [0, T].$$

By hypothesis, for $r \le r_0$ small enough, there is $A_r \subset B_r(p)$ with

$$\mathfrak{m}(A_r) \ge \tfrac{1}{2}\mathfrak{m}(B_r(p)), \quad \text{and} \quad \boldsymbol{X}_t(A_r) \subset B_{2r}(\boldsymbol{X}_t(p)) \quad \text{for all } t \in [0, T].$$

Then if $\tilde{A}_r \subset \widetilde{X}$ denotes the intersection of the preimage of A_r with $B_r(\tilde{p})$, one has

$$\tilde{\mathfrak{m}}(\tilde{A}_r) \ge \tfrac{1}{2}\tilde{\mathfrak{m}}(B_r(\tilde{p})), \quad \text{and} \quad \widetilde{\boldsymbol{X}}_t(\tilde{A}_r) \subset B_{2r}(\widetilde{\boldsymbol{X}}_t(\tilde{p})) \quad \text{for all } t \in [0, T].$$ □

8. Rescaling theorem

In this section we prove the following result, following the lines of [Kapovitch and Wilking 2011, Section 5].

Theorem 8.1. *For each i, let $(X_i, d_i, \mathfrak{m}_i, p_i)$ be an $\mathrm{RCD}\big(-\frac{1}{i}, N\big)$ space of rectifiable dimension n, $(\widetilde{X}_i, \tilde{d}_i, \widetilde{\mathfrak{m}}_i, \tilde{p}_i)$ be its universal cover, and define $\Gamma_i := \mathcal{G}(\pi_1(X_i), \widetilde{X}_i, \tilde{p}_i, 1)$. If (X_i, p_i) converges in the pointed Gromov–Hausdorff sense to $(\mathbb{R}^k, 0)$ with $k < n$, then, after taking a subsequence, there are sets $\Theta_i \subset B_{1/2}(p_i)$ such that $\mathfrak{m}_i(\Theta_i)/\mathfrak{m}_i(B_{1/2}(p_i)) \to 1$ as $i \to \infty$, a sequence $\lambda_i \to \infty$, and a compact space (Y, y) such that $Y \neq \{*\}$, $\mathrm{diam}(Y) \leq \frac{1}{10}$, and*

(1) *for all $x_i \in \Theta_i$, after taking a subsequence, $(\lambda_i X_i, x_i)$ converges to $(\mathbb{R}^k \times Y, (0, y_1))$ in the pointed Gromov–Hausdorff sense (y_1 may depend on the x_i, but Y doesn't), and, for any lift $\tilde{x}_i \in B_{1/2}(\tilde{p}_i)$,*

$$\Gamma_i = \mathcal{G}(\pi_1(X_i), \lambda_i X_i, x_i, 1),$$

(2) *for all $a_i, b_i \in \Theta_i$ and lifts $\tilde{a}_i, \tilde{b}_i \in B_{1/2}(\tilde{p}_i)$, there are sequences*

$$h_i : [\lambda_i X_i, a_i] \to [\lambda_i X_i, b_i],$$
$$f_i : [\lambda_i \widetilde{X}_i, \tilde{a}_i] \to [\lambda_i \widetilde{X}_i, \tilde{b}_i]$$

of maps with the GS property such that the f_i are deck maps with $(f_i)_* \in (\Gamma_i)_*$ for all i, where $(\Gamma_i)_* := \{g_* : \pi_1(X_i) \to \pi_1(X_i) \mid g \in \Gamma_i\}$.

Lemma 8.2. *For each $N \geq 1$, $L \geq 1$, there are $R > 1$, $\delta \leq \frac{1}{100}$, such that the following holds. If $r \leq 1$, $(X, d, \mathfrak{m}, p)$ is a pointed $\mathrm{RCD}(-\delta, N)$ space with $d_{GH}((r^{-1}X, p), (\mathbb{R}^k, 0)) \leq \delta$, and $f \in H^{1,2}(X; \mathbb{R}^k)$ is an L-Lipschitz function with $f(p) = 0$ such that ∇f_j is divergence free in $B_R(p)$ for each $j \in \{1, \ldots, k\}$, and for all $s \in [r, R]$, one has*

$$\fint_{B_s(p)} \Bigg[\sum_{j_1, j_2 = 1}^{k} |\langle \nabla f_{j_1}, \nabla f_{j_2} \rangle - \delta_{j_1, j_2}| + \sum_{j=1}^{k} |\nabla\nabla(f_j)|^2 \Bigg] d\mathfrak{m} \leq \delta.$$

Then:

(1) *For all $x_1, x_2 \in B_{10^k r}(p)$, and $r_1, r_2 \in \big[\frac{r}{4}, (10^k + 1)r\big]$, one has*

$$\mathfrak{m}(B_{r_1}(x_1)) \leq 2 \cdot \frac{r_1^k}{r_2^k} \cdot \mathfrak{m}(B_{r_2}(x_2)).$$

(2) *For all $x \in B_{10^k r}(p)$, if $\boldsymbol{X} : [0, 1] \times X \to X$ denotes the RLF of the vector field*

$$V_x := -\sum_{j=1}^{k} f_j(x) \nabla f_j, \tag{8.3}$$

then there is a set $A \subset B_{r/10}(x)$ of points of essential stability of $\boldsymbol{X}$ with

$$\mathfrak{m}(A) \geq \tfrac{1}{2}\mathfrak{m}(B_{r/10}(x)) \quad \textit{and} \quad \boldsymbol{X}_1(A) \subset B_{r/5}(p). \tag{8.4}$$

Proof. By replacing X by $r^{-1}X$ and f by $r^{-1}f$, we can assume $r=1$, and without loss of generality we can also assume $(X, d, \mathfrak{m}, p)$ is normalized. Arguing by contradiction, we get sequences $R_i \to \infty$, $\delta_i \to 0$, a sequence $(X_i, d_i, \mathfrak{m}_i, p_i)$ of normalized RCD$(-\delta_i, N)$ spaces for which (X_i, p_i) converges to $(\mathbb{R}^k, 0)$ in the pointed Gromov–Hausdorff sense and L-Lipschitz functions $f^i \in H^{1,2}(X_i; \mathbb{R}^k)$ with $f^i(p_i) = 0$ such that

- ∇f_i^i is divergence free in $B_{R_i}(p_i)$ for each $j \in \{1, \dots, k\}$, $i \in \mathbb{N}$,
- for all $s \in [1, R_i]$, one has

$$\fint_{B_s(p_i)} \left[\sum_{j_1, j_2=1}^{k} |\langle \nabla f_{j_1}^i, \nabla f_{j_2}^i \rangle - \delta_{j_1, j_2}| + \sum_{j=1}^{k} |\nabla\nabla(f_j^i)|^2 \right] d\mathfrak{m}_i \leq \delta_i.$$

