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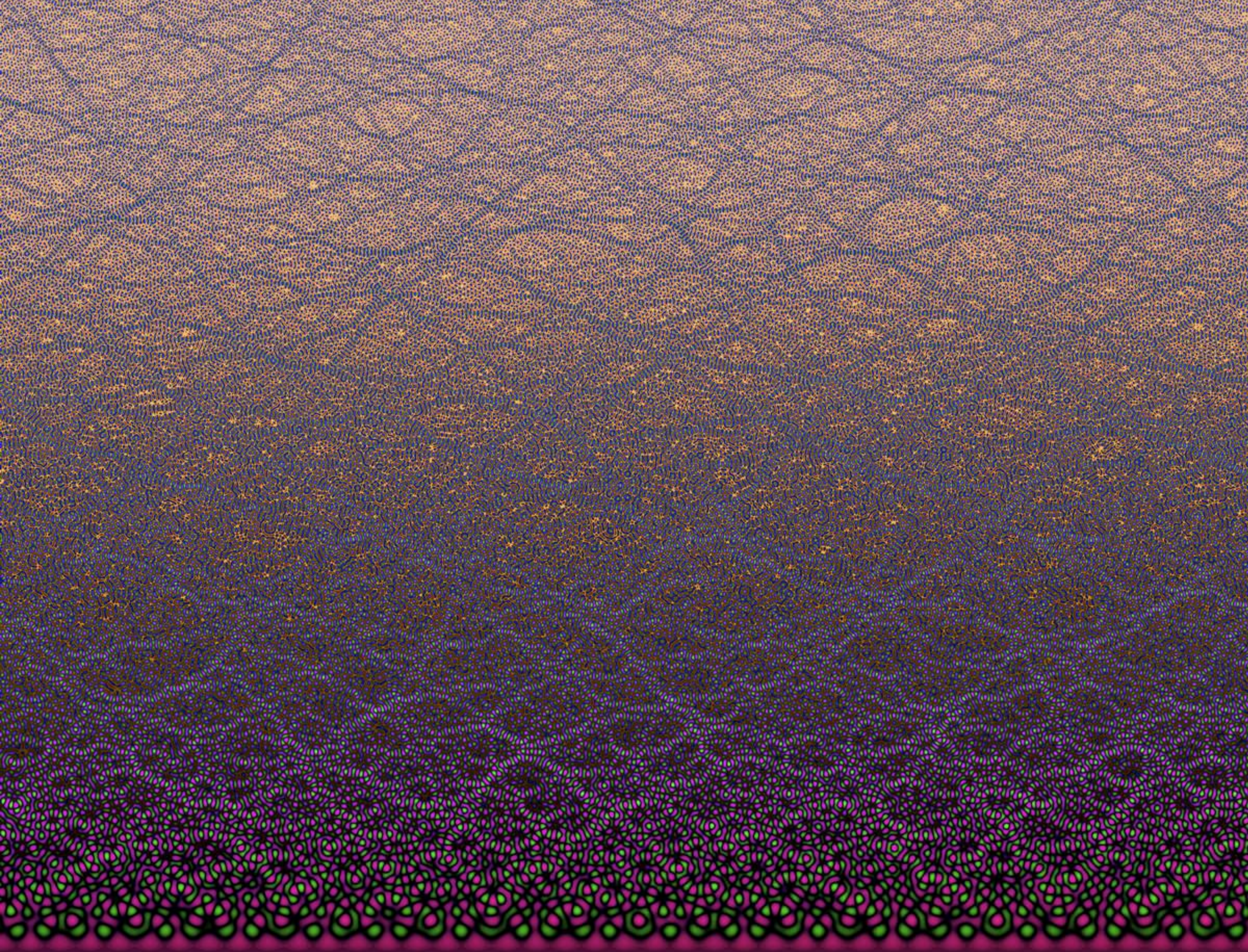

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RANDOM SCHRÖDINGER OPERATORS WITH COMPLEX DECAYING POTENTIALS

JEAN-CLAUDE CUENIN AND KONSTANTIN MERZ

We prove that the eigenvalues of a continuum random Schrödinger operator $-\Delta + V_\omega$ of Anderson-type, with complex decaying potential, can be bounded (with high probability) in terms of an L^q norm of the potential for all $q \le d+1$. This shows that, in the random setting, the exponent q can be essentially doubled compared to the deterministic bounds of Frank (*Bull. Lond. Math. Soc.* **43**:4 (2011), 745–750). This improvement is based on ideas of Bourgain (*Discrete Contin. Dyn. Syst.* **8**:1, (2002), 1–15) related to almost-sure scattering for lattice Schrödinger operators.

1. Introduction and main result

Consider a Schrödinger operator $-\Delta + V$ on $L^2(\mathbb{R}^d)$. Frank [2011] proved the scale-invariant bounds

$$|z|^{q-d/2} \lesssim \int_{\mathbb{R}^d} |V(x)|^q \,\mathrm{d}x \tag{1}$$

for eigenvalues z of $-\Delta + V$, when $q \le \frac{1}{2}(d+1)$ (we call such V short range). The short range condition is best possible, i.e., (1) is generally not true for $q > \frac{1}{2}(d+1)$. Counterexamples for $z > 0$ were constructed by Frank and Simon [2017], and for $\operatorname{Im} z \neq 0$ by Bögli and the first author [Bögli and Cuenin 2023]. These counterexamples settle the Laptev–Safronov conjecture [Laptev and Safronov 2009] in the negative.

The aim of this paper is to show that, for random potentials, the short range exponent can be essentially doubled, from $\frac{1}{2}(d+1)$ to $d+1$, compared to the deterministic case. We consider Anderson-type Schrödinger operators of the form $-\Delta + V_\omega$, where

$$V_\omega(x) = \sum_{j \in h\mathbb{Z}^d} \omega_j v_j \mathbf{1}_Q((x-j)/h), \quad Q = [0,1)^d, \quad h > 0. \tag{2}$$

More generally, given a deterministic potential V, consider its randomization at scale $h > 0$, given by

$$V_\omega(x) = \sum_{j \in h\mathbb{Z}^d} \omega_j V(x) \mathbf{1}_Q((x-j)/h). \tag{3}$$

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One could also replace $\mathbf{1}_{Q_j}$ with some rapidly decaying function. Note that, in both cases (2) and (3), the L^q norm of V_ω is deterministic:

$$\|V_\omega\|_{L^q(\mathbb{R}^d)} = \Big(h^d \sum_{j\in h\mathbb{Z}^d} |v_j|^q\Big)^{1/q}, \tag{4}$$

where v_j is the L^q-average of V over $j+hQ$ in the general case (3), and we have $\|V_\omega\|_{L^q(\mathbb{R}^d)} = \|V\|_{L^q(\mathbb{R}^d)}$. For this reason, we also denote the norm (4) by $\|V\|_{L^q(\mathbb{R}^d)}$ in case (2). Crucially, we assume that $(\omega_j)_{j\in h\mathbb{Z}^d}$ are *independent*, *mean-zero Gaussian or symmetric Bernoulli random variables* (real- or complex-valued). In the following, V_ω will always denote the randomization (3) of a given deterministic potential V, and $\langle x\rangle = 2+|x|$. The standard assumptions on the local singularities of $V \in L^q(\mathbb{R}^d)$,

$$q \ge 1 \quad \text{if } d = 1, \qquad q > 1 \quad \text{if } d = 2, \qquad q \ge \tfrac{1}{2}d \quad \text{if } d \ge 3, \tag{5}$$

ensure that $-\Delta + V$ can be defined as an m-sectorial operator. These can be slightly weakened (see Remark 2 (ii)) and only play a minor role here. In contrast, the average decay of the potential (i.e., an upper bound on q) — to be stated in the assumptions of the following theorems — is of central importance. We tacitly assume $q > \frac{1}{2}(d+1)$, since the case $q \le \frac{1}{2}(d+1)$ is already covered by the deterministic bound (1).

Theorem 1. *There exist constants M_0, $c > 0$ such that the following holds. For any R, $\lambda > 0$, $0 < h < R$, $|\varepsilon| \ll \lambda$, $q \le d+1$, for any $V \in L^q(\mathbb{R}^d)$ supported in a ball of radius R, and, for any $M \ge M_0$, each eigenvalue $z = (\lambda + \mathrm{i}\varepsilon)^2$ of $-\Delta + V_\omega$ satisfies*

$$\frac{\lambda^{2-d/q}}{\langle \lambda h\rangle^{d/2} (\log\langle \lambda R\rangle)^{7/2}} \le M \|V\|_{L^q(\mathbb{R}^d)},$$

except for ω in a set of measure at most $\exp(-cM^2)$.

Remark 2. (i) Outside the set $\lambda > 0$, $|\varepsilon| \ll \lambda$, obvious estimates (as in the case of real potentials) are available. These even hold for sums of powers of eigenvalues as in the classical Lieb–Thirring inequalities; see Frank, Laptev, Lieb, and Seiringer [Frank et al. 2006].

(ii) As in [Cuenin and Merz 2021] (see also [Ionescu and Schlag 2006]), one could weaken the local singularity assumption to $V \in L^{q_0}_{\mathrm{loc}}(\mathbb{R}^d)$, with q_0 satisfying (5), and then replace $\|V\|_{L^q(\mathbb{R}^d)}$ by the right-hand side of (4), where v_j is now the L^{q_0}-average of V over $j+hQ$.

Remark 3. There are three scales in the problem:

- the energy scale λ^2,
- the scale R measuring the support of the potential,
- the randomization scale $h < R$.

In addition, we have introduced an arbitrary (dimensionless) parameter M that appears in the large deviation bound. There is a separation of scales at $\lambda h = 1$ (and to a lesser extent at $\lambda R = 1$ but we ignore logarithms for the purpose of this remark). All eigenvalues with $|z|^{1/2} \le h^{-1}$ are contained in a ball of radius proportional to $\|V\|_{L^q}^{q/(2q-d)}$. By Hölder's inequality, the deterministic bound (6) shows that $|z|^{1/2} \le h^{-1}$ is satisfied whenever $h \ll R^{d((d+1)/(2q)-1)} \|V\|_{L^q}^{-(d+1)/2}$.

Remark 4. Of course, a compactly supported potential of the form (2) is in any L^q space. The point of the estimate is the very weak dependence on R (logarithmic) compared to what one would get by using Hölder's inequality and the deterministic bound (1), namely

$$|z|^{1/(d+1)} \lesssim R^{d(2/(d+1)-1/q)} \|V\|_{L^q}. \tag{6}$$

Moreover, compactly supported potentials are interesting in view of the counterexample to the Laptev–Safronov conjecture of Bögli and the first author [Bögli and Cuenin 2023]. The counterexample yields a sequence of potentials $V_\varepsilon,\ \varepsilon > 0$ small, with $|V_\varepsilon| \lesssim \varepsilon\chi_\varepsilon$, where χ_ε is the indicator function of the tube

$$T_\varepsilon = \{(x_1, x') : |x_1| < \varepsilon^{-1},\ |x'| < \varepsilon^{-1/2}\}$$

such that $1 + i\varepsilon$ is an eigenvalue of $-\Delta + V_\varepsilon$. Since

$$\|V_\varepsilon\|_{L^q(\mathbb{R}^d)} \lesssim \varepsilon^{1-(d+1)/(2q)},$$

this shows that (1) cannot hold for $q > \frac{1}{2}(d+1)$. In this context, Theorem 1 says that, after randomization on the scale

$$h \le [\varepsilon^{(d+1)/(2q)-1} \log(1/\varepsilon)^{-7/2}]^{2/d},$$

the counterexample for $\frac{1}{2}(d+1) < q \le d+1$ is almost surely destroyed.

Remark 5. Safronov [2023] has recently considered eigenvalue sums for random Schrödinger operators with complex potentials of the same form as (2), but without the assumption on the distribution of ω_j. However, these results do not give any new information about individual eigenvalues beyond what is known in the deterministic case [Frank 2011; 2018]. Moreover, Safronov's results only apply to the smaller range $q < \frac{1}{2}(d+1) + 1/(2d-4)$ compared to $q \le d+1$. Our results are of a quite different character and therefore a direct comparison is not possible.

The compact support assumption can be removed at the price of a tiny bit of pointwise decay.

Theorem 6. *For any $\delta > 0$, there exist constants $M_0,\ c > 0$ such that the following holds. For any h, $\lambda > 0,\ |\varepsilon| \ll \lambda,\ q \le d+1$, for any $V \in \langle x\rangle^{-\delta} L^q(\mathbb{R}^d)$, and for any $M \ge M_0$, each eigenvalue $z = (\lambda + \mathrm{i}\varepsilon)^2$ of $-\Delta + V_\omega$ satisfies*

$$\frac{\lambda^{2-d/q}}{\langle\lambda h\rangle^{d/2}(\log\langle\lambda h\rangle)^2} \le M \|\langle\lambda x\rangle^\delta V\|_{L^q(\mathbb{R}^d)},$$

except for ω in a set of measure at most $\exp(-cM^2)$.

In fact, if we sacrifice the endpoint, we can also remove the pointwise decay assumption.

Theorem 7. *For any $q < d+1$, there exist constants $M_0,\ c > 0$ such that the following holds. For any h, $\lambda > 0,\ |\varepsilon| \ll \lambda$, for any $V \in L^q(\mathbb{R}^d)$, and for any $M \ge M_0$, each eigenvalue $z = (\lambda + \mathrm{i}\varepsilon)^2$ of $-\Delta + V_\omega$ satisfies*

$$\frac{\lambda^{2-d/q}}{\langle\lambda h\rangle^{d/2}(\log\langle\lambda h\rangle)^2} \le M \|V\|_{L^q(\mathbb{R}^d)},$$

except for ω in a set of measure at most $\exp(-cM^2)$.

Remark 8. (i) For fixed h (and up to logarithms) and large λ, the left-hand side of the inequality is $\lambda^{2-d/q-d/2}$. The exponent is negative for $d \geq 3$ and $q \leq d+1$. Therefore, the estimate gives no information about large eigenvalues in this case. We believe that a factor of the form $\langle \lambda h \rangle^{\kappa}$ in the denominator is unavoidable, and is an expression of the fact that randomization only occurs down to scale h but not below (meaning that at scale h and below, V is deterministic). The example in Appendix B suggests that one should expect a loss of at least $\kappa = \frac{1}{4}(d+1)$. Our method of proof only yields $\kappa = \frac{1}{2}d$. It is an interesting question whether this can be improved. It would also be interesting to study the case where V is random at all scales (i.e., V is a random field). In particular, under which assumptions on the randomization is the estimate true with $\kappa = 0$?

(ii) For $d = 2$ and $2 < q < 3$ the exponent is positive. Bounding the logarithm by an arbitrary small power of λh, we see that if $\lambda h \geq 1$, then

$$h^{-1} \leq \lambda \lesssim (M h^{1+\varepsilon} \|V\|_q)^{q/(q-2-\varepsilon q)},$$

and hence $h \gtrsim (M\|V\|_q)^{-q/(2q-2)}$. Conversely, if $h \ll (M\|V\|_q)^{-q/(2q-2)}$, then the case $\lambda h \geq 1$ does not occur and we have $\lambda^{2-2/q} \lesssim M\|V\|_q$.

(iii) The techniques we use were originally developed in the discrete setting. In this case the spectrum is compact, and the issue of large eigenvalues does not arise (for operators on $h\mathbb{Z}^d$ the largest frequencies are of order h^{-1}).

Corollary 9. *Let $J \subset (0, \infty)$ be a compact interval, $q < d+1$, and $h > 0$. Then we have*

$$\sup_{\operatorname{Re} z \in J} \frac{|z|^{q-d/2}}{\|V\|_q^q} < \infty$$

almost surely. The supremum is taken over eigenvalues $z = (\lambda + \mathrm{i}\varepsilon)^2$ of $-\Delta + V_\omega$ with $|\varepsilon| \ll \lambda$.

Proof. Denote the supremum by S, and consider the events $E_M = \{S^{1/q} > M\}$. Since $E_M \supset E_{M+1}$ and $\boldsymbol{P}(E_{M_0}) < \infty$, we have

$$\boldsymbol{P}(S = \infty) = \lim_{M \to \infty} \boldsymbol{P}(E_M) = 0. \qquad \square$$

Remark 10. The proof shows that Theorem 7 (and hence Corollary 9) actually hold with $\|V\|_{L^q}$ replaced by the (smaller) Lorentz norm $\|V\|_{L^{q,\infty}}$.

The key technical elements in this work are estimates on certain "elementary operators", roughly of the form

$$R_0^{1/2} V_\omega R_0^{1/2}, \tag{7}$$

where R_0 is the free resolvent at a fixed (complex) energy and V_ω is supported on a ball of radius $R > 1$. In dimension 2 and in the discrete case (i.e., when Δ is replaced by the discrete Laplacian), Schlag, Shubin, and Wolff [Schlag et al. 2002] proved[1] that the norm of these operators is bounded by a power of $\log R$. Their proof used in an essential way that the level sets corresponding to the symbol

[1] This is roughly the content of [Schlag et al. 2002, Lemma 3.9]. Strictly speaking, the half powers of the resolvent are replaced by Fourier restriction and extension operators (or some mollified versions thereof); see also [Bourgain 2002, (1.12)].

of Δ (the discrete Laplacian) are curved. Bourgain [2002] gave a different proof using entropy bounds. His result is stated in dimension 2 but works in any dimension since it does not require curvature of the level sets (for the discrete Laplacian, these sets are not curved in higher dimensions). Motivated by work of Rodnianski and Schlag [2003], he uses these bounds to prove almost-sure existence and uniqueness of wave operators and absolutely continuous spectrum (for energies away from the edges of the spectrum and zero). The result shaves off half a power of pointwise decay compared to the classical (deterministic) Agmon–Kato–Kuroda theory. In a follow-up work, Bourgain [2003] combined his method with the two-dimensional Stein–Tomas restriction theorem to obtain the same conclusion for potentials in $\langle x\rangle^{-\delta}\ell^3(\mathbb{Z}^2)$ ($\delta > 0$ arbitrary). Note that there is a gap between the pointwise decay $\langle x\rangle^{-1/2}$ and $\ell^3(\mathbb{Z}^2)$. Bourgain [2003] observes that this gap cannot be overcome if one works with operators of the form (7) since the corresponding bounds (involving the $\ell^{3/2}(\mathbb{Z}^2)$ norm of the potential) are saturated (up to logarithms) by a Knapp example. Since the argument in [Bourgain 2003] is only stated in the two-dimensional discrete case, we will provide a similar, but more detailed, argument suggesting the optimality of our operator norm estimates (for the continuum multidimensional case) in Appendix B.[2] A representative (and simplified) example of these estimates, when λ and h are of unit size, is that

$$\|R_0^{1/2} V_\omega R_0^{1/2}\| \lesssim (\log R)^{\mathcal{O}(1)} \|V\|_{d+1}$$

with high probability (see Lemma 31 for a precise statement). Via a Born series argument (see Section 2 for details) this bound leads to a proof of Theorem 1. The proof of Theorem 6 then follows by a straightforward decomposition of the potential into dyadic shells $|x| \asymp 2^k$, similar to techniques of Bourgain [2002; 2003]. The proof of Theorem 7 requires more effort and the argument presented in Section 7 is new to the best of our knowledge. The technique[3] is reminiscent of an "epsilon removal lemma" in the context of Fourier restriction theory (see, e.g., [Tao 1999]). However, the technical implementation is a bit different since we are working with multilinear bounds (and with the resolvent instead of the Fourier restriction operator).

While the bounds (7) are optimal (up to logarithms) in the sense that the Lebesgue exponent $d+1$ cannot be increased, it is an interesting open problem whether our eigenvalue estimates (say in the form of Corollary 9) are optimal. This problem is connected to a remark of Bourgain [2003] that contains the idea of renormalizing away the self-energy interactions and then controlling the Born series via the sharp two-dimensional Fourier restriction theory of Carleson–Sjölin and Zygmund. This would amount to an $\ell^4(\mathbb{Z}^2)$ bound on the potential and would be natural and optimal from the point of view of restriction theory. A rigorous implementation of this idea seems difficult and has not been done so far, to the best of our knowledge.

Notation. We write $A \lesssim B$ for two nonnegative quantities $A, B \geq 0$ to indicate that there is a constant $C > 0$ such that $A \leq CB$. The dependence of the constant on fixed parameters like d and q is usually omitted (except in Section 7). The notation $A \asymp B$ means $A \lesssim B \lesssim A$. The product measure associated to the ω_j is denoted by $\boldsymbol{P}$ and the expectation by $\boldsymbol{E}$. We denote the L^p norm of a function f in $\mathbb{R}^d$ by $\|f\|_{L^p(\mathbb{R}^d)}$.

[2] Bourgain's ideas and his Knapp example were also explained in a talk of Wilhelm Schlag at the Institute for Advanced Study on March 29, 2017.

[3] Although Bourgain was almost certainly aware of these techniques, he did not bother to remove the logarithmic losses.

If the function is defined on a countable set Λ, we write $\|f\|_{\ell^p(\Lambda)} = \big(\sum_{\nu\in\Lambda} |f(\nu)|^p\big)^{1/p}$. If Λ is finite, we also set $\|f\|_{\ell^p_{\mathrm{av}}(\Lambda)} = \big(|\Lambda|^{-1} \sum_{\nu\in\Lambda} |f(\nu)|^p\big)^{1/p}$. If it is clear from the context which norm is meant, we sometimes use the abbreviation $\|f\|_p$. If $T : X \to Y$ is a bounded linear operator between two Banach spaces X and Y, we denote its operator norm by $\|T\|_{X\to Y}$. The indicator function of a set $\Omega \subset \mathbb{R}^d$ is denoted by $\mathbf{1}_\Omega$. For $1 \le p \le \infty$, we denote its Hölder conjugate by $p' = (1-1/p)^{-1}$. An arbitrary ball of radius R will be denoted by B_R, without specifying its center. We use the convention $\hat{f}(\xi) = \int_{\mathbb{R}^d} \mathrm{e}(-x\cdot\xi) f(x)\,\mathrm{d}x$ for the Fourier transform of f, where $\mathrm{e}(x) = \mathrm{e}^{2\pi i x}$, and $(f)^\vee(x) = \int_{\mathbb{R}^d} \mathrm{e}(x\cdot\xi) f(\xi)\,\mathrm{d}\xi$ for the inverse Fourier transform. Moreover, we recall the notation $\langle x\rangle = 2 + |x|$.

Organization. In Section 2 we outline the rough top-down strategy to prove our main results (see Proposition 11 for a summary). In Section 3, we collect basic facts related to the uncertainty principle and recall the Stein–Tomas theorem for a discrete version of the Fourier extension operator that will play a major technical role in the proofs of the estimates in Section 6. Section 4 is a short summary of probabilistic tools that will be used in the article. Section 5 fleshes out Bourgain's key idea of using entropy bounds. Section 6 contains the main local estimates and the completion of the proof of Theorem 1. Finally, in Section 7, the local estimates are converted to global ones, leading to the proofs of Theorems 6 and 7.

2. Born series

The proof of the eigenvalue estimates starts with the standard observation that $z \in \mathbb{C}\setminus[0,\infty)$ is an eigenvalue if and only if $I + R_0(z)V$ fails to be invertible as a bounded operator. This follows from the identity

$$-\Delta + V - z = (-\Delta - z)(I + R_0(z)V). \tag{8}$$

Here we denoted the free resolvent operator $(-\Delta - z)^{-1}$ by $R_0(z)$ and we omitted the subscript ω on V. Similarly, we will denote the perturbed resolvent operator $(-\Delta + V - z)^{-1}$ by $R(z)$. There are several variations of this argument based on variations of the identity (8). Perhaps the most well-known version is the so called *Birman–Schwinger principle*: z is an eigenvalue of $-\Delta+V$ if and only if -1 is an eigenvalue of the Birman–Schwinger operator $BS(z) = |V|^{1/2} R_0(z) V^{1/2}$. In particular, the norm of $BS(z)$ must be at least 1. This is perhaps the most commonly used approach in the literature since the seminal work of Frank [2011]. In the random case, this approach does not work so well because the sign (or phase) of the potential is of crucial importance. To exploit cancellations, we will work with the spectral radius (which must also be at least 1 but is in general smaller and harder to estimate than the norm). Although we could work with $BS(z)$, we prefer to work directly with the Born series; our approach may also be viewed as a multilinear version of the Birman–Schwinger principle.

In the following, to avoid confusion between the deterministic and the random potential, we focus our attention on the Anderson-type potentials (2). In this case, the assumption that $V \in L^q(\mathbb{R}^d)$ already implies that V is bounded (this follows from (4) and the fact that the ℓ^p spaces are nested). In particular, $R_0(z)V$ is a bounded operator. In the general case (3), one truncates the potential at some fixed large level. Since the estimates of Theorems 1–7 are independent of the L^∞ norm of V and the truncated Schrödinger operator converges to the untruncated one in the norm resolvent sense, there is no loss of generality in assuming that the deterministic potential is bounded. In the following, we assume that V is supported on a ball of radius R, i.e., the setting of Theorem 1. The case where V is not compactly supported (Theorems 6 and 7) will be considered in Section 7.

Returning to (8), we see that z cannot be an eigenvalue if the Born series

$$R(z) = \sum_{n\in\mathbb{N}} (-1)^n [R_0(z)V]^n R_0(z)$$

converges, which is the case if the spectral radius of $R_0 V$ is less than 1. Consider the following multilinear expansion (omitting z):

$$[R_0 V]^n = \sum_{\sigma_1,\dots,\sigma_n} R_0^{\sigma_1} V R_0^{\sigma_2} V \cdots R_0^{\sigma_n} V, \tag{9}$$

where $\sigma_j \in \{\text{low}, \text{high}\}$. Here, R_0^{low} is the resolvent (smoothly) localized to frequencies in $B(0,2)$ and $R_0^{\text{high}} = R_0 - R_0^{\text{low}}$. Since we are dealing with scale-invariant estimates, we may assume without loss of generality that $\lambda = 1$, hence $z = (1+\mathrm{i}\varepsilon)^2$ (see Remark 14 for more details). Then each summand is a composition of operators of the form $C^{(\delta_2)} V C^{(\delta_1)}$, where $C^{(\delta)}$ denotes a function satisfying a bound

$$|C^{(\delta)}(\xi)| \le \big(\big||2\pi\xi|^2 - 1\big| + \delta\big)^{-1/2}, \tag{10}$$

and the corresponding Fourier multiplier is denoted by the same symbol. Clearly, the bound (10) holds with $\delta = 1$ for $C^{(\delta)} = (R_0^{\text{high}})^{1/2}$ or $C^{(\delta)} = |R_0^{\text{high}}|^{1/2}$. In Section 6.2, we will show that (10) holds with $\delta = 1/R$ if $C^{(\delta)}$ is a mollification of $(R_0^{\text{low}})^{1/2}$ or $|R_0^{\text{low}}|^{1/2}$ at scale $1/R$. Such a mollification can always be performed (except for the first resolvent in the Born series, but this does not affect convergence), due to the localizing effect of the potential, which we assumed to be supported in a ball of radius R. The spectral radius is given by Gelfand's formula: $\operatorname{spr}(R_0 V) = \lim_{n\to\infty} \|[R_0 V]^n\|^{1/n}$. Thus, in view of the previous discussion, we have $\operatorname{spr}(R_0 V) \le \sup \|C^{(\delta_2)} V C^{(\delta_1)}\|$, where the supremum is taken over all functions satisfying (10). We will ignore the high frequency part of the resolvent R_0^{high} from now on since there are obvious elliptic estimates available for this part. We may thus restrict our attention to functions as in (10) that are compactly supported in $B(0,2)$. We summarize the observations of this section in the following proposition.

Proposition 11. *Let $z = (1+\mathrm{i}\varepsilon)^2$, with $|\varepsilon| \ll 1$. Let V be supported in a ball of radius R. If (for a given realization of ω)*

$$\|C^{(\delta_2)} V C^{(\delta_1)}\| \le c < 1 \tag{11}$$

for all functions $C^{(\delta_i)}$, $i = 1, 2$, satisfying (10) with $\delta_1, \delta_2 = 1/R$ and supported in $B(0,2)$, then z is not an eigenvalue of $-\Delta + V$.

We refer to operators of the form (11) as "elementary operators" since they form the building blocks of the Born series. We prove norm estimates on these and related operators in Section 6. These estimates are the key technical elements in this work.

Remark 12. Strictly speaking, the previous argument is only valid for $\varepsilon \ne 0$, but there are techniques to extend this to embedded eigenvalues ($\varepsilon = 0$); see, e.g., [Frank and Simon 2017, Proposition 3.1].

Remark 13. Later on, we will assume that all functions $C^{(\delta)}$ are supported in a small neighborhood of the unit sphere. This does not affect the validity of the above argument.

Remark 14. To restore the λ-dependence in the inequalities one can, e.g., use dimensional analysis: Since h and R have the dimension of a length L, the eigenvalue z and the potential V have the dimension L^{-2}, i.e., λ has the dimension L^{-1} and $\|V\|_q$ has the dimension $L^{d/q-2}$. Therefore, once an inequality for an eigenvalue $(1+i\varepsilon)^2$ involving $h,\ R,\ \|V\|_q$ has been proved, the λ-dependence is restored by multiplying h and R by λ and $\|V\|_q$ by $\lambda^{d/q-2}$.

3. Localization and discretization

3.1. ***Localization in momentum space.*** Denote by $\mathcal{Q}_h$ the collection of all cubes Q_h of sidelength h. Define the weight function

$$w_{Q_h}(x) = (1+h^{-1}\operatorname{dist}(x, Q_h))^{-100d}, \quad x \in \mathbb{R}^d, \quad Q_h \in \mathcal{Q}_h.$$

Lemma 15. *Let $v \in \mathcal{S}(\mathbb{R}^d)$, and assume that $\hat{v}$ is supported in $B(0, 1/h)$. Then v is locally constant on cubes Q_h of sidelength h in the sense that*

$$\|v\|_{L^\infty(Q_h)} \lesssim |Q_h|^{-1} \|v\|_{L^1(w_{Q_h})}.$$

Proof. By scaling, it suffices to prove this for $h = 1$. Choose $\eta \in \mathcal{S}(\mathbb{R}^d)$ such that $\eta = 1$ on $B(0, 1)$. Then we have $\hat{v} = \eta\hat{v}$, and hence $v = (\eta)^\vee * v$. Since $(\eta)^\vee \in \mathcal{S}(\mathbb{R}^d)$, it follows that

$$\kappa_w = \sup_{Q\in\mathcal{Q}_1} \sup_{(x,y)\in Q\times\mathbb{R}^d} |(\eta)^\vee(x-y)| w_Q(y)^{-1} < \infty,$$

where the first supremum is taken over all cubes of sidelength 1. Thus, for any cube Q of sidelength 1 and for $x \in Q$, we have

$$|v(x)| \le \int_{\mathbb{R}^d} |(\eta)^\vee(x-y)||v(y)|\,\mathrm{d}y \le \kappa_w \|v\|_{L^1(w_Q)}.$$

Taking the supremum over $x \in Q$ proves the claim. □

Lemma 16. *Let $v \in \mathcal{S}(\mathbb{R}^d)$, and assume that $\hat{v}$ is supported in $B(0, 1/h)$. Let $\Lambda_h \subset \mathbb{R}^d$ be a set of h-separated points. Then, for any $p \ge 1$, we have*

$$\|v\|_{\ell^p(\Lambda_h)} \lesssim h^{-d/p} \|v\|_{L^p(\mathbb{R}^d)}.$$

Proof. Again by scaling, we can assume $h = 1$. Thus, let $\Lambda \subset \mathbb{R}^d$ be a set of 1-separated points. Pick a collection of cubes Q of sidelength 1 that cover Λ. By Lemma 15,

$$\|v\|_{\ell^p(\Lambda)}^p = \sum_{\nu\in\Lambda} |v(\nu)|^p \lesssim \sum_Q \|v\|_{L^1(w_Q)}^p.$$

Write $v = \sum_{Q'} v_{Q'}$, where $v_{Q'}$ is supported on Q'. Then

$$\|v_{Q'}\|_{L^1(w_Q)} \le (1+\operatorname{dist}(Q, Q'))^{-100d} \|v_{Q'}\|_{L^1(\mathbb{R}^d)}.$$

By Hölder, $\|v_{Q'}\|_{L^1(\mathbb{R}^d)} \le \|v_{Q'}\|_{L^p(\mathbb{R}^d)}$. Hence,

$$\sum_Q \|v\|^p_{L^1(w_Q)} \lesssim \sum_{Q,Q'} (1+\operatorname{dist}(Q, Q'))^{-100dp} \|v_{Q'}\|^p_{L^p(\mathbb{R}^d)} \lesssim \|v\|^p_{L^p(\mathbb{R}^d)},$$

where we summed a geometric series in Q. □

3.2. ***Localization in position space.*** We will make use of the following standard device in local restriction theory (see, e.g., [Demeter 2020, Lemma 1.26]).

Lemma 17. *There exists a bump function ϕ on $\mathbb{R}^d$ with* $\operatorname{supp}\phi \subset B(0, 1)$ *and with nonnegative Fourier transform satisfying $\mathbf{1}_{B(0,1)} \le \hat{\phi}$. Moreover, $\hat{\phi}$ is an even function.*

It is clear that the rescaled function $\phi_R(\xi) = R^d \phi(R\xi)$ satisfies

$$\operatorname{supp}\phi_R \subset B(0, R^{-1}), \quad \mathbf{1}_{B(0,R)} \le \hat{\phi}_R.$$

Let $M_\lambda = \{\xi \in \mathbb{R}^d : |\xi| = \lambda\}$, and consider the extension operator

$$\mathcal{E}_\lambda : L^2(M_\lambda, \mathrm{d}\sigma_\lambda) \to L^\infty(\mathbb{R}^d), \quad (\mathcal{E}_\lambda g)(x) = (g\, \mathrm{d}\sigma_\lambda)^\vee(x),$$

where σ_λ is the surface measure on M_λ. We write $\mathcal{E} \equiv \mathcal{E}_1$ and $M \equiv M_1$, $\sigma \equiv \sigma_1$.

3.3. ***Discrete Fourier extension operator.***

Definition 18. Let $\operatorname{Discres}(M, p, 2)$ be the best constant such that the following holds for each $R \ge 2$, each collection Λ^*_R consisting of $1/R$-separated points on M, each sequence $a_\nu \subset \mathbb{C}$, each ball B_R, and each collection Λ_1 of 1-separated points in $\mathbb{R}^d$:

$$\Big\| \sum_{\nu \in \Lambda^*_R} a_\nu \mathrm{e}(\nu \cdot x) \Big\|_{\ell^{p'}(\Lambda_1 \cap B_R)} \le \operatorname{Discres}(M, p, 2) R^{(d-1)/2} \|a_\nu\|_{\ell^2(\Lambda^*_R)}. \tag{12}$$

Proposition 19. *If* $1 \le p \le \infty$, *then*

$$\operatorname{Discres}(M, p, 2) \lesssim \|\mathcal{E}\|_{L^2(M,\mathrm{d}\sigma) \to L^{p'}(\mathbb{R}^d)}. \tag{13}$$

Moreover, if $p \ge 2$, *then the reverse inequality also holds.*

Proof. The claim is a special case of [Demeter 2020, Proposition 1.29], with one small difference. There, $\operatorname{Discres}(M, p, 2)$ is defined with the $L^{p'}(B_R)$ norm in the left-hand side of (12). Thus, let $\operatorname{Discres}'(M, p, 2)$ be the best constant in the inequality

$$\Big\| \sum_{\nu \in \Lambda^*_R} a_\nu \mathrm{e}(\nu \cdot x) \Big\|_{L^{p'}(B_R)} \le \operatorname{Discres}'(M, p, 2) R^{(d-1)/2} \|a_\nu\|_{\ell^2(\Lambda^*_R)}. \tag{14}$$

Then [Demeter 2020, Proposition 1.29] asserts that the proposition holds with $\operatorname{Discres}'(M, p, 2)$ in place of $\operatorname{Discres}(M, p, 2)$. Thus, (13) follows once we show that

$$\operatorname{Discres}'(M, p, 2) \gtrsim \operatorname{Discres}(M, p, 2). \tag{15}$$

Without loss of generality we may assume that $B_R = B(0, R)$. If we set

$$f(x) = \sum_{\nu\in\Lambda_R^*} a_\nu \mathrm{e}(\nu\cdot x), \quad \text{then } \mathcal{F}(f\hat{\phi}_R)(\xi) = \sum_{\nu\in\Lambda_R^*} a_\nu \phi_R(\xi+\nu),$$

where ϕ_R is as before and $\mathcal{F}$ denotes the Fourier transform. Note that $\mathcal{F}(f\hat{\phi}_R) = \hat{f} * \phi_R$ is supported in an $1/R$-neighborhood of M. In particular, it is supported on the ball $B(0, 2)$. Thus, for any collection Λ_1 of 1-separated points in $\mathbb{R}^d$,

$$\|f\|_{\ell^{p'}(\Lambda_1\cap B_R)} \le \|f\hat{\phi}_R\|_{\ell^{p'}(\Lambda_1)} \lesssim \|f\hat{\phi}_R\|_{L^{p'}(\mathbb{R}^d)},$$

where we used $\hat{\phi}_R \ge \mathbf{1}_{B_R}$ in the first inequality and Lemma 16 in the second. By a partition of unity and a sparsification argument, we may assume that f is supported on a disjoint union of balls of radius R. By the rapid decay of $\hat{\phi}_R$ and by the definition of $\mathrm{Discres}'(M, p, 2)$,

$$\|\hat{\phi}_R f\|_{L^{p'}(\mathbb{R}^d)} \lesssim_N \sum_{j=1}^{\infty} j^{-N} \|f\|_{L^{p'}(B(x_j,R))} \lesssim \mathrm{Discres}'(M, p, 2) R^{(d-1)/2} \|a_\nu\|_{\ell^2(\Lambda_R^*)},$$

where we used that (14) holds uniformly in the centers of the balls. Combining the last two estimates yields (15).

To prove the reverse inequality to (13), we may assume that $B_R = B(0, R)$. By [Demeter 2020, Proposition 1.29] it suffices to prove the reverse inequality to (15). Let Λ_1 be a 1-net of points $x_j \in B_R$. Let $f(x)$ be defined as above. Without loss of generality we may assume that f is supported on a disjoint collection of balls $B(x_j, 10)$. Then

$$\begin{aligned}\|f\|_{L^{p'}(B_R)} &= \Big(\sum_j \|f\|^{p'}_{L^{p'}(B(x_j,10))}\Big)^{1/p'} = \Big(\int_{B(0,10)} \sum_j |f(x_j+y)|^{p'}\,\mathrm{d}y\Big)^{1/p'} \\ &\lesssim \mathrm{Discres}(M, p, 2) R^{(d-1)/2} \|a_\nu\|_{\ell^2(\Lambda_R^*)},\end{aligned}$$

where we used that (12) holds for each collection $x_j + y$ of 1-separated points, uniformly in y. □

3.4. ***Stein–Tomas theorem.*** The following is an immediate consequence of the Stein–Tomas theorem and Proposition 19 (see also [Demeter 2020, Corollary 1.30]).

Proposition 20. *Let* $p' \ge 2(d+1)/(d-1)$. *Then* $\mathrm{Discres}(M, p, 2) \lesssim 1$.

4. Randomization

4.1. ***Subgaussian random variables.*** We recall that a (complex) scalar random variable X is called *subgaussian* if it has finite subgaussian norm:

$$\|X\|_{\psi_2} = \inf\{t > 0 : \boldsymbol{E}\exp(|X|^2/t^2) \le 2\} < \infty.$$

We will need the following elementary properties of subgaussian (e.g., Gaussian or symmetric Bernoulli) random variables (see, e.g., [Vershynin 2018, Proposition 2.6.1 and Exercise 2.5.10]).

Proposition 21. *Assume that $(X_j)_{j=1}^N$, $N \geq 2$, is a finite collection of i.i.d. mean-zero subgaussian random variables.*

(i) *Then $\sum_{j=1}^N X_j$ is also subgaussian, and*

$$\Big\|\sum_{j=1}^N X_j\Big\|_{\psi_2}^2 \lesssim \sum_{j=1}^N \|X_j\|_{\psi_2}^2.$$

(ii) *We have*

$$\boldsymbol{E} \max_{j\leq N} |X_j| \lesssim \sqrt{\log N} \max_{j\leq N} \|X_j\|_{\psi_2}.$$

Proof. The claim follows by applying [Vershynin 2018, Proposition 2.6.1 and Exercise 2.5.10] to Re X_j and Im X_j separately. ☐

4.2. *Tail bounds.* We now consider tail bounds for *vector-valued* Gaussian or Bernoulli random variables X. We have $(\boldsymbol{E}\|X\|^p)^{1/p} \asymp (\boldsymbol{E}\|X\|^q)^{1/q}$ for all $p, q > 0$ (see [Ledoux and Talagrand 1991, Corollary 3.2 and Theorem 4.7]), which, combined with [Ledoux and Talagrand 1991, (3.5), (4.12)], implies

$$\boldsymbol{P}(\|X\| > t) \leq \exp\biggl(-\frac{ct^2}{(\boldsymbol{E}\|X\|)^2}\biggr)$$

for some $c > 0$. Thus the following lemma is obvious.

Lemma 22. *If $\boldsymbol{E}\|X\| \leq C$, then*

$$\boldsymbol{P}(\|X\| > MC) \leq \exp(-cM^2)$$

for any $M > 0$.

5. Entropy bound

Consider a linear operator $S : \mathcal{H} \to \ell_m^\infty$, where $\mathcal{H}$ is a finite-dimensional Hilbert space and $\ell_m^\infty = \ell^\infty(\{1, \dots, m\})$. For $\varepsilon > 0$, let $\mathcal{N}(\varepsilon)$ be the minimal number of balls in ℓ_m^∞ of radius ε needed to cover the set $\{Sx : x \in \mathcal{H},\ \|x\|_{\mathcal{H}} \leq 1\}$. Here we use the convention that the centers of the balls are contained in the set they cover (i.e., $\mathcal{N}(t)$ is the *covering number* as opposed to the *exterior covering number*; see, e.g., [Vershynin 2018, Section 4.2]). Using an entropy bound known as the "dual Sudakov inequality"—which is attributed to Pajor and Tomczak-Jaegermann [1986]—Bourgain [2002, (4.2)] shows that

$$\log \mathcal{N}(\varepsilon) \lesssim (\log m)\varepsilon^{-2}\|S\|_{\mathcal{H}\to\ell_m^\infty}^2. \tag{16}$$

The quantity $\log \mathcal{N}(\varepsilon)$ is called the *entropy number* of the image of the unit ball in $\mathcal{H}$ under the map S. The crucial observation is that (16) is independent of dim $\mathcal{H}$. We apply this bound to the operator featuring in (12), i.e.,

$$S : \mathcal{H} \to \ell^\infty(\Lambda_1 \cap B_R), \quad \{a_\nu\} \mapsto \biggl\{\sum_{\nu\in\Lambda_R^*} a_\nu \mathrm{e}(\nu \cdot x)\biggr\}_x. \tag{17}$$

In this case, $\mathcal{H} = \ell^2(\Lambda_R^*)$ with norm $\|a\|_{\mathcal{H}} := R^{(d-1)/2}\big(\sum_{\nu\in\Lambda_R^*}|a_\nu|^2\big)^{1/2}$ and $\ell_m^\infty = \ell^\infty(\Lambda_1\cap B_R)$. In particular, we have $m \asymp R^d$. Here and in the following we always assume $R\geq 2$. Proposition 20 gives

$$\|S\|_{\mathcal{H}\to\ell^{p'}(\Lambda_1\cap B_R)} \lesssim 1 \quad \text{for } p'\geq 2(d+1)/(d-1). \tag{18}$$

In particular, we have the trivial bound ($p'=\infty$)

$$\|S\|_{\mathcal{H}\to\ell^{\infty}(\Lambda_1\cap B_R)} \lesssim 1. \tag{19}$$

Combining the latter with (16) yields the following entropy bound.

Proposition 23. *Let S be given by* (17). *The entropy number satisfies the bound*

$$\log\mathcal{N}(\varepsilon) \lesssim (\log R)\varepsilon^{-2}.$$

Corollary 24. *Let $p'\geq 2(d+1)/(d-1)$. For every $k\in\mathbb{Z}_+$, there exist sets $\mathcal{F}_k\subset\ell^\infty(\Lambda_1\cap B_R)$ with the following properties*:

(a) $\log|\mathcal{F}_k| \lesssim \log(R)4^k$ (*here* $|\cdot|$ *denotes the counting measure*).

(b) *For $\xi\in\mathcal{F}_k$,*

$$\|\xi\|_{\ell^\infty(\Lambda_1)} \lesssim 2^{-k}, \quad \|\xi\|_{\ell^{p'}(\Lambda_1)} \lesssim 1.$$

(c) *For each $a\in\mathcal{H}$ with $\|a\|_{\mathcal{H}}\leq 1$, there is a representation*

$$Sa = \sum_{k\in\mathbb{Z}_+}\xi^{(k)} \quad \textit{for some } \xi^{(k)}\in\mathcal{F}_k.$$

Proof. We follow Bourgain [2003, pages 75-76], but provide more details (note also that there is a misprint in (3.13) in that work; it should be 4^r, not 4^{-r}). This is a standard chaining argument.

We start by noting that, in view of (16) and (19), we have $\mathcal{N}(C)=1$ for C sufficiently large. In the following (and only in this proof), denote the unit ball in $\mathcal{H}$ by B_1. Similarly, $B(\xi,\varepsilon)$ denotes a ball centered at ξ and with radius ε in $\ell^\infty(\Lambda_1\cap B_R)$. We also write $\|\cdot\|_p = \|\cdot\|_{\ell^p(\Lambda_1)}$ here. By possibly rescaling SB_1 by a constant, we may assume that $C=1$. Thus, we have $\mathcal{N}(1)=1$. We get, by Proposition 23,

$$\log\mathcal{N}(2^{-k}) \lesssim \log(R)4^k.$$

Thus, for each $k\geq 0$, there exist subsets $\mathcal{E}_k\subset\ell^\infty(\Lambda_1\cap B_R)$ of cardinality $\mathcal{N}(2^{-k})$ satisfying

$$SB_1 \subset \bigcup_{\xi\in\mathcal{E}_k} B(\xi,2^{-k}).$$

Applying these nets for each k, we can assign to each element $Sa\in SB_1$ a chain $\{\xi_k\}$ converging to Sa, with $\xi_k\in\mathcal{E}_k$ and

$$\|\xi_k-\xi_{k-1}\|_\infty \leq 2^{-k}+2^{1-k} \tag{20}$$

for all k. By telescoping, we have

$$Sa = \xi_0 + \lim_{N\to\infty}\sum_{k=1}^{N}(\xi_k-\xi_{k-1}).$$

Thus, we may choose $\mathcal{F}_0 = \mathcal{E}_0$ and $\mathcal{F}_k \subset \mathcal{E}_k - \mathcal{E}_{k-1},\ k > 0$, as the collection of all vectors $\xi^{(k)} = \xi_k - \xi_{k-1}$ for which (20) holds. Since the difference set $\mathcal{E}_k - \mathcal{E}_{k-1}$ has cardinality $|\mathcal{E}_k||\mathcal{E}_{k-1}|$, the claimed properties hold by construction. ☐

6. Local bounds on elementary operators

6.1. ***Local extension bound.*** Let $h,\ R > 0$. Consider V_ω of the form (3), where V is a given deterministic potential supported in B_R. Also fix $p' \geq 2(d+1)/(d-1)$, and define q by $1/q = 1/p - 1/p'$. Note that this convention differs from that in the main theorems by a change of variables $q \to 2q$.

Lemma 25. *Under the above assumptions, we have*

$$\boldsymbol{E}\|\mathcal{E}^* V_\omega \mathcal{E}\|_{L^2(M,\mathrm{d}\sigma)\to L^2(M,\mathrm{d}\sigma)} \lesssim \langle h\rangle^{d/2}(\log\langle R\rangle)^{1/2}(\log\langle h\rangle + \log\langle R\rangle)^2 \|V\|_{L^{2q}(\mathbb{R}^d)}.$$

Proof. Since the right-hand side only gets larger if we replace R and h by $R+2$ and $h+2$, respectively, we may assume $R, h \geq 2$. We first observe that

$$\mathcal{E}^* V_\omega \mathcal{E} = \mathcal{E}^*(V_\omega * \varphi)\mathcal{E} \tag{21}$$

for any Schwartz function φ satisfying $\hat{\varphi} = 1$ on $B(0, 2)$. We can thus assume without loss of generality that V is smooth on the unit scale. Let g, g' be unit vectors in $L^2(M, \mathrm{d}\sigma)$. Then

$$\langle \mathcal{E}^* V_\omega \mathcal{E} g, g'\rangle = \sum_{j\in h\mathbb{Z}^d} \omega_j \int_{Q_h+j} \overline{V(x)(\mathcal{E}g)(x)}(\mathcal{E}g')(x)\,\mathrm{d}x,$$

where $Q_h = [0, h)^d$. Let $\Lambda_R^* = \{\eta_\nu\}$ be a $1/R$-net in M. By working with a partition of unity, we may assume that g is supported on a collection of disjoint balls $B(\eta_\nu, 10/R)$. After a change of variables $g(\eta) = g(\eta_\nu + \tau)$, we may write

$$\mathcal{E}g(x) = \sum_\nu \int_{M\cap B(0,10/R)} \mathrm{e}(x\cdot(\eta_\nu + \tau))g(\eta_\nu + \tau)\,\mathrm{d}\tau,$$

where $\mathrm{d}\tau$ denotes the surface measure, and similarly (summing over a possibly different index set)

$$\mathcal{E}g'(x) = \sum_{\nu'} \int_{M\cap B(0,10/R)} \mathrm{e}(x\cdot(\eta_{\nu'} + \tau'))g'(\eta_{\nu'} + \tau')\,\mathrm{d}\tau'.$$

Similar to the change of variables $\eta = \eta_\nu + \tau$ in the domain, we change variables $x = x_i + y$ in the target. Here, $\Lambda_1 = \{x_i\}$ is a 1-net in $\mathbb{R}^d$. Hence, for any integrable function $F : \mathbb{R}^d \to \mathbb{C}$ supported on a disjoint collection of balls $B(x_i, 10)$,

$$\int_{\mathbb{R}^d} F(x)\,\mathrm{d}x = \sum_i \int_{B(0,10)} F(x_i + y)\,\mathrm{d}y.$$

Using a partition of unity we may sparsify the potential, so that the above holds for

$$F_j(x) = \overline{V(x)(\mathcal{E}g)(x)}(\mathcal{E}g')(x)\mathbf{1}_{Q_h+j}(x).$$

Note that in this case the sum is restricted to those i satisfying $x_i \in B(j, 10+h)$. For fixed $\tau \in B(0, 10/R)$ and $y \in B(0, 10)$, we consider the discrete extension operator

$$S : \mathcal{H} \to \ell^\infty(\Lambda_1 \cap B_R), \quad \{g(\eta_\nu + \tau)\}_\nu \mapsto \left\{ \sum_\nu \mathrm{e}((x_i + y) \cdot (\eta_\nu + \tau)) g(\eta_\nu + \tau) \right\}_i .$$

Note that the points $\mu_\nu = \eta_\nu + \tau$ and $z_i = x_i + y$ form a $1/R$-separated set in M and a 1-separated set in $\mathbb{R}^d$, respectively, so that (18) and (19) hold. Using Corollary 24, we can find a representation (note that the vectors $\xi^{(k)}$ depend on τ, y)

$$\sum_\nu \mathrm{e}((x_i + y) \cdot (\eta_\nu + \tau)) g(\eta_\nu + \tau) = \sum_{k \in \mathbb{Z}_+} \xi_i^{(k)}, \quad \xi^{(k)} \in \mathcal{F}_k,$$

with bounds

$$\|\xi^{(k)}\|_\infty \lesssim 2^{-k} \|g(\eta_\nu + \tau)\|_{\ell^2_{\nu,\mathrm{av}}}, \quad \|\xi^{(k)}\|_{p'} \lesssim \|g(\eta_\nu + \tau)\|_{\ell^2_{\nu,\mathrm{av}}} \tag{22}$$

for all $k \in \mathbb{Z}_+$ and $y \in B(0, 10)$. Similarly, there is a representation

$$\sum_{\nu'} \mathrm{e}((x_i + y) \cdot (\eta_{\nu'} + \tau')) g'(\eta_{\nu'} + \tau') = \sum_{k' \in \mathbb{Z}_+} \xi_i^{(k')}, \quad \xi^{(k')} \in \mathcal{F}_{k'},$$

with bounds

$$\|\xi^{(k')}\|_\infty \lesssim 2^{-k'} \|g'(\eta_{\nu'} + \tau')\|_{\ell^2_{\nu',\mathrm{av}}}, \quad \|\xi^{(k')}\|_{p'} \lesssim \|g'(\eta_{\nu'} + \tau')\|_{\ell^2_{\nu',\mathrm{av}}}. \tag{23}$$

The above observations lead to the estimate

$$\sup_{g,g'} |\langle \mathcal{E}^* V_\omega \mathcal{E} g, g' \rangle| \le \sum_{k,k' \in \mathbb{Z}_+} \int \max_{(\xi,\xi') \in \mathcal{F}_k \times \mathcal{F}_{k'}} \left| \sum_{j \in h\mathbb{Z}^d} \sum_i \omega_j \overline{V(x_i + y) \xi_i} \xi_i' \right| \mathrm{d}y \, \mathrm{d}\tau \, \mathrm{d}\tau',$$

where the integral is taken over $(y, \tau, \tau') \in B(0, 10) \times (M \cap B(0, 10/R))^2$ and the sum over i is restricted to $x_i + y \in Q_h + j$ (we recall that y is fixed). By monotonicity of the expectation,

$$\boldsymbol{E} \sup_{g,g'} |\langle \mathcal{E}^* V_\omega \mathcal{E} g, g' \rangle| \le \sum_{k,k' \in \mathbb{Z}_+} \int \boldsymbol{E} \max_{\mathcal{F}_k \times \mathcal{F}_{k'}} |X_{\xi,\xi'}| \, \mathrm{d}y \, \mathrm{d}\tau \, \mathrm{d}\tau',$$

where (suppressing the dependence on y, τ, τ')

$$X_{\xi,\xi'} = \sum_{j \in h\mathbb{Z}^d} \omega_j \sum_i \overline{V(x_i + y) \xi_i} \xi_i'.$$

The conclusion follows by Lemmas 26 and 38 (details of the calculation are provided in Appendix A). □

Lemma 26. *Let $R, h \ge 2$. Then we have the bounds*

$$\int \boldsymbol{E} \max_{\mathcal{F}_k \times \mathcal{F}_{k'}} |X_{\xi,\xi'}| \, \mathrm{d}y \, \mathrm{d}\tau \, \mathrm{d}\tau' \lesssim (\log R)^{1/2} h^{d/2} \|V\|_{L^{2q}(\mathbb{R}^d)},$$

$$\int \max_{\mathcal{F}_k \times \mathcal{F}_{k'}} |X_{\xi,\xi'}| \, \mathrm{d}y \, \mathrm{d}\tau \, \mathrm{d}\tau' \lesssim R^{d - d/(2q)} 2^{-k-k'} \|V\|_{L^{2q}(\mathbb{R}^d)}.$$

Proof. Note first that the index set of $X_{\xi,\xi'}$ is finite and has cardinality N, satisfying

$$\log N = \log|\mathcal{F}_k \times \mathcal{F}_{k'}| \lesssim \log R \max(4^k, 4^{k'}) \tag{24}$$

by Corollary 24 (a). Proposition 21 implies that $X_{\xi,\xi'}$ are (scalar) subgaussian random variables, and

$$\boldsymbol{E} \max_{\mathcal{F}_k\times\mathcal{F}_{k'}} |X_{\xi,\xi'}| \lesssim \sqrt{\log N}\bigg(\sum_{j\in h\mathbb{Z}^d}\bigg|\sum_i \overline{V(x_i+y)\xi_i}\xi_i'\bigg|^2\bigg)^{1/2},$$

where we recall that we are assuming $\|\omega_j\|_{\psi_2} \lesssim 1$. Using Hölder's inequality twice, it follows that

$$\begin{aligned}\boldsymbol{E} \max_{\mathcal{F}_k\times\mathcal{F}_{k'}} |X_{\xi,\xi'}| &\lesssim \sqrt{\log N}\big\| \|V(x_i+y)\|_{\ell_i^q}\|\xi_i\|_{\ell_i^{p'}}\|\xi_i'\|_{\ell_i^{p'}}\big\|_{\ell_j^2}\\ &\lesssim \sqrt{\log N}\big\| \|V(x_i+y)\|_{\ell_i^q}\big\|_{\ell_j^{2q}}\big\|\|\xi_i\|_{\ell_i^{p'}}\|\xi_i'\|_{\ell_i^{p'}}\big\|_{\ell_j^{p'}},\end{aligned} \tag{25}$$

where we recall that i is restricted to $x_i+y\in Q_h+j$ and y is fixed. In particular, we have

$$|\{j\in h\mathbb{Z}^d : x_i+y\in Q_h+j\}| = 1 \quad \text{for each } i \tag{26}$$

and

$$|\{i : x_i+y\in Q_h+j\}| \le h^d \quad \text{for each } j\in h\mathbb{Z}^d. \tag{27}$$

We will show that

$$\big\|\|\xi_i\|_{\ell_i^{p'}}\|\xi_i'\|_{\ell_i^{p'}}\big\|_{\ell_j^{p'}} \lesssim h^{d/p'}\min(2^{-k},2^{-k'})\|g(\eta_\nu+\tau)\|_{\ell^2_{\nu,\mathrm{av}}}\|g'(\eta_{\nu'}+\tau')\|_{\ell^2_{\nu',\mathrm{av}}}. \tag{28}$$

By symmetry in ξ and ξ', it suffices to prove this in the case $k\ge k'$. Using Hölder once more, we have

$$\big\|\|\xi_i\|_{\ell_i^{p'}}\|\xi_i'\|_{\ell_i^{p'}}\big\|_{\ell_j^{p'}} \le \big\|\|\xi_i\|_{\ell_i^{p'}}\big\|_{\ell_j^\infty}\big\|\|\xi_i'\|_{\ell_i^{p'}}\big\|_{\ell_j^{p'}}.$$

By Fubini's theorem and (26),

$$\big\|\|\xi_i'\|_{\ell_i^{p'}}\big\|_{\ell_j^{p'}} = \bigg(\sum_i\sum_{j\in h\mathbb{Z}^d}|\xi_i'|^{p'}\bigg)^{1/p'} = \bigg(\sum_i|\xi_i'|^{p'}\bigg)^{1/p'} = \|\xi'\|_{p'}.$$

Similarly, by (27), we have

$$\big\|\|\xi_i\|_{\ell_i^{p'}}\big\|_{\ell_j^\infty} \le h^{d/p'}\|\xi\|_\infty.$$

Combining these estimates with (22) and (23) yields (28). Next, we have (again by Hölder, Fubini and (27))

$$\begin{aligned}\big\|\|V(x_i+y)\|_{\ell_i^q}\big\|_{\ell_j^{2q}} &\le h^{d/(2q)}\big\|\|V(x_i+y)\|_{\ell_i^{2q}}\big\|_{\ell_j^{2q}}\\ &= h^{d/(2q)}\big\|\|V(x_i+y)\|_{\ell_j^{2q}}\big\|_{\ell_i^{2q}} = h^{d/(2q)}\|V(x_i+y)\|_{\ell_i^{2q}}.\end{aligned} \tag{29}$$

Integrating (25) over $y,\ \tau,\ \tau'$ and using (28) and (29), we obtain

$$\int \boldsymbol{E}\max_{\mathcal{F}_k\times\mathcal{F}_{k'}}|X_{\xi,\xi'}|\,\mathrm{d}y\,\mathrm{d}\tau\,\mathrm{d}\tau' \lesssim \sqrt{\log N}\min(2^{-k},2^{-k'})h^{d/2}\|V\|_{L^{2q}(\mathbb{R}^d)},$$

where we used $\frac{1}{2} = 1/(2q) + 1/p'$, $\|V(x_i + y)\|_{L^{2q}_y \ell^{2q}_i} \lesssim \|V\|_{L^{2q}(\mathbb{R}^d)}$, and

$$R^{-(d-1)}\|g(\eta_\nu + \tau)\|_{L^2_\tau \ell^2_{\nu,\mathrm{av}}}\|g'(\eta_{\nu'} + \tau')\|_{L^2_{\tau'} \ell^2_{\nu',\mathrm{av}}} \lesssim \|g\|_{L^2(M,\mathrm{d}\sigma)}\|g'\|_{L^2(M,\mathrm{d}\sigma)} = 1.$$

Combining this with (24) yields the first bound of the lemma. The second bound follows from the estimate

$$\begin{aligned}
|X_{\xi,\xi'}| &\le \sum_{j\in h\mathbb{Z}^d}\left|\sum_i \overline{V(x_i+y)\xi_i}\xi'_i\right| \le \sum_{j\in h\mathbb{Z}^d}\|V(x_i+y)\|_{\ell^1_i}\|\xi_i\|_{\ell^\infty_i}\|\xi'_i\|_{\ell^\infty_i}\\
&\le \|V(x_i+y)\|_{\ell^1_j\ell^1_i}\|\xi\|_\infty\|\xi'\|_\infty\\
&= \|V(x_i+y)\|_{\ell^1_i\ell^1_j}\|\xi\|_\infty\|\xi'\|_\infty\\
&= \|V(x_i+y)\|_{\ell^1_i}\|\xi\|_\infty\|\xi'\|_\infty\\
&\lesssim R^{d-d/(2q)}\|V(x_i+y)\|_{\ell^{2q}_i}\|\xi\|_\infty\|\xi'\|_\infty\\
&\lesssim R^{d-d/(2q)}\|V(x_i+y)\|_{\ell^{2q}_i}2^{-k-k'}\|g(\eta_\nu+\tau)\|_{\ell^2_{\nu,\mathrm{av}}}\|g'(\eta_{\nu'}+\tau')\|_{\ell^2_{\nu',\mathrm{av}}},
\end{aligned}$$

where we used Hölder in the first, second and fifth line, Fubini in the third line, (26) in the fourth, $\operatorname{supp} V \subset B_R$ in the fifth and (22), (23) in the last line. Integrating over y, τ, τ' and using Hölder as before yields the second bound in the lemma. □

Remark 27. If we restore the frequency in the extension operator, i.e., if we consider $\mathcal{E}^*_\lambda V_\omega \mathcal{E}_{\lambda'}$, then it is obvious from the proof of Lemma 25 that the same estimate holds for this operator, locally uniformly in $\lambda, \lambda' \asymp 1$. Explicitly,

$$\begin{aligned}
\sup_{\lambda,\lambda'\asymp 1} \boldsymbol{E}\|\mathcal{E}^*_\lambda V_\omega \mathcal{E}_{\lambda'}\|_{L^2(M_\lambda,\mathrm{d}\sigma_\lambda)\to L^2(M_{\lambda'},\mathrm{d}\sigma_{\lambda'})} &\le A(h, R, V),\\
A(h, R, V) \lesssim \langle h\rangle^{d/2}(\log\langle R\rangle)^{1/2}(\log\langle h\rangle + \log\langle R\rangle)^2&\|V\|_{L^{2q}(\mathbb{R}^d)}.
\end{aligned} \tag{30}$$

6.2. *Smoothing.* We observe that if $m(D)$ is a Fourier multiplier and B_{R_1} and B_{R_2} are two balls with the same center, then

$$\mathbf{1}_{B_{R_1}} m(D)\mathbf{1}_{B_{R_2}} = \mathbf{1}_{B_{R_1}} m_R(D)\mathbf{1}_{B_{R_2}}, \quad m_R := \gamma_R * m, \tag{31}$$

whenever $R > R_1 + R_2$, $\gamma_R(\xi) = R^d\gamma(R\xi)$, and $(\gamma)^\vee$ is a bump function such that $(\gamma)^\vee(x) = 1$ for $|x| \le 1$. This can be checked by comparing the kernels of both sides in (31) and using the convolution theorem. The convolution with γ_R can be considered a smoothing operator at scale R^{-1}. We recall from Section 2 that $C^{(\delta)}$ denotes a generic function satisfying a bound

$$|C^{(\delta)}(\xi)| \lesssim \left(\left||2\pi\xi|^2 - 1\right| + \delta\right)^{-1/2}. \tag{32}$$

We will apply (31) to

$$m(\xi) = (|2\pi\xi|^2 - (1 + \mathrm{i}0)^2)^{-1} \tag{33}$$

to produce a product of two functions $C^{(\delta)}(\xi)$ satisfying (32) with $\delta = R^{-1}$.

Lemma 28. *For $R \ge 1$, we have*

$$|\gamma_R * m| \lesssim R.$$

In particular, $(\gamma_R * m)^{1/2}$ *satisfies* (32) *with* $\delta = R^{-1}$.

Proof. By a partition of unity we may assume that m is supported in a small conic neighborhood of the first coordinate axis. The implicit function theorem then allows us to reduce the proof to the bound

$$\left|\gamma_R * \frac{1}{\xi_1 + \mathrm{i}0}\right| \lesssim R,$$

where $\gamma_R(\xi_1) = R\gamma(R\xi_1)$ is a function of one variable. By the convolution theorem,

$$\left|\gamma_R * \frac{1}{\xi_1 + \mathrm{i}0}\right| \lesssim \|\hat{\gamma}_R\|_1 \lesssim R,$$

where we used that the Fourier transform of $(\xi_1 + \mathrm{i}0)^{-1}$ is bounded. See also [Ruiz 2002, Lemma 5.2] for an alternative proof. □

Remark 29. The boundary value in (33) is defined in the usual way (in the sense of tempered distributions, see, e.g., [Hörmander 1990]). The analogue expression with $(1 - \mathrm{i}0)^2$ clearly satisfies the same bound. A similar argument (using the Malgrange preparation theorem) also works for ε nonzero and fixed. This argument is presented in the proof of Lemma 23 in [Bögli and Cuenin 2023]. Alternatively, one can work with the boundary values throughout and appeal to the Phragmén–Lindelöf maximum principle to extend the results to nonzero ε (see, e.g., [Cuenin 2017, Appendix A; Guillarmou et al. 2020; Ruiz 2002]). We will not pursue this issue.

In practice, we are working with a localized version of (33), supported near the singular manifold M. Even though $\gamma_R * m$ loses compact support, it decays rapidly away from M on the $1/R$ scale. Neglecting the tail (which can be bounded in a straightforward way), we assume that all functions $C^{(\delta)}$ that appear from now on are compactly supported in a small neighborhood of M. Alternatively, one could avoid tails by smoothing the resolvent first and then perform the low/high decomposition as in Section 2.

6.3. ***Foliation by level sets.*** In the following we will assume that $C^{(\delta)}$ is supported in a c-neighborhood (c small and fixed) of M and satisfies (32). We will also assume that $\lambda \in [1-c, 1+c]$ and denote the constant $A(h, R, V)$ appearing in (30) by A.

Lemma 30. *Assume that* (30) *and* (32) *hold. Then we have*

$$\boldsymbol{E}\|\mathcal{E}_\lambda^* V C^{(\delta)}\|_{L^2(\mathbb{R}^d)\to L^2(M_\lambda)} \lesssim A\Big(\log\frac{1}{\delta}\Big)^{1/2}. \tag{34}$$

Moreover, if (32) *holds for* $C^{(\delta_1)}$ *and* $C^{(\delta_2)}$, *then*

$$\boldsymbol{E}\|C^{(\delta_1)} V C^{(\delta_2)}\|_{L^2(\mathbb{R}^d)\to L^2(\mathbb{R}^d)} \lesssim A\Big(\log\frac{1}{\delta_1}\Big)^{1/2}\Big(\log\frac{1}{\delta_2}\Big)^{1/2}. \tag{35}$$

Proof. For $f \in L^2(\mathbb{R}^d)$, we foliate by level sets M_λ,

$$C^{(\delta)} f(x) = \int_{1-c}^{1+c}\int_{M_{\lambda'}} \mathrm{e}(x\cdot\xi) C^{(\delta)}(\xi)\hat{f}(\xi)\,\mathrm{d}\sigma_{\lambda'}(\xi)\,\mathrm{d}\lambda', \tag{36}$$

up to an innocuous Jacobian factor. Without loss of generality we now assume that f has Fourier support in $1-c \le |\xi| \le 1+c$. Using (32) and the fact that $(\mathrm{d}\sigma_\lambda)^\vee * f$ is a constant multiple of $\mathcal{E}_\lambda\mathcal{E}_\lambda^* f$, we get, by

Cauchy–Schwarz,

$$\boldsymbol{E}\|\mathcal{E}_\lambda^* V C^{(\delta)} f\|_{L^2(M_\lambda)} \le A\Big(\int_{1-c}^{1+c} \mathrm{d}\lambda'(|\lambda'-1|+\delta)^{-1}\Big)^{1/2}\Big(\int_{1-c}^{1+c} \mathrm{d}\lambda' \|\mathcal{E}_{\lambda'}^* f\|_{L^2(M_{\lambda'})}^2\Big)^{1/2} \lesssim A\Big(\log\frac{1}{\delta}\Big)^{1/2}\|f\|_2,$$

where we used

$$\int_{1-c}^{1+c} \mathrm{d}\lambda' \|\mathcal{E}_{\lambda'}^* f\|_{L^2(M_{\lambda'})}^2 = \int_{1-c}^{1+c} \mathrm{d}\lambda' \int_{M_{\lambda'}} |\hat{f}(\xi)|^2 \,\mathrm{d}\sigma_{\lambda'}(\xi) \lesssim \|f\|_{L^2(\mathbb{R}^d)}^2 \tag{37}$$

and

$$\int_{1-c}^{1+c} \mathrm{d}\lambda'(|\lambda'-1|+\delta)^{-1} \lesssim \log\frac{1}{\delta}. \tag{38}$$

This proves (34). To prove (35), we use the dual estimate to (37), which is

$$\Big\|\int_{1-c}^{1+c} \mathcal{E}_{\lambda'} g(\lambda')\,\mathrm{d}\lambda'\Big\|_{L^2(\mathbb{R}^d)} \lesssim \Big(\int_{1-c}^{1+c} \|g(\lambda')\|_{L^2(M_{\lambda'})}^2 \,\mathrm{d}\lambda'\Big)^{1/2} \tag{39}$$

for $g(\lambda') \in L^2(M_{\lambda'})$. This follows from

$$\int_{1-c}^{1+c} \langle \mathcal{E}_{\lambda'}^* f, g(\lambda')\rangle_{L^2(M_{\lambda'})} \,\mathrm{d}\lambda' = \Big\langle f, \int_{1-c}^{1+c} \mathcal{E}_{\lambda'} g(\lambda')\,\mathrm{d}\lambda'\Big\rangle_{L^2(\mathbb{R}^d)}.$$

Using the foliation (36) for the $C^{(\delta_1)}$ factor and using (34) and (38), inequality (39) gives, with $g(\lambda') = (|\lambda'-1|+\delta_1)^{-1/2}\mathcal{E}_{\lambda'}^* V C^{(\delta_2)} f$,

$$\begin{aligned}\boldsymbol{E}\|C^{(\delta_1)} V C^{(\delta_2)} f\|_{L^2(\mathbb{R}^d)} &\lesssim \Big\|\int_{1-c}^{1+c} \mathcal{E}_{\lambda'} g(\lambda')\,\mathrm{d}\lambda'\Big\|_{L^2(\mathbb{R}^d)} \lesssim \Big(\int_{1-c}^{1+c} \|g(\lambda')\|_{L^2(M_{\lambda'})}^2 \,\mathrm{d}\lambda'\Big)^{1/2} \\ &\lesssim A\Big(\log\frac{1}{\delta_1}\Big)^{1/2}\Big(\log\frac{1}{\delta_2}\Big)^{1/2}\|f\|_{L^2(\mathbb{R}^d)}. \qquad \square\end{aligned}$$

6.4. ***Local resolvent bound.*** We use the same conventions as in the previous section. Additionally, in the following, the norm is the $L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d)$ operator norm. Recall that, by the discussion at the end of Section 6.2, the square root of the localized resolvent R_0^{low} can be replaced by a compactly supported multiplier satisfying the bound (32) with $\delta = 1/R$. As a consequence of Lemma 25, (35), and the discussion in Section 2, we immediately obtain the following resolvent bound.

Lemma 31. *Assume that* (32) *holds for* $C^{(\delta_1)}$ *and* $C^{(\delta_2)}$, *with* $\delta_1, \delta_2 \asymp 1/R$. *Then we have*

$$\boldsymbol{E}\|C^{(\delta_2)} V_\omega C^{(\delta_1)}\| \lesssim \langle h\rangle^{d/2}(\log\langle R\rangle)^{3/2}(\log\langle h\rangle + \log\langle R\rangle)^2\|V\|_{L^{2q}(\mathbb{R}^d)}.$$

By using the tail bound of Lemma 22 and rescaling, we obtain the following corollary.

Corollary 32. *Let* $h, R, \lambda, M > 0$, *and let* $|\varepsilon| \ll \lambda$. *Then the spectral radius of* $R_0((\lambda+\mathrm{i}\varepsilon)^2)V_\omega$ *is bounded by*

$$\operatorname{spr}(R_0 V) \lesssim M\langle \lambda h\rangle^{d/2}(\log\langle\lambda R\rangle)^{3/2}(\log\langle\lambda h\rangle + \log\langle\lambda R\rangle)^2\lambda^{d/(2q)-2}\|V\|_{L^{2q}(\mathbb{R}^d)},$$

except for ω *in a set of measure at most* $\exp(-cM^2)$.

6.5. ***Completion of the proof of Theorem 1.*** We first undo the change of variables $q \to 2q$. Theorem 1 then follows immediately from Proposition 11 and Corollary 32. □

7. Local to global arguments

7.1. ***Proof of Theorem 6.*** To complete the proof of Theorem 6, we rescale again to $\lambda = 1$. We decompose $V = \sum_{k \in \mathbb{Z}^+} V_k$ into dyadic pieces with support in $\{0 \le |x| \le 1\}$ for $k = 0$, and in $\{2^{k-1} \le |x| \le 2^k\}$ for $k \ge 1$. The assumption on V guarantees that $\|V_k\|_q \le 2^{-\delta k} \|\langle x\rangle^\delta V\|_q$. Instead of (9), we consider the multilinear expansion

$$[R_0 V]^n = \sum_{\sigma_1,\dots,\sigma_n} \sum_{k_1,\dots,k_n} R_0^{\sigma_1} V_{k_1} R_0^{\sigma_2} V_{k_2} \cdots R_0^{\sigma_n} V_{k_n},$$

where we again omitted the spectral parameter z, and we are assuming, as we may, that $z = (1 + \mathrm{i}\varepsilon)^2$, $|\varepsilon| \ll 1$. By the same arguments in Section 2, it suffices to estimate the norms of elementary blocks of the form $C^{(\delta_{l-1})} V_{k_l} C^{(\delta_l)}$, where $\delta_l = (2^{k_l} + 2^{k_{l-1}})^{-1}$. Lemmas 25 and 30 and an analogue of Lemma 28 with $\delta = \delta_l$ or δ_{l-1} yield (again undoing the change of variables $q \to 2q$)

$$\boldsymbol{E}\|C^{(\delta_{l-1})} V_{k_l} C^{(\delta_l)}\| \lesssim (k_{l-1} + k_l + k_{l+1}) \langle h\rangle^{d/2} \langle k_l\rangle^{1/2} (\log\langle h\rangle + \langle k_l\rangle)^2 2^{-\delta k_l} \|\langle x\rangle^\delta V\|_q.$$

Applying the tail bound of Lemma 22 yields

$$\|C^{(\delta_{l-1})} V_{k_l} C^{(\delta_l)}\| \le M_1 (k_{l-1} + k_l + k_{l+1}) \langle h\rangle^{d/2} \langle k_l\rangle^{1/2} (\log\langle h\rangle + \langle k_l\rangle)^2 2^{-\delta k_l} \|\langle x\rangle^\delta V\|_q,$$

except for ω in a set of measure at most $\exp(-c' M_1^2)$. Choosing $M_1 = M(k_{l-1} + k_l + k_{l+1})$ and summing the previous bound over $k_1, \dots, k_n$ yields

$$\operatorname{spr}(R_0 V) = \lim_{n\to\infty} \|[R_0 V]^n\|^{1/n} \lesssim \langle h\rangle^{d/2} (\log\langle h\rangle)^2 \|\langle x\rangle^\delta V\|_q,$$

except for ω in a set of measure at most

$$\sum_{k_{l-1}, k_l, k_{l+1}} \exp(-c' M_1^2) \le \exp(-c M^2).$$

This concludes the proof of Theorem 6. □

7.2. ***Sparse decomposition.*** To prove Theorem 7, we use a device reminiscent of an "epsilon removal lemma" (see, e.g., [Tao 1999]) but adapted to our multilinear bounds (and the resolvent as opposed to the Fourier restriction operator). For this reason, we need to perform several decompositions simultaneously:

(1) We first decompose V dyadically:

$$V = \sum_{i \in \mathbb{Z}_+} V_i, \quad V_i = V \mathbf{1}_{H_i \ge |V| \ge H_{i+1}}, \quad H_i = \inf\{t > 0 : |\{|V| > t\}| \le 2^{i-1}\}.$$

This is a "horizontal" dyadic decomposition since the widths of the supports of V_i are approximately 2^i. Here we are assuming that V is constant on the unit scale (hence $i \ge 0$ in the sum above). In view of (21), there is no loss of generality in this assumption for the purpose of proving estimates (this is the same argument as explained in the paragraph before [Tao 1999, Lemma 3.3]). Note that we have

$$\|H_i 2^{i/q}\|_{\ell_i^r(\mathbb{Z}_+)} \asymp \|V\|_{L^{q,r}},$$

where $L^{q,r}$ denotes a Lorentz space (see, e.g., [Tao 2006, Theorem 6.6]). Also note that $L^{q,q} = L^q$.

(2) Next, split each dyadic piece into a sum of "sparse families",

$$V_i = \sum_{j=1}^{K_i} \sum_{k=1}^{N_i} V_{ijk}, \tag{40}$$

where, for fixed i and j, the V_{ijk} are supported on a "sparse collection" of balls $\{B(x_k, R_i)\}_{k=1}^{N_i}$. By this we mean that the support of V_{ijk} is contained in $B(x_k, R_i)$ and that the following definition is satisfied (see [Tao 1999, Definition 3.1]) for some sufficiently large γ (to be chosen later).

Definition 33. A collection $\{B(x_k, R)\}_{k=1}^{N}$ is γ-sparse if the centers x_k are $(RN)^\gamma$-separated.

For fixed $\gamma > 0$ and $K \geq 1$, [Tao 1999, Lemma 3.3] asserts that (40) holds with

$$K_i = \mathcal{O}(K 2^{i/K}), \quad N_i = \mathcal{O}(2^i), \quad R_i = \mathcal{O}(2^{i\gamma^K}). \tag{41}$$

7.3. ***Spectral radius estimates.*** The preceding decompositions produce a multilinear expansion of the Born series,

$$[R_0 V]^n = \sum_{\alpha_1,\dots,\alpha_n} R_0 V_{\alpha_1} R_0 V_{\alpha_2} \cdots R_0 V_{\alpha_n}, \tag{42}$$

where $\alpha_l = (i_l, j_l, k_l)$ and $i_l \in \mathbb{Z}_+$, $1 \leq j_l \leq K_{i_l}$, $1 \leq k_l \leq N_{i_l}$. To estimate the spectral radius of $R_0 V$, we estimate the summands in (42) in two different ways. For the first estimate, we follow a similar strategy as before. However, since the smoothing of the resolvent (see Section 6.2) now depends on the mutual positions of the supports of V_{α_l}, we consider the (slightly more general) elementary operators

$$C^{(\delta_1)} \mathbf{1}_{B_2} W C^{(\delta_2)}, \tag{43}$$

where the $B_k = B(x_k, R_k)$ are arbitrary balls and W is a bounded potential. As before, the $C^{(\delta)}$ are Fourier multipliers satisfying (32), now with

$$\delta_1 = \langle \mathrm{d}(B_1, B_2) + 2R_1 + 2R_2 \rangle^{-1}, \quad \delta_2 = \langle \mathrm{d}(B_2, B_3) + 2R_2 + 2R_3 \rangle^{-1}.$$

The operators (43) arise from an analogue of (31) and Lemma 28 for balls with different centers. In the same way that Lemma 31 and its corollary follow from Lemma 25, (35), and the tail bound of Lemma 22, we obtain

$$\|C^{(\delta_1)} \mathbf{1}_{B_2} W_\omega C^{(\delta_2)}\| \leq M_1 h^{d/2} (\log h)^2 \Big[\log\Big(\frac{1}{\delta_1} + \frac{1}{\delta_2}\Big)\Big]^{\mathcal{O}(1)} \|W\|_{L^q(B_2)} \tag{44}$$

for any $q \leq d+1$ and for all ω except for a set of measure at most $\exp(-c' M_1^2)$. Here we have assumed again, as we may, that $\lambda = 1$ and $R, h > 2$. For the remainder of this section we omit the (obvious) dependence on h. We also switch from the (modified) Vinogradov notation $A \lesssim B$ to the Hardy notation

$A \le CB$ or Landau notation $A = \mathcal{O}(B)$, and we indicate the dependence of constants on q (since q will no longer be in a compact interval) or other related parameters. It is also convenient to use the letter A for quantities (norms, constants) containing $\mathcal{O}(1)$ terms that are bounded uniformly in n (and may change from line to line).

The case of interest is of course when the balls in (44) contain the supports of the potentials in (42), and W is one of these potentials. Similar to the proof of Theorem 6, we choose $M_1 = M[\log(1/\delta_1 + 1/\delta_2)]^{\mathcal{O}(1)}$ without qualitatively changing the estimate (44). In this way, the union bound for the probability of the complementary event yields

$$\begin{aligned}\boldsymbol{P}\Big(\bigcup_{\alpha_1,\alpha_2,\alpha_3}\{\omega : (44)\text{ does not hold}\}\Big) &\le \sum_{\alpha_1,\alpha_2,\alpha_3} \exp(-c' M_1^2)\\ &\le \sum_{i_1,i_2,i_3} N_{i_1}K_{i_1}N_{i_2}K_{i_2}N_{i_3}K_{i_3}\exp(-c'M_1^2)\\ &\le \exp(-cM^2),\end{aligned}$$

and hence we have

$$\|R_0 V_{\alpha_1} R_0 V_{\alpha_2}\cdots R_0 V_{\alpha_n}\| \le AM^n \prod_{l=1}^{n}\Big[\log\Big(\frac{1}{\delta_{\alpha_l}}\Big) + \log\Big(\frac{1}{\delta_{\alpha_{l+1}}}\Big)\Big]^{\mathcal{O}(1)} \|V_{\alpha_l}\|_q, \tag{45}$$

except for ω in a set of measure at most $\exp(-cM^2)$.

For the second estimate, we observe that, by the triangle inequality and Cauchy–Schwarz,

$$\|[R_0V]^n\| \le \sum_{\alpha_1,\ldots,\alpha_n} \|R_0|V_{\alpha_1}|^{1/2}\|\,\|V_{\alpha_1}^{1/2}R_0|V_{\alpha_2}|^{1/2}\|\cdots\|V_{\alpha_{n-1}}^{1/2}R_0|V_{\alpha_n}|^{1/2}\|\,\|V_{\alpha_n}^{1/2}\|.$$

Here we are again assuming, as we may, that V is bounded. The operator norm $\|V_{\alpha_n}^{1/2}\|$ (equal to the L^∞ norm) will be annihilated by taking the n-th root at the end and letting n tend to infinity. Let

$$L_{\alpha,\beta} := \delta_{\alpha,\beta} + \mathrm{d}(B_\alpha, B_\beta),$$

where the balls B_α contain the support of V_α.

Lemma 34. *For $q \le \frac{1}{2}(d+1)$,*

$$\|V_\alpha^{1/2}R_0|V_\beta|^{1/2}\| \le C_q L_{\alpha,\beta}^{1-(d+1)/(2q)}\|V_\alpha\|_q^{1/2}\|V_\beta\|_q^{1/2}. \tag{46}$$

Proof. To prove this, one uses the well known pointwise bound

$$|R_0^{(a+\mathrm{i}t)}(x-y)| \le C_1 \mathrm{e}^{C_2 t^2}|x-y|^{-(d+1)/2+a} \tag{47}$$

for $a \in \big[\frac{1}{2}(d-1), \frac{1}{2}(d+1)\big]$ and $d \ge 2$ (see, e.g., [Lee and Seo 2019, (2.5)]), or the explicit formula for the resolvent kernel in $d = 1$. More precisely, consider the analytic family $V_\alpha^{\zeta/2}R_0^\zeta|V_\beta|^{\zeta/2}$. Then (47) implies that, for $\operatorname{Re}\zeta = q$, the kernel is bounded by

$$|V_\alpha(x)^{\zeta/2}R_0^\zeta(x-y)|V_\beta(y)|^{\zeta/2}| \le C_1\mathrm{e}^{C_2(\operatorname{Im}\zeta)^2}L_{\alpha,\beta}^{-\eta}|V_\alpha(x)|^{q/2}|V_\beta(y)|^{q/2},$$

where $\eta = \frac{1}{2}(d+1) - q \geq 0$, leading to the Hilbert–Schmidt bound

$$\|V_\alpha^{\zeta/2} R_0^\zeta |V_\beta|^{\zeta/2}\| \leq C_\eta L_{\alpha,\beta}^{-\eta} \|V_\alpha\|_q^{q/2} \|V_\beta\|_q^{q/2}$$

for some constant C_η (allowed to change from line to line). Interpolating this with the trivial bound $\|V_\alpha^{\zeta/2} R_0^\zeta |V_\beta|^{\zeta/2}\| \leq C_1 \mathrm{e}^{C_2(\operatorname{Im}\zeta)^2}$ for $\operatorname{Re}\zeta = 0$ yields (46). □

The previous lemma yields the second estimate

$$\|R_0 V_{\alpha_1} R_0 V_{\alpha_2} \cdots R_0 V_{\alpha_n}\| \leq A C_\eta^n \prod_{l=1}^n \|V_{\alpha_l}\|_{q_\eta} L_{\alpha_l,\alpha_{l+1}}^{-\eta'},$$

where $\eta' = \eta / \big(\frac{1}{2}(d+1) - \eta\big)$ and $q_\eta = \frac{1}{2}(d+1) - \eta$. Interpolating this with (45), we get, for $0 < \theta < 1$,

$$\|R_0 V_{\alpha_1} R_0 V_{\alpha_2} \cdots R_0 V_{\alpha_n}\| \leq A (C_\eta M)^n \prod_{l=1}^n [\log(1 + R_{i_{l-1}} + R_{i_l} + R_{i_{l+1}})]^{\mathcal{O}(1)} L_{\alpha_l,\alpha_{l+1}}^{-\theta\eta'/2} \|V_{\alpha_l}\|_q^{(1-\theta)} \|V_{\alpha_l}\|_{q_\eta}^{\theta},$$

except on an exceptional set of measure at most $\exp(-cM^2)$. (Here we used $L_{\alpha_l,\alpha_{l+1}}^{-\theta\eta'/2}$ to control $\mathrm{d}(B_{\alpha_l}, B_{\alpha_{l+1}})$ appearing in $\log(1/\delta_{\alpha_l})$.) Using that

$$\|V_{\alpha_l}\|_q \lesssim H_{i_l} 2^{i_l/q}$$

for all $q \geq 1$ and summing the resulting estimate first over k_1, then continuing up to k_{n-1}, yields

$$\sum_{k_1,\dots,k_{n-1}} \|R_0 V_{\alpha_1} R_0 V_{\alpha_2} \cdots R_0 V_{\alpha_n}\| \leq A (C_\eta M)^n \prod_{l=1}^{n-1} [\log(1 + R_{i_{l-1}} + R_{i_l} + R_{i_{l+1}})]^{\mathcal{O}(1)} H_{i_l} 2^{i_l((1-\theta)/q + \theta/q_\eta)}.$$

Here we have used that, for $\alpha_1 = (i_1, j_1, k_1)$, $\alpha_2 = (i_2, j_2, k_2)$ and $i_1,\ j_1,\ i_2,\ j_2,\ k_2$ fixed, the sum over k_1 is bounded:

$$\sum_{k_1 \leq N_{i_1}} \langle \mathrm{d}(B(x_{k_1}, R_{i_1}), B_{\alpha_2}) \rangle^{-\theta\eta'/2} = \mathcal{O}_{\gamma_0}(1), \tag{48}$$

uniformly in $i_1,\ j_1,\ i_2,\ j_2,\ k_2$, provided $\frac{1}{2}\theta\eta'\gamma_0 > 1$ and $\gamma \geq \gamma_0$. We will momentarily fix η and θ, and then choose $\gamma_0 = 4/(\eta'\theta)$. See also [Cho et al. 2022] for a precise version of Tao's lemma; there, it is clear that γ can be chosen. Note that, even though the balls in (48) may belong to different sparse families, we have that

$$\mathrm{d}(B(x_{k_1}, R_{i_1}), B_{\alpha_2}) \geq \tfrac{1}{2}(N_{i_1} R_{i_1})^\gamma$$

for all but at most one k_1. Indeed, suppose for contradiction that this does not hold for two distinct $k_1,\ k_1'$. Then by the triangle inequality,

$$\mathrm{d}(B(x_{k_1}, R_{i_1}), B(x_{k_1'}, R_{i_1})) < (N_{i_1} R_{i_1})^\gamma,$$

which contradicts the sparsity of the collection $\{B(x_{k_1}, R_{i_1})\}$.

Note that the last summation over k_n produces a $\mathcal{O}(2^{i_n})$ factor, but this can be absorbed into the constant A after summing over i_n and hence we do dot display it.

Summing over $j_1, \dots, j_n$ yields

$$\sum_{j_1,\dots,j_n} \sum_{k_1,\dots,k_n} \| R_0 V_{\alpha_1} R_0 V_{\alpha_2} \cdots R_0 V_{\alpha_n} \| \\ \le A(C_\eta M)^n \prod_{l=1}^{n} [\log(1 + R_{i_{l-1}} + R_{i_l} + R_{i_{l+1}})]^{\mathcal{O}(1)} K_{i_l} H_{i_l} 2^{i_l((1-\theta)/q+\theta/q_\eta)},$$

where K_i is as in (41). Finally, summing over $i_1, \dots, i_n$ yields

$$\|[R_0 V]^n\| \le A(C_\eta M K)^n \biggl(\sum_{i \in \mathbb{Z}_+} \langle i \rangle^{\mathcal{O}(1)} H_i 2^{i((1-\theta)/q+\theta/q_\eta+1/K)} \biggr)^n.$$

Once K is fixed, we choose η and θ such that $0 < \theta(1/q_\eta - 1/q) < 1/K$. Then

$$\operatorname{spr}(R_0 V_\omega) = \lim_{n\to\infty} \|[R_0 V]^n\|^{1/n} \le C_{\eta,K} M \sum_{i \in \mathbb{Z}_+} H_i 2^{i/q} 2^{3i/K}, \tag{49}$$

where we used that $\langle i \rangle^{\mathcal{O}(1)} \le C_K 2^{i/K}$.

7.4. ***Completion of the proof of Theorem 7.*** We use (49) for $\tilde{q} > q$ instead of q; that is, we now regard $\frac{1}{2}(d+1) < q < d+1$ as given and choose $\tilde{q} < d+1$ and K such that $1/\tilde{q} + 3/K < 1/q$. Then

$$\operatorname{spr}(R_0 V_\omega) \lesssim \sup_{i \in \mathbb{Z}^+} H_i 2^{i/q} \sum_{i \in \mathbb{Z}_+} 2^{i(1/\tilde{q}-1/q+3/K)} \le C_{\tilde{q},K} M \|V\|_{L^{q,\infty}}.$$

Clearly, the choice of $\tilde{q}$ depends only on $q,\ K,\ d$ and $\|V\|_{L^q} \le \|V\|_{L^{q,\infty}}$. We have thus proved the main estimate of this section, which also completes the proof of Theorem 7.

Lemma 35. *Let $q < d+1$. Then there exists c and M_0 such that, for all $M \ge M_0$, $z = (\lambda + \mathrm{i}\varepsilon)^2$, $\lambda \asymp 1$, $|\varepsilon| \ll 1$, and $V \in L^q(\mathbb{R}^d)$,*

$$\operatorname{spr}(R_0(z) V_\omega) \le M \|V\|_q,$$

except for ω in a set of measure at most $\exp(-cM^2)$.

7.5. ***Global extension bound.*** For potential future reference we include a similar bound to that proved in Lemma 35 but for the norms of the elementary operators (11) instead of the spectral radius of $R_0 V$.

Proposition 36. *Let $q < d+1$. Then there exist constants M_0 and c such that, for any $M \ge M_0$, $\lambda, \lambda' \asymp 1$, and $V \in L^q(\mathbb{R}^d)$,*

$$\|\mathcal{E}_\lambda^* V_\omega \mathcal{E}_{\lambda'}\| \le M \langle h \rangle^{d/2} (\log\langle h \rangle)^2 \|V\|_{L^q},$$

except for ω in a set of measure at most $\exp(-cM^2)$.

We refer to [Cuenin and Merz 2023, Theorem 5] for novel bounds on $\mathcal{E}_\lambda^* V_\omega \mathcal{E}_{\lambda'}$ in Schatten norms.

In the following, we use the notation $\|V\|_{\ell^\infty L^q} = \sup_{j \le N} \|V\|_{L^q(B(x_j,R))}$ and $V_j = V \mathbf{1}_{(B(x_j,R))}$, whenever V is supported on a γ-sparse collection $\{B(x_j, R)\}_{j=1}^N$. We will show that Proposition 36 follows from the subsequent lemma.

Lemma 37. *There exist constants $M_0,\ c,\ \gamma_0 > 0$ such that the following holds. For any $R > 0,\ 0 < h < R$, $\lambda, \lambda' \asymp 1,\ q < d+1,\ N \in \mathbb{N},\ \gamma \geq \gamma_0$, for any $V \in L^q(\mathbb{R}^d)$ supported on a γ-sparse collection $\{B(x_j, R)\}_{j=1}^N$, and, for any $M \geq M_0,\ \varepsilon > 0$,*

$$\|\mathcal{E}_\lambda^* V_\omega \mathcal{E}_{\lambda'}\| \leq C_{q,\varepsilon}(M^2 + \log N)^{1/2} \langle h\rangle^{d/2} (\log\langle h\rangle)^2 \langle R\rangle^\varepsilon \|V\|_{\ell^\infty L^q},$$

except for ω in a set of measure at most $\exp(-cM^2)$.

Proof. We may assume without loss of generality that $\lambda, \lambda' = 1$ and $R > 2$. We omit the subscripts in $\mathcal{E}_\lambda^*$ and $\mathcal{E}_{\lambda'}$ as well as the (obvious) h-dependence (i.e., we set $h = 1$). Consider the operators

$$T_j = \mathcal{E}^* V_j \mathcal{E}, \quad 1 \leq j \leq N,$$

where we omitted ω from the notation. Then

$$T_i T_j^* = \mathcal{E}^* V_i \mathcal{E}\mathcal{E}^* \overline{V_j} \mathcal{E}, \quad T_i^* T_j = \mathcal{E}^* \overline{V_i} \mathcal{E}\mathcal{E}^* V_j \mathcal{E}.$$

As in the endpoint proof of the Stein–Tomas theorem (see, e.g., [Stein 1993, IX.1.2.2]) we embed $\mathcal{E}\mathcal{E}^*$ into an analytic family of operators U_s in the strip $\frac{1}{2}(1-d) \leq \operatorname{Re} s \leq 1$, satisfying

$$\|U_s\|_{L^2 \to L^2} \lesssim 1, \quad \operatorname{Re} s = 1, \qquad \|U_s\|_{L^1 \to L^\infty} \lesssim 1, \quad \operatorname{Re} s = \tfrac{1}{2}(1-d),$$

and $U_0 = \mathcal{E}\mathcal{E}^*$. Similar to the proof of Lemma 34, we then use complex interpolation on the family $|V_i|^{(1-s)/2} U_s |V_j|^{(1-s)/2}$ to obtain the bound

$$\||V_i|^{1/2} \mathcal{E}\mathcal{E}^* |V_j|^{1/2}\| \lesssim L_{ij}^{-\eta'} \|V_i\|_{L^{q_\eta}}^{1/2} \|V_j\|_{L^{q_\eta}}^{1/2}$$

for $\eta' = \eta/q_\eta,\ q_\eta = \frac{1}{2}(d+1) - \eta$, and $0 < \eta \ll 1$. By the Stein–Tomas theorem and Hölder's inequality, we also have

$$\|\mathcal{E}^* V_i^{1/2}\| \lesssim \|V_j\|_{L^{q_\eta}}^{1/2}, \quad \|V_i^{1/2} \mathcal{E}\| \lesssim \|V_j\|_{L^{q_\eta}}^{1/2}.$$

Combining the last two displayed formulas yields the deterministic bound

$$\|T_i T_j^*\|^{1/2} + \|T_i^* T_j\|^{1/2} \lesssim L_{ij}^{-\eta'} \|V\|_{\ell^\infty L^{q_\eta}}$$

for all $i, j \leq N$. On the other hand, the bound of Lemma 25 (and changing variables $2q \to q$) yields

$$\|T_i T_j^*\|^{1/2} + \|T_i^* T_j\|^{1/2} \leq M_1 (\log R)^{5/2} \|V\|_{\ell^\infty L^q}$$

for all $i, j \leq N$, and for all ω except for an exceptional set of measure at most $N\exp(-cM_1^2)$. Interpolating the previous two estimates as in the proof of Lemma 35, we get, by the Cotlar–Stein lemma and (48),

$$\|\mathcal{E}^* V \mathcal{E}\| \leq C_{\eta,\gamma_0} [(\log R)^{5/2} \|V\|_{\ell^\infty L^q}]^{1-\theta} \|V\|_{\ell^\infty L^{q_\eta}}^\theta$$

for any $\theta \in (0, 1)$ and for all ω except for an exceptional set, provided $\frac{1}{2}\theta\eta'\gamma_0 > 1$ and $\gamma \geq \gamma_0$. Finally, we use Hölder's inequality

$$\|V\|_{\ell^\infty L^{q_\eta}} \lesssim R^{d/s_\eta} \|V\|_{\ell^\infty L^q}, \quad \frac{1}{q_\eta} = \frac{1}{s_\eta} + \frac{1}{q},$$

to convert the previous estimate to

$$\|\mathcal{E}^* V \mathcal{E}\| \leq C_{\eta,\gamma_0} [\log R]^{5(1-\theta)/2} R^{\theta d/s_\eta} \|V\|_{\ell^\infty L^q}.$$

We now fix $0 < \eta \ll 1$ (small, but independent of ε) and choose $\theta \in (0, 1)$ such that

$$[\log R]^{5(1-\theta)/2} R^{\theta d/s} \leq R^{\varepsilon}.$$

Moreover, we choose $M_1 = (M^2 + c^{-1} \log N)^{1/2}$, which ensures that the exceptional set has measure at most $\exp(-cM^2)$. Then the claim holds with the choice $\gamma_0 = 4/(\eta'\theta)$. The remainder of the proof is the same as that of Lemma 35. □

Proof of Proposition 36. We again use the sparse decomposition of Section 7.2 and recall the bounds (41) on K_i, N_i, R_i. As before, we also set $\lambda, \lambda', h = 1$. Lemma 37 yields the estimate

$$\|\mathcal{E}^* V_{ij} \mathcal{E}\| \leq C_{q,\varepsilon} (M_i^2 + \log N_i)^{1/2} R_i^{\varepsilon} \|V_{ij}\|_q$$

for all $q < d+1$, uniformly in i and j, and for ω outside of a set of measure $\exp(-cM_i^2)$. Here we are assuming, as we may, that $M_i, N_i, R_i > 2$, say. We may choose M_i freely, and we take $M_i = 2M\langle i\rangle^{\delta}$, with $\delta > 0$. Summing over j yields, by the triangle inequality,

$$\|\mathcal{E}^* V_i \mathcal{E}\| \leq C_{q,\varepsilon} K_i (M_i^2 + \log N_i)^{1/2} R_i^{\varepsilon} \|V_i\|_q.$$

Summing over i,

$$\|\mathcal{E}^* V_i \mathcal{E}\| \leq C_{q,\varepsilon,K} \sum_{i \in \mathbb{Z}_+} H_i 2^{i(1/q + 2/K + \varepsilon\gamma^K)}.$$

Here we also used (41), $\|V_i\|_q \lesssim H_i 2^{i/q}$, and $(M^2 + \log N_i)^{1/2} \leq C_K M 2^{i/K}$. We again apply this bound for $\tilde{q} > q$ instead of q, this time with $\tilde{q} < d+1$ and K, ε such that $1/\tilde{q} + 2/K + \varepsilon\gamma^K < 1/q$. Then the claimed bound again follows by summing a geometric series. The union bound yields that this bound holds outside an exceptional set of measure at most

$$\sum_{i,j} \exp(-c' M_i^2) \leq \sum_i K_i \exp(-c' M_i^2) \leq \exp(-cM^2),$$

due to the choice of M_i. □

Appendix A: Geometric series estimate

Lemma 38. *Let $A > 0$. Then we have*

$$\sum_{k,k' \in \mathbb{Z}_+} \min(2^{-k-k'}, A) \lesssim \begin{cases} A(1 + (\log A)^2) & \text{if } A < 1, \\ 1 & \text{if } A \geq 1. \end{cases}$$

Proof. The case $A \geq 1$ is trivial. Assume $A < 1$. We split the double sum into the obvious regions $\Sigma_1 = \{(k, k') : 2^{-k-k'} \leq A\}$ and $\Sigma_2 = \{(k, k') : 2^{-k-k'} > A\}$. Then we have

$$\sum_{(k,k') \in \Sigma_1} \min(2^{-k-k'}, A) = \sum_{k' \in \mathbb{Z}_+} 2^{-k'} \sum_{k : 2^{-k} \leq 2^{k'} A} 2^{-k} \lesssim \sum_{k' \in \mathbb{Z}_+} 2^{-k'} \min(1, 2^{k'} A).$$

Splitting the last sum again in the obvious way yields

$$\sum_{(k,k') \in \Sigma_1} \min(2^{-k-k'}, A) \lesssim A(1 + \log A^{-1}).$$

Turning to the contribution of Σ_2, we have

$$\sum_{(k,k')\in\Sigma_2} \min(2^{-k-k'}, A) = A \sum_{k'\in\mathbb{Z}_+} |\{k \in \mathbb{Z}_+ : 2^{-k} > 2^{k'} A\}| \le A \sum_{k'\in\mathbb{Z}_+} (\log A - k')_+ \le A(\log A)^2.$$

The claim follows since $\log A^{-1} \le 1 + (\log A)^2$. □

We now provide details of the calculation at the end of the proof of Lemma 25. Without loss of generality we may assume that $\|V\|_{2q} = 1$. By Lemma 26, we have

$$\sum_{k,k'\in\mathbb{Z}_+} \int \boldsymbol{E} \max_{\mathcal{F}_k\times\mathcal{F}_{k'}} |X_{\xi,\xi'}| \,\mathrm{d}y\,\mathrm{d}\tau\,\mathrm{d}\tau' \lesssim R^{d-d/(2q)} \sum_{k,k'\in\mathbb{Z}_+} \min(2^{-k-k'}, A),$$

with $A = R^{-d+d/(2q)}(\log R)^{1/2} h^{d/2}$, where we recall that we are assuming that $R, h > 2$. Since we may always assume that $R \gg 1$ and $h < R$ (otherwise there is no randomization), we have $A \ll 1$, and thus

$$R^{d-d/(2q)} \sum_{k,k'\in\mathbb{Z}_+} \min(2^{-k-k'}, A) \lesssim (\log R)^{1/2} h^{d/2} (\log h + \log R)^2$$

by Lemma 38.

Appendix B: Knapp example

As mentioned in the introduction, we give an example that suggests optimality of the key bounds of Lemmas 25 and 31 with respect to the Lebesgue exponent q. (Here we work with second moments whereas in these lemmas we used first moments.)

In view of the foliation (36) it is sufficient to prove optimality of Lemma 25. To this end, let V be the indicator function of the tube

$$T_R = \{(x_1, x') : |x_1| < R,\ |x'| < R^{1/2}\},$$

normalized in L^q, i.e., $V = R^{-(d+1)/(2q)} \mathbf{1}_{T_R}$ (we will mollify this later). Here $R > 1$ is a large parameter. We consider the randomization V_ω (as in (3)) of this potential. We assume in the following that $\lambda = 1$ in Lemma 25 and that h is sufficiently small (to be fixed later). It is easy to see that we have

$$\begin{aligned}\boldsymbol{E}\|\mathcal{E}^* V_\omega \mathcal{E}\|^2 = \boldsymbol{E}\|\mathcal{E}^* \overline{V}_\omega \mathcal{E}\mathcal{E}^* V_\omega \mathcal{E}\| &= \boldsymbol{E} \sup_{\|f\|_{L^2(M)}=1} |\langle \mathcal{E}\mathcal{E}^* V_\omega \mathcal{E} f, V_\omega \mathcal{E} f\rangle| \\ &\ge \sup_{\|f\|_{L^2(M)}=1} |\boldsymbol{E}\langle \mathcal{E}\mathcal{E}^* V_\omega \mathcal{E} f, V_\omega \mathcal{E} f\rangle| \ge \sup_{\|f\|_{L^2(M)}=1} |\operatorname{Re} \boldsymbol{E}\langle \mathcal{E}\mathcal{E}^* V_\omega \mathcal{E} f, V_\omega \mathcal{E} f\rangle|,\end{aligned}$$

where we recall that M is the unit sphere in $\mathbb{R}^d$. In order to estimate the last expression from below, we consider a Knapp example (see, e.g., [Demeter 2020, Example 1.8])

$$f_R(\xi) := R^{(d-1)/4} \eta(R\xi_1, R^{1/2}\xi'),$$

where $\xi = (\xi_1, \xi') \in \mathbb{R} \times \mathbb{R}^{d-1}$ and $\eta \in C_0^\infty(B(0, 2))$ is a nonnegative bump function equal to 1 on $B(0, 1)$. The normalization is chosen such that (up to an R-independent constant) $\|f_R\|_{L^2(M)} = 1$. Assuming, as we may, that $\boldsymbol{E}\omega_i\omega_j = \delta_{ij}$, we have

$$\boldsymbol{E}\langle \mathcal{E}\mathcal{E}^* V_\omega \mathcal{E} f, V_\omega \mathcal{E} f\rangle = \sum_{j\in h\mathbb{Z}^d} \int_{\mathbb{R}^d\times\mathbb{R}^d} (\mathcal{E}\mathcal{E}^*)(x-y) \overline{V_j(y)(\mathcal{E} f)(y)} V_j(x)(\mathcal{E} f)(x) \,\mathrm{d}y\,\mathrm{d}x,$$

where we wrote $V_j = V_\omega \mathbf{1}_{Q_j}$ and $Q_j = j + hQ$. Since $\mathcal{E}\mathcal{E}^*$ is proportional to convolution with $(\mathrm{d}\sigma)^\vee$ and the latter oscillates on the unit scale, there are positive constants r and c such that $\mathrm{Re}(\mathrm{d}\sigma)^\vee(u) \geq c$ for $|u| \leq r$ (this follows from standard stationary phase asymptotics). Assume now that $2h < r$. Then, using the above Knapp example f_R as a test function and changing variables $u = x - y$, we obtain

$$\boldsymbol{E}\|\mathcal{E}^* V_\omega \mathcal{E}\|^2 \gtrsim \mathrm{Re} \sum_{j \in h\mathbb{Z}^d} \int_{\mathbb{R}^d \times \mathbb{R}^d} \overline{F_j(x-u)} F_j(x)\,\mathrm{d}u\,\mathrm{d}x \quad (F_j = V_j \mathcal{E} f_R)$$

up to an error involving the imaginary part $\overline{F_j(x-u)} F_j(x)$ (which is small as we will see). At this point we consider a smooth (at the scale of T_R) version of the potential; this does not affect the previous arguments. What we gain by this is that now $|\nabla F_j(x)| = \mathcal{O}(R^{-1/2})|F_j(x)|$, whence, by Taylor expansion,

$$\sum_{j \in h\mathbb{Z}^d} \int_{\mathbb{R}^d \times \mathbb{R}^d} \overline{F_j(x-u)} F_j(x)\,\mathrm{d}u\,\mathrm{d}x = (2h)^d (1 - \mathcal{O}(R^{-1/2})) \sum_{j \in h\mathbb{Z}^d} \int_{\mathbb{R}^d} |F_j(x)|^2\,\mathrm{d}x.$$

Computing the integral, this shows that

$$\boldsymbol{E}\|\mathcal{E}^* V_\omega \mathcal{E}\|^2 \gtrsim h^d R^{1-(d+1)/q} \|V\|_q,$$

which implies that $q \leq d+1$ is necessary for Lemma 25 to hold (since R is arbitrarily large). If $h \gg 1$, one uses $\mathrm{Re}(\mathrm{d}\sigma)^\vee(u) \geq c|u|^{-(d-1)/2}$ for at least one percent of the u in $B(0, 2h)$. Then the u integration gives $h^{(d+1)/2}$ instead of h^d.

Added in proof

Recently, we have proved estimates for Schatten norms of the elementary operators $C^{(\delta_2)} V_\omega C^{(\delta_1)}$ for pointwise decaying potentials. They allowed us to prove estimates for sums over functions of the distances of the eigenvalues of $-\Delta + V_\omega$ to the origin or to the positive real axis, which quantify the eigenvalue accummulation. These results appear in [Cuenin and Merz 2023].

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ROTATING WAVES IN NONLINEAR MEDIA AND CRITICAL DEGENERATE SOBOLEV INEQUALITIES

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We investigate the presence of rotating wave solutions of the nonlinear wave equation $\partial_t^2 v - \Delta v + mv = |v|^{p-2}v$ in $\mathbb{R} \times \boldsymbol{B}$, where $\boldsymbol{B} \subset \mathbb{R}^N$ is the unit ball, complemented with Dirichlet boundary conditions on $\mathbb{R} \times \partial\boldsymbol{B}$. Depending on the prescribed angular velocity α of the rotation, this leads to a Dirichlet problem for a semilinear elliptic or degenerate elliptic equation. We show that this problem is governed by an associated critical degenerate Sobolev inequality in the half-space. After proving this inequality and the existence of associated extremal functions, we then deduce necessary and sufficient conditions for the existence of ground state solutions. Moreover, we analyze under which conditions on α, m and p these ground states are nonradial and therefore give rise to truly rotating waves. Our approach carries over to the corresponding Dirichlet problems in an annulus and in more general Riemannian models with boundary, including the hemisphere. We briefly discuss these problems and show that they are related to a larger family of associated critical degenerate Sobolev inequalities.

1. Introduction

Within a simple model, the analysis of wave propagation in an ambient medium with nonlinear response leads to the study of a nonlinear wave equation of the type

$$\partial_t^2 v - \Delta v + mv = f(v) \quad \text{in } \mathbb{R} \times \Omega, \tag{1-1}$$

in an ambient domain $\Omega \subset \mathbb{R}^N$ with mass parameter $m \geq 0$ and nonlinear response function f. In the case $m = 0$, (1-1) is the classical nonlinear wave equation, while the case $m > 0$ is also known as a nonlinear

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Klein–Gordon equation. For nonlinearities of the form $f(v) = g(|v|^2)v$ with a real-valued function g, standing wave solutions can be found by the ansatz

$$v(t,x) = e^{-ikt}u(x), \quad k > 0, \tag{1-2}$$

with a real-valued function u. Depending on the frequency parameter k, this reduces (1-1) either to a stationary nonlinear Schrödinger or a nonlinear Helmholtz equation (see, e.g., [Evéquoz and Weth 2015] for more details).

The resulting stationary nonlinear Schrödinger equation has been studied extensively in the past four decades by variational methods, see, e.g., the monograph [Ambrosetti and Malchiodi 2006]. Due to a lack of a direct variational framework, the nonlinear Helmholtz equation requires a different approach and has been studied more recently, e.g., in [Chen et al. 2021; Evéquoz and Weth 2015; Gutiérrez 2004; Mandel et al. 2017; 2021] by dual variational methods and bifurcation theory.

Clearly, the amplitude $|v|$ of a solution v of (1-1) given by the ansatz (1-2) remains time-independent. As a consequence, the analysis of standing wave solutions does not lead to a full understanding of (1-1) from a dynamical point of view and should be complemented, in particular, by the study of nonstationary real-valued time-periodic solutions, traveling wave solutions and scattering solutions. We stress that the ansatz (1-2) does not give rise to nonstationary real-valued time-periodic solutions since the nonlinearity of the problem does not allow to pass to real and imaginary parts.

In the case where $\Omega = \mathbb{R}^N$ and $f(v)$ in (1-1) is replaced by $q(x)f(v)$ with a compactly supported weight function q, spatially localized real-valued time-periodic solutions, also called breathers, have attracted increasing attention recently, see, e.g., [Hirsch and Reichel 2019; Mandel and Scheider 2021]. In the case where Ω is a radial domain, a further interesting type of real-valued time-periodic solution is given by *rotating wave solutions*. In particular, if Ω is a bounded radial domain and (1-1) is complemented with the Dirichlet boundary condition $v = 0$ on $\mathbb{R} \times \partial\Omega$, the existence of rotating waves and their variational characterization arises as a natural question which, to our knowledge, has not been addressed systematically so far.

The main purpose of the present paper is to provide such a systematic study. While we mainly focus on the case where $\Omega = \boldsymbol{B}$ is the open unit ball in $\mathbb{R}^N$, we will also address the case where Ω is an annulus or a Riemannian model with boundary; see Sections 5 and 6 below. Specifically, we study the case of a focusing nonlinearity of the form $f(v) = |v|^{p-2}v$, which leads to the superlinear problem

$$\begin{cases} \partial_t^2 v - \Delta v + mv = |v|^{p-2}v & \text{in } \mathbb{R} \times \boldsymbol{B}, \\ v = 0 & \text{on } \mathbb{R} \times \partial\boldsymbol{B}, \end{cases} \tag{1-3}$$

for $N \geq 2$, where

$$2 < p < 2^* \quad \text{and} \quad m > -\lambda_1(\boldsymbol{B}).$$

Here, $\lambda_1(\boldsymbol{B})$ denotes the first Dirichlet eigenvalue of $-\Delta$ on $\boldsymbol{B}$ and 2^* denotes the critical Sobolev exponent given by

$$2^* = \frac{2N}{N-2} \quad \text{for } N \geq 3 \qquad \text{and} \qquad 2^* = \infty \quad \text{for } N = 2.$$

The ansatz for time-periodic rotating solutions of (1-3) is given by

$$v(t,x)=u(R_{\alpha t}(x)), \tag{1-4}$$

where, for $\theta\in\mathbb{R}$, we let $R_\theta\in O(N)$ denote a planar rotation in $\mathbb{R}^N$ with angle θ, so the constant $\alpha>0$ in (1-4) is the angular velocity of the rotation. Without loss of generality, we may assume that

$$R_\theta(x)=(x_1\cos\theta+x_2\sin\theta,\,-x_1\sin\theta+x_2\cos\theta,\,x_3,\dots,x_N)\quad\text{for } x\in\mathbb{R}^N,$$

so R_θ is the rotation in the (x_1,x_2)-plane with fixed point set $\{0_{\mathbb{R}^2}\}\times\mathbb{R}^{N-2}$. In the following, we call a function u on the unit ball (x_1,x_2)-*nonradial* if it is not R_θ-invariant for at least one angle $\theta\in\mathbb{R}$. If the profile function u in (1-4) is (x_1,x_2)-nonradial, then the corresponding solution v can be interpreted as a rotating wave in a medium with nonlinear response given by the right-hand side of (1-3). The ansatz (1-4) reduces (1-3) to

$$\begin{cases}-\Delta u+\alpha^2\,\partial_\theta^2 u+mu=|u|^{p-2}u & \text{in } \boldsymbol{B},\\ u=0 & \text{on } \partial\boldsymbol{B},\end{cases} \tag{1-5}$$

where $\partial_\theta=x_1\,\partial_{x_2}-x_2\,\partial_{x_1}$ denotes the associated angular derivative operator. We point out that a seemingly closely related equation, with the term $\alpha^2\,\partial_\theta^2 u$ replaced by $-\alpha^2\,\partial_\theta^2 u$, arises in an ansatz for solutions of nonlinear Schrödinger equations in $\mathbb{R}^3$ with invariance with respect to screw motion, see [Agudelo et al. 2022] and also [del Pino et al. 2012] for a related work on Allen–Cahn equations. Note, however, that the positive sign of the term $\alpha^2\,\partial_\theta^2 u$ results in a drastic change of the nature of the problem, as the operator $-\Delta+\alpha^2\,\partial_\theta^2$ loses uniform ellipticity in $\boldsymbol{B}$ if $\alpha\ge 1$. For balls of arbitrary radius, the threshold for α corresponds to the inverse of the radius. In our case, for $\alpha=1$, we will observe that ellipticity is lost on the great circle

$$\gamma:=\{x\in\partial\boldsymbol{B}: x_3=\cdots=x_N=0\}, \tag{1-6}$$

which equals $\partial\boldsymbol{B}$ in the case $N=2$. This also distinguishes the study of (1-5) from the related study of rotating solutions to nonlinear Schrödinger equations, where the angular velocity α appears within a first-order term which does not affect the ellipticity of the associated Schrödinger operator, see, e.g., [Lieb and Seiringer 2006; Seiringer 2002].

If a solution u of (1-5) satisfies $\partial_\theta u\equiv 0$ in $\boldsymbol{B}$, then u solves the classical stationary nonlinear Schrödinger equation $-\Delta u+mu=|u|^{p-2}u$ in $\boldsymbol{B}$ with Dirichlet boundary conditions on $\partial\boldsymbol{B}$, so it satisfies (1-5) with $\alpha=0$. If, in addition, u is positive, then u has to be a radial function as a consequence of the symmetry result of Gidas, Ni and Nirenberg [Gidas et al. 1979]. Thus, the ansatz (1-4) then merely gives rise to a radial stationary solution of (1-3). We mention here that radially symmetric nonstationary solutions of (1-1) in $\Omega=\boldsymbol{B}$ were first studied by Ben-Naoum and Mawhin [1993] for sublinear nonlinearities, and more recently by Chen and Zhang [2014; 2016; 2017]. In this problem, the spectral properties of the radial wave operator lead to delicate assumptions on the dimension as well as the ratio between the radius of the ball and the period length. The main purpose of the present paper is to analyze for which range of parameters α, m and p ground state solutions of (1-5) exist and to distinguish under which assumptions on α, m and p they are radial or (x_1,x_2)-nonradial and therefore correspond to rotating waves via the ansatz (1-4).

By a ground state solution of (1-5), we mean a solution characterized as a minimizer of the minimization problem for

$$\mathscr{C}_{\alpha,m,p}(\boldsymbol{B}) := \inf_{u \in H_0^1(\boldsymbol{B})\setminus\{0\}} R_{\alpha,m,p}(u), \tag{1-7}$$

where, for $m \in \mathbb{R}$, $\alpha \geq 0$ and $p \in [2, 2^*)$, we consider the associated Rayleigh quotient $R_{\alpha,m,p}$ given by

$$R_{\alpha,m,p}(u) = \frac{\int_{\boldsymbol{B}} (|\nabla u|^2 - \alpha^2 |\partial_\theta u|^2 + m u^2)\, dx}{\left(\int_{\boldsymbol{B}} |u|^p \, dx\right)^{2/p}}, \quad u \in H_0^1(\boldsymbol{B}) \setminus \{0\}. \tag{1-8}$$

As we shall see in Remark 4.20 below, this minimization problem is only meaningful for $0 \leq \alpha \leq 1$, since, for every $p \in [2, 2^*)$ and $m \in \mathbb{R}$, we have

$$\mathscr{C}_{\alpha,m,p}(\boldsymbol{B}) = -\infty \quad \text{for } \alpha > 1.$$

Moreover, for every $p \in [2, 2^*)$ and $m \in \mathbb{R}$,

$$\text{the function } \alpha \mapsto \mathscr{C}_{\alpha,m,p}(\boldsymbol{B}) \text{ is continuous and nonincreasing on } [0, 1]. \tag{1-9}$$

In the case $0 < \alpha < 1$, the operator $-\Delta + \alpha^2\, \partial_\theta^2$ is uniformly elliptic, as can be seen by writing the operator in polar coordinates as

$$-\Delta + \alpha^2\, \partial_\theta^2 = -\Delta_r u - \frac{1}{r^2} \Delta_{\mathbb{S}^{N-1}} u + \alpha^2\, \partial_\theta^2 u, \tag{1-10}$$

where $\Delta_{\mathbb{S}^{N-1}}$ denotes the Laplace–Beltrami operator on the unit sphere $\mathbb{S}^{N-1}$. In this case the existence of minimizers of $R_{\alpha,m,p}$ on $H_0^1(\boldsymbol{B}) \setminus \{0\}$ follows by a standard compactness and weak lower-semicontinuity argument. However, even in this case it is difficult to decide in general whether minimizers are radial or nonradial functions. This is due to competing effects. Firstly, the additional term $-\alpha^2 \|\partial_\theta u\|^2_{L^2(\boldsymbol{B})}$ favors (x_1, x_2)-nonradial functions as energy minimizers. On the other hand, the Pólya–Szegö inequality yields

$$\int_{\boldsymbol{B}} |\nabla u^*|^2 \, dx \leq \int_{\boldsymbol{B}} |\nabla u|^2 \, dx,$$

where u^* denotes the (radial) Schwarz symmetrization of a function $u \in H_0^1(\boldsymbol{B})$.

Since $R_{\alpha,m,p}(u) = R_{0,m,p}(u)$ for every radial function $u \in H_0^1(\boldsymbol{B}) \setminus \{0\}$ and every $\alpha \in [0, 1]$, a sufficient condition for the (x_1, x_2)-nonradiality of all ground state solutions is the inequality

$$\mathscr{C}_{\alpha,m,p}(\boldsymbol{B}) < \mathscr{C}_{0,m,p}(\boldsymbol{B}). \tag{1-11}$$

In particular, we will be interested in proving this inequality for α close to 1. As mentioned already, the borderline case $\alpha = 1$ differs significantly from the case $0 \leq \alpha < 1$, since in this case $-\Delta + \partial_\theta^2$ fails to be uniformly elliptic in a neighborhood of the great circle γ defined in (1-6). We shall see in this paper that the minimization problem in the case $\alpha = 1$ is essentially governed by a degenerate anisotropic critical Sobolev inequality in the half-space. The corresponding critical exponent in this Sobolev inequality is given by

$$2_1^* := \frac{4N+2}{2N-3}.$$

This exponent's relevance is indicated by our first main result which yields the following characterization.

Theorem 1.1. *Let $m > -\lambda_1(\boldsymbol{B})$ and $p \in (2, 2^*)$.*

(i) *If $\alpha \in (0,1)$, then there exists a ground state solution of* (1-5).

(ii) *We have*

$$\mathscr{C}_{1,m,p}(\boldsymbol{B}) = 0 \quad \textit{for } p > 2_1^* \qquad \textit{and} \qquad \mathscr{C}_{1,m,p}(\boldsymbol{B}) > 0 \quad \textit{for } p \le 2_1^*. \tag{1-12}$$

Moreover, for any $p \in (2_1^*, 2^*)$, there exists $\alpha_p \in (0,1)$ with the property that

$$\mathscr{C}_{\alpha,m,p}(\boldsymbol{B}) < \mathscr{C}_{0,m,p}(\boldsymbol{B}) \quad \textit{for } \alpha \in (\alpha_p, 1],$$

and therefore every ground state solution of (1-5) *is (x_1, x_2)-nonradial for $\alpha \in (\alpha_p, 1)$.*

The following new degenerate Sobolev inequality is an immediate consequence of the special case $m = 0,\ \alpha = 1$ in Theorem 1.1.

Corollary 1.2. *We have*

$$\left(\int_{\boldsymbol{B}} |u|^{2_1^*}\,dx\right)^{2/2_1^*} \le \frac{1}{\mathscr{C}_{1,0,2_1^*}(\boldsymbol{B})} \int_{\boldsymbol{B}} (|\nabla u|^2 - |\partial_\theta u|^2)\,dx \quad \textit{for } u \in H_0^1(\boldsymbol{B}).$$

Moreover, the exponent 2_1^* is optimal in the sense that no such inequality holds for $p > 2_1^*$.

Theorem 1.1 yields symmetry-breaking of ground states for suitable parameter values of p, α and m, but the precise parameter range giving rise to this symmetry-breaking remains largely open. To shed further light on this question, we state the following result which establishes uniqueness and radial symmetry of ground state solutions for α close to zero and every $m \ge 0,\ 2 < p < 2^*$.

Theorem 1.3. *Let $m \ge 0$ and $2 < p < 2^*$. Then there exists $\alpha_0 > 0$ such that*

$$\mathscr{C}_{\alpha,m,p}(\boldsymbol{B}) = \mathscr{C}_{0,m,p}(\boldsymbol{B}) \quad \textit{for } \alpha \in [0, \alpha_0).$$

Moreover, for $\alpha \in [0, \alpha_0)$, there is, up to sign, a unique ground state solution of (1-5) *which is a radial function.*

Our proof of this theorem relies on the uniqueness and nondegeneracy of the radial positive solution of (1-5) in the case $\alpha = 0$. Combining Theorems 1.1 and 1.3, we find that, for fixed $p > 2_1^*$, symmetry-breaking of ground state solutions occurs when passing a critical parameter $\alpha = \alpha(p)$ which lies in the interval $[\alpha_0, \alpha_p]$. However, so far it remains unclear whether symmetry-breaking also occurs in the case $p \le 2_1^*$. Before stating a partial answer to this question for $2 < p < 2_1^*$, we first note that symmetry-breaking does not occur in the linear case $p = 2$. More precisely, we shall observe in Section 4 below that

$$\mathscr{C}_{\alpha,m,2}(\boldsymbol{B}) = \mathscr{C}_{0,m,2}(\boldsymbol{B}) = \lambda_1(\boldsymbol{B}) + m \quad \text{for all } \alpha \in [0,1],\ m \in \mathbb{R}.$$

Moreover, if $\alpha \in (0,1)$ and $m \ge 0$ are fixed, then every minimizer of (1-7) is radial if $p \ge 2$ is sufficiently close to 2; see Remark 4.16 below. On the other hand, for every p strictly greater than 2, symmetry-breaking occurs for sufficiently large values of the parameter m, as the following result shows.

Theorem 1.4. *Let $\alpha \in (0,1)$ and $2 < p < 2^*$. Then there exists $m_0 > 0$ with the property that* (1-11) *holds for $m \geq m_0$, and therefore every ground state solution of* (1-5) *is (x_1, x_2)-nonradial for $m \geq m_0$.*

As symmetry-breaking occurs, for fixed $p \in (2, 2^*)$, both in the parameter α and m, it is tempting to guess that there exists a unique curve in the (α, m)-plane separating the parameter region of symmetry-breaking from the one where radial symmetry of ground state solutions is preserved. A bifurcation analysis might be useful to detect the precise symmetry-breaking regime, but this seems far from straightforward, and we leave it for future research.

Next, we discuss the limit case $\alpha = 1$ in the minimization problem (1-7). We may study this limit case based on Corollary 1.2, but we need to look for minimizers in a space larger than $H_0^1(\boldsymbol{B})$. More precisely, we let $\mathcal{H}_1$ be given as the completion of $C_c^1(\boldsymbol{B})$ with respect to the norm $\|\cdot\|_{\mathcal{H}_1}$ given by

$$\|u\|_{\mathcal{H}_1}^2 := \int_{\boldsymbol{B}} (|\nabla u|^2 - |\partial_\theta u|^2)\, dx.$$

Here we recall that Corollary 1.2 gives the norm property of $\|\cdot\|_{\mathcal{H}_1}$ on $C_c^1(\boldsymbol{B}) \subset H_0^1(\boldsymbol{B})$, and it also implies that $\mathcal{H}_1$ is embedded in $L^{2_1^*}(\boldsymbol{B})$. We then have the following result, which complements Theorems 1.1 and 1.4 in the case $\alpha = 1$.

Theorem 1.5. *Let $2 < p < 2_1^*$ and $\alpha = 1$.*

(i) *For every $m > -\lambda_1(\boldsymbol{B})$, there exists a ground state solution of* (1-5).

(ii) *There exists $m_0 > 0$ with the property that* (1-11) *holds for $m \geq m_0$, and therefore every ground state solution $u \in \mathcal{H}_1$ of* (1-5) *is (x_1, x_2)-nonradial for $m \geq m_0$.*

The critical case $\alpha = 1$, $p = 2_1^*$ remains largely open, but we have a partial result on the existence of ground state solutions which relates problem (1-5) to a degenerate Sobolev inequality of the form

$$\|u\|_{L^{2_s^*}(\mathbb{R}_+^N)} \leq C\left(\int_{\mathbb{R}_+^N} \sum_{i=1}^{N-1} |\partial_i u|^2 + x_1^s |\partial_N u|^2\, dx\right)^{1/2} \tag{1-13}$$

in the half-space

$$\mathbb{R}_+^N := \{x \in \mathbb{R}^N : x_1 > 0\}.$$

This inequality seems new and of independent interest, and it is the key ingredient in the proof of Theorem 1.1. Our main result related to this half-space inequality is the following.

Theorem 1.6. *Let $s > 0$, and set $2_s^* := (4N + 2s)/(2N - 4 + s)$. Then we have*

$$\mathcal{S}_s(\mathbb{R}_+^N) := \inf_{u \in C_c^1(\mathbb{R}_+^N)} \frac{\int_{\mathbb{R}_+^N} \sum_{i=1}^{N-1} |\partial_i u|^2 + x_1^s |\partial_N u|^2\, dx}{\left(\int_{\mathbb{R}_+^N} |u|^{2_s^*}\, dx\right)^{2/2_s^*}} > 0. \tag{1-14}$$

Moreover, the value $\mathcal{S}_s(\mathbb{R}_+^N)$ is attained in $H_s \setminus \{0\}$, where H_s denotes the closure of $C_c^1(\mathbb{R}_+^N)$ in the space

$$\left\{u \in L^{2_s^*}(\mathbb{R}_+^N) : \|u\|_{H_s}^2 := \int_{\mathbb{R}_+^N} \sum_{i=1}^{N-1} |\partial_i u|^2 + x_1^s |\partial_N u|^2\, dx < \infty\right\} \tag{1-15}$$

with respect to the norm $\|\cdot\|_{H_s}$.

Here, distributional derivatives are considered in (1-15). Moreover, we note that $\|\cdot\|_{H_s}$ defines a norm on the space defined in (1-15), as the vanishing of $\|u\|_{H_s}$ implies that the distributional gradient ∇u vanishes a.e. in $\mathbb{R}^N_+$. This, in turn, implies that u must be constant on $\mathbb{R}^N_+$, and therefore $u = 0$ since $u \in L^{2^*_s}(\mathbb{R}^N_+)$.

Several remarks regarding Theorem 1.6 are in order. First, we point out that the criticality of the exponent $2^*_s := (4N+2s)/(2N-4+s)$ in Theorem 1.6 corresponds to the fact that the quotient in (1-14) is invariant under an anisotropic rescaling given by $u \mapsto u_\lambda$ for $\lambda > 0$, with

$$u_\lambda(x) := u(\lambda x_1, \lambda x_2, \dots, \lambda x_{N-1}, \lambda^{1+s/2} x_N).$$

This invariance leads to a lack of compactness, and we have to apply concentration-compactness methods to deduce the existence of minimizers. We further note that the existence of minimizers in the half-space problem is in striking contrast to the case $s = 0$ which is excluded in Theorem 1.6. Indeed, the case $s = 0$ in Theorem 1.6 corresponds to the classical Sobolev inequality which only admits extremal functions in the entire space $\mathbb{R}^N$; see, e.g., [Struwe 2008, Chapter III, Theorem 1.2].

We have already noted that the case $s = 1$ in Theorem 1.6 is of key importance in the proof of Theorem 1.1. The more general case $s \in (0, 2]$ arises in a similar way when (1-5) is studied in Riemannian models with boundary in place of $\boldsymbol{B}$, and we will discuss this case in Section 6 below. We point out that the setting of Riemannian models includes hypersurfaces of revolution with boundary in $\mathbb{R}^{N+1}$, and that the particular case of a hemisphere corresponds to the case $s = 2$. The latter is no surprise in view of the recent work of Taylor [2016] and Mukherjee [2017; 2018], who studied the problem of rotating solutions on the unit sphere. In particular, their work relies on degenerate Sobolev embeddings on the unit sphere where also the value $2^*_2 = 2(N+1)/(N-1)$ appears as a critical exponent; see [Taylor 2016, Proposition 3.2] and [Mukherjee 2017, Proposition 1.2 and Lemma 1.3]. In fact, our approach allows to use the case $s = 2$ in Theorem 1.6 and the corresponding inequality in $\mathbb{R}^N$ (see Theorem 2.2 below) to give new proofs of these degenerate Sobolev embeddings which do not rely on the Fourier analytic arguments used in [Taylor 2016].

Next we remark that degenerate Sobolev-type inequalities have been studied extensively in the context of Grushin operators which take the form

$$\mathcal{L} = \Delta_x + c|x|^s \Delta_y$$

on $\mathbb{R}^N = \mathbb{R}^m \times \mathbb{R}^k$, where $x \in \mathbb{R}^m$, $y \in \mathbb{R}^k$ and $s > 0$. For a comprehensive survey of the properties of these operators, see, e.g., [Hajłasz and Koskela 2000]. In particular, an associated Sobolev-type inequality of the type

$$\|u\|_{L^{\frac{4m+2k(s+2)}{2m+k(s+2)-4}}(\mathbb{R}^N)} \le C\left(\int_{\mathbb{R}^N} |\nabla_x u|^2 + c|x|^s |\nabla_y u|^2 \, d(x, y)\right)^{1/2}, \quad u \in C^1_c(\mathbb{R}^N) \tag{1-16}$$

has been established. Here, the associated critical exponent is related to the homogeneous dimension in the context of more general weighted Sobolev inequalities. We also mention symmetry results for positive entire solutions to semilinear problems involving $\mathcal{L}$ in [Monti and Morbidelli 2006], as well as the existence of extremal functions on $\mathbb{R}^N$ shown in [Beckner 2001; Monti 2006].

We point out that the restriction of inequality (1-16) to the half-space coincides with the inequality (1-13) in the case $N = 2,\ m = k = 1$. On the other hand, for $N \geq 3,\ m = N - 1$ and $k = 1$, the critical exponents in (1-13) and (1-16) still coincide, but (1-13) is a strict improvement of (1-16) since the weight x_1^s is strictly smaller than $|(x_1, \ldots, x_{N-1})|^s$ away from the x_1-axis. More closely related to Theorem 1.6 in the case $N \geq 3$ is [Filippas et al. 2008, Theorem 1.7], where a more general family of Grushin-type operators and their associated inequalities has been considered. However, the inequality (1-13) associated to (1-11) is a limit case which is not part of the family of inequalities considered in [Filippas et al. 2008]. More precisely, inequality (1-13) extends [Filippas et al. 2008, Theorem 1.7] to the case $A = B = 0$ (with A and B given in [Filippas et al. 2008]).

Coming back to the existence of ground state solutions of (1-5) in the critical case $\alpha = 1,\ p = 2_1^*$, we state the following result.

Theorem 1.7. (i) *If*

$$\mathscr{C}_{1,m,2_1^*}(\boldsymbol{B}) < 2^{1/2-1/2_1^*} \mathcal{S}_1(\mathbb{R}^N_+) \tag{1-17}$$

for some $m > -\lambda_1(\boldsymbol{B})$, then the value $\mathscr{C}_{1,m,2_1^*}(\boldsymbol{B})$ is attained in $\mathcal{H} \setminus \{0\}$ by a ground state solution of (1-5).

(ii) *There exists $\varepsilon > 0$ with the property that* (1-17) *holds for every $m \in (-\lambda_1(\boldsymbol{B}), -\lambda_1(\boldsymbol{B}) + \varepsilon)$.*

Here, the factor $2^{1/2-1/2_1^*}$ is due to the scaling properties of a more general quotient related to (1-14); see Remark 2.3 (ii) below. We note that criterion (1-17) prevents, with the help of Theorem 1.6 and a blow-up argument, the concentration of minimizing sequences close to the great circle γ defined in (1-6).

The paper is organized as follows. We first study the degenerate Sobolev inequality (1-13) and hence prove Theorem 1.6 in Section 2. This is subsequently used in Section 3 to prove the second part of Theorem 1.1. In Section 4 we then discuss the properties of ground state solutions of (1-5) in detail and give the proofs of Theorems 1.3 and 1.4. This also includes the degenerate case $\alpha = 1$ and the proofs of Theorems 1.5 and 1.7. Section 5 is then devoted to the properties of rotating waves when $\boldsymbol{B}$ is replaced by an annulus. In this case, our methods give rise to an analogue of Theorem 1.1 with more explicit conditions for (x_1, x_2)-nonradiality of ground states. In Section 6 we discuss how the general degenerate Sobolev inequality (1-13) can be used to give an analogue of Theorem 1.1 for Riemannian models. Finally, in Appendix A, we prove uniform L^∞-bounds for weak solutions of (1-5) in the case $\alpha = 1$. Moreover, we recall in Appendix B useful formulas related to the round metric on the unit sphere in angular coordinates.

We finally remark that the general approach of the present paper also allows to analyze (x_1, x_2)-nonradial solutions of (1-5) on domains of the type

$$\{(x_1, x_2, x') \in \mathbb{R}^N : x_1^2 + x_2^2 < 1,\ |x'| < \psi(x_1^2 + x_2^2)\}$$

for suitable functions $\psi : [0, 1) \to (0, \infty)$ satisfying $\lim_{r \to 1^-} \psi(r) = 0$. However, the underlying analysis will be more involved, and limiting Sobolev inequalities different from (1-13) might arise. We shall leave this open problem for future research.[1]

[1] We wish to thank the referee for pointing out this question.

2. A family of degenerate Sobolev inequalities

In this section, we give the proof of Theorem 1.6. More precisely, in the first part of the section, we prove the corresponding degenerate Sobolev inequality

$$\left(\int_{\mathbb{R}^N} |u|^{2_s^*}\,dx\right)^{2/2_s^*} \le C\int_{\mathbb{R}^N}\left(\sum_{i=1}^{N-1}|\partial_i u|^2 + |x_1|^s|\partial_N u|^2\right)dx \quad \text{for } u\in C_c^1(\mathbb{R}^N) \tag{2-1}$$

in the entire space with a constant $C>0$, from which the positivity of $\mathcal{S}_s(\mathbb{R}^N_+)$ in (1-14) follows.

In the second part of the section, we then prove the existence of minimizers of the quotient in (1-14) in the larger space H_s defined in Theorem 1.6.

2.1. ***A degenerate Sobolev inequality on*** $\mathbb{R}^N$**.** The first step in the proof of (2-1) is the following inequality.

