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ROTATING WAVES IN NONLINEAR MEDIA AND CRITICAL DEGENERATE SOBOLEV INEQUALITIES

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We investigate the presence of rotating wave solutions of the nonlinear wave equation $\partial_t^2 v - \Delta v + mv = |v|^{p-2}v$ in $\mathbb{R} \times \boldsymbol{B}$, where $\boldsymbol{B} \subset \mathbb{R}^N$ is the unit ball, complemented with Dirichlet boundary conditions on $\mathbb{R} \times \partial \boldsymbol{B}$. Depending on the prescribed angular velocity α of the rotation, this leads to a Dirichlet problem for a semilinear elliptic or degenerate elliptic equation. We show that this problem is governed by an associated critical degenerate Sobolev inequality in the half-space. After proving this inequality and the existence of associated extremal functions, we then deduce necessary and sufficient conditions for the existence of ground state solutions. Moreover, we analyze under which conditions on α, m and p these ground states are nonradial and therefore give rise to truly rotating waves. Our approach carries over to the corresponding Dirichlet problems in an annulus and in more general Riemannian models with boundary, including the hemisphere. We briefly discuss these problems and show that they are related to a larger family of associated critical degenerate Sobolev inequalities.

1. Introduction

Within a simple model, the analysis of wave propagation in an ambient medium with nonlinear response leads to the study of a nonlinear wave equation of the type

$$\partial_t^2 v - \Delta v + mv = f(v) \quad \text{in } \mathbb{R} \times \Omega, \tag{1-1}$$

in an ambient domain $\Omega \subset \mathbb{R}^N$ with mass parameter $m \geq 0$ and nonlinear response function f. In the case $m = 0$, (1-1) is the classical nonlinear wave equation, while the case $m > 0$ is also known as a nonlinear

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Klein–Gordon equation. For nonlinearities of the form $f(v) = g(|v|^2)v$ with a real-valued function g, standing wave solutions can be found by the ansatz

$$v(t, x) = e^{-ikt}u(x), \quad k > 0, \tag{1-2}$$

with a real-valued function u. Depending on the frequency parameter k, this reduces (1-1) either to a stationary nonlinear Schrödinger or a nonlinear Helmholtz equation (see, e.g., [Evéquoz and Weth 2015] for more details).

The resulting stationary nonlinear Schrödinger equation has been studied extensively in the past four decades by variational methods, see, e.g., the monograph [Ambrosetti and Malchiodi 2006]. Due to a lack of a direct variational framework, the nonlinear Helmholtz equation requires a different approach and has been studied more recently, e.g., in [Chen et al. 2021; Evéquoz and Weth 2015; Gutiérrez 2004; Mandel et al. 2017; 2021] by dual variational methods and bifurcation theory.

Clearly, the amplitude $|v|$ of a solution v of (1-1) given by the ansatz (1-2) remains time-independent. As a consequence, the analysis of standing wave solutions does not lead to a full understanding of (1-1) from a dynamical point of view and should be complemented, in particular, by the study of nonstationary real-valued time-periodic solutions, traveling wave solutions and scattering solutions. We stress that the ansatz (1-2) does not give rise to nonstationary real-valued time-periodic solutions since the nonlinearity of the problem does not allow to pass to real and imaginary parts.

In the case where $\Omega = \mathbb{R}^N$ and $f(v)$ in (1-1) is replaced by $q(x)f(v)$ with a compactly supported weight function q, spatially localized real-valued time-periodic solutions, also called breathers, have attracted increasing attention recently, see, e.g., [Hirsch and Reichel 2019; Mandel and Scheider 2021]. In the case where Ω is a radial domain, a further interesting type of real-valued time-periodic solution is given by *rotating wave solutions*. In particular, if Ω is a bounded radial domain and (1-1) is complemented with the Dirichlet boundary condition $v = 0$ on $\mathbb{R} \times \partial\Omega$, the existence of rotating waves and their variational characterization arises as a natural question which, to our knowledge, has not been addressed systematically so far.

The main purpose of the present paper is to provide such a systematic study. While we mainly focus on the case where $\Omega = \boldsymbol{B}$ is the open unit ball in $\mathbb{R}^N$, we will also address the case where Ω is an annulus or a Riemannian model with boundary; see Sections 5 and 6 below. Specifically, we study the case of a focusing nonlinearity of the form $f(v) = |v|^{p-2}v$, which leads to the superlinear problem

$$\begin{cases} \partial_t^2 v - \Delta v + mv = |v|^{p-2}v & \text{in } \mathbb{R} \times \boldsymbol{B}, \\ v = 0 & \text{on } \mathbb{R} \times \partial\boldsymbol{B}, \end{cases} \tag{1-3}$$

for $N \geq 2$, where

$$2 < p < 2^* \quad \text{and} \quad m > -\lambda_1(\boldsymbol{B}).$$

Here, $\lambda_1(\boldsymbol{B})$ denotes the first Dirichlet eigenvalue of $-\Delta$ on $\boldsymbol{B}$ and 2^* denotes the critical Sobolev exponent given by

$$2^* = \frac{2N}{N-2} \quad \text{for } N \geq 3 \qquad \text{and} \qquad 2^* = \infty \quad \text{for } N = 2.$$

The ansatz for time-periodic rotating solutions of (1-3) is given by

$$v(t,x) = u(R_{\alpha t}(x)), \tag{1-4}$$

where, for $\theta \in \mathbb{R}$, we let $R_\theta \in O(N)$ denote a planar rotation in $\mathbb{R}^N$ with angle θ, so the constant $\alpha > 0$ in (1-4) is the angular velocity of the rotation. Without loss of generality, we may assume that

$$R_\theta(x) = (x_1 \cos\theta + x_2 \sin\theta, -x_1 \sin\theta + x_2 \cos\theta, x_3, \dots, x_N) \quad \text{for } x \in \mathbb{R}^N,$$

so R_θ is the rotation in the (x_1, x_2)-plane with fixed point set $\{0_{\mathbb{R}^2}\} \times \mathbb{R}^{N-2}$. In the following, we call a function u on the unit ball (x_1, x_2)-*nonradial* if it is not R_θ-invariant for at least one angle $\theta \in \mathbb{R}$. If the profile function u in (1-4) is (x_1, x_2)-nonradial, then the corresponding solution v can be interpreted as a rotating wave in a medium with nonlinear response given by the right-hand side of (1-3). The ansatz (1-4) reduces (1-3) to

$$\begin{cases} -\Delta u + \alpha^2\, \partial_\theta^2 u + mu = |u|^{p-2}u & \text{in } \boldsymbol{B}, \\ u = 0 & \text{on } \partial \boldsymbol{B}, \end{cases} \tag{1-5}$$

where $\partial_\theta = x_1\, \partial_{x_2} - x_2\, \partial_{x_1}$ denotes the associated angular derivative operator. We point out that a seemingly closely related equation, with the term $\alpha^2\, \partial_\theta^2 u$ replaced by $-\alpha^2\, \partial_\theta^2 u$, arises in an ansatz for solutions of nonlinear Schrödinger equations in $\mathbb{R}^3$ with invariance with respect to screw motion, see [Agudelo et al. 2022] and also [del Pino et al. 2012] for a related work on Allen–Cahn equations. Note, however, that the positive sign of the term $\alpha^2\, \partial_\theta^2 u$ results in a drastic change of the nature of the problem, as the operator $-\Delta + \alpha^2\, \partial_\theta^2$ loses uniform ellipticity in $\boldsymbol{B}$ if $\alpha \geq 1$. For balls of arbitrary radius, the threshold for α corresponds to the inverse of the radius. In our case, for $\alpha = 1$, we will observe that ellipticity is lost on the great circle

$$\gamma := \{x \in \partial \boldsymbol{B} : x_3 = \cdots = x_N = 0\}, \tag{1-6}$$

which equals $\partial \boldsymbol{B}$ in the case $N = 2$. This also distinguishes the study of (1-5) from the related study of rotating solutions to nonlinear Schrödinger equations, where the angular velocity α appears within a first-order term which does not affect the ellipticity of the associated Schrödinger operator, see, e.g., [Lieb and Seiringer 2006; Seiringer 2002].

If a solution u of (1-5) satisfies $\partial_\theta u \equiv 0$ in $\boldsymbol{B}$, then u solves the classical stationary nonlinear Schrödinger equation $-\Delta u + mu = |u|^{p-2}u$ in $\boldsymbol{B}$ with Dirichlet boundary conditions on $\partial \boldsymbol{B}$, so it satisfies (1-5) with $\alpha = 0$. If, in addition, u is positive, then u has to be a radial function as a consequence of the symmetry result of Gidas, Ni and Nirenberg [Gidas et al. 1979]. Thus, the ansatz (1-4) then merely gives rise to a radial stationary solution of (1-3). We mention here that radially symmetric nonstationary solutions of (1-1) in $\Omega = \boldsymbol{B}$ were first studied by Ben-Naoum and Mawhin [1993] for sublinear nonlinearities, and more recently by Chen and Zhang [2014; 2016; 2017]. In this problem, the spectral properties of the radial wave operator lead to delicate assumptions on the dimension as well as the ratio between the radius of the ball and the period length. The main purpose of the present paper is to analyze for which range of parameters α, m and p ground state solutions of (1-5) exist and to distinguish under which assumptions on α, m and p they are radial or (x_1, x_2)-nonradial and therefore correspond to rotating waves via the ansatz (1-4).

By a ground state solution of (1-5), we mean a solution characterized as a minimizer of the minimization problem for

$$\mathscr{C}_{\alpha,m,p}(\boldsymbol{B}) := \inf_{u \in H_0^1(\boldsymbol{B})\setminus\{0\}} R_{\alpha,m,p}(u), \tag{1-7}$$

where, for $m \in \mathbb{R}$, $\alpha \geq 0$ and $p \in [2, 2^*)$, we consider the associated Rayleigh quotient $R_{\alpha,m,p}$ given by

$$R_{\alpha,m,p}(u) = \frac{\int_{\boldsymbol{B}} (|\nabla u|^2 - \alpha^2 |\partial_\theta u|^2 + m u^2)\, dx}{\left(\int_{\boldsymbol{B}} |u|^p\, dx\right)^{2/p}}, \quad u \in H_0^1(\boldsymbol{B}) \setminus \{0\}. \tag{1-8}$$

As we shall see in Remark 4.20 below, this minimization problem is only meaningful for $0 \leq \alpha \leq 1$, since, for every $p \in [2, 2^*)$ and $m \in \mathbb{R}$, we have

$$\mathscr{C}_{\alpha,m,p}(\boldsymbol{B}) = -\infty \quad \text{for } \alpha > 1.$$

Moreover, for every $p \in [2, 2^*)$ and $m \in \mathbb{R}$,

$$\text{the function } \alpha \mapsto \mathscr{C}_{\alpha,m,p}(\boldsymbol{B}) \text{ is continuous and nonincreasing on } [0, 1]. \tag{1-9}$$

In the case $0 < \alpha < 1$, the operator $-\Delta + \alpha^2\, \partial_\theta^2$ is uniformly elliptic, as can be seen by writing the operator in polar coordinates as

$$-\Delta + \alpha^2\, \partial_\theta^2 = -\Delta_r u - \frac{1}{r^2} \Delta_{\mathbb{S}^{N-1}} u + \alpha^2\, \partial_\theta^2 u, \tag{1-10}$$

where $\Delta_{\mathbb{S}^{N-1}}$ denotes the Laplace–Beltrami operator on the unit sphere $\mathbb{S}^{N-1}$. In this case the existence of minimizers of $R_{\alpha,m,p}$ on $H_0^1(\boldsymbol{B}) \setminus \{0\}$ follows by a standard compactness and weak lower-semicontinuity argument. However, even in this case it is difficult to decide in general whether minimizers are radial or nonradial functions. This is due to competing effects. Firstly, the additional term $-\alpha^2 \|\partial_\theta u\|^2_{L^2(\boldsymbol{B})}$ favors (x_1, x_2)-nonradial functions as energy minimizers. On the other hand, the Pólya–Szegö inequality yields

$$\int_{\boldsymbol{B}} |\nabla u^*|^2\, dx \leq \int_{\boldsymbol{B}} |\nabla u|^2\, dx,$$

where u^* denotes the (radial) Schwarz symmetrization of a function $u \in H_0^1(\boldsymbol{B})$.

Since $R_{\alpha,m,p}(u) = R_{0,m,p}(u)$ for every radial function $u \in H_0^1(\boldsymbol{B}) \setminus \{0\}$ and every $\alpha \in [0, 1]$, a sufficient condition for the (x_1, x_2)-nonradiality of all ground state solutions is the inequality

$$\mathscr{C}_{\alpha,m,p}(\boldsymbol{B}) < \mathscr{C}_{0,m,p}(\boldsymbol{B}). \tag{1-11}$$

In particular, we will be interested in proving this inequality for α close to 1. As mentioned already, the borderline case $\alpha = 1$ differs significantly from the case $0 \leq \alpha < 1$, since in this case $-\Delta + \partial_\theta^2$ fails to be uniformly elliptic in a neighborhood of the great circle γ defined in (1-6). We shall see in this paper that the minimization problem in the case $\alpha = 1$ is essentially governed by a degenerate anisotropic critical Sobolev inequality in the half-space. The corresponding critical exponent in this Sobolev inequality is given by

$$2_1^* := \frac{4N+2}{2N-3}.$$

This exponent's relevance is indicated by our first main result which yields the following characterization.

Theorem 1.1. *Let* $m > -\lambda_1(\boldsymbol{B})$ *and* $p \in (2, 2^*)$.

(i) *If* $\alpha \in (0, 1)$, *then there exists a ground state solution of* (1-5).

(ii) *We have*

$$\mathscr{C}_{1,m,p}(\boldsymbol{B}) = 0 \quad \text{for } p > 2_1^* \qquad \text{and} \qquad \mathscr{C}_{1,m,p}(\boldsymbol{B}) > 0 \quad \text{for } p \le 2_1^*. \tag{1-12}$$

Moreover, for any $p \in (2_1^*, 2^*)$, *there exists* $\alpha_p \in (0, 1)$ *with the property that*

$$\mathscr{C}_{\alpha,m,p}(\boldsymbol{B}) < \mathscr{C}_{0,m,p}(\boldsymbol{B}) \quad \text{for } \alpha \in (\alpha_p, 1],$$

and therefore every ground state solution of (1-5) *is* (x_1, x_2)*-nonradial for* $\alpha \in (\alpha_p, 1)$.

The following new degenerate Sobolev inequality is an immediate consequence of the special case $m = 0,\ \alpha = 1$ in Theorem 1.1.

Corollary 1.2. *We have*

$$\left(\int_{\boldsymbol{B}} |u|^{2_1^*}\,dx\right)^{2/2_1^*} \le \frac{1}{\mathscr{C}_{1,0,2_1^*}(\boldsymbol{B})} \int_{\boldsymbol{B}} (|\nabla u|^2 - |\partial_\theta u|^2)\,dx \quad \text{for } u \in H_0^1(\boldsymbol{B}).$$

Moreover, the exponent 2_1^* *is optimal in the sense that no such inequality holds for* $p > 2_1^*$.

Theorem 1.1 yields symmetry-breaking of ground states for suitable parameter values of p, α and m, but the precise parameter range giving rise to this symmetry-breaking remains largely open. To shed further light on this question, we state the following result which establishes uniqueness and radial symmetry of ground state solutions for α close to zero and every $m \ge 0,\ 2 < p < 2^*$.

Theorem 1.3. *Let* $m \ge 0$ *and* $2 < p < 2^*$. *Then there exists* $\alpha_0 > 0$ *such that*

$$\mathscr{C}_{\alpha,m,p}(\boldsymbol{B}) = \mathscr{C}_{0,m,p}(\boldsymbol{B}) \quad \text{for } \alpha \in [0, \alpha_0).$$

Moreover, for $\alpha \in [0, \alpha_0)$, *there is, up to sign, a unique ground state solution of* (1-5) *which is a radial function.*

Our proof of this theorem relies on the uniqueness and nondegeneracy of the radial positive solution of (1-5) in the case $\alpha = 0$. Combining Theorems 1.1 and 1.3, we find that, for fixed $p > 2_1^*$, symmetry-breaking of ground state solutions occurs when passing a critical parameter $\alpha = \alpha(p)$ which lies in the interval $[\alpha_0, \alpha_p]$. However, so far it remains unclear whether symmetry-breaking also occurs in the case $p \le 2_1^*$. Before stating a partial answer to this question for $2 < p < 2_1^*$, we first note that symmetry-breaking does not occur in the linear case $p = 2$. More precisely, we shall observe in Section 4 below that

$$\mathscr{C}_{\alpha,m,2}(\boldsymbol{B}) = \mathscr{C}_{0,m,2}(\boldsymbol{B}) = \lambda_1(\boldsymbol{B}) + m \quad \text{for all } \alpha \in [0, 1],\ m \in \mathbb{R}.$$

Moreover, if $\alpha \in (0, 1)$ and $m \ge 0$ are fixed, then every minimizer of (1-7) is radial if $p \ge 2$ is sufficiently close to 2; see Remark 4.16 below. On the other hand, for every p strictly greater than 2, symmetry-breaking occurs for sufficiently large values of the parameter m, as the following result shows.

Theorem 1.4. *Let $\alpha \in (0, 1)$ and $2 < p < 2^*$. Then there exists $m_0 > 0$ with the property that* (1-11) *holds for $m \geq m_0$, and therefore every ground state solution of* (1-5) *is (x_1, x_2)-nonradial for $m \geq m_0$.*

As symmetry-breaking occurs, for fixed $p \in (2, 2^*)$, both in the parameter α and m, it is tempting to guess that there exists a unique curve in the (α, m)-plane separating the parameter region of symmetry-breaking from the one where radial symmetry of ground state solutions is preserved. A bifurcation analysis might be useful to detect the precise symmetry-breaking regime, but this seems far from straightforward, and we leave it for future research.

Next, we discuss the limit case $\alpha = 1$ in the minimization problem (1-7). We may study this limit case based on Corollary 1.2, but we need to look for minimizers in a space larger than $H_0^1(\boldsymbol{B})$. More precisely, we let $\mathcal{H}_1$ be given as the completion of $C_c^1(\boldsymbol{B})$ with respect to the norm $\|\cdot\|_{\mathcal{H}_1}$ given by

$$\|u\|_{\mathcal{H}_1}^2 := \int_{\boldsymbol{B}} (|\nabla u|^2 - |\partial_\theta u|^2)\, dx.$$

Here we recall that Corollary 1.2 gives the norm property of $\|\cdot\|_{\mathcal{H}_1}$ on $C_c^1(\boldsymbol{B}) \subset H_0^1(\boldsymbol{B})$, and it also implies that $\mathcal{H}_1$ is embedded in $L^{2_1^*}(\boldsymbol{B})$. We then have the following result, which complements Theorems 1.1 and 1.4 in the case $\alpha = 1$.

Theorem 1.5. *Let $2 < p < 2_1^*$ and $\alpha = 1$.*

(i) *For every $m > -\lambda_1(\boldsymbol{B})$, there exists a ground state solution of* (1-5).

(ii) *There exists $m_0 > 0$ with the property that* (1-11) *holds for $m \geq m_0$, and therefore every ground state solution $u \in \mathcal{H}_1$ of* (1-5) *is (x_1, x_2)-nonradial for $m \geq m_0$.*

The critical case $\alpha = 1$, $p = 2_1^*$ remains largely open, but we have a partial result on the existence of ground state solutions which relates problem (1-5) to a degenerate Sobolev inequality of the form

$$\|u\|_{L^{2_s^*}(\mathbb{R}_+^N)} \leq C\biggl(\int_{\mathbb{R}_+^N} \sum_{i=1}^{N-1} |\partial_i u|^2 + x_1^s |\partial_N u|^2\, dx\biggr)^{1/2} \tag{1-13}$$

in the half-space

$$\mathbb{R}_+^N := \{x \in \mathbb{R}^N : x_1 > 0\}.$$

This inequality seems new and of independent interest, and it is the key ingredient in the proof of Theorem 1.1. Our main result related to this half-space inequality is the following.

Theorem 1.6. *Let $s > 0$, and set $2_s^* := (4N + 2s)/(2N - 4 + s)$. Then we have*

$$\mathcal{S}_s(\mathbb{R}_+^N) := \inf_{u \in C_c^1(\mathbb{R}_+^N)} \frac{\int_{\mathbb{R}_+^N} \sum_{i=1}^{N-1} |\partial_i u|^2 + x_1^s |\partial_N u|^2\, dx}{\bigl(\int_{\mathbb{R}_+^N} |u|^{2_s^*}\, dx\bigr)^{2/2_s^*}} > 0. \tag{1-14}$$

Moreover, the value $\mathcal{S}_s(\mathbb{R}_+^N)$ is attained in $H_s \setminus \{0\}$, where H_s denotes the closure of $C_c^1(\mathbb{R}_+^N)$ in the space

$$\biggl\{u \in L^{2_s^*}(\mathbb{R}_+^N) : \|u\|_{H_s}^2 := \int_{\mathbb{R}_+^N} \sum_{i=1}^{N-1} |\partial_i u|^2 + x_1^s |\partial_N u|^2\, dx < \infty\biggr\} \tag{1-15}$$

with respect to the norm $\|\cdot\|_{H_s}$.

Here, distributional derivatives are considered in (1-15). Moreover, we note that $\|\cdot\|_{H_s}$ defines a norm on the space defined in (1-15), as the vanishing of $\|u\|_{H_s}$ implies that the distributional gradient ∇u vanishes a.e. in $\mathbb{R}^N_+$. This, in turn, implies that u must be constant on $\mathbb{R}^N_+$, and therefore $u = 0$ since $u \in L^{2^*_s}(\mathbb{R}^N_+)$.

Several remarks regarding Theorem 1.6 are in order. First, we point out that the criticality of the exponent $2^*_s := (4N+2s)/(2N-4+s)$ in Theorem 1.6 corresponds to the fact that the quotient in (1-14) is invariant under an anisotropic rescaling given by $u \mapsto u_\lambda$ for $\lambda > 0$, with

$$u_\lambda(x) := u(\lambda x_1, \lambda x_2, \ldots, \lambda x_{N-1}, \lambda^{1+s/2} x_N).$$

This invariance leads to a lack of compactness, and we have to apply concentration-compactness methods to deduce the existence of minimizers. We further note that the existence of minimizers in the half-space problem is in striking contrast to the case $s = 0$ which is excluded in Theorem 1.6. Indeed, the case $s = 0$ in Theorem 1.6 corresponds to the classical Sobolev inequality which only admits extremal functions in the entire space $\mathbb{R}^N$; see, e.g., [Struwe 2008, Chapter III, Theorem 1.2].

We have already noted that the case $s = 1$ in Theorem 1.6 is of key importance in the proof of Theorem 1.1. The more general case $s \in (0, 2]$ arises in a similar way when (1-5) is studied in Riemannian models with boundary in place of $\boldsymbol{B}$, and we will discuss this case in Section 6 below. We point out that the setting of Riemannian models includes hypersurfaces of revolution with boundary in $\mathbb{R}^{N+1}$, and that the particular case of a hemisphere corresponds to the case $s = 2$. The latter is no surprise in view of the recent work of Taylor [2016] and Mukherjee [2017; 2018], who studied the problem of rotating solutions on the unit sphere. In particular, their work relies on degenerate Sobolev embeddings on the unit sphere where also the value $2^*_2 = 2(N+1)/(N-1)$ appears as a critical exponent; see [Taylor 2016, Proposition 3.2] and [Mukherjee 2017, Proposition 1.2 and Lemma 1.3]. In fact, our approach allows to use the case $s = 2$ in Theorem 1.6 and the corresponding inequality in $\mathbb{R}^N$ (see Theorem 2.2 below) to give new proofs of these degenerate Sobolev embeddings which do not rely on the Fourier analytic arguments used in [Taylor 2016].

Next we remark that degenerate Sobolev-type inequalities have been studied extensively in the context of Grushin operators which take the form

$$\mathcal{L} = \Delta_x + c|x|^s \Delta_y$$

on $\mathbb{R}^N = \mathbb{R}^m \times \mathbb{R}^k$, where $x \in \mathbb{R}^m$, $y \in \mathbb{R}^k$ and $s > 0$. For a comprehensive survey of the properties of these operators, see, e.g., [Hajłasz and Koskela 2000]. In particular, an associated Sobolev-type inequality of the type

$$\|u\|_{L^{\frac{4m+2k(s+2)}{2m+k(s+2)-4}}(\mathbb{R}^N)} \le C\left(\int_{\mathbb{R}^N} |\nabla_x u|^2 + c|x|^s |\nabla_y u|^2 \, d(x,y)\right)^{1/2}, \quad u \in C^1_c(\mathbb{R}^N) \tag{1-16}$$

has been established. Here, the associated critical exponent is related to the homogeneous dimension in the context of more general weighted Sobolev inequalities. We also mention symmetry results for positive entire solutions to semilinear problems involving $\mathcal{L}$ in [Monti and Morbidelli 2006], as well as the existence of extremal functions on $\mathbb{R}^N$ shown in [Beckner 2001; Monti 2006].

We point out that the restriction of inequality (1-16) to the half-space coincides with the inequality (1-13) in the case $N = 2,\ m = k = 1$. On the other hand, for $N \geq 3,\ m = N - 1$ and $k = 1$, the critical exponents in (1-13) and (1-16) still coincide, but (1-13) is a strict improvement of (1-16) since the weight x_1^s is strictly smaller than $|(x_1, \dots, x_{N-1})|^s$ away from the x_1-axis. More closely related to Theorem 1.6 in the case $N \geq 3$ is [Filippas et al. 2008, Theorem 1.7], where a more general family of Grushin-type operators and their associated inequalities has been considered. However, the inequality (1-13) associated to (1-11) is a limit case which is not part of the family of inequalities considered in [Filippas et al. 2008]. More precisely, inequality (1-13) extends [Filippas et al. 2008, Theorem 1.7] to the case $A = B = 0$ (with A and B given in [Filippas et al. 2008]).

Coming back to the existence of ground state solutions of (1-5) in the critical case $\alpha = 1,\ p = 2_1^*$, we state the following result.

Theorem 1.7. (i) *If*

$$\mathscr{C}_{1,m,2_1^*}(\boldsymbol{B}) < 2^{1/2 - 1/2_1^*} \mathcal{S}_1(\mathbb{R}^N_+) \tag{1-17}$$

for some $m > -\lambda_1(\boldsymbol{B})$, then the value $\mathscr{C}_{1,m,2_1^*}(\boldsymbol{B})$ is attained in $\mathcal{H} \setminus \{0\}$ by a ground state solution of (1-5).

(ii) *There exists $\varepsilon > 0$ with the property that* (1-17) *holds for every $m \in (-\lambda_1(\boldsymbol{B}), -\lambda_1(\boldsymbol{B}) + \varepsilon)$.*

Here, the factor $2^{1/2-1/2_1^*}$ is due to the scaling properties of a more general quotient related to (1-14); see Remark 2.3 (ii) below. We note that criterion (1-17) prevents, with the help of Theorem 1.6 and a blow-up argument, the concentration of minimizing sequences close to the great circle γ defined in (1-6).

The paper is organized as follows. We first study the degenerate Sobolev inequality (1-13) and hence prove Theorem 1.6 in Section 2. This is subsequently used in Section 3 to prove the second part of Theorem 1.1. In Section 4 we then discuss the properties of ground state solutions of (1-5) in detail and give the proofs of Theorems 1.3 and 1.4. This also includes the degenerate case $\alpha = 1$ and the proofs of Theorems 1.5 and 1.7. Section 5 is then devoted to the properties of rotating waves when $\boldsymbol{B}$ is replaced by an annulus. In this case, our methods give rise to an analogue of Theorem 1.1 with more explicit conditions for (x_1, x_2)-nonradiality of ground states. In Section 6 we discuss how the general degenerate Sobolev inequality (1-13) can be used to give an analogue of Theorem 1.1 for Riemannian models. Finally, in Appendix A, we prove uniform L^∞-bounds for weak solutions of (1-5) in the case $\alpha = 1$. Moreover, we recall in Appendix B useful formulas related to the round metric on the unit sphere in angular coordinates.