And for each i, at least one of the conditions (1) or (2) fails. Notice however, that (1) holds as by Corollary 2.26, $(X_i, d_i, \mathfrak{m}_i, p_i)$ converges to $(\mathbb{R}^k, d^{\mathbb{R}^k}, \mathcal{H}^k, 0)$ in the pointed measured Gromov–Hausdorff sense. For a sequence $x_i \in B_{10^k}(p_i)$, let

$$V^i := -\sum_{j=1}^{k} f_j^i(x_i) \nabla f_j^i,$$

and $\boldsymbol{X}^i : [0, 1] \times X_i \to X_i$ its RLF. Then, for $s = (k+1) \cdot 10^k L^2$ and i large enough,

$$\begin{aligned}
&\fint_{B_{1/10}(x_i)} \int_0^1 \sum_{j_1, j_2=1}^{k} |\langle \nabla f_{j_1}^i, \nabla f_{j_2}^i \rangle - \delta_{j_1, j_2}|(\boldsymbol{X}_t^i(y))\, dt\, d\mathfrak{m}_i(y) \\
&\qquad = \int_0^1 \fint_{\boldsymbol{X}_t^i(B_{1/10}(x_i))} \sum_{j_1, j_2=1}^{k} |\langle \nabla f_{j_1}^i, \nabla f_{j_2}^i \rangle - \delta_{j_1, j_2}|(y)\, d\mathfrak{m}_i(y)\, dt \\
&\qquad \leq \frac{\mathfrak{m}_i(B_s(p_i))}{\mathfrak{m}_i(B_{1/10}(x_i))} \fint_{B_s(p_i)} \sum_{j_1, j_2=1}^{k} |\langle \nabla f_{j_1}^i, \nabla f_{j_2}^i \rangle - \delta_{j_1, j_2}|(y)\, d\mathfrak{m}_i(y) \\
&\qquad \leq C(N, L)\delta_i.
\end{aligned}$$

Then, from the definition of RLF,

$$\begin{aligned}
&\fint_{B_{1/10}(x_i)} |(f^i(\boldsymbol{X}_1^i(y)) - f^i(y)) + f^i(x_i)|\, d\mathfrak{m}_i(y) \\
&\quad \leq \sum_{j=1}^{k} \fint_{B_{1/10}(x_i)} |(f_j^i(\boldsymbol{X}_1^i(y)) - f_j^i(y)) + f_j^i(x_i)|\, d\mathfrak{m}_i(y) \\
&\quad \leq \sum_{j=1}^{k} \fint_{B_{1/10}(x_i)} |f_j^i(x_i)| \int_0^1 \big| |\nabla f_j^i|^2 - 1 \big| (\boldsymbol{X}_t^i(y))\, dt\, d\mathfrak{m}_i(y) \\
&\qquad + \sum_{\substack{j_1, j_2=1 \\ j_1 \neq j_2}}^{k} \fint_{B_{1/10}(x_i)} |f_{j_1}^i(x_i)| \int_0^1 \langle \nabla f_{j_1}^i, \nabla f_{j_2}^i \rangle (\boldsymbol{X}_t^i(y))\, dt\, d\mathfrak{m}_i(y) \\
&\quad \leq C(N, L)\delta_i.
\end{aligned}$$

From the last assertion of Lemma 2.36, $|f^i(y) - f^i(x_i)| \le \frac{3}{20}$ for all $y \in B_{1/10}(x_i)$ if i is large enough, so the set

$$A_i' := \left\{ y \in B_{1/10}(x_i) \;\middle|\; |\boldsymbol{X}_1^i(y)| < \tfrac{1}{5} \right\}$$

satisfies $\mathfrak{m}_i(A_i')/\mathfrak{m}_i(B_{1/10}(x_i)) \to 1$. Then by Corollary 1.8, if we define A_i to be the points of A_i' that are of essential stability of $\boldsymbol{X}^i$, we get that $\mathfrak{m}_i(A_i) \ge \frac{1}{2}\mathfrak{m}_i(B_{1/10}(x_i))$ for i large enough, implying condition (2); a contradiction. □

Lemma 8.5. *For $N \ge 1$, $L \ge 1$, let $\delta > 0$ be given by Lemma 8.2. Then there are $R > 1$, $C_0 > 1$, $\varepsilon_0 > 0$ such that the following holds. Let $r \le 1$, $(X, d, \mathfrak{m}, p)$ be a pointed* RCD$(-\delta, N)$ *space with $d_{GH}((r^{-1}X, p), (\mathbb{R}^k, 0)) \le \delta$, and $f \in H^{1,2}(X; \mathbb{R}^k)$ an L-Lipschitz function with $f(p) = 0$ such that ∇f_j is divergence free in $B_R(p)$ for each $j \in \{1, \dots, k\}$, and*

$$Mx_R\left(\sum_{j_1, j_2 = 1}^{k} |\langle \nabla f_{j_1}, \nabla f_{j_2} \rangle - \delta_{j_1, j_2}| + \sum_{j=1}^{k} |\nabla\nabla(f_j)|^2 \right)(p) \le \varepsilon^2 \le \varepsilon_0^2.$$

For $x \in B_{10^k r}(p)$, let V_x be given by (8.3), *and let $\boldsymbol{X} : [0,1] \times X \to X$ be the RLF of V_x. Then there is a subset $B'_{r/2}(x) \subset B_{r/2}(x)$ of points of essential stability of $\boldsymbol{X}$ satisfying*

$$\boldsymbol{X}_1(B'_{r/2}(x)) \subset B_r(p), \tag{8.6}$$

$$\mathfrak{m}(B'_{r/2}(x)) \ge (1 - C_0 \varepsilon r)\mathfrak{m}(B_{r/2}(x)), \tag{8.7}$$

$$\frac{1}{\mathfrak{m}(B_{r/2}(x))} \int_{B'_{r/2}(x)} \int_0^1 Mx(|\nabla V_x|^{3/2})^{2/3}(\boldsymbol{X}_t(y))\, dt\, d\mathfrak{m}(y) \le C_0 \varepsilon r. \tag{8.8}$$

