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CONTINUOUS SYMMETRIZATIONS AND UNIQUENESS OF SOLUTIONS TO NONLOCAL EQUATIONS

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We show that nonlocal seminorms are strictly decreasing under the continuous Steiner rearrangement. This implies that all solutions to nonlocal equations which arise as critical points of nonlocal energies are radially symmetric and decreasing. Moreover, we show uniqueness of solutions by exploiting the convexity of the energies under a tailored interpolation in the space of radially symmetric and decreasing functions. As an application, we consider the long-time dynamics of a higher-order nonlocal equation which models the growth of symmetric cracks in an elastic medium.

1. Introduction

In [Carrillo et al. 2019], Carrillo, Hittmeir, Volzone and Yao used continuous Steiner symmetrization to show that all critical points of

$$\mathcal{E}[\rho] = \underbrace{\frac{1}{p-1}\|\rho\|^p_{L^p(\mathbb{R}^n)}}_{\text{local repulsion}} + \underbrace{\frac{1}{2}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \rho(x)\rho(y)W(x-y)\,dx\,dy}_{\text{nonlocal attraction}} \tag{1-1}$$

are radially symmetric and decreasing as long as $W:\mathbb{R}^n \to \mathbb{R}$ is isotropic and attractive, meaning $W(z) = w(|z|)$ with $w' > 0$. Noticing that the nonlinear aggregation-diffusion equation

$$\partial_t \rho = \Delta \rho^p + \nabla\cdot(\rho \nabla W * \rho) \quad \text{in } \mathbb{R}^n \times (0,T) \tag{1-2}$$

is the gradient flow of $\mathcal{E}$ in (1-1), they were able to conclude that all steady states of (1-2) are radially symmetric and decreasing; see also [Carrillo et al. 2018; Huang et al. 2024] for the extension of this result to more singular potentials. Nonlocal attraction-repulsion of interacting particle models have recently garnered a lot of attention in the mathematical community, noting in particular the case of the Patlak–Keller–Segel model [Blanchet et al. 2006; Dolbeault and Perthame 2004; Yao 2014]. Under an appropriate scaling limit, these models converge towards the higher-order degenerate Cahn–Hilliard equation [Topaz et al. 2006; Delgadino 2018; Carrillo et al. 2024; 2025; Elbar and Skrzeczkowski 2023], where the local repulsion potential is given by the Dirichlet energy or the H^1 energy.

The main aim of this paper is to extend the methods in [Carrillo et al. 2019] to more singular nonlocal equations. More specifically, we consider the models that arise when the repulsion potential energy is

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given by a fractional Gagliardo seminorm

$$[f]^p_{W^{s,p}(\mathbb{R}^n)} := \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{|f(x)-f(y)|^p}{|x-y|^{n+sp}}\,dx\,dy, \tag{1-3}$$

with $s \in (0,1)$. In the case $p=2$, we define $H^s := W^{s,2}$.

As an application, we study the long-time behavior of the fractional thin-film equation

$$\partial_t u - \operatorname{div}(u^m \nabla(-\Delta)^s u) = 0 \quad \text{in } \mathbb{R}^n \times (0,T), \tag{1-4}$$

where $m \in \mathbb{R}$ and $(-\Delta)^s$ denotes the fractional Laplacian of order $0 < 2s < 2$. It was recently proved by Lisini that (1-4) with $m=1$ is the 2-Wasserstein gradient flow of the square of the H^s seminorm up to multiplying by an explicit constant depending on dimension n and $s \in (0,1)$; see [Lisini 2024]. For $m \neq 1$, interpreting (1-4) is an open problem; we reference [Dolbeault et al. 2009] for $m \in (0,1)$. This equation arises as a model for the propagation of symmetric hydraulic fractures in an elastic medium; see below for more details.

We finally bring attention to the fact that modeling attraction and repulsion isotropically does not necessarily imply radial symmetry of steady state solutions. When the repulsion potential is nonlocal there are several examples of nonradial energy minimizers; see for instance [Kolokolnikov et al. 2011].

1.1. ***Symmetric decreasing rearrangements.*** Symmetric rearrangements are invaluable tools in the study of symmetry of solutions to partial differential equations. Thanks to the famous inequalities of Riesz [1930], Pólya and Szegö [1951], Almgren and Lieb [1989], see also [Lieb and Loss 2001; Burchard 2009], we know that the absolute minimizer of many physical energies needs to be radially symmetric and decreasing. Hence it follows that ground state solutions associated to partial differential equations that arise as first variations of these energies need to be radially symmetric and decreasing. However, this does not imply directly the symmetry of nonminimizing critical points, if they exist.

Continuous symmetrizations provide a useful way to deal with critical points; see [Kawohl 1989]. The continuous Steiner symmetrization f^τ, $0 \le \tau \le \infty$, is a continuous interpolation between the original function f and its Steiner symmetrization; see Figure 1. We write the precise definition with more details in Section 2. Brock [1995; 2000] used this method to show radial symmetry of any positive solution to the nonlinear p-Laplace equations. More recently, Carrillo, Hittmeir, Volzone, and Yao [Carrillo et al. 2019] revisited this technique to show symmetry of steady states of isotropic aggregation equations; see also Proposition 3.1 below. We further mention that the continuous Steiner symmetrization is used in [Bonacini et al. 2022] to establish a discrete isoperimetric inequality in $\mathbb{R}^2$ for Riesz-type nonlocal energies.

Our first main result is that the Gagliardo seminorms are decreasing under continuous Steiner symmetrizations. It is well known that the Gagliardo seminorms (1-3) are the natural energies associated to fractional p-Laplacians and thus are linked to free energies arising from fractional equations, such as (1-4). They also arise in game theory [Caffarelli 2012], anomalous diffusion [Metzler and Klafter 2000], minimal surfaces [Caffarelli et al. 2010], to name only a few. We refer the reader to [Di Nezza et al. 2012] for more on fractional Sobolev spaces and fractional p-Laplacians.

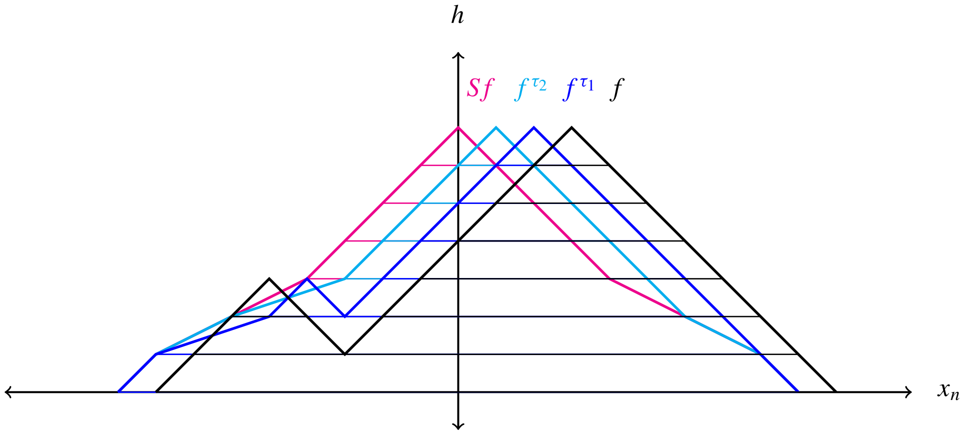

Figure 1. The continuous Steiner symmetrization f^τ for $0 < \tau_1 < \tau_2 < \infty$ as it interpolates between the function f and its Steiner symmetrization Sf.

Theorem 1.1. *Let $0 < s < 1$ and $1 < p < \infty$. For any positive $f \in L^1(\mathbb{R}^n) \cap C(\mathbb{R}^n)$ that is not radially decreasing about any center, there are constants $\gamma = \gamma(n, s, p, f) > 0$ and $\tau_0 = \tau_0(f) > 0$ and a hyperplane H such that*

$$[f^\tau]^p_{W^{s,p}(\mathbb{R}^n)} \le [f]^p_{W^{s,p}(\mathbb{R}^n)} - \gamma\tau \quad \text{for all } 0 \le \tau \le \tau_0, \tag{1-5}$$

where f^τ is the continuous Steiner symmetrization of f about H.

Our result extends to a more general class of kernels as highlighted in the next remark.

Remark 1.2. As a direct consequence of the proof, Theorem 1.1 holds for any fractional seminorm of the form

$$\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{|f(x) - f(y)|^p}{K(|x-y|)}\,dx\,dy,$$

where $K : [0, \infty) \to [0, \infty]$ is increasing.

To recover the corresponding local results as $s \to 1^-$ and $s \to 0^+$, one must normalize the energy by multiplying (1-5) by $s(1-s)$; see [Bourgain et al. 2001; Maz'ya and Shaposhnikova 2002]. We will showcase in Remark 3.6 that the constant $\gamma = \gamma(s)$ in Theorem 1.1 remains strictly positive and bounded as $s \to 0^+, 1^-$. Consequently, $s(1-s)\gamma \to 0$ as $s \to 0^+, 1^-$. Regarding $s = 0$, it is known that continuous Steiner symmetrizations preserve the L^p norms; see [Carrillo et al. 2019]. As for $s = 1$, we have the following.

Corollary 1.3. *Let $1 < p < \infty$. For any nonnegative $f \in L^1(\mathbb{R}^n) \cap C(\mathbb{R}^n)$ that is not radially decreasing about any center, there is a constant $\tau_0 = \tau_0(f) > 0$ such that*

$$[f^\tau]^p_{W^{1,p}(\mathbb{R}^n)} \le [f]^p_{W^{1,p}(\mathbb{R}^n)} \quad \text{for all } 0 \le \tau \le \tau_0.$$

Consequently,

$$[f^\tau]_{\mathrm{Lip}(\mathbb{R}^n)} \le [f]_{\mathrm{Lip}(\mathbb{R}^n)} \quad \text{for all } 0 \le \tau \le \tau_0. \tag{1-6}$$

The control on the Lipschitz norm in (1-6) was first established by Brock [1995, Theorem 11] for a different variant of continuous Steiner symmetrizations but can be adapted to our setting. We echo his observation in Remark 8 of the same work, which states that Lipschitz continuity is the best regularity one can expect under continuous Steiner symmetrizations as kinks can form when symmetrizing a C^1 function that is not quasiconvex. The inequality in Corollary 1.3 is not strict as a simple counterexample can be constructed using that the norm is local; see Example 4.9.

Since Steiner symmetrizations are rearrangements in $\mathbb{R}^n$ with respect to a single direction, the proof of Theorem 1.1 relies on a corresponding one-dimensional result for characteristic functions (see Lemma 3.4). In fact, the definition of continuous Steiner symmetrization of a set (and hence a function) is understood first in terms of open intervals, then finite unions of open intervals, and lastly infinite unions. Accordingly, we have found it insightful to make a special study of those functions whose level sets can be expressed as a finite union of open intervals, also known as *good functions*. We establish an explicit version of Theorem 1.1 for good functions. Here, we will explain the simplest setting and delay the detailed result until Section 4.

Let $p = 2$ and consider a function with a simple geometry, that is, a positive function $f : \mathbb{R} \to \mathbb{R}$ whose level sets are each a single open interval. In particular, for each $h > 0$, there are at most two solutions to $f(x) = h$, which we denote by $x_- = x_-(h)$ and $x_+ = x_+(h)$. Note that if f is radially decreasing, then each level set (x_-, x_+) is centered at the origin. We use a continuous Steiner symmetrization with constant speed towards the origin:

$$x_\pm^\tau = x_\pm - \tau \operatorname{sgn}(x_+ + x_-) \quad \text{for all } 0 \le \tau \le \tfrac{1}{2}|x_+ + x_-|. \tag{1-7}$$

The energy $[f^\tau]^2_{H^s(\mathbb{R})}$ is a double integral involving f^τ over the spatial variables $x, y \in \mathbb{R}$. Formally, we can make a change of variables to write the energy instead as a double integral involving $x_\pm^\tau$ and $y_\pm^\tau$ over the heights $h, u > 0$, where $f^\tau(x_\pm) = h$ and $f^\tau(y_\pm) = u$; see Lemma 4.14. With this, we can write the derivative of the energy for f^τ in terms of the level sets of f as

$$\begin{aligned}\frac{d[f^\tau]^2_{H^s(\mathbb{R})}}{d\tau}\bigg|_{\tau=0} = c_s \int_0^\infty\!\!\int_0^\infty &(\operatorname{sgn}(x_+ + x_-) - \operatorname{sgn}(y_+ + y_-)) \\ &\times \left[\frac{\operatorname{sgn}(x_+ - y_+)}{|x_+ - y_+|^{2s}} - \frac{\operatorname{sgn}(x_+ - y_-)}{|x_+ - y_-|^{2s}} - \frac{\operatorname{sgn}(x_- - y_+)}{|x_- - y_+|^{2s}} + \frac{\operatorname{sgn}(x_- - y_-)}{|x_- - y_-|^{2s}}\right] dh\, du.\end{aligned}$$

In the integrand, we see the derivative in τ of (1-7) multiplied by an antiderivative of the kernel at the endpoints of the corresponding level sets. We will show in the proof of Proposition 4.10 that this product is negative when the level sets (x_-, x_+) and (y_-, y_+) are not centered. We should note that a different expression for the derivative can already be found in [Carrillo et al. 2019, (2.23)] for more regular kernels. In Section 4, we also write an explicit formula for the derivative $d/d\tau|\nabla f^\tau|^p_{L^p}$; see Proposition 4.6 and Corollary 4.7.

1.2. *Uniqueness.* Uniqueness of critical points within the class of positive and fixed-mass functions does not follow immediately from the fact that these are radially symmetric and decreasing. For the specific case of the nonlinear aggregation equation (1-2), when $p \in (1, 2)$ one can construct an ad hoc isotropic attractive interaction potential such that there are an infinite amount of steady state solutions; see [Delgadino et al. 2022, Theorem 1.2]. On the other hand, in the case $p \in [2, \infty)$, [Delgadino et al. 2022] also shows uniqueness of critical points by introducing a height function interpolation curve over radially symmetric profiles and showing that the associated energy (1-1) is strictly convex under the interpolation. See Section 5 for definitions and details.

We show that the square of the H^s seminorms are strictly convex under the height function interpolation.

Theorem 1.4. *Fix $0 < s < 1$. Let $f_0, f_1 \in C(\mathbb{R}^n)$ be two distinct, nonnegative, symmetric decreasing functions with unit mass, and let $\{f_t\}_{t\in[0,1]}$ be the height function interpolation between f_0 and f_1. Then*

$$t \mapsto \|f_t\|^2_{H^s(\mathbb{R}^n)}$$

is strictly convex for all $0 < t < 1$.

The uniqueness of solutions to fractional Laplace equations is a deep and active area of research; see for instance [Frank and Lenzmann 2013; Frank et al. 2016; Chan et al. 2020; de Pablo et al. 2011; Vázquez 2014; Cabré and Sire 2015; Bonforte et al. 2017; Caffarelli and Silvestre 2009]. Currently, the methods to show uniqueness within the class of radially symmetric states are quite involved and at times only address the uniqueness of global minimizers and not of general critical points [Frank and Lenzmann 2013; Frank et al. 2016]. The spirit of Theorem 1.4 is to try to simplify the theory, when possible.

The uniqueness methods presented here do not cover the general Gagliardo seminorm $W^{s,p}$ for $p \neq 2$. Still, we are able to show the convexity under the interpolant of $W^{1,p}$ seminorms for $p \geq 2n/(n+1)$; see Proposition 5.2. Moreover, we also cover the case of the potential energy when the potential is radial and increasing, which we use in the next section; see Proposition 5.4.

1.3. ***Application to fractional thin-film equations.*** As an application of Theorems 1.1 and 1.4, we study the uniqueness of stationary solutions and the long-time asymptotic of fractional thin-film equations given by (1-4). The fractional thin-film equation with exponent $s = \frac{1}{2}$ and mobility $m = 3$ was originally derived to model the growth of symmetric hydraulic fractures in an elastic material arising from the pressure of a viscous fluid pumped into the opening; see the original references [Geertsma and De Klerk 1969; Zheltov and Khristianovich 1955]. A practical man-made application of this phenomenon is commonly known as fracking, which enhances oil or gas extraction from a well. In nature, this process occurs in volcanic dikes when magma causes fracture propagation through the earth's crust and also when water opens fractures in ice shelf.

Nonetheless, due to the nonlocal and higher-order nature of this equation, there is a striking lack of mathematical analysis regarding solutions to fractional thin-film equations. Indeed, (1-4) is an interpolation between the second-order porous medium equation ($s = 0$), see [Vázquez 2007], and the fourth-order thin-film equation ($s = 1$), see [Bertozzi and Pugh 1994; Otto 1998]. We mention that the study of self-similar solutions was first started by Spence and Sharp [1985], but rigorous existence of solutions was only recently shown by Imbert and Mellet [2011; 2015].

Even more recently, Segatti and Vazquez [2020] studied the long-time behavior of (1-4) with linear mobility $m(u) = u$ by studying the rescalings of Barenblatt [1962]. Namely, if u is a solution to (1-4), we consider v defined by the rescaling

$$u(x,t)=\frac{1}{(1+t)^{\alpha}}v\left(\frac{x}{(1+t)^{\beta}},\log(1+t)\right)\quad\text{in }\mathbb{R}^n\times(0,T),$$

where $\alpha,\beta>0$ are given explicitly by

$$\alpha=\frac{n}{n+2(1+s)},\qquad \beta=\frac{1}{n+2(1+s)}.$$

The function $v=v(y,\tau)$ satisfies the rescaled equation

$$\partial_\tau v=\nabla\cdot\left(v\nabla_y\left((-\Delta)^s v+\beta\frac{|y|^2}{2}\right)\right),\tag{1-8}$$

which contains an extra confining term. Under an extra qualitative assumption on the integrability of the gradient, Segatti and Vazquez [2020, Theorem 5.9] showed that v converges as $\tau\to\infty$ to a solution of

$$\begin{cases}(-\Delta)^s v=\sum_i\lambda_i\chi_{\mathcal{P}_i}(y)-\frac{1}{2}\beta|y|^2 & \text{in }\operatorname{supp}(v)\subset\mathbb{R}^n,\\ v\geq 0 & \text{in }\mathbb{R}^n,\end{cases}\tag{1-9}$$

where $\mathcal{P}_i$ are the connected components of $\operatorname{supp}(v)$ and λ_i are the corresponding Lagrange multipliers, which can change from one connected component to another.

As is the case for higher-order equations, like for instance the classical thin-film equation, uniqueness results that do not assume strict positivity of the functions are rare. We mention the work of Majdoub, Masmoudi and Tayachi [Majdoub et al. 2018] as one of the few available examples on uniqueness of source solutions to the thin-film equation. With respect to the problem at hand, Segatti and Vazquez showed the solution to (1-9) is unique under the extra assumption that the solution has a single connected component. In this work, we instead use the rearrangement techniques described above to show that the solution to (1-9) is first radially symmetric and then unique by using Theorems 1.1 and 1.4, respectively.

Theorem 1.5. *Fix $0<s<\frac{1}{2}$, and let $v\in C^{0,1}(\mathbb{R}^n)$ be a compactly supported solution to (1-9); then v is radially decreasing. Moreover, up to scaling, it is uniquely given by*

$$v(x)=\frac{1}{\lambda^s\kappa}(1-\lambda|x|^2)_+^{1+s},\tag{1-10}$$

where $\lambda>0$ and $\kappa=4^s\Gamma(s+2)\Gamma\left(s+\frac{1}{2}n\right)/\Gamma\left(\frac{1}{2}n\right)$.

Remark 1.6. The function v in (1-10) belongs to C^{1+s}; hence the Lipschitz assumption in Theorem 1.5 is natural, but it is not currently known.

Following [Lisini 2024] (see also [Otto 1998]), (1-8) is the 2-Wasserstein gradient flow of the energy functional

$$\mathcal{E}(v)=c_{n,s}[v]^2_{H^s(\mathbb{R}^n)}+\frac{1}{2}\beta\int_{\mathbb{R}^n}|y|^2v(y)\,dy,\tag{1-11}$$

where $c_{n,s}\simeq s(1-s)>0$ is a normalizing constant, depending only on $n\in\mathbb{N}$ and $s\in(0,1)$, such that $c_{n,s}[v]^2_{H^s}=\langle(-\Delta)^s v,v\rangle_{L^2(\mathbb{R}^n)}$. Hence any steady state is a critical point of (1-11). By Theorem 1.1, we can show that if v is not radially decreasing, then (1-11) is decreasing to first-order under continuous

Steiner symmetrization. However, notice from Figure 1 that f^τ does not necessarily preserve the support of f, which we need for the proof of Theorem 1.5 since (1-9) is only satisfied in $\operatorname{supp}(v)$. To address this issue, we slow down the speed of the level sets near the base of the solution v. In particular, for $h > 0$, if $x_\pm = x_\pm(h)$ are the boundary of the superlevel set $\{f > h\}$, then we replace (1-7) with

$$x_\pm^\tau = x_\pm - \tau \operatorname{sgn}(x_+ + x_-) \min\left\{1, \frac{h}{h_0}\right\}$$

for some small, fixed $h_0 > 0$. Unlike (1-7), the perturbation only makes sense for superlevel sets. The regularity assumption on v ensures that the perturbation is well-defined, as in general level sets can fall, see Figure 5. This idea first appeared in the work of Carrillo, Hittmeir, Volzone, and Yao [Carrillo et al. 2019] and is fundamental for these types of free boundary problems.

We should note that our uniqueness result does not complete the full characterization of the long-time asymptotic of the fractional thin-film equation. Our methods only cover the range $s \in \left(0, \frac{1}{2}\right)$ and require a Lipschitz regularity assumption, which is not currently known. Moreover, the convergence result of Segatti and Vazquez [2020, Theorem 5.9] requires an extra qualitative condition on the integrability of the gradient of the solution, which is also not currently known.

Lastly, we note that the symmetrization methods in Theorem 1.5 hold for any equation that arises as a positive mass-constrained critical point of energies that are a combination of:

- isotropic local first-order seminorms:

$$\int_{\mathbb{R}^n} G(|\nabla v|)\,dx \quad \text{with } G : [0, \infty] \to \mathbb{R} \text{ convex},$$

- isotropic nonlocal seminorms:

$$\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{|v(x) - v(y)|^p}{K(|x - y|)}\,dx\,dy \quad \text{with } K : [0, \infty] \to [0, \infty] \text{ increasing},$$

- local functionals:

$$\int_{\mathbb{R}^n} F(v)\,dx \quad \text{with } F : [0, \infty] \to \mathbb{R},$$

- isotropic interaction energies:

$$\frac{1}{2}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} W(|x - y|)v(x)v(y)\,dx\,dy \quad \text{with } W : [0, \infty] \to \mathbb{R} \text{ increasing},$$

- radial potential functionals:

$$\int_{\mathbb{R}^n} U(|x|)v(x)\,dx \quad \text{with } U : [0, \infty] \to \mathbb{R} \text{ increasing}.$$

The uniqueness strategy presented here is a bit more finicky and only holds for a strict subset of these equations.

1.4. *Organization of the paper.* The rest of the paper is organized as follows. Section 2 contains background and preliminary results on continuous Steiner symmetrizations. Theorem 1.1 is proved in Section 3. An explicit version of Theorem 1.1 for good functions is discussed in Section 4. Section 5 presents the height function interpolation and proves Theorem 1.4. Finally, Section 6 establishes properties of truncated continuous Steiner symmetrizations which are then used to prove Theorem 1.5.

2. Continuous Steiner symmetrization

In this section, we present background and preliminaries on continuous Steiner symmetrizations with constant speed in the direction $e \in \mathbb{S}^{n-1}$. For simplicity in the presentation, we will assume that $e = e_n$ and write $x = (x', x_n) \in \mathbb{R}^{n-1} \times \mathbb{R}$.

Given $x' \in \mathbb{R}^{n-1}$, we denote the section of an open subset $U \subset \mathbb{R}^n$ in the direction e_n by

$$U_{x'} = \{x_n \in \mathbb{R} : (x', x_n) \in U\}.$$

The Steiner symmetrization of U with respect to the direction e_n is defined by

$$S(U) = \{x = (x', x_n) \in \mathbb{R}^n : x_n \in U^*_{x'}\},$$

where $U^*_{x'} = \left\{x_n \in \mathbb{R} : |x_n| < \frac{1}{2}|U_{x'}|\right\}$ is the symmetric rearrangement of $U_{x'}$ in $\mathbb{R}$. Note that $|U^*_{x'}| = |U_{x'}|$. To define Sf for a nonnegative function $f \in L^1(\mathbb{R}^n) \cap C(\mathbb{R}^n)$, we denote the $h > 0$ level sets of f in the direction of e_n by

$$U^h_{x'} = \{x_n \in \mathbb{R} : f(x', x_n) > h\}.$$

Then, the Steiner symmetrization Sf of f in the direction e_n is given by

$$Sf(x) = \int_0^\infty \chi_{S(U^h_{x'})}(x_n)\, dh.$$

Definition 2.1. The continuous Steiner symmetrization of an open set $U \subset \mathbb{R}$ is denoted by $M^\tau(U)$, $\tau \geq 0$, and defined as follows.

(1) *Intervals.* If $U = (y_-, y_+)$, then $M^\tau(U) = (y^\tau_-, y^\tau_+)$, where

$$\begin{cases} y^\tau_- = y_- - \tau \operatorname{sgn}(y_+ + y_-), \\ y^\tau_+ = y_+ - \tau \operatorname{sgn}(y_+ + y_-) \end{cases} \qquad \text{for } 0 < \tau \leq \tfrac{1}{2}|y_+ + y_-|$$

and

$$\begin{cases} y^\tau_- = -\frac{1}{2}(y_+ - y_-), \\ y^\tau_+ = \frac{1}{2}(y_+ - y_-) \end{cases} \qquad \text{for } \tau > \tfrac{1}{2}|y_+ + y_-|.$$

(2) *Finite union of intervals.* If $U = \bigcup_{i=1}^m I_i$, $m \in \mathbb{N}$, where I_i are disjoint, open intervals, then

$$M^\tau(U) = \bigcup_{i=1}^m M^\tau(I_i) \quad \text{for } 0 \leq \tau < \tau_1,$$

where τ_1 is the first time that two intervals $M^\tau(I_i)$ touch. At τ_1, we merge the two intervals and start again.

(3) *Countably infinite union of intervals*. If $U = \bigcup_{i=1}^{\infty} I_i$, where I_i are disjoint, open intervals, then

$$M^\tau(U) = \bigcup_{i=1}^{\infty} M^\tau(U_i), \quad \text{where } U_m = \bigcup_{i=1}^{m} I_i,\ m \in \mathbb{N}.$$

Definition 2.2. The continuous Steiner symmetrization of a nonnegative function $f \in L^1(\mathbb{R}^n) \cap C(\mathbb{R}^n)$ in the direction e_n is denoted by f^τ, $\tau \geq 0$, and defined as

$$f^\tau(x) = \int_0^\infty \chi_{M^\tau(U_{x'}^h)}(x_n)\,dh \quad \text{for } x = (x', x_n) \in \mathbb{R}^n.$$

By definition, f^τ interpolates continuously between $f = f^0$ and $Sf = f^\infty$; see Figure 1. As a consequence of the layer-cake representation, the continuous Steiner symmetrization of f preserves the L^p norm, see [Carrillo et al. 2019, Lemma 2.14],

$$\|f\|_{L^p(\mathbb{R}^n)} = \|f^\tau\|_{L^p(\mathbb{R}^n)}, \quad 1 \leq p \leq \infty. \tag{2-1}$$

We also have the semigroup property presented in the next lemma.

Lemma 2.3 [Carrillo et al. 2019, Lemma 2.1]. *The collection of operators $(M^\tau)_{\tau \geq 0}$ satisfies the semigroup property. That is, for each $\tau_1, \tau_2 \geq 0$ and any open set $U \subset \mathbb{R}$,*

$$M^{\tau_1}(M^{\tau_2}(U)) = M^{\tau_1+\tau_2}(U).$$

Consequently, f^τ satisfies the semigroup property: $(f^{\tau_1})^{\tau_2} = f^{\tau_1+\tau_2}$ for $\tau_1, \tau_2 \geq 0$.

A priori, one could define M^τ with any sufficiently smooth speed $V = V(y,h) : \mathbb{R} \times \mathbb{R}_+ \to \mathbb{R}$ by replacing $\operatorname{sgn}(y)$ with $V(y,h)$ in Definition 2.1(1). With a different speed however, $(M^\tau)_{\tau \geq 0}$ will not necessarily satisfy the desired semigroup property. Instead, one should replace Definition 2.1(1) with the ODE

$$\begin{cases} \frac{d}{d\tau}[y_\pm^\tau] = -V(y_+^\tau + y_-^\tau, h), & \tau > 0, \\ y_\pm^\tau = y_\pm, & \tau = 0. \end{cases} \tag{2-2}$$

With this modification, $(M^\tau)_{\tau \geq 0}$ satisfies the semigroup property, as long as the level sets remain ordered; see Section 6.1.

Remark 2.4. The continuous symmetrization considered by Brock [2000; 1995] is equivalent to taking $V(y,h) = y$.

Lastly, we note the following consequence of the semigroup property.

Lemma 2.5. *Let $h > 0$ and $\tau_1, \tau_2 \geq 0$. For a nonnegative function $f \in L^1(\mathbb{R}^n) \cap C(\mathbb{R}^n)$, we have*

$$\operatorname{dist}(\partial\{f^{\tau_1} > h\}, \partial\{f^{\tau_2} > h\}) \leq |\tau_1 - \tau_2|.$$

Proof. If $\tau_2 = 0$, then it is clear from the definition that

$$\operatorname{dist}(\partial\{f^{\tau_1} > h\}, \partial\{f > h\}) \leq \tau_1.$$

The result follows from Lemma 2.3. ∎

3. On Theorem 1.1

This section contains the proof of Theorem 1.1. We begin by presenting a simplified version of the result in [Carrillo et al. 2019].

Proposition 3.1 [Carrillo et al. 2019, Proposition 2.15]. *Consider $W \in C^1(\mathbb{R}^n)$ an increasing radially symmetric kernel with associated interaction energy*

$$\mathcal{I}[f] = \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} f(x)f(y)W(x-y)\,dx\,dy.$$

Assume $f \in L^1(\mathbb{R}^n)$ is positive and not radially decreasing. Then, there exist constants $\gamma = \gamma(W, f) > 0$ and $\tau_0 = \tau_0(f) > 0$ and a hyperplane H such that

$$\mathcal{I}[f^\tau] \le \mathcal{I}[f] - \gamma\tau \quad \text{for all } 0 \le \tau \le \tau_0,$$

where f^τ is the continuous Steiner symmetrization about H.

The original result in [Carrillo et al. 2019] allows for more singular kernels, but our prototype kernels $W(x) \approx -|x|^{-n-sp}$ are too singular to directly apply their result. In fact, for $\mathcal{I}(f)$ to be well-defined in our setting, one must replace $f(x)f(y)$ by $|f(x)-f(y)|^p$. To see this, consider the case $p=2$ and $W(x) = c_{n,s}|x|^{-2s}$, where $c_{n,s} > 0$ is the normalizing constant for the fractional Laplacian. Using the Fourier transform, we can formally write

$$\mathcal{I}[f] = \int_{\mathbb{R}^n} |\hat{f}(\xi)|^2 \widehat{W}(\xi)\,d\xi,$$

but $\widehat{W}(\xi)$ is not defined. On the other hand, using the definition of $(-\Delta)^s$,

$$\frac{1}{2}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} |f(x)-f(y)|^2 W(x-y)\,dy\,dx = \int_{\mathbb{R}^n} f(x)(-\Delta)^s f(x)\,dx = \int_{\mathbb{R}^n} |\hat{f}(\xi)|^2|\xi|^{2s}\,d\xi,$$

which is a well-defined seminorm.

We use an ε-regularization of W for which Proposition 3.1 holds. For each $0 < \varepsilon \le 1$, we consider the energy given by

$$\mathcal{F}^p_\varepsilon(f) = \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} |f(x)-f(y)|^p W_\varepsilon(x-y)\,dx\,dy, \qquad W_\varepsilon(x) := \frac{1}{|x|^{n+sp}+\varepsilon}. \tag{3-1}$$

Notice that the kernel associated to $\mathcal{F}^p_\varepsilon$ is integrable for each fixed $\varepsilon > 0$ and that

$$\lim_{\varepsilon\to 0} \mathcal{F}^p_\varepsilon(f) = \sup_{0<\varepsilon\le 1} \mathcal{F}^p_\varepsilon(f) = [f]^p_{W^{s,p}(\mathbb{R}^n)}.$$

Using that W_ε is radially symmetric, the energy $\mathcal{F}^p_\varepsilon(f)$ can be written as

$$\begin{aligned}\mathcal{F}^p_\varepsilon(f) &= \int_{\mathbb{R}^{2n}} (|f(x)-f(y)|^p - |f(x)|^p - |f(y)|^p) W_\varepsilon(x-y)\,dx\,dy \\ &\qquad\qquad + \int_{\mathbb{R}^{2n}} (|f(x)|^p + |f(y)|^p) W_\varepsilon(x-y)\,dx\,dy \\ &= \int_{\mathbb{R}^{2n}} (|f(x)-f(y)|^p - |f(x)|^p - |f(y)|^p) W_\varepsilon(x-y)\,dx\,dy + C_\varepsilon \|f\|^p_{L^p(\mathbb{R}^n)},\end{aligned}$$

where the constant $C_\varepsilon = C_\varepsilon(s, p)$ satisfies

$$C_\varepsilon = 2\int_{\mathbb{R}^n} W_\varepsilon(y)\,dy \to \infty \quad \text{as } \varepsilon \to 0^+. \tag{3-2}$$

For convenience, we define

$$\begin{aligned}\mathcal{I}_\varepsilon^p(f) &:= \mathcal{F}_\varepsilon^p(f) - C_\varepsilon \|f\|_{L^p(\mathbb{R}^n)}^p \\ &= \int_{\mathbb{R}^{2n}} (|f(x)-f(y)|^p - |f(x)|^p - |f(y)|^p) W_\varepsilon(x-y)\,dx\,dy.\end{aligned} \tag{3-3}$$

Consider the continuous Steiner symmetrization f^τ of f. As a consequence of (2-1),

$$\frac{d}{d\tau}[\mathcal{F}_\varepsilon^p(f^\tau)] = \frac{d}{d\tau}[\mathcal{I}_\varepsilon^p(f^\tau) + C_\varepsilon \|f\|_{L^p(\mathbb{R}^n)}^p] = \frac{d}{d\tau}[\mathcal{I}_\varepsilon^p(f^\tau)].$$

In the special case of $p = 2$, the integrand in $\mathcal{I}_\varepsilon^2(f)$ simplifies nicely, and we get

$$\frac{d}{d\tau}[\mathcal{F}_\varepsilon^2(f^\tau)] = \frac{d}{d\tau}\left[-2\int_{\mathbb{R}^{2n}} f(x)f(y)W_\varepsilon(x-y)\,dx\,dy\right] = -2\frac{d}{d\tau}\langle f^\tau, W_\varepsilon * f^\tau\rangle_{L^2(\mathbb{R}^n)}.$$

Since $\widetilde{W}_\varepsilon := -2W_\varepsilon \in C^1(\mathbb{R}^n)$ is symmetric and increasing along its rays, we apply Proposition 3.1 to find constants $\gamma_\varepsilon, \tau_0 > 0$ such that

$$\langle f^\tau, \widetilde{W}_\varepsilon * f^\tau\rangle_{L^2(\mathbb{R}^n)} \le \langle f, \widetilde{W}_\varepsilon * f\rangle_{L^2(\mathbb{R}^n)} - \gamma_\varepsilon \tau \quad \text{for all } 0 \le \tau \le \tau_0.$$

We will show that γ_ε can be bounded uniformly from below in $0 < \varepsilon \le 1$ and also that we can handle all $1 < p < \infty$. More precisely, we prove the following result.

Proposition 3.2. *Let $0 < s < 1$, $1 < p < \infty$, and $f \in L^1(\mathbb{R}^n) \cap C(\mathbb{R}^n)$ be nonnegative. Assume that, up to translation or rotation, the nonlinear center of mass is at the origin*

$$\int_{\mathbb{R}^n} \tan^{-1}(x_n) f(x)\,dx = 0 \tag{3-4}$$

and f is not symmetric decreasing across the plane $\{x_n = 0\}$. Then, there are constants $\gamma = \gamma(n, s, p, f)$, $\tau_0 = \tau_0(f) > 0$, independent of ε, such that

$$\mathcal{I}_\varepsilon^p(f^\tau) \le \mathcal{I}_\varepsilon^p(f) - \gamma\tau \quad \textit{for all } 0 \le \tau \le \tau_0.$$

Remark 3.3. In the original reference [Carrillo et al. 2019], the condition of the nonlinear center of mass (3-4) is replaced by the hyperplane $\{x_n = 0\}$ dividing the mass in half. We impose condition (3-4) to simplify some measure-theoretical aspects of the proof. We chose the function $\tan^{-1}(x_n)$ because it is odd, strictly monotone, and bounded, which makes the integral well-defined under the assumption $f \in L^1(\mathbb{R}^n)$.

Since f^τ is defined as an integral in terms of the one-dimensional level sets $U_{x'}^h \subset \mathbb{R}$ of f, roughly speaking, one can reduce the proof of Proposition 3.2 to a one-dimensional setting. For a fixed $\ell > 0$, consider the one-dimensional kernel

$$K_\varepsilon(r) = K_{\varepsilon,\ell}(r) = \frac{1}{(\ell^2 + r^2)^{(n+sp)/2} + \varepsilon} \quad \text{for } 0 < \varepsilon \le 1,$$

so that $W_\varepsilon(x', x_n) = K_{\varepsilon,|x'|}(x_n)$. With the layer-cake-type representation

$$|f(x)-f(y)|^p - |f(x)|^p - |f(y)|^p = -p(p-1)\int_0^\infty\!\!\int_0^\infty |h-u|^{p-2}\chi_{U^h_{x'}}(x_n)\chi_{U^h_{y'}}(y_n)\,dh\,du,$$

we write (3-3) as

$$\begin{aligned}\mathcal{I}^p_\varepsilon(f) &= \int_{\mathbb{R}^{2n}} (|f(x',x_n)-f(y',y_n)|^p - |f(x',x_n)|^p - |f(y',y_n)|^p)K_{\varepsilon,|x'-y'|}(x_n-y_n)\,dx\,dy\\ &= -p(p-1)\int_{\mathbb{R}^{2n}}\int_{\mathbb{R}^2_+} |h-u|^{p-2}\chi_{U^h_{x'}}(x_n)\chi_{U^h_{y'}}(y_n)K_{\varepsilon,|x'-y'|}(x_n-y_n)\,dh\,du\,dx\,dy\\ &= -p(p-1)\int_{\mathbb{R}^{2(n-1)}}\int_{\mathbb{R}^2_+}|h-u|^{p-2}\int_{\mathbb{R}^2}\chi_{U^h_{x'}}(x_n)\chi_{U^h_{y'}}(y_n)\\ &\qquad\qquad \times K_{\varepsilon,|x'-y'|}(x_n-y_n)\,dx_n\,dy_n\,dh\,du\,dx'\,dy'. \end{aligned}\tag{3-5}$$

Consequently,

$$\begin{aligned}\frac{d}{d\tau}[\mathcal{I}^p_\varepsilon(f^\tau)] = -p(p-1)\int_{\mathbb{R}^{2(n-1)}}\int_{\mathbb{R}^2_+}|h-u|^{p-2}\frac{d}{d\tau}\int_{\mathbb{R}^2}\big[\chi_{M^\tau(U^h_{x'})}(x_n)\chi_{M^\tau(U^h_{y'})}(y_n)\\ \times K_{\varepsilon,|x'-y'|}(x_n-y_n)\big]\,dx_n\,dy_n\,dh\,du\,dx'\,dy'.\end{aligned}$$

Hence, to establish Proposition 3.2, we first study the corresponding problem in one dimension.

For open sets $U_1, U_2 \subset \mathbb{R}$, define

$$I_\varepsilon(\tau) = I_\varepsilon[U_1,U_2](\tau) := \int_{\mathbb{R}^2}\chi_{M^\tau(U_1)}(x)\chi_{M^\tau(U_2)}(y)K_\varepsilon(x-y)\,dx\,dy,$$

where $K_\varepsilon = K_{\varepsilon,\ell}$ for a fixed $\ell > 0$. The main lemma of this section establishes that $I_\varepsilon(\tau)$ is strictly increasing in $\tau > 0$ when U_1 and U_2 are sufficiently separated.

For a function $g = g(\tau)$, we denote the upper and lower Dini derivatives of g respectively by

$$\frac{d^+}{d\tau}g(\tau) = \limsup_{\delta\to 0^+}\frac{g(\tau+\delta)-g(\tau)}{\delta} \quad\text{and}\quad \frac{d^-}{d\tau}g(\tau) = \limsup_{\delta\to 0^-}\frac{g(\tau+\delta)-g(\tau)}{\delta}.\tag{3-6}$$

Lemma 3.4. *Let $U_1, U_2 \subset \mathbb{R}$ be open sets with finite measure. Then*

$$\frac{d^+}{d\tau}I_\varepsilon(\tau) \ge 0 \quad \textit{for all } \tau \ge 0.\tag{3-7}$$

If, in addition, there exist $0 < a < 1$ and $R > \max\{|U_1|, |U_2|\}$ such that $\big|U_1 \cap \big(\tfrac12|U_1|, R\big)\big| > a$ and $\big|U_2 \cap \big(-R, -\tfrac12|U_2|\big)\big| > a$, then

$$\frac{d^+}{d\tau}I_\varepsilon(\tau) \ge \frac{1}{128}ca^3 > 0 \quad \textit{for all } 0 \le \tau \le \frac14 a,\tag{3-8}$$

where

$$c = \min\big\{|K_1'(r)| : r \in \big[\tfrac14 a, 4R\big]\big\}.$$

To prove Lemma 3.4, we apply Propositions 2.16 and 2.17 in [Carrillo et al. 2019] to I_ε and show that the upper Dini derivative of I_ε can be uniformly bounded below. For the sake of the reader, we first provide a brief, formal argument in the simplest setting. Indeed, if $U_i = [c_i - r_i, c_i + r_i]$, $i = 1, 2$, then we can write

$$I_\varepsilon(\tau) = \int_{-r_1+c_1-\tau\,\mathrm{sgn}(c_1)}^{r_1+c_1-\tau\,\mathrm{sgn}(c_1)} \int_{-r_2+c_2-\tau\,\mathrm{sgn}(c_2)}^{r_2+c_2-\tau\,\mathrm{sgn}(c_2)} K_\varepsilon(x-y)\,dy\,dx. \tag{3-9}$$

By the semigroup property, it is enough to take the derivative at $\tau = 0$ and estimate

$$\frac{d^+}{d\tau} I_\varepsilon(0) = (\mathrm{sgn}(c_2) - \mathrm{sgn}(c_1)) \int_{-r_1}^{r_1} \int_{-r_2+c_2-c_1}^{r_2+c_2-c_1} K'_\varepsilon(x-y)\,dy\,dx. \tag{3-10}$$

If $c_2 > c_1$, then $Q = [-r_1, r_1] \times [-r_2 + c_2 - c_1, r_2 + c_2 - c_1]$ is a rectangle in the xy-plane centered across $\{x = 0\}$. Since K_ε is increasing in $\{y > 0\}$ and decreasing in $\{y < 0\}$, one can show that $d^+/d\tau\, I_\varepsilon(0) > 0$. The more refined lower bound is roughly controlled by the size of the excess strip $[-r_1, r_1] \times \left[r_2 + \frac{1}{2}(c_2 - c_1), r_2 + c_2 - c_1\right]$ and by K'_ε.

Proof of Lemma 3.4. First consider when $U_i = [c_i - r_i, c_i + r_i]$, $i = 1, 2$, are intervals. Then, following the proof of [Carrillo et al. 2019, Lemma 2.16], we can show that

$$\frac{d^+}{d\tau} I_\varepsilon(0) \geq d_\varepsilon \min\{r_1, r_2\}|c_2 - c_1|,$$

where

$$d_\varepsilon = \min\left\{|K'_\varepsilon(r)| : r \in \left[\tfrac{1}{2}|c_2 - c_1|, r_1 + r_2 + |c_2 - c_1|\right]\right\}.$$

Since

$$K'_\varepsilon(r) = -\frac{n+sp}{((\ell^2 + r^2)^{(n+sp)/2} + \varepsilon)^2} (\ell^2 + r^2)^{(n+sp)/2-1} r$$

for all $0 < \varepsilon \leq 1$, we have

$$|K'_\varepsilon(r)| \geq |K'_1(r)|. \tag{3-11}$$

Hence $d_\varepsilon \geq d_1$ and

$$\frac{d^+}{d\tau} I_\varepsilon(0) \geq d_1 \min\{r_1, r_2\}|c_2 - c_1| \quad \text{for all } 0 < \varepsilon \leq 1.$$

With this, we follow the proof of [Carrillo et al. 2019, Lemma 2.17] for U_1, U_2 finite open sets to show

$$\frac{d^+}{d\tau} I_\varepsilon(\tau) \geq \frac{1}{128} c_\varepsilon a^3 > 0 \quad \text{for all } 0 \leq \tau \leq \frac{1}{4}a,$$

where

$$c_\varepsilon = \min\left\{|K'_\varepsilon(r)| : r \in \left[\tfrac{1}{4}a, 4R\right]\right\} \geq \min\left\{|K'_1(r)| : r \in \left[\tfrac{1}{4}a, 4R\right]\right\} = c. \qquad \square$$

Before proceeding with the proof of Proposition 3.2, we will need the following technical lemma for the case $p \neq 2$.

Lemma 3.5. *Under the assumptions of Proposition 3.2, for $a > 0$ small and $R > 0$ large, define the sets B^a_+, B^a_-:*

$$\begin{aligned} B^a_+ &= \left\{(x', h) \in \mathbb{R}^{n-1} \times (0, \infty) : \left|U^h_{x'} \cap \left(\tfrac{1}{2}|U^h_{x'}|, R\right)\right| > a \text{ and } |x'|, h \leq R\right\}, \\ B^a_- &= \left\{(x', h) \in \mathbb{R}^{n-1} \times (0, \infty) : \left|U^h_{x'} \cap \left(-R, -\tfrac{1}{2}|U^h_{x'}|\right)\right| > a \text{ and } |x'|, h \leq R\right\}. \end{aligned} \tag{3-12}$$

If B^a_+ has positive measure, then there are heights $0 < h_1 < h_2 < \infty$ such that both

$$B^a_+ \cap \{h < h_1\} \quad \text{and} \quad B^a_+ \cap \{h > h_2\}$$

have positive measure. Similarly for B^a_-.

Proof. Consider a density point $(x', h) \in B_+^a$ and the rectangles

$$\mathrm{Rec}_\delta = \{(y', u) : |y'| < R,\ |u - h| < \delta\}.$$

Note that $|\mathrm{Rec}_\delta| = \omega_n \delta R^{n-1}$, where ω_n is the volume of the unit ball in $\mathbb{R}^{n-1}$. By density,

$$|B_+^a \cap \mathrm{Rec}_\delta| > 0 \quad \text{for every } \delta > 0.$$

Let $\delta \ll 1$ be small enough to guarantee

$$|B_+^a \cap (\mathrm{Rec}_\delta)^c| > 0.$$

For such a small δ, it follows that

$$\min(|B_+^a \cap \mathrm{Rec}_{\delta/2}|, |B_+^a \cap (\mathrm{Rec}_\delta)^c|) > 0.$$

The lemma holds by choosing $h_1 = h' - \delta$ and $h_2 = h' - \frac{1}{2}\delta$, or $h_1 = h' + \frac{1}{2}\delta$ and $h_2 = h' + \delta$. □

Proof of Proposition 3.2. The proof follows along the same lines as the proof of [Carrillo et al. 2019, Proposition 2.15]. We will sketch the idea in order to showcase where we need Lemma 3.5 and where we apply the estimates in Lemma 3.4, which we have already established to be independent of $0 < \varepsilon \le 1$.

First, since f is not symmetric across $H = \{x_n = 0\}$, there exist $a > 0$ small and $R > 0$ large enough to guarantee that at least one of the sets B_+^a, B_-^a defined in (3-12) has positive measure. Due to the nonlinear center of mass condition (3-4), we know that both of them need to have positive measure.

As a consequence of Lemma 3.5, there exist $0 < h_1 < h_2 < \infty$ and $0 < u_1 < u_2 < \infty$ such that the sets

$$B_+^a \cap \{h < h_1\}, \quad B_+^a \cap \{h > h_2\}, \quad B_-^a \cap \{u < u_1\}, \quad B_-^a \cap \{u > u_2\}$$

all have positive measure. Without loss of generality, assume that $u_1 < h_2$. Otherwise $h_1 < u_2$ and the proof is analogous.

Next, we use (3-5) and the definitions of f^τ and $I_\varepsilon^p[U_{x'}^h, U_{y'}^u](\tau)$ to write

$$\mathcal{I}_\varepsilon^p(f^\tau) = -p(p-1) \int_{\mathbb{R}^{2(n-1)}} \int_{\mathbb{R}_+^2} |h-u|^{p-2} I_\varepsilon^p[U_{x'}^h, U_{y'}^u](\tau)\, dh\, du\, dx'\, dy'.$$

Using (3-7), we can estimate

$$\begin{aligned} -\frac{d^+}{d\tau}[\mathcal{I}_\varepsilon^p(f^\tau)] &\ge p(p-1) \int_{B_-^a \cap \{u<u_1\}} \int_{B_+^a \cap \{h>h_2\}} |h-u|^{p-2} \frac{d}{d\tau}[I_\varepsilon^p[U_{x'}^h, U_{y'}^u](\tau)]\, dh\, dx'\, du\, dy' \\ &\ge m_{p,f} \int_{B_-^a \cap \{u<u_1\}} \int_{B_+^a \cap \{h>h_2\}} \frac{d}{d\tau}[I_\varepsilon^p[U_{x'}^h, U_{y'}^u](\tau)]\, dh\, dx'\, du\, dy', \end{aligned}$$

where

$$m_{p,f} := p(p-1) \begin{cases} |h_2 - u_1|^{p-2} & \text{if } p > 2, \\ 1 & \text{if } p = 2, \\ (2R)^{p-2} & \text{if } 1 < p < 2. \end{cases}$$

Applying now (3-8) and following the proof of [Carrillo et al. 2019, Proposition 2.15], we obtain

$$-\frac{d^+}{d\tau}\mathcal{I}_\varepsilon^p(f^\tau)\Big|_{\tau=0} \ge \frac{1}{6000} m_{p,f}|B_+^a\cap\{h>h_2\}||B_-^a\cap\{u<u_1\}| \min_{r\in[a/4,4R]}|W_1'(r)|a^4>0$$

for all $0\le\tau\le\frac{1}{4}a$ where, with an abuse of notation, $W_1(r)$ is such that $W_1(|x|):=W_1(x)$. □

We conclude this section with the proofs of Theorem 1.1 and Corollary 1.3.

Proof of Theorem 1.1. Without loss of generality, assume that $H=\{x_n=0\}$ coincides with the nonlinear center of mass condition and that f is not symmetric decreasing across H. For $0<\varepsilon\le 1$, let $\mathcal{F}_\varepsilon^p$ and $\mathcal{I}_\varepsilon^p$ be as in (3-1) and (3-3), respectively. Using (2-1), we have

$$\begin{aligned}\mathcal{F}_\varepsilon^p(f^\tau) &= C_\varepsilon\|f\|_{L^p(\mathbb{R}^n)}^p+\mathcal{I}_\varepsilon^p(f^\tau)\\ &= \mathcal{F}_\varepsilon^p(f)+\mathcal{I}_\varepsilon^p(f^\tau)-\mathcal{I}_\varepsilon^p(f).\end{aligned}$$

By Proposition 3.2, there are constants $\gamma=\gamma(n,s,p,f)$ and $\tau_0=\tau_0(f)>0$, independent of ε, such that

$$\mathcal{F}_\varepsilon^p(f^\tau)\le\mathcal{F}_\varepsilon^p(f)-\gamma\tau\quad\text{for all } 0\le\tau\le\tau_0.$$

The statement follows by taking $\varepsilon\to 0^+$. □

Remark 3.6. Notice from the proof of Proposition 3.2 that

$$\lim_{s\to1^-}\min_{r\in[a/4,4R]}|W_1'(r)|=\min_{r\in[a/4,4R]}\frac{(n+p)r^{n+p-1}}{(r^{n+p}+1)^2}>0$$

and also

$$\lim_{s\to0^+}\min_{r\in[a/4,4R]}|W_1'(r)|=\min_{r\in[a/4,4R]}\frac{nr^{n-1}}{(r^n+1)^2}>0.$$

Therefore, we have

$$\begin{aligned}&\lim_{s\to1^-}\gamma(n,s,p,f)>0,\\ &\lim_{s\to0^+}\gamma(n,s,p,f)>0,\end{aligned}$$

and we have $s(1-s)\gamma\to0$ as $s\to0^+,1^-$. After multiplying both sides of (1-5) by $s(1-s)$ and taking the limit as $s\to1^-$, we obtain Corollary 1.3. Note that if we instead take $s\to0^+$, we do not contradict (2-1).

4. Explicit representations for good functions

Here, we define and establish preliminary results for good functions, then we prove an explicit version of Theorem 1.1 for good functions.

4.1. ***Good functions and local energies.*** We begin by presenting the definition of *good functions* and highlight their uses, which can be found in the work of Brock [2000; 1995]. We note that Definition 2.1 is broken down into cases for which the open set $U\subset\mathbb{R}$ is either an interval, a finite union of intervals, or an infinite union of intervals. Good functions are those functions whose sections $U_{x'}^h$ are a finite union of intervals which allows for explicit computation of both the fractional energy for f^τ and its derivative in τ.

Definition 4.1. A nonnegative, piecewise smooth function $f = f(x', x_n)$, $x' \in \mathbb{R}^{n-1}$, $x_n \in \mathbb{R}$, with compact support is called a *good function* if:

(1) for every $x' \in \mathbb{R}^{n-1}$ and every $h > 0$ except a finite set, the equation $f(x', x_n) = h$ has exactly $2m$ solutions, denoted by $x_n = x_n^\ell(x, h)$, satisfying $x_n^\ell < x_n^{\ell+1}$, $\ell = 1, \dots, 2m$, where $m = m(x', h) < \infty$, and

(2) $\inf\left\{\left|\dfrac{\partial f}{\partial x_n}(x', x_n)\right| : x' \in \mathbb{R}^{n-1},\ x_n \in \mathbb{R},\ \text{and } \dfrac{\partial f}{\partial x_n}(x', x_n) \text{ exists}\right\} > 0.$

The functions illustrated in Figure 1 and below in Figure 2 are good functions. In general, one might think of good functions as a collection of peaks, creating a mountain range.

Notation 4.2. We denote the solutions x_n to $f(x', x_n) = h$ using subscript notation $x_{2k-1} < x_{2k}$ for $k = 1, \dots, m = m_h$. (This is not to be confused with the subscripts in $x' = (x_1, \dots, x_{n-1})$.) In the case of $m = 1$ or when considering an arbitrary interval (x_{2k-1}, x_{2k}), we will commonly adopt the notation $x_+ := x_{2k}$ and $x_- := x_{2k-1}$. We will also denote solutions y_n to $f(y', y_n) = u$ by $y_{2\ell-1} < y_{2\ell}$ for $\ell = 1, \dots, m = m_u$.

Remark 4.3. If f is a good function that is symmetric and decreasing across $\{x_n = 0\}$, then it must be that $m = 1$ and $x_- = -x_+$ for all $0 < h < \|f\|_{L^\infty(\mathbb{R}^n)}$.

Just as finite union of intervals can be used to approximate open sets in $\mathbb{R}$, good functions can be used to approximate Sobolev functions.

Lemma 4.4 (see [Brock 2000]). (1) *Good functions are dense in* $W_+^{1,p}(\mathbb{R}^n)$ *for every* $1 \le p < \infty$.

(2) *If f is a good function, then f^τ is a good function for* $0 \le \tau \le \infty$.

Good functions are a powerful tool for continuous Steiner symmetrizations as they allow us to take the τ-derivative directly and expose a quantification of asymmetry. Even in the local setting (see Corollary 1.3), we can explicitly estimate the derivative in τ of $\|f^\tau\|^p_{W^{1,p}(\mathbb{R}^n)}$ when f is a good function. This is in contrast to the original approach of Brock [2000], which relies on convexity to estimate the difference in norms of f and f^τ.

We present the following discussion for the interested reader to showcase the convenience of using good functions in computations.

Since the sign function is not differentiable at the origin, fix $\varepsilon > 0$, let δ_ε be an ε-regularization of the usual Dirac delta

$$\delta_\varepsilon(x) = \begin{cases} \frac{1}{2\varepsilon} & \text{if } |x| < \varepsilon, \\ 0 & \text{if } |x| > \varepsilon, \end{cases}$$

and consider V_ε in (2-2) such that $V_\varepsilon' = 2\delta_\varepsilon$. Let $M^{\tau,\varepsilon}(U)$ denote the continuous Steiner symmetrization of an open set $U \subset \mathbb{R}$ with speed V_ε, and let $f^{\tau,\varepsilon}$ denote the corresponding continuous Steiner symmetrization of f. As the regularization parameter ε goes to 0^+, we recover the original rearrangement:

Lemma 4.5. *Let $f : \mathbb{R}^n \to \mathbb{R}$ be a good function. For any continuous function $g : \mathbb{R}^n \to \mathbb{R}$, we have*

$$\lim_{\varepsilon \to 0^+} \int_{\mathbb{R}^n} f^{\tau,\varepsilon} g \, dx = \int_{\mathbb{R}^n} f^\tau g \, dx.$$

Proof. Step 1. We consider

$$U = \bigcup_{i=1}^{r} (x_{2i-1}, x_{2i})$$

an open set which is the union of $r \in \mathbb{N}$ intervals. Then, we can show the following bound:

$$|M^{\tau,\varepsilon}(U) \triangle M^{\tau}(U)| \le (r+3)\varepsilon \quad \text{for all } \tau \ge 0. \tag{4-1}$$

Proof of Step 1. We start by perturbing the set U by considering

$$U^{\varepsilon} = U \cup (-\varepsilon, \varepsilon) = \bigcup_{j=1}^{s} (x^{\varepsilon}_{2i-1}, x^{\varepsilon}_{2i}),$$

with $s \le r+1$. As we added the set $(-\varepsilon, \varepsilon)$, there can exist at most one interval $I_{i_0} = (x^{\varepsilon}_{2i_0-1}, x^{\varepsilon}_{2i_0})$ satisfying $|x^{\varepsilon}_{2i_0-1} + x^{\varepsilon}_{2i_0}| \le \varepsilon$. We expand this interval to center it; namely we consider the new interval

$$\widetilde{I}_{i_0} = (\tilde{x}^{\varepsilon}_{2i_0-1}, \tilde{x}^{\varepsilon}_{2i_0}) = I_{i_0} \cup P$$

such that $|P| \le \varepsilon$ and $|\tilde{x}^{\varepsilon}_{2i_0-1} - \tilde{x}^{\varepsilon}_{2i_0}| = 0$. If there are any new nontrivial intersections with $\widetilde{I}_{i_0}$, we relabel the intervals and repeat the process. This procedure finishes in k_0 steps with $1 \le k_0 \le r+1$, where r is the original number of intervals. Hence we constructed an open set $S(U^{\varepsilon})$ that satisfies $U^{\varepsilon} \subset S(U^{\varepsilon})$ with at most $r - k_0 + 2$ disjoint intervals and

$$|U^{\varepsilon} \triangle S(U^{\varepsilon})| \le k_0 \varepsilon.$$

The advantage of the set $S(U^{\varepsilon})$ is that the rearrangements coincide:

$$M^{\tau,\varepsilon}(S(U^{\varepsilon})) = M^{\tau}(S(U^{\varepsilon})) \quad \text{for all } \tau \le \tau_1,$$

where τ_1 is larger than the first time when two intervals meet. At τ_1, when there exist an interval $I_{i_1} = (x^{\varepsilon}_{2i_1-1}, x^{\varepsilon}_{2i_1})$ satisfying $|x^{\varepsilon}_{2i_1-1} + x^{\varepsilon}_{2i_1}| \le \varepsilon$, we repeat the procedure of enlargement and centering from before. This process ends in k_1 steps with $1 \le k_1 \le n - k_0 + 1$ and produces a new set $S(M^{\tau_1}(S(U^{\varepsilon}))$ that satisfies $M^{\tau_1}(S(U^{\varepsilon})) \subset S(M^{\tau_1}(S(U^{\varepsilon})))$ with at most $r - k_0 - k_1 + 2$ disjoint intervals and

$$|M^{\tau_1}(S(U^{\varepsilon})) \triangle S(M^{\tau_1}(S(U^{\varepsilon})))| \le k_1 \varepsilon.$$

Again, the rearrangements coincide for the set $S(M^{\tau_1}(S(U^{\varepsilon})))$:

$$M^{\tau,\varepsilon}(S(M^{\tau_1}(S(U^{\varepsilon}))) = M^{\tau}(S(M^{\tau_1}(S(U^{\varepsilon}))) \quad \text{for all } \tau \le \tau_2,$$

where τ_2 is larger than the time when two intervals meet. Continuing this process inductively, we can produce a *discontinuous* family of open set $\{U^{\varepsilon}_{\tau}\}_{\tau>0}$. Using that the rearrangements preserve containment, we can obtain

$$M^{\tau}(U) \cup M^{\tau,\varepsilon}(U) \subset U^{\varepsilon}_{\tau} \quad \text{for all } \tau > 0.$$

This process adds at most $r+3$ intervals of size ε. Hence

$$|M^{\tau}(U) \triangle M^{\tau,\varepsilon}(U)| \le |M^{\tau}(U) \triangle U^{\varepsilon}_{\tau}| \le (r+3)\varepsilon.$$

Step 2. We show that, for any good function f_r satisfying that $U^h_{x'}$ has at most $r \in \mathbb{N}$ intervals for every $x' \in \mathbb{R}^{n-1}$ and every $h \geq 0$, we have for any $g \in C(\mathbb{R}^n)$ that

$$\lim_{\varepsilon \to 0^+} \int_{\mathbb{R}^n} f_r^{\tau,\varepsilon} g\, dx = \int_{\mathbb{R}^n} f_r^{\tau} g\, dx.$$

Proof of Step 2. Using the representation,

$$\begin{aligned}\left|\int_{\mathbb{R}^n} g(x) f^{\tau,\varepsilon}(x)\, dx - \int_{\mathbb{R}^n} g(x) f^{\tau}(x)\, dx\right| &= \left|\int_{\operatorname{supp} f} (f^{\tau,\varepsilon}(x) - f^{\tau}(x)) g(x)\, dx\right| \\ &= \left|\int_0^{\|f\|_\infty}\int_{\operatorname{supp} f} (\chi_{M^{\tau,\varepsilon}(U^h_{x'})} - \chi_{M^{\tau}(U^h_{x'})}) g(x)\, dx\, dh\right| \\ &\leq \|g\|_{L^1(\operatorname{supp} f)} \|f\|_\infty \sup_{x',h} |M^{\tau,\varepsilon}(U^h_{x'}) \triangle M^{\tau}(U^h_{x'})| \\ &\leq \|g\|_{L^1(\operatorname{supp} f)} \|f\|_\infty (r+3)\varepsilon \to 0,\end{aligned}$$

where we have used Step 1 for the last bound.

Step 3. To conclude the proof, we can approximate any good function f by a sequence $\{f_r\}_{r\in\mathbb{N}}$ of good functions such that f_r satisfies the properties of Step 2. □

We now present an explicit estimate on the derivative of $\|f^\tau\|^p_{W^{1,p}(\mathbb{R}^n)}$. For simplicity, we will only state the case of $n = 2$.

Proposition 4.6. *Assume that $f = f(x, y)$ is a nonnegative good function; then*

$$\begin{aligned}&\frac{d}{d\tau}[f^\tau]^p_{W^{1,p}(\mathbb{R}^2)}\Big|_{\tau=0} \\ &\quad\leq -\liminf_{\varepsilon\to 0^+} \int_{\mathbb{R}}\int_0^\infty \sum_{k=1}^{m_h} \delta_\varepsilon(y_{2k} + y_{2k-1}) \left|\frac{\partial y_{2k}}{\partial h}\right|^{-p} \left|\frac{\partial y_{2k-1}}{\partial h}\right|^{-p} \\ &\qquad\times\Bigg[p\left(\left(\frac{\partial y_{2k}}{\partial x}\right)^2 + 1\right)^{p/2-1}\left(\frac{\partial y_{2k}}{\partial x}\right)\left(\frac{\partial y_{2k}}{\partial x} + \frac{\partial y_{2k-1}}{\partial x}\right)\left|\frac{\partial y_{2k}}{\partial h}\right|\left|\frac{\partial y_{2k-1}}{\partial h}\right|^p \\ &\qquad + p\left(\left(\frac{\partial y_{2k-1}}{\partial x}\right)^2 + 1\right)^{p/2-1}\left(\frac{\partial y_{2k-1}}{\partial x}\right)\left(\frac{\partial y_{2k}}{\partial x} + \frac{\partial y_{2k-1}}{\partial x}\right)\left|\frac{\partial y_{2k-1}}{\partial h}\right|\left|\frac{\partial y_{2k}}{\partial h}\right|^p \\ &\qquad - (p-1)\left(\left(\frac{\partial y_{2k}}{\partial x}\right)^2 + 1\right)^{p/2}\left|\frac{\partial y_{2k-1}}{\partial h}\right|^p\left(\left|\frac{\partial y_{2k}}{\partial h}\right| - \left|\frac{\partial y_{2k-1}}{\partial h}\right|\right) \\ &\qquad + (p-1)\left(\left(\frac{\partial y_{2k-1}}{\partial x}\right)^2 + 1\right)^{p/2}\left|\frac{\partial y_{2k}}{\partial h}\right|^p\left(\left|\frac{\partial y_{2k}}{\partial h}\right| - \left|\frac{\partial y_{2k-1}}{\partial h}\right|\right)\Bigg] dh\, dx,\end{aligned}$$

where $f(x, y_i(h)) = h$ for $i = 1, \ldots, 2m_h$ and $m_h = m(x, h)$.

When $p = 2$, we can further factor the integrand to readily check the sign of the derivative.

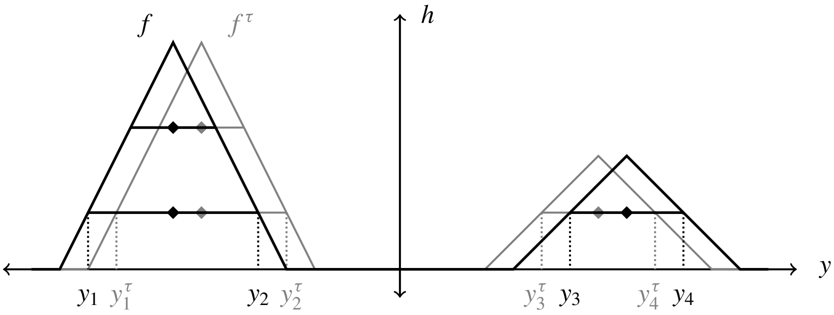

Figure 2. Graph of f and f^τ in Example 4.9 with $\tau = .25,\ x = 0$.

Corollary 4.7. *Assume that $f = f(x, y)$ is a nonnegative good function; then*

$$\begin{aligned}\frac{d}{d\tau}[f^\tau]^2_{H^1(\mathbb{R}^2)}\Big|_{\tau=0} &\le -\liminf_{\varepsilon\to 0^+}\int_{\mathbb{R}}\int_0^\infty \sum_{k=1}^{m_h} \delta_\varepsilon(y_{2k}+y_{2k-1})\left|\frac{\partial y_{2k}}{\partial h}\right|^{-2}\left|\frac{\partial y_{2k-1}}{\partial h}\right|^{-2}\left(\left|\frac{\partial y_{2k}}{\partial h}\right|+\left|\frac{\partial y_{2k-1}}{\partial h}\right|\right)\\ &\qquad\times\left[\left(\left|\frac{\partial y_{2k}}{\partial h}\right|-\left|\frac{\partial y_{2k-1}}{\partial h}\right|\right)^2 + A_k(h)\left\langle\frac{\partial y_{2k}}{\partial x},\frac{\partial y_{2k-1}}{\partial x}\right\rangle\cdot\left\langle\frac{\partial y_{2k}}{\partial x},\frac{\partial y_{2k-1}}{\partial x}\right\rangle\right] dh\,dx \le 0,\end{aligned}$$

where $A_k(h)$ is the positive semidefinite matrix

$$A_k(h) = \begin{pmatrix} \left|\frac{\partial y_{2k-1}}{\partial h}\right|^2 & \left|\frac{\partial y_{2k}}{\partial h}\right|\left|\frac{\partial y_{2k-1}}{\partial h}\right| \\ \left|\frac{\partial y_{2k}}{\partial h}\right|\left|\frac{\partial y_{2k-1}}{\partial h}\right| & \left|\frac{\partial y_{2k}}{\partial h}\right|^2 \end{pmatrix}.$$

Remark 4.8. In the case of $p \neq 2$, we have checked numerically that the integrand in the expression for the derivative is indeed negative, but due to the nonlinearity, we have not been able to analytically observe the sign as cleanly as in the case of $p = 2$.

Notice in Corollary 4.7 that the derivative is strictly negative if and only if

$$\frac{\partial y_{2k}}{\partial h} \neq -\frac{\partial y_{2k-1}}{\partial h} \tag{4-2}$$

on a set of positive measure. Indeed, as mentioned after Corollary 1.3, the strict inequality does not hold in general. For instance, the derivative in τ can be zero when $\operatorname{supp} f$ is not connected but f is radially decreasing in each connected component. We illustrate this with a simple example.

Example 4.9. Consider the good function $f : \mathbb{R}^2 \to \mathbb{R}$ given by

$$f(x, y) = \begin{cases} -|(x, y-2)| + 1 & \text{if } |(x, y-2)| \le 1, \\ -2|(x, y+2)| + 2 & \text{if } |(x, y+2)| \le 1, \\ 0 & \text{otherwise} \end{cases}$$

and illustrated in Figure 2 at $x = 0$. One can readily check that *equality* holds in (4-2), so that the integrand in Proposition 4.6 is exactly zero.

Proof of Proposition 4.6. Begin by writing the energy as

$$[f^\tau]^p_{W^{1,p}(\mathbb{R}^2)} = \int_{\mathbb{R}^2} |\nabla f^\tau|^p \, dy\, dx = \int_{\mathbb{R}^2} \left(\left(\frac{\partial f^\tau}{\partial x}\right)^2 + \left(\frac{\partial f^\tau}{\partial y}\right)^2 \right)^{p/2} dy\, dx.$$

For each fixed $x \in \mathbb{R}$, we make the change of variables $f(x, y) = h$ using the coarea formula in the variable y to write

$$\begin{aligned}[f^\tau]^p_{W^{1,p}(\mathbb{R}^n)} &= \int_{\mathbb{R}^2} \left(\left(\frac{\partial f^\tau}{\partial x}\right)^2 + \left(\frac{\partial f^\tau}{\partial y}\right)^2 \right)^{p/2} \left|\frac{\partial f}{\partial y}\right|^{-1} \left|\frac{\partial f}{\partial y}\right| dy\, dx \\ &= \int_{\mathbb{R}} \int_0^\infty \left(\left(\frac{\partial f^\tau}{\partial x}\right)^2 + \left(\frac{\partial f^\tau}{\partial y}\right)^2 \right)^{p/2} \left|\frac{\partial f}{\partial y}\right|^{-1} \bigg|_{\{f(x,y)=h\}} dh\, dx.\end{aligned}$$

Since f is a good function, for each $h > 0$ except a finite set, there are at most $2m_h$ solutions to $f(x, y) = h$. We denote these by $y_i = y_i(x, h)$, $i = 1, \ldots, 2m_h$. In the new variables, we have

$$\frac{\partial f^\tau}{\partial y}(x, y_+) = \left(\frac{\partial y_+^\tau}{\partial h}\right)^{-1} < 0, \quad \frac{\partial f^\tau}{\partial y}(x, y_-) = \left(\frac{\partial y_-^\tau}{\partial h}\right)^{-1} > 0 \tag{4-3}$$

and

$$\frac{\partial f^\tau}{\partial x}(x, y_+) = \frac{\partial y_+^\tau}{\partial x}\left(\frac{\partial y_+^\tau}{\partial h}\right)^{-1}, \quad \frac{\partial f^\tau}{\partial x}(x, y_-) = \frac{\partial y_-^\tau}{\partial x}\left(\frac{\partial y_-^\tau}{\partial h}\right)^{-1}$$

for an arbitrary $y_- < y_+$ (recall Notation 4.2). Then, we write

$$\begin{aligned}[f^\tau]^p_{W^{1,p}(\mathbb{R}^n)} = \int_{\mathbb{R}} \int_0^\infty \sum_{k=1}^{m_h} \Bigg[&\left(\left(\frac{\partial y_{2k}^\tau}{\partial x}\right)^2 \left(\frac{\partial y_{2k}^\tau}{\partial h}\right)^{-2} + \left(\frac{\partial y_{2k}^\tau}{\partial h}\right)^{-2} \right)^{p/2} \left|\frac{\partial y_{2k}^\tau}{\partial h}\right| \\ &+ \left(\left(\frac{\partial y_{2k-1}^\tau}{\partial x}\right)^2 \left(\frac{\partial y_{2k-1}^\tau}{\partial h}\right)^{-2} + \left(\frac{\partial y_{2k-1}^\tau}{\partial h}\right)^{-2} \right)^{p/2} \left|\frac{\partial y_{2k-1}^\tau}{\partial h}\right| \Bigg] dh\, dx.\end{aligned}$$

For simplicity in the proof, let us assume that $m_h = 1$ or $m_h = 0$ for all h except a finite set and write

$$\begin{aligned}[f^\tau]^p_{W^{1,p}(\mathbb{R}^n)} &= \int_{\mathbb{R}} \int_0^\infty \Bigg[\left(\left(\frac{\partial y_+^\tau}{\partial x}\right)^2 \left(\frac{\partial y_+^\tau}{\partial h}\right)^{-2} + \left(\frac{\partial y_+^\tau}{\partial h}\right)^{-2} \right)^{p/2} \left|\frac{\partial y_+^\tau}{\partial h}\right| \\ &\qquad + \left(\left(\frac{\partial y_-^\tau}{\partial x}\right)^2 \left(\frac{\partial y_-^\tau}{\partial h}\right)^{-2} + \left(\frac{\partial y_-^\tau}{\partial h}\right)^{-2} \right)^{p/2} \left|\frac{\partial y_-^\tau}{\partial h}\right| \Bigg] dh\, dx \\ &= \int_{\mathbb{R}} \int_0^\infty \left[\left(\left(\frac{\partial y_+^\tau}{\partial x}\right)^2 + 1 \right)^{p/2} \left|\frac{\partial y_+^\tau}{\partial h}\right|^{1-p} + \left(\left(\frac{\partial y_-^\tau}{\partial x}\right)^2 + 1 \right)^{p/2} \left|\frac{\partial y_-^\tau}{\partial h}\right|^{1-p} \right] dh\, dx.\end{aligned}$$

We have now arrived at a useful expression for taking the derivative in τ. Indeed, let V_ε be an approximation of the speed $V(y) = \operatorname{sgn}(y)$ such that $\delta_\varepsilon = V_\varepsilon'$. Then, from (2-2) with speed V_ε, we obtain

$$\frac{\partial^2 y_\pm^\tau}{\partial \tau \partial h} = -\delta_\varepsilon(y_+ + y_-)\left(\frac{\partial y_+}{\partial h} + \frac{\partial y_-}{\partial h}\right)$$

and similarly

$$\frac{\partial^2 y_\pm^\tau}{\partial\tau\,\partial x} = -\delta_\varepsilon(y_+ + y_-)\left(\frac{\partial y_+}{\partial x} + \frac{\partial y_-}{\partial x}\right).$$

With this we can use the lower-semicontinuity of the $W^{1,p}$ seminorm to have

$$\begin{aligned}
&\frac{d}{d\tau}[f^\tau]^p_{W^{1,p}(\mathbb{R}^n)}\Big|_{\tau=0} \\
&\quad\le \liminf_{\varepsilon\to 0^+} \frac{d}{d\tau}[f^{\tau,\varepsilon}]^p_{W^{1,p}(\mathbb{R}^n)}\Big|_{\tau=0} \\
&\quad\le \liminf_{\varepsilon\to 0^+} \int_{\mathbb{R}}\int_0^\infty \Bigg[-p\left(\left(\frac{\partial y_+}{\partial x}\right)^2+1\right)^{p/2-1}\left(\frac{\partial y_+}{\partial x}\right)\delta_\varepsilon(y_+ + y_-)\left(\frac{\partial y_+}{\partial x} + \frac{\partial y_-}{\partial x}\right)\left|\frac{\partial y_+}{\partial h}\right|^{1-p} \\
&\qquad\qquad - p\left(\left(\frac{\partial y_-}{\partial x}\right)^2+1\right)^{p/2-1}\left(\frac{\partial y_-}{\partial x}\right)\delta_\varepsilon(y_+ + y_-)\left(\frac{\partial y_+}{\partial x} + \frac{\partial y_-}{\partial x}\right)\left|\frac{\partial y_-}{\partial h}\right|^{1-p} \\
&\qquad\qquad + (p-1)\left(\left(\frac{\partial y_+}{\partial x}\right)^2+1\right)^{p/2}\left|\frac{\partial y_+}{\partial h}\right|^{-p-1}\left(\frac{\partial y_+}{\partial h}\right)\delta_\varepsilon(y_+ + y_-)\left(\frac{\partial y_+}{\partial h} + \frac{\partial y_-}{\partial h}\right) \\
&\qquad\qquad + (p-1)\left(\left(\frac{\partial y_-}{\partial x}\right)^2+1\right)^{p/2}\left|\frac{\partial y_-}{\partial h}\right|^{-p-1}\left(\frac{\partial y_-}{\partial h}\right)\delta_\varepsilon(y_+ + y_-)\left(\frac{\partial y_+}{\partial h} + \frac{\partial y_-}{\partial h}\right)\Bigg]\,dh\,dx.
\end{aligned}$$

Recalling (4-3) and factoring gives the desired expression. □

4.2. ***Explicit derivative computation for good functions.*** The objective of this section is to compute the derivative of the nonlocal energy explicitly. With the same notation as in Section 3, we consider again $\mathcal{F}^p_\varepsilon$ and $\mathcal{I}^p_\varepsilon$ defined in (3-1) and (3-3), respectively. When f is a good function, we can write each $U^h_{x'}$ in (3-5) as a finite union of open intervals. This provides a useful expression of $\mathcal{I}^p_\varepsilon$ which allows for more explicit computation.

It will be useful to notate the first and second antiderivatives of $K(r) = K_{\varepsilon,|x'|}(r)$ in r respectively by

$$\overline{K}(r) := \int_0^r K(\xi)\,d\xi, \qquad \overline{\overline{K}}(r) := \int_0^r\int_0^\rho K(\xi)\,d\xi\,d\rho. \tag{4-4}$$

We now present the main result of this section.

Proposition 4.10. *Let $\varepsilon \ge 0$. Assume that f is a nonnegative good function. Then,*

$$\begin{aligned}
&\frac{d}{d\tau}\mathcal{F}^p_\varepsilon(f^\tau)\Big|_{\tau=0} \\
&\quad = -p(p-1)\int_{\mathbb{R}^{n-1}}\int_{\mathbb{R}^{n-1}}\int_0^\infty\int_0^\infty \sum_{\ell=1}^{m_u}\sum_{k=1}^{m_h} |h-u|^{p-2} \\
&\qquad\qquad \times\big[\overline{K}_{\varepsilon,|x'-y'|}(x_{2k}-y_{2\ell}) - \overline{K}_{\varepsilon,|x'-y'|}(x_{2k}-y_{2\ell-1}) \\
&\qquad\qquad\quad - \overline{K}_{\varepsilon,|x'-y'|}(x_{2k-1}-y_{2\ell}) + \overline{K}_{\varepsilon,|x'-y'|}(x_{2k-1}-y_{2\ell-1})\big] \\
&\qquad\qquad \times(\operatorname{sgn}(x_{2k}+x_{2k-1}) - \operatorname{sgn}(y_{2\ell}+y_{2\ell-1}))\,dh\,du\,dx'\,dy' \le 0,
\end{aligned}\tag{4-5}$$

where $f(x', x_i(h)) = h$, $f(y', y_i(u)) = u$ for $i = 1, \dots, 2m$ and $m_h = m(x', h)$, $m_u = m(y', u)$.

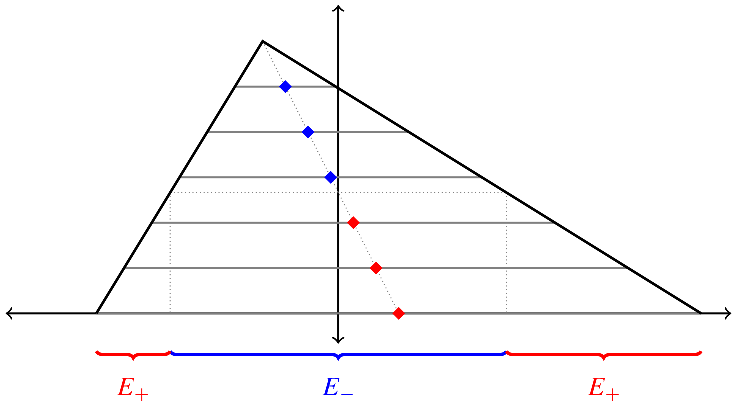

Figure 3. Decomposition of supp f based on the center of mass of the level sets.

Even more, we can explicitly write down the derivative in real variables.

Notation 4.11. For an interval $I = (a, b) \subset \mathbb{R}$, we write

$$I = \begin{cases} I^+ & \text{if } a+b>0, \\ I^0 & \text{if } a+b=0, \\ I^- & \text{if } a+b<0. \end{cases}$$

For a fixed $x \in \mathbb{R}^n$, we write the sections as a finite union of intervals:

$$U_{x'}^h = \bigcup_{i=1}^m I_i = \left(\bigcup_{i=1}^{m_+} I_i^+\right) \cup \left(\bigcup_{i=1}^{m_0} I_i^0\right) \cup \left(\bigcup_{i=1}^{m_-} I_i^-\right) \quad \text{for } h = f(x)$$

and where $m = m_+ + m_0 + m_-$. We define E_+, E_0, $E_- \subset \partial U_{x'}^{f(x)}$ as the set of points that belong to a piece of a boundary of a level set that is moving to the left, centered, and moving to the right, respectively; see Figure 3. More precisely,

$$E_+ = \{x : x_n \in \partial I^+\}, \quad E_0 = \{x : x_n \in \partial I^0\}, \quad E_- = \{x : x_n \in \partial I^-\}.$$

The following corollary of Proposition 4.10 follows from undoing the change of variables.

Corollary 4.12. *Assume that f is a good function. Then,*

$$\frac{d}{d\tau}\mathcal{F}_\varepsilon^p(f^\tau)\Big|_{\tau=0} = 2p(p-1)\biggl(\int_{E_+}\int_{E_-\cup E_0} \overline{K}_{\varepsilon,|x'-y'|}(x_n-y_n) f_{x_n}(x) f_{y_n}(y)\,dx\,dy \\ - \int_{E_-}\int_{E_+\cup E_0} \overline{K}_{\varepsilon,|x'-y'|}(x_n-y_n) f_{x_n}(x) f_{y_n}(y)\,dx\,dy\biggr). \quad (4\text{-}6)$$

One can view the expression on the right-hand side of (4-6) as a quantification of asymmetry. Indeed, if f is radially symmetric across $\{x_n = 0\}$, then $E_+ = E_- = \varnothing$ and the derivative is zero. Note that if

$|E_0| = 0$, then (4-6) can be written concisely as

$$\frac{d}{d\tau}\mathcal{F}_\varepsilon^p(f^\tau)\Big|_{\tau=0} = 4p(p-1)\int_{E_+}\int_{E_-}\overline{K}_{\varepsilon,|x'-y'|}(x_n-y_n)\,f_{x_n}(x)\,f_{y_n}(y)\,dx\,dy. \tag{4-7}$$

Unfortunately the integrand of this expression does not have a clear sign, and the sign only appears when we are considering the level-set representation. We illustrate this in the following example.

Example 4.13. Consider the good function $f:\mathbb{R}^2\to\mathbb{R}$ given in Example 4.9, and let

$$B_1(0,t) = \{(x',x_n)\in\mathbb{R}^{n-1}\times\mathbb{R} : |(x',x_n-t)|<1\}.$$

From (4-7), we have

$$\begin{aligned}\frac{d}{d\tau}\mathcal{F}_\varepsilon^p(f^\tau)\Big|_{\tau=0} &= 4p(p-1)\int_{B_1(0,2)}\int_{B_1(0,-2)}\overline{K}_{\varepsilon,|x'-y'|}(x_n-y_n)\frac{x_n+2}{|(x',x_n+2)|}\frac{y_n-2}{|(y',y_n-2)|}\,dx\,dy\\ &= -4p(p-1)\int_{B_1(0,2)}\int_{B_1(0,-2)}\int_0^{y_n-x_n}K_{\varepsilon,|x'-y'|}(r)\frac{x_n+2}{|(x',x_n+2)|}\frac{y_n-2}{|(y',y_n-2)|}\,dr\,dx\,dy\end{aligned}$$

since $x_n<0<y_n$ for all $|(x',x_n+2)|<1$, $|(y',y_n-2)|<1$. Notice that the sign of the integrand in this expression is not positive for each fixed (x',x_n), (y',y_n), and r in the domain of integration. The sign is instead observed by studying the endpoints of the level sets as done in the proof of Proposition 4.10.

4.2.1. *Proof of Proposition 4.10.* We begin with an expression for the fractional energy in terms of its level sets.

Lemma 4.14. *Assume that f is a nonnegative, good function. Then,*

$$\begin{aligned}&\mathcal{F}_\varepsilon^p(f) - C_\varepsilon\|f\|^p_{L^p(\mathbb{R}^n)}\\ &\quad= p(p-1)\int_{\mathbb{R}^{n-1}}\int_{\mathbb{R}^{n-1}}\int_0^\infty\int_0^\infty\sum_{\ell=1}^{m_u}\sum_{k=1}^{m_h}|h-u|^{p-2}\\ &\qquad\qquad\times\Big[\overline{\overline{K}}_{\varepsilon,|x'-y'|}(x_{2k}-y_{2\ell}) - \overline{\overline{K}}_{\varepsilon,|x'-y'|}(x_{2k}-y_{2\ell-1})\\ &\qquad\qquad\qquad - \overline{\overline{K}}_{\varepsilon,|x'-y'|}(x_{2k-1}-y_{2\ell}) + \overline{\overline{K}}_{\varepsilon,|x'-y'|}(x_{2k-1}-y_{2\ell-1})\Big]\,dh\,du\,dx'\,dy',\end{aligned}$$

where $f(x',x_i(h))=h$, $f(y',y_i(u))=u$ for $i=1,\dots,2m$ and $m_h=m(x',h)$, $m_u=m(y',u)$.

Proof. As in (3-5), we begin by writing

$$\begin{aligned}&\mathcal{F}_\varepsilon^p(f) - C_\varepsilon\|f\|^p_{L^p(\mathbb{R}^n)}\\ &\quad= -p(p-1)\int_{\mathbb{R}^{2(n-1)}}\int_{\mathbb{R}^2_+}|h-u|^{p-2}\int_{\mathbb{R}^2}\chi_{U^h_{x'}}(x_n)\chi_{U^h_{y'}}(y_n)K_{\varepsilon,|x'-y'|}(x_n-y_n)\,dx_n\,dy_n\,dh\,du\,dx'\,dy'.\end{aligned}$$

Fix $x',y'\in\mathbb{R}^{n-1}$ and let $h,u>0$. Since f is a good function, there are at most m_h solutions to $f(x',x_n)=h$, which we denote by $x_{2k-1}\le x_{2k}$, $k=1,\dots,m_h$. We similarly denote the solutions to

$f(y', y_n) = u$ by $y_{2\ell-1} \le y_{2\ell}$, $\ell = 1, \dots, m_u$. Then we have

$$\begin{aligned}
\int_{\mathbb{R}^2} \chi_{U^h_{x'}}(x_n)\chi_{U^h_{y'}}(y_n) K_{\varepsilon,|x'-y'|}(x_n - y_n)\,dx_n\,dy_n \\
&= \sum_{\ell=1}^{m_u}\sum_{k=1}^{m_h} \int_{y_{2\ell-1}}^{y_{2\ell}} \int_{x_{2k-1}}^{x_{2k}} K_{\varepsilon,|x'-y'|}(x_n - y_n)\,dx_n\,dy_n \\
&= \sum_{\ell=1}^{m_u}\sum_{k=1}^{m_h} \int_{y_{2\ell-1}}^{y_{2\ell}} \big[\overline{K}_{\varepsilon,|x'-y'|}(x_{2k} - y_n) - \overline{K}_{\varepsilon,|x'-y'|}(x_{2k-1} - y_n)\big]\,dy_n \\
&= -\sum_{\ell=1}^{m_u}\sum_{k=1}^{m_h} \big[\overline{\overline{K}}_{\varepsilon,|x'-y'|}(x_{2k} - y_{2\ell}) - \overline{\overline{K}}_{\varepsilon,|x'-y'|}(x_{2k} - y_{2\ell-1}) \\
&\qquad - \overline{\overline{K}}_{\varepsilon,|x'-y'|}(x_{2k-1} - y_{2\ell}) + \overline{\overline{K}}_{\varepsilon,|x'-y'|}(x_{2k-1} - y_{2\ell-1})\big]. \qquad \square
\end{aligned}$$

We are now prepared to present the proof of Proposition 4.10.

Proof of Proposition 4.10. First fix $\varepsilon > 0$. Since f is a good function, we know that f^τ is also a good function. By Lemma 4.14 for f^τ and applying (2-1), we have

$$\begin{aligned}
\mathcal{F}^p_\varepsilon(f^\tau) = C_\varepsilon \|f\|^p_{L^p(\mathbb{R}^n)} + p(p-1) \int_{\mathbb{R}^{n-1}}\int_{\mathbb{R}^{n-1}}\int_0^\infty\int_0^\infty \sum_{\ell=1}^{m_u}\sum_{k=1}^{m_h} |h-u|^{p-2} \\
\times \big[\overline{\overline{K}}_{\varepsilon,|x'-y'|}(x^\tau_{2k} - y^\tau_{2\ell}) - \overline{\overline{K}}_{\varepsilon,|x'-y'|}(x^\tau_{2k} - y^\tau_{2\ell-1}) \\
- \overline{\overline{K}}_{\varepsilon,|x'-y'|}(x^\tau_{2k-1} - y^\tau_{2\ell}) + \overline{\overline{K}}_{\varepsilon,|x'-y'|}(x^\tau_{2k-1} - y^\tau_{2\ell-1})\big]\,dh\,du\,dx'\,dy'.
\end{aligned}$$

From Definition 2.1(1), we have

$$\frac{d}{d\tau}(x^\tau_i - y^\tau_j) = -(\operatorname{sgn}(x_{2k} + x_{2k-1}) - \operatorname{sgn}(y_{2\ell} + y_{2\ell-1})) \quad \text{for } i = 2k, 2k-1,\ j = 2\ell, 2\ell-1.$$

Therefore, taking the derivative of $\mathcal{F}^p_\varepsilon(f^\tau)$ with respect to τ gives

$$\begin{aligned}
\frac{d}{d\tau}\mathcal{F}^p_\varepsilon(f^\tau) &= p(p-1) \int_{\mathbb{R}^{2(n-1)}}\int_0^\infty\int_0^\infty \sum_{k=1}^{m}\sum_{\ell=1}^{m} |h-u|^{p-2} \\
&\qquad \times \frac{d}{d\tau}\big[\overline{\overline{K}}_{\varepsilon,|x'-y'|}(x^\tau_{2k} - y^\tau_{2\ell}) - \overline{\overline{K}}_{\varepsilon,|x'-y'|}(x^\tau_{2k} - y^\tau_{2\ell-1}) \\
&\qquad\qquad - \overline{\overline{K}}_{\varepsilon,|x'-y'|}(x^\tau_{2k-1} - y^\tau_{2\ell}) + \overline{\overline{K}}_{\varepsilon,|x'-y'|}(x^\tau_{2k-1} - y^\tau_{2\ell-1})\big]\,dh\,du\,dx'\,dy' \\
&= -p(p-1) \int_{\mathbb{R}^{2(n-1)}}\int_0^\infty\int_0^\infty \sum_{k=1}^{m}\sum_{\ell=1}^{m} |h-u|^{p-2} \\
&\qquad \times \big[\overline{K}_{\varepsilon,|x'-y'|}(x^\tau_{2k} - y^\tau_{2\ell}) - \overline{K}_{\varepsilon,|x'-y'|}(x^\tau_{2k} - y^\tau_{2\ell-1}) \\
&\qquad\qquad - \overline{K}_{\varepsilon,|x'-y'|}(x^\tau_{2k-1} - y^\tau_{2\ell}) + \overline{K}_{\varepsilon,|x'-y'|}(x^\tau_{2k-1} - y^\tau_{2\ell-1})\big] \\
&\qquad\qquad \times (\operatorname{sgn}(x_{2k} + x_{2k-1}) - \operatorname{sgn}(y_{2\ell} + y_{2\ell-1}))\,dh\,du\,dx'\,dy'.
\end{aligned}$$

Evaluating at $\tau = 0$, we obtain (4-5).

We next show that the integrand in (4-5) is positive for each fixed x', y', h, u, k, ℓ. For this, we use the simplified notation

$$x_+ = x_{2k}, \quad x_- = x_{2k-1}, \quad y_+ = y_{2\ell}, \quad y_- = y_{2\ell-1} \tag{4-8}$$

and $K_\varepsilon(r) = K_{\varepsilon,|x'-y'|}(r)$. Assume, without loss of generality, that

$$(x_+ + x_-) - (y_+ + y_-) > 0.$$

Then, it is enough to check that

$$\left[\overline{K}_\varepsilon(x_+ - y_+) - \overline{K}_\varepsilon(x_+ - y_-) - \overline{K}_\varepsilon(x_- - y_+) + \overline{K}_\varepsilon(x_- - y_-)\right] > 0. \tag{4-9}$$

We break into three cases based on the interaction of the intervals (x_-, x_+) and (y_-, y_+). In the following, we will use the antisymmetry of $\overline{K}_\varepsilon(r)$ and that $K_\varepsilon'(r) < 0$ for $r > 0$.

Case 1. Embedded intervals: $(y_-, y_+) \subset (x_-, x_+)$.

Since $x_- < y_- < y_+ < x_+$, we have

$$\begin{aligned}
&\overline{K}_\varepsilon(x_+ - y_+) - \overline{K}_\varepsilon(x_+ - y_-) - \overline{K}_\varepsilon(x_- - y_+) + \overline{K}_\varepsilon(x_- - y_-)\\
&\qquad = \int_{-(x_- - y_-)}^{x_+ - y_+} K_\varepsilon(r)\,dr - \int_{-(x_- - y_+)}^{x_+ - y_-} K_\varepsilon(r)\,dr\\
&\qquad = \int_0^{(x_+ + x_-)-(y_+ + y_-)} [K_\varepsilon(r + (y_- - x_-)) - K_\varepsilon(r + (y_+ - x_-))]\,dr\\
&\qquad = \int_0^{(x_+ + x_-)-(y_+ + y_-)} \int_{y_+ - x_-}^{y_- - x_-} K_\varepsilon'(r+\xi)\,d\xi\,dr\\
&\qquad = -\int_0^{(x_+ + x_-)-(y_+ + y_-)} \int_0^{y_+ - y_-} K_\varepsilon'(r + \xi + y_- - x_-)\,d\xi\,dr\\
&\qquad \geq [(x_+ + x_-) - (y_+ + y_-)](y_+ - y_-) \min_{y_- - x_- < r < x_+ - y_-} |K_\varepsilon'(r)| > 0.
\end{aligned}$$

Case 2. Separated intervals: $(y_-, y_+) \cap (x_-, x_+) = \varnothing$.

Since $x_- < y_- < y_+ < x_+$, we have

$$\begin{aligned}
&\overline{K}_\varepsilon(x_+ - y_+) - \overline{K}_\varepsilon(x_+ - y_-) - \overline{K}_\varepsilon(x_- - y_+) + \overline{K}_\varepsilon(x_- - y_-)\\
&\qquad = \int_{x_- - y_+}^{x_+ - y_+} K_\varepsilon(r)\,dr - \int_{x_- - y_-}^{x_+ - y_-} K_\varepsilon(r)\,dr\\
&\qquad = \int_0^{x_+ - x_-} [K_\varepsilon(r + x_- - y_+) - K(r + x_- - y_-)]\,dr\\
&\qquad = \int_0^{x_+ - x_-} \int_{x_- - y_-}^{x_- - y_+} K_\varepsilon'(r+\xi)\,d\xi\,dr\\
&\qquad = -\int_0^{x_+ - x_-} \int_0^{y_+ - y_-} K_\varepsilon'(r + \xi + x_- - y_+)\,d\xi\,dr\\
&\qquad \geq (x_+ - x_-)(y_+ - y_-) \min_{x_- - y_+ < r < x_+ - y_-} |K_\varepsilon'(r)| > 0.
\end{aligned}$$

Case 3. Overlapping intervals: $(y_-, y_+) \not\subset (x_-, x_+)$ and $(y_-, y_+) \cap (x_-, x_+) \neq \varnothing$.

Since $y_- < x_- < y_+ < x_+$, we have

$$x_+ - y_- = (x_+ - y_+) + (y_+ - x_-) + (x_- - y_-),$$

so that

$$\begin{aligned}
&\overline{K}_\varepsilon(x_+ - y_+) - \overline{K}_\varepsilon(x_+ - y_-) - \overline{K}_\varepsilon(x_- - y_+) + \overline{K}_\varepsilon(x_- - y_-) \\
&\quad = \int_0^{x_+ - y_+} K_\varepsilon(r)\,dr + \int_0^{y_+ - x_-} K_\varepsilon(r)\,dr + \int_0^{x_- - y_-} K_\varepsilon(r)\,dr - \int_0^{x_+ - y_-} K_\varepsilon(r)\,dr \\
&\quad = \int_0^{y_+ - x_-} K_\varepsilon(r)\,dr + \int_0^{x_- - y_-} K_\varepsilon(r)\,dr - \int_{x_+ - y_+}^{x_+ - y_-} K_\varepsilon(r)\,dr \\
&\quad = \int_0^{y_+ - x_-} K(r)\,dr + \int_0^{x_- - y_-} K_\varepsilon(r)\,dr - \int_0^{y_+ - y_-} K_\varepsilon(r + x_+ - y_+)\,dr \\
&\quad = \int_0^{y_+ - x_-} K_\varepsilon(r)\,dr + \int_0^{x_- - y_-} K_\varepsilon(r)\,dr \\
&\qquad\qquad - \int_0^{x_- - y_-} K_\varepsilon(r + x_+ - y_+)\,dr - \int_{x_- - y_-}^{y_+ - y_-} K_\varepsilon(r + x_+ - y_+)\,dr \\
&\quad = \int_0^{y_+ - x_-} K_\varepsilon(r)\,dr + \int_0^{x_- - y_-} K_\varepsilon(r)\,dr \\
&\qquad\qquad - \int_0^{x_- - y_-} K_\varepsilon(r + x_+ - y_+)\,dr - \int_0^{y_+ - x_-} K_\varepsilon(r + x_+ - y_+ + x_- - y_-)\,dr \\
&\quad = -\left[\int_0^{y_+ - x_-}\int_0^{(x_+ + x_-) - (y_+ + y_-)} K_\varepsilon'(r + \xi)\,d\xi\,dr + \int_0^{x_- - y_-}\int_0^{x_+ - y_+} K_\varepsilon'(r + \xi)\,d\xi\,dr\right] \\
&\quad > [(x_+ + x_-) - (y_+ + y_-)](y_+ - x_-) \min_{0 < r < x_+ - y_-} |K_\varepsilon'(r)| \\
&\qquad\qquad + (x_- - y_-)(x_+ - y_+) \min_{0 < r < (x_+ + x_-) - (y_+ - y_-)} |K_\varepsilon'(r)| = 0.
\end{aligned}$$

We have established (4-9) in all possible cases. Recalling (3-11) and a further analysis of the final expressions in Cases 1–3 shows that they are monotone as $\varepsilon \to 0^+$. Moreover, $\mathcal{F}_\varepsilon^p(f) \to \mathcal{F}^p(f)$ as $\varepsilon \to 0^+$. Hence, in this case, we can conclude that

$$\lim_{\varepsilon \to 0^+} \frac{d}{d\tau} \mathcal{F}_\varepsilon^p(f^\tau)\Big|_{\tau=0} = \frac{d}{d\tau} \mathcal{F}^p(f^\tau)\Big|_{\tau=0}$$

and that the sign of the derivative is preserved. □

5. Interpolation between symmetric decreasing functions

Let us now review the interpolation between symmetric decreasing functions of unit mass introduced in [Delgadino et al. 2022]. Consider a nonnegative, symmetric decreasing function $f \in L^1(\mathbb{R}^n) \cap C(\mathbb{R}^n)$ with mass 1. The associated height function $H : (0, 1) \to (0, \|f\|_{L^\infty(\mathbb{R}^n)})$ is defined implicitly by

$$\int_{\mathbb{R}^n} \min\{f(x), H(m)\}\,dx = m \quad \text{for } m \in (0, 1). \tag{5-1}$$

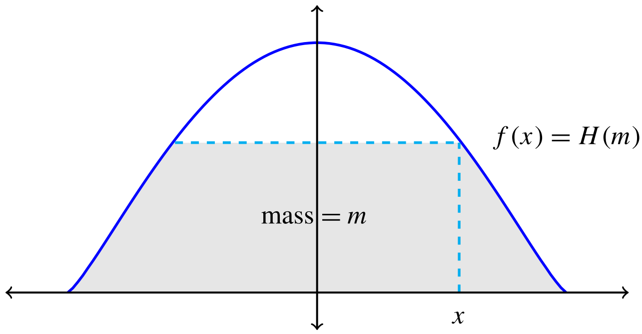

Figure 4. The height function $H(m)$ associated to $f(x)$.

That is, $H(m)$ is the unique value such that the mass under the plane $f(x) = H(m)$ has mass $m \in (0, 1)$; see Figure 4. The height function H satisfies the following properties.

Lemma 5.1 (see [Delgadino et al. 2022]). *Let $f \in L^1(\mathbb{R}^n) \cap C(\mathbb{R}^n)$ and H be its associated height function defined in (5-1). Then*:

(1) $H = H(m) \in (0, \|f\|_{L^\infty(\mathbb{R}^n)})$ *is continuous*, *strictly increasing*, *and convex on* $(0, 1)$.

(2) *If in addition we assume that f has compact support and is strictly decreasing in the radial variable*, *then*
$$\lim_{m\to 0^+} H'(m) = |\{f > 0\}|^{-1} \quad \textit{and} \quad \lim_{m\to 1^-} H'(m) = +\infty.$$

(3) *The function H fully determines f as*
$$f(x) = \int_0^1 \chi_{(c_n H'(m))^{-1/n}}(x) H'(m)\, dm, \quad \textit{where } \chi_r(x) := \chi_{B(0,r)}(x). \tag{5-2}$$

(4) *For almost every $m \in (0, 1)$*, *we have*
$$-\Phi'((c_n H'(m))^{-1/n}) = n c_n^{1/n} \frac{(H'(m))^{2+1/n}}{H''(m)} \geq 0, \tag{5-3}$$
where $\Phi : [0, \infty) \to [0, \infty)$ satisfies $f(x) = \Phi(|x|)$.

Proof. Properties (1)–(3) are established in [Delgadino et al. 2022, Lemma 2.1]. Property (4), established in [Delgadino et al. 2022, Lemma 4.2], follows from differentiating the identity
$$\Phi((c_n H'(m))^{-1/n}) = H(m). \qquad \square$$

Now, consider two symmetric decreasing functions $f_0, f_1 \in L^1(\mathbb{R}^n) \cap C(\mathbb{R}^n)$, both with unit mass, and let H_0, H_1 denote their associated height functions. Let H_t, $0 \le t \le 1$, be a linear interpolation between H_0 and H_1:
$$H_t(m) = (1-t) H_0(m) + t H_1(m). \tag{5-4}$$

By Lemma 5.1 statement (3), the height function H_t uniquely determines a radially decreasing function $f_t \in L^1(\mathbb{R}^n) \cap C(\mathbb{R}^n)$ with unit mass. In particular, $\{f_t\}_{t\in[0,1]}$ is an interpolation between f_0 and f_1.

It is shown in [Delgadino et al. 2022, Proposition 2.3] that the p-th power of the L^p norms are convex under this interpolation if and only if $p \geq 2$. More precisely,

$$\frac{d^2}{dt^2}\|f_t\|^p_{L^p(\mathbb{R}^n)} \begin{cases} < 0 & \text{if } 1 \leq p < 2, \\ = 0 & \text{if } p = 2, \\ > 0 & \text{if } p > 2 \end{cases} \qquad \text{for } 0 < t < 1. \tag{5-5}$$

In our next result, we determine when the $W^{1,p}$ seminorms are convex under the height function interpolation.

Proposition 5.2. *Fix $1 < p < \infty$. Let f_0, $f_1 \in W^{1,p}(\mathbb{R}^n)$ be two distinct, nonnegative, symmetric decreasing functions with unit mass, and let f_t, $0 \leq t \leq 1$, be the height function interpolation between f_0 and f_1. Then,*

$$t \mapsto [f_t]^p_{W^{1,p}(\mathbb{R}^n)} = \int_{\mathbb{R}^n} |\nabla f_t|^p \, dx$$

is convex if $p \geq 2n/(n+1)$.

Consequently, the following a priori estimate on the interpolation holds when $p \geq 2n/(n+1)$:

$$\max_{t\in[0,1]} [f_t]^p_{W^{1,p}(\mathbb{R}^n)} \leq \max\{[f_0]^p_{W^{1,p}(\mathbb{R}^n)}, [f_1]^p_{W^{1,p}(\mathbb{R}^n)}\}. \tag{5-6}$$

Remark 5.3. It is not known if the condition $p \geq 2n/(n+1)$ is sharp for the convexity in Proposition 5.2.

Proof. Given a symmetric, radially decreasing function $f \in W^{1,p}(\mathbb{R}^n)$, we let $\Phi : [0,\infty) \to [0,\infty)$ be such that $f(x) = \Phi(|x|)$ and write

$$\int_{\mathbb{R}^n} |\nabla f|^p \, dx = nc_n \int_0^\infty r^{n-1} |\Phi'(r)|^p \, dr. \tag{5-7}$$

As a consequence of (5-2), the radial variable r can be expressed in terms of the height function H as

$$r = (c_n H'(m))^{-1/n},$$

which gives

$$dr = -\frac{c_n^{-1/n}}{n} \frac{H''(m)}{H'(m)^{1+1/n}} \, dm.$$

Moreover, using (5-3), we can write

$$\Phi'(r) = -nc_n^{1/n} \frac{(H'(m))^{2+1/n}}{H''(m)}.$$

Therefore, applying the change of variable $r \mapsto m$ in (5-7) gives

$$\begin{aligned} \int_{\mathbb{R}^n} |\nabla f|^p \, dx &= nc_n \int_0^1 (c_n H'(m))^{-1+1/n} n^p c_n^{p/n} \frac{(H'(m))^{p(2+1/n)}}{(H''(m))^p} \frac{c_n^{-1/n}}{n} \frac{H''(m)}{H'(m)^{1+1/n}} \, dm \\ &= C_{n,p} \int_0^1 (H'(m))^{p(2+1/n)-2} (H''(m))^{1-p} \, dm. \end{aligned}$$

We consider the function $\Psi : \mathbb{R} \times [0, \infty) \to [0, \infty)$ given by

$$\Psi(a, b) = a^k b^\ell \quad \text{for fixed } k, \ell \in \mathbb{R}.$$

To check the convexity or concavity of Ψ, we find the Hessian matrix

$$D^2\Psi(a, b) = \begin{pmatrix} k(k-1)a^{k-2}b^\ell & k\ell a^{k-1}b^{\ell-1} \\ k\ell a^{k-1}b^{\ell-1} & \ell(\ell-1)a^k b^{\ell-2} \end{pmatrix}.$$

Momentarily, we will set

$$k = p\Big(2+\frac{1}{n}\Big) - 2 \quad \text{and} \quad \ell = 1-p, \tag{5-8}$$

so we assume that $\ell < 0$ and $k > 0$ since $p > 1$. Now, we have two cases, $0 < k < 1$ and $1 < k$, which correspond to $1 < p < 3n/(2n+1)$ and $p > 3n/(2n+1)$, respectively. In the case $0 < k < 1$, the first minor is negative; hence if the determinant $\det D^2\Psi$ is positive, then the matrix $D^2\Psi$ is negative definite. In the case $1 < k$, the first minor is positive, so if the determinant is positive, then the matrix $D^2\Psi$ is positive definite.

Hence, to determine the convexity or concavity of Ψ, we need to check the positivity of the determinant of the Hessian, which is given by

$$\det(D^2\Psi(a, b)) = a^{2(k-1)}b^{2(\ell-1)}(k(k-1)\ell(\ell-1) - k^2\ell^2) = a^{2k-2}b^{2\ell-2}k\ell(1-k-\ell).$$

Now taking k and ℓ as in (5-8), we find that

$$\begin{aligned} k\ell(1-k-\ell) &= \Big(p\Big(2+\frac{1}{n}\Big)-2\Big)(1-p)\Big(1-\Big(p\Big(2+\frac{1}{n}\Big)-2\Big)-(1-p)\Big) \\ &= \Big(p\Big(2+\frac{1}{n}\Big)-2\Big)(p-1)\Big(p\Big(1+\frac{1}{n}\Big)-2\Big) \end{aligned}$$

is nonnegative if and only if

$$p \geq \frac{2n}{n+1}.$$

The stated result follows after writing

$$\int_{\mathbb{R}^n} |\nabla f_t|^2\, dx = C_{n,p} \int_0^1 \Psi(H_t'(m), H_t''(m))\, dm. \qquad \square$$

Strict convexity under the height function interpolation also holds for potential energies with symmetric increasing potentials.

Proposition 5.4. *Consider V a smooth, bounded, increasing radially symmetric potential. Let $f_0, f_1 \in C(\mathbb{R}^n)$ be two distinct, nonnegative, symmetric decreasing functions with unit mass, and let f_t, $0 \leq t \leq 1$, be the height function interpolation. Then*

$$t \mapsto \int_{\mathbb{R}^n} V(x) f_t(x)\, dx$$

is strictly convex for all $0 < t < 1$.

Proof. Let $v : [0,\infty) \to \mathbb{R}$ be such that $V(x) = v(|x|)$. With this and (5-2) for f_t, we rewrite

$$\begin{aligned}\int_{\mathbb{R}^n} V(x) f_t(x)\,dx &= \int_{\mathbb{R}^n} V(x) \int_0^1 \chi_{(c_n H_t'(m))^{-1/n}}(x) H_t'(m)\,dm\,dx \\ &= \int_0^1 \biggl(nc_n \int_0^{(c_n H_t'(m))^{-1/n}} v(r) r^{n-1}\,dr \biggr) H_t'(m)\,dm \\ &= \int_0^1 F_V((c_n H_t'(m))^{-1/n}) H_t'(m)\,dm, \end{aligned} \tag{5-9}$$

where we define

$$F_V(\xi) := nc_n \int_0^{\xi} v(r) r^{n-1}\,dr, \quad \xi \ge 0.$$

Differentiating this function we obtain

$$F_V'(\xi) = nc_n v(\xi) \xi^{n-1},$$

so that, differentiating the potential energy (5-9), we obtain

$$\begin{aligned}&\frac{d}{dt} \int_{\mathbb{R}^n} V(x) f_t(x)\,dx \\ &\quad = \int_0^1 \biggl(-\frac{1}{n} \frac{F_V'((c_n H_t'(m))^{-1/n})}{(c_n H_t'(m))^{1/n}} + F_V((c_n H_t'(m))^{-1/n}) \biggr) (H_1'(m) - H_0'(m))\,dm \\ &\quad = \int_0^1 \biggl(-\frac{v((c_n H_t'(m))^{-1/n})}{H_t'(m)} + F_V((c_n H_t'(m))^{-1/n}) \biggr) (H_1'(m) - H_0'(m))\,dm. \end{aligned} \tag{5-10}$$

Differentiating again gives the desired result:

$$\frac{d^2}{dt^2} \int_{\mathbb{R}^n} V(x) f_t(x)\,dx = \int_0^1 \frac{1}{c_n^{1/n} n} \frac{v'((c_n H_t'(m))^{-1/n})}{(H_t'(m))^{2+1/n}} (H_1'(m) - H_0'(m))^2\,dm > 0. \qquad \square$$

Lastly, we turn our attention to the proof of Theorem 1.4. For reference, we state a simplified version of the result in [Delgadino et al. 2022].

Proposition 5.5 [Delgadino et al. 2022, Proposition 4.5]. *Consider W a smooth, bounded, increasing radially symmetric kernel. Let $f_0, f_1 \in C(\mathbb{R}^n)$ be two distinct, nonnegative, symmetric decreasing functions with unit mass, and let f_t, $0 \le t \le 1$, be the height function interpolation. Then*

$$t \mapsto \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} f_t(x) f_t(y) W(x-y)\,dx\,dy$$

is strictly convex for all $0 < t < 1$. Moreover, the convexity is monotonic on the derivative with respect to the radial variable W' of the potential.

While the original result in [Delgadino et al. 2022] allows for more singular kernels, the kernels in the Gagliardo seminorms are not included. We again utilize the ε-regularization in (3-1).

Proof of Theorem 1.4. Fix $\varepsilon > 0$ and let $\mathcal{F}_\varepsilon^2$ be as in (3-1). Recalling (3-3), we write

$$\mathcal{F}_\varepsilon^2(f_t) = C_\varepsilon \| f_t \|^2_{L^2(\mathbb{R}^n)} - 2 \int_{\mathbb{R}^{2n}} f_t(x) f_t(x) W_\varepsilon(x-y)\, dx\, dy.$$

Note that $\widetilde{W}_\varepsilon = -2W_\varepsilon$ is a smooth, bounded, increasing radially symmetric kernel. Consequently, we may apply (5-5) and Proposition 5.5 to obtain

$$\frac{d^2}{dt^2}\mathcal{F}_\varepsilon^2(f_t) = 0 + \frac{d^2}{dt^2}\left[\int_{\mathbb{R}^{2n}} f_t(x) f_t(x) \widetilde{W}_\varepsilon(x-y)\, dx\, dy\right] > 0.$$

The convexity now follows by taking $\varepsilon \to 0^+$, using the monotonicity with respect to ε of W'_ε, and the monotonicity of convexity under Proposition 5.5. □

6. On Theorem 1.5

In this section, we use Theorems 1.1 and 1.4 to establish Theorem 1.5. Notice from Figures 1 and 2 that the continuous Steiner symmetrization v^τ of v does not preserve the support of v, so we cannot directly compare v and v^τ using (1-9) to establish uniqueness. To preserve the support of v, we slow down the speed of the level sets near $h = 0$ in Definition 2.1.

Before proceeding with the proof of Theorem 1.5, we present truncated continuous Steiner symmetrizations and their properties.

6.1. *Truncated symmetrizations.* Fix $h_0 > 0$ and let $v_0(h) = \min\{1, h/h_0\}$ for $h \geq 0$. The continuous Steiner symmetrization truncated at height h_0 of a superlevel set $U = \{f > h\} \subset \mathbb{R}$ of a function f at height $h > 0$ is given by $M^{v_0(h)\tau}(U)$. The continuous Steiner symmetrization truncated at height h_0 of a nonnegative function $f \in L^1(\mathbb{R}^n)$ in the direction of e_n is denoted by $\tilde{f}^\tau$ and defined as

$$\tilde{f}^\tau(x) = \int_0^\infty \chi_{M^{v_0(h)\tau}(U^h_{x'})}(x_n)\, dh \quad \text{for } x = (x', x_n) \in \mathbb{R}^n,\ h > 0.$$

Given a Lipschitz function f, we know by Corollary 1.3 that f^τ is also Lipschitz. However, the corresponding truncated symmetrization, $\tilde{f}^\tau$, is not necessarily Lipschitz for all τ since the level sets near $h = 0$ move slower than those above. In particular, the higher level sets may "drop"; see Figure 5. We will show that, when τ is sufficiently small, this does not happen and that $\tilde{f}^\tau$ is Lipschitz with the same support as f.

