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JEAN-CLAUDE CUENIN AND KONSTANTIN MERZ

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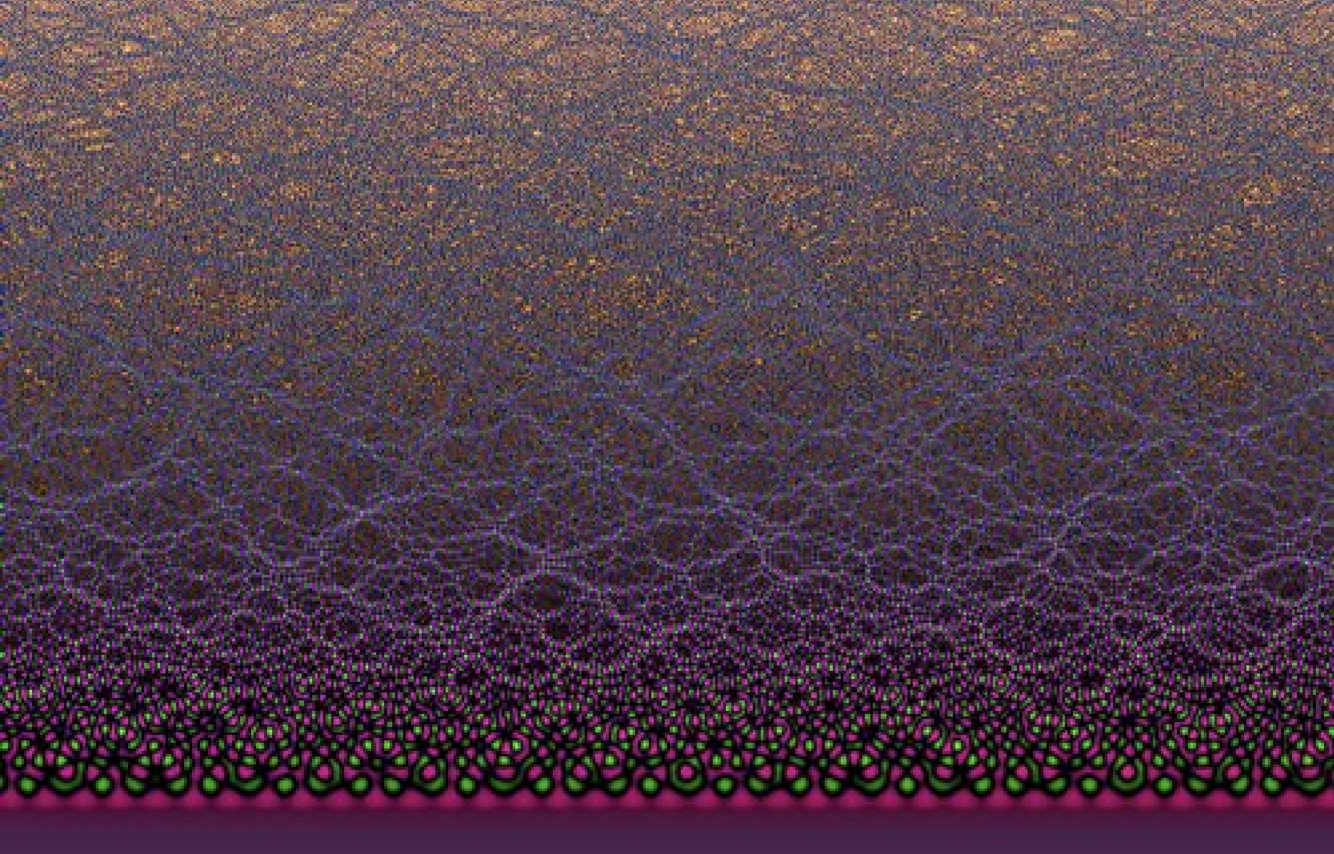

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RANDOM SCHRÖDINGER OPERATORS WITH COMPLEX DECAYING POTENTIALS

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We prove that the eigenvalues of a continuum random Schrödinger operator $-\Delta + V_\omega$ of Anderson-type, with complex decaying potential, can be bounded (with high probability) in terms of an L^q norm of the potential for all $q \le d+1$. This shows that, in the random setting, the exponent q can be essentially doubled compared to the deterministic bounds of Frank (*Bull. Lond. Math. Soc.* **43**:4 (2011), 745–750). This improvement is based on ideas of Bourgain (*Discrete Contin. Dyn. Syst.* **8**:1, (2002), 1–15) related to almost-sure scattering for lattice Schrödinger operators.

1. Introduction and main result

Consider a Schrödinger operator $-\Delta + V$ on $L^2(\mathbb{R}^d)$. Frank [2011] proved the scale-invariant bounds

$$|z|^{q-d/2} \lesssim \int_{\mathbb{R}^d} |V(x)|^q \, \mathrm{d}x \tag{1}$$

for eigenvalues z of $-\Delta + V$, when $q \le \frac{1}{2}(d+1)$ (we call such V short range). The short range condition is best possible, i.e., (1) is generally not true for $q > \frac{1}{2}(d+1)$. Counterexamples for $z > 0$ were constructed by Frank and Simon [2017], and for $\operatorname{Im} z \neq 0$ by Bögli and the first author [Bögli and Cuenin 2023]. These counterexamples settle the Laptev–Safronov conjecture [Laptev and Safronov 2009] in the negative.

The aim of this paper is to show that, for random potentials, the short range exponent can be essentially doubled, from $\frac{1}{2}(d+1)$ to $d+1$, compared to the deterministic case. We consider Anderson-type Schrödinger operators of the form $-\Delta + V_\omega$, where

$$V_\omega(x) = \sum_{j \in h\mathbb{Z}^d} \omega_j v_j \mathbf{1}_Q((x-j)/h), \qquad Q = [0,1)^d, \quad h > 0. \tag{2}$$

More generally, given a deterministic potential V, consider its randomization at scale $h > 0$, given by

$$V_\omega(x) = \sum_{j \in h\mathbb{Z}^d} \omega_j V(x) \mathbf{1}_Q((x-j)/h). \tag{3}$$

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One could also replace $\mathbf{1}_{Q_j}$ with some rapidly decaying function. Note that, in both cases (2) and (3), the L^q norm of V_ω is deterministic:

$$\|V_\omega\|_{L^q(\mathbb{R}^d)} = \biggl(h^d \sum_{j\in h\mathbb{Z}^d} |v_j|^q\biggr)^{1/q}, \tag{4}$$

where v_j is the L^q-average of V over $j+hQ$ in the general case (3), and we have $\|V_\omega\|_{L^q(\mathbb{R}^d)} = \|V\|_{L^q(\mathbb{R}^d)}$. For this reason, we also denote the norm (4) by $\|V\|_{L^q(\mathbb{R}^d)}$ in case (2). Crucially, we assume that $(\omega_j)_{j\in h\mathbb{Z}^d}$ are *independent*, *mean-zero Gaussian or symmetric Bernoulli random variables* (real- or complex-valued). In the following, V_ω will always denote the randomization (3) of a given deterministic potential V, and $\langle x\rangle = 2+|x|$. The standard assumptions on the local singularities of $V \in L^q(\mathbb{R}^d)$,

$$q \ge 1 \quad \text{if } d=1, \qquad q>1 \quad \text{if } d=2, \qquad q \ge \tfrac{1}{2}d \quad \text{if } d \ge 3, \tag{5}$$

ensure that $-\Delta + V$ can be defined as an m-sectorial operator. These can be slightly weakened (see Remark 2 (ii)) and only play a minor role here. In contrast, the average decay of the potential (i.e., an upper bound on q) — to be stated in the assumptions of the following theorems — is of central importance. We tacitly assume $q > \frac{1}{2}(d+1)$, since the case $q \le \frac{1}{2}(d+1)$ is already covered by the deterministic bound (1).

Theorem 1. *There exist constants M_0, $c>0$ such that the following holds. For any R, $\lambda>0$, $0<h<R$, $|\varepsilon| \ll \lambda$, $q \le d+1$, for any $V \in L^q(\mathbb{R}^d)$ supported in a ball of radius R, and, for any $M \ge M_0$, each eigenvalue $z = (\lambda + \mathrm{i}\varepsilon)^2$ of $-\Delta + V_\omega$ satisfies*

$$\frac{\lambda^{2-d/q}}{\langle \lambda h\rangle^{d/2}(\log\langle \lambda R\rangle)^{7/2}} \le M\|V\|_{L^q(\mathbb{R}^d)},$$

except for ω in a set of measure at most $\exp(-cM^2)$.

Remark 2. (i) Outside the set $\lambda > 0$, $|\varepsilon| \ll \lambda$, obvious estimates (as in the case of real potentials) are available. These even hold for sums of powers of eigenvalues as in the classical Lieb–Thirring inequalities; see Frank, Laptev, Lieb, and Seiringer [Frank et al. 2006].

(ii) As in [Cuenin and Merz 2021] (see also [Ionescu and Schlag 2006]), one could weaken the local singularity assumption to $V \in L^{q_0}_{\mathrm{loc}}(\mathbb{R}^d)$, with q_0 satisfying (5), and then replace $\|V\|_{L^q(\mathbb{R}^d)}$ by the right-hand side of (4), where v_j is now the L^{q_0}-average of V over $j+hQ$.

Remark 3. There are three scales in the problem:

- the energy scale λ^2,
- the scale R measuring the support of the potential,
- the randomization scale $h < R$.

In addition, we have introduced an arbitrary (dimensionless) parameter M that appears in the large deviation bound. There is a separation of scales at $\lambda h = 1$ (and to a lesser extent at $\lambda R = 1$ but we ignore logarithms for the purpose of this remark). All eigenvalues with $|z|^{1/2} \le h^{-1}$ are contained in a ball of radius proportional to $\|V\|_{L^q}^{q/(2q-d)}$. By Hölder's inequality, the deterministic bound (6) shows that $|z|^{1/2} \le h^{-1}$ is satisfied whenever $h \ll R^{d((d+1)/(2q)-1)}\|V\|_{L^q}^{-(d+1)/2}$.

Remark 4. Of course, a compactly supported potential of the form (2) is in any L^q space. The point of the estimate is the very weak dependence on R (logarithmic) compared to what one would get by using Hölder's inequality and the deterministic bound (1), namely

$$|z|^{1/(d+1)} \lesssim R^{d(2/(d+1)-1/q)} \|V\|_{L^q}. \tag{6}$$

Moreover, compactly supported potentials are interesting in view of the counterexample to the Laptev–Safronov conjecture of Bögli and the first author [Bögli and Cuenin 2023]. The counterexample yields a sequence of potentials V_ε, $\varepsilon > 0$ small, with $|V_\varepsilon| \lesssim \varepsilon\chi_\varepsilon$, where χ_ε is the indicator function of the tube

$$T_\varepsilon = \{(x_1, x') : |x_1| < \varepsilon^{-1},\ |x'| < \varepsilon^{-1/2}\}$$

such that $1 + i\varepsilon$ is an eigenvalue of $-\Delta + V_\varepsilon$. Since

$$\|V_\varepsilon\|_{L^q(\mathbb{R}^d)} \lesssim \varepsilon^{1-(d+1)/(2q)},$$

this shows that (1) cannot hold for $q > \frac{1}{2}(d+1)$. In this context, Theorem 1 says that, after randomization on the scale

$$h \leq [\varepsilon^{(d+1)/(2q)-1} \log(1/\varepsilon)^{-7/2}]^{2/d},$$

the counterexample for $\frac{1}{2}(d+1) < q \leq d+1$ is almost surely destroyed.

Remark 5. Safronov [2023] has recently considered eigenvalue sums for random Schrödinger operators with complex potentials of the same form as (2), but without the assumption on the distribution of ω_j. However, these results do not give any new information about individual eigenvalues beyond what is known in the deterministic case [Frank 2011; 2018]. Moreover, Safronov's results only apply to the smaller range $q < \frac{1}{2}(d+1) + 1/(2d-4)$ compared to $q \leq d+1$. Our results are of a quite different character and therefore a direct comparison is not possible.

The compact support assumption can be removed at the price of a tiny bit of pointwise decay.

Theorem 6. *For any* $\delta > 0$, *there exist constants* M_0, $c > 0$ *such that the following holds. For any* h, $\lambda > 0$, $|\varepsilon| \ll \lambda$, $q \leq d+1$, *for any* $V \in \langle x\rangle^{-\delta} L^q(\mathbb{R}^d)$, *and for any* $M \geq M_0$, *each eigenvalue* $z = (\lambda + \mathrm{i}\varepsilon)^2$ *of* $-\Delta + V_\omega$ *satisfies*

$$\frac{\lambda^{2-d/q}}{\langle\lambda h\rangle^{d/2}(\log\langle\lambda h\rangle)^2} \leq M \|\langle\lambda x\rangle^\delta V\|_{L^q(\mathbb{R}^d)},$$

except for ω *in a set of measure at most* $\exp(-cM^2)$.

In fact, if we sacrifice the endpoint, we can also remove the pointwise decay assumption.

Theorem 7. *For any* $q < d+1$, *there exist constants* M_0, $c > 0$ *such that the following holds. For any* h, $\lambda > 0$, $|\varepsilon| \ll \lambda$, *for any* $V \in L^q(\mathbb{R}^d)$, *and for any* $M \geq M_0$, *each eigenvalue* $z = (\lambda + \mathrm{i}\varepsilon)^2$ *of* $-\Delta + V_\omega$ *satisfies*

$$\frac{\lambda^{2-d/q}}{\langle\lambda h\rangle^{d/2}(\log\langle\lambda h\rangle)^2} \leq M \|V\|_{L^q(\mathbb{R}^d)},$$

except for ω *in a set of measure at most* $\exp(-cM^2)$.