Proof. Set $s = (k+1) \cdot 10^k L^2 r$ and compute, provided $R \ge 2s + 8$,

$$\begin{aligned}
⨍_{B_{r/2}(x)} \int_0^1 \mathrm{Mx}_4(|\nabla V_x|^{3/2})^{2/3}(\boldsymbol{X}_t(y))\, dt\, d\mathfrak{m}(y) &= \int_0^1 ⨍_{\boldsymbol{X}_t(B_{r/2}(x))} \mathrm{Mx}_4(|\nabla V_x|^{3/2})^{2/3}\, d\mathfrak{m}\, dt \\
&\le \frac{1}{\mathfrak{m}(B_{r/2}(x))} \int_{B_s(p)} \mathrm{Mx}_4(|\nabla V_x|^{3/2})^{2/3}\, d\mathfrak{m} \\
&= \frac{\mathfrak{m}(B_s(p))}{\mathfrak{m}(B_{r/2}(x))} ⨍_{B_s(p)} \mathrm{Mx}_4(|\nabla V_x|^{3/2})^{2/3}\, d\mathfrak{m} \\
&\le C(N, L) \cdot \mathrm{Mx}_s(\mathrm{Mx}_4(|\nabla V_x|^{3/2})^{2/3})(p) \\
&\le C(N, L) \cdot \mathrm{Mx}_R(|\nabla V_x|^2)^{1/2}(p) \\
&\le C(N, L) \sum_{j=1}^{k} |f_j(x)|^2 \mathrm{Mx}_R(|\nabla\nabla f_j|^2)^{1/2}(p) \\
&\le C(N, L) \cdot \varepsilon \cdot r.
\end{aligned} \tag{8.9}$$

Combining this with Theorem 1.5, if

$$A_0 := \{ y \in B_{r/2}(x) \mid y \text{ is of essential stability of } \boldsymbol{X} \},$$

then

$$\mathfrak{m}(A_0) \ge (1 - C(N, L)\varepsilon r)\mathfrak{m}(B_{r/2}(x)). \tag{8.10}$$

From Lemma 8.2, if ε is small enough, there is a set $A \subset B_{r/10}(x) \cap A_0$ satisfying (8.4). By (8.9), there is $q \in A$ with $\int_0^1 \mathrm{Mx}(|\nabla V_x|)(\boldsymbol{X}_t(q))\, dt \le C(N, L)\varepsilon r$, and by Proposition 5.19, this implies

$$\fint_{B_r(q)^{\times 2}} \mathrm{dt}_r(1)\, d(\mathfrak{m} \times \mathfrak{m}) \le C(N, L)\varepsilon r^2,$$

so

$$\fint_{A \times A_0} \mathrm{dt}_r(1)\, d(\mathfrak{m} \times \mathfrak{m}) \le C(N, L)\varepsilon r^2.$$

Hence there is $y \in A$ such that

$$\fint_{A_0} \mathrm{dt}_r(1)(y, z)\, d\mathfrak{m}(z) \le C(N, L)\varepsilon r^2, \tag{8.11}$$

so we define

$$B'_{r/2}(x) := \{z \in A_0 \mid \mathrm{dt}_r(1)(y, z) < r/10\}.$$

Then for all $z \in B'_{r/2}(x)$ we have

$$\begin{aligned} d(\boldsymbol{X}_1(z), p) &\le d(\boldsymbol{X}_1(z), \boldsymbol{X}_1(y)) + d(\boldsymbol{X}_1(y), p) \\ &\le d(z, y) + r/10 + r/5 \\ &\le r/2 + r/10 + r/10 + r/5 < r, \end{aligned}$$

so (8.6) holds. (8.10) and (8.11) imply (8.7), and (8.9) implies (8.8). □

Lemma 8.12. *For $N \ge 1$, $L \ge 1$, let $\delta > 0$ be given by Lemma 8.2. Then there are $R \ge 1$, $C_0 \ge 1$, $\varepsilon_0 > 0$ such that the following holds. Assume $(X, d, \mathfrak{m}, p)$ is a pointed* RCD$(-\delta, N)$ *space with $d_{GH}((X, p), (\mathbb{R}^k, 0)) \le \delta$, and $f \in H^{1,2}(X; \mathbb{R}^k)$ is an L-Lipschitz function with $f(p) = 0$ such that ∇f_j is divergence free in $B_R(p)$ for each $j \in \{1, \dots, k\}$, and*

$$Mx_R\Bigg(\sum_{j_1, j_2=1}^{k} |\langle \nabla f_{j_1}, \nabla f_{j_2}\rangle - \delta_{j_1, j_2}| + \sum_{j=1}^{k} |\nabla\nabla(f_j)|^2\Bigg)(p) \le \varepsilon^2 \le \varepsilon_0^2.$$

Assume p is an n-regular point with $n > k$ and let

$$\rho \ge \sup\{r \in (0, 1] \mid d_{GH}((r^{-1}X, p), (\mathbb{R}^k, 0)) = \delta\}.$$

Then there is a set $G \subset B_1(p)$ with

$$\mathfrak{m}(G) \ge (1 - C_0\varepsilon)\mathfrak{m}(B_1(p)),$$

a finite number of divergence-free on $B_{100C_0}(p)$ vector fields

$$V_1, \dots, V_m \in L^1([0, 1]; H^{1,2}_{C,s}(TX))$$

with $\|V_j(t)\|_\infty \le C_0$ for all $t \in [0, 1]$, $j \in \{1, \dots, m\}$, with RLFs

$$\boldsymbol{X}^1, \dots, \boldsymbol{X}^m : [0, 1] \times X \to X,$$

and a measurable map $\theta : G \to \{1, \dots, m\}$ such that for all $y \in G$, y is a point of weak essential stability of $\boldsymbol{X}^{\theta(y)}$, $\boldsymbol{X}_1^{\theta(y)}(y) \in B_\rho(p)$, and

$$\fint_G \int_0^1 Mx(|\nabla V_{\theta(y)}(t)|^{3/2})^{2/3}(\boldsymbol{X}_t^{\theta(y)}(y))\, dt\, d\mathfrak{m}(y) \le C_0\varepsilon.$$

Proof. We will show that for each $r \leq 1$, there is $G_r \subset B_r(p)$ with

$$\mathfrak{m}(G_r) \geq (1 - C_0 \varepsilon r)\mathfrak{m}(B_r(p)),$$

a finite number of divergence-free on $B_{100C_0}(p)$ vector fields

$$W_1, \dots, W_m \in L^1([0,1]; H^{1,2}_{C,s}(TX))$$

with $\|W_j(t)\|_\infty \leq C_0 r$ for all $t \in [0,1]$, $j \in \{1, \dots, m\}$, with RLFs

$$\Phi^1, \dots, \Phi^m : [0,1] \times X \to X,$$

and a measurable map $\theta_r : G_r \to \{1, \dots, m\}$ satisfying that for all $y \in G_r$, y is a point of weak essential stability of $\Phi^{\theta_r(y)}$, $\Phi_1^{\theta_r(y)}(y) \in B_\rho(p)$, and

$$\fint_{G_r} \int_0^1 (\mathrm{Mx}(|\nabla W_{\theta_r(y)}(t)|)^{3/2})^{2/3}(\Phi_t^{\theta_r(y)})(y)\, dt\, d\mathfrak{m}(y) \leq C_0 \varepsilon r.$$

Clearly, the claim holds for $r \leq \rho$ with $G_r = B_r(p)$ and the zero vector field. Now we check that if the claim holds for some $r \leq 10^{-k}$, then it also holds for $10^k r$.

