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LIOUVILLE THEOREM FOR MINIMAL GRAPHS OVER MANIFOLDS OF NONNEGATIVE RICCI CURVATURE

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Let Σ be a complete Riemannian manifold of nonnegative Ricci curvature. We prove a Liouville-type theorem: every smooth solution u to the minimal hypersurface equation on Σ is a constant provided u has sublinear growth for its negative part. Here, the sublinear growth condition is sharp. Our proof relies on a gradient estimate for minimal graphs over Σ with small linear growth of the negative parts of graphic functions via iteration.

1. Introduction

Let Σ be a complete noncompact Riemannian manifold. Let D and div_Σ be the Levi-Civita connection and the divergence operator (in terms of the Riemannian metric of Σ), respectively. In this paper, we study the minimal hypersurface equation on Σ,

$$\mathrm{div}_\Sigma\left(\frac{Du}{\sqrt{1+|Du|^2}}\right)=0, \tag{1-1}$$

which is a nonlinear partial differential equation describing the minimal graph

$$M=\{(x,u(x))\in\Sigma\times\mathbb{R}\mid x\in\Sigma\}$$

over Σ. The smooth solution u to (1-1) is the height function of the minimal graph M in $\Sigma\times\mathbb{R}$. Therefore, we call u a *minimal graphic function* on Σ.

When Σ is Euclidean space $\mathbb{R}^n$, equation (1-1) has been studied successfully by many mathematicians. Bombieri, De Giorgi, and Miranda [Bombieri et al. 1969b] (see also [Gilbarg and Trudinger 1983]) proved interior gradient estimates for solutions to the minimal hypersurface equation on $\mathbb{R}^n$, where the 2-dimensional case had already been obtained in [Finn 1954]. As a corollary, they immediately got a Liouville-type theorem in [Bombieri et al. 1969b] as follows.

Theorem 1.1. *If a minimal graphic function u on $\mathbb{R}^n$ satisfies sublinear growth for its negative part*, *i.e.*,

$$\limsup_{x\to\infty}\frac{\max\{-u(x),0\}}{|x|}=0, \tag{1-2}$$

then u is a constant.

The condition (1-2) is sharp since any affine function is a minimal graphic function on $\mathbb{R}^n$. When the minimal graphic function u on $\mathbb{R}^n$ has the uniformly bounded gradient, Moser [1961] proved u is affine

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using Harnack's inequalities for uniformly elliptic equations. The gradient estimate of u on $\mathbb{R}^n$ can also be derived by the maximum principle (see [Korevaar 1986; Wang 1998] for instance). Without the "uniformly bounded gradient" condition, it is the celebrated Bernstein theorem; see [Fleming 1962; De Giorgi 1965; Almgren 1966; Simons 1968] and the counterexample in [Bombieri et al. 1969a]. Specifically, any minimal graphic function on $\mathbb{R}^n$ is affine for $n \le 7$ by Simons [1968].

Let us review Liouville-type theorems for nonnegative minimal graphic functions on manifolds briefly. From Fischer-Colbrie and Schoen [1980], any positive minimal graphic function on a Riemann surface S of nonnegative curvature is constant (see [Rosenberg 2002] for the case of minimal surfaces in $S \times \mathbb{R}$). Rosenberg, Schulze, and Spruck [Rosenberg et al. 2013] proved that every nonnegative minimal graphic function on a complete manifold of nonnegative Ricci curvature and sectional curvature uniformly bounded below is a constant. Casteras, Heinonen, and Holopainen [Casteras et al. 2020] showed that every nonnegative minimal graphic function u on a complete manifold of asymptotically nonnegative sectional curvature is a constant provided u has at most linear growth. In [Ding 2021], the author proved that every nonnegative minimal graphic function on a complete manifold of nonnegative Ricci curvature is constant, which was also obtained independently by Colombo, Magliaro, Mari, and Rigoli [Colombo et al. 2022]. In fact, the "nonnegative Ricci curvature" condition can be further weakened to the volume doubling property and the Neumann–Poincaré inequality in [Ding 2021].

In some situations, the above "nonnegative" condition for the solution u on a manifold Σ can be weakened to the condition of "sublinear growth for its negative part", i.e.,

$$\limsup_{\Sigma \ni x \to \infty} \frac{\max\{-u(x), 0\}}{d(x,p)} = 0 \tag{1-3}$$

for some $p \in \Sigma$, where $d(x,p)$ denotes the distance function on Σ between x and p. Motivated by Theorem 1.1, for brevity, we say the *strong Liouville theorem for minimal graphs over* Σ *holds* if every minimal graphic function u on Σ is a constant provided u admits sublinear growth for its negative part.

In [Rosenberg et al. 2013], the strong Liouville theorem for minimal graphs over complete manifolds of nonnegative sectional curvature was proved. Ding, Jost, and Xin [Ding et al. 2016] proved the strong Liouville theorem for minimal graphs over complete manifolds of nonnegative Ricci curvature, Euclidean volume growth and quadratic curvature decay. In [Ding 2025], the author proved the same without the above quadratic curvature decay condition, which is a biproduct of Poincaré inequality on minimal graphs; see [Bombieri and Giusti 1972] for the Euclidean case. Colombo, Gama, Mari, and Rigoli [Colombo et al. 2024] proved the strong Liouville theorem for minimal graphs over complete manifolds of nonnegative Ricci curvature and that the $(n-2)$-th Ricci curvature in the radial direction from a fixed origin has a lower bound decaying quadratically to zero.

Colombo, Mari, and Rigoli [Colombo et al. 2023] proved an interesting theorem: if a minimal graphic function u on a complete noncompact Riemannian manifold Σ of nonnegative Ricci curvature satisfies

$$\limsup_{\Sigma \ni x \to \infty} \frac{\log d(x,p)}{d(x,p)} \max\{-u(x), 0\} < \infty \tag{1-4}$$

for some $p \in \Sigma$, then u is a constant.