Lemma 2.1. *Let $\alpha>0$ and $p>2$ be given. Then we have*

$$\int_{\mathbb{R}^N}|u|^p\,dx \le \kappa\left(\int_{\mathbb{R}^N}|x_1|^\alpha|u|^q\,dx\right)^{2/(2+\alpha)}\left(\int_{\mathbb{R}^N}|\partial_1 u|^2\,dx\right)^{\alpha/(2+\alpha)} \quad \textit{for } u\in C_c^1(\mathbb{R}^N),$$

with

$$q=\tfrac{1}{2}(p(2+\alpha)-2\alpha)>2$$

and

$$\kappa=\begin{cases}\left(\frac{q+2}{\alpha}\right)^{2\alpha/(2+\alpha)}, & 0<\alpha\le 2,\\ p^{2\alpha/(2+\alpha)}, & \alpha>2.\end{cases}$$

Proof. We first consider the case $\alpha\in(0,2)$, and we let $u\in C_c^1(\mathbb{R}^N)$. By Hölder's inequality, we have

$$\begin{aligned}\int_{\mathbb{R}^N}|u|^p\,dx &\le \left(\int_{\mathbb{R}^N}|x_1|^{s\sigma'}|u|^{r\sigma'}\,dx\right)^{1/\sigma'}\left(\int_{\mathbb{R}^N}|x_1|^{-s\sigma}|u|^{(p-r)\sigma}\,dx\right)^{1/\sigma}\\ &=\left(\int_{\mathbb{R}^N}|x_1|^{\alpha}|u|^{q}\,dx\right)^{1/\sigma'}\left(\int_{\mathbb{R}^N}|x_1|^{-s\sigma}|u|^{(p-r)\sigma}\,dx\right)^{1/\sigma},\end{aligned} \tag{2-2}$$

with

$$\sigma:=\frac{2+\alpha}{2\alpha},\quad \sigma'=\frac{\sigma}{\sigma-1}=\frac{2+\alpha}{2-\alpha}\in(1,\infty),\quad s:=\frac{\alpha}{\sigma'}\quad\text{and}\quad r:=\frac{q}{\sigma'}.$$

Here we used that $0<r<p$. Indeed we have

$$p=\frac{2q+2\alpha}{2+\alpha}=\frac{q}{\sigma'}+\frac{\alpha(q+2)}{2+\alpha}=r+\frac{q+2}{2\sigma},$$

which furthermore implies that

$$(p-r)\sigma-1=\tfrac{1}{2}q>0. \tag{2-3}$$

Since also

$$s\sigma=\alpha(\sigma-1)=\tfrac{1}{2}(2-\alpha)\in(0,1), \tag{2-4}$$

we may integrate by parts and use Hölder's inequality again to get

$$\begin{aligned}\int_{\mathbb{R}^N} |x_1|^{-s\sigma}|u|^{(p-r)\sigma}\,dx &= -\frac{(p-r)\sigma}{1-s\sigma}\int_{\mathbb{R}^N} x_1|x_1|^{-s\sigma}|u|^{(p-r)\sigma-1}\,\partial_1 u\,dx\\ &\le \frac{(p-r)\sigma}{1-s\sigma}\int_{\mathbb{R}^N}|x_1|^{1-s\sigma}|u|^{(p-r)\sigma-1}|\partial_1 u|\,dx\\ &\le \frac{q+2}{\alpha}\Big(\int_{\mathbb{R}^N}|x_1|^{\alpha}|u|^q\,dx\Big)^{1/2}\Big(\int_{\mathbb{R}^N}|\partial_1 u|^2\,dx\Big)^{1/2}. \end{aligned}\tag{2-5}$$

Here we have used (2-3), (2-4) and the identity

$$\frac{(p-r)\sigma}{1-s\sigma} = \frac{q+2}{\alpha}$$

in the last step. Combining (2-2) and (2-5) gives

$$\begin{aligned}\int_{\mathbb{R}^N}|u|^p\,dx &\le \Big(\frac{q+2}{\alpha}\Big)^{1/\sigma}\Big(\int_{\mathbb{R}^N}|x_1|^{\alpha}|u|^q\,dx\Big)^{1/\sigma'+1/(2\sigma)}\Big(\int_{\mathbb{R}^N}|\partial_1 u|^2\,dx\Big)^{1/(2\sigma)}\\ &= \Big(\frac{q+2}{\alpha}\Big)^{2\alpha/(2+\alpha)}\Big(\int_{\mathbb{R}^N}|x_1|^{\alpha}|u|^q\,dx\Big)^{2/(2+\alpha)}\Big(\int_{\mathbb{R}^N}|\partial_1 u|^2\,dx\Big)^{\alpha/(2+\alpha)},\end{aligned}$$

as claimed. Next we note that the case $\alpha = 2$ follows by continuity, which gives

$$\Big(\int_{\mathbb{R}^N}|u|^p\,dx\Big)^2 \le p^2\Big(\int_{\mathbb{R}^N}|x_1|^2|u|^{2(p-1)}\,dx\Big)\Big(\int_{\mathbb{R}^N}|\partial_1 u|^2\,dx\Big).\tag{2-6}$$

From this we now deduce the claim in the case $\alpha > 2$. Indeed, writing

$$|x_1|^2|u|^{2(p-1)} = (|x_1|^2|u|^{2q/\alpha})|u|^{2(p-1-q/\alpha)}$$

we get, by Hölder's inequality,

$$\int_{\mathbb{R}^N}|x_1|^2|u|^{2(p-1)}\,dx \le \Big(\int_{\mathbb{R}^N}|x_1|^{\alpha}|u|^q\,dx\Big)^{2/\alpha}\Big(\int_{\mathbb{R}^N}|u|^{(2\alpha/(\alpha-2))\cdot(p-1-q/\alpha)}\,dx\Big)^{(\alpha-2)/\alpha}.\tag{2-7}$$

Since $(2\alpha/(\alpha-2))\cdot(p-1-q/\alpha) = p$, we deduce from (2-6) and (2-7) that

$$\Big(\int_{\mathbb{R}^N}|u|^p\,dx\Big)^{(\alpha+2)/\alpha} \le p^2\Big(\int_{\mathbb{R}^N}|x_1|^{\alpha}|u|^q\,dx\Big)^{2/\alpha}\Big(\int_{\mathbb{R}^N}|\partial_1 u|^2\,dx\Big),$$

and hence

$$\int_{\mathbb{R}^N}|u|^p\,dx \le p^{2\alpha/(\alpha+2)}\Big(\int_{\mathbb{R}^N}|x_1|^{\alpha}|u|^q\,dx\Big)^{2/(\alpha+2)}\Big(\int_{\mathbb{R}^N}|\partial_1 u|^2\,dx\Big)^{\alpha/(\alpha+2)}. \qquad \square$$

We may now complete the proof of the main result of this section, given as follows.

Theorem 2.2. *Let $s > 0$ and $2_s^* = (4N+2s)/(2N-4+s)$ as in Theorem 1.6. Then inequality* (2-1) *holds with some constant $C > 0$.*

We remark that this may be proven by combining Lemma 2.1 with a suitable adaptation of the inequality on the half-space given in [Filippas et al. 2008, Theorem 1.7] to the setting of the entire space $\mathbb{R}^N$. For the convenience of the reader, we give a self-contained proof.

Proof. In the following, the letter $c > 0$ stands for a constant which may change from line to line. Let $\alpha = s/(2(N-1))$. Then Lemma 2.1 yields

$$\int_{\mathbb{R}^N} |u|^{2_s^*}\,dx \le \kappa\left(\int_{\mathbb{R}^N} |x_1|^\alpha |u|^{q_s}\,dx\right)^{\frac{2}{2+\alpha}}\left(\int_{\mathbb{R}^N} |\partial_1 u|^2\,dx\right)^{\frac{\alpha}{2+\alpha}} \quad \text{for } u \in C_c^1(\mathbb{R}^N),$$

with $q_s := N(2_s^* + 2)/(2(N-1))$. To estimate the term $\int_{\mathbb{R}^N} |x_1|^\alpha |u|^{q_s}\,dx$, we define, for $i = 1, \dots, N$, the functions $a_i \in C_c(\mathbb{R}^{N-1})$ by

$$a_i(\hat{x}_i) := \int_{\mathbb{R}} |u|^{\frac{q_s(N-1)}{N}-1} |\partial_i u|\,dx_i,$$

where

$$\hat{x}_i := (x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_N) \in \mathbb{R}^{N-1} \quad \text{for } x \in \mathbb{R}^N \text{ and } i = 1, \dots, N.$$

Integrating the derivative $\partial_i(|u|^{q_s(N-1)/N})$ in the x_i-direction, we find that $|u(x)|^{q_s(N-1)/N} \le c a_i(\hat{x}_i)$ for all $x \in \mathbb{R}^N$, $i = 1, \dots, N$, and therefore

$$|u(x)|^{q_s(N-1)} \le c \prod_{i=1}^{N} a_i(\hat{x}_i) \quad \text{for } x \in \mathbb{R}^N.$$

Applying Gagliardo's lemma [1958, Lemma 4.1] to the functions $a_1^{1/(N-1)}, \dots, a_{N-1}^{1/(N-1)}$ and the function $x \mapsto |x_1|^\alpha a_N^{1/(N-1)}(x)$, we thus find that

$$\begin{aligned}
\int_{\mathbb{R}^N} |x_1|^\alpha |u|^{q_s}\,dx &\le c\left(\int_{\mathbb{R}^{N-1}} |x_1|^{(N-1)\alpha} a_N(\hat{x}_N)\,d\hat{x}_N \prod_{i=1}^{N-1}\int_{\mathbb{R}^{N-1}} a_i(\hat{x}_i)\,d\hat{x}_i\right)^{\frac{1}{N-1}} \\
&= c\left(\int_{\mathbb{R}^N} |x_1|^{s/2}|u|^{\frac{q_s(N-1)}{N}-1}|\partial_N u|\,dx \prod_{i=1}^{N-1}\int_{\mathbb{R}^N} |u|^{\frac{q_s(N-1)}{N}-1}|\partial_i u|\,dx\right)^{\frac{1}{N-1}} \\
&\le c\left(\int_{\mathbb{R}^N} |u|^{2\frac{q_s(N-1)}{N}-2}\,dx\right)^{\frac{N}{2(N-1)}}\left(\int_{\mathbb{R}^N} |x_1|^s|\partial_N u|^2\,dx \prod_{i=1}^{N-1}\int_{\mathbb{R}^N} |\partial_i u|^2\,dx\right)^{\frac{1}{2(N-1)}}.
\end{aligned}$$

Since $2(N-1)q_s/N - 2 = 2_s^*$, we conclude that

$$\begin{aligned}
&\int_{\mathbb{R}^N} |u|^{2_s^*}\,dx \\
&\quad\le c\left(\int_{\mathbb{R}^N} |x_1|^\alpha|u|^{q_s}\,dx\right)^{\frac{2}{2+\alpha}}\left(\int_{\mathbb{R}^N} |\partial_1 u|^2\,dx\right)^{\frac{\alpha}{2+\alpha}} \\
&\quad\le c\left(\left(\int_{\mathbb{R}^N} |u|^{2_s^*}\,dx\right)^{\frac{N}{2(N-1)}}\left(\int_{\mathbb{R}^N} |x_1|^s|\partial_N u|^2\,dx \prod_{i=1}^{N-1}\int_{\mathbb{R}^N} |\partial_i u|^2\,dx\right)^{\frac{1}{2(N-1)}}\right)^{\frac{2}{2+\alpha}}\left(\int_{\mathbb{R}^N} |\partial_1 u|^2\,dx\right)^{\frac{\alpha}{2+\alpha}} \\
&\quad= c\left(\int_{\mathbb{R}^N} |u|^{2_s^*}\,dx\right)^{\frac{N}{2(N-1)+s/2}}\left(\int_{\mathbb{R}^N} |x_1|^s|\partial_N u|^2\,dx \prod_{i=2}^{N-1}\int_{\mathbb{R}^N} |\partial_i u|^2\,dx\right)^{\frac{1}{2(N-1)+s/2}}\left(\int_{\mathbb{R}^N} |\partial_1 u|^2\,dx\right)^{\frac{1+s/2}{2(N-1)+s/2}},
\end{aligned}$$

and therefore

$$\left(\int_{\mathbb{R}^N} |u|^{2_s^*}\,dx\right)^{\frac{N-2+s/2}{2(N-1)+s/2}} \le c\left(\int_{\mathbb{R}^N} |x_1|^s|\partial_N u|^2\,dx \prod_{i=2}^{N-1}\int_{\mathbb{R}^N} |\partial_i u|^2\,dx\right)^{\frac{1}{2(N-1)+s/2}}\left(\int_{\mathbb{R}^N} |\partial_1 u|^2\,dx\right)^{\frac{1+s/2}{2(N-1)+s/2}}.$$

Finally, Young's inequality gives

$$\begin{aligned}\Big(\int_{\mathbb{R}^N}|u|^{2_s^*}\,dx\Big)^{2/2_s^*} &\le c\Big(\int_{\mathbb{R}^N}|x_1|^s|\partial_N u|^2\,dx\prod_{i=2}^{N-1}\int_{\mathbb{R}^N}|\partial_i u|^2\,dx\Big)^{\frac{2}{2N+s}}\Big(\int_{\mathbb{R}^N}|\partial_1 u|^2\,dx\Big)^{\frac{2+s}{2N+s}}\\ &\le c\Big(\int_{\mathbb{R}^N}|x_1|^s|\partial_N u|^2\,dx+\sum_{i=1}^{N-1}\int_{\mathbb{R}^N}|\partial_i u|^2\,dx\Big).\end{aligned}$$ □

In particular, this implies

$$\mathcal{S}_s(\mathbb{R}^N_+)=\inf_{u\in C^1_c(\mathbb{R}^N_+)}\frac{\int_{\mathbb{R}^N_+}\sum_{i=1}^{N-1}|\partial_i u|^2+x_1^s|\partial_N u|^2\,dx}{\big(\int_{\mathbb{R}^N_+}|u|^{2_s^*}\,dx\big)^{2/2_s^*}}>0,$$

and thus the first part of Theorem 1.6.

Remark 2.3 (optimality and variants). (i) The exponent 2_s^* in (1-14) is optimal in the sense that

$$\inf_{u\in C^1_c(\mathbb{R}^N)}\frac{\int_{\mathbb{R}^N}\big(\sum_{i=1}^{N-1}|\partial_i u|^2+|x_1|^s|\partial_N u|^2\big)\,dx}{\|u\|^2_{L^p(\mathbb{R}^N)}}=0\quad\text{for } p\neq 2_s^*. \tag{2-8}$$

This follows by considering the rescaling $u\mapsto u_\lambda,\ \lambda>0$, with

$$u_\lambda(x):=u(\lambda x_1,\lambda x_2,\dots,\lambda x_{N-1},\lambda^{1+s/2}x_N).$$

Indeed, for $u\in C^1_c(\mathbb{R}^N)$, we have

$$\int_{\mathbb{R}^N}\Big(\sum_{i=1}^{N-1}|\partial_i u_\lambda|^2+|x_1|^s|\partial_N u_\lambda|^2\Big)\,dx=\lambda^{-(2N+s-4)/2}\int_{\mathbb{R}^N}\Big(\sum_{i=1}^{N-1}|\partial_i v|^2+|x_1|^s|\partial_N u|^2\Big)\,dx$$

and, for $1<p<\infty$,

$$\Big(\int_{\mathbb{R}^N}|u_\lambda|^p\,dx\Big)^{2/p}=\lambda^{-(2/p)(N+s/2)}\Big(\int_{\mathbb{R}^N}|u|^p\,dx\Big)^{2/p}.$$

Since $\frac{1}{2}(2N+s-4)=(2/p)\big(N+\frac{1}{2}s\big)$ if and only if $p=2_s^*$, (2-8) follows.

(ii) For $\kappa>0,\ u\in C^1_c(\mathbb{R}^N)$, we may consider a rescaled function of the form

$$v(x)=u\Big(x_1,\dots,x_{N-1},\frac{x_N}{\sqrt{\kappa}}\Big).$$

Comparing the associated quotients then yields

$$\inf_{u\in C^1_c(\mathbb{R}^N)}\frac{\int_{\mathbb{R}^N}\big(\sum_{i=1}^{N-1}|\partial_i u|^2+\kappa|x_1|^s|\partial_N u|^2\big)\,dx}{\|u\|^2_{L^{2_s^*}(\mathbb{R}^N)}}=\kappa^{1/2-1/2_s^*}\mathcal{S}_s(\mathbb{R}^N_+). \tag{2-9}$$

In the special case $\kappa=2$, this quotient will appear later when we connect $\mathscr{C}_{1,m,2_1^*}(\boldsymbol{B})$ and $\mathcal{S}_1(\mathbb{R}^N_+)$, in particular in the proof of Theorem 1.7.

Recalling the space H_s defined in Theorem 1.6, we see that Theorem 2.2 immediately implies that H_s is continuously embedded into $L^{2_s^*}(\mathbb{R}^N_+)$.

2.2. ***Existence of minimizers.*** In the following, we fix $s > 0$ and study minimizing sequences for

$$\mathcal{S} := \mathcal{S}_s(\mathbb{R}^N_+) = \inf_{u \in H_s \setminus \{0\}} \frac{\int_{\mathbb{R}^N_+} \left(\sum_{i=1}^{N-1} |\partial_i u|^2 + x_1^s |\partial_N u|^2 \right) dx}{\left(\int_{\mathbb{R}^N_+} |u|^{2_s^*} \, dx \right)^{2/2_s^*}} > 0.$$

First, consider the following classical lemma due to Lions [1984], which we give in the form presented in [Struwe 2008].

Lemma 2.4 (concentration-compactness lemma). *Suppose $(\mu_n)_n$ is a sequence of probability measures on $\mathbb{R}^N$. Then, after passing to a subsequence, one of the following three conditions holds*:

(i) <u>Compactness</u>: *There exists a sequence $(x_n)_n \subset \mathbb{R}^N$ such that, for any $\varepsilon > 0$, there exists $R > 0$ such that*

$$\int_{B_R(x_n)} d\mu_n \geq 1 - \varepsilon.$$

(ii) <u>Vanishing</u>: *For all $R > 0$,*

$$\lim_{n \to \infty} \left(\sup_{x \in \mathbb{R}^N} \int_{B_R(x)} d\mu_n \right) = 0.$$

(iii) <u>Dichotomy</u>: *There exists $\lambda \in (0, 1)$ such that, for any $\varepsilon > 0$, there exists $R > 0$ and $(x_n)_n \subset \mathbb{R}^N$ with the following property: given $R' > R$ there are nonnegative measures μ_n^1, μ_n^2 such that*

$$0 \leq \mu_n^1 + \mu_n^2 \leq \mu_n, \quad \operatorname{supp} \mu_n^1 \subset B_R(x_n), \quad \operatorname{supp} \mu_n^2 \subset \mathbb{R}^N \setminus B_{R'}(x_n),$$
$$\limsup_{n \to \infty} \left(\left| \lambda - \int_{\mathbb{R}^N} d\mu_n^1 \right| + \left| (1 - \lambda) - \int_{\mathbb{R}^N} d\mu_n^2 \right| \right) \leq \varepsilon.$$

A characterization of minimizing sequences in the sense of measures is given in the following lemma, which is a straightforward adaption of [Struwe 2008, Lemma 4.8].

Lemma 2.5 (concentration-compactness lemma II). *Let $s > 0$, and suppose $u_n \rightharpoonup u$ in H_s and $\mu_n := \left(\sum_{i=1}^{N-1} |\partial_i u_n|^2 + x_1^s |\partial_N u_n|^2 \right) dx \rightharpoonup \mu$, $\nu_n := |u_n|^{2_s^*} \, dx \rightharpoonup \nu$ weakly in the sense of measures, where μ and ν are finite measures on $\mathbb{R}^N_+$. Then*:

(i) *There exists an at most countable set J and sets $\{x^j : j \in J\} \subset \mathbb{R}^N_+$ and $\{\nu^j : j \in J\} \subset (0, \infty)$ such that*

$$\nu = |u|^{2_s^*} \, dx + \sum_{j \in J} \nu^j \delta_{x^j}.$$

(ii) *There exists a set $\{\mu^j : j \in J\} \subset (0, \infty)$ such that*

$$\mu \geq \left(\sum_{i=1}^{N-1} |\partial_i u|^2 + x_1^s |\partial_N u|^2 \right) dx + \sum_{j \in J} \mu^j \delta_{x^j},$$

where

$$\mathcal{S} (\nu^j)^{2/2_s^*} \leq \mu^j$$

for $j \in J$. In particular, $\sum_{j \in J} (\nu^j)^{2/2_s^*} < \infty$.

Our main result then states that $\mathcal{S}$ is attained in H_s and completes the proof of Theorem 1.6.

Theorem 2.6. *Let $s > 0$, and suppose $(u_n)_n$ is a minimizing sequence for*

$$\mathcal{S} = \inf_{u \in H_s \setminus \{0\}} \frac{\int_{\mathbb{R}^N_+} \left(\sum_{i=1}^{N-1} |\partial_i u|^2 + x_1^s |\partial_N u|^2 \right) dx}{\left(\int_{\mathbb{R}^N_+} |u|^{2^*_s} \, dx \right)^{2/2^*_s}},$$

with $\|u_n\|_{L^{2^*_s}} = 1$. Then, up to translations orthogonal to x_1 and anisotropic scaling, $(u_n)_n$ is relatively compact in H_s.

Proof. The proof consists of four steps: In the first step, we use a suitable anisotropic scaling and translations to exclude vanishing in the sense of Lemma 2.4. This is adapted from the classical case $s = 0$ with adjustments based on the different scaling properties appearing in this case.

In the second step, we similarly adapt the classical arguments to show that dichotomy cannot occur.

The third step then uses Lemma 2.5 to deduce further information on potential concentration behavior of the minimizing sequence in order to exclude the existence of multiple concentration points.

In the fourth step we then show that the sequence cannot concentrate in a single point either. Compared to the classical case, the scaling and translation properties are much weaker in our setting, making this step much more involved. A crucial idea is the following: If the sequence concentrates in a single point, its L^q-norm will blow up for any $q > 2^*_s$ in a neighborhood of this point. If the concentration point is not on the boundary however, the H_s-norm is comparable to the Sobolev-norm in a neighborhood and can be used to bound the L^q-norm for $q < 2^*$. Since $2^*_s < 2^*$, this can be brought to a contradiction.

Step 1: *After rescaling and translation, the sequence cannot vanish.*

For $r > 0$ we define the family of rectangles

$$\mathcal{Q}_r := \{(0, r^2) \times (y + (-r^2, r^2)^{N-2} \times (-r^{2+s}, r^{2+s})) : y \in \mathbb{R}^{N-1}\}.$$

It is important to note that, for $R > 0$, with respect to the transformation

$$\tau_R(x) = (R^2 x_1, R^2 x_2, \ldots, R^2 x_{N-1}, R^{2+s} x_N), \tag{2-10}$$

these sets satisfy

$$\tau_R(\mathcal{Q}_r) = \mathcal{Q}_{rR}.$$

Moreover, the functions

$$Q_n(r) := \sup_{E \in \mathcal{Q}_r} \int_E |u_n|^{2^*_s} \, dx$$

are continuous on $[0, \infty)$ and satisfy

$$\lim_{r \to 0} Q_n(r) = 0, \qquad \lim_{r \to \infty} Q_n(r) = 1.$$

Moreover, the supremum in the definition of Q_n is attained. Indeed, by definition there exists a sequence $(y_k)_k \subset \mathbb{R}^{N-1}$ such that $\int_{E_k} |u_n|^{2^*_s} \, dx \to Q_n(r)$ as $k \to \infty$, where

$$E_k := (0, r^2) \times (y_k + (-r^2, r^2)^{N-2} \times (-r^{2+s}, r^{2+s})).$$

Since $|u_n|^{2^*_s} \, dx$ is a finite measure, $(y_k)_k$ must be bounded so we may pass to a convergent subsequence, whose limit attains the supremum.

Hence we may choose $A_n > 0,\ y_n \in \mathbb{R}^{N-1}$ such that the rescaled sequence $v_n \in H_s$ given by

$$v_n(x) := A_n^{(2N-4+s)/2} u_n(A_n^2 x_1, A_n^2(x_2 + (y_n)_1), \dots, A_n^{2+s}(x_N + (y_n)_{N-1}))$$

satisfies

$$Q_n(1) = \sup_{E \in \mathcal{Q}_1} \int_E |v_n|^{2_s^*}\, dx = \int_{(0,1)\times(-1,1)^{N-1}} |v_n|^{2_s^*}\, dx = \tfrac{1}{2}.$$

After passing to a subsequence, we may assume $v_n \rightharpoonup v$ in H_s and in $L^{2_s^*}(\mathbb{R}^N_+)$. We now consider the measures

$$\mu_n := \left(\sum_{i=1}^{N-1} |\partial_i v_n|^2 + x_1^s |\partial_N v_n|^2\right) dx \quad \text{and} \quad \nu_n := |v_n|^{2_s^*}\, dx$$

and apply Lemma 2.4 to $(\nu_n)_n$, where we note that μ_n and ν_n are initially measures on $\mathbb{R}^N_+$ but can trivially be extended to $\mathbb{R}^N$. By our normalization, vanishing cannot occur.

Step 2: *Exclusion of dichotomy.*

We argue by contradiction and assume that we have dichotomy, and thus let $\lambda \in (0, 1)$ be as in Lemma 2.4 (iii). Then, considering a sequence $\varepsilon_n \downarrow 0$, for any $n \in \mathbb{N}$, there exist $R_n > 0,\ x_n \in \mathbb{R}^N_+$ as well as nonnegative measures ν_n^1 and ν_n^2 on $\mathbb{R}^N_+$ such that

$$0 \le \nu_n^1 + \nu_n^2 \le \nu_n, \quad \operatorname{supp} \nu_n^1 \subset \mathbb{R}^N_+ \cap B_{R_n}(x_n), \quad \operatorname{supp} \nu_n^2 \subset \mathbb{R}^N_+ \setminus B_{2R_n^{(2+s)/2}+1}(x_n),$$
$$\left|\lambda - \int_{\mathbb{R}^N_+} d\nu_n^1\right| + \left|(1-\lambda) - \int_{\mathbb{R}^N_+} d\nu_n^2\right| \le 2\varepsilon_n,$$

and thus

$$\limsup_{n\to\infty} \left(\left|\lambda - \int_{\mathbb{R}^N_+} d\nu_n^1\right| + \left|(1-\lambda) - \int_{\mathbb{R}^N_+} d\nu_n^2\right| \right) = 0.$$

From the proof of Lemma 2.4 (see [Struwe 2008]) we can assume $R_n \to \infty$ and, in particular, $R_n \ge 1$.

For $r > 0$, let the anisotropic scaling τ_r be defined as in (2-10). We crucially note that

$$B_{R_n}(0) \subset \tau_{\sqrt{R_n}}(B_1(0))$$

and

$$\mathbb{R}^N_+ \setminus B_{2R_n^{(2+s)/2}+1}(0) \subset \mathbb{R}^N_+ \setminus \tau_{\sqrt{R_n}}(B_2(0)).$$

We take $\varphi \in C_c^\infty(B_2(0))$ with $0 \le \varphi \le 1$ and $\varphi \equiv 1$ in $B_1(0)$. For $n \in \mathbb{N}$, let $\varphi_n(x) := \varphi(\tau^{-1}_{\sqrt{R_n}}(x - x_n))$, so that

$$\varphi_n \equiv 1 \quad \text{on } x_n + \tau_{\sqrt{R_n}}(B_1(0)), \qquad \varphi_n \equiv 0 \quad \text{on } \mathbb{R}^N \setminus (x_n + \tau_{\sqrt{R_n}}(B_2(0))),$$

and thus, in particular,

$$\varphi_n \equiv 1 \quad \text{on } \operatorname{supp} \nu_n^1, \qquad \varphi_n \equiv 0 \quad \text{on } \operatorname{supp} \nu_n^2.$$

Note that

$$|\partial_i v_n|^2 + x_1^s |\partial_N v_n|^2 \ge (|\partial_i v_n|^2 + x_1^s |\partial_N v_n|^2)(\varphi_n^2 + (1-\varphi_n)^2) \quad \text{for } i = 1, \dots, N-1.$$

We have

$$\Bigl(\int_{\mathbb{R}^N_+}\Bigl(\sum_{i=1}^{N-1}|\partial_i(\varphi_n v_n)|^2+x_1^s|\partial_N(\varphi_n v_n)|^2\Bigr)\,dx\Bigr)^{1/2}$$
$$\le\Bigl(\int_{\mathbb{R}^N_+}\varphi_n^2\Bigl(\sum_{i=1}^{N-1}|\partial_i v_n|^2+x_1^s|\partial_N v_n|^2\Bigr)\,dx\Bigr)^{1/2}+\Bigl(\int_{\mathbb{R}^N_+}v_n^2\Bigl(\sum_{i=1}^{N-1}|\partial_i\varphi_n|^2+x_1^s|\partial_N\varphi_n|^2\Bigr)\,dx\Bigr)^{1/2}$$

and analogously for $(1-\varphi_n)$ instead of φ_n. Squaring and adding these estimates gives

$$\|\varphi_n v_n\|_{H_s}^2+\|(1-\varphi_n)v_n\|_{H_s}^2$$
$$\le\int_{\mathbb{R}^N_+}\Bigl(\sum_{i=1}^{N-1}|\partial_i v_n|^2+x_1^s|\partial_N v_n|^2\Bigr)\,dx+2\int_{\mathbb{R}^N_+}v_n^2\Bigl(\sum_{i=1}^{N-1}|\partial_i\varphi_n|^2+x_1^s|\partial_N\varphi_n|^2\Bigr)\,dx$$
$$+4\Bigl(\int_{\mathbb{R}^N_+}\Bigl(\sum_{i=1}^{N-1}|\partial_i v_n|^2+x_1^s|\partial_N v_n|^2\Bigr)\,dx\Bigr)^{1/2}\Bigl(\int_{\mathbb{R}^N_+}v_n^2\Bigl(\sum_{i=1}^{N-1}|\partial_i\varphi_n|^2+x_1^s|\partial_N\varphi_n|^2\Bigr)\,dx\Bigr)^{1/2}.$$

Setting

$$\beta_n:=2\int_{\mathbb{R}^N_+}v_n^2\Bigl(\sum_{i=1}^{N-1}|\partial_i\varphi_n|^2+x_1^s|\partial_N\varphi_n|^2\Bigr)\,dx$$
$$+4\Bigl(\int_{\mathbb{R}^N_+}\Bigl(\sum_{i=1}^{N-1}|\partial_i v_n|^2+x_1^s|\partial_N v_n|^2\Bigr)\,dx\Bigr)^{1/2}\Bigl(\int_{\mathbb{R}^N_+}v_n^2\Bigl(\sum_{i=1}^{N-1}|\partial_i\varphi_n|^2+x_1^s|\partial_N\varphi_n|^2\Bigr)\,dx\Bigr)^{1/2},$$

we thus have

$$\int_{\mathbb{R}^N_+}\Bigl(\sum_{i=1}^{N-1}|\partial_i v_n|^2+x_1^s|\partial_N v_n|^2\Bigr)\,dx\ge\|\varphi_n v_n\|_{H_s}^2+\|(1-\varphi_n)v_n\|_{H_s}^2-\beta_n.$$

Next, we define the anisotropic annulus

$$A_n:=x_n+\tau_{\sqrt{R_n}}(B_2(0))\setminus\tau_{\sqrt{R_n}}(B_1(0))$$

and consider $\delta>0$. Using Young's inequality and the fact that any derivative of φ_n vanishes outside of A_n, we can estimate

$$\beta_n\le\delta\int_{\mathbb{R}^N_+}\Bigl(\sum_{i=1}^{N-1}|\partial_i v_n|^2+x_1^s|\partial_N v_n|^2\Bigr)\,dx+C(\delta)\int_{A_n}v_n^2\Bigl(\sum_{i=1}^{N-1}|\partial_i\varphi_n|^2+x_1^s|\partial_N\varphi_n|^2\Bigr)\,dx.$$

Note that

$$\sum_{i=1}^{N-1}|\partial_i\varphi_n|^2+x_1^s|\partial_N\varphi_n|^2=R_n^{-2}\sum_{i=1}^{N-1}|[\partial_i\varphi](\tau^{-1}_{\sqrt{R_n}}(x))|^2+x_1^sR_n^{-2-s}|[\partial_N\varphi](\tau^{-1}_{\sqrt{R_n}}(x))|^2$$
$$=R_n^{-2}\Bigl(\sum_{i=1}^{N-1}|[\partial_i\varphi]|^2+(\,\cdot\,)_1^s|[\partial_N\varphi|^2\Bigr)\circ\tau^{-1}_{\sqrt{R_n}},$$

and thus

$$\sum_{i=1}^{N-1}|\partial_i\varphi_n|^2+x_1^s|\partial_N\varphi_n|^2\le CR_n^{-2}$$

for some $C > 0$ independent of n. This gives

$$\int_{A_n} v_n^2 \left(\sum_{i=1}^{N-1} |\partial_i \varphi_n|^2 + x_1^s |\partial_N \varphi_n|^2 \right) dx \le C R_n^{-2} \|v_n\|^2_{L^2(A_n)}.$$

Using Hölder's inequality then further yields

$$\begin{aligned} R_n^{-1} \|v_n\|_{L^2(A_n)} &\le R_n^{-1} |A_n|^{2/(2N+s)} \|v_n\|_{L^{2_s^*}(A_n)} \\ &\le C \|v_n\|_{L^{2_s^*}(A_n)} \\ &\le C \left(\int_{\mathbb{R}^N} d\nu_n - \left(\int_{\mathbb{R}^N} d\nu_n^1 + \int_{\mathbb{R}^N} d\nu_n^2 \right) \right)^{1/2_s^*} \to 0 \end{aligned}$$

as $n \to \infty$. Here we used

$$|A_n| = |\tau_{\sqrt{R_n}}(B_2(x_n))| - |\tau_{\sqrt{R_n}}(B_1(x_n))| = R_n^{(2N+s)/2}(|B_2(0)| - |B_1(0)|).$$

Overall, we find that, for any $\delta > 0$,

$$\limsup_{n\to\infty} \beta_n \le \delta \sup_n \|v_n\|_H^2,$$

and since $(v_n)_n$ remains bounded in H_s, we conclude

$$\begin{aligned} \int_{\mathbb{R}^N_+} \left(\sum_{i=1}^{N-1} |\partial_i v_n|^2 + x_1^s |\partial_N v_n|^2 \right) dx &\ge \|\varphi_n v_n\|^2_{H_s} + \|(1-\varphi_n) v_n\|^2_{H_s} - \beta_n \\ &\ge \mathcal{S}(\|\varphi_n v_n\|^2_{L^{2_s^*}(\mathbb{R}^N_+)} + \|(1-\varphi_n) v_n\|^2_{L^{2_s^*}(\mathbb{R}^N_+)}) + o(1) \\ &\ge \mathcal{S}\left(\left(\int_{B_{R_n}(x_n)} d\nu_n \right)^{2/2_s^*} + \left(\int_{\mathbb{R}^N_+ \setminus B_{R_n'}(x_n)} d\nu_n \right)^{2/2_s^*} \right) + o(1) \\ &\ge \mathcal{S}\left(\left(\int_{\mathbb{R}^N_+} d\nu_n^1 \right)^{2/2_s^*} + \left(\int_{\mathbb{R}^N_+} d\nu_n^2 \right)^{2/2_s^*} \right) + o(1) \\ &\ge \mathcal{S}(\lambda^{2/2_s^*} + (1-\lambda)^{2/2_s^*}) + o(1). \end{aligned}$$

But since $\lambda \in (0, 1)$, we have $\lambda^{2/2_s^*} + (1-\lambda)^{2/2_s^*} > 1$, and thus

$$\mathcal{S} = \lim_{n\to\infty} \int_{\mathbb{R}^N_+} \left(\sum_{i=1}^{N-1} |\partial_i v_n|^2 + x_1^s |\partial_N v_n|^2 \right) dx \ge \liminf_{n\to\infty} (\mathcal{S}(\lambda^{2/2_s^*} + (1-\lambda)^{2/2_s^*}) + o(1)) > \mathcal{S},$$

a contradiction. Hence we cannot have dichotomy.

Step 3: *The sequence cannot concentrate in multiple points.*

Since we are therefore in condition (i) of Lemma 2.4, there exists a sequence $(x_n)_n$ such that, for any $\varepsilon > 0$, there exists $R = R(\varepsilon) > 0$ with

$$\int_{B_R(x_n)} d\nu_n = \int_{B_R(x_n) \cap \mathbb{R}^N_+} d\nu_n \ge 1 - \varepsilon.$$

Since we normalized so that

$$\int_{(0,1)\times(-1,1)^{N-1}} |v_n|^{2_s^*}\,dx = \tfrac{1}{2},$$

we must have $(0,1)\times(-1,1)^{N-1}\cap B_R(x_n)\neq\varnothing$ if $\varepsilon<\frac{1}{2}$. By making R larger if necessary, we can thus assume

$$\int_{B_R(0)} d\nu_n = \int_{B_R(0)\cap\mathbb{R}^N_+} d\nu_n \geq 1-\varepsilon.$$

In particular, we may therefore pass to a subsequence such that $\nu_n \rightharpoonup \nu$ weakly in the sense of measure, where ν is a finite measure on $\mathbb{R}^N_+$. By weak lower- (and upper-) semicontinuity (of measures), we then have

$$\int_{\mathbb{R}^N_+} d\nu = 1.$$

By Lemma 2.5, we may now assume

$$\mu_n \rightharpoonup \mu \geq \sum_{i=1}^{N-1}(|\partial_i v|^2 + x_1^s|\partial_N v|^2)\,dx + \sum_{j\in J}\mu^j\delta_{x^j} \quad\text{and}\quad \nu_n \rightharpoonup |v|^{2_s^*}\,dx + \sum_{j\in J}\nu^j\delta_{x^j}$$

for points $x^j\in\mathbb{R}^N_+$ and positive $\mu^j,\ \nu^j$ satisfying $\mathcal{S}(\nu^j)^{2/2_s^*}\leq\mu^j$. We have

$$\begin{aligned}
\mathcal{S}+o(1) = \|v_n\|^2_{H_s} &= \int_{\mathbb{R}^N_+} d\mu_n \geq \int_{\mathbb{R}^N_+} d\mu + o(1) \geq \|v\|^2_{H_s} + \sum_{j\in J}\mu^j + o(1)\\
&\geq \mathcal{S}\biggl(\|v\|^2_{L^{2_s^*}(\mathbb{R}^N_+)} + \sum_j (\nu^j)^{2/2_s^*}\biggr) + o(1) \geq \mathcal{S}\biggl(\|v\|^{2_s^*}_{L^{2_s^*}(\mathbb{R}^N_+)} + \sum_j \nu^j\biggr)^{2/2_s^*} + o(1)\\
&= \mathcal{S}\biggl(\int_{\mathbb{R}^N_+} d\nu\biggr)^{2/2_s^*} + o(1) = \mathcal{S}+o(1)
\end{aligned} \tag{2-11}$$

as $n\to\infty$. In the second inequality, we used the fact that the map $t\mapsto t^{2/2_s^*}$ is strictly concave and hence subadditive. Moreover, the strict concavity implies that equality can only hold, if at most one of the terms $\|v\|^{2_s^*}_{L^{2_s^*}(\mathbb{R}^N_+)}$ and $\nu^j,\ j\in J$, is nonzero.

<u>Step 4</u>: *The sequence cannot concentrate in a single point, i.e., we have $\nu^j=0$ for all j.*

Assuming that this is false, we have $\nu_n \rightharpoonup \delta_{x^1}$ for some $x^1\in\overline{\mathbb{R}}^N_+$. By our normalization and weak lower-semicontinuity (of measures), $x^1\notin Q := (0,1)\times(-1,1)^{N-1}$ since

$$\delta_{x^1}(Q) \leq \liminf_{n\to\infty}\nu_n(Q) = \tfrac{1}{2}.$$

Moreover, if $\operatorname{dist}(x^1,Q)>0$, there exists $\varepsilon>0$ such that $B_\varepsilon(x_1)\cap Q\neq\varnothing$, and thus

$$1 = \delta_{x^1}(B_\varepsilon(x^1)) \leq \liminf_{n\to\infty}\nu_n(B_\varepsilon(x^1)) \leq \tfrac{1}{2},$$

which is a contradiction. Hence it only remains to consider the case $x^1\in\partial Q$. Due to the normalization

$$\sup_{E\in\mathcal{Q}_1}\int_E |v_n|^{2_s^*}\,dx = \int_{(0,1)\times(-1,1)^{N-1}} |v_n|^{2_s^*}\,dx = \tfrac{1}{2},$$

we have $x^1 \notin ((0, y) + Q)$ for all $y \in \mathbb{R}^{N-1}$, so x^1 must be of the form $x^1 = (1, y)$ or $(0, y)$ for some $y \in (-1, 1)^{N-1}$. The latter case can be excluded, since, for $\varepsilon \in \left(0, \frac{1}{2}\right)$,

$$\delta_{x^1}(B_\varepsilon(0, y)) \le \liminf_{n\to\infty} \nu_n(B_\varepsilon(0, y)) \le \liminf_{n\to\infty} \nu_n((0, y) + Q) \le \tfrac{1}{2}.$$

After a translation orthogonal to the x_1-direction, we may therefore assume $x^1 = (1, 0, \dots, 0)$ and first note that $v \equiv 0$ and hence $\mu \ge \mathcal{S}\delta_{x^1}$ by (2-11). On the other hand,

$$\int_{\mathbb{R}^N} d\mu \le \liminf_{n\to\infty} \int_{\mathbb{R}^N} d\mu_n = \mathcal{S},$$

whence we conclude $\mu = \mathcal{S}\delta_{x^1}$.

For any $0 < \delta < \frac{1}{2}$, $B_\delta := B_\delta(x_1)$ is a continuity set of $\nu = \delta_{x_1}$; hence

$$\nu_n(B_\delta) \to 1$$

and similarly

$$\mu_n(B_\delta) \to S$$

as $n \to \infty$. In particular, for fixed $\varepsilon > 0$, we find $n_0 = n_0(\varepsilon, \delta)$ such that

$$\int_{B_\delta} |v_n|^{2_s^*}\, dx \ge 1 - \varepsilon, \quad \mathcal{S} - \varepsilon \le \int_{B_\delta} \left(\sum_{i=1}^{N-1} |\partial_i v_n|^2 + x_1^s |\partial_N v_n|^2 \right) dx \le \mathcal{S} + \varepsilon$$

for $n \ge n_0$. Furthermore,

$$\frac{1}{1+\delta} \int_{B_\delta} \left(\sum_{i=1}^{N-1} |\partial_i v_n|^2 + x_1^s |\partial_N v_n|^2 \right) dx \le \int_{B_\delta} \sum_{i=1}^{N} |\partial_i v_n|^2\, dx$$

and

$$\int_{B_\delta} \sum_{i=1}^{N} |\partial_i v_n|^2\, dx \le \frac{1}{1-\delta} \int_{B_\delta} \left(\sum_{i=1}^{N-1} |\partial_i v_n|^2 + x_1^s |\partial_N v_n|^2 \right) dx$$

imply

$$\frac{\mathcal{S} - \varepsilon}{1+\delta} \le \int_{B_\delta} \sum_{i=1}^{N} |\partial_i v_n|^2\, dx \le \frac{\mathcal{S} + \varepsilon}{1-\delta}$$

for $n \ge n_0$. It is important to note that the weak convergence $\nu_n \rightharpoonup \delta_{x^1}$ implies that, for any $t \in (0, \delta)$ and $q \in (2_s^*, 2^*)$, we have

$$1 = \liminf_{n\to\infty} \int_{B_t} |v_n|^{2_s^*}\, dx \le |B_t|^{1-2_s^*/q} \liminf_{n\to\infty} \left(\int_{B_t} |v_n|^q\, dx \right)^{2_s^*/q} \le |B_t|^{1-2_s^*/q} \liminf_{n\to\infty} \left(\int_{B_\delta} |v_n|^q\, dx \right)^{2_s^*/q}.$$

In particular, this implies

$$\liminf_{n\to\infty} \left(\int_{B_\delta} |v_n|^q\, dx \right)^{2_s^*/q} \ge |B_t|^{2_s^*/q-1}, \tag{2-12}$$

and since $t \in (0, \delta)$ was arbitrary, we conclude that $\|v_n\|_{L^q(B_\delta)} \to \infty$ as $n \to \infty$ for any $q \in (2_s^*, 2^*)$.

Now let $\varphi \in C_c^\infty(\mathbb{R}^N)$ such that $\varphi \equiv 1$ on $B_1(0)$ and $\varphi \equiv 0$ on $\mathbb{R}^N \setminus B_2(0)$, and set

$$\varphi_\delta(x) := \varphi\Big(\frac{x - x^1}{\delta}\Big),$$

so that $\varphi_\delta \equiv 1$ on $B_\delta(x^1)$ and $\varphi \equiv 0$ on $\mathbb{R}^N \setminus B_{2\delta}(x^1)$. Then, by Sobolev's inequality,

$$\Big(\int_{\mathbb{R}^N_+} |\varphi_\delta v_n|^q\, dx\Big)^{2/q} \le C_q \Big(\int_{\mathbb{R}^N_+} \sum_{i=1}^N |\partial_i(\varphi_\delta v_n)|^2\, dx + \int_{\mathbb{R}^N_+} |\varphi_\delta v_n|^2\, dx\Big). \tag{2-13}$$

Note that (2-12) implies that the left-hand side goes to infinity as $n \to \infty$ since

$$\int_{B_\delta} |v_n|^q\, dx \le \int_{\mathbb{R}^N} |\varphi_\delta v_n|^q\, dx.$$

On the other hand,

$$\int_{\mathbb{R}^N_+} |\varphi_\delta v_n|^2\, dx \le |B_{2\delta}|^{1-2/2_s^*} \Big(\int_{B_{2\delta}} |v_n|^{2_s^*}\, dx\Big)^{2/2_s^*} \le |B_2|^{1-2/2_s^*}$$

and, noting that $\nabla\varphi_\delta(x) = \delta^{-1}[\nabla\varphi]((x - x^1)/\delta)$,

$$\begin{aligned}\Big(\int_{\mathbb{R}^N_+} \sum_{i=1}^N |\partial_i(\varphi_\delta v_n)|^2\, dx\Big)^{1/2} &\le \Big(\int_{\mathbb{R}^N_+} \varphi_\delta^2 \sum_{i=1}^N |\partial_i v_n|^2\, dx\Big)^{1/2} + \Big(\int_{\mathbb{R}^N_+} v_n^2 \sum_{i=1}^N |\partial_i \varphi_\delta|^2\, dx\Big)^{1/2}\\
&\le \Big(\int_{B_{2\delta}} \sum_{i=1}^N |\partial_i v_n|^2\, dx\Big)^{1/2} + \sqrt{N}\delta^{-1}\|\nabla\varphi\|_\infty \Big(\int_{B_{2\delta}\setminus B_\delta} |v_n|^2\, dx\Big)^{1/2}\\
&\le \sqrt{\frac{\mathcal{S}+\varepsilon}{1-2\delta}} + \sqrt{N}\delta^{-1}\|\nabla\varphi\|_\infty |B_{2\delta}\setminus B_\delta|^{1/2-1/2_s^*} \Big(\int_{B_{2\delta}\setminus B_\delta} |v_n|^{2_s^*}\, dx\Big)^{1/2_s^*}\\
&\le \sqrt{\frac{\mathcal{S}+\varepsilon}{1-2\delta}} + \sqrt{N}\delta^{-1}\|\nabla\varphi\|_\infty |B_{2\delta}\setminus B_\delta|^{1/2-1/2_s^*}.\end{aligned}$$

This implies that the right-hand side of (2-13) remains bounded as $n \to \infty$, a contradiction.

We conclude $\nu^j = 0$ for all j, and hence $\|v\|_{L^{2_s^*}(\mathbb{R}^N_+)} = 1$. Since $L^{2_s^*}(\mathbb{R}^N_+)$ is uniformly convex, this implies $v_n \to v$ in $L^{2_s^*}(\mathbb{R}^N_+)$. Moreover, since $\|v\|_{H_s}^2 \ge \mathcal{S}$, weak lower-semicontinuity gives $\|v_n\|_{H_s}^2 \to \mathcal{S} = \|v\|_{H_s}^2$, and hence $v_n \to v$ in H_s again by uniform convexity of the Hilbert space H_s. This completes the proof. □

Remark 2.7 (existence of minimizers on $\mathbb{R}^N$). We note that Theorem 2.2 implies

$$\mathcal{S}_s(\mathbb{R}^N) := \inf_{u \in C_c^1(\mathbb{R}^N)} \frac{\int_{\mathbb{R}^N} \big(\sum_{i=1}^{N-1} |\partial_i u|^2 + |x_1|^s |\partial_N u|^2\big)\, dx}{\big(\int_{\mathbb{R}^N} |u|^{2_s^*}\, dx\big)^{2/2_s^*}} > 0.$$

Consequently, we can look for minimizers in the closure of $C_c^1(\mathbb{R}^N)$ in

$$\Big\{u \in L^{2_s^*}(\mathbb{R}^N) : \int_{\mathbb{R}^N} \sum_{i=1}^{N-1} |\partial_i u|^2 + |x_1|^s |\partial_N u|^2\, dx < \infty\Big\}.$$

The previous arguments can then easily be adapted to prove the existence of minimizers of $\mathcal{S}_s(\mathbb{R}^N)$ similar to Theorem 2.6.

3. A degenerate Sobolev inequality on $\boldsymbol{B}$

In this section we shall prove the second part of Theorem 1.1, namely the properties of $\mathscr{C}_{1,m,p}(\boldsymbol{B})$ given in (1-12).

We first use the scaling properties discussed in Remark 2.3 (i) to prove the following.

Proposition 3.1. *Let $p > 2_1^*$ and $m > -\lambda_1(\boldsymbol{B})$. Then $\mathscr{C}_{1,m,p}(\boldsymbol{B}) = 0$, i.e.,*

$$\inf_{u \in C_c^1(\boldsymbol{B}) \setminus \{0\}} \frac{\|\nabla u\|_2^2 - \|\partial_\theta u\|_2^2 + m\|u\|_2^2}{\|u\|_p^2} = 0.$$

Proof. Let $v \in C_c^1(\mathbb{R}_+^N) \setminus \{0\}$ be arbitrary and, for $\lambda \in (0,1)$, let

$$\tau_\lambda : \boldsymbol{B} \to \mathbb{R}_+^N, \quad \tau_\lambda(x) = (\lambda^{-2}(x_1+1), \lambda^{-2}x_3, \ldots, \lambda^{-2}x_N, \lambda^{-3}x_2), \tag{3-1}$$

and set $u := v \circ \tau_\lambda$. If λ is chosen sufficiently small, we have $u \in C_c^1(\boldsymbol{B})$ and

$$\begin{aligned}
&\|\nabla u\|_{L^2(\boldsymbol{B})}^2 - \|\partial_\theta u\|_{L^2(\boldsymbol{B})}^2 \\
&= \int_{\boldsymbol{B}} \biggl(\sum_{i=1}^N |\partial_i u|^2 - |x_1\,\partial_2 u - x_2\,\partial_1 u|^2 \biggr)\,dx \\
&= \int_{\boldsymbol{B}} \biggl(\sum_{i=1}^{N-1} |\lambda^{-2}[\partial_i v] \circ \tau_\lambda|^2 + |\lambda^{-3}[\partial_N v] \circ \tau_\lambda|^2 - |x_1 \lambda^{-3}[\partial_N v] \circ \tau_\lambda - x_2 \lambda^{-2}[\partial_1 v] \circ \tau_\lambda|^2 \biggr)\,dx \\
&= \lambda^{2N+1} \int_{\mathbb{R}_+^N} \biggl(\sum_{i=1}^{N-1} \lambda^{-4} |\partial_i v|^2 + \lambda^{-6} |\partial_N v|^2 - |(\lambda^2 x_1 - 1)\lambda^{-3}\,\partial_N v - \lambda^3 x_2 \lambda^{-2}\,\partial_1 v|^2 \biggr)\,dx \\
&= \lambda^{2N-3} \int_{\mathbb{R}_+^N} \biggl(\sum_{i=1}^{N-1} |\partial_i v|^2 + 2x_1 |\partial_N v|^2 \biggr)\,dx \\
&\qquad + \lambda^{2N-3} \int_{\mathbb{R}_+^N} \Bigl(-\lambda^2 x_1^2 |\partial_N v|^2 - 2x_2 \lambda^2 (\lambda^2 x_1 - 1)\,\partial_1 v\,\partial_N v + \lambda^6 x_2^2 |\partial_1 v|^2 \Bigr)\,dx,
\end{aligned}$$

while

$$\|u\|_{L^2(\boldsymbol{B})}^2 = \lambda^{2N+1} \|v\|_{L^2(\mathbb{R}_+^N)}^2 \quad \text{and} \quad \|u\|_{L^p(\boldsymbol{B})}^2 = \lambda^{(4N+2)/p} \|v\|_{L^p(\mathbb{R}_+^N)}^2.$$

We conclude that

$$\begin{aligned}
\mathscr{C}_{1,m,p}(\boldsymbol{B}) &\le \frac{\|\nabla u\|_{L^2(\boldsymbol{B})}^2 - \|\partial_\theta u\|_{L^2(\boldsymbol{B})}^2 + m\|u\|_{L^2(\boldsymbol{B})}^2}{\|u\|_{L^p(\boldsymbol{B})}^2} \\
&= \lambda^{(p(2N-3)-(4N+2))/p} \frac{\int_{\mathbb{R}_+^N} \bigl(\sum_{i=1}^{N-1} |\partial_i v|^2 + 2x_1 |\partial_N v|^2 \bigr)\,dx}{\|v\|_{L^p(\mathbb{R}_+^N)}^2} + o(\lambda^{(p(2N-3)-(4N+2))/p}) \to 0
\end{aligned}$$

as $\lambda \to 0$, since $p > 2_1^* = (4N+2)/(2N-3)$. □

To prove the second assertion on $\mathscr{C}_{1,m,p}(\boldsymbol{B})$ in (1-12), we now transfer the information given by Theorem 1.6 in the case $s = 1$ to the ball $\boldsymbol{B}$.

Lemma 3.2. *Consider the great circle γ defined in* (1-6), *and let $\varepsilon > 0$. Then there exists $\delta > 0$ with the property that*, *for any $x_0 \in \gamma$*,

$$\frac{\int_{\Omega_{x_0,\delta}} (|\nabla u|^2 - |\partial_\theta u|^2)\,dx}{\|u\|^2_{L^{2_1^*}(\Omega_{x_0,\delta})}} \geq (1-\varepsilon) 2^{1/2 - 1/2_1^*} \mathcal{S}_1(\mathbb{R}^N_+) \quad \textit{for } u \in C^1_c(\Omega_{x_0,\delta}) \setminus \{0\},$$

where $\mathcal{S}_1(\mathbb{R}^N_+)$ is given in Theorem 1.6 and

$$\Omega_{x_0,\delta} := \boldsymbol{B} \cap B_\delta(x_0) = \{x \in \boldsymbol{B} : |x - x_0| < \delta\}. \tag{3-2}$$

Proof. We may assume $x_0 = e_2 = (0, 1, 0, \dots, 0)$ is the second coordinate vector. We fix $\delta > 0$ and consider a function $u \in C^1_c(\Omega_{e_2,\delta})$ which we extend trivially to a function $u \in C^1_c(\mathbb{R}^N)$. Moreover, we write u in N-dimensional polar coordinates, so we consider $U := [0, 1] \times (-\pi, \pi) \times (0, \pi)^{N-2}$ and the function

$$v = u \circ P : U \to \mathbb{R},$$

with $P : U \to \mathbb{R}^N$ given by

$$\begin{aligned} P(r, \theta, \vartheta_1, \dots, \vartheta_{N-2}) = (&r \sin\vartheta_1 \cdots \sin\vartheta_{N-2} \cos\theta,\ r \sin\vartheta_1 \cdots \sin\vartheta_{N-2} \sin\theta, \\ &r \cos\vartheta_1, r \sin\vartheta_1 \cos\vartheta_2, \dots, r \sin\vartheta_1 \cdots \sin\vartheta_{N-3} \cos\vartheta_{N-2}). \end{aligned} \tag{3-3}$$

We emphasize here that we use the angular variable $\theta \in (-\pi, \pi)$ for the angle of the (x_1, x_2)-coordinate of $x \in S^{N-1}$ relative to the positive x_1-axis in $\mathbb{R}^2$ (in the literature, this is usually done for the (x_{N-1}, x_N)-coordinate). Noting that

$$|\nabla u(r\Theta)|^2 = |\partial_r u(r\Theta)|^2 + \frac{1}{r^2} |\nabla_\Theta u(r\Theta)|^2 \quad \text{for } r > 0,\ \Theta \in S^{N-1},$$

we then have, by (B-3) from Appendix B,

$$\begin{aligned} &\int_{\boldsymbol{B}} (|\nabla u|^2 - |\partial_\theta u|^2)\,dx \\ &= \int_0^1 \int_{-\pi}^{\pi} \int_0^{\pi} \cdots \int_0^{\pi} \left(|\partial_r v|^2 + \frac{1}{r^2} \sum_{i=1}^{N-2} h_i |\partial_{\vartheta_i} v|^2 + \left(\frac{h_{N-1}}{r^2} - 1 \right) |\partial_\theta v|^2 \right) h \, d\vartheta_1 \cdots d\vartheta_{N-2}\, d\theta\, dr, \end{aligned} \tag{3-4}$$

with the functions $h, h_i : U \to \mathbb{R},\ i = 1, \dots, N-1$, given by

$$h(r, \theta, \vartheta_1, \dots, \vartheta_{N-2}) = r^{N-1} \prod_{k=1}^{N-2} \sin^{N-1-k} \vartheta_k, \quad h_i(r, \theta, \vartheta_1, \dots, \vartheta_{N-2}) = \prod_{k=1}^{i-1} \frac{1}{\sin^2 \vartheta_k}. \tag{3-5}$$

In particular, we have $h \leq 1$ and $h_i \geq 1$ in U for $i = 1, \dots, N-1$. Moreover, since

$$P^{-1}(e_2) = \left(1, \tfrac{\pi}{2}, \dots, \tfrac{\pi}{2}\right) \quad \text{and} \quad h\left(1, \tfrac{\pi}{2}, \dots, \tfrac{\pi}{2}\right) = 1,$$

we can choose $\delta > 0$ small enough that

$$P^{-1}(\Omega_{e_2,\delta}) \subset (0, 1) \times (0, \pi)^{N-1} \qquad \text{and} \qquad h \geq (1 - \varepsilon) \quad \text{in } P^{-1}(\Omega_{e_2,\delta}). \tag{3-6}$$

Therefore

$$\int_{\boldsymbol{B}}(|\nabla u|^2-|\partial_\theta u|^2)\,dx \\ \geq (1-\varepsilon)\int_0^1\int_{-\pi}^{\pi}\int_0^{\pi}\cdots\int_0^{\pi}\left(|\partial_r v|^2+\sum_{i=1}^{N-2}|\partial_{\vartheta_i}v|^2+\frac{(1-r)(1+r)}{r^2}|\partial_\theta v|^2\right)d\vartheta_1\cdots d\vartheta_{N-2}\,d\theta\,dr.$$

Noting that

$$\frac{(1-r)(1+r)}{r^2}\geq\frac{(2-\delta)(1-r)}{(1-\delta)^2}\geq 2(1-r)$$

and substituting $t=1-r$, we thus find that

$$\int_{\boldsymbol{B}}(|\nabla u|^2-|\partial_\theta u|^2)\,dx\geq(1-\varepsilon)\int_0^1\int_{-\pi}^{\pi}\int_0^{\pi}\cdots\int_0^{\pi}\left(|\partial_t\tilde{v}|^2+\sum_{i=1}^{N-2}|\partial_{\vartheta_i}\tilde{v}|^2+2t|\partial_\theta\tilde{v}|^2\right)d\vartheta_1\cdots d\vartheta_{N-2}\,d\theta\,dt,$$

with

$$\tilde{v}:U\to\mathbb{R},\qquad \tilde{v}(t,\vartheta_1,\dots,\vartheta_{N-2},\theta)=v(1-t,\vartheta_1,\dots,\vartheta_{N-2},\theta).$$

Note that $u\in C_c^1(\Omega_{e_2,\delta})$ implies, by (3-6), that $\tilde{v}$ is compactly supported in $(0,1)\times(0,\pi)^{N-1}\subset\mathbb{R}^N_+$, so we may regard $\tilde{v}$ as a function in $C_c^1(\mathbb{R}^N_+)$ and deduce that

$$\int_{\boldsymbol{B}}(|\nabla u|^2-|\partial_\theta u|^2)\,dx\geq(1-\varepsilon)\int_{\mathbb{R}^N_+}\left(\sum_{i=1}^{N-1}|\partial_i\tilde{v}|^2+2x_1|\partial_N\tilde{v}|^2\right)dx.$$

Rather directly, we also find that, by a change of variables,

$$\int_{\Omega_{e_2,\delta}}|u|^{2_1^*}\,dx=\int_U|v|^{2_1^*}h\,d(r,\theta,\vartheta_1,\dots,\vartheta_{N-2})\leq\int_U|v|^{2_1^*}\,d(r,\theta,\vartheta_1,\dots,\vartheta_{N-2})\\ =\int_U|\tilde{v}|^{2_1^*}\,d(r,\theta,\vartheta_1,\dots,\vartheta_{N-2})=\int_{\mathbb{R}^N_+}|\tilde{v}|^{2_1^*}\,dx.$$

Using (2-9) with $\kappa=2$, we conclude that

$$\frac{\int_{\Omega_{e_2,\delta}}(|\nabla u|^2-|\partial_\theta u|^2)\,dx}{\|u\|^2_{L^{2_1^*}(\Omega_{e_2,\delta})}}\geq(1-\varepsilon)\frac{\int_{\mathbb{R}^N_+}\left(\sum_{i=1}^{N-1}|\partial_i\tilde{v}|^2+2x_1|\partial_N\tilde{v}|^2\right)dx}{\left(\int_{\mathbb{R}^N_+}|\tilde{v}|^{2_1^*}\,dx\right)^{2/2_1^*}}\geq(1-\varepsilon)2^{1/2-1/2_1^*}\mathcal{S}_1(\mathbb{R}^N_+)$$

as claimed. □

We can now prove the main result of this section.

Theorem 3.3. *For any* $1\leq p\leq 2_1^*$, *there exists* $C>0$ *such that any* $u\in C_c^1(\boldsymbol{B})$ *satisfies*

$$\|u\|^2_{L^p(\boldsymbol{B})}\leq C\int_{\boldsymbol{B}}(|\nabla u|^2-|\partial_\theta u|^2)\,dx. \tag{3-7}$$

Moreover, *in the case* $p=2$, (3-7) *holds with* $C=1/\lambda_1(\boldsymbol{B})$.

Recall here that $\lambda_1(\boldsymbol{B})$ is the first Dirichlet eigenvalue of $-\Delta$ on $\boldsymbol{B}$.

Proof. Let $u \in C_c^1(\boldsymbol{B})$. We first show that

$$\int_{\boldsymbol{B}} (|\nabla u|^2 - |\partial_\theta u|^2)\,dx \geq \lambda_1(\boldsymbol{B})\|u\|^2_{L^2(\boldsymbol{B})}. \tag{3-8}$$

In the following, for every integer $\ell \geq 0$, we let $\{Y_{\ell,k} : \ell \in \mathbb{N}\cup\{0\},\, k = 1, \dots, d_\ell\}$ denote an L^2-orthonormal basis of $L^2(\mathbb{S}^{N-1})$ of spherical harmonics of degree ℓ. More precisely, we can choose $Y_{\ell,k}$ in such a way that, for every $\ell \in \mathbb{N}\cup\{0\}$, the functions $Y_{\ell,k},\ k = 1, \dots, d_\ell$ form a basis of the eigenspace of the Laplace Beltrami operator $-\Delta_{\mathbb{S}^{N-1}}$ corresponding to the eigenvalue $\ell(\ell+N-2)$ and such that

$$-\partial_\theta^2 Y_{\ell,k} = \ell_k^2 Y_{\ell,k} \quad \text{for } k = 1, \dots, d_\ell,$$

where $|\ell_k| \leq \ell$; see, e.g., [Higuchi 1987]. Next, let $u_{\ell,k} \in C^1([0,1])$ be the angular Fourier coefficient functions defined by

$$u_{\ell,k}(r) = \int_{\mathbb{S}^{N-1}} u(r\omega) Y_{\ell,k}(\omega)\,d\omega, \quad 0 \leq r \leq 1.$$

For fixed $r \in [0,1]$, we then have the Parseval identities

$$\begin{aligned}
\|u(r\,\cdot)\|^2_{L^2(\mathbb{S}^{N-1})} &= \sum_{\ell,k} |u_{\ell,k}(r)|^2 \|Y_{\ell,k}\|^2_{L^2(\mathbb{S}^{N-1})},\\
\|\partial_r u(r\,\cdot)\|^2_{L^2(\mathbb{S}^{N-1})} &= \sum_{\ell,k} |\partial_r u_{\ell,k}(r)|^2 \|Y_{\ell,k}\|^2_{L^2(\mathbb{S}^{N-1})},\\
\|\nabla_{\mathbb{S}^{N-1}} u(r\,\cdot)\|^2_{L^2(\mathbb{S}^{N-1})} &= \sum_{\ell,k} (\ell+N-2)|u_{\ell,k}(r)|^2 \|Y_{\ell,k}\|^2_{L^2(\mathbb{S}^{N-1})} \quad \text{and}\\
\|\partial_\theta u(r\,\cdot)\|^2_{L^2(\mathbb{S}^{N-1})} &= \sum_{\ell,k} \ell_k^2 |u_{\ell,k}(r)|^2 \|Y_{\ell,k}\|^2_{L^2(\mathbb{S}^{N-1})}
\end{aligned}$$

in $L^2(\mathbb{S}^{N-1})$. Hereafter, we simply write $\sum_{\ell,k}$ in place of $\sum_{\ell=0}^\infty \sum_{k=1}^{d_\ell}$. Since $\ell(\ell+N-2)/r^2 \geq \ell_k^2$ for $r \in [0,1]$ and every $\ell,\ k$, we estimate that

$$\begin{aligned}
\int_{\boldsymbol{B}} (|\nabla u|^2 - |\partial_\theta u|^2)\,dx &= \int_0^1 r^{N-1} \int_{\mathbb{S}^{N-1}} \Big(|\partial_r u(r\omega)|^2 + \frac{1}{r^2} |\nabla_{\mathbb{S}^{N-1}} u(r\omega)|^2 - |\partial_\theta u(r\omega)|^2 \Big)\,d\omega\,dr\\
&= \sum_{\ell,k} \|Y_{\ell,k}\|^2_{L^2(\mathbb{S}^{N-1})} \int_0^1 r^{N-1} \Big(|\partial_r u_{\ell,k}(r)|^2 + \Big(\frac{\ell(\ell+N-2)}{r^2} - \ell_k^2 \Big) |u_{\ell,k}(r)|^2 \Big)\,dr\\
&\geq \sum_{\ell,k} \|Y_{\ell,k}\|^2_{L^2(\mathbb{S}^{N-1})} \int_0^1 r^{N-1} |\partial_r u_{\ell,k}(r)|^2\,dr\\
&\geq \lambda_1(\boldsymbol{B}) \sum_{\ell,k} \|Y_{\ell,k}\|^2_{L^2(\mathbb{S}^{N-1})} \int_0^1 r^{N-1} |u_{\ell,k}(r)|^2\,dr\\
&= \lambda_1(-\Delta, \boldsymbol{B}) \int_0^1 r^{N-1} \|u(r\,\cdot)\|^2_{L^2(\mathbb{S}^{N-1})}\,dr\\
&= \lambda_1(\boldsymbol{B}) \int_{\boldsymbol{B}} |u|^2\,dx.
\end{aligned}$$

Hence (3-8) holds. To show (3-7), it suffices to consider the case $p = 2_1^*$. In the following, $C > 0$ is a constant independent of u which may change from line to line. Fix $\varepsilon \in \left(0, \frac{1}{2}\right)$ and let $\delta > 0$ be given as in Lemma 3.2. Take points $x_1, \dots, x_m \in \gamma$ such that the sets $U_k := B_\delta(x_k)$ satisfy

$$\gamma \subset \bigcup_{k=1}^{m} U_k,$$

and let $\delta_0 := \operatorname{dist}\bigl(\gamma, \boldsymbol{B} \setminus \bigcup_{k=1}^m U_k\bigr)$. We then let $U_0 := \bigl\{x \in \boldsymbol{B} : \operatorname{dist}(x, \gamma) > \frac{1}{2}\delta_0\bigr\}$, and thus we have $\boldsymbol{B} \subset \bigcup_{k=0}^m U_k$. We may then choose a partition of unity $\eta_0, \dots, \eta_m$ subordinate to this covering. Then

$$\|u\|_{L^{2_1^*}(\boldsymbol{B})} \le \sum_{k=0}^{m} \|\eta_k u\|_{L^{2_1^*}(U_k)} \le C \sum_{k=0}^{m} \biggl(\int_{U_k} (|\nabla(\eta_k u)|^2 - |\partial_\theta(\eta_k u)|^2)\,dx\biggr)^{1/2},$$

where we used Lemma 3.2 and the fact that $v \mapsto \int_{U_0} (|\nabla v|^2 - |\partial_\theta v|^2)\,dx$ induces an equivalent norm on $H_0^1(U_0)$. Indeed, recall that $\partial_\theta = x_1\,\partial_{x_2} - x_2\,\partial_{x_1}$, and hence

$$|(\partial_\theta v)| \le |(x_1, x_2)|\,|\nabla v| \quad \text{a.e. in } \boldsymbol{B}, \tag{3-9}$$

which implies

$$\int_{U_0} (|\nabla v|^2 - |\partial_\theta v|^2)\,dx \ge \int_{U_0} (1 - |(x_1, x_2)|^2)|\nabla v|^2\,dx.$$

Letting $x = (x_1, x_2, x') \in U_0$ with $x' \in \mathbb{R}^{N-2}$, we then find that

$$\tfrac{1}{4}\delta_0^2 < \operatorname{dist}(x, \gamma)^2 = (1 - |(x_1, x_2)|)^2 + |x'|^2 \le (1 - |(x_1, x_2)|)^2 + 1 - |(x_1, x_2)|^2 \le 2(1 - |(x_1, x_2)|^2),$$

and hence

$$\int_{U_0} (|\nabla v|^2 - |\partial_\theta v|^2)\,dx \ge \tfrac{1}{8}\delta_0^2 \int_{U_0} |\nabla v|^2\,dx,$$

i.e., $v \mapsto \int_{U_0} (|\nabla v|^2 - |\partial_\theta v|^2)\,dx$ induces an equivalent norm on $H_0^1(U_0)$, as claimed.

Note that, for $k = 0, \dots, m$, we have

$$\begin{aligned}\int_{U_k} (|\nabla(\eta_k u)|^2 - |\partial_\theta(\eta_k u)|^2)\,dx &\le 2\biggl(\int_{U_k} \eta_k^2(|\nabla u|^2 - |\partial_\theta u|^2)\,dx + \int_{U_k} u^2(|\nabla \eta_k|^2 - |\partial_\theta \eta_k|^2)\,dx\biggr)\\ &\le C \int_{U_k} (|\nabla u|^2 - |\partial_\theta u|^2 + u^2)\,dx,\end{aligned}$$

with some fixed $C > 0$. Here we used the fact that

$$\begin{aligned}2(\nabla \eta_k \cdot \nabla u - \partial_\theta \eta_k\, \partial_\theta u) &= -(|\nabla(\eta_k - u)|^2 - |\partial_\theta(\eta_k - u)|^2) + (|\nabla \eta_k|^2 - |\partial_\theta \eta_k|^2) + (|\nabla u|^2 - |\partial_\theta u|^2)\\ &\le (|\nabla \eta_k|^2 - |\partial_\theta \eta_k|^2) + (|\nabla u|^2 - |\partial_\theta u|^2)\end{aligned}$$

pointwisely on $\boldsymbol{B}$ again by (3-9). We conclude that

$$\|u\|_{L^{2_1^*}(\boldsymbol{B})} \le C \sum_{k=0}^{m} \biggl(\int_{U_k} (|\nabla u|^2 - |\partial_\theta u|^2 + u^2)\,dx\biggr)^{1/2},$$

and thus

$$\|u\|^2_{L^{2_1^*}(\boldsymbol{B})} \le C \sum_{k=0}^{m} \int_{U_k} (|\nabla u|^2 - |\partial_\theta u|^2 + u^2)\, dx \le C \int_{\boldsymbol{B}} (|\nabla u|^2 - |\partial_\theta u|^2 + u^2)\, dx$$
$$\le C \int_{\boldsymbol{B}} (|\nabla u|^2 - |\partial_\theta u|^2)\, dx, \tag{3-10}$$

where we used (3-8) in the last step. The proof is finished. □

4. The variational setting for and main results on ground state solutions

In this section, we set up the variational framework for (1-5) and discuss the notions of weak and ground state solutions of (1-5). Then, we shall complete the proofs of Theorems 1.3, 1.4, 1.5 and 1.7.

4.1. *The variational setting.* We need to fix some notation. Let $0 \le \alpha \le 1$. It then follows from Theorem 3.3 that

$$(u, v) \mapsto \langle u, v\rangle_{\mathcal{H}_\alpha} := \int_{\boldsymbol{B}} (\nabla u \cdot \nabla v - \alpha^2\, \partial_\theta u\, \partial_\theta v)\, dx$$

defines a scalar product on $C^1_c(\boldsymbol{B})$. The induced norm will be denoted by $\|\cdot\|_{\mathcal{H}_\alpha}$. We then let $\mathcal{H}_\alpha$ be the Hilbert space defined as the completion of $C^1_c(\boldsymbol{B})$ with respect to the norm $\|\cdot\|_{\mathcal{H}_\alpha}$. Since

$$\|u\|^2_{\mathcal{H}_\alpha} = \alpha^2 \|u\|^2_{\mathcal{H}_1} + (1-\alpha^2)\|\nabla u\|^2_{L^2(\boldsymbol{B})} \quad \text{for } u \in C^1_c(\boldsymbol{B}),$$

it follows that $\|\cdot\|_{\mathcal{H}_\alpha}$ is equivalent to $\|\cdot\|_{H^1_0(\boldsymbol{B})}$ for $\alpha \in [0, 1)$, and therefore

$$\mathcal{H}_\alpha = H^1_0(\boldsymbol{B}) \quad \text{for } \alpha \in [0, 1).$$

As a consequence, we have embeddings

$$\mathcal{H}_\alpha \hookrightarrow L^p(\boldsymbol{B}) \quad \text{for } \alpha \in [0, 1),\ 1 \le p \le 2^*,$$

which are compact in the Sobolev subcritical case $p < 2^*$. The next lemma is concerned with the exceptional case $\alpha = 1$.

Lemma 4.1. *$\mathcal{H}_1$ is embedded in $L^p(\boldsymbol{B})$ for $p \in [1, 2_1^*]$. Moreover, if $1 \le p < 2_1^*$, then the embedding $\mathcal{H}_1 \hookrightarrow L^p(\boldsymbol{B})$ is compact.*

Proof. The embedding $\mathcal{H}_1 \hookrightarrow L^p(\boldsymbol{B})$ for $p \in [1, 2_1^*]$ is an immediate consequence of Theorem 3.3 and the fact that $L^p(\boldsymbol{B}) \subset L^{2_1^*}(\boldsymbol{B})$ for $p \in [1, 2_1^*]$. To prove the compactness of the embedding for fixed $p \in [1, 2_1^*)$, we let $(u_n)_n \subset \mathcal{H}_1$ be a bounded sequence. Moreover, we put $B_m := B_{1-1/m}(0) \subset \boldsymbol{B}$ for $m \ge 2$. Then $u_n^m := \mathbb{1}_{B_m} u_n$ defines a bounded sequence in $H^1(B_m)$ for every $m \ge 2$. After passing to a subsequence, $(u_n^m)_n$ converges in $L^p(B_m)$ by Rellich–Kondrachov. After passing to a diagonal sequence we may therefore assume that there exists $u \in L^p(\boldsymbol{B})$ with the property that $u_n \to u$ for $m \in \mathbb{N}$ pointwisely in $\boldsymbol{B}$. Moreover,

$$\|u - u_n\|_{L^p(\boldsymbol{B})} \le \|u - u_n\|_{L^p(B_m)} + \|u - u_n\|_{L^{2_1^*}(\boldsymbol{B}\setminus B_m)} |\boldsymbol{B} \setminus B_m|^{1/p - 1/2_1^*}.$$

Since $\|u-u_n\|_{L^{2_1^*}(\boldsymbol{B}\setminus B_m)} \le \|u-u_n\|_{L^{2_1^*}(\boldsymbol{B})}$ remains bounded independently of m and n, this gives

$$\limsup_{n\to\infty} \|u-u_n\|_{L^p(\boldsymbol{B})} \le C|\boldsymbol{B}\setminus B_m|^{1/p-1/2_1^*}$$

for some $C>0$ independent of m, where the right-hand side tends to zero as $m\to\infty$. This proves that $u_n \to u$ in $L^p(\boldsymbol{B})$. □

Remark 4.2. Let $\alpha \in [0,1]$. We first note that, for any $f \in C^1(\mathbb{R})$ such that f' is bounded and $f(0)=0$, we have $f\circ u \in \mathcal{H}_\alpha$. Indeed, recall that by the definition of $\mathcal{H}_\alpha$, there exists a sequence $(u_n)_n \subset C_c^1(\boldsymbol{B})$ such that $\|u-u_n\|_{\mathcal{H}_\alpha} \to 0$ as $n\to\infty$, and the differentiability of f readily implies $f\circ u_n \in C_c^1(\boldsymbol{B}) \subset \mathcal{H}_\alpha$ for all n. Using the chain rule and the boundedness of f', it can then be shown that $f\circ u_n \to f\circ u$ in $\mathcal{H}_\alpha$ as $n\to\infty$. Via approximation, this observation can be used to show $u^\pm, |u| \in \mathcal{H}_\alpha$, which is a classical fact in the case $\alpha<1$, where $\mathcal{H}_\alpha = H_0^1(\boldsymbol{B})$.

Definition 4.3. Let $\alpha\in[0,1]$ and $f\in L^{2_1^\sharp}(\boldsymbol{B})$, where $2_1^\sharp = 2_1^*/(2_1^*-1)$ denotes the conjugate of 2_1^*.

(i) We call $u\in\mathcal{H}_\alpha$ a weak solution of

$$-\Delta u + \alpha^2\,\partial_\theta^2 u = f \tag{4-1}$$

if

$$\langle u, v\rangle_{\mathcal{H}_\alpha} = \int_{\boldsymbol{B}} f v\,dx \quad \text{for every } v\in\mathcal{H}_\alpha. \tag{4-2}$$

(ii) We call $u\in\mathcal{H}_\alpha$ a weak supersolution of (4-1) if (4-2) holds with $\ge$ in place of $=$ for every $v\in\mathcal{H}_\alpha$, $v\ge 0$.

We have the following useful properties.

Lemma 4.4. *Let $\alpha\in[0,1]$ and $f\in L^{2_1^\sharp}(\boldsymbol{B})$.*

(i) *If $f\ge 0$ and $u\in\mathcal{H}_\alpha$ is a weak supersolution of* (4-1), *then $u\ge 0$.*

(ii) *If $f\in L^\infty(\boldsymbol{B})$ and $u\in\mathcal{H}_\alpha$ is a weak solution of* (4-1), *then $u\in C^1_{\mathrm{loc}}(\overline{\boldsymbol{B}}\setminus\gamma)\cap C(\overline{\boldsymbol{B}})$ with $u\equiv 0$ on $\partial\boldsymbol{B}$, where γ is the great circle defined in* (1-6). *Additionally, if $f\in C^\sigma_{\mathrm{loc}}(\overline{\boldsymbol{B}}\setminus\gamma)$ for some $\sigma\in(0,1)$, then $u\in C^{2,\sigma}_{\mathrm{loc}}(\overline{\boldsymbol{B}}\setminus\gamma)$.*

Moreover, $\overline{\boldsymbol{B}}\setminus\gamma$ can be replaced by $\overline{\boldsymbol{B}}$ in these statements if $\alpha<1$.

Proof. (i) Using $v=u^-=-\min\{0,u\}$ in the definition of a weak supersolution, we find that

$$-\|u^-\|^2_{\mathcal{H}_\alpha} = \langle u, v\rangle_{\mathcal{H}_\alpha} = \int_{\boldsymbol{B}} f u^-\,dx \ge 0,$$

and thus $u^-\equiv 0$.

(ii) Since the operator $-\Delta u+\alpha^2\,\partial_\theta^2$ is uniformly elliptic on $\overline{\boldsymbol{B}}$ if $\alpha\in[0,1)$ and locally uniformly elliptic on $\overline{\boldsymbol{B}}\setminus\gamma$ if $\alpha=1$, all statements follow from standard elliptic regularity theory, with the exception of the claim

$$u\in C(\overline{\boldsymbol{B}}) \qquad \text{with } u\equiv 0 \quad \text{on } \partial\boldsymbol{B}. \tag{4-3}$$

To prove (4-3), we let $c := \|f\|_{L^\infty(\boldsymbol{B})}$, and we note that $u_c \in \mathcal{H}_\alpha$ defined by $u_c(x) = c(1-|x|^2)/(2N)$ is a classical solution of

$$(-\Delta + \alpha^2\, \partial_\theta^2)u_c = -\Delta u_c = c \quad \text{in } \boldsymbol{B}, \qquad u \equiv 0 \quad \text{on } \partial \boldsymbol{B}.$$

Hence $u_c - u \in \mathcal{H}_\alpha$ is a weak supersolution of (4-1) with f replaced by $c - f \geq 0$, so $u_c - u \geq 0$ by (i). Similarly, we see that $u_c + u \geq 0$, and therefore

$$|u(x)| \leq u_c(x) = \frac{c}{2N}(1-|x|^2) \quad \text{for } x \in \boldsymbol{B}.$$

This shows the continuity of u at all points $x_0 \in \partial \boldsymbol{B}$ and that necessarily $u(x_0) = 0$. $\square$

Remark 4.5. Let $\alpha \in [0, 1]$, $V \in L^\infty(\boldsymbol{B})$, and let $u \in \mathcal{H}_\alpha$ be a weak solution of (4-1) with $f = Vu$. If u is nonnegative in $\boldsymbol{B}$, then either $u \equiv 0$, or u is strictly positive in $\boldsymbol{B}$. This follows from the strong maximum principle, since the operator $-\Delta + \alpha^2\, \partial_\theta^2 - V$ is uniformly elliptic in every compactly contained subset of $\boldsymbol{B}$.

The following proposition extends the Poincaré-type estimate for the case $p = 2$ given in Theorem 3.3. Recall again that $\lambda_1(\boldsymbol{B})$ is the first Dirichlet eigenvalue of $-\Delta$ on $\boldsymbol{B}$.

Proposition 4.6. *For* $0 \leq \alpha \leq 1$, *we have*

$$\mathscr{C}_{\alpha,0,2}(\boldsymbol{B}) = \inf_{u \in \mathcal{H}_\alpha \setminus \{0\}} \frac{\|u\|^2_{\mathcal{H}_\alpha}}{\int_{\boldsymbol{B}} u^2\, dx} = \lambda_1(\boldsymbol{B}). \tag{4-4}$$

Moreover, the minimizers are precisely the Dirichlet eigenfunctions of $-\Delta$ *on* $\boldsymbol{B}$ *corresponding to the eigenvalue* $\lambda_1(\boldsymbol{B})$ *and are therefore radial.*

Proof. Let $\alpha \in [0, 1]$. Since $\lambda_1(\boldsymbol{B}) \leq \mathscr{C}_{1,0,2}(\boldsymbol{B}) \leq \mathscr{C}_{\alpha,0,2}(\boldsymbol{B}) \leq \mathscr{C}_{0,0,2}(\boldsymbol{B}) = \lambda_1(\boldsymbol{B})$ by Theorem 3.3 and the variational characterization of $\lambda_1(\boldsymbol{B})$, we obtain (4-4). Moreover, it follows that every minimizer of (4-4) also minimizes (4-4) with $\alpha = 0$; hence it is a Dirichlet eigenfunction of $-\Delta$ on $\boldsymbol{B}$ corresponding to the eigenvalue $\lambda_1(\boldsymbol{B})$ and therefore radial. $\square$

Remark 4.7. Once the existence and compactness of the embedding $\mathcal{H}_\alpha \hookrightarrow L^2(\boldsymbol{B})$ is established, a direct proof of Proposition 4.6 without expansion in spherical harmonics, as used in the proof of inequality (3-8), can be given at least in the case $N = 2$. Indeed, one may then show by weak lower-semicontinuity of the $\mathcal{H}_\alpha$-norm that the infimum in (4-4) is attained. Moreover, by standard variational arguments, a function $u \in \mathcal{H}_\alpha \setminus \{0\}$ is a minimizer of (4-4) if and only if u is a weak solution of

$$-\Delta u + \alpha^2\, \partial_\theta^2 u = \mathscr{C}_{\alpha,0,2}(\boldsymbol{B})u. \tag{4-5}$$

Additionally, if $u \in \mathcal{H}_\alpha \setminus \{0\}$ solves (4-5), then also $|u| \in \mathcal{H}_\alpha$ is a minimizer of (4-4) and therefore a solution of (4-5). Consequently, $|u| > 0$ by Remark 4.5. Thus every weak solution of (4-5) does not change sign in $\boldsymbol{B}$, which shows that the solutions of (4-5) form a one-dimensional subspace of $\mathcal{H}_\alpha$. Combining this information with the fact that (4-5) remains invariant under transformations $A \in O(2) \times O(N-2)$, we conclude that the (up to a factor unique) solution u of (4-5) must satisfy $u \circ A = u$ for every $A \in O(2) \times O(N-2)$. In the case $N = 2$ this implies that u is radial. Thus u is a Dirichlet eigenfunction of $-\Delta$ corresponding to the eigenvalue $\mathscr{C}_{\alpha,0,2}(\boldsymbol{B})$. Since u does not change sign, it must correspond to the first Dirichlet eigenvalue of $-\Delta$ on $\boldsymbol{B}$, so a posteriori we conclude that $\mathscr{C}_{\alpha,0,2}(\boldsymbol{B}) = \lambda_1(\boldsymbol{B})$.

Corollary 4.8. *Let $\alpha \in [0, 1]$ and $m \in \mathbb{R}$.*

(i) *We have $\mathscr{C}_{\alpha,m,2}(\boldsymbol{B}) = \mathscr{C}_{0,m,2}(\boldsymbol{B}) = \lambda_1(\boldsymbol{B}) + m$.*

(ii) *If $m > -\lambda_1(\boldsymbol{B})$ and $2 \le p \le 2^*$, we have $R_{\alpha,m,p}(u) > 0$ for $u \in \mathcal{H}_\alpha \setminus \{0\}$.*

Here we recall that the quantities $R_{\alpha,m,p}(u)$ and $\mathscr{C}_{\alpha,m,p}(\boldsymbol{B})$ have been defined in (1-7) and (1-8).

Proof. (i) This follows immediately from Proposition 4.6.

(ii) Since $m \ge -\lambda_1(\boldsymbol{B})$, we have, by (i), for $u \in \mathcal{H}_\alpha \setminus \{0\}$,

$$R_{\alpha,m,p}(u) = R_{\alpha,m,2}(u)\frac{\|u\|^2_{L^2(\boldsymbol{B})}}{\|u\|^2_{L^p(\boldsymbol{B})}} \ge \mathscr{C}_{\alpha,m,2}(\boldsymbol{B})\frac{\|u\|^2_{L^2(\boldsymbol{B})}}{\|u\|^2_{L^p(\boldsymbol{B})}} = (\lambda_1(\boldsymbol{B}) + m)\frac{\|u\|^2_{L^2(\boldsymbol{B})}}{\|u\|^2_{L^p(\boldsymbol{B})}} > 0. \qquad \square$$

Definition 4.9. Let $m > -\lambda_1(\boldsymbol{B})$, $p \in (2, 2_1^*]$ and $\alpha \in [0, 1]$.

(i) We call $u \in \mathcal{H}_\alpha$ a weak solution of (1-5) if u is a weak solution of (4-1), with $f = |u|^{p-2}u - mu$.

(ii) A weak solution $u \in \mathcal{H}_\alpha \setminus \{0\}$ of (1-5) will be called a ground state solution if u is a minimizer for $R_{\alpha,m,p}$, i.e., we have $R_{\alpha,m,p}(u) = \mathscr{C}_{\alpha,m,p}(\boldsymbol{B})$.

Lemma 4.10. *Let $0 \le \alpha \le 1$, $2 < p \le 2^*$ and $m > -\lambda_1(\boldsymbol{B})$, and let $u \in \mathcal{H}_\alpha$ be a weak solution of* (1-5).

(i) *If $\alpha < 1$, then $u \in C^{2,\sigma}(\overline{\boldsymbol{B}})$ with $u \equiv 0$ on $\partial\boldsymbol{B}$ for all $\sigma \in (0, 1)$.*

(ii) *If $\alpha = 1$ and $2 \le p < 2_1^*$, then $u \in C^{2,\sigma}_{\mathrm{loc}}(\overline{\boldsymbol{B}} \setminus \gamma) \cap C(\overline{\boldsymbol{B}})$ for all $\sigma \in (0, 1)$ with $u \equiv 0$ on $\partial\boldsymbol{B}$, where γ is the great circle defined in* (1-6).

(iii) *If u is a ground state solution, then u does not change sign in $\boldsymbol{B}$.*

Proof. The regularity results in (i) and (ii) follow immediately from Lemma 4.4 once we have shown that $u \in L^\infty(\boldsymbol{B})$. This is a well-known consequence of classical Moser iteration in the case $\alpha < 1$, which can be performed similarly also in the case $\alpha = 1$; see Lemma A.1 in Appendix A for a detailed proof.

It thus remains to prove (iii). If u is a ground state solution, then

$$R_{\alpha,m,p}(|u|) = R_{\alpha,m,p}(u) = \mathscr{C}_{\alpha,m,p}(\boldsymbol{B}),$$

and therefore $|u|$ is also a ground state solution of (1-5). By Remark 4.5 and since $u \not\equiv 0$, it follows that $|u| > 0$ in $\boldsymbol{B}$, so u does not change sign in $\boldsymbol{B}$. $\square$

Lemma 4.11. *Let $0 \le \alpha \le 1$, $2 \le p < 2^*$ and $m > -\lambda_1(\boldsymbol{B})$. If*

$$\alpha < 1 \qquad \text{or} \qquad \alpha = 1 \quad \text{and} \quad p < 2_1^*, \tag{4-6}$$

then

$$\mathscr{C}_{\alpha,m,p}(\boldsymbol{B}) > 0, \tag{4-7}$$

and this value is attained in $\mathcal{H}_\alpha \setminus \{0\}$. Moreover, up to multiplication by a positive constant, all minimizers are ground state solutions of (1-5), *so they have the properties in Lemma 4.10.*

Proof. We first note that, following from Proposition 4.6, the quadratic form

$$u \mapsto \int_{\boldsymbol{B}} (|\nabla u|^2 - \alpha^2 |\partial_\theta u|^2 + m|u|^2)\, dx$$

is positive on $\mathcal{H}_\alpha \setminus \{0\}$, so it is weakly lower-semicontinuous on $\mathcal{H}_\alpha$. Moreover, by the assumption (4-6), Sobolev embeddings (in the case $\alpha < 1$) and Lemma 4.1 (in the case $\alpha = 1$), the embedding $\mathcal{H}_\alpha \hookrightarrow L^p(\boldsymbol{B})$ is compact. Hence, by a standard analysis of minimizing sequences, the value $\mathscr{C}_{\alpha,m,p}(\boldsymbol{B})$ is attained in $\mathcal{H}_\alpha \setminus \{0\}$, and thus it is positive. Moreover, standard variational arguments show that every L^p-normalized minimizer u_0 must be a weak solution of

$$-\Delta u + \alpha^2\, \partial_\theta^2 u + mu = \mathscr{C}_{\alpha,m,p}(\boldsymbol{B})|u|^{p-2}u \quad \text{in } \boldsymbol{B}. \tag{4-8}$$

We then conclude that $[\mathscr{C}_{\alpha,m,p}(\boldsymbol{B})]^{1/(p-2)}u_0$ weakly solves (1-5). □

Next, we treat the critical case $p = 2_1^*$. We first show that $\mathscr{C}_{1,m,2_1^*}(\boldsymbol{B})$ is attained, provided it is small enough, as stated in Theorem 1.7 (i). The strategy of the proof is inspired by [Frank et al. 2018] and requires the following characterization of sequences in $\mathcal{H}_1$.

Lemma 4.12. *Let*

$$Z(v) := \int_{\boldsymbol{B}} (|\nabla v|^2 - |\partial_\theta v|^2 + mv^2)\, dx \quad \textit{for } v \in \mathcal{H}_1$$

and

$$N(v) := \int_{\boldsymbol{B}} |v|^{2_1^*}\, dx \qquad \textit{for } v \in \mathcal{H}_1.$$

Then we have

$$2^{1/2-1/2_1^*}\mathcal{S}_1(\mathbb{R}^N_+) \le \inf\bigl\{\liminf_{n\to\infty} Z(w_n) : (w_n)_n \subset \mathcal{H},\ N(w_n) = 1,\ w_n \rightharpoonup 0 \textit{ in } \mathcal{H}_1\bigr\}.$$

Proof. Let $(w_n)_n \subset \mathcal{H}_1$ such that $N(w_n) = 1,\ w_n \rightharpoonup 0$ in $\mathcal{H}_1$. Let $\varepsilon > 0$, and choose $U_0, \dots, U_m \subset \boldsymbol{B}$ as in the proof of Theorem 3.3, so that

$$\boldsymbol{B} \subset \bigcup_{k=0}^m U_k.$$

We may then choose functions $\eta_0, \dots, \eta_m \in C_c^2(\boldsymbol{B})$ such that $\operatorname{supp} \eta_k \subset U_k$ and $\sum_{k=0}^m \eta_k^2 \equiv 1$ on $\boldsymbol{B}$. Then

$$\begin{aligned}\int_{\boldsymbol{B}} (|\nabla(\eta_k w_n)|^2 - |\partial_\theta(\eta_k w_n)|^2)\, dx &= \int_{\boldsymbol{B}} (\eta_k^2 |\nabla w_n|^2 + 2 w_n \eta_k \nabla w_n \cdot \nabla \eta_k + w_n^2 |\nabla \eta_k|^2)\, dx \\ &\quad - \int_{\boldsymbol{B}} (\eta_k^2 |\partial_\theta w_n|^2 + 2 w_n \eta_k\, \partial_\theta w_n \cdot \partial_\theta \eta_k + w_n^2 |\partial_\theta \eta_k|^2)\, dx,\end{aligned}$$

and thus

$$\int_{\boldsymbol{B}} (|\nabla w_n|^2 - |\partial_\theta w_n|^2 + m w_n^2)\, dx \ge \sum_{k=0}^m \int_{\boldsymbol{B}} (|\nabla(\eta_k w_n)|^2 - |\partial_\theta(\eta_k w_n)|^2)\, dx - C \int_{\boldsymbol{B}} w_n^2\, dx$$

with a constant $C > 0$ independent of n. Here we used the fact that the mixed terms can be estimated as follows:

$$\int_{\boldsymbol{B}} w_n^2(|\nabla\eta_k|^2 - |\partial_\theta\eta_k|^2)\,dx \le 2 \sup_{k\in\{0,\dots,m\}} \|\nabla\eta_k\|_\infty^2 \int_{\boldsymbol{B}} w_n^2\,dx,$$

$$\begin{aligned}\int_{\boldsymbol{B}} \eta_k w_n(\nabla w_n\cdot\nabla\eta_k - \partial_\theta w_n\,\partial_\theta\eta_k)\,dx &\le \int_{\boldsymbol{B}} \eta_k w_n^2|-\Delta\eta_k + \partial_\theta^2\eta_k|\,dx\\ &\le \sup_{k\in\{0,\dots,m\}} \|-\Delta\eta_k+\partial_\theta^2\eta_k\|_\infty \int_{\boldsymbol{B}} |w_n|^2\,dx.\end{aligned}$$

We first note that $w_n \to 0$ in $L^2(\boldsymbol{B})$ since the embedding $\mathcal{H} \hookrightarrow L^2(\boldsymbol{B})$ is compact by Lemma 4.1. Moreover, it is easy to see that $\|\cdot\|_{\mathcal{H}_1}$ induces an equivalent norm on $H_0^1(U_0)$, which implies that $\eta_0 w_n \rightharpoonup 0$ in $H_0^1(U_0)$. In particular, noting that by $2_1^* < 2^*$ the classical Rellich–Kondrachov theorem implies $\eta_0 w_n \to 0$ in $L^{2_1^*}(\boldsymbol{B})$, we conclude

$$\liminf_{n\to\infty}\int_{\boldsymbol{B}}(|\nabla(\eta_0 w_n)|^2 - |\partial_\theta(\eta_0 w_n)|^2 + m(\eta_0 w_n)^2)\,dx \ge (1-\varepsilon)2^{1/2-1/2_1^*}\mathcal{S}_1(\mathbb{R}_+^N)\lim_{n\to\infty}\left(\int_{\boldsymbol{B}}|\eta_0 w_n|^{2_1^*}\,dx\right)^{2/2_1^*}$$

since the limit on the right-hand side is zero. On the other hand, Lemma 3.2 gives

$$\int_{\boldsymbol{B}}(|\nabla(\eta_k w_n)|^2 - |\partial_\theta(\eta_k w_n)|^2)\,dx \ge (1-\varepsilon)2^{1/2-1/2_1^*}\mathcal{S}_1(\mathbb{R}_+^N)\left(\int_{\boldsymbol{B}}|\eta_k w_n|^{2_1^*}\,dx\right)^{2/2_1^*}$$

for $k = 1,\dots,m$, and hence

$$\begin{aligned}&\liminf_{n\to\infty}\int_{\boldsymbol{B}}(|\nabla w_n|^2 - |\partial_\theta w_n|^2 + m w_n^2)\,dx\\
&\quad\ge \liminf_{n\to\infty}\sum_{k=0}^m\int_{\boldsymbol{B}}(|\nabla(\eta_k w_n)|^2 - |\partial_\theta(\eta_k w_n)|^2)\,dx\\
&\quad\ge (1-\varepsilon)2^{1/2-1/2_1^*}\mathcal{S}_1(\mathbb{R}_+^N)\liminf_{n\to\infty}\sum_{k=0}^m\left(\int_{\boldsymbol{B}}|\eta_k w_n|^{2_1^*}\,dx\right)^{2/2_1^*}\\
&\quad= (1-\varepsilon)2^{1/2-1/2_1^*}\mathcal{S}_1(\mathbb{R}_+^N)\liminf_{n\to\infty}\sum_{k=0}^m\|\eta_k^2 w_n^2\|_{2_1^*/2} \ge (1-\varepsilon)2^{1/2-1/2_1^*}\mathcal{S}_1(\mathbb{R}_+^N)\liminf_{n\to\infty}\left\|\sum_{k=0}^m\eta_k^2 w_n^2\right\|_{2_1^*/2}\\
&\quad= (1-\varepsilon)2^{1/2-1/2_1^*}\mathcal{S}_1(\mathbb{R}_+^N)\liminf_{n\to\infty}\|w_n\|_{2_1^*/2} = (1-\varepsilon)2^{1/2-1/2_1^*}\mathcal{S}_1(\mathbb{R}_+^N).\end{aligned}$$

Since $\varepsilon > 0$ was arbitrary, we conclude that

$$\liminf_{n\to\infty}\int_{\boldsymbol{B}}(|\nabla w_n|^2 - |\partial_\theta w_n|^2 + m w_n^2)\,dx \ge 2^{1/2-1/2_1^*}\mathcal{S}_1(\mathbb{R}_+^N). \qquad \square$$

We may now complete the proof of our main result.

Proof of Theorem 1.7 (i). Consider a minimizing sequence $(u_n)_n \subset \mathcal{H}_1$ for $\mathscr{C}_{1,m,2_1^*}(\boldsymbol{B})$ with $\|u_n\|_{2_1^*} = 1$. Then $(u_n)_n$ is bounded in $\mathcal{H}_1$; hence, after passing to a subsequence, we may assume $u_n \rightharpoonup u_0$ in $\mathcal{H}_1$. We set $v_n := u_n - u_0$ and note that, by Lemma 4.1,

$$v_n \to 0 \quad \text{in } L^q(\boldsymbol{B})$$

for $1 \le q < 2_1^*$. Moreover, we may pass to a subsequence such that $u_n \to u$ almost everywhere. Weak convergence implies

$$\mathscr{C}_{1,m,2_1^*}(\boldsymbol{B}) = \lim_{n\to\infty} Z(u_n) = Z(u_0) + \lim_{n\to\infty} Z(v_n),$$

whereas the Brezis–Lieb lemma yields

$$1 = N(u_n) = N(u_0) + N(v_n) + o(1).$$

In particular, the limits $T := \lim_{n\to\infty} N(v_n)$ and $M := \lim_{n\to\infty} Z(v_n)$ exist, and

$$M \ge 2^{1/2-1/2_1^*} \mathcal{S}_1(\mathbb{R}_+^N) T^{2/2_1^*}.$$

Indeed, this is trivial if $T = 0$ and follows from Lemma 4.12 in the case $T > 0$. Moreover, by definition we have

$$Z(u_0) \ge \mathscr{C}_{1,m,2_1^*}(\boldsymbol{B}) N(u_0)^{2/2_1^*}. \tag{4-9}$$

Hence

$$\begin{aligned}
\mathscr{C}_{1,m,2_1^*}(\boldsymbol{B}) &= Z(u_0) + M \ge Z(u_0) + 2^{1/2-1/2_1^*} \mathcal{S}_1(\mathbb{R}_+^N) T^{2/2_1^*} \\
&\ge Z(u_0) + (2^{1/2-1/2_1^*} \mathcal{S}_1(\mathbb{R}_+^N) - \mathscr{C}_{1,m,2_1^*}(\boldsymbol{B})) T^{2/2_1^*} + \mathscr{C}_{1,m,2_1^*}(\boldsymbol{B})(1 - N(u_0))^{2/2_1^*} \\
&\ge Z(u_0) + (2^{1/2-1/2_1^*} \mathcal{S}_1(\mathbb{R}_+^N) - \mathscr{C}_{1,m,2_1^*}(\boldsymbol{B})) T^{2/2_1^*} + \mathscr{C}_{1,m,2_1^*}(\boldsymbol{B}) - \mathscr{C}_{1,m,2_1^*}(\boldsymbol{B}) N(u_0)^{2/2_1^*} \\
&\ge (2^{1/2-1/2_1^*} \mathcal{S}_1(\mathbb{R}_+^N) - \mathscr{C}_{1,m,2_1^*}(\boldsymbol{B})) T^{2/2_1^*} + \mathscr{C}_{1,m,2_1^*}(\boldsymbol{B}),
\end{aligned} \tag{4-10}$$

where we used the inequality $(a-b)^\tau \ge a^\tau - b^\tau$ for $a \ge b \ge 0$ and $0 \le \tau \le 1$. It follows that

$$(2^{1/2-1/2_1^*} \mathcal{S}_1(\mathbb{R}_+^N) - \mathscr{C}_{1,m,2_1^*}(\boldsymbol{B})) T^{2/2_1^*} \le 0.$$

We assumed $2^{1/2-1/2_1^*} \mathcal{S}_1(\mathbb{R}_+^N) - \mathscr{C}_{1,m,2_1^*}(\boldsymbol{B}) > 0$, so we must have $T = 0$. Hence $N(u_0) = 1$, and therefore $Z(u_0) = \mathscr{C}_{1,m,2_1^*}(\boldsymbol{B})$ by (4-9) and the second line of (4-10), which implies that u_0 is a minimizer. □

We note the following consequence of Theorem 1.7 (i), which extends (4-7) to the critical case.

Corollary 4.13. *We have $\mathscr{C}_{1,m,2_1^*}(\boldsymbol{B}) > 0$ for $m > -\lambda_1(\boldsymbol{B})$.*

Proof. If the value $\mathscr{C}_{1,m,2_1^*}(\boldsymbol{B})$ is attained in $\mathcal{H} \setminus \{0\}$, then we have $\mathscr{C}_{1,m,2_1^*}(\boldsymbol{B}) > 0$ by Corollary 4.8 (ii). If not, we have $\mathscr{C}_{1,m,2_1^*}(\boldsymbol{B}) \ge 2^{1/2-1/2_1^*} \mathcal{S}_1(\mathbb{R}_+^N) > 0$ by Theorem 1.7 (i) and Theorem 1.6. □

In general, the existence of ground state solutions in the case $\alpha = 1$, $p = 2_1^*$ remains an open problem and might depend on the parameter $m > -\lambda_1(\boldsymbol{B})$. We have the following partial existence result in the critical case.

Theorem 4.14. *There exists $\varepsilon > 0$ such that, for $m \in (-\lambda_1(\boldsymbol{B}), -\lambda_1(\boldsymbol{B}) + \varepsilon)$, there exists $u_0 \in \mathcal{H} \setminus \{0\}$ such that*

$$R_{1,m,2_1^*}(u_0) = \inf_{u \in \mathcal{H}\setminus\{0\}} R_{1,m,2_1^*}(u),$$

i.e., u_0 minimizes $R_{1,m,2_1^*}$. Furthermore, after multiplication by a positive constant, u_0 is a weak solution of

$$-\Delta u + \partial_\theta^2 u + mu = |u|^{2_1^*-2} u \quad \text{in } \boldsymbol{B}.$$

Proof. For a (necessarily radial) eigenfunction φ_1 of $-\Delta$ on $\boldsymbol{B}$ corresponding to $\lambda_1(\boldsymbol{B})$, we have

$$\mathscr{C}_{1,m,2_1^*}(\boldsymbol{B}) \le R_{1,m,2_1^*}(\varphi_1) = \frac{(\lambda_1(\boldsymbol{B})+m)\int_{\boldsymbol{B}} \varphi_1^2\,dx}{\left(\int_{\boldsymbol{B}} |\varphi_1|^{2_1^*}\,dx\right)^{2/2_1^*}},$$

which implies $\mathscr{C}_{1,m,2_1^*}(\boldsymbol{B}) \to 0$ as $m \to -\lambda_1(\boldsymbol{B})^+$. In particular, it follows that there exists $\varepsilon > 0$ such that

$$\mathscr{C}_{1,m,2_1^*}(\boldsymbol{B}) < 2^{1/2-1/2_1^*}\mathcal{S}_1(\mathbb{R}^N_+)$$

holds for $m \in (-\lambda_1(\boldsymbol{B}), -\lambda_1(\boldsymbol{B})+\varepsilon)$. By Theorem 1.7 (i), this finishes the proof. □

Note that this implies Theorem 1.7 (ii).

4.2. *Radiality versus (x_1, x_2)-nonradiality of ground state solutions.* As outlined in the introduction, it is in general difficult to decide whether ground states of (1-5) are (x_1, x_2)-nonradial or not. In this section, we follow several approaches to this problem.

The first approach is based on the continuity of the ground state energy and the sufficient condition (1-11) to make use of the results of Section 3 for the endpoint case $\alpha = 1$. To this end, we recall that, in the case $m \ge 0$, by classical results due to [Kwong 1989; Kwong and Zhang 1991; McLeod and Serrin 1987] (see also [Damascelli et al. 1999]), the problem

$$\begin{cases} -\Delta u + mu = |u|^{p-2}u & \text{in } \boldsymbol{B}, \\ u = 0 & \text{on } \partial\boldsymbol{B}, \end{cases} \tag{4-11}$$

has a unique radial positive solution $u_{\mathrm{rad}} \in H_0^1(\boldsymbol{B})$, which is a minimizer for $\mathscr{C}_{0,m,p}(\boldsymbol{B})$. Clearly, u_{rad} is also a weak solution of (1-5) for every $\alpha > 0$, but it might not be a ground state solution, as we see next.

Lemma 4.15. *Let $2 < p < 2^*$ and $m > -\lambda_1(\boldsymbol{B})$ be fixed.*

(i) *The map*

$$[0, 1] \to \mathbb{R}, \quad \alpha \mapsto \mathscr{C}_{\alpha,m,p}(\boldsymbol{B})$$

is continuous and nonincreasing.

(ii) *Let $\alpha \in (0, 1]$, and suppose that $p \le 2_1^*$ in the case $\alpha = 1$. Then the following properties are equivalent*:

- (ii)$_1$ $\mathscr{C}_{\alpha,m,p}(\boldsymbol{B}) < \mathscr{C}_{0,m,p}(\boldsymbol{B})$.
- (ii)$_2$ *Every ground state solution of* (1-5) *is (x_1, x_2)-nonradial.*

Proof. (i) The monotonicity of $\mathscr{C}_{\alpha,m,p}(\boldsymbol{B})$ in α follows immediately from the definition. In order to prove continuity, we first consider $\alpha_0 \in (0, 1]$ and let $\varepsilon > 0$. Moreover, we let $u_0 \in H_0^1(\boldsymbol{B}) \setminus \{0\}$ be a function with $R_{\alpha_0,m,p}(u_0) < \mathscr{C}_{\alpha_0,m,p}(\boldsymbol{B}) + \varepsilon$. For $\alpha \le \alpha_0$, we then have

$$\begin{aligned} \mathscr{C}_{\alpha_0,m,p}(\boldsymbol{B}) \le \mathscr{C}_{\alpha,m,p}(\boldsymbol{B}) \le R_{\alpha,m,p}(u_0) &\le R_{\alpha_0,m,p}(u_0) + (\alpha_0^2-\alpha^2)\frac{\int_{\boldsymbol{B}} |\partial_\theta u_0|^2\,dx}{\left(\int_{\boldsymbol{B}} |u_0|^p\,dx\right)^{2/p}} \\ &\le \mathscr{C}_{\alpha_0,m,p}(\boldsymbol{B}) + \varepsilon + (\alpha_0^2-\alpha^2)\frac{\int_{\boldsymbol{B}} |\partial_\theta u_0|^2\,dx}{\left(\int_{\boldsymbol{B}} |u_0|^p\,dx\right)^{2/p}}, \end{aligned}$$

which implies that $\limsup_{\alpha\to\alpha_0^-} |\mathscr{C}_{\alpha,m,p}(\boldsymbol{B}) - \mathscr{C}_{\alpha_0,m,p}(\boldsymbol{B})| \le \varepsilon$. This shows continuity from the left in α_0.

Next we let $\alpha_0 \in [0, 1)$ and show continuity from the right in α_0. For this we fix $\delta > 0$ such that $(\alpha_0, \alpha_0 + \delta) \subset (0, 1)$. For $\alpha \in (\alpha_0, \alpha_0 + \delta)$, Lemma 4.11 implies that the value $\mathscr{C}_{\alpha,m,p}(\boldsymbol{B})$ is attained at a function $u_\alpha \in H_0^1(\boldsymbol{B}) \setminus \{0\}$ with $\int_{\boldsymbol{B}} |u_\alpha|^p \, dx = 1$. Therefore

$$\begin{aligned}\mathscr{C}_{\alpha_0,m,p}(\boldsymbol{B}) \geq \mathscr{C}_{\alpha,m,p}(\boldsymbol{B}) = R_{\alpha,m,p}(u_\alpha) &= R_{\alpha_0,m,p}(u_\alpha) + (\alpha_0^2 - \alpha^2) \int_{\boldsymbol{B}} |\partial_\theta u_\alpha|^2 \, dx \\ &\geq \mathscr{C}_{\alpha_0,m,p}(\boldsymbol{B}) - |\alpha_0^2 - \alpha^2| \int_{\boldsymbol{B}} |\nabla u_\alpha|^2 \, dx,\end{aligned}$$

whence, using the fact that

$$(1 - \alpha^2) \int_{\boldsymbol{B}} |\nabla u_\alpha|^2 \, dx \leq \int_{\boldsymbol{B}} (|\nabla u_\alpha|^2 - \alpha^2 |\partial_\theta u_\alpha|^2) \, dx = \mathscr{C}_{\alpha,m,p}(\boldsymbol{B}) \leq \mathscr{C}_{0,m,p}(\boldsymbol{B}),$$

we conclude

$$\begin{aligned}\mathscr{C}_{\alpha_0,m,p}(\boldsymbol{B}) \geq \mathscr{C}_{\alpha,m,p}(\boldsymbol{B}) &\geq \mathscr{C}_{\alpha_0,m,p}(\boldsymbol{B}) - \frac{|\alpha_0^2 - \alpha^2|}{1 - \alpha^2} \mathscr{C}_{0,m,p}(\boldsymbol{B}) \\ &\geq \mathscr{C}_{\alpha_0,m,p}(\boldsymbol{B}) - \frac{|\alpha_0^2 - \alpha^2|}{1 - (\alpha_0 + \delta)^2} \mathscr{C}_{0,m,p}(\boldsymbol{B}).\end{aligned}$$

This shows continuity from the right in α_0.

(ii) As noted above, $\mathscr{C}_{0,m,p}(\boldsymbol{B})$ is attained by a radial positive solution u_{rad} of (4-11), and we have $R_{0,m,p}(u_{\mathrm{rad}}) = R_{\alpha,m,p}(u_{\mathrm{rad}})$. Hence, if $\mathscr{C}_{0,m,p}(\boldsymbol{B}) = \mathscr{C}_{\alpha,m,p}(\boldsymbol{B})$, then u_{rad} is also a radial ground state solution of (1-5). Hence (ii)$_2$ and (i) imply that $\mathscr{C}_{\alpha,m,p}(\boldsymbol{B}) < \mathscr{C}_{0,m,p}(\boldsymbol{B})$. If, conversely, there exists a radial ground state solution u of (1-5), then we have

$$\mathscr{C}_{0,m,p}(\boldsymbol{B}) \leq R_{0,m,p}(u) = R_{\alpha,m,p}(u) = \mathscr{C}_{\alpha,m,p}(\boldsymbol{B}),$$

and therefore equality holds by (i). Consequently, the inequality $\mathscr{C}_{\alpha,m,p}(\boldsymbol{B}) < \mathscr{C}_{0,m,p}(\boldsymbol{B})$ implies that every ground state solution of (1-5) is (x_1, x_2)-nonradial. □

We now turn to the proof of Theorem 1.3, which yields radiality of ground state solutions for α close to zero. This essentially relies on the implicit function theorem and the fact that the case $\alpha = 0$ corresponds to the classical nonlinear Schrödinger equation (4-11), where nondegeneracy results are available.

Proof of Theorem 1.3. We fix $m \geq 0$ and $2 < p < 2^*$. Moreover, we consider a sequence of numbers $\alpha_n \in (0, 1)$, $\alpha_n \to 0$, and, for every $n \in \mathbb{N}$, a positive ground state solution $u_n \in H_0^1(\boldsymbol{B})$ of (1-5) with $\alpha = \alpha_n$. Recall that the existence of u_n is proved in Lemma 4.11. In order to prove the theorem, it then suffices to show that

$$u_n = u_{\mathrm{rad}} \quad \text{for } n \text{ sufficiently large,} \tag{4-12}$$

where u_{rad} is the unique positive solution of (4-11).

Step 1: We claim that

$$u_n \to u_{\mathrm{rad}} \quad \text{in } H_0^1(\boldsymbol{B}) \quad \text{as } n \to \infty. \tag{4-13}$$

To this end, we put $v_n := u_n / \|u_n\|_{L^p(\boldsymbol{B})}$, so v_n is an L^p-normalized minimizer for $\mathscr{C}_{\alpha_n,m,p}(\boldsymbol{B})$. Then $(v_n)_n$ is bounded in $H_0^1(\boldsymbol{B})$ by definition of $\mathscr{C}_{\alpha_n,m,p}(\boldsymbol{B})$. Consequently, we have $v_n \rightharpoonup v_0$ in $H_0^1(\boldsymbol{B})$ after passing to a subsequence, which implies that $v_n \to v_0$ in $L^p(\boldsymbol{B})$, and therefore $\int_{\boldsymbol{B}} |v_0|^p\,dx = 1$. We show that v_0 is a minimizer for $\mathscr{C}_{0,m,p}(\boldsymbol{B})$. Indeed, by weak lower-semicontinuity, we have

$$\begin{aligned}\mathscr{C}_{0,m,p}(\boldsymbol{B}) \le R_{0,m,p}(v_0) \le \liminf_{n\to\infty} R_{0,m,p}(v_n) &= \lim_{n\to\infty} \big(R_{\alpha_n,m,p}(v_n) + \alpha_n^2 \|\partial_\theta v_n\|^2_{L^2(\boldsymbol{B})}\big)\\ &\le \lim_{n\to\infty} \big(\mathscr{C}_{\alpha_n,m,p}(\boldsymbol{B}) + \alpha_n^2 \|v_n\|^2_{H^1(\boldsymbol{B})}\big) = \mathscr{C}_{0,m,p}(\boldsymbol{B}),\end{aligned}$$

where we used Lemma 4.15 in the last step. Hence v_0 is a minimizer of $\mathscr{C}_{0,m,p}(\boldsymbol{B})$, and a posteriori we find that

$$\begin{aligned}\|\nabla v_n\|^2_{L^2(\boldsymbol{B})} + m\|v_n\|^2_{L^2(\boldsymbol{B})} &= R_{\alpha_n,m,p}(v_n) + \alpha_n^2 \|\partial_\theta v_n\|^2_{L^2(\boldsymbol{B})}\\ &\to R_{0,m,p}(v_0) = \|\nabla v_0\|^2_{L^2(\boldsymbol{B})} + m\|v_0\|^2_{L^2(\boldsymbol{B})} \quad \text{as } n\to\infty.\end{aligned}$$

By uniform convexity of $H^1(\boldsymbol{B})$, we thus conclude that $v_n \to v_0$ in $H^1_0(\boldsymbol{B})$. Next we recall that, as noted in the proof of Lemma 4.11, we have

$$u_n := [\mathscr{C}_{\alpha_n,m,p}(\boldsymbol{B})]^{1/(p-2)} v_n \quad \text{and, by uniqueness,} \quad u_{\mathrm{rad}} = [\mathscr{C}_{\alpha_n,m,p}(\boldsymbol{B})]^{1/(p-2)} v_0.$$

Hence Lemma 4.15 implies that $u_n \to u_{\mathrm{rad}}$ in $H^1_0(\boldsymbol{B})$. Although we have proved this only after passing to a subsequence, the convergence of the full sequence now follows from the uniqueness of u_{rad} and yields (4-13).

Step 2: Next, we improve this convergence by noting that

$$u_n \to u_{\mathrm{rad}} \quad \text{in } H^2(\boldsymbol{B}). \tag{4-14}$$

This follows in a standard way from (4-13) and standard elliptic regularity theory (see, e.g., [Gilbarg and Trudinger 1977, Theorem 8.12]) since $w_n = u_{\mathrm{rad}} - u_n \in H^1_0(\boldsymbol{B})$ is a weak solution of

$$\begin{cases} -\Delta w_n + \alpha_n^2\,\partial_\theta^2 w_n + m w_n = |v_{\mathrm{rad}}|^{p-2} v_{\mathrm{rad}} - |v_n|^{p-2} v_n & \text{in } \boldsymbol{B},\\ w_n = 0 & \text{on } \partial\boldsymbol{B},\end{cases}$$

and the coefficients of the differential operator $-\Delta + \alpha_n^2\,\partial_\theta$ are uniformly bounded and elliptic in $n \in \mathbb{N}$.

Step 3 (conclusion): To complete the proof of (4-12), we consider the map

$$F : (-1,1) \times H^2(\boldsymbol{B}) \cap H^1_0(\boldsymbol{B}) \to L^2(\boldsymbol{B}), \qquad F(\alpha,u) := -\Delta u + \alpha^2\,\partial_\theta^2 u + mu - |u|^{p-2}u,$$

and we note that weak solutions of (1-5) correspond to zeroes of F. We also note that $F(\alpha, u_{\mathrm{rad}}) = 0$ for all α. We wish to apply the implicit function theorem at $(0, u_{\mathrm{rad}})$, so we need to check that

$$[\partial_u F](0, u_{\mathrm{rad}}) = -\Delta + m - (p-1)|u_{\mathrm{rad}}|^{p-2}$$

is invertible as a map $H^2(\boldsymbol{B}) \cap H^1_0(\boldsymbol{B}) \to L^2(\boldsymbol{B})$. This is equivalent to the nondegeneracy of u_{rad} as a solution of (4-11) which is noted, e.g., in [Damascelli et al. 1999, Theorem 4.2] for $m = 0$ and in [Aftalion and Pacella 2003, Theorem 1.1] in the case $m > 0$. Now the implicit function theorem yields $\varepsilon > 0$ with the following property: if $u \in H^2(\boldsymbol{B}) \cap H^1_0(\boldsymbol{B})$ satisfies $\|u - u_{\mathrm{rad}}\|_{H^2(\boldsymbol{B})} < \varepsilon$ and $F(\alpha, u) = 0$ for some $\alpha \in (-\varepsilon, \varepsilon)$, then $u = u_{\mathrm{rad}}$.

Hence (4-14) implies that $u_n = u_{\mathrm{rad}}$ for n sufficiently large, which shows (4-12), as claimed. $\square$

Remark 4.16. In a similar way, the following result can be shown: *for fixed $\alpha \in (0,1)$ and $m \geq 0$, there exists $p_0 > 2$ with the property that, for $2 \leq p \leq p_0$, there exists a unique positive L^p-normalized minimizer for $\mathscr{C}_{\alpha,m,p}(\boldsymbol{B})$ in $\mathcal{H}_\alpha$ which is a radial function.*

To see this, we first show, similar to the proof of Lemma 4.15, that the map

$$[2, 2^*) \to \mathbb{R}, \quad p \mapsto \mathscr{C}_{\alpha,m,p}(\boldsymbol{B}),$$

is continuous. Then we argue by contradiction again and assume that, for some sequence of numbers $p_n > 2$ with $p_n \to 2$ as $n \to \infty$, there exists nonradial minimizers $u_n \in \mathcal{H}_\alpha$ for $\mathscr{C}_{\alpha,m,p}(\boldsymbol{B})$ with $\|u_n\|_{L^{p_n}(\boldsymbol{B})} = 1$ for $n \in \mathbb{N}$. Similar to the proof of Theorem 1.3 above, one can then show with the help of Proposition 4.6 that, after passing to a subsequence,

$$u_n \to u_* \text{ in } \mathcal{H}_\alpha \quad \text{and pointwisely a.e. on } \boldsymbol{B} \quad \text{as } n \to \infty, \tag{4-15}$$

where u_* is the unique L^2-normalized positive eigenfunction of the Dirichlet Laplacian on $\boldsymbol{B}$. The nonradiality of u_n then allows us to distinguish two cases.

Case 1: After passing to a subsequence, u_n is (x_1, x_2)-nonradial for every $n \in \mathbb{N}$.

Case 2: After passing to a subsequence, there exists, for every $n \in \mathbb{N}$, a reflection σ_n at a hyperplane containing the (x_1, x_2)-plane with $u_n \neq \tilde{u}_n$, where $\tilde{u}_n := u_n \circ \sigma_n$.

We then define $v_n : \boldsymbol{B} \to \mathbb{R}$ by $v_n := \partial_\theta u_n$ in Case 1 and $v_n = \tilde{u}_n - u_n$ in Case 2. Then $v_n \in \mathcal{H}_\alpha = H_0^1(\boldsymbol{B})$ by Lemma 4.10 (i) since $\alpha < 1$. Moreover, since u_* is a radial function, we have

$$\int_{\boldsymbol{B}} v_n u_* \, dx = 0 \quad \text{for every } n \in \mathbb{N}. \tag{4-16}$$

We then consider $w_n := v_n / \|v_n\|_{\mathcal{H}_\alpha}$, which is a weak solution of

$$-\Delta w_n + \alpha^2 \, \partial_\theta^2 w_n + m w_n = \mathscr{C}_{\alpha,m,p_n}(\boldsymbol{B}) c_n(x) w_n \quad \text{in } \boldsymbol{B},$$

with

$$c_n = (p_n - 1) u_n^{p_n - 2} \quad \text{in Case 1}$$

and

$$c_n = (p_n - 1) \int_0^1 ((1-\tau) u_n + \tau \tilde{u}_n)^{p_n - 2} \, d\tau \quad \text{in Case 2.}$$

In particular, this implies that

$$\mathscr{C}_{\alpha,m,p_n}(\boldsymbol{B}) \int_{\boldsymbol{B}} c_n(x) |w_n|^2 \, dx = \|w_n\|_{\mathcal{H}_\alpha}^2 + m \|w_n\|_{L^2(\boldsymbol{B})}^2 = 1 + m \|w_n\|_{L^2(\boldsymbol{B})}^2 \tag{4-17}$$

for $n \in \mathbb{N}$. Since w_n is bounded in $\mathcal{H}_\alpha$, we may, since $\alpha < 1$, pass to a subsequence such that $w_n \rightharpoonup w$ in $\mathcal{H}_\alpha$, $w_n \to w$ strongly in $L^p(\boldsymbol{B})$ for $p \in [2, 2^*)$ and $w_n \to w$ pointwisely a.e. on $\boldsymbol{B}$. Moreover, from (4-15), it is not difficult to see that $c_n \to 1$ in $L^q(\boldsymbol{B})$ for every $q \in [2, \infty)$. By Hölder's inequality, we may therefore pass to the limit in (4-17) to see that

$$\mathscr{C}_{\alpha,m,2}(\boldsymbol{B}) \|w\|_{L^2(\boldsymbol{B})}^2 = 1 + m \|w\|_{L^2(\boldsymbol{B})}^2;$$

hence $w \neq 0$ and

$$\mathscr{C}_{\alpha,0,2}(\boldsymbol{B}) \geq \frac{1}{\|w\|^2_{L^2(\boldsymbol{B})}} \geq \frac{\|w\|^2_{\mathcal{H}_\alpha}}{\|w\|^2_{L^2(\boldsymbol{B})}}.$$

From Proposition 4.6, it then follows that $w = cu_*$ for some $c \in \mathbb{R} \setminus \{0\}$, which contradicts the fact that

$$\int_{\boldsymbol{B}} wu_*\,dx = \lim_{n\to\infty} \int_{\boldsymbol{B}} w_n u_*\,dx = 0$$

by (4-16). The contradiction allows us to conclude that there exists $p_0 > 2$ with the property that all minimizers for $\mathscr{C}_{\alpha,m,p}(\boldsymbol{B})$ are radial functions for $2 \leq p \leq p_0$. The uniqueness statement then follows from the uniqueness of positive radial solutions of (4-11).

In the remainder of this section, we show the existence of (x_1, x_2)-nonradial ground states for large m, as claimed in Theorem 1.4. This is based on the scaling property

$$\partial_\theta[u(\varepsilon(\,\cdot\,))] = [\partial_\theta u](\varepsilon(\,\cdot\,))$$

for $\varepsilon > 0$, which is used to relate (1-5) to a similar problem on larger balls. Localized ground states of the associated classical nonlinear Schrödinger on $\mathbb{R}^N$ can then be used to construct suitable test functions and disprove symmetry via energy estimates for small ε, which translates into a large mass term. We first restate Theorem 1.4 here in an equivalent form.

Theorem 4.17. *Let $\alpha \in (0, 1]$ and $2 < p < 2^*$. Then there exists $\varepsilon_0 > 0$ such that the ground states of*

$$\begin{cases} -\Delta u + \alpha^2\,\partial_\theta^2 u + u/\varepsilon^2 = |u|^{p-2}u & \text{in } \boldsymbol{B}, \\ u = 0 & \text{on } \partial\boldsymbol{B}, \end{cases} \tag{4-18}$$

are (x_1, x_2)-nonradial for $\varepsilon \in (0, \varepsilon_0)$. Moreover, if $p < 2_1^*$, the same result holds for $\alpha = 1$.

Proof. We first treat the case $\alpha \in (0, 1)$. In the following, for $u \in H_0^1(\boldsymbol{B})$ and $\varepsilon > 0$, we consider $B_{1/\varepsilon} := B_{1/\varepsilon}(0)$ and the rescaled function $u_\varepsilon \in H_0^1(B_{1/\varepsilon})$, $u_\varepsilon(x) = u(\varepsilon x)$. A direct computation then shows that

$$\frac{\int_{B_{1/\varepsilon}} (|\nabla u_\varepsilon|^2 - \alpha^2\varepsilon^2|\partial_\theta u_\varepsilon|^2 + u_\varepsilon^2)\,dx}{\left(\int_{B_{1/\varepsilon}} |u_\varepsilon|^p\,dx\right)^{2/p}} = \varepsilon^{2-N+2N/p} R_{\alpha,1/\varepsilon^2,p}(u). \tag{4-19}$$

As a consequence, we have

$$\mathscr{C}_{\alpha\varepsilon,1,p}(B_{1/\varepsilon}) := \inf_{v\in H_0^1(B_{1/\varepsilon})\setminus\{0\}} \frac{\int_{B_{1/\varepsilon}} (|\nabla v|^2 - \alpha^2\varepsilon^2|\partial_\theta v|^2 + v^2)\,dx}{\left(\int_{B_{1/\varepsilon}} |v|^p\,dx\right)^{2/p}} = \varepsilon^{2-N+2N/p}\mathscr{C}_{\alpha,1/\varepsilon^2,p}(\boldsymbol{B}).$$

It suffices to show that there exists $\varepsilon_0 > 0$ such that all minimizers for $\mathscr{C}_{\alpha\varepsilon,1,p}(B_{1/\varepsilon})$ in $H_0^1(B_{1/\varepsilon}) \setminus \{0\}$ are (x_1, x_2)-nonradial if $\varepsilon \in (0, \varepsilon_0)$. We argue by contradiction and suppose that there exists a sequence $\varepsilon_n \to 0$ and, for every $n \in \mathbb{N}$, a minimizer $v_{\varepsilon_n} \in H_0^1(B_{1/\varepsilon_n}) \setminus \{0\}$ for $\mathscr{C}_{\alpha\varepsilon_n,1,p}(B_{1/\varepsilon_n})$ which satisfies

$$\partial_\theta v_{\varepsilon_n} \equiv 0 \quad \text{in } B_{1/\varepsilon_n}. \tag{4-20}$$

To simplify the notation, we continue writing ε in place of ε_n in the following. From (4-20) and the inclusion $H_0^1(B_{1/\varepsilon}) \subset H^1(\mathbb{R}^N)$, we then deduce that

$$\mathscr{C}_{\alpha\varepsilon,1,p}(B_{1/\varepsilon}) = \frac{\int_{B_{1/\varepsilon}}(|\nabla v_\varepsilon|^2 + v^2)\,dx}{\left(\int_{B_{1/\varepsilon}} |v|^p\,dx\right)^{2/p}} \geq \inf_{v \in H^1(\mathbb{R}^N)\setminus\{0\}} \frac{\int_{\mathbb{R}^N}(|\nabla v|^2 + v^2)\,dx}{\left(\int_{\mathbb{R}^N} |v|^p\,dx\right)^{2/p}} =: \mathscr{C}_{0,1,p}(\mathbb{R}^N). \tag{4-21}$$

We will now derive a contradiction to this inequality by constructing suitable functions in $H_0^1(B_{1/\varepsilon} \setminus \{0\})$ to estimate $\mathscr{C}_{\alpha\varepsilon,1,p}(B_{1/\varepsilon})$. To this end, we first note that the value $\mathscr{C}_{0,1,p}(\mathbb{R}^N)$ is attained by any translation of the unique positive radial solution $\tilde{u}_0 \in H^1(\mathbb{R}^N)$ of the nonlinear Schrödinger equation

$$-\Delta u + u = |u|^{p-2}u \quad \text{in } \mathbb{R}^N.$$

Now take a radial function $\eta \in C_c^1(\boldsymbol{B})$ such that $0 \leq \eta \leq 1$ and $\eta \equiv 1$ in $B_{1/2}$, and let $u_0(x) := \tilde{u}_0(x - e_1)$, where $e_1 = (1, 0, \dots, 0)$. We then define

$$\eta_\varepsilon, w_\varepsilon \in C_c^1(B_{1/\varepsilon}) \quad \text{by } \eta_\varepsilon(x) = \eta(\varepsilon x),\ w_\varepsilon(x) = \eta_\varepsilon(x)u_0(x).$$

Then we have $w_\varepsilon \equiv u_0$ in $B_{1/(2\varepsilon)}$, and

$$\begin{aligned}\mathscr{C}_{\alpha\varepsilon,1,p}(B_{1/\varepsilon}) &\leq \frac{\int_{B_{1/\varepsilon}}(|\nabla w_\varepsilon|^2 - \alpha^2\varepsilon^2|\partial_\theta w_\varepsilon|^2 + w_\varepsilon^2)\,dx}{\left(\int_{B_{1/\varepsilon}} |w_\varepsilon|^p\,dx\right)^{2/p}} \\ &= \frac{\int_{B_{1/\varepsilon}} \eta_\varepsilon^2(|\nabla u_0|^2 + u_0^2)\,dx}{\left(\int_{B_{1/\varepsilon}} \eta_\varepsilon^p |u_0|^p\,dx\right)^{2/p}} + \frac{\int_{B_{1/\varepsilon}}(u_0^2|\nabla_\varepsilon \eta|^2 + 2\eta_\varepsilon u_0 \nabla\eta_\varepsilon \cdot \nabla u_0 - \alpha^2\varepsilon^2\eta_\varepsilon^2|\partial_\theta u_0|^2)\,dx}{\left(\int_{B_{1/\varepsilon}} \eta_\varepsilon^p|u_0|^p\,dx\right)^{2/p}}.\end{aligned} \tag{4-22}$$

We first estimate the second term and note that classical results (see [Berestycki and Lions 1983]) imply that there exist $C_0, \delta_0 > 0$ such that

$$|u_0(x)|, |\nabla u_0(x)| \leq C_0 e^{-\delta_0|x|} \quad \text{for } x \in \mathbb{R}^N. \tag{4-23}$$

Noting that $\nabla\eta_\varepsilon \equiv 0$ on $B_{1/(2\varepsilon)}$, this readily implies

$$\int_{B_{1/\varepsilon}} (u_0^2|\nabla\eta_\varepsilon|^2 + 2\eta_\varepsilon u_0 \nabla\eta_\varepsilon \cdot \nabla u_0)\,dx \leq C_1 e^{-\delta_1/\varepsilon}$$

for some constants $C_1, \delta_1 > 0$. Moreover, for $\varepsilon \in \left(0, \frac{1}{2}\right)$, we have

$$\alpha^2\varepsilon^2 \int_{B_{1/\varepsilon}} \eta_\varepsilon^2|\partial_\theta u_0|^2\,dx \geq C_2\varepsilon^2, \quad \text{with } C_2 := \alpha^2 \int_{\boldsymbol{B}} |\partial_\theta u_0|^2\,dx > 0,$$

since u_0 is an (x_1, x_2)-nonradial function. After possibly modifying $C_1, C_2 > 0$, this gives

$$\frac{\int_{B_{1/\varepsilon}}(u_0^2|\nabla\eta_\varepsilon|^2 + 2\eta_\varepsilon u_0 \nabla\eta_\varepsilon \cdot \nabla u_0 - \alpha^2\varepsilon^2\eta_\varepsilon^2|\partial_\theta u_0|^2)\,dx}{\left(\int_{B_{1/\varepsilon}} \eta_\varepsilon^p|u_0|^p\,dx\right)^{2/p}} \leq C_1 e^{-\delta_1/\varepsilon} - C_2\varepsilon^2.$$

Next we consider the first term in (4-22) and note that

$$\frac{\int_{B_{1/\varepsilon}} \eta_\varepsilon^2(|\nabla u_0|^2+u_0^2)\,dx}{\left(\int_{B_{1/\varepsilon}} \eta_\varepsilon^p |u_0|^p\,dx\right)^{2/p}} \le \frac{\int_{\mathbb{R}^N}(|\nabla u_0|^2+u_0^2)\,dx}{\left(\int_{B_{1/(2\varepsilon)}} |u_0|^p\,dx\right)^{2/p}},$$

while (4-23) implies

$$\int_{\mathbb{R}^N\setminus B_{1/(2\varepsilon)}} |u_0|^p\,dx \le C_3 e^{-\delta_2/\varepsilon}$$

for some $C_3, \delta_2 > 0$. It thus follows that

$$\begin{aligned}\frac{\int_{B_{1/\varepsilon}} \eta_\varepsilon^2(|\nabla u_0|^2+u_0^2)\,dx}{\left(\int_{B_{1/\varepsilon}} \eta_\varepsilon^p |u_0|^p\,dx\right)^{2/p}} &\le \frac{\int_{\mathbb{R}^N}(|\nabla u_0|^2+u_0^2)\,dx}{\left(\int_{B_{1/(2\varepsilon)}} |u_0|^p\,dx\right)^{2/p}} \le \frac{\int_{\mathbb{R}^N}(|\nabla u_0|^2+u_0^2)\,dx}{\left(\int_{\mathbb{R}^N} |u_0|^p\,dx - C_3e^{-\delta_2/\varepsilon}\right)^{2/p}}\\ &\le \frac{\int_{\mathbb{R}^N}(|\nabla u_0|^2+u_0^2)\,dx}{\left(\int_{\mathbb{R}^N} |u_0|^p\,dx\right)^{2/p}} + C_4e^{-\delta_2/\varepsilon} = \mathscr{C}_{0,1,p}(\mathbb{R}^N) + C_4e^{-2\delta_2/(p\varepsilon)}\end{aligned}$$

for $\varepsilon > 0$ sufficiently small with some constant $C_4 > 0$, since u_0 attains $\mathscr{C}_{0,1,p}(\mathbb{R}^N)$. In view of (4-21) and (4-22), this yields that

$$\mathscr{C}_{0,1,p}(\mathbb{R}^N) \le \mathscr{C}_{\alpha\varepsilon,1,p}(B_{1/\varepsilon}) \le \mathscr{C}_{0,1,p}(\mathbb{R}^N) - C_2\varepsilon^2 + C_1e^{-\delta_1/\varepsilon} + C_4e^{-2\delta_2/(p\varepsilon)},$$

and the right-hand side of this inequality is smaller than $\mathscr{C}_{0,1,p}(\mathbb{R}^N)$ if $\varepsilon > 0$ is sufficiently small. This is a contradiction, and thus the claim follows in this case.

In the case $\alpha = 1$, the argument is the same up to replacing $H_0^1(\boldsymbol{B})$ by $\mathcal{H}_1$ and by considering the corresponding rescaled function space $\mathcal{H}_\varepsilon$ on $B_{1/\varepsilon}$. Then the contradiction argument can be carried out in the same way, since radial functions in $\mathcal{H}_\varepsilon$ belong to $H_0^1(B_{1/\varepsilon}) \subset H^1(\mathbb{R}^N)$. □

4.3. *Additional remarks.*

Remark 4.18. Let $0 \le \alpha \le 1$, $2 < p \le 2^*$ and $m > -\lambda_1(\boldsymbol{B})$. While we have seen that ground state solutions of (1-5) are not radially symmetric in general, it is reasonable to expect that, in the case $N \ge 3$, they are invariant under rotations which leave the (x_1, x_2)-plane fixed. This is indeed the case, and we give a brief sketch of the proof in the following. By the $O(2) \times O(N-2)$-equivariance of (1-5), it suffices to show that

$$\begin{cases}\text{any ground state solution } u \text{ of (1-5) is symmetric}\\ \qquad\qquad\text{with respect to the reflection } x \mapsto (x_1, \dots, x_{N-1}, -x_N).\end{cases} \tag{4-24}$$

Then it follows that any such ground state solution is symmetric with respect to reflection at any hyperplane which contains the (x_1, x_2)-plane, so $u(x)$ only depends on (x_1, x_2) and $|(x_3, \dots, x_N)|$.

To prove (4-24), we fix a positive ground state solution $u \in \mathcal{H}_\alpha$ of (1-5), and we introduce some notation. For fixed $\lambda \in (0, 1)$, we consider the open affine half-space $\Sigma_\lambda := \{x \in \mathbb{R}^N : x_N < \lambda\}$ and the reflection at $\partial\Sigma_\lambda$ given by

$$x \mapsto x^\lambda := (x_1, \dots, x_{N-1}, 2\lambda - x_N).$$

Moreover, we define the *polarization* u_λ *of* u with respect to Σ_λ by

$$u_\lambda(x) = \begin{cases} \min\{u(x), u(x_\lambda)\} & \text{if } x \in \boldsymbol{B} \setminus \Sigma_\lambda, \\ \max\{u(x), u(x_\lambda)\} & \text{if } x \in \boldsymbol{B} \cap \Sigma_\lambda \text{ and } x^\lambda \in \boldsymbol{B}, \\ u(x) & \text{if } x \in \boldsymbol{B} \cap \Sigma_\lambda \text{ and } x^\lambda \notin \boldsymbol{B}. \end{cases}$$

By the same argument as given, for example, in Section 3 of the survey paper [Weth 2010], we then find that $u_\lambda \in \mathcal{H}_\alpha$ and $R_{\alpha,m,p}(u_\lambda) = R_{\alpha,m,p}(u) = \mathscr{C}_{\alpha,m,p}(\boldsymbol{B})$. Consequently, both u and u_λ solve (4-8), so $w_\lambda := u_\lambda - u$ solves

$$-\Delta w_\lambda + \alpha^2\, \partial_\theta^2 w_\lambda = c(x) w_\lambda \quad \text{in } \boldsymbol{B},$$

with a function $c \in L^\infty(\boldsymbol{B})$. Since $w_\lambda \geq 0$ in $\boldsymbol{B} \setminus \Sigma_\lambda$ by definition, it follows from the strong maximum principle that either $w_\lambda \equiv 0$ or $w_\lambda > 0$ in $\boldsymbol{B} \setminus \overline{\Sigma}_\lambda$. Here we note again that the operator $-\Delta + \alpha^2\, \partial_\theta^2 - c$ is uniformly elliptic in every compactly contained subset of the open set $\boldsymbol{B} \setminus \overline{\Sigma}_\lambda$. Since $w_\lambda(x) = u(x^\lambda) > 0$ on $\partial \boldsymbol{B} \setminus \overline{\Sigma}_\lambda$, we can exclude the case $w_\lambda \equiv 0$ in $\boldsymbol{B} \setminus \overline{\Sigma}_\lambda$. Hence $w_\lambda > 0$ in $\boldsymbol{B} \setminus \overline{\Sigma}_\lambda$, so $u(x) \leq u(x^\lambda)$ for $x \in \boldsymbol{B} \setminus \Sigma_\lambda$. Since $\lambda \in (0, 1)$ was fixed arbitrarily, we may pass to the limit $\lambda \to 0^+$ in this inequality and see that $u(x) \leq u(x_1, \dots, x_{N-1}, -x_N)$ for all $x \in \boldsymbol{B}$ with $x_N \geq 0$. Applying the same argument to the reflection of u with respect to the x_N-variable, we also find that $u(x) \leq u(x_1, \dots, x_{N-1}, -x_N)$ for all $x \in \boldsymbol{B}$ with $x_N \leq 0$. Consequently, (4-24) holds, as required.

Remark 4.19. Let $m > -\lambda_1(\boldsymbol{B})$. The compactness of the embedding $\mathcal{H}_\alpha \hookrightarrow L^p(\boldsymbol{B})$ in the cases

$$0 \leq \alpha < 1, \quad 2 < p < 2^*$$

and

$$\alpha = 1, \quad 2 < p < 2_1^*$$

suggests that one may show via Lusternik–Schnirelmann theory (or by using the symmetric mountain-pass theorem [Ambrosetti and Rabinowitz 1973]) that (1-5) admits infinitely many solutions under these assumptions. This is indeed the case, but it does not provide new information as it is well known that (1-5) admits infinitely many *radial* solutions if p is Sobolev subcritical; see, e.g., [Struwe 1982]. On the other hand, one might ask how many geometrically distinct (x_1, x_2)-nonradial solutions of (1-5) exist. Here we call two solutions of (1-5) geometrically distinct if they do not coincide up to rotation. We leave this question for future work.

