ANALYSIS & PDE

Volume 18 No. 2 2025

ALEXANDER STROHMAIER AND ALDEN WATERS

THE RELATIVE TRACE FORMULA IN ELECTROMAGNETIC SCATTERING AND BOUNDARY LAYER OPERATORS

msp

ANALYSIS AND PDE
Vol. 18 (2025), No. 2, pp. 361–408
DOI: 10.2140/apde.2025.18.361

THE RELATIVE TRACE FORMULA IN ELECTROMAGNETIC SCATTERING AND BOUNDARY LAYER OPERATORS

ALEXANDER STROHMAIER AND ALDEN WATERS

This paper establishes trace formulae for a class of operators defined in terms of the functional calculus for the Laplace operator on divergence-free vector fields with relative and absolute boundary conditions on Lipschitz domains in $\mathbb{R}^3$. Spectral and scattering theory of the absolute and relative Laplacian is equivalent to the spectral analysis and scattering theory for Maxwell equations. The trace formulae allow for unbounded functions in the functional calculus that are not admissible in the Birman–Krein formula. In special cases, the trace formula reduces to a determinant formula for the Casimir energy that is used in the physics literature for the computation of the Casimir energy for objects with metallic boundary conditions. Our theorems justify these formulae in the case of electromagnetic scattering on Lipschitz domains, give a rigorous meaning to them as the trace of certain trace-class operators, and clarify the function spaces on which the determinants need to be taken.

1. Introduction

In this paper we establish several trace formulae for operators governing the time-harmonic Maxwell equations on an open set $X = \Omega \cup M \subset \mathbb{R}^3$ of the form $\mathbb{R}^3 \setminus \partial\Omega$, where Ω is a bounded (strongly) Lipschitz

Supported by Leverhulme grant RPG-2017-329.
MSC2020: 11F72, 35P25, 35Q61, 81T55.
Keywords: Maxwell equations, layer potential, Casimir energy, trace formula.

domain. Here we will refer to Ω as the interior and to M as the exterior domain. We denote by E and H the electric and magnetic fields, respectively. The time-harmonic Maxwell system is given by

$$\begin{aligned} \operatorname{curl} E - \mathrm{i}\lambda H &= 0, \\ \operatorname{div} E &= 0, \\ \operatorname{curl} H + \mathrm{i}\lambda E &= 0, \\ \operatorname{div} H &= 0, \\ \nu \times E &= A \quad \text{on } \partial\Omega, \\ \langle \nu, H\rangle &= f \quad \text{on } \partial\Omega, \end{aligned} \tag{1}$$

where the first four equations are considered in either Ω or M separately, or simultaneously by considering this as an equation on X. Here ν is the almost everywhere defined outward-pointing unit normal vector field on $\partial\Omega$. This system is well-posed on suitable function spaces under natural consistency conditions on A and f. In particular, if A is sufficiently regular and tangential and $\lambda \neq 0$, the function f is determined by A. For the interior problem, given a tangential A, the system then has a unique solution for λ away from a discrete set of points. For the exterior problem and $\operatorname{Im}\lambda > 0$, one imposes that E and H are square-integrable and then obtains a unique solution for any sufficiently regular tangential A. In both cases, the solution E can be expressed as

$$E = \widetilde{\mathcal{L}}_\lambda \mathcal{L}_\lambda^{-1} A,$$

where $\widetilde{\mathcal{L}}_\lambda$ is the electric field boundary layer potential operator and $\mathcal{L}_\lambda$ is the electric field boundary layer operator. For a continuous tangential vector field A, one has

$$(\widetilde{\mathcal{L}}_\lambda A)(x) = \operatorname{curl}\operatorname{curl} \int_{\partial\Omega} \frac{e^{\mathrm{i}\lambda|x-y|}}{4\pi|x-y|} A(y)\,dy,$$

and $\mathcal{L}_\lambda A$ is obtained by taking the boundary value of $\nu \times \widetilde{\mathcal{L}}$. These operators extend to suitable function spaces and we refer to Section 6 for the precise definitions. The vector field H and the function f are then determined by $H = -(\mathrm{i}/\lambda)\operatorname{curl} E$. As usual, this layer potential operator creates a solution of the Maxwell system by placing certain sources on the boundary, and the choice of $\widetilde{\mathcal{L}}$ is now a standard operator in computational electrodynamics.

For $\lambda \neq 0$, the system for E becomes

$$\begin{aligned} -\Delta E - \lambda^2 E &= 0, \\ \operatorname{div} E &= 0, \\ \nu \times E &= A \quad \text{on } \partial\Omega. \end{aligned}$$

The associated spectral problem is therefore that of the Laplace–Beltrami operator Δ on divergence-free vector fields with the corresponding boundary condition. For the electric field, the boundary condition $\nu \times E = 0$ on $\partial\Omega$ leads to the relative Laplacian Δ_{rel} by quadratic form considerations. Similarly, for the magnetic field, the boundary condition $\nu \cdot H = 0$ leads to the absolute Laplace operator Δ_{abs}. Both are self-adjoint operators on $L^2(\mathbb{R}^3, \mathbb{C}^3) = L^2(\Omega, \mathbb{C}^3) \oplus L^2(M, \mathbb{C}^3)$, and their definitions and properties are explained in detail in Sections 3 and 4. Functional calculus for the relative Laplacian determines the solutions E of the time-harmonic Maxwell system, whereas functional calculus for the absolute Laplacian determines the solutions H of the system. Here we use the more mathematical notation that is inspired by Hodge theory. The harmonic forms satisfying relative boundary conditions give rise to relative de Rham cohomology classes, and the ones satisfying absolute boundary conditions give rise to absolute de Rham cohomology classes.