Remark 8. (i) For fixed h (and up to logarithms) and large λ, the left-hand side of the inequality is $\lambda^{2-d/q-d/2}$. The exponent is negative for $d \geq 3$ and $q \leq d+1$. Therefore, the estimate gives no information about large eigenvalues in this case. We believe that a factor of the form $\langle \lambda h\rangle^{\kappa}$ in the denominator is unavoidable, and is an expression of the fact that randomization only occurs down to scale h but not below (meaning that at scale h and below, V is deterministic). The example in Appendix B suggests that one should expect a loss of at least $\kappa = \frac{1}{4}(d+1)$. Our method of proof only yields $\kappa = \frac{1}{2}d$. It is an interesting question whether this can be improved. It would also be interesting to study the case where V is random at all scales (i.e., V is a random field). In particular, under which assumptions on the randomization is the estimate true with $\kappa = 0$?

(ii) For $d = 2$ and $2 < q < 3$ the exponent is positive. Bounding the logarithm by an arbitrary small power of λh, we see that if $\lambda h \geq 1$, then

$$h^{-1} \leq \lambda \lesssim (M h^{1+\varepsilon} \|V\|_q)^{q/(q-2-\varepsilon q)},$$

and hence $h \gtrsim (M\|V\|_q)^{-q/(2q-2)}$. Conversely, if $h \ll (M\|V\|_q)^{-q/(2q-2)}$, then the case $\lambda h \geq 1$ does not occur and we have $\lambda^{2-2/q} \lesssim M\|V\|_q$.

(iii) The techniques we use were originally developed in the discrete setting. In this case the spectrum is compact, and the issue of large eigenvalues does not arise (for operators on $h\mathbb{Z}^d$ the largest frequencies are of order h^{-1}).

Corollary 9. *Let $J \subset (0, \infty)$ be a compact interval, $q < d+1$, and $h > 0$. Then we have*

$$\sup_{\operatorname{Re} z \in J} \frac{|z|^{q-d/2}}{\|V\|_q^q} < \infty$$

almost surely. The supremum is taken over eigenvalues $z = (\lambda + \mathrm{i}\varepsilon)^2$ of $-\Delta + V_\omega$ with $|\varepsilon| \ll \lambda$.

Proof. Denote the supremum by S, and consider the events $E_M = \{S^{1/q} > M\}$. Since $E_M \supset E_{M+1}$ and $\boldsymbol{P}(E_{M_0}) < \infty$, we have

$$\boldsymbol{P}(S = \infty) = \lim_{M\to\infty} \boldsymbol{P}(E_M) = 0. \qquad \square$$

Remark 10. The proof shows that Theorem 7 (and hence Corollary 9) actually hold with $\|V\|_{L^q}$ replaced by the (smaller) Lorentz norm $\|V\|_{L^{q,\infty}}$.

The key technical elements in this work are estimates on certain "elementary operators", roughly of the form

$$R_0^{1/2} V_\omega R_0^{1/2}, \tag{7}$$

where R_0 is the free resolvent at a fixed (complex) energy and V_ω is supported on a ball of radius $R > 1$. In dimension 2 and in the discrete case (i.e., when Δ is replaced by the discrete Laplacian), Schlag, Shubin, and Wolff [Schlag et al. 2002] proved[1] that the norm of these operators is bounded by a power of $\log R$. Their proof used in an essential way that the level sets corresponding to the symbol

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[1] This is roughly the content of [Schlag et al. 2002, Lemma 3.9]. Strictly speaking, the half powers of the resolvent are replaced by Fourier restriction and extension operators (or some mollified versions thereof); see also [Bourgain 2002, (1.12)].

of Δ (the discrete Laplacian) are curved. Bourgain [2002] gave a different proof using entropy bounds. His result is stated in dimension 2 but works in any dimension since it does not require curvature of the level sets (for the discrete Laplacian, these sets are not curved in higher dimensions). Motivated by work of Rodnianski and Schlag [2003], he uses these bounds to prove almost-sure existence and uniqueness of wave operators and absolutely continuous spectrum (for energies away from the edges of the spectrum and zero). The result shaves off half a power of pointwise decay compared to the classical (deterministic) Agmon–Kato–Kuroda theory. In a follow-up work, Bourgain [2003] combined his method with the two-dimensional Stein–Tomas restriction theorem to obtain the same conclusion for potentials in $\langle x\rangle^{-\delta}\ell^3(\mathbb{Z}^2)$ ($\delta > 0$ arbitrary). Note that there is a gap between the pointwise decay $\langle x\rangle^{-1/2}$ and $\ell^3(\mathbb{Z}^2)$. Bourgain [2003] observes that this gap cannot be overcome if one works with operators of the form (7) since the corresponding bounds (involving the $\ell^{3/2}(\mathbb{Z}^2)$ norm of the potential) are saturated (up to logarithms) by a Knapp example. Since the argument in [Bourgain 2003] is only stated in the two-dimensional discrete case, we will provide a similar, but more detailed, argument suggesting the optimality of our operator norm estimates (for the continuum multidimensional case) in Appendix B.[2] A representative (and simplified) example of these estimates, when λ and h are of unit size, is that

$$\|R_0^{1/2}V_\omega R_0^{1/2}\| \lesssim (\log R)^{\mathcal{O}(1)}\|V\|_{d+1}$$

with high probability (see Lemma 31 for a precise statement). Via a Born series argument (see Section 2 for details) this bound leads to a proof of Theorem 1. The proof of Theorem 6 then follows by a straightforward decomposition of the potential into dyadic shells $|x| \asymp 2^k$, similar to techniques of Bourgain [2002; 2003]. The proof of Theorem 7 requires more effort and the argument presented in Section 7 is new to the best of our knowledge. The technique[3] is reminiscent of an "epsilon removal lemma" in the context of Fourier restriction theory (see, e.g., [Tao 1999]). However, the technical implementation is a bit different since we are working with multilinear bounds (and with the resolvent instead of the Fourier restriction operator).

While the bounds (7) are optimal (up to logarithms) in the sense that the Lebesgue exponent $d+1$ cannot be increased, it is an interesting open problem whether our eigenvalue estimates (say in the form of Corollary 9) are optimal. This problem is connected to a remark of Bourgain [2003] that contains the idea of renormalizing away the self-energy interactions and then controlling the Born series via the sharp two-dimensional Fourier restriction theory of Carleson–Sjölin and Zygmund. This would amount to an $\ell^4(\mathbb{Z}^2)$ bound on the potential and would be natural and optimal from the point of view of restriction theory. A rigorous implementation of this idea seems difficult and has not been done so far, to the best of our knowledge.

Notation. We write $A \lesssim B$ for two nonnegative quantities $A, B \geq 0$ to indicate that there is a constant $C > 0$ such that $A \leq CB$. The dependence of the constant on fixed parameters like d and q is usually omitted (except in Section 7). The notation $A \asymp B$ means $A \lesssim B \lesssim A$. The product measure associated to the ω_j is denoted by $\boldsymbol{P}$ and the expectation by $\boldsymbol{E}$. We denote the L^p norm of a function f in $\mathbb{R}^d$ by $\|f\|_{L^p(\mathbb{R}^d)}$.

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[2] Bourgain's ideas and his Knapp example were also explained in a talk of Wilhelm Schlag at the Institute for Advanced Study on March 29, 2017.

[3] Although Bourgain was almost certainly aware of these techniques, he did not bother to remove the logarithmic losses.

If the function is defined on a countable set Λ, we write $\|f\|_{\ell^p(\Lambda)} = \bigl(\sum_{\nu\in\Lambda} |f(\nu)|^p\bigr)^{1/p}$. If Λ is finite, we also set $\|f\|_{\ell^p_{\mathrm{av}}(\Lambda)} = \bigl(|\Lambda|^{-1} \sum_{\nu\in\Lambda} |f(\nu)|^p\bigr)^{1/p}$. If it is clear from the context which norm is meant, we sometimes use the abbreviation $\|f\|_p$. If $T : X \to Y$ is a bounded linear operator between two Banach spaces X and Y, we denote its operator norm by $\|T\|_{X\to Y}$. The indicator function of a set $\Omega \subset \mathbb{R}^d$ is denoted by $\mathbf{1}_\Omega$. For $1 \le p \le \infty$, we denote its Hölder conjugate by $p' = (1-1/p)^{-1}$. An arbitrary ball of radius R will be denoted by B_R, without specifying its center. We use the convention $\hat{f}(\xi) = \int_{\mathbb{R}^d} \mathrm{e}(-x\cdot\xi) f(x)\,\mathrm{d}x$ for the Fourier transform of f, where $\mathrm{e}(x) = \mathrm{e}^{2\pi \mathrm{i} x}$, and $(f)^\vee(x) = \int_{\mathbb{R}^d} \mathrm{e}(x\cdot\xi) f(\xi)\,\mathrm{d}\xi$ for the inverse Fourier transform. Moreover, we recall the notation $\langle x\rangle = 2 + |x|$.

Organization. In Section 2 we outline the rough top-down strategy to prove our main results (see Proposition 11 for a summary). In Section 3, we collect basic facts related to the uncertainty principle and recall the Stein–Tomas theorem for a discrete version of the Fourier extension operator that will play a major technical role in the proofs of the estimates in Section 6. Section 4 is a short summary of probabilistic tools that will be used in the article. Section 5 fleshes out Bourgain's key idea of using entropy bounds. Section 6 contains the main local estimates and the completion of the proof of Theorem 1. Finally, in Section 7, the local estimates are converted to global ones, leading to the proofs of Theorems 6 and 7.

2. Born series

The proof of the eigenvalue estimates starts with the standard observation that $z \in \mathbb{C} \setminus [0,\infty)$ is an eigenvalue if and only if $I + R_0(z)V$ fails to be invertible as a bounded operator. This follows from the identity

$$-\Delta + V - z = (-\Delta - z)(I + R_0(z)V). \tag{8}$$

Here we denoted the free resolvent operator $(-\Delta - z)^{-1}$ by $R_0(z)$ and we omitted the subscript ω on V. Similarly, we will denote the perturbed resolvent operator $(-\Delta + V - z)^{-1}$ by $R(z)$. There are several variations of this argument based on variations of the identity (8). Perhaps the most well-known version is the so called *Birman–Schwinger principle*: z is an eigenvalue of $-\Delta + V$ if and only if -1 is an eigenvalue of the Birman–Schwinger operator $BS(z) = |V|^{1/2} R_0(z) V^{1/2}$. In particular, the norm of $BS(z)$ must be at least 1. This is perhaps the most commonly used approach in the literature since the seminal work of Frank [2011]. In the random case, this approach does not work so well because the sign (or phase) of the potential is of crucial importance. To exploit cancellations, we will work with the spectral radius (which must also be at least 1 but is in general smaller and harder to estimate than the norm). Although we could work with $BS(z)$, we prefer to work directly with the Born series; our approach may also be viewed as a multilinear version of the Birman–Schwinger principle.

In the following, to avoid confusion between the deterministic and the random potential, we focus our attention on the Anderson-type potentials (2). In this case, the assumption that $V \in L^q(\mathbb{R}^d)$ already implies that V is bounded (this follows from (4) and the fact that the ℓ^p spaces are nested). In particular, $R_0(z)V$ is a bounded operator. In the general case (3), one truncates the potential at some fixed large level. Since the estimates of Theorems 1–7 are independent of the L^∞ norm of V and the truncated Schrödinger operator converges to the untruncated one in the norm resolvent sense, there is no loss of generality in assuming that the deterministic potential is bounded. In the following, we assume that V is supported on a ball of radius R, i.e., the setting of Theorem 1. The case where V is not compactly supported (Theorems 6 and 7) will be considered in Section 7.

Returning to (8), we see that z cannot be an eigenvalue if the Born series

$$R(z)=\sum_{n\in\mathbb{N}}(-1)^n[R_0(z)V]^nR_0(z)$$

converges, which is the case if the spectral radius of R_0V is less than 1. Consider the following multilinear expansion (omitting z):

$$[R_0V]^n=\sum_{\sigma_1,\dots,\sigma_n}R_0^{\sigma_1}VR_0^{\sigma_2}V\cdots R_0^{\sigma_n}V, \tag{9}$$

where $\sigma_j\in\{\text{low},\text{high}\}$. Here, R_0^{low} is the resolvent (smoothly) localized to frequencies in $B(0,2)$ and $R_0^{\text{high}}=R_0-R_0^{\text{low}}$. Since we are dealing with scale-invariant estimates, we may assume without loss of generality that $\lambda=1$, hence $z=(1+\mathrm{i}\varepsilon)^2$ (see Remark 14 for more details). Then each summand is a composition of operators of the form $C^{(\delta_2)}VC^{(\delta_1)}$, where $C^{(\delta)}$ denotes a function satisfying a bound

$$|C^{(\delta)}(\xi)|\leq\bigl(\bigl||2\pi\xi|^2-1\bigr|+\delta\bigr)^{-1/2}, \tag{10}$$

and the corresponding Fourier multiplier is denoted by the same symbol. Clearly, the bound (10) holds with $\delta=1$ for $C^{(\delta)}=(R_0^{\text{high}})^{1/2}$ or $C^{(\delta)}=|R_0^{\text{high}}|^{1/2}$. In Section 6.2, we will show that (10) holds with $\delta=1/R$ if $C^{(\delta)}$ is a mollification of $(R_0^{\text{low}})^{1/2}$ or $|R_0^{\text{low}}|^{1/2}$ at scale $1/R$. Such a mollification can always be performed (except for the first resolvent in the Born series, but this does not affect convergence), due to the localizing effect of the potential, which we assumed to be supported in a ball of radius R. The spectral radius is given by Gelfand's formula: $\operatorname{spr}(R_0V)=\lim_{n\to\infty}\|[R_0V]^n\|^{1/n}$. Thus, in view of the previous discussion, we have $\operatorname{spr}(R_0V)\leq\sup\|C^{(\delta_2)}VC^{(\delta_1)}\|$, where the supremum is taken over all functions satisfying (10). We will ignore the high frequency part of the resolvent R_0^{high} from now on since there are obvious elliptic estimates available for this part. We may thus restrict our attention to functions as in (10) that are compactly supported in $B(0,2)$. We summarize the observations of this section in the following proposition.