Choose $\{q_1, \dots, q_\ell\}$, a maximal $r/2$-separated set in $B_{10^k r}(p)$. By Lemma 8.2.(1), one has

$$\sum_{j=1}^{\ell} \mathfrak{m}(B_{r/2}(q_j)) \leq 2^{k+1} \sum_{j=1}^{\ell} \mathfrak{m}(B_{r/4}(q_j)) \leq 2^{k+3}\mathfrak{m}(B_{10^k r}(p)).$$

By Lemma 8.5, if ε is small enough and R is large enough, for each $j \in \{1, \dots, \ell\}$ there is a divergence-free vector field $\overline{W}_j \in H^{1,2}_{C,s}(TX)$ such that $\|\overline{W}_j\|_\infty \leq C(N,L)r$, with RLF

$$\overline{\Phi}^j : [0,1] \times X \to X,$$

and a set $B'_{r/2}(q_j)$ of points of essential stability of $\overline{\Phi}^j$ such that $\overline{\Phi}_1^j(B'_{r/2}(q_j)) \subset B_r(p)$, and

$$\mathfrak{m}(B'_{r/2}(q_j)) \geq (1 - C(N,L)\varepsilon r)\mathfrak{m}(B_{r/2}(q_j)),$$
$$\frac{1}{\mathfrak{m}(B_{r/2}(q_j))} \int_{B'_{r/2}(q_j)} \int_0^1 \mathrm{Mx}(|\nabla \overline{W}_j|^{3/2})^{2/3}(\overline{\Phi}_t^j(y))\, dt\, d\mathfrak{m}(y) \leq C(N,L)\varepsilon r.$$

Set

$$G_{10^k r} := B_{10^k r}(p) \cap \bigcup_{j=1}^{\ell} (B'_{r/2}(q_j) \cap ((\overline{\Phi}_1^j)^{-1}(G_r))).$$

Then

$$\begin{aligned}
\mathfrak{m}(G_{10^k r}) &\geq \mathfrak{m}(B_{10^k r}(p)) - \sum_{j=1}^{\ell} (\mathfrak{m}(B_{r/2}(q_j)) - \mathfrak{m}(B'_{r/2}(q_j)) + \mathfrak{m}(B_r(p)) - \mathfrak{m}(G_r)) \\
&\geq \mathfrak{m}(B_{10^k r}(p)) - \sum_{j=1}^{\ell} \big(C(N,L)\varepsilon r \mathfrak{m}(B_{r/2}(q_j)) + 2^{k+3} C_0 \varepsilon r \mathfrak{m}(B_{r/2}(q_j))\big) \\
&\geq \big(1 - 2^{k+2}(C(N,L)\varepsilon r + 2^{k+3} C_0 \varepsilon r)\big)\mathfrak{m}(B_{10^k r}(p)) \\
&\geq (1 - C_0 \varepsilon 10^k r)\mathfrak{m}(B_{10^k r}(p)),
\end{aligned}$$

provided C_0 was chosen large enough, depending on N and L. For each $y \in G_{10^k r}$, set V_y as follows: let $j \in \{1, \dots, \ell\}$ be the smallest index for which $y \in B'_{r/2}(q_j) \cap ((\overline{\Phi}^j_1)^{-1}(G_r))$. Then define

$$V_y(t) := \begin{cases} 2\overline{W}_j & \text{if } t \in \left[0, \frac{1}{2}\right) \\ 2W_{\theta_r(\overline{\Phi}^j_1(y))}(2t-1) & \text{if } t \in \left[\frac{1}{2}, 1\right]. \end{cases}$$

Notice that $\|V_y(t)\|_\infty \le \max\{C(N, L)r, 2C_0 r\} \le C_0 10^k r$ for all $t \in [0, 1]$, provided C_0 was chosen large enough. Set Ψ^y be the RLF of V_y, and set

$$B''_{r/2}(q_j) := G_{10^k r} \cap B'_{r/2}(q_j) \setminus \bigcup_{\alpha=1}^{j-1} B'_{r/2}(q_\alpha).$$

Then

$$\begin{aligned} &\fint_{G_{10^k r}} \int_0^1 \mathrm{Mx}(|\nabla V_y|^{3/2})^{2/3}(\Psi^y_t(y))\, dt\, d\mathfrak{m}(y) \\ &\qquad \le \frac{1}{\mathfrak{m}(B_{10^k r}(p))} \sum_{j=1}^{\ell} \int_{B''_{r/2}(q_j)} \int_0^1 \mathrm{Mx}(|\nabla V_y|^{3/2})^{2/3}(\Psi^y_t(t))\, dt\, d\mathfrak{m}(y) \\ &\qquad \le \frac{1}{\mathfrak{m}(B_{10^k r}(p))} \sum_{j=1}^{\ell} \big(C(N, L)\varepsilon r \mathfrak{m}(B_{r/2}(q_j)) + 2^{k+2} C_0 \varepsilon r \mathfrak{m}(B_{r/2}(q_j))\big) \\ &\qquad \le C_0 \varepsilon 10^k r, \end{aligned}$$

again provided C_0 was chosen large enough, depending on N and L. □

Proof of Theorem 8.1. By Lemma 2.34, there are sequences $\delta_i \to 0$, $R_i \to \infty$, and a sequence of $L(N)$-Lipschitz maps $h^i \in H^{1,2}(X_i; \mathbb{R}^k)$ with $h^i(p_i) = 0$ for all i, ∇h^i_j divergence free in $B_{R_i}(p_i)$ for each $j \in \{1, \dots, k\}$, and such that if

$$u_i := \sum_{j_1, j_2 = 1}^{k} |\langle \nabla h^i_{j_1}, \nabla h^i_{j_2} \rangle - \delta_{j_1, j_2}| + \sum_{j=1}^{k} |\nabla \nabla h^i_j|^2,$$

then for all $r \in [1, R_i]$, one has

$$\fint_{B_r(p_i)} u_i \, d\mathfrak{m}_i \le \delta_i^3.$$

Set $\Theta_i := \{x \in B_{1/2}(p_i) \mid x \text{ is } n\text{-regular}, \mathrm{Mx}(u_i) \le \delta_i^2\}$. By Proposition 2.5(1), $\mathfrak{m}_i(\Theta_i)/\mathfrak{m}_i(B_{1/2}(p_i)) \to 1$. Notice that, possibly after modifying δ_i and R_i, we may assume

$$\mathrm{Mx}_{R_i}(u_i)(x) \le \delta_i^2 \quad \text{for all } x \in \Theta_i.$$