Remark 4.20 (the case $\alpha > 1$). We finally discuss the natural question of what happens for $\alpha > 1$. In fact, in this case, the infimum $\mathscr{C}_{\alpha,m,p}(\boldsymbol{B})$ in (1-7) satisfies

$$\mathscr{C}_{\alpha,m,p}(\boldsymbol{B}) = -\infty \quad \text{for every } m \in \mathbb{R},\ p \in [2, \infty). \tag{4-25}$$

To see this, we fix $\varepsilon \in (0, 1)$ and nonzero functions $\varphi \in C_c^1(1-\varepsilon, 1)$, $\psi \in C_c^1\big(\frac{\pi}{2} - \varepsilon, \frac{\pi}{2} + \varepsilon\big)$. Moreover, we consider the sequence of functions $u_k \in C_c^1(\boldsymbol{B})$ which, in the polar coordinates from (3-3), are given by

$$(r, \theta, \vartheta_1, \dots, \vartheta_{N-2}) \mapsto \varphi(r)\psi(\vartheta_1) \cdots \psi(\vartheta_{N-2}) X_k(\theta), \quad \text{where } X_k(\theta) = \sin(k\theta).$$

Similar to (3-4), we then find, with $U_\varepsilon := (1-\varepsilon, 1)\times(-\pi,\pi)\times\left(\frac{\pi}{2}-\varepsilon, \frac{\pi}{2}+\varepsilon\right)^{N-2}$, that

$$\begin{aligned}\int_{\boldsymbol{B}} (|\nabla u_k|^2 - \alpha^2|\partial_\theta u_k|^2)\,dx \\ = \int_{U_\varepsilon}\Bigg(&|\varphi'(r)|^2|X_k(\theta)|^2\prod_{i=1}^{N-2}|\psi(\vartheta_i)|^2 \\ &+\frac{1}{r^2}\sum_{i=1}^{N-2} h_i|\psi'(\vartheta_i)|^2|\varphi(r)|^2|X_k(\theta)|^2\prod_{j=1,\, j\neq i}^{N-2}|\psi(\vartheta_j)|^2 \\ &+\left(\frac{h_{N-1}}{r^2}-\alpha^2\right)|X_k'(\theta)|^2|\varphi(r)|^2\prod_{i=1}^{N-2}|\psi(\vartheta_i)|^2\Bigg)h\,d(r,\theta,\vartheta_1,\dots,\vartheta_{N-2}),\end{aligned}$$

with the functions $h, h_i : U \to \mathbb{R},\ i = 1,\dots,N-1$, given in (3-5). Since $\alpha > 1$, we may now choose $\varepsilon = \varepsilon(\alpha) > 0$ small enough that

$$\tfrac{1}{2} \le h \le 1 \quad\text{and}\quad \alpha^2 - \frac{h_{N-1}}{r^2} \ge \varepsilon \qquad \text{on } U_\varepsilon.$$

Since also $|X_k| \le 1$ by definition, we estimate

$$\int_{\boldsymbol{B}} (|\nabla u_k|^2 - \alpha^2|\partial_\theta u_k|^2)\,dx \le c - d(k),$$

where

$$c := \int_{U_\varepsilon}\left(|\varphi'(r)|^2\prod_{i=1}^{N-2}|\psi(\vartheta_i)|^2 + \frac{1}{r^2}\sum_{i=1}^{N-2} h_i|\psi'(\vartheta_i)|^2|\varphi(r)|^2\prod_{j=1,\, j\neq i}^{N-2}|\psi(\vartheta_j)|^2\right)d(r,\theta,\vartheta_1,\dots,\vartheta_{N-2})$$

and

$$\begin{aligned} d(k) &:= \int_{U_\varepsilon}\left(\alpha^2 - \frac{h_{N-1}}{r^2}\right)|X_k'(\theta)|^2|\varphi(r)|^2\prod_{i=1}^{N-2}|\psi(\vartheta_i)|^2 g\,d(r,\theta,\vartheta_1,\dots,\vartheta_{N-2}) \\ &\ge \frac{k^2\varepsilon}{2}\int_{1-\varepsilon}^{1}|\varphi(r)|^2\,dr\int_{-\pi}^{\pi}\cos^2(k\theta)\,d\theta\left(\int_{\pi/2-\varepsilon}^{\pi/2+\varepsilon}|\psi(\vartheta)|^2\,d\vartheta\right)^{N-2} = \frac{\varepsilon\pi}{2}d_2k^2,\end{aligned}$$

with

$$d_2 := \int_{1-\varepsilon}^{1}|\varphi(r)|^2\,dr\left(\int_{\pi/2-\varepsilon}^{\pi/2+\varepsilon}|\psi(\vartheta)|^2\,d\vartheta\right)^{N-2}.$$

Hence $d(k) \to \infty$ as $k \to \infty$. Moreover, for every $p \in [2,\infty)$, we have

$$\int_{\boldsymbol{B}} |u_k|^p\,dx = \int_{U_\varepsilon}|\varphi(r)|^p|X_k(\theta)|^p\prod_{i=1}^{N-2}|\psi(\vartheta_i)|^p g\,d(r,\theta,\vartheta_1,\dots,\vartheta_{N-2}) \le d_p,$$

with

$$d_p := 2\pi\int_{1-\varepsilon}^{1}|\varphi(r)|^p\,dr\left(\int_{\pi/2-\varepsilon}^{\pi/2+\varepsilon}|\psi(\vartheta)|^p\,d\vartheta\right)^{N-2} < \infty.$$

It thus follows that

$$\frac{\int_{\boldsymbol{B}}(|\nabla u_k|^2-\alpha^2|\partial_\theta u_k|^2+m|u_k|^2)\,dx}{\left(\int_{\boldsymbol{B}}|u_k|^p\,dx\right)^{2/p}}\le\frac{c-d(k)-md_2}{(d_p)^{2/p}}\to-\infty\quad\text{as } k\to\infty$$

for every $p\in[2,\infty)$, $m\in\mathbb{R}$. This shows (4-25).

Consequently, the study of ground state solutions of (1-5) requires a completely different approach in the case $\alpha>1$. This is further treated in [Kübler 2023].

5. The case of an annulus

In this section, we consider rotating solutions of (1-3) in the case where $\boldsymbol{B}$ is replaced by an annulus

$$\boldsymbol{A}_r:=\{x\in\mathbb{R}^N : r<|x|<1\}$$

for some $r\in(0,1)$. The ansatz (1-4) then leads to the reduced problem

$$\begin{cases}-\Delta u+\alpha^2\,\partial_\theta^2 u+mu=|u|^{p-2}u & \text{in } \boldsymbol{A}_r,\\ u=0 & \text{on } \partial\boldsymbol{A}_r,\end{cases}\tag{5-1}$$

where $m>-\lambda_1(\boldsymbol{A}_r)$, $p\in(2,2^*)$ and $\partial_\theta=x_1\,\partial_{x_2}-x_2\,\partial_{x_1}$ as before. Here, $\lambda_1(\boldsymbol{A}_r)$ denotes the first Dirichlet eigenvalue of $-\Delta$ on $\boldsymbol{A}_r$. As in (1-7), we may then define

$$\mathscr{C}_{\alpha,m,p}(\boldsymbol{A}_r):=\inf_{u\in H_0^1(\boldsymbol{A}_r)\setminus\{0\}}R_{\alpha,m,p}(u),\tag{5-2}$$

with the Rayleigh quotient $R_{\alpha,m,p}(u)$ given by (1-8) for functions $u\in H_0^1(\boldsymbol{A}_r)$. In the following, a weak solution of (5-1) will be called a ground state solution if it is a minimizer for (5-2). We then have the following analogue of Theorem 1.1.

Theorem 5.1. *Let $r\in(0,1)$, $m>-\lambda_1(\boldsymbol{A}_r)$ and $p\in(2,2^*)$.*

(i) *If $\alpha\in(0,1)$, then there exists a ground state solution of* (5-1).

(ii) *We have*

$$\mathscr{C}_{1,m,p}(\boldsymbol{A}_r)=0\quad\textit{for } p>2_1^*\qquad\textit{and}\qquad\mathscr{C}_{1,m,p}(\boldsymbol{A}_r)>0\quad\textit{for } p\le 2_1^*.$$

Moreover, for any $p\in(2_1^*,2^*)$, there exists $\alpha_p\in(0,1)$ with the property that

$$\mathscr{C}_{\alpha,m,p}(\boldsymbol{A}_r)<\mathscr{C}_{0,m,p}(\boldsymbol{A}_r)\quad\textit{for } \alpha\in(\alpha_p,1],$$

and therefore every ground state solution of (5-1) *is (x_1,x_2)-nonradial for $\alpha\in(\alpha_p,1]$.*

This theorem does not come as a surprise and is proved by precisely the same arguments as Theorem 1.1, so we omit the proof. Instead, we now discuss an interesting additional feature of the annulus $\boldsymbol{A}_r$. Unlike in the case of the ball, we can formulate *explicit* sufficient conditions for the parameters p, α, m and r which guarantee that every ground state solution of (5-1) is (x_1,x_2)-nonradial. This is the content of the following theorem.

Theorem 5.2. *Let* $N \geq 2,\ m \geq 0,\ r, \alpha \in (0, 1)$, *and assume*

$$2 + \frac{N-1-r^2\alpha^2}{\kappa(r,m)} < p < 2^*,$$

with

$$\kappa(r,m) = \begin{cases} mr^2 + \max\bigl\{\bigl(\frac{N-2}{2}\bigr)^2, \bigl(\frac{\pi}{1-r}\bigr)^2 r^{N-1}\bigr\}, & N \geq 3, \\ mr^2 + \bigl(\frac{\pi}{1-r}\bigr)^2 r^N, & N = 2. \end{cases} \tag{5-3}$$

Then every ground state solution of (5-1) *is* (x_1, x_2)*-nonradial.*

We point out that $\kappa(m,r) \to \infty$ if $m \to \infty$ or $r \to 1^-$. Consequently, for given $p > 2$, ground states of (5-1) are nonradial if either m is large or the annulus is thin, i.e., r is close to 1. The proof is based on the following lemma.

Lemma 5.3. *Suppose that* $m \geq 0,\ \alpha \in (0, 1),\ p \in (2, 2^*)$ *and that there exists a function* $v \in H_0^1(\boldsymbol{A}_r)$ *satisfying*

$$\int_{\mathbb{S}^{N-1}} v(s(\,\cdot\,))\, d\sigma = 0 \quad \textit{for every } s \in (r, 1) \tag{5-4}$$

and

$$\int_{\boldsymbol{A}_r} (|\nabla v|^2 - \alpha^2 |\partial_\theta v|^2 + m v^2)\, dx - (p-1) \int_{\boldsymbol{A}_r} |u_0|^{p-2} v^2\, dx < 0. \tag{5-5}$$

Then we have

$$\mathscr{C}_{\alpha,m,p}(\boldsymbol{A}_r) < R_{\alpha,m,p}(u_0), \tag{5-6}$$

where $u_0 \in H_0^1(\boldsymbol{A}_r)$ *is the unique positive radial solution of* (5-1).

Here we note that, in the case $m = 0$, the uniqueness of the positive radial solution u_0 of (5-1) has been first proved by Ni and Nussbaum [1985]. In the case $m > 0$, the uniqueness is due to Tang [2003] and Felmer, Martínez and Tanaka [Felmer et al. 2008] for $N \geq 3$ and $N = 2$, respectively.

Proof. We argue by contradiction and assume that equality holds in (5-6). Then u_0 is a minimizer for the C^2-functional $R_{\alpha,m,p} : H_0^1(\boldsymbol{A}_r) \setminus \{0\}) \to \mathbb{R}$, which implies, in particular, that

$$R'_{\alpha,m,p}(u_0)\tilde{v} = 0 \quad \text{and} \quad R''_{\alpha,m,p}(u_0)(\tilde{v}, \tilde{v}) \geq 0 \qquad \text{for all } \tilde{v} \in H_0^1(\boldsymbol{A}_r). \tag{5-7}$$

In the following, we write $R_{\alpha,m,p} = Z(u)/N(u)$ for $u \in H^1(\boldsymbol{A}_r) \setminus \{0\}$, with

$$Z(u) := \int_{\boldsymbol{A}_r} (|\nabla u|^2 - \alpha^2 |\partial_\theta u|^2 + m u^2)\, dx \quad \text{and} \quad N(u) := \biggl(\int_{\boldsymbol{A}_r} |u|^p\, dx\biggr)^{2/p}.$$

The first property in (5-7), applied with $\tilde{v} = v$, then gives $N(u_0)Z'(u_0)v = Z(u_0)N'(u_0)v$ and consequently

$$N(u_0)^3 [R_{\alpha,m,p}]''(u_0)(v,v) = N(u_0)^2 Z''(u_0)(v,v) - Z(u_0)N(u_0)N''(u_0)(v,v)$$

for $v \in H_0^1(\boldsymbol{A}_r)$. Therefore, applying the second property in (5-7) with $\tilde{v} = v$ yields

$$Z''(u_0)(v,v) - \frac{Z(u_0)}{N(u_0)} N''(u_0)(v,v) \geq 0.$$

Moreover, noting that u_0 is a weak solution of (5-1) and therefore $Z(u_0) = N(u_0)^{p/2}$, we conclude that

$$
\begin{aligned}
0 &\le \frac{1}{2}\left(Z''(u_0)(v,v) - \frac{Z(u_0)}{N(u_0)} N''(u_0)(v,v)\right) \\
&= \int_{A_r} (|\nabla v|^2 - \alpha^2 |\partial_\theta v|^2 + m v^2)\,dx \\
&\qquad\qquad - (p-1)\int_{A_r} |u_0|^{p-2} v^2\,dx + (p-2) N(u_0)^{-p/2}\left(\int_{A_r} |u_0|^{p-2} u_0 v\,dx\right)^2.
\end{aligned}
$$

This, however, contradicts (5-5), since $\int_{A_r} |u_0|^{p-2} u_0 v\,dx = 0$ by (5-4). The proof is thus finished. □

Proof of Theorem 5.2. Our goal is to construct a function that satisfies the conditions of Lemma 5.3. To this end, let μ_1 be the first eigenvalue of the weighted eigenvalue problem

$$
\begin{cases}
-w_{\rho\rho} - \frac{N-1}{\rho} w_\rho + m w - (p-1)|u_0(\rho)|^{p-2} w = \frac{\mu}{\rho^2} w & \text{in } (r,1), \\
w(r) = w(1) = 0,
\end{cases}
$$

and let w be the unique positive eigenfunction up to normalization. Moreover, let $Y \in C^\infty(\mathbb{S}^{N-1})$ be given by $Y(x) = x_2$. Then Y is a spherical harmonic of degree 1 on $\mathbb{S}^{N-1}$, which in the polar coordinates from (3-3) is written as $Y(r,\theta,\vartheta_1,\dots,\vartheta_{N-2}) = r \sin\theta \sin\vartheta_1 \cdots \sin\vartheta_{N-2}$, and therefore satisfies $\partial_\theta^2 Y = -Y$ on $\mathbb{S}^{N-1}$. Moreover, set $v(\rho,\omega) := w(\rho) Y(\omega)$. Then condition (5-4) of Lemma 5.3 is satisfied. By construction, v also satisfies

$$
-\Delta v + \alpha^2 \partial_\theta^2 v + m v - (p-1)|u_0|^{p-2} v = \frac{\mu_1 + N - 1}{|x|^2} v - \alpha^2 v,
$$

and testing this equation with v itself yields

$$
\begin{aligned}
\int_{A_r} (|\nabla v|^2 - \alpha^2 |\partial_\theta v|^2 + m v^2 - (p-1)|u_0|^{p-2} v^2)\,dx &= (\mu_1 + (N-1)) \int_{A_r} \frac{v^2}{|x|^2}\,dx - \alpha^2 \int_{A_r} v^2\,dx \\
&\le (\mu_1 + (N-1) - r^2\alpha^2) \int_{A_r} \frac{v^2}{|x|^2}\,dx. \qquad (5\text{-}8)
\end{aligned}
$$

We recall that μ_1 can be characterized by

$$
\mu_1 = \min_{\varphi \in H^1_{0,\mathrm{rad}}(A_r)\setminus\{0\}} \frac{\int_{A_r} (|\nabla\varphi|^2 + m\varphi^2)\,dx - (p-1)\int_{A_r} |u_0|^{p-2}\varphi^2\,dx}{\int_{A_r} \varphi^2/|x|^2\,dx}.
$$

Taking $\varphi = u_0$ in this quotient, we obtain the estimate

$$
\begin{aligned}
\mu_1 &\le \frac{\int_{A_r} (|\nabla u_0|^2 + m u_0^2)\,dx - (p-1)\int_{A_r} |u_0|^p\,dx}{\int_{A_r} u_0^2/|x|^2\,dx} \\
&= -(p-2) \frac{\int_{A_r} (|\nabla u_0|^2 + m u_0^2)\,dx}{\int_{A_r} u_0^2/|x|^2\,dx} \\
&\le -(p-2)\left(\frac{\int_{A_r} |\nabla u_0|^2\,dx}{\int_{A_r} u_0^2/|x|^2\,dx} + m r^2\right) \qquad (5\text{-}9)
\end{aligned}
$$

We now distinguish the cases $N \geq 3$ and $N = 2$. If $N \geq 3$, Hardy's inequality gives

$$\int_{\boldsymbol{A}_r} |\nabla u_0|^2 \, dx \geq \left(\frac{N-2}{2}\right)^2 \int_{\boldsymbol{A}_r} \frac{u_0^2}{|x|^2} \, dx. \tag{5-10}$$

Alternatively, we may also estimate, since u_0 is radial,

$$\begin{aligned} \int_{\boldsymbol{A}_r} |\nabla u_0|^2 \, dx &= |\mathbb{S}^{N-1}| \int_r^1 \rho^{N-1} |\partial_r u_0(\rho)|^2 \, d\rho \geq |\mathbb{S}^{N-1}| r^{N-1} \int_r^1 |\partial_r u_0(\rho)|^2 \, d\rho \\ &\geq |\mathbb{S}^{N-1}| \left(\frac{\pi}{1-r}\right)^2 r^{N-1} \int_r^1 u_0^2(\rho) \, d\rho \geq |\mathbb{S}^{N-1}| \left(\frac{\pi}{1-r}\right)^2 r^{N-1} \int_r^1 \rho^{N-3} u_0^2(\rho) \, d\rho \\ &= \left(\frac{\pi}{1-r}\right)^2 r^{N-1} \int_{\boldsymbol{A}_r} \frac{u_0^2}{|x|^2} \, dx. \end{aligned} \tag{5-11}$$

Thus (5-9) gives $\mu_1 < -(p-2)\kappa(r, m)$, with $\kappa(r, m)$ given in (5-3) for $N \geq 3$. Inserting this into (5-8) yields

$$\int_{\boldsymbol{A}_r} (|\nabla v|^2 - \alpha^2 |\partial_\theta v|^2 + m v^2 - (p-1)|u_0|^{p-2} v^2) \, dx < -(p-2)\kappa + N - 1 - r^2\alpha^2,$$

i.e., condition (5-5) of Lemma 5.3 is satisfied if $p > (N - 1 - r^2\alpha^2)/\kappa + 2$, which holds by assumption.

Hence v satisfies the assumptions of Lemma 5.3, which implies that (5-6) holds and therefore every minimizer for (5-2) is nonradial. Let u denote such a nonradial ground state solution, and suppose by contradiction that $\partial_\theta u_0 \equiv 0$. The nonradiality of u implies that there exists an isometry $A \in O(N)$ such that $\tilde{u} := u \circ A \in H_0^1(\boldsymbol{A}_r)$ satisfies $\partial_\theta \tilde{u} \not\equiv 0$. Since A is an isometry, this implies

$$R_{\alpha,m,p}(\tilde{u}) = R_{\alpha,m,p}(u) - \alpha^2 \frac{\int_{\boldsymbol{A}_r} |\partial_\theta \tilde{u}|^2 \, dx}{\left(\int_{\boldsymbol{A}_r} |u|^p \, dx\right)^{2/p}} < R_{\alpha,m,p}(u) = \mathscr{C}_{1,m,p}(\boldsymbol{A}_r),$$

which contradicts (5-2). Consequently, we have $\partial_\theta u_0 \not\equiv 0$, which yields that u_0 is (x_1, x_2)-nonradial. This finishes the proof in the case $N \geq 3$.

It remains to consider the case $N = 2$. In this case, we replace the estimates (5-10) and (5-11) by

$$\int_{\boldsymbol{A}_r} |\nabla u_0|^2 \, dx \geq |\mathbb{S}^1| \left(\frac{\pi}{1-r}\right)^2 r^{N-1} \int_r^1 u_0^2(\rho) \, d\rho \geq \left(\frac{\pi}{1-r}\right)^2 r^N \int_{\boldsymbol{A}_r} \frac{u_0^2}{|x|^2} \, dx.$$

Combining this with (5-9) we again get $\mu_1 < -(p-2)\kappa(r, m)$, with $\kappa(r, m)$ given in (5-3) for $N = 2$. We may thus complete the proof as above. $\square$

6. Riemannian models

So far we only used the inequality stated in Theorem 2.2 in the case $s = 1$. We shall now consider an application for general $s \in (0, 2]$ by considering (1-3) on some Riemannian manifolds with boundary. More precisely, we consider a class of Riemannian models given by $(\boldsymbol{B}, g)$, where, as before, $\boldsymbol{B}$ denotes the open ball of radius 1 centered at zero, and the metric g on $\boldsymbol{B}$ is written, in polar coordinates, as

$$ds^2 = dr^2 + (\psi(r))^2\, d\Theta^2 \tag{6-1}$$

for $r > 0$, $\Theta \in \mathbb{S}^{N-1}$. Here $d\Theta^2$ denotes the canonical metric on $\mathbb{S}^{N-1}$ and ψ is a smooth function that is positive on $(0, \infty)$. Moreover, we assume

$$\psi'(0) > 0 \qquad \text{and} \qquad \psi^{(2k)}(0) = 0 \quad \text{for } k \in \mathbb{N}_0. \tag{6-2}$$

We note that the second condition in (6-2) ensures smoothness of g at the origin. For such a Riemannian model, the associated Laplace–Beltrami operator becomes

$$\Delta_g f = \frac{1}{\psi^{N-1}}\, \partial_r(\psi^{N-1}\, \partial_r f) + \frac{1}{\psi^2} \Delta_{\mathbb{S}^{N-1}} f,$$

where $\Delta_{\mathbb{S}^{N-1}}$ denotes the Laplace–Beltrami operator on $\mathbb{S}^{N-1}$. Riemannian models are of independent geometric interest; we refer to [Berchio et al. 2014] for a more detailed discussion.

We again study the problem

$$\begin{cases} \partial_t^2 v - \Delta_g v + mv = |v|^{p-2} v & \text{in } M, \\ v = 0 & \text{on } \partial M, \end{cases} \tag{6-3}$$

where $2 < p < 2N/(N-2)$ and $m > -\lambda_1(M)$, with $\lambda_1(M)$ denoting the first Dirichlet eigenvalue of $-\Delta_g$ on M. We stress that the case $\psi(r) = r$ corresponds to the classical flat metric on $\boldsymbol{B}$ considered in detail in the previous sections. A further example is the hemisphere $\mathbb{S}^N_{\tau,+} := \{x \in \mathbb{R}^{N+1} : |x| = \tau,\, x_{N+1} > 0\}$ of radius $\tau > 0$. Indeed, using polar coordinates $(r, \omega) \in (0, 1) \times \mathbb{S}^{N-1}$, a parametrization $\boldsymbol{B} \to \mathbb{S}^N_{\tau,+}$ is given by $(r, \omega) \mapsto \tau\big(\sin\big(\frac{\pi}{2} r\big)\omega, \cos\big(\frac{\pi}{2} r\big)\big)$. This yields (6-1) with $\psi(r) = \tau \sin\big(\frac{\pi}{2} r\big)$. Similarly, spherical caps can be considered.

As in the flat case, we restrict our attention to solutions of (6-3) of the form $v(t, x) = u(R_{\alpha t}(x))$, where R_θ is the rotation in the (x_1, x_2)-plane with angle θ. This leads to the reduced equation

$$\begin{cases} -\Delta_g u + \alpha^2\, \partial_\theta^2 u + mu = |u|^{p-2} u & \text{in } M, \\ u = 0 & \text{on } \partial M, \end{cases} \tag{6-4}$$

with the differential operator $\partial_\theta = x_1\, \partial_{x_2} - x_2\, \partial_{x_1}$ associated to the Killing vector field $x \mapsto (-x_2, x_1, 0, \dots, 0)$ on M. We may then again study the quotient

$$R^M_{\alpha,m,p} : H^1_0(M) \setminus \{0\} \to \mathbb{R}, \qquad R^M_{\alpha,m,p}(u) := \frac{\int_M (|\nabla_g u|^2 - \alpha^2 |\partial_\theta u|^2 + mu^2)\, dg}{\|u\|^2_{L^p(M)}},$$

and its minimizers, i.e.,

$$\mathscr{C}_{\alpha,m,p}(M) := \inf_{u \in C^1_c(\boldsymbol{B}) \setminus \{0\}} R^M_{\alpha,m,p}(u).$$

Analogously to Theorem 1.1, we can use the general inequality stated in Theorem 2.2 to give the following result, recalling that we set $2^*_s = (4N + 2s)/(2N - 4 + s)$.

Theorem 6.1. *Let $s \in (0, 2]$, and let (M, g) be a Riemannian model, with $M = \boldsymbol{B}$ and associated function $\psi \in C^\infty[0, 1)$ satisfying* (6-2) *and*

$$c_1(1-r)^s \le 1 - \psi(r) \le c_2(1-r)^s \quad \text{for } r \in (0,1) \text{ with constants } c_1, c_2 > 0. \tag{6-5}$$

Moreover, let $m > -\lambda_1(M)$, and let $2 < p < 2^*$.

(i) *If $\alpha \in (0, 1)$, then there exists a ground state solution of* (6-4).

(ii) *We have*

$$\mathscr{C}_{1,m,p}(M) = 0 \quad \text{for } p > 2^*_s \qquad \text{and} \qquad \mathscr{C}_{1,m,p}(M) > 0 \quad \text{for } p \le 2^*_s. \tag{6-6}$$

Moreover, for any $p \in (2^*_s, 2^*)$, there exists $\alpha_p \in (0, 1)$ with the property that

$$\mathscr{C}_{\alpha,m,p}(M) < \mathscr{C}_{0,m,p}(M) \quad \text{for } \alpha \in (\alpha_p, 1],$$

and therefore every ground state solution of (6-4) *is (x_1, x_2)-nonradial for $\alpha \in (\alpha_p, 1)$.*

Proof. Since the proof is completely parallel to the proof of Theorem 1.1, we omit some details and focus our attention on showing where condition (6-5) enters. It is again useful to introduce polar coordinates $(r, \theta, \vartheta_1, \dots, \vartheta_{N-2}) \in U := (0, 1) \times (-\pi, \pi) \times (0, \pi)^{N-2}$ given by

$$\begin{aligned}(x_1, \dots, x_N) = (&r \sin\vartheta_1 \cdots \sin\vartheta_{N-2} \cos\theta,\ r \sin\vartheta_1 \cdots \sin\vartheta_{N-2} \sin\theta,\ r\cos\vartheta_1,\\ &r\sin\vartheta_1\cos\vartheta_2, \dots, r\sin\vartheta_1 \cdots \sin\vartheta_{N-3}\cos\vartheta_{N-2},\ r\sin\vartheta_1 \cdots \cos\vartheta_{N-2}).\end{aligned} \tag{6-7}$$

In the following, we will abbreviate the coordinates $(\theta, \vartheta_1, \dots, \vartheta_{N-2})$ to Θ for simplicity. Using (B-1) from Appendix B, we see that the metric (6-1) is written in these coordinates as

$$dg = dr^2 + (\psi(r))^2 \biggl(\sum_{i=1}^{N-2} \biggl(\prod_{k=1}^{i-1} \sin^2\vartheta_k \biggr) d\vartheta_i^2 + \biggl(\prod_{k=1}^{N-1} \sin^2\vartheta_k \biggr) d\theta^2 \biggr).$$

Therefore, by (B-3), the quadratic form associated to the operator $-\Delta_g + \partial_\theta^2$ is given by

$$\int_M (|\nabla_g u|^2 - |\partial_\theta u|^2)\, dg = \int_U \biggl(|\partial_r u|^2 + \frac{1}{\psi^2} \sum_{i=1}^{N-2} h_i |\partial_{\vartheta_i} u|^2 + \biggl(\frac{h_{N-1}}{\psi^2} - 1 \biggr) |\partial_\theta u|^2 \biggr) |g|\, d(r, \Theta)$$

for $u \in C^1_c(M)$, with

$$|g|(r, \Theta) = (\psi(r))^{N-1} \prod_{k=1}^{N-2} \sin^{N-1-k}\vartheta_k, \quad h_i(r, \Theta) = \prod_{k=1}^{i-1} \frac{1}{\sin^2\vartheta_k}.$$

Moreover,

$$\int_M |u|^p\, dg = \int_U |u|^p |g|\, d(r, \Theta) \quad \text{for } u \in C^1_c(M) \text{ and } p > 1.$$

Next we note that, as a consequence of (6-5), we have

$$|g|(\Theta_0) = 1 \quad \text{and} \quad h_i(\Theta_0) = 1 \ \text{ for } i = 1, \dots, N-1, \quad \text{with } \Theta_0 := \bigl(1, 0, \tfrac{\pi}{2}, \dots, \tfrac{\pi}{2}\bigr). \tag{6-8}$$

Setting

$$U_0 := \left(\tfrac{1}{2}, 1\right) \times (-\pi, \pi) \times \left(\tfrac{\pi}{4}, \tfrac{3}{4}\pi\right)^{N-2} \subset U,$$

we now claim that assumption (6-5) implies that the function $h_{N-1}/\psi^2 - 1$ satisfies

$$\tilde{c}_1\bigg((1-r)^s + \sum_{k=1}^{N-2}\big(\vartheta_k - \tfrac{\pi}{2}\big)^2\bigg) \le \frac{h_{N-1}}{\psi^2}(r, \Theta) - 1 \le \tilde{c}_2\bigg((1-r)^s + \sum_{k=1}^{N-2}\big(\vartheta_k - \tfrac{\pi}{2}\big)^2\bigg) \tag{6-9}$$

for $(r, \theta, \vartheta_1, \dots, \vartheta_{N-2}) \in U_0$, with suitable constants $\tilde{c}_1, \tilde{c}_2 > 0$. Indeed, note that

$$\frac{h_{N-1}}{\psi^2} - 1 = \frac{1}{\psi^2}(h_{N-1} - \psi^2) = \frac{1}{\psi^2}(\sqrt{h_{N-1}} + \psi)(\sqrt{h_{N-1}} - \psi)$$

and that, in U_0, the factor $(\sqrt{h_{N-1}} + \psi)/\psi^2$ can clearly be bounded from above and below by positive constants. Moreover, since the first-order derivatives of $\sqrt{h_{N-1}} = \prod_{k=1}^{N-2} 1/\sin\vartheta_k$ vanish in Θ_0, a Taylor expansion yields

$$1 + C_1 \sum_{k=1}^{N-2}\big(\vartheta_k - \tfrac{\pi}{2}\big)^2 \le \sqrt{h_{N-1}} \le 1 + C_2 \sum_{k=1}^{N-2}\big(\vartheta_k - \tfrac{\pi}{2}\big)^2 \quad \text{in } U_0,$$

with constants $C_1, C_2 > 0$. We thus find that

$$1 - \psi(r) + C_1 \sum_{k=1}^{N-2}\big(\vartheta_k - \tfrac{\pi}{2}\big)^2 \ge \sqrt{h_{N-1}} - \psi \le 1 - \psi(r) + C_2 \sum_{k=1}^{N-2}\big(\vartheta_k - \tfrac{\pi}{2}\big)^2$$

holds in U_0, and (6-9) can finally be deduced from (6-5).

We now consider a fixed function $u \in C^1_c(U_0) \setminus \{0\} \subset C^1_c(U) \setminus \{0\}$, which, regarded as a function of polar coordinates, gives rise to a function in $C^1_c(M)$. For $\lambda \in (0, 1)$, we consider the map

$$\Lambda_\lambda : U_0 \to U_0, \quad (r, \Theta) \mapsto \big(1 + \lambda(r-1), \lambda^{1+s/2}\theta, \tfrac{\pi}{2} + \lambda\big(\vartheta_1 - \tfrac{\pi}{2}\big), \dots, \tfrac{\pi}{2} + \lambda\big(\vartheta_{N-2} - \tfrac{\pi}{2}\big)\big),$$

and we define $u_\lambda := u \circ \Lambda_\lambda^{-1} \in C^1_c(U_0) \setminus \{0\}$ for $\lambda \in (0, 1)$. Note that Λ_λ shrinks U_0 to the point Θ_0, which we may show similarly to the arguments in the proof of Proposition 3.1.

Using (6-8) and (6-9), we find that

$$\lambda^{-(2N+s)/p}\bigg(\int_U |u_\lambda|^p |g|\, d(r, \Theta)\bigg)^{2/p} = \bigg(\int_U |u|^p (|g| \circ \Lambda_\lambda)\, d(r, \Theta)\bigg)^{2/p} \to \bigg(\int_U |u|^p\, d(r, \Theta)\bigg)^{2/p} =: c_u(p)$$

as $\lambda \to 0^+$ and

$$\begin{aligned}
&\limsup_{\lambda \to 0^+} \lambda^{2-s/2-N} \int_U \bigg(|\partial_r u_\lambda|^2 + \frac{1}{\psi^2}\sum_{i=1}^{N-2} h_i |\partial_{\vartheta_i} u_\lambda|^2 + \Big(\frac{h_{N-1}}{\psi^2} - 1\Big)|\partial_\theta u_\lambda|^2\bigg)|g|\, d(r, \Theta) \\
&= \limsup_{\lambda \to 0^+} \int_U \bigg(|\partial_r u|^2 + \frac{1}{\psi^2} \circ \Lambda_\lambda \sum_{i=1}^{N-2} (h_i \circ \Lambda_\lambda)|\partial_{\vartheta_i} u|^2 + \lambda^{-s}\Big(\Big(\frac{h_{N-1}}{\psi^2}\Big) \circ \Lambda_\lambda - 1\Big)|\partial_\theta u|^2\bigg)(|g| \circ \Lambda_\lambda)\, d(r, \Theta) \\
&\le d_u^1 + d_u^2,
\end{aligned} \tag{6-10}$$

with

$$d_u^1 := \int_U \Big(|\partial_r u|^2 + \sum_{i=1}^{N-2} |\partial_{\vartheta_i} u|^2 \Big)\, d(r,\Theta)$$

and

$$\begin{aligned} d_u^2 &= \tilde{c}_2 \limsup_{\lambda\to 0^+} \int_U \Big((1-r)^s + \lambda^{2-s} \sum_{k=1}^{N-2} \big(\vartheta_k - \tfrac{\pi}{2}\big)^2 \Big) |\partial_\theta u|^2 \, d(r,\Theta) \\ &= \begin{cases} \tilde{c}_2 \int_U (1-r)^s |\partial_\theta u|^2 \, d(r,\Theta), & s \in (0,2), \\ \tilde{c}_2 \int_U \big((1-r)^2 + \sum_{i=1}^{N-2} \big(\vartheta_k - \tfrac{\pi}{2}\big)^2\big) |\partial_\theta u|^2 \, d(r,\Theta), & s = 2. \end{cases} \end{aligned}$$

It thus follows that

$$\begin{aligned} \mathscr{C}_{1,m,p}(M) &\le \limsup_{\lambda\to 0^+} R^M_{1,m,p}(u_\lambda) \\ &= \limsup_{\lambda\to 0^+} \frac{\lambda^{N+s/2-2}(d_u^1 + d_u^2) + \lambda^{(2N+s)/2} c_u(2)}{\lambda^{(2N+s)/p} c_u(p)} = 0 \quad \text{if } p > 2_s^*. \end{aligned}$$

This shows the first identity in (6-6). To see the second identity in (6-6), we argue as in Section 3. More precisely, we first note that it is sufficient to consider the case $p = 2_s^*$, and then we show the inequality

$$\Big(\int_U |g|\, |u|^{2_s^*} \, d(r,\Theta) \Big)^{2/2_s^*} \le C \int_U \Big(|\partial_r u|^2 + \frac{1}{\psi^2} \sum_{i=1}^{N-2} h_i |\partial_{\vartheta_i} u|^2 + \Big(\frac{h_{N-1}}{\psi^2} - 1 \Big) |\partial_\theta u|^2 \Big) |g| \, d(r,\Theta)$$

for functions $u \in C_c^1(U_0)$, with a suitable constant $C > 0$. For this, we use Theorem 1.6 and the first inequality in (6-9). The argument is then completed by using the rotation invariance of the problem and a partition of unity argument to localize the problem. ☐

Remark 6.2. (i) As noted before, the case of a hemisphere $\mathbb{S}^N_{1,+}$ of radius 1 corresponds to $\psi(r) = \sin\big(\frac{\pi}{2} r\big)$. In this case Theorem 6.1 applies with $s = 2$, and it yields nonradial ground state solutions for $p > 2_2^* = 2(N+1)/(N-1)$. Notably, this corresponds to the critical exponent for generalized traveling waves on the sphere $\mathbb{S}^N$ found in [Mukherjee 2017; 2018; Taylor 2016]. In fact, our approach based on Theorem 1.6 can be used to give an alternative proof for the existence of nontrivial solutions and the embeddings stated in [Taylor 2016, Proposition 3.2] and [Mukherjee 2017, Proposition 1.2 and Lemma 1.3].

(ii) Theorem 6.1 leaves open the case $s > 2$. Note that the two-sided estimate (6-9) needs to be analyzed more carefully if $s > 2$ and $N \ge 3$, as the leading-order term is then 2 in place of s. In this case, if (6-5) holds for some $s > 2$, Theorem 6.1 (ii) holds with 2_s^* replaced by 2_2^*, i.e.,

$$\mathscr{C}_{1,m,p}(M) = 0 \quad \text{for } p > 2_2^* \qquad \text{and} \qquad \mathscr{C}_{1,m,p}(M) > 0 \quad \text{for } p \le 2_2^*.$$

For $N = 2$, on the other hand, no angular terms appear in (6-9). Consequently, Theorem 6.1 holds for arbitrary $s > 0$ in this case.

Appendix A: Boundedness of solutions

In the proof of the regularity properties of weak solutions of (1-5) in the case $\alpha = 1$ stated in Lemma 4.10, we used the following.

Lemma A.1. *Let* $2 < p < 2_1^*$, $m > -\lambda_1$, *and let* $u \in \mathcal{H}_1$ *be a weak solution of*

$$-\Delta u + \partial_\theta^2 u + mu = |u|^{p-2}u \quad \textit{in } \boldsymbol{B}. \tag{A-1}$$

Then $u \in L^\infty(\boldsymbol{B})$. *Furthermore, there exist constants* $C = C(N, m)$, $\sigma > 0$ *such that*

$$|u|_\infty \le C\|u\|_{\mathcal{H}}^\sigma. \tag{A-2}$$

For $m \ge 0$, *the constant* $C = C(N) > 0$ *can be chosen independent of* m.

Proof. The proof is based on a Moser iteration scheme and essentially identical to the classical arguments with the Sobolev critical exponent replaced by 2_1^*; see [Struwe 2008, Appendix B].

We fix $L, s \ge 2$ and consider auxiliary functions $h, g \in C^1([0, \infty))$ defined by

$$h(t) := s\int_0^t \min\{\tau^{s-1}, L^{s-1}\}\,d\tau \quad \text{and} \quad g(t) := \int_0^t [h'(\tau)]^2\,d\tau.$$

We note that

$$h(t) = t^s \quad \text{for } t \le L \qquad \text{and} \qquad g(t) \le tg'(t) = t(h'(t))^2 \quad \text{for } t \ge 0 \tag{A-3}$$

since the function $t \mapsto h'(t) = s\min\{t^{s-1}, L^{s-1}\}$ is nondecreasing. We now show that $w := u^+ \in L^\infty(\boldsymbol{B})$ and that $\|w\|_\infty$ is bounded by the right-hand side of (A-2). Since we may replace u with $-u$, the claim will then follow.

We now note that $w \in \mathcal{H}_1$ and $\varphi := g(w) \in \mathcal{H}_1$, with

$$\nabla w = \mathbb{1}_{\{u>0\}}\nabla u, \quad \nabla\varphi = g'(w)\nabla w, \quad \partial_\theta w = \mathbb{1}_{\{u>0\}}\,\partial_\theta u, \quad \partial_\theta\varphi = g'(w)\,\partial_\theta w.$$

As outlined in Remark 4.2, this follows from the boundedness of g' and the estimate $g(t) \le s^2 t^{2s-1}$ for $t \ge 0$. Testing (A-1) with φ gives

$$\int_{\boldsymbol{B}} (\nabla u \cdot \nabla\varphi - (\partial_\theta u\,\partial_\theta\varphi) + mu\varphi)\,dx = \int_{\boldsymbol{B}} |u|^{p-2}u\varphi\,dx,$$

from where we estimate, using $h'(w)^2 = g'(w)$,

$$\begin{aligned}\int_{\boldsymbol{B}} (|\nabla(h(w))|^2 - (\partial_\theta(h(w)))^2 + mwg(w))\,dx &= \int_{\boldsymbol{B}} (g'(w)(|\nabla w|^2 - (\partial_\theta w)^2) + mug(w))\,dx \\ &= \int_{\boldsymbol{B}} |u|^{p-2}ug(w)\,dx \le \int_{\boldsymbol{B}} w^p(h'(w))^2\,dx.\end{aligned} \tag{A-4}$$

Here we used (A-3) in the last step. Combining (A-4) with Proposition 4.6 and Theorem 3.3, we obtain the inequality

$$|h(w)|_{2_1^*}^2 \le c_0 \int_{\boldsymbol{B}} w^p(h'(w))^2\,dx, \tag{A-5}$$

with a constant $c_0 = c_0(N, m) > 0$. Note that, for $m \ge 0$, c_0 only depends on N. Since

$$h(t) = t^s, \quad h'(t) = st^{s-1} \quad \text{and} \quad g(t) = s^2 \int_0^t \tau^{2s-2}\, d\tau = \frac{s^2}{2s-1} t^{2s-1} \qquad \text{for } t \le L,$$

we may let $L \to \infty$ in (A-5) and apply Lebesgue's theorem to obtain

$$|w^s|_{2_1^*}^2 \le c_0 s^2 \int_{\boldsymbol{B}} w^{p+2s-2}\, dx \le c_0 s^2 |w|_{2_1^*}^{p-2} |w|_{2sq}^{2s},$$

where $q = 2_1^*/(2_1^* - p + 2)$ is the conjugated exponent to $2_1^*/(p-2)$. This yields

$$|w|_{s2_1^*} \le (c_1 s)^{1/s} |w|_{2sq}, \quad \text{with } c_1 := (c_0 |w|_{2_1^*}^{p-2})^{1/2}, \tag{A-6}$$

whenever $w \in L^{2sq}(\boldsymbol{B})$. We now consider $s = s_n = \rho^n$ for $n \in \mathbb{N}$ with $\rho := 2_1^*/(2q) = \frac{1}{2}(2 + 2_1^* - p) > 1$, so that

$$2s_1 q = 2_1^* \qquad \text{and} \qquad 2s_{n+1} q = s_n 2_1^* \quad \text{for } n \in \mathbb{N}.$$

Iterating (A-6) then gives

$$|w|_{\rho^n 2_1^*} = |w|_{s_n 2_1^*} \le |w|_{2_1^*} \prod_{j=1}^{n} (c_1 \rho^j)^{\rho^{-j}} \le c_1^{\rho/(\rho-1)} c_2 |w|_{2_1^*}$$

for all n, with

$$c_2 := \rho^{\sum_{j=1}^\infty j\rho^{-j}} < \infty.$$

It follows that

$$|w|_\infty = \lim_{n\to\infty} |w|_{\rho^n 2_1^*} \le c_1^{\rho/(\rho-1)} c_2 |w|_{2_1^*}. \tag{A-7}$$

Moreover, by (A-6) and Theorem 3.3, we have

$$c_1 \le c_1' \|w\|_{\mathcal{H}}^{(p-2)/2} \le c_1' \|u\|_{\mathcal{H}}^{(p-2)/2} \quad \text{and} \quad |w|_{2_1^*} \le \tilde{c} \|w\|_{\mathcal{H}} \le \tilde{c} \|u\|_{\mathcal{H}},$$

with constants $c_1', \tilde{c} > 0$ depending only on N. It thus follows from (A-7) that

$$|w|_\infty \le C \|u\|_{\mathcal{H}}^{(p-2)\rho/(2(\rho-1))+1} \quad \text{with } C := c_2 (c_1')^{\rho/(\rho-1)} \tilde{c}.$$

The proof is thus finished. □

Appendix B: Round metric on spheres in angular coordinates

Let $U := (-\pi, \pi) \times (0, \pi)^{N-2}$, and consider angular coordinates $U \to S^{N-1}$ given by

$$(\theta, \vartheta_1, \dots, \vartheta_{N-2}) \mapsto (\sin\vartheta_1 \cdots \sin\vartheta_{N-2} \cos\theta,\ \sin\vartheta_1 \cdots \sin\vartheta_{N-2} \sin\theta,$$
$$\cos\vartheta_1,\ \sin\vartheta_1 \cos\vartheta_2, \dots, \sin\vartheta_1 \cdots \sin\vartheta_{N-3} \cos\vartheta_{N-2}).$$

As in (3-3) and (6-7), we use the angular variable $\theta \in (-\pi, \pi)$ for the angle of the (x_1, x_2)-coordinate of $x \in S^{N-1}$ relative to the positive x_1-axis in $\mathbb{R}^2$, which differs from most of the literature (see, e.g., [Blumenson 1960]). The standard round metric on S^{N-1} (induced by the embedding $S^{N-1} \hookrightarrow \mathbb{R}^N$) with respect to these orthogonal coordinates is then written as

$$\sum_{i=1}^{N-2}\left(\prod_{k=1}^{i-1}\sin^2\vartheta_k\right)d\vartheta_i^2+\left(\prod_{k=1}^{N-1}\sin^2\vartheta_k\right)d\theta^2, \tag{B-1}$$

see, e.g., [Campos and Silva 2020, Section 2.2]. Moreover, the associated volume element is given by

$$\left(\prod_{i=1}^{N-1}\prod_{k=1}^{i-1}\sin\vartheta_k\right)d\vartheta_1\cdots d\vartheta_{N-2}\,d\theta=\left(\prod_{k=1}^{N-2}\sin^{N-1-k}\vartheta_k\right)d\vartheta_1\cdots d\vartheta_{N-2}\,d\theta. \tag{B-2}$$

The Dirichlet energy of a function $f \in H^1(S^{N-1})$ with respect to the round metric is therefore written in these coordinates as

$$\int_{S^{N-1}}|\nabla f|^2\,d\sigma=\int_U\left(\sum_{i=1}^{N-2}h_i|\partial_{\vartheta_i}v|^2+h_{N-1}|\partial_\theta v|^2\right)\left(\prod_{k=1}^{N-2}\sin^{N-1-k}\vartheta_k\right)d\vartheta_1\cdots d\vartheta_{N-2}\,d\theta, \tag{B-3}$$

with $h_i := \prod_{k=1}^{i-1} 1/\sin^2\vartheta_k$ for $i = 1, \dots, N-1$.

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THE RELATIVE TRACE FORMULA IN ELECTROMAGNETIC SCATTERING AND BOUNDARY LAYER OPERATORS

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This paper establishes trace formulae for a class of operators defined in terms of the functional calculus for the Laplace operator on divergence-free vector fields with relative and absolute boundary conditions on Lipschitz domains in $\mathbb{R}^3$. Spectral and scattering theory of the absolute and relative Laplacian is equivalent to the spectral analysis and scattering theory for Maxwell equations. The trace formulae allow for unbounded functions in the functional calculus that are not admissible in the Birman–Krein formula. In special cases, the trace formula reduces to a determinant formula for the Casimir energy that is used in the physics literature for the computation of the Casimir energy for objects with metallic boundary conditions. Our theorems justify these formulae in the case of electromagnetic scattering on Lipschitz domains, give a rigorous meaning to them as the trace of certain trace-class operators, and clarify the function spaces on which the determinants need to be taken.

1. Introduction

In this paper we establish several trace formulae for operators governing the time-harmonic Maxwell equations on an open set $X = \Omega \cup M \subset \mathbb{R}^3$ of the form $\mathbb{R}^3 \setminus \partial\Omega$, where Ω is a bounded (strongly) Lipschitz

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domain. Here we will refer to Ω as the interior and to M as the exterior domain. We denote by E and H the electric and magnetic fields, respectively. The time-harmonic Maxwell system is given by

$$\begin{aligned} \operatorname{curl} E - \mathrm{i}\lambda H &= 0, \\ \operatorname{div} E &= 0, \\ \operatorname{curl} H + \mathrm{i}\lambda E &= 0, \\ \operatorname{div} H &= 0, \\ \nu \times E &= A \quad \text{on } \partial\Omega, \\ \langle \nu, H \rangle &= f \quad \text{on } \partial\Omega, \end{aligned} \tag{1}$$

where the first four equations are considered in either Ω or M separately, or simultaneously by considering this as an equation on X. Here ν is the almost everywhere defined outward-pointing unit normal vector field on $\partial\Omega$. This system is well-posed on suitable function spaces under natural consistency conditions on A and f. In particular, if A is sufficiently regular and tangential and $\lambda \neq 0$, the function f is determined by A. For the interior problem, given a tangential A, the system then has a unique solution for λ away from a discrete set of points. For the exterior problem and $\operatorname{Im}\lambda > 0$, one imposes that E and H are square-integrable and then obtains a unique solution for any sufficiently regular tangential A. In both cases, the solution E can be expressed as

$$E = \widetilde{\mathcal{L}}_\lambda \mathcal{L}_\lambda^{-1} A,$$

where $\widetilde{\mathcal{L}}_\lambda$ is the electric field boundary layer potential operator and $\mathcal{L}_\lambda$ is the electric field boundary layer operator. For a continuous tangential vector field A, one has

$$(\widetilde{\mathcal{L}}_\lambda A)(x) = \operatorname{curl}\operatorname{curl} \int_{\partial\Omega} \frac{e^{\mathrm{i}\lambda|x-y|}}{4\pi|x-y|} A(y)\, dy,$$

and $\mathcal{L}_\lambda A$ is obtained by taking the boundary value of $\nu \times \widetilde{\mathcal{L}}$. These operators extend to suitable function spaces and we refer to Section 6 for the precise definitions. The vector field H and the function f are then determined by $H = -(\mathrm{i}/\lambda)\operatorname{curl} E$. As usual, this layer potential operator creates a solution of the Maxwell system by placing certain sources on the boundary, and the choice of $\widetilde{\mathcal{L}}$ is now a standard operator in computational electrodynamics.

For $\lambda \neq 0$, the system for E becomes

$$\begin{aligned} -\Delta E - \lambda^2 E &= 0, \\ \operatorname{div} E &= 0, \\ \nu \times E &= A \quad \text{on } \partial\Omega. \end{aligned}$$

The associated spectral problem is therefore that of the Laplace–Beltrami operator Δ on divergence-free vector fields with the corresponding boundary condition. For the electric field, the boundary condition $\nu \times E = 0$ on $\partial\Omega$ leads to the relative Laplacian Δ_{rel} by quadratic form considerations. Similarly, for the magnetic field, the boundary condition $\nu \cdot H = 0$ leads to the absolute Laplace operator Δ_{abs}. Both are self-adjoint operators on $L^2(\mathbb{R}^3, \mathbb{C}^3) = L^2(\Omega, \mathbb{C}^3) \oplus L^2(M, \mathbb{C}^3)$, and their definitions and properties are explained in detail in Sections 3 and 4. Functional calculus for the relative Laplacian determines the solutions E of the time-harmonic Maxwell system, whereas functional calculus for the absolute Laplacian determines the solutions H of the system. Here we use the more mathematical notation that is inspired by Hodge theory. The harmonic forms satisfying relative boundary conditions give rise to relative de Rham cohomology classes, and the ones satisfying absolute boundary conditions give rise to absolute de Rham cohomology classes.

Before we describe the general case, we would like to explain and motivate this in an important special case and when the bounded Lipschitz domain $\Omega \subset \mathbb{R}^3$ consists of two connected components Ω_1 and Ω_2. We then construct the self-adjoint operator Δ_{rel} out of the Laplace operator on $\mathbb{R}^3 \setminus \partial\Omega$ by imposing relative boundary conditions in each side of $\partial\Omega$. The operators $\Delta_{j,\mathrm{rel}}$ are obtained in the same way from the Laplace operator on $\mathbb{R}^3 \setminus \partial\Omega_j$ with boundary conditions only imposed on each side of $\partial\Omega_j$. The operators Δ_{abs} and $\Delta_{j,\mathrm{abs}}$ are defined analogously with absolute boundary conditions. It is a special case of our result that the two operators

$$\begin{aligned} C_E &= (-\Delta_{\mathrm{rel}})^{-1/2}\delta\,\mathrm{d} - (-\Delta_{1,\mathrm{rel}})^{-1/2}\delta\,\mathrm{d} - (-\Delta_{2,\mathrm{rel}})^{-1/2}\delta\,\mathrm{d} + (-\Delta_{\mathrm{free}})^{-1/2}\delta\,\mathrm{d}, \\ C_H &= (-\Delta_{\mathrm{abs}})^{-1/2}\delta\,\mathrm{d} - (-\Delta_{1,\mathrm{abs}})^{-1/2}\delta\,\mathrm{d} - (-\Delta_{2,\mathrm{abs}})^{-1/2}\delta\,\mathrm{d} + (-\Delta_{\mathrm{free}})^{-1/2}\delta\,\mathrm{d} \end{aligned}$$

defined on smooth compactly supported vector fields on $X = \mathbb{R}^3 \setminus \partial\Omega$ extend to trace-class operators on $L^2(\mathbb{R}^3, \mathbb{C}^3)$, and their trace can be expressed in terms of the determinant of a combination of Maxwell boundary layer operators (see Theorems 1.1 and 1.3). In fact, we will see that their traces coincide, i.e., $\mathrm{tr}(C_E) = \mathrm{tr}(C_H)$. We have used here differential form notation, with d being the exterior derivative and δ being the coderivative. The trace-class property is due to several cancellations. Any linear combination of operators appearing in the expressions above that is not proportional to this expression is not trace-class. This statement remains true even if one introduces an artificial boundary, thereby compactifying the problem.

In terms of vector calculus, the above two operators can also be written as

$$\begin{aligned} C_E &= (-\Delta_{\mathrm{rel}})^{-1/2}\mathrm{curl}\,\mathrm{curl} - (-\Delta_{1,\mathrm{rel}})^{-1/2}\mathrm{curl}\,\mathrm{curl} - (-\Delta_{2,\mathrm{rel}})^{-1/2}\mathrm{curl}\,\mathrm{curl} + (-\Delta_{\mathrm{free}})^{-1/2}\,\mathrm{curl}\,\mathrm{curl}, \\ C_H &= \mathrm{curl}(-\Delta_{\mathrm{rel}})^{-1/2}\mathrm{curl} - \mathrm{curl}(-\Delta_{1,\mathrm{rel}})^{-1/2}\mathrm{curl} - \mathrm{curl}(-\Delta_{2,\mathrm{rel}})^{-1/2}\mathrm{curl} + \mathrm{curl}(-\Delta_{\mathrm{free}})^{-1/2}\,\mathrm{curl}. \end{aligned}$$

Apart from being interesting from the point of view of spectral analysis, these operators also have a direct physical significance. Namely, $\frac{1}{4}\,\mathrm{tr}(C_E + C_H) = \frac{1}{2}\,\mathrm{tr}(C_E)$ represents the Casimir energy of the two Lipschitz obstacles Ω_1 and Ω_2. Indeed, as shown in [Strohmaier 2021] in a general rigorous framework of quantum field theory, the local trace, i.e., the trace of the integral kernel restricted to the diagonal, of the operator

$$\tfrac{1}{4}((-\Delta_{\mathrm{rel}})^{-1/2}\mathrm{curl}\,\mathrm{curl} - (-\Delta_{\mathrm{free}})^{-1/2}\,\mathrm{curl}\,\mathrm{curl}) + \tfrac{1}{4}((-\Delta_{\mathrm{abs}})^{-1/2}\mathrm{curl}\,\mathrm{curl} - (-\Delta_{\mathrm{free}})^{-1/2}\,\mathrm{curl}\,\mathrm{curl})$$

is the renormalised energy density obtained from the electromagnetic quantum field theory. The relative resolvent differences C_E and C_H then describe differences of energies. It was shown in [Fang and Strohmaier 2022], again in a rigorous quantum field theory framework, that in the case of a scalar field, such "energy differences" lead to a Casimir force as determined from the quantum stress energy tensor as in [Candelas 1982; Kay 1979]. The same statement is expected to hold for the electromagnetic field, but this will be discussed elsewhere.

The mathematical statements above can therefore also be interpreted as a rigorous proof that the Casimir energy as derived from spectral quantities is well defined in this framework and can be computed from determinants of boundary layer operators. It also clarifies the function spaces needed to compute these quantities for nonsmooth boundaries.

We focus in this paper on Maxwell's equations in dimension three, and we will mostly use vector calculus notation rather than differential forms. This has the advantage of keeping the notation and exposition more accessible, and we can then also rely on a wealth of previous results on boundary layer operators [Buffa and Hiptmair 2003; Buffa et al. 2002; Claeys and Hiptmair 2019; Costabel 1988; 1990; Gol'dshtein et al. 2011; Kirsch and Hettlich 2015; Mitrea 1995; 2000; Mitrea and Mitrea 2002; Mitrea et al. 1997]. Focussing on dimension three also avoids complications with the free Green's function having more complicated expressions or a logarithmic singularity at 0. More importantly, the focus on dimension three allows us to stay close to the classical notation in Maxwell theory without having to distract the reader with more complicated notation.

Although this is a mathematical paper, we also try to give the physics background for the interested reader. To our knowledge, a determinant formula for the Casimir energy first appeared in the physics literature [Renne 1971], where this was derived microscopically and without reference to spectral theory. Physics derivations have also appeared in various contexts based on path integrals and fluctuations of configurations on the surface on the obstacles [Emig et al. 2007; Kenneth and Klich 2006; 2008] and have led to numerical schemes [Johnson 2011] and asymptotic formulae. The spectral side, often favouring a zeta function regularisation approach as in Casimir's original work [1948], was developed somewhat independently. We refer to [Bordag et al. 2009] and [Kirsten 2002] for a comprehensive overview of the subject. The relation between the various approaches remained unclear even in the physics world (for a very recent report on this see [Bimonte and Emig 2021]). We also mention the approach of [Balian and Duplantier 1978], which is also based on a reduction to the boundary.

1.1. ***Statement of main results.*** We now describe the general setting of our results. We assume that $\Omega \subset \mathbb{R}^3$ is an open and bounded (strongly) Lipschitz domain in $\mathbb{R}^3$ in the sense that the boundary of Ω is locally congruent to the graph of a Lipschitz function. The finitely many connected components will be denoted by Ω_j with some index j, which ranges from 1 to N. We will think of the closure $\overline{\Omega}$ as a collection of disjoint compact obstacles $\overline{\Omega}_j$ placed in $\mathbb{R}^3$ (see Figure 1). Removing these obstacles from $\mathbb{R}^3$ results in a noncompact open domain $M = \mathbb{R}^3 \backslash \overline{\Omega}$ with Lipschitz boundary $\partial\Omega$. We will assume throughout that M is connected. It will also be convenient to introduce $X = \mathbb{R}^3 \setminus \partial\Omega = M \cup \Omega$.

On the boundary, one has well-defined anisotropic Sobolev spaces $H^{-1/2}(\mathrm{Div}, \partial\Omega)$ (see Section 2) and the Maxwell electric field operator $\mathcal{L}_\lambda$ is a bounded operator $H^{-1/2}(\mathrm{Div}, \partial\Omega) \to H^{-1/2}(\mathrm{Div}, \partial\Omega)$ (see Section 6). This can be done for each object separately, and one can assemble the individual parts $\mathcal{L}_{\lambda,\partial\Omega_j} : H^{-1/2}(\mathrm{Div}, \partial\Omega_j) \to H^{-1/2}(\mathrm{Div}, \partial\Omega_j)$ into an operator $\mathcal{L}_{D,\lambda} = \bigoplus_{j=1}^N \mathcal{L}_{\lambda,\partial\Omega_j}$ acting on $H^{-1/2}(\mathrm{Div}, \partial\Omega)$.

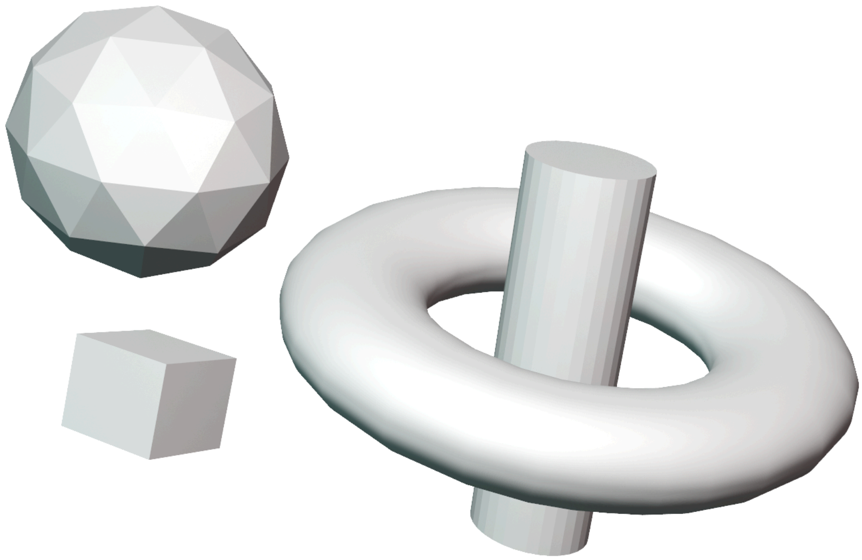

Figure 1. A Lipschitz domain Ω consisting of four connected components $\Omega_1,\ \Omega_2,\ \Omega_3,\ \Omega_4$.

Theorem 1.1. *The operator $\mathcal{L}_\lambda \mathcal{L}_{D,\lambda}^{-1}$ is well defined and a trace-class perturbation of the identity for any complex λ with $\operatorname{Im}\lambda > 0$. It therefore has a well-defined Fredholm determinant $\det(\mathcal{L}_\lambda \mathcal{L}_{D,\lambda}^{-1})$ on the space $H^{-1/2}(\mathrm{Div}, \partial\Omega)$. Let δ be the minimal distance between separate objects. Then, for any $0 < \delta' < \delta$, the function*

$$\Xi(\lambda) = \log\det(\mathcal{L}_\lambda \mathcal{L}_{D,\lambda}^{-1}),$$

where the branch of the logarithm has been fixed by continuity, extends to a holomorphic function in a neighbourhood of the closed upper half-space, and it satisfies the bound

$$|\Xi(\lambda)| \le C e^{-\delta' \operatorname{Im}\lambda}$$

for λ in any sector about the positive imaginary axis of angle strictly less than π.

We note that the operators $\mathcal{L}_\lambda^{-1}$ and $\mathcal{L}_{D,\lambda}^{-1}$ have singularities at 0, and it is due to a variety of cancellations that the quotient is regular at 0, in particular when the objects have nontrivial topology. Our proof is based on a careful analysis of these singularities.

Before we formulate the trace formula, we need to define a large class of functions to which it applies. These will be analytic functions in certain sectors, and we start by describing these sectors. Assume $0 < \epsilon \le \pi$, and let $\mathfrak{S}_\epsilon$ be the open sector

$$\mathfrak{S}_\epsilon = \{z \in \mathbb{C} \mid z \neq 0,\ |\arg(z)| < \epsilon\}$$

containing the real axis (see Figure 2). Associated to these we define the following spaces of functions. The space $\mathcal{E}_\epsilon$ will be defined by

$$\mathcal{E}_\epsilon = \{f : \mathfrak{S}_\epsilon \to \mathbb{C} \mid f \text{ is holomorphic in } \mathfrak{S}_\epsilon,\ \exists\alpha > 0,\ \forall\epsilon_0 > 0,\ |f(z)| = O(|z|^\alpha e^{\epsilon_0 |z|})\}.$$

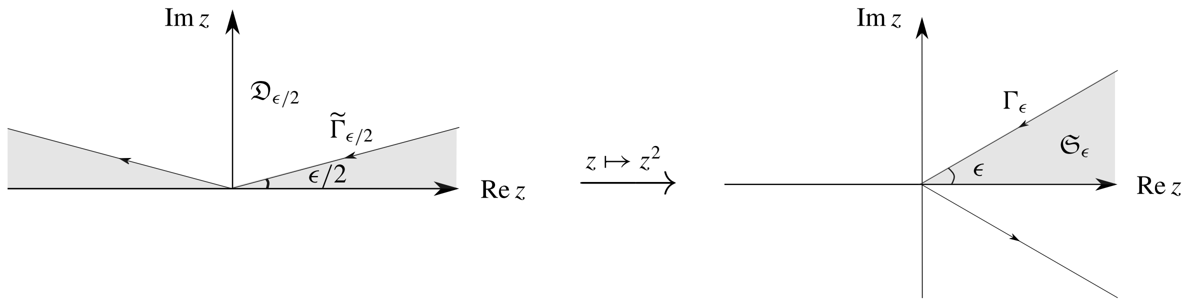

Figure 2. The sectors $\mathfrak{S}_\epsilon$, $\mathfrak{D}_{\epsilon/2}$ and the corresponding contours.

We define the space $\mathcal{P}_\epsilon$ as the set of functions in $\mathcal{E}_\epsilon$ whose restriction to $[0, \infty)$ is polynomially bounded and that extend continuously to the boundary of $\mathfrak{S}_\epsilon$ in the logarithmic cover of the complex plane. Reference to the logarithmic cover of the complex plane is only needed in the case $\epsilon = \pi$. In this case functions in $\mathcal{P}_\pi$ are required to have continuous limits from above and below on the negative real axis. We do not however require that these limits coincide. The space $\mathcal{P}_\epsilon$ contains, in particular, $f(z) = z^\alpha$, $\alpha > 0$, for any $0 < \epsilon \leq \pi$.

When working with the Laplace operator, it is often convenient to change variables and use λ^2 as a spectral parameter, and in the context of Maxwell theory it turns out to be beneficial to introduce an extra λ^{-2} factor. For notational brevity we therefore introduce another class of functions as follows.

Definition 1.2. The space $\widetilde{\mathcal{P}}_\epsilon$ is defined to be the space of functions f such that $f(\lambda) = \lambda^{-2} g(\lambda^2)$ for some $g \in \mathcal{P}_\epsilon$. In particular, $f(\lambda) = O(\lambda^a)$ for some $a > -2$ near $\lambda = 0$.

Generally, the operator Δ_{rel} decomposes into a direct sum of unbounded operators $\Delta_{\mathrm{rel}} = 0 \oplus d\delta \oplus \delta\, d$ under the weak Hodge–Helmholtz decomposition (see Section 4, (9)), and we have

$$f((-\Delta_{\mathrm{rel}})^{1/2}) \operatorname{curl}\operatorname{curl} = f((-\Delta_{\mathrm{rel}})^{1/2}) \delta\, d = f((\delta\, d)^{1/2}) \delta\, d$$

for any Borel function f. From this we have that, for a function $f \in \widetilde{\mathcal{P}}_\epsilon$, the unbounded operator $f((-\Delta_{\mathrm{rel}})^{1/2}) \operatorname{curl}\operatorname{curl}$ contains $C_0^\infty(X, \mathbb{C}^3)$ in its domain. Indeed, for $\psi \in C_0^\infty(X, \mathbb{C}^3)$ and $k \in \mathbb{N}$ large enough, we have the factorisation $f((\delta\, d)^{1/2})\delta\, d = h((\delta\, d)^{1/2})(\delta\, d + 1)^k \psi$, where $(\delta\, d + 1)^k \psi \in C_0^\infty(X, \mathbb{C}^3)$ and the function $h(\lambda) = (1 + \lambda^2)^{-k} \lambda^2 f(\lambda)$ is bounded on the real line.

For $0 < \epsilon \leq \pi$, we also define the contours Γ_ϵ in the complex plane as the boundary curves of the sectors $\mathfrak{S}_\epsilon$. In the case $\epsilon = \pi$, the contour is defined as a contour in the logarithmic cover of the complex plane. We also let $\widetilde{\Gamma}_{\epsilon/2}$ be the corresponding contour after the change of variables, i.e., the preimage in the upper half-space under the map $z \to z^2$ of Γ_ϵ (see Figure 2).

For $f \in \widetilde{\mathcal{P}}_\epsilon$, we define the *relative operator*

$$D_{\mathrm{rel},f} = f((-\Delta_{\mathrm{rel}})^{1/2}) \operatorname{curl}\operatorname{curl} - f((-\Delta_{\mathrm{free}})^{1/2}) \operatorname{curl}\operatorname{curl} - \sum_{j=1}^{N} \bigl(f((-\Delta_{j,\mathrm{rel}})^{1/2}) \operatorname{curl}\operatorname{curl} - f((-\Delta_{\mathrm{free}})^{1/2}) \operatorname{curl}\operatorname{curl} \bigr),$$

where $f(\lambda) = g(\lambda^2)$. Similarly,

$$\begin{aligned}
D_{\mathrm{abs},f} &= f((-\Delta_{\mathrm{abs}})^{1/2})\mathrm{curl}\,\mathrm{curl} - f((-\Delta_{\mathrm{free}})^{1/2})\,\mathrm{curl}\,\mathrm{curl} \\
&\qquad - \sum_{j=1}^{N} \big(f((-\Delta_{j,\mathrm{abs}})^{1/2})\mathrm{curl}\,\mathrm{curl} - f((-\Delta_{\mathrm{free}})^{1/2})\,\mathrm{curl}\,\mathrm{curl}\big) \\
&= \mathrm{curl}\, f((-\Delta_{\mathrm{rel}})^{1/2})\mathrm{curl} - \mathrm{curl}\, f((-\Delta_{\mathrm{free}})^{1/2})\,\mathrm{curl} \\
&\qquad - \sum_{j=1}^{N} \big(\mathrm{curl}\, f((-\Delta_{j,\mathrm{rel}})^{1/2})\mathrm{curl} - \mathrm{curl}\, f((-\Delta_{\mathrm{free}})^{1/2})\,\mathrm{curl}\big).
\end{aligned}$$

Since these operators contain $C_0^\infty(X, \mathbb{C}^3)$ in their domain, they are densely defined.

We refer to taking these differences as the *relative setting*, indicating that this compares interacting quantities to noninteracting ones. It is unfortunate that the word relative is also used to denote the relative boundary conditions. We alert the reader that these two uses of the word relative are unrelated, but to avoid confusion we have used the symbol D for "difference" to denote relative objects.

Our main result reads as follows.

Theorem 1.3. *If $f \in \widetilde{\mathcal{P}}_\epsilon$, then operators $D_{\mathrm{rel},f}$ and $D_{\mathrm{abs},f}$ extend to trace-class operators $L^2(\mathbb{R}^3, \mathbb{C}^3) \to L^2(\mathbb{R}^3, \mathbb{C}^3)$, and*

$$\mathrm{tr}(D_{\mathrm{rel},f}) = \mathrm{tr}(D_{\mathrm{abs},f}) = \frac{\mathrm{i}}{2\pi} \int_{\widetilde{\Gamma}_{\epsilon/2}} \Xi(\lambda) \frac{\mathrm{d}}{\mathrm{d}\lambda}(\lambda^2 f(\lambda))\,\mathrm{d}\lambda,$$

where the contour $\widetilde{\Gamma}_{\epsilon/2}$ is the clockwise-oriented boundary of a sector that includes the imaginary axis.

We would like to mention that expressions formally similar to the relative trace-formula have appeared in the context of multichannel scattering theory and were introduced by Buslaev and Merkur' ev [1969] (see also [Vasy and Wang 2002]) to prove Birman–Krein-type formulae. In this context, the test function f is still required to decay sufficiently fast.

An interesting application of the relative trace is that it allows one to define a relative zeta function, namely,

$$\zeta_D(s) = \mathrm{tr}(D_{f_s}), \qquad f_s(\lambda) = \frac{1}{\lambda^{2s+2}}$$

for $\mathrm{Re}\, s < 0$. As a consequence of Theorem 1.3, this relative zeta function then satisfies

$$\zeta_D(s) = \frac{2s}{\pi} \sin(\pi s) \int_0^\infty \lambda^{-2s-1}\, \Xi(\mathrm{i}\lambda)\,\mathrm{d}\lambda.$$

This formula allows for a meromorphic continuation of ζ_D with poles of order at most one and residues related to the Taylor coefficients of $\Xi(\mathrm{i}\lambda)$ at 0. These coefficients are interesting in their own right and will be investigated elsewhere. In the special case when $f(\lambda) = 1/\lambda$, this gives the expression

$$\frac{1}{4}\,\mathrm{tr}(C_E + C_H) = \frac{1}{2\pi} \int_0^\infty \Xi(\mathrm{i}\lambda)\,\mathrm{d}\lambda$$

for the Casimir energy.

Under our more general assumptions on f, the operators

$$B_{\mathrm{rel},f} = f((-\Delta_{\mathrm{rel}})^{1/2})\operatorname{curl}\operatorname{curl} - f((-\Delta_{\mathrm{free}})^{1/2})\operatorname{curl}\operatorname{curl}, \tag{2}$$

$$B_{\mathrm{abs},f} = f((-\Delta_{\mathrm{abs}})^{1/2})\operatorname{curl}\operatorname{curl} - f((-\Delta_{\mathrm{free}})^{1/2})\operatorname{curl}\operatorname{curl} \tag{3}$$

are not trace-class. One has however the following theorem about the smoothness and integrability properties of their integral kernels.

Theorem 1.4. *Let B_f be either $B_{\mathrm{rel},f}$, defined by* (2), *or $B_{\mathrm{rel},f}$, defined by* (3). *Then B_f has an integral kernel $\kappa \in C^\infty(X \times X, \mathrm{Mat}(3, \mathbb{C}))$, which is smooth away from the boundary. If $\Omega_0 \subset X$ has positive distance to the boundary $\partial\Omega$ and p_{Ω_0} is the orthogonal projection $L^2(\mathbb{R}^3, \mathbb{C}^3) \to L^2(\Omega_0, \mathbb{C}^3)$, then $p_{\Omega_0} B_f p_{\Omega_0}$ extends to a trace-class operator with trace equal to the convergent integral*

$$\int_{\Omega_0} \mathrm{tr}(\kappa(x, x))\, \mathrm{d}x.$$

If $f(z) = O(|z|^a)$ for $|z| < 1$, we have for large $|x|$ the estimate

$$\|\kappa(x, x)\| \le C_f \frac{1}{|x|^{6+a}}.$$

1.2. ***Discussion.*** The theorems presented here are the Maxwell analogue of [Hanisch et al. 2022], where a similar statement was proved for the scalar Laplacian in the case of smooth boundary. The Maxwell system on a Lipschitz domain is different in several regards and introduces challenges that are absent in the scalar case:

• Maxwell's equations arise from an abelian gauge theory, and the gauge freedom results in the loss of ellipticity of the equations for the electromagnetic field. On the analysis side, this manifests itself as the equations taking place on the space of divergence-free vector fields rather than the space of sections of the vector bundle. This can however be fixed by considering the spectral decomposition of the Laplace operator and then employing the Helmholtz–Hodge decomposition to project onto the subspace of divergence-free vector fields. Projecting works well in cases with and without boundary as long as the geometric configuration is fixed. The projector constructed from the Helmholtz decomposition is roughly of the form $-\Delta^{-1}\delta\, \mathrm{d} = -\Delta^{-1}\operatorname{curl}\operatorname{curl}$, and it involves the nonlocal functional calculus of the Laplace operator. It therefore depends on the geometric configuration and also the boundary conditions imposed on the Laplace operator. This makes it much harder to directly apply scattering theory which requires an identification of the involved Hilbert spaces. The same problem appears in the context of the Birman–Krein formula in electromagnetic scattering. We have proved a variant of the Birman–Krein formula in [Strohmaier and Waters 2022] and we will follow the same formulation here.

• Unlike the Dirichlet–Laplacian, the Laplace operator on the space of vector fields with relative boundary conditions has a nontrivial kernel in the exterior domain. This leads to singularities of the resolvent near 0 and manifests itself in the presence of singularities of the boundary layer operators. Additional singularities of the boundary layer operators appear if the obstacles have nontrivial topology, which we do not exclude. To overcome this, we carefully analyse the singularities of various Maxwell boundary layer operators at 0, and we show that there are various cancellations that render a final result without singularities.

• An additional complication arises in this paper since we are considering Lipschitz domains instead of smooth ones. This requires more sophisticated harmonic analysis techniques. We rely here on a lot of progress in this subject that has been made during the past several decades, in particular with the identification of the appropriate function spaces.

As explained, the spectral theory of Δ_{rel} and Δ_{abs} determines the Maxwell system. Suitably interpreted, the curl operator intertwines these two operators in the sense that $\operatorname{curl}\Delta_{\mathrm{abs}} = \Delta_{\mathrm{rel}}\operatorname{curl}$. In the interior, the relative Laplacian on a suitable closed subspace consisting of divergence-free vector fields has the Maxwell eigenvalues as its spectrum, and the eigenfunctions describe modes of photons that are confined to Ω. The exterior relative Laplacian on a suitable closed subspace of divergence-free vector fields describes the scattering of electromagnetic waves or photons by the obstacles Ω. The functional calculus on Δ_{rel} on this subspace can be understood in terms of the operators $f(\Delta_{\mathrm{rel}})\operatorname{curl}\operatorname{curl}$. The following Birman–Krein formula has been proved.

1.3. *Relation to the Birman–Krein formula.* In the case that f is an even Schwartz function, we have that

$$(\operatorname{curl}\operatorname{curl}(f((-\Delta_{\mathrm{rel}})^{1/2}) - f((-\Delta_{\mathrm{free}})^{1/2})))$$

is trace-class, and its trace can be computed by the Birman–Krein-type formula

$$\operatorname{tr}(\operatorname{curl}\operatorname{curl}(f((-\Delta_{\mathrm{rel}})^{1/2}) - f((-\Delta_{\mathrm{free}})^{1/2}))) = \frac{1}{2\pi\mathrm{i}}\int_0^\infty \lambda^2 \operatorname{tr}(S_\lambda^{-1}(S_\lambda)')f(\lambda)\,\mathrm{d}\lambda + \sum_{j=1}^\infty f(\mu_j)\mu_j^2,$$

where S_λ is the scattering matrix for the Maxwell equation and μ_j are the Maxwell eigenvalues of the interior. As a consequence of this formula,

$$\operatorname{tr} D_{\mathrm{rel},f} = -\frac{1}{2\pi\mathrm{i}}\int_0^\infty \log\frac{\det S_\lambda}{\det(S_{1,\lambda})\cdots\det(S_{N,\lambda})}\frac{\mathrm{d}}{\mathrm{d}\lambda}(\lambda^2 f(\lambda))\,\mathrm{d}\lambda,$$

which is valid only under very restrictive assumptions on f. The same formula and statements hold for absolute instead of relative boundary conditions.

In the motivating example, one cannot use this formula. It would require $f(\lambda) = 1/\lambda$, which does not satisfy the assumptions of the Birman–Krein formula. In fact it can be shown that the integrand on the right-hand side is not integrable in that case. One has however the following relation between the function Ξ and the scattering matrices.

Theorem 1.5. *We have*

$$\log\frac{\det S_\lambda}{\det(S_{1,\lambda})\cdots\det(S_{N,\lambda})} = -(\Xi(\lambda) - \Xi(-\lambda))$$

for $\lambda \in \mathbb{R}$.

This theorem reflects the relation between the spectral shift function and zeta regularised determinants as discovered by Carron [2002, Theorem 1.3], generalising a formula by Gesztesy and Simon [1996, Theorem 1.1].

1.4. ***Organisation of the paper.*** The paper is organised as follows. Sections 2- 6 provide the required theoretical background for the paper and consist of essentially known material. Section 2 sets up the basic function spaces needed for boundary layer theory on Lipschitz domains. Section 3 summarises the spectral properties of the interior relative and absolute Laplace operators, and Section 4 reviews the scattering theory for the relative and absolute Laplacians on the exterior. Both are combined into one operator in Section 5. In this section we also discuss the Birman–Krein formula in the context of our setting. Section 6 introduces the basic Maxwell boundary layer operators and their properties.

The basic estimates and expansions for the layer potential operators needed for the proofs are covered in Section 7. This section is presented independently of the main results as its content is interesting in its own right. It covers various aspects of low-energy expansions for the electric and magnetic boundary layer operators and inverses. Section 8 gives formulae of the resolvent differences in terms of layer potential operators and thereby provides estimates for these differences. Such formulae are sometimes referred to as Krein-type resolvent formulae, and this section provides a Maxwell analogue of these. Sections 9 and 10 take on the main subject of this paper, namely function Ξ, the relative resolvent, and its trace. Section 11 finally contains the proofs of the main theorems.

2. Function spaces on Lipschitz domains

Since Ω is a Lipschitz domain, we have, by Rademacher's theorem, an almost everywhere defined exterior unit vector field $\nu \in L^\infty(\partial\Omega, \mathbb{R}^3)$. We will use the following spaces that now are standard in Maxwell theory:

- $H(\mathrm{curl}, M) = \{f \in L^2(M, \mathbb{C}^3) \mid \mathrm{curl}\, f \in L^2(M, \mathbb{C}^3)\}$.
- $H(\mathrm{div}, M) = \{f \in L^2(M, \mathbb{C}^3) \mid \mathrm{div}\, f \in L^2(M)\}$.
- $L^2_{\tan}(\partial\Omega) = \{f \in L^2(\partial\Omega, \mathbb{C}^3) \mid \nu \cdot f = 0 \text{ a.e. on } \partial\Omega\}$.
- $H^{-1/2}(\mathrm{Div}, \partial\Omega)$, $H^{-1/2}(\mathrm{Curl}, \partial\Omega)$.
- $H^{-1/2}(\mathrm{Div}\, 0, \partial\Omega)$, $H^{-1/2}(\mathrm{Curl}\, 0, \partial\Omega)$.

These spaces were introduced in [Buffa et al. 2002] and provide a convenient framework for dealing with Maxwell's equations on Lipschitz domains. We refer to the Appendix of [Kirsch and Hettlich 2015] for an extensive discussion, and we only summarise the basic properties.

In the case that $\partial\Omega$ is smooth, we have

$$\begin{aligned}
H^{-1/2}(\mathrm{Div}, \partial\Omega) &= \{f \in H^{-1/2}(\partial\Omega; T\partial\Omega) \mid \mathrm{Div}\, f \in H^{-1/2}(\partial\Omega)\},\\
H^{-1/2}(\mathrm{Curl}, \partial\Omega) &= \{f \in H^{-1/2}(\partial\Omega; T\partial\Omega) \mid \mathrm{Curl}\, f \in H^{-1/2}(\partial\Omega)\},\\
H^{-1/2}(\mathrm{Div}\, 0, \partial\Omega) &= \{f \in H^{-1/2}(\partial\Omega; T\partial\Omega) \mid \mathrm{Div}\, f = 0\},\\
H^{-1/2}(\mathrm{Curl}\, 0, \partial\Omega) &= \{f \in H^{-1/2}(\partial\Omega; T\partial\Omega) \mid \mathrm{Curl}\, f = 0\},
\end{aligned}$$

where Div is the surface divergence on $\partial\Omega$ and Curl is the surface curl. On a general Lipschitz domain, this can be defined via Lipschitz coordinate charts, thus locally reducing it to the smooth case. Note that the spaces $H^s_{\mathrm{loc}}(\mathbb{R}^d)$ are invariant under bi-Lipschitz maps if $|s| \le 1$. We refer to [Kirsch and Hettlich 2015] for a detailed discussion of the definition via coordinate charts. We also have the corresponding spaces for the interior domains. Namely we have

$$H(\mathrm{curl}, \Omega) = \{f \in L^2(\Omega, \mathbb{C}^3) \mid \mathrm{curl}\, f \in L^2(\Omega, \mathbb{C}^3)\},$$
$$H(\mathrm{div}, \Omega) = \{f \in L^2(\Omega, \mathbb{C}^3) \mid \mathrm{div}\, f \in L^2(\Omega)\}.$$

On $H(\mathrm{curl}, M)$, there are two distinguished and well-defined continuous trace maps

$$\gamma_{T,-} : H(\mathrm{curl}, M) \to H^{-1/2}(\mathrm{Curl}, \partial\Omega),$$
$$\gamma_{t,-} : H(\mathrm{curl}, M) \to H^{-1/2}(\mathrm{Div}, \partial\Omega),$$

which continuously extend the maps $f \mapsto (\nu \times f|_{\partial\Omega}) \times \nu$ and $f \mapsto (\nu \times f|_{\partial\Omega})$, respectively, defined on $C_0(\overline{M}, \mathbb{C}^3)$. Note that, for $x \in \partial\Omega$ such that ν_x is defined, the map $v \mapsto (\nu_x \times v) \times \nu_x$ is the orthogonal projection onto the tangent space of $\partial\Omega$ at x. Similarly, we have the map

$$\gamma_{\nu,-} : H(\mathrm{div}, M) \to H^{-1/2}(\partial\Omega)$$

continuously extending the normal restriction map $f \mapsto \nu \cdot f|_{\partial\Omega}$. On the interior domain Ω, we have the analogous maps

$$\gamma_{T,+} : H(\mathrm{curl}, \Omega) \to H^{-1/2}(\mathrm{Curl}, \partial\Omega),$$
$$\gamma_{t,+} : H(\mathrm{curl}, \Omega) \to H^{-1/2}(\mathrm{Div}, \partial\Omega),$$
$$\gamma_{\nu,+} : H(\mathrm{div}, \Omega) \to H^{-1/2}(\partial\Omega).$$

There is a well-defined dual pairing between $H^{-1/2}(\mathrm{Curl}, \partial\Omega)$ and $H^{-1/2}(\mathrm{Div}, \partial\Omega)$ that extends the L^2-inner product on $H^{1/2}(\partial\Omega) \cap L^2_{\tan}(\partial\Omega)$. We will denote this pairing by $\langle \cdot\,, \cdot \rangle_{L^2(\partial\Omega)}$, irrespective of the Sobolev order and mildly abusing notation. The map $\phi \mapsto \nu \times \phi$ extends to a continuous isomorphism from $H^{-1/2}(\mathrm{Div}, \partial\Omega)$ to $H^{-1/2}(\mathrm{Curl}, \partial\Omega)$ and vice versa. Moreover, the L^2-pairing induces an antilinear isomorphism between $H^{-1/2}(\mathrm{Div}, \partial\Omega)$ and $H^{-1/2}(\mathrm{Curl}, \partial\Omega)$ (see, for example, [Kirsch and Hettlich 2015, Lemma 5.61] for both statements). In other words, the antisymmetric bilinear form $\langle \cdot\,, \nu \times \cdot \rangle$ on $H^{-1/2}(\mathrm{Div}, \partial\Omega)$ is nondegenerate. Note here that, since $\nu \in L^\infty(\partial\Omega, \mathbb{R}^3)$, it is not immediately obvious that it is defined as a map between Sobolev spaces.

We recall Stokes' theorem for $\phi, E \in H(\mathrm{curl}, \Omega)$:

$$\langle \gamma_{t,+} E, \gamma_{T,+}\phi \rangle_{L^2(\partial\Omega)} = \langle \mathrm{curl}\, E, \phi \rangle_{L^2(\Omega)} - \langle E, \mathrm{curl}\, \phi \rangle_{L^2(\Omega)}, \tag{4}$$
$$\langle \mathrm{curl}\,\mathrm{curl}\, E, \phi \rangle_{L^2(\Omega)} - \langle E, \mathrm{curl}\,\mathrm{curl}\, \phi \rangle_{L^2(\Omega)} = \langle \gamma_{t,+}\, \mathrm{curl}\, E, \gamma_{T,+}\phi \rangle_{L^2(\partial\Omega)} + \langle \gamma_{t,+} E, \gamma_{T,+}\, \mathrm{curl}\, \phi \rangle_{L^2(\partial\Omega)}.$$

As before, we slightly abuse notation and write $\langle \cdot\,, \cdot \rangle_{L^2(\Omega)}$ for pairings extending the L^2-inner product. We define $H_0(\mathrm{curl}, M)$ and $H_0(\mathrm{div}, M)$ as the kernels of $\gamma_{t,-}$ and $\gamma_{\nu,-}$, respectively. These spaces play a similar role to the Sobolev space of functions $H^1_0(M)$, which can also be characterised as the kernel of the trace map $\gamma : H^1(M) \to H^{1/2}(\partial M)$. The spaces $H_0(\mathrm{curl}, \Omega)$ and $H_0(\mathrm{div}, \Omega)$, as well as $H^1_0(\Omega)$, are defined analogously.

If there is no danger of confusion, we will omit the $\pm$ and simply write γ_t and γ_ν, respectively.

We also have surface divergence Div and surface curl Curl. They satisfy

$$\mathrm{Div} \circ \gamma_{t,+} = -\gamma_{\nu,+} \circ \mathrm{curl}. \tag{5}$$

3. Laplace operators on the interior domain

3.1. ***The relative Laplacian.*** The operator

$$\mathrm{curl}_{\min} = \mathrm{curl}\,|_{H_0(\mathrm{curl},\Omega)} : H_0(\mathrm{curl}, \Omega) \to L^2(\Omega, \mathbb{C}^3)$$

is a closed densely defined operator. It coincides with the closure of the operator curl on the space of compactly supported smooth vector fields on Ω [Kirsch and Hettlich 2015, Theorem 5.25] and therefore equals the minimal closed extension of curl.

Its adjoint is the maximal extension, i.e., the closed operator

$$\mathrm{curl}_{\max} : H(\mathrm{curl}, \Omega) \to L^2(\Omega, \mathbb{C}^3). \tag{6}$$

For any closed densely defined operator A, the operator A^*A is automatically self-adjoint. If in addition $\mathrm{rg}(A) \subset \ker(A)$, then $A^*A + AA^*$ is self-adjoint if it is densely defined; see for example [Strohmaier and Waters 2022, Section 2]. It follows that $\mathrm{curl}_{\max}\,\mathrm{curl}_{\min}$ with domain

$$\{f \in H_0(\mathrm{curl}, \Omega) \mid \mathrm{curl}\, f \in H(\mathrm{curl}, \Omega)\}$$

is a nonnegative self-adjoint operator. Similarly, $\mathrm{div}_{\max} : H(\mathrm{div}, \Omega) \to L^2(\Omega)$ is a closed operator with adjoint $-\mathrm{grad}_{\min} : H_0^1(\Omega) \to L^2(\Omega)$. Therefore, the operator $-\mathrm{grad}_{\min}\,\mathrm{div}_{\max}$ is a nonnegative self-adjoint operator with domain

$$\{f \in H(\mathrm{div}, \Omega) \mid \mathrm{div}\, f \in H_0(\Omega)\}.$$

Their sum $\Delta_{\Omega,\mathrm{rel}} = \mathrm{curl}_{\max}\,\mathrm{curl}_{\min} - \mathrm{grad}_{\min}\,\mathrm{div}_{\max}$ is again self-adjoint and has domain

$$\{f \in H(\mathrm{div}, \Omega) \cap H_0(\mathrm{curl}, \Omega) \mid \mathrm{div}\, f \in H_0(\Omega),\ \mathrm{curl}\, f \in H(\mathrm{curl}, \Omega)\},$$

and on this domain $-\Delta_{\Omega,\mathrm{rel}}$ is given by $\mathrm{curl}\,\mathrm{curl} - \mathrm{grad}\,\mathrm{div}$. The implied boundary conditions of this operator are the so-called relative boundary conditions

$$\gamma_{t,+}(f) = 0, \quad \mathrm{div}\, f|_{\partial\Omega} = 0.$$

In the case of smooth boundary, the form domain of the interior relative Laplace operator is contained in $H^1(\Omega, \mathbb{C}^3)$. In the more general Lipschitz case this is no longer true, but it is known that the form domain is contained in $H^{1/2}(\Omega, \mathbb{C}^3)$; see [Costabel 1990, Theorem 2] and also [Mitrea and Mitrea 2002]. This is compactly embedded in $L^2(\Omega, \mathbb{C}^3)$, and therefore the interior relative Laplace operator has purely discrete spectrum. We have the classical Hodge–Helmholtz decomposition

$$L^2(\Omega) = \mathcal{H}^1(\Omega) \oplus \overline{\mathrm{rg}(\mathrm{grad}_{\min})} \oplus \overline{\mathrm{rg}(\mathrm{curl}_{\max})}$$

into an orthogonal direct sum. Here $\mathcal{H}^1(\Omega) = \ker(\Delta_{\Omega,\mathrm{rel}})$ is the finite-dimensional space of harmonic vector fields satisfying the relative boundary conditions. We will see in Section 3.3 that in fact the assumption that M is connected implies that $\mathcal{H}^1(\Omega) = \{0\}$.

We now describe the spectrum of the relative Laplace operator. On Ω we can choose an orthonormal basis (v_j) of Dirichlet eigenfunctions v_j in the domain of the Dirichlet Laplacian with eigenvalues λ_j^2:

$$-\Delta v_j = \lambda_{D,j}^2 v_j, \quad \lambda_{D,j} > 0, \quad v_j \in \{v \in H_0^1(\Omega, \mathbb{C}^3) \mid \nabla v \in H(\mathrm{div}, \Omega)\}.$$

We have $\lambda_j \to \infty$, and we arrange the eigenfunctions such that $\lambda_j \nearrow \infty$. Then $(1/\lambda_j)\nabla v_j$ form an orthonormal basis of eigenfunctions in $\overline{\mathrm{rg}(\mathrm{grad}_{\min})}$ of $-\Delta_\Omega$ with eigenvalues λ_j^2. One has the usual Weyl law for Lipschitz domains which can easily be inferred from the Weyl law for smooth domains using domain monotonicity and an approximation by smooth domains:

$$\lambda_{D,k} \sim \left(\frac{6\pi^2}{\mathrm{Vol}(\Omega)}\right)^{1/3} k^{1/3}, \quad k \to \infty.$$

The space $\overline{\mathrm{rg}(\mathrm{curl}_{\max})}$ on the other hand is the closure of the subspace spanned by ϕ_j, where (ϕ_j) is an orthonormal basis in $\ker(\mathrm{div}_{\max}) \subset L^2(\Omega, \mathbb{C}^3)$ satisfying the eigenvalue equation

$$-\Delta_{\Omega,\mathrm{rel}} \phi_j = \mu_j^2 \phi_j, \quad \mathrm{div}\, \phi_j = 0,$$

with boundary condition $\gamma_t(\phi_j) = 0$. Therefore 0 is not an eigenvalue. The numbers $\mu_j > 0$ are the Maxwell eigenvalues, and we again assume these are arranged such that $\mu_j \nearrow \infty$. The Maxwell eigenvalues are known to satisfy a Weyl law (see [Birman and Solomyak 1987] for Lipschitz domains, but also [Filonov 2013] and references for a general statement in arbitrary dimension):

$$\mu_k \sim \left(\frac{3\pi^2}{\mathrm{Vol}(\Omega)}\right)^{1/3} k^{1/3}, \quad k \to \infty.$$

The family $(\phi_j)_{\mu_j>0}$ then forms an orthonormal basis in $\overline{\mathrm{rg}(\mathrm{curl}_{\max})}$ consisting of eigenfunctions of $-\Delta_{\Omega,\mathrm{rel}}$ with nonzero eigenvalues μ_j^2. Summarising, there is an orthonormal basis of eigenfunctions $\Delta_{\Omega,\mathrm{rel}}$ of the form

$$\left\{\frac{1}{\lambda_j} \,\mathrm{grad}\, v_j \;\middle|\; j \in \mathbb{N}\right\} \cup \{\phi_j \mid \mu_j > 0\},$$

where v_j are the Dirichlet eigenfunctions and ϕ_j the Maxwell eigenfunctions with Maxwell eigenvalues μ_j.

3.2. ***The absolute Laplacian.*** It will also be convenient to consider another operator $\Delta_{\Omega,\mathrm{abs}}$, which is defined by

$$-\Delta_{\Omega,\mathrm{abs}} = \mathrm{curl}_{\min} \mathrm{curl}_{\max} - \mathrm{grad}_{\max} \mathrm{div}_{\min},$$

with domain

$$\{f \in H_0(\mathrm{div}, \Omega) \cap H(\mathrm{curl}, \Omega) \mid \mathrm{div}\, f \in H^1(\Omega),\ \mathrm{curl}\, f \in H_0(\mathrm{curl}, \Omega)\}.$$

Again, it is known that the form domain is contained in $H^{1/2}(\Omega, \mathbb{C}^3)$ [Costabel 1990, Theorem 2] and the domain is therefore compactly embedded into $L^2(\Omega, \mathbb{C}^3)$. In the same way as for the relative Laplacian, there is an explicit description of the spectrum which we now give. Let (u_j) be an orthonormal basis consisting of eigenfunctions of the Neumann Laplacian with eigenvalues $\lambda_{N,j}$. Hence

$$-\Delta u_j = \lambda_{N,j}^2 u_j, \quad \partial_\nu u_j|_{\partial\Omega} = 0, \quad u_j \in \{u \in H^1(\Omega) \mid \nabla u \in H_0^1(\mathrm{div}, \Omega)\}.$$

Then the functions $(1/\lambda_{N,j})\nabla u_j$ form an orthonormal set consisting of eigenfunctions of $\Delta_{\Omega,\mathrm{abs}}$.

We can construct another orthogonal set (ψ_j) from the Maxwell eigenfunctions ϕ_j of the relative Laplace operator by defining

$$\psi_j = \frac{1}{\mu_j} \operatorname{curl} \phi_j .$$

Since the spectrum is discrete, standard Hodge theory applies for the absolute Laplacian, and we obtain an orthogonal decomposition

$$L^2(\Omega, \mathbb{C}^3) = \mathcal{H}^1_{\mathrm{abs}}(\Omega) \oplus \overline{\operatorname{span}\left\{ \frac{1}{\lambda_{N,j}} \operatorname{grad} u_j \right\}} \oplus \overline{\operatorname{span}\{\psi_j\}},$$

where $\mathcal{H}^1_{\mathrm{abs}}(\Omega) = \ker \Delta_{\Omega,\mathrm{abs}}$. Unlike in the case of the relative Laplace operator, this space is in general nontrivial. We will in the following choose an orthonormal basis $(\psi_{0,k})_k$, where $1 \le k \le \dim(\mathcal{H}^1_{\mathrm{abs}}(\Omega))$. Therefore an orthonormal basis in $L^2(\Omega, \mathbb{C}^3)$ consisting of eigenfunctions of the absolute Laplacian is

$$\{\psi_{0,k} \mid 1 \le k \le \dim(H^1_{\mathrm{abs}}(\Omega))\} \cup \left\{ \frac{1}{\lambda_{N,j}} \nabla u_j \;\middle|\; j \in \mathbb{N} \right\} \cup \{\psi_j \mid j \in \mathbb{N}\}.$$

3.3. ***Relation to singular and de Rham cohomology groups.*** Since Ω is an oriented smooth manifold, we have, by de Rham's theorem, a natural isomorphism identifying $H^p_{\mathrm{dR}}(\Omega, \mathbb{C})$ with $H^p_{\mathrm{sing}}(\Omega, \mathbb{C}) = H^p_{\mathrm{sing}}(\Omega, \mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{C}$. Hodge theory is also applicable for Lipschitz domains in the sense that the natural map from $\ker \Delta_{\Omega,\mathrm{abs}}$ to the first de Rham cohomology group $H^1_{\mathrm{dR}}(\Omega, \mathbb{C})$ is an isomorphism. This can for example be inferred from the statement of [Mitrea et al. 2001, Theorems 11.1 and 11.2] together with the universal coefficient theorem and de Rham's theorem. This theorem also applies to the absolute Laplacian on 2-forms as defined in [Mitrea et al. 2001]. Since this operator is obtained by conjugation of the relative Laplacian on 1-forms with the Hodge star operator $*$, we therefore have that $* \ker \Delta_{\Omega,\mathrm{rel}}$ is isomorphic to $H^2_{\mathrm{dR}}(\Omega, \mathbb{C})$. Because the inner product is nondegenerate on these spaces, we have the following nondegenerate dual pairing

$$\ker \Delta_{\Omega,\mathrm{rel}} \times (* \ker \Delta_{\Omega,\mathrm{rel}}) \to \mathbb{C}, \quad (f_1, f_2) \mapsto \int_\Omega f_1 \wedge f_2 .$$

We also have, as a consequence of Poincaré duality, the nondegenerate dual pairing

$$H^1_{\mathrm{c,dR}}(\Omega, \mathbb{C}) \times H^2_{\mathrm{dR}}(\Omega, \mathbb{C}) \to \mathbb{C}, \quad (f_1, f_2) \mapsto \int_\Omega f_1 \wedge f_2 .$$

This establishes an isomorphism $\ker \Delta_{\Omega,\mathrm{rel}} \to H^1_{\mathrm{c,dR}}(\Omega, \mathbb{C})$, which relates the harmonic forms to the de Rham cohomology groups with compact support. Since elements in $\ker \Delta_{\Omega,\mathrm{rel}}$ are not compactly supported, this map is defined indirectly by duality.

Our assumptions imply that in fact $H^1_{\mathrm{c,dR}}(\Omega, \mathbb{C})$ is trivial and therefore $\ker \Delta_{\Omega,\mathrm{rel}} = \{0\}$. This reflects the observation that a domain with connected exterior cannot have homologically nontrivial 2-cycles (inclusions).

Lemma 3.1. *Let U be an open C^0-domain with compact closure in $\mathbb{R}^d$ with $d \ge 2$ such that $\mathbb{R}^d \setminus \overline{U}$ is connected. Then $H^1_{\mathrm{c,dR}}(U) = \{0\}$.*

Proof. Let α be a smooth closed 1-form with compact support in U. By the Poincaré lemma, there is a smooth function $f : \mathbb{R}^d \to \mathbb{R}$ with $\alpha = \mathrm{d}f$. Since f is locally constant in the complement of the support of α, it must be constant in $\mathbb{R}^d \setminus \overline{U}$, as this set was assumed to be connected. By continuity, f is constant in $\mathbb{R}^d \setminus \overline{U}$ and, since locally constant, it is constant in a neighbourhood of $\mathbb{R}^d \setminus \overline{U}$. It follows that $f - c$ is compactly supported in U. Since $\alpha = \mathrm{d}(f - c)$, the class α vanishes $H^1_{\mathrm{c,dR}}(U)$, and therefore $H^1_{\mathrm{c,dR}}(U) = \{0\}$. □

4. Laplace operators on the exterior domain

As in the interior case, the operator

$$\mathrm{curl}_{\min} = \mathrm{curl}\,|_{H_0(\mathrm{curl},M)} : H_0(\mathrm{curl}, M) \to L^2(M, \mathbb{C}^3)$$

is a closed densely defined operator with adjoint

$$\mathrm{curl}_{\max} : H(\mathrm{curl}, M) \to L^2(M, \mathbb{C}^3).$$

It follows that $\mathrm{curl}_{\max}\,\mathrm{curl}_{\min}$ with domain

$$\{f \in H_0(\mathrm{curl}, M) \mid \mathrm{curl}\, f \in H(\mathrm{curl}, M)\}$$

is a nonnegative self-adjoint operator. Similarly, $\mathrm{div}_{\max} : H(\mathrm{div}, M) \to L^2(M)$ is a closed operator with adjoint $-\mathrm{grad}_{\min} : H^1_0(M) \to L^2(M)$. Therefore, the operator $-\mathrm{grad}_{\min}\,\mathrm{div}_{\max}$ is a nonnegative self-adjoint operator with domain

$$\{f \in H(\mathrm{div}, M) \mid \mathrm{div}\, f \in H_0(M)\}.$$

Their sum $-\Delta_{M,\mathrm{rel}} = \mathrm{curl}_{\max}\,\mathrm{curl}_{\min} - \mathrm{grad}_{\min}\,\mathrm{div}_{\max}$ then has domain

$$\{f \in H(\mathrm{div}, M) \cap H_0(\mathrm{curl}, M) \mid \mathrm{div}\, f \in H_0(M),\ \mathrm{curl}\, f \in H(\mathrm{curl}, M)\}.$$

The implied boundary conditions are the exterior relative boundary conditions

$$\gamma_{t,-}(f) = 0, \quad \mathrm{div}\, f|_{\partial\Omega} = 0.$$

The spectrum of the operator $\Delta_{M,\mathrm{rel}}$ consists of a finite multiplicity eigenvalue at 0 and a purely absolutely continuous part. This is the consequence of the finite-type meromorphic continuation of the resolvent and Rellich's theorem. We have described this in detail in [Strohmaier and Waters 2020] for smooth domains, but this part of the paper carries over to Lipschitz domains without change; see [Strohmaier and Waters 2022] for a discussion of this point. The absolutely continuous part of the spectrum can be described well by stationary scattering theory. For each $\Phi \in C^\infty(\mathbb{S}^2, \mathbb{C}^3)$ and $\lambda > 0$, there exists a unique generalised eigenfunction $E_\lambda(\Phi) \in C^\infty(M, \mathbb{C}^3)$ satisfying the boundary conditions of $\Delta_{M,\mathrm{rel}}$ near $\partial\Omega$ such that

$$(-\Delta - \lambda^2)E_\lambda(\Phi) = 0, \tag{7}$$

$$E_\lambda(\Phi) = \frac{\mathrm{e}^{-\mathrm{i}\lambda r}}{r}\Phi - \frac{\mathrm{e}^{\mathrm{i}\lambda r}}{r}\Psi_\lambda(\Phi) + O\left(\frac{1}{r^2}\right) \quad \text{for } r \to \infty \tag{8}$$

uniformly in the angular variables on the sphere for some $\Psi_\lambda(\Phi) \in C^\infty(\mathbb{S}^2, \mathbb{C}^3)$. The expansion (8) may be differentiated; see Proposition 2.6 and Appendix E in [Strohmaier and Waters 2020] for a justification. Here, satisfying the boundary conditions near $\partial\Omega$ means that $\chi E_\lambda(\Phi) \in \mathrm{dom}(\Delta_M)$ for any compactly supported smooth χ on M such that $\chi = 1$ near $\partial\Omega$.

The above implicitly defines the *scattering matrix* as a map $\widetilde{S}_\lambda : C^\infty(\mathbb{S}^2, \mathbb{C}^3) \to C^\infty(\mathbb{S}^2, \mathbb{C}^3)$ by $\Psi_\lambda(\Phi) = \tau \widetilde{S}_\lambda \Phi$, where $\tau : C^\infty(\mathbb{S}^2; \mathbb{C}^3) \to C^\infty(\mathbb{S}^2; \mathbb{C}^3)$ is the pullback of the antipodal map. It extends continuously as $\widetilde{S}_\lambda : L^2(\mathbb{S}^2, \mathbb{C}^3) \to L^2(\mathbb{S}^2, \mathbb{C}^3)$. The map $\tilde{A}_\lambda = \widetilde{S}_\lambda - \mathrm{id}$ is called the scattering amplitude. We have the equations

$$\operatorname{curl}\operatorname{curl} E_\lambda(\Phi) = \lambda^2 E_\lambda(\boldsymbol{r} \times \Phi \times \boldsymbol{r}), \quad \operatorname{div} E_\lambda(\Phi) = -\mathrm{i}\lambda E^0_\lambda(\boldsymbol{r} \cdot \Phi),$$

where $\boldsymbol{r}$ is the radius vector, i.e., the outward-pointing unit vector on the sphere. Here $E^0_\lambda(\boldsymbol{r} \cdot \Phi)$ is the generalised eigenfunction for the exterior Dirichlet problem on scalar-valued functions defined in an analogous way; see Proposition 4.7 in [Strohmaier and Waters 2022]. In particular this means that in the case that Φ is purely tangential, $\boldsymbol{r} \cdot \Phi = 0$, the generalised eigenfunction is a solution of the stationary Maxwell equation

$$\operatorname{curl}\operatorname{curl} E_\lambda(\Phi) = \lambda^2 E_\lambda(\Phi),$$
$$\operatorname{div} E_\lambda(\Phi) = 0$$

that satisfies the boundary conditions near $\partial\Omega$. These equations also imply that the scattering matrix is of the form

$$\widetilde{S}_\lambda = \begin{pmatrix} S^D_\lambda & 0 \\ 0 & S_\lambda \end{pmatrix}$$

if $L^2(\mathbb{S}^2, \mathbb{C}^3)$ is decomposed into $L^2(\mathbb{S}^2)\boldsymbol{r} \oplus L^2_{\tan}(\mathbb{S}^2, \mathbb{C}^3)$. Here $L^2_{\tan}(\mathbb{S}^2, \mathbb{C}^3)$ is the space of tangential square-integrable vector fields on the sphere. The operator S^D_λ is the scattering operator for scalar-valued functions with Dirichlet conditions imposed on $\partial\Omega$, and S_λ is the Maxwell scattering operator, describing the scattering of electromagnetic waves. Note that we have the weak Hodge–Helmholtz decomposition

$$L^2(M) = \mathcal{H}^1_{\mathrm{rel}}(M) \oplus \overline{\mathrm{rg}(\mathrm{grad}_{\min})} \oplus \overline{\mathrm{rg}(\mathrm{curl}_{\max})}, \tag{9}$$

which holds very generally in the abstract context of Hilbert complexes [Brüning and Lesch 1992]. The first summand is the discrete spectral subspace, and the splitting of its orthogonal complement into the last two subspaces corresponds to the above decomposition of the scattering matrix.

4.1. ***The exterior absolute Laplacian.*** In the same way as for the interior problem, there is also an exterior absolute Laplacian $\Delta_{M,\mathrm{abs}}$ defined by

$$-\Delta_{M,\mathrm{abs}} = \mathrm{curl}_{\min}\mathrm{curl}_{\max} - \mathrm{grad}_{\max}\mathrm{div}_{\min}.$$

The spectrum of $\Delta_{M,\mathrm{abs}}$ consists of a finite multiplicity eigenvalue at 0 and an absolutely continuous part. The absolutely continuous part is described by generalised eigenfunctions $E_{\mathrm{abs},\lambda}(\Phi)$ which are related to the generalised eigenfunctions $E_\lambda(\Phi)$ of the relative Laplacian by

$$E_{\mathrm{abs},\lambda}(\boldsymbol{r} \times \Phi) = -\frac{\mathrm{i}}{\lambda}\operatorname{curl} E_\lambda(\Phi). \tag{10}$$

One checks easily that

$$(-\Delta_{M,\mathrm{rel}} - \lambda^2)^{-1}\,\mathrm{curl} = \mathrm{curl}(-\Delta_{M,\mathrm{abs}} - \lambda^2)^{-1}$$

on the dense set of compactly supported smooth functions and, appropriately interpreted, extends by continuity to a larger space. This will allow us to reduce to statements about the absolute Laplace operator to statements about the relative Laplace operator. For the purposes of this paper, it will therefore not be necessary to introduce separate notation for the spectral decomposition. For example, the scattering matrix

$$\widetilde{S}_{\mathrm{abs},\lambda} = \begin{pmatrix} S_\lambda^N & 0 \\ 0 & S_{\mathrm{abs},\lambda} \end{pmatrix}$$

for the absolute Laplacian is defined by the expansion of $E_{\mathrm{abs},\lambda}(\Phi)$. Here S_λ^N is the scattering matrix for the Neumann Laplace operator on M acting on functions. We then have the equation

$$S_{\mathrm{abs},\lambda}(g) = \boldsymbol{r} \times S_\lambda(g \times \boldsymbol{r}) \tag{11}$$

for $g \in L^2_{\mathrm{tan}}(\mathbb{S}^2, \mathbb{C}^3)$. This follows by applying curl to the expansion (8), the uniqueness of the generalised eigenfunctions, and (10).

5. The combined relative operators and the Birman–Krein formula

In the following, it will be convenient to combine the operators $\Delta_{M,\mathrm{rel}}$ and $\Delta_{\Omega,\mathrm{rel}}$ into a single operator acting on the Hilbert space $L^2(\mathbb{R}^3, \mathbb{C}^3)$. We have $L^2(\mathbb{R}^3, \mathbb{C}^3) = L^2(M, \mathbb{C}^3) \oplus L^2(\Omega, \mathbb{C}^3)$, and we define the operator $\Delta_{\mathrm{rel}} := \Delta_{M,\mathrm{rel}} \oplus \Delta_{\Omega,\mathrm{rel}}$. In contrast to this, we also have the free Laplace operator Δ_{free} with domain $H^2(\mathbb{R}^3, \mathbb{C}^3)$. Following the paper [Hanisch et al. 2022], on the relative trace we also define the operator $\Delta_{j,\mathrm{rel}}$ for each boundary component Ω_j. This will correspond to the operator Δ_{rel} when all the other boundary components are absent, i.e., when $\Omega = \Omega_j$. As in [Hanisch et al. 2022], we would like to consider an analogue of the relative trace for the Laplace operator acting on divergence-free vector fields. In this section we assume that $f \in \mathcal{S}(\mathbb{R})$ is an even Schwartz function, but later on we will focus on another function class. We would like to compute the relative trace

$$\begin{aligned}\mathrm{tr}\biggl(\mathrm{curl}\,\mathrm{curl}\biggl(f((-\Delta_{\mathrm{rel}})^{1/2}) - f((-\Delta_{\mathrm{free}})^{1/2}) - \biggl(\sum_{j=1}^N f((-\Delta_{j,\mathrm{rel}})^{1/2}) - f((-\Delta_{\mathrm{free}})^{1/2})\biggr)\biggr)\biggr) \\ = \mathrm{tr}\biggl(\mathrm{curl}\,\mathrm{curl}\biggl(f((-\Delta_{\mathrm{rel}})^{1/2}) - \sum_{j=1}^N f((-\Delta_{j,\mathrm{rel}})^{1/2}) + (N-1) f((-\Delta_{\mathrm{free}})^{1/2})\biggr)\biggr),\end{aligned}$$

which is the trace of the operator

$$D_{\mathrm{rel},f} = \mathrm{curl}\,\mathrm{curl}\biggl(f((-\Delta_{\mathrm{rel}})^{1/2}) - f((-\Delta_{\mathrm{free}})^{1/2}) - \biggl(\sum_{j=1}^N f((-\Delta_{j,\mathrm{rel}})^{1/2}) - f((-\Delta_{\mathrm{free}})^{1/2})\biggr)\biggr).$$

We have the following Birman–Krein-type formula, proved recently in [Strohmaier and Waters 2022] and its simple consequence for the relative trace.

Theorem 5.1 [Strohmaier and Waters 2022, Theorem 1.5]. *Let $f \in C_0^\infty(\mathbb{R})$ be an even function. Then the operator*

$$\operatorname{curl}\operatorname{curl}(f((-\Delta_{\mathrm{rel}})^{1/2}) - f((-\Delta_{\mathrm{free}})^{1/2}))$$

extends to a trace-class operator on $L^2(\mathbb{R}^3, \mathbb{C}^3)$, and its trace equals

$$\operatorname{tr}(\operatorname{curl}\operatorname{curl}(f((-\Delta_{\mathrm{rel}})^{1/2}) - f((-\Delta_{\mathrm{free}})^{1/2}))) = \frac{1}{2\pi \mathrm{i}} \int_0^\infty \lambda^2 \operatorname{tr}(S_\lambda^{-1}(S_\lambda)') f(\lambda)\, \mathrm{d}\lambda + \sum_{j=1}^\infty f(\mu_j)\mu_j^2.$$

Moreover,

$$\operatorname{tr}(D_f) = -\int_0^\infty \xi_D(\lambda)(f(\lambda)\lambda^2)'\, \mathrm{d}\lambda,$$

where

$$\xi_D(\lambda) = \frac{1}{2\pi \mathrm{i}} \log \frac{\det S_\lambda}{\det(S_{1,\lambda}) \cdots \det(S_{N,\lambda})}.$$

A similar statement holds for the absolute Laplacian. Using (11) and

$$\operatorname{curl} f((-\Delta_{\mathrm{rel}})^{1/2}) = f((-\Delta_{\mathrm{abs}})^{1/2}) \operatorname{curl}$$

one obtains

$$\begin{aligned} \operatorname{tr}(\operatorname{curl}(f((-\Delta_{\mathrm{rel}})^{1/2}) - f((-\Delta_{\mathrm{free}})^{1/2})) \operatorname{curl}) &= \operatorname{tr}(\operatorname{curl}\operatorname{curl}(f((-\Delta_{\mathrm{abs}})^{1/2}) - f((-\Delta_{\mathrm{free}})^{1/2}))) \\ &= \operatorname{tr}(\operatorname{curl}\operatorname{curl}(f((-\Delta_{\mathrm{rel}})^{1/2}) - f((-\Delta_{\mathrm{free}})^{1/2}))) \\ &= \frac{1}{2\pi \mathrm{i}} \int_0^\infty \lambda^2 \operatorname{tr}(S_\lambda^{-1}(S_\lambda)') f(\lambda)\, \mathrm{d}\lambda + \sum_{j=1}^\infty f(\mu_j)\mu_j^2. \end{aligned}$$

The Birman–Krein formula can be proved for a slightly larger function class than the space of even Schwartz functions, but nondecaying functions are not admissible. The rest of the paper is devoted to dealing with exactly the trace-class properties of D_f when f is in a different function class that contains possibly growing functions.

6. Maxwell boundary layer operators

Maxwell boundary layer theory for Lipschitz domains is a well-developed subject in mathematics, and in this section we summarise the material that we are going to need. The distributional kernel of the resolvent of the operator $(-\Delta_{\mathrm{free}} - \lambda^2)^{-1}$ is called the Green's function and in dimension three is given explicitly by

$$G_{\lambda,\mathrm{free}}(x, y) = \frac{1}{4\pi} \frac{\mathrm{e}^{\mathrm{i}\lambda|x-y|}}{|x-y|}. \tag{12}$$

Note that this kernel is holomorphic at 0. As usual we define the single layer potential operator $\widetilde{\mathcal{S}}_\lambda : H^{-1/2}(\partial\Omega) \to H^1_{\mathrm{loc}}(\mathbb{R}^3)$ by

$$\widetilde{\mathcal{S}}_\lambda = (-\Delta_{\mathrm{free}} - \lambda^2)^{-1}\gamma^*.$$

This is defined for any $\lambda \in \mathbb{C}$ and a holomorphic family of operators. The single layer operator is defined by taking the trace $\mathcal{S}_\lambda = \gamma_+\widetilde{\mathcal{S}}_\lambda = \gamma_+(-\Delta_{\mathrm{free}} - \lambda^2)^{-1}\gamma^*$. The interior trace γ_+ and the exterior trace γ_-

coincide on the range of $\widetilde{\mathcal{S}}_\lambda$ and therefore we could also have used γ_- to define this operator. The operator $\mathcal{S}_\lambda$ is a holomorphic family of maps $H^{-1/2}(\partial\Omega) \to H^{1/2}(\partial\Omega)$. Both operators $\widetilde{\mathcal{S}}_\lambda$ and $\mathcal{S}_\lambda$ act componentwise on $H^{-1/2}(\partial\Omega, \mathbb{C}^3)$ and define maps to $H^1_{\mathrm{loc}}(\mathbb{R}^3, \mathbb{C}^3)$ and $H^{-1/2}(\partial\Omega, \mathbb{C}^3)$, respectively. We will distinguish this notationally from the map on functions.

We will also need the double layer operator $\mathcal{K}_\lambda$ and its transpose (complex conjugate-adjoint) $\mathcal{K}^t_\lambda$. The latter is given by

$$\mathcal{K}^t_\lambda u = \tfrac{1}{2}(\gamma_+ \nabla_\nu \mathcal{S}_\lambda u + \gamma_- \nabla_\nu \mathcal{S}_\lambda u)$$

and defines a continuous map $\mathcal{K}^t_\lambda : H^{-1/2}(\partial\Omega) \to H^{-1/2}(\partial\Omega)$. Its transpose $\mathcal{K}_\lambda$ therefore defines a continuous map $\mathcal{K}_\lambda : H^{1/2}(\partial\Omega) \to H^{1/2}(\partial\Omega)$. The following jump relations are characteristic:

$$\gamma_+ \mathcal{S}_\lambda u = \gamma_- \mathcal{S}_\lambda u, \quad \gamma_\pm \nabla_\nu \mathcal{S}_\lambda u = \left(\mp\tfrac{1}{2} + \mathcal{K}^t_\lambda\right) u.$$

We have the following representation formulae for divergence-free solutions $\phi \in H(\mathrm{curl}, M) \oplus H(\mathrm{curl}, \Omega)$ of the vector-valued Helmholtz equation

$$(-\Delta - \lambda^2)\phi = 0, \quad \operatorname{div}\phi = 0$$

by single layer potential operators:

$$\phi|_M = -\operatorname{curl} \widetilde{\mathcal{S}}_\lambda(\gamma_{t,-}\phi) + \nabla \widetilde{\mathcal{S}}_\lambda(\gamma_{\nu,-}\phi) - \widetilde{\mathcal{S}}_\lambda(\gamma_{t,-} \operatorname{curl}\phi) \tag{13}$$

and likewise

$$\phi|_\Omega = -\operatorname{curl} \widetilde{\mathcal{S}}_\lambda(\gamma_{t,+}\phi) + \nabla \widetilde{\mathcal{S}}_\lambda(\gamma_{\nu,+}\phi) - \widetilde{\mathcal{S}}_\lambda(\gamma_{t,+} \operatorname{curl}\phi); \tag{14}$$

see Corollary 3.3 in [Mitrea et al. 1997]

In Maxwell theory one defines additional layer potential operators as follows. Let L be the distribution defined by

$$L_\lambda(x, y) = \operatorname{curl}_x \operatorname{curl}_x G_{\lambda,\mathrm{free}}(x, y).$$

This is the kernel of the operator $(-\Delta_{\mathrm{free}} - \lambda^2)^{-1} \operatorname{curl}\operatorname{curl} = \operatorname{curl}\operatorname{curl}(-\Delta_{\mathrm{free}} - \lambda^2)^{-1}$. It is again holomorphic at $\lambda = 0$ as a kernel. The corresponding operator L_λ is related to the operator

$$(\lambda^2 + \operatorname{grad}\operatorname{div})(-\Delta_{\mathrm{free}} - \lambda^2)^{-1},$$

whose distributional integral kernel equals the so-called dyadic Green's function

$$K_\lambda(x, y) = (\lambda^2 + \operatorname{grad}_x \operatorname{div}_x) \frac{1}{4\pi} \frac{\mathrm{e}^{\mathrm{i}\lambda|x-y|}}{|x-y|},$$

which is more commonly used in computational electrodynamics. However, we also have the equality

$$L_\lambda(x, y) - K_\lambda(x, y) = \delta(x - y);$$

hence the kernels agree outside the diagonal. We define now the *Maxwell single layer potential operator* for $u \in H^{1/2}(\partial\Omega, \mathbb{C}^3) \cap L^2_{\tan}(\partial\Omega)$ as

$$u \mapsto \widetilde{\mathcal{L}}_\lambda u, \quad (\widetilde{\mathcal{L}}_\lambda u)(x) = \int_{\partial\Omega} L_\lambda(x, y) u(y)\, \mathrm{d}y = \int_{\partial\Omega} K_\lambda(x, y) u(y)\, \mathrm{d}y.$$

Therefore this can also be written as $\widetilde{\mathcal{L}}_\lambda u = \operatorname{curl}\operatorname{curl}\widetilde{\mathcal{S}}_\lambda u$. Similarly one defines the Maxwell magnetic layer potential operator $\widetilde{\mathcal{M}}_\lambda$ as $\widetilde{\mathcal{M}}_\lambda u = \operatorname{curl}\widetilde{\mathcal{S}}_\lambda u$. For all $\lambda \in \mathbb{C}$, these maps extend continuously to maps as follows:

$$\widetilde{\mathcal{L}}_\lambda : H^{-1/2}(\mathrm{Div}, \partial\Omega) \to H_{\mathrm{loc}}(\mathrm{curl}, M) \oplus H_{\mathrm{loc}}(\mathrm{curl}, \Omega),$$
$$\widetilde{\mathcal{M}}_\lambda : H^{-1/2}(\mathrm{Div}, \partial\Omega) \to H_{\mathrm{loc}}(\mathrm{curl}, M) \oplus H_{\mathrm{loc}}(\mathrm{curl}, \Omega).$$

It will be convenient to distinguish notationally between the exterior part $\widetilde{\mathcal{M}}_{-,\lambda}$ and the interior part $\widetilde{\mathcal{M}}_{+,\lambda}$ of $\widetilde{\mathcal{M}}_\lambda$. The boundedness of these maps is established in [Kirsch and Hettlich 2015] for $\operatorname{Im}\lambda \geq 0,\ \lambda \neq 0$, but these maps extend to holomorphic families on the entire complex plane as we will see later.

The *Maxwell single layer operator* $\mathcal{L}_\lambda$ is then defined for all $\lambda \in \mathbb{C}$ as a map

$$\mathcal{L}_\lambda : H^{-1/2}(\mathrm{Div}, \partial\Omega) \to H^{-1/2}(\mathrm{Div}, \partial\Omega), \quad u \mapsto \gamma_t \widetilde{\mathcal{L}}_\lambda$$

and is a holomorphic family of bounded operators on $H^{-1/2}(\mathrm{Div}, \partial\Omega)$ in λ. With respect to the above splitting, we then have

$$\widetilde{\mathcal{M}}_\lambda = \widetilde{\mathcal{M}}_{-,\lambda} \oplus \widetilde{\mathcal{M}}_{+,\lambda}.$$

One defines the *magnetic dipole operator* $\mathcal{M}_\lambda$ for all $\lambda \in \mathbb{C}$ by

$$\mathcal{M}_\lambda : H^{-1/2}(\mathrm{Div}, \partial\Omega) \to H^{-1/2}(\mathrm{Div}, \partial\Omega), \quad \mathcal{M}_\lambda = \tfrac{1}{2}(\gamma_t \widetilde{\mathcal{M}}_{-,\lambda} + \gamma_t \widetilde{\mathcal{M}}_{+,\lambda}).$$

By [Kirsch and Hettlich 2015, Theorem 5.52], this is a family of bounded operators on the space $H^{-1/2}(\mathrm{Div}, \partial\Omega)$ when $\operatorname{Im}\lambda > 0$. If $u = \widetilde{\mathcal{M}}a = \operatorname{curl}\widetilde{\mathcal{S}}_\lambda a$ then we have the jump conditions

$$\gamma_{t,\pm} u = \mp\tfrac{1}{2}a + \mathcal{M}_\lambda a, \quad \gamma_{t,\pm}\operatorname{curl} u = \mathcal{L}_\lambda a. \tag{15}$$

Moreover, the operator $\widetilde{\mathcal{L}}_\lambda a$ can be written as

$$\widetilde{\mathcal{L}}_\lambda a = \nabla\widetilde{\mathcal{S}}_\lambda \operatorname{Div} a + \lambda^2 \widetilde{\mathcal{S}}_\lambda a, \quad a \in H^{-1/2}(\mathrm{Div}, \partial\Omega). \tag{16}$$

We refer to [Kirsch and Hettlich 2015, Theorem 5.4] for both statements.

If $\operatorname{Im}\lambda \geq 0$ is nonzero then there exists a unique solution of the exterior boundary value problem for every $A \in H^{-1/2}(\mathrm{Div}, \partial\Omega)$, which satisfies the Silver–Müller radiation condition [Kirsch and Hettlich 2015, Theorem 5.64]. For the interior problem there exists a similar statement. If $\lambda \in \mathbb{C} \setminus \{0\}$ is not a Maxwell eigenvalue then there exists a unique solution of the interior boundary value problem for every $A \in H^{-1/2}(\mathrm{Div}, \partial\Omega)$. In both cases, if $\lambda \neq 0$ the solution can be written as a boundary layer potential of the form

$$E(x) = (\widetilde{\mathcal{L}}a)(x) = \operatorname{curl}^2\langle a, G_\lambda(x, \cdot)\rangle_{\partial\Omega}, \quad H(x) = \frac{\mathrm{i}\operatorname{curl} E}{-\lambda}, \quad x \notin \partial\Omega, \tag{17}$$

with the density $a \in H^{-1/2}(\mathrm{Div}, \partial\Omega)$, which satisfies $\mathcal{L}_\lambda a = A$; see again Theorem 5.60 in [Kirsch and Hettlich 2015].

The space of boundary data $(\gamma_t(E), \gamma_t(H))$ of solutions of Maxwell's equations is described by the Calderon projector. To describe this we first observe that, given $a, b \in H^{-1/2}(\mathrm{Div}, \partial\Omega)$, we obtain for any nonzero λ a solution of the interior Maxwell's equation $E, H \in H(\mathrm{curl}, \Omega)$ given by

$$E = -\widetilde{\mathcal{M}}_\lambda a + \frac{1}{\mathrm{i}\lambda}\widetilde{\mathcal{L}}_\lambda b, \quad H = -\widetilde{\mathcal{M}}_\lambda b - \frac{1}{\mathrm{i}\lambda}\widetilde{\mathcal{L}}_\lambda a,$$

and therefore, using (15), the boundary data $(\gamma_t(E), \gamma_t(H))$ is described as

$$\begin{pmatrix} \gamma_t(E) \\ \gamma_t(H) \end{pmatrix} = \begin{pmatrix} \frac{1}{2} - \mathcal{M}_\lambda & \frac{1}{\mathrm{i}\lambda}\mathcal{L}_\lambda \\ -\frac{1}{\mathrm{i}\lambda}\mathcal{L}_\lambda & \frac{1}{2} - \mathcal{M}_\lambda \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix}.$$

By the Stratton–Chu representation formula [Kirsch and Hettlich 2015, Theorem 5.49], we have that if (E, H) solves Maxwell's equations then E and H can be recovered from the boundary data as

$$E = -\widetilde{\mathcal{M}}_\lambda(\gamma_t E) + \frac{1}{\mathrm{i}\lambda}\widetilde{\mathcal{L}}_\lambda(\gamma_t H), \quad H = -\widetilde{\mathcal{M}}_\lambda(\gamma_t H) - \frac{1}{\mathrm{i}\lambda}\widetilde{\mathcal{L}}_\lambda(\gamma_t E).$$

Hence the operator

$$P_+ = \begin{pmatrix} \frac{1}{2} - \mathcal{M}_\lambda & \frac{1}{\mathrm{i}\lambda}\mathcal{L}_\lambda \\ -\frac{1}{\mathrm{i}\lambda}\mathcal{L}_\lambda & \frac{1}{2} - \mathcal{M}_\lambda \end{pmatrix}$$

acting on $H^{-1/2}(\mathrm{Div}, \partial\Omega) \oplus H^{-1/2}(\mathrm{Div}, \partial\Omega)$ is a projection onto the space of boundary data of solutions of Maxwell's equation in $H(\mathrm{curl}, \Omega) \oplus H(\mathrm{curl}, \Omega)$. This map is called the interior Calderon projector. In the same way, the exterior Calderon projector P_- acting on $H^{-1/2}(\mathrm{Div}, \partial\Omega) \oplus H^{-1/2}(\mathrm{Div}, \partial\Omega)$ is given by

$$P_- = \begin{pmatrix} \frac{1}{2} + \mathcal{M}_\lambda & -\frac{1}{\mathrm{i}\lambda}\mathcal{L}_\lambda \\ \frac{1}{\mathrm{i}\lambda}\mathcal{L}_\lambda & \frac{1}{2} + \mathcal{M}_\lambda \end{pmatrix}.$$

It projects onto the space of boundary data of solutions of Maxwell's equation in $H(\mathrm{curl}, \Omega) \oplus H(\mathrm{curl}, \Omega)$ when $\mathrm{Im}\,\lambda > 0$ and more generally solutions satisfying a radiation condition for nonzero real λ. As usual one has $P_+ + P_- = \mathrm{id}$.

We now define the voltage-to-current mappings $\Lambda_\lambda^\pm : H^{-1/2}(\mathrm{Div}, \partial\Omega) \to H^{-1/2}(\mathrm{Div}, \partial\Omega)$ by

$$\Lambda_\lambda^\pm : \gamma_t(E) \to \gamma_t(H), \tag{18}$$

where E and H are solutions to the interior and exterior boundary value problem for the Maxwell system (1), respectively, whenever these solutions are unique. The graphs of $\Lambda_\lambda^\pm$ in $H^{-1/2}(\mathrm{Div}, \partial\Omega) \oplus H^{-1/2}(\mathrm{Div}, \partial\Omega)$ are therefore by definition the ranges of the Calderon projectors $P_\pm$. The voltage-to-current maps are henceforth the Maxwell analogues of the interior and exterior Helmholtz Dirichlet-to-Neumann maps.

The mapping Λ_λ^+ is well defined for any $\lambda \in \mathbb{C}$ which is not a Maxwell eigenvalue or 0. The mapping Λ_λ^- is well defined for all nonzero λ in the closed upper half-space. In this case these are bounded operators on $H^{-1/2}(\mathrm{Div}, \partial\Omega)$. We will see later that these operators extend meromorphically to the complex plane. In anticipation of this we will not explicitly state the domains when dealing with algebraic identities. As a consequence of the symmetry $(E, H) \mapsto (H, -E)$ of the Maxwell system and the above relations, one obtains the formulae

$$(\Lambda_\lambda^\pm)^2 = -\mathrm{id} \quad \text{and} \quad \mathcal{L}_\lambda = \mathrm{i}\lambda\Lambda_\lambda^\pm\big(\mp\tfrac{1}{2} + \mathcal{M}_\lambda\big) = -\mathrm{i}\lambda\big(\pm\tfrac{1}{2} + \mathcal{M}_\lambda\big)\Lambda_\lambda^\pm, \tag{19}$$

and as a consequence

$$-\mathrm{i}\lambda^{-1}\mathcal{L}_\lambda(\Lambda_\lambda^+ - \Lambda_\lambda^-) = \mathrm{id} \quad \text{and} \quad \mathcal{L}_\lambda^2 = -\lambda^2\big(-\tfrac{1}{2}+\mathcal{M}_\lambda\big)\big(\tfrac{1}{2}+\mathcal{M}_\lambda\big). \tag{20}$$

These are also manifestations of the Calderon projector being a projection mapping, i.e., $P_\pm^2 = P_\pm$. We refer to [Mitrea et al. 1997, Lemma 5.10] for these and more statements in the L^2-setting. Notice that we are using the opposite sign convention for $\widetilde{\mathcal{S}}_\lambda$ than in [Mitrea et al. 1997].

For later reference and completeness we also state the following identities.

Lemma 6.1. *For $A \in H^{-1/2}(\mathrm{Div}, \partial\Omega)$ and $f \in H^{1/2}(\partial\Omega)$, we have*

$$\operatorname{div} \widetilde{\mathcal{S}}_\lambda A = \widetilde{\mathcal{S}}_\lambda \operatorname{Div} A, \tag{21}$$

$$\operatorname{curl} \widetilde{\mathcal{S}}_\lambda \nu f = -\widetilde{\mathcal{S}}_\lambda(\nu \times \nabla f), \tag{22}$$

$$\operatorname{Div} \mathcal{M}_\lambda A = -\lambda^2 \nu \cdot \mathcal{S}_\lambda A - \mathcal{K}_\lambda^{\mathrm{t}}(\operatorname{Div} A), \tag{23}$$

$$(\nu \times \nabla)\mathcal{K}_\lambda f = \lambda^2 \nu \times \mathcal{S}_\lambda(\nu f) + \mathcal{M}_\lambda(\nu \times \nabla f), \tag{24}$$

$$(\nu \times \nabla)\mathcal{K}_0 f = \mathcal{M}_0(\nu \times \nabla f). \tag{25}$$

These identities were for example proved in [Mitrea et al. 1997, Lemmas 4.2, 4.3, 4.4, and 5.11] in slightly different function spaces containing the image of $C_0^\infty(\mathbb{R}^3, \mathbb{C}^3)$ under the tangential restriction map γ_t. Since $C_0^\infty(\mathbb{R}^3, \mathbb{C}^3)$ is a dense subspace in $H(\mathrm{curl}, \mathbb{R}^3)$, the space $\gamma_t C_0^\infty(\mathbb{R}^3, \mathbb{C}^3)$ is dense in $H^{-1/2}(\mathrm{Div}, \partial\Omega)$. Hence these equations extend by continuity to the claimed larger space if we use the continuous mapping properties of the potential layer operators. We note here that the gradient ∇ defines a continuous map $H^{1/2}(\partial\Omega) \to H^{-1/2}(\mathrm{Curl}, \partial\Omega)$ and the map $\nu \times \nabla$ is continuous from $H^{1/2}(\partial\Omega) \to H^{-1/2}(\mathrm{Div}, \partial\Omega)$.

Lemma 6.2. *The map $\mathcal{S}_\lambda$ satisfies $\mathcal{S}_\lambda^* = \mathcal{S}_{\bar\lambda}$, where the adjoint is taken with respect to the L^2-induced dual pairing between $H^{1/2}(\partial\Omega)$ and $H^{-1/2}(\partial\Omega)$. In other words it is its own transpose: $\mathcal{S}_\lambda^{\mathrm{t}} = \mathcal{S}_\lambda$. We also have $(\mathcal{L}_\lambda(\nu\, \times))^{\mathrm{t}} = \mathcal{L}_\lambda(\nu\, \times)$, i.e., $\mathcal{L}_\lambda$ is symmetric with respect to the bilinear form induced by $\langle\,\cdot\,, \nu \times \cdot\,\rangle$.*

Proof. The symmetry of the operator $\mathcal{S}_\lambda$ with respect to the real inner product are classical and follow from the symmetry properties of the integral kernel. See for example Theorem 5.44 in [Kirsch and Hettlich 2015]. The statement about $\mathcal{L}_\lambda^{\mathrm{t}}$ is Lemma 5.6.1 in [Kirsch and Hettlich 2015]. □

The following lemma is implicit in [Kirsch and Hettlich 2015].

Lemma 6.3. *The operator $\pm\frac{1}{2} + \mathcal{M}_\lambda$ is for any $\operatorname{Im}\lambda > 0$ an isomorphism from $H^{-1/2}(\mathrm{Div}, \partial\Omega)$ to $H^{-1/2}(\mathrm{Div}, \partial\Omega)$.*

Proof. Assume that $\operatorname{Im}\lambda > 0$. It was shown in [Kirsch and Hettlich 2015, Theorem 5.52 (d)] that $\mathcal{L}_\lambda$ is invertible modulo compact operators and therefore is a Fredholm operator of index 0. Moreover, by [Kirsch and Hettlich 2015, Theorem 5.59], we know that $\mathcal{L}_\lambda$ is injective and hence invertible. Since $\Lambda_\lambda^\pm$ are invertible, it follows from (19) that $\pm\frac{1}{2} + \mathcal{M}_\lambda$ is also. As usual the inverse is continuous by the open mapping theorem. □

Invertibility of operators $\pm\frac{1}{2}+\mathcal{M}_\lambda$ on several other L^p-spaces has been shown in the works of M. Mitrea and D. Mitrea; see, for example, Theorem 4.1 in [Mitrea 1995].