Proposition 11. *Let $z=(1+\mathrm{i}\varepsilon)^2$, with $|\varepsilon|\ll 1$. Let V be supported in a ball of radius R. If (for a given realization of ω)*

$$\|C^{(\delta_2)}VC^{(\delta_1)}\|\leq c<1 \tag{11}$$

for all functions $C^{(\delta_i)}$, $i=1,2$, satisfying (10) with $\delta_1,\delta_2=1/R$ and supported in $B(0,2)$, then z is not an eigenvalue of $-\Delta+V$.

We refer to operators of the form (11) as "elementary operators" since they form the building blocks of the Born series. We prove norm estimates on these and related operators in Section 6. These estimates are the key technical elements in this work.

Remark 12. Strictly speaking, the previous argument is only valid for $\varepsilon\neq 0$, but there are techniques to extend this to embedded eigenvalues ($\varepsilon=0$); see, e.g., [Frank and Simon 2017, Proposition 3.1].

Remark 13. Later on, we will assume that all functions $C^{(\delta)}$ are supported in a small neighborhood of the unit sphere. This does not affect the validity of the above argument.

Remark 14. To restore the λ-dependence in the inequalities one can, e.g., use dimensional analysis: Since h and R have the dimension of a length L, the eigenvalue z and the potential V have the dimension L^{-2}, i.e., λ has the dimension L^{-1} and $\|V\|_q$ has the dimension $L^{d/q-2}$. Therefore, once an inequality for an eigenvalue $(1+i\varepsilon)^2$ involving $h,\ R,\ \|V\|_q$ has been proved, the λ-dependence is restored by multiplying h and R by λ and $\|V\|_q$ by $\lambda^{d/q-2}$.

3. Localization and discretization

3.1. ***Localization in momentum space.*** Denote by $\mathcal{Q}_h$ the collection of all cubes Q_h of sidelength h. Define the weight function

$$w_{Q_h}(x) = (1+h^{-1}\operatorname{dist}(x, Q_h))^{-100d}, \quad x \in \mathbb{R}^d, \quad Q_h \in \mathcal{Q}_h.$$

Lemma 15. *Let $v \in \mathcal{S}(\mathbb{R}^d)$, and assume that $\hat{v}$ is supported in $B(0, 1/h)$. Then v is locally constant on cubes Q_h of sidelength h in the sense that*

$$\|v\|_{L^\infty(Q_h)} \lesssim |Q_h|^{-1}\|v\|_{L^1(w_{Q_h})}.$$

Proof. By scaling, it suffices to prove this for $h = 1$. Choose $\eta \in \mathcal{S}(\mathbb{R}^d)$ such that $\eta = 1$ on $B(0, 1)$. Then we have $\hat{v} = \eta\hat{v}$, and hence $v = (\eta)^\vee * v$. Since $(\eta)^\vee \in \mathcal{S}(\mathbb{R}^d)$, it follows that

$$\kappa_w = \sup_{Q\in\mathcal{Q}_1} \sup_{(x,y)\in Q\times\mathbb{R}^d} |(\eta)^\vee(x-y)|w_Q(y)^{-1} < \infty,$$

where the first supremum is taken over all cubes of sidelength 1. Thus, for any cube Q of sidelength 1 and for $x \in Q$, we have

$$|v(x)| \le \int_{\mathbb{R}^d} |(\eta)^\vee(x-y)|\,|v(y)|\,\mathrm{d}y \le \kappa_w\|v\|_{L^1(w_Q)}.$$

Taking the supremum over $x \in Q$ proves the claim. □

Lemma 16. *Let $v \in \mathcal{S}(\mathbb{R}^d)$, and assume that $\hat{v}$ is supported in $B(0, 1/h)$. Let $\Lambda_h \subset \mathbb{R}^d$ be a set of h-separated points. Then, for any $p \ge 1$, we have*

$$\|v\|_{\ell^p(\Lambda_h)} \lesssim h^{-d/p}\|v\|_{L^p(\mathbb{R}^d)}.$$

Proof. Again by scaling, we can assume $h = 1$. Thus, let $\Lambda \subset \mathbb{R}^d$ be a set of 1-separated points. Pick a collection of cubes Q of sidelength 1 that cover Λ. By Lemma 15,

$$\|v\|_{\ell^p(\Lambda)}^p = \sum_{\nu\in\Lambda} |v(\nu)|^p \lesssim \sum_Q \|v\|_{L^1(w_Q)}^p.$$

Write $v = \sum_{Q'} v_{Q'}$, where $v_{Q'}$ is supported on Q'. Then

$$\|v_{Q'}\|_{L^1(w_Q)} \le (1+\operatorname{dist}(Q, Q'))^{-100d}\|v_{Q'}\|_{L^1(\mathbb{R}^d)}.$$

By Hölder, $\|v_{Q'}\|_{L^1(\mathbb{R}^d)} \le \|v_{Q'}\|_{L^p(\mathbb{R}^d)}$. Hence,

$$\sum_{Q} \|v\|_{L^1(w_Q)}^p \lesssim \sum_{Q,Q'} (1+\operatorname{dist}(Q,Q'))^{-100dp} \|v_{Q'}\|_{L^p(\mathbb{R}^d)}^p \lesssim \|v\|_{L^p(\mathbb{R}^d)}^p,$$

where we summed a geometric series in Q. □

3.2. ***Localization in position space.*** We will make use of the following standard device in local restriction theory (see, e.g., [Demeter 2020, Lemma 1.26]).

Lemma 17. *There exists a bump function ϕ on $\mathbb{R}^d$ with $\operatorname{supp}\phi \subset B(0,1)$ and with nonnegative Fourier transform satisfying $\mathbf{1}_{B(0,1)} \le \hat{\phi}$. Moreover, $\hat{\phi}$ is an even function.*

It is clear that the rescaled function $\phi_R(\xi) = R^d \phi(R\xi)$ satisfies

$$\operatorname{supp}\phi_R \subset B(0,R^{-1}), \quad \mathbf{1}_{B(0,R)} \le \hat{\phi}_R.$$

Let $M_\lambda = \{\xi \in \mathbb{R}^d : |\xi| = \lambda\}$, and consider the extension operator

$$\mathcal{E}_\lambda : L^2(M_\lambda, \mathrm{d}\sigma_\lambda) \to L^\infty(\mathbb{R}^d), \quad (\mathcal{E}_\lambda g)(x) = (g\,\mathrm{d}\sigma_\lambda)^\vee(x),$$

where σ_λ is the surface measure on M_λ. We write $\mathcal{E} \equiv \mathcal{E}_1$ and $M \equiv M_1$, $\sigma \equiv \sigma_1$.

3.3. ***Discrete Fourier extension operator.***

Definition 18. Let $\operatorname{Discres}(M,p,2)$ be the best constant such that the following holds for each $R \ge 2$, each collection Λ_R^* consisting of $1/R$-separated points on M, each sequence $a_\nu \subset \mathbb{C}$, each ball B_R, and each collection Λ_1 of 1-separated points in $\mathbb{R}^d$:

$$\Big\| \sum_{\nu\in\Lambda_R^*} a_\nu \mathrm{e}(\nu\cdot x) \Big\|_{\ell^{p'}(\Lambda_1\cap B_R)} \le \operatorname{Discres}(M,p,2) R^{(d-1)/2} \|a_\nu\|_{\ell^2(\Lambda_R^*)}. \tag{12}$$

Proposition 19. *If $1 \le p \le \infty$, then*

$$\operatorname{Discres}(M,p,2) \lesssim \|\mathcal{E}\|_{L^2(M,\mathrm{d}\sigma)\to L^{p'}(\mathbb{R}^d)}. \tag{13}$$

Moreover, if $p \ge 2$, then the reverse inequality also holds.

Proof. The claim is a special case of [Demeter 2020, Proposition 1.29], with one small difference. There, $\operatorname{Discres}(M,p,2)$ is defined with the $L^{p'}(B_R)$ norm in the left-hand side of (12). Thus, let $\operatorname{Discres}'(M,p,2)$ be the best constant in the inequality

$$\Big\| \sum_{\nu\in\Lambda_R^*} a_\nu \mathrm{e}(\nu\cdot x) \Big\|_{L^{p'}(B_R)} \le \operatorname{Discres}'(M,p,2) R^{(d-1)/2} \|a_\nu\|_{\ell^2(\Lambda_R^*)}. \tag{14}$$

Then [Demeter 2020, Proposition 1.29] asserts that the proposition holds with $\operatorname{Discres}'(M,p,2)$ in place of $\operatorname{Discres}(M,p,2)$. Thus, (13) follows once we show that

$$\operatorname{Discres}'(M,p,2) \gtrsim \operatorname{Discres}(M,p,2). \tag{15}$$

Without loss of generality we may assume that $B_R = B(0, R)$. If we set

$$f(x) = \sum_{\nu\in\Lambda_R^*} a_\nu \mathrm{e}(\nu\cdot x), \quad \text{then } \mathcal{F}(f\hat{\phi}_R)(\xi) = \sum_{\nu\in\Lambda_R^*} a_\nu \phi_R(\xi+\nu),$$

where ϕ_R is as before and $\mathcal{F}$ denotes the Fourier transform. Note that $\mathcal{F}(f\hat{\phi}_R) = \hat{f} * \phi_R$ is supported in an $1/R$-neighborhood of M. In particular, it is supported on the ball $B(0, 2)$. Thus, for any collection Λ_1 of 1-separated points in $\mathbb{R}^d$,

$$\|f\|_{\ell^{p'}(\Lambda_1\cap B_R)} \le \|f\hat{\phi}_R\|_{\ell^{p'}(\Lambda_1)} \lesssim \|f\hat{\phi}_R\|_{L^{p'}(\mathbb{R}^d)},$$

where we used $\hat{\phi}_R \ge \mathbf{1}_{B_R}$ in the first inequality and Lemma 16 in the second. By a partition of unity and a sparsification argument, we may assume that f is supported on a disjoint union of balls of radius R. By the rapid decay of $\hat{\phi}_R$ and by the definition of $\mathrm{Discres}'(M, p, 2)$,

$$\|\hat{\phi}_R f\|_{L^{p'}(\mathbb{R}^d)} \lesssim_N \sum_{j=1}^{\infty} j^{-N} \|f\|_{L^{p'}(B(x_j,R))} \lesssim \mathrm{Discres}'(M, p, 2) R^{(d-1)/2} \|a_\nu\|_{\ell^2(\Lambda_R^*)},$$

where we used that (14) holds uniformly in the centers of the balls. Combining the last two estimates yields (15).

To prove the reverse inequality to (13), we may assume that $B_R = B(0, R)$. By [Demeter 2020, Proposition 1.29] it suffices to prove the reverse inequality to (15). Let Λ_1 be a 1-net of points $x_j \in B_R$. Let $f(x)$ be defined as above. Without loss of generality we may assume that f is supported on a disjoint collection of balls $B(x_j, 10)$. Then

$$\begin{aligned}\|f\|_{L^{p'}(B_R)} &= \Big(\sum_j \|f\|^{p'}_{L^{p'}(B(x_j,10))}\Big)^{1/p'} = \Big(\int_{B(0,10)} \sum_j |f(x_j+y)|^{p'}\,\mathrm{d}y\Big)^{1/p'} \\ &\lesssim \mathrm{Discres}(M, p, 2) R^{(d-1)/2} \|a_\nu\|_{\ell^2(\Lambda_R^*)},\end{aligned}$$

where we used that (12) holds for each collection $x_j + y$ of 1-separated points, uniformly in y. □

3.4. ***Stein–Tomas theorem.*** The following is an immediate consequence of the Stein–Tomas theorem and Proposition 19 (see also [Demeter 2020, Corollary 1.30]).

Proposition 20. *Let* $p' \ge 2(d+1)/(d-1)$. *Then* $\mathrm{Discres}(M, p, 2) \lesssim 1$.

4. Randomization

4.1. ***Subgaussian random variables.*** We recall that a (complex) scalar random variable X is called *subgaussian* if it has finite subgaussian norm:

$$\|X\|_{\psi_2} = \inf\{t > 0 : \boldsymbol{E}\exp(|X|^2/t^2) \le 2\} < \infty.$$

We will need the following elementary properties of subgaussian (e.g., Gaussian or symmetric Bernoulli) random variables (see, e.g., [Vershynin 2018, Proposition 2.6.1 and Exercise 2.5.10]).

Proposition 21. *Assume that* $(X_j)_{j=1}^N$, $N \geq 2$, *is a finite collection of i.i.d. mean-zero subgaussian random variables.*

(i) *Then* $\sum_{j=1}^N X_j$ *is also subgaussian, and*

$$\Bigl\| \sum_{j=1}^N X_j \Bigr\|_{\psi_2}^2 \lesssim \sum_{j=1}^N \|X_j\|_{\psi_2}^2.$$

(ii) *We have*

$$\boldsymbol{E} \max_{j \leq N} |X_j| \lesssim \sqrt{\log N} \max_{j \leq N} \|X_j\|_{\psi_2}.$$

Proof. The claim follows by applying [Vershynin 2018, Proposition 2.6.1 and Exercise 2.5.10] to $\operatorname{Re} X_j$ and $\operatorname{Im} X_j$ separately. ☐

4.2. ***Tail bounds.*** We now consider tail bounds for *vector-valued* Gaussian or Bernoulli random variables X. We have $(\boldsymbol{E}\|X\|^p)^{1/p} \asymp (\boldsymbol{E}\|X\|^q)^{1/q}$ for all $p, q > 0$ (see [Ledoux and Talagrand 1991, Corollary 3.2 and Theorem 4.7]), which, combined with [Ledoux and Talagrand 1991, (3.5), (4.12)], implies

$$\boldsymbol{P}(\|X\| > t) \leq \exp\biggl(-\frac{ct^2}{(\boldsymbol{E}\|X\|)^2}\biggr)$$

for some $c > 0$. Thus the following lemma is obvious.

Lemma 22. *If* $\boldsymbol{E}\|X\| \leq C$, *then*

$$\boldsymbol{P}(\|X\| > MC) \leq \exp(-cM^2)$$

for any $M > 0$.