From now on, we always let Σ denote a complete noncompact Riemannian manifold of nonnegative Ricci curvature (without extra assumptions). In this paper, we can weaken the condition (1-4) to (1-3) and prove the strong Liouville theorem for minimal graphs over Σ as follows.

Theorem 1.2. *Any minimal graphic function u on Σ is a constant provided u has sublinear growth for its negative part.*

The condition of "sublinear growth for its negative part", i.e., (1-3), is sharp from the Euclidean case and the manifolds case; see Proposition 9 in [Colombo et al. 2024]. To arrive at Theorem 1.2, we prove a stronger result: a gradient estimate for small linear growth of the negative part of u (without the upper bound condition of u) as follows.

Theorem 1.3. *There exists a constant $\beta_* > 0$ depending only on n such that if a minimal graphic function u on Σ satisfies*

$$\liminf_{x\to\infty}\frac{u(x)}{d(x,p)}\geq-\beta_* \tag{1-5}$$

for some $p\in\Sigma$, then there is a constant $c>0$ depending only on n such that

$$\sup_{x\in\Sigma}|Du|(x)\leq c\limsup_{x\to\infty}\frac{\max\{-u(x),0\}}{d(x,p)}. \tag{1-6}$$

The key ingredient in the proof of Theorem 1.3 is to get an integral estimate of v^k on geodesic balls in Σ for a large constant k by an iteration (on l) of an integral of $(\log v)^l v$, where v is the volume function of the minimal graphic function u. Then using the Sobolev inequality on Σ, we can carry out a (modified) De Giorgi–Nash–Moser iteration on geodesic balls in Σ starting from an integral of v^{2k} with $k>n$ and get the bound of v; see Theorem 4.3 since Harnack's inequality holds in Theorem 4.3 of [Ding 2021].

Once we get the uniform gradient estimate (1-6), from Theorem 8 (or Theorem 6(ii)) in [Colombo et al. 2024], we can conclude that any tangent cone of Σ at infinity splits off a line isometrically; compare with the harmonic case by Cheeger, Colding, and Minicozzi [Cheeger et al. 1995]. It's worth pointing out that Σ may not split off any line from a counterexample in Proposition 9 of [Colombo et al. 2024].

Without (1-5), we have the gradient estimates without the "entire" condition of M or Σ, where the estimates depend on the lower bound of the volume of geodesic balls of Σ; see [Ding 2025]. In [Colombo et al. 2024], the authors obtained gradient estimates for minimal graphs over manifolds of nonnegative Ricci curvature and that the $(n-2)$-th Ricci curvature of Σ in radial direction from a fixed origin has a lower bound decaying quadratically to zero.

2. Preliminaries

Let Σ be an n-dimensional complete Riemannian manifold of nonnegative Ricci curvature. For any $R>0$ and $p\in\Sigma$, let $B_R(p)$ be the geodesic ball in Σ centered at p with radius R. For each integer $k\geq 0$, let $\mathcal{H}^k$ denote the k-dimensional Hausdorff measure. From the Bishop–Gromov volume comparison theorem,

$$\frac{1}{n}r^{1-n}\mathcal{H}^{n-1}(\partial B_r(p))\leq r^{-n}\mathcal{H}^n(B_r(p))\leq s^{-n}\mathcal{H}^n(B_s(p)) \tag{2-1}$$

for all $0<s<r$. Let D be the Levi-Civita connection of Σ. From [Anderson 1992] or [Croke 1980], the Sobolev inequality

$$\frac{(\mathcal{H}^n(B_r(p)))^{\frac{1}{n}}}{r}\left(\int_{B_r(p)}|\phi|^{\frac{n}{n-1}}\right)^{\frac{n-1}{n}}\le\Theta\int_{B_r(p)}|D\phi| \tag{2-2}$$

holds for any Lipschitz function ϕ on $B_r(p)$ with compact support in $B_r(p)$, where $\Theta>0$ is a constant depending only n.

Let Φ be a Lipschitz function on $B_{r+s}(p)$, $s\in(0,r]$, and ζ be a nonnegative Lipschitz function such that $\zeta\equiv 1$ on $B_r(p)$, $\zeta\equiv 0$ outside $B_{r+s}(p)$ and $|D\zeta|\le 1/s$. Then, from the Cauchy–Schwarz inequality, we have

$$\begin{aligned}\int_{B_r(p)}|D(\Phi^2\zeta)|&\le 2\int_{B_r(p)}|\Phi|\zeta|D\Phi|+\int_{B_r(p)}\Phi^2|D\zeta|\\&\le r\int_{B_r(p)}|D\Phi|^2\zeta+\frac{1}{r}\int_{B_r(p)}\Phi^2\zeta+\frac{1}{s}\int_{B_{r+s}(p)}\Phi^2.\end{aligned} \tag{2-3}$$

From (2-2), it follows that

$$(\mathcal{H}^n(B_r(p)))^{\frac{1}{n}}\left(\int_{B_r(p)}|\Phi|^{\frac{2n}{n-1}}\right)^{\frac{n-1}{n}}\le\Theta r^2\int_{B_r(p)}|D\Phi|^2+\frac{2\Theta r}{s}\int_{B_{r+s}(p)}\Phi^2. \tag{2-4}$$

From [Buser 1982] or [Cheeger and Colding 1996], we have the Neumann–Poincaré inequality on geodesic balls of Σ. Namely, we have (up to a choice of Θ)

$$\int_{B_r(p)}|\varphi-\varphi_{B_r(p)}|\le\Theta r\int_{B_r(p)}|D\varphi| \tag{2-5}$$

for any Lipschitz function φ on $B_r(p)$, where

$$\varphi_{B_r(p)}=\fint_{B_r(p)}\varphi:=\frac{1}{\mathcal{H}^n(B_r(p))}\int_{B_r(p)}\varphi.$$