Proposition 6.4. *The family $\pm\frac{1}{2}+\mathcal{M}_\lambda$ is a holomorphic family of Fredholm operators of index* 0 *from $H^{-1/2}(\mathrm{Div}, \partial\Omega)$ to $H^{-1/2}(\mathrm{Div}, \partial\Omega)$. The derivative $\mathcal{M}'_\lambda = \frac{\mathrm{d}}{\mathrm{d}\lambda}\mathcal{M}_\lambda$ is a continuous family of Hilbert–Schmidt operators on $H^{-1/2}(\mathrm{Div}, \partial\Omega)$.*

Proof. We will show that $\mathcal{M}_\lambda$ is complex-differentiable as a family of bounded operators $H^{-1/2}(\mathrm{Div}, \partial\Omega)$ and its derivative is compact. The first part of the theorem then follows from

$$\left(\pm\tfrac{1}{2}+\mathcal{M}_\lambda\right)-\left(\pm\tfrac{1}{2}+\mathcal{M}_{\mathrm{i}}\right)=\int_{\mathrm{i}}^{\lambda}\mathcal{M}'_\mu\,\mathrm{d}\mu$$

and the proposition above. We have used here that Fredholm operators are stable under compact perturbations; see, for example, Lemma 8.6 in [Shubin 1987]. It is therefore sufficient to show that $\mathcal{M}'_\lambda$ exists and is Hilbert–Schmidt. First choose a compactly supported smooth cut-off function χ supported in $(-2R, 2R)$ and which equals 1 on $[-R, R]$ for sufficiently large $R > 0$. The integral kernel of $\widetilde{\mathcal{M}}_{\pm,\lambda}$ is given by $\mathrm{curl}_x\, \mathrm{e}^{\mathrm{i}\lambda|x-y|}/(4\pi|x-y|)$. For x not far from $\partial\Omega$, we can replace this by

$$\chi(|x-y|)\,\mathrm{curl}_x\,\frac{\mathrm{e}^{\mathrm{i}\lambda|x-y|}}{4\pi|x-y|}.$$

Consider the Taylor expansion

$$\begin{aligned}\chi(|x-y|)\,\mathrm{curl}_x\,\frac{\mathrm{e}^{\mathrm{i}\lambda|x-y|}}{4\pi|x-y|}=\chi(|x-y|)\,\mathrm{curl}_x\,\frac{\mathrm{e}^{\mathrm{i}\mu|x-y|}}{4\pi|x-y|}+\chi(|x-y|)\,\mathrm{curl}_x\,\frac{\mathrm{e}^{\mathrm{i}\mu|x-y|}}{4\pi}(\lambda-\mu)\\+\chi(|x-y|)T_\lambda(x-y)(\lambda-\mu)^2\end{aligned}$$

with remainder term T_λ. This gives rise to an operator expansion

$$\mathcal{M}_{\pm,\lambda}=\mathcal{M}_{\pm,\mu}+A_\lambda(\lambda-\mu)+B_\lambda(\lambda-\mu)^2.$$

Here the operators A_λ and B_λ arise as compositions as

$$H^{-1/2}(\mathrm{Div}, \partial\Omega)\xrightarrow{\gamma_T^*}H^{-1}(U)\xrightarrow{K_A,K_B}H^1(\mathbb{R}^d)\longrightarrow H(\mathrm{curl}, M)\xrightarrow{\gamma_t}H^{-1/2}(\mathrm{Div}, \partial\Omega),$$

where K_A or K_B is the integral operator with kernel $\chi(|x-y|)\,\mathrm{curl}_x\,\mathrm{e}^{\mathrm{i}\mu|x-y|}/(4\pi)$ or $\chi(|x-y|)T_\lambda(x-y)$, respectively. Here U is a bounded open neighbourhood of $\partial\Omega$. It is now sufficient to show that the operators K_A and K_B are bounded as Hilbert–Schmidt operators. In view of Lemma A.3, we would like to bound the $H^2(\mathbb{R}^d\times\mathbb{R}^d)$-norm of the kernels. Taking two derivatives gives in both cases an integrable convolution kernel in $L^1(\mathbb{R}^d)$ and the $H^2(\mathbb{R}^d\times\mathbb{R}^d)$-norm is then, by Young's inequality, bounded by the L^1-norm of this kernel. □

Definition 6.5. The spaces $\mathcal{B}^{\pm}_{\partial\Omega}\subset H^{-1/2}(\mathrm{Div}, \partial\Omega)$ of interior/exterior boundary data of absolute harmonic forms are defined as

$$\mathcal{B}^{+}_{\partial\Omega}=\{\gamma_{t,+}(\phi)\mid\phi\in\mathcal{H}^1_{\mathrm{abs}}(\Omega)\}\quad\text{and}\quad\mathcal{B}^{-}_{\partial\Omega}=\{\gamma_{t,-}(\phi)\mid\phi\in\mathcal{H}^1_{\mathrm{abs}}(M)\}.$$

It is then obvious that $\mathcal{B}^{+}_{\partial\Omega}=\mathcal{B}^{+}_{\partial\Omega_1}\oplus\cdots\oplus\mathcal{B}^{+}_{\partial\Omega_N}$ with respect to the decomposition

$$H^{-1/2}(\mathrm{Div}, \partial\Omega)=H^{-1/2}(\mathrm{Div}, \partial\Omega_1)\oplus\cdots\oplus H^{-1/2}(\mathrm{Div}, \partial\Omega_N).$$

This is not true for the space $\mathcal{B}^-_{\partial\Omega}$. The spaces $\mathcal{B}^+_{\partial\Omega}$ are also known to be subspaces of $L^2(\partial\Omega, \mathbb{C}^3)$, see [Mitrea et al. 2001, Theorem 11.2], but this will not be needed.

The following was announced by D. Mitrea [2000] in the context of L^p-spaces, with p sufficiently close to 2. It is a reflection of general Hodge theory for Lipschitz domains, and we restate and prove this here for our choice of function spaces.

Proposition 6.6. *We have*

$$\mathcal{B}^{\pm}_{\partial\Omega} = \ker\bigl(\pm\tfrac{1}{2} + \mathcal{M}_0\bigr) \subset H^{-1/2}(\operatorname{Div} 0, \partial\Omega). \tag{26}$$

Proof. We will prove this only for $\mathcal{B}^+_{\partial\Omega}$ since the proof for $\mathcal{B}^-_{\partial\Omega}$, when supplemented by Lemma 3.1, is exactly the same. Suppose that $u \in \ker\bigl(\frac{1}{2} + \mathcal{M}_0\bigr)$, and define $\phi = -\widetilde{\mathcal{M}}_0 u$. Then ϕ is divergence-free and harmonic on M and on Ω. The jump relations (15) hold by analytic continuation for all $\lambda \in \mathbb{C}$, and they show that $\gamma_{t,-}\phi = 0$, $\gamma_{t,+}\phi = u$, and $\gamma_{\nu,+}\phi = \gamma_{\nu,-}\phi$. We first show that $q = \gamma_{\nu,+}\phi$ vanishes, thus establishing the inclusion $\phi|_\Omega \in \mathcal{H}^1_{\text{abs}}(\Omega)$, $\gamma_{t,+}\phi = u$. The proof uses similar arguments as in [Verchota 1984] and reflects the mapping properties of the adjoint double layer operator.

On the exterior, ϕ is a harmonic vector field satisfying relative boundary conditions. The decay of $\operatorname{curl}(1/|x-y|)$ implies that ϕ is square-integrable. This shows that $\operatorname{curl}\phi$ must vanish in the exterior. From the representation (13) we obtain, using the jump relations and $\gamma_{t,-}\phi = 0$,

$$\phi|_M = \nabla\widetilde{\mathcal{S}}_0 q.$$

Taking the normal trace, one gets $q = \gamma_{\nu,-}\nabla\widetilde{\mathcal{S}}_0 q$. Taking the tangential trace, one obtains, from the jump relations,

$$\gamma_{t,-}\nabla\widetilde{\mathcal{S}}_0(\gamma_{\nu,-}\phi) = \nabla_{\partial\Omega}\mathcal{S}_0 q = 0.$$

This shows that $w = \mathcal{S}_0 q$ is locally constant (and in particular in $L^2(\partial\Omega)$). Using the divergence theorem on the interior of each of the components Ω_j one finds that $\int_{\partial\Omega_j} q = 0$. This gives $\langle \mathcal{S}_0 q, q\rangle_{L^2(\partial\Omega)} = 0$ and therefore

$$\langle \mathcal{S}_0 q, \nabla_\nu \widetilde{\mathcal{S}}_0 q\rangle_{L^2(\partial\Omega)} = 0.$$

Since this is the boundary term in the integration by parts formula for $\langle \nabla\widetilde{\mathcal{S}}_0 q, \nabla\widetilde{\mathcal{S}}_0 q\rangle = 0$, we can then imply that $\widetilde{\mathcal{S}}_0 q$ is constant. Since it decays we must have $\widetilde{\mathcal{S}}_0 q = 0$ and therefore $\mathcal{S}_0 q = 0$. By invertibility of the single layer operator, one obtains $q = 0$ as claimed.

We now show the inclusion in the other direction. Suppose that $u = \gamma_{t,+}(h)$, where $h \in \mathcal{H}^1_{\text{abs}}(\Omega)$. This means in particular that h is divergence-free, curl-free, and $\gamma_{\nu,+}h = 0$. Taking the tangential trace in representation (14), we obtain

$$u = \bigl(\tfrac{1}{2} - \mathcal{M}_0\bigr)u,$$

and therefore $\bigl(\frac{1}{2} + \mathcal{M}_0\bigr)u = 0$ as claimed.

It finally remains to show that $\{\gamma_{t,+}(\phi) \mid \phi \in \mathcal{H}^1_{\text{abs}}(\Omega)\} \subset H^{-1/2}(\operatorname{Div} 0, \partial\Omega)$. This follows immediately from the fact that $\operatorname{curl}\phi = 0$ and $\operatorname{Div}\circ\gamma_{t,+} = -\gamma_{\nu,+}\circ\operatorname{curl}$. □

A similar but easier argument applies to other elements of the real line and gives the following.

Proposition 6.7. *If* $\lambda \in \mathbb{R} \setminus \{0\}$ *then* $\ker\bigl(\frac{1}{2} + \mathcal{M}_\lambda\bigr) = \{0\}$ *when* $|\lambda| \neq \mu_k$ *for all* $k \in \mathbb{N}$, *i.e.*, $|\lambda|$ *is not a Maxwell eigenvalue. Moreover*,

$$\ker\bigl(\tfrac{1}{2} + \mathcal{M}_{\mu_k}\bigr) = \{\gamma_{t,+}(u) \mid u \in V_{\mu_k}\}, \tag{27}$$

with V_{μ_k} *the eigenspace of* $\Delta_{\Omega,\mathrm{abs}}$ *for the eigenvalue* μ_k^2 *on the subspace of divergence-free vector fields.*

Proof. The proof is very similar to the proof of the previous proposition, and we therefore only give a brief sketch. As before let $u \in \ker\bigl(\frac{1}{2} + \mathcal{M}_{\mu_k}\bigr)$ and $\phi = -\widetilde{\mathcal{M}}_\lambda u$. Then $\phi|_M$ is a purely incoming or outgoing solution of the Helmholtz equation (see, e.g., [Strohmaier and Waters 2020, Appendix C] for details) satisfying relative boundary conditions. It therefore vanishes. By the jump relations (15), the function $\phi|_\Omega$ satisfies absolute boundary conditions, is divergence-free, and is a Maxwell eigenfunction with Maxwell eigenvalue μ_k. Moreover, again by the jump relation, $\gamma_{t,+}\phi = u$. This proves the inclusion in one direction. Conversely, assume that $u = \gamma_{t,+}\phi$, where ϕ is divergence-free, satisfies absolute boundary conditions, and $-\Delta\phi = \mu_k^2\phi$. Taking the tangential trace in representation (14), we obtain

$$u = \bigl(\tfrac{1}{2} - \mathcal{M}_{\mu_k}\bigr)u,$$

and therefore $\bigl(\frac{1}{2} + \mathcal{M}_{\mu_k}\bigr)u = 0$ as claimed. □

7. Estimates and low-energy expansions for the layer potential operators

For $0 < \epsilon < \frac{\pi}{2}$, define the sector $\mathfrak{D}_\epsilon$ in the upper half-plane by

$$\mathfrak{D}_\epsilon := \{z \in \mathbb{C} \mid \epsilon < \arg(z) < \pi - \epsilon\}.$$

The next proposition establishes properties of the single layer operator $\widetilde{\mathcal{S}}_\lambda$ and the operator $\widetilde{\mathcal{L}}_\lambda$.

Proposition 7.1. *For* $\epsilon \in \bigl(0, \frac{\pi}{2}\bigr)$ *and for all* $\lambda \in \mathfrak{D}_\epsilon$, *we have the following bounds*:

(1) *Let* $\Omega_0 \subset \mathbb{R}^d$ *be an open subset and assume* $\delta = \mathrm{dist}(\Omega_0, \partial\Omega) > 0$. *Let* $0 < \delta' < \delta$. *Assume that* $\varphi \in C_b^1(\mathbb{R}^3)$ *is bounded with bounded derivative and supported in* Ω_0. *For each* $\lambda \in \mathfrak{D}_\epsilon$, *the operators*

$$\begin{aligned}
\varphi\widetilde{\mathcal{L}}_\lambda &: H^{-1/2}(\mathrm{Div}, \partial\Omega) \to H(\mathrm{curl}, \mathbb{R}^3),\\
\varphi\widetilde{\mathcal{S}}_\lambda &: H^{-1/2}(\partial\Omega) \to H^1(\mathbb{R}^3),\\
\varphi\nabla\widetilde{\mathcal{S}}_\lambda &: H^{-1/2}(\partial\Omega) \to L^2(\mathbb{R}^3),\\
\varphi\widetilde{\mathcal{M}}_\lambda &: H^{-1/2}(\mathrm{Div}, \partial\Omega) \to H(\mathrm{div}, \mathbb{R}^3)
\end{aligned}$$

are Hilbert–Schmidt operators. There exists $C_{\delta',\epsilon} > 0$ *such that, for all* $\lambda \in \mathfrak{D}_\epsilon$, *we have the following bounds on the Hilbert–Schmidt norms between these spaces*:

$$\|\varphi\widetilde{\mathcal{L}}_\lambda\|_{\mathrm{HS}} \le C_{\delta',\epsilon}\mathrm{e}^{-\delta'\,\mathrm{Im}\,\lambda}, \tag{28}$$

$$\|\varphi\widetilde{\mathcal{S}}_\lambda\|_{\mathrm{HS}} \le |\lambda|^{-1/2}C_{\delta',\epsilon}\mathrm{e}^{-\delta'\,\mathrm{Im}\,\lambda}, \tag{29}$$

$$\|\varphi\nabla\widetilde{\mathcal{S}}_\lambda\|_{\mathrm{HS}} \le C_{\delta',\epsilon}\mathrm{e}^{-\delta'\,\mathrm{Im}\,\lambda}, \tag{30}$$

$$\|\varphi\widetilde{\mathcal{M}}_\lambda\|_{\mathrm{HS}} \le C_{\delta',\epsilon}\mathrm{e}^{-\delta'\,\mathrm{Im}\,\lambda}, \tag{31}$$

$$\|\varphi\widetilde{\mathcal{S}}_\lambda\,\mathrm{Div}\,\|_{\mathrm{HS}} \le C_{\delta',\epsilon}\mathrm{e}^{-\delta'\,\mathrm{Im}\,\lambda}. \tag{32}$$

(2) *For $\lambda \in \mathfrak{D}_\epsilon$, we have the operator norm bound*

$$\|\widetilde{\mathcal{L}}_\lambda\|_{H^{-1/2}(\mathrm{Div},\partial\Omega)\to H(\mathrm{curl},\mathbb{R}^3)} \le C_\epsilon(1+|\lambda|^2). \tag{33}$$

(3) *For $\lambda \in \mathfrak{D}_\epsilon$, we have the operator norm bound*

$$\|\widetilde{\mathcal{M}}_\lambda\|_{H^{-1/2}(\mathrm{Div},\partial\Omega)\to L^2(\mathbb{R}^3,\mathbb{C}^3)} \le C_\epsilon. \tag{34}$$

(4) *For $\lambda \in \mathfrak{D}_\epsilon$, we have the operator norm bounds*

$$\|\widetilde{\mathcal{S}}_\lambda\|_{H^{-1/2}(\partial\Omega)\to H^1(\mathbb{R}^3)} \le C_\epsilon|\lambda|^{-1/2}(1+|\lambda|^{1/2}), \tag{35}$$

$$\|\nabla\widetilde{\mathcal{S}}_\lambda\|_{H^{-1/2}(\partial\Omega)\to L^2(\mathbb{R}^3,\mathbb{C}^3)} \le C_\epsilon, \tag{36}$$

(5) *On the space of functions of mean zero, $H_0^{-1/2}(\partial\Omega) = \{u \in H_0^{-1/2}(\partial\Omega) \mid \langle u, 1\rangle = 0\}$, we have for $\lambda \in \mathfrak{D}_\epsilon$ the improved estimate*

$$\|\widetilde{\mathcal{S}}_\lambda|_{H_0^{-1/2}(\partial\Omega)}\|_{\mathrm{HS}} \le C_\epsilon. \tag{37}$$

Proof. The operator $\varphi\widetilde{\mathcal{L}}_\lambda$ can be written as $\varphi\,\mathrm{curl}\,\mathrm{curl}\,G_{\lambda,0}\gamma_T^*$. Similarly, we have $\varphi\widetilde{\mathcal{M}}\gamma_T^*$ and $\varphi\widetilde{\mathcal{S}}\gamma_T^*$. We choose a bounded open neighbourhood U of $\partial\Omega$ such that $\mathrm{dist}(\Omega_0, U) > \delta'$. Since γ_T^* continuously maps $H^{-1/2}(\partial\Omega)$ to $H^{-1}(U)$, we only need to show that the map $\mathrm{curl}\,\mathrm{curl}\,G_\lambda$ is a Hilbert–Schmidt operator from $H^{-1}(U)$ to $H^1(\Omega_0)$ and establish the corresponding bound on its Hilbert–Schmidt norm. By Lemma A.3, the Hilbert–Schmidt norm can be bounded by the $H^2(\Omega_0 \times U)$-norm of the kernel of $\mathrm{curl}\,\mathrm{curl}\,G_{\lambda,0}$ on $\Omega_0 \times U$. The corresponding bound has been established in Lemma A.1. The same argument works for $\varphi\widetilde{\mathcal{M}}_\lambda$ and $\varphi\widetilde{\mathcal{S}}_\lambda$. This concludes the proof of the estimates (28), (29), (31), (30), (32).

Since the operator norm is bounded in terms of the Hilbert–Schmidt norm and by the estimates (28), (29), (30), (31), it is sufficient to prove the estimates (33), (34), (35), and (36) for the operators $\chi\widetilde{\mathcal{L}}_\lambda$, $\chi\widetilde{\mathcal{S}}_\lambda$, $\chi\widetilde{\mathcal{M}}_\lambda$, $\chi\nabla\widetilde{\mathcal{S}}_\lambda$, where $\chi \in C_0^\infty(\mathbb{R}^3)$ is a compactly supported function that equals 1 near $\partial\Omega$. We write

$$\chi\widetilde{\mathcal{L}}_\lambda = \chi\nabla\widetilde{\mathcal{S}}_\lambda\mathrm{Div} + \lambda^2\chi\widetilde{\mathcal{S}}_\lambda.$$

The map γ_t^* is from $H^{-1/2}(\partial\Omega)$ to $H_c^{-1}(U)$, where U is an open neighbourhood of $\partial\Omega$. To prove all the bounds (33), (34), (35), and (36), it is therefore sufficient to show that the resolvent $(-\Delta_{\mathrm{free}} - \lambda^2)^{-1}$ is a bounded map from $H^{-1}_{\mathrm{comp}}(\mathbb{R}^3)$ to $H^1_{\mathrm{loc}}(\mathbb{R}^3)$ uniformly in λ for all $\lambda \in \mathfrak{D}_\epsilon$. This means that we need to show that the cut-off resolvent $\chi(-\Delta_{\mathrm{free}} - \lambda^2)^{-1}\chi$ is a uniformly bounded map from $H^{-1}(\mathbb{R}^3)$ to $H^1(\mathbb{R}^3)$ for all $\lambda \in \mathfrak{D}_\epsilon$. To see this, let $\eta \in C_0^\infty(\mathbb{R})$ be a function that is 1 near $[-R_1, R_1]$, where R_1 is the diameter of the support of χ. Let R be large enough that $\mathrm{supp}\,\eta \in (-R, R)$. This implies that $\chi(-\Delta - \lambda^2)^{-1}\chi = \chi R_{\eta,\lambda}\chi$, where $R_{\eta,\lambda}$ is the operator with integral kernel

$$\eta(|x-y|)\frac{1}{4\pi|x-y|}\mathrm{e}^{\mathrm{i}\lambda|x-y|} =: k_\lambda(x-y).$$

It is therefore sufficient to show that $R_{\eta,\lambda}$ is uniformly bounded for all $\lambda \in \mathfrak{D}_\epsilon$ as a map $H^s(\mathbb{R}^3)$ to $H^{s+2}(\mathbb{R}^3)$. Since this is a convolution operator, it commutes with the Laplace operator, and therefore it is sufficient to show that $R_{\eta,\lambda}$ is uniformly bounded as a map $L^2(\mathbb{R}^3)$ to $H^2(\mathbb{R}^3)$. We will show that $(-\Delta+1)R_{\eta,\lambda}$ is uniformly bounded as a map from $L^2(\mathbb{R}^3)$ to $L^2(\mathbb{R}^3)$. Using

$$(-\Delta+1)(-\Delta-\lambda^2)^{-1} = \mathrm{id} + (1+\lambda^2)(-\Delta-\lambda^2)^{-1},$$

one obtains that the integral kernel of $(-\Delta+1)R_{\eta,\lambda} - \mathrm{id}$ equals

$$(-(\Delta_x \eta(|x-y|)) + (1+\lambda^2)\eta(|x-y|))\frac{1}{4\pi|x-y|}\mathrm{e}^{\mathrm{i}\lambda|x-y|} - 2\nabla_x \eta(|x-y|)\nabla_x \frac{1}{4\pi|x-y|}\mathrm{e}^{\mathrm{i}\lambda|x-y|}.$$

This is a convolution operator, and we can use Young's inequality to estimate its operator norm. In particular, using spherical coordinates, the estimates

$$\int_0^R \frac{1}{4\pi r}|\mathrm{e}^{\mathrm{i}\lambda r}|r^2\,\mathrm{d}r \le C_R \frac{1}{1+|\operatorname{Im}\lambda|^2} \quad \text{and} \quad \int_0^R \frac{1}{4\pi r^2}|\mathrm{e}^{\mathrm{i}\lambda r}|r^2\,\mathrm{d}r \le C_R \frac{1}{1+|\operatorname{Im}\lambda|}$$

show that the convolution kernel is uniformly bounded in $L^1(\mathbb{R}^3)$ for $\lambda \in \mathfrak{D}_\epsilon$. Thus $R_{\eta,\lambda}$ is uniformly bounded as a map from $L^2(\mathbb{R}^3)$ to $H^2(\mathbb{R}^3)$ for $\lambda \in \mathfrak{D}_\epsilon$.

It remains to show the improved estimate (37). We again choose cutoffs χ, ψ as above, and we arrange them so that $\psi + \phi = 1$. Since the cut-off resolvent $\chi(-\Delta_{\mathrm{free}} - \lambda^2)^{-1}\chi$ is regular near 0 as a map from $H^{-1}(\mathbb{R}^3)$ to $H^1(\mathbb{R}^3)$, we know that $\chi\widetilde{\mathcal{S}}_\lambda : H^{-1/2}(\partial\Omega) \to H(\mathrm{curl}, \mathbb{R}^3)$ is regular near 0. It is therefore sufficient to establish the bound for $\phi\widetilde{\mathcal{S}}_\lambda$ as a map from $H_0^{-1/2}(\partial\Omega)$ to $H^1(\mathbb{R}^3)$. We argue similarly as above choosing an open neighbourhood U such that the support of ϕ has positive distance from U. For convenience, we will also assume that the support of ϕ is sufficiently separated from Ω; more precisely, we assume that the support of ϕ has positive distance to the convex hull of Ω. With $u \in H_0^{-1/2}(\partial\Omega)$, the distribution $\gamma^* u$ is in the space distributions $H_0^{-1}(U) = \{v \in H_{\mathrm{c}}^{-1}(U) \mid \langle v, 1\rangle = 0\}$ of mean zero. We therefore only need to bound $\phi(-\Delta_{\mathrm{free}} - \lambda^2)^{-1}$ as a map from $H_0^{-1}(U)$ to $H^1(\mathbb{R}^n)$. This map is the restriction of the integral operator with smooth kernel

$$g(x,y) = \phi(x)\left(\frac{\mathrm{e}^{\mathrm{i}\lambda|x-y|}}{4\pi|x-y|} - \frac{\mathrm{e}^{\mathrm{i}\lambda|x-z|}}{4\pi|x-z|}\right)$$

to $H_0^{-1}(U)$, where z is any fixed point on $\partial\Omega$. One shows that this kernel is in the Sobolev space $H^2(\mathbb{R}^3 \times U)$ and is uniformly bounded in $\lambda \in \mathfrak{D}_\epsilon$. This kernel and its derivatives are easily bounded using the mean-value inequality

$$|\partial_x^\alpha g(x,y)| \le |y-z| \sup_{\tilde{y}\in K} \|\partial^\alpha \nabla_x g(x,\tilde{y})\| \le C \sup_{\tilde{y}\in K} \|\partial^\alpha \nabla_x g(x,\tilde{y})\|,$$

where K is the closure of the convex hull of $\partial\Omega$. The L^2-norm of this expression is uniformly bounded for all $\lambda \in \mathfrak{D}_\epsilon$ by the same estimate as in (88). This works essentially because, with repeated application of the product rule, the terms either have improved decay or have an extra λ-factor. □

The proof above can also be applied directly to $\chi\widetilde{\mathcal{L}}_\lambda$ in the entire complex plane to bound the operator norm, the norm of the derivative, and the norm of the remainder term. This gives the following result. We will not repeat the proof but simply state the result.

Lemma 7.2. *The families*

$$\widetilde{\mathcal{L}}_\lambda : H^{-1/2}(\mathrm{Div}, \partial\Omega) \to H_{\mathrm{loc}}(\mathrm{curl}, M) \oplus H_{\mathrm{loc}}(\mathrm{curl}, \Omega),$$
$$\mathcal{L}_\lambda : H^{-1/2}(\mathrm{Div}, \partial\Omega) \to H^{-1/2}(\mathrm{Div}, \partial\Omega)$$

are holomorphic families of bounded operators in the complex plane.

Lemma 7.3. *The families $\mathcal{L}_\lambda^{-1}$ and $\Lambda_\lambda^\pm$ are meromorphic in λ as families of bounded operators on $H^{-1/2}(\mathrm{Div}, \partial\Omega)$. The family Λ_λ^- has no poles in $\mathbb{R} \setminus \{0\}$ and in the upper half-plane.*

Proof. By Proposition 6.4 and Lemma 6.3, the operator $\left(\frac{1}{2} + \mathcal{M}_\lambda\right)$ is an analytic family of Fredholm operators which is invertible for $\operatorname{Im} \lambda > 0$. By the analytic Fredholm theorem, the inverse $\left(\frac{1}{2} + \mathcal{M}_\lambda\right)^{-1}$ is a meromorphic family of finite type, i.e., the negative Laurent coefficients are finite-rank operators. We have

$$\mathcal{L}_\lambda^{-2} = -\lambda^{-2}\left(\tfrac{1}{2} + \mathcal{M}_\lambda\right)^{-1}\left(\tfrac{1}{2} - \mathcal{M}_\lambda\right)^{-1},$$

which shows that $\mathcal{L}_\lambda^{-2}$ is meromorphic. Since $\mathcal{L}_\lambda$ is holomorphic, this shows that $\mathcal{L}_\lambda^{-1}$ is meromorphic. Finally $\Lambda^\pm$ is meromorphic by (19). Poles of Λ_λ^- are absent in the closed upper half-space because of the uniqueness of the exterior boundary value problem. Indeed, the most negative Laurent coefficient would give rise to an outgoing solution of the Helmholtz equation satisfying relative boundary conditions. But such an outgoing solution vanishes. □

Remark 7.4. The above cannot be easily concluded from analytic Fredholm theory since the operators $\mathcal{L}_\lambda$ and $\Lambda^\pm$ are not Fredholm operators. Indeed, the singular Laurent coefficients are not finite-rank operators.

We now aim to show a new formula for the voltage-to-current map in order to find bounds on $\mathcal{L}_\lambda^{-1}$ where it is well defined.

Theorem 7.5. *The interior voltage-to-current mapping Λ_λ^+ satisfies*

$$\mathrm{i}\Lambda_\lambda^+ = \frac{1}{\lambda} T + \lambda U_\lambda,$$

where T is a bounded operator on $H^{-1/2}(\mathrm{Div}, \partial\Omega)$ and U_λ is a meromorphic family of bounded operators on $H^{-1/2}(\mathrm{Div}, \partial\Omega)$ which is regular at $\lambda = 0$. We have explicitly

$$TA = \sum_{k=1}^{\beta_1} \langle A, \gamma_T \psi_{0,k} \rangle_{L^2(\partial\Omega)} \gamma_t(\psi_{0,k}) + \sum_{\lambda_{N,k}>0} \frac{1}{\lambda_{N,k}^2} \langle A, \gamma_T \nabla v_k \rangle_{L^2(\partial\Omega)} \gamma_t(\nabla v_k),$$
$$U_\lambda A = \sum_{k=1}^{\infty} \frac{1}{\lambda^2 - \mu_k^2} \langle A, \gamma_T \psi_k \rangle_{L^2(\partial\Omega)} \gamma_t(\psi_k)$$

for $A \in H^{-1/2}(\mathrm{Div}, \partial\Omega)$. Both sums converge in $H^{-1/2}(\mathrm{Div}, \partial\Omega)$. Here $\beta_1 = \dim \mathcal{H}^1_{\mathrm{abs}}(\Omega)$ is the first Betti number of the domain. We have $T^2 = 0$ and $TU_\lambda + U_\lambda T = \mathrm{id} - \lambda^2 U_\lambda$.

Proof. We start with an interior solution $E \in H(\text{curl}, \Omega)$ of the Maxwell system and assume $A = \gamma_t(E) \in H^{-1/2}(\text{Div}, \partial\Omega)$. First note that E satisfies div $E = 0$, but it is not in general in ker(div$_0$) because it may not satisfy the correct boundary conditions. We have

$$L^2(\Omega, \mathbb{C}^3) = \mathcal{H}^1_{\text{abs}}(\Omega) \oplus \{\psi_j \mid \mu_j > 0\} \oplus \{\nabla v_k | \lambda_{N,k}\}, \tag{38}$$

where v_k is an orthonormal basis of Neumann eigenfunctions on Ω. Define

$$\tilde{\psi}_k = \frac{1}{\lambda_{N,k}} \operatorname{grad} v_k. \tag{39}$$

Now we can write

$$E = \sum \langle E, \psi_k\rangle \psi_k + \sum \langle E, \tilde{\psi}_k\rangle \tilde{\psi}_k, \tag{40}$$

which we need to show converges in $H(\text{curl}, \Omega)$. We have

$$\begin{aligned} \langle E, \psi_k\rangle_{L^2(\Omega)} &= \frac{1}{\lambda^2 - \mu_k^2}(\langle -\Delta E, \psi_k\rangle_{L^2(\partial\Omega)} - \langle E, -\Delta\psi_k\rangle_{L^2(\partial\Omega)}) \\ &= \frac{1}{\lambda^2 - \mu_k^2}(\langle \gamma_t \operatorname{curl} E, \gamma_T \psi_k\rangle_{L^2(\partial\Omega)} + \langle \gamma_t E, \gamma_T \operatorname{curl} \psi_k\rangle_{L^2(\partial\Omega)}) \\ &= \frac{1}{\lambda^2 - \mu_k^2}\langle \gamma_t \operatorname{curl} E, \gamma_T \psi_k\rangle_{L^2(\partial\Omega)} \\ &= \frac{\mathrm{i}\lambda}{\lambda^2 - \mu_k^2}\langle \gamma_t H, \gamma_T \psi_k\rangle_{L^2(\partial\Omega)} = \frac{\mathrm{i}\lambda}{\lambda^2 - \mu_k^2}\langle \Lambda_\lambda^+ A, \gamma_T \psi_k\rangle_{L^2(\partial\Omega)}, \end{aligned} \tag{41}$$

where we have used Stokes' theorem (4) as well as Maxwell system properties in (1) repeatedly. Since $E \in L^2(\Omega, \mathbb{C}^3)$, the sum $\sum \langle E, \psi_k\rangle \psi_k$ converges in $L^2(\Omega, \mathbb{C}^3)$. Let ϕ_k denote an orthonormal basis of eigenfunctions of Δ_{rel}. We now note that

$$\sum_{\lambda_k \neq 0} \langle E, \psi_k\rangle \operatorname{curl} \psi_k = \sum_{\mu_k \neq 0} \langle E, \psi_k\rangle \operatorname{curl} \frac{1}{\mu_k} \operatorname{curl} \phi_k = \sum_{\lambda_k \neq 0} \langle E, \psi_k\rangle \mu_k \phi_k$$

converges in $L^2(\Omega, \mathbb{C}^3)$ whenever $(\langle E, \psi_k\rangle_{L^2(\Omega)} \mu_k)_k \in \ell^2$. The latter is true because

$$\mu_k \langle E, \psi_k\rangle_{L^2(\Omega)} = \langle E, \operatorname{curl} \phi_k\rangle_{L^2(\Omega)} = \langle \operatorname{curl} E, \phi_k\rangle_{L^2(\Omega)} \in \ell^2, \tag{42}$$

where we have used the fact curl $E \in L^2(\Omega, \mathbb{C}^3)$. Therefore

$$\sum_{k=1}^{\infty} \langle E, \psi_k\rangle \psi_k \tag{43}$$

converges in $H(\text{curl}, \Omega)$. For the second term, now we have

$$\begin{aligned} \langle E, \tilde{\psi}_k\rangle_{L^2(\Omega)} &= \frac{\mathrm{i}}{\lambda}\Big\langle \operatorname{curl} H, \frac{1}{\lambda_{N,k}} \operatorname{grad} v_k\Big\rangle_{L^2(\Omega)} \\ &= \frac{\mathrm{i}}{\lambda_{N,k}\lambda}\langle \operatorname{curl} H, \operatorname{grad} v_k\rangle_{L^2(\Omega)} = \frac{\mathrm{i}}{\lambda_{N,k}\lambda}\langle \gamma_t H, \gamma_T \operatorname{grad} v_k\rangle_{L^2(\partial\Omega)}. \end{aligned} \tag{44}$$

This also gives that

$$\sum_{\lambda_k \neq 0} \langle E, \tilde{\psi}_k \rangle \tilde{\psi}_k \tag{45}$$

converges in $H(\mathrm{curl}, \Omega)$ as $\lambda_{N,k}^{-2}$ is summable. Therefore we have

$$E = \sum_{k=0}^{\infty} \frac{\mathrm{i}\lambda}{\lambda^2 - \mu_k^2} \langle \gamma_t H, \gamma_T \psi_k \rangle_{L^2(\partial\Omega)} \psi_k + \sum_{\lambda_{N,k} \neq 0} \frac{\mathrm{i}}{\lambda_k^2 \lambda} \langle \gamma_t H, \gamma_T \operatorname{grad} v_k \rangle_{L^2(\partial\Omega)} \operatorname{grad} v_k, \tag{46}$$

and this representation converges in $H(\mathrm{curl}, \Omega)$. As a result, we have convergence in $H^{-1/2}(\mathrm{Div}, \partial\Omega)$ of

$$\begin{aligned} A &= \nu \times E|_{\partial\Omega} \\ &= \sum_{\mu_k \geq 0} \frac{\mathrm{i}\lambda}{\lambda^2 - \mu_k^2} \langle \gamma_t H, \gamma_T \psi_k \rangle_{L^2(\partial\Omega)} \gamma_t(\psi_k) + \sum_{\lambda_{N,k} \neq 0} \frac{\mathrm{i}}{\lambda_{N,k}^2 \lambda} \langle \gamma_t H, \gamma_T \operatorname{grad} v_k \rangle_{L^2(\partial\Omega)} \gamma_t(\operatorname{grad} v_k). \end{aligned} \tag{47}$$

Then using the fact that $(\gamma_t(H)) = \Lambda_\lambda^+(\gamma_t(E)) = \Lambda_\lambda^+(A)$ and remarking that $(\Lambda^+)^2 = -\,\mathrm{id}$, we obtain the desired result. Expanding the formula $(\mathrm{i}\Lambda^+)^2 = \mathrm{id}$ also gives the claimed identities. □

We now aim to show operator bounds on the electric dipole map in order to find bounds on the large $|\lambda|$ behaviour of $\mathcal{L}_\lambda^{-1}$. Note that, for $\lambda \in \mathfrak{D}_\epsilon$, we have the estimate

$$\operatorname{Im} \lambda = |\operatorname{Im} \lambda| \leq |\lambda| \leq C_\epsilon \operatorname{Im} \lambda,$$

where $C_\epsilon := \sin(\epsilon)^{-1}$ is independent of $\lambda \in \mathfrak{D}_\epsilon$.

Theorem 7.6. *There exists a constant C such that, for all $\operatorname{Im} \lambda > 0$, we have the estimate*

$$\|\Lambda_\lambda^\pm\|_{H^{-1/2}(\mathrm{Div},\partial\Omega) \mapsto H^{-1/2}(\mathrm{Div},\partial\Omega)} \leq C \frac{1}{|\lambda|} \left(1 + \frac{|\lambda|(1+|\lambda|^2)}{\operatorname{Im} \lambda}\right). \tag{48}$$

Proof. We first consider the case $\operatorname{Re} \lambda^2 < 0$, i.e., $|\operatorname{Im} \lambda| > |\operatorname{Re} \lambda|$. We have the integral identity

$$\langle v, \operatorname{curl} u \rangle_{L^2(M)} - \langle \operatorname{curl} v, u \rangle_{L^2(M)} = \langle \gamma_t v, \gamma_T u \rangle_{L^2(\partial\Omega)} \tag{49}$$

for $u, v \in H(\mathrm{curl}, M)$. Applying this integral identity with E and H gives

$$\begin{aligned} \mathrm{i}\lambda \langle \gamma_t H, \gamma_T E \rangle_{L^2(\partial\Omega)} &= \mathrm{i}\lambda (\langle H, \operatorname{curl} E \rangle_{L^2(M)} - \langle \operatorname{curl} H, E \rangle_{L^2(M)}) \\ &= \langle \operatorname{curl} E, \operatorname{curl} E \rangle - \lambda^2 \langle E, E \rangle. \end{aligned}$$

Taking the real part we obtain

$$|\operatorname{Re} \lambda^2| \langle E, E \rangle_{L^2(M)} \leq |\lambda| \cdot |\langle \gamma_t H, \gamma_T E \rangle_{L^2(\partial\Omega)}|.$$

The antisymmetric bilinear form $\langle \nu \times u, v \rangle_{L^2(\partial\Omega)}$ extends continuously to $H^{-1/2}(\mathrm{Div}, \partial\Omega)$ (see for example Lemma 5.61 in [Kirsch and Hettlich 2015]), and we therefore have

$$\|E\|_{L^2(M)}^2 \leq C_1 |\lambda| (-\operatorname{Re} \lambda^2)^{-1} \|\gamma_t E\|_{H^{-1/2}(\mathrm{Div},\partial\Omega)} \|\gamma_t(H)\|_{H^{-1/2}(\mathrm{Div},\partial\Omega)}. \tag{50}$$

Now we use the continuity of the tangential trace map and obtain

$$\begin{aligned}\|\gamma_t(\operatorname{curl} E)\|^2_{H^{-1/2}(\mathrm{Div},\partial\Omega)} &\le C_2\|(\operatorname{curl} E)\|^2_{H(\mathrm{curl},M)} = C_2(\|\operatorname{curl} E\|^2_{L^2(M)} + |\lambda|^4\|E\|^2_{L^2(M)})\\ &= C_2(|\lambda|^2\|E\|^2_{L^2(M)} + |\lambda|^4\|E\|^2_{L^2(M)} + \langle\gamma_t E, \gamma_T \operatorname{curl} E\rangle_{L^2(\partial\Omega)})\\ &\le C_3(|\lambda|^2(1+|\lambda|^2)\|E\|^2_{L^2(M)} + \|\gamma_t E\|_{H^{-1/2}(\mathrm{Div},\partial\Omega)}\|\gamma_t \operatorname{curl} E\|_{H^{-1/2}(\mathrm{Div},\partial\Omega)}).\end{aligned}$$

Choosing

$$a = C_3\|\gamma_t E\|_{H^{-1/2}(\mathrm{Div},\partial\Omega)} \quad \text{and} \quad b = \|\gamma_t \operatorname{curl} E\|_{H^{-1/2}(\mathrm{Div},\partial\Omega)}$$

and using the inequality $|ab| \le \frac{1}{2}(a^2+b^2)$, one obtains

$$\|\gamma_t(\operatorname{curl} E)\|^2_{H^{-1/2}(\mathrm{Div},\partial\Omega)} \le (2C_3|\lambda|^2(1+|\lambda|^2)\|E\|^2_{L^2(M)} + C_3^2\|\gamma_t E\|^2_{H^{-1/2}(\mathrm{Div},\partial\Omega)}).$$

Using (50), this further gives

$$\begin{aligned}&\|\gamma_t(\operatorname{curl} E)\|^2_{H^{-1/2}(\mathrm{Div},\partial\Omega)}\\ &\qquad\le C_4\left(\frac{|\lambda|^2(1+|\lambda|^2)}{-\operatorname{Re}\lambda^2}\|\gamma_t E\|_{H^{-1/2}(\mathrm{Div},\partial\Omega)}\|\gamma_t \operatorname{curl} E\|_{H^{-1/2}(\mathrm{Div},\partial\Omega)} + \|\gamma_t E\|^2_{H^{-1/2}(\mathrm{Div},\partial\Omega)}\right).\end{aligned}$$

The same trick as before with

$$a = C_4\frac{|\lambda|^2(1+|\lambda|^2)}{-\operatorname{Re}\lambda^2}\|\gamma_t E\|_{H^{-1/2}(\mathrm{Div},\partial\Omega)} \quad \text{and} \quad b = \|\gamma_t \operatorname{curl} E\|_{H^{-1/2}(\mathrm{Div},\partial\Omega)}$$

yields

$$\|\gamma_t(\operatorname{curl} E)\|^2_{H^{-1/2}(\mathrm{Div},\partial\Omega)} \le \left(C_4^2\frac{|\lambda|^4(1+|\lambda|^2)^2}{(-\operatorname{Re}\lambda^2)^2} + 2C_4\right)\|\gamma_t E\|^2_{H^{-1/2}(\mathrm{Div},\partial\Omega)},$$

which finally gives

$$\|\gamma_t(\operatorname{curl} E)\|_{H^{-1/2}(\mathrm{Div},\partial\Omega)} \le C\left(1 + \frac{|\lambda|^2(1+|\lambda|^2)}{-\operatorname{Re}\lambda^2}\right)\|\gamma_t E\|_{H^{-1/2}(\mathrm{Div},\partial\Omega)}. \tag{51}$$

Next consider the case $\operatorname{Im}\lambda^2 < 0$. The same proof with imaginary parts taken instead of real parts gives the estimate

$$\|\gamma_t(\operatorname{curl} E)\|_{H^{-1/2}(\mathrm{Div},\partial\Omega)} \le C\left(1 + \frac{|\lambda|^2(1+|\lambda|^2)}{-\operatorname{Im}\lambda^2}\right)\|\gamma_t E\|_{H^{-1/2}(\mathrm{Div},\partial\Omega)}. \tag{52}$$

These two estimates cover the upper half-space and are combined into

$$\|\gamma_t(\operatorname{curl} E)\|_{H^{-1/2}(\mathrm{Div},\partial\Omega)} \le C\left(1 + \frac{|\lambda|(1+|\lambda|^2)}{\operatorname{Im}\lambda}\right)\|\gamma_t E\|_{H^{-1/2}(\mathrm{Div},\partial\Omega)}, \tag{53}$$

which holds in the upper half-space except when $\operatorname{Im}\lambda < \operatorname{Re}\lambda$. The estimate holds in this region too as can be seen by replacing λ by $-\bar{\lambda}$, which is a symmetry operation of the Maxwell system that preserves the radiation condition. Hence, the estimate holds in the upper half-space. Since $\mathrm{i}\lambda H = \operatorname{curl} E$, this proves the claimed estimate. The same proof works for the interior with M replaced by Ω. ☐

Lemma 7.7. *The operator* $\left(\frac{1}{2}+\mathcal{M}_\lambda\right)^{-1}$ *is meromorphic of finite type, and we have near* 0 *the expansion*

$$\left(\tfrac{1}{2}+\mathcal{M}_\lambda\right)^{-1}=\frac{P}{\lambda^2}+\frac{B}{\lambda}+Q_\lambda, \tag{54}$$

where P *and* B *are finite-rank operators and* Q_λ *is analytic near* $\lambda=0$ *taking values in the bounded operators on* $H^{-1/2}(\mathrm{Div},\partial\Omega)$. *We also have*

$$\operatorname{image}(P)\cup\operatorname{image}(B)\subseteq\mathcal{B}_{\partial\Omega},\qquad P(\nu\times\nabla u)=B(\nu\times\nabla u)=0$$

for all $u\in H^{1/2}(\partial\Omega)$.

Proof. By the proof of Lemma 7.3, we know that $\left(\frac{1}{2}+\mathcal{M}_\lambda\right)^{-1}$ is a meromorphic family of finite type. The order of the singularity at 0 is at most 2 since, for $\lambda\in\mathfrak{D}_\epsilon$, $\lambda\neq 0$, we have

$$\left(\tfrac{1}{2}+\mathcal{M}_\lambda\right)^{-1}=-\Lambda_\lambda^{+}(\Lambda_\lambda^{+}-\Lambda_\lambda^{-}), \tag{55}$$

and the bound in Theorem 7.6 holds. Hence $\left(\frac{1}{2}+\mathcal{M}_\lambda\right)^{-1}$ has the claimed form:

$$\left(\tfrac{1}{2}+\mathcal{M}_\lambda\right)^{-1}=\frac{P}{\lambda^2}+\frac{B}{\lambda}+Q_\lambda,$$

with P and B of finite rank.

We must naturally have for these λ

$$\left(\tfrac{1}{2}+\mathcal{M}_\lambda\right)^{-1}\left(\tfrac{1}{2}+\mathcal{M}_\lambda\right)=\left(\tfrac{1}{2}+\mathcal{M}_\lambda\right)\left(\tfrac{1}{2}+\mathcal{M}_\lambda\right)^{-1}=\mathrm{id}. \tag{56}$$

Expanding $\left(\frac{1}{2}+\mathcal{M}_\lambda\right)$ around $\lambda=0$, we see that it has operator kernel

$$\frac{1}{2}+\frac{1}{4\pi}\gamma_{t,x}\gamma_{T,y}^{*}\operatorname{curl}\left(\frac{1}{|x-y|}\right)+O(\lambda^2) \tag{57}$$

since the first-order term in the expansion distributional kernel of the free Green's function is constant and therefore curl-free. Hence,

$$\left(\tfrac{1}{2}+\mathcal{M}_\lambda\right)=\tfrac{1}{2}+\mathcal{M}_0+O(\lambda^2)$$

near $\lambda=0$. Inserting this into (56) and comparing coefficients, one obtains

$$\left(\tfrac{1}{2}+\mathcal{M}_0\right)P=0,\quad \left(\tfrac{1}{2}+\mathcal{M}_0\right)B=0,\quad P\left(\tfrac{1}{2}+\mathcal{M}_0\right)=0,\quad B\left(\tfrac{1}{2}+\mathcal{M}_0\right)=0.$$

By Proposition 6.6, we therefore obtain $\operatorname{image}(P)$, $\operatorname{image}(B)\subseteq\mathcal{B}_{\partial\Omega}$ as claimed. It remains to show that

$$P(\nu\times\nabla u)=B(\nu\times\nabla u)=0.$$

To see this, it is sufficient to show that $\nu\times\nabla u$ is in the range of $\frac{1}{2}+\mathcal{M}_0$. To see this we use a classical result in potential layer theory, namely the invertibility of $\left(\frac{1}{2}+\mathcal{K}_0\right)$; see [Verchota 1984]. We then have by (25)

$$\nu\times\nabla u=\nu\times\nabla\left(\tfrac{1}{2}+\mathcal{K}_0\right)\left(\tfrac{1}{2}+\mathcal{K}_0\right)^{-1}u=\left(\tfrac{1}{2}+\mathcal{M}_0\right)\left(\nu\times\nabla\left(\tfrac{1}{2}+\mathcal{K}_0\right)^{-1}u\right). \qquad\square$$

Lemma 7.8. *The nonzero poles of* $\left(\frac{1}{2}+\mathcal{M}_\lambda\right)^{-1}$ *in the closed upper half-space are precisely the Maxwell eigenvalues of* Ω. *Near a Maxwell eigenvalue* $\mu=\mu_k$, *we have the expansion*

$$\left(\tfrac{1}{2}+\mathcal{M}_\lambda\right)^{-1}=\frac{P_\mu}{(\lambda-\mu)^2}+\frac{B_\mu}{\lambda-\mu}+Q_{\mu,\lambda}, \tag{58}$$

where P_μ *and* B_μ *are finite-rank operators with range in* $\ker\left(\frac{1}{2}+\mathcal{M}_\mu\right)^{-1}$ *and* $Q_{\mu,\lambda}$ *is holomorphic in* λ *near* μ.

Proof. The poles are precisely where $\left(\frac{1}{2}+\mathcal{M}_\lambda\right)$ is not injective. On the closed upper half-space, this means that the only poles are at 0 and at the Maxwell eigenvalues by Propositions 6.6 and 6.7. The statement now follows immediately from the formula

$$\left(\tfrac{1}{2}+\mathcal{M}_\lambda\right)^{-1}=-\Lambda_\lambda^+(\Lambda_\lambda^+-\Lambda_\lambda^-), \tag{59}$$

the expansion of Theorem 7.5, and the fact that Λ^- is holomorphic near $\mathbb{R}\setminus\{0\}$ by Lemma 7.3. □

Theorem 7.9. *For any* $\epsilon>0$, *we have there exists a constant* $C>0$ *such that*

$$\|\mathcal{L}_\lambda^{-1}\|_{H^{-1/2}(\mathrm{Div},\partial\Omega)\to H^{-1/2}(\mathrm{Div},\partial\Omega)}\le\frac{1+|\lambda|^2}{|\lambda|^2}C(1+|\lambda|^2),$$
$$\|\mathrm{Div}\circ(\mathcal{L}_\lambda^{-1})\circ(\nu\times\nabla)\|_{H^{-1/2}(\partial\Omega)\to H^{-1/2}(\partial\Omega)}\le C(1+|\lambda|^2)$$

for all λ *in the sector* $\mathfrak{D}_\epsilon$.

Proof. We use the identity, derived from (20),

$$\mathcal{L}_\lambda^{-1}=-\frac{\mathrm{i}(\Lambda_\lambda^+-\Lambda_\lambda^-)}{\lambda} \tag{60}$$

to reduce the analysis to that of $\Lambda_\lambda^\pm$. The bounds on the operator norm on the space $H^{-1/2}(\mathrm{Div},\partial\Omega)$ then follow immediately from Theorem 7.6. By (19), we have the identity

$$\mathcal{L}_\lambda^{-1}=\frac{1}{\mathrm{i}\lambda}\Lambda_\lambda^+\left(\tfrac{1}{2}+\mathcal{M}_\lambda\right)^{-1}.$$

Using Theorem 7.5, we obtain

$$\begin{aligned}\mathcal{L}_\lambda^{-1}&=-\left(\frac{1}{\lambda^2}T+U_\lambda\right)\left(\frac{1}{\lambda^2}P+\frac{1}{\lambda}B+Q_\lambda\right)\\&=-\frac{1}{\lambda^2}(TQ_\lambda+U_\lambda P)-\frac{1}{\lambda}U_\lambda B+U_\lambda Q_\lambda.\end{aligned} \tag{61}$$

We have used that $TP=TB=0$, which follows from Lemma 7.7 and Theorem 7.5. Since $\mathrm{Div}\circ T=0$, $P\circ(\nu\times\nabla)=0$, and $B\circ(\nu\times\nabla)=0$, we then obtain

$$\mathrm{Div}\circ(\mathcal{L}_\lambda^{-1})\circ(\nu\times\nabla)=\mathrm{Div}\circ U_\lambda Q_\lambda\circ(\nu\times\nabla),$$

which is regular at 0. □

8. Resolvent formulae and estimates

Proposition 8.1. *Assume that* $\operatorname{Im}\lambda > 0$. *For* $f \in C_0^\infty(\mathbb{R}^3, \mathbb{C}^3)$, *we have the following formulae for the difference of resolvents*:

$$((-\Delta_{\mathrm{rel}} - \lambda^2)^{-1} - (-\Delta_{\mathrm{free}} - \lambda^2)^{-1}) \operatorname{curl}\operatorname{curl} f = -\widetilde{\mathcal{L}}_\lambda (\mathcal{L}_\lambda)^{-1} (\nu \times) \widetilde{\mathcal{L}}_\lambda^{\mathtt{t}} f, \tag{62}$$

$$((-\Delta_{\mathrm{rel}} - \lambda^2)^{-1} - (-\Delta_{\mathrm{free}} - \lambda^2)^{-1}) \operatorname{curl} f = -\widetilde{\mathcal{L}}_\lambda (\mathcal{L}_\lambda)^{-1} (\nu \times) \widetilde{\mathcal{M}}_\lambda^{\mathtt{t}} f, \tag{63}$$

$$\operatorname{curl}((-\Delta_{\mathrm{rel}} - \lambda^2)^{-1} - (-\Delta_{\mathrm{free}} - \lambda^2)^{-1}) \operatorname{curl} f = -\lambda^2 \widetilde{\mathcal{M}}_\lambda (\mathcal{L}_\lambda)^{-1} (\nu \times) \widetilde{\mathcal{M}}_\lambda^{\mathtt{t}} f. \tag{64}$$

Here $\widetilde{\mathcal{L}}_\lambda^{\mathtt{t}}$ *is the transpose operator to* $\widetilde{\mathcal{L}}_\lambda$ *obtained from the real* L^2*-inner product, i.e.,* $\widetilde{\mathcal{L}}_\lambda^{\mathtt{t}} f = \overline{\widetilde{\mathcal{L}}_\lambda^* \overline{f}}$. *Similarly,* $\widetilde{\mathcal{M}}_\lambda^{\mathtt{t}}$ *is the transpose of* $\widetilde{\mathcal{M}}_\lambda$.

Proof. We begin with the first formula. We know that $\widetilde{\mathcal{L}}_\lambda$ maps to functions satisfying the Helmholtz equation $(-\Delta - \lambda^2)v = 0$. Therefore we only need to show that, given $f \in C_0^\infty(\mathbb{R}^3, \mathbb{C}^3)$, the function

$$u = (-\Delta_{\mathrm{free}} - \lambda^2)^{-1} \operatorname{curl}\operatorname{curl} f - \widetilde{\mathcal{L}}_\lambda (\mathcal{L}_\lambda)^{-1} (\nu \times) \widetilde{\mathcal{L}}_\lambda^{\mathtt{t}} f$$

satisfies relative boundary conditions. Since clearly $\operatorname{div} u = 0$ we only need to check that $\gamma_t u = 0$. One computes

$$\begin{aligned}\gamma_t u &= \gamma_t \operatorname{curl}\operatorname{curl}(-\Delta_{\mathrm{free}} - \lambda^2)^{-1} f - \mathcal{L}_\lambda (\mathcal{L}_\lambda)^{-1} (\nu \times) \widetilde{\mathcal{L}}_\lambda^{\mathtt{t}} f \\ &= \gamma_t \operatorname{curl}\operatorname{curl}(-\Delta_{\mathrm{free}} - \lambda^2)^{-1} f - (\nu \times) \gamma_T \operatorname{curl}\operatorname{curl}(-\Delta_{\mathrm{free}} - \lambda^2)^{-1} f = 0,\end{aligned}$$

which gives the result.

Next consider the second formula. We again only need to check that $\gamma_{t,\pm}(u) = 0$, where

$$u = (-\Delta_{\mathrm{free}} - \lambda^2)^{-1} \operatorname{curl} f - \widetilde{\mathcal{L}}_\lambda (\mathcal{L}_\lambda)^{-1} (\nu \times) \widetilde{\mathcal{M}}_\lambda^{\mathtt{t}} f.$$

The third formula follows from the second by applying the curl operator from the left and using $\operatorname{curl}\operatorname{curl}\operatorname{curl} \widetilde{\mathcal{S}}_\lambda = \lambda^2 \operatorname{curl} \widetilde{\mathcal{S}}_\lambda$. ☐

This can be used to show the following.

Theorem 8.2. *Let* $\epsilon > 0$, *and also suppose that* Ω_0 *is a smooth open set in* $\mathbb{R}^3$ *whose complement contains* $\overline{\Omega}$. *Let* $\delta = \operatorname{dist}(\partial\Omega, \Omega_0)$. *If* p *is the projection onto* $L^2(\Omega_0; \mathbb{C}^3)$ *in* $L^2(\mathbb{R}^3; \mathbb{C}^3)$ *then the operators*

$$p(-\Delta_{\mathrm{rel}} - \lambda^2)^{-1} \operatorname{curl}\operatorname{curl} p - p(-\Delta_{\mathrm{free}} - \lambda^2)^{-1} \operatorname{curl}\operatorname{curl} p,$$
$$p(-\Delta_{\mathrm{abs}} - \lambda^2)^{-1} \operatorname{curl}\operatorname{curl} p - p(-\Delta_{\mathrm{free}} - \lambda^2)^{-1} \operatorname{curl}\operatorname{curl} p,$$

are trace-class for all $\lambda \in \mathfrak{D}_\epsilon$ *as operators on* $L^2(\mathbb{R}^3; \mathbb{C}^3)$. *Moreover, for any* $\delta' \in (0, \delta)$, *their trace norms satisfy the bounds*

$$\| p(-\Delta_{\mathrm{rel}} - \lambda^2)^{-1} \operatorname{curl}\operatorname{curl} p - p(-\Delta_{\mathrm{free}} - \lambda^2)^{-1} \operatorname{curl}\operatorname{curl} p \|_1 \le C_{\delta',\epsilon} \mathrm{e}^{-\delta' \operatorname{Im}\lambda}, \tag{65}$$

$$\| p(-\Delta_{\mathrm{abs}} - \lambda^2)^{-1} \operatorname{curl}\operatorname{curl} p - p(-\Delta_{\mathrm{free}} - \lambda^2)^{-1} \operatorname{curl}\operatorname{curl} p \|_1 \le C_{\delta',\epsilon} \mathrm{e}^{-\delta' \operatorname{Im}\lambda} \tag{66}$$

for all $\lambda \in \mathfrak{D}_\epsilon$. *Moreover, both operators have integral kernels* $\kappa_{\mathrm{rel},\lambda}$, $\kappa_{\mathrm{abs},\lambda}$ *that are smooth on* $\Omega_0 \times \Omega_0$ *for all* $\lambda \in \mathfrak{D}_\epsilon$. *There exists* $C_{\Omega_0,\epsilon} > 0$ *depending on* Ω_0 *and* ϵ *such that*

$$\|k_{\mathrm{rel},\lambda}(x, x)\| + \|k_{\mathrm{abs},\lambda}(x, x)\| \le \left(C_{\Omega_0,\epsilon} \frac{\mathrm{e}^{-\operatorname{dist}(x,\partial\Omega) \operatorname{Im}\lambda}}{(\operatorname{dist}(x, \partial\Omega))^4} \right). \tag{67}$$

Proof. Given $\delta' \in (0, \delta)$, we choose a compactly supported smooth cut-off function χ which vanishes in Ω_0 such that the support of $\varphi = 1 - \chi$ has distance at least δ' from Ω. Then, since $\varphi p = p$, it is sufficient to show the estimates with p replaced by φ. From (62), we have

$$\begin{aligned}\varphi(-\Delta_{\mathrm{rel}} - \lambda^2)^{-1}(\operatorname{curl}\operatorname{curl})\varphi - \varphi(-\Delta_{\mathrm{free}} - \lambda^2)^{-1}(\operatorname{curl}\operatorname{curl})\varphi \\ = -\varphi\widetilde{\mathcal{L}}_\lambda \mathcal{L}_\lambda^{-1}(\nu\,\times)\widetilde{\mathcal{L}}_\lambda^{\mathrm{t}}\varphi = -(\varphi\widetilde{\mathcal{L}}_\lambda)\mathcal{L}_\lambda^{-1}(\nu\,\times)(\varphi\widetilde{\mathcal{L}}_\lambda)^{\mathrm{t}}.\end{aligned} \tag{68}$$

The operator $\varphi\widetilde{\mathcal{L}}_\lambda$ is Hilbert–Schmidt by Proposition 7.1. Since $\mathcal{L}_\lambda^{-1}$ is bounded by Corollary 7.9 on the correct domains, this factorises the right-hand side of (68) into a product of the two Hilbert–Schmidt operators $(\varphi\widetilde{\mathcal{L}}_\lambda)$, $(\varphi\widetilde{\mathcal{L}}_\lambda)^{\mathrm{t}}$ and a bounded operator $\mathcal{L}_\lambda^{-1}(\nu\,\times)$. This shows it is trace-class; see for example [Shubin 1987, (A.3.4) and (A.3.2)]. We need to show the bound for the trace-norm. We now employ the more explicit description of $\varphi\widetilde{\mathcal{L}}_\lambda = \varphi(\nabla\widetilde{\mathcal{S}}_\lambda \operatorname{Div} + \lambda^2\widetilde{\mathcal{S}}_\lambda)$. This gives

$$\begin{aligned}&(\varphi\widetilde{\mathcal{L}}_\lambda)\mathcal{L}_\lambda^{-1}(\nu\,\times)(\varphi\widetilde{\mathcal{L}}_\lambda)^{\mathrm{t}}\\ &\quad= (\varphi\nabla\widetilde{\mathcal{S}}_\lambda\operatorname{Div} + \lambda^2\varphi\widetilde{\mathcal{S}}_\lambda)\mathcal{L}_\lambda^{-1}((\nu\,\times)\nabla\widetilde{\mathcal{S}}_\lambda^{\mathrm{t}}\operatorname{div}\varphi + \lambda^2(\nu\,\times)(\varphi\widetilde{\mathcal{S}}_\lambda)^{\mathrm{t}})\\ &\quad= (\varphi\nabla\widetilde{\mathcal{S}}_\lambda\operatorname{Div} + \lambda^2\varphi\widetilde{\mathcal{S}}_\lambda)\mathcal{L}_\lambda^{-1}((\nu\,\times)\nabla\widetilde{\mathcal{S}}_\lambda^{\mathrm{t}}\operatorname{div}\varphi + \lambda^2(\nu\,\times)(\varphi\widetilde{\mathcal{S}}_\lambda)^{\mathrm{t}})\\ &\quad= \varphi\nabla\widetilde{\mathcal{S}}_\lambda\operatorname{Div}\mathcal{L}_\lambda^{-1}(\nu\,\times)\nabla\widetilde{\mathcal{S}}_\lambda^{\mathrm{t}}\operatorname{div}\varphi + \lambda^4\varphi\widetilde{\mathcal{S}}_\lambda\mathcal{L}_\lambda^{-1}(\nu\,\times)(\varphi\widetilde{\mathcal{M}}_\lambda)^{\mathrm{t}} + \lambda^2\varphi\nabla\widetilde{\mathcal{S}}_\lambda\operatorname{Div}\mathcal{L}_\lambda^{-1}(\nu\,\times)(\varphi\widetilde{\mathcal{S}}_\lambda)^{\mathrm{t}}\\ &\qquad\qquad + \lambda^2\varphi\widetilde{\mathcal{S}}_\lambda\mathcal{L}_\lambda^{-1}(\nu\,\times)\nabla\widetilde{\mathcal{S}}_\lambda^{\mathrm{t}}\operatorname{div}\varphi\\ &\quad= (\mathrm{I}) + (\mathrm{II}) + (\mathrm{III}) + (\mathrm{IV}).\end{aligned} \tag{69}$$

We will show that the estimate holds for the individual terms. The trace-norm of (I) is bounded by $\|\varphi\nabla\widetilde{\mathcal{S}}_\lambda\|_{\mathrm{HS}}^2\|\operatorname{Div}\mathcal{L}_\lambda^{-1}(\nu\,\times)\nabla\|$ using the fact that the Hilbert–Schmidt norm is invariant under transposition. This is bounded by $C\mathrm{e}^{-\delta'\operatorname{Im}\lambda}$ in the sector by Theorem 7.9 and estimate (30) of Proposition 7.1.

The trace-norm of term (II) is bounded by $|\lambda|^4\|\varphi\widetilde{\mathcal{S}}_\lambda\|_{\mathrm{HS}}^2\|\mathcal{L}_\lambda^{-1}\|$. This is again bounded by $C\mathrm{e}^{-\delta'\operatorname{Im}\lambda}$ by Theorem 7.9 and (29) of Proposition 7.1. Expression (III) is the transpose of (IV) as one computes easily from Lemma 6.2. It is therefore sufficient to bound the trace-norm of (IV). We have

$$\begin{aligned}(\mathrm{IV}) &= \lambda^2\varphi\widetilde{\mathcal{S}}_\lambda\Big(-\frac{1}{\lambda^2}(TQ_\lambda + U_\lambda P) - \frac{1}{\lambda}U_\lambda B + U_\lambda Q_\lambda\Big)(\nu\,\times)\nabla(\varphi\widetilde{\mathcal{S}}_\lambda)^{\mathrm{t}}\\ &= \lambda^2(\varphi\widetilde{\mathcal{S}}_\lambda)\Big(\frac{1}{\lambda^2}TQ_\lambda + U_\lambda Q_\lambda\Big)(\nu\times\nabla)(\varphi\widetilde{\mathcal{S}}_\lambda)^{\mathrm{t}},\end{aligned} \tag{70}$$

where we have used Lemma 7.7, the expansion (61), and the fact that $P(\nu\times\nabla) = 0$ and $B(\nu\times\nabla) = 0$. The range of T consists of distributions in $H_0^{-1/2}(\partial\Omega, \mathbb{C}^3) \cap H^{-1/2}(\operatorname{Div}, \partial\Omega)$. To see this, note that the range of T consists, by Theorem 7.5, of limits in $H^{-1/2}(\operatorname{Div}, \partial\Omega)$ of boundary values of curl-free vector fields. Applying the integration by parts formula (4) with $\phi \in \operatorname{rg}(T)$ and E a constant unit vector field, noting that $\operatorname{curl}\phi = \operatorname{curl} E = 0$, one obtains that $\langle\gamma_t\phi, \gamma E\rangle_{L^2(\partial\Omega,\mathbb{C}^3)} = \langle\gamma_t\phi, \gamma_T E\rangle_{L^2(\partial\Omega,\mathbb{C}^3)} = 0$ as claimed. It follows that the trace-norm of (III) and (IV) are bounded by $C\mathrm{e}^{-\delta'\operatorname{Im}\lambda}$ by Theorem 7.9 and by the estimates (30), (29).

Next we use (64) to obtain

$$\begin{aligned}\varphi(-\Delta_{\mathrm{abs}} - \lambda^2)^{-1}(\operatorname{curl}\operatorname{curl})\varphi - \varphi(-\Delta_{\mathrm{free}} - \lambda^2)^{-1}(\operatorname{curl}\operatorname{curl})\varphi \\ = -\lambda^2\varphi\widetilde{\mathcal{M}}_\lambda\mathcal{L}_\lambda^{-1}(\nu\,\times)\widetilde{\mathcal{M}}_\lambda^{\mathrm{t}}\varphi = -\lambda^2(\varphi\widetilde{\mathcal{M}}_\lambda)\mathcal{L}_\lambda^{-1}(\nu\,\times)(\varphi\widetilde{\mathcal{M}}_\lambda)^{\mathrm{t}}.\end{aligned} \tag{71}$$

The operators $\varphi\widetilde{\mathcal{M}}_\lambda$, $(\varphi\widetilde{\mathcal{M}}_\lambda)^{\mathrm{t}}$ are Hilbert–Schmidt, and their Hilbert–Schmidt norms are bounded by $\mathrm{e}^{-\delta' \operatorname{Im}\lambda}$ by Proposition 7.1 (31). This gives the claimed estimate for the trace-norm since the operator $\lambda^2\mathcal{L}_\lambda^{-1}$ is polynomially bounded in any sector by Theorem 7.9.

It remains to show the estimate on the diagonal of the integral kernel. This is done the same way using the pointwise estimate

$$\|\partial_x^\alpha \widetilde{\mathcal{S}}_\lambda(x,\cdot)\|_{H^{-1}(\partial\Omega)} \le C\frac{1}{(\operatorname{dist}(x,\partial\Omega))^{1+|\alpha|}}\mathrm{e}^{-\operatorname{Im}\lambda \operatorname{dist}(x,\partial\Omega)/2},$$

which is easily obtained directly from the integral kernel, noting that differentiation in the x or y-variable gives a linear combination of terms that are bounded by

$$\frac{\lambda^k(\operatorname{dist}(x,\partial\Omega))^k}{(\operatorname{dist}(x,\partial\Omega))^{1+|\alpha|}}\mathrm{e}^{-\operatorname{Im}\lambda \operatorname{dist}(x,\partial\Omega)} \le C_{k,\epsilon}\frac{1}{(\operatorname{dist}(x,\partial\Omega))^{1+|\alpha|}}\mathrm{e}^{-\operatorname{Im}\lambda \operatorname{dist}(x,\partial\Omega)/2},$$

with $0 \le k \le \alpha$. One now applies this estimate to each of the four terms (I), (II), (III), (IV) and observes that every factor of λ can be absorbed using the bound

$$|\lambda|^k\mathrm{e}^{-\operatorname{Im}\lambda \operatorname{dist}(x,\partial\Omega)} = C_{k,\epsilon}\frac{1}{\operatorname{dist}(x,\partial\Omega)^k}\mathrm{e}^{-\operatorname{Im}\lambda \operatorname{dist}(x,\partial\Omega)/2}.$$

This gives the first claimed estimate. The second estimate follows the same way, since the above implies

$$\|\widetilde{\mathcal{M}}_\lambda(x,\cdot)\|_{H^{-1}(\partial\Omega)} \le C\frac{1}{(\operatorname{dist}(x,\partial\Omega))^2}\mathrm{e}^{-\operatorname{Im}\lambda \operatorname{dist}(x,\partial\Omega)/2}. \qquad \square$$

9. The function Ξ

Recall that the boundary $\partial\Omega$ consists of N connected components $\partial\Omega_j$. To keep the discussion meaningful, we will assume throughout this section that $N \ge 2$. This gives a natural decomposition

$$H^{-1/2}(\mathrm{Div},\partial\Omega) = \bigoplus_{j=1}^N H^{-1/2}(\mathrm{Div},\partial\Omega_j).$$

Let q_j be the orthogonal projection $H^{-1/2}(\mathrm{Div},\partial\Omega) \to H^{-1/2}(\mathrm{Div},\partial\Omega_j)$ and $\mathcal{L}_{j,\lambda} = q_j\mathcal{L}_\lambda q_j$. We then can write

$$\mathcal{L}_\lambda = \sum_{j=1}^N \mathcal{L}_{j,\lambda} + \sum_{j\ne k} q_j\mathcal{L}_\lambda q_k = \mathcal{L}_{D,\lambda} + \mathcal{T}_\lambda. \tag{72}$$

We remark that $\mathcal{L}_{j,\lambda}$, which is regarded as a map from $H^{-1/2}(\mathrm{Div},\partial\Omega) \to H^{-1/2}(\mathrm{Div},\partial\Omega)$, is independent of the other components. The sum $\mathcal{L}_{D,\lambda}$ describes the diagonal part of the operator $\mathcal{L}$ with respect to the decomposition above.

We have a similar decomposition for the operator

$$\mathcal{M}_\lambda = \mathcal{M}_{D,\lambda} + \mathcal{J}_\lambda.$$

We set

$$\delta = \min_{j\ne k} \operatorname{dist}(\partial\Omega_j,\partial\Omega_k) > 0. \tag{73}$$

Then we have the following proposition.

Proposition 9.1. *The families $\mathcal{T}_\lambda, \mathcal{J}_\lambda : H^{-1/2}(\mathrm{Div}, \partial\Omega) \to H^{-1/2}(\mathrm{Div}, \partial\Omega)$ are holomorphic families of trace-class operators in the complex plane. For any $\epsilon > 0$ and any $\delta' \in (0, \delta)$, the following estimates for their trace-norms $\|\cdot\|_1$ hold*:

$$\|\mathcal{T}_\lambda\|_1 \le C_{\delta',\epsilon} \mathrm{e}^{-\delta' \operatorname{Im}\lambda}, \qquad \|\mathcal{J}_\lambda\|_1 \le C_{\delta',\epsilon} \mathrm{e}^{-\delta' \operatorname{Im}\lambda}, \tag{74}$$

$$\Big\|\frac{\mathrm{d}}{\mathrm{d}\lambda}\mathcal{T}_\lambda\Big\|_1 \le C_{\delta',\epsilon} \mathrm{e}^{-\delta' \operatorname{Im}\lambda}, \quad \Big\|\frac{\mathrm{d}}{\mathrm{d}\lambda}\mathcal{J}_\lambda\Big\|_1 \le C_{\delta',\epsilon} \mathrm{e}^{-\delta' \operatorname{Im}\lambda} \tag{75}$$

for all λ in the sector $\mathfrak{D}_\epsilon$. We also have

$$\|\mathcal{T}_\lambda|_{H^{-1/2}(\mathrm{Div}\,0,\partial\Omega)}\|_1 \le C_{\delta',\epsilon} |\lambda|^2 \mathrm{e}^{-\delta' \operatorname{Im}\lambda}, \tag{76}$$

$$\Big\|\frac{\mathrm{d}}{\mathrm{d}\lambda}\mathcal{T}_\lambda|_{H^{-1/2}(\mathrm{Div}\,0,\partial\Omega)}\Big\|_1 \le C_{\delta',\epsilon} |\lambda| \mathrm{e}^{-\delta' \operatorname{Im}\lambda}. \tag{77}$$

Proof. We will prove this estimate only for $\mathcal{T}_\lambda$ as the estimate for $\mathcal{J}_\lambda$ is proved in the same way. It is sufficient to show this for the individual terms $q_j \mathcal{L}_\lambda q_k$ with $j \neq k$. We choose an open bounded neighbourhood U of $\partial\Omega_j$ and an open bounded neighbourhood V of $\partial\Omega_k$ such that $\mathrm{dist}(U, V) > \delta'$. The first two estimates are implied by Lemma A.4 by observing that the operator is the composition

$$H^{-1/2}(\mathrm{Div}, \partial\Omega) \to H^{-1}(V) \to H^1(U) \to H^{-1/2}(\mathrm{Div}, \partial\Omega),$$

and the map $H^{-1}(V) \to H^1(U)$ has smooth integral kernel

$$\chi(x, y)\,\mathrm{curl}\,\mathrm{curl}_x \frac{\mathrm{e}^{\mathrm{i}\lambda|x-y|}}{4\pi|x-y|}$$

for a suitable cut-off function that is compactly supported in $U \times V$. The same argument applies to the λ-derivative.

To show the bounds on the restriction to $H^{-1/2}(\mathrm{Div}\,0, \partial\Omega)$, one uses that $\mathcal{L}_\lambda = \gamma_t \nabla \mathcal{S}_\lambda \mathrm{Div} + \lambda^2 \mathcal{S}_\lambda$. To bound the trace-norm of $\lambda^2 q_j \mathcal{S}_\lambda q_k$, one uses exactly the same argument as above applied to the kernel

$$\lambda^2 \chi(x, y) \frac{\mathrm{e}^{\mathrm{i}\lambda|x-y|}}{4\pi|x-y|}$$

and its λ-derivative. □

Proposition 9.2. *Fix $\epsilon > 0$. Then $(\mathcal{L}_\lambda \mathcal{L}_{D,\lambda}^{-1} - \mathrm{id}) : H^{-1/2}(\mathrm{Div}, \partial\Omega) \to H^{-1/2}(\mathrm{Div}, \partial\Omega)$ is a meromorphic family of trace-class operators with no poles in the closed upper half-plane. In the sector*, *we have*, *for any $\delta' \in (0, \delta)$, the estimate*

$$\|\mathcal{L}_{D,\lambda}^{-1} \mathcal{L}_\lambda - \mathrm{id}\|_1 = \|\mathcal{L}_\lambda \mathcal{L}_{D,\lambda}^{-1} - \mathrm{id}\|_1 \le C_{\delta',\epsilon} \mathrm{e}^{-\delta' \operatorname{Im}\lambda}. \tag{78}$$

Proof. We use (19) and obtain

$$\mathcal{L}_\lambda \mathcal{L}_{D,\lambda}^{-1} - \mathrm{id} = \big(\tfrac{1}{2} + \mathcal{M}_\lambda\big)\big(\tfrac{1}{2} + \mathcal{M}_{D,\lambda}\big)^{-1} - \mathrm{id},$$

bearing in mind that $\Lambda_\lambda^+ = \Lambda_{D,\lambda}^+$. With

$$\big(\tfrac{1}{2} + \mathcal{M}_{D,\lambda}\big)^{-1} = \frac{1}{\lambda^2} P_D + \frac{1}{\lambda} B_D + Q_\lambda,$$

we remark that

$$\left(\tfrac{1}{2}+\mathcal{M}_{D,0}\right)P_D=\left(\tfrac{1}{2}+\mathcal{M}_{D,0}\right)B_D=0,$$

but then also

$$\left(\tfrac{1}{2}+\mathcal{M}_{0}\right)P_D=\left(\tfrac{1}{2}+\mathcal{M}_{0}\right)B_D=0$$

because, according to Proposition 6.6, we know that the kernels of $\left(\tfrac{1}{2}+\mathcal{M}_0\right)$ and $\left(\tfrac{1}{2}+\mathcal{M}_{D,0}\right)$ coincide. We have used here, as in the proof of Lemma 7.7, that the first-order terms in the expansion of $\mathcal{M}_\lambda$ vanish at $\lambda=0$, i.e., $\left(\frac{d}{d\lambda}\mathcal{M}_\lambda\right)\big|_{\lambda=0}=\left(\frac{d}{d\lambda}\mathcal{M}_{D,\lambda}\right)\big|_{\lambda=0}=0$. Using the abbreviation $\mathcal{J}_\lambda=\mathcal{M}_\lambda-\mathcal{M}_{D,\lambda}$, this implies $\mathcal{J}_0P_D=\mathcal{J}_0B_D=0$. Moreover, $\mathcal{J}_\lambda$ is trace-class. This shows that

$$\left(\tfrac{1}{2}+\mathcal{M}_\lambda\right)\left(\tfrac{1}{2}+\mathcal{M}_{D,\lambda}\right)^{-1}-\mathrm{id}$$

is a meromorphic family of trace-class operators and 0 is not a pole. Interior Maxwell eigenvalues are not poles by the same argument, since the kernel of $\left(\tfrac{1}{2}+\mathcal{M}_\mu\right)$ coincides with the kernel of $\left(\tfrac{1}{2}+\mathcal{M}_{D,\mu}\right)$ and by the expansion of Lemma 7.8.

Moreover, $\left(\tfrac{1}{2}+\mathcal{M}_{D,\lambda}\right)$ is invertible for all the other points in the closed upper half-space, and hence there are no poles there. To show the estimate in the sector, we note that

$$\mathcal{L}_\lambda\mathcal{L}_{D,\lambda}^{-1}-\mathrm{id}=\mathcal{T}_\lambda\mathcal{L}_{D,\lambda}^{-1}. \tag{79}$$

Then the bound for large $|\lambda|$ is a result of Theorem 7.9 and Proposition 9.1. □

Proposition 9.3. *The Fredholm determinant* $\det(\mathcal{L}_\lambda\mathcal{L}_{D,\lambda}^{-1})$ *in the space* $H^{-1/2}(\mathrm{Div},\partial\Omega)$ *is well defined and holomorphic in a neighbourhood of the closed upper half-space. For any* $\epsilon>0$ *and* $\delta'\in(0,\delta)$, *we have the bound*

$$|\det(\mathcal{L}_\lambda\mathcal{L}_{D,\lambda}^{-1})-1|\leq C_{\delta',\epsilon}e^{-\delta'\operatorname{Im}\lambda} \tag{80}$$

for all λ *in the sector* $\mathfrak{D}_\epsilon$. *Moreover*, $\det(\mathcal{L}_\lambda\mathcal{L}_{D,\lambda}^{-1})$ *is nonzero in the closed upper half-space.*

Proof. The trace of $(\mathcal{L}_\lambda\mathcal{L}_{D,\lambda}^{-1}-\mathrm{id})$ is bounded by Proposition 9.2. Using the bound

$$|\det(1+A)-1|\leq\|A\|_1e^{1+\|A\|_1}$$

for the Fredholm determinant (see for example [Simon 1977, (3.7)]), one obtains

$$|\Xi(\lambda)|\leq|\log\det(\mathcal{L}_\lambda\mathcal{L}_{D,\lambda}^{-1})|\leq C_{\delta',\epsilon}e^{-\delta'\operatorname{Im}\lambda}. \tag{81}$$

By analyticity of $(\mathcal{L}_\lambda\mathcal{L}_{D,\lambda}^{-1}-\mathrm{id})=\mathcal{J}_\lambda\left(\tfrac{1}{2}+\mathcal{M}_{D,\lambda}\right)^{-1}$ as a family of trace-class operators in the upper half-space and near 0, the determinant also depends analytically on λ (e.g., [Simon 1977, Theorem 3.3]). By invertibility of the operator in the closed upper half-space, the determinant never vanishes [Simon 1977, Theorem 3.9], and therefore log det is analytic in the union of the upper half-space and a neighbourhood of 0. □

Since the determinant does not vanish near the closed upper half-space, we can choose a simply connected open neighbourhood $\mathcal{U}$ of the closed upper half-space, and it then defines a holomorphic function $\mathcal{U}\to\mathbb{C}\setminus\{0\}$ which we can lift to a holomorphic function on the logarithmic cover of the complex plane, where we choose the branch cut to be the negative real line $(-\infty,0)$. Composition with log is then well defined, and we write $\log\det(\mathcal{L}_\lambda \mathcal{L}_{D,\lambda}^{-1})$ to mean this composition. This means that this function and the branch of the logarithm is fixed by requiring this to be a holomorphic function that decays exponentially fast along the positive imaginary axis.

Definition 9.4. The function Ξ is defined in a sufficiently small simply connected open neighbourhood of the closed upper half-space by

$$\Xi(\lambda) = \log\det(\mathcal{L}_\lambda \mathcal{L}_{D,\lambda}^{-1}),$$

where the branch of the logarithm is chosen as explained above.

Theorem 9.5. *The function $\Xi(\lambda)$ is holomorphic near the closed upper half-space and*, *for any $\epsilon > 0$ and $\delta' \in (0, \delta)$*, *we have the bounds*

$$|\Xi(\lambda)| \le C_{\delta',\epsilon} \mathrm{e}^{-\delta' \operatorname{Im} \lambda}, \quad |\Xi'(\lambda)| \le C_{\delta',\epsilon} \mathrm{e}^{-\delta' \operatorname{Im} \lambda} \tag{82}$$

for λ in the sector $\mathfrak{D}_\epsilon$.

Proof. The first bound is a direct consequence of the proposition above. The second bound is a direct consequence of the maximum modulus principle. □

10. Relative trace formula

We consider the two Maxwell resolvent differences

$$R_{D,\mathrm{rel},\lambda} = \left(((-\Delta_{\mathrm{rel}} - \lambda^2)^{-1} - (-\Delta_{\mathrm{free}} - \lambda^2)^{-1}) - \sum_{j=1}^{N} ((-\Delta_{\mathrm{rel},j} - \lambda^2)^{-1} - (-\Delta_{\mathrm{free}} - \lambda^2)^{-1}) \right) \operatorname{curl}\operatorname{curl},$$

$$R_{D,\mathrm{abs},\lambda} = \left(((-\Delta_{\mathrm{abs}} - \lambda^2)^{-1} - (-\Delta_{\mathrm{free}} - \lambda^2)^{-1}) - \sum_{j=1}^{N} ((-\Delta_{\mathrm{abs},j} - \lambda^2)^{-1} - (-\Delta_{\mathrm{free}} - \lambda^2)^{-1}) \right) \operatorname{curl}\operatorname{curl}.$$

Using (62) and (64), we conclude

$$((-\Delta_{\mathrm{rel},j} - \lambda^2)^{-1} - (-\Delta_{\mathrm{free}} - \lambda^2)^{-1}) \operatorname{curl}\operatorname{curl} = -\widetilde{\mathcal{L}}_\lambda \mathcal{L}_{j,\lambda}^{-1} (\nu\times) \widetilde{\mathcal{L}}_\lambda^{\mathfrak{t}},$$
$$((-\Delta_{\mathrm{abs},j} - \lambda^2)^{-1} - (-\Delta_{\mathrm{free}} - \lambda^2)^{-1}) \operatorname{curl}\operatorname{curl} = -\lambda^2 \widetilde{\mathcal{M}}_\lambda \mathcal{L}_{j,\lambda}^{-1} (\nu\times) \widetilde{\mathcal{M}}_\lambda^{\mathfrak{t}},$$

and hence

$$R_{D,\mathrm{rel},\lambda} = -\widetilde{\mathcal{L}}_\lambda \mathcal{L}_\lambda^{-1} (\nu\times) \widetilde{\mathcal{L}}_\lambda^{\mathfrak{t}} + \widetilde{\mathcal{L}}_\lambda \mathcal{L}_{D,\lambda}^{-1} (\nu\times) \widetilde{\mathcal{L}}_\lambda^{\mathfrak{t}},$$
$$R_{D,\mathrm{abs},\lambda} = -\widetilde{\mathcal{M}}_\lambda \mathcal{L}_\lambda^{-1} (\nu\times) \widetilde{\mathcal{M}}_\lambda^{\mathfrak{t}} + \widetilde{\mathcal{M}}_\lambda \mathcal{L}_{D,\lambda}^{-1} (\nu\times) \widetilde{\mathcal{M}}_\lambda^{\mathfrak{t}}.$$

We have the following improvement of Theorem 8.2 in the relative setting.

Proposition 10.1. *Let $\epsilon > 0$, and let $\delta' > 0$ be smaller than $\delta = \operatorname{dist}(\partial\Omega_j, \partial\Omega_k)$. Then the operators $R_{D,\mathrm{rel},\lambda}, R_{D,\mathrm{abs},\lambda} : L^2(\mathbb{R}^3, \mathbb{C}^3) \to L^2(\mathbb{R}^3, \mathbb{C}^3)$ are trace-class for all $\lambda \in \mathfrak{D}_\epsilon$, and their trace norm can be estimated by*

$$\|R_{D,\mathrm{rel},\lambda}\|_1 + \|R_{D,\mathrm{abs},\lambda}\|_1 \le C_{\delta',\epsilon} \mathrm{e}^{-\delta' \operatorname{Im}\lambda}, \quad \lambda \in \mathfrak{D}_\epsilon.$$

Proof. First note that

$$(\mathcal{L}_\lambda^{-1} - \mathcal{L}_{\lambda,D}^{-1}) = -(\mathcal{L}_\lambda^{-1} \mathcal{T}_\lambda \mathcal{L}_{\lambda,D}^{-1})$$

is a meromorphic family of trace-class operators $H^{-1/2}(\mathrm{Div}, \partial\Omega) \to H^{-1/2}(\mathrm{Div}, \partial\Omega)$ in the complex plane. For $|\lambda| > 1$, the bound then follows from Proposition 9.1 and the bounds in Theorem 7.9. In particular, the expansion

$$(\mathcal{L}_\lambda^{-1} - \mathcal{L}_{\lambda,D}^{-1}) = \frac{1}{\lambda^2} L_2 + \frac{1}{\lambda} L_1 + L_{0,\lambda}$$

resulting from (61) is in terms of trace-class operators $L_2,\ L_1$ and the holomorphic family of trace-class operators $L_{0,\lambda}$. Specifically,

$$\begin{aligned} L_2 &= -T(Q^{(0)} - Q_D^{(0)}) - (U^{(0)}P - U_D^{(0)}P_D) = TW_2 + V_2, \\ L_1 &= -T(Q^{(1)} - Q_D^{(1)}) - (U^{(0)}B - U_D^{(0)}B_D) - (U^{(1)}P - U_D^{(1)}P_D) = TW_1 + V_1, \end{aligned}$$

where $Q^{(0)}$ and $Q^{(1)}$ are the expansion coefficients of

$$Q_\lambda = Q^{(0)} + Q^{(1)}\lambda + O(|\lambda|^2)$$

near $\lambda = 0$. The same notation is used for the expansion coefficients of $Q_{D,\lambda},\ U_\lambda,\ U_{D,\lambda}$. Since the operator

$$\left(\tfrac{1}{2} + \mathcal{M}_\lambda\right)^{-1} - \left(\tfrac{1}{2} + \mathcal{M}_{D,\lambda}\right)^{-1} = -\left(\tfrac{1}{2} + \mathcal{M}_\lambda\right)^{-1} \mathcal{J}_\lambda \left(\tfrac{1}{2} + \mathcal{M}_{D,\lambda}\right)^{-1}$$

is a meromorphic family of trace-class operators, we know that the expansion coefficients $W_2 = Q^{(0)} - Q_D^{(0)}$ and $W_1 = Q^{(1)} - Q_D^{(1)}$ are trace-class. We also record that $V_2(\nu \times \nabla) = 0$ and $V_1(\nu \times \nabla) = 0$ and recall that $\mathrm{Div} \circ T = 0$. Now we are ready to estimate the resolvent differences. We first focus on $R_{D,\mathrm{rel},\lambda}$. We have

$$R_{D,\mathrm{rel},\lambda} = -\widetilde{\mathcal{L}}_\lambda (\mathcal{L}_\lambda^{-1} - \mathcal{L}_{D,\lambda}^{-1})(\nu \times) \widetilde{\mathcal{L}}_\lambda^{\mathtt{t}} = -\widetilde{\mathcal{L}}_\lambda \left(\frac{1}{\lambda^2}(TW_2 + V_2) + \frac{1}{\lambda}(TW_1 + V_1) + L_{0,\lambda} \right)(\nu \times) \widetilde{\mathcal{L}}_\lambda^{\mathtt{t}}.$$

We expand this further using $\widetilde{\mathcal{L}}_\lambda = \nabla \widetilde{\mathcal{S}}_\lambda \mathrm{Div} + \lambda^2 \widetilde{\mathcal{S}}_\lambda$ to obtain that, modulo terms that have bounded trace-norm near $\lambda = 0$, the operator $R_{D,\lambda}$ equals

$$\begin{aligned} &(\nabla \widetilde{\mathcal{S}}_\lambda \mathrm{Div} + \lambda^2 \widetilde{\mathcal{S}}_\lambda) \left(\frac{1}{\lambda^2}(TW_2 + V_2) + \frac{1}{\lambda}(TW_1 + V_1) \right) ((\nu \times) \nabla \widetilde{\mathcal{S}}_\lambda^{\mathtt{t}} \mathrm{div} + \lambda^2 (\nu \times)(\widetilde{\mathcal{S}}_\lambda)^{\mathtt{t}}) \\ &\quad = \nabla \widetilde{\mathcal{S}}_\lambda \,\mathrm{Div}((TW_2 + V_2) + \lambda(TW_1 + V_1))(\nu \times)(\widetilde{\mathcal{S}}_\lambda)^{\mathtt{t}} + \widetilde{\mathcal{S}}_\lambda((TW_2 + V_2) + \lambda(TW_1 + V_1))(\nu \times) \nabla \widetilde{\mathcal{S}}_\lambda^{\mathtt{t}} \,\mathrm{div} \\ &\qquad\qquad + \lambda^4 \widetilde{\mathcal{S}}_\lambda \left(\frac{1}{\lambda^2}(TW_2 + V_2) + \frac{1}{\lambda}(TW_1 + V_1) \right)(\nu \times)(\widetilde{\mathcal{S}}_\lambda)^{\mathtt{t}} \\ &\quad = \text{(I)} + \text{(II)} + \text{(III)}. \end{aligned}$$

Since $\mathcal{L}_{D,\lambda}$ and $\mathcal{L}_\lambda$ are self-adjoint with respect to the antisymmetric bilinear and since

$$\mathcal{L}_\lambda^{-1} - \mathcal{L}_{D,\lambda}^{-1} = \left(\frac{1}{\lambda^2}(TW_2 + V_2) + \frac{1}{\lambda}(TW_1 + V_1) + L_{0,\lambda} \right),$$

one obtains

$$\left(\left(\frac{1}{\lambda^2}(TW_2 + V_2) + \frac{1}{\lambda}(TW_1 + V_1) \right)(\nu \times) \right)^{\mathtt{t}} = \left(\frac{1}{\lambda^2}(TW_2 + V_2) + \frac{1}{\lambda}(TW_1 + V_1) \right)(\nu \times),$$

and therefore (II) is the transpose of (I). (III) has bounded trace-norm near $\lambda = 0$. Finally

$$\text{(II)} = \widetilde{\mathcal{S}}_\lambda((TW_2 + V_2) + \lambda(TW_1 + V_1))(\nu \times) \nabla \widetilde{\mathcal{S}}_\lambda^{\mathtt{t}} \,\mathrm{div} = \widetilde{\mathcal{S}}_\lambda((TW_2) + \lambda(TW_1))(\nu \times) \nabla \widetilde{\mathcal{S}}_\lambda^{\mathtt{t}} \,\mathrm{div}$$

has bounded trace-norm near $\lambda = 0$ as $\widetilde{\mathcal{S}}_\lambda T$ and $\nabla \widetilde{\mathcal{S}}_\lambda^{\mathtt{t}}$ div have bounded operator norm, and W_2 and W_1 are trace-class. Finally we consider $R_{D,\mathrm{abs},\lambda}$. We compute as above

$$\begin{aligned} R_{D,\mathrm{abs},\lambda} &= -\lambda^2 \widetilde{\mathcal{M}}_\lambda (\mathcal{L}_\lambda^{-1} - \mathcal{L}_{D,\lambda}^{-1})(\nu \times) \widetilde{\mathcal{M}}_\lambda^{\mathtt{t}} \\ &= -\lambda^2 \widetilde{\mathcal{M}}_\lambda \left(\frac{1}{\lambda^2}(T W_2 + V_2) + \frac{1}{\lambda}(T W_1 + V_1) + L_{0,\lambda} \right) (\nu \times) \widetilde{\mathcal{M}}_\lambda^{\mathtt{t}} \\ &= -\widetilde{\mathcal{M}}_\lambda ((T W_2 + V_2) + \lambda (T W_1 + V_1) + \lambda^2 L_{0,\lambda})(\nu \times) \widetilde{\mathcal{M}}_\lambda^{\mathtt{t}}, \end{aligned}$$

whose trace-norm is bounded near 0 since $\widetilde{\mathcal{M}}_\lambda$ is uniformly bounded. ☐

Lemma 10.2. *We have*

$$\operatorname{tr}(R_{D,\mathrm{rel},\lambda}) = \operatorname{tr}(R_{D,\mathrm{abs},\lambda}) = -\frac{\lambda}{2} \Xi'(\lambda).$$

Proof. One has

$$\begin{aligned} (\nu \times) \widetilde{\mathcal{L}}_\lambda^{\mathtt{t}} \widetilde{\mathcal{L}}_\lambda &= \gamma_t \operatorname{curl}\operatorname{curl}\operatorname{curl}\operatorname{curl}(-\Delta_{\mathrm{free}} - \lambda^2)^{-2} \gamma_T^{\mathtt{t}} = \gamma_t \operatorname{curl}\operatorname{curl}(-\Delta_{\mathrm{free}})(-\Delta_{\mathrm{free}} - \lambda^2)^{-2} \gamma_T^{\mathtt{t}} \\ &= \gamma_t \operatorname{curl}\operatorname{curl}(-\Delta_{\mathrm{free}} - \lambda^2)^{-1} \gamma_T^{\mathtt{t}} + \gamma_t \operatorname{curl}\operatorname{curl} \lambda^2 (-\Delta_{\mathrm{free}} - \lambda^2)^{-2} \gamma_T^{\mathtt{t}} = \mathcal{L}_\lambda + \frac{\lambda}{2} \frac{\mathrm{d}}{\mathrm{d}\lambda} \mathcal{L}_\lambda. \end{aligned}$$

Similarly, we also have

$$\lambda^2 (\nu \times) \widetilde{\mathcal{M}}_\lambda^{\mathtt{t}} \widetilde{\mathcal{M}}_\lambda = \lambda^2 \gamma_t \operatorname{curl}\operatorname{curl}(-\Delta_{\mathrm{free}} - \lambda^2)^{-2} \gamma_T^{\mathtt{t}} = \frac{\lambda}{2} \frac{\mathrm{d}}{\mathrm{d}\lambda} \mathcal{L}_\lambda.$$

Using invariance of the trace in $H^{-1/2}(\mathrm{Div}, \partial\Omega)$ under cyclic permutations, we get

$$\begin{aligned} \operatorname{tr}(R_{D,\mathrm{rel},\lambda}) &= -\operatorname{tr}(-\widetilde{\mathcal{L}}_\lambda (\mathcal{L}_\lambda^{-1} - \mathcal{L}_{D,\lambda}^{-1})(\nu \times) \widetilde{\mathcal{L}}_\lambda^{\mathtt{t}}) = -\operatorname{tr}\Big(\Big(\mathcal{L}_\lambda + \frac{\lambda}{2} \frac{\mathrm{d}}{\mathrm{d}\lambda} \mathcal{L}_\lambda \Big) (\mathcal{L}_\lambda^{-1} - \mathcal{L}_{D,\lambda}^{-1}) \Big) \\ &= -\operatorname{tr}(\mathrm{id} - \mathcal{L}_\lambda \mathcal{L}_{D,\lambda}^{-1}) - \frac{\lambda}{2} \frac{\mathrm{d}}{\mathrm{d}\lambda} \log\det(\mathcal{L}_\lambda \mathcal{L}_{D,\lambda}^{-1}) = -\frac{\lambda}{2} \frac{\mathrm{d}}{\mathrm{d}\lambda} \log\det(\mathcal{L}_\lambda \mathcal{L}_{D,\lambda}^{-1}). \end{aligned}$$

Here we have used that $\operatorname{tr}\big(\mathcal{L}_{D,\lambda}^{-1} \frac{\mathrm{d}}{\mathrm{d}\lambda}(\mathcal{T}_\lambda)\big) = 0$ and $\operatorname{tr}(\mathcal{L}_{D,\lambda}^{-1} \mathcal{T}_\lambda) = 0$. Indeed this follows as

$$\operatorname{Tr}\Big(\mathcal{L}_{\lambda,D}^{-1} \Big(\frac{\mathrm{d}}{\mathrm{d}\lambda} \mathcal{T}_\lambda \Big) \Big) = \sum_{j \neq k} \operatorname{Tr}\Big(\mathcal{L}_{\lambda,D}^{-1} \Big(q_j \Big(\frac{\mathrm{d}}{\mathrm{d}\lambda} \mathcal{L}_\lambda \Big) q_k \Big) \Big) = \sum_{j \neq k} \operatorname{Tr}\Big(\Big(q_j \mathcal{L}_{\lambda,D}^{-1} \frac{\mathrm{d}}{\mathrm{d}\lambda} \mathcal{L}_\lambda \Big) q_k \Big) = 0.$$

We have also used the fact that, for a holomorphic family of trace-class operators $A(\lambda)$, we have that $\log\det(\mathrm{id} + A(\lambda))$ is holomorphic and we have the identity

$$\frac{\mathrm{d}}{\mathrm{d}\lambda} \log\det(\mathrm{id} + A(\lambda)) = \operatorname{tr}\Big((\mathrm{id} + A(\lambda))^{-1} \frac{\mathrm{d}}{\mathrm{d}\lambda} A(\lambda) \Big),$$

so that

$$\frac{\mathrm{d}}{\mathrm{d}\lambda} \log\det(\mathcal{L}_\lambda \mathcal{L}_{D,\lambda}^{-1}) = \operatorname{tr}\Big(\mathcal{L}_\lambda^{-1} \frac{\mathrm{d}}{\mathrm{d}\lambda}(\mathcal{L}_\lambda) - \mathcal{L}_{D,\lambda}^{-1} \frac{\mathrm{d}}{\mathrm{d}\lambda}(\mathcal{L}_{D,\lambda}) \Big).$$

In the same way,

$$\begin{aligned} \operatorname{tr}(R_{D,\mathrm{abs},\lambda}) &= -\operatorname{tr}(\lambda^2 \widetilde{\mathcal{M}}_\lambda (\mathcal{L}_\lambda^{-1} - \mathcal{L}_{D,\lambda}^{-1})(\nu \times) \widetilde{\mathcal{M}}_\lambda^{\mathtt{t}}) = -\operatorname{tr}\Big(\Big(\frac{\lambda}{2} \frac{\mathrm{d}}{\mathrm{d}\lambda} \mathcal{L}_\lambda \Big) (\mathcal{L}_\lambda^{-1} - \mathcal{L}_{D,\lambda}^{-1}) \Big) \\ &= -\frac{\lambda}{2} \frac{\mathrm{d}}{\mathrm{d}\lambda} \log\det(\mathcal{L}_\lambda \mathcal{L}_{D,\lambda}^{-1}) \end{aligned}$$

☐

11. Proof of the main theorems

Proof of Theorem 1.1. This theorem is the combination of Proposition 9.3 and Theorem 9.5 in Section 9. □

Proof of Theorem 1.3. We set $f(z) = z^{-2}g(z^2)$, where $g \in \mathcal{P}_\epsilon$. By the decay properties of Ξ, it is sufficient to show equality for small ϵ, so we assume $\epsilon < \frac{\pi}{4}$. Then the function e^{-1/nz^2} is holomorphic in the sector $\mathfrak{S}_\epsilon$ and decays faster than exponentially. The function $g_n(z) = \mathrm{e}^{-1/nz^2} g(z)$ is therefore an admissible function for the Riesz–Dunford functional calculus, and we therefore have

$$f_n((-\Delta_{\mathrm{rel}})^{1/2})\,\mathrm{curl}\,\mathrm{curl} = f_n((\delta\,\mathrm{d})^{1/2})\delta\,\mathrm{d} = g_n(\delta\,\mathrm{d}) = -\frac{1}{2\pi\mathrm{i}}\int_{\Gamma_\epsilon}(\delta\,\mathrm{d} - z)^{-1} g_n(z)\,\mathrm{d}z$$

and similarly for the other terms appearing in $R_{D,\mathrm{rel},\lambda}$. The integral converges despite the pole of order 1 at 0 since $g \in \mathcal{P}_\epsilon$ implies that $g_n(z) = O(|z|^\alpha)$ for some $\alpha > 0$ near $z = 0$. Here $f_n(z) = z^{-2} g_n(z^2)$. If $h \in C_0^\infty(X, \mathbb{C}^3)$ then we have convergence of $g_n(\delta\,\mathrm{d})$ to $g(\delta\,\mathrm{d})$ in L^2. Indeed, by our definition of the function class $\mathcal{P}_\epsilon$, the function g is polynomially bounded on the real line and therefore h is in the domain of the operator $g(\delta\,\mathrm{d})$. Consequently the function g is square-integrable with respect to the measure $\langle \mathrm{d}E_\lambda h, h\rangle$, where $\mathrm{d}E_\lambda$ is the spectral measure of $\delta\,\mathrm{d}$. Then we have

$$\|(g(\delta\,\mathrm{d}) - g_n(\delta\,\mathrm{d}))h\|_{L^2} = \int_{\mathbb{R}} (1 - \mathrm{e}^{-x^2/n})^2 |g|^2(x)\,\langle \mathrm{d}E_\lambda h, h\rangle,$$

which tends to 0 as $n \to \infty$ by the dominated convergence theorem.

We note now that

$$(-\Delta_{\mathrm{rel}} - z)^{-1}\delta\,\mathrm{d} = (\delta\,\mathrm{d} - z)^{-1}\delta\,\mathrm{d} = \mathrm{id} + z(\delta\,\mathrm{d} - z)^{-1},$$

and again this formula applies to the other terms in $R_{D,\mathrm{rel},\lambda}$. This gives

$$D_{\mathrm{rel},f_n} = -\frac{1}{2\pi\mathrm{i}}\int_{\Gamma_\epsilon} R_{D,\mathrm{rel}}(\sqrt{z})\frac{1}{z} g_n(z)\,\mathrm{d}z = -\frac{1}{\mathrm{i}\pi}\int_{\widetilde{\Gamma}_{\epsilon/2}} R_{D,\mathrm{rel}}(\lambda)\lambda f_n(\lambda)\,\mathrm{d}\lambda.$$

Moreover, $D_{\mathrm{rel},f_n}h$ converges in L^2 to $D_{\mathrm{rel},f}h$ for any $h \in C_0^\infty(X, \mathbb{C}^3)$. By the decay properties of $R_{D,\mathrm{rel}}(\lambda)$, Proposition 10.1, the integral converges in the Banach space of trace-class operators, and the sequence D_{f_n} is Cauchy in the Banach space of trace-class operators. We conclude that $D_{\mathrm{rel},f}$ is trace-class.

To compute the trace, we can again use the convergence of the integral in the space of trace-class operators and therefore, using Lemma 10.2, we obtain

$$D_{\mathrm{rel},f} = -\frac{1}{\mathrm{i}\pi}\int_{\widetilde{\Gamma}_{\epsilon/2}} R_{\mathrm{rel},D}(\lambda)\lambda f(\lambda)\,\mathrm{d}\lambda = \frac{1}{2\pi\mathrm{i}}\int_{\widetilde{\Gamma}_{\epsilon/2}} (\Xi'(\lambda))\lambda^2 f(\lambda)\,\mathrm{d}\lambda.$$

Integration by parts and the decay of Ξ, Theorem 9.5, then completes the proof for $D_{\mathrm{rel},f}$. The proof for $D_{\mathrm{abs},f}$ is exactly the same. □

Proof of Theorem 1.4. We first establish the smoothness away from the objects. To see this we again use the Riesz–Dunford functional calculus. Let $\kappa_\lambda(x, y)$ be the integral kernel of the difference

$$(-\Delta_{\mathrm{rel}} - \lambda^2)^{-1}\mathrm{curl}\,\mathrm{curl} - (-\Delta_{\mathrm{free}} - \lambda^2)^{-1}\,\mathrm{curl}\,\mathrm{curl}.$$

Let U be an open neighbourhood of $\partial\Omega$ such that $\operatorname{dist}(U, \Omega_0) > \delta' \in (0, \delta)$. Then, on $\Omega_0 \times \Omega_0$, the integral kernel of $\kappa_\lambda(x, y)$ satisfies the estimate

$$\|\kappa_\lambda(x, y)\|_{C^k(K)} \leq C_{k,K} \mathrm{e}^{-\delta' \operatorname{Im} \lambda}$$

for any compact subset $K \subset \Omega_0 \times \Omega_0$. This can be seen directly from (62), as Lemma A.1 implies that the integral kernel of $\widetilde{\mathcal{L}}_\lambda$ is smooth and C^∞-seminorms satisfy an exponential decay estimate on $\Omega_0 \times U$, whereas the norm of $\mathcal{L}_\lambda^{-1}$ is polynomially bounded by Theorem 7.9. By the same argument as in the proof of Theorem 1.3 above, the integral

$$2 \int_{\widetilde{\Gamma}_{\epsilon/2}} \kappa_\lambda(x, y) \lambda f(\lambda) \, \mathrm{d}\lambda$$

then converges in $C^\infty(\Omega_0 \times \Omega_0)$ to the integral kernel of B_f restricted to $\Omega_0 \times \Omega_0$. Hence this kernel is smooth on $\Omega_0 \times \Omega_0$. It remains to show the decay estimate. For large $|x|$, we have by (67) the estimate

$$\|\kappa_\lambda(x, x)\| \leq \frac{C}{|x|^4} \mathrm{e}^{-\delta' |x| \operatorname{Im} \lambda}.$$

Then, using functional calculus as before, we have the representation

$$\kappa(x, x) = \frac{\mathrm{i}}{\pi} \int_{\widetilde{\Gamma}_{\epsilon/2}} \kappa_\lambda(x, x) \lambda f(\lambda) \, \mathrm{d}\lambda,$$

which gives the estimate

$$\|\kappa(x, x)\| \leq \int_1^\infty \frac{C}{|x|^4} \lambda \mathrm{e}^{-\delta_1 \lambda |x|} \, \mathrm{d}\lambda + \int_0^1 \frac{C}{|x|^4} \lambda \mathrm{e}^{-\delta_1 \lambda |x|} \lambda^a \, \mathrm{d}\lambda \leq \frac{C_1}{|x|^{6+a}}.$$

This shows that $\kappa(x, x)$ is integrable and by Mercer's theorem the integral of $\operatorname{tr}(\kappa(x, x))$ is equal to the trace, as claimed. □

Proof of Theorem 1.5. Define the relative spectral shift function

$$\xi_D(\lambda) = \frac{1}{2\pi \mathrm{i}} \log \frac{\det S_\lambda}{\det(S_{1,\lambda}) \cdots \det(S_{N,\lambda})}.$$

By the Birman–Krein formula we have

$$\operatorname{tr} D_{\mathrm{rel}, f} = -\int_0^\infty \xi_D(\lambda) \frac{\mathrm{d}}{\mathrm{d}\lambda} (\lambda^2 f(\lambda)) \, \mathrm{d}\lambda$$

for any even Schwartz function f.

Recall that Ξ' has a meromorphic extension to the complex plane and is holomorphic on the real line. Now assume that f is a compactly supported even test function, and let $\tilde{f}$ be a compactly supported almost analytic extension; see for example [Davies 1995, p. 169-170]. Let $\mathrm{d}m(z) = \mathrm{d}x \, \mathrm{d}y$ be the Lebesgue measure on $\mathbb{C}$. By the Helffer–Sjöstrand formula [Davies 1995; Helffer and Sjöstrand 1989] combined with the substitution $z \mapsto z^2$, we have

$$\operatorname{curl} \operatorname{curl} f(\Delta_{\mathrm{rel}}^{1/2}) = \frac{2}{\pi} \operatorname{curl} \operatorname{curl} \int_{\operatorname{Im} z > 0} z \frac{\partial \tilde{f}}{\partial \bar{z}} (\Delta_{\mathrm{rel}} - z^2)^{-1} \, \mathrm{d}m(z).$$

Therefore

$$D_{\mathrm{rel},f} = \frac{2}{\pi}\int_{\operatorname{Im} z>0} z\frac{\partial \tilde{f}}{\partial \bar{z}} R_{D,\mathrm{rel}}(z)\,\mathrm{d}m(z),$$

and hence, by Lemma 10.2, we have

$$\operatorname{Tr}(D_{\mathrm{rel},f}) = -\frac{1}{\pi}\int_{\operatorname{Im} z>0} z^2\frac{\partial \tilde{f}}{\partial \bar{z}}\,\Xi'(z)\,\mathrm{d}m(z).$$

Using Stokes' theorem in the form of [Hörmander 2003, p. 62-63], we therefore obtain

$$\operatorname{Tr}(D_f) = \frac{\mathrm{i}}{2\pi}\int_{\mathbb{R}} (\Xi'(x)+\Xi'(-x))x^2 f(x)\,\mathrm{d}x.$$

Comparing this with the Birman–Krein formula in Theorem 5.1 gives $\frac{\mathrm{i}}{2\pi}(\Xi'(x)+\Xi'(-x)) = \xi_D'(x)$. Since both functions are meromorphic, we have that this identity holds everywhere. We conclude that $\frac{\mathrm{i}}{2\pi}(\Xi(\lambda)-\Xi(-\lambda))-\xi_D(\lambda)$ is constant. Clearly, $(\Xi(\lambda)-\Xi(-\lambda))$ vanishes at 0, so the statement follows if we can show that $\xi_D(0)=0$. The estimate [Strohmaier and Waters 2020, Theorem 1.10] shows that $S_0 = S_{1,0} = \cdots = S_{N,0} = \mathrm{id}$, which then indeed implies $\xi_D(0)=0$. The paper [Strohmaier and Waters 2020] assumes the boundary of Ω to be smooth, but the section on the expansions in this paper carry over unmodified to the Lipschitz case (see also the remarks in [Strohmaier and Waters 2022] where this is made explicit). □

Appendix

A.1. *Norm estimates.* In the following we assume that Ω and M are as in the main body of the text. Recall that the integral kernel of the free resolvent is given by (12). We will subsequently prove norm and pointwise estimates for $G_{\lambda,0}$ and its derivatives, which are used in the main body of the text.

Lemma A.1. *Let $\Omega_0 \subset M$ be an open set with $\operatorname{dist}(\Omega_0, \partial\Omega) = \delta > 0$, and choose $\epsilon \in (0,\pi]$. Let U be a bounded open neighbourhood of the boundary $\partial\Omega$ such that $\operatorname{dist}(\Omega_0, U) > 0$, and fix $\delta' > 0$ such that $\delta' < \operatorname{dist}(\Omega_0, U) \le \delta$. Then, for any $k \in \mathbb{N}_0$, there exists $C_{k,\delta',\epsilon} > 0 > 0$ such that*

$$\|G_{\lambda,0}\|^2_{H^k(\Omega_0\times U)} \le C_{k,\delta',\epsilon}\frac{(1+\operatorname{Im}\lambda)\mathrm{e}^{-2\delta'\operatorname{Im}\lambda}}{\operatorname{Im}\lambda}, \tag{83}$$

$$\|\nabla_x G_{\lambda,0}\|^2_{H^k(\Omega_0\times U)} \le C_{k,\delta',\epsilon}\mathrm{e}^{-2\delta'\operatorname{Im}\lambda} \tag{84}$$

for all $\lambda \in \mathfrak{D}_\epsilon$. Here ∇_x denotes differentiation in the first variable, i.e., $(\nabla_x G_\lambda)(x,y) = \nabla_x G_\lambda(x,y)$.

Proof. Let us set $\lambda = \theta|\lambda|$ and note that $\operatorname{Im}\theta \ge \sin(\epsilon) > 0$. Since the kernel $G_{\lambda,0}$ satisfies the Helmholtz equation in both variables away from the diagonal, we have $((-\Delta_x)^k + (-\Delta_y)^k)G_{\lambda,0}(x,y) = 2\lambda^{2k}G_{\lambda,0}(x,y)$. We then change variables, so that $r := |x-y| \ge \delta_0$. By homogeneity, all of the integration will be carried out in this variable, with the angular variables only contributing a constant. Substituting $s := \operatorname{Im}\lambda r$ into the formula for the Green's function implies, for all $k \in \mathbb{N}$, that

$$\|\Delta^k G_{\lambda,0}\|^2_{L^2(\Omega_0'\times U')} \le C_k(\operatorname{Im}\lambda)^{4k}\int_{\delta_0}^\infty |G_{\lambda,0}(r)|^2 r^2\,\mathrm{d}r \le C_k(\operatorname{Im}\lambda)^{4k}\int_{\delta_0}^\infty \mathrm{e}^{-2\operatorname{Im}\lambda r}\,\mathrm{d}r. \tag{85}$$

Here we have enlarged the domains slightly, so that $\Omega_0' \times U'$ has positive distance from $\Omega_0 \times U$ and $\operatorname{dist}(\Omega_0', U') > \delta'$. This allows us to estimate the Sobolev norms using Lemma A.5. We then have

$$\int_{\delta_0}^{\infty} \mathrm{e}^{-2\operatorname{Im}\lambda r}\,\mathrm{d}r = \frac{\mathrm{e}^{-2\delta_0 \operatorname{Im}\lambda}}{-2\operatorname{Im}\lambda}. \tag{86}$$

Let $C_{\delta',\epsilon,k}$ denote a generic constant depending on $\delta',\ \epsilon,\ k$. Using (85) and interpolation, we can conclude, for all $k \geq 0$, that we have

$$\|G_{\lambda,0}\|^2_{H^k(\Omega_0\times U)} \leq C_{\delta',\epsilon,k}\frac{(1+\operatorname{Im}\lambda)\mathrm{e}^{-2\delta'\operatorname{Im}\lambda}}{\operatorname{Im}\lambda}. \tag{87}$$

The second inequality follows by replacing $G_{\lambda,0}$ by $\nabla_x G_{\lambda,0}$ in (85). We then have

$$\begin{aligned}\|\Delta_x^k \nabla_x G_{\lambda,0}\|^2_{L^2(\Omega_0'\times U')} &\leq C_k(\operatorname{Im}\lambda)^{4k}\int_{\delta_0}^{\infty} |\nabla_x G_{\lambda,0}(r)|^2 r^2\,\mathrm{d}r \\ &\leq C_k(\operatorname{Im}\lambda)^{4k}\int_{\delta_0}^{\infty}\left(|\operatorname{Im}\lambda|^2+\frac{1}{r^2}\right)\mathrm{e}^{-2\operatorname{Im}\lambda r}\,\mathrm{d}r \leq C_k \mathrm{e}^{-2\delta'\operatorname{Im}\lambda}.\end{aligned} \tag{88}$$

The proof is complete. □

We now combine these estimates to get an estimate on the Maxwell layer potential operator.

Lemma A.2. *Let* $\Omega_0 \subset M$ *be an open set with* $\operatorname{dist}(\Omega_0,\Omega) = \delta > 0$ *and* $\lambda \in \mathfrak{D}_\epsilon$. *Then, for any* $0 < \delta' < \delta$, *there exists* $C_{\delta',\epsilon} > 0$ *such that*

$$\|\widetilde{\mathcal{L}}_\lambda\|^2_{H^{-1/2}(\mathrm{Div},\partial\Omega)\to H(\mathrm{curl},\Omega_0)} \leq C_{\delta',\epsilon}\mathrm{e}^{-2\delta'\operatorname{Im}\lambda} \tag{89}$$

and

$$\|\widetilde{\mathcal{L}}_\lambda\|^2_{H^{-1/2}(\mathrm{Div}\,0,\partial\Omega)\to H(\mathrm{curl},\Omega_0)} \leq C_{\delta',\epsilon}|\operatorname{Im}\lambda|^3\mathrm{e}^{-2\delta'\operatorname{Im}\lambda}. \tag{90}$$

Proof. We choose as in Lemma A.1 a bounded open neighbourhood of $\partial\Omega$. For $a \in H^{-1/2}(\partial\Omega)$, the distribution $\gamma_t^*(a)$ is, by duality, in $H_{\mathrm{c}}^{-1}(U)$. The first inequality then follows by using Lemma A.1 and bearing in mind that integration defines a continuous map

$$H^k(\Omega_0\times U)\times H_{\mathrm{c}}^{-s}(U) \to H^{k-s}(\Omega_0)$$

for k large enough. The second inequality follows from the identity (16), namely that we can write

$$\widetilde{\mathcal{L}}_\lambda a = \nabla\widetilde{\mathcal{S}}_\lambda \operatorname{Div} a + \lambda^2\widetilde{\mathcal{S}}_\lambda a, \quad a \in H^{-1/2}(\mathrm{Div},\,\partial\Omega), \tag{91}$$

and again using Lemma A.1 in the same way as above. □

Lemma A.3. *Let* $k \in H^2(\mathbb{R}^d\times\mathbb{R}^d)$. *Then* k *is the integral kernel of a Hilbert–Schmidt operator*

$$K : H^{-1}(\mathbb{R}^d) \to H^1(\mathbb{R}^d),$$

with Hilbert–Schmidt norm bounded by $\|k\|_{H^2(\mathbb{R}^d\times\mathbb{R}^d)}$.

Proof. Let K be the integral operator with kernel k. Since $(-\Delta+1)^{1/2}$ is an isometry from $L^2(\mathbb{R}^d)$ to $H^{-1}(\mathbb{R}^d)$ and from $H^1(\mathbb{R}^d)$ to $L^2(\mathbb{R}^d)$, it suffices to show that $(-\Delta+1)^{1/2}K(-\Delta+1)^{1/2}$ is Hilbert–Schmidt from $L^2(\mathbb{R}^d)$ to $L^2(\mathbb{R}^d)$ and bound its Hilbert–Schmidt norm. This is equivalent to the distributional integral kernel of $(-\Delta+1)^{1/2}K(-\Delta+1)^{1/2}$ being in $L^2(\mathbb{R}^d\times\mathbb{R}^d)$; see for example [Shubin 1987]. The Hilbert–Schmidt norm is equal to the L^2-norm of the kernel. The Fourier transform is given by $(\xi^2+1)^{1/2}(\eta^2+1)^{1/2}\hat{k}(\xi,\eta)$ and this is in L^2 with the L^2-norm bounded by $\|k\|_{H^2(\mathbb{R}^d\times\mathbb{R}^d)}$ thanks to the inequality

$$\frac{(\xi^2+1)^{1/2}(\eta^2+1)^{1/2}}{\xi^2+\eta^2+1}\le 1. \qquad \square$$

Lemma A.4. *Let $k\in H^4_{\mathrm{c}}(\mathbb{R}^3\times\mathbb{R}^3)$ be supported in a compact set $Q\times Q\subset\mathbb{R}^3\times\mathbb{R}^3$. Then k is the integral kernel of a nuclear operator*

$$K: H^{-1}(\mathbb{R}^3)\to H^1(\mathbb{R}^3),$$

with trace norm bounded by $C_Q\|k\|_{H^4(\mathbb{R}^d\times\mathbb{R}^d)}$.

Proof. Since k is compactly supported in Q, we can assume without loss of generality that Q is a subset of a torus $\mathbb{T}^n$ by imposing periodic boundary conditions on a sufficiently large rectangle and remarking that the Sobolev norms on the torus restricted to a neighbourhood of Q are then equivalent to those of $\mathbb{R}^d$ restricted to that neighbourhood. We can therefore assume without loss of generality that we are on a compact manifold Y. We can then write K as $K=(-\Delta_Y+1)^{-1}(-\Delta_Y+1)K$. The operator $(-\Delta_Y+1)^{-1}$ is Hilbert–Schmidt from $H^1(Y)$ to $H^1(Y)$, as for example can be seen from Weyl's law. The operator $(-\Delta_Y+1)K$ is Hilbert–Schmidt by Lemma A.3. Since we have written the operator as a product of two Hilbert–Schmidt operators, it is nuclear and the corresponding estimate for the nuclear norm follows by estimating in terms of the Hilbert–Schmidt norms. $\square$

Lemma A.5. *Suppose that $\Omega\subset\mathbb{R}^d$ is an open subset, and assume that $\Omega'\subset\mathbb{R}^d$ is a larger subset such that $\overline{\Omega}\subset\Omega'$ and $\operatorname{dist}(\partial\Omega,\partial\Omega')>0$. Let $N\in\mathbb{N}$. Then, for any $f\in L^2(\Omega')$ with $(-\Delta)^k f\in L^2(\Omega')$ for all $k=0,1,\dots,N$, we have $f|_\Omega\in H^{2N}(\Omega)$, and there exists a constant $C_{N,\Omega',\Omega}>0$, independent of f, such that $\|f|_\Omega\|_{H^{2N}(\Omega)}\le C_{N,\Omega',\Omega}\sum_{k\le N}\|(-\Delta)^k f\|_{L^2(\Omega')}$.*

Proof. This is the usual proof of interior regularity applied to the possibly noncompact domain Ω'. We will show that $f\in H^s(\Omega')$, $\Delta f\in H^s(\Omega')$ implies $f\in H^{s+2}(\Omega)$ with the corresponding norm-estimates. The result then follows from this statement by iterating using a sequence of intermediate domains $\Omega\subset\Omega_1\subset\cdots\subset\Omega_{N-1}\subset\Omega'$. We will choose U such that $\Omega\subset U\subset\Omega'$ while we still have $\operatorname{dist}(\partial U,\partial\Omega')>0$, $\operatorname{dist}(\partial U,\partial\Omega)>0$. We can choose a regularised distance function and construct a function $\chi\in C^\infty_b(\mathbb{R}^d)$ which is compactly supported in Ω' which equals 1 in a neighbourhood of U. Then, if $f\in H^s(\Omega')$ and $\Delta f\in H^s(\Omega')$, we have

$$(1-\Delta)(\chi f)=(\chi-\Delta(\chi))f-\chi\Delta f-2(\nabla\chi)\nabla f.$$

From this we see that $(-\Delta+1)(\chi f)\in H^{s-1}(\mathbb{R}^d)$ and therefore $\chi f\in H^{s+1}(\mathbb{R}^d)$. Hence the restriction of f to U is in $H^{s+1}(\Omega_1)$. Now we choose another cut-off function η in $C^\infty_b(\mathbb{R}^d)$ supported in U that equals 1 near Ω. Then $(\nabla\eta)\nabla f$ is in $H^s(U)$, and we now conclude in the same way that $f|_\Omega\in H^{s+2}(\Omega)$. $\square$

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ON THE SINGULARITIES OF THE SPECTRAL AND BERGMAN PROJECTIONS ON COMPLEX MANIFOLDS WITH BOUNDARY

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We show that the spectral kernel of the $\bar{\partial}$-Neumann Laplacian acting on $(0, q)$-forms on a smooth relatively compact domain admits a full asymptotic expansion near the nondegenerate part of the boundary. We show further that the Bergman projection admits an asymptotic expansion under certain local closed range condition. In particular, if condition $Z(q)$ fails but conditions $Z(q-1)$ and $Z(q+1)$ hold, the Bergman projection on $(0, q)$-forms admits an asymptotic expansion. As applications, we establish Bergman kernel asymptotic expansions near nondegenerate points of some domains with weakly pseudoconvex boundary and S^1-equivariant asymptotic expansions and embedding theorems for domains with holomorphic S^1-action.

1. Introduction

1.1. ***Setting and statement of the main results.*** Let M be a relatively compact open subset with smooth boundary X of a complex manifold M' of complex dimension $n \geq 2$. The study of the $\bar{\partial}$-Neumann Laplacian on M is a classical subject in several complex variables. For $q \in \{0, 1, \dots, n-1\}$, let $\Box^{(q)}$ be the $\bar{\partial}$-Neumann Laplacian for $(0, q)$-forms on M. The domain M is said to satisfy condition $Z(q)$ $(0 \leq q \leq n-1)$ at $p \in X$ if the Levi form of a (and hence any) defining function of M near p has at least $n-q$ positive or at least $q+1$ negative eigenvalues on the holomorphic tangential space to ∂M at p. When condition $Z(q)$ holds at each point of X, Kohn proved subelliptic estimates with gain of one

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derivative in Sobolev norms for the solutions of $\square^{(q)}u = f$ (see [Chen and Shaw 2001; Folland and Kohn 1972; Hörmander 1965; Kohn 1963; 1964; Kohn and Nirenberg 1965]). This means that for each $(0,q)$-form f orthogonal to $\operatorname{Ker}\square^{(q)}$ with derivatives of order $\leq s$ in L^2 the equation $\square^{(q)}u = f$ has a solution u with derivatives of order $\leq s+1$ in L^2. Moreover, $\operatorname{Ker}\square^{(q)}$ is a finite-dimensional subspace of $\Omega^{0,q}(\overline{M})$. A closely related notion to the condition $Z(q)$ is the notion of q-convexity (and q-concavity) in the sense of [Andreotti and Grauert 1962] and is one of the basic tools in the study of the geometry of noncompact complex manifolds.

The Bergman projection $B^{(q)}$ is the orthogonal projection onto the kernel of $\square^{(q)}$ in the L^2 space. The Schwartz kernel $B^{(q)}(\cdot,\cdot)$ of $B^{(q)}$ is called the Bergman kernel. If $Z(q)$ holds, the above results show that the Bergman projection $B^{(q)}$ is a smoothing operator on $\overline{M}$ and $B^{(q)}(\cdot,\cdot)$ is smooth on $\overline{M}\times\overline{M}$. When $Z(q)$ fails at some point of X, the study of the boundary behavior of the Bergman kernel $B^{(q)}(\cdot,\cdot)$ is a very interesting problem.

The case when $q=0$ and the Levi form is positive definite on X (so $Z(0)$ fails) is especially a classical subject with a rich history. After the seminal paper [Bergmann 1933], Hörmander [1965, Theorem 3.5.1] (see also [Diederich 1970]) determined the limit of $B^{(0)}(x,x)$ when x approaches a strictly pseudoconvex point of the boundary of a domain for which the maximal $\bar\partial$ operator acting on functions has closed range.

More precisely, let $\rho\in\mathscr{C}^\infty(M')$ be a defining function of M, that is, $M=\{\rho<0\}$, $X=\{\rho=0\}$, and $d\rho\neq 0$ near X. We can and will assume that $|d\rho|=1$ on the boundary X. Let $x_0\in X$ be a point where the Levi form $\mathcal{L}_{x_0}(\rho)$ is positive definite. Then we have[1]

$$(-\rho(x))^{n+1}B^{(0)}(x,x)\to 2^{\frac{-n+1}{2}}\,\frac{n!}{4\pi^n}\det\mathcal{L}_{x_0}(\rho),\quad x\to x_0. \tag{1-1}$$

There are many extensions and variations of Hörmander's asymptotics for weakly pseudoconvex or hyperconvex domains; see, e.g., [Boas et al. 1995; Catlin 1989; Hsiao and Savale 2022; Nagel et al. 1989; Ohsawa 1984].

The existence of the complete asymptotic expansion $B^{(0)}(x,x)$ at the boundary was obtained by Fefferman [1974] on the diagonal; namely, there are functions $a,b\in\mathscr{C}^\infty(\overline{M})$ such that

$$B^{(0)}(x,x)=a(x)(-\rho(x))^{-(n+1)}+b(x)\log(-\rho(x)) \tag{1-2}$$

in M. Subsequently, Boutet de Monvel and Sjöstrand [1976] described the singularity of the full Bergman kernel $B^{(0)}(x,y)$ by showing that it is a Fourier integral operator with complex phase (see (1-15), (1-19)).

If $q=n-1$ and the Levi form is negative definite (so $Z(n-1)$ fails), Hörmander [2004, Theorem 4.6] obtained the corresponding asymptotics for the Bergman projection for $(0,n-1)$-forms in the distribution sense. For general $q>0$, the first author showed in [Hsiao 2010, Part II] that if $Z(q)$ fails, the Levi form is nondegenerate on X and $\square^{(q)}$ has L^2 closed range, the singularities of the Bergman projection on $(0,q)$-forms admits a full asymptotic expansion.

The developments about the Bergman projection mentioned above regard the points of the boundary where the Levi form is nondegenerate. For points where the Levi form is degenerate there are fewer

[1] The constant before the determinant of the Levi form here differs by rescaling from the corresponding constant in [Hörmander 1965, Theorem 3.5.1], since in this reference ρ satisfies $|d\rho|=1/\sqrt{2}$ on the boundary.

results. For example, in [Hsiao and Savale 2022] a pointwise asymptotic expansion of the Bergman kernel of a weakly pseudoconvex domain of finite type in $\mathbb{C}^2$ was obtained.

Fix a point $p \in X$. Suppose that $Z(q)$ fails at p and the Levi form is nondegenerate near p (the Levi form can be degenerate away from p). In this work, we show that the spectral kernel of $\Box^{(q)}$ admits a full asymptotic expansion near p and the Bergman projection for $(0,q)$-forms admits an asymptotic expansion near p under a certain closed range condition. Our results are natural generalizations of the asymptotics of the Bergman kernel for strictly pseudoconvex domains by Fefferman [1974] and Boutet de Monvel and Sjöstrand [1976].

Another motivation to study the spectral kernel of $\Box^{(q)}$ comes from geometric quantization. An important question in the presence of a Lie group G acting on M' is "quantization commutes with reduction" [Guillemin and Sternberg 1982]; see [Ma 2010] for a survey. The study of G-invariant Bergman projection plays an important role in geometric quantization. If we consider a manifold with boundary as above, the $\bar\partial$-Neumann Laplacian may not have L^2 closed range but the G-invariant $\bar\partial$-Neumann Laplacian has L^2 closed range. In these cases, we can use the asymptotic expansion for the spectral kernel of $\Box^{(q)}$ to study G-invariant Bergman projection. Therefore, our results about spectral kernels for the $\bar\partial$-Neumann Laplacian could have applications in geometric quantization on complex manifolds with boundary. In [Hsiao et al. 2023], we used the asymptotic expansions of the spectral kernels for the Kohn Laplacian to study the geometric quantization on CR manifolds.

We now formulate the main results. We refer to Section 2 for some notation and terminology used here. Let (M', J) be a complex manifold of dimension n with complex structure J. We denote by $T^{1,0}M'$ the holomorphic and antiholomorphic tangent bundles of M', and by $T^{*p,q}M'$ the bundle of (p,q)-forms. We fix a J-invariant Riemannian metric $g^{TM'}$ on TM' and let $dv_{M'}$ be its volume form. We denote by $\langle \cdot \mid \cdot \rangle$ the pointwise Hermitian product induced by $g^{TM'}$ on the fibers of $\mathbb{C}TM'$ and by duality on $\mathbb{C}T^*M'$; hence on $T^{*p,q}M'$.

Let M be a relatively compact open subset with $\mathscr{C}^\infty$ boundary of M'. We denote by $X = \partial M$ the boundary of M. Let $\rho \in \mathscr{C}^\infty(M', \mathbb{R})$ be a defining function of M with $|d\rho| = 1$ on X. Let $\frac{\partial}{\partial\rho} \in \mathscr{C}^\infty(M', TM')$ be the gradient of ρ with respect to the metric $g^{TM'}$. Then

$$d\rho\Big(\frac{\partial}{\partial\rho}\Big) = 1 \quad \text{on } X, \qquad \Big\langle \frac{\partial}{\partial\rho}(x) \,\Big|\, v \Big\rangle = 0 \quad \text{at every } x \in X, \text{ for every } v \in T_xX. \tag{1-3}$$

Put

$$T = J\Big(\frac{\partial}{\partial\rho}\Big) \in \mathscr{C}^\infty(M', TM'). \tag{1-4}$$

It is easy to see that T is orthogonal to $T^{1,0}X \oplus T^{0,1}X$ and $|T| = 1$ on X. We consider the 1-form on M',

$$\omega_0 = -d\rho \circ J = i(\bar\partial\rho - \partial\rho). \tag{1-5}$$

We have

$$\begin{aligned} &\omega_0(x)(u) = 0 \quad \text{for every } x \in X \text{ and every } u \in T^{1,0}_xX \oplus T^{0,1}_xX, \\ &\omega_0(T) = 1 \quad \text{on } X. \end{aligned} \tag{1-6}$$

For $x \in X$, the Levi form $\mathcal{L}_x$ is the Hermitian quadratic form on $T^{1,0}_xX$ given by

$$\mathcal{L}_x(Z, \overline{W}) = \frac{1}{2i} d\omega_0(x)(Z, \overline{W}) = \partial\bar\partial\rho(x)(Z, \overline{W}), \quad Z, W \in T^{1,0}_xX. \tag{1-7}$$

For a given point $x \in X$ let $\{W_j\}_{j=1}^{n-1}$ be an orthonormal frame of $(T^{1,0}X, \langle \cdot \,|\, \cdot \rangle)$ near x for which the Levi form is diagonal at x. We define the eigenvalues $\mu_j(x)$, $j = 1, \dots, n-1$, of the Levi form $\mathcal{L}_x$ by

$$\mathcal{L}_x(W_j, \overline{W}_\ell) = \mu_j(x)\delta_{j\ell}, \quad j, \ell = 1, \dots, n-1. \tag{1-8}$$

The determinant of the Levi form at x is denoted by

$$\det \mathcal{L}_x = \prod_{j=1}^{n-1} \mu_j(x). \tag{1-9}$$

For every $q = 0, 1, \dots, n-1$, let $T^{*0,q}X$ be the bundle of $(0,q)$-forms on X. We assume that $\mu_j(x) < 0$ if $1 \le j \le n_-$ and $\mu_j(x) > 0$ if $n_- + 1 \le j \le n-1$. Let $\{e_j\}_{j=1}^{n-1}$ denote the basis of $T^{*0,1}X$, dual to $\{\overline{W}_j\}_{j=1}^{n-1}$. Put

$$\mathcal{N}(x, n_-) := \mathbb{C}e_1(x) \wedge \cdots \wedge e_{n_-}(x), \tag{1-10}$$

and let

$$\tau_{x,n_-} : T^{*0,n_-}_x X \to \mathcal{N}(x, n_-) \tag{1-11}$$

be the orthogonal projection onto $\mathcal{N}(x_0, n_-)$ with respect to $\langle \cdot \,|\, \cdot \rangle$.

Fix $x \in M'$. Let $L \in T^{*0,1}_x M'$ and let $L^\wedge : T^{*0,q}_x M' \to T^{*0,q+1}_x M'$ be the operator with wedge multiplication by L and let $L^{\wedge,*} : T^{*0,q+1}_x M' \to T^{*0,q}_x M'$ be its adjoint with respect to $\langle \cdot \,|\, \cdot \rangle$, that is,

$$\langle\, L \wedge u \mid v \,\rangle = \langle\, u \mid L^{\wedge,*} v \,\rangle, \quad u \in T^{*0,q}_x M',\ v \in T^{*0,q+1}_x M'. \tag{1-12}$$

Let $(\cdot \,|\, \cdot)_M$ be the L^2 inner product on $\Omega^{0,q}(\overline{M})$ induced by $\langle \cdot \,|\, \cdot \rangle$ (see (2-7)). Let $L^2_{(0,q)}(M)$ be the completion of $\Omega^{0,q}(\overline{M})$ with respect to $(\cdot \,|\, \cdot)_M$. Let

$$\Box^{(q)} : \operatorname{Dom} \Box^{(q)} \subset L^2_{(0,q)}(M) \to L^2_{(0,q)}(M), \quad q \in \{0, 1, \dots, n-1\},$$

be the $\bar\partial$-Neumann Laplacian on $(0,q)$-forms (see (2-8)). The operator $\Box^{(q)}$ is a nonnegative self-adjoint operator. We denote by $\mathcal{E}^{(q)}$ the spectral measure of $\Box^{(q)}$. For a Borel set $B \subset \mathbb{R}$, $\mathcal{E}^{(q)}(B)$ is the spectral projection of $\Box^{(q)}$ corresponding to the set B. For $\lambda \ge 0$ we consider the spectral projectors,

$$B^{(q)}_{\le\lambda} := \mathcal{E}^{(q)}\big((-\infty, \lambda]\big) : L^2_{(0,q)}(M) \to \mathcal{H}^q_{\le\lambda}(\overline{M}) := \operatorname{Ran} B^{(q)}_{\le\lambda}, \tag{1-13}$$

and denote by

$$B^{(q)}_{\le\lambda}(x, y) \in \mathscr{D}'(M \times M, T^{*0,q}M \boxtimes (T^{*0,q}M)^*)$$

their distribution kernels. For $\lambda = 0$ we obtain the *Bergman projection* $B^{(q)} := B^{(q)}_{\le 0}$, the *Bergman kernel* $B^{(q)}(x, y) := B^{(q)}_{\le 0}(x, y)$ and the space of harmonic forms $\mathcal{H}^q(\overline{M}) := \mathcal{H}^q_{\le 0}(\overline{M}) = \operatorname{Ker} \Box^{(q)}$. Let us define

$$\Lambda^{(0,q)|(0,q)}_{M' \times M'} := T^{*0,q}M' \boxtimes (T^{*0,q}M')^*$$

and set, for $W \subset M' \times M'$ open,

$$\Omega^{(0,q)|(0,q)}(W) := \mathscr{C}^\infty(W, \Lambda^{(0,q)|(0,q)}_{M' \times M'}) = \mathscr{C}^\infty(W, T^{*0,q}M' \boxtimes (T^{*0,q}M')^*),$$
$$\Omega^{(0,q)|(0,q)}(W \cap (\overline{M} \times \overline{M})) := \mathscr{C}^\infty(W \cap (\overline{M} \times \overline{M}), \Lambda^{(0,q)|(0,q)}_{M' \times M'}).$$

Let U be an open set of M' with $U \cap X \neq \emptyset$. We shall consider $B^{(q)}_{\leq\lambda}$ as a continuous operator,

$$B^{(q)}_{\leq\lambda} : \Omega^{0,q}_c(U \cap M) \to \mathscr{D}'(U \cap M, T^{*0,q}M'),$$

and let $B^{(q)}_{\leq\lambda}(x, y) \in \mathscr{D}'((U \times U) \cap (M \times M), \Lambda^{(0,q)|(0,q)}_{M'\times M'})$ be the distribution kernel of $B^{(q)}_{\leq\lambda}$. We denote in the sequel by $S^n_{1,0}$ the Hörmander symbol space. Our first main result is the following.

Theorem 1.1. *Let $M = \{\rho < 0\}$ be a relatively compact open subset with smooth boundary X of a complex manifold M' of complex dimension n. Let U be an open set of M' with $U \cap X \neq \emptyset$. Suppose that the Levi form is nondegenerate of constant signature (n_-, n_+) on $U \cap X$, where n_- denotes the number of negative eigenvalues of the Levi form on $U \cap X$. Fix $\lambda > 0$. If $q \neq n_-$, we have*

$$B^{(q)}_{\leq\lambda}(x, y) \in \Omega^{(0,q)|(0,q)}((U \times U) \cap (\overline{M} \times \overline{M})), \tag{1-14}$$

and for $q = n_-$ the operator $B^{(q)}_{\leq\lambda}$ is a Fourier integral operator with complex phase. More precisely,

$$B^{(q)}_{\leq\lambda}(x, y) - \int_0^\infty e^{i\phi(x,y)t} b(x, y, t)\, dt \in \Omega^{(0,q)|(0,q)}((U \times U) \cap (\overline{M} \times \overline{M})), \tag{1-15}$$

where $b(x, y, t) \in S^n_{1,0}((U \times U) \cap (\overline{M} \times \overline{M}) \times (0, \infty), \Lambda^{(0,q)|(0,q)}_{M'\times M'})$ has asymptotic expansion $b(x, y, t) \sim \sum_{j=0}^\infty b_j(x, y)t^{n-j}$ in $S^n_{1,0}$, $b_j(x, y) \in \mathscr{C}^\infty((U \times U) \cap (\overline{M} \times \overline{M}), \Lambda^{(0,q)|(0,q)}_{M'\times M'})$, $j = 0, 1, \dots,$ and the leading term is given by

$$b_0(x, x) = 2\pi^{-n}|\det \mathcal{L}_x|\tau_{x,n_-} \circ (\bar\partial\rho(x))^{\wedge,*}(\bar\partial\rho(x))^\wedge \quad \textit{for every } x \in U \cap X. \tag{1-16}$$

Moreover,

$$\begin{gathered} \phi(x, y) \in \mathscr{C}^\infty((U \times U) \cap (\overline{M} \times \overline{M})), \quad \operatorname{Im}\phi \geq 0, \\ \phi(x, x) = 0, \quad x \in U \cap X, \quad \phi(x, y) \neq 0 \quad \textit{if } (x, y) \notin \operatorname{diag}((U \times U) \cap (X \times X)), \\ \operatorname{Im}\phi(x, y) > 0 \quad \textit{if } (x, y) \notin (U \times U) \cap (X \times X), \\ \phi(x, y) = -\overline{\phi(y, x)}, \\ d_x\phi(x, y)|_{x=y} = -2i\,\partial\rho(x) \quad \textit{for every } x \in U \cap X. \end{gathered} \tag{1-17}$$

Moreover, we can describe the phase function ϕ from (1-15) in the following complement to Theorem 1.1. Let $\bar\partial^*_f$ denote the formal adjoint of $\bar\partial$, and let $\Box^{(q)}_f := \bar\partial^*_f\,\bar\partial + \bar\partial\,\bar\partial^*_f$ be the $\bar\partial$-Laplacian acting on $\Omega^{0,*}(M')$. We denote by $\sigma(\Box^{(q)}_f)$ its principal symbol.