Before we describe the general case, we would like to explain and motivate this in an important special case and when the bounded Lipschitz domain $\Omega \subset \mathbb{R}^3$ consists of two connected components Ω_1 and Ω_2. We then construct the self-adjoint operator Δ_{rel} out of the Laplace operator on $\mathbb{R}^3 \setminus \partial\Omega$ by imposing relative boundary conditions in each side of $\partial\Omega$. The operators $\Delta_{j,\text{rel}}$ are obtained in the same way from the Laplace operator on $\mathbb{R}^3 \setminus \partial\Omega_j$ with boundary conditions only imposed on each side of $\partial\Omega_j$. The operators Δ_{abs} and $\Delta_{j,\text{abs}}$ are defined analogously with absolute boundary conditions. It is a special case of our result that the two operators

$$C_E = (-\Delta_{\text{rel}})^{-1/2}\delta\,\mathrm{d} - (-\Delta_{1,\text{rel}})^{-1/2}\delta\,\mathrm{d} - (-\Delta_{2,\text{rel}})^{-1/2}\delta\,\mathrm{d} + (-\Delta_{\text{free}})^{-1/2}\delta\,\mathrm{d},$$
$$C_H = (-\Delta_{\text{abs}})^{-1/2}\delta\,\mathrm{d} - (-\Delta_{1,\text{abs}})^{-1/2}\delta\,\mathrm{d} - (-\Delta_{2,\text{abs}})^{-1/2}\delta\,\mathrm{d} + (-\Delta_{\text{free}})^{-1/2}\delta\,\mathrm{d}$$

defined on smooth compactly supported vector fields on $X = \mathbb{R}^3 \setminus \partial\Omega$ extend to trace-class operators on $L^2(\mathbb{R}^3, \mathbb{C}^3)$, and their trace can be expressed in terms of the determinant of a combination of Maxwell boundary layer operators (see Theorems 1.1 and 1.3). In fact, we will see that their traces coincide, i.e., $\mathrm{tr}(C_E) = \mathrm{tr}(C_H)$. We have used here differential form notation, with d being the exterior derivative and δ being the coderivative. The trace-class property is due to several cancellations. Any linear combination of operators appearing in the expressions above that is not proportional to this expression is not trace-class. This statement remains true even if one introduces an artificial boundary, thereby compactifying the problem.

In terms of vector calculus, the above two operators can also be written as

$$C_E = (-\Delta_{\text{rel}})^{-1/2}\mathrm{curl}\,\mathrm{curl} - (-\Delta_{1,\text{rel}})^{-1/2}\mathrm{curl}\,\mathrm{curl} - (-\Delta_{2,\text{rel}})^{-1/2}\mathrm{curl}\,\mathrm{curl} + (-\Delta_{\text{free}})^{-1/2}\,\mathrm{curl}\,\mathrm{curl},$$
$$C_H = \mathrm{curl}(-\Delta_{\text{rel}})^{-1/2}\mathrm{curl} - \mathrm{curl}(-\Delta_{1,\text{rel}})^{-1/2}\mathrm{curl} - \mathrm{curl}(-\Delta_{2,\text{rel}})^{-1/2}\mathrm{curl} + \mathrm{curl}(-\Delta_{\text{free}})^{-1/2}\,\mathrm{curl}.$$

Apart from being interesting from the point of view of spectral analysis, these operators also have a direct physical significance. Namely, $\frac{1}{4}\,\mathrm{tr}(C_E + C_H) = \frac{1}{2}\,\mathrm{tr}(C_E)$ represents the Casimir energy of the two Lipschitz obstacles Ω_1 and Ω_2. Indeed, as shown in [Strohmaier 2021] in a general rigorous framework of quantum field theory, the local trace, i.e., the trace of the integral kernel restricted to the diagonal, of the operator

$$\tfrac{1}{4}((-\Delta_{\text{rel}})^{-1/2}\mathrm{curl}\,\mathrm{curl} - (-\Delta_{\text{free}})^{-1/2}\,\mathrm{curl}\,\mathrm{curl}) + \tfrac{1}{4}((-\Delta_{\text{abs}})^{-1/2}\mathrm{curl}\,\mathrm{curl} - (-\Delta_{\text{free}})^{-1/2}\,\mathrm{curl}\,\mathrm{curl})$$

is the renormalised energy density obtained from the electromagnetic quantum field theory. The relative resolvent differences C_E and C_H then describe differences of energies. It was shown in [Fang and Strohmaier 2022], again in a rigorous quantum field theory framework, that in the case of a scalar field, such “energy differences” lead to a Casimir force as determined from the quantum stress energy tensor as in [Candelas 1982; Kay 1979]. The same statement is expected to hold for the electromagnetic field, but this will be discussed elsewhere.

The mathematical statements above can therefore also be interpreted as a rigorous proof that the Casimir energy as derived from spectral quantities is well defined in this framework and can be computed from determinants of boundary layer operators. It also clarifies the function spaces needed to compute these quantities for nonsmooth boundaries.

We focus in this paper on Maxwell's equations in dimension three, and we will mostly use vector calculus notation rather than differential forms. This has the advantage of keeping the notation and exposition more accessible, and we can then also rely on a wealth of previous results on boundary layer operators [Buffa and Hiptmair 2003; Buffa et al. 2002; Claeys and Hiptmair 2019; Costabel 1988; 1990; Gol'dshtein et al. 2011; Kirsch and Hettlich 2015; Mitrea 1995; 2000; Mitrea and Mitrea 2002; Mitrea et al. 1997]. Focussing on dimension three also avoids complications with the free Green's function having more complicated expressions or a logarithmic singularity at 0. More importantly, the focus on dimension three allows us to stay close to the classical notation in Maxwell theory without having to distract the reader with more complicated notation.