We finally remark that the general approach of the present paper also allows to analyze (x_1, x_2)-nonradial solutions of (1-5) on domains of the type

$$\{(x_1, x_2, x') \in \mathbb{R}^N : x_1^2 + x_2^2 < 1,\ |x'| < \psi(x_1^2 + x_2^2)\}$$

for suitable functions $\psi : [0, 1) \to (0, \infty)$ satisfying $\lim_{r \to 1^-} \psi(r) = 0$. However, the underlying analysis will be more involved, and limiting Sobolev inequalities different from (1-13) might arise. We shall leave this open problem for future research.[1]

[1]We wish to thank the referee for pointing out this question.

2. A family of degenerate Sobolev inequalities

In this section, we give the proof of Theorem 1.6. More precisely, in the first part of the section, we prove the corresponding degenerate Sobolev inequality

$$\left(\int_{\mathbb{R}^N} |u|^{2_s^*}\,dx\right)^{2/2_s^*} \le C\int_{\mathbb{R}^N}\left(\sum_{i=1}^{N-1}|\partial_i u|^2+|x_1|^s|\partial_N u|^2\right)dx \quad \text{for } u\in C_c^1(\mathbb{R}^N) \tag{2-1}$$

in the entire space with a constant $C>0$, from which the positivity of $\mathcal{S}_s(\mathbb{R}^N_+)$ in (1-14) follows.

In the second part of the section, we then prove the existence of minimizers of the quotient in (1-14) in the larger space H_s defined in Theorem 1.6.

2.1. ***A degenerate Sobolev inequality on*** $\mathbb{R}^N$**.** The first step in the proof of (2-1) is the following inequality.

Lemma 2.1. *Let $\alpha>0$ and $p>2$ be given. Then we have*

$$\int_{\mathbb{R}^N}|u|^p\,dx \le \kappa\left(\int_{\mathbb{R}^N}|x_1|^\alpha|u|^q\,dx\right)^{2/(2+\alpha)}\left(\int_{\mathbb{R}^N}|\partial_1 u|^2\,dx\right)^{\alpha/(2+\alpha)} \quad \textit{for } u\in C_c^1(\mathbb{R}^N),$$

with

$$q=\tfrac{1}{2}(p(2+\alpha)-2\alpha)>2$$

and

$$\kappa=\begin{cases}\left(\frac{q+2}{\alpha}\right)^{2\alpha/(2+\alpha)}, & 0<\alpha\le 2,\\ p^{2\alpha/(2+\alpha)}, & \alpha>2.\end{cases}$$

Proof. We first consider the case $\alpha\in(0,2)$, and we let $u\in C_c^1(\mathbb{R}^N)$. By Hölder's inequality, we have

$$\begin{aligned}\int_{\mathbb{R}^N}|u|^p\,dx &\le \left(\int_{\mathbb{R}^N}|x_1|^{s\sigma'}|u|^{r\sigma'}\,dx\right)^{1/\sigma'}\left(\int_{\mathbb{R}^N}|x_1|^{-s\sigma}|u|^{(p-r)\sigma}\,dx\right)^{1/\sigma}\\ &=\left(\int_{\mathbb{R}^N}|x_1|^{\alpha}|u|^{q}\,dx\right)^{1/\sigma'}\left(\int_{\mathbb{R}^N}|x_1|^{-s\sigma}|u|^{(p-r)\sigma}\,dx\right)^{1/\sigma},\end{aligned} \tag{2-2}$$

with

$$\sigma:=\frac{2+\alpha}{2\alpha},\quad \sigma'=\frac{\sigma}{\sigma-1}=\frac{2+\alpha}{2-\alpha}\in(1,\infty),\quad s:=\frac{\alpha}{\sigma'}\quad\text{and}\quad r:=\frac{q}{\sigma'}.$$

Here we used that $0<r<p$. Indeed we have

$$p=\frac{2q+2\alpha}{2+\alpha}=\frac{q}{\sigma'}+\frac{\alpha(q+2)}{2+\alpha}=r+\frac{q+2}{2\sigma},$$

which furthermore implies that

$$(p-r)\sigma-1=\tfrac{1}{2}q>0. \tag{2-3}$$

Since also

$$s\sigma=\alpha(\sigma-1)=\tfrac{1}{2}(2-\alpha)\in(0,1), \tag{2-4}$$

we may integrate by parts and use Hölder's inequality again to get

$$\begin{aligned}\int_{\mathbb{R}^N} |x_1|^{-s\sigma}|u|^{(p-r)\sigma}\,dx &= -\frac{(p-r)\sigma}{1-s\sigma}\int_{\mathbb{R}^N} x_1|x_1|^{-s\sigma}|u|^{(p-r)\sigma-1}\,\partial_1 u\,dx\\ &\le \frac{(p-r)\sigma}{1-s\sigma}\int_{\mathbb{R}^N} |x_1|^{1-s\sigma}|u|^{(p-r)\sigma-1}|\partial_1 u|\,dx\\ &\le \frac{q+2}{\alpha}\Big(\int_{\mathbb{R}^N}|x_1|^{\alpha}|u|^q\,dx\Big)^{1/2}\Big(\int_{\mathbb{R}^N}|\partial_1 u|^2\,dx\Big)^{1/2}. \end{aligned} \tag{2-5}$$

Here we have used (2-3), (2-4) and the identity

$$\frac{(p-r)\sigma}{1-s\sigma} = \frac{q+2}{\alpha}$$

in the last step. Combining (2-2) and (2-5) gives

$$\begin{aligned}\int_{\mathbb{R}^N}|u|^p\,dx &\le \Big(\frac{q+2}{\alpha}\Big)^{1/\sigma}\Big(\int_{\mathbb{R}^N}|x_1|^{\alpha}|u|^q\,dx\Big)^{1/\sigma'+1/(2\sigma)}\Big(\int_{\mathbb{R}^N}|\partial_1 u|^2\,dx\Big)^{1/(2\sigma)}\\ &= \Big(\frac{q+2}{\alpha}\Big)^{2\alpha/(2+\alpha)}\Big(\int_{\mathbb{R}^N}|x_1|^{\alpha}|u|^q\,dx\Big)^{2/(2+\alpha)}\Big(\int_{\mathbb{R}^N}|\partial_1 u|^2\,dx\Big)^{\alpha/(2+\alpha)}, \end{aligned}$$

as claimed. Next we note that the case $\alpha = 2$ follows by continuity, which gives

$$\Big(\int_{\mathbb{R}^N}|u|^p\,dx\Big)^2 \le p^2\Big(\int_{\mathbb{R}^N}|x_1|^2|u|^{2(p-1)}\,dx\Big)\Big(\int_{\mathbb{R}^N}|\partial_1 u|^2\,dx\Big). \tag{2-6}$$

From this we now deduce the claim in the case $\alpha > 2$. Indeed, writing

$$|x_1|^2|u|^{2(p-1)} = (|x_1|^2|u|^{2q/\alpha})|u|^{2(p-1-q/\alpha)}$$

we get, by Hölder's inequality,

$$\int_{\mathbb{R}^N}|x_1|^2|u|^{2(p-1)}\,dx \le \Big(\int_{\mathbb{R}^N}|x_1|^{\alpha}|u|^q\,dx\Big)^{2/\alpha}\Big(\int_{\mathbb{R}^N}|u|^{(2\alpha/(\alpha-2))\cdot(p-1-q/\alpha)}\,dx\Big)^{(\alpha-2)/\alpha}. \tag{2-7}$$

Since $(2\alpha/(\alpha-2))\cdot(p-1-q/\alpha) = p$, we deduce from (2-6) and (2-7) that

$$\Big(\int_{\mathbb{R}^N}|u|^p\,dx\Big)^{(\alpha+2)/\alpha} \le p^2\Big(\int_{\mathbb{R}^N}|x_1|^{\alpha}|u|^q\,dx\Big)^{2/\alpha}\Big(\int_{\mathbb{R}^N}|\partial_1 u|^2\,dx\Big),$$

and hence

$$\int_{\mathbb{R}^N}|u|^p\,dx \le p^{2\alpha/(\alpha+2)}\Big(\int_{\mathbb{R}^N}|x_1|^{\alpha}|u|^q\,dx\Big)^{2/(\alpha+2)}\Big(\int_{\mathbb{R}^N}|\partial_1 u|^2\,dx\Big)^{\alpha/(\alpha+2)}. \qquad \square$$

We may now complete the proof of the main result of this section, given as follows.

Theorem 2.2. *Let $s > 0$ and $2_s^* = (4N+2s)/(2N-4+s)$ as in Theorem 1.6. Then inequality* (2-1) *holds with some constant $C > 0$.*

We remark that this may be proven by combining Lemma 2.1 with a suitable adaptation of the inequality on the half-space given in [Filippas et al. 2008, Theorem 1.7] to the setting of the entire space $\mathbb{R}^N$. For the convenience of the reader, we give a self-contained proof.

Proof. In the following, the letter $c > 0$ stands for a constant which may change from line to line. Let $\alpha = s/(2(N-1))$. Then Lemma 2.1 yields

$$\int_{\mathbb{R}^N} |u|^{2_s^*}\,dx \le \kappa\Big(\int_{\mathbb{R}^N} |x_1|^{\alpha}|u|^{q_s}\,dx\Big)^{\frac{2}{2+\alpha}}\Big(\int_{\mathbb{R}^N} |\partial_1 u|^2\,dx\Big)^{\frac{\alpha}{2+\alpha}} \quad \text{for } u \in C_c^1(\mathbb{R}^N),$$

with $q_s := N(2_s^*+2)/(2(N-1))$. To estimate the term $\int_{\mathbb{R}^N} |x_1|^{\alpha}|u|^{q_s}\,dx$, we define, for $i = 1, \dots, N$, the functions $a_i \in C_c(\mathbb{R}^{N-1})$ by

$$a_i(\hat{x}_i) := \int_{\mathbb{R}} |u|^{\frac{q_s(N-1)}{N}-1}|\partial_i u|\,dx_i,$$

where

$$\hat{x}_i := (x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_N) \in \mathbb{R}^{N-1} \quad \text{for } x \in \mathbb{R}^N \text{ and } i = 1, \dots, N.$$

Integrating the derivative $\partial_i(|u|^{q_s(N-1)/N})$ in the x_i-direction, we find that $|u(x)|^{q_s(N-1)/N} \le c a_i(\hat{x}_i)$ for all $x \in \mathbb{R}^N$, $i = 1, \dots, N$, and therefore

$$|u(x)|^{q_s(N-1)} \le c\prod_{i=1}^{N} a_i(\hat{x}_i) \quad \text{for } x \in \mathbb{R}^N.$$

Applying Gagliardo's lemma [1958, Lemma 4.1] to the functions $a_1^{1/(N-1)}, \dots, a_{N-1}^{1/(N-1)}$ and the function $x \mapsto |x_1|^{\alpha} a_N^{1/(N-1)}(x)$, we thus find that

$$\begin{aligned}\int_{\mathbb{R}^N} |x_1|^{\alpha}|u|^{q_s}\,dx &\le c\Big(\int_{\mathbb{R}^{N-1}} |x_1|^{(N-1)\alpha} a_N(\hat{x}_N)\,d\hat{x}_N \prod_{i=1}^{N-1}\int_{\mathbb{R}^{N-1}} a_i(\hat{x}_i)\,d\hat{x}_i\Big)^{\frac{1}{N-1}}\\ &= c\Big(\int_{\mathbb{R}^N} |x_1|^{s/2}|u|^{\frac{q_s(N-1)}{N}-1}|\partial_N u|\,dx \prod_{i=1}^{N-1}\int_{\mathbb{R}^N} |u|^{\frac{q_s(N-1)}{N}-1}|\partial_i u|\,dx\Big)^{\frac{1}{N-1}}\\ &\le c\Big(\int_{\mathbb{R}^N} |u|^{2\frac{q_s(N-1)}{N}-2}\,dx\Big)^{\frac{N}{2(N-1)}}\Big(\int_{\mathbb{R}^N} |x_1|^{s}|\partial_N u|^2\,dx \prod_{i=1}^{N-1}\int_{\mathbb{R}^N} |\partial_i u|^2\,dx\Big)^{\frac{1}{2(N-1)}}.\end{aligned}$$

Since $2(N-1)q_s/N - 2 = 2_s^*$, we conclude that

$$\begin{aligned}&\int_{\mathbb{R}^N} |u|^{2_s^*}\,dx\\ &\quad\le c\Big(\int_{\mathbb{R}^N} |x_1|^{\alpha}|u|^{q_s}\,dx\Big)^{\frac{2}{2+\alpha}}\Big(\int_{\mathbb{R}^N} |\partial_1 u|^2\,dx\Big)^{\frac{\alpha}{2+\alpha}}\\ &\quad\le c\bigg(\Big(\int_{\mathbb{R}^N} |u|^{2_s^*}\,dx\Big)^{\frac{N}{2(N-1)}}\Big(\int_{\mathbb{R}^N} |x_1|^{s}|\partial_N u|^2\,dx \prod_{i=1}^{N-1}\int_{\mathbb{R}^N} |\partial_i u|^2\,dx\Big)^{\frac{1}{2(N-1)}}\bigg)^{\frac{2}{2+\alpha}}\Big(\int_{\mathbb{R}^N} |\partial_1 u|^2\,dx\Big)^{\frac{\alpha}{2+\alpha}}\\ &\quad= c\Big(\int_{\mathbb{R}^N} |u|^{2_s^*}\,dx\Big)^{\frac{N}{2(N-1)+s/2}}\Big(\int_{\mathbb{R}^N} |x_1|^{s}|\partial_N u|^2\,dx \prod_{i=2}^{N-1}\int_{\mathbb{R}^N} |\partial_i u|^2\,dx\Big)^{\frac{1}{2(N-1)+s/2}}\Big(\int_{\mathbb{R}^N} |\partial_1 u|^2\,dx\Big)^{\frac{1+s/2}{2(N-1)+s/2}},\end{aligned}$$

and therefore

$$\Big(\int_{\mathbb{R}^N} |u|^{2_s^*}\,dx\Big)^{\frac{N-2+s/2}{2(N-1)+s/2}} \le c\Big(\int_{\mathbb{R}^N} |x_1|^{s}|\partial_N u|^2\,dx \prod_{i=2}^{N-1}\int_{\mathbb{R}^N} |\partial_i u|^2\,dx\Big)^{\frac{1}{2(N-1)+s/2}}\Big(\int_{\mathbb{R}^N} |\partial_1 u|^2\,dx\Big)^{\frac{1+s/2}{2(N-1)+s/2}}.$$

Finally, Young's inequality gives

$$\left(\int_{\mathbb{R}^N} |u|^{2_s^*}\,dx\right)^{2/2_s^*} \le c\left(\int_{\mathbb{R}^N} |x_1|^s |\partial_N u|^2\,dx \prod_{i=2}^{N-1}\int_{\mathbb{R}^N} |\partial_i u|^2\,dx\right)^{\frac{2}{2N+s}}\left(\int_{\mathbb{R}^N} |\partial_1 u|^2\,dx\right)^{\frac{2+s}{2N+s}}$$
$$\le c\left(\int_{\mathbb{R}^N} |x_1|^s |\partial_N u|^2\,dx + \sum_{i=1}^{N-1}\int_{\mathbb{R}^N} |\partial_i u|^2\,dx\right). \qquad \square$$

In particular, this implies

$$\mathcal{S}_s(\mathbb{R}^N_+) = \inf_{u\in C^1_c(\mathbb{R}^N_+)} \frac{\int_{\mathbb{R}^N_+} \sum_{i=1}^{N-1} |\partial_i u|^2 + x_1^s |\partial_N u|^2\,dx}{\left(\int_{\mathbb{R}^N_+} |u|^{2_s^*}\,dx\right)^{2/2_s^*}} > 0,$$

and thus the first part of Theorem 1.6.

Remark 2.3 (optimality and variants). (i) The exponent 2_s^* in (1-14) is optimal in the sense that

$$\inf_{u\in C^1_c(\mathbb{R}^N)} \frac{\int_{\mathbb{R}^N} \left(\sum_{i=1}^{N-1} |\partial_i u|^2 + |x_1|^s |\partial_N u|^2\right) dx}{\|u\|^2_{L^p(\mathbb{R}^N)}} = 0 \quad \text{for } p \ne 2_s^*. \tag{2-8}$$

This follows by considering the rescaling $u \mapsto u_\lambda,\ \lambda > 0$, with

$$u_\lambda(x) := u(\lambda x_1, \lambda x_2, \dots, \lambda x_{N-1}, \lambda^{1+s/2} x_N).$$

Indeed, for $u \in C^1_c(\mathbb{R}^N)$, we have

$$\int_{\mathbb{R}^N} \left(\sum_{i=1}^{N-1} |\partial_i u_\lambda|^2 + |x_1|^s |\partial_N u_\lambda|^2\right) dx = \lambda^{-(2N+s-4)/2} \int_{\mathbb{R}^N} \left(\sum_{i=1}^{N-1} |\partial_i v|^2 + |x_1|^s |\partial_N u|^2\right) dx$$

and, for $1 < p < \infty$,

$$\left(\int_{\mathbb{R}^N} |u_\lambda|^p\,dx\right)^{2/p} = \lambda^{-(2/p)(N+s/2)} \left(\int_{\mathbb{R}^N} |u|^p\,dx\right)^{2/p}.$$

Since $\frac{1}{2}(2N+s-4) = (2/p)\big(N+\frac{1}{2}s\big)$ if and only if $p = 2_s^*$, (2-8) follows.

(ii) For $\kappa > 0,\ u \in C^1_c(\mathbb{R}^N)$, we may consider a rescaled function of the form

$$v(x) = u\left(x_1, \dots, x_{N-1}, \frac{x_N}{\sqrt{\kappa}}\right).$$

Comparing the associated quotients then yields

$$\inf_{u\in C^1_c(\mathbb{R}^N)} \frac{\int_{\mathbb{R}^N} \left(\sum_{i=1}^{N-1} |\partial_i u|^2 + \kappa |x_1|^s |\partial_N u|^2\right) dx}{\|u\|^2_{L^{2_s^*}(\mathbb{R}^N)}} = \kappa^{1/2-1/2_s^*} \mathcal{S}_s(\mathbb{R}^N_+). \tag{2-9}$$

In the special case $\kappa = 2$, this quotient will appear later when we connect $\mathscr{C}_{1,m,2_1^*}(\boldsymbol{B})$ and $\mathcal{S}_1(\mathbb{R}^N_+)$, in particular in the proof of Theorem 1.7.

Recalling the space H_s defined in Theorem 1.6, we see that Theorem 2.2 immediately implies that H_s is continuously embedded into $L^{2_s^*}(\mathbb{R}^N_+)$.

2.2. *Existence of minimizers.* In the following, we fix $s > 0$ and study minimizing sequences for

$$\mathcal{S} := \mathcal{S}_s(\mathbb{R}^N_+) = \inf_{u \in H_s \setminus \{0\}} \frac{\int_{\mathbb{R}^N_+} \big(\sum_{i=1}^{N-1} |\partial_i u|^2 + x_1^s |\partial_N u|^2\big)\, dx}{\big(\int_{\mathbb{R}^N_+} |u|^{2_s^*}\, dx\big)^{2/2_s^*}} > 0.$$

First, consider the following classical lemma due to Lions [1984], which we give in the form presented in [Struwe 2008].

Lemma 2.4 (concentration-compactness lemma). *Suppose $(\mu_n)_n$ is a sequence of probability measures on $\mathbb{R}^N$. Then, after passing to a subsequence, one of the following three conditions holds*:

(i) <u>Compactness</u>: *There exists a sequence $(x_n)_n \subset \mathbb{R}^N$ such that, for any $\varepsilon > 0$, there exists $R > 0$ such that*

$$\int_{B_R(x_n)} d\mu_n \geq 1 - \varepsilon.$$

(ii) <u>Vanishing</u>: *For all $R > 0$,*

$$\lim_{n\to\infty} \left(\sup_{x \in \mathbb{R}^N} \int_{B_R(x)} d\mu_n \right) = 0.$$

(iii) <u>Dichotomy</u>: *There exists $\lambda \in (0, 1)$ such that, for any $\varepsilon > 0$, there exists $R > 0$ and $(x_n)_n \subset \mathbb{R}^N$ with the following property: given $R' > R$ there are nonnegative measures μ_n^1, μ_n^2 such that*

$$0 \leq \mu_n^1 + \mu_n^2 \leq \mu_n, \quad \operatorname{supp} \mu_n^1 \subset B_R(x_n), \quad \operatorname{supp} \mu_n^2 \subset \mathbb{R}^N \setminus B_{R'}(x_n),$$
$$\limsup_{n\to\infty} \left(\left| \lambda - \int_{\mathbb{R}^N} d\mu_n^1 \right| + \left| (1-\lambda) - \int_{\mathbb{R}^N} d\mu_n^2 \right| \right) \leq \varepsilon.$$

A characterization of minimizing sequences in the sense of measures is given in the following lemma, which is a straightforward adaption of [Struwe 2008, Lemma 4.8].

Lemma 2.5 (concentration-compactness lemma II). *Let $s > 0$, and suppose $u_n \rightharpoonup u$ in H_s and $\mu_n := \big(\sum_{i=1}^{N-1} |\partial_i u_n|^2 + x_1^s |\partial_N u_n|^2\big)\, dx \rightharpoonup \mu$, $\nu_n := |u_n|^{2_s^*}\, dx \rightharpoonup \nu$ weakly in the sense of measures, where μ and ν are finite measures on $\mathbb{R}^N_+$. Then*:

(i) *There exists an at most countable set J and sets $\{x^j : j \in J\} \subset \mathbb{R}^N_+$ and $\{\nu^j : j \in J\} \subset (0, \infty)$ such that*

$$\nu = |u|^{2_s^*}\, dx + \sum_{j \in J} \nu^j \delta_{x^j}.$$

(ii) *There exists a set $\{\mu^j : j \in J\} \subset (0, \infty)$ such that*

$$\mu \geq \left(\sum_{i=1}^{N-1} |\partial_i u|^2 + x_1^s |\partial_N u|^2 \right) dx + \sum_{j \in J} \mu^j \delta_{x^j},$$

where

$$\mathcal{S} (\nu^j)^{2/2_s^*} \leq \mu^j$$

for $j \in J$. In particular, $\sum_{j \in J} (\nu^j)^{2/2_s^*} < \infty$.

Our main result then states that $\mathcal{S}$ is attained in H_s and completes the proof of Theorem 1.6.

Theorem 2.6. *Let $s > 0$, and suppose $(u_n)_n$ is a minimizing sequence for*

$$\mathcal{S} = \inf_{u \in H_s \setminus \{0\}} \frac{\int_{\mathbb{R}^N_+} \left(\sum_{i=1}^{N-1} |\partial_i u|^2 + x_1^s |\partial_N u|^2 \right) dx}{\left(\int_{\mathbb{R}^N_+} |u|^{2_s^*} \, dx \right)^{2/2_s^*}},$$

with $\|u_n\|_{L^{2_s^*}} = 1$. Then, up to translations orthogonal to x_1 and anisotropic scaling, $(u_n)_n$ is relatively compact in H_s.

Proof. The proof consists of four steps: In the first step, we use a suitable anisotropic scaling and translations to exclude vanishing in the sense of Lemma 2.4. This is adapted from the classical case $s = 0$ with adjustments based on the different scaling properties appearing in this case.

In the second step, we similarly adapt the classical arguments to show that dichotomy cannot occur.

The third step then uses Lemma 2.5 to deduce further information on potential concentration behavior of the minimizing sequence in order to exclude the existence of multiple concentration points.

In the fourth step we then show that the sequence cannot concentrate in a single point either. Compared to the classical case, the scaling and translation properties are much weaker in our setting, making this step much more involved. A crucial idea is the following: If the sequence concentrates in a single point, its L^q-norm will blow up for any $q > 2_s^*$ in a neighborhood of this point. If the concentration point is not on the boundary however, the H_s-norm is comparable to the Sobolev-norm in a neighborhood and can be used to bound the L^q-norm for $q < 2^*$. Since $2_s^* < 2^*$, this can be brought to a contradiction.

Step 1: *After rescaling and translation, the sequence cannot vanish.*

For $r > 0$ we define the family of rectangles

$$\mathcal{Q}_r := \{(0, r^2) \times (y + (-r^2, r^2)^{N-2} \times (-r^{2+s}, r^{2+s})) : y \in \mathbb{R}^{N-1}\}.$$

It is important to note that, for $R > 0$, with respect to the transformation

$$\tau_R(x) = (R^2 x_1, R^2 x_2, \dots, R^2 x_{N-1}, R^{2+s} x_N), \tag{2-10}$$

these sets satisfy

$$\tau_R(\mathcal{Q}_r) = \mathcal{Q}_{rR}.$$

Moreover, the functions

$$Q_n(r) := \sup_{E \in \mathcal{Q}_r} \int_E |u_n|^{2_s^*} \, dx$$

are continuous on $[0, \infty)$ and satisfy

$$\lim_{r \to 0} Q_n(r) = 0, \qquad \lim_{r \to \infty} Q_n(r) = 1.$$

Moreover, the supremum in the definition of Q_n is attained. Indeed, by definition there exists a sequence $(y_k)_k \subset \mathbb{R}^{N-1}$ such that $\int_{E_k} |u_n|^{2_s^*} \, dx \to Q_n(r)$ as $k \to \infty$, where

$$E_k := (0, r^2) \times (y_k + (-r^2, r^2)^{N-2} \times (-r^{2+s}, r^{2+s})).$$

Since $|u_n|^{2_s^*} \, dx$ is a finite measure, $(y_k)_k$ must be bounded so we may pass to a convergent subsequence, whose limit attains the supremum.

Hence we may choose $A_n > 0$, $y_n \in \mathbb{R}^{N-1}$ such that the rescaled sequence $v_n \in H_s$ given by

$$v_n(x) := A_n^{(2N-4+s)/2} u_n(A_n^2 x_1, A_n^2(x_2 + (y_n)_1), \dots, A_n^{2+s}(x_N + (y_n)_{N-1}))$$

satisfies

$$Q_n(1) = \sup_{E \in \mathcal{Q}_1} \int_E |v_n|^{2_s^*}\, dx = \int_{(0,1)\times(-1,1)^{N-1}} |v_n|^{2_s^*}\, dx = \tfrac{1}{2}.$$

After passing to a subsequence, we may assume $v_n \rightharpoonup v$ in H_s and in $L^{2_s^*}(\mathbb{R}^N_+)$. We now consider the measures

$$\mu_n := \left(\sum_{i=1}^{N-1} |\partial_i v_n|^2 + x_1^s |\partial_N v_n|^2\right) dx \quad \text{and} \quad \nu_n := |v_n|^{2_s^*}\, dx$$

and apply Lemma 2.4 to $(\nu_n)_n$, where we note that μ_n and ν_n are initially measures on $\mathbb{R}^N_+$ but can trivially be extended to $\mathbb{R}^N$. By our normalization, vanishing cannot occur.