Zusatz 1.2. Fix $p \in U \cap X$ and choose local holomorphic coordinates $z = (z_1, \dots, z_n)$, $z_j = x_{2j-1} + ix_{2j}$, $j = 1, \dots, n$, vanishing at p such that the metric on $T^{1,0}M'$ is $\sum_{j=1}^n dz_j \otimes d\bar z_j$ at p and $\rho(z) = \sqrt{2}\operatorname{Im} z_n + \sum_{j=1}^{n-1}\mu_j|z_j|^2 + O(|z|^3)$, where μ_j, $j = 1, \dots, n-1$, are the eigenvalues of $\mathcal{L}_p$. We also write $w = (w_1, \dots, w_n)$, $w_j = y_{2j-1} + iy_{2j}$, $j = 1, \dots, n$. Then, we can take $\phi(z, w)$ in (1-15) so that in some neighborhood of (p,p) in $M'\times M'$ we have

$$\begin{aligned}\phi(z,w)=-\sqrt{2}x_{2n-1}+\sqrt{2}y_{2n-1}-i\rho(z)\Big(1+\sum_{j=1}^{2n-1}a_jx_j+\frac{1}{2}a_{2n}x_{2n}\Big)-i\rho(w)\Big(1+\sum_{j=1}^{2n-1}\bar a_jy_j+\frac{1}{2}\bar a_{2n}y_{2n}\Big)\\+i\sum_{j=1}^{n-1}|\mu_j||z_j-w_j|^2+\sum_{j=1}^{n-1}i\mu_j(\bar z_jw_j-z_j\bar w_j)+O\big(|(z,w)|^3\big),\quad\text{(1-18)}\end{aligned}$$

where $a_j=\frac{1}{2}\partial_{x_j}\sigma(\Box_f^{(q)})(p,-2i\partial\rho(p)),\ j=1,\dots,2n$.

The essential step in the proof of Theorem 1.1 is the construction of a microlocal Hodge decomposition (Theorems 5.9, 5.23) up to smoothing operators. Namely, there exists an approximate Neumann operator $N^{(q)}$ and an approximate Bergman operator Π^q on $U\cap\overline{M}$ such that $\Box^{(q)}N^{(q)}+\Pi^{(q)}-I$, $N^{(q)}\Box^{(q)}+\Pi^{(q)}-I$, $\Box^{(q)}\Pi^{(q)}$ are smoothing on $U\cap\overline{M}$ (here I denotes the identity) and $\Pi^{(q)}$ differs from the Fourier integral operator $\int_0^\infty e^{i\phi(x,y)t}b(x,y,t)\,dt$ by a smoothing operator on $U\cap\overline{M}$. In Theorem 6.7 we prove that, for every $\lambda>0$, $B^{(q)}_{\le\lambda}-\Pi^{(q)}$ is smoothing on $U\cap\overline{M}$. Since $\Pi^{(q)}$ is independent of λ, the complex Fourier integral operator $\int_0^\infty e^{i\phi(x,y)t}b(x,y,t)\,dt$ in (1-15) can be taken to be independent of λ. Hence, for every $\lambda_1>\lambda>0$, $B^{(q)}_{\le\lambda_1}(x,y)$ and $B^{(q)}_{\le\lambda}(x,y)$ differ by a smooth section on $(U\times U)\cap(\overline{M}\times\overline{M})$.

By integrating over t in the oscillatory integral $\int_0^\infty e^{i\phi(x,y)t}b(x,y,t)\,dt$ in (1-15), we have the following corollary of Theorem 1.1.

Corollary 1.3. *Under the assumptions of Theorem 1.1, let U be an open set of M' with $U\cap X\neq\varnothing$. Suppose that the Levi form is nondegenerate of constant signature (n_-,n_+) on $U\cap X$. Let $q=n_-$. There exist forms $F,G\in\Omega^{(0,q)|(0,q)}((U\times U)\cap(\overline{M}\times\overline{M}))$ such that for every $\lambda>0$ we have*

$$B^{(q)}_{\le\lambda}(x,y)=F(x,y)(-i(\phi(x,y)+i0))^{-n-1}+G(x,y)\log(-i(\phi(x,y)+i0))+R_\lambda(x,y),\quad\text{(1-19)}$$

where $R_\lambda(x,y)\in\Omega^{(0,q)|(0,q)}((U\times U)\cap(\overline{M}\times\overline{M}))$ is a λ-dependent smooth form. Moreover, we have

$$\begin{aligned}F(x,y)&=\sum_{j=0}^{n}(n-j)!b_j(x,y)(-i\phi(x,y))^j,\\G(x,y)&\sim\sum_{j=0}^{\infty}\frac{(-1)^{j+1}}{j!}b_{n+j+1}(x,y)(-i\phi(x,y))^j\ \mathrm{mod}\,\mathscr{C}^\infty((U\times U)\cap(\overline{M}\times\overline{M})),\end{aligned}\quad\text{(1-20)}$$

where $b_j(x,y)$, $j\in\mathbb{N}_0$, and $\phi(x,y)$ are as in Theorem 1.1.

We introduce now a condition which allows us to pass from spectral projections $B^{(q)}_{\le\lambda}$ with $\lambda>0$ to the Bergman projector $B^{(q)}=B^{(q)}_{\le 0}$.

Definition 1.4. Let U be an open set in M' with $U\cap X\neq\varnothing$. We say that $\Box^{(q)}$ has local closed range in U if, for every open set $W\subset U$ with $W\cap X\neq\varnothing$, $\overline{W}\subset U$, there is a constant $C_W>0$ such that

$$\|(I-B^{(q)})u\|_M\le C_W\|\Box^{(q)}u\|_M,\quad u\in\Omega_c^{0,q}(W\cap\overline{M})\cap\mathrm{Dom}\,\Box^{(q)}.$$

Note that if $\Box^{(q)}$ has closed range then $\Box^{(q)}$ has local closed range in U for any U.

Our second main result is the following.

Theorem 1.5. *Under the assumptions of Theorem 1.1, let U be an open set of M' with $U \cap X \neq \varnothing$. Suppose that the Levi form is nondegenerate of constant signature (n_-, n_+) on $U \cap X$. Let $q = n_-$. Suppose that $\Box^{(q)}$ has local closed range in U. Then*

$$B^{(q)}(x, y) - \int_0^\infty e^{i\phi(x,y)t} b(x, y, t)\, dt \in \Omega^{(0,q)|(0,q)}((U \times U) \cap (\overline{M} \times \overline{M})), \tag{1-21}$$

where $b(x, y, t)$ and $\phi(x, y)$ are as in Theorem 1.1. In particular, $B^{(q)}(x, y)$ has the asymptotics (1-19).

Hörmander [2004, Theorem 4.6] determined the leading asymptotics of $B^{(n-1)}(x, y)$ near a boundary point where the Levi form is negative definite under the condition that $\Box^{(n-1)}$ has closed range. Theorem 1.5 thus generalizes this result and gives the full asymptotics.

Remark 1.6. Let (E, h^E) be a Hermitian holomorphic vector bundle over M'. As in (2-8) below, we can consider the $\bar\partial$-Neumann Laplacian on $(0, q)$-forms with values in E:

$$\Box^{(q)} = \bar\partial\, \bar\partial^* + \bar\partial^*\, \bar\partial : \operatorname{Dom} \Box^{(q)} \subset L^2_{(0,q)}(M, E) \to L^2_{(0,q)}(M, E), \tag{1-22}$$

where $L^2_{(0,q)}(M, E)$ denotes the L^2 space of $(0, q)$-forms with values in E. We can define $B^{(q)}_{\leq\lambda}(x, y)$ in the same way as above and by the same proofs, Theorems 1.1 and 1.5 hold also in the presence of a vector bundle E.

In particular, we can consider the trivial line bundle $E = \mathbb{C}$ with the metric $h^E = e^{-\varphi}$, where $\varphi \in \mathscr{C}^\infty(M')$ is a weight function. In this case the space $L^2_{(0,q)}(M, E)$ is the completion of $\Omega^{0,q}(\overline{M})$ with respect to the weighted L^2 inner product $(u|v)_\varphi = \int_M \langle u|v\rangle e^{-\varphi}\, dv_{M'}$ and is denoted by $L^2_{(0,q)}(M, \varphi)$. The Bergman projection is denoted by $B^{(q)}_\varphi$ and the Bergman kernel by $B^{(q)}_\varphi(\cdot\,,\cdot)$. So all the results above have versions for *weighted Bergman kernels $B^{(q)}_\varphi(\cdot\,,\cdot)$*.

We now give some applications of the results above.

Corollary 1.7. (i) *Let M be a bounded domain of holomorphy in $\mathbb{C}^n$ with smooth boundary and let φ be any function in $\mathscr{C}^\infty(\overline{M})$. Let U be an open set in $\mathbb{C}^n$ such that $U \cap \partial M$ is strictly pseudoconvex. Then the weighted Bergman kernel $B^{(0)}_\varphi(\cdot\,,\cdot)$ has the asymptotics* (1-21) *on $U \cap \overline{M}$.*

(ii) *Let M be an open relatively compact domain with smooth boundary X in a complex manifold M' of dimension n. Assume that X satisfies condition $Z(1)$, i.e., the Levi form of X has everywhere either $n-1$ positive or two negative eigenvalues. Let U be an open set in M' such that $U \cap X$ is strictly pseudoconvex. Then the Bergman kernel $B^{(0)}(\cdot\,,\cdot)$ has the asymptotics* (1-21) *on $U \cap \overline{M}$.*

(iii) *Let M be a pseudoconvex domain with smooth boundary in $\mathbb{P}^n$. Let U be an open set in $\mathbb{P}^n$ such that $U \cap \partial M$ is strictly pseudoconvex. Then the Bergman kernel $B^{(0)}(\cdot\,,\cdot)$ has the asymptotics* (1-21) *on $U \cap \overline{M}$.*

(iv) *Let M be an open relatively compact domain with smooth boundary X in a complex manifold M' of dimension n. Fix $p \in X$ and assume that the Levi form is nondegenerate of constant signature (n_-, n_+) at every point of $U \cap X$, where U is an open set of p in M'. Let $q = n_-$. Assume that $Z(q-1)$, $Z(q+1)$ hold of every point of X. The Bergman kernel $B^{(q)}(\cdot\,,\cdot)$ has the asymptotics* (1-21) *on $U \cap \overline{M}$.*

Note that by the solution of the Levi problem [Range 1986, Theorem V.1.5], for a bounded domain $M \subset \mathbb{C}^n$ with smooth boundary the notions of domain of holomorphy and weak (Levi) pseudoconvexity are equivalent. We can apply the L^2 estimates for $\bar\partial$ of [Hörmander 1965, Theorem 2.2.1′] to obtain that $\Box^{(0)}$ has closed range in L^2, and hence settle case (i). Note that the analogous L^2 estimate for $\bar\partial_b$ along the boundary was done in [Shaw 1985]. Moreover, it follows from [Folland and Kohn 1972, Theorem 3.1.19], [Hörmander 1965, Theorem 3.4.1] in case (ii), and [Henkin and Iordan 2000, Corollary 3.6] in case (iii), that $\Box^{(0)}$ has closed range. Note that these assertions are independent of the choice of the function $\varphi \in \mathscr{C}^\infty(\overline{M})$, since changing φ only means introducing equivalent norms in the Hilbert spaces concerned. Obviously, the items (i) and (ii) hold also if we work with Bergman kernels of holomorphic sections in a Hermitian holomorphic vector bundle (E, h^E) defined in a neighborhood of $\overline{M}$ (see Remark 1.6).

We now explain point (iv). Let M be an open relatively compact domain with smooth boundary X in a complex manifold M' of dimension n. We recall that X satisfies condition $Z(q)$ if the Levi from of X has at least $n-q$ positive eigenvalues or at least $q+1$ negative eigenvalues at every point of X. It was proved in [Folland and Kohn 1972, Proposition 3.1.18] that if $Z(q-1)$, $Z(q+1)$ hold at every point of X, then $\Box^{(q)}$ has closed range. If the Levi form is nondegenerate of signature (n_-, n_+) then $Z(q)$ holds if and only if $q \neq n_-$. We call n_- the critical degree.

Next we consider Bergman kernels on shell domains. These are domains with two boundary components, one pseudoconvex, the other pseudoconcave They appear for example in Andreotti–Grauert theory, e.g., as $(1,1)$-convex-concave domains (roughly speaking of the form $M = \{c \leq \varphi \leq d\}$, where $\varphi : M' \to \mathbb{R}$ is a strictly plurisubharmonic exhaustion function on M'). Such domains play an important role in problems of compactification of complex manifolds; see, e.g., [Andreotti and Siu 1970].

Corollary 1.8. *Let $M \Subset \mathbb{C}^n$ be the shell domain $M = M_0 \setminus \overline{M}_1$ between two pseudoconvex domains M_0 and M_1 with smooth boundary and $M_1 \Subset M_0$. Let U an open set such that $U \cap \partial M_1$ is strictly pseudoconvex and $U \cap \partial M_0 = \varnothing$. Then the Bergman kernel $B^{(n-1)}(x, y)$ on $(0, n-1)$-forms has the asymptotics* (1-21) *and* (1-19).

By [Shaw 2010, Theorem 3.5], the operator $\Box^{(n-1)}$ has closed range in L^2 for a shell domain between two pseudoconvex domains as above. Moreover, the Levi form of ∂M is negative definite on $U \cap \partial M$, so the corollary follows from Theorem 1.5.

We consider further shell domains $M = M_0 \setminus \overline{M}_1$ in a complex manifold M'. For general shell domains, e.g., $(1,1)$-convex-concave domains, the associated $\bar\partial$-Neumann Laplacian may not have closed range. This happens for example for domains which cannot be compactified on the pseudoconcave end [Andreotti and Siu 1970] (the pseudoconcave boundary component is not embeddable in the Euclidean space). To overcome this difficulty, we consider a holomorphic line bundle L over M'. In Theorem 1.9 below, we will see that the associated $\bar\partial$-Neumann Laplacian with values in L^k has closed range if k is large and the curvature of L is positive. We refer to [Ma and Marinescu 2007] for a comprehensive study of Bergman kernel asymptotics for high tensor powers of a line bundles.

Suppose that there is a holomorphic line bundle (L, h^L) over M', where h^L denotes a Hermitian metric of L and R^L is the curvature of L induced by h^L. For every $k \in \mathbb{N}$, let (L^k, h^{L^k}) be the k-th power of (L, h^L). Let $(\cdot \mid \cdot)_k$ be the L^2 inner product on $\Omega^{0,q}(M, L^k)$ induced by the given Hermitian metric

$\langle\,\cdot\,|\,\cdot\,\rangle$ on $\mathbb{C}TM'$ and h^L and let $L^2_{(0,q)}(M,L^k)$ be the completion of $\Omega^{0,q}(M,L^k)$. Let

$$\Box^{(q)}_k:\bar\partial\,\bar\partial^*+\bar\partial^*\,\bar\partial:\mathrm{Dom}\,\Box^{(q)}\subset L^2_{(0,q)}(M,L^k)\to L^2_{(0,q)}(M,L^k)$$

be the $\bar\partial$-Neumann operator on M with values in L^k and let

$$B^{(q)}_k:L^2_{(0,q)}(M,L^k)\to\mathrm{Ker}\,\Box^{(q)}_k$$

be the orthogonal projection with respect to $(\,\cdot\,|\,\cdot\,)_k$ and let $B^{(q)}_k(\,\cdot\,,\cdot\,)$ be the distribution kernel of $B^{(q)}_k$.

Theorem 1.9. *Let $M=M_0\setminus\overline{M}_1$ be the shell domain between two pseudoconvex domains M_0 and M_1 with smooth boundary, $M_1\Subset M_0\Subset M'$. Let $X_0=\partial M_0$ and $X_1=\partial M_1$. Assume that (L,h^L) is a positive line bundle in a neighborhood of $\overline{M}_0$. Let U be an open set in M' with of $U\cap X_0\neq\varnothing$ and $U\cap X_1=\varnothing$. There exists $k_0\in\mathbb{N}$ such that, for every $k\in\mathbb{N}$, $k\geq k_0$,*

$$\Box^{(0)}_k \text{ has local closed range in } U. \tag{1-23}$$

Moreover, for every $k\in\mathbb{N}$, $k\geq k_0$, the Bergman kernel of M with values in L^k satisfies

$$B^{(0)}_k(x,y)\equiv\int_0^\infty e^{i\phi(x,y)t}b(x,y,t)\,dt \mod \mathscr{C}^\infty((U\times U)\cap(\overline{M}\times\overline{M}),L^k\boxtimes(L^k)^*), \tag{1-24}$$

where $\phi(x,y)\in\mathscr{C}^\infty((U\times U)\cap(\overline{M}\times\overline{M}))$ and $b(x,y,t)\in S^n_{1,0}(((U\times U)\cap(\overline{M}\times\overline{M}))\times(0,\infty),L^k\boxtimes(L^k)^*)$ are as in Theorem 1.1.

The next applications concerns the asymptotics of the S^1-equivariant Bergman kernel and embedding theorems. We assume that M' admits a holomorphic S^1-action

$$S^1\times M'\to M',\quad (e^{i\theta},x)\mapsto e^{i\theta}\circ x.$$

The S^1-action preserves the complex structure J of M'. Let $T_0\in\mathscr{C}^\infty(M',TM')$ be the infinitesimal generator of the S^1-action on M', that is $(T_0u)(x)=\frac{\partial}{\partial\theta}u(e^{i\theta}\circ x)\big|_{\theta=0}$ for every $u\in\mathscr{C}^\infty(M')$.

We take the Hermitian metric $\langle\,\cdot\,|\,\cdot\,\rangle$ on $\mathbb{C}TM'$ to be S^1-invariant and $|T_0|=1$ on X. We take an S^1-invariant defining function ρ so that $|d\rho|=1$ on X. Fix an open connected component X_0 of X. Suppose that

$$\omega_0(T_0)>0\quad\text{on } X_0, \tag{1-25}$$

where J is the complex structure map on T^*M'. From (1-4), (1-25) and noting that $|T_0|=|d\rho|=1$ on X, it is easy to see that

$$T_0=T\quad\text{on } X_0. \tag{1-26}$$

For every $m\in\mathbb{Z}$, put

$$\Omega^{0,q}_m(M')=\{u\in\Omega^{0,q}(M'):Tu=imu\}, \tag{1-27}$$

where Tu is the Lie derivative of u along direction T. Similarly, let $\Omega^{0,q}_m(\overline{M})$ denote the space of restrictions to M of elements in $\Omega^{0,q}_m(M')$. We write $\mathscr{C}^\infty_m(\overline{M}):=\Omega^{0,0}_m(\overline{M})$. Let $L^2_{(0,q),m}(M)$ be the completion of $\Omega^{0,q}_m(\overline{M})$ with respect to $(\cdot\,|\,\cdot)_M$. For $q=0$, we write $L^2_m(M) := L^2_{(0,0),m}(M)$. Fix $\lambda \geq 0$ and $m \in \mathbb{Z}$. Put

$$\mathcal{H}^q_{\leq\lambda,m}(\overline{M}) := \mathcal{H}^q_{\leq\lambda}(\overline{M}) \cap L^2_{(0,q),m}(M), \tag{1-28}$$

where $\mathcal{H}^q_{\leq\lambda}(\overline{M})$ is given by (1-13). Let

$$B^{(q)}_{\leq\lambda,m} : L^2_{(0,q)}(M) \to \mathcal{H}^q_{\leq\lambda,m}(\overline{M}) \tag{1-29}$$

be the orthogonal projection with respect to $(\cdot\,|\,\cdot)_M$ and let

$$B^{(q)}_{\leq\lambda,m}(x,y) \in \mathscr{D}'(M \times M, \Lambda^{(0,q)|(0,q)}_{M'\times M'})$$

be the distribution kernel of $B^{(q)}_{\leq\lambda,m}$. For $\lambda = 0$, we write $\mathcal{H}^q_m(\overline{M}) := \mathcal{H}^q_{\leq 0,m}(\overline{M})$, $B^{(q)}_m := B^{(q)}_{\leq 0,m}$, $B^{(q)}_m(x,y) := B^{(q)}_{\leq 0,m}(x,y)$. From [Hsiao et al. 2020, Theorem 3.3], we see that $\mathcal{H}^q_{\leq\lambda,m}(\overline{M})$ is a finite-dimensional subspace of $\Omega^{0,q}_m(\overline{M})$ and hence

$$B^{(q)}_{\leq\lambda,m}(x,y) \in \Omega^{(0,q)|(0,q)}(\overline{M} \times \overline{M}).$$

Moreover, it is straightforward to see that

$$B^{(q)}_{\leq\lambda,m}(x,y) = \frac{1}{2\pi}\int_{-\pi}^{\pi} B^{(q)}_{\leq\lambda}(x, e^{i\theta}\circ y)e^{im\theta}\,d\theta. \tag{1-30}$$

We have the following asymptotic expansion for the S^1-equivariant Bergman kernel. We use here the symbol spaces S^n_{loc}; see Definition 2.1 and the discussion after (2-6).

Theorem 1.10. *Assume that M' admits a holomorphic S^1-action that is boundary preserving, locally free and transversal to the CR structure on the boundary. Let X_0 be a connected component of X such that* (1-26) *holds, let $p \in X_0$ and let U be an open set of p in M' with $U \cap X_0 \neq \varnothing$. Suppose that $Z(1)$ holds on X and that the Levi form is positive $U \cap X_0$. Let $N_p := \{g \in S^1 : g \circ p = p\} = \{g_0 := e, g_1, \dots, g_r\}$, where e denotes the identity element in S^1 and $g_j \neq g_\ell$ if $j \neq \ell$ for every $j, \ell = 0, 1, \dots, r$. Then*

$$B^{(0)}_m(x,y) \equiv \sum_{\alpha=0}^{r} g^m_\alpha e^{im\phi(x, g_\alpha\circ y)} b_\alpha(x,y,m) \bmod O(m^{-\infty}) \quad \text{on } U \cap \overline{M}, \tag{1-31}$$

where, for every $\alpha = 0, 1, \dots, r$,

$$\begin{gathered} b_\alpha(x,y,m) \in S^n_{\mathrm{loc}}((U\times U)\cap(\overline{M}\times\overline{M})), \\ b_\alpha(x,y,m) \sim \textstyle\sum_{j=0}^{\infty} b_{\alpha,j}(x,y)m^{n-j} \quad \text{in } S^n_{\mathrm{loc}}((U\times U)\cap(\overline{M}\times\overline{M})), \\ b_{\alpha,0}(x,x) = b_0(x,x), \end{gathered} \tag{1-32}$$

where $b_0(x,x)$ is given by (5-124) *and $\phi(x,y) \in \mathscr{C}^\infty((U\times U)\cap(\overline{M}\times\overline{M}))$ is as in* (1-17).

Actually, we have more general results than Theorem 1.10. In Theorem 8.2, we get an asymptotic expansion for $B^{(q)}_{\leq\lambda,m}$ in m for every $\lambda > 0$, and in Theorem 8.3, we get an asymptotic expansion for $B^{(q)}_m$ in m under the local closed range condition of $\Box^{(q)}$. Moreover, when $Z(q-1)$ and $Z(q+1)$ hold, then $\Box^{(q)}$ has closed range and an analogous statement to Theorem 1.10 holds for $B^{(q)}_m$.

For every $m \in \mathbb{N}$, let

$$\Phi_m : \overline{M} \to \mathbb{C}^{d_m}, \quad x \mapsto (f_1(x), \dots, f_{d_m}(x)), \tag{1-33}$$

where $\{f_1(x), \dots, f_{d_m}(x)\}$ is an orthonormal basis for $\mathcal{H}^0_m(\overline{M})$ with respect to $(\cdot \mid \cdot)_M$ and $d_m = \dim \mathcal{H}^0_m(\overline{M})$. We have the following S^1-equivariant embedding theorem

Theorem 1.11. *Assume that M' admits a holomorphic S^1-action that is boundary preserving, locally free and transversal to the CR structure on the boundary. Let X_0 be a connected component of X such that* (1-26) *holds. Assume that the Levi form is positive definite on X_0 and $Z(1)$ holds on X. For every $m_0 \in \mathbb{N}$, there exist $m_1, \dots, m_k \in \mathbb{N}$, with $m_j \geq m_0$, $j = 1, \dots, k$, and an S^1-invariant open neighborhood V of X_0 such that the map*

$$\Phi_{m_1,\dots,m_k} : V \cap \overline{M} \to \mathbb{C}^{\hat{d}_m}, \quad x \mapsto (\Phi_{m_1}(x), \dots, \Phi_{m_k}(x)), \tag{1-34}$$

is a holomorphic embedding, where Φ_{m_j} is given by (1-33) *and $\hat{d}_m = d_{m_1} + \cdots + d_{m_k}$.*

Without the $Z(1)$ condition, we can still formulate the following S^1-equivariant embedding theorem.

Theorem 1.12. *Assume that M' admits a holomorphic S^1-action that is boundary preserving, locally free and transversal to the CR structure on the boundary. Let X_0 be a connected component of X such that* (1-26) *holds and the Levi form is positive definite on X_0. For every $m_0 \in \mathbb{N}$, there exist an S^1-invariant open neighborhood V of X_0 and $f_j \in \mathscr{C}^\infty(V \cap \overline{M})$ with $\bar{\partial} f_j = 0$ on $V \cap \overline{M}$, $f_j(e^{i\theta}x) = e^{im_j\theta} f(x)$, $j = 1, \dots, k$, for $e^{i\theta} \in S^1$ and every $x \in V$ and some $m_j \geq m_0$, such that the map*

$$\Phi : V \cap \overline{M} \to \mathbb{C}^k, \quad x \mapsto (f_1(x), \dots, f_k(x)), \tag{1-35}$$

is a holomorphic embedding.

1.2. ***Methods and further previous results.*** In [Hsiao 2010] the first author extended the results of the fundamental paper [Boutet de Monvel and Sjöstrand 1976] on the off-diagonal and boundary asymptotics of the Szegő and Bergman kernels to the case of domains whose Levi form is everywhere nondegenerate on the boundary. Building on [Hsiao 2010] we constructed in [Hsiao and Marinescu 2017] a parametrix for the Szegő kernel on the boundary, and extended the above results in several directions: (i) the global nondegeneracy condition on the Levi form was relaxed to local nondegeneracy near the point where the parametrix is being constructed; (ii) a more general projector onto low-energy eigenspaces of the Kohn Laplacian was considered; (iii) the boundary and domain were allowed to be noncompact. In the present paper we achieve the passage from the Szegő parametrix on the boundary to the Bergman parametrix in the interior.

The main technical part of this paper is the construction of the microlocal parametrices for the $\bar{\partial}$-Neumann problem done in Sections 4 and 5 (see Theorems 4.7, 5.9, 5.23). More precisely, in Section 4, we construct parametrices for $\Box^{(q)}$ near a point $p \in X$ under the assumptions that $Z(q)$ holds at p and the Levi form is nondegenerate at p. Our result generalizes the global result [Folland and Kohn 1972, Theorem 3.1.14] (see also [Kohn 1963; 1964]) about the solution of the $\bar{\partial}$-Neumann problem under the hypothesis that the $Z(q)$ condition holds on the whole boundary. In this case $\Box^{(q)}$ has a parametrix $N^{(q)}$, the $\bar{\partial}$-Neumann operator, which has a local character. Our method uses a reduction to the analysis on the boundary and the use of a boundary pseudodifferential operator $\Box^{(q)}_{-}$ which is elliptic along the negative component $\Sigma_- \subset T^*X$ of the characteristic cone (see Section 3).

In Section 5, we construct microlocal Hodge decomposition theorems for $\Box^{(q)}$ near a point $p \in X$ under the assumptions that $Z(q)$ fails at p and the Levi form is nondegenerate at p. This is the most technical part of the paper. Again, this is the local counterpart of the global result [Folland and Kohn 1972, Proposition 3.1.17] saying that if $Z(q)$ fails but $Z(q-1)$ and $Z(q+1)$ hold on X, there exists a global Hodge decomposition theorem for $\Box^{(q)}$. Our method is to first construct a parametrix $N^{(q)}$ of the $\bar\partial$-Neumann Laplacian and an approximate Bergman projector $\Pi^{(q)}$, then to link $\Pi^{(q)}$ to an approximate Szegő projector, which turns out to be a Fourier operator with complex phase, on the boundary via the Poisson operator. Note that already in [Boutet de Monvel and Sjöstrand 1976] the analysis of the Bergman projector on a strictly pseudoconvex domain was done by reduction to the Szegő projector.

The localization of the $\bar\partial$-Neumann operator was observed in several papers under global assumptions. It was remarked in [Folland and Kohn 1972, p. 52] that the $\bar\partial$-Neumann operator localizes assuming that $\Box^{(q)}$ has globally closed range (see also Theorem 3.6 and Remark (ii) on page 70 of [Straube 2010]). Near a strictly pseudoconvex point ($n_- = 0$), the existence of the localized $\bar\partial$-Neumann operator in Theorems 4.7 and 5.9 follows from the main results of [Henkin et al. 1996; Henkin and Iordan 1997; Michel and Shaw 1998], under various hypotheses, such as piecewise smooth boundary. The generalizations of these articles for higher q have been considered in [Hefer and Lieb 2000, Theorem 3.16].

As mentioned above, a geometric counterpart of the condition $Z(q)$ is the notion of q-convexity [Andreotti and Grauert 1962]. A manifold M of dimension n is called q-convex ($1 \le q \le n$) if there exists an exhaustion function $\varphi : M \to \mathbb{R}$ such that its Levi form $i\partial\bar\partial\varphi$ has $n-q+1$ positive eigenvalues outside a compact set K. If $c \in \mathbb{R}$ is a regular value of φ such that $M_c := \{x \in M : \varphi(x) \le c\} \Subset M$ contains K, then M_c satisfies condition $Z(\ell)$ for every $\ell \ge q$. By [Andreotti and Grauert 1962], if M is q-convex then the cohomology $H^\ell(M, E)$ with values in any holomorphic bundle E is finite-dimensional for any $\ell \ge q$. This can be also deduced from the fact that the $\dim \operatorname{Ker} \Box^{(\ell)} < \infty$ for $\ell \ge q$ and from Hodge theory of the $\bar\partial$-Neumann Laplacian; see [Hörmander 1965]. If M is a domain such that the Levi form of the boundary is nondegenerate of signature (n_-, n_+), it follows from Andreotti–Grauert theory and [Andreotti and Hill 1972] that $\dim H^\ell(M, E) < \infty$ for $\ell \ne n_-$ and $\dim H^\ell(M, E) = \infty$ for $\ell = n_-$. This reflects the fact that in this case the Bergman projector on $(0, n_-)$-forms has infinite-dimensional range.

1.3. ***Organization of the paper.*** The paper is organized as follows. In Section 2, we collect some standard notation, terminology, definitions and statements we use throughout. To construct parametrices for $\Box^{(q)}$, we introduce in Section 3 the boundary operator $\Box^{(q)}_{-}$. In Section 4, we construct parametrices for $\Box^{(q)}$ near a point $p \in X$ under the assumption that $Z(q)$ holds at p. Up to the authors' knowledge, the parametrices construction in Section 4 under no global assumptions is also a new result. In Section 5, we obtain microlocal Hodge decomposition theorems for $\Box^{(q)}$ near a point $p \in X$ under the assumption that $Z(q)$ fails at p. By using the results in Sections 4 and 5, we prove Theorems 1.1 and 1.5 in Section 6. In Section 7, we prove Theorem 1.9. In Section 8, we prove Theorems 1.10, 1.11 and 1.12 about the asymptotic expansions of the S^1-equivariant Bergman kernel and embedding theorems for domains with holomorphic S^1-action.

2. Preliminaries

2.1. ***Notions from microlocal and semiclassical analysis.*** We shall use the following notation: $\mathbb{N} = \{1, 2, \dots\}$ is the set of natural numbers, $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$, $\mathbb{R}$ is the set of real numbers, and $\overline{\mathbb{R}}_+ := \{x \in \mathbb{R} : x \geq 0\}$. For a multiindex $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}_0^n$ we denote by $|\alpha| = \alpha_1 + \cdots + \alpha_n$ its norm and by $l(\alpha) = n$ its length. For $m \in \mathbb{N}$, write $\alpha \in \{1, \dots, m\}^n$ if $\alpha_j \in \{1, \dots, m\}$, $j = 1, \dots, n$. A multi-index α is strictly increasing if $\alpha_1 < \alpha_2 < \cdots < \alpha_n$. For $x = (x_1, \dots, x_n)$ we write

$$
\begin{gathered}
x^\alpha = x_1^{\alpha_1} \cdots x_n^{\alpha_n}, \\
\partial_{x_j} = \frac{\partial}{\partial x_j}, \quad \partial_x^\alpha = \partial_{x_1}^{\alpha_1} \cdots \partial_{x_n}^{\alpha_n} = \frac{\partial^{|\alpha|}}{\partial x^\alpha}, \\
D_{x_j} = \frac{1}{i}\partial_{x_j}, \quad D_x^\alpha = D_{x_1}^{\alpha_1} \cdots D_{x_n}^{\alpha_n}, \quad D_x = \frac{1}{i}\partial_x.
\end{gathered}
$$

Let $z = (z_1, \dots, z_n)$, $z_j = x_{2j-1} + i x_{2j}$, $j = 1, \dots, n$, be coordinates of $\mathbb{C}^n$. We write

$$
\begin{gathered}
z^\alpha = z_1^{\alpha_1} \cdots z_n^{\alpha_n}, \quad \bar{z}^\alpha = \bar{z}_1^{\alpha_1} \cdots \bar{z}_n^{\alpha_n}, \\
\partial_{z_j} = \frac{\partial}{\partial z_j} = \frac{1}{2}\Big(\frac{\partial}{\partial x_{2j-1}} - i\frac{\partial}{\partial x_{2j}}\Big), \quad \partial_{\bar{z}_j} = \frac{\partial}{\partial \bar{z}_j} = \frac{1}{2}\Big(\frac{\partial}{\partial x_{2j-1}} + i\frac{\partial}{\partial x_{2j}}\Big), \\
\partial_z^\alpha = \partial_{z_1}^{\alpha_1} \cdots \partial_{z_n}^{\alpha_n} = \frac{\partial^{|\alpha|}}{\partial z^\alpha}, \quad \partial_{\bar{z}}^\alpha = \partial_{\bar{z}_1}^{\alpha_1} \cdots \partial_{\bar{z}_n}^{\alpha_n} = \frac{\partial^{|\alpha|}}{\partial \bar{z}^\alpha}.
\end{gathered}
$$

For $j, s \in \mathbb{Z}$, set $\delta_{j,s} = 1$ if $j = s$, and $\delta_{j,s} = 0$ if $j \neq s$.

Let $\mathcal{M}$ be a smooth paracompact manifold. We let $T\mathcal{M}$ and $T^*\mathcal{M}$ denote the tangent bundle of $\mathcal{M}$ and the cotangent bundle of $\mathcal{M}$ respectively. The complexified tangent bundle of $\mathcal{M}$ and the complexified cotangent bundle of $\mathcal{M}$ are denoted by $\mathbb{C}T\mathcal{M}$ and $\mathbb{C}T^*\mathcal{M}$, respectively. Write $\langle \cdot, \cdot \rangle$ to denote the pointwise duality between $T\mathcal{M}$ and $T^*\mathcal{M}$. We extend $\langle \cdot, \cdot \rangle$ bilinearly to $\mathbb{C}T\mathcal{M} \times \mathbb{C}T^*\mathcal{M}$. Let E be a $\mathscr{C}^\infty$ vector bundle over $\mathcal{M}$. The fiber of E at $x \in \mathcal{M}$ will be denoted by E_x. Let $\widehat{E}$ be another vector bundle over $\mathcal{M}$. We write $E \boxtimes \widehat{E}^*$ to denote the vector bundle over $\mathcal{M} \times \mathcal{M}$ with fiber over $(x, y) \in \mathcal{M} \times \mathcal{M}$ consisting of the linear maps from $\widehat{E}_y$ to E_x. Let $Y \subset \mathcal{M}$ be an open set. From now on, the spaces of distribution sections of E over Y and smooth sections of E over Y will be denoted by $\mathscr{D}'(Y, E)$ and $\mathscr{C}^\infty(Y, E)$ respectively. Let $\mathscr{E}'(Y, E)$ be the subspace of $\mathscr{D}'(Y, E)$ whose elements have compact support in Y and let $\mathscr{C}_c^\infty(Y, E)$ be the subspace of $\mathscr{C}^\infty(Y, E)$ whose elements have compact support in Y. For $m \in \mathbb{R}$, let $H^m(Y, E)$ denote the Sobolev space of order m of sections of E over Y. Put

$$
\begin{aligned}
&H^m_{\mathrm{loc}}(Y, E) = \{u \in \mathscr{D}'(Y, E) : \varphi u \in H^m(Y, E) \text{ for every } \varphi \in \mathscr{C}_c^\infty(Y)\}, \\
&H^m_c(Y, E) = H^m_{\mathrm{loc}}(Y, E) \cap \mathscr{E}'(Y, E).
\end{aligned}
$$

Let E and $\widehat{E}$ be $\mathscr{C}^\infty$ vector bundles over a paracompact orientable $\mathscr{C}^\infty$ manifold $\mathcal{M}$ equipped with a smooth density of integration. If $A : \mathscr{C}_c^\infty(\mathcal{M}, E) \to \mathscr{D}'(\mathcal{M}, \widehat{E})$ is continuous, we write $A(x, y)$ to denote the distribution kernel of A. The following two statements are equivalent:

(a) A is continuous: $\mathscr{E}'(\mathcal{M}, E) \to \mathscr{C}^\infty(\mathcal{M}, \widehat{E})$.

(b) $A(x, y) \in \mathscr{C}^\infty(\mathcal{M} \times \mathcal{M}, \widehat{E} \boxtimes E^*)$.

If A satisfies (a) or (b), we say that A is smoothing on $\mathcal{M}$. Let $A, B : \mathscr{C}_c^\infty(\mathcal{M}, E) \to \mathscr{D}'(\mathcal{M}, \widehat{E})$ be continuous operators. We write

$$A \equiv B \text{ (on } \mathcal{M}) \tag{2-1}$$

if $A - B$ is a smoothing operator.

We say that A is properly supported if the restrictions of the two projections $(x, y) \mapsto x,\ (x, y) \mapsto y$ to $\operatorname{supp} A(x, y)$ are proper.

Let $H(x, y) \in \mathscr{D}'(\mathcal{M} \times \mathcal{M}, \widehat{E} \boxtimes E^*)$. We denote by H the unique continuous operator $\mathscr{C}_c^\infty(\mathcal{M}, E) \to \mathscr{D}'(\mathcal{M}, \widehat{E})$ with distribution kernel $H(x, y)$. In this work, we identify H with $H(x, y)$.

Let D be an open set of a smooth manifold X and let E be a vector bundle over X. Let

$$L^m_{\frac{1}{2},\frac{1}{2}}(D, E \boxtimes E^*), \quad L^m_{\mathrm{cl}}(D, E \boxtimes E^*)$$

denote the space of pseudodifferential operators on D of order m and type $\left(\frac{1}{2}, \frac{1}{2}\right)$ from sections of E to sections of E and the space of classical pseudodifferential operators on D of order m from sections of E to sections of E respectively. The classical result of Calderon and Vaillancourt [Hörmander 1985, Chapter 18] tells us that any $A \in L^m_{1/2,1/2}(D, E \boxtimes E^*)$ induces for any $s \in \mathbb{R}$ a continuous operator

$$A : H^s_c(D, E) \to H^{s-m}_{\mathrm{loc}}(D, E). \tag{2-2}$$

Let $A \in L^m_{1/2,1/2}(D, E \boxtimes E^*)$, $B \in L^{m_1}_{1/2,1/2}(D, E \boxtimes E^*)$, where $m, m_1 \in \mathbb{R}$. If A or B is properly supported, then the composition of A and B is well-defined. Moreover, we can repeat the proof of [Boutet de Monvel 1974, Proposition 3.2] and conclude that

$$AB \in L^{m+m_1}_{\frac{1}{2},\frac{1}{2}}(D, E \boxtimes E^*). \tag{2-3}$$

For $m \in \mathbb{R},\ \rho, \delta \in \mathbb{R},\ 0 \le \rho, \delta \le 1$, let

$$S^m_{\rho,\delta}(T^*D, E \boxtimes E^*)$$

be the Hörmander symbol space on T^*D with values in $E \boxtimes E^*$ of order m and type (ρ, δ); see [Hörmander 1983, Definition 7.8.1]. Let

$$S^{-\infty}_{\rho,\delta}(T^*D, E \boxtimes E^*) := \bigcap_{m \in \mathbb{R}} S^m_{\rho,\delta}(T^*D, E \boxtimes E^*).$$

Let $a_j \in S^{m_j}_{\rho,\delta}(T^*D, E \boxtimes E^*)$ with $m_j \searrow -\infty$, $j \to \infty$. Then there exists $a \in S^{m_0}_{\rho,\delta}(T^*D, E \boxtimes E^*)$ such that, for any $k \in \mathbb{N}$,

$$a - \sum_{j=0}^{k-1} a_j \in S^{m_k}_{1,0}(T^*D, E \boxtimes E^*).$$

In this case we write

$$a \sim \sum_{j=0}^{+\infty} a_j \quad \text{in } S^{m_0}_{\rho,\delta}(T^*D, E \boxtimes E^*).$$

The symbol a is unique modulo $S^{-\infty}_{\rho,\delta}(T^*D, E \boxtimes E^*)$.

Let W_1 be an open set in $\mathbb{R}^{N_1}$ and let W_2 be an open set in $\mathbb{R}^{N_2}$. Let E_1 and E_2 be vector bundles over W_1 and W_2, respectively. An m-dependent continuous operator $F_m : \mathscr{C}_c^\infty(W_2, E_2) \to \mathscr{D}'(W_1, E_1)$ is called m-negligible on $W_1 \times W_2$ if, for m large enough, F_m is smoothing and, for any $K \Subset W_1 \times W_2$, any multi-indices α, β and any $N \in \mathbb{N}$, there exists $C_{K,\alpha,\beta,N} > 0$ such that

$$|\partial_x^\alpha \partial_y^\beta F_m|(x, y) \le C_{K,\alpha,\beta,N}\, m^{-N} \quad \text{on } K \text{ for } m \gg 1. \tag{2-4}$$

In that case we write

$$F_m(x, y) = O(m^{-\infty}) \quad \text{or} \quad F_m = O(m^{-\infty}) \quad \text{on } W_1 \times W_2.$$

If $F_m, G_m : \mathscr{C}_c^\infty(W_2, E_2) \to \mathscr{D}'(W_1, E_1)$ are m-dependent continuous operators, we write $F_m = G_m + O(m^{-\infty})$ on $W_1 \times W_2$ or $F_m(x, y) = G_m(x, y) + O(m^{-\infty})$ on $W_1 \times W_2$ if $F_m - G_m = O(m^{-\infty})$ on $W_1 \times W_2$. When $W = W_1 = W_2$, we sometimes write "on W".

Let $\mathcal{M}_1$ and $\mathcal{M}_2$ be smooth manifolds and let E_1 and E_2 be vector bundles over $\mathcal{M}_1$ and $\mathcal{M}_2$, respectively. Let $F_m, G_m : \mathscr{C}^\infty(\mathcal{M}_2, E_2) \to \mathscr{C}^\infty(\mathcal{M}_1, E_1)$ be m-dependent smoothing operators. We write $F_m = G_m + O(m^{-\infty})$ on $\mathcal{M}_1 \times \mathcal{M}_2$ if, on every local coordinate patch D of $\mathcal{M}_1$ and local coordinate patch D_1 of $\mathcal{M}_2$, $F_m = G_m + O(m^{-\infty})$ on $D \times D_1$. When $\mathcal{M}_1 = \mathcal{M}_2$, we sometimes write "on $\mathcal{M}_2$".

We recall the definition of the semiclassical symbol spaces.

Definition 2.1. Let W be an open set in $\mathbb{R}^N$. Let

$$S(W) := \big\{a \in \mathscr{C}^\infty(W) \;\big|\; \text{for every } \alpha \in \mathbb{N}_0^N : \sup_{x \in W} |\partial^\alpha a(x)| < \infty\big\},$$
$$S^0_{\mathrm{loc}}(W) := \big\{(a(\cdot, m))_{m \in \mathbb{R}} \;\big|\; \text{for all } \alpha \in \mathbb{N}_0^N,\ \chi \in \mathscr{C}_c^\infty(W),\ \sup_{m \ge 1} \sup_{x \in W} |\partial^\alpha(\chi a(x, m))| < \infty\big\}.$$

For $k \in \mathbb{R}$, let

$$S^k_{\mathrm{loc}} = S^k_{\mathrm{loc}}(W) = \{(a(\cdot, m))_{m \in \mathbb{R}} \mid (m^{-k} a(\cdot, m)) \in S^0_{\mathrm{loc}}(W)\}.$$

Hence $a(\cdot, m) \in S^k_{\mathrm{loc}}(W)$ if, for every $\alpha \in \mathbb{N}_0^N$ and $\chi \in \mathscr{C}_c^\infty(W)$, there exists $C_\alpha > 0$ independent of m such that $|\partial^\alpha(\chi a(\cdot, m))| \le C_\alpha m^k$ holds on W.

Consider a sequence $a_j \in S^{k_j}_{\mathrm{loc}}$, $j \in \mathbb{N}_0$, where $k_j \searrow -\infty$, and let $a \in S^{k_0}_{\mathrm{loc}}$. We say that

$$a(\cdot, m) \sim \sum_{j=0}^{\infty} a_j(\cdot, m) \quad \text{in } S^{k_0}_{\mathrm{loc}}$$

if, for every $\ell \in \mathbb{N}_0$, we have $a - \sum_{j=0}^{\ell} a_j \in S^{k_{\ell+1}}_{\mathrm{loc}}$. For a given sequence a_j as above, we can always find such an asymptotic sum a, which is unique up to an element in $S^{-\infty}_{\mathrm{loc}} = S^{-\infty}_{\mathrm{loc}}(W) := \bigcap_k S^k_{\mathrm{loc}}$.

Similarly, we can define $S^k_{\mathrm{loc}}(Y, A)$ in the standard way, where Y is a smooth manifold and A is a vector bundle over Y.

2.2. ***Manifolds with smooth boundary.*** Let M be a relatively compact open subset with smooth boundary X of a smooth manifold M'. Let A be a $\mathscr{C}^\infty$ vector bundle over M'. Let U be an open set in M'. Let

$$\mathscr{C}^\infty(U \cap \overline{M}, A), \quad \mathscr{D}'(U \cap \overline{M}, A), \quad \mathscr{C}_c^\infty(U \cap \overline{M}, A), \quad \mathscr{E}'(U \cap \overline{M}, A),$$
$$H^s(U \cap \overline{M}, A), \quad H_c^s(U \cap \overline{M}, A), \quad H^s_{\mathrm{loc}}(U \cap \overline{M}, A)$$

(where $s \in \mathbb{R}$) denote the spaces of restrictions to $U \cap \overline{M}$ of elements in

$$\mathscr{C}^\infty(U \cap M', A), \quad \mathscr{D}'(U \cap M', A), \quad \mathscr{C}^\infty(U \cap M', A), \quad \mathscr{E}'(U \cap M', A),$$
$$H^s(M', A), \quad H^s_c(M', A), \quad H^s_{\mathrm{loc}}(M', A),$$

respectively. Write

$$L^2(U \cap \overline{M}, A) := H^0(U \cap \overline{M}, A), \quad L^2_c(U \cap \overline{M}, A) := H^0_c(U \cap \overline{M}, A),$$
$$L^2_{\mathrm{loc}}(U \cap \overline{M}, A) := H^0_{\mathrm{loc}}(U \cap \overline{M}, A).$$

Let A and B be $\mathscr{C}^\infty$ vector bundles over M'. Let U be an open set in M'. Let

$$F_1, F_2 : \mathscr{C}^\infty_c(U \cap M, A) \to \mathscr{D}'(U \cap M, B)$$

be continuous operators. Let $F_1(x, y), F_2(x, y) \in \mathscr{D}'((U \times U) \cap (M \times M), A \boxtimes B^*)$ be the distribution kernels of F_1 and F_2 respectively. We write

$$F_1 \equiv F_2 \mod \mathscr{C}^\infty((U \times U) \cap (\overline{M} \times \overline{M}))$$

or $F_1(x, y) \equiv F_2(x, y) \mod \mathscr{C}^\infty((U \times U) \cap (\overline{M} \times \overline{M}))$ if $F_1(x, y) = F_2(x, y) + r(x, y)$, where $r(x, y) \in \mathscr{C}^\infty((U \times U) \cap (\overline{M} \times \overline{M}), A \boxtimes B^*)$. Similarly, let

$$\hat{F}_1, \hat{F}_2 : \mathscr{C}^\infty_c(U \cap M, A) \to \mathscr{D}'(U \cap X, B)$$

be continuous operators. Let

$$\hat{F}_1(x, y), \hat{F}_2(x, y) \in \mathscr{D}'((U \times U) \cap (X \times M), A \boxtimes B^*)$$

be the distribution kernels of $\hat{F}_1$ and $\hat{F}_2$ respectively. We write $\hat{F}_1 \equiv \hat{F}_2 \mod \mathscr{C}^\infty((U \times U) \cap (X \times \overline{M}))$ or $\hat{F}_1(x, y) \equiv \hat{F}_2(x, y) \mod \mathscr{C}^\infty((U \times U) \cap (X \times \overline{M}))$ if $\hat{F}_1(x, y) = \hat{F}_2(x, y) + \hat{r}(x, y)$, where $\hat{r}(x, y) \in \mathscr{C}^\infty((U \times U) \cap (X \times \overline{M}), A \boxtimes B^*)$. Similarly, let $\tilde{F}_1, \tilde{F}_2 : \mathscr{C}^\infty_c(U \cap X, A) \to \mathscr{D}'(U \cap M, B)$ be continuous operators. Let

$$\tilde{F}_1(x, y), \tilde{F}_2(x, y) \in \mathscr{D}'((U \times U) \cap (M \times X), A \boxtimes B^*)$$

be the distribution kernels of $\tilde{F}_1$ and $\tilde{F}_2$ respectively. We write $\tilde{F}_1 \equiv \tilde{F}_2 \mod \mathscr{C}^\infty((U \times U) \cap (\overline{M} \times X))$ or $\tilde{F}_1(x, y) \equiv \tilde{F}_2(x, y) \mod \mathscr{C}^\infty((U \times U) \cap (\overline{M} \times X))$ if $\tilde{F}_1(x, y) = \tilde{F}_2(x, y) + \tilde{r}(x, y)$, where $\tilde{r}(x, y) \in \mathscr{C}^\infty((U \times U) \cap (\overline{M} \times X), A \boxtimes B^*)$.

Let

$$F_m, G_m : \mathscr{C}^\infty_c(U \cap M, A) \to \mathscr{D}'(U \cap M, B)$$

be m-dependent continuous operators. Let

$$F_m(x, y), G_m(x, y) \in \mathscr{D}'((U \times U) \cap (M \times M), A \boxtimes B^*)$$

be the distribution kernels of F_m and G_m respectively. We write

$$F_m \equiv G_m \mod O(m^{-\infty}) \quad \text{on } U \cap \overline{M} \tag{2-5}$$

if there is a $r_m(x, y) \in \mathscr{C}^\infty(U \times U, A \boxtimes B^*)$ with $r_m(x, y) = O(m^{-\infty})$ on $U \times U$ such that

$$r_m(x, y)|_{(U \times U) \cap (\overline{M} \times \overline{M})} = F_m(x, y) - G_m(x, y) \quad \text{for } m \gg 1.$$

Let $k \in \mathbb{R}$. Let U be an open set in M' and let E be a vector bundle over $M' \times M'$. Let

$$S^k_{\mathrm{loc}}((U \times U) \cap (\overline{M} \times \overline{M}), E) \tag{2-6}$$

denote the space of restrictions to $U \cap \overline{M}$ of elements in $S^k_{\mathrm{loc}}(U \times U, E)$. Let

$$a_j \in S^{k_j}_{\mathrm{loc}}((U \times U) \cap (\overline{M} \times \overline{M}), E), \quad j = 0, 1, 2, \dots,$$

with $k_j \searrow -\infty$, $j \to \infty$. Then there exists $a \in S^{k_0}_{\mathrm{loc}}((U \times U) \cap (\overline{M} \times \overline{M}), E)$ such that, for every $\ell \in \mathbb{N}$,

$$a - \sum_{j=0}^{\ell-1} a_j \in S^{k_\ell}_{\mathrm{loc}}((U \times U) \cap (\overline{M} \times \overline{M}), E).$$

If a and a_j have the properties above, we write

$$a \sim \sum_{j=0}^{\infty} a_j \quad \text{in } S^{k_0}_{\mathrm{loc}}((U \times U) \cap (\overline{M} \times \overline{M}), E).$$

If E is trivial, then we write $S^{k_0}_{\mathrm{loc}}((U \times U) \cap (\overline{M} \times \overline{M}))$ to denote $S^{k_0}_{\mathrm{loc}}((U \times U) \cap (\overline{M} \times \overline{M}), E)$.

2.3. *The $\bar{\partial}$-Neumann Laplacian.* Let M be a relatively compact open subset with $\mathscr{C}^\infty$ boundary X of a complex manifold M' of dimension n. Let $T^{1,0}M'$ and $T^{0,1}M'$ be the holomorphic tangent bundle of M' and the antiholomorphic tangent bundle of M'. We fix a Hermitian metric $\langle \cdot \mid \cdot \rangle$ on $\mathbb{C}TM'$ so that $T^{1,0}M' \perp T^{0,1}M'$. For $p, q \in \mathbb{N}$, let $T^{*p,q}M'$ be the vector bundle of (p,q)-forms on M'. The Hermitian metric $\langle \cdot \mid \cdot \rangle$ on $\mathbb{C}TM'$ induces by duality a Hermitian metric $\langle \cdot \mid \cdot \rangle$ on $\bigoplus_{p,q=1}^{p,q=n} T^{*p,q}M'$. Let $|\cdot|$ be the corresponding pointwise norm with respect to $\langle \cdot \mid \cdot \rangle$. Let $\rho \in \mathscr{C}^\infty(M', \mathbb{R})$ be a defining function of X, that is, $\rho = 0$ on X, $\rho < 0$ on M and $d\rho \neq 0$ near X. From now on, we take a defining function ρ so that $|d\rho| = 1$ on X. Let U be an open set of M'. For every $p, q = 0, \dots, n$, we define

$$\Omega^{p,q}(U \cap \overline{M}) := \mathscr{C}^\infty(U \cap \overline{M}, T^{*p,q}M'), \quad \Omega^{p,q}(M') := \mathscr{C}^\infty(M', T^{*p,q}M'),$$
$$\Omega^{p,q}_c(U \cap \overline{M}) := \mathscr{C}^\infty_c(U \cap \overline{M}, T^{*p,q}M'),$$
$$\Omega^{p,q}_c(M') := \mathscr{C}^\infty_c(M', T^{*p,q}M'), \quad \Omega^{p,q}_c(M) := \mathscr{C}^\infty_c(M, T^{*p,q}M').$$

Let $dv_{M'}$ be the volume form on M' induced by the Hermitian metric $\langle \cdot \mid \cdot \rangle$ on $\mathbb{C}TM'$ and let $(\cdot \mid \cdot)_M$ and $(\cdot \mid \cdot)_{M'}$ be the inner products on $\Omega^{0,q}(\overline{M})$ and $\Omega^{0,q}_c(M')$ defined by

$$\begin{aligned} (f \mid h)_M &= \int_M \langle f \mid h \rangle \, dv_{M'}, \quad f, h \in \Omega^{0,q}(\overline{M}), \\ (f \mid h)_{M'} &= \int_{M'} \langle f \mid h \rangle \, dv_{M'}, \quad f, h \in \Omega^{0,q}_c(M'). \end{aligned} \tag{2-7}$$

Let $\|\cdot\|_M$ and $\|\cdot\|_{M'}$ be the corresponding norms with respect to $(\cdot \mid \cdot)_M$ and $(\cdot \mid \cdot)_{M'}$ respectively. Let $L^2_{(0,q)}(M)$ be the completion of $\Omega^{0,q}(\overline{M})$ with respect to $(\cdot \mid \cdot)_M$. We extend $(\cdot \mid \cdot)_M$ to $L^2_{(0,q)}(M)$ in the standard way. Let $\bar{\partial} : \Omega^{0,q}(M') \to \Omega^{0,q+1}(M')$ be the part of the exterior differential operator which maps forms of type $(0,q)$ to forms of type $(0,q+1)$ and we denote by $\bar{\partial}^*_f : \Omega^{0,q+1}(M') \to \Omega^{0,q}(M')$ the formal adjoint of $\bar{\partial}$. That is,

$$(\bar{\partial} f \mid h)_{M'} = (f \mid \bar{\partial}^*_f h)_{M'},$$

$f \in \Omega_c^{0,q}(M')$, $h \in \Omega^{0,q+1}(M')$. We shall also use the notation $\bar\partial$ for the closure in L^2 of the $\bar\partial$ operator, initially defined on $\Omega^{0,q}(\overline{M})$ and $\bar\partial^*$ for the Hilbert space adjoint of $\bar\partial$. Recall that for $u \in L^2_{(0,q)}(M)$, we say that $u \in \operatorname{Dom}\bar\partial$ if we can find a sequence $u_j \in \Omega^{0,q}(\overline{M})$, $j = 1, 2, \dots$, with $\lim_{j\to\infty}\|u_j - u\|_M = 0$ such that $\lim_{j\to\infty}\|\bar\partial u_j - v\|_M = 0$ for some $v \in L^2_{(0,q+1)}(M)$. We set $\bar\partial u = v$. The $\bar\partial$-Neumann Laplacian on $(0,q)$-forms is then the nonnegative self-adjoint operator in the space $L^2_{(0,q)}(M)$ (see [Folland and Kohn 1972, Chapter 1]):

$$\Box^{(q)} = \bar\partial\,\bar\partial^* + \bar\partial^*\,\bar\partial : \operatorname{Dom}\Box^{(q)} \subset L^2_{(0,q)}(M) \to L^2_{(0,q)}(M), \tag{2-8}$$

where

$$\operatorname{Dom}\Box^{(q)} = \{u \in L^2_{(0,q)}(M),\ u \in \operatorname{Dom}\bar\partial^* \cap \operatorname{Dom}\bar\partial,\ \bar\partial^* u \in \operatorname{Dom}\bar\partial,\ \bar\partial u \in \operatorname{Dom}\bar\partial^*\} \tag{2-9}$$

and $\Omega^{0,q}(\overline{M}) \cap \operatorname{Dom}\Box^{(q)}$ is dense in $\operatorname{Dom}\Box^{(q)}$ for the norm

$$\operatorname{Dom}\Box^{(q)} \ni u \mapsto \|u\|_M + \|\bar\partial u\|_M + \|\bar\partial^* u\|_M;$$

see [Folland and Kohn 1972, p. 14]. We denote by $\operatorname{Spec}\Box^{(q)}$ the spectrum of $\Box^{(q)}$.

Now, we consider the boundary X of M. The boundary X is a compact CR manifold of dimension $2n-1$ with natural CR structure $T^{1,0}X := T^{1,0}M' \cap \mathbb{C}TX$. Let $T^{0,1}X := \overline{T^{1,0}X}$. The Hermitian metric on $\mathbb{C}TM'$ induces Hermitian metrics $\langle\cdot\,|\,\cdot\rangle$ on $\mathbb{C}TX$ and also on the bundle $\bigoplus_{j=1}^{2n-1}\Lambda^j(\mathbb{C}T^*X)$. Let dv_X be the volume form on X induced by the Hermitian metric $\langle\cdot\,|\,\cdot\rangle$ on $\mathbb{C}TX$ and let $(\cdot\,|\,\cdot)_X$ be the L^2 inner product on $\mathscr{C}^\infty\big(X, \bigoplus_{j=1}^{2n-1}\Lambda^j(\mathbb{C}T^*X)\big)$ induced by dv_X and the Hermitian metric $\langle\cdot\,|\,\cdot\rangle$ on $\bigoplus_{j=0}^{2n-1}\Lambda^j(\mathbb{C}T^*X)$. Put

$$T^{*1,0}X := (T^{0,1}X \oplus \mathbb{C}T)^\perp \subset \mathbb{C}T^*X, \quad T^{*0,1}X := (T^{1,0}X \oplus \mathbb{C}T)^\perp \subset \mathbb{C}T^*X.$$

We have the pointwise orthogonal decomposition (see (1-5))

$$\begin{aligned} \mathbb{C}T^*X &= T^{*1,0}X \oplus T^{*0,1}X \oplus \{\lambda\omega_0 : \lambda \in \mathbb{C}\}, \\ \mathbb{C}TX &= T^{1,0}X \oplus T^{0,1}X \oplus \{\lambda T : \lambda \in \mathbb{C}\}. \end{aligned} \tag{2-10}$$

Define the vector bundle of $(0,q)$-forms by $T^{*0,q}X := \Lambda^q T^{*0,1}X$. Let $D \subset X$ be an open set. Let $\Omega^{0,q}(D)$ denote the space of smooth sections of $T^{*0,q}X$ over D and let $\Omega_c^{0,q}(D)$ be the subspace of $\Omega^{0,q}(D)$ whose elements have compact support in D.

In order to describe the $\bar\partial$-Neumann boundary conditions we introduce the operator of restriction to the boundary X: let γ denote the operator of restriction to the boundary X,

$$\gamma : \Omega^{0,\bullet}(\overline{M}) \to \mathscr{C}^\infty(X, T^{*0,\bullet}M'|_X), \quad u \mapsto \gamma u := u|_X. \tag{2-11}$$

We have $\{u \in \Omega^{0,q}(\overline{M}) : (\bar\partial\rho)^{\wedge,*}\gamma u = 0\} = \operatorname{Dom}\bar\partial^* \cap \Omega^{0,q}(\overline{M})$. We have thus

$$u \in \operatorname{Dom}\Box^{(q)} \cap \Omega^{0,q}(\overline{M}) \iff (\bar\partial\rho)^{\wedge,*}\gamma u = 0, (\bar\partial\rho)^{\wedge,*}\gamma\bar\partial u = 0. \tag{2-12}$$

The conditions on the right-hand side are called first and second $\bar\partial$-Neumann boundary conditions.

3. The boundary operator $\Box_-^{(q)}$

In this section, we introduce a boundary operator on $X = \partial M$ defined for a form u on X as the complex tangential component of the form $\bar\partial v$, where $v = \widetilde{P}u$ is the extension of u from X to M by the Poisson operator $\widetilde{P}$. This operator will play a central role in Section 4 for the construction of the parametrix of the $\bar\partial$-Neumann problem and Section 5 (Lemma 5.18). We fix $q \in \{0, 1, \dots, n-1\}$. Let

$$\Box_f^{(q)} = \bar\partial\,\bar\partial_f^* + \bar\partial_f^*\,\bar\partial : \Omega^{0,q}(M') \to \Omega^{0,q}(M')$$

denote the complex Laplace–Beltrami operator on $(0,q)$-forms. The subscript f indicates that the operator is not subject to any boundary conditions. The boundary problem $(\Box_f^{(q)}, \gamma)$ on $\overline{M}$ is the Dirichlet boundary problem, which is a regular elliptic boundary problem; see, e.g., [Taylor 2011, Chapter 5, Proposition 11.10]. Let us consider the map

$$F^{(q)} : H^2(\overline{M}, T^{*0,q}M') \to L^2_{(0,q)}(M) \oplus H^{\frac{3}{2}}(X, T^{*0,q}M'), \quad u \mapsto (\Box_f^{(q)}u, \gamma u). \tag{3-1}$$

By the general theory of regular elliptic boundary problems [Boutet de Monvel 1971; Taylor 2011, Chapter 5, Proposition 11.16], we know that $\dim \operatorname{Ker} F^{(q)} < \infty$ and $\operatorname{Ker} F^{(q)} \subset \Omega^{0,q}(\overline{M})$. Let

$$K^{(q)} : H^2(\overline{M}, T^{*0,q}M') \to \operatorname{Ker} F^{(q)} \tag{3-2}$$

be the orthogonal projection with respect to $(\cdot\,|\,\cdot)_M$. Put $\widetilde{\Box}_f^{(q)} = \Box_f^{(q)} + K^{(q)}$ and consider the map

$$\widetilde{F}^{(q)} : H^2(\overline{M}, T^{*0,q}M') \to L^2_{(0,q)}(M) \oplus H^{\frac{3}{2}}(X, T^{*0,q}M'), \quad u \mapsto (\widetilde{\Box}_f^{(q)}u, \gamma u). \tag{3-3}$$

It is easy to see that $\widetilde{F}^{(q)}$ is injective. Let

$$\widetilde{P} : \mathscr{C}^\infty(X, T^{*0,q}M') \to \Omega^{0,q}(\overline{M}) \tag{3-4}$$

be the Poisson operator for $\widetilde{\Box}_f^{(q)}$ which is well-defined since (3-3) is injective. The Poisson operator $\widetilde{P}$ satisfies

$$\widetilde{\Box}_f^{(q)}\widetilde{P}u = 0, \quad \gamma\widetilde{P}u = u \quad \text{for every } u \in \mathscr{C}^\infty(X, T^{*0,q}M'). \tag{3-5}$$

By [Boutet de Monvel 1971, p. 29] the operator $\widetilde{P}$ extends continuously

$$\widetilde{P} : H^s(X, T^{*0,q}M') \to H^{s+\frac{1}{2}}(\overline{M}, T^{*0,q}M') \quad \text{for all } s \in \mathbb{R}, \tag{3-6}$$

and there is a continuous operator

$$D^{(q)} : H^s(\overline{M}, T^{*0,q}M') \to H^{s+2}(\overline{M}, T^{*0,q}M') \quad \text{for all } s \in \mathbb{R} \tag{3-7}$$

such that

$$D^{(q)}\widetilde{\Box}_f^{(q)} + \widetilde{P}\gamma = I \quad \text{on } \Omega^{0,q}(\overline{M}). \tag{3-8}$$

Let $\widehat{\mathscr{E}}'(\overline{M}, T^{*0,q}M')$ denote the space of continuous linear map from $\Omega^{0,q}(\overline{M})$ to $\mathbb{C}$ with respect to $(\cdot\,|\,\cdot)_M$. Let

$$\widetilde{P}^* : \widehat{\mathscr{E}}'(\overline{M}, T^{*0,q}M') \to \mathscr{D}'(X, T^{*0,q}M') \tag{3-9}$$

be the operator defined by

$$(\widetilde{P}^*u \mid v)_X = (u \mid \widetilde{P}v)_M, \quad u \in \widehat{\mathscr{E}}'(\overline{M}, T^{*0,q}M'),\ v \in \mathscr{C}^\infty(X, T^{*0,q}M').$$

By [Boutet de Monvel 1971, p. 30] the operator

$$\widetilde{P}^* : H^s(\overline{M}, T^{*0,q}M') \to H^{s+\frac{1}{2}}(X, T^{*0,q}M') \tag{3-10}$$

is continuous for every $s \in \mathbb{R}$ and

$$\widetilde{P}^* : \Omega^{0,q}(\overline{M}) \to \mathscr{C}^\infty(X, T^{*0,q}M').$$

Let

$$\Box_-^{(q)} := (\bar\partial\rho)^{\wedge,*}\gamma\bar\partial\widetilde{P} : \Omega^{0,q}(X) \to \Omega^{0,q}(X). \tag{3-11}$$

In this section, we will construct a parametrix for $\Box_-^{(q)}$ under certain Levi curvature assumptions. Put

$$\begin{aligned} \Sigma^- &= \{(x, \lambda\omega_0(x)) \in T^*X : \lambda < 0\}, \\ \Sigma^+ &= \{(x, \lambda\omega_0(x)) \in T^*X : \lambda > 0\}. \end{aligned} \tag{3-12}$$

Note that we use here a different sign convention than in [Hsiao 2010], where ω_0 equals $d\rho \circ J$ (compare [loc. cit., (1.9), p. 84], (1-5)), thus we swap here the roles of Σ^+ and Σ^- compared to [loc. cit.].

Definition 3.1. Let $A \in L^m_{1/2,1/2}(D, T^{*0,q}X \boxtimes (T^{*0,q}X)^*)$, where $m \in \mathbb{R}$. We write

$$A \equiv 0 \quad \text{near } \Sigma^+ \cap T^*D$$

if there exists $A' \in L^m_{1/2,1/2}(D, T^{*0,q}X \boxtimes (T^{*0,q}X)^*)$ with full symbol

$$a(x,\eta) \in S^m_{1/2,1/2}(T^*D, T^{*0,q}X \boxtimes (T^{*0,q}X)^*)$$

such that

$$A \equiv A' \text{ on } D$$

and $a(x,\eta)$ vanishes in an open neighborhood of $\Sigma^+ \cap T^*D$.

For A as in Definition 3.1 we have $\mathrm{WF}(A) \cap \Sigma^+ = \varnothing$, where $\mathrm{WF}(A)$ denotes the wave front set of the pseudodifferential operator A; see [Grigis and Sjöstrand 1994, Chapter 7].

Let us consider the Hodge–de Rham Laplacian

$$\triangle_X := dd^* + d^*d : \mathscr{C}^\infty(X, \Lambda^q(\mathbb{C}T^*X)) \to \mathscr{C}^\infty(X, \Lambda^q(\mathbb{C}T^*X)), \tag{3-13}$$

where $d^* : \mathscr{C}^\infty(X, \Lambda^{q+1}(\mathbb{C}T^*X)) \to \mathscr{C}^\infty(X, \Lambda^q(\mathbb{C}T^*X))$ is the formal adjoint of the exterior derivative d with respect to $(\cdot \mid \cdot)_X$. Let $\sqrt{\triangle_X}$ be the nonnegative square root of $\triangle_X$.

Theorem 3.2 [Hsiao 2010, Part II, Proposition 4.1]. *The operator $\Box_-^{(q)}$ from* (3-11) *is a classical pseudodifferential operator of order* 1 *and we have*

$$\Box_-^{(q)} = \tfrac{1}{2}(iT + \sqrt{\triangle_X}) + \Psi^0, \tag{3-14}$$

where $\Psi^0 \in L^0_{\mathrm{cl}}(X, T^{*0,q}X \boxtimes (T^{*0,q}X)^*)$. In particular, $\Box_-^{(q)}$ is elliptic outside Σ^+.

Let $\bar\partial_b : \Omega^{0,q}(X) \to \Omega^{0,q+1}(X)$ be the tangential Cauchy–Riemann operator. It is not difficult to see that

$$\bar\partial_b = 2(\bar\partial\rho)^{\wedge,*}(\bar\partial\rho)^\wedge \gamma\bar\partial\widetilde{P} : \Omega^{0,q}(X) \to \Omega^{0,q+1}(X). \tag{3-15}$$

We notice that, for $u \in \mathscr{C}^\infty(X, \Lambda^q(\mathbb{C}T^*X))$,

$$u \in \Omega^{0,q}(X) \quad \text{if and only if} \quad u = 2(\bar\partial\rho)^{\wedge,*}(\bar\partial\rho)^\wedge u \ \text{ on } X \tag{3-16}$$

and

$$2(\bar\partial\rho)^{\wedge,*}(\bar\partial\rho)^\wedge + 2(\bar\partial\rho)^\wedge(\bar\partial\rho)^{\wedge,*} = I \quad \text{on } \mathscr{C}^\infty(X, \Lambda^q(\mathbb{C}T^*X)). \tag{3-17}$$

Consider

$$\gamma\bar\partial_f^*\widetilde{P} : \mathscr{C}^\infty(X, \Lambda^{q+1}(\mathbb{C}T^*X)) \to \mathscr{C}^\infty(X, \Lambda^q(\mathbb{C}T^*X)).$$

It is not difficult to check that (see [Hsiao 2010, Part II, Lemma 2.2])

$$\gamma\bar\partial_f^*\widetilde{P} : \Omega^{0,q+1}(X) \to \Omega^{0,q}(X). \tag{3-18}$$

Put

$$\widetilde{\Box}_b^{(q)} := \gamma\bar\partial_f^*\widetilde{P}\bar\partial_b + \bar\partial_b\gamma\bar\partial_f^*\widetilde{P} : \Omega^{0,q}(X) \to \Omega^{0,q}(X). \tag{3-19}$$

Lemma 3.3. *We have*

$$\widetilde{\Box}_b^{(q)} = -4(\bar\partial\rho)^{\wedge,*}(\bar\partial\rho)^\wedge\gamma\bar\partial_f^*\widetilde{P}(\bar\partial\rho)^\wedge\Box_-^{(q)} + R^{(q)} \quad \text{on } \Omega^{0,q}(X),$$

where $R^{(q)} : \Omega^{0,q}(X) \to \Omega^{0,q}(X)$ is a smoothing operator.

Proof. From (3-5), (3-15), (3-16), (3-17), (3-18), we have

$$\begin{aligned}
\widetilde{\Box}_b^{(q)} &= 2(\bar\partial\rho)^{\wedge,*}(\bar\partial\rho)^\wedge\widetilde{\Box}_b^{(q)} \\
&= 2(\bar\partial\rho)^{\wedge,*}(\bar\partial\rho)^\wedge\big(\gamma\bar\partial_f^*\widetilde{P}\bar\partial_b + \bar\partial_b\gamma\bar\partial_f^*\widetilde{P}\big) \\
&= 2(\bar\partial\rho)^{\wedge,*}(\bar\partial\rho)^\wedge\gamma\bar\partial_f^*\widetilde{P}\bar\partial_b + 2(\bar\partial\rho)^{\wedge,*}(\bar\partial\rho)^\wedge\bar\partial_b\gamma\bar\partial_f^*\widetilde{P} \\
&= 2(\bar\partial\rho)^{\wedge,*}(\bar\partial\rho)^\wedge\gamma\bar\partial_f^*\widetilde{P}\bar\partial_b + 2(\bar\partial\rho)^{\wedge,*}(\bar\partial\rho)^\wedge\gamma\bar\partial\widetilde{P}\gamma\bar\partial_f^*\widetilde{P} \\
&= 2(\bar\partial\rho)^{\wedge,*}(\bar\partial\rho)^\wedge\gamma\bar\partial_f^*\widetilde{P}\big(\gamma\bar\partial\widetilde{P} - 2(\bar\partial\rho)^\wedge(\bar\partial\rho)^{\wedge,*}\gamma\bar\partial\widetilde{P}\big) + 2(\bar\partial\rho)^{\wedge,*}(\bar\partial\rho)^\wedge\gamma\bar\partial\widetilde{P}\gamma\bar\partial_f^*\widetilde{P}.
\end{aligned} \tag{3-20}$$

From (3-8), we have

$$\begin{aligned}
\bar\partial\widetilde{P} &= \widetilde{P}\gamma\bar\partial\widetilde{P} + D^{(q+1)}\widetilde{\Box}_f^{(q+1)}\bar\partial\widetilde{P} \\
&= \widetilde{P}\gamma\bar\partial\widetilde{P} + D^{(q+1)}(\Box_f^{(q+1)} + K^{(q+1)})\bar\partial\widetilde{P} \\
&= \widetilde{P}\gamma\bar\partial\widetilde{P} + D^{(q+1)}\bar\partial(\Box_f^{(q)} + K^{(q)})\widetilde{P} - D^{(q+1)}\bar\partial K^{(q)}\widetilde{P} + D^{(q+1)}K^{(q+1)}\bar\partial\widetilde{P} \\
&\equiv \widetilde{P}\gamma\bar\partial\widetilde{P} \mod \mathscr{C}^\infty(\overline{M}\times X).
\end{aligned}$$

Similarly, we have

$$\bar\partial_f^*\widetilde{P} \equiv \widetilde{P}\gamma\bar\partial_f^*\widetilde{P} \mod \mathscr{C}^\infty(\overline{M}\times X).$$

Thus,

$$\gamma\bar\partial_f^*\widetilde{P}\gamma\bar\partial\widetilde{P} + \gamma\bar\partial\widetilde{P}\gamma\bar\partial_f^*\widetilde{P} \equiv \gamma(\bar\partial_f^*\,\bar\partial + \bar\partial\,\bar\partial_f^*)\widetilde{P}. \tag{3-21}$$

From (3-11), (3-20) and (3-21), we get

$$\begin{aligned}\widetilde{\Box}_b^{(q)} &\equiv 2(\bar\partial\rho)^{\wedge,*}(\bar\partial\rho)^{\wedge}\gamma\Box_f^{(q)}\widetilde{P}-4(\bar\partial\rho)^{\wedge,*}(\bar\partial\rho)^{\wedge}\gamma\bar\partial_f^*\widetilde{P}(\bar\partial\rho)^{\wedge}\Box_-^{(q)}\\ &\equiv -2(\bar\partial\rho)^{\wedge,*}(\bar\partial\rho)^{\wedge}\gamma K^{(q)}\widetilde{P}-4(\bar\partial\rho)^{\wedge,*}(\bar\partial\rho)^{\wedge}\gamma\bar\partial_f^*\widetilde{P}(\bar\partial\rho)^{\wedge}\Box_-^{(q)},\end{aligned}\tag{3-22}$$

where $K^{(q)}$ is as in (3-2). Note that $K^{(q)}\equiv 0 \bmod \mathscr{C}^\infty(\overline{M}\times\overline{M})$. From this observation and (3-6), we deduce that

$$-2(\bar\partial\rho)^{\wedge,*}(\bar\partial\rho)^{\wedge}\gamma K^{(q)}\widetilde{P}:H^s(X,T^{*0,q}M')\to H^{s+N}(X,T^{*0,q}M'),$$

for every $s\in\mathbb{R}$ and every $N\in\mathbb{N}$. Hence, $-2(\bar\partial\rho)^{\wedge,*}(\bar\partial\rho)^{\wedge}\gamma K^{(q)}\widetilde{P}$ is smoothing. From this observation and (3-20), the lemma follows. □

Lemma 3.3 gives a relation between $\widetilde{\Box}_b^{(q)}$ and $\Box_-^{(q)}$. Put

$$A^{(q)}:=-4(\bar\partial\rho)^{\wedge,*}(\bar\partial\rho)^{\wedge}\gamma\bar\partial_f^*\widetilde{P}(\bar\partial\rho)^{\wedge}:\Omega^{0,q}(X)\to\Omega^{0,q}(X).\tag{3-23}$$

Then, $\widetilde{\Box}_b^{(q)}\equiv A^{(q)}\Box_-^{(q)}$. We are going to show that $A^{(q)}$ is an elliptic classical pseudodifferential operator near Σ^+. We pause and introduce some notation. Near X, put

$$\widetilde{T}_z^{*0,1}M'=\{u\in T_z^{*0,1}M':\bar\partial\rho(u)=0\},\tag{3-24}$$

$$\widetilde{T}_z^{0,1}M'=\Big\{u\in T_z^{0,1}M':\Big(iT+\frac{\partial}{\partial\rho}\Big)(u)=0\Big\}.\tag{3-25}$$

We have the orthogonal decompositions with respect to $\langle\cdot\,|\,\cdot\rangle$ for every $z\in M'$, z is near X:

$$\begin{aligned}T_z^{*0,1}M'&=\widetilde{T}_z^{*0,1}M'\oplus\{\lambda(\bar\partial\rho)(z):\lambda\in\mathbb{C}\},\\ T_z^{0,1}M'&=\widetilde{T}_z^{0,1}M'\oplus\Big\{\lambda\Big(iT+\frac{\partial}{\partial\rho}\Big)(z):\lambda\in\mathbb{C}\Big\}.\end{aligned}\tag{3-26}$$

Note that $\widetilde{T}_z^{*0,1}M'=T_z^{*0,1}X$ and $\widetilde{T}_z^{0,1}M'=T_z^{0,1}X$ for every $z\in X$. Fix $z_0\in X$. We can choose an orthonormal frame $t_1(z),\dots,t_{n-1}(z)$ for $\widetilde{T}_z^{*,0,1}M'$ varying smoothly with z in a neighborhood U of z_0 in M'. Then

$$t_1(z),\dots,t_{n-1}(z),t_n(z):=\frac{\bar\partial\rho(z)}{|\bar\partial\rho(z)|}$$

is an orthonormal frame for $T_z^{*0,1}M'$. Let

$$T_1(z),\ \dots,\ T_{n-1}(z),\ T_n(z)$$

denote the basis of $T_z^{0,1}M'$ which is dual to $t_1(z),\dots,t_n(z)$. We have $T_j(z)\in\widetilde{T}_z^{0,1}M'$, $j=1,\dots,n-1$, and

$$T_n=\frac{iT+\frac{\partial}{\partial\rho}}{\big|iT+\frac{\partial}{\partial\rho}\big|}.$$

From now on, we write Ψ^0 to denote any element in $L^0_{\mathrm{cl}}(X,T^{*0,q}X\boxtimes(T^{*0,q}X)^*)$. By [Hsiao 2010, Part II, (4.11)] we have

$$\gamma\bar\partial_f^*\widetilde{P}=\sum_{j=1}^{n-1}t_j^{\wedge,*}\circ T_j^*+(\bar\partial\rho)^{\wedge,*}\circ(iT-\sqrt{\triangle_X})+\Psi^0,\tag{3-27}$$

where T_j^* is the adjoint of T_j with respect to $(\cdot \mid \cdot)$, i.e., $(T_j f \mid g)_X = (f \mid T_j^* g)_X$ for every $f, g \in \mathscr{C}_c^\infty(U \cap X)$, $j = 1, \dots, n-1$, and $\Psi^0 \in L^0_{\mathrm{cl}}(X, T^{*0,q}X \boxtimes (T^{*0,q}X)^*)$.

Theorem 3.4. *The operator $A^{(q)}$ from* (3-23) *is a classical pseudodifferential operator with*

$$A^{(q)} = -(iT - \sqrt{\triangle_X}) + \Psi^0 \quad \text{on } \Omega^{0,q}(X), \tag{3-28}$$

where $\Psi^0 \in L^0_{\mathrm{cl}}(X, T^{*0,q}X \boxtimes (T^{*0,q}X)^*)$. Hence $A^{(q)}$ is elliptic near Σ^+.

Proof. From (3-23) and (3-27), we have

$$A^{(q)} = -4(\bar\partial\rho)^{\wedge,*}(\bar\partial\rho)^{\wedge}\Big(\sum_{j=1}^{n-1} t_j^{\wedge,*}(\bar\partial\rho)^{\wedge}T_j^* + (\bar\partial\rho)^{\wedge,*}(\bar\partial\rho)^{\wedge}(iT - \sqrt{\triangle_X}) + \Psi^0\Big). \tag{3-29}$$

We notice that

$$\begin{aligned} (\bar\partial\rho)^{\wedge,*}(\bar\partial\rho)^{\wedge}(\bar\partial\rho)^{\wedge}T_j^* &= 0 \quad \text{on } \Omega^{0,q}(X) \text{ for every } j = 1, \dots, n-1, \\ 4(\bar\partial\rho)^{\wedge,*}(\bar\partial\rho)^{\wedge}(\bar\partial\rho)^{\wedge,*}(\bar\partial\rho)^{\wedge} &= I \quad \text{on } \Omega^{0,q}(X). \end{aligned} \tag{3-30}$$

From (3-29) and (3-30), we get (3-28). ☐

Let $D \subset X$ be an open coordinate patch with local coordinates $x = (x_1, \dots, x_{2n-1})$. Assume that the Levi form is nondegenerate of constant signature (n_-, n_+) on D. Note that $(\bar\partial\rho)^{\wedge,*}u = 0$, $u \in \Omega^{0,\bullet}(X)$. From this observation and (3-27), we deduce that

$$\gamma\bar\partial_f^*\widetilde{P} = \sum_{j=1}^{n-1} t_j^{\wedge,*} \circ T_j^* + \Psi^0 \quad \text{on } \Omega^{0,\bullet}(X),$$

and hence

$$\gamma\bar\partial_f^*\widetilde{P} = \bar\partial_b^* + \Psi^0 \quad \text{on } \Omega^{0,\bullet}(X),$$

where $\Psi^0 \in L^0_{\mathrm{cl}}(X, T^{*0,q}X \boxtimes (T^{*0,q}X)^*)$.

We can apply the method in [Sjöstrand 1974] to construct a parametrix of $\widetilde{\Box}_b^{(q)}$ near Σ^+ (see also [Hsiao 2010, Part I, Proposition 6.3]) and deduce the following.

Theorem 3.5. *Let $D \subset X$ be an open coordinate patch such that the Levi form is nondegenerate of constant signature (n_-, n_+) on D. Then for any $q \neq n_-$ there exists a properly supported operator $E^{(q)} \in L^{-1}_{1/2,1/2}(D, T^{*0,q}X \boxtimes (T^{*0,q}X)^*)$ such that*

$$\widetilde{\Box}_b^{(q)}E^{(q)} \equiv I + R \quad \text{on } D, \tag{3-31}$$

where $R \in L^1_{1/2,1/2}(D, T^{*0,q}X \boxtimes (T^{*0,q}X)^*)$ with $R \equiv 0$ near $\Sigma^+ \cap T^*D$.

If $q \neq n_-, n_+$, then $\widetilde{\Box}_b^{(q)}$ is hypoelliptic with loss of one derivative and from [Sjöstrand 1974], we can find $\widetilde{E}^{(q)}$ so that $\widetilde{\Box}_b^{(q)}\widetilde{E}^{(q)} \equiv I$. In Theorem 3.5, q could be equal to n_+ and $\widetilde{\Box}_b^{(n_+)}$ is not hypoelliptic; therefore we have R in (3-31). In [Hsiao 2010, Part I, Proposition 6.3], we do not have R since $q \neq n_-, n_+$.

We can now prove the main result of this section. We will use it in the proof of Theorem 4.3 for the definition of the operator $N_5^{(q)}$; see (4-15).

Theorem 3.6. *Let $D \subset X$ be an open coordinate patch such that the Levi form is nondegenerate of constant signature (n_-, n_+) on D. Then for any $q \neq n_-$ there exists a properly supported operator $G^{(q)} \in L^0_{1/2,1/2}(D, T^{*0,q}X \boxtimes (T^{*0,q}X)^*)$ such that*

$$\Box^{(q)}_- G^{(q)} \equiv I \text{ on } D. \tag{3-32}$$

Proof. Let $A^{(q)} \in L^1_{\mathrm{cl}}(D, T^{*0,q}X \boxtimes (T^{*0,q}X)^*)$ be as in (3-23). Since $A^{(q)}$ is elliptic near Σ^+ (see Theorem 3.4), there are properly supported elliptic pseudodifferential operators $H^{(q)}, H^{(q)}_1 \in L^{-1}_{\mathrm{cl}}(D, T^{*0,q}X \boxtimes (T^{*0,q}X)^*)$ such that

$$\begin{aligned} A^{(q)}H^{(q)} - I &\equiv 0 \quad \text{near } \Sigma^+ \cap T^*D, \\ H^{(q)}_1 A^{(q)} - I &\equiv 0 \quad \text{near } \Sigma^+ \cap T^*D. \end{aligned} \tag{3-33}$$

From Lemma 3.3, (3-23) and (3-33), we have $\widetilde{\Box}^{(q)}_b \equiv A^{(q)}\Box^{(q)}_-$, $H^{(q)}_1\widetilde{\Box}^{(q)}_b \equiv H^{(q)}_1 A^{(q)}\Box^{(q)}_-$ and hence

$$\Box^{(q)}_- \equiv H^{(q)}_1\widetilde{\Box}^{(q)}_b \quad \text{near } \Sigma^+ \cap T^*D. \tag{3-34}$$

Let $E^{(q)} \in L^{-1}_{1/2,1/2}(D, T^{*0,q}X \boxtimes (T^{*0,q}X)^*)$ be as in Theorem 3.5. From (3-34), we have

$$\Box^{(q)}_- E^{(q)}A^{(q)} - I \equiv H^{(q)}_1\widetilde{\Box}^{(q)}_b E^{(q)}A^{(q)} - I \quad \text{near } \Sigma^+ \cap T^*D. \tag{3-35}$$

From (3-31), we have $H^{(q)}_1\widetilde{\Box}^{(q)}_b E^{(q)}A^{(q)} - I \equiv H^{(q)}_1(I+R)A^{(q)} - I$ and hence

$$H^{(q)}_1\widetilde{\Box}^{(q)}_b E^{(q)}A^{(q)} - I \equiv H^{(q)}_1 A^{(q)} - I \quad \text{near } \Sigma^+ \cap T^*D. \tag{3-36}$$

From (3-36) and (3-33), we get

$$H^{(q)}_1\widetilde{\Box}^{(q)}_b E^{(q)}A^{(q)} - I \equiv 0 \quad \text{near } \Sigma^+ \cap T^*D. \tag{3-37}$$

From (3-35), (3-36) and (3-37), we conclude that

$$\Box^{(q)}_- E^{(q)}A^{(q)} = I + r,$$

where $r \in L^1_{1/2,1/2}(D, T^{*0,q}X \boxtimes (T^{*0,q}X)^*)$ with $r \equiv 0$ near $\Sigma^+ \cap T^*D$. Since $\Box^{(q)}_-$ is elliptic outside Σ^+, we can find a properly supported operator $r_1 \in L^1_{1/2,1/2}(D, T^{*0,q}X \boxtimes (T^{*0,q}X)^*)$ such that $\Box^{(q)}_- r_1 \equiv -r$ on D. Let $G^{(q)} \in L^0_{1/2,1/2}(D, T^{*0,q}X \boxtimes (T^{*0,q}X)^*)$ be a properly supported operator so that $G^{(q)} \equiv E^{(q)}A^{(q)} + r_1$ on D. Hence $\Box^{(q)}_- G^{(q)} \equiv I$ on D. □

4. Parametrices for the $\bar\partial$-Neumann Laplacian outside the critical degree

In this section we consider boundary points where the Levi form is nondegenerate of constant signature (n_-, n_+) on D. In the neighborhood of such points we construct a local parametrix of the $\bar\partial$-Neumann Laplacian on $(0,q)$-forms with $q \neq n_-$.

We briefly recall the global situation [Chen and Shaw 2001; Folland and Kohn 1972; Kohn 1963; 1964]. If $Z(q)$ holds at each point of the boundary X, then $\operatorname{Ker}\Box^{(q)}$ is a finite-dimensional subspace of $\Omega^{0,q}(\overline{M})$, $\Box^{(q)}$ has closed range in L^2 and the Bergman projector $B^{(q)}$ on $\operatorname{Ker}\Box^{(q)}$ is a smoothing operator on $\overline{M}$. Moreover, there exists a continuous partial inverse $N^{(q)} : L^2(M, T^{*0,q}M) \to \operatorname{Dom} \Box^{(q)}$ of $\Box^{(q)}$, called the Neumann operator, such that we have the Hodge decomposition at the operator level, $\Box^{(q)} N^{(q)} + B^{(q)} = I$ on $L^2_{(0,q)}(M)$ and $N^{(q)}\Box^{(q)} + B^{(q)} = I$ on $\operatorname{Dom} \Box^{(q)}$. Moreover, the Neumann operator $N^{(q)}$ maps continuously the Sobolev spaces H^s to H^{s+1} for every $s \in \mathbb{Z}$, and maps the space of smooth forms on $\overline{M}$ into itself. If the Levi form is nondegenerate of signature (n_-, n_+) on X, then $Z(q)$ holds if and only if $q \neq n_-$. We will show in this section a local version of these global results, in which case the Neumann operator will be a local parametrix of the $\bar{\partial}$-Neumann operator.

Let D be a local coordinate patch of X with local coordinates $x = (x_1, \dots, x_{2n-1})$. Then $\hat{x} := (x_1, \dots, x_{2n-1}, \rho)$ are local coordinates of M' defined in an open set U of M' with $U \cap X = D$. Until further notice, we work on U.

Let $\widehat{\mathscr{E}}'(U \cap \overline{M}, T^{*0,q}M')$ be the space of continuous linear forms from $\Omega^{0,q}(U \cap \overline{M})$ to $\mathbb{C}$. Let $F : \Omega^{0,q}_c(U \cap \overline{M}) \to \mathscr{D}'(U \cap \overline{M}, T^{*0,q}M')$ be a continuous operator. We say that F is properly supported on $U \cap \overline{M}$ if, for every $\chi \in \mathscr{C}^\infty_c(U \cap \overline{M})$, there are $\chi_1 \in \mathscr{C}^\infty_c(U \cap \overline{M})$, $\chi_2 \in \mathscr{C}^\infty_c(U \cap \overline{M})$, such that $F\chi u = \chi_2 F u$, $\chi F u = F\chi_1 u$ for every $u \in \Omega^{0,q}_c(U \cap \overline{M})$. We say that F is smoothing away the diagonal on $U \cap \overline{M}$ if, for every $\chi, \chi_1 \in \mathscr{C}^\infty_c(U \cap \overline{M})$ with $\operatorname{supp} \chi \cap \operatorname{supp} \chi_1 = \varnothing$, we have

$$\chi F \chi_1 \equiv 0 \mod \mathscr{C}^\infty((U \times U) \cap (\overline{M} \times \overline{M})).$$

Lemma 4.1. *Let* $\tau_1 \in \mathscr{C}^\infty(X)$, $\tau \in \mathscr{C}^\infty(\overline{M})$ *with* $\operatorname{supp} \tau \cap \operatorname{supp} \tau_1 = \varnothing$. *Then,*

$$\tau \widetilde{P} \tau_1 \equiv 0 \mod \mathscr{C}^\infty(\overline{M} \times X).$$

Proof. Since $\gamma \tau \widetilde{P} \tau_1 = \tau|_X \tau_1 = 0$, we have $\widetilde{P} \gamma \tau \widetilde{P} \tau_1 = 0$. From this observation and (3-8), we have

$$\tau \widetilde{P} \tau_1 = (D^{(q)} \widetilde{\Box}^{(q)}_f + \widetilde{P}\gamma) \tau \widetilde{P} \tau_1 = D^{(q)} \widetilde{\Box}^{(q)}_f \tau \widetilde{P} \tau_1 = -D^{(q)} [\tau, \widetilde{\Box}^{(q)}_f] \widetilde{P} \tau_1. \tag{4-1}$$

By (3-7) the operator

$$D^{(q)} [\tau, \widetilde{\Box}^{(q)}_f] : H^s(\overline{M}, T^{*0,q}M') \to H^{s+1}(\overline{M}, T^{*0,q}M')$$

is continuous, for every $s \in \mathbb{Z}$. Using this observation, (3-6) and (4-1), we have

$$\tau \widetilde{P} \tau_1 : H^s(X, T^{*0,q}M') \to H^{s+\frac{3}{2}}(\overline{M}, T^{*0,q}M')$$

is continuous for every $s \in \mathbb{Z}$. We have proved that, for any $\tilde{\tau} \in \mathscr{C}^\infty(\overline{M})$ with $\operatorname{supp} \tilde{\tau} \cap \operatorname{supp} \tau_1 = \varnothing$,

$$\tilde{\tau} \widetilde{P} \tau_1 : H^s(X, T^{*0,q}M') \to H^{s+\frac{3}{2}}(\overline{M}, T^{*0,q}M') \tag{4-2}$$

is continuous for every $s \in \mathbb{Z}$. Let $\tilde{\tau} \in \mathscr{C}^\infty(\overline{M})$ with $\tilde{\tau} = 1$ near $\operatorname{supp} \tau$ and $\operatorname{supp} \tilde{\tau} \cap \operatorname{supp} \tau_1 = \varnothing$. From (4-1), we have

$$\tau \widetilde{P} \tau_1 = D^{(q)} [\tau, \widetilde{\Box}^{(q)}_f] \tilde{\tau} \widetilde{P} \tau_1. \tag{4-3}$$

From (4-3), (4-2) and (3-8), $\tau \widetilde{P} \tau_1 : H^s(X, T^{*0,q}M') \to H^{s+5/2}(\overline{M}, T^{*0,q}M')$ is continuous for every $s \in \mathbb{Z}$. Continuing in this way, we conclude that

$$\tau \widetilde{P} \tau_1 : H^s(X, T^{*0,q}M') \to H^{s+\frac{2N+1}{2}}(\overline{M}, T^{*0,q}M')$$

is continuous for every $s \in \mathbb{Z}$ and $N > 0$. The lemma follows. $\square$

From Lemma 4.1 we obtain the following result for the adjoint $\widetilde{P}^*$ given by (3-9)

Lemma 4.2. *Let* $\tau_1 \in \mathscr{C}^\infty(X)$, $\tau \in \mathscr{C}^\infty(\overline{M})$ *with* $\operatorname{supp}\tau \cap \operatorname{supp}\tau_1 = \varnothing$. *Then,*

$$\tau_1 \widetilde{P}^* \tau \equiv 0 \bmod \mathscr{C}^\infty(X \times \overline{M}).$$

We come back to our situation. Until further notice, we assume that the Levi form is nondegenerate of constant signature (n_-, n_+) on $D \subset X$. In the following theorem we construct a local parametrix $N^{(q)}$ for the $\bar\partial$-Neumann Laplacian on $(0,q)$-forms for $q \neq n_-$.

Theorem 4.3. *We assume that the Levi form is nondegenerate of constant signature* (n_-, n_+) *on* D *and let* $q \neq n_-$. *Then there exist a properly supported operator* $N^{(q)}$ *on* $U \cap \overline{M}$ *that is continuous for every* $s \in \mathbb{Z}$ *between*

$$N^{(q)} : H^s_{\mathrm{loc}}(U \cap \overline{M}, T^{*0,q}M') \to H^{s+1}_{\mathrm{loc}}(U \cap \overline{M}, T^{*0,q}M') \quad \textit{for every } s \in \mathbb{Z}, \tag{4-4}$$

and such that $N^{(q)}u$ *satisfies the* $\bar\partial$*-Neumann conditions*

$$(\bar\partial\rho)^{\wedge,*}\gamma N^{(q)}u|_D = 0, \quad u \in \Omega^{0,q}(U \cap \overline{M}), \tag{4-5}$$

$$(\bar\partial\rho)^{\wedge,*}\gamma\bar\partial N^{(q)}u|_D = 0, \quad u \in \Omega^{0,q}(U \cap \overline{M}), \tag{4-6}$$

$$\Box^{(q)}_f N^{(q)} = I + F^{(q)} \quad \textit{on } \Omega^{0,q}_c(U \cap M), \tag{4-7}$$

where $F^{(q)} : \mathscr{D}'(U \cap M) \to \Omega^{0,q}(U \cap M)$ *is a properly supported smoothing operator on* $U \cap \overline{M}$.

Hence for $u \in \Omega^{0,q}_c(U \cap M)$ we have $N^{(q)}u \in \operatorname{Dom}\Box^{(q)}$ and $\Box^{(q)}N^{(q)} = I + F^{(q)}$, with $F^{(q)}$ a smoothing operator on $U \cap \overline{M}$.

Proof. Since $\Box^{(q)}_f$ is an elliptic operator on M', we can find a properly supported continuous operator

$$N^{(q)}_1 : H^s_{\mathrm{loc}}(U \cap \overline{M}, T^{*0,q}M') \to H^{s+2}_{\mathrm{loc}}(U \cap \overline{M}, T^{*0,q}M') \quad \text{for every } s \in \mathbb{Z}$$

such that $N^{(q)}_1$ is smoothing away the diagonal on $U \cap \overline{M}$ and

$$\Box^{(q)}_f N^{(q)}_1 = I + F_1 \quad \text{on } \Omega^{0,q}_c(U \cap M'), \tag{4-8}$$

where $F_1 \equiv 0 \bmod \mathscr{C}^\infty((U \times U) \cap (\overline{M} \times \overline{M}))$.

For $u \in \Omega^{0,q}(U \cap \overline{M})$ the form $N^{(q)}_1 u$ doesn't necessarily satisfy the $\bar\partial$-Neumann conditions (4-5), (4-6). We will now construct corrections $N^{(q)}_j$, $j = 2, \dots, 7$, and finally $N^{(q)}$, starting with $N^{(q)}_1$, such that at the end the operator $N^{(q)}$ satisfies (4-4)–(4-8). Consider, for every $s \in \mathbb{Z}$,

$$N^{(q)}_2 := N^{(q)}_1 - \widetilde{P}\gamma N^{(q)}_1 : H^s_c(U \cap \overline{M}, T^{*0,q}M') \to H^{s+2}_{\mathrm{loc}}(U \cap \overline{M}, T^{*0,q}M').$$

From (3-5) and (4-8), we see that

$$\gamma N^{(q)}_2 u|_D = 0 \quad \text{for every } u \in \Omega^{0,q}_c(U \cap \overline{M}) \tag{4-9}$$

and

$$\Box^{(q)}_f N^{(q)}_2 = I + F_2 \quad \text{on } \Omega^{0,q}_c(U \cap M'), \tag{4-10}$$

where $F_2 \equiv 0 \mod \mathscr{C}^\infty((U\times U)\cap(\overline{M}\times\overline{M}))$. From Lemma 4.1, it is not difficult to check that $N_2^{(q)}$ is smoothing away the diagonal on $U\cap\overline{M}$. Hence, we can find a properly supported continuous operator

$$N_3^{(q)} : H^s_{\mathrm{loc}}(U\cap\overline{M}, T^{*0,q}M') \to H^{s+2}_{\mathrm{loc}}(U\cap\overline{M}, T^{*0,q}M') \quad \text{for every } s\in\mathbb{Z}$$

such that

$$N_3^{(q)} \equiv N_2^{(q)} \mod \mathscr{C}^\infty((U\times U)\cap(\overline{M}\times\overline{M})). \tag{4-11}$$

From (4-9) and (4-11), we conclude that

$$\gamma N_3^{(q)} \equiv 0 \mod \mathscr{C}^\infty((U\times U)\cap(X\times\overline{M})). \tag{4-12}$$

Let $E^{(q)} \equiv 0 \mod \mathscr{C}^\infty((U\times U)\cap(\overline{M}\times\overline{M}))$ be any smoothing properly supported extension of $\gamma N_3^{(q)}$, that is, $\gamma E^{(q)}u|_D = \gamma N_3^{(q)}u|_D$, for every $u\in\Omega^{0,q}(U\cap\overline{M})$ and $E^{(q)}$ is properly supported on $U\cap\overline{M}$. For every $s\in\mathbb{Z}$ let

$$N_4^{(q)} := N_3^{(q)} - E^{(q)} : H^s_{\mathrm{loc}}(U\cap\overline{M}, T^{*0,q}M') \to H^{s+2}_{\mathrm{loc}}(U\cap\overline{M}, T^{*0,q}M'). \tag{4-13}$$

Then $N_4^{(q)}$ is properly supported on $U\cap\overline{M}$ and

$$\begin{aligned} &\gamma N_4^{(q)}u|_D = 0 && \text{for every } u\in\Omega^{0,q}(U\cap\overline{M}),\\ &\Box_f^{(q)} N_4^{(q)} = I + F_3 && \text{on } \Omega_c^{0,q}(U\cap M), \end{aligned} \tag{4-14}$$

where $F_3 \equiv 0 \mod \mathscr{C}^\infty((U\times U)\cap(\overline{M}\times\overline{M}))$. Let $G^{(q)}\in L^0_{1/2,1/2}(D, T^{*0,q}X\boxtimes(T^{*0,q}X)^*)$ be as in Theorem 3.6. Put, for every $s\in\mathbb{Z}$,

$$\begin{aligned} N_5^{(q)} &: H^s_c(U\cap\overline{M}, T^{*0,q}M') \to H^{s+1}_{\mathrm{loc}}(U\cap\overline{M}, T^{*0,q}M'),\\ N_5^{(q)} &:= N_4^{(q)} - \widetilde{P}G^{(q)}(\bar\partial\rho)^{\wedge,*}\gamma\bar\partial N_4^{(q)}. \end{aligned} \tag{4-15}$$

From Theorem 3.6, (3-11), (3-32) and (4-14), we can check that

$$\begin{aligned} &(\bar\partial\rho)^{\wedge,*}\gamma N_5^{(q)}u|_D = 0 \quad \text{for every } u\in\Omega^{0,q}(U\cap\overline{M}),\\ &(\bar\partial\rho)^{\wedge,*}\gamma\bar\partial N_5^{(q)} \equiv 0 \mod \mathscr{C}^\infty((U\times U)\cap(X\times\overline{M})),\\ &\Box_f^{(q)} N_5^{(q)} = I + F_4 \quad \text{on } \Omega_c^{0,q}(U\cap M), \end{aligned} \tag{4-16}$$

where $F_4 \equiv 0 \mod \mathscr{C}^\infty((U\times U)\cap(\overline{M}\times\overline{M}))$. We explain the first equation in (4-16). From (4-14), we have $(\bar\partial\rho)^{\wedge,*}\gamma N_5^{(q)}u = -(\bar\partial\rho)^{\wedge,*}G^{(q)}(\bar\partial\rho)^{\wedge,*}\gamma\bar\partial N_4^{(q)}u = 0$ since $G^{(q)}$ maps $\Omega^{0,q}(X)$ to $\Omega^{0,q}(X)$. It is not difficult to check that $N_5^{(q)}$ is smoothing away the diagonal on $U\cap\overline{M}$. Hence, we can find a properly supported continuous operator

$$N_6^{(q)} : H^s_{\mathrm{loc}}(U\cap\overline{M}, T^{*0,q}M') \to H^{s+1}_{\mathrm{loc}}(U\cap\overline{M}, T^{*0,q}M') \quad \text{for every } s\in\mathbb{Z}$$

such that

$$N_5^{(q)} \equiv N_6^{(q)} \mod \mathscr{C}^\infty((U\times U)\cap(\overline{M}\times\overline{M})). \tag{4-17}$$

Let $R^{(q)} \equiv 0 \bmod \mathscr{C}^\infty((U \times U) \cap (\overline{M} \times \overline{M}))$ be any smoothing properly supported extension of $2(\bar\partial\rho)^\wedge(\bar\partial\rho)^{\wedge,*}\gamma N_6^{(q)}$. For every $s \in \mathbb{Z}$ put

$$N_7^{(q)} := N_6^{(q)} - R^{(q)} : H^s_{\mathrm{loc}}(U \cap \overline{M}, T^{*0,q}M') \to H^{s+1}_{\mathrm{loc}}(U \cap \overline{M}, T^{*0,q}M'). \tag{4-18}$$

From (3-17), we have

$$\begin{aligned}(\bar\partial\rho)^{\wedge,*}\gamma N_7^{(q)} &= (\bar\partial\rho)^{\wedge,*}\gamma N_6^{(q)} - (\bar\partial\rho)^{\wedge,*}\gamma R^{(q)}\\ &= (\bar\partial\rho)^{\wedge,*}\gamma N_6^{(q)} - 2(\bar\partial\rho)^{\wedge,*}(\bar\partial\rho)^\wedge(\bar\partial\rho)^{\wedge,*}\gamma N_6^{(q)}\\ &= (\bar\partial\rho)^{\wedge,*}\gamma N_6^{(q)} - (\bar\partial\rho)^{\wedge,*}\gamma N_6^{(q)} = 0.\end{aligned} \tag{4-19}$$

From (4-16) and (4-19), we have

$$\begin{gathered}(\bar\partial\rho)^{\wedge,*}\gamma N_7^{(q)}u|_D = 0 \quad \text{for every } u \in \Omega^{0,q}(U \cap \overline{M}),\\ (\bar\partial\rho)^{\wedge,*}\gamma\bar\partial N_7^{(q)} \equiv 0 \bmod \mathscr{C}^\infty((U \times U) \cap (X \times \overline{M})),\\ \Box_f^{(q)} N_7^{(q)} = I + F_5 \quad \text{on } \Omega_c^{0,q}(U \cap M),\end{gathered} \tag{4-20}$$

where $F_5 \equiv 0 \bmod \mathscr{C}^\infty((U \times U) \cap (\overline{M} \times \overline{M}))$. Let $J^{(q)}$ be any smoothing properly supported extension of $(\bar\partial\rho)^{\wedge,*}\gamma\bar\partial N_7^{(q)}$. Let $\chi \in \mathscr{C}_c^\infty((-\varepsilon, \varepsilon))$ with $\chi \equiv 1$ near 0, where $\varepsilon > 0$ is a sufficiently small constant. For every $s \in \mathbb{Z}$ put

$$N^{(q)} := N_7^{(q)} - 2\chi(\rho)\rho J^{(q)} : H^s_{\mathrm{loc}}(U \cap \overline{M}, T^{*0,q}M') \to H^{s+1}_{\mathrm{loc}}(U \cap \overline{M}, T^{*0,q}M'). \tag{4-21}$$

It is not difficult to see that $N^{(q)}$ is properly supported on $U \cap \overline{M}$,

$$N^{(q)} \equiv N_7^{(q)} \bmod \mathscr{C}^\infty((U \times U) \cap (\overline{M} \times \overline{M}))$$

and

$$(\bar\partial\rho)^{\wedge,*}\gamma N^{(q)}u|_D = (\bar\partial\rho)^{\wedge,*}\gamma N_7^{(q)}u|_D = 0 \quad \text{for every } u \in \Omega^{0,q}(U \cap \overline{M}).$$

From (3-17), we have, for every $u \in \Omega^{0,q}(U \cap M)$,

$$\begin{aligned}(\bar\partial\rho)^{\wedge,*}\gamma\bar\partial N^{(q)}u|_D &= (\bar\partial\rho)^{\wedge,*}\gamma\bar\partial N_7^{(q)}u|_D - 2(\bar\partial\rho)^{\wedge,*}(\bar\partial\rho)^\wedge\gamma J^{(q)}u|_D\\ &= (\bar\partial\rho)^{\wedge,*}\gamma\bar\partial N_7^{(q)}u|_D - 2(\bar\partial\rho)^{\wedge,*}(\bar\partial\rho)^\wedge(\bar\partial\rho)^{\wedge,*}\gamma\bar\partial N_7^{(q)}u|_D\\ &= (\bar\partial\rho)^{\wedge,*}\gamma\bar\partial N_7^{(q)}u|_D - (\bar\partial\rho)^{\wedge,*}\gamma\bar\partial N_7^{(q)}u|_D = 0.\end{aligned} \tag{4-22}$$

We have proved that $N^{(q)}$ satisfies (4-5), (4-6) and (4-7). The theorem follows. □

Let $N^{(q)}$ be as in Theorem 4.3 and let $(N^{(q)})^* : \Omega_c^{0,q}(U \cap M) \to \mathscr{D}'(U \cap M, T^{*0,q}M')$ be the formal adjoint of $N^{(q)}$ given by

$$((N^{(q)})^*u \mid v)_M = (u \mid N^{(q)}v)_M \quad \text{for every } u, v \in \Omega_c^{0,q}(U \cap M).$$

The following result shows that $N^{(q)}$ is formally self-adjoint up to a smoothing operator.

Lemma 4.4. *With the assumptions and notation used above, we have*

$$(N^{(q)})^*u = N^{(q)}u + H^{(q)}u \quad \textit{for every } u \in \Omega_c^{0,q}(U \cap M), \tag{4-23}$$

where $H^{(q)} : \mathscr{D}'(U \cap M, T^{*0,q}M') \to \Omega^{0,q}(U \cap M)$ *is a properly supported continuous operator on* $U \cap \overline{M}$ *with* $H^{(q)} \equiv 0 \bmod \mathscr{C}^\infty((U \times U) \cap (\overline{M} \times \overline{M}))$.

Proof. Let $u, v \in \Omega_c^{0,q}(U \cap M)$. From (4-7), we have

$$\begin{aligned}((N^{(q)})^*u \mid v)_M &= ((N^{(q)})^*(\Box_f^{(q)} N^{(q)} - F^{(q)})u \mid v)_M \\ &= (\Box_f^{(q)} N^{(q)}u \mid N^{(q)}v)_M - (F^{(q)}u \mid N^{(q)}v)_M.\end{aligned} \tag{4-24}$$

From (4-5) and (4-6), we can integrate by parts and get

$$(\Box_f^{(q)} N^{(q)}u \mid N^{(q)}v)_M = (N^{(q)}u \mid \Box_f^{(q)} N^{(q)}v)_M = (N^{(q)}u \mid (I + F^{(q)})v)_M, \tag{4-25}$$

where we used (4-7). From (4-24) and (4-25), we deduce that

$$((N^{(q)})^*u \mid v)_M = ((N^{(q)} + ((F^{(q)})^* N^{(q)}))u \mid v)_M - (u \mid (F^{(q)})^* N^{(q)}v)_M, \tag{4-26}$$

where $(F^{(q)})^* : \Omega_c^{0,q}(U \cap M) \to \mathscr{D}'(U \cap M, T^{*0,q}M')$ is the formal adjoint of $F^{(q)}$ with respect to $(\cdot \mid \cdot)_M$. It is clear that $(F^{(q)})^*$ is a properly supported continuous operator on $U \cap \overline{M}$ with $(F^{(q)})^* \equiv 0 \bmod \mathscr{C}^\infty((U \times U) \cap (\overline{M} \times \overline{M}))$.

It is not difficult to check that $(F^{(q)})^* N^{(q)}$ is a properly supported continuous operator on $U \cap \overline{M}$ with $(F^{(q)})^* N^{(q)} \equiv 0 \bmod \mathscr{C}^\infty((U \times U) \cap (\overline{M} \times \overline{M}))$. Let

$$((F^{(q)})^* N^{(q)})^* : \Omega_c^{0,q}(U \cap M) \to \mathscr{D}'(U \cap M, T^{*0,q}M')$$

be the formal adjoint of $(F^{(q)})^* N^{(q)}$ with respect to $(\cdot \mid \cdot)_M$. Then $((F^{(q)})^* N^{(q)})^*$ is a properly supported continuous operator on $U \cap \overline{M}$ with

$$((F^{(q)})^* N^{(q)})^* \equiv 0 \bmod \mathscr{C}^\infty((U \times U) \cap (\overline{M} \times \overline{M})).$$

From this observation and (4-26), we have

$$((N^{(q)})^*u \mid v)_M = \big((N^{(q)} + (F^{(q)})^* N^{(q)} - ((F^{(q)})^* N^{(q)})^*)u \mid v\big)_M.$$

Relation (4-23) follows. ☐

From (4-23), we can extend $(N^{(q)})^*$ to

$$(N^{(q)})^* : L^2_{\mathrm{loc}}(U \cap \overline{M}, T^{*0,q}M') \to L^2_{\mathrm{loc}}(U \cap \overline{M}, T^{*0,q}M') \quad \text{for every } s \in \mathbb{Z}$$

as a properly supported continuous operator on $U \cap \overline{M}$ and we have

$$(N^{(q)})^*u = N^{(q)}u + H^{(q)}u \quad \text{for every } u \in L^2_{\mathrm{loc}}(U \cap M, T^{*0,q}M'), \tag{4-27}$$

where $H^{(q)}$ is as in (4-23). Moreover, for every $g \in L^2_c(U \cap M, T^{*0,q}M')$ and $u \in L^2_{\mathrm{loc}}(U \cap M, T^{*0,q}M')$, we have

$$((N^{(q)})^*u \mid g)_M = (u \mid N^{(q)}g)_M, \quad ((N^{(q)})^*g \mid u)_M = (g \mid N^{(q)}u)_M. \tag{4-28}$$

We can now improve Theorem 4.3.

Theorem 4.5. *With the assumptions and notation used above, let $q \neq n_-$. We have*

$$N^{(q)}\Box^{(q)}u = u + F_1^{(q)}u \quad \text{on } U\cap M \text{ for every } u \in \operatorname{Dom}\Box^{(q)}, \tag{4-29}$$

$$\Box_f^{(q)}N^{(q)}u = u + F_2^{(q)}u \quad \text{on } U\cap M \text{ for every } u \in \Omega^{0,q}(U\cap \overline{M}), \tag{4-30}$$

where $F_1^{(q)}, F_2^{(q)} : \mathscr{D}'(U\cap M) \to \Omega^{0,q}(U\cap M)$ are properly supported smoothing operators on $U\cap\overline{M}$.

Remark 4.6. Let $u \in \operatorname{Dom}\Box^{(q)}$. By (4-29) we have, for every $g \in \Omega_c^{0,q}(U\cap M)$,

$$(N^{(q)}\Box^{(q)}u \mid g)_M = (u + F_1^{(q)}u \mid g)_M. \tag{4-31}$$

Since $N^{(q)}$ and $F_1^{(q)}$ are properly supported operators on $U\cap\overline{M}$, (4-31) makes sense. For $u \in \Omega^{0,q}(U\cap M)$, equation (4-30) means that, for every $g \in \Omega_c^{0,q}(U\cap M)$, we have

$$(\Box_f^{(q)}N^{(q)}u \mid g)_M = (u + F_2^{(q)}u \mid g)_M. \tag{4-32}$$

Proof of Theorem 4.5. Let $u \in \operatorname{Dom}\Box^{(q)}$. Then, $\Box^{(q)}u \in L^2_{(0,q)}(M) \subset L^2_{\mathrm{loc}}(U\cap\overline{M}, T^{*0,q}M')$. Let $g \in \Omega_c^{0,q}(U\cap M)$. From (4-27) and (4-28), we have

$$\begin{aligned}(N^{(q)}\Box^{(q)}u \mid g)_M &= ((N^{(q)})^* - H^{(q)})\Box^{(q)}u \mid g)_M\\ &= (\Box^{(q)}u \mid N^{(q)}g)_M - (H^{(q)}\Box_f^{(q)}u \mid g)_M.\end{aligned} \tag{4-33}$$

Since $u \in \operatorname{Dom}\Box^{(q)}$ and by (4-5), (4-6), $N^{(q)}g \in \operatorname{Dom}\Box^{(q)}$, we can integrate by parts and get

$$(\Box^{(q)}u \mid N^{(q)}g)_M = (u \mid \Box^{(q)}N^{(q)}g)_M = (u \mid (I + F^{(q)})g)_M = (u + (F^{(q)})^*u \mid g)_M, \tag{4-34}$$

where $F^{(q)}$ is as in (4-7) and $(F^{(q)})^*$ is the formal adjoint of $F^{(q)}$. From (4-33) and (4-34), we have

$$(N^{(q)}\Box^{(q)}u \mid g)_M = (u + (F^{(q)})^*u - H^{(q)}\Box_f^{(q)}u \mid g)_M. \tag{4-35}$$

From (4-35), we get (4-29) with $F_1^{(q)} = (F^{(q)})^* - H^{(q)}\Box_f^{(q)}$.

Let $u \in \Omega^{0,q}(U\cap\overline{M})$ and let $g \in \Omega_c^{0,q}(U\cap M)$. From (4-27), (4-28), (4-29), and since $N^{(q)}$ is properly supported on $U\cap\overline{M}$, we have

$$\begin{aligned}(\Box_f^{(q)}N^{(q)}u \mid g)_M &= (N^{(q)}u \mid \Box_f^{(q)}g)_M = (u \mid (N^{(q)})^*\Box_f^{(q)}g)_M\\ &= (u \mid (N^{(q)} + H^{(q)})\Box_f^{(q)}g)_M = (u \mid g + F_1^{(q)}g + H^{(q)}\Box_f^{(q)}g)_M\\ &= (u + (F_1^{(q)})^*u + (H^{(q)}\Box_f^{(q)})^*u \mid g)_M,\end{aligned} \tag{4-36}$$

where $(F_1^{(q)})^*$ and $(H^{(q)}\Box_f^{(q)})^*$ are the formal adjoints of $F_1^{(q)}$ and $H^{(q)}\Box_f^{(q)}$ respectively. From (4-36), we get (4-30) with $F_2^{(q)} = (F_1^{(q)})^* + (H^{(q)}\Box_f^{(q)})^*$. □

From Theorems 4.3 and 4.5, we get the main result of this section about the local parametrix of the $\bar\partial$-Neumann Laplacian.

Theorem 4.7. *Let U be an open set of M' with $U \cap X \neq \varnothing$. Suppose that the Levi form is nondegenerate of constant signature (n_-, n_+) on $U \cap X$. Let $q \neq n_-$. We can find properly supported continuous operators on $U \cap \overline{M}$:*

$$N^{(q)} : H^s_{\mathrm{loc}}(U \cap \overline{M}, T^{*0,q}M') \to H^{s+1}_{\mathrm{loc}}(U \cap \overline{M}, T^{*0,q}M') \quad \textit{for every } s \in \mathbb{Z}$$

such that (4-5), (4-6), (4-23), (4-28), (4-29) *and* (4-30) *hold.*

5. Microlocal Hodge decomposition in the critical degree

In this section we will construct a local parametrix of $N^{(q)}$ of the $\bar\partial$-Neumann Laplacian acting on $(0, q)$-forms and a local approximate Bergman operator $\Pi^{(q)}$ in the critical degree $q = n_-$.

We briefly recall the global situation [Folland and Kohn 1972, (3.1.7)–(3.1.19)]. Assume that $Z(q-1)$ and $Z(q+1)$ hold everywhere on X (but $Z(q)$ does not necessarily hold). Then $\Box^{(q)}$ is bounded away from zero on $(\operatorname{Ker} \Box^{(q)})^\perp$, so $\Box^{(q)}$ has closed range in L^2 and one can define a bounded operator $N^{(q)} : L^2(M, T^{*0,q}M') \to \operatorname{Dom} \Box^{(q)}$ (the $\bar\partial$-Neumann operator) such that

$$\begin{gathered} u = \bar\partial\bar\partial^* N^{(q)}u + \bar\partial^*\bar\partial N^{(q)}u + B^{(q)}u, \quad u \in L^2(M, T^{*0,q}M'), \\ B^{(q)}N^{(q)} = N^{(q)}B^{(q)} = 0, \quad N^{(q)}\Box^{(q)} = \Box^{(q)}N^{(q)} = I - B^{(q)} \quad \text{on } \operatorname{Dom} \Box^{(q)}, \\ B^{(q)} = I - \bar\partial N^{(q-1)}\bar\partial^* - \bar\partial^* N^{(q+1)}\bar\partial \quad \text{on } \operatorname{Dom} \bar\partial \cap \operatorname{Dom} \bar\partial^*, \\ B^{(q)}(\operatorname{Dom} \bar\partial^* \cap \Omega^{0,q}(\overline{M})) \subset \operatorname{Dom} \Box^{(q)} \cap \Omega^{0,q}(\overline{M}). \end{gathered} \tag{5-1}$$

If the Levi form is nondegenerate of signature (n_-, n_+) on an open set $D \subset X$, then $Z(q)$ holds on D if and only if $q \neq n_-$. We will give in this section a (micro-)local version of the above global results in the critical degree $q = n_-$, in which case the Neumann operator will be a local parametrix of the $\bar\partial$-Neumann operator and the Bergman projection $B^{(q)}$ will be replaced by an approximate Bergman projection $\Pi^{(q)}$.

5.1. ***The parametrix and the approximate Bergman operator.*** We recall the following lemma about integration by parts.

Lemma 5.1 [Folland and Kohn 1972, p. 13]. *For all $f \in \Omega^{0,q}(\overline{M})$, $g \in \Omega^{0,q+1}(\overline{M})$, we have*

$$(g \mid \bar\partial f)_M = (\bar\partial^*_f g \mid f)_M + ((\bar\partial\rho)^{\wedge,*}\gamma g \mid \gamma f)_X. \tag{5-2}$$

Let D be a local coordinate patch of X with local coordinates $x = (x_1, \dots, x_{2n-1})$. Then, $\hat{x} := (x_1, \dots, x_{2n-1}, \rho)$ are local coordinates of M' defined in an open set U of M' with $U \cap X = D$. Until further notice, we work on U.

Lemma 5.2. *Let $u \in \Omega^{0,q}(U \cap \overline{M})$. Assume that $(\bar\partial\rho)^{\wedge,*}\gamma u|_D = 0$. Then,*

$$(\bar\partial\rho)^{\wedge,*}\gamma\bar\partial^*_f u|_D = 0. \tag{5-3}$$

Proof. Let $g \in \Omega^{0,q-2}_c(U \cap \overline{M})$. From (5-2), we have

$$(\bar\partial^*_f u \mid \bar\partial g)_M = ((\bar\partial^*_f)^2 u \mid g)_M + ((\bar\partial\rho)^{\wedge,*}\gamma\bar\partial^*_f u \mid \gamma g)_X = ((\bar\partial\rho)^{\wedge,*}\gamma\bar\partial^*_f u \mid \gamma g)_X. \tag{5-4}$$

On the other hand, from (5-2) again, we have

$$0=(u\mid\bar\partial^2 g)_M=(\bar\partial^*_f u\mid\bar\partial g)_M+((\bar\partial\rho)^{\wedge,*}\gamma u\mid\gamma\bar\partial g)_X=(\bar\partial^*_f u\mid\bar\partial g)_M \tag{5-5}$$

since $(\bar\partial\rho)^{\wedge,*}\gamma u|_D=0$. From (5-4) and (5-5), we conclude that

$$((\bar\partial\rho)^{\wedge,*}\gamma\bar\partial^*_f u\mid\gamma g)_X=0.$$

Since g is arbitrary, $(\bar\partial\rho)^{\wedge,*}\gamma\bar\partial^*_f u|_D=0$. □

We now assume that the Levi form is nondegenerate of constant signature (n_-,n_+) on $D=U\cap X$. Let $q=n_-$. Let $N^{(q+1)}$ and $N^{(q-1)}$ be local parametrices of the $\bar\partial$-Neumann Laplacian as in Theorem 4.7. We define, for every $s\in\mathbb{Z}$,

$$\begin{gathered}\widehat{N}^{(q)}:H^s_{\mathrm{loc}}(U\cap\overline{M},T^{*0,q}M')\to H^s_{\mathrm{loc}}(U\cap\overline{M},T^{*0,q}M'),\\ \widehat{N}^{(q)}:=\bar\partial^*_f(N^{(q+1)})^2\bar\partial+\bar\partial(N^{(q-1)})^2\bar\partial^*_f.\end{gathered}\tag{5-6}$$

Put

$$A^{0,q}(U\cap\overline{M}):=\{u\in\Omega^{0,q}(U\cap\overline{M}):(\bar\partial\rho)^{\wedge,*}\gamma u|_D=0\}=\operatorname{Dom}\bar\partial^*\cap\Omega^{0,q}(U\cap\overline{M}).\tag{5-7}$$

We define

$$\widehat{\Pi}^{(q)}:=I-\bar\partial^*_f N^{(q+1)}\bar\partial-\bar\partial N^{(q-1)}\bar\partial^*_f:A^{0,q}(U\cap\overline{M})\to\Omega^{0,q}(U\cap\overline{M}).\tag{5-8}$$

We show in Theorem 5.3 below that the operators $\widehat{N}^{(q)}$ and $\widehat{\Pi}^{(q)}$ provide a rough version of the microlocal Hodge decomposition. By (5-9) the operator $\widehat{N}^{(q)}$ satisfies the first $\bar\partial$-Neumann condition. However, by (5-10), the second $\bar\partial$-Neumann condition is satisfied only modulo a smoothing operator (analogously for $\widehat{\Pi}^{(q)}$ by (5-12)). In the sequel we will modify these operators in order to obtain operators $N^{(q)}$ (the parametrix of the $\bar\partial$-Neumann Laplacian) and $\Pi^{(q)}$ (the approximate Bergman projector) which satisfy exactly the $\bar\partial$-Neumann condition (see Theorems 5.9, 5.11, 5.23).

Theorem 5.3. *With the assumptions and notation above, let $q=n_-$. We have*

$$(\bar\partial\rho)^{\wedge,*}\gamma\widehat{N}^{(q)}u=0\quad\text{for every }u\in\Omega^{0,q}(U\cap\overline{M}),\tag{5-9}$$

$$(\bar\partial\rho)^{\wedge,*}\gamma\bar\partial\widehat{N}^{(q)}u=H^{(q)}_1u\quad\text{for every }u\in\Omega^{0,q}(U\cap\overline{M}),\tag{5-10}$$

$$\widehat{\Pi}^{(q)}u\in A^{0,q}(U\cap\overline{M})\quad\text{for every }u\in A^{0,q}(U\cap\overline{M}),\tag{5-11}$$

$$(\bar\partial\rho)^{\wedge,*}\gamma\bar\partial\widehat{\Pi}^{(q)}u=H^{(q)}_2u\quad\text{for every }u\in A^{0,q}(U\cap\overline{M}),\tag{5-12}$$

$$\Box^{(q)}_f\widehat{N}^{(q)}u+\widehat{\Pi}^{(q)}u=u+H^{(q)}_3u\quad\text{for every }u\in A^{0,q}(U\cap\overline{M}),\tag{5-13}$$

$$\bar\partial\widehat{\Pi}^{(q)}u=H^{(q)}_4u\quad\text{for every }u\in A^{0,q}(U\cap\overline{M})\cap\Omega^{0,q}_c(U\cap\overline{M}),\tag{5-14}$$

$$\bar\partial^*_f\widehat{\Pi}^{(q)}u=H^{(q)}_5u\quad\text{for every }u\in A^{0,q}(U\cap\overline{M})\cap\Omega^{0,q}_c(U\cap\overline{M}),\tag{5-15}$$

where $H^{(q)}_j$, $j=1,\dots,5$, are properly supported on $U\cap\overline{M}$ and

$$\begin{aligned}H^{(q)}_j&\equiv0\bmod\mathscr{C}^\infty((U\times U)\cap(X\times\overline{M})),\quad j=1,2,\\ H^{(q)}_j&\equiv0\bmod\mathscr{C}^\infty((U\times U)\cap(\overline{M}\times\overline{M})),\quad j=3,4,5.\end{aligned}$$

Proof. From (4-5), (4-6), Lemma 5.2 and the definitions of $\widehat{N}^{(q)}$, $\widehat{\Pi}^{(q)}$, we get (5-9) and (5-11). Let $u \in \Omega^{0,q}(U \cap \overline{M})$. From (4-30) and (5-6), we have

$$\begin{aligned}
&(\bar\partial\rho)^{\wedge,*}\gamma\bar\partial\widehat{N}^{(q)}u \\
&\qquad= (\bar\partial\rho)^{\wedge,*}\gamma\bar\partial\,\bar\partial^*_f(N^{(q+1)})^2\bar\partial u \\
&\qquad= (\bar\partial\rho)^{\wedge,*}\gamma\Box^{(q+1)}_f(N^{(q+1)})^2\bar\partial u-(\bar\partial\rho)^{\wedge,*}\gamma\bar\partial^*_f\bar\partial(N^{(q+1)})^2\bar\partial u \\
&\qquad= (\bar\partial\rho)^{\wedge,*}\gamma(I+F_2^{(q+1)})N^{(q+1)}\bar\partial u-(\bar\partial\rho)^{\wedge,*}\gamma\bar\partial^*_f\bar\partial(N^{(q+1)})^2\bar\partial u \\
&\qquad= (\bar\partial\rho)^{\wedge,*}\gamma F_2^{(q+1)}N^{(q+1)}\bar\partial u+(\bar\partial\rho)^{\wedge,*}\gamma N^{(q+1)}\bar\partial u-(\bar\partial\rho)^{\wedge,*}\gamma\bar\partial^*_f\bar\partial(N^{(q+1)})^2\bar\partial u, \qquad (5\text{-}16)
\end{aligned}$$

where $F_2^{(q+1)} \equiv 0 \mod \mathscr{C}^\infty((U\times U)\cap(\overline{M}\times\overline{M}))$ is as in (4-30). Again, from (4-5), (4-6) and Lemma 5.2, we see that

$$(\bar\partial\rho)^{\wedge,*}\gamma N^{(q+1)}\bar\partial u|_D = 0,$$
$$(\bar\partial\rho)^{\wedge,*}\gamma\bar\partial^*_f\bar\partial(N^{(q+1)})^2\bar\partial u|_D = 0.$$

From this observation, (5-16) and noticing that

$$(\bar\partial\rho)^{\wedge,*}\gamma F_2^{(q+1)}N^{(q+1)}\bar\partial \equiv 0 \mod \mathscr{C}^\infty((U\times U)\cap(X\times\overline{M})),$$

we get (5-10). We now prove (5-12). From (5-8), we have

$$\begin{aligned}
\bar\partial\widehat{\Pi}^{(q)} &= \bar\partial-\bar\partial\,\bar\partial^*_f N^{(q+1)}\bar\partial \\
&= \bar\partial-\Box^{(q+1)}_f N^{(q+1)}\bar\partial+\bar\partial^*_f\,\bar\partial N^{(q+1)}\bar\partial \\
&= -F_2^{(q+1)}\bar\partial+\bar\partial^*_f\,\bar\partial N^{(q+1)}\bar\partial, \qquad (5\text{-}17)
\end{aligned}$$

where $F_2^{(q+1)}$ is as in (4-30). Let $u \in A^{0,q}(U \cap \overline{M})$. From (5-17), we have

$$\bar\partial\widehat{\Pi}^{(q)}u = -F_2^{(q+1)}\bar\partial u+\bar\partial^*_f\,\bar\partial N^{(q+1)}\bar\partial u. \qquad (5\text{-}18)$$

From (4-6) and (5-3), we see that $(\bar\partial\rho)^{\wedge,*}\gamma\bar\partial^*_f\,\bar\partial N^{(q+1)}\bar\partial u|_D = 0$. From this observation and (5-18), we get (5-12).

Let $u \in A^{0,q}(U \cap \overline{M})$. From (4-30), (5-6) and (5-8), we have

$$\begin{aligned}
\Box^{(q)}_f\widehat{N}^{(q)}u &= \Box^{(q)}_f\big(\bar\partial^*_f(N^{(q+1)})^2\bar\partial+\bar\partial(N^{(q-1)})^2\bar\partial^*_f\big)u \\
&= \bar\partial^*_f\Box^{(q+1)}_f(N^{(q+1)})^2\bar\partial u+\bar\partial\Box^{(q-1)}_f(N^{(q-1)})^2\bar\partial^*_f u \\
&= \bar\partial^*_f(I+F_2^{(q+1)})N^{(q+1)}\bar\partial u+\bar\partial(I+F_2^{(q-1)})N^{(q-1)}\bar\partial^*_f u \\
&= \bar\partial^*_f N^{(q+1)}\bar\partial u+\bar\partial N^{(q-1)}\bar\partial^*_f u+\bar\partial^*_f F_2^{(q+1)}N^{(q+1)}\bar\partial u+\bar\partial F_2^{(q-1)}N^{(q-1)}\bar\partial^*_f u \\
&= (I-\widehat{\Pi}^{(q)})u+\big(\bar\partial^*_f F_2^{(q+1)}N^{(q+1)}\bar\partial+\bar\partial F_2^{(q-1)}N^{(q-1)}\bar\partial^*_f\big)u, \qquad (5\text{-}19)
\end{aligned}$$

where

$$F_2^{(q+1)} \equiv 0 \bmod \mathscr{C}^\infty((U\times U)\cap(\overline{M}\times\overline{M})),$$
$$F_2^{(q-1)} \equiv 0 \bmod \mathscr{C}^\infty((U\times U)\cap(\overline{M}\times\overline{M}))$$

are as in (4-30). It is clear that

$$\bar\partial_f^* F_2^{(q+1)} N^{(q+1)}\bar\partial + \bar\partial F_2^{(q-1)} N^{(q-1)}\bar\partial_f^* \equiv 0 \bmod \mathscr{C}^\infty((U\times U)\cap(\overline{M}\times\overline{M})).$$

From this observation and (5-19), we get (5-13).

Let $u\in A^{0,q}(U\cap\overline{M})\cap\Omega_c^{0,q}(U\cap\overline{M})$, from (4-29), (4-30), (5-6) and (5-8), we have

$$\begin{aligned}\bar\partial_f^*\widehat{\Pi}^{(q)}u &= \bar\partial_f^* u - \bar\partial_f^*(\bar\partial_f^* N^{(q+1)}\bar\partial u - \bar\partial N^{(q-1)}\bar\partial_f^* u)\\ &= \bar\partial_f^* u - \bar\partial_f^*\,\bar\partial N^{(q-1)}\bar\partial_f^* u\\ &= \bar\partial_f^* u - (\Box_f^{(q-1)} - \bar\partial\,\bar\partial_f^*)N^{(q-1)}\bar\partial_f^* u\\ &= \bar\partial_f^* u - (I+F_2^{(q-1)})\bar\partial_f^* u + \bar\partial\,\bar\partial_f^* N^{(q-1)}\bar\partial_f^* u\\ &= -F_2^{(q-1)}\bar\partial_f^* u + \bar\partial\,\bar\partial_f^* N^{(q-1)}\bar\partial_f^* u.\end{aligned}\tag{5-20}$$

For every $g\in A^{0,q}(U\cap\overline{M})\cap\Omega_c^{0,q}(U\cap\overline{M})$, from (4-6), (4-30) and (5-3), we have

$$\begin{aligned}(\bar\partial\rho)^{\wedge,*}\gamma\bar\partial\,\bar\partial_f^* N^{(q-1)}\bar\partial_f^* g &= (\bar\partial\rho)^{\wedge,*}\gamma(\Box_f^{(q-1)} - \bar\partial_f^*\,\bar\partial)N^{(q-1)}\bar\partial_f^* g\\ &= (\bar\partial\rho)^{\wedge,*}\gamma(I+F_2^{(q-1)})\bar\partial_f^* g - (\bar\partial\rho)^{\wedge,*}\gamma\bar\partial_f^*\,\bar\partial N^{(q-1)}\bar\partial_f^* g\\ &= (\bar\partial\rho)^{\wedge,*}\gamma F_2^{(q-1)}\bar\partial_f^* g.\end{aligned}\tag{5-21}$$

Thus,

$$(\bar\partial\rho)^{\wedge,*}\gamma\bar\partial\,\bar\partial_f^* N^{(q-1)} \equiv 0 \bmod \mathscr{C}^\infty((U\times U)\cap(X\times\overline{M})).$$

Let $\hat\varepsilon^{(q-1)} \equiv 0 \bmod \mathscr{C}^\infty((U\times U)\cap(\overline{M}\times\overline{M}))$ be any smoothing properly supported extension of $(\bar\partial\rho)^{\wedge,*}\gamma\bar\partial\,\bar\partial_f^* N^{(q-1)}$. Put

$$\varepsilon^{(q-1)} := 2\chi(\rho)\rho\hat\varepsilon^{(q-1)} : \mathscr{D}'(U\cap M, T^{*0,q-1}M') \to \Omega^{0,q-2}(U\cap M),$$

where $\chi\in\mathscr{C}_c^\infty((-\varepsilon,\varepsilon))$, $\chi\equiv 1$ near $0\in\mathbb{R}$, for a sufficiently small constant $\varepsilon>0$. We have

$$\begin{gathered}(\bar\partial\rho)^{\wedge,*}\gamma(\bar\partial_f^* N^{(q-1)} - \varepsilon^{(q-1)})\bar\partial_f^* g = 0,\\ (\bar\partial\rho)^{\wedge,*}\gamma\bar\partial\,(\bar\partial_f^* N^{(q-1)} - \varepsilon^{(q-1)})\bar\partial_f^* g = 0\end{gathered}\tag{5-22}$$

for every $g\in A^{0,q}(U\cap\overline{M})\cap\Omega_c^{0,q}(U\cap\overline{M})$ and hence

$$(\bar\partial_f^* N^{(q-1)} - \varepsilon^{(q-1)})\bar\partial_f^* g \in \operatorname{Dom}\Box^{(q-2)}\tag{5-23}$$

for every $g\in A^{0,q}(U\cap\overline{M})\cap\Omega_c^{0,q}(U\cap\overline{M})$. From (4-29), (4-30), (5-20) and (5-23), we have

$$\begin{aligned}\bar\partial_f^*\widehat{\Pi}^{(q)}u &= -F_2^{(q-1)}\bar\partial_f^* u + \bar\partial\,\bar\partial_f^* N^{(q-1)}\bar\partial_f^* u\\ &= -F_2^{(q-1)}\bar\partial_f^* u + \bar\partial\,(\bar\partial_f^* N^{(q-1)} - \varepsilon^{(q-1)})\bar\partial_f^* u + \bar\partial\varepsilon^{(q-1)}\bar\partial_f^* u\end{aligned}\tag{5-24}$$

$$
\begin{aligned}
&= -F_2^{(q-1)}\bar\partial_f^* u + \bar\partial\,(N^{(q-2)}\Box^{(q-2)} - F_1^{(q-2)})(\bar\partial_f^* N^{(q-1)} - \varepsilon^{(q-1)})\bar\partial_f^* u + \bar\partial\varepsilon^{(q-1)}\bar\partial_f^* u \\
&= -F_2^{(q-1)}\bar\partial_f^* u + \bar\partial N^{(q-2)}\Box_f^{(q-2)}\bar\partial_f^* N^{(q-1)}\bar\partial_f^* u - \bar\partial N^{(q-2)}\Box_f^{(q-2)}\varepsilon^{(q-1)}\bar\partial_f^* u \\
&\qquad\qquad - \bar\partial F_1^{(q-2)}(\bar\partial_f^* N^{(q-1)} - \varepsilon^{(q-1)})\bar\partial_f^* u + \bar\partial\varepsilon^{(q-1)}\bar\partial_f^* u \\
&= -F_2^{(q-1)}\bar\partial_f^* u + \bar\partial N^{(q-2)}\bar\partial_f^*\Box_f^{(q-1)} N^{(q-1)}\bar\partial_f^* u - \bar\partial N^{(q-2)}\Box_f^{(q-2)}\varepsilon^{(q-1)}\bar\partial_f^* u \\
&\qquad\qquad - \bar\partial F_1^{(q-2)}(\bar\partial_f^* N^{(q-1)} - \varepsilon^{(q-1)})\bar\partial_f^* u + \bar\partial\varepsilon^{(q-1)}\bar\partial_f^* u \\
&= -F_2^{(q-1)}\bar\partial_f^* u + \bar\partial N^{(q-2)}\bar\partial_f^*(I + F_2^{(q-1)})\bar\partial_f^* u - \bar\partial N^{(q-2)}\Box_f^{(q-2)}\varepsilon^{(q-1)}\bar\partial_f^* u \\
&\qquad\qquad - \bar\partial F_1^{(q-2)}(\bar\partial_f^* N^{(q-1)} - \varepsilon^{(q-1)})\bar\partial_f^* u + \bar\partial\varepsilon^{(q-1)}\bar\partial_f^* u \\
&= -F_2^{(q-1)}\bar\partial_f^* u + \bar\partial N^{(q-2)}\bar\partial_f^* F_2^{(q-1)}\bar\partial_f^* u - \bar\partial N^{(q-2)}\Box_f^{(q-2)}\varepsilon^{(q-1)}\bar\partial_f^* u \\
&\qquad\qquad - \bar\partial F_1^{(q-2)}(\bar\partial_f^* N^{(q-1)} - \varepsilon^{(q-1)})\bar\partial_f^* u + \bar\partial\varepsilon^{(q-1)}\bar\partial_f^* u, \qquad \text{(5-24 cont.)}
\end{aligned}
$$

where $u \in A^{0,q}(U\cap\overline{M}) \cap \Omega_c^{0,q}(U\cap\overline{M})$. It is clear that

$$
\begin{aligned}
&-F_2^{(q-1)}\bar\partial_f^* + \bar\partial N^{(q-2)}\bar\partial_f^* F_2^{(q-1)}\bar\partial_f^* - \bar\partial N^{(q-2)}\Box_f^{(q-2)}\varepsilon^{(q-1)}\bar\partial_f^* \\
&\qquad - \bar\partial F_1^{(q-2)}(\bar\partial_f^* N^{(q-1)} - \varepsilon^{(q-1)})\bar\partial_f^* + \bar\partial\varepsilon^{(q-1)}\bar\partial_f^* \equiv 0 \bmod \mathscr{C}^\infty((U\times U)\cap(\overline{M}\times\overline{M})).
\end{aligned}
$$

From this observation and (5-24), we get (5-15). The proof of (5-14) is similar but simpler and therefore we omit the details. □

From (5-14) and (5-15), we get

$$
\Box_f^{(q)}\widehat{\Pi}^{(q)} u = H_6^{(q)} u \quad \text{for every } u \in A^{0,q}(U\cap\overline{M})\cap\Omega_c^{0,q}(U\cap\overline{M}), \tag{5-25}
$$

where $H_6^{(q)} \equiv 0 \bmod \mathscr{C}^\infty((U\times U)\cap(\overline{M}\times\overline{M}))$ and $H_6^{(q)}$ is properly supported on $U\cap\overline{M}$.