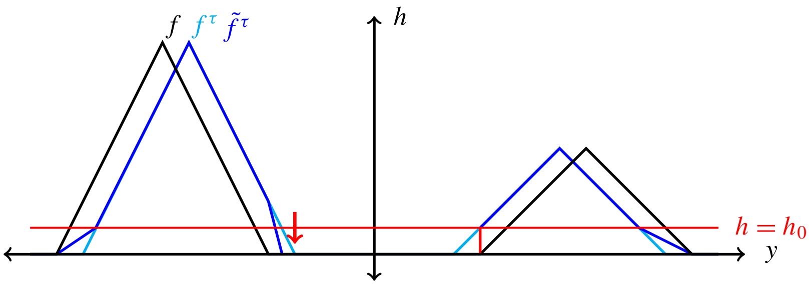

Figure 5. The graphs of f, f^τ, and $\tilde{f}^\tau$ in Example 4.9 at $x = 0$ with $h_0 = \tau = .25$ illustrating how the level sets below the line $h = h_0$ have dropped.

Proposition 6.1. *Let $f : \mathbb{R}^n \to [0, \infty)$ be Lipschitz with $c_0 = [f]_{\mathrm{Lip}}$. Then, for each $0 \le \tau < h_0/c_0$, the function $\tilde{f}^\tau$ is Lipschitz with*

$$[\tilde{f}^\tau]_{\mathrm{Lip}} \le \frac{c_0 h_0}{h_0 - c_0\tau} \tag{6-1}$$

and satisfies

$$\operatorname{supp} \tilde{f}^\tau = \operatorname{supp} f \quad \text{and} \quad \tilde{f}^\tau = f^\tau \qquad \text{in } \{f^\tau > h_0\}. \tag{6-2}$$

Consequently, the upper Dini derivative of $\tilde{f}^\tau$ with respect to τ satisfies

$$\frac{d^+}{d\tau}[\tilde{f}^\tau]_{\mathrm{Lip}}\bigg|_{\tau=0} \le \frac{c_0^2}{h_0}. \tag{6-3}$$

First, we prove a characterization of Lipschitz functions with respect to their level sets.

Lemma 6.2. *For a function $f : \mathbb{R}^n \to [0, \infty)$, we have*

$$[f]_{\mathrm{Lip}} = \sup_{0<h_1<h_2} \frac{h_2 - h_1}{\operatorname{dist}(\partial\{f \ge h_2\}, \partial\{f \le h_1\})}. \tag{6-4}$$

Proof. For ease, set

$$c_0 := \sup_{0<h_1<h_2} \frac{h_2 - h_1}{\operatorname{dist}(\partial\{f \ge h_2\}, \partial\{f \le h_1\})}.$$

We will show that $c_0 = [f]_{\mathrm{Lip}}$.

First, we claim that $c_0 \le [f]_{\mathrm{Lip}}$. If $[f]_{\mathrm{Lip}} = +\infty$, there is nothing to show, so assume that f is Lipschitz. Fix $0 < h_1 < h_2$. If $x, y \in \mathbb{R}^n$ are such that $f(y) = h_1$ and $f(x) = h_2$, then

$$h_2 - h_1 = |f(x) - f(y)| \le [f]_{\mathrm{Lip}}|x - y|.$$

Taking the infimum over all $x \in \{f = h_1\}$ and $y \in \{f = h_2\}$, we have that

$$h_2 - h_1 = |f(x) - f(y)| \le [f]_{\mathrm{Lip}} \operatorname{dist}(\partial\{f \ge h_2\}, \partial\{f \le h_1\}).$$

Equivalently,

$$\frac{h_2 - h_1}{\operatorname{dist}(\partial\{f \ge h_2\}, \partial\{f \le h_1\})} \le [f]_{\mathrm{Lip}} \quad \text{for all } 0 < h_1 < h_2,$$

and we have that $c_0 \le [f]_{\mathrm{Lip}}$.

Let us now show that $[f]_{\mathrm{Lip}} \le c_0$. We may assume that $c_0 < \infty$; otherwise we are done. Let $x, y \in \mathbb{R}^n$ and set $h_1 = f(y)$ and $h_2 = f(x)$. Without loss of generality, assume $0 < h_1 < h_2$. Then

$$|f(x) - f(y)| = h_2 - h_1 \le c_0 \operatorname{dist}(\partial\{f \ge h_2\}, \partial\{f \le h_1\}) \le c_0|x - y|,$$

which shows that f is Lipschitz with $[f]_{\mathrm{Lip}} \le c_0$. This completes the proof of (6-4). □

Remark 6.3. Following the proof of Lemma 6.2, one can show for $\alpha \in (0, 1)$ that

$$[f]_{C^\alpha} = \sup_{0<h_1<h_2} \frac{h_2 - h_1}{|\operatorname{dist}(\partial\{f \ge h_2\}, \partial\{f \le h_1\})|^\alpha}.$$

We expect that a result similar to Proposition 6.1 holds for all C^α-seminorms.

Note in the following that if f is a good function, then

$$\partial\{f \geq h\} = \partial\{f > h\} = \partial\{f \leq h\} \quad \text{for } h > 0.$$

Proof of Proposition 6.1. By Corollary 1.3, we have that f^τ is Lipschitz with $[f^\tau]_{\mathrm{Lip}} \leq c_0$. Fix $h_2 > h_1 > 0$. By Lemma 6.2, we have that

$$\operatorname{dist}(\partial\{f^\tau \geq h_2\}, \partial\{f^\tau \leq h_1\}) \geq \frac{h_2 - h_1}{c_0} \quad \text{for all } \tau \geq 0.$$

Assume for now that f, and consequently f^τ, is a good function, so that

$$\operatorname{dist}(\partial\{f^\tau > h_2\}, \partial\{f^\tau > h_1\}) \geq \frac{h_2 - h_1}{c_0} \quad \text{for all } \tau \geq 0.$$

With this and Lemma 2.5, we have

$$\begin{aligned}
&\operatorname{dist}(\partial\{f^{\tau v_0(h_2)} \geq h_2\}, \partial\{f^{\tau v_0(h_1)} \leq h_1\}) \\
&\qquad = \operatorname{dist}(\partial\{f^{\tau v_0(h_2)} > h_2\}, \partial\{f^{\tau v_0(h_1)} > h_1\}) \\
&\qquad \geq \operatorname{dist}(\partial\{f^{\tau v_0(h_1)} > h_2\}, \partial\{f^{\tau v_0(h_1)} > h_1\}) - \operatorname{dist}(\partial\{f^{\tau v_0(h_2)} > h_2\}, \partial\{f^{\tau v_0(h_1)} > h_2\}) \\
&\qquad \geq \frac{h_2 - h_1}{c_0} - |v_0(h_2)\tau - v_0(h_1)\tau| \\
&\qquad \geq \frac{h_2 - h_1}{c_0} - (h_2 - h_1)\frac{\tau}{h_0} = \left(\frac{c_0 h_0}{h_0 - c_0\tau}\right)^{-1}(h_2 - h_1).
\end{aligned}$$

For each fixed $x' \in \mathbb{R}^{n-1}$, we can similarly show that

$$\operatorname{dist}(\partial\{f^{\tau v_0(h_2)}(x', \cdot) \geq h_2\}, \partial\{f^{\tau v_0(h_1)}(x', \cdot) \leq h_1\}) \geq \left(\frac{c_0 h_0}{h_0 - c_0\tau}\right)^{-1}(h_2 - h_1) > 0$$

for all $\tau < h_0/c_0$. Consequently,

$$M^{v_0(h)\tau}(U_{x'}^{h_2}) \subset M^{v_0(h)\tau}(U_{x'}^{h_1}) \quad \text{for all } 0 < h_1 < h_2 \text{ and } x' \in \mathbb{R}^{n-1}.$$

That is, the sections $U_{x'}^h$ remain ordered and we have (6-2). Therefore, $\tilde{f}^\tau = f^{\tau v_0(h_1)}$ for all $\tau < h_0/c_0$ and $h > 0$, so we have

$$\operatorname{dist}(\partial\{\tilde{f}^\tau \geq h_2\}, \partial\{\tilde{f}^\tau \leq h_1\}) = \operatorname{dist}(\partial\{f^{\tau v_0(h_2)} \geq h_2\}, \partial\{f^{\tau v_0(h_1)} \leq h_1\}) \geq \left(\frac{c_0 h_0}{h_0 - c_0\tau}\right)^{-1}(h_2 - h_1).$$

It follows from Lemma 6.2 that $\tilde{f}^\tau$ is Lipschitz for $\tau < h_0/c_0$ with (6-1).

Suppose now that f is a Lipschitz function but not a good function. In light of Lemma 4.4, there is an approximating sequence of functions f_k that are both good and Lipschitz with $[f_k]_{\mathrm{Lip}(\mathbb{R}^n)} \leq c_0$. By the above, we have that $\tilde{f}_k^\tau$ are also good and Lipschitz. Consequently,

$$\begin{aligned}
[\tilde{f}^\tau]_{\mathrm{Lip}(\mathbb{R}^n)} &\leq [\tilde{f}_k^\tau - \tilde{f}^\tau]_{\mathrm{Lip}(\mathbb{R}^n)} + [\tilde{f}_k^\tau]_{\mathrm{Lip}(\mathbb{R}^n)} \\
&\leq [\tilde{f}_k^\tau - \tilde{f}^\tau]_{\mathrm{Lip}(\mathbb{R}^n)} + \frac{c_0 h_0}{h_0 - c_0\tau} \to \frac{c_0 h_0}{h_0 - c_0\tau} \quad \text{as } k \to \infty,
\end{aligned}$$

and the result holds.

To prove (6-3), we simply use (6-1) to estimate

$$\frac{d^+}{d\tau}[\tilde{f}^\tau]_{\mathrm{Lip}}\Big|_{\tau=0} = \limsup_{\tau\to 0^+} \frac{[\tilde{f}^\tau]_{\mathrm{Lip}} - [f]_{\mathrm{Lip}}}{\tau} \le \limsup_{\tau\to 0^+} \frac{\frac{c_0 h_0}{h_0 - c_0\tau} - c_0}{\tau} = \limsup_{\tau\to 0^+} \frac{c_0^2}{h_0 - c_0\tau} = \frac{c_0^2}{h_0}. \qquad \square$$

We will also need the following estimate on the distance between f and $\tilde{f}^\tau$ in L^1. See [Brock 2000, Theorem 4.2] for a similar result in the setting of Remark 2.4.

Lemma 6.4. *Let $f : \mathbb{R}^n \to [0,\infty)$ have compact support. If $f \in L^\infty(\mathbb{R}^n)$ is Lipschitz with $c_0 = [f]_{\mathrm{Lip}}$, then*

$$\|f - \tilde{f}^\tau\|_{L^\infty(\mathbb{R}^n)} \le \tau [f]_{\mathrm{Lip}(\mathbb{R}^n)} \quad \textit{for all } \tau < h_0/c_0. \tag{6-5}$$

Consequently,

$$\|f - \tilde{f}^\tau\|_{L^1(\mathbb{R}^n)} \le \tau [f]_{\mathrm{Lip}(\mathbb{R}^n)} |\mathrm{supp}\, f| \quad \textit{for all } \tau < h_0/c_0.$$

Moreover, the same bounds also hold for the standard symmetrization f^τ.

Proof. Assume, up to an approximation, that f is a good function. Fix $x = (x', x_n) \in \mathrm{supp}\, f$ and $0 \le \tau < h_0/c_0$. Let $h_1 = \tilde{f}^\tau(x)$ and $h_2 = f(x)$, and assume, without loss of generality, that $0 < h_1 < h_2$. Note that there is a $y_n \in \mathbb{R}$ such that $\tilde{f}^\tau(x', x_n) = f(x', y_n)$, which implies that $\partial\{f(x',\cdot) \le h_1\}$ is nonempty. Then, from Lemma 6.2,

$$|f(x) - \tilde{f}^\tau(x)| = (h_2 - h_1) \le c_0 \,\mathrm{dist}(\partial\{f(x',\cdot) \ge h_2\}, \partial\{f(x',\cdot) \le h_1\}),$$

and with Lemma 2.5, we obtain

$$\begin{aligned}
|f(x) - \tilde{f}^\tau(x)| &\le c_0 \,\mathrm{dist}(\partial\{f(x',\cdot) \ge h_2\}, \partial\{f(x',\cdot) \le h_1\}) \\
&= c_0 \,\mathrm{dist}(\partial\{f(x',\cdot) > h_2\}, \partial\{f(x',\cdot) > h_1\}) \\
&\le c_0 (\mathrm{dist}(\partial\{f(x',\cdot) > h_2\}, \partial\{\tilde{f}^\tau(x',\cdot) > h_1\}) + \mathrm{dist}(\partial\{\tilde{f}^\tau(x',\cdot) > h_1\}, \partial\{f(x',\cdot) > h_1\})) \\
&\le c_0 (0 + v_0(h_1)\tau) \le c_0 \tau.
\end{aligned}$$

Hence the L^∞ estimate holds. With Proposition 6.1, we conclude that

$$\|f - \tilde{f}^\tau\|_{L^1(\mathbb{R}^n)} = \int_{\mathrm{supp}\, f} |f(x) - \tilde{f}^\tau(x)|\, dx \le c_0 \tau |\mathrm{supp}\, f|. \qquad \square$$

We conclude this section with an estimate proving that the H^s norms of f^τ and $\tilde{f}^\tau$ can be made arbitrarily close for sufficiently small $h_0 > 0$ and in the case $0 < s < \frac{1}{2}$.

Lemma 6.5. *Let $0 < s < \frac{1}{2}$. Assume that f is Lipschitz with $[f]_{\mathrm{Lip}(\mathbb{R}^n)} \le c_0$ and not radially decreasing across $\{x_n = 0\}$. Then, for any $\varepsilon > 0$, there is a $h_0 = h_0(\varepsilon, n, s, f) > 0$ and $\tau_0 = \tau_0(h_0, f) > 0$ such that*

$$\Big| \|\tilde{f}^\tau\|^2_{H^s(\mathbb{R}^n)} - \|f^\tau\|^2_{H^s(\mathbb{R}^n)} \Big| < \varepsilon\tau \quad \textit{for all } 0 < \tau \le \tau_0.$$

Proof. Fix $\varepsilon > 0$, and let $h_0 > 0$ to be determined. From Proposition 6.1, we have

$$[\tilde{f}^\tau]_{\mathrm{Lip}(\mathbb{R}^n)} \le \frac{c_0 h_0}{h_0 - c_0\tau} \le 2c_0 \quad \text{for all } 0 \le \tau \le \frac{h_0}{2c_0} =: \tau_0.$$

Moreover, $\tilde{f}^\tau = f^\tau$ in $\{f^\tau > h_0\} \cup (\{\tilde{f}^\tau = 0\} \cap \{f^\tau = 0\})$, so that

$$\|\tilde{f}^\tau\|^2_{H^s(\mathbb{R}^n)} - \|f^\tau\|^2_{H^s(\mathbb{R}^n)} = \int_{\mathbb{R}^n}\int_{A_{h_0}} \frac{(\tilde{f}^\tau(x) - \tilde{f}^\tau(y))^2 - (f^\tau(x) - f^\tau(y))^2}{|x-y|^{n+2s}}\,dx\,dy,$$

where

$$A_{h_0} := (\{\tilde{f}^\tau < h_0\} \cup \{f^\tau < h_0\}) \cap (\operatorname{supp} \tilde{f}^\tau \cup \operatorname{supp} f^\tau).$$

That is, A_{h_0} is the set in which $\tilde{f}^\tau \neq f^\tau$. To estimate the integral, we split into short and long-range interactions. Let $R > 0$ be such that $\operatorname{supp} f^\tau \cup \operatorname{supp} \tilde{f}^\tau \subset B_R$, and write

$$\begin{aligned} \|\tilde{f}^\tau\|^2_{H^s(\mathbb{R}^n)} - \|f^\tau\|^2_{H^s(\mathbb{R}^n)} &= \int_{A_{h_0}}\int_{|x-y|<R} \frac{(\tilde{f}^\tau(x) - \tilde{f}^\tau(y))^2 - (f^\tau(x) - f^\tau(y))^2}{|x-y|^{n+2s}}\,dy\,dx \\ &\quad + \int_{A_{h_0}}\int_{|x-y|\geq R} \frac{(\tilde{f}^\tau(x) - \tilde{f}^\tau(y))^2 - (f^\tau(x) - f^\tau(y))^2}{|x-y|^{n+2s}}\,dy\,dx \\ &=: I + II. \end{aligned}$$

First considering I, notice that the support of the integrand is contained in the set

$$\begin{aligned} &\{(x,y) : |x| < R \text{ or } |y| < R\} \cap \{(x,y) : |x-y| < R\} \\ &\qquad \subset \{(x,y) : |x| < 2R \text{ and } |y| < 2R\} \cap \{(x,y) : |x-y| < R\}. \end{aligned}$$

Therefore,

$$I = \int_{A_{h_0}\cap B_{2R}}\int_{B_{2R}\cap B_R(x)} \frac{(\tilde{f}^\tau(x) - \tilde{f}^\tau(y))^2 - (f^\tau(x) - f^\tau(y))^2}{|x-y|^{n+2s}}\,dy\,dx.$$

With the Lipschitz bounds for $\tilde{f}^\tau$ and f^τ, we estimate

$$\begin{aligned} &|(\tilde{f}^\tau(x) - \tilde{f}^\tau(y))^2 - (f^\tau(x) - f^\tau(y))^2| \\ &\qquad = |(\tilde{f}^\tau(x) - \tilde{f}^\tau(y)) + (f^\tau(x) - f^\tau(y))|\,|(\tilde{f}^\tau(x) - f^\tau(x)) - (\tilde{f}^\tau(y) - f^\tau(y))| \\ &\qquad \leq ([\tilde{f}^\tau]_{\mathrm{Lip}(\mathbb{R}^n)} + [f^\tau]_{\mathrm{Lip}(\mathbb{R}^n)})|x-y|\,|(\tilde{f}^\tau(x) - f^\tau(x)) - (\tilde{f}^\tau(y) - f^\tau(y))| \\ &\qquad \leq 3c_0|x-y|\,|(\tilde{f}^\tau(x) - f^\tau(x)) - (\tilde{f}^\tau(y) - f^\tau(y))|. \end{aligned}$$

Therefore,

$$\begin{aligned} |I| &\leq 3c_0 \int_{A_{h_0}\cap B_{2R}}\int_{B_{2R}\cap B_R(x)} \frac{|(\tilde{f}^\tau(x) - f^\tau(x)) - (\tilde{f}^\tau(y) - f^\tau(y))|}{|x-y|^{n+2s-1}}\,dy\,dx \\ &\leq 3c_0\biggl[\int_{A_{h_0}\cap B_{2R}} |\tilde{f}^\tau(x) - f^\tau(x)|\biggl(\int_{B_{2R}\cap B_R(x)} \frac{1}{|x-y|^{n+2s-1}}\,dy\biggr)\,dx \\ &\qquad + \int_{B_{2R}} |\tilde{f}^\tau(y) - f^\tau(y)|\biggl(\int_{A_{h_0}\cap B_{2R}\cap B_R(y)} \frac{1}{|x-y|^{n+2s-1}}\,dx\biggr)\,dy\biggr]. \end{aligned} \tag{6-6}$$

Since $0 < s < \frac{1}{2}$, we have

$$\int_{B_{2R}\cap B_R(x)} \frac{1}{|x-y|^{n+2s-1}}\,dy \leq \int_{B_{R(x)}} \frac{1}{|x-y|^{n+2s-1}}\,dy = \int_{B_R} \frac{1}{|z|^{n+2s-1}}\,dz = c_{n,s,R} < \infty, \tag{6-7}$$

and similarly

$$\int_{A_{h_0}\cap B_{2R}\cap B_R(y)} \frac{1}{|x-y|^{n+2s-1}}\,dx \le \int_{B_R(y)} \frac{1}{|x-y|^{n+2s-1}}\,dx = c_{n,s,R} < \infty. \tag{6-8}$$

Using again that $\tilde{f}^\tau = f^\tau$ in $\mathbb{R}^n \setminus A_{h_0}$, we note that

$$\int_{B_{2R}} |\tilde{f}^\tau(y) - f^\tau(y)|\,dy = \int_{A_{h_0}\cap B_{2R}} |\tilde{f}^\tau(y) - f^\tau(y)|\,dy. \tag{6-9}$$

Therefore, from (6-6)–(6-9),

$$\begin{aligned}|I| &\le C_{n,s,f,R}\biggl[\int_{A_{h_0}\cap B_{2R}} |\tilde{f}^\tau(x) - f^\tau(x)|\,dx + \int_{B_{2R}} |\tilde{f}^\tau(y) - f^\tau(y)|\,dy\biggr]\\ &\le 2C_{n,s,f,R}\int_{A_{h_0}} |\tilde{f}^\tau(x) - f^\tau(x)|\,dx.\end{aligned}$$

By Lemma 6.4, we arrive at

$$|I| \le C_{n,s,f}|A_{h_0}|\tau.$$

Regarding II, we expand the squares to obtain

$$\begin{aligned}II = \int_{A_{h_0}}\int_{|x-y|\ge R} \frac{(\tilde{f}^\tau(x))^2 - (f^\tau(x))^2}{|x-y|^{n+2s}}\,dy\,dx &+ \int_{A_{h_0}}\int_{|x-y|\ge R} \frac{(\tilde{f}^\tau(y))^2 - (f^\tau(y))^2}{|x-y|^{n+2s}}\,dy\,dx\\ &- 2\int_{A_{h_0}}\int_{|x-y|\ge R} \frac{\tilde{f}^\tau(x)\tilde{f}^\tau(y) - f^\tau(x)f^\tau(y)}{|x-y|^{n+2s}}\,dy\,dx.\end{aligned} \tag{6-10}$$

First observe that

$$\begin{aligned}\int_{A_{h_0}}\int_{|x-y|\ge R} \frac{(\tilde{f}^\tau(x))^2 - (f^\tau(x))^2}{|x-y|^{n+2s}}\,dy\,dx &= \int_{A_{h_0}} [(\tilde{f}^\tau(x))^2 - (f^\tau(x))^2]\biggl(\int_{|z|\ge R} \frac{1}{|z|^{n+2s}}\,dy\biggr)\,dx\\ &= C_{n,s,R}(\|\tilde{f}^\tau\|^2_{L^2(A_{h_0})} - \|f^\tau\|^2_{L^2(A_{h_0})}),\end{aligned} \tag{6-11}$$

and similarly, using Lemma 6.4,

$$\begin{aligned}&\int_{A_{h_0}}\int_{|x-y|\ge R} \frac{(\tilde{f}^\tau(y))^2 - (f^\tau(y))^2}{|x-y|^{n+2s}}\,dy\,dx\\ &\qquad= \int_{A_{h_0}}\int_{|x-y|\ge R} \frac{(\tilde{f}^\tau(y))^2 - (f^\tau(y))^2}{|x-y|^{n+2s}}\,dx\,dy\\ &\qquad= \int_{A_{h_0}} [(\tilde{f}^\tau(y) - f^\tau(y))(\tilde{f}^\tau(y) + f^\tau(y))]\biggl(\int_{|z|\ge R} \frac{1}{|z|^{n+2s}}\,dz\biggr)\,dy\\ &\qquad\le C_{n,s,R}\|f\|_{L^\infty}\|\tilde{f}^\tau - f^\tau\|_{L^1(A_{h_0})}\\ &\qquad\le C_{n,s,f,R}|A_{h_0}|\tau.\end{aligned} \tag{6-12}$$

Consequently, from (6-10)–(6-12),

$$|II| \le C_{n,s,f,R}|A_{h_0}|\tau + \biggl|\int_{A_{h_0}} \tilde{f}^\tau(x)(W_R * \tilde{f}^\tau)(x) - f^\tau(x)(W_R * f^\tau)(x)\,dy\,dx\biggr|,$$

where $W_R(x) = |x|^{-n-2s}\chi_{\mathbb{R}^n\setminus B_R(0)}(x)$. Using that

$$|\nabla(W_R * \tilde{f}^\tau)(x)| \le \int_{\mathbb{R}^n} W_R(y)|\nabla \tilde{f}^\tau(x-y)|\,dy \le 2c_0 \int_{\mathbb{R}^n} W_R(y)\,dy = c_{n,s,f,R},$$

and similarly for $|\nabla W_R * f^\tau|$, we can follow the proof of [Carrillo et al. 2019, Proposition 2.8] to show that

$$\left|\int_{A_{h_0}} \tilde{f}^\tau(x)(W_R * \tilde{f}^\tau)(x) - f^\tau(x)(W_R * f^\tau)(x)\,dy\,dx\right| \le C_{n,s,f,R}\|\min\{f, h_0\}\|_{L^1(\mathbb{R}^n)}\tau.$$

Summarizing, we have

$$\left|\|\tilde{f}^\tau\|^2_{H^s(\mathbb{R}^n)} - \|f^\tau\|^2_{H^s(\mathbb{R}^n)}\right| \le |I| + |II| \le C_{n,s,f,R}(|A_{h_0}| + \|\min\{f, h_0\}\|_{L^1(\mathbb{R}^n)})\tau < \varepsilon\tau$$

for h_0 sufficiently small. □

6.2. ***Proof of Theorem 1.5.*** The proof relies on two results regarding the nonlocal energy

$$\mathcal{E}_s(v) = c_{n,s}[v]^2_{H^s} + \int_{\mathbb{R}^n} |x|^2 v(x)\,dx,$$

where we recall that $c_{n,s}[v]^2_{H^s} = \langle(-\Delta)^s v, v\rangle_{L^2(\mathbb{R}^n)}$. First, we will show that small perturbations of stationary solutions v to the fractional thin-film equation that preserve the support of v correspond to small perturbations in the energy.

Lemma 6.6. *Assume that v is Lipschitz and satisfies the stationary equation*

$$\begin{cases} (-\Delta)^s v = \sum_i \lambda_i \chi_{\mathcal{P}_i}(y) - \frac{1}{2}\beta|y|^2 & \text{in } \operatorname{supp}(v) \subset \mathbb{R}^n, \\ v \ge 0 & \text{in } \mathbb{R}^n. \end{cases} \tag{6-13}$$

Let v^τ be a perturbation of v which is continuous in the C^α norm for every $0 < \alpha < 1$ and preserves mass in each connected component. Then

$$\lim_{\tau\to 0^+} \frac{\mathcal{E}_s(v^\tau) - \mathcal{E}_s(v)}{\tau} = 0.$$

Proof. From the definition of $\mathcal{E}_s$ and the fact that $(-\Delta)^s$ is self-adjoint, we have

$$\begin{aligned} \frac{\mathcal{E}_s(v^\tau) - \mathcal{E}_s(v)}{\tau} &= \int_{\mathbb{R}^n} \left(\frac{1}{2}(-\Delta)^s(v^\tau + v) + \frac{1}{2}\beta|y|^2\right)\frac{(v^\tau - v)}{\tau}\,dy \\ &= \int_{\mathbb{R}^n} \left((-\Delta)^s v + \frac{1}{2}\beta|y|^2\right)\frac{(v^\tau - v)}{\tau}\,dy + \frac{1}{2}\int_{\mathbb{R}^n} (-\Delta)^s(v^\tau - v)\frac{(v^\tau - v)}{\tau}\,dy. \end{aligned}$$

Using that v solves (6-13) and that v^τ preserves the mass of v in each $\mathcal{P}_i$, we notice that the first term vanishes:

$$\int_{\mathbb{R}^n} \left((-\Delta)^s v + \frac{1}{2}\beta|y|^2\right)\frac{(v^\tau - v)}{\tau}\,dy = \sum_i \lambda_i \frac{1}{\tau}\int_{\mathcal{P}_i} (v^\tau - v)\,dy = 0.$$

Using Lemma 6.4 and that

$$(-\Delta)^s(v^\tau - v) \to 0 \quad \text{in } C^0(\mathbb{R}^n),$$

we take $\tau \to 0^+$ in the second term to complete the proof. □

Next, we show that if the perturbation is precisely the truncated Steiner symmetrization of v, then the energy is in fact strictly decreasing.

Proposition 6.7. *Assume $0 < s < \frac{1}{2}$ and v is Lipschitz, nonnegative with compact support. If v is not radially decreasing, then there exist constants $h_0, \gamma, \tau_0 > 0$ such that*

$$\mathcal{E}_s(\tilde{v}^\tau) \le \mathcal{E}_s(v) - \tfrac{1}{2}c_{n,s}\gamma\tau \quad \textit{for all } 0 < \tau < \tau_0,$$

where $\tilde{v}^\tau$ is the continuous Steiner symmetrization truncated at height h_0.

Proof. Begin by writing

$$\mathcal{E}_s(\tilde{v}^\tau) - \mathcal{E}_s(v) = c_{n,s}([\tilde{v}^\tau]_{H^s(\mathbb{R}^n)} - [v]_{H^s(\mathbb{R}^n)}) + \frac{1}{2}\beta\int_{\mathbb{R}^n}|y|^2(\tilde{v}^\tau - v)\,dy.$$

Rearranging, we have

$$\begin{aligned}\int_{\mathbb{R}^n}|y|^2(\tilde{v}^\tau(y) - v(y))\,dy &= \int_{\mathbb{R}^n}|y|^2\int_0^\infty(\chi_{M^{v_0(h)\tau}(U^h_{y'})}(y_n) - \chi_{U^h_{y'}}(y_n))\,dh\,dy\\ &= \int_{\mathbb{R}^{n-1}}\int_0^\infty\biggl(\int_{M^{v_0(h)\tau}(U^h_{y'})}|y_n|^2\,dy_n - \int_{U^h_{y'}}|y_n|^2\,dy_n\biggr)\,dh\,dy'\\ &\le 0,\end{aligned}$$

where the last inequality follows by the definition of the symmetrization. On the other hand, by Theorem 1.1 and Lemma 6.5 with $\varepsilon = \frac{1}{2}\gamma$, there are $h_0, \tau_0 > 0$ such that

$$\begin{aligned}[\tilde{v}^\tau]^2_{H^s(\mathbb{R}^n)} - [v]^2_{H^s(\mathbb{R}^n)} &= ([\tilde{v}^\tau]^2_{H^s(\mathbb{R}^n)} - [v^\tau]^2_{H^s(\mathbb{R}^n)}) + ([v^\tau]^2_{H^s(\mathbb{R}^n)} - [v]^2_{H^s(\mathbb{R}^n)})\\ &\le \tfrac{1}{2}\gamma\tau - \gamma\tau = -\tfrac{1}{2}\gamma\tau \quad \text{for any } \tau < \tau_0. \qquad \square\end{aligned}$$

Proof of Theorem 1.5. Assume, by way of contradiction, that v is not radially decreasing, and let $\tilde{v}^\tau$ denote the continuous Steiner symmetrization of v truncated at height $h_0 > 0$. By Proposition 6.1, we have that $\tilde{v}^\tau$ is Lipschitz for sufficiently small $0 \le \tau < c_0/h_0$ and preserves the mass of v in each connected component. Moreover, by Lemma 6.4, $\tilde{v}^\tau$ is continuous in τ in the C^α norm for any $0 < \alpha < 1$. Hence the hypotheses of Lemma 6.6 are satisfied, so that, for all $\varepsilon > 0$, there is $\delta > 0$ such that

$$-\varepsilon\tau < \mathcal{E}_s(\tilde{v}^\tau) - \mathcal{E}(v) < \tau\varepsilon \quad \text{for all } 0 \le \tau < \delta.$$

On the other hand, Proposition 6.7 guarantees that

$$\mathcal{E}_s(\tilde{v}^\tau) - \mathcal{E}(v) < -\tfrac{1}{2}c_{n,s}\gamma\tau \quad \text{for all } 0 \le \tau < \tau_0.$$

We arrive at a contradiction by choosing $0 < \varepsilon < c_{n,s}\frac{1}{2}\gamma$. Therefore, it must be that v is radially decreasing. Consequently, $\operatorname{supp} v$ is a single connected component.

To show uniqueness, up to the scaling, we follow the argument in the proof of [Delgadino et al. 2022, Theorem 1.1]. Consider two radially symmetric critical points v_0 and v_1 that are Lipschitz, and let $\{v_t\}_{t\in[0,1]}$ be the height function interpolation presented in Section 5. Using that $v_0, v_1 \in C^{0,1}(\mathbb{R}^n)$, we can use Proposition 5.2 to conclude that $\{v_t\}_{t\in[0,1]}$ is continuous in $C^\alpha(\mathbb{R}^n)$ for any $0<\alpha<1$. Recall the upper and lower Dini derivatives in (3-6). We claim that

$$\frac{d^+}{dt}\mathcal{E}_s(v_t)\Big|_{t=0} = \lim_{t\to 0^+}\frac{\mathcal{E}_s(v_t)-\mathcal{E}(v_0)}{t} = 0 = \lim_{t\to 1^-}\frac{\mathcal{E}_s(v_t)-\mathcal{E}(v_1)}{1-t} = \frac{d^-}{dt}\mathcal{E}_s(v_t)\Big|_{t=1}.$$

Following the proof of [Delgadino et al. 2022, Proposition 4.4], it is enough to show that

$$\frac{d^+}{dt}[v_t]^2_{H^s}\Big|_{t=0} = \int_{\mathbb{R}^n}(-\Delta)^s v_0 \frac{dv_\tau}{d\tau}\Big|_{\tau=0}\,dx \tag{6-14}$$

as the potential part of the energy follows in the same way. Since $(-\Delta)^s$ is self-adjoint,

$$\frac{d^+}{dt}[v_t]^2_{H^s}\Big|_{t=0} = \lim_{t\to 0^+}\frac{1}{t}\int_0^t\int_{\mathbb{R}^n}(-\Delta)^s v_\tau\frac{dv_\tau}{d\tau}\,dx\,d\tau.$$

Then, (6-14) holds as long as the pairing

$$\int_{\mathbb{R}^n}(-\Delta)^s v_\tau\frac{dv_\tau}{d\tau}\,dx$$

is continuous in τ. From the Lipschitz a priori estimate, we know that

$$(-\Delta)^s v_\tau \to (-\Delta)^s v_0 \quad \text{strongly in continuous functions as } \tau\to 0,$$

so we only need to check that weakly

$$\frac{dv_\tau}{d\tau} \rightharpoonup \frac{dv_\tau}{d\tau}\Big|_{\tau=0}.$$

This follows directly from [Delgadino et al. 2022, Lemma 4.3], after noticing that both v_0 and v_1 are not degenerate. More specifically, v_0 and v_1 are twice differentiable around zero, and there exists a $c>0$ such that

$$\max\{\Delta v_0(0), \Delta v_1(0)\} < -c.$$

This follows because both v_0 and v_1 solve a fractional elliptic equation in a neighborhood of zero.

However, by the strict convexity of $\mathcal{E}_s(v_t)$, see Theorem 1.4 and Proposition 5.4, we know that

$$\frac{d^+}{dt}\mathcal{E}_s(v_t)\Big|_{t=0} < \frac{d^-}{dt}\mathcal{E}_s(v_t)\Big|_{t=1},$$

which is a contradiction. Therefore, for any given mass, there is a unique critical point to $\mathcal{E}_s$, and it is given by (1-10); see [Dyda 2012]. □

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ROBUST NONLOCAL TRACE AND EXTENSION THEOREMS

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We prove trace and extension results for Sobolev-type function spaces that are well suited for nonlocal Dirichlet and Neumann problems including those for the fractional p-Laplacian. Our results are robust with respect to the order of differentiability. In this sense they align with the classical trace and extension theorems.

1. Introduction

We are concerned with well-posedness of nonlinear nonlocal equations in bounded domains, such as

$$\begin{aligned} (-\Delta)_p^s u &= f \quad \text{in } \Omega, \\ u &= g \quad \text{in } \mathbb{R}^d \setminus \Omega, \end{aligned} \tag{1-1}$$

where the fractional p-Laplacian is defined via

$$(-\Delta)_p^s u(x) = (1-s)\,\text{p.v.}\int_{\mathbb{R}^d} |u(x)-u(y)|^{p-2}(u(x)-u(y))\frac{\mathrm{d}y}{|x-y|^{d+sp}}.$$

A standard approach to problems like (1-1) is the variational approach, which is based on an energy functional and corresponding function spaces. Since the operator $(-\Delta)_p^s u$ is nonlocal, it is necessary to prescribe values $u(x)$ for $x \in \mathbb{R}^d \setminus \Omega$ in order for (1-1) to be well-posed. A possible yet restrictive option is to work in the Sobolev–Slobodeckij space $W^{s,p}(\mathbb{R}^d)$. Note that an assumption of the type $g \in W^{s,p}(\mathbb{R}^d)$ imposes unnatural restrictions since problem (1-1) does not involve any regularity of g in $\mathbb{R}^d \setminus \overline{\Omega}$ other than some weighted integrability. Popular workarounds include assumptions of the type $g \in W^{s,p}(\Omega_\varepsilon) \cap L^p(\mathbb{R}^d; (1+|x|)^{-d-sp}\,\mathrm{d}x)$ for some enlarged domain $\Omega_\varepsilon = \{x \in \mathbb{R}^d \mid \operatorname{dist}(x, \overline{\Omega}) < \varepsilon\}$.

We introduce and study trace spaces on $\mathbb{R}^d \setminus \Omega$ that allow for a natural variational approach to nonlocal nonlinear problems. An important feature of our approach is the robustness of our results as $s \to 1^-$. This allows for a theory of well-posedness for problems like (1-1) that is continuous in the parameter s at $s = 1$. In this case, problem (1-1) reduces to

$$\begin{aligned} -\operatorname{div}(|\nabla u|^{p-2}\nabla u) &= f \quad \text{in } \Omega, \\ u &= g \quad \text{on } \partial\Omega. \end{aligned}$$

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In order to derive the setting of the variational approach, let us explain the definition of a weak solution to our model example (1-1). Given a sufficiently regular solution u to (1-1) and a regular test function $\varphi : \mathbb{R}^d \to \mathbb{R}$ with compact support in Ω, the following should hold:

$$\int_\Omega (-\Delta)_p^s u\, \varphi = \int_\Omega f\varphi,$$

which, after an application of Fubini's theorem, reads

$$\frac{1-s}{2} \iint_{(\Omega^c\times\Omega^c)^c} |u(x)-u(y)|^{p-2}(u(x)-u(y))(\varphi(x)-\varphi(y)) \frac{\mathrm{d}y\,\mathrm{d}x}{|x-y|^{d+sp}} = \int_\Omega f\varphi. \tag{1-2}$$

This line motivates the following definition of an energy space. For a bounded open set $\Omega \subset \mathbb{R}^d$ and $1 \le p < \infty$, we consider the fractional Sobolev-type space

$$V^{s,p}(\Omega \,|\, \mathbb{R}^d) := \{u : \mathbb{R}^d \to \mathbb{R} \text{ measurable} \mid [u]_{V^{s,p}(\Omega \,|\, \mathbb{R}^d)} < \infty\}, \tag{1-3}$$

$$[u]^p_{V^{s,p}(A \,|\, B)} := (1-s) \int_A \int_B \frac{|u(x)-u(y)|^p}{|x-y|^{d+sp}}\,\mathrm{d}x\,\mathrm{d}y, \qquad A, B \in \mathcal{B}(\mathbb{R}^d). \tag{1-4}$$

We endow this space with the norm $\|u\|^p_{V^{s,p}(\Omega \,|\, \mathbb{R}^d)} := \|u\|^p_{L^p(\Omega)} + [u]^p_{V^{s,p}(\Omega \,|\, \mathbb{R}^d)}$. The space $V^{s,p}(\Omega \,|\, \mathbb{R}^d)$ is a separable Banach space and reflexive for $p > 1$; see, e.g., [Foghem Gounoue 2020, Chapter 3.4]. It is well known that this space converges to $W^{1,p}(\Omega)$ for $1 < p < \infty$ as $s \to 1^-$; see [Bourgain et al. 2001, Theorem 2] and [Foghem Gounoue 2023, Theorems 1.1, 1.3, 1.5]. In his famous work, Gagliardo [1957] proved that the classical trace $\gamma : W^{1,p}(\Omega) \to W^{1-1/p,p}(\partial\Omega)$ is linear and continuous and has a continuous right inverse. We are concerned with the search for a trace theorem and extension result for the fractional Sobolev spaces of type $V^{s,p}(\Omega \,|\, \mathbb{R}^d)$ onto the nonlocal boundary Ω^c such that the result is robust in the limit $s \to 1^-$.

Remark 1.1. In some more applied fields such as peridynamics, one studies nonlocal problems in bounded open sets Ω, where data are prescribed in a bounded open set $E \supset \Omega$; see [Mengesha and Du 2016]. Then, there is no need to discuss the decay at infinity, but the main challenge remains: quantify local behavior of functions across the boundary $\partial\Omega$. Our results apply to such problems directly as $\mathbb{R}^d$ can be replaced by a general set E.

Main results. We introduce a space of functions $\mathcal{T}^{s,p}(\Omega^c)$ defined on Ω^c, see (1-6), and prove trace and extension results which are robust in the parameter s; see Theorems 1.2 and 1.3. Lastly, we prove the asymptotic of the spaces $\mathcal{T}^{s,p}(\Omega^c)$ as well as some related weighted L^p spaces as $s \to 1^-$; see Theorem 1.4.

Due to the nonlocality of the operators under consideration, problems like (1-1) can be formulated in open sets, which are not necessarily connected. Since our main results do not require Ω to be connected, we define $\Omega \subset \mathbb{R}^d$ to be a Lipschitz domain if it is open and has a uniform Lipschitz boundary; see Section 2. We define measures

$$\mu_s(\mathrm{d}x) := \mathbb{1}_{\Omega^c}(x)(1-s)d_x^{-s}(1+d_x)^{-d-s(p-1)}\,\mathrm{d}x \tag{1-5}$$

on the Borel σ-algebra $\mathcal{B}(\mathbb{R}^d)$, $s \in (0,1)$, $1 \le p < \infty$, where $d_x := \operatorname{dist}(x, \partial\Omega)$ for $x \in \mathbb{R}^d$. We simply write $\mu_s(x)$ for the density of the measure μ_s with respect to the Lebesgue measure on $\mathbb{R}^d$. Given an open bounded set $A \subset \mathbb{R}^d$, note that $\mu_s(A) \asymp (1-s)\int_{A\cap\Omega^c} d_x^{-s}\,\mathrm{d}x$ and μ_s converges weakly for $s \to 1^-$ to the Hausdorff measure on $\partial\Omega \cap A$; see Lemma 5.1.

We introduce for $s \in (0,1)$, $1 \le p < \infty$, our trace spaces

$$\begin{aligned} \mathcal{T}^{s,p}(\Omega^c) &:= \{g : \Omega^c \to \mathbb{R} \text{ measurable} \mid \|g\|_{\mathcal{T}^{s,p}(\Omega^c)} < \infty\},\\ \|g\|^p_{\mathcal{T}^{s,p}(\Omega^c)} &:= \|g\|^p_{L^p(\Omega^c;\mu_s)} + [g]^p_{\mathcal{T}^{s,p}(\Omega^c)},\\ [f,g]^p_{\mathcal{T}^{s,p}(\Omega^c)} &:= \int_{\Omega^c}\int_{\Omega^c} \frac{|f(x)-f(y)|^{p-2}(f(x)-f(y))(g(x)-g(y))}{((|x-y|+d_x+d_y)\wedge 1)^{d+s(p-2)}}\,\mu_s(\mathrm{d}x)\mu_s(\mathrm{d}y). \end{aligned} \tag{1-6}$$

Here, we use the convention $[g]_{\mathcal{T}^{s,p}(\Omega^c)} = [g,g]_{\mathcal{T}^{s,p}(\Omega^c)}$. The space $\mathcal{T}^{s,p}(\Omega^c)$ is a separable Banach space (Hilbert space for $p=2$) and reflexive for $p>1$; see Lemma 2.2. Now we state the trace result and extension result for $p>1$.

Theorem 1.2. *Let $\Omega \subset \mathbb{R}^d$ be a bounded Lipschitz domain, $s \in (0,1)$, $1 < p < \infty$. Then the trace operator*

$$\operatorname{Tr_s} : V^{s,p}(\Omega \,|\, \mathbb{R}^d) \to \mathcal{T}^{s,p}(\Omega^c), \quad u \mapsto u|_{\Omega^c},$$

is continuous and linear and there exists a continuous linear right inverse

$$\operatorname{Ext_s} : \mathcal{T}^{s,p}(\Omega^c) \to V^{s,p}(\Omega \,|\, \mathbb{R}^d), \quad g \mapsto \operatorname{Ext_s}(g),$$

which we call the nonlocal extension operator. Moreover, the continuity constants of the linear trace and extension operator only depend on Ω and a lower bound on s, as well as a lower and upper bound on p.

An extension of Theorem 1.2 to the case $p = 1$ requires a refined consideration. Analogously to the case $p > 1$, one might guess that the limit space of $V^{s,1}(\Omega \,|\, \mathbb{R}^d)$ as $s \to 1^-$ is $W^{1,1}(\Omega)$. But, in fact, the Sobolev space $W^{1,1}(\Omega)$ is too small to capture all functions such that $\liminf_{s\to 1^-}\|f\|_{V^{s,1}(\Omega\,|\,\mathbb{R}^d)}$ is finite. The limit space of $V^{s,1}(\Omega \,|\, \mathbb{R}^d)$ as $s \to 1^-$ turns out to be the space of functions of bounded variation $BV(\Omega)$; see [Dávila 2002, Theorem 1; Bourgain et al. 2001, Theorem 3′, Corollaries 2 and 5; Foghem Gounoue 2023, Theorems 1.3 and 1.4′]. It is well known that functions in $BV(\Omega)$ have a trace to the boundary $\partial\Omega$ that is integrable and the trace map to $L^1(\partial\Omega)$ is surjective; see [Gagliardo 1957], [Dávila 2002, Theorem 1] or [Leoni 2017, Theorem 18.13]. Theorem 1.2 may be extended to the case $p = 1$ as follows.

Theorem 1.3. *Let $\Omega \subset \mathbb{R}^d$ be a bounded Lipschitz domain, $s \in (0,1)$. Then the trace operator*

$$\operatorname{Tr_s} : V^{s,1}(\Omega \,|\, \mathbb{R}^d) \to L^1(\Omega^c; \mu_s(\mathrm{d}x)), \quad u \mapsto u|_{\Omega^c},$$

is continuous and linear. There exists a continuous linear right inverse

$$\operatorname{Ext_s} : \mathcal{T}^{s,1}(\Omega^c) \to V^{s,1}(\Omega \,|\, \mathbb{R}^d), \quad g \mapsto \operatorname{ext}(g).$$

The continuity constants of the linear trace and extension operator only depend on Ω and a lower bound on s. In addition, the norm of the extension operator in dimension $d = 1$ also depends on a lower bound on $1 - s$.

This result is analogous to the local setting where $L^1(\Omega^c;\mu_s)$ is a suitable replacement for $L^1(\partial\Omega)$. In particular, a direct analog of the trace result from Theorem 1.2 for $p=1$, i.e., $\|u\|_{\mathcal{T}^{s,1}(\Omega^c)} \lesssim \|u\|_{V^{s,1}(\Omega\,|\,\mathbb{R}^d)}$, cannot hold; see the counterexample in Remark 3.11. This is in alignment with the local setting. Recall that there exists a nonlinear bounded extension operator from $L^1(\partial\Omega)$ to $BV(\Omega)$; see, e.g., [Malý et al. 2018, Theorem 1.2]. It was shown in [Peetre 1979] that a continuous extension map of integrable functions on $\partial\Omega$ to a function of bounded variation in Ω cannot be linear. If we restrict ourselves to the Besov space $B^0_{1,1}(\partial\Omega)\subset L^1(\partial\Omega)$, then a continuous linear extension to functions $BV(\Omega)$ that is a right inverse to the trace map exists; see [Malý et al. 2018, Theorem 1.1]. A function $f\in L^1(\partial\Omega)$ is in the Besov space $B^0_{1,1}(\partial\Omega)$ whenever the seminorm $[f]_{B^0_{1,1}(\partial\Omega)}$ is finite, where

$$[f]_{B^0_{1,1}(\partial\Omega)} := \int_{\partial\Omega\times\partial\Omega} \frac{|f(x)-f(y)|}{|x-y|^{d-1}}(\sigma\otimes\sigma)(\mathrm{d}(x,y)).$$

Here, the measure σ is the surface measure on $\partial\Omega$. The Besov space $B^0_{1,1}(\partial\Omega)$ is a Banach space endowed with the norm $\|f\|_{B^0_{1,1}(\partial\Omega)} := \|f\|_{L^1(\partial\Omega)} + [f]_{B^0_{1,1}(\partial\Omega)}$. In Step 1 of the proof of Theorem 1.4, see Section 5, we show that our trace norm $\|f\|_{\mathcal{T}^{s,1}(\Omega^c)}$ converges to $\|f\|_{B^0_{1,1}(\partial\Omega)}$ as $s\to 1^-$ for any $f\in C^{0,1}_c(\mathbb{R}^d)$. In this regard, we recover the local extension theorem to $BV(\Omega)$ functions in the limit $s\to 1^-$ as the extension operator in Theorem 1.3 has a uniformly bounded norm in the same limit.

As mentioned above, the spaces $V^{s,p}(\Omega\,|\,\mathbb{R}^d)$, $1<p<\infty$, converge to the traditional Sobolev space $W^{1,p}(\Omega)$ as the order of differentiability s reaches 1. Having established the robust continuity of the trace and extension operators from Theorems 1.2 and 1.3, our second goal is to study the limiting properties of the spaces $\mathcal{T}^{s,p}(\Omega^c)$ for $s\to 1^-$ and to recover the classical trace and extension results for Sobolev spaces.

Theorem 1.4. *Let $\Omega\subset\mathbb{R}^d$ be a bounded Lipschitz domain, $s\in(0,1)$, $1<p<\infty$. Then*

$$\begin{aligned}
\|\mathrm{Tr_s}\, u\|_{L^p(\Omega^c;\mu_s)} &\to \|\gamma u\|_{L^p(\partial\Omega)}, && u\in W^{1,p}(\mathbb{R}^d),\\
[\mathrm{Tr_s}\, u]_{\mathcal{T}^{s,p}(\Omega^c)} &\to [\gamma u]_{W^{1-1/p,p}(\partial\Omega)}, && u\in W^{1,p}(\mathbb{R}^d),\\
\|\mathrm{Tr_s}\, u\|_{L^1(\Omega^c;\mu_s)} &\to \|\gamma u\|_{L^1(\partial\Omega)}, && u\in BV(\mathbb{R}^d),\\
[\mathrm{Tr_s}\, u]_{\mathcal{T}^{s,1}(\Omega^c)} &\to [\gamma u]_{B^0_{1,1}(\partial\Omega)}, && u\in C^{0,1}_c(\mathbb{R}^d),
\end{aligned}$$

as $s\to 1^-$. Here, γ denotes the classical trace operator and $B^0_{1,1}(\partial\Omega)$ is the Besov space defined above.

Remark 1.5. In the case of a bounded connected $C^{1,1}$-domain Ω and $p=2$, Theorems 1.2 and 1.4 have been established in [Grube and Hensiek 2024]; see the discussion of related literature below.

Applications to the Dirichlet problem. Let us present a well-posedness result for (1-1). We define the space of test functions for the Dirichlet problem as follows:

$$V^{s,p}_0(\Omega\,|\,\mathbb{R}^d) = \{v\in V^{s,p}(\Omega\,|\,\mathbb{R}^d) \mid v=0 \text{ a.e. on } \mathbb{R}^d\setminus\Omega\} \tag{1-7}$$

Definition 1.6. Let $\Omega\subset\mathbb{R}^d$ be a bounded Lipschitz domain, $s\in(0,1)$, $1<p<\infty$. Let $g\in\mathcal{T}^{s,p}(\Omega^c)$ and $f\in V^{s,p}(\Omega\,|\,\mathbb{R}^d)'\supset L^{p'}(\Omega)$. We say that $u\in V^{s,p}(\Omega\,|\,\mathbb{R}^d)$ is a weak solution of (1-1) if, for every $\varphi\in V^{s,p}_0(\Omega\,|\,\mathbb{R}^d)$, the equation (1-2) holds.

Here is our result on well-posedness of the Dirichlet problem.

Corollary 1.7. *Let $\Omega \subset \mathbb{R}^d$ be a bounded Lipschitz domain, $s_\star \le s < 1$, $1 < p < \infty$. Let $g \in \mathcal{T}^{s,p}(\Omega^c)$ and $f \in V^{s,p}(\Omega \,|\, \mathbb{R}^d)' \supset L^{p'}(\Omega)$. Then there exists a unique weak solution $u \in V^{s,p}(\Omega \,|\, \mathbb{R}^d)$ to problem (1-1). Moreover, there is a constant $c > 0$, depending only on p, Ω, $s_\star$, such that*

$$\|u\|_{V^{s,p}(\Omega \,|\, \mathbb{R}^d)} \le c(\|g\|_{\mathcal{T}^{s,p}(\Omega^c)} + \|f\|_{V^{s,p}(\Omega \,|\, \mathbb{R}^d)'}). \tag{1-8}$$

Proof. Let $V^{s,p}_g(\Omega \,|\, \mathbb{R}^d)$ be the set of all functions v of the form $v = \mathrm{Ext}_s(g) + v_0$ with $v_0 \in V^{s,p}_0(\Omega \,|\, \mathbb{R}^d)$ and $\mathrm{Ext}_s(g)$ as in Theorem 1.2. This set is a closed convex subset of $V^{s,p}(\Omega \,|\, \mathbb{R}^d)$. Let $I : V^{s,p}_g(\Omega \,|\, \mathbb{R}^d) \to \mathbb{R}$ be defined by

$$I(v) = \frac{1-s}{2p} \iint_{(\Omega^c \times \Omega^c)^c} \frac{|v(x) - v(y)|^p}{|x-y|^{d+sp}} \,\mathrm{d}y\,\mathrm{d}x - f(v).$$

We note that $f(v)$ is the duality pairing between the functional $f \in V^{s,p}(\Omega \,|\, \mathbb{R}^d)'$ and the function $v \in V^{s,p}_g(\Omega \,|\, \mathbb{R}^d)$. The functional I is strictly convex and weakly lower semicontinuous on the reflexive, separable Banach space $V^{s,p}_g(\Omega \,|\, \mathbb{R}^d)$. Since

$$|f(v)| \le \|f\|_{V^{s,p}(\Omega \,|\, \mathbb{R}^d)'} \|v\|_{V^{s,p}(\Omega \,|\, \mathbb{R}^d)} \le \delta \|v\|^p_{V^{s,p}(\Omega \,|\, \mathbb{R}^d)} + (p')^{-1}(\delta p)^{-1/(p-1)} \|f\|^{p'}_{V^{s,p}(\Omega \,|\, \mathbb{R}^d)'},$$

for every $\delta \in (0,1)$, we can apply the Poincaré inequality, see Proposition 2.1, to the function $v - \mathrm{Ext}_s(g)$ to obtain

$$I(v) \ge \frac{1}{4p}[v]^p_{V^{s,p}(\Omega \,|\, \mathbb{R}^d)} + c_1^{-1}\|v\|^p_{L^p(\Omega)} - c_1\|f\|^{p'}_{V^{s,p}(\Omega \,|\, \mathbb{R}^d)'} - c_1\|\mathrm{Ext}_s(g)\|^p_{V^{s,p}(\Omega \,|\, \mathbb{R}^d)}$$

for some constant c_1 depending on p and on the constant from Proposition 2.1. Thus, the functional I is coercive in the sense that $I(v) \to +\infty$ for $\|v\|_{V^{s,p}(\Omega \,|\, \mathbb{R}^d)} \to +\infty$. We have shown that I attains a unique minimizer u on the set $V^{s,p}_g(\Omega \,|\, \mathbb{R}^d)$. It is now straightforward to show that the function u solves problem (1-1). The claimed estimate follows from $I(u) \le I(\mathrm{Ext}_s(g))$, the above estimate and Theorem 1.2. □

Let us quickly review some related results on problems for nonlocal operators in bounded domains with given exterior data. Note that there are also approaches to nonlocal problems in bounded domains Ω with data given on $\partial\Omega$ such as [Grubb 2015], which we do not take into account here.

Some early well-posedness results for variational nonlocal problems of the type (1-1) can be found in [Servadei and Valdinoci 2012; 2013; Felsinger et al. 2015]. The case of homogeneous problems, i.e., when $g = 0$, is particularly simple and has been treated by several authors. Note that the vector space $\widetilde{D}^{s,p}(\Omega)$ in [Piersanti and Pucci 2017] equals the space $V^{s,p}_0(\Omega \,|\, \mathbb{R}^d)$. Existing results for nonzero data g often assume g to be regular in all of $\mathbb{R}^d$, e.g., [Di Castro et al. 2016, Theorem 2.3; Lindgren and Lindqvist 2017, Theorem 8; Acosta et al. 2019, Proposition 2.2]. As [Korvenpää et al. 2017, Example 1] shows, optimal results require extra care and more regularity than just suitable integrability of g in $\mathbb{R}^d$. Also, $g \in W^{s,p}(\Omega) \cap L^p(\mathbb{R}^d; (1+|x|)^{-d-sp}\,\mathrm{d}x)$ does not imply well-posedness as claimed in [Palatucci 2018], which is not essential at all for the main results of that work. Workarounds avoiding global $W^{s,p}(\mathbb{R}^d)$-regularity are used in [Korvenpää et al. 2016; 2017, Lemma 6; Brasco et al. 2018, Definition 2.10]. These approaches assume $g \in W^{s,p}(\Omega_\varepsilon) \cap L^p(\mathbb{R}^d; (1+|x|)^{-d-sp}\,\mathrm{d}x)$ for some enlarged domain $\Omega_\varepsilon = \{x \in \mathbb{R}^d \mid \operatorname{dist}(x, \overline{\Omega}) < \varepsilon\}$. Concerning the case $p = 1$, we refer to [Bucur et al. 2023] for results on existence and regularity of solutions to (1-1) with given exterior data.

Note that well-posedness and energy estimates similar to (1-8) are proved for $p = 2$ in [Foghem and Kassmann 2024] and for general p in [Foghem 2025]. The present work resolves the matter of optimal assumptions on exterior data g, which has been achieved for $p = 2$ and $C^{1,1}$-domains in [Grube and Hensiek 2024].

Remark 1.8. Note that the fractional p-Laplacian is well defined at a point $x \in \mathbb{R}^d$ if u is sufficiently regular in a neighborhood of x and $u \in L^{p-1}(\mathbb{R}^d; (1+|x|)^{-d-sp}\,\mathrm{d}x)$. For a variational approach, the tail space $L^p(\mathbb{R}^d; (1+|x|)^{-d-sp}\,\mathrm{d}x)$ is more natural, but modifications are possible.

Remark 1.9. For demonstration purposes, we have presented the well-posedness result for the fractional p-Laplacian. It is straightforward to extend to more general nonlinear operators of the form

$$\text{p.v.} \int_{\mathbb{R}^d} \Phi(x, |u(x)-u(y)|)(u(x)-u(y)) k_s(x,y)\,\mathrm{d}y$$

for appropriate functions Φ and kernels k_s, $s \in (0,1)$.

Related results. Let us discuss related results concerning function spaces, in particular trace theorems. As explained above, the main new feature of the energy space $V^{s,p}(\Omega \mid \mathbb{R}^d)$ is that functions in $V^{s,p}(\Omega \mid \mathbb{R}^d)$ satisfy some incremental regularity across the boundary plus some integrability at infinity. Dyda and Kassmann [2019] provide trace and extension results for $V^{s,p}(\Omega \mid \mathbb{R}^d)$ for rather general domains Ω.[1] The proof is based on a Whitney decomposition of Ω and Ω^c, which we employ here, too. However, the construction of the extension operator in [Dyda and Kassmann 2019] is much simpler and uses the Lebesgue measure. Thus, for $s \to 1^-$, one does not recover the classical extension result. In order to resolve this problem, we introduce the measure μ_s on Ω^c, which converges to the surface measure on $\partial\Omega$.

In [Bogdan et al. 2020], the authors prove a version of the Douglas identity and provide trace and extensions results for spaces like $V^{s,2}(\Omega \mid \mathbb{R}^d)$, where they allow for a large class of Lévy measures $\nu(\mathrm{d}h)$ instead of $|h|^{-d-2s}\,\mathrm{d}h$. The proof is based on a careful study of the Poisson kernel and provides a representation of the energy of the solution u to problems like (1-1) ($p = 2$) in terms of its trace on Ω^c. The article leaves open the question of robustness as $s \to 1^-$. Unlike [Bogdan et al. 2020], we define the trace space for general $p \geq 1$ with the help of explicitly given norms that allow for robustness and limit results as $s \to 1^-$. Extensions of the results in [Bogdan et al. 2020] to some nonlinear cases, still based on L^2-Lévy integrable kernels, can be found in [Bogdan et al. 2023].

A systematic study of generalizations of the energy space $V^{s,p}(\Omega \mid \mathbb{R}^d)$ in the case of $p = 2$ and a Lévy measure $\nu(\mathrm{d}h)$ can be found in [Foghem and Kassmann 2024], where functional inequalities, well-posedness results and some nonlocal-to-local convergence results are provided. The trace space is shown to contain a certain weighted L^2-space of functions on Ω^c. Foghem [2025] provided extensions to the general

[1] Note that in [Dyda and Kassmann 2019] the domain of integration in (1.6) and (1.7) has to be changed from $\Omega^c \times \Omega^c$ to $M \times \Omega^c$ with $M = \{x \in \Omega^c \mid \operatorname{dist}(x, \partial\Omega) < 1\}$.

case $p > 1$. Nonlocal energy spaces appear also in the context of Markov jump processes in [Vondraček 2021]. Here, the author considers the intersection with $L^2(\mathbb{R}^d; m)$, where $m(x) = \mathbb{1}_\Omega(x) + \mu(x)\mathbb{1}_{\Omega^c}(x)$ and $\mu(x)$ behaves like $\operatorname{dist}(x, \partial\Omega)^{-2s}$ for x close to $\partial\Omega$; see Remark 2.37 in [Foghem and Kassmann 2024] for detailed comments. This approach together with functional inequalities and questions of well-posedness has been studied for more general kernels in [Frerick et al. 2025].

The present work can be seen as an extension of results in [Grube and Hensiek 2024]. Here we treat general bounded Lipschitz domains and the full range $p \geq 1$ instead of bounded $C^{1,1}$-domains and $p = 2$ in the aforementioned work. Both works use the measure μ_s, but the construction of the extension operator is different. In the present work we employ the Whitney decomposition technique and not the Poisson extension. The study of nonlocal Neumann problems as in [Grube and Hensiek 2024] together with the asymptotic behavior for $s \to 1^-$ is possible in our framework, too. In order to keep the scope of this work reasonable, we defer this line of research until a later date.

Last, let us mention recent trace and extension results for nonlocal function spaces, where problems similar to ours occur but the setup is conceptually different. In [Tian and Du 2017] the trace space $H^{1/2}(\partial\Omega)$ is recovered as the trace space of a certain $L^2(\Omega)$-space with a nonlocal interaction kernel that has a localizing property at the boundary $\partial\Omega$. The analogous result for $W^{s-1/p,p}(\partial\Omega)$ is proved in [Du et al. 2022a]. The result is extended further to domains with very rough boundaries including those with spatially varying dimension in [Foss 2021]. See [Scott and Du 2024] for applications to nonlocal equations with Dirichlet data given on $\partial\Omega$. Given a localization parameter $\delta > 0$ and a domain Ω, the authors of [Du et al. 2022b] study trace and extension operators between the domain and a layer $\{x \in \Omega^c \mid \operatorname{dist}(x, \partial\Omega) < \delta\}$. The operators are shown to be robust as $\delta \to 0$, which makes it possible to recover classical trace results. For more details we refer to the discussion in [Grube and Hensiek 2024, Section 1.2].

The development of nonlocal function spaces and related trace and extension results benefits greatly from classical results for Sobolev, Sobolev–Slobodeckij, or Besov spaces. Early results on trace spaces for $W^{1,p}(\Omega)$ can be found in [Aronszajn 1955; Slobodeckiĭ and Babič 1956; Prodi 1956; Gagliardo 1957; Slobodeckiĭ 1958] and the monograph [Nečas 1967]. See [Nečas 2012] for an English translation and, in particular, Chapter 2.5 therein. Lipschitz domains and fractional-order spaces are covered in [Grisvard 2011], e.g., in Theorems 1.5.1.3 and 1.5.2.1. For domains with corners see also [Yakovlev 1967]. The corresponding state-of-the-art around this time is summarized in Chapter 1, Sections 7–9 of [Lions and Magenes 1972]. Another standard reference focusing on contributions of researchers from the Soviet Union is [Besov et al. 1975, Chapter IV]. Another important monograph in this direction is [Triebel 1983], in particular Chapters 3.3.3 and 3.3.4. Trace and extensions results are provided in [Marschall 1987] under minimal regularity assumptions on the domains. A survey of results on boundary value problems for higher-order elliptic equations with degeneracies along the boundary is given in [Nikolskiĭ et al. 1988]. Kim [2007] extends well-known trace assertions for weighted Sobolev spaces. The aforementioned list is rather selective and not complete at all. Even some fundamental problems such as a trace result for $H^s(\Omega)$, $1 < s < \frac{3}{2}$, and Ω a bounded Lipschitz domain are not covered in the list above; see [Ding 1996].

Very useful references for our work are contributions of A. Jonsson and H. Wallin [Jonsson and Wallin 1978; 1984; Jonsson 1994]. The setting in the aforementioned references includes results for subsets of the Euclidean space endowed with general doubling measures. This is related to our framework because we consider measure spaces $(\Omega^c; \mu_s)$ with μ_s as in (1-5). Moreover, the construction used in the proof of the extension result Theorem 1.2 is inspired by the corresponding results Theorems 3.1 and 4.1 in [Jonsson and Wallin 1978].

Outline. In Section 2 we fix the notation and shortly introduce function spaces used throughout this work. The trace embeddings are studied in Section 3. We divide the proofs of the trace results from Theorems 1.2 and 1.3 into the L^p-embedding, see Proposition 3.9, and the seminorm inequality, see Proposition 3.10. We construct the extension operator in Section 4. The extension theorems are proven in Proposition 4.5 as well as Proposition 4.6 with precise dependencies of the operator norms. Lastly, the limiting properties of the spaces $\mathcal{T}^{s,p}(\Omega^c)$, see Theorem 1.4, are proven in Section 5.

2. Preliminaries

2.1. ***Notation.*** For two real numbers $a, b \in \mathbb{R}$, we write $a \wedge b = \min\{a, b\}$, $a \vee b := \max\{a, b\}$ and $\lfloor a \rfloor = \max(-\infty, a] \cap \mathbb{Z}$. The ball of radius $r > 0$ centered at $x \in \mathbb{R}^d$ in the d-dimensional Euclidean space is written as $B_r(x)$ or $B_r^{(d)}$ whenever we want to specify the dimension. For a set A, we denote by $\mathbb{1}_A$ the indicator function on A. An open set $\Omega \subset \mathbb{R}^d$ is said to have a uniform Lipschitz boundary if there exists a localization radius $r > 0$ and a constant $L > 0$ such that, for any boundary point $z \in \partial\Omega \neq \varnothing$, there exists a translation and rotation $T_z : \mathbb{R}^d \to \mathbb{R}^d$ satisfying $T_z(z) = 0$ as well as a Lipschitz continuous function $\phi_z : \mathbb{R}^{d-1} \to \mathbb{R}$ whose Lipschitz constant is bounded by L such that

$$T_z(\Omega \cap B_r(z)) = \{(x', x_d) \in B_r(0) \mid \phi_z(x') > x_d\};$$

see, e.g., [Leoni 2017, Definition 13.11] and the discussion in [Grisvard 2011, Chapter 1.2.1]. An open set $\varnothing \neq B \subset \mathbb{R}^d$ is said to satisfy the uniform exterior cone condition if we find an opening angle α and a height $h_0 > 0$ such that, for any $z \in \partial\Omega$, there exists an exterior cone $\mathcal{C}_z \subset \overline{\Omega}^c$ with apex at z and height h_0. The notion of the uniform interior cone condition is defined analogously. Note that an open set with a uniform Lipschitz boundary satisfies both uniform interior and exterior cone conditions. The interior cones (resp. the exterior cones) can simply be constructed via

$$\mathcal{C}_z := T_z^{-1}\left\{(x', x_d) \in B_r(0) \mid x_d < -\tfrac{1}{2}L|x'|\right\}$$

for $z \in \partial\Omega$. We say that $\Omega \subset \mathbb{R}^d$ is a Lipschitz domain if it is open and has a uniform Lipschitz boundary. Notice that we do not assume Ω to be connected. Nevertheless, a bounded Lipschitz domain has finitely many connected components since the uniform interior cone condition bounds the volume of each connected component from below by a uniform positive constant. We denote the distance of x to a closed set $A \subset \mathbb{R}^d$ by $\operatorname{dist}(x, A) = \inf\{|x - a| \mid a \in A\}$. When the dependencies are clear, we write for short $d_x := \operatorname{dist}(x, \partial\Omega)$ for any $x \in \mathbb{R}^d$. Furthermore, we use for $r > 0$ the notation

$$\Omega_r^{\text{ext}} := \{x \in \overline{\Omega}^c \mid d_x < r\}, \qquad \Omega^r_{\text{ext}} := \{x \in \overline{\Omega}^c \mid d_x \geq r\}. \tag{2-1}$$

We denote by $\mathcal{H}^{(d-l)}$ the normalized $(d-l)$-dimensional Hausdorff measure on $\mathbb{R}^d$. The surface measure of the $(d-1)$-dimensional unit sphere will be written for short as $\mathcal{H}^{(d-1)}(\partial B_1) = \omega_{d-1}$. To shorten the notation, we write σ for the surface measure on $\partial\Omega$. The inner radius of the domain Ω we denote by

$$\operatorname{inr}(\Omega) := \sup\{r \mid B_r \subset \Omega\}.$$

We will use lowercase letters $c_1, c_2, \ldots$ with running indices as constants in our proofs and reset them after every proof. When we introduce a new constant, we write $C = C(\ldots)$ to indicate what the constant depends on, i.e., $C = C(d, \Omega) > 0$ depends only on the dimension d and the set Ω.

2.2. *Function spaces.* The following function spaces will be used throughout this work. We denote by $W^{s,p}(\Omega)$ (resp. $W^{s,p}(\partial\Omega)$), $s \in (0, 1)$, $p \geq 1$, the Sobolev–Slobodeckij space of functions in $u \in L^p(\Omega)$ satisfying

$$[u]_{W^{s,p}(\Omega)} := [u]_{V^{s,p}(\Omega \mid \Omega)} < \infty$$

endowed with the norm $\|u\|^p_{W^{s,p}(\Omega)} := \|u\|^p_{L^p(\Omega)} + [u]^p_{W^{s,p}(\Omega)}$ (resp. $\partial\Omega$ with the surface measure). See (1-4) for the definition of the seminorm $[\,\cdot\,]_{V^{s,p}(A \mid B)}$. We write $BV(\Omega)$ for the space of functions $u \in L^1(\Omega)$ with bounded variation endowed with the norm $\|u\|_{BV(\Omega)} := \|u\|_{L^1(\Omega)} + |\nabla u|(\Omega)$. The Bessel potential spaces $H^{s,p}(\mathbb{R}^d)$ are defined in (3-5). As mentioned in the introduction, a variational approach to equations like (1-1) leads naturally to function spaces like $V^{s,p}(\Omega \mid \mathbb{R}^d)$ which we introduced in (1-3). These function spaces are the focus of our study. They were first introduced in [Servadei and Valdinoci 2012; 2014; Felsinger et al. 2015] for the case $p = 2$. We also refer to [Dipierro et al. 2017], in which the nonlocal normal operator was introduced, and [Foghem Gounoue 2020; Foghem and Kassmann 2024; Foghem 2025] for an intensive study of these spaces for general p. It is well known that $V^{s,p}(\Omega \mid \mathbb{R}^d)$ is a separable Banach space (Hilbert space for $p = 2$) which is reflexive for $1 < p < \infty$; see, e.g., [Foghem Gounoue 2020, Chapter 3.4].

The spaces $V^{s,p}(\Omega \mid \mathbb{R}^d)$ allow for a Poincaré inequality, which is an important ingredient for the proof of well-posedness for the Dirichlet problem (1-1) together with an energy estimate; see Corollary 1.7. We will need a version of the Poincaré inequality that is robust as s reaches 1.

Proposition 2.1 [Foghem 2025, Theorem 10.1]. *Let $p > 1$ and $s_\star \in (0, 1)$. Let $\Omega \subset \mathbb{R}^d$ be a bounded Lipschitz domain. Then there exists $c > 0$ such that, for all $s_\star \leq s < 1$ and $u \in V_0^{s,p}(\Omega \mid \mathbb{R}^d)$,*

$$\|u\|_{L^p(\Omega)} \leq c\|u\|_{V^{s,p}(\Omega \mid \mathbb{R}^d)}. \tag{2-2}$$

Let us recall our trace spaces $\mathcal{T}^{s,p}(\Omega^c)$, which are introduced in (1-6). For $s \in (0, 1)$, $1 \leq p < \infty$ and $A, B \in \mathcal{B}(\Omega^c)$, we define

$$[f, g]^p_{\mathcal{T}^{s,p}(A \mid B)} := \int_A \int_B \frac{|f(x) - f(y)|^{p-2}(f(x) - f(y))(g(x) - g(y))}{((|x - y| + d_x + d_y) \wedge 1)^{d+s(p-2)}} \mu_s(\mathrm{d}x)\mu_s(\mathrm{d}y) \tag{2-3}$$

with the convention $[g]_{\mathcal{T}^{s,p}(A \mid B)} = [g, g]_{\mathcal{T}^{s,p}(A \mid B)}$. Note that $[f, g]_{\mathcal{T}^{s,p}(\Omega^c)} = [f, g]_{\mathcal{T}^{s,p}(\Omega^c \mid \Omega^c)}$. We employ standard techniques to prove that these spaces are separable Banach spaces (resp. Hilbert spaces if $p = 2$).

Lemma 2.2. *Let Ω be an open set. The space $\mathcal{T}^{s,p}(\Omega^c)$ is a separable Banach space, reflexive for $1 < p < \infty$, and in the case $p = 2$, it is a separable Hilbert space with inner product*

$$(u, v)_{\mathcal{T}^{s,2}(\Omega^c)} = (u, v)_{L^2(\Omega^c,\mu_s)} + [u, v]^2_{\mathcal{T}^{s,2}(\Omega^c)}.$$

Proof. To prove completeness, we take a Cauchy sequence $\{u_n\}_n \subset \mathcal{T}^{s,p}(\Omega^c)$. Then $v_n(x) := u_n(x)\mu_s(x)^{1/p}$ is Cauchy in $L^p(\Omega^c)$ with limit $v \in L^p(\Omega^c)$. Define $u(x) := v(x)\mu_s(x)^{-1/p}$. Then u is the limit of u_n with respect to $\|\cdot\|_{L^p(\Omega^c;\mu_s)}$. Take a subsequence $\{u_{n_l}\}_l$ converging a.e. to u on $\mathbb{R}^d$. Then, by Fatou's lemma, we have

$$[u - u_{n_l}]^p_{\mathcal{T}^{s,p}(\Omega^c)} \le \liminf_{k\to\infty} [u_{n_k} - u_{n_l}]^p_{\mathcal{T}^{s,p}(\Omega^c)} \to 0 \quad \text{as } l \to \infty.$$

Separability follows from the fact that the map $\iota : \mathcal{T}^{s,p}(\Omega^c) \to L^p(\Omega^c) \times L^p(\Omega^c \times \Omega^c)$,

$$u \mapsto \left(x \mapsto u(x)\mu_s(x)^{1/p},\ (x, y) \mapsto \frac{u(x) - u(y)}{((|x - y| + d_x + d_y) \wedge 1)^{d/p+s(p-2)/p}} \mu_s(x)^{1/p}\mu_s(y)^{1/p} \right),$$

is an isometric injection. As $\iota(\mathcal{T}^{s,p}(\Omega^c))$ is closed and since $L^p(\Omega^c) \times L^p(\Omega^c \times \Omega^c)$ is separable, so is $\mathcal{T}^{s,p}(\Omega^c)$. In the same manner, as $L^p(\Omega^c) \times L^p(\Omega^c \times \Omega^c)$ is reflexive for $1 < p < \infty$, so is $\mathcal{T}^{s,p}(\Omega^c)$. □

The functions from $\mathcal{T}^{s,p}(\Omega^c)$ have some regularity at the boundary because the weight in the seminorm $[\,\cdot\,,\cdot\,]_{\mathcal{T}^{s,p}(\Omega^c)}$ becomes $((|x - y|) \wedge 1)^{-d-s(p-2)}$ as $x, y \to \partial\Omega$. Thereby, for sufficiently large s, the functions in the trace space $\mathcal{T}^{s,p}(\Omega^c)$ themselves have a trace onto the boundary $\partial\Omega$. This is a direct consequence of Theorem 1.2.

Corollary 2.3. *The space $\mathcal{T}^{s,p}(\Omega^c)$ is continuously embedded in $W^{s-1/p,p}(\partial\Omega)$ for any $s \in (1/p, 1)$ and $p \in (1, \infty)$. The embedding is surjective. The continuity constant depends only on Ω, p and a lower bound on s.*

Proof. By Theorem 1.2, the extension $\mathrm{Ext}_s : \mathcal{T}^{s,p}(\Omega^c) \to V^{s,p}(\Omega \,|\, \mathbb{R}^d)$ is continuous and the continuity constant $c_1 > 0$ depends only on Ω, p and a lower bound on s. The space $V^{s,p}(\Omega \,|\, \mathbb{R}^d)$ is embedded in $W^{s,p}(\Omega)$ with the embedding constant depending only on a lower bound on s. The result follows from the classical trace result $W^{s,p}(\Omega) \to W^{s-1/p,p}(\partial\Omega)$. The embedding is surjective since we can extend a function from $W^{s-1/p,p}(\partial\Omega)$ to an element from $W^{s,p}(\mathbb{R}^d) \hookrightarrow V^{s,p}(\Omega \,|\, \mathbb{R}^d) \hookrightarrow \mathcal{T}^{s,p}(\Omega^c)$. □

3. Trace theorem

Here we aim to prove the trace parts of Theorems 1.2 and 1.3. This proof is carried out in Propositions 3.9 and 3.10. Essential building blocks in the respective proofs are an approximation to the classical L^p-trace embedding in Theorem 3.5 and, for $p = 1$, a Hardy-type inequality provided in Theorem 3.6. On a more technical level, we use upper and lower bounds of the distance function; see Lemmas 3.7 and 3.8.

To prove Theorem 3.5 we apply techniques developed in [Jonsson and Wallin 1984]. In particular, we use the interpolation between Bessel potential spaces on $\mathbb{R}^d$. For this reason we need a Sobolev extension operator for fractional Sobolev spaces $W^{s,p}(\Omega)$ whose continuity constant is independent of s. The existence of such an extension is well known in the literature. We provide this result in the following theorem for the convenience of the reader.

Theorem 3.1 [Jonsson and Wallin 1984, Chapter VI.2, Theorem 3; Triebel 1995]. *Let $\Omega \subset \mathbb{R}^d$ be a connected Lipschitz domain. There exists a linear map E, which extends measurable functions $f : \Omega \to \mathbb{R}$ such that $E : L^p(\Omega) \to L^p(\mathbb{R}^d)$ for all $p \geq 1$ and, with some constant $C = C(d, \Omega, p) > 0$, for any $0 < s \leq 1$,*

$$\|Ef\|_{W^{s,p}(\mathbb{R}^d)} \leq C\|f\|_{W^{s,p}(\Omega)}.$$

Remark 3.2. The extension is constructed via a Whitney decomposition of $\overline{\Omega}^c$, a smooth partition of unity and copying mean values of f from inside to respective Whitney cubes. The construction of the extension Ef is independent of s and p and satisfies $E : W^{1,p}(\Omega) \to W^{1,p}(\mathbb{R}^d)$. Real interpolation allows us to choose the constant $C(d, \Omega, p)$ in the theorem independent of s.

Analogously to the measure μ_s from (1-5), we define for $s \in (0, 1)$ the measure

$$\tau_s(\mathrm{d}x) = \frac{1-s}{d_x^s} \mathbb{1}_\Omega(x)\, \mathrm{d}x \tag{3-1}$$

on the σ-algebra $\mathcal{B}(\mathbb{R}^d)$. Recall that $d_x = \operatorname{dist}(x, \partial\Omega)$. The measure $\tau_s(\mathrm{d}x)$ plays the same role as μ_s but is supported inside Ω. We use it in Theorem 3.5 for the proof of the trace part of our main theorems; see also Propositions 3.9 and 3.10. In contrast to μ_s, the measure τ_s does not need the additional term $(1 + d_x)^{-d-s(p-1)}$ for the decay at infinity since the open set Ω is assumed to be bounded throughout this work. The following lemma shows how balls scale under τ_s. This scaling plays a crucial role in Theorem 3.5.

Lemma 3.3. *Let $\Omega \subset \mathbb{R}^d$ be a bounded Lipschitz domain with a localization radius $r_0 > 0$. There exists a constant $C = C(d, \Omega) > 0$ such that, for any $s \in (0, 1)$, $0 < r \leq \frac{1}{2}r_0$ and $x \in \Omega$,*

$$\tau_s(B_r(x)) \leq C r^{d-s}. \tag{3-2}$$

Proof. Let $d \geq 2$. If $r \leq d_x$, i.e., $B_r(x) \Subset \Omega$, then $d_y \geq r - |x-y|$ for any $y \in B_r(x)$ and, thus,

$$\tau_s(B_r(x)) = \int_{B_r(x)} \frac{1-s}{d_y^s}\, \mathrm{d}y \leq \int_{B_r(x)} \frac{1-s}{(r-|x-y|)^s}\, \mathrm{d}y = \omega_{d-1} \int_0^r \frac{1-s}{(r-t)^s} t^{d-1}\, \mathrm{d}t \leq \omega_{d-1} r^{d-s}.$$

Now we consider the case $r > d_x$, i.e., $B_r(x) \cap \partial\Omega \neq \varnothing$. Without loss of generality we assume that $0 \in \partial\Omega$ is a minimizer of d_x. Since Ω is a Lipschitz domain, we find a Lipschitz map $\phi : \mathbb{R}^{d-1} \to \mathbb{R}$ such that $\Omega \cap B_{r_0} = \{(y', y_d) \in B_{r_0} \mid y_d < \phi(y')\}$. The Lipschitz constant of ϕ is bounded by $L > 0$ independent of x. A simple calculation yields, for any $y = (y', y_d) \in B_r(x) \cap \Omega$,

$$\begin{aligned} |y| &\leq |x| + |y-x| \leq 2r, \\ |y_d - \phi(y')| &\leq \inf_{(\tilde{y}', \phi(\tilde{y}')) \in B_{r_0}} |y_d - \phi(\tilde{y}')| + |\phi(y') - \phi(\tilde{y}')| \\ &\leq 2^{1/2}(1+L) \inf_{(\tilde{y}', \phi(\tilde{y}')) \in B_{r_0}} |y - (\tilde{y}', \phi(\tilde{y}'))| \\ &= 2^{1/2}(1+L) d_y. \end{aligned} \tag{3-3}$$

In the case that the minimizer of d_y is not in the graph of ϕ, we simply pick a smaller r_0 depending only on the constant L. Therefore,

$$\tau_s(B_r(x)) \le 2^{s/2}(1+L)^s \int_{B_{2r}\cap\{y_d<\phi(y')\}} \frac{1-s}{|y_d-\phi(y')|^s}\,\mathrm{d}(y',y_d)$$
$$\le 2(1+L)^s\omega_{d-2}(2r)^{d-1}\int_0^{(2+L)r}\frac{1-s}{y_d^s}\,\mathrm{d}y_d \le 2^{d+1}(2+L)\omega_{d-2}r^{d-s}. \tag{3-4}$$

The proof in the case $d=1$ is straightforward. Note that similar arguments as in this proof are employed in the proof of Lemma 4.1. □

In the proof of Theorem 3.5, we use interpolation results, which we explain now. Let G_α, $\alpha>0$ be the Bessel potential kernel. We introduce the Bessel potential spaces

$$H^{\alpha,p}(\mathbb{R}^d) := \{g:\mathbb{R}^d\to\mathbb{R} \mid \exists f\in L^p(\mathbb{R}^d) : g = G_\alpha * f\} \tag{3-5}$$

with the canonical norm $\|g\|_{H^{\alpha,p}(\mathbb{R}^d)} := \|f\|_{L^p(\mathbb{R}^d)}$ if $g\in H^{\alpha,p}(\mathbb{R}^d)$ and $g=G_\alpha * f$. The convolution of the Bessel potential kernel with the function f can be written as $G_\alpha * f = \mathscr{F}^{-1}((1+|\cdot|^2)^{-\alpha/2}\mathscr{F}f)$, where $\mathscr{F}$ is the Fourier transformation; see [Bergh and Löfström 1976, p. 139, Definition 6.2.3]. We refer the reader to [Aronszajn and Smith 1961] for more details on the kernel G_α. We recall the real interpolation result

$$[H^{\alpha_0,p}(\mathbb{R}^d), H^{\alpha_1,p}(\mathbb{R}^d)]_{p\theta} = W^{s,p}(\mathbb{R}^d), \tag{3-6}$$

where $0<\alpha_0<\alpha_1$, $\theta\in(0,1)$, $s=(1-\theta)\alpha_0+\theta\alpha_1$ and $p\ge 1$; see, e.g., [Bergh and Löfström 1976, Theorem 6.2.4]. Analogously to [Jonsson and Wallin 1984, Chapter V], we calculate bounds on the Bessel potential kernel G_{α_i} for some $0<\alpha_0<s<\alpha_1$, see Lemma 3.4, and prove an approximate trace result inside Ω; see Theorem 3.5.

The following lemma is a slight modification of [Jonsson and Wallin 1984, Lemma C] that fits our setting. The well-known estimates of the Bessel potential kernel, its gradient and decay at infinity are crucial in the proof. For more details on the Bessel potential we refer to [Taibleson 1964, Chapter IV]. In particular, we need to pay attention to the constants and their dependencies.

Lemma 3.4 [Jonsson and Wallin 1984, Chapter V, Lemma C]. *Let $\Omega\subset\mathbb{R}^d$ be a bounded connected Lipschitz domain, $0<s_\star\le s<1$ and $1<p_\star\le p\le p^\star<\infty$. We set*

$$\alpha_0 := s\frac{1+p}{2p}, \qquad \alpha_1 := 1+\frac{s}{2p}, \tag{3-7}$$

and $\beta_i := \alpha_i - s/p$ for $i\in\{0,1\}$. There exists a constant $C=C(d,\Omega,p_\star,p^\star,s_\star)>0$ such that, for all $0<r\le\frac{1}{2}r_0$ and $f\in L^p(\mathbb{R}^d)$, we have

$$\frac{1}{r^{d-s}}\iint_{\substack{\Omega\times\Omega\\|x-y|<r}} |G_{\alpha_i}*f(x)-G_{\alpha_i}*f(y)|^p\tau_s(\mathrm{d}y)\tau_s(\mathrm{d}x) \le Cr^{p\beta_i}\|f\|^p_{L^p(\mathbb{R}^d)}, \tag{3-8}$$

$$\int_\Omega |G_{\alpha_i}*f(x)|^p\tau_s(\mathrm{d}x)\le C\|f\|^p_{L^p(\mathbb{R}^d)}. \tag{3-9}$$

Proof. In [Jonsson and Wallin 1984, Chapter V, Lemma C], the statement is proven for doubling measures satisfying (3-2) under the assumptions $0 < \beta_i < 1$ and $0 < \alpha_i \neq d$. The proof uses estimates of the Bessel potential kernel G_α; see [loc. cit., Chapter V, Lemmas 1, A, B]. Carefully inspecting the proof of [loc. cit., Chapter V, Lemma C], we find that the resulting constant depends on the constant C from (3-2), a lower bound $0 < \beta_{i,\star} \leq \beta_i$ and an upper bound $\beta_i \leq \beta_i^\star < 1$, as well as a lower bound on $|d - \alpha_i|$. We calculate

$$0 < s_\star \frac{p_\star - 1}{2p_\star} \leq s \frac{p-1}{2p} = \beta_0 \leq \frac{1}{2} < 1,$$
$$0 < 1 - \frac{1}{p_\star} < 1 - \frac{s}{2p} = \beta_1 \leq 1 - \frac{s_\star}{p^\star} < 1.$$

Furthermore, we have

$$|d - \alpha_0| = d - s\frac{1+p}{2p} \geq (d-1) + \frac{p-1}{2p} \geq \frac{p_\star - 1}{2p_\star} > 0$$

and

$$|d - \alpha_1| = \begin{cases} d - 1 - \dfrac{s}{2p} \geq 1 - \dfrac{1}{2p_\star} > 0, & d \geq 2, \\ \dfrac{s}{2p} \geq \dfrac{s_\star}{2p^\star} > 0, & d = 1. \end{cases}$$

This yields the estimates with dependencies of the constants as claimed. □

Theorem 3.5 (approximate trace inequality). *Let $\Omega \subset \mathbb{R}^d$ be a bounded Lipschitz domain, $1 < p_\star < p^\star < \infty$ and $s_\star \in (0, 1)$. There exists a constant $C = C(d, \Omega, p_\star, p^\star, s_\star) > 0$ such that, for every $s \in (s_\star, 1)$, $p_\star \leq p \leq p^\star$ and $u \in W^{s,p}(\Omega)$,*

$$\int_\Omega |u(x)|^p \tau_s(\mathrm{d}x) + \int_\Omega \int_\Omega \frac{|u(x) - u(y)|^p}{((|x-y| + d_x + d_y) \wedge 1)^{d+s(p-2)}} \tau_s(\mathrm{d}y)\tau_s(\mathrm{d}x) \leq C \|u\|^p_{W^{s,p}(\Omega)}. \tag{3-10}$$

Before we give the proof of this theorem we want to motivate it. In anticipation of Section 5, the left-hand side of (3-10) converges in the limit $s \to 1$ to

$$\int_{\partial\Omega} |u|^p \,\mathrm{d}\sigma + \int_{\partial\Omega \times \partial\Omega} \frac{|u(x) - u(y)|^p}{(|x-y| \wedge 1)^{d-1+p(1-1/p)}} (\sigma \otimes \sigma)(x, y) \asymp \|u\|^p_{W^{1-1/p,p}(\partial\Omega)}.$$

Thereby, we retrieve the classical trace result $W^{1,p}(\Omega) \to W^{1-1/p,p}(\partial\Omega)$ in the limit $s \to 1^-$.

Proof. Since Ω is a bounded open set with a uniform Lipschitz boundary, Ω decomposes into finitely many connected components Ω_i, $i \in \{1, \ldots, I\}$, and each Ω_i is a connected bounded Lipschitz domain. First, we prove (3-10) for each Ω_i.

We define α_0 and α_1 as in (3-7) depending on p and s. We set $\theta := \frac{s(p-1)}{(2+s)p} \in (0, 1)$ and notice that

$$0 < s_\star \frac{p_\star - 1}{(2 + s_\star)p_\star} \leq \theta \leq \frac{p^\star - 1}{3p^\star} < 1.$$

Most importantly, the relation $s = (1-\theta)\alpha_0 + \theta\alpha_1$ is true. By Theorem 3.1, it is sufficient to prove the existence of a constant $C > 0$ such that

$$\int_{\Omega_i} |u(x)|^p \tau_s(\mathrm{d}x) + \int_{\Omega_i} \int_{\Omega_i} \frac{|u(x) - u(y)|^p}{((|x-y| + d_x + d_y) \wedge 1)^{d+s(p-2)}} \tau_s(\mathrm{d}y)\tau_s(\mathrm{d}x) \leq C \|u\|^p_{W^{s,p}(\mathbb{R}^d)}$$

for any $u \in W^{s,p}(\mathbb{R}^d)$. Let $c_1 > 0$ be the constant from Lemma 3.4. The equality (3-9) proves the continuity of the restriction operator $Ru(x) = u|_{\Omega_i}$ as a map from $R : H^{\alpha_i,p}(\mathbb{R}^d) \to L^p(\Omega_i, \tau_s(\mathrm{d}x))$, $i = 0, 1$. Real interpolation yields the continuity of

$$R : [H^{\alpha_0,p}(\mathbb{R}^d), H^{\alpha_1,p}(\mathbb{R}^d)]_{p\theta} = W^{s,p}(\mathbb{R}^d) \to [L^p(\Omega_i, \tau_s), L^p(\Omega_i, \tau_s)]_{p\theta} = L^p(\Omega_i, \tau_s)$$

with the continuity constant c_1; see, e.g., [Bergh and Löfström 1976]. Now we consider the second term on the left-hand side of (3-10) with Ω_i in place of Ω. Let $u \in W^{s,p}(\mathbb{R}^d)$. We write

$$\begin{aligned}\int_{\Omega_i}\int_{\Omega_i} &\frac{|u(x)-u(y)|^p}{((|x-y|+d_x+d_y)\wedge 1)^{d+s(p-2)}}\tau_s(\mathrm{d}y)\tau_s(\mathrm{d}x)\\ &\le 2\sum_{n=0}^{\infty} 2^{ns(p-1)} \iint_{\substack{\Omega_i\times\Omega_i\\ 2^{-n-1}\le|x-y|<2^{-n}}} |u(x)-u(y)|^p \frac{(\tau_s\otimes\tau_s)(\mathrm{d}(y,x))}{|x-y|^{d-s}}\\ &\qquad + \iint_{\substack{\Omega_i\times\Omega_i\\ 1\le|x-y|}} |u(x)-u(y)|^p \tau_s(\mathrm{d}y)\tau_s(\mathrm{d}x) =: \mathrm{(I)} + \mathrm{(II)}.\end{aligned}$$

We estimate (II) using the continuity of R shown above. We have

$$\mathrm{(II)} \le 2^p\tau_s(\Omega_i)\int_{\Omega_i} |Ru(x)|^p\tau_s(\mathrm{d}x) \le c_1^p 2^p \tau_s(\Omega_i)\|u\|^p_{W^{s,p}(\mathbb{R}^d)}.$$

We set $H := L^p(\Omega_i\times\Omega_i, |x-y|^{-d+s}\tau_s(\mathrm{d}y)\tau_s(\mathrm{d}x))$ and define for any $1 < q \le \infty$, $\beta > 0$ the space of sequences

$$l^{\beta,q} := \{(h_n)_n \mid h_n \in H\}, \qquad \|(h_n)\|_{l^{\beta,q}} := \big\|(2^{n\beta}\|h_n\|_H)_n\big\|_{l^q(\mathbb{N})}.$$

Notice that

$$\mathrm{(I)} = \|((u(x)-u(y))\mathbb{1}_{2^{-n-1}\le|x-y|<2^{-n}})_n\|^p_{l^{s-s/p,p}}. \tag{3-11}$$

We define the linear map

$$Tf(x,y) := ((f(x)-f(y))\mathbb{1}_{2^{-n-1}\le|x-y|<2^{-n}})_n, \qquad f : \mathbb{R}^d \to \mathbb{R}.$$

Lemma 3.4, in particular (3-8), shows the continuity of $T : H^{\alpha_i,s} \to l^{\beta_i,\infty}$ with $\beta_i = \alpha_i - s/p$ and the continuity constant c_1, $i = 0, 1$. Real interpolation yields the continuity of $T : [H^{\alpha_0,p}(\mathbb{R}^d), H^{\alpha_1,p}(\mathbb{R}^d)]_{p\theta} = W^{s,p}(\mathbb{R}^d) \to [l^{\beta_0,\infty}, l^{\beta_1,\infty}]_{p\theta} = l^{(1-\theta)\beta_0+\theta\beta_1,p}$ with the continuity constant c_1; see, e.g., [Bergh and Löfström 1976]. This proves the claimed inequality for each connected component Ω_i by (3-11) and

$$(1-\theta)\beta_0 + \theta\beta_1 = (1-\theta)\alpha_0 + \theta\alpha_1 - \frac{s}{p} = s - \frac{s}{p}.$$

Simply summing over $i \in \{1, \dots, I\}$ yields a constant $c_2 = c_2(d, \Omega, p_\star, p^\star, s_\star) > 0$ such that

$$\|u\|^p_{L^p(\Omega;\tau_s)} \le c_2\|u\|^p_{W^{s,p}(\Omega)}. \tag{3-12}$$

It remains to prove the existence of a constant $c_3 = c_3(d, \Omega, p_\star, p^\star, s_\star) > 0$ such that, for any $i \ne j$,

$$\int_{\Omega_i}\int_{\Omega_j} \frac{|u(x)-u(y)|^p}{((|x-y|+d_x+d_y)\wedge 1)^{d+s(p-2)}}\tau_s(\mathrm{d}y)\tau_s(\mathrm{d}x) \le c_3\|u\|^p_{W^{s,p}(\Omega)}.$$

Since the distance between any two connected components is bounded from below by a uniform constant, this is an immediate consequence of the triangle inequality and Lemma 3.3, as well as (3-12). ☐

Theorem 3.5 is not true in the case $p = 1$; see Remark 3.11. If we only keep the first term on the left-hand side in the estimate (3-10), then it is a fractional Hardy inequality; see, e.g., [Dyda 2004]. In [Dyda and Kijaczko 2022, Theorem 4], such a Hardy inequality is proven with a constant whose dependency on the parameter s is not known. Since the dependency on s is crucial in our setup, we prove the following theorem based on a Hardy inequality on the half-space with the best constant; see Theorem B.1.

Theorem 3.6 (Hardy inequality). *Let $\Omega \subset \mathbb{R}^d$ be a bounded Lipschitz domain and $s \in (0, 1)$. There exists a constant $C = C(d, \Omega) > 0$ such that*

$$(1-s)\int_\Omega \frac{|u(x)|}{d_x^s}\,\mathrm{d}x \le C\left(\|u\|_{L^1(\Omega)} + s(1-s)\int_{\Omega\times\Omega}\frac{|u(x)-u(y)|}{|x-y|^{d+s}}\,\mathrm{d}(x,y)\right)$$

for any $u \in W^{s,1}(\Omega)$.

Before we state the proof of the theorem, let us remark that the previous inequality, in the limit $s \to 1^-$, yields the classical trace embedding $W^{1,1}(\partial\Omega) \to L^1(\partial\Omega)$ since the measure τ_s converges weakly to the surface measure on $\partial\Omega$.

Proof. It is sufficient to prove the statement for any connected component of Ω in place of Ω. Thus, we can assume without loss of generality that Ω is a connected bounded Lipschitz domain. Therefore, we can cover the boundary with finitely many neighborhoods U_i and bi-Lipschitz maps $\phi_i : U_i \to B_1(0)$ such that $\phi_i(U_i \cap \Omega) = B_1(0)_+ := \{(x', x_d) \in B_1(0) \mid x_d > 0\}$, $i \in \{1, \dots, N\}$; see, e.g., [Grisvard 2011, Chapter 1.2.1]. We denote the distance of $\Omega \cap \bigcap_{i=1}^N U_i^c$ to the boundary $\partial\Omega$ by $2r_0 > 0$. We fix

$$U_0 := \left\{x \in \mathbb{R}^d \;\middle|\; \operatorname{dist}\left(x, \Omega \cap \bigcap_{i=1}^N U_i^c\right) < r_0\right\} \subset \Omega.$$

Notice that $\{U_i \mid i = 0, \dots, N\}$ is an open cover of $\overline{\Omega}$ and $\operatorname{dist}(U_0, \partial\Omega) \ge r_0$. Next, we pick a partition of unity $\eta_i \in C_c^\infty(U_i)$ adapted to U_i, i.e., $\sum_{i=0}^N \eta_i = 1$ on $\overline{\Omega}$. We define $\tilde{\eta}_i := \eta_i \circ \phi_i^{-1} \in C_c^{0,1}(B_1(0))$. Let $c_1 = c_1(\tilde{\eta}_1, \dots, \tilde{\eta}_N) \ge 1$ such that $[\tilde{\eta}_i]_{C^{0,1}} \le c_1$ for all $i = 1, \dots, N$. Without loss of generality, we assume that $\tilde{\eta}_i = 1$ in $B_{1/2}(0)$ for all $i = 1, \dots, N$. Then

$$\begin{aligned}\int_\Omega \frac{|u(x)|}{d_x^s}\,\mathrm{d}x \\ &= \sum_{i=1}^N \int_{B_1(0)_+} \frac{|u(\phi_i^{-1}(x))|}{d^s_{\phi_i^{-1}(x)}}\eta_i(\phi_i^{-1}(x))|\det(D\phi_i^{-1}(x))|\,\mathrm{d}x + \int_{U_0}\eta_0(x)\frac{|u(x)|}{d_x^s}\,\mathrm{d}x =: \sum_{i=1}^N (\mathrm{I}_i) + (\mathrm{II}).\end{aligned}$$

We define $u_i := u \circ \phi_i^{-1}$ for all $i = 1, \dots, N$. By the bi-Lipschitz continuity of the ϕ_i, we find a constant $c_2 = c_2(\phi_1, \dots, \phi_N) > 1$ such that $c_2^{-1}x_d \le d_{\phi_i^{-1}(x)} \le c_2 x_d$ for any $x \in B_1(0)_+$ and $i \in \{1, \dots, N\}$; see (3-3). Further, we find a constant $c_3 = c_3(\phi_1, \dots, \phi_N) \ge 1$ such that both $[\phi_i^{-1}]_{C^{0,1}}$ and $[\phi_i]_{C^{0,1}}$ are bounded from above by c_3 and from below by c_3^{-1} for all i. We apply Theorem B.1 to the function $\tilde{\eta}_i u_i$ to find

$$\begin{aligned}
(\mathrm{I}_i) &\le c_2 c_3^d \int_{\mathbb{R}^d_+} \frac{|\tilde{\eta}_i(x) u_i(x)|}{x_d^s} \,\mathrm{d}x \\
&\le c_2 c_3^d \mathcal{D}_{s,1}^{-1} \int_{\mathbb{R}^d_+ \times \mathbb{R}^d_+} \frac{|\tilde{\eta}_i(x) u_i(x) - \tilde{\eta}_i(y) u_i(y)|}{|x-y|^{d+s}} \,\mathrm{d}(x,y) \\
&\le c_2 c_3^d \mathcal{D}_{s,1}^{-1} \Bigg(\int_{B_1(0)_+ \times B_1(0)_+} \tilde{\eta}_i(x) \frac{|u_i(x) - u_i(y)|}{|x-y|^{d+s}} \,\mathrm{d}(x,y) \\
&\qquad\qquad + \int_{B_1(0)_+ \times B_1(0)_+} |u_i(y)| \frac{|\tilde{\eta}_i(x) - \tilde{\eta}_i(y)|}{|x-y|^{d+s}} \,\mathrm{d}(x,y) \\
&\qquad\qquad\qquad + 2 \int_{B_1(0)_+} \tilde{\eta}_i(x) |u_i(x)| \int_{B_1(0)^c} |x-y|^{-d-s} \,\mathrm{d}y \,\mathrm{d}x \Bigg) =: (\mathrm{III}_i) + (\mathrm{IV}_i) + (\mathrm{V}_i).
\end{aligned}$$

The first term in the previous estimate, i.e., (III_i), can be simply estimated using a change of variables and the bi-Lipschitz continuity of ϕ_i:

$$(\mathrm{III}_i) \le c_2 c_3^{4d+s} \mathcal{D}_{s,1}^{-1} \int_{(U_i \cap \Omega) \times (U_i \cap \Omega)} \frac{|u(x) - u(y)|}{|x-y|^{d+s}} \,\mathrm{d}(x,y).$$

To estimate (IV_i), we calculate

$$\int_{B_1(0)} \frac{|\tilde{\eta}_i(x) - \tilde{\eta}_i(y)|}{|x-y|^{d+s}} \,\mathrm{d}x \le c_1 \frac{\omega_{d-1} 2^{1-s}}{1-s}.$$

Using this, we find

$$(\mathrm{IV}_i) \le c_1 c_2 c_3^d \mathcal{D}_{s,1}^{-1} \frac{2\omega_{d-1}}{1-s} \int_{B_1(0)_+} |u_i(y)| \,\mathrm{d}y \le c_1 c_2 c_3^{2d} \mathcal{D}_{s,1}^{-1} \frac{2\omega_{d-1}}{1-s} \int_{U_i \cap \Omega} |u(x)| \,\mathrm{d}x.$$

Now, we estimate (V_i). Since $\tilde{\eta}_i \in C_c^{0,1}(B_1)$, we find a constant $c_4 \ge 1$ such that $\tilde{\eta}_i(x) \le c_4(1-|x|)$ for all $i = 1, \dots, N$. We notice for any $x \in B_1(0)$

$$\int_{B_1(0)^c} |x-y|^{-d-s} \,\mathrm{d}y \le \int_{B_{(1-|x|)}(x)^c} |x-y|^{-d-s} \,\mathrm{d}y = \omega_{d-1} \frac{(1-|x|)^{-s}}{s}$$

and, thus,

$$(\mathrm{V}_i) \le 2 c_2 c_3^d c_4 \frac{\mathcal{D}_{s,1}^{-1}}{s} \omega_{d-1} \int_{B_1(0)_+} |u_i(x)| \,\mathrm{d}x \le 2 c_2 c_3^{2d} c_4 \frac{\mathcal{D}_{s,1}^{-1}}{s} \omega_{d-1} \int_{U_i \cap \Omega} |u(x)| \,\mathrm{d}x.$$

To estimate (II), we simply notice that the distance function is bounded from below by r_0 on U_0. So, finally, we put everything together. This yields

$$\int_\Omega \frac{|u(x)|}{d_x^s} \,\mathrm{d}x \le \frac{c_6}{1-s} \int_\Omega |u(x)| \,\mathrm{d}x + c_7 s \int_{\Omega \times \Omega} \frac{|u(x) - u(y)|}{|x-y|^{d+s}} \,\mathrm{d}(x,y).$$

Here

$$c_6 := (r_0 \wedge 1)^{-1} + 2\omega_{d-1} N c_1 c_2 c_3^{2d} c_4 c_5, \quad c_7 := N c_2 c_3^{4d+1} c_5$$

and c_5 is the constant from Lemma B.2. $\square$

The next two lemmas are technical tools which we employ in the proof of the trace result; see Propositions 3.9 and 3.10. They allow us to rewrite the distance functions appearing in the measure μ_s as an integral over Ω. This enables us to use the regularity of the functions from $V^{s,p}(\Omega\,|\,\mathbb{R}^d)$ in Ω when we prove the trace result.

Lemma 3.7. *Let $\varnothing \neq B \subset \mathbb{R}^d$ be an open set, $s > 0$ and $f : [0,\infty) \to [0,\infty)$ be a nonincreasing function. For any $x \in \overline{B}^c$, we have*

$$\int_B \frac{f(|x-z|)}{|x-z|^{d+s}}\,\mathrm{d}z \le \frac{\omega_{d-1}}{s}\frac{f(\operatorname{dist}(x,B))}{\operatorname{dist}(x,B)^s}.$$

If B is bounded, then there exists a constant $C = C(d,B)$ such that, for any $x \in \overline{B}^c$,

$$\int_B \frac{f(|x-z|)}{|x-z|^{d+s}}\,\mathrm{d}z \le \frac{C}{s}\frac{f(\operatorname{dist}(x,B))}{\operatorname{dist}(x,B)^s(1+\operatorname{dist}(x,B))^d}.$$

Proof. Fix $x \in \overline{B}^c$. We use $B \subset B_{\operatorname{dist}(x,B)}(x)^c$ and apply polar coordinates to get

$$\int_B \frac{f(|x-z|)}{|x-z|^{d+s}}\,\mathrm{d}z \le \int_{B_{\operatorname{dist}(x,B)}(x)^c} \frac{f(|x-z|)}{|x-z|^{d+s}}\,\mathrm{d}z = \omega_{d-1}\int_{\operatorname{dist}(x,B)}^\infty f(t)t^{-1-s}\,\mathrm{d}t \le \omega_{d-1}\frac{f(\operatorname{dist}(x,B))}{s\operatorname{dist}(x,B)^s}.$$

In the case that $\operatorname{dist}(x,B) < 1$, the second claim for bounded B is a direct consequence of the first statement. If B is bounded and $\operatorname{dist}(x,B) \ge 1$, then

$$\int_B \frac{f(|x-z|)}{|x-z|^{d+s}}\,\mathrm{d}z \le |B|\frac{f(\operatorname{dist}(x,B))}{\operatorname{dist}(x,B)^{d+s}} \le |B|2^d\frac{f(\operatorname{dist}(x,B))}{\operatorname{dist}(x,B)^s(1+\operatorname{dist}(x,B))^d}. \qquad \square$$

Lemma 3.8. *Let $\varnothing \neq B \subset \mathbb{R}^d$ be an open set satisfying the uniform interior cone condition with a compact boundary. Then there exists a constant $C = C(d,B) > 0$ such that, for any nonincreasing function $f : [0,\infty) \to [0,\infty)$ and any $s > 0$,*

$$\frac{f(2\operatorname{dist}(x,\overline{B}))}{\operatorname{dist}(x,\overline{B})^s(1+\operatorname{dist}(x,\overline{B}))^d} \le C\int_B \frac{f(|x-z|)}{|x-z|^{d+s}}\mathbb{1}_{B_{1+\operatorname{dist}(x,\overline{B})}(x)}(z)\,\mathrm{d}z$$

for all $x \in \overline{B}^c$.

Proof. Fix $x \in \overline{B}^c$ and a minimizer $x_0 \in \partial B$ of the distance $\operatorname{dist}(x,\overline{B})$. Since B satisfies the uniform interior cone condition we find an interior cone $\mathcal{C}$ with apex at x_0 whose height h_0 and open angle are independent of x_0. Without loss of generality, we assume $h_0 \le 1$. Let $\widetilde{\mathcal{C}} := \{z \in \mathcal{C} \mid |z-x_0| < \operatorname{dist}(x,\overline{B})\}$ be a subcone with a reduced height. Notice $\widetilde{\mathcal{C}} \subset B_{1+\operatorname{dist}(x,\overline{B})}(x) \cap B$. For any $z \in \widetilde{\mathcal{C}}$, we have

$$|x-z| \le |x-x_0| + |x-z| \le \operatorname{dist}(x,\overline{B}) + \min\{\operatorname{dist}(x,\overline{B}),h_0\} \le 2\operatorname{dist}(x,\overline{B}).$$

Thus, the claim simply follows from

$$\begin{aligned}\int_B \frac{f(|x-z|)}{|x-z|^{d+s}}\mathbb{1}_{B_{1+\operatorname{dist}(x,\overline{B})}(x)}(z)\,\mathrm{d}z &\ge \frac{f(2\operatorname{dist}(x,\overline{B}))}{(2\operatorname{dist}(x,\overline{B}))^{d+s}}|\widetilde{\mathcal{C}}| = f(2\operatorname{dist}(x,\overline{B}))\frac{c_1(\min\{\operatorname{dist}(x,\overline{B}),h_0\})^d}{(2\operatorname{dist}(x,\overline{B}))^{d+s}}\\ &\ge \frac{c_1h_0^d}{2^{d+1}}\frac{f(2\operatorname{dist}(x,\overline{B}))}{\operatorname{dist}(x,\overline{B})^s(1+\operatorname{dist}(x,\overline{B}))^d}.\end{aligned}$$

Here we used $|\widetilde{\mathcal{C}}| = c_1(\min\{\operatorname{dist}(x,\overline{B}),h_0\})^d$, where $c_1 > 0$ depends only on d and the opening angle of $\widetilde{\mathcal{C}}$, which is independent of x_0. $\square$

We are now in the position to prove the trace part in Theorems 1.2 and 1.3. We split the proof into two propositions. The following proposition contains the trace embedding $V^{s,p}(\Omega\,|\,\mathbb{R}^d) \to L^p(\Omega^c;\mu_s(\mathrm{d}x))$ for all $1 \le p < \infty$. The estimates of the seminorm $[\,\cdot\,]_{\mathcal{T}^{s,p}(\Omega^c)}$ for $1 < p < \infty$ are proven thereafter in Proposition 3.10. Recall the definition of the sets Ω_r^{ext} and Ω^r_{ext} in (2-1) for given $r > 0$.