Step 2: *Exclusion of dichotomy.*

We argue by contradiction and assume that we have dichotomy, and thus let $\lambda \in (0,1)$ be as in Lemma 2.4 (iii). Then, considering a sequence $\varepsilon_n \downarrow 0$, for any $n \in \mathbb{N}$, there exist $R_n > 0$, $x_n \in \mathbb{R}^N_+$ as well as nonnegative measures ν_n^1 and ν_n^2 on $\mathbb{R}^N_+$ such that

$$0 \le \nu_n^1 + \nu_n^2 \le \nu_n, \quad \operatorname{supp} \nu_n^1 \subset \mathbb{R}^N_+ \cap B_{R_n}(x_n), \quad \operatorname{supp} \nu_n^2 \subset \mathbb{R}^N_+ \setminus B_{2R_n^{(2+s)/2}+1}(x_n),$$
$$\left|\lambda - \int_{\mathbb{R}^N_+} d\nu_n^1\right| + \left|(1-\lambda) - \int_{\mathbb{R}^N_+} d\nu_n^2\right| \le 2\varepsilon_n,$$

and thus

$$\limsup_{n\to\infty}\left(\left|\lambda - \int_{\mathbb{R}^N_+} d\nu_n^1\right| + \left|(1-\lambda) - \int_{\mathbb{R}^N_+} d\nu_n^2\right|\right) = 0.$$

From the proof of Lemma 2.4 (see [Struwe 2008]) we can assume $R_n \to \infty$ and, in particular, $R_n \ge 1$.

For $r > 0$, let the anisotropic scaling τ_r be defined as in (2-10). We crucially note that

$$B_{R_n}(0) \subset \tau_{\sqrt{R_n}}(B_1(0))$$

and

$$\mathbb{R}^N_+ \setminus B_{2R_n^{(2+s)/2}+1}(0) \subset \mathbb{R}^N_+ \setminus \tau_{\sqrt{R_n}}(B_2(0)).$$

We take $\varphi \in C_c^\infty(B_2(0))$ with $0 \le \varphi \le 1$ and $\varphi \equiv 1$ in $B_1(0)$. For $n \in \mathbb{N}$, let $\varphi_n(x) := \varphi(\tau_{\sqrt{R_n}}^{-1}(x - x_n))$, so that

$$\varphi_n \equiv 1 \quad \text{on } x_n + \tau_{\sqrt{R_n}}(B_1(0)), \qquad \varphi_n \equiv 0 \quad \text{on } \mathbb{R}^N \setminus (x_n + \tau_{\sqrt{R_n}}(B_2(0))),$$

and thus, in particular,

$$\varphi_n \equiv 1 \quad \text{on } \operatorname{supp} \nu_n^1, \qquad \varphi_n \equiv 0 \quad \text{on } \operatorname{supp} \nu_n^2.$$

Note that

$$|\partial_i v_n|^2 + x_1^s |\partial_N v_n|^2 \ge (|\partial_i v_n|^2 + x_1^s |\partial_N v_n|^2)(\varphi_n^2 + (1-\varphi_n)^2) \quad \text{for } i = 1, \dots, N-1.$$

We have

$$\Big(\int_{\mathbb{R}^N_+}\Big(\sum_{i=1}^{N-1}|\partial_i(\varphi_n v_n)|^2+x_1^s|\partial_N(\varphi_n v_n)|^2\Big)\,dx\Big)^{1/2} \le \Big(\int_{\mathbb{R}^N_+}\varphi_n^2\Big(\sum_{i=1}^{N-1}|\partial_i v_n|^2+x_1^s|\partial_N v_n|^2\Big)\,dx\Big)^{1/2}+\Big(\int_{\mathbb{R}^N_+}v_n^2\Big(\sum_{i=1}^{N-1}|\partial_i\varphi_n|^2+x_1^s|\partial_N\varphi_n|^2\Big)\,dx\Big)^{1/2}$$

and analogously for $(1-\varphi_n)$ instead of φ_n. Squaring and adding these estimates gives

$$\begin{aligned}&\|\varphi_n v_n\|^2_{H_s}+\|(1-\varphi_n)v_n\|^2_{H_s}\\ &\quad\le \int_{\mathbb{R}^N_+}\Big(\sum_{i=1}^{N-1}|\partial_i v_n|^2+x_1^s|\partial_N v_n|^2\Big)\,dx+2\int_{\mathbb{R}^N_+}v_n^2\Big(\sum_{i=1}^{N-1}|\partial_i\varphi_n|^2+x_1^s|\partial_N\varphi_n|^2\Big)\,dx\\ &\qquad+4\Big(\int_{\mathbb{R}^N_+}\Big(\sum_{i=1}^{N-1}|\partial_i v_n|^2+x_1^s|\partial_N v_n|^2\Big)\,dx\Big)^{1/2}\Big(\int_{\mathbb{R}^N_+}v_n^2\Big(\sum_{i=1}^{N-1}|\partial_i\varphi_n|^2+x_1^s|\partial_N\varphi_n|^2\Big)\,dx\Big)^{1/2}.\end{aligned}$$

Setting

$$\begin{aligned}\beta_n:=2\int_{\mathbb{R}^N_+}v_n^2\Big(\sum_{i=1}^{N-1}|\partial_i\varphi_n|^2+x_1^s|\partial_N\varphi_n|^2\Big)\,dx\\ +4\Big(\int_{\mathbb{R}^N_+}\Big(\sum_{i=1}^{N-1}|\partial_i v_n|^2+x_1^s|\partial_N v_n|^2\Big)\,dx\Big)^{1/2}\Big(\int_{\mathbb{R}^N_+}v_n^2\Big(\sum_{i=1}^{N-1}|\partial_i\varphi_n|^2+x_1^s|\partial_N\varphi_n|^2\Big)\,dx\Big)^{1/2},\end{aligned}$$

we thus have

$$\int_{\mathbb{R}^N_+}\Big(\sum_{i=1}^{N-1}|\partial_i v_n|^2+x_1^s|\partial_N v_n|^2\Big)\,dx\ge\|\varphi_n v_n\|^2_{H_s}+\|(1-\varphi_n)v_n\|^2_{H_s}-\beta_n.$$

Next, we define the anisotropic annulus

$$A_n:=x_n+\tau_{\sqrt{R_n}}(B_2(0))\setminus\tau_{\sqrt{R_n}}(B_1(0))$$

and consider $\delta>0$. Using Young's inequality and the fact that any derivative of φ_n vanishes outside of A_n, we can estimate

$$\beta_n\le\delta\int_{\mathbb{R}^N_+}\Big(\sum_{i=1}^{N-1}|\partial_i v_n|^2+x_1^s|\partial_N v_n|^2\Big)\,dx+C(\delta)\int_{A_n}v_n^2\Big(\sum_{i=1}^{N-1}|\partial_i\varphi_n|^2+x_1^s|\partial_N\varphi_n|^2\Big)\,dx.$$

Note that

$$\begin{aligned}\sum_{i=1}^{N-1}|\partial_i\varphi_n|^2+x_1^s|\partial_N\varphi_n|^2&=R_n^{-2}\sum_{i=1}^{N-1}|[\partial_i\varphi](\tau^{-1}_{\sqrt{R_n}}(x))|^2+x_1^sR_n^{-2-s}|[\partial_N\varphi](\tau^{-1}_{\sqrt{R_n}}(x))|^2\\ &=R_n^{-2}\Big(\sum_{i=1}^{N-1}|[\partial_i\varphi]|^2+(\,\cdot\,)_1^s|[\partial_N\varphi|^2\Big)\circ\tau^{-1}_{\sqrt{R_n}},\end{aligned}$$

and thus

$$\sum_{i=1}^{N-1}|\partial_i\varphi_n|^2+x_1^s|\partial_N\varphi_n|^2\le CR_n^{-2}$$

for some $C > 0$ independent of n. This gives

$$\int_{A_n} v_n^2 \left(\sum_{i=1}^{N-1} |\partial_i \varphi_n|^2 + x_1^s |\partial_N \varphi_n|^2 \right) dx \le C R_n^{-2} \|v_n\|_{L^2(A_n)}^2.$$

Using Hölder's inequality then further yields

$$\begin{aligned} R_n^{-1} \|v_n\|_{L^2(A_n)} &\le R_n^{-1} |A_n|^{2/(2N+s)} \|v_n\|_{L^{2_s^*}(A_n)} \\ &\le C \|v_n\|_{L^{2_s^*}(A_n)} \\ &\le C \left(\int_{\mathbb{R}^N} d\nu_n - \left(\int_{\mathbb{R}^N} d\nu_n^1 + \int_{\mathbb{R}^N} d\nu_n^2 \right) \right)^{1/2_s^*} \to 0 \end{aligned}$$

as $n \to \infty$. Here we used

$$|A_n| = |\tau_{\sqrt{R_n}}(B_2(x_n))| - |\tau_{\sqrt{R_n}}(B_1(x_n))| = R_n^{(2N+s)/2}(|B_2(0)| - |B_1(0)|).$$

Overall, we find that, for any $\delta > 0$,

$$\limsup_{n\to\infty} \beta_n \le \delta \sup_n \|v_n\|_H^2,$$

and since $(v_n)_n$ remains bounded in H_s, we conclude

$$\begin{aligned} \int_{\mathbb{R}_+^N} \left(\sum_{i=1}^{N-1} |\partial_i v_n|^2 + x_1^s |\partial_N v_n|^2 \right) dx &\ge \|\varphi_n v_n\|_{H_s}^2 + \|(1-\varphi_n) v_n\|_{H_s}^2 - \beta_n \\ &\ge \mathcal{S}(\|\varphi_n v_n\|_{L^{2_s^*}(\mathbb{R}_+^N)}^2 + \|(1-\varphi_n) v_n\|_{L^{2_s^*}(\mathbb{R}_+^N)}^2) + o(1) \\ &\ge \mathcal{S}\left(\left(\int_{B_{R_n}(x_n)} d\nu_n \right)^{2/2_s^*} + \left(\int_{\mathbb{R}_+^N \setminus B_{R_n'}(x_n)} d\nu_n \right)^{2/2_s^*} \right) + o(1) \\ &\ge \mathcal{S}\left(\left(\int_{\mathbb{R}_+^N} d\nu_n^1 \right)^{2/2_s^*} + \left(\int_{\mathbb{R}_+^N} d\nu_n^2 \right)^{2/2_s^*} \right) + o(1) \\ &\ge \mathcal{S}(\lambda^{2/2_s^*} + (1-\lambda)^{2/2_s^*}) + o(1). \end{aligned}$$

But since $\lambda \in (0, 1)$, we have $\lambda^{2/2_s^*} + (1-\lambda)^{2/2_s^*} > 1$, and thus

$$\mathcal{S} = \lim_{n\to\infty} \int_{\mathbb{R}_+^N} \left(\sum_{i=1}^{N-1} |\partial_i v_n|^2 + x_1^s |\partial_N v_n|^2 \right) dx \ge \liminf_{n\to\infty} (\mathcal{S}(\lambda^{2/2_s^*} + (1-\lambda)^{2/2_s^*}) + o(1)) > \mathcal{S},$$

a contradiction. Hence we cannot have dichotomy.

Step 3: *The sequence cannot concentrate in multiple points.*

Since we are therefore in condition (i) of Lemma 2.4, there exists a sequence $(x_n)_n$ such that, for any $\varepsilon > 0$, there exists $R = R(\varepsilon) > 0$ with

$$\int_{B_R(x_n)} d\nu_n = \int_{B_R(x_n) \cap \mathbb{R}_+^N} d\nu_n \ge 1 - \varepsilon.$$

Since we normalized so that

$$\int_{(0,1)\times(-1,1)^{N-1}} |v_n|^{2_s^*}\,dx = \tfrac{1}{2},$$

we must have $(0,1)\times(-1,1)^{N-1} \cap B_R(x_n) \neq \varnothing$ if $\varepsilon < \frac{1}{2}$. By making R larger if necessary, we can thus assume

$$\int_{B_R(0)} d\nu_n = \int_{B_R(0)\cap\mathbb{R}^N_+} d\nu_n \geq 1-\varepsilon.$$

In particular, we may therefore pass to a subsequence such that $\nu_n \rightharpoonup \nu$ weakly in the sense of measure, where ν is a finite measure on $\mathbb{R}^N_+$. By weak lower- (and upper-) semicontinuity (of measures), we then have

$$\int_{\mathbb{R}^N_+} d\nu = 1.$$

By Lemma 2.5, we may now assume

$$\mu_n \rightharpoonup \mu \geq \sum_{i=1}^{N-1}(|\partial_i v|^2 + x_1^s|\partial_N v|^2)\,dx + \sum_{j\in J}\mu^j\delta_{x^j} \quad\text{and}\quad \nu_n \rightharpoonup |v|^{2_s^*}\,dx + \sum_{j\in J}\nu^j\delta_{x^j}$$

for points $x^j \in \mathbb{R}^N_+$ and positive $\mu^j,\ \nu^j$ satisfying $\mathcal{S}(\nu^j)^{2/2_s^*} \leq \mu^j$. We have

$$\begin{aligned}\mathcal{S}+o(1) = \|v_n\|^2_{H_s} &= \int_{\mathbb{R}^N_+} d\mu_n \geq \int_{\mathbb{R}^N_+} d\mu + o(1) \geq \|v\|^2_{H_s} + \sum_{j\in J}\mu^j + o(1)\\ &\geq \mathcal{S}\Big(\|v\|^2_{L^{2_s^*}(\mathbb{R}^N_+)} + \sum_j (\nu^j)^{2/2_s^*}\Big) + o(1) \geq \mathcal{S}\Big(\|v\|^{2_s^*}_{L^{2_s^*}(\mathbb{R}^N_+)} + \sum_j \nu^j\Big)^{2/2_s^*} + o(1)\\ &= \mathcal{S}\Big(\int_{\mathbb{R}^N_+} d\nu\Big)^{2/2_s^*} + o(1) = \mathcal{S} + o(1) \end{aligned} \tag{2-11}$$

as $n\to\infty$. In the second inequality, we used the fact that the map $t \mapsto t^{2/2_s^*}$ is strictly concave and hence subadditive. Moreover, the strict concavity implies that equality can only hold, if at most one of the terms $\|v\|^{2_s^*}_{L^{2_s^*}(\mathbb{R}^N_+)}$ and $\nu^j,\ j\in J$, is nonzero.

Step 4: *The sequence cannot concentrate in a single point, i.e., we have* $\nu^j = 0$ *for all* j.

Assuming that this is false, we have $\nu_n \rightharpoonup \delta_{x^1}$ for some $x^1 \in \overline{\mathbb{R}}^N_+$. By our normalization and weak lower-semicontinuity (of measures), $x^1 \notin Q := (0,1)\times(-1,1)^{N-1}$ since

$$\delta_{x^1}(Q) \leq \liminf_{n\to\infty} \nu_n(Q) = \tfrac{1}{2}.$$

Moreover, if $\operatorname{dist}(x^1, Q) > 0$, there exists $\varepsilon > 0$ such that $B_\varepsilon(x_1)\cap Q \neq \varnothing$, and thus

$$1 = \delta_{x^1}(B_\varepsilon(x^1)) \leq \liminf_{n\to\infty} \nu_n(B_\varepsilon(x^1)) \leq \tfrac{1}{2},$$

which is a contradiction. Hence it only remains to consider the case $x^1 \in \partial Q$. Due to the normalization

$$\sup_{E\in\mathcal{Q}_1}\int_E |v_n|^{2_s^*}\,dx = \int_{(0,1)\times(-1,1)^{N-1}} |v_n|^{2_s^*}\,dx = \tfrac{1}{2},$$

we have $x^1 \notin ((0, y) + Q)$ for all $y \in \mathbb{R}^{N-1}$, so x^1 must be of the form $x^1 = (1, y)$ or $(0, y)$ for some $y \in (-1, 1)^{N-1}$. The latter case can be excluded, since, for $\varepsilon \in \big(0, \frac{1}{2}\big)$,

$$\delta_{x^1}(B_\varepsilon(0, y)) \le \liminf_{n\to\infty} \nu_n(B_\varepsilon(0, y)) \le \liminf_{n\to\infty} \nu_n((0, y) + Q) \le \tfrac{1}{2}.$$

After a translation orthogonal to the x_1-direction, we may therefore assume $x^1 = (1, 0, \dots, 0)$ and first note that $v \equiv 0$ and hence $\mu \ge \mathcal{S}\delta_{x^1}$ by (2-11). On the other hand,

$$\int_{\mathbb{R}^N} d\mu \le \liminf_{n\to\infty} \int_{\mathbb{R}^N} d\mu_n = \mathcal{S},$$

whence we conclude $\mu = \mathcal{S}\delta_{x^1}$.

For any $0 < \delta < \frac{1}{2}$, $B_\delta := B_\delta(x_1)$ is a continuity set of $\nu = \delta_{x_1}$; hence

$$\nu_n(B_\delta) \to 1$$

and similarly

$$\mu_n(B_\delta) \to S$$

as $n \to \infty$. In particular, for fixed $\varepsilon > 0$, we find $n_0 = n_0(\varepsilon, \delta)$ such that

$$\int_{B_\delta} |v_n|^{2_s^*}\, dx \ge 1 - \varepsilon, \quad \mathcal{S} - \varepsilon \le \int_{B_\delta} \bigg(\sum_{i=1}^{N-1} |\partial_i v_n|^2 + x_1^s |\partial_N v_n|^2 \bigg)\, dx \le \mathcal{S} + \varepsilon$$

for $n \ge n_0$. Furthermore,

$$\frac{1}{1+\delta} \int_{B_\delta} \bigg(\sum_{i=1}^{N-1} |\partial_i v_n|^2 + x_1^s |\partial_N v_n|^2 \bigg)\, dx \le \int_{B_\delta} \sum_{i=1}^{N} |\partial_i v_n|^2\, dx$$

and

$$\int_{B_\delta} \sum_{i=1}^{N} |\partial_i v_n|^2\, dx \le \frac{1}{1-\delta} \int_{B_\delta} \bigg(\sum_{i=1}^{N-1} |\partial_i v_n|^2 + x_1^s |\partial_N v_n|^2 \bigg)\, dx$$

imply

$$\frac{\mathcal{S} - \varepsilon}{1+\delta} \le \int_{B_\delta} \sum_{i=1}^{N} |\partial_i v_n|^2\, dx \le \frac{\mathcal{S} + \varepsilon}{1-\delta}$$

for $n \ge n_0$. It is important to note that the weak convergence $\nu_n \rightharpoonup \delta_{x^1}$ implies that, for any $t \in (0, \delta)$ and $q \in (2_s^*, 2^*)$, we have

$$1 = \liminf_{n\to\infty} \int_{B_t} |v_n|^{2_s^*}\, dx \le |B_t|^{1-2_s^*/q} \liminf_{n\to\infty} \bigg(\int_{B_t} |v_n|^q\, dx \bigg)^{2_s^*/q} \le |B_t|^{1-2_s^*/q} \liminf_{n\to\infty} \bigg(\int_{B_\delta} |v_n|^q\, dx \bigg)^{2_s^*/q}.$$

In particular, this implies

$$\liminf_{n\to\infty} \bigg(\int_{B_\delta} |v_n|^q\, dx \bigg)^{2_s^*/q} \ge |B_t|^{2_s^*/q-1}, \tag{2-12}$$

and since $t \in (0, \delta)$ was arbitrary, we conclude that $\|v_n\|_{L^q(B_\delta)} \to \infty$ as $n \to \infty$ for any $q \in (2_s^*, 2^*)$.

Now let $\varphi \in C_c^\infty(\mathbb{R}^N)$ such that $\varphi \equiv 1$ on $B_1(0)$ and $\varphi \equiv 0$ on $\mathbb{R}^N \setminus B_2(0)$, and set

$$\varphi_\delta(x) := \varphi\left(\frac{x - x^1}{\delta}\right),$$

so that $\varphi_\delta \equiv 1$ on $B_\delta(x^1)$ and $\varphi \equiv 0$ on $\mathbb{R}^N \setminus B_{2\delta}(x^1)$. Then, by Sobolev's inequality,

$$\left(\int_{\mathbb{R}^N_+} |\varphi_\delta v_n|^q \, dx\right)^{2/q} \le C_q \left(\int_{\mathbb{R}^N_+} \sum_{i=1}^N |\partial_i(\varphi_\delta v_n)|^2 \, dx + \int_{\mathbb{R}^N_+} |\varphi_\delta v_n|^2 \, dx\right). \tag{2-13}$$

Note that (2-12) implies that the left-hand side goes to infinity as $n \to \infty$ since

$$\int_{B_\delta} |v_n|^q \, dx \le \int_{\mathbb{R}^N} |\varphi_\delta v_n|^q \, dx.$$

On the other hand,

$$\int_{\mathbb{R}^N_+} |\varphi_\delta v_n|^2 \, dx \le |B_{2\delta}|^{1-2/2_s^*} \left(\int_{B_{2\delta}} |v_n|^{2_s^*} \, dx\right)^{2/2_s^*} \le |B_2|^{1-2/2_s^*}$$

and, noting that $\nabla \varphi_\delta(x) = \delta^{-1}[\nabla\varphi]((x - x^1)/\delta)$,

$$\begin{aligned}
\left(\int_{\mathbb{R}^N_+} \sum_{i=1}^N |\partial_i(\varphi_\delta v_n)|^2 \, dx\right)^{1/2} &\le \left(\int_{\mathbb{R}^N_+} \varphi_\delta^2 \sum_{i=1}^N |\partial_i v_n|^2 \, dx\right)^{1/2} + \left(\int_{\mathbb{R}^N_+} v_n^2 \sum_{i=1}^N |\partial_i \varphi_\delta|^2 \, dx\right)^{1/2} \\
&\le \left(\int_{B_{2\delta}} \sum_{i=1}^N |\partial_i v_n|^2 \, dx\right)^{1/2} + \sqrt{N}\delta^{-1} \|\nabla\varphi\|_\infty \left(\int_{B_{2\delta}\setminus B_\delta} |v_n|^2 \, dx\right)^{1/2} \\
&\le \sqrt{\frac{\mathcal{S}+\varepsilon}{1-2\delta}} + \sqrt{N}\delta^{-1}\|\nabla\varphi\|_\infty |B_{2\delta}\setminus B_\delta|^{1/2-1/2_s^*} \left(\int_{B_{2\delta}\setminus B_\delta} |v_n|^{2_s^*} \, dx\right)^{1/2_s^*} \\
&\le \sqrt{\frac{\mathcal{S}+\varepsilon}{1-2\delta}} + \sqrt{N}\delta^{-1}\|\nabla\varphi\|_\infty |B_{2\delta}\setminus B_\delta|^{1/2-1/2_s^*}.
\end{aligned}$$

This implies that the right-hand side of (2-13) remains bounded as $n \to \infty$, a contradiction.

We conclude $\nu^j = 0$ for all j, and hence $\|v\|_{L^{2_s^*}(\mathbb{R}^N_+)} = 1$. Since $L^{2_s^*}(\mathbb{R}^N_+)$ is uniformly convex, this implies $v_n \to v$ in $L^{2_s^*}(\mathbb{R}^N_+)$. Moreover, since $\|v\|^2_{H_s} \ge \mathcal{S}$, weak lower-semicontinuity gives $\|v_n\|^2_{H_s} \to \mathcal{S} = \|v\|^2_{H_s}$, and hence $v_n \to v$ in H_s again by uniform convexity of the Hilbert space H_s. This completes the proof. □

Remark 2.7 (existence of minimizers on $\mathbb{R}^N$). We note that Theorem 2.2 implies

$$\mathcal{S}_s(\mathbb{R}^N) := \inf_{u \in C_c^1(\mathbb{R}^N)} \frac{\int_{\mathbb{R}^N} \left(\sum_{i=1}^{N-1} |\partial_i u|^2 + |x_1|^s |\partial_N u|^2\right) dx}{\left(\int_{\mathbb{R}^N} |u|^{2_s^*} \, dx\right)^{2/2_s^*}} > 0.$$

Consequently, we can look for minimizers in the closure of $C_c^1(\mathbb{R}^N)$ in

$$\left\{u \in L^{2_s^*}(\mathbb{R}^N) : \int_{\mathbb{R}^N} \sum_{i=1}^{N-1} |\partial_i u|^2 + |x_1|^s |\partial_N u|^2 \, dx < \infty\right\}.$$

The previous arguments can then easily be adapted to prove the existence of minimizers of $\mathcal{S}_s(\mathbb{R}^N)$ similar to Theorem 2.6.

3. A degenerate Sobolev inequality on $\boldsymbol{B}$

In this section we shall prove the second part of Theorem 1.1, namely the properties of $\mathscr{C}_{1,m,p}(\boldsymbol{B})$ given in (1-12).

We first use the scaling properties discussed in Remark 2.3 (i) to prove the following.

Proposition 3.1. *Let $p > 2_1^*$ and $m > -\lambda_1(\boldsymbol{B})$. Then $\mathscr{C}_{1,m,p}(\boldsymbol{B}) = 0$, i.e.,*

$$\inf_{u \in C_c^1(\boldsymbol{B})\setminus\{0\}} \frac{\|\nabla u\|_2^2 - \|\partial_\theta u\|_2^2 + m\|u\|_2^2}{\|u\|_p^2} = 0.$$

Proof. Let $v \in C_c^1(\mathbb{R}_+^N) \setminus \{0\}$ be arbitrary and, for $\lambda \in (0, 1)$, let

$$\tau_\lambda : \boldsymbol{B} \to \mathbb{R}_+^N, \qquad \tau_\lambda(x) = (\lambda^{-2}(x_1+1), \lambda^{-2}x_3, \dots, \lambda^{-2}x_N, \lambda^{-3}x_2), \tag{3-1}$$

and set $u := v \circ \tau_\lambda$. If λ is chosen sufficiently small, we have $u \in C_c^1(\boldsymbol{B})$ and

$$\begin{aligned}
&\|\nabla u\|_{L^2(\boldsymbol{B})}^2 - \|\partial_\theta u\|_{L^2(\boldsymbol{B})}^2 \\
&= \int_{\boldsymbol{B}} \left(\sum_{i=1}^{N} |\partial_i u|^2 - |x_1\, \partial_2 u - x_2\, \partial_1 u|^2 \right) dx \\
&= \int_{\boldsymbol{B}} \left(\sum_{i=1}^{N-1} |\lambda^{-2}[\partial_i v] \circ \tau_\lambda|^2 + |\lambda^{-3}[\partial_N v] \circ \tau_\lambda|^2 - |x_1 \lambda^{-3}[\partial_N v] \circ \tau_\lambda - x_2 \lambda^{-2}[\partial_1 v] \circ \tau_\lambda|^2 \right) dx \\
&= \lambda^{2N+1} \int_{\mathbb{R}_+^N} \left(\sum_{i=1}^{N-1} \lambda^{-4} |\partial_i v|^2 + \lambda^{-6} |\partial_N v|^2 - |(\lambda^2 x_1 - 1)\lambda^{-3}\, \partial_N v - \lambda^3 x_2 \lambda^{-2}\, \partial_1 v|^2 \right) dx \\
&= \lambda^{2N-3} \int_{\mathbb{R}_+^N} \left(\sum_{i=1}^{N-1} |\partial_i v|^2 + 2x_1 |\partial_N v|^2 \right) dx \\
&\qquad + \lambda^{2N-3} \int_{\mathbb{R}_+^N} \left(-\lambda^2 x_1^2 |\partial_N v|^2 - 2x_2 \lambda^2 (\lambda^2 x_1 - 1)\, \partial_1 v\, \partial_N v + \lambda^6 x_2^2 |\partial_1 v|^2 \right) dx,
\end{aligned}$$

while

$$\|u\|_{L^2(\boldsymbol{B})}^2 = \lambda^{2N+1} \|v\|_{L^2(\mathbb{R}_+^N)}^2 \quad \text{and} \quad \|u\|_{L^p(\boldsymbol{B})}^2 = \lambda^{(4N+2)/p} \|v\|_{L^p(\mathbb{R}_+^N)}^2.$$

We conclude that

$$\begin{aligned}
\mathscr{C}_{1,m,p}(\boldsymbol{B}) &\le \frac{\|\nabla u\|_{L^2(\boldsymbol{B})}^2 - \|\partial_\theta u\|_{L^2(\boldsymbol{B})}^2 + m\|u\|_{L^2(\boldsymbol{B})}^2}{\|u\|_{L^p(\boldsymbol{B})}^2} \\
&= \lambda^{(p(2N-3)-(4N+2))/p} \frac{\int_{\mathbb{R}_+^N} \left(\sum_{i=1}^{N-1} |\partial_i v|^2 + 2x_1 |\partial_N v|^2 \right) dx}{\|v\|_{L^p(\mathbb{R}_+^N)}^2} + o(\lambda^{(p(2N-3)-(4N+2))/p}) \to 0
\end{aligned}$$

as $\lambda \to 0$, since $p > 2_1^* = (4N+2)/(2N-3)$. □

To prove the second assertion on $\mathscr{C}_{1,m,p}(\boldsymbol{B})$ in (1-12), we now transfer the information given by Theorem 1.6 in the case $s = 1$ to the ball $\boldsymbol{B}$.