Let M be a minimal graph over Σ with the graphic function u on Σ, where M has the induced metric from $\Sigma\times\mathbb{R}$ equipped with the standard product metric. By Stokes' formula, M is area-minimizing in $\Sigma\times\mathbb{R}$ by an argument analog to the case of Euclidean space. Let ∇ and Δ denote the Levi-Civita connection and Laplacian of M, respectively. We also see u as a function on M by projection $M\to\Sigma$; i.e., $u(x,u(x))=u(x)$ for any $x\in\Sigma$. Then equation (1-1) is equivalent to the condition that u is harmonic on M, i.e.,

$$\Delta u=0. \tag{2-6}$$

Let

$$v=\sqrt{1+|Du|^2}$$

be the volume function of M (as mentioned above); we see v as a function on M by identifying $v(x,u(x))=v(x)$. Recall the following Bochner-type formula:

$$\Delta v^{-1}=-(|A|^2+v^{-2}\operatorname{Ric}(Du,Du))v^{-1}. \tag{2-7}$$

Here, A denotes the second fundamental form of M in $\Sigma\times\mathbb{R}$, and Ric denotes the Ricci curvature of Σ. From (2-7), it follows that

$$\Delta\log v=|A|^2+v^{-2}\operatorname{Ric}(Du,Du)+|\nabla\log v|^2\ge|\nabla\log v|^2. \tag{2-8}$$

For a C^1-function f on an open set of Σ, we can see f as a function on M: $f(x,u(x))=f(x)$. Then

$$|\nabla f|^2=|Df|^2-\frac{1}{v^2}|\langle Du,Df\rangle|^2\ge|Df|^2-\frac{|Du|^2}{v^2}|Df|^2=\frac{1}{v^2}|Df|^2. \tag{2-9}$$

Notational convention. When we write an integral over a subset of a Riemannian manifold with respect to its standard metric of the manifold, we always omit the volume element for simplicity.

3. Integral estimates of powers of the volume function

Lemma 3.1. *Let ξ be a Lipschitz function on Σ with compact support. For all constants $l\ge1$ and $q,\theta>0$, we have*

$$\int_\Sigma|D(\log v)^l|\xi^{q+1}\le l\theta r\int_\Sigma(\log v)^{l-1}v|D\xi|^2+\frac{l}{\theta r}\int_\Sigma(\log v)^{l-1}v\xi^{2q}. \tag{3-1}$$

Proof. We also see ξ as a function on M by letting $\xi(x,u(x))=\xi(x)$. From (2-8), for each $l'\ge0$, from the Cauchy–Schwarz inequality we have

$$\begin{aligned}\int_M(\log v)^{l'}\xi^2|\nabla\log v|^2&\le\int_M(\log v)^{l'}\xi^2\Delta\log v\le-2\int_M(\log v)^{l'}\xi\nabla\xi\cdot\nabla\log v\\&\le\frac12\int_M(\log v)^{l'}\xi^2|\nabla\log v|^2+2\int_M(\log v)^{l'}|\nabla\xi|^2,\end{aligned} \tag{3-2}$$

which implies

$$\int_M(\log v)^{l'}|\nabla\log v|^2\xi^2\le4\int_M(\log v)^{l'}|\nabla\xi|^2. \tag{3-3}$$

From (2-9) and (3-3), for all constants $q,\theta>0$ and $l\ge1$, we have

$$\begin{aligned}\int_\Sigma|D(\log v)^l|\xi^{q+1}&\le\int_M|\nabla(\log v)^l|\xi^{q+1}=l\int_M(\log v)^{l-1}|\nabla\log v|\xi^{q+1}\\&\le\frac{l\theta r}{4}\int_M(\log v)^{l-1}|\nabla\log v|^2\xi^2+\frac{l}{\theta r}\int_M(\log v)^{l-1}\xi^{2q}\\&\le l\theta r\int_M(\log v)^{l-1}|\nabla\xi|^2+\frac{l}{\theta r}\int_M(\log v)^{l-1}\xi^{2q}.\end{aligned} \tag{3-4}$$

This gives (3-1) by combining with (2-9) again. □

Given two constants $\beta,r_0>0$, we assume

$$|u(x)|\le\beta\max\{r_0,d(x,p)\}\quad\text{for each }x\in\Sigma. \tag{3-5}$$

For each $r\ge r_0$, it's clear that

$$|u(x)|\le\beta\max\{r,d(x,p)\}\quad\text{for each }x\in\Sigma. \tag{3-6}$$

We fix the point p and write $\rho(x)=d(x,p)$ for each $x\in\Sigma$.

Lemma 3.2. *Given a constant $\theta\in(0,1]$ and a constant $0<\delta\ll 1$, for each constant $l\ge 1$, we have*

$$\fint_{B_r(p)}(\log v)^l v\le(1+\delta)\beta l\frac{(1+\theta)^{n+1}}{\theta}\fint_{B_{(1+\theta)r}(p)}(\log v)^{l-1}v+\frac{2^n(1+c_\delta\beta)}{\theta}\fint_{B_{(1+\theta)r}(p)}(\log v)^l,\quad(3\text{-}7)$$

where $c_\delta\ge 1$ is a constant depending only on n and δ.

Proof. Let δ be a positive constant, $\delta\ll 1$, and ξ be a Lipschitz function on Σ with $\operatorname{supp}\xi\subset B_{(1+\theta)r}(p)$, $\xi\equiv 1$ on $B_r(p)$ and

$$\xi(x)=\begin{cases}\cos\dfrac{\rho(x)-r}{\theta r} & \text{for } x\in B_{(1+\delta\theta/4)r}(p)\setminus B_r(p),\\ \dfrac{\cos(\delta/4)}{1-\delta/4}\Big(1-\dfrac{\rho(x)-r}{\theta r}\Big) & \text{for } x\in B_{(1+\theta)r}(p)\setminus B_{(1+\delta\theta/4)r}(p).\end{cases}\tag{3-8}$$

Then

$$\theta r|D\xi|(x)=\begin{cases}\sin\dfrac{\rho(x)-r}{\theta r} & \text{for } x\in B_{(1+\delta\theta/4)r}(p)\setminus B_r(p),\\ \dfrac{\cos(\delta/4)}{1-\delta/4} & \text{for } x\in B_{(1+\theta)r}(p)\setminus B_{(1+\delta\theta/4)r}(p).\end{cases}\tag{3-9}$$