Proposition 3.9. *Let $\Omega \subset \mathbb{R}^d$, $d \in \mathbb{N}$, be a bounded Lipschitz domain, $s_\star \in (0,1)$ and $1 < p^\star < \infty$. There exists a constant $C = C(\Omega, p^\star, s_\star) > 0$ such that*

$$\|\mathrm{Tr_s}\, u\|_{L^p(\Omega^c;\mu_s)} \le C\|u\|_{V^{s,p}(\Omega\,|\,\mathbb{R}^d)} \tag{3-13}$$

for any $s \in (s_\star, 1)$, $1 \le p \le p^\star$ and $u \in V^{s,p}(\Omega\,|\,\mathbb{R}^d)$

Proof. We split the integration domain of $\|\mathrm{Tr_s}\, u\|^p_{L^p(\Omega^c;\mu_s)}$ into Ω_1^{ext} and Ω^1_{ext}. Let $c_1 = c_1(d,\Omega) > 0$ be the constant from Lemma 3.8 when applied to $B = \Omega$ and $f = 1$. We have

$$\begin{aligned}\|\mathrm{Tr_s}\, u\|^p_{L^p(\Omega_1^{\mathrm{ext}};\mu_s)} &\le (1-s)\int_{\Omega_1^{\mathrm{ext}}} \frac{|u(x)|^p}{\mathrm{dist}(x,\Omega)^s}\,\mathrm{d}x \le 2^d(1-s)c_1 \int_{\Omega_1^{\mathrm{ext}}}\int_\Omega \frac{|u(x)|^p}{|x-z|^{d+s}}\,\mathrm{d}z\,\mathrm{d}x \\ &\le 2^{d+p}(1-s)c_1\biggl[\int_{\Omega_1^{\mathrm{ext}}}\int_\Omega \frac{|u(x)-u(z)|^p}{|x-z|^{d+s}}\,\mathrm{d}z\,\mathrm{d}x + \int_{\Omega_1^{\mathrm{ext}}}\int_\Omega \frac{|u(z)|^p}{|x-z|^{d+s}}\,\mathrm{d}z\,\mathrm{d}x\biggr] \\ &=: (\mathrm{I}) + (\mathrm{II}).\end{aligned}$$

The term (I) is estimated easily via

$$(\mathrm{I}) \le 2^{d+p^\star} c_1(\mathrm{diam}\,\Omega + 1)^{p^\star-1}[u]^p_{V^{s,p}(\Omega\,|\,\Omega_1^{\mathrm{ext}})}.$$

An application of Lemma 3.7 with $B = \Omega_1^{\mathrm{ext}}$ and Theorem 3.5 (resp. Theorem 3.6 in the case $p = 1$) yields the following bound on the term (II):

$$(\mathrm{II}) \le 2^{d+p}(1-s)c_1\frac{\omega_{d-1}}{s}\int_\Omega \frac{|u(z)|^p}{d_z^s}\,\mathrm{d}z \le 2^{d+p^\star}\frac{\omega_{d-1}}{s_\star}c_1c_2([u]^p_{W^{s,p}(\Omega)} + \|u\|^p_{L^p(\Omega)}).$$

Here $c_2 > 0$ is the constant from Theorem 3.5 in the case $p > 1$. In the case $p = 1$ let c_2 be the constant from Theorem 3.6. For the estimate of Ω^1_{ext}, we define $\mathrm{diam}\,\Omega + 1 =: c_3 \ge 1$ and notice that $d_x \ge |x-z|/c_3$ as well as $\Omega^1_{\mathrm{ext}} \subset B_1(z)^c$ holds for any $z \in \Omega$ and $x \in \Omega^1_{\mathrm{ext}}$. Thus,

$$\begin{aligned}\|\mathrm{Tr_s}\, u\|^p_{L^p(\Omega^1_{\mathrm{ext}};\mu_s)} &\le 2^{p^\star}(1-s)\int_{\Omega^1_{\mathrm{ext}}}\fint_\Omega \frac{|u(x)-u(z)|^p + |u(z)|^p}{d_x^s(1+d_x)^{d+s(p-1)}}\,\mathrm{d}z\,\mathrm{d}x \\ &\le 2^{p^\star}c_3^{d+sp^\star}(1-s)\int_{\Omega^1_{\mathrm{ext}}}\fint_\Omega \frac{|u(x)-u(z)|^p}{|x-z|^{d+sp}}\,\mathrm{d}z\,\mathrm{d}x + 2^{p^\star}(1-s)\int_{\Omega^1_{\mathrm{ext}}}\fint_\Omega \frac{|u(z)|^p}{d_x^{d+sp}}\,\mathrm{d}z\,\mathrm{d}x \\ &\le \frac{2^{p^\star}c_3^{d+sp^\star}}{|\Omega|}[u]^p_{V^{s,p}(\Omega\,|\,\Omega^1_{\mathrm{ext}})} + \frac{\omega_{d-1}2^{p^\star}c_3^{d+sp^\star}(1-s)}{sp|\Omega|}\|u\|^p_{L^p(\Omega)}.\end{aligned} \tag{3-14}$$

In the last step we used

$$\int_{\Omega^1_{\mathrm{ext}}}\frac{1}{d_x^{d+sp}}\,\mathrm{d}x \le c_3^{d+sp}\int_{B_1(x_0)^c}\frac{1}{|x_0-x|^{d+sp}}\,\mathrm{d}x = c_3^{d+sp}\frac{\omega_{d-1}}{sp}, \tag{3-15}$$

where $x_0 \in \Omega$ is a fixed point. Combining the estimates of (I) and (II), as well as (3-14) yields (3-13). □

Proposition 3.10. *Let $\Omega \subset \mathbb{R}^d$, $d \in \mathbb{N}$, be a bounded Lipschitz domain, $s_\star \in (0,1)$ and $1 < p_\star < p^\star < \infty$. There exists a constant $C = C(\Omega, p_\star, p^\star, s_\star) > 0$ such that, for any $s \in (s_\star, 1)$, $p_\star \le p \le p^\star$ and $u \in V^{s,p}(\Omega \,|\, \mathbb{R}^d)$,*

$$[\mathrm{Tr}_s\, u]_{\mathcal{T}^{s,p}(\Omega^c)} \le C \|u\|_{V^{s,p}(\Omega \,|\, \mathbb{R}^d)}. \tag{3-16}$$

Proof. We fix $\rho := \mathrm{inr}(\Omega) > 0$ and divide the integration domain of $[\mathrm{Tr}_s\, u]^p_{\mathcal{T}^{s,p}(\Omega^c)}$ into $\Omega^{\mathrm{ext}}_\rho \times \Omega^{\mathrm{ext}}_\rho$, $\Omega^c \times \Omega^\rho_{\mathrm{ext}}$ and $\Omega^\rho_{\mathrm{ext}} \times \Omega^c$. By symmetry the estimates for $\Omega^c \times \Omega^\rho_{\mathrm{ext}}$ and $\Omega^\rho_{\mathrm{ext}} \times \Omega^c$ are equivalent. Thus, we settle on $\Omega^\rho_{\mathrm{ext}} \times \Omega^c$. Since $|x-y| + d_x + d_y \ge \rho$ for any $x \in \Omega^c$ and $y \in \Omega^\rho_{\mathrm{ext}}$, we have

$$\begin{aligned}
[\mathrm{Tr}_s\, u]^p_{\mathcal{T}^{s,p}(\Omega^\rho_{\mathrm{ext}} \,|\, \Omega^c)} &\le \int_{\Omega^\rho_{\mathrm{ext}}} \int_{\Omega^c} \frac{|u(x)-u(y)|^p}{(1\wedge\rho)^{d+p^\star}} \mu_s(\mathrm{d}x)\mu_s(\mathrm{d}y) \\
&\le \frac{2^p(1-s)^2}{(1\wedge\rho)^{d+p^\star}} \Bigg[\int_{\Omega^c} \frac{|u(x)|^p}{d_x^s(1+d_x)^{d+s(p-1)}} \int_{\Omega^\rho_{\mathrm{ext}}} \frac{1}{d_y^s(1+d_y)^{d+s(p-1)}} \,\mathrm{d}y\, \mathrm{d}x \\
&\qquad + \int_{\Omega^\rho_{\mathrm{ext}}} \frac{|u(y)|^p}{d_y^s(1+d_y)^{d+s(p-1)}} \int_{\Omega^c} \frac{1}{d_x^s(1+d_x)^{d+s(p-1)}} \,\mathrm{d}x\, \mathrm{d}y \Bigg].
\end{aligned} \tag{3-17}$$

After covering Ω^{ext}_1 by finitely many balls, a calculation similar to (3-4) yields a constant $c_1 > 0$, independent of s, such that

$$\int_{\Omega^{\mathrm{ext}}_1} d_x^{-s} \,\mathrm{d}x \le c_1(1-s)^{-1}.$$

By possibly enlarging the constant, we assume $c_1 \ge \omega_{d-1}(1+\mathrm{diam}(\Omega))^{d+p^\star}$. With this observation and (3-15), we find

$$\int_{\Omega^c} \frac{1}{d_x^s(1+d_x)^{d+s(p-1)}} \,\mathrm{d}x \le \int_{\Omega^{\mathrm{ext}}_1} \frac{1}{d_x^s} \,\mathrm{d}x + \int_{\Omega^1_{\mathrm{ext}}} d_x^{-d-sp} \,\mathrm{d}x \le \frac{c_1}{s(1-s)}. \tag{3-18}$$

Now, we combine this estimate with Proposition 3.9:

$$\begin{aligned}
[\mathrm{Tr}_s\, u]^p_{\mathcal{T}^{s,p}(\Omega^\rho_{\mathrm{ext}} \,|\, \Omega^c)} &\le \frac{2^{p^\star+1}}{(1\wedge\rho)^{d+p^\star}} \frac{c_1}{s} \|\mathrm{Tr}_s\, u\|^p_{L^p(\Omega^c;\mu_s)} \\
&\le \frac{2^{p^\star+1}}{(1\wedge\rho)^{d+p^\star}} \frac{c_1 c_2}{s_\star} \|u\|^p_{V^{s,p}(\Omega \,|\, \mathbb{R}^d)}.
\end{aligned}$$

Here $c_2 = c_2(\Omega, p^\star, s_\star) > 0$ is the constant from Proposition 3.9.

Lastly, we prove the inequality for the more delicate part of the seminorm, where higher-order singularities close to the boundary may occur. For $x, y \in \Omega^{\mathrm{ext}}_\rho$, we have

$$(|x-y| + d_x + d_y) \le 4\rho + \mathrm{diam}(\Omega) + 1 =: c_3$$

and, thus,

$$(|x-y| + d_x + d_y) \wedge 1 \ge c_3^{-1}(|x-y| + d_x + d_y).$$

We apply Lemma 3.8 twice with the monotone decreasing function $f(r) = \big(|x-y| + \tfrac{1}{2}r + d_y\big)^{-d-s(p-2)}$ and then again with the function $f(r) = \big(|x-y| + \tfrac{1}{2}|x-z| + \tfrac{1}{2}r\big)^{-d-s(p-2)}$. We now let $c_4 = c_4(\Omega, d) > 0$

and $r > 0$ be the constants from Lemma 3.8. This yields

$$
\begin{aligned}
&[\mathrm{Tr}_s\, u]^p_{\mathcal{T}^{s,p}(\Omega_\rho^{\mathrm{ext}}\,|\,\Omega_\rho^{\mathrm{ext}})} \\
&\quad\le (1-s)c_3^{d+p}c_4 \int_{\Omega_\rho^{\mathrm{ext}}}\int_{\Omega_\rho^{\mathrm{ext}}}\int_{\Omega} \frac{|u(x)-u(y)|^p}{|x-z|^{d+s}(|x-y|+|x-z|/2+d_y)^{d+s(p-2)}}\,\mathrm{d}z\,\mathrm{d}x\,\mu_s(\mathrm{d}y) \\
&\quad\le (1-s)^2c_3^{d+p}c_4^2 \int_{\Omega_\rho^{\mathrm{ext}}}\int_{\Omega_\rho^{\mathrm{ext}}}\int_{\Omega}\int_{\Omega} \frac{|u(x)-u(y)|^p|x-z|^{-d-s}|y-w|^{-d-s}}{(|x-y|+(|x-z|+|y-w|)/2)^{d+s(p-2)}}\,\mathrm{d}w\,\mathrm{d}z\,\mathrm{d}x\,\mathrm{d}y \\
&\quad\le 2^{p^\star}4^{d+p^\star}c_3^{d+p^\star}c_4^2((\mathrm{III})+2(\mathrm{IV})).
\end{aligned}
$$

Here we added $\pm u(z)\pm u(w)$ and used the triangle inequality. The terms are

$$
\begin{aligned}
(\mathrm{III}) &:= (1-s)^2\int_{\Omega}\int_{\Omega}|u(z)-u(w)|^p a(z,w)\,\mathrm{d}w\,\mathrm{d}z, \\
a(z,w) &:= \int_{\Omega_\rho^{\mathrm{ext}}}\int_{\Omega_\rho^{\mathrm{ext}}} \frac{1}{|x-z|^{d+s}|y-w|^{d+s}(|x-y|+2|x-z|+2|y-w|)^{d+s(p-2)}}\,\mathrm{d}x\,\mathrm{d}y, \\
(\mathrm{IV}) &:= (1-s)^2\int_{\Omega_\rho^{\mathrm{ext}}}\int_{\Omega}|u(x)-u(w)|^p b(x,w)\,\mathrm{d}w\,\mathrm{d}x, \\
b(x,w) &:= \int_{\Omega_\rho^{\mathrm{ext}}}\int_{\Omega} \frac{1}{|x-z|^{d+s}|y-w|^{d+s}(|x-y|+2|x-z|+2|y-w|)^{d+s(p-2)}}\,\mathrm{d}z\,\mathrm{d}y.
\end{aligned}
$$

Estimate of (III): Our goal is to find an appropriate estimate of kernel a to apply Theorem 3.5. Notice that

$$|x-y|+2|x-z|+2|y-w| \ge |z-w|+|x-z|+|y-w| \quad \text{for any } x,y,w,z\in\mathbb{R}^d.$$

Thus, for any $z,w\in\Omega$,

$$a(z,w) \le \int_{\Omega_\rho^{\mathrm{ext}}}\int_{\Omega_\rho^{\mathrm{ext}}} \frac{1}{|x-z|^{d+s}|y-w|^{d+s}(|z-w|+|x-z|+|y-w|)^{d+s(p-2)}}\,\mathrm{d}x\,\mathrm{d}y.$$

Now, we apply Lemma 3.7 twice with $B=\Omega_\rho^{\mathrm{ext}}$. We use the function $f(t_1,t_2)=(|z-w|+t_1+t_2)^{-d-s(p-2)}$ which is decreasing in both t_1 and t_2. Thereby,

$$a(z,w) \le \frac{\omega_{d-1}^2}{s^2}\frac{1}{d_z^s d_w^s(|z-w|+d_z+d_w)^{d+s(p-2)}}. \tag{3-19}$$

This yields the desired estimate for (III) via Theorem 3.5:

$$
\begin{aligned}
(\mathrm{III}) &\le \frac{\omega_{d-1}^2}{s^2}(1-s)^2\int_{\Omega}\int_{\Omega}\frac{|u(z)-u(w)|^p}{d_z^s d_w^s(|z-w|+d_z+d_w)^{d+s(p-2)}}\,\mathrm{d}w\,\mathrm{d}z \\
&\le c_5\frac{\omega_{d-1}^2}{s^2}\|u\|^p_{W^{s,p}(\Omega)} \\
&\le c_5\frac{\omega_{d-1}^2}{s_\star^2}\|u\|^p_{V^{s,p}(\Omega\,|\,\mathbb{R}^d)}.
\end{aligned}
$$

Here $c_5=c_5(d,\Omega,p_\star,p^\star,s_\star)>0$ is the constant from Theorem 3.5.

Estimate of (IV): Our approach to estimate (IV) is similar to the proof of [Dyda and Kassmann 2019, Theorem 5]. Analogously to the estimate of a, see (3-19), we find

$$b(x,w) \le \frac{\omega_{d-1}^2}{s^2} \frac{1}{d_x^s d_w^s (|x-w|+d_x+d_w)^{d+s(p-2)}}$$

for any $x \in \Omega_\rho^{\mathrm{ext}}$ and $w \in \Omega$. We define

$$(\mathrm{V}) := (1-s)^2 \int_{\Omega_\rho^{\mathrm{ext}}} \int_\Omega \frac{|u(x)-u(w)|^p}{d_x^s d_w^s (|x-w|+d_x+d_w)^{d+s(p-2)}} \,\mathrm{d}w\,\mathrm{d}x.$$

The previous estimate of b yields

$$(\mathrm{IV}) \le \omega_{d-1}^2 s_\star^{-2} (\mathrm{V}).$$

Claim: We will show that there exists a constant $c_6 = c_6(d, \Omega, \rho, p^\star) > 0$ such that

$$(\mathrm{V}) \le c_6 \biggl(\frac{1}{s^2(p_\star - 1)} [u]^p_{V^{s,p}(\Omega \,|\, \Omega_\rho^{\mathrm{ext}})} + (1-s)^2 \int_\Omega \int_\Omega \frac{|u(z)-u(w)|^p}{d_z^s d_w^s (|z-w|+d_z+d_w)^{d+s(p-2)}} \,\mathrm{d}w\,\mathrm{d}z \biggr).$$

Let $\mathcal{W}(\mathbb{R}^d \setminus \overline{\Omega})$ be the Whitney decomposition from Appendix A and recall that $\rho = \mathrm{inr}(\Omega)$. We define the set of Whitney cubes $Q \in \mathcal{W}(\mathbb{R}^d \setminus \overline{\Omega})$ with $\mathrm{diam}(Q) \le \rho$ by $\mathcal{W}_\rho(\mathbb{R}^d \setminus \overline{\Omega})$. As in Appendix A and [Dyda and Kassmann 2019], we denote by $\widetilde{Q} \subset \Omega$ the reflected Whitney cube for any cube $Q \in \mathcal{W}_\rho(\mathbb{R}^d \setminus \overline{\Omega})$. The collection of reflected Whitney cubes satisfies a bounded overlap property; see (A-4). Let $N \in \mathbb{N}$ be the constant from the bounded overlap property. Furthermore, the distance to the boundary as well as the diameter of the reflected cubes are comparable to the original cubes with the constant $M > 0$ from (A-2). By the covering properties of the Whitney cubes (A-1) and the reflecting cubes (A-3), we find

$$(\mathrm{V}) \le \sum_{Q_1, Q_2 \in \mathcal{W}_\rho(\mathbb{R}^d \setminus \overline{\Omega})} (1-s)^2 \int_{Q_1} \int_{\widetilde{Q}_2} \frac{|u(x)-u(w)|^p}{d_x^s d_w^s (|x-w|+d_x+d_w)^{d+s(p-2)}} \,\mathrm{d}w\,\mathrm{d}x. \tag{3-20}$$

For the moment we fix two cubes $Q_1, Q_2 \in \mathcal{W}(\mathbb{R}^d \setminus \overline{\Omega})$ satisfying $\mathrm{diam}\, Q_1, \mathrm{diam}\, Q_2 \le \rho$. For each $x \in Q_1$ and $w \in \widetilde{Q}_2$, we define

$$z(x,w) := q_{\widetilde{Q}_1} + \biggl(\frac{x - q_{Q_1}}{2l(Q_1)} + \frac{w - q_{\widetilde{Q}_2}}{2l(\widetilde{Q}_2)} \biggr) l(\widetilde{Q}_1) \in \widetilde{Q}_1.$$

The map z connects points in $Q_1, \widetilde{Q}_2$ with points in $\widetilde{Q}_1$ in a continuous way. We will use it for a change of variables in either x or w. Therefore,

$$\begin{aligned}
&\int_{Q_1} \int_{\widetilde{Q}_2} \frac{|u(x)-u(w)|^p}{d_x^s d_w^s (|x-w|+d_x+d_w)^{d+s(p-2)}} \,\mathrm{d}w\,\mathrm{d}x \\
&\le 2^p \int_{Q_1} \int_{\widetilde{Q}_2} \frac{|u(z(x,w))-u(w)|^p}{d_x^s d_w^s (|x-w|+d_x+d_w)^{d+s(p-2)}} \,\mathrm{d}w\,\mathrm{d}x + 2^p \int_{Q_1} \int_{\widetilde{Q}_2} \frac{|u(x)-u(z(x,w))|^p}{d_x^s d_w^s (|x-w|+d_x+d_w)^{d+s(p-2)}} \,\mathrm{d}w\,\mathrm{d}x \\
&=: 2^p ((\mathrm{VI}) + (\mathrm{VII})).
\end{aligned} \tag{3-21}$$

We make a few observations before estimating (VI). For $x \in Q_1$ and $w \in \widetilde{Q}_2$, we have $d_x \geq M^{-1} d_{z(x,w)}$ as well as

$$
\begin{aligned}
|x-w|+d_x+d_w &\geq M^{-1}(|x-w|+\operatorname{dist}(Q_1, \widetilde{Q}_1))+(1-M^{-1})|x-w|+d_w \\
&\geq M^{-1}(\operatorname{dist}(Q_1, \widetilde{Q}_2)+\operatorname{dist}(Q_1, \widetilde{Q}_1))+(1-M^{-1})(\operatorname{dist}(Q_1, \partial\Omega)+\operatorname{dist}(\widetilde{Q}_2, \partial\Omega))+d_w \\
&\geq M^{-1} \operatorname{dist}(\widetilde{Q}_1, \widetilde{Q}_2)+(1-M^{-1})M^{-1} \operatorname{dist}(\widetilde{Q}_1, \partial\Omega)+(2-M^{-1}) \operatorname{dist}(\widetilde{Q}_2, \partial\Omega) \\
&\geq \frac{(1-M^{-1})M^{-1}\big(\operatorname{dist}(\widetilde{Q}_1, \widetilde{Q}_2)+\sum_{i=1}^2 \operatorname{dist}(\widetilde{Q}_i, \partial\Omega)\big)(|z(x,w)-w|+d_{z(x,w)}+d_w)}{(\operatorname{diam} \widetilde{Q}_1+\operatorname{dist}(\widetilde{Q}_1, \widetilde{Q}_2)+\operatorname{diam} \widetilde{Q}_2)+\sum_{i=1}^2(\operatorname{dist}(\widetilde{Q}_i, \partial\Omega)+\operatorname{diam} \widetilde{Q}_i)} \\
&\geq (1-M^{-1})M^{-1}\tfrac{2}{3}(|z(x,w)-w|+d_{z(x,w)}+d_w).
\end{aligned}
$$

We define $c_7 := \frac{3}{2}M^2(M-1)^{-1} > 1$ and use the previous calculation to estimate (VI) by

$$
\begin{aligned}
\text{(VI)} &\leq M^s c_7^{d+s(p-2)} \int_{\widetilde{Q}_2}\int_{Q_1} \frac{|u(z(x,w))-u(w)|^p}{d_{z(x,w)}^s d_w^s(|z(x,w)-w|+d_{z(x,w)}+d_w)^{d+s(p-2)}} \,\mathrm{d}x\,\mathrm{d}w \\
&\leq M^s c_7^{d+s(p-2)} \left(\frac{2l(Q_1)}{l(\widetilde{Q}_1)}\right)^d \int_{\widetilde{Q}_2}\int_{\widetilde{Q}_1} \frac{|u(z)-u(w)|^p}{d_z^s d_w^s(|z-w|+d_z+d_w)^{d+s(p-2)}} \,\mathrm{d}z\,\mathrm{d}w \\
&\leq M c_7^{d+(0\vee(p^\star-2))}(2M)^d \int_{\widetilde{Q}_1}\int_{\widetilde{Q}_2} \frac{|u(z)-u(w)|^p}{d_z^s d_w^s(|z-w|+d_z+d_w)^{d+s(p-2)}} \,\mathrm{d}w\,\mathrm{d}z. \qquad (3\text{-}22)
\end{aligned}
$$

Here we used the change of variables $z = z(x,w)$. We set $c_8 := 2^{d+p^\star} c_7^{d+(0\vee(p^\star-2))} M^{d+1}$. Now we sum (VI) over all Whitney cubes in $\mathcal{W}_\rho(\mathbb{R}^d \setminus \overline{\Omega})$. By the bounded overlap property of the Whitney decomposition, see (A-4), we have

$$
\begin{aligned}
\sum_{Q_1,Q_2\in\mathcal{W}_\rho(\mathbb{R}^d\setminus\overline{\Omega})} (1-s)^2 2^p \text{(VI)} &\leq c_8(1-s)^2 \sum_{Q_1,Q_2\in\mathcal{W}_\rho(\mathbb{R}^d\setminus\overline{\Omega})} \int_{\widetilde{Q}_1}\int_{\widetilde{Q}_2} \frac{|u(z)-u(w)|^p}{d_z^s d_w^s(|z-w|+d_z+d_w)^{d+s(p-2)}} \,\mathrm{d}w\,\mathrm{d}z \\
&\leq N^2 c_8(1-s)^2 \int_\Omega\int_\Omega \frac{|u(z)-u(w)|^p}{d_z^s d_w^s(|z-w|+d_z+d_w)^{d+s(p-2)}} \,\mathrm{d}w\,\mathrm{d}z. \qquad (3\text{-}23)
\end{aligned}
$$

Now we estimate (VII). We make a few observations upon the choice of the Whitney decomposition and reflected cubes in Appendix A. For $x \in Q_1$ and $w \in \widetilde{Q}_2$, we have

$$
\begin{aligned}
d_x &\geq (1+M^{-1})^{-1}(d_x+d_{z(x,w)}) \geq (1+M^{-1})^{-1}|x-z(x,w)|, \\
d_w &\geq \operatorname{dist}(\widetilde{Q}_2, \partial\Omega), \\
d_x+d_w+|x-w| &\geq \operatorname{dist}(Q_1, \partial\Omega)+\operatorname{dist}(\widetilde{Q}_2, \partial\Omega)+\operatorname{dist}(Q_1, \widetilde{Q}_2), \\
|x-z(x,w)| &\leq \operatorname{dist}(Q_1, \widetilde{Q}_1) \leq M \operatorname{dist}(Q_1, \partial\Omega).
\end{aligned}
$$

We set

$$
J(Q_1, Q_2) := \frac{\operatorname{dist}(Q_1, \partial\Omega)^{d+s(p-1)}}{\operatorname{dist}(\widetilde{Q}_2, \partial\Omega)^s(\operatorname{dist}(Q_1, \partial\Omega)+\operatorname{dist}(\widetilde{Q}_2, \partial\Omega)+\operatorname{dist}(Q_1, \widetilde{Q}_2))^{d+s(p-2)}}.
$$

Therefore,

$$\begin{aligned}\text{(VII)} &\le (1+M^{-1})^s M^{d+s(p-1)} J(Q_1, \widetilde{Q}_2) \int_{Q_1}\int_{\widetilde{Q}_2} \frac{|u(x)-u(z(x,w))|^p}{|x-z(x,w)|^{d+sp}}\,\mathrm{d}w\,\mathrm{d}x\\ &\le (1+M^{-1})M^{d+p-1} J(Q_1, \widetilde{Q}_2)\Big(\frac{2l(\widetilde{Q}_2)}{l(\widetilde{Q}_1)}\Big)^d \int_{Q_1}\int_{\widetilde{Q}_1} \frac{|u(x)-u(z)|^p}{|x-z|^{d+sp}}\,\mathrm{d}z\,\mathrm{d}x.\end{aligned}$$

Here we used the change of variables $z := z(x, w)$. Set $c_9 := (1+M^{-1})M^{d+p^\star-1}2^{d+p^\star}$. By (A-4),

$$\begin{aligned}(1-s)^2 &\sum_{Q_1,Q_2\in\mathcal{W}_\rho(\mathbb{R}^d\setminus\bar{\Omega})} 2^p\text{(VII)}\\ &\le c_9(1-s)^2 \sum_{Q_1,Q_2\in\mathcal{W}_\rho(\mathbb{R}^d\setminus\bar{\Omega})} J(Q_1, \widetilde{Q}_2)\Big(\frac{l(\widetilde{Q}_2)}{l(\widetilde{Q}_1)}\Big)^d \int_{Q_1}\int_{\widetilde{Q}_1} \frac{|u(x)-u(z)|^p}{|x-z|^{d+sp}}\,\mathrm{d}z\,\mathrm{d}x. \qquad (3\text{-}24)\end{aligned}$$

Therefore, it is sufficient to prove that

$$I(Q_1) := \sum_{Q_2\in\mathcal{W}_\rho(\mathbb{R}^d\setminus\bar{\Omega})} J(Q_1, \widetilde{Q}_2)\Big(\frac{l(\widetilde{Q}_2)}{l(\widetilde{Q}_1)}\Big)^d$$

is bounded independent of Q_1. We fix $Q_1 \in \mathcal{W}_\rho(\mathbb{R}^d \setminus \Omega)$ and set $a := \operatorname{dist}(Q_1, \partial\Omega)$. Let $\hat{q}_{Q_1} \in \partial\Omega$ be a minimizer of the distance of q_{Q_1} to $\partial\Omega$; i.e., $|\hat{q}_{Q_1} - q_{Q_1}| = \operatorname{dist}(q_{Q_1}, \partial\Omega)$. By the properties of the Whitney cubes, we have for any $w \in \widetilde{Q}_2$

$$\operatorname{dist}(\widetilde{Q}_2, \partial\Omega) \ge \tfrac{1}{2}(\operatorname{dist}(\widetilde{Q}_2, \partial\Omega) + \operatorname{diam}\widetilde{Q}_2) \ge \tfrac{1}{2}d_w, \qquad (3\text{-}25)$$

$$\begin{aligned}|w-b| &\le |w-q_{Q_1}| + |q_{Q_1}-\hat{q}_{Q_1}| \le \operatorname{diam} Q_1 + \operatorname{dist}(\widetilde{Q}_2, Q_1) + \operatorname{dist}(q_{Q_1}, \partial\Omega)\\ &\le \operatorname{dist}(\widetilde{Q}_2, Q_1) + 3\operatorname{dist}(Q_1, \partial\Omega) \le 4\operatorname{dist}(\widetilde{Q}_2, Q_1). \qquad (3\text{-}26)\end{aligned}$$

We estimate $I(Q_1)$ using the properties of the Whitney cubes, (3-25), (3-26) and (A-4):

$$\begin{aligned}I(Q_1) &\le M^d 4^d \sum_{Q_2\in\mathcal{W}_\rho(\mathbb{R}^d\setminus\bar{\Omega})} \frac{\operatorname{diam}(\widetilde{Q}_2)^d a^{s(p-1)}}{\operatorname{dist}(\widetilde{Q}_2, \partial\Omega)^s(a+\operatorname{dist}(\widetilde{Q}_2, \partial\Omega)+\operatorname{dist}(Q_1, \widetilde{Q}_2))^{d+s(p-2)}}\\ &= M^d 4^d \sum_{Q_2\in\mathcal{W}_\rho(\mathbb{R}^d\setminus\bar{\Omega})} \int_{\widetilde{Q}_2} \frac{a^{s(p-1)}}{\operatorname{dist}(\widetilde{Q}_2, \partial\Omega)^s(a+\operatorname{dist}(\widetilde{Q}_2, \partial\Omega)+\operatorname{dist}(Q_1, \widetilde{Q}_2))^{d+s(p-2)}}\,\mathrm{d}w\\ &\le 2^s 4^{d+s(p-2)} M^d 4^d N \int_\Omega \frac{a^{s(p-1)}}{d_w^s(a+d_w+|w-\hat{q}_{Q_1}|)^{d+s(p-2)}}\,\mathrm{d}w. \qquad (3\text{-}27)\end{aligned}$$

We claim that the integral in the last line is bounded independent of a and $\hat{q}_{Q_1}$.

We localize the boundary in a neighborhood of $\hat{q}_{Q_1}$. Let $r_0 > 0$ be the localization radius, and let $\phi : B_{r_0}(\hat{q}_{Q_1}) \to B_1(0)$ be a bi-Lipschitz flattening of the boundary since Ω has a uniform Lipschitz boundary. A change of variables and an estimate similar to (3-3) yields a constant $c_{10} = c_{10}(d, \Omega) \ge 1$ such that

$$\int_{\Omega\cap B_{r_0}(\hat{q}_{Q_1})} \frac{a^{s(p-1)}}{d_w^s(a+d_w+|w-\hat{q}_{Q_1}|)^{d+s(p-2)}}\,\mathrm{d}w \le c_{10}\int_{B_1(0)_+} \frac{a^{s(p-1)}}{w_d^s(a+|w|)^{d+s(p-2)}}\,\mathrm{d}w.$$

To calculate this integral, we apply the coarea formula; see, e.g., [Federer 1969] with $(r,t)=(w_d,|w|)$ in the case $d\ge 2$:

$$\begin{aligned}\int_{B_1(0)_+}\frac{a^{s(p-1)}}{w_d^s(a+|w|)^{d+s(p-2)}}\,\mathrm{d}w &\le \omega_{d-2}\int_0^1\int_0^t \frac{a^{s(p-1)}t^{d-2}}{r^s(a+t)^{d+s(p-2)}}\,\mathrm{d}r\,\mathrm{d}t\\ &\le \frac{\omega_{d-2}}{1-s}\int_0^1\frac{a^{s(p-1)}t^{d-1-s}}{(a+t)^{d+s(p-2)}}\,\mathrm{d}t\\ &\le \frac{\omega_{d-2}}{1-s}\int_0^1\frac{a^{s(p-1)}}{(a+t)^{1+s(p-1)}}\,\mathrm{d}t \le \frac{\omega_{d-2}}{(1-s)s(p-1)}.\end{aligned}\tag{3-28}$$

Here we used

$$\mathcal{H}^{(d-2)}(\{w\in\Omega\mid w_d=r,\,|w|=t\})\le \omega_{d-2}\mathbb{1}_{r\le t}t^{d-2}.$$

A similar calculation shows the same estimate in the case $d=1$. Furthermore, the remainder of the integral on the right-hand side of (3-27), i.e.,

$$\int_{\Omega\cap B_{r_0}(\hat q_{Q_1})^c}\frac{a^{s(p-1)}}{d_w^s(a+d_w+|w-\hat q_{Q_1}|)^{d+s(p-2)}}\,\mathrm{d}w,$$

is easily bounded independent of a since $a+d_w+|w-\hat q_{Q_1}|\ge r_0$ for $w\in\Omega\cap B_{r_0}(\hat q_{Q_1})^c$ and $a\le 4\rho$.

Therefore $(1-s)I(Q_1)$ is bounded independent of Q_1 by a constant $c_{11}=c_{11}(d,\Omega,p_\star,p^\star,s_\star)>0$. We combine (3-24), (3-27) and (3-28) as well as (A-4) to obtain

$$\begin{aligned}(1-s)^2\sum_{Q_1,Q_2\in\mathcal{W}_\rho(\mathbb{R}^d\setminus\bar\Omega)}2^p(\mathrm{VII}) &\le c_9c_{11}N(1-s)\int_{\Omega_\rho^{\mathrm{ext}}}\int_\Omega\frac{|u(x)-u(z)|^p}{|x-z|^{d+sp}}\,\mathrm{d}z\,\mathrm{d}x\\ &= c_9c_{11}N[u]^p_{V^{s,p}(\Omega\,|\,\Omega_\rho^{\mathrm{ext}})}.\end{aligned}\tag{3-29}$$

Finally, we combine (3-20), (3-21), (3-23) and (3-29) and the claim follows. The constant is given by $c_6:=N\max\{Nc_8,c_9c_{11}\}$.

We finish the estimate of (IV) using the previous claim and Theorem 3.5:

$$(\mathrm{V})\le c_6\biggl(\frac{1}{s^2(p_\star-1)}[u]^p_{V^{s,p}(\Omega\,|\,\Omega_\rho^{\mathrm{ext}})}+c_5\|u\|^p_{W^{s,p}(\Omega)}\biggr).$$

Combining the estimates of (III) and (IV) yields (3-16). □

Remark 3.11. As mentioned in the introduction, the trace embedding $V^{s,1}(\Omega\,|\,\mathbb{R}^d)\to\mathcal{T}^{s,1}(\Omega^c)$ cannot be continuous. This may be seen as follows: Consider the sequence of functions

$$u_n(x):=\begin{cases}0, & x\in\Omega,\\ n^{1-s}, & x\in\Omega^{\mathrm{ext}}_{1/n},\\ 0, & x\in\Omega_{\mathrm{ext}}^{1/n},\end{cases}$$

for $n\in\mathbb{N}$. By Lemmas 3.7 and 3.8 one easily sees that $\|u_n\|_{V^{s,1}(\Omega\,|\,\mathbb{R}^d)}\asymp\|u_n\|_{L^1(\Omega^c;\mu_s)}\asymp 1$, but a simple calculation yields $[u_n]_{\mathcal{T}^{s,1}(\Omega^c)}\asymp\ln(n)\to\infty$ as $n\to\infty$. A similar sequence of functions proves Theorem 3.5 to be false for $p=1$.

4. Extension results

The aim of this section is to prove the extension parts of Theorems 1.2 and 1.3. This proof is carried out in Propositions 4.5 and 4.6. We are able to treat the cases $1 < p < \infty$ and $p = 1$ together.

The method used in this section is essentially inspired by [Jonsson and Wallin 1984, Chapter V]. We decompose the domain Ω into Whitney cubes and consider neighborhoods of each cube intersected with Ω^c. The extension is constructed by copying weighted mean values of the exterior data g from this intersection into the respective cube; see (4-11). The weights are taken with respect to a measure that behaves like μ_s close to the boundary $\partial\Omega$.

Throughout this section we fix an open nonempty proper subset $\Omega \subset \mathbb{R}^d$. We will introduce additional assumptions on Ω when needed. Further, we fix a dyadic Whitney-decomposition $\mathcal{W}(\Omega)$ of Ω consisting of cubes with parallel sides to the axes of $\mathbb{R}^d$ such that

(a) $\Omega = \bigcup_{Q\in\mathcal{W}(\Omega)} Q$,

(b) the interior of the cubes are mutually disjoint,

(c) for all cubes $Q \in \mathcal{W}(\Omega)$, the diameter is comparable to the distance to the boundary of Ω; i.e.,

$$\operatorname{diam} Q \le d(Q, \partial\Omega) \le 4 \operatorname{diam} Q. \tag{4-1}$$

We set $s_Q \in 2^{\mathbb{Z}}$ to be the side length of a cube Q and let $q_Q \in Q$ be the center of the cube Q. We denote the diameter of the cube Q by $l_Q = \operatorname{diam}(Q) = \sqrt{d}s_Q$. This decomposition satisfies the following: Suppose $Q_1, Q_2 \in \mathcal{W}(\Omega)$ touch each other. Then

$$\tfrac{1}{4} \operatorname{diam} Q_1 \le \operatorname{diam} Q_2 \le 4 \operatorname{diam} Q_1. \tag{4-2}$$

Additionally, we denote by $Q^\star$ a cube having the same center as $Q \in \mathcal{W}(\Omega)$ but side length $\bigl(1 + \frac{1}{8}\bigr)s_Q$. We denote the collection of these scaled cubes by

$$\mathcal{W}^\star(\Omega) := \{Q^\star \mid Q \in \mathcal{W}(\Omega)\}.$$

These scaled cubes satisfy a finite overlap property; i.e., $\sum_{Q^\star\in\mathcal{W}^\star(\Omega)} \mathbb{1}_{Q^\star} \le N$, where $N \in \mathbb{N}$ is a fixed number for the remainder of this section. Additionally, two cubes $Q_1^\star, Q_2^\star \in \mathcal{W}^\star(\Omega)$ have nonempty intersection if and only if Q_1 and Q_2 touch. We define $J_i \subset \mathcal{W}(\Omega)$ to be the set of all cubes with side lengths 2^{-i}, and set

$$D_i := \bigcup_{Q\in J_i} Q, \qquad D_{\ge i} := \bigcup_{j\ge i} D_j. \tag{4-3}$$

Analogously to [Jonsson and Wallin 1984, Section 1.2], we introduce a specific partition of unity on Ω which we will use to construct an extension operator $\mathcal{T}^{s,p}(\Omega^c) \to V^{s,p}(\Omega \mid \mathbb{R}^d)$. We emphasize that the construction of the extension is independent of p. Let $\psi \in C_c^\infty(\mathbb{R}^d)$ be a bump function such that $\psi = 1$ on $\bigl[-\frac{1}{2}, \frac{1}{2}\bigr]^d$ and $\psi = 0$ on $\bigl(\bigl[-\frac{1}{2}\bigl(1+\frac{1}{8}\bigr), \frac{1}{2}\bigl(1+\frac{1}{8}\bigr)\bigr]^d\bigr)^c$, $0 \le \psi \le 1$. Then we define for any $Q \in \mathcal{W}(\Omega)$ a translated and rescaled version of ψ,

$$\psi_Q(x) := \psi\left(\frac{x - q_Q}{s_Q}\right),$$

and our partition functions

$$\phi_Q(x) := \frac{\psi_Q(x)}{\sum_{\widetilde{Q}\in\mathcal{W}(\Omega)} \psi_{\widetilde{Q}(x)}}. \tag{4-4}$$

Then obviously $\sum_Q \phi_Q = \mathbb{1}_\Omega$ holds. Furthermore, we set

$$\begin{aligned} \rho &:= \tfrac{1}{2}\operatorname{inr}(\Omega) \wedge \tfrac{1}{2} > 0, \\ \kappa &:= \left\lfloor \log_2 \frac{\rho}{\sqrt{d}} \right\rfloor, \\ \mathcal{W}_{\leq\kappa}(\Omega) &:= \{Q \in \mathcal{W}(\Omega) \mid s_Q \leq 2^\kappa\}. \end{aligned} \tag{4-5}$$

Since $l_Q = \sqrt{d}s_Q$ for any $Q \in \mathcal{W}$ and the cubes have dyadic side lengths, $l_Q \leq \rho$ for any $Q \in \mathcal{W}_{\leq\kappa}$. For any $x \in Q \in \mathcal{W}$, we know

$$\begin{aligned} d_x &\leq l_Q + \operatorname{dist}(Q, \partial\Omega) \leq 5l_Q, \\ d_x &\geq \operatorname{dist}(Q, \partial\Omega) \geq \tfrac{1}{4}l_Q. \end{aligned}$$

Therefore,

$$\left\{x \in \Omega \mid d_x < \tfrac{1}{4}\rho\right\} \subset \bigcup_{Q\in\mathcal{W}_{\leq\kappa}(\Omega)} Q \subset \{x \in \Omega \mid d_x < 5\rho\}. \tag{4-6}$$

For any cube $Q \in \mathcal{W}_{\leq\kappa}$ such that all neighboring cubes are in $\mathcal{W}_{\leq\kappa}$, we have

$$\sum_{Q'\in\mathcal{W}_{\leq\kappa}} \phi_{Q'}(x) = 1, \quad x \in Q.$$

By (4-2), all cubes Q with a side length that is at most $2^{\kappa-2}$ only have neighboring cubes in $\mathcal{W}_{\leq\kappa}$. Therefore,

$$\sum_{Q\in\mathcal{W}_{\leq\kappa}} \phi_Q(x) = 1, \quad x \in D_{\geq-\kappa+2}. \tag{4-7}$$

We define for $s \in (0, 1)$ the measure on $\mathcal{B}(\mathbb{R}^d)$

$$\tilde{\mu}_s(\mathrm{d}z) = \mathbb{1}_{\Omega^c}(z)\frac{1-s}{d_z^s}\,\mathrm{d}z. \tag{4-8}$$

This measure behaves like μ_s, see (1-5), near the boundary $\partial\Omega$. We will use $\tilde{\mu}_s$ to construct the extension of a function $g : \Omega^c \to \mathbb{R}$; see (4-11). In particular, the value of the extension $\operatorname{Ext_s}(g)$ in a cube $Q \in \mathcal{W}_{\leq\kappa}$ will depend on a $\tilde{\mu}_s$-mean of g in a neighborhood of Q. Since $\tilde{\mu}_s$ converges weakly to the surface measure on $\partial\Omega$, we recover a classical Whitney extension of functions in $W^{1-1/p,p}(\partial\Omega)$ in the limit $s \to 1^-$. We set for $Q \in \mathcal{W}(\Omega)$

$$a_{Q,s} := (\tilde{\mu}_s(B_{6l_Q}(q_Q)))^{-1}. \tag{4-9}$$

Since $\operatorname{dist}(q_Q, \partial\Omega) \leq 5l_Q$, the intersection $B_{6l_Q}(q_Q) \cap \Omega^c$ has nonempty interior. The following lemma shows the order of $a_{Q,s}$ in terms of l_Q and s for Lipschitz domains. The estimate (4-10) is essential in Propositions 4.5 and 4.6.

Lemma 4.1. *Let $\varnothing \neq \Omega \subset \mathbb{R}^d$ be a Lipschitz domain. There exists a constant $C = C(d, \Omega) > 1$ such that, for any $s \in (0, 1)$ and $Q \in \mathcal{W}_{\leq \kappa+6}(\Omega)$,*

$$C^{-1} l_Q^{s-d} \leq a_{Q,s} \leq C l_Q^{s-d}. \tag{4-10}$$

Proof. Let $z_Q \in \partial\Omega$ be a minimizer of the distance of q_Q to the boundary $\partial\Omega$. Since the distance $\operatorname{dist}(q_Q, \partial\Omega)$ is bounded from above by $5l_Q$, we have $B_{l_Q}(z_Q) \subset B_{6l_Q}(q_Q) \subset B_{11l_Q}(z_Q)$. Since Ω has a uniform Lipschitz boundary, we find a radius $r = r(\Omega) > 0$ and a constant $L = L(\Omega) > 0$, both independent of z_Q, and a rotation and translation $T_{z_Q} : \mathbb{R}^d \to \mathbb{R}^d$ as well as a Lipschitz continuous function $\phi : \mathbb{R}^{d-1} \to \mathbb{R}$ such that

$$T_{z_Q}(\Omega \cap B_r(z_q)) = \{(x', x_d) \in B_r(z_Q) \mid x_d > \phi(x')\} \quad \text{and} \quad [\phi]_{C^{0,1}} \leq L.$$

Without loss of generality, we assume T_{z_Q} to be the identity map, in particular, $z_Q = 0$. By arguments similar to (3-3), we find $2(1+L)d_x \geq |x_d - \phi(x')|$ for any $(x', x_d) \in B_r(0)$ such that $x_d > \phi(x')$. First we assume that $11l_Q \leq r$. Then proceeding as in (3-4) yields

$$a_{Q,s}^{-1} \leq (2(1+L))^s \int_{B_{11l_Q} \cap \{x_d \geq \phi(x')\}} \frac{1-s}{(x_d - \phi(x'))^s} \, \mathrm{d}(x', x_d) \leq 2(11+L)(11)^d l_Q^{d-s}.$$

If $11l_Q > r$, then we simply cover $B_{11l_Q}(z_Q) \cap \partial\Omega$ by finitely many balls $B_1, \dots, B_N$. Since l_Q is bounded from above by $2^6\rho$, the number of balls of radius r which are needed can be picked uniformly. Set $A := \Omega^c \cap B_{11lQ}(z_Q) \cap \bigcap_j B_j^c$ and $r_1 := \operatorname{dist}(\partial\Omega, A)$. We pick the balls $B_1, \dots, B_N$ such that $r_1 > \frac{1}{2}r$. Then, by a similar calculation as above,

$$\begin{aligned} a_{Q,s}^{-1} \leq \sum_{j=1}^{N} \tilde{\mu}_s(B_j) + \tilde{\mu}_s(A) &\leq N2(11+L)(11)^d r^{d-s} + 2^7 11^d \omega_{d-1}(r \wedge 1)^{-1}(\rho \vee 1) l_Q^{d-s} \\ &\leq (N2(11+L)(11)^{2d} + 2^7 11^d \omega_{d-1}(r \wedge 1)^{-1}(\rho \vee 1)) l_Q^{d-s}. \end{aligned}$$

For the upper bound in (4-10), we simply notice that

$$a_{Q,s}^{-1} \geq \int_{B_{l_Q \wedge r} \cap \{x_d \geq \phi(x')\}} \frac{1-s}{(x_d - \phi(x'))^s} \, \mathrm{d}(x', x_d)$$

and proceed in a similar fashion. □

For $g \in L^p_{\mathrm{loc}}(\mathbb{R}^d)$, we define the extension $\operatorname{Ext_s}(g)$ as

$$\operatorname{Ext_s}(g)(x) := \begin{cases} \sum_{Q \in \mathcal{W}_{\leq\kappa}(\Omega)} \phi_Q(x) a_{Q,s} \int_{\Omega^c \cap B_{6l_Q}(q_Q)} g(z)\tilde{\mu}_s(\mathrm{d}z) & \text{for } x \in \Omega, \\ g(x) & \text{for } x \in \Omega^c. \end{cases} \tag{4-11}$$

For any $Q \in \mathcal{W}_{\leq\kappa}(\Omega)$, we have

$$\sup\{\operatorname{dist}(x, \partial\Omega) \mid x \in Q^\star\} \leq \operatorname{dist}(Q^\star, \partial\Omega) + \operatorname{diam}(Q^\star) \leq 4\rho + \tfrac{9}{8}\rho \leq 6\rho.$$

Therefore, $\operatorname{Ext_s}(g)(x) = 0$ for $x \in \Omega$ such that $d_x > 6\rho$. Additionally, the definition of $\operatorname{Ext_s}(g)$ inside Ω depends only on the values of g on $\Omega^{\mathrm{ext}}_{6\rho} \subset \Omega^{\mathrm{ext}}_{3\,\mathrm{inr}(\Omega)}$. We could use the measure μ_s introduced in (1-5) instead of $\tilde{\mu}_s$ in the definition of the extension because $\operatorname{Ext_s}(g)|_\Omega$ does not depend on the values of g far away from the boundary. But the benefit of $\tilde{\mu}_s$ is that the extension is independent of the parameter p.

We begin by proving some properties of $\mathrm{Ext_s}$ analogous to [Jonsson 1994, Lemma 1]. The proof follows the same lines as [Jonsson and Wallin 1984, Chapter V, Lemma D]. We define for cubes $Q_1, Q_2 \in \mathcal{W}_{\leq\kappa}(\Omega)$ and $g \in L^p_{\mathrm{loc}}(\mathbb{R}^d)$

$$J_p(q_{Q_1}, q_{Q_2}) := \left(a_{Q_1,s}a_{Q_2,s}\int_{B_{30l_{Q_1}}(q_{Q_1})}\int_{B_{30l_{Q_2}}(q_{Q_2})}|g(z_1)-g(z_2)|^p\tilde{\mu}_s(\mathrm{d}z_2)\tilde{\mu}_s(\mathrm{d}z_1)\right)^{1/p}. \tag{4-12}$$

Lemma 4.2. *Let $s \in (0,1)$ and $1 \leq p < \infty$, and assume that Ω satisfies (4-10). Further, let $Q_1, Q_2 \in \mathcal{W}_{\leq\kappa-2}(\Omega)$. There exists a constant $C = C(d, \Omega, \psi, \mathcal{W}(\Omega)) > 0$ such that, for any $g \in L^p(\Omega^c)$ and $x \in Q_1$, $y \in Q_2$ as well as $b \in \mathbb{R}$:*

(a) $|\mathrm{Ext_s}(g)(x) - \mathrm{Ext_s}(g)(y)| \leq C J_p(q_{Q_1}, q_{Q_2})$,

(b) $|\nabla \mathrm{Ext_s}(g)(x)| \leq C l_{Q_1}^{-1} J_p(q_{Q_1}, q_{Q_2})$,

(c) $|\mathrm{Ext_s}(g)(x) - b| \leq C\big(a_{Q_1,s}\int_{B_{30l_{Q_1}}(q_{Q_1})}|g(z_1)-b|^p\tilde{\mu}_s(\mathrm{d}z_1)\big)^{1/p}$,

(d) $|\nabla \mathrm{Ext_s}(g)(w)| \leq C l_Q^{-1}\big(a_{Q,s}\int_{B_{30l_Q}(q_Q)}|g(z)|^p\tilde{\mu}_s(\mathrm{d}z)\big)^{1/p}$ *for any $w \in Q \in \mathcal{W}(\Omega)$.*

Proof. (a) By (4-7),

$$\sum_{Q\in\mathcal{W}_{\leq\kappa}}\phi_Q(x) = 1 = \sum_{Q\in\mathcal{W}_{\leq\kappa}}\phi_Q(y).$$

By the choice of $a_{Q,s}$, we find

$$\begin{aligned}
&\mathrm{Ext_s}(g)(x) - \mathrm{Ext_s}(g)(y)\\
&\qquad= \sum_{Q\in\mathcal{W}_{\leq\kappa}(\Omega)}\phi_Q(x)a_{Q,s}\int_{B_{6l_Q}(q_Q)}(g(z_1)-\mathrm{Ext_s}(g)(y))\tilde{\mu}_s(\mathrm{d}z_1)\\
&\qquad= \sum_{Q,\widetilde{Q}\in\mathcal{W}_{\leq\kappa}(\Omega)}\phi_Q(x)\phi_{\widetilde{Q}}(y)a_{Q,s}a_{\widetilde{Q},s}\int_{B_{6l_Q}(q_Q)}\int_{B_{6l_{\widetilde{Q}}}(q_{\widetilde{Q}})}(g(z_1)-g(z_2))\tilde{\mu}_s(\mathrm{d}z_2)\tilde{\mu}_s(\mathrm{d}z_1).
\end{aligned}$$

We apply Hölder's inequality to find

$$\begin{aligned}
&|\mathrm{Ext_s}(g)(x) - \mathrm{Ext_s}(g)(y)|\\
&\quad\leq \sum_{Q,\widetilde{Q}\in\mathcal{W}_{\leq\kappa}(\Omega)}\phi_Q(x)\phi_{\widetilde{Q}}(y)a_{Q,s}a_{\widetilde{Q},s}\big(\tilde{\mu}_s(B_{6l_Q}(q_Q))\tilde{\mu}_s(B_{6l_{\widetilde{Q}}}(q_{\widetilde{Q}}))\big)^{1-1/p}\\
&\qquad\qquad\times\left(\int_{B_{6l_Q}(q_Q)}\int_{B_{6l_{\widetilde{Q}}}(q_{\widetilde{Q}})}|g(z_1)-g(z_2)|^p\tilde{\mu}_s(\mathrm{d}z_2)\tilde{\mu}_s(\mathrm{d}z_1)\right)^{1/p}\\
&\quad= \sum_{Q,\widetilde{Q}\in\mathcal{W}_{\leq\kappa}(\Omega)}\phi_Q(x)\phi_{\widetilde{Q}}(y)\left(a_{Q,s}a_{\widetilde{Q},s}\int_{B_{6l_Q}(q_Q)}\int_{B_{6l_{\widetilde{Q}}}(q_{\widetilde{Q}})}|g(z_1)-g(z_2)|^p\tilde{\mu}_s(\mathrm{d}z_2)\tilde{\mu}_s(\mathrm{d}z_1)\right)^{1/p}.
\end{aligned}$$

Let $Q \in \mathcal{W}_{\leq\kappa}$ be a cube such that $\phi_Q(x) \neq 0$. Then Q touches Q_1 by the definition of ϕ_Q. By (4-2), we find

$$|t - q_{Q_1}| \leq |t - q_Q| + |q_Q - q_{Q_1}| \leq 6l_Q + (l_Q + l_{Q_1}) \leq (6\cdot 4 + 4 + 1)l_{Q_1} \leq 30 l_{Q_1}$$

for any $t \in B_{6l_Q}(q_Q)$. Let $c_1 = c_1(d, \Omega, p) > 1$ be the constant from (4-10); then

$$a_{Q,s} \leq c_1 l_Q^{s-d} \leq c_1 4^{d-s} l_{Q_1}^{s-d} \leq c_1^2 4^d a_{Q_1,s}.$$

By the finite overlap property, there exist at most $N-1$ cubes touching Q_1. The same holds for Q_1 replaced by Q_2. Therefore,

$$
\begin{aligned}
&|\mathrm{Ext}_s(g)(x)-\mathrm{Ext}_s(g)(y)|\\
&\qquad\le (N-1)^2(c_1^2 4^d)^{2/p}\Bigl(a_{Q_1,s}a_{Q_2,s}\int_{B_{30l_{Q_1}}(q_{Q_1})}\int_{B_{30l_{Q_2}}(q_{Q_2})}|g(z_1)-g(z_2)|^p\tilde{\mu}_s(\mathrm{d}z_2)\tilde{\mu}_s(\mathrm{d}z_1)\Bigr)^{1/p}.
\end{aligned}
$$

(b) By (4-7), $\sum_{Q\in\mathcal{W}_{\le\kappa}}\phi_Q=1$ on $D_{\ge 2-\kappa}$, and thus $\sum_{Q\in\mathcal{W}_{\le\kappa}}\nabla\phi_Q=0$ on $D_{\ge 2-\kappa}$. We write

$$
\begin{aligned}
\nabla\,\mathrm{Ext}_s(g)(x)&=\sum_{Q\in\mathcal{W}_{\le\kappa}(\Omega)}\nabla\phi_Q(x)\Bigl(a_{Q,s}\int_{B_{6l_Q}(q_Q)}g(z_1)\tilde{\mu}_s(\mathrm{d}z_1)-\mathrm{Ext}_s(g)(y)\Bigr)\\
&=\sum_{Q,\tilde{Q}\in\mathcal{W}_{\le\kappa}(\Omega)}\nabla\phi_Q(x)\phi_{\tilde{Q}}(y)a_{Q,s}a_{\tilde{Q},s}\int_{B_{6l_Q}(q_Q)}\int_{B_{6l_{\tilde{Q}}}(q_{\tilde{Q}})}(g(z_1)-g(z_2))\tilde{\mu}_s(\mathrm{d}z_2)\tilde{\mu}_s(\mathrm{d}z_1).
\end{aligned}
$$

By definition of the partition of unity $\phi_{(\cdot)}$, see (4-4), there exists a positive constant $c_2=c_2(\mathcal{W}(\Omega),d,\psi)$ such that, for any $Q\in\mathcal{W}_{\le\kappa}$ touching Q_1 or Q_1 itself, we have

$$|\nabla\phi_Q(x)|\le c_2 s_Q^{-1}\le 4\sqrt{d}c_2 l_{Q_1}^{-1}.$$

We calculate using the same arguments as in the proof of (a) and get

$$|\nabla\,\mathrm{Ext}_s(g)(x)|\le l_{Q_1}^{-1}4\sqrt{d}c_2(N-1)^2(c_1^2 4^d)^{2/p}J_p(q_{Q_1},q_{Q_2}).$$

The proofs of (c) and (d) follow the same lines as the proofs of (a) and (b). Therefore, we omit them. □

Lemma 4.3 [Jonsson 1994, Lemma 2; Jonsson and Wallin 1984, Chapter V, Lemma 2]. *Let $s\in(0,1)$, $a>0$, $h:\Omega^c\to\mathbb{R}$. Set, for $x\in\Omega$,*

$$f(x)=\int_{B_{al_Q}(q_Q)}h(t)\tilde{\mu}_s(\mathrm{d}t)$$

if $x\in\mathring{Q}$ for $Q\in J_i$, $i\in\mathbb{N}$, and $f(x)=0$ otherwise. There exist constants $C=C(d,a)>0$ and $a_0=a_0(d,a)>0$ such that, for any $x_0\in\mathbb{R}^d$ and $0<r\le\infty$,

$$\int_{D_i\cap B_r(x_0)}f(x)\,\mathrm{d}x\le C2^{-id}\int_{\Omega^{\mathrm{ext}}_{\sqrt{d}a2^{-i}}\cap B_{r+a_02^{-i}}(x_0)}h(t)\tilde{\mu}_s(\mathrm{d}t). \tag{4-13}$$

Lemma 4.4. *Assume $a,b>0$ and $p^\star\ge 1$. There exists a constant $C=C(a,b,d,p_\star)>0$ such that, for $s\in(0,1)$ and $1\le p\le p^\star$,*

$$
\begin{aligned}
&\sum_{j=0}^{\infty}2^{j(d+s(p-2))}\iint_{\substack{|x-y|\le a2^{-j}\\ d_x\le b2^{-j}}}|g(x)-g(y)|^p\tilde{\mu}_s(\mathrm{d}y)\tilde{\mu}_s(\mathrm{d}x)\\
&\qquad\qquad\le\frac{C}{d+s(p-2)}\iint_{\substack{|x-y|\le a\\ d_x\le b}}\frac{|g(x)-g(y)|^p}{((|x-y|+d_x+d_y)\wedge 1)^{d+s(p-2)}}\mu_s(\mathrm{d}y)\mu_s(\mathrm{d}x).
\end{aligned}
$$

It is only due to this lemma that the norm of the extension operator in Theorem 1.3 in the case $d = 1$ depends on a lower bound of $(1-s)$.

Proof. The left-hand side of the inequality is equal to

$$\sum_{j=0}^{\infty}\sum_{k,n\geq j} 2^{j(d+s(p-2))}\iint_{\substack{a2^{-n-1}\leq|x-y|\leq a2^{-n}\\ b2^{-k-1}\leq d_x\leq b2^{-k}}}|g(x)-g(y)|^p\tilde{\mu}_s(\mathrm{d}y)\tilde{\mu}_s(\mathrm{d}x)$$

$$=\Bigg(\sum_{n\geq k\geq 0}\sum_{j=0}^{k}+\sum_{k>n\geq 0}\sum_{j=0}^{n}\Bigg)2^{j(d+s(p-2))}\iint_{\substack{a2^{-n-1}\leq|x-y|\leq a2^{-n}\\ b2^{-k-1}\leq d_x\leq b2^{-k}}}|g(x)-g(y)|^p\tilde{\mu}_s(\mathrm{d}y)\tilde{\mu}_s(\mathrm{d}x)$$

$$\leq 2^{d+p+1}\sum_{n,k\geq 0}\frac{2^{(k\wedge n)(d+s(p-2))}}{d+s(p-2)}(1+a+b)^{2d+2s(p-1)}\iint_{\substack{a2^{-n-1}\leq|x-y|\leq a2^{-n}\\ b2^{-k-1}\leq d_x\leq b2^{-k}}}|g(x)-g(y)|^p\mu_s(\mathrm{d}y)\mu_s(\mathrm{d}x)$$

$$=: (\mathrm{I}).$$

For $x, y \in \Omega^c$ satisfying $|x-y| \leq a2^{-n}$ and $d_x \leq b2^{-k}$, we have $|x-y| + d_x + d_y \leq 2(a+b)2^{-(k\wedge n)}$. Therefore,

$$(\mathrm{I}) \leq \frac{(2(a+b)+1)^{4d+4p+1}}{d+s(p-2)}\iint_{\substack{|x-y|\leq a\\ d_x\leq b}}\frac{|g(x)-g(y)|^p}{((|x-y|+d_x+d_y)\wedge 1)^{d+s(p-2)}}\mu_s(\mathrm{d}y)\mu_s(\mathrm{d}x). \qquad \square$$

Now we are in the position to prove the continuity of the L^p part. Recall the definition of the sets Ω_r^{ext} and Ω_{ext}^r in (2-1) for given $r > 0$.

Proposition 4.5. *Let $s \in (0,1)$ and $1 \leq p \leq p^\star < \infty$, and assume that Ω satisfies (4-10). Then, for every measurable $g : \Omega^c \to \mathbb{R}$,*

$$\|\mathrm{Ext_s}(g)\|_{L^p(\Omega)} \leq \frac{C}{s^{1/p}}\|g\|_{L^p(\Omega^{\text{ext}}_{3\,\mathrm{inr}(\Omega)};\mu_s)}$$

with a constant $C = C(d, \Omega, p^\star) > 0$ which is independent of s and p.

Proof. Firstly, $\int_\Omega \phi_Q(x)\,\mathrm{d}x \leq |Q^\star| \leq \left(1+\frac{1}{8}\right)^d s_Q^d$ for any $Q \in \mathcal{W}(\Omega)$. Let $c_1 = c_1(d,\Omega) > 1$ be the constant from (4-10). Recall the definitions of ρ and κ in (4-5). For any $Q \in \mathcal{W}_{\leq\kappa}$ and $z \in \Omega^c \cap B_{6l_Q}(q_Q)$, we know $d_z \leq 6l_Q = 6\sqrt{d}s_Q \leq 6\sqrt{d}2^\kappa \leq 6\rho$. We use the finite overlap property of the Whitney decomposition to estimate

$$\|\mathrm{Ext_s}(g)\|^p_{L^p(\Omega)} \leq \left(1+\tfrac{1}{8}\right)^d N^p \sum_{Q\in\mathcal{W}_{\leq\kappa}(\Omega)} s_Q^d\bigg(\fint_{B_{6l_Q}(q_Q)}|g(z)|\tilde{\mu}_s(\mathrm{d}z)\bigg)^p$$

$$\leq 2^d c_1 N^p(6\rho+1)^{d+p-1}\sqrt{d}^{\,s-d}\sum_{Q\in\mathcal{W}_{\leq\kappa}(\Omega)} s_Q^s\int_{B_{6l_Q}(q_Q)}|g(z)|^p\mu_s(\mathrm{d}z) =: (\star).$$

Now, we consider $i \in \mathbb{N}$ and two cubes $Q_1, Q_2 \in \mathcal{W}_{\leq\kappa}(\Omega)$ such that $s_{Q_1} = s_{Q_2} = 2^{-i}$. If

$$|q_{Q_1} - q_{Q_2}| \geq 6(l_{Q_1} + l_{Q_2}) = 12\sqrt{d}2^{-i},$$

then $B_{6l_{Q_1}(q_{Q_1})} \cap B_{6l_{Q_2}(q_{Q_2})} = \varnothing$. The number of cubes $Q \in \mathcal{W}_{\le\kappa}$ with side length 2^{-i} that fit in the ball $B_{12\sqrt{d}2^{-i}}(q_{Q_1})$ is bounded from above by $\lceil 12\sqrt{d}2^{-i}/s_Q\rceil^d = \lceil 12\sqrt{d}\rceil^d$. Therefore,

$$\#\{Q \in \mathcal{W}_{\le\kappa} \mid s_Q = 2^{-i},\, B_{6l_{Q_1}(q_{Q_1})} \cap B_{6l_Q(q_Q)} \neq \varnothing\} \le \lceil 12\sqrt{d}\rceil^d. \tag{4-14}$$

We set $c_2 := 2^d c_1 N^p (6\rho+1)^{d+p-1}$. For any $z \in \Omega^c$ such that there exists a cube $Q \in \mathcal{W}_{\le\kappa}$ which satisfies $z \in B_{6l_Q}(q_Q)$, we have

$$d_z \le |z - q_Q| \le 6l_Q \le 6\rho.$$

We change the order of summation and use the above consideration to estimate

$$\begin{aligned}(\star) &\le c_2 \sum_{i=0}^{\infty} \sum_{Q\in\mathcal{W}_{\le\kappa}(\Omega)\cap J_i} 2^{-is} \int_{B_{6\cdot 2^{-i}}(q_Q)} |g(z)|^p \mu_s(\mathrm{d}z) \\ &\le c_2 \lceil 12\sqrt{d}\rceil^d \frac{1}{1-2^{-s}} \int_{\Omega^{\mathrm{ext}}_{6\rho}} |g(z)|^p \mu_s(\mathrm{d}z) \le c_2\lceil 12\sqrt{d}\rceil^d \frac{2}{s} \|g\|^p_{L^p(\Omega^{\mathrm{ext}}_{3\,\mathrm{inr}(\Omega)};\mu_s)}.\end{aligned}$$

Thus, the proposition is proven with the constant

$$C := 2^{d+1} c_1 N^{p^\star} (6\rho+1)^{d+p^\star-1} \lceil 12\sqrt{d}\rceil^d. \qquad \square$$

Proposition 4.6. *Let $s \in (0, 1)$ and $1 \le p \le p^\star < \infty$. We assume that Ω is bounded and satisfies (4-10). Then, for every $g \in \mathcal{T}^{s,p}(\Omega^c)$,*

$$[\mathrm{Ext_s}(g)]_{V^{s,p}(\Omega\,|\,\mathbb{R}^d)} \le \frac{C}{(d+s(p-2))^{1/p} s^{2/p}} \|g\|_{\mathcal{T}^{s,p}(\Omega^c)}$$

with a constant $C = C(d, \Omega, p^\star) > 0$ which is independent of s and p.

Proof. We set $c_1 := (\sqrt{d}2^{\kappa+1}) \wedge \frac{1}{2^4} < 2\rho \wedge \frac{1}{8}$ and $j_0 := -\kappa - 6$, where κ and ρ are as in (4-5) and split the integration domain of the seminorm into

$$\begin{aligned}[\mathrm{Ext_s}(g)]^p_{V^{s,p}(\Omega\,|\,\mathbb{R}^d)} &= (1-s)\int_\Omega\int_{\{|h|\ge c_1\}} \frac{|\mathrm{Ext_s}(g)(x) - \mathrm{Ext_s}(g)(x+h)|^p}{|h|^{d+sp}}\,\mathrm{d}h\,\mathrm{d}x \\ &\quad + \left(\sum_{j=j_0}^{\infty}(1-s)\int_{D_j}\int_{\{|h|<c_1 2^{-j}\}} \frac{|\mathrm{Ext_s}(g)(x) - \mathrm{Ext_s}(g)(x+h)|^p}{|h|^{d+sp}}\,\mathrm{d}h\,\mathrm{d}x\right) \\ &\quad + \left(\sum_{j=j_0}^{\infty}(1-s)\int_{D_j}\int_{\{c_1 2^{-j}\le|h|<c_1\}} \frac{|\mathrm{Ext_s}(g)(x) - \mathrm{Ext_s}(g)(x+h)|^p}{|h|^{d+sp}}\,\mathrm{d}h\,\mathrm{d}x\right) \\ &\quad + \left(\sum_{j<j_0}(1-s)\int_{D_j}\int_{\{|h|<c_1\}} \frac{|\mathrm{Ext_s}(g)(x) - \mathrm{Ext_s}(g)(x+h)|^p}{|h|^{d+sp}}\,\mathrm{d}h\,\mathrm{d}x\right) \\ &=: \mathrm{(I)} + \mathrm{(II)} + \mathrm{(III)} + \mathrm{(IV)}.\end{aligned}$$

Recall that D_j is the collection of Whitney cubes with side length 2^{-j}; see (4-3). We handle these four terms separately.

Estimate of (IV): For any $x \in D_j$, $j < j_0$ and $|h| < c_1$, the distance of x as well as $x+h$ to the boundary $\partial\Omega$ is bigger or equal to

$$\begin{aligned} d_x - |h| &\geq \operatorname{dist}(D_j, \partial\Omega) - c_1 \\ &\geq \sqrt{d}2^{-j-2} - c_1 \\ &\geq \sqrt{d}(2^{\kappa+4} - 2^{\kappa+1}) \\ &> 6\rho. \end{aligned}$$

Therefore,

$$\mathrm{Ext_s}(g)(x) = 0 = \mathrm{Ext_s}(g)(x+h) \quad \text{and} \quad (\mathrm{IV}) = 0.$$

Estimate of (I): Using the triangle inequality and the definition of the extension $\mathrm{Ext_s}(g)$, we estimate (I) as follows:

$$\begin{aligned} (\mathrm{I}) &\leq 2^p(1-s)\bigg(\int_\Omega \int_{|h|\geq c_1} \frac{|\mathrm{Ext_s}(g)(x)|^p}{|h|^{d+sp}}\,\mathrm{d}h\,\mathrm{d}x + \int_\Omega \int_{|h|\geq c_1} \frac{|\mathrm{Ext_s}(g)(x+h)|^p}{|h|^{d+sp}}\,\mathrm{d}h\,\mathrm{d}x\bigg) \\ &\leq 2\frac{2^p(1-s)}{sp} c_1^{-sp} \omega_{d-1} \|\mathrm{Ext_s}(g)\|^p_{L^p(\Omega)} + 2^p 2^{-d-s(p-1)} |\Omega| \int_{\Omega^c} |g(y)|^p \mu_s(\mathrm{d}y). \end{aligned}$$

By Proposition 4.5 and the previous inequality, we find a constant $c_2 = c_2(d, \Omega, p^\star) > 0$ such that

$$(\mathrm{I}) \leq \frac{1-s}{s^2} \frac{2^{p+1}\omega_{d-1} c_2^p}{p c_1^p} \|g\|^p_{L^p(\Omega^c;\mu_s)} + 2^p |\Omega| \|g\|^p_{L^p(\Omega^c;\mu_s)}.$$

This is the desired estimate for (I).

Estimate of (II): For the moment we fix $j \in \mathbb{N}$, $j \geq j_0$, $Q_j \in J_j$, $x \in Q_j$ and $|h| < c_1 2^{-j} \leq \frac{1}{2^4} s_{Q_j}$. Under these assumptions, $x + h \in Q_j^\star$, where $Q_j^\star$ is the cube with the same center as Q_j but side length $\big(1+\frac{1}{8}\big)s_{Q_j}$. Thus, for any $t \in [0, 1]$, the vector $x + th$ is either in Q_j or in a neighboring cube, say Q, touching Q_j. By (4-2),

$$\tfrac{1}{4} l_{Q_j} \leq l_Q \leq 4 l_{Q_j} \quad \text{and} \quad |q_{Q_j} - q_Q| \leq \operatorname{diam}(Q_j) + \operatorname{diam}(Q) \leq (1+4) l_{Q_j}.$$

Further, for $z \in B_{30 l_Q}(q_Q)$, we find

$$|z - q_{Q_j}| \leq (30 \cdot 4 + 5) l_{Q_j}.$$

Set $c_3 := 30 \cdot 4 + 5$, and let $c_4 > 0$ be the constant from Lemma 4.2 and $c_5 > 0$ be the constant from (4-10). We want to apply Lemma 4.2(b) and (d) to estimate (II). We set $j_1 := j_0 + 8 = -\kappa + 2$ and write

$$\begin{aligned} (\mathrm{II}) = \sum_{j=j_1}^\infty (1-s) \int_{D_j} \int_{\{|h|<c_1 2^{-j}\}} &\frac{|\mathrm{Ext_s}(g)(x) - \mathrm{Ext_s}(g)(x+h)|^p}{|h|^{d+sp}}\,\mathrm{d}h\,\mathrm{d}x \\ &+ \sum_{j=j_0}^{j_1-1} (1-s) \int_{D_j} \int_{\{|h|<c_1 2^{-j}\}} \frac{|\mathrm{Ext_s}(g)(x) - \mathrm{Ext_s}(g)(x+h)|^p}{|h|^{d+sp}}\,\mathrm{d}h\,\mathrm{d}x \\ =: (\mathrm{II}_1) + (\mathrm{II}_2).& \end{aligned}$$

For all $j \geq j_1$, all neighboring cubes of Q_j have side lengths at most 2^κ, and thus $B_{c_1 2^{-j}}(x) \subset D_{\geq -\kappa}$. Since $\mathrm{Ext_s}(g)$ is smooth in Ω, the fundamental theorem of calculus and Lemma 4.2(b) yield

$$
\begin{aligned}
&|\mathrm{Ext_s}(g)(x) - \mathrm{Ext_s}(g)(x+h)| \\
&\quad = \left| \int_0^1 \nabla \mathrm{Ext_s}(g)(x+th) \cdot h \, \mathrm{d}t \right| \\
&\quad \leq |h| \sup_{B_{c_1 2^{-j}}(x)} |\nabla \mathrm{Ext_s}(g)| \\
&\quad \leq c_4 |h| \frac{2^{j+2}}{\sqrt{d}} \left(c_5^2 \sqrt{d}^{2(s-d)} 2^{2(j+2)(d-s)} \int_{(B_{c_3 l_{Q_j}}(q_{Q_j}))^2} |g(z_1) - g(z_2)|^p (\tilde{\mu}_s \otimes \tilde{\mu}_s)(\mathrm{d}(z_1, z_2)) \right)^{1/p}.
\end{aligned}
$$

Using this, we estimate

$$
\begin{aligned}
(\mathrm{II}_1) &\leq (1-s) c_4^p 4^{2(d-s)+p} c_5^2 \sum_{j=j_0}^{\infty} 2^{2j(d-s)+jp} \sum_{Q \in J_j} \int_{|h|<c_1 2^{-j}} |h|^{-d+p(1-s)} \, \mathrm{d}h \\
&\qquad \times \int_Q \int_{(B_{c_3 l_Q}(q_Q))^2} |g(z_1) - g(z_2)|^p (\tilde{\mu}_s \otimes \tilde{\mu}_s)(\mathrm{d}(z_1, z_2)) \, \mathrm{d}x \\
&\leq \frac{\omega_{d-1} c_1^{p(1-s)} c_4^p 4^{2(d-s)+p} c_5^2}{p} \sum_{j=j_0}^{\infty} 2^{2j(d-s)+jp-jp(1-s)} \sum_{Q \in J_j} \\
&\qquad \times \int_Q \int_{\substack{(B_{c_3 l_Q}(q_Q))^2 \\ |z_1 - z_2| \leq 2\sqrt{d} c_3 2^{-j}}} |g(z_1) - g(z_2)|^p (\tilde{\mu}_s \otimes \tilde{\mu}_s)(\mathrm{d}(z_1, z_2)) \, \mathrm{d}x.
\end{aligned}
$$

We define the functions

$$
f_j : \Omega \to \mathbb{R} \quad \text{and} \quad h_j : \Omega^c \to \mathbb{R}
$$

via

$$
\begin{aligned}
h_j(z_1) &:= \int_{|z_1 - z_2| \leq 2\sqrt{d} c_3 2^{-j}} |g(z_1) - g(z_2)|^p \tilde{\mu}_s(\mathrm{d}z_2), && z_1 \in \Omega^c, \\
f_j(x) &:= \int_{B_{c_3 l_Q}(q_Q)} h_j(z_1) \tilde{\mu}_s(\mathrm{d}z_1), && x \in \Omega,
\end{aligned}
$$

whenever there exists $Q \in J_j$ such that $x \in \mathring{Q}$, otherwise we set $f_j = 0$. With this notation we estimate (II_1) using Lemma 4.3 and $d_z \leq c_3 \sqrt{d} 2^\kappa$ for $z \in B_{c_3 l_Q}(q_Q) \cap \Omega^c$ and $Q \in \mathcal{W}_{\leq \kappa}$:

$$
\begin{aligned}
(\mathrm{II}_1) &\leq \frac{\omega_{d-1} c_1^{p(1-s)} c_4^p 4^{2(d-s)+p} c_5^2}{p} \sum_{j=j_0}^{\infty} 2^{2j(d-s)+jps} \int_{D_j} f_j(x) \, \mathrm{d}x \\
&\leq \frac{\omega_{d-1} (c_1 \vee 1)^p c_4^p 4^{2d+p} c_5^2}{p} c_6 \sum_{j=j_0}^{\infty} 2^{j(d+s(p-2))} \int_{\Omega^{\mathrm{ext}}_{\sqrt{d} c_3 2^{-j}}} h_j(z_1) \tilde{\mu}_s(\mathrm{d}z_1) \\
&\leq \frac{c_8}{d+s(p-2)} \int_{\substack{\Omega^{\mathrm{ext}}_{\sqrt{d} c_3} \times \Omega^c \\ |z_1 - z_2| \leq 2\sqrt{d} c_3}} \frac{|g(z_1) - g(z_2)|^p}{((|z_1 - z_2| + d_{z_1} + d_{z_2}) \wedge 1)^{d+s(p-2)}} (\tilde{\mu}_s \otimes \tilde{\mu}_s)(\mathrm{d}(z_1, z_2)).
\end{aligned}
$$

In the last inequality we used Lemma 4.4. Here $c_6 = c_6(d, c_3) > 0$ is the constant from Lemma 4.3, $c_7 = c_7(d, p, c_3) > 0$ is the constant from Lemma 4.4 and

$$c_8 = c_8(d, p, c_1, c_3, c_5, c_6, c_7) := \frac{\omega_{d-1}(c_1 \vee 1)^p c_4^p 4^{2d+p} c_5^2}{p} (c_3\sqrt{d}2^{\kappa} + 1)^{2d+2(p-1)} c_6 c_7.$$

Just as in the proof of the estimate of (II_1), we apply the fundamental theorem of calculus and Lemma 4.2(d), (4-10), and Lemma 4.3 to estimate (II_2):

$$\begin{aligned}
(\mathrm{II}_2) &\leq \frac{\omega_{d-1}(c_1 2^{-j_0})^{p(1-s)}}{p} \sum_{j=j_0}^{j_1-1} \sum_{Q\in J_j} \int_Q \max_{y\in Q^\star} |\nabla \operatorname{Ext}_s(g)(y)|^p \,\mathrm{d}x \\
&\leq \frac{\omega_{d-1}(c_1 2^{-j_0})^{p(1-s)}}{p} \left(c_4 \frac{2^{(j_1+1)}}{\sqrt{d}}\right)^p c_5(\sqrt{d}2^{-j_1+1})^{s-d} \sum_{j=j_0}^{j_1-1} \sum_{Q\in J_j} \int_Q \int_{B_{c_3 l_Q}(q_Q)} |g(z)|^p \tilde{\mu}_s(\mathrm{d}z)\,\mathrm{d}x \\
&\leq \frac{\omega_{d-1}(c_1 2^{-j_0})^{p(1-s)}}{p} \left(c_4 \frac{2^{(j_1+1)}}{\sqrt{d}}\right)^p c_5(\sqrt{d}2^{-j_1+1})^{s-d} \sum_{j=j_0}^{j_1-1} c_6 \int_{\Omega^{\mathrm{ext}}_{\sqrt{d}c_3 2^{-j}}} |g(z)|^p \tilde{\mu}_s(\mathrm{d}z) \\
&\leq \frac{\omega_{d-1}(c_1 2^{-j_0} \vee 1)^p}{p} \left(c_4 \frac{2^{(j_1+1)}}{\sqrt{d}}\right)^p c_5 \frac{(1+\sqrt{d}c_3 2^{-j_0})^{d+p-1}}{(\sqrt{d}2^{-j_1+1} \wedge 1)^d} c_6 (j_1 - j_0) \|g\|^p_{L^p(\Omega^{\mathrm{ext}}_{\sqrt{d}c_3 2^{-j_0}};\mu_s)}.
\end{aligned}$$

Estimate of (III): We put $h_m := c_1 2^{-m}$ and write

$$\begin{aligned}
(\mathrm{III}) &= (1-s) \sum_{j=j_0}^{\infty} \sum_{m=0}^{j-1} \int_{D_j} \int_{\{h_{m+1}\leq |h| < h_m\}} \frac{|\operatorname{Ext}_s(g)(x) - \operatorname{Ext}_s(g)(x+h)|^p}{|h|^{d+sp}} \,\mathrm{d}h\,\mathrm{d}x \\
&\leq (1-s) \sum_{m=0}^{\infty} \sum_{j=m+j_0}^{\infty} \int_{D_j} \int_{\{h_{m+1}\leq |h| < h_m\}} \frac{|\operatorname{Ext}_s(g)(x) - \operatorname{Ext}_s(g)(x+h)|^p}{|h|^{d+sp}} \,\mathrm{d}h\,\mathrm{d}x \\
&= (1-s) \sum_{m=0}^{\infty} \int_{D_{\geq m+j_0}} \int_{\{h_{m+1}\leq |h| < h_m\}} \frac{|\operatorname{Ext}_s(g)(x) - \operatorname{Ext}_s(g)(x+h)|^p}{|h|^{d+sp}} \,\mathrm{d}h\,\mathrm{d}x \\
&\leq (1-s) \sum_{m=0}^{\infty} h_{m+1}^{-d-sp} \Bigg(\int_{D_{\geq m+j_0}} \int_{\Omega} |\operatorname{Ext}_s(g)(x) - \operatorname{Ext}_s(g)(y)|^p \mathbb{1}_{|x-y|\leq h_m} \,\mathrm{d}y\,\mathrm{d}x \\
&\qquad\qquad + \int_{D_{\geq m+j_0}} \int_{\Omega^c} |\operatorname{Ext}_s(g)(x) - g(y)|^p \mathbb{1}_{|x-y|\leq h_m} \,\mathrm{d}y\,\mathrm{d}x \Bigg) \\
&=: (1-s)((\mathrm{III}_1) + (\mathrm{III}_2)).
\end{aligned}$$

Estimate of (III_1): For $x \in Q \subset D_{\geq m+j_0}$ and $y \in \Omega$ such that $|x-y| \leq h_m$, we have

$$\begin{aligned}
d_y \leq d_x + |x-y| &\leq \operatorname{dist}(Q, \partial\Omega) + \operatorname{diam}(Q) + h_m \\
&\leq (5\sqrt{d} + c_1) 2^{-m-j_0} \\
&\leq \sqrt{d} 2^{3-j_0-m}
\end{aligned}$$

and, therefore, $y \in D_{\geq m+j_0-3}$ by (4-1). We calculate

$$\begin{aligned}
(\mathrm{III}_1) &\leq \sum_{m=0}^{\infty}\sum_{k=m+j_0}^{\infty}\sum_{n=m+j_0-3}^{\infty} h_{m+1}^{-d-sp}\int_{D_n}\int_{D_k\cap B_{h_m}(y)} |\mathrm{Ext_s}(g)(x)-\mathrm{Ext_s}(g)(y)|^p\,\mathrm{d}x\,\mathrm{d}y \\
&= \sum_{m=j_1-j_0+3}^{\infty}\sum_{k=m+j_0}^{\infty}\sum_{n=m+j_0-3}^{\infty} h_{m+1}^{-d-sp}\int_{D_n}\int_{D_k\cap B_{h_m}(y)} |\mathrm{Ext_s}(g)(x)-\mathrm{Ext_s}(g)(y)|^p\,\mathrm{d}x\,\mathrm{d}y \\
&\quad + \sum_{m=0}^{j_1-j_0+2}\sum_{k=m+j_0}^{\infty}\sum_{n=m+j_0-3}^{\infty} h_{m+1}^{-d-sp}\int_{D_n}\int_{D_k\cap B_{h_m}(y)} |\mathrm{Ext_s}(g)(x)-\mathrm{Ext_s}(g)(y)|^p\,\mathrm{d}x\,\mathrm{d}y \\
&=: (\mathrm{III}_{1,1})+(\mathrm{III}_{1,2}).
\end{aligned} \tag{4-15}$$

We estimate $(\mathrm{III}_{1,1})$ first. For

$$m \geq j_1-j_0+3 = 11, \quad x \in Q_k \in J_k \text{ and } y \in Q_n \in J_n, \quad n \geq m+j_0-3, \quad k \geq m+j_0$$

such that $|x-y| \leq h_m$ and $z_1 \in B_{30l_{Q_k}}(q_{Q_k})$, $z_2 \in B_{30l_{Q_n}}(q_{Q_n})$, we have $n, k \geq j_1$ and

$$\begin{aligned}
|z_1-z_2| &\leq |z_1-q_{Q_k}|+|q_{Q_k}-x|+|x-y|+|y-q_{Q_n}|+|q_{Q_n}-z_2| \\
&\leq 31\sqrt{d}2^{-k}+c_1 2^{-m}+31\sqrt{d}2^{-n} \leq c_9 2^{-m},
\end{aligned}$$

where $c_9 := 31\sqrt{d}2^{-j_0}+c_1+31\sqrt{d}2^{3-j_0}$. Note that $s_{Q_k}, s_{Q_n} \leq 2^{\kappa-2}$. Lemma 4.2(a) yields

$$\begin{aligned}
&|\mathrm{Ext_s}(g)(x)-\mathrm{Ext_s}(g)(y)|^p \\
&\qquad \leq c_4^p a_{Q_k,s}a_{Q_n,s}\int_{B_{30l_{Q_k}}(q_{Q_k})}\int_{B_{30l_{Q_n}}(q_{Q_n})} |g(z_1)-g(z_2)|^p \mathbb{1}_{|z_1-z_2|\leq c_9 2^{-m}}\tilde{\mu}_s(\mathrm{d}z_2)\tilde{\mu}_s(\mathrm{d}z_1) \\
&\qquad = c_4^p a_{Q_k,s}a_{Q_n,s}\tilde{f}_m(x),
\end{aligned} \tag{4-16}$$

where $\tilde{f}_m : \mathbb{R}^d \to \mathbb{R}$ is defined by

$$\tilde{f}_m(x) := \int_{B_{30l_Q}(q_Q)} \tilde{h}_m(z_1)\tilde{\mu}_s(\mathrm{d}z_1)$$

for $x \in \mathbb{R}^d$ whenever there exists $Q \in J_k$ such that $x \in \mathring{Q}$, otherwise we set $\tilde{f}_m = 0$. Here $\tilde{h}_m : \Omega^c \to \mathbb{R}$ is defined by

$$\tilde{h}_m(z_1) := \int_{B_{30l_{Q_n}}(q_{Q_n})} |g(z_1)-g(z_2)|^p \mathbb{1}_{|z_1-z_2|\leq c_9 2^{-m}}\tilde{\mu}_s(\mathrm{d}z_2), \quad z_1 \in \Omega^c.$$

Thus, Lemma 4.3 together with (4-16) yields

$$\begin{aligned}
\int_{D_n}\int_{D_k\cap B_{h_m}(y)} |\mathrm{Ext_s}(g)(x)-\mathrm{Ext_s}(g)(y)|^p\,\mathrm{d}y\,\mathrm{d}x &\leq c_4^p a_{Q_k,s}a_{Q_n,s}\int_{D_n}\int_{D_k\cap B_{h_m}(y)} \tilde{f}_m(x)\,\mathrm{d}x\,\mathrm{d}y \\
&\leq c_4^p c_{10}2^{-kd}a_{Q_k,s}a_{Q_n,s}\int_{D_n}\int_{B_{h_m+a_1 2^{-k}}(y)} \tilde{h}_m(z_1)\tilde{\mu}_s(\mathrm{d}z_1)\,\mathrm{d}y \\
&=: (\widetilde{\mathrm{III}}_{1,1}).
\end{aligned} \tag{4-17}$$

Here $c_{10} = c_{10}(d) > 0$ and $a_1 = a_1(d) > 0$ are the constants from Lemma 4.3. For $y \in D_n$ and $z_1 \in B_{h_m + a_1 2^{-k}}(y) \cap \Omega^c$, we find

$$d_{z_1} \le |y - z_1| \le (c_1 + a_1 2^{-j_0}) 2^{-m}$$

and set $c_{11} := (c_1 + a_1 2^{-j_0})$. Then, $(\widetilde{\mathrm{III}}_{1,1})$ becomes, after applying Lemma 4.3 again,

$$\begin{aligned}
(\widetilde{\mathrm{III}}_{1,1}) &\le \frac{c_4^p c_{10}}{2^{kd}} a_{Q_k,s} a_{Q_n,s} \int_{D_n} \int_{B_{30 l_{Q_n}}(q_{Q_n})} \int_{\Omega^{\mathrm{ext}}_{c_{11} 2^{-m}}} |g(z_1) - g(z_2)|^p \mathbb{1}_{|z_1 - z_2| \le c_9 2^{-m}} \tilde{\mu}_s(\mathrm{d}z_1) \tilde{\mu}_s(\mathrm{d}z_2)\, \mathrm{d}y \\
&\le \frac{c_4^p c_{12} c_{10}}{2^{(k+n)d}} a_{Q_k,s} a_{Q_n,s} \int_{\Omega^c} \int_{\Omega^{\mathrm{ext}}_{c_{11} 2^{-m}}} |g(z_1) - g(z_2)|^p \mathbb{1}_{|z_1 - z_2| \le c_9 2^{-m}} \tilde{\mu}_s(\mathrm{d}z_1) \tilde{\mu}_s(\mathrm{d}z_2) \\
&\le \frac{c_4^p c_5 c_{12} c_{10}}{2^{s(k+n)}} \int_{\Omega^c} \int_{\Omega^{\mathrm{ext}}_{c_{11} 2^{-m}}} |g(z_1) - g(z_2)|^p \mathbb{1}_{|z_1 - z_2| \le c_9 2^{-m}} \tilde{\mu}_s(\mathrm{d}z_1) \tilde{\mu}_s(\mathrm{d}z_2).
\end{aligned} \tag{4-18}$$

In the last inequality, we used (4-10) and $c_{12} = c_{12}(d) > 0$ is the constant from Lemma 4.3. We set $c_{13} := c_4^p c_5 c_{10} c_{12} 2^{d+p} (c_0 \wedge 1)^{-d-p}$. Recall that $j_1 - j_0 = 8$. The estimates (4-17) and (4-18) yield

$$\begin{aligned}
(\mathrm{III}_{1,1}) &\le c_{13} \sum_{m=j_1-j_0+3}^{\infty} \sum_{k=m+j_0}^{\infty} \sum_{n=m+j_0-3}^{\infty} 2^{m(d+sp)} 2^{-s(k+n)} \\
&\qquad\qquad \times \int_{\Omega^c} \int_{\Omega^{\mathrm{ext}}_{c_{11} 2^{-m}}} |g(z_1) - g(z_2)|^p \mathbb{1}_{|z_1 - z_2| \le c_9 2^{-m}} \tilde{\mu}_s(\mathrm{d}z_1) \tilde{\mu}_s(\mathrm{d}z_2) \\
&= \frac{c_{13}}{2^{2sj_0 - 3s}} \left(\frac{2^s}{2^s - 1} \right)^2 \sum_{m=11}^{\infty} 2^{m(d+s(p-2))} \int_{\Omega^c} \int_{\Omega^{\mathrm{ext}}_{c_{11} 2^{-m}}} |g(z_1) - g(z_2)|^p \mathbb{1}_{|z_1 - z_2| \le c_9 2^{-m}} \tilde{\mu}_s(\mathrm{d}z_1) \tilde{\mu}_s(\mathrm{d}z_2) \\
&\le c_{14} \frac{(1 + c_9 + c_{11})^{2d + 2(p-1)}}{s^2 (d + s(p-2))} \int_{\Omega^{\mathrm{ext}}_{c_{11}}} \int_{\Omega^c \cap B_{c_9}(z_2)} \frac{|g(z_1) - g(z_2)|^p}{((|z_1 - z_2| + d_{z_1} + d_{z_2}) \wedge 1)^{d + s(p-2)}} \mu_s(\mathrm{d}z_1) \mu_s(\mathrm{d}z_2).
\end{aligned}$$

In the last estimate we used Lemma 4.4. Here $c_{14} := 2^{5 - 2(j_0 \wedge 0)} c_{13} c_{15}$ and $c_{15} = c_{15}(d, p, c_9, c_{11}) > 0$ is the constant from Lemma 4.4. This is the desired estimate for $(\mathrm{III}_{1,1})$. To estimate $(\mathrm{III}_{1,2})$, we calculate

$$\begin{aligned}
(\mathrm{III}_{1,2}) &\le \frac{j_1 - j_0 + 2}{(c_1 2^{-(j_1 - j_0 - 1)})^{d+sp}} \sum_{k=j_0}^{\infty} \sum_{n=j_0-3}^{\infty} \int_{D_k} \int_{D_n \cap B_{c_1}(y)} |\mathrm{Ext}_s(g)(x) - \mathrm{Ext}_s(g)(y)|^p \,\mathrm{d}x\, \mathrm{d}y \\
&\le 2^{p+1} \frac{10}{(c_1 2^{-7})^{d+sp}} \sum_{k,n=j_0-3}^{\infty} \int_{D_k} \int_{D_n \cap B_{c_1}(y)} |\mathrm{Ext}_s(g)(x)|^p \,\mathrm{d}x\, \mathrm{d}y \\
&\le \frac{2^{p+5}}{(c_1 2^{-7})^{d+sp}} \omega_{d-1} c_1^d \|\mathrm{Ext}_s(g)\|^p_{L^p(\Omega)} \\
&\le \frac{2^{p+5} \omega_{d-1} c_1^d}{(c_1 2^{-7} \wedge 1)^{d+p}} \frac{c_2^p}{s} \|g\|^p_{L^p(\Omega^{\mathrm{ext}}_{3\,\mathrm{inr}(\Omega)}; \mu_s)}.
\end{aligned}$$

Here we used Proposition 4.5. Combining the estimates of $(\mathrm{III}_{1,1})$ and $(\mathrm{III}_{1,2})$ yields the desired estimate of (III_1).

Estimate of (III_2): In contrast to (III_1), we integrate over Ω^c where the extension is just g. For $x \in Q \in J_k$, $k \geq m + j_0$ and $y \in \Omega^c$ such that $|x-y| \leq h_m = c_1 2^{-m}$, the distance of y to the center of the cube q_Q is smaller than

$$d_y \leq |y - q_Q| \leq |y-x| + |x - q_Q| \leq c_1 2^{-m} + \sqrt{d}\, 2^{-k} \leq \sqrt{d}(c_1 + 2^{-j_0}) 2^{-m}.$$

Set $c_{16} := 2c_1^{-1}\sqrt{d}(c_1 + 2^{-j_0} + 1)$. Thus $(1-s)c_{16}^{-s} h_{m+1}^{-s} \leq (1-s) d_y^{-s}$. With this we split $(1-s)(\mathrm{III}_2)$ into

$$\begin{aligned}
&(1-s)(\mathrm{III}_2)\\
&\quad \leq \sum_{m=0}^{j_1-j_0-1} \frac{(1-s)}{h_{m+1}^{d+sp}} \sum_{k=m+j_0}^{\infty} \sum_{Q\in J_k} \int_Q \int_{B_{c_{16}2^{-m}}(q_Q)} |\mathrm{Ext}_s(g)(x) - g(y)|^p \mathbb{1}_{|x-y|\leq h_m}\, \mathrm{d}y\, \mathrm{d}x\\
&\qquad + c_{16}^s \sum_{m=j_1-j_0}^{\infty} h_{m+1}^{-d-s(p-1)} \sum_{k=m+j_0}^{\infty} \sum_{Q\in J_k} \int_Q \int_{B_{c_{16}2^{-m}}(q_Q)} |\mathrm{Ext}_s(g)(x) - g(y)|^p \mathbb{1}_{|x-y|\leq h_m} \tilde{\mu}_s(\mathrm{d}y)\, \mathrm{d}x\\
&\quad =: (\mathrm{III}_{2,1}) + (\mathrm{III}_{2,2}).
\end{aligned} \tag{4-19}$$

We estimate $(\mathrm{III}_{2,1})$ by

$$\begin{aligned}
(\mathrm{III}_{2,1}) &\leq 2^p (j_1 - j_0) \Bigg((1-s) \sum_{k=j_0}^{\infty} \sum_{Q\in J_k} \int_Q \int_{B_{c_{16}}(q_Q)} |\mathrm{Ext}_s(g)(x)|^p \mathbb{1}_{|x-y|\leq c_1}\, \mathrm{d}y\, \mathrm{d}x\\
&\qquad\qquad + c_{16}^{d+sp} \sum_{k=j_0}^{\infty} \sum_{Q\in J_k} \int_Q \int_{B_{c_{16}}(q_Q)} |g(y)|^p \mathbb{1}_{|x-y|\leq c_1} \mu_s(\mathrm{d}y)\, \mathrm{d}x \Bigg)\\
&\leq 2^p 8 \big((1-s)\omega_{d-1} c_1^d \|\mathrm{Ext}_s(g)\|^p_{L^p(\Omega)} + c_{16}^{d+sp} \omega_{d-1} c_1^d \|g\|^p_{L^p(\Omega^{\mathrm{ext}}_{c_{16}};\mu_s)} \big)\\
&\leq 2^p 8 \bigg((1-s)\omega_{d-1} c_1^d \frac{c_2}{s} \|g\|^p_{L^p(\Omega^{\mathrm{ext}}_{3\,\mathrm{inr}(\Omega)};\mu_s)} + c_{16}^{d+sp} \omega_{d-1} c_1^d \|g\|^p_{L^p(\Omega^{\mathrm{ext}}_{c_{16}};\mu_s)} \bigg).
\end{aligned}$$

Here, we used Proposition 4.5. We apply Lemma 4.2(c) and (4-10) to estimate $(\mathrm{III}_{2,2})$:

$$\begin{aligned}
&(\mathrm{III}_{2,2})\\
&\leq \sum_{m=j_1-j_0}^{\infty} \frac{c_4^p c_5 c_{16}^s}{h_{m+1}^{d+s(p-1)}} \sum_{k=m+j_0}^{\infty} 2^{k(d-s)} \sum_{Q\in J_k} \int_Q \int_{B_{30 l_Q}(q_Q)} \int_{B_{c_{16}/2^m}(q_Q)} |g(z) - g(y)|^p \tilde{\mu}_s(\mathrm{d}y) \tilde{\mu}_s(\mathrm{d}z)\, \mathrm{d}x.
\end{aligned} \tag{4-20}$$

For $z \in B_{30\sqrt{d}2^{-k}}(q_Q)$ and $y \in \Omega^c$ with $|q_Q - y| \leq c_{16} 2^{-m}$, we have

$$|y - z| \leq (c_{16} + \sqrt{d}\, 30) 2^{-m}.$$

We write $c_{17} := c_{16} + \sqrt{d}\,30$. Now, we apply Lemma 4.3 with $r = +\infty$ and conclude, with a positive constant $c_{18} = c_{18}(d)$,

$$\begin{aligned}
&\sum_{Q\in J_k} \int_Q \int_{B_{30 l_Q}(q_Q)} \int_{B_{c_{16}2^{-m}}(q_Q)} |g(z) - g(y)|^p \tilde{\mu}_s(\mathrm{d}y) \tilde{\mu}_s(\mathrm{d}z)\, \mathrm{d}x\\
&\qquad\qquad \leq c_{18} 2^{-kd} \int_{\Omega^{\mathrm{ext}}_{30\sqrt{d}2^{-m}}} \int_{B_{c_{17}2^{-m}}(z)} |g(z) - g(y)|^p \tilde{\mu}_s(\mathrm{d}y) \tilde{\mu}_s(\mathrm{d}z).
\end{aligned} \tag{4-21}$$

By (4-20), (4-21) and Lemma 4.4,

$$
\begin{aligned}
(\mathrm{III}_{2,2}) &\leq \frac{c_4^p c_5 c_{16}^s}{c_1^{d+s(p-1)}} \frac{2^{-s(j_0\wedge 0)}}{1-2^{-s}} c_{18} \sum_{m=0}^{\infty} 2^{m(d+s(p-2))} \int_{\Omega^{\mathrm{ext}}_{30\sqrt{d}2^{-m}}} \int_{B_{c_{17}2^{-m}}(z)} |g(z)-g(y)|^p \tilde{\mu}_s(\mathrm{d}y)\tilde{\mu}_s(\mathrm{d}z) \\
&\leq \frac{c_{20}}{s(d+s(p-2))} \int_{\Omega^{\mathrm{ext}}_{30\sqrt{d}}} \int_{B_{c_{17}}(z)} \frac{|g(z)-g(y)|^p}{((|y-z|+d_z+d_y)\wedge 1)^{d+s(p-2)}} \mu_s(\mathrm{d}y)\mu_s(\mathrm{d}z).
\end{aligned}
$$

Here the constant is

$$c_{20} := 2^{2-(j_0\wedge 0)}(c_1\wedge 1)^{-d-(p-1)} c_4^p c_5 c_{16} (60\sqrt{d}+2c_{17})^{d+|p-2|} c_{19},$$

where $c_{19} = c_{19}(d, p, c_{17}) > 0$ is the constant from Lemma 4.4. Combining $(\mathrm{III}_{2,1})$ and $(\mathrm{III}_{2,2})$ yields the desired estimate of (III_2). Further, the previous estimates of (III_1) and (III_2) finish the proof of the bound on (III). □

Proof of Theorems 1.2 and 1.3. The proof of Theorem 1.2 follows from Propositions 3.9 and 3.10, Lemma 4.1, and Propositions 4.5 and 4.6. The proof of Theorem 1.2 does not require Proposition 3.10. □

5. Nonlocal to local

In this section, we prove the convergence of the trace spaces $\mathcal{T}^{s,p}(\Omega^c) \to W^{1-1/p,p}(\partial\Omega)$ as $s \to 1^-$ as claimed in Theorem 1.4.

The following lemma is a minor extension of [Grube and Hensiek 2024, Lemma 4.1] to Lipschitz domains. Note that the term $(1-s)/d_x^s$ in the measure μ_s is responsible for the reduction of Ω^c to $\partial\Omega$. Recall the definition of the sets Ω_r^{ext} and Ω^r_{ext} in (2-1) for given $r > 0$.

Lemma 5.1. *Let $\Omega \subset \mathbb{R}^d$ be a Lipschitz domain, $d \geq 2$, and $0 < r \leq \infty$. We define a family of measures $\bar{\mu}_s$ on $(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))$ via*

$$\bar{\mu}_s(x) := \frac{1-s}{d_x^s} \mathbb{1}_{\Omega_r^{\mathrm{ext}}}(x).$$

Let σ be the surface measure on the Lipschitz submanifold $\partial\Omega$, and set $\sigma(D) := \sigma(\partial\Omega \cap D)$ for all sets $D \in \mathcal{B}(\mathbb{R}^d)$. Then $\{\bar{\mu}_s\}_s$ converges weakly to σ as $s \to 1^-$, i.e., when integrated against test functions in $C_c(\mathbb{R}^d)$.

Remark. (1) In dimension $d = 1$, the previous convergence result reads

$$\bar{\mu}_s \to \sum_{x_0\in\partial\Omega} \delta_{x_0} \text{ weakly,}$$

i.e., when tested against $C_c(\mathbb{R})$ functions. Here δ_{x_0} is the Dirac measure in the boundary point $x_0 \in \partial\Omega$.

(2) In [Grube and Hensiek 2024, Lemma 4.1], the first author and his coauthor proved Lemma 5.1 for bounded C^1-domains. Below, we adopt this proof and explain necessary differences to that result.

Proof. Let $f \in C_c(\mathbb{R}^d)$. We have shown in [Grube and Hensiek 2024, Lemma 4.1] without using that the boundary $\partial\Omega$ is C^1 that $\int_{\Omega^{\varepsilon}_{\text{ext}}} |f|\bar{\mu}_s \to 0$ as $s \to 1^-$ for any $\varepsilon > 0$. Thus, the problem localizes. Without loss of generality, we find a cube $Q = (-\rho, \rho)^d$ and a Lipschitz continuous map $\phi : \mathbb{R}^{d-1} \to \mathbb{R}$ such that $\Omega \cap Q = \{(x', x_d) \mid x_d < \phi(x')\} \cap Q$. Furthermore, on this cube we have

$$\int_{Q\cap\Omega^c} f \,\mathrm{d}\bar{\mu}_s = \int_{(-\rho,\rho)^{d-1}} \int_{\phi(x')}^{\rho} f(x', x_d) \frac{1-s}{d^s_{(x',x_d)}} \,\mathrm{d}x_d \,\mathrm{d}x'.$$

Since $\frac{1-s}{x_d - \phi(x')}$ is an approximate identity evaluating at $x_d = \phi(x')$ as $s \to 1^-$, it remains to show that

$$\frac{x_d - \phi(x')}{d_{(x',x_d)}} \to \sqrt{1 + |\nabla\phi(x')|^2} \quad \text{as } x_d \to \phi(x') \tag{5-1}$$

for almost every $x' \in (-\rho, \rho)^{d-1}$. Since ϕ is Lipschitz continuous, it is differentiable at almost every point $x' \in (-\rho, \rho)^{d-1}$ by Rademacher's theorem. In [Grube and Hensiek 2024, Lemma 4.1], we used continuous differentiability of ϕ to show (5-1). Here we only assume ϕ to be Lipschitz continuous. We fix $x' \in (-\rho, \rho)^{d-1}$ such that ϕ is differentiable at x'. For $x_d > \phi(x')$ such that $x_d < \rho$, we pick a minimizer of the distance of $x = (x', x_d)$ to the surface $\partial\Omega$ and call it $y = y(x_d) = (y', \phi(y'))$. Analogously to the estimate (3-3), we have

$$|x_d - \phi(x')| \le 2(1 + [\phi]_{C^{0,1}}) d_x. \tag{5-2}$$

Furthermore, we define the hyperplane

$$P := \{(z', \phi(x') + (z' - x') \cdot \nabla\phi(x')) \mid z' \in (-\rho, \rho)^{d-1}\}$$

which is tangential to the surface at x. Let $z = z(x_d) = (z', z_d)$ be the minimizer of $x = x(x', x_d)$ to the plane P. A small calculation yields

$$\begin{aligned}
z' &= x' + \frac{x_d - \phi(x')}{1 + |\phi(x')|^2} \nabla\phi(x'), \\
z_d &= \phi(x') + \frac{|\nabla\phi(x')|^2}{1 + |\nabla\phi(x')|^2}(x_d - \phi(x')), \\
\operatorname{dist}(x, P) = |x - z| &= \frac{x_d - \phi(x')}{\sqrt{1 + |\nabla\phi(x')|^2}}.
\end{aligned}$$

Since ϕ is differentiable, the error function $r : (-\rho, \rho)^{d-1} \to \mathbb{R}$ given by

$$r(w') := \phi(w') - \phi(x') - (w' - x') \cdot \nabla\phi(x')$$

satisfies

$$\frac{r(w')}{|w' - x'|} \to 0 \quad \text{as } w' \to x'.$$

Since $|y' - x'| \le d_x \le |x_d - \phi(x')|$, we know

$$\frac{r(y')}{|x_d - \phi(x')|} \to 0 \quad \text{as } x_d \to \phi(x'). \tag{5-3}$$

Now, we will estimate $\mathrm{dist}(x, P)$ by d_x and an error and vice versa. Let $y_d \in \mathbb{R}$ such that $(y', y_d) \in P$. By the triangle inequality, we have

$$|(y', y_d) - (x', x_d)| \le |(y', \phi(y')) - (x', x_d)| + |(y', y_d) - (y', \phi(y'))| = d_x + r(y').$$

Since $(y', y_d) \in P$ and z is the minimizer of the distance of x to P, we have $\mathrm{dist}(x, P) \le d_x + r(y')$. Again by the triangle inequality,

$$d_x \le |(z', \phi(z')) - (x', x_d)| \le |(z', z_d) - (x', x_d)| + |(z', \phi(z')) - (z', z_d)| = \mathrm{dist}(x, P) + r(z').$$

Therefore,

$$\begin{aligned}\left|1 - \frac{d_x}{\mathrm{dist}(x, P)}\right| &= \frac{|\mathrm{dist}(x, P) - d_x|}{\mathrm{dist}(x, P)} \le \frac{\max\{|r(y')|, |r(z')|\}}{\mathrm{dist}(x, P)} \\ &= \sqrt{1 + |\nabla\phi(x')|^2}\,\frac{\max\{|r(y')|, |r(z')|\}}{|x_d - \phi(x')|} \to 0\end{aligned}$$

as $x_d \to \phi(x')$ by (5-3), the choice of z' and the properties of the error function r. By the previous calculation of $\mathrm{dist}(x, P)$ and this convergence, (5-1) follows. Since $(1-s)/|x_d - \phi(x')|^s$ is an approximate identity, we have for any $x' \in (-\rho, \rho)^{d-1}$ such that ϕ is differentiable at x'

$$\int_{\phi(x')}^{\rho} \frac{1-s}{d^s_{(x',x_d)}} f(x', x_d)\,\mathrm{d}x_d \to f(x', 0)\sqrt{1 + |\nabla\phi(x')|^2} \quad \text{as } s \to 1^-.$$

Since f has compact support, there exists $R > 0$ such that $\mathrm{supp}(f) \subset B_R(0)$. By (5-2), we have

$$\begin{aligned}\left|\int_{\phi(x')}^{\rho} \frac{1-s}{d^s_{(x',x_d)}} f(x', x_d)\,\mathrm{d}x_d\right| &\le \|f\|_{L^\infty} \mathbb{1}_{B_R(0)}(x') \int_{\phi(x')}^{\rho} \frac{1-s}{d^s_{(x',x_d)}}\,\mathrm{d}x_d \\ &\le 2(1 + [\phi]_{C^{0,1}})\|f\|_{L^\infty} \mathbb{1}_{B_R(0)}(x') \int_{\phi(x')}^{\rho} \frac{1-s}{|x_d - \phi(x')|^s}\,\mathrm{d}x_d \\ &\le 2(1 + [\phi]_{C^{0,1}})(\rho + 1)\|f\|_{L^\infty} \mathbb{1}_{B_R(0)}(x').\end{aligned}$$

By dominated convergence,

$$\int_{Q\cap\Omega^c} f\,\mathrm{d}\bar{\mu}_s \to \int_{\partial\Omega\cap Q} f\,\mathrm{d}\sigma \quad \text{as } s \to 1^-.$$

Following the proof of [Grube and Hensiek 2024, Lemma 4.1], the result follows. □

We are now in the position to prove the convergence theorem. We use similar arguments as in the proof of [Grube and Hensiek 2024, Proposition 4.3, Theorem 1.4].

Proof of Theorem 1.4. First, we prove the convergence result for $u \in C_c^{0,1}(\mathbb{R}^d)$. Then we apply a density argument.

Step 1: Let us assume $u \in C_c^{0,1}(\mathbb{R}^d)$.

L^p part: We split Ω^c into the union $\Omega_1^{\mathrm{ext}} \cup \Omega^1_{\mathrm{ext}}$. On Ω^1_{ext} we apply the estimates from the proof of the trace continuity. By (3-14), there exists a constant $c_1 = c_1(d, \Omega, p) > 0$ such that

$$\lim_{s\to 1^-} \|\mathrm{Tr}_s\, u\|^p_{L^p(\Omega^1_{\mathrm{ext}};\mu_s)} \le c_1 \lim_{s\to 1^-} [u]^p_{V^{s,p}(\Omega\,|\,\Omega^1_{\mathrm{ext}})}.$$

Notice that

$$\begin{aligned}
[u]^p_{V^{s,p}(\Omega\,|\,\Omega^1_{\mathrm{ext}})} &\le (1-s)\int_{B_1(0)^c}\int_\Omega \frac{|u(y)-u(y+h)|^p}{|h|^{d+sp}}\,\mathrm{d}y\,\mathrm{d}h \\
&\le 2^p(1-s)\|u\|^p_{L^p(\mathbb{R}^d)}\int_{B_1(0)^c}\frac{1}{|h|^{d+sp}}\,\mathrm{d}h = \frac{2^p\omega_{d-1}}{p}(1-s)\|u\|_{L^p(\mathbb{R}^d)}
\end{aligned} \tag{5-4}$$

and, thus, $\lim_{s\to 1^-}\|\mathrm{Tr_s}\,u\|^p_{L^p(\Omega^1_{\mathrm{ext}};\mu_s)}=0$. On Ω^{ext}_1 we consider the family of measures $\{\bar{\mu}_s\}$ from Lemma 5.1 with $r=1$. This family converges weakly to the surface measure on $\partial\Omega$, which we denote by σ. Thus, we conclude the claim via

$$\|\mathrm{Tr_s}\,u\|^p_{L^p(\Omega^{\mathrm{ext}}_1;\mu_s)} = \int_{\mathbb{R}^d}\frac{|u(x)|^p}{(1+d_x)^{d+s(p-1)}}\bar{\mu}_s(\mathrm{d}x) \le \int_{\mathbb{R}^d}|u(x)|^p\bar{\mu}_s(\mathrm{d}x) \to \int_{\partial\Omega}|u(x)|^p\,\mathrm{d}\sigma(x)$$

and similarly

$$\|\mathrm{Tr_s}\,u\|^p_{L^p(\Omega^{\mathrm{ext}}_1;\mu_s)} \ge \int_{\mathbb{R}^d}\frac{|u(x)|^p}{(1+d_x)^{d+p-1}}\bar{\mu}_s(\mathrm{d}x) \to \int_{\partial\Omega}|u(x)|^p\,\mathrm{d}\sigma(x) \quad \text{as } s\to 1^-$$

because $x\mapsto(1+d_x)^{-d-p+1}$ is continuous.