5. Entropy bound

Consider a linear operator $S : \mathcal{H} \to \ell_m^\infty$, where $\mathcal{H}$ is a finite-dimensional Hilbert space and $\ell_m^\infty = \ell^\infty(\{1, \dots, m\})$. For $\varepsilon > 0$, let $\mathcal{N}(\varepsilon)$ be the minimal number of balls in ℓ_m^∞ of radius ε needed to cover the set $\{Sx : x \in \mathcal{H},\ \|x\|_{\mathcal{H}} \leq 1\}$. Here we use the convention that the centers of the balls are contained in the set they cover (i.e., $\mathcal{N}(t)$ is the *covering number* as opposed to the *exterior covering number*; see, e.g., [Vershynin 2018, Section 4.2]). Using an entropy bound known as the "dual Sudakov inequality" — which is attributed to Pajor and Tomczak-Jaegermann [1986] — Bourgain [2002, (4.2)] shows that

$$\log \mathcal{N}(\varepsilon) \lesssim (\log m)\varepsilon^{-2} \|S\|_{\mathcal{H} \to \ell_m^\infty}^2. \tag{16}$$

The quantity $\log \mathcal{N}(\varepsilon)$ is called the *entropy number* of the image of the unit ball in $\mathcal{H}$ under the map S. The crucial observation is that (16) is independent of $\dim \mathcal{H}$. We apply this bound to the operator featuring in (12), i.e.,

$$S : \mathcal{H} \to \ell^\infty(\Lambda_1 \cap B_R), \quad \{a_\nu\} \mapsto \biggl\{ \sum_{\nu \in \Lambda_R^*} a_\nu \mathrm{e}(\nu \cdot x) \biggr\}_x. \tag{17}$$

In this case, $\mathcal{H} = \ell^2(\Lambda_R^*)$ with norm $\|a\|_{\mathcal{H}} := R^{(d-1)/2}\big(\sum_{\nu\in\Lambda_R^*}|a_\nu|^2\big)^{1/2}$ and $\ell_m^\infty = \ell^\infty(\Lambda_1\cap B_R)$. In particular, we have $m \asymp R^d$. Here and in the following we always assume $R \geq 2$. Proposition 20 gives

$$\|S\|_{\mathcal{H}\to\ell^{p'}(\Lambda_1\cap B_R)} \lesssim 1 \quad \text{for } p' \geq 2(d+1)/(d-1). \tag{18}$$

In particular, we have the trivial bound ($p' = \infty$)

$$\|S\|_{\mathcal{H}\to\ell^{\infty}(\Lambda_1\cap B_R)} \lesssim 1. \tag{19}$$

Combining the latter with (16) yields the following entropy bound.

Proposition 23. *Let S be given by (17). The entropy number satisfies the bound*

$$\log\mathcal{N}(\varepsilon) \lesssim (\log R)\varepsilon^{-2}.$$

Corollary 24. *Let $p' \geq 2(d+1)/(d-1)$. For every $k \in \mathbb{Z}_+$, there exist sets $\mathcal{F}_k \subset \ell^\infty(\Lambda_1\cap B_R)$ with the following properties*:

(a) $\log|\mathcal{F}_k| \lesssim \log(R)4^k$ (*here $|\cdot|$ denotes the counting measure*).

(b) *For $\xi \in \mathcal{F}_k$,*

$$\|\xi\|_{\ell^\infty(\Lambda_1)} \lesssim 2^{-k}, \qquad \|\xi\|_{\ell^{p'}(\Lambda_1)} \lesssim 1.$$

(c) *For each $a \in \mathcal{H}$ with $\|a\|_{\mathcal{H}} \leq 1$, there is a representation*

$$Sa = \sum_{k\in\mathbb{Z}_+} \xi^{(k)} \quad \textit{for some } \xi^{(k)} \in \mathcal{F}_k.$$

Proof. We follow Bourgain [2003, pages 75-76], but provide more details (note also that there is a misprint in (3.13) in that work; it should be 4^r, not 4^{-r}). This is a standard chaining argument.

We start by noting that, in view of (16) and (19), we have $\mathcal{N}(C) = 1$ for C sufficiently large. In the following (and only in this proof), denote the unit ball in $\mathcal{H}$ by B_1. Similarly, $B(\xi,\varepsilon)$ denotes a ball centered at ξ and with radius ε in $\ell^\infty(\Lambda_1\cap B_R)$. We also write $\|\cdot\|_p = \|\cdot\|_{\ell^p(\Lambda_1)}$ here. By possibly rescaling SB_1 by a constant, we may assume that $C = 1$. Thus, we have $\mathcal{N}(1) = 1$. We get, by Proposition 23,

$$\log\mathcal{N}(2^{-k}) \lesssim \log(R)4^k.$$

Thus, for each $k \geq 0$, there exist subsets $\mathcal{E}_k \subset \ell^\infty(\Lambda_1\cap B_R)$ of cardinality $\mathcal{N}(2^{-k})$ satisfying

$$SB_1 \subset \bigcup_{\xi\in\mathcal{E}_k} B(\xi, 2^{-k}).$$

Applying these nets for each k, we can assign to each element $Sa \in SB_1$ a chain $\{\xi_k\}$ converging to Sa, with $\xi_k \in \mathcal{E}_k$ and

$$\|\xi_k - \xi_{k-1}\|_\infty \leq 2^{-k} + 2^{1-k} \tag{20}$$

for all k. By telescoping, we have

$$Sa = \xi_0 + \lim_{N\to\infty}\sum_{k=1}^{N}(\xi_k - \xi_{k-1}).$$

Thus, we may choose $\mathcal{F}_0 = \mathcal{E}_0$ and $\mathcal{F}_k \subset \mathcal{E}_k - \mathcal{E}_{k-1}$, $k > 0$, as the collection of all vectors $\xi^{(k)} = \xi_k - \xi_{k-1}$ for which (20) holds. Since the difference set $\mathcal{E}_k - \mathcal{E}_{k-1}$ has cardinality $|\mathcal{E}_k||\mathcal{E}_{k-1}|$, the claimed properties hold by construction. ☐

6. Local bounds on elementary operators

6.1. ***Local extension bound.*** Let $h, R > 0$. Consider V_ω of the form (3), where V is a given deterministic potential supported in B_R. Also fix $p' \geq 2(d+1)/(d-1)$, and define q by $1/q = 1/p - 1/p'$. Note that this convention differs from that in the main theorems by a change of variables $q \to 2q$.

Lemma 25. *Under the above assumptions, we have*

$$\boldsymbol{E}\|\mathcal{E}^* V_\omega \mathcal{E}\|_{L^2(M,\mathrm{d}\sigma)\to L^2(M,\mathrm{d}\sigma)} \lesssim \langle h\rangle^{d/2}(\log\langle R\rangle)^{1/2}(\log\langle h\rangle + \log\langle R\rangle)^2 \|V\|_{L^{2q}(\mathbb{R}^d)}.$$

Proof. Since the right-hand side only gets larger if we replace R and h by $R+2$ and $h+2$, respectively, we may assume $R, h \geq 2$. We first observe that

$$\mathcal{E}^* V_\omega \mathcal{E} = \mathcal{E}^*(V_\omega * \varphi)\mathcal{E} \tag{21}$$

for any Schwartz function φ satisfying $\hat{\varphi} = 1$ on $B(0,2)$. We can thus assume without loss of generality that V is smooth on the unit scale. Let g, g' be unit vectors in $L^2(M, \mathrm{d}\sigma)$. Then

$$\langle \mathcal{E}^* V_\omega \mathcal{E} g, g'\rangle = \sum_{j\in h\mathbb{Z}^d} \omega_j \int_{Q_h+j} \overline{V(x)(\mathcal{E}g)(x)}(\mathcal{E}g')(x)\,\mathrm{d}x,$$

where $Q_h = [0,h)^d$. Let $\Lambda_R^* = \{\eta_\nu\}$ be a $1/R$-net in M. By working with a partition of unity, we may assume that g is supported on a collection of disjoint balls $B(\eta_\nu, 10/R)$. After a change of variables $g(\eta) = g(\eta_\nu + \tau)$, we may write

$$\mathcal{E}g(x) = \sum_\nu \int_{M\cap B(0,10/R)} \mathrm{e}(x\cdot(\eta_\nu+\tau))g(\eta_\nu+\tau)\,\mathrm{d}\tau,$$

where $\mathrm{d}\tau$ denotes the surface measure, and similarly (summing over a possibly different index set)

$$\mathcal{E}g'(x) = \sum_{\nu'} \int_{M\cap B(0,10/R)} \mathrm{e}(x\cdot(\eta_{\nu'}+\tau'))g'(\eta_{\nu'}+\tau')\,\mathrm{d}\tau'.$$

Similar to the change of variables $\eta = \eta_\nu + \tau$ in the domain, we change variables $x = x_i + y$ in the target. Here, $\Lambda_1 = \{x_i\}$ is a 1-net in $\mathbb{R}^d$. Hence, for any integrable function $F : \mathbb{R}^d \to \mathbb{C}$ supported on a disjoint collection of balls $B(x_i, 10)$,

$$\int_{\mathbb{R}^d} F(x)\,\mathrm{d}x = \sum_i \int_{B(0,10)} F(x_i+y)\,\mathrm{d}y.$$

Using a partition of unity we may sparsify the potential, so that the above holds for

$$F_j(x) = \overline{V(x)(\mathcal{E}g)(x)}(\mathcal{E}g')(x)\mathbf{1}_{Q_h+j}(x).$$

Note that in this case the sum is restricted to those i satisfying $x_i \in B(j, 10+h)$. For fixed $\tau \in B(0, 10/R)$ and $y \in B(0, 10)$, we consider the discrete extension operator

$$S : \mathcal{H} \to \ell^\infty(\Lambda_1 \cap B_R), \quad \{g(\eta_\nu + \tau)\}_\nu \mapsto \left\{ \sum_\nu \mathrm{e}((x_i + y) \cdot (\eta_\nu + \tau)) g(\eta_\nu + \tau) \right\}_i .$$

Note that the points $\mu_\nu = \eta_\nu + \tau$ and $z_i = x_i + y$ form a $1/R$-separated set in M and a 1-separated set in $\mathbb{R}^d$, respectively, so that (18) and (19) hold. Using Corollary 24, we can find a representation (note that the vectors $\xi^{(k)}$ depend on τ, y)

$$\sum_\nu \mathrm{e}((x_i + y) \cdot (\eta_\nu + \tau)) g(\eta_\nu + \tau) = \sum_{k \in \mathbb{Z}_+} \xi_i^{(k)}, \quad \xi^{(k)} \in \mathcal{F}_k,$$

with bounds

$$\|\xi^{(k)}\|_\infty \lesssim 2^{-k} \|g(\eta_\nu + \tau)\|_{\ell^2_{\nu,\mathrm{av}}}, \quad \|\xi^{(k)}\|_{p'} \lesssim \|g(\eta_\nu + \tau)\|_{\ell^2_{\nu,\mathrm{av}}} \tag{22}$$

for all $k \in \mathbb{Z}_+$ and $y \in B(0, 10)$. Similarly, there is a representation

$$\sum_{\nu'} \mathrm{e}((x_i + y) \cdot (\eta_{\nu'} + \tau')) g'(\eta_{\nu'} + \tau') = \sum_{k' \in \mathbb{Z}_+} \xi_i^{(k')}, \quad \xi^{(k')} \in \mathcal{F}_{k'},$$

with bounds

$$\|\xi^{(k')}\|_\infty \lesssim 2^{-k'} \|g'(\eta_{\nu'} + \tau')\|_{\ell^2_{\nu',\mathrm{av}}}, \quad \|\xi^{(k')}\|_{p'} \lesssim \|g'(\eta_{\nu'} + \tau')\|_{\ell^2_{\nu',\mathrm{av}}}. \tag{23}$$

The above observations lead to the estimate

$$\sup_{g,g'} |\langle \mathcal{E}^* V_\omega \mathcal{E} g, g' \rangle| \le \sum_{k,k' \in \mathbb{Z}_+} \int \max_{(\xi,\xi') \in \mathcal{F}_k \times \mathcal{F}_{k'}} \left| \sum_{j \in h\mathbb{Z}^d} \sum_i \omega_j \overline{V(x_i + y) \xi_i} \xi'_i \right| \mathrm{d}y \, \mathrm{d}\tau \, \mathrm{d}\tau',$$

where the integral is taken over $(y, \tau, \tau') \in B(0, 10) \times (M \cap B(0, 10/R))^2$ and the sum over i is restricted to $x_i + y \in Q_h + j$ (we recall that y is fixed). By monotonicity of the expectation,

$$\boldsymbol{E} \sup_{g,g'} |\langle \mathcal{E}^* V_\omega \mathcal{E} g, g' \rangle| \le \sum_{k,k' \in \mathbb{Z}_+} \int \boldsymbol{E} \max_{\mathcal{F}_k \times \mathcal{F}_{k'}} |X_{\xi,\xi'}| \, \mathrm{d}y \, \mathrm{d}\tau \, \mathrm{d}\tau',$$

where (suppressing the dependence on y, τ, τ')

$$X_{\xi,\xi'} = \sum_{j \in h\mathbb{Z}^d} \omega_j \sum_i \overline{V(x_i + y) \xi_i} \xi'_i.$$

The conclusion follows by Lemmas 26 and 38 (details of the calculation are provided in Appendix A). □

Lemma 26. *Let $R, h \ge 2$. Then we have the bounds*

$$\int \boldsymbol{E} \max_{\mathcal{F}_k \times \mathcal{F}_{k'}} |X_{\xi,\xi'}| \, \mathrm{d}y \, \mathrm{d}\tau \, \mathrm{d}\tau' \lesssim (\log R)^{1/2} h^{d/2} \|V\|_{L^{2q}(\mathbb{R}^d)},$$

$$\int \max_{\mathcal{F}_k \times \mathcal{F}_{k'}} |X_{\xi,\xi'}| \, \mathrm{d}y \, \mathrm{d}\tau \, \mathrm{d}\tau' \lesssim R^{d-d/(2q)} 2^{-k-k'} \|V\|_{L^{2q}(\mathbb{R}^d)}.$$