Let $q=q_\delta>1$ such that

$$\cos^q(\delta/4)=\frac{\sin(\delta/4)}{1-\delta/4}.$$

Noting that $(1-\delta/4)^{-2}<1+\delta$ as $0<\delta\ll 1$, with (3-9) we have

$$\theta^2r^2|D\xi|^2+\xi^{2q}\le(1-\delta/4)^{-2}<1+\delta\quad\text{on }\Sigma.\tag{3-10}$$

In [Bombieri et al. 1969b], the authors gave an estimate of an integral of $v\log v$ using (1-1) in the Euclidean case; see also [Gilbarg and Trudinger 1983] and [Ding et al. 2016] for manifolds. Enlightened by this, we further estimate an integral of $(\log v)^l v$ on geodesic balls of Σ using (1-1) for each $l>0$. Integrating by parts, for each $r\ge r_0$ with (3-6) we have

$$\begin{aligned}0&=\int_\Sigma\frac{Du}{v}\cdot D(u(\log v)^l\xi^{q+1})\\&=\int_\Sigma\frac{|Du|^2}{v}(\log v)^l\xi^{q+1}+\int_\Sigma u\xi^{q+1}\frac{Du}{v}\cdot D(\log v)^l+\int_\Sigma u(\log v)^l\frac{Du}{v}\cdot D\xi^{q+1}\\&\ge\int_\Sigma\frac{|Du|^2}{v}(\log v)^l\xi^{q+1}-(1+\theta)\beta r\int_\Sigma\xi^{q+1}|D(\log v)^l|-\frac{c_\delta\beta}{\theta}\int_{B_{(1+\theta)r}(p)}(\log v)^l.\end{aligned}\tag{3-11}$$

Here, $c_\delta\ge 1$ is a constant depending only on n and $q=q_\delta$. Then it follows that

$$\begin{aligned}\int_\Sigma(\log v)^l v\xi^{q+1}&\le\int_\Sigma\Big(\frac{|Du|^2}{v}+1\Big)(\log v)^l\xi^{q+1}\\&\le(1+\theta)\beta r\int_\Sigma|D(\log v)^l|\xi^{q+1}+\frac{1+c_\delta\beta}{\theta}\int_{B_{(1+\theta)r}(p)}(\log v)^l.\end{aligned}\tag{3-12}$$

For each $l \geq 1$, from (3-1) and (3-10) we have

$$\int_{\Sigma} |D(\log v)^l|\xi^{q+1} \leq \frac{l}{\theta r}\int_{B_{(1+\theta)r}(p)} (\log v)^{l-1} v(\theta^2 r^2 |D\xi|^2 + \xi^{2q}) \leq \frac{(1+\delta)l}{\theta r}\int_{B_{(1+\theta)r}(p)} (\log v)^{l-1} v. \tag{3-13}$$

Substituting (3-13) into (3-12), we get

$$\int_{B_r(p)} (\log v)^l v \leq (1+\delta)\beta l \frac{(1+\theta)}{\theta}\int_{B_{(1+\theta)r}(p)} (\log v)^{l-1} v + \frac{1+c_\delta\beta}{\theta}\int_{B_{(1+\theta)r}(p)} (\log v)^l. \tag{3-14}$$

This finishes the proof with (2-1). □

Now we further assume $\beta \leq 1$. Write $\gamma_\delta = (1+\delta)n(1+1/n)^{n+1}$. By taking $\theta = 1/n$ in (3-7), for each $l \geq 1$ (up to a choice of $c_\delta \geq 1$), we have

$$⨍_{B_r(p)} (\log v)^l v \leq \gamma_\delta \beta l ⨍_{B_{(n+1)r/n}(p)} (\log v)^{l-1} v + c_\delta ⨍_{B_{(n+1)r/n}(p)} (\log v)^l. \tag{3-15}$$

Since M is area-minimizing in $\Sigma \times \mathbb{R}$, with (2-1) we get

$$\int_{B_r(p)} v = \mathcal{H}^n(M \cap (B_r(p) \times \mathbb{R})) \leq \mathcal{H}^n(B_r(p)) + \int_{\partial B_r(p)} |u| \leq \mathcal{H}^n(B_r(p)) + \beta r \mathcal{H}^n(\partial B_r(p)) \leq (1+n\beta)\mathcal{H}^n(B_r(p)). \tag{3-16}$$

Let us iterate the estimate (3-15) on l.

Lemma 3.3. *Let c_δ be the constant in (3-15) with the given $0 < \delta \ll 1$. For each integer $j \geq 0$, we have*

$$\sup_{r \geq r_0} ⨍_{B_r(p)} (\log v)^j v \leq j!\, \gamma_\delta^j \beta^j \binom{j+m}{m}(1+n\beta), \tag{3-17}$$

where $m = \left[\frac{c_\delta}{\gamma_\delta \beta}\right] + 1 \in \mathbb{N}$ depends on n, δ, β, and

$$\binom{j+m}{m} = \frac{(m+j)!}{j!\,m!}.$$

Proof. Let us prove it by induction. From (3-15) and $\log v \leq v$, for each $j \geq 1$, we have

$$\sup_{r \geq r_0} ⨍_{B_r(p)} (\log v)^j v \leq \gamma_\delta \beta j \sup_{r \geq r_0} ⨍_{B_r(p)} (\log v)^{j-1} v + c_\delta \sup_{r \geq r_0} ⨍_{B_r(p)} (\log v)^{j-1} v. \tag{3-18}$$

Let $m = \left[\frac{c_\delta}{\gamma_\delta \beta}\right] + 1 \in \mathbb{N}$ depend on n, δ, β, and let $\{a_j\}_{j \in \mathbb{N}}$ be a sequence defined by

$$a_j = \sup_{r \geq r_0} ⨍_{B_r(p)} (\log v)^j v. \tag{3-19}$$

From (3-18), for each integer $j \geq 1$, one has

$$a_j \leq \gamma_\delta \beta j a_{j-1} + c_\delta a_{j-1} \leq \gamma_\delta \beta (j+m) a_{j-1}. \tag{3-20}$$