For $\delta \le \frac{1}{100}$ given by Lemma 8.2, set

$$\lambda_i := \inf_{x \in \Theta_i} \inf\{\lambda \ge 1 \mid d_{GH}((\lambda X_i, x), (\mathbb{R}^k, 0)) \ge \delta\}.$$

First we check that λ_i is finite. Fix $i \in \mathbb{N}$ and take $z \in \Theta_i$. Since z is n-regular, as $\lambda \to \infty$ the distance $d_{GH}((\lambda X_i, z), (\mathbb{R}^k, 0))$ converges to $d_{GH}((\mathbb{R}^n, 0), (\mathbb{R}^k, 0)) > \delta$, so $\lambda_i < \infty$.

Now we show that $\lambda_i \to \infty$. If after passing to a subsequence, the sequence λ_i converges to a number $\lambda_\infty \in [1, \infty)$, then for any choice of $x_i \in \Theta_i$ the sequence $(\lambda_i X_i, x_i)$ converges to $(\lambda_\infty \mathbb{R}^k, 0) = (\mathbb{R}^k, 0)$ in the pointed Gromov–Hausdorff sense. However, by the definition of λ_i there are $y_i \in \Theta_i$ with $d_{GH}((\lambda_i X_i, y_i), (\mathbb{R}^k, 0)) \geq \delta/2$; a contradiction.

We claim there is a sequence $\mu_i \to \infty$ with the property that for all $x_i \in \Theta_i$, and any lift $\tilde{x}_i \in B_{1/2}(\tilde{p}_i)$,

$$\Gamma_i = \mathcal{G}(\pi_1(X_i), \widetilde{X}_i, \tilde{x}_i, 1/\lambda_i) = \mathcal{G}(\pi_1(X_i), \widetilde{X}_i, \tilde{x}_i, \mu_i). \tag{8.13}$$

Otherwise, by Corollary 2.55, after taking a subsequence, there would be a sequence

$$r_i \in \sigma(\pi_1(X_i), \widetilde{X}_i, \tilde{x}_i) \cap [1/\lambda_i, M] \quad \text{for some } M > 0.$$

After again taking a subsequence, by Lemma 2.36 and Proposition 2.33, we can assume $(r_i^{-1} X_i, x_i)$, $(r_i^{-1} \widetilde{X}_i, \tilde{x}_i)$ converge to $(\mathbb{R}^k \times Z, (0, z))$, $(\mathbb{R}^k \times \widetilde{Z}, (0, \tilde{z}))$, respectively, for some spaces $Z, \widetilde{Z}$ with $\operatorname{diam}(Z) \leq \frac{1}{10}$ (see Remark 2.20) in such a way that the sequence Γ_i converges equivariantly to some group $\Gamma \leq \operatorname{Iso}(\mathbb{R}^k \times \widetilde{Z})$ that acts trivially on the first factor and such that $\widetilde{Z}/\Gamma = Z$. By Lemma 2.56, $r \leq \frac{1}{2}$ for all $r \in \sigma(\Gamma)$, but by construction $1 \in \sigma(\Gamma_i, r_i^{-1} \widetilde{X}_i, \tilde{x}_i)$ for all i, contradicting Proposition 2.52 and proving (8.13).

Let $x_i \in \Theta_i$ be such that

$$\inf\{\lambda \geq 1 \mid d_{GH}((\lambda X_i, x_i), (\mathbb{R}^k, 0)) \geq \delta\} \leq \lambda_i + 1.$$

After passing to a subsequence, by Lemma 2.36, we can assume $(\lambda_i X_i, x_i)$ converges to $(\mathbb{R}^k \times Y, (0, y))$ for some compact space (Y, y) with $\operatorname{diam}(Y) \in \left(0, \frac{1}{10}\right]$.

For $a_i, b_i \in \Theta_i$, let $G(a_i) \subset B_1(a_i)$, $G(b_i) \subset B_1(b_i)$ be the sets given by Lemma 8.12, and let $U_i := G(a_i) \cap G(b_i)$. Notice that for i large enough we have

$$\max\{\mathfrak{m}_i(B_1(a_i)), \mathfrak{m}_i(B_1(b_i))\} \leq C \cdot \mathfrak{m}_i(U_i) \quad \text{for some } C(N).$$

For each $y \in U_i$, let $V_y^{a_i}$, $V_y^{b_i} \in L^1([0, 1]; H^{1,2}_{C,s}(TX_i))$ denote the vector fields given by Lemma 8.12, define $V_y \in L^1([0, 1]; H^{1,2}(TX_i))$ as

$$V_y(t) := \begin{cases} -2V_y^{a_i}(1 - 2t) & \text{if } t \in \left[0, \frac{1}{2}\right], \\ 2V_y^{b_i}(2t - 1) & \text{if } t \in \left[\frac{1}{2}, 1\right], \end{cases}$$

and let $\boldsymbol{X}^y : [0, 1] \times X_i \to X_i$ be its RLF. Then there are measurable maps

$$V : U_i \to L^\infty([0, 1]; H^{1,2}_{C,s}(TX_i))$$

with finite image such that, for all $y \in U_i$, V_y is a divergence-free on $B_{100C(N)}(p_i)$ vector field with $\|V_y(t)\|_\infty \leq C(N)$, there is a point $y' \in B_{1/\lambda_i}(a_i)$ of weak essential stability of $\boldsymbol{X}^y$ with $\boldsymbol{X}^y_1(y') \in B_{1/\lambda_i}(b_i)$, and

$$\fint_{U_i} \int_0^1 \operatorname{Mx}(|\nabla V_y(t)|^{3/2})^{2/3}(\boldsymbol{X}^y_t(y'))\, dt\, d\mathfrak{m}_i(y) \leq C(N)\delta_i.$$