Lemma 5.4. *With the assumptions and notation above, let $q = n_-$. We have*

$$
(\widehat{N}^{(q)} u \mid v)_M = (u \mid \widehat{N}^{(q)} v)_M + (u \mid \widehat{\Gamma}^{(q)} v)_M
$$

for every $u \in L_c^2(U\cap\overline{M}, T^{*0,q}M')$, $v \in L^2_{\mathrm{loc}}(U\cap\overline{M}, T^{*0,q}M')$, where $\widehat{\Gamma}^{(q)}$ is properly supported on $U\cap\overline{M}$ and $\widehat{\Gamma}^{(q)} \equiv 0 \bmod \mathscr{C}^\infty((U\times U)\cap(\overline{M}\times\overline{M}))$.

Proof. Let $u \in L_c^2(U\cap\overline{M}, T^{*0,q}M')$, $v \in L^2_{\mathrm{loc}}(U\cap\overline{M}, T^{*0,q}M')$. Let $u_j \in \Omega_c^{0,q}(U\cap\overline{M})$, $v_j \in \Omega_c^{0,q}(U\cap\overline{M})$, $j = 1, 2, \dots$, such that $u_j \to u$ in $L_c^2(U\cap\overline{M}, T^{*0,q}M')$ as $j\to\infty$ and $v_j \to v$ in $L^2_{\mathrm{loc}}(U\cap\overline{M}, T^{*0,q}M')$ as $j\to\infty$. From (5-6), we see that

$$
(\widehat{N}^{(q)} u \mid v)_M = \lim_{j\to+\infty} (\widehat{N}^{(q)} u_j \mid v_j)_M. \tag{5-26}
$$

We infer from (4-28) that for every $j\in\mathbb{N}$ we have $(\widehat{N}^{(q)} u_j \mid v_j)_M = (u_j \mid (\widehat{N}^{(q)})^* v_j)_M$. From (4-23), we see that $(\widehat{N}^{(q)})^* = N^{(q)} + \widehat{\Gamma}^{(q)}$ on $\Omega_c^{0,q}(U\cap\overline{M})$, where $\widehat{\Gamma}^{(q)} \equiv 0 \bmod \mathscr{C}^\infty((U\times U)\cap(\overline{M}\times\overline{M}))$ and $\widehat{\Gamma}^{(q)}$ is properly supported on $U\cap\overline{M}$. From this observation, we conclude that

$$
(\widehat{N}^{(q)} u_j \mid v_j)_M = (u_j \mid \widehat{N}^{(q)} v_j)_M + (u_j \mid \widehat{\Gamma}^{(q)} v_j)_M \quad \text{for every } j\in\mathbb{N}. \tag{5-27}
$$

From (5-26) and (5-27), the lemma follows. □

Lemma 5.5. *With the assumptions and notation used above, let $q = n_-$. Fix an open set $W \subset U$ with $\overline{W}$ a compact subset of U. There is a constant $C_W > 0$ such that, for every $u \in A^{0,q}(U \cap \overline{M}) \cap \Omega_c^{0,q}(W \cap \overline{M})$,*

$$\|\hat{\Pi}^{(q)} u\|_M \le C_W \|u\|_M. \tag{5-28}$$

Proof. Let $u \in A^{0,q}(U \cap \overline{M}) \cap \Omega_c^{0,q}(W \cap \overline{M})$. From (5-13), we have

$$\begin{aligned}(\hat{\Pi}^{(q)} u \mid \hat{\Pi}^{(q)} u)_M &= (\hat{\Pi}^{(q)} u \mid u)_M - (\hat{\Pi}^{(q)} u \mid (I - \hat{\Pi}^{(q)}) u)_M \\ &= (\hat{\Pi}^{(q)} u \mid u)_M - (\hat{\Pi}^{(q)} u \mid (\Box_f^{(q)} \hat{N}^{(q)} - H_3^{(q)}) u)_M. \end{aligned} \tag{5-29}$$

From (5-9) and (5-10), we can repeat the proof of Theorem 4.3 and deduce that there is a properly supported operator $N^{(q)} : \mathscr{D}'(U \cap M, T^{*0,q} M') \to \Omega^{0,q}(U \cap M)$ on $U \cap \overline{M}$ with $N^{(q)} - \hat{N}^{(q)} \equiv 0 \bmod \mathscr{C}^\infty((U \times U) \cap (\overline{M} \times \overline{M}))$ such that

$$N^{(q)} g \in \operatorname{Dom} \Box^{(q)} \tag{5-30}$$

for every $g \in A^{0,q}(U \cap \overline{M}) \cap \Omega_c^{0,q}(W \cap \overline{M})$. From (5-11), (5-13), (5-14), (5-15), (5-29), (5-30), we have

$$\begin{aligned}(\hat{\Pi}^{(q)} u \mid \hat{\Pi}^{(q)} u)_M &= (\hat{\Pi}^{(q)} u \mid u)_M - (\hat{\Pi}^{(q)} u \mid (\Box_f^{(q)} \hat{N}^{(q)} - H_3^{(q)}) u)_M \\ &= (\hat{\Pi}^{(q)} u \mid u)_M - (\hat{\Pi}^{(q)} u \mid (\Box_f^{(q)} N^{(q)} - H_3^{(q)}) u)_M + (\hat{\Pi}^{(q)} u \mid \Box_f^{(q)} (N^{(q)} - \hat{N}^{(q)}) u)_M \\ &= (\hat{\Pi}^{(q)} u \mid u)_M - (\bar\partial \hat{\Pi}^{(q)} u \mid \bar\partial N^{(q)} u)_M - (\bar\partial_f^* \hat{\Pi}^{(q)} u \mid \bar\partial^* N^{(q)} u)_M \\ &\qquad + (\hat{\Pi}^{(q)} u \mid H_3^{(q)} u)_M + (\hat{\Pi}^{(q)} u \mid \Box_f^{(q)} (N^{(q)} - \hat{N}^{(q)}) u)_M \\ &= (\hat{\Pi}^{(q)} u \mid u)_M - (H_4^{(q)} u \mid \bar\partial N^{(q)} u)_M - (H_5^{(q)} u \mid \bar\partial^* N^{(q)} u)_M \\ &\qquad + (\hat{\Pi}^{(q)} u \mid H_3^{(q)} u)_M + (\hat{\Pi}^{(q)} u \mid \Box_f^{(q)} (N^{(q)} - \hat{N}^{(q)}) u)_M \\ &= (\hat{\Pi}^{(q)} u \mid u)_M - (u \mid ((H_4^{(q)})^* \bar\partial N^{(q)} + (H_5^{(q)})^* \bar\partial^* N^{(q)}) u)_M \\ &\qquad + (\hat{\Pi}^{(q)} u \mid H_3^{(q)} u)_M + (\hat{\Pi}^{(q)} u \mid \Box_f^{(q)} (N^{(q)} - \hat{N}^{(q)}) u)_M, \end{aligned} \tag{5-31}$$

where

$$H_3^{(q)}, H_4^{(q)}, H_5^{(q)} \equiv 0 \bmod \mathscr{C}^\infty((U \times U) \cap (\overline{M} \times \overline{M}))$$

are as in (5-13), (5-14), (5-15), and $(H_4^{(q)})^*$ and $(H_5^{(q)})^*$ are the formal adjoints of $H_4^{(q)}$ and $H_5^{(q)}$, respectively. Note that the operators

$$(H_4^{(q)})^* \bar\partial N^{(q)} + (H_5^{(q)})^* \bar\partial^* N^{(q)}, \quad H_3^{(q)}, \quad \Box_f^{(q)} (N^{(q)} - \hat{N}^{(q)})$$

map $L^2_{\mathrm{loc}}(U \cap \overline{M}, T^{*0,q} M')$ into itself continuously. From this observation and (5-31), we deduce that there exists $\hat{C} > 0$ such that

$$\|\hat{\Pi}^{(q)} u\|_M^2 \le \hat{C} (\|\hat{\Pi}^{(q)} u\|_M \|u\|_M + \|u\|_M^2), \quad u \in A^{0,q}(U \cap \overline{M}) \cap \Omega_c^{0,q}(W \cap \overline{M}). \tag{5-32}$$

From (5-32), we get (5-28). □

As a comment regarding the proof of Lemma 5.5, one could try to use $\hat{N}^{(q)}$ directly, since $\bar\partial \hat{N}^{(q)}$, $\bar\partial^* \hat{N}^{(q)}$ are also bounded in L^2_{loc}. However, the range of $\hat{N}^{(q)}$ is not contained in $\operatorname{Dom} \Box^{(q)}$, since $(\bar\partial \rho)^{\wedge,*} \gamma \hat{N}^{(q)}$ and $(\bar\partial \rho)^{\wedge,*} \gamma \bar\partial \hat{N}^{(q)}$ do not necessarily vanish on the boundary (we only know that they are smoothing operators). Thus, we use the operator $N^{(q)}$ which satisfies (5-30).

Remark 5.6. Since $N^{(q-1)}$ and $N^{(q+1)}$ are properly supported on $U \cap \overline{M}$, $\widehat{\Pi}$ is properly supported on $U \cap \overline{M}$. Hence for every $\chi \in \mathscr{C}_c^\infty(U \cap \overline{M})$, there are $\chi_1 \in \mathscr{C}_c^\infty(U \cap \overline{M})$, $\chi_2 \in \mathscr{C}_c^\infty(U \cap \overline{M})$, such that

$$\begin{aligned} \widehat{\Pi}^{(q)}\chi u &= \chi_2 \widehat{\Pi}^{(q)} u \quad \text{for every } u \in A^{0,q}(U \cap \overline{M}), \\ \chi \widehat{\Pi}^{(q)} u &= \widehat{\Pi}^{(q)}\chi_1 u \quad \text{for every } u \in A^{0,q}(U \cap \overline{M}). \end{aligned}$$

By Lemma 5.5 we can extend $\widehat{\Pi}^{(q)}$ to $L^2_c(U \cap \overline{M}, T^{*0,q}M')$ by density. More precisely, let $u \in L^2_c(U \cap \overline{M}, T^{*0,q}M')$. Suppose that $\operatorname{supp} u \subset W$, where $W \subset U$ is an open set with $\overline{W} \Subset U$. Take any sequence $(u_j)_j$ in $A^{0,q}(U \cap \overline{M}) \cap \Omega_c^{0,q}(W \cap \overline{M})$, with $\lim_{j\to+\infty}\|u_j - u\|_M = 0$. Since $\widehat{\Pi}^{(q)}$ is properly supported on $U \cap \overline{M}$, we have

$$\widehat{\Pi}^{(q)} u := \lim_{j\to+\infty} \widehat{\Pi}^{(q)} u_j \quad \text{in } L^2_c(U \cap \overline{M}, T^{*0,q}M'). \tag{5-33}$$

By using that $\widehat{\Pi}^{(q)}$ is properly supported on $U \cap \overline{M}$, we extend $\widehat{\Pi}^{(q)}$ to $L^2_{\mathrm{loc}}(U \cap \overline{M}, T^{*0,q}M')$ and the extensions

$$\begin{aligned} \widehat{\Pi}^{(q)} &: L^2_c(U \cap \overline{M}, T^{*0,q}M') \to L^2_c(U \cap \overline{M}, T^{*0,q}M'), \\ \widehat{\Pi}^{(q)} &: L^2_{\mathrm{loc}}(U \cap \overline{M}, T^{*0,q}M') \to L^2_{\mathrm{loc}}(U \cap \overline{M}, T^{*0,q}M') \end{aligned} \tag{5-34}$$

are continuous.

Lemma 5.7. *With the assumptions and notation above, let $q = n_-$. We have*

$$(\widehat{\Pi}^{(q)} u \mid v)_M = (u \mid \widehat{\Pi}^{(q)} v)_M + (u \mid \widehat{\Gamma}_1^{(q)} v)_M \tag{5-35}$$

for every $u \in L^2_c(U \cap \overline{M}, T^{*0,q}M')$, $v \in L^2_{\mathrm{loc}}(U \cap \overline{M}, T^{*0,q}M')$, where $\widehat{\Gamma}_1^{(q)}$ is a properly supported continuous operator on $U \cap \overline{M}$ and $\widehat{\Gamma}_1^{(q)} \equiv 0 \bmod \mathscr{C}^\infty((U \times U) \cap (\overline{M} \times \overline{M}))$.

Proof. From (4-23), (4-28) and (5-8), we get (5-35) for $u, v \in A^{0,q}(U \cap \overline{M}) \cap \Omega_c^{0,q}(U \cap \overline{M})$. By using a density argument and noticing that $\widehat{\Pi}^{(q)}$ is properly supported on $U \cap \overline{M}$, we get (5-35). □

Theorem 5.8. *We have*

$$\chi \widehat{\Pi}^{(q)} u \in \operatorname{Dom} \bar\partial^* \quad \textit{for every } \chi \in \mathscr{C}_c^\infty(U \cap \overline{M}),\ u \in L^2_{\mathrm{loc}}(U \cap \overline{M}, T^{*0,q}M'), \tag{5-36}$$

$$\bar\partial \widehat{\Pi}^{(q)} u = H_4^{(q)} u \quad \textit{for every } u \in L^2_{\mathrm{loc}}(U \cap \overline{M}, T^{*0,q}M'), \tag{5-37}$$

$$\bar\partial_f^* \widehat{\Pi}^{(q)} u = H_5^{(q)} u \quad \textit{for every } u \in L^2_{\mathrm{loc}}(U \cap \overline{M}, T^{*0,q}M'), \tag{5-38}$$

$$\Box_f^{(q)} \widehat{N}^{(q)} u + \widehat{\Pi}^{(q)} u = u + H_3^{(q)} u \quad \textit{for every } u \in \Omega^{0,q}(U \cap \overline{M}), \tag{5-39}$$

where $H_j^{(q)} \equiv 0 \bmod \mathscr{C}^\infty((U \times U) \cap (\overline{M} \times \overline{M}))$, $j = 3, 4, 5$, are as in Theorem 5.3.

Proof. Let $u \in L^2_{\mathrm{loc}}(U \cap \overline{M}, T^{*0,q}M')$ and let $\chi \in \mathscr{C}_c^\infty(U \cap \overline{M})$. Since $\widehat{\Pi}^{(q)}$ is properly supported on $U \cap \overline{M}$ (see Remark 5.6), there is a $\chi_1 \in \mathscr{C}_c^\infty(U \cap \overline{M})$ such that $\chi \widehat{\Pi}^{(q)} = \widehat{\Pi}^{(q)} \chi_1$ on $L^2_{\mathrm{loc}}(U \cap \overline{M}, T^{*0,q}M')$. Let $g \in \operatorname{Dom} \bar\partial \cap L^2_{(0,q)}(\overline{M})$. Let $u_j \in A^{0,q}(U \cap \overline{M}) \cap \Omega_c^{0,q}(U \cap \overline{M})$, $j = 1, 2, \dots$, with $\lim_{j\to+\infty}\|u_j - \chi_1 u\|_M = 0$. Then,

$$\begin{aligned}(\chi\widehat{\Pi}^{(q)}u \mid \bar\partial g)_M &= (\widehat{\Pi}^{(q)}\chi_1 u \mid \bar\partial g)_M = \lim_{j\to+\infty}(\widehat{\Pi}^{(q)}u_j \mid \bar\partial g)_M \\ &= \lim_{j\to+\infty}(\bar\partial^*\widehat{\Pi}^{(q)}u_j \mid g)_M = \lim_{j\to+\infty}(H_5^{(q)}u_j \mid g)_M = (H_5^{(q)}u \mid g)_M,\end{aligned}\tag{5-40}$$

where $H_5^{(q)} \equiv 0 \bmod \mathscr{C}^\infty((U\times U)\cap(\overline{M}\times\overline{M}))$ is as in (5-15).

From (5-40), we deduce that $\chi\widehat{\Pi}^{(q)}u \in \operatorname{Dom}\bar\partial^*$, we get (5-36) and we also get (5-38). The proof of (5-37) is similar. We now prove (5-39).

Let $u \in \Omega^{0,q}(U\cap\overline{M})$ and let $g \in \Omega_c^{0,q}(U\cap M)$. Since $\widehat{\Pi}^{(q)}$, $\widehat{N}^{(q)}$ and $H_3^{(q)}$ are properly supported on $U\cap\overline{M}$, there is a $\tau \in \mathscr{C}_c^\infty(U\cap\overline{M})$ such that

$$\begin{gathered}((\Box_f^{(q)}\widehat{N}^{(q)} + \widehat{\Pi}^{(q)})u \mid g)_M = ((\Box_f^{(q)}\widehat{N}^{(q)} + \widehat{\Pi}^{(q)})\tau u \mid g)_M, \\ ((I + H_3^{(q)})u \mid g)_M = ((I + H_3^{(q)})\tau u \mid g)_M.\end{gathered}\tag{5-41}$$

Let $u_j \in A^{0,q}(U\cap\overline{M})\cap\Omega_c^{0,q}(U\cap\overline{M}),\ j = 1, 2, \dots$, with $\lim_{j\to+\infty}\|u_j - \tau u\|_M = 0$. From (5-13) and (5-41), we have

$$\begin{aligned}((\Box_f^{(q)}\widehat{N}^{(q)} + \widehat{\Pi}^{(q)})u \mid g)_M &= ((\Box_f^{(q)}\widehat{N}^{(q)} + \widehat{\Pi}^{(q)})\tau u \mid g)_M \\ &= (\widehat{N}^{(q)}\tau u \mid \Box_f^{(q)}g)_M + (\widehat{\Pi}^{(q)}\tau u \mid g)_M \\ &= \lim_{j\to+\infty}\big((\widehat{N}^{(q)}u_j \mid \Box_f^{(q)}g)_M + (\widehat{\Pi}^{(q)}u_j \mid g)_M\big) \\ &= \lim_{j\to+\infty}((\Box_f^{(q)}\widehat{N}^{(q)} + \widehat{\Pi}^{(q)})u_j \mid g)_M = \lim_{j\to+\infty}((I + H_3^{(q)})u_j \mid g)_M \\ &= ((I + H_3^{(q)})\tau u \mid g)_M = ((I + H_3^{(q)})u \mid g)_M.\end{aligned}\tag{5-42}$$

Let $h \in \Omega_c^{0,q}(U\cap\overline{M})$. Take $h_j \in \Omega_c^{0,q}(U\cap M),\ j = 1, 2, \dots$, so that $\lim_{j\to+\infty}\|h_j - h\|_M = 0$. From (5-34) and (5-42), we have

$$\begin{aligned}((\Box_f^{(q)}\widehat{N}^{(q)} + \widehat{\Pi}^{(q)})u \mid h)_M &= \lim_{j\to+\infty}((\Box_f^{(q)}\widehat{N}^{(q)} + \widehat{\Pi}^{(q)})u \mid h_j)_M \\ &= \lim_{j+\infty}((I + H_3^{(q)})u \mid h_j)_M = ((I + H_3^{(q)})u \mid h)_M.\end{aligned}\tag{5-43}$$

From (5-43), we get (5-39). □

The following result is the first version of the local approximate Hodge decomposition for the $\bar\partial$-Neumann Laplacian in the critical degree $q = n_-$.

Theorem 5.9. *With the assumptions and notation used above, let $q = n_-$. We can find properly supported continuous operators on $U\cap\overline{M}$,*

$$\begin{gathered}N^{(q)} : H^s_{\mathrm{loc}}(U\cap\overline{M}, T^{*0,q}M') \to H^s_{\mathrm{loc}}(U\cap\overline{M}, T^{*0,q}M') \quad \text{for every } s\in\mathbb{Z}, \\ \Pi^{(q)} : L^2_{\mathrm{loc}}(U\cap\overline{M}, T^{*0,q}M') \to L^2_{\mathrm{loc}}(U\cap\overline{M}, T^{*0,q}M')\end{gathered}\tag{5-44}$$

such that

$$\begin{gathered}N^{(q)} - \widehat{N}^{(q)} \equiv 0 \bmod \mathscr{C}^\infty((U\times U)\cap(\overline{M}\times\overline{M})), \\ \Pi^{(q)} - \widehat{\Pi}^{(q)} \equiv 0 \bmod \mathscr{C}^\infty((U\times U)\cap(\overline{M}\times\overline{M})),\end{gathered}\tag{5-45}$$

$$\begin{aligned}
\Box_f^{(q)} N^{(q)} u + \Pi^{(q)} u &= u + R_0^{(q)} u && \text{for every } u \in \Omega^{0,q}(U \cap \overline{M}), \\
\Box_f^{(q)} \Pi^{(q)} u &= R_1^{(q)} u && \text{for every } u \in L^2_{\mathrm{loc}}(U \cap M), \\
\bar\partial \Pi^{(q)} u &= R_2^{(q)} u && \text{for every } u \in L^2_{\mathrm{loc}}(U \cap \overline{M}, T^{*0,q}M'), \\
\bar\partial_f^* \Pi^{(q)} u &= R_3^{(q)} u && \text{for every } u \in L^2_{\mathrm{loc}}(U \cap \overline{M}, T^{*0,q}M'),
\end{aligned} \tag{5-46}$$

$$\begin{aligned}
(\bar\partial\rho)^{\wedge,*}\gamma N^{(q)} u|_D &= 0 && \text{for every } u \in \Omega^{0,q}(U \cap \overline{M}), \\
\chi \Pi^{(q)} u &\in \mathrm{Dom}\,\bar\partial^* && \text{for every } \chi \in \mathscr{C}_c^\infty(U \cap \overline{M}),\ u \in L^2_{\mathrm{loc}}(U \cap \overline{M}, T^{*0,q}M'),
\end{aligned} \tag{5-47}$$

$$\begin{aligned}
(\bar\partial\rho)^{\wedge,*}\gamma \bar\partial N^{(q)} u|_D &= 0 && \text{for every } u \in \Omega^{0,q}(U \cap \overline{M}), \\
(\bar\partial\rho)^{\wedge,*}\gamma \bar\partial \Pi^{(q)} u|_D &= 0 && \text{for every } u \in L^2_{\mathrm{loc}}(U \cap \overline{M}, T^{*0,q}M'),
\end{aligned} \tag{5-48}$$

where $R_j^{(q)} : \mathscr{D}'(U \cap M) \to \Omega^{0,q}(U \cap M)$ is a properly supported continuous operator on $U \cap \overline{M}$ with $R_j^{(q)} \equiv 0 \bmod \mathscr{C}^\infty((U \times U) \cap (\overline{M} \times \overline{M}))$, $j = 0, 1, 2, 3$.

Proof. We define, following (4-21), $N^{(q)} := \widehat{N}_7^{(q)} - 2\chi(\rho)\rho\widetilde{H}_1^{(q)}$, $\Pi^{(q)} := \widehat{\Pi}_7^{(q)} - 2\chi(\rho)\rho\widetilde{H}_2^{(q)}$, where $\widetilde{H}_1^{(q)}$ is a smoothing extension of $H_1^{(q)}$ from (5-10), and $\widetilde{H}_2^{(q)}$ is a smoothing extension of $H_2^{(q)}$ from (5-12). We show as in the proof of Theorem 4.3 that $N^{(q)}$ and $\Pi^{(q)}$ satisfy the $\bar\partial$-Neumann conditions and by using Theorems 5.3 and 5.8 we conclude the result. □

From Lemmas 5.4 and 5.7, we get:

Theorem 5.10. *With the assumptions and notation used above, let $q = n_-$. We have*

$$(N^{(q)}u \mid v)_M = (u \mid N^{(q)}v)_M + (u \mid \Gamma^{(q)}v)_M, \tag{5-49}$$

$$(\Pi^{(q)}u \mid v)_M = (u \mid \Pi^{(q)}v)_M + (u \mid \Gamma_1^{(q)}v)_M \tag{5-50}$$

for every $u \in L^2_c(U \cap \overline{M}, T^{*0,q}M')$, $v \in L^2_{\mathrm{loc}}(U \cap \overline{M}, T^{*0,q}M')$, where $N^{(q)}$ and $\Pi^{(q)}$ are as in Theorem 5.9, $\Gamma^{(q)}, \Gamma_1^{(q)} \equiv 0 \bmod \mathscr{C}^\infty((U \times U) \cap (\overline{M} \times \overline{M}))$, $\Gamma^{(q)}$ and $\Gamma_1^{(q)}$ are properly supported on $U \cap \overline{M}$.

Theorem 5.11. *With the assumptions and notation used above, let $q = n_-$. Let $N^{(q)}$ and $\Pi^{(q)}$ be as in Theorem 5.9. Then we have on $U \cap \overline{M}$, for every $u \in \mathrm{Dom}\,\Box^{(q)}$,*

$$\Pi^{(q)}\Box^{(q)}u = \Lambda_0^{(q)}u, \tag{5-51}$$

$$N^{(q)}\Box^{(q)}u + \Pi^{(q)}u = u + \Lambda^{(q)}u, \tag{5-52}$$

where $\Lambda_0^{(q)}, \Lambda^{(q)}$ are properly supported on $U \cap \overline{M}$ and $\Lambda_0^{(q)}, \Lambda^{(q)} \equiv 0 \bmod \mathscr{C}^\infty((U \times U) \cap (\overline{M} \times \overline{M}))$.

Proof. Let $u \in \mathrm{Dom}\,\Box^{(q)}$ and let $v \in \Omega_c^{0,q}(U \cap M)$. From (5-46), (5-47), (5-48) and (5-50), we have

$$\begin{aligned}
(\Pi^{(q)}\Box^{(q)}u \mid v)_M &= (\Box^{(q)}u \mid \Pi^{(q)}v)_M + (\Box^{(q)}u \mid \Gamma_1^{(q)}v)_M \\
&= (u \mid \Box^{(q)}\Pi^{(q)}v)_M + (\Box^{(q)}u \mid \Gamma_1^{(q)}v)_M \\
&= (u \mid R_1^{(q)}v)_M + (\Box^{(q)}u \mid \Gamma_1^{(q)}v)_M \\
&= (((R_1^{(q)})^* + (\Gamma_1^{(q)})^*\Box_f^{(q)})u \mid v)_M,
\end{aligned} \tag{5-53}$$

where $R_1^{(q)}$, $\Gamma_1^{(q)}$ are as in (5-46) and (5-50) respectively and $(R_1^{(q)})^*$ and $(\Gamma_1^{(q)})^*$ are the formal adjoints of $R_1^{(q)}$ and $\Gamma_1^{(q)}$ with respect to $(\cdot\,|\,\cdot)_M$ respectively. It is clear that $(R_1^{(q)})^* + (\Gamma_1^{(q)})^*\Box_f^{(q)} \equiv 0 \mod \mathscr{C}^\infty((U\times U)\cap(\overline{M}\times\overline{M}))$. From this observation and (5-53), we get (5-51).

Let $u \in \operatorname{Dom}\Box^{(q)}$ and let $v \in \Omega_c^{0,q}(U\cap M)$. From (5-46), (5-47), (5-48), (5-49) and (5-50), we have

$$\begin{aligned}(N^{(q)}\Box^{(q)}u+\Pi^{(q)}u\,|\,v)_M &= (\Box^{(q)}u\,|\,N^{(q)}v)_M+(\Box^{(q)}u\,|\,\Gamma^{(q)}v)_M+(u\,|\,\Pi^{(q)}v)_M+(u\,|\,\Gamma_1^{(q)}v)_M\\ &= (u\,|\,\Box^{(q)}N^{(q)}v)_M+((\Gamma^{(q)})^*\Box_f^{(q)}u\,|\,v)_M+(u\,|\,\Pi^{(q)}v)_M+(u\,|\,\Gamma_1^{(q)}v)_M\\ &= (u\,|\,(\Box^{(q)}N^{(q)}+\Pi^{(q)})v)_M+((\Gamma^{(q)})^*\Box_f^{(q)}u\,|\,v)_M+(u\,|\,\Gamma_1^{(q)}v)_M\\ &= (u\,|\,R_0^{(q)}v)_M+((\Gamma^{(q)})^*\Box_f^{(q)}u\,|\,v)_M+(u\,|\,\Gamma_1^{(q)}v)_M\\ &= (((R_0^{(q)}+\Gamma_1^{(q)})^*+(\Gamma^{(q)})^*\Box_f^{(q)})u\,|\,v)_M,\end{aligned} \tag{5-54}$$

where $R_0^{(q)}$, $\Gamma^{(q)}$, $\Gamma_1^{(q)}$ are as in (5-46), (5-49) and (5-50) respectively, $(\Gamma^{(q)})^*$ is the formal adjoint of $\Gamma^{(q)}$ with respect to $(\cdot\,|\,\cdot)_M$ and $(R_0^{(q)}+\Gamma_1^{(q)})^*$ is the formal adjoint of $R_0^{(q)}+\Gamma_1^{(q)}$ with respect to $(\cdot\,|\,\cdot)_M$. It is clear that $(\Gamma^{(q)})^*\Box_f^{(q)} \equiv 0 \mod \mathscr{C}^\infty((U\times U)\cap(\overline{M}\times\overline{M}))$ and $(R_0^{(q)}+\Gamma_1^{(q)})^* \equiv 0 \mod \mathscr{C}^\infty((U\times U)\cap(\overline{M}\times\overline{M}))$. From this observation and (5-54), we get (5-52). $\Box$

5.2. ***The distribution kernel of the approximate Bergman kernel.*** In this section, we will study the distribution kernel of $\Pi^{(q)}$ and regularity properties of the operators $\Pi^{(q)}$ and $N^{(q)}$. We will refine in this way the Hodge decomposition from Theorem 5.9 in Theorem 5.23.

Let $[\cdot\,|\,\cdot]_X$ be the L^2 inner product on $H^{-1/2}(X, T^{*0,q}M')$ given by

$$[u\,|\,v]_X := (\widetilde{P}u\,|\,\widetilde{P}v)_M, \tag{5-55}$$

where $\widetilde{P}$ is the Poisson operator given by (3-4). Let $\widetilde{P}^* : \Omega^{0,q}(\overline{M}) \to \mathscr{C}^\infty(X, T^{*0,q}M')$ be the adjoint of $\widetilde{P}$ as defined in (3-9). Then,

$$\widetilde{P}^*\widetilde{P} : \mathscr{C}^\infty(X, T^{*0,q}M') \to \mathscr{C}^\infty(X, T^{*0,q}M')$$

is an injective continuous operator. Let

$$(\widetilde{P}^*\widetilde{P})^{-1} : \mathscr{C}^\infty(X, T^{*0,q}M') \to \mathscr{C}^\infty(X, T^{*0,q}M')$$

be the inverse of $\widetilde{P}^*\widetilde{P}$. It is well known that $(\widetilde{P}^*\widetilde{P})^{-1}$ is a classical pseudodifferential operator of order 1 on X (see [Boutet de Monvel 1971]).

Sections of $T^{*0,q}M'$ over X annihilated by $(\bar\partial\rho)^{\wedge,*}$ can be identified with sections of $T^{*0,q}X$, so they are called tangential. We have

$$\operatorname{Ker}(\bar\partial\rho)^{\wedge,*} := \{u \in H^{-\frac12}(X, T^{*0,q}M') : (\bar\partial\rho)^{\wedge,*}u = 0\} = H^{-\frac12}(X, T^{*0,q}X).$$

Let

$$Q^{(q)} : H^{-\frac12}(X, T^{*0,q}M') \to \operatorname{Ker}(\bar\partial\rho)^{\wedge,*} \tag{5-56}$$

be the orthogonal projection with respect to $[\cdot\,|\,\cdot]_X$.

Theorem 5.12 [Hsiao 2010, Part II, Lemma 3.3]. *$Q^{(q)}$ is a classical pseudodifferential operator of order* 0 *with principal symbol* $2(\bar\partial\rho)^{\wedge,*}(\bar\partial\rho)^{\wedge}$. *Moreover,*

$$I - Q^{(q)} = (\widetilde{P}^*\widetilde{P})^{-1}(\bar\partial\rho)^{\wedge}R, \tag{5-57}$$

where $R:\mathscr{C}^\infty(X, T^{*0,q}M') \to \mathscr{C}^\infty(X, T^{*0,q-1}M')$ *is a classical pseudodifferential operator of order* -1.

Let $u \in \Omega_c^{0,q}(U\cap M)$. From Theorem 4.3, (5-8) and Theorem 5.9, we see that $\Pi^{(q)}u \in \Omega_c^{0,q}(U\cap \overline{M})$ and $\gamma\Pi^{(q)}u \in \mathscr{C}^\infty(X, T^{*0,q}M')$.

Theorem 5.13. *Under the assumptions and notation used before we have, for* $q = n_-$,

$$\Pi^{(q)}u = \widetilde{P}\gamma\Pi^{(q)}u + \varepsilon^{(q)}u \quad \text{for every } u \in \Omega_c^{0,q}(U\cap M), \tag{5-58}$$

where $\varepsilon^{(q)} \equiv 0 \bmod \mathscr{C}^\infty((U\times U)\cap(\overline{M}\times\overline{M}))$.

Proof. Let $u \in \Omega_c^{0,q}(U\cap M)$. Since $\Pi^{(q)}$ is properly supported on $U\cap\overline{M}$,

$$\Pi^{(q)}u \in \Omega_c^{0,q}(U\cap\overline{M}) \subset \Omega^{0,q}(\overline{M}).$$

From (3-8), we have

$$D^{(q)}\widetilde{\Box}_f^{(q)}\Pi^{(q)}u + \widetilde{P}\gamma\Pi^{(q)}u = \Pi^{(q)}u. \tag{5-59}$$

From (5-46) and $\widetilde{\Box}_f^{(q)} - \Box_f^{(q)} \equiv 0 \bmod \mathscr{C}^\infty(\overline{M}\times\overline{M})$, we see that

$$D^{(q)}\widetilde{\Box}_f^{(q)}\Pi^{(q)} \equiv 0 \bmod \mathscr{C}^\infty((U\times U)\cap(\overline{M}\times\overline{M})).$$

From this observation and (5-59), we get (5-58). □

From (5-59), we have

$$\begin{aligned}(\widetilde{P}^*\widetilde{P})^{-1}\widetilde{P}^*\Pi^{(q)}u &= (\widetilde{P}^*\widetilde{P})^{-1}\widetilde{P}^*\widetilde{P}\gamma\Pi^{(q)}u + (\widetilde{P}^*\widetilde{P})^{-1}\widetilde{P}^*\varepsilon^{(q)}u \\ &= \gamma\Pi^{(q)}u + (\widetilde{P}^*\widetilde{P})^{-1}\widetilde{P}^*\varepsilon^{(q)}u\end{aligned} \tag{5-60}$$

and

$$\Pi^{(q)}u = \widetilde{P}(\widetilde{P}^*\widetilde{P})^{-1}\widetilde{P}^*\Pi^{(q)}u + \varepsilon_1^{(q)}u \tag{5-61}$$

for every $u \in \Omega_c^{0,q}(U\cap M)$, where $\varepsilon_1^{(q)} = -\widetilde{P}(\widetilde{P}^*\widetilde{P})^{-1}\widetilde{P}^*\varepsilon^{(q)}u \equiv 0 \bmod \mathscr{C}^\infty((U\times U)\cap(\overline{M}\times\overline{M}))$. From (3-6) and (3-10), we see that $\widetilde{P}(\widetilde{P}^*\widetilde{P})^{-1}\widetilde{P}^*\Pi^{(q)}$ is well-defined as a continuous operator

$$\widetilde{P}(\widetilde{P}^*\widetilde{P})^{-1}\widetilde{P}^*\Pi^{(q)} : L_c^2(U\cap\overline{M}, T^{*0,q}M') \to L^2_{\mathrm{loc}}(U\cap\overline{M}, T^{*0,q}M').$$

From this observation, (5-61) and by using a density argument, we conclude that

$$\Pi^{(q)} - \widetilde{P}(\widetilde{P}^*\widetilde{P})^{-1}\widetilde{P}^*\Pi^{(q)} \equiv 0 \bmod \mathscr{C}^\infty((U\times U)\cap(\overline{M}\times\overline{M})). \tag{5-62}$$

Similarly, from (3-6) and (3-10), we see that $\Pi^{(q)}\widetilde{P}(\widetilde{P}^*\widetilde{P})^{-1}\widetilde{P}^*$ is well-defined as a continuous operator

$$\Pi^{(q)}\widetilde{P}(\widetilde{P}^*\widetilde{P})^{-1}\widetilde{P}^* : L^2(\overline{M}, T^{*0,q}M') \to L^2_{\mathrm{loc}}(U\cap\overline{M}, T^{*0,q}M').$$

Lemma 5.14. *Under the assumptions and notation used before we have, for $q = n_-$,*

$$\Pi^{(q)} \widetilde{P}(\widetilde{P}^* \widetilde{P})^{-1} \widetilde{P}^* - \Pi^{(q)} \equiv 0 \mod \mathscr{C}^\infty((U \times U) \cap (\overline{M} \times \overline{M})).$$

Proof. Let $u \in L^2_{(0,q)}(\overline{M})$ and let $v \in \Omega^{0,q}_c(U \cap \overline{M})$. From (5-50) and (5-61), we have

$$\begin{aligned}(\Pi^{(q)} u \mid v)_M &= (u \mid \Pi^{(q)} v)_M + (u \mid \Gamma^{(q)}_1 v)_M \\ &= (u \mid \widetilde{P}(\widetilde{P}^* \widetilde{P})^{-1} \widetilde{P}^* \Pi^{(q)} v)_M + (u \mid \varepsilon^{(q)}_1 v)_M + (u \mid \Gamma^{(q)}_1 v)_M \\ &= (\widetilde{P}(\widetilde{P}^* \widetilde{P})^{-1} \widetilde{P}^* u \mid \Pi^{(q)} v)_M + (u \mid \varepsilon^{(q)}_1 v)_M + (u \mid \Gamma^{(q)}_1 v)_M \\ &= (\Pi^{(q)} \widetilde{P}(\widetilde{P}^* \widetilde{P})^{-1} \widetilde{P}^* u \mid v)_M \\ &\qquad - (\widetilde{P}(\widetilde{P}^* \widetilde{P})^{-1} \widetilde{P}^* u \mid \Gamma^{(q)}_1 v)_M + (u \mid \varepsilon^{(q)}_1 v)_M + (u \mid \Gamma^{(q)}_1 v)_M \\ &= (\Pi^{(q)} \widetilde{P}(\widetilde{P}^* \widetilde{P})^{-1} \widetilde{P}^* u \mid v)_M \\ &\qquad - ((\Gamma^{(q)}_1)^* \widetilde{P}(\widetilde{P}^* \widetilde{P})^{-1} \widetilde{P}^* u \mid v)_M + ((\varepsilon^{(q)}_1 + \Gamma^{(q)}_1)^* u \mid v)_M, \end{aligned} \tag{5-63}$$

where $(\Gamma^{(q)}_1)^*$ and $(\varepsilon^{(q)}_1 + \Gamma^{(q)}_1)^*$ are the formal adjoints of $\Gamma^{(q)}_1$ and $\varepsilon^{(q)}_1 + \Gamma^{(q)}_1$ respectively. Note that

$$(\Gamma^{(q)}_1)^* \widetilde{P}(\widetilde{P}^* \widetilde{P})^{-1} \widetilde{P}^*, (\varepsilon^{(q)}_1 + \Gamma^{(q)}_1)^* \equiv 0 \mod \mathscr{C}^\infty((U \times U) \cap (\overline{M} \times \overline{M})).$$

From this observation and (5-63), the lemma follows. □

Theorem 5.15. *With the assumptions and notation used before, we have*

$$\Pi^{(q)} - \Pi^{(q)} \widetilde{P} Q^{(q)} (\widetilde{P}^* \widetilde{P})^{-1} \widetilde{P}^* \equiv 0 \mod \mathscr{C}^\infty((U \times U) \cap (\overline{M} \times \overline{M})), \tag{5-64}$$

$$\Pi^{(q)} - \widetilde{P} Q^{(q)} (\widetilde{P}^* \widetilde{P})^{-1} \widetilde{P}^* \Pi^{(q)} \equiv 0 \mod \mathscr{C}^\infty((U \times U) \cap (\overline{M} \times \overline{M})). \tag{5-65}$$

Proof. Let $u \in L^2_{(0,q)}(\overline{M})$ and let $v \in \Omega^{0,q}_c(U \cap M)$. From (5-50) and (5-58), we have

$$\begin{aligned}&(\Pi^{(q)} \widetilde{P}(I - Q^{(q)})(\widetilde{P}^* \widetilde{P})^{-1} \widetilde{P}^* u \mid v)_M \\ &\quad = (\widetilde{P}(I - Q^{(q)})(\widetilde{P}^* \widetilde{P})^{-1} \widetilde{P}^* u \mid \Pi^{(q)} v)_M + (\widetilde{P}(I - Q^{(q)})(\widetilde{P}^* \widetilde{P})^{-1} \widetilde{P}^* u \mid \Gamma^{(q)}_1 v)_M \\ &\quad = (\widetilde{P}(I - Q^{(q)})(\widetilde{P}^* \widetilde{P})^{-1} \widetilde{P}^* u \mid \widetilde{P}\gamma \Pi^{(q)} v)_M + (\widetilde{P}(I - Q^{(q)})(\widetilde{P}^* \widetilde{P})^{-1} \widetilde{P}^* u \mid \varepsilon^{(q)} v)_M \\ &\qquad\qquad + (\widetilde{P}(I - Q^{(q)})(\widetilde{P}^* \widetilde{P})^{-1} \widetilde{P}^* u \mid \Gamma^{(q)}_1 v)_M \\ &\quad = [(I - Q^{(q)})(\widetilde{P}^* \widetilde{P})^{-1} \widetilde{P}^* u \mid \gamma \Pi^{(q)} v]_X + ((\varepsilon^{(q)})^* \widetilde{P}(I - Q^{(q)})(\widetilde{P}^* \widetilde{P})^{-1} \widetilde{P}^* u \mid v)_M \\ &\qquad\qquad + ((\Gamma^{(q)}_1)^* \widetilde{P}(I - Q^{(q)})(\widetilde{P}^* \widetilde{P})^{-1} \widetilde{P}^* u \mid v)_M, \end{aligned} \tag{5-66}$$

where $(\varepsilon^{(q)})^*$ and $(\Gamma^{(q)}_1)^*$ are the formal adjoints of $\varepsilon^{(q)}$ and $\Gamma^{(q)}_1$ respectively. From the second formula of (5-47) and noticing that $\Pi^{(q)}$ is properly supported on $U \cap \overline{M}$, we get $(\bar\partial\rho)^{\wedge,*}\gamma\Pi^{(q)} v = 0$; hence $\gamma\Pi^{(q)} v \in \operatorname{Ker}(\bar\partial\rho)^{\wedge,*}$. Thus, $[(I - Q^{(q)})(\widetilde{P}^* \widetilde{P})^{-1} \widetilde{P}^* u \mid \gamma\Pi^{(q)} v]_X = 0$. From this observation, (5-66) and noticing that

$$(\varepsilon^{(q)})^* \widetilde{P}(I - Q^{(q)})(\widetilde{P}^* \widetilde{P})^{-1} \widetilde{P}^*, (\Gamma^{(q)}_1)^* \widetilde{P}(I - Q^{(q)})(\widetilde{P}^* \widetilde{P})^{-1} \widetilde{P}^* \equiv 0 \mod \mathscr{C}^\infty((U \times U) \cap (\overline{M} \times \overline{M})),$$

we get

$$\Pi^{(q)} \widetilde{P}(I - Q^{(q)})(\widetilde{P}^* \widetilde{P})^{-1} \widetilde{P}^* \equiv 0 \mod \mathscr{C}^\infty((U \times U) \cap (\overline{M} \times \overline{M})). \tag{5-67}$$

From (5-67) and Lemma 5.14, we get (5-64).

Let $u \in L^2_c(\overline{M})$ and let $v \in \Omega^{0,q}_c(U \cap M)$. From (5-50), we have

$$
\begin{aligned}
&(\widetilde{P}(I-Q^{(q)})(\widetilde{P}^*\widetilde{P})^{-1}\widetilde{P}^*\Pi^{(q)}u \mid v)_M \\
&\qquad = ((I-Q^{(q)})(\widetilde{P}^*\widetilde{P})^{-1}\widetilde{P}^*\Pi^{(q)}u \mid \widetilde{P}^*v)_X \\
&\qquad = ((I-Q^{(q)})(\widetilde{P}^*\widetilde{P})^{-1}\widetilde{P}^*\Pi^{(q)}u \mid (\widetilde{P}^*\widetilde{P})(\widetilde{P}^*\widetilde{P})^{-1}\widetilde{P}^*v)_X \\
&\qquad = [(I-Q^{(q)})(\widetilde{P}^*\widetilde{P})^{-1}\widetilde{P}^*\Pi^{(q)}u \mid (\widetilde{P}^*\widetilde{P})^{-1}\widetilde{P}^*v]_X \\
&\qquad = [(\widetilde{P}^*\widetilde{P})^{-1}\widetilde{P}^*\Pi^{(q)}u \mid (I-Q^{(q)})(\widetilde{P}^*\widetilde{P})^{-1}\widetilde{P}^*v]_X \\
&\qquad = (\Pi^{(q)}u \mid \widetilde{P}(I-Q^{(q)})(\widetilde{P}^*\widetilde{P})^{-1}\widetilde{P}^*v)_M \\
&\qquad = (u \mid \Pi^{(q)}\widetilde{P}(I-Q^{(q)})(\widetilde{P}^*\widetilde{P})^{-1}\widetilde{P}^*v)_M + (u \mid \Gamma^{(q)}_1\widetilde{P}(I-Q^{(q)})(\widetilde{P}^*\widetilde{P})^{-1}\widetilde{P}^*v)_M. \qquad (5\text{-}68)
\end{aligned}
$$

From (5-68) and (5-67), we deduce that

$$\widetilde{P}(I-Q^{(q)})(\widetilde{P}^*\widetilde{P})^{-1}\widetilde{P}^*\Pi^{(q)} \equiv 0 \bmod \mathscr{C}^\infty((U\times U)\cap(\overline{M}\times\overline{M})).$$

From this observation and (5-62), we get (5-65). □

We can now prove the following regularity property for $\Pi^{(q)}$.

Theorem 5.16. *With the assumptions and notation used before*, $\Pi^{(q)}$ *can be continuously extended to*

$$
\begin{aligned}
&\Pi^{(q)} : H^s_{\mathrm{loc}}(U\cap\overline{M}, T^{*0,q}M') \to H^{s-1}_{\mathrm{loc}}(U\cap\overline{M}, T^{*0,q}M') \quad \textit{for every } s\in\mathbb{Z},\\
&\Pi^{(q)} : H^s_c(U\cap\overline{M}, T^{*0,q}M') \to H^{s-1}_c(U\cap\overline{M}, T^{*0,q}M') \quad \textit{for every } s\in\mathbb{Z}.
\end{aligned} \qquad (5\text{-}69)
$$

Proof. Let $u \in \Omega^{0,q}_c(U\cap\overline{M})$. From (5-64), we see that

$$\Pi^{(q)}u = \Pi^{(q)}\widetilde{P}Q^{(q)}(\widetilde{P}^*\widetilde{P})^{-1}\widetilde{P}^*u + \gamma^{(q)}u, \qquad (5\text{-}70)$$

where $\gamma^{(q)} : \Omega^{0,q}_c(U\cap M) \to \mathscr{D}'(U\cap M, T^{*0,q}M')$ is a continuous operator with

$$\gamma^{(q)} \equiv 0 \bmod \mathscr{C}^\infty((U\times U)\cap(\overline{M}\times\overline{M})).$$

From Theorem 4.3, (5-8), Theorem 5.9 and noticing that

$$\widetilde{P}Q^{(q)}(\widetilde{P}^*\widetilde{P})^{-1}\widetilde{P}^*u \in A^{0,q}(U\cap\overline{M}),$$

we conclude that

$$\Pi^{(q)}u = (I - \bar\partial^*_f N^{(q+1)}\bar\partial - \bar\partial N^{(q-1)}\bar\partial^*_f)\widetilde{P}Q^{(q)}(\widetilde{P}^*\widetilde{P})^{-1}\widetilde{P}^*u + \gamma^{(q)}_1 u, \qquad (5\text{-}71)$$

where $\gamma^{(q)}_1 : \Omega^{0,q}_c(U\cap M) \to \mathscr{D}'(U\cap M, T^{*0,q}M')$ is a continuous operator with

$$\gamma^{(q)}_1 \equiv 0 \bmod \mathscr{C}^\infty((U\times U)\cap(\overline{M}\times\overline{M})).$$

From (5-71),

$$
\begin{aligned}
&N^{(q-1)} : H^s_c(U\cap\overline{M}, T^{*0,q}M') \to H^{s+1}_c(U\cap\overline{M}, T^{*0,q}M') \quad \text{for every } s\in\mathbb{Z},\\
&N^{(q+1)} : H^s_c(U\cap\overline{M}, T^{*0,q}M') \to H^{s+1}_c(U\cap\overline{M}, T^{*0,q}M') \quad \text{for every } s\in\mathbb{Z}
\end{aligned}
$$

are continuous and note that $\Omega_c^{0,q}(U\cap\overline{M})$ is dense in $H_c^s(U\cap\overline{M},T^{*0,q}M')$ for every $s\in\mathbb{Z}$, and thus we get (5-69). □

The reason why in the proof of Theorem 5.16 we do not use $\widehat{\Pi}^{(q)}$ directly is the following: In (5-8), $\widehat{\Pi}^{(q)}$ is just defined on the space $A^{0,q}(U\cap\overline{M})$. If $u\in\Omega_c^{0,q}(U\cap\overline{M})$, we cannot define $\widehat{\Pi}^{(q)}u$ by using (5-8) since in general

$$(I-\bar\partial_f^*N^{(q+1)}\bar\partial-\bar\partial N^{(q-1)}\bar\partial_f^*)u\notin\operatorname{Dom}\Box^{(q)}.$$

We extend $\widehat{\Pi}^{(q)}$ to $L^2_{\mathrm{loc}}(U\cap\overline{M},T^{*0,q}M')$ by density and we have (5-71) for the relation between $\Pi^{(q)}$ and (5-8).

5.3. ***Reduction to the analysis on the boundary.*** In order to refine the approximate Hodge decomposition of Theorem 5.9 and show that $\Pi^{(q)}$ is a Fourier integral operator we will bring in an approximate Szegő projector on the boundary, which is a Fourier integral operator, and link it to $\Pi^{(q)}$ by means of the Poisson operator. The approximate Szegő projector appears in the microlocal Hodge decomposition of the boundary Laplacian $\Box_\beta^{(q)}$, which is a perturbation of the Kohn Laplacian.

We recall the operators $\bar\partial_\beta$ and $\Box_\beta^{(q)}$ introduced in [Hsiao 2010, Part II, Chapter 5]. Recall that $Q^{(q+1)}$ is given by (5-56). The operator $\bar\partial_\beta$ is defined by

$$\bar\partial_\beta=Q^{(q+1)}\gamma\bar\partial\widetilde{P}:\Omega^{0,q}(X)\to\Omega^{0,q+1}(X)\tag{5-72}$$

and it is obtained by taking the $\bar\partial$ derivative of the extension of a form to the interior by the Poisson operator and then taking the projection on the space of the tangential forms to the boundary. It is a classical pseudodifferential operator of order 1 which is a perturbation of the $\bar\partial_b$ operator by a zeroth-order operator. It has the advantage that it involves directly the Poisson operator. Let

$$\bar\partial_\beta^\dagger:\Omega^{0,q+1}(X)\to\Omega^{0,q}(X)\tag{5-73}$$

be the formal adjoint of $\bar\partial_\beta$ with respect to $[\cdot\mid\cdot]_X$, that is, $[\bar\partial_\beta f\mid h]=[f\mid\bar\partial_\beta^\dagger h]_X$, $f\in\Omega^{0,q}(X)$, $h\in\Omega^{0,q+1}(X)$. Then $\bar\partial_\beta^\dagger$ is a classical pseudodifferential operator of order 1 and we have

$$\bar\partial_\beta^\dagger=\gamma\bar\partial_f^*\widetilde{P}\quad\text{on }\Omega^{0,q}(X)\text{ for }q=1,\dots,n-1;\tag{5-74}$$

see [Hsiao 2010, Part II, Chapter 5]. Set

$$\Box_\beta^{(q)}=\bar\partial_\beta^\dagger\,\bar\partial_\beta+\bar\partial_\beta\,\bar\partial_\beta^\dagger:\mathscr{D}'(X,T^{*0,q}X)\to\mathscr{D}'(X,T^{*0,q}X).\tag{5-75}$$

It was shown in [Hsiao 2010, Part II, Chapter 5] that $\Box_\beta^{(q)}$ is a classical pseudodifferential operator of order 2 and the characteristic manifold of $\Box_\beta^{(q)}$ is given by $\Sigma=\Sigma^+\cup\Sigma^-$, where Σ^+, Σ^- are as in (3-12). Roughly speaking, forms annihilated by $\Box_\beta^{(q)}$ on the boundary are microlocally boundary values of harmonic forms. More precisely, if S_β is the orthogonal projection onto the kernel of $\Box_\beta^{(q)}$, then $\widetilde{P}S_\beta$ is in the kernel of the $\bar\partial$-Neumann Laplacian up to a smoothing operator. If S is the orthogonal projection onto the kernel of $\Box_b^{(q)}$ (the Szegő projector), then $\widetilde{P}S$ does not have this property.

Let D be a local coordinate patch of X with local coordinates $x=(x_1,\dots,x_{2n-1})$ and we assume the Levi form is nondegenerate of constant signature (n_-,n_+) on D. Let $H\in L_{\mathrm{cl}}^{-1}(D,T^{*0,q}X\boxtimes(T^{*0,q}X)^*)$

be a properly supported pseudodifferential operator of order -1 on D such that

$$H - \widetilde{P}^* \widetilde{P} \equiv 0 \quad \text{on } D. \tag{5-76}$$

The following microlocal Hodge decomposition for $\Box^{(q)}_\beta$ was established in [Hsiao 2010, Part II, Theorem 6.15].

Theorem 5.17. *With the assumptions and notation above, let $q = n_-$. Then there exist properly supported operators*

$$A \in L^{-1}_{\frac{1}{2},\frac{1}{2}}(D, T^{*0,q}X \boxtimes (T^{*0,q}X)^*),$$
$$S_-, S_+ \in L^{0}_{\frac{1}{2},\frac{1}{2}}(D, T^{*0,q}X \boxtimes (T^{*0,q}X)^*)$$

such that

$$\begin{aligned} \mathrm{WF}'(S_-(x,y)) &= \mathrm{diag}((\Sigma^+ \cap T^*D) \times (\Sigma^+ \cap T^*D)), \\ \mathrm{WF}'(S_+(x,y)) &\subset \mathrm{diag}((\Sigma^- \cap T^*D) \times (\Sigma^- \cap T^*D)) \end{aligned} \tag{5-77}$$

and

$$A\Box^{(q)}_\beta + S_- + S_+ = I, \tag{5-78}$$
$$\bar\partial_\beta S_- \equiv 0, \quad \bar\partial^\dagger_\beta S_- \equiv 0, \tag{5-79}$$
$$S_- \equiv S_-^\dagger \equiv S_-^2, \tag{5-80}$$
$$S_+ \equiv 0 \quad \text{if } q \neq n_+, \tag{5-81}$$

where

$$S_-^\dagger := 2Q^{(q)}(\widetilde{P}^*\widetilde{P})^{-1} S_-^* (\bar\partial\rho)^{\wedge,*}(\bar\partial\rho)^\wedge H : \Omega^{0,q}_c(D) \to \Omega^{0,q}(X), \tag{5-82}$$

H is given by (5-76) *and S_-^* is the formal adjoint of S_- with respect to $(\cdot\,|\,\cdot)_X$. Moreover, the kernel $S_-(x,y)$ satisfies*

$$S_-(x,y) \equiv \int_0^\infty e^{i\varphi_-(x,y)t} a(x,y,t)\,dt,$$

with

$$\begin{aligned} &a(x,y,t) \in S^{n-1}_{1,0}(D \times D \times (0,\infty), T^{*0,q}X \boxtimes (T^{*0,q}X)^*), \\ &a(x,y,t) \sim \sum_{j=0}^\infty a_j(x,y) t^{n-1-j} \quad \text{in } S^{n-1}_{1,0}(D \times D \times (0,\infty), T^{*0,q}X \boxtimes (T^{*0,q}X)^*) \end{aligned} \tag{5-83}$$

and

$$a_0(x,x) = \frac{1}{2\pi^n} |\det \mathcal{L}_x| \tau_{x,n_-} \quad \text{for every } x \in D, \tag{5-84}$$

where $a_j(x,y) \in \mathscr{C}^\infty(D \times D; T^{*0,q}X \boxtimes (T^{*0,q}X)^*)$, $j = 0, 1, \dots,$ and the phase function φ_- is the same as the phase function appearing in the description of the singularities of the Szegő kernels for lower-energy forms in [Hsiao and Marinescu 2017, Theorems 3.3, 3.4]. *In particular, we have*

$$\varphi_-(x,y) \in \mathscr{C}^\infty(X \times X), \quad \mathrm{Im}\,\varphi_-(x,y) \geq 0, \tag{5-85}$$
$$\varphi_-(x,x) = 0, \quad \varphi_-(x,y) \neq 0 \quad \text{if } x \neq y, \tag{5-86}$$
$$d_x\varphi_- \neq 0, \quad d_y\varphi_- \neq 0 \quad \text{where } \mathrm{Im}\,\varphi_- = 0, \tag{5-87}$$

$$d_x\varphi_-(x,y)|_{x=y} = -d_y\varphi_-(x,y)|_{x=y} = \omega_0(x), \tag{5-88}$$
$$\varphi_-(x,y) = -\bar{\varphi}_-(y,x). \tag{5-89}$$

We have denoted by $\mathrm{WF}(S_\pm(x,y))$ the wave front set in the sense of Hörmander of the distributions $S_\pm(x,y)$ and

$$\mathrm{WF}'(S_\pm(x,y)) := \{(x,\xi,y,\eta)\in T^*X\times T^*X : (x,\xi,y,-\eta)\in \mathrm{WF}(S_\pm(x,y))\}.$$

The leading coefficient $a_0(x,x)$ from (5-83) was obtained in [Hsiao 2010, Part II, Proposition 6.17].

We come back to our situation. In view of Lemmas 4.1, 4.2 and Theorem 5.12, we see that $Q^{(q)}(\widetilde{P}^*\widetilde{P})^{-1}\widetilde{P}^*$ is smoothing away the diagonal. Hence, there is a continuous operator $L^{(q)}$: $\Omega^{0,q}_c(U\cap\overline{M})\to\Omega^{0,q}(D)$ such that

$$L^{(q)} - Q^{(q)}(\widetilde{P}^*\widetilde{P})^{-1}\widetilde{P}^* \equiv 0 \mod \mathscr{C}^\infty((U\times U)\cap(X\times\overline{M})) \tag{5-90}$$

and $L^{(q)}$ is properly supported on $U\cap\overline{M}$, that is, for every $\chi\in\mathscr{C}^\infty_c(U\cap\overline{M})$, there is a $\tau\in\mathscr{C}^\infty_c(D)$ such that $L^{(q)}\chi=\tau L^{(q)}$ on $\Omega^{0,q}_c(U\cap\overline{M})$ and, for every $\tau_1\in\mathscr{C}^\infty_c(D)$, there is a $\chi_1\in\mathscr{C}^\infty_c(U\cap\overline{M})$ such that $\tau_1L^{(q)}=L^{(q)}\chi_1$ on $\Omega^{0,q}_c(U\cap\overline{M})$. We can extend $L^{(q)}$ to a continuous operator

$$L^{(q)}:\Omega^{0,q}(U\cap\overline{M})\to\Omega^{0,q}(D),\quad L^{(q)}:\Omega^{0,q}_c(U\cap\overline{M})\to\Omega^{0,q}_c(D).$$

From Theorem 5.15, we have

$$\Pi^{(q)} - \widetilde{P}L^{(q)}\Pi^{(q)} \equiv 0 \mod \mathscr{C}^\infty((U\times U)\cap(\overline{M}\times\overline{M})). \tag{5-91}$$

Lemma 5.18. *With the notation and assumptions above, we have*

$$S_+L^{(q)}\Pi^{(q)} \equiv 0 \mod \mathscr{C}^\infty((U\times U)\cap(X\times\overline{M})), \tag{5-92}$$

where S_+ is as in Theorem 5.17.

Proof. Since $\mathrm{WF}'(S_+(x,y))\subset\mathrm{diag}\big((\Sigma^-\cap T^*D)\times(\Sigma^-\cap T^*D)\big)$ and by Theorem 3.2 the operator $\Box^{(q)}_-$ is elliptic near Σ^-, there is a classical pseudodifferential operator $E^{(q)}\in L^{-1}_{\mathrm{cl}}(D,T^{*0,q}X\boxtimes(T^{*0,q}X)^*)$ such that

$$S_+ - S_+E^{(q)}\Box^{(q)}_- \equiv 0. \tag{5-93}$$

From (5-46) and (5-91), we deduce that

$$\Box^{(q)}_-L^{(q)}\Pi^{(q)} \equiv 0 \mod \mathscr{C}^\infty((U\times U)\cap(X\times\overline{M})). \tag{5-94}$$

From (5-93) and (5-94), we get (5-92). □

Theorem 5.19. *With the notation and assumptions above, we have*

$$S_-L^{(q)}\Pi^{(q)} - L^{(q)}\Pi^{(q)} \equiv 0 \mod \mathscr{C}^\infty((U\times U)\cap(X\times\overline{M})), \tag{5-95}$$
$$\widetilde{P}S_-L^{(q)}\Pi^{(q)} - \Pi^{(q)} \equiv 0 \mod \mathscr{C}^\infty((U\times U)\cap(\overline{M}\times\overline{M})), \tag{5-96}$$
$$\Pi^{(q)}\widetilde{P}S_-L^{(q)} - \Pi^{(q)} \equiv 0 \mod \mathscr{C}^\infty((U\times U)\cap(\overline{M}\times\overline{M})). \tag{5-97}$$

Proof. From (5-46) and (5-91), we see that

$$\Box^{(q)}_{\beta} L^{(q)} \Pi^{(q)} \equiv 0 \bmod \mathscr{C}^{\infty}((U \times U) \cap (X \times \overline{M})). \tag{5-98}$$

From (5-78), (5-92) and (5-98), we have

$$\begin{aligned} L^{(q)} \Pi^{(q)} &= (A \Box^{(q)}_{\beta} + S_- + S_+) L^{(q)} \Pi^{(q)} \\ &\equiv S_- L^{(q)} \Pi^{(q)} \bmod \mathscr{C}^{\infty}((U \times U) \cap (X \times \overline{M})) \end{aligned}$$

and we get (5-95). From (5-95) and (5-91), we get (5-96). We now prove (5-97). Put

$$\begin{aligned} &\gamma^{(q)} := \Pi^{(q)} - \widetilde{P} L^{(q)} \Pi^{(q)} : \Omega^{0,q}_c(U \cap \overline{M}) \to \Omega^{0,q}(\overline{M}), \\ &\gamma^{(q)}_0 := \widetilde{P}^* \widetilde{P} - H : \Omega^{0,q}_c(D) \to \Omega^{0,q}(X), \\ &\gamma^{(q)}_1 := S^{\dagger}_- - S_- : \Omega^{0,q}_c(D) \to \Omega^{0,q}(X), \\ &\gamma^{(q)}_2 := L^{(q)} - Q^{(q)} (\widetilde{P}^* \widetilde{P})^{-1} \widetilde{P}^* : \Omega^{0,q}_c(U \cap \overline{M}) \to \Omega^{0,q}(X), \\ &\gamma^{(q)}_3 := S_- L^{(q)} \Pi^{(q)} - L^{(q)} \Pi^{(q)} : \Omega^{0,q}_c(U \cap \overline{M}) \to \Omega^{0,q}_c(D), \end{aligned}$$

where $S^{\dagger}_-$ is given by (5-82). From (5-80), (5-90), (5-91) and (5-95), we see that

$$\begin{gathered} \gamma^{(q)} \equiv 0 \bmod \mathscr{C}^{\infty}((U \times U) \cap (\overline{M} \times \overline{M})), \\ \gamma^{(q)}_2 \equiv 0 \bmod \mathscr{C}^{\infty}((U \times U) \cap (X \times \overline{M})), \\ \gamma^{(q)}_3 \equiv 0 \bmod \mathscr{C}^{\infty}((U \times U) \cap (X \times \overline{M})), \\ \gamma^{(q)}_1 \equiv 0, \quad \gamma^{(q)}_0 \equiv 0. \end{gathered} \tag{5-99}$$

Let

$$(\gamma^{(q)})^* : \Omega^{0,q}(\overline{M}) \to \Omega^{0,q}(U \cap \overline{M})$$

be the formal adjoint of $\gamma^{(q)}$ with respect to $(\cdot \mid \cdot)_M$ and let

$$(\gamma^{(q)}_2)^* : \Omega^{0,q}(X) \to \Omega^{0,q}(U \cap \overline{M})$$

be the formal adjoint of $\gamma^{(q)}_2$ with respect to $(\cdot \mid \cdot)_M$ and $(\cdot \mid \cdot)_X$, that is,

$$(\gamma^{(q)}_2 u \mid v)_X = (u \mid (\gamma^{(q)}_2)^* v)_M \quad \text{for every } u \in \Omega^{0,q}_c(U \cap \overline{M}),\ v \in \Omega^{0,q}(X).$$

It is obvious that

$$(\gamma^{(q)})^* \equiv 0, \ \ (\gamma^{(q)}_2)^* \equiv 0 \bmod \mathscr{C}^{\infty}((U \times U) \cap (\overline{M} \times X)). \tag{5-100}$$

Let $u, v \in \Omega^{0,q}_c(U \cap \overline{M})$. From (5-50), it is straightforward to check that

$$\begin{aligned} &(\Pi^{(q)} \widetilde{P} S_- L^{(q)} u \mid v)_M \\ &\quad = (\widetilde{P} S_- L^{(q)} u \mid \Pi^{(q)} v)_M + (\widetilde{P} S_- L^{(q)} u \mid \Gamma^{(q)}_1 v)_M \\ &\quad = (\widetilde{P} S_- L^{(q)} u \mid \widetilde{P} L^{(q)} \Pi^{(q)} v)_M + (\widetilde{P} S_- L^{(q)} u \mid \gamma^{(q)} v)_M + (\widetilde{P} S_- L^{(q)} u \mid \Gamma^{(q)}_1 v)_M \\ &\quad = (S_- L^{(q)} u \mid H L^{(q)} \Pi^{(q)} v)_X + (S_- L^{(q)} u \mid \gamma^{(q)}_0 L^{(q)} \Pi^{(q)} v)_X \\ &\qquad\qquad + (\widetilde{P} S_- L^{(q)} u \mid \gamma^{(q)} v)_M + (\widetilde{P} S_- L^{(q)} u \mid \Gamma^{(q)}_1 v)_M \end{aligned} \tag{5-101}$$

$$
\begin{aligned}
&= [L^{(q)}u \mid S_-^{\dagger} L^{(q)}\Pi^{(q)}v]_X + (S_-L^{(q)}u \mid \gamma_0^{(q)} L^{(q)}\Pi^{(q)}v)_X \\
&\qquad\qquad + (\widetilde{P}S_-L^{(q)}u \mid \gamma^{(q)}v)_M + (\widetilde{P}S_-L^{(q)}u \mid \Gamma_1^{(q)}v)_M \\
&= [L^{(q)}u \mid S_-L^{(q)}\Pi^{(q)}v]_X + [L^{(q)}u \mid \gamma_1^{(q)} L^{(q)}\Pi^{(q)}v]_X + (S_-L^{(q)}u \mid \gamma_0^{(q)} L^{(q)}\Pi^{(q)}v)_X \\
&\qquad\qquad + (\widetilde{P}S_-L^{(q)}u \mid \gamma^{(q)}v)_M + (\widetilde{P}S_-L^{(q)}u \mid \Gamma_1^{(q)}v)_M \\
&= [Q^{(q)}(\widetilde{P}^*\widetilde{P})^{-1}\widetilde{P}^*u \mid S_-L^{(q)}\Pi^{(q)}v]_X + [\gamma_2^{(q)}u \mid S_-L^{(q)}\Pi^{(q)}v]_X + [L^{(q)}u \mid \gamma_1^{(q)}L^{(q)}\Pi^{(q)}v]_X \\
&\qquad + (S_-L^{(q)}u \mid \gamma_0^{(q)}L^{(q)}\Pi^{(q)}v)_X + (\widetilde{P}S_-L^{(q)}u \mid \gamma^{(q)}v)_M + (\widetilde{P}S_-L^{(q)}u \mid \Gamma_1^{(q)}v)_M \\
&= (u \mid \widetilde{P}S_-L^{(q)}\Pi^{(q)}v)_M + (\widetilde{P}\gamma_2^{(q)}u \mid \widetilde{P}S_-L^{(q)}\Pi^{(q)}v)_M + (\widetilde{P}L^{(q)}u \mid \widetilde{P}\gamma_1^{(q)}L^{(q)}\Pi^{(q)}v)_M \\
&\quad + (\widetilde{P}S_-L^{(q)}u \mid \widetilde{P}(\widetilde{P}^*\widetilde{P})^{-1}\gamma_0^{(q)}L^{(q)}\Pi^{(q)}v)_M + (\widetilde{P}S_-L^{(q)}u \mid \gamma^{(q)}v)_M + (\widetilde{P}S_-L^{(q)}u \mid \Gamma_1^{(q)}v)_M \\
&= (u \mid \widetilde{P}L^{(q)}\Pi^{(q)}v)_M + (u \mid \widetilde{P}\gamma_3^{(q)}v)_M + (u \mid (\gamma_2^{(q)})^*\widetilde{P}^*\widetilde{P}S_-L^{(q)}\Pi^{(q)}v)_M \\
&\qquad + (u \mid (L^{(q)})^*\widetilde{P}^*\widetilde{P}\gamma_1^{(q)}L^{(q)}\Pi^{(q)}v)_M + (u \mid (L^{(q)})^*S_-^*\gamma_0^{(q)}L^{(q)}\Pi^{(q)}v)_M \\
&\qquad\qquad + (u \mid (L^{(q)})^*(S_-)^*(\widetilde{P})^*\gamma^{(q)}v)_M + (u \mid (L^{(q)})^*(S_-)^*(\widetilde{P})^*\Gamma_1^{(q)}v)_M \\
&= (u \mid \Pi^{(q)}v)_M - (u \mid \gamma^{(q)}v)_M + (u \mid \widetilde{P}\gamma_3^{(q)}v)_M + (u \mid (\gamma_2^{(q)})^*\widetilde{P}^*\widetilde{P}S_-L^{(q)}\Pi^{(q)}v)_M \\
&\qquad + (u \mid (L^{(q)})^*\widetilde{P}^*\widetilde{P}\gamma_1^{(q)}L^{(q)}\Pi^{(q)}v)_M + (u \mid (L^{(q)})^*S_-^*\gamma_0^{(q)}L^{(q)}\Pi^{(q)}v)_M \\
&\qquad + (u \mid (L^{(q)})^*(S_-)^*(\widetilde{P})^*\gamma^{(q)}v)_M + (u \mid (L^{(q)})^*(S_-)^*(\widetilde{P})^*\Gamma_1^{(q)}v)_M, \qquad \text{(5-101 cont.)}
\end{aligned}
$$

where $(L^{(q)})^* : \Omega^{0,q}(D) \to \Omega^{0,q}(U \cap \overline{M})$ is the formal adjoint of $L^{(q)}$ with respect to $(\cdot \mid \cdot)_M$ and $(\cdot \mid \cdot)_X$. We explain the third-to-last equality of (5-101). Since $S_-L^{(q)}\Pi^{(q)}v \in \mathrm{Ker}(\bar\partial\rho)^{\wedge,*}$, we have

$$
[\,Q^{(q)}(\widetilde{P}^*\widetilde{P})^{-1}\widetilde{P}^*u \mid S_-L^{(q)}\Pi^{(q)}v\,]_X = [\,(\widetilde{P}^*\widetilde{P})^{-1}\widetilde{P}^*u \mid S_-L^{(q)}\Pi^{(q)}v\,]_X. \tag{5-102}
$$

From (5-102), we get the third-to-last equality of (5-101).

Note that $(L^{(q)})^*$ is properly supported. From (5-101), we conclude that there is a continuous operator $\varepsilon^{(q)} : \Omega_c^{0,q}(U \cap \overline{M}) \to \Omega^{0,q}(U \cap \overline{M})$ with $\varepsilon^{(q)} \equiv 0 \bmod \mathscr{C}^\infty((U \times U) \cap (\overline{M} \times \overline{M}))$ such that

$$
(\Pi^{(q)}\widetilde{P}S_-L^{(q)}u \mid v\,)_M = (u \mid \Pi^{(q)}v\,)_M + (u \mid \varepsilon^{(q)}v\,)_M \tag{5-103}
$$

for every $u, v \in \Omega_c^{0,q}(U \cap \overline{M})$. From (5-50) and (5-103), we get

$$
(\Pi^{(q)}\widetilde{P}S_-L^{(q)}u \mid v\,)_M = (\Pi^{(q)}u \mid v\,)_M - ((\Gamma_1^{(q)})^*u \mid v\,)_M + ((\varepsilon^{(q)})^*u \mid v\,)_M \tag{5-104}
$$

for every $u, v \in \Omega_c^{0,q}(U \cap \overline{M})$, where $(\Gamma_1^{(q)})^*, (\varepsilon^{(q)})^* : \Omega_c^{0,q}(U \cap \overline{M}) \to \Omega^{0,q}(U \cap \overline{M})$ are the formal adjoints of $\Gamma_1^{(q)}$ and $\varepsilon^{(q)}$ with respect to $(\cdot \mid \cdot)_M$ respectively. Note that $(\Gamma_1^{(q)})^*, (\varepsilon^{(q)})^* \equiv 0 \bmod \mathscr{C}^\infty((U \times U) \cap (\overline{M} \times \overline{M}))$. From this observation and (5-104), we get (5-97). $\square$

Theorem 5.20. *With the notation and assumptions used above, we have*

$$
\bar\partial\widetilde{P}S_-L^{(q)} \equiv 0 \bmod \mathscr{C}^\infty((U \times U) \cap (\overline{M} \times \overline{M})). \tag{5-105}
$$

Proof. From [Hsiao 2010, Part II, Proposition 6.18], we have

$$
\gamma\bar\partial\widetilde{P}S_- \equiv 0. \tag{5-106}
$$

From (3-8), we have

$$
D^{(q+1)}\widetilde{\Box}_f^{(q+1)}\bar\partial\widetilde{P}S_- + \widetilde{P}\gamma\bar\partial\widetilde{P}S_- = \bar\partial\widetilde{P}S_-. \tag{5-107}
$$

Now,

$$\begin{aligned}D^{(q+1)}\widetilde{\Box}_f^{(q+1)}\bar\partial\widetilde{P}S_- &= D^{(q+1)}(\Box_f^{(q+1)}+K^{(q+1)})\bar\partial\widetilde{P}S_-\\ &= D^{(q+1)}\bar\partial\Box_f^{(q)}\widetilde{P}S_- + D^{(q+1)}K^{(q+1)}\bar\partial\widetilde{P}S_-\\ &= D^{(q+1)}\bar\partial\widetilde{\Box}_f^{(q)}\widetilde{P}S_- - D^{(q+1)}\bar\partial K^{(q)}\widetilde{P}S_- + D^{(q+1)}K^{(q+1)}\bar\partial\widetilde{P}S_-\\ &= -D^{(q+1)}\bar\partial K^{(q)}\widetilde{P}S_- + D^{(q+1)}K^{(q+1)}\bar\partial\widetilde{P}S_-\\ &\equiv 0 \bmod \mathscr{C}^\infty((U\times U)\cap(\overline{M}\times X)). \end{aligned}\tag{5-108}$$

From (5-106), (5-107) and (5-108), we get (5-105). □

Let $\delta^{(q)} := 2\rho\widetilde{P}((\bar\partial\rho)^{\wedge,*}\gamma\bar\partial\widetilde{P}S_-L^{(q)}) : \Omega_c^{0,q}(U\cap\overline{M}) \to \Omega^{0,q}(\overline{M})$. By (5-105) we have

$$\delta^{(q)} \equiv 0 \bmod \mathscr{C}^\infty((U\times U)\cap(\overline{M}\times\overline{M})). \tag{5-109}$$

Moreover, it is easy to check that

$$(\widetilde{P}S_-L^{(q)}-\delta^{(q)})u \in \operatorname{Dom}\Box^{(q)}\cap\Omega^{0,q}(\overline{M}) \quad \text{for every } u\in\Omega_c^{0,q}(U\cap\overline{M}). \tag{5-110}$$

We come now to the crucial relation between the approximate Bergman and Szegő kernels via the Poisson operator.

Theorem 5.21. *With the notation and assumptions used above, we have*

$$\Pi^{(q)} - \widetilde{P}S_-L^{(q)} \equiv 0 \bmod \mathscr{C}^\infty((U\times U)\cap(\overline{M}\times\overline{M})). \tag{5-111}$$

Proof. We first claim that

$$\Pi^{(q)}\widetilde{P}S_-L^{(q)} - \widetilde{P}S_-L^{(q)} \equiv 0 \bmod \mathscr{C}^\infty((U\times U)\cap(\overline{M}\times\overline{M})). \tag{5-112}$$

From (5-52) and (5-110), we have

$$\begin{aligned}N^{(q)}\Box^{(q)}(\widetilde{P}S_-L^{(q)}-\delta^{(q)})u &+ \Pi^{(q)}(\widetilde{P}S_-L^{(q)}-\delta^{(q)})u\\ &= (\widetilde{P}S_-L^{(q)}-\delta^{(q)})u + \Lambda^{(q)}(\widetilde{P}S_-L^{(q)}-\delta^{(q)})u\end{aligned} \tag{5-113}$$

for every $u\in\Omega_c^{0,q}(U\cap\overline{M})$, where $\Lambda^{(q)}\equiv 0 \bmod \mathscr{C}^\infty((U\times U)\cap(\overline{M}\times\overline{M}))$ is as in (5-52). From (5-79), (5-105) and (5-109), we have

$$N^{(q)}\Box^{(q)}(\widetilde{P}S_-L^{(q)}-\delta^{(q)})u = N^{(q)}\Box_f^{(q)}(\widetilde{P}S_-L^{(q)}-\delta^{(q)})u = \Lambda_1^{(q)}u \tag{5-114}$$

for every $u\in\Omega_c^{0,q}(U\cap\overline{M})$, where $\Lambda_1^{(q)}\equiv 0 \bmod \mathscr{C}^\infty((U\times U)\cap(\overline{M}\times\overline{M}))$. From (5-109), (5-113) and (5-114) we get the claim (5-112).

From (5-97) and (5-112), we get (5-111). □

Note that $S_-\in L^0_{1/2,1/2}(D, T^{*0,q}X\boxtimes(T^{*0,q}X)^*)$. From this observation and the classical result of Calderon and Vaillancourt (see (2-2)), (3-6), (3-10) and (5-111), we can improve Theorem 5.16 as follows.

Theorem 5.22. *With the notation used above*, $\Pi^{(q)}$ *can be continuously extended to*

$$\begin{aligned}\Pi^{(q)} &: H^s_{\mathrm{loc}}(U\cap\overline{M}, T^{*0,q}M') \to H^s_{\mathrm{loc}}(U\cap\overline{M}, T^{*0,q}M') \quad \textit{for every } s\in\mathbb{Z},\\ \Pi^{(q)} &: H^s_c(U\cap\overline{M}, T^{*0,q}M') \to H^s_c(U\cap\overline{M}, T^{*0,q}M') \quad \textit{for every } s\in\mathbb{Z}.\end{aligned}\tag{5-115}$$

5.4. ***Final version of the microlocal Hodge decomposition.*** We can now prove the our final version of the approximate Hodge decomposition by constructing a parametrix $N^{(q)}$ and an approximate Bergman projector $\Pi^{(q)}$, which is a Fourier integral operator with complex phase.

Theorem 5.23. *Let U be an open set of M' with $U\cap X\neq\varnothing$. Suppose that the Levi form is nondegenerate of constant signature (n_-,n_+) on $U\cap X$. Let $q=n_-$. There exist properly supported continuous operators on $U\cap\overline{M}$,*

$$\begin{aligned} N^{(q)} &: H^s_{\mathrm{loc}}(U\cap\overline{M},T^{*0,q}M')\to H^s_{\mathrm{loc}}(U\cap\overline{M},T^{*0,q}M') \quad \text{for every } s\in\mathbb{Z},\\ \Pi^{(q)} &: H^s_{\mathrm{loc}}(U\cap\overline{M},T^{*0,q}M')\to H^s_{\mathrm{loc}}(U\cap\overline{M},T^{*0,q}M') \quad \text{for every } s\in\mathbb{Z}, \end{aligned} \tag{5-116}$$

such that

$$\begin{aligned} N^{(q)}u &\in \operatorname{Dom}\Box^{(q)} \quad \text{for every } u\in\Omega^{0,q}_c(U\cap\overline{M}),\\ \Pi^{(q)}u &\in \operatorname{Dom}\Box^{(q)} \quad \text{for every } u\in\Omega^{0,q}_c(U\cap\overline{M}), \end{aligned} \tag{5-117}$$

and on $U\cap\overline{M}$, we have

$$\begin{aligned} \Box^{(q)}_f N^{(q)}u+\Pi^{(q)}u &= u+r^{(q)}_0u && \text{for every } u\in\Omega^{0,q}(U\cap M),\\ N^{(q)}\Box^{(q)}u+\Pi^{(q)}u &= u+r^{(q)}_1u && \text{for every } u\in\operatorname{Dom}\Box^{(q)},\\ \bar\partial\Pi^{(q)}u &= r^{(q)}_2u && \text{for every } u\in L^2_{\mathrm{loc}}(U\cap\overline{M},T^{*0,q}M'),\\ \bar\partial^*_f\Pi^{(q)}u &= r^{(q)}_3u && \text{for every } u\in L^2_{\mathrm{loc}}(U\cap\overline{M},T^{*0,q}M'),\\ \Pi^{(q)}\Box^{(q)}u &= r^{(q)}_4u && \text{for every } u\in\operatorname{Dom}\Box^{(q)},\\ \Box^{(q)}_f\Pi^{(q)}u &= r^{(q)}_5u && \text{for every } u\in\Omega^{0,q}(U\cap\overline{M}),\\ (\Pi^{(q)})^2u-\Pi^{(q)}u &= r^{(q)}_6u && \text{for every } u\in\Omega^{0,q}(U\cap\overline{M}), \end{aligned} \tag{5-118}$$

where $r^{(q)}_j$ is properly supported on $U\cap\overline{M}$ with $r^{(q)}_j\equiv 0 \bmod \mathscr{C}^\infty((U\times U)\cap(\overline{M}\times\overline{M}))$ for every $j=0,\dots,6$, and the distribution kernel of $\Pi^{(q)}$ satisfies

$$\Pi^{(q)}(z,w)\equiv\int_0^\infty e^{i\phi(z,w)t}b(z,w,t)\,dt \mod \mathscr{C}^\infty((U\times U)\cap(\overline{M}\times\overline{M})), \tag{5-119}$$

with

$$\begin{aligned} &b(z,w,t)\in S^n_{1,0}((U\times U)\cap(\overline{M}\times\overline{M})\times(0,\infty),\Lambda^{(0,q)\mid(0,q)}_{M'\times M'}),\\ &b(z,w,t)\sim\sum_{j=0}^\infty b_j(z,w)t^{n-j} \quad \text{in } S^n_{1,0}((U\times U)\cap(\overline{M}\times\overline{M})\times(0,\infty),\Lambda^{(0,q)\mid(0,q)}_{M'\times M'}), \end{aligned} \tag{5-120}$$

with $b_0(z,z)$ given by (5-124) *below. Moreover,*

$$\begin{gathered} \phi(z,w)\in\mathscr{C}^\infty((U\times U)\cap(\overline{M}\times\overline{M})), \quad \operatorname{Im}\phi\geq 0,\\ \phi(z,z)=0, \quad z\in U\cap X, \quad \phi(z,w)\neq 0 \quad \text{if } (z,w)\notin\operatorname{diag}((U\times U)\cap(X\times X)),\\ \operatorname{Im}\phi(z,w)>0 \quad \text{if } (z,w)\notin(U\times U)\cap(X\times X),\\ \phi(z,w)=-\bar\phi(w,z),\\ d_x\phi(x,y)|_{x=y}=-2i\,\partial\rho(x) \quad \text{for every } x\in U\cap X, \end{gathered} \tag{5-121}$$

$\phi(z,w)\in\mathscr{C}^\infty((U\times U)\cap(\overline{M}\times\overline{M}))$ is as in [Hsiao 2010, Part II, Theorem 1.4] *and $\phi(z,w)=\varphi_-(z,w)$ if $z,w\in U\cap X$, where $\varphi_-\in\mathscr{C}^\infty((U\times U)\cap(X\times X))$ is as in Theorem 5.17.*

Proof. Let $N^{(q)}$ and $\Pi^{(q)}$ be as in Theorem 5.9. From (5-47), (5-48) and noticing that $N^{(q)}$ and $\Pi^{(q)}$ are properly supported on $U \cap \overline{M}$, we get (5-117).

From (5-46), (5-51) and (5-52), we get the first six equations in (5-118). From the second and sixth equations in (5-118), we have

$$\Pi^{(q)} \equiv N^{(q)} \Box^{(q)} \Pi^{(q)} + (\Pi^{(q)})^2 \equiv (\Pi^{(q)})^2 \mod \mathscr{C}^\infty((U \times U) \cap (\overline{M} \times \overline{M})).$$

We get (5-118). We now study distribution kernel of $\Pi^{(q)}$. From Theorem 5.21, we see that

$$\Pi^{(q)} - \widetilde{P} S_- L^{(q)} \equiv 0 \mod \mathscr{C}^\infty((U \times U) \cap (\overline{M} \times \overline{M})).$$

We just need to study distribution kernel of $\widetilde{P} S_- L^{(q)}$. Let $x = (x_1, \dots, x_{2n-1})$ be local coordinates of X and extend $x_1, \dots, x_{2n-1}$ to real smooth functions in some neighborhood of X. We may assume that $z = (x, \rho) = (x_1, \dots, x_{2n-1}, \rho)$ are local coordinates of U. In view of Theorem 5.17, we have

$$S_-(x, y) \equiv \int_0^{+\infty} e^{i\varphi_-(x,y)t} a(x, y, t)\, dt.$$

We can repeat the proof of [Hsiao 2010, Part II, Proposition 7.6] and find a phase

$$\tilde{\phi}(z, y) \in \mathscr{C}^\infty((U \times U) \cap (\overline{M} \times X))$$

such that

$$\tilde{\phi}(x, y) = \varphi_-(x, y),\ (d_z\tilde{\phi})(x, x) = -\omega_0(x) - i\, d\rho(x) \quad \text{for all } (x, y) \in (U \times U) \cap (X \times X),$$

$\operatorname{Im} \tilde{\phi}(z, y) > 0$ if $\rho \neq 0$ and $q_0(z, \tilde{\phi}'_z)$ vanishes to infinite order at $\rho = 0$, where q_0 denotes the principal symbol of $\Box_f^{(q)}$. We can repeat the procedure in the proof of [Hsiao 2010, Part II, Proposition 7.8] and deduce that the distribution kernel of $\widetilde{P} S_-$ is of the form

$$\begin{gathered}\widetilde{P} S_-(z, y) \equiv \int_0^{\infty} e^{i\tilde{\phi}(z,y)t} \tilde{b}(z, y, t)\, dt \mod \mathscr{C}^\infty((U \times U) \cap (\overline{M} \times X)),\\ \tilde{b}(z, y, t) \in S^{n-1}_{\mathrm{cl}}((U \times U) \cap (\overline{M} \times X)) \times (0, +\infty), \Lambda^{(0,q)\,|\,(0,q)}_{M' \times M'}),\\ \tilde{b}(x, y, t) = a(x, y, t) \quad \text{for all } (x, y) \in (U \times U) \cap (X \times X).\end{gathered}$$

Similarly, we can repeat the procedure above and deduce that

$$\widetilde{P} S_- L^{(q)}(z, w) \equiv \int_0^{\infty} e^{i\phi(z,w)t} b(z, w, t)\, dt \mod \mathscr{C}^\infty((U \times U) \cap (\overline{M} \times \overline{M})), \tag{5-122}$$

where $\phi(z, w) \in \mathscr{C}^\infty((U \times U) \cap (\overline{M} \times \overline{M}))$ satisfies (5-121),

$$b(z, w, t) \in S^{n}_{\mathrm{cl}}(((U \times U) \cap (\overline{M} \times \overline{M})) \times (0, +\infty), \Lambda^{(0,q)\,|\,(0,q)}_{M' \times M'}).$$

Since

$$(\widetilde{P}^* \widetilde{P})^{-1} = 2\sqrt{\triangle_X} + \Psi^0$$

and

$$Q^{(q)} = 2((\bar{\partial}\rho)(x))^{\wedge,*}((\bar{\partial}\rho)(x))^{\wedge} + \Psi^0$$

for some elements $\Psi^0 \in L^0_{\mathrm{cl}}(X, T^{*0,q}X \boxtimes (T^{*0,q}X)^*)$, we deduce as in [Hsiao 2010, Part II, (7.22)],

$$b_0(x,x) = 4a_0(x,x)((\bar\partial\rho)(x))^{\wedge,*}((\bar\partial\rho)(x))^{\wedge}, \quad x \in U \cap X,$$

where $a_0(x,x)$ is as in (5-84).

From Theorems 5.9, 5.10, 5.11, 5.22, (5-111) and (5-122), the theorem follows. □

The following result describes the phase function ϕ (see (5-119)) of the Fourier integral operator $\Pi^{(q)}$.

Theorem 5.24 [Hsiao 2010, Part II, Theorem 1.4]. *Under the assumptions and notation of Theorem 5.23, fix $p \in U \cap X$. We choose local holomorphic coordinates $z = (z_1, \dots, z_n)$, $z_j = x_{2j-1} + ix_{2j}$, $j = 1, \dots, n$, vanishing at p such that the metric on $T^{1,0}M'$ is $\sum_{j=1}^n dz_j \otimes d\bar z_j$ at p and $\rho(z) = \sqrt{2}\,\mathrm{Im}\, z_n + \sum_{j=1}^{n-1} \lambda_j |z_j|^2 + O(|z|^3)$, where λ_j, $j = 1, \dots, n-1$, are the eigenvalues of $\mathcal{L}_p$. We also write $w = (w_1, \dots, w_n)$, $w_j = y_{2j-1} + iy_{2j}$, $j = 1, \dots, n$. Then, we can take $\phi(z,w)$ in* (5-119) *so that*

$$\begin{aligned}\phi(z,w) = -\sqrt{2}x_{2n-1} + \sqrt{2}y_{2n-1} - i\rho(z)\Bigl(1 + \sum_{j=1}^{2n-1} a_j x_j + \frac{1}{2}a_{2n}x_{2n}\Bigr)\\ - i\rho(w)\Bigl(1 + \sum_{j=1}^{2n-1} \bar a_j y_j + \frac{1}{2}\bar a_{2n} y_{2n}\Bigr) + i\sum_{j=1}^{n-1} |\lambda_j|\,|z_j - w_j|^2\\ + \sum_{j=1}^{n-1} i\lambda_j(\bar z_j w_j - z_j \bar w_j) + O(|(z,w)|^3)\end{aligned} \tag{5-123}$$

in some neighborhood of (p,p) in $M' \times M'$, where $a_j = \frac{1}{2}\partial_{x_j}\sigma(\Box_f^{(q)})(p, -2i\partial\rho(p))$ for $j = 1, \dots, 2n$, and $\sigma(\Box_f^{(q)})$ denotes the principal symbol of $\Box_f^{(q)}$.

The following result describes the restriction to the diagonal of the coefficient b_0 from the expansion of the symbol $b(z,w,t)$ of $\Pi^{(q)}$; see (5-119), (5-120).

Theorem 5.25 [Hsiao 2010, Part II, Proposition 1.6]. *Under the assumptions and notation of Theorem 5.23, fix $p \in U \cap X$. The coefficient $b_0(z,w)$ from* (5-120) *satisfies*

$$b_0(x,x) = 2\pi^{-n}|\det \mathcal{L}_x|\tau_{x,n_-} \circ (\bar\partial\rho(x))^{\wedge,*}(\bar\partial\rho(x))^{\wedge} \quad \textit{for every } x \in U \cap X, \tag{5-124}$$

where $\det \mathcal{L}_x$, τ_{x,n_-} are given by (1-9), (1-11) *respectively and $(\bar\partial\rho(x))^{\wedge,*}$ is given by* (1-12).

6. Microlocal spectral theory for the $\bar\partial$-Neumann Laplacian

In this section, we will apply the approximate Hodge decomposition theorems for the $\bar\partial$-Neumann Laplacian $\Box^{(q)}$ from Sections 4 and 5 to study the singularities for the kernel $B^{(q)}_{\le\lambda}(x,y)$ near the nondegenerate part of the Levi form. In particular, we give the proof of Theorem 1.1.

Until further notice, we fix $\lambda > 0$. Since $\Box^{(q)}$ is bounded below by $\lambda > 0$ on $\mathrm{Ker}\, B^{(q)}_{\le\lambda}$ there exists a continuous operator

$$A^{(q)}_\lambda : L^2_{(0,q)}(M) \to \mathrm{Dom}\,\Box^{(q)}$$

such that

$$\begin{aligned}\Box^{(q)}A^{(q)}_\lambda + B^{(q)}_{\le\lambda} = I \quad \text{on } L^2_{(0,q)}(M),\\ A^{(q)}_\lambda\Box^{(q)} + B^{(q)}_{\le\lambda} = I \quad \text{on } \mathrm{Dom}\,\Box^{(q)}.\end{aligned} \tag{6-1}$$

Let U be an open set of M' with $U \cap X \neq \varnothing$. Suppose that the Levi form is nondegenerate of constant signature (n_-, n_+) on $U \cap X$. Until further notice, we let $q = n_-$.

Theorem 6.1. *Let $q = n_-$. The operators*

$$\bar\partial B^{(q)}_{\leq\lambda} : L^2_{(0,q)}(M) \to H^s_{\mathrm{loc}}(U \cap \overline{M}, T^{*0,q+1}M'), \tag{6-2}$$

$$\bar\partial^* B^{(q)}_{\leq\lambda} : L^2_{(0,q)}(M) \to H^s_{\mathrm{loc}}(U \cap \overline{M}, T^{*0,q}M'), \tag{6-3}$$

$$\Box^{(q)} B^{(q)}_{\leq\lambda} : L^2_{(0,q)}(M) \to H^s_{\mathrm{loc}}(U \cap \overline{M}, T^{*0,q}M') \tag{6-4}$$

are continuous for every $s \in \mathbb{N}$.

Proof. Let $u \in L^2(M, T^{*0,q}M')$. Since $B^{(q)}_{\leq\lambda}u \in \mathrm{Dom}\,\Box^{(q)}$, $\bar\partial B^{(q)}_{\leq\lambda}u \in L^2_{(0,q+1)}(M)$. We claim that

$$\bar\partial B^{(q)}_{\leq\lambda}u \in \mathrm{Dom}\,\Box^{(q+1)}. \tag{6-5}$$

It is clear that $\bar\partial B^{(q)}_{\leq\lambda}u \in \mathrm{Dom}\,\bar\partial \cap \mathrm{Dom}\,\bar\partial^*$ and $\bar\partial^2 B^{(q)}_{\leq\lambda}u = 0$. Hence, $\bar\partial^2 B^{(q)}_{\leq\lambda}u \in \mathrm{Dom}\,\bar\partial^*$. We only need to show that $\bar\partial^*\,\bar\partial B^{(q)}_{\leq\lambda}u \in \mathrm{Dom}\,\bar\partial$. We have

$$\bar\partial^*\,\bar\partial B^{(q)}_{\leq\lambda}u = \Box^{(q)} B^{(q)}_{\leq\lambda}u - \bar\partial\,\bar\partial^* B^{(q)}_{\leq\lambda}u. \tag{6-6}$$

By spectral theory [Ma and Marinescu 2007, Theorem C.2.1], we see that $\Box^{(q)} B^{(q)}_{\leq\lambda}u \in \mathrm{Dom}\,\Box^{(q)}$ and hence $\Box^{(q)} B^{(q)}_{\leq\lambda}u \in \mathrm{Dom}\,\bar\partial$. Note that $\bar\partial^2\bar\partial^* B^{(q)}_{\leq\lambda}u = 0$, $\bar\partial\,\bar\partial^* B^{(q)}_{\leq\lambda}u \in \mathrm{Dom}\,\bar\partial$. From this observation and (6-6), we get (6-5). From (4-29), we have

$$N^{(q+1)}\Box^{(q+1)}\bar\partial B^{(q)}_{\leq\lambda}u = \bar\partial B^{(q)}_{\leq\lambda}u + F^{(q+1)}_1\bar\partial B^{(q)}_{\leq\lambda}u, \tag{6-7}$$

where $F^{(q+1)}_1 \equiv 0 \mod \mathscr{C}^\infty((U \times U) \cap (\overline{M} \times \overline{M}))$ is as in (4-29). It is clear that

$$F^{(q+1)}_1\bar\partial : L^2_{\mathrm{loc}}(U \cap \overline{M}, T^{*0,q}M') \to H^s_{\mathrm{loc}}(U \cap \overline{M}, T^{*0,q+1}M') \tag{6-8}$$

is continuous for every $s \in \mathbb{Z}$. We have

$$N^{(q+1)}\Box^{(q+1)}\bar\partial B^{(q)}_{\leq\lambda} = N^{(q+1)}\bar\partial\Box^{(q)} B^{(q)}_{\leq\lambda} \quad \text{on } L^2_{(0,q)}(M). \tag{6-9}$$

By spectral theory,

$$\Box^{(q)} B^{(q)}_{\leq\lambda} : L^2_{(0,q)}(M) \to L^2_{(0,q)}(M) \tag{6-10}$$

is continuous. In view of Theorem 4.3, we see that

$$N^{(q+1)}\bar\partial : H^s_{\mathrm{loc}}(U \cap \overline{M}, T^{*0,q}M') \to H^s_{\mathrm{loc}}(U \cap \overline{M}, T^{*0,q+1}M') \tag{6-11}$$

is continuous for every $s \in \mathbb{Z}$. From (6-7), (6-8), (6-9), (6-10) and (6-11), we deduce that

$$\bar\partial B^{(q)}_{\leq\lambda} : L^2_{(0,q)}(M) \to L^2_{\mathrm{loc}}(U \cap \overline{M}, T^{*0,q+1}M') \tag{6-12}$$

is continuous. We have

$$N^{(q+1)}\Box^{(q+1)}\bar\partial B^{(q)}_{\leq\lambda}u = N^{(q+1)}\bar\partial\Box^{(q)} B^{(q)}_{\leq\lambda}u = N^{(q+1)}\bar\partial B^{(q)}_{\leq\lambda}\Box^{(q)} B^{(q)}_{\leq\lambda}u.$$

From this observation and (6-7), we have

$$N^{(q+1)}\bar\partial B^{(q)}_{\leq\lambda}\Box^{(q)}B^{(q)}_{\leq\lambda}u=\bar\partial B^{(q)}_{\leq\lambda}u+F^{(q)}_1\bar\partial B^{(q)}_{\leq\lambda}u. \tag{6-13}$$

From (6-8), (6-10), (6-12), (6-13) and since that

$$N^{(q+1)}:H^s_{\rm loc}(U\cap\overline{M},T^{*0,q+1}M')\to H^{s+1}_{\rm loc}(U\cap\overline{M},T^{*0,q+1}M') \tag{6-14}$$

is continuous for every $s\in\mathbb{Z}$, we deduce that

$$\bar\partial B^{(q)}_{\leq\lambda}:L^2_{(0,q)}(M)\to H^1_{\rm loc}(U\cap\overline{M},T^{*0,q+1}M') \tag{6-15}$$

is continuous. The continuity of (6-2) follows by induction. The proof of the continuity of (6-3) is analogous, and that of (6-4) follows then immediately. $\Box$

Lemma 6.2. *Let $q=n_-$. For every $m\in\mathbb{N}$, the operator $B^{(q)}_{\leq\lambda}\bar\partial(\Box^{(q-1)}_f)^m:\Omega^{0,q-1}_c(M)\to L^2_{(0,q)}(M)$ can be continuously extended to*

$$B^{(q)}_{\leq\lambda}\bar\partial(\Box^{(q-1)}_f)^m:L^2_{(0,q-1)}(M)\to L^2_{(0,q)}(M). \tag{6-16}$$

Proof. Let $u\in\Omega^{0,q-1}_c(M)$, $v\in L^2_{(0,q)}(M)$. We have

$$(B^{(q)}_{\leq\lambda}\bar\partial(\Box^{(q-1)}_f)^m u\mid v)_M=(B^{(q)}_{\leq\lambda}(\Box^{(q)}_f)^m\bar\partial u\mid v)_M=(u\mid\bar\partial^*(\Box^{(q)})^m B^{(q)}_{\leq\lambda}v)_M. \tag{6-17}$$

We have

$$\begin{aligned}\|\bar\partial^*(\Box^{(q)})^m B^{(q)}_{\leq\lambda}v\|^2_M&\leq\|\bar\partial^*(\Box^{(q)})^m B^{(q)}_{\leq\lambda}v\|^2_M+\|\bar\partial(\Box^{(q)})^m B^{(q)}_{\leq\lambda}v\|^2_M\\&=\big((\Box^{(q)})^{m+1}B^{(q)}_{\leq\lambda}v\mid(\Box^{(q)})^m B^{(q)}_{\leq\lambda}v\big)_M\leq\lambda^{2m+1}\|v\|^2_M.\end{aligned} \tag{6-18}$$

From (6-17), (6-18) and taking $v=B^{(q)}_{\leq\lambda}\bar\partial(\Box^{(q-1)}_f)^m u$, it is straightforward to see that

$$\|B^{(q)}_{\leq\lambda}\bar\partial(\Box^{(q-1)}_f)^m u\|_M\leq\lambda^{m+\frac12}\|u\|_M. \tag{6-19}$$

From (6-19) and noticing that $\Omega^{0,q-1}_c(M)$ is dense in $L^2_{(0,q-1)}(M)$, the lemma follows. $\Box$

Theorem 6.3. (i) *The operator $B^{(q)}_{\leq\lambda}\bar\partial:\Omega^{0,q-1}_c(U\cap\overline{M})\to L^2_{(0,q)}(M)$ can be continuously extended to*

$$B^{(q)}_{\leq\lambda}\bar\partial:H^{-s}_c(U\cap\overline{M},T^{*0,q-1}M')\to L^2_{(0,q)}(M)\quad\text{for every } s\in\mathbb{N}. \tag{6-20}$$

(ii) *The operator $B^{(q)}_{\leq\lambda}\bar\partial^*_f:\Omega^{0,q+1}_c(U\cap\overline{M})\to L^2_{(0,q)}(M)$ can be continuously extended to*

$$B^{(q)}_{\leq\lambda}\bar\partial^*_f:H^{-s}_c(U\cap\overline{M},T^{*0,q+1}M')\to L^2_{(0,q)}(M)\quad\text{for every } s\in\mathbb{N}. \tag{6-21}$$

(iii) *The operator $B^{(q)}_{\leq\lambda}\Box^{(q)}_f:\Omega^{0,q}_c(U\cap\overline{M})\to L^2_{(0,q)}(M)$ can be continuously extended to*

$$B^{(q)}_{\leq\lambda}\Box^{(q)}_f:H^{-s}_c(U\cap\overline{M},T^{*0,q}M')\to L^2_{(0,q)}(M)\quad\text{for every } s\in\mathbb{N}. \tag{6-22}$$

Proof. Let $u \in \Omega_c^{0,q-1}(U \cap \overline{M})$. From (4-30), we have

$$B_{\leq\lambda}^{(q)}\bar{\partial}\Box_f^{(q-1)}N^{(q-1)}u = B_{\leq\lambda}^{(q)}\bar{\partial}u + B_{\leq\lambda}^{(q)}\bar{\partial}F_2^{(q-1)}u, \tag{6-23}$$

where $F_2^{(q-1)} \equiv 0$ on $U \cap \overline{M}$. From Theorem 4.3, (6-16), (6-23) and since

$$N^{(q-1)} : H_c^s(U \cap \overline{M}, T^{*0,q-1}M') \to H_c^{s+1}(U \cap \overline{M}, T^{*0,q-1}M') \tag{6-24}$$

is continuous for every $s \in \mathbb{Z}$, we deduce that $B_{\leq\lambda}^{(q)}\bar{\partial}$ can be continuously extended to

$$B_{\leq\lambda}^{(q)}\bar{\partial} : H_c^{-1}(U \cap \overline{M}, T^{*0,q-1}M') \to L^2_{(0,q)}(M). \tag{6-25}$$

From Lemma 6.2, we can repeat the proof of (6-25) and deduce that $B_{\leq\lambda}^{(q)}\bar{\partial}\Box_f^{(q-1)}$ can be continuously extended to

$$B_{\leq\lambda}^{(q)}\bar{\partial}\Box_f^{(q-1)} : H_c^{-1}(U \cap \overline{M}, T^{*0,q-1}M') \to L^2_{(0,q)}(M). \tag{6-26}$$

From (6-23), (6-24) and (6-26), we deduce that $B_{\leq\lambda}^{(q)}\bar{\partial}$ can be continuously extended to

$$B_{\leq\lambda}^{(q)}\bar{\partial} : H_c^{-2}(U \cap \overline{M}, T^{*0,q-1}M') \to L^2_{(0,q)}(M).$$

Continuing by induction we get (i). Item (ii) follows analogously and (iii) follows from (i) and (ii). □

We consider

$$\Box^{(q)}B_{\leq\lambda}^{(q)}\Box_f^{(q)} : \Omega_c^{0,q}(U \cap \overline{M}) \to L^2_{(0,q)}(M) \subset L^2_{\mathrm{loc}}(U \cap \overline{M}, T^{*0,q}M'),$$
$$(\Box^{(q)})^2B_{\leq\lambda}^{(q)} : \Omega_c^{0,q}(U \cap \overline{M}) \to L^2_{(0,q)}(M) \subset L^2_{\mathrm{loc}}(U \cap \overline{M}, T^{*0,q}M').$$

Theorem 6.4. *We have*

$$\Box^{(q)}B_{\leq\lambda}^{(q)}\Box_f^{(q)} \equiv 0 \bmod \mathscr{C}^\infty((U \times U) \cap (\overline{M} \times \overline{M})), \tag{6-27}$$
$$(\Box^{(q)})^2B_{\leq\lambda}^{(q)} \equiv 0 \bmod \mathscr{C}^\infty((U \times U) \cap (\overline{M} \times \overline{M})). \tag{6-28}$$

Proof. From (6-4) and (6-22), we have

$$\Box^{(q)}B_{\leq\lambda}^{(q)}\Box_f^{(q)} : H_c^{-s}(U \cap \overline{M}, T^{*0,q}M') \to H_{\mathrm{loc}}^s(U \cap \overline{M}, T^{*0,q}M')$$

for every $s \in \mathbb{N}$. This proves (6-27). Let $u \in L^2_{(0,q)}(M)$. Take $u_j \in \Omega_c^{0,q}(M)$, $j = 1, 2, \ldots$, so that $\lim_{j\to+\infty}\|u_j - u\|_M = 0$. Since $(\Box^{(q)})^2B_{\leq\lambda}^{(q)}$ is L^2 continuous, we have

$$(\Box^{(q)})^2B_{\leq\lambda}^{(q)}u = \lim_{j\to+\infty}(\Box^{(q)})^2B_{\leq\lambda}^{(q)}u_j \quad \text{in } L^2_{(0,q)}(M). \tag{6-29}$$

From the fact that $u_j \in \operatorname{Dom}\Box^{(q)}$ for every $j = 1, 2, \ldots$, we can check that

$$(\Box^{(q)})^2B_{\leq\lambda}^{(q)}u_j = \Box^{(q)}B_{\leq\lambda}^{(q)}\Box^{(q)}u_j = \Box^{(q)}B_{\leq\lambda}^{(q)}\Box_f^{(q)}u_j \quad \text{for every } j = 1, 2, \ldots. \tag{6-30}$$

From (6-29) and (6-30), we conclude that

$$(\Box^{(q)})^2B_{\leq\lambda}^{(q)} = \Box^{(q)}B_{\leq\lambda}^{(q)}\Box_f^{(q)} \quad \text{on } L^2_{(0,q)}(M). \tag{6-31}$$

From (6-27) and (6-31), we get (6-28). □

Lemma 6.5. *The operator $B^{(q)}_{\leq\lambda}$ can be continuously extended to*

$$B^{(q)}_{\leq\lambda}: H^{-s}_c(U\cap\overline{M}, T^{*0,q}M')\to H^{-s}_{\mathrm{loc}}(U\cap\overline{M}, T^{*0,q}M') \tag{6-32}$$

for every $s\in\mathbb{N}$.

Proof. Let $u\in\Omega^{0,q}_c(U\cap\overline{M})$. From (5-118), we have

$$B^{(q)}_{\leq\lambda}\Box^{(q)}_f N^{(q)}u+B^{(q)}_{\leq\lambda}\Pi^{(q)}u=B^{(q)}_{\leq\lambda}u+B^{(q)}_{\leq\lambda}r^{(q)}_0u, \tag{6-33}$$

where $r^{(q)}_0\equiv 0 \mod \mathscr{C}^\infty((U\times U)\cap(\overline{M}\times\overline{M}))$ is as in (5-118). From (5-116), (6-22) and (6-33) and noting that $\Omega^{0,q}_c(U\cap\overline{M})$ is dense in $H^{-s}_c(U\cap\overline{M}, T^{*0,q}M')$ for every $s\in\mathbb{N}$, we deduce that $B^{(q)}_{\leq\lambda}-B^{(q)}_{\leq\lambda}\Pi^{(q)}$ can be continuously extended to

$$B^{(q)}_{\leq\lambda}-B^{(q)}_{\leq\lambda}\Pi^{(q)}: H^{-s}_c(U\cap\overline{M}, T^{*0,q}M')\to L^2_{(0,q)}(M)\quad\text{for every } s\in\mathbb{N}. \tag{6-34}$$

On the other hand, from (6-1) and (5-118), we have

$$\begin{aligned}\Pi^{(q)}u&=(A^{(q)}_\lambda\Box^{(q)}+B^{(q)}_{\leq\lambda})\Pi^{(q)}u\\&=A^{(q)}_\lambda\Box^{(q)}_f\Pi^{(q)}u+B^{(q)}_{\leq\lambda}\Pi^{(q)}u\\&=A^{(q)}_\lambda r^{(q)}_5u+B^{(q)}_{\leq\lambda}\Pi^{(q)}u\end{aligned} \tag{6-35}$$

for every $u\in\Omega^{0,q}_c(U\cap\overline{M})$, where $r^{(q)}_5\equiv 0 \mod \mathscr{C}^\infty((U\times U)\cap(\overline{M}\times\overline{M}))$ is as in (5-118). From (6-35), we conclude that $\Pi^{(q)}-B^{(q)}_{\leq\lambda}\Pi^{(q)}$ can be continuously extended to

$$\Pi^{(q)}-B^{(q)}_{\leq\lambda}\Pi^{(q)}: H^{-s}_c(U\cap\overline{M}, T^{*0,q}M')\to L^2_{(0,q)}(M)\quad\text{for every } s\in\mathbb{N}. \tag{6-36}$$

From (6-34) and (6-36), we deduce that $\Pi^{(q)}-B^{(q)}_{\leq\lambda}$ can be continuously extended to

$$\Pi^{(q)}-B^{(q)}_{\leq\lambda}: H^{-s}_c(U\cap\overline{M}, T^{*0,q}M')\to L^2_{(0,q)}(M)\quad\text{for every } s\in\mathbb{N}. \tag{6-37}$$

From (5-116) and (6-37), we get (6-32). ☐

Theorem 6.6. *We have*

$$\Box^{(q)}B^{(q)}_{\leq\lambda}\equiv 0 \mod \mathscr{C}^\infty((U\times U)\cap(\overline{M}\times\overline{M})). \tag{6-38}$$

Proof. By (6-28), $\varepsilon^{(q)}:=(\Box^{(q)})^2B^{(q)}_{\leq\lambda}$ is smoothing on $U\cap\overline{M}$. Let $u\in\Omega^{0,q}_c(U\cap\overline{M})$. From the second equation in (5-118), we have

$$\begin{aligned}\Box^{(q)}B^{(q)}_{\leq\lambda}u&=N^{(q)}(\Box^{(q)})^2B^{(q)}_{\leq\lambda}u+\Pi^{(q)}\Box^{(q)}B^{(q)}_{\leq\lambda}u-r^{(q)}_1\Box^{(q)}B^{(q)}_{\leq\lambda}u\\&=N^{(q)}\varepsilon^{(q)}u+r^{(q)}_4B^{(q)}_{\leq\lambda}u-r^{(q)}_1\Box^{(q)}B^{(q)}_{\leq\lambda}u,\end{aligned} \tag{6-39}$$

where $r^{(q)}_1$, $r^{(q)}_4$ are the smoothing operators from (5-118). From (6-32), we see that

$$r^{(q)}_1\Box^{(q)}B^{(q)}_{\leq\lambda},\ r^{(q)}_4B^{(q)}_{\leq\lambda}: H^{-s}_c(U\cap\overline{M}, T^{*0,q}M')\to H^{s}_{\mathrm{loc}}(U\cap\overline{M}, T^{*0,q}M')$$

are continuous for every $s \in \mathbb{N}$, and hence they are smoothing on $U \cap \overline{M}$. From this observation and (6-39), we get (6-38). $\square$

We can now prove one of the main results of this work.

Theorem 6.7. *Let U be an open set of M' with $U \cap X \neq \varnothing$. Suppose that the Levi form is nondegenerate of constant signature (n_-, n_+) on $U \cap X$. Let $q = n_-$ and fix $\lambda > 0$. We have*

$$B^{(q)}_{\leq\lambda} - \Pi^{(q)} \equiv 0 \mod \mathscr{C}^\infty((U \times U) \cap (\overline{M} \times \overline{M})),$$

where $\Pi^{(q)}$ is as in Theorem 5.23.

Proof. From the second equation in (5-118), we have

$$N^{(q)}\Box^{(q)}B^{(q)}_{\leq\lambda}u + \Pi^{(q)}B^{(q)}_{\leq\lambda}u = r_1^{(q)}B^{(q)}_{\leq\lambda}u + B^{(q)}_{\leq\lambda}u \tag{6-40}$$

for every $u \in \Omega^{0,q}_c(U \cap \overline{M})$, where $r_1^{(q)} \equiv 0 \mod \mathscr{C}^\infty((U \times U) \cap (\overline{M} \times \overline{M}))$ is as in (5-118). From (5-116), (6-32), (6-38) and (6-40), we deduce that

$$B^{(q)}_{\leq\lambda} - \Pi^{(q)}B^{(q)}_{\leq\lambda} =: \varepsilon^{(q)} \equiv 0 \mod \mathscr{C}^\infty((U \times U) \cap (\overline{M} \times \overline{M})). \tag{6-41}$$

Similarly, from the first equation in (5-118), we have

$$B^{(q)}_{\leq\lambda}\Box^{(q)}_f N^{(q)}u + B^{(q)}_{\leq\lambda}\Pi^{(q)}u = B^{(q)}_{\leq\lambda}u + B^{(q)}_{\leq\lambda}r_0^{(q)}u \tag{6-42}$$

for every $u \in \Omega^{0,q}_c(U \cap \overline{M})$, where $r_0^{(q)} \equiv 0 \mod \mathscr{C}^\infty((U \times U) \cap (\overline{M} \times \overline{M}))$ is as in (5-118). Since $N^{(q)}u \in \operatorname{Dom}\Box^{(q)}$, we have

$$B^{(q)}_{\leq\lambda}\Box^{(q)}_f N^{(q)}u = B^{(q)}_{\leq\lambda}\Box^{(q)}N^{(q)}u = \Box^{(q)}B^{(q)}_{\leq\lambda}N^{(q)}u$$

for every $u \in \Omega^{0,q}_c(U \cap \overline{M})$. From this observation and (6-42), we deduce that

$$\Box^{(q)}B^{(q)}_{\leq\lambda}N^{(q)}u + B^{(q)}_{\leq\lambda}\Pi^{(q)}u = B^{(q)}_{\leq\lambda}u + B^{(q)}_{\leq\lambda}r_0^{(q)}u \tag{6-43}$$

for every $u \in \Omega^{0,q}_c(U \cap \overline{M})$. From (6-32), (6-38) and (6-43), we deduce that

$$B^{(q)}_{\leq\lambda} - B^{(q)}_{\leq\lambda}\Pi^{(q)} =: \varepsilon_1^{(q)} \equiv 0 \mod \mathscr{C}^\infty((U \times U) \cap (\overline{M} \times \overline{M})). \tag{6-44}$$

Let $u \in \Omega^{0,q}_c(U \cap \overline{M})$. From (6-1), we have

$$\Pi^{(q)}\Box^{(q)}A^{(q)}_\lambda u + \Pi^{(q)}B^{(q)}_{\leq\lambda}u = \Pi^{(q)}u \quad \text{on } U \cap \overline{M}, \tag{6-45}$$

$$A^{(q)}_\lambda\Box^{(q)}\Pi^{(q)}u + B^{(q)}_{\leq\lambda}\Pi^{(q)}u = \Pi^{(q)}u \quad \text{on } U \cap \overline{M}. \tag{6-46}$$

From (5-118), we have

$$\Pi^{(q)}\Box^{(q)}A^{(q)}_\lambda u = r_4^{(q)}A^{(q)}_\lambda u \quad \text{on } U \cap X, \tag{6-47}$$

$$A^{(q)}_\lambda\Box^{(q)}\Pi^{(q)}u = A^{(q)}_\lambda r_5^{(q)}u \quad \text{on } U \cap X, \tag{6-48}$$

where $r_4^{(q)} \equiv 0 \bmod \mathscr{C}^\infty((U\times U)\cap(\overline{M}\times\overline{M}))$ and $r_5^{(q)} \equiv 0 \bmod \mathscr{C}^\infty((U\times U)\cap(\overline{M}\times\overline{M}))$ are as in (5-118). From (6-47), (6-48) and (6-46), we deduce that

$$\begin{aligned}\Pi^{(q)} - \Pi^{(q)} B^{(q)}_{\le\lambda} &= r_4^{(q)} A^{(q)}_\lambda,\\ \Pi^{(q)} - B^{(q)}_{\le\lambda}\Pi^{(q)} &= A^{(q)}_\lambda r_5^{(q)}.\end{aligned}\tag{6-49}$$

From (6-41), (6-44) and (6-49), we get

$$\begin{aligned}\Pi^{(q)} - B^{(q)}_{\le\lambda} &= r_4^{(q)} A^{(q)}_\lambda - \varepsilon^{(q)},\\ \Pi^{(q)} - B^{(q)}_{\le\lambda} &= A^{(q)}_\lambda r_5^{(q)} - \varepsilon_1^{(q)}.\end{aligned}\tag{6-50}$$

From (6-50), we have

$$\begin{aligned}&(\Pi^{(q)} - B^{(q)}_{\le\lambda})(\Pi^{(q)} - B^{(q)}_{\le\lambda})\\ &\qquad= (r_4^{(q)} A^{(q)}_\lambda - \varepsilon^{(q)})(A^{(q)}_\lambda r_5^{(q)} - \varepsilon_1^{(q)})\\ &\qquad= r_4^{(q)}(A^{(q)}_\lambda)^2 r_5^{(q)} - r_4^{(q)} A^{(q)}_\lambda \varepsilon_1^{(q)} - \varepsilon^{(q)} A^{(q)}_\lambda r_5^{(q)} + \varepsilon^{(q)}\varepsilon_1^{(q)} \quad \text{on } \Omega^{0,q}_c(U\cap\overline{M}).\end{aligned}\tag{6-51}$$

Note that $r_5^{(q)}$ and $r_4^{(q)}$ are properly supported on $U\cap\overline{M}$ and $r_4^{(q)}(A^{(q)}_\lambda)^2 r_5^{(q)}$, $r_4^{(q)} A^{(q)}_\lambda \varepsilon_1^{(q)}$, $\varepsilon^{(q)} A^{(q)}_\lambda r_5^{(q)}$, $\varepsilon^{(q)}\varepsilon_1^{(q)}$ are well-defined as continuous operators: $\Omega^{0,q}_c(U\cap\overline{M}) \to \Omega^{0,q}(U\cap\overline{M})$. Now,

$$\begin{aligned}r_4^{(q)}(A^{(q)}_\lambda)^2 r_5^{(q)} : H^{-s}_c(U\cap\overline{M}, T^{*0,q}M') \to H^s_c(U\cap\overline{M}, T^{*0,q}M') \subset L^2_{(0,q)}(M)\\ \to L^2_{(0,q)}(M) \to H^s_{\rm loc}(U\cap\overline{M}, T^{*0,q}M')\end{aligned}$$

is continuous for every $s\in\mathbb{N}$. Hence, $r_4^{(q)}(A^{(q)}_\lambda)^2 r_5^{(q)} \equiv 0 \bmod \mathscr{C}^\infty((U\times U)\cap(\overline{M}\times\overline{M}))$. Similarly, $r_4^{(q)} A^{(q)}_\lambda \varepsilon_1^{(q)}, \varepsilon^{(q)} A^{(q)}_\lambda r_5^{(q)}, \varepsilon^{(q)}\varepsilon_1^{(q)} \equiv 0 \bmod \mathscr{C}^\infty((U\times U)\cap(\overline{M}\times\overline{M}))$. From this observation and (6-51), we get

$$(\Pi^{(q)} - B^{(q)}_{\le\lambda})(\Pi^{(q)} - B^{(q)}_{\le\lambda}) \equiv 0 \bmod \mathscr{C}^\infty((U\times U)\cap(\overline{M}\times\overline{M})).\tag{6-52}$$

Now,

$$\begin{aligned}(\Pi^{(q)} - B^{(q)}_{\le\lambda})(\Pi^{(q)} - B^{(q)}_{\le\lambda}) &= (\Pi^{(q)})^2 - \Pi^{(q)} B^{(q)}_{\le\lambda} - B^{(q)}_{\le\lambda}\Pi^{(q)} + (B^{(q)}_{\le\lambda})^2\\ &= \Pi^{(q)} + r_6^{(q)} - B^{(q)}_{\le\lambda} + \varepsilon^{(q)} - B^{(q)}_{\le\lambda} + \varepsilon_1^{(q)} + B^{(q)}_{\le\lambda}\\ &= \Pi^{(q)} - B^{(q)}_{\le\lambda} + r_6^{(q)} + \varepsilon^{(q)} + \varepsilon_1^{(q)},\end{aligned}\tag{6-53}$$

where $r_6^{(q)}, \varepsilon^{(q)}, \varepsilon_1^{(q)} \equiv 0 \bmod \mathscr{C}^\infty((U\times U)\cap(\overline{M}\times\overline{M}))$ are as in (5-118), (6-41) and (6-44) respectively. From (6-52) and (6-53), the theorem follows. □

By using Theorem 4.7, we can repeat the proof of Theorem 6.7 with minor changes and deduce:

Theorem 6.8. *Let U be an open set of M' with $U\cap X \neq \varnothing$. Suppose that the Levi form is nondegenerate of constant signature (n_-, n_+) on $U\cap X$. Let $q\neq n_-$. Fix $\lambda > 0$. We have*

$$B^{(q)}_{\le\lambda} \equiv 0 \bmod \mathscr{C}^\infty((U\times U)\cap(\overline{M}\times\overline{M})).$$

Proof of Theorem 1.1. This follows immediately from Theorems 5.23, 6.7 and 6.8. □

We remind the reader that the local closed range condition is given by Definition 1.4. The following is our second main result.

Theorem 6.9. *Let U be an open set of M' with $U\cap X\neq\varnothing$. Assume that the Levi form is nondegenerate of constant signature (n_-,n_+) on $U\cap X$. Let $q=n_-$. Suppose that $\Box^{(q)}$ has local closed range in U. Then the Bergman projection $B^{(q)}$ satisfies*

$$B^{(q)}-\Pi^{(q)}\equiv 0 \bmod \mathscr{C}^\infty((U\times U)\cap(\overline{M}\times\overline{M})),$$

where $\Pi^{(q)}$ is as in Theorem 5.23.

Proof. Let W be any open set of U with $W\cap U\neq\varnothing$, $\overline{W}\Subset U$. Since $\Pi^{(q)}$ is properly supported on $U\cap\overline{M}$, there is an open set $W'\subset U$ with $W'\cap X\neq\varnothing$, $\overline{W}'\Subset U$, such that $\Pi^{(q)}u\in\Omega_c^{0,q}(W'\cap\overline{M})\cap\operatorname{Dom}\Box^{(q)}$ for every $u\in\Omega_c^{0,q}(W\cap\overline{M})$. Since $\Box^{(q)}$ has local closed range on U, there is a constant $C_{W'}>0$ such that, for every $u\in\Omega_c^{0,q}(W\cap\overline{M})$,

$$\|(I-B^{(q)})\Pi^{(q)}u\|_M\leq C_{W'}\|\Box^{(q)}\Pi^{(q)}u\|_M=\|r_5^{(q)}u\|_M\,, \tag{6-54}$$

where $r_5^{(q)}$ is as in (5-118). Let $u\in H_c^{-s}(W\cap\overline{M},T^{*0,q}M')$. Let $u_j\in\Omega_c^{0,q}(W\cap\overline{M})$, $\lim_{j\to\infty}u_j=u$ in $H_c^{-s}(W\cap\overline{M},T^{*0,q}M')$. Since $r_5^{(q)}$ is smoothing on $U\cap\overline{M}$ the sequence $r_5^{(q)}u_j$ is Cauchy, so by (6-54) $(\Pi^{(q)}-B^{(q)}\Pi^{(q)})u_j$ converges in $L^2_{(0,q)}(M)$, as $j\to\infty$. Thus, u is in the domain of $\Pi^{(q)}-B^{(q)}\Pi^{(q)}$ and $(\Pi^{(q)}-B^{(q)}\Pi^{(q)})u\in L^2_{(0,q)}(M)$. We conclude that $\Pi^{(q)}-B^{(q)}\Pi^{(q)}$ can be extended continuously to

$$\Pi^{(q)}-B^{(q)}\Pi^{(q)}:H_c^{-s}(U\cap\overline{M},T^{*0,q}M')\to L^2_{(0,q)}(M)\quad\text{for every } s\in\mathbb{N}. \tag{6-55}$$

From the first two equations in (5-118) we have, on $U\cap X$,

$$\begin{aligned}\Pi^{(q)}B^{(q)}u&=N^{(q)}\Box^{(q)}B^{(q)}u+\Pi^{(q)}B^{(q)}u=B^{(q)}u+r_1^{(q)}B^{(q)}u,\quad u\in L^2_{(0,q)}(M),\\ B^{(q)}\Pi^{(q)}u&=B^{(q)}\Box^{(q)}N^{(q)}u+B^{(q)}\Pi^{(q)}u=B^{(q)}u+B^{(q)}r_0^{(q)}u,\quad u\in\Omega_c^{0,q}(U\cap\overline{M}),\end{aligned} \tag{6-56}$$

where $r_0^{(q)},r_1^{(q)}\equiv 0\bmod\mathscr{C}^\infty((U\times U)\cap(\overline{M}\times\overline{M}))$ are as in (5-118). From (6-56), we conclude that $B^{(q)}-\Pi^{(q)}B^{(q)}$ and $B^{(q)}-B^{(q)}\Pi^{(q)}$ can be extended continuously to

$$B^{(q)}-\Pi^{(q)}B^{(q)}:L^2_{(0,q)}(M)\to H^s_{\mathrm{loc}}(U\cap\overline{M},T^{*0,q}M')\quad\text{for every } s\in\mathbb{N}, \tag{6-57}$$

$$B^{(q)}-B^{(q)}\Pi^{(q)}:H_c^{-s}(U\cap\overline{M},T^{*0,q}M')\to L^2_{(0,q)}(M)\quad\text{for every } s\in\mathbb{N}. \tag{6-58}$$

Form (6-55) and (6-58), we deduce that $\Pi^{(q)}-B^{(q)}$ can be extended continuously to

$$\Pi^{(q)}-B^{(q)}:H_c^{-s}(U\cap\overline{M},T^{*0,q}M')\to L^2_{(0,q)}(M)\quad\text{for every } s\in\mathbb{N}. \tag{6-59}$$

Since $\Pi^{(q)}:H_c^s(U\cap\overline{M},T^{*0,q}M')\to H_c^s(U\cap\overline{M},T^{*0,q}M')$ is continuous for every $s\in\mathbb{Z}$, we deduce that $B^{(q)}$ can be extended continuously to

$$B^{(q)}:H_c^{-s}(U\cap\overline{M},T^{*0,q}M')\to H^{-s}_{\mathrm{loc}}(U\cap\overline{M},T^{*0,q}M')\quad\text{for every } s\in\mathbb{N}. \tag{6-60}$$

From (6-60), we deduce that

$$r_1^{(q)}B^{(q)},(r_0^{(q)})^*B^{(q)}:H_c^{-s}(U\cap\overline{M},T^{*0,q}M')\to H^s_{\mathrm{loc}}(U\cap\overline{M},T^{*0,q}M')\quad\text{for every } s\in\mathbb{N},$$

where $r_1^{(q)},r_0^{(q)}$ are as in (5-118) and $(r_0^{(q)})^*$ is the formal adjoint of $r_0^{(q)}$ with respect to $(\cdot\,|\,\cdot)_M$. Hence,

$$r_1^{(q)}B^{(q)},(r_0^{(q)})^*B^{(q)}\equiv 0\bmod\mathscr{C}^\infty((U\times U)\cap(\overline{M}\times\overline{M})). \tag{6-61}$$

By taking adjoint of $(r_0^{(q)})^* B^{(q)}$, we get

$$B^{(q)} r_0^{(q)} \equiv 0 \bmod \mathscr{C}^\infty((U \times U) \cap (\overline{M} \times \overline{M})). \tag{6-62}$$

From (6-56), (6-61) and (6-62), we get

$$\begin{aligned} &\Pi^{(q)} B^{(q)} u - B^{(q)} u = f_1^{(q)} u \quad \text{on } U \cap X \text{ for every } u \in L^2_{(0,q)}(M), \\ &B^{(q)} \Pi^{(q)} u - B^{(q)} u = f_2^{(q)} u \quad \text{on } U \cap X \text{ for every } u \in \Omega_c^{0,q}(U \cap \overline{M}), \end{aligned} \tag{6-63}$$

where

$$\begin{aligned} f_1^{(q)} &: L^2_{(0,q)}(M) \to \Omega^{0,q}(U \cap \overline{M}), \quad f_1^{(q)} \equiv 0 \bmod \mathscr{C}^\infty((U \times U) \cap (\overline{M} \times \overline{M})), \\ f_2^{(q)} &: \Omega_c^{0,q}(U \cap \overline{M}) \to L^2_{(0,q)}(M), \qquad f_2^{(q)} \equiv 0 \bmod \mathscr{C}^\infty((U \times U) \cap (\overline{M} \times \overline{M})). \end{aligned}$$

Taking adjoint in (6-59), we conclude that $(\Pi^{(q)})^* - B^{(q)}$ can be extended continuously to

$$(\Pi^{(q)})^* - B^{(q)} : L^2_{(0,q)}(M) \to H^s_{\mathrm{loc}}(U \cap \overline{M}, T^{*0,q} M') \quad \text{for every } s \in \mathbb{N}, \tag{6-64}$$

where $(\Pi^{(q)})^*$ is the formal adjoint of $\Pi^{(q)}$ with respect to $(\cdot \mid \cdot)_M$. From (5-50), we see that

$$(\Pi^{(q)})^* = \Pi^{(q)} + \Gamma_1^{(q)},$$

where $\Gamma_1^{(q)} \equiv 0 \bmod \mathscr{C}^\infty((U \times U) \cap (\overline{M} \times \overline{M}))$. From this observation and (6-64), we deduce that $\Pi^{(q)} - B^{(q)}$ can be extended continuously to

$$\Pi^{(q)} - B^{(q)} : L^2_{(0,q)}(M) \to H^s_{\mathrm{loc}}(U \cap \overline{M}, T^{*0,q} M') \quad \text{for every } s \in \mathbb{N}. \tag{6-65}$$

From (6-59) and (6-65), we get

$$(\Pi^{(q)} - B^{(q)})(\Pi^{(q)} - B^{(q)}) : H_c^{-s}(U \cap \overline{M}, T^{*0,q} M') \to H^s_{\mathrm{loc}}(U \cap \overline{M}, T^{*0,q} M')$$

is continuous for every $s \in \mathbb{N}$. Hence,

$$(\Pi^{(q)} - B^{(q)})(\Pi^{(q)} - B^{(q)}) \equiv 0 \bmod \mathscr{C}^\infty((U \times U) \cap (\overline{M} \times \overline{M})). \tag{6-66}$$

On the other hand, we have

$$\begin{aligned} &(\Pi^{(q)} - B^{(q)})(\Pi^{(q)} - B^{(q)}) u \\ &\quad = (\Pi^{(q)})^2 u - \Pi^{(q)} B^{(q)} u - B^{(q)} \Pi^{(q)} u + (B^{(q)})^2 u \\ &\quad = \Pi^{(q)} u - B^{(q)} u - B^{(q)} u + B^{(q)} u + ((\Pi^{(q)})^2 - \Pi^{(q)}) u + (B^{(q)} - \Pi^{(q)} B^{(q)}) u + (B^{(q)} - B^{(q)} \Pi^{(q)}) u \\ &\quad = \Pi^{(q)} u - B^{(q)} u + r_6^{(q)} u - f_1^{(q)} u - f_2^{(q)} u \quad \text{for every } u \in \Omega_c^{0,q}(U \cap \overline{M}), \end{aligned} \tag{6-67}$$

where $r_6^{(q)} \equiv 0 \bmod \mathscr{C}^\infty((U \times U) \cap (\overline{M} \times \overline{M}))$ is as in (5-118), $f_1^{(q)}, f_2^{(q)} \equiv 0 \bmod \mathscr{C}^\infty((U \times U) \cap (\overline{M} \times \overline{M}))$ are as in (6-63). From (6-66) and (6-67), the theorem follows. ☐

7. Proof of Theorem 1.9

To prove Theorem 1.9, we need a result of [Takegoshi 1983], which is a generalization of [Kohn 1973]. Consider an open relatively compact subset $M_0 := \{z \in M' : \rho(z) < 0\}$ with smooth boundary X_0 of M'. We have the following (see [Takegoshi 1983, Section 3, Theorem N]).

Theorem 7.1. *Let M_0 be a pseudoconvex domain with smooth boundary X_0 in a complex manifold M' and let L be a holomorphic line bundle on M' which is positive on a neighborhood of M_0. Then there exists $k_0 \in \mathbb{N}$ such that the following statement holds for every $k \in \mathbb{N}$, $k \geq k_0$: For every $f \in L^2_{(0,1)}(M_0, L^k)$ with $\bar\partial f = 0$ on M_0 there exists $g \in L^2(M_0, L^k)$ such that $\bar\partial g = f$ on M_0 and*

$$\int_{M_0} |g|^2_{h^{L^k}}\, dv_{M'} \leq C_k \int_{M_0} |f|^2_{h^{L^k}}\, dv_{M'}, \tag{7-1}$$

where $C_k > 0$ is a constant independent of f and g and $|\cdot|_{h^{L^k}}$ denotes the pointwise norm on $\bigoplus_{q=0}^n T^{*0,q}M' \otimes L^k$ induced by the given Hermitian metric $\langle\cdot\,|\,\cdot\rangle$ on $\mathbb{C}TM'$ and h^L.

Proof of (1-23). Let $k_0 \in \mathbb{N}$ be as in Theorem 7.1. Let $k \geq k_0$, $k \in \mathbb{N}$, and let U be any open set of X_0 with $U \cap X_1 = \varnothing$. Let $u \in \mathscr{C}^\infty_c(U \cap \overline{M}, L^k) \cap \operatorname{Dom} \Box^{(0)}_k$ and let $f := \bar\partial u \in \Omega^{0,1}_c(U \cap M, L^k) \subset L^2_{(0,1)}(M_0, L^k)$. From Theorem 7.1, we see that there is a $g \in L^2(M_0, L^k) \subset L^2(M, L^k)$ such that $\bar\partial g = \bar\partial u$ on M_0 (hence on M) and

$$\int_{M_0} |g|^2_{h^{L^k}}\, dv_{M'} \leq C_k \int_{M_0} |\bar\partial u|^2_{h^{L^k}}\, dv_{M'}, \tag{7-2}$$

where $C_k > 0$ is a constant independent of u and g. Since $(I - B^{(0)}_k)u$ is the solution of $\bar\partial g = \bar\partial u$ on M of minimal L^2 norm, we have

$$\int_M |(I - B^{(0)}_k)u|^2_{h^{L^k}}\, dv_{M'} \leq \int_M |g|^2_{h^{L^k}}\, dv_{M'}. \tag{7-3}$$

From (7-2) and (7-3), we get

$$\int_M |(I - B^{(0)}_k)u|^2_{h^{L^k}}\, dv_{M'} \leq C_k \int_{M_0} |\bar\partial u|^2_{h^{L^k}}\, dv_{M'}. \tag{7-4}$$

Since $\bar\partial u$ has compact support in $U \cap \overline{M}$, we have

$$\int_{M_0} |\bar\partial u|^2_{h^{L^k}}\, dv_{M'} = \int_M |\bar\partial u|^2_{h^{L^k}}\, dv_{M'}. \tag{7-5}$$

From (7-4) and (7-5), we get

$$\int_M |(I - B^{(0)}_k)u|^2_{h^{L^k}}\, dv_{M'} \leq C_k \int_M |\bar\partial u|^2_{h^{L^k}}\, dv_{M'}. \tag{7-6}$$

Since $u \in \operatorname{Dom} \Box^{(0)}_k$, we can check that

$$\begin{aligned}\int_M |\bar\partial u|^2_{h^{L^k}}\, dv_{M'} &= (\bar\partial u \mid \bar\partial u\,)_k = (\bar\partial u \mid \bar\partial(I - B^{(0)}_k)u\,)_k \\ &= (\Box^{(0)}_k u \mid (I - B^{(0)}_k)u\,)_k \leq \|\Box^{(0)}_k u\|_k \|(I - B^{(0)}_k)u\|_k.\end{aligned} \tag{7-7}$$

Since (1-23) follows from (7-6) and (7-7), we are done. ☐

From Theorem 1.5, Remark 1.6 and Theorem 1.9, we immediately get (1-24).