Seminorm part: The main task is to show

$$\begin{aligned}
\iint_{\Omega^c\times\Omega^c} &\frac{|u(x)-u(y)|^p(1+d_x)^{-d-s(p-1)}(1+d_y)^{-d-s(p-1)}}{((|x-y|+d_x+d_y)\wedge 1)^{d+s(p-2)}}\,\mathrm{d}(\bar{\mu}_s\otimes\bar{\mu}_s)((x,y)) \\
&\to \iint_{\partial\Omega\times\partial\Omega}\frac{|u(x)-u(y)|^p}{|x-y|^{d+p-2}}\,\mathrm{d}(\sigma\otimes\sigma)(x,y) \quad \text{as } s\to 1^-
\end{aligned}$$

for some $r>0$ in the definition of $\bar{\mu}_s$. The choice of r is arbitrary and will be made later. Let $\bar{\mu}_s$ be the measure from Lemma 5.1 to the parameter r. As in the proof of Proposition 3.10, we split $\Omega^c\times\Omega^c$ into the union $\Omega^{\mathrm{ext}}_r\times\Omega^{\mathrm{ext}}_r\cup\Omega^c\times\Omega^r_{\mathrm{ext}}\cup\Omega^r_{\mathrm{ext}}\times\Omega^c$. The proof in the case $\Omega^c\times\Omega^r_{\mathrm{ext}}$ is the same as the one in the case $\Omega^r_{\mathrm{ext}}\times\Omega^c$ and shows that $[\mathrm{Tr_s}\,u]^p_{\mathcal{T}^{s,p}(\Omega^r_{\mathrm{ext}}\,|\,\Omega^c)}$ converges to zero. By (3-17), (5-4) and a calculation similar to (3-18), we find a constant $c_2=c_2(\Omega,p)>0$ such that

$$\begin{aligned}
[\mathrm{Tr_s}\,u]^p_{\mathcal{T}^{s,p}(\Omega^r_{\mathrm{ext}}\,|\,\Omega^c)} &\le 2^p\left((1-s)\|\mathrm{Tr_s}\,u\|^p_{L^p(\Omega^c;\mu_s)}\int_{\Omega^r_{\mathrm{ext}}}d_x^{-d-sp}\,\mathrm{d}x + \|\mathrm{Tr_s}\,u\|^p_{L^p(\Omega^r_{\mathrm{ext}};\mu_s)}\mu_s(\Omega^c)\right) \\
&\le 2^p\left(\frac{(1-s)c_2r^{-sp}}{s}\|\mathrm{Tr_s}\,u\|^p_{L^p(\Omega^c;\mu_s)} + \frac{c_2}{s}\|\mathrm{Tr_s}\,u\|^p_{L^p(\Omega^r_{\mathrm{ext}};\mu_s)}\right)\to 0
\end{aligned}$$

as $s\to 1-$. Now, we consider the interesting part $\Omega^{\mathrm{ext}}_r\times\Omega^{\mathrm{ext}}_r$. We would like to apply Lemma 5.1 to the function

$$h_s(x,y) := \frac{|u(x)-u(y)|^p(1+d_x)^{-d-s(p-1)}(1+d_y)^{-d-s(p-1)}}{((|x-y|+d_x+d_y)\wedge 1)^{d+s(p-2)}}$$

since

$$[\mathrm{Tr_s}\,u]^p_{\mathcal{T}^{s,p}(\Omega^{\mathrm{ext}}_r\,|\,\Omega^{\mathrm{ext}}_r)} = \iint_{\Omega^{\mathrm{ext}}_r\times\Omega^{\mathrm{ext}}_r} h_s(x,y)\,\mathrm{d}(\bar{\mu}_s\otimes\bar{\mu}_s)((x,y))$$

and

$$h_s(x,y) \to \frac{|u(w)-u(z)|^p}{|w-z|^{d+p-2}} \quad \text{for } x \to w \in \partial\Omega,\ y \to z \in \partial\Omega,\ s \to 1^- \quad \text{for } w \neq z.$$

Lemma 5.1 cannot be applied directly because h_s is neither continuous on $\Omega^c \times \Omega^c$ nor independent of s. We resolve this issue by arguments similar to the ones used in [Grube and Hensiek 2024, Proposition 4.3]. Let us fix a radial bump function $\varphi \in C_c^\infty(\overline{B_2(0)})$, $0 \le \varphi \le 1$, $\varphi = 1$ in B_1, and define $\varphi_\varepsilon(x,y) := \varphi(|x-y|/\varepsilon)$ for $\varepsilon \in (0,1)$. We set

$$a_s^\star := \begin{cases} 1, & p \ge 2, \\ \varepsilon^{-(1-s)(2-p)}, & 1 \le p < 2, \end{cases} \qquad a_{s,\star} := \begin{cases} \varepsilon^{(1-s)(p-2)}, & p \ge 2, \\ 1, & 1 \le p < 2, \end{cases}$$

$$h_\varepsilon^\star(x,y) := \frac{|u(x)-u(y)|^p}{(|x-y| \wedge 1)^{d+p-2}}(1-\varphi_\varepsilon(x,y)),$$

$$h_{\varepsilon,\star}(x,y) := \frac{|u(x)-u(y)|^p((1+d_x)(1+d_y))^{-d-p+1}}{((|x-y|+d_x+d_y)\wedge 1)^{d+p-2}}(1-\varphi_\varepsilon(x,y)),$$

$$g_{s,\varepsilon}(x,y) := h_s(x,y)\varphi_\varepsilon(x,y).$$

For every $s \in (0,1)$ and every $\varepsilon > 0$, we have $a_{s,\star}h_{\varepsilon,\star} + g_{s,\varepsilon} \le h_s \le a_s^\star h_\varepsilon^\star + g_{s,\varepsilon}$. We will now consider the limit $s \to 1^-$ and subsequently $\varepsilon \to 0^+$ of the upper and lower bound in this inequality integrated against $\bar{\mu}_s \otimes \bar{\mu}_s$. Both $h_{\varepsilon,\star}$ and $h_\varepsilon^\star$ are continuous and bounded on $\mathbb{R}^d \times \mathbb{R}^d$. By Lemma 5.1, $\{\mu_s\}_s$ converges weakly to σ, and thus the sequence of product measures $\{\mu_s \otimes \mu_s\}$ converges weakly to $\sigma \otimes \sigma$. Therefore,

$$\begin{aligned} &\iint h_{\varepsilon,\star}(x,y)\,\mathrm{d}(\bar{\mu}_s \otimes \bar{\mu}_s)((x,y)) \\ &\qquad \to \iint_{\partial\Omega\times\partial\Omega} \frac{|u(x)-u(y)|^p}{|x-y|^{d+p-2}}(1-\varphi_\varepsilon(x,y))\,\mathrm{d}(\sigma\otimes\sigma)(x,y) \quad \text{as } s \to 1^-, \\ &\qquad \to \iint_{\partial\Omega\times\partial\Omega} \frac{|u(x)-u(y)|^p}{|x-y|^{d+p-2}}\,\mathrm{d}(\sigma\otimes\sigma)(x,y) \quad \text{as } \varepsilon \to 0+. \end{aligned}$$

The same is true for $h_\varepsilon^\star$. Furthermore, $a_s^\star, a_{s,\star} \to 1^-$. Now, we show that

$$\lim_{s\to 1^-} \iint g_{s,\varepsilon}\,\mathrm{d}(\bar{\mu}_s \otimes \bar{\mu}_s)(x,y) \to 0 \quad \text{as } \varepsilon \to 0+.$$

Note

$$\iint g_{s,\varepsilon}(x,y)\,\mathrm{d}(\bar{\mu}_s\otimes\bar{\mu}_s)(x,y) \le [u]_{C^{0,1}} \iint_{\substack{\Omega_r^{\mathrm{ext}}\times\Omega_r^{\mathrm{ext}} \\ |x-y|<2\varepsilon}} |x-y|^{-d-s(p-2)+2}\,\mathrm{d}(\bar{\mu}_s\otimes\bar{\mu}_s)(x,y).$$

The problem localizes. Since Ω is a bounded Lipschitz domain, we find a uniform localization radius $r_0 > 0$. Let $\mathcal{Q}$ be a cover of $\partial\Omega$ with balls $B \in \mathcal{Q}$ with radius $r_0 > 0$ such that the union of these balls with half their radii still contains $\partial\Omega$. Now, we fix $r > 0$ from the beginning of the proof such that $\bigcup_{B\in\mathcal{Q}} \frac{1}{2}B \supset \Omega_r^{\mathrm{ext}}$, where $\frac{1}{2}B$ is the ball with half the radius. We assume $\varepsilon < \frac{1}{4}r_0$ such that, for any $x, y \in \Omega_r^{\mathrm{ext}}$ satisfying $x \in \frac{1}{2}B$, $B \in \mathcal{Q}$, $|x-y| < 2\varepsilon$, we have $y \in B$. Thus, we only need to consider one ball $B \in \mathcal{Q}$. After translation we assume, with loss of generality, $B = B_{r_0}(0)$. We flatten the boundary $\partial\Omega$ that lies in B with the function $\phi \in C^{0,1}(\mathbb{R}^{d-1})$ such that $\Omega \cap B = \{(x', x_d) \in B \mid x_d < \phi(x')\}$. Since

Ω has a uniform Lipschitz boundary, the Lipschitz constant of ϕ does not depend on the position of B. For any $x \in B \cap \Omega^c$, we have $d_x \geq (1+\|\phi\|_{C^{0,1}})^{-1}|x_d - \phi(x')|$. Therefore,

$$\begin{aligned}
&\iint_{(\Omega^c\cap Q)\times(\Omega^c\cap Q)} \mathbb{1}_{B_{2\varepsilon}(x)}(y)|x-y|^{-d-s(p-2)+2p}\,\mathrm{d}(\bar{\mu}_s\otimes\bar{\mu}_s)(x,y)\\
&\qquad\leq (1+\|\phi\|_{C^{0,1}})^2 \int_{B^{(d-1)}_{r_0}(0)}\int_{\phi(x')}^{2r_0}\int_{B^{(d-1)}_{2\varepsilon}(x')}\int_{\phi(y')}^{2r_0} \frac{(1-s)^2|x'-y'|^{-d-s(p-2)+2p}}{(x_d-\phi(x'))^s(y_d-\phi(y'))^s}\,\mathrm{d}y_d\,\mathrm{d}y'\,\mathrm{d}x_d\,\mathrm{d}x'\\
&\qquad\leq r_0^{2-2s}(1+\|\phi\|_{C^{0,1}})^2\frac{\omega_{d-2}^2}{d-1}r_0^{d-1}\int_0^{2\varepsilon} t^{(1-s)(p-2)}\,\mathrm{d}t\\
&\qquad= r_0^{2-2s}(1+\|\phi\|_{C^{0,1}})^2\frac{\omega_{d-2}^2}{d-1}r_0^{d-1}\frac{(2\varepsilon)^{(1-s)(p-2)+1}}{(1-s)(p-2)+1} \to (1+\|\phi\|_{C^{0,1}})^2\frac{\omega_{d-2}^2}{d-1}r_0^{d-1}2\varepsilon \to 0.
\end{aligned}$$

In the last line, we first consider the limit $s\to 1^-$ and then the limit $\varepsilon \to 0+$. Thus,

$$\iint g_{s,\varepsilon}(x,y)\,\mathrm{d}(\bar{\mu}_s\otimes\bar{\mu}_s)(x,y) \leq \sum_{B\in\mathcal{Q}}\iint_{Q\times Q}\mathbb{1}_{B_{2\varepsilon}(x)}(y)|x-y|^{-d-s(p-2)+2p}\,\mathrm{d}(\bar{\mu}_s\otimes\bar{\mu}_s)(x,y)\to 0.$$

The result for $C_c^{0,1}(\mathbb{R}^d)$ functions follows from

$$a_{s,\star}h_{\varepsilon,\star} + g_{s,\varepsilon} \leq h_s \leq a_s^\star h_\varepsilon^\star + g_{s,\varepsilon}.$$

The proof of $[u]_{\mathcal{T}^{s,1}(\Omega^c)} \to [u]_{B^0_{1,1}(\partial\Omega)}$ follows analogously.

Step 2: Let $1<p<\infty$. We generalize the result from Step 1 to all functions in $W^{1,p}(\mathbb{R}^d)$ via a density argument. Firstly, there exists a constant $c_3 = c_3(d,p) > 0$ such that $\|u\|_{V^{s,p}(\Omega\,|\,\mathbb{R}^d)} \leq c_3\|u\|_{W^{s,p}(\mathbb{R}^d)}$. Combining this estimate with (3-13) and (3-16) yields

$$\|\mathrm{Tr_s}\,u\|_{\mathcal{T}^{s,p}(\Omega^c)} \leq c_4\|u\|_{V^{s,p}(\Omega\,|\,\mathbb{R}^d)} \leq c_3c_4\|u\|_{W^{1,p}(\mathbb{R}^d)}.$$

Here $c_4>0$ is the sum of the constants from Propositions 3.9 and 3.10. Now take any $u\in W^{1,p}(\mathbb{R}^d)$. Since $C_c^{0,1}(\mathbb{R}^d)$ is dense in $W^{1,p}(\mathbb{R}^d)$, we find a sequence $u_n \in C_c^{0,1}(\mathbb{R}^d)$ such that $\|u-u_n\|_{W^{1,p}(\mathbb{R}^d)}\to 0$ as $n\to\infty$. Since the classical trace is continuous, the mapping

$$\gamma_0 : W^{1,p}(\mathbb{R}^d) \xhookrightarrow{\text{cts.}} W^{1,p}(\Omega) \xrightarrow{\gamma} W^{1-1/p,p}(\partial\Omega)$$

is linear and continuous. Thus, uniformly in $s\in(0,1)$,

$$\begin{aligned}
\|\mathrm{Tr_s}\,u - \mathrm{Tr_s}\,u_n\|_{\mathcal{T}^{s,p}(\Omega^c)} &\leq c_1\|u-u_n\|_{W^{1,p}(\mathbb{R}^d)} \to 0,\\
\|\gamma_0 u - \gamma_0 u_n\|_{W^{1-1/p,p}(\partial\Omega)} &\leq C_3\|u-u_n\|_{W^{1,p}(\mathbb{R}^d)} \to 0 \quad \text{as } n\to\infty.
\end{aligned}$$

Finally, Step 1 yields $\|u_n\|_{\mathcal{T}^{s,p}(\Omega^c)} \to \|u_n\|_{W^{1-1/p,p}(\partial\Omega)}$ as $s\to 1^-$. The proof of the statement for $d=1$ follows with minor changes and we omit it. Notice in this case that functions in $W^{s,p}(\Omega)$ for $s>1/p$ are continuous by Morrey's inequality. Lastly, the proof of $\|\mathrm{Tr_s}\,u\|_{L^1(\Omega;\mu_s)} \to \|\gamma u\|_{L^1(\partial\Omega)}$ follows analogously to the proof in Step 2 using the uniform trace embedding Proposition 3.9. ∎

Appendix A: Reflected Whitney cubes

The following results are taken from [Dyda and Kassmann 2019, Section 3.2]. Throughout this section, we fix a Lipschitz domain $\Omega \subset \mathbb{R}^d$. We fix a Whitney decomposition $\mathcal{W}(\mathbb{R}^d \setminus \overline{\Omega})$ of the open set with Lipschitz boundary $\mathbb{R}^d \setminus \overline{\Omega}$; i.e., each cube $Q \in \mathcal{W}(\mathbb{R}^d \setminus \overline{\Omega})$ satisfies $\operatorname{diam} Q \le \operatorname{dist}(Q, \partial\Omega) \le 4 \operatorname{diam} Q$. We denote the center of a Whitney cube $Q \in \mathcal{W}(\mathbb{R}^d \setminus \overline{\Omega})$ by $q_Q \in Q$. These cubes satisfy

$$\sum_{Q \in \mathcal{W}(\mathbb{R}^d \setminus \overline{\Omega})} \mathbb{1}_Q = \mathbb{1}_{\mathbb{R}^d \setminus \overline{\Omega}}. \tag{A-1}$$

Bounded Lipschitz domains are both interior and exterior thick; see [loc. cit., Definition 14 and 15]. Thereby, we find a constant $M > 1$ and a reflected Whitney cube $\widetilde{Q} \subset \Omega$ for any $Q \in \mathcal{W}(\mathbb{R}^d \setminus \overline{\Omega})$ such that $\operatorname{diam} Q \le \operatorname{inr}(\Omega) = \sup\{r \mid B_r \subset \Omega\}$ satisfying

$$\begin{gathered} \operatorname{diam} \widetilde{Q} \le \operatorname{dist}(\widetilde{Q}, \partial\Omega) \le 4 \operatorname{diam} \widetilde{Q}, \\ M^{-1} \operatorname{diam} Q \le \operatorname{diam} \widetilde{Q} \le M \operatorname{diam} Q, \\ \operatorname{dist}(Q, \widetilde{Q}) \le M \operatorname{dist}(Q, \partial\Omega). \end{gathered} \tag{A-2}$$

Again we denote the centers of the reflected cubes by $q_{\widetilde{Q}} \in \widetilde{Q}$. The collection of these reflected cubes cover the domain Ω; i.e.,

$$\bigcup_{\substack{Q \in \mathcal{W}(\mathbb{R}^d \setminus \overline{\Omega}) \\ \operatorname{diam} Q \le \operatorname{inr}(\Omega)}} \widetilde{Q} = \Omega. \tag{A-3}$$

Additionally, the reflected cubes satisfy the bounded overlap property; i.e., there exists a constant $N \ge 1$ such that

$$\sum_{\substack{Q \in \mathcal{W}(\mathbb{R}^d \setminus \overline{\Omega}) \\ \operatorname{diam} Q \le \operatorname{inr}(\Omega)}} \mathbb{1}_{\widetilde{Q}} \le N \mathbb{1}_\Omega; \tag{A-4}$$

see [loc. cit., Remark 19]. We define

$$\mathcal{W}_{\operatorname{inr}(\Omega)}(\mathbb{R}^d \setminus \Omega) := \{Q \in \mathcal{W}(\mathbb{R}^d \setminus \Omega) \mid \operatorname{diam} Q \le \operatorname{inr}(\Omega)\}.$$

Appendix B: Hardy inequality for the half-space

The following Hardy inequality for the half-space is proven in [Frank and Seiringer 2010; Bogdan and Dyda 2011] in the case $p = 2$. See [Dyda and Kijaczko 2024] for a corresponding weighted Hardy inequality. Note that the constant $\mathcal{D}_{s,p}$ is optimal.

Theorem B.1 [Frank and Seiringer 2010, Theorem 1.1; Dyda and Kijaczko 2024, Theorem 1]. *Let* $0 < s < 1$, $d \in \mathbb{N}$, $p \in [1, \infty)$ *with* $ps \ne 1$. *Then*

$$\mathcal{D}_{s,p} \int_{\mathbb{R}^d_+} \frac{|u(x)|^p}{x_d^{sp}} \,\mathrm{d}x \le \int_{\mathbb{R}^d_+ \times \mathbb{R}^d_+} \frac{|u(x) - u(y)|^p}{|x - y|^{d+sp}} \,\mathrm{d}(x, y) \tag{B-1}$$

for any $u \in W_0^{s,p}(\mathbb{R}_+^d) = \overline{C_c^\infty(\mathbb{R}_+^d)}^{\|\cdot\|_{W^{s,p}}}$. *The constant is given by*

$$\mathcal{D}_{s,p} := 2\pi^{(d-1)/2}\frac{\Gamma\left(\frac{1+sp}{2}\right)}{\Gamma\left(\frac{d+sp}{2}\right)}\int_0^1 \frac{|1-t^{(ps-1)/p}|^p}{(1-t)^{1+ps}}\,\mathrm{d}t. \tag{B-2}$$

Furthermore, in the case $p = 1$ *and* $d = 1$, *the inequality only holds for functions that are proportional to a nonincreasing function.*

Lemma B.2. *There exists a constant* $C = C(d) \geq 1$ *such that, for all* $0 < s < 1$,

$$C^{-1} \leq s\mathcal{D}_{s,1} \leq C,$$

where $\mathcal{D}_{s,1}$ *is the constant defined in* (B-2).

Proof. We split the integral in (B-2) into two parts. First,

$$\int_0^{1/2} \frac{|1-t^{s-1}|}{(1-t)^{1+s}}\,\mathrm{d}t \leq 2^{1+s}\int_0^{1/2} t^{s-1}\,\mathrm{d}t \leq 4\frac{\left(\frac{1}{2}\right)^s}{s} \leq \frac{4}{s}.$$

A lower bound in the same term is calculated similarly:

$$\int_0^{1/2} \frac{|1-t^{s-1}|}{(1-t)^{1+s}}\,\mathrm{d}t \geq \left(1-\left(\frac{1}{2}\right)^{1-s}\right)\int_0^{1/2} t^{s-1}\,\mathrm{d}t = \frac{2^{1-s}-1}{2s} \geq \frac{1-s}{4s}.$$

We move to the remaining part of the integral:

$$\int_{1/2}^1 \frac{|1-t^{s-1}|}{(1-t)^{1+s}}\,\mathrm{d}t \leq 2^{1-s}\int_{1/2}^1 \frac{1}{(1-t)^{1+s}}\left((1-s)\int_t^1 r^{-s}\,\mathrm{d}r\right)\mathrm{d}t \leq 2\int_{1/2}^1 \frac{1-s}{(1-t)^s}\,\mathrm{d}t \leq 2.$$

And a lower bound is calculated in a similar fashion:

$$\int_{1/2}^1 \frac{|1-t^{s-1}|}{(1-t)^{1+s}}\,\mathrm{d}t \geq \int_{1/2}^1 \frac{1}{(1-t)^{1+s}}\left((1-s)\int_t^1 r^{-s}\,\mathrm{d}r\right)\mathrm{d}t \geq \int_{1/2}^1 \frac{1-s}{(1-t)^s}\,\mathrm{d}t = \left(\frac{1}{2}\right)^{1-s} \geq \frac{1}{2}.$$

Therefore,

$$2\pi^{(d-1)/2}\frac{1}{\Gamma\left(\frac{d}{2}\right)\vee\Gamma\left(\frac{(d+1)}{2}\right)}\frac{1}{4s} \leq \mathcal{D}_{s,1} \leq 2\pi^{(d-1)/2}\frac{\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{d}{2}\right)\wedge\Gamma\left(\frac{(d+1)}{2}\right)}\frac{6}{s}.$$

□

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QUANTIZED SLOW BLOW-UP DYNAMICS FOR THE ENERGY-CRITICAL COROTATIONAL WAVE MAP PROBLEM

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We study the blow-up dynamics for the energy-critical 1-corotational wave map problem with target the 2-sphere. Raphaël and Rodnianski (*Publ. Math. Inst. Hautes Études Sci.* **115** (2012), 1–122) exhibited stable finite-time blow-up dynamics arising from smooth initial data. In this paper, we exhibit a sequence of new finite-time blow-up rates (quantized rates), which can still arise from well-localized smooth initial data. We closely follow the strategy of Raphaël and Schweyer (*Anal. PDE* **7**:8 (2014), 1713–1805), who exhibited a similar construction of the quantized blow-up rates for the harmonic map heat flow. The main difficulty in our wave map setting stems from the lack of dissipation and its critical nature, which we overcome by a systematic identification of correction terms in higher-order energy estimates.

1. Introduction

1.1. *Wave map problem.* For a map $\Phi : \mathbb{R}^{n+1} \to \mathbb{S}^n$, the wave map problem is given by

$$\partial_{tt}\Phi - \Delta\Phi = \Phi(|\nabla\Phi|^2 - |\partial_t\Phi|^2), \quad \vec{\Phi}(t) := (\Phi, \partial_t\Phi)(t) \in \mathbb{S}^n \times T_\Phi\mathbb{S}^n. \tag{1-1}$$

Problem (1-1) has an intrinsic derivation from the Lagrangian action

$$\frac{1}{2}\int_{\mathbb{R}^{n+1}} (|\nabla\Phi(x,t)|^2 - |\partial_t\Phi(x,t)|^2)\, dx\, dt, \tag{1-2}$$

which yields the energy conservation

$$E(\vec{\Phi}(t)) = \frac{1}{2}\int_{\mathbb{R}^n} |\nabla\Phi|^2 + |\partial_t\Phi|^2\, dx = E(\vec{\Phi}(0)). \tag{1-3}$$

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In particular, for the case $n = 2$, (1-1) is called *energy-critical* since the conserved energy is invariant under the scaling symmetry: if $\vec{\Phi}(t, x)$ is a solution to (1-1), then $\vec{\Phi}_\lambda(t, x)$ is also a solution to (1-1):

$$\vec{\Phi}_\lambda(t, x) := \left(\Phi\left(\frac{t}{\lambda}, \frac{x}{\lambda}\right), \frac{1}{\lambda}\partial_t \Phi\left(\frac{t}{\lambda}, \frac{x}{\lambda}\right)\right),$$

and $\vec{\Phi}_\lambda(t, x)$ satisfies $E(\vec{\Phi}_\lambda) = E(\vec{\Phi})$.

When observing a complicated model, it makes sense from a physics perspective to extract the essential dynamics of the problem by reducing the degrees of freedom. Especially for field theories such as (1-1), the *geodesic approximation* — that is, a method of approximating the dynamics of the full problem as a geodesic motion over a space of static solutions — is prevalent (see [Manton and Sutcliffe 2004]).

To discuss static solutions in more detail, we focus on the solutions with finite energy. This assumption extends the spatial domain of Φ to $\mathbb{S}^2$ and allows the *topological degree* of Φ to be well-defined:

$$k = \frac{1}{|\mathbb{S}^2|}\int_{\mathbb{R}^2} \Phi^*(dw) = \frac{1}{4\pi}\int_{\mathbb{R}^2} \Phi \cdot (\partial_x \Phi \times \partial_y \Phi)\, dx\, dy.$$

Here, dw is the area form on $\mathbb{S}^2$ and k is given only as an integer. We also remark that k is conserved over time.

We now consider static solutions to (1-1),

$$\Delta\Phi + \Phi|\nabla\Phi|^2 = 0, \tag{1-4}$$

so-called *harmonic maps*. Recalling our Lagrangian action (1-2), harmonic maps are characterized as the (local) minimizer of the Dirichlet energy

$$\frac{1}{2}\int_{\mathbb{R}^2} |\nabla\Phi|^2\, dx\, dy.$$

Assume the topological degree of a harmonic map Φ is $k \in \mathbb{Z}$. Then we have the inequality

$$\begin{aligned}
\frac{1}{2}\int_{\mathbb{R}^2} |\nabla\Phi|^2\, dx\, dy &= \frac{1}{2}\int_{\mathbb{R}^2} |\partial_x\Phi|^2 + |\partial_y\Phi|^2\, dx\, dy \\
&= \frac{1}{2}\int_{\mathbb{R}^2} |\partial_x\Phi \pm \Phi \times \partial_y\Phi|^2\, dx\, dy \mp \int_{\mathbb{R}^2} \partial_x\Phi \cdot (\Phi \times \partial_y\Phi)\, dx\, dy \\
&\geq \pm\int_{\mathbb{R}^2} \Phi \cdot (\partial_x\Phi \times \partial_y\Phi)\, dx\, dy = 4\pi|k|.
\end{aligned}$$

Hence, in a given topological sector k, Φ satisfies the *Bogomolny equation* [1976]

$$\partial_x\Phi \pm \Phi \times \partial_y\Phi = 0 \quad \text{for } \pm k \geq 0. \tag{1-5}$$

That is, the field equation (1-4) can be reduced from a second-order PDE to a first-order PDE. From the stereographic projection, we can see that equation (1-5) is equivalent to the Cauchy–Riemann equation,[1] which clearly identifies the space of harmonic maps as the space of rational maps of degree k.

[1] If k is negative, we adopt the *conjugate* Cauchy–Riemann equation instead of the Cauchy–Riemann equation. Thence, harmonic maps can be represented as rational maps with $\bar{z}$ as a complex variable.

Under the L^2 metric induced naturally from the kinetic energy formula, it is well known that the space of static solutions is *geodesically incomplete*, which leads us to expect a blow-up scenario of the low-energy problem.

1.2. *Corotational symmetry.* We consider an ansatz of solutions to (1-1) with k-corotational symmetry:

$$\Phi(t,r,\theta)=\begin{pmatrix}\sin(u(t,r))\cos k\theta\\ \sin(u(t,r))\sin k\theta\\ \cos(u(t,r))\end{pmatrix}, \tag{1-6}$$

where (r,θ) are polar coordinates on $\mathbb{R}^2$.

Under the k-corotational symmetry assumption, $u(t,r)$ satisfies

$$\begin{cases}\partial_{tt}u-\partial_{rr}u-(1/r)\partial_r u+k^2 f(u)/r^2=0,\\ u|_{t=0}=u_0,\quad \partial_t u|_{t=0}=\dot{u}_0,\end{cases}\qquad f(u)=\frac{\sin 2u}{2}. \tag{1-7}$$

It is known that the flow (1-1) preserves such corotational symmetry (1-6) with smooth initial data at least locally in time; see [Raphaël and Rodnianski 2012].

Also, the energy functional (1-3) can be rewritten as

$$E(u,\dot{u}):=\pi\int_0^\infty\left(|\dot{u}|^2+|\partial_r u|^2+k^2\frac{\sin^2 u}{r^2}\right)r\,dr=E(u_0,\dot{u}_0). \tag{1-8}$$

From the above expression, we can observe that a solution to (1-7) with finite energy must satisfy the boundary conditions

$$\lim_{r\to 0}u(r)=m\pi\ \text{ and }\ \lim_{r\to\infty}u(r)=n\pi,\quad m,n\in\mathbb{Z}. \tag{1-9}$$

We have additional symmetries from the geometry of the target domain $\mathbb{S}^2$:

$$-u(t,r),\quad u(t,r)+\pi \tag{1-10}$$

are also solutions to (1-7). Thus, we restrict our solution space to a set of functions $(u,\dot{u})$ that have finite energy and satisfy the boundary conditions (1-9) with $m=0,\ n=1$, which provides the local well-posedness of (1-7) (see also [Klainerman and Machedon 1993; 1995; Krieger 2004; Tao 2001; Tataru 2005]).

1.3. *Harmonic map.* With this restriction, the harmonic map is uniquely determined (up to scaling) and can be written explicitly as

$$Q(r)=2\tan^{-1}r^k. \tag{1-11}$$

Based on the geodesic approximation, it can be said that observing the vicinity of Q under the corotational symmetry assumption facilitates the analysis of blow-up dynamics. This has been proven as a rigorous statement in several past global regularity works (see [Christodoulou and Tahvildar-Zadeh 1993; Shatah and Tahvildar-Zadeh 1992; 1994; Struwe 2003]).

The above results proved that if a wave map blows up in finite time, such a singularity should be created by bubbling off of a nontrivial harmonic map (strictly) inside the backward light cone.

This statement has inspired other researches studying global behaviors of solutions, and many of the results have been developed based on the existence of nontrivial harmonic maps.

Firstly, there is global existence, which is a consequence of the preceding blow-up criteria. If the initial data cannot form a nontrivial harmonic map — that is, if the energy is less than the ground state energy — it can be naturally predicted that the solution exists globally in time, and mathematical proof is also contained in the previously mentioned global regularity results.

This study also allows us to consider the problems of energy threshold (see [Côte et al. 2008] for the symmetric case and [Krieger and Schlag 2012; Sterbenz and Tataru 2010a; 2010b; Tao 2008a; 2008b; 2008c; 2009a; 2009b] for the general case). In this case, it is also important to set an appropriate threshold value and the ground state energy is suitable for our problem setting. However, for other boundary conditions or other topological degrees, it is often given as an integer multiple of $E(Q, 0)$. The heuristic reason is that the degree condition cannot be satisfied with just one bubble (see [Côte et al. 2015a; Lawrie and Oh 2016]). This goes beyond suggesting the existence of a multibubble solution [Jendrej and Lawrie 2018; 2022a; 2022b; 2023; Rodriguez 2021] and serves as an opportunity to verify the soliton resolution conjecture [Duyckaerts et al. 2022; Jendrej and Lawrie 2025] (see also [Côte 2015; Côte et al. 2015a; 2015b; Jia and Kenig 2017]).

The most recent soliton resolution result [Jendrej and Lawrie 2025] fully characterizes the profile decomposition of the solution in all equivariant classes. Thus, our interest is to observe how the scale of the profile given by the harmonic map changes over time within the lifespan of the solution. In particular, for the case of low energy — that is, when the energy is slightly greater than the ground state energy — the geodesic approximation discussed earlier leads us to focus on the situation of having only one harmonic map as the blow-up profile.

1.4. ***Blow-up near Q.*** From a methodological perspective, studies investigating the blow-up of a single bubble can be broadly divided into the backward construction starting from Krieger, Schlag and Tataru [Krieger et al. 2008] and the forward construction inspired by Rodnianski and Sterbenz [2010] and Raphaël and Rodnianski [2012].

The former work obtained a continuum of blow-up rates for the case $k = 1$ via the iteration method and inspired other extended results such as stability under regular perturbations [Krieger and Miao 2020; Krieger et al. 2020] and the construction of more exotic solutions [Pillai 2023b; 2023a]. Beyond direct extensions of this approach, there is a classification result [Jendrej et al. 2022] via configuring radiations appropriately at the blow-up time. These constructions inevitably involve some constraints on regularity and degeneracy of the initial data.

The latter case adopts a method that accurately describes the initial data set that drives blowup. Although it is difficult (probably impossible) to form a family of blow-up rates as in the previous results, the emphasis is on being able to observe the construction of blow-up solutions with smooth initial data. Especially in [Raphaël and Rodnianski 2012], the authors explicitly describe an initial data set that is open under H^2 topology around Q and prove the so-called stable blowup, in which the solutions starting from that set blow up with a universal rate that slightly misses the self-similar one for all $k \geq 1$.

We note that the initial data set in the above result does not imply a universal blowup of all well-localized smooth data. Our main theorem says that there exist other smooth solutions that blow up in finite time with quantized rates corresponding to the excited regime.

1.5. ***Main theorem.*** We focus on the solution to (1-7) with 1-corotational initial data, i.e., $k = 1$. Let us restate the stable blowup result.

Theorem 1.1 (stable blowup for 1-corotational wave maps [Kim 2023; Raphaël and Rodnianski 2012]). *There exists a constant $\varepsilon_0 > 0$ such that, for all 1-corotational initial data $(u_0, \dot{u}_0)$ with*

$$\|u_0 - Q, \dot{u}_0\|_{\mathcal{H}^2} < \varepsilon_0, \tag{1-12}$$

the corresponding solutions to (1-7) blow up in finite time $0 < T = T(u_0, \dot{u}_0) < \infty$ as follows: for some $(u^*, \dot{u}^*) \in \mathcal{H}$,

$$\left\| u(t,r) - Q\left(\frac{r}{\lambda(t)}\right) - u^*, \partial_t u(t,r) - \dot{u}^* \right\|_{\mathcal{H}} \to 0 \quad \text{as } t \to T \tag{1-13}$$

with the universal blow-up speed

$$\lambda(t) = 2e^{-1}(1 + o_{t\to T}(1))(T-t)e^{-\sqrt{|\log(T-t)|}}. \tag{1-14}$$

Here, $\mathcal{H}$ and $\mathcal{H}^2$ are given by (1-24) and (1-25), respectively.

Remark (1-corotational symmetry). Raphaël and Rodnianski [2012] mentioned that the nature of the harmonic map, which varies depending on whether k is equal to 1 or not, leads to distinctive blow-up rates. As a result of the logarithmic calculation that occurs additionally only when $k = 1$, the universality of the blow-up rate in this case was unclear. The sharp constant $2e^{-1}$ in (1-14) was later obtained by Kim [2023] using a refined modulation analysis.

Nevertheless, the slow decaying nature of the harmonic map is rather an advantage in our analysis, which allows us to exhibit the following smooth blowup with the quantized blow-up rates corresponding to the excited regime.

Theorem 1.2 (quantized blowup for 1-corotational wave map). *For a natural number $\ell \geq 2$ and an arbitrarily small constant $\varepsilon_0 > 0$, there exists a smooth 1-corotational initial data $(u_0, \dot{u}_0)$ with*

$$\|u_0 - Q, \dot{u}_0\|_{\mathcal{H}} < \varepsilon_0 \tag{1-15}$$

such that the corresponding solution to (1-7) blows up in finite time $0 < T = T(u_0, \dot{u}_0) < \infty$ and satisfies (1-13), with the quantized blow-up speed

$$\lambda(t) = c(u_0, \dot{u}_0)(1 + o_{t\to T}(1))\frac{(T-t)^\ell}{|\log(T-t)|^{\ell/(\ell-1)}}, \quad c(u_0, \dot{u}_0) > 0. \tag{1-16}$$

Remark (further regularity of asymptotic profile). The asymptotic profile $(u^*, \dot{u}^*)$ also has $\dot{H}^\ell \times \dot{H}^{\ell-1}$ regularity in the sense that certain ℓ-fold (resp. $(\ell-1)$-fold) derivatives of u^* (resp. $\dot{u}^*$) belong to L^2. This is a consequence of the fact that the ℓ-th-order energy of the radiative part of the solution satisfies the scaling invariance bound ($\mathcal{E}_\ell \leq C\lambda^{2(\ell-1)}$; see (4-13)) similar to [Raphaël and Schweyer 2014].

Remark (quantized blowup). The existence of (type-II) blow-up solutions with quantized blow-up rates has also been well studied in parabolic equations, especially for nonlinear heat equations. Starting with the discovery of formal mechanisms [Filippas et al. 2000; Herrero and Velázquez 1992; 1994], there are classification works [Mizoguchi 2007; 2011] in the energy-supercritical regime. The proofs in this literature are based on the maximum principle (see [Matano and Merle 2004; 2009]).

Through modulation analysis, not relying on maximum principle, there have been some (type-II) quantized rate constructions in the critical parabolic equations such as [Hadžić and Raphaël 2019; Raphaël and Schweyer 2014] for the energy-critical case and [Collot et al. 2022] for the mass-critical case. See also [del Pino et al. 2020; Harada 2020], which rely on the inner-outer gluing method. Raphaël and Schweyer [2014] expected that their modulation technique could be robust enough to be propagated to dispersive models including the wave map problem, and the quantized rate constructions have been established in the energy-supercritical dispersive equations [Collot 2018; Ghoul et al. 2018; Merle et al. 2015]. To our knowledge, Theorem 1.2 provides the first rigorous quantized rate constructions for critical dispersive equations. We expect that our analysis can also be extended to other energy-critical dispersive equations such as the nonlinear wave equation.

Remark (instability of blowup). In contrast to Theorem 1.1, our initial data set is of codimension $(\ell-1)$, similar to [Raphaël and Schweyer 2014], due to unstable directions inherent in the ODE system driving the blow-up dynamics. This similarity follows from the fact that the wave map problem and the harmonic map heat flow share the same ground states and linearized Hamiltonian under the 1-corotational symmetry. We also expect the stability formulated by constructing a smooth manifold of initial data leading to our quantized blow-up scenario.

1.6. *Notation.* We introduce some notation needed for the proof before going into the strategy of the proof. We first use the bold notation for vectors in $\mathbb{R}^2$:

$$\boldsymbol{u} := \begin{pmatrix} u \\ \dot{u} \end{pmatrix}, \quad \boldsymbol{u}(r) := \begin{pmatrix} u(r) \\ \dot{u}(r) \end{pmatrix}. \tag{1-17}$$

For $\lambda > 0$, the $\dot{H}^1 \times L^2$ scaling is defined by

$$\boldsymbol{u}_\lambda(r) = \begin{pmatrix} u_\lambda(r) \\ \lambda^{-1}\dot{u}_\lambda(r) \end{pmatrix} := \begin{pmatrix} u(y) \\ \lambda^{-1}\dot{u}(y) \end{pmatrix}, \quad y := \frac{r}{\lambda}, \tag{1-18}$$

and the corresponding generator is denoted by

$$\boldsymbol{\Lambda u} := \begin{pmatrix} \Lambda u \\ \Lambda_0 \dot{u} \end{pmatrix} := -\frac{d\boldsymbol{u}_\lambda(r)}{d\lambda}\bigg|_{\lambda=1} = \begin{pmatrix} r\partial_r u(r) \\ (1+r\partial_r)\dot{u}(r) \end{pmatrix}. \tag{1-19}$$

In general, we employ the $\dot{H}^k$ scaling generator

$$\Lambda_k u := -\frac{d}{d\lambda}(\lambda^{k-1}u_\lambda(r))\bigg|_{\lambda=1} = (-k+1+r\partial_r)u(r). \tag{1-20}$$

We now reformulate (1-7) using the vector-valued function $\boldsymbol{F} : \mathbb{R}^2 \to \mathbb{R}^2$:

$$\begin{cases} \partial_t \boldsymbol{u} = \boldsymbol{F}(\boldsymbol{u}), \\ \boldsymbol{u}|_{t=0} = \boldsymbol{u}_0, \end{cases} \qquad \boldsymbol{u} = \boldsymbol{u}(t,r), \quad \boldsymbol{F}(\boldsymbol{u}) := \begin{pmatrix} \dot{u} \\ \Delta u - f(u)/r^2 \end{pmatrix}, \tag{1-21}$$

where $\Delta = \partial_{rr} + (1/r)\partial_r$.

We use two subsets of the real line:

$$\mathbb{R}_+ = \{r \in \mathbb{R} : x \geq 0\},$$
$$\mathbb{R}_+^* = \{r \in \mathbb{R} : x > 0\}.$$

We denote by χ a C^∞ radial cut-off function on $\mathbb{R}_+$:

$$\chi(r) = \begin{cases} 1 & \text{for } r \leq 1, \\ 0 & \text{for } r \geq 2. \end{cases}$$

We let $\chi_B(r) := \chi(r/B)$ for $B > 0$. Similarly, we denote by $\mathbf{1}_A(y)$ the indicator function on the set A. In particular, $\mathbf{1}_{B \leq y \leq 2B}$ will be rewritten as $\mathbf{1}_{y \sim B}$, or abusively as simply $\mathbf{1}_B$. The cut-off boundary B will often be chosen as the constant multiples of

$$B_0 := \frac{1}{b_1}, \quad B_1 := \frac{|\log b_1|^\gamma}{b_1}, \qquad b_1 > 0. \tag{1-22}$$

Later, we will choose $\gamma = 1 + \bar{\ell}$, where ℓ appeared in Theorem 1.2. Here, we denote by $\bar{i}$ the remainder of dividing i by 2, i.e., $\bar{i} = i \mod 2$ for an integer i. We also write $L = \ell + \overline{\ell + 1}$, i.e., L is the smallest odd integer greater than or equal to ℓ. We also abuse the indicator notation $\mathbf{1}_{\{l \geq m\}}$:

$$\mathbf{1}_{\{l \geq m\}} = \begin{cases} 1 & \text{if } l \geq m, \\ 0 & \text{if } l < m, \end{cases} \qquad l, m \in \mathbb{Z}.$$

We adopt the following $L^2(\mathbb{R}^2)$ inner product for radial functions u, v:

$$\langle u, v \rangle := \int_0^\infty u(r) v(r) r \, dr,$$

and the $L^2 \times L^2$ inner product for vector-valued functions $\boldsymbol{u}$, $\boldsymbol{v}$:

$$\langle \boldsymbol{u}, \boldsymbol{v} \rangle := \langle u, v \rangle + \langle \dot{u}, \dot{v} \rangle. \tag{1-23}$$

We introduce two Sobolev spaces $\mathcal{H}$ and $\mathcal{H}^2$ with the following norms:

$$\|u, \dot{u}\|_{\mathcal{H}}^2 := \int |\partial_y u|^2 + \frac{|u|^2}{y^2} + |\dot{u}|^2, \tag{1-24}$$

$$\|u, \dot{u}\|_{\mathcal{H}^2}^2 := \|u, \dot{u}\|_{\mathcal{H}}^2 + \int |\partial_y^2 u|^2 + |\partial_y \dot{u}|^2 + \frac{|\dot{u}|^2}{y^2} + \int_{|y| \leq 1} \frac{1}{y^2} \Big(\partial_y u - \frac{u}{y} \Big)^2, \tag{1-25}$$

where the above shorthand for integrals is given by $\int = \int_{\mathbb{R}^2}$.

For any $x := (x_1, \ldots, x_n) \in \mathbb{R}^n$, we set $|x|^2 = x_1^2 + \cdots + x_n^2$ and

$$\mathcal{B}^n := \{x \in \mathbb{R}^n, \ |x| \leq 1\},$$
$$\mathcal{S}^n := \partial \mathcal{B}^n = \{x \in \mathbb{R}^n, \ |x| = 1\}.$$

We use the Kronecker delta notation: $\delta_{ij} = 1$ for $i = j$ and $\delta_{ij} = 0$ for $i \neq j$.

1.7. ***Strategy of the proof.*** Our proof is based on the general modulation analysis scheme developed by Raphaël and Rodnianski [2012], Merle, Raphaël and Rodnianski [Merle et al. 2013] and Raphaël and Schweyer [2014], which also have difficulties arising from the energy-critical nature and the small equivariance index, including logarithmic computations. We closely follow the main strategy of [Raphaël and Schweyer 2014]. However, notable differences stem from the lack of dissipation in the higher-order (H^{L+1}, $L \gg 1$) energy estimates due to the dispersive nature of our problem. We overcome this difficulty by carefully correcting the higher-order energy functional to uncover the repulsive property (to identify terms with good sign), generalizing the computation in the H^2 energy estimates of [Raphaël and Rodnianski 2012].

Given an odd integer $L \geq 3$, we first construct the blow-up profile $\boldsymbol{Q}_b$ of the form

$$\boldsymbol{Q}_b := \boldsymbol{Q} + \boldsymbol{\alpha}_b := \begin{pmatrix} Q \\ 0 \end{pmatrix} + \sum_{i=1}^{L} b_i \boldsymbol{T}_i + \sum_{i=2}^{L+2} \boldsymbol{S}_i, \tag{1-26}$$

where $b = (b_1, \dots, b_L)$ is a set of modulation parameters and $\boldsymbol{T}_i$ and $\boldsymbol{S}_i$ are deformation directions so that $(\boldsymbol{Q}_{b(t)})_{\lambda(t)}$ solves (1-21) approximately. Equivalently, $\boldsymbol{Q}_b$ satisfies

$$\partial_s \boldsymbol{Q}_b - \boldsymbol{F}(\boldsymbol{Q}_b) - \frac{\lambda_s}{\lambda} \boldsymbol{\Lambda} \boldsymbol{Q}_b \approx 0, \qquad \frac{ds}{dt} = \frac{1}{\lambda(t)}. \tag{1-27}$$

From the imposed relations (1-27), the blow-up dynamics are determined by the evolution of the modulation parameters $b = (b_1, \dots, b_L)$. The leading dynamics of b and $\boldsymbol{T}_i$ are determined by considering the linearized flow of (1-27) near $\boldsymbol{Q}$:

$$\begin{aligned} 0 \approx \partial_s \boldsymbol{Q}_b - \boldsymbol{F}(\boldsymbol{Q}_b) - \frac{\lambda_s}{\lambda} \boldsymbol{\Lambda} \boldsymbol{Q}_b &= \partial_s (\boldsymbol{Q}_b - \boldsymbol{Q}) - \boldsymbol{F}(\boldsymbol{Q}_b) + \boldsymbol{F}(\boldsymbol{Q}) - \frac{\lambda_s}{\lambda} \boldsymbol{\Lambda} \boldsymbol{Q}_b \\ &\approx \partial_s \boldsymbol{\alpha}_b + \boldsymbol{H} \boldsymbol{\alpha}_b - \frac{\lambda_s}{\lambda} \boldsymbol{\Lambda} (\boldsymbol{Q} + \boldsymbol{\alpha}_b), \end{aligned} \tag{1-28}$$

where $\boldsymbol{H}$ denotes the linearized Hamiltonian:

$$\boldsymbol{H} := \begin{pmatrix} 0 & -1 \\ H & 0 \end{pmatrix}, \qquad H = -\Delta + \frac{f'(Q)}{y^2}. \tag{1-29}$$

After defining $\boldsymbol{T}_i$ inductively,

$$\boldsymbol{H} \boldsymbol{T}_{i+1} = -\boldsymbol{T}_i, \qquad \boldsymbol{T}_0 := \boldsymbol{\Lambda} \boldsymbol{Q}, \tag{1-30}$$

equation (1-28) and the asymptotics $\boldsymbol{\Lambda} \boldsymbol{T}_i \sim (i-1) \boldsymbol{T}_i$ yield the leading dynamics of b:

$$-\frac{\lambda_s}{\lambda} = b_1, \qquad (b_k)_s = b_{k+1} - (k-1) b_1 b_k, \quad b_{L+1} := 0, \quad 1 \leq k \leq L. \tag{1-31}$$

$\boldsymbol{S}_i$ appears to correct (1-28) to (1-27) containing some radiative terms from the difference $\boldsymbol{\Lambda} \boldsymbol{T}_i - (i-1) \boldsymbol{T}_i$ and the nonlinear effect from $\boldsymbol{F}(\boldsymbol{Q}_b) - \boldsymbol{F}(\boldsymbol{Q}) + \boldsymbol{H} \boldsymbol{\alpha}_b$. Then b drives the ODE system

$$(b_k)_s = b_{k+1} - \left(k - 1 + \frac{1}{(1 + \delta_{1k}) \log s}\right) b_1 b_k, \qquad b_{L+1} := 0, \qquad 1 \leq k \leq L. \tag{1-32}$$

We then choose a special solution of (1-32),

$$b_1(s) \sim \frac{\ell}{\ell-1}\left(\frac{1}{s} - \frac{(\ell-1)^{-1}}{s\log s}\right), \tag{1-33}$$

which leads to (1-16) from the relations

$$-\lambda_t = b_1 \quad \text{and} \quad \frac{ds}{dt} = \frac{1}{\lambda}.$$

Since the special solution we choose is formally codimension $(\ell-1)$ stable, we control the unstable directions in the vicinity of these special solutions to ODE system (1-32) by Brouwer's fixed point theorem.

Now, we decompose the solution $\boldsymbol{u} = \boldsymbol{u}(t,r)$ to (1-21) as

$$\boldsymbol{u} = (\boldsymbol{Q}_{b(t)} + \boldsymbol{\varepsilon})_{\lambda(t)} = (\boldsymbol{Q}_{b(t)})_{\lambda(t)} + \boldsymbol{w}, \quad \langle \boldsymbol{H}^i \boldsymbol{\varepsilon}, \boldsymbol{\Phi}_M\rangle = 0, \quad 0 \le i \le L, \tag{1-34}$$

where $\boldsymbol{\Phi}_M$ is defined in (3-1). The orthogonality conditions in (1-34) uniquely determine the decomposition by the implicit function theorem. Then we derive the evolution equation of $\boldsymbol{\varepsilon}$ from (1-21), which contains the formal modulation ODE (1-32) with some errors in terms of $\boldsymbol{\varepsilon}$.

To justify the formal modulation ODE (1-32), we need sufficient smallness of $\boldsymbol{\varepsilon}$ and we need to propagate it. For this purpose, we consider the higher-order energy associated to the linearized Hamiltonian H:

$$\mathcal{E}_{L+1} = \langle H^{(L+1)/2}\varepsilon, H^{(L+1)/2}\varepsilon\rangle + \langle H H^{(L-1)/2}\dot{\varepsilon}, H^{(L-1)/2}\dot{\varepsilon}\rangle. \tag{1-35}$$

This energy is coercive thanks to the orthogonality conditions in (1-34).

Thus, our analysis boils down to estimating the time derivative of $\mathcal{E}_{L+1}$. Unlike in [Raphaël and Schweyer 2014], we cannot employ dissipation to control the time derivative of $\mathcal{E}_{L+1}$ due to the dispersive nature of our problem. Instead, we use the repulsive property of the (supersymmetric) conjugated Hamiltonian $\widetilde{H}$ of H observed in [Raphaël and Rodnianski 2012; Rodnianski and Sterbenz 2010]. To illuminate the repulsive property in the energy estimate, we consider the linearized flow in terms of w from $\boldsymbol{w} = (w, \dot{w})$ and the well-known factorization:

$$w_{tt} + H_\lambda w = 0, \quad H_\lambda = A_\lambda^* A_\lambda, \quad A_\lambda = -\partial_r + \frac{\cos Q_\lambda}{r}.$$

Defining the higher-order derivatives adapted to H_λ and its corresponding operator

$$w_k := \mathcal{A}_\lambda^k w, \quad \mathcal{A}_\lambda = A_\lambda, \quad \mathcal{A}_\lambda^2 = A_\lambda^* A_\lambda, \quad \dots, \quad \mathcal{A}_\lambda^k = \underbrace{\cdots A_\lambda^* A_\lambda A_\lambda^* A_\lambda}_{k \text{ times}},$$

the higher-order energy (1-35) can essentially be written as

$$\begin{aligned}\mathcal{E}_{L+1} &\approx \lambda^{2L}(\langle w_{L+1}, w_{L+1}\rangle + \langle \partial_t w_L, \partial_t w_L\rangle) \\ &= \lambda^{2L}(\langle \widetilde{H}_\lambda w_L, w_L\rangle + \langle \partial_t w_L, \partial_t w_L\rangle)\end{aligned}$$

where $\widetilde{H}_\lambda = A_\lambda A_\lambda^*$ is the conjugated Hamiltonian of H_λ. As an advantage of the adoption of the Leibniz rule notation between an operator and a function,

$$\partial_t(Pf) = \partial_t(P)f + Pf_t, \quad \partial_t(P) := [\partial_t, P],$$

we can express the energy estimate for $\mathcal{E}_{L+1}$ succinctly:

$$\begin{aligned}\frac{d}{dt}\left\{\frac{\mathcal{E}_{L+1}}{2\lambda^{2L}}\right\} &\approx \frac{1}{2}\langle\partial_t(\widetilde{H}_\lambda)w_L, w_L\rangle + \langle\widetilde{H}_\lambda w_L, \partial_t w_L\rangle + \langle\partial_{tt} w_L, \partial_t w_L\rangle \\ &\approx \frac{1}{2}\langle\partial_t(\widetilde{H}_\lambda)w_L, w_L\rangle + 2\langle\partial_t w_L, \partial_t(\mathcal{A}_\lambda^L)w_t\rangle.\end{aligned}$$

Integrating the second term by parts in time, we get

$$\frac{d}{dt}\left\{\frac{\mathcal{E}_{L+1}}{2\lambda^{2L}} - 2\langle w_L, \partial_t(\mathcal{A}_\lambda^L)w_t\rangle\right\} \approx \frac{1}{2}\langle\partial_t(\widetilde{H}_\lambda)w_L, w_L\rangle + 2\langle w_L, \partial_t(\mathcal{A}_\lambda^L)w_2\rangle.$$

Raphaël and Rodnianski [2012] exhibited the repulsive property by directly calculating the following identity with the advantage of $L = 1$:

$$\langle w_1, \partial_t(\mathcal{A}_\lambda)w_2\rangle = \tfrac{1}{2}\langle\partial_t(\widetilde{H}_\lambda)w_1, w_1\rangle \le 0.$$

However, this computation does not seem to directly extend to our case $L \ge 3$. We overcome this problem by first writing $\mathcal{A}_\lambda^L = \widetilde{H}_\lambda\mathcal{A}_\lambda^{L-2}$ and pulling out the repulsive term using the Leibniz rule:

$$\begin{aligned}\langle w_L, \partial_t(\mathcal{A}_\lambda^L)w_2\rangle &= \langle w_L, \partial_t(\widetilde{H}_\lambda)w_L\rangle + \langle\widetilde{H}_\lambda w_L, \partial_t(\mathcal{A}_\lambda^{L-2})w_2\rangle \\ &\approx \langle w_L, \partial_t(\widetilde{H}_\lambda)w_L\rangle - \langle\partial_{tt} w_L, \partial_t(\mathcal{A}_\lambda^{L-2})w_2\rangle.\end{aligned}$$

Again integrating by parts in time, we obtain

$$\begin{aligned}\frac{d}{dt}\left\{\frac{\mathcal{E}_{L+1}}{2\lambda^{2L}} - 2\big(\langle w_L, \partial_t(\mathcal{A}_\lambda^L)w_t\rangle - \langle\partial_t w_L, \partial_t(\mathcal{A}_\lambda^{L-2})w_2\rangle + \langle w_L, \partial_t(\mathcal{A}_\lambda^{L-2})\partial_t w_2\rangle\big)\right\} \\ \approx \frac{5}{2}\langle\partial_t(\widetilde{H}_\lambda)w_L, w_L\rangle + 2\langle w_L, \partial_t(\mathcal{A}_\lambda^{L-2})w_4\rangle.\end{aligned}$$

Repeating the above correction procedure, we arrive at the term with good sign:

$$\begin{aligned}\frac{d}{dt}\left\{\frac{\mathcal{E}_{L+1}}{2\lambda^{2L}} + \text{corrections}\right\} &\approx \frac{2L-1}{2}\langle\partial_t(\widetilde{H}_\lambda)w_L, w_L\rangle + 2\langle w_L, \partial_t(\mathcal{A}_\lambda)w_{L+1}\rangle \\ &\approx \frac{2L+1}{2}\langle\partial_t(\widetilde{H}_\lambda)w_L, w_L\rangle \le 0.\end{aligned}$$

In the actual energy estimate, there are also error terms such as the profile equation error and nonlinear terms in $\boldsymbol{\varepsilon}$. For these nonlinear terms, we also estimate the intermediate energies $\mathcal{E}_k$, which can be defined similarly to $\mathcal{E}_{L+1}$.

Organization of the paper. In Section 2, we construct the approximate blow-up profile with the description of the ODE dynamics of the modulation equations. Section 3 is devoted to the decomposition of the solution into the blow-up profile constructed in the previous section and the remaining error. We also introduce the bootstrap setting to control the error and establish a Lyapunov-type monotonicity for the higher-order energy with respect to such error. Section 4 provides the proof of Theorem 1.2 by closing the bootstrap with some standard topological arguments.

2. Construction of the approximate solution

In this section, we construct the approximate blow-up profile $\boldsymbol{Q}_b$, represented by a deformation of the harmonic map $\boldsymbol{Q}$ through modulation parameters $b = (b_1, \dots, b_L)$. We also derive formal dynamical laws of b, which leads to our desired blow-up rate.

2.1. ***The linearized dynamics.*** It is natural to look into the linearized dynamics of our system near the stationary solution Q. Let $\boldsymbol{u} = \boldsymbol{Q} + \boldsymbol{\varepsilon}$, where $\boldsymbol{Q} = (Q, 0)^t$ and $\boldsymbol{u}$ is the solution to (1-21). Then $\boldsymbol{\varepsilon}$ satisfies

$$\partial_t \boldsymbol{\varepsilon} = \boldsymbol{F}(\boldsymbol{Q}+\boldsymbol{\varepsilon}) - \boldsymbol{F}(\boldsymbol{Q}) = \begin{pmatrix} \dot{\varepsilon} \\ \Delta\varepsilon - (f(Q+\varepsilon) - f(Q))/r^2 \end{pmatrix}$$
$$= \begin{pmatrix} \dot{\varepsilon} \\ \Delta\varepsilon - r^{-2} f'(Q)\varepsilon \end{pmatrix} - \frac{1}{r^2} \begin{pmatrix} 0 \\ f(Q+\varepsilon) - f(Q) - f'(Q)\varepsilon \end{pmatrix}.$$

Ignoring higher-order terms for $\boldsymbol{\varepsilon}$ and setting $\lambda = 1$ (i.e., $r = y$), we roughly obtain the linearized system

$$\partial_t \boldsymbol{\varepsilon} + \boldsymbol{H}\boldsymbol{\varepsilon} = 0, \quad \boldsymbol{H}\boldsymbol{\varepsilon} = \begin{pmatrix} 0 & -1 \\ H & 0 \end{pmatrix} \begin{pmatrix} \varepsilon \\ \dot{\varepsilon} \end{pmatrix}, \tag{2-1}$$

where H is the Schrödinger operator with explicitly computable potential $f'(Q)$ from (1-7) and (1-11),

$$H := -\Delta + \frac{V}{y^2}, \quad V = f'(Q) = \frac{y^4 - 6y^2 + 1}{(y^2+1)^2}. \tag{2-2}$$

Due to the scaling invariance, we have $H\Lambda Q = 0$, where

$$\Lambda Q = \frac{2y}{1+y^2}. \tag{2-3}$$

However, ΛQ slightly fails to belong to $L^2(\mathbb{R}^2)$, so we call ΛQ the *resonance* of H. The positivity of ΛQ on $\mathbb{R}^*_+$ allows us to factorize H:

$$H = A^* A, \quad A = -\partial_y + \frac{Z}{y}, \quad A^* = \partial_y + \frac{1+Z}{y}, \quad Z(y) = \cos Q = \frac{1-y^2}{1+y^2}. \tag{2-4}$$

The above factorization facilitates examining the formal kernel of H on $\mathbb{R}^*_+$, denoted by $\operatorname{Ker}(H)$. More precisely, the equivalent form

$$Au = -\partial_y u + \partial_y(\log \Lambda Q) u = -\Lambda Q \partial_y \Big(\frac{u}{\Lambda Q}\Big), \tag{2-5}$$

$$A^* u = \frac{1}{y}\partial_y(yu) + \partial_y(\log\Lambda Q)u = \frac{1}{y\Lambda Q}\partial_y(uy\Lambda Q) \tag{2-6}$$

yields, for $y > 0$, $\operatorname{Ker}(H) = \operatorname{Span}(\Lambda Q, \Gamma)$, where

$$\Gamma(y) = \Lambda Q \int_1^y \frac{dx}{x(\Lambda Q(x))^2} = \begin{cases} O(1/y) & \text{as } y \to 0, \\ y/4 + O(\log y/y) & \text{as } y \to \infty. \end{cases} \tag{2-7}$$

From variation of parameters, we obtain the formal inverse of H:

$$H^{-1} f = \Lambda Q \int_0^y f\Gamma x\, dx - \Gamma \int_0^y f \Lambda Q x\, dx, \tag{2-8}$$

so the inverse of $\boldsymbol{H}$ is given by

$$\boldsymbol{H}^{-1} := \begin{pmatrix} 0 & H^{-1} \\ -1 & 0 \end{pmatrix}.$$

We remark that the inverse formula (2-8) is uniquely determined by the boundary condition at the origin: for any smooth function f with $f = O(1)$, we have $H^{-1} f = O(y^2)$ near the origin.

On the other hand, the supersymmetric conjugate operator $\widetilde{H}$ is given by

$$\widetilde{H} := AA^* = -\Delta + \frac{\widetilde{V}}{y^2}, \quad \widetilde{V}(y) = (1+Z)^2 - \Lambda Z = \frac{4}{y^2+1}. \tag{2-9}$$

We note that $\widetilde{H}$ has a repulsive property represented by its potential

$$\widetilde{V} = \frac{4}{y^2+1} > 0, \quad \Lambda \widetilde{V} = -\frac{8y^2}{(y^2+1)^2} \leq 0. \tag{2-10}$$

Based on the commutation relation

$$AH = \widetilde{H} A,$$

we can naturally define higher-order derivatives adapted to the linearized Hamiltonian H inductively:

$$f_0 := f, \quad f_{k+1} := \begin{cases} Af_k & \text{for } k \text{ even}, \\ A^* f_k & \text{for } k \text{ odd}. \end{cases} \tag{2-11}$$

For the sake of simplicity, we denote the corresponding operator as follows:

$$\mathcal{A} := A, \quad \mathcal{A}^2 := A^*A, \quad \mathcal{A}^3 := AA^*A, \quad \dots, \quad \mathcal{A}^k := \underbrace{\cdots A^*AA^*A}_{k \text{ times}}. \tag{2-12}$$

We observe that f needs an odd parity condition near the origin to define f_k. More precisely, for any smooth function f, (2-5) implies

$$f_1 = Af \sim -y\partial_y(y^{-1} f) \tag{2-13}$$

near $y = 0$. Thus, f must degenerate near the origin as $f = cy + O(y^2)$, and so $Af = c'y + O(y^2)$. Here, the leading term $c'y$ comes from the cancellation

$$Ay = O(y^2), \tag{2-14}$$

which is a direct consequence of (2-13). However, f_2 does not degenerate near the origin like f, since A^* does not have any cancellation like (2-14). Hence, f should be more degenerate near the origin as $f = cy + c'y^3 + O(y^4)$. Furthermore, if f_k is to be well-defined for all $k \in \mathbb{N}$, f must satisfy the following condition: for all $p \in \mathbb{N}$, f has a Taylor expansion near the origin:

$$f(y) = \sum_{k=0}^{p} c_k y^{2k+1} + O(y^{2p+3}). \tag{2-15}$$

In Appendix A of [Raphaël and Schweyer 2014], it is proved that, for a well-localized smooth 1-corotational map $\Phi(r,\theta)$, the corresponding u is a smooth function that satisfies (2-15).

2.2. ***Admissible functions.*** As mentioned earlier, the leading dynamics of the blow-up are determined by the leading growth of tails from the blow-up profile. In the same way as [Collot 2018; Raphaël and Schweyer 2014], we first define an "admissible" vector-valued function characterized by three different indices, which represent a certain behavior near the origin and infinity, and the position of a nonzero coordinate.

Definition 2.1 (admissible functions). We say that a smooth vector-valued function $\boldsymbol{f}:\mathbb{R}_+\to\mathbb{R}^2$ is admissible of degree $(p_1, p_2, \iota)\in\mathbb{N}\times\mathbb{Z}\times\{0,1\}$ if it satisfies the following:

(i) $\boldsymbol{f}$ is situated on the $(\iota+1)$-th coordinate, i.e.,

$$\boldsymbol{f}=\begin{pmatrix} f\\ 0\end{pmatrix}\ \text{ if } \iota=0 \quad\text{and}\quad \boldsymbol{f}=\begin{pmatrix} 0\\ f\end{pmatrix}\ \text{ if } \iota=1. \tag{2-16}$$

In such cases, we use f and $\boldsymbol{f}$ interchangeably.

(ii) We can expand f near $y=0$: for all $2p\geq p_1$,

$$f(y)=\sum_{\substack{k=p_1-\iota\\ k \text{ is even}}}^{2p} c_k y^{k+1}+O(y^{2p+3}), \tag{2-17}$$

and similar expansions hold after taking derivatives.

(iii) The adapted derivatives f_k have the following bounds: for all $k\geq 0$ and $y\geq 1$,

$$|f_k(y)|\lesssim y^{p_2-1-\iota-k}(1+|\log y|\mathbf{1}_{p_2-k-\iota\geq 1}). \tag{2-18}$$

Remark. The logarithmic term in (2-18) comes from integrating y^{-1}.

From (2-3), we can easily check that $\boldsymbol{\Lambda Q}=(\Lambda Q,0)^t$ is admissible of degree $(0,0,0)$. The next lemma says that admissible functions are designed to be compatible with the linearized operator $\boldsymbol{H}$.

Lemma 2.2 (action of $\boldsymbol{H}$ and $\boldsymbol{H}^{-1}$ on admissible functions). *Let $\boldsymbol{f}$ be an admissible function of degree (p_1, p_2, ι). Recall $\overline{i}=i \mod 2$. Then,*

(i) *for all $k\in\mathbb{N}$, $\boldsymbol{H}^k\boldsymbol{f}$ is admissible of degree*

$$(\max(p_1-k,\iota),\ p_2-k,\ \overline{\iota+k}), \tag{2-19}$$

(ii) *for all $k\in\mathbb{N}$ and $p_2\geq\iota$, $\boldsymbol{H}^{-k}\boldsymbol{f}$ is admissible of degree*

$$(p_1+k,\ p_2+k,\ \overline{\iota+k}). \tag{2-20}$$

Proof. (i) This claim directly comes from the facts

$$\boldsymbol{H}=\begin{pmatrix} 0 & -1\\ H & 0\end{pmatrix},\qquad \boldsymbol{H}^2=\begin{pmatrix} -H & 0\\ 0 & -H\end{pmatrix}.$$

More precisely, the maximum choice, $\max(p_1-k,\iota)$, appears from the cancellation (2-14) near the origin. Near infinity, the degree condition p_2-k is a consequence of the simple relation $Hf=f_2$.

(ii) It suffices to calculate the case $k=1$ by induction. For $\iota=0$,

$$\boldsymbol{H}^{-1}\boldsymbol{f}=\begin{pmatrix}0 & H^{-1}\\ -1 & 0\end{pmatrix}\begin{pmatrix}f\\ 0\end{pmatrix}=\begin{pmatrix}0\\ -f\end{pmatrix},$$

and $\boldsymbol{H}^{-1}\boldsymbol{f}$ is admissible of degree $(p_1+1,\, p_2+1,\, 1)$. For $\iota=1$, we have

$$\boldsymbol{H}^{-1}\boldsymbol{f}=\begin{pmatrix}0 & H^{-1}\\ -1 & 0\end{pmatrix}\begin{pmatrix}0\\ f\end{pmatrix}=\begin{pmatrix}H^{-1}f\\ 0\end{pmatrix}.$$

Instead of using the formal inverse formula (2-8) directly, we utilize the relation (2-6) as

$$AH^{-1}f=\frac{1}{y\Lambda Q}\int_0^y f\Lambda Qx\,dx, \tag{2-21}$$

and the relation (2-5) as

$$H^{-1}f=-\Lambda Q\int_0^y \frac{AH^{-1}f}{\Lambda Q}\,dx. \tag{2-22}$$

Near the origin, (2-21) gives the following expansion for $AH^{-1}f$:

$$AH^{-1}f=\sum_{\substack{k=p_1-1\\ k\text{ is even}}}^{2p}\tilde{c}_k y^{k+2}+O(y^{2p+4}), \tag{2-23}$$

and thus $H^{-1}f$ satisfies the Taylor expansion

$$H^{-1}f=\sum_{\substack{k=p_1-1\\ k\text{ is even}}}^{2p}\tilde{c}_k y^{k+3}+O(y^{2p+5})=\sum_{\substack{k=p_1+1-0\\ k\text{ is even}}}^{2p}\tilde{c}_k y^{k+1}+O(y^{2p+3}). \tag{2-24}$$

For $y\ge 1$, (2-21) and (2-22) imply

$$\begin{aligned}|AH^{-1}f|\lesssim\int_0^y |f|\,dx&\lesssim\int_1^y x^{p_2-2}(1+|\log x|\mathbf{1}_{p_2\ge 2})\,dx\\ &\lesssim y^{(p_2+1)-1-0-1}(1+|\log y|\mathbf{1}_{p_2\ge 1}),\end{aligned} \tag{2-25}$$

$$\begin{aligned}|H^{-1}f|\lesssim\frac{1}{y}\int_0^y |xAH^{-1}f|\,dx&\lesssim\frac{1}{y}\int_1^y x^{p_2}(1+|\log x|\mathbf{1}_{p_2\ge 1})\,dx\\ &\lesssim y^{(p_2+1)-0-1}(1+|\log y|\mathbf{1}_{p_2\ge 0}),\end{aligned} \tag{2-26}$$

and we obtain (2-18) for f and f_1. The higher derivative results come from $H(H^{-1}f)=f$. Hence $\boldsymbol{H}^{-1}\boldsymbol{f}$ is admissible of degree $(p_1+1,\, p_2+1,\, 0)$. □

Lemma 2.2 yields the presence of the admissible functions which generate the generalized null space of $\boldsymbol{H}$, which we now define formally.

Definition 2.3 (generalized kernel of $\boldsymbol{H}$). For each $i\ge 0$, we define an admissible function $\boldsymbol{T}_i$ of degree $(i, i, \bar{i})$ as

$$\boldsymbol{T}_i := (-\boldsymbol{H})^{-i}\boldsymbol{\Lambda Q}. \tag{2-27}$$

Remark. By the definition of the admissible functions, we will use the notation T_i as a scalar function.

2.3. b_1-admissible functions. We will keep track of the logarithmic weight $|\log b_1|$ from the blow-up profiles to be constructed later. In this sense, the logarithmic loss of $\boldsymbol{T}_i$ hinders our analysis, so we settle this problem by introducing a new class of functions.

Definition 2.4 (b_1-admissible functions). We say that a smooth vector-valued function $\boldsymbol{f}:\mathbb{R}_+^*\times\mathbb{R}_+\to\mathbb{R}^2$ is b_1-admissible of degree $(p_1, p_2, \iota)\in\mathbb{N}\times\mathbb{Z}\times\{0,1\}$ if it satisfies the following:

(i) $\boldsymbol{f}$ is situated on the $(\iota+1)$-th coordinate (so we use f and $\boldsymbol{f}$ interchangeably).

(ii) $f=f(b_1,y)$ can be expressed as a finite sum of smooth functions of the form $h(b_1)\tilde{f}(y)$, where $\tilde{f}(y)$ has a Taylor expansion (2-17) and $h(b_1)$ satisfies,

$$\text{for all } l\ge 0,\qquad \left|\frac{\partial^l h_j}{\partial b_1^l}\right|\lesssim\frac{1}{b_1^l},\quad b_1>0. \tag{2-28}$$

(iii) f and its adapted derivatives f_k given by (2-11) have the following bounds: there exists a constant $c_{p_2}>0$ such that, for all $k\ge 0$ and $y\ge 1$,

$$|f_k(b_1,y)|\lesssim y^{p_2-k-1-\iota}\left(g_{p_2-k-\iota}(b_1,y)+\frac{|\log y|^{c_{p_2}}}{y^2}+\frac{\mathbf{1}_{\{p_2\ge k+3+\iota,\,y\ge 3B_0\}}}{y^2b_1^2|\log b_1|}\right), \tag{2-29}$$

and, for all $l\ge 1$,

$$\left|\frac{\partial^l}{\partial b_1^l}f_k(b_1,y)\right|\lesssim\frac{y^{p_2-k-1-\iota}}{b_1^l|\log b_1|}\left(\tilde{g}_{p_2-k-\iota}(b_1,y)+\frac{|\log y|^{c_{p_2}}}{y^2}+\frac{\mathbf{1}_{\{p_2\ge k+3+\iota,\,y\ge 3B_0\}}}{y^2b_1^2}\right), \tag{2-30}$$

where B_0 is given by (1-22) and g_l, $\tilde{g}_l$ are defined as

$$g_l(b_1,y)=\frac{1+|\log(b_1y)|\mathbf{1}_{\{l\ge1\}}}{|\log b_1|}\mathbf{1}_{y\le 3B_0},\qquad \tilde{g}_l(b_1,y)=\frac{1+|\log y|\mathbf{1}_{\{l\ge1\}}}{|\log b_1|}\mathbf{1}_{y\le 3B_0}. \tag{2-31}$$

Remark. One may think that the asymptotics (2-29) and (2-30) are quite artificial, however, the functions $g_\ell(b_1,y)$ and $\tilde{g}_\ell(b_1,y)$ will appear in the cancellation by the radiation in Lemma 2.6. Then the indicator part $\mathbf{1}_{p_2\ge k+3+\iota,\,y\ge 3B_0}$ comes from integrating g_ℓ in the region $1\le y\le 3B_0$ to take $\boldsymbol{H}^{-1}$, which can be seen in more detail in the proof of the following lemma.

Lemma 2.5 (action of $\boldsymbol{H}$ and $\boldsymbol{H}^{-1}$ on b_1-admissible functions). *Let $\boldsymbol{f}$ be a b_1-admissible function of degree (p_1,p_2,ι). Then,*

(i) *for all $k\in\mathbb{N}$, $\boldsymbol{H}^k\boldsymbol{f}$ is b_1-admissible of degree*

$$(\max(p_1-k,\iota),\,p_2-k,\,\overline{\iota+k}), \tag{2-32}$$

(ii) *for all $k\in\mathbb{N}$ and $p_2\ge\iota$, $\boldsymbol{H}^{-k}\boldsymbol{f}$ is b_1-admissible of degree*

$$(p_1+k,\,p_2+k,\,\overline{\iota+k}), \tag{2-33}$$

(iii) *the operators*

$$\boldsymbol{\Lambda}:\boldsymbol{f}\mapsto\boldsymbol{\Lambda f}\quad\textit{and}\quad b_1\frac{\partial}{\partial b_1}:\boldsymbol{f}\mapsto b_1\frac{\partial\boldsymbol{f}}{\partial b_1}$$

preserve the degree.

Proof. (i) We can borrow the proof of Lemma 2.2 since b_1 is independent of H.

(ii) Similar to the proof of Lemma 2.2, it suffices to consider the case $\iota = 1$ and $k = 1$. Near the origin, we still use (2-23) and (2-24) for $\tilde{f}$ from $h(b_1)\tilde{f}(y)$ in Definition 2.4.

However, for $y \geq 1$, we need a subtle calculation to integrate the terms containing g_l and $\tilde{g}_l$, defined in (2-31). More precisely, (2-25) implies, for $1 \leq y \leq 3B_0$,

$$\begin{aligned}
|AH^{-1}f| &\lesssim \int_1^y x^{p_2-2} g_{p_2-1}(b_1, x) + x^{p_2-4}|\log x|^{c_{p_2}}\,dx \\
&\lesssim \int_1^y x^{p_2-2}\frac{1+|\log(b_1x)|\mathbf{1}_{\{p_2\geq 2\}}}{|\log b_1|}\,dx + y^{p_2-3}|\log y|^{1+c_{p_2}} \\
&\lesssim \frac{1}{b_1^{p_2-1}|\log b_1|}\int_0^{b_1y} x^{p_2-2}(1+|\log x|\mathbf{1}_{\{p_2\geq 2\}})\,dx + y^{p_2-3}|\log y|^{1+c_{p_2}} \\
&\lesssim y^{p_2-1}\frac{1+|\log(b_1y)|\mathbf{1}_{\{p_2\geq 1\}}}{|\log b_1|} + y^{p_2-3}|\log y|^{1+c_{p_2}} \\
&= y^{(p_2+1)-1-1-0}\left(g_{(p_2+1)-1}(b_1, y) + \frac{|\log y|^{1+c_{p_2}}}{y^2}\right),
\end{aligned} \tag{2-34}$$

and, for $y \geq 3B_0$,

$$\begin{aligned}
|AH^{-1}f| &\lesssim \int_1^y x^{p_2-2} g_{p_2-1}(b_1, x) + x^{p_2-4}|\log x|^{c_{p_2}} + \frac{x^{p_2-4}\mathbf{1}_{\{p_2\geq 4, x\geq 3B_0\}}}{b_1^2|\log b_1|}\,dx \\
&\lesssim \frac{1}{b_1^{p_2-1}|\log b_1|} + \frac{y^{p_2-3}\mathbf{1}_{\{p_2\geq 4\}}}{b_1^2|\log b_1|} + y^{p_2-3}|\log y|^{1+c_{p_2}} \\
&\lesssim y^{(p_2+1)-1-1-0}\left(\frac{\mathbf{1}_{\{p_2\geq 1+3, y\geq 3B_0\}}}{y^2 b_1^2|\log b_1|} + \frac{|\log y|^{1+c_{p_2}}}{y^2}\right).
\end{aligned} \tag{2-35}$$

Once again, (2-26) and (2-34) yield, for $1 \leq y \leq 3B_0$,

$$\begin{aligned}
|H^{-1}f| &\lesssim \frac{1}{y}\int_1^y x^{p_2} g_{p_2}(b_1, x) + x^{p_2-3}|\log x|^{1+c_{p_2}}\,dx \\
&= y^{(p_2+1)-1-0}\left(g_{p_2+1}(b_1, y) + \frac{|\log y|^{2+c_{p_2}}}{y^2}\right),
\end{aligned}$$

and (2-35) implies, for $y \geq 3B_0$,

$$\begin{aligned}
|H^{-1}f| &\lesssim \frac{1}{y}\int_1^y x^{p_2-2}|\log x|^{1+c_{p_2}} + \frac{x^{p_2-2}\mathbf{1}_{\{p_2\geq 4, x\geq 3B_0\}}}{b_1^2|\log b_1|}\,dx \\
&\lesssim y^{(p_2+1)-1-0}\left(\frac{\mathbf{1}_{\{p_2\geq 3, y\geq 3B_0\}}}{y^2 b_1^2|\log b_1|} + \frac{|\log y|^{2+c_{p_2}}}{y^2}\right),
\end{aligned}$$

and we obtain (2-29) for f and f_1. The higher derivative results come from $H(H^{-1}f) = f$. We can easily prove (2-30) by replacing g_l with $\tilde{g}_l$ and dividing by $b_1^l|\log b_1|$. Hence $\boldsymbol{H^{-1}f}$ is b_1-admissible of degree $(p_1+1, p_2+1, 0)$.

(iii) Note that

$$\boldsymbol{\Lambda} \boldsymbol{f} = \begin{cases} (\Lambda f, 0)^t & \text{if } \iota = 0, \\ (0, \Lambda_0 f)^t & \text{if } \iota = 1, \end{cases}$$

and $\Lambda_0 f = f + \Lambda f$; therefore we get the desired result since Λ preserves the parity of f and its adapted derivative satisfies the bound

$$|(\Lambda f)_k| \lesssim |y f_{k+1}| + |f_k| + y^{p_2 - k - 3 - \iota}, \quad y \geq 1,$$

which was established in [Raphaël and Schweyer 2014].

Near the origin, the property of the operator $b_1(\partial/\partial b_1)$ comes from the fact that $b_1(\partial/\partial b_1)$ preserves the parity of f. For $y \geq 1$, (2-30) multiplied by b_1 with $l = 1$ is bounded by (2-29) with the bound

$$\frac{\tilde{g}_l(b_1, y)}{|\log b_1|} \lesssim g_l(b_1, y). \qquad \square$$

2.4. *Control of the extra growth.* The elements of the null space of $\boldsymbol{H}$, which was defined in (2-27), serve as a kind of tail in our blow-up profile. Since we basically plan a bubbling off of the blowup by scaling, the situation where the scaling generator $\boldsymbol{\Lambda}$ is taken by the tails $\boldsymbol{T}_i$ naturally emerges. Especially for $i \geq 2$, the leading asymptotics of $\boldsymbol{\Lambda T}_i$ matches that of $(i-1)\boldsymbol{T}_i$ and determines the leading dynamical laws. However, the extra growth of $\boldsymbol{\Lambda T}_i - (i-1)\boldsymbol{T}_i$ is inadequate to close our analysis. We will eliminate it by adding some radiations, which were first introduced in [Merle et al. 2013].

We now define the radiation situated on the first coordinate as follows: for small $b_1 > 0$,

$$\boldsymbol{\Sigma}_{b_1} = \begin{pmatrix} \Sigma_{b_1} \\ 0 \end{pmatrix}, \quad \Sigma_{b_1} = H^{-1}\{-c_{b_1} \chi_{B_0/4} \Lambda Q + d_{b_1} H[(1 - \chi_{B_0}) \Lambda Q]\}, \tag{2-36}$$

where

$$c_{b_1} = \frac{4}{\int \chi_{B_0/4} (\Lambda Q)^2} = \frac{1}{|\log b_1|} + O\left(\frac{1}{|\log b_1|^2}\right), \tag{2-37}$$

$$d_{b_1} = c_{b_1} \int_0^{B_0} \chi_{B_0/4} \Lambda Q \Gamma y \, dy = O\left(\frac{1}{b_1^2 |\log b_1|}\right). \tag{2-38}$$

From the inverse formula (2-8), we obtain the asymptotics near the origin and infinity:

$$\Sigma_{b_1} = \begin{cases} c_{b_1} T_2 & \text{for } y \leq \frac{1}{4} B_0, \\ 4\Gamma & \text{for } y \geq 3 B_0. \end{cases} \tag{2-39}$$

To deal with $\boldsymbol{T}_1$, which is radiative itself, we further define

$$\tilde{c}_{b_1} := \frac{\langle \Lambda_0 \Lambda Q, \Lambda Q \rangle}{\langle \chi_{B_0/4} \Lambda Q, \Lambda Q \rangle} = \frac{1}{2|\log b_1|} + O\left(\frac{1}{|\log b_1|^2}\right). \tag{2-40}$$

Lemma 2.6 (cancellation by the radiation). *For $i \geq 1$, let $\boldsymbol{\Theta}_i$ be defined as*

$$\boldsymbol{\Theta}_1 := \boldsymbol{\Lambda T}_1 - \tilde{c}_{b_1} \chi_{B_0/4} \boldsymbol{T}_1, \tag{2-41}$$

$$\text{for } i \geq 2, \quad \boldsymbol{\Theta}_i := \boldsymbol{\Lambda T}_i - (i-1)\boldsymbol{T}_i - (-\boldsymbol{H})^{-i+2} \boldsymbol{\Sigma}_{b_1}, \tag{2-42}$$

where $\boldsymbol{T}_i$ is given by (2-27). Then $\boldsymbol{\Theta}_i$ is b_1-admissible of degree $(i, i, \bar{i})$.

Remark. As mentioned earlier, our radiation Σ_{b_1} cancels the extra growth of $\Lambda T_2 - T_2 \sim y$ from the asymptotics

$$T_2 = y\log y + cy + O\left(\frac{|\log y|^2}{y}\right), \quad \Lambda T_2 = y\log y + (c+1)y + O\left(\frac{|\log y|^2}{y}\right)$$

by 4Γ in (2-39). Since T_2 and Γ are elements of the generalized null space of H, the above cancellation holds for all $\boldsymbol{\Theta}_i$, $i \geq 2$.

Proof. Step 1: $i = 1$. Note that $\boldsymbol{\Theta}_1 = (0, \Theta_1)^t$ and

$$\Theta_1 = \Lambda_0 \Lambda Q - \tilde{c}_{b_1} \Lambda Q \chi_{B_0/4},$$

and we have that $\boldsymbol{\Theta}_1$ is b_1-admissible of degree $(1, 1, 1)$ from the explicit formulae

$$\Lambda Q(y) = \frac{2y}{1+y^2}, \quad \Lambda_0 \Lambda Q(y) = \frac{4y}{(1+y^2)^2}$$

and the bounds, for $l \geq 1$,

$$\left|\frac{\partial^l c_{b_1}}{\partial b_1^l}\right| + \left|\frac{\partial^l \tilde{c}_{b_1}}{\partial b_1^l}\right| \lesssim \frac{1}{b_1^l |\log b_1|^2}, \quad \left|\frac{\partial^l d_{b_1}}{\partial b_1^l}\right| \lesssim \frac{1}{b_1^{l+2} |\log b_1|}, \quad \left|\frac{\partial^l \chi_{B_0}}{\partial b_1^l}\right| \lesssim \frac{\mathbf{1}_{y\sim B_0}}{b_1^l}. \tag{2-43}$$

Step 2: $i = 2$. Now, we use induction on $i \geq 2$. For $i = 2$, (2-39) and the admissibility of $\boldsymbol{T}_2$ imply that Θ_2 satisfies the desired condition near zero (2-17) since

$$\boldsymbol{\Theta}_2 = \begin{pmatrix} \Theta_2 \\ 0 \end{pmatrix} = \begin{pmatrix} \Lambda T_2 - T_2 - \Sigma_{b_1} \\ 0 \end{pmatrix}. \tag{2-44}$$

To exhibit the behavior near infinity, we deal with the cases $1 \leq y \leq 3B_0$ and $y \geq 3B_0$ separately. The inverse formula (2-8) yields, for $1 \leq y \leq 3B_0$,

$$\begin{aligned} \Sigma_{b_1}(y) &= \Gamma \int_0^y c_{b_1} \chi_{B_0/4} (\Lambda Q)^2 x\, dx - \Lambda Q \int_0^y c_{b_1} \chi_{B_0/4} \Lambda Q \Gamma x\, dx + d_{b_1}(1 - \chi_{B_0}) \Lambda Q \\ &= y \frac{\int_0^y \chi_{B_0/4} (\Lambda Q)^2 x}{\int \chi_{B_0/4} (\Lambda Q)^2 x} + O\left(\frac{1+y}{|\log b_1|}\right), \end{aligned} \tag{2-45}$$

$$\begin{aligned} \Theta_2(y) &= y + O\left(\frac{|\log y|^2}{y}\right) - y \frac{\int_0^y \chi_{B_0/4} (\Lambda Q)^2 x}{\int \chi_{B_0/4} (\Lambda Q)^2} + O\left(\frac{1+y}{|\log b_1|}\right) \\ &= y \frac{\int_y^{B_0} \chi_{B_0/4} (\Lambda Q)^2 x}{\int \chi_{B_0/4} (\Lambda Q)^2} + O\left(\frac{1+y}{|\log b_1|}\right) + O\left(\frac{|\log y|^2}{y}\right) \\ &= O\left(\frac{1+y}{|\log b_1|} (1 + |\log(b_1 y)|)\right). \end{aligned} \tag{2-46}$$

For $y \geq 3B_0$, (2-7) implies

$$\Sigma_{b_1}(y) = \Gamma \int_0^y c_{b_1} \chi_{B_0/4} (\Lambda Q)^2 x\, dx = y + O\left(\frac{\log y}{y}\right). \tag{2-47}$$

Hence, for $y \ge 1$, Θ_2 satisfies (2-29) for the case $k = 0$ as

$$|\Theta_2(y)| \lesssim y^{2-0-1-0} g_2(b_1, y) + y^{2-0-3-0}(\log y)^2. \tag{2-48}$$

The higher derivatives, namely f_k and $\partial^l f_k / \partial b_1^l$, can also be estimated by using (2-21), the bounds of the coefficients (2-37), (2-38), (2-43) and the commutator relation

$$A(\Lambda f) = Af + \Lambda A f - \frac{\Lambda Z}{y} f, \quad H(\Lambda f) = 2Hf + \Lambda H f - \frac{\Lambda V}{y^2} f,$$

where Z and V are given by (2-2) and (2-4), respectively. Here, we can easily check that $\Lambda Z/y$ is an odd function and $\Lambda V/y^2$ is an even function. Furthermore, for $y \ge 1$,

$$\left| \frac{\partial^k}{\partial y^k} \left(\frac{\Lambda Z}{y} \right) \right| \lesssim \frac{1}{1 + y^{k+3}}, \quad \left| \frac{\partial^k}{\partial y^k} \left(\frac{\Lambda V}{y} \right) \right| \lesssim \frac{1}{1 + y^{k+4}}. \tag{2-49}$$

Therefore, $\boldsymbol{\Theta}_2$ is b_1-admissible of degree $(2, 2, 0)$.

<u>Step 3</u>: *Induction on i.* Suppose that $\boldsymbol{\Theta}_i$ is b_1-admissible of degree $(i, i, \bar{i})$. For even i, we have that $\boldsymbol{\Theta}_{i+1}$ is b_1-admissible of degree $(i+1, i+1, \overline{i+1})$ since

$$\boldsymbol{\Theta}_{i+1} = \begin{pmatrix} 0 \\ \Lambda_0 T_{i+1} - i T_{i+1} - (-H)^{-i/2+1} \Sigma_{b_1} \end{pmatrix} = \begin{pmatrix} 0 \\ \Lambda T_i - (i-1) T_i - (-H)^{-i/2+1} \Sigma_{b_1} \end{pmatrix} = \begin{pmatrix} 0 \\ \Theta_i \end{pmatrix}.$$

For odd i, we have

$$\begin{aligned}
\boldsymbol{H} \boldsymbol{\Theta}_{i+1} &= \begin{pmatrix} 0 & 1 \\ H & 0 \end{pmatrix} \begin{pmatrix} \Theta_{i+1} \\ 0 \end{pmatrix} \\
&= \begin{pmatrix} 0 \\ H \Lambda T_{i+1} - i H T_{i+1} - H(-H)^{-(i+1)/2+1} \Sigma_{b_1} \end{pmatrix} \\
&= \begin{pmatrix} 0 \\ \Lambda H T_{i+1} - (i-2) H T_{i+1} - y^{-2} \Lambda V T_{i+1} + (-H)^{-(i-1)/2+1} \Sigma_{b_1} \end{pmatrix} \\
&= - \begin{pmatrix} 0 \\ \Lambda T_i - (i-2) T_i - (-H)^{-(i-1)/2+1} \Sigma_{b_1} + y^{-2} \Lambda V T_{i+1} \end{pmatrix} \\
&= - \begin{pmatrix} 0 \\ \Lambda_0 T_i - (i-1) T_i - (-H)^{-(i-1)/2+1} \Sigma_{b_1} \end{pmatrix} + \begin{pmatrix} 0 \\ y^{-2} \Lambda V T_{i+1} \end{pmatrix} \\
&= -\boldsymbol{\Theta}_i + \begin{pmatrix} 0 \\ y^{-2} \Lambda V T_{i+1} \end{pmatrix}.
\end{aligned}$$

The Taylor expansion condition (2-17) of $(0, y^{-2} \Lambda V T_{i+1})^t$ comes from the definition of $\boldsymbol{T}_i$ and the cancellation $\Lambda V = O(y^2)$ near $y = 0$.

For $y \ge 1$, (2-49) implies

$$\mathcal{A}^k \left(\frac{\Lambda V}{y^2} T_{i+1} \right) \lesssim \sum_{j=0}^{k} \frac{1}{y^{j+4}} y^{i-(k-j)} |\log y|^{c_i} \lesssim y^{i-3-k-1} |\log y|^{c_i}.$$

Hence $(0, y^{-2} \Lambda V T_{i+1})^t$ is b_1-admissible of degree $(i, i, 1)$; the desired result comes from Lemma 2.5. $\square$

2.5. ***Adapted norms of b_1 admissible functions.*** The next lemma yields some suitable norms corresponding to the adapted derivatives of b_1-admissible functions.

Lemma 2.7 (adapted norms of b_1-admissible function). *For $i \geq 1$, a b_1-admissible function f of degree $(i, i, \bar{i})$ has the following bounds*:

(i) *Global bounds*:

$$\|f_{k-\bar{i}}\|_{L^2(|y|\leq 2B_1)} \lesssim \begin{cases} b_1^{k-i}|\log b_1|^{\gamma(i-k-2)-1} & \text{if } k \leq i-3, \\ b_1^{k-i}/|\log b_1| & \text{if } k = i-2,\ i-1, \\ 1 & \text{if } k \geq i. \end{cases} \tag{2-50}$$

(ii) *Logarithmic weighted bounds*:

$$\sum_{k=0}^{m} \left\| \frac{1+|\log y|}{1+y^{m-k}} f_{k-\bar{i}} \right\|_{L^2(|y|\leq 2B_1)} \lesssim \begin{cases} b_1^{m-i}|\log b_1|^C & \text{for } m \leq i-1, \\ |\log b_1|^C & \text{for } m \geq i. \end{cases} \tag{2-51}$$

(iii) *Improved global bounds*:

$$\sum_{j=0}^{k-\bar{i}} \|y^{-(k-\bar{i}-j)} f_j\|_{L^2(y\sim B_1)} \lesssim b_1^{k-i}|\log b_1|^{\gamma(i-k-2)-1}. \tag{2-52}$$

Here, $B_1 = |\log b_1|^\gamma / b_1$ *and* $\gamma = 1 + \bar{\ell}$.

Remark. Due to the growth in (2-29), it is indispensable to restrict the integration domain by taking the L^2 norm. Later, we will attach a cutoff function χ_{B_1} to the profile modifications. Considering Leibniz's rule, the adapted derivative $\mathcal{A}^k$ can be taken on such modifications or the cutoff function. Then the global bounds (2-50) yield some estimates for the former case and (2-52) gives those for the latter case. The choice of cutoff region B_1 will be determined by the localization of our blow-up profile, which can be seen in more detail in Proposition 2.10.

Proof. (i) From (2-29), $f_{k-\bar{i}}$ satisfies the following estimate for $y \geq 2$:

$$|f_{k-\bar{i}}| \lesssim y^{i-k-1}\left(g_{i-k}(b_1, y) + \frac{|\log y|^{c_{p_2}}}{y^2} + \frac{\mathbf{1}_{\{i\geq k+3,\, y\geq 3B_0\}}}{y^2 b_1^2 |\log b_1|} \right).$$

Therefore, we obtain (2-50) for $i \geq k+1$:

$$\begin{aligned} \|f_{k-\bar{i}}\|_{L^2(|y|\leq 2B_1)} &\lesssim \|\mathbf{1}_{|y|\leq 2}\|_{L^2} + \left\| y^{i-k-1} \frac{1+|\log(b_1 y)|}{|\log b_1|} \right\|_{L^2(2\leq|y|\leq 3B_0)} \\ &\qquad + \|y^{i-k-3}|\log y|^{c_i}\|_{L^2(2\leq|y|\leq 2B_1)} + \left\| \frac{y^{i-k-3}\mathbf{1}_{\{i\geq k+3\}}}{b_1^2|\log b_1|} \right\|_{L^2(3B_0\leq|y|\leq 2B_1)} \\ &\lesssim 1 + \frac{b_1^{k-i}}{|\log b_1|} + b_1^{(k-i+2)\mathbf{1}_{\{i\geq k+2\}}}|\log b_1|^C + \frac{B_1^{i-k-2}}{b_1^2|\log b_1|}\mathbf{1}_{\{i\geq k+3\}} \\ &\lesssim \frac{b_1^{k-i}}{|\log b_1|}|\log b_1|^{\gamma(i-k-2)\mathbf{1}_{\{i\geq k+3\}}}, \end{aligned}$$

and the case $i \leq k$ also holds similarly.

(ii) The logarithmic weighted bounds (2-51) are nothing but (2-50) multiplied by the logarithmic loss $|\log b_1|^C$ and then using the fact that $|\log y|/|\log b_1| \lesssim 1$ on $2 \le |y| \le 3B_0$.

(iii) We can prove (2-52) from a pointwise estimate in the region $y \sim B_1$:

$$|y^{-(k-\bar{i}-j)} f_j| \lesssim y^{i-k-3}\left(|\log y|^C + \frac{\mathbf{1}_{\{i\ge\bar{i}+j+3\}}}{b_1^2|\log b_1|}\right) \lesssim \frac{y^{i-k-1}}{|\log b_1|^{2\gamma+1}}, \tag{2-53}$$

and the proof is complete. □

2.6. *Approximate blow-up profiles.* From now on, we fix

$$\ell \ge 2 \quad \text{and} \quad L = \ell + \overline{\ell+1}.$$

We construct the blow-up profiles based on the generalized kernels $\boldsymbol{T}_i$. To be more specific, our blow-up scenario is created by bubbling off $\boldsymbol{Q}$ via scaling and adding $b_i\boldsymbol{T}_i$; the evolution of λ is determined by the system of dynamical laws for $b = (b_1, \dots, b_L)$. Here, we are faced with unnecessary growth caused by linear and nonlinear terms. To minimize this growth, we define the homogeneous functions, which do not affect the evolution of b (i.e., b_iT_i). We note that this kind of construction was introduced in [Raphaël and Schweyer 2014].

Definition 2.8 (homogeneous functions). Write $J = (J_1, \dots, J_L)$ and $|J|_2 = \sum_{k=1}^{L} kJ_k$. We say that a smooth vector-valued function $\boldsymbol{S}(b, y) = \boldsymbol{S}(b_1, \dots, b_L, y)$ is homogeneous of degree $(p_1, p_2, \iota, p_3) \in \mathbb{N}\times\mathbb{Z}\times\{0, 1\}\times\mathbb{N}$ if it can be expressed as a finite sum of smooth functions of the form $\big(\prod_{i=1}^{L} b_i^{J_i}\big)\boldsymbol{S}_J(y)$, where $\boldsymbol{S}_J(y)$ is a b_1-admissible function of degree (p_1, p_2, ι) with $|J|_2 = p_3$.

Proposition 2.9 (construction of the approximate profile). *Given a large constant* $M > 0$, *there exists a small constant* $0 < b^*(M) \ll 1$ *such that a* C^1 *map*

$$b : s \mapsto (b_1(s), \dots, b_L(s)) \in \mathbb{R}_+^* \times \mathbb{R}^{L-1}$$

verifies the existence of a slowly modulated profile $\boldsymbol{Q}_b$ *given by*

$$\boldsymbol{Q}_b := \boldsymbol{Q} + \boldsymbol{\alpha}_b, \ \boldsymbol{\alpha}_b := \sum_{i=1}^{L} b_i\boldsymbol{T}_i + \sum_{i=2}^{L+2} \boldsymbol{S}_i, \tag{2-54}$$

which drives the equation

$$\partial_s \boldsymbol{Q}_b - \boldsymbol{F}(\boldsymbol{Q}_b) + b_1\boldsymbol{\Lambda}\boldsymbol{Q}_b = \mathbf{Mod}(t) + \boldsymbol{\psi}_b, \tag{2-55}$$

where $\mathbf{Mod}(t) := (\mathrm{Mod}(t), \dot{\mathrm{Mod}}(t))^t$ *establishes the dynamical law of* b:

$$\mathbf{Mod}(t) = \sum_{i=1}^{L}((b_i)_s + (i-1+c_{b_1,i})b_1b_i - b_{i+1})\left(\boldsymbol{T}_i + \sum_{j=i+1}^{L+2}\frac{\partial \boldsymbol{S}_j}{\partial b_i}\right), \tag{2-56}$$

where we set $b_{L+1} = 0$ *for convenience and* $c_{b_1,i}$ *is defined by*

$$c_{b_1,i} = \begin{cases} \tilde{c}_{b_1} = \dfrac{\langle \Lambda_0\Lambda Q, \Lambda Q\rangle}{\langle \chi_{B_0/4}\Lambda Q, \Lambda Q\rangle} & \text{for } i = 1, \\ c_{b_1} = \dfrac{4}{\int \chi_{B_0/4}(\Lambda Q)^2} & \text{for } i \ne 1. \end{cases} \tag{2-57}$$

Here, $\boldsymbol{T}_i$ *is given by* (2-27) *and* $\boldsymbol{S}_i$ *is a homogeneous function of degree* $(i, i, \bar{i}, i)$ *satisfying*

$$\boldsymbol{S}_1 = 0, \qquad \frac{\partial \boldsymbol{S}_i}{\partial b_j} = 0 \ \text{ for } 2 \le i \le j \le L. \tag{2-58}$$

Moreover, the restrictions $|b_k| \lesssim b_1^k$ *and* $0 < b_1 < b^*(M)$ *yield the estimates below for* $\boldsymbol{\psi}_b = (\psi_b, \dot{\psi}_b)^t$:

(i) *Global bound*: *for* $2 \le k \le L-1$,

$$\|\mathcal{A}^k \psi_b\|_{L^2(|y|\le 2B_1)} + \|\mathcal{A}^{k-1}\dot{\psi}_b\|_{L^2(|y|\le 2B_1)} \lesssim b_1^{k+1}|\log b_1|^C, \tag{2-59}$$

$$\|\mathcal{A}^L \psi_b\|_{L^2(|y|\le 2B_1)} + \|\mathcal{A}^{L-1}\dot{\psi}_b\|_{L^2(|y|\le 2B_1)} \lesssim \frac{b_1^{L+1}}{|\log b_1|^{1/2}}, \tag{2-60}$$

$$\|\mathcal{A}^{L+1} \psi_b\|_{L^2(|y|\le 2B_1)} + \|\mathcal{A}^{L}\dot{\psi}_b\|_{L^2(|y|\le 2B_1)} \lesssim \frac{b_1^{L+2}}{|\log b_1|}. \tag{2-61}$$

(ii) *Logarithmic weighted bound*: *for* $m \ge 1$ *and* $0 \le k \le m$,

$$\left\| \frac{1+|\log y|}{1+y^{m-k}} \mathcal{A}^k \psi_b \right\|_{L^2(|y|\le 2B_1)} \lesssim b_1^{m+1}|\log b_1|^C, \quad m \le L+1, \tag{2-62}$$

$$\left\| \frac{1+|\log y|}{1+y^{m-k}} \mathcal{A}^k \dot{\psi}_b \right\|_{L^2(|y|\le 2B_1)} \lesssim b_1^{m+2}|\log b_1|^C, \quad m \le L. \tag{2-63}$$

(iii) *Improved local bound*: *for all* $2 \le k \le L+1$,

$$\|\mathcal{A}^k \psi_b\|_{L^2(|y|\le 2M)} + \|\mathcal{A}^{k-1}\dot{\psi}_b\|_{L^2(|y|\le 2M)} \lesssim C(M) b_1^{L+3}. \tag{2-64}$$

Here, $B_0 = 1/b_1$ *and* $B_1 = |\log b_1|^\gamma / b_1$.

Remark. As can be seen in the following proof, the homogeneous profile $\boldsymbol{S}_i$ is eventually derived from the b_1-admissible function $\boldsymbol{\Theta}_{i-1}$ with some nonlinear effects.

Proof. Step 1: Linearization. We pull out the modulation law of b from linearizing the renormalized equation. Recall

$$\boldsymbol{F}(\boldsymbol{u}) := \begin{pmatrix} \dot{u} \\ \Delta u - f(u)/r^2 \end{pmatrix}.$$

Since $\boldsymbol{F}(\boldsymbol{Q}) = 0$, we have

$$\begin{aligned} \partial_s \boldsymbol{Q}_b + b_1 \boldsymbol{\Lambda} \boldsymbol{Q}_b - \boldsymbol{F}(\boldsymbol{Q}_b) &= \partial_s \boldsymbol{\alpha}_b + b_1 \boldsymbol{\Lambda}(\boldsymbol{Q} + \boldsymbol{\alpha}_b) - (\boldsymbol{F}(\boldsymbol{Q}+\boldsymbol{\alpha}_b) - \boldsymbol{F}(\boldsymbol{Q})) \\ &=: b_1 \boldsymbol{\Lambda} \boldsymbol{Q} + (\partial_s + b_1 \boldsymbol{\Lambda})\boldsymbol{\alpha}_b + \boldsymbol{H}\boldsymbol{\alpha}_b + \boldsymbol{N}(\boldsymbol{\alpha}_b), \end{aligned}$$

where $\boldsymbol{N}$ denotes the higher-order terms:

$$\boldsymbol{N}(\boldsymbol{\alpha}_b) := \frac{1}{y^2} \begin{pmatrix} 0 \\ f(Q+\alpha_b) - f(Q) - f'(Q)\alpha_b \end{pmatrix}, \quad \boldsymbol{\alpha}_b = \begin{pmatrix} \alpha_b \\ \dot{\alpha}_b \end{pmatrix}. \tag{2-65}$$

Note that

$$\partial_s \boldsymbol{\alpha}_b = \sum_{i=1}^{L} \left[(b_i)_s \boldsymbol{T}_i + \sum_{j=i+1}^{L+2} (b_i)_s \frac{\partial \boldsymbol{S}_j}{\partial b_i} \right] = \sum_{i=1}^{L} \left[(b_i)_s \boldsymbol{T}_i + \sum_{j=1}^{i-1} (b_j)_s \frac{\partial \boldsymbol{S}_i}{\partial b_j} \right] + \sum_{i=1}^{L} (b_i)_s \frac{\partial \boldsymbol{S}_{L+1}}{\partial b_i} + \sum_{i=1}^{L} (b_i)_s \frac{\partial \boldsymbol{S}_{L+2}}{\partial b_i}.$$

Rearranging the linear terms to the degree with respect to b_1 and using the fact $\boldsymbol{H}\boldsymbol{T}_{i+1} = -\boldsymbol{T}_i$ for $1 \le i \le L-1$,

$$\begin{aligned}
&b_1 \boldsymbol{\Lambda} \boldsymbol{Q} + (\partial_s + b_1 \boldsymbol{\Lambda})\boldsymbol{\alpha}_b + \boldsymbol{H}\boldsymbol{\alpha}_b \\
&= \sum_{i=1}^{L} [(b_i)_s \boldsymbol{T}_i + b_1 b_i \boldsymbol{\Lambda} \boldsymbol{T}_i - b_{i+1}\boldsymbol{T}_i] + \sum_{i=1}^{L} \biggl[\boldsymbol{H}\boldsymbol{S}_{i+1} + b_1 \boldsymbol{\Lambda} \boldsymbol{S}_i + \sum_{j=1}^{i-1} (b_j)_s \frac{\partial \boldsymbol{S}_i}{\partial b_j} \biggr] \\
&\qquad + b_1 \boldsymbol{\Lambda} \boldsymbol{S}_{L+1} + \boldsymbol{H}\boldsymbol{S}_{L+2} + \sum_{i=1}^{L} (b_i)_s \frac{\partial \boldsymbol{S}_{L+1}}{\partial b_i} + b_1 \boldsymbol{\Lambda} \boldsymbol{S}_{L+2} + \sum_{i=1}^{L} (b_i)_s \frac{\partial \boldsymbol{S}_{L+2}}{\partial b_i}. \qquad (2\text{-}66)
\end{aligned}$$

From Lemma 2.6,

$$(b_1)_s \boldsymbol{T}_1 + b_1^2 \boldsymbol{\Lambda} \boldsymbol{T}_1 - b_2 \boldsymbol{T}_1 = ((b_1)_s + b_1^2 \tilde{c}_{b_1} - b_2)\boldsymbol{T}_1 - b_1^2 \tilde{c}_{b_1}(1 - \chi_{B_0/4})\boldsymbol{T}_1 + b_1^2 \boldsymbol{\Theta}_1,$$

and, for $2 \le i \le L$,

$$\begin{aligned}
&(b_i)_s \boldsymbol{T}_i + b_1 b_i \boldsymbol{\Lambda} \boldsymbol{T}_i - b_{i+1} \boldsymbol{T}_i \\
&\qquad = ((b_i)_s + (i - 1 + c_{b_1}) b_1 b_i - b_{i+1})\boldsymbol{T}_i + b_1 b_i (-\boldsymbol{H})^{-i+2}(\boldsymbol{\Sigma}_{b_1} - c_{b_1}\boldsymbol{T}_2) + b_1 b_i \boldsymbol{\Theta}_i. \qquad (2\text{-}67)
\end{aligned}$$

Hence, we can separate $\mathbf{Mod}(t)$ from the right-hand side of (2-66) to get the expression

$$\begin{aligned}
&\mathbf{Mod}(t) - b_1^2 \tilde{c}_{b_1}(1 - \chi_{B_0/4})\boldsymbol{T}_1 + \sum_{i=2}^{L} b_1 b_i (-\boldsymbol{H})^{-i+2}(\boldsymbol{\Sigma}_{b_1} - c_{b_1}\boldsymbol{T}_2) \\
&\quad + \sum_{i=1}^{L} \biggl[\boldsymbol{H}\boldsymbol{S}_{i+1} + b_1 b_i \boldsymbol{\Theta}_i + b_1 \boldsymbol{\Lambda} \boldsymbol{S}_i - \sum_{j=1}^{i-1} ((j - 1 + c_{b_1,j}) b_1 b_j - b_{j+1}) \frac{\partial \boldsymbol{S}_i}{\partial b_j} \biggr] \\
&\qquad + \boldsymbol{H}\boldsymbol{S}_{L+2} + b_1 \boldsymbol{\Lambda} \boldsymbol{S}_{L+1} - \sum_{i=1}^{L} ((i - 1 + c_{b_1,i}) b_1 b_i - b_{i+1}) \frac{\partial \boldsymbol{S}_{L+1}}{\partial b_i} \\
&\qquad\qquad + b_1 \boldsymbol{\Lambda} \boldsymbol{S}_{L+2} - \sum_{i=1}^{L} ((i - 1 + c_{b_1,i}) b_1 b_i - b_{i+1}) \frac{\partial \boldsymbol{S}_{L+2}}{\partial b_i}. \qquad (2\text{-}68)
\end{aligned}$$

Step 2: Construction of $\boldsymbol{S}_i$. One can observe that the second and third lines of (2-68) provide the definition of the homogeneous profiles $\boldsymbol{S}_i$ inductively. We need to pull out the additional homogeneous functions from $\boldsymbol{N}(\boldsymbol{\alpha}_b) = (0, N(\alpha_b))^t$ via Taylor's theorem:

$$N(\alpha_b) = \frac{1}{y^2} \biggl\{ \sum_{j=2}^{(L+1)/2} \frac{f^{(j)}(Q)}{j!} \alpha_b^j + N_0(\alpha_b) \alpha_b^{(L+3)/2} \biggr\},$$

where $N_0(\alpha_b)$ is the coefficient of the remainder term:

$$N_0(\alpha_b) = \frac{1}{((L+1)/2)!} \int_0^1 (1-\tau)^{(L+1)/2} f^{((L+3)/2)}(Q + \tau \alpha_b)\, d\tau.$$

Roughly speaking, $N_0(\alpha_b) = O(b_1^{L+3})$. We also rewrite the Taylor polynomial part of $N(\alpha_b)$ in terms of the degree of b_1: for the L-tuple $J := (J_2, J_4, \dots, J_{L-1}, \tilde{J}_2, \tilde{J}_4, \dots, \tilde{J}_{L+1})$,

$$\sum_{j=2}^{(L+1)/2} \frac{f^{(j)}(Q)}{j!} \alpha_b^j = \sum_{i=1}^{(L+1)/2} P_{2i} + R',$$

where

$$P_i := \sum_{j=2}^{(L+1)/2} \sum_{|J|_1=j}^{|J|_2=i} c_{j,J} \prod_{k=1}^{(L-1)/2} (b_{2k}T_{2k})^{J_{2k}} \prod_{k=1}^{(L+1)/2} S_{2k}^{\tilde{J}_{2k}},$$

$$R' := \sum_{j=2}^{(L+1)/2} \sum_{|J|_1=j}^{|J|_2\geq L+3} c_{j,J} \prod_{k=1}^{(L-1)/2} (b_{2k}T_{2k})^{J_{2k}} \prod_{k=1}^{(L+1)/2} S_{2k}^{\tilde{J}_{2k}}, \qquad c_{j,J} = \frac{f^{(j)}(Q)}{\prod_{k=1}^{(L-1)/2} J_{2k}!\prod_{k=1}^{(L+1)/2} \tilde{J}_{2k}!},$$

with two distinct counting notations

$$|J|_1 := \sum_{k=1}^{(L-1)/2} J_{2k} + \sum_{k=1}^{(L+1)/2} \tilde{J}_{2k}, \quad |J|_2 := \sum_{k=1}^{(L-1)/2} 2kJ_{2k} + \sum_{k=1}^{(L+1)/2} 2k\tilde{J}_{2k}.$$

In short, $P_{2i} = O(b_1^{2i})$ and $R' = O(b_1^{L+3})$. We collect all $O(b_1^{L+3})$ terms as follows:

$$R := N_0(\alpha_b)\alpha_b^{(L+3)/2} + R'. \tag{2-69}$$

We claim that $\boldsymbol{P}_{2i}/y^2 = (0, P_{2i}/y^2)$ is homogeneous of degree $(2i-1, 2i-1, 1, 2i)$ for $1 \leq i \leq \frac{1}{2}(L+1)$. The case $i = 1$ is trivial since $P_2 = 0$. For $2 \leq i \leq \frac{1}{2}(L+1)$, we recall that P_{2i}/y^2 is a linear combination of the following monomials: for $|J|_1 = j$, $|J|_2 = 2i$ and $2 \leq j \leq i$,

$$\frac{f^{(j)}(Q)}{y^2} \prod_{k=1}^{i} (b_{2k}T_{2k})^{J_{2k}} \prod_{k=1}^{i} S_{2k}^{\tilde{J}_{2k}}.$$

Near the origin, we observe that T_{2k} and S_{2k} are odd functions, and the parity of a function $f^{(j)}(Q)$ is determined by the parity of j, so each monomial is either an odd or even function. Hence it suffices to calculate the leading power of the Taylor expansion of each function constituting the monomial:

$$T_{2k} \sim y^{2k+1}, \quad S_{2k} \sim O(b_1^{2k})y^{2k+1} \quad \text{and} \quad f^{(j)}(Q) \sim y^{\overline{j+1}},$$

and the leading power of each monomial is given by

$$\begin{aligned} b_1^{\sum_{k=1}^{i} 2kJ_{2k}} \cdot b_1^{\sum_{k=1}^{i} 2k\tilde{J}_{2k}} &= b_1^{2i}, \\ y^{-2}y^{\overline{j+1}}y^{\sum_{k=1}^{i}(2k+1)J_{2k}}y^{\sum_{k=1}^{i}(2k+1)\tilde{J}_{2k}} &= y^{2i+j-1-\bar{j}}. \end{aligned} \tag{2-70}$$

Therefore, the Taylor expansion condition (2-17) comes from the fact that $j-1-\bar{j}$ is a positive odd integer when $j \geq 2$.

Similarly, for $y \geq 1$, we have that $|T_{2k}| \lesssim y^{2k-1}\log y$, $|S_{2k}| \lesssim b_1^{2k}y^{2k-1}$ and $|f^{(j)}(Q)| \lesssim y^{-1+\bar{j}}$ imply

$$\begin{aligned} \left|\frac{f^{(j)}(Q)}{y^2} \prod_{k=1}^{i} b_{2k}^{J_{2k}} T_{2k}^{J_{2k}} \prod_{k=1}^{i} S_{2k}^{\tilde{J}_{2k}}\right| &\lesssim b_1^{2i}|y^{-3+\bar{j}}| \prod_{k=1}^{i} |y^{2k-1}\log y|^{J_{2k}} \prod_{k=1}^{i} |y^{2k-1}|^{\tilde{J}_{2k}} \\ &\lesssim b_1^{2i}y^{2i-j-3+\bar{j}}|\log y|^C \\ &\lesssim b_1^{2i}y^{2i-5}|\log y|^C \end{aligned} \tag{2-71}$$

with the fact that $j-\bar{j} \geq 2$. We can easily estimate the higher derivatives of each monomial.

Under the setting $\boldsymbol{P}_{2k+1} := (0,0)^t$ for $k \in \mathbb{N}$, we obtain the final definition of $\boldsymbol{S}_i$: $\boldsymbol{S}_1 := 0$ and, for $i = 1, \dots, L+1$,

$$\boldsymbol{S}_{i+1} := (-\boldsymbol{H})^{-1}\biggl(b_1 b_i \boldsymbol{\Theta}_i + b_1 \boldsymbol{\Lambda} \boldsymbol{S}_i + \frac{\boldsymbol{P}_{i+1}}{y^2} - \sum_{j=1}^{i-1}((j-1+c_{b_1,j})b_1 b_j - b_{j+1})\frac{\partial \boldsymbol{S}_i}{\partial b_j}\biggr). \tag{2-72}$$

From the homogeneity of $\boldsymbol{P}_i/y^2$ established above and Lemmas 2.5 and 2.6, we can prove that $\boldsymbol{S}_i$ is homogeneous of degree $(i, i, \bar{i}, i)$ for $1 \le i \le L+2$ with (2-58) via induction. To sum up, we get (2-55) by collecting remaining errors into $\boldsymbol{\psi}_b$:

$$\begin{aligned}\boldsymbol{\psi}_b := -b_1^2 \tilde{c}_{b_1}(1-\chi_{B_0/4})\boldsymbol{T}_1 + \sum_{i=2}^{L} b_1 b_i (-\boldsymbol{H})^{-i+2}\widetilde{\boldsymbol{\Sigma}}_{b_1} \\ + b_1 \boldsymbol{\Lambda} \boldsymbol{S}_{L+2} - \sum_{i=1}^{L}((i-1+c_{b_1,i})b_1 b_i - b_{i+1})\frac{\partial \boldsymbol{S}_{L+2}}{\partial b_i} + \frac{\boldsymbol{R}}{y^2},\end{aligned} \tag{2-73}$$

where $\widetilde{\boldsymbol{\Sigma}}_{b_1} := \boldsymbol{\Sigma}_{b_1} - c_{b_1}\boldsymbol{T}_2$ and $\boldsymbol{R} = (0, R)^t$ from (2-69).

<u>Step 3</u>: Error bounds. Now, it remains to prove the Sobolev bounds (2-59)–(2-64). We can treat the errors involving $\boldsymbol{S}_{L+2}$ in (2-73) easily. Since $\boldsymbol{S}_{L+2}$ is homogeneous of degree $(L+2, L+2, 1, L+2)$, Lemma 2.5 ensures that the functions containing $\boldsymbol{S}_{L+2}$ are homogeneous of degree $(L+2, L+2, 1, L+3)$, and thus the desired bounds come from Lemma 2.7.