By iteration,

$$a_j \le \gamma_\delta^j \beta^j \frac{(j+m)!}{m!} a_0 = j!\,\gamma_\delta^j \beta^j \binom{j+m}{m} a_0. \tag{3-21}$$

From (3-16), $a_0 \le 1+n\beta$. This completes the proof. □

Theorem 3.4. *Let u be a minimal graphic function on Σ satisfying (3-6) for some constant $\beta \in (0,1]$. There is a constant $c(n,\delta,\beta) > 0$ depending only on n, δ, β such that, for each constant $\lambda \in (0, 1/(\gamma_\delta \beta))$, we have*

$$\sup_{r \ge r_0} ⨍_{B_r(p)} v^{\lambda+1} \le c(n,\delta,\beta)(1-\lambda\gamma_\delta\beta)^{-m-1}. \tag{3-22}$$

Proof. Let $\lambda < \dfrac{1}{\gamma_\delta \beta}$ be a positive constant. From Taylor's expansion

$$v^\lambda = e^{\lambda \log v} = \sum_{j=0}^{\infty} \frac{\lambda^j}{j!} (\log v)^j, \tag{3-23}$$

combining with (3-17) we get

$$\begin{aligned} ⨍_{B_r(p)} v^{\lambda+1} &= \sum_{j=0}^{\infty} \frac{\lambda^j}{j!} ⨍_{B_r(p)} (\log v)^j v \\ &\le \sum_{j=0}^{\infty} \frac{\lambda^j}{j!} j!\,\gamma_\delta^j \beta^j \binom{j+m}{m} (1+n\beta) \\ &= (1+n\beta) \sum_{j=0}^{\infty} (\lambda\gamma_\delta\beta)^j \binom{j+m}{m}. \end{aligned} \tag{3-24}$$

From

$$\sum_{j=0}^{\infty} \binom{j+m}{m} t^j = \frac{1}{m!} \frac{d^m}{dt^m} \sum_{j=0}^{\infty} t^{j+m} = \frac{1}{m!} \frac{d^m}{dt^m} \left(\frac{t^m}{1-t} \right) \tag{3-25}$$

for each $t \in (0,1)$, we complete the proof. □

4. Mean value inequality and gradient estimate

For each nonnegative measurable function f on Σ and each constant $q > 0$, we define

$$\|f\|_{q,r} = \left(⨍_{B_r(p)} f^q \right)^{1/q}.$$

Now, let us carry out a (modified) De Giorgi–Nash–Moser iteration to get the mean value inequality for the volume function v with the help of the Sobolev inequality on Σ.

Lemma 4.1. *For each constant $k > n$ and $\sigma \in (0,1)$, there is a constant $c_{\sigma,k}$ depending only on n, σ, k such that*

$$\|v\|_{\infty,\sigma r} \le c_{\sigma,k} (\|v\|_{2k,r})^{e^{n/(k-n)}} \tag{4-1}$$

for any $r > 0$.

Proof. Let η be a Lipschitz function on Σ with compact support which will be defined later. Write $\eta(x)=\eta(x,u(x))$. From (2-8), we have $\Delta v\geq 0$ on M clearly. For any constant $\ell\geq 1$, we have

$$\begin{aligned}0\geq-\int_M v^{2\ell}\eta^2\Delta v&=2\ell\int_M v^{2\ell-1}\eta^2|\nabla v|^2+2\int_M v^{2\ell}\eta\nabla v\cdot\nabla\eta\\&\geq 2\ell\int_M v^{2\ell-1}\eta^2|\nabla v|^2-\ell\int_M v^{2\ell-1}\eta^2|\nabla v|^2-\frac{1}{\ell}\int_M v^{2\ell+1}|\nabla\eta|^2\\&=\ell\int_M v^{2\ell-1}\eta^2|\nabla v|^2-\frac{1}{\ell}\int_M v^{2\ell+1}|\nabla\eta|^2.\end{aligned}\tag{4-2}$$

From (2-9) and (4-2), we infer that

$$\int_\Sigma|Dv^\ell|^2\eta^2\leq\int_M|\nabla v^\ell|^2v\eta^2=\ell^2\int_M|\nabla v|^2v^{2\ell-1}\eta^2\leq\int_M v^{2\ell+1}|\nabla\eta|^2\leq\int_\Sigma v^{2\ell+2}|D\eta|^2.\tag{4-3}$$

For each $r\geq\tau>0$, let η be defined by $\eta\equiv 1$ on $B_{r+\tau/2}(p)$, $\eta=(2/\tau)(r+\tau-\rho)$ on $B_{r+\tau}(p)\setminus B_{r+\tau/2}(p)$, $\eta\equiv 0$ outside $B_{r+\tau}(p)$. Then $|D\eta|\leq 2/\tau$. Combining (2-4) and (4-3), we have

$$\begin{aligned}\|v^{2\ell}\|_{\frac{n}{n-1},r}&\leq\Theta\Big(r^2\|Dv^\ell\|^2_{2,r+\frac{\tau}{2}}+\frac{4r}{\tau}\|v^{2\ell}\|_{1,r+\frac{\tau}{2}}\Big)\\&\leq\Theta\Big(r^2\int_\Sigma v^{2\ell+2}|D\eta|^2+\frac{4r}{\tau}\|v^{2\ell}\|_{1,r+\tau}\Big)\\&\leq\Theta\Big(\frac{4r^2}{\tau^2}\|v^{2\ell+2}\|_{1,r+\tau}+\frac{4r}{\tau}\|v^{2\ell}\|_{1,r+\tau}\Big)\\&\leq c\frac{r^2}{\tau^2}\|v^{2\ell+2}\|_{1,r+\tau}=c\frac{r^2}{\tau^2}\|v\|^{2\ell+2}_{2\ell+2,r+\tau}.\end{aligned}\tag{4-4}$$