8. S^1-equivariant Bergman kernel asymptotics and embedding theorems

In this section, we assume that M' admits a holomorphic S^1-action $e^{i\theta}$, $\theta \in [0, 2\pi)$, $e^{i\theta} : M' \to M'$, $x \in M' \to e^{i\theta} \circ x \in M'$. Recall that X_0 is an open connected component of X such that (1-26) holds and we work with the following assumption.

Assumption 8.1. For every $x \in X$ we have $\mathbb{C}T_0(x) \oplus T_x^{1,0}X \oplus T_x^{0,1}X = \mathbb{C}T_xX$, and the S^1-action preserves the boundary X, that is, there exists a defining function $\rho \in \mathscr{C}^\infty(M', \mathbb{R})$ of X such that $\rho(e^{i\theta} \circ x) = \rho(x)$ for every $x \in M'$ and every $\theta \in [0, 2\pi]$.

Theorem 8.2. *Assume that M' admits a holomorphic S^1-action and Assumption 8.1 holds. Let X_0 be a connected component of X such that* (1-26) *holds, let $p \in X_0$ and let U be an open set of p in M' with $U \cap X_0 \neq \varnothing$. Suppose that the Levi form is nondegenerate of constant signature (n_-, n_+) on $U \cap X_0$, where n_- denotes the number of the negative eigenvalues of the Levi form on $U \cap X_0$. Fix $\lambda > 0$. If $q \neq n_-$, then as $m \to +\infty$,*

$$B^{(q)}_{\leq\lambda,m} \equiv 0 \bmod O(m^{-\infty}) \quad \text{on } U \cap \overline{M}. \tag{8-1}$$

Let $q = n_-$. Let $N_p := \{g \in S^1 : g \circ p = p\} = \{g_0 := e, g_1, \dots, g_r\}$, where e denotes the identify element in S^1 and $g_j \neq g_\ell$ if $j \neq \ell$ for every $j, \ell \in \{0, 1, \dots, r\}$. We have

$$B^{(q)}_{\leq\lambda,m}(x, y) \equiv \sum_{\alpha=0}^{r} g_\alpha^m e^{im\phi(x, g_\alpha y)} b_\alpha(x, y, m) \bmod O(m^{-\infty}) \quad \text{on } U \cap \overline{M}, \tag{8-2}$$

where, for every $\alpha = 0, 1, \dots, r$,

$$\begin{gathered}
b_\alpha(x, y, m) \in S^n_{\mathrm{loc}}((U \times U) \cap (\overline{M} \times \overline{M}), \Lambda^{(0,q)\,|\,(0,q)}_{M' \times M'}),\\
b_\alpha(x, y, m) \sim \textstyle\sum_{j=0}^\infty b_{\alpha,j}(x, y) m^{n-j} \quad \text{in } S^n_{\mathrm{loc}}((U \times U) \cap (\overline{M} \times \overline{M}), \Lambda^{(0,q)\,|\,(0,q)}_{M' \times M'}),\\
b_{\alpha,j}(x, y) \in \mathscr{C}^\infty((U \times U) \cap (\overline{M} \times \overline{M}), \Lambda^{(0,q)\,|\,(0,q)}_{M' \times M'}), \quad j = 0, 1, \dots,\\
b_{\alpha,0}(x, x) = b_0(x, x), \quad b_0(x, x) \text{ is given by (5-124)},
\end{gathered} \tag{8-3}$$

and $\phi(x, y) \in \mathscr{C}^\infty((U \times U) \cap (\overline{M} \times \overline{M}))$ is as in (1-17).

Proof. From (1-14) and (1-30), we can integrate by parts in θ and get (8-1). We now prove (8-2). From Theorem 6.7 and (5-111), it is straightforward to see that

$$B^{(q)}_{\leq\lambda} \equiv \widetilde{P} S_{-,m} L^{(q)} \bmod O(m^{-\infty}) \quad \text{on } U \cap \overline{M}, \tag{8-4}$$

where $S_{-,m}(x, y) = \frac{1}{2\pi}\int_{-\pi}^{\pi} S_-(x, e^{i\theta} y) e^{im\theta}\, d\theta$ and $S_-(x, y)$ is as in Theorem 5.17. From Theorem 5.17, we can repeat the proof of [Hsiao and Marinescu 2014, Theorem 3.12] with minor changes and deduce that

$$S_{-,m}(x, y) \equiv \sum_{\alpha=0}^{r} g_\alpha^m e^{im\varphi_-(x, g_\alpha y)} a_\alpha(x, y, m) \bmod O(m^{-\infty}) \quad \text{on } U \cap X, \tag{8-5}$$

where, for every $\alpha = 0, 1, \dots, r$,

$$\begin{gathered}
a_\alpha(x, y, m) \in S^{n-1}_{\mathrm{loc}}((U \times U) \cap (X \times X), \Lambda^{(0,q)\,|\,(0,q)}_{M' \times M'}),\\
a_\alpha(x, y, m) \sim \textstyle\sum_{j=0}^\infty a_{\alpha,j}(x, y) m^{n-1-j} \quad \text{in } S^{n-1}_{\mathrm{loc}}((U \times U) \cap (X \times X), \Lambda^{(0,q)\,|\,(0,q)}_{M' \times M'}),\\
a_{\alpha,j}(x, y) \in \mathscr{C}^\infty((U \times U) \cap (\overline{M} \times \overline{M}), \Lambda^{(0,q)\,|\,(0,q)}_{M' \times M'}), \quad j = 0, 1, \dots,\\
a_{\alpha,0}(x, x) = a_0(x, x),
\end{gathered} \tag{8-6}$$

where $a_0(x, x)$ is given by (5-84) and $\varphi_-(x, y) \in \mathscr{C}^\infty((U \times U) \cap (\overline{M} \times \overline{M}))$ is as in Theorem 5.17. From (8-5), we can repeat the procedure in the proof of [Hsiao 2010, Part II, Proposition 7.8] and deduce that the distribution kernel of $\widetilde{P} S_{-,m} L^{(q)}$ is of the form (8-2). $\square$

By Theorem 1.5, we can repeat the proof of Theorem 8.2 and deduce:

Theorem 8.3. *Assume that M' admits a holomorphic S^1-action and Assumption 8.1 holds. Let X_0 be a connected component of X such that* (1-26) *holds, let $p \in X_0$ and let U be an open set of p in M' with $U \cap X_0 \neq \varnothing$. Suppose that the Levi form is nondegenerate of constant signature (n_-, n_+) on $U \cap X_0$, where n_- denotes the number of the negative eigenvalues of the Levi form on $U \cap X_0$. Suppose that $\square^{(q)}$ has local closed range in U. If $q \neq n_-$, then*

$$B_m^{(q)} \equiv 0 \bmod O(m^{-\infty}) \quad \text{on } U \cap \overline{M}. \tag{8-7}$$

Let $q = n_-$. Let $N_p := \{g \in S^1 : g \circ p = p\} = \{g_0 := e, g_1, \dots, g_r\}$, where e denotes the identify element in S^1 and $g_j \neq g_\ell$ if $j \neq \ell$ for every $j, \ell = 0, 1, \dots, r$. We have

$$B_m^{(q)}(x, y) \equiv \sum_{\alpha=0}^{r} g_\alpha^m e^{im\phi(x, g_\alpha y)} b_\alpha(x, y, m) \bmod O(m^{-\infty}) \quad \text{on } U \cap \overline{M}, \tag{8-8}$$

where $\phi(x, y) \in \mathscr{C}^\infty((U \times U) \cap (\overline{M} \times \overline{M}))$ and $b_\alpha(x, y, m) \in S^n_{\mathrm{loc}}((U \times U) \cap (\overline{M} \times \overline{M}), \Lambda^{(0,q)|(0,q)}_{M' \times M'})$, $\alpha = 0, 1, \dots, r$, are as in Theorem 8.2.

Proof of Theorem 1.10. We now consider the case $q = 0$. When $Z(1)$ holds on X, it is well known (see [Folland and Kohn 1972]) that $\square^{(0)}$ has L^2 closed range. From this observation and Theorem 8.3, we deduce Theorem 1.10. ☐

We will now prove Theorem 1.11 about the S^1-equivariant embedding theorem.

Proof of Theorem 1.11. Fix $m_0 \in \mathbb{N}$. By using Theorem 1.10, we can repeat the proof of [Herrmann et al. 2018, Theorem 1.2] with minor changes and conclude that we can find $m_1 \in \mathbb{N}, \dots, m_k \in \mathbb{N}$, with $m_j \geq m_0$, $j = 1, \dots, k$, such that

$$\Phi_{m_1,\dots,m_k} : X_0 \to \mathbb{C}^{\hat{d}_m} \text{ is an embedding} \tag{8-9}$$

and there is an S^1-invariant open set U of X_0 such that

$$\Phi_{m_1,\dots,m_k} : U \cap \overline{M} \to \mathbb{C}^{\hat{d}_m} \text{ is an immersion.} \tag{8-10}$$

Fix $x_0 \in X_0$. From (8-10), it is straightforward to see that there are S^1-invariant open sets $\Omega_{x_0} \Subset W_{x_0} \Subset U_{x_0}$ of x_0 in M' such that

$$\Phi_{m_1,\dots,m_k} : U_{x_0} \cap \overline{M} \to \mathbb{C}^{\hat{d}_m} \text{ is injective.} \tag{8-11}$$

Let

$$\delta_{x_0} := \inf\{|\Phi_{m_1,\dots,m_k}(x) - \Phi_{m_1,\dots,m_k}(y)| : x \in \Omega_{x_0} \cap X_0,\ y \in X_0,\ y \notin W_{x_0} \cap X_0\}. \tag{8-12}$$

From (8-9), we see that $\delta_{x_0} > 0$. Let V^{x_0} be a small S^1-invariant open set of X_0 in M' such that, for every $x \in V^{x_0} \cap \overline{M}$, $x \notin U_{x_0}$, there is a $y \in X_0$, $y \notin W_{X_0} \cap X_0$, such that

$$|\Phi_{m_1,\dots,m_k}(x) - \Phi_{m_1,\dots,m_k}(y)| \leq \tfrac{1}{2}\delta_{x_0}. \tag{8-13}$$

Assume that $X_0=\bigcup_{j=1}^N(\Omega_{x_j}\cap X_0)$, $N\in\mathbb{N}$, and let

$$V:=U\cap\Big(\bigcap_{j=1}^N V^{x_j}\Big)\cap\Big(\bigcup_{j=1}^N\Omega_{x_j}\Big),$$

where Ω_{x_j}, V^{x_j}, $j=1,\dots,N$, are as above, and U is as in (8-10). From (8-10), we see that $\Phi_{m_1,\dots,m_k}:V\cap\overline{M}\to\mathbb{C}^{\hat{d}_m}$ is an immersion. We claim that $\Phi_{m_1,\dots,m_k}:V\cap\overline{M}\to\mathbb{C}^{\hat{d}_m}$ is injective. Let $p,q\in V\cap\overline{M}$, $p\neq q$. We may assume that $p\in\Omega_{x_1}\cap\overline{M}$. If $q\in U_{x_1}$ we see from (8-11) that $\Phi_{m_1,\dots,m_k}(p)\neq\Phi_{m_1,\dots,m_k}(q)$. Assume that $q\notin U_{x_1}$. From the discussion before (8-13), we see that there is $y_0\in X_0$, $y_0\notin W_{x_1}\cap X_0$ such that

$$|\Phi_{m_1,\dots,m_k}(q)-\Phi_{m_1,\dots,m_k}(y_0)|\le\tfrac12\delta_{x_1}. \tag{8-14}$$

From (8-14) and (8-12), we have

$$\begin{aligned}|\Phi_{m_1,\dots,m_k}(p)-\Phi_{m_1,\dots,m_k}(q)|&\ge|\Phi_{m_1,\dots,m_k}(p)-\Phi_{m_1,\dots,m_k}(y_0)|-|\Phi_{m_1,\dots,m_k}(y_0)-\Phi_{m_1,\dots,m_k}(q)|\\&\ge\delta_{x_1}-\tfrac12\delta_{x_1}>0.\end{aligned}$$

Hence $\Phi_{m_1,\dots,m_k}(p)\neq\Phi_{m_1,\dots,m_k}(q)$, so $\Phi_{m_1,\dots,m_k}$ is injective and the theorem follows. □

Proof of Theorem 1.12. We may assume that $X_0=\{x\in M':\rho(x)=0\}$. Consider the shell domain

$$\widehat{M}:=\{x\in M':-\varepsilon<\rho(x)<0\},$$

where $\varepsilon>0$ is a small constant. Then $\widehat{M}$ is a complex manifold with smooth boundary $\widehat{X}$. Moreover, it is easy to see that X_0 is an open connected component of $\widehat{X}$ and $Z(1)$ holds on $\widehat{X}$. Hence, we can apply Theorem 1.11 to get Theorem 1.12. □

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A RESTRICTED 2-PLANE TRANSFORM RELATED TO FOURIER RESTRICTION FOR SURFACES OF CODIMENSION 2

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We draw a connection between the affine invariant surface measures constructed by P. Gressman (*Duke Math. J.* **168**:11 (2019), 2075–2126) and the boundedness of a certain geometric averaging operator associated to surfaces of codimension 2 and related to the Fourier restriction problem for such surfaces. For a surface given by $(\xi, Q_1(\xi), Q_2(\xi))$, with Q_1, Q_2 quadratic forms on $\mathbb{R}^d$, the particular operator in question is the 2-plane transform restricted to directions normal to the surface, that is,

$$\mathcal{T} f(x,\xi) := \iint_{|s|,|t|\le 1} f(x - s\nabla Q_1(\xi) - t\nabla Q_2(\xi), s, t)\, ds\, dt,$$

where $x, \xi \in \mathbb{R}^d$. We show that when the surface is well-curved in the sense of Gressman (that is, the associated affine invariant surface measure does not vanish) the operator satisfies sharp $L^p \to L^q$ inequalities for p, q up to the critical point. We also show that the well-curvedness assumption is necessary to obtain the full range of estimates. The proof relies on two main ingredients: a characterisation of well-curvedness in terms of properties of the polynomial $\det(s\nabla^2 Q_1 + t\nabla^2 Q_2)$, obtained with geometric invariant theory techniques, and Christ's method of refinements. With the latter, matters are reduced to a sublevel set estimate, which is proven by a linear programming argument.

1. Introduction

The k-plane transform in $\mathbb{R}^n$ is the operator $T_{n,k}$ defined by

$$T_{n,k} f(\pi) := \int_\pi f\, d\mathcal{L}_\pi,$$

where π is any affine k-plane in $\mathbb{R}^n$ and $d\mathcal{L}_\pi$ denotes the Lebesgue measure on π. Such operators are generalisations of the X-ray transform and of the Radon transform, with which they coincide when $k = 1$ and $k = n-1$ respectively. The strongest results for the boundedness of $T_{n,k}$ for (n, k) generic were obtained by M. Christ [15], who proved a range of mixed-norm estimates (building upon work of S. W. Drury [23; 24]); see also [45] for some improvements for a subset of (n, k) values and [25] for a survey of further developments. The particular case of $k = 1$ has been the object of considerable attention due to its relationship with the Kakeya maximal function — see T. Wolff's influential paper [52] for the $n = 3$ case, [39] for generic n and again [45] for other improvements.

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In this paper we will be concerned with the restriction of the 2-plane transform to particular sets of directions — ones that arise as normals to surfaces of codimension 2 that are "well-curved", in a sense that will be made precise later on (we regard the identification of the correct notion of well-curvedness as one of the main aims of this paper). A number of instances of restricted $T_{n,k}$ transforms exist in the literature, particularly when $k = 1$:

(i) The restriction of the X-ray transform $T_{n,1}$ to a 1-dimensional set of directions of the form $(\gamma(t), 1)$, with $\gamma : [-1, 1] \to \mathbb{R}^{n-1}$ a curve, was first considered by M. Christ and B. Erdoğan [19] for the moment curve $(t, t^2, \dots, t^{n-1})$; those results were later extended to the sharp mixed-norm range by the first author and B. Stovall [21; 22]. In this case, in order to obtain estimates for the largest range of exponents it is vital to assume that the curve γ is well-curved in the sense of having nonvanishing torsion. The latter condition is equivalent to the nonvanishing of the affine invariant surface measure on γ as introduced by Gressman [30].[1]

(ii) The restriction of the X-ray transform $T_{n,1}$ to 2-dimensional sets of directions was studied by B. Erdoğan and R. Oberlin [26]; the authors considered directions of the form $(\varphi(u, v), 1)$ for various examples of maps $\varphi : [-1, 1]^2 \to \mathbb{R}^{n-1}$. It can be verified by the methods of [30] (in particular, by Theorem 6 in that paper) that in all their examples the affine invariant surface measure on the surface $\varphi([-1, 1]^2)$ is nonvanishing.

(iii) The restriction of the X-ray transform $T_{n,1}$ to the $(n-2)$-dimensional set of directions given by light-rays (that is, directions of the form $(\omega, 1)$ with $\omega \in \mathbb{S}^{n-2}$) was studied by T. Wolff [53], who proved mixed-norm estimates in a certain range (not believed to be sharp). In this case the set of directions possesses curvature because the sphere $\mathbb{S}^{n-2}$ is curved.

(iv) The restriction of the Radon transform $T_{n,n-1}$ to hyperplanes orthogonal to directions of the form $(\Gamma(\xi), 1)$, with $\Gamma : [-1, 1]^m \to \mathbb{R}^{n-1}$ the parametrisation of an m-dimensional submanifold of $\mathbb{R}^{n-1}$, was considered by P. Gressman [31].[2] Combining the methods of that paper with those of [30], one obtains nontrivial $L^p \to L^q$ estimates under the assumption that the image of Γ has affine invariant surface measure (as per [30]) that is nonvanishing.

We are not aware of restrictions of $T_{n,k}$ transforms for k other than 1 or $n - 1$ that have been studied in the literature;[3] ours seems to be the first such instance.

We will now introduce the restriction of the 2-plane transform $T_{n,2}$ that we are going to consider in this paper. Besides fitting in well within the aforementioned literature, the operators we are about to introduce arise naturally in the study of Fourier restriction for surfaces of codimension 2, as will be illustrated in Section 2. Let $d \geq 2$ and take a compact quadratic surface of codimension 2 in $\mathbb{R}^{d+2}$, given as a graph by the parametrisation

$$\phi(\xi) := (\xi, Q_1(\xi), Q_2(\xi)), \quad \xi \in [-1, 1]^d,$$

[1] This measure further coincides with the well-known affine arclength from affine geometry.

[2] More precisely, the operator here described is the dual operator to the one described in Example 3, Section 6 of [31].

[3] Save perhaps for [46], which however has a measure-theoretic flavour rather than the geometric flavour we are interested in.

where Q_1, Q_2 are quadratic forms on $\mathbb{R}^d$; we use $\Sigma(Q_1, Q_2)$ to denote the surface $\phi([-1,1]^d)$. It will also be convenient to introduce the real symmetric $d \times d$ matrices A, B that correspond to the Hessians $\nabla^2 Q_1, \nabla^2 Q_2$, that is, the matrices given by

$$A\xi := \nabla Q_1(\xi), \quad B\xi := \nabla Q_2(\xi) \quad \text{for all } \xi \in \mathbb{R}^d.$$

Remark 1. We concentrate on quadratic surfaces for simplicity of exposition, but the main result that will be given in Section 1.1 (Theorem 5) holds for more general surfaces, as will be explained there.

To any such pair of quadratic forms (or equivalently, to any surface $\Sigma(Q_1, Q_2)$) we associate the operator $\mathcal{T} = \mathcal{T}_{Q_1,Q_2}$, acting on (Schwartz) functions $f : \mathbb{R}^{d+2} \to \mathbb{C}$, given by

$$\mathcal{T} f(x, \xi) := \iint_{|s|,|t|\le 1} f(x - s\,\nabla Q_1(\xi) - t\,\nabla Q_2(\xi), s, t)\, ds\, dt, \tag{1}$$

where $x \in \mathbb{R}^d$, $\xi \in [-1,1]^d$. The operator $\mathcal{T}$ is a (local) 2-plane transform in a restricted set of directions parametrised by ξ: indeed, the 2-plane in question is given by

$$\pi_{x,\xi} := \{(x, 0, 0) + s(-\nabla Q_1(\xi), 1, 0) + t(-\nabla Q_2(\xi), 0, 1) : s, t \in \mathbb{R}\};$$

moreover, it is readily verified that $\pi_{x,\xi}$ is normal to the tangent plane of $\Sigma(Q_1, Q_2)$ at the point $\phi(\xi)$. Notice that in (1) we are not integrating with respect to the Lebesgue measure on the 2-plane as one does in $T_{n,2}$, but the $ds\,dt$ measure is nevertheless comparable to it since ξ is bounded, so that, if we were to extend the integration in (1) to all $s, t \in \mathbb{R}$, we would have

$$\mathcal{T} f(x, \xi) \le T_{d+2,2} f(\pi_{x,\xi}) \lesssim_{Q_1,Q_2} \mathcal{T} f(x, \xi).$$

To gauge the severity of the restriction in directions, notice that the Grassmannian $\mathrm{Gr}(2, d+2)$ of 2-dimensional linear subspaces of $\mathbb{R}^{d+2}$ has dimension $2d$, whereas the submanifold of $\mathrm{Gr}(2, d+2)$ given by the directions of the family of 2-planes $\pi_{x,\xi}$ above is parametrised by ξ and thus has dimension at most d.

We are interested in the boundedness properties of $\mathcal{T}$ and how these relate to how well-curved the surface $\Sigma(Q_1, Q_2)$ is. We next introduce the general collection of mixed-norm estimates. Let $q, r \ge 1$ and define for any $F : \mathbb{R}^d \times [-1,1]^d \to \mathbb{C}$ its $L^q(L^r)$ mixed-norm[4] to be

$$\|F\|_{L^q(L^r)} := \left(\int_{[-1,1]^d} \left(\int_{\mathbb{R}^d} |F(x, \xi)|^r\, dx \right)^{\frac{q}{r}} d\xi \right)^{\frac{1}{q}} \tag{2}$$

(notice that when $q = r$ the $L^q(L^q)$-norm is simply the usual L^q-norm). For exponents $p, q, r \ge 1$, we say that $\mathcal{T}$ satisfies the mixed-norm estimate $L^p \to L^q(L^r)$ if we have the a priori estimate

$$\|\mathcal{T} f\|_{L^q(L^r)} \lesssim_{p,q,r} \|f\|_{L^p}. \tag{3}$$

[4]With this order of integration, this is sometimes called the *Kakeya-order* mixed-norm.

For the rest of the paper we will make the assumption that f is supported on, say, $B(0, C) \times [-1, 1]^2$ for some $C > 0$. Due to the local nature of the operator $\mathcal{T}$, this assumption can be removed when $r \geq q \geq p$ by a standard localisation argument.

Remark 2. By a standard duality and discretisation argument, any estimate of the form (3) translates into a Kakeya-type bound for collections of $\delta \times \cdots \times \delta \times 1 \times 1$ slabs associated to $\Sigma(Q_1, Q_2)$; see Section 2 for details (in particular Corollary 10 there) and an application.

Testing the mixed-norm inequalities (3) against some simple geometric examples leads to a conjectural range of boundedness, as will now be described. Let $0 < \delta < 1$ and let $B_n(r)$ denote the n-dimensional ball of radius r centred at the origin. We use A, B in place of $\nabla^2 Q_1$, $\nabla^2 Q_2$ for convenience. Observe that for $|s| \lesssim \|A\|^{-1}\delta$ and $|t| \lesssim \|B\|^{-1}\delta$ we have $|x - sA\xi - tB\xi| \leq \delta$ for all $|x| \lesssim \delta$ and all $\xi \in [-1, 1]^d$. Therefore

$$\mathcal{T}\mathbf{1}_{B_{d+2}(\delta)}(x, \xi) \gtrsim \delta^2\, \mathbf{1}_{B_d(O(\delta))}(x)\, \mathbf{1}_{[-1,1]^d}(\xi),$$

so that for (3) to hold as $\delta \to 0$ we see with a simple computation that we must have

$$2 + \frac{d}{r} \geq \frac{d+2}{p}.$$

For our second example, let S_δ denote the "slab"

$$S_\delta := \{(x - sA\xi - tB\xi, s, t) : |s|, |t| \sim 1,\ x, \xi \in B_d(\delta)\},$$

and observe that $|S_\delta| \lesssim \delta^d$ by similar considerations as above. Clearly we have

$$\mathcal{T}\mathbf{1}_{S_\delta}(x, \xi) \gtrsim \mathbf{1}_{B_d(\delta)}(x)\, \mathbf{1}_{B_d(\delta)}(\xi),$$

and thus if estimate (3) is to hold as $\delta \to 0$ we obtain a second necessary condition. The two conditions together are then

$$\begin{cases} 2 + \dfrac{d}{r} \geq \dfrac{d+2}{p}, \\ \dfrac{1}{r} + \dfrac{1}{q} \geq \dfrac{1}{p}. \end{cases} \tag{4}$$

Remark 3. We record the following trivial facts about certain exponents in the range allowed by (4):

(i) Inequality (3) is certainly satisfied for $p = \infty$ and for every $1 \leq q, r \leq \infty$ (recall that we are assuming f is supported in $B(0, C) \times [-1, 1]^2$).

(ii) Inequality (3) is certainly satisfied for $p = r = 1$ and for every $1 \leq q \leq \infty$.

(iii) If inequality (3) holds for exponents (p, q, r) then it also holds for any exponents $(p, \tilde{q}, r)$ with $1 \leq \tilde{q} \leq q$ (by the Hölder inequality).

We conjecture that when $\Sigma(Q_1, Q_2)$ is well-curved (in a sense to be made precise shortly; see Definition 4 of next subsection) then the necessary conditions (4) are also sufficient, with the possible exception of the endpoint $L^{(d+2)/2} \to L^{(d+2)/2}(L^\infty)$. In this paper we will concern ourselves mainly with nonmixed-norm estimates, that is, estimates with $q = r$ (this is because mixed-norm estimates are not accessible with the methods we employ, at least not without significant reworking); in this case the necessary conditions are rewritten as

$$2+\frac{d}{q} \geq \frac{d+2}{p}, \quad \frac{2}{q} \geq \frac{1}{p}.$$

As described in the next subsection, we are able to confirm the conjecture in the nonmixed-norm range given by these conditions, with the exclusion of a critical line. By interpolation with the trivial inequalities observed above, one also obtains a range of mixed-norm inequalities as a consequence.

1.1. *Main results.* In order to state our main results, we will now clarify the notion of curvature that we are going to employ. It is based upon P. Gressman's work [30], in which a construction was provided that, given a submanifold $\mathcal{M}$ of $\mathbb{R}^n$, produces a unique (up to multiplicative constants) surface measure $\nu_{\mathcal{M}}$ (that is, a measure with support on $\mathcal{M}$ and absolutely continuous with respect to the standard surface measure) which is equi-affine invariant.[5] Moreover, the measure $\nu_{\mathcal{M}}$ satisfies an affine curvature condition of the form $\nu_{\mathcal{M}}(R) \lesssim |R|^{\alpha}$ for every rectangle R in $\mathbb{R}^n$ (for a specific value of α that depends only on n and $\dim \mathcal{M}$), and is the largest such measure up to multiplicative constants. Details on Gressman's construction will be provided in Section 3.

Definition 4. We say that a submanifold $\mathcal{M}$ of $\mathbb{R}^n$ is *well-curved* if the density of its affine invariant surface measure $\nu_{\mathcal{M}}$ (with respect to the standard surface measure $d\sigma$) does not vanish anywhere on $\mathcal{M}$. If the density of $\nu_{\mathcal{M}}$ vanishes identically, we say that $\mathcal{M}$ is *flat*.

When the submanifold $\mathcal{M} \subset \mathbb{R}^n$ has codimension 1 or $n-1$, the measure $\nu_{\mathcal{M}}$ corresponds respectively to the affine hypersurface measure and the affine arclength (see Theorem 1 (4) of [30]). In these two extremal cases, the submanifold is then well-curved if the Gaussian curvature is nonvanishing or if the torsion is nonvanishing, respectively — thus recovering the common notions of well-curvedness for such codimensions present in the literature. Definition 4 should also be compared to the curvature assumptions present in the examples of restricted k-plane transforms listed at the beginning of this section.

In the case of the compact quadratic surfaces $\mathcal{M} = \Sigma(Q_1, Q_2)$ we have that $d\xi/d\sigma$ is bounded away from zero, and therefore $\Sigma(Q_1, Q_2)$ is well-curved according to our definition if and only if $d\nu_{\mathcal{M}}/d\xi$ does not vanish. However, it is shown in [30] (see also Section 3) that, for a surface in such a form, the density $d\nu_{\mathcal{M}}/d\xi$ is actually a constant that depends only on Q_1, Q_2, and therefore $\Sigma(Q_1, Q_2)$ is well-curved if and only if that constant is nonzero — and if it is zero, then the surface is flat. Thus in our quadratic case the well-curved/flat distinction of Definition 4 will be a perfect dichotomy.

We can now state our main result, which connects the boundedness properties of operators (1) to the curvature of $\Sigma(Q_1, Q_2)$.

Theorem 5 (well-curved surfaces). *Let Q_1, Q_2 be quadratic forms on $\mathbb{R}^d$ and suppose that the quadratic surface $\Sigma(Q_1, Q_2)$ is well-curved. Then, for every $1 \leq p, q \leq \infty$ such that*

$$2+\frac{d}{q} > \frac{d+2}{p} \quad \text{and} \quad \frac{2}{q} \geq \frac{1}{p},$$

[5] That is, if T is an affine transformation of $\mathbb{R}^n$ that preserves volumes, one has $\nu_{T(\mathcal{M})}(T(E)) = \nu_{\mathcal{M}}(E)$ for all Borel sets E.

we have

$$\|\mathcal{T} f\|_{L^q} \lesssim_{p,q,Q_1,Q_2} \|f\|_{L^p}$$

for every function f supported in $B(0, C) \times [-1, 1]^2$.

If instead the surface $\Sigma(Q_1, Q_2)$ is not well-curved (hence flat), then every $L^p \to L^q$ estimate with (p, q) sufficiently close to the endpoint $((d+4)/4, (d+4)/2)$ is false.

The examples that yield the conjectural range (4) show that the range of exponents in the theorem above is sharp, save perhaps for the missing critical line $2 + d/q = (d+2)/p$. The theorem is obtained by interpolating the trivial $L^p \to L^1$ and $L^\infty \to L^q$ estimates from Remark 3 with restricted weak-type estimates along the critical line $2/q = 1/p$ and arbitrarily near the endpoint estimate $L^{(d+4)/4} \to L^{(d+4)/2}$. The latter are obtained using Christ's method of refinements, but alternative proofs can be given using techniques of Gressman from either [31] or [32]; see Remark 26 in this regard.

We observe that in general it is possible with our methods to obtain the restricted weak-type endpoint estimate as well, unless the surface $\Sigma(Q_1, Q_2)$ belongs to a certain class that can be described explicitly; this description relies upon Theorem 7 below and will be given in Remark 35 of Section 6. As stated, the range of exponents is also sharp in the curvature condition, in the sense that the range of true estimates is necessarily smaller when the surface is flat (this will be proven in Section 7 — see also Theorem 9 below). In particular, Theorem 5 shows that any $L^p \to L^q$ estimate for $\mathcal{T}$ with (p, q) near the endpoint is equivalent to the well-curvedness of $\Sigma(Q_1, Q_2)$ (see [36] for a result of similar flavour in the context of Fourier restriction for hypersurfaces).

Our methods are sufficiently stable under perturbation that we are also able to extend Theorem 5 to more general codimension-2 surfaces. Indeed, let $\varphi_1, \varphi_2 : \mathbb{R}^d \to \mathbb{R}$ be C^2 functions such that $\nabla\varphi_1(0) = \nabla\varphi_2(0) = 0$ and let $\Sigma(\varphi_1, \varphi_2)$ denote the surface parametrised by

$$(\xi, \varphi_1(\xi), \varphi_2(\xi)), \quad \xi \in [-\epsilon, \epsilon]^d,$$

where $\epsilon > 0$ is sufficiently small depending on φ_1, φ_2. The analogue of operator (1), denoted by $\mathcal{T}_{\varphi_1,\varphi_2}$, is given by

$$\mathcal{T}_{\varphi_1,\varphi_2} f(x, \xi) := \iint_{|s|,|t|\le 1} f(x - s\,\nabla\varphi_1(\xi) - t\,\nabla\varphi_2(\xi), s, t)\, ds\, dt.$$

Theorem 5′ (general well-curved surfaces). *Let φ_1, φ_2 be as above and suppose that $\Sigma(\varphi_1, \varphi_2)$ is well-curved at $\xi = 0$. Then, for every $1 \le p, q \le \infty$ such that*

$$2 + \frac{d}{q} > \frac{d+2}{p} \quad \textit{and} \quad \frac{2}{q} \ge \frac{1}{p},$$

we have

$$\|\mathcal{T}_{\varphi_1,\varphi_2} f\|_{L^q} \lesssim_{p,q,\varphi_1,\varphi_2} \|f\|_{L^p}$$

for every function f supported in $B(0, C) \times [-1, 1]^2$.

The range of exponents above is identical to the one given in Theorem 5. To show Theorem 5′, only small adjustments need to be made to the argument for the quadratic surface case — the necessary modifications will be sketched in the Appendix.

By standard interpolation theory for mixed-norm spaces (see, e.g., [3]), one obtains from the strong-type inequalities of Theorem 5 a whole range of mixed-norm estimates of the form (3), upon interpolation with the (strong-type) trivial estimates in Remark 3.

Corollary 6 (mixed-norm range). *Let Q_1, Q_2 be quadratic forms on $\mathbb{R}^d$ and suppose that the quadratic surface $\Sigma(Q_1, Q_2)$ is well-curved. Then, for every $1 \le p, q, r \le \infty$ such that*

$$2 + \frac{d}{r} > \frac{d+2}{p}, \quad \frac{1}{r} + \frac{1}{q} \ge \frac{1}{p} \quad \text{and} \quad \frac{2}{r} \ge \frac{1}{p},$$

we have

$$\|\mathcal{T} f\|_{L^q(L^r)} \lesssim_{p,q,r,Q_1,Q_2} \|f\|_{L^p}$$

for every function f supported in $B(0, C) \times [-1, 1]^d$.

The proof of Theorem 5 rests on an algebraic characterisation of well-curvedness which is enabled by a connection between Gressman's affine invariant measures and geometric invariant theory; it is of independent interest. Specifically, we prove the following fact.

Theorem 7. *Let Q_1, Q_2 be quadratic forms on $\mathbb{R}^d$. The quadratic surface $\Sigma(Q_1, Q_2)$ is well-curved if and only if the following condition is satisfied*:

the homogeneous polynomial in s, t given by $\det(s\nabla^2 Q_1 + t\nabla^2 Q_2)$ does not vanish identically and does not admit any root of multiplicity larger than $d/2$. (M)

Here by *root* of a homogeneous polynomial in $\mathbb{R}[s, t]$ we mean a homogeneous linear divisor $as + bt$ in $\mathbb{C}[s, t]$, and by its (algebraic) multiplicity we mean the largest power m such that $(as + bt)^m$ is still a divisor. Theorem 7 is stated for quadratic forms, but it holds "pointwise" for arbitrary $\Sigma(\varphi_1, \varphi_2)$ surfaces: the surface is well-curved if $\det(s\nabla^2\varphi_1(\xi) + t\nabla^2\varphi_2(\xi))$ satisfies (M) for every ξ.

Example 8. Consider the quadratic surfaces $\Sigma(Q_1, Q_2)$ given by

$$Q_1(\xi) := \frac{1}{2}\sum_{j=1}^{d} \lambda_j \xi_j^2, \quad Q_2(\xi) := \frac{1}{2}\sum_{j=1}^{d} \mu_j \xi_j^2,$$

where the λ_j, μ_j are real coefficients that for any j are not simultaneously zero. We have

$$\det(s\nabla^2 Q_1 + t\nabla^2 Q_2) = \prod_{j=1}^{d} (s\lambda_j + t\mu_j)$$

and thus by Theorem 7 the surface $\Sigma(Q_1, Q_2)$ is well-curved if $\#\{j : [\lambda_j : \mu_j] = [\lambda : \mu]\} \le d/2$ for all $[\lambda : \mu] \in \mathbb{P}(\mathbb{R}^2)$.

This is not the first instance in which the object $\det(s\nabla^2 Q_1 + t\nabla^2 Q_2)$ and condition (M) have made their appearance in harmonic analysis: readers familiar with M. Christ's Ph.D. thesis [14] will recognise (M) above as being precisely the condition that yields the sharp $L^p \to L^2$ estimates for the operator of Fourier restriction to surfaces $\Sigma(Q_1, Q_2)$. Thus, in light of Theorem 7, M. Christ's result can be retroactively reformulated as saying that the Fourier restriction operator $Rf := \hat{f}|_{\Sigma(Q_1,Q_2)}$ satisfies optimal $L^p \to L^2$ estimates if and only if $\Sigma(Q_1, Q_2)$ is well-curved in the sense of Definition 4. See Section 2 for additional details.

The characterisation of well-curvedness provided above is quantitative to some extent, and in particular it gives us a way to gauge the "flatness" of surfaces which are not well-curved. Indeed, a flat $\Sigma(Q_1, Q_2)$ surface must be such that $\det(s\nabla^2 Q_1 + t\nabla^2 Q_2)$ has a root of multiplicity $m_* > d/2$, which in particular is the largest of all the root multiplicities. Intuitively, we expect that as the largest multiplicity m_* increases, the surface gets flatter (with the most extreme case being that in which $\det(s\nabla^2 Q_1 + t\nabla^2 Q_2)$ vanishes identically); consequently, we expect the $L^p \to L^q$ mapping properties of operator (1) to worsen. It turns out that indeed this largest multiplicity m_* controls the surviving range of boundedness of the operators (1), particularly along the critical line $2/q = 1/p$. We have the following partial analogue of Theorem 5 for flat surfaces.

Theorem 9 (flat surfaces). *Let Q_1, Q_2 be quadratic forms on $\mathbb{R}^d$ and suppose that the quadratic surface $\Sigma(Q_1, Q_2)$ is flat but $\det(s\nabla^2 Q_1 + t\nabla^2 Q_2)$ is not identically vanishing. Let $m_* > d/2$ denote the largest multiplicity among its roots. Then for every $1 \le p, q \le \infty$ such that*

$$1 + \frac{m_*}{q} \ge \frac{m_*+1}{p} \quad \textit{and} \quad \frac{2}{q} \ge \frac{1}{p},$$

with the exception of $p = (m_* + 2)/2,\ q = m_* + 2$, we have

$$\|\mathcal{T} f\|_{L^q} \lesssim_{p,q,Q_1,Q_2} \|f\|_{L^p}$$

for every function f supported in $B(0, C) \times [-1, 1]^d$. Moreover, every $L^p \to L^q$ estimate with $1+m_*/q < (m_* + 1)/p$ and $2/q = 1/p$ is false.

If instead $\det(s\nabla^2 Q_1 + t\nabla^2 Q_2)$ vanishes identically, then there is an $\epsilon = \epsilon_{Q_1,Q_2}$ with $0 < \epsilon < 1$ such that every $L^p \to L^q$ estimate with $(2-\epsilon)/q < 1/p$ is false (this includes in particular estimates with $2/q = 1/p$ for $(p, q) \neq (\infty, \infty)$).

The statement above does not paint the full picture: our counterexamples rule out a range of exponents beyond those on the line $2/q = 1/p$; however, which exponents we are able to rule out depends on properties of Q_1, Q_2 (or rather, of the associated Hessian matrices A, B) that go beyond the single value m_*. We direct the reader to Section 7 for the more precise picture, and particularly to condition (30) and Figure 3 there.

The ranges given in Theorem 9 are strict subsets of that given in Theorem 5, and the aforementioned counterexamples of Section 7 show that this is necessarily the case. Moreover, these ranges become smaller as m_* increases. We do not know whether the given ranges are sharp for all flat surfaces $\Sigma(Q_1, Q_2)$ outside of the line $2/q = 1/p$, but we are able to show that they are for some classes of surfaces. This will also be detailed in Section 7.

1.2. ***Structure of the paper.*** In Section 2 we provide context for the study of operators (1) by describing how they relate to the Fourier restriction problem for surfaces of codimension 2 such as $\Sigma(Q_1, Q_2)$; the connection passes through Kakeya-type estimates, and some application of these is also discussed. In Section 3 we recall Gressman's construction of affine invariant surface measures from [30] in the special case of a surface of codimension 2, and we describe how the well-curvedness of such surfaces can be interpreted in algebraic terms via geometric invariant theory. In Section 4 we harness this connection to prove an algebraic characterisation of well-curvedness in terms of the multiplicity of the roots of polynomials $\det(s\nabla^2 Q_1 + t\nabla^2 Q_2)$ — this is Theorem 7. The argument is split in two parts, as the case in which $\det(s\nabla^2 Q_1 + t\nabla^2 Q_2)$ vanishes identically needs to be treated separately. With this preliminary work done, in Section 5 we prove Theorems 5 and 9 with a particularly simple instance of Christ's method of refinements from [16]. The latter reduces matters to proving sharp sublevel set estimates for the polynomial $\det(s\nabla^2 Q_1 + t\nabla^2 Q_2)$, which are the subject of Section 6. The proof is somewhat unusual in that it employs a simple linear programming argument; it might be of independent interest. In Section 7 we discuss the case of flat surfaces of codimension 2; we provide counterexamples that rule out various $L^p \to L^q$ estimates that are instead true for well-curved surfaces. Finally, in the Appendix we sketch the modification needed to prove Theorem 5′.

Notation. For M a matrix, we let $M^\top$ denote its transpose and $\|M\|$ denote its operator norm. For $E \subset \mathbb{R}^n$ a set, we let $\mathbf{1}_E$ denote its characteristic function and $|E|$ denote its Lebesgue measure. For nonnegative quantities A, B, we write $A \lesssim B$ if there exists a constant $C > 0$ such that $A \leq CB$. If the value of the constant C depends on a list of parameters $\mathcal{P}$ we write $A \lesssim_{\mathcal{P}} B$ to highlight this fact. If $A \lesssim B$ and $B \lesssim A$, we write $A \sim B$. In conditional statements we will write $A \ll B$ to denote the inequality $A \leq cB$ for some sufficiently small constant $c > 0$.

2. Motivation and applications

In this section we will provide motivation for the study of the operators $\mathcal{T}$ given by (1). Such motivation arises most prominently from the study of the Fourier restriction problem and related matters such as the study of Kakeya/Besicovitch-type sets and the Mizohata–Takeuchi conjecture; we will review these in the context of codimension-2 surfaces, as this will allow us to compare conditions present in the literature with our definition of well-curvedness.

2.1. ***Fourier restriction.*** The Fourier restriction problem for a submanifold $\mathcal{M} \subset \mathbb{R}^n$ (with surface measure $d\sigma$), in its equivalent adjoint formulation known as the Fourier extension problem, is concerned with the boundedness properties of the Fourier extension operator given by

$$E_{\mathcal{M}} g(x) := \int_{\mathcal{M}} g(\xi) e^{2\pi i \xi \cdot x} \, d\sigma(\xi).$$

More specifically, one is interested in determining the full set of exponents p, q for which estimates

$$\|E_{\mathcal{M}} g\|_{L^q(\mathbb{R}^n)} \lesssim_{p,q} \|g\|_{L^p(\mathcal{M}, d\sigma)} \tag{6}$$

hold. The literature on this problem is immense (particularly in the case of codimension 1) and we do not attempt to review it here; rather, we concentrate on (a selection of) works on the case of submanifolds of codimension 2, which is most directly relevant to us and has been studied in a number of instances.

The first such instance addressing codimension 2 specifically occurred in M. Christ's Ph.D. thesis [14], in which he studied inequalities (6) for $p = 2$; such results are commonly known as L^2-restriction theorems or as Tomas–Stein theorems. For quadratic surfaces $\Sigma(Q_1, Q_2)$ he proved[6] that under condition (M) of

[6] See Section 12 of [14].

Section 1.1 the extension operator $E_{\Sigma(Q_1,Q_2)}$ satisfies the $L^2 \to L^q$ estimates (6) for every $q \geq q_0 := (2d+8)/d$ (which is sharp), with the exception of the case of d even and $\det(s\nabla^2 Q_1 + t\nabla^2 Q_2)$ having a root of multiplicity exactly $d/2$, in which case $q > q_0$ instead. Moreover, he showed[7] that (M) is also necessary, in the sense that if $\Sigma(Q_1, Q_2)$ violates the condition then the $L^2 \to L^q$ estimates are false for any q sufficiently close to the endpoint q_0. In this work, condition (M) came about as the condition that would guarantee the appropriate decay of $\hat{\mu}$, where μ is the measure given by

$$\mu(f) := \int_{[-1,1]^d} f(\xi, Q_1(\xi), Q_2(\xi))\, d\xi;$$

such decay is a fundamental ingredient in L^2-restriction arguments à la Tomas–Stein. In retrospect, it should come as no surprise that the endpoint or near-endpoint $L^2 \to L^{q_0}$ Fourier extension estimate — and hence condition (M) — is equivalent to the well-curvedness of the surface $\Sigma(Q_1, Q_2)$, as it was shown in [36] that this is also the case for hypersurfaces. The interpretation of (M) as a type of curvature condition was noted in [14].

Christ's L^2-restriction results were later extended by G. Mockenhaupt [41] to flat quadratic surfaces and by L. De Carli and A. Iosevich [20] to some flat nonquadratic surfaces. D. Oberlin [43] proved Fourier restriction estimates beyond the Tomas–Stein range for $d = 3$ and for the surface given by $Q_1(\xi) = \xi_1^2 + \xi_2^2,\ Q_2(\xi) = \xi_1^2 + \xi_3^2$. More recently, S. Guo and C. Oh [33] have addressed the Fourier restriction problem for general quadratic surfaces of codimension 2 in $\mathbb{R}^5$, proving estimates of type (6) that go beyond the Tomas–Stein range and are sharp for some classes of surfaces (all of them flat). Their only assumptions on the pair (Q_1, Q_2) are that the quadratic forms are linearly independent and that $\ker \nabla^2 Q_1 \cap \ker \nabla^2 Q_2 = \{0\}$ — in particular, this excludes only a rather degenerate subclass of the set of pairs (Q_1, Q_2) for which $\det(s\nabla^2 Q_1 + t\nabla^2 Q_2)$ vanishes identically (see Section 4.3 for more general pairs with vanishing determinant). Interestingly, the range of exponents they obtain is the same for all pairs of quadratic forms considered; it is expected that a larger range could be obtained for well-curved surfaces.

Having provided some context, we will now describe how the operator $\mathcal{T}$ makes its appearance in the Fourier restriction problem. We will keep the discussion light by not worrying too much about rigour.

The most successful approaches to the Fourier restriction problem to date are all based on wavepacket decompositions. In the case of codimension 2 specifically (we will use Σ for $\Sigma(Q_1, Q_2)$ for shortness), in order to study the extension operator E_Σ one can equivalently study the modified extension operator

$$E^\delta_\Sigma\, g(x) := \int_{\mathcal{N}_\delta(\Sigma)} g(\xi) e^{2\pi i \xi\cdot x}\, d\xi,$$

where $\mathcal{N}_\delta(\Sigma)$ is the δ-neighbourhood of Σ and g is supported on this neighbourhood (thus $E^\delta_\Sigma g = \hat{g}$). Estimates (6) are then replaced by local-type estimates of the form

$$\|E^\delta_\Sigma\, g\|_{L^q(B(\delta^{-1}))} \lesssim_{p,q,\alpha} \delta^{\frac{2}{p'}-\alpha} \|g\|_{L^p(\mathcal{N}_\delta(\Sigma))} \tag{7}$$

for every $\delta \leq 1$ and every $\alpha \geq 0$, where $B(\delta^{-1})$ is the ball of radius δ^{-1} centred at 0. These estimates are known to imply estimates of type (6) (see, e.g., Section 4 of [33]). The reason for passing to local-type

[7] See Section 3 of [14] and in particular Proposition 3.1 therein.

estimates is that $\mathcal{N}_\delta(\Sigma)$ can be neatly partitioned into parabolic boxes adapted to the geometry of Σ, and such a partition automatically yields a geometrically meaningful way to partition g and $E^\delta_\Sigma g$. The parabolic box that approximates $\mathcal{N}_\delta(\Sigma)$ in the vicinity of point $\phi(\xi) = (\xi, Q_1(\xi), Q_2(\xi))$ must have dimensions $\sim \delta^{1/2} \times \cdots \times \delta^{1/2} \times \delta \times \delta$ (this can be seen by a Taylor expansion). It can be described as the set of points given by

$$\phi(\xi) + \sum_{j=1}^{d} \delta^{\frac{1}{2}} \lambda_j \boldsymbol{v}_j(\xi) + \delta \nu_1 \boldsymbol{n}_1(\xi) + \delta \nu_2 \boldsymbol{n}_2(\xi)$$

for arbitrary $|\lambda_j|, |\nu_1|, |\nu_2| \lesssim 1$, where[8]

$$\begin{aligned} \boldsymbol{v}_j(\xi) &:= (\boldsymbol{e}_j, \partial_j Q_1(\xi), \partial_j Q_2(\xi)), \quad j = 1, \dots, d, \\ \boldsymbol{n}_1(\xi) &:= (-\nabla Q_1(\xi), 1, 0), \\ \boldsymbol{n}_2(\xi) &:= (-\nabla Q_2(\xi), 0, 1); \end{aligned}$$

here the $\boldsymbol{v}_j$ span the directions tangent to Σ and $\boldsymbol{n}_1, \boldsymbol{n}_2$ span the normal ones. Given a collection $\mathcal{F}$ of boundedly overlapping boxes θ of the form above covering $\mathcal{N}_\delta(\Sigma)$, one can form an associated partition of unity by smooth functions χ_θ and consequently decompose g as

$$g = \sum_{\theta \in \mathcal{F}} g_\theta := \sum_{\theta \in \mathcal{F}} g \chi_\theta.$$

By the uncertainty principle, $|\hat{g}_\theta|$ (that is, $|E^\delta_\Sigma g_\theta|$) is approximately constant on any translate of the box dual[9] to the box θ, denoted by θ^*, which has dimensions $\sim \delta^{-1/2} \times \cdots \times \delta^{-1/2} \times \delta^{-1} \times \delta^{-1}$ and long directions spanning the same 2-plane as $\boldsymbol{n}_1, \boldsymbol{n}_2$. Thus geometrically θ^* is roughly the intersection of a cube of sidelength $\sim \delta^{-1}$ with the $O(\delta^{-1/2})$-neighbourhood of a 2-plane normal to Σ at some point; we call these objects *slabs* (of length δ^{-1} and thickness $\delta^{-1/2}$). Denote by $\mathcal{S}_\theta$ a collection of boundedly overlapping copies of θ^* (i.e., slabs) that covers $\mathbb{R}^{d+2}$; then we can further partition each $\hat{g}_\theta$ by localising it[10] to every $S \in \mathcal{S}_\theta$, writing $\hat{g}_\theta = \sum_{S \in \mathcal{S}_\theta} \hat{g}_\theta \chi_S$. In this way we effectively resolve $E^\delta_\Sigma g$ into wavepackets that are frequency-supported on some box θ, concentrated on a translate of θ^* and approximately constant (in magnitude) there.

To obtain Fourier extension estimates, the strategy typically involves controlling the interactions between different wavepackets by various means; by the observations above, such control can be achieved by studying the overlap of slabs coming from different $\mathcal{S}_\theta$'s. The celebrated Bourgain–Guth argument (also referred to as broad/narrow analysis), originating in [9], employs precisely such a strategy to prove estimates of the form (7). It is beyond the scope of this article to present the argument in any amount of detail, but we remark that it can take as input Kakeya-type inequalities, which are functionally of the form

$$\Big\| \sum_{S \in \mathcal{S}} \mathbf{1}_S \Big\|_{L^r} \lesssim_r \delta^{-\beta},$$

[8] $\boldsymbol{e}_j$ denotes the j-th element in the standard basis of $\mathbb{R}^d$.

[9] Recall given a parallelepiped P in $\mathbb{R}^n$ centred at 0, its dual P^* is the parallelepiped $P^* := \{\boldsymbol{u} \in \mathbb{R}^n : |\boldsymbol{u} \cdot \boldsymbol{v}| \le 1 \text{ for all } \boldsymbol{v} \in P\}$.

[10] This can only be done approximately, as it is good to keep the frequency localisation intact.

where $\mathcal{S}$ is (for example) a collection of slabs containing a single element from each $\mathcal{S}_\theta$ and $\beta \geq 0$. Such inequalities can be deduced from estimates (3) via duality and discretisation, as the proof of the following corollary will show. In order to avoid technicalities, we work with some simpler slabs which are rescaled to have length 1 and thickness δ: using A, B for $\nabla^2 Q_1$, $\nabla^2 Q_2$, for $x \in \mathbb{R}^d$ and $\xi \in [-1,1]^d$ we let $S_\delta(x,\xi)$ denote the slab

$$S_\delta(x,\xi) := \{(y,s,t) \in \mathbb{R}^d \times [-1,1]^2 : |y - x + sA\xi + tB\xi| < \delta\}$$

(notice that this is indeed the $O(\delta)$-neighbourhood of the 2-plane spanned by $\boldsymbol{n}_1(\xi)$, $\boldsymbol{n}_2(\xi)$, intersected with a cube of sidelength ~ 1).

Corollary 10 (Kakeya-type estimate). *Let Q_1, Q_2 be quadratic forms on $\mathbb{R}^d$ and suppose that the operator $\mathcal{T}$ is $L^p \to L^q$ bounded. If $(x_j,\xi_j)_{j\in J}$ are points in $\mathbb{R}^d \times [-1,1]^d$ such that the ξ_j are δ-separated, we have*

$$\Big\| \sum_{j\in J} a_j \mathbf{1}_{S_\delta(x_j,\xi_j)} \Big\|_{L^{p'}} \lesssim_{Q_1,Q_2,p,q} \delta^{-d+\frac{2d}{q'}} \Big(\sum_{j\in J} |a_j|^{q'} \Big)^{\frac{1}{q'}}. \tag{8}$$

In particular, if $\Sigma(Q_1,Q_2)$ is well-curved, we have for every $\epsilon > 0$

$$\Big\| \sum_{j\in J} \mathbf{1}_{S_\delta(x_j,\xi_j)} \Big\|_{L^{(d+4)/d}} \lesssim_{Q_1,Q_2,\epsilon} \delta^{\frac{d^2}{d+4}-\epsilon} (\#J)^{\frac{d+2}{d+4}}.$$

In the next subsection we will provide an application of Corollary 10 to a problem in geometric measure theory.

Remark 11. It is well known that by a standard randomisation argument it is possible to deduce estimates such as those encountered in Corollary 10 from Fourier restriction estimates such as (6) (see for instance Section 22.3 of [40]). However, away from the restriction endpoint these estimates are not necessarily as efficient as those deduced from $L^p \to L^q$ bounds for the operator $\mathcal{T}$. To wit, using Christ's Fourier restriction estimate one can deduce the inequality

$$\Big\| \sum_{j\in J} \mathbf{1}_{S_\delta(x_j,\xi_j)} \Big\|_{L^{(d+4)/d}} \lesssim_{Q_1,Q_2,\epsilon} \delta^{\frac{d^2}{d+4}-\epsilon} (\#J),$$

which is weaker than the one obtained in Corollary 10.

Proof of Corollary 10. From the hypothesis we have by duality $\|\mathcal{T}^* g\|_{L^{p'}} \lesssim \|g\|_{L^{q'}}$, where the adjoint $\mathcal{T}^*$ is given by

$$\mathcal{T}^* g(y,s,t) = \int_{[-1,1]^d} g(y + sA\xi + tB\xi, \xi)\, d\xi.$$

The statement is a consequence of following simple fact: with $K := \|A\| + \|B\|$, we have

$$\mathcal{T}^*(\mathbf{1}_{B(x,2\delta)} \mathbf{1}_{B(\xi,K^{-1}\delta)}) \gtrsim_{A,B} \delta^d\, \mathbf{1}_{S_\delta(x,\xi)}.$$

Taking $g(x,\xi) = \sum_{j\in J} a_j \mathbf{1}_{B(x_j,2\delta)}(x) \mathbf{1}_{B(\xi_j,K^{-1}\delta)}(\xi)$ and using the δ-separation of the ξ_j, estimate (8) follows readily from the dual estimate above.

For the well-curved case, apply (8) with $a_j = 1$ and (p,q) along the $2/q = 1/p$ line and arbitrarily close to the endpoint $(p,q) = ((d+4)/4, (d+4)/2)$ (these are the estimates afforded by Theorem 5).

Finally, interpolate with the trivial $\big\|\sum_{j\in J}\mathbf{1}_{S_\delta(x_j,\xi_j)}\big\|_{L^\infty}\lesssim \#J$ estimate to upgrade the norm to an $L^{(d+4)/d}$ one (this costs us a $\delta^{-\epsilon}$ loss, since $\#J\lesssim\delta^{-d}$). □

The above discussion thus motivates the study of restricted 2-plane transforms (1) in the context of the Fourier restriction problem. We plan to pursue this connection further in the near future.

2.2. *(n,k)-Kakeya sets.* Kakeya sets are subsets of $\mathbb{R}^n$ that contain a unit segment in every possible direction; a Kakeya set of measure zero is usually called a Besicovitch set (such sets exist). The Kakeya conjecture in geometric measure theory states that Besicovitch sets in $\mathbb{R}^n$ have necessarily Hausdorff dimension equal to n. More generally, (n,k)-Kakeya sets are subsets $E\subset\mathbb{R}^n$ such that for any k-dimensional subspace V (or "k-plane") there exists an affine translate $V+p$ such that $B(p,1)\cap(V+p)\subset E$ (where $B(p,1)$ denotes a ball in $\mathbb{R}^n$ of radius 1 centred at p); Kakeya sets then coincide with $(n,1)$-Kakeya sets. Analogously, an (n,k)-Besicovitch set is an (n,k)-Kakeya set of measure zero. Even the existence of (n,k)-Besicovitch sets for $k>1$ is an open problem, but it is generally believed that no such sets exist, as the numerology of the dimensions involved is not favourable — and for some (n,k) pairs this has indeed been proven. We direct the reader to Chapter 24 of [40] for details and an overview of the problem.

In order to obtain a more favourable situation, one might restrict the directions of the k-planes to lie in a submanifold $\mathcal{G}$ of the Grassmannian[11] $G(n,k)$ and define a $\mathcal{G}$-Kakeya set to be a set $E\subset\mathbb{R}^n$ such that for every $V\in\mathcal{G}$ there exists an affine translate $V+p$ such that $B(p,1)\cap(V+p)\subset E$. Some works exist in this direction — see [27; 44; 46] for some general types of submanifolds. Heuristically however, the most favourable situation appears to be that in which $\mathcal{G}$ satisfies $\dim\mathcal{G}+k=n$. This was the approach taken by K. Rogers [48], in which he considered $\mathcal{G}$-Kakeya sets for $\mathcal{G}$ a d-dimensional submanifold of $G(d+2,2)$, a case that is directly relevant to us. Indeed, the set

$$N(Q_1,Q_2):=\{\pi_\xi:\xi\in[-1,1]^d\},$$

where

$$\pi_\xi:=\operatorname{Span}\{(-\nabla Q_1(\xi),1,0),(-\nabla Q_2(\xi),0,1)\}$$

is the set of 2-planes that are normal to $\Sigma(Q_1,Q_2)$ at some point; under the very mild assumption $\ker\nabla^2Q_1\cap\ker\nabla^2Q_2=\{0\}$, this set is precisely a d-dimensional submanifold of $G(d+2,2)$. Rogers proved that when the submanifold $\mathcal{G}$ satisfies a certain curvature condition (akin to the Wolff axioms[12]) and $d=1$ then a $\mathcal{G}$-Kakeya set has Hausdorff dimension 3 (thus equal to the ambient dimension $d+2$), and when $d=2$ it has Hausdorff dimension at least $\frac{7}{2}$. Using Corollary 10, we can prove a similar statement for arbitrary $d\geq 2$ and Kakeya sets with respect to directions normal to surfaces $\Sigma(Q_1,Q_2)$.

Proposition 12 ($N(Q_1,Q_2)$-Kakeya sets). *Let $d\geq 2$ and let Q_1,Q_2 be quadratic forms on $\mathbb{R}^d$ with the property that the polynomial* $\det(s\nabla^2Q_1+t\nabla^2Q_2)$ *does not vanish identically. If E is an $N(Q_1,Q_2)$-Kakeya set in $\mathbb{R}^{d+2}$, then*

$$\dim_H E\geq\tfrac{1}{2}(d+4).$$

[11] The manifold of all linear subspaces of $\mathbb{R}^n$ of dimension k.

[12] See, e.g., Definition 13.1 in [38].

Proof. We will present the argument for Minkowski dimension for simplicity of exposition — the extension of the proof to Hausdorff dimension follows a standard argument that can be found in Section 4 of [48].

Let $(\xi_j)_{j\in J}$ be a maximal collection of δ-separated points in $[-1,1]^d$ and let $(x_j)_{j\in J}$ be arbitrary points in $\mathbb{R}^d$. It will suffice to show that to cover

$$E_\delta := \bigcup_{j\in J} S_\delta(x_j,\xi_j)$$

one needs at least $\gtrsim \delta^{-(d+4)/2}$ balls of radius δ. Observe that

$$\sum_{j\in J} |S_\delta(x_j,\xi_j)| \sim \delta^d \# J \sim 1,$$

and therefore by the Hölder inequality

$$|E_\delta|^{\frac{1}{p}} \Big\| \sum_{j\in J} \mathbf{1}_{S_\delta(x_j,\xi_j)} \Big\|_{L^{p'}} \gtrsim 1.$$

Since $\det(s\nabla^2 Q_1 + t\nabla^2 Q_2)$ does not vanish, Theorems 5 and 9 show that $\mathcal{T}$ is $L^p \to L^q$ bounded for some nontrivial (p,q) on the line $2/q = 1/p$. Applying Corollary 10 with any such estimate (and taking $a_j = 1$ in (8)) we obtain after some rearrangement

$$|E_\delta| \gtrsim \delta^{\frac{d}{2}},$$

which implies the claim (since $|B_{d+2}(\delta)| \sim \delta^{d+2}$). □

It is natural to want to compare the curvature assumptions, and in particular to wonder whether all $N(Q_1,Q_2)$ submanifolds are curved in the sense of [48]. We claim that they are, under the hypotheses of Proposition 12. The curvature condition would be somewhat cumbersome to state in here, so we omit it; however, in our case it boils down to the condition that for every $V \in G(d+2,2)$ with $\dim V > 2$ one has

$$\dim\{\pi \in \mathcal{G} : \pi \subset V\} \le \dim V - 2.$$

We will verify that this is the case when $\mathcal{G} = N(Q_1,Q_2)$. Let $\dim V = d+2-\ell$ and write $V = \{\boldsymbol{x} \in \mathbb{R}^{d+2} : \boldsymbol{v}_1 \cdot \boldsymbol{x} = \cdots = \boldsymbol{v}_\ell \cdot \boldsymbol{x} = 0\}$ for some linearly independent $\boldsymbol{v}_1,\dots,\boldsymbol{v}_\ell$ (the case $\ell = 0$ is trivial, so we can assume $\ell \ge 1$). Write $\boldsymbol{v}_j = (u_j, a_j, b_j) \in \mathbb{R}^d \times \mathbb{R} \times \mathbb{R}$ and observe that $\pi_\xi \subset V$ if and only if

$$Au_j \cdot \xi = a_j, \quad Bu_j \cdot \xi = b_j \quad \text{for all } j \in \{1,\dots,\ell\}$$

so that the dimension of $\{\pi \in N(Q_1,Q_2) : \pi \subset V\}$ is the same as the dimension of the space of solutions to these equations. If the $u_1,\dots,u_\ell$ are not linearly independent then the equations do not have a solution (as this would make the $\boldsymbol{v}_j$ linearly dependent as well); hence we can assume that they are linearly independent. Letting $U := (u_1 \ \cdots \ u_\ell)$ we see that the dimension is bounded by $\dim\ker\binom{AU}{BU}$. To show that this is $\le \dim V - 2 = d - \ell$ it is equivalent to show that $\operatorname{rk}\binom{AU}{BU} \ge \ell$; but by assumption there exists (s,t) such that $\det(sA+tB) \ne 0$, and since $\operatorname{rk} U = \ell$ we see that $\operatorname{rk}(sA+tB)U = \ell$ and thus the rank condition is satisfied. This finishes the proof of the claim.

2.3. ***Mizohata–Takeuchi conjecture.*** In this last motivational subsection we show how operators of the form (1) appear naturally in the context of the Mizohata–Takeuchi conjecture for surfaces of codimension 2.

The Mizohata–Takeuchi conjecture is a variant of the Fourier restriction problem that concerns weighted L^2 estimates for the Fourier extension operator (it originated in the study of dispersive and hyperbolic PDEs). For a hypersurface $\Sigma \subset \mathbb{R}^n$ with surface measure $d\sigma$ the conjecture takes the form

$$\int_{\mathbb{R}^n} |E_\Sigma\, g(x)|^2 w(x)\, dx \lesssim \|Xw\|_{L^\infty} \int_\Sigma |g|^2\, d\sigma,$$

where $X = T_{n,1}$ is the X-ray transform and w is a nonnegative function. The conjecture has been verified in the special case of $\Sigma = \mathbb{S}^{n-1}$ and weight-w radial, and this was done independently in [1] and [10]; it can also be proven by the methods of [11] but this was not realised at the time.[13] The single-scale version of the result was treated in [2]. The case of weights concentrated on a circle in the plane — the opposite case to radial weights in some sense — was treated in [5]. The conjecture is otherwise open in all dimensions n, including in $n = 2$, and the topic has been attracting increasing attention lately: see [4] and [6] for some variants involving tomographic bounds (that is, bounds on objects such as $X(|E_\Sigma\, g|^2)$, where X can later be transferred to the weight w via the X-ray inversion formula), [7] for connections with smoothing estimates, [49] for some results in $n = 2$, and [13] for a result for general n but with a loss in the scale.

For surfaces of codimension other than 1 one can generalise the conjecture as follows. For a submanifold $\mathcal{M} \subset \mathbb{R}^n$ of codimension k, denote by $N(\mathcal{M})$ the set of k-planes π such that, for some point $p \in \mathcal{M}$, π is orthogonal to $T_p\mathcal{M}$ (thus $N(\mathcal{M})$ is the set of normal directions of $\mathcal{M}$); then one conjectures that for every nonnegative weight w

$$\int_{\mathbb{R}^n} |E_{\mathcal{M}}\, g(x)|^2 w(x)\, dx \lesssim \sup_{\substack{\pi \in N(\mathcal{M}),\\ x \in \mathbb{R}^n}} |T_{n,k} w(\pi + x)| \int_{\mathcal{M}} |g|^2\, d\sigma.$$

The first factor on the right-hand side is effectively the L^∞ norm of the restriction of the k-plane transform $T_{n,k}$ to the set of normal directions to $\mathcal{M}$ — which is precisely the same type of operator as (1). We offer some modest evidence for this generalisation of the Mizohata–Takeuchi conjecture in all codimensions by proving the weak version stated in the proposition below (which has a worse norm on the weight). We prelude some definitions: for $Q_1, \dots, Q_k$ quadratic forms on $\mathbb{R}^d$ we let $\boldsymbol{Q}(\xi) := (Q_1(\xi), \dots, Q_k(\xi))$; we denote by $\Sigma(\boldsymbol{Q})$ the compact quadratic surface of codimension k in $\mathbb{R}^{d+k}$ parametrised by

$$\phi_{\boldsymbol{Q}}(\xi) := (\xi, \boldsymbol{Q}(\xi)), \quad \xi \in [-1, 1]^d.$$

We let

$$\mathcal{T}_{\boldsymbol{Q}} f(x, \xi) := \int_{\mathbb{R}^k} f(x - \nabla(\boldsymbol{s} \cdot \boldsymbol{Q})(\xi), \boldsymbol{s})\, d\boldsymbol{s},$$

where $\nabla = \nabla_\xi$ is applied componentwise, that is, $\nabla(\boldsymbol{s} \cdot \boldsymbol{Q}) = \sum_{j=1}^k s_j \nabla Q_j$; notice that when $k = 2$ this is precisely the nonlocal version of operator (1). This operator is pointwise comparable to the restriction of

[13] This was communicated to us by A. Carbery.

$T_{n,k}$ to directions normal to $\Sigma(\boldsymbol{Q})$. Finally, for simplicity we will work with the slightly modified Fourier extension operator

$$\mathsf{E}_{\Sigma(\boldsymbol{Q})}\, g(\boldsymbol{x}) := \int_{[-1,1]^d} g(\xi) e^{2\pi i \boldsymbol{x}\cdot\phi_{\boldsymbol{Q}}(\xi)}\, d\xi.$$

Proposition 13. *Let $k \geq 1$ and let $\boldsymbol{Q} = (Q_1, \dots, Q_k)$ be a vector of k quadratic forms on $\mathbb{R}^d$. For every integrable weight $w : \mathbb{R}^{d+k} \to [0, \infty)$ we have*[14]

$$\int_{\mathbb{R}^{d+k}} |\mathsf{E}_{\Sigma(\boldsymbol{Q})}\, g(\boldsymbol{x})|^2\, w(\boldsymbol{x})\, d\boldsymbol{x} \lesssim \|\mathcal{T}_{\boldsymbol{Q}} w\|_{L^\infty(L^2)} \int_{[-1,1]^d} |g(\xi)|^2\, d\xi \tag{9}$$

for every function $g \in L^2$.

Proof. We note the following Radon duality formula, which will be useful later:

$$\begin{aligned}\mathcal{T}_{\boldsymbol{Q}} f(x,\xi) &= \int_{\mathbb{R}^k}\int_{\mathbb{R}^{d+k}} \hat{f}(\eta,\alpha) e^{2\pi i[\eta\cdot(x-\nabla(\boldsymbol{s}\cdot\boldsymbol{Q})(\xi))+\alpha\cdot\boldsymbol{s}]}\, d\eta\, d\alpha\, d\boldsymbol{s} \\ &= \int_{\mathbb{R}^{d+k}} \hat{f}(\eta,\alpha) e^{2\pi i\eta\cdot x} \int_{\mathbb{R}^k} e^{2\pi i\boldsymbol{s}\cdot(\alpha-\eta\cdot\nabla\boldsymbol{Q}(\xi))}\, d\boldsymbol{s}\, d\eta\, d\alpha \\ &= \int_{\mathbb{R}^{d+k}} e^{2\pi i\eta\cdot x}\, \hat{f}(\eta, \eta\cdot\nabla\boldsymbol{Q}(\xi))\, d\eta,\end{aligned} \tag{10}$$

where $\eta\cdot\nabla\boldsymbol{Q}(\xi) = (\eta\cdot\nabla Q_1(\xi), \dots, \eta\cdot\nabla Q_k(\xi))$.

Expanding the square in the left-hand side of (9), we have by Fubini

$$\begin{aligned}\int_{\mathbb{R}^{d+k}} |\mathsf{E}_{\Sigma(\boldsymbol{Q})}\, g(\boldsymbol{x})|^2 w(\boldsymbol{x})\, d\boldsymbol{x} &= \iiint g(\eta)\overline{g(\xi)} e^{2\pi i\boldsymbol{x}\cdot(\phi_{\boldsymbol{Q}}(\eta)-\phi_{\boldsymbol{Q}}(\xi))} w(\boldsymbol{x})\, d\eta\, d\xi\, d\boldsymbol{x} \\ &= \iint g(\eta)\overline{g(\xi)}\hat{w}(\phi_{\boldsymbol{Q}}(\xi)-\phi_{\boldsymbol{Q}}(\eta))\, d\eta\, d\xi.\end{aligned}$$

Now using the polarisation identity

$$Q(\xi) - Q(\eta) = \tfrac{1}{2}(\xi-\eta)\cdot\nabla Q(\xi+\eta),$$

we see by a change of variables that the last integral is equal to

$$\iint g\Big(\xi - \frac{\eta}{2}\Big)\overline{g\Big(\xi+\frac{\eta}{2}\Big)}\hat{w}(\eta, \eta\cdot\nabla\boldsymbol{Q}(\xi))\, d\xi\, d\eta.$$

As g is supported in $[-1,1]^d$ we see that we can insert in this expression a localisation factor $\mathbf{1}_{[-1,1]^d}(\xi)$ for free. By the Fourier inversion formula applied to $g, \bar{g}$ (which can be assumed to be Schwartz by a standard approximation argument) and a second change of variables we see that the expression can then be rearranged to be

$$\iint\bigg(\int \hat{g}\Big(\frac{y}{2}-x\Big)\overline{\hat{g}\Big(\frac{y}{2}+x\Big)} e^{2\pi i\xi\cdot y}\, dy\bigg)\bigg(\int e^{2\pi i\eta\cdot x}\, \hat{w}(\eta, \eta\cdot\nabla\boldsymbol{Q}(\xi))\, d\eta\bigg)\mathbf{1}_{[-1,1]^d}(\xi)\, d\xi\, dx.$$

[14] The mixed-norm is as in (2), that is, $L^\infty(L^2) = L^\infty_\xi(L^2_x)$.

In the second factor at the integrand we recognise $\mathcal{T}_{\boldsymbol{Q}} w(x, \xi)$ via the Radon duality formula (10). For the first factor, define the bilinear operator[15]

$$W(F_1, F_2)(x, \xi) := \int F_1\Big(\frac{y}{2} - x\Big) F_2\Big(\frac{y}{2} + x\Big) e^{2\pi i \xi \cdot y}\, dy;$$

then we see that the expression has become

$$\iint W(\hat{g}, \bar{\hat{g}})(x, \xi) \mathcal{T}_{\boldsymbol{Q}} w(x, \xi) \mathbf{1}_{[-1,1]^d}(\xi)\, d\xi\, dx.$$

By two applications of Cauchy–Schwarz (and using the fact that ξ is localised) this is bounded by

$$\begin{aligned}
&\leq \int \Big(\int |W(\hat{g}, \bar{\hat{g}})(x, \xi)|^2\, dx \Big)^{\frac{1}{2}} \Big(\int |\mathcal{T}_{\boldsymbol{Q}} w(x, \xi)|^2\, dx \Big)^{\frac{1}{2}} \mathbf{1}_{[-1,1]^d}(\xi)\, d\xi \\
&\leq \|\mathcal{T}_{\boldsymbol{Q}} w\|_{L^\infty_\xi(L^2_x)} \int \Big(\int |W(\hat{g}, \bar{\hat{g}})(x, \xi)|^2\, dx \Big)^{\frac{1}{2}} \mathbf{1}_{[-1,1]^d}(\xi)\, d\xi \\
&\lesssim \|\mathcal{T}_{\boldsymbol{Q}} w\|_{L^\infty(L^2)} \|W(\hat{g}, \bar{\hat{g}})\|_{L^2(L^2)}.
\end{aligned}$$

Finally, by identifying $W(F_1, F_2)$ with a Fourier transform, we see by Plancherel that $\|W(F_1, F_2)\|_{L^2(L^2)} = \|F_1\|_{L^2} \|F_2\|_{L^2}$, so that by a further application of Plancherel we have $\|W(\hat{g}, \bar{\hat{g}})\|_{L^2(L^2)} \leq \|g\|_{L^2}^2$. Inequality (9) follows. $\square$

3. Affine invariant measures and GIT

In this section we will briefly illustrate the construction of the affine invariant measures of Gressman [30] that are foundational to the definition of well-curvedness adopted here. In particular, we will explain how the nonvanishing of these measures is connected to the concept of semistability in geometric invariant theory (abbreviated GIT, from here onwards).

3.1. ***Construction of the affine invariant measure.*** In order to keep things simple, we will describe Gressman's construction only in the context of surfaces of codimension 2. The construction here given can extend easily to surfaces of other sufficiently low codimension (see Remark 15, but for the most general construction we refer the reader to [30].

The construction rests on two elements, the first being a lemma that allows one to construct a density from an arbitrary m-linear functional and the second being a choice of a suitable m-linear functional that captures curvature and enjoys affine invariance. We begin from the lemma, for which we introduce the following notation: letting Φ be an m-linear functional on the real finite-dimensional vector space V (that is, $\Phi \in (V^*)^{\otimes m}$), we denote by ρ the action of the special linear group $\mathrm{SL}(V)$ on $(V^*)^{\otimes m}$ given by

$$(\rho_M \Phi)(\boldsymbol{v}_1, \ldots, \boldsymbol{v}_m) := \Phi(M^\top \boldsymbol{v}_1, \ldots, M^\top \boldsymbol{v}_m) \tag{11}$$

[15]This operator is variously known as *ambiguity function* (in signal processing) or as *cross-Wigner distribution* (in quantum mechanics).

for any $M \in \mathrm{SL}(V)$ and any $\boldsymbol{v}_j \in V$. For $(\boldsymbol{v}_1, \dots, \boldsymbol{v}_d)$ an ordered choice of d vectors in V (where $d = \dim V$), we let

$$\|\Phi\|_{(\boldsymbol{v}_1,\dots,\boldsymbol{v}_d)} := \|(\Phi(\boldsymbol{v}_{j_1}, \dots, \boldsymbol{v}_{j_m}))_{j_1,\dots,j_m\in\{1,\dots,d\}}\|,$$

where $\|\cdot\|$ denotes an arbitrary norm on $\mathbb{R}^{dm}$ (say, the ℓ^2 norm for the sake of fixing one). The lemma is then as follows.

Lemma 14 [30, Proposition 1]. *Let V be a real vector space with $d = \dim V$ and let $\Phi \in (V^*)^{\otimes m}$ be an m-linear functional on V. Then there is a constant $c_\Phi \geq 0$ such that for every $\boldsymbol{v}_1, \dots, \boldsymbol{v}_d$*

$$\inf_{M\in\mathrm{SL}(V)} \|\rho_M \Phi\|^{\frac{d}{m}}_{(\boldsymbol{v}_1,\dots,\boldsymbol{v}_d)} = c_\Phi |\det(\boldsymbol{v}_1 \ \cdots \ \boldsymbol{v}_d)|.$$

The lemma comes with the important caveat that the constant c_Φ could vanish (this will correspond to the surface being "flat" at a point).

The multilinear functional to which Lemma 14 will be applied is called the *affine curvature tensor* and in the case of surfaces of codimension 2 it is defined as follows. Let $\phi : \Omega \to \mathbb{R}^{d+2}$ be an embedding of a d-dimensional manifold into $\mathbb{R}^{d+2}$ and for a fixed $p \in \Omega$ consider vector fields $X_1, \dots, X_d, Y_1, Y_2, Z_1, Z_2$ defined in a neighbourhood of p. Then we define the affine curvature tensor $\mathcal{A}^\phi_p$ to be

$$\mathcal{A}^\phi_p(X_1, \dots, X_d, Y_1, Y_2, Z_1, Z_2) := \det\big(X_1\phi(p) \ \cdots \ X_d\phi(p) \ \ Y_1Y_2\phi(p) \ \ Z_1Z_2\phi(p)\big).$$

It can be shown that $\mathcal{A}^\phi_p$ is indeed a tensor, in the sense that its value depends only on the value of the vector fields at p (see Proposition 2 of [30]); therefore $\mathcal{A}^\phi_p$ can be identified with an element of $((T_p\Omega)^*)^{\otimes(d+4)}$, that is, with a $(d+4)$-linear functional on the tangent space at p. Heuristically, the affine curvature tensor probes the Taylor expansion of ϕ around any given point (hence the second derivatives $Y_1Y_2\phi$ and $Z_1Z_2\phi$ in the definition, which detect the quadratic terms). It has moreover the important property of being *equi-affine invariant*, meaning that if T is any affine transformation of $\mathbb{R}^{d+2}$ that preserves volumes, then we have $\mathcal{A}^{T\circ\phi}_p = \mathcal{A}^\phi_p$.

Combining Lemma 14 with the affine curvature tensor one can then construct a surface measure on $\Sigma = \phi(\Omega)$ as follows. Define first of all the density

$$\delta^p_{\mathcal{A}}(X_1, \dots, X_d) := \inf_{M\in\mathrm{SL}(T_p\Omega)} \|\rho_M \mathcal{A}^\phi_p\|^{\frac{d}{d+4}}_{(X_1,\dots,X_d)}.$$

Then one can define the surface measure ν_Σ via push-forward: for a ball $B \subset \mathbb{R}^d$ and a coordinate chart $\varphi : B \to \Omega$, we let

$$\int_{\varphi(B)} g \, d\mu_{\mathcal{A}} := \int_B g(\varphi(y)) \, \delta^{\varphi(y)}_{\mathcal{A}}(d\varphi(\partial_{y_1}), \dots, d\varphi(\partial_{y_d})) \, dy_1 \cdots dy_d;$$

finally, we define the *affine invariant surface measure* ν_Σ by

$$\int_\Sigma f \, d\nu_\Sigma := \int_\Omega f \circ \phi \, d\mu_{\mathcal{A}}.$$

By Lemma 14, the definition of $\mu_{\mathcal{A}}$ is consistent on overlapping charts, giving a measure on the whole Ω (and thus on the whole Σ); moreover, it is not hard to see that the definition is independent of the particular embedding and that ν_Σ inherits the equi-affine invariance of $\mathcal{A}^\phi_p$.

Remark 15. The construction above is readily extended to d-dimensional submanifolds of $\mathbb{R}^{d+r}$ such that the codimension satisfies $r \le d(d+1)/2$. Indeed, it suffices to modify the affine curvature tensor to be

$$\mathcal{A}_p^\phi(X_1, \dots, X_d, Y_1, Z_1, \dots, Y_r, Z_r) := \det\big(X_1\phi(p) \ \cdots \ X_d\phi(p) \ \ Y_1Z_1\phi(p) \ \cdots \ Y_rZ_r\phi(p)\big);$$

then the density $\delta_{\mathcal{A}}^p$ is given by

$$\delta_{\mathcal{A}}^p(X_1, \dots, X_d) := \inf_{M \in \mathrm{SL}(T_p\Omega)} \|\rho_M \mathcal{A}_p^\phi\|_{(X_1,\dots,X_d)}^{\frac{d}{d+2r}}$$

and the rest of the construction is the same. The codimension condition $r \le d(d+1)/2$ has to do with the Taylor expansion of ϕ and in particular with the fact that there are exactly $d(d+1)/2$ monomials of degree 2 in d many variables; to deal with higher codimensions yet, the tensor $\mathcal{A}_p^\phi$ needs to be modified by introducing derivatives of progressively higher orders. The fully general construction is presented in [30].

The case in which we are interested is $\phi(\xi) = (\xi, Q_1(\xi), Q_2(\xi))$ (with $\Omega = [-1,1]^d$); we see then that the measure ν_Σ on $\Sigma = \Sigma(Q_1, Q_2)$ is given by

$$\int_{\Sigma(Q_1,Q_2)} f \, d\nu_\Sigma = \int_{[-1,1]^d} f(\phi(\xi)) \, \delta_{\mathcal{A}}^\xi(\partial_1, \dots, \partial_d) \, d\xi.$$

Let us write $(M\partial)_j := M^\top \partial_j$; thus if M_{ij} denotes the (i,j)-entry of M, we have $(M\partial)_j = \sum_{k=1}^d M_{jk}\partial_k$. Expanding the definitions, we have for the density $d\nu_\Sigma/d\xi$

$$\begin{aligned}\frac{d\nu_\Sigma}{d\xi} &= \delta_{\mathcal{A}}^\xi(\partial_1, \dots, \partial_d) \\ &= \Bigg[\inf_{M \in \mathrm{SL}(\mathbb{R}^d)} \Bigg(\sum_{\substack{i_1,\dots,i_d \\ j_1,j_2,k_1,k_2}} \big|\det\big((M\partial)_{i_1}\phi(\xi) \ \cdots \ (M\partial)_{i_d}\phi(\xi) \ \ (M\partial)_{j_1}(M\partial)_{j_2}\phi(\xi) \ \ (M\partial)_{k_1}(M\partial)_{k_2}\phi(\xi)\big)\big|^2\Bigg)^{\frac{1}{2}}\Bigg]^{\frac{d}{d+4}}.\end{aligned}$$

The expression simplifies significantly due to the special form of ϕ. Indeed, observe that the first d components of $(M\partial)_i\phi$ are simply the i-th column of $M^\top$, and the first d components of $(M\partial)_{j_1}(M\partial)_{j_2}\phi$ are identically zero; therefore the determinant vanishes unless $i_1, \dots, i_d$ is a permutation of $1, \dots, d$. Since $\det M = 1$, we obtain for the sum of determinants in the last expression

$$\begin{aligned}&\sum_{\substack{i_1,\dots,i_d \\ j_1,j_2, \\ k_1,k_2}} \big|(M\partial)_{i_1}\phi(\xi) \ \cdots \ (M\partial)_{i_d}\phi(\xi) \ \ (M\partial)_{j_1}(M\partial)_{j_2}\phi(\xi) \ \ (M\partial)_{k_1}(M\partial)_{k_2}\phi(\xi)\big|^2 \\ &\qquad = d! \sum_{j_1,j_2,k_1,k_2} \begin{vmatrix} (M\partial)_{j_1}(M\partial)_{j_2}Q_1(\xi) & (M\partial)_{k_1}(M\partial)_{k_2}Q_1(\xi) \\ (M\partial)_{j_1}(M\partial)_{j_2}Q_2(\xi) & (M\partial)_{k_1}(M\partial)_{k_2}Q_2(\xi) \end{vmatrix}^2.\end{aligned}$$

Remark 16. When Q_1, Q_2 are quadratic forms, the last expression is clearly independent of ξ and thus we see that $d\nu_\Sigma/d\xi$ is a constant, as claimed in Section 1.1. According to Definition 4 the surface $\Sigma(Q_1, Q_2)$ is well-curved if this constant is nonzero, and flat otherwise.

The expression can be massaged further: it is immediate that

$$(M\partial)_j(M\partial)_k Q_i(\xi) = (M\nabla^2 Q_i(\xi) M^\top)_{j,k},$$

and therefore the sum above coincides with

$$\sum_{j_1,j_2,k_1,k_2} \begin{vmatrix} (M\nabla^2 Q_1 M^\top)_{j_1,j_2} & (M\nabla^2 Q_1 M^\top)_{k_1,k_2} \\ (M\nabla^2 Q_2 M^\top)_{j_1,j_2} & (M\nabla^2 Q_2 M^\top)_{k_1,k_2} \end{vmatrix}^2 .$$

We can summarise the above as follows. Using again A, B in place of $\nabla^2 Q_1$, $\nabla^2 Q_2$, define the quadrilinear functional

$$\mathscr{A}_{A,B}(Y_1, Y_2; Z_1, Z_2) := \begin{vmatrix} \langle AY_1, Y_2\rangle & \langle AZ_1, Z_2\rangle \\ \langle BY_1, Y_2\rangle & \langle BZ_1, Z_2\rangle \end{vmatrix}$$

and notice that ρ given by (11) acts on this functional by

$$\rho_M \mathscr{A}_{A,B} = \mathscr{A}_{MAM^\top, MBM^\top}.$$

Then the density of ν_Σ for $\Sigma = \Sigma(Q_1, Q_2)$ is given by

$$\frac{d\nu_\Sigma}{d\xi} = c_d \inf_{M\in \mathrm{SL}(\mathbb{R}^d)} \|\mathscr{A}_{MAM^\top, MBM^\top}\|_\partial^{\frac{d}{d+4}},$$

where c_d is an absolute constant and we have shortened $\|\cdot\|_\partial := \|\cdot\|_{(\partial_1,\dots,\partial_d)}$. The fact that this quantity depends only on the Hessians has been made explicit. The reparametrisation and equi-affine invariances have also been made explicit in the following way: firstly, it is obvious that the density, as a function of the Hessians A, B, is invariant with respect to the "reparametrisation action" of $\mathrm{SL}(\mathbb{R}^d)$ on pairs of symmetric matrices given by (with a little abuse of notation)

$$\rho_M(A, B) := (MAM^\top, MBM^\top). \tag{12}$$

Secondly (and slightly less obviously), the density is also invariant as a function of A, B with respect to the action σ of $\mathrm{SL}(\mathbb{R}^2)$ given by,

$$\text{for } N = \begin{pmatrix} \lambda & \mu \\ \lambda' & \mu' \end{pmatrix} \in \mathrm{SL}(\mathbb{R}^2), \quad \sigma_N(A, B) := (\lambda A + \mu B, \lambda' A + \mu' B); \tag{13}$$

this is a consequence of the fact that $\mathscr{A}_{\cdot,\cdot}$ itself is σ-invariant, as can be seen by a straightforward calculation. These observations about invariances lead us directly into the next subsection.

3.2. ***Connection to GIT.*** GIT is the branch of algebraic geometry that studies group actions on algebraic varieties (of which vector spaces are a particularly simple instance); it provides a way to construct well-behaved quotient spaces via the study of polynomials that are invariant under these actions. One of the deepest insights of [30] is the realisation that the nonvanishing of the affine invariant surface measure is equivalent to the concept of semistability in GIT. Below we will explain this connection, limiting ourselves to the bare minimum of theory in order not to encumber the exposition.

We dive right in by stating a lemma from [30] that connects the density $d\nu_\Sigma/d\xi$ to certain invariant polynomials; the statement will be customised to our particular situation. Recall that the quadrilinear form $\mathscr{A}_{A,B}$ is an element of the vector space of quadrilinear functionals on $\mathbb{R}^d$, that is, $V := ((\mathbb{R}^d)^*)^{\otimes 4}$, and that the ρ action given by (11) is defined over the whole of V. A real polynomial P on V (that is, a polynomial in the coefficients of elements $\mathscr{A} \in V$) is *ρ-invariant* if for every $\mathscr{A} \in V$ and every $M \in \mathrm{SL}(\mathbb{R}^d)$

$$P(\rho_M \mathscr{A}) = P(\mathscr{A}).$$

These invariant polynomials form a ring, which is moreover finitely generated (this is a celebrated theorem of Hilbert [34]). It turns out that one can estimate the density via any set of homogeneous generators[16] of the invariant polynomials.

Lemma 17 [30, Lemma 2]. *Let $P_1, \ldots, P_N$ be homogeneous polynomials on $V = ((\mathbb{R}^d)^*)^{\otimes 4}$ that generate the ring of ρ-invariant polynomials. Then for every $\mathscr{A} \in V$ we have*

$$\inf_{M \in \mathrm{SL}(\mathbb{R}^d)} \|\rho_M \mathscr{A}\|_\partial \sim \max_{j \in \{1,\ldots,N\}} |P_j(\mathscr{A})|^{\frac{1}{\deg P_j}}.$$

By taking $\mathscr{A} = \mathscr{A}_{A,B}$, the left-hand side becomes (a multiple of) $(d\nu_\Sigma/d\xi)^{(d+4)/d}$, so that the lemma provides a way to estimate the density in terms of the generators via the expression at the right-hand side. The implicit constants depend on the choice of generators.

In proving the characterisation of well-curvedness given by Theorem 7, we will make use of a straightforward consequence of Lemma 17. Let $\mathrm{Sym}^2(\mathbb{R}^d)$ denote the space of real symmetric $d \times d$ matrices; then the actions ρ, σ, given by (12), (13) respectively, combine into an action of $\mathrm{SL}(\mathbb{R}^d) \times \mathrm{SL}(\mathbb{R}^2)$ on $\mathrm{Sym}^2(\mathbb{R}^d) \times \mathrm{Sym}^2(\mathbb{R}^d)$ denoted by $\rho \times \sigma$ and given by

$$(\rho \times \sigma)_{M,N}(A, B) := \rho_M(\sigma_N(A, B))$$

for any $M \in \mathrm{SL}(\mathbb{R}^d)$, $N \in \mathrm{SL}(\mathbb{R}^2)$ (observe that ρ and σ commute, so the order is inconsequential). We say that a polynomial Q on $\mathrm{Sym}^2(\mathbb{R}^d) \times \mathrm{Sym}^2(\mathbb{R}^d)$ is $(\rho \times \sigma)$-invariant if for every pair of real symmetric matrices A, B and every $M \in \mathrm{SL}(\mathbb{R}^d)$, $N \in \mathrm{SL}(\mathbb{R}^2)$

$$Q((\rho \times \sigma)_{M,N}(A, B)) = Q(A, B).$$

The lemma we will use is then the following.

Lemma 18. *Let Q_1, Q_2 be quadratic forms on $\mathbb{R}^d$, with associated surface $\Sigma = \Sigma(Q_1, Q_2)$, and let A, B be the Hessians $\nabla^2 Q_1, \nabla^2 Q_2$ respectively. Then the density $d\nu_\Sigma/d\xi$ is nonzero if and only if there exists a $(\rho \times \sigma)$-invariant polynomial Q on $\mathrm{Sym}^2(\mathbb{R}^d) \times \mathrm{Sym}^2(\mathbb{R}^d)$ such that $Q(0, 0) = 0$ but*

$$Q(A, B) \neq 0.$$

We point out that since the density is defined pointwise, the lemma extends to arbitrary surfaces parametrised by $(\xi, \varphi_1(\xi), \varphi_2(\xi))$ — just replace A, B with the Hessians of φ_1, φ_2 at the desired point.

Proof. By Lemma 17, if the density $d\nu_\Sigma/d\xi$ is nonzero then there exists a ρ-invariant homogeneous polynomial P on $V = ((\mathbb{R}^d)^*)^{\otimes 4}$ such that $P(\mathscr{A}_{A,B}) \neq 0$; but since $\mathscr{A}_{X,Y}$ is σ-invariant, we see that the polynomial $Q(X, Y) := P(\mathscr{A}_{X,Y})$ is $(\rho \times \sigma)$-invariant, $Q(0, 0) = 0$ and $Q(A, B) \neq 0$.

[16] It is easy to see that, since ρ commutes with dilations, given any set of generators one can form a set of homogeneous generators.

Conversely, assume that there exists such a polynomial $Q(X, Y)$ as per the statement. The polynomial is in particular σ-invariant, and it is a well-known fact that σ-invariant polynomials are generated by determinants

$$\begin{vmatrix} X_{j_1,j_2} & X_{k_1,k_2} \\ Y_{j_1,j_2} & Y_{k_1,k_2} \end{vmatrix}$$

(this is known as the first fundamental theorem for SL(2)-invariants; see for example Chapter II of [29]). However, the above is nothing but the coefficient $\mathscr{A}_{A,B}(\partial_{j_1}, \partial_{j_2}; \partial_{k_1}, \partial_{k_2})$, and therefore there exists some polynomial P such that $Q(X, Y) = P(\mathscr{A}_{X,Y})$ for all $(X, Y) \in \mathrm{Sym}^2(\mathbb{R}^d) \times \mathrm{Sym}^2(\mathbb{R}^d)$. Since Q is also ρ-invariant, we see that

$$P(\rho_M \mathscr{A}_{X,Y}) = P(\mathscr{A}_{X,Y}). \tag{14}$$

Assume now by way of contradiction that $d\nu_\Sigma/d\xi = 0$, which in particular means that

$$\inf_{M \in \mathrm{SL}(\mathbb{R}^d)} \|\rho_M \mathscr{A}_{A,B}\|_\partial = 0.$$

Thus there exists a sequence $(M_k)_{k\in\mathbb{N}} \subset \mathrm{SL}(\mathbb{R}^d)$ such that $\|\rho_{M_k} \mathscr{A}_{A,B}\|_\partial \to 0$ as $k \to \infty$; in particular, every component of $\rho_{M_k}\mathscr{A}_{A,B}$ tends to zero. By (14) this implies by continuity that $Q(A, B) = P(\mathscr{A}_{A,B}) = 0$, but this is a contradiction. $\square$

The existence of a nonconstant invariant polynomial that does not vanish on (A, B) is equivalent, in GIT language, to (A, B) being semistable. More precisely, consider an affine variety $\mathscr{C}$ given as the zero set of a finite collection of homogeneous polynomials; observe that $0 \in \mathscr{C}$ and that if $x \in \mathscr{C}$ then $\lambda x \in \mathscr{C}$ for every $\lambda \in \mathbb{R}$. We will call $\mathscr{C}$ a *cone*. Given an action $\theta : G \times \mathscr{C} \to \mathscr{C}$ of a linearly reductive algebraic group G on the cone $\mathscr{C}$, and assuming that the action commutes with dilations,[17] a point $x \in \mathscr{C}$ is said to be θ-*semistable* if

$$0 \notin \mathrm{Cl}_{\mathrm{Zar}}(\{\theta_g(x) : g \in G\}),$$

that is, if 0 is not contained in the Zariski closure of the orbit of x; else the point is called θ-*unstable*. Notice that semistability is a property of the orbit and not of the particular point. It is immediate to see that if there exists a θ-invariant polynomial P such that $P(x) \neq 0$ then x is θ-semistable; the opposite implication is also true but nontrivial, and is the content of the so-called fundamental theorem of GIT (see Theorem 1.1 in Chapter 1, Section 2 of [42] or Section 3.4.1 of [51]). Thus we have the equivalent definition of semistability: $x \in \mathscr{C}$ is θ-semistable if and only if there exists a θ-invariant polynomial P on $\mathscr{C}$ such that $P(0) = 0$ but $P(x) \neq 0$.

Remark 19. Effectively, we could have simply defined semistability in terms of nonvanishing invariant polynomials. However, in the next section we will need to use tools from GIT that are better phrased in terms of orbits, and therefore decided to provide here the more standard definition of semistability.

Since $\mathrm{Sym}^2(\mathbb{R}^d) \times \mathrm{Sym}^2(\mathbb{R}^d)$ is a cone and $\mathrm{SL}(\mathbb{R}^d) \times \mathrm{SL}(\mathbb{R}^2)$ is a linearly reductive group, we can rephrase Lemma 18 informally as

$d\nu_\Sigma/d\xi$ is nonzero if and only if (A, B) is $(\rho \times \sigma)$-semistable.

[17] Any such action is always assumed to be algebraic, in the sense that there exist embeddings of $G, \mathscr{C}$ as affine varieties in affine spaces such that the action is given by a polynomial map in the resulting affine coordinates.

4. Characterisation of well-curvedness

In this section we will provide the following algebraic characterisation of the semistability of a pair of symmetric matrices (A, B) under the $\rho \times \sigma$ action introduced in the previous section.

Proposition 20. *Let $A, B \in \mathrm{Sym}^2(\mathbb{R}^d)$. The pair (A, B) is $(\rho \times \sigma)$-semistable if and only if the homogeneous polynomial $s, t \mapsto \det(sA + tB)$ does not vanish identically and has no root of multiplicity $> d/2$.*

Together with Lemma 18, this proposition immediately implies Theorem 7, as the root condition above is precisely condition (M) when $(A, B) = (\nabla^2 Q_1, \nabla^2 Q_2)$. The rest of the section is dedicated to the proof of the proposition, which is articulated in three subsections.

4.1. ***Preliminaries.*** In the proof of Proposition 20 we will make use of a fundamental GIT result — the so-called Hilbert–Mumford criterion, which provides a characterisation of semistable/unstable points. The classical Hilbert–Mumford criterion (like much of GIT) is formulated over the complex numbers: this means that below $\mathscr{C}$ is an affine variety in some $\mathbb{C}^n$ and G is an algebraic subgroup[18] of $\mathrm{GL}(\mathbb{C}^n)$.

Lemma 21 (Hilbert–Mumford criterion). *Let $\mathscr{C}$ be a cone and let $\theta : G \times \mathscr{C} \to \mathscr{C}$ be the action of a linearly reductive group G, which we assume commutes with dilations. If $x \in \mathscr{C}$ is θ-unstable, then there exists a one-parameter subgroup of G given by an algebraic homomorphism $\eta : \mathbb{C}^\times \to G$ such that*

$$\lim_{\lambda \to 0} \theta_{\eta(\lambda)}(x) = 0,$$

where the limit is taken in the standard topology of $\mathscr{C}$ (the one inherited from the standard topology of $\mathbb{C}^n$).

The real version of the Hilbert–Mumford criterion is due to Birkes [8]: its statement is exactly the same, but $\mathbb{C}$ is replaced everywhere by $\mathbb{R}$. An easy consequence of the real Hilbert–Mumford criterion is that $x \in \mathscr{C}$ is θ-semistable if and only if it is semistable for the complexification of θ (which entails complexifying $\mathscr{C}$, G as well). Indeed, if x is θ-unstable then by the real Hilbert–Mumford criterion 0 is in the standard closure of the orbit of x, and therefore 0 is also in the Zariski closure of the orbit under the complexified action; vice versa, if x is θ-semistable then for some θ-invariant polynomial P such that $P(0) = 0$ we have $P(x) \neq 0$, but P is also invariant with respect to the complexified action.

For us the above means that a pair of real symmetric matrices (A, B) is semistable under the action $\rho \times \sigma$ of $\mathrm{SL}(\mathbb{R}^d) \times \mathrm{SL}(\mathbb{R}^2)$ if and only if it is semistable under the same action of group $\mathrm{SL}(\mathbb{C}^d) \times \mathrm{SL}(\mathbb{C}^2)$ instead. This will afford us some convenient technical simplifications later on, but is by no means necessary.

Remark 22. Lemma 17 is a direct consequence of the real Hilbert–Mumford criterion.

Let us write

$$\Delta_{A,B}(s, t) := \det(sA + tB)$$

for convenience; thus Δ can be regarded as a map $\mathrm{Sym}^2(\mathbb{C}^d) \times \mathrm{Sym}^2(\mathbb{C}^d) \to \mathbb{C}[s, t]$. Some observations about the symmetries enjoyed by this map are in order. The first observation is that Δ is invariant under

[18] An algebraic subgroup of $\mathrm{GL}(\mathbb{C}^n)$ is a subgroup that is also a subvariety of $\mathrm{GL}(\mathbb{C}^n)$.

the action ρ: indeed,

$$\det(sMAM^\top + tMBM^\top) = \det(M(sA+tB)M^\top) = \det(sA+tB);$$

therefore

$$\Delta_{\rho_M(A,B)} = \Delta_{A,B}.$$

The second observation is that Δ is not invariant under the action σ, but it is nevertheless equivariant: indeed,

$$\det(s(\lambda A+\mu B)+t(\lambda' A+\mu' B)) = \det((\lambda s+\lambda' t)A+(\mu s+\mu' t)B),$$

so if we let $\tilde{\sigma}$ denote the action on polynomials of two variables defined by

$$\tilde{\sigma}_N P\begin{pmatrix} s \\ t \end{pmatrix} := P\left(N^\top \begin{pmatrix} s \\ t \end{pmatrix}\right)$$

for any $N \in \mathrm{SL}(\mathbb{C}^2)$, we have

$$\Delta_{\sigma_N(A,B)} = \tilde{\sigma}_N(\Delta_{A,B}).$$

We are of course only interested in the action of $\tilde{\sigma}$ on homogeneous polynomials of two variables and degree d. It will be very useful to identify which polynomials are semistable under this action; we can do so very easily with the Hilbert–Mumford criterion. By Lemma 21, $P \in \mathbb{C}[s,t]$ (homogeneous of degree d) will be $\tilde{\sigma}$-unstable if and only if there exists a one-parameter subgroup $(N_\lambda)_{\lambda\in\mathbb{C}^\times}$ of $\mathrm{SL}(\mathbb{C}^2)$ such that

$$\lim_{\lambda\to 0} \tilde{\sigma}_{N_\lambda} P = 0,$$

where the limit is taken in the standard vector space topology of $\mathbb{C}[s,t]$. The one-parameter (algebraic) subgroups of the special linear groups $\mathrm{SL}(\mathbb{C}^n)$ are well known: they are all of the form

$$N_\lambda = G\begin{pmatrix} \lambda^{a_1} & & \\ & \ddots & \\ & & \lambda^{a_n} \end{pmatrix} G^{-1},$$

where $G \in \mathrm{SL}(\mathbb{C}^n)$ and the exponents a_j are integers that satisfy $\sum_{j=1}^n a_j = 0$ (but are otherwise unconstrained). In our case $n = 2$, so the one-parameter subgroups are simply conjugates of $\begin{pmatrix} \lambda & \\ & \lambda^{-1} \end{pmatrix}$, and therefore if we let

$$\begin{pmatrix} \hat{s} \\ \hat{t} \end{pmatrix} = G^{-1}\begin{pmatrix} s \\ t \end{pmatrix}$$

we can write

$$\tilde{\sigma}_{N_\lambda} P = \sum_{k=0}^{d} c_k \lambda^{2k-d} \hat{s}^k \hat{t}^{d-k},$$

where the c_k are the coefficients of $P \circ G$. This expression can only tend to zero as $\lambda \to 0$ if the coefficients c_k vanish for all $k \le d/2$; but this means in particular that $\hat{s}^m$ divides $P \circ G$ for some $m > d/2$, or in other words that P has a root of multiplicity $> d/2$. The argument can be run in reverse, and therefore we have shown the following known fact.

Lemma 23. *Let P be a homogeneous polynomial of degree d in $\mathbb{C}[s,t]$. Then P is $\tilde{\sigma}$-semistable if and only if P has no root of multiplicity $> d/2$.*

In light of this lemma, we could rephrase Proposition 20 as

(A, B) *is* $(\rho \times \sigma)$*-semistable if and only if* $\Delta_{A,B}$ *is* $\tilde{\sigma}$*-semistable.*

Now we are ready to begin the proof of Proposition 20. One implication is easy: suppose that (A, B) is $(\rho \times \sigma)$-unstable, and therefore by Lemma 21 there exists a one-parameter subgroup $((M_\lambda, N_\lambda))_{\lambda \in \mathbb{C}^\times} \subset \mathrm{SL}(\mathbb{C}^d) \times \mathrm{SL}(\mathbb{C}^2)$ such that

$$\lim_{\lambda \to 0} \rho_{M_\lambda} \sigma_{N_\lambda}(A, B) = (0, 0).$$

By the invariance of Δ under ρ and equivariance under σ, we have then that

$$\lim_{\lambda \to 0} \tilde{\sigma}_{N_\lambda} \Delta_{A,B} = 0,$$

that is, the polynomial $\Delta_{A,B}$ is $\tilde{\sigma}$-unstable. By Lemma 23 we have then that $\Delta_{A,B}$ has a root of multiplicity larger than $d/2$, thus proving one side of the equivalence.

It remains to prove the opposite implication: we will assume in the rest of the section that $\Delta_{A,B}$ has a root of multiplicity strictly larger than $d/2$, and show that this makes (A, B) unstable. There is a relevant dichotomy here: either $\Delta_{A,B}$ is a nonvanishing polynomial in s, t or it is identically zero. We treat each case on its own.

4.2. *Case I*: $\boldsymbol{\Delta_{A,B}}$ *is not identically vanishing.* Since the determinant is nonvanishing, for some (s_0, t_0) we have that $s_0 A + t_0 B$ is invertible. We may assume without loss of generality that $(s_0, t_0) = (0, 1)$, or in other words that $\det B \neq 0$. Indeed, observe that if $s_0 \neq 0$, we can let

$$N_0 := \begin{pmatrix} 0 & -1/s_0 \\ s_0 & t_0 \end{pmatrix} \in \mathrm{SL}_2(\mathbb{C})$$

and we have

$$\sigma_{N_0}(A, B) = ((-1/s_0)B, s_0 A + t_0 B);$$

(A, B) is $(\rho \times \sigma)$-unstable if and only if the pair $((-1/s_0)B, s_0 A + t_0 B)$ is, and therefore it is just a matter of relabelling $A' := (-1/s_0)B,\ B' := s_0 A + t_0 B$ in the arguments below.

We can thus assume $\det B \neq 0$ and write

$$\det(sA + tB) = \det(B) \det(sAB^{-1} + tI).$$

We put AB^{-1} in Jordan normal form: for any r, λ denote by $J_r(\lambda)$ the $r \times r$ Jordan block of eigenvalue λ, that is,

$$J_r(\lambda) := \begin{pmatrix} \lambda & 1 & & & \\ & \lambda & 1 & & \\ & & \ddots & \ddots & \\ & & & \lambda & 1 \\ & & & & \lambda \end{pmatrix}$$

(if $r = 1$ we have simply $J_1(\lambda) = (\lambda)$); then there exists a matrix $Q \in \mathrm{GL}(\mathbb{C}^d)$ such that $AB^{-1} = Q\boldsymbol{J}Q^{-1}$, where

$$\boldsymbol{J} = \begin{pmatrix} \boxed{J_{r_1}(\lambda_1)} & & \\ & \ddots & \\ & & \boxed{J_{r_\ell}(\lambda_\ell)} \end{pmatrix}$$

for some r_j and λ_j. We have

$$\det(sAB^{-1}+tI)=\det(sQ\boldsymbol{J}Q^{-1}+tI)=\det(s\boldsymbol{J}+tI),$$

so that matters are reduced to the Jordan normal form of AB^{-1}. With I_r denoting the $r\times r$ identity matrix, we have

$$s\boldsymbol{J}+tI=\begin{pmatrix}\boxed{sJ_{r_1}(\lambda_1)+tI_{r_1}} & & \\ & \ddots & \\ & & \boxed{sJ_{r_\ell}(\lambda_\ell)+tI_{r_\ell}}\end{pmatrix},$$

where in particular

$$sJ_{r_j}(\lambda_j)+tI_{r_j}=\begin{pmatrix}s\lambda_j+t & s & & \\ & \ddots & \ddots & \\ & & s\lambda_j+t & s \\ & & & s\lambda_j+t\end{pmatrix}.$$

We then see that the above has produced the factorisation

$$\det(sA+tB)=\det(B)\prod_{j=1}^{\ell}(s\lambda_j+t)^{r_j};$$

we caution the reader that the λ_j are not necessarily distinct and therefore the r_j are not exactly the multiplicities. If we want to highlight the correct multiplicities, we let $\lambda_1^*,\ldots,\lambda_n^*$ be all the distinct values the λ_j take and we write

$$\det(sA+tB)=\det(B)\prod_{j=1}^{n}(s\lambda_j^*+t)^{m_j},$$

where

$$m_j=\sum_{k:\,\lambda_k=\lambda_j^*}r_k.$$

One of the m_j is larger than $d/2$ by assumption — let it be m_1 for convenience. Then we have deduced that $\boldsymbol{J}$, the Jordan form of AB^{-1}, has an eigenvalue that is repeated more than $d/2$ times. We will now see how to connect this fact to the original pair (A,B) of symmetric matrices.

Observe that every block $J_r(\lambda)$ can be written as the product of two symmetric matrices: indeed, if we let

$$\tilde{J}_r(\lambda):=\begin{pmatrix} & & & 1 & \lambda \\ & & 1 & \lambda & \\ & ⋰ & ⋰ & & \\ 1 & \lambda & & & \\ \lambda & & & & \end{pmatrix},\quad \tilde{I}_r:=\begin{pmatrix} & & & 1 \\ & & 1 & \\ & ⋰ & & \\ 1 & & & \end{pmatrix}, \tag{15}$$

then it is immediate to verify that

$$J_r(\lambda)=\tilde{J}_r(\lambda)\tilde{I}_r.$$

We can therefore factorise

$$\boldsymbol{J}=\tilde{\boldsymbol{J}}\tilde{\boldsymbol{I}},$$

where

$$\tilde{\boldsymbol{J}} = \begin{pmatrix} \boxed{\tilde{J}_{r_1}(\lambda_1)} & & \\ & \ddots & \\ & & \boxed{\tilde{J}_{r_\ell}(\lambda_\ell)} \end{pmatrix}, \quad \tilde{\boldsymbol{I}} = \begin{pmatrix} \boxed{\tilde{I}_{r_1}} & & \\ & \ddots & \\ & & \boxed{\tilde{I}_{r_\ell}} \end{pmatrix}. \tag{16}$$

We claim that (A, B) and $(\tilde{\boldsymbol{J}}, \tilde{\boldsymbol{I}})$ belong to the same $(\rho\times\sigma)$-orbit, and therefore they are either both unstable or both semistable. Indeed, since B is invertible we can write

$$(A, B) = (AB^{-1}B, B) = (Q\boldsymbol{J}Q^{-1}B, B) = (Q\tilde{\boldsymbol{J}}\tilde{\boldsymbol{I}}Q^{-1}B, B);$$

since B is also symmetric, acting with $\rho_{\mu B^{-1}}$ (where μ is such that $\det(\mu B^{-1}) = 1$) we have that the orbit of (A, B) contains

$$\mu^2\ (B^{-1}Q\tilde{\boldsymbol{J}}\tilde{\boldsymbol{I}}Q^{-1}, B^{-1}).$$

Acting with $\rho_{\mu' Q^\top}$ (where μ' is such that $\det(\mu' Q^\top) = 1$) we see that

$$\mu^2\mu'^2\ (Q^\top B^{-1}Q\tilde{\boldsymbol{J}}\tilde{\boldsymbol{I}}, Q^\top B^{-1}Q)$$

is also in the orbit of (A, B); moreover, since $\tilde{\boldsymbol{I}}$ is symmetric and its own inverse, we have in the orbit of (A, B) also the element

$$\mu^2\mu'^2\mu''^2\ (\tilde{\boldsymbol{I}}(Q^\top B^{-1}Q)\tilde{\boldsymbol{J}}, \tilde{\boldsymbol{I}}(Q^\top B^{-1}Q)\tilde{\boldsymbol{I}})$$

(where μ'' is such that $\det(\mu''\tilde{\boldsymbol{I}}) = 1$). Letting $N := \mu\mu'^2\mu''\ \tilde{\boldsymbol{I}}(Q^\top B^{-1}Q) \in \mathrm{SL}(\mathbb{C}^d)$, we see that the last element is simply $\mu\mu''\ (N\tilde{\boldsymbol{J}}, N\tilde{\boldsymbol{I}})$ (notice that $N\tilde{\boldsymbol{J}}$ and $N\tilde{\boldsymbol{I}}$ are both symmetric). We will show that there exists a matrix $M \in \mathrm{SL}(\mathbb{C}^d)$ such that $\rho_M(N\tilde{\boldsymbol{J}}, N\tilde{\boldsymbol{I}}) = (\tilde{\boldsymbol{J}}, \tilde{\boldsymbol{I}})$, and this will prove the claim at hand. This fact is an immediate consequence of the following lemma.

Lemma 24. *Let (A_1, A_2) be a pair of symmetric $d \times d$ matrices, of which at least one is invertible, and assume that $N \in \mathrm{SL}(\mathbb{C}^d)$ is such that (NA_1, NA_2) is also a pair of symmetric matrices. Then there exists $M \in \mathrm{SL}(\mathbb{C}^d)$ such that*

$$(NA_1, NA_2) = (MA_1M^\top, MA_2M^\top).$$

We remark that the lemma can be extended to general n-tuples of symmetric matrices by essentially the same proof.

Proof. Assume A_2 is invertible, without loss of generality. We will show that it suffices to take M to be a square root of N.

Since $NA_2 = (NA_2)^\top = A_2N^\top$, we have

$$N^\top = A_2^{-1}NA_2, \tag{17}$$

and therefore $N(A_1A_2^{-1}) = A_1N^\top A_2^{-1} = (A_1A_2^{-1})N$. In other words, $A_1A_2^{-1}$ commutes with N. Since N is a (complex) invertible matrix, it has a square root $N^{1/2}$ that commutes with $A_1A_2^{-1}$ too. Indeed, this can be constructed via holomorphic calculus as follows: let $\log z$ denote a branch of the logarithm such that the branch cut does not contain any eigenvalue of N; then we define by Cauchy's formula

$$\operatorname{Log} N := \frac{1}{2\pi i}\int_{\gamma} \log z\,(zI - N)^{-1}\,dz,$$

where γ is the boundary of a domain that encloses the spectrum of N and avoids the branch cut of $\log z$; finally, we define

$$N^{\frac{1}{2}} := \operatorname{Exp}\bigl(\tfrac{1}{2}\operatorname{Log} N\bigr).$$

It is easy to see that $N^{1/2}$ is indeed a square root of N and that, thanks to the formula above, $N^{1/2}$ commutes with $A_1A_2^{-1}$ as well. Notice that we also have the analogue of (17) for $N^{1/2}$, that is, we have $(N^{1/2})^{\top} = A_2^{-1}N^{1/2}A_2$. As a consequence we have

$$N^{\frac{1}{2}}A_1(N^{\frac{1}{2}})^{\top} = N^{\frac{1}{2}}A_1(A_2^{-1}N^{\frac{1}{2}}A_2) = N^{\frac{1}{2}}N^{\frac{1}{2}}(A_1A_2^{-1})A_2 = NA_1;$$

similarly,

$$N^{\frac{1}{2}}A_2(N^{\frac{1}{2}})^{\top} = N^{\frac{1}{2}}A_2(A_2^{-1}N^{\frac{1}{2}}A_2) = NA_2,$$

and the lemma follows by taking $M = N^{1/2}$. □

We have therefore proven that (A, B) and $(\tilde{\boldsymbol{J}}, \tilde{\boldsymbol{I}})$ belong to the same orbit, and in particular to the same ρ-orbit (we omit the constant factor $\mu\mu''$ from now on). Now we take into account the action σ as well by observing that $(\tilde{\boldsymbol{J}}, \tilde{\boldsymbol{I}})$ is unstable if and only if the element $(\tilde{\boldsymbol{J}} - \lambda_1^*\tilde{\boldsymbol{I}}, \tilde{\boldsymbol{I}})$ is, since for

$$N_0 := \begin{pmatrix} 1 & -\lambda_1^* \\ 0 & 1 \end{pmatrix}$$

we have

$$\sigma_{N_0}(\tilde{\boldsymbol{J}}, \tilde{\boldsymbol{I}}) = (\tilde{\boldsymbol{J}} - \lambda_1^*\tilde{\boldsymbol{I}}, \tilde{\boldsymbol{I}}).$$

Evaluating the expression $\tilde{\boldsymbol{J}} - \lambda_1^*\tilde{\boldsymbol{I}}$ block by block, we see that the above is a pair of matrices of the same form as $(\tilde{\boldsymbol{J}}, \tilde{\boldsymbol{I}})$ but where the eigenvalue of highest multiplicity has been replaced by 0 (more precisely, each $\tilde{J}_r(\lambda_1^*)$ block has been replaced by $\tilde{J}_r(0)$). We will now show that the pair $(\tilde{\boldsymbol{J}} - \lambda_1^*\tilde{\boldsymbol{I}}, \tilde{\boldsymbol{I}})$ is unstable in two steps:

(i) First we will exhibit a one-parameter subgroup of $\mathrm{SL}(\mathbb{C}^d)$ that leaves $\tilde{\boldsymbol{I}}$ fixed but is such that in the limit $\lambda \to 0$ every $\tilde{J}_r(0)$ block in $\tilde{\boldsymbol{J}} - \lambda_1^*\tilde{\boldsymbol{I}}$ is replaced by a block of zeroes.

(ii) Then we will exhibit a one-parameter subgroup of $\mathrm{SL}(\mathbb{C}^d) \times \mathrm{SL}(\mathbb{C}^2)$ that shows that the latter is unstable (here is where we finally make use of the fact that $m_1 > d/2$).

This is enough to conclude: indeed, if (C, D) is $(\rho \times \sigma)$-unstable and for a one-parameter subgroup $((M_\lambda, N_\lambda))_{\lambda \in \mathbb{C}^\times}$ we have $\lim_{\lambda\to 0}\rho_{M_\lambda}\sigma_{N_\lambda}(A, B) = (C, D)$, we have by continuity that $Q(A, B) = Q(C, D)$ for all $(\rho \times \sigma)$-invariant polynomials (with $Q(0, 0) = 0$); but $Q(C, D) = 0$ always, and so the same holds for (A, B), which is thus unstable as well.

Consider any $\tilde{J}_r(0)$ block in $\tilde{\boldsymbol{J}} - \lambda_1^* \tilde{\boldsymbol{I}}$, with $r > 1$ (if $r = 1$ we do not need to do anything); the corresponding block in $\tilde{\boldsymbol{I}}$ is $\tilde{I}_r$. If we define

$$M_\lambda = \begin{pmatrix} \lambda^{a_1} & & \\ & \ddots & \\ & & \lambda^{a_r} \end{pmatrix}$$

then we see that

$$M_\lambda \tilde{J}_r(0) M_\lambda^\top = \begin{pmatrix} & & & \lambda^{a_1+a_{r-1}} & 0 \\ & & \iddots & \iddots & \\ & \lambda^{a_{r-2}+a_2} & 0 & & \\ \lambda^{a_{r-1}+a_1} & 0 & & & \\ 0 & & & & \end{pmatrix},$$

$$M_\lambda \tilde{I}_r M_\lambda^\top = \begin{pmatrix} & & \lambda^{a_1+a_r} \\ & \iddots & \\ \lambda^{a_{r-1}+a_2} & & \\ \lambda^{a_r+a_1} & & \end{pmatrix}.$$

If r is even, we choose

$$(a_1, \dots, a_r) = \left(\frac{r}{2}, \frac{r}{2} - 1, \dots, 1 - \frac{r}{2}, -\frac{r}{2}\right)$$

and if r is odd we choose

$$a_j := \left\lfloor \frac{r}{2} \right\rfloor - (j-1);$$

these choices satisfy the condition $\sum_{j=1}^r a_j = 0$; moreover they satisfy $a_{r-j} + a_j > 0$ and $a_{r-j} + a_{j+1} = 0$ for every j. Thus it is immediate that

$$\lim_{\lambda \to 0} M_\lambda \tilde{J}_r(0) M_\lambda^\top = 0, \quad M_\lambda \tilde{I}_r M_\lambda^\top = \tilde{I}_r.$$

It is then clear that we can construct (block by block) a one-parameter subgroup $(M_\lambda)_{\lambda \in \mathbb{C}^\times} \subset \mathrm{SL}_d(\mathbb{C})$ such that

$$\lim_{\lambda \to 0} \rho_{M_\lambda}(\tilde{\boldsymbol{J}} - \lambda_1^* \tilde{\boldsymbol{I}}, \tilde{\boldsymbol{I}}) = (\boldsymbol{J}_0, \tilde{\boldsymbol{I}}),$$

where $\boldsymbol{J}_0$ is the matrix obtained from $\tilde{\boldsymbol{J}} - \lambda_1^* \tilde{\boldsymbol{I}}$ by replacing every $\tilde{J}_r(0)$ block with a block of zeroes of the same $r \times r$ size (notice that we choose ρ_{M_λ} to act trivially on the blocks of nonzero eigenvalue).

Finally, we show that $(\boldsymbol{J}_0, \tilde{\boldsymbol{I}})$ is $(\rho \times \sigma)$-unstable. By reordering the blocks (something that can be easily achieved via ρ) we may assume that $\boldsymbol{J}_0, \tilde{\boldsymbol{I}}$ are of the form

$$\boldsymbol{J}_0 = \begin{pmatrix} \boxed{0} & \\ & \boxed{\boldsymbol{J}_1} \end{pmatrix}, \quad \tilde{\boldsymbol{I}} = \begin{pmatrix} \boxed{\tilde{\boldsymbol{I}}_1} & \\ & \boxed{\tilde{\boldsymbol{I}}_2} \end{pmatrix},$$

where $\boldsymbol{J}_1$ is a matrix consisting of the remaining nonzero diagonal blocks of type $\tilde{J}_r(\lambda_j)$ and $\tilde{\boldsymbol{I}}_1, \tilde{\boldsymbol{I}}_2$ are matrices consisting of the corresponding $\tilde{I}_r$ diagonal blocks (in particular, $\boldsymbol{J}_1$ and $\tilde{\boldsymbol{I}}_2$ have the same size).

Observe that $\tilde{\boldsymbol{I}}_1$ has size $m_1 \times m_1$, while $\boldsymbol{J}_1$, $\tilde{\boldsymbol{I}}_2$ have size $(d - m_1) \times (d - m_1)$. If we let M_λ denote the matrix

$$M_\lambda := \begin{pmatrix} \boxed{\lambda^{-(d-m_1)} I_{m_1}} & \\ & \boxed{\lambda^{m_1} I_{d-m_1}} \end{pmatrix}$$

then we see that the M_λ form a one-parameter subgroup of $\mathrm{SL}(\mathbb{C}^d)$ and moreover we have by a direct computation that

$$\rho_{M_\lambda}(\boldsymbol{J}_0, \tilde{\boldsymbol{I}}) = \left(\begin{pmatrix} \boxed{0} & \\ & \boxed{\lambda^{2m_1} \boldsymbol{J}_1} \end{pmatrix}, \begin{pmatrix} \boxed{\lambda^{-2(d-m_1)} \tilde{\boldsymbol{I}}_1} & \\ & \boxed{\lambda^{2m_1} \tilde{\boldsymbol{I}}_2} \end{pmatrix} \right).$$

Consider also the one-parameter subgroup of $\mathrm{SL}(\mathbb{C}^2)$ given by

$$N_\lambda := \begin{pmatrix} \lambda^{-2(d-m_1)-1} & 0 \\ 0 & \lambda^{2(d-m_1)+1} \end{pmatrix}$$

and observe that

$$\rho_{M_\lambda} \sigma_{N_\lambda}(\boldsymbol{J}_0, \tilde{\boldsymbol{I}}) = \left(\begin{pmatrix} \boxed{0} & \\ & \boxed{\lambda^{2(2m_1-d)-1} \boldsymbol{J}_1} \end{pmatrix}, \begin{pmatrix} \boxed{\lambda \tilde{\boldsymbol{I}}_1} & \\ & \boxed{\lambda^{2d+1} \tilde{\boldsymbol{I}}_2} \end{pmatrix} \right).$$

Since $m_1 > d/2$, we have $2(2m_1 - d) - 1 > 0$, and therefore

$$\lim_{\lambda \to 0} \sigma_{N_\lambda} \rho_{M_\lambda}(\boldsymbol{J}_0, \tilde{\boldsymbol{I}}) = (0, 0),$$

thus completing the proof that (A, B) is $(\rho \times \sigma)$-unstable if $\Delta_{A,B}$ is not identically vanishing and $\tilde{\sigma}$-unstable.

4.3. ***Case II*: $\boldsymbol{\Delta_{A,B}}$ *vanishes identically.*** Here we assume that $(A, B) \in \mathrm{Sym}^2(\mathbb{C}^d) \times \mathrm{Sym}^2(\mathbb{C}^d)$ is such that

$$\det(sA + tB) \equiv 0,$$

or in other words that $\ker(sA + tB) \neq \{0\}$ for all $(s, t) \in \mathbb{C}^2$.

We perform a first reduction. Suppose that for two linearly independent pairs (s_1, t_1), (s_2, t_2) we have

$$\ker(s_1 A + t_1 B) \cap \ker(s_2 A + t_2 B) \neq \{0\}$$

(that is, the kernels have nontrivial intersection); we claim that (A, B) is automatically $(\rho \times \sigma)$-unstable as a consequence. Notice that we can assume for simplicity that $(s_1, t_1) = (1, 0)$ and $(s_2, t_2) = (0, 1)$ by using the action σ (this is essentially the same argument that was given before). Thus we are assuming that there exists a vector $\boldsymbol{v} \neq 0$ such that $A\boldsymbol{v} = B\boldsymbol{v} = 0$. Pick then vectors $\boldsymbol{u}_2, \ldots, \boldsymbol{u}_d$ so that $\{\boldsymbol{v}, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_d\}$ forms a basis of $\mathbb{C}^d$ and moreover normalise them so that the matrix

$$M := \begin{pmatrix} \boldsymbol{v}^\top \\ \boldsymbol{u}_2^\top \\ \vdots \\ \boldsymbol{u}_d^\top \end{pmatrix}$$

is in $\mathrm{SL}(\mathbb{C}^d)$. We then see by direct computation that $\rho_M(A, B)$ consists of a pair of matrices each of the form

$$\begin{pmatrix} 0 & 0 & \cdots & 0 \\ 0 & * & \cdots & * \\ \vdots & \vdots & \ddots & \vdots \\ 0 & * & \cdots & * \end{pmatrix}$$

(where the asterisks denote possibly nonzero entries). If we consider now the one-parameter subgroup of $\mathrm{SL}(\mathbb{C}^d)$ given by

$$M_\lambda := \begin{pmatrix} \lambda^{-(d-1)} & & & \\ & \lambda & & \\ & & \ddots & \\ & & & \lambda \end{pmatrix},$$

a computation reveals immediately that the effect of ρ_{M_λ} on $\rho_M(A, B)$ is multiplication of every nonzero entry by λ^2 (because of the particular form of the matrices). Therefore we have

$$\lim_{\lambda \to 0} \rho_{M_\lambda}(\rho_M(A, B)) = (0, 0)$$

and thus (A, B) is indeed $(\rho \times \sigma)$-unstable.

In light of the above, we will assume in the rest of the argument that for every pair of linearly independent $(s_1, t_1), (s_2, t_2) \in \mathbb{C}^2$ we have

$$\ker(s_1 A + t_1 B) \cap \ker(s_2 A + t_2 B) = \{0\}. \tag{18}$$

Letting $I, J \subset \{1, \dots, d\}$ with $|I| = |J|$, we denote by $\det_{I,J} M$ the minor of the matrix M obtained by selecting the rows with index in I and the columns with index in J. If $(A, B) \neq (0, 0)$, some minors of $sA + tB$ will be not identically vanishing. We can then find I_*, J_* of maximal cardinality such that $\det_{I_*,J_*}(sA + tB)$ does not vanish identically (and therefore it is nonzero for all (s, t) except for a finite number of directions $as + bt = 0$). We define the set of *generic* (s, t) to be

$$\mathscr{G} := \{(s, t) \in \mathbb{C}^2 : \det_{I_*,J_*}(sA + tB) \neq 0\}.$$

Notice that for (s, t) generic we have that the dimension of $\ker(sA + tB)$ is constant and equal to d minus the size of the minor; for $(s, t) \notin \mathscr{G}$ the dimension of the kernel is larger instead. It will be useful to consider the vector space generated by the kernels of $sA + tB$ for generic (s, t), that is,

$$V := \mathrm{Span}\Bigg\{ \bigcup_{(s,t) \in \mathscr{G}} \ker(sA + tB) \Bigg\}.$$

We let $k := \dim V$ and notice that by assumption (18) we have $k \geq 2$. For convenience, we choose a basis $\{\boldsymbol{v}_1, \dots, \boldsymbol{v}_k\}$ of V such that for every $j \in \{1, \dots, k\}$

$$\boldsymbol{v}_j \in \ker(\tilde{s}_j A + \tilde{t}_j B)$$

for some $(\tilde{s}_j, \tilde{t}_j) \in \mathscr{G}$.

The first important observation to make is that all the images $(sA+tB)V$ for $(s,t)\in\mathscr{G}$ consist of a same vector space H. To begin with, all such images have the same dimension: indeed, for each $(s,t)\in\mathscr{G}$ we have $\ker(sA+tB)\le V$ and $\dim\ker(sA+tB)$ is a constant; therefore $\dim(sA+tB)V=\dim V-\dim\ker(sA+tB)$ is a constant too. To conclude the claim, it will suffice to verify that for two linearly independent $(s_1,t_1),(s_2,t_2)\in\mathscr{G}$ we have

$$(s_1A+t_1B)V=(s_2A+t_2B)V=:H;$$

for if this is true, then by linear independence we will have $(sA+tB)V\le H$ for every other $(s,t)\in\mathscr{G}$, and since the dimensions must be the same, we will have actually $(sA+tB)V=H$ too. Take then $(s_1,t_1),(s_2,t_2)$ that are linearly independent and not multiples of any of the $(\tilde{s}_j,\tilde{t}_j)$ associated to the basis chosen above. For any $j\in\{1,\dots,k\}$ there exist coefficients a_j,b_j (both nonzero) such that

$$\tilde{s}_jA+\tilde{t}_jB=a_j(s_1A+t_1B)+b_j(s_2A+t_2B),$$

and since $(\tilde{s}_jA+\tilde{t}_jB)\boldsymbol{v}_j=0$ we have

$$a_j(s_1A+t_1B)\boldsymbol{v}_j=-b_j(s_2A+t_2B)\boldsymbol{v}_j.$$

Therefore

$$\begin{aligned}(s_1A+t_1B)V&=\operatorname{span}\{(s_1A+t_1B)\boldsymbol{v}_j:1\le j\le k\}\\&=\operatorname{span}\{(s_2A+t_2B)\boldsymbol{v}_j:1\le j\le k\}=(s_2A+t_2B)V,\end{aligned}$$

as desired.

The second observation to make (which is a consequence of the first) is that V and H are actually orthogonal to each other. Indeed, letting $\boldsymbol{u}\in H$, it suffices to show that $\langle\boldsymbol{u},\boldsymbol{v}_j\rangle=0$ for all j. This is however easy to see: since $\boldsymbol{u}\in H$ and $H=(\tilde{s}_jA+\tilde{t}_jB)V$, there is a vector $\boldsymbol{v}\in V$ such that $(\tilde{s}_jA+\tilde{t}_jB)\boldsymbol{v}=\boldsymbol{u}$, and since the matrices are symmetric we have

$$\langle(\tilde{s}_jA+\tilde{t}_jB)\boldsymbol{v},\boldsymbol{v}_j\rangle=\langle\boldsymbol{v},(\tilde{s}_jA+\tilde{t}_jB)\boldsymbol{v}_j\rangle=\langle\boldsymbol{v},0\rangle=0.$$

Thus V and H are orthogonal, and besides $\dim H<k$ we have therefore $\dim H\le d-k$ too.

We now claim that, as a consequence of the above observations, the $(\sigma\times\rho)$-orbit of (A,B) contains a pair of symmetric matrices both of the form indicated in Figure 1.

Remark 25. We caution the reader that in the matrix diagram of Figure 1 the shape of the blocks of nonzero entries could be slightly misleading for large k and $\dim H=d-k$ (more precisely, for $k>d/2$), but the block dimensions as stated are correct for all values of $k\ge 2$. For example, when $k=d-2$ we have $\dim H\le 2$, and thus if $\dim H=2$ the matrix looks like

$$\begin{pmatrix} & & & * & * \\ & & & \vdots & \vdots \\ & & & * & * \\ * & \cdots & * & * & * \\ * & \cdots & * & * & * \end{pmatrix};$$

it is evident that the block dimensions here are still as indicated in Figure 1.

$$
\begin{array}{rcl}
 & \hfill \overbrace{\qquad\qquad\quad}^{\dim H} & \\
\begin{matrix} \\ \\ \\ \\ \\ \dim H \left\{ \begin{matrix} \\ \\ \\ \end{matrix} \right. \end{matrix} &
\begin{pmatrix}
 & & & & & * & \cdots & * \\
 & & & & & \vdots & \ddots & \vdots \\
 & & & & & \vdots & \ddots & \vdots \\
 & & & & & * & \cdots & * \\
 & & & & * & \cdots & \cdots & * \\
* & \cdots & \cdots & * & \vdots & \ddots & & \vdots \\
\vdots & \ddots & \ddots & \vdots & \vdots & & \ddots & \vdots \\
* & \cdots & \cdots & * & * & \cdots & \cdots & *
\end{pmatrix} &
\begin{matrix} \left. \begin{matrix} \\ \\ \\ \\ \end{matrix} \right\} k \\ \left. \begin{matrix} \\ \\ \\ \\ \end{matrix} \right\} d-k \end{matrix} \\
 & \underbrace{\qquad\qquad\qquad}_{k}\underbrace{\qquad\qquad\qquad}_{d-k} &
\end{array}
$$

Figure 1. The special form of the matrices A, B in the appropriate basis.

By the usual argument using the action σ, we assume that $(1, 0), (0, 1) \in \mathscr{G}$ (that is, we can work with (A, B)). In order to show that there exists $M \in \mathrm{SL}(\mathbb{C}^d)$ such that $\rho_M(A, B) = (MAM^\top, MBM^\top)$ consists of a pair of matrices both of the same form given in Figure 1, begin by observing that if we write

$$M = \begin{pmatrix} \boldsymbol{u}_1^\top \\ \vdots \\ \boldsymbol{u}_d^\top \end{pmatrix}$$

then the (i, j)-entry of $MAM^\top$ is $\langle \boldsymbol{u}_i, A\boldsymbol{u}_j \rangle$ (and the same holds for B). We then construct a basis of $\mathbb{C}^d$ in the following way: choose first $\boldsymbol{u}_1, \dots, \boldsymbol{u}_k$ to be a basis of V, and then complete it to a basis of $H^\perp$ by further choosing linearly independent $\boldsymbol{u}_{k+1}, \dots, \boldsymbol{u}_{d-\dim H}$; finally, complete the list to a basis of the whole $\mathbb{C}^d$ by choosing $\boldsymbol{u}_{d-\dim H+1}, \dots, \boldsymbol{u}_d$ to be a basis of H (normalised so that $\det M = 1$). We have thus by construction (recall that $H = AV = BV$) that for all $1 \le i \le d - \dim H$ and $1 \le j \le k$

$$\langle \boldsymbol{u}_i, A\boldsymbol{u}_j \rangle = \langle \boldsymbol{u}_i, B\boldsymbol{u}_j \rangle = 0;$$

since the matrices are symmetric, the fact that $MAM^\top$ and $MBM^\top$ are in the form of Figure 1 follows.

We assume therefore that A and B are both of the form given in Figure 1 and proceed to make the block structure more explicit. Letting $\ell := \dim H$ for convenience, we can decompose A and B into (possibly rectangular) blocks as indicated in Figure 2, with dimensions as given there.

We will now show that such a pair of matrices is necessarily ρ-unstable by producing an explicit one-parameter subgroup of $\mathrm{SL}(\mathbb{C}^d)$ that sends (A, B) to $(0, 0)$ in the limit $\lambda \to 0$. This subgroup can be taken to be as follows: set

$$
\begin{aligned}
a_1 &:= -((d-1)\ell + d - k), \\
a_2 &:= k, \\
a_3 &:= dk,
\end{aligned}
$$

$$\begin{array}{cc} \left(\begin{array}{ccc} & & A_1 \\ & A_2 & A_3 \\ A_4 & A_5 & A_6 \end{array}\right) & \begin{array}{l} \}\,k \\ \}\,d-k-\ell \\ \}\,\ell \end{array} \\ \begin{array}{ccc} \underbrace{\qquad}_{k} & \underbrace{\qquad}_{d-k-\ell} & \underbrace{\qquad}_{\ell} \end{array} & \end{array}$$

Figure 2. The decomposition of A into rectangular blocks of dimensions as indicated. The decomposition of B has the exact same shape. We remark that it might be the case that $d-k-\ell=0$, in which case the blocks with the corresponding dimension are omitted (e.g., A would contain only blocks A_1, A_4, A_6, which would be adjacent to each other).

then define the block matrix

$$M_\lambda = \begin{pmatrix} \boxed{\lambda^{a_1} I_k} & & \\ & \boxed{\lambda^{a_2} I_{d-k-\ell}} & \\ & & \boxed{\lambda^{a_3} I_\ell} \end{pmatrix}$$

(once again, if $d-k-\ell=0$, the middle block is omitted). The M_λ's form a one-parameter subgroup of $\mathrm{SL}(\mathbb{C}^d)$ because the sum of all the exponents involved is

$$a_1 k + a_2(d-k-\ell) + a_3\ell = -((d-1)\ell + d - k)k + k(d-k-\ell) + dk\ell = 0.$$

By inspection, the effect of ρ_{M_λ} on matrices of the form given in Figure 2 is

$$\rho_{M_\lambda}(A, B) = \left(\begin{pmatrix} & & \lambda^{a_1+a_3} A_1 \\ & \lambda^{2a_2} A_2 & \lambda^{a_2+a_3} A_3 \\ \lambda^{a_1+a_3} A_4 & \lambda^{a_2+a_3} A_5 & \lambda^{2a_3} A_6 \end{pmatrix}, \begin{pmatrix} & & \lambda^{a_1+a_3} B_1 \\ & \lambda^{2a_2} B_2 & \lambda^{a_2+a_3} B_3 \\ \lambda^{a_1+a_3} B_4 & \lambda^{a_2+a_3} B_5 & \lambda^{2a_3} B_6 \end{pmatrix} \right).$$

Notice that $a_2, a_3 > 0$ and moreover, since $k > \ell$,

$$a_1 + a_3 = -((d-1)\ell + d - k) + dk = d(k-\ell) - (d-k-\ell) \geq k + \ell > 0;$$

therefore we obtain

$$\lim_{\lambda \to 0} \rho_{M_\lambda}(A, B) = (0, 0),$$

and the proof of Proposition 20 (and hence of Theorem 7) is concluded.

5. Proof of Theorems 5 and 9

We will now prove our main results by a simple instance of Christ's method of refinements. The method will reduce matters to sublevel set estimates for the polynomial $\det(s\nabla^2 Q_1 + t\nabla^2 Q_2)$, and these will be proven in Section 6.

Remark 26. The result can also be proven by different methods — in particular, the inflation technique in [31] and the testing conditions in [32] (both due to Gressman) can each be employed to provide an alternative proof. Proceeding with either of those methods, the boundedness of the operator $\mathcal{T}$ is reduced to verifying respectively a nonconcentration inequality and an integrability condition that explicitly involves $\det(s\nabla^2 Q_1 + t\nabla^2 Q_2)$; Theorem 7 provides the information needed to conclude either of these. In this paper we have chosen to use Christ's method of refinements mainly in the interest of providing a more self-contained exposition and because the condition to be verified (the sublevel set estimate) is slightly simpler.