The other errors require separate integration to conclude. We start with the first line of (2-73). Noting that $\boldsymbol{T}_1 = (0, T_1)^t$ and $\Lambda Q \sim 1/y$ on $y \ge 1$, we have, for $k \ge 0$,

$$|\mathcal{A}^k(1-\chi_{B_0/4})T_1| \lesssim y^{-(k+1)}\mathbf{1}_{y \ge B_0/4}, \tag{2-74}$$

which imply (2-59), (2-60) and (2-61): for $2 \le k \le L+1$,

$$\|b_1^2 \tilde{c}_{b_1}\mathcal{A}^{k-1}(1-\chi_{B_0/4})T_1\|_{L^2(|y|\le 2B_1)} \lesssim \frac{b_1^2}{|\log b_1|}\|y^{-k}\|_{L^2(B_0/4 \le |y| \le 2B_1)} \lesssim \frac{b_1^{k+1}}{|\log b_1|}. \tag{2-75}$$

For $2 \le i \le L$, we rewrite

$$(-\boldsymbol{H})^{i+2}\widetilde{\boldsymbol{\Sigma}}_{b_1} = \begin{cases} ((-H)^{-i/2+1}\widetilde{\Sigma}_{b_1}, 0)^t & \text{for even } i, \\ (0, -(-H)^{-(i-1)/2+1}\widetilde{\Sigma}_{b_1})^t & \text{for odd } i \end{cases} \tag{2-76}$$

using the fact $\boldsymbol{H}^{-2} = -H^{-1}$. Moreover, $\operatorname{supp}(\widetilde{\Sigma}_{b_1}) \subset \bigl\{|y| \ge \frac{1}{4}B_0\bigr\}$ and, for $k \ge 0$, we have the following crude bound: for $\frac{1}{4}B_0 \le y \le 2B_1$,

$$|\mathcal{A}^{k-\bar{i}}H^{-(i-\bar{i})/2+1}\widetilde{\Sigma}_{b_1}| \lesssim y^{i-k-1}\frac{|\log y|}{|\log b_1|} \lesssim y^{i-k-1}. \tag{2-77}$$

Hence, for $1 \le k < i \le L$, we obtain (2-59) from the estimation

$$\|b_1 b_i \mathcal{A}^{k-\bar{i}}H^{-(i-\bar{i})/2+1}\widetilde{\Sigma}_{b_1}\|_{L^2(|y|\le 2B_1)} \lesssim b_1^{i+1}\|y^{i-k-1}\|_{L^2(B_0/4 \le |y| \le 2B_1)} \lesssim b_1^{k+1}|\log b_1|^{\gamma(i-k)}. \tag{2-78}$$

We also observe, for $k \ge i$,

$$\mathcal{A}^{k-\bar{i}}H^{-(i-\bar{i})/2+1}\widetilde{\Sigma}_{b_1} = \mathcal{A}^{k-i}H\widetilde{\Sigma}_{b_1}, \tag{2-79}$$

and together with the sharp bounds

$$|H\widetilde{\Sigma}_{b_1}| \lesssim \frac{\mathbf{1}_{y\geq B_0/4}}{|\log b_1|}\frac{1}{y}, \quad |\mathcal{A}^j H\widetilde{\Sigma}_{b_1}| \lesssim \frac{\mathbf{1}_{y\sim B_0}}{B_0^{j+1}|\log b_1|}, \quad j\geq 1, \tag{2-80}$$

this implies (2-59), (2-60) and (2-61):

$$\|b_1 b_i \mathcal{A}^{k-i} H\widetilde{\Sigma}_{b_1}\|_{L^2(|y|\leq 2B_1)} \lesssim \frac{b_1^{i+1}}{|\log b_1|}\|y^{i-k-1}\|_{L^2(B_0/4\leq|y|\leq 2B_1)} \lesssim \frac{b_1^{k+1}}{|\log b_1|^{1/2}},$$

$$\|b_1 b_i \mathcal{A}^{L+1-i} H\widetilde{\Sigma}_{b_1}\|_{L^2(|y|\leq 2B_1)} \lesssim \frac{b_1^{i+1}}{B_0^{L+1-i}|\log b_1|} \lesssim \frac{b_1^{L+2}}{|\log b_1|}.$$

The logarithmic weighted bounds (2-62) and (2-63) come from the above estimation with the trivial bound $|\log y/\log b_1| \lesssim 1$ on $\frac{1}{4}B_0 \leq y \leq 2B_1$ and the fact that the errors in the first line of (2-73) are supported in $y \geq \frac{1}{4}B_0$. This support property also yields the improved local bound (2-64) by choosing $b^*(M)$ small enough.

Now, we move to the last error: $\boldsymbol{R}/y^2$. Recalling (2-69), we observe that $\boldsymbol{R}/y^2 = (0, R/y^2)$ has two parts: a sum of monomials like P_{2i}/y^2 and nonlinear terms

$$\frac{1}{y^2}N_0(\alpha_b)\alpha_b^{(L+3)/2}.$$

For the monomial part, we borrow the calculation of P_{2i}/y^2: (2-70), (2-71). Under the range $|J|_1 = j$, $|J|_2 \geq L+3,\ 2\leq j\leq \frac{1}{2}(L+1)$, those k-th suitable derivatives (i.e., $\mathcal{A}^k$) have the pointwise bounds

$$\begin{cases} b_1^{L+3} & \text{for } y\leq 1,\\ b_1^{|J|_2} y^{|J|_2-k-5}|\log y|^C & \text{for } 1\leq y\leq 2B_1,\end{cases} \tag{2-81}$$

and we simply obtain the bounds (2-59)–(2-64) via integrating the above bound. It remains to estimate the nonlinear term. For $y\leq 1$, we utilize the parity of $f^{((L+3)/2)}(Q)$ and α_b. We already know that α_b is an odd function with the leading term $O(b_1^2)y^3$, the parity of $f^{((L+3)/2)}(Q)$ is opposite that of $\frac{1}{2}(L+3)$, and $N_0(\alpha_b)\alpha_b^{(L+3)/2}/y^2$ is an odd function with the leading term $O(b_1^{L+3})y^{3(L+3)/2-1-\overline{(L+3)/2}}$. Hence, for $1\leq k\leq L$,

$$\left\|\mathcal{A}^k\left(\frac{N_0(\alpha_b)}{y^2}\alpha_b^{(L+3)/2}\right)\right\|_{L^\infty(y\leq 1)} \lesssim b_1^{L+3}.$$

For $1\leq y\leq 2B_1$, the simple bound

$$|\partial_y^k(Q+\tau\alpha_b)| \lesssim \frac{|\log b_1|^C}{y^{k+1}}, \quad k\geq 1,$$

implies

$$|N_0(\alpha_b)| \lesssim 1, \quad |\partial_y^k N_0(\alpha_b)| \lesssim \frac{|\log b_1|^C}{y^{k+1}} \text{ for } k\geq 1.$$

From the Leibniz rule and the crude bound $|\partial_y^k\alpha_b| \lesssim b_1^2|\log b_1|y^{1-k}$, we have

$$\left|\mathcal{A}^k\left(\frac{N_0(\alpha_b)}{y^2}\alpha_b^{(L+3)/2}\right)\right| \lesssim \sum_{j=0}^{k}\frac{|\partial_y^j(N_0(\alpha_b)\alpha_b^{(L+3)/2})|}{y^{2+k-j}} \lesssim b_1^{L+3}|\log b_1|^C y^{(L+3)/2-2-k} \tag{2-82}$$

for $0\leq k\leq L$, and the above pointwise bound yields (2-59)–(2-64) via integration. ☐

2.7. ***Localization of the approximate profile.*** In the previous construction, we observe that the blow-up profile does not approximate the solution of (2-55) on the region $y \geq 2B_1$. Hence it is necessary to cut off the overgrowth of each tail.

Proposition 2.10 (localization of the approximate profile). *Assume the hypotheses of Proposition 2.9, and assume moreover the a priori bounds*

$$|(b_1)_s| \lesssim b_1^2, \quad |b_L| \lesssim \frac{b_1^L}{|\log b_1|} \text{ when } \ell = L-1. \tag{2-83}$$

Then the localized profile $\widetilde{\boldsymbol{Q}}_b$ *given by*

$$\widetilde{\boldsymbol{Q}}_b = \boldsymbol{Q} + \chi_{B_1}\boldsymbol{\alpha}_b \tag{2-84}$$

drives the equation

$$\partial_s \widetilde{\boldsymbol{Q}}_b - \boldsymbol{F}(\widetilde{\boldsymbol{Q}}_b) + b_1 \boldsymbol{\Lambda} \widetilde{\boldsymbol{Q}}_b = \chi_{B_1}\mathbf{Mod}(t) + \tilde{\boldsymbol{\psi}}_b, \tag{2-85}$$

where $\mathbf{Mod}(t)$ *was defined in* (2-56) *and* $\tilde{\boldsymbol{\psi}}_b = (\tilde{\psi}_b, \dot{\tilde{\psi}}_b)^t$ *satisfies the following bounds*:

(i) *Global bound*: *for all* $2 \leq k \leq L-1$,

$$\|\mathcal{A}^k \tilde{\psi}_b\|_{L^2} + \|\mathcal{A}^{k-1}\dot{\tilde{\psi}}_b\|_{L^2} \lesssim b_1^{k+1}|\log b_1|^C, \tag{2-86}$$

$$\|\mathcal{A}^L \tilde{\psi}_b\|_{L^2} + \|\mathcal{A}^{L-1}\dot{\tilde{\psi}}_b\|_{L^2} \lesssim b_1^{L+1}|\log b_1|, \tag{2-87}$$

$$\|\mathcal{A}^{L+1} \tilde{\psi}_b\|_{L^2} + \|\mathcal{A}^{L}\dot{\tilde{\psi}}_b\|_{L^2} \lesssim \frac{b_1^{L+2}}{|\log b_1|}. \tag{2-88}$$

(ii) *Logarithmic weighted bound*: *for* $m \geq 1$ *and* $0 \leq k \leq m$,

$$\left\| \frac{1+|\log y|}{1+y^{m-k}} \mathcal{A}^k \tilde{\psi}_b \right\|_{L^2} \lesssim b_1^{m+1}|\log b_1|^C, \quad m \leq L+1, \tag{2-89}$$

$$\left\| \frac{1+|\log y|}{1+y^{m-k}} \mathcal{A}^k \dot{\tilde{\psi}}_b \right\|_{L^2} \lesssim b_1^{m+2}|\log b_1|^C, \quad m \leq L. \tag{2-90}$$

(iii) *Improved local bound*: *for all* $2 \leq k \leq L+1$,

$$\|\mathcal{A}^k \tilde{\psi}_b\|_{L^2(|y|\leq 2M)} + \|\mathcal{A}^{k-1}\dot{\tilde{\psi}}_b\|_{L^2(|y|\leq 2M)} \lesssim C(M) b_1^{L+3}. \tag{2-91}$$

Remark. This proposition says that our cutoff function χ_{B_1} does not affect the estimates (2-59)–(2-64) in Proposition 2.9. Although such bounds came from integrating over the region $|y| \leq 2B_1$, there are two main reasons why this is possible. First, we do not need to keep track of the logarithmic weight $|\log b_1|$ except for (2-61) corresponding to the highest-order derivative. Second, (2-61) was derived from the sharp pointwise bound (2-80), which only depends on B_0. Thus, $B_1 = |\log b_1|^\gamma / b_1$ just needs to be large enough to obtain (2-88) by increasing γ.

Proof. Noting that $\tilde{\boldsymbol{\psi}}_b = \boldsymbol{\psi}_b$ on $|y| \leq B_1$, we see that (2-64) directly implies the local bound (2-91). For the other estimates, we will prove the global bounds (2-86) and (2-88) first, and the less demanding logarithmic weighted bounds (2-89) and (2-90) later. By a straightforward calculation, $\tilde{\boldsymbol{\psi}}_b$ is given by

$$\tilde{\boldsymbol{\psi}}_b = \chi_{B_1}\boldsymbol{\psi}_b + (\partial_s(\chi_{B_1}) + b_1(y\chi')_{B_1})\boldsymbol{\alpha}_b + b_1(1-\chi_{B_1})\boldsymbol{\Lambda Q} - \begin{pmatrix} 0 \\ \Delta(\chi_{B_1}\alpha_b) - \chi_{B_1}\Delta(\alpha_b) \end{pmatrix} - \frac{1}{y^2}\begin{pmatrix} 0 \\ f(\widetilde{Q}_b) - f(Q) - \chi_{B_1}(f(Q_b) - f(Q)) \end{pmatrix}. \tag{2-92}$$

Before we estimate $\chi_{B_1}\boldsymbol{\psi}_b$ in (2-92), we introduce a useful asymptotics of cutoff:

$$\mathcal{A}^k(\chi_{B_1} f) = \chi_{B_1}\mathcal{A}^k f + \mathbf{1}_{y\sim B_1}\sum_{j=0}^{k-1} O(y^{-(k-j)})\mathcal{A}^j f. \tag{2-93}$$

Applying the above asymptotics to $\chi_{B_1}\boldsymbol{\psi}_b$, we get from Proposition 2.9 that we only need to estimate the errors localized in $y\sim B_1$. From (2-53), (2-74), (2-77), (2-81) and (2-82), we obtain the following pointwise bounds: for $y\sim B_1$ and $0\le j\le k$,

$$|y^{-(k-j)}\mathcal{A}^j\psi_{b_1}| \lesssim \sum_{i=1}^{(L-1)/2} b_1^{2i+1}y^{2i-k-1} \lesssim b_1^{k+1}|\log b_1|^{\gamma(L-1-k)}B_1^{-1} \tag{2-94}$$

and

$$\begin{aligned}|y^{-(k-1-j)}\mathcal{A}^j\dot{\psi}_{b_1}| &\lesssim \sum_{i=1}^{(L+1)/2} b_1^{2i}y^{2i-k-2} + \frac{b_1^{L+3}y^{L+1-k}}{|\log b_1|^{2\gamma+1}} + (b_1^{k+4} + b_1^{(L+3)/2+k+1})|\log b_1|^C \\ &\lesssim b_1^{k+1}|\log b_1|^{\gamma(L-k)}B_1^{-1}.\end{aligned}$$

These pointwise bounds directly imply the global bounds (2-86), (2-87) and (2-88) if we choose $\gamma\ge 1$.

For the second term in the right-hand side of (2-92), we recall

$$\boldsymbol{\alpha}_b = \begin{pmatrix}\alpha_b \\ \dot{\alpha}_b\end{pmatrix} = \begin{pmatrix}\sum_{i=1,\text{even}}^{L} b_i T_i + \sum_{i=2,\text{even}}^{L+2} S_i \\ \sum_{i=1,\text{odd}}^{L} b_i T_i + \sum_{i=2,\text{odd}}^{L+2} S_i\end{pmatrix}.$$

From the a priori bound $|b_{1,s}|\lesssim b_1^2$,

$$|\partial_s(\chi_{B_1}) + b_1(y\chi')_{B_1}| \lesssim \left(\frac{|b_{1,s}|}{b_1} + b_1\right)|(y\chi')_{B_1}| \lesssim b_1\mathbf{1}_{y\sim B_1}. \tag{2-95}$$

One can easily check that (2-93) still holds even if we replace the cutoff function χ_{B_1} with other cutoff functions supported in $y\sim B_1$. Hence the cutoff asymptotics (2-93) and the admissibility of $\boldsymbol{T}_i$ imply, for $1\le i\le L$,

$$\begin{aligned}\|b_i\mathcal{A}^{k-\bar{i}}(\partial_s(\chi_{B_1}) + b_1(y\chi')_{B_1})T_i\|_{L^2} &\lesssim \sum_{j=0}^{k-\bar{i}} b_1|b_i|\|y^{-(k-j-\bar{i})}\mathcal{A}^jT_i\|_{L^2(y\sim B_1)} \lesssim b_1|b_i|\|y^{i-k-1}|\log y|\|_{L^2(y\sim B_1)} \\ &\lesssim b_1^{k+1-i}|b_i||\log b_1|^{\gamma(i-k)+1},\end{aligned} \tag{2-96}$$

and, for $2\le i\le L+2$, Lemma 2.7 implies

$$\|\mathcal{A}^{k-\bar{i}}(\partial_s(\chi_{B_1}) + b_1(y\chi')_{B_1})S_i\|_{L^2} \lesssim b_1\sum_{j=0}^{k-\bar{i}}\|y^{-(k-j-\bar{i})}\mathcal{A}^jS_i\|_{L^2(y\sim B_1)} \lesssim b_1^{k+1}|\log b_1|^{\gamma(i-k-2)-1}, \tag{2-97}$$

so we obtain the global bounds (2-86) and (2-87). In (2-96), we cannot cancel $\log y$ from T_i: the additional $|\log b_1|$ appears. Thus, we need to choose $\gamma = 1+\bar{\ell}$ for the case $(k,i)=(L+1,L)$, which corresponds to (2-88). We note that $\gamma = 1$ when $\ell = L-1$ since we have the additional $|\log b_1|$ gain of b_L from (2-83).

The third term in the right-hand side of (2-92) can be estimated as

$$\|b_1\mathcal{A}^k(1-\chi_{B_1})\Lambda Q\|_{L^2} \lesssim b_1\|y^{-k-1}\|_{L^2(y\geq B_1)} \lesssim \frac{b_1^{k+1}}{|\log b_1|^{\gamma k}}.$$

Finally, we compute (2-92):

$$\Delta(\chi_{B_1}\alpha_b)-\chi_{B_1}\Delta(\alpha_b) = (\Delta\chi_{B_1})\alpha_b + 2\partial_y(\chi_{B_1})\partial_y(\alpha_b),$$

$$f(\widetilde{Q}_b)-f(Q)-\chi_{B_1}(f(Q_b)-f(Q)) = \chi_{B_1}\alpha_b\int_0^1 [f'(Q+\tau\chi_{B_1}\alpha_b)-f'(Q+\tau\alpha_b)]\,d\tau,$$

and we can easily check that each term is localized in $y\sim B_1$. In this region, the rough bounds

$$|f^{(k)}|\lesssim 1 \quad\text{and}\quad |\partial_y^k Q|+|\partial_y^k\chi_{B_1}|\lesssim y^{-k}$$

yield

$$\left|\frac{\partial^k}{\partial y^k}\left(\Delta(\chi_{B_1}\alpha_b)-\chi_{B_1}\Delta(\alpha_b)+\frac{f(\widetilde{Q}_b)-f(Q)-\chi_{B_1}(f(Q_b)-f(Q))}{y^2}\right)\right| \lesssim \frac{|\alpha_b|}{y^{k+2}},$$

and we can borrow the estimation of $\partial_s(\chi_{B_1})\alpha_b$, namely (2-96) and (2-97).

The logarithmic weighted bounds (2-89) and (2-90) basically come from the fact that $|\log y|\sim|\log b_1|$ on $y\sim B_1$. We further use the decay property $|\log y|^C/y\to 0$ as $y\to\infty$ for the third term in the right-hand side of (2-92). $\square$

We also introduce another localization that depends on ℓ to verify the further regularity found in the remark after Theorem 1.2 on page 2419.

Proposition 2.11 (localization for the case when $\ell = L$). *Assume the hypotheses of Proposition 2.10. Then the localized profile $\widehat{\boldsymbol{Q}}_b$ given by*

$$\widehat{\boldsymbol{Q}}_b = \widetilde{\boldsymbol{Q}}_b+\boldsymbol{\zeta}_b := \widetilde{\boldsymbol{Q}}_b+(\chi_{B_0}-\chi_{B_1})b_L\boldsymbol{T}_L \tag{2-98}$$

drives the equation

$$\partial_s\widehat{\boldsymbol{Q}}_b-\boldsymbol{F}(\widehat{\boldsymbol{Q}}_b)+b_1\boldsymbol{\Lambda}\widehat{\boldsymbol{Q}}_b = \widehat{\mathbf{Mod}}(t)+\hat{\boldsymbol{\psi}}_b, \tag{2-99}$$

where $\widehat{\mathbf{Mod}}(t)$ is given by

$$\widehat{\mathbf{Mod}}(t) = \chi_{B_1}\mathbf{Mod}(t)+(\chi_{B_0}-\chi_{B_1})((b_L)_s+(L-1+c_{b,L})b_1b_L)\boldsymbol{T}_L \tag{2-100}$$

and $\hat{\boldsymbol{\psi}}_b = (\hat{\psi}_b,\dot{\hat{\psi}}_b)^t$ satisfies the bounds

$$\|\mathcal{A}^L(\hat{\psi}_b-(\chi_{B_1}-\chi_{B_0})b_LT_{L-1})\|_{L^2}\lesssim b_1^{L+1}, \tag{2-101}$$

$$\|\mathcal{A}^{L-1}(\dot{\hat{\psi}}_b-(\partial_s\chi_{B_0}+b_1(y\chi')_{B_0})b_LT_L)\|_{L^2}\lesssim b_1^{L+1}. \tag{2-102}$$

Proof. Note that $\boldsymbol{F}(\widetilde{\boldsymbol{Q}}_b+\boldsymbol{\zeta}_b)-\boldsymbol{F}(\widetilde{\boldsymbol{Q}}_b)=(\chi_{B_0}-\chi_{B_1})b_L\boldsymbol{T}_{L-1}$. From (2-67) and (2-56), we have

$$\begin{aligned}\partial_s\widehat{\boldsymbol{Q}}_b-\boldsymbol{F}(\widehat{\boldsymbol{Q}}_b)+b_1\boldsymbol{\Lambda}\widehat{\boldsymbol{Q}}_b&=\chi_{B_1}\mathbf{Mod}(t)+\tilde{\boldsymbol{\psi}}_b+\partial_s\boldsymbol{\zeta}_b-(\boldsymbol{F}(\widetilde{\boldsymbol{Q}}_b+\boldsymbol{\zeta}_b)-\boldsymbol{F}(\widetilde{\boldsymbol{Q}}_b))+b_1\boldsymbol{\Lambda}\boldsymbol{\zeta}_b\\&=\widehat{\mathbf{Mod}}(t)+b_1b_L(\chi_{B_0}-\chi_{B_1})\{(-\boldsymbol{H})^{L+2}\widetilde{\Sigma}_{b_1}+\boldsymbol{\theta}_L\}\\&\quad+\tilde{\boldsymbol{\psi}}_b-(\partial_s(\chi_{B_1})+b_1(y\chi')_{B_1})b_L\boldsymbol{T}_L\\&\quad\quad+(\partial_s(\chi_{B_0})+b_1(y\chi')_{B_0})b_L\boldsymbol{T}_L+(\chi_{B_1}-\chi_{B_0})b_L\boldsymbol{T}_{L-1}.\end{aligned}\tag{2-103}$$

From the above identity, we can see that the last line of (2-103) is exactly subtracted from $\hat{\boldsymbol{\psi}}_b$ in (2-101) and (2-102). Hence we need to estimate the second term and second line of the right-hand side of (2-103). We point out that the logarithm weight $|\log b_1|$ in (2-87) comes from the estimate (2-96) when $i=L$, which is eliminated in the second line of the right-hand side of (2-103). For the second term of the right-hand side of (2-103), we can borrow the bound (2-80) and use Lemma 2.7. ☐

Proposition 2.12 (localization for the case when $\ell=L-1$). *Assume the hypotheses of Proposition 2.10. Then the localized profile $\widehat{\boldsymbol{Q}}_b$ given by*

$$\widehat{\boldsymbol{Q}}_b=\widetilde{\boldsymbol{Q}}_b+\boldsymbol{\zeta}_b:=\widetilde{\boldsymbol{Q}}_b+(\chi_{B_0}-\chi_{B_1})(b_{L-1}\boldsymbol{T}_{L-1}+b_L\boldsymbol{T}_L)\tag{2-104}$$

drives the equation

$$\partial_s\widehat{\boldsymbol{Q}}_b-\boldsymbol{F}(\widehat{\boldsymbol{Q}}_b)+b_1\boldsymbol{\Lambda}\widehat{\boldsymbol{Q}}_b=\widehat{\mathbf{Mod}}(t)+\hat{\boldsymbol{\psi}}_b,\tag{2-105}$$

where $\widehat{\mathbf{Mod}}(t)$ is given by

$$\begin{aligned}\widehat{\mathbf{Mod}}(t)=\chi_{B_1}\mathbf{Mod}(t)+(\chi_{B_0}-\chi_{B_1})((b_{L-1})_s+(L-2+c_{b,L-1})b_1b_{L-1})\boldsymbol{T}_{L-1}&\\+(\chi_{B_0}-\chi_{B_1})((b_L)_s+(L-1+c_{b,L})b_1b_L)\boldsymbol{T}_L&\end{aligned}\tag{2-106}$$

and $\hat{\boldsymbol{\psi}}_b=(\hat{\psi}_b,\dot{\hat{\psi}}_b)^t$ satisfies the bounds

$$\|\mathcal{A}^{L-1}(\hat{\psi}_b-(\partial_s\chi_{B_0}+b_1(y\chi')_{B_0})b_{L-1}T_{L-1}-(\chi_{B_1}-\chi_{B_0})b_LT_{L-1})\|_{L^2}\lesssim b_1^L,\tag{2-107}$$

$$\|\mathcal{A}^{L-2}(\dot{\hat{\psi}}_b-(\partial_s\chi_{B_0}+b_1(y\chi')_{B_0})b_LT_L+b_{L-1}H(\chi_{B_1}-\chi_{B_0})T_L)\|_{L^2}\lesssim b_1^L.\tag{2-108}$$

Remark. We point out that Propositions 2.11 and 2.12 provide improved bounds (2-101), (2-102), (2-107) and (2-108) compared to (2-86) and (2-87) in Proposition 2.10. These improved bounds will be essential to prove the monotonicity formula (4-12) later.

Proof. Note that

$$\boldsymbol{F}(\widetilde{\boldsymbol{Q}}_b+\boldsymbol{\zeta}_b)-\boldsymbol{F}(\widetilde{\boldsymbol{Q}}_b)=-\boldsymbol{H}\boldsymbol{\zeta}_b-\boldsymbol{NL}(\boldsymbol{\zeta}_b)-\boldsymbol{L}(\boldsymbol{\zeta}_b),\tag{2-109}$$

where

$$\boldsymbol{NL}(\boldsymbol{\zeta}_b)=\begin{pmatrix}0\\NL(\zeta_b)\end{pmatrix}:=\frac{1}{y^2}\begin{pmatrix}0\\f(\widetilde{Q}_b+\zeta_b)-f(\widetilde{Q}_b)-f'(\widetilde{Q}_b)\zeta_b\end{pmatrix},\tag{2-110}$$

$$\boldsymbol{L}(\boldsymbol{\zeta}_b)=\begin{pmatrix}0\\L(\zeta_b)\end{pmatrix}:=\frac{1}{y^2}\begin{pmatrix}0\\(f'(\widetilde{Q}_b)-f'(Q))\zeta_b\end{pmatrix}.\tag{2-111}$$

From (2-67) and (2-56), we have

$$
\begin{aligned}
\partial_s \widehat{\boldsymbol{Q}}_b - \boldsymbol{F}(\widehat{\boldsymbol{Q}}_b) + b_1 \boldsymbol{\Lambda} \widehat{\boldsymbol{Q}}_b &= \chi_{B_1} \mathbf{Mod}(t) + \tilde{\boldsymbol{\psi}}_b + \partial_s \boldsymbol{\zeta}_b - (\boldsymbol{F}(\widetilde{\boldsymbol{Q}}_b + \boldsymbol{\zeta}_b) - \boldsymbol{F}(\widetilde{\boldsymbol{Q}}_b)) + b_1 \boldsymbol{\Lambda} \boldsymbol{\zeta}_b \\
&= \widehat{\mathbf{Mod}}(t) + b_1 b_{L-1} (\chi_{B_0} - \chi_{B_1}) \{(-\boldsymbol{H})^{L+1} \widetilde{\Sigma}_{b_1} + \boldsymbol{\theta}_{L-1}\} \\
&\quad + b_1 b_L (\chi_{B_0} - \chi_{B_1}) \{(-\boldsymbol{H})^{L+2} \widetilde{\Sigma}_{b_1} + \boldsymbol{\theta}_L\} + \boldsymbol{NL}(\boldsymbol{\zeta}_b) + \boldsymbol{L}(\boldsymbol{\zeta}_b) \\
&\quad + \tilde{\boldsymbol{\psi}}_b - (\partial_s(\chi_{B_1}) + b_1 (y\chi')_{B_1})(b_{L-1} \boldsymbol{T}_{L-1} + b_L \boldsymbol{T}_L) \\
&\quad + (\partial_s(\chi_{B_0}) + b_1 (y\chi')_{B_0}) b_L \boldsymbol{T}_L + (\chi_{B_1} - \chi_{B_0}) b_L \boldsymbol{T}_{L-1} + \boldsymbol{H} \boldsymbol{\zeta}_b.
\end{aligned}
\tag{2-112}
$$

Based on the proof of the previous proposition, it suffices to show that

$$
\|\mathcal{A}^{L-2} NL(\zeta_b)\|_{L^2} + \|\mathcal{A}^{L-2} L(\zeta_b)\|_{L^2} \lesssim b_1^L,
$$

which comes from the following crude pointwise bounds on $B_0 \le y \le 2B_1$: for $k \ge 0$,

$$
|\mathcal{A}^k NL(\zeta_b)| \lesssim b_1^{2L-2} y^{2L-6-k} |\log b_1|^C, \quad |\mathcal{A}^k L(\zeta_b)| \lesssim b_1^L y^{L-4-k} |\log b_1|^C. \qquad \square
$$

2.8. *Dynamical laws of $b = (b_1, \dots, b_L)$.* As previously mentioned, the blow-up rate is determined by the evolution of the vector b, so we will figure out its dynamical laws from (2-56): for $1 \le k \le L$,

$$
(b_k)_s = b_{k+1} - \left(k - 1 + \frac{1}{(1+\delta_{1k}) \log s}\right) b_1 b_k, \quad b_{L+1} = 0. \tag{2-113}
$$

One can check that the above system has L independent solutions characterized by the number of nonzero coordinates: for $1 \le k \le L$, we have $b = (b_1, \dots, b_k, 0, \dots, 0)$. Here, we adopt two special solutions (recall that there are two ℓs that can achieve the same L) among them.

Lemma 2.13 (special solutions for the b system). *For all $\ell \ge 2$, the vector of functions*

$$
b_k^e(s) = \frac{c_k}{s^k} + \frac{d_k}{s^k \log s} \quad \text{for } 1 \le k \le \ell, \qquad b_k^e \equiv 0 \quad \text{for } k > \ell \tag{2-114}
$$

solves (2-113) approximately: for $1 \le k \le L$,

$$
(b_k^e)_s + \left(k - 1 + \frac{1}{(1+\delta_{1k}) \log s}\right) b_1^e b_k^e - b_{k+1}^e = O\left(\frac{1}{s^{k+1} (\log s)^2}\right) \quad \text{as } s \to +\infty, \tag{2-115}
$$

where the sequence $(c_k, d_k)_{k=1,\dots,\ell}$ is given by

$$
c_1 = \frac{\ell}{\ell - 1}, \quad c_{k+1} = -\frac{\ell - k}{\ell - 1} c_k, \quad 1 \le k \le \ell, \tag{2-116}
$$

and, for $2 \le k \le \ell - 1$,

$$
d_1 = -\frac{\ell}{(\ell-1)^2}, \quad d_2 = -d_1 + \frac{1}{2} c_1^2, \quad d_{k+1} = -\frac{\ell - k}{\ell - 1} d_k + \frac{\ell(\ell - k)}{(\ell - 1)^2} c_k. \tag{2-117}
$$

Remark. The recurrence relations (2-116) and (2-117) are obtained by substituting (2-114) into (2-115) and comparing the coefficients of s^{-k} and $(s^k \log s)^{-1}$, yielding the proof.

For our solution b to drive the system like the special solution b^e, we should control the fluctuation

$$\frac{U_k(s)}{s^k(\log s)^\beta} := b_k(s) - b_k^e(s) \quad \text{for } 1 \le k \le \ell. \tag{2-118}$$

Here, (2-114) and (2-115) restrict the range of β to $1 < \beta < 2$; we will choose $\beta = \frac{5}{4}$ later. The next lemma provides the evolution of $U = (U_1, \dots, U_\ell)$ from (2-113).

Lemma 2.14 (evolution of U). *Let $b_k(s)$ be a solution to (2-113) and U be defined by (2-118). Then U solves*

$$s(U)_s = A_\ell U + O\left(\frac{1}{(\log s)^{2-\beta}} + \frac{|U| + |U|^2}{\log s}\right), \tag{2-119}$$

where the $\ell \times \ell$ matrix A_ℓ has the form

$$A_\ell = \begin{pmatrix} 1 & 1 & & & & \\ -c_2 & \frac{\ell-2}{\ell-1} & 1 & & (0) & \\ -2c_3 & & \frac{\ell-3}{\ell-1} & 1 & & \\ \vdots & & & \ddots & \ddots & \\ -(\ell-2)c_{\ell-1} & & (0) & & \frac{1}{\ell-1} & 1 \\ -(\ell-1)c_\ell & & & & & 0 \end{pmatrix}. \tag{2-120}$$

Moreover, there exists an invertible matrix P_ℓ such that $A_\ell = P_\ell^{-1} D_\ell P_\ell$ with

$$D_\ell = \begin{pmatrix} -1 & & & & & \\ & \frac{2}{\ell-1} & & & (0) & \\ & & \frac{3}{\ell-1} & & & \\ & & & \ddots & & \\ & (0) & & & 1 & \\ & & & & & \frac{\ell}{\ell-1} \end{pmatrix}. \tag{2-121}$$

Proof. Observing the relation

$$(k-1)c_1 - k = \frac{(k-1)\ell}{\ell-1} - k = -\frac{\ell-k}{\ell-1},$$

we obtain (2-119) and (2-120) since

$$\begin{aligned}
&(b_k)_s + \left(k - 1 + \frac{1}{(1+\delta_{1k})\log s}\right) b_1 b_k - b_{k+1} \\
&= \frac{1}{s^{k+1}(\log s)^\beta}\left[s(U_k)_s - kU_k + O\left(\frac{|U|}{\log s}\right)\right] + O\left(\frac{1}{s^{k+1}(\log s)^2}\right) \\
&\qquad + \frac{1}{s^{k+1}(\log s)^\beta}\left[(k-1)c_k U_1 + (k-1)c_1 U_k - U_{k+1} + O\left(\frac{|U| + |U|^2}{\log s}\right)\right] \\
&= \frac{1}{s^{k+1}(\log s)^\beta}\left[s(U_k)_s + (k-1)c_k U_1 - \frac{\ell-k}{\ell-1} U_k - U_{k+1}\right] \\
&\qquad + O\left(\frac{1}{s^{k+1}(\log s)^2} + \frac{|U| + |U^2|}{s^{k+1}(\log s)^{1+\beta}}\right).
\end{aligned} \tag{2-122}$$

Equation (2-121) is obtained by substituting $\alpha = 1$ in [Collot 2018, Lemma 2.17]. □

Remark. Since the above process can be seen as linearizing (2-113) around our special solution b^e, the appearance of the matrix A_ℓ is quite natural. We also note that $\ell-1$ unstable directions corresponding to $\ell-1$ positive eigenvalues yield the (formal) codimension $(\ell-1)$ restriction of our initial data.

3. The trapped solutions

Our goal in this section is to implement the blow-up dynamics constructed in the previous section into the real solution $\boldsymbol{u}$. To do this, we first decompose the solution $\boldsymbol{u}$ as the blow-up profile and the error, i.e., $\boldsymbol{u} = (\widetilde{\boldsymbol{Q}}_b + \boldsymbol{\varepsilon})_\lambda = \widetilde{\boldsymbol{Q}}_{b,\lambda} + \boldsymbol{w}$. For the term "error" to be meaningful, we need to control the "direction" and "size" of $\boldsymbol{w} = \boldsymbol{\varepsilon}_\lambda$.

Here, $\boldsymbol{\varepsilon}$ must be orthogonal to the directions that provoke blowup from $\widetilde{\boldsymbol{Q}}_{b,\lambda}$. Such orthogonal conditions determine the system modulation equations of the dynamical parameters b as designed in Section 2.8.

In this process, $\boldsymbol{\varepsilon}$ appears as an error that is small in some suitable norms. The smallness is required in order to keep the leading-order evolution laws unchanged (2-113). We describe the set of initial data and the trapped conditions represented by some bootstrap bounds for such suitable norms, i.e, the higher-order energies.

After establishing estimates of modulation parameters, we also establish a Lyapunov-type monotonicity of the higher-order energies to close our bootstrap assumptions.

3.1. ***Decomposition of the flow.*** We recall the approximate direction $\boldsymbol{\Phi}_M$ which was defined in [Collot 2018]. For a large constant $M > 0$, we define

$$\boldsymbol{\Phi}_M = \sum_{p=0}^{L} c_{p,M} \boldsymbol{H}^{*p}(\chi_M \boldsymbol{\Lambda Q}), \qquad \boldsymbol{H}^* = \begin{pmatrix} 0 & H \\ -1 & 0 \end{pmatrix}, \tag{3-1}$$

where $c_{p,M}$ is given by

$$c_{0,M} = 1, \qquad c_{k,M} = (-1)^{k+1} \frac{\sum_{p=0}^{k-1} c_{p,M} \langle \boldsymbol{H}^{*p}(\chi_M \boldsymbol{\Lambda Q}), \boldsymbol{T}_k\rangle}{\langle \chi_M \boldsymbol{\Lambda Q}, \boldsymbol{\Lambda Q}\rangle}, \quad 1 \le k \le L. \tag{3-2}$$

One can easily verify (see [Collot 2018, Section 3.1.1]) that $\boldsymbol{H}^*$ is an adjoint operator of $\boldsymbol{H}$ in the sense that

$$\langle \boldsymbol{H}\boldsymbol{u}, \boldsymbol{v}\rangle = \langle \boldsymbol{u}, \boldsymbol{H}^*\boldsymbol{v}\rangle,$$

and $\boldsymbol{\Phi}_M = (\Phi_M, 0)$ satisfies

$$\langle \boldsymbol{\Phi}_M, \boldsymbol{\Lambda Q}\rangle = \langle \chi_M \boldsymbol{\Lambda Q}, \boldsymbol{\Lambda Q}\rangle \sim 4\log M, \quad |c_{p,M}| \lesssim M^p, \quad \|\Phi_M\|_{L^2}^2 \sim c\log M. \tag{3-3}$$

We then obtain our desired decomposition by imposing a collection of orthogonal directions, which approximates the generalized kernel defined in Definition 2.3.

Lemma 3.1 (decomposition). *Let $\boldsymbol{u}(t)$ be a solution to (1-21) starting close enough to $\boldsymbol{Q}$ in $\mathcal{H}$. Then there exist C^1 functions $\lambda(t)$ and $b(t) = (b_1, \dots, b_L)$ such that $\boldsymbol{u}$ can be decomposed as*

$$\boldsymbol{u} = (\widetilde{\boldsymbol{Q}}_{b(t)} + \boldsymbol{\varepsilon})_{\lambda(t)}, \tag{3-4}$$

where $\widetilde{\boldsymbol{Q}}_b$ *is given in Proposition 2.10 and* $\boldsymbol{\varepsilon}$ *satisfies the orthogonality conditions*

$$\langle \boldsymbol{\varepsilon}, \boldsymbol{H}^{*i}\boldsymbol{\Phi}_M\rangle = 0, \quad \textit{for } 0 \le i \le L. \tag{3-5}$$

and an orbital stability estimate

$$|b(t)| + \|\boldsymbol{\varepsilon}\|_{\mathcal{H}} \ll 1. \tag{3-6}$$

Remark. Equation (3-7) says that the elements of $\{\langle\,\cdot\,, \boldsymbol{H}^{*i}\boldsymbol{\Phi}_M\rangle\}_{i\ge 0}$ serve as coordinate functions on the space $\operatorname{Span}\{\boldsymbol{T}_i\}_{i\ge 0}$.

Proof. It is clear that $\boldsymbol{H}^i\boldsymbol{T}_j = 0$ for $i > j$. For $0 \le i \le j$,

$$\begin{aligned}\langle \boldsymbol{\Phi}_M, \boldsymbol{H}^i\boldsymbol{T}_j\rangle &= (-1)^i\langle \boldsymbol{\Phi}_M, \boldsymbol{T}_{j-i}\rangle \\ &= (-1)^i\sum_{p=0}^{j-i-1} c_{p,M}\langle \boldsymbol{H}^{*p}(\chi_M\boldsymbol{\Lambda}\boldsymbol{Q}), \boldsymbol{T}_{j-i}\rangle + (-1)^j c_{j-i,M}\langle \chi_M\boldsymbol{\Lambda}\boldsymbol{Q}, \boldsymbol{\Lambda}\boldsymbol{Q}\rangle \\ &= (-1)^j\langle \chi_M\boldsymbol{\Lambda}\boldsymbol{Q}, \boldsymbol{\Lambda}\boldsymbol{Q}\rangle\delta_{i,j}.\end{aligned} \tag{3-7}$$

Now, we consider $\boldsymbol{\varepsilon} := \boldsymbol{u}_{1/\lambda} - \widetilde{\boldsymbol{Q}}_b$ as a map in the $(\lambda, b, \boldsymbol{u})$ basis. By the implicit function theorem, (3-4) is deduced from the nondegeneracy of the Jacobian

$$\begin{aligned}\left|\left(\frac{\partial}{\partial(\lambda, b)}\langle \boldsymbol{\varepsilon}, \boldsymbol{H}^{*i}\boldsymbol{\Phi}_M\rangle\right)_{0\le i\le L}\right|_{(\lambda,b,\boldsymbol{u})=(1,0,\boldsymbol{Q})} &= (-1)^{L+1}|(\langle \boldsymbol{T}_j, \boldsymbol{H}^{*i}\boldsymbol{\Phi}_M\rangle)_{0\le i,j\le L}| \\ &= |(\langle \boldsymbol{\Phi}_M, \boldsymbol{H}^i\boldsymbol{T}_j\rangle)_{0\le i,j\le L}| \\ &= |((-1)^j\langle \chi_M\boldsymbol{\Lambda}\boldsymbol{Q}, \boldsymbol{\Lambda}\boldsymbol{Q}\rangle\delta_{i,j})_{0\le i,j\le L}| \\ &= (-1)^{(L+1)/2}\langle \chi_M\boldsymbol{\Lambda}\boldsymbol{Q}, \boldsymbol{\Lambda}\boldsymbol{Q}\rangle^{L+1} \ne 0.\end{aligned}$$

□

3.2. ***Equation for the error.*** Based on the previously established decomposition

$$\boldsymbol{u} = \widetilde{\boldsymbol{Q}}_{b(t),\lambda(t)} + \boldsymbol{w} = (\widetilde{\boldsymbol{Q}}_{b(s)} + \boldsymbol{\varepsilon}(s))_{\lambda(s)},$$

(1-21) turns into the following evolution equation of $\boldsymbol{\varepsilon}$:

$$\begin{aligned}\partial_s\boldsymbol{\varepsilon} - \frac{\lambda_s}{\lambda}\boldsymbol{\Lambda}\boldsymbol{\varepsilon} + \boldsymbol{H}\boldsymbol{\varepsilon} &= -\left(\partial_s\widetilde{\boldsymbol{Q}}_b - \frac{\lambda_s}{\lambda}\boldsymbol{\Lambda}\widetilde{\boldsymbol{Q}}_b\right) + \boldsymbol{F}(\widetilde{\boldsymbol{Q}}_b + \boldsymbol{\varepsilon}) + \boldsymbol{H}\boldsymbol{\varepsilon} \\ &= -(\partial_s\widetilde{\boldsymbol{Q}}_b - \boldsymbol{F}(\widetilde{\boldsymbol{Q}}_b) + b_1\boldsymbol{\Lambda}\widetilde{\boldsymbol{Q}}_b) + \left(\frac{\lambda_s}{\lambda} + b_1\right)\boldsymbol{\Lambda}\widetilde{\boldsymbol{Q}}_b + \boldsymbol{F}(\widetilde{\boldsymbol{Q}}_b + \boldsymbol{\varepsilon}) - \boldsymbol{F}(\widetilde{\boldsymbol{Q}}_b) + \boldsymbol{H}\boldsymbol{\varepsilon} \\ &= -\widetilde{\mathbf{Mod}}(t) - \tilde{\boldsymbol{\psi}}_b - \boldsymbol{NL}(\boldsymbol{\varepsilon}) - \boldsymbol{L}(\boldsymbol{\varepsilon}),\end{aligned} \tag{3-8}$$

where

$$\widetilde{\mathbf{Mod}}(t) := \chi_{B_1}\mathbf{Mod}(t) - \left(\frac{\lambda_s}{\lambda} + b_1\right)\boldsymbol{\Lambda}\widetilde{\boldsymbol{Q}}_b, \quad \widetilde{\mathbf{Mod}}(t) := \begin{pmatrix}\widetilde{\mathrm{Mod}}(t)\\ \dot{\widetilde{\mathrm{Mod}}}(t)\end{pmatrix} \tag{3-9}$$

$$\boldsymbol{NL}(\boldsymbol{\varepsilon}) := \frac{1}{y^2}\begin{pmatrix}0\\ f(\widetilde{Q}_b + \varepsilon) - f(\widetilde{Q}_b) - f'(\widetilde{Q}_b)\varepsilon\end{pmatrix}, \quad \boldsymbol{L}(\boldsymbol{\varepsilon}) := \frac{1}{y^2}\begin{pmatrix}0\\ (f'(\widetilde{Q}_b) - f'(Q))\varepsilon\end{pmatrix}. \tag{3-10}$$

For later analysis, we also employ the following evolution equation of $\boldsymbol{w}$:

$$\partial_t\boldsymbol{w} + \boldsymbol{H}_\lambda\boldsymbol{w} = \frac{1}{\lambda}\boldsymbol{\mathcal{F}}_\lambda, \quad \boldsymbol{\mathcal{F}} = -\widetilde{\mathbf{Mod}}(t) - \tilde{\boldsymbol{\psi}}_b - \boldsymbol{NL}(\boldsymbol{\varepsilon}) - \boldsymbol{L}(\boldsymbol{\varepsilon}), \tag{3-11}$$

where

$$\boldsymbol{H}_\lambda = \begin{pmatrix} 0 & -1 \\ H_\lambda & +0 \end{pmatrix} := \begin{pmatrix} 0 & -1 \\ -\Delta + r^{-2} f'(Q_\lambda) & +0 \end{pmatrix}, \tag{3-12}$$

We notice that the $\boldsymbol{NL}$ and $\boldsymbol{L}$ terms are situated on the second coordinate:

$$\boldsymbol{NL}(\boldsymbol{\varepsilon}) = \begin{pmatrix} 0 \\ NL(\varepsilon) \end{pmatrix}, \quad \boldsymbol{L}(\boldsymbol{\varepsilon}) = \begin{pmatrix} 0 \\ L(\varepsilon) \end{pmatrix}. \tag{3-13}$$

We also introduce another decomposition

$$\boldsymbol{u} = \widehat{\boldsymbol{Q}}_{b(t),\lambda(t)} + \hat{\boldsymbol{w}} = (\widehat{\boldsymbol{Q}}_{b(s)} + \hat{\boldsymbol{\varepsilon}}(s))_{\lambda(s)},$$

which depends on whether $\ell = L$ (Proposition 2.11) or $\ell = L-1$ (Proposition 2.12). The evolution equation of $\hat{\boldsymbol{\varepsilon}}$ is given by

$$\partial_s \hat{\boldsymbol{\varepsilon}} - \frac{\lambda_s}{\lambda} \boldsymbol{\Lambda} \hat{\boldsymbol{\varepsilon}} + \boldsymbol{H} \hat{\boldsymbol{\varepsilon}} = -\widehat{\mathbf{Mod}}'(t) - \hat{\boldsymbol{\psi}}_b - \widehat{\boldsymbol{NL}}(\hat{\boldsymbol{\varepsilon}}) - \widehat{\boldsymbol{L}}(\hat{\boldsymbol{\varepsilon}}), \tag{3-14}$$

where

$$\widehat{\mathbf{Mod}}'(t) := \widehat{\mathbf{Mod}}(t) - \left(\frac{\lambda_s}{\lambda} + b_1\right) \boldsymbol{\Lambda} \widehat{\boldsymbol{Q}}_b, \tag{3-15}$$

$$\widehat{\boldsymbol{NL}}(\hat{\boldsymbol{\varepsilon}}) := \frac{1}{y^2} \begin{pmatrix} 0 \\ f(\widehat{Q}_b + \hat{\varepsilon}) - f(\widehat{Q}_b) - f'(\widehat{Q}_b)\hat{\varepsilon} \end{pmatrix}, \quad \widehat{\boldsymbol{L}}(\hat{\boldsymbol{\varepsilon}}) := \frac{1}{y^2} \begin{pmatrix} 0 \\ (f'(\widehat{Q}_b) - f'(Q))\hat{\varepsilon} \end{pmatrix}. \tag{3-16}$$

We also employ the evolution equation of $\hat{\boldsymbol{w}}$:

$$\partial_t \hat{\boldsymbol{w}} + \boldsymbol{H}_\lambda \hat{\boldsymbol{w}} = \frac{1}{\lambda} \widehat{\boldsymbol{\mathcal{F}}}_\lambda, \quad \widehat{\boldsymbol{\mathcal{F}}} = -\widehat{\mathbf{Mod}}'(t) - \hat{\boldsymbol{\psi}}_b - \widehat{\boldsymbol{NL}}(\hat{\boldsymbol{\varepsilon}}) - \widehat{\boldsymbol{L}}(\hat{\boldsymbol{\varepsilon}}). \tag{3-17}$$

3.3. ***Initial data setting for the bootstrap.*** In this subsection, we describe our initial data and the bootstrap assumption. To do this, we recall the fluctuation (2-118), i.e., $U = (U_1, \dots, U_\ell)$,

$$U_k(s) = s^k (\log s)^\beta (b_k(s) - b_k^e(s)).$$

We also define the adapted higher-order energies given by

$$\mathcal{E}_k := \langle \varepsilon_k, \varepsilon_k \rangle + \langle \dot{\varepsilon}_{k-1}, \dot{\varepsilon}_{k-1} \rangle, \quad 2 \le k \le L+1. \tag{3-18}$$

We set our renormalized space-time variables (s, y) as follows: for a large enough $s_0 \gg 1$,

$$y = \frac{r}{\lambda(t)}, \quad s(t) = s_0 + \int_0^t \frac{d\tau}{\lambda(\tau)}.$$

For the sake of simplicity, we use a transformed fluctuation $V = (V_1(s), \dots, V_\ell(s))$,

$$V = P_\ell U, \tag{3-19}$$

where P_ℓ yields the diagonalization (2-121). Then we illustrate the modulation parameters b as a sum of the exact solutions $b^e(s)$ and $V(s)$: for $\ell = L-1$ or L,

$$b(s) = b^e(s) + \left(\frac{(P_\ell^{-1} V(s))_1}{s (\log s)^\beta}, \dots, \frac{(P_\ell^{-1} V(s))_\ell}{s^\ell (\log s)^\beta}, b_{\ell+1}(s), \dots, b_L(s) \right).$$

Now, we assume some smallness conditions for our initial data $\boldsymbol{u}_0(s_0) = (u_0, \dot{u}_0)$ as follows: for large constants $M = M(L),\ K = K(L, M),\ s_0 = s_0(L, M, K)$, we set the initial data $\boldsymbol{u}_0 = \boldsymbol{u}(s_0)$ as

$$\boldsymbol{u}_0 = (\widetilde{\boldsymbol{Q}}_{b(s_0)} + \boldsymbol{\varepsilon}(s_0))_{\lambda(s_0)}, \tag{3-20}$$

where $\boldsymbol{\varepsilon}(s_0)$ satisfies the orthogonality conditions (3-5), we have the smallness of higher-order energies

$$\mathcal{E}_k(s_0) \le b_1^{2L+4}(s_0), \tag{3-21}$$

and $b(s_0)$ satisfies the smallness of the stable modes:

$$\begin{aligned} |V_1(s_0)| &\le \frac{1}{4}, \\ |b_L(s_0)| &\le \frac{1}{s_0^{(L-1)c_1}(\log s_0)^{3/2}} \quad \text{for } \ell = L-1, \end{aligned} \tag{3-22}$$

where $c_1 = \ell/(\ell-1)$. Furthermore, we may assume

$$\lambda(s_0) = 1 \tag{3-23}$$

up to rescaling.

Proposition 3.2 (existence of trapped solutions). *Given* $\boldsymbol{u}(s_0)$ *of the form* (3-20) *satisfying* (3-5), (3-21) *and* (3-22), *there exists an initial direction of the unstable modes*

$$(V_2(s_0), \dots, V_\ell(s_0)) \in \mathcal{B}^{\ell-1} \tag{3-24}$$

such that the corresponding solution to (1-21) *becomes* ***trapped***; *namely*, *it satisfies the following bounds for all* $s \ge s_0$:

- *Control of the higher-order energies*: *for* $2 \le k \le \ell - 1$,

$$\begin{aligned} \mathcal{E}_k(s) &\le b_1^{2(k-1)c_1}|\log b_1|^K, \\ \mathcal{E}_{L+1}(s) &\le K\frac{b_1^{2L+2}}{|\log b_1|^2}, \end{aligned} \tag{3-25}$$

$$\mathcal{E}_L(s) \le \begin{cases} K\lambda^{2(L-1)} & \textit{when } \ell = L, \\ b_1^{2L}|\log b_1|^K & \textit{when } \ell = L-1, \end{cases} \tag{3-26}$$

$$\mathcal{E}_{L-1}(s) \le K\lambda^{2(L-2)} \qquad \textit{when } \ell = L-1. \tag{3-27}$$

- *Control of the stable modes*:

$$\begin{aligned} |V_1(s)| &\le 1, \\ |b_L(s)| &\le \frac{1}{s^L(\log s)^\beta} \quad \textit{when } \ell = L-1. \end{aligned} \tag{3-28}$$

- *Control of the unstable modes*:

$$(V_2(s), \dots, V_\ell(s)) \in \mathcal{B}^{\ell-1}. \tag{3-29}$$

Under the initial setting of $(\boldsymbol{\varepsilon}(s_0), V(s_0), b_{\ell+1}(s_0), \dots, b_L(s_0))$ (see (3-20)–(3-22) and (3-24)), we define an exit time

$$s^* = \sup\{s \geq s_0 : \text{(3-25)–(3-29) hold on } [s_0, s]\}. \tag{3-30}$$

From (3-20)–(3-22) and (3-24), it is clear that (3-25)–(3-29) hold at $s = s_0$. We will prove Proposition 3.2 in Section 4 by contradiction, assume that

$$s^* < \infty \quad \text{for all } (V_2(s_0), \dots, V_\ell(s_0)) \in \mathcal{B}^{\ell-1}. \tag{3-31}$$

At the exit time s^*, we claim that only (3-29) fails among the bootstrap bounds in Proposition 3.2 through establishing estimates of modulation parameters and some monotonicity formulae of the higher-order energies. Then, the codimension $(\ell-1)$ stability (2-121) leads to a contradiction by Brouwer's fixed point theorem.

3.4. ***Modulation equations.*** Now we provide the evolution of the modulation parameters from the orthogonality conditions (3-5).

Lemma 3.3 (modulation equations). *The modulation parameters* $(\lambda, b_1, \dots, b_L)$ *satisfy the bounds*

$$\left|\frac{\lambda_s}{\lambda} + b_1\right| + \sum_{i=1}^{L-1} |(b_i)_s + (i-1+c_{b_1,i}) b_1 b_i - b_{i+1}| \lesssim C(M) b_1 (\sqrt{\mathcal{E}_{L+1}} + b_1^{L+2}), \tag{3-32}$$

$$|(b_L)_s + (L-1+c_{b_1,L}) b_1 b_L| \lesssim \frac{\sqrt{\mathcal{E}_{L+1}}}{\sqrt{\log M}} + C(M) b_1^{L+3}. \tag{3-33}$$

Remark. The bounds (3-32) and (3-25) allow us to obtain the a priori assumption (2-83).

Proof. Step 1: *Modulation identity*. Write $D(t) = (D_0(t), \dots, D_L(t))$, where $D_i(t)$ is given by

$$D_0(t) := -\left(\frac{\lambda_s}{\lambda} + b_1\right), \qquad D_i(t) := (b_i)_s + (i-1+c_{b_1,i}) b_1 b_i - b_{i+1}, \quad b_{L+1} = 0.$$

We take the vector-valued inner product (1-23) of (3-8) with $\boldsymbol{H}^{*k} \boldsymbol{\Phi}_M$ for $0 \leq k \leq L$. Then we have the identity

$$\begin{aligned}\langle \widetilde{\mathbf{Mod}}(t), \boldsymbol{H}^{*k} \boldsymbol{\Phi}_M \rangle &+ \langle \boldsymbol{H} \boldsymbol{\varepsilon}, \boldsymbol{H}^{*k} \boldsymbol{\Phi}_M \rangle \\ &= \frac{\lambda_s}{\lambda} \langle \boldsymbol{\Lambda} \boldsymbol{\varepsilon}, \boldsymbol{H}^{*k} \boldsymbol{\Phi}_M \rangle - \langle \tilde{\boldsymbol{\psi}}_b, \boldsymbol{H}^{*k} \boldsymbol{\Phi}_M \rangle - \langle \boldsymbol{NL}(\boldsymbol{\varepsilon}) + \boldsymbol{L}(\boldsymbol{\varepsilon}), \boldsymbol{H}^{*k} \boldsymbol{\Phi}_M \rangle. \end{aligned} \tag{3-34}$$

Step 2: *Estimates for each term in* (3-34). We claim that the left-hand side of (3-34) gives the main contribution needed to prove (3-32) and (3-33).

(i) $\widetilde{\mathbf{Mod}}(t)$ *terms*. First, $\chi_{B_1} \boldsymbol{\alpha}_b = \boldsymbol{\alpha}_b$ holds on $|y| \leq 2M$ for small enough b_1. We also have the pointwise bound

$$|\boldsymbol{\Lambda} \boldsymbol{\alpha}_b| + \sum_{i=1}^{L} \sum_{j=i+1}^{L+2} \left|\frac{\partial \boldsymbol{S}_j}{\partial b_i}\right| \lesssim b_1 C(M) \quad \text{for } |y| \leq 2M$$

from our blow-up profile construction. Hence we estimate the $\widetilde{\mathbf{Mod}}(t)$ term in (3-34) by the transversality (3-7) and the compact support property of $\mathbf{\Phi}_M$:

$$\begin{aligned}\langle \widetilde{\mathbf{Mod}}(t), \boldsymbol{H}^{*k}\mathbf{\Phi}_M\rangle &= D_0(t)\langle \mathbf{\Lambda}\boldsymbol{Q}_b, \boldsymbol{H}^{*k}\mathbf{\Phi}_M\rangle + \sum_{i=1}^{L} D_i(t)\Big\langle \boldsymbol{T}_i + \sum_{j=i+1}^{L+2}\frac{\partial \boldsymbol{S}_j}{\partial b_i}, \boldsymbol{H}^{*k}\mathbf{\Phi}_M\Big\rangle \\ &= \sum_{i=0}^{L} D_i(t)\langle \boldsymbol{T}_i, \boldsymbol{H}^{*k}\mathbf{\Phi}_M\rangle + \Big\langle D_0(t)\mathbf{\Lambda}\boldsymbol{\alpha}_b + \sum_{i=1}^{L}\sum_{j=i+1}^{L+2} D_i(t)\frac{\partial \boldsymbol{S}_j}{\partial b_i}, \boldsymbol{H}^{*k}\mathbf{\Phi}_M\Big\rangle \\ &= (-1)^k D_k(t)\langle \mathbf{\Lambda}\boldsymbol{Q}, \mathbf{\Phi}_M\rangle + O(C(M)b_1|D(t)|). \end{aligned} \tag{3-35}$$

(ii) *Linear terms*. For $0 \le k \le L-1$, we have

$$\langle \boldsymbol{H}\boldsymbol{\varepsilon}, \boldsymbol{H}^{*k}\mathbf{\Phi}_M\rangle = \langle \boldsymbol{\varepsilon}, \boldsymbol{H}^{*(k+1)}\mathbf{\Phi}_M\rangle = 0$$

from the orthogonal conditions (3-5). For $k = L$, the Cauchy–Schwarz inequality implies

$$|\langle \boldsymbol{\varepsilon}, \boldsymbol{H}^{*(L+1)}\mathbf{\Phi}_M\rangle| = |\langle \boldsymbol{H}^{L+1}\boldsymbol{\varepsilon}, \mathbf{\Phi}_M\rangle| \lesssim \sqrt{\log M}\sqrt{\mathcal{E}_{L+1}}. \tag{3-36}$$

(iii) *Scaling terms*. We can estimate the scaling term in (3-34) from the compact support property of $\mathbf{\Phi}_M$ and the coercivity bound (A-15):

$$\begin{aligned}\left|\frac{\lambda_s}{\lambda}\langle \mathbf{\Lambda}\boldsymbol{\varepsilon}, \boldsymbol{H}^{*k}\mathbf{\Phi}_M\rangle\right| &\le (b_1 + |D_0(t)|)|\langle \mathbf{\Lambda}\boldsymbol{\varepsilon}, \boldsymbol{H}^{*k}\mathbf{\Phi}_M\rangle| \\ &\lesssim (b_1 + |D_0(t)|)C(M)\sqrt{\mathcal{E}_{L+1}}. \end{aligned} \tag{3-37}$$

(iv) $\tilde{\boldsymbol{\psi}}_b$ *terms*. Here, the improved local bound (2-91) implies

$$|\langle \tilde{\boldsymbol{\psi}}_b, \boldsymbol{H}^{*k}\mathbf{\Phi}_M\rangle| \lesssim C(M)b_1^{L+3}. \tag{3-38}$$

(v) $\boldsymbol{NL}(\boldsymbol{\varepsilon})$ and $\boldsymbol{L}(\boldsymbol{\varepsilon})$ *terms*. Using the coercivity bound (A-15) with the crude bound $|NL(\varepsilon)| \lesssim |\varepsilon|^2/y^2$ and $|L(\varepsilon)| \lesssim b_1^2|\varepsilon|/y$,

$$|\langle \boldsymbol{NL}(\boldsymbol{\varepsilon}), \boldsymbol{H}^{*i}\mathbf{\Phi}_M\rangle| \lesssim C(M)\mathcal{E}_{L+1}, \quad |\langle \boldsymbol{L}(\boldsymbol{\varepsilon}), \boldsymbol{H}^{*i}\mathbf{\Phi}_M\rangle| \lesssim C(M)b_1^2\sqrt{\mathcal{E}_{L+1}}. \tag{3-39}$$

<u>Step 3</u>: *Conclusion*. Injecting the estimates from (3-35)–(3-39) into (3-34), we obtain

$$\begin{aligned}&(-1)^k D_k(t)\langle \mathbf{\Lambda}\boldsymbol{Q}, \mathbf{\Phi}_M\rangle + O(C(M)b_1|D(t)|) \\ &\qquad = O(\sqrt{\log M}\sqrt{\mathcal{E}_{L+1}})\delta_{kL} + O(C(M)b_1(\sqrt{\mathcal{E}_{L+1}} + b_1^{L+2})) \end{aligned} \tag{3-40}$$

for $0 \le k \le L$. Dividing the above equation by $\langle \mathbf{\Lambda}\boldsymbol{Q}, \mathbf{\Phi}_M\rangle$, (3-3) implies

$$D_k(t) + O(C(M)b_1|D(t)|) = O\left(\frac{\sqrt{\mathcal{E}_{L+1}}}{\sqrt{\log M}}\right)\delta_{kL} + O(C(M)b_1(\sqrt{\mathcal{E}_{L+1}} + b_1^{L+2})),$$

which yields (3-32) and (3-33). □

3.5. ***Improved modulation equation of b_L.*** At first glance, (3-33) seems sufficient to close the modulation equation for b_L because of the presence of $\sqrt{\log M}$. However, our desired blow-up scenario comes from the exact solution b_L^e, and (3-33) is inadequate to close the bootstrap bounds for stable/unstable modes $V(s)$. Thus, we need to obtain further logarithmic room by adding some correction to b_L.

Lemma 3.4 (improved modulation equation of b_L). *Let $B_\delta = B_0^\delta$ and*

$$\tilde{b}_L = b_L + (-1)^L \frac{\langle \boldsymbol{H}^L \boldsymbol{\varepsilon}, \chi_{B_\delta} \boldsymbol{\Lambda} \boldsymbol{Q} \rangle}{4\delta |\log b_1|} \tag{3-41}$$

for some small enough universal constant $0 < \delta \ll 1$. Then $\tilde{b}_L$ satisfies

$$|\tilde{b}_L - b_L| \lesssim b_1^{L+1-C\delta} \tag{3-42}$$

and

$$|(\tilde{b}_L)_s + (L-1+c_{b,L}) b_1 \tilde{b}_L| \lesssim \frac{\sqrt{\mathcal{E}_{L+1}}}{\sqrt{|\log b_1|}}. \tag{3-43}$$

Remark. We point out that $\tilde{b}_L$ is well-defined at time $s = s_0$ since $\tilde{b}_L - b_L$ only depends on b_1 and $\boldsymbol{\varepsilon}$.

Proof. We obtain (3-42) from the coercivity bound (A-15) and (3-32):

$$|\langle \boldsymbol{H}^L \boldsymbol{\varepsilon}, \chi_{B_\delta} \boldsymbol{\Lambda} \boldsymbol{Q} \rangle| \lesssim |\langle H^{(L-1)/2} \dot{\varepsilon}, \chi_{B_\delta} \Lambda Q \rangle| \lesssim C(M) \delta b_1^{-C\delta} \sqrt{\mathcal{E}_{L+1}} \lesssim b_1^{L+1-C\delta}, \tag{3-44}$$

We also know

$$\frac{d}{ds} \langle \boldsymbol{H}^L \boldsymbol{\varepsilon}, \chi_{B_\delta} \boldsymbol{\Lambda} \boldsymbol{Q} \rangle = \langle \boldsymbol{H}^L \boldsymbol{\varepsilon}_s, \chi_{B_\delta} \boldsymbol{\Lambda} \boldsymbol{Q} \rangle + \langle \boldsymbol{H}^L \boldsymbol{\varepsilon}, (\chi_{B_\delta})_s \boldsymbol{\Lambda} \boldsymbol{Q} \rangle. \tag{3-45}$$

We compute the last inner product in (3-45) similar to (3-44):

$$|\langle \boldsymbol{H}^L \boldsymbol{\varepsilon}, (\chi_{B_\delta})_s \boldsymbol{\Lambda} \boldsymbol{Q} \rangle| = |\delta (b_1)_s b_1^{-1}| |\langle H^{(L-1)/2} \dot{\varepsilon}, (y \partial_y \chi)_{B_\delta} \Lambda Q \rangle| \lesssim C(M) \delta b_1^{1-\delta} \sqrt{\mathcal{E}_{L+1}}. \tag{3-46}$$

Using (3-8), we obtain an identity similar to (3-34):

$$\begin{aligned}\langle \boldsymbol{H}^L \boldsymbol{\varepsilon}_s, \chi_{B_\delta} \boldsymbol{\Lambda} \boldsymbol{Q} \rangle = -\langle \boldsymbol{H}^L \widetilde{\mathbf{Mod}}(t), \chi_{B_\delta} \boldsymbol{\Lambda} \boldsymbol{Q} \rangle - \langle \boldsymbol{H}^{L+1} \boldsymbol{\varepsilon}, \chi_{B_\delta} \boldsymbol{\Lambda} \boldsymbol{Q} \rangle + \frac{\lambda_s}{\lambda} \langle \boldsymbol{H}^L \boldsymbol{\Lambda} \boldsymbol{\varepsilon}, \chi_{B_\delta} \boldsymbol{\Lambda} \boldsymbol{Q} \rangle \\ - \langle \boldsymbol{H}^L \tilde{\boldsymbol{\psi}}_b, \chi_{B_\delta} \boldsymbol{\Lambda} \boldsymbol{Q} \rangle - \langle \boldsymbol{H}^L \boldsymbol{NL}(\boldsymbol{\varepsilon}), \chi_{B_\delta} \boldsymbol{\Lambda} \boldsymbol{Q} \rangle - \langle \boldsymbol{H}^L \boldsymbol{L}(\boldsymbol{\varepsilon}), \chi_{B_\delta} \boldsymbol{\Lambda} \boldsymbol{Q} \rangle.\end{aligned}$$

Considering the support of $\chi_{B_\delta} \boldsymbol{\Lambda} \boldsymbol{Q}$, we can borrow all the estimates in Step 2 of the proof of Lemma 3.3 by replacing the weight $\log M$ and $C(M)$ with $|\log b_1|$ and $b_1^{-C\delta}$, respectively. Hence Lemma 3.3 and (3-46) give a "B_δ version" of (3-40):

$$\begin{aligned}\frac{d}{ds} \langle \boldsymbol{H}^L \boldsymbol{\varepsilon}, \chi_{B_\delta} \boldsymbol{\Lambda} \boldsymbol{Q} \rangle &= (-1)^{L+1} D_L(t) \langle \boldsymbol{\Lambda} \boldsymbol{Q}, \chi_{B_\delta} \boldsymbol{\Lambda} \boldsymbol{Q} \rangle + O(b_1^{1-C\delta} |D(t)|) \\ &\quad + O(\sqrt{|\log b_1|} \sqrt{\mathcal{E}_{L+1}}) + O(b_1^{1-C\delta} (\sqrt{\mathcal{E}_{L+1}} + b_1^{L+2})) \\ &= (-1)^{L+1} 4\delta |\log b_1| D_L(t) + O(\sqrt{|\log b_1|} \sqrt{\mathcal{E}_{L+1}}).\end{aligned}$$

Hence we obtain (3-43) as follows:

$$\begin{aligned}|(\tilde{b}_L)_s + (L-1+c_{b,L}) b_1 \tilde{b}_L| &\lesssim |\langle \boldsymbol{H}^L \boldsymbol{\varepsilon}, \chi_{B_\delta} \boldsymbol{\Lambda} \boldsymbol{Q} \rangle| \left| b_1 + \frac{d}{ds} \left\{ \frac{1}{4\delta \log b_1} \right\} \right| + \frac{\sqrt{\mathcal{E}_{L+1}}}{\sqrt{|\log b_1|}} \\ &\lesssim \frac{\sqrt{\mathcal{E}_{L+1}}}{\sqrt{|\log b_1|}} + b_1^{L+2-C\delta}.\end{aligned}$$

□

3.6. *Lyapunov monotonicity for $\mathcal{E}_{L+1}$*. A simple way to control the adapted higher-order energy $\mathcal{E}_{L+1}$ is to estimate its time derivative. However, we cannot obtain enough estimates to close the bootstrap bound (3-25) with $\mathcal{E}_{L+1}$ by itself, i.e., with $b_1 \sim -\lambda_t$:

$$\frac{d}{dt}\left\{\frac{\mathcal{E}_{L+1}}{\lambda^{2L}}\right\} \le C b_1 \frac{\mathcal{E}_{L+1}}{\lambda^{2L+1}}, \quad \frac{\mathcal{E}_{L+1}(t)}{\lambda^{2L}(t)} \le \frac{\mathcal{E}_{L+1}(0)}{\lambda^{2L}(0)} + C\int_0^t b_1(\tau)\frac{\mathcal{E}_{L+1}(\tau)}{\lambda^{2L+1}(\tau)}\,d\tau$$
$$\le K\int_0^t \frac{b_1(\tau)}{\lambda^{2L+1}(\tau)}\frac{b_1^{2(L+1)}(\tau)}{|\log b_1(\tau)|^2}\,d\tau$$
$$\lesssim \frac{K}{\lambda^{2L}(t)}\frac{b_1^{2(L+1)}(t)}{|\log b_1(t)|^2}.$$

Thus, we use the repulsive property of the conjugated Hamiltonian $\widetilde{H}$ of H observed in [Raphaël and Rodnianski 2012; Rodnianski and Sterbenz 2010] with some additional integration by parts to pull out the accurate corrections.

Proposition 3.5 (Lyapunov monotonicity for $\mathcal{E}_{L+1}$). *We have the bound*

$$\frac{d}{dt}\left\{\frac{\mathcal{E}_{L+1}}{\lambda^{2L}} + O\left(\frac{b_1 C(M)\mathcal{E}_{L+1}}{\lambda^{2L}}\right)\right\} \le C\frac{b_1}{\lambda^{2L+1}}\left[\frac{b_1^{L+1}}{|\log b_1|}\sqrt{\mathcal{E}_{L+1}} + \frac{\mathcal{E}_{L+1}}{\sqrt{\log M}}\right]. \tag{3-47}$$

Proof. Step 1: *Evolution of adapted derivatives.* We start by introducing the rescaled version of the operators A and A^*:

$$A_\lambda := -\partial_r + \frac{Z_\lambda}{r}, \quad A_\lambda^* := \partial_r + \frac{1+Z_\lambda}{r}, \quad Z_\lambda(r) = Z\left(\frac{r}{\lambda}\right) = \frac{1-(r/\lambda)^2}{1+(r/\lambda)^2}.$$

We also recall H_λ in (3-12) and define its conjugate operator $\widetilde{H}_\lambda$ as the rescaled version of the linearized operator H and its conjugate $\widetilde{H}$:

$$H_\lambda := A_\lambda^* A_\lambda = -\Delta + \frac{V_\lambda}{r^2}, \quad V(y) = \frac{y^4 - 6y^2 + 1}{(y^2+1)^2},$$
$$\widetilde{H}_\lambda := A_\lambda A_\lambda^* = -\Delta + \frac{\widetilde{V}_\lambda}{r^2}, \quad \widetilde{V}(y) = \frac{4}{y^2+1}.$$

In the same manner as (2-12), we define the rescaled version of the adapted derivative operator

$$\mathcal{A}_\lambda := A_\lambda, \quad \mathcal{A}_\lambda^2 := A_\lambda^* A_\lambda, \quad \mathcal{A}_\lambda^3 := A_\lambda A_\lambda^* A_\lambda, \quad \dots, \quad \mathcal{A}_\lambda^k := \underbrace{\cdots A_\lambda^* A_\lambda A_\lambda^* A_\lambda}_{k \text{ times}}, \tag{3-48}$$

so the higher-order derivatives of $\boldsymbol{w} = (w, \dot{w})^t$ adapted to the Hamiltonian H_λ are given by

$$w_k := \mathcal{A}_\lambda^k w, \quad \dot{w}_k := \mathcal{A}_\lambda^k \dot{w}.$$

One can easily check that $w_k = (\varepsilon_k)_\lambda/\lambda^k$ and $\dot{w}_k = (\dot{\varepsilon}_k)_\lambda/\lambda^{k+1}$, and our target energy can be written as

$$\frac{\mathcal{E}_{L+1}}{\lambda^{2L}} = \langle w_{L+1}, w_{L+1}\rangle + \langle \dot{w}_L, \dot{w}_L\rangle = \langle \widetilde{H}_\lambda w_L, w_L\rangle + \langle \dot{w}_L, \dot{w}_L\rangle. \tag{3-49}$$

To describe the evolution of w_k and $\dot{w}_k$, we first rewrite the flow (3-11) of $\boldsymbol{w} = (w, \dot{w})$ componentwise:

$$\begin{cases} w_t - \dot{w} = \mathcal{F}_1, \\ \dot{w}_t + H_\lambda w = \mathcal{F}_2, \end{cases} \qquad \begin{pmatrix} \mathcal{F}_1 \\ \mathcal{F}_2 \end{pmatrix} := \frac{1}{\lambda} \boldsymbol{\mathcal{F}}_\lambda = \frac{1}{\lambda} \begin{pmatrix} \mathcal{F} \\ \dot{\mathcal{F}} \end{pmatrix}_\lambda . \tag{3-50}$$

Substituting $\mathcal{A}_\lambda^k$ given by (3-48) into (3-50), we obtain the evolution equation of w_k

$$\begin{cases} \partial_t w_k - \dot{w}_k = [\partial_t, \mathcal{A}_\lambda^k] w + \mathcal{A}_\lambda^k \mathcal{F}_1, \\ \partial_t \dot{w}_k + w_{k+2} = [\partial_t, \mathcal{A}_\lambda^k] \dot{w} + \mathcal{A}_\lambda^k \mathcal{F}_2. \end{cases} \tag{3-51}$$

Lastly, we employ the following notation: for any time-dependent operator P,

$$\partial_t(P) := [\partial_t, P],$$

which yields the Leibniz rule between the operator and function:

$$\partial_t(Pf) = \partial_t(P) f + P f_t . \tag{3-52}$$

<u>Step 2</u>: *First energy identity*. Recalling (3-49), we compute the energy identity

$$\begin{aligned} \partial_t \left(\frac{\mathcal{E}_{L+1}}{2\lambda^{2L}} \right) &= \frac{1}{2} \langle \partial_t(\widetilde{H}_\lambda) w_L, w_L \rangle + \langle \widetilde{H}_\lambda w_L, \partial_t w_L \rangle + \langle \dot{w}_L, \partial_t \dot{w}_L \rangle \\ &= \frac{1}{2} \langle \partial_t(\widetilde{H}_\lambda) w_L, w_L \rangle + \langle \widetilde{H}_\lambda w_L, \partial_t(\mathcal{A}_\lambda^L) w \rangle + \langle \dot{w}_L, \partial_t(\mathcal{A}_\lambda^L) \dot{w} \rangle \\ &\qquad + \langle \widetilde{H}_\lambda w_L, \mathcal{A}_\lambda^L \mathcal{F}_1 \rangle + \langle \dot{w}_L, \mathcal{A}_\lambda^L \mathcal{F}_2 \rangle . \end{aligned} \tag{3-53}$$

We will check that the last two terms of (3-53) satisfy the desired bound (3-47) later. Unlike the last two terms of (3-53), when the first three terms of (3-53) are estimated using coercivity (A-15) directly, we obtain the insufficient bound

$$\frac{b_1}{\lambda^{2L+1}} C(M) \mathcal{E}_{L+1} . \tag{3-54}$$

One can employ repulsive property (2-10) for the first term of (3-53) with the modulation equation (3-32):

$$\partial_t(\widetilde{H}_\lambda) = -\frac{\lambda_t}{\lambda} \frac{(\Lambda \widetilde{V})_\lambda}{r^2} = -\frac{b_1 + O(b_1^{L+2})}{\lambda^3} \frac{8}{(1+y^2)^2} \quad \Longrightarrow \quad \langle \partial_t(\widetilde{H}_\lambda) w_L, w_L \rangle < 0. \tag{3-55}$$

We claim that the sum $\langle \widetilde{H}_\lambda w_L, \partial_t(\mathcal{A}_\lambda^L) w \rangle + \langle \dot{w}_L, \partial_t(\mathcal{A}_\lambda^L) \dot{w} \rangle$ in (3-53) is eventually negative like (3-55) by adding some corrections. For this, we start by employing (3-51) to exchange $\widetilde{H}_\lambda w_L$ for $-\partial_t \dot{w}_L$:

$$\langle \widetilde{H}_\lambda w_L, \partial_t(\mathcal{A}_\lambda^L) w \rangle = -\langle \partial_t \dot{w}_L, \partial_t(\mathcal{A}_\lambda^L) w \rangle + \langle \partial_t(\mathcal{A}_\lambda^L) \dot{w}, \partial_t(\mathcal{A}_\lambda^L) w \rangle + \langle \mathcal{A}_\lambda^L \mathcal{F}_2, \partial_t(\mathcal{A}_\lambda^L) w \rangle , \tag{3-56}$$

and we can treat the first term on the right-hand side of (3-56) via integration by parts in time with (3-50):

$$\begin{aligned} -\langle \partial_t \dot{w}_L, \partial_t(\mathcal{A}_\lambda^L) w \rangle + \partial_t \langle \dot{w}_L, \partial_t(\mathcal{A}_\lambda^L) w \rangle &= \langle \dot{w}_L, \partial_{tt}(\mathcal{A}_\lambda^L) w \rangle + \langle \dot{w}_L, \partial_t(\mathcal{A}_\lambda^L) w_t \rangle \\ &= \langle \dot{w}_L, \partial_t(\mathcal{A}_\lambda^L) \dot{w} \rangle + \langle \dot{w}_L, \partial_{tt}(\mathcal{A}_\lambda^L) w \rangle + \langle \dot{w}_L, \partial_t(\mathcal{A}_\lambda^L) \mathcal{F}_1 \rangle . \end{aligned} \tag{3-57}$$

In short, we add a correction to the energy identity to transform the inner product $\langle \widetilde{H}_\lambda w_L, \partial_t(\mathcal{A}_\lambda^L) w \rangle$ to the inner product $\langle \dot{w}_L, \partial_t(\mathcal{A}_\lambda^L) \dot{w} \rangle$ in (3-53) up to some errors from (3-56) and (3-57):

$$\langle \widetilde{H}_\lambda w_L, \partial_t(\mathcal{A}_\lambda^L) w \rangle + \partial_t D_{0,1,1} = \langle \dot{w}_L, \partial_t(\mathcal{A}_\lambda^L) \dot{w} \rangle + E_{0,1,1} + E_{0,1,2} + F_{0,1,1} + F_{0,1,2}, \tag{3-58}$$

where

$$\begin{aligned}
D_{0,1,1} &= \langle \dot{w}_L, \partial_t(\mathcal{A}_\lambda^L)w\rangle,\\
E_{0,1,1} &= \langle \dot{w}_L, \partial_{tt}(\mathcal{A}_\lambda^L)w\rangle,\\
E_{0,1,2} &= \langle \partial_t(\mathcal{A}_\lambda^L)\dot{w}, \partial_t(\mathcal{A}_\lambda^L)w\rangle,\\
F_{0,1,1} &= \langle \dot{w}_L, \partial_t(\mathcal{A}_\lambda^L)\mathcal{F}_1\rangle,\\
F_{0,1,2} &= \langle \mathcal{A}_\lambda^L\mathcal{F}_2, \partial_t(\mathcal{A}_\lambda^L)w\rangle.
\end{aligned}$$

However, the inner product $\langle \dot{w}_L, \partial_t(\mathcal{A}_\lambda^L)\dot{w}\rangle$ in (3-53) is also not small enough to close our bootstrap by itself. Thus, we use (3-51) again to exchange $\dot{w}_L$ for $\partial_t w_L$:

$$\langle \dot{w}_L, \partial_t(\mathcal{A}_\lambda^L)\dot{w}\rangle = \langle \partial_t w_L, \partial_t(\mathcal{A}_\lambda^L)\dot{w}\rangle - \langle \partial_t(\mathcal{A}_\lambda^L)w, \partial_t(\mathcal{A}_\lambda^L)\dot{w}\rangle - \langle \mathcal{A}_\lambda^L\mathcal{F}_1, \partial_t(\mathcal{A}_\lambda^L)\dot{w}\rangle. \tag{3-59}$$

Integrating by parts in time once more,

$$\begin{aligned}
\langle \partial_t w_L, \partial_t(\mathcal{A}_\lambda^L)\dot{w}\rangle - \partial_t\langle w_L, \partial_t(\mathcal{A}_\lambda^L)\dot{w}\rangle &= -\langle w_L, \partial_{tt}(\mathcal{A}_\lambda^L)\dot{w}\rangle - \langle w_L, \partial_t(\mathcal{A}_\lambda^L)\dot{w}_t\rangle\\
&= \langle w_L, \partial_t(\mathcal{A}_\lambda^L)w_2\rangle - \langle w_L, \partial_{tt}(\mathcal{A}_\lambda^L)\dot{w}\rangle - \langle w_L, \partial_t(\mathcal{A}_\lambda^L)\mathcal{F}_2\rangle.
\end{aligned}\tag{3-60}$$

To sum it up, we obtain a relation similar to (3-58):

$$\langle \dot{w}_L, \partial_t(\mathcal{A}_\lambda^L)\dot{w}\rangle + \partial_t D_{0,2,1} = \langle w_L, \partial_t(\mathcal{A}_\lambda^L)w_2\rangle + E_{0,2,1} + E_{0,2,2} + F_{0,2,1} + F_{0,2,2}, \tag{3-61}$$

where

$$\begin{aligned}
D_{0,2,1} &= -\langle w_L, \partial_t(\mathcal{A}_\lambda^L)\dot{w}\rangle,\\
E_{0,2,1} &= -\langle w_L, \partial_{tt}(\mathcal{A}_\lambda^L)\dot{w}\rangle,\\
E_{0,2,2} &= -\langle \partial_t(\mathcal{A}_\lambda^L)w, \partial_t(\mathcal{A}_\lambda^L)\dot{w}\rangle,\\
F_{0,2,1} &= -\langle \mathcal{A}_\lambda^L\mathcal{F}_1, \partial_t(\mathcal{A}_\lambda^L)\dot{w}\rangle,\\
F_{0,2,2} &= -\langle w_L, \partial_t(\mathcal{A}_\lambda^L)\mathcal{F}_2\rangle.
\end{aligned}$$

Raphaël and Rodnianski [2012] directly checked that $\langle w_1, \partial_t(\mathcal{A}_\lambda^L)w_2\rangle < 0$ in the case $L = 1$. In contrast, when $L \geq 3$, we cannot obtain similar information from $\langle w_L, \partial_t(\mathcal{A}_\lambda^L)w_2\rangle$ by itself. We pull out the repulsive terms using the Leibniz rule:

$$\begin{aligned}
\langle w_L, \partial_t(\mathcal{A}_\lambda^L)w_2\rangle &= \langle w_L, \partial_t(\widetilde{H}_\lambda)w_L\rangle + \langle w_L, \widetilde{H}_\lambda\partial_t(\mathcal{A}_\lambda^{L-2})w_2\rangle\\
&= \langle w_L, \partial_t(\widetilde{H}_\lambda)w_L\rangle + \langle \widetilde{H}_\lambda w_L, \partial_t(\mathcal{A}_\lambda^{L-2})w_2\rangle.
\end{aligned}\tag{3-62}$$

We observe that the second inner product in (3-62) has the same form as the first inner product in (3-58); we can iterate integration by parts, which leads to the following recurrence equations: for $0 \leq k \leq \frac{1}{2}(L-1)$,

$$\langle \widetilde{H}_\lambda w_L, \partial_t(\mathcal{A}_\lambda^{L-2k})w_{2k}\rangle + \partial_t D_{k,1,1} = \langle \dot{w}_L, \partial_t(\mathcal{A}_\lambda^{L-2k})\dot{w}_{2k}\rangle + E_{k,1,1} + E_{k,1,2} + F_{k,1,1} + F_{k,1,2}, \tag{3-63}$$

where

$$\begin{aligned}
D_{k,1,1} &= \langle \dot{w}_L, \partial_t(\mathcal{A}_\lambda^{L-2k})w_{2k}\rangle,\\
E_{k,1,1} &= \langle \dot{w}_L, \partial_{tt}(\mathcal{A}_\lambda^{L-2k})w_{2k}\rangle,\\
E_{k,1,2} &= \langle \partial_t(\mathcal{A}_\lambda^L)\dot{w}, \partial_t(\mathcal{A}_\lambda^{L-2k})w_{2k}\rangle + \langle \dot{w}_L, \partial_t(\mathcal{A}_\lambda^{L-2k})\partial_t(H_\lambda^k)w\rangle,\\
F_{k,1,1} &= \langle \dot{w}_L, \partial_t(\mathcal{A}_\lambda^{L-2k})H_\lambda^k\mathcal{F}_1\rangle,\\
F_{k,1,2} &= \langle \mathcal{A}_\lambda^L\mathcal{F}_2, \partial_t(\mathcal{A}_\lambda^{L-2k})w_{2k}\rangle
\end{aligned}$$

and

$$\langle \dot{w}_L, \partial_t(\mathcal{A}_\lambda^{L-2k})\dot{w}_{2k}\rangle + \partial_t D_{k,2,1} = \langle w_L, \partial_t(\mathcal{A}_\lambda^{L-2k}) w_{2k+2}\rangle + E_{k,2,1} + E_{k,2,2} + F_{k,2,1} + F_{k,2,2}, \tag{3-64}$$

where

$$\begin{aligned}
D_{k,2,1} &= -\langle w_L, \partial_t(\mathcal{A}_\lambda^{L-2k})\dot{w}_{2k}\rangle,\\
E_{k,2,1} &= -\langle w_L, \partial_{tt}(\mathcal{A}_\lambda^{L-2k})\dot{w}_{2k}\rangle,\\
E_{k,2,2} &= -\langle \partial_t(\mathcal{A}_\lambda^{L}) w, \partial_t(\mathcal{A}_\lambda^{L-2k})\dot{w}_{2k}\rangle - \langle w_L, \partial_t(\mathcal{A}_\lambda^{L-2k})\partial_t(H_\lambda^k)\dot{w}\rangle,\\
F_{k,2,1} &= -\langle \mathcal{A}_\lambda^{L}\mathcal{F}_1, \partial_t(\mathcal{A}_\lambda^{L-2k})\dot{w}_{2k}\rangle,\\
F_{k,2,2} &= -\langle w_L, \partial_t(\mathcal{A}_\lambda^{L-2k})\mathcal{F}_2\rangle.
\end{aligned}$$

We can also pull out the repulsive term like (3-62) from (3-64): for $0 \le k \le \frac{1}{2}(L-3)$,

$$\langle w_L, \partial_t(\mathcal{A}_\lambda^{L-2k}) w_{2k+2}\rangle = \langle w_L, \partial_t(\widetilde{H}_\lambda) w_L\rangle + \langle \widetilde{H}_\lambda w_L, \partial_t(\mathcal{A}_\lambda^{L-2k-2}) w_{2k+2}\rangle. \tag{3-65}$$

The displays (3-63), (3-64) and (3-65) allow us to iterate our recurrence relations. For $k = \frac{1}{2}(L-1)$, we can verify that (3-64) is negative from the facts

$$\begin{aligned}
\partial_t(A_\lambda) = \partial_t(A_\lambda^*) &= \frac{-\lambda_t}{\lambda}\frac{(\Lambda Z)_\lambda}{r},\\
\langle \partial_t(\widetilde{H}_\lambda) w_L, w_L\rangle &= \langle \partial_t(A_\lambda A_\lambda^*) w_L, w_L\rangle\\
&= \langle \partial_t(A_\lambda) A_\lambda^* w_L, w_L\rangle + \langle A_\lambda \partial_t(A_\lambda^*) w_L, w_L\rangle\\
&= 2\langle \partial_t(A_\lambda) w_{L+1}, w_L\rangle.
\end{aligned}$$

Hence we write the following decomposition of the term $\langle \widetilde{H}_\lambda w_L, \partial_t(\mathcal{A}_\lambda^L) w\rangle$ of (3-53):

$$\langle \widetilde{H}_\lambda w_L, \partial_t(\mathcal{A}_\lambda^L) w\rangle + \sum_{k=0}^{(L-1)/2}\sum_{i=1}^{2} \partial_t D_{k,i,1} = \frac{L}{2}\langle \partial_t(\widetilde{H}_\lambda) w_L, w_L\rangle + \sum_{k=0}^{(L-1)/2}\sum_{i,j=1}^{2}(E_{k,i,j} + F_{k,i,j}).$$

Similarly, we write the following decomposition the term $\langle \dot{w}_L, \partial_t(\mathcal{A}_\lambda^L)\dot{w}\rangle$ of (3-53):

$$\begin{aligned}
\langle \dot{w}_L, \partial_t(\mathcal{A}_\lambda^L)\dot{w}\rangle &+ \sum_{k=0}^{(L-1)/2}\sum_{i=1}^{2} \partial_t(1-\delta_{k,0}\delta_{i,1}) D_{k,i,1}\\
&= \frac{L}{2}\langle \partial_t(\widetilde{H}_\lambda) w_L, w_L\rangle + \sum_{k=0}^{(L-1)/2}\sum_{i,j=1}^{2}(1-\delta_{k,0}\delta_{i,1})(E_{k,i,j} + F_{k,i,j}).
\end{aligned}$$

Together with the first term and the last two terms of (3-53), we obtain the following initial identity of $\mathcal{E}_{L+1}$:

$$\begin{aligned}
\partial_t\biggl\{\frac{\mathcal{E}_{L+1}}{2\lambda^{2L}} + \sum_{k=0}^{(L-1)/2}\sum_{i=1}^{2}(2-\delta_{k,0}\delta_{i,1}) D_{k,i,1}\biggr\} &= \frac{2L+1}{2}\langle \partial_t(\widetilde{H}_\lambda) w_L, w_L\rangle + \langle \widetilde{H}_\lambda w_L, \mathcal{A}_\lambda^L \mathcal{F}_1\rangle + \langle \dot{w}_L, \mathcal{A}_\lambda^L \mathcal{F}_2\rangle\\
&\quad + \sum_{k=0}^{(L-1)/2}\sum_{i,j=1}^{2}(2-\delta_{k,0}\delta_{i,1})(E_{k,i,j} + F_{k,i,j}).
\end{aligned} \tag{3-66}$$

Step 3: *Second energy identity*. We find additional corrections from $E_{k,i,1}$, which contain $\partial_{tt}(\mathcal{A}_\lambda^{L-2k})$. More precisely, from Lemma C.1,

$$E_{k,1,1}=\langle \dot w_L,\partial_{tt}(\mathcal{A}_\lambda^{L-2k})w_{2k}\rangle =\sum_{m=2k}^{L-1}\frac{\lambda_{tt}}{\lambda^{L+1-m}}\langle (\Phi^{(1)}_{m,L,k})_\lambda w_m,\dot w_L\rangle+\sum_{m=2k}^{L-1}\frac{O(b_1^2)}{\lambda^{L+2-m}}\langle (\Phi^{(2)}_{m,L,k})_\lambda w_m,\dot w_L\rangle$$

and

$$E_{k,2,1}=-\langle w_L,\partial_{tt}(\mathcal{A}_\lambda^{L-2k})\dot w_{2k}\rangle =-\sum_{m=2k}^{L-1}\frac{\lambda_{tt}}{\lambda^{L+1-m}}\langle (\Phi^{(1)}_{m,L,k})_\lambda \dot w_m,w_L\rangle-\sum_{m=2k}^{L-1}\frac{O(b_1^2)}{\lambda^{L+2-m}}\langle (\Phi^{(2)}_{m,L,k})_\lambda \dot w_m,w_L\rangle,$$

where $\Phi^{(j_1)}_{m,L,k}(y):=\Phi^{(j_1)}_{m-2k,L-2k}(y)$ with $j_1=1,2$, so that

$$|\Phi^{(j_1)}_{m,L,k}(y)|\lesssim\frac{1}{1+y^{L+2-m}}.$$

Here, we cannot treat λ_{tt} directly because we do not have estimates on second derivatives of the modulation parameters (and we did not set $\lambda_t=-b_1$). Thus, we add $(b_1)_t$ to λ_{tt} and use (3-32):

$$\frac{\lambda_{tt}}{\lambda^{L+1-m}}\langle (\Phi^{(1)}_{m,L,k})_\lambda w_m,\dot w_L\rangle=\frac{(\lambda_t+b_1)_t}{\lambda^{L+1-m}}\langle (\Phi^{(1)}_{m,L,k})_\lambda w_m,\dot w_L\rangle+\frac{O(b_1^2)}{\lambda^{L+2-m}}\langle (\Phi^{(1)}_{m,L,k})_\lambda w_m,\dot w_L\rangle. \quad (3\text{-}67)$$

We then correct (3-67) via integration by parts in time with (3-51):

$$\begin{aligned}&\frac{(\lambda_t+b_1)_t}{\lambda^{L+1-m}}\langle (\Phi^{(1)}_{m,L,k})_\lambda w_m,\dot w_L\rangle-\partial_t\left(\frac{\lambda_t+b_1}{\lambda^{L+1-m}}\langle (\Phi^{(1)}_{m,L,k})_\lambda w_m,\dot w_L\rangle\right)\\&=(\lambda_t+b_1)\left\langle\partial_t\left(\frac{1}{\lambda^{L+1-m}}(\Phi^{(1)}_{m,L,k})_\lambda\right)w_m,\dot w_L\right\rangle+\frac{\lambda_t+b_1}{\lambda^{L+1-m}}[\langle (\Phi^{(1)}_{m,L,k})_\lambda\partial_t w_m,\dot w_L\rangle+\langle (\Phi^{(1)}_{m,L,k})_\lambda w_m,\partial_t\dot w_L\rangle]\\&=-\frac{\lambda_t(\lambda_t+b_1)}{\lambda^{L+2-m}}\langle (\Lambda_{m-L}\Phi^{(1)}_{m,L,k})_\lambda w_m,\dot w_L\rangle-\frac{\lambda_t+b_1}{\lambda^{L+1-m}}\langle (\Phi^{(1)}_{m,L,k})_\lambda(\dot w_m+\partial_t(\mathcal{A}_\lambda^m)w+\mathcal{A}_\lambda^m\mathcal{F}_1),\dot w_L\rangle\\&\qquad+\frac{\lambda_t+b_1}{\lambda^{L+1-m}}\langle (\Phi^{(1)}_{m,L,k})_\lambda w_m,w_{L+2}-\partial_t(\mathcal{A}_\lambda^L)\dot w-\mathcal{A}_\lambda^L\mathcal{F}_2\rangle.\end{aligned}$$

We can also obtain the same correction for $E_{k,2,1}$:

$$\begin{aligned}&\frac{(\lambda_t+b_1)_t}{\lambda^{L+1-m}}\langle (\Phi^{(1)}_{m,L,k})_\lambda \dot w_m,w_L\rangle-\partial_t\left(\frac{\lambda_t+b_1}{\lambda^{L+1-m}}\langle (\Phi^{(1)}_{m,L,k})_\lambda \dot w_m,w_L\rangle\right)\\&=-\frac{\lambda_t(\lambda_t+b_1)}{\lambda^{L+2-m}}\langle (\Lambda_{m-L}\Phi^{(1)}_{m,L,k})_\lambda \dot w_m,w_L\rangle-\frac{\lambda_t+b_1}{\lambda^{L+1-m}}\langle (\Phi^{(1)}_{m,L,k})_\lambda(w_{m+2}-\partial_t(\mathcal{A}_\lambda^m)\dot w-\mathcal{A}_\lambda^m\mathcal{F}_2),w_L\rangle\\&\qquad+\frac{\lambda_t+b_1}{\lambda^{L+1-m}}\langle (\Phi^{(1)}_{m,L,k})_\lambda \dot w_m,\dot w_L+\partial_t(\mathcal{A}_\lambda^L)w+\mathcal{A}_\lambda^L\mathcal{F}_1\rangle.\end{aligned}$$

Rearranging the existing errors $E_{k,i,j}$, $F_{k,i,j}$, while introducing a new correction notation $D_{k,i,2}$ and new error notation $E^*_{k,i,j}$, $F^*_{k,i,j}$ for $0\le k\le\frac12(L-1)$ and $i=1,2$, we have

$$E_{k,i,1}-\partial_t D_{k,i,2}+E_{k,i,2}+F_{k,i,1}+F_{k,i,2}=E^*_{k,i,1}+E^*_{k,i,2}+F^*_{k,i,1}+F^*_{k,i,2}, \quad (3\text{-}68)$$

where

$$D_{k,1,2}=\sum_{m=2k}^{L-1}\frac{\lambda_t+b_1}{\lambda^{L+1-m}}\langle(\Phi^{(1)}_{m,L,k})_\lambda w_m,\dot{w}_L\rangle,$$

$$E^*_{k,1,1}=-\sum_{m=2k}^{L-1}\frac{\lambda_t(\lambda_t+b_1)}{\lambda^{L+2-m}}\langle(\Lambda_{m-L}\Phi^{(1)}_{m,L,k})_\lambda w_m,\dot{w}_L\rangle-\sum_{m=2k}^{L-1}\frac{\lambda_t+b_1}{\lambda^{L+1-m}}\langle(\Phi^{(1)}_{m,L,k})_\lambda(\dot{w}_m+\partial_t(\mathcal{A}^m_\lambda)w),\dot{w}_L\rangle$$
$$+\sum_{m=2k}^{L-1}\frac{\lambda_t+b_1}{\lambda^{L+1-m}}\langle(\Phi^{(1)}_{m,L,k})_\lambda w_m,w_{L+2}-\partial_t(\mathcal{A}^L_\lambda)\dot{w}\rangle,$$

$$E^*_{k,1,2}=E_{k,1,2}+\sum_{m=2k}^{L-1}\frac{O(b_1^2)}{\lambda^{L+2-m}}\langle(\Phi^{(2)}_{m,L,k})_\lambda w_m,\dot{w}_L\rangle,$$

$$F^*_{k,1,1}=F_{k,1,1}-\sum_{m=2k}^{L-1}\frac{\lambda_t+b_1}{\lambda^{L+1-m}}\langle(\Phi^{(1)}_{m,L,k})_\lambda\mathcal{A}^m_\lambda\mathcal{F}_1,\dot{w}_L\rangle,$$

$$F^*_{k,1,2}=F_{k,1,2}-\sum_{m=2k}^{L-1}\frac{\lambda_t+b_1}{\lambda^{L+1-m}}\langle(\Phi^{(1)}_{m,L,k})_\lambda w_m,\mathcal{A}^L_\lambda\mathcal{F}_2\rangle,$$

and

$$D_{k,2,2}=-\sum_{m=2k}^{L-1}\frac{\lambda_t+b_1}{\lambda^{L+1-m}}\langle(\Phi^{(1)}_{m,L,k})_\lambda\dot{w}_m,w_L\rangle,$$

$$E^*_{k,2,1}=\sum_{m=2k}^{L-1}\frac{\lambda_t(\lambda_t+b_1)}{\lambda^{L+2-m}}\langle(\Lambda_{m-L}\Phi^{(1)}_{m,L,k})_\lambda\dot{w}_m,w_L\rangle+\sum_{k=2m}^{L-1}\frac{\lambda_t+b_1}{\lambda^{L+1-m}}\langle(\Phi^{(1)}_{m,L,k})_\lambda(w_{m+2}-\partial_t(\mathcal{A}^m_\lambda)\dot{w}),w_L\rangle$$
$$-\sum_{m=2k}^{L-1}\frac{\lambda_t+b_1}{\lambda^{L+1-m}}\langle(\Phi^{(1)}_{m,L,k})_\lambda\dot{w}_m,\dot{w}_L+\partial_t(\mathcal{A}^L_\lambda)w\rangle,$$

$$E^*_{k,2,2}=E_{k,2,2}-\sum_{m=2k}^{L-1}\frac{O(b_1^2)}{\lambda^{L+2-m}}\langle(\Phi^{(2)}_{m,L,k})_\lambda\dot{w}_m,w_L\rangle,$$

$$F^*_{k,2,1}=F_{k,2,1}-\sum_{m=2k}^{L-1}\frac{\lambda_t+b_1}{\lambda^{L+1-m}}\langle(\Phi^{(1)}_{m,L,k})_\lambda\dot{w}_m,\mathcal{A}^L_\lambda\mathcal{F}_1\rangle,$$

$$F^*_{k,2,2}=F_{k,2,2}-\sum_{m=2k}^{L-1}\frac{\lambda_t+b_1}{\lambda^{L+1-m}}\langle(\Phi^{(1)}_{m,L,k})_\lambda\mathcal{A}^m_\lambda\mathcal{F}_2,w_L\rangle.$$

Hence we obtain the modified energy identity

$$\partial_t\left\{\frac{\mathcal{E}_{L+1}}{2\lambda^{2L}}+\sum_{k=0}^{(L-1)/2}\sum_{i,j=1}^{2}(2-\delta_{k,0}\delta_{i,1})D_{k,i,j}\right\}=\frac{2L+1}{2}\langle\partial_t(\widetilde{H}_\lambda)w_L,w_L\rangle+\langle\widetilde{H}_\lambda w_L,\mathcal{A}^L_\lambda\mathcal{F}_1\rangle+\langle\dot{w}_L,\mathcal{A}^L_\lambda\mathcal{F}_2\rangle$$
$$+\sum_{k=0}^{(L-1)/2}\sum_{i,j=1}^{2}(2-\delta_{k,0}\delta_{i,1})(E^*_{k,i,j}+F^*_{k,i,j}).\quad(3\text{-}69)$$

Step 4: *Error estimation.* All we need is to estimate all inner products except the repulsive one $\langle \partial_t(\widetilde{H}_\lambda) w_L, w_L\rangle$. We can classify such inner products into two main categories: quadratic terms with respect to w (i.e., $D_{k,i,j}$ and $E^*_{k,i,j}$), or those involving $\mathcal{F}_i$, $i=1,2$ (i.e., $F^*_{k,i,j}$ and the last terms of (3-53)).

(i) $D_{k,i,j}$ *terms*. From (C-1) and Lemma C.1, all inner products of $D_{k,i,j}$ can be written as sums of terms of the form, for $0\le m\le L-1$,

$$\frac{O(b_1)}{\lambda^{2L}}\langle \Phi_{m,L}\varepsilon_m, \dot{\varepsilon}_L\rangle, \quad \frac{O(b_1)}{\lambda^{2L}}\langle \Phi_{m,L}\dot{\varepsilon}_m, \varepsilon_L\rangle, \quad |\Phi_{m,L}(y)|\lesssim \frac{1}{1+y^{L+2-m}}.$$

Indeed, the $\Phi_{m,L}$ included in each of the above inner products are different functions (e.g., $\Phi^{(j_1)}_{m-2k,L-2k}$, $\Phi^{(j_2)}_{m,L,k}$, $\Lambda_{m-L}\Phi^{(1)}_{m,L,k}, \dots$), but we abuse the notation because they are all rational functions with the same asymptotics. From the coercive property (A-15), we obtain the desired bound for the correction in (3-47):

$$|\langle \Phi_{m,L}\varepsilon_m, \dot{\varepsilon}_L\rangle| \lesssim \left\| \frac{\varepsilon_m}{1+y^{L+2-m}}\right\|_{L^2}\sqrt{\mathcal{E}_{L+1}} \lesssim C(M)\mathcal{E}_{L+1},$$
$$|\langle \Phi_{m,L}\dot{\varepsilon}_m, \varepsilon_L\rangle| \lesssim \left\| \frac{1+|\log y|}{1+y^{L+1-m}}\dot{\varepsilon}_m\right\|_{L^2}\sqrt{\mathcal{E}_{L+1}} \lesssim C(M)\mathcal{E}_{L+1}.$$

(ii) $E^*_{k,i,j}$ *terms*. Similarly, all inner products of $E^*_{k,i,j}$ can be written as sums of terms of the form, for $0\le m,n\le L-1$,

$$\frac{O(b_1^2)}{\lambda^{2L+1}}\langle \Phi_{m,L}\varepsilon_m, \dot{\varepsilon}_L\rangle, \quad \frac{O(b_1^2)}{\lambda^{2L+1}}\langle \Phi_{m,L}\dot{\varepsilon}_m, \varepsilon_L\rangle, \quad \frac{O(b_1^2)}{\lambda^{2L+1}}\langle \Phi_{m,L}\dot{\varepsilon}_m, \Phi_{n,L}\varepsilon_n\rangle,$$
$$\frac{O(b_1^2)}{\lambda^{2L+1}}\langle \Phi_{m,L}\dot{\varepsilon}_m, \dot{\varepsilon}_L\rangle, \quad \frac{O(b_1^2)}{\lambda^{2L+1}}\langle \Phi_{m,L}\varepsilon_m, \varepsilon_{L+2}\rangle, \quad \frac{O(b_1^2)}{\lambda^{2L+1}}\langle \Phi_{m,L}\varepsilon_{m+2}, \varepsilon_L\rangle,$$

which are bounded by

$$\frac{b_1^2}{\lambda^{2L+1}}C(M)\mathcal{E}_{L+1}.$$

(iii) $F^*_{k,i,j}$ *and the last two terms of* (3-53). Recalling $\mathcal{F}_1=\lambda^{-1}\mathcal{F}_\lambda$ and $\mathcal{F}_2=\lambda^{-2}\dot{\mathcal{F}}_\lambda$, all inner products of $F^*_{k,i,j}$ can be written as sums of terms of the form, for $0\le m\le L-1$,

$$\frac{O(b_1)}{\lambda^{2L+1}}\langle \Phi_{m,L}\mathcal{A}^m\mathcal{F}, \dot{\varepsilon}_L\rangle, \quad \frac{O(b_1)}{\lambda^{2L+1}}\langle \Phi_{m,L}\dot{\varepsilon}_m, \mathcal{A}^L\mathcal{F}\rangle, \quad \frac{O(b_1)}{\lambda^{2L+1}}\langle \Phi_{m,L}\varepsilon_m, \mathcal{A}^L\dot{\mathcal{F}}\rangle, \tag{3-70}$$
$$\frac{O(b_1)}{\lambda^{2L+1}}\langle \Phi_{m,L}\mathcal{A}^m\dot{\mathcal{F}}, \varepsilon_L\rangle, \quad \frac{1}{\lambda^{2L+1}}\langle \varepsilon_{L+1}, \mathcal{A}^{L+1}\mathcal{F}\rangle, \quad \frac{1}{\lambda^{2L+1}}\langle \dot{\varepsilon}_L, \mathcal{A}^L\dot{\mathcal{F}}\rangle. \tag{3-71}$$

We claim that $\mathcal{F}$ and $\dot{\mathcal{F}}$ satisfy the following estimates: for $0\le k\le L-1$,

$$\|\mathcal{A}^{L+1}\mathcal{F}\|_{L^2}+\|\mathcal{A}^L\dot{\mathcal{F}}\|_{L^2}\lesssim b_1\left[\frac{b_1^{L+1}}{|\log b_1|}+\sqrt{\frac{\mathcal{E}_{L+1}}{\log M}}\right], \tag{3-72}$$
$$\left\|\frac{1+|\log y|}{1+y^{L+1-k}}\mathcal{A}^k\mathcal{F}\right\|_{L^2}\lesssim b_1^{L+2}|\log b_1|^C, \tag{3-73}$$
$$\left\|\frac{1+|\log y|}{1+y^{L+1-k}}\mathcal{A}^k\dot{\mathcal{F}}\right\|_{L^2}\lesssim \frac{b_1^{L+1}}{|\log b_1|}+\sqrt{\frac{\mathcal{E}_{L+1}}{\log M}}. \tag{3-74}$$

Assuming the claims (3-72)–(3-74) with the coercivity (A-15), we can estimate $F^*_{k,i,j}$ terms as follows: the three inner products in (3-70) are bounded by

$$\frac{b_1}{\lambda^{2L+1}} C(M) b_1^{L+2} |\log b_1|^C \sqrt{\mathcal{E}_{L+1}}.$$

For the three inner products in (3-71), we obtain the sharp bound

$$\frac{b_1}{\lambda^{2L+1}} \left(\frac{b_1^{L+1}}{|\log b_1|} + \sqrt{\frac{\mathcal{E}_{L+1}}{\log M}} \right) \sqrt{\mathcal{E}_{L+1}}$$

from (3-72), (3-74) and the sharp coercivity bound

$$\left\| \frac{\varepsilon_L}{y(1+|\log y|)} \right\|_{L^2}^2 \leq C \langle \widetilde{H} \varepsilon_L, \varepsilon_L \rangle \leq C \mathcal{E}_{L+1}.$$

Hence it remains to prove (3-72)–(3-74).

Step 5: Proof of (3-72), (3-73) and (3-74). Recalling (3-11), we have $\boldsymbol{\mathcal{F}} = (\mathcal{F}, \dot{\mathcal{F}})^t$ and

$$\begin{pmatrix} \mathcal{F} \\ \dot{\mathcal{F}} \end{pmatrix} = -\widetilde{\mathbf{Mod}}(t) - \tilde{\boldsymbol{\psi}}_b - \boldsymbol{NL}(\boldsymbol{\varepsilon}) - \boldsymbol{L}(\boldsymbol{\varepsilon}), \quad \boldsymbol{NL}(\boldsymbol{\varepsilon}) = \begin{pmatrix} 0 \\ NL(\varepsilon) \end{pmatrix}, \quad \boldsymbol{L}(\boldsymbol{\varepsilon}) = \begin{pmatrix} 0 \\ L(\varepsilon) \end{pmatrix}.$$

Thus, we will estimate each of the above four errors.

(i) $\tilde{\boldsymbol{\psi}}_b$ *term*. It directly follows from the global and logarithmic weighted bounds of Proposition 2.10.

(ii) $\widetilde{\mathbf{Mod}}(t)$ *term*. Recall (3-9): we have

$$\widetilde{\mathbf{Mod}}(t) = -\left(\frac{\lambda_s}{\lambda} + b_1 \right) \left(\boldsymbol{\Lambda Q} + \sum_{i=1}^{L} b_i \boldsymbol{\Lambda}(\chi_{B_1} \boldsymbol{T}_i) + \sum_{i=2}^{L+2} \boldsymbol{\Lambda}(\chi_{B_1} \boldsymbol{S}_i) \right) + \sum_{i=1}^{L} ((b_i)_s + (i-1+c_{b,i}) b_1 b_i - b_{i+1}) \chi_{B_1} \left(\boldsymbol{T}_i + \sum_{j=i+1}^{L+2} \frac{\partial \boldsymbol{S}_j}{\partial b_i} \right). \quad (3\text{-}75)$$

Due to Lemma 3.3, the logarithmic weighted bounds (3-73) and (3-74) are derived from the finiteness of the integrals

$$\int \left| \frac{1+|\log y|}{1+y^{L+1-k}} \mathcal{A}^k \left[\Lambda Q + \sum_{i=1}^{L} b_i \Lambda_{1-\bar{i}} (\chi_{B_1} T_i) + \sum_{i=2}^{L+2} \Lambda_{1-\bar{i}} (\chi_{B_1} S_i) \right] \right|^2 \lesssim 1,$$

$$\sum_{i=1}^{L} \int \left| \frac{1+|\log y|}{1+y^{L+1-k}} \mathcal{A}^k \left[\chi_{B_1} T_i + \chi_{B_1} \sum_{j=i+1}^{L+2} \frac{\partial S_j}{\partial b_i} \right] \right|^2 \lesssim 1,$$

which comes from the admissibility of $\boldsymbol{T}_i$ and Lemma 2.7. For the global bounds (3-72), we need to gain one extra b_1 as follows: since $A\Lambda Q = 0$, the admissibility of $\boldsymbol{T}_i$ and Lemma 2.7 imply

$$\int \left| \mathcal{A}^{L+1} \Lambda Q + \sum_{i=1}^{L} b_i \mathcal{A}^{L+1-\bar{i}} [\Lambda_{1-\bar{i}} (\chi_{B_1} T_i)] + \sum_{i=2}^{L+2} \mathcal{A}^{L+1-\bar{i}} [\Lambda_{1-\bar{i}} (\chi_{B_1} S_i)] \right|^2 \lesssim \sum_{i=1}^{L} \int_{y \leq 2B_1} b_1^i \left| \frac{(1+|\log y|) y^{i-2}}{1+y^L} \right|^2 + \sum_{i=2}^{L+1} b_1^{2i} + \frac{b_1^{2(L+1)}}{|\log b_1|^2} \lesssim b_1^2.$$

For (3-75), we additionally use the cancellation $\mathcal{A}^L T_i = 0$ for $1 \le i \le L$ to estimate

$$\sum_{i=1}^{L} \int |\mathcal{A}^{L+1-\bar{i}}(\chi_{B_1} T_i)|^2 \lesssim \sum_{i=1}^{L} \int_{y\sim B_1} \left| \frac{y^{i-2}\log y}{y^L} \right|^2 \lesssim b_1^2,$$

$$\sum_{j=i+1}^{L+2} \int \left| \mathcal{A}^{L+1-\bar{i}} \left[\chi_{B_1} \frac{\partial S_j}{\partial b_i} \right] \right|^2 \lesssim \sum_{j=i+1}^{L+2} b_1^{2(j-i)} + \frac{b_1^{2(L+1-i)}}{|\log b_1|^2} \lesssim b_1^2.$$

Hence (3-72) comes from Lemma 3.3:

$$\|\mathcal{A}^{L+1}\widetilde{\mathrm{Mod}}(t)\|_{L^2} + \|\mathcal{A}^L \dot{\widetilde{\mathrm{Mod}}}(t)\|_{L^2} \lesssim b_1 \left[\frac{b_1^{L+1}}{|\log b_1|} + \sqrt{\frac{\mathcal{E}_{L+1}}{\log M}} \right].$$

For the remaining two terms, $\boldsymbol{NL}(\boldsymbol{\varepsilon})$ and $\boldsymbol{L}(\boldsymbol{\varepsilon})$, we follow the approach developed in [Raphaël and Schweyer 2014]. We deal with the cases $y \le 1$ and $y \ge 1$ separately.