Lemma 3.2. *Consider the great circle γ defined in* (1-6), *and let $\varepsilon > 0$. Then there exists $\delta > 0$ with the property that, for any $x_0 \in \gamma$,*

$$\frac{\int_{\Omega_{x_0,\delta}} (|\nabla u|^2 - |\partial_\theta u|^2)\,dx}{\|u\|^2_{L^{2^*_1}(\Omega_{x_0,\delta})}} \ge (1-\varepsilon)2^{1/2-1/2^*_1}\mathcal{S}_1(\mathbb{R}^N_+) \quad \text{for } u \in C^1_c(\Omega_{x_0,\delta}) \setminus \{0\},$$

where $\mathcal{S}_1(\mathbb{R}^N_+)$ is given in Theorem 1.6 and

$$\Omega_{x_0,\delta} := \boldsymbol{B} \cap B_\delta(x_0) = \{x \in \boldsymbol{B} : |x - x_0| < \delta\}. \tag{3-2}$$

Proof. We may assume $x_0 = e_2 = (0, 1, 0, \dots, 0)$ is the second coordinate vector. We fix $\delta > 0$ and consider a function $u \in C^1_c(\Omega_{e_2,\delta})$ which we extend trivially to a function $u \in C^1_c(\mathbb{R}^N)$. Moreover, we write u in N-dimensional polar coordinates, so we consider $U := [0, 1] \times (-\pi, \pi) \times (0, \pi)^{N-2}$ and the function

$$v = u \circ P : U \to \mathbb{R},$$

with $P : U \to \mathbb{R}^N$ given by

$$P(r, \theta, \vartheta_1, \dots, \vartheta_{N-2}) = (r \sin\vartheta_1 \cdots \sin\vartheta_{N-2} \cos\theta,\ r \sin\vartheta_1 \cdots \sin\vartheta_{N-2} \sin\theta,$$
$$r\cos\vartheta_1, r\sin\vartheta_1\cos\vartheta_2, \dots, r\sin\vartheta_1 \cdots \sin\vartheta_{N-3}\cos\vartheta_{N-2}). \tag{3-3}$$

We emphasize here that we use the angular variable $\theta \in (-\pi, \pi)$ for the angle of the (x_1, x_2)-coordinate of $x \in S^{N-1}$ relative to the positive x_1-axis in $\mathbb{R}^2$ (in the literature, this is usually done for the (x_{N-1}, x_N)-coordinate). Noting that

$$|\nabla u(r\Theta)|^2 = |\partial_r u(r\Theta)|^2 + \frac{1}{r^2}|\nabla_\Theta u(r\Theta)|^2 \quad \text{for } r > 0,\ \Theta \in S^{N-1},$$

we then have, by (B-3) from Appendix B,

$$\int_{\boldsymbol{B}} (|\nabla u|^2 - |\partial_\theta u|^2)\,dx$$
$$= \int_0^1 \int_{-\pi}^{\pi} \int_0^{\pi} \cdots \int_0^{\pi} \biggl(|\partial_r v|^2 + \frac{1}{r^2}\sum_{i=1}^{N-2} h_i |\partial_{\vartheta_i} v|^2 + \Bigl(\frac{h_{N-1}}{r^2} - 1\Bigr)|\partial_\theta v|^2\biggr) h\,d\vartheta_1 \cdots d\vartheta_{N-2}\,d\theta\,dr, \tag{3-4}$$

with the functions $h, h_i : U \to \mathbb{R}$, $i = 1, \dots, N-1$, given by

$$h(r, \theta, \vartheta_1, \dots, \vartheta_{N-2}) = r^{N-1}\prod_{k=1}^{N-2} \sin^{N-1-k}\vartheta_k, \quad h_i(r, \theta, \vartheta_1, \dots, \vartheta_{N-2}) = \prod_{k=1}^{i-1} \frac{1}{\sin^2\vartheta_k}. \tag{3-5}$$

In particular, we have $h \le 1$ and $h_i \ge 1$ in U for $i = 1, \dots, N-1$. Moreover, since

$$P^{-1}(e_2) = \bigl(1, \tfrac{\pi}{2}, \dots, \tfrac{\pi}{2}\bigr) \quad \text{and} \quad h\bigl(1, \tfrac{\pi}{2}, \dots, \tfrac{\pi}{2}\bigr) = 1,$$

we can choose $\delta > 0$ small enough that

$$P^{-1}(\Omega_{e_2,\delta}) \subset (0,1) \times (0,\pi)^{N-1} \qquad \text{and} \qquad h \ge (1-\varepsilon) \quad \text{in } P^{-1}(\Omega_{e_2,\delta}). \tag{3-6}$$

Therefore

$$\int_{\boldsymbol{B}} (|\nabla u|^2 - |\partial_\theta u|^2)\,dx \\ \geq (1-\varepsilon) \int_0^1 \int_{-\pi}^{\pi} \int_0^{\pi} \cdots \int_0^{\pi} \left(|\partial_r v|^2 + \sum_{i=1}^{N-2} |\partial_{\vartheta_i} v|^2 + \frac{(1-r)(1+r)}{r^2} |\partial_\theta v|^2 \right) d\vartheta_1 \cdots d\vartheta_{N-2}\, d\theta\, dr.$$

Noting that

$$\frac{(1-r)(1+r)}{r^2} \geq \frac{(2-\delta)(1-r)}{(1-\delta)^2} \geq 2(1-r)$$

and substituting $t = 1-r$, we thus find that

$$\int_{\boldsymbol{B}} (|\nabla u|^2 - |\partial_\theta u|^2)\,dx \geq (1-\varepsilon) \int_0^1 \int_{-\pi}^{\pi} \int_0^{\pi} \cdots \int_0^{\pi} \left(|\partial_t \tilde{v}|^2 + \sum_{i=1}^{N-2} |\partial_{\vartheta_i} \tilde{v}|^2 + 2t |\partial_\theta \tilde{v}|^2 \right) d\vartheta_1 \cdots d\vartheta_{N-2}\, d\theta\, dt,$$

with

$$\tilde{v} : U \to \mathbb{R}, \quad \tilde{v}(t, \vartheta_1, \dots, \vartheta_{N-2}, \theta) = v(1-t, \vartheta_1, \dots, \vartheta_{N-2}, \theta).$$

Note that $u \in C_c^1(\Omega_{e_2,\delta})$ implies, by (3-6), that $\tilde{v}$ is compactly supported in $(0, 1) \times (0, \pi)^{N-1} \subset \mathbb{R}^N_+$, so we may regard $\tilde{v}$ as a function in $C_c^1(\mathbb{R}^N_+)$ and deduce that

$$\int_{\boldsymbol{B}} (|\nabla u|^2 - |\partial_\theta u|^2)\,dx \geq (1-\varepsilon) \int_{\mathbb{R}^N_+} \left(\sum_{i=1}^{N-1} |\partial_i \tilde{v}|^2 + 2x_1 |\partial_N \tilde{v}|^2 \right) dx.$$

Rather directly, we also find that, by a change of variables,

$$\int_{\Omega_{e_2,\delta}} |u|^{2_1^*}\,dx = \int_U |v|^{2_1^*} h\, d(r, \theta, \vartheta_1, \dots, \vartheta_{N-2}) \leq \int_U |v|^{2_1^*}\, d(r, \theta, \vartheta_1, \dots, \vartheta_{N-2}) \\ = \int_U |\tilde{v}|^{2_1^*}\, d(r, \theta, \vartheta_1, \dots, \vartheta_{N-2}) = \int_{\mathbb{R}^N_+} |\tilde{v}|^{2_1^*}\,dx.$$

Using (2-9) with $\kappa = 2$, we conclude that

$$\frac{\int_{\Omega_{e_2,\delta}} (|\nabla u|^2 - |\partial_\theta u|^2)\,dx}{\|u\|^2_{L^{2_1^*}(\Omega_{e_2,\delta})}} \geq (1-\varepsilon) \frac{\int_{\mathbb{R}^N_+} \left(\sum_{i=1}^{N-1} |\partial_i \tilde{v}|^2 + 2x_1 |\partial_N \tilde{v}|^2 \right) dx}{\left(\int_{\mathbb{R}^N_+} |\tilde{v}|^{2_1^*}\,dx \right)^{2/2_1^*}} \geq (1-\varepsilon) 2^{1/2 - 1/2_1^*} \mathcal{S}_1(\mathbb{R}^N_+)$$

as claimed. □

We can now prove the main result of this section.

Theorem 3.3. *For any $1 \leq p \leq 2_1^*$, there exists $C > 0$ such that any $u \in C_c^1(\boldsymbol{B})$ satisfies*

$$\|u\|^2_{L^p(\boldsymbol{B})} \leq C \int_{\boldsymbol{B}} (|\nabla u|^2 - |\partial_\theta u|^2)\,dx. \tag{3-7}$$

Moreover, in the case $p = 2$, (3-7) holds with $C = 1/\lambda_1(\boldsymbol{B})$.

Recall here that $\lambda_1(\boldsymbol{B})$ is the first Dirichlet eigenvalue of $-\Delta$ on $\boldsymbol{B}$.

Proof. Let $u \in C^1_c(\boldsymbol{B})$. We first show that

$$\int_{\boldsymbol{B}} (|\nabla u|^2 - |\partial_\theta u|^2)\,dx \ge \lambda_1(\boldsymbol{B})\|u\|^2_{L^2(\boldsymbol{B})}. \tag{3-8}$$

In the following, for every integer $\ell \ge 0$, we let $\{Y_{\ell,k} : \ell \in \mathbb{N}\cup\{0\},\, k = 1, \dots, d_\ell\}$ denote an L^2-orthonormal basis of $L^2(\mathbb{S}^{N-1})$ of spherical harmonics of degree ℓ. More precisely, we can choose $Y_{\ell,k}$ in such a way that, for every $\ell \in \mathbb{N}\cup\{0\}$, the functions $Y_{\ell,k},\ k = 1, \dots, d_\ell$ form a basis of the eigenspace of the Laplace Beltrami operator $-\Delta_{\mathbb{S}^{N-1}}$ corresponding to the eigenvalue $\ell(\ell+N-2)$ and such that

$$-\partial^2_\theta Y_{\ell,k} = \ell_k^2 Y_{\ell,k} \quad \text{for } k = 1, \dots, d_\ell,$$

where $|\ell_k| \le \ell$; see, e.g., [Higuchi 1987]. Next, let $u_{\ell,k} \in C^1([0,1])$ be the angular Fourier coefficient functions defined by

$$u_{\ell,k}(r) = \int_{\mathbb{S}^{N-1}} u(r\omega) Y_{\ell,k}(\omega)\,d\omega, \quad 0 \le r \le 1.$$

For fixed $r \in [0,1]$, we then have the Parseval identities

$$\begin{aligned}
\|u(r\,\cdot)\|^2_{L^2(\mathbb{S}^{N-1})} &= \sum_{\ell,k} |u_{\ell,k}(r)|^2 \|Y_{\ell,k}\|^2_{L^2(\mathbb{S}^{N-1})},\\
\|\partial_r u(r\,\cdot)\|^2_{L^2(\mathbb{S}^{N-1})} &= \sum_{\ell,k} |\partial_r u_{\ell,k}(r)|^2 \|Y_{\ell,k}\|^2_{L^2(\mathbb{S}^{N-1})},\\
\|\nabla_{\mathbb{S}^{N-1}} u(r\,\cdot)\|^2_{L^2(\mathbb{S}^{N-1})} &= \sum_{\ell,k} (\ell+N-2)|u_{\ell,k}(r)|^2 \|Y_{\ell,k}\|^2_{L^2(\mathbb{S}^{N-1})} \quad \text{and}\\
\|\partial_\theta u(r\,\cdot)\|^2_{L^2(\mathbb{S}^{N-1})} &= \sum_{\ell,k} \ell_k^2 |u_{\ell,k}(r)|^2 \|Y_{\ell,k}\|^2_{L^2(\mathbb{S}^{N-1})}
\end{aligned}$$

in $L^2(\mathbb{S}^{N-1})$. Hereafter, we simply write $\sum_{\ell,k}$ in place of $\sum_{\ell=0}^{\infty}\sum_{k=1}^{d_\ell}$. Since $\ell(\ell+N-2)/r^2 \ge \ell_k^2$ for $r \in [0,1]$ and every $\ell,\ k$, we estimate that

$$\begin{aligned}
\int_{\boldsymbol{B}} (|\nabla u|^2 - |\partial_\theta u|^2)\,dx &= \int_0^1 r^{N-1} \int_{\mathbb{S}^{N-1}} \Big(|\partial_r u(r\omega)|^2 + \frac{1}{r^2}|\nabla_{\mathbb{S}^{N-1}} u(r\omega)|^2 - |\partial_\theta u(r\omega)|^2\Big)\,d\omega\,dr\\
&= \sum_{\ell,k} \|Y_{\ell,k}\|^2_{L^2(\mathbb{S}^{N-1})} \int_0^1 r^{N-1}\Big(|\partial_r u_{\ell,k}(r)|^2 + \Big(\frac{\ell(\ell+N-2)}{r^2} - \ell_k^2\Big)|u_{\ell,k}(r)|^2\Big)\,dr\\
&\ge \sum_{\ell,k} \|Y_{\ell,k}\|^2_{L^2(\mathbb{S}^{N-1})} \int_0^1 r^{N-1} |\partial_r u_{\ell,k}(r)|^2\,dr\\
&\ge \lambda_1(\boldsymbol{B}) \sum_{\ell,k} \|Y_{\ell,k}\|^2_{L^2(\mathbb{S}^{N-1})} \int_0^1 r^{N-1} |u_{\ell,k}(r)|^2\,dr\\
&= \lambda_1(-\Delta, \boldsymbol{B}) \int_0^1 r^{N-1} \|u(r\,\cdot)\|^2_{L^2(\mathbb{S}^{N-1})}\,dr\\
&= \lambda_1(\boldsymbol{B}) \int_{\boldsymbol{B}} |u|^2\,dx.
\end{aligned}$$

Hence (3-8) holds. To show (3-7), it suffices to consider the case $p = 2_1^*$. In the following, $C > 0$ is a constant independent of u which may change from line to line. Fix $\varepsilon \in \big(0, \frac{1}{2}\big)$ and let $\delta > 0$ be given as in Lemma 3.2. Take points $x_1, \dots, x_m \in \gamma$ such that the sets $U_k := B_\delta(x_k)$ satisfy

$$\gamma \subset \bigcup_{k=1}^m U_k,$$

and let $\delta_0 := \operatorname{dist}\big(\gamma, \boldsymbol{B} \setminus \bigcup_{k=1}^m U_k\big)$. We then let $U_0 := \big\{x \in \boldsymbol{B} : \operatorname{dist}(x, \gamma) > \frac{1}{2}\delta_0\big\}$, and thus we have $\boldsymbol{B} \subset \bigcup_{k=0}^m U_k$. We may then choose a partition of unity $\eta_0, \dots, \eta_m$ subordinate to this covering. Then

$$\|u\|_{L^{2_1^*}(\boldsymbol{B})} \le \sum_{k=0}^m \|\eta_k u\|_{L^{2_1^*}(U_k)} \le C \sum_{k=0}^m \biggl(\int_{U_k} (|\nabla(\eta_k u)|^2 - |\partial_\theta(\eta_k u)|^2)\, dx\biggr)^{1/2},$$

where we used Lemma 3.2 and the fact that $v \mapsto \int_{U_0} (|\nabla v|^2 - |\partial_\theta v|^2)\, dx$ induces an equivalent norm on $H_0^1(U_0)$. Indeed, recall that $\partial_\theta = x_1\, \partial_{x_2} - x_2\, \partial_{x_1}$, and hence

$$|(\partial_\theta v)| \le |(x_1, x_2)|\, |\nabla v| \quad \text{a.e. in } \boldsymbol{B}, \tag{3-9}$$

which implies

$$\int_{U_0} (|\nabla v|^2 - |\partial_\theta v|^2)\, dx \ge \int_{U_0} (1 - |(x_1, x_2)|^2) |\nabla v|^2\, dx.$$

Letting $x = (x_1, x_2, x') \in U_0$ with $x' \in \mathbb{R}^{N-2}$, we then find that

$$\tfrac{1}{4}\delta_0^2 < \operatorname{dist}(x, \gamma)^2 = (1 - |(x_1, x_2)|)^2 + |x'|^2 \le (1 - |(x_1, x_2)|)^2 + 1 - |(x_1, x_2)|^2 \le 2(1 - |(x_1, x_2)|^2),$$

and hence

$$\int_{U_0} (|\nabla v|^2 - |\partial_\theta v|^2)\, dx \ge \tfrac{1}{8}\delta_0^2 \int_{U_0} |\nabla v|^2\, dx,$$

i.e., $v \mapsto \int_{U_0} (|\nabla v|^2 - |\partial_\theta v|^2)\, dx$ induces an equivalent norm on $H_0^1(U_0)$, as claimed.

Note that, for $k = 0, \dots, m$, we have

$$\begin{aligned}\int_{U_k} (|\nabla(\eta_k u)|^2 - |\partial_\theta(\eta_k u)|^2)\, dx &\le 2\biggl(\int_{U_k} \eta_k^2 (|\nabla u|^2 - |\partial_\theta u|^2)\, dx + \int_{U_k} u^2 (|\nabla \eta_k|^2 - |\partial_\theta \eta_k|^2)\, dx\biggr)\\ &\le C \int_{U_k} (|\nabla u|^2 - |\partial_\theta u|^2 + u^2)\, dx,\end{aligned}$$

with some fixed $C > 0$. Here we used the fact that

$$\begin{aligned}2(\nabla \eta_k \cdot \nabla u - \partial_\theta \eta_k\, \partial_\theta u) &= -(|\nabla(\eta_k - u)|^2 - |\partial_\theta(\eta_k - u)|^2) + (|\nabla \eta_k|^2 - |\partial_\theta \eta_k|^2) + (|\nabla u|^2 - |\partial_\theta u|^2)\\ &\le (|\nabla \eta_k|^2 - |\partial_\theta \eta_k|^2) + (|\nabla u|^2 - |\partial_\theta u|^2)\end{aligned}$$

pointwisely on $\boldsymbol{B}$ again by (3-9). We conclude that

$$\|u\|_{L^{2_1^*}(\boldsymbol{B})} \le C \sum_{k=0}^m \biggl(\int_{U_k} (|\nabla u|^2 - |\partial_\theta u|^2 + u^2)\, dx\biggr)^{1/2},$$

and thus

$$\|u\|^2_{L^{2_1^*}(\boldsymbol{B})} \le C \sum_{k=0}^{m} \int_{U_k} (|\nabla u|^2 - |\partial_\theta u|^2 + u^2)\,dx \le C \int_{\boldsymbol{B}} (|\nabla u|^2 - |\partial_\theta u|^2 + u^2)\,dx$$
$$\le C \int_{\boldsymbol{B}} (|\nabla u|^2 - |\partial_\theta u|^2)\,dx, \tag{3-10}$$

where we used (3-8) in the last step. The proof is finished. □

4. The variational setting for and main results on ground state solutions

In this section, we set up the variational framework for (1-5) and discuss the notions of weak and ground state solutions of (1-5). Then, we shall complete the proofs of Theorems 1.3, 1.4, 1.5 and 1.7.

4.1. *The variational setting.* We need to fix some notation. Let $0 \le \alpha \le 1$. It then follows from Theorem 3.3 that

$$(u, v) \mapsto \langle u, v\rangle_{\mathcal{H}_\alpha} := \int_{\boldsymbol{B}} (\nabla u \cdot \nabla v - \alpha^2\, \partial_\theta u\, \partial_\theta v)\,dx$$

defines a scalar product on $C^1_c(\boldsymbol{B})$. The induced norm will be denoted by $\|\cdot\|_{\mathcal{H}_\alpha}$. We then let $\mathcal{H}_\alpha$ be the Hilbert space defined as the completion of $C^1_c(\boldsymbol{B})$ with respect to the norm $\|\cdot\|_{\mathcal{H}_\alpha}$. Since

$$\|u\|^2_{\mathcal{H}_\alpha} = \alpha^2 \|u\|^2_{\mathcal{H}_1} + (1-\alpha^2)\|\nabla u\|^2_{L^2(\boldsymbol{B})} \quad \text{for } u \in C^1_c(\boldsymbol{B}),$$

it follows that $\|\cdot\|_{\mathcal{H}_\alpha}$ is equivalent to $\|\cdot\|_{H^1_0(\boldsymbol{B})}$ for $\alpha \in [0, 1)$, and therefore

$$\mathcal{H}_\alpha = H^1_0(\boldsymbol{B}) \quad \text{for } \alpha \in [0, 1).$$

As a consequence, we have embeddings

$$\mathcal{H}_\alpha \hookrightarrow L^p(\boldsymbol{B}) \quad \text{for } \alpha \in [0, 1),\ 1 \le p \le 2^*,$$

which are compact in the Sobolev subcritical case $p < 2^*$. The next lemma is concerned with the exceptional case $\alpha = 1$.

Lemma 4.1. *$\mathcal{H}_1$ is embedded in $L^p(\boldsymbol{B})$ for $p \in [1, 2_1^*]$. Moreover, if $1 \le p < 2_1^*$, then the embedding $\mathcal{H}_1 \hookrightarrow L^p(\boldsymbol{B})$ is compact.*

Proof. The embedding $\mathcal{H}_1 \hookrightarrow L^p(\boldsymbol{B})$ for $p \in [1, 2_1^*]$ is an immediate consequence of Theorem 3.3 and the fact that $L^p(\boldsymbol{B}) \subset L^{2_1^*}(\boldsymbol{B})$ for $p \in [1, 2_1^*]$. To prove the compactness of the embedding for fixed $p \in [1, 2_1^*)$, we let $(u_n)_n \subset \mathcal{H}_1$ be a bounded sequence. Moreover, we put $B_m := B_{1-1/m}(0) \subset \boldsymbol{B}$ for $m \ge 2$. Then $u_n^m := \mathbb{1}_{B_m} u_n$ defines a bounded sequence in $H^1(B_m)$ for every $m \ge 2$. After passing to a subsequence, $(u_n^m)_n$ converges in $L^p(B_m)$ by Rellich–Kondrachov. After passing to a diagonal sequence we may therefore assume that there exists $u \in L^p(\boldsymbol{B})$ with the property that $u_n \to u$ for $m \in \mathbb{N}$ pointwisely in $\boldsymbol{B}$. Moreover,

$$\|u - u_n\|_{L^p(\boldsymbol{B})} \le \|u - u_n\|_{L^p(B_m)} + \|u - u_n\|_{L^{2_1^*}(\boldsymbol{B}\setminus B_m)} |\boldsymbol{B} \setminus B_m|^{1/p - 1/2_1^*}.$$

Since $\|u-u_n\|_{L^{2_1^*}(\boldsymbol{B}\setminus B_m)} \le \|u-u_n\|_{L^{2_1^*}(\boldsymbol{B})}$ remains bounded independently of m and n, this gives

$$\limsup_{n\to\infty} \|u-u_n\|_{L^p(\boldsymbol{B})} \le C|\boldsymbol{B}\setminus B_m|^{1/p-1/2_1^*}$$

for some $C>0$ independent of m, where the right-hand side tends to zero as $m\to\infty$. This proves that $u_n \to u$ in $L^p(\boldsymbol{B})$. □

Remark 4.2. Let $\alpha\in[0,1]$. We first note that, for any $f\in C^1(\mathbb{R})$ such that f' is bounded and $f(0)=0$, we have $f\circ u\in\mathcal{H}_\alpha$. Indeed, recall that by the definition of $\mathcal{H}_\alpha$, there exists a sequence $(u_n)_n\subset C_c^1(\boldsymbol{B})$ such that $\|u-u_n\|_{\mathcal{H}_\alpha}\to 0$ as $n\to\infty$, and the differentiability of f readily implies $f\circ u_n\in C_c^1(\boldsymbol{B})\subset\mathcal{H}_\alpha$ for all n. Using the chain rule and the boundedness of f', it can then be shown that $f\circ u_n\to f\circ u$ in $\mathcal{H}_\alpha$ as $n\to\infty$. Via approximation, this observation can be used to show $u^\pm, |u|\in\mathcal{H}_\alpha$, which is a classical fact in the case $\alpha<1$, where $\mathcal{H}_\alpha=H_0^1(\boldsymbol{B})$.

Definition 4.3. Let $\alpha\in[0,1]$ and $f\in L^{2_1^\sharp}(\boldsymbol{B})$, where $2_1^\sharp = 2_1^*/(2_1^*-1)$ denotes the conjugate of 2_1^*.

(i) We call $u\in\mathcal{H}_\alpha$ a weak solution of

$$-\Delta u+\alpha^2\,\partial_\theta^2 u = f \tag{4-1}$$

if

$$\langle u,v\rangle_{\mathcal{H}_\alpha} = \int_{\boldsymbol{B}} fv\,dx \quad \text{for every } v\in\mathcal{H}_\alpha. \tag{4-2}$$

(ii) We call $u\in\mathcal{H}_\alpha$ a weak supersolution of (4-1) if (4-2) holds with $\ge$ in place of $=$ for every $v\in\mathcal{H}_\alpha$, $v\ge 0$.

We have the following useful properties.

Lemma 4.4. *Let $\alpha\in[0,1]$ and $f\in L^{2_1^\sharp}(\boldsymbol{B})$.*

(i) *If $f\ge 0$ and $u\in\mathcal{H}_\alpha$ is a weak supersolution of* (4-1), *then $u\ge 0$.*

(ii) *If $f\in L^\infty(\boldsymbol{B})$ and $u\in\mathcal{H}_\alpha$ is a weak solution of* (4-1), *then $u\in C^1_{\mathrm{loc}}(\overline{\boldsymbol{B}}\setminus\gamma)\cap C(\overline{\boldsymbol{B}})$ with $u\equiv 0$ on $\partial\boldsymbol{B}$, where γ is the great circle defined in* (1-6). *Additionally, if $f\in C^\sigma_{\mathrm{loc}}(\overline{\boldsymbol{B}}\setminus\gamma)$ for some $\sigma\in(0,1)$, then $u\in C^{2,\sigma}_{\mathrm{loc}}(\overline{\boldsymbol{B}}\setminus\gamma)$.*

Moreover, $\overline{\boldsymbol{B}}\setminus\gamma$ can be replaced by $\overline{\boldsymbol{B}}$ in these statements if $\alpha<1$.

Proof. (i) Using $v=u^-=-\min\{0,u\}$ in the definition of a weak supersolution, we find that

$$-\|u^-\|^2_{\mathcal{H}_\alpha} = \langle u,v\rangle_{\mathcal{H}_\alpha} = \int_{\boldsymbol{B}} fu^-\,dx \ge 0,$$

and thus $u^-\equiv 0$.