Proof. Note first that the index set of $X_{\xi,\xi'}$ is finite and has cardinality N, satisfying

$$\log N = \log|\mathcal{F}_k \times \mathcal{F}_{k'}| \lesssim \log R \max(4^k, 4^{k'}) \tag{24}$$

by Corollary 24 (a). Proposition 21 implies that $X_{\xi,\xi'}$ are (scalar) subgaussian random variables, and

$$\boldsymbol{E} \max_{\mathcal{F}_k\times\mathcal{F}_{k'}} |X_{\xi,\xi'}| \lesssim \sqrt{\log N}\Big(\sum_{j\in h\mathbb{Z}^d}\Big|\sum_i \overline{V(x_i+y)\xi_i\xi_i'}\Big|^2\Big)^{1/2},$$

where we recall that we are assuming $\|\omega_j\|_{\psi_2}\lesssim 1$. Using Hölder's inequality twice, it follows that

$$\begin{aligned}\boldsymbol{E} \max_{\mathcal{F}_k\times\mathcal{F}_{k'}} |X_{\xi,\xi'}| &\lesssim \sqrt{\log N}\big\|\|V(x_i+y)\|_{\ell_i^q}\|\xi_i\|_{\ell_i^{p'}}\|\xi_i'\|_{\ell_i^{p'}}\big\|_{\ell_j^2}\\ &\lesssim \sqrt{\log N}\big\|\|V(x_i+y)\|_{\ell_i^q}\big\|_{\ell_j^{2q}}\big\|\|\xi_i\|_{\ell_i^{p'}}\|\xi_i'\|_{\ell_i^{p'}}\big\|_{\ell_j^{p'}},\end{aligned} \tag{25}$$

where we recall that i is restricted to $x_i+y\in Q_h+j$ and y is fixed. In particular, we have

$$|\{j\in h\mathbb{Z}^d : x_i+y\in Q_h+j\}| = 1 \quad \text{for each } i \tag{26}$$

and

$$|\{i : x_i+y\in Q_h+j\}| \le h^d \quad \text{for each } j\in h\mathbb{Z}^d. \tag{27}$$

We will show that

$$\big\|\|\xi_i\|_{\ell_i^{p'}}\|\xi_i'\|_{\ell_i^{p'}}\big\|_{\ell_j^{p'}} \lesssim h^{d/p'}\min(2^{-k},2^{-k'})\|g(\eta_\nu+\tau)\|_{\ell^2_{\nu,\mathrm{av}}}\|g'(\eta_{\nu'}+\tau')\|_{\ell^2_{\nu',\mathrm{av}}}. \tag{28}$$

By symmetry in ξ and ξ', it suffices to prove this in the case $k\ge k'$. Using Hölder once more, we have

$$\big\|\|\xi_i\|_{\ell_i^{p'}}\|\xi_i'\|_{\ell_i^{p'}}\big\|_{\ell_j^{p'}} \le \big\|\|\xi_i\|_{\ell_i^{p'}}\big\|_{\ell_j^\infty}\big\|\|\xi_i'\|_{\ell_i^{p'}}\big\|_{\ell_j^{p'}}.$$

By Fubini's theorem and (26),

$$\big\|\|\xi_i'\|_{\ell_i^{p'}}\big\|_{\ell_j^{p'}} = \Big(\sum_i\sum_{j\in h\mathbb{Z}^d}|\xi_i'|^{p'}\Big)^{1/p'} = \Big(\sum_i|\xi_i'|^{p'}\Big)^{1/p'} = \|\xi'\|_{p'}.$$

Similarly, by (27), we have

$$\big\|\|\xi_i\|_{\ell_i^{p'}}\big\|_{\ell_j^\infty} \le h^{d/p'}\|\xi\|_\infty.$$

Combining these estimates with (22) and (23) yields (28). Next, we have (again by Hölder, Fubini and (27))

$$\begin{aligned}\big\|\|V(x_i+y)\|_{\ell_i^q}\big\|_{\ell_j^{2q}} &\le h^{d/(2q)}\big\|\|V(x_i+y)\|_{\ell_i^{2q}}\big\|_{\ell_j^{2q}}\\ &= h^{d/(2q)}\big\|\|V(x_i+y)\|_{\ell_j^{2q}}\big\|_{\ell_i^{2q}} = h^{d/(2q)}\|V(x_i+y)\|_{\ell_i^{2q}}.\end{aligned} \tag{29}$$

Integrating (25) over $y,\ \tau,\ \tau'$ and using (28) and (29), we obtain

$$\int \boldsymbol{E} \max_{\mathcal{F}_k\times\mathcal{F}_{k'}} |X_{\xi,\xi'}|\,\mathrm{d}y\,\mathrm{d}\tau\,\mathrm{d}\tau' \lesssim \sqrt{\log N}\min(2^{-k},2^{-k'})h^{d/2}\|V\|_{L^{2q}(\mathbb{R}^d)},$$

where we used $\frac{1}{2} = 1/(2q) + 1/p'$, $\|V(x_i + y)\|_{L_y^{2q}\ell_i^{2q}} \lesssim \|V\|_{L^{2q}(\mathbb{R}^d)}$, and

$$R^{-(d-1)}\|g(\eta_\nu + \tau)\|_{L_\tau^2 \ell_{\nu,\mathrm{av}}^2} \|g'(\eta_{\nu'} + \tau')\|_{L_{\tau'}^2 \ell_{\nu',\mathrm{av}}^2} \lesssim \|g\|_{L^2(M,\mathrm{d}\sigma)} \|g'\|_{L^2(M,\mathrm{d}\sigma)} = 1.$$

Combining this with (24) yields the first bound of the lemma. The second bound follows from the estimate

$$\begin{aligned}
|X_{\xi,\xi'}| &\le \sum_{j\in h\mathbb{Z}^d} \Big|\sum_i \overline{V(x_i + y)\xi_i}\xi_i'\Big| \le \sum_{j\in h\mathbb{Z}^d} \|V(x_i + y)\|_{\ell_i^1} \|\xi_i\|_{\ell_i^\infty} \|\xi_i'\|_{\ell_i^\infty} \\
&\le \|V(x_i + y)\|_{\ell_j^1 \ell_i^1} \|\xi\|_\infty \|\xi'\|_\infty \\
&= \|V(x_i + y)\|_{\ell_i^1 \ell_j^1} \|\xi\|_\infty \|\xi'\|_\infty \\
&= \|V(x_i + y)\|_{\ell_i^1} \|\xi\|_\infty \|\xi'\|_\infty \\
&\lesssim R^{d-d/(2q)} \|V(x_i + y)\|_{\ell_i^{2q}} \|\xi\|_\infty \|\xi'\|_\infty \\
&\lesssim R^{d-d/(2q)} \|V(x_i + y)\|_{\ell_i^{2q}} 2^{-k-k'} \|g(\eta_\nu + \tau)\|_{\ell_{\nu,\mathrm{av}}^2} \|g'(\eta_{\nu'} + \tau')\|_{\ell_{\nu',\mathrm{av}}^2},
\end{aligned}$$

where we used Hölder in the first, second and fifth line, Fubini in the third line, (26) in the fourth, $\operatorname{supp} V \subset B_R$ in the fifth and (22), (23) in the last line. Integrating over y, τ, τ' and using Hölder as before yields the second bound in the lemma. ☐

Remark 27. If we restore the frequency in the extension operator, i.e., if we consider $\mathcal{E}_\lambda^* V_\omega \mathcal{E}_{\lambda'}$, then it is obvious from the proof of Lemma 25 that the same estimate holds for this operator, locally uniformly in $\lambda, \lambda' \asymp 1$. Explicitly,

$$\begin{gathered}
\sup_{\lambda,\lambda'\asymp 1} \boldsymbol{E}\|\mathcal{E}_\lambda^* V_\omega \mathcal{E}_{\lambda'}\|_{L^2(M_\lambda,\mathrm{d}\sigma_\lambda)\to L^2(M_{\lambda'},\mathrm{d}\sigma_{\lambda'})} \le A(h, R, V), \\
A(h, R, V) \lesssim \langle h\rangle^{d/2} (\log\langle R\rangle)^{1/2} (\log\langle h\rangle + \log\langle R\rangle)^2 \|V\|_{L^{2q}(\mathbb{R}^d)}.
\end{gathered} \tag{30}$$

6.2. *Smoothing.* We observe that if $m(D)$ is a Fourier multiplier and B_{R_1} and B_{R_2} are two balls with the same center, then

$$\mathbf{1}_{B_{R_1}} m(D) \mathbf{1}_{B_{R_2}} = \mathbf{1}_{B_{R_1}} m_R(D) \mathbf{1}_{B_{R_2}}, \quad m_R := \gamma_R * m, \tag{31}$$

whenever $R > R_1 + R_2$, $\gamma_R(\xi) = R^d \gamma(R\xi)$, and $(\gamma)^\vee$ is a bump function such that $(\gamma)^\vee(x) = 1$ for $|x| \le 1$. This can be checked by comparing the kernels of both sides in (31) and using the convolution theorem. The convolution with γ_R can be considered a smoothing operator at scale R^{-1}. We recall from Section 2 that $C^{(\delta)}$ denotes a generic function satisfying a bound

$$|C^{(\delta)}(\xi)| \lesssim \big(\big||2\pi\xi|^2 - 1\big| + \delta\big)^{-1/2}. \tag{32}$$

We will apply (31) to

$$m(\xi) = (|2\pi\xi|^2 - (1 + \mathrm{i}0)^2)^{-1} \tag{33}$$

to produce a product of two functions $C^{(\delta)}(\xi)$ satisfying (32) with $\delta = R^{-1}$.

Lemma 28. *For $R \ge 1$, we have*

$$|\gamma_R * m| \lesssim R.$$

In particular, $(\gamma_R * m)^{1/2}$ satisfies (32) with $\delta = R^{-1}$.

Proof. By a partition of unity we may assume that m is supported in a small conic neighborhood of the first coordinate axis. The implicit function theorem then allows us to reduce the proof to the bound

$$\left|\gamma_R * \frac{1}{\xi_1 + \mathrm{i}0}\right| \lesssim R,$$

where $\gamma_R(\xi_1) = R\gamma(R\xi_1)$ is a function of one variable. By the convolution theorem,

$$\left|\gamma_R * \frac{1}{\xi_1 + \mathrm{i}0}\right| \lesssim \|\hat{\gamma}_R\|_1 \lesssim R,$$

where we used that the Fourier transform of $(\xi_1 + \mathrm{i}0)^{-1}$ is bounded. See also [Ruiz 2002, Lemma 5.2] for an alternative proof. ☐

Remark 29. The boundary value in (33) is defined in the usual way (in the sense of tempered distributions, see, e.g., [Hörmander 1990]). The analogue expression with $(1 - \mathrm{i}0)^2$ clearly satisfies the same bound. A similar argument (using the Malgrange preparation theorem) also works for ε nonzero and fixed. This argument is presented in the proof of Lemma 23 in [Bögli and Cuenin 2023]. Alternatively, one can work with the boundary values throughout and appeal to the Phragmén–Lindelöf maximum principle to extend the results to nonzero ε (see, e.g., [Cuenin 2017, Appendix A; Guillarmou et al. 2020; Ruiz 2002]). We will not pursue this issue.

In practice, we are working with a localized version of (33), supported near the singular manifold M. Even though $\gamma_R * m$ loses compact support, it decays rapidly away from M on the $1/R$ scale. Neglecting the tail (which can be bounded in a straightforward way), we assume that all functions $C^{(\delta)}$ that appear from now on are compactly supported in a small neighborhood of M. Alternatively, one could avoid tails by smoothing the resolvent first and then perform the low/high decomposition as in Section 2.

6.3. *Foliation by level sets.* In the following we will assume that $C^{(\delta)}$ is supported in a c-neighborhood (c small and fixed) of M and satisfies (32). We will also assume that $\lambda \in [1-c, 1+c]$ and denote the constant $A(h, R, V)$ appearing in (30) by A.