Here, $c=8\Theta$ is a constant depending only on n. Given a constant $k>n$, we set

$$\alpha=\frac{n(k-1)}{(n-1)k}>1.\tag{4-5}$$

For $\ell+1\geq k$, we have

$$\frac{2\ell n}{n-1}-(2\ell+2)\alpha=\frac{2n}{(n-1)k}(\ell+1-k)\geq 0.\tag{4-6}$$

From the Hölder inequality and (4-4), one has

$$\|v\|_{(2\ell+2)\alpha,r}\leq\|v\|_{\frac{2\ell n}{n-1},r}\leq c^{\frac{1}{2\ell}}r^{\frac{1}{\ell}}\tau^{-\frac{1}{\ell}}\|v\|^{\frac{\ell+1}{\ell}}_{2\ell+2,r+\tau}.\tag{4-7}$$

For any $\sigma\in(0,1)$ and any integer $i\geq -1$, set $m_i=2k\alpha^i$, $\ell_i=\frac{1}{2}m_i-1$, $\tau_i=2^{-(1+i)}(1-\sigma)r$ and $r_{i+1}=r_i-\tau_{i+1}$ with $r_{-1}=r$. Then

$$r_{i+1}=r-\sum_{j=0}^{i+1}\tau_j=\sigma r+\tau_{i+1}\leq r,$$

and $\lim_{i\to\infty}r_i=\sigma r$. By iterating (4-7), for each $i\geq 0$, we have

$$\|v\|_{\alpha m_i,r_i}\leq c^{\frac{1}{2\ell_i}}r_i^{\frac{1}{\ell_i}}\tau_i^{-\frac{1}{\ell_i}}\|v\|^{\frac{\ell_i+1}{\ell_i}}_{\alpha m_{i-1},r_{i-1}}.\tag{4-8}$$

Set $\xi_i = \log \|v\|_{\alpha m_i, r_i}$ for each integer $i \ge -1$, and set $b_\sigma = c/(1-\sigma)^2$. Note that $\tau_i/r_i \ge 2^{-(1+i)}(1-\sigma)$ and $\ell_i \ge k\alpha^i - 1 \ge (k-1)\alpha^i$ for every $i \ge 0$. Then

$$\xi_i \le \frac{1}{2\ell_i}\log c + \frac{1}{\ell_i}\log\frac{r_i}{\tau_i} + \frac{\ell_i+1}{\ell_i}\xi_{i-1} \le \frac{1}{2\ell_i}\log b_\sigma + \frac{1+i}{\ell_i}\log 2 + e^{\frac{1}{\ell_i}}\xi_{i-1}$$
$$\le \frac{1}{2(k-1)\alpha^i}\log b_\sigma + \frac{1+i}{(k-1)\alpha^i}\log 2 + e^{\frac{\alpha^{-i}}{k-1}}\xi_{i-1}. \tag{4-9}$$

For all $0 \le i_0 \le i$, we have

$$\prod_{j=i_0}^{i} e^{\frac{\alpha^{-j}}{k-1}} = e^{\frac{1}{k-1}\sum_{j=i_0}^{i}\alpha^{-j}} \le e^{\frac{\alpha^{1-i_0}}{(k-1)(\alpha-1)}}.$$

Hence, for each $i \ge 1$,

$$\xi_i \le \frac{\log b_\sigma}{2(k-1)\alpha^i} + \frac{(1+i)\log 2}{(k-1)\alpha^i} + e^{\frac{\alpha^{-i}}{k-1}}\left(\frac{\log b_\sigma}{2(k-1)\alpha^{i-1}} + \frac{i\log 2}{(k-1)\alpha^{i-1}} + e^{\frac{\alpha^{1-i}}{k-1}}\xi_{i-2}\right)$$
$$\le \cdots \le \sum_{j=0}^{i}\left(\frac{\log b_\sigma}{2(k-1)\alpha^j} + \frac{1+j}{(k-1)\alpha^j}\log 2\right)\prod_{J=j+1}^{i} e^{\frac{\alpha^{-J}}{k-1}} + \xi_{-1}\prod_{j=0}^{i} e^{\frac{\alpha^{-j}}{k-1}}$$
$$\le e^{\frac{1}{(k-1)(\alpha-1)}}\sum_{j=0}^{i}\left(\frac{\log b_\sigma}{2(k-1)}\frac{1}{\alpha^j} + \frac{\log 2}{k-1}\frac{1+j}{\alpha^j}\right) + e^{\frac{\alpha}{(k-1)(\alpha-1)}}\xi_{-1}. \tag{4-10}$$

Since

$$\sum_{j=0}^{\infty}\frac{j+1}{\alpha^j} = \frac{\alpha^2}{(\alpha-1)^2}, \tag{4-11}$$

we have

$$\xi_i \le e^{\frac{1}{(k-1)(\alpha-1)}}\left(\frac{\log b_\sigma}{2(k-1)}\frac{\alpha}{\alpha-1} + \frac{\log 2}{k-1}\frac{\alpha^2}{(\alpha-1)^2}\right) + e^{\frac{\alpha}{(k-1)(\alpha-1)}}\xi_{-1}. \tag{4-12}$$

From $\alpha - 1 = \frac{k-n}{(n-1)k}$ and $\frac{\alpha}{\alpha-1} = \frac{n(k-1)}{k-n}$, we obtain

$$\xi_i \le e^{\frac{n}{k-n}}\left(\frac{n\log b_\sigma}{2(k-n)} + \frac{n^2k\log 2}{(k-n)^2}\right) + e^{\frac{n}{k-n}}\xi_{-1}. \tag{4-13}$$

Namely,

$$\|v\|_{\alpha m_i, r_i} \le \exp\left(e^{\frac{n}{k-n}}\left(\frac{n\log b_\sigma}{2(k-n)} + \frac{n^2k\log 2}{(k-n)^2}\right)\right)(\|v\|_{2k,r})^{e^{n/(k-n)}}. \tag{4-14}$$