(ii) Since the operator $-\Delta u+\alpha^2\,\partial_\theta^2$ is uniformly elliptic on $\overline{\boldsymbol{B}}$ if $\alpha\in[0,1)$ and locally uniformly elliptic on $\overline{\boldsymbol{B}}\setminus\gamma$ if $\alpha=1$, all statements follow from standard elliptic regularity theory, with the exception of the claim

$$u\in C(\overline{\boldsymbol{B}}) \qquad \text{with } u\equiv 0 \quad \text{on } \partial\boldsymbol{B}. \tag{4-3}$$

To prove (4-3), we let $c := \|f\|_{L^\infty(\boldsymbol{B})}$, and we note that $u_c \in \mathcal{H}_\alpha$ defined by $u_c(x) = c(1-|x|^2)/(2N)$ is a classical solution of

$$(-\Delta + \alpha^2\, \partial_\theta^2)u_c = -\Delta u_c = c \quad \text{in } \boldsymbol{B}, \qquad u \equiv 0 \quad \text{on } \partial\boldsymbol{B}.$$

Hence $u_c - u \in \mathcal{H}_\alpha$ is a weak supersolution of (4-1) with f replaced by $c - f \geq 0$, so $u_c - u \geq 0$ by (i). Similarly, we see that $u_c + u \geq 0$, and therefore

$$|u(x)| \leq u_c(x) = \frac{c}{2N}(1-|x|^2) \quad \text{for } x \in \boldsymbol{B}.$$

This shows the continuity of u at all points $x_0 \in \partial\boldsymbol{B}$ and that necessarily $u(x_0) = 0$. □

Remark 4.5. Let $\alpha \in [0, 1]$, $V \in L^\infty(\boldsymbol{B})$, and let $u \in \mathcal{H}_\alpha$ be a weak solution of (4-1) with $f = Vu$. If u is nonnegative in $\boldsymbol{B}$, then either $u \equiv 0$, or u is strictly positive in $\boldsymbol{B}$. This follows from the strong maximum principle, since the operator $-\Delta + \alpha^2\, \partial_\theta^2 - V$ is uniformly elliptic in every compactly contained subset of $\boldsymbol{B}$.

The following proposition extends the Poincaré-type estimate for the case $p = 2$ given in Theorem 3.3. Recall again that $\lambda_1(\boldsymbol{B})$ is the first Dirichlet eigenvalue of $-\Delta$ on $\boldsymbol{B}$.

Proposition 4.6. *For* $0 \leq \alpha \leq 1$, *we have*

$$\mathscr{C}_{\alpha,0,2}(\boldsymbol{B}) = \inf_{u \in \mathcal{H}_\alpha \setminus \{0\}} \frac{\|u\|^2_{\mathcal{H}_\alpha}}{\int_{\boldsymbol{B}} u^2\, dx} = \lambda_1(\boldsymbol{B}). \tag{4-4}$$

Moreover, the minimizers are precisely the Dirichlet eigenfunctions of $-\Delta$ *on* $\boldsymbol{B}$ *corresponding to the eigenvalue* $\lambda_1(\boldsymbol{B})$ *and are therefore radial.*

Proof. Let $\alpha \in [0, 1]$. Since $\lambda_1(\boldsymbol{B}) \leq \mathscr{C}_{1,0,2}(\boldsymbol{B}) \leq \mathscr{C}_{\alpha,0,2}(\boldsymbol{B}) \leq \mathscr{C}_{0,0,2}(\boldsymbol{B}) = \lambda_1(\boldsymbol{B})$ by Theorem 3.3 and the variational characterization of $\lambda_1(\boldsymbol{B})$, we obtain (4-4). Moreover, it follows that every minimizer of (4-4) also minimizes (4-4) with $\alpha = 0$; hence it is a Dirichlet eigenfunction of $-\Delta$ on $\boldsymbol{B}$ corresponding to the eigenvalue $\lambda_1(\boldsymbol{B})$ and therefore radial. □

Remark 4.7. Once the existence and compactness of the embedding $\mathcal{H}_\alpha \hookrightarrow L^2(\boldsymbol{B})$ is established, a direct proof of Proposition 4.6 without expansion in spherical harmonics, as used in the proof of inequality (3-8), can be given at least in the case $N = 2$. Indeed, one may then show by weak lower-semicontinuity of the $\mathcal{H}_\alpha$-norm that the infimum in (4-4) is attained. Moreover, by standard variational arguments, a function $u \in \mathcal{H}_\alpha \setminus \{0\}$ is a minimizer of (4-4) if and only if u is a weak solution of

$$-\Delta u + \alpha^2\, \partial_\theta^2 u = \mathscr{C}_{\alpha,0,2}(\boldsymbol{B})u. \tag{4-5}$$

Additionally, if $u \in \mathcal{H}_\alpha \setminus \{0\}$ solves (4-5), then also $|u| \in \mathcal{H}_\alpha$ is a minimizer of (4-4) and therefore a solution of (4-5). Consequently, $|u| > 0$ by Remark 4.5. Thus every weak solution of (4-5) does not change sign in $\boldsymbol{B}$, which shows that the solutions of (4-5) form a one-dimensional subspace of $\mathcal{H}_\alpha$. Combining this information with the fact that (4-5) remains invariant under transformations $A \in O(2) \times O(N-2)$, we conclude that the (up to a factor unique) solution u of (4-5) must satisfy $u \circ A = u$ for every $A \in O(2) \times O(N-2)$. In the case $N = 2$ this implies that u is radial. Thus u is a Dirichlet eigenfunction of $-\Delta$ corresponding to the eigenvalue $\mathscr{C}_{\alpha,0,2}(\boldsymbol{B})$. Since u does not change sign, it must correspond to the first Dirichlet eigenvalue of $-\Delta$ on $\boldsymbol{B}$, so a posteriori we conclude that $\mathscr{C}_{\alpha,0,2}(\boldsymbol{B}) = \lambda_1(\boldsymbol{B})$.

Corollary 4.8. *Let* $\alpha \in [0, 1]$ *and* $m \in \mathbb{R}$.

(i) *We have* $\mathscr{C}_{\alpha,m,2}(\boldsymbol{B}) = \mathscr{C}_{0,m,2}(\boldsymbol{B}) = \lambda_1(\boldsymbol{B}) + m$.

(ii) *If* $m > -\lambda_1(\boldsymbol{B})$ *and* $2 \le p \le 2^*$, *we have* $R_{\alpha,m,p}(u) > 0$ *for* $u \in \mathcal{H}_\alpha \setminus \{0\}$.

Here we recall that the quantities $R_{\alpha,m,p}(u)$ and $\mathscr{C}_{\alpha,m,p}(\boldsymbol{B})$ have been defined in (1-7) and (1-8).

Proof. (i) This follows immediately from Proposition 4.6.

(ii) Since $m \ge -\lambda_1(\boldsymbol{B})$, we have, by (i), for $u \in \mathcal{H}_\alpha \setminus \{0\}$,

$$R_{\alpha,m,p}(u) = R_{\alpha,m,2}(u) \frac{\|u\|^2_{L^2(\boldsymbol{B})}}{\|u\|^2_{L^p(\boldsymbol{B})}} \ge \mathscr{C}_{\alpha,m,2}(\boldsymbol{B}) \frac{\|u\|^2_{L^2(\boldsymbol{B})}}{\|u\|^2_{L^p(\boldsymbol{B})}} = (\lambda_1(\boldsymbol{B}) + m) \frac{\|u\|^2_{L^2(\boldsymbol{B})}}{\|u\|^2_{L^p(\boldsymbol{B})}} > 0. \qquad \square$$

Definition 4.9. Let $m > -\lambda_1(\boldsymbol{B})$, $p \in (2, 2_1^*]$ and $\alpha \in [0, 1]$.

(i) We call $u \in \mathcal{H}_\alpha$ a weak solution of (1-5) if u is a weak solution of (4-1), with $f = |u|^{p-2}u - mu$.

(ii) A weak solution $u \in \mathcal{H}_\alpha \setminus \{0\}$ of (1-5) will be called a ground state solution if u is a minimizer for $R_{\alpha,m,p}$, i.e., we have $R_{\alpha,m,p}(u) = \mathscr{C}_{\alpha,m,p}(\boldsymbol{B})$.

Lemma 4.10. *Let* $0 \le \alpha \le 1$, $2 < p \le 2^*$ *and* $m > -\lambda_1(\boldsymbol{B})$, *and let* $u \in \mathcal{H}_\alpha$ *be a weak solution of* (1-5).

(i) *If* $\alpha < 1$, *then* $u \in C^{2,\sigma}(\overline{\boldsymbol{B}})$ *with* $u \equiv 0$ *on* $\partial \boldsymbol{B}$ *for all* $\sigma \in (0, 1)$.

(ii) *If* $\alpha = 1$ *and* $2 \le p < 2_1^*$, *then* $u \in C^{2,\sigma}_{\text{loc}}(\overline{\boldsymbol{B}} \setminus \gamma) \cap C(\overline{\boldsymbol{B}})$ *for all* $\sigma \in (0, 1)$ *with* $u \equiv 0$ *on* $\partial \boldsymbol{B}$, *where* γ *is the great circle defined in* (1-6).

(iii) *If* u *is a ground state solution*, *then* u *does not change sign in* $\boldsymbol{B}$.

Proof. The regularity results in (i) and (ii) follow immediately from Lemma 4.4 once we have shown that $u \in L^\infty(\boldsymbol{B})$. This is a well-known consequence of classical Moser iteration in the case $\alpha < 1$, which can be performed similarly also in the case $\alpha = 1$; see Lemma A.1 in Appendix A for a detailed proof.

It thus remains to prove (iii). If u is a ground state solution, then

$$R_{\alpha,m,p}(|u|) = R_{\alpha,m,p}(u) = \mathscr{C}_{\alpha,m,p}(\boldsymbol{B}),$$

and therefore $|u|$ is also a ground state solution of (1-5). By Remark 4.5 and since $u \not\equiv 0$, it follows that $|u| > 0$ in $\boldsymbol{B}$, so u does not change sign in $\boldsymbol{B}$. $\square$

Lemma 4.11. *Let* $0 \le \alpha \le 1$, $2 \le p < 2^*$ *and* $m > -\lambda_1(\boldsymbol{B})$. *If*

$$\alpha < 1 \qquad \text{or} \qquad \alpha = 1 \quad \text{and} \quad p < 2_1^*, \tag{4-6}$$

then

$$\mathscr{C}_{\alpha,m,p}(\boldsymbol{B}) > 0, \tag{4-7}$$

and this value is attained in $\mathcal{H}_\alpha \setminus \{0\}$. *Moreover*, *up to multiplication by a positive constant*, *all minimizers are ground state solutions of* (1-5), *so they have the properties in Lemma 4.10.*

Proof. We first note that, following from Proposition 4.6, the quadratic form

$$u \mapsto \int_{\boldsymbol{B}} (|\nabla u|^2 - \alpha^2 |\partial_\theta u|^2 + m|u|^2)\, dx$$

is positive on $\mathcal{H}_\alpha \setminus \{0\}$, so it is weakly lower-semicontinuous on $\mathcal{H}_\alpha$. Moreover, by the assumption (4-6), Sobolev embeddings (in the case $\alpha < 1$) and Lemma 4.1 (in the case $\alpha = 1$), the embedding $\mathcal{H}_\alpha \hookrightarrow L^p(\boldsymbol{B})$ is compact. Hence, by a standard analysis of minimizing sequences, the value $\mathscr{C}_{\alpha,m,p}(\boldsymbol{B})$ is attained in $\mathcal{H}_\alpha \setminus \{0\}$, and thus it is positive. Moreover, standard variational arguments show that every L^p-normalized minimizer u_0 must be a weak solution of

$$-\Delta u + \alpha^2\, \partial_\theta^2 u + mu = \mathscr{C}_{\alpha,m,p}(\boldsymbol{B})|u|^{p-2}u \quad \text{in } \boldsymbol{B}. \tag{4-8}$$

We then conclude that $[\mathscr{C}_{\alpha,m,p}(\boldsymbol{B})]^{1/(p-2)}u_0$ weakly solves (1-5). □

Next, we treat the critical case $p = 2_1^*$. We first show that $\mathscr{C}_{1,m,2_1^*}(\boldsymbol{B})$ is attained, provided it is small enough, as stated in Theorem 1.7 (i). The strategy of the proof is inspired by [Frank et al. 2018] and requires the following characterization of sequences in $\mathcal{H}_1$.

Lemma 4.12. *Let*

$$Z(v) := \int_{\boldsymbol{B}} (|\nabla v|^2 - |\partial_\theta v|^2 + mv^2)\, dx \quad \text{for } v \in \mathcal{H}_1$$

and

$$N(v) := \int_{\boldsymbol{B}} |v|^{2_1^*}\, dx \qquad \text{for } v \in \mathcal{H}_1.$$

Then we have

$$2^{1/2-1/2_1^*}\mathcal{S}_1(\mathbb{R}_+^N) \le \inf\bigl\{\liminf_{n\to\infty} Z(w_n) : (w_n)_n \subset \mathcal{H},\ N(w_n) = 1,\ w_n \rightharpoonup 0 \text{ in } \mathcal{H}_1\bigr\}.$$

Proof. Let $(w_n)_n \subset \mathcal{H}_1$ such that $N(w_n) = 1,\ w_n \rightharpoonup 0$ in $\mathcal{H}_1$. Let $\varepsilon > 0$, and choose $U_0, \dots, U_m \subset \boldsymbol{B}$ as in the proof of Theorem 3.3, so that

$$\boldsymbol{B} \subset \bigcup_{k=0}^m U_k.$$

We may then choose functions $\eta_0, \dots, \eta_m \in C_c^2(\boldsymbol{B})$ such that $\operatorname{supp} \eta_k \subset U_k$ and $\sum_{k=0}^m \eta_k^2 \equiv 1$ on $\boldsymbol{B}$. Then

$$\begin{aligned}\int_{\boldsymbol{B}} (|\nabla(\eta_k w_n)|^2 - |\partial_\theta(\eta_k w_n)|^2)\, dx &= \int_{\boldsymbol{B}} (\eta_k^2 |\nabla w_n|^2 + 2w_n\eta_k \nabla w_n \cdot \nabla\eta_k + w_n^2 |\nabla\eta_k|^2)\, dx \\ &\quad - \int_{\boldsymbol{B}} (\eta_k^2 |\partial_\theta w_n|^2 + 2w_n\eta_k\, \partial_\theta w_n \cdot \partial_\theta \eta_k + w_n^2 |\partial_\theta\eta_k|^2)\, dx,\end{aligned}$$

and thus

$$\int_{\boldsymbol{B}} (|\nabla w_n|^2 - |\partial_\theta w_n|^2 + mw_n^2)\, dx \ge \sum_{k=0}^m \int_{\boldsymbol{B}} (|\nabla(\eta_k w_n)|^2 - |\partial_\theta(\eta_k w_n)|^2)\, dx - C\int_{\boldsymbol{B}} w_n^2\, dx$$

with a constant $C > 0$ independent of n. Here we used the fact that the mixed terms can be estimated as follows:

$$\int_{\boldsymbol{B}} w_n^2(|\nabla\eta_k|^2 - |\partial_\theta \eta_k|^2)\,dx \le 2 \sup_{k\in\{0,\dots,m\}} \|\nabla\eta_k\|_\infty^2 \int_{\boldsymbol{B}} w_n^2\,dx,$$

$$\begin{aligned}\int_{\boldsymbol{B}} \eta_k w_n(\nabla w_n\cdot\nabla\eta_k - \partial_\theta w_n\,\partial_\theta\eta_k)\,dx &\le \int_{\boldsymbol{B}} \eta_k w_n^2|-\Delta\eta_k + \partial_\theta^2\eta_k|\,dx \\ &\le \sup_{k\in\{0,\dots,m\}} \|-\Delta\eta_k + \partial_\theta^2\eta_k\|_\infty \int_{\boldsymbol{B}} |w_n|^2\,dx.\end{aligned}$$

We first note that $w_n \to 0$ in $L^2(\boldsymbol{B})$ since the embedding $\mathcal{H} \hookrightarrow L^2(\boldsymbol{B})$ is compact by Lemma 4.1. Moreover, it is easy to see that $\|\cdot\|_{\mathcal{H}_1}$ induces an equivalent norm on $H_0^1(U_0)$, which implies that $\eta_0 w_n \rightharpoonup 0$ in $H_0^1(U_0)$. In particular, noting that by $2_1^* < 2^*$ the classical Rellich–Kondrachov theorem implies $\eta_0 w_n \to 0$ in $L^{2_1^*}(\boldsymbol{B})$, we conclude

$$\liminf_{n\to\infty} \int_{\boldsymbol{B}} (|\nabla(\eta_0 w_n)|^2 - |\partial_\theta(\eta_0 w_n)|^2 + m(\eta_0 w_n)^2)\,dx \ge (1-\varepsilon)2^{1/2-1/2_1^*}\mathcal{S}_1(\mathbb{R}_+^N) \lim_{n\to\infty}\left(\int_{\boldsymbol{B}} |\eta_0 w_n|^{2_1^*}\,dx\right)^{2/2_1^*}$$

since the limit on the right-hand side is zero. On the other hand, Lemma 3.2 gives

$$\int_{\boldsymbol{B}} (|\nabla(\eta_k w_n)|^2 - |\partial_\theta(\eta_k w_n)|^2)\,dx \ge (1-\varepsilon)2^{1/2-1/2_1^*}\mathcal{S}_1(\mathbb{R}_+^N)\left(\int_{\boldsymbol{B}} |\eta_k w_n|^{2_1^*}\,dx\right)^{2/2_1^*}$$

for $k = 1, \dots, m$, and hence

$$\begin{aligned}&\liminf_{n\to\infty} \int_{\boldsymbol{B}} (|\nabla w_n|^2 - |\partial_\theta w_n|^2 + m w_n^2)\,dx \\ &\quad\ge \liminf_{n\to\infty} \sum_{k=0}^m \int_{\boldsymbol{B}} (|\nabla(\eta_k w_n)|^2 - |\partial_\theta(\eta_k w_n)|^2)\,dx \\ &\quad\ge (1-\varepsilon)2^{1/2-1/2_1^*}\mathcal{S}_1(\mathbb{R}_+^N) \liminf_{n\to\infty} \sum_{k=0}^m \left(\int_{\boldsymbol{B}} |\eta_k w_n|^{2_1^*}\,dx\right)^{2/2_1^*} \\ &\quad= (1-\varepsilon)2^{1/2-1/2_1^*}\mathcal{S}_1(\mathbb{R}_+^N) \liminf_{n\to\infty} \sum_{k=0}^m \|\eta_k^2 w_n^2\|_{2_1^*/2} \ge (1-\varepsilon)2^{1/2-1/2_1^*}\mathcal{S}_1(\mathbb{R}_+^N) \liminf_{n\to\infty} \left\|\sum_{k=0}^m \eta_k^2 w_n^2\right\|_{2_1^*/2} \\ &\quad= (1-\varepsilon)2^{1/2-1/2_1^*}\mathcal{S}_1(\mathbb{R}_+^N) \liminf_{n\to\infty} \|w_n\|_{2_1^*/2} = (1-\varepsilon)2^{1/2-1/2_1^*}\mathcal{S}_1(\mathbb{R}_+^N).\end{aligned}$$

Since $\varepsilon > 0$ was arbitrary, we conclude that

$$\liminf_{n\to\infty} \int_{\boldsymbol{B}} (|\nabla w_n|^2 - |\partial_\theta w_n|^2 + m w_n^2)\,dx \ge 2^{1/2-1/2_1^*}\mathcal{S}_1(\mathbb{R}_+^N). \qquad \square$$

We may now complete the proof of our main result.

Proof of Theorem 1.7 (i). Consider a minimizing sequence $(u_n)_n \subset \mathcal{H}_1$ for $\mathscr{C}_{1,m,2_1^*}(\boldsymbol{B})$ with $\|u_n\|_{2_1^*} = 1$. Then $(u_n)_n$ is bounded in $\mathcal{H}_1$; hence, after passing to a subsequence, we may assume $u_n \rightharpoonup u_0$ in $\mathcal{H}_1$. We set $v_n := u_n - u_0$ and note that, by Lemma 4.1,

$$v_n \to 0 \quad \text{in } L^q(\boldsymbol{B})$$

for $1 \le q < 2_1^*$. Moreover, we may pass to a subsequence such that $u_n \to u$ almost everywhere. Weak convergence implies

$$\mathscr{C}_{1,m,2_1^*}(\boldsymbol{B}) = \lim_{n\to\infty} Z(u_n) = Z(u_0) + \lim_{n\to\infty} Z(v_n),$$

whereas the Brezis–Lieb lemma yields

$$1 = N(u_n) = N(u_0) + N(v_n) + o(1).$$

In particular, the limits $T := \lim_{n\to\infty} N(v_n)$ and $M := \lim_{n\to\infty} Z(v_n)$ exist, and

$$M \ge 2^{1/2-1/2_1^*}\mathcal{S}_1(\mathbb{R}^N_+)T^{2/2_1^*}.$$

Indeed, this is trivial if $T = 0$ and follows from Lemma 4.12 in the case $T > 0$. Moreover, by definition we have

$$Z(u_0) \ge \mathscr{C}_{1,m,2_1^*}(\boldsymbol{B})N(u_0)^{2/2_1^*}. \tag{4-9}$$

Hence

$$\begin{aligned}\mathscr{C}_{1,m,2_1^*}(\boldsymbol{B}) &= Z(u_0) + M \ge Z(u_0) + 2^{1/2-1/2_1^*}\mathcal{S}_1(\mathbb{R}^N_+)T^{2/2_1^*}\\ &\ge Z(u_0) + (2^{1/2-1/2_1^*}\mathcal{S}_1(\mathbb{R}^N_+) - \mathscr{C}_{1,m,2_1^*}(\boldsymbol{B}))T^{2/2_1^*} + \mathscr{C}_{1,m,2_1^*}(\boldsymbol{B})(1-N(u_0))^{2/2_1^*}\\ &\ge Z(u_0) + (2^{1/2-1/2_1^*}\mathcal{S}_1(\mathbb{R}^N_+) - \mathscr{C}_{1,m,2_1^*}(\boldsymbol{B}))T^{2/2_1^*} + \mathscr{C}_{1,m,2_1^*}(\boldsymbol{B}) - \mathscr{C}_{1,m,2_1^*}(\boldsymbol{B})N(u_0)^{2/2_1^*}\\ &\ge (2^{1/2-1/2_1^*}\mathcal{S}_1(\mathbb{R}^N_+) - \mathscr{C}_{1,m,2_1^*}(\boldsymbol{B}))T^{2/2_1^*} + \mathscr{C}_{1,m,2_1^*}(\boldsymbol{B}),\end{aligned} \tag{4-10}$$

where we used the inequality $(a-b)^\tau \ge a^\tau - b^\tau$ for $a \ge b \ge 0$ and $0 \le \tau \le 1$. It follows that

$$(2^{1/2-1/2_1^*}\mathcal{S}_1(\mathbb{R}^N_+) - \mathscr{C}_{1,m,2_1^*}(\boldsymbol{B}))T^{2/2_1^*} \le 0.$$

We assumed $2^{1/2-1/2_1^*}\mathcal{S}_1(\mathbb{R}^N_+) - \mathscr{C}_{1,m,2_1^*}(\boldsymbol{B}) > 0$, so we must have $T = 0$. Hence $N(u_0) = 1$, and therefore $Z(u_0) = \mathscr{C}_{1,m,2_1^*}(\boldsymbol{B})$ by (4-9) and the second line of (4-10), which implies that u_0 is a minimizer. □

We note the following consequence of Theorem 1.7 (i), which extends (4-7) to the critical case.

Corollary 4.13. *We have* $\mathscr{C}_{1,m,2_1^*}(\boldsymbol{B}) > 0$ *for* $m > -\lambda_1(\boldsymbol{B})$.

Proof. If the value $\mathscr{C}_{1,m,2_1^*}(\boldsymbol{B})$ is attained in $\mathcal{H} \setminus \{0\}$, then we have $\mathscr{C}_{1,m,2_1^*}(\boldsymbol{B}) > 0$ by Corollary 4.8 (ii). If not, we have $\mathscr{C}_{1,m,2_1^*}(\boldsymbol{B}) \ge 2^{1/2-1/2_1^*}\mathcal{S}_1(\mathbb{R}^N_+) > 0$ by Theorem 1.7 (i) and Theorem 1.6. □

In general, the existence of ground state solutions in the case $\alpha = 1,\ p = 2_1^*$ remains an open problem and might depend on the parameter $m > -\lambda_1(\boldsymbol{B})$. We have the following partial existence result in the critical case.

Theorem 4.14. *There exists* $\varepsilon > 0$ *such that, for* $m \in (-\lambda_1(\boldsymbol{B}), -\lambda_1(\boldsymbol{B}) + \varepsilon)$, *there exists* $u_0 \in \mathcal{H} \setminus \{0\}$ *such that*

$$R_{1,m,2_1^*}(u_0) = \inf_{u\in\mathcal{H}\setminus\{0\}} R_{1,m,2_1^*}(u),$$

i.e., u_0 *minimizes* $R_{1,m,2_1^*}$. *Furthermore, after multiplication by a positive constant*, u_0 *is a weak solution of*

$$-\Delta u + \partial_\theta^2 u + mu = |u|^{2_1^*-2}u \quad \text{in } \boldsymbol{B}.$$

Proof. For a (necessarily radial) eigenfunction φ_1 of $-\Delta$ on $\boldsymbol{B}$ corresponding to $\lambda_1(\boldsymbol{B})$, we have

$$\mathscr{C}_{1,m,2_1^*}(\boldsymbol{B}) \le R_{1,m,2_1^*}(\varphi_1) = \frac{(\lambda_1(\boldsymbol{B})+m)\int_{\boldsymbol{B}} \varphi_1^2\,dx}{\left(\int_{\boldsymbol{B}} |\varphi_1|^{2_1^*}\,dx\right)^{2/2_1^*}},$$

which implies $\mathscr{C}_{1,m,2_1^*}(\boldsymbol{B}) \to 0$ as $m \to -\lambda_1(\boldsymbol{B})^+$. In particular, it follows that there exists $\varepsilon > 0$ such that

$$\mathscr{C}_{1,m,2_1^*}(\boldsymbol{B}) < 2^{1/2-1/2_1^*}\mathcal{S}_1(\mathbb{R}^N_+)$$

holds for $m \in (-\lambda_1(\boldsymbol{B}), -\lambda_1(\boldsymbol{B})+\varepsilon)$. By Theorem 1.7 (i), this finishes the proof. □

Note that this implies Theorem 1.7 (ii).

4.2. ***Radiality versus (x_1, x_2)-nonradiality of ground state solutions.*** As outlined in the introduction, it is in general difficult to decide whether ground states of (1-5) are (x_1, x_2)-nonradial or not. In this section, we follow several approaches to this problem.

The first approach is based on the continuity of the ground state energy and the sufficient condition (1-11) to make use of the results of Section 3 for the endpoint case $\alpha = 1$. To this end, we recall that, in the case $m \ge 0$, by classical results due to [Kwong 1989; Kwong and Zhang 1991; McLeod and Serrin 1987] (see also [Damascelli et al. 1999]), the problem

$$\begin{cases} -\Delta u + mu = |u|^{p-2}u & \text{in } \boldsymbol{B}, \\ u = 0 & \text{on } \partial\boldsymbol{B}, \end{cases} \tag{4-11}$$

has a unique radial positive solution $u_{\text{rad}} \in H_0^1(\boldsymbol{B})$, which is a minimizer for $\mathscr{C}_{0,m,p}(\boldsymbol{B})$. Clearly, u_{rad} is also a weak solution of (1-5) for every $\alpha > 0$, but it might not be a ground state solution, as we see next.

Lemma 4.15. *Let $2 < p < 2^*$ and $m > -\lambda_1(\boldsymbol{B})$ be fixed.*

(i) *The map*

$$[0, 1] \to \mathbb{R}, \quad \alpha \mapsto \mathscr{C}_{\alpha,m,p}(\boldsymbol{B})$$

is continuous and nonincreasing.