Lemma 30. *Assume that (30) and (32) hold. Then we have*

$$\boldsymbol{E}\|\mathcal{E}_\lambda^* V C^{(\delta)}\|_{L^2(\mathbb{R}^d)\to L^2(M_\lambda)} \lesssim A\Big(\log\frac{1}{\delta}\Big)^{1/2}. \tag{34}$$

Moreover, if (32) holds for $C^{(\delta_1)}$ and $C^{(\delta_2)}$, then

$$\boldsymbol{E}\|C^{(\delta_1)} V C^{(\delta_2)}\|_{L^2(\mathbb{R}^d)\to L^2(\mathbb{R}^d)} \lesssim A\Big(\log\frac{1}{\delta_1}\Big)^{1/2}\Big(\log\frac{1}{\delta_2}\Big)^{1/2}. \tag{35}$$

Proof. For $f \in L^2(\mathbb{R}^d)$, we foliate by level sets M_λ,

$$C^{(\delta)} f(x) = \int_{1-c}^{1+c}\int_{M_{\lambda'}} \mathrm{e}(x\cdot\xi) C^{(\delta)}(\xi)\hat{f}(\xi)\,\mathrm{d}\sigma_{\lambda'}(\xi)\,\mathrm{d}\lambda', \tag{36}$$

up to an innocuous Jacobian factor. Without loss of generality we now assume that f has Fourier support in $1 - c \le |\xi| \le 1 + c$. Using (32) and the fact that $(\mathrm{d}\sigma_\lambda)^\vee * f$ is a constant multiple of $\mathcal{E}_\lambda\mathcal{E}_\lambda^* f$, we get, by

Cauchy–Schwarz,

$$\boldsymbol{E}\|\mathcal{E}_\lambda^* V C^{(\delta)} f\|_{L^2(M_\lambda)} \le A\Big(\int_{1-c}^{1+c} \mathrm{d}\lambda' (|\lambda'-1|+\delta)^{-1}\Big)^{1/2}\Big(\int_{1-c}^{1+c} \mathrm{d}\lambda' \|\mathcal{E}_{\lambda'}^* f\|^2_{L^2(M_{\lambda'})}\Big)^{1/2} \lesssim A\Big(\log\frac{1}{\delta}\Big)^{1/2}\|f\|_2,$$

where we used

$$\int_{1-c}^{1+c} \mathrm{d}\lambda' \|\mathcal{E}_{\lambda'}^* f\|^2_{L^2(M_{\lambda'})} = \int_{1-c}^{1+c} \mathrm{d}\lambda' \int_{M_{\lambda'}} |\hat f(\xi)|^2 \,\mathrm{d}\sigma_{\lambda'}(\xi) \lesssim \|f\|^2_{L^2(\mathbb{R}^d)} \tag{37}$$

and

$$\int_{1-c}^{1+c} \mathrm{d}\lambda' (|\lambda'-1|+\delta)^{-1} \lesssim \log\frac{1}{\delta}. \tag{38}$$

This proves (34). To prove (35), we use the dual estimate to (37), which is

$$\Big\|\int_{1-c}^{1+c} \mathcal{E}_{\lambda'} g(\lambda')\,\mathrm{d}\lambda'\Big\|_{L^2(\mathbb{R}^d)} \lesssim \Big(\int_{1-c}^{1+c} \|g(\lambda')\|^2_{L^2(M_{\lambda'})}\,\mathrm{d}\lambda'\Big)^{1/2} \tag{39}$$

for $g(\lambda') \in L^2(M_{\lambda'})$. This follows from

$$\int_{1-c}^{1+c} \langle \mathcal{E}_{\lambda'}^* f, g(\lambda')\rangle_{L^2(M_{\lambda'})}\,\mathrm{d}\lambda' = \Big\langle f, \int_{1-c}^{1+c} \mathcal{E}_{\lambda'} g(\lambda')\,\mathrm{d}\lambda'\Big\rangle_{L^2(\mathbb{R}^d)}.$$

Using the foliation (36) for the $C^{(\delta_1)}$ factor and using (34) and (38), inequality (39) gives, with $g(\lambda') = (|\lambda'-1|+\delta_1)^{-1/2}\mathcal{E}_{\lambda'}^* V C^{(\delta_2)} f$,

$$\begin{aligned}\boldsymbol{E}\|C^{(\delta_1)} V C^{(\delta_2)} f\|_{L^2(\mathbb{R}^d)} &\lesssim \Big\|\int_{1-c}^{1+c} \mathcal{E}_{\lambda'} g(\lambda')\,\mathrm{d}\lambda'\Big\|_{L^2(\mathbb{R}^d)} \lesssim \Big(\int_{1-c}^{1+c} \|g(\lambda')\|^2_{L^2(M_{\lambda'})}\,\mathrm{d}\lambda'\Big)^{1/2}\\ &\lesssim A\Big(\log\frac{1}{\delta_1}\Big)^{1/2}\Big(\log\frac{1}{\delta_2}\Big)^{1/2}\|f\|_{L^2(\mathbb{R}^d)}.\end{aligned}$$

□

6.4. *Local resolvent bound.* We use the same conventions as in the previous section. Additionally, in the following, the norm is the $L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d)$ operator norm. Recall that, by the discussion at the end of Section 6.2, the square root of the localized resolvent R_0^{low} can be replaced by a compactly supported multiplier satisfying the bound (32) with $\delta = 1/R$. As a consequence of Lemma 25, (35), and the discussion in Section 2, we immediately obtain the following resolvent bound.

Lemma 31. *Assume that (32) holds for $C^{(\delta_1)}$ and $C^{(\delta_2)}$, with $\delta_1, \delta_2 \asymp 1/R$. Then we have*

$$\boldsymbol{E}\|C^{(\delta_2)} V_\omega C^{(\delta_1)}\| \lesssim \langle h\rangle^{d/2}(\log\langle R\rangle)^{3/2}(\log\langle h\rangle + \log\langle R\rangle)^2\|V\|_{L^{2q}(\mathbb{R}^d)}.$$

By using the tail bound of Lemma 22 and rescaling, we obtain the following corollary.

Corollary 32. *Let $h, R, \lambda, M > 0$, and let $|\varepsilon| \ll \lambda$. Then the spectral radius of $R_0((\lambda+\mathrm{i}\varepsilon)^2)V_\omega$ is bounded by*

$$\mathrm{spr}(R_0 V) \lesssim M\langle \lambda h\rangle^{d/2}(\log\langle \lambda R\rangle)^{3/2}(\log\langle\lambda h\rangle + \log\langle\lambda R\rangle)^2\lambda^{d/(2q)-2}\|V\|_{L^{2q}(\mathbb{R}^d)},$$

except for ω in a set of measure at most $\exp(-cM^2)$.

6.5. ***Completion of the proof of Theorem 1.*** We first undo the change of variables $q \to 2q$. Theorem 1 then follows immediately from Proposition 11 and Corollary 32. ☐

7. Local to global arguments

7.1. ***Proof of Theorem 6.*** To complete the proof of Theorem 6, we rescale again to $\lambda = 1$. We decompose $V = \sum_{k\in\mathbb{Z}^+} V_k$ into dyadic pieces with support in $\{0 \le |x| \le 1\}$ for $k = 0$, and in $\{2^{k-1} \le |x| \le 2^k\}$ for $k \ge 1$. The assumption on V guarantees that $\|V_k\|_q \le 2^{-\delta k}\|\langle x\rangle^\delta V\|_q$. Instead of (9), we consider the multilinear expansion

$$[R_0V]^n = \sum_{\sigma_1,\dots,\sigma_n}\sum_{k_1,\dots,k_n} R_0^{\sigma_1}V_{k_1}R_0^{\sigma_2}V_{k_2}\cdots R_0^{\sigma_n}V_{k_n},$$

where we again omitted the spectral parameter z, and we are assuming, as we may, that $z = (1 + \mathrm{i}\varepsilon)^2$, $|\varepsilon| \ll 1$. By the same arguments in Section 2, it suffices to estimate the norms of elementary blocks of the form $C^{(\delta_{l-1})}V_{k_l}C^{(\delta_l)}$, where $\delta_l = (2^{k_l} + 2^{k_{l-1}})^{-1}$. Lemmas 25 and 30 and an analogue of Lemma 28 with $\delta = \delta_l$ or δ_{l-1} yield (again undoing the change of variables $q \to 2q$)

$$\boldsymbol{E}\|C^{(\delta_{l-1})}V_{k_l}C^{(\delta_l)}\| \lesssim (k_{l-1} + k_l + k_{l+1})\langle h\rangle^{d/2}\langle k_l\rangle^{1/2}(\log\langle h\rangle + \langle k_l\rangle)^2 2^{-\delta k_l}\|\langle x\rangle^\delta V\|_q.$$

Applying the tail bound of Lemma 22 yields

$$\|C^{(\delta_{l-1})}V_{k_l}C^{(\delta_l)}\| \le M_1(k_{l-1} + k_l + k_{l+1})\langle h\rangle^{d/2}\langle k_l\rangle^{1/2}(\log\langle h\rangle + \langle k_l\rangle)^2 2^{-\delta k_l}\|\langle x\rangle^\delta V\|_q,$$

except for ω in a set of measure at most $\exp(-c'M_1^2)$. Choosing $M_1 = M(k_{l-1} + k_l + k_{l+1})$ and summing the previous bound over $k_1, \dots, k_n$ yields

$$\operatorname{spr}(R_0V) = \lim_{n\to\infty}\|[R_0V]^n\|^{1/n} \lesssim \langle h\rangle^{d/2}(\log\langle h\rangle)^2\|\langle x\rangle^\delta V\|_q,$$

except for ω in a set of measure at most

$$\sum_{k_{l-1},k_l,k_{l+1}} \exp(-c'M_1^2) \le \exp(-cM^2).$$

This concludes the proof of Theorem 6. ☐

7.2. ***Sparse decomposition.*** To prove Theorem 7, we use a device reminiscent of an “epsilon removal lemma” (see, e.g., [Tao 1999]) but adapted to our multilinear bounds (and the resolvent as opposed to the Fourier restriction operator). For this reason, we need to perform several decompositions simultaneously:

(1) We first decompose V dyadically:

$$V = \sum_{i\in\mathbb{Z}_+} V_i, \quad V_i = V\mathbf{1}_{H_i \ge |V| \ge H_{i+1}}, \quad H_i = \inf\{t > 0 : |\{|V| > t\}| \le 2^{i-1}\}.$$

This is a “horizontal” dyadic decomposition since the widths of the supports of V_i are approximately 2^i. Here we are assuming that V is constant on the unit scale (hence $i \ge 0$ in the sum above). In view of (21), there is no loss of generality in this assumption for the purpose of proving estimates (this is the same argument as explained in the paragraph before [Tao 1999, Lemma 3.3]). Note that we have

$$\|H_i 2^{i/q}\|_{\ell_i^r(\mathbb{Z}_+)} \asymp \|V\|_{L^{q,r}},$$

where $L^{q,r}$ denotes a Lorentz space (see, e.g., [Tao 2006, Theorem 6.6]). Also note that $L^{q,q} = L^q$.

(2) Next, split each dyadic piece into a sum of “sparse families”,

$$V_i = \sum_{j=1}^{K_i} \sum_{k=1}^{N_i} V_{ijk}, \tag{40}$$

where, for fixed i and j, the V_{ijk} are supported on a “sparse collection” of balls $\{B(x_k, R_i)\}_{k=1}^{N_i}$. By this we mean that the support of V_{ijk} is contained in $B(x_k, R_i)$ and that the following definition is satisfied (see [Tao 1999, Definition 3.1]) for some sufficiently large γ (to be chosen later).

Definition 33. A collection $\{B(x_k, R)\}_{k=1}^N$ is γ-sparse if the centers x_k are $(RN)^\gamma$-separated.

For fixed $\gamma > 0$ and $K \geq 1$, [Tao 1999, Lemma 3.3] asserts that (40) holds with

$$K_i = \mathcal{O}(K 2^{i/K}), \quad N_i = \mathcal{O}(2^i), \quad R_i = \mathcal{O}(2^{i\gamma^K}). \tag{41}$$

7.3. *Spectral radius estimates.* The preceding decompositions produce a multilinear expansion of the Born series,

$$[R_0 V]^n = \sum_{\alpha_1,\dots,\alpha_n} R_0 V_{\alpha_1} R_0 V_{\alpha_2} \cdots R_0 V_{\alpha_n}, \tag{42}$$

where $\alpha_l = (i_l, j_l, k_l)$ and $i_l \in \mathbb{Z}_+$, $1 \leq j_l \leq K_{i_l}$, $1 \leq k_l \leq N_{i_l}$. To estimate the spectral radius of $R_0 V$, we estimate the summands in (42) in two different ways. For the first estimate, we follow a similar strategy as before. However, since the smoothing of the resolvent (see Section 6.2) now depends on the mutual positions of the supports of V_{α_l}, we consider the (slightly more general) elementary operators

$$C^{(\delta_1)} \mathbf{1}_{B_2} W C^{(\delta_2)}, \tag{43}$$

where the $B_k = B(x_k, R_k)$ are arbitrary balls and W is a bounded potential. As before, the $C^{(\delta)}$ are Fourier multipliers satisfying (32), now with

$$\delta_1 = \langle \mathrm{d}(B_1, B_2) + 2R_1 + 2R_2 \rangle^{-1}, \quad \delta_2 = \langle \mathrm{d}(B_2, B_3) + 2R_2 + 2R_3 \rangle^{-1}.$$

The operators (43) arise from an analogue of (31) and Lemma 28 for balls with different centers. In the same way that Lemma 31 and its corollary follow from Lemma 25, (35), and the tail bound of Lemma 22, we obtain

$$\|C^{(\delta_1)} \mathbf{1}_{B_2} W_\omega C^{(\delta_2)}\| \leq M_1 h^{d/2} (\log h)^2 \Big[\log\Big(\frac{1}{\delta_1} + \frac{1}{\delta_2}\Big)\Big]^{\mathcal{O}(1)} \|W\|_{L^q(B_2)} \tag{44}$$

for any $q \leq d+1$ and for all ω except for a set of measure at most $\exp(-c' M_1^2)$. Here we have assumed again, as we may, that $\lambda = 1$ and $R, h > 2$. For the remainder of this section we omit the (obvious) dependence on h. We also switch from the (modified) Vinogradov notation $A \lesssim B$ to the Hardy notation

$A \leq CB$ or Landau notation $A = \mathcal{O}(B)$, and we indicate the dependence of constants on q (since q will no longer be in a compact interval) or other related parameters. It is also convenient to use the letter A for quantities (norms, constants) containing $\mathcal{O}(1)$ terms that are bounded uniformly in n (and may change from line to line).

The case of interest is of course when the balls in (44) contain the supports of the potentials in (42), and W is one of these potentials. Similar to the proof of Theorem 6, we choose $M_1 = M[\log(1/\delta_1 + 1/\delta_2)]^{\mathcal{O}(1)}$ without qualitatively changing the estimate (44). In this way, the union bound for the probability of the complementary event yields

$$\begin{aligned} \boldsymbol{P}\Big(\bigcup_{\alpha_1,\alpha_2,\alpha_3} \{\omega : (44) \text{ does not hold}\}\Big) &\leq \sum_{\alpha_1,\alpha_2,\alpha_3} \exp(-c' M_1^2) \\ &\leq \sum_{i_1,i_2,i_3} N_{i_1} K_{i_1} N_{i_2} K_{i_2} N_{i_3} K_{i_3} \exp(-c' M_1^2) \\ &\leq \exp(-cM^2), \end{aligned}$$

and hence we have

$$\|R_0 V_{\alpha_1} R_0 V_{\alpha_2} \cdots R_0 V_{\alpha_n}\| \leq A M^n \prod_{l=1}^{n} \Big[\log\Big(\frac{1}{\delta_{\alpha_l}}\Big) + \log\Big(\frac{1}{\delta_{\alpha_{l+1}}}\Big)\Big]^{\mathcal{O}(1)} \|V_{\alpha_l}\|_q, \tag{45}$$

except for ω in a set of measure at most $\exp(-cM^2)$.