(iii) $\boldsymbol{NL}(\boldsymbol{\varepsilon})$ *term*: (a) $y \le 1$. From a Taylor Lagrange formula in Lemma B.1, $NL(\varepsilon)$ also satisfies a Taylor Lagrange formula

$$NL(\varepsilon) = \sum_{i=0}^{(L-1)/2} c_i y^{2i+1} + r_\varepsilon, \tag{3-76}$$

where

$$|c_i| \lesssim C(M)\mathcal{E}_{L+1}, \quad |\mathcal{A}^k r_\varepsilon| \lesssim y^{L-k} |\log y| C(M)\mathcal{E}_{L+1}, \quad 0 \le k \le L. \tag{3-77}$$

Since the expansion part of $NL(\varepsilon)$ is an odd function, that of $\mathcal{A}^k NL(\varepsilon)$ also has a single parity from the cancellation $A(y) = O(y^2)$. Using (3-77), we obtain

$$|\mathcal{A}^k NL(\varepsilon)(y)| \lesssim C(M)|\log y|\mathcal{E}_{L+1}, \quad 0 \le k \le L, \tag{3-78}$$

and thus we conclude

$$\|\mathcal{A}^L NL(\varepsilon)\|_{L^2(y\le 1)} + \left\| \frac{1+|\log y|^C}{1+y^{L+1-k}} \mathcal{A}^k NL(\varepsilon) \right\|_{L^2(y\le 1)} \lesssim C(M)\mathcal{E}_{L+1} \lesssim b_1^{2L+1}.$$

(b) $y \ge 1$. Let

$$NL(\varepsilon) = \zeta^2 N_1(\varepsilon), \quad \zeta = \frac{\varepsilon}{y}, \quad N_1(\varepsilon) = \int_0^1 (1-\tau) f''(\widetilde{Q}_b + \tau\varepsilon)\, d\tau. \tag{3-79}$$

We have the following bounds for $i \ge 0$, $j \ge 1$ and $1 \le i+j \le L$:

$$\left\| \frac{\partial_y^i \zeta}{y^{j-1}} \right\|_{L^\infty(y\ge 1)} + \left\| \frac{\partial_y^i \zeta}{y^j} \right\|_{L^2(y\ge 1)} \lesssim |\log b_1|^{C(K)} b_1^{m_{i+j+1}}, \quad \|\zeta\|_{L^2(y\ge 1)} \lesssim 1, \tag{3-80}$$

$$|N_1(\varepsilon)| \lesssim 1, \quad |\partial_y^k N_1(\varepsilon)| \lesssim |\log b_1|^{C(K)} \left[\frac{1}{y^{k+1}} + b_1^{m_{k+1}} \right], \quad 1 \le k \le L, \tag{3-81}$$

where

$$m_{k+1} = \begin{cases} kc_1 & \text{if } 1 \le k \le L-2, \\ L & \text{if } k = L-1, \\ L+1 & \text{if } k = L. \end{cases} \tag{3-82}$$

The estimates (3-80) are consequences of Lemma B.1 and the orbital stability (3-6). We can prove the estimates (3-81) by borrowing the proof of (3-77) in [Raphaël and Schweyer 2014] (see page 1768 line 1 in that work), since we can obtain the crude bound

$$
\begin{aligned}
|\partial_y^k \widetilde{Q}_b| &\lesssim |\log b_1|^C \Bigg[\frac{1}{y^{k+1}} + \sum_{i=1}^{(L+1)/2} b_1^{2i} y^{2i-1-k} \mathbf{1}_{y\le 2B_1} \Bigg] \\
&\lesssim \frac{|\log b_1|^C}{y^{k+1}}.
\end{aligned}
$$

Returning to the estimates for $\boldsymbol{NL}(\boldsymbol{\varepsilon})$, we have the trivial bound,

$$
\text{for } 0 \le k \le L, \qquad \left| \frac{1+|\log y|^C}{y^{L+1-k}} \mathcal{A}^k NL(\varepsilon) \right| \lesssim \left| \frac{\mathcal{A}^k NL(\varepsilon)}{y^{L-k}} \right|,
$$

and (3-79) and (3-81) imply

$$
\begin{aligned}
\left| \frac{\mathcal{A}^k NL(\varepsilon)}{y^{L-k}} \right| &\lesssim \sum_{k=0}^{L} \frac{|\partial_y^k NL(\varepsilon)|}{y^{L-k}} \\
&\lesssim \sum_{k=0}^{L} \frac{1}{y^{L-k}} \sum_{i=0}^{k} |\partial_y^i \zeta^2| |\partial_y^{k-i} N_1(\varepsilon)| \\
&\lesssim \sum_{k=0}^{L} \frac{|\log b_1|^{C(K)}}{y^{L-k}} \Bigg[|\partial_y^k \zeta^2| + \sum_{i=0}^{k-1} b_1^{m_{k-i+1}} |\partial_y^i \zeta^2| \Bigg] \\
&\lesssim \sum_{k=0}^{L} \frac{|\log b_1|^{C(K)}}{y^{L-k}} \Bigg[\sum_{i=0}^{k} |\partial_y^i \zeta| |\partial_y^{k-i} \zeta| + \sum_{i=0}^{k-1} \sum_{j=0}^{i} b_1^{m_{k-i+1}} |\partial_y^j \zeta| |\partial_y^{i-j} \zeta| \Bigg].
\end{aligned}
$$

Writing $I_1 = k - i$, $I_2 = i$, there exists $J_2 \in \mathbb{N}$ such that

$$
\max(0, 1-i) \le J_2 \le \min(L+1-k, L-i), \qquad J_1 = L+1-k-J_2,
$$

and we have

$$
1 \le I_1 + J_1 \le L, \quad 1 \le I_2 + J_2 \le L, \quad I_1 + I_2 + J_1 + J_2 = L+1.
$$

Thus,

$$
\begin{aligned}
\left\| \frac{\partial_y^i \zeta \cdot \partial_y^{k-i} \zeta}{y^{L-k}} \right\|_{L^2(y\ge 1)} &\le \left\| \frac{\partial_y^{I_1} \zeta}{y^{J_1-1}} \right\|_{L^\infty(y\ge 1)} \left\| \frac{\partial_y^{I_2} \zeta}{y^{J_2}} \right\|_{L^2(y\ge 1)} \\
&\lesssim |\log b_1|^{C(K)} b_1^{m_{I_1+J_1+1}} b_1^{m_{I_2+J_2+1}} \lesssim b_1^{\delta(L)} b_1^{L+2}
\end{aligned}
$$

since

$$
\begin{aligned}
m_{I_1+J_1+1} + m_{I_2+J_2+1} &= \begin{cases} (L+1)c_1 & \text{if } I_1+J_1 < L-1 \text{ and } I_2+J_2 < L-1, \\ L+2c_1 & \text{if } I_1+J_1 = L-1 \text{ or } I_2+J_2 = L-1, \\ L+1+c_1 & \text{if } I_1+J_1 = L \text{ or } I_2+J_2 = L, \end{cases} \\
&> L+2.
\end{aligned}
$$

We calculate the latter term similarly except for the cases $k=L$ and $0\le i=j\le k-1$. Here, we use the energy bound $\|\zeta\|_{L^2(y\ge 1)}\lesssim 1$ and obtain

$$\begin{aligned}|\log b_1|^{C(K)}b_1^{m_{L-i+1}}\|\partial_y^i\zeta\cdot\zeta\|_{L^2(y\ge 1)} &\lesssim |\log b_1|^{C(K)}b_1^{m_{L-i+1}}\|\partial_y^i\zeta\|_{L^\infty(y\ge 1)}\\ &\lesssim \begin{cases}|\log b_1|^{C(K)}b_1^{(L+1)c_1} & \text{if } 0<i<L-1,\\ |\log b_1|^{C(K)}b_1^{L+2c_1} & \text{if } i=1,\ L-2,\\ |\log b_1|^{C(K)}b_1^{L+1+c_1} & \text{if } i=0,\ L-1,\end{cases}\\ &\lesssim b_1^{\delta(L)}b_1^{L+2}.\end{aligned}$$

The remaining case can be estimated by the following inequalities: since $k-i\ge 1,\ I_1+J_1\ge 1,\ I_2+J_2\ge 1$ and $I_1+I_2+J_1+J_2=L+1-(k-i)$,

$$\begin{aligned}|\log b_1|^{C(K)}b_1^{m_{k-i+1}+m_{I_1+J_1+1}+m_{I_2+J_2+1}} &\lesssim \begin{cases}|\log b_1|^{C(K)}b_1^{(L+1)c_1} & \text{if } k-i<L-1,\\ |\log b_1|^{C(K)}b_1^{L+2c_1} & \text{if } k-i=L-1,\end{cases}\\ &\lesssim b_1^{\delta(L)}b_1^{L+2}.\end{aligned}$$

(iv) $\boldsymbol{L(\varepsilon)}$ *term*: (a) $y\le 1$. Similar to the case $\boldsymbol{NL(\varepsilon)}$, we obtain a Taylor Lagrange formula for $\boldsymbol{L(\varepsilon)}$:

$$L(\varepsilon)=b_1^2\Biggl[\sum_{i=0}^{(L-1)/2}\tilde{c}_i y^{2i+1}+\tilde{r}_\varepsilon\Biggr], \tag{3-83}$$

where

$$|\tilde{c}_i|\lesssim C(M)\sqrt{\mathcal{E}_{L+1}},\quad |\mathcal{A}^k\tilde{r}_\varepsilon|\lesssim y^{L-k}|\log y|C(M)\sqrt{\mathcal{E}_{L+1}},\ \ 0\le k\le L. \tag{3-84}$$

Using the cancellation $A(y)=O(y^2)$ and (3-84), we obtain

$$|\mathcal{A}^kL(\varepsilon)(y)|\lesssim C(M)b_1^2|\log y|\sqrt{\mathcal{E}_{L+1}},\quad 0\le k\le L, \tag{3-85}$$

and thus we conclude

$$\|\mathcal{A}^LL(\varepsilon)\|_{L^2(y\le 1)}+\left\|\frac{1+|\log y|^C}{1+y^{L+1-k}}\mathcal{A}^kL(\varepsilon)\right\|_{L^2(y\le 1)}\lesssim C(M)b_1^2\sqrt{\mathcal{E}_{L+1}}.$$

(b) $y\ge 1$. Let

$$L(\varepsilon)=\varepsilon N_2(\alpha_b),\quad N_2(\alpha_b)=\frac{f'(\widetilde{Q}_b)-f'(Q)}{y^2}=\frac{\chi_{B_1}\alpha_b}{y^2}\int_0^1 f''(Q+\tau\chi_{B_1}\alpha_b)\,d\tau.$$

Similar to (3-81), we have the bound

$$|\partial_y^kN_2|\lesssim \frac{b_1^2|\log b_1|^C}{y^{k+1}},\quad 0\le k\le L. \tag{3-86}$$

Since $L(\varepsilon)$ satisfies the pointwise bound

$$\left|\frac{\mathcal{A}^kL(\varepsilon)}{y^{L-k}}\right|\lesssim\sum_{i=0}^k\frac{|\partial_y^i\varepsilon||\partial_y^{k-i}N_2|}{y^{L-k}}\lesssim b_1^2|\log b_1|^C\sum_{i=0}^k\frac{|\partial_y^i\varepsilon|}{y^{L+1-i}}, \tag{3-87}$$

this yields the desired result. □

4. Proof of the main theorem

4.1. ***Proof of Proposition 3.2.*** Step 1: *Control of the scaling law.* We have the bound

$$-\frac{\lambda_s}{\lambda}=\frac{c_1}{s}+\frac{d_1}{s\log s}+O\biggl(\frac{1}{s(\log s)^{\beta}}\biggr).$$

We rewrite this as

$$\biggl|\frac{d}{ds}(\log(s^{c_1}(\log s)^{d_1}\lambda(s)))\biggr|\lesssim\frac{1}{s(\log s)^{\beta}};$$

integration and (3-23) give

$$\lambda(s)=\frac{s_0^{c_1}(\log s_0)^{d_1}}{s^{c_1}(\log s)^{d_1}}\biggl(1+O\biggl(\frac{1}{(\log s_0)^{\beta-1}}\biggr)\biggr).\tag{4-1}$$

Note that

$$\frac{d}{ds}\biggl(\frac{b_1^{2n}(\log b_1)^{2m}}{\lambda^{2k-2}}\biggr)=2\frac{b_1^{2n-1}(\log b_1)^{2m}}{\lambda^{2k-2}}\biggl[(k-1)b_1^2+b_{1s}\biggl(n+\frac{m}{\log b_1}\biggr)+O(b_1^{L+2})\biggr].\tag{4-2}$$

From Lemma 3.3 with (2-118), (2-115) and (3-28),

$$\begin{aligned}(k-1)b_1^2+b_{1s}\biggl(n+\frac{m}{\log b_1}\biggr)&=(k-1)b_1^2+\bigl(b_2-c_{b_1,1}b_1^2\bigr)\biggl(n+\frac{m}{\log b_1}\biggr)+O(b_1^{L+2})\\&=(k-1)b_1^2+nb_2+\frac{2mb_2-nb_1^2}{2\log b_1}+O\biggl(\frac{b_1^2}{(\log b_1)^2}\biggr)\\&=\frac{(k-1)c_1^2+nc_2}{s^2}+\frac{2(k-1)c_1d_1-nd_2-mc_2+\frac12 nc_1^2}{s^2\log s}+O\biggl(\frac{1}{s^2(\log s)^{\beta}}\biggr).\end{aligned}$$

The recurrence relations (2-116) and (2-117) imply

$$(k-1)c_1^2+nc_2=c_1\biggl((k-1)\frac{\ell}{\ell-1}-n\biggr)$$

and

$$2(k-1)c_1d_1-nd_2+\tfrac12 nc_1^2=d_1(2(k-1)c_1+n)<0.$$

Hence, if we set $n=L+1$ and $m=-1$ for $k=L+1$, $c_1\geq L/(L-1)$ implies

$$(k-1)b_1^2+b_{1s}\biggl(n+\frac{m}{\log b_1}\biggr)\geq\frac{1}{s^2}\biggl(\frac{c_1}{L-1}+O\biggl(\frac{1}{\log s}\biggr)\biggr)>0,$$

and, if we set $n=(k-1)c_1$ and $m=m(k,L)$ large enough for $k\leq L$,

$$(k-1)b_1^2+b_{1s}\biggl(n+\frac{m}{\log b_1}\biggr)\geq\frac{c_1}{s^2\log s}\biggl(\frac{m}{2}+O\biggl(\frac{1}{(\log s)^{\beta-1}}\biggr)\biggr)>0$$

for all $s\in[s_0,s^*)$ with sufficiently large s_0. Thus,

$$\frac{b_1^{2(L+1)}(0)}{(\log b_1(0))^2\lambda^{2L}(0)}\leq\frac{b_1^{2(L+1)}(t)}{(\log b_1(t))^2\lambda^{2L}(t)}\tag{4-3}$$

and

$$\frac{b_1^{2(k-1)c_1}(0)|\log b_1(0)|^m}{\lambda^{2(k-1)}(0)}\leq\frac{b_1^{2(k-1)c_1}(t)|\log b_1(t)|^m}{\lambda^{2(k-1)}(t)}.\tag{4-4}$$

Step 2: *Improved bound on $\mathcal{E}_{L+1}$.* We integrate the Lyapunov monotonicity (3-47) and inject the bootstrap bounds (3-21) and (3-25):

$$\begin{aligned}
\mathcal{E}_{L+1}(t) &\lesssim \frac{\lambda^{2L}(t)}{\lambda^{2L}(0)}(1+b_1C(M))\mathcal{E}_{L+1}(0)+b_1C(M)\mathcal{E}_{L+1}(t)\\
&\qquad\qquad+\left[\frac{K}{\sqrt{\log M}}+\sqrt{K}\right]\lambda^{2L}(t)\int_0^t\frac{b_1}{\lambda^{2L+1}}\frac{b_1^{2(L+1)}}{|\log b_1|^2}\,d\tau\\
&\lesssim \frac{b_1^{2(L+1)}(t)}{|\log b_1(t)|^2}+\left[\frac{K}{\sqrt{\log M}}+\sqrt{K}\right]\lambda^{2L}(t)\int_0^t\frac{b_1}{\lambda^{2L+1}}\frac{b_1^{2(L+1)}}{|\log b_1|^2}.
\end{aligned}\tag{4-5}$$

To deal with the integral in (4-5), one can directly replace λ and b_1 with functions of s using (4-1) and (2-118). However, the fact that s_0 in (4-1) depends on the bootstrap constant K requires (more) care in direct substitution. On behalf of this approach, we integrate by parts using (4-2), (4-3) and the fact $c_1\ge L/(L-1)$:

$$\begin{aligned}
\int_0^t\frac{b_1}{\lambda^{2L+1}}\frac{b_1^{2(L+1)}}{|\log b_1|^2} &= -\int_0^t\frac{\lambda_t}{\lambda^{2L+1}}\frac{b_1^{2(L+1)}}{|\log b_1|^2}+\int_0^t O(b_1^{L+2})\frac{b_1^{2(L+1)}}{\lambda^{2L+1}|\log b_1|^2}\\
&=\frac{1}{2L}\left[\frac{b_1^{2(L+1)}(t)}{\lambda^{2L}(t)|\log b_1(t)|^2}-\frac{b_1^{2(L+1)}(0)}{\lambda^{2L}(0)|\log b_1(0)|^2}\right]\\
&\qquad\qquad-\frac{1}{2L}\int_0^t\frac{1}{\lambda^{2L}}\left(\frac{b_1^{2(L+1)}}{|\log b_1|^2}\right)_t+\int_0^t O(b_1^{L+2})\frac{b_1^{2(L+1)}}{\lambda^{2L+1}|\log b_1|^2}\\
&\le \frac{b_1^{2(L+1)}(t)}{\lambda^{2L}(t)|\log b_1(t)|^2}+\int_0^t\frac{b_1}{\lambda^{2L+1}}\left(\frac{L^2-1}{L^2}+\frac{C}{|\log b_1|}\right)\frac{b_1^{2(L+1)}}{|\log b_1|^2},
\end{aligned}$$

and we obtain the bound

$$\int_0^t\frac{b_1}{\lambda^{2L+1}}\frac{b_1^{2(L+1)}}{|\log b_1|^2}\lesssim\frac{b_1^{2(L+1)}(t)}{\lambda^{2L}(t)|\log b_1(t)|^2},$$

and therefore,

$$\mathcal{E}_{L+1}(t)\lesssim\left[1+\frac{K}{\sqrt{\log M}}+\sqrt{K}\right]\frac{b_1^{2(L+1)}(t)}{|\log b_1(t)|^2}\le\frac{K}{2}\frac{b_1^{2(L+1)}(t)}{|\log b_1(t)|^2}.\tag{4-6}$$

Step 3: *Improved bound on $\mathcal{E}_k$.* We now claim the improved bound on the intermediate energies: for $2\le k\le L$,

$$\mathcal{E}_k\le b_1^{2(k-1)c_1}|\log b_1|^{C+K/2}.\tag{4-7}$$

This follows from the monotonicity formula, for $2\le k\le L$,

$$\frac{d}{dt}\left\{\frac{\mathcal{E}_k}{\lambda^{2k-2}}\right\}\le C\frac{b_1|\log b_1|^C}{\lambda^{2k-1}}(\sqrt{\mathcal{E}_{k+1}}+b_1^k+b_1^{\delta(k)+(k-1)c_1})\sqrt{\mathcal{E}_k}\tag{4-8}$$

for some universal constants $C,\delta>0$ independent of the bootstrap constant K. Estimate (4-7) will be proved in Appendix D. We integrate the above monotonicity formula ($K/2$ comes from $\sqrt{\mathcal{E}_k}$) and obtain

$$\mathcal{E}_k\lesssim b_1^{2(k-1)c_1}|\log b_1|^{C+K/2}+\lambda^{2k-2}(t)\int_0^t\frac{b_1^{1+2(k-1)c_1}}{\lambda^{2k-1}}|\log b_1|^{C+K/2}.\tag{4-9}$$

In this case, we directly substitute λ and b_1 with functions of s since the possible large coefficient can be absorbed by $|\log b_1|^C$. From (4-1), (2-114) and (2-118),

$$\begin{aligned}\lambda^{2k-2}(t)\int_0^t \frac{b_1^{1+2(k-1)c_1}}{\lambda^{2k-1}}|\log b_1|^{C+K/2}\,d\tau &= \lambda^{2k-2}(s)\int_{s_0}^s \frac{b_1^{1+2(k-1)c_1}}{\lambda^{2k-2}}|\log b_1|^{C+K/2}\,d\sigma \\ &\lesssim \frac{(\log s)^{C+K/2}}{s^{2(k-1)c_1}}\int_{s_0}^s \frac{1}{\sigma}\,d\sigma \\ &\lesssim b_1^{2(k-1)c_1}|\log b_1|^{C+K/2}. \end{aligned} \tag{4-10}$$

However, these improved bounds (4-7) are inadequate to close the bootstrap bounds when $\ell = L$ (3-26) and when $\ell = L-1$ (3-27) due to the logarithm factor. In these cases, we employ alternative energies defined by

$$\widehat{\mathcal{E}}_\ell := \langle \hat{\varepsilon}_\ell, \hat{\varepsilon}_\ell\rangle + \langle \dot{\hat{\varepsilon}}_{\ell-1}, \dot{\hat{\varepsilon}}_{\ell-1}\rangle. \tag{4-11}$$

We can easily check that

$$\widehat{\mathcal{E}}_\ell = \mathcal{E}_\ell + O(b_1^{2\ell}|\log b_1|^2).$$

Then we have the monotonicity formulae

$$\frac{d}{dt}\left\{\frac{\widehat{\mathcal{E}}_\ell}{\lambda^{2\ell-2}} + O\left(\frac{b_1^{2\ell}|\log b_1|^2}{\lambda^{2\ell-2}}\right)\right\} \le \frac{b_1^{\ell+1}|\log b_1|^\delta}{\lambda^{2\ell-1}}(b_1^\ell|\log b_1| + \sqrt{\mathcal{E}_\ell}). \tag{4-12}$$

Integrating (4-12), the initial bounds (3-21) and the bootstrap bounds (3-26), (3-27) imply

$$\begin{aligned}\frac{\widehat{\mathcal{E}}_\ell(t)}{\lambda^{2(\ell-1)}(t)} &\lesssim \frac{b_1^{2\ell}|\log b_1|^2(t)}{\lambda^{2\ell-2}(t)} + \frac{\widehat{\mathcal{E}}_\ell(0) + b_1^{2\ell}(0)|\log b_1(0)|^2}{\lambda^{2(\ell-1)}(0)} + \int_0^t \frac{b_1^{\ell+1}|\log b_1|^\delta}{\lambda^{2\ell-1}}(b_1^\ell|\log b_1| + \sqrt{\mathcal{E}_\ell})\,d\tau \\ &\lesssim 1 + \int_0^t \frac{b_1^{\ell+1}|\log b_1|^{\delta'}}{\lambda^\ell}\,d\tau \lesssim 1 + \int_{s_0}^s \frac{1}{\sigma(\log\sigma)^{\ell/(\ell-1)-\delta'}}\,d\sigma \lesssim \frac{K}{2}.\end{aligned}$$

The monotonicity formulae (4-8), (4-12) are proved in Appendix D.

Remark. We remark that the exponent $1+2(k-1)c_1$ of b_1 in (4-9) can be replaced by $1+\delta+2(k-1)c_1$ for some small $\delta > 0$ when $2 \le k \le \ell-1$, so we can improve the bound (4-10) to $b_1^{2(k-1)c_1+\delta}|\log b_1|^C$. Hence, for $2 \le k \le \ell$, we get the uniform bounds

$$\mathcal{E}_k \lesssim \lambda^{2k-2}. \tag{4-13}$$

Step 4: *Control of stable/unstable parameters*. We make use of the modified modulation parameters $\tilde{b} = (b_1, \dots, b_{L-1}, \tilde{b}_L)$ with $\tilde{b}_L$ given by (3-41) and the corresponding fluctuation $\widetilde{V} = P_\ell \widetilde{U}$, where $\widetilde{U} = (\widetilde{U}_1, \dots, \widetilde{U}_\ell)$ is defined by

$$\frac{\widetilde{U}_k}{s^k(\log s)^\beta} = \tilde{b}_k - b_k^e, \quad 1 \le k \le \ell.$$

We note that the existence of $V(s_0)$ in Proposition 3.2 is equivalent to the existence of $\widetilde{V}(s_0)$ from the remark on page 2453 and (3-42) in view of

$$|V - \widetilde{V}| \lesssim s^L|\log s|^\beta|b_L - \tilde{b}_L| \lesssim s^L|\log s|^\beta b_1^{L+1-C\delta} \lesssim \frac{1}{s^{1/2}}. \tag{4-14}$$

Hence we can replace $\widetilde{V}$ for all the V of the initial assumptions (3-22), (3-24) and bootstrap bounds (3-28), (3-29) in Section 3.3. In particular, we replace the assumption (3-31) with

$$\tilde{s}^* < \infty \quad \text{for all } (V_2(s_0), \dots, V_\ell(s_0)) \in \mathcal{B}^{\ell-1}, \tag{4-15}$$

where $\tilde{s}^*$ denotes the modified exit time to indicate that V has been changed to $\widetilde{V}$.

We start by closing the bootstrap bounds for the stable parameters b_L (for the case $\ell = L-1$) and $\widetilde{V}_1$, then we rule out the assumption of the unstable parameters $(\widetilde{V}_2(s), \dots, \widetilde{V}_\ell(s))$ via showing a contradiction by Brouwer's fixed point theorem.

(i) *Stable parameter b_L when $\ell = L-1$*: Recalling Lemma 3.4, we have

$$|(\tilde{b}_L)_s + (L-1+c_{b,L})b_1\tilde{b}_L| \lesssim \frac{\sqrt{\mathcal{E}_{L+1}}}{\sqrt{|\log b_1|}}. \tag{4-16}$$

Note that $c_1 = (L-1)/(L-2)$ and $b_1 \sim c_1/s + d_1/(s\log s)$. Then, from (3-28) and (4-16),

$$\begin{aligned}
\frac{d}{ds}(s^{(L-1)c_1}(\log s)^{3/2}\tilde{b}_L) &= s^{(L-1)c_1-1}(\log s)^{3/2}\left((L-1)c_1 + \frac{3/2}{\log s}\right)\tilde{b}_L \\
&\quad - s^{(L-1)c_1}(\log s)^{3/2}\left((L-1+c_{b,L})b_1\tilde{b}_L + O\left(\frac{\sqrt{\mathcal{E}_{L+1}}}{\sqrt{|\log b_1|}}\right)\right) \\
&= s^{(L-1)c_1-1}(\log s)^{3/2} O\left(\frac{1}{s^L(\log s)^{1+\beta}} + \frac{1}{s^L(\log s)^{3/2}}\right) \\
&= O(s^{(L-1)c_1-L-1}).
\end{aligned}$$

We integrate the above equation and estimate using the initial condition (3-22)

$$|b_L(s)| \lesssim b_1^{L+1-C\delta} + \frac{s_0^{(L-1)c_1}(\log s_0)^{3/2}|\tilde{b}_L(s_0)|}{s^{(L-1)c_1}(\log s)^{3/2}} + \frac{1+(s_0/s)^{(L-1)c_1-L}}{s^L(\log s)^{3/2}} \le \frac{1/2}{s^L(\log s)^\beta}$$

with the fact $(L-1)c_1 > L$. Here, we choose $\beta = \frac{5}{4}$.

To control the modes $\widetilde{V}$, we rewrite (2-119) for our $\tilde{b}$ as follows:

$$s(\widetilde{U})_s - A_\ell\widetilde{U} = O\left(\frac{1}{(\log s)^{3/2-\beta}}\right) \tag{4-17}$$

using (2-122) and Lemmas 3.3 and 3.4. Here, the reduced exponent $\frac{3}{2}$ comes from (4-16). By the definition of $\widetilde{V}$, (4-17) is equivalent to

$$s(\widetilde{V})_s - D_\ell\widetilde{V} = O\left(\frac{1}{(\log s)^{3/2-\beta}}\right), \tag{4-18}$$

where D_ℓ is given by (2-121).

(ii) *Stable mode $\widetilde{V}_1$*: The first coordinate of (4-18) can be written as

$$s(\widetilde{V}_1)_s + \widetilde{V}_1 = (s\widetilde{V}_1)_s = O\left(\frac{1}{(\log s)^{3/2-\beta}}\right).$$

Hence we improve the bound for $\widetilde{V}_1(s)$ from the initial assumption (3-22):

$$|\widetilde{V}_1(s)| \lesssim \frac{s_0}{s}|\widetilde{V}_1(s_0)| + \frac{C}{s}\int_{s_0}^{s} \frac{d\tau}{(\log\tau)^{3/2-\beta}} \le \frac{1}{2}.$$

(iii) *Unstable mode* $\widetilde{V}_k$, $2 \le k \le \ell$: Our goal is to construct a continuous map $f : \mathcal{B}^{\ell-1} \to \mathcal{S}^{\ell-1}$ defined as

$$f(\widetilde{V}_2(s_0), \dots, \widetilde{V}_\ell(s_0)) = (\widetilde{V}_2(\tilde{s}^*), \dots, \widetilde{V}_\ell(\tilde{s}^*)).$$

The assumption (4-15) yields that f can be well-defined on $\mathcal{B}^{\ell-1}$ and the improved bootstrap bounds give the exit condition $(\widetilde{V}_2(\tilde{s}^*), \dots, \widetilde{V}_\ell(\tilde{s}^*)) \in \mathcal{S}^{\ell-1}$.

We obtain the outgoing behavior of the flow map $s \mapsto (\widetilde{V}_2, \dots, \widetilde{V}_\ell)$ from (4-18): for all time $s \in [s_0, \tilde{s}^*]$ such that $\sum_{i=2}^{\ell} \widetilde{V}_i^2 \ge \frac{1}{2}$,

$$\frac{d}{ds}\biggl(\sum_{i=2}^{\ell} \widetilde{V}_i^2\biggr) = 2\sum_{i=2}^{\ell}(\widetilde{V}_i)_s\widetilde{V}_i = \frac{2}{s}\sum_{i=2}^{\ell}\biggl[\frac{i}{\ell-1}\widetilde{V}_i^2 + O\biggl(\frac{1}{(\log s)^{3/2-\beta}}\biggr)\biggr] > 0. \tag{4-19}$$

We note that (4-19) implies two key results. First, (4-19) allows us to prove the continuity of f by showing the continuity of the map $(\widetilde{V}_2(s_0), \dots, \widetilde{V}_\ell(s_0)) \mapsto \tilde{s}^*$ with some standard arguments (see [Côte et al. 2011, Lemma 6]).

Second, if we choose $s = s_0$ and $(\widetilde{V}_2(s_0), \dots, \widetilde{V}_\ell(s_0)) \in \mathcal{S}^{\ell-1}$, we have $\sum_{i=2}^{\ell} \widetilde{V}_i^2(s) > 1$ for any $s > s_0$, and so $\tilde{s}^* = s_0$. Hence f is an identity map on $\mathcal{S}^{\ell-1}$ itself, which contradicts to Brouwer's fixed point theorem. □

4.2. ***Proof of Theorem 1.2.*** Recall that there exists $c(u_0, \dot{u}_0) > 0$ such that

$$\lambda(s) = \frac{c(u_0, \dot{u}_0)}{s^{c_1}(\log s)^{d_1}}\biggl[1 + O\biggl(\frac{1}{(\log s_0)^{\beta-1}}\biggr)\biggr].$$

Using $T - t = \int_s^\infty \lambda(s)\,ds < \infty$, we have $T < \infty$ and

$$(T-t)^{\ell-1} = c'(u_0, \dot{u}_0)s^{-1}(\log s)^{\ell/(\ell-1)}[1 + o_{t\to T}(1)] = c''(u_0, \dot{u}_0)\lambda(s)^{(\ell-1)/\ell}(\log s)[1 + o_{t\to T}(1)].$$

Therefore, we obtain

$$\lambda(t) = c'''(u_0, \dot{u}_0)\frac{(T-t)^\ell}{|\log(T-t)|^{\ell/(\ell-1)}}[1 + o_{t\to T}(1)].$$

The strong convergence (1-13) follows as in [Raphaël and Rodnianski 2012].

Appendix A: Coercive properties

We recall that $\boldsymbol{\Phi}_M = (\Phi_M, 0)^t$, and hence the orthogonality conditions (3-5) are equivalent to

$$\langle \varepsilon, H^i\Phi_M\rangle = \langle \dot{\varepsilon}, H^i\Phi_M\rangle = 0, \quad 0 \le i \le \tfrac{1}{2}(L-1). \tag{A-1}$$

In this section, we claim that the above equivalent orthogonality conditions yield the coercive property of the higher-order energy $\mathcal{E}_{k+1}$:

$$\mathcal{E}_{k+1} = \langle \varepsilon_{k+1}, \varepsilon_{k+1}\rangle + \langle \dot{\varepsilon}_k, \dot{\varepsilon}_k\rangle, \quad 1 \le k \le L. \tag{A-2}$$

Our desired result is deduced from the coercivity of $\{\|v_m\|_{L^2}^2\}_{m=1}^{L+1}$ under the orthogonality conditions

$$\langle v, H^i\Phi_M\rangle = 0, \quad 0\le i\le \left\lfloor \tfrac{1}{2}(m-1)\right\rfloor. \tag{A-3}$$

First, we restate Lemma B.5 of [Raphaël and Schweyer 2014], which established the coercivity of $\|v_m\|_{L^2}^2$ when m is even. For the rest of the paper, we use [RS14] to abbreviate this work.

Lemma A.1 (coercivity of $\|v_{2k+2}\|_{L^2}^2$). *Let $0\le k\le \frac{1}{2}(L-1)$ and $M=M(L)>0$ be a large constant. Then there exists $C(M)>0$ such that the following holds. For all radially symmetric v with*

$$\int |v_{2k+2}|^2+\int \frac{|v_{2k+1}|^2}{y^2(1+y^2)} + \sum_{i=0}^{k}\int \frac{|v_{2i-1}|^2}{y^6(1+|\log y|^2)(1+y^{4(k-i)})}+\frac{|v_{2i}|^2}{y^4(1+|\log y|^2)(1+y^{4(k-i)})}<\infty \tag{A-4}$$

(we write $v_{-1}=0$) and (A-3) *for $m=2k+2$, we have*

$$\int |v_{2k+2}|^2\ge C(M)\biggl\{\int \frac{|v_{2k+1}|^2}{y^2(1+|\log y|^2)} + \sum_{i=0}^{k}\int\biggl[\frac{|v_{2i-1}|^2}{y^6(1+|\log y|^2)(1+y^{4(k-i)})}+\frac{|v_{2i}|^2}{y^4(1+|\log y|^2)(1+y^{4(k-i)})}\biggr]\biggr\}. \tag{A-5}$$

We additionally prove the coercivity of $\|v_m\|_{L^2}^2$ when m is odd, which is an unnecessary step in [RS14].

Lemma A.2 (coercivity of $\|v_{2k+1}\|_{L^2}^2$). *Let $1\le k\le \frac{1}{2}(L-1)$ and $M=M(L)>0$ be a large constant. Then there exists $C(M)>0$ such that the following holds. For all radially symmetric v with*

$$\int |v_{2k+1}|^2+\int \frac{|v_{2k}|^2}{y^2}+\int\frac{|v_{2k-1}|^2}{y^4(1+|\log y|^2)} + \sum_{i=0}^{k-1}\int \frac{|v_{2i-1}|^2}{y^6(1+|\log y|^2)(1+y^{4(k-i)-2})}+\frac{|v_{2i}|^2}{y^4(1+|\log y|^2)(1+y^{4(k-i)-2})}<\infty \tag{A-6}$$

(we write $v_{-1}=0$) and (A-3) *for $m=2k+1$, we have*

$$\int |v_{2k+1}|^2\ge C(M)\biggl\{\int \frac{|v_{2k}|^2}{y^2}+\frac{|v_{2k-1}|^2}{y^4(1+|\log y|^2)} + \sum_{i=0}^{k-1}\int\biggl[\frac{|v_{2i-1}|^2}{y^6(1+|\log y|^2)(1+y^{4(k-i)-2})}+\frac{|v_{2i}|^2}{y^4(1+|\log y|^2)(1+y^{4(k-i)-2})}\biggr]\biggr\}. \tag{A-7}$$

Remark. The case $k=0$ is nothing but the coercivity of H described in Lemma B.1 of [RS14].

Based on the induction on k introduced in the proof of Lemma B.5 of [RS14], Lemma A.2 can be deduced from the following two lemmas, corresponding to the cases $k=1$ and $k\to k+1$.

Lemma A.3 (coercivity of $\|v_3\|_{L^2}^2$). *Let $M = M(L) > 0$ be a large constant. Then there exists $C(M) > 0$ such that the following holds: for all radially symmetric v with*

$$\int |v_3|^2 + \int \frac{|v_2|^2}{y^2} + \int \frac{|v_1|^2}{y^4(1+|\log y|^2)} + \int \frac{|v|^2}{y^4(1+|\log y|^2)(1+y^2)} < \infty$$

(*we write $v_{-1} = 0$*) *and* (A-3) *for $m = 3$, we have*

$$\int |v_3|^2 \geq C(M)\biggl\{\int \frac{|v_2|^2}{y^2} + \frac{|v_1|^2}{y^4(1+|\log y|^2)} + \int \frac{|v|^2}{y^4(1+|\log y|^2)(1+y^2)}\biggr\}. \tag{A-8}$$

Proof. From the coercivity of H, we have

$$\int |v_3|^2 = \langle Hv_2, v_2\rangle \geq C(M)\int \frac{|v_2|^2}{y^2}. \tag{A-9}$$

To prove the rest of (A-8), we claim the following weighted coercive bound:

$$\int \frac{|Hv|^2}{y^2(1+|\log y|^2)} \geq C(M)\biggl\{\int \frac{|v|^2}{y^4(1+|\log y|^2)(1+y^2)} + \frac{|Av|^2}{y^4(1+|\log y|^2)}\biggr\}. \tag{A-10}$$

By proving Lemma B.4 in [RS14], it is sufficient for (A-10) to prove only the subcoercivity estimate

$$\begin{aligned}\int \frac{|Hv|^2}{y^2(1+|\log y|^2)} \gtrsim \int \frac{|\partial_y^2 v|^2}{y^2(1+|\log y|^2)} &+ \int \frac{|\partial_y v|^2}{y^2(1+|\log y|^2)(1+y^2)} \\ &+ \int \frac{|v|^2}{y^4(1+|\log y|^2)(1+y^2)} - C\biggl[\int \frac{|\partial_y v|^2}{1+y^6} + \int \frac{|v|^2}{1+y^8}\biggr].\end{aligned} \tag{A-11}$$

Unlike the region $y \leq 1$, which can be directly proved by borrowing the proof of Lemma B.4 in [RS14], we remark that (A-11) required some cautious estimates in the region $y \geq 1$: we have

$$\begin{aligned}\int_{y\geq 1} \frac{|Hv|^2}{y^2(1+|\log y|^2)} \geq \int_{y\geq 1} \frac{|\partial_y(y\partial_y v)|^2}{y^4(1+|\log y|^2)} &- \int_{y\geq 1} |v|^2 \Delta\biggl(\frac{V}{y^4(1+|\log y|^2)}\biggr) \\ &+ \int_{y\geq 1} \frac{V^2|v|^2}{y^6(1+|\log y|^2)} - C\int_{1\leq y\leq 2} [|\partial_y v|^2 + |v|^2],\end{aligned} \tag{A-12}$$

where $V(y) = 1 - 8y^2/(1+y^2)^2$ is the potential part of H. Using the sharp logarithmic Hardy inequality employed in the proof of Lemma B.4 of [RS14], we obtain

$$\int_{y\geq 1} \frac{|\partial_y(y\partial_y v)|^2}{y^4(1+|\log y|^2)} - \int_{y\geq 1} |v|^2 \Delta\biggl(\frac{1}{y^4(1+|\log y|^2)}\biggr) \geq -C\int_{1\leq y\leq 2} [|\partial_y v|^2 + |v|^2].$$

Now we employ the additional positive term in (A-12) with the asymptotics of the potential $V(y) = 1 + O(y^{-2})$ for $y \geq 1$,

$$\int_{y\geq 1} \frac{V^2|v|^2}{y^6(1+|\log y|^2)} \geq (1-\delta)\int_{y\geq 1} \frac{|v|^2}{y^6(1+|\log y|^2)} - C\int \frac{|v|^2}{1+y^8}.$$

□

Lemma A.4 (weighted coercivity bound). *For $k \ge 1$ and radially symmetric v with*

$$\int \frac{|v|^2}{y^4(1+|\log y|^2)(1+y^{4k+2})} + \frac{|Av|^2}{y^6(1+|\log y|^2)(1+y^{4k-2})} < \infty \tag{A-13}$$

and

$$\langle v, \Phi_M \rangle = 0,$$

we have

$$\int \frac{|Hv|^2}{y^4(1+|\log y|^2)(1+y^{4k-2})} \ge C(M)\biggl\{\int \frac{|v|^2}{y^4(1+|\log y|^2)(1+y^{4k+2})} + \frac{|Av|^2}{y^6(1+|\log y|^2)(1+y^{4k-2})}\biggr\}. \tag{A-14}$$

Proof. We can prove (A-14) easily by replacing all $4k$ in the proof of Lemma B.4 of [RS14] with $4k-2$ since the range of our k is $k \ge 1$. □

From the previous lemmas, we obtain the coercivity of $\mathcal{E}_{k+1}$.

Lemma A.5 (coercivity of $\mathcal{E}_{k+1}$). *Let $1 \le k \le L$ and $M = M(L) > 0$ be a large constant. Then there exists $C(M) > 0$ such that*

$$\begin{aligned}\mathcal{E}_{k+1} &= \langle \varepsilon_{k+1}, \varepsilon_{k+1}\rangle + \langle \dot{\varepsilon}_k, \dot{\varepsilon}_k\rangle \\ &\ge C(M)\biggl[\sum_{i=0}^{k}\int \frac{|\varepsilon_i|^2}{y^2(1+y^{2(k-i)})(1+|\log y|^2)} + \sum_{i=0}^{k-1}\int \frac{|\dot{\varepsilon}_i|^2}{y^2(1+y^{2(k-1-i)})(1+|\log y|^2)}\biggr].\end{aligned} \tag{A-15}$$

Remark. The finiteness assumptions (A-4), (A-6) and (A-13) for (A-15) are satisfied from the well-localized smoothness of the 1-corotational map $(\Phi, \partial_t\Phi)$ (see Lemma A.1 in [RS14]).

Appendix B: Interpolation estimates

In this section, we provide some interpolation estimates for ε, i.e., the first coordinate part of $\boldsymbol{\varepsilon}$. We will employ these bounds to deal with $\boldsymbol{NL}(\boldsymbol{\varepsilon})$ and $\boldsymbol{L}(\boldsymbol{\varepsilon})$ terms in the evolution equation of $\boldsymbol{\varepsilon}$ (3-8).

Lemma B.1 (interpolation estimates). (i) *For $y \le 1$, ε has a Taylor–Lagrange expansion*

$$\varepsilon = \sum_{i=1}^{(L+1)/2} c_i T_{L+1-2i} + r_\varepsilon,$$

where T_{2i} is the first coordinate part of $\boldsymbol{T}_{2i}$ and

$$|c_i| \lesssim C(M)\sqrt{\mathcal{E}_{L+1}}, \quad |\partial_y^k r_\varepsilon| \lesssim C(M) y^{L-k}|\log y|\sqrt{\mathcal{E}_{L+1}}, \quad 0 \le k \le L.$$

(ii) *For $y \le 1$, ε satisfies the pointwise bounds*

$$\begin{aligned}|\varepsilon_k| &\lesssim C(M) y^{1+\bar{k}}|\log y|\sqrt{\mathcal{E}_{L+1}}, \quad 0 \le k \le L-1,\\ |\varepsilon_L| &\lesssim C(M)\sqrt{\mathcal{E}_{L+1}},\\ |\partial_y^k \varepsilon| &\lesssim C(M) y^{\overline{k+1}}|\log y|\sqrt{\mathcal{E}_{L+1}}, \quad 0 \le k \le L.\end{aligned}$$

(iii) *For* $1 \le k \le L$ *and* $0 \le i \le k$,

$$\int \frac{1+|\log y|^C}{1+y^{2(k-i+1)}}(|\varepsilon_i|^2+|\partial_y^i\varepsilon|^2)+\left\|\frac{\partial_y^i\varepsilon}{y^{k-i}}\right\|^2_{L^\infty(y\ge 1)} \lesssim |\log b_1|^C b_1^{2m_{k+1}},$$

where

$$m_{k+1}=\begin{cases} kc_1 & \text{if } 1\le k\le L-2,\\ L & \text{if } k=L-1,\\ L+1 & \text{if } k=L.\end{cases}$$

Proof. It is provided by the proof of Lemma C.1 in [RS14]. □

Appendix C: Leibniz rule for $\mathcal{A}^k$

Unlike [RS14], we encounter some terms in which ∂_t is applied more than once to $\mathcal{A}_\lambda^k$, such as $\partial_{tt}(\mathcal{A}_\lambda^k)$, $\partial_t(\mathcal{A}_\lambda^i)\partial_t(H_\lambda^j)$, etc. To control those terms, we recall the asymptotics

$$\partial_t(\mathcal{A}_\lambda^k)f_\lambda(r)=\frac{\lambda_t}{\lambda^{k+1}}\sum_{i=0}^{k-1}\Phi_{i,k}^{(1)}(y)f_i(y),\quad |\Phi_{i,k}^{(1)}(y)|\lesssim\frac{1}{1+y^{k+2-i}},\tag{C-1}$$

which were introduced in Appendices D and E of [RS14]. We note that near, the origin, $\Phi_{i,k}^{(1)}$ satisfies

$$\Phi_{i,k}^{(1)}(y)=\begin{cases}\sum_{p=0}^N c_{i,k,p}y^{2p}+O(y^{2N+2}), & k-i \text{ is even},\\ \sum_{p=0}^N c_{i,k,p}y^{2p+1}+O(y^{2N+3}), & k-i \text{ is odd}.\end{cases}\tag{C-2}$$

Based on the above facts, we can obtain the following lemma.

Lemma C.1. *Let* $1\le k\le \frac12(L-1)$. *Then*

$$\partial_{tt}(\mathcal{A}_\lambda^k)f_\lambda(r)=\frac{\lambda_{tt}}{\lambda^{k+1}}\sum_{i=0}^{k-1}\Phi_{i,k}^{(1)}(y)f_i(y)+\frac{O(b_1^2)}{\lambda^{k+2}}\sum_{i=0}^{k-1}\Phi_{i,k}^{(2)}(y)f_i(y),\tag{C-3}$$

$$\partial_t(\mathcal{A}_\lambda^{L-2k})\partial_t(H_\lambda^k)f_\lambda(r)=\frac{O(b_1^2)}{\lambda^{L+2}}\sum_{i=0}^{L-1}\Phi_{i,L}^{(3)}(y)f_i(y),\tag{C-4}$$

where

$$|\Phi_{i,k}^{(2)}(y)|\lesssim\frac{1}{1+y^{k+2-i}},\qquad |\Phi_{i,L}^{(3)}(y)|\lesssim\frac{1}{1+y^{L+3-i}}.$$

Proof. Recalling that $\partial_{tt}(\mathcal{A}_\lambda^k)f_\lambda=[\partial_t,\partial_t(\mathcal{A}_\lambda^k)]f_\lambda$ and

$$\frac{\lambda_t}{\lambda^{k+1}}\Phi_{i,k}^{(1)}(y)f_i(y)=\frac{\lambda_t}{\lambda^{k+1-i}}(\Phi_{i,k}^{(1)})_\lambda(r)\mathcal{A}_\lambda^i f_\lambda(r),\quad \partial_t\Phi_\lambda=-\frac{\lambda_t}{\lambda}(\Lambda\Phi)_\lambda,$$

we get (C-3) since

$$\begin{aligned}\left[\partial_t,\frac{\lambda_t}{\lambda^{k+1-i}}(\Phi_{i,k}^{(1)})_\lambda\mathcal{A}_\lambda^i\right]f_\lambda&=\frac{\lambda_{tt}}{\lambda^{k+1-i}}(\Phi_{i,k}^{(1)})_\lambda\mathcal{A}_\lambda^i f_\lambda-\frac{(\lambda_t)^2}{\lambda^{k+2-i}}(\Lambda_{i-k}\Phi_{i,k}^{(1)})_\lambda\mathcal{A}_\lambda^i f_\lambda+\frac{\lambda_t}{\lambda^{k+1-i}}(\Phi_{i,k}^{(1)})_\lambda\partial_t(\mathcal{A}_\lambda^i)f_\lambda\\ &=\frac{\lambda_{tt}}{\lambda^{k+1}}\Phi_{i,k}^{(1)}(y)f_i(y)+\frac{O(b_1^2)}{\lambda^{k+2}}\sum_{j=0}^{i}\Phi_{i,j,k}(y)f_j(y),\end{aligned}$$

where

$$|\Phi_{i,j,k}(y)| \lesssim \frac{1}{1+y^{k+2-j}}.$$

Moreover, we can easily check that $\Phi^{(2)}_{i,k}$ satisfies (C-2) because the scaling generator Λ preserves the asymptotics near the origin as well as at infinity.

To prove (C-4), we need to justify the terms of the form $\mathcal{A}^i \circ \Phi\mathcal{A}^j$. When j is an even number, we can use the Leibniz rule from Appendix D of [RS14]. However, when j is odd, terms such as $A \circ \Phi A$ appear, making the problem a bit more tricky.

Fortunately, our Φ from the terms of the form $\mathcal{A}^i \circ \Phi\mathcal{A}^{2j+1}$ have an expansion

$$\Phi(y) = \sum_{p=0}^{N} c_p y^{2p+1} + O(y^{2N+3})$$

near the origin since each $\Phi\mathcal{A}^{2j+1}$ comes from $\partial_t(H^k_\lambda)$ or $\partial_{tt}(H^k_\lambda)$, satisfying (C-2). Hence

$$(A \circ \Phi\mathcal{A}^{2j+1})f = (A\Phi)f_{2j+1} - \Phi\partial_y f_{2j+1} = \left(-\partial_y + \frac{1+2Z}{y}\right)\Phi \cdot f_{2j+1} - \Phi f_{2j+2} =: \Phi_1 f_{2j+1} - \Phi f_{2j+2},$$

where Φ_1 satisfies

$$\Phi_1(y) = \sum_{p=0}^{N} c_p y^{2p} + O(y^{2N+2})$$

near the origin. If we take A^* here,

$$\begin{aligned}
(H \circ \Phi\mathcal{A}^{2j+1})f &= A^*(\Phi_1 f_{2j+1} - \Phi f_{2j+2}) \\
&= (\partial_y\Phi_1)f_{2j+1} + (\Phi_1 - A^*\Phi)f_{2j+2} - \Phi\partial_y f_{2j+2} \\
&= (\partial_y\Phi_1)f_{2j+1} + \left(\Phi_1 - \partial_y\Phi - \frac{1+2Z}{y}\Phi\right)f_{2j+2} + \Phi f_{2j+3},
\end{aligned}$$

we can justify $\mathcal{A}^i \circ \Phi\mathcal{A}^{2j+1}$ by iterating the above calculation. □

Appendix D: Monotonicity for the intermediate energy

Proposition D.1 (Lyapunov monotonicity for $\mathcal{E}_k$). *Let $2 \le k \le L$. We have*

$$\frac{d}{dt}\left\{\frac{\mathcal{E}_k}{\lambda^{2k-2}}\right\} \le \frac{b_1|\log b_1|^{C(k)}}{\lambda^{2k-1}}\left(\sqrt{\mathcal{E}_{k+1}} + b_1^k + b_1^{\delta(k)+(k-1)c_1}\right)\sqrt{\mathcal{E}_k}, \tag{D-1}$$

where $C(k)$, $\delta(k) > 0$ are constants that depend only on k and L.

Proof. We compute the energy identity

$$\begin{aligned}
\partial_t\left(\frac{\mathcal{E}_k}{2\lambda^{2(k-1)}}\right) &= \langle \partial_t w_k, w_k\rangle + \langle \partial_t \dot{w}_{k-1}, \dot{w}_{k-1}\rangle \\
&= \langle \partial_t(\mathcal{A}^k_\lambda)w, w_k\rangle + \langle \partial_t(\mathcal{A}^{k-1}_\lambda)\dot{w}, \dot{w}_{k-1}\rangle + \langle \mathcal{A}^k_\lambda\mathcal{F}_1, w_k\rangle + \langle \mathcal{A}^{k-1}_\lambda\mathcal{F}_2, \dot{w}_{k-1}\rangle.
\end{aligned} \tag{D-2}$$

We can directly estimate the first two terms of the right-hand side of (D-2) by Lemma C.1:

$$|\langle \partial_t(\mathcal{A}^k_\lambda) w, w_k\rangle| \lesssim \frac{b_1}{\lambda^{2k-1}} \sum_{m=0}^{k-1} |\langle \Phi^{(1)}_{m,k}\varepsilon_m, \varepsilon_k\rangle| \lesssim \frac{b_1}{\lambda^{2k-1}} \sum_{m=0}^{k-1} \left\| \frac{\varepsilon_m}{1+y^{k+2-m}} \right\|_{L^2} \sqrt{\mathcal{E}_k} \lesssim \frac{b_1 C(M)}{\lambda^{2k-1}} \sqrt{\mathcal{E}_{k+1}\mathcal{E}_k}, \tag{D-3}$$

$$|\langle \partial_t(\mathcal{A}^{k-1}_\lambda) \dot{w}, \dot{w}_{k-1}\rangle| \lesssim \frac{b_1}{\lambda^{2k-1}} \sum_{m=0}^{k-2} |\langle \Phi^{(1)}_{m,k-1}\dot{\varepsilon}_m, \dot{\varepsilon}_{k-1}\rangle| \lesssim \frac{b_1 C(M)}{\lambda^{2k-1}} \sqrt{\mathcal{E}_{k+1}\mathcal{E}_k}. \tag{D-4}$$

Then we conclude (D-1) from the bounds

$$\|\mathcal{A}^k \mathcal{F}\|_{L^2} + \|\mathcal{A}^{k-1}\dot{\mathcal{F}}\|_{L^2} \lesssim b_1 |\log b_1|^C [b_1^k + b_1^{\delta(k)+(k-1)c_1}]. \tag{D-5}$$

The last two terms of the right-hand side of (D-2) is bounded by

$$\frac{b_1|\log b_1|^C}{\lambda^{2k-1}} (b_1^k + b_1^{\delta(k)+(k-1)c_1}) \sqrt{\mathcal{E}_k}. \tag{D-6}$$

Now, it remains to prove (D-5), and we address it by separating $\boldsymbol{\mathcal{F}} = (\mathcal{F}, \dot{\mathcal{F}})^t$ into four types, as we did for Step 5 in the proof of Proposition 3.5.

(i) $\tilde{\boldsymbol{\psi}}_b$ *terms*. The contribution of $\tilde{\boldsymbol{\psi}}_b$ terms to the above inequalities is estimated from the global weighted bounds of Proposition 2.10.

(ii) $\widetilde{\mathbf{Mod}}(t)$ *terms*. Similar to (ii) of Step 5 in the proof of Proposition 3.5 with the cancellation $\mathcal{A}^k T_i = 0$ for $1 \le i \le k$ and Lemma 2.7, we obtain

$$\int \left| \sum_{i=1}^{L} b_i \mathcal{A}^{k-\bar{i}} [\Lambda_{1-\bar{i}}(\chi_{B_1} T_i)] + \sum_{i=2}^{L+2} \mathcal{A}^{k-\bar{i}} [\Lambda_{1-\bar{i}}(\chi_{B_1} S_i)] \right|^2 \lesssim b_1^2,$$

$$\sum_{i=1}^{L} \int \left| \mathcal{A}^{k-\bar{i}} \left[\chi_{B_1} T_i + \chi_{B_1} \sum_{j=i+1}^{L+2} \frac{\partial S_j}{\partial b_i} \right] \right|^2 \lesssim b_1^{2(k-L)} |\log b_1|^{2\gamma(L-k)+2}.$$

Hence Lemma 3.3 and the bootstrap bound (3-25) imply

$$\|\mathcal{A}^k \widetilde{\mathrm{Mod}}(t)\|_{L^2} + \|\mathcal{A}^{k-1} \dot{\widetilde{\mathrm{Mod}}}(t)\|_{L^2} \lesssim b_1^{k-L} |\log b_1|^{\gamma(L-k)+1} \frac{b_1^{L+1}}{|\log b_1|} \lesssim b_1^{k+1} |\log b_1|^{\gamma(L-k)}.$$

(iii) $\boldsymbol{NL}(\boldsymbol{\varepsilon})$ *term*: We can utilize the bound (3-78) near the origin. For $y \ge 1$, we recall the calculation and estimates from (iii) of Step 5 in the proof of Proposition 3.5: $\|\mathcal{A}^{k-1} NL(\varepsilon)\|_{L^2(y\ge 1)}$ is bounded by

$$|\log b_1|^C b_1^{m_I+1} b_1^{m_J+1} + |\log b_1|^C b_1^{m_X+1} b_1^{m_Y+1} b_1^{m_J+1},$$

where $I, J, X, Y, Z \ge 1$, $I + J = k$ and $X + Y + Z = k$. From the bootstrap bounds (3-25), (3-27) and the fact that $c_1 > 1$, we obtain

$$\|\mathcal{A}^{k-1} NL(\varepsilon)\|_{L^2(y\ge 1)} \lesssim |\log b_1|^{C(K)} b_1^{kc_1} \lesssim b_1^{1+\delta(k)+(k-1)c_1}.$$

(iv) $\boldsymbol{L}(\boldsymbol{\varepsilon})$ *term*: With some modifications (replacing L by $k-1$, for instance), it is proved by (3-85) and (3-87). □

Remark. In step (iii) when $k=L$, we can avoid the case that either $I=L-1$ or $J=L-1$ by estimating $\|\partial_y^{L-1}N_1(\varepsilon)\|_{L^2(y\geq 1)}$ instead of $\|\partial_y^{L-1}N_1(\varepsilon)\|_{L^\infty(y\geq 1)}$.

Recall the modified higher-order energies

$$\widehat{\mathcal{E}}_\ell := \langle \hat{\varepsilon}_\ell, \hat{\varepsilon}_\ell\rangle + \langle \dot{\hat{\varepsilon}}_{\ell-1}, \dot{\hat{\varepsilon}}_{\ell-1}\rangle.$$

We rewrite the flow (3-17) componentwise: for $1\leq k\leq \ell$,

$$\begin{cases} \partial_t \hat{w}_k - \dot{\hat{w}}_k = \partial_t(\mathcal{A}_\lambda^k)\hat{w} + \mathcal{A}_\lambda^k \widehat{\mathcal{F}}_1, \\ \partial_t \dot{\hat{w}}_k + \hat{w}_{k+2} = \partial_t(\mathcal{A}_\lambda^k)\dot{\hat{w}} + \mathcal{A}_\lambda^k \widehat{\mathcal{F}}_2, \end{cases} \qquad \begin{pmatrix} \widehat{\mathcal{F}}_1 \\ \widehat{\mathcal{F}}_2 \end{pmatrix} := \frac{1}{\lambda}\widehat{\boldsymbol{\mathcal{F}}}_\lambda = \frac{1}{\lambda}\begin{pmatrix} \widehat{\mathcal{F}} \\ \dot{\widehat{\mathcal{F}}} \end{pmatrix}_\lambda. \tag{D-7}$$

Proposition D.2 (Lyapunov monotonicity for $\mathcal{E}_L$). *Let $\ell = L$. Then we have*

$$\frac{d}{dt}\left\{\frac{\widehat{\mathcal{E}}_L}{\lambda^{2L-2}} + O\left(\frac{b_1^{2L}|\log b_1|^2}{\lambda^{2L-2}}\right)\right\} \leq \frac{b_1^{L+1}|\log b_1|^\delta}{\lambda^{2L-1}}(b_1^L|\log b_1| + \sqrt{\mathcal{E}_L}), \tag{D-8}$$

where $0<\delta\ll 1$ is a sufficient small constant that depend only on L.

Proof. We compute the energy identity

$$\partial_t\left(\frac{\widehat{\mathcal{E}}_L}{2\lambda^{2(L-1)}}\right) = \langle \partial_t(\mathcal{A}_\lambda^L)\hat{w}, \hat{w}_L\rangle + \langle \partial_t(\mathcal{A}_\lambda^{L-1})\dot{\hat{w}}, \dot{\hat{w}}_{L-1}\rangle + \langle \mathcal{A}_\lambda^L \widehat{\mathcal{F}}_1, \hat{w}_L\rangle + \langle \mathcal{A}_\lambda^{L-1}\widehat{\mathcal{F}}_2, \dot{\hat{w}}_{L-1}\rangle. \tag{D-9}$$

We can directly estimate the first two terms of the right-hand side of (D-9) from the bounds (D-3), (D-4) and the fact $\boldsymbol{\varepsilon} - \hat{\boldsymbol{\varepsilon}} = \boldsymbol{\zeta}_b$: we obtain the upper bound

$$\frac{b_1 C(M)}{\lambda^{2L-1}}\sqrt{\mathcal{E}_{L+1}\mathcal{E}_L} + \frac{b_1^{L+3}|\log b_1|^C}{\lambda^{2L-1}}\sqrt{\mathcal{E}_L} + \frac{b_1^{2L+3}|\log b_1|^C}{\lambda^{2L-1}}. \tag{D-10}$$

We can borrow steps (ii), (iii) and (iv) in the proof of Proposition D.1 to estimate the last two terms of the right-hand side of (D-9) except for the $\hat{\boldsymbol{\psi}}_b$ terms. Also, by Proposition 2.11, all the inner products we have to deal with are

$$b_L\langle \mathcal{A}^L(\chi_{B_1}-\chi_{B_0})T_{L-1}, \hat{\varepsilon}_L\rangle, \quad b_L\langle \mathcal{A}^{L-1}(\partial_s\chi_{B_0} + b_1(y\chi')_{B_0})T_L, \dot{\hat{\varepsilon}}_{L-1}\rangle. \tag{D-11}$$

From the fact $\hat{\varepsilon} = \varepsilon$ and $\mathcal{A}^{L-1}T_{L-1} = (-1)^{(L-1)/2}\Lambda Q$, we obtain

$$\mathcal{A}^{L-1}(\chi_{B_1}-\chi_{B_0})T_{L-1} = (-1)^{(L-1)/2}(\chi_{B_1}-\chi_{B_0})\Lambda Q + (\mathbf{1}_{y\sim B_1} + \mathbf{1}_{y\sim B_0})O(y^{-1}|\log y|).$$

Hence the bootstrap bound (3-25) yields

$$\begin{aligned} |\langle \mathcal{A}^L(\chi_{B_1}-\chi_{B_0})T_{L-1}, \hat{\varepsilon}_L\rangle| &= |\langle \mathcal{A}^{L-1}(\chi_{B_1}-\chi_{B_0})T_{L-1}, \hat{\varepsilon}_{L+1}\rangle| \\ &\leq |\langle y^{-1}\mathbf{1}_{B_0\leq y\leq 2B_1} + (\mathbf{1}_{y\sim B_1} + \mathbf{1}_{y\sim B_0})y^{-1}|\log y|, \varepsilon_{L+1}\rangle| \\ &\leq (|\log b_1|^{1/2} + |\log b_1|)\sqrt{\mathcal{E}_{L+1}} \leq b_1^{L+1}|\log b_1|^\delta. \end{aligned}$$

Note that $\dot{\hat{\varepsilon}} = \dot{\varepsilon} + b_L(\chi_{B_1}-\chi_{B_0})T_L$. The asymptotics (2-95) imply

$$\begin{aligned} |\langle \mathcal{A}^{L-1}(\partial_s\chi_{B_0} + b_1(y\chi')_{B_0})T_L, \dot{\varepsilon}_{L-1}\rangle| &\leq b_1|\langle \mathcal{A}^{L-2}(\mathbf{1}_{y\sim B_0}y^{L-2}|\log y|), \dot{\varepsilon}_L\rangle| \\ &\leq |\log b_1|\sqrt{\mathcal{E}_{L+1}} \leq b_1^{L+1}|\log b_1|^\delta. \end{aligned}$$

To estimate the last inner product, we employ the sharp asymptotics

$$b_1(y\chi')_{B_0} = -c_1\partial_s\chi_{B_0} + O\left(\frac{b_1\mathbf{1}_{y\sim B_0}}{|\log b_1|}\right)$$

from the fact $(b_1)_s = b_2 + O(b_1^2/|\log b_1|)$. Using the cancellation $\mathcal{A}^L T_L = 0$ and $\chi_{B_1} = 1$ on $y \sim B_0$, the remaining inner product can be written as

$$\frac{1}{L-1}b_L^2\langle\mathcal{A}^{L-1}\partial_s(\chi_{B_0}T_L), \mathcal{A}^{L-1}(\chi_{B_0}T_L)\rangle + O\left(\frac{b_1^{2L+1}}{|\log b_1|}\|\mathcal{A}^{L-1}(\mathbf{1}_{y\sim B_0}T_L)\|_{L^2}^2\right). \tag{D-12}$$

We can easily check that the second term in (D-12) is bounded by $b_1^{2L+1}|\log b_1|$. For the first term in (D-12), we use integration by parts in time to find out the correction for $\widehat{\mathcal{E}}_L$:

$$\begin{aligned}\frac{b_L^2}{\lambda^{2L-1}}\langle\mathcal{A}^{L-1}\partial_s(\chi_{B_0}T_L), \mathcal{A}^{L-1}(\chi_{B_0}T_L)\rangle &= \frac{b_L^2}{2\lambda^{2L-1}}\partial_s\langle\mathcal{A}^{L-1}(\chi_{B_0}T_L), \mathcal{A}^{L-1}(\chi_{B_0}T_L)\rangle \\ &= \frac{b_L^2}{2\lambda^{2L-2}}\partial_t\|\mathcal{A}^{L-1}(\chi_{B_0}T_L)\|_{L^2}^2.\end{aligned}$$

By Lemma 3.3, we conclude (D-8):

$$\begin{aligned}&\frac{b_L^2}{2\lambda^{2L-2}}\partial_t\|\mathcal{A}^{L-1}(\chi_{B_0}T_L)\|_{L^2}^2 - \partial_t\left(\frac{b_L^2}{2\lambda^{2L-2}}\|\mathcal{A}^{L-1}(\chi_{B_0}T_L)\|_{L^2}^2\right) \\ &\qquad = -\partial_t\left(\frac{b_L^2}{2\lambda^{2L-2}}\right)\|\mathcal{A}^{L-1}(\chi_{B_0}T_L)\|_{L^2}^2 = \left(\frac{(L-1)b_L^2\lambda_t}{\lambda^{2L-1}} - \frac{b_L(b_L)_t}{\lambda^{2L-2}}\right)\|\mathcal{A}^{L-1}(\chi_{B_0}T_L)\|_{L^2}^2 \\ &\qquad = -\frac{b_L}{\lambda^{2L-1}}((b_L)_s + (L-1)b_1b_L)O(|\log b_1|^2) = O\left(\frac{b_1^{2L+1}}{\lambda^{2L-1}}|\log b_1|\right).\end{aligned}$$

□

Proposition D.3 (Lyapunov monotonicity for $\mathcal{E}_{L-1}$). *Let $\ell = L-1$. Then we have*

$$\frac{d}{dt}\left\{\frac{\widehat{\mathcal{E}}_{L-1}}{\lambda^{2L-4}} + O\left(\frac{b_1^{2L-2}|\log b_1|^2}{\lambda^{2L-4}}\right)\right\} \le \frac{b_1^L|\log b_1|^\delta}{\lambda^{2L-3}}(b_1^{L-1}|\log b_1| + \sqrt{\mathcal{E}_{L-1}}), \tag{D-13}$$

where $0 < \delta \ll 1$ is a sufficient small constant that depends only on L.

Proof. Based on the proof of Proposition D.2 with Proposition 2.12, all the inner products we have to deal with are

$$\begin{aligned}&b_L\langle\mathcal{A}^{L-1}(\chi_{B_1}-\chi_{B_0})T_{L-1}, \hat{\varepsilon}_{L-1}\rangle, \quad b_{L-1}\langle\mathcal{A}^{L-1}(\partial_s\chi_{B_0} + b_1(y\chi')_{B_0})T_{L-1}, \hat{\varepsilon}_{L-1}\rangle \\ &b_{L-1}\langle\mathcal{A}^{L-2}H(\chi_{B_1}-\chi_{B_0})T_L, \dot{\hat{\varepsilon}}_{L-2}\rangle, \quad b_L\langle\mathcal{A}^{L-2}(\partial_s\chi_{B_0} + b_1(y\chi')_{B_0})T_L, \dot{\hat{\varepsilon}}_{L-2}\rangle.\end{aligned}$$

By additionally considering $\hat{\varepsilon} = \varepsilon + b_{L-1}(\chi_{B_1}-\chi_{B_0})T_{L-1}$, we can estimate the above inner products similarly to (D-12) due to the derivative gain $\mathcal{A}^{L-2}H = \mathcal{A}^L$ and the logarithmic gain $|\log b_1|^{-\beta}$ from the bootstrap bound (3-28) for b_L when $\ell = L-1$. The exact correction term is given by

$$-\partial_t\left(\frac{b_{L-1}^2}{2(L-2)\lambda^{2L-4}}\|\mathcal{A}^{L-1}(\chi_{B_0}T_{L-1})\|_{L^2}^2\right).$$

□

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MARGULIS LEMMA ON RCD(K, N) SPACES

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We extend the Margulis lemma for manifolds with lower Ricci curvature bounds to the RCD(K, N) setting. As one of our main tools, we obtain improved regularity estimates for regular Lagrangian flows on these spaces.

1. Introduction

The main result of this paper extends the Margulis lemma to RCD(K, N) spaces. Recall that for a group G, we say an (ordered) generating set $\beta = \{\gamma_1, \dots, \gamma_n\} \subset G$ is a *nilpotent basis of length n* if for all $i, j \in \{1, \dots, n\}$ one has $[\gamma_i, \gamma_j] \in \langle\{\gamma_1, \dots, \gamma_{i-1}\}\rangle$.

Theorem 1.1. *For each $K \in \mathbb{R}$, $N \geq 1$, there exist $\varepsilon > 0$ and $C \in \mathbb{N}$ such that if $(X, d, \mathfrak{m}, p)$ is a pointed* $\mathrm{RCD}(K, N)$ *space of rectifiable dimension n, the image of the map*

$$\pi_1(B_\varepsilon(p), p) \to \pi_1(X, p)$$

induced by inclusion contains a subgroup of index $\leq C$ that admits a nilpotent basis of length $\leq n$.

From [Kapovitch and Wilking 2011], Theorem 1.1 is known to hold when X is a smooth Riemannian manifold. On the other hand, Breuillard, Green and Tao [Breuillard et al. 2012, Corollary 11.17] proved that, after quotienting by a finite normal subgroup, Theorem 1.1 holds in more general metric spaces with nice packing properties.

The proof strategy of Theorem 1.1 is similar to that of Kapovitch and Wilking, including a reverse induction argument (see Theorem 1.14). Nevertheless, there are quite a few technical challenges to generalizing their arguments to a nonsmooth framework. An important tool used in [Kapovitch and Wilking 2011] is the gradient flow of smooth functions with suitable integral Hessian bounds and their

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associated regularity estimates. In the nonsmooth framework, these gradient flows are by necessity replaced by the regular Lagrangian flows (RLFs) of Sobolev vector fields. Intuitively, RLFs are the appropriate notion of flows in a context where pointwise defined flows do not make sense and might not be unique. When restricted to smooth vector fields on Riemannian manifolds, RLFs coincide with the classical flows almost everywhere.

Several regularity results have been obtained for RLFs in [Brué and Semola 2020a; 2020b; Brué et al. 2022], but are not quite strong enough to give the necessary estimates; see the discussion at the end of Section 1.1 for more details. The main technical contribution of this paper, therefore, is to establish new regularity estimates for regular Lagrangian flows on the $\mathrm{RCD}(K, N)$ spaces. These estimates match the effective estimates known for smooth manifolds with a Ricci curvature lower bound. We mention that related estimates of this type have also been employed successfully in other works to study the structure of Ricci limit spaces (see [Cheeger and Colding 1996; Colding and Naber 2012; Kapovitch and Li 2018]) and $\mathrm{RCD}(K, N)$ spaces (see [Brué and Semola 2020b; Deng 2020]).

For the rest of the paper we shall assume some basic familiarity with the theory of $\mathrm{RCD}(K, N)$ spaces, and in particular that of its first and second order calculus framework. We refer to [Sturm 2006a; 2006b; Lott and Villani 2009; Ambrosio et al. 2014a; 2014b; 2015; Savaré 2014; Gigli 2015; Mondino and Naber 2019; Gigli 2018; Brué and Semola 2020b], among others, for a detailed treatment.

1.1. *Main regularity estimates on RLFs.* Let us first define regular Lagrangian flows [Ambrosio 2004; Ambrosio and Trevisan 2014] and maximal functions [Stein 1993].

Definition 1.2. Let $(X, d, \mathfrak{m})$ be an $\mathrm{RCD}(K, N)$ space, $T > 0$, and let $V : [0, T] \to L^2_{\mathrm{loc}}(TX)$ be a time-dependent vector field. A Borel map $\boldsymbol{X} : [0, T] \times X \to X$ is called a *regular Lagrangian flow* (*RLF*) to V if the following holds:

$R.1$ $\boldsymbol{X}_0(x) = x$ and $[0, T] \ni t \mapsto \boldsymbol{X}_t(x)$ is continuous for every $x \in X$.

$R.2$ For every $f \in \mathrm{TestF}(X)$ and $\mathfrak{m}$-a.e. $x \in X$, $t \mapsto f(\boldsymbol{X}_t(x))$ is in $W^{1,1}([0, T])$ and

$$\frac{d}{dt} f(\boldsymbol{X}_t(x)) = df(V(t))(\boldsymbol{X}_t(x)) \quad \text{for a.e. } t \in [0, T]. \tag{1.3}$$

$R.3$ There exists a constant $C(V)$ such that $(\boldsymbol{X}_t)_* m \le C\mathfrak{m}$ for all t in $[0, T]$.

Definition 1.4. Let $(X, d, \mathfrak{m})$ be an $\mathrm{RCD}(K, N)$ space, $R > 0$, and $h : X \to \mathbb{R}^+$ measurable. The *R-maximal function* $\mathrm{Mx}_R(h) : X \to \mathbb{R}$ is defined as

$$\mathrm{Mx}_R(h)(x) := \sup_{0<r\le R} \fint_{B_r(x)} h \, d\mathfrak{m}.$$

For simplicity, we denote Mx_1 by Mx.

The following regularity result is our substitute for smoothness in the context of regular Lagrangian flows. Roughly speaking, it establishes that for a vector field V, if one has enough integral control on $\mathrm{Mx}(|\nabla V|)$ along most flow lines that start in a ball B, then the RLF of V maps most points of B to a ball of similar scale. Theorem 1.5 will be later used as a base of induction to produce stronger quantitative estimates along flow lines in Section 5.

Theorem 1.5. *Let $\rho > 0$, $T > 0$, $L \geq 1$, $D \geq 0$, $(X, d, \mathfrak{m})$ an* RCD$(-(N-1), N)$ *space, $V \in L^1([0,T]; H^{1,2}_{C,s}(TX))$ a vector field with $\|V(t)\|_\infty \leq L$ and $\|\mathrm{div}(V(t))\|_\infty \leq D$ for all $t \in [0,T]$, $\boldsymbol{X} : [0,T] \times X \to X$ its RLF, and define $H : X \to \mathbb{R}$ as $H(y) := \int_0^T Mx_\rho(|\nabla V(t)|)(\boldsymbol{X}_t(y))\,dt$. Then there are $\delta(D, T, N) > 0$, $M(D, T, N) > 0$, such that if $x \in X$ satisfies*

$$\limsup_{r \to 0} \frac{\mathfrak{m}(\{y \in B_r(x) | H(y) > \delta\})}{\mathfrak{m}(B_r(x))} < \tfrac{1}{2}, \tag{1.6}$$

then there is $r_x \leq \rho/100$ and a representative $\widetilde{\boldsymbol{X}} : [0,T] \times X \to X$ of the RLF to V such that for all $r \leq r_x$ the following holds:

*S.*1 *There is $A_r \subset B_r(x)$ with $\mathfrak{m}(A_r) \geq \frac{1}{M}\mathfrak{m}(B_r(x))$ and*

$$\widetilde{\boldsymbol{X}}_t(A_r) \subset B_{2r}(\widetilde{\boldsymbol{X}}_t(x)) \quad \text{for all } t \in [0,T].$$

*S.*2 *For all $t \in [0,T]$,*

$$\frac{1}{M}\mathfrak{m}(B_r(x)) \leq \mathfrak{m}(B_r(\widetilde{\boldsymbol{X}}_t(x))) \leq M\mathfrak{m}(B_r(x)).$$

Moreover, $\widetilde{\boldsymbol{X}}$ can be chosen so that any point $x \in X$ satisfying (1.6) also satisfies S.1 and S.2 for r sufficiently small (depending on x).

For smooth vector fields on Riemannian manifolds, the previous result follows immediately from the infinitesimal to local property in differential calculus (see [Kapovitch and Wilking 2011, Lemma 3.7; Colding and Naber 2012, Proposition 3.6]). This issue is far more delicate in the nonsmooth setting since one cannot perform infinitesimal calculus pointwise. To overcome this, we directly obtain quantitative estimates on all scales using some new technical arguments developed in [Deng 2020], which builds on the ideas of [Kapovitch and Wilking 2011; Colding and Naber 2012].

The proof of Theorem 1.5 uses a similar technique as [Deng 2020, Lemma 5.1], generalizing it to a wider class of flows. However, in order to successfully use it to perform the topological arguments required for Theorem 1.1, we need to adjust the RLF to obtain the appropriate representative $\widetilde{\boldsymbol{X}}$ mentioned at the end of the theorem, while in [Deng 2020] the flow one initially works with is already good enough for the required application (roughly this is because, in [Deng 2020], one can show that the flows starting from close to a given point should always limit in some sense to a geodesic, which can be identified canonically and without ambiguity, whereas in this work, all objects considered are defined "almost-everywhere" and there was no natural canonical limit to begin with).

Definition 1.7. Let $M(1, T, N) > 0$ be given by Theorem 1.5, $(X, d, \mathfrak{m})$ an RCD$(-(N-1), N)$ space, $V : [0,T] \to L^2_{\mathrm{loc}}(TX)$ a vector field, and $\boldsymbol{X} : [0,T] \times X \to X$ its RLF. We say that $x \in X$ is a *point of essential stability* of $\boldsymbol{X}$ if there is $r_x > 0$ such that S.1 and S.2 hold for all $r \leq r_x$.

Corollary 1.8. *For each $N \geq 1$, $T \geq 0$, $D \geq 0$, $r \geq 0$, $L \geq 0$ and $\varepsilon > 0$, there are $R \geq 1$, $\eta > 0$, such that the following holds. Let $(X, d, \mathfrak{m}, p)$ be an* RCD$(-(N-1), N)$ *space, $V \in H^{1,2}_{C,s}(TX)$ a vector field with $\|V\|_\infty \leq L$, $\|\mathrm{div}(V)\|_\infty \leq D$, and $\boldsymbol{X} : [0,T] \times X \to X$ its RLF. Assume that for all $s \in [1, R]$ one has*

$$\fint_{B_s(p)} |\nabla V|^2 \, d\mathfrak{m} \leq \eta.$$

Then if $G \subset X$ denotes the set of points of essential stability of $\mathbf{X}$, one has

$$(G \cap B_r(p)) \geq (1-\varepsilon)\mathfrak{m}(B_r(p)).$$

We remark that for noncollapsed RCD(K, N) spaces, a version of these regularity results were obtained in [Brué et al. 2022] using alternative methods relying on estimates of the Green's function, which cannot be readily applied in collapsed cases. Moreover, the use of the Green's function in [Brué et al. 2022] resulted in the dependence of various estimates on nonstructural information such as the space itself, somewhat inevitably since the Green's function naturally contains global information about X. This is undesirable for the application at present since we will need to consider sequences of RCD spaces and therefore cannot make use of estimates which depend on the space. Indeed this dependence can be avoided by adapting the scheme of [Deng 2020]. We point out that the advantage of using the Green's function in the noncollapsed setting is that one obtains optimal infinitesimal Lipschitz estimates [Brué et al. 2022, Theorem 1.6], which does not seem to be readily obtainable using the methods employed here.

1.2. *Induction theorem.* In this subsection we state Theorem 1.14; our main technical result from which Theorem 1.1 follows. Recall that for a semilocally simply connected space X, we can identify its fundamental group $\pi_1(X)$ with the group of deck transformations of its universal cover $\widetilde{X}$.

Definition 1.9. Let X be a semilocally simply connected geodesic space and $\widetilde{X}$ its universal cover. We say a function $f : \widetilde{X} \to \widetilde{X}$ is of *deck type* if there is an automorphism $f_* \in \mathrm{Aut}(\pi_1(X))$ such that for all $g \in \pi_1(X)$ and $x \in \widetilde{X}$, one has $f(g(x)) = f_*(g)(f(x))$.

Example 1.10. If $f \in \pi_1(X)$, then it is of deck type with $f_*(g) := f \circ g \circ f^{-1}$.

Definition 1.11. For metric spaces X, Y, a function $f : X \to Y$, and $r > 0$, the *distortion at scale r* is defined as the map $\mathrm{dt}_r(f) : X \times X \to [0, r]$ with

$$\mathrm{dt}_r(f)(x_1, x_2) := \min\{r, |d_X(x_1, x_2) - d_Y(f(x_1), f(x_2))|\}.$$

If X is equipped with a measure $\mathfrak{m}$, we say that $x \in X$ is a *point of essential continuity* of f if there exists $r_0 > 0$ such that for all $r \leq r_0$ there is a subset $A_r \subset B_r(x)$ with $\mathfrak{m}(A_r) \geq \frac{1}{2}\mathfrak{m}(B_r(x))$ and $f(A_r) \subset B_{2r}(f(x))$.

The next definition is a nonsmooth version of the *maps with zoom-in property* from [Kapovitch and Wilking 2011]. Although the notion is very technical, these are precisely the properties present in gradient flows of δ-splittings (and as we will show, also in the RLFs of δ-splittings in the RCD setting).

Definition 1.12. Let $(X_i^j, d_i^j, \mathfrak{m}_i^j, p_i^j)$, $j \in \{1, 2\}$ be two sequences of pointed RCD(K, N) spaces. We say that a sequence of measurable functions $f_i : [X_i^1, p_i^1] \to [X_i^2, p_i^2]$ is *good at all scales* (GS) if there is a sequence of measurable functions $f_i^{-1} : [X_i^2, p_i^2] \to [X_i^1, p_i^1]$ such that $f_i^{-1} \circ f_i = \mathrm{Id}_{X_i^1}$ almost everywhere and $f_i \circ f_i^{-1} = \mathrm{Id}_{X_i^2}$ almost everywhere, satisfying the following:

(1) $(f_i)_*(\mathfrak{m}_i^1) \ll \mathfrak{m}_i^2$ and $(f_i^{-1})_*(\mathfrak{m}_i^2) \ll \mathfrak{m}_i^1$ for all i.

(2) There is $R_0 > 0$ and sequences $S_i^j \subset B_1(p_i^j)$ for $j \in \{1, 2\}$ with $\mathfrak{m}_i^j(S_i^j) \geq \frac{1}{2}\mathfrak{m}_i^j(B_1(p_i^j))$ and $f_i(S_i^1) \subset B_{R_0}(p_i^2)$, $f_i^{-1}(S_i^2) \subset B_{R_0}(p_i^1)$.

(3) There is a sequence $\varepsilon_i \to 0$ and sequences of subsets $U_i^j \subset X_i^j$ for $j \in \{1, 2\}$ such that:
   (a) The points of U_i^1 (resp. U_i^2) are of essential continuity of f_i (resp. f_i^{-1}).
   (b) f_i (resp. f_i^{-1}) restricted to U_i^1 (resp. U_i^2) is measure preserving.
   (c) For all $R > 0$ and $j \in \{1, 2\}$, one has

$$\lim_{i\to\infty} \frac{\mathfrak{m}_i^j(U_i^j \cap B_R(p_i^j))}{\mathfrak{m}_i^j(B_R(p_i^j))} = 1.$$

   (d) For all $x_i^1 \in U_i^1$, $x_i^2 \in U_i^2$, $r \le 1$, one has

$$\fint_{B_r(x_i^1)^{\times 2}} \mathrm{dt}_r(f_i)(a, b)\, d(\mathfrak{m}_i^1 \times \mathfrak{m}_i^1)(a, b) \le \varepsilon_i r,$$

$$\fint_{B_r(x_i^2)^{\times 2}} \mathrm{dt}_r(f_i^{-1})(a, b)\, d(\mathfrak{m}_i^2 \times \mathfrak{m}_i^2)(a, b) \le \varepsilon_i r.$$

Definition 1.13. Let Γ be a group, $G \le \Gamma$ a subgroup admitting a nilpotent basis $\beta = \{\gamma_1, \dots, \gamma_n\}$, and $\varphi \in \mathrm{Aut}(\Gamma)$. We say that φ *respects* β if it preserves $\langle\{\gamma_1, \dots, \gamma_m\}\rangle$ for each m, and acts trivially on $\langle\{\gamma_1, \dots, \gamma_m\}\rangle/\langle\{\gamma_1, \dots, \gamma_{m-1}\}\rangle$ for each m.

Theorem 1.14. *Let $(X_i, d_i, \mathfrak{m}_i, p_i)$ be a sequence of pointed* RCD$\big(-\frac{1}{i}, N\big)$ *spaces of rectifiable dimension n and a pointed compact metric space (Y, y) of diameter D for which the sequence (X_i, p_i) converges in the pointed Gromov–Hausdorff sense to $(\mathbb{R}^k \times Y, (0, y))$. Let $\widetilde{X}_i$ be the sequence of universal covers, $\tilde{p}_i \in \widetilde{X}_i$ in the preimage of p_i, $\Gamma_i \le \pi_1(X_i)$ be the group generated by the elements $g \in \pi_1(X_i)$ with $d(g\tilde{p}_i, \tilde{p}_i) \le 2D+1$, and for each $j \in \{1, \dots, \ell\}$, $f_{j,i} : [\widetilde{X}_i, \tilde{p}_i] \to [\widetilde{X}_i, \tilde{p}_i]$ a sequence of deck type maps with the GS property. Then for some $C > 0$ and i large enough, Γ_i contains a subgroup $G_i \le \Gamma_i$ with the following properties*:

- $[\Gamma_i, G_i] \le C$.
- *G_i admits a nilpotent basis β_i of length $\le n-k$.*
- *$(f_{j,i})_*^{C!}$ respects β_i for each j.*

Naber and Zhang [2016, Appendix A] proved a blown-down version of Theorem 1.14 for Riemannian manifolds. The techniques they used to obtain this version from Theorem 1.14 (also present in [Kapovitch and Wilking 2011]) apply to RCD(K, N) spaces, giving the following result.

Corollary 1.15. *Let $(X, d, \mathfrak{m}, p)$ be a pointed* RCD(K, N) *space of rectifiable dimension k. Then there is $\varepsilon > 0$ such that if a pointed* RCD(K, N) *space $(X', d', \mathfrak{m}', p')$ of rectifiable dimension n satisfies*

$$d_{GH}((X', p'), (X, p)) < \varepsilon,$$

then the image of the map

$$\pi_1(B_\varepsilon(p'), p') \to \pi_1(X', p')$$

induced by inclusion contains a subgroup of index $\le C(X, p)$ that admits a nilpotent basis of length $\le n-k$.

1.3. ***Main ideas of the proof.*** A natural approach to prove a result like Theorem 1.1 is to consider a contradicting sequence of pointed $\mathrm{RCD}(K, N)$ spaces $(X_i, d_i, \mathfrak{m}_i, p_i)$ of rectifiable dimension n, and $\varepsilon_i \to 0$ for which the group $\Gamma_i := j_*(\pi_1(B_{\varepsilon_i}(p_i), p_i))$ does not contain a subgroup of index $\leq i$ admitting a nilpotent basis of length $\leq n$, where $j_* : \pi_1(B_{\varepsilon_i}(p_i), p_i) \to \pi_1(X_i, p_i)$ is the natural map induced by inclusion. After slowly blowing up and taking a subsequence, one can assume the universal covers $(\widetilde{X}_i, \tilde{d}_i, \widetilde{\mathfrak{m}}_i, \tilde{p}_i)$ converge in the pointed measured Gromov–Hausdorff sense to a pointed $\mathrm{RCD}(0, N)$ space $(X, d, \mathfrak{m}, p)$, and the actions of Γ_i on these spaces converge equivariantly to a Lie group $\Gamma \leq \mathrm{Iso}(X)$.

From here, it would be easy to obtain via well-established techniques that the identity connected component $\Gamma_0 \leq \Gamma$ is nilpotent and $[\Gamma : \Gamma_0] < \infty$. This nice behavior can be traced back to the groups Γ_i using Gromov–Hausdorff approximations $\psi_i : \Gamma_i \to \Gamma$. This would finish the proof, if not for the possibility that there may be subgroups $H_i \leq \Gamma_i$ too small for the Gromov–Hausdorff approximations to detect them. Recall that while a sequence of Gromov–Hausdorff approximations describes very well the geometry of a sequence of spaces at a certain scale, it fails to see

- features that are too small,
- features that are too far from the basepoints.

To remedy the issue of having subgroups $H_i \leq \Gamma_i$ which are too small, one could blow up the sequence by factors $\lambda_i \to \infty$ to a scale at which the groups H_i are visible, and again take a subsequence in such a way that the actions of Γ_i on the spaces $(\widetilde{X}_i, \lambda_i \tilde{d}_i, \widetilde{\mathfrak{m}}_i, \tilde{p}_i)$ converge equivariantly to a Lie group Γ' acting by isometries on a new limit space $(X', d', \mathfrak{m}', p')$. The problem with doing so is that relevant elements of the original group Γ_i may be sent too far for the new Gromov–Hausdorff approximations $\psi' : \Gamma_i \to \Gamma'$ to see them.

In order to understand how the elements of H_i interact with the elements lost due the blow-up, we need to bring these elements back by homotopy. For this purpose, the gradient flow of semiconcave (resp. harmonic) functions is used in [Kapovitch et al. 2010] (resp. [Kapovitch and Wilking 2011]). In the setting of $\mathrm{RCD}(K, N)$ spaces, the regular Lagrangian flows play the role of such tools. However, since this process has to be done multiple times, without prior knowledge about scale and location, one needs to control the regularity of such flows at all small scales. This is the reason for the technical nature of Theorems 1.5 and 1.14. The maps $f_{j,i}$ in Theorem 1.14 are precisely these isometries in Γ_i that were sent too far and then brought back by composing them with an appropriate regular Lagrangian flow.

1.4. ***Open problems.*** In the context of Theorem 1.1, it has been conjectured that the nilpotent group can be taken so that its torsion lies in its center. This is not known even for Riemannian manifolds of sectional curvature $\geq K/(N-1)$ [Kapovitch et al. 2010; 2018] (see also [Fukaya and Yamaguchi 1992, Conjecture 0.16]).

Conjecture 1.16. *For each $K \in \mathbb{R}$, $N \geq 1$, there exist $\varepsilon > 0$ and $C \in \mathbb{N}$ such that if $(X, d, \mathfrak{m}, p)$ is a pointed* $\mathrm{RCD}(K, N)$ *space of rectifiable dimension n, the image of the map*

$$\pi_1(B_\varepsilon(p), p) \to \pi_1(X, p)$$

induced by inclusion contains a subgroup of index $\leq C$ whose torsion elements are contained in its center.

On the other hand, it is also a very challenging problem to find an explicit expression for $C(K, N)$ in Theorem 1.1. Such an expression hasn't been found even for Riemannian manifolds of sectional curvature $\geq K/(N-1)$ (see [Kapovitch et al. 2010]).

1.5. ***Structure of the paper.*** In Section 2, we cover the background material we will need. In Section 3, we prove Theorem 3.1, which provides us with subgroups $\Upsilon_i \triangleleft \Gamma_i$ that play the role of identity connected components in the discrete groups Γ_i.

In Section 4 we prove Theorem 1.5 and Corollary 1.8, allowing us to find points of essential stability, and in Section 5 we study how essential stability allows one to obtain stronger estimates. In Section 6 we prove properties of GS maps, and in Section 7 we give two ways to construct GS maps (cf. [Kapovitch and Wilking 2011, Section 3]).

In Section 8 we show Theorem 8.1, reducing Theorem 1.14 to the case $Y \neq \{*\}$ (cf. [Kapovitch and Wilking 2011, Section 5]). In Section 9 we prove Theorem 1.14 and with it Theorem 1.1 and Corollary 1.15.

2. Preamble

2.1. ***Notation.*** For a set A, we denote by $A^{\times 2}$ the set $A \times A$. If $A \subset X$, we denote by $\chi_A : X \to [0, 1]$ the characteristic function of A. For a group G and $g \in G$, we denote by $g_* \in \mathrm{Aut}(G)$ the map $h \mapsto ghg^{-1}$. For metric spaces (X, d_X) and (Y, d_Y), we denote by $X \times Y$ the L^2 product. That is, for $x_1, x_2 \in X,\ y_1, y_2 \in Y$,

$$d_{X\times Y}((x_1, y_1), (x_2, y_2)) := \sqrt{d_X(x_1, x_2)^2 + d_Y(y_1, y_2)^2}.$$

We say a pointed metric measure space $(X, d, \mathfrak{m}, p)$ is *normalized* if

$$\int_{B_1(p)} (1 - d(p, \cdot))\, d\mathfrak{m} = 1.$$

For $m \in \mathbb{N}$, we denote by $\mathbb{R}^m$ the m-dimensional Euclidean space equipped with its usual metric, and by $\mathcal{H}^m$ the m-dimensional Hausdorff measure for which the metric measure space $(\mathbb{R}^m, \mathcal{H}^m, 0)$ is normalized.

To a metric space (X, d), we can adjoin a point $*$ at infinite distance from any point of X to get a new space we denote by $X \cup \{*\}$. Similarly, to any group G we can adjoin an element $*$ whose product with any element of G is defined as $*$, obtaining a binary operation on $G \cup \{*\}$.

We write $C(\alpha, \beta, \gamma)$ to denote a constant C that depends only on the quantities α, β, γ.

2.2. **RCD(*K*, *N*)** ***spaces; doubling, isometries, covers, and geodesics.*** The main objects of this text are RCD(K, N) spaces. We note that a large number of papers in the literature work with a condition known as RCD$^*(K, N)$, originally introduced in [Bacher and Sturm 2010]. Since it is now known that this condition is equivalent to the RCD(K, N) condition [Cavalletti and Milman 2021; Li 2024], we will make no distinction between them.

One of the most powerful tools in the study of RCD(K, N) spaces is the Bishop–Gromov inequality [Sturm 2006b].

Theorem 2.1 (Sturm). *For each $K \in \mathbb{R}$, $N \geq 1$, $R > 0$, $\lambda > 1$ there is $C(K, N, R, \lambda) > 0$ such that for any pointed* $\mathrm{RCD}(K, N)$ *space $(X, d, \mathfrak{m}, p)$, and any $r \leq R$, one has*

$$\mathfrak{m}(B_{\lambda r}(p)) \leq C \cdot \mathfrak{m}(B_r(p)).$$

Moreover, for fixed K, N, R, if $\lambda \to 1$ then $C \to 1$.

Corollary 2.2. *Let $(X_i, d_i, \mathfrak{m}_i, p_i)$ be a sequence of pointed* $\mathrm{RCD}(K, N)$ *spaces and consider a sequence of subsets $U_i \subset X_i$. Then the following are equivalent:*

- *For all $R > 0$, there is a sequence $\eta_i \to 0$ such that*

$$\mathfrak{m}_i(U_i \cap B_R(p_i)) \geq (1 - \eta_i)\mathfrak{m}_i(B_R(p_i)).$$

- *For all $R \geq \delta > 0$, there is a sequence $\varepsilon_i \to 0$ such that if $x \in B_R(p_i)$, one has*

$$\mathfrak{m}_i(U_i \cap B_\delta(x)) \geq (1 - \varepsilon_i)\mathfrak{m}_i(B_\delta(x)).$$

In either case, we say that the sequence U_i has asymptotically full measure.

Proof. Assume the first condition holds. If the second condition fails for some $R \geq \delta > 0$, then after passing to a subsequence, there would be $\varepsilon > 0$ and $x_i \in B_R(p_i)$ with

$$\mathfrak{m}_i(B_\delta(x_i) \backslash U_i) \geq \varepsilon \cdot \mathfrak{m}_i(B_\delta(x_i)). \tag{2.3}$$

By the triangle inequality and Theorem 2.1, there is $C(K, N, R, \delta) > 0$ with

$$\mathfrak{m}_i(B_{R+\delta}(p_i)) \leq \mathfrak{m}_i(B_{2R+\delta}(x_i)) \leq C \cdot \mathfrak{m}_i(B_\delta(x_i)). \tag{2.4}$$

Since $B_\delta(x_i) \subset B_{R+\delta}(p_i)$, combining (2.3) and (2.4) we get

$$\mathfrak{m}_i(B_{R+\delta}(p_i) \backslash U_i) \geq \varepsilon \cdot \mathfrak{m}_i(B_{R+\delta}(p_i))/C,$$

contradicting our hypothesis.

The other implication is evident by taking $\delta = R$ and $x = p_i$. □

The following well-known facts follow from Theorem 2.1 (see [Stein 1993, p. 12; Kapovitch and Wilking 2011, p. 6]). For definition of maximal function, see for example Definition 1.4.

Proposition 2.5. *Let $(X, d, \mathfrak{m})$ be an* $\mathrm{RCD}(K, N)$ *space, $h : X \to \mathbb{R}^+$ measurable, and $R > 0$.*

(1) *For all $\delta > 0$,*

$$\mathfrak{m}(\{x \in X \mid Mx_R(h)(x) \geq \delta\}) \leq \frac{C(K, N, R)}{\delta} \int_X h \, d\mathfrak{m}.$$

(2) *For all $\alpha > 1$,*

$$\|Mx_R(h)\|_\alpha \leq C(K, N, R, \alpha)\|h\|_\alpha.$$

(3) *For all $\alpha > 1$, $s \leq R/2$,*

$$Mx_s(Mx_s(h)^\alpha) \leq C(K, N, R, \alpha)Mx_R(h^\alpha).$$

For a proper metric space X, the topology that we use on its group of isometries $\mathrm{Iso}(X)$ is the compact-open topology, which in this setting coincides with both the topology of pointwise convergence and the topology of uniform convergence on compact sets. This topology makes $\mathrm{Iso}(X)$ a locally compact second countable metric group. In the case $(X, d, \mathfrak{m})$ is an RCD(K, N) space, $\mathrm{Iso}(X)$ is a Lie group [Guijarro and Santos-Rodríguez 2019; Sosa 2018].

Theorem 2.6 (Sosa, Guijarro and Santos-Rodríguez). *Let $(X, d, \mathfrak{m})$ be an* RCD(K, N) *space. Then* $\mathrm{Iso}(X)$ *is a Lie group.*

The RCD(K, N) condition can be checked locally (see [Erbar et al. 2015, Section 3]). Hence if $(X, d, \mathfrak{m})$ is an RCD(K, N) space and $\rho : \widetilde{X} \to X$ is a covering space, $\widetilde{X}$ admits a unique measure making it an RCD(K, N) space, and for which ρ is a local isomorphism of metric measure spaces (see [Mondino and Wei 2019, Section 2.3]). Whenever we have a covering space of an RCD(K, N) space, we assume it is equipped with such measure. This allows one to lift estimates on maximal functions [Kapovitch and Wilking 2011, Lemma 1.6].

Proposition 2.7. *Let $(X, d, \mathfrak{m})$ be an* RCD(K, N), *$\rho : (\widetilde{X}, \tilde{d}, \tilde{\mathfrak{m}}) \to (X, d, \mathfrak{m})$ a covering space, $x \in X$, $\tilde{x} \in \rho^{-1}(x)$, $f : X \to \mathbb{R}^+$ measurable. Then for all $r \leq R$, one has*

$$\fint_{B_r(\tilde{x})} (f \circ \rho)\, d\tilde{\mathfrak{m}} \leq C(K, N, R) \fint_{B_r(x)} f\, d\mathfrak{m}.$$

In particular,

$$Mx_R(f \circ \rho) \leq C(K, N, R) \cdot Mx_R(f) \circ \rho.$$

An important topological property of RCD(K, N) spaces is that they are semilocally simply connected [Wang 2024].

Theorem 2.8 (Wang). *Let $(X, d, \mathfrak{m})$ be an* RCD(K, N) *space. Then X is semilocally simply connected, so its universal cover $\widetilde{X}$ is simply connected and we can identify $\pi_1(X)$ with the group of deck transformations $\widetilde{X} \to \widetilde{X}$.*

The following is a well-known equivalence of semilocal simple-connectedness (see for example [Calcut and McCarthy 2009]). We include its proof for completeness.

Proposition 2.9. *Let X be a semilocally simply connected geodesic space. Then for each compact set $K \subset X$ there is $\delta > 0$ with the property that any two curves $\alpha, \beta : [0, 1] \to K$ sharing endpoints and at uniform distance $\leq \delta$ are homotopic relative to their endpoints.*

Proof. By hypothesis, X admits an open cover $\mathcal{U}$ with the property that each loop contained in an element of $\mathcal{U}$ is contractible in X. It is an easy exercise to check that if $U \in \mathcal{U}$, and $\sigma_1, \sigma_2 : [0, 1] \to U$ are two paths with the same endpoints, then σ_1 and σ_2 are homotopic relative to their endpoints as curves in X.

Consider a compact set $K \subset X$. Then there is $\delta > 0$ such that 2δ is a Lebesgue number of $\mathcal{U}$ as a cover of K. That is, for any $x \in K$ there is an element of $\mathcal{U}$ that contains $B_{2\delta}(x)$. Then take two curves $\alpha, \beta : [0, 1] \to K$ with the same endpoints and at uniform distance $\leq \delta$. We claim that they are homotopic relative to their endpoints as curves in X.

To see this, take a partition $0 = t_0 < t_1 < \cdots < t_k = 1$ with the property that $\alpha(t) \in B_\delta(\alpha(t_j))$ for all $t \in [t_{j-1}, t_j]$, $j \in \{1, \ldots, k\}$. For $j \in \{1, \ldots, k\}$ define $\gamma_j : [0, 1] \to X$ to be a curve that agrees with β along $[0, t_j]$, with α along $[t_{j+1}, 1]$, and along $[t_j, t_{j+1}]$ is a minimizing curve connecting $\beta(t_j)$ with $\alpha(t_{j+1})$. Notice that $\gamma_k = \beta$.

It is then easy to see by induction that α and γ_j are homotopic relative to their endpoints. Indeed, if we set $\gamma_0 := \alpha$, then for each $j \in \{1, \ldots, k\}$ the curves γ_{j-1} and γ_j are identical except along an interval where their images are contained in $B_{2\delta}(\alpha(t_j))$, and hence in an element of $\mathcal{U}$ and consequently homotopic relative to their endpoints. □

To conclude this subsection, we note that by the Kuratowski–Ryll-Nardzewski measurable selection theorem, for any $\mathrm{RCD}(K, N)$ space $(X, d, \mathfrak{m})$, there is a measurable map

$$\gamma_{\cdot,\cdot}(\cdot) : X \times X \times [0, 1] \to X$$

such that, for all $x, y \in X$, the map $[0, 1] \ni s \mapsto \gamma_{x,y}(s)$ is a constant speed geodesic from x to y. For the rest of this paper, for each $(X, d, \mathfrak{m})$ we fix such a choice of γ. This allows us to state the segment inequality for $\mathrm{RCD}(K, N)$ spaces [Deng 2020, Theorem 3.22].

Theorem 2.10. *Let $(X, d, \mathfrak{m})$ be an $\mathrm{RCD}(K, N)$ space, $h : X \to \mathbb{R}^+$ measurable, $p \in X$, and $r \leq R$. Then*

$$\fint_{B_r(p)^{\times 2}} d(x, y)\left[\int_0^1 h(\gamma_{x,y}(s))\, ds\right] d(\mathfrak{m} \times \mathfrak{m})(x, y) \leq r \cdot C(K, N, R) \fint_{B_{2r}(p)} h\, d\mathfrak{m}.$$

We will also need the following variation of the Lebesgue differentiation theorem (see [Stein and Shakarchi 2005; Heinonen et al. 2015]).

Definition 2.11. Let $(X, d, \mathfrak{m})$ be a metric measure space. We say that a family of measures $\mathcal{V}$ on X has *bounded eccentricity* if there are $M \geq 1 \geq \eta > 0$ such that $\nu \leq M\mathfrak{m}$ for all $\nu \in \mathcal{V}$, and a map $\theta : \mathcal{V} \to X$ such that for all $\nu \in \mathcal{V}$ there is $r(\nu) > 0$ with $\mathrm{supp}(\nu) \subset B_r(\theta(\nu))$ and $\nu(B_r(\theta(\nu))) \geq \eta\, \mathfrak{m}(B_r(\theta(\nu)))$. We then say that a net $\nu_i \in \mathcal{V}$ converges to $x \in X$ if $\theta(\nu_i) = x$ for all large i and $r(\nu_i) \to 0$.

Lemma 2.12. *Let $(X, d, \mathfrak{m})$ be an $\mathrm{RCD}(K, N)$ space, $f \in L^1(\mathfrak{m})$, and $\mathcal{V}$ a family of measures of bounded eccentricity. Then for $\mathfrak{m}$-almost every $x \in X$ we have*

$$f(x) = \lim_{\nu \to x} \frac{1}{\nu(X)} \int_X f\, d\nu.$$

Proof. For $\alpha > 0$, define

$$E_\alpha := \left\{ x \in X : \limsup_{\nu \to x} \frac{1}{\nu(X)} \left| \int_X f(y) - f(x)\, d\nu \right| > 2\alpha \right\}.$$

Given $\varepsilon > 0$, pick a continuous function $g \in L^1(\mathfrak{m})$ with

$$\|f - g\|_{L^1(\mathfrak{m})} \leq \varepsilon. \tag{2.13}$$

For $\nu \in \mathcal{V}$ with $r(\nu) \leq 1$ and $\theta(\nu) = x$ we have

$$\frac{1}{\nu(X)}\left|\int_X (f(y)-f(x))\,d\nu(y)\right| \leq \frac{1}{\nu(X)}\left|\int_X (f(y)-g(y))\,d\nu(y)\right| + \frac{1}{\nu(X)}\left|\int_X (g(y)-g(x))\,d\nu(y)\right| + |g(x)-f(x)|. \tag{2.14}$$

Since g is continuous, for all $x \in X$ we have

$$\lim_{\nu\to x} \frac{1}{\nu(X)}\left|\int_X (g(y)-g(x))\,d\nu(y)\right| = 0. \tag{2.15}$$

To deal with the first summand, we compute

$$\frac{1}{\nu(X)}\left|\int_X (f(y)-g(y))\,d\nu(y)\right| \leq \frac{M}{\eta\,\mathfrak{m}(B_r(x))}\int_{B_r(x)} |f(y)-g(y)|\,d\mathfrak{m} \leq \frac{M}{\eta}\mathrm{Mx}(|f-g|)(x). \tag{2.16}$$

Combining (2.14), (2.15), and (2.16), we get

$$E_\alpha \subset \left\{\mathrm{Mx}(|f-g|) \geq \frac{\eta\,\alpha}{M}\right\} \cup \{|f-g| \geq \alpha\}.$$

Then from (2.13) and Proposition 2.5(1) we obtain

$$\mathfrak{m}(E_\alpha) \leq \frac{C(K,N)M}{\eta\alpha}\varepsilon.$$

Since ε was arbitrary we get $\mathfrak{m}(E_\alpha) = 0$, and hence the result. $\square$

2.3. *Gromov–Hausdorff topology.*

Definition 2.17. Let (X_i, p_i) be a sequence of pointed proper metric spaces. We say that it *converges in the pointed Gromov–Hausdorff sense* to a proper pointed metric space (X, p) if there is a sequence of functions $\varphi_i : X_i \to X \cup \{*\}$ with $\varphi_i(p_i) \to p$ such that, for each $R > 0$,

$$\varphi_i^{-1}(B_R(p)) \subset B_{2R}(p_i) \text{ for } i \text{ large enough},$$
$$\lim_{i\to\infty} \sup_{x_1,x_2\in B_{2R}(p_i)} |d(\varphi_i(x_1), \varphi_i(x_2)) - d(x_1,x_2)| = 0,$$
$$\lim_{i\to\infty} \sup_{y\in B_R(p)} \inf_{x\in B_{2R}(p_i)} d(\varphi_i(x), y) = 0.$$

If, in addition, $(X_i, d_i, \mathfrak{m}_i)$, $(X, d, \mathfrak{m})$ are metric measure spaces, the maps φ_i are Borel measurable, and

$$\int_X f\cdot d((\varphi_i)_*\mathfrak{m}_i) \to \int_X f\cdot d\mathfrak{m}$$

for all $f : X \to \mathbb{R}$ bounded continuous with compact support, then we say that $(X_i, d_i, \mathfrak{m}_i, p_i)$ converges to $(X, d, \mathfrak{m}, p)$ in the *pointed measured Gromov–Hausdorff sense*.

Remark 2.18. Whenever a sequence of pointed spaces (X_i, p_i) converges in the pointed Gromov–Hausdorff sense to some pointed space (X, p), we implicitly assume the existence of the maps φ_i, called *Gromov–Hausdorff approximations* satisfying the above conditions, and if a sequence $x_i \in X_i$ is such that $\varphi_i(x_i) \to x \in X$, by an abuse of notation we say that x_i *converges* to x.

The topology induced by this convergence is also given by a metric [Gromov 2007].

Theorem 2.19 (Gromov). *There is a metric d_{GH} in the class of pointed proper metric spaces modulo pointed isometry with the property that a sequence (X_i, p_i) converges to a space (X, p) in the pointed Gromov–Hausdorff sense if and only if $d_{GH}((X_i, p_i), (X, p)) \to 0$.*

Remark 2.20. The only property we will need about this metric is that if (Y, y) is a pointed compact geodesic space for which

$$d_{GH}((\mathbb{R}^k \times Y, (0, y)), (\mathbb{R}^k, 0)) \leq \tfrac{1}{100} \quad \text{for some } k \in \mathbb{N},$$

then $\operatorname{diam}(Y) \leq \frac{1}{10}$.

One of the main features of the class of $\mathrm{RCD}(K, N)$ spaces is the compactness property. Theorem 2.21 follows immediately from Gromov's compactness criterion [2007, Proposition 5.2], and Theorem 2.22 was proven in [Gigli et al. 2015] building upon [Lott and Villani 2009; Sturm 2006a; 2006b; Ambrosio et al. 2014b].

Theorem 2.21. *If $(X_i, d_i, \mathfrak{m}_i, p_i)$ is a sequence of pointed $\mathrm{RCD}(K, N)$ spaces, then one can find a subsequence for which (X_i, p_i) converges in the pointed Gromov–Hausdorff sense to some pointed proper geodesic space (X, p).*

Notice that for any pointed $\mathrm{RCD}(K, N)$ space $(X, d, \mathfrak{m}, p)$, there is a unique $c > 0$ for which $(X, d, c\mathfrak{m}, p)$ is normalized.

Theorem 2.22. *The class of pointed normalized $\mathrm{RCD}(K, N)$ spaces is closed under pointed measured Gromov–Hausdorff convergence. Moreover, if $(X_i, d_i, \mathfrak{m}_i, p_i)$ is a sequence of $\mathrm{RCD}(K - \varepsilon_i, N)$ spaces such that $\varepsilon_i \to 0$ and (X_i, p_i) converges in the pointed Gromov–Hausdorff sense to a pointed proper metric space (X, p), then X admits a measure $\mathfrak{m}$ that makes it a normalized $\mathrm{RCD}(K, N)$ space, and after passing to a subsequence, there are $c_i > 0$ for which $(X_i, d_i, c_i\mathfrak{m}_i, p_i)$ converges in the pointed measured Gromov–Hausdorff sense to $(X, d, \mathfrak{m}, p)$.*

Definition 2.23. Let $(X, d, \mathfrak{m})$ be an $\mathrm{RCD}(K, N)$ space and $m \in \mathbb{N}$. We say that $p \in X$ is an *m-regular* point if for each $\lambda_i \to \infty$, the sequence $(\lambda_i X, p)$ converges in the pointed Gromov–Hausdorff sense to $(\mathbb{R}^m, 0)$.

Mondino and Naber [2019] showed that the set of regular points in an $\mathrm{RCD}(K, N)$ space has full measure. This result was refined by Brué and Semola [2020b] who showed that most points have the same local dimension.

Theorem 2.24 (Brué and Semola). *Let $(X, d, \mathfrak{m})$ be an $\mathrm{RCD}(K, N)$ space. Then there is a unique $m \in \mathbb{N} \cap [0, N]$ such that the set of m-regular points in X has full measure. This number m is called the rectifiable dimension of X.*

The Cheeger–Gromoll splitting theorem was extended by Gigli [2014] to this setting.

Theorem 2.25 (Gigli). *Let $(X, d, \mathfrak{m})$ be an $\mathrm{RCD}(0, N)$ space of rectifiable dimension n and assume the metric space (X, d) contains an isometric copy of $\mathbb{R}^m$, then there is $c > 0$ and an $\mathrm{RCD}(0, N - m)$ space (Y, d^Y, ν) of rectifiable dimension $n - m$ such that $(X, d, c\mathfrak{m})$ is isomorphic to the product $(\mathbb{R}^m \times Y, d^{\mathbb{R}^m} \times d^Y, \mathcal{H}^m \otimes \nu)$. In particular $m \leq n$, and if $m = n$ then Y is a point.*

Corollary 2.26. *Let $(X_i, d_i, \mathfrak{m}_i, p_i)$ be a sequence of pointed normalized* $\mathrm{RCD}(-\delta_i, N)$ *spaces with $\delta_i \to 0$. If (X_i, p_i) converges in the pointed Gromov–Hausdorff sense to $(\mathbb{R}^k, 0)$, then $(X_i, d_i, \mathfrak{m}_i, p_i)$ converges to $(\mathbb{R}^k, d^{\mathbb{R}^k}, \mathcal{H}^k, 0)$ in the pointed measured Gromov–Hausdorff sense as well.*

Corollary 2.27 below follows from Theorem 2.25 the same way [Cheeger and Gromoll 1971/72, Theorem 3] follows from the splitting theorem for smooth manifolds.

Corollary 2.27. *Let $(\widetilde{Y}, d, \mathfrak{m})$ be an* $\mathrm{RCD}(0, N)$ *space of rectifiable dimension n for which $\widetilde{Y}/\,\mathrm{Iso}(\widetilde{Y})$ is compact. Then there are $m \leq n$ and a compact metric space Z for which $\widetilde{Y}$ is isometric to the product $\mathbb{R}^m \times Z$.*

The rectifiable dimension is lower semicontinuous [Kitabeppu 2019].

Theorem 2.28 (Kitabeppu). *Let $(X_i, d_i, \mathfrak{m}_i, p_i)$ be a sequence of pointed* $\mathrm{RCD}(K, N)$ *spaces of rectifiable dimension m. Assume (X_i, p_i) converges in the pointed Gromov–Hausdorff sense to (X, p). If $\mathfrak{m}$ is a measure on X that makes it an* $\mathrm{RCD}(K, N)$ *space, then $(X, d, \mathfrak{m})$ has rectifiable dimension at most m.*

2.4. ***Equivariant Gromov–Hausdorff convergence.*** In the setting of Gromov–Hausdorff convergence, there is a notion of convergence of group actions [Fukaya and Yamaguchi 1992, Section 3]. For a pointed proper metric space (X, p), we equip its isometry group $\mathrm{Iso}(X)$ with the metric d_0^p given by

$$d_0^p(h_1, h_2) := \inf_{r>0}\left\{\frac{1}{r} + \sup_{x\in B_r(p)} d(h_1x, h_2x)\right\} \tag{2.29}$$

for $h_1, h_2 \in \mathrm{Iso}(X)$. It is easy to see that this metric is left invariant, induces the compact-open topology, and makes $\mathrm{Iso}(X)$ a proper metric space.