(ii) *Let $\alpha \in (0, 1]$, and suppose that $p \le 2_1^*$ in the case $\alpha = 1$. Then the following properties are equivalent*:

- (ii)$_1$ $\mathscr{C}_{\alpha,m,p}(\boldsymbol{B}) < \mathscr{C}_{0,m,p}(\boldsymbol{B})$.
- (ii)$_2$ *Every ground state solution of* (1-5) *is (x_1, x_2)-nonradial.*

Proof. (i) The monotonicity of $\mathscr{C}_{\alpha,m,p}(\boldsymbol{B})$ in α follows immediately from the definition. In order to prove continuity, we first consider $\alpha_0 \in (0, 1]$ and let $\varepsilon > 0$. Moreover, we let $u_0 \in H_0^1(\boldsymbol{B}) \setminus \{0\}$ be a function with $R_{\alpha_0,m,p}(u_0) < \mathscr{C}_{\alpha_0,m,p}(\boldsymbol{B}) + \varepsilon$. For $\alpha \le \alpha_0$, we then have

$$\begin{aligned}\mathscr{C}_{\alpha_0,m,p}(\boldsymbol{B}) \le \mathscr{C}_{\alpha,m,p}(\boldsymbol{B}) \le R_{\alpha,m,p}(u_0) &\le R_{\alpha_0,m,p}(u_0) + (\alpha_0^2-\alpha^2)\frac{\int_{\boldsymbol{B}} |\partial_\theta u_0|^2\,dx}{\left(\int_{\boldsymbol{B}} |u_0|^p\,dx\right)^{2/p}} \\ &\le \mathscr{C}_{\alpha_0,m,p}(\boldsymbol{B}) + \varepsilon + (\alpha_0^2-\alpha^2)\frac{\int_{\boldsymbol{B}} |\partial_\theta u_0|^2\,dx}{\left(\int_{\boldsymbol{B}} |u_0|^p\,dx\right)^{2/p}},\end{aligned}$$

which implies that $\limsup_{\alpha\to\alpha_0^-} |\mathscr{C}_{\alpha,m,p}(\boldsymbol{B}) - \mathscr{C}_{\alpha_0,m,p}(\boldsymbol{B})| \le \varepsilon$. This shows continuity from the left in α_0.

Next we let $\alpha_0 \in [0, 1)$ and show continuity from the right in α_0. For this we fix $\delta > 0$ such that $(\alpha_0, \alpha_0 + \delta) \subset (0, 1)$. For $\alpha \in (\alpha_0, \alpha_0 + \delta)$, Lemma 4.11 implies that the value $\mathscr{C}_{\alpha,m,p}(\boldsymbol{B})$ is attained at a function $u_\alpha \in H_0^1(\boldsymbol{B}) \setminus \{0\}$ with $\int_{\boldsymbol{B}} |u_\alpha|^p \, dx = 1$. Therefore

$$\begin{aligned}\mathscr{C}_{\alpha_0,m,p}(\boldsymbol{B}) \geq \mathscr{C}_{\alpha,m,p}(\boldsymbol{B}) = R_{\alpha,m,p}(u_\alpha) = R_{\alpha_0,m,p}(u_\alpha) + (\alpha_0^2 - \alpha^2) \int_{\boldsymbol{B}} |\partial_\theta u_\alpha|^2 \, dx \\ \geq \mathscr{C}_{\alpha_0,m,p}(\boldsymbol{B}) - |\alpha_0^2 - \alpha^2| \int_{\boldsymbol{B}} |\nabla u_\alpha|^2 \, dx,\end{aligned}$$

whence, using the fact that

$$(1 - \alpha^2) \int_{\boldsymbol{B}} |\nabla u_\alpha|^2 \, dx \leq \int_{\boldsymbol{B}} (|\nabla u_\alpha|^2 - \alpha^2 |\partial_\theta u_\alpha|^2) \, dx = \mathscr{C}_{\alpha,m,p}(\boldsymbol{B}) \leq \mathscr{C}_{0,m,p}(\boldsymbol{B}),$$

we conclude

$$\begin{aligned}\mathscr{C}_{\alpha_0,m,p}(\boldsymbol{B}) \geq \mathscr{C}_{\alpha,m,p}(\boldsymbol{B}) \geq \mathscr{C}_{\alpha_0,m,p}(\boldsymbol{B}) - \frac{|\alpha_0^2 - \alpha^2|}{1 - \alpha^2} \mathscr{C}_{0,m,p}(\boldsymbol{B}) \\ \geq \mathscr{C}_{\alpha_0,m,p}(\boldsymbol{B}) - \frac{|\alpha_0^2 - \alpha^2|}{1 - (\alpha_0 + \delta)^2} \mathscr{C}_{0,m,p}(\boldsymbol{B}).\end{aligned}$$

This shows continuity from the right in α_0.

(ii) As noted above, $\mathscr{C}_{0,m,p}(\boldsymbol{B})$ is attained by a radial positive solution u_{rad} of (4-11), and we have $R_{0,m,p}(u_{\mathrm{rad}}) = R_{\alpha,m,p}(u_{\mathrm{rad}})$. Hence, if $\mathscr{C}_{0,m,p}(\boldsymbol{B}) = \mathscr{C}_{\alpha,m,p}(\boldsymbol{B})$, then u_{rad} is also a radial ground state solution of (1-5). Hence (ii)$_2$ and (i) imply that $\mathscr{C}_{\alpha,m,p}(\boldsymbol{B}) < \mathscr{C}_{0,m,p}(\boldsymbol{B})$. If, conversely, there exists a radial ground state solution u of (1-5), then we have

$$\mathscr{C}_{0,m,p}(\boldsymbol{B}) \leq R_{0,m,p}(u) = R_{\alpha,m,p}(u) = \mathscr{C}_{\alpha,m,p}(\boldsymbol{B}),$$

and therefore equality holds by (i). Consequently, the inequality $\mathscr{C}_{\alpha,m,p}(\boldsymbol{B}) < \mathscr{C}_{0,m,p}(\boldsymbol{B})$ implies that every ground state solution of (1-5) is (x_1, x_2)-nonradial. □

We now turn to the proof of Theorem 1.3, which yields radiality of ground state solutions for α close to zero. This essentially relies on the implicit function theorem and the fact that the case $\alpha = 0$ corresponds to the classical nonlinear Schrödinger equation (4-11), where nondegeneracy results are available.

Proof of Theorem 1.3. We fix $m \geq 0$ and $2 < p < 2^*$. Moreover, we consider a sequence of numbers $\alpha_n \in (0, 1)$, $\alpha_n \to 0$, and, for every $n \in \mathbb{N}$, a positive ground state solution $u_n \in H_0^1(\boldsymbol{B})$ of (1-5) with $\alpha = \alpha_n$. Recall that the existence of u_n is proved in Lemma 4.11. In order to prove the theorem, it then suffices to show that

$$u_n = u_{\mathrm{rad}} \quad \text{for } n \text{ sufficiently large,} \tag{4-12}$$

where u_{rad} is the unique positive solution of (4-11).

Step 1: We claim that

$$u_n \to u_{\mathrm{rad}} \quad \text{in } H_0^1(\boldsymbol{B}) \quad \text{as } n \to \infty. \tag{4-13}$$

To this end, we put $v_n := u_n / \|u_n\|_{L^p(\boldsymbol{B})}$, so v_n is an L^p-normalized minimizer for $\mathscr{C}_{\alpha_n,m,p}(\boldsymbol{B})$. Then $(v_n)_n$ is bounded in $H_0^1(\boldsymbol{B})$ by definition of $\mathscr{C}_{\alpha_n,m,p}(\boldsymbol{B})$. Consequently, we have $v_n \rightharpoonup v_0$ in $H_0^1(\boldsymbol{B})$ after passing to a subsequence, which implies that $v_n \to v_0$ in $L^p(\boldsymbol{B})$, and therefore $\int_{\boldsymbol{B}} |v_0|^p\,dx = 1$. We show that v_0 is a minimizer for $\mathscr{C}_{0,m,p}(\boldsymbol{B})$. Indeed, by weak lower-semicontinuity, we have

$$\begin{aligned}\mathscr{C}_{0,m,p}(\boldsymbol{B}) \le R_{0,m,p}(v_0) \le \liminf_{n\to\infty} R_{0,m,p}(v_n) &= \lim_{n\to\infty} \big(R_{\alpha_n,m,p}(v_n) + \alpha_n^2 \|\partial_\theta v_n\|^2_{L^2(\boldsymbol{B})}\big) \\ &\le \lim_{n\to\infty} \big(\mathscr{C}_{\alpha_n,m,p}(\boldsymbol{B}) + \alpha_n^2 \|v_n\|^2_{H^1(\boldsymbol{B})}\big) = \mathscr{C}_{0,m,p}(\boldsymbol{B}),\end{aligned}$$

where we used Lemma 4.15 in the last step. Hence v_0 is a minimizer of $\mathscr{C}_{0,m,p}(\boldsymbol{B})$, and a posteriori we find that

$$\begin{aligned}\|\nabla v_n\|^2_{L^2(\boldsymbol{B})} + m\|v_n\|^2_{L^2(\boldsymbol{B})} &= R_{\alpha_n,m,p}(v_n) + \alpha_n^2 \|\partial_\theta v_n\|^2_{L^2(\boldsymbol{B})} \\ &\to R_{0,m,p}(v_0) = \|\nabla v_0\|^2_{L^2(\boldsymbol{B})} + m\|v_0\|^2_{L^2(\boldsymbol{B})} \quad \text{as } n \to \infty.\end{aligned}$$

By uniform convexity of $H^1(\boldsymbol{B})$, we thus conclude that $v_n \to v_0$ in $H^1_0(\boldsymbol{B})$. Next we recall that, as noted in the proof of Lemma 4.11, we have

$$u_n := [\mathscr{C}_{\alpha_n,m,p}(\boldsymbol{B})]^{1/(p-2)} v_n \quad \text{and, by uniqueness,} \quad u_{\mathrm{rad}} = [\mathscr{C}_{\alpha_n,m,p}(\boldsymbol{B})]^{1/(p-2)} v_0.$$

Hence Lemma 4.15 implies that $u_n \to u_{\mathrm{rad}}$ in $H^1_0(\boldsymbol{B})$. Although we have proved this only after passing to a subsequence, the convergence of the full sequence now follows from the uniqueness of u_{rad} and yields (4-13).

Step 2: Next, we improve this convergence by noting that

$$u_n \to u_{\mathrm{rad}} \quad \text{in } H^2(\boldsymbol{B}). \tag{4-14}$$

This follows in a standard way from (4-13) and standard elliptic regularity theory (see, e.g., [Gilbarg and Trudinger 1977, Theorem 8.12]) since $w_n = u_{\mathrm{rad}} - u_n \in H^1_0(\boldsymbol{B})$ is a weak solution of

$$\begin{cases} -\Delta w_n + \alpha_n^2\,\partial_\theta^2 w_n + m w_n = |v_{\mathrm{rad}}|^{p-2} v_{\mathrm{rad}} - |v_n|^{p-2} v_n & \text{in } \boldsymbol{B}, \\ w_n = 0 & \text{on } \partial\boldsymbol{B}, \end{cases}$$

and the coefficients of the differential operator $-\Delta + \alpha_n^2\,\partial_\theta$ are uniformly bounded and elliptic in $n \in \mathbb{N}$.

Step 3 (conclusion): To complete the proof of (4-12), we consider the map

$$F : (-1, 1) \times H^2(\boldsymbol{B}) \cap H^1_0(\boldsymbol{B}) \to L^2(\boldsymbol{B}), \qquad F(\alpha, u) := -\Delta u + \alpha^2\,\partial_\theta^2 u + mu - |u|^{p-2}u,$$

and we note that weak solutions of (1-5) correspond to zeroes of F. We also note that $F(\alpha, u_{\mathrm{rad}}) = 0$ for all α. We wish to apply the implicit function theorem at $(0, u_{\mathrm{rad}})$, so we need to check that

$$[\partial_u F](0, u_{\mathrm{rad}}) = -\Delta + m - (p-1)|u_{\mathrm{rad}}|^{p-2}$$

is invertible as a map $H^2(\boldsymbol{B}) \cap H^1_0(\boldsymbol{B}) \to L^2(\boldsymbol{B})$. This is equivalent to the nondegeneracy of u_{rad} as a solution of (4-11) which is noted, e.g., in [Damascelli et al. 1999, Theorem 4.2] for $m = 0$ and in [Aftalion and Pacella 2003, Theorem 1.1] in the case $m > 0$. Now the implicit function theorem yields $\varepsilon > 0$ with the following property: if $u \in H^2(\boldsymbol{B}) \cap H^1_0(\boldsymbol{B})$ satisfies $\|u - u_{\mathrm{rad}}\|_{H^2(\boldsymbol{B})} < \varepsilon$ and $F(\alpha, u) = 0$ for some $\alpha \in (-\varepsilon, \varepsilon)$, then $u = u_{\mathrm{rad}}$.

Hence (4-14) implies that $u_n = u_{\mathrm{rad}}$ for n sufficiently large, which shows (4-12), as claimed. □

Remark 4.16. In a similar way, the following result can be shown: *for fixed $\alpha \in (0,1)$ and $m \geq 0$, there exists $p_0 > 2$ with the property that, for $2 \leq p \leq p_0$, there exists a unique positive L^p-normalized minimizer for $\mathscr{C}_{\alpha,m,p}(\boldsymbol{B})$ in $\mathcal{H}_\alpha$ which is a radial function.*

To see this, we first show, similar to the proof of Lemma 4.15, that the map

$$[2, 2^*) \to \mathbb{R}, \quad p \mapsto \mathscr{C}_{\alpha,m,p}(\boldsymbol{B}),$$

is continuous. Then we argue by contradiction again and assume that, for some sequence of numbers $p_n > 2$ with $p_n \to 2$ as $n \to \infty$, there exists nonradial minimizers $u_n \in \mathcal{H}_\alpha$ for $\mathscr{C}_{\alpha,m,p}(\boldsymbol{B})$ with $\|u_n\|_{L^{p_n}(\boldsymbol{B})} = 1$ for $n \in \mathbb{N}$. Similar to the proof of Theorem 1.3 above, one can then show with the help of Proposition 4.6 that, after passing to a subsequence,

$$u_n \to u_* \ \text{ in } \mathcal{H}_\alpha \quad \text{and pointwisely a.e. on } \boldsymbol{B} \quad \text{as } n \to \infty, \tag{4-15}$$

where u_* is the unique L^2-normalized positive eigenfunction of the Dirichlet Laplacian on $\boldsymbol{B}$. The nonradiality of u_n then allows us to distinguish two cases.

Case 1: After passing to a subsequence, u_n is (x_1, x_2)-nonradial for every $n \in \mathbb{N}$.

Case 2: After passing to a subsequence, there exists, for every $n \in \mathbb{N}$, a reflection σ_n at a hyperplane containing the (x_1, x_2)-plane with $u_n \neq \tilde{u}_n$, where $\tilde{u}_n := u_n \circ \sigma_n$.

We then define $v_n : \boldsymbol{B} \to \mathbb{R}$ by $v_n := \partial_\theta u_n$ in Case 1 and $v_n = \tilde{u}_n - u_n$ in Case 2. Then $v_n \in \mathcal{H}_\alpha = H_0^1(\boldsymbol{B})$ by Lemma 4.10 (i) since $\alpha < 1$. Moreover, since u_* is a radial function, we have

$$\int_{\boldsymbol{B}} v_n u_* \, dx = 0 \quad \text{for every } n \in \mathbb{N}. \tag{4-16}$$

We then consider $w_n := v_n / \|v_n\|_{\mathcal{H}_\alpha}$, which is a weak solution of

$$-\Delta w_n + \alpha^2 \, \partial_\theta^2 w_n + m w_n = \mathscr{C}_{\alpha,m,p_n}(\boldsymbol{B}) c_n(x) w_n \quad \text{in } \boldsymbol{B},$$

with

$$c_n = (p_n - 1) u_n^{p_n - 2} \quad \text{in Case 1}$$

and

$$c_n = (p_n - 1) \int_0^1 ((1-\tau) u_n + \tau \tilde{u}_n)^{p_n - 2} \, d\tau \quad \text{in Case 2.}$$

In particular, this implies that

$$\mathscr{C}_{\alpha,m,p_n}(\boldsymbol{B}) \int_{\boldsymbol{B}} c_n(x) |w_n|^2 \, dx = \|w_n\|_{\mathcal{H}_\alpha}^2 + m \|w_n\|_{L^2(\boldsymbol{B})}^2 = 1 + m \|w_n\|_{L^2(\boldsymbol{B})}^2 \tag{4-17}$$

for $n \in \mathbb{N}$. Since w_n is bounded in $\mathcal{H}_\alpha$, we may, since $\alpha < 1$, pass to a subsequence such that $w_n \rightharpoonup w$ in $\mathcal{H}_\alpha$, $w_n \to w$ strongly in $L^p(\boldsymbol{B})$ for $p \in [2, 2^*)$ and $w_n \to w$ pointwisely a.e. on $\boldsymbol{B}$. Moreover, from (4-15), it is not difficult to see that $c_n \to 1$ in $L^q(\boldsymbol{B})$ for every $q \in [2, \infty)$. By Hölder's inequality, we may therefore pass to the limit in (4-17) to see that

$$\mathscr{C}_{\alpha,m,2}(\boldsymbol{B}) \|w\|_{L^2(\boldsymbol{B})}^2 = 1 + m \|w\|_{L^2(\boldsymbol{B})}^2;$$

hence $w \neq 0$ and

$$\mathscr{C}_{\alpha,0,2}(\boldsymbol{B}) \geq \frac{1}{\|w\|^2_{L^2(\boldsymbol{B})}} \geq \frac{\|w\|^2_{\mathcal{H}_\alpha}}{\|w\|^2_{L^2(\boldsymbol{B})}}.$$

From Proposition 4.6, it then follows that $w = cu_*$ for some $c \in \mathbb{R} \setminus \{0\}$, which contradicts the fact that

$$\int_{\boldsymbol{B}} wu_*\,dx = \lim_{n\to\infty} \int_{\boldsymbol{B}} w_n u_*\,dx = 0$$

by (4-16). The contradiction allows us to conclude that there exists $p_0 > 2$ with the property that all minimizers for $\mathscr{C}_{\alpha,m,p}(\boldsymbol{B})$ are radial functions for $2 \leq p \leq p_0$. The uniqueness statement then follows from the uniqueness of positive radial solutions of (4-11).

In the remainder of this section, we show the existence of (x_1, x_2)-nonradial ground states for large m, as claimed in Theorem 1.4. This is based on the scaling property

$$\partial_\theta[u(\varepsilon(\,\cdot\,))] = [\partial_\theta u](\varepsilon(\,\cdot\,))$$

for $\varepsilon > 0$, which is used to relate (1-5) to a similar problem on larger balls. Localized ground states of the associated classical nonlinear Schrödinger on $\mathbb{R}^N$ can then be used to construct suitable test functions and disprove symmetry via energy estimates for small ε, which translates into a large mass term. We first restate Theorem 1.4 here in an equivalent form.

Theorem 4.17. *Let $\alpha \in (0, 1]$ and $2 < p < 2^*$. Then there exists $\varepsilon_0 > 0$ such that the ground states of*

$$\begin{cases} -\Delta u + \alpha^2\,\partial_\theta^2 u + u/\varepsilon^2 = |u|^{p-2}u & \text{in } \boldsymbol{B},\\ u = 0 & \text{on } \partial\boldsymbol{B}, \end{cases} \tag{4-18}$$

are (x_1, x_2)-nonradial for $\varepsilon \in (0, \varepsilon_0)$. Moreover, if $p < 2_1^*$, the same result holds for $\alpha = 1$.

Proof. We first treat the case $\alpha \in (0, 1)$. In the following, for $u \in H_0^1(\boldsymbol{B})$ and $\varepsilon > 0$, we consider $B_{1/\varepsilon} := B_{1/\varepsilon}(0)$ and the rescaled function $u_\varepsilon \in H_0^1(B_{1/\varepsilon})$, $u_\varepsilon(x) = u(\varepsilon x)$. A direct computation then shows that

$$\frac{\int_{B_{1/\varepsilon}} (|\nabla u_\varepsilon|^2 - \alpha^2\varepsilon^2|\partial_\theta u_\varepsilon|^2 + u_\varepsilon^2)\,dx}{\bigl(\int_{B_{1/\varepsilon}} |u_\varepsilon|^p\,dx\bigr)^{2/p}} = \varepsilon^{2-N+2N/p} R_{\alpha,1/\varepsilon^2,p}(u). \tag{4-19}$$

As a consequence, we have

$$\mathscr{C}_{\alpha\varepsilon,1,p}(B_{1/\varepsilon}) := \inf_{v\in H_0^1(B_{1/\varepsilon})\setminus\{0\}} \frac{\int_{B_{1/\varepsilon}} (|\nabla v|^2 - \alpha^2\varepsilon^2|\partial_\theta v|^2 + v^2)\,dx}{\bigl(\int_{B_{1/\varepsilon}} |v|^p\,dx\bigr)^{2/p}} = \varepsilon^{2-N+2N/p}\mathscr{C}_{\alpha,1/\varepsilon^2,p}(\boldsymbol{B}).$$

It suffices to show that there exists $\varepsilon_0 > 0$ such that all minimizers for $\mathscr{C}_{\alpha\varepsilon,1,p}(B_{1/\varepsilon})$ in $H_0^1(B_{1/\varepsilon}) \setminus \{0\}$ are (x_1, x_2)-nonradial if $\varepsilon \in (0, \varepsilon_0)$. We argue by contradiction and suppose that there exists a sequence $\varepsilon_n \to 0$ and, for every $n \in \mathbb{N}$, a minimizer $v_{\varepsilon_n} \in H_0^1(B_{1/\varepsilon_n}) \setminus \{0\}$ for $\mathscr{C}_{\alpha\varepsilon_n,1,p}(B_{1/\varepsilon_n})$ which satisfies

$$\partial_\theta v_{\varepsilon_n} \equiv 0 \quad \text{in } B_{1/\varepsilon_n}. \tag{4-20}$$

To simplify the notation, we continue writing ε in place of ε_n in the following. From (4-20) and the inclusion $H_0^1(B_{1/\varepsilon}) \subset H^1(\mathbb{R}^N)$, we then deduce that

$$\mathscr{C}_{\alpha\varepsilon,1,p}(B_{1/\varepsilon}) = \frac{\int_{B_{1/\varepsilon}} (|\nabla v_\varepsilon|^2 + v^2)\,dx}{\left(\int_{B_{1/\varepsilon}} |v|^p\,dx\right)^{2/p}} \geq \inf_{v \in H^1(\mathbb{R}^N)\setminus\{0\}} \frac{\int_{\mathbb{R}^N} (|\nabla v|^2 + v^2)\,dx}{\left(\int_{\mathbb{R}^N} |v|^p\,dx\right)^{2/p}} =: \mathscr{C}_{0,1,p}(\mathbb{R}^N). \tag{4-21}$$

We will now derive a contradiction to this inequality by constructing suitable functions in $H_0^1(B_{1/\varepsilon} \setminus \{0\})$ to estimate $\mathscr{C}_{\alpha\varepsilon,1,p}(B_{1/\varepsilon})$. To this end, we first note that the value $\mathscr{C}_{0,1,p}(\mathbb{R}^N)$ is attained by any translation of the unique positive radial solution $\tilde{u}_0 \in H^1(\mathbb{R}^N)$ of the nonlinear Schrödinger equation

$$-\Delta u + u = |u|^{p-2}u \quad \text{in } \mathbb{R}^N.$$

Now take a radial function $\eta \in C_c^1(\boldsymbol{B})$ such that $0 \leq \eta \leq 1$ and $\eta \equiv 1$ in $B_{1/2}$, and let $u_0(x) := \tilde{u}_0(x - e_1)$, where $e_1 = (1, 0, \dots, 0)$. We then define

$$\eta_\varepsilon, w_\varepsilon \in C_c^1(B_{1/\varepsilon}) \quad \text{by } \eta_\varepsilon(x) = \eta(\varepsilon x),\ w_\varepsilon(x) = \eta_\varepsilon(x)u_0(x).$$

Then we have $w_\varepsilon \equiv u_0$ in $B_{1/(2\varepsilon)}$, and

$$\begin{aligned}
\mathscr{C}_{\alpha\varepsilon,1,p}(B_{1/\varepsilon}) &\leq \frac{\int_{B_{1/\varepsilon}} (|\nabla w_\varepsilon|^2 - \alpha^2\varepsilon^2|\partial_\theta w_\varepsilon|^2 + w_\varepsilon^2)\,dx}{\left(\int_{B_{1/\varepsilon}} |w_\varepsilon|^p\,dx\right)^{2/p}} \\
&= \frac{\int_{B_{1/\varepsilon}} \eta_\varepsilon^2(|\nabla u_0|^2 + u_0^2)\,dx}{\left(\int_{B_{1/\varepsilon}} \eta_\varepsilon^p|u_0|^p\,dx\right)^{2/p}} + \frac{\int_{B_{1/\varepsilon}} (u_0^2|\nabla_\varepsilon \eta|^2 + 2\eta_\varepsilon u_0 \nabla\eta_\varepsilon \cdot \nabla u_0 - \alpha^2\varepsilon^2\eta_\varepsilon^2|\partial_\theta u_0|^2)\,dx}{\left(\int_{B_{1/\varepsilon}} \eta_\varepsilon^p|u_0|^p\,dx\right)^{2/p}}.
\end{aligned} \tag{4-22}$$

We first estimate the second term and note that classical results (see [Berestycki and Lions 1983]) imply that there exist $C_0, \delta_0 > 0$ such that

$$|u_0(x)|, |\nabla u_0(x)| \leq C_0 e^{-\delta_0|x|} \quad \text{for } x \in \mathbb{R}^N. \tag{4-23}$$

Noting that $\nabla\eta_\varepsilon \equiv 0$ on $B_{1/(2\varepsilon)}$, this readily implies

$$\int_{B_{1/\varepsilon}} (u_0^2|\nabla\eta_\varepsilon|^2 + 2\eta_\varepsilon u_0 \nabla\eta_\varepsilon \cdot \nabla u_0)\,dx \leq C_1 e^{-\delta_1/\varepsilon}$$

for some constants $C_1, \delta_1 > 0$. Moreover, for $\varepsilon \in \left(0, \frac{1}{2}\right)$, we have

$$\alpha^2\varepsilon^2 \int_{B_{1/\varepsilon}} \eta_\varepsilon^2|\partial_\theta u_0|^2\,dx \geq C_2\varepsilon^2, \quad \text{with } C_2 := \alpha^2 \int_{\boldsymbol{B}} |\partial_\theta u_0|^2\,dx > 0,$$

since u_0 is an (x_1, x_2)-nonradial function. After possibly modifying $C_1, C_2 > 0$, this gives

$$\frac{\int_{B_{1/\varepsilon}} (u_0^2|\nabla\eta_\varepsilon|^2 + 2\eta_\varepsilon u_0 \nabla\eta_\varepsilon \cdot \nabla u_0 - \alpha^2\varepsilon^2\eta_\varepsilon^2|\partial_\theta u_0|^2)\,dx}{\left(\int_{B_{1/\varepsilon}} \eta_\varepsilon^p|u_0|^p\,dx\right)^{2/p}} \leq C_1 e^{-\delta_1/\varepsilon} - C_2\varepsilon^2.$$

Next we consider the first term in (4-22) and note that

$$\frac{\int_{B_{1/\varepsilon}} \eta_\varepsilon^2(|\nabla u_0|^2+u_0^2)\,dx}{\left(\int_{B_{1/\varepsilon}} \eta_\varepsilon^p|u_0|^p\,dx\right)^{2/p}} \le \frac{\int_{\mathbb{R}^N}(|\nabla u_0|^2+u_0^2)\,dx}{\left(\int_{B_{1/(2\varepsilon)}}|u_0|^p\,dx\right)^{2/p}},$$

while (4-23) implies

$$\int_{\mathbb{R}^N\setminus B_{1/(2\varepsilon)}} |u_0|^p\,dx \le C_3 e^{-\delta_2/\varepsilon}$$

for some $C_3, \delta_2 > 0$. It thus follows that

$$\begin{aligned}\frac{\int_{B_{1/\varepsilon}} \eta_\varepsilon^2(|\nabla u_0|^2+u_0^2)\,dx}{\left(\int_{B_{1/\varepsilon}} \eta_\varepsilon^p|u_0|^p\,dx\right)^{2/p}} &\le \frac{\int_{\mathbb{R}^N}(|\nabla u_0|^2+u_0^2)\,dx}{\left(\int_{B_{1/(2\varepsilon)}}|u_0|^p\,dx\right)^{2/p}} \le \frac{\int_{\mathbb{R}^N}(|\nabla u_0|^2+u_0^2)\,dx}{\left(\int_{\mathbb{R}^N}|u_0|^p\,dx - C_3e^{-\delta_2/\varepsilon}\right)^{2/p}}\\ &\le \frac{\int_{\mathbb{R}^N}(|\nabla u_0|^2+u_0^2)\,dx}{\left(\int_{\mathbb{R}^N}|u_0|^p\,dx\right)^{2/p}} + C_4e^{-\delta_2/\varepsilon} = \mathscr{C}_{0,1,p}(\mathbb{R}^N) + C_4e^{-2\delta_2/(p\varepsilon)}\end{aligned}$$

for $\varepsilon > 0$ sufficiently small with some constant $C_4 > 0$, since u_0 attains $\mathscr{C}_{0,1,p}(\mathbb{R}^N)$. In view of (4-21) and (4-22), this yields that

$$\mathscr{C}_{0,1,p}(\mathbb{R}^N) \le \mathscr{C}_{\alpha\varepsilon,1,p}(B_{1/\varepsilon}) \le \mathscr{C}_{0,1,p}(\mathbb{R}^N) - C_2\varepsilon^2 + C_1e^{-\delta_1/\varepsilon} + C_4e^{-2\delta_2/(p\varepsilon)},$$

and the right-hand side of this inequality is smaller than $\mathscr{C}_{0,1,p}(\mathbb{R}^N)$ if $\varepsilon > 0$ is sufficiently small. This is a contradiction, and thus the claim follows in this case.