Although this is a mathematical paper, we also try to give the physics background for the interested reader. To our knowledge, a determinant formula for the Casimir energy first appeared in the physics literature [Renne 1971], where this was derived microscopically and without reference to spectral theory. Physics derivations have also appeared in various contexts based on path integrals and fluctuations of configurations on the surface on the obstacles [Emig et al. 2007; Kenneth and Klich 2006; 2008] and have led to numerical schemes [Johnson 2011] and asymptotic formulae. The spectral side, often favouring a zeta function regularisation approach as in Casimir's original work [1948], was developed somewhat independently. We refer to [Bordag et al. 2009] and [Kirsten 2002] for a comprehensive overview of the subject. The relation between the various approaches remained unclear even in the physics world (for a very recent report on this see [Bimonte and Emig 2021]). We also mention the approach of [Balian and Duplantier 1978], which is also based on a reduction to the boundary.

1.1. ***Statement of main results.*** We now describe the general setting of our results. We assume that $\Omega \subset \mathbb{R}^3$ is an open and bounded (strongly) Lipschitz domain in $\mathbb{R}^3$ in the sense that the boundary of Ω is locally congruent to the graph of a Lipschitz function. The finitely many connected components will be denoted by Ω_j with some index j, which ranges from 1 to N. We will think of the closure $\overline{\Omega}$ as a collection of disjoint compact obstacles $\overline{\Omega}_j$ placed in $\mathbb{R}^3$ (see Figure 1). Removing these obstacles from $\mathbb{R}^3$ results in a noncompact open domain $M = \mathbb{R}^3 \backslash \overline{\Omega}$ with Lipschitz boundary $\partial\Omega$. We will assume throughout that M is connected. It will also be convenient to introduce $X = \mathbb{R}^3 \setminus \partial\Omega = M \cup \Omega$.

On the boundary, one has well-defined anisotropic Sobolev spaces $H^{-1/2}(\mathrm{Div}, \partial\Omega)$ (see Section 2) and the Maxwell electric field operator $\mathcal{L}_\lambda$ is a bounded operator $H^{-1/2}(\mathrm{Div}, \partial\Omega) \to H^{-1/2}(\mathrm{Div}, \partial\Omega)$ (see Section 6). This can be done for each object separately, and one can assemble the individual parts $\mathcal{L}_{\lambda,\partial\Omega_j} : H^{-1/2}(\mathrm{Div}, \partial\Omega_j) \to H^{-1/2}(\mathrm{Div}, \partial\Omega_j)$ into an operator $\mathcal{L}_{D,\lambda} = \bigoplus_{j=1}^{N} \mathcal{L}_{\lambda,\partial\Omega_j}$ acting on $H^{-1/2}(\mathrm{Div}, \partial\Omega)$.

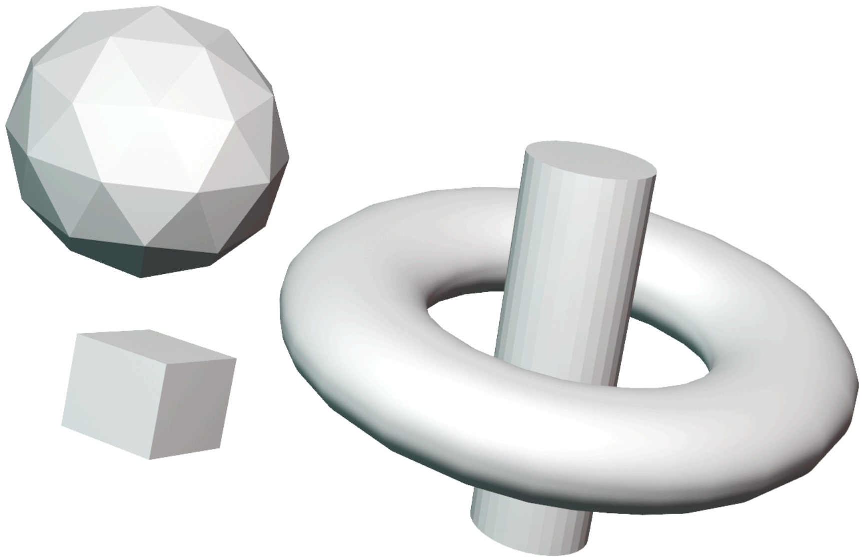

Figure 1. A Lipschitz domain Ω consisting of four connected components $\Omega_1,\ \Omega_2,\ \Omega_3,\ \Omega_4$.

Theorem 1.1. *The operator* $\mathcal{L}_\lambda \mathcal{L}_{D,\lambda}^{-1}$ *is well defined and a trace-class perturbation of the identity for any complex* λ *with* $\operatorname{Im}\lambda > 0$. *It therefore has a well-defined Fredholm determinant* $\det(\mathcal{L}_\lambda \mathcal{L}_{D,\lambda}^{-1})$ *on the space* $H^{-1/2}(\mathrm{Div}, \partial\Omega)$. *Let* δ *be the minimal distance between separate objects. Then, for any* $0 < \delta' < \delta$, *the function*

$$\Xi(\lambda) = \log\det(\mathcal{L}_\lambda \mathcal{L}_{D,\lambda}^{-1}),$$

where the branch of the logarithm has been fixed by continuity, extends to a holomorphic function in a neighbourhood of the closed upper half-space, and it satisfies the bound

$$|\Xi(\lambda)| \leq C e^{-\delta' \operatorname{Im}\lambda}$$

for λ *in any sector about the positive imaginary axis of angle strictly less than* π.