This implies there is a sequence $y_i \in B_{1/\lambda_i}(a_i)$ and vector fields $W_i \in L^1([0, 1]; H^{1,2}_{C,s}(TX_i))$ with RLFs $\boldsymbol{X}^i : [0, 1] \times X_i \to X_i$ such that y_i is a point of weak essential stability of $\boldsymbol{X}^i$, $\boldsymbol{X}^i_1(y_i) \in B_{1/\lambda_i}(b_i)$, and

$$\int_0^1 \operatorname{Mx}(|\nabla W_i(t)|^{3/2})^{2/3}(\boldsymbol{X}^i_t(y_i))\, dt \leq C(N)\delta_i.$$

By Proposition 5.21, y_i is a point of essential stability of X^i for i large enough, so by Lemma 7.7, we get a sequence of GS maps $h_i : [\lambda_i X_i, a_i] \to [\lambda_i X_i, b_i]$. This implies, by Lemma 6.4, that the pointed measured Gromov–Hausdorff limit of the sequence $(\lambda_i X_i, a_i)$ does not depend on the choice of $a_i \in \Theta_i$.

By Propositions 2.7, 7.13, and Lemma 7.7, for lifts $\tilde{a}_i, \tilde{b}_i \in B_{1/2}(\tilde{p}_i)$, we also get a sequence of deck type GS maps $f_i' : [\lambda_i \widetilde{X}_i, \tilde{a}_i] \to [\lambda_i \widetilde{X}_i, b_i']$ with $(f_i')_* = \mathrm{Id}_{\pi_1(X_i)}$ for some b_i' in the preimage of b_i with

$$\tilde{d}(\tilde{a}_i, b_i') \leq C(N) \tag{8.14}$$

and such that $(f_i')_* = \mathrm{Id}_{\pi_1(X_i)}$. From (8.13) and (8.14), there are $g_i \in \Gamma_i$ with $g_i(b_i') = \tilde{b}_i$. Composing f_i' with g_i, we get a sequence of deck type GS maps $f_i : [\lambda_i \widetilde{X}_i, \tilde{a}_i] \to [\lambda_i \widetilde{X}_i, \tilde{b}_i]$ with $(f_i)_* = (g_i)_* \in (\Gamma_i)_*$. □

9. Proof of main theorems

We now prove Theorem 1.14 by reverse induction on k. This is done by contradiction; after passing to a subsequence, we assume the following.

Assumption 9.1. There is a sequence of integers $\xi_i \to \infty$ such that no subgroup of Γ_i of index $\leq \xi_i$ admits a nilpotent basis of length $\leq n - k$ respected by $(f_{j,i})_*^{\xi_i!}$ for each $j \in \{1, \dots, \ell\}$.

The base of induction consists of Proposition 9.2. The induction step is first proved assuming that the sequences $f_{j,i}$ converge to the identity and $Y \neq \{*\}$ (Proposition 9.3). Then we drop the assumption that the maps $f_{j,i}$ converge to the identity (Proposition 9.4), and the last step consists on dropping the assumption $Y \neq \{*\}$ (Proposition 9.5).

After taking a subsequence we may assume $(\widetilde{X}_i, \tilde{p}_i)$ converges to a space $(\mathbb{R}^k \times \widetilde{Y}, (0, \tilde{y}))$ and Γ_i converges equivariantly to some closed group $\Gamma \leq \mathrm{Iso}(\mathbb{R}^k \times \widetilde{Y})$, which by Proposition 2.33, acts trivially on the $\mathbb{R}^k$ factor. By Corollary 2.27, $\widetilde{Y}$ splits as a product $\mathbb{R}^m \times Z$ for some compact space Z, and by Corollary 2.65, Γ / Γ_0 has an abelian subgroup of finite index generated by at most m elements. After passing to a subsequence, by Lemma 6.4 we can also assume that for each $j \in \{1, \dots, \ell\}$, $f_{j,i}$ converges weakly to an isometry $f_{j,\infty} : \mathbb{R}^k \times \widetilde{Y} \to \mathbb{R}^k \times \widetilde{Y}$.

Proposition 9.2. *Theorem 1.14 holds if $k = n$.*

Proof. By dimensionality, $\widetilde{Y}$ is trivial and so is Γ. By Lemma 3.2, the sequence Γ_i is trivial as well. □

Proposition 9.3. *In the induction step, Assumption 9.1 leads to a contradiction if $Y \neq \{*\}$ and $f_{j,\infty} = \mathrm{Id}_{\mathbb{R}^k \times \widetilde{Y}}$ for all j.*

Proof. Let $v_1, \dots, v_m \in \Gamma$ be such that $\{v_1 \Gamma_0, \dots, v_m \Gamma_0\}$ generates a finite index abelian subgroup of Γ / Γ_0. For each $j \in \{1, \dots, m\}$ pick $w_j^i \in \Gamma_i$ with $w_j^i \to v_j$, and define $\Upsilon_i \triangleleft \Gamma_i$ to be the subgroups given by Theorem 3.1. Then from the proof of Theorem 3.1 one has, for i large enough, that

- $\langle \Upsilon_i, w_1^i, \dots, w_m^i \rangle$ is a finite index subgroup of Γ_i,
- $[w_{j_1}^i, w_{j_2}^i] \in \Upsilon_i$ for $j_1, j_2 \in \{1, \dots, m\}$.

Furthermore, as $f_{j,i} \to \mathrm{Id}_{\mathbb{R}^k \times \widetilde{Y}}$ for each $j \in \{1, \dots, \ell\}$, we also have

- $[f_{j,i}, w_{j_1}^i] \in \Upsilon_i$ for all $j_1 \in \{1, \dots, m\}$ and large enough i.

Case 1: The sequence $[\Gamma_i : \Upsilon_i]$ is bounded.

By Lemma 2.57 and Proposition 2.58, there are characteristic subgroups $H_i \triangleleft \Gamma_i$ contained in Υ_i such that the sequence $[\Upsilon_i : H_i]$ is bounded. After slightly shifting the basepoints p_i, we may assume y is an α-regular point of Y. Let $\lambda_i \to \infty$ so slowly that

- $(\lambda_i X_i, p_i) \to (\mathbb{R}^{k+\alpha}, 0)$ in the pointed Gromov–Hausdorff sense,
- $f_{j,i} : [\lambda_i \widetilde{X}_i, \tilde{p}_i] \to [\lambda_i \widetilde{X}_i, \tilde{p}_i]$ still is GS and converges to $\mathrm{Id}_{\mathbb{R}^{m+\alpha}}$,
- $\lim_{i\to\infty} \sup \sigma(H_i, \lambda_i \widetilde{X}_i, \tilde{p}_i) < \infty$.