Letting $i \to \infty$, it follows that

$$\|v\|_{\infty,\sigma r} \le \exp\left(e^{\frac{n}{k-n}}\left(\frac{n\log b_\sigma}{2(k-n)} + \frac{n^2k\log 2}{(k-n)^2}\right)\right)(\|v\|_{2k,r})^{e^{n/(k-n)}}. \tag{4-15}$$

This completes the proof. □

Remark 4.2. The factor $e^{n/(k-n)}$ in (4-1) comes from (4-3), which transforms an estimate on M to another estimate on Σ with a slight but definite "loss". In fact, the factor could be smaller if we choose a larger factor than α in (4-7) for large ℓ. However, we cannot reduce the constant k to a constant $\le n$, since we need $\alpha > 1$ in (4-5). Hence, unlike the classic De Giorgi–Nash–Moser iteration, here we are not able to obtain $\sup_{B_r(p)} v$ bounded by a multiple of an integral of v^γ with $\gamma \le 2n$ on $B_{2r}(p)$.

Put

$$\beta_n = \frac{1}{n(2n-1)}\Big(1+\frac{1}{n}\Big)^{-n-1}. \tag{4-16}$$

To prove Theorem 1.3, we only need to show the following theorem since we have Harnack's inequality in Theorem 4.3 of [Ding 2021] (or (A-8) in the Appendix directly).

Theorem 4.3. *If a minimal graphic function u on Σ satisfies*

$$\limsup_{x\to\infty} \frac{|u(x)|}{d(x,p)} < \beta_n \tag{4-17}$$

for some $p\in\Sigma$, then there is a constant $c>0$ depending only on n such that

$$\sup_{x\in\Sigma} |Du|(x) \le c \limsup_{x\to\infty} \frac{|u(x)|}{d(x,p)}. \tag{4-18}$$

Proof. From (4-17), there is a constant $\beta\in(0,\beta_n)$ such that

$$\limsup_{x\to\infty} \frac{|u(x)|}{d(x,p)} < \beta. \tag{4-19}$$

Then there is a constant $r_\beta>0$ such that

$$|u(x)| \le \beta \max\{r_\beta, d(x,p)\} \quad \text{for each } x\in\Sigma. \tag{4-20}$$

We fix a positive constant $\delta=\delta(\beta)\ll 1$ satisfying $\beta(1+\delta)<\beta_n$. Recall $\gamma_\delta=(1+\delta)n(1+1/n)^{n+1}$. From Theorem 3.4, there is a constant

$$\lambda_\beta = \Big(1+\frac{\beta_n}{\beta(1+\delta)}\Big)\Big(n-\frac{1}{2}\Big)+1$$

such that

$$\fint_{B_r(p)} v^{\lambda_\beta} \le \frac{c(n,\delta,\beta)}{(1-(\lambda_\beta-1)\gamma_\delta\beta)^{m+1}} = c(n,\delta,\beta)\Big(\frac{2\beta_n}{\beta_n-(1+\delta)\beta}\Big)^{m+1} \tag{4-21}$$

for all $r\ge r_\beta$. From Lemma 4.1, we get

$$\sup_{B_{r/2}(p)} v = \|v\|_{\infty,\frac{r}{2}} \le c_{\frac12,\frac{\lambda_\beta}{2}} (\|v\|_{\lambda_\beta,r})^{e^{n/(\lambda_\beta/2-n)}} \le \psi(n,\beta), \tag{4-22}$$

where $\psi=\psi(n,\beta)$ is a positive function depending only on n and $\beta<\beta_n$ satisfying $\lim_{\beta\to\beta_n}\psi(n,\beta)=\infty$, which may change from line to line. In other words, we have concluded that v is uniformly bounded on Σ. In the following, let us give a better bound of v than (4-22).

Let $\bar p=(p,u(p))$, and let $B_r(\bar p)$ denote the geodesic ball in $\Sigma\times\mathbb{R}$ with radius r centered at $\bar p$. From [Ding 2023, (3.5)], (2-1) and (3-16), we get

$$2\mathcal{H}^n(B_r(p)) \ge \mathcal{H}^n(M\cap B_r(\bar p)) \ge \frac{1}{r}\mathcal{H}^{n+1}(B_{r/2}(\bar p)) \ge \frac{1}{c}\mathcal{H}^n(B_r(p)) \tag{4-23}$$

for each $r>0$. Here, $c\ge 1$ is a constant depending only on n, which may change from line to line. Combining (2-2) and (4-22), (by projection from $\Sigma\times\mathbb{R}$ into Σ) we have the Sobolev inequality on M; i.e.,

$$\Big(\fint_{M\cap B_r(\bar p)} |\phi|^{\frac{n}{n-1}}\Big)^{\frac{n-1}{n}} \le \psi r \fint_{M\cap B_r(\bar p)} |D\phi| \tag{4-24}$$

holds for any Lipschitz function ϕ on $M\cap B_r(\bar p)$ with compact support in $M\cap B_r(\bar p)$. Combining (2-5) and (4-22), we have the Neumann–Poincaré inequality on exterior geodesic balls of M; i.e.,

$$\int_{M\cap B_r(\bar p)}|\varphi-\bar\varphi_{p,r}|\le\psi r\int_{M\cap B_r(\bar p)}|D\varphi| \tag{4-25}$$

for any Lipschitz function φ on $M\cap B_r(\bar p)$ with $\bar\varphi_{p,r}=\fint_{M\cap B_r(\bar p)}\varphi$. From De Giorgi–Nash–Moser iteration, the mean value inequalities hold on M for sub- and superharmonic functions on M. Define $|Du|_0=\sup_\Sigma|Du|$. Since $|Du|^2$ is subharmonic on M from (2-7), we conclude that $|Du|_0^2-|Du|^2$ is nonnegative superharmonic on M. Then (see page 42 in [Ding 2025] or Lemma 3.5 in [Ding et al. 2016] up to a suitable modification)

$$|Du|_0^2=\sup_\Sigma|Du|^2=\lim_{r\to\infty}\fint_{M\cap B_r(\bar p)}|Du|^2. \tag{4-26}$$