We note that the operators $\mathcal{L}_\lambda^{-1}$ and $\mathcal{L}_{D,\lambda}^{-1}$ have singularities at 0, and it is due to a variety of cancellations that the quotient is regular at 0, in particular when the objects have nontrivial topology. Our proof is based on a careful analysis of these singularities.

Before we formulate the trace formula, we need to define a large class of functions to which it applies. These will be analytic functions in certain sectors, and we start by describing these sectors. Assume $0 < \epsilon \leq \pi$, and let $\mathfrak{S}_\epsilon$ be the open sector

$$\mathfrak{S}_\epsilon = \{z \in \mathbb{C} \mid z \neq 0,\ |\arg(z)| < \epsilon\}$$

containing the real axis (see Figure 2). Associated to these we define the following spaces of functions. The space $\mathcal{E}_\epsilon$ will be defined by

$$\mathcal{E}_\epsilon = \{f : \mathfrak{S}_\epsilon \to \mathbb{C} \mid f \text{ is holomorphic in } \mathfrak{S}_\epsilon,\ \exists \alpha > 0,\ \forall \epsilon_0 > 0,\ |f(z)| = O(|z|^\alpha e^{\epsilon_0 |z|})\}.$$

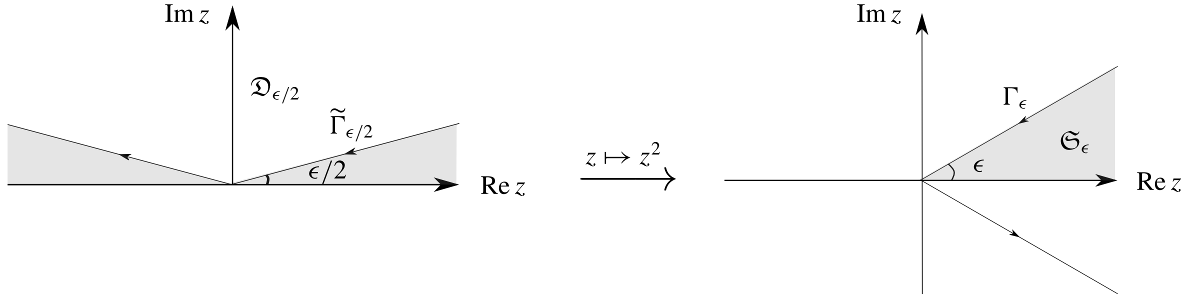

Figure 2. The sectors $\mathfrak{S}_\epsilon$, $\mathfrak{D}_{\epsilon/2}$ and the corresponding contours.

We define the space $\mathcal{P}_\epsilon$ as the set of functions in $\mathcal{E}_\epsilon$ whose restriction to $[0, \infty)$ is polynomially bounded and that extend continuously to the boundary of $\mathfrak{S}_\epsilon$ in the logarithmic cover of the complex plane. Reference to the logarithmic cover of the complex plane is only needed in the case $\epsilon = \pi$. In this case functions in $\mathcal{P}_\pi$ are required to have continuous limits from above and below on the negative real axis. We do not however require that these limits coincide. The space $\mathcal{P}_\epsilon$ contains, in particular, $f(z) = z^\alpha$, $\alpha > 0$, for any $0 < \epsilon \le \pi$.

When working with the Laplace operator, it is often convenient to change variables and use λ^2 as a spectral parameter, and in the context of Maxwell theory it turns out to be beneficial to introduce an extra λ^{-2} factor. For notational brevity we therefore introduce another class of functions as follows.

Definition 1.2. The space $\widetilde{\mathcal{P}}_\epsilon$ is defined to be the space of functions f such that $f(\lambda) = \lambda^{-2} g(\lambda^2)$ for some $g \in \mathcal{P}_\epsilon$. In particular, $f(\lambda) = O(\lambda^a)$ for some $a > -2$ near $\lambda = 0$.

Generally, the operator Δ_{rel} decomposes into a direct sum of unbounded operators $\Delta_{\mathrm{rel}} = 0 \oplus \mathrm{d}\delta \oplus \delta\,\mathrm{d}$ under the weak Hodge–Helmholtz decomposition (see Section 4, (9)), and we have

$$f((-\Delta_{\mathrm{rel}})^{1/2})\,\mathrm{curl}\,\mathrm{curl} = f((-\Delta_{\mathrm{rel}})^{1/2})\delta\,\mathrm{d} = f((\delta\,\mathrm{d})^{1/2})\delta\,\mathrm{d}$$

for any Borel function f. From this we have that, for a function $f \in \widetilde{\mathcal{P}}_\epsilon$, the unbounded operator $f((-\Delta_{\mathrm{rel}})^{1/2})\,\mathrm{curl}\,\mathrm{curl}$ contains $C_0^\infty(X, \mathbb{C}^3)$ in its domain. Indeed, for $\psi \in C_0^\infty(X, \mathbb{C}^3)$ and $k \in \mathbb{N}$ large enough, we have the factorisation $f((\delta\,\mathrm{d})^{1/2})\delta\,\mathrm{d} = h((\delta\,\mathrm{d})^{1/2})(\delta\,\mathrm{d}+1)^k\psi$, where $(\delta\,\mathrm{d}+1)^k\psi \in C_0^\infty(X, \mathbb{C}^3)$ and the function $h(\lambda) = (1+\lambda^2)^{-k}\lambda^2 f(\lambda)$ is bounded on the real line.