By Proposition 2.33, any pointed Gromov–Hausdorff limit of $(\lambda_i \widetilde{X}_i / H_i, [\tilde{p}_i])$ splits off $\mathbb{R}^{k+\alpha}$, and as $H_i \triangleleft \Gamma_i$ is characteristic, it is preserved by $(f_{j,i})_*$ for each j, so the induction hypothesis applies to the spaces $(\widetilde{X}_i / H_i, \lambda_i d_i, \mathfrak{m}_i, [\tilde{p}_i])$, contradicting Assumption 9.1.

Case 2: After passing to a subsequence, $[\Gamma_i : \Upsilon_i] \to \infty$.

For each $j \in \{1, \dots, m\}$, set $\Gamma_{i,j} := \langle \Upsilon_i, w^i_j, \dots, w^i_m \rangle$ and $\Gamma_{i,m+1} := \Upsilon_i$. Let $j_0 \in \{1, \dots, m\}$ be the smallest number such that, after passing to a subsequence, we get $[\Gamma_{i,j_0} : \Gamma_{i,j_0+1}] \to \infty$, and let $\Gamma'_i := \Gamma_{i,j_0}$, $\Upsilon'_i := \Gamma_{i,j_0+1}$. Notice that by our choice of w^i_j's, Υ'_i is normal in Γ'_i.

Let $X'_i := \widetilde{X}_i / \Upsilon'_i$, $p'_i \in X'_i$ the image of $\tilde{p}_i$, and $f_{\ell+1,i} := w^i_{j_0} \in \mathrm{Iso}(\widetilde{X}_i)$. After taking a subsequence, we can assume (X'_i, p'_i) converges to a space $(\mathbb{R}^k \times Y', (0, y'))$, Υ'_i converges to a closed group $\Upsilon' \leq \Gamma$, and Γ'_i converges to a closed group $\Gamma' \leq \Gamma$ with $[\Gamma : \Gamma'] < \infty$. By Theorem 3.1, $[\Gamma' : \Upsilon'] = \infty$, so the group Γ'/Υ' is noncompact.

Since Γ'/Υ' acts on Y' with compact quotient $\widetilde{Y}/\Gamma'$, Corollary 2.27 applies, and since Γ'/Υ' is noncompact, Y' contains a nontrivial Euclidean factor. Therefore the induction hypothesis applies to the sequence of spaces (X'_i, p'_i), the groups Υ'_i, and the maps $f_{j,i}$ for $j \in \{1, \dots, \ell+1\}$ (as $f_{j,i} \to \mathrm{Id}_{\mathbb{R}^k \times \widetilde{Y}}$, $(f_{j,i})_*$ preserves Υ'_i). This means there is $C > 0$ and subgroups $G'_i \leq \Upsilon'_i$ such that, for i large enough,

- $[\Upsilon'_i : G'_i] \leq C$,
- G'_i admits a nilpotent basis β'_i of length $\leq n-k-1$,
- $(f_{j,i})^{C!}_*$ respects β'_i for $j \in \{1, \dots, \ell+1\}$.

By Lemma 2.57 and Proposition 2.58, we can assume G'_i is characteristic in Υ'_i. Then we define $G_i := \langle G'_i, f^{(2C)!}_{\ell+1,i} \rangle$. Notice that G_i admits the nilpotent chain β_i obtained by appending $f^{(2C)!}_{\ell+1,i}$ to β'_i. From the fact that $[f_{j,i}, f_{\ell+1,i}] \in \Upsilon'_i$ for $j \in \{1, \dots, \ell\}$ and Proposition 2.61, we have that $(f_{j,i})^{C!}_*$ respects β for $j \in \{1, \dots, \ell\}$. Finally, the sequence

$$[\Gamma_i : G_i] = [\Gamma_i : \Gamma'_i][\Gamma'_i : G_i] \leq [\Gamma_i : \Gamma'_i](2C)!C$$

is bounded, contradicting Assumption 9.1. □

Proposition 9.4. *In the induction step, Assumption 9.1 leads to a contradiction if* $Y \neq \{*\}$.

Proof. Fix $j \in \{1, \dots, \ell\}$. Then $f_{j,\infty}(0, \tilde{y}) = (s, y')$ for some $s \in \mathbb{R}^k$, $y' \in \widetilde{Y}$. By Proposition 6.8, after composing $f_{j,i}$ with maps given by Lemma 7.1, we can assume $f_{j,\infty}(0, \tilde{y}) = (0, y')$. Since Γ acts co-compactly on $\widetilde{Y}$, there is a sequence $\gamma_\nu \in \Gamma$ such that the sequence $f^{\nu}_{j,\infty}(\gamma_\nu(0, \tilde{y}))$ is bounded.

As $\mathrm{Iso}(\mathbb{R}^k \times \widetilde{Y})$ is proper, there is a sequence $\nu_\alpha \to \infty$ such that $f_{j,\infty}^{\nu_\alpha}(\gamma_{\nu_\alpha})$ is a Cauchy sequence in $\mathrm{Iso}(\mathbb{R}^k \times \widetilde{Y})$. This implies that the sequence

$$(f_{j,\infty}^{\nu_\alpha}\gamma_{\nu_\alpha})^{-1}(f_{j,\infty}^{\nu_{\alpha+1}}\gamma_{\nu_{\alpha+1}}) = (f_{j,\infty}^{\nu_{\alpha+1}-\nu_\alpha})[(f_{j,\infty}^{\nu_{\alpha+1}-\nu_\alpha})_*^{-1}(\gamma_{\nu_\alpha}^{-1})](\gamma_{\nu_{\alpha+1}})$$

converges to $\mathrm{Id}_{\mathbb{R}^k\times\widetilde{Y}}$.