For the second estimate, we observe that, by the triangle inequality and Cauchy–Schwarz,

$$\|[R_0 V]^n\| \leq \sum_{\alpha_1,\ldots,\alpha_n} \|R_0 |V_{\alpha_1}|^{1/2}\| \, \|V_{\alpha_1}^{1/2} R_0 |V_{\alpha_2}|^{1/2}\| \cdots \|V_{\alpha_{n-1}}^{1/2} R_0 |V_{\alpha_n}|^{1/2}\| \, \|V_{\alpha_n}^{1/2}\|.$$

Here we are again assuming, as we may, that V is bounded. The operator norm $\|V_{\alpha_n}^{1/2}\|$ (equal to the L^∞ norm) will be annihilated by taking the n-th root at the end and letting n tend to infinity. Let

$$L_{\alpha,\beta} := \delta_{\alpha,\beta} + \mathrm{d}(B_\alpha, B_\beta),$$

where the balls B_α contain the support of V_α.

Lemma 34. *For* $q \leq \frac{1}{2}(d+1)$,

$$\|V_\alpha^{1/2} R_0 |V_\beta|^{1/2}\| \leq C_q L_{\alpha,\beta}^{1-(d+1)/(2q)} \|V_\alpha\|_q^{1/2} \|V_\beta\|_q^{1/2}. \tag{46}$$

Proof. To prove this, one uses the well known pointwise bound

$$|R_0^{(a+\mathrm{i}t)}(x-y)| \leq C_1 \mathrm{e}^{C_2 t^2} |x-y|^{-(d+1)/2+a} \tag{47}$$

for $a \in \big[\frac{1}{2}(d-1), \frac{1}{2}(d+1)\big]$ and $d \geq 2$ (see, e.g., [Lee and Seo 2019, (2.5)]), or the explicit formula for the resolvent kernel in $d = 1$. More precisely, consider the analytic family $V_\alpha^{\zeta/2} R_0^\zeta |V_\beta|^{\zeta/2}$. Then (47) implies that, for $\operatorname{Re}\zeta = q$, the kernel is bounded by

$$|V_\alpha(x)^{\zeta/2} R_0^\zeta(x-y) |V_\beta(y)|^{\zeta/2}| \leq C_1 \mathrm{e}^{C_2 (\operatorname{Im}\zeta)^2} L_{\alpha,\beta}^{-\eta} |V_\alpha(x)|^{q/2} |V_\beta(y)|^{q/2},$$

where $\eta = \frac{1}{2}(d+1) - q \geq 0$, leading to the Hilbert–Schmidt bound

$$\| V_\alpha^{\zeta/2} R_0^\zeta |V_\beta|^{\zeta/2} \| \leq C_\eta L_{\alpha,\beta}^{-\eta} \| V_\alpha \|_q^{q/2} \| V_\beta \|_q^{q/2}$$

for some constant C_η (allowed to change from line to line). Interpolating this with the trivial bound $\| V_\alpha^{\zeta/2} R_0^\zeta |V_\beta|^{\zeta/2} \| \leq C_1 \mathrm{e}^{C_2 (\operatorname{Im} \zeta)^2}$ for $\operatorname{Re} \zeta = 0$ yields (46). □

The previous lemma yields the second estimate

$$\| R_0 V_{\alpha_1} R_0 V_{\alpha_2} \cdots R_0 V_{\alpha_n} \| \leq A C_\eta^n \prod_{l=1}^{n} \| V_{\alpha_l} \|_{q_\eta} L_{\alpha_l, \alpha_{l+1}}^{-\eta'},$$

where $\eta' = \eta / \bigl(\frac{1}{2}(d+1) - \eta\bigr)$ and $q_\eta = \frac{1}{2}(d+1) - \eta$. Interpolating this with (45), we get, for $0 < \theta < 1$,

$$\| R_0 V_{\alpha_1} R_0 V_{\alpha_2} \cdots R_0 V_{\alpha_n} \| \leq A (C_\eta M)^n \prod_{l=1}^{n} [\log(1 + R_{i_{l-1}} + R_{i_l} + R_{i_{l+1}})]^{\mathcal{O}(1)} L_{\alpha_l, \alpha_{l+1}}^{-\theta\eta'/2} \| V_{\alpha_l} \|_q^{(1-\theta)} \| V_{\alpha_l} \|_{q_\eta}^{\theta},$$

except on an exceptional set of measure at most $\exp(-cM^2)$. (Here we used $L_{\alpha_l, \alpha_{l+1}}^{-\theta\eta'/2}$ to control $\mathrm{d}(B_{\alpha_l}, B_{\alpha_{l+1}})$ appearing in $\log(1/\delta_{\alpha_l})$.) Using that

$$\| V_{\alpha_l} \|_q \lesssim H_{i_l} 2^{i_l/q}$$

for all $q \geq 1$ and summing the resulting estimate first over k_1, then continuing up to k_{n-1}, yields

$$\sum_{k_1, \dots, k_{n-1}} \| R_0 V_{\alpha_1} R_0 V_{\alpha_2} \cdots R_0 V_{\alpha_n} \| \leq A (C_\eta M)^n \prod_{l=1}^{n-1} [\log(1 + R_{i_{l-1}} + R_{i_l} + R_{i_{l+1}})]^{\mathcal{O}(1)} H_{i_l} 2^{i_l((1-\theta)/q + \theta/q_\eta)}.$$

Here we have used that, for $\alpha_1 = (i_1, j_1, k_1)$, $\alpha_2 = (i_2, j_2, k_2)$ and $i_1,\ j_1,\ i_2,\ j_2,\ k_2$ fixed, the sum over k_1 is bounded:

$$\sum_{k_1 \leq N_{i_1}} \langle \mathrm{d}(B(x_{k_1}, R_{i_1}), B_{\alpha_2}) \rangle^{-\theta\eta'/2} = \mathcal{O}_{\gamma_0}(1), \tag{48}$$

uniformly in $i_1,\ j_1,\ i_2,\ j_2,\ k_2$, provided $\frac{1}{2}\theta\eta'\gamma_0 > 1$ and $\gamma \geq \gamma_0$. We will momentarily fix η and θ, and then choose $\gamma_0 = 4/(\eta'\theta)$. See also [Cho et al. 2022] for a precise version of Tao's lemma; there, it is clear that γ can be chosen. Note that, even though the balls in (48) may belong to different sparse families, we have that

$$\mathrm{d}(B(x_{k_1}, R_{i_1}), B_{\alpha_2}) \geq \tfrac{1}{2}(N_{i_1} R_{i_1})^\gamma$$

for all but at most one k_1. Indeed, suppose for contradiction that this does not hold for two distinct $k_1,\ k_1'$. Then by the triangle inequality,

$$\mathrm{d}(B(x_{k_1}, R_{i_1}), B(x_{k_1'}, R_{i_1})) < (N_{i_1} R_{i_1})^\gamma,$$

which contradicts the sparsity of the collection $\{B(x_{k_1}, R_{i_1})\}$.

Note that the last summation over k_n produces a $\mathcal{O}(2^{i_n})$ factor, but this can be absorbed into the constant A after summing over i_n and hence we do dot display it.

Summing over $j_1, \dots, j_n$ yields

$$\sum_{j_1,\dots,j_n} \sum_{k_1,\dots,k_n} \|R_0 V_{\alpha_1} R_0 V_{\alpha_2} \cdots R_0 V_{\alpha_n}\| \\ \leq A(C_\eta M)^n \prod_{l=1}^{n} [\log(1 + R_{i_{l-1}} + R_{i_l} + R_{i_{l+1}})]^{\mathcal{O}(1)} K_{i_l} H_{i_l} 2^{i_l((1-\theta)/q+\theta/q_\eta)},$$

where K_i is as in (41). Finally, summing over $i_1, \dots, i_n$ yields

$$\|[R_0 V]^n\| \leq A(C_\eta M K)^n \biggl(\sum_{i \in \mathbb{Z}_+} \langle i \rangle^{\mathcal{O}(1)} H_i 2^{i((1-\theta)/q+\theta/q_\eta+1/K)} \biggr)^n.$$

Once K is fixed, we choose η and θ such that $0 < \theta(1/q_\eta - 1/q) < 1/K$. Then

$$\operatorname{spr}(R_0 V_\omega) = \lim_{n\to\infty} \|[R_0 V]^n\|^{1/n} \leq C_{\eta,K} M \sum_{i \in \mathbb{Z}_+} H_i 2^{i/q} 2^{3i/K}, \tag{49}$$

where we used that $\langle i \rangle^{\mathcal{O}(1)} \leq C_K 2^{i/K}$.

7.4. ***Completion of the proof of Theorem 7.*** We use (49) for $\tilde{q} > q$ instead of q; that is, we now regard $\frac{1}{2}(d+1) < q < d+1$ as given and choose $\tilde{q} < d+1$ and K such that $1/\tilde{q} + 3/K < 1/q$. Then

$$\operatorname{spr}(R_0 V_\omega) \lesssim \sup_{i \in \mathbb{Z}^+} H_i 2^{i/q} \sum_{i \in \mathbb{Z}_+} 2^{i(1/\tilde{q}-1/q+3/K)} \leq C_{\tilde{q},K} M \|V\|_{L^{q,\infty}}.$$

Clearly, the choice of $\tilde{q}$ depends only on q, K, d and $\|V\|_{L^q} \leq \|V\|_{L^{q,\infty}}$. We have thus proved the main estimate of this section, which also completes the proof of Theorem 7.

Lemma 35. *Let $q < d+1$. Then there exists c and M_0 such that, for all $M \geq M_0$, $z = (\lambda + \mathrm{i}\varepsilon)^2$, $\lambda \asymp 1$, $|\varepsilon| \ll 1$, and $V \in L^q(\mathbb{R}^d)$,*

$$\operatorname{spr}(R_0(z) V_\omega) \leq M \|V\|_q,$$

except for ω in a set of measure at most $\exp(-cM^2)$.

7.5. ***Global extension bound.*** For potential future reference we include a similar bound to that proved in Lemma 35 but for the norms of the elementary operators (11) instead of the spectral radius of $R_0 V$.

Proposition 36. *Let $q < d+1$. Then there exist constants M_0 and c such that, for any $M \geq M_0$, $\lambda, \lambda' \asymp 1$, and $V \in L^q(\mathbb{R}^d)$,*

$$\|\mathcal{E}_\lambda^* V_\omega \mathcal{E}_{\lambda'}\| \leq M \langle h \rangle^{d/2} (\log\langle h \rangle)^2 \|V\|_{L^q},$$

except for ω in a set of measure at most $\exp(-cM^2)$.

We refer to [Cuenin and Merz 2023, Theorem 5] for novel bounds on $\mathcal{E}_\lambda^* V_\omega \mathcal{E}_{\lambda'}$ in Schatten norms.

In the following, we use the notation $\|V\|_{\ell^\infty L^q} = \sup_{j \leq N} \|V\|_{L^q(B(x_j,R))}$ and $V_j = V \mathbf{1}_{(B(x_j,R))}$, whenever V is supported on a γ-sparse collection $\{B(x_j, R)\}_{j=1}^N$. We will show that Proposition 36 follows from the subsequent lemma.

Lemma 37. *There exist constants* $M_0,\ c,\ \gamma_0>0$ *such that the following holds. For any* $R>0,\ 0<h<R$, $\lambda,\lambda'\asymp 1,\ q<d+1,\ N\in\mathbb{N},\ \gamma\geq\gamma_0$, *for any* $V\in L^q(\mathbb{R}^d)$ *supported on a* γ*-sparse collection* $\{B(x_j,R)\}_{j=1}^N$, *and, for any* $M\geq M_0,\ \varepsilon>0$,

$$\|\mathcal{E}_\lambda^* V_\omega\mathcal{E}_{\lambda'}\|\leq C_{q,\varepsilon}(M^2+\log N)^{1/2}\langle h\rangle^{d/2}(\log\langle h\rangle)^2\langle R\rangle^\varepsilon\|V\|_{\ell^\infty L^q},$$

except for ω *in a set of measure at most* $\exp(-cM^2)$.