For $0 < \epsilon \le \pi$, we also define the contours Γ_ϵ in the complex plane as the boundary curves of the sectors $\mathfrak{S}_\epsilon$. In the case $\epsilon = \pi$, the contour is defined as a contour in the logarithmic cover of the complex plane. We also let $\widetilde{\Gamma}_{\epsilon/2}$ be the corresponding contour after the change of variables, i.e., the preimage in the upper half-space under the map $z \to z^2$ of Γ_ϵ (see Figure 2).

For $f \in \widetilde{\mathcal{P}}_\epsilon$, we define the *relative operator*

$$D_{\mathrm{rel},f} = f((-\Delta_{\mathrm{rel}})^{1/2})\mathrm{curl}\,\mathrm{curl} - f((-\Delta_{\mathrm{free}})^{1/2})\,\mathrm{curl}\,\mathrm{curl} - \sum_{j=1}^{N}\bigl(f((-\Delta_{j,\mathrm{rel}})^{1/2})\mathrm{curl}\,\mathrm{curl} - f((-\Delta_{\mathrm{free}})^{1/2})\,\mathrm{curl}\,\mathrm{curl}\bigr),$$

where $f(\lambda) = g(\lambda^2)$. Similarly,

$$\begin{aligned} D_{\mathrm{abs},f} &= f((-\Delta_{\mathrm{abs}})^{1/2})\operatorname{curl}\operatorname{curl} - f((-\Delta_{\mathrm{free}})^{1/2})\operatorname{curl}\operatorname{curl} \\ &\qquad - \sum_{j=1}^{N} \big(f((-\Delta_{j,\mathrm{abs}})^{1/2})\operatorname{curl}\operatorname{curl} - f((-\Delta_{\mathrm{free}})^{1/2})\operatorname{curl}\operatorname{curl}\big) \\ &= \operatorname{curl} f((-\Delta_{\mathrm{rel}})^{1/2})\operatorname{curl} - \operatorname{curl} f((-\Delta_{\mathrm{free}})^{1/2})\operatorname{curl} \\ &\qquad - \sum_{j=1}^{N} \big(\operatorname{curl} f((-\Delta_{j,\mathrm{rel}})^{1/2})\operatorname{curl} - \operatorname{curl} f((-\Delta_{\mathrm{free}})^{1/2})\operatorname{curl}\big). \end{aligned}$$

Since these operators contain $C_0^\infty(X, \mathbb{C}^3)$ in their domain, they are densely defined.

We refer to taking these differences as the *relative setting*, indicating that this compares interacting quantities to noninteracting ones. It is unfortunate that the word relative is also used to denote the relative boundary conditions. We alert the reader that these two uses of the word relative are unrelated, but to avoid confusion we have used the symbol D for "difference" to denote relative objects.

Our main result reads as follows.

Theorem 1.3. *If $f \in \widetilde{\mathcal{P}}_\epsilon$, then operators $D_{\mathrm{rel},f}$ and $D_{\mathrm{abs},f}$ extend to trace-class operators $L^2(\mathbb{R}^3, \mathbb{C}^3) \to L^2(\mathbb{R}^3, \mathbb{C}^3)$, and*

$$\operatorname{tr}(D_{\mathrm{rel},f}) = \operatorname{tr}(D_{\mathrm{abs},f}) = \frac{\mathrm{i}}{2\pi} \int_{\widetilde{\Gamma}_{\epsilon/2}} \Xi(\lambda) \frac{\mathrm{d}}{\mathrm{d}\lambda}(\lambda^2 f(\lambda))\, \mathrm{d}\lambda,$$

where the contour $\widetilde{\Gamma}_{\epsilon/2}$ is the clockwise-oriented boundary of a sector that includes the imaginary axis.

We would like to mention that expressions formally similar to the relative trace-formula have appeared in the context of multichannel scattering theory and were introduced by Buslaev and Merkur' ev [1969] (see also [Vasy and Wang 2002]) to prove Birman–Krein-type formulae. In this context, the test function f is still required to decay sufficiently fast.

An interesting application of the relative trace is that it allows one to define a relative zeta function, namely,

$$\zeta_D(s) = \operatorname{tr}(D_{f_s}), \qquad f_s(\lambda) = \frac{1}{\lambda^{2s+2}}$$

for $\operatorname{Re} s < 0$. As a consequence of Theorem 1.3, this relative zeta function then satisfies

$$\zeta_D(s) = \frac{2s}{\pi} \sin(\pi s) \int_0^\infty \lambda^{-2s-1}\, \Xi(\mathrm{i}\lambda)\, \mathrm{d}\lambda.$$

This formula allows for a meromorphic continuation of ζ_D with poles of order at most one and residues related to the Taylor coefficients of $\Xi(\mathrm{i}\lambda)$ at 0. These coefficients are interesting in their own right and will be investigated elsewhere. In the special case when $f(\lambda) = 1/\lambda$, this gives the expression

$$\frac{1}{4}\operatorname{tr}(C_E + C_H) = \frac{1}{2\pi} \int_0^\infty \Xi(\mathrm{i}\lambda)\, \mathrm{d}\lambda$$

for the Casimir energy.