Recall that if a sequence of pointed proper metric spaces (X_i, p_i) converges in the pointed Gromov–Hausdorff sense to the pointed proper metric space (X, p), one has Gromov–Hausdorff approximations $\varphi_i : X_i \to X \cup \{*\}$.

Definition 2.30. Consider a sequence of pointed proper metric spaces (X_i, p_i) that converges in the pointed Gromov–Hausdorff sense to a pointed proper metric space (X, p), a sequence of closed groups of isometries $\Gamma_i \leq \mathrm{Iso}(X_i)$, and a closed group $\Gamma \leq \mathrm{Iso}(X)$. Equip Γ_i with the metric $d_0^{p_i}$ and Γ with the metric d_0^p. We say that the sequence Γ_i *converges equivariantly* to Γ if there is a sequence of Gromov–Hausdorff approximations $\psi_i : \Gamma_i \to \Gamma \cup \{*\}$ such that for each $R > 0$ one has

$$\lim_{i\to\infty} \sup_{g\in B_R(\mathrm{Id}_{X_i})} \sup_{x\in B_R(p_i)} d(\varphi_i(gx), \psi_i(g)(\varphi_i x)) = 0.$$

Isometry groups of proper spaces satisfy a compactness property [Fukaya and Yamaguchi 1992, Proposition 3.6].

Theorem 2.31 (Fukaya and Yamaguchi). *Let (Y_i, q_i) be a sequence of proper metric spaces that converges in the pointed Gromov–Hausdorff sense to a proper space (Y, q), and take a sequence $\Gamma_i \leq \mathrm{Iso}(Y_i)$ of closed groups of isometries. Then, after taking a subsequence, Γ_i converges equivariantly to a closed group $\Gamma \leq \mathrm{Iso}(Y)$, and the sequence $(Y_i/\,\Gamma_i, [q_i])$ converges in the pointed Gromov–Hausdorff sense* *to $(Y/\Gamma, [q])$. Moreover, if $\rho_i : Y_i \to Y_i/\Gamma_i$, $\rho : Y \to Y/\Gamma$ are the projections, there are $\delta_i \to 0$, $R_i \to \infty$, and Gromov–Hausdorff approximations $\tilde{\varphi}_i : Y_i \to Y \cup \{*\}$, $\varphi_i : Y_i/\Gamma_i \to Y/\Gamma \cup \{*\}$ such that for all $x \in B_{R_i}(q_i)$ one has*

$$d(\varphi_i(\rho_i(x)), \rho(\tilde{\varphi}_i(x))) \leq \delta_i. \tag{2.32}$$

As a consequence of Theorems 2.25 and 2.31, one gets the following well-known result.

Proposition 2.33. *For each $i \in \mathbb{N}$, let $(X_i, d_i, \mathfrak{m}_i, p_i)$ be a pointed $\mathrm{RCD}\big(-\frac{1}{i}, N\big)$ space of rectifiable dimension n. Assume (X_i, p_i) converges in the pointed Gromov–Hausdorff sense to (X, p), there is a sequence of closed groups of isometries $\Gamma_i \leq \mathrm{Iso}(X_i)$ that converges equivariantly to $\Gamma \leq \mathrm{Iso}(X)$, and the sequence of pointed metric spaces $(X_i/\Gamma_i, [p_i])$ converges in the pointed Gromov–Hausdorff sense to $(\mathbb{R}^k \times Y, (0, q))$ for some pointed proper metric space (Y, q).*

Then there is a pointed metric space $(\widetilde{Y}, \tilde{q})$ for which X is isomorphic to the product $\mathbb{R}^k \times \widetilde{Y}$, the Γ-action respects the splitting $\mathbb{R}^k \times \widetilde{Y}$, and acts trivially on the first factor. In particular, if $k = n$, then $\widetilde{Y}$ is a point.

Proof. By Theorem 2.31, $X/\Gamma = \mathbb{R}^k \times Y$, and one can use the submetry $\rho : X \to X/\Gamma$ to lift the lines of $\mathbb{R}^k$ to lines in X passing through p. By Theorem 2.22, X admits a measure that makes it an $\mathrm{RCD}(0, N)$ space, so by Theorems 2.25 and 2.28, we get the desired splitting $X = \mathbb{R}^k \times \widetilde{Y}$ with the property that $\rho(x, \tilde{q}) = (x, q)$ for all $x \in \mathbb{R}^k$.

Now we show that the action of Γ respects the $\widetilde{Y}$-fibers. Let $g \in \Gamma$ and assume $g(x_1, \tilde{q}) = (x_2, y)$ for some $x_1, x_2 \in \mathbb{R}^k$, $y \in \widetilde{Y}$. Then for all $t \geq 1$, one has

$$\begin{aligned} t|x_1 - x_2| &= d(\rho(x_1 + t(x_2 - x_1), \tilde{q}), \rho(x_1, \tilde{q})) \\ &= d(\rho(x_1 + t(x_2 - x_1), \tilde{q}), \rho(x_2, y)) \\ &\leq d^X((x_1 + t(x_2 - x_1), \tilde{q}), (x_2, y)) \\ &= \sqrt{|(t-1)(x_2 - x_1)|^2 + d^{\widetilde{Y}}(\tilde{q}, y)^2}. \end{aligned}$$

As $t \to \infty$, this is only possible if $x_1 = x_2$. This shows that $g(x, \tilde{q}) = (x, y)$ for some $y \in \widetilde{Y}$ independent of $x \in \mathbb{R}^k$. Now assume $g(x_1, z) = (x_2, z')$ for some $z, z' \in \widetilde{Y}$. Then

$$\begin{aligned} t^2|x_1 - x_2|^2 + d^{\widetilde{Y}}(\tilde{q}, z)^2 &= d^X((x_1 + t(x_2 - x_1), \tilde{q}), (x_1, z))^2 \\ &= d^X((x_1 + t(x_2 - x_1), y), (x_2, z'))^2 \\ &= |(t-1)(x_2 - x_1)|^2 + d^{\widetilde{Y}}(y, z')^2. \end{aligned}$$

This is only possible if $x_1 = x_2$, showing that Γ acts trivially on the $\mathbb{R}^k$-factor. □

2.5. *δ-splittings.* Let us recall some results on δ-splittings. For proof and detailed discussions see for example [Bruè et al. 2023, Section 3.1].

Lemma 2.34. *Let $(X_i, d_i, \mathfrak{m}_i, p_i)$ be a sequence of $\mathrm{RCD}\big(-\frac{1}{i}, N\big)$ spaces for which (X_i, p_i) converges in the pointed Gromov–Hausdorff sense to $(\mathbb{R}^k \times Y, (0, y))$ for some metric space (Y, y). Then for any* *sequence of Gromov–Hausdorff approximations* $\varphi_i : X_i \to \mathbb{R}^k \times Y \cup \{*\}$, *there are sequences* $\delta_i \to 0$, $R_i \to \infty$, *and a sequence of* $L(N)$*-Lipschitz functions* $h^i \in H^{1,2}(X_i; \mathbb{R}^k)$ *such that*

- h^i *is harmonic* (*equivalently*, ∇h^i *is divergence free*) *in* $B_{R_i}(p_i)$,
- *for all* $r \in [1, R_i]$, *one has*

$$\fint_{B_r(p_i)} \Bigg[\sum_{j_1, j_2 = 1}^{k} |\langle \nabla h^i_{j_1}, \nabla h^i_{j_2} \rangle - \delta_{j_1, j_2}| + \sum_{j=1}^{k} |\nabla \nabla h^i_j|^2 \Bigg] \, d\mathfrak{m}_i \le \delta_i^2,$$

- *for all* $x \in B_{R_i}(p_i)$ *one has*

$$|h^i(x) - \pi(\varphi_i x)| \le \delta_i, \tag{2.35}$$

where $\pi : \mathbb{R}^k \times Y \to \mathbb{R}^k$ *is the projection.*

Lemma 2.36. *Let* $(X_i, d_i, \mathfrak{m}_i, p_i)$ *be a sequence of* $\mathrm{RCD}\big(-\frac{1}{i}, N\big)$. *Assume there are sequences* $\delta_i \to 0$, $R_i \to \infty$, *and a sequence of* L*-Lipschitz functions* $h^i \in H^{1,2}(X_i; \mathbb{R}^k)$ *with* $h^i(p_i) = 0$ *for all* i *and such that*

- h^i *is harmonic* (*equivalently*, ∇h^i *is divergence free*) *in* $B_{R_i}(p_i)$,
- *for all* $r \in [1, R_i]$, *one has*

$$\fint_{B_r(p_i)} \Bigg[\sum_{j_1, j_2 = 1}^{k} |\langle \nabla h^i_{j_1}, \nabla h^i_{j_2} \rangle - \delta_{j_1, j_2}| + \sum_{j=1}^{k} |\nabla \nabla h^i_j|^2 \Bigg] \, d\mathfrak{m}_i \le \delta_i^2.$$

Then, *after taking a subsequence*, *there is a metric space* (Y, y) *and a sequence of Gromov–Hausdorff approximations* $\varphi_i : X_i \to \mathbb{R}^k \times Y \cup \{*\}$ *for which*

$$\sup_{x \in B_{R_i}(p_i)} |h^i(x) - \pi(\varphi_i x)| \to 0 \quad \text{as } i \to \infty, \tag{2.37}$$

where $\pi : \mathbb{R}^k \times Y \to \mathbb{R}^k$ *is the projection.*

Remark 2.38. In the literature, Lemmas 2.34 and 2.36 are often stated without equations (2.35) and (2.37). However, these equations follow from how the functions h_i (resp. φ_i) are constructed in the proof of Lemma 2.34 (resp. Lemma 2.36). Similarly, the maps h_i are usually only defined on balls around p_i with radii going to infinity, but thanks to the existence of good cut-off functions [Mondino and Naber 2019, Lemma 3.1], we can assume they are fully defined on the spaces X_i.

2.6. ***Regular Lagrangian flows.*** In $\mathrm{RCD}(K, N)$ spaces, there exist flows of certain Sobolev vector fields. For the definition of RLFs, see for example Definition 1.2. For sufficiently regular vector fields, RLFs satisfy an existence and uniqueness property [Ambrosio and Trevisan 2014].

Theorem 2.39. *Let* $(X, d, \mathfrak{m})$ *be an* $\mathrm{RCD}(K, N)$ *space*, *and assume* $V \in L^1([0, T], L^2(TX))$ *satisfies* $V(t) \in D(\mathrm{div})$ *for a.e.* $t \in [0, T]$ *with*

$$\mathrm{div}(V(\cdot)) \in L^1([0, T], L^2(\mathfrak{m})), \quad (\mathrm{div}(V(\cdot)))^- \in L^1([0, T], L^\infty(\mathfrak{m})), \quad \nabla V(\cdot) \in L^1([0, T], L^2(T^{\otimes 2} X)).$$

Then there exists a unique (up to $\mathfrak{m}$-a.e. equality) RLF $\boldsymbol{X} : [0, T] \times X \to X$ for V satisfying

$$(\boldsymbol{X}_t)_*(\mathfrak{m}) \le \exp\biggl(\int_0^t \|\mathrm{div}(V(s))^-\|_{L^\infty(\mathfrak{m})}\, ds\biggr)\mathfrak{m} \tag{2.40}$$

for every $t \in [0, T]$.

The estimate (2.40) can be localized [Gigli and Violo 2023, Proposition 5.3].

Proposition 2.41. *Let $(X, d, \mathfrak{m})$, T, V, and $\boldsymbol{X}$ be as in Theorem 2.39. Then for any $S \in \mathcal{B}(X)$ and $t \in [0, T]$ one has*

$$(\boldsymbol{X}_t)_*(\mathfrak{m}|_S) \le \exp\biggl(\int_0^t \|\mathrm{div}(V(s))^-\|_{L^\infty((\boldsymbol{X}_s)_*(\mathfrak{m}|_S))}\, ds\biggr)\mathfrak{m}.$$

Remark 2.42. From *R*.2, we get that if $\|V(t)\|_\infty \le L$ for all $t \in [0, T]$ and some $L > 0$, then for $\mathfrak{m}$-a.e. $x \in X$, the map

$$[0, T] \ni t \mapsto \boldsymbol{X}_t(x) \quad \text{is } L\text{-Lipschitz}. \tag{2.43}$$

Thus, after modifying $\boldsymbol{X}$ on a set of measure zero, we can always assume (2.43) holds for all $x \in X$ (see [Gigli and Tamanini 2021, Theorem A.4]).

For nice vector fields, there is a reverse flow [Deng 2020, Proposition 3.12].

Proposition 2.44. *Let $(X, d, \mathfrak{m})$, V, $\boldsymbol{X}$, be as in Theorem 2.39, and define $\overline{V} : [0, T] \to L^2(TX)$ as*

$$\overline{V}(t)(x) := -V(T-t)(x)$$

for each $t \in [0, T]$, $x \in X$. Then there is a map $\overline{\boldsymbol{X}} : [0, T] \times X \to X$ which is an RLF for $\overline{V}$ and for $\mathfrak{m}$-a.e. $x \in X$ one has

$$\overline{\boldsymbol{X}}_t(\boldsymbol{X}_T(x)) = \boldsymbol{X}_{T-t}(x) \quad \textit{for all } t \in [0, T].$$

Remark 2.45. If $\|\mathrm{div}(V(t))\|_\infty \le D$ for all $t \in [0, T]$ and some $D > 0$, (2.40) implies

$$e^{-DT}\mathfrak{m} \le (\boldsymbol{X}_t(\,\cdot\,))_*(\mathfrak{m}) \le e^{DT}\mathfrak{m} \quad \text{for all } t \in [0, T]. \tag{2.46}$$

The integral first variation formula extends to $\mathrm{RCD}(K, N)$ spaces [Brué et al. 2022, Corollary 4.2].

Theorem 2.47. *Let $r > 0$, $(X, d, \mathfrak{m})$ an $\mathrm{RCD}(K, N)$ space, and V a time-dependent vector field satisfying the conditions of Theorem 2.39. Set*

$$\begin{aligned} &\mathrm{dt}_r : [0, T] \times X \times X \to [0, r], \\ &\mathrm{dt}_r(t)(a, b) := \sup_{s \in [0,t]} \mathrm{dt}_r(\boldsymbol{X}_s)(a, b). \end{aligned} \tag{2.48}$$

Let S_1, S_2 be Borel subsets of X with finite positive measure, and define

$$\Gamma(t) := \{(a, b) \in S_1 \times S_2 \mid \mathrm{dt}_r(t)(a, b) < r\}. \tag{2.49}$$

Then the map $t \mapsto \int_{S_1 \times S_2} \mathrm{dt}_r(t)(a, b)\, d(\mathfrak{m} \times \mathfrak{m})(a, b)$ is Lipschitz on $[0, T]$ and for a.e. $t \in [0, T]$ one has

$$\frac{d}{dt}\int_{S_1 \times S_2} \mathrm{dt}_r(t)(a, b)\, d(\mathfrak{m}\times\mathfrak{m})(a, b) \le \int_0^1 \int_{\Gamma(t)} d(\boldsymbol{X}_t(a), \boldsymbol{X}_t(b)) |\nabla V(t)|(\gamma_{\boldsymbol{X}_t(a), \boldsymbol{X}_t(b)}(s))\, d(\mathfrak{m}\times\mathfrak{m})(a, b)\, ds.$$

Remark 2.50. Although [Brué et al. 2022, Corollary 4.2] was stated only for the noncollapsed case (i.e., $\mathfrak{m} = \mathscr{H}^N$), its proof follows that of [Deng 2020, Proposition 3.27] (see also [Brué et al. 2022, Proposition 4.1] for additional comments) and in particular works without the noncollapsed assumption.

2.7. *Group norms.* Let (X, p) be a pointed proper geodesic space and $\Gamma \leq \mathrm{Iso}(X)$ a closed group of isometries. The *norm* $\|\cdot\|_p : \Gamma \to \mathbb{R}$ associated to p is defined as $\|g\|_p := d(gp, p)$. We denote as $\mathcal{G}(\Gamma, X, p, r)$ the subgroup of Γ generated by the elements of norm $\|\cdot\|_p \leq r$. The *norm spectrum* $\sigma(\Gamma)$ is defined as the set of $r \geq 0$ for which $\mathcal{G}(\Gamma, X, p, r) \neq \mathcal{G}(\Gamma, X, p, r-\varepsilon)$ for all $\varepsilon > 0$. Notice that we always have $0 \in \sigma(\Gamma)$. If we want redundancy we sometimes write $\sigma(\Gamma, X, p)$ to denote the spectrum of the action of Γ on the pointed space (X, p). See also [Sormani and Wei 2004; 2015; Plaut 2021] for similar notions of group spectra and their relationship.

Proposition 2.51. *If Γ is equipped with the metric d_0^p from (2.29), and $\Gamma = \mathcal{G}(\Gamma, X, p, D)$ for some $D > 0$, then $\Gamma = \langle B_{D+2\sqrt{2}+\varepsilon}(\mathrm{Id}_X)\rangle$ for all $\varepsilon > 0$.*

Proof. From (2.29) with $r = 1/\sqrt{2}$, for all $g \in \Gamma$ one gets

$$\|g\|_p \leq d_0^p(g, I_X) \leq \|g\|_p + 2\sqrt{2}.$$

Then $\{g \in \Gamma \mid \|g\|_p \leq D\} \subset B_{D+2\sqrt{2}+\varepsilon}(\mathrm{Id}_X)$ for all $\varepsilon > 0$. □

It also satisfies a continuity property [Santos-Rodríguez and Zamora 2023, Proposition 47].

Proposition 2.52. *Let (X_i, p_i) be a sequence of pointed proper metric spaces that converges in the pointed Gromov–Hausdorff sense to (X, p) and consider a sequence of closed isometry groups $\Gamma_i \leq \mathrm{Iso}(X_i)$ that converges equivariantly to a closed group $\Gamma \leq \mathrm{Iso}(X)$. Then for any convergent sequence of real numbers $r_i \in \sigma(\Gamma_i)$, the limit $\lim_{i\to\infty} r_i$ lies in $\sigma(\Gamma)$.*

Remark 2.53. It is possible that an element in $\sigma(\Gamma)$ is not a limit of elements in $\sigma(\Gamma_i)$, so this spectrum is not necessarily continuous with respect to equivariant convergence (see [Kapovitch and Wilking 2011, Example 1]).

Proposition 2.54. *For any $a > 0$, one has $\mathcal{G}(\Gamma, X, p, a) = \mathcal{G}(\Gamma, X, p, a+\varepsilon)$ for $\varepsilon > 0$ small enough.*

Proof. Assuming the proposition fails, there is a sequence of elements g_i not in $\mathcal{G}(\Gamma, X, p, a)$ with $\|g_i\|_p \to a$. As the sequence $\|g_i\|_p$ is bounded, after taking a subsequence we can assume $g_i \to g$ for some $g \in \Gamma$ with $\|g\|_p = a$. Then for large enough i, $\|g^{-1}g_i\|_p < a$, so $g_i = (g)(g^{-1}g_i) \in \mathcal{G}(\Gamma, X, p, a)$, which is a contradiction. □

Corollary 2.55. *For any $[a, b] \subset (0, \infty)$, the following are equivalent*:

- $\sigma(\Gamma) \cap \big(a, b] = \varnothing$.
- $\mathcal{G}(\Gamma, X, p, a) = \mathcal{G}(\Gamma, X, p, b)$.

It is well known that when a group action is co-compact, the spectrum is bounded [Gromov 2007, Proposition 5.28].

Lemma 2.56. *Let (X, p) be a pointed proper geodesic space and $\Gamma \leq \mathrm{Iso}(X)$ a closed group of isometries. Then $r \leq 2 \cdot \mathrm{diam}(X/\Gamma)$ for all $r \in \sigma(\Gamma, X, p)$.*

To prove Theorem 1.14, one needs to control the number of generators of the groups Γ_i. This was done in [Santos-Rodríguez and Zamora 2023, Theorem 80] after [Kapovitch and Wilking 2011, Theorem 2.5].

Lemma 2.57. *Let $(X, d, \mathfrak{m}, p)$ be a pointed $\mathrm{RCD}(K, N)$ space, and $\Gamma \leq \mathrm{Iso}(X)$ a discrete group of measure preserving isometries with $\Gamma = \mathcal{G}(\Gamma, X, p, D)$. Then Γ can be generated by at most $C(K, N, D)$ elements.*

2.8. *Group theory.* In this section we cover basic group theory results needed later. Proofs of Propositions 2.58 and 2.60 below can be found in [Fukaya and Yamaguchi 1992, Section 4].

Proposition 2.58. *Let G be a group generated by k elements and $H \leq G$ a subgroup of index $[G : H] \leq M$. Then there is a characteristic subgroup $H' \triangleleft G$ with $H' \leq H$ and $[G : H'] \leq C(M, k)$.*

Remark 2.59. By Lemma 2.57 and Proposition 2.58, whenever Theorem 1.14 holds, we may assume the subgroups $G_i \triangleleft \Gamma_i$ are characteristic.

Proposition 2.60. *Let A be an abelian group generated by m elements, and $\varphi : G \to A$ a surjective morphism with finite kernel. Then G contains a finite index abelian subgroup generated by m elements.*

Proposition 2.61. *Let G be a group, $H \triangleleft G$ a normal subgroup, $a, b \in G$ such that $[a, b] \in H$, and $H_0 \triangleleft H$ a characteristic subgroup of H with $[H : H_0] \leq M$. Then for all $C \geq 2M$ one has $[a^{C!}, b] \in H_0$.*

Proof. In the group G/H_0, set $\alpha := aH_0$ and $\beta := bH_0$. Then $\alpha\beta\alpha^{-1} = \beta h$ for some $h \in H/H_0$. A direct computation shows that $\alpha^k\beta\alpha^{-k} = \beta(h)(\alpha h\alpha^{-1}) \cdots (\alpha^{k-1}h\alpha^{-k+1})$. As H/H_0 is normal in G/H_0 and $|H/H_0| \leq M$, one gets that $\alpha^{M!}h\alpha^{-M!} = h$, so

$$\alpha^{M!M}\beta\alpha^{-M!M} = \beta \prod_{j=0}^{M!M-1}(\alpha^j h\alpha^{-j}) = \beta\left(\prod_{j=0}^{M!-1}(\alpha^j h\alpha^{-j})\right)^M = \beta.$$

If $C \geq 2M$, then $C!$ is a multiple of $M!M$, and

$$[\alpha^{C!}, \beta] = \alpha^{C!}\beta\alpha^{-C!}\beta^{-1} = \alpha^{M!M}(\cdots(\alpha^{M!M}\beta\alpha^{-M!M})\cdots\alpha^{-M!M})\beta^{-1} = \beta\beta^{-1} = e_{G/H_0}$$

This shows that $[a^{C!}, b] \in H_0$. ☐

Proposition 2.62. *Let Γ be a group, $G \triangleleft \Gamma$ a characteristic subgroup admitting a nilpotent basis, $g \in \Gamma$, $\varphi \in \mathrm{Aut}(\Gamma)$, and $C \in 2\mathbb{Z}$. If $[\Gamma : G] \leq C/2$, then the nilpotent basis in G is preserved by $\varphi^{C!}$ if and only if it is preserved by $(\varphi \circ g_*)^{C!}$.*

Proof. First we observe that for any $k \in \mathbb{N}$ we have

$$\begin{aligned}(\varphi \circ g_*)^k &= (\varphi \circ g_* \circ \varphi^{-1})(\varphi^2 \circ g_* \circ \varphi^{-2}) \cdots (\varphi^k \circ g_* \circ \varphi^{-k})\varphi^k \\ &= (\varphi(g))_*(\varphi^2(g))_* \cdots (\varphi^k(g))_*\varphi^k \\ &= (\varphi(g)\varphi^2(g) \cdots \varphi^k(g))_*\varphi^k. \end{aligned} \tag{2.63}$$

On the other hand, as G is characteristic in Γ, the group $G_* := \{x_* : \Gamma \to \Gamma \mid x \in G\}$ is normal in $\mathrm{Aut}(\Gamma)$, so one has $y_*^C G_* = G_*$ for all $y \in \Gamma$. Also, notice that $\varphi^{(C/2)!}(g)G = gG$ in Γ/G, so $(\varphi^{(C/2)!}(g))_* G_* = g_* G_*$ in $\mathrm{Aut}(\Gamma)/G_*$. Thus if $\ell = (C-1)!/(C/2)!$, using (2.63) we have

$$(\varphi \circ g_*)^{C!} G_* = ((\varphi(g)\varphi^2(g) \cdots \varphi^{(C/2)!}(g))_*)^{C\ell} \varphi^{C!} G_* = \varphi^{C!} G_*.$$

This implies that $(\varphi \circ g_*)^{C!}$ and $\varphi^{C!}$ differ only by an element of G_*, which clearly respects the nilpotent basis in G. □

We will also need the following version of the Bieberbach theorem [Fukaya and Yamaguchi 1992, Section 4].

Theorem 2.64 (Fukaya and Yamaguchi). *Let $G \leq \mathrm{Iso}(\mathbb{R}^m)$ be a closed group of isometries and $G_0 \leq G$ its identity connected component. Then G/G_0 contains a finite index abelian subgroup generated by at most m elements.*

Corollary 2.65. *Let Z be a compact metric space, $\Gamma \leq \mathrm{Iso}(\mathbb{R}^m \times Z)$ a closed group of isometries and $\Gamma_0 \leq \Gamma$ its identity connected component. If $\mathrm{Iso}(Z)$ is a Lie group, then Γ/Γ_0 contains a finite index abelian subgroup generated by at most m elements.*

Proof. Notice that for each $(x, z) \in \mathbb{R}^m \times Z$, the $\mathbb{R}^m$-fiber passing through (x, z) can be characterized as the union of the images of all infinite geodesics passing through (x, z). This implies that $\mathrm{Iso}(\mathbb{R}^m \times Z)$ respects the splitting $\mathbb{R}^m \times Z$ and decomposes as $\mathrm{Iso}(\mathbb{R}^m \times Z) = \mathrm{Iso}(\mathbb{R}^m) \times \mathrm{Iso}(Z)$. Let $G \leq \mathrm{Iso}(\mathbb{R}^m)$ be the image of Γ under the projection $\pi : \mathrm{Iso}(\mathbb{R}^m \times Z) \to \mathrm{Iso}(\mathbb{R}^m)$. As Γ is closed and $\mathrm{Iso}(Z) = \mathrm{Ker}(\pi)$ is compact, G is closed in $\mathrm{Iso}(\mathbb{R}^m)$.

We claim that $\pi(\Gamma_0) = G_0$. Assuming the contrary, as G_0 is connected, there would be a sequence $x_i \in G_0 \backslash \pi(\Gamma_0)$ with $x_i \to e_{G_0}$. Pick $g_i \in \Gamma$ with $\pi(g_i) = x_i$. Since $\mathrm{Iso}(Z)$ is compact, after passing to a subsequence we can assume $g_i \to g_\infty$ for some $g_\infty \in \mathrm{Ker}(\pi)$. Then $g_\infty^{-1} g_i \to e_\Gamma$, so for i large enough one has $g_\infty^{-1} g_i \in \Gamma_0$. This would mean that $\pi(g_\infty^{-1} g_i) = \pi(g_i) = x_i \in \pi(\Gamma_0)$, which is a contradiction.

Let $H := \pi^{-1}(G_0) \cap \Gamma$. We claim that $[H : \Gamma_0] < \infty$. Otherwise, there would be a sequence $h_i \in H \cap \mathrm{Ker}(\pi)$ with $h_i^{-1} h_j \in H \backslash \Gamma_0$ for all $i \neq j$. As $\mathrm{Ker}(\pi)$ is compact, after taking a subsequence we can assume $h_i \to h_\infty$ for some $h_\infty \in \mathrm{Ker}(\pi)$. This would mean that for i, j large enough, one has $h_i^{-1} h_j \in \Gamma_0$, which is a contradiction.

The above implies that Γ/Γ_0 is a finite extension of $(\Gamma/\Gamma_0)/(H/\Gamma_0) = \Gamma/H \cong G/G_0$, so the result follows from Theorem 2.64 and Proposition 2.60. □

3. Groups of connected components

The goal of this section is to prove the following result (cf. [Fukaya and Yamaguchi 1992, Theorem 3.10] and [Santos-Rodríguez and Zamora 2023, Lemma 58]). The groups Υ_i play the role of "connected component of the identity" in the groups Γ_i.

Theorem 3.1. *Let (X_i, p_i) be a sequence of proper geodesic spaces that converges in the pointed Gromov–Hausdorff sense to a space (X, p), $\Gamma_i \leq \mathrm{Iso}(X_i)$ a sequence of closed groups of isometries that converges* *equivariantly to a closed group* $\Gamma \leq \mathrm{Iso}(X)$, *and* $\psi_i : \Gamma_i \to \Gamma \cup \{*\}$ *the Gromov Hausdorff approximations given by Definition 2.30. Assume*

- $\Gamma_i = \mathcal{G}(\Gamma_i, X_i, p_i, D)$ *for some* $D > \sqrt{2}$,
- Γ_0, *the connected component of the identity of* Γ, *is open*,
- Γ/Γ_0 *is finitely presented.*

Then there are subgroups $\Upsilon_i \triangleleft \Gamma_i$ *such that*

- Υ_i *is normal in* Γ_i *for* i *large enough*,
- *for any* $R > 0$, $\Upsilon_i = \langle \psi_i^{-1}(B_R(\mathrm{Id}_X) \cap \Gamma_0) \rangle$ *for* i *large enough*,
- *for* i *large enough, there are surjective morphisms* $\Gamma/\Gamma_0 \to \Gamma_i/\Upsilon_i$.

Proof. Let $r > 0$ be such that $B_{2r}(\mathrm{Id}_X) \subset \Gamma_0$. First we show that for any fixed $R \geq r$ and $\delta \in (0, r]$, the subgroup of Γ_i generated by $\psi_i^{-1}(B_\delta(\mathrm{Id}_X))$ in Γ_i coincides with $\langle \psi_i^{-1}(B_R(\mathrm{Id}_X) \cap \Gamma_0) \rangle$ for large enough i. To see this, first take a collection $y_1, \ldots, y_n \in B_R(\mathrm{Id}_X) \cap \Gamma_0$ with

$$B_R(\mathrm{Id}_X) \cap \Gamma_0 \subset \bigcup_{j=1}^{n} B_{\delta/10}(y_j).$$

By connectedness, for each $j \in \{1, \ldots, n\}$ we can construct a sequence $e = z_{j,0}, \ldots, z_{j,k_j} = y_j$ in Γ with $d(z_{j,\ell-1}, z_{j,\ell}) \leq \delta/10$ for each $\ell \in \{1, \ldots, k_j\}$. Since all $z_{j,\ell}$ are contained in a compact subset of Γ_0, if i is large enough, for any element $x \in \psi_i^{-1}(B_R(\mathrm{Id}_X) \cap \Gamma_0)$ we can find y_j with $d(y_j, \psi_i(x)) \leq \delta/10$, and $e = x_0, \ldots, x_{k_j} = x$ in Γ_i with $d(z_{j,\ell}, \psi_i(x_\ell)) \leq \delta/10$ for each ℓ. This allows us to write $x = (x_1)(x_1^{-1}x_2) \cdots (x_{k_j-1}^{-1}x_{k_j})$ as a product of k_j elements in $\psi_i^{-1}(B_\delta(\mathrm{Id}_X))$, proving our claim. Set Υ_i to be the subgroup of Γ_i generated by $\psi_i^{-1}(B_r(\mathrm{Id}_X))$.

Choose $\delta > 0$ small enough so that for all $g \in B_{3D}(\mathrm{Id}_X)$, $h \in B_\delta(\mathrm{Id}_X)$ one has $ghg^{-1} \in B_{r/2}(\mathrm{Id}_X)$. Then for large enough i, the conjugate of an element in $\psi_i^{-1}(B_\delta(\mathrm{Id}_X))$ by an element in $\psi_i^{-1}(B_{3D}(\mathrm{Id}_X))$ lies in $\psi_i^{-1}(B_r(\mathrm{Id}_X))$. By Proposition 2.51, $\psi_i^{-1}(B_{3D}(\mathrm{Id}_X))$ generates Γ_i and $\psi_i^{-1}(B_\delta(\mathrm{Id}_X))$ generates Υ_i for large enough i, implying that Υ_i is normal in Γ_i.

Let $S_0 = \{\bar{s}_1, \ldots, \bar{s}_k\} \subset \Gamma/\Gamma_0$ be a finite symmetric generating set containing all connected components intersecting $B_{4D}(\mathrm{Id}_X)$, $S = \{s_1, \ldots, s_k\} \subset \Gamma$ a set of representatives, and for each $j \in \{1, \ldots, k\}$, pick a sequence $g_i^j \in \Gamma_i$ with $\psi_i(g_i^j) \to s_j$. Then define $h_i' : S_0 \to \Gamma_i/\Upsilon_i$ as $h_i'(\bar{s}_j) := g_i^j \Upsilon_i \in \Gamma_i/\Upsilon_i$. It is easy to check that $h_i'(s_j)$ does not depend on the choices of the representatives s_j nor the sequences g_i^j for i large enough.

By hypothesis, Γ/Γ_0 admits a presentation $\langle S_0, W \rangle$ with W a finite set of words. For $\bar{s}_{i_1} \ldots \bar{s}_{i_\ell} \in W$, one has $d_0^p(\psi_i(g_i^{i_1}) \cdots \psi_i(g_i^{i_\ell}), \psi_i(g_i^{i_1} \cdots g_i^{i_\ell})) < r$ for i large enough, and hence $\psi_i(g_i^{i_1} \cdots g_i^{i_\ell}) \in \Gamma_0$. This means, again for i large enough, that $g_i^{i_1} \cdots g_i^{i_\ell} \in \Upsilon_i$, and $h_i'(s_{i_1}) \cdots h_i'(s_{i_\ell}) = g_i^{i_1} \cdots g_i^{i_\ell} \Upsilon_i = \Upsilon_i \in \Gamma_i/\Upsilon_i$. As there are only finitely many words in W, the functions $h_i' : S_0 \to \Gamma_i/\Upsilon_i$ extend to group morphisms $h_i : \Gamma/\Gamma_0 \to \Gamma_i/\Upsilon_i$. As S_0 intersects each connected component in $B_{4D}(\mathrm{Id}_X)$ and Γ_i is generated by $B_{3D}(\mathrm{Id}_{X_i})$, the maps h_i are surjective. $\square$

The following result deals with the base of induction in the proof of Theorem 1.14.

Lemma 3.2. *Let $(X_i, d_i, \mathfrak{m}_i, p_i)$ be a sequence of* RCD(K, N) *spaces of rectifiable dimension n, and $\Gamma_i \leq \mathrm{Iso}(X_i)$ a sequence of closed groups of isometries. Assume the sequence $(X_i, d_i, \mathfrak{m}_i, p_i)$ converges in the pointed measured Gromov–Hausdorff sense to a pointed* RCD(K, N) *space $(X, d, \mathfrak{m}, p)$ of rectifiable dimension n. If there is $D > 0$ such that $\Gamma_i = \mathcal{G}(\Gamma_i, X_i, p_i, D)$ for all i, and Γ_i converges equivariantly to the trivial group, then the groups Γ_i are trivial for i large enough.*

Proof. Clearly, we can assume $D > \sqrt{2}$. Let $\Upsilon_i \leq \Gamma_i$ be the subgroups given by Theorem 3.1. Then $\Gamma_i = \Upsilon_i = \langle \psi_i^{-1}(\mathrm{Id}_X) \rangle$ for i large enough. From the definition of equivariant convergence, it is easy to see that the ψ_i-preimage of an open compact subgroup of Γ is a subgroup in Γ_i for i large enough; hence $\Gamma_i = \psi_i^{-1}(\mathrm{Id}_X)$. This means that Γ_i are small subgroups in the sense of [Santos-Rodríguez and Zamora 2023, Definition 66 and Remark 75], so by [loc. cit., Theorem 93] the result follows. □

4. Proof of main regularity estimates on RLFs

In this section we prove Theorem 1.5 and Corollary 1.8, the key step being Lemma 4.1.

Proof of Theorem 1.5. Let $r_x \leq \rho/100$ be such that for all $r \leq r_x$ one has

$$\mathfrak{m}(\{y \in B_r(x) \mid H(y) \leq \delta\}) \geq \tfrac{1}{2}\mathfrak{m}(B_r(x)).$$

Lemma 4.1. *Fix $r \leq r_x$. If δ is small enough, there is $x_r \in B_r(x) \cap \{H \leq \delta\}$ and a constant $C_0(N) > 1$ which is independent of r such that*:

$S_r.1$ *There is $B_r'(x_r) \subseteq B_r(x_r)$ such that $\mathfrak{m}(B_r'(x_r)) \geq (1 - \sqrt{\delta})\mathfrak{m}(B_r(x_r))$ and*

$$\boldsymbol{X}_t(B_r'(x_r)) \subseteq B_{2r}(\boldsymbol{X}_t(x_r)) \quad \textit{for all } t \in [0, T].$$

$S_r.2$ *For all $t \in [0, T]$,*

$$\frac{1}{C_0}\mathfrak{m}(B_r(x_r)) \leq \mathfrak{m}(B_r(\boldsymbol{X}_t(x_r))) \leq C_0\mathfrak{m}(B_r(x_r)).$$

Lemma 4.1 is proven by an induction on time following the scheme of [Deng 2020, Section 5]. We now give an outline of this proof.

First choose $x_{r,0}$ so that $H(x_{r,0}) \leq \delta$. Then by the Bishop–Gromov inequality, the estimates $S_r.1$ and $S_r.2$ trivially hold for $x_r = x_{r,0}$ up to time $r/(10L)$. This serves as the base of induction. We then assume there is $x_{r,k}$ with $H(x_{r,k}) \leq \delta$ and such that $S_r.1$ and $S_r.2$ hold for $x_r = x_{r,k}$ along the interval $[0, kr/(10L)]$. The goal is then to show there is $x_{r,k+1}$ with $H(x_{r,k+1}) \leq \delta$ and such that $S_r.1$ and $S_r.2$ hold for $x_r = x_{r,k+1}$ along the interval $[0, t_k]$, where $t_k := \min\{(k+1)r/(10L), T\}$.

In order to achieve this, we first combine the fact that flow lines are L-Lipschitz with the inductive hypothesis to obtain integral estimates on $\mathrm{dt}_r(t_k)$ over carefully chosen sets (see (4.5) and (4.7)), from which we deduce that a significant portion of $B_{2r}(x)$ stays within $7r$ of $\boldsymbol{X}_t(x_{r,k})$ up to time t_k. This allows us to choose $x_{r,k+1} \in B_r(x) \cap \{H \leq \delta\}$ so that along the interval $[0, t_k]$, most of the flow lines starting at $B_r(x_{r,k+1})$ stay within $2r$ of $\boldsymbol{X}_t(x_{r,k+1})$. This is enough to guarantee that both $S_r.1$ and the first inequality of $S_r.2$ hold up to time t_k.

By the Bishop–Gromov inequality, we also have the other inequality, but with a worse constant. In order to improve this constant back to the original one, we perform the analysis of the previous paragraph but in the reverse direction using the flow $\overline{\boldsymbol{X}}$ given by Proposition 2.44. We show that for each $t \in [0, t_k]$, under the reverse flow $\overline{\boldsymbol{X}}_t$ a significant portion of $B_r(\boldsymbol{X}_t(x_{r,k+1}))$ returns to $B_{2r}(x_{r,k+1})$, and hence close to x. This is enough to improve the volume ratio to the desired constant.

We now turn to the actual proof, where for the sake of detail, we also present the case $k = 0$.

Proof of Lemma 4.1. Let $I_0 = [0, r/(10L)]$ and fix some $x_{r,0} \in B_r(x)$ with $H(x_{r,0}) \leq \delta$. By Remark 2.42, for any $y \in B_r(x)$ and any $t \in I_0$, we have

$$d(\boldsymbol{X}_t(y), \boldsymbol{X}_t(x_{r,0})) \leq d(\boldsymbol{X}_t(y), y) + d(y, x) + d(x, x_{r,0}) + d(x_{r,0}, \boldsymbol{X}_t(x_{r,0})) \leq 2L\frac{r}{10L} + 2r < 3r. \tag{4.2}$$

Define $\mathrm{dt}_r(t)$ as in (2.48), and set $S_1 = B_r(x) \cap \{H \leq \delta\}$, $S_2 = B_{2r}(x)$, and $\Gamma(t)$ as in (2.49). By Theorem 2.47, we obtain

$$\begin{aligned}\int_{S_1\times S_2} \mathrm{dt}_r\left(\frac{r}{10L}\right)(y,z)\, d(\mathfrak{m}\times\mathfrak{m})(y,z) \\ &= \int_{I_0} \frac{d}{dt} \int_{S_1\times S_2} \mathrm{dt}_r(t)(y,z)\, d(\mathfrak{m}\times\mathfrak{m})(y,z)\, dt \\ &\leq \int_{I_0}\int_0^1\int_{\Gamma(t)} d(\boldsymbol{X}_t(y), \boldsymbol{X}_t(z))|\nabla V(t)|(\gamma_{\boldsymbol{X}_t(y),\boldsymbol{X}_t(z)}(s))\, d(\mathfrak{m}\times\mathfrak{m})(y,z)\, ds\, dt. \end{aligned} \tag{4.3}$$

Using (2.46) and a change of variables, for any $t \in I_0$ we have

$$\begin{aligned}\int_0^1\int_{\Gamma(t)} d(\boldsymbol{X}_t(y), \boldsymbol{X}_t(z))|\nabla V(t)|(\gamma_{\boldsymbol{X}_t(y),\boldsymbol{X}_t(z)}(s))\, d(\mathfrak{m}\times\mathfrak{m})(y,z)\, ds \\ \leq e^{DT}\int_0^1\int_{\boldsymbol{X}_t(\Gamma(t))} d(y,z)|\nabla V(t)|(\gamma_{y,z}(s))\, d(\mathfrak{m}\times\mathfrak{m})(y,z)\, ds.\end{aligned}$$

Furthermore,

$$\begin{aligned} e^{DT}\int_0^1\int_{\boldsymbol{X}_t(\Gamma(t))} d(y,z)|\nabla V(t)|(\gamma_{y,z}(s))\, d(\mathfrak{m}\times\mathfrak{m})(y,z)\, ds \\ &\leq e^{DT}\int_0^1\int_{B_{6r}(\boldsymbol{X}_t(x_{r,0}))^{\times 2}} d(y,z)|\nabla V(t)|(\gamma_{y,z}(s))\, d(\mathfrak{m}\times\mathfrak{m})(y,z)\, ds \\ &\leq e^{DT}C(N)r\mathfrak{m}(B_{6r}(\boldsymbol{X}_t(x_{r,0})))^2 \fint_{B_{12r}(\boldsymbol{X}_t(x_{r,0}))} |\nabla V(t)|\, d\mathfrak{m} \\ &\leq e^{DT}C(N)r\mathfrak{m}(B_r(x))^2 \fint_{B_{12r}(\boldsymbol{X}_t(x_{r,0}))} |\nabla V(t)|\, d\mathfrak{m}, \end{aligned}$$

where we used (4.2) for the second line, Theorem 2.10 for the third line, and Theorem 2.1 for the fourth line. Combining the above estimates starting from (4.3), we obtain

$$\begin{aligned}\int_{S_1\times S_2} \mathrm{dt}_r\left(\frac{r}{10L}\right)(y,z)\, d(\mathfrak{m}\times\mathfrak{m})(y,z) &\leq e^{DT}C(N)r\mathfrak{m}(B_r(x))^2\int_{I_0}\fint_{B_{12r}(\boldsymbol{X}_t(x_{r,0}))} |\nabla V(t)|\, d\mathfrak{m}\, dt \\ &\leq e^{DT}C(N)r\mathfrak{m}(B_r(x))^2\int_{I_0} \mathrm{Mx}_\rho(|\nabla V(t)|)(\boldsymbol{X}_t(x_{r,0}))\, dt \\ &\leq e^{DT}C(N)r\mathfrak{m}(B_r(x))^2 H(x_{r,0}) \leq e^{DT}C(N)r\mathfrak{m}(B_r(x))^2\delta.\end{aligned}$$

By Chebyshev's inequality, there is $x_{r,1} \in B_r(x) \cap \{H \le \delta\}$ with

$$\int_{B_{2r}(x)} \mathrm{dt}_r\left(\frac{r}{10L}\right)(x_{r,1}, y)\, d\mathfrak{m}(y) \le e^{DT} C(N) r \mathfrak{m}(B_r(x))\delta.$$

By Theorem 2.1, we have

$$\mathfrak{m}(B_r(x_{r,1})) \ge \frac{1}{C(N)}\mathfrak{m}(B_r(x)),$$

thus another instance of Chebyshev's inequality implies there is $B_{r,1}(x_{r,1}) \subseteq B_r(x_{r,1})$ with $\mathfrak{m}(B_{r,1}(x_{r,1})) \ge (1-\sqrt{\delta})\mathfrak{m}(B_r(x_{r,1}))$ and

$$\mathrm{dt}_r\left(\frac{r}{10L}\right)(x_{r,1}, z) \le e^{DT} C(N) r\sqrt{\delta} \quad \text{for all } z \in B_{r,1}(x_{r,1}).$$

Hence if $e^{DT}C(N)\sqrt{\delta} < 1$, then by the definition of $\mathrm{dt}_r(t)$, for all $t \in I_0$ and $z \in B_{r,1}(x_{r,1})$ we have

$$d(\boldsymbol{X}_t(x_{r,1}), \boldsymbol{X}_t(z)) < d(x, z) + e^{DT}C(N)r\sqrt{\delta} < 2r,$$

so $\boldsymbol{X}_t(B_{r,1}(x_{r,1})) \subset B_{2r}(\boldsymbol{X}_t(x_{r,1}))$ for all $t \in I_0$. As

$$B_{r/2}(x_{r,1}) \subset B_r(\boldsymbol{X}_t(x_{r,1})) \subset B_{3r/2}(x_{r,1}) \quad \text{for all } t \in I_0,$$

from Theorem 2.1 we have, for all $t \in I_0$,

$$\frac{1}{C}\mathfrak{m}(B_r(x_{r,1})) \le \mathfrak{m}(B_r(\boldsymbol{X}_t(x_{r,1}))) \le C\mathfrak{m}(B_r(x_{r,1})).$$

The argument above establishes $S_r.1$ and $S_r.2$ up to time $r/(10L)$. Now we show we can establish the same estimate up to time T provided δ is small enough.

Let $k \in \mathbb{N}$ with $k < \lceil 10TL/r \rceil$, and assume there is $x_{r,k} \in B_r(x)$ such that

$S_{r,k}.1$ There exists $B'_r(x_{r,k}) \subseteq B_r(x_{r,k})$ with $\mathfrak{m}(B'_r(x_{r,k})) \ge (1-\sqrt{\delta})\mathfrak{m}(B_r(x_{r,k}))$ and

$$\boldsymbol{X}(B'_r(x_{r,k})) \subset B_{2r}(x_{r,k}) \quad \text{for all } t \in \left[0, \frac{kr}{10L}\right].$$

$S_{r,k}.2$ For all $t \in [0, kr/(10L)]$,

$$\frac{1}{C_0}\mathfrak{m}(B_r(x_{r,k})) \le \mathfrak{m}(B_r(\boldsymbol{X}_t(x_{r,k}))) \le C_0\mathfrak{m}(B_r(x_{r,k})).$$

Set

$$t_k := \min\left\{\frac{(k+1)r}{10L}, T\right\} \quad \text{and} \quad I_k := [0, t_k].$$

From $S_{r,k}.1$ and Remark 2.42, for all $t \in I_k$ we have

$$\boldsymbol{X}_t(B'_r(x_{r,k})) \subset B_{2r+r/2}(\boldsymbol{X}_t(x_{r,k})). \tag{4.4}$$

Let $S_1 := B'_r(x_{r,k})$, $S_2 := B_{2r}(x)$, and $\Gamma(t)$ be given by (2.49). By Theorem 2.47,

$$\begin{aligned}
\int_{S_1\times S_2} &\mathrm{dt}_r(t_k)(y,z)\,d(\mathfrak{m}\times\mathfrak{m})(y,z)\\
&= \int_{I_k}\frac{d}{dt}\int_{S_1\times S_2}\mathrm{dt}_r(t)(y,z)\,d(\mathfrak{m}\times\mathfrak{m})(y,z)\,dt\\
&\le \int_{I_k}\int_0^1\int_{\Gamma(t)} d(\boldsymbol{X}_t(y),\boldsymbol{X}_t(z))|\nabla V(t)|(\gamma_{\boldsymbol{X}_t(y),\boldsymbol{X}_t(z)}(s))\,d(\mathfrak{m}\times\mathfrak{m})(y,z)\,ds\,dt\\
&\le \int_{I_k} e^{DT}\int_0^1\int_{\boldsymbol{X}_t(\Gamma(t))} d(y,z)|\nabla V(t)|(\gamma_{y,z}(s))\,d(\mathfrak{m}\times\mathfrak{m})(y,z)\,ds\,dt\\
&\le \int_{I_k} e^{DT}\int_0^1\int_{B_{6r}(\boldsymbol{X}_t(x_{r,k}))^{\times 2}} d(y,z)|\nabla V(t)|(\gamma_{y,z}(s))\,d(\mathfrak{m}\times\mathfrak{m})(y,z)\,ds\,dt\\
&\le \int_{I_k} e^{DT}C(N)r\mathfrak{m}(B_{12r}(\boldsymbol{X}_t(x_{r,k})))^2 \fint_{B_{6r}(\boldsymbol{X}_t(x_{r,k}))}|\nabla V(t)|\,d\mathfrak{m}\,dt\\
&\le \int_{I_k} e^{DT}C_0^2C(N)r\mathfrak{m}(B_r(x))^2 \fint_{B_{12r}(\boldsymbol{X}_t(x_{r,k}))}|\nabla V(t)|\,d\mathfrak{m}\,dt. \qquad (4.5)
\end{aligned}$$

where we used (2.46), (4.4), Theorem 2.10, and Theorem 2.1 with (4.4). From the above estimates we get

$$\begin{aligned}
\int_{S_1\times S_2}\mathrm{dt}_r(t_k)(y,z)\,d(\mathfrak{m}\times\mathfrak{m})(y,z) &\le e^{DT}C_0^2C(N)r\mathfrak{m}(B_r(x))^2\int_{I_k}\fint_{B_{12r}(\boldsymbol{X}_t(x_{r,k}))}|\nabla V(t)|\,d\mathfrak{m}\,dt\\
&\le e^{DT}C_0^2C(N)r\mathfrak{m}(B_r(x))^2\int_{I_k}\mathrm{Mx}_\rho(|\nabla V(t)|)(\boldsymbol{X}_t(x_{r,k}))\,dt\\
&\le e^{DT}C_0^2C(N)r\mathfrak{m}(B_r(x))^2 H(x_{r,k})\\
&\le e^{DT}C_0^2C(N)r\mathfrak{m}(B_r(x))^2\delta.
\end{aligned}$$

By Chebyshev's inequality, there is some $x' \in B'_r(x_{r,k})$ with

$$\int_{B_{2r}(x)}\mathrm{dt}_r(t_k)(x',y)\,d\mathfrak{m}(y) \le e^{DT}C_0^2C(N)r\mathfrak{m}(B_r(x))\delta.$$

Thus there are $C(D,T,N) > 1$ and $B_k \subset B_{2r}(x)$ with

$$\begin{aligned}
\mathrm{dt}_r(t_k)(x',y) &\le Cr\sqrt{\delta} \quad \text{for all } y\in B_k,\\
\mathfrak{m}(B_k) &\ge (1-\sqrt{\delta}/2)\mathfrak{m}(B_{2r}(x)). \qquad (4.6)
\end{aligned}$$

As $x' \in B'_r(x_{r,k})$, from (4.4) we have $\boldsymbol{X}_t(x') \in B_{2r+r/2}(\boldsymbol{X}_t(x_{r,k}))$ for all $t \in I_k$, provided $C\sqrt{\delta} < 1$. Thus for all $t \in I_k$, we also have

$$\boldsymbol{X}_t(B_k) \subset B_{7r}(\boldsymbol{X}_t(x_{r,k})).$$

Define $S_1 = B_k$, $S_2 = B_r(x) \cap \{H \le \delta\}$, and $\Gamma(t)$ as in (2.49). Similar to before, we have

$$\begin{aligned}\int_{S_1\times S_2} \mathrm{dt}_r(t_k)(y,z)\,d(\mathfrak{m}\times\mathfrak{m})(y,z) &\le e^{DT}C_0^2C(N)r\mathfrak{m}(B_r(x))^2\int_{I_k}\fint_{B_{20r}(\boldsymbol{X}_t(x_{r,k}))}|\nabla V(t)|\,d\mathfrak{m}\,dt\\ &\le e^{DT}C_0^2C(N)r\mathfrak{m}(B_r(x))^2\int_{I_k}\mathrm{Mx}_\rho(|\nabla V|)(\boldsymbol{X}_t(x_{r,k}))\,dt\\ &\le e^{DT}C_0^2C(N)r\mathfrak{m}(B_r(x))^2\delta. \end{aligned}\tag{4.7}$$

Thus there is $x_{r,k+1} \in B_r(x)\cap\{H \le \delta\}$ such that

$$\int_{B_k}\mathrm{dt}_r(t_k)(x_{r,k+1},y)\,d\mathfrak{m}(y) \le e^{DT}C_0^2C(N)r\mathfrak{m}(B_r(x))\delta.$$

From (4.6) and Theorem 2.1, there are $C(D,T,N) > 1$ and $B'_r(x_{r,k+1}) \subset B_r(x_{r,k+1})$ with

$$\begin{aligned}\mathrm{dt}_r(t_k)(x_{r,k+1},y) \le C\sqrt{\delta}r \quad \text{for all } y \in B'_r(x_{r,k+1}),\\ \mathfrak{m}(B'_r(x_{r,k+1})) \ge (1-\sqrt{\delta})\mathfrak{m}(B_r(x_{r,k+1})).\end{aligned}$$

Thus if $C\sqrt{\delta} < 1$, for all $t \in I_k$ we have

$$\boldsymbol{X}_t(B'_r(x_{r,k+1})) \subseteq B_{2r}(\boldsymbol{X}_t(x_{r,k+1})).$$

Also, from Theorem 2.1 and (2.46) we have, for some $C(D,T,N) > 1$,

$$\begin{aligned}\mathfrak{m}(B_r(\boldsymbol{X}_t(x_{r,k+1}))) &\ge \frac{1}{C}\mathfrak{m}(B_{2r}(\boldsymbol{X}_t(x_{r,k+1}))) \ge \frac{1}{C}\mathfrak{m}(\boldsymbol{X}_t(B'_r(x_{r,k+1})))\\ &\ge \frac{1}{C}\mathfrak{m}(B_r(x_{r,k+1})).\end{aligned}\tag{4.8}$$

To obtain the other direction of the volume estimate corresponding to S_r.2, we consider the reversal of the flow. Fix $t \in I_k$, define $\overline{V} \in L^1([0,t]; H^{1,2}_{C,s}(TX))$ as

$$\overline{V}(s) := -V(t-s) \quad \text{for all } s \in [0,t],$$

and let $\overline{\boldsymbol{X}} : [0,t]\times X \to X$ be its RLF. Define $\mathrm{dt}'_r : [0,t]\times X\times X \to [0,r]$ as

$$\mathrm{dt}'_r(s)(y,z) := \sup_{0\le u\le s}\mathrm{dt}_r(\overline{\boldsymbol{X}}_u)(y,z),$$

$S_1 = \boldsymbol{X}_t(B'_r(x_{r,k+1}))$, and $S_2 = B_r(\boldsymbol{X}_t(x_{r,k+1}))$. Similar to before we have

$$\begin{aligned}\int_{S_1\times S_2}\mathrm{dt}'_r(t)(y,z)\,d(\mathfrak{m}\times\mathfrak{m})(y,z) &\le e^{DT}C_0^2C(N)r\mathfrak{m}(B_r(x_{r,k+1}))^2\int_0^t\fint_{B_{20r}(\boldsymbol{X}_{t-s}(x_{r,k+1}))}|\nabla\overline{V}(s)|\,d\mathfrak{m}\,ds\\ &\le e^{DT}C_0^2C(N)r\mathfrak{m}(B_r(x))^2\int_0^t\mathrm{Mx}_\rho(|\nabla V(s)|)(\boldsymbol{X}_{t-s}(x_{r,k+1}))\,ds\\ &\le e^{DT}C_0^2C(N)r\mathfrak{m}(B_r(x))^2\delta.\end{aligned}$$

Thus we have $x'' \in \boldsymbol{X}_t(B'_r(x_{r,k+1}))$ with

$$\int_{B_r(\boldsymbol{X}_t(x_{r,k+1}))} \mathrm{dt}'_r(t)(x'', y)\, d\mathfrak{m}(y) \le e^{DT} C_0^2 C(N) r \mathfrak{m}(B_r(x))\delta.$$

Hence there are $C(D, T, N) > 0$ and $A' \subset B_r(\boldsymbol{X}_t(x_{r,k+1}))$ such that

$$\begin{aligned} \mathrm{dt}'_r(t)(x'', y) &\le Cr\sqrt{\delta} \quad \text{for all } y \in A', \\ \mathfrak{m}(A') &\ge (1-\sqrt{\delta})\mathfrak{m}(B_r(\boldsymbol{X}_t(x_{r,k+1}))). \end{aligned}$$

Thus we have, for some $C(D, T, N) > 1$,

$$\begin{aligned} \mathfrak{m}(B_r(x_{r,k+1})) &\ge \frac{1}{C}\mathfrak{m}(B_{2r}(x_{r,k+1})) \ge \frac{1}{C}\mathfrak{m}(\overline{\boldsymbol{X}}_t(A')) \\ &\ge \frac{1}{C}\mathfrak{m}(B_r(\boldsymbol{X}_t(x_{r,k+1}))). \end{aligned}$$

Combining with (4.8) we have the desired volume bound, concluding the induction step. The result follows by taking $x_r := x_{r,k}$ with $k = \lfloor 10TL/r \rfloor$. □

Take

$$\begin{aligned} x_r &\in B_r(x) \cap \{H \le \delta\}, & B'_r(x_r) &\subset B_r(x_r), \\ x_{r/2} &\in B_{r/2}(x) \cap \{H \le \delta\}, & B'_{r/2}(x_{r/2}) &\subset B_{r/2}(x_{r/2}) \end{aligned}$$

given by Lemma 4.1. That is, they satisfy $S_r.1$ and $S_r.2$ with r and $r/2$ respectively. We claim that

$$d(\boldsymbol{X}_t(x_r), \boldsymbol{X}_t(x_{r/2})) \le 20r \quad \text{for all } t \in [0, T]. \tag{4.9}$$

Take $S_1 = B_r(x)$ and $S_2 = B'_{r/2}(x_{r/2})$. Arguing as before, we can find $x' \in B'_{r/2}(x_{r/2})$ with

$$\int_{B_r(x)} \mathrm{dt}_r(T)(x', y)\, d\mathfrak{m}(y) \le e^{DT} C(N) r \mathfrak{m}(B_r(x))\delta,$$

and $B''_r(x) \subset B_r(x)$ such that

$$\mathrm{dt}_r(T)(x', y) < r \quad \text{for all } y \in B''_r(x), \tag{4.10}$$

$$\mathfrak{m}(B''_r(x)) \ge (1-\sqrt{\delta})\mathfrak{m}(B_r(x)). \tag{4.11}$$

Hence for all $t \in [0, T]$, $y \in B'_r(x)$, using (4.10) we have

$$\begin{aligned} d(\boldsymbol{X}_t(x_{r/2}), \boldsymbol{X}_t(y)) &\le d(\boldsymbol{X}_t(x_{r/2}), \boldsymbol{X}_t(x')) + d(\boldsymbol{X}_t(x'), \boldsymbol{X}_t(y)) \\ &\le r + r + d(x', y) \le 4r. \end{aligned} \tag{4.12}$$

In a similar fashion, one can find a subset $B'''_r(x) \subset B_r(x)$ with

$$\mathbf{X}_t(B'''_r(x)) \subset B_{10r}(\boldsymbol{X}_t(x_r)), \tag{4.13}$$

$$\mathfrak{m}(B'''_r(x)) \ge (1-\sqrt{\delta})\mathfrak{m}(B_r(x)). \tag{4.14}$$

From (4.11) and (4.14), one can find $z \in B''_r(x) \cap B'''_r(x)$. Then from (4.12) and (4.13) applied to z, we conclude (4.9).

Notice that by iterated applications of (4.9), for $r_1, r_2 \leq r_x$ and $t \in [0, T]$, one gets

$$d(\boldsymbol{X}_t(x_{r_1}), \boldsymbol{X}_t(x_{r_2})) \leq 100 \max\{r_1, r_2\}. \tag{4.15}$$

Now we will use what we have proven so far to construct an adjusted representative $\widetilde{\boldsymbol{X}}$ of the RLF to V with the property that any $x \in X$ satisfying (1.6) also satisfies $S.1$ and $S.2$ for r sufficiently small. Let $S \subset X$ denote the set of x satisfying (1.6) and for each $x \in S$ define $r_x \leq \rho/100$ such that for all $r \leq r_x$ one has

$$\frac{\mathfrak{m}(\{H > \delta\} \cap B_r(x))}{\mathfrak{m}(B_r(x))} \leq \tfrac{1}{2}.$$

As the construction of r_x only uses measurable functions, guaranteed from Kuratowski–Ryll-Nardzewski measurable selection theorem (see for example [Deng 2020, Remark 2.26]) we can take a measurable choice of $r_x : S \to \mathbb{R}$. Moreover, the same is true for x_r given by Lemma 4.1, allowing us to define a measurable map $\tilde{x} : \mathbb{R}^+ \times X \to X$ as

$$\tilde{x}(r, x) := \begin{cases} x_r & \text{if } x \in S,\ r \leq r_x, \\ x & \text{otherwise.} \end{cases}$$

Then let us define the adjusted flow $\widetilde{\boldsymbol{X}} : [0, T] \times X \to X$ as

$$\widetilde{\boldsymbol{X}}(x, t) = \lim_{r \to 0} \boldsymbol{X}(\tilde{x}(r, x), t).$$

By (4.15), the limit exists and satisfies $S.1$ and $S.2$ for all $x \in S$ and $r \leq r_x$. Now we need to verify that $\widetilde{\boldsymbol{X}}$ is also a regular Lagrangian flow.

$R.1$ holds as (2.43) passes to the limit trajectories. Given $x \in S$, $r \leq r_x$, choose a set $A_r(x) \subset B_r(x)$ satisfying $S.1$, and consider the probability measures

$$\mu_{r,x}(t) = (\boldsymbol{X}_t)_* \left(\frac{\chi_{A_r(x)}}{\mathfrak{m}(A_r(x))} \mathfrak{m} \right).$$

From the definition of $\boldsymbol{X}$, for all $f \in \operatorname{TestF}(X)$ we have

$$\frac{d}{dt} \int_X f \, d\mu_{r,x}(t) = \int_X df(V(t)) \, d\mu_{r,x}(t). \tag{4.16}$$

Also notice that, by $S.1$, for all $t \in [0, T]$ we have

$$\operatorname{supp}(\mu_{r,x}(t)) \subset B_{2r}(\widetilde{\boldsymbol{X}}_t(x)), \tag{4.17}$$

and, by $S.2$, the map $\mu_{r,x}(t) \xmapsto{\theta} \widetilde{\boldsymbol{X}}_t(x)$ makes

$$\{\mu_{r,x}(t) \mid x \in S, r \leq r_x, t \in [0, T]\}$$

a family of bounded eccentricity. Let $f \in \operatorname{TestF}(X)$ and $t \in [0, T]$. From Lemma 2.12 we have, for $\mathfrak{m}$-a.e. $x \in S$,

$$df(V(t))(\widetilde{\boldsymbol{X}}_t(x)) = \lim_{r \to 0} \int_X df(V(t)) \, d\mu_{r,x}(t).$$

Hence, for all $t_0, t_1 \in [0, T]$ and $\mathfrak{m}$-a.e. $x \in S$,

$$\begin{aligned}\int_{t_0}^{t_1} df(V(t))(\widetilde{\boldsymbol{X}}_t(x))dt &= \lim_{r\to 0}\int_{t_0}^{t_1}\int_X df(V(t))\, d\mu_{r,x}(t)\\ &= \lim_{r\to 0}\int_X f\, d\mu_{r,x}(t_1) - \int_X f\, d\mu_{r,x}(t_0)\\ &= f(\widetilde{\boldsymbol{X}}_{t_1}(x)) - f(\widetilde{\boldsymbol{X}}_{t_0}(x)),\end{aligned}$$

where we used dominated convergence on the first two lines, (4.16) on the second, and (4.17) on the third. This implies $\widetilde{\boldsymbol{X}}$ satisfies $R.2$ for $\mathfrak{m}$-almost all $x \in S$. Since $\widetilde{\boldsymbol{X}}$ and $\boldsymbol{X}$ coincide on $X\backslash S$, and $\boldsymbol{X}$ satisfies $R.2$, we deduce $\widetilde{\boldsymbol{X}}$ satisfies $R.2$ as well.

Let $S_k := \left\{x \in S \mid r_x \geq \frac{1}{k}\right\}$. For all $r \leq \frac{1}{k}$, $y \in X$, we have

$$\begin{aligned}\int_{S_k} \frac{\chi_{\boldsymbol{X}_t(A_r(x))}(y)}{\mathfrak{m}(A_r(x))}\, d\mathfrak{m}(x) &\leq M^2 \int_{S_k} \frac{\chi_{B_{2r}(y)}(\boldsymbol{X}_t(x))}{\mathfrak{m}(B_r(\boldsymbol{X}_t(x)))}\, d\mathfrak{m}(x)\\ &\leq M^2 \int_{B_{2r}(y)} \frac{1}{\mathfrak{m}(B_r(z))}\, d((\boldsymbol{X}_t)_*(\mathfrak{m}))(z)\\ &\leq M^2 e^{DT} \int_{B_{2r}(y)} \frac{1}{\mathfrak{m}(B_r(z))}\, d\mathfrak{m}(z)\\ &\leq M^2 e^{DT} C(N),\end{aligned} \tag{4.18}$$

where the first line follows from $S.1$ and $S.2$, the second from a change of variables, the third from (2.46), and the fourth from Theorem 2.1. Then, given any $0 \leq f \in \mathrm{TestF}(X)$, we compute

$$\begin{aligned}\int_X f\, d((\widetilde{\boldsymbol{X}}_t)_*(\mathfrak{m}|_{S_k})) &= \int_{S_k} (f \circ \widetilde{\boldsymbol{X}}_t)\, d\mathfrak{m}\\ &\leq \lim_{r\to 0}\int_{S_k}\int_X f(y)\, d\mu_{r,x}(t)(y)\, d\mathfrak{m}(x)\\ &= \lim_{r\to 0}\int_X f(y) \int_{S_k} \frac{\chi_{X_t(A_r(x))}(y)}{\mathfrak{m}(A_r(x))}\, d\mathfrak{m}(x)\, d((\boldsymbol{X}_t)_*\mathfrak{m})(y)\\ &\leq M^2 e^{DT} C(N) \int_X f(y)\, d((\boldsymbol{X}_t)_*\mathfrak{m})(y)\\ &\leq M^2 e^{2DT} C(N) \int_X f\, d\mathfrak{m},\end{aligned} \tag{4.19}$$

where we used (4.17) on the second line, Tonelli's theorem on the third, (4.18) on the fourth, and (2.46) on the fifth. Since f was arbitrary, (4.19) implies that $(\widetilde{\boldsymbol{X}}_t)_*\mathfrak{m}|_{S_k} \leq \mathfrak{C}\mathfrak{m}$ for some $\mathfrak{C}(D, T, N) > 0$. Hence

$$\begin{aligned}(\widetilde{\boldsymbol{X}}_t)_*\mathfrak{m} &= (\widetilde{\boldsymbol{X}}_t)_*(\mathfrak{m}|_{X\backslash S}) + (\widetilde{\boldsymbol{X}}_t)_*(\mathfrak{m}|_S)\\ &= (\boldsymbol{X}_t)_*(\mathfrak{m}|_{X\backslash S}) + \lim_{k\to\infty}(\widetilde{\boldsymbol{X}}_t)_*(\mathfrak{m}|_{S_k})\\ &\leq (e^{DT} + \mathfrak{C})\mathfrak{m},\end{aligned}$$

establishing $R.3$ for $\widetilde{\boldsymbol{X}}$, so we conclude $\widetilde{\boldsymbol{X}}$ is an RLF for b. ☐

Proof of Corollary 1.8. By Proposition 2.5(2), for all $s \in [1, R-1]$, one has

$$\fint_{B_s(p)} \mathrm{Mx}(|\nabla V|)^2 \, d\mathfrak{m} \leq C(N)\eta.$$

Define $H : X \to \mathbb{R}$ as $H(x) := \int_0^T \mathrm{Mx}(|\nabla V|)(\boldsymbol{X}_t(x))\, dt$. Then by the Cauchy-Schwarz inequality and (2.46) one gets

$$\begin{aligned}
\left[\fint_{B_r(p)} H(x)\, d\mathfrak{m}(x)\right]^2 &\leq \fint_{B_r(p)} \left[\int_0^T \mathrm{Mx}(|\nabla V|)(\boldsymbol{X}_t(x))\, dt\right]^2 d\mathfrak{m}(x) \\
&\leq T \fint_{B_r(p)} \int_0^T \mathrm{Mx}(|\nabla V|)^2(\boldsymbol{X}_t(x))\, dt\, d\mathfrak{m}(x) \\
&\leq Te^{2DT} \int_0^T \fint_{\boldsymbol{X}_t(B_r(x))} \mathrm{Mx}(|\nabla V|)^2(\boldsymbol{X}_t(x))\, d\mathfrak{m}(x)\, dt \\
&\leq C(D, T, N) \int_0^T \fint_{B_{r+LT}(x)} \mathrm{Mx}(|\nabla V|)^2(\boldsymbol{X}_t(x))\, d\mathfrak{m}(x)\, dt \\
&\leq C(D, T, N)\eta.
\end{aligned} \tag{4.20}$$

Let $\delta(D, T, N) > 0$ be given by Theorem 1.5. From (4.20), there is $C(D, T, N) > 1$ such that

$$\mathfrak{m}(\{H \leq \delta\} \cap B_r(p)) \geq (1 - C\sqrt{\eta})\, \mathfrak{m}(B_r(p)). \tag{4.21}$$

By Theorem 1.5, G contains the density points of $\{H \leq \delta\}$, so the result follows from (4.21) provided $\eta \leq \varepsilon^2/C^2$. □

5. Self-improving stability

In this section, we show that combining essential stability with integral control on the covariant derivative of the corresponding vector field, one can improve the conditions of essential stability to much better estimates. For example, one could compare Corollary 5.13 with $S.2$ and Proposition 5.14 with $S.1$.

This improvement is attained by induction on the radius. Roughly speaking, if for some small $r > 0$ one has $S.1$, $S.2$, and enough control on the covariant derivative of the vector field, then at a scale slightly larger than r, one can obtain conditions similar to $S.1$ and $S.2$ but with better constants. This is the content of Lemma 5.1 (cf. [Kapovitch and Wilking 2011, Lemma 3.7; Colding and Naber 2012, Proposition 3.6]).

Lemma 5.1. *For each $N \geq 1$, $M \geq 1$, there are $\lambda(N) \geq 4$, $\varepsilon(N, M) > 0$, such that the following holds. Let $(X, d, \mathfrak{m})$ be an* RCD$(-(N-1), N)$ *space, $x \in X$, $V \in L^1([0, T]; H^{1,2}_{C,s}(TX))$ a divergence-free vector field, and $\boldsymbol{X} : [0, T] \times X \to X$ the RLF of V. Assume*

$$\int_0^T Mx_4(|\nabla V(t)|)(\boldsymbol{X}_t(x))\, dt \leq \varepsilon.$$

For some $r \leq 1/\lambda$ one has

$$\frac{1}{M}\mathfrak{m}(B_r(x)) \leq \mathfrak{m}(B_r(\boldsymbol{X}_t(x))) \leq M\mathfrak{m}(B_r(x)) \quad \textit{for all } t \in [0, T], \tag{5.2}$$

and there is $S_r \subset B_r(x)$ with

$$\mathfrak{m}(S_r) \geq \frac{1}{M}\mathfrak{m}(B_r(x)),$$
$$\boldsymbol{X}_t(S_r) \subset B_{2r}(\boldsymbol{X}_t(x)) \quad \text{for all } t \in [0, T].$$

Then

$$\tfrac{1}{2}\mathfrak{m}(B_{\lambda r}(x)) \leq \mathfrak{m}(B_{\lambda r}(\boldsymbol{X}_t(x))) \leq 2\mathfrak{m}(B_{\lambda r}(x)) \quad \text{for all } t \in [0, T], \tag{5.3}$$

and there is $A_{\lambda r} \subset B_{\lambda r}(x)$ with

$$\boldsymbol{X}_t(A_{\lambda r}) \subset B_{(\lambda+4)r}(\boldsymbol{X}_t(x)) \qquad \text{for all } t \in [0, T], \tag{5.4}$$

$$\mathfrak{m}(\boldsymbol{X}_t(A_{\lambda r}) \cap B_{\lambda r}(\boldsymbol{X}_t(x))) \geq \tfrac{9}{10}\mathfrak{m}(B_{\lambda r}(\boldsymbol{X}_t(x))) \quad \text{for all } t \in [0, T]. \tag{5.5}$$

Proof. Pick $\lambda(N) \geq 5$ such that

$$\mathfrak{m}(B_{(\lambda+5)s}(y)) \leq \tfrac{101}{100}\mathfrak{m}(B_{\lambda s}(y)) \tag{5.6}$$

for all $y \in X$, $s \leq 10/\lambda$. With this choice of λ, (5.3) will follow from (5.4) and (5.5). Let $\mathrm{dt}_r(t)$ be given by (2.48) and

$$\Gamma(t) := \{(a, b) \in S_r \times B_{\lambda r}(x) \mid \mathrm{dt}_r(t)(a, b) < r\}.$$

Notice that for each $t \in [0, T]$, $(a, b) \in \Gamma(t)$, one has $d(\boldsymbol{X}_t(a), \boldsymbol{X}_t(x)) \leq 2r$, and

$$\begin{aligned} d(\boldsymbol{X}_t(b), \boldsymbol{X}_t(x)) &\leq d(\boldsymbol{X}_t(b), \boldsymbol{X}_t(a)) + d(\boldsymbol{X}_t(a), \boldsymbol{X}_t(x)) \\ &\leq d(a, b) + r + 2r \\ &\leq \lambda r + 4r. \end{aligned} \tag{5.7}$$

Then

$$\begin{aligned} &\fint_{S_r \times B_{\lambda r}(x)} \mathrm{dt}_r(T)\, d(\mathfrak{m} \times \mathfrak{m}) \\ &\quad \leq \frac{M}{\mathfrak{m}(B_r(x))^2} \int_0^T \int_{\Gamma(t)} d(\boldsymbol{X}_t(y), \boldsymbol{X}_t(z)) \left[\int_0^1 |\nabla V(t)|(\gamma_{\boldsymbol{X}_t(y), \boldsymbol{X}_t(z)}(s))\, ds\right] d(\mathfrak{m} \times \mathfrak{m})(y, z)\, dt \\ &\quad \leq \frac{M}{\mathfrak{m}(B_r(x))^2} \int_0^T \int_{\boldsymbol{X}_t(\Gamma(t))} d(y, z) \left[\int_0^1 |\nabla V(t)|(\gamma_{y,z}(s))\, ds\right] d(\mathfrak{m} \times \mathfrak{m})(y, z)\, dt \\ &\quad \leq \frac{M}{\mathfrak{m}(B_r(x))^2} \int_0^T \int_{B_{(\lambda+4)r}(\boldsymbol{X}_t(x))^{\times 2}} d(y, z) \left[\int_0^1 |\nabla V(t)|(\gamma_{y,z}(s))\, ds\right] d(\mathfrak{m} \times \mathfrak{m})(y, z)\, dt \\ &\quad \leq C(N) \cdot M^3 \cdot r \cdot \int_0^T \fint_{B_{(2\lambda+8)r}(\boldsymbol{X}_t(x))} |\nabla V(t)|(y)\, d\mathfrak{m}(y)\, dt \\ &\quad \leq C(N) \cdot M^3 \cdot r \cdot \int_0^T \mathrm{Mx}_4(|\nabla V(t)|)(\boldsymbol{X}_t(x))\, dt \leq C(N) M^3 \varepsilon r, \end{aligned} \tag{5.8}$$

where the first inequality follows from Theorem 2.47, the second from Tonelli's theorem, the third from (5.7), and the fourth from Theorem 2.10 and (5.2). Hence there is $y \in S_r$ with

$$\fint_{B_{\lambda r}(x)} \mathrm{dt}_r(T)(y,z)\,d\mathfrak{m}(z) \leq C(N)M^3\varepsilon r. \tag{5.9}$$

We can then define $A_{\lambda r} := \{z \in B_{\lambda r}(x) \mid \mathrm{dt}_r(T)(y,z) < r\}$, which by (5.7) satisfies (5.4), and by (5.9) satisfies

$$\mathfrak{m}(A_{\lambda r}) \geq (1 - C(N)M^3\varepsilon)\mathfrak{m}(B_{\lambda r}(x)) \geq \tfrac{99}{100}\mathfrak{m}(B_{\lambda r}(x)), \tag{5.10}$$

provided ε is small enough. To verify (5.5), fix $t \in [0, T]$, consider the vector field $\overline{V} \in L^1([0,t], H^{1,2}_{C,s}(TX))$ given by $\overline{V}(s) := -V(t-s)$, and $\overline{\boldsymbol{X}} : [0,t] \times X \to X$ its RLF. Also set

$$\overline{\mathrm{dt}}_r(\,\cdot\,)(\,\cdot\,,\,\cdot\,) : [0,t] \times X \times X \to \mathbb{R},$$
$$\overline{\mathrm{dt}}_r(s)(y,z) := \sup_{u\in[0,s]} \mathrm{dt}_r(\overline{\boldsymbol{X}}_u)(y,z),$$

and define $\overline{\Gamma}(s) := \{(a,b) \in \boldsymbol{X}_t(S_r) \times B_{\lambda r}(\boldsymbol{X}_t(x)) \mid \overline{\mathrm{dt}}_r(s)(a,b) < r\}$. Then for all $s \in [0,t]$ and $(a,b) \in \overline{\Gamma}(s)$, one has $d(\overline{\boldsymbol{X}}_s(a), \boldsymbol{X}_{t-s}(x)) \leq 2r$, and

$$\begin{aligned} d(\overline{\boldsymbol{X}}_s(b), \boldsymbol{X}_{t-s}(x)) &\leq d(\overline{\boldsymbol{X}}_s(b), \overline{\boldsymbol{X}}_s(a)) + d(\overline{\boldsymbol{X}}_s(a), \boldsymbol{X}_{t-s}(x)) \\ &\leq d(a,b) + r + 2r \\ &\leq \lambda r + 5r. \end{aligned} \tag{5.11}$$

As in (5.8), using (5.11) instead of (5.7) we get

$$\begin{aligned} &\fint_{\boldsymbol{X}_t(S_r)\times B_{\lambda r}(\boldsymbol{X}_t(x))} \overline{\mathrm{dt}}_r(t)\,d(\mathfrak{m}\times\mathfrak{m}) \\ &\quad\leq \frac{2M}{\mathfrak{m}(B_r(x))^2}\int_0^t\int_{\overline{\Gamma}(s)} d(\overline{\boldsymbol{X}}_s(y), \overline{\boldsymbol{X}}_s(z))\left[\int_0^1 |\nabla\overline{V}(s)|(\gamma_{\overline{\boldsymbol{X}}_s(y),\overline{\boldsymbol{X}}_s(z)}(u))\,du\right] d(\mathfrak{m}\times\mathfrak{m})(y,z)\,ds \\ &\quad\leq \frac{2M}{\mathfrak{m}(B_r(x))^2}\int_0^t\int_{\overline{\boldsymbol{X}}_s(\overline{\Gamma}(s))} d(y,z)\left[\int_0^1 |\nabla\overline{V}(s)|(\gamma_{y,z}(u))\,du\right] d(\mathfrak{m}\times\mathfrak{m})(y,z)\,ds \\ &\quad\leq \frac{2M}{\mathfrak{m}(B_r(x))^2}\int_0^t\int_{B_{(\lambda+5)r}(\boldsymbol{X}_{t-s}(x))^{\times 2}} d(y,z)\left[\int_0^1 |\nabla\overline{V}(s)|(\gamma_{y,z}(u))\,du\right] d(\mathfrak{m}\times\mathfrak{m})(y,z)\,ds \\ &\quad\leq C(N)M^3 r\int_0^t\fint_{B_{(2\lambda+10)r}(\boldsymbol{X}_{t-s}(x))} |\nabla\overline{V}(s)|(y)\,d\mathfrak{m}(y)\,ds \\ &\quad\leq C(N)M^3 r\int_0^t \mathrm{Mx}_4(|\nabla V(t-s)|)(\boldsymbol{X}_{t-s}(x))\,ds \leq C(N)M^3\varepsilon r. \end{aligned}$$

Pick $y_t \in \boldsymbol{X}_t(S_r)$ such that

$$\fint_{B_{\lambda r}(\boldsymbol{X}_t(x))} \overline{\mathrm{dt}}_r(t)(y_t,z)\,d\mathfrak{m}(z) \leq C(N)M^3\varepsilon r.$$

Then the set $A^t_{\lambda r} := \{z \in B_{\lambda r}(\boldsymbol{X}_t(x)) \mid \overline{\mathrm{dt}}_r(t)(y_t, z) < r\}$ satisfies

$$\mathfrak{m}(A^t_{\lambda r}) \geq (1 - C(N)M^3\varepsilon)\mathfrak{m}(B_{\lambda r}(\boldsymbol{X}_t(x))).$$

Since $\overline{X}_t(A^t_{\lambda r}) \subset B_{(\lambda+5)r}(x)$ and $\overline{X}_t$ is measure preserving, using (5.6) we have

$$\mathfrak{m}(B_{\lambda r}(x)) \geq \tfrac{98}{100}\mathfrak{m}(B_{\lambda r}(\boldsymbol{X}_t(x))) \tag{5.12}$$

provided ε is small enough. We conclude

$$\begin{aligned}\mathfrak{m}(\boldsymbol{X}_t(A_{\lambda r}) \cap B_{\lambda r}(\boldsymbol{X}_t(x))) &\geq \mathfrak{m}(\boldsymbol{X}_t(A_{\lambda r})) - \mathfrak{m}(B_{(\lambda+4)r}(\boldsymbol{X}_t(x)) \backslash B_{\lambda r}(\boldsymbol{X}_t(x)))\\ &\geq \mathfrak{m}(A_{\lambda r}) - \tfrac{1}{100}\mathfrak{m}(B_{\lambda r}(\boldsymbol{X}_t(x)))\\ &\geq \tfrac{99}{100}\mathfrak{m}(B_{\lambda r}(x)) - \tfrac{1}{100}\mathfrak{m}(B_{\lambda r}(\boldsymbol{X}_t(x)))\\ &\geq \tfrac{9}{10}\mathfrak{m}(B_{\lambda r}(\boldsymbol{X}_t(x))),\end{aligned}$$

where we used (5.4) on the first inequality, (5.6) on the second, (5.10) on the third, and (5.12) on the fourth. □

Corollary 5.13. *Let $(X, d, \mathfrak{m})$ be an* RCD$(-(N-1), N)$ *space, $x \in X$, $V \in L^1([0,T]; H^{1,2}_{C,s}(TX))$ a divergence-free vector field, and $\boldsymbol{X} : [0,T] \times X \to X$ the RLF of V. Assume x is a point of essential stability of $\boldsymbol{X}$ and*

$$\int_0^T Mx_4(|\nabla V(t)|)(\boldsymbol{X}_t(x))\, dt \leq \varepsilon.$$

If ε is small enough, depending only on N, then for all $r \leq 1$, $t \in [0,T]$, one has

$$\tfrac{1}{2}\mathfrak{m}(B_r(x)) \leq \mathfrak{m}(B_r(\boldsymbol{X}_t(x))) \leq 2\mathfrak{m}(B_r(x)).$$

Proof. By the definition of essential stability, there is $M(N) > 0$ for which the hypotheses of Lemma 5.1 hold for small enough $r \leq 1$. By Lemma 5.1, if they hold for a certain r, then they hold for λr so we can apply Lemma 5.1 repeatedly, and (5.3) is valid for all $r \leq 1/\lambda$. □

Proposition 5.14. *There is $C_0(N) > 0$ such that, under the conditions of Corollary 5.13, for all $r \leq 1$ there is $A_r \subset B_r(x)$ such that*

$$\mathfrak{m}(A_r) \geq (1 - C_0\varepsilon)\mathfrak{m}(B_r(x)), \tag{5.15}$$

$$\boldsymbol{X}_t(A_r) \subset B_{2r}(\boldsymbol{X}_t(x)) \quad \text{for all } t \in [0,T]. \tag{5.16}$$

Proof. By the definition of essential stability, for $r \leq 1$ sufficiently small, there is $S_{r/10} \subset B_{r/10}(x)$ with $\mathfrak{m}(S_{r/10}) \geq \frac{1}{M(N)}\mathfrak{m}(B_{r/10}(x))$ and $\boldsymbol{X}_t(S_{r/10}) \subset B_{r/5}(\boldsymbol{X}_t(x))$ for all $t \in [0,T]$. Set

$$\mathrm{dist}_{r/10}(\cdot)(\cdot,\cdot) : [0,T] \times X \times X \to [0, r/10],$$

$$\mathrm{dist}_{r/10}(t)(y,z) := \sup_{s \in [0,t]} \mathrm{dist}_{r/10}(\boldsymbol{X}_s)(y,z),$$

$$\Gamma(t) := \{(a,b) \in S_{r/10} \times B_r(x) \mid \mathrm{dist}_{r/10}(t)(a,b) < r/10\}.$$

Then for $(y, z) \in \Gamma(t)$, for each $t \in [0, T]$ we have

$$\begin{aligned} d(\boldsymbol{X}_t(b), \boldsymbol{X}_t(x)) &\leq d(\boldsymbol{X}_t(b), \boldsymbol{X}_t(a)) + d(\boldsymbol{X}_t(a), \boldsymbol{X}_t(x)) \\ &\leq d(a, b) + r/10 + r/5 \\ &\leq r/2 + r/10 + r/10 + r/5 < r, \end{aligned} \tag{5.17}$$

so as in (5.8), using Corollary 5.13 one gets

$$\fint_{S_{r/10}\times B_r(x)} \mathrm{dt}_{r/10}(T)\, d(\mathfrak{m}\times\mathfrak{m}) \leq C(N)\cdot r\cdot \int_0^T \mathrm{Mx}_4(|\nabla V(t)|)(\boldsymbol{X}_t(x))\, dt \leq C(N)\varepsilon r.$$

Then there is $y \in S_{r/10}$ with

$$\fint_{B_r(x)} \mathrm{dt}_{r/10}(T)(y, z)\, d\mathfrak{m}(z) \leq C(N)\varepsilon r. \tag{5.18}$$

We can then define $A_r := \{z \in B_r(x) \mid \mathrm{dt}_{r/10}(T)(y, z) < r/10\}$, which by (5.18) satisfies (5.15) and by (5.17) also (5.16). The above analysis shows that (5.15) and (5.16) hold for all r small enough. An identical argument (using $A_{r/10}$ instead of $S_{r/10}$) shows that if (5.15) and (5.16) hold for some $r/10 \leq 1/10$, then they hold for r. $\square$

Proposition 5.19. *There is $C_0(N) > 0$ such that under the conditions of Corollary 5.13, for all $r \leq 1$, one has*

$$\fint_{B_r(x)^{\times 2}} \mathrm{dt}_r(\boldsymbol{X}_T)\, d(\mathfrak{m}\times\mathfrak{m}) \leq C_0\varepsilon r.$$

Proof. For $A_r \subset B_r(x)$ given by Proposition 5.14, a computation analogous to (5.8) yields

$$\fint_{A_r^{\times 2}} \mathrm{dt}_r(\boldsymbol{X}_T)\, d(\mathfrak{m}\times\mathfrak{m}) \leq C(N)\varepsilon r.$$

Combining this with (5.15), we get the result. $\square$

Definition 5.20. Let $(X, d, \mathfrak{m})$ be an RCD$(-(N-1), N)$ space, $V_1, \dots, V_k \in L^1([0,1]; H^{1,2}_{C,s}(TX))$ divergence-free vector fields, $\boldsymbol{X}^j : [0,1]\times X \to X$ their RLFs, and $x_1, \dots, x_k \in X$ such that

- x_j is a point of essential stability of $\boldsymbol{X}^j$ for each $j \in \{1, \dots, k\}$.
- $\boldsymbol{X}^j_1(x_j) = x_{j+1}$ for each $j \in \{1, \dots, k-1\}$.

If $V \in L^1([0,1]; H^{1,2}_{C,s}(TX))$ is given by

$$V(t) := k\cdot V_j(kt - j + 1) \quad \text{for } t \in \left[\frac{j-1}{k}, \frac{j}{k}\right],$$

and $\boldsymbol{X} : [0,1]\times X \to X$ is its RLF, then we say x_1 is a point of *weak essential stability* of $\boldsymbol{X}$.

The following proposition shows that, under suitable conditions, weak essential stability can be upgraded to essential stability, allowing one to concatenate well behaved flows; a crucial step in the proof of the rescaling theorem.

Proposition 5.21. *Under the conditions of Definition 5.20, there is $\eta(N) > 0$ such that if*

$$\int_0^1 Mx(|\nabla V(t)|)(\boldsymbol{X}_t(x_1))\,dt = \sum_{j=1}^{k} \int_0^1 Mx(|\nabla V_j(t)|)(\boldsymbol{X}_t^j(x_j))\,dt \le \eta,$$

then x_1 is a point of essential stability of $\boldsymbol{X}$.

Proof. By induction we can assume $k = 2$. By Corollary 5.13, and Proposition 5.14, we can apply Lemma 5.1 to both X^1 and X^2, provided η is small enough. By (5.3), for each $r \le 1$, $t \in [0, 1]$, we get

$$\tfrac{1}{4}\mathfrak{m}(B_r(x_1)) \le \mathfrak{m}(B_r(\boldsymbol{X}_t(x_1))) \le 4\mathfrak{m}(B_r(x_1)).$$

Let $\overline{V}_1 \in L^1([0, 1]; H^{1,2}_{C,s}(TX))$ be given by $\overline{V}_1(t) := -V_1(1-t)$ and let $\overline{\boldsymbol{X}}^1 : [0, 1] \times X \to X$ be its RLF. Again by Lemma 5.1, for r small enough there are sets $A_r^1, A_r^2 \subset B_r(x_2)$ such that

$$\begin{gathered} \overline{\boldsymbol{X}}_t^1(A_r^1) \subset B_{2r}(\overline{\boldsymbol{X}}_t^1(x_2)), \quad \boldsymbol{X}_t^2(A_r^2) \subset B_{2r}(\boldsymbol{X}_t^2(x_2)), \\ \mathfrak{m}(\overline{\boldsymbol{X}}_t^1(A_r^1) \cap B_r(\overline{\boldsymbol{X}}_t^1(x_2))) \ge \tfrac{9}{10}\mathfrak{m}(B_r(\overline{\boldsymbol{X}}_t^1(x_2))), \\ \mathfrak{m}(\boldsymbol{X}_t^2(A_r^2) \cap B_r(\boldsymbol{X}_t^2(x_2))) \ge \tfrac{9}{10}\mathfrak{m}(B_r(\boldsymbol{X}_t^2(x_2))) \end{gathered}$$

for all $t \in [0, 1]$. Then $A_r := B_r(x_1) \cap \overline{\boldsymbol{X}}_1^1(A_r^1 \cap A_r^2)$ satisfies

$$\boldsymbol{X}_t(A_r) \subset B_{2r}(\boldsymbol{X}_t(x)) \quad \text{for all } t \in [0, 1],$$

and using (5.3) we conclude

$$\begin{aligned} \mathfrak{m}(A_r) &\ge \mathfrak{m}(B_r(x_1) \cap \overline{\boldsymbol{X}}_1^1(A_r^1)) - \mathfrak{m}(A_r^1 \backslash A_r^2) \\ &\ge \tfrac{9}{10}\mathfrak{m}(B_r(x_1)) - \tfrac{1}{10}\mathfrak{m}(B_r(x_2)) \\ &\ge \left[\tfrac{9}{10} - \tfrac{1}{5}\right]\mathfrak{m}(B_r(x_1)) \\ &\ge \tfrac{1}{2}\mathfrak{m}(B_r(x_1)). \end{aligned}$$

□

6. Properties of GS maps

In this section we prove the main properties of GS maps; they converge weakly to an isometry (Lemma 6.4), have the zoom-in property (Proposition 6.7), and can be concatenated (Proposition 6.8).

Remark 6.1. From condition (1) of Definition 1.12, we can assume that for all $x \in U_i^1$, $y \in U_i^2$, we have $(f_i^{-1})^{-1}(x) = \{f_i(x)\}$ and $(f_i)^{-1}(y) = \{f_i^{-1}(y)\}$.

Definition 6.2. For $j \in \{1, 2\}$, let $(X_i^j, d_i^j, \mathfrak{m}_i^j, p_i^j)$, be a sequence of pointed RCD(K, N) spaces for which (X_i^j, p_i^j) converges to (X^j, p^j) in the pointed Gromov–Hausdorff sense. We say a sequence of maps $f_i : X_i^1 \to X_i^2$ converges weakly to $f_\infty : X^1 \to X^2$ if there is a sequence of subsets $U_i \subset X_i^1$ with asymptotically full measure such that

$$\lim_{i\to\infty} \sup_{x \in U_i} d(\varphi_i^2 f_i(x), f_\infty \varphi_i^1(x)) = 0, \tag{6.3}$$

where $\varphi_i^j : X_i^j \to X^j \cup \{*\}$ are Gromov–Hausdorff approximations for $j \in \{1, 2\}$.

Lemma 6.4. *For $j \in \{1, 2\}$, let $(X_i^j, d_i^j, \mathfrak{m}_i^j, p_i^j)$, be a sequence of pointed* RCD(K, N) *spaces for which (X_i^j, p_i^j) converges to (X^j, p^j) in the pointed Gromov–Hausdorff sense. If $f_i : [X_i^1, p_i^1] \to [X_i^2, p_i^2]$ is a sequence of GS maps, then, after taking a subsequence, f_i converges weakly to an isometry $f_\infty : X^1 \to X^2$.*

Proof. Let $R_0 > 0$, $\varepsilon_i \to 0$, and $S_i^j, U_i^j \subset X_i^j$ be given by Definition 1.12 (see Remark 6.1).

Step 1: For i large enough, $x \in U_i^1$, $r \leq 1$, there is $A \subset B_r(x)$ such that

$$f_i(A) \subset B_{2r}(f_i(x)) \quad \text{and} \quad \mathfrak{m}_i^1(A) \geq \tfrac{1}{2}\mathfrak{m}_i^1(B_r(x)).$$

From the definition of essential continuity, the statement holds for r small enough (depending on x). We now see that if i is large enough, and there is $A_0 \subset B_{r/10}(x)$ such that $f_i(A_0) \subset B_{r/5}(f_i(x))$, and $\mathfrak{m}_i^1(A_0) \geq \frac{1}{2}\mathfrak{m}_i^1(B_{r/5}(x))$, then there is $A \subset B_r(x)$ such that $f_i(A) \subset B_{2r}(f_i(x))$, and $\mathfrak{m}_i^1(A) \geq \frac{1}{2}\mathfrak{m}_i^1(B_r(x))$. Since there is $C(K, N) > 0$ such that $\mathfrak{m}_i^1(B_r(x)) \leq C\mathfrak{m}_i^1(A_0)$, we have

$$\fint_{B_r(x_i^1)\times A_0} \mathrm{dt}_r(f_i)\, d(\mathfrak{m}_i^1 \times \mathfrak{m}_i^1) \leq Cr\varepsilon_i.$$

Hence if $A := \{y \in B_r(x) \mid d(fy, fx) < 2r\}$, one gets

$$r \cdot \frac{\mathfrak{m}_i^1(B_r(x)\backslash A)}{\mathfrak{m}_i^1(B_r(x))} = \frac{\int_{(B_r(x)\backslash A)\times A_0} \mathrm{dt}_r(f_i)\, d(\mathfrak{m}_i^1 \times \mathfrak{m}_i^1)}{\mathfrak{m}_i^1(B_r(x)) \cdot \mathfrak{m}_i^1(A_0)} \leq Cr\varepsilon_i,$$

implying that $\mathfrak{m}_i^1(A) \geq \frac{1}{2}\mathfrak{m}_i^1(B_r(x))$ provided $\varepsilon_i \leq \frac{1}{2C}$.

Step 2: For all distinct $x_i, y_i \in U_i^1$ with $d(x_i, y_i) \leq \frac{1}{2}$, one has

$$\limsup_{i\to\infty} \frac{d(f_i x_i, f_i y_i)}{d(x_i, y_i)} \leq 1.$$

Set $r_i := d(x_i, y_i)$ and assume, after taking a subsequence, that $d(f_i x_i, f_i y_i) \geq (1+\delta)r_i$ for some $\delta > 0$ and all i. By Step 1, there are subsets $A_i \subset B_{\delta r_i/10}(x_i)$, $B_i \subset B_{\delta r_i/10}(y_i)$ with $f_i(A_i) \subset B_{\delta r_i/5}(f_i x_i)$, $f_i(B_i) \subset B_{\delta r_i/5}(f_i y_i)$, $\mathfrak{m}_i^1(A_i) \geq \frac{1}{2}\mathfrak{m}_i^1(B_{\delta r_i/10}(x_i))$, and $\mathfrak{m}_i^1(B_i) \geq \frac{1}{2}\mathfrak{m}_i^1(B_{\delta r_i/10}(y_i))$. Since there is $C(K, N) > 0$ such that

$$\mathfrak{m}_i^1(B_{2r_i}(x_i)) \leq C \cdot \min\{\mathfrak{m}_i^1(A_i), \mathfrak{m}_i^1(B_i)\},$$

one has

$$\frac{\delta r_i}{10 \cdot C^2} \leq \frac{\int_{A_i \times B_i} \mathrm{dt}_r(f_i)\, d(\mathfrak{m}_i^1 \times \mathfrak{m}_i^1)}{\mathfrak{m}_i^1(B_{2r_i}(x_i))^2} \leq \fint_{B_{2r_i}(x_i)^{\times 2}} \mathrm{dt}_r(f)\, d(\mathfrak{m}_i^1 \times \mathfrak{m}_i^1) \leq 2r_i\varepsilon_i,$$

which is impossible as $\varepsilon_i \to 0$.

Step 3: For $R > 0$ and distinct $x_i, y_i \in U_i^1$ with $d(x_i, p_i^1), d(y_i, p_i^1) \leq R$, one has

$$\limsup_{i\to\infty} \frac{d(f_i x_i, f_i y_i)}{d(x_i, y_i)} \leq 1.$$

By Step 2, we can assume $d(x_i, y_i) \geq \frac{1}{2}$ for all i. For each i, choose a sequence $x_i = z_i^0, \ldots, z_i^k = y_i \in X_i^1$ with $d(z_i^{j-1}, z_i^j) \leq \frac{1}{3}$ for each $j \in \{1, \ldots, k\}$, $d(x_i, y_i) = \sum_{j=1}^k d(z_i^{j-1}, z_i^j)$, and $k = \lfloor 10R \rfloor$. For each $i \in \mathbb{N}$ and $j \in \{1, \ldots, k\}$, let $w_i^j \in U_i^1$ be such that

$$d(w_i^j, z_i^j) \leq 2 \cdot \inf\{d(w, z_i^j) \mid w \in U_i^1\}.$$

As the sets U_i^1 have asymptotically full measure, $\sup_j d(w_i^j, z_i^j) \to 0$ as $i \to \infty$, and the claim follows from Step 2 applied to pairs (w_i^{j-1}, w_i^j).

Step 4: For $R > 0$ and $x_i \in U_i^1$ with $d(x_i, p_i^1) \leq R$, one has

$$\limsup_{i \to \infty} d(f_i x_i, p_i^2) \leq R_0 + R + 1.$$

As the sets U_i^1 have asymptotically full measure, for i large enough one can pick $y_i \in U_i^1 \cap S_i^1$. Then the result follows from Step 3 and the fact that $d(x_i, y_i) \leq R + 1$ for all i.

Step 5: For $R > 0$, $x_i \in U_i^1$ with $d(x_i, p_i^1) \leq R$, and $\delta > 0$, for large enough i there is

$$y_i \in B_\delta(x_i) \cap U_i^1 \cap f_i^{-1}(U_i^2).$$

Without loss of generality assume $\delta < \frac{1}{2}$. As the sets U_i^1 have asymptotically full measure, the sets $A_i := B_\delta(x_i) \cap U_i^1$ satisfy $\mathfrak{m}_i^1(A_i) \geq \frac{1}{2}\mathfrak{m}_i^1(B_\delta(x_i))$ for i large enough. Assuming the claim fails, one has from Step 2, after taking a subsequence, that $f_i(A_i) \subset B_{2\delta}(f_i x_i) \backslash U_i^2$ for all i. As f_i restricted to A_i is measure preserving, and the sets U_i^2 have asymptotically full measure, this means that

$$\frac{\mathfrak{m}_i^1(B_\delta(x_i))}{\mathfrak{m}_i^2(B_{2\delta}(f_i x_i))} \to 0 \quad \text{as } i \to \infty. \tag{6.5}$$

From Step 4 we know that $B_{2\delta}(f_i x_i) \subset B_{R_0+R+2}(p_i)$ for large i, and from the Bishop–Gromov inequality, there is $C(K, N, R_0, R, \delta) > 0$ such that

- $\mathfrak{m}_i^2(B_{R_0+R+2}(p_i^2)) \leq C \cdot \mathfrak{m}_i^2(S_i^2 \cap U_i^2)$ for large enough i,
- $\mathfrak{m}_i^1(B_{R_0}(p_i^1)) \leq C \cdot \mathfrak{m}_i^1(B_\delta(x_i))$.

Combining this with the fact that $f_i^{-1}(S_i^2 \cap U_i^2) \subset B_{R_0}(p_i^1)$, we get that

$$\frac{\mathfrak{m}_i^2(B_{2\delta}(f_i x_i))}{\mathfrak{m}_i^1(B_\delta(x_i))} \leq C^2$$

for i large enough, contradicting (6.5).

Step 6: For $R > 0$ and distinct $x_i, y_i \in U_i^1$ with $d(x_i, p_i^1), d(y_i, p_i^1) \leq R$, one has

$$\lim_{i \to \infty} |d(f_i x_i, f_i y_i) - d(x_i, y_i)| = 0.$$

From Step 3, one gets

$$\limsup_{i \to \infty} (d(f_i x_i, f_i y_i) - d(x_i, y_i)) \leq 0.$$

By Step 5, there are sequences $w_i, z_i \in U_i^1 \cap h_i(U_i^2)$ with $d(w_i, x_i), d(z_i, y_i) \to 0$. By Step 2, we have $d(f_i w_i, f_i x_i), d(f_i z_i, f_i y_i) \to 0$, and by Step 3 applied to h_i, one gets

$$\limsup_{i\to\infty}(d(w_i, z_i) - d(f_i w_i, f_i z_i)) \leq 0.$$

Hence

$$\limsup_{i\to\infty}(d(x_i, y_i) - d(f_i x_i, f_i y_i)) \leq 0.$$

Step 7: Lemma 6.4 holds.

Let $\varphi_i^j : X_i^j \to X^j \cup \{*\}$ be Gromov–Hausdorff approximations and fix $\mathcal{D} \subset X^1$ a countable dense set. For $x \in \mathcal{D}$, choose $x_i \in U_i^1$ converging to x. By Step 4 we can define (after taking a subsequence) $f'_\infty(x) \in X^2$ as

$$f'_\infty(x) := \lim_{i\to\infty} \varphi_i^2 f_i(x_i).$$

By a diagonal argument, this can be done simultaneously for all $x \in \mathcal{D}$. It is easy to see from Step 6 that $f'_\infty : \mathcal{D} \to X^2$ extends to an isometry $f_\infty : X^1 \to X^2$ and satisfies (6.3). $\square$

Proposition 6.6. *For $j \in \{1,2\}$, let $(X_i^j, d_i^j, \mathfrak{m}_i^j, p_i^j)$, be a sequence of pointed* RCD(K,N) *spaces that converges in the pointed measured Gromov–Hausdorff sense to a pointed* RCD(K,N) *space $(X^j, d^j, \mathfrak{m}^j, p^j)$, and assume there is a sequence $f_i : [X_i^1, p_i^1] \to [X_i^2, p_i^2]$ of GS maps. If sequences of sets $V_i^1 \subset X_i^1$ and $V_i^2 \subset X_i^2$ have asymptotically full measure, then the sequences $f_i(V_i^1) \subset X_i^2$ and $f_i^{-1}(V_i^2) \subset X_i^1$ have asymptotically full measure as well.*

Proof. Let $U_i^j \subset X_i^j$ be sets given by condition (3) of Definition 1.12. By replacing U_i^j and V_i^j by $U_i^j \cap V_i^j$, we can assume $U_i^j = V_i^j$ for all $j \in \{1,2\}$, $i \in \mathbb{N}$. Fix $R > \delta > 0$, and consider a sequence $x_i \in B_R(p_i^1)$. As the sets U_i^1 have asymptotically full measure, by Step 5 above, there is a sequence $y_i^1 \in U_i^1 \cap f_i^{-1}(V_i^2)$ with $d(x_i, y_i^1) \to 0$. Define $y_i^2 := f_i y_i^1$, $A_i^j := U_i^j \cap B_\delta(y_i^j)$ for $j \in \{1,2\}$. By Step 6 above, there is a sequence $\varepsilon_i \to 0$ such that

$$f_i(A_i^1) \subset B_{\delta+\varepsilon_i}(y_i^2),\ f_i^{-1}(A_i^2) \subset B_{\delta+\varepsilon_i}(y_i^1).$$

Then

$$\begin{aligned}\lim_{i\to\infty} \frac{\mathfrak{m}_i^1(f_i^{-1}(A_i^2) \cap B_\delta(x_i^1))}{\mathfrak{m}_i^1(B_\delta(x_i^1))} &\geq \lim_{i\to\infty} \frac{\mathfrak{m}_i^1(f_i^{-1}(A_i^2))}{\mathfrak{m}_i^1(A_i^1)} \geq \lim_{i\to\infty} \frac{\mathfrak{m}_i^2(A_i^2)}{\mathfrak{m}_i^2(f_i(A_i^1))}\\ &\geq \lim_{i\to\infty} \frac{\mathfrak{m}_i^2(B_\delta(y_i^2))}{\mathfrak{m}_i^2(B_\delta(y_i^2))} = 1.\end{aligned}$$

This shows that $f_i^{-1}(U_i^2)$ has asymptotically full measure. The result for $f_i(U_i^1)$ is analogous. $\square$

Proposition 6.7. *Let $(X_i^j, d_i^j, \mathfrak{m}_i^j, p_i^j)$, $j \in \{1,2\}$ be a pair of sequences of pointed* RCD(K,N) *spaces and $f_i : [X_i^1, p_i^1] \to [X_i^2, p_i^2]$ is a sequence of GS maps. Then there is a sequence of subsets $W_i^1 \subset X_i^1$ of asymptotically full measure with the property that for all $w_i \in W_i^1$ and $\lambda_i \to \infty$, the sequence $f_i : [\lambda_i X_i^1, w_i] \to [\lambda_i X_i^2, f_i(w_i)]$ is GS.*

Proof. Let $U_i^j \subset X_i^j$ be given by Definition 1.12 and consider a sequence $\delta_i \to 0$. Set

$$\chi_i^j := 1 - \chi_{U_i^j} : X_i^j \to \mathbb{R},$$

and $V_i^j := \{x \in U_i^j \mid \mathrm{Mx}(\chi_i^j)(x) \leq \delta_i\}$. Then by Proposition 2.5(1) and Proposition 6.6, if $\delta_i \to 0$ slowly enough, the sets

$$W_i^1 := V_i^1 \cap f_i^{-1}(V_i^2), \; W_i^2 := V_i^2 \cap f_i(V_i^1),$$

have asymptotically full measure. Moreover, by construction, for any sequences $\lambda_i \to \infty$ and $w_i \in W_i^1$, the sets W_i^1 and W_i^2 also have asymptotically full measure when regarded as subsets of the spaces $(X_i^1, \lambda_i d_i^1, \mathfrak{m}_i^1, w_i)$ and $(X_i^2, \lambda_i d_i^2, \mathfrak{m}_i^2, f_i(w_i))$, respectively.

Using the sets W_i^j as a replacement for U_i^j, all the properties of Definition 1.12 for $f_i : [\lambda_i X_i^1, w_i] \to [\lambda_i X_i^2, f_i(w_i)]$ follow from the ones of the original sequence, except for condition (2), which follows from Step 1 in the proof of Lemma 6.4. □

Proposition 6.8. *Let $(X_i^1 j, d_i^j, \mathfrak{m}_i^j, p_i^j)$, $j \in \{1, 2, 3\}$ be sequences of pointed* $\mathrm{RCD}(K, N)$ *spaces, and $f_i : [X_i^1, p_i^1] \to [X_i^2, p_i^2]$, $h_i : [X_i^2, p_i^2] \to [X_i^3, p_i^3]$ be sequences of GS maps. Then $h_i \circ f_i : [X_i^1, p_i^1] \to [X_i^3, p_i^3]$ is GS. Moreover, if f_i converges weakly to f and h_i converges weakly to h, then $h_i f_i$ converges weakly to hf.*

Proof. Let $U_i^1 \subset X_i^1$, $U_i^2 \subset X_i^2$, $V_i^2 \subset X_i^2$, $V_i^3 \subset X_i^3$ be given by Definition 1.12 applied to f_i and h_i respectively. Set $W_i' := U_i^1 \cap f_i^{-1}(U_i^2 \cap V_i^2 \cap h_i^{-1}(V_i^3))$, $\chi_i := 1 - \chi_{W_i'}$, and for a sequence $\delta_i \to 0$, define

$$W_i := \{x \in U_i^1 \mid \mathrm{Mx}(\chi_i)(x) \leq \delta_i\}.$$

By Proposition 2.5(1), if $\delta_i \to 0$ slowly enough, the sets $W_i \subset X_i^1$ have asymptotically full measure, so from Lemma 6.4, $S_i^1 := W_i \cap B_1(p_i^1)$ and $h_i f_i$ satisfy condition (2) from Definition 1.12. By Step 2 in the proof of Lemma 6.4 (applied to both f_i and f_i^{-1}), there is $\eta_i \to 0$ such that for all $r < 10$, $a, b \in W_i'$ with $d(a, b) \leq 2r$, one has

$$\mathrm{dt}_r(f_i)(a, b), \mathrm{dt}_r(h_i)(f_i a, f_i b) \leq \eta_i \cdot d(a, b).$$

This implies that W_i consists of essential continuity points of $h_i f_i$ provided $\delta_i, \eta_i \leq \frac{1}{2}$. Also, for $x \in W_i$, $r \leq \frac{1}{10}$, set $Z = B_r(x) \cap W_i'$. Then

$$\begin{aligned}
\frac{1}{r} \fint_{B_r(x)^{\times 2}} \mathrm{dt}_r(h_i f_i)\, d(\mathfrak{m}_i^1 \times \mathfrak{m}_i^1) &\leq 2\frac{\mathfrak{m}_i^1(B_r(x) \backslash Z)}{\mathfrak{m}_i^1(B_r(x))} + \frac{1}{\mathfrak{m}_i^1(B_r(x))^2} \int_{Z^{\times 2}} \frac{\mathrm{dt}_r(h_i f_i)}{r}\, d(\mathfrak{m}_i^1 \times \mathfrak{m}_i^1) \\
&\leq 2\delta_i + \fint_{Z^{\times 2}} \frac{\mathrm{dt}_r(h_i)(f_i \cdot, f_i \cdot) + \mathrm{dt}_r(f_i)(\cdot, \cdot)}{r}\, d(\mathfrak{m}_i^1 \times \mathfrak{m}_i^1) \\
&\leq 2\delta_i + 4\eta_i.
\end{aligned}$$

This shows that $h_i f_i$ satisfies condition (3)(d) from Definition 1.12, with $r \leq \frac{1}{10}$ instead of $r \leq 1$. Identical arguments show that $f_i^{-1} h_i^{-1}$ also satisfy the corresponding properties in Definition 1.12. Conditions (1) and (3)(b) for $h_i f_i$ follow from the corresponding conditions for h_i and f_i.

Notice that the proof of Lemma 6.4 still goes through if we replace 1 by $\frac{1}{10}$ in (3)(d). In particular, by Step 6, if we replace W_i by $W_i \cap B_{R_i}(p_i^1)$ for some sequence $R_i \to \infty$ diverging slowly enough, then (3)(d) holds for $r \in \left[\frac{1}{10}, 1\right]$ as well, and the sequence $h_i f_i$ is GS.

To verify the last claim, let $(X^j, d_j, \mathfrak{m}^j, p^j)$ be pointed RCD(K, N) spaces and $\varphi_i^j : X_i^j \to X^j \cup \{*\}$ Gromov–Hausdorff approximations for $j \in \{1, 2, 3\}$. Then by hypothesis, there are $\varepsilon_i \to 0$ and sets $A_i^j \subset X_i^j$ with $j \in \{1, 2\}$ having asymptotically full measure such that for $x \in A_i^1$, $y \in A_i^2$ one has

$$d(\varphi_i^2 f_i(x), f_\infty \varphi_i^1(x)), d(\varphi_i^3 h_i(y), h_\infty \varphi_i^2(y)) \le \varepsilon_i.$$

Then by Proposition 6.6, the sets $A_i := A_i^1 \cap f_i^{-1}(A_i^2)$ have asymptotically full measure, and for all $x \in A_i$, one has

$$d(\varphi_i^3 h_i f_i(x), h_\infty f_\infty \varphi_i^1(x)) \le d(\varphi_i^3 h_i f_i(x), h_\infty \varphi_i^2 f_i(x)) + d(h_\infty \varphi_i^2 f_i(x), h_\infty f_\infty \varphi_i^1(x)) \le 2\varepsilon_i,$$

so $h_i f_i$ converges weakly to $h_\infty f_\infty$. □

7. Construction of GS maps

In this section we follow [Kapovitch and Wilking 2011] closely in order to construct GS maps out of RLFs of suitable functions. Specifically, Lemmas 7.1 and 7.7 are adaptations of Lemmas 3.8 and 3.6 of [Kapovitch and Wilking 2011], respectively. Both proofs make heavy use of the estimates obtained in Section 5.

Roughly speaking, Lemma 7.1 establishes that if a sequence of RCD spaces X_i converges to a space of the form $\mathbb{R}^k \times Y$, and the universal covers $\widetilde{X}_i$ converge to $\mathbb{R}^k \times \widetilde{Y}$, then any translation on the first factor of $\mathbb{R}^k \times \widetilde{Y}$ is a limit of GS maps $f_i : \widetilde{X}_i \to \widetilde{X}_i$ which are lifts of maps $X_i \to X_i$ homotopic to the identity Id_{X_i}. These maps are constructed via RLFs of the gradient vector fields of the δ-splittings given by Lemma 2.34.