Proof. We may assume without loss of generality that $\lambda,\lambda'=1$ and $R>2$. We omit the subscripts in $\mathcal{E}_\lambda^*$ and $\mathcal{E}_{\lambda'}$ as well as the (obvious) h-dependence (i.e., we set $h=1$). Consider the operators

$$T_j=\mathcal{E}^*V_j\mathcal{E},\quad 1\leq j\leq N,$$

where we omitted ω from the notation. Then

$$T_iT_j^*=\mathcal{E}^*V_i\mathcal{E}\mathcal{E}^*\overline{V}_j\mathcal{E},\quad T_i^*T_j=\mathcal{E}^*\overline{V}_i\mathcal{E}\mathcal{E}^*V_j\mathcal{E}.$$

As in the endpoint proof of the Stein–Tomas theorem (see, e.g., [Stein 1993, IX.1.2.2]) we embed $\mathcal{E}\mathcal{E}^*$ into an analytic family of operators U_s in the strip $\frac{1}{2}(1-d)\leq\operatorname{Re}s\leq 1$, satisfying

$$\|U_s\|_{L^2\to L^2}\lesssim 1,\quad \operatorname{Re}s=1,\qquad \|U_s\|_{L^1\to L^\infty}\lesssim 1,\quad \operatorname{Re}s=\tfrac{1}{2}(1-d),$$

and $U_0=\mathcal{E}\mathcal{E}^*$. Similar to the proof of Lemma 34, we then use complex interpolation on the family $|V_i|^{(1-s)/2}U_s|V_j|^{(1-s)/2}$ to obtain the bound

$$\||V_i|^{1/2}\mathcal{E}\mathcal{E}^*|V_j|^{1/2}\|\lesssim L_{ij}^{-\eta'}\|V_i\|_{L^{q_\eta}}^{1/2}\|V_j\|_{L^{q_\eta}}^{1/2}$$

for $\eta'=\eta/q_\eta,\ q_\eta=\frac{1}{2}(d+1)-\eta$, and $0<\eta\ll 1$. By the Stein–Tomas theorem and Hölder's inequality, we also have

$$\|\mathcal{E}^*V_i^{1/2}\|\lesssim\|V_j\|_{L^{q_\eta}}^{1/2},\quad \|V_i^{1/2}\mathcal{E}\|\lesssim\|V_j\|_{L^{q_\eta}}^{1/2}.$$

Combining the last two displayed formulas yields the deterministic bound

$$\|T_iT_j^*\|^{1/2}+\|T_i^*T_j\|^{1/2}\lesssim L_{ij}^{-\eta'}\|V\|_{\ell^\infty L^{q_\eta}}$$

for all $i,j\leq N$. On the other hand, the bound of Lemma 25 (and changing variables $2q\to q$) yields

$$\|T_iT_j^*\|^{1/2}+\|T_i^*T_j\|^{1/2}\leq M_1(\log R)^{5/2}\|V\|_{\ell^\infty L^q}$$

for all $i,j\leq N$, and for all ω except for an exceptional set of measure at most $N\exp(-cM_1^2)$. Interpolating the previous two estimates as in the proof of Lemma 35, we get, by the Cotlar–Stein lemma and (48),

$$\|\mathcal{E}^*V\mathcal{E}\|\leq C_{\eta,\gamma_0}[(\log R)^{5/2}\|V\|_{\ell^\infty L^q}]^{1-\theta}\|V\|_{\ell^\infty L^{q_\eta}}^\theta$$

for any $\theta\in(0,1)$ and for all ω except for an exceptional set, provided $\frac{1}{2}\theta\eta'\gamma_0>1$ and $\gamma\geq\gamma_0$. Finally, we use Hölder's inequality

$$\|V\|_{\ell^\infty L^{q_\eta}}\lesssim R^{d/s_\eta}\|V\|_{\ell^\infty L^q},\quad \frac{1}{q_\eta}=\frac{1}{s_\eta}+\frac{1}{q},$$

to convert the previous estimate to

$$\|\mathcal{E}^*V\mathcal{E}\|\leq C_{\eta,\gamma_0}[\log R]^{5(1-\theta)/2}R^{\theta d/s_\eta}\|V\|_{\ell^\infty L^q}.$$

We now fix $0 < \eta \ll 1$ (small, but independent of ε) and choose $\theta \in (0,1)$ such that

$$[\log R]^{5(1-\theta)/2} R^{\theta d/s} \leq R^{\varepsilon}.$$

Moreover, we choose $M_1 = (M^2 + c^{-1}\log N)^{1/2}$, which ensures that the exceptional set has measure at most $\exp(-cM^2)$. Then the claim holds with the choice $\gamma_0 = 4/(\eta'\theta)$. The remainder of the proof is the same as that of Lemma 35. ☐

Proof of Proposition 36. We again use the sparse decomposition of Section 7.2 and recall the bounds (41) on $K_i,\ N_i,\ R_i$. As before, we also set $\lambda, \lambda', h = 1$. Lemma 37 yields the estimate

$$\|\mathcal{E}^* V_{ij}\mathcal{E}\| \leq C_{q,\varepsilon}(M_i^2 + \log N_i)^{1/2} R_i^{\varepsilon} \|V_{ij}\|_q$$

for all $q < d+1$, uniformly in i and j, and for ω outside of a set of measure $\exp(-cM_i^2)$. Here we are assuming, as we may, that $M_i, N_i, R_i > 2$, say. We may choose M_i freely, and we take $M_i = 2M\langle i\rangle^{\delta}$, with $\delta > 0$. Summing over j yields, by the triangle inequality,

$$\|\mathcal{E}^* V_i\mathcal{E}\| \leq C_{q,\varepsilon} K_i (M_i^2 + \log N_i)^{1/2} R_i^{\varepsilon} \|V_i\|_q.$$

Summing over i,

$$\|\mathcal{E}^* V_i\mathcal{E}\| \leq C_{q,\varepsilon,K} \sum_{i\in\mathbb{Z}_+} H_i 2^{i(1/q+2/K+\varepsilon\gamma^K)}.$$

Here we also used (41), $\|V_i\|_q \lesssim H_i 2^{i/q}$, and $(M^2 + \log N_i)^{1/2} \leq C_K M 2^{i/K}$. We again apply this bound for $\tilde{q} > q$ instead of q, this time with $\tilde{q} < d+1$ and $K,\ \varepsilon$ such that $1/\tilde{q} + 2/K + \varepsilon\gamma^K < 1/q$. Then the claimed bound again follows by summing a geometric series. The union bound yields that this bound holds outside an exceptional set of measure at most

$$\sum_{i,j} \exp(-c'M_i^2) \leq \sum_i K_i \exp(-c'M_i^2) \leq \exp(-cM^2),$$

due to the choice of M_i. ☐

Appendix A: Geometric series estimate

Lemma 38. *Let* $A > 0$. *Then we have*

$$\sum_{k,k'\in\mathbb{Z}_+} \min(2^{-k-k'}, A) \lesssim \begin{cases} A(1+(\log A)^2) & \text{if } A < 1, \\ 1 & \text{if } A \geq 1. \end{cases}$$

Proof. The case $A \geq 1$ is trivial. Assume $A < 1$. We split the double sum into the obvious regions $\Sigma_1 = \{(k,k') : 2^{-k-k'} \leq A\}$ and $\Sigma_2 = \{(k,k') : 2^{-k-k'} > A\}$. Then we have

$$\sum_{(k,k')\in\Sigma_1} \min(2^{-k-k'}, A) = \sum_{k'\in\mathbb{Z}_+} 2^{-k'} \sum_{k:2^{-k}\leq 2^{k'}A} 2^{-k} \lesssim \sum_{k'\in\mathbb{Z}_+} 2^{-k'} \min(1, 2^{k'}A).$$

Splitting the last sum again in the obvious way yields

$$\sum_{(k,k')\in\Sigma_1} \min(2^{-k-k'}, A) \lesssim A(1+\log A^{-1}).$$

Turning to the contribution of Σ_2, we have

$$\sum_{(k,k')\in\Sigma_2} \min(2^{-k-k'}, A) = A \sum_{k'\in\mathbb{Z}_+} |\{k \in \mathbb{Z}_+ : 2^{-k} > 2^{k'} A\}| \le A \sum_{k'\in\mathbb{Z}_+} (\log A - k')_+ \le A(\log A)^2.$$

The claim follows since $\log A^{-1} \le 1 + (\log A)^2$. □

We now provide details of the calculation at the end of the proof of Lemma 25. Without loss of generality we may assume that $\|V\|_{2q} = 1$. By Lemma 26, we have

$$\sum_{k,k'\in\mathbb{Z}_+} \int \boldsymbol{E} \max_{\mathcal{F}_k\times\mathcal{F}_{k'}} |X_{\xi,\xi'}| \, dy \, d\tau \, d\tau' \lesssim R^{d-d/(2q)} \sum_{k,k'\in\mathbb{Z}_+} \min(2^{-k-k'}, A),$$

with $A = R^{-d+d/(2q)}(\log R)^{1/2} h^{d/2}$, where we recall that we are assuming that $R, h > 2$. Since we may always assume that $R \gg 1$ and $h < R$ (otherwise there is no randomization), we have $A \ll 1$, and thus

$$R^{d-d/(2q)} \sum_{k,k'\in\mathbb{Z}_+} \min(2^{-k-k'}, A) \lesssim (\log R)^{1/2} h^{d/2} (\log h + \log R)^2$$

by Lemma 38.

Appendix B: Knapp example

As mentioned in the introduction, we give an example that suggests optimality of the key bounds of Lemmas 25 and 31 with respect to the Lebesgue exponent q. (Here we work with second moments whereas in these lemmas we used first moments.)

In view of the foliation (36) it is sufficient to prove optimality of Lemma 25. To this end, let V be the indicator function of the tube

$$T_R = \{(x_1, x') : |x_1| < R, \ |x'| < R^{1/2}\},$$

normalized in L^q, i.e., $V = R^{-(d+1)/(2q)} \mathbf{1}_{T_R}$ (we will mollify this later). Here $R > 1$ is a large parameter. We consider the randomization V_ω (as in (3)) of this potential. We assume in the following that $\lambda = 1$ in Lemma 25 and that h is sufficiently small (to be fixed later). It is easy to see that we have

$$\begin{aligned} \boldsymbol{E}\|\mathcal{E}^* V_\omega \mathcal{E}\|^2 &= \boldsymbol{E}\|\mathcal{E}^* \overline{V}_\omega \mathcal{E}\mathcal{E}^* V_\omega \mathcal{E}\| = \boldsymbol{E} \sup_{\|f\|_{L^2(M)}=1} |\langle \mathcal{E}\mathcal{E}^* V_\omega \mathcal{E} f, V_\omega \mathcal{E} f\rangle| \\ &\ge \sup_{\|f\|_{L^2(M)}=1} |\boldsymbol{E}\langle \mathcal{E}\mathcal{E}^* V_\omega \mathcal{E} f, V_\omega \mathcal{E} f\rangle| \ge \sup_{\|f\|_{L^2(M)}=1} |\operatorname{Re} \boldsymbol{E}\langle \mathcal{E}\mathcal{E}^* V_\omega \mathcal{E} f, V_\omega \mathcal{E} f\rangle|, \end{aligned}$$

where we recall that M is the unit sphere in $\mathbb{R}^d$. In order to estimate the last expression from below, we consider a Knapp example (see, e.g., [Demeter 2020, Example 1.8])

$$f_R(\xi) := R^{(d-1)/4} \eta(R\xi_1, R^{1/2}\xi'),$$

where $\xi = (\xi_1, \xi') \in \mathbb{R} \times \mathbb{R}^{d-1}$ and $\eta \in C_0^\infty(B(0, 2))$ is a nonnegative bump function equal to 1 on $B(0, 1)$. The normalization is chosen such that (up to an R-independent constant) $\|f_R\|_{L^2(M)} = 1$. Assuming, as we may, that $\boldsymbol{E}\omega_i\omega_j = \delta_{ij}$, we have

$$\boldsymbol{E}\langle \mathcal{E}\mathcal{E}^* V_\omega \mathcal{E} f, V_\omega \mathcal{E} f\rangle = \sum_{j\in h\mathbb{Z}^d} \int_{\mathbb{R}^d\times\mathbb{R}^d} (\mathcal{E}\mathcal{E}^*)(x-y) \overline{V_j(y)(\mathcal{E} f)(y)} V_j(x)(\mathcal{E} f)(x) \, dy \, dx,$$

where we wrote $V_j = V_\omega \mathbf{1}_{Q_j}$ and $Q_j = j + hQ$. Since $\mathcal{E}\mathcal{E}^*$ is proportional to convolution with $(\mathrm{d}\sigma)^\vee$ and the latter oscillates on the unit scale, there are positive constants r and c such that $\mathrm{Re}(\mathrm{d}\sigma)^\vee(u) \geq c$ for $|u| \leq r$ (this follows from standard stationary phase asymptotics). Assume now that $2h < r$. Then, using the above Knapp example f_R as a test function and changing variables $u = x - y$, we obtain

$$\boldsymbol{E}\|\mathcal{E}^* V_\omega \mathcal{E}\|^2 \gtrsim \mathrm{Re} \sum_{j \in h\mathbb{Z}^d} \int_{\mathbb{R}^d \times \mathbb{R}^d} \overline{F_j(x-u)} F_j(x)\, \mathrm{d}u\, \mathrm{d}x \quad (F_j = V_j \mathcal{E} f_R)$$

up to an error involving the imaginary part $\overline{F_j(x-u)} F_j(x)$ (which is small as we will see). At this point we consider a smooth (at the scale of T_R) version of the potential; this does not affect the previous arguments. What we gain by this is that now $|\nabla F_j(x)| = \mathcal{O}(R^{-1/2})|F_j(x)|$, whence, by Taylor expansion,

$$\sum_{j \in h\mathbb{Z}^d} \int_{\mathbb{R}^d \times \mathbb{R}^d} \overline{F_j(x-u)} F_j(x)\, \mathrm{d}u\, \mathrm{d}x = (2h)^d (1 - \mathcal{O}(R^{-1/2})) \sum_{j \in h\mathbb{Z}^d} \int_{\mathbb{R}^d} |F_j(x)|^2\, \mathrm{d}x.$$

Computing the integral, this shows that

$$\boldsymbol{E}\|\mathcal{E}^* V_\omega \mathcal{E}\|^2 \gtrsim h^d R^{1-(d+1)/q} \|V\|_q,$$

which implies that $q \leq d+1$ is necessary for Lemma 25 to hold (since R is arbitrarily large). If $h \gg 1$, one uses $\mathrm{Re}(\mathrm{d}\sigma)^\vee(u) \geq c|u|^{-(d-1)/2}$ for at least one percent of the u in $B(0, 2h)$. Then the u integration gives $h^{(d+1)/2}$ instead of h^d.

Added in proof

Recently, we have proved estimates for Schatten norms of the elementary operators $C^{(\delta_2)} V_\omega C^{(\delta_1)}$ for pointwise decaying potentials. They allowed us to prove estimates for sums over functions of the distances of the eigenvalues of $-\Delta + V_\omega$ to the origin or to the positive real axis, which quantify the eigenvalue accummulation. These results appear in [Cuenin and Merz 2023].

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JEAN-CLAUDE CUENIN: `j.cuenin@lboro.ac.uk`
Department of Mathematical Sciences, Loughborough University, Loughborough, United Kingdom

KONSTANTIN MERZ: `k.merz@tu-bs.de`
Current address: Institut für Analysis und Algebra, Technische Universität Braunschweig, Braunschweig, Germany
Department of Mathematics, Graduate School of Science, Osaka University, Osaka, Japan

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