Under our more general assumptions on f, the operators

$$B_{\mathrm{rel},f} = f((-\Delta_{\mathrm{rel}})^{1/2})\operatorname{curl}\operatorname{curl} - f((-\Delta_{\mathrm{free}})^{1/2})\operatorname{curl}\operatorname{curl}, \tag{2}$$

$$B_{\mathrm{abs},f} = f((-\Delta_{\mathrm{abs}})^{1/2})\operatorname{curl}\operatorname{curl} - f((-\Delta_{\mathrm{free}})^{1/2})\operatorname{curl}\operatorname{curl} \tag{3}$$

are not trace-class. One has however the following theorem about the smoothness and integrability properties of their integral kernels.

Theorem 1.4. *Let B_f be either $B_{\mathrm{rel},f}$, defined by* (2), *or $B_{\mathrm{rel},f}$, defined by* (3). *Then B_f has an integral kernel $\kappa \in C^\infty(X \times X, \mathrm{Mat}(3, \mathbb{C}))$, which is smooth away from the boundary. If $\Omega_0 \subset X$ has positive distance to the boundary $\partial\Omega$ and p_{Ω_0} is the orthogonal projection $L^2(\mathbb{R}^3, \mathbb{C}^3) \to L^2(\Omega_0, \mathbb{C}^3)$, then $p_{\Omega_0} B_f p_{\Omega_0}$ extends to a trace-class operator with trace equal to the convergent integral*

$$\int_{\Omega_0} \operatorname{tr}(\kappa(x, x))\,\mathrm{d}x.$$

If $f(z) = O(|z|^a)$ for $|z| < 1$, we have for large $|x|$ the estimate

$$\|\kappa(x, x)\| \le C_f \frac{1}{|x|^{6+a}}.$$

1.2. ***Discussion.*** The theorems presented here are the Maxwell analogue of [Hanisch et al. 2022], where a similar statement was proved for the scalar Laplacian in the case of smooth boundary. The Maxwell system on a Lipschitz domain is different in several regards and introduces challenges that are absent in the scalar case:

- Maxwell's equations arise from an abelian gauge theory, and the gauge freedom results in the loss of ellipticity of the equations for the electromagnetic field. On the analysis side, this manifests itself as the equations taking place on the space of divergence-free vector fields rather than the space of sections of the vector bundle. This can however be fixed by considering the spectral decomposition of the Laplace operator and then employing the Helmholtz–Hodge decomposition to project onto the subspace of divergence-free vector fields. Projecting works well in cases with and without boundary as long as the geometric configuration is fixed. The projector constructed from the Helmholtz decomposition is roughly of the form $-\Delta^{-1}\delta\,\mathrm{d} = -\Delta^{-1}\operatorname{curl}\operatorname{curl}$, and it involves the nonlocal functional calculus of the Laplace operator. It therefore depends on the geometric configuration and also the boundary conditions imposed on the Laplace operator. This makes it much harder to directly apply scattering theory which requires an identification of the involved Hilbert spaces. The same problem appears in the context of the Birman–Krein formula in electromagnetic scattering. We have proved a variant of the Birman–Krein formula in [Strohmaier and Waters 2022] and we will follow the same formulation here.

- Unlike the Dirichlet–Laplacian, the Laplace operator on the space of vector fields with relative boundary conditions has a nontrivial kernel in the exterior domain. This leads to singularities of the resolvent near 0 and manifests itself in the presence of singularities of the boundary layer operators. Additional singularities of the boundary layer operators appear if the obstacles have nontrivial topology, which we do not exclude. To overcome this, we carefully analyse the singularities of various Maxwell boundary layer operators at 0, and we show that there are various cancellations that render a final result without singularities.

• An additional complication arises in this paper since we are considering Lipschitz domains instead of smooth ones. This requires more sophisticated harmonic analysis techniques. We rely here on a lot of progress in this subject that has been made during the past several decades, in particular with the identification of the appropriate function spaces.

As explained, the spectral theory of Δ_{rel} and Δ_{abs} determines the Maxwell system. Suitably interpreted, the curl operator intertwines these two operators in the sense that $\operatorname{curl} \Delta_{\mathrm{abs}} = \Delta_{\mathrm{rel}} \operatorname{curl}$. In the interior, the relative Laplacian on a suitable closed subspace consisting of divergence-free vector fields has the Maxwell eigenvalues as its spectrum, and the eigenfunctions describe modes of photons that are confined to Ω. The exterior relative Laplacian on a suitable closed subspace of divergence-free vector fields describes the scattering of electromagnetic waves or photons by the obstacles Ω. The functional calculus on Δ_{rel} on this subspace can be understood in terms of the operators $f(\Delta_{\mathrm{rel}}) \operatorname{curl}\operatorname{curl}$. The following Birman–Krein formula has been proved.

1.3. *Relation to the Birman–Krein formula.* In the case that f is an even Schwartz function, we have that

$$(\operatorname{curl}\operatorname{curl}(f((-\Delta_{\mathrm{rel}})^{1/2}) - f((-\Delta_{\mathrm{free}})^{1/2})))$$

is trace-class, and its trace can be computed by the Birman–Krein-type formula

$$\operatorname{tr}(\operatorname{curl}\operatorname{curl}(f((-\Delta_{\mathrm{rel}})^{1/2}) - f((-\Delta_{\mathrm{free}})^{1/2}))) = \frac{1}{2\pi \mathrm{i}} \int_0^\infty \lambda^2 \operatorname{tr}(S_\lambda^{-1}(S_\lambda)') f(\lambda)\, \mathrm{d}\lambda + \sum_{j=1}^\infty f(\mu_j)\mu_j^2,$$

where S_λ is the scattering matrix for the Maxwell equation and μ_j are the Maxwell eigenvalues of the interior. As a consequence of this formula,

$$\operatorname{tr} D_{\mathrm{rel},f} = -\frac{1}{2\pi \mathrm{i}} \int_0^\infty \log \frac{\det S_\lambda}{\det(S_{1,\lambda}) \cdots \det(S_{N,\lambda})} \frac{\mathrm{d}}{\mathrm{d}\lambda}(\lambda^2 f(\lambda))\, \mathrm{d}\lambda,$$

which is valid only under very restrictive assumptions on f. The same formula and statements hold for absolute instead of relative boundary conditions.