In the case $\alpha = 1$, the argument is the same up to replacing $H_0^1(\boldsymbol{B})$ by $\mathcal{H}_1$ and by considering the corresponding rescaled function space $\mathcal{H}_\varepsilon$ on $B_{1/\varepsilon}$. Then the contradiction argument can be carried out in the same way, since radial functions in $\mathcal{H}_\varepsilon$ belong to $H_0^1(B_{1/\varepsilon}) \subset H^1(\mathbb{R}^N)$. □

4.3. *Additional remarks.*

Remark 4.18. Let $0 \le \alpha \le 1$, $2 < p \le 2^*$ and $m > -\lambda_1(\boldsymbol{B})$. While we have seen that ground state solutions of (1-5) are not radially symmetric in general, it is reasonable to expect that, in the case $N \ge 3$, they are invariant under rotations which leave the (x_1, x_2)-plane fixed. This is indeed the case, and we give a brief sketch of the proof in the following. By the $O(2)\times O(N-2)$-equivariance of (1-5), it suffices to show that

$$\begin{cases}\text{any ground state solution } u \text{ of (1-5) is symmetric}\\ \qquad\qquad\text{with respect to the reflection } x \mapsto (x_1, \dots, x_{N-1}, -x_N).\end{cases} \tag{4-24}$$

Then it follows that any such ground state solution is symmetric with respect to reflection at any hyperplane which contains the (x_1, x_2)-plane, so $u(x)$ only depends on (x_1, x_2) and $|(x_3, \dots, x_N)|$.

To prove (4-24), we fix a positive ground state solution $u \in \mathcal{H}_\alpha$ of (1-5), and we introduce some notation. For fixed $\lambda \in (0, 1)$, we consider the open affine half-space $\Sigma_\lambda := \{x \in \mathbb{R}^N : x_N < \lambda\}$ and the reflection at $\partial\Sigma_\lambda$ given by

$$x \mapsto x^\lambda := (x_1, \dots, x_{N-1}, 2\lambda - x_N).$$

Moreover, we define the *polarization* u_λ *of* u with respect to Σ_λ by

$$u_\lambda(x) = \begin{cases} \min\{u(x), u(x_\lambda)\} & \text{if } x \in \boldsymbol{B} \setminus \Sigma_\lambda, \\ \max\{u(x), u(x_\lambda)\} & \text{if } x \in \boldsymbol{B} \cap \Sigma_\lambda \text{ and } x^\lambda \in \boldsymbol{B}, \\ u(x) & \text{if } x \in \boldsymbol{B} \cap \Sigma_\lambda \text{ and } x^\lambda \notin \boldsymbol{B}. \end{cases}$$

By the same argument as given, for example, in Section 3 of the survey paper [Weth 2010], we then find that $u_\lambda \in \mathcal{H}_\alpha$ and $R_{\alpha,m,p}(u_\lambda) = R_{\alpha,m,p}(u) = \mathscr{C}_{\alpha,m,p}(\boldsymbol{B})$. Consequently, both u and u_λ solve (4-8), so $w_\lambda := u_\lambda - u$ solves

$$-\Delta w_\lambda + \alpha^2\, \partial_\theta^2 w_\lambda = c(x) w_\lambda \quad \text{in } \boldsymbol{B},$$

with a function $c \in L^\infty(\boldsymbol{B})$. Since $w_\lambda \geq 0$ in $\boldsymbol{B} \setminus \Sigma_\lambda$ by definition, it follows from the strong maximum principle that either $w_\lambda \equiv 0$ or $w_\lambda > 0$ in $\boldsymbol{B} \setminus \overline{\Sigma}_\lambda$. Here we note again that the operator $-\Delta + \alpha^2\, \partial_\theta^2 - c$ is uniformly elliptic in every compactly contained subset of the open set $\boldsymbol{B} \setminus \overline{\Sigma}_\lambda$. Since $w_\lambda(x) = u(x^\lambda) > 0$ on $\partial\boldsymbol{B} \setminus \overline{\Sigma}_\lambda$, we can exclude the case $w_\lambda \equiv 0$ in $\boldsymbol{B} \setminus \overline{\Sigma}_\lambda$. Hence $w_\lambda > 0$ in $\boldsymbol{B} \setminus \overline{\Sigma}_\lambda$, so $u(x) \leq u(x^\lambda)$ for $x \in \boldsymbol{B} \setminus \Sigma_\lambda$. Since $\lambda \in (0, 1)$ was fixed arbitrarily, we may pass to the limit $\lambda \to 0^+$ in this inequality and see that $u(x) \leq u(x_1, \dots, x_{N-1}, -x_N)$ for all $x \in \boldsymbol{B}$ with $x_N \geq 0$. Applying the same argument to the reflection of u with respect to the x_N-variable, we also find that $u(x) \leq u(x_1, \dots, x_{N-1}, -x_N)$ for all $x \in \boldsymbol{B}$ with $x_N \leq 0$. Consequently, (4-24) holds, as required.

Remark 4.19. Let $m > -\lambda_1(\boldsymbol{B})$. The compactness of the embedding $\mathcal{H}_\alpha \hookrightarrow L^p(\boldsymbol{B})$ in the cases

$$0 \leq \alpha < 1, \quad 2 < p < 2^*$$

and

$$\alpha = 1, \quad 2 < p < 2_1^*$$

suggests that one may show via Lusternik–Schnirelmann theory (or by using the symmetric mountain-pass theorem [Ambrosetti and Rabinowitz 1973]) that (1-5) admits infinitely many solutions under these assumptions. This is indeed the case, but it does not provide new information as it is well known that (1-5) admits infinitely many *radial* solutions if p is Sobolev subcritical; see, e.g., [Struwe 1982]. On the other hand, one might ask how many geometrically distinct (x_1, x_2)-nonradial solutions of (1-5) exist. Here we call two solutions of (1-5) geometrically distinct if they do not coincide up to rotation. We leave this question for future work.

Remark 4.20 (the case $\alpha > 1$). We finally discuss the natural question of what happens for $\alpha > 1$. In fact, in this case, the infimum $\mathscr{C}_{\alpha,m,p}(\boldsymbol{B})$ in (1-7) satisfies

$$\mathscr{C}_{\alpha,m,p}(\boldsymbol{B}) = -\infty \quad \text{for every } m \in \mathbb{R},\ p \in [2, \infty). \tag{4-25}$$

To see this, we fix $\varepsilon \in (0, 1)$ and nonzero functions $\varphi \in C_c^1(1-\varepsilon, 1)$, $\psi \in C_c^1\left(\frac{\pi}{2} - \varepsilon, \frac{\pi}{2} + \varepsilon\right)$. Moreover, we consider the sequence of functions $u_k \in C_c^1(\boldsymbol{B})$ which, in the polar coordinates from (3-3), are given by

$$(r, \theta, \vartheta_1, \dots, \vartheta_{N-2}) \mapsto \varphi(r)\psi(\vartheta_1) \cdots \psi(\vartheta_{N-2}) X_k(\theta), \quad \text{where } X_k(\theta) = \sin(k\theta).$$

Similar to (3-4), we then find, with $U_\varepsilon := (1-\varepsilon, 1) \times (-\pi, \pi) \times \left(\frac{\pi}{2}-\varepsilon, \frac{\pi}{2}+\varepsilon\right)^{N-2}$, that

$$
\begin{aligned}
&\int_{\boldsymbol{B}} (|\nabla u_k|^2 - \alpha^2 |\partial_\theta u_k|^2)\,dx \\
&\quad = \int_{U_\varepsilon} \Bigg(|\varphi'(r)|^2 |X_k(\theta)|^2 \prod_{i=1}^{N-2} |\psi(\vartheta_i)|^2 \\
&\qquad\qquad + \frac{1}{r^2} \sum_{i=1}^{N-2} h_i |\psi'(\vartheta_i)|^2 |\varphi(r)|^2 |X_k(\theta)|^2 \prod_{j=1,\, j\neq i}^{N-2} |\psi(\vartheta_j)|^2 \\
&\qquad\qquad\qquad + \left(\frac{h_{N-1}}{r^2} - \alpha^2\right) |X_k'(\theta)|^2 |\varphi(r)|^2 \prod_{i=1}^{N-2} |\psi(\vartheta_i)|^2 \Bigg) h \, d(r, \theta, \vartheta_1, \dots, \vartheta_{N-2}),
\end{aligned}
$$

with the functions $h, h_i : U \to \mathbb{R},\ i = 1, \dots, N-1$, given in (3-5). Since $\alpha > 1$, we may now choose $\varepsilon = \varepsilon(\alpha) > 0$ small enough that

$$
\tfrac{1}{2} \le h \le 1 \quad \text{and} \quad \alpha^2 - \frac{h_{N-1}}{r^2} \ge \varepsilon \qquad \text{on } U_\varepsilon.
$$

Since also $|X_k| \le 1$ by definition, we estimate

$$
\int_{\boldsymbol{B}} (|\nabla u_k|^2 - \alpha^2 |\partial_\theta u_k|^2)\,dx \le c - d(k),
$$

where

$$
c := \int_{U_\varepsilon} \left(|\varphi'(r)|^2 \prod_{i=1}^{N-2} |\psi(\vartheta_i)|^2 + \frac{1}{r^2} \sum_{i=1}^{N-2} h_i |\psi'(\vartheta_i)|^2 |\varphi(r)|^2 \prod_{j=1,\, j\neq i}^{N-2} |\psi(\vartheta_j)|^2 \right) d(r, \theta, \vartheta_1, \dots, \vartheta_{N-2})
$$

and

$$
\begin{aligned}
d(k) &:= \int_{U_\varepsilon} \left(\alpha^2 - \frac{h_{N-1}}{r^2}\right) |X_k'(\theta)|^2 |\varphi(r)|^2 \prod_{i=1}^{N-2} |\psi(\vartheta_i)|^2 g \, d(r, \theta, \vartheta_1, \dots, \vartheta_{N-2}) \\
&\ge \frac{k^2 \varepsilon}{2} \int_{1-\varepsilon}^{1} |\varphi(r)|^2 \, dr \int_{-\pi}^{\pi} \cos^2(k\theta)\, d\theta \left(\int_{\pi/2-\varepsilon}^{\pi/2+\varepsilon} |\psi(\vartheta)|^2 \, d\vartheta \right)^{N-2} = \frac{\varepsilon \pi}{2} d_2 k^2,
\end{aligned}
$$

with

$$
d_2 := \int_{1-\varepsilon}^{1} |\varphi(r)|^2 \, dr \left(\int_{\pi/2-\varepsilon}^{\pi/2+\varepsilon} |\psi(\vartheta)|^2 \, d\vartheta \right)^{N-2}.
$$

Hence $d(k) \to \infty$ as $k \to \infty$. Moreover, for every $p \in [2, \infty)$, we have

$$
\int_{\boldsymbol{B}} |u_k|^p \, dx = \int_{U_\varepsilon} |\varphi(r)|^p |X_k(\theta)|^p \prod_{i=1}^{N-2} |\psi(\vartheta_i)|^p g \, d(r, \theta, \vartheta_1, \dots, \vartheta_{N-2}) \le d_p,
$$

with

$$
d_p := 2\pi \int_{1-\varepsilon}^{1} |\varphi(r)|^p \, dr \left(\int_{\pi/2-\varepsilon}^{\pi/2+\varepsilon} |\psi(\vartheta)|^p \, d\vartheta \right)^{N-2} < \infty.
$$

It thus follows that

$$\frac{\int_{\boldsymbol{B}}(|\nabla u_k|^2-\alpha^2|\partial_\theta u_k|^2+m|u_k|^2)\,dx}{\left(\int_{\boldsymbol{B}}|u_k|^p\,dx\right)^{2/p}}\le\frac{c-d(k)-md_2}{(d_p)^{2/p}}\to-\infty\quad\text{as }k\to\infty$$

for every $p\in[2,\infty)$, $m\in\mathbb{R}$. This shows (4-25).

Consequently, the study of ground state solutions of (1-5) requires a completely different approach in the case $\alpha>1$. This is further treated in [Kübler 2023].

5. The case of an annulus

In this section, we consider rotating solutions of (1-3) in the case where $\boldsymbol{B}$ is replaced by an annulus

$$\boldsymbol{A}_r:=\{x\in\mathbb{R}^N : r<|x|<1\}$$

for some $r\in(0,1)$. The ansatz (1-4) then leads to the reduced problem

$$\begin{cases}-\Delta u+\alpha^2\,\partial_\theta^2 u+mu=|u|^{p-2}u & \text{in }\boldsymbol{A}_r,\\ u=0 & \text{on }\partial\boldsymbol{A}_r,\end{cases}\tag{5-1}$$

where $m>-\lambda_1(\boldsymbol{A}_r)$, $p\in(2,2^*)$ and $\partial_\theta=x_1\,\partial_{x_2}-x_2\,\partial_{x_1}$ as before. Here, $\lambda_1(\boldsymbol{A}_r)$ denotes the first Dirichlet eigenvalue of $-\Delta$ on $\boldsymbol{A}_r$. As in (1-7), we may then define

$$\mathscr{C}_{\alpha,m,p}(\boldsymbol{A}_r):=\inf_{u\in H_0^1(\boldsymbol{A}_r)\setminus\{0\}}R_{\alpha,m,p}(u),\tag{5-2}$$

with the Rayleigh quotient $R_{\alpha,m,p}(u)$ given by (1-8) for functions $u\in H_0^1(\boldsymbol{A}_r)$. In the following, a weak solution of (5-1) will be called a ground state solution if it is a minimizer for (5-2). We then have the following analogue of Theorem 1.1.

Theorem 5.1. *Let $r\in(0,1)$, $m>-\lambda_1(\boldsymbol{A}_r)$ and $p\in(2,2^*)$.*

(i) *If $\alpha\in(0,1)$, then there exists a ground state solution of* (5-1).

(ii) *We have*

$$\mathscr{C}_{1,m,p}(\boldsymbol{A}_r)=0\quad\textit{for }p>2_1^*\qquad\textit{and}\qquad\mathscr{C}_{1,m,p}(\boldsymbol{A}_r)>0\quad\textit{for }p\le 2_1^*.$$

Moreover, for any $p\in(2_1^*,2^*)$, there exists $\alpha_p\in(0,1)$ with the property that

$$\mathscr{C}_{\alpha,m,p}(\boldsymbol{A}_r)<\mathscr{C}_{0,m,p}(\boldsymbol{A}_r)\quad\textit{for }\alpha\in(\alpha_p,1],$$

and therefore every ground state solution of (5-1) *is (x_1,x_2)-nonradial for $\alpha\in(\alpha_p,1]$.*

This theorem does not come as a surprise and is proved by precisely the same arguments as Theorem 1.1, so we omit the proof. Instead, we now discuss an interesting additional feature of the annulus $\boldsymbol{A}_r$. Unlike in the case of the ball, we can formulate *explicit* sufficient conditions for the parameters p, α, m and r which guarantee that every ground state solution of (5-1) is (x_1,x_2)-nonradial. This is the content of the following theorem.

Theorem 5.2. *Let* $N \ge 2,\ m \ge 0,\ r, \alpha \in (0, 1)$, *and assume*

$$2 + \frac{N - 1 - r^2\alpha^2}{\kappa(r, m)} < p < 2^*,$$

with

$$\kappa(r, m) = \begin{cases} mr^2 + \max\left\{\left(\frac{N-2}{2}\right)^2, \left(\frac{\pi}{1-r}\right)^2 r^{N-1}\right\}, & N \ge 3, \\ mr^2 + \left(\frac{\pi}{1-r}\right)^2 r^N, & N = 2. \end{cases} \tag{5-3}$$

Then every ground state solution of (5-1) *is* (x_1, x_2)*-nonradial.*

We point out that $\kappa(m, r) \to \infty$ if $m \to \infty$ or $r \to 1^-$. Consequently, for given $p > 2$, ground states of (5-1) are nonradial if either m is large or the annulus is thin, i.e., r is close to 1. The proof is based on the following lemma.

Lemma 5.3. *Suppose that* $m \ge 0,\ \alpha \in (0, 1),\ p \in (2, 2^*)$ *and that there exists a function* $v \in H_0^1(\boldsymbol{A}_r)$ *satisfying*

$$\int_{\mathbb{S}^{N-1}} v(s(\,\cdot\,))\, d\sigma = 0 \quad \textit{for every } s \in (r, 1) \tag{5-4}$$

and

$$\int_{\boldsymbol{A}_r} (|\nabla v|^2 - \alpha^2 |\partial_\theta v|^2 + m v^2)\, dx - (p-1) \int_{\boldsymbol{A}_r} |u_0|^{p-2} v^2\, dx < 0. \tag{5-5}$$

Then we have

$$\mathscr{C}_{\alpha,m,p}(\boldsymbol{A}_r) < R_{\alpha,m,p}(u_0), \tag{5-6}$$

where $u_0 \in H_0^1(\boldsymbol{A}_r)$ *is the unique positive radial solution of* (5-1).

Here we note that, in the case $m = 0$, the uniqueness of the positive radial solution u_0 of (5-1) has been first proved by Ni and Nussbaum [1985]. In the case $m > 0$, the uniqueness is due to Tang [2003] and Felmer, Martínez and Tanaka [Felmer et al. 2008] for $N \ge 3$ and $N = 2$, respectively.

Proof. We argue by contradiction and assume that equality holds in (5-6). Then u_0 is a minimizer for the C^2-functional $R_{\alpha,m,p} : H_0^1(\boldsymbol{A}_r) \setminus \{0\}) \to \mathbb{R}$, which implies, in particular, that

$$R'_{\alpha,m,p}(u_0)\tilde{v} = 0 \quad \text{and} \quad R''_{\alpha,m,p}(u_0)(\tilde{v}, \tilde{v}) \ge 0 \qquad \text{for all } \tilde{v} \in H_0^1(\boldsymbol{A}_r). \tag{5-7}$$

In the following, we write $R_{\alpha,m,p} = Z(u)/N(u)$ for $u \in H^1(\boldsymbol{A}_r) \setminus \{0\}$, with

$$Z(u) := \int_{\boldsymbol{A}_r} (|\nabla u|^2 - \alpha^2 |\partial_\theta u|^2 + m u^2)\, dx \quad \text{and} \quad N(u) := \left(\int_{\boldsymbol{A}_r} |u|^p\, dx\right)^{2/p}.$$

The first property in (5-7), applied with $\tilde{v} = v$, then gives $N(u_0)Z'(u_0)v = Z(u_0)N'(u_0)v$ and consequently

$$N(u_0)^3 [R_{\alpha,m,p}]''(u_0)(v, v) = N(u_0)^2 Z''(u_0)(v, v) - Z(u_0)N(u_0)N''(u_0)(v, v)$$

for $v \in H_0^1(\boldsymbol{A}_r)$. Therefore, applying the second property in (5-7) with $\tilde{v} = v$ yields

$$Z''(u_0)(v, v) - \frac{Z(u_0)}{N(u_0)} N''(u_0)(v, v) \ge 0.$$

Moreover, noting that u_0 is a weak solution of (5-1) and therefore $Z(u_0) = N(u_0)^{p/2}$, we conclude that

$$\begin{aligned} 0 &\le \frac{1}{2}\left(Z''(u_0)(v,v) - \frac{Z(u_0)}{N(u_0)} N''(u_0)(v,v)\right) \\ &= \int_{A_r} (|\nabla v|^2 - \alpha^2 |\partial_\theta v|^2 + m v^2)\, dx \\ &\qquad - (p-1)\int_{A_r} |u_0|^{p-2} v^2\, dx + (p-2) N(u_0)^{-p/2}\left(\int_{A_r} |u_0|^{p-2} u_0 v\, dx\right)^2. \end{aligned}$$

This, however, contradicts (5-5), since $\int_{A_r} |u_0|^{p-2} u_0 v\, dx = 0$ by (5-4). The proof is thus finished. □

Proof of Theorem 5.2. Our goal is to construct a function that satisfies the conditions of Lemma 5.3. To this end, let μ_1 be the first eigenvalue of the weighted eigenvalue problem

$$\begin{cases} -w_{\rho\rho} - \frac{N-1}{\rho} w_\rho + m w - (p-1)|u_0(\rho)|^{p-2} w = \frac{\mu}{\rho^2} w & \text{in } (r,1), \\ w(r) = w(1) = 0, \end{cases}$$

and let w be the unique positive eigenfunction up to normalization. Moreover, let $Y \in C^\infty(\mathbb{S}^{N-1})$ be given by $Y(x) = x_2$. Then Y is a spherical harmonic of degree 1 on $\mathbb{S}^{N-1}$, which in the polar coordinates from (3-3) is written as $Y(r,\theta,\vartheta_1,\dots,\vartheta_{N-2}) = r\sin\theta\sin\vartheta_1\cdots\sin\vartheta_{N-2}$, and therefore satisfies $\partial_\theta^2 Y = -Y$ on $\mathbb{S}^{N-1}$. Moreover, set $v(\rho,\omega) := w(\rho)Y(\omega)$. Then condition (5-4) of Lemma 5.3 is satisfied. By construction, v also satisfies

$$-\Delta v + \alpha^2\,\partial_\theta^2 v + m v - (p-1)|u_0|^{p-2} v = \frac{\mu_1 + N - 1}{|x|^2} v - \alpha^2 v,$$

and testing this equation with v itself yields

$$\begin{aligned} \int_{A_r} (|\nabla v|^2 - \alpha^2 |\partial_\theta v|^2 + m v^2 - (p-1)|u_0|^{p-2} v^2)\, dx &= (\mu_1 + (N-1)) \int_{A_r} \frac{v^2}{|x|^2}\, dx - \alpha^2 \int_{A_r} v^2\, dx \\ &\le (\mu_1 + (N-1) - r^2\alpha^2) \int_{A_r} \frac{v^2}{|x|^2}\, dx. \end{aligned} \tag{5-8}$$

We recall that μ_1 can be characterized by

$$\mu_1 = \min_{\varphi \in H^1_{0,\mathrm{rad}}(A_r)\setminus\{0\}} \frac{\int_{A_r} (|\nabla\varphi|^2 + m\varphi^2)\, dx - (p-1)\int_{A_r} |u_0|^{p-2}\varphi^2\, dx}{\int_{A_r} \varphi^2/|x|^2\, dx}.$$

Taking $\varphi = u_0$ in this quotient, we obtain the estimate

$$\begin{aligned} \mu_1 &\le \frac{\int_{A_r} (|\nabla u_0|^2 + m u_0^2)\, dx - (p-1)\int_{A_r} |u_0|^p\, dx}{\int_{A_r} u_0^2/|x|^2\, dx} \\ &= -(p-2)\frac{\int_{A_r} (|\nabla u_0|^2 + m u_0^2)\, dx}{\int_{A_r} u_0^2/|x|^2\, dx} \\ &\le -(p-2)\left(\frac{\int_{A_r} |\nabla u_0|^2\, dx}{\int_{A_r} u_0^2/|x|^2\, dx} + m r^2\right) \end{aligned} \tag{5-9}$$

We now distinguish the cases $N \geq 3$ and $N = 2$. If $N \geq 3$, Hardy's inequality gives

$$\int_{\boldsymbol{A}_r} |\nabla u_0|^2 \, dx \geq \left(\frac{N-2}{2}\right)^2 \int_{\boldsymbol{A}_r} \frac{u_0^2}{|x|^2} \, dx. \tag{5-10}$$

Alternatively, we may also estimate, since u_0 is radial,

$$\begin{aligned}\int_{\boldsymbol{A}_r} |\nabla u_0|^2 \, dx &= |\mathbb{S}^{N-1}| \int_r^1 \rho^{N-1} |\partial_r u_0(\rho)|^2 \, d\rho \geq |\mathbb{S}^{N-1}| r^{N-1} \int_r^1 |\partial_r u_0(\rho)|^2 \, d\rho \\ &\geq |\mathbb{S}^{N-1}| \left(\frac{\pi}{1-r}\right)^2 r^{N-1} \int_r^1 u_0^2(\rho) \, d\rho \geq |\mathbb{S}^{N-1}| \left(\frac{\pi}{1-r}\right)^2 r^{N-1} \int_r^1 \rho^{N-3} u_0^2(\rho) \, d\rho \\ &= \left(\frac{\pi}{1-r}\right)^2 r^{N-1} \int_{\boldsymbol{A}_r} \frac{u_0^2}{|x|^2} \, dx. \end{aligned} \tag{5-11}$$

Thus (5-9) gives $\mu_1 < -(p-2)\kappa(r, m)$, with $\kappa(r, m)$ given in (5-3) for $N \geq 3$. Inserting this into (5-8) yields

$$\int_{\boldsymbol{A}_r} (|\nabla v|^2 - \alpha^2 |\partial_\theta v|^2 + m v^2 - (p-1)|u_0|^{p-2} v^2) \, dx < -(p-2)\kappa + N - 1 - r^2 \alpha^2,$$

i.e., condition (5-5) of Lemma 5.3 is satisfied if $p > (N - 1 - r^2\alpha^2)/\kappa + 2$, which holds by assumption.

Hence v satisfies the assumptions of Lemma 5.3, which implies that (5-6) holds and therefore every minimizer for (5-2) is nonradial. Let u denote such a nonradial ground state solution, and suppose by contradiction that $\partial_\theta u_0 \equiv 0$. The nonradiality of u implies that there exists an isometry $A \in O(N)$ such that $\tilde{u} := u \circ A \in H_0^1(\boldsymbol{A}_r)$ satisfies $\partial_\theta \tilde{u} \not\equiv 0$. Since A is an isometry, this implies

$$R_{\alpha,m,p}(\tilde{u}) = R_{\alpha,m,p}(u) - \alpha^2 \frac{\int_{\boldsymbol{A}_r} |\partial_\theta \tilde{u}|^2 \, dx}{\left(\int_{\boldsymbol{A}_r} |u|^p \, dx\right)^{2/p}} < R_{\alpha,m,p}(u) = \mathscr{C}_{1,m,p}(\boldsymbol{A}_r),$$

which contradicts (5-2). Consequently, we have $\partial_\theta u_0 \not\equiv 0$, which yields that u_0 is (x_1, x_2)-nonradial. This finishes the proof in the case $N \geq 3$.