Let $\tilde\eta$ be a Lipschitz function on Σ with $\operatorname{supp}\tilde\eta\subset B_{2r}(p)$, $\tilde\eta\equiv1$ on $B_r(p)$ and $|D\tilde\eta|\le1/r$. We see $\tilde\eta$ as a function on M by letting $\tilde\eta(x,u(x))=\tilde\eta(x)$. From (2-6) and the Cauchy–Schwarz inequality, it follows that

$$\begin{aligned}0=\int_M\nabla u\cdot\nabla(u\tilde\eta^2)&=\int_M|\nabla u|^2\tilde\eta^2+2\int_M u\tilde\eta\nabla u\cdot\nabla\tilde\eta\\&\ge\int_M|\nabla u|^2\tilde\eta^2-\frac12\int_M|\nabla u|^2\tilde\eta^2-2\int_M u^2|\nabla\tilde\eta|^2.\end{aligned} \tag{4-27}$$

Combining this with (2-1) and (3-16), we get

$$\begin{aligned}\int_{B_r(p)}|\nabla u|^2v\le\int_M|\nabla u|^2\tilde\eta^2\le4\int_M u^2|\nabla\tilde\eta|^2\le16\beta^2\int_{B_{2r}(p)}v&\le16(1+n\beta)\beta^2\mathcal{H}^n(B_{2r}(p))\\&\le16(1+n\beta)2^n\beta^2\mathcal{H}^n(B_r(p)).\end{aligned} \tag{4-28}$$

Since $M\cap B_r(\bar p)\subset B_r(p)\times\mathbb{R}$, combining with (2-9), (4-23), (4-26), and (4-28), we get

$$\begin{aligned}\frac{|Du|_0^2}{1+|Du|_0^2}&\le\limsup_{r\to\infty}\fint_{M\cap B_r(\bar p)}\frac{|Du|^2}{v^2}\le\limsup_{r\to\infty}\fint_{M\cap B_r(\bar p)}|\nabla u|^2\\&\le\limsup_{r\to\infty}\frac{1}{\mathcal{H}^n(M\cap B_r(\bar p))}\int_{B_r(p)}|\nabla u|^2v\le c\limsup_{r\to\infty}\fint_{B_r(p)}|\nabla u|^2v\le c\beta^2.\end{aligned} \tag{4-29}$$

Letting $\beta\to\limsup_{x\to\infty}d^{-1}(x,p)|u(x)|$, we deduce (4-18), which completes the proof. □

Appendix

Let Σ be an n-dimensional complete Riemannian manifold of nonnegative Ricci curvature. Let M be a minimal graph over Σ with the graphic function u on Σ. Suppose u is not a constant. For any $r>0$ and $\bar x=(x,t_x)\in\Sigma\times\mathbb{R}$, we define

$$\mathfrak{D}_{\bar x,r}=\{(y,s)\in\Sigma\times\mathbb{R}\mid d(y,x)+|s-t_x|<r\}$$

and $\mathscr{B}_r(\bar{x}) = M \cap \mathfrak{D}_{\bar{x},r}$. For each $s \le \inf_{B_{4R}(p)} u$, write $\bar{p}_s = (p, u(p) - s)$. From Theorem 4.3 in [Ding 2021], $u - s$ satisfies Harnack's inequality as follows:

$$\sup_{\mathscr{B}_{2R}(\bar{p}_s)} (u - s) \le \vartheta \inf_{\mathscr{B}_{2R}(\bar{p}_s)} (u - s) \tag{A-1}$$

for some constant $\vartheta \ge 2$ depending only on n.

We suppose that there is a positive constant $\beta_* < \dfrac{\beta_n}{4(\vartheta - 1)}$ with β_n defined as in (4-16) such that

$$\liminf_{x \to \infty} \frac{u(x)}{d(x, p)} \ge -\beta_* \tag{A-2}$$

for some $p \in \Sigma$. Write $\epsilon = \dfrac{\beta_n}{8\beta_*(\vartheta - 1)} - \dfrac{1}{2} > 0$. There is a constant $r_\epsilon > 0$ such that

$$u(x) \ge -(1+\epsilon)\beta_* \max\{d(x, p), r_\epsilon\} \tag{A-3}$$

for all $x \in \Sigma$. For each $R \ge r_\epsilon$, let $\hat{u}_R = u + 4(1+\epsilon)\beta_* R$ and $\hat{p}_R = (p, \hat{u}_R(p)) \in \Sigma \times \mathbb{R}$. Then $\hat{u}_R > 0$ on $B_{4R}(p)$, which implies that

$$\sup_{\mathscr{B}_{2R}(\hat{p}_R)} \hat{u}_R \le \vartheta \inf_{\mathscr{B}_{2R}(\hat{p}_R)} \hat{u}_R \le \vartheta \hat{u}_R(p) \tag{A-4}$$

from (A-1). Since

$$(\vartheta - 1)\hat{u}_R(p) = (\vartheta - 1)(u(p) + 4(1+\epsilon)\beta_* R) = (\vartheta - 1)u(p) + (\beta_n + 4(\vartheta - 1)\beta_*)\frac{R}{2}, \tag{A-5}$$

we get

$$(\vartheta - 1)\hat{u}_R(p) < \beta_n R \tag{A-6}$$

for sufficiently large $R \ge r_\epsilon$. Note that $B_R(p) \times (-R + \hat{u}_R(p), R + \hat{u}_R(p)) \subset \mathfrak{D}_{\hat{p}_R, 2R}$. From (A-4), we conclude that

$$\sup_{B_R(p)} \hat{u}_R < \beta_n R \tag{A-7}$$

for all sufficiently large $R \ge r_\epsilon$. From (A-7) and the definition of $\hat{u}_R$, it follows that

$$\sup_{B_R(p)} u < \sup_{B_R(p)} \hat{u}_R - 4\beta_* R < (\beta_n - 4\beta_*)R. \tag{A-8}$$

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