Lemma 7.1. *Let $(X_i, d_i, \mathfrak{m}_i, p_i)$ be a sequence of* RCD$\left(-\frac{1}{i}, N\right)$ *spaces,*

$$\rho_i : (\widetilde{X}_i, \tilde{d}_i, \tilde{\mathfrak{m}}_i, \tilde{p}_i) \to (X_i, d_i, \mathfrak{m}_i, p_i)$$

their universal covers, (Y, y), $(\widetilde{Y}, \tilde{y})$ a pair of pointed metric spaces, and a closed group $\Gamma \le \mathrm{Iso}(\mathbb{R}^k \times \widetilde{Y})$ that acts trivially on the $\mathbb{R}^k$ factor with $\widetilde{Y}/\Gamma = Y$. Assume the sequences (X_i, p_i) and $(\widetilde{X}_i, \tilde{p}_i)$ converge in the pointed Gromov–Hausdorff sense to $(\mathbb{R}^k \times Y, (0, y))$ and $(\mathbb{R}^k \times \widetilde{Y}, (0, \tilde{y}))$, respectively, and the sequence of groups $\pi_1(X_i)$ converges to Γ. Let $\tilde{\varphi}_i : \widetilde{X}_i \to \mathbb{R}^k \times \widetilde{Y} \cup \{*\}$, $\varphi_i : X_i \to \mathbb{R}^k \times Y \cup \{*\}$ be the Gromov–Hausdorff approximations given by Theorem 2.31. Then for all $s \in \mathbb{R}^k$, there is a sequence $f_i : [\widetilde{X}_i, \tilde{p}_i] \to [\widetilde{X}_i, \tilde{p}_i]$ of deck type GS maps with $(f_i)_* = \mathrm{Id}_{\pi_1(X_i)}$, and such that f_i converges weakly to the map $\bar{s} : \mathbb{R}^k \times \widetilde{Y} \to \mathbb{R}^k \times \widetilde{Y}$, where $\bar{s}(x, y) := (x + s, y)$.

Proof. Notice that by replacing Y by $s^\perp \times Y$, where $s^\perp \le \mathbb{R}^k$ denotes the orthogonal complement of s, we can assume $k = 1$ and $s > 0$. By Lemma 2.34, there are $\delta_i \to 0$, $R_i \to \infty$, and a sequence of $L(N)$-Lipschitz functions $h^i \in H^{1,2}(X_i)$ such that

- ∇h^i is divergence free in $B_{R_i}(p_i)$,

- for all $r \in [1, R_i]$, one has

$$\fint_{B_r(p_i)} \left[\left| |\nabla h^i|^2 - 1 \right| + |\nabla \nabla h^i|^2 \right] d\mathfrak{m}_i \leq \delta_i^2,$$

- for all $x \in B_{R_i}(p_i)$, one has

$$d(h^i(x), \pi(\varphi_i(x))) \leq \delta_i,$$

  where $\pi : \mathbb{R} \times Y \to \mathbb{R}$ is the projection.

Set $\tilde{h}^i : \widetilde{X}_i \to \mathbb{R}$ as $\tilde{h}^i := h^i \circ \rho_i$, and $\tilde{\pi} : \mathbb{R} \times \widetilde{Y} \to \mathbb{R}$ as $\tilde{\pi} := \pi \circ \rho$. By (2.32) one gets for $x \in B_{R_i}(\tilde{p}_i)$, after possibly updating δ_i and R_i, that

$$\begin{aligned} d(\tilde{h}^i(x), \tilde{\pi}(\tilde{\varphi}_i(x))) &\leq d\big(h^i(\rho_i(x)), \pi(\varphi_i(\rho_i(x)))\big) + d\big(\pi(\varphi_i(\rho_i(x))), \pi(\rho(\tilde{\varphi}_i(x)))\big) \\ &\leq \delta_i + \delta_i. \end{aligned}$$

Then by Proposition 2.7 one gets, after possibly updating δ_i and R_i, that

- $\nabla \tilde{h}^i$ is divergence free in $B_{R_i}(\tilde{p}_i)$,
- for all $r \in [1, R_i]$, one has

$$\fint_{B_r(\tilde{p}_i)} \left[\left| |\nabla \tilde{h}^i|^2 - 1 \right| + |\nabla \nabla \tilde{h}^i|^2 \right] d\tilde{\mathfrak{m}}_i \leq \delta_i^2,$$

- for all $x \in B_{R_i}(\tilde{p}_i)$, one has

$$d(\tilde{h}^i(x), \tilde{\pi}(\tilde{\varphi}_i(x))) \leq \delta_i. \tag{7.2}$$

Set $V_i := s\nabla \tilde{h}^i$, $\boldsymbol{X}^i : [0,1] \times \widetilde{X}_i \to \widetilde{X}_i$ the corresponding RLF, and $f_i := \boldsymbol{X}_1^i$. For $r \geq 1$, and i large enough, using the Cauchy-Schwarz inequality and Proposition 2.5(2), we have

$$\begin{aligned} \fint_{B_r(\tilde{p}_i)} \left[\int_0^1 \mathrm{Mx}_4(|\nabla V_i|)(\boldsymbol{X}_t^i(x))\, dt \right] d\tilde{\mathfrak{m}}_i(x) &= \int_0^1 \left[\fint_{\boldsymbol{X}_t^i(B_r(\tilde{p}_i))} \mathrm{Mx}_4(|\nabla V_i|)\, d\tilde{\mathfrak{m}}_i \right] dt \\ &\leq C(N,s,r) \fint_{B_{r+sL}(\tilde{p}_i)} \mathrm{Mx}_4(|\nabla V_i|)\, d\tilde{\mathfrak{m}}_i \\ &\leq C(N,s,r) \sqrt{\fint_{B_{r+sL}(\tilde{p}_i)} \mathrm{Mx}_4(|\nabla V_i|)^2\, d\tilde{\mathfrak{m}}_i} \\ &\leq C(N,s,r)\delta_i. \end{aligned} \tag{7.3}$$

Set

$$U_i' := \left\{ x \in \widetilde{X}_i \;\middle|\; \int_0^1 \mathrm{Mx}_4(|\nabla V_i|)(\boldsymbol{X}_t^i(x))\, dt \leq \sqrt{\delta_i} \right\},$$

and let $U_i \subset U_i'$ be the density points of U_i'. From (7.3), the sets U_i have asymptotically full measure. By Theorem 1.5, for i large enough, U_i consists of points of essential stability of $\boldsymbol{X}^i$, and hence of essential continuity of f_i. To verify part (2) of Definition 1.12, we notice that for all i we have

$$f_i(B_1(\tilde{p}_i)) \subset B_{1+sL}(\tilde{p}_i).$$

Applying Proposition 5.19 to the points in U_i, we see that part (3)(d) of Definition 1.12 holds. The corresponding properties for f_i^{-1} follow by identical arguments applied to the reverse flow, so we get that the maps f_i are good at all scales and converge, by Lemma 6.4, to a measure preserving isometry $f_\infty : \mathbb{R} \times \widetilde{Y} \to \mathbb{R} \times \widetilde{Y}$. It remains to show that f_∞ coincides with the translation $\bar{s}$.

For $q \in \mathbb{R} \times \widetilde{Y}$ with $d(q, (0, y)) < R$, choose $q_i \in U_i \cap f_i^{-1}(U_i)$ converging to q, and $\eta < \frac{1}{4}$. Then, for i large enough we have

$$\begin{aligned}
\fint_{B_\eta(q_i)} \left| \int_0^1 |V_i|(\boldsymbol{X}_t^i(x))\, dt - s \right| d\widetilde{\mathfrak{m}}_i(x) &\leq s \fint_{B_\eta(q_i)} \left[\int_0^1 \big| |\nabla h_i| - 1 \big| (\boldsymbol{X}_t^i(x))\, dt \right] d\widetilde{\mathfrak{m}}_i(x) \\
&\leq s \int_0^1 \left[\fint_{\boldsymbol{X}_t^i(B_\eta(q_i))} \big| |\nabla h_i| - 1 \big|\, d\widetilde{\mathfrak{m}}_i \right] dt \\
&\leq C(N, R, s, \eta) \fint_{B_{\eta+R+sL}(p_i)} \big| |\nabla h_i| - 1 \big|\, d\widetilde{\mathfrak{m}}_i \\
&\leq C(N, R, s, \eta) \sqrt{\fint_{B_{\eta+R+sL}(p_i)} \big| |\nabla h_i|^2 - 1 \big| d\widetilde{\mathfrak{m}}_i} \\
&\leq C(N, R, s, \eta) \delta_i.
\end{aligned}$$

Hence, from the derivative formula [Deng 2020, Proposition 3.6], and using the fact that U_i have asymptotically full measure, we have

$$\fint_{B_\eta(q_i) \cap U_i} \max\{d(f_i(x), x) - s, 0\}\, d\widetilde{\mathfrak{m}}_i(x) \leq C(N, R, s, \eta)\delta_i. \tag{7.4}$$

From Step 2 of Lemma 6.4, we know that for i large enough, $d(f_i(x), x)$ varies by at most 5η for $x \in B_\eta(q_i) \cap U_i$. Since η was arbitrary, (7.4) implies that

$$d(f_\infty(q), q) = \lim_{i\to\infty} d(f_i(q_i), q_i) \leq s. \tag{7.5}$$

Similarly, by the definition of RLF, if $\eta < \frac{1}{4}$,

$$\begin{aligned}
\fint_{B_\eta(q_i)} \big| (\tilde{h}_i(f_i(x)) - \tilde{h}_i(x)) - s \big|\, d\widetilde{\mathfrak{m}}_i(x) &\leq s \fint_{B_\eta(q_i)} \left[\int_0^1 \big| |\nabla \tilde{h}_i|^2 - 1 \big| (\boldsymbol{X}_t^i(x))\, dt \right] d\widetilde{\mathfrak{m}}_i(x) \\
&\leq C(N, s, R, \eta) \cdot \delta_i^2.
\end{aligned}$$

Then, as η was arbitrary, from (7.2) we get

$$\tilde{\pi}(f_\infty q) - \tilde{\pi}(q) = \lim_{i\to\infty} [\tilde{h}_i(f_i(q_i)) - \tilde{h}_i(q_i)] = s. \tag{7.6}$$

Since $\bar{s}$ is the only map $\mathbb{R} \times \widetilde{Y} \to \mathbb{R} \times \widetilde{Y}$ satisfying (7.5) and (7.6), we get $f_\infty = \bar{s}$. $\square$

Lemma 7.7 gives another way of constructing GS maps. One needs a sequence of vector fields V_i and for each i a point x_i of essential stability of the flow of V_i. If one has enough control on the covariant derivative ∇V_i along the trajectory of x_i, then after blowing up around x_i, one obtains GS maps as the endpoint maps of the flows of the vector fields V_i.

Lemma 7.7. *Let $(X_i, d_i, \mathfrak{m}_i)$ be a sequence of* $\mathrm{RCD}(-(N-1), N)$ *spaces, $V_i \in L^1([0,1]; H^{1,2}_{C,s}(TX_i))$ a sequence of piecewise constant on time, divergence-free vector fields, $\boldsymbol{X}^i : [0,1] \times X_i \to X_i$ their RLFs, and $x_i \in X_i$ a sequence such that x_i is a point of essential stability of $\boldsymbol{X}^i$, and*

$$\int_0^1 Mx(|\nabla V_i(t)|^{3/2})^{2/3}(\boldsymbol{X}^i_t(x_i))\,dt = \varepsilon_i.$$

If $\varepsilon_i \to 0$, then for all $\lambda_i \to \infty$, the sequence of maps

$$\boldsymbol{X}^i_1 : [\lambda_i X_i, x_i] \to [\lambda_i X_i, \boldsymbol{X}^i_1(x_i)]$$

has the GS property.

Proof. For $r \leq \frac{1}{4}$, let

$$A^i_r := \{y \in B_r(x_i) \mid \boldsymbol{X}^i_t(y) \in B_{2r}(\boldsymbol{X}^i_t(x_i)) \text{ for all } t \in [0,1]\}.$$

By Corollary 5.13 and Proposition 5.14, we have, for i large enough,

$$\mathfrak{m}_i(B_r(\boldsymbol{X}_t(x_i))) \leq 2\mathfrak{m}_i(B_r(x_i)). \tag{7.8}$$

$$\mathfrak{m}_i(A^i_r) \geq (1 - C(N)\varepsilon_i)\mathfrak{m}_i(B_r(x_i)), \tag{7.9}$$

Also, using (7.8) and Proposition 2.5(3),

$$\begin{aligned}
\fint_{A^i_r}\int_0^1 \mathrm{Mx}_{1/2}(|\nabla V_i(t)|)(\boldsymbol{X}^i_t(y))\,dt\,d\mathfrak{m}_i(y) &= \int_0^1 \fint_{\boldsymbol{X}^i_t(A^i_r)} \mathrm{Mx}_{1/2}(|\nabla V_i(t)|)(y)\,d\mathfrak{m}_i(y)\,dt \\
&\leq C(N)\int_0^1 \fint_{B_{2r}(\boldsymbol{X}^i_t(x_i))} \mathrm{Mx}_{1/2}(|\nabla V_i(t)|)(y)\,d\mathfrak{m}_i(y)\,dt \\
&\leq C(N)\int_0^1 \mathrm{Mx}_{1/2}(\mathrm{Mx}_{1/2}(|\nabla V_i(t)|))(\boldsymbol{X}^i_t(x_i))\,dt \\
&\leq C(N)\int_0^1 \mathrm{Mx}(|\nabla V_i(t)|^{3/2})^{2/3}(\boldsymbol{X}^i_t(x_i))\,dt \\
&\leq C(N)\varepsilon_i.
\end{aligned} \tag{7.10}$$

Let $U_i(r)$ be the density points of the set

$$\left\{ y \in A^i_r \,\middle|\, \int_0^1 \mathrm{Mx}_{1/2}(|\nabla V_i(t)|)(\boldsymbol{X}^i_t(y))\,dt \leq \sqrt{\varepsilon_i} \right\}.$$

By Theorem 1.5, the set $U_i(r)$ consists of points of essential stability of $\boldsymbol{X}^i$ for i large enough. From (7.9) and (7.10), we have

$$\mathfrak{m}_i(U_i(r)) \geq (1 - C(N)\sqrt{\varepsilon_i})\mathfrak{m}_i(B_r(x_i)). \tag{7.11}$$

Given $\lambda_i \to \infty$ and $r_i \to 0$, by (7.11) and Theorem 2.1, if $\lambda_i r_i \to \infty$ slowly enough, the sets $U_i(r_i)$ have asymptotically full measure in the spaces $(X_i, \lambda_i d_i, \mathfrak{m}_i, x_i)$. By Proposition 5.19, for $y \in U_i(r_i)$, $r < 1/\lambda_i$, we get

$$\fint_{B_r(y)^{\times 2}} \mathrm{dt}_r(\boldsymbol{X}^i_1)d(\mathfrak{m}_i \times \mathfrak{m}_i) \leq C(N)\sqrt{\varepsilon_i}\,r,$$

verifying part (3)(d) of Definition 1.12. The analogue properties for $\boldsymbol{X}^i_{-1} : [\lambda_i X_i, \boldsymbol{X}^i_1(x_i)] \to [\lambda_i X_i, x_i]$ follow from an identical argument. Property (2) of Definition 1.12 follows from the definition of essential stability. □

Definition 7.12. Let X be a geodesic space, $\rho : Y \to X$ a covering map, and $\varphi : [0, T] \times X \to X$ be a function such that for each $x \in X$, the map $t \mapsto \varphi(t, x)$ is continuous, and $\varphi(0, x) = x$. The *lift* of φ is defined to be the unique map $\psi : [0, T] \times Y \to Y$ such that for each $y \in Y$, the map $t \mapsto \psi(t, y)$ is continuous, $\psi(0, y) = y$, and $\rho(\psi(t, y)) = \varphi(t, y)$ for all $t \in [0, T]$.

Notice that if Y is the universal cover of X, then ψ is a deck type map with $\psi_* = \mathrm{Id}_{\pi_1(X)}$.

Proposition 7.13. *Let $(X, d, \mathfrak{m}, p)$ be a pointed* RCD$(-(N-1), N)$ *space, $(\widetilde{X}, \tilde{d}, \widetilde{\mathfrak{m}}, \tilde{p})$ its universal cover, $V \in L^1([0, T]; L^2(TX))$ a vector field satisfying the conditions of Theorem 2.39, $\boldsymbol{X} : [0, T] \times X \to X$ its RLF, and $\widetilde{V} : [0, T] \to L^2_{\mathrm{loc}}(T\widetilde{X})$ its lift. Then $\widetilde{\boldsymbol{X}} : [0, T] \times \widetilde{X} \to \widetilde{X}$, the lift of $\boldsymbol{X}$, is the RLF of $\widetilde{V}$. Moreover, if p is a point of essential stability of $\boldsymbol{X}$, then $\tilde{p}$ is a point of essential stability of $\widetilde{\boldsymbol{X}}$.*

Proof. Let $\rho : \widetilde{X} \to X$ be the projection. $R.1$ holds by construction. To verify $R.2$, notice that by linearity, it is enough to check it for $\tilde{f} \in \mathrm{TestF}(\widetilde{X})$ supported in a ball $\tilde{B} \subset \widetilde{X}$ sent isomorphically as a metric measure space to a ball $B = \rho(\tilde{B})$. For such $\tilde{f}$, it induces a function $f \in \mathrm{TestF}(X)$ supported in B with $\tilde{f}|_{\tilde{B}} = f \circ \rho|_{\tilde{B}}$. Then $R.2$ holds for $\widetilde{\boldsymbol{X}}$ and $\tilde{f}$ since it holds for $\boldsymbol{X}$ and f by locality of (1.3).

To verify $R.3$, consider a Borel partition $\{E_k\}_{k \in \mathbb{N}}$ of $\widetilde{X}$ consisting of subsets sent isomorphically by ρ as metric measure spaces to subsets of X. For a Borel set $A \subset \widetilde{X}$, and $t \in [0, T]$, setting

$$A_{k,\ell} := A \cap E_k \cap \widetilde{\boldsymbol{X}}_t^{-1}(E_\ell),$$

and using that $\boldsymbol{X}$ satisfies $R.3$, we get

$$\begin{aligned}\widetilde{\mathfrak{m}}(\widetilde{\boldsymbol{X}}_t(A)) &= \sum_{k,\ell \in \mathbb{N}} \widetilde{\mathfrak{m}}(\widetilde{\boldsymbol{X}}_t(A_{k,\ell})) = \sum_{k,\ell \in \mathbb{N}} \mathfrak{m}(\boldsymbol{X}_t(\rho(A_{k,\ell}))) \\ &\leq \sum_{k,\ell \in \mathbb{N}} C\mathfrak{m}(\rho(A_{k,\ell})) = \sum_{k,\ell} C\widetilde{\mathfrak{m}}(A_{k,\ell}) = C\widetilde{\mathfrak{m}}(A),\end{aligned}$$

and hence $\widetilde{X}$ is the RLF of $\widetilde{V}$.

Now assume p is a point of essential stability. Let $R \geq 1$ be such that $\boldsymbol{X}([0, T] \times \{p\}) \subset B_R(p)$. By Proposition 2.9, there is $r_0 \leq \frac{1}{10}$ such that any two curves $\alpha, \beta : [a, b] \to B_{2R}(p)$ sharing endpoints and at uniform distance $\leq 10r_0$, are homotopic relative to their endpoints. Then for each $t \in [0, T]$, the ball $B_{2r_0}(\widetilde{\boldsymbol{X}}_t(\tilde{p}))$ is isomorphic as a metric measure space to $B_{2r_0}(\boldsymbol{X}_t(p))$, so for $r \leq r_0$ small enough one has

$$\frac{1}{M}\widetilde{\mathfrak{m}}(B_r(\tilde{p})) \leq \widetilde{\mathfrak{m}}(B_r(\widetilde{\boldsymbol{X}}_t(\tilde{p}))) \leq M\widetilde{\mathfrak{m}}(B_r(\tilde{p})) \quad \text{for all } t \in [0, T].$$

By hypothesis, for $r \leq r_0$ small enough, there is $A_r \subset B_r(p)$ with

$$\mathfrak{m}(A_r) \geq \tfrac{1}{2}\mathfrak{m}(B_r(p)), \quad \text{and} \quad \boldsymbol{X}_t(A_r) \subset B_{2r}(\boldsymbol{X}_t(p)) \quad \text{for all } t \in [0, T].$$

Then if $\tilde{A}_r \subset \widetilde{X}$ denotes the intersection of the preimage of A_r with $B_r(\tilde{p})$, one has

$$\widetilde{\mathfrak{m}}(\tilde{A}_r) \geq \tfrac{1}{2}\widetilde{\mathfrak{m}}(B_r(\tilde{p})), \quad \text{and} \quad \widetilde{\boldsymbol{X}}_t(\tilde{A}_r) \subset B_{2r}(\widetilde{\boldsymbol{X}}_t(\tilde{p})) \quad \text{for all } t \in [0, T].$$ □

8. Rescaling theorem

In this section we prove the following result, following the lines of [Kapovitch and Wilking 2011, Section 5].

Theorem 8.1. *For each i, let $(X_i, d_i, \mathfrak{m}_i, p_i)$ be an* $\mathrm{RCD}\big(-\frac{1}{i}, N\big)$ *space of rectifiable dimension n, $(\widetilde{X}_i, \tilde{d}_i, \widetilde{\mathfrak{m}}_i, \tilde{p}_i)$ be its universal cover, and define $\Gamma_i := \mathcal{G}(\pi_1(X_i), \widetilde{X}_i, \tilde{p}_i, 1)$. If (X_i, p_i) converges in the pointed Gromov–Hausdorff sense to $(\mathbb{R}^k, 0)$ with $k < n$, then, after taking a subsequence, there are sets $\Theta_i \subset B_{1/2}(p_i)$ such that $\mathfrak{m}_i(\Theta_i)/\mathfrak{m}_i(B_{1/2}(p_i)) \to 1$ as $i \to \infty$, a sequence $\lambda_i \to \infty$, and a compact space (Y, y) such that $Y \neq \{*\}$,* $\mathrm{diam}(Y) \leq \frac{1}{10}$, *and*

(1) *for all $x_i \in \Theta_i$, after taking a subsequence, $(\lambda_i X_i, x_i)$ converges to $(\mathbb{R}^k \times Y, (0, y_1))$ in the pointed Gromov–Hausdorff sense (y_1 may depend on the x_i, but Y doesn't), and, for any lift $\tilde{x}_i \in B_{1/2}(\tilde{p}_i)$,*

$$\Gamma_i = \mathcal{G}(\pi_1(X_i), \lambda_i X_i, x_i, 1),$$

(2) *for all $a_i, b_i \in \Theta_i$ and lifts $\tilde{a}_i, \tilde{b}_i \in B_{1/2}(\tilde{p}_i)$, there are sequences*

$$h_i : [\lambda_i X_i, a_i] \to [\lambda_i X_i, b_i],$$
$$f_i : [\lambda_i \widetilde{X}_i, \tilde{a}_i] \to [\lambda_i \widetilde{X}_i, \tilde{b}_i]$$

of maps with the GS property such that the f_i are deck maps with $(f_i)_* \in (\Gamma_i)_*$ for all i, where $(\Gamma_i)_* := \{g_* : \pi_1(X_i) \to \pi_1(X_i) \mid g \in \Gamma_i\}$.

Lemma 8.2. *For each $N \geq 1$, $L \geq 1$, there are $R > 1$, $\delta \leq \frac{1}{100}$, such that the following holds. If $r \leq 1$, $(X, d, \mathfrak{m}, p)$ is a pointed* $\mathrm{RCD}(-\delta, N)$ *space with $d_{GH}((r^{-1}X, p), (\mathbb{R}^k, 0)) \leq \delta$, and $f \in H^{1,2}(X; \mathbb{R}^k)$ is an L-Lipschitz function with $f(p) = 0$ such that ∇f_j is divergence free in $B_R(p)$ for each $j \in \{1, \dots, k\}$, and for all $s \in [r, R]$, one has*

$$\fint_{B_s(p)} \Bigg[\sum_{j_1, j_2 = 1}^{k} |\langle \nabla f_{j_1}, \nabla f_{j_2} \rangle - \delta_{j_1, j_2}| + \sum_{j=1}^{k} |\nabla \nabla (f_j)|^2 \Bigg] d\mathfrak{m} \leq \delta.$$

Then:

(1) *For all $x_1, x_2 \in B_{10^k r}(p)$, and $r_1, r_2 \in \big[\frac{r}{4}, (10^k + 1)r\big]$, one has*

$$\mathfrak{m}(B_{r_1}(x_1)) \leq 2 \cdot \frac{r_1^k}{r_2^k} \cdot \mathfrak{m}(B_{r_2}(x_2)).$$

(2) *For all $x \in B_{10^k r}(p)$, if $\boldsymbol{X} : [0, 1] \times X \to X$ denotes the RLF of the vector field*

$$V_x := -\sum_{j=1}^{k} f_j(x) \nabla f_j, \tag{8.3}$$

then there is a set $A \subset B_{r/10}(x)$ of points of essential stability of $\boldsymbol{X}$ with

$$\mathfrak{m}(A) \geq \tfrac{1}{2} \mathfrak{m}(B_{r/10}(x)) \quad \text{and} \quad \boldsymbol{X}_1(A) \subset B_{r/5}(p). \tag{8.4}$$

Proof. By replacing X by $r^{-1}X$ and f by $r^{-1}f$, we can assume $r = 1$, and without loss of generality we can also assume $(X, d, \mathfrak{m}, p)$ is normalized. Arguing by contradiction, we get sequences $R_i \to \infty$, $\delta_i \to 0$, a sequence $(X_i, d_i, \mathfrak{m}_i, p_i)$ of normalized RCD$(-\delta_i, N)$ spaces for which (X_i, p_i) converges to $(\mathbb{R}^k, 0)$ in the pointed Gromov–Hausdorff sense and L-Lipschitz functions $f^i \in H^{1,2}(X_i; \mathbb{R}^k)$ with $f^i(p_i) = 0$ such that

- ∇f^i_i is divergence free in $B_{R_i}(p_i)$ for each $j \in \{1, \dots, k\}$, $i \in \mathbb{N}$,
- for all $s \in [1, R_i]$, one has

$$\fint_{B_s(p_i)} \Bigg[\sum_{j_1, j_2 = 1}^{k} |\langle \nabla f^i_{j_1}, \nabla f^i_{j_2} \rangle - \delta_{j_1, j_2}| + \sum_{j=1}^{k} |\nabla\nabla(f^i_j)|^2 \Bigg] \, d\mathfrak{m}_i \le \delta_i .$$

And for each i, at least one of the conditions (1) or (2) fails. Notice however, that (1) holds as by Corollary 2.26, $(X_i, d_i, \mathfrak{m}_i, p_i)$ converges to $(\mathbb{R}^k, d^{\mathbb{R}^k}, \mathcal{H}^k, 0)$ in the pointed measured Gromov–Hausdorff sense. For a sequence $x_i \in B_{10^k}(p_i)$, let

$$V^i := -\sum_{j=1}^{k} f^i_j(x_i) \nabla f^i_j ,$$

and $\boldsymbol{X}^i : [0, 1] \times X_i \to X_i$ its RLF. Then, for $s = (k+1) \cdot 10^k L^2$ and i large enough,

$$\begin{aligned} &\fint_{B_{1/10}(x_i)} \int_0^1 \sum_{j_1, j_2 = 1}^{k} |\langle \nabla f^i_{j_1}, \nabla f^i_{j_2} \rangle - \delta_{j_1, j_2}|(\boldsymbol{X}^i_t(y)) \, dt \, d\mathfrak{m}_i(y) \\ &\qquad = \int_0^1 \fint_{\boldsymbol{X}^i_t(B_{1/10}(x_i))} \sum_{j_1, j_2 = 1}^{k} |\langle \nabla f^i_{j_1}, \nabla f^i_{j_2} \rangle - \delta_{j_1, j_2}|(y) \, d\mathfrak{m}_i(y) \, dt \\ &\qquad \le \frac{\mathfrak{m}_i(B_s(p_i))}{\mathfrak{m}_i(B_{1/10}(x_i))} \fint_{B_s(p_i)} \sum_{j_1, j_2 = 1}^{k} |\langle \nabla f^i_{j_1}, \nabla f^i_{j_2} \rangle - \delta_{j_1, j_2}|(y) \, d\mathfrak{m}_i(y) \\ &\qquad \le C(N, L) \delta_i . \end{aligned}$$

Then, from the definition of RLF,

$$\begin{aligned} &\fint_{B_{1/10}(x_i)} |(f^i(\boldsymbol{X}^i_1(y)) - f^i(y)) + f^i(x_i)| \, d\mathfrak{m}_i(y) \\ &\quad \le \sum_{j=1}^{k} \fint_{B_{1/10}(x_i)} |(f^i_j(\boldsymbol{X}^i_1(y)) - f^i_j(y)) + f^i_j(x_i)| \, d\mathfrak{m}_i(y) \\ &\quad \le \sum_{j=1}^{k} \fint_{B_{1/10}(x_i)} |f^i_j(x_i)| \int_0^1 \big| |\nabla f^i_j|^2 - 1 \big| (\boldsymbol{X}^i_t(y)) \, dt \, d\mathfrak{m}_i(y) \\ &\qquad\qquad + \sum_{\substack{j_1, j_2 = 1 \\ j_1 \ne j_2}}^{k} \fint_{B_{1/10}(x_i)} |f^i_{j_1}(x_i)| \int_0^1 \langle \nabla f^i_{j_1}, \nabla f^i_{j_2} \rangle (\boldsymbol{X}^i_t(y)) \, dt \, d\mathfrak{m}_i(y) \\ &\quad \le C(N, L) \delta_i . \end{aligned}$$

From the last assertion of Lemma 2.36, $|f^i(y) - f^i(x_i)| \le \frac{3}{20}$ for all $y \in B_{1/10}(x_i)$ if i is large enough, so the set

$$A_i' := \left\{ y \in B_{1/10}(x_i) \;\middle|\; |\boldsymbol{X}_1^i(y)| < \tfrac{1}{5} \right\}$$

satisfies $\mathfrak{m}_i(A_i')/\mathfrak{m}_i(B_{1/10}(x_i)) \to 1$. Then by Corollary 1.8, if we define A_i to be the points of A_i' that are of essential stability of $\boldsymbol{X}^i$, we get that $\mathfrak{m}_i(A_i) \ge \frac{1}{2}\mathfrak{m}_i(B_{1/10}(x_i))$ for i large enough, implying condition (2); a contradiction. □

Lemma 8.5. *For $N \ge 1$, $L \ge 1$, let $\delta > 0$ be given by Lemma 8.2. Then there are $R > 1$, $C_0 > 1$, $\varepsilon_0 > 0$ such that the following holds. Let $r \le 1$, $(X, d, \mathfrak{m}, p)$ be a pointed* $\mathrm{RCD}(-\delta, N)$ *space with $d_{GH}((r^{-1}X, p), (\mathbb{R}^k, 0)) \le \delta$, and $f \in H^{1,2}(X; \mathbb{R}^k)$ an L-Lipschitz function with $f(p) = 0$ such that ∇f_j is divergence free in $B_R(p)$ for each $j \in \{1, \dots, k\}$, and*

$$Mx_R\Bigg(\sum_{j_1, j_2 = 1}^{k} |\langle \nabla f_{j_1}, \nabla f_{j_2}\rangle - \delta_{j_1, j_2}| + \sum_{j=1}^{k} |\nabla\nabla(f_j)|^2 \Bigg)(p) \le \varepsilon^2 \le \varepsilon_0^2.$$

For $x \in B_{10^k r}(p)$, let V_x be given by (8.3), and let $\boldsymbol{X} : [0, 1] \times X \to X$ be the RLF of V_x. Then there is a subset $B'_{r/2}(x) \subset B_{r/2}(x)$ of points of essential stability of $\boldsymbol{X}$ satisfying

$$\boldsymbol{X}_1(B'_{r/2}(x)) \subset B_r(p), \tag{8.6}$$

$$\mathfrak{m}(B'_{r/2}(x)) \ge (1 - C_0 \varepsilon r)\mathfrak{m}(B_{r/2}(x)), \tag{8.7}$$

$$\frac{1}{\mathfrak{m}(B_{r/2}(x))} \int_{B'_{r/2}(x)} \int_0^1 Mx(|\nabla V_x|^{3/2})^{2/3}(\boldsymbol{X}_t(y))\, dt\, d\mathfrak{m}(y) \le C_0 \varepsilon r. \tag{8.8}$$

Proof. Set $s = (k+1) \cdot 10^k L^2 r$ and compute, provided $R \ge 2s + 8$,

$$\begin{aligned}
\fint_{B_{r/2}(x)} \int_0^1 \mathrm{Mx}_4(|\nabla V_x|^{3/2})^{2/3}(\boldsymbol{X}_t(y))\, dt\, d\mathfrak{m}(y) &= \int_0^1 \fint_{\boldsymbol{X}_t(B_{r/2}(x))} \mathrm{Mx}_4(|\nabla V_x|^{3/2})^{2/3}\, d\mathfrak{m}\, dt \\
&\le \frac{1}{\mathfrak{m}(B_{r/2}(x))} \int_{B_s(p)} \mathrm{Mx}_4(|\nabla V_x|^{3/2})^{2/3}\, d\mathfrak{m} \\
&= \frac{\mathfrak{m}(B_s(p))}{\mathfrak{m}(B_{r/2}(x))} \fint_{B_s(p)} \mathrm{Mx}_4(|\nabla V_x|^{3/2})^{2/3}\, d\mathfrak{m} \\
&\le C(N, L) \cdot \mathrm{Mx}_s(\mathrm{Mx}_4(|\nabla V_x|^{3/2})^{2/3})(p) \\
&\le C(N, L) \cdot \mathrm{Mx}_R(|\nabla V_x|^2)^{1/2}(p) \\
&\le C(N, L) \sum_{j=1}^{k} |f_j(x)|^2 \mathrm{Mx}_R(|\nabla\nabla f_j|^2)^{1/2}(p) \\
&\le C(N, L) \cdot \varepsilon \cdot r.
\end{aligned} \tag{8.9}$$

Combining this with Theorem 1.5, if

$$A_0 := \{ y \in B_{r/2}(x) \mid y \text{ is of essential stability of } \boldsymbol{X} \},$$

then

$$\mathfrak{m}(A_0) \ge (1 - C(N, L)\varepsilon r)\mathfrak{m}(B_{r/2}(x)). \tag{8.10}$$

From Lemma 8.2, if ε is small enough, there is a set $A \subset B_{r/10}(x) \cap A_0$ satisfying (8.4). By (8.9), there is $q \in A$ with $\int_0^1 \mathrm{Mx}(|\nabla V_x|)(\boldsymbol{X}_t(q))\, dt \leq C(N, L)\varepsilon r$, and by Proposition 5.19, this implies

$$\fint_{B_r(q)^{\times 2}} \mathrm{dt}_r(1)\, d(\mathfrak{m} \times \mathfrak{m}) \leq C(N, L)\varepsilon r^2,$$

so

$$\fint_{A \times A_0} \mathrm{dt}_r(1)\, d(\mathfrak{m} \times \mathfrak{m}) \leq C(N, L)\varepsilon r^2.$$

Hence there is $y \in A$ such that

$$\fint_{A_0} \mathrm{dt}_r(1)(y, z)\, d\mathfrak{m}(z) \leq C(N, L)\varepsilon r^2, \tag{8.11}$$

so we define

$$B'_{r/2}(x) := \{z \in A_0 \mid \mathrm{dt}_r(1)(y, z) < r/10\}.$$

Then for all $z \in B'_{r/2}(x)$ we have

$$\begin{aligned} d(\boldsymbol{X}_1(z), p) &\leq d(\boldsymbol{X}_1(z), \boldsymbol{X}_1(y)) + d(\boldsymbol{X}_1(y), p) \\ &\leq d(z, y) + r/10 + r/5 \\ &\leq r/2 + r/10 + r/10 + r/5 < r, \end{aligned}$$

so (8.6) holds. (8.10) and (8.11) imply (8.7), and (8.9) implies (8.8). □

Lemma 8.12. *For $N \geq 1$, $L \geq 1$, let $\delta > 0$ be given by Lemma 8.2. Then there are $R \geq 1$, $C_0 \geq 1$, $\varepsilon_0 > 0$ such that the following holds. Assume $(X, d, \mathfrak{m}, p)$ is a pointed* RCD$(-\delta, N)$ *space with $d_{GH}((X, p), (\mathbb{R}^k, 0)) \leq \delta$, and $f \in H^{1,2}(X; \mathbb{R}^k)$ is an L-Lipschitz function with $f(p) = 0$ such that ∇f_j is divergence free in $B_R(p)$ for each $j \in \{1, \ldots, k\}$, and*

$$Mx_R\Bigg(\sum_{j_1, j_2=1}^{k} |\langle \nabla f_{j_1}, \nabla f_{j_2}\rangle - \delta_{j_1, j_2}| + \sum_{j=1}^{k} |\nabla\nabla(f_j)|^2\Bigg)(p) \leq \varepsilon^2 \leq \varepsilon_0^2.$$

Assume p is an n-regular point with $n > k$ and let

$$\rho \geq \sup\{r \in (0, 1] \mid d_{GH}((r^{-1}X, p), (\mathbb{R}^k, 0)) = \delta\}.$$

Then there is a set $G \subset B_1(p)$ with

$$\mathfrak{m}(G) \geq (1 - C_0\varepsilon)\mathfrak{m}(B_1(p)),$$

a finite number of divergence-free on $B_{100C_0}(p)$ vector fields

$$V_1, \ldots, V_m \in L^1([0, 1]; H^{1,2}_{C,s}(TX))$$

with $\|V_j(t)\|_\infty \leq C_0$ for all $t \in [0, 1]$, $j \in \{1, \ldots, m\}$, with RLFs

$$\boldsymbol{X}^1, \ldots, \boldsymbol{X}^m : [0, 1] \times X \to X,$$

and a measurable map $\theta : G \to \{1, \ldots, m\}$ such that for all $y \in G$, y is a point of weak essential stability of $\boldsymbol{X}^{\theta(y)}$, $\boldsymbol{X}_1^{\theta(y)}(y) \in B_\rho(p)$, and

$$\fint_G \int_0^1 Mx(|\nabla V_{\theta(y)}(t)|^{3/2})^{2/3}(\boldsymbol{X}_t^{\theta(y)}(y))\, dt\, d\mathfrak{m}(y) \leq C_0\varepsilon.$$

Proof. We will show that for each $r \leq 1$, there is $G_r \subset B_r(p)$ with

$$\mathfrak{m}(G_r) \geq (1 - C_0 \varepsilon r)\mathfrak{m}(B_r(p)),$$

a finite number of divergence-free on $B_{100C_0}(p)$ vector fields

$$W_1, \dots, W_m \in L^1([0,1]; H^{1,2}_{C,s}(TX))$$

with $\|W_j(t)\|_\infty \leq C_0 r$ for all $t \in [0,1]$, $j \in \{1, \dots, m\}$, with RLFs

$$\Phi^1, \dots, \Phi^m : [0,1] \times X \to X,$$

and a measurable map $\theta_r : G_r \to \{1, \dots, m\}$ satisfying that for all $y \in G_r$, y is a point of weak essential stability of $\Phi^{\theta_r(y)}$, $\Phi_1^{\theta_r(y)}(y) \in B_\rho(p)$, and

$$\fint_{G_r} \int_0^1 (\mathrm{Mx}(|\nabla W_{\theta_r(y)}(t)|)^{3/2})^{2/3}(\Phi_t^{\theta_r(y)})(y)\, dt\, d\mathfrak{m}(y) \leq C_0 \varepsilon r.$$

Clearly, the claim holds for $r \leq \rho$ with $G_r = B_r(p)$ and the zero vector field. Now we check that if the claim holds for some $r \leq 10^{-k}$, then it also holds for $10^k r$.

Choose $\{q_1, \dots, q_\ell\}$, a maximal $r/2$-separated set in $B_{10^k r}(p)$. By Lemma 8.2.(1), one has

$$\sum_{j=1}^{\ell} \mathfrak{m}(B_{r/2}(q_j)) \leq 2^{k+1} \sum_{j=1}^{\ell} \mathfrak{m}(B_{r/4}(q_j)) \leq 2^{k+3}\mathfrak{m}(B_{10^k r}(p)).$$

By Lemma 8.5, if ε is small enough and R is large enough, for each $j \in \{1, \dots, \ell\}$ there is a divergence-free vector field $\overline{W}_j \in H^{1,2}_{C,s}(TX)$ such that $\|\overline{W}_j\|_\infty \leq C(N,L)r$, with RLF

$$\overline{\Phi}^j : [0,1] \times X \to X,$$

and a set $B'_{r/2}(q_j)$ of points of essential stability of $\overline{\Phi}^j$ such that $\overline{\Phi}^j_1(B'_{r/2}(q_j)) \subset B_r(p)$, and

$$\mathfrak{m}(B'_{r/2}(q_j)) \geq (1 - C(N,L)\varepsilon r)\mathfrak{m}(B_{r/2}(q_j)),$$

$$\frac{1}{\mathfrak{m}(B_{r/2}(q_j))} \int_{B'_{r/2}(q_j)} \int_0^1 \mathrm{Mx}(|\nabla \overline{W}_j|^{3/2})^{2/3}(\overline{\Phi}^j_t(y))\, dt\, d\mathfrak{m}(y) \leq C(N,L)\varepsilon r.$$

Set

$$G_{10^k r} := B_{10^k r}(p) \cap \bigcup_{j=1}^{\ell} (B'_{r/2}(q_j) \cap ((\overline{\Phi}^j_1)^{-1}(G_r))).$$

Then

$$\begin{aligned}
\mathfrak{m}(G_{10^k r}) &\geq \mathfrak{m}(B_{10^k r}(p)) - \sum_{j=1}^{\ell} (\mathfrak{m}(B_{r/2}(q_j)) - \mathfrak{m}(B'_{r/2}(q_j)) + \mathfrak{m}(B_r(p)) - \mathfrak{m}(G_r)) \\
&\geq \mathfrak{m}(B_{10^k r}(p)) - \sum_{j=1}^{\ell} \big(C(N,L)\varepsilon r \mathfrak{m}(B_{r/2}(q_j)) + 2^{k+3} C_0 \varepsilon r \mathfrak{m}(B_{r/2}(q_j))\big) \\
&\geq \big(1 - 2^{k+2}(C(N,L)\varepsilon r + 2^{k+3} C_0 \varepsilon r)\big)\mathfrak{m}(B_{10^k r}(p)) \\
&\geq (1 - C_0 \varepsilon 10^k r)\mathfrak{m}(B_{10^k r}(p)),
\end{aligned}$$

provided C_0 was chosen large enough, depending on N and L. For each $y \in G_{10^k r}$, set V_y as follows: let $j \in \{1, \dots, \ell\}$ be the smallest index for which $y \in B'_{r/2}(q_j) \cap ((\overline{\Phi}^j_1)^{-1}(G_r))$. Then define

$$V_y(t) := \begin{cases} 2\overline{W}_j & \text{if } t \in \left[0, \frac{1}{2}\right) \\ 2W_{\theta_r(\overline{\Phi}^j_1(y))}(2t-1) & \text{if } t \in \left[\frac{1}{2}, 1\right]. \end{cases}$$

Notice that $\|V_y(t)\|_\infty \le \max\{C(N, L)r, 2C_0 r\} \le C_0 10^k r$ for all $t \in [0, 1]$, provided C_0 was chosen large enough. Set Ψ^y be the RLF of V_y, and set

$$B''_{r/2}(q_j) := G_{10^k r} \cap B'_{r/2}(q_j) \setminus \bigcup_{\alpha=1}^{j-1} B'_{r/2}(q_\alpha).$$

Then

$$\begin{aligned}
&\fint_{G_{10^k r}} \int_0^1 \mathrm{Mx}(|\nabla V_y|^{3/2})^{2/3}(\Psi^y_t(y))\, dt\, d\mathfrak{m}(y) \\
&\qquad \le \frac{1}{\mathfrak{m}(B_{10^k r}(p))} \sum_{j=1}^{\ell} \int_{B''_{r/2}(q_j)} \int_0^1 \mathrm{Mx}(|\nabla V_y|^{3/2})^{2/3}(\Psi^y_t(t))\, dt\, d\mathfrak{m}(y) \\
&\qquad \le \frac{1}{\mathfrak{m}(B_{10^k r}(p))} \sum_{j=1}^{\ell} \big(C(N, L)\varepsilon r \mathfrak{m}(B_{r/2}(q_j)) + 2^{k+2} C_0 \varepsilon r \mathfrak{m}(B_{r/2}(q_j))\big) \\
&\qquad \le C_0 \varepsilon 10^k r,
\end{aligned}$$

again provided C_0 was chosen large enough, depending on N and L. □

Proof of Theorem 8.1. By Lemma 2.34, there are sequences $\delta_i \to 0$, $R_i \to \infty$, and a sequence of $L(N)$-Lipschitz maps $h^i \in H^{1,2}(X_i; \mathbb{R}^k)$ with $h^i(p_i) = 0$ for all i, ∇h^i_j divergence free in $B_{R_i}(p_i)$ for each $j \in \{1, \dots, k\}$, and such that if

$$u_i := \sum_{j_1, j_2 = 1}^{k} |\langle \nabla h^i_{j_1}, \nabla h^i_{j_2}\rangle - \delta_{j_1, j_2}| + \sum_{j=1}^{k} |\nabla\nabla h^i_j|^2,$$

then for all $r \in [1, R_i]$, one has

$$\fint_{B_r(p_i)} u_i\, d\mathfrak{m}_i \le \delta_i^3.$$

Set $\Theta_i := \{x \in B_{1/2}(p_i) \mid x \text{ is } n\text{-regular}, \mathrm{Mx}(u_i) \le \delta_i^2\}$. By Proposition 2.5(1), $\mathfrak{m}_i(\Theta_i)/\mathfrak{m}_i(B_{1/2}(p_i)) \to 1$. Notice that, possibly after modifying δ_i and R_i, we may assume

$$\mathrm{Mx}_{R_i}(u_i)(x) \le \delta_i^2 \quad \text{for all } x \in \Theta_i.$$

For $\delta \le \frac{1}{100}$ given by Lemma 8.2, set

$$\lambda_i := \inf_{x \in \Theta_i} \inf\{\lambda \ge 1 \mid d_{GH}((\lambda X_i, x), (\mathbb{R}^k, 0)) \ge \delta\}.$$

First we check that λ_i is finite. Fix $i \in \mathbb{N}$ and take $z \in \Theta_i$. Since z is n-regular, as $\lambda \to \infty$ the distance $d_{GH}((\lambda X_i, z), (\mathbb{R}^k, 0))$ converges to $d_{GH}((\mathbb{R}^n, 0), (\mathbb{R}^k, 0)) > \delta$, so $\lambda_i < \infty$.

Now we show that $\lambda_i \to \infty$. If after passing to a subsequence, the sequence λ_i converges to a number $\lambda_\infty \in [1, \infty)$, then for any choice of $x_i \in \Theta_i$ the sequence $(\lambda_i X_i, x_i)$ converges to $(\lambda_\infty \mathbb{R}^k, 0) = (\mathbb{R}^k, 0)$ in the pointed Gromov–Hausdorff sense. However, by the definition of λ_i there are $y_i \in \Theta_i$ with $d_{GH}((\lambda_i X_i, y_i), (\mathbb{R}^k, 0)) \geq \delta/2$; a contradiction.

We claim there is a sequence $\mu_i \to \infty$ with the property that for all $x_i \in \Theta_i$, and any lift $\tilde{x}_i \in B_{1/2}(\tilde{p}_i)$,

$$\Gamma_i = \mathcal{G}(\pi_1(X_i), \widetilde{X}_i, \tilde{x}_i, 1/\lambda_i) = \mathcal{G}(\pi_1(X_i), \widetilde{X}_i, \tilde{x}_i, \mu_i). \tag{8.13}$$

Otherwise, by Corollary 2.55, after taking a subsequence, there would be a sequence

$$r_i \in \sigma(\pi_1(X_i), \widetilde{X}_i, \tilde{x}_i) \cap [1/\lambda_i, M] \quad \text{for some } M > 0.$$

After again taking a subsequence, by Lemma 2.36 and Proposition 2.33, we can assume $(r_i^{-1} X_i, x_i)$, $(r_i^{-1} \widetilde{X}_i, \tilde{x}_i)$ converge to $(\mathbb{R}^k \times Z, (0, z))$, $(\mathbb{R}^k \times \widetilde{Z}, (0, \tilde{z}))$, respectively, for some spaces $Z, \widetilde{Z}$ with $\operatorname{diam}(Z) \leq \frac{1}{10}$ (see Remark 2.20) in such a way that the sequence Γ_i converges equivariantly to some group $\Gamma \leq \operatorname{Iso}(\mathbb{R}^k \times \widetilde{Z})$ that acts trivially on the first factor and such that $\widetilde{Z}/\Gamma = Z$. By Lemma 2.56, $r \leq \frac{1}{2}$ for all $r \in \sigma(\Gamma)$, but by construction $1 \in \sigma(\Gamma_i, r_i^{-1} \widetilde{X}_i, \tilde{x}_i)$ for all i, contradicting Proposition 2.52 and proving (8.13).

Let $x_i \in \Theta_i$ be such that

$$\inf\{\lambda \geq 1 \mid d_{GH}((\lambda X_i, x_i), (\mathbb{R}^k, 0)) \geq \delta\} \leq \lambda_i + 1.$$

After passing to a subsequence, by Lemma 2.36, we can assume $(\lambda_i X_i, x_i)$ converges to $(\mathbb{R}^k \times Y, (0, y))$ for some compact space (Y, y) with $\operatorname{diam}(Y) \in \left(0, \frac{1}{10}\right]$.

For $a_i, b_i \in \Theta_i$, let $G(a_i) \subset B_1(a_i)$, $G(b_i) \subset B_1(b_i)$ be the sets given by Lemma 8.12, and let $U_i := G(a_i) \cap G(b_i)$. Notice that for i large enough we have

$$\max\{\mathfrak{m}_i(B_1(a_i)), \mathfrak{m}_i(B_1(b_i))\} \leq C \cdot \mathfrak{m}_i(U_i) \quad \text{for some } C(N).$$

For each $y \in U_i$, let $V_y^{a_i}$, $V_y^{b_i} \in L^1([0, 1]; H^{1,2}_{C,s}(TX_i))$ denote the vector fields given by Lemma 8.12, define $V_y \in L^1([0, 1]; H^{1,2}(TX_i))$ as

$$V_y(t) := \begin{cases} -2V_y^{a_i}(1 - 2t) & \text{if } t \in \left[0, \frac{1}{2}\right], \\ 2V_y^{b_i}(2t - 1) & \text{if } t \in \left[\frac{1}{2}, 1\right], \end{cases}$$

and let $\boldsymbol{X}^y : [0, 1] \times X_i \to X_i$ be its RLF. Then there are measurable maps

$$V : U_i \to L^\infty([0, 1]; H^{1,2}_{C,s}(TX_i))$$

with finite image such that, for all $y \in U_i$, V_y is a divergence-free on $B_{100C(N)}(p_i)$ vector field with $\|V_y(t)\|_\infty \leq C(N)$, there is a point $y' \in B_{1/\lambda_i}(a_i)$ of weak essential stability of $\boldsymbol{X}^y$ with $\boldsymbol{X}_1^y(y') \in B_{1/\lambda_i}(b_i)$, and

$$\fint_{U_i} \int_0^1 \operatorname{Mx}(|\nabla V_y(t)|^{3/2})^{2/3}(\boldsymbol{X}_t^y(y'))\, dt\, d\mathfrak{m}_i(y) \leq C(N)\delta_i.$$

This implies there is a sequence $y_i \in B_{1/\lambda_i}(a_i)$ and vector fields $W_i \in L^1([0, 1]; H^{1,2}_{C,s}(TX_i))$ with RLFs $\boldsymbol{X}^i : [0, 1] \times X_i \to X_i$ such that y_i is a point of weak essential stability of $\boldsymbol{X}^i$, $\boldsymbol{X}_1^i(y_i) \in B_{1/\lambda_i}(b_i)$, and

$$\int_0^1 \operatorname{Mx}(|\nabla W_i(t)|^{3/2})^{2/3}(\boldsymbol{X}_t^i(y_i))\, dt \leq C(N)\delta_i.$$

By Proposition 5.21, y_i is a point of essential stability of X^i for i large enough, so by Lemma 7.7, we get a sequence of GS maps $h_i : [\lambda_i X_i, a_i] \to [\lambda_i X_i, b_i]$. This implies, by Lemma 6.4, that the pointed measured Gromov–Hausdorff limit of the sequence $(\lambda_i X_i, a_i)$ does not depend on the choice of $a_i \in \Theta_i$.

By Propositions 2.7, 7.13, and Lemma 7.7, for lifts $\tilde{a}_i, \tilde{b}_i \in B_{1/2}(\tilde{p}_i)$, we also get a sequence of deck type GS maps $f_i' : [\lambda_i \widetilde{X}_i, \tilde{a}_i] \to [\lambda_i \widetilde{X}_i, b_i']$ with $(f_i')_* = \mathrm{Id}_{\pi_1(X_i)}$ for some b_i' in the preimage of b_i with

$$\tilde{d}(\tilde{a}_i, b_i') \leq C(N) \tag{8.14}$$

and such that $(f_i')_* = \mathrm{Id}_{\pi_1(X_i)}$. From (8.13) and (8.14), there are $g_i \in \Gamma_i$ with $g_i(b_i') = \tilde{b}_i$. Composing f_i' with g_i, we get a sequence of deck type GS maps $f_i : [\lambda_i \widetilde{X}_i, \tilde{a}_i] \to [\lambda_i \widetilde{X}_i, \tilde{b}_i]$ with $(f_i)_* = (g_i)_* \in (\Gamma_i)_*$. $\square$

9. Proof of main theorems

We now prove Theorem 1.14 by reverse induction on k. This is done by contradiction; after passing to a subsequence, we assume the following.

Assumption 9.1. There is a sequence of integers $\xi_i \to \infty$ such that no subgroup of Γ_i of index $\leq \xi_i$ admits a nilpotent basis of length $\leq n - k$ respected by $(f_{j,i})_*^{\xi_i !}$ for each $j \in \{1, \ldots, \ell\}$.

The base of induction consists of Proposition 9.2. The induction step is first proved assuming that the sequences $f_{j,i}$ converge to the identity and $Y \neq \{*\}$ (Proposition 9.3). Then we drop the assumption that the maps $f_{j,i}$ converge to the identity (Proposition 9.4), and the last step consists on dropping the assumption $Y \neq \{*\}$ (Proposition 9.5).

After taking a subsequence we may assume $(\widetilde{X}_i, \tilde{p}_i)$ converges to a space $(\mathbb{R}^k \times \widetilde{Y}, (0, \tilde{y}))$ and Γ_i converges equivariantly to some closed group $\Gamma \leq \mathrm{Iso}(\mathbb{R}^k \times \widetilde{Y})$, which by Proposition 2.33, acts trivially on the $\mathbb{R}^k$ factor. By Corollary 2.27, $\widetilde{Y}$ splits as a product $\mathbb{R}^m \times Z$ for some compact space Z, and by Corollary 2.65, Γ / Γ_0 has an abelian subgroup of finite index generated by at most m elements. After passing to a subsequence, by Lemma 6.4 we can also assume that for each $j \in \{1, \ldots, \ell\}$, $f_{j,i}$ converges weakly to an isometry $f_{j,\infty} : \mathbb{R}^k \times \widetilde{Y} \to \mathbb{R}^k \times \widetilde{Y}$.

Proposition 9.2. *Theorem 1.14 holds if $k = n$.*

Proof. By dimensionality, $\widetilde{Y}$ is trivial and so is Γ. By Lemma 3.2, the sequence Γ_i is trivial as well. $\square$

Proposition 9.3. *In the induction step, Assumption 9.1 leads to a contradiction if $Y \neq \{*\}$ and $f_{j,\infty} = \mathrm{Id}_{\mathbb{R}^k \times \widetilde{Y}}$ for all j.*

Proof. Let $v_1, \ldots, v_m \in \Gamma$ be such that $\{v_1 \Gamma_0, \ldots, v_m \Gamma_0\}$ generates a finite index abelian subgroup of Γ / Γ_0. For each $j \in \{1, \ldots, m\}$ pick $w_j^i \in \Gamma_i$ with $w_j^i \to v_j$, and define $\Upsilon_i \triangleleft \Gamma_i$ to be the subgroups given by Theorem 3.1. Then from the proof of Theorem 3.1 one has, for i large enough, that

- $\langle \Upsilon_i, w_1^i, \ldots, w_m^i \rangle$ is a finite index subgroup of Γ_i,
- $[w_{j_1}^i, w_{j_2}^i] \in \Upsilon_i$ for $j_1, j_2 \in \{1, \ldots, m\}$.

Furthermore, as $f_{j,i} \to \mathrm{Id}_{\mathbb{R}^k \times \widetilde{Y}}$ for each $j \in \{1, \ldots, \ell\}$, we also have

- $[f_{j,i}, w_{j_1}^i] \in \Upsilon_i$ for all $j_1 \in \{1, \ldots, m\}$ and large enough i.

Case 1: The sequence $[\Gamma_i : \Upsilon_i]$ is bounded.

By Lemma 2.57 and Proposition 2.58, there are characteristic subgroups $H_i \triangleleft \Gamma_i$ contained in Υ_i such that the sequence $[\Upsilon_i : H_i]$ is bounded. After slightly shifting the basepoints p_i, we may assume y is an α-regular point of Y. Let $\lambda_i \to \infty$ so slowly that

- $(\lambda_i X_i, p_i) \to (\mathbb{R}^{k+\alpha}, 0)$ in the pointed Gromov–Hausdorff sense,
- $f_{j,i} : [\lambda_i \widetilde{X}_i, \tilde{p}_i] \to [\lambda_i \widetilde{X}_i, \tilde{p}_i]$ still is GS and converges to $\mathrm{Id}_{\mathbb{R}^{m+\alpha}}$,
- $\lim_{i\to\infty} \sup \sigma(H_i, \lambda_i \widetilde{X}_i, \tilde{p}_i) < \infty$.

By Proposition 2.33, any pointed Gromov–Hausdorff limit of $(\lambda_i \widetilde{X}_i / H_i, [\tilde{p}_i])$ splits off $\mathbb{R}^{k+\alpha}$, and as $H_i \triangleleft \Gamma_i$ is characteristic, it is preserved by $(f_{j,i})_*$ for each j, so the induction hypothesis applies to the spaces $(\widetilde{X}_i / H_i, \lambda_i d_i, \mathfrak{m}_i, [\tilde{p}_i])$, contradicting Assumption 9.1.

Case 2: After passing to a subsequence, $[\Gamma_i : \Upsilon_i] \to \infty$.

For each $j \in \{1, \dots, m\}$, set $\Gamma_{i,j} := \langle \Upsilon_i, w_j^i, \dots, w_m^i \rangle$ and $\Gamma_{i,m+1} := \Upsilon_i$. Let $j_0 \in \{1, \dots, m\}$ be the smallest number such that, after passing to a subsequence, we get $[\Gamma_{i,j_0} : \Gamma_{i,j_0+1}] \to \infty$, and let $\Gamma_i' := \Gamma_{i,j_0}$, $\Upsilon_i' := \Gamma_{i,j_0+1}$. Notice that by our choice of w_j^i's, Υ_i' is normal in Γ_i'.

Let $X_i' := \widetilde{X}_i / \Upsilon_i'$, $p_i' \in X_i'$ the image of $\tilde{p}_i$, and $f_{\ell+1,i} := w_{j_0}^i \in \mathrm{Iso}(\widetilde{X}_i)$. After taking a subsequence, we can assume (X_i', p_i') converges to a space $(\mathbb{R}^k \times Y', (0, y'))$, Υ_i' converges to a closed group $\Upsilon' \leq \Gamma$, and Γ_i' converges to a closed group $\Gamma' \leq \Gamma$ with $[\Gamma : \Gamma'] < \infty$. By Theorem 3.1, $[\Gamma' : \Upsilon'] = \infty$, so the group Γ'/Υ' is noncompact.

Since Γ'/Υ' acts on Y' with compact quotient $\widetilde{Y}/\Gamma'$, Corollary 2.27 applies, and since Γ'/Υ' is noncompact, Y' contains a nontrivial Euclidean factor. Therefore the induction hypothesis applies to the sequence of spaces (X_i', p_i'), the groups Υ_i', and the maps $f_{j,i}$ for $j \in \{1, \dots, \ell+1\}$ (as $f_{j,i} \to \mathrm{Id}_{\mathbb{R}^k \times \widetilde{Y}}$, $(f_{j,i})_*$ preserves Υ_i'). This means there is $C > 0$ and subgroups $G_i' \leq \Upsilon_i'$ such that, for i large enough,

- $[\Upsilon_i' : G_i'] \leq C$,
- G_i' admits a nilpotent basis β_i' of length $\leq n-k-1$,
- $(f_{j,i})_*^{C!}$ respects β_i' for $j \in \{1, \dots, \ell+1\}$.

By Lemma 2.57 and Proposition 2.58, we can assume G_i' is characteristic in Υ_i'. Then we define $G_i := \langle G_i', f_{\ell+1,i}^{(2C)!} \rangle$. Notice that G_i admits the nilpotent chain β_i obtained by appending $f_{\ell+1,i}^{(2C)!}$ to β_i'. From the fact that $[f_{j,i}, f_{\ell+1,i}] \in \Upsilon_i'$ for $j \in \{1, \dots, \ell\}$ and Proposition 2.61, we have that $(f_{j,i})_*^{C!}$ respects β for $j \in \{1, \dots, \ell\}$. Finally, the sequence

$$[\Gamma_i : G_i] = [\Gamma_i : \Gamma_i'][\Gamma_i' : G_i] \leq [\Gamma_i : \Gamma_i'](2C)!C$$

is bounded, contradicting Assumption 9.1. □

Proposition 9.4. *In the induction step, Assumption 9.1 leads to a contradiction if $Y \neq \{*\}$.*

Proof. Fix $j \in \{1, \dots, \ell\}$. Then $f_{j,\infty}(0, \tilde{y}) = (s, y')$ for some $s \in \mathbb{R}^k$, $y' \in \widetilde{Y}$. By Proposition 6.8, after composing $f_{j,i}$ with maps given by Lemma 7.1, we can assume $f_{j,\infty}(0, \tilde{y}) = (0, y')$. Since Γ acts co-compactly on $\widetilde{Y}$, there is a sequence $\gamma_\nu \in \Gamma$ such that the sequence $f_{j,\infty}^{\nu}(\gamma_\nu(0, \tilde{y}))$ is bounded.

As $\mathrm{Iso}(\mathbb{R}^k \times \widetilde{Y})$ is proper, there is a sequence $\nu_\alpha \to \infty$ such that $f_{j,\infty}^{\nu_\alpha}(\gamma_{\nu_\alpha})$ is a Cauchy sequence in $\mathrm{Iso}(\mathbb{R}^k \times \widetilde{Y})$. This implies that the sequence

$$(f_{j,\infty}^{\nu_\alpha}\gamma_{\nu_\alpha})^{-1}(f_{j,\infty}^{\nu_{\alpha+1}}\gamma_{\nu_{\alpha+1}}) = (f_{j,\infty}^{\nu_{\alpha+1}-\nu_\alpha})[(f_{j,\infty}^{\nu_{\alpha+1}-\nu_\alpha})_*^{-1}(\gamma_{\nu_\alpha}^{-1})](\gamma_{\nu_{\alpha+1}})$$

converges to $\mathrm{Id}_{\mathbb{R}^k\times\widetilde{Y}}$.

Set $\mu_\alpha := \nu_{\alpha+1} - \nu_\alpha$, $g_\alpha := [(f_{j,\infty}^{\mu_\alpha})_*^{-1}(\gamma_{\nu_\alpha}^{-1})](\gamma_{\nu_{\alpha+1}})$, and choose $g_{\alpha,i} \in \Gamma_i$ such that $g_{\alpha,i}$ converges to g_α as $i \to \infty$. By Proposition 6.8 and a diagonal argument, if a function $i \mapsto \alpha(i)$ diverges to infinity slowly enough, the maps $f_{j,i}^{\mu_\alpha} g_{\alpha,i} : [\widetilde{X}_i, \tilde{p}_i] \to [\widetilde{X}_i, \tilde{p}_i]$ are GS and converge to $\mathrm{Id}_{\mathbb{R}^k\times\widetilde{Y}}$. By Proposition 2.62, if a function $i \mapsto \alpha(i)$ diverges slowly enough, we can replace $f_{j,i}$ by $f_{j,i}^{\mu_\alpha} g_{\alpha,i}$ and still have Assumption 9.1. By doing this independently for each j, we can assume $f_{j,\infty} = \mathrm{Id}_{\mathbb{R}^k\times\widetilde{Y}}$ for all $j \in \{1, \dots, \ell\}$ and Proposition 9.3 applies. □

Proposition 9.5. *In the induction step, Assumption 9.1 leads to a contradiction.*

Proof. After rescaling down each X_i by a fixed factor, we can assume $f_{j,\infty}$ displaces $(0, \tilde{y})$ at most $1/10$. If $Y = \{*\}$, let λ_i and $\Theta_i \subset B_{1/2}(p_i)$ be given by Theorem 8.1, and $\tilde{\Theta}_i \subset B_{1/2}(\tilde{p}_i)$ their lifts. For each $j \in \{1, \dots, \ell\}$, let $W_{j,i}^1 \subset \widetilde{X}_i$ be the sets obtained by applying Proposition 6.7 to each $f_{j,i}$, and set

$$W_i := \tilde{\Theta}_i \cap \bigcap_j W_{j,i}^1.$$

Then for large enough i, we can take $a_i \in W_i$ such that $f_{j,i}(a_i) \in W_i$ for each j. Let

$$\varphi_{j,i} : [\lambda_i \widetilde{X}_i, f_{j,i}(a_i)] \to [\lambda_i \widetilde{X}_i, a_i]$$

be the maps given by part (2) of Theorem 8.1. By Remark 2.59 and Proposition 2.62, if we replace $f_{j,i}$ by $\varphi_{j,i} f_{j,i}$, we still have Assumption 9.1. Then by part (1) of Theorem 8.1, Proposition 9.4 applies to the spaces $(\lambda_i X_i, a_i)$ and the GS maps $\varphi_{j,i} f_{j,i} : [\lambda \widetilde{X}_i, a_i] \to [\lambda_i \widetilde{X}_i, a_i]$. □

Proof of Theorem 1.1. Assuming the result fails, there is a sequence $(X_i, d_i, \mathfrak{m}_i, p_i)$ of pointed RCD(K, N) spaces, $\varepsilon_i \to 0$, and integers $\xi_i \to \infty$, such that if $H_i \leq \pi_1(X_i, p_i)$ denotes the image of the map $\pi_1(B_{\varepsilon_i}(p_i), p_i) \to \pi_1(X_i, p_i)$ induced by the inclusion, then no subgroup of H_i of index $\leq \xi_i$ admits a nilpotent basis of length $\leq n$.

Taking the pointed universal covers $(\widetilde{X}_i, \tilde{d}_i, \tilde{\mathfrak{m}}_i, \tilde{p}_i)$, for each i one can identify H_i with a subgroup of $\mathcal{G}(\pi_1(X_i), \widetilde{X}_i, \tilde{p}_i, 2\varepsilon_i)$. After taking a subsequence, we can assume (X_i, p_i) and $(\widetilde{X}_i, \tilde{p}_i)$ converge in the pointed Gromov–Hausdorff sense to spaces (X, p) and $(\widetilde{X}, \tilde{p})$, respectively, and the sequence $\pi_1(X_i)$ converges equivariantly to a closed group of isometries $G \leq \mathrm{Iso}(\widetilde{X})$.

Let $K \leq G$ be the stabilizer of $\tilde{p}$, and let m be the number of connected components of G it intersects. Fix $\varepsilon > 0$ such that the set $\{g \in G \mid d(d\tilde{p}, \tilde{p}) \leq 2\varepsilon\}$ intersects the same m connected components of G as K, and define

$$H_i' := \langle \{g \in \pi_1(X_i) \mid d(g\tilde{p}_i, \tilde{p}_i) \leq \varepsilon\} \rangle.$$

After taking a subsequence, we can assume H_i' converges equivariantly to a closed group $H' \leq G$, and let $\Upsilon_i \triangleleft H_i'$ be given by Theorem 3.1. Then for i large enough, $H_i \leq H_i'$ and by Theorem 3.1, $[H_i' : \Upsilon_i] \leq m$. Hence no subgroup of Υ_i of index $\leq \xi_i/m$ admits a nilpotent basis of length $\leq n$.

Pick $q \in B_1(p)$ a k-regular point, $\tilde{q} \in B_1(\tilde{p})$ a lift, and $\tilde{q}_i \in B_1(\tilde{p}_i)$ converging to $\tilde{q}$. If we equip $\pi_1(X_i)$ with the metric $d_0^{\tilde{p}_i}$ from (2.29), then for any $g \in B_\delta(\mathrm{Id}_{\widetilde{X}_i})$ with $\delta < 1$ one has $d(g\tilde{q}_i, \tilde{q}_i) < \delta$. Hence for all $\delta < 1$ we have

$$B_\delta(\mathrm{Id}_{\widetilde{X}_i}) \subset \{g \in \pi_1(X_i) | d(g\tilde{q}_i, \tilde{q}_i) < \delta\}. \tag{9.6}$$

For a sequence $\delta_i \to 0$, define $\Gamma_i := \mathcal{G}(\pi_1(X_i), \widetilde{X}_i, \tilde{q}_i, \delta_i)$. By (9.6) and Theorem 3.1, if $\delta_i \to 0$ slowly enough, for all i large we have $\Upsilon_i \leq \Gamma_i$. Finally, consider a sequence $\lambda_i \to \infty$ diverging so slowly that $\lambda_i \delta_i \to 0$, and such that $(\lambda_i X_i, q_i)$ converges to $(\mathbb{R}^k, 0)$ in the pointed Gromov–Hausdorff sense. Then $\Gamma_i = \mathcal{G}(\Gamma_i, \lambda_i \widetilde{X}_i, \tilde{q}_i, 1)$, contradicting Theorem 1.14 with $\ell = 0$. □

Finally, we point out that the proof of Corollary 1.15 is the same as of Theorem 1.1 almost verbatim. The only differences are that the contradicting sequence a priori converges to (X, p), and the nilpotent bases we are seeking are of length $\leq n - k$ instead of $\leq n$.

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LIOUVILLE THEOREM FOR MINIMAL GRAPHS OVER MANIFOLDS OF NONNEGATIVE RICCI CURVATURE

QI DING

Let Σ be a complete Riemannian manifold of nonnegative Ricci curvature. We prove a Liouville-type theorem: every smooth solution u to the minimal hypersurface equation on Σ is a constant provided u has sublinear growth for its negative part. Here, the sublinear growth condition is sharp. Our proof relies on a gradient estimate for minimal graphs over Σ with small linear growth of the negative parts of graphic functions via iteration.

1. Introduction

Let Σ be a complete noncompact Riemannian manifold. Let D and div_Σ be the Levi-Civita connection and the divergence operator (in terms of the Riemannian metric of Σ), respectively. In this paper, we study the minimal hypersurface equation on Σ,

$$\mathrm{div}_\Sigma\left(\frac{Du}{\sqrt{1+|Du|^2}}\right)=0, \tag{1-1}$$

which is a nonlinear partial differential equation describing the minimal graph

$$M=\{(x,u(x))\in\Sigma\times\mathbb{R}\mid x\in\Sigma\}$$

over Σ. The smooth solution u to (1-1) is the height function of the minimal graph M in $\Sigma\times\mathbb{R}$. Therefore, we call u a *minimal graphic function* on Σ.

When Σ is Euclidean space $\mathbb{R}^n$, equation (1-1) has been studied successfully by many mathematicians. Bombieri, De Giorgi, and Miranda [Bombieri et al. 1969b] (see also [Gilbarg and Trudinger 1983]) proved interior gradient estimates for solutions to the minimal hypersurface equation on $\mathbb{R}^n$, where the 2-dimensional case had already been obtained in [Finn 1954]. As a corollary, they immediately got a Liouville-type theorem in [Bombieri et al. 1969b] as follows.

Theorem 1.1. *If a minimal graphic function u on $\mathbb{R}^n$ satisfies sublinear growth for its negative part*, *i.e.*,

$$\limsup_{x\to\infty}\frac{\max\{-u(x),0\}}{|x|}=0, \tag{1-2}$$

then u is a constant.

The condition (1-2) is sharp since any affine function is a minimal graphic function on $\mathbb{R}^n$. When the minimal graphic function u on $\mathbb{R}^n$ has the uniformly bounded gradient, Moser [1961] proved u is affine

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using Harnack's inequalities for uniformly elliptic equations. The gradient estimate of u on $\mathbb{R}^n$ can also be derived by the maximum principle (see [Korevaar 1986; Wang 1998] for instance). Without the "uniformly bounded gradient" condition, it is the celebrated Bernstein theorem; see [Fleming 1962; De Giorgi 1965; Almgren 1966; Simons 1968] and the counterexample in [Bombieri et al. 1969a]. Specifically, any minimal graphic function on $\mathbb{R}^n$ is affine for $n \le 7$ by Simons [1968].

Let us review Liouville-type theorems for nonnegative minimal graphic functions on manifolds briefly. From Fischer-Colbrie and Schoen [1980], any positive minimal graphic function on a Riemann surface S of nonnegative curvature is constant (see [Rosenberg 2002] for the case of minimal surfaces in $S \times \mathbb{R}$). Rosenberg, Schulze, and Spruck [Rosenberg et al. 2013] proved that every nonnegative minimal graphic function on a complete manifold of nonnegative Ricci curvature and sectional curvature uniformly bounded below is a constant. Casteras, Heinonen, and Holopainen [Casteras et al. 2020] showed that every nonnegative minimal graphic function u on a complete manifold of asymptotically nonnegative sectional curvature is a constant provided u has at most linear growth. In [Ding 2021], the author proved that every nonnegative minimal graphic function on a complete manifold of nonnegative Ricci curvature is constant, which was also obtained independently by Colombo, Magliaro, Mari, and Rigoli [Colombo et al. 2022]. In fact, the "nonnegative Ricci curvature" condition can be further weakened to the volume doubling property and the Neumann–Poincaré inequality in [Ding 2021].

In some situations, the above "nonnegative" condition for the solution u on a manifold Σ can be weakened to the condition of "sublinear growth for its negative part", i.e.,

$$\limsup_{\Sigma \ni x \to \infty} \frac{\max\{-u(x), 0\}}{d(x, p)} = 0 \tag{1-3}$$

for some $p \in \Sigma$, where $d(x, p)$ denotes the distance function on Σ between x and p. Motivated by Theorem 1.1, for brevity, we say the *strong Liouville theorem for minimal graphs over* Σ *holds* if every minimal graphic function u on Σ is a constant provided u admits sublinear growth for its negative part.

In [Rosenberg et al. 2013], the strong Liouville theorem for minimal graphs over complete manifolds of nonnegative sectional curvature was proved. Ding, Jost, and Xin [Ding et al. 2016] proved the strong Liouville theorem for minimal graphs over complete manifolds of nonnegative Ricci curvature, Euclidean volume growth and quadratic curvature decay. In [Ding 2025], the author proved the same without the above quadratic curvature decay condition, which is a biproduct of Poincaré inequality on minimal graphs; see [Bombieri and Giusti 1972] for the Euclidean case. Colombo, Gama, Mari, and Rigoli [Colombo et al. 2024] proved the strong Liouville theorem for minimal graphs over complete manifolds of nonnegative Ricci curvature and that the $(n-2)$-th Ricci curvature in the radial direction from a fixed origin has a lower bound decaying quadratically to zero.

Colombo, Mari, and Rigoli [Colombo et al. 2023] proved an interesting theorem: if a minimal graphic function u on a complete noncompact Riemannian manifold Σ of nonnegative Ricci curvature satisfies

$$\limsup_{\Sigma \ni x \to \infty} \frac{\log d(x, p)}{d(x, p)} \max\{-u(x), 0\} < \infty \tag{1-4}$$

for some $p \in \Sigma$, then u is a constant.

From now on, we always let Σ denote a complete noncompact Riemannian manifold of nonnegative Ricci curvature (without extra assumptions). In this paper, we can weaken the condition (1-4) to (1-3) and prove the strong Liouville theorem for minimal graphs over Σ as follows.

Theorem 1.2. *Any minimal graphic function u on Σ is a constant provided u has sublinear growth for its negative part.*

The condition of "sublinear growth for its negative part", i.e., (1-3), is sharp from the Euclidean case and the manifolds case; see Proposition 9 in [Colombo et al. 2024]. To arrive at Theorem 1.2, we prove a stronger result: a gradient estimate for small linear growth of the negative part of u (without the upper bound condition of u) as follows.

Theorem 1.3. *There exists a constant $\beta_* > 0$ depending only on n such that if a minimal graphic function u on Σ satisfies*

$$\liminf_{x\to\infty} \frac{u(x)}{d(x,p)} \geq -\beta_* \tag{1-5}$$

for some $p \in \Sigma$, then there is a constant $c > 0$ depending only on n such that

$$\sup_{x\in\Sigma} |Du|(x) \leq c \limsup_{x\to\infty} \frac{\max\{-u(x),0\}}{d(x,p)}. \tag{1-6}$$

The key ingredient in the proof of Theorem 1.3 is to get an integral estimate of v^k on geodesic balls in Σ for a large constant k by an iteration (on l) of an integral of $(\log v)^l v$, where v is the volume function of the minimal graphic function u. Then using the Sobolev inequality on Σ, we can carry out a (modified) De Giorgi–Nash–Moser iteration on geodesic balls in Σ starting from an integral of v^{2k} with $k > n$ and get the bound of v; see Theorem 4.3 since Harnack's inequality holds in Theorem 4.3 of [Ding 2021].

Once we get the uniform gradient estimate (1-6), from Theorem 8 (or Theorem 6(ii)) in [Colombo et al. 2024], we can conclude that any tangent cone of Σ at infinity splits off a line isometrically; compare with the harmonic case by Cheeger, Colding, and Minicozzi [Cheeger et al. 1995]. It's worth pointing out that Σ may not split off any line from a counterexample in Proposition 9 of [Colombo et al. 2024].

Without (1-5), we have the gradient estimates without the "entire" condition of M or Σ, where the estimates depend on the lower bound of the volume of geodesic balls of Σ; see [Ding 2025]. In [Colombo et al. 2024], the authors obtained gradient estimates for minimal graphs over manifolds of nonnegative Ricci curvature and that the $(n-2)$-th Ricci curvature of Σ in radial direction from a fixed origin has a lower bound decaying quadratically to zero.

2. Preliminaries

Let Σ be an n-dimensional complete Riemannian manifold of nonnegative Ricci curvature. For any $R > 0$ and $p \in \Sigma$, let $B_R(p)$ be the geodesic ball in Σ centered at p with radius R. For each integer $k \geq 0$, let $\mathcal{H}^k$ denote the k-dimensional Hausdorff measure. From the Bishop–Gromov volume comparison theorem,

$$\frac{1}{n} r^{1-n} \mathcal{H}^{n-1}(\partial B_r(p)) \leq r^{-n}\mathcal{H}^n(B_r(p)) \leq s^{-n}\mathcal{H}^n(B_s(p)) \tag{2-1}$$

for all $0 < s < r$. Let D be the Levi-Civita connection of Σ. From [Anderson 1992] or [Croke 1980], the Sobolev inequality

$$\frac{(\mathcal{H}^n(B_r(p)))^{\frac{1}{n}}}{r}\left(\int_{B_r(p)}|\phi|^{\frac{n}{n-1}}\right)^{\frac{n-1}{n}} \le \Theta\int_{B_r(p)}|D\phi| \tag{2-2}$$

holds for any Lipschitz function ϕ on $B_r(p)$ with compact support in $B_r(p)$, where $\Theta > 0$ is a constant depending only n.

Let Φ be a Lipschitz function on $B_{r+s}(p)$, $s \in (0, r]$, and ζ be a nonnegative Lipschitz function such that $\zeta \equiv 1$ on $B_r(p)$, $\zeta \equiv 0$ outside $B_{r+s}(p)$ and $|D\zeta| \le 1/s$. Then, from the Cauchy–Schwarz inequality, we have

$$\begin{aligned}\int_{B_r(p)}|D(\Phi^2\zeta)| &\le 2\int_{B_r(p)}|\Phi|\zeta|D\Phi| + \int_{B_r(p)}\Phi^2|D\zeta| \\ &\le r\int_{B_r(p)}|D\Phi|^2\zeta + \frac{1}{r}\int_{B_r(p)}\Phi^2\zeta + \frac{1}{s}\int_{B_{r+s}(p)}\Phi^2. \end{aligned}\tag{2-3}$$

From (2-2), it follows that

$$(\mathcal{H}^n(B_r(p)))^{\frac{1}{n}}\left(\int_{B_r(p)}|\Phi|^{\frac{2n}{n-1}}\right)^{\frac{n-1}{n}} \le \Theta r^2\int_{B_r(p)}|D\Phi|^2 + \frac{2\Theta r}{s}\int_{B_{r+s}(p)}\Phi^2. \tag{2-4}$$

From [Buser 1982] or [Cheeger and Colding 1996], we have the Neumann–Poincaré inequality on geodesic balls of Σ. Namely, we have (up to a choice of Θ)

$$\int_{B_r(p)}|\varphi - \varphi_{B_r(p)}| \le \Theta r\int_{B_r(p)}|D\varphi| \tag{2-5}$$

for any Lipschitz function φ on $B_r(p)$, where

$$\varphi_{B_r(p)} = \fint_{B_r(p)}\varphi := \frac{1}{\mathcal{H}^n(B_r(p))}\int_{B_r(p)}\varphi.$$

Let M be a minimal graph over Σ with the graphic function u on Σ, where M has the induced metric from $\Sigma \times \mathbb{R}$ equipped with the standard product metric. By Stokes' formula, M is area-minimizing in $\Sigma \times \mathbb{R}$ by an argument analog to the case of Euclidean space. Let ∇ and Δ denote the Levi-Civita connection and Laplacian of M, respectively. We also see u as a function on M by projection $M \to \Sigma$; i.e., $u(x, u(x)) = u(x)$ for any $x \in \Sigma$. Then equation (1-1) is equivalent to the condition that u is harmonic on M, i.e.,

$$\Delta u = 0. \tag{2-6}$$

Let

$$v = \sqrt{1 + |Du|^2}$$

be the volume function of M (as mentioned above); we see v as a function on M by identifying $v(x, u(x)) = v(x)$. Recall the following Bochner-type formula:

$$\Delta v^{-1} = -(|A|^2 + v^{-2}\,\mathrm{Ric}(Du, Du))v^{-1}. \tag{2-7}$$

Here, A denotes the second fundamental form of M in $\Sigma\times\mathbb{R}$, and Ric denotes the Ricci curvature of Σ. From (2-7), it follows that

$$\Delta\log v=|A|^2+v^{-2}\operatorname{Ric}(Du,Du)+|\nabla\log v|^2\geq|\nabla\log v|^2. \tag{2-8}$$

For a C^1-function f on an open set of Σ, we can see f as a function on M: $f(x,u(x))=f(x)$. Then

$$|\nabla f|^2=|Df|^2-\frac{1}{v^2}|\langle Du,Df\rangle|^2\geq|Df|^2-\frac{|Du|^2}{v^2}|Df|^2=\frac{1}{v^2}|Df|^2. \tag{2-9}$$

Notational convention. When we write an integral over a subset of a Riemannian manifold with respect to its standard metric of the manifold, we always omit the volume element for simplicity.

3. Integral estimates of powers of the volume function

Lemma 3.1. *Let ξ be a Lipschitz function on Σ with compact support. For all constants $l\geq 1$ and $q,\theta>0$, we have*

$$\int_\Sigma|D(\log v)^l|\xi^{q+1}\leq l\theta r\int_\Sigma(\log v)^{l-1}v|D\xi|^2+\frac{l}{\theta r}\int_\Sigma(\log v)^{l-1}v\xi^{2q}. \tag{3-1}$$

Proof. We also see ξ as a function on M by letting $\xi(x,u(x))=\xi(x)$. From (2-8), for each $l'\geq 0$, from the Cauchy–Schwarz inequality we have

$$\begin{aligned}\int_M(\log v)^{l'}\xi^2|\nabla\log v|^2&\leq\int_M(\log v)^{l'}\xi^2\Delta\log v\leq-2\int_M(\log v)^{l'}\xi\nabla\xi\cdot\nabla\log v\\&\leq\frac12\int_M(\log v)^{l'}\xi^2|\nabla\log v|^2+2\int_M(\log v)^{l'}|\nabla\xi|^2,\end{aligned} \tag{3-2}$$

which implies

$$\int_M(\log v)^{l'}|\nabla\log v|^2\xi^2\leq 4\int_M(\log v)^{l'}|\nabla\xi|^2. \tag{3-3}$$

From (2-9) and (3-3), for all constants $q,\theta>0$ and $l\geq 1$, we have

$$\begin{aligned}\int_\Sigma|D(\log v)^l|\xi^{q+1}&\leq\int_M|\nabla(\log v)^l|\xi^{q+1}=l\int_M(\log v)^{l-1}|\nabla\log v|\xi^{q+1}\\&\leq\frac{l\theta r}{4}\int_M(\log v)^{l-1}|\nabla\log v|^2\xi^2+\frac{l}{\theta r}\int_M(\log v)^{l-1}\xi^{2q}\\&\leq l\theta r\int_M(\log v)^{l-1}|\nabla\xi|^2+\frac{l}{\theta r}\int_M(\log v)^{l-1}\xi^{2q}.\end{aligned} \tag{3-4}$$

This gives (3-1) by combining with (2-9) again. □

Given two constants $\beta,r_0>0$, we assume

$$|u(x)|\leq\beta\max\{r_0,d(x,p)\}\quad\text{for each }x\in\Sigma. \tag{3-5}$$

For each $r\geq r_0$, it's clear that

$$|u(x)|\leq\beta\max\{r,d(x,p)\}\quad\text{for each }x\in\Sigma. \tag{3-6}$$

We fix the point p and write $\rho(x)=d(x,p)$ for each $x\in\Sigma$.

Lemma 3.2. *Given a constant $\theta \in (0,1]$ and a constant $0<\delta \ll 1$, for each constant $l \ge 1$, we have*

$$\fint_{B_r(p)} (\log v)^l v \le (1+\delta)\beta l \frac{(1+\theta)^{n+1}}{\theta} \fint_{B_{(1+\theta)r}(p)} (\log v)^{l-1} v + \frac{2^n(1+c_\delta\beta)}{\theta} \fint_{B_{(1+\theta)r}(p)} (\log v)^l, \tag{3-7}$$

where $c_\delta \ge 1$ is a constant depending only on n and δ.

Proof. Let δ be a positive constant, $\delta \ll 1$, and ξ be a Lipschitz function on Σ with $\operatorname{supp}\xi \subset B_{(1+\theta)r}(p)$, $\xi \equiv 1$ on $B_r(p)$ and

$$\xi(x) = \begin{cases} \cos \dfrac{\rho(x)-r}{\theta r} & \text{for } x \in B_{(1+\delta\theta/4)r}(p) \setminus B_r(p), \\ \dfrac{\cos(\delta/4)}{1-\delta/4}\Big(1-\dfrac{\rho(x)-r}{\theta r}\Big) & \text{for } x \in B_{(1+\theta)r}(p) \setminus B_{(1+\delta\theta/4)r}(p). \end{cases} \tag{3-8}$$

Then

$$\theta r |D\xi|(x) = \begin{cases} \sin \dfrac{\rho(x)-r}{\theta r} & \text{for } x \in B_{(1+\delta\theta/4)r}(p) \setminus B_r(p), \\ \dfrac{\cos(\delta/4)}{1-\delta/4} & \text{for } x \in B_{(1+\theta)r}(p) \setminus B_{(1+\delta\theta/4)r}(p). \end{cases} \tag{3-9}$$

Let $q = q_\delta > 1$ such that

$$\cos^q(\delta/4) = \frac{\sin(\delta/4)}{1-\delta/4}.$$

Noting that $(1-\delta/4)^{-2} < 1+\delta$ as $0<\delta \ll 1$, with (3-9) we have

$$\theta^2 r^2 |D\xi|^2 + \xi^{2q} \le (1-\delta/4)^{-2} < 1+\delta \quad \text{on } \Sigma. \tag{3-10}$$

In [Bombieri et al. 1969b], the authors gave an estimate of an integral of $v \log v$ using (1-1) in the Euclidean case; see also [Gilbarg and Trudinger 1983] and [Ding et al. 2016] for manifolds. Enlightened by this, we further estimate an integral of $(\log v)^l v$ on geodesic balls of Σ using (1-1) for each $l>0$. Integrating by parts, for each $r \ge r_0$ with (3-6) we have

$$\begin{aligned} 0 &= \int_\Sigma \frac{Du}{v} \cdot D(u(\log v)^l \xi^{q+1}) \\ &= \int_\Sigma \frac{|Du|^2}{v} (\log v)^l \xi^{q+1} + \int_\Sigma u\xi^{q+1} \frac{Du}{v} \cdot D(\log v)^l + \int_\Sigma u(\log v)^l \frac{Du}{v} \cdot D\xi^{q+1} \\ &\ge \int_\Sigma \frac{|Du|^2}{v} (\log v)^l \xi^{q+1} - (1+\theta)\beta r \int_\Sigma \xi^{q+1} |D(\log v)^l| - \frac{c_\delta \beta}{\theta} \int_{B_{(1+\theta)r}(p)} (\log v)^l. \end{aligned} \tag{3-11}$$

Here, $c_\delta \ge 1$ is a constant depending only on n and $q = q_\delta$. Then it follows that

$$\begin{aligned} \int_\Sigma (\log v)^l v \xi^{q+1} &\le \int_\Sigma \Big(\frac{|Du|^2}{v} + 1\Big)(\log v)^l \xi^{q+1} \\ &\le (1+\theta)\beta r \int_\Sigma |D(\log v)^l| \xi^{q+1} + \frac{1+c_\delta\beta}{\theta} \int_{B_{(1+\theta)r}(p)} (\log v)^l. \end{aligned} \tag{3-12}$$

For each $l \ge 1$, from (3-1) and (3-10) we have

$$\int_\Sigma |D(\log v)^l|\xi^{q+1} \le \frac{l}{\theta r}\int_{B_{(1+\theta)r}(p)}(\log v)^{l-1}v(\theta^2r^2|D\xi|^2+\xi^{2q}) \\ \le \frac{(1+\delta)l}{\theta r}\int_{B_{(1+\theta)r}(p)}(\log v)^{l-1}v. \tag{3-13}$$

Substituting (3-13) into (3-12), we get

$$\int_{B_r(p)}(\log v)^l v \le (1+\delta)\beta l\frac{(1+\theta)}{\theta}\int_{B_{(1+\theta)r}(p)}(\log v)^{l-1}v+\frac{1+c_\delta\beta}{\theta}\int_{B_{(1+\theta)r}(p)}(\log v)^l. \tag{3-14}$$

This finishes the proof with (2-1). □

Now we further assume $\beta \le 1$. Write $\gamma_\delta = (1+\delta)n(1+1/n)^{n+1}$. By taking $\theta = 1/n$ in (3-7), for each $l \ge 1$ (up to a choice of $c_\delta \ge 1$), we have

$$\fint_{B_r(p)}(\log v)^l v \le \gamma_\delta\beta l\fint_{B_{(n+1)r/n}(p)}(\log v)^{l-1}v+c_\delta\fint_{B_{(n+1)r/n}(p)}(\log v)^l. \tag{3-15}$$

Since M is area-minimizing in $\Sigma\times\mathbb{R}$, with (2-1) we get

$$\int_{B_r(p)} v = \mathcal{H}^n(M\cap(B_r(p)\times\mathbb{R})) \le \mathcal{H}^n(B_r(p))+\int_{\partial B_r(p)}|u| \\ \le \mathcal{H}^n(B_r(p))+\beta r\mathcal{H}^n(\partial B_r(p)) \le (1+n\beta)\mathcal{H}^n(B_r(p)). \tag{3-16}$$

Let us iterate the estimate (3-15) on l.

Lemma 3.3. *Let c_δ be the constant in (3-15) with the given $0<\delta\ll 1$. For each integer $j \ge 0$, we have*

$$\sup_{r\ge r_0}\fint_{B_r(p)}(\log v)^j v \le j!\,\gamma_\delta^j\beta^j\binom{j+m}{m}(1+n\beta), \tag{3-17}$$

where $m = \left[\frac{c_\delta}{\gamma_\delta\beta}\right]+1 \in \mathbb{N}$ depends on n, δ, β, and

$$\binom{j+m}{m} = \frac{(m+j)!}{j!\,m!}.$$

Proof. Let us prove it by induction. From (3-15) and $\log v \le v$, for each $j \ge 1$, we have

$$\sup_{r\ge r_0}\fint_{B_r(p)}(\log v)^j v \le \gamma_\delta\beta j\sup_{r\ge r_0}\fint_{B_r(p)}(\log v)^{j-1}v+c_\delta\sup_{r\ge r_0}\fint_{B_r(p)}(\log v)^{j-1}v. \tag{3-18}$$

Let $m = \left[\frac{c_\delta}{\gamma_\delta\beta}\right]+1 \in \mathbb{N}$ depend on n, δ, β, and let $\{a_j\}_{j\in\mathbb{N}}$ be a sequence defined by

$$a_j = \sup_{r\ge r_0}\fint_{B_r(p)}(\log v)^j v. \tag{3-19}$$

From (3-18), for each integer $j \ge 1$, one has

$$a_j \le \gamma_\delta\beta j a_{j-1}+c_\delta a_{j-1} \le \gamma_\delta\beta(j+m)a_{j-1}. \tag{3-20}$$

By iteration,

$$a_j \le \gamma_\delta^j \beta^j \frac{(j+m)!}{m!} a_0 = j!\, \gamma_\delta^j \beta^j \binom{j+m}{m} a_0. \tag{3-21}$$

From (3-16), $a_0 \le 1 + n\beta$. This completes the proof. □

Theorem 3.4. *Let u be a minimal graphic function on Σ satisfying (3-6) for some constant $\beta \in (0,1]$. There is a constant $c(n,\delta,\beta) > 0$ depending only on n, δ, β such that, for each constant $\lambda \in (0, 1/(\gamma_\delta \beta))$, we have*

$$\sup_{r \ge r_0} \fint_{B_r(p)} v^{\lambda+1} \le c(n,\delta,\beta)(1-\lambda\gamma_\delta\beta)^{-m-1}. \tag{3-22}$$

Proof. Let $\lambda < \dfrac{1}{\gamma_\delta \beta}$ be a positive constant. From Taylor's expansion

$$v^\lambda = e^{\lambda \log v} = \sum_{j=0}^\infty \frac{\lambda^j}{j!} (\log v)^j, \tag{3-23}$$

combining with (3-17) we get

$$\begin{aligned} \fint_{B_r(p)} v^{\lambda+1} &= \sum_{j=0}^\infty \frac{\lambda^j}{j!} \fint_{B_r(p)} (\log v)^j v \\ &\le \sum_{j=0}^\infty \frac{\lambda^j}{j!} j!\, \gamma_\delta^j \beta^j \binom{j+m}{m} (1+n\beta) \\ &= (1+n\beta) \sum_{j=0}^\infty (\lambda\gamma_\delta\beta)^j \binom{j+m}{m}. \end{aligned} \tag{3-24}$$

From

$$\sum_{j=0}^\infty \binom{j+m}{m} t^j = \frac{1}{m!} \frac{d^m}{dt^m} \sum_{j=0}^\infty t^{j+m} = \frac{1}{m!} \frac{d^m}{dt^m} \left(\frac{t^m}{1-t} \right) \tag{3-25}$$

for each $t \in (0,1)$, we complete the proof. □

4. Mean value inequality and gradient estimate

For each nonnegative measurable function f on Σ and each constant $q > 0$, we define

$$\|f\|_{q,r} = \left(\fint_{B_r(p)} f^q \right)^{1/q}.$$

Now, let us carry out a (modified) De Giorgi–Nash–Moser iteration to get the mean value inequality for the volume function v with the help of the Sobolev inequality on Σ.

Lemma 4.1. *For each constant $k > n$ and $\sigma \in (0,1)$, there is a constant $c_{\sigma,k}$ depending only on n, σ, k such that*

$$\|v\|_{\infty,\sigma r} \le c_{\sigma,k} (\|v\|_{2k,r})^{e^{n/(k-n)}} \tag{4-1}$$

for any $r > 0$.

Proof. Let η be a Lipschitz function on Σ with compact support which will be defined later. Write $\eta(x)=\eta(x,u(x))$. From (2-8), we have $\Delta v\ge 0$ on M clearly. For any constant $\ell\ge 1$, we have

$$\begin{aligned}0\ge-\int_M v^{2\ell}\eta^2\Delta v&=2\ell\int_M v^{2\ell-1}\eta^2|\nabla v|^2+2\int_M v^{2\ell}\eta\nabla v\cdot\nabla\eta\\&\ge 2\ell\int_M v^{2\ell-1}\eta^2|\nabla v|^2-\ell\int_M v^{2\ell-1}\eta^2|\nabla v|^2-\frac{1}{\ell}\int_M v^{2\ell+1}|\nabla\eta|^2\\&=\ell\int_M v^{2\ell-1}\eta^2|\nabla v|^2-\frac{1}{\ell}\int_M v^{2\ell+1}|\nabla\eta|^2.\end{aligned}\tag{4-2}$$

From (2-9) and (4-2), we infer that

$$\int_\Sigma |Dv^\ell|^2\eta^2\le\int_M|\nabla v^\ell|^2v\eta^2=\ell^2\int_M|\nabla v|^2v^{2\ell-1}\eta^2\le\int_M v^{2\ell+1}|\nabla\eta|^2\le\int_\Sigma v^{2\ell+2}|D\eta|^2.\tag{4-3}$$

For each $r\ge\tau>0$, let η be defined by $\eta\equiv 1$ on $B_{r+\tau/2}(p)$, $\eta=(2/\tau)(r+\tau-\rho)$ on $B_{r+\tau}(p)\setminus B_{r+\tau/2}(p)$, $\eta\equiv 0$ outside $B_{r+\tau}(p)$. Then $|D\eta|\le 2/\tau$. Combining (2-4) and (4-3), we have

$$\begin{aligned}\|v^{2\ell}\|_{\frac{n}{n-1},r}&\le\Theta\Big(r^2\|Dv^\ell\|^2_{2,r+\frac{\tau}{2}}+\frac{4r}{\tau}\|v^{2\ell}\|_{1,r+\frac{\tau}{2}}\Big)\\&\le\Theta\Big(r^2\int_\Sigma v^{2\ell+2}|D\eta|^2+\frac{4r}{\tau}\|v^{2\ell}\|_{1,r+\tau}\Big)\\&\le\Theta\Big(\frac{4r^2}{\tau^2}\|v^{2\ell+2}\|_{1,r+\tau}+\frac{4r}{\tau}\|v^{2\ell}\|_{1,r+\tau}\Big)\\&\le c\frac{r^2}{\tau^2}\|v^{2\ell+2}\|_{1,r+\tau}=c\frac{r^2}{\tau^2}\|v\|^{2\ell+2}_{2\ell+2,r+\tau}.\end{aligned}\tag{4-4}$$

Here, $c=8\Theta$ is a constant depending only on n. Given a constant $k>n$, we set

$$\alpha=\frac{n(k-1)}{(n-1)k}>1.\tag{4-5}$$

For $\ell+1\ge k$, we have

$$\frac{2\ell n}{n-1}-(2\ell+2)\alpha=\frac{2n}{(n-1)k}(\ell+1-k)\ge 0.\tag{4-6}$$

From the Hölder inequality and (4-4), one has

$$\|v\|_{(2\ell+2)\alpha,r}\le\|v\|_{\frac{2\ell n}{n-1},r}\le c^{\frac{1}{2\ell}}r^{\frac{1}{\ell}}\tau^{-\frac{1}{\ell}}\|v\|^{\frac{\ell+1}{\ell}}_{2\ell+2,r+\tau}.\tag{4-7}$$

For any $\sigma\in(0,1)$ and any integer $i\ge-1$, set $m_i=2k\alpha^i$, $\ell_i=\frac{1}{2}m_i-1$, $\tau_i=2^{-(1+i)}(1-\sigma)r$ and $r_{i+1}=r_i-\tau_{i+1}$ with $r_{-1}=r$. Then

$$r_{i+1}=r-\sum_{j=0}^{i+1}\tau_j=\sigma r+\tau_{i+1}\le r,$$

and $\lim_{i\to\infty}r_i=\sigma r$. By iterating (4-7), for each $i\ge 0$, we have

$$\|v\|_{\alpha m_i,r_i}\le c^{\frac{1}{2\ell_i}}r_i^{\frac{1}{\ell_i}}\tau_i^{-\frac{1}{\ell_i}}\|v\|^{\frac{\ell_i+1}{\ell_i}}_{\alpha m_{i-1},r_{i-1}}.\tag{4-8}$$

Set $\xi_i = \log \|v\|_{\alpha m_i, r_i}$ for each integer $i \ge -1$, and set $b_\sigma = c/(1-\sigma)^2$. Note that $\tau_i/r_i \ge 2^{-(1+i)}(1-\sigma)$ and $\ell_i \ge k\alpha^i - 1 \ge (k-1)\alpha^i$ for every $i \ge 0$. Then

$$\begin{aligned}\xi_i &\le \frac{1}{2\ell_i}\log c + \frac{1}{\ell_i}\log\frac{r_i}{\tau_i} + \frac{\ell_i+1}{\ell_i}\xi_{i-1} \le \frac{1}{2\ell_i}\log b_\sigma + \frac{1+i}{\ell_i}\log 2 + e^{\frac{1}{\ell_i}}\xi_{i-1}\\ &\le \frac{1}{2(k-1)\alpha^i}\log b_\sigma + \frac{1+i}{(k-1)\alpha^i}\log 2 + e^{\frac{\alpha^{-i}}{k-1}}\xi_{i-1}.\end{aligned} \tag{4-9}$$

For all $0 \le i_0 \le i$, we have

$$\prod_{j=i_0}^{i} e^{\frac{\alpha^{-j}}{k-1}} = e^{\frac{1}{k-1}\sum_{j=i_0}^{i}\alpha^{-j}} \le e^{\frac{\alpha^{1-i_0}}{(k-1)(\alpha-1)}}.$$

Hence, for each $i \ge 1$,

$$\begin{aligned}\xi_i &\le \frac{\log b_\sigma}{2(k-1)\alpha^i} + \frac{(1+i)\log 2}{(k-1)\alpha^i} + e^{\frac{\alpha^{-i}}{k-1}}\left(\frac{\log b_\sigma}{2(k-1)\alpha^{i-1}} + \frac{i\log 2}{(k-1)\alpha^{i-1}} + e^{\frac{\alpha^{1-i}}{k-1}}\xi_{i-2}\right)\\ &\le \cdots \le \sum_{j=0}^{i}\left(\frac{\log b_\sigma}{2(k-1)\alpha^j} + \frac{1+j}{(k-1)\alpha^j}\log 2\right)\prod_{J=j+1}^{i} e^{\frac{\alpha^{-J}}{k-1}} + \xi_{-1}\prod_{j=0}^{i} e^{\frac{\alpha^{-j}}{k-1}}\\ &\le e^{\frac{1}{(k-1)(\alpha-1)}}\sum_{j=0}^{i}\left(\frac{\log b_\sigma}{2(k-1)}\frac{1}{\alpha^j} + \frac{\log 2}{k-1}\frac{1+j}{\alpha^j}\right) + e^{\frac{\alpha}{(k-1)(\alpha-1)}}\xi_{-1}.\end{aligned} \tag{4-10}$$

Since

$$\sum_{j=0}^{\infty}\frac{j+1}{\alpha^j} = \frac{\alpha^2}{(\alpha-1)^2}, \tag{4-11}$$

we have

$$\xi_i \le e^{\frac{1}{(k-1)(\alpha-1)}}\left(\frac{\log b_\sigma}{2(k-1)}\frac{\alpha}{\alpha-1} + \frac{\log 2}{k-1}\frac{\alpha^2}{(\alpha-1)^2}\right) + e^{\frac{\alpha}{(k-1)(\alpha-1)}}\xi_{-1}. \tag{4-12}$$

From $\alpha - 1 = \dfrac{k-n}{(n-1)k}$ and $\dfrac{\alpha}{\alpha-1} = \dfrac{n(k-1)}{k-n}$, we obtain

$$\xi_i \le e^{\frac{n}{k-n}}\left(\frac{n\log b_\sigma}{2(k-n)} + \frac{n^2k\log 2}{(k-n)^2}\right) + e^{\frac{n}{k-n}}\xi_{-1}. \tag{4-13}$$

Namely,

$$\|v\|_{\alpha m_i, r_i} \le \exp\left(e^{\frac{n}{k-n}}\left(\frac{n\log b_\sigma}{2(k-n)} + \frac{n^2k\log 2}{(k-n)^2}\right)\right)(\|v\|_{2k,r})^{e^{n/(k-n)}}. \tag{4-14}$$

Letting $i \to \infty$, it follows that

$$\|v\|_{\infty,\sigma r} \le \exp\left(e^{\frac{n}{k-n}}\left(\frac{n\log b_\sigma}{2(k-n)} + \frac{n^2k\log 2}{(k-n)^2}\right)\right)(\|v\|_{2k,r})^{e^{n/(k-n)}}. \tag{4-15}$$

This completes the proof. □

Remark 4.2. The factor $e^{n/(k-n)}$ in (4-1) comes from (4-3), which transforms an estimate on M to another estimate on Σ with a slight but definite "loss". In fact, the factor could be smaller if we choose a larger factor than α in (4-7) for large ℓ. However, we cannot reduce the constant k to a constant $\le n$, since we need $\alpha > 1$ in (4-5). Hence, unlike the classic De Giorgi–Nash–Moser iteration, here we are not able to obtain $\sup_{B_r(p)} v$ bounded by a multiple of an integral of v^γ with $\gamma \le 2n$ on $B_{2r}(p)$.

Put

$$\beta_n = \frac{1}{n(2n-1)}\Big(1+\frac{1}{n}\Big)^{-n-1}. \tag{4-16}$$

To prove Theorem 1.3, we only need to show the following theorem since we have Harnack's inequality in Theorem 4.3 of [Ding 2021] (or (A-8) in the Appendix directly).

Theorem 4.3. *If a minimal graphic function u on Σ satisfies*

$$\limsup_{x\to\infty} \frac{|u(x)|}{d(x,p)} < \beta_n \tag{4-17}$$

for some $p \in \Sigma$, then there is a constant $c > 0$ depending only on n such that

$$\sup_{x\in\Sigma} |Du|(x) \le c \limsup_{x\to\infty} \frac{|u(x)|}{d(x,p)}. \tag{4-18}$$

Proof. From (4-17), there is a constant $\beta \in (0, \beta_n)$ such that

$$\limsup_{x\to\infty} \frac{|u(x)|}{d(x,p)} < \beta. \tag{4-19}$$

Then there is a constant $r_\beta > 0$ such that

$$|u(x)| \le \beta \max\{r_\beta, d(x,p)\} \quad \text{for each } x \in \Sigma. \tag{4-20}$$

We fix a positive constant $\delta = \delta(\beta) \ll 1$ satisfying $\beta(1+\delta) < \beta_n$. Recall $\gamma_\delta = (1+\delta)n(1+1/n)^{n+1}$. From Theorem 3.4, there is a constant

$$\lambda_\beta = \Big(1+\frac{\beta_n}{\beta(1+\delta)}\Big)\Big(n-\frac{1}{2}\Big)+1$$

such that

$$\fint_{B_r(p)} v^{\lambda_\beta} \le \frac{c(n,\delta,\beta)}{(1-(\lambda_\beta-1)\gamma_\delta\beta)^{m+1}} = c(n,\delta,\beta)\Big(\frac{2\beta_n}{\beta_n-(1+\delta)\beta}\Big)^{m+1} \tag{4-21}$$

for all $r \ge r_\beta$. From Lemma 4.1, we get

$$\sup_{B_{r/2}(p)} v = \|v\|_{\infty,\frac{r}{2}} \le c_{\frac{1}{2},\frac{\lambda_\beta}{2}} (\|v\|_{\lambda_\beta,r})^{e^{n/(\lambda_\beta/2-n)}} \le \psi(n,\beta), \tag{4-22}$$

where $\psi = \psi(n,\beta)$ is a positive function depending only on n and $\beta < \beta_n$ satisfying $\lim_{\beta\to\beta_n} \psi(n,\beta) = \infty$, which may change from line to line. In other words, we have concluded that v is uniformly bounded on Σ. In the following, let us give a better bound of v than (4-22).

Let $\bar p = (p, u(p))$, and let $B_r(\bar p)$ denote the geodesic ball in $\Sigma\times\mathbb{R}$ with radius r centered at $\bar p$. From [Ding 2023, (3.5)], (2-1) and (3-16), we get

$$2\mathcal{H}^n(B_r(p)) \ge \mathcal{H}^n(M\cap B_r(\bar p)) \ge \frac{1}{r}\mathcal{H}^{n+1}(B_{r/2}(\bar p)) \ge \frac{1}{c}\mathcal{H}^n(B_r(p)) \tag{4-23}$$

for each $r > 0$. Here, $c \ge 1$ is a constant depending only on n, which may change from line to line. Combining (2-2) and (4-22), (by projection from $\Sigma\times\mathbb{R}$ into Σ) we have the Sobolev inequality on M; i.e.,

$$\Big(\fint_{M\cap B_r(\bar p)} |\phi|^{\frac{n}{n-1}}\Big)^{\frac{n-1}{n}} \le \psi r \fint_{M\cap B_r(\bar p)} |D\phi| \tag{4-24}$$

holds for any Lipschitz function ϕ on $M\cap B_r(\bar p)$ with compact support in $M\cap B_r(\bar p)$. Combining (2-5) and (4-22), we have the Neumann–Poincaré inequality on exterior geodesic balls of M; i.e.,

$$\int_{M\cap B_r(\bar p)}|\varphi-\bar\varphi_{p,r}|\le\psi r\int_{M\cap B_r(\bar p)}|D\varphi| \tag{4-25}$$

for any Lipschitz function φ on $M\cap B_r(\bar p)$ with $\bar\varphi_{p,r}=\fint_{M\cap B_r(\bar p)}\varphi$. From De Giorgi–Nash–Moser iteration, the mean value inequalities hold on M for sub- and superharmonic functions on M. Define $|Du|_0=\sup_\Sigma|Du|$. Since $|Du|^2$ is subharmonic on M from (2-7), we conclude that $|Du|_0^2-|Du|^2$ is nonnegative superharmonic on M. Then (see page 42 in [Ding 2025] or Lemma 3.5 in [Ding et al. 2016] up to a suitable modification)

$$|Du|_0^2=\sup_\Sigma|Du|^2=\lim_{r\to\infty}\fint_{M\cap B_r(\bar p)}|Du|^2. \tag{4-26}$$

Let $\tilde\eta$ be a Lipschitz function on Σ with $\operatorname{supp}\tilde\eta\subset B_{2r}(p)$, $\tilde\eta\equiv1$ on $B_r(p)$ and $|D\tilde\eta|\le1/r$. We see $\tilde\eta$ as a function on M by letting $\tilde\eta(x,u(x))=\tilde\eta(x)$. From (2-6) and the Cauchy–Schwarz inequality, it follows that

$$\begin{aligned}0=\int_M\nabla u\cdot\nabla(u\tilde\eta^2)&=\int_M|\nabla u|^2\tilde\eta^2+2\int_M u\tilde\eta\nabla u\cdot\nabla\tilde\eta\\&\ge\int_M|\nabla u|^2\tilde\eta^2-\frac12\int_M|\nabla u|^2\tilde\eta^2-2\int_M u^2|\nabla\tilde\eta|^2.\end{aligned} \tag{4-27}$$

Combining this with (2-1) and (3-16), we get

$$\begin{aligned}\int_{B_r(p)}|\nabla u|^2v\le\int_M|\nabla u|^2\tilde\eta^2\le4\int_M u^2|\nabla\tilde\eta|^2\le16\beta^2\int_{B_{2r}(p)}v&\le16(1+n\beta)\beta^2\mathcal{H}^n(B_{2r}(p))\\&\le16(1+n\beta)2^n\beta^2\mathcal{H}^n(B_r(p)).\end{aligned} \tag{4-28}$$

Since $M\cap B_r(\bar p)\subset B_r(p)\times\mathbb{R}$, combining with (2-9), (4-23), (4-26), and (4-28), we get

$$\begin{aligned}\frac{|Du|_0^2}{1+|Du|_0^2}&\le\limsup_{r\to\infty}\fint_{M\cap B_r(\bar p)}\frac{|Du|^2}{v^2}\le\limsup_{r\to\infty}\fint_{M\cap B_r(\bar p)}|\nabla u|^2\\&\le\limsup_{r\to\infty}\frac{1}{\mathcal{H}^n(M\cap B_r(\bar p))}\int_{B_r(p)}|\nabla u|^2v\le c\limsup_{r\to\infty}\fint_{B_r(p)}|\nabla u|^2v\le c\beta^2.\end{aligned} \tag{4-29}$$

Letting $\beta\to\limsup_{x\to\infty}d^{-1}(x,p)|u(x)|$, we deduce (4-18), which completes the proof. □

Appendix

Let Σ be an n-dimensional complete Riemannian manifold of nonnegative Ricci curvature. Let M be a minimal graph over Σ with the graphic function u on Σ. Suppose u is not a constant. For any $r>0$ and $\bar x=(x,t_x)\in\Sigma\times\mathbb{R}$, we define

$$\mathfrak{D}_{\bar x,r}=\{(y,s)\in\Sigma\times\mathbb{R}\mid d(y,x)+|s-t_x|<r\}$$

and $\mathscr{B}_r(\bar{x}) = M \cap \mathfrak{D}_{\bar{x},r}$. For each $s \le \inf_{B_{4R}(p)} u$, write $\bar{p}_s = (p, u(p) - s)$. From Theorem 4.3 in [Ding 2021], $u - s$ satisfies Harnack's inequality as follows:

$$\sup_{\mathscr{B}_{2R}(\bar{p}_s)} (u-s) \le \vartheta \inf_{\mathscr{B}_{2R}(\bar{p}_s)} (u-s) \tag{A-1}$$

for some constant $\vartheta \ge 2$ depending only on n.

We suppose that there is a positive constant $\beta_* < \dfrac{\beta_n}{4(\vartheta-1)}$ with β_n defined as in (4-16) such that

$$\liminf_{x\to\infty} \frac{u(x)}{d(x,p)} \ge -\beta_* \tag{A-2}$$

for some $p \in \Sigma$. Write $\epsilon = \dfrac{\beta_n}{8\beta_*(\vartheta-1)} - \dfrac{1}{2} > 0$. There is a constant $r_\epsilon > 0$ such that

$$u(x) \ge -(1+\epsilon)\beta_* \max\{d(x,p), r_\epsilon\} \tag{A-3}$$

for all $x \in \Sigma$. For each $R \ge r_\epsilon$, let $\hat{u}_R = u + 4(1+\epsilon)\beta_* R$ and $\hat{p}_R = (p, \hat{u}_R(p)) \in \Sigma \times \mathbb{R}$. Then $\hat{u}_R > 0$ on $B_{4R}(p)$, which implies that

$$\sup_{\mathscr{B}_{2R}(\hat{p}_R)} \hat{u}_R \le \vartheta \inf_{\mathscr{B}_{2R}(\hat{p}_R)} \hat{u}_R \le \vartheta \hat{u}_R(p) \tag{A-4}$$

from (A-1). Since

$$(\vartheta-1)\hat{u}_R(p) = (\vartheta-1)(u(p) + 4(1+\epsilon)\beta_* R) = (\vartheta-1)u(p) + (\beta_n + 4(\vartheta-1)\beta_*)\frac{R}{2}, \tag{A-5}$$

we get

$$(\vartheta-1)\hat{u}_R(p) < \beta_n R \tag{A-6}$$

for sufficiently large $R \ge r_\epsilon$. Note that $B_R(p) \times (-R + \hat{u}_R(p), R + \hat{u}_R(p)) \subset \mathfrak{D}_{\hat{p}_R, 2R}$. From (A-4), we conclude that

$$\sup_{B_R(p)} \hat{u}_R < \beta_n R \tag{A-7}$$

for all sufficiently large $R \ge r_\epsilon$. From (A-7) and the definition of $\hat{u}_R$, it follows that

$$\sup_{B_R(p)} u < \sup_{B_R(p)} \hat{u}_R - 4\beta_* R < (\beta_n - 4\beta_*)R. \tag{A-8}$$

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