5.1. ***Preliminaries and refinements.*** We begin by reformulating the desired estimates in a combinatorial fashion. Let $1 \leq p, q < \infty$ be exponents such that $2/q = 1/p$; the restricted weak-type version of inequality $\|\mathcal{T} f\|_{L^q} \lesssim_{p,q} \|f\|_{L^p}$ is then

$$\langle \mathcal{T}\mathbf{1}_E, \mathbf{1}_F\rangle \lesssim_q |E|^{\frac{2}{q}} |F|^{\frac{1}{q'}},$$

where $E \subset \mathbb{R}^d \times [-1, 1]^2$ and $F \subset \mathbb{R}^d \times [-1, 1]^d$ have finite measure. Introducing the quantities

$$\alpha := \frac{\langle \mathcal{T}\mathbf{1}_E, \mathbf{1}_F\rangle}{|F|}, \quad \beta := \frac{\langle \mathbf{1}_E, \mathcal{T}^*\mathbf{1}_F\rangle}{|E|}, \tag{19}$$

the restricted weak-type inequality above can be rewritten with a little algebra as

$$\alpha^{q-1}\beta \lesssim_q |E|. \tag{20}$$

The problem has then been reduced to that of providing a lower bound for the measure of E in terms of α, β. When the surface $\Sigma(Q_1, Q_2)$ is well-curved, we will prove this lower bound for q arbitrarily close to the critical value $q_0 = (d+4)/2$ (recall that the strong-type endpoint inequality is $L^{(d+4)/4} \to L^{(d+4)/2}$); and when we are in the situation described in the statement of Theorem 9, we will prove the lower bound for $q = m_* + 2$. Theorems 5 and 9 will then follow by entirely standard interpolation arguments.

We now introduce some "refinements" of the sets E, F with improved behaviour (this is what gives the method its name). Observe that if we let

$$F' := \Big\{(x, \xi) \in F : \mathcal{T}\mathbf{1}_E(x, \xi) > \frac{\alpha}{2}\Big\}$$

then we have $\langle \mathcal{T}\mathbf{1}_E, \mathbf{1}_{F'}\rangle \geq \frac{1}{2}\langle \mathcal{T}\mathbf{1}_E, \mathbf{1}_F\rangle$: indeed, clearly

$$\langle \mathcal{T}\mathbf{1}_E, \mathbf{1}_{F\setminus F'}\rangle \leq \frac{\alpha}{2}|F| = \frac{1}{2}\langle \mathcal{T}\mathbf{1}_E, \mathbf{1}_F\rangle,$$

and the claim follows; notice that $F' \neq \varnothing$, as a consequence. Thus in F' we have enforced a lower bound on $\mathcal{T}\mathbf{1}_E$. Next we observe that we can enforce an analogous lower bound in a refinement of E (but with respect to $\mathcal{T}^*\mathbf{1}_{F'}$ instead): we let

$$E' := \Big\{(y, s, t) \in E : \mathcal{T}^*\mathbf{1}_{F'}(y, s, t) > \frac{\beta}{4}\Big\},$$

and by a repetition of the argument above we see that we have

$$\langle \mathbf{1}_{E'}, \mathcal{T}^*\mathbf{1}_{F'}\rangle \geq \langle \mathcal{T}\mathbf{1}_E, \mathbf{1}_{F'}\rangle - \tfrac{1}{4}\langle \mathcal{T}\mathbf{1}_E, \mathbf{1}_F\rangle \geq \tfrac{1}{4}\langle \mathcal{T}\mathbf{1}_E, \mathbf{1}_F\rangle$$

(so that $E' \neq \varnothing$ too). Summarising, we have shown the following lemma.

Lemma 27. *Let $E \subset \mathbb{R}^d \times [-1,1]^2$ and $F \subset \mathbb{R}^d \times [-1,1]^d$ be sets of finite positive measure, and let α, β be as in* (19). *Then there exist nonempty subsets $E' \subseteq E$, $F' \subseteq F$ such that*

(i) *for every $(x,\xi) \in F'$ we have $\mathcal{T}\mathbf{1}_E(x,\xi) \gtrsim \alpha$,*

(ii) *for every $(y,s,t) \in E'$ we have $\mathcal{T}^*\mathbf{1}_{F'}(y,s,t) \gtrsim \beta$.*

The reason why these properties are remarkable is that they translate into (uniform) lower bounds for the size of certain sets. To see this, let us introduce some notation: we let

$$\gamma((x,\xi),(s,t)) := (x - s\nabla Q_1(\xi) - t\nabla Q_2(\xi), s, t),$$

so that $\mathcal{T}f(x,\xi) = \iint_{|s|,|t|\le 1} f(\gamma((x,\xi),(s,t)))\,ds\,dt$; moreover, we let

$$\gamma^*((y,s,t),\eta) = (y + s\nabla Q_1(\eta) + t\nabla Q_2(\eta), \eta),$$

so that $\mathcal{T}^*g(y,s,t) := \int_{[-1,1]^d} g(\gamma^*((y,s,t),\eta))\,d\eta$. Now observe that

$$\mathcal{T}^*\mathbf{1}_{F'}(y,s,t) = |\{\eta \in [-1,1]^d : \gamma^*((y,s,t),\eta) \in F'\}|,$$

so that if we pick $(y_0,s_0,t_0) \in E'$ and we let

$$\mathcal{B} := \{\eta \in [-1,1]^d : \gamma^*((y_0,s_0,t_0),\eta) \in F'\},$$

we have by Lemma 27

$$|\mathcal{B}| \gtrsim \beta.$$

Similarly, we see that if $(x,\xi) \in F'$, we have (again by Lemma 27)

$$|\{(s,t) \in [-1,1]^2 : \gamma((x,\xi),(s,t)) \in E\}| \gtrsim \alpha;$$

we can then define for $\eta \in \mathcal{B}$

$$\mathcal{A}_\eta := \{(s,t) \in [-1,1]^2 : \gamma(\gamma^*((y_0,s_0,t_0),\eta),(s,t)) \in E\}$$

and have uniformly

$$|\mathcal{A}_\eta| \gtrsim \alpha.$$

5.2. ***Change of variables and conclusion.*** We can see from the above discussion that the function

$$\Psi(\eta,s,t) := \gamma(\gamma^*((y_0,s_0,t_0),\eta),(s,t))$$

maps the set

$$\bigcup_{\eta\in\mathcal{B}} (\{\eta\} \times \mathcal{A}_\eta)$$

into the set E, thus providing a way to obtain lower bounds on $|E|$; moreover, it is a map from $\mathbb{R}^{d+2}$ into itself, which will enable us to use the change of variables formula to obtain explicit lower bounds. To make use of these ideas and in anticipation of the technical challenges, we introduce for every $\eta \in \mathcal{B}$ subsets $\mathcal{A}'_\eta \subseteq \mathcal{A}_\eta$, which will be specified later; these are assembled into the set

$$S := \bigcup_{\eta\in\mathcal{B}} (\{\eta\} \times \mathcal{A}'_\eta), \tag{21}$$

and we stress that we have $\Psi(S) \subset E$. By the change of variables formula we have then

$$|E| \geq \mu_{\Psi}^{-1} \int_{S} |J\Psi(\eta, s, t)|\, d\eta\, ds\, dt,$$

where $\mu_{\Psi} = \max_{(\eta,s,t)\in S} \#\Psi^{-1}(\eta, s, t)$ is the multiplicity of the map Ψ and $J\Psi$ its Jacobian determinant, which we will now calculate. Observe that

$$\Psi(\eta, s, t) = (y_0 - (s - s_0)\nabla Q_1(\eta) - (t - t_0)\nabla Q_2(\eta), s, t),$$

so that the Jacobian of Ψ is given by

$$-\begin{pmatrix} (s-s_0)\nabla^2 Q_1(\eta)+(t-t_0)\nabla^2 Q_2(\eta) & \nabla Q_1(\eta) & \nabla Q_2(\eta) \\ 0\cdots 0 & -1 & 0 \\ 0\cdots 0 & 0 & -1 \end{pmatrix}$$

and it is immediate that[19]

$$J\Psi(\eta, s, t) = (-1)^d \det((s - s_0)\nabla^2 Q_1(\eta) + (t - t_0)\nabla^2 Q_2(\eta)); \tag{22}$$

crucially, this is the same object that characterises the well-curvedness of $\Sigma(Q_1, Q_2)$. As for μ_Ψ, we have $\Psi(\eta, s, t) = \Psi(\eta', s', t')$ only if $s = s'$, $t = t'$; moreover, Q_1, Q_2 are quadratic forms and therefore we must have (switching again to Hessian matrices A, B)

$$(s - s_0)A(\eta - \eta') + (t - t_0)B(\eta - \eta') = 0.$$

If we choose S so as to impose $\det((s - s_0)A + (t - t_0)B) \neq 0$ (which we will), we see that the above equation is solved only by $\eta = \eta'$, and thus we will have $\mu_\Psi = 1$.

Assume now that the surface $\Sigma(Q_1, Q_2)$ is well-curved and fix $\epsilon > 0$ arbitrarily small. We claim that we can choose subsets $\mathcal{A}'_\eta$ so that

(i) $|\mathcal{A}'_\eta| \gtrsim \alpha$ for every $\eta \in \mathcal{B}$,

(ii) for every $(\eta, s, t) \in S$ we have $|J\Psi(\eta, s, t)| \gtrsim_\epsilon \alpha^{d/2+\epsilon}$.

If these conditions are satisfied we see immediately from (21) that $|S| \gtrsim \alpha\beta$ and moreover that

$$|E| \geq \int_{S} |J\Psi(\eta, s, t)|\, d\eta\, ds\, dt \gtrsim_\epsilon \alpha^{\frac{d+2}{2}+\epsilon}\beta,$$

which is precisely the desired inequality (20) for $q = (d + 4)/2 + \epsilon$; since ϵ is arbitrary, this proves Theorem 5. To obtain the conditions above, simply choose

$$\mathcal{A}'_\eta := \mathcal{A}_\eta \setminus \{(s, t) \in [-1, 1]^2 : |\det((s - s_0)A + (t - t_0)B)| < C_\epsilon \alpha^{\frac{d}{2}+\epsilon}\}$$

for $C_\epsilon > 0$; then by (22) we see that condition (ii) is automatically satisfied. As for condition (i), Theorem 7 and Proposition 29 (which will be proven in Section 6) imply the sublevel set estimate

$$|\{(s, t) \in [-1, 1]^2 : |\det((s - s_0)A + (t - t_0)B)| < C_\epsilon \alpha^{\frac{d}{2}+\epsilon}\}| \ll \alpha$$

(provided C_ϵ is chosen sufficiently small), from which condition (i) follows at once.

[19] Notice that when Q_1, Q_2 are quadratic forms, the Jacobian determinant is independent of η.

Suppose instead that the surface $\Sigma(Q_1, Q_2)$ is flat, but $\det(sA + tB)$ does not vanish identically and has a root of multiplicity $m_* > d/2$ (these are the hypotheses of Theorem 9). In this case we claim that we can find subsets $\mathcal{A}'_\eta$ so that

(i) $|\mathcal{A}'_\eta| \gtrsim \alpha$ for every $\eta \in \mathcal{B}$ (as before),

(ii) for every $(\eta, s, t) \in S$ we have $|J\Psi(\eta, s, t)| \gtrsim \alpha^{m_*}$.

This is achieved in exactly the same way, with the only difference being that we appeal to Proposition 30 instead to obtain the sublevel set estimate

$$|\{(s,t) \in [-1,1]^2 : |\det((s - s_0)A + (t - t_0)B)| < C\alpha^{m_*}\}| \ll \alpha.$$

Then the same argument as before shows that

$$|E| \gtrsim \alpha^{m_*+1}\beta,$$

which is inequality (20) for $q = m_* + 2$, as claimed. The proofs of Theorems 5 and 9 are thus concluded, conditionally on Propositions 29 and 30 (recall also that the negative parts of the statements will be proven in Section 7).

Remark 28. In Theorem 9 and in certain cases of Theorem 5, it is possible to refine the restricted weak-type inequalities to restricted strong-type inequalities by using the inflation method instead (also originating in M. Christ's work; see [17; 18]); however, the range of exponents obtained by interpolation is the same in either case.

6. Sublevel set estimates

In this section we will prove the sublevel set estimates that are needed to close the argument of Section 5. There are two types of estimates (one for the well-curved case, one for the flat case), which are encapsulated in the two propositions below, stated for general homogeneous polynomials of two variables. Recall that by a root of a homogeneous polynomial in $\mathbb{R}[s, t]$ we mean a homogeneous linear divisor in $\mathbb{C}[s, t]$.

Proposition 29. *Let $P(s, t)$ be a real homogeneous polynomial of degree d. If all the roots of P have multiplicity $\leq d/2$ then we have for every $\delta > 0$*

$$|\{(s,t) : |s|, |t| \lesssim 1,\ |P(s,t)| < \delta\}| \lesssim_P \delta^{2/d} \log^+ 1/\delta. \tag{23}$$

Proposition 30. *Let $P(s, t)$ be a real homogeneous polynomial of degree d. If P has a root of multiplicity $m_* > d/2$ then we have for every $\delta > 0$*

$$|\{(s,t) : |s|, |t| \lesssim 1,\ |P(s,t)| < \delta\}| \lesssim_P \delta^{\frac{1}{m_*}}. \tag{24}$$

These sublevel set estimates are sharp in several ways. First of all, it is not possible to improve the exponent $2/d$ in (23): indeed, if $|s|, |t| \lesssim \delta^{1/d}$ then each monomial in $P(s, t)$ is $\lesssim \delta$, and therefore the sublevel set contains the set $\{(s, t) : |s|, |t| \lesssim \delta^{1/d}\}$, which has measure $\gtrsim \delta^{2/d}$. Secondly, if the root multiplicity assumption of Proposition 29 is violated, (23) can no longer hold: since we can write

$P(s,t) = (as+bt)^m Q(s,t)$ for some $a, b \in \mathbb{C}$ and some homogeneous polynomial Q of degree $d-m$, we see that $|Q(s,t)| \lesssim 1$ and therefore the sublevel set contains the set $\{(s,t) : |s|, |t| \lesssim 1,\ |as+bt|^m \lesssim \delta\}$, which is seen to have measure $\gtrsim \delta^{1/m} \gg \delta^{2/d}$. This also shows that it is not possible to improve the exponent $1/m_*$ in (24). Finally, it is not possible in general to remove the logarithmic factor in (23): consider for example polynomials $P(s,t) = s^{d/2}t^{d/2}$ when d is even.[20]

There is a rich and well-developed theory of sublevel set estimates for polynomials (and more generally for analytic functions) which runs in parallel to an analogous theory of oscillatory integral estimates with polynomial phases. The two are intimately related: indeed, it is well known that it is possible to deduce sublevel set estimates from estimates for the corresponding oscillatory integrals (see, e.g., Section 1 of [12]). For multivariable phases, the oscillatory integrals theory was developed by A. N. Varchenko in his foundational work [50]. The main takeaway of this theory is that the rate of decay is controlled by the *height* of the phase, which is the supremum of the Newton distance[21] taken over all locally smooth (or analytic) coordinate systems. One could therefore prove Propositions 29 and 30 from the corresponding oscillatory integral estimates of Varchenko by computing the height of P, given the multiplicity assumption. This computation has been carried out already by I. A. Ikromov and D. Müller [35] (Corollary 3.4), who showed that in our case the height is $\max\{m_*, d/2\}$, where m_* denotes the largest root multiplicity; thus one obtains the desired proofs. Alternatively, one could use the same corollary of [35] and an integration argument in polar coordinates to obtain a direct proof that does not require the oscillatory integrals theory of Varchenko.[22] Here however we will offer our own independent proofs that rely on a simple but interesting linear programming argument (that such arguments are powerful enough to deal with sublevel set and oscillatory integral estimates was already observed in [28]). Besides the inherent interest, the method we employ is conveniently stable under perturbations of P, due to the fact that the constants involved are sufficiently explicit; this will come in handy when we prove Theorem 5′ in the Appendix. The estimates of Varchenko are also stable under analytic perturbations in the case of two variables, as was shown by V. N. Karpushkin [37]. By contrast, the aforementioned integration argument in polar coordinates produces a constant that depends on the separation between the roots, which is not stable under perturbations.

Proof of Proposition 29. Since $P \in \mathbb{R}[s,t]$ is homogeneous of degree d, it can be factored over $\mathbb{C}$ as

$$P(s,t) = C \prod_{j=1}^{d} \theta_j(s,t),$$

where the θ_j are homogeneous linear forms (that is, $\theta_j(s,t) = a_j s + b_j t$). Since P is a real polynomial, we can arrange things so that the θ_j are either real or occur in complex conjugate pairs. We furthermore choose a normalisation of the θ_j's so that if $[a_j : b_j] = [a_k : b_k]$ (as points of $\mathbb{P}(\mathbb{C}^2)$) then $\theta_j = \theta_k$; thus the multiplicity of a root of $P(s,t)$ is simply the number of occurrences of a same factor θ in the product

[20] This polynomial can be realised as $\det(sA+tB)$ for block matrices $A = \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix}$, $B = \begin{pmatrix} 0 & 0 \\ 0 & I \end{pmatrix}$; thus the log-loss cannot be avoided even in our case of interest.

[21] The Newton distance of an analytic function f is the smallest $d \geq 0$ such that $(d, \dots, d)$ belongs to the Newton diagram of f.

[22] The argument proceeds by rewriting $|\{(s,t) : s^2+t^2 \leq 1,\ |P(s,t)| < \delta\}| = \int_0^{2\pi} \int_0^1 \mathbf{1}_{[-\delta,\delta]}(r^d |P(\cos\alpha, \sin\alpha)|)\, r\, dr\, d\alpha$, which is then equal to $\frac{1}{2}\delta^{2/d} \int_{\{\alpha : |Q(\alpha)| > \delta\}} |Q(\alpha)|^{-2/d}\, d\alpha + \frac{1}{2} |\{\alpha : |Q(\alpha)| < \delta\}|$ (letting $Q(\alpha) := P(\cos\alpha, \sin\alpha)$); both terms can be estimated by factoring $Q(\alpha)$ and using [35]. The argument was pointed out to us by J. Wright.

above. Notice that C ends up depending on P. If the distinct factors are $\theta_1, \dots, \theta_\ell$ (in particular, they are all pairwise linearly independent) and the respective multiplicities are m_j (thus $\sum_{j=1}^{\ell} m_j = d$ and $m_j \le d/2$), we can write

$$P(s,t) = C \prod_{j=1}^{\ell} \theta_j(s,t)^{m_j}.$$

First of all, we will need to control sublevel sets of polynomials with only two distinct roots; this is achieved by the next lemma.

Lemma 31. *Let $\mu, \nu > 0$ and let $\theta, \theta' \in \mathbb{C}[s,t]$ be linear forms that are $\mathbb{C}$-linearly independent. Then for every $\delta > 0$*

$$|\{s,t : |s|, |t| \lesssim 1,\ |\theta(s,t)^{\mu}\theta'(s,t)^{\nu}| \lesssim \delta\}| \lesssim_{\theta,\theta'} \begin{cases} \delta^{\frac{1}{\max\{\mu,\nu\}}} & \text{if } \mu \neq \nu \\ \delta^{\frac{1}{\mu}} \log^{+}(1/\delta) & \text{if } \mu = \nu. \end{cases}$$

Proof of Lemma 31. From $\mathbb{C}$-linear independence we see in fact that we can pick real linear forms $\hat{\theta} \in \{\operatorname{Re}\theta, \operatorname{Im}\theta\}$ and $\hat{\theta}' \in \{\operatorname{Re}\theta', \operatorname{Im}\theta'\}$ so that $\hat{\theta}, \hat{\theta}'$ are $\mathbb{R}$-linearly independent. Since $|\hat{\theta}| \le |\theta|$ and $|\hat{\theta}'| \le |\theta'|$, we have then

$$\left|\left\{s,t : |s|, |t| \lesssim 1,\ |\theta(s,t)^{\mu}\theta'(s,t)^{\nu}| \lesssim \delta\right\}\right| \le \left|\left\{s,t : |s|, |t| \lesssim 1,\ |\hat{\theta}(s,t)|^{\mu}|\hat{\theta}'(s,t)|^{\nu} \lesssim \delta\right\}\right|,$$

and by a linear change of variables the latter is

$$\lesssim_{\theta,\theta'} |\{s,t : |s|, |t| \lesssim_{\theta,\theta'} 1,\ |s|^{\mu}|t|^{\nu} \lesssim \delta\}|.$$

By a simple integration we see that if $\mu \neq \nu$ then the last expression is dominated by $\lesssim \delta^{1/\max\{\mu,\nu\}}$, and if $\mu = \nu$ then it is dominated by $\lesssim \delta^{1/\mu} \log^{+}(1/\delta)$. □

Remark 32. The implicit constant in the estimate of Lemma 31 can be made explicit: it is simply $O(|\det(\hat{\theta}\ \ \hat{\theta}')|^{-1})$, where $\det(\hat{\theta}\ \ \hat{\theta}')$ denotes the Jacobian determinant of the map $(s,t) \mapsto (\hat{\theta}(s,t), \hat{\theta}'(s,t))$, and $\hat{\theta}, \hat{\theta}'$ are as in the proof just given.

We will show that for a general polynomial that satisfies the multiplicity assumption of Proposition 29, we can always reduce at least to the second case of the lemma.

As a step in the direction indicated, we claim that we can always rewrite the polynomial P as a product of pairs of the form $(\theta_j\theta_k)^{\mu}$: more precisely, we will show that there exist quantities $\mu_{jk} \ge 0$ such that

$$\prod_{j=1}^{\ell} \theta_j(s,t)^{m_j} = \prod_{j=1}^{\ell} \prod_{j<k\le\ell} (\theta_j(s,t)\theta_k(s,t))^{\mu_{jk}}. \tag{25}$$

Indeed, looking at the exponents, the equality translates immediately into the existence of a nonnegative solution $(\mu_{jk})_{1\le j<k\le\ell}$ to the linear equations[23]

$$L_j : \sum_{i:\, i<j} \mu_{ij} + \sum_{k:\, k>j} \mu_{jk} = m_j, \quad j \in \{1, \dots, \ell\}. \tag{26}$$

[23]Notice that the resulting system of equations has $\ell(\ell-1)/2$ variables and ℓ equations, and is therefore severely underdetermined.

In order to treat such a system of linear equations, we recall the following fundamental linear programming lemma. For convenience, given a vector $\boldsymbol{v}$ we write $\boldsymbol{v} \geq 0$ to denote the fact that all components of $\boldsymbol{v}$ are nonnegative.

Lemma 33 (Farkas' lemma [47]). *If M is an $m \times n$ real matrix and $\boldsymbol{b} \in \mathbb{R}^m$, then exactly one of the following mutually exclusive cases holds*:

(i) *there exists $\boldsymbol{x} \in \mathbb{R}^n$ such that $M\boldsymbol{x} = \boldsymbol{b}$ with $\boldsymbol{x} \geq 0$, or*

(ii) *there exists $\boldsymbol{y} \in \mathbb{R}^m$ such that $M^\top \boldsymbol{y} \geq 0$ and $\boldsymbol{b} \cdot \boldsymbol{y} < 0$.*

Remark 34. The statement might appear somewhat cryptic at first, but the geometric content is actually elementary: if we let $\Gamma_+ := \{\boldsymbol{x} \in \mathbb{R}^n : \boldsymbol{x} \geq 0\}$, we observe that Γ_+ is a closed convex cone and therefore so is $M\Gamma_+$; then Farkas' lemma simply states that either $\boldsymbol{b}$ belongs to $M\Gamma_+$ or not, in which case the two can be separated by a hyperplane ($\boldsymbol{y}$ is an element orthogonal to this hyperplane and on the opposite side to $\boldsymbol{b}$).

We will show that case (ii) of Lemma 33 is impossible in our situation (in which $\boldsymbol{b} = (m_1, \dots, m_\ell)$ and M can be read off of the system of equations (26)), and thus the desired $(\mu_{jk})_{j<k}$ exist. Assume by contradiction that there is such a vector $\boldsymbol{y} = (y_1, \dots, y_\ell)$ as in case (ii). Inspecting the system (26) we see that the condition $M^\top \boldsymbol{y} \geq 0$ translates into the system of inequalities

$$y_j + y_k \geq 0 \tag{27}$$

for all $1 \leq j < k \leq \ell$ (indeed, observe that each variable μ_{jk} appears only in equations L_j and L_k, always with coefficient $+1$); the condition $\boldsymbol{b} \cdot \boldsymbol{y} < 0$ is simply the statement that

$$y_1 m_1 + \cdots + y_\ell m_\ell < 0.$$

On the one hand, since the m_j are all positive, from the last inequality we see that at least one of the y_j must be negative. On the other hand, from inequalities (27) we see that there can be at most a single index j_* such that $y_{j_*} < 0$ and that all other y_j must be strictly positive instead; in particular, $y_j \geq |y_{j_*}| > 0$. But then we have

$$\sum_{j \neq j_*} m_j \leq \sum_{j \neq j_*} \frac{y_j}{|y_{j_*}|} m_j < m_{j_*},$$

and this implies that $m_{j_*} > d/2$, which is a contradiction.

The above has shown that the desired structural factorisation of P can be achieved — and notice in particular that we have necessarily $\sum_{j<k} \mu_{jk} = d/2$. Now consider only those indices j, k such that $\mu_{jk} > 0$. By the pigeonhole principle and factorisation (25) we have that if $|P(s,t)| \leq \delta$ then for at least one pair of indices $j < k$ we have

$$|\theta_j(s,t)\theta_k(s,t)|^{\mu_{jk}} \lesssim_P \delta^{2\mu_{jk}/d};$$

it follows that $|\{s, t : |s|, |t| \lesssim 1, |P(s,t)| \leq \delta\}|$ is dominated by the sum in indices $j < k$ of

$$|\{s, t : |s|, |t| \lesssim 1, |\theta_j(s,t)\theta_k(s,t)|^{\mu_{jk}} \lesssim_P \delta^{2\mu_{jk}/d}\}|.$$

However, since the θ_j's are normalised and distinct, they are linearly independent in pairs; by Lemma 31 this measure is dominated by $\lesssim_P (\delta^{2\mu_{jk}/d})^{1/\mu_{jk}} \log^+(1/\delta^{2\mu_{jk}/d}) \sim \delta^{2/d} \log^+ 1/\delta$, and we are done. □

The proof of (24) follows similar lines but is much simpler.

Proof of Proposition 30. As in the proof of (23), we can factorise P as

$$P(s,t) = C\theta_*(s,t)^{m_*} \prod_{j=1}^{\ell} \theta_j(s,t)^{m_j},$$

where $m_* > d/2$ is the largest multiplicity and $\theta_*, \theta_1, \dots, \theta_\ell$ are linearly independent linear forms. Since $m_* > \sum_{j=1}^{\ell} m_j$, we can find μ_j such that $\mu_j > m_j$ and $\sum_{j=1}^{\ell} \mu_j = m_*$; as a consequence, we can rearrange the factorisation of P as

$$P(s,t) = C \prod_{j=1}^{\ell} (\theta_*(s,t)^{\mu_j} \theta_j(s,t)^{m_j}).$$

By the pigeonhole principle, if $|P(s,t)| < \delta$ then for at least one index j we have

$$|\theta_*(s,t)^{\mu_j} \theta_j(s,t)^{m_j}| \lesssim_P \delta^{\mu_j/m_*};$$

therefore the sublevel set $\{s,t : |s|, |t| \lesssim 1, |P(s,t)| < \delta\}$ is contained in the union over j of sublevel sets

$$\{s,t : |s|, |t| \lesssim 1, |\theta_*(s,t)^{\mu_j} \theta_j(s,t)^{m_j}| \lesssim_P \delta^{\mu_j/m_*}\}.$$

By Lemma 31, each of these has measure $\lesssim_P (\delta^{\mu_j/m_*})^{1/\max\{\mu_j, m_j\}} = \delta^{1/m_*}$, concluding the proof. □

Remark 35. While it is not possible in general to remove the logarithmic loss in (23) even in the case of polynomials $P(s,t) = \det(sA + tB)$, the class of polynomials for which we incur such a loss can be narrowed down significantly. Indeed, with a more precise argument (such as, e.g., the aforementioned integration argument in polar coordinates using Corollary 3.4 of [35]) one incurs logarithmic losses only when the polynomial P has a root of multiplicity exactly equal to $d/2$. It follows that for well-curved surfaces $\Sigma(Q_1, Q_2)$ we can always obtain the restricted weak-type endpoint $L^{(d+4)/4} \to L^{(d+4)/2}$, provided all the roots have multiplicity strictly smaller than $d/2$. In particular, one recovers in these cases the critical line that is missing from the statement of Theorem 5.

7. Flat surfaces

In this final section we will give counterexamples that show the necessity of the curvature assumptions of Theorems 5 and 9. More specifically, for flat $\Sigma(Q_1, Q_2)$ surfaces:

- We will show that if $\det(s\nabla^2 Q_1 + t\nabla^2 Q_2)$ does not vanish identically but has a root of multiplicity $m_* > d/2$, then for any (p,q) sufficiently close to the endpoint $((d+4)/4, (d+4)/2)$ the $L^p \to L^q$ estimate for the operator $\mathcal{T}$ given by (1) is false; in particular, we will show that any estimate with $2/q = 1/p$ and $q < m_* + 2$ is false.

- We will show that if $\det(s\nabla^2 Q_1 + t\nabla^2 Q_2)$ vanishes identically then any estimate with $2/q = 1/p$ is false (except for $p = q = \infty$); more generally, we will rule out every estimate for which $(2-\epsilon)/q < 1/p$ for some $\epsilon > 0$ (this range intersects nontrivially the conjectural nonmixed range given by (4)).

We will deal with each case in a separate subsection. Once again we resort to writing A, B for $\nabla^2 Q_1, \nabla^2 Q_2$.

7.1. ***Case I*: $\det(sA + tB)$ *is not identically vanishing.*** In order to allow for a cleaner argument, we begin by making some reductions that are entirely analogous to those operated in Section 4; some care is needed because of the local nature of $\mathcal{T}$. For added precision, we introduce operators

$$\mathcal{T}_{\Omega}^{A,B} f(x,\xi) := \iint_{\Omega} f(x - (sA + tB)\xi, s, t)\, ds\, dt,$$

in which the subscript Ω specifies the integration domain; thus for the operator given by (1) we have $\mathcal{T} = \mathcal{T}_{[-1,1]^2}^{A,B}$.

First of all, we claim that we can assume that B is invertible. Indeed, otherwise there exists some τ_0 such that $B_0 := -A - \tau_0 B$ is invertible, and we can write

$$sA + tB = s' A_0 + t' B_0$$

for $A_0 := B$ and s', t' given by

$$\begin{pmatrix} s' \\ t' \end{pmatrix} = N \begin{pmatrix} s \\ t \end{pmatrix}, \quad N = \begin{pmatrix} -\tau_0 & 1 \\ -1 & 0 \end{pmatrix} \in \mathrm{SL}(\mathbb{R}^2).$$

If for any function f we let $f_{\tau_0}(y, s, t) := f(y, t - s\tau_0, -s)$, we see by a change of variables that

$$\mathcal{T}_{[-1,1]^2}^{A,B} f_{\tau_0} = \mathcal{T}_{N([-1,1]^2)}^{A_0,B_0} f;$$

therefore it will suffice to show that $\mathcal{T}_{N([-1,1]^2)}^{A_0,B_0}$ is unbounded, where now B_0 is invertible. Notice that since the operators are positive it will suffice to show that $\mathcal{T}_{[-\epsilon,\epsilon]^2}^{A_0,B_0}$ is unbounded for some $\epsilon > 0$ such that $[-\epsilon,\epsilon]^2 \subset N([-1,1]^2)$; by a rescaling, it then suffices to show that $\mathcal{T}_{[-1,1]^2}^{\epsilon A_0,\epsilon B_0}$ is unbounded.

Assuming then that B is invertible, we further claim that we can assume that (A, B) is in the form $(\tilde{\boldsymbol{J}}, \tilde{\boldsymbol{I}})$ given by (16) of Section 4. Indeed, using the notation of that section, we see that

$$\begin{aligned} sA + tB &= (sAB^{-1} + tI)B = (sQJQ^{-1} + tI)B \\ &= Q(s\tilde{\boldsymbol{J}}\tilde{\boldsymbol{I}} + t\tilde{\boldsymbol{I}}^2)Q^{-1}B = Q(s\tilde{\boldsymbol{J}} + t\tilde{\boldsymbol{I}})\tilde{\boldsymbol{I}}Q^{-1}B \end{aligned}$$

(recall that Q is an invertible matrix such that $Q^{-1}AB^{-1}Q$ is in Jordan normal form). If for any function f we let $f_{Q^{-1}}(y, s, t) := f(Q^{-1}y, s, t)$, we see by a straightforward calculation that

$$\mathcal{T}_{[-1,1]^2}^{A,B} f_{Q^{-1}}(Qx, B^{-1}Q\tilde{\boldsymbol{I}}\xi) = \mathcal{T}_{[-1,1]^2}^{\tilde{\boldsymbol{J}},\tilde{\boldsymbol{I}}} f(x,\xi).$$

As a consequence, it will suffice to show that $\mathcal{T}_{[-1,1]^2}^{\tilde{\boldsymbol{J}},\tilde{\boldsymbol{I}}}$ is unbounded from $L^p(B(0, C) \times [-1,1]^2)$ to $L^q(\mathbb{R}^d \times [-\epsilon',\epsilon']^d)$, where $\epsilon' > 0$ is chosen sufficiently small to ensure $B^{-1}Q\tilde{\boldsymbol{I}}([-\epsilon',\epsilon']^d) \subset [-1,1]^d$.

Finally, assuming that the matrices are of the form $(\tilde{\boldsymbol{J}}, \tilde{\boldsymbol{I}})$, we can further assume that the eigenvalue of $\tilde{\boldsymbol{J}}$ of highest multiplicity is $\lambda_* = 0$: this can be achieved by a repetition of the argument given to show that we could assume B to be invertible, and thus we omit the details. Associated to eigenvalue 0 we have the generalised eigenspaces of $\tilde{\boldsymbol{J}}$: let V_0 be the span of all the generalised eigenspaces of dimension 1, and let $V_1, \dots, V_\ell$ be the generalised eigenspaces of dimension larger than 1. For $j \in \{0, \dots, \ell\}$ we let $\boldsymbol{e}_1^{(j)}, \dots, \boldsymbol{e}_{n_j}^{(j)}$ be the generalised eigenvectors that span V_j, where $n_j := \dim V_j$. Moreover, we let W denote the span of the generalised eigenspaces of nonzero eigenvalue — thus $\mathbb{R}^d = V_0 \oplus \cdots \oplus V_\ell \oplus W$. In the resulting basis of generalised eigenvectors the matrix $\tilde{\boldsymbol{J}}$ has the form

$$\tilde{\boldsymbol{J}} = \begin{pmatrix} 0 & & & & & & & & \\ & \ddots & & & & & & & \\ & & 0 & & & & & & \\ & & & \tilde{J}_{n_1}(0) & & & & & \\ & & & & \ddots & & & & \\ & & & & & \tilde{J}_{n_\ell}(0) & & & \\ & & & & & & * & \cdots & * \\ & & & & & & \vdots & \ddots & \vdots \\ & & & & & & * & \cdots & * \end{pmatrix},$$

where $\tilde{J}_r(0)$ is given by (15); matrix $\tilde{\boldsymbol{I}}$ has analogous form but $\tilde{J}_r(0)$ is replaced by $\tilde{I}_r$ (also given by (15)). In particular, we have

$$\tilde{\boldsymbol{J}} \boldsymbol{e}_k^{(0)} = 0, \quad \tilde{\boldsymbol{I}} \boldsymbol{e}_k^{(0)} = \boldsymbol{e}_k^{(0)} \tag{28}$$

for every $k \leq n_0$, and for $1 \leq j \leq \ell$

$$\begin{aligned} \tilde{\boldsymbol{J}} \boldsymbol{e}_k^{(j)} &= \boldsymbol{e}_{n_j - k}^{(j)}, & \tilde{\boldsymbol{I}} \boldsymbol{e}_k^{(j)} &= \boldsymbol{e}_{n_j + 1 - k}^{(j)} \quad \text{for } 1 \leq k \leq n_j - 1, \\ \tilde{\boldsymbol{J}} \boldsymbol{e}_{n_j}^{(j)} &= 0, & \tilde{\boldsymbol{I}} \boldsymbol{e}_{n_j}^{(j)} &= \boldsymbol{e}_1^{(j)}. \end{aligned} \tag{29}$$

We introduce two types of parabolic boxes adapted to the generalised eigenspaces: for any $j \in \{0, \dots, \ell\}$ and $\delta > 0$ (an arbitrarily small parameter) we let

$$R(\delta, V_j) := \left\{ \sum_{k=1}^{n_j} \alpha_k^{(j)} \delta^{n_j - k} \boldsymbol{e}_k^{(j)} : |\alpha_k^{(j)}| < \epsilon' \text{ for all } k \right\},$$

and for $\epsilon > 0$ we let also

$$\widetilde{R}(\delta, \epsilon, V_j) := \left\{ \sum_{k=1}^{n_j} \beta_k^{(j)} \delta^k \boldsymbol{e}_k^{(j)} : |\beta_k^{(j)}| < \epsilon \text{ for all } k \right\}$$

(notice how $R(\delta, V_j)$ is a parabolic box of dimensions $\sim \delta^{n_j - 1} \times \cdots \times \delta \times 1$, whereas $\widetilde{R}(\delta, \epsilon, V_j)$ is a parabolic box of dimensions $\sim \delta \times \delta^2 \times \cdots \times \delta^{n_j}$). Consider now parameters (s, t) restricted to the strip

$$S_\delta := \{(s, t) \in [-1, 1]^2 : |t| < \delta\}$$

and let us study how $s\tilde{\boldsymbol{J}} + t\tilde{\boldsymbol{I}}$ acts on the parabolic boxes. If $\boldsymbol{v} \in R(1, V_0)$, we have $\boldsymbol{v} = \sum_{k=1}^{n_0} \alpha_k^{(0)} \boldsymbol{e}_k^{(0)}$ and thus by (28)

$$(s\tilde{\boldsymbol{J}} + t\tilde{\boldsymbol{I}}) \boldsymbol{v} = \sum_{k=1}^{n_0} t \alpha_k^{(0)} \boldsymbol{e}_k^{(0)};$$

as a consequence, we have $(s\tilde{\boldsymbol{J}} + t\tilde{\boldsymbol{I}}) R(1, V_0) \subset \widetilde{R}(1, \delta\epsilon', V_0)$. If $\boldsymbol{v} \in R(\delta, V_j)$ for $1 \leq j \leq \ell$ we have $\boldsymbol{v} = \sum_{k=1}^{n_j} \alpha_k^{(j)} \delta^{n_j - k} \boldsymbol{e}_k^{(j)}$ and thus by (29)

$$(s\tilde{\boldsymbol{J}} + t\tilde{\boldsymbol{I}}) \boldsymbol{v} = t \alpha_1^{(j)} \delta^{n_j - 1} \boldsymbol{e}_{n_j}^{(j)} + \sum_{k=1}^{n_j - 1} (s \alpha_{n_j - k}^{(j)} \delta + t \alpha_{n_j + 1 - k}^{(j)}) \delta^{k-1} \boldsymbol{e}_k^{(j)};$$

therefore $(s\tilde{\boldsymbol{J}} + t\tilde{\boldsymbol{I}})R(\delta, V_j) \subset \widetilde{R}(\delta, 2\epsilon', V_j)$. Such inclusions have the following consequences: define (with a little abuse of notation) subsets of $\mathbb{R}^d$

$$E_\delta := R(1, V_0) \times \left(\prod_{j=1}^{\ell} R(\delta, V_j)\right) \times \{\boldsymbol{w} \in W : \|\boldsymbol{w}\|_{\ell^\infty} < \epsilon'\},$$
$$F_\delta := \widetilde{R}(1, \delta\epsilon', V_0) \times \left(\prod_{j=1}^{\ell} \widetilde{R}(\delta, 2\epsilon', V_j)\right) \times \{\boldsymbol{w} \in W : \|\boldsymbol{w}\|_{\ell^\infty} \lesssim_{\tilde{\boldsymbol{J}}, \tilde{\boldsymbol{I}}} \epsilon'\};$$

then we have $(s\tilde{\boldsymbol{J}} + t\tilde{\boldsymbol{I}})E_\delta \subset F_\delta$ and $F_\delta - F_\delta \subset 2F_\delta$, which in particular implies

$$\mathcal{T}\mathbf{1}_{2F_\delta \times S_\delta} \geq |S_\delta| \mathbf{1}_{F_\delta \times E_\delta}$$

(where we wrote $\mathcal{T}$ for $\mathcal{T}^{\tilde{\boldsymbol{J}}, \tilde{\boldsymbol{I}}}_{[-1,1]^2}$ to ease the notation a little). If $\mathcal{T}$ were $L^p \to L^q$ bounded, the last inequality would imply (with some rearranging)

$$|S_\delta|^{\frac{1}{p'}} |E_\delta|^{\frac{1}{q}} \lesssim |F_\delta|^{\frac{1}{p} - \frac{1}{q}}.$$

However, it is easy to see that in terms of δ

$$|S_\delta| \sim \delta, \quad |E_\delta| \sim \delta^{\sum_{j=1}^{\ell} \frac{1}{2} n_j(n_j - 1)}, \quad |F_\delta| \sim \delta^{n_0 + \sum_{j=1}^{\ell} \frac{1}{2} n_j(n_j+1)},$$

and letting $\delta \to 0$ we obtain the necessary condition (after further rearranging)

$$1 + \left(n_0 + \sum_{j=1}^{\ell} n_j^2\right)\frac{1}{q} \geq \left(1 + n_0 + \sum_{j=1}^{\ell} \frac{n_j(n_j+1)}{2}\right)\frac{1}{p}. \tag{30}$$

Observe that $m_* = n_0 + \sum_{j=1}^{\ell} n_j$, so that if we restrict ourselves to exponents such that $2/q = 1/p$, we see with some algebra that (30) yields the same set of exponents as the condition

$$1 + \frac{m_*}{q} \geq \frac{m_* + 1}{p}$$

stated in Theorem 9. On the other hand, the general condition excludes a range of exponents beyond those strictly on the critical line $2/q = 1/p$, as illustrated in Figure 3. The figure also illustrates that the reduced range provided by (30) does not quite coincide with the range of true estimates afforded by Theorem 9; notice however that the two ranges coincide when $m_* = n_0$, that is, when the generalised eigenspaces of eigenvalue λ_* are all of dimension 1 (Theorem 9 is then sharp in such cases, save perhaps for the endpoint).

7.2. ***Case II*: $\det(sA + tB)$ *vanishes identically.*** We consider first the case in which $\ker(s_1A + t_1B) \cap \ker(s_2A + t_2B) = \{0\}$ for any linearly independent (s_1, t_1), (s_2, t_2) (equivalently, $\ker A \cap \ker B = \{0\}$). As in Section 4.3, we can locate a maximal nonvanishing minor $\det_{I_*, J_*}(sA + tB)$ (where $I_*, J_* \subset \{1, \dots, d\}$ and $|I_*| = |J_*|$) and use it to define the set of generic (s, t):

$$\mathscr{G} := \{(s, t) \in \mathbb{R}^2 : \det_{I_*, J_*}(sA + tB) \neq 0\}$$

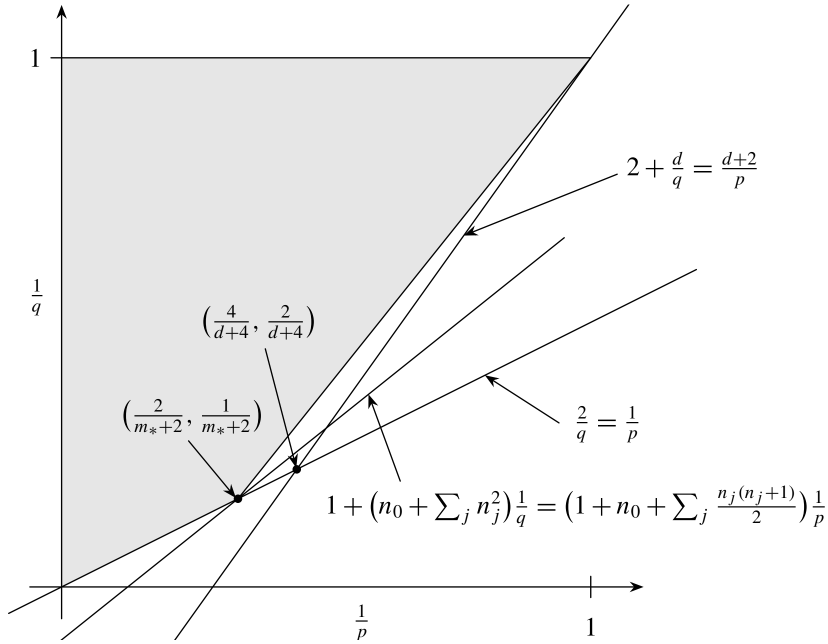

Figure 3. The shaded area corresponds to the range of boundedness afforded by Theorem 9, that is, when the surface $\Sigma(Q_1, Q_2)$ is flat but $\det(s\nabla^2 Q_1 + t\nabla^2 Q_2)$ does not vanish identically. The critical lines given by (4) and (30) are indicated: as one can see, the range of Theorem 9 is sharp when $2/q = 1/p$. The endpoint $(4/(d+4), 2/(d+4))$ for the well-curved case is also indicated, and one can see that for these surfaces all $L^p \to L^q$ estimates for $(1/p, 1/q)$ close to this endpoint are false.

(notice that, unlike in Section 4.3, we are considering real parameters only). Observe that we can find a set $S \subset [-1, 1]^2 \cap \mathscr{G}$ such that $|S| > \frac{1}{2}$, since $\mathscr{G}$ is simply $\mathbb{R}^2$ with some lines removed. We define then the subspace of $\mathbb{R}^d$

$$V := \operatorname{Span}\Big\{ \bigcup_{(s,t)\in\mathscr{G}} \ker(sA + tB) \Big\};$$

by the same arguments given in Section 4.3 we have that for every $(s, t) \in \mathscr{G}$ the image $(sA + tB)V$ consists of a common subspace H, which is a strict subspace of V. As a consequence, if $\xi \in \mathcal{N}_\delta(V)$ (the δ-neighbourhood of V), we see that for $(s, t) \in S$ we have $(sA+tB)\xi \in \mathcal{N}_{K\delta}(H)$, where $K := \|A\| + \|B\|$. Define then sets

$$E_\delta := \mathcal{N}_\delta(V) \cap [-1, 1]^d,$$
$$F_\delta := \mathcal{N}_{K\delta}(H) \cap [-K, K]^d;$$

by the discussion above we have

$$\mathcal{T}\mathbf{1}_{2F_\delta \times S} \gtrsim \mathbf{1}_{F_\delta \times E_\delta},$$

and therefore if $\mathcal{T}$ is $L^p \to L^q$ bounded we have from the last inequality (after some rearranging)

$$|E_\delta|^{\frac{1}{q}} \lesssim |F_\delta|^{\frac{1}{p}-\frac{1}{q}}.$$

It is easy to see that

$$|E_\delta| \sim \delta^{d-\dim V}, \quad |F_\delta| \sim \delta^{d-\dim H},$$

so that letting $\delta \to 0$ we obtain the necessary condition

$$\frac{d-\dim V}{q} \geq (d-\dim H)\Big(\frac{1}{p}-\frac{1}{q}\Big),$$

which after some rearranging is rewritten as

$$\Big(2-\frac{\dim V-\dim H}{d-\dim H}\Big)\frac{1}{q} \geq \frac{1}{p},$$

as claimed in Theorem 9. Since $\dim V > \dim H$, the condition shows that every $L^p \to L^q$ estimate with $2/q = 1/p$ is false in this case (with the exclusion of $(p,q) = (\infty,\infty)$).

It remains to treat the case in which $\ker A \cap \ker B \neq \{0\}$, in which case $\mathcal{T}$ does not satisfy any nontrivial estimate. Indeed, there exists a strict subspace $W \subsetneq \mathbb{R}^d$ such that $(sA+tB)\mathbb{R}^d \subset W$ for all (s,t). If we let

$$F_\delta := \mathcal{N}_\delta(W) \cap [-K,K]^d,$$

we see easily that

$$\mathcal{T}\mathbf{1}_{2F_\delta \times [-1,1]^2} \geq \mathbf{1}_{F_\delta \times [-1,1]^d};$$

if $\mathcal{T}$ is $L^p \to L^q$ bounded we have then

$$|F_\delta|^{\frac{1}{q}} \lesssim |F_\delta|^{\frac{1}{p}},$$

and since $|F_\delta| \sim \delta^{d-\dim W}$, it is immediate to deduce the necessary condition $1/q \geq 1/p$. Thus every estimate beyond those obtained from interpolation of the trivial estimates of Remark 3 is false.

Appendix: General well-curved surfaces

In this appendix we sketch the modifications of the arguments presented in this paper that allow us to extend Theorem 5 to Theorem 5′, that is, to general well-curved surfaces $\Sigma(\varphi_1,\varphi_2)$ of the form

$$(\xi, \varphi_1(\xi), \varphi_2(\xi)), \quad \xi \in [-\epsilon,\epsilon]^d,$$

where φ_1, φ_2 are C^2 functions such that $\nabla\varphi_1(0) = \nabla\varphi_2(0) = 0$, and ϵ will be taken sufficiently small depending on φ_1, φ_2. We will borrow heavily from other sections and their notation to keep the appendix short.

The first observation is that if $\Sigma(\varphi_1,\varphi_2)$ is well-curved at $\xi = 0$ then it is well-curved in a neighbourhood of 0 as well. Indeed, this is a consequence of the fact that condition (M) is stable under small perturbations: observe that the coefficients of the polynomial $\det(s\nabla^2\varphi_1(\xi) + t\nabla^2\varphi_2(\xi))$ are continuous functions of ξ. It is well known that the roots of a univariate polynomial are continuous functions of its coefficients, and it is not hard to see that this fact extends to homogeneous polynomials of two variables (for example, by passing to the projectivisation). Thus the roots of $\det(s\nabla^2\varphi_1(\xi) + t\nabla^2\varphi_2(\xi))$ are continuous functions of ξ and we see that if (M) is satisfied at $\xi = 0$ then it is satisfied for $\xi \in [-\epsilon,\epsilon]^d$ for some $\epsilon > 0$ (this is because the maximal algebraic multiplicity of the roots of $\det(sA+tB)$ is an upper semicontinuous function of the matrices A, B).

The bulk of the argument of Section 5 goes through without major changes: in particular, the Jacobian determinant of the map Ψ is still given by (22) — that is, by $\det((s-s_0)\nabla^2\varphi_1(\eta)+(t-t_0)\nabla^2\varphi_2(\eta))$, which unlike the quadratic case is now a function of η too. For ϵ sufficiently small, the multiplicity μ_Ψ of the map Ψ is still 1. Indeed, we see that $\Psi(\eta,s,t)=\Psi(\eta',s',t')$ only if $s=s',\ t=t'$ and

$$\hat{s}(\nabla\varphi_1(\eta)-\nabla\varphi_1(\eta'))+\hat{t}(\nabla\varphi_2(\eta)-\nabla\varphi_2(\eta'))=0$$

(where $\hat{s}=s-s_0,\ \hat{t}=t-t_0$ for shortness); this can be rewritten as

$$\left(\int_0^1[\hat{s}\nabla^2\varphi_1+\hat{t}\nabla^2\varphi_2](\theta\eta+(1-\theta)\eta')\,d\theta\right)(\eta-\eta')=0,$$

so that the matrix in brackets must have determinant zero if $\eta\neq\eta'$. However, expanding the determinant we see that it equals

$$\int_{[0,1]^d}\sum_{\sigma\in S_d}\operatorname{sgn}\sigma\prod_{j=1}^{d}\partial_j\partial_{\sigma(j)}(\hat{s}\varphi_1+\hat{t}\varphi_2)(\theta_j\eta+(1-\theta_j)\eta')\,d\theta_1\cdots d\theta_d;$$

the integrand is seen to be the determinant of a matrix that is a small perturbation of $\hat{s}\nabla^2\varphi_1(\eta)+\hat{t}\nabla^2\varphi_2(\eta)$. If we impose — as we do — that for $(\eta,s,t)\in S$ (where S is given by (21)) this is nonzero, then the integrand is never zero and in particular single-signed (provided ϵ is small), and therefore the determinant above is not zero and $\eta=\eta'$.

To complete the proof given in Section 5 all that remains to show is that we can make the sublevel set estimate (23) uniform in η; this is the most delicate part. First of all, recall as observed in Remark 32 that the implicit constant in Lemma 31 can be made explicit: with θ,θ' normalised linear forms (which for simplicity we assume real, without loss of generality), we have

$$|\{(s,t):|s|,|t|\leq 1,|\theta(s,t)^\mu\theta'(s,t)^\nu|<\delta\}|\lesssim\frac{|\{(s,t):|s|,|t|\lesssim 1,|s^\mu t^\nu|<\delta\}|}{|\det(\theta\ \ \theta')|},$$

where $\det(\theta\ \ \theta')$ is the Jacobian determinant of the map $(s,t)\mapsto(\theta(s,t),\theta'(s,t))$; thus the implicit constant is $O(|\det(\theta\ \ \theta')|^{-1})$. Secondly, by continuity of the roots we have the following: if

$$\theta_1^{m_1},\ldots,\theta_\ell^{m_\ell}$$

are the distinct normalised roots of $\det(s\nabla^2\varphi_1(0)+t\nabla^2\varphi_2(0))$ with respective multiplicities, then for a fixed $\eta\in[-\epsilon,\epsilon]^d$ and ϵ sufficiently small the distinct normalised roots of $\det(s\nabla^2\varphi_1(\eta)+t\nabla^2\varphi_2(\eta))$ are

$$\tilde{\theta}_{11}^{m_{11}},\ldots,\tilde{\theta}_{1n_1}^{m_{1n_1}},\ldots,\tilde{\theta}_{\ell 1}^{m_{\ell 1}},\ldots,\tilde{\theta}_{\ell n_\ell}^{m_{\ell n_\ell}},$$

where each $\tilde{\theta}_{ji}$ for $1\leq i\leq n_j$ is a small perturbation of θ_j and for each j we have $\sum_{i=1}^{n_j}m_{ji}=m_j$. In particular, for any j,k we have

$$|\det(\theta_j\ \ \theta_k)|\sim|\det(\tilde{\theta}_{ji}\ \ \tilde{\theta}_{ki'})|$$

for all $1 \le i \le n_j$ and $1 \le i' \le n_k$. To obtain a sublevel set estimate that is uniform in $\eta \in [-\epsilon, \epsilon]^d$ it will then suffice to show that we can find coefficients $\mu_{jiki'} \ge 0$ such that we have the structural factorisation

$$\prod_{j=1}^{\ell}\prod_{i=1}^{n_j} \tilde{\theta}_{ji}^{m_{ji}} = \prod_{j=1}^{\ell}\prod_{j<k\le\ell}\prod_{i=1}^{n_j}\prod_{i'=1}^{n_k} (\tilde{\theta}_{ji}\tilde{\theta}_{ki'})^{\mu_{jiki'}}$$

(in this way in our constants we will avoid terms like $|\det(\tilde{\theta}_{ji}\ \ \tilde{\theta}_{ji'})|^{-1}$, which could be arbitrarily large). This can be achieved by a variation of the argument used in the proof of Proposition 29, as we now illustrate. As in there, the existence of such a factorisation translates into the existence of a nonnegative solution to the equations

$$L_{ji}: \quad \sum_{k<j}\sum_{i'=1}^{n_k} \mu_{ki'ji} + \sum_{k'>j}\sum_{i''=1}^{n_{k'}} \mu_{jik'i''} = m_{ji}$$

for $1 \le j \le \ell$ and $1 \le i \le n_j$. Appealing once again to Lemma 33, it suffices to show that there is no simultaneous solution $(y_{ji})_{j\le\ell, i\le n_j}$ to the inequalities

$$\begin{cases} y_{ji} + y_{ki'} \ge 0 \quad \text{for all } 1 \le j < k \le \ell,\ 1 \le i \le n_j,\ 1 \le i' \le n_k, \\ \displaystyle\sum_{j=1}^{\ell}\sum_{i=1}^{n_j} m_{ji} y_{ji} < 0. \end{cases}$$

Since $m_{ji} > 0$ the second inequality implies that for some j_* one coefficient $y_{j_* i}$ is negative; let $y_{j_* i_*}$ be the most negative of such coefficients. From the first inequality we see that for every $j \ne j_*$ we must have $y_{ji} \ge |y_{j_* i_*}| > 0$ for all $1 \le i \le n_j$, and therefore we have

$$\begin{aligned} \sum_{j\ne j_*} m_j &= \sum_{j\ne j_*}\sum_{i=1}^{n_j} m_{ji} \le \sum_{j\ne j_*}\sum_{i=1}^{n_j} m_{ji}\frac{y_{ji}}{|y_{j_* i_*}|} \\ &= \frac{1}{|y_{j_* i_*}|}\sum_{j=1}^{\ell}\sum_{i=1}^{n_j} m_{ji}y_{ji} - \frac{1}{|y_{j_* i_*}|}\sum_{i=1}^{n_{j_*}} m_{j_* i}(y_{j_* i} - y_{j_* i_*}) + \sum_{i=1}^{n_{j_*}} m_{j_* i} \\ &< -\frac{1}{|y_{j_* i_*}|}\sum_{i=1}^{n_{j_*}} m_{j_* i}(y_{j_* i} - y_{j_* i_*}) + \sum_{i=1}^{n_{j_*}} m_{j_* i} \le \sum_{i=1}^{n_{j_*}} m_{j_* i} = m_{j_*}; \end{aligned}$$

this would imply $m_{j_*} > d/2$, a contradiction because $\Sigma(\varphi_1, \varphi_2)$ is well-curved at $\xi = 0$. This concludes the proof.

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THE PROJECTION CONSTANT FOR THE TRACE CLASS

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We study the projection constant of the space of operators on n-dimensional Hilbert spaces with the trace norm $\mathscr{S}_1(n)$. We show an integral formula for the projection constant of $\mathscr{S}_1(n)$; namely, $\boldsymbol{\lambda}(\mathscr{S}_1(n)) = n\int_{\mathscr{U}_n}|\mathrm{tr}(U)|\,dU$, where the integration is with respect to the Haar probability measure on the group $\mathscr{U}_n$ of unitary operators. Using a probabilistic approach, we derive the limit formula $\lim_{n\to\infty}\boldsymbol{\lambda}(\mathscr{S}_1(n))/n = \sqrt{\pi}/2$.

Introduction

The projection constant is a fundamental concept in Banach spaces and their local theory. It has its origins in the study of complemented subspaces of Banach spaces. If X is a complemented subspace of a Banach space Y, then the relative projection constant of X in Y is defined by

$$\begin{aligned}\boldsymbol{\lambda}(X,Y) &= \inf\{\|P\| : P\in\mathscr{L}(Y,X),\ P|_X = \mathrm{Id}_X\}\\ &= \inf\{c>0 : \forall\, T\in\mathscr{L}(X,Z)\ \exists \text{ an extension } \widetilde{T}\in\mathscr{L}(Y,Z) \text{ with } \|\widetilde{T}\|\le c\|T\|\},\end{aligned}$$

where Id_X denotes the identity operator on X and as usual $\mathscr{L}(U,V)$ denotes the Banach space of all bounded linear operators between the Banach spaces U and V with the uniform norm. In what follows $\mathscr{L}(U) := \mathscr{L}(U,U)$. We use here the convention that $\inf\varnothing = \infty$.

The (absolute) projection constant of X is given by

$$\boldsymbol{\lambda}(X) := \sup \boldsymbol{\lambda}(I(X),Y),$$

where the supremum is taken over all Banach spaces Y and isometric embeddings $I : X\to Y$.

It is well known that any Banach space X embeds isometrically into $\ell_\infty(\Gamma)$, where Γ is a nonempty set depending on X (and $\ell_\infty(\Gamma)$ as usual stands for the Banach space of all bounded scalar-valued functions on Γ), and then

$$\boldsymbol{\lambda}(X) = \boldsymbol{\lambda}(X,\ell_\infty(\Gamma)). \tag{1}$$

Thus finding $\boldsymbol{\lambda}(X)$ is equivalent to finding the norm of a minimal projection from $\ell_\infty(\Gamma)$ onto an isometric copy of X in $\ell_\infty(\Gamma)$. Note also the well-known fact that if X is a finite-dimensional Banach and X_1 is a subspace of some $C(K)$-space isometric to X, then $\boldsymbol{\lambda}(X) = \boldsymbol{\lambda}(X_1, C(K))$.

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Let us recall a few concrete cases relevant for our purposes — for an extensive treatment on all of this we refer to the excellent monographs [Diestel et al. 1995; Lindenstrauss and Tzafriri 1977; Pisier 1986; Tomczak-Jaegermann 1989; Wojtaszczyk 1991]. We use standard notation from (local) Banach space theory and note that, throughout the article, all Banach spaces are assumed to be complex. As usual $\mathscr{L}(X)$ denotes the Banach space of all (bounded) linear operators T on X together with the operator norm. For $1 \le p \le \infty$ and $n \in \mathbb{N}$, the symbol ℓ_p^n denotes the Banach space $\mathbb{C}^n$ equipped with the Minkowski norm $\|x\|_p = \left(\sum_{k=1}^n |x_k|^p\right)^{1/p}$ for $1 \le p < \infty$, and $\|x\|_\infty = \sup_{1 \le k \le n} |x_k|$ for $p = \infty$.

A well-known simple application of the Hahn–Banach theorem shows that

$$\boldsymbol{\lambda}(\ell_\infty^n) = 1.$$

The exact values of $\boldsymbol{\lambda}(\ell_2^n)$ and $\boldsymbol{\lambda}(\ell_1^n)$ were computed by Grünbaum [1960] and Rutovitz [1965]: If $d\sigma$ stands for the normalised surface measure on the sphere $\mathbb{S}_n(\mathbb{C})$, then

$$\boldsymbol{\lambda}(\ell_2^n) = n \int_{\mathbb{S}_n} |x_1|\, d\sigma = \frac{\sqrt{\pi}}{2} \frac{n!}{\Gamma\left(n + \frac{1}{2}\right)}. \tag{2}$$

On the other hand, if dz denotes the normalised Lebesgue measure on the distinguished boundary $\mathbb{T}^n$ in $\mathbb{C}^n$ and J_0 is the zero Bessel function defined by $J_0(t) = \frac{1}{2\pi} \int_0^\infty \cos(t \cos\varphi)\, d\varphi$, then

$$\boldsymbol{\lambda}(\ell_1^n) = \int_{\mathbb{T}^n} \left| \sum_{k=1}^n z_k \right| dz = \int_0^\infty \frac{1 - J_0(t)^n}{t^2}\, dt. \tag{3}$$

Moreover, König, Schütt and Tomczak-Jagermann [König et al. 1999] proved that, for $1 \le p \le 2$,

$$\lim_{n \to \infty} \frac{\boldsymbol{\lambda}(\ell_p^n)}{\sqrt{n}} = \frac{\sqrt{\pi}}{2}. \tag{4}$$

Let us turn to the noncommutative analogs of these results. The operator analog of ℓ_∞^n is the Banach space $\mathscr{L}(\ell_2^n)$. By [Gordon and Lewis 1974, Theorem 5.6] it is known that

$$\boldsymbol{\lambda}(\mathscr{L}(\ell_2^n)) = \frac{\pi}{4} \frac{n!^2}{\Gamma\left(n + \frac{1}{2}\right)^2}.$$

The space of Hilbert–Schmidt operators $\mathscr{H}_2(n)$ on ℓ_2^n is a Hilbert space, and we may deduce from (2) that

$$\boldsymbol{\lambda}(\mathscr{H}_2(n)) = \frac{\sqrt{\pi}}{2} \frac{n^2!}{\Gamma\left(n^2 + \frac{1}{2}\right)},$$

which in particular leads to the two limits

$$\lim_{n \to \infty} \frac{\boldsymbol{\lambda}(\mathscr{H}_2(n))}{n} = \frac{\sqrt{\pi}}{2} \quad \text{and} \quad \lim_{n \to \infty} \frac{\boldsymbol{\lambda}(\mathscr{L}(\ell_2^n))}{n} = \frac{\pi}{4}. \tag{5}$$

Finite dimensional Schatten classes form the building blocks of a variety of natural objects in noncommutative functional analysis. Recall that the singular numbers $(s_k(u))_{k=1}^n$ of $u \in \mathscr{L}(\ell_2^n)$ are given by the eigenvalues of $|u| = (u^*u)^{1/2}$, and that the Schatten p-class $\mathscr{S}_p(n)$, $1 \le p \le \infty$, by definition is the Banach space of all operators on ℓ_2^n endowed with the norm $\|u\|_p = \left(\sum_{k=1}^n |s_k(u)|^p\right)^{1/p}$ (for $p = \infty$

we here take the maximum over all $1 \le k \le n$). It is well known that the equalities $\mathscr{S}_\infty(n) = \mathscr{L}(\ell_2^n)$ and $\mathscr{S}_2(n) = \mathscr{H}_2(n)$ hold isometrically. We remark that the space $\mathscr{S}_1(n)$ is usually referred to as trace class.

For the noncommutative analog of (3) in the case of $\mathscr{S}_1(n)$, the best known estimate seems to be

$$\frac{n}{3} \le \boldsymbol{\lambda}(\mathscr{S}_1(n)) \le n. \tag{6}$$

The lower bound was proved by Gordon and Lewis [1974], while the upper bound is a consequence of the famous Kadets–Snobar inequality [Kadets and Snobar 1971].

As pointed out in (3), there is a useful integral formula for $\boldsymbol{\lambda}(\ell_1^n)$. Our main aim is to show a noncommutative analog for $\boldsymbol{\lambda}(\mathscr{S}_1(n))$ and to employ it to get the missing limit from (5). More precisely, we prove that

$$\boldsymbol{\lambda}(\mathscr{S}_1(n)) = n \int_{\mathscr{U}_n} |\mathrm{tr}(U)| \, dU,$$

where $\mathrm{tr}(U)$ denotes the trace of the matrix U and the integration is with respect to the Haar probability measure on the unitary group $\mathscr{U}_n$, and then we apply a probabilistic approach (within the so-called Weingarten calculus) to derive

$$\lim_{n\to\infty} \frac{\boldsymbol{\lambda}(\mathscr{S}_1(n))}{n} = \frac{\sqrt{\pi}}{2}.$$

We finish this introduction with a few words on the technique used. An important tool to calculate projection constants, and more generally to obtain minimal projections, is due to [Rudin 1962]; see also [Wojtaszczyk 1991, Chapter III.B]. This technique is sometimes called Rudin's averaging technique, and it for example may be used to prove (2) as well as (3).

Given an isometric subspace X of Y, we outline the main steps of the strategy to find the relative projection constant $\boldsymbol{\lambda}(X, Y)$. First, one selects a possible "natural candidate" $\boldsymbol{P} : Y \to X$ for a minimal projection. Then, one identifies a topological group G acting on $\mathscr{L}(Y)$ such that every $g \in G$ defines an operator T_g acting in a "compatible way" on Y. Next, it is shown that $\boldsymbol{P}$ is the unique projection commuting with all operators T_g, $g \in G$. Afterward, an arbitrary projection $\boldsymbol{Q} : Y \to X$ is considered, and all operators $T_g^{-1} \boldsymbol{Q} T_g$ are averaged with respect to the Haar measure on G. This average commutes with all operators T_g, $g \in G$, and must coincide with $\boldsymbol{P}$. A simple convexity argument is then employed to establish that $\boldsymbol{\lambda}(X, Y) = \|\boldsymbol{P} : Y \to X\|$, and this norm is subsequently analyzed to refine the formula for $\boldsymbol{\lambda}(X, Y)$.

Thus, if here $Y = \ell_\infty(\Gamma)$, then (1) shows that these steps may lead to a formula/estimate of $\boldsymbol{\lambda}(X)$. Let us see how our object of desire $\mathscr{S}_1(n)$ naturally embeds in some reasonable $\ell_\infty(\Gamma)$. It is well known that $\mathscr{L}(\ell_2^n)$ and $\mathscr{S}_1(n)$ are in trace duality; that is, the mapping

$$\mathscr{S}_1(n) \to \mathscr{L}(\ell_2^n)^*, \quad u \mapsto [v \mapsto \mathrm{tr}(uv)], \tag{7}$$

defines a linear and isometric bijection. To go one step further, we may compose this mapping with the restriction map $\mathscr{L}(\ell_2^n)^* \to C(\mathscr{U}_n)$, $u \mapsto u|_{\mathscr{U}_n}$, where $\mathscr{U}_n$ stands for the group of all unitary $n \times n$ matrices, and in fact this leads to an isometric embedding $\mathscr{S}_1(n) \hookrightarrow C(\mathscr{U}_n)$; see Proposition 2.11. So our aim in the following will be to analyze the relative projection constant $\boldsymbol{\lambda}(\mathscr{S}_1(n), C(\mathscr{U}_n))$.

In order to apply Rudin's averaging technique, we first need to develop what we call "unitary harmonics" on the unitary group $\mathscr{U}_n$, which (roughly speaking) are harmonic polynomials in finitely many "matrix variables" z and $\bar{z}$ from the unitary group $\mathscr{U}_n$. All this is deeply inspired by the classical theory of spherical harmonics (see, e.g., [Rudin 1980]), that is, the study of harmonic polynomials in finitely many complex variables z and $\bar{z}$ on the n-dimensional euclidean sphere $\mathbb{S}_n$. Unitary harmonics and their density in $C(\mathscr{U}_n)$ are described in Section 1.

In Section 2 we formulate and prove our main Theorem 2.1. And although this is the sole focus of this work, structuring the proof of Theorem 2.1 carefully shows that parts of it extend to a more abstract version given in Theorem 2.6.

1. Unitary harmonics and their density

We need to extend a few aspects of the theory of spherical harmonics on the sphere $\mathbb{S}_n$ (as developed for example in [Rudin 1980, Chapter 12] and [Atkinson and Han 2012, Chapter 2]) to what we call unitary harmonics on the unitary group $\mathscr{U}_n$.

1.1. ***Unitaries.*** Denote by M_n the space of all $n \times n$ matrices $Z = (z_{k\ell})$ with entries from $\mathbb{C}$. The group $\mathscr{U}_n$ of all unitary $n \times n$ matrices $U = (u_{ij})_{1 \le i,j \le n}$ endowed with the topology induced by $\mathscr{L}(\ell_2^n)$ forms a nonabelian compact group. It is unimodular, and we denote the integral, with respect to the Haar measure on $\mathscr{U}_n$, of a function $f \in L_2(\mathscr{U}_n)$ by

$$\int_{\mathscr{U}_n} f(U)\, dU.$$

Integrals of this type form the so-called Weingarten calculus, which is of outstanding importance in random matrix theory, mathematical physics, and the theory of quantum information; see, e.g., [Collins and Śniady 2006; Köstenberger 2021].

Basically, we will only need the precise values of two concrete integrals from the Weingarten calculus. The first is

$$\int_{\mathscr{U}_n} u_{i,j} \overline{u_{k,\ell}}\, dU = \frac{1}{n} \delta_{i,k} \delta_{j,\ell} \tag{8}$$

for all possible $1 \le i, j, k, \ell \le n$, and the second is

$$\int_{\mathscr{U}_n} |\operatorname{tr}(AU)|^2\, dU = \frac{1}{n} \operatorname{tr}(AA^*) \tag{9}$$

for every $A \in M_n$; see, e.g., [Cerezo et al. 2021, p. 16], [Collins and Śniady 2006], or [Zhang 2014, Corollary 3.6].

Every operator $T \colon M_n \to M_n$ that leaves $\mathscr{U}_n$ invariant (i.e., $T\mathscr{U}_n \subset \mathscr{U}_n$), defines a composition operator

$$C_T \colon L_2(\mathscr{U}_n) \to L_2(\mathscr{U}_n), \quad f \mapsto f \circ T.$$

There are in fact two such operators T (leaving $\mathscr{U}_n$ invariant) of special interest: the left and right multiplication operators L_V and R_V with respect to $V \in \mathscr{U}_n$ are given by

$$L_V(U) := VU \quad \text{and} \quad R_V(U) := UV, \quad U \in M_n.$$

A subspace $S \subset L_2(\mathscr{U}_n)$ is said to be $\mathscr{U}_n$-invariant whenever it is invariant under all possible composition operators C_{L_V} and C_{R_V} with $V \in \mathscr{U}_n$.

For any closed subspace $S \subset L_2(\mathscr{U}_n)$, we denote by $\pi_S \colon L_2(\mathscr{U}_n) \to L_2(\mathscr{U}_n)$ the orthogonal projection on $L_2(\mathscr{U}_n)$ with range S.

1.2. ***Spherical harmonics.*** The symbol $\mathscr{P}(\mathbb{R}^N)$ stands for all polynomials $f \colon \mathbb{R}^N \to \mathbb{C}$ of the form

$$f(x) = \sum_{\alpha \in J} c_\alpha x^\alpha, \tag{10}$$

where $J \subset \mathbb{N}_0^N$ is finite and $(c_\alpha)_{\alpha \in J}$ are complex coefficients. Moreover, given $k \in \mathbb{N}_0$, we write $\mathscr{P}_k(\mathbb{R}^N)$ for all k-homogeneous polynomials f of this type; that is, f has a representation like that in (10) with $|\alpha| := \sum \alpha_i = k$ for each $\alpha \in J$.

Observe that in (10) one has $c_\alpha = \partial^\alpha f(0)/\alpha!$ for each $\alpha \in J$, which in particular shows the uniqueness of the coefficients for each $f \in \mathscr{P}(\mathbb{R}^N)$; in the following we often write $c_\alpha = c_\alpha(f)$. A particular consequence is that the linear space $\mathscr{P}(\mathbb{R}^N)$ carries a natural inner product given by

$$\langle f, g \rangle_{\mathscr{P}} := \sum_\alpha \alpha!\, c_\alpha(f)\overline{c_\alpha(g)}, \quad f, g \in \mathscr{P}(\mathbb{R}^N). \tag{11}$$

This scalar product has a useful reformulation. To see this, note first that every $f \in \mathscr{P}(\mathbb{R}^N)$ defines the differential operator $f(D) \colon \mathscr{P}(\mathbb{R}^N) \to \mathscr{P}(\mathbb{R}^N)$ given by

$$f(D)g := \sum_\alpha c_\alpha(f)\partial^\alpha g, \quad g \in \mathscr{P}(\mathbb{R}^N).$$

And then it is straightforward to verify for every $f, g \in \mathscr{P}(\mathbb{R}^N)$ the formula

$$\langle f, g \rangle_{\mathscr{P}} = [f(D)\bar{g}](0). \tag{12}$$

The polynomial $\boldsymbol{t} \in \mathscr{P}_2(\mathbb{R}^N)$ defined by

$$\boldsymbol{t}(x) := \|x\|_2^2, \quad x \in \mathbb{R}^N, \tag{13}$$

is of special importance, since then

$$\Delta := \boldsymbol{t}(D) = \sum_{j=1}^N \frac{\partial^2}{\partial x_j^2} \colon \mathscr{P}(\mathbb{R}^N) \to \mathscr{P}(\mathbb{R}^N)$$

is the Laplace operator. A polynomial $f \in \mathscr{P}(\mathbb{R}^N)$ is said to be harmonic whenever $\Delta f = 0$, and we write $\mathscr{H}(\mathbb{R}^N)$ for the subspace of all harmonic polynomials in $\mathscr{P}(\mathbb{R}^N)$ and $\mathscr{H}_k(\mathbb{R}^N)$ for all k-homogeneous, harmonic polynomials. For each $N \in \mathbb{N}$, one has

$$\mathscr{H}(\mathbb{R}^N) = \operatorname{span}_k \mathscr{H}_k(\mathbb{R}^N). \tag{14}$$

To see this, fix $f = \sum_{\alpha \in J} c_\alpha(f)x^\alpha \in \mathscr{H}(\mathbb{R}^N)$ with degree $d = \max_{\alpha \in J} |\alpha|$. For each $k \in \{0, 1, \dots, d\}$, let $f_k = \sum_{|\alpha|=k} c_\alpha(f)x^\alpha$ be the k-homogeneous part of f. Since $f = \sum_{k=0}^d f_k$, it remains to show that each f_k is harmonic. Clearly, $\sum_{k=0}^d \triangle f_k = \triangle f = 0$. Since all f_k are supported on disjoint index sets of multi-indices, we conclude that $\triangle f_k = 0$ for each $0 \le k \le d$.

Much of what follows is based on the following well-known decomposition of $\mathscr{P}_k(\mathbb{R}^N)$ into harmonic subspaces; see, e.g., [Atkinson and Han 2012, Theorem 2.18]. For the sake of completeness we include a proof.

Proposition 1.1. *For each $k \in \mathbb{N}_0$ and $N \in \mathbb{N}$,*

$$\mathscr{P}_k(\mathbb{R}^N) = \mathscr{H}_k(\mathbb{R}^N) \oplus \boldsymbol{t} \cdot \mathscr{H}_{k-2}(\mathbb{R}^N) \oplus \boldsymbol{t}^2 \cdot \mathscr{H}_{k-4}(\mathbb{R}^N) \oplus \cdots,$$

where the orthogonal sum, taken with respect to the inner product from (12), *stops when the subscript reaches* 1 *or* 0.

Proof. Given $g \in \mathscr{P}_{k-2}(\mathbb{R}^N)$, we let $h(x) := \boldsymbol{t}(x)g(x)$ for all $x \in \mathbb{R}^N$. Since $\boldsymbol{t}(D) = \Delta$, this implies that $h(D) = \Delta \circ g(D) = g(D) \circ \Delta$. Clearly, if now $f \in \mathscr{P}_k(\mathbb{R}^N)$, then by (12)

$$\langle h, f\rangle_{\mathscr{P}} = [h(D)\bar{f}](0) = [g(D)(\Delta\bar{f})](0) = \langle g, \Delta f\rangle_{\mathscr{P}}.$$

Thus, the condition $f \perp \boldsymbol{t}g$ for every $g \in \mathscr{P}_{k-2}(\mathbb{R}^N)$ is equivalent to $\Delta f \perp g$ for every $g \in \mathscr{P}_{k-2}(\mathbb{R}^N)$, which is also equivalent to $f \in \mathscr{H}_k(\mathbb{R}^N)$. As a consequence, we get

$$\mathscr{P}_k(\mathbb{R}^N) = \mathscr{H}_k(\mathbb{R}^N) \oplus \boldsymbol{t} \cdot \mathscr{P}_{k-2}(\mathbb{R}^N).$$

The proof finishes by repeating this procedure for $\mathscr{H}_{k-2}(\mathbb{R}^N)$, $\mathscr{H}_{k-4}(\mathbb{R}^N)$, and so on. □

By $\mathbb{S}_N^{\mathbb{R}}$ we denote the sphere in the real Hilbert space $\ell_2^N(\mathbb{R})$. We write $\mathscr{P}(\mathbb{S}_N^{\mathbb{R}})$ for the linear space of all restrictions $f|_{\mathbb{S}_N^{\mathbb{R}}}$ of polynomials $f \in \mathscr{P}(\mathbb{R}^N)$ and $\mathscr{P}_k(\mathbb{S}_N^{\mathbb{R}})$ whenever we only consider restrictions of k-homogeneous polynomials.

All restrictions of harmonic polynomials on $\mathbb{R}^N$ (so polynomials in $\mathscr{H}(\mathbb{R}^N)$) to the sphere $\mathbb{S}_N^{\mathbb{R}}$ are denoted by $\mathscr{H}(\mathbb{S}_N^{\mathbb{R}})$, and such polynomials are called spherical harmonics. Similarly, we denote by $\mathscr{H}_k(\mathbb{S}_N^{\mathbb{R}})$ the space collecting all k-homogeneous polynomials restricted to $\mathbb{S}_N^{\mathbb{R}}$. Endowed with the supremum norm taken on $\mathbb{S}_N^{\mathbb{R}}$, both spaces $\mathscr{H}(\mathbb{S}_N^{\mathbb{R}})$ and $\mathscr{H}_k(\mathbb{S}_N^{\mathbb{R}})$ form subspaces of $C(\mathbb{S}_N^{\mathbb{R}})$.

An important fact, not needed here, is that the spaces $\mathscr{H}_k(\mathbb{S}_N^{\mathbb{R}})$ are pairwise orthogonal in $L_2(\mathbb{S}_N^{\mathbb{R}})$; see, e.g., [Atkinson and Han 2012, Corollary 2.15].

1.3. ***Unitary harmonics.*** Going one step further, we extend the notion of spherical harmonics on the real sphere $\mathbb{S}_N^{\mathbb{R}}$ to what we call unitary harmonics on the unitary group $\mathscr{U}_n$.

Recall that M_n here stands for the space of all $n \times n$ matrices $Z = (z_{k\ell})$ with entries from $\mathbb{C}$. The subset of such matrices $\alpha = (\alpha_{k\ell})$ with entries from $\mathbb{N}_0$ is denoted by $M_n(\mathbb{N}_0)$. For $Z \in M_n$ and $\alpha = (\alpha_{k\ell}) \in M_n(\mathbb{N}_0)$, we define

$$Z^\alpha = \prod_{k,\ell=1}^{n} z_{k\ell}^{\alpha_{k\ell}}.$$

We identify M_n with $\mathbb{R}^{2n^2}$ in the canonical way through the bijective mapping

$$\boldsymbol{I}_n \colon M_n \longrightarrow \mathbb{R}^{2n^2}, \tag{15}$$

which assigns to every matrix $Z = (z_{k\ell})_{k\ell} = (x_{k\ell} + i y_{k\ell})_{k\ell} \in M_n$ the element

$$(x_{11}, y_{11}, \dots, x_{1n}, y_{1n}, x_{21}, y_{21}, \dots, x_{2n}, y_{2n}, \dots, x_{n1}, y_{n1}, \dots, x_{nn}, y_{nn}) \in \mathbb{R}^{2n^2}.$$

Then $\mathfrak{P}(M_n)$ denotes the linear space of all polynomials $f = g \circ \boldsymbol{I}_n$ with $g \in \mathscr{P}(\mathbb{R}^{2n^2})$. Hence, by definition, the mapping

$$\mathscr{P}(\mathbb{R}^{2n^2}) = \mathfrak{P}(M_n), \quad g \mapsto g \circ \boldsymbol{I}_n, \tag{16}$$

identifies both spaces as vector spaces.

We collect a couple of useful facts. Note first that if $f = g \circ \boldsymbol{I}_n \in \mathfrak{P}(M_n)$ with $g \in \mathscr{P}(\mathbb{R}^{2n^2})$, then

$$\Delta f = \sum_{i,j=1}^{n} \frac{\partial^2 f}{\partial z_{ij} \partial \bar{z}_{ij}} = \frac{1}{4} \sum_{i,j=1}^{n} \left(\frac{\partial^2 g}{\partial x_{ij}^2} + \frac{\partial^2 g}{\partial y_{ij}^2} \right);$$

a formula that follows directly from the definition of $\partial_{z_{ij}} = \frac{1}{2}(\partial_{x_{ij}} - i \partial_{y_{ij}})$ and $\partial_{\bar{z}_{ij}} = \frac{1}{2}(\partial_{x_{ij}} + i \partial_{y_{ij}})$.

Secondly, a function $f : M_n \to \mathbb{C}$ belongs to $\mathfrak{P}(M_n)$ if and only if it has a representation

$$f(Z) = \sum_{(\alpha,\beta) \in J} c_{(\alpha,\beta)} Z^{\alpha} \bar{Z}^{\beta}, \quad Z \in M_n, \tag{17}$$

where J is a finite index set in $M_n(\mathbb{N}_0) \times M_n(\mathbb{N}_0)$ and $c_{(\alpha,\beta)} \in \mathbb{C}$, $(\alpha, \beta) \in J$. Moreover, in this case this representation is unique.

Indeed, if f is given by (17), then $g = f \circ \boldsymbol{I}_n^{-1} \in \mathscr{P}(\mathbb{R}^{2n^2})$ and $f = g \circ \boldsymbol{I}_n \in \mathfrak{P}(M_n)$. Conversely, if $f = g \circ \boldsymbol{I}_n \in \mathfrak{P}(M_n)$ with $g = \sum_{\alpha} c_{\alpha} x^{\alpha} \in \mathscr{P}(\mathbb{R}^{2n^2})$, then the desired representation easily follows from the substitution $\operatorname{Re} z_{ij} = \frac{1}{2}(z_{ij} + \bar{z}_{ij})$ and $\operatorname{Im} z_{ij} = \frac{1}{2}(z_{ij} - \bar{z}_{ij})$. To see the uniqueness of the representation in (17), observe that if $f = 0$, then, given $(\alpha, \beta) \neq (0, 0)$, an application of the differential operator

$$\partial_{z_{11}}^{\alpha_{11}} \cdots \partial_{z_{1n}}^{\alpha_{1n}} \partial_{\bar{z}_{11}}^{\beta_{11}} \cdots \partial_{\bar{z}_{1n}}^{\beta_{1n}} \cdots \partial_{z_{n1}}^{\alpha_{n1}} \cdots \partial_{z_{nn}}^{\alpha_{nn}} \partial_{\bar{z}_{n1}}^{\beta_{n1}} \cdots \partial_{\bar{z}_{nn}}^{\beta_{nn}}$$

to f (and evaluating at zero), shows that $c_{(\alpha,\beta)} = 0$.

We again use the identification $g \mapsto g \circ \boldsymbol{I}_n$ from (15) to define the spaces

$$\mathfrak{P}_k(M_n) := \mathscr{P}_k(\mathbb{R}^{2n^2}), \quad \mathfrak{H}_k(M_n) := \mathscr{H}_k(\mathbb{R}^{2n^2}) \quad \text{and} \quad \mathfrak{H}(M_n) := \mathscr{H}(\mathbb{R}^{2n^2}). \tag{18}$$

The following lemma gives a simple description of the elements of $\mathfrak{P}_k(M_n)$.

Lemma 1.2. *Let $f \in \mathfrak{P}(M_n)$ and $k \in \mathbb{N}$. Then $f \in \mathfrak{P}_k(M_n)$ if and only if $f(\lambda Z) = \lambda^k f(Z)$ for all $\lambda \in \mathbb{R}$ and $Z \in M_n$.*

Proof. For $f \in \mathfrak{P}_k(M_n)$ there is $g \in \mathscr{P}_k(\mathbb{R}^{2n^2})$ such that $f = g \circ \boldsymbol{I}_n$. Clearly, $f(\lambda Z) = \lambda^k f(Z)$ for all $\lambda \in \mathbb{R}$ and $Z \in M_n$. Assume conversely that f is k-homogeneous in the meaning of the statement. Since $f \in \mathfrak{P}(M_n)$, there is a finite polynomial $g(x) = \sum_J c_{\alpha}(g) x^{\alpha}$, $x \in \mathbb{R}^{2n^2}$, such that $f = g \circ \boldsymbol{I}_n$. Since $g = f \circ \boldsymbol{I}_n^{-1}$, it follows that $g(\lambda x) = \lambda^k g(x)$ for all $\lambda \in \mathbb{R}$ and $x \in \mathbb{R}^{2n^2}$. But this by the uniqueness of the coefficients $c_{\alpha}(g)$ necessarily implies that $c_{\alpha}(g) \neq 0$ only if $|\alpha| = k$, so as desired $g \in \mathscr{P}_k(\mathbb{R}^{2n^2})$. $\square$

Obviously,

$$\mathfrak{P}(M_n) = \operatorname{span}_k \mathfrak{P}_k(M_n) \tag{19}$$

(consider the polynomials on $\mathbb{R}^{2n^2}$ defining these spaces), and less trivially (as an immediate consequences of (14)) we have

$$\mathfrak{H}(M_n) = \operatorname{span}_k \mathfrak{H}_k(M_n). \tag{20}$$

The polynomial $\boldsymbol{t}_{M_n} \in \mathfrak{P}_2(M_n)$ given by

$$\boldsymbol{t}_{M_n}(Z) := \operatorname{tr}(ZZ^*), \quad Z \in M_n,$$

where $\operatorname{tr}\colon M_n \to \mathbb{C}$ denotes the trace, is of special importance. It is easily seen that under the identification from (16) the image of the polynomial $\boldsymbol{t} \in \mathscr{P}_2(\mathbb{R}^{2n^2})$ (see again (13)) is $\boldsymbol{t}_{M_n} \in \mathfrak{P}_2(M_n)$; that is,

$$\boldsymbol{t}_{M_n}(Z) = \boldsymbol{t}(\boldsymbol{I}_n Z), \quad Z \in M_n. \tag{21}$$

Recall again that $\mathscr{P}(\mathbb{R}^{2n^2})$ carries the natural inner product from (11), which then by the identification in (16) transfers to a natural inner product on $\mathfrak{P}(M_n)$; that is,

$$\langle f, g\rangle_{\mathfrak{P}} := \langle f \circ \boldsymbol{I}_n^{-1}, g \circ \boldsymbol{I}_n^{-1}\rangle_{\mathscr{P}}, \quad f, g \in \mathfrak{P}(M_n). \tag{22}$$

Using (18) and (21), we deduce from Proposition 1.1 its matrix analog, which is going to be of great value later on.

Proposition 1.3. *For all $k \in \mathbb{N}_0$ and $n \in \mathbb{N}$,*

$$\mathfrak{P}_k(M_n) = \mathfrak{H}_k(M_n) \oplus \boldsymbol{t}_{M_n} \cdot \mathfrak{H}_{k-2}(M_n) \oplus \boldsymbol{t}_{M_n}^2 \cdot \mathfrak{H}_{k-4}(M_n) \oplus \cdots,$$

where the last term of the orthogonal sum is the span of $\boldsymbol{t}_{M_n}^{k/2}$ for even k and $\boldsymbol{t}_{M_n}^{(k-1)/2} \cdot \mathfrak{H}_1(M_n)$ for odd k.

We need two more lemmas.

Lemma 1.4. *Let $f \in \mathfrak{H}(M_n)$ and $U \in \mathscr{U}_n$. Then $f \circ L_U \in \mathfrak{H}(M_n)$. Moreover, if $f \in \mathfrak{H}_k(M_n)$, then also $f \circ L_U \in \mathfrak{H}_k(M_n)$.*

Proof. Recall the well-known fact that, for every harmonic function $F\colon \mathbb{C}^{n^2} \to \mathbb{C}$ and every $W \in \mathscr{U}_{n^2}$, we have $\triangle(F \circ \Phi_W) = \triangle F \circ \Phi_W$, where $\Phi_W z = Wz$ for $z \in \mathbb{C}^{n^2}$. Now identify M_n and $\mathbb{C}^{n^2}$ in the natural way by

$$\boldsymbol{J}_n(Z) = (z_{11}, \dots, z_{1n}, z_{21}, \dots, z_{2n}, \dots, z_{n1}, \dots, z_{nn}), \quad Z \in M_n, \tag{23}$$

and define $g = f \circ \boldsymbol{J}_n^{-1} : \mathbb{C}^{n^2} \to \mathbb{C}$. Then obviously $\triangle g = 0$, and moreover a simple calculation shows

$$f \circ L_U = g \circ \Phi_{U \otimes \mathrm{id}_{\mathbb{C}^n}} \circ \boldsymbol{J}_n.$$

Since $U \otimes \mathrm{id}_{\mathbb{C}^n} \in \mathscr{U}_{n^2}$ is unitary, it follows that

$$\triangle(f \circ L_U) = \triangle(g \circ \Phi_{U \otimes \mathrm{id}_{\mathbb{C}^n}}) = \triangle g \circ \Phi_{U \otimes \mathrm{id}_{\mathbb{C}^n}} = 0.$$

For the second statement, note that if $f \in \mathfrak{H}_k(M_n)$, then by the first statement $f \circ L_U \in \mathfrak{H}(M_n)$. But $f \circ L_U(\lambda Z) = \lambda f \circ L_U(Z)$ for all $Z \in M_n$ and $\lambda \in \mathbb{R}$, and hence the claim follows from Lemma 1.2. $\square$

For $p, q \in \mathbb{N}_0$, let $\mathfrak{H}_{(p,q)}(M_n) \subset \mathfrak{H}(M_n)$ be the subspace of all harmonic polynomials which are p-homogeneous in $Z = (z_{ij})$ and q-homogeneous in $\bar{Z} = (\bar{z}_{ij})$; that is, all polynomials $f \in \mathfrak{H}(M_n)$ of the form

$$f(Z) = \sum_{|\alpha|=p, |\beta|=q} c_{(\alpha,\beta)} Z^\alpha \bar{Z}^\beta, \quad Z \in M_n. \tag{24}$$

By Lemma 1.2, we immediately see that

$$\mathfrak{H}_{(p,q)}(M_n) \subset \mathfrak{H}_{p+q}(M_n). \tag{25}$$

The following result is crucial for our purpose; see also Lemma 1.7.

Lemma 1.5. *For all $f \in \mathfrak{H}_{(p,q)}(M_n)$ and $U \in \mathscr{U}_n$, one has*

$$f \circ L_U \in \mathfrak{H}_{(p,q)}(M_n) \quad \text{and} \quad f \circ R_U \in \mathfrak{H}_{(p,q)}(M_n).$$

Moreover,

$$\mathfrak{H}(M_n) = \operatorname{span}_{p,q} \mathfrak{H}_{(p,q)}(M_n). \tag{26}$$

Proof. Taking for f a representation as in (24), we have

$$f \circ L_U(Z) = \sum_{|\alpha|=p, |\beta|=q} c_{(\alpha,\beta)} (UZ)^\alpha (\overline{UZ})^\beta, \quad Z \in M_n.$$

Now, for each $1 \le i, j \le n$, we use the multinomial formula for $\left(\sum_\ell u_{i\ell} z_{\ell j}\right)^{\alpha_{ij}}$ to get

$$(UZ)^\alpha (\overline{UZ})^\beta = \sum_{|\gamma| \le p, |\zeta| \le q} d_{(\gamma,\delta)} Z^\gamma \bar{Z}^\zeta, \quad Z = (z_{k\ell})_{k,\ell} \in M_n.$$

Combining, we conclude that $f \circ L_U$ has a representation

$$f \circ L_U(Z) = \sum_{|\eta| \le p, |\sigma| \le q} e_{(\eta,\sigma)} Z^\eta \bar{Z}^\sigma, \quad Z \in M_n.$$

On the other hand, by (25) and Lemma 1.4, it follows that $f \circ L_U \in \mathfrak{H}_{p+q}(M_n)$, and hence, for all $\lambda \in \mathbb{R}$ and $Z \in M_n$, one has

$$\sum_{|\eta| \le p, |\sigma| \le q} e_{(\eta,\sigma)} \lambda^{|\eta|+|\sigma|} Z^\eta \bar{Z}^\sigma = (f \circ L_U)(\lambda Z) = \lambda^{p+q} (f \circ L_U)(Z) = \sum_{|\eta| \le p, |\sigma| \le q} c_{(\eta,\sigma)} \lambda^{p+q} Z^\eta \bar{Z}^\sigma.$$

Inserting $Z = \mathrm{id} \in M_n$ shows that $e_{(\eta,\sigma)} \neq 0$ only if $|\eta| + |\sigma| = p + q$, and since $|\eta| \le p$ and $|\sigma| \le q$, this is only possible whenever $|\eta| = p$ and $|\sigma| = q$. This as desired proves $f \circ L_U \in \mathfrak{H}_{(p,q)}(M_n)$.

The equality (26) follows from (20) since it may easily be seen that $\mathfrak{H}_k(M_n) = \operatorname{span}_{p+q=k} \mathfrak{H}_{(p,q)}(M_n)$; see also, e.g., [Rudin 1980, Proposition 12.2.2].

In order to prove that $f \circ R_U \in \mathfrak{H}_{(p,q)}(M_n)$, define $f^*(Z) = f(Z^*)$ for $Z \in M_n$. Since the mapping $\ell_2^{n^2} \to \ell_2^{n^2}$, $Z \mapsto Z^*$, is unitary (it is an isometry), the function f^* is harmonic. Now, looking at the representation of f as in (24), we see that $f^* \in \mathfrak{H}_{(q,p)}(M_n)$. This, by what is already proved, gives that $f^* \circ L_{U^*} \in \mathfrak{H}_{(q,p)}(M_n)$. But, for $Z \in M_n$,

$$f \circ R_U(Z) = f(ZU) = f((U^*Z^*)^*) = f^*(U^*Z^*) = f^* \circ L_{U^*}(Z^*) = (f^* \circ L_{U^*})^*(Z),$$

and hence $f \circ R_U = (f^* \circ L_{U^*})^* \in \mathfrak{H}_{(p,q)}(M_n)$. $\square$

1.4. ***Unitarily invariant subspaces of $C(\mathscr{U}_n)$.*** By $\mathfrak{P}(\mathscr{U}_n)$ and $\mathfrak{P}_k(\mathscr{U}_n)$ we denote the linear space of all restrictions $f|_{\mathscr{U}_n} : \mathscr{U}_n \to \mathbb{C}$ of polynomials $f \in \mathfrak{P}(M_n)$ and $f \in \mathfrak{P}_k(M_n)$, respectively.

Similarly, for all restrictions to $\mathscr{U}_n$ of harmonic polynomials from $\mathfrak{H}(M_n)$ and $\mathfrak{H}_k(M_n)$, we write $\mathfrak{H}(\mathscr{U}_n)$ and $\mathfrak{H}_k(\mathscr{U}_n)$, respectively, and the elements therein we address as unitary harmonics. All these constitute important subspaces of $C(\mathscr{U}_n)$.

Lemma 1.6. *For each k,*

$$\mathfrak{P}_k(\mathscr{U}_n) = \operatorname{span}_{\ell \le k} \mathfrak{H}_\ell(\mathscr{U}_n) \tag{27}$$

and

$$\mathfrak{P}(\mathscr{U}_n) = \operatorname{span}_k \mathfrak{P}_k(\mathscr{U}_n) = \operatorname{span}_\ell \mathfrak{H}_\ell(\mathscr{U}_n) = \mathfrak{H}(\mathscr{U}_n). \tag{28}$$

Proof. Proposition 1.3 and the fact that the function $\boldsymbol{t}_{M_n} = n$ on $\mathscr{U}_n$ imply (27). To prove (28), note that the first equality is a consequence of (19), the second of (27), and the last of (20). □

Moreover, for $p, q \in \mathbb{N}_0$, we write $\mathfrak{H}_{(p,q)}(\mathscr{U}_n)$ for all restrictions to $\mathscr{U}_n$ of functions in $\mathfrak{H}_{(p,q)}(M_n)$. Observe that a function $f : \mathscr{U}_n \to \mathbb{C}$ belongs to $\mathfrak{H}_{(p,q)}(\mathscr{U}_n)$ if and only if it has on $\mathscr{U}_n$ a representation like in (24). All needed information on these subspaces of $C(\mathscr{U}_n)$ is included in the following lemma, which is an immediate consequence of Lemma 1.5.

Lemma 1.7. *Each space $\mathfrak{H}_{(p,q)}(\mathscr{U}_n)$ is a $\mathscr{U}_n$-invariant subspace of $C(\mathscr{U}_n)$; that is, for all $f \in \mathfrak{H}_{(p,q)}(\mathscr{U}_n)$ and $U \in \mathscr{U}_n$, we have $f \circ L_U$, $f \circ R_U \in \mathfrak{H}_{(p,q)}(\mathscr{U}_n)$. Moreover,*

$$\mathfrak{H}(\mathscr{U}_n) = \operatorname{span}_{p,q} \mathfrak{H}_{(p,q)}(\mathscr{U}_n). \tag{29}$$

We finish with the following density result as it is crucial for our purposes.

Theorem 1.8. *$\mathfrak{H}(\mathscr{U}_n)$ is dense in $C(\mathscr{U}_n)$. In particular, the span of the union of all $\mathfrak{H}_k(\mathscr{U}_n)$ as well as the span of the union of all $\mathfrak{H}_{(p,q)}(\mathscr{U}_n)$ are dense in $C(\mathscr{U}_n)$.*

Proof. Observe first that $\mathfrak{P}(\mathscr{U}_n)$ is a subalgebra of $C(\mathscr{U}_n)$, which is closed under conjugation, and that the collection of all coordinate functions e_{ij} separates the points of $\mathscr{U}_n$. Thus, by the Stone–Weierstrass theorem, $\mathfrak{P}(\mathscr{U}_n)$ is dense in $C(\mathscr{U}_n)$. The rest follows from (28) and (29). □

Remark 1.9. An important difference between spherical harmonics and unitary harmonics is that, for the case of the sphere, the corresponding spaces $\mathfrak{H}_{(p,q)}(\mathbb{S}_n^{\mathbb{C}})$ are mutually orthogonal in $L_2(\mathbb{S}_n^{\mathbb{C}})$; see [Rudin 1980, Theorem 12.2.3]. But for the subspaces $\mathfrak{H}_{(p,q)}(\mathscr{U}_n)$ of $L^2(\mathscr{U}_n)$ this is no longer true. To see an example, take $f \in \mathfrak{H}_{(1,0)}(\mathscr{U}_n)$ and $g \in \mathfrak{H}_{(2,1)}(\mathscr{U}_n)$ defined by $f(U) = u_{1,1}$ and $g(U) = \overline{u_{2,2}}u_{1,2}u_{2,1}$. Then (see, e.g., [Hiai and Petz 2000, Section 4.2])

$$\langle f, g\rangle_{L_2} = \int_{\mathscr{U}_n} u_{1,1}u_{2,2}\overline{u_{1,2}u_{2,1}}\, dU = -\frac{1}{(n-1)n(n+1)}. \tag{30}$$

On the other hand, using basic properties of the Haar measure on $\mathscr{U}_n$, it is not difficult to prove that

$$\mathfrak{H}_{(p,q)}(\mathscr{U}_n) \perp \mathfrak{H}_{(p',q')}(\mathscr{U}_n) \quad \text{whenever} \quad p+q = p'+q' \quad \text{and} \quad (p,q) \neq (p',q'); \tag{31}$$

see [Hewitt and Ross 1963, §29] or [Köstenberger 2021].

It is worth noting the following conclusion from (31) — not needed for our further purposes — which states that

$$\mathfrak{H}_k(\mathscr{U}_n) = \mathfrak{H}_{(k,0)}(\mathscr{U}_n) \oplus \mathfrak{H}_{(k-1,1)}(\mathscr{U}_n) \oplus \cdots \oplus \mathfrak{H}_{(0,k)}(\mathscr{U}_n),$$

where $\oplus$ indicates the orthogonal sum in $L_2(\mathscr{U}_n)$. We conclude with the observation that, in contrast to (30), we have $\langle f, g\rangle_{\mathfrak{P}} = 0$, so the euclidean structure, which $\mathfrak{H}_{(p,q)}(\mathscr{U}_n)$ inherits from $L_2(\mathscr{U}_n)$, is different from that induced by the inner product from (22).

2. Projection constants

As explained in the introduction, the main goal of this work is to prove the following result.

Theorem 2.1. *For each $n \in \mathbb{N}$,*

$$\boldsymbol{\lambda}(\mathscr{S}_1(n)) = \|\pi_{(1,0)} : C(\mathscr{U}_n) \to \mathscr{S}_1(n)\| = n \int_{\mathscr{U}_n} |\mathrm{tr}(V)|\, dV. \tag{32}$$

Moreover,

$$\lim_{n\to\infty} \frac{\boldsymbol{\lambda}(\mathscr{S}_1(n))}{n} = \frac{\sqrt{\pi}}{2}. \tag{33}$$

The proof of this theorem is presented in Section 2.5. It is based on preliminary results we prove in the following, which require some preliminary arguments.

2.1. ***Rudin's averaging technique.*** Given a topological group G and a Banach space Y, we say that G acts on Y (through T) whenever there is a mapping

$$T : G \to \mathscr{L}(Y), \quad g \mapsto T_g,$$

such that

$$T_e = I_Y, \quad T_{gh} = T_g T_h, \quad g, h \in G,$$

and all mappings

$$g \ni G \mapsto T_g(y) \in Y, \quad y \in Y,$$

are continuous. If in addition all operators T_g, $g \in G$, are isometries, then we say that G acts isometrically on Y. We say that $S \in \mathscr{L}(Y)$ commutes with the action T of G on Y whenever S commutes with all T_h, $h \in G$.

The following theorem was presented in [Rudin 1962]; see also [Wojtaszczyk 1991, Theorem III.B.13].

Theorem 2.2. *Let Y be a Banach space, X a complemented subspace of Y, and $\boldsymbol{Q} : Y \to Y$ a projection onto X. Suppose that G is a compact group with Haar measure* m *which acts on Y through T such that X is invariant under the action of G; that is, $T_g(X) \subset X$ for all $g \in G$. Then $\boldsymbol{P} : Y \to Y$ given by*

$$\boldsymbol{P}(y) := \int_G T_{g^{-1}} \boldsymbol{Q} T_g(y)\, d\mathrm{m}(g), \quad y \in Y, \tag{34}$$

is a projection onto X which commutes with the action of G on Y (meaning that $T_g\boldsymbol{P} = \boldsymbol{P}T_g$ for all $g \in G$) and satisfies

$$\|\boldsymbol{P}\| \leq \|\boldsymbol{Q}\| \sup_{g \in G} \|T_g\|^2.$$

Moreover, if there is a unique projection on Y onto X that commutes with the action of G on Y, and if G acts isometrically on Y, then $\boldsymbol{P}$ given in (34) *is minimal, i.e.,*

$$\boldsymbol{\lambda}(X, Y) = \|\boldsymbol{P}\|.$$

In order to be able to apply Rudin's technique, we need to endow $\mathscr{U}_n \times \mathscr{U}_n$ with a special group structure, which allows us to represent the resulting group in $\mathscr{L}(C(\mathscr{U}_n))$. To do so, consider on $\mathscr{U}_n \times \mathscr{U}_n$ the multiplication

$$(U_0, V_0) \cdot (U_1, V_1) := (U_1 U_0, V_0 V_1).$$

With this multiplication and endowed with the product topology, $\mathscr{U}_n \times \mathscr{U}_n$ turns into a compact topological group, and it may be seen easily that the Haar measure on $\mathscr{U}_n \times \mathscr{U}_n$ is given by the product measure of the Haar measure on $\mathscr{U}_n$ with itself.

Further, for any $(U, V) \in \mathscr{U}_n \times \mathscr{U}_n$ and any $f \in L_2(\mathscr{U}_n)$, we define

$$\rho_{(U,V)} f := (C_{L_U} \circ C_{R_V}) f = f \circ L_U \circ R_V,$$

which leads to an action of $\mathscr{U}_n \times \mathscr{U}_n$ on $C(\mathscr{U}_n)$ given by

$$\mathscr{U}_n \times \mathscr{U}_n \to \mathscr{L}(C(\mathscr{U}_n)), \quad (U, V) \mapsto [\rho_{(U,V)} : f \mapsto f \circ L_U \circ R_V]. \tag{35}$$

We say that a mapping $T : S_1 \to S_2$, where S_1 and S_2 both are $\mathscr{U}_n$-invariant subspaces of $L_2(\mathscr{U}_n)$, commutes with the action of $\mathscr{U}_n \times \mathscr{U}_n$ on $C(\mathscr{U}_n)$ whenever

$$(C_{L_U} \circ C_{R_V})(Tf) = T((C_{L_U} \circ C_{R_V}) f) \quad \text{for every } (U, V) \in \mathscr{U}_n \times \mathscr{U}_n \text{ and } f \in S_1.$$

2.2. *Convolution.* Recall from Section 1.1 that $\pi_S : L_2(\mathscr{U}_n) \to S$ denotes the orthogonal projection on $L_2(\mathscr{U}_n)$ onto a given closed subspace S. The following result shows that, under the assumption of $\mathscr{U}_n$-invariance of S, this projection is a convolution operator with respect to some kernel in S.

Theorem 2.3. *Let S be a $\mathscr{U}_n$-invariant subspace of $C(\mathscr{U}_n)$ which is closed in $L_2(\mathscr{U}_n)$. Then the following statements hold:*

(i) *There is a unique function $\mathrm{t}_S \in S$ such that, for all $f \in L_2(\mathscr{U}_n)$,*

$$\pi_S f = f * \mathrm{t}_S.$$

(ii) *π_S commutes with all L_U and R_U for $U \in \mathscr{U}_n$; that is, π_S commutes with the action of $\mathscr{U}_n \times \mathscr{U}_n$ on $C(\mathscr{U}_n)$.*

(iii) $\|\pi_S : C(\mathscr{U}_n) \to S\| = \int_{\mathscr{U}_n} |\mathrm{t}_S(V)| \, dV.$

The proof is an easy consequence of the following lemma.

Lemma 2.4. *Let S be a $\mathscr{U}_n$-invariant subspace of $C(\mathscr{U}_n)$ which is closed in $L_2(\mathscr{U}_n)$. Then, for every $U \in \mathscr{U}_n$, there exists a unique function $K_U^S \in S$ such that, for all $f \in L_2(\mathscr{U}_n)$,*

(i) $(\pi_S f)(U) = \langle f, K_U^S\rangle_{L_2} = \int_{\mathscr{U}_n} f(V)\overline{K_U^S(V)}\,dV$,

and moreover, for every choice of $U, V \in \mathscr{U}_n$, we have

(ii) $K_U^S(V) = \langle K_U^S, K_V^S\rangle_{L_2} = \overline{K_V^S(U)}$,

(iii) $K_U^S \circ L_{V^{-1}} = K_{VU}^S = K_V^S \circ R_{U^{-1}}$,

(iv) $K_V^S(V) = K_{\mathrm{Id}}^S(\mathrm{Id}) > 0$.

Proof. The claim from (i) is an immediate consequence of the Riesz representation theorem applied to the continuous linear functional $L_2(\mathscr{U}_n) \to \mathbb{C},\ f \mapsto (\pi_S f)(U)$.

(ii) $K_U^S(V) = \pi_S(K_U^S)(V) = \langle K_U^S, K_V^S\rangle_{L_2} = \overline{\langle K_V^S, K_U^S\rangle_{L_2}} = \overline{K_V^S(U)}$ for all $V \in \mathscr{U}_n$.

(iii) Fix some $V \in \mathscr{U}_n$ and $f \in L_2(\mathscr{U}_n)$, and note first that $S^\perp$ is also $\mathscr{U}_n$-invariant. Then

$$(\mathrm{Id} - \pi_S)(f) \circ L_V \in S^\perp \quad \text{and} \quad f \circ L_V = \pi_S(f) \circ L_V + (\mathrm{Id} - \pi_S)(f) \circ L_V,$$

and hence

$$\pi_S(f \circ L_V) = \pi_S(\pi_S(f) \circ L_V) + \pi_S((\mathrm{Id} - \pi_S)(f) \circ L_V) = \pi_S(f) \circ L_V. \tag{36}$$

Then

$$\langle f, K_{VU}^S\rangle_{L_2} = \pi_S(f)(VU) = \pi_S(f) \circ L_V(U) = \pi_S(f \circ L_V)(U),$$

and thus, by (i),

$$\langle f, K_{VU}^S\rangle_{L_2} = \langle f \circ L_V, K_U^S\rangle_{L_2} = \langle C_{L_V} f, K_U^S\rangle_{L_2} = \langle f, C_{L_{V^{-1}}} K_U^S\rangle_{L_2} = \langle f, K_U^S \circ L_{V^{-1}}\rangle_{L_2}.$$

Since $f \in L_2(\mathscr{U}_n)$ was chosen arbitrarily, we obtain that $K_{VU}^S = K_U^S \circ L_{V^{-1}}$. The other identity follows similarly.

(iv) Let $V \in \mathscr{U}_n$. Then

$$K_V^S(V) = \langle K_V^S, K_V^S\rangle_{L_2} = \langle K_{\mathrm{Id}}^S \circ L_{V^{-1}}, K_V^S\rangle_{L_2} = \langle K_{\mathrm{Id}}^S, K_V^S \circ L_V\rangle_{L_2} = \langle K_{\mathrm{Id}}^S, K_{\mathrm{Id}}^S\rangle_{L_2} = K_{\mathrm{Id}}^S(\mathrm{Id}) > 0. \quad \square$$

It remains to prove Theorem 2.3. Defining

$$\mathrm{t}_S := K_{\mathrm{Id}}^S, \tag{37}$$

this proof is in fact a straightforward consequence of the preceding lemma. But before we do this, we collect two elementary properties of the kernel t_S.

Remark 2.5. Let S be a $\mathscr{U}_n$-invariant subspace of $C(\mathscr{U}_n)$ which is closed in $L_2(\mathscr{U}_n)$. Then $\mathrm{t}_S = K_{\mathrm{Id}}^S$ satisfies

- $\mathrm{t}_S(V^*) = \overline{\mathrm{t}_S(V)}$ for all $V \in \mathscr{U}_n$,
- $\mathrm{t}_S(V^*UV) = \mathrm{t}_S(U)$ for all $U, V \in \mathscr{U}_n$; that is, t_S is a so-called class function.

Indeed, for the first equality, note that

$$\mathrm{t}_S(V^*) = (K^S_{\mathrm{Id}} \circ L_{V^{-1}})(\mathrm{Id}) = K^S_V(\mathrm{Id}) = \overline{K^{\mathrm{Id}}_V(S)} = \overline{\mathrm{t}_S(V)},$$

and together with this we get

$$\begin{aligned}\mathrm{t}_S(V^{-1}UV) &= \overline{\mathrm{t}_S(V^{-1}U^*V)} = \overline{(K^S_{\mathrm{Id}} \circ L_{V^{-1}})(U^*V)}\\ &= \overline{K^S_V(U^*V)} = K^S_{U^*V}(V) = K^S_{\mathrm{Id}} \circ R_{V^{-1}U}(V) = \mathrm{t}_S(U).\end{aligned}$$

Proof of Theorem 2.3. By Lemma 2.4, for all $U \in \mathscr{U}_n$ and $f \in L_2(\mathscr{U}_n)$,

$$\begin{aligned}(\pi_S f)(U) &= \int_{\mathscr{U}_n} f(V)\overline{K^S_U(V)}\,dV = \int_{\mathscr{U}_n} f(V)K^S_V(U)\,dV\\ &= \int_{\mathscr{U}_n} f(V)K^S_{\mathrm{Id}}(UV^{-1})\,dV = \int_{\mathscr{U}_n} f(V)\mathrm{t}_S(UV^*)\,dV = (f * \mathrm{t}_S)(U),\end{aligned}$$

which proves (i). Statement (ii) was already shown in (36), and it remains to check (iii). Obviously, we have that

$$\|\pi_S : C(\mathscr{U}_n) \to S\| = \sup_{U\in\mathscr{U}_n} \int_{\mathscr{U}_n} |\mathrm{t}_S(UV^*)|\,dV,$$

and, for every $U \in \mathscr{U}_n$ by Remark 2.5,

$$\int_{\mathscr{U}_n} |\mathrm{t}_S(UV^*)|\,dV = \int_{\mathscr{U}_n} |\mathrm{t}_S(V^*)|\,dV = \int_{\mathscr{U}_n} |\mathrm{t}_S(V)|\,dV.$$

This completes the argument. □

2.3. ***Accessibility.*** Let S a be $\mathscr{U}_n$-invariant subspace of $C(\mathscr{U}_n)$ which is closed in $L_2(\mathscr{U}_n)$. Then S is called *accessible* if every projection Q on $C(\mathscr{U}_n)$ onto S which commutes with the action of $\mathscr{U}_n \times \mathscr{U}_n$ on $C(\mathscr{U}_n)$ equals $\pi_S|_{C(\mathscr{U}_n)}$.

Theorem 2.6. *Let S be a $\mathscr{U}_n$-invariant and accessible subspace of $C(\mathscr{U}_n)$ which is closed in $L_2(\mathscr{U}_n)$. Then*

$$\boldsymbol{\lambda}(S) = \|\pi_S : C(\mathscr{U}_n) \to S\| = \int_{\mathscr{U}_n} |\mathrm{t}_S(V)|\,dV.$$

Proof. The proof is an immediate consequence of Rudin's Theorem 2.2 and the assumptions on S, taking into account that we know (ii) and (iii) from Theorem 2.3 as well as (1). □

We say that a $\mathscr{U}_n$-invariant subspace S of $C(\mathscr{U}_n)$ which is closed in $L_2(\mathscr{U}_n)$ is *strongly accessible* whenever every $f \in S$ for which $f(VUV^*) = f(U)$ for all $U, V \in \mathscr{U}_n$ is a scalar multiple of t_S. In other words, every class function in S is a multiple of t_S.

As the name in the previous definition suggests, we have the following key result.

Proposition 2.7. *Let S be a $\mathscr{U}_n$-invariant subspace of $C(\mathscr{U}_n)$ which is closed in $L_2(\mathscr{U}_n)$. Then S is accessible whenever it is strongly accessible.*

The proof requires the next statement.

Lemma 2.8. *Let H and S be $\mathscr{U}_n$-invariant subspaces of $C(\mathscr{U}_n)$ which are both closed in $L_2(\mathscr{U}_n)$. Then, if S is strongly accessible, every operator $T : H \to S$ that commutes with the action of $\mathscr{U}_n \times \mathscr{U}_n$ on $C(\mathscr{U}_n)$ is a scalar multiple of $\pi_S|_H$.*

Moreover, if H is orthogonal to S and Q is a projection on $H \oplus S$ onto S that commutes with the action of $\mathscr{U}_n \times \mathscr{U}_n$ on $C(\mathscr{U}_n)$, then $Q = \pi_S|_{H\oplus S}$.

Proof. By the assumption on T and Lemma 2.4 (iii), for every $V \in \mathscr{U}_n$,

$$(C_{L_V} \circ C_{R_{V^{-1}}})(T\mathrm{t}_H) = T((C_{L_V} \circ C_{R_{V^{-1}}})\mathrm{t}_H) = Tt_H.$$

This implies that $(T\mathrm{t}_H)(V^*UV) = (T\mathrm{t}_H)(U)$ for all $U, V \in \mathscr{U}_n$. Since S is strongly accessible, we have that $T\mathrm{t}_H = \gamma \mathrm{t}_S$ for some $\gamma \in \mathbb{C}$. But from Theorem 2.3 we know that, for all $h \in H$,

$$h = \pi_H h = h * \mathrm{t}_H,$$

and hence

$$Th = h * T\mathrm{t}_H = \gamma h * \mathrm{t}_S = \gamma \pi_S h.$$

To see the second assertion, note that, by the first part of the lemma, we have $Q|_H = \gamma \pi_S|_H$ for some $\gamma \in \mathbb{C}$. But since by assumption $H \subset S^\perp$, this implies $Q|_H = 0 = \pi_S|_H$. On the other hand, since Q is a projection onto S, we see that $Q|_S = \mathrm{Id}_S = \pi_S|_S$, which finishes the proof. □

We are now ready to give the following.

Proof of Proposition 2.7. Let Q be a projection on $C(\mathscr{U}_n)$ onto S which commutes with the action of $\mathscr{U}_n \times \mathscr{U}_n$ on $C(\mathscr{U}_n)$. By Theorem 1.8, it suffices to show that, for each pair $(p, q) \in \mathbb{N}_0 \times \mathbb{N}_0$,

$$Q|_{\mathfrak{H}_{(p,q)}} = \pi_S|_{\mathfrak{H}_{(p,q)}}.$$

Given such a pair (p, q), we define the subspace

$$H := \{f - \pi_S f : f \in \mathfrak{H}_{(p,q)}\} \subset C(\mathscr{U}_n).$$

Then H is $\mathscr{U}_n$-invariant; indeed, by Theorem 2.3 (ii) and the fact that $\mathfrak{H}_{(p,q)}$ is $\mathscr{U}_n$-invariant (proved in Lemma 1.7), for every $f \in \mathfrak{H}_{(p,q)}$ and $U \in \mathscr{U}_n$, we have

$$(f - \pi_S f) \circ L_U = f \circ L_U - \pi_S f \circ L_U = f \circ L_U - \pi_S(f \circ L_U) \in H,$$

and the invariance under right multiplication follows similarly. Since $H \perp S$ and Q commutes with the action of $\mathscr{U}_n \times \mathscr{U}_n$ on $C(\mathscr{U}_n)$, Lemma 2.8 (the second part applied to the restriction of Q to $H \oplus S$) shows that

$$Q|_{H\oplus S} = \pi_S|_{H\oplus S},$$

so in particular $Q|_H = \pi_S|_H = 0$. But then, for every $f \in \mathfrak{H}_{(p,q)}(\mathscr{U}_n)$,

$$Q(f) = Q(f - \pi_S f) + Q(\pi_S f) = \pi_S f,$$

which completes the argument. □

2.4. ***The special case $S = \mathfrak{H}_{(1,0)}(\mathscr{U}_n)$.*** Recall from Section 1.4 the definition of the $\mathscr{U}_n$-invariant subspace $\mathfrak{H}_{(1,0)}(\mathscr{U}_n)$ of $C(\mathscr{U}_n)$ of all polynomials $f \in C(\mathscr{U}_n)$ of the form

$$f(U) = \sum_{1 \le i,j \le n} c_{i,j} u_{i,j},$$

where $U = (u_{i,j})_{1 \le i,j \le n} \in \mathscr{U}_n$.

In Theorem 2.3 we showed that the orthogonal projection $\pi_{(1,0)} = \pi_{\mathfrak{H}_{(1,0)}(\mathscr{U}_n)}$ on $L_2(\mathscr{U}_n)$ onto $\mathfrak{H}_{(1,0)}(\mathscr{U}_n)$ is a convolution operator with respect to the kernel $t_{(1,0)} = t_{\mathfrak{H}_{(1,0)}(\mathscr{U}_n)}$. We need an alternative description of this projection in terms of the canonical orthonormal basis of $\mathfrak{H}_{(1,0)}(\mathscr{U}_n)$.

By (8), the collection of all normalised functions $\sqrt{n}e_{ij}$, $1 \le i, j \le n$, forms an orthonormal system in $L_2(\mathscr{U}_n)$, and hence an orthonormal basis of $\mathfrak{H}_{(1,0)}(\mathscr{U}_n)$ considered as a subspace of $L_2(\mathscr{U}_n)$. Consequently, for each $f \in L_2(\mathscr{U}_n)$,

$$\pi_{(1,0)}(f) = \sum_{1 \le i,j \le n} \langle f, \sqrt{n}e_{ij}\rangle_{L_2} \sqrt{n}e_{ij} = n \sum_{1 \le i,j \le n} \langle f, e_{ij}\rangle_{L_2} e_{ij}, \tag{38}$$

where $e_{ij} \in \mathfrak{H}_{(1,0)}(\mathscr{U}_n)$ is defined by $e_{ij}(U) = u_{i,j}$ for $U \in \mathscr{U}_n$.

Comparing the two representations of $\pi_{(1,0)}$ we now have leads to the following.

Proposition 2.9. *For each $n \in \mathbb{N}$, we have $\mathrm{t}_{(1,0)} = n\,\mathrm{tr}$, and moreover*

$$\pi_{(1,0)} f = n(f * \mathrm{tr}), \quad f \in L_2(\mathscr{U}_n),$$

and

$$\|\pi_{(1,0)} : C(\mathscr{U}_n) \to \mathfrak{H}_{(1,0)}(\mathscr{U}_n)\| = n \int_{\mathscr{U}_n} |\mathrm{tr}(V)|\, dV.$$

Proof. To check the equality $\mathrm{t}_{(1,0)} = n\,\mathrm{tr}$, recall that, by Lemma 2.4 (i) and the definition of $\mathrm{t}_{(1,0)}$ from (37), for all $f \in L_2(\mathscr{U}_n)$, one gets

$$(\pi_{(1,0)} f)(\mathrm{Id}) = \langle f, \mathrm{t}_{(1,0)}\rangle_{L_2}.$$

On the other hand, by (38), for all $f \in L_2(\mathscr{U}_n)$,

$$(\pi_{(1,0)} f)(\mathrm{Id}) = n \sum_{i,j} \langle f, e_{ij}\rangle_{L_2} e_{ij}(\mathrm{Id}) = n \sum_{i} \langle f, e_{ii}\rangle_{L_2} = n\langle f, \mathrm{tr}\rangle_{L_2},$$

which together with the preceding equality is what we were looking for. Deducing the second and third claim is then immediate from Theorem 2.3 (iii). □

Proposition 2.10. *$\mathfrak{H}_{(1,0)}(\mathscr{U}_n)$ is a strongly accessible $\mathscr{U}_n$-invariant subspace of $C(\mathscr{U}_n)$.*

Proof. Take $f = \sum_{1 \le i,j \le n} c_{i,j} e_{i,j} \in \mathfrak{H}_{(1,0)}(\mathscr{U}_n)$ such that $f(V^{-1}UV) = f(U)$ for every $U, V \in \mathscr{U}_n$. Clearly, f can be considered as a linear functional on M_n. This implies that there exists $A \in M_n$ such that $f(U) = \mathrm{tr}(AU)$ for all $U \in M_n$. Then, from the assumption on f, it follows that, for all $U, V \in \mathscr{U}_n$,

$$\mathrm{tr}(AU) = f(U) = f(V^{-1}UV) = \mathrm{tr}(AV^{-1}UV) = \mathrm{tr}(VAV^{-1}U).$$

Combining this with the fact that any matrix in M_n is a linear combination of unitary matrices, we deduce that $A = VAV^{-1}$ for every $V \in \mathscr{U}_n$, and so A commutes with all matrices in M_n. This implies that $A = \gamma \mathrm{Id}$ for some $\gamma \in \mathbb{C}$, and hence as desired $f = \gamma\, \mathrm{tr}$. □

A comment is in order: if $p+q > 1$, then $g_1(A) := \mathrm{tr}(A^p(A^*)^q)$ and $g_2(A) := \mathrm{tr}(A)^p\, \mathrm{tr}(A^*)^q$ are different class functions. Thus, in this case, $\mathfrak{H}_{(p,q)}(\mathscr{U}_n)$ is not strongly accessible.

The following result identifies $\mathfrak{H}_{(1,0)}(\mathscr{U}_n)$ with the trace class $\mathscr{S}_1(n)$.

Proposition 2.11. *The space $\mathfrak{H}_{(1,0)}(\mathscr{U}_n)$ is isometrically isomorphic to $\mathscr{S}_1(n)$. More precisely,*

$$\mathscr{S}_1(n) \to \mathfrak{H}_{(1,0)}(\mathscr{U}_n), \quad A \mapsto [f : U \mapsto \mathrm{tr}(AU)], \tag{39}$$

is a surjective isometry.

Proof. Obviously, the mapping in (39) is a linear bijection. Indeed, as a linear space $\mathscr{S}_1(n)$ equals M_n, and $\mathfrak{H}_{(1,0)}(\mathscr{U}_n)$ equals the algebraic dual $M_n^\times$ of M_n. Moreover, it is well known that the mapping $A \mapsto [f : U \mapsto \mathrm{tr}(AU)]$ identifies M_n and $M_n^\times$. So it remains to prove that the mapping in (39) is isometric. To prove this, we use a result of Nelson [1961] (see also [Harris 1997, Theorem 1]) showing that, for any complex-valued function f which is continuous on the closed and analytic on the open unit ball of $\mathscr{L}(\ell_2^n)$, we have $\sup_{\|T\|\le 1} |f(T)| = \sup_{U\in\mathscr{U}_n} |f(U)|$. But then, by (7), for every $A \in \mathscr{S}_1(n)$,

$$\|A\|_1 = \sup_{\|T\|\le 1} |\mathrm{tr}(AT)| = \sup_{U\in\mathscr{U}_n} |\mathrm{tr}(AU)|,$$

completing the argument. □

2.5. ***Proof of the main result.*** We begin with the following presentation.

Proof of the integral formula from (32). We use the identification from Proposition 2.11 and combine it with Proposition 2.10 and Theorem 2.6. Then Proposition 2.9 completes the argument. □

Now we deal with the limit formula from (33). For this we need to recall some well-known results from probability theory; for more on this see [Billingsley 1999]. We are going to use that, given any sequence (Y_n) of random variables which converges in distribution to the random variable Y and any continuous real-valued function f, the sequence $(f(Y_n))$ converges in distribution to $f(Y)$. Recall also that a sequence $(Y_n)_n$ of random variables is said to be uniformly integrable whenever

$$\lim_{a\to\infty} \sup_{n\ge 1} \int_{|Y_n|\ge a} |Y_n|\, dP = 0.$$

Uniform integrability will be useful for us due to the fact (see for example [Billingsley 1999, Theorem 3.5]) that if $(Y_n)_n$ is a uniformly integrable sequence of random variables and $Y_n \xrightarrow{D} Y$, then Y is integrable and

$$\mathbb{E}(Y_n) \to \mathbb{E}(Y). \tag{40}$$

To check uniform integrability we cite a standard criterion.

Remark 2.12. If $\sup_n \mathbb{E}(|Y_n|^{1+\varepsilon}) \leq C$ for some $\varepsilon, C > 0$, then $(Y_n)_n$ is uniformly integrable; indeed, this is a consequence of

$$\lim_{a\to\infty} \sup_{n\geq 1} \int_{|Y_n|\geq a} |Y_n|\, dP \leq \lim_{a\to\infty} \frac{1}{a^{\varepsilon}} C.$$

We are now ready to provide the following.

Proof of the limit formula from (33). Consider the sequence $(\mathrm{tr}(U(n)))$ of random variables on $\mathscr{U}_n$, where $U(n)$ is a unitary matrix uniformly Haar distributed. Then, by [Johansson 1997, Corollary 2.4] (see also [Diaconis and Shahshahani 1994] or [Pastur and Shcherbina 2011, Problem 8.5.5]), the previous sequence converges in distribution to the standard Gaussian complex random variable $\boldsymbol{\gamma}$. Indeed, the random variables $\sqrt{2}\,\mathrm{Re}[\mathrm{tr}(U(n))]$ and $\sqrt{2}\,\mathrm{Im}[\mathrm{tr}(U(n))]$ converge in distribution to a standard real Gaussian random variable.

Thus, the sequence $(\sqrt{2}|\mathrm{tr}(U(n))|)$ of random variables on $\mathscr{U}_n$ converges in distribution to a Rayleigh random variable. Moreover, since, as mentioned in (9), for each n,

$$\mathbb{E}(|\mathrm{tr}(U(n))|^2) = \int_{\mathscr{U}_n} |\mathrm{tr}(V)|^2\, dV = 1,$$

the sequence of random variables $\mathrm{tr}(U(n))$ by Remark 2.12 is uniformly integrable. Consequently, we deduce from (40) that $(\mathbb{E}(\sqrt{2}|\mathrm{tr}(U(n))|))$ converges to the expectation of a Rayleigh random variable. That is,

$$\lim_{n\to\infty} \mathbb{E}(\sqrt{2}|\mathrm{tr}(U(n))|) \to \sqrt{\frac{\pi}{2}}.$$

Using (32), we arrive at

$$\lim_{n\to\infty} \frac{1}{n} \boldsymbol{\lambda}(\mathscr{S}_1(n)) = \frac{1}{\sqrt{2}} \lim_{n\to\infty} \mathbb{E}(\sqrt{2}|\mathrm{tr}(U(n))|) = \frac{\sqrt{\pi}}{2},$$

which completes the proof. □

2.6. ***Other examples.*** In this final subsection, we give some other examples where the theory developed to reach our main objective (Theorem 2.1) could be applied.

The first result shows that examples of accessible $\mathscr{U}_n$-invariant subspaces come in pairs. To see this we define the linear and isometric bijection

$$\phi : C(\mathscr{U}_n) \to C(\mathscr{U}_n), \quad f \mapsto [U \mapsto f(U^*)].$$

For any subspace S in $C(\mathscr{U}_n)$, we write $S_* := \phi S$. As a first example we mention that, isometrically,

$$(\mathfrak{H}_{(1,0)}(\mathscr{U}_n))_* = \phi(\mathfrak{H}_{(1,0)}(\mathscr{U}_n)) = \mathfrak{H}_{(0,1)}(\mathscr{U}_n).$$

Proposition 2.13. *Let S be a $\mathscr{U}_n$-invariant subspace of $C(\mathscr{U}_n)$ which is closed in $L_2(\mathscr{U}_n)$. Then S_* is $\mathscr{U}_n$-invariant and $\mathrm{t}_{S_*} = \overline{\mathrm{t}_S}$. Moreover, S is strongly accessible* (*resp. accessible*) *if and only if S_* is strongly accessible* (*resp. accessible*).

Proof. Obviously, S_* is $\mathscr{U}_n$-invariant. In order to show that $\mathrm{t}_{S_*} = \overline{\mathrm{t}_S}$ note first that $\pi_{S_*} = \phi \circ \pi_S \circ \phi$. Then, for every $f \in L_2(\mathscr{U}_n)$ and $U \in \mathscr{U}_n$, it follows by Theorem 2.3 and Remark 2.5 that

$$\begin{aligned}(\pi_{S_*} f)(U) &= ((\pi_S \phi f))(U^*) \\ &= (\phi f * \mathrm{t}_S)(U^*) = \int_{\mathscr{U}_n} f(V^*)\mathrm{t}_S(U^*V^*)\,dV \\ &= \int_{\mathscr{U}_n} f(V^*)\overline{\mathrm{t}_S}(VU)\,dV = \int_{\mathscr{U}_n} f(V)\overline{\mathrm{t}_S}(V^*U)\,dV \\ &= \int_{\mathscr{U}_n} f(V)\overline{\mathrm{t}_S}(UV^*)\,dV = (f * \overline{\mathrm{t}_S})(U),\end{aligned}$$

which by the uniqueness of t_{S_*} leads to the claim. Let us turn to the "moreover part". It is immediate that strong accessibility of S is equivalent to strong accessibility of S_*. So let us assume that S is accessible and show that then S_* is accessible. Take any projection $Q : C(\mathscr{U}_n) \to S_*$ which commutes with the action of $\mathscr{U}_n \times \mathscr{U}_n$ on $C(\mathscr{U}_n)$. Since $\phi \circ C_{L_V} = R_{V^*} \circ \phi$ and $\phi \circ C_{R_V} = L_{V^*} \circ \phi$ for all $V \in \mathscr{U}_n$, the projection $\phi \circ Q \circ \phi$ onto S commutes with the action of $\mathscr{U}_n \times \mathscr{U}_n$ on $C(\mathscr{U}_n)$, and hence by assumption $\phi \circ Q \circ \phi = \pi_S$. But then clearly $Q = \phi \circ \pi_S \circ \phi = \pi_{S_*}$, which is the desired conclusion. ☐

Note that, in particular, $\mathfrak{H}_{(0,1)}(\mathscr{U}_n)$ is $\mathscr{U}_n$-invariant and accessible, and $\mathrm{t}_{(0,1)} = \overline{\mathrm{tr}}$; therefore, by Theorem 2.6,

$$\boldsymbol{\lambda}(\mathfrak{H}_{(0,1)}(\mathscr{U}_n)) = \|\pi_{(0,1)} : C(\mathscr{U}_n) \to \mathfrak{H}_{(0,1)}(\mathscr{U}_n)\| = n \int_{\mathscr{U}_n} |\mathrm{tr}(V)|\,dV.$$

Also,

$$\lim_{n\to\infty} \frac{\boldsymbol{\lambda}(\mathfrak{H}_{(0,1)}(\mathscr{U}_n))}{n} = \frac{\sqrt{\pi}}{2}.$$

Of course, this is also a simple consequence of Theorem 2.1 using that ϕ identifies $\mathfrak{H}_{(0,1)}(\mathscr{U}_n)$ and $\mathfrak{H}_{(1,0)}(\mathscr{U}_n)$ isometrically.

We continue with another simple stability property of accessible subspaces.

Proposition 2.14. *Let S_1 and S_2 be accessible, $\mathscr{U}_n$-invariant subspaces of $C(\mathscr{U}_n)$ which in $L_2(\mathscr{U}_n)$ are closed and orthogonal. Then $S_1 \oplus S_2$ is accessible and $\mathscr{U}_n$-invariant, and moreover $t_{S_1 \oplus S_2} = \mathrm{t}_{S_1} + \mathrm{t}_{S_2}$. Consequently,*

$$\boldsymbol{\lambda}(S_1 \oplus S_2) = \|\pi_{S_1} + \pi_{S_2} : C(\mathscr{U}_n) \to S_1 \oplus S_2\| = \int_{\mathscr{U}_n} |\mathrm{t}_{S_1}(V) + \mathrm{t}_{S_2}(V)|\,dV. \tag{41}$$

Proof. That $S_1 \oplus S_2$ is $\mathscr{U}_n$-invariant is straightforward. Note that $\pi_{S_1 \oplus S_2} = \pi_{S_1} + \pi_{S_2}$ is the orthogonal projection on $L_2(\mathscr{U}_n)$ onto $S_1 \oplus S_2$. Then, by Theorem 2.3, for all $f \in L_2(\mathscr{U}_n)$, we have

$$\pi_{S_1 \oplus S_2} f = \pi_{S_1} f + \pi_{S_2} f = f * \mathrm{t}_{S_1} + f * \mathrm{t}_{S_2} = f * (\mathrm{t}_{S_1} + \mathrm{t}_{S_2}).$$

Hence by the uniqueness of $t_{S_1 \oplus S_2}$, we get

$$\mathrm{t}_{S_1 \oplus S_2} = \mathrm{t}_{S_1} + \mathrm{t}_{S_2}.$$

Let us now show that $S_1 \oplus S_2$ is accessible. So let Q be a projection on $C(\mathscr{U}_n)$ onto $S_1 \oplus S_2$ which commutes with the action of $\mathscr{U}_n \times \mathscr{U}_n$ on $C(\mathscr{U}_n)$. We claim that $Q = \pi_{S_1 \oplus S_2}$. Indeed, consider the two projections

$$Q_{S_1} = \pi_{S_1} \circ Q \quad \text{and} \quad Q_{S_2} = \pi_{S_2} \circ Q$$

on $C(\mathscr{U}_n)$ onto S_1 and S_2, respectively. Since π_{S_1} and π_{S_2} both commute with the action of $\mathscr{U}_n \times \mathscr{U}_n$ on $C(\mathscr{U}_n)$, we have that Q_{S_1} and Q_{S_2} also do. Then, by the accessibility of S_1 and S_2, we see that

$$Q_{S_1} = \pi_{S_1} \quad \text{and} \quad Q_{S_2} = \pi_{S_2},$$

and hence, for all $f \in C(\mathscr{U}_n)$, as desired,

$$Qf = \pi_{S_1}(Qf) + \pi_{S_2}(Qf) = \pi_{S_1} f + \pi_{S_2} f = \pi_{S_1 \oplus S_2} f.$$

To conclude the proof just note that (41) is then a direct consequence of Theorem 2.6. □

Combining the previous two propositions we obtain the following.

Corollary 2.15. *For each $n \in \mathbb{N}$,*

$$\begin{aligned}\boldsymbol{\lambda}(\mathfrak{H}_{(1,0)}(\mathscr{U}_n) \oplus \mathfrak{H}_{(0,1)}(\mathscr{U}_n)) &= \|\pi_{(1,0)} \oplus \pi_{(0,1)} : C(\mathscr{U}_n) \to \mathfrak{H}_{(1,0)}(\mathscr{U}_n) \oplus \mathfrak{H}_{(0,1)}(\mathscr{U}_n)\| \\ &= 2n \int_{\mathscr{U}_n} |\mathrm{Re}(\mathrm{tr}(V))| \, dV.\end{aligned}$$

Moreover,

$$\lim_{n\to\infty} \frac{\boldsymbol{\lambda}(\mathfrak{H}_{(1,0)}(\mathscr{U}_n) \oplus \mathfrak{H}_{(0,1)}(\mathscr{U}_n))}{\sqrt{2}n} = \sqrt{\frac{2}{\pi}}. \tag{42}$$

Before giving a proof of this, we mention that the denominator of the fraction above (so $\sqrt{2}n$) is exactly the square root of the dimension of the sum space $\mathfrak{H}_{(1,0)}(\mathscr{U}_n) \oplus \mathfrak{H}_{(0,1)}(\mathscr{U}_n)$.

Proof of Corollary 2.15. We only have to prove (42) since the integral formula for the projection constant follows directly from (41) and Proposition 2.9.

We repeat an argument similar to the proof of (33). We know, by [Johansson 1997, Corollary 2.4], that the sequence $(\sqrt{2}\,\mathrm{Re}[\mathrm{tr}(U(n))])$ of random variables converges in distribution to a standard real Gaussian random variable g. In particular, $(\sqrt{2}|\mathrm{Re}[\mathrm{tr}(U(n))]|)$ converges in distribution to $|g|$. Note that the sequence $(\sqrt{2}|\mathrm{Re}[\mathrm{tr}(U(n))]|)$ is uniformly integrable. Indeed,

$$\mathbb{E}(|\mathrm{Re}[\mathrm{tr}(U(n))]|^2) \le \mathbb{E}(|\mathrm{tr}(U(n))|^2) = 1;$$

see again Remark 2.12. Thus, by (40),

$$\begin{aligned}\lim_{n\to\infty} \frac{\boldsymbol{\lambda}(\mathfrak{H}_{(1,0)}(\mathscr{U}_n) \oplus \mathfrak{H}_{(0,1)}(\mathscr{U}_n))}{\sqrt{2}n} &= \lim_{n\to\infty} \mathbb{E}(\sqrt{2}|\mathrm{Re}[\mathrm{tr}(U(n))]|) \\ &= \mathbb{E}|g| = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} |x| e^{-x^2/2} \, dx = \sqrt{\frac{2}{\pi}}.\end{aligned}$$

□

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