It remains to consider the case $N = 2$. In this case, we replace the estimates (5-10) and (5-11) by

$$\int_{\boldsymbol{A}_r} |\nabla u_0|^2 \, dx \geq |\mathbb{S}^1| \left(\frac{\pi}{1-r}\right)^2 r^{N-1} \int_r^1 u_0^2(\rho) \, d\rho \geq \left(\frac{\pi}{1-r}\right)^2 r^N \int_{\boldsymbol{A}_r} \frac{u_0^2}{|x|^2} \, dx.$$

Combining this with (5-9) we again get $\mu_1 < -(p-2)\kappa(r, m)$, with $\kappa(r, m)$ given in (5-3) for $N = 2$. We may thus complete the proof as above. □

6. Riemannian models

So far we only used the inequality stated in Theorem 2.2 in the case $s = 1$. We shall now consider an application for general $s \in (0, 2]$ by considering (1-3) on some Riemannian manifolds with boundary. More precisely, we consider a class of Riemannian models given by $(\boldsymbol{B}, g)$, where, as before, $\boldsymbol{B}$ denotes the open ball of radius 1 centered at zero, and the metric g on $\boldsymbol{B}$ is written, in polar coordinates, as

$$ds^2 = dr^2 + (\psi(r))^2 \, d\Theta^2 \tag{6-1}$$

for $r > 0$, $\Theta \in \mathbb{S}^{N-1}$. Here $d\Theta^2$ denotes the canonical metric on $\mathbb{S}^{N-1}$ and ψ is a smooth function that is positive on $(0, \infty)$. Moreover, we assume

$$\psi'(0) > 0 \qquad \text{and} \qquad \psi^{(2k)}(0) = 0 \quad \text{for } k \in \mathbb{N}_0. \tag{6-2}$$

We note that the second condition in (6-2) ensures smoothness of g at the origin. For such a Riemannian model, the associated Laplace–Beltrami operator becomes

$$\Delta_g f = \frac{1}{\psi^{N-1}} \, \partial_r (\psi^{N-1} \, \partial_r f) + \frac{1}{\psi^2} \Delta_{\mathbb{S}^{N-1}} f,$$

where $\Delta_{\mathbb{S}^{N-1}}$ denotes the Laplace–Beltrami operator on $\mathbb{S}^{N-1}$. Riemannian models are of independent geometric interest; we refer to [Berchio et al. 2014] for a more detailed discussion.

We again study the problem

$$\begin{cases} \partial_t^2 v - \Delta_g v + mv = |v|^{p-2} v & \text{in } M, \\ v = 0 & \text{on } \partial M, \end{cases} \tag{6-3}$$

where $2 < p < 2N/(N-2)$ and $m > -\lambda_1(M)$, with $\lambda_1(M)$ denoting the first Dirichlet eigenvalue of $-\Delta_g$ on M. We stress that the case $\psi(r) = r$ corresponds to the classical flat metric on $\boldsymbol{B}$ considered in detail in the previous sections. A further example is the hemisphere $\mathbb{S}^N_{\tau,+} := \{x \in \mathbb{R}^{N+1} : |x| = \tau,\ x_{N+1} > 0\}$ of radius $\tau > 0$. Indeed, using polar coordinates $(r, \omega) \in (0, 1) \times \mathbb{S}^{N-1}$, a parametrization $\boldsymbol{B} \to \mathbb{S}^N_{\tau,+}$ is given by $(r, \omega) \mapsto \tau\big(\sin\big(\tfrac{\pi}{2} r\big)\omega, \cos\big(\tfrac{\pi}{2} r\big)\big)$. This yields (6-1) with $\psi(r) = \tau \sin\big(\tfrac{\pi}{2} r\big)$. Similarly, spherical caps can be considered.

As in the flat case, we restrict our attention to solutions of (6-3) of the form $v(t, x) = u(R_{\alpha t}(x))$, where R_θ is the rotation in the (x_1, x_2)-plane with angle θ. This leads to the reduced equation

$$\begin{cases} -\Delta_g u + \alpha^2 \, \partial_\theta^2 u + mu = |u|^{p-2} u & \text{in } M, \\ u = 0 & \text{on } \partial M, \end{cases} \tag{6-4}$$

with the differential operator $\partial_\theta = x_1 \, \partial_{x_2} - x_2 \, \partial_{x_1}$ associated to the Killing vector field $x \mapsto (-x_2, x_1, 0, \dots, 0)$ on M. We may then again study the quotient

$$R^M_{\alpha,m,p} : H^1_0(M) \setminus \{0\} \to \mathbb{R}, \qquad R^M_{\alpha,m,p}(u) := \frac{\int_M (|\nabla_g u|^2 - \alpha^2 |\partial_\theta u|^2 + mu^2) \, dg}{\|u\|^2_{L^p(M)}},$$

and its minimizers, i.e.,

$$\mathscr{C}_{\alpha,m,p}(M) := \inf_{u \in C^1_c(\boldsymbol{B}) \setminus \{0\}} R^M_{\alpha,m,p}(u).$$

Analogously to Theorem 1.1, we can use the general inequality stated in Theorem 2.2 to give the following result, recalling that we set $2^*_s = (4N + 2s)/(2N - 4 + s)$.

Theorem 6.1. *Let $s \in (0, 2]$, and let (M, g) be a Riemannian model, with $M = \boldsymbol{B}$ and associated function $\psi \in C^\infty[0, 1)$ satisfying* (6-2) *and*

$$c_1(1-r)^s \le 1 - \psi(r) \le c_2(1-r)^s \quad \text{for } r \in (0,1) \text{ with constants } c_1, c_2 > 0. \tag{6-5}$$

Moreover, let $m > -\lambda_1(M)$, and let $2 < p < 2^*$.

(i) *If $\alpha \in (0, 1)$, then there exists a ground state solution of* (6-4).

(ii) *We have*

$$\mathscr{C}_{1,m,p}(M) = 0 \quad \text{for } p > 2_s^* \qquad \text{and} \qquad \mathscr{C}_{1,m,p}(M) > 0 \quad \text{for } p \le 2_s^*. \tag{6-6}$$

Moreover, for any $p \in (2_s^*, 2^*)$, there exists $\alpha_p \in (0, 1)$ with the property that

$$\mathscr{C}_{\alpha,m,p}(M) < \mathscr{C}_{0,m,p}(M) \quad \text{for } \alpha \in (\alpha_p, 1],$$

and therefore every ground state solution of (6-4) *is (x_1, x_2)-nonradial for $\alpha \in (\alpha_p, 1)$.*

Proof. Since the proof is completely parallel to the proof of Theorem 1.1, we omit some details and focus our attention on showing where condition (6-5) enters. It is again useful to introduce polar coordinates $(r, \theta, \vartheta_1, \dots, \vartheta_{N-2}) \in U := (0, 1) \times (-\pi, \pi) \times (0, \pi)^{N-2}$ given by

$$\begin{aligned}(x_1, \dots, x_N) = (&r \sin\vartheta_1 \cdots \sin\vartheta_{N-2} \cos\theta,\ r \sin\vartheta_1 \cdots \sin\vartheta_{N-2} \sin\theta,\ r\cos\vartheta_1, \\ &r \sin\vartheta_1 \cos\vartheta_2, \dots, r\sin\vartheta_1 \cdots \sin\vartheta_{N-3}\cos\vartheta_{N-2},\ r \sin\vartheta_1 \cdots \cos\vartheta_{N-2}).\end{aligned} \tag{6-7}$$

In the following, we will abbreviate the coordinates $(\theta, \vartheta_1, \dots, \vartheta_{N-2})$ to Θ for simplicity. Using (B-1) from Appendix B, we see that the metric (6-1) is written in these coordinates as

$$dg = dr^2 + (\psi(r))^2 \Biggl(\sum_{i=1}^{N-2} \Biggl(\prod_{k=1}^{i-1} \sin^2\vartheta_k \Biggr) d\vartheta_i^2 + \Biggl(\prod_{k=1}^{N-1} \sin^2\vartheta_k \Biggr) d\theta^2 \Biggr).$$

Therefore, by (B-3), the quadratic form associated to the operator $-\Delta_g + \partial_\theta^2$ is given by

$$\int_M (|\nabla_g u|^2 - |\partial_\theta u|^2)\, dg = \int_U \Biggl(|\partial_r u|^2 + \frac{1}{\psi^2} \sum_{i=1}^{N-2} h_i |\partial_{\vartheta_i} u|^2 + \Bigl(\frac{h_{N-1}}{\psi^2} - 1 \Bigr) |\partial_\theta u|^2 \Biggr) |g|\, d(r, \Theta)$$

for $u \in C_c^1(M)$, with

$$|g|(r, \Theta) = (\psi(r))^{N-1} \prod_{k=1}^{N-2} \sin^{N-1-k}\vartheta_k, \qquad h_i(r, \Theta) = \prod_{k=1}^{i-1} \frac{1}{\sin^2\vartheta_k}.$$

Moreover,

$$\int_M |u|^p\, dg = \int_U |u|^p |g|\, d(r, \Theta) \quad \text{for } u \in C_c^1(M) \text{ and } p > 1.$$

Next we note that, as a consequence of (6-5), we have

$$|g|(\Theta_0) = 1 \quad \text{and} \quad h_i(\Theta_0) = 1 \ \text{ for } i = 1, \dots, N-1, \quad \text{with } \Theta_0 := \bigl(1, 0, \tfrac{\pi}{2}, \dots, \tfrac{\pi}{2}\bigr). \tag{6-8}$$

Setting

$$U_0 := \left(\tfrac{1}{2}, 1\right) \times (-\pi, \pi) \times \left(\tfrac{\pi}{4}, \tfrac{3}{4}\pi\right)^{N-2} \subset U,$$

we now claim that assumption (6-5) implies that the function $h_{N-1}/\psi^2 - 1$ satisfies

$$\tilde{c}_1\left((1-r)^s + \sum_{k=1}^{N-2}\left(\vartheta_k - \tfrac{\pi}{2}\right)^2\right) \le \frac{h_{N-1}}{\psi^2}(r,\Theta) - 1 \le \tilde{c}_2\left((1-r)^s + \sum_{k=1}^{N-2}\left(\vartheta_k - \tfrac{\pi}{2}\right)^2\right) \tag{6-9}$$

for $(r, \theta, \vartheta_1, \dots, \vartheta_{N-2}) \in U_0$, with suitable constants $\tilde{c}_1, \tilde{c}_2 > 0$. Indeed, note that

$$\frac{h_{N-1}}{\psi^2} - 1 = \frac{1}{\psi^2}(h_{N-1} - \psi^2) = \frac{1}{\psi^2}(\sqrt{h_{N-1}} + \psi)(\sqrt{h_{N-1}} - \psi)$$

and that, in U_0, the factor $(\sqrt{h_{N-1}} + \psi)/\psi^2$ can clearly be bounded from above and below by positive constants. Moreover, since the first-order derivatives of $\sqrt{h_{N-1}} = \prod_{k=1}^{N-2} 1/\sin\vartheta_k$ vanish in Θ_0, a Taylor expansion yields

$$1 + C_1 \sum_{k=1}^{N-2}\left(\vartheta_k - \tfrac{\pi}{2}\right)^2 \le \sqrt{h_{N-1}} \le 1 + C_2 \sum_{k=1}^{N-2}\left(\vartheta_k - \tfrac{\pi}{2}\right)^2 \quad \text{in } U_0,$$

with constants $C_1, C_2 > 0$. We thus find that

$$1 - \psi(r) + C_1 \sum_{k=1}^{N-2}\left(\vartheta_k - \tfrac{\pi}{2}\right)^2 \ge \sqrt{h_{N-1}} - \psi \le 1 - \psi(r) + C_2 \sum_{k=1}^{N-2}\left(\vartheta_k - \tfrac{\pi}{2}\right)^2$$

holds in U_0, and (6-9) can finally be deduced from (6-5).

We now consider a fixed function $u \in C_c^1(U_0) \setminus \{0\} \subset C_c^1(U) \setminus \{0\}$, which, regarded as a function of polar coordinates, gives rise to a function in $C_c^1(M)$. For $\lambda \in (0, 1)$, we consider the map

$$\Lambda_\lambda : U_0 \to U_0, \quad (r, \Theta) \mapsto \left(1 + \lambda(r-1), \lambda^{1+s/2}\theta, \tfrac{\pi}{2} + \lambda\left(\vartheta_1 - \tfrac{\pi}{2}\right), \dots, \tfrac{\pi}{2} + \lambda\left(\vartheta_{N-2} - \tfrac{\pi}{2}\right)\right),$$

and we define $u_\lambda := u \circ \Lambda_\lambda^{-1} \in C_c^1(U_0) \setminus \{0\}$ for $\lambda \in (0, 1)$. Note that Λ_λ shrinks U_0 to the point Θ_0, which we may show similarly to the arguments in the proof of Proposition 3.1.

Using (6-8) and (6-9), we find that

$$\lambda^{-(2N+s)/p}\left(\int_U |u_\lambda|^p |g|\, d(r,\Theta)\right)^{2/p} = \left(\int_U |u|^p (|g| \circ \Lambda_\lambda)\, d(r,\Theta)\right)^{2/p} \to \left(\int_U |u|^p\, d(r,\Theta)\right)^{2/p} =: c_u(p)$$

as $\lambda \to 0^+$ and

$$\begin{aligned}
&\limsup_{\lambda \to 0^+} \lambda^{2-s/2-N} \int_U \left(|\partial_r u_\lambda|^2 + \frac{1}{\psi^2}\sum_{i=1}^{N-2} h_i |\partial_{\vartheta_i} u_\lambda|^2 + \left(\frac{h_{N-1}}{\psi^2} - 1\right)|\partial_\theta u_\lambda|^2\right)|g|\, d(r,\Theta) \\
&= \limsup_{\lambda \to 0^+} \int_U \left(|\partial_r u|^2 + \frac{1}{\psi^2} \circ \Lambda_\lambda \sum_{i=1}^{N-2} (h_i \circ \Lambda_\lambda)|\partial_{\vartheta_i} u|^2 + \lambda^{-s}\left(\left(\frac{h_{N-1}}{\psi^2}\right) \circ \Lambda_\lambda - 1\right)|\partial_\theta u|^2\right)(|g| \circ \Lambda_\lambda)\, d(r,\Theta) \\
&\le d_u^1 + d_u^2,
\end{aligned} \tag{6-10}$$

with

$$d_u^1 := \int_U \Big(|\partial_r u|^2 + \sum_{i=1}^{N-2} |\partial_{\vartheta_i} u|^2 \Big) \, d(r,\Theta)$$

and

$$\begin{aligned} d_u^2 &= \tilde{c}_2 \limsup_{\lambda \to 0^+} \int_U \Big((1-r)^s + \lambda^{2-s} \sum_{k=1}^{N-2} \big(\vartheta_k - \tfrac{\pi}{2}\big)^2 \Big) |\partial_\theta u|^2 \, d(r,\Theta) \\ &= \begin{cases} \tilde{c}_2 \int_U (1-r)^s |\partial_\theta u|^2 \, d(r,\Theta), & s \in (0,2), \\ \tilde{c}_2 \int_U \big((1-r)^2 + \sum_{i=1}^{N-2} \big(\vartheta_k - \tfrac{\pi}{2}\big)^2 \big) |\partial_\theta u|^2 \, d(r,\Theta), & s = 2. \end{cases} \end{aligned}$$

It thus follows that

$$\begin{aligned} \mathscr{C}_{1,m,p}(M) &\le \limsup_{\lambda \to 0^+} R^M_{1,m,p}(u_\lambda) \\ &= \limsup_{\lambda \to 0^+} \frac{\lambda^{N+s/2-2}(d_u^1 + d_u^2) + \lambda^{(2N+s)/2} c_u(2)}{\lambda^{(2N+s)/p} c_u(p)} = 0 \quad \text{if } p > 2_s^*. \end{aligned}$$

This shows the first identity in (6-6). To see the second identity in (6-6), we argue as in Section 3. More precisely, we first note that it is sufficient to consider the case $p = 2_s^*$, and then we show the inequality

$$\Big(\int_U |g|\,|u|^{2_s^*} \, d(r,\Theta) \Big)^{2/2_s^*} \le C \int_U \Big(|\partial_r u|^2 + \frac{1}{\psi^2} \sum_{i=1}^{N-2} h_i |\partial_{\vartheta_i} u|^2 + \Big(\frac{h_{N-1}}{\psi^2} - 1 \Big) |\partial_\theta u|^2 \Big) |g| \, d(r,\Theta)$$

for functions $u \in C_c^1(U_0)$, with a suitable constant $C > 0$. For this, we use Theorem 1.6 and the first inequality in (6-9). The argument is then completed by using the rotation invariance of the problem and a partition of unity argument to localize the problem. ☐

Remark 6.2. (i) As noted before, the case of a hemisphere $\mathbb{S}^N_{1,+}$ of radius 1 corresponds to $\psi(r) = \sin\big(\frac{\pi}{2} r\big)$. In this case Theorem 6.1 applies with $s = 2$, and it yields nonradial ground state solutions for $p > 2_2^* = 2(N+1)/(N-1)$. Notably, this corresponds to the critical exponent for generalized traveling waves on the sphere $\mathbb{S}^N$ found in [Mukherjee 2017; 2018; Taylor 2016]. In fact, our approach based on Theorem 1.6 can be used to give an alternative proof for the existence of nontrivial solutions and the embeddings stated in [Taylor 2016, Proposition 3.2] and [Mukherjee 2017, Proposition 1.2 and Lemma 1.3].

(ii) Theorem 6.1 leaves open the case $s > 2$. Note that the two-sided estimate (6-9) needs to be analyzed more carefully if $s > 2$ and $N \ge 3$, as the leading-order term is then 2 in place of s. In this case, if (6-5) holds for some $s > 2$, Theorem 6.1 (ii) holds with 2_s^* replaced by 2_2^*, i.e.,

$$\mathscr{C}_{1,m,p}(M) = 0 \quad \text{for } p > 2_2^* \qquad \text{and} \qquad \mathscr{C}_{1,m,p}(M) > 0 \quad \text{for } p \le 2_2^*.$$

For $N = 2$, on the other hand, no angular terms appear in (6-9). Consequently, Theorem 6.1 holds for arbitrary $s > 0$ in this case.

Appendix A: Boundedness of solutions

In the proof of the regularity properties of weak solutions of (1-5) in the case $\alpha = 1$ stated in Lemma 4.10, we used the following.

Lemma A.1. *Let* $2 < p < 2_1^*$, $m > -\lambda_1$, *and let* $u \in \mathcal{H}_1$ *be a weak solution of*

$$-\Delta u + \partial_\theta^2 u + mu = |u|^{p-2}u \quad \text{in } \boldsymbol{B}. \tag{A-1}$$

Then $u \in L^\infty(\boldsymbol{B})$. *Furthermore, there exist constants* $C = C(N, m)$, $\sigma > 0$ *such that*

$$|u|_\infty \le C\|u\|_{\mathcal{H}}^{\sigma}. \tag{A-2}$$

For $m \ge 0$, *the constant* $C = C(N) > 0$ *can be chosen independent of* m.

Proof. The proof is based on a Moser iteration scheme and essentially identical to the classical arguments with the Sobolev critical exponent replaced by 2_1^*; see [Struwe 2008, Appendix B].

We fix $L, s \ge 2$ and consider auxiliary functions $h, g \in C^1([0, \infty))$ defined by

$$h(t) := s\int_0^t \min\{\tau^{s-1}, L^{s-1}\}\, d\tau \quad \text{and} \quad g(t) := \int_0^t [h'(\tau)]^2\, d\tau.$$

We note that

$$h(t) = t^s \quad \text{for } t \le L \qquad \text{and} \qquad g(t) \le tg'(t) = t(h'(t))^2 \quad \text{for } t \ge 0 \tag{A-3}$$

since the function $t \mapsto h'(t) = s\min\{t^{s-1}, L^{s-1}\}$ is nondecreasing. We now show that $w := u^+ \in L^\infty(\boldsymbol{B})$ and that $\|w\|_\infty$ is bounded by the right-hand side of (A-2). Since we may replace u with $-u$, the claim will then follow.

We now note that $w \in \mathcal{H}_1$ and $\varphi := g(w) \in \mathcal{H}_1$, with

$$\nabla w = \mathbb{1}_{\{u>0\}}\nabla u, \quad \nabla\varphi = g'(w)\nabla w, \quad \partial_\theta w = \mathbb{1}_{\{u>0\}}\,\partial_\theta u, \quad \partial_\theta\varphi = g'(w)\,\partial_\theta w.$$

As outlined in Remark 4.2, this follows from the boundedness of g' and the estimate $g(t) \le s^2 t^{2s-1}$ for $t \ge 0$. Testing (A-1) with φ gives

$$\int_{\boldsymbol{B}} (\nabla u \cdot \nabla\varphi - (\partial_\theta u\, \partial_\theta\varphi) + mu\varphi)\, dx = \int_{\boldsymbol{B}} |u|^{p-2}u\varphi\, dx,$$

from where we estimate, using $h'(w)^2 = g'(w)$,

$$\begin{aligned}\int_{\boldsymbol{B}} (|\nabla(h(w))|^2 - (\partial_\theta(h(w)))^2 + mwg(w))\, dx &= \int_{\boldsymbol{B}} (g'(w)(|\nabla w|^2 - (\partial_\theta w)^2) + mug(w))\, dx \\ &= \int_{\boldsymbol{B}} |u|^{p-2}ug(w)\, dx \le \int_{\boldsymbol{B}} w^p(h'(w))^2\, dx.\end{aligned} \tag{A-4}$$

Here we used (A-3) in the last step. Combining (A-4) with Proposition 4.6 and Theorem 3.3, we obtain the inequality

$$|h(w)|_{2_1^*}^2 \le c_0 \int_{\boldsymbol{B}} w^p(h'(w))^2\, dx, \tag{A-5}$$

with a constant $c_0 = c_0(N, m) > 0$. Note that, for $m \geq 0$, c_0 only depends on N. Since

$$h(t) = t^s, \quad h'(t) = st^{s-1} \quad \text{and} \quad g(t) = s^2 \int_0^t \tau^{2s-2}\, d\tau = \frac{s^2}{2s-1} t^{2s-1} \qquad \text{for } t \leq L,$$

we may let $L \to \infty$ in (A-5) and apply Lebesgue's theorem to obtain

$$|w^s|_{2_1^*}^2 \leq c_0 s^2 \int_{\boldsymbol{B}} w^{p+2s-2}\, dx \leq c_0 s^2 |w|_{2_1^*}^{p-2} |w|_{2sq}^{2s},$$

where $q = 2_1^*/(2_1^* - p + 2)$ is the conjugated exponent to $2_1^*/(p-2)$. This yields

$$|w|_{s2_1^*} \leq (c_1 s)^{1/s} |w|_{2sq}, \quad \text{with } c_1 := (c_0 |w|_{2_1^*}^{p-2})^{1/2}, \tag{A-6}$$

whenever $w \in L^{2sq}(\boldsymbol{B})$. We now consider $s = s_n = \rho^n$ for $n \in \mathbb{N}$ with $\rho := 2_1^*/(2q) = \frac{1}{2}(2 + 2_1^* - p) > 1$, so that

$$2s_1 q = 2_1^* \qquad \text{and} \qquad 2s_{n+1} q = s_n 2_1^* \quad \text{for } n \in \mathbb{N}.$$

Iterating (A-6) then gives

$$|w|_{\rho^n 2_1^*} = |w|_{s_n 2_1^*} \leq |w|_{2_1^*} \prod_{j=1}^{n} (c_1 \rho^j)^{\rho^{-j}} \leq c_1^{\rho/(\rho-1)} c_2 |w|_{2_1^*}$$

for all n, with

$$c_2 := \rho^{\sum_{j=1}^{\infty} j\rho^{-j}} < \infty.$$

It follows that

$$|w|_\infty = \lim_{n\to\infty} |w|_{\rho^n 2_1^*} \leq c_1^{\rho/(\rho-1)} c_2 |w|_{2_1^*}. \tag{A-7}$$

Moreover, by (A-6) and Theorem 3.3, we have

$$c_1 \leq c_1' \|w\|_{\mathcal{H}}^{(p-2)/2} \leq c_1' \|u\|_{\mathcal{H}}^{(p-2)/2} \quad \text{and} \quad |w|_{2_1^*} \leq \tilde{c} \|w\|_{\mathcal{H}} \leq \tilde{c} \|u\|_{\mathcal{H}},$$

with constants $c_1', \tilde{c} > 0$ depending only on N. It thus follows from (A-7) that

$$|w|_\infty \leq C \|u\|_{\mathcal{H}}^{(p-2)\rho/(2(\rho-1))+1} \quad \text{with } C := c_2 (c_1')^{\rho/(\rho-1)} \tilde{c}.$$

The proof is thus finished. □

Appendix B: Round metric on spheres in angular coordinates

Let $U := (-\pi, \pi) \times (0, \pi)^{N-2}$, and consider angular coordinates $U \to S^{N-1}$ given by

$$(\theta, \vartheta_1, \ldots, \vartheta_{N-2}) \mapsto (\sin\vartheta_1 \cdots \sin\vartheta_{N-2} \cos\theta,\ \sin\vartheta_1 \cdots \sin\vartheta_{N-2} \sin\theta,$$
$$\cos\vartheta_1,\ \sin\vartheta_1 \cos\vartheta_2,\ \ldots,\ \sin\vartheta_1 \cdots \sin\vartheta_{N-3} \cos\vartheta_{N-2}).$$

As in (3-3) and (6-7), we use the angular variable $\theta \in (-\pi, \pi)$ for the angle of the (x_1, x_2)-coordinate of $x \in S^{N-1}$ relative to the positive x_1-axis in $\mathbb{R}^2$, which differs from most of the literature (see, e.g., [Blumenson 1960]). The standard round metric on S^{N-1} (induced by the embedding $S^{N-1} \hookrightarrow \mathbb{R}^N$) with respect to these orthogonal coordinates is then written as

$$\sum_{i=1}^{N-2}\left(\prod_{k=1}^{i-1}\sin^2\vartheta_k\right)d\vartheta_i^2+\left(\prod_{k=1}^{N-1}\sin^2\vartheta_k\right)d\theta^2, \tag{B-1}$$

see, e.g., [Campos and Silva 2020, Section 2.2]. Moreover, the associated volume element is given by

$$\left(\prod_{i=1}^{N-1}\prod_{k=1}^{i-1}\sin\vartheta_k\right)d\vartheta_1\cdots d\vartheta_{N-2}\,d\theta=\left(\prod_{k=1}^{N-2}\sin^{N-1-k}\vartheta_k\right)d\vartheta_1\cdots d\vartheta_{N-2}\,d\theta. \tag{B-2}$$

The Dirichlet energy of a function $f \in H^1(S^{N-1})$ with respect to the round metric is therefore written in these coordinates as

$$\int_{S^{N-1}}|\nabla f|^2\,d\sigma=\int_U\left(\sum_{i=1}^{N-2}h_i|\partial_{\vartheta_i}v|^2+h_{N-1}|\partial_\theta v|^2\right)\left(\prod_{k=1}^{N-2}\sin^{N-1-k}\vartheta_k\right)d\vartheta_1\cdots d\vartheta_{N-2}\,d\theta, \tag{B-3}$$

with $h_i := \prod_{k=1}^{i-1} 1/\sin^2\vartheta_k$ for $i = 1, \dots, N-1$.

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