In the motivating example, one cannot use this formula. It would require $f(\lambda) = 1/\lambda$, which does not satisfy the assumptions of the Birman–Krein formula. In fact it can be shown that the integrand on the right-hand side is not integrable in that case. One has however the following relation between the function Ξ and the scattering matrices.

Theorem 1.5. *We have*

$$\log \frac{\det S_\lambda}{\det(S_{1,\lambda}) \cdots \det(S_{N,\lambda})} = -(\Xi(\lambda) - \Xi(-\lambda))$$

for $\lambda \in \mathbb{R}$.

This theorem reflects the relation between the spectral shift function and zeta regularised determinants as discovered by Carron [2002, Theorem 1.3], generalising a formula by Gesztesy and Simon [1996, Theorem 1.1].

1.4. ***Organisation of the paper.*** The paper is organised as follows. Sections 2- 6 provide the required theoretical background for the paper and consist of essentially known material. Section 2 sets up the basic function spaces needed for boundary layer theory on Lipschitz domains. Section 3 summarises the spectral properties of the interior relative and absolute Laplace operators, and Section 4 reviews the scattering theory for the relative and absolute Laplacians on the exterior. Both are combined into one operator in Section 5. In this section we also discuss the Birman–Krein formula in the context of our setting. Section 6 introduces the basic Maxwell boundary layer operators and their properties.

The basic estimates and expansions for the layer potential operators needed for the proofs are covered in Section 7. This section is presented independently of the main results as its content is interesting in its own right. It covers various aspects of low-energy expansions for the electric and magnetic boundary layer operators and inverses. Section 8 gives formulae of the resolvent differences in terms of layer potential operators and thereby provides estimates for these differences. Such formulae are sometimes referred to as Krein-type resolvent formulae, and this section provides a Maxwell analogue of these. Sections 9 and 10 take on the main subject of this paper, namely function Ξ, the relative resolvent, and its trace. Section 11 finally contains the proofs of the main theorems.

2. Function spaces on Lipschitz domains

Since Ω is a Lipschitz domain, we have, by Rademacher's theorem, an almost everywhere defined exterior unit vector field $\nu \in L^\infty(\partial\Omega, \mathbb{R}^3)$. We will use the following spaces that now are standard in Maxwell theory:

- $H(\mathrm{curl}, M) = \{f \in L^2(M, \mathbb{C}^3) \mid \mathrm{curl}\, f \in L^2(M, \mathbb{C}^3)\}$.
- $H(\mathrm{div}, M) = \{f \in L^2(M, \mathbb{C}^3) \mid \mathrm{div}\, f \in L^2(M)\}$.
- $L^2_{\tan}(\partial\Omega) = \{f \in L^2(\partial\Omega, \mathbb{C}^3) \mid \nu \cdot f = 0 \text{ a.e. on } \partial\Omega\}$.
- $H^{-1/2}(\mathrm{Div}, \partial\Omega)$, $H^{-1/2}(\mathrm{Curl}, \partial\Omega)$.
- $H^{-1/2}(\mathrm{Div}\, 0, \partial\Omega)$, $H^{-1/2}(\mathrm{Curl}\, 0, \partial\Omega)$.

These spaces were introduced in [Buffa et al. 2002] and provide a convenient framework for dealing with Maxwell's equations on Lipschitz domains. We refer to the Appendix of [Kirsch and Hettlich 2015] for an extensive discussion, and we only summarise the basic properties.

In the case that $\partial\Omega$ is smooth, we have

$$\begin{aligned} H^{-1/2}(\mathrm{Div}, \partial\Omega) &= \{f \in H^{-1/2}(\partial\Omega; T\partial\Omega) \mid \mathrm{Div}\, f \in H^{-1/2}(\partial\Omega)\}, \\ H^{-1/2}(\mathrm{Curl}, \partial\Omega) &= \{f \in H^{-1/2}(\partial\Omega; T\partial\Omega) \mid \mathrm{Curl}\, f \in H^{-1/2}(\partial\Omega)\}, \\ H^{-1/2}(\mathrm{Div}\, 0, \partial\Omega) &= \{f \in H^{-1/2}(\partial\Omega; T\partial\Omega) \mid \mathrm{Div}\, f = 0\}, \\ H^{-1/2}(\mathrm{Curl}\, 0, \partial\Omega) &= \{f \in H^{-1/2}(\partial\Omega; T\partial\Omega) \mid \mathrm{Curl}\, f = 0\}, \end{aligned}$$

where Div is the surface divergence on $\partial\Omega$ and Curl is the surface curl. On a general Lipschitz domain, this can be defined via Lipschitz coordinate charts, thus locally reducing it to the smooth case. Note that the spaces $H^s_{\mathrm{loc}}(\mathbb{R}^d)$ are invariant under bi-Lipschitz maps if $|s| \le 1$. We refer to [Kirsch and Hettlich 2015] for a detailed discussion of the definition via coordinate charts. We also have the corresponding spaces for the interior domains. Namely we have

$$H(\mathrm{curl}, \Omega) = \{f \in L^2(\Omega, \mathbb{C}^3) \mid \mathrm{curl}\, f \in L^2(\Omega, \mathbb{C}^3)\},$$
$$H(\mathrm{div}, \Omega) = \{f \in L^2(\Omega, \mathbb{C}^3) \mid \mathrm{div}\, f \in L^2(\Omega)\}.$$