Set $\mu_\alpha := \nu_{\alpha+1} - \nu_\alpha$, $g_\alpha := [(f_{j,\infty}^{\mu_\alpha})_*^{-1}(\gamma_{\nu_\alpha}^{-1})](\gamma_{\nu_{\alpha+1}})$, and choose $g_{\alpha,i} \in \Gamma_i$ such that $g_{\alpha,i}$ converges to g_α as $i \to \infty$. By Proposition 6.8 and a diagonal argument, if a function $i \mapsto \alpha(i)$ diverges to infinity slowly enough, the maps $f_{j,i}^{\mu_\alpha} g_{\alpha,i} : [\widetilde{X}_i, \tilde{p}_i] \to [\widetilde{X}_i, \tilde{p}_i]$ are GS and converge to $\mathrm{Id}_{\mathbb{R}^k\times\widetilde{Y}}$. By Proposition 2.62, if a function $i \mapsto \alpha(i)$ diverges slowly enough, we can replace $f_{j,i}$ by $f_{j,i}^{\mu_\alpha} g_{\alpha,i}$ and still have Assumption 9.1. By doing this independently for each j, we can assume $f_{j,\infty} = \mathrm{Id}_{\mathbb{R}^k\times\widetilde{Y}}$ for all $j \in \{1, \dots, \ell\}$ and Proposition 9.3 applies. □

Proposition 9.5. *In the induction step, Assumption 9.1 leads to a contradiction.*

Proof. After rescaling down each X_i by a fixed factor, we can assume $f_{j,\infty}$ displaces $(0, \tilde{y})$ at most $1/10$. If $Y = \{*\}$, let λ_i and $\Theta_i \subset B_{1/2}(p_i)$ be given by Theorem 8.1, and $\tilde{\Theta}_i \subset B_{1/2}(\tilde{p}_i)$ their lifts. For each $j \in \{1, \dots, \ell\}$, let $W_{j,i}^1 \subset \widetilde{X}_i$ be the sets obtained by applying Proposition 6.7 to each $f_{j,i}$, and set

$$W_i := \tilde{\Theta}_i \cap \bigcap_j W_{j,i}^1.$$

Then for large enough i, we can take $a_i \in W_i$ such that $f_{j,i}(a_i) \in W_i$ for each j. Let

$$\varphi_{j,i} : [\lambda_i \widetilde{X}_i, f_{j,i}(a_i)] \to [\lambda_i \widetilde{X}_i, a_i]$$

be the maps given by part (2) of Theorem 8.1. By Remark 2.59 and Proposition 2.62, if we replace $f_{j,i}$ by $\varphi_{j,i} f_{j,i}$, we still have Assumption 9.1. Then by part (1) of Theorem 8.1, Proposition 9.4 applies to the spaces $(\lambda_i X_i, a_i)$ and the GS maps $\varphi_{j,i} f_{j,i} : [\lambda \widetilde{X}_i, a_i] \to [\lambda_i \widetilde{X}_i, a_i]$. □

Proof of Theorem 1.1. Assuming the result fails, there is a sequence $(X_i, d_i, \mathfrak{m}_i, p_i)$ of pointed RCD(K, N) spaces, $\varepsilon_i \to 0$, and integers $\xi_i \to \infty$, such that if $H_i \leq \pi_1(X_i, p_i)$ denotes the image of the map $\pi_1(B_{\varepsilon_i}(p_i), p_i) \to \pi_1(X_i, p_i)$ induced by the inclusion, then no subgroup of H_i of index $\leq \xi_i$ admits a nilpotent basis of length $\leq n$.

Taking the pointed universal covers $(\widetilde{X}_i, \tilde{d}_i, \tilde{\mathfrak{m}}_i, \tilde{p}_i)$, for each i one can identify H_i with a subgroup of $\mathcal{G}(\pi_1(X_i), \widetilde{X}_i, \tilde{p}_i, 2\varepsilon_i)$. After taking a subsequence, we can assume (X_i, p_i) and $(\widetilde{X}_i, \tilde{p}_i)$ converge in the pointed Gromov–Hausdorff sense to spaces (X, p) and $(\widetilde{X}, \tilde{p})$, respectively, and the sequence $\pi_1(X_i)$ converges equivariantly to a closed group of isometries $G \leq \mathrm{Iso}(\widetilde{X})$.

Let $K \leq G$ be the stabilizer of $\tilde{p}$, and let m be the number of connected components of G it intersects. Fix $\varepsilon > 0$ such that the set $\{g \in G \mid d(d\tilde{p}, \tilde{p}) \leq 2\varepsilon\}$ intersects the same m connected components of G as K, and define

$$H_i' := \langle \{g \in \pi_1(X_i) \mid d(g\tilde{p}_i, \tilde{p}_i) \leq \varepsilon\} \rangle.$$

After taking a subsequence, we can assume H_i' converges equivariantly to a closed group $H' \leq G$, and let $\Upsilon_i \triangleleft H_i'$ be given by Theorem 3.1. Then for i large enough, $H_i \leq H_i'$ and by Theorem 3.1, $[H_i' : \Upsilon_i] \leq m$. Hence no subgroup of Υ_i of index $\leq \xi_i/m$ admits a nilpotent basis of length $\leq n$.

Pick $q \in B_1(p)$ a k-regular point, $\tilde{q} \in B_1(\tilde{p})$ a lift, and $\tilde{q}_i \in B_1(\tilde{p}_i)$ converging to $\tilde{q}$. If we equip $\pi_1(X_i)$ with the metric $d_0^{\tilde{p}_i}$ from (2.29), then for any $g \in B_\delta(\mathrm{Id}_{\widetilde{X}_i})$ with $\delta < 1$ one has $d(g\tilde{q}_i, \tilde{q}_i) < \delta$. Hence for all $\delta < 1$ we have

$$B_\delta(\mathrm{Id}_{\widetilde{X}_i}) \subset \{g \in \pi_1(X_i) | d(g\tilde{q}_i, \tilde{q}_i) < \delta\}. \tag{9.6}$$

For a sequence $\delta_i \to 0$, define $\Gamma_i := \mathcal{G}(\pi_1(X_i), \widetilde{X}_i, \tilde{q}_i, \delta_i)$. By (9.6) and Theorem 3.1, if $\delta_i \to 0$ slowly enough, for all i large we have $\Upsilon_i \leq \Gamma_i$. Finally, consider a sequence $\lambda_i \to \infty$ diverging so slowly that $\lambda_i \delta_i \to 0$, and such that $(\lambda_i X_i, q_i)$ converges to $(\mathbb{R}^k, 0)$ in the pointed Gromov–Hausdorff sense. Then $\Gamma_i = \mathcal{G}(\Gamma_i, \lambda_i \widetilde{X}_i, \tilde{q}_i, 1)$, contradicting Theorem 1.14 with $\ell = 0$. □

Finally, we point out that the proof of Corollary 1.15 is the same as of Theorem 1.1 almost verbatim. The only differences are that the contradicting sequence a priori converges to (X, p), and the nilpotent bases we are seeking are of length $\leq n - k$ instead of $\leq n$.

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