On $H(\mathrm{curl}, M)$, there are two distinguished and well-defined continuous trace maps

$$\gamma_{T,-} : H(\mathrm{curl}, M) \to H^{-1/2}(\mathrm{Curl}, \partial\Omega),$$
$$\gamma_{t,-} : H(\mathrm{curl}, M) \to H^{-1/2}(\mathrm{Div}, \partial\Omega),$$

which continuously extend the maps $f \mapsto (\nu \times f|_{\partial\Omega}) \times \nu$ and $f \mapsto (\nu \times f|_{\partial\Omega})$, respectively, defined on $C_0(\overline{M}, \mathbb{C}^3)$. Note that, for $x \in \partial\Omega$ such that ν_x is defined, the map $v \mapsto (\nu_x \times v) \times \nu_x$ is the orthogonal projection onto the tangent space of $\partial\Omega$ at x. Similarly, we have the map

$$\gamma_{\nu,-} : H(\mathrm{div}, M) \to H^{-1/2}(\partial\Omega)$$

continuously extending the normal restriction map $f \mapsto \nu \cdot f|_{\partial\Omega}$. On the interior domain Ω, we have the analogous maps

$$\gamma_{T,+} : H(\mathrm{curl}, \Omega) \to H^{-1/2}(\mathrm{Curl}, \partial\Omega),$$
$$\gamma_{t,+} : H(\mathrm{curl}, \Omega) \to H^{-1/2}(\mathrm{Div}, \partial\Omega),$$
$$\gamma_{\nu,+} : H(\mathrm{div}, \Omega) \to H^{-1/2}(\partial\Omega).$$

There is a well-defined dual pairing between $H^{-1/2}(\mathrm{Curl}, \partial\Omega)$ and $H^{-1/2}(\mathrm{Div}, \partial\Omega)$ that extends the L^2-inner product on $H^{1/2}(\partial\Omega) \cap L^2_{\tan}(\partial\Omega)$. We will denote this pairing by $\langle \cdot\,, \cdot \rangle_{L^2(\partial\Omega)}$, irrespective of the Sobolev order and mildly abusing notation. The map $\phi \mapsto \nu \times \phi$ extends to a continuous isomorphism from $H^{-1/2}(\mathrm{Div}, \partial\Omega)$ to $H^{-1/2}(\mathrm{Curl}, \partial\Omega)$ and vice versa. Moreover, the L^2-pairing induces an antilinear isomorphism between $H^{-1/2}(\mathrm{Div}, \partial\Omega)$ and $H^{-1/2}(\mathrm{Curl}, \partial\Omega)$ (see, for example, [Kirsch and Hettlich 2015, Lemma 5.61] for both statements). In other words, the antisymmetric bilinear form $\langle \cdot\,, \nu \times \cdot \rangle$ on $H^{-1/2}(\mathrm{Div}, \partial\Omega)$ is nondegenerate. Note here that, since $\nu \in L^\infty(\partial\Omega, \mathbb{R}^3)$, it is not immediately obvious that it is defined as a map between Sobolev spaces.

We recall Stokes' theorem for $\phi, E \in H(\mathrm{curl}, \Omega)$:

$$\langle \gamma_{t,+} E, \gamma_{T,+}\phi \rangle_{L^2(\partial\Omega)} = \langle \mathrm{curl}\, E, \phi \rangle_{L^2(\Omega)} - \langle E, \mathrm{curl}\, \phi \rangle_{L^2(\Omega)}, \tag{4}$$
$$\langle \mathrm{curl}\,\mathrm{curl}\, E, \phi \rangle_{L^2(\Omega)} - \langle E, \mathrm{curl}\,\mathrm{curl}\, \phi \rangle_{L^2(\Omega)} = \langle \gamma_{t,+}\, \mathrm{curl}\, E, \gamma_{T,+}\phi \rangle_{L^2(\partial\Omega)} + \langle \gamma_{t,+} E, \gamma_{T,+}\, \mathrm{curl}\, \phi \rangle_{L^2(\partial\Omega)}.$$

As before, we slightly abuse notation and write $\langle \cdot\,, \cdot \rangle_{L^2(\Omega)}$ for pairings extending the L^2-inner product. We define $H_0(\mathrm{curl}, M)$ and $H_0(\mathrm{div}, M)$ as the kernels of $\gamma_{t,-}$ and $\gamma_{\nu,-}$, respectively. These spaces play a similar role to the Sobolev space of functions $H^1_0(M)$, which can also be characterised as the kernel of the trace map $\gamma : H^1(M) \to H^{1/2}(\partial M)$. The spaces $H_0(\mathrm{curl}, \Omega)$ and $H_0(\mathrm{div}, \Omega)$, as well as $H^1_0(\Omega)$, are defined analogously.

If there is no danger of confusion, we will omit the $\pm$ and simply write γ_t and γ_ν, respectively.

We also have surface divergence Div and surface curl Curl. They satisfy

$$\mathrm{Div} \circ \gamma_{t,+} = -\gamma_{\nu,+} \circ \mathrm{curl}. \tag{5}$$

3. Laplace operators on the interior domain

3.1. ***The relative Laplacian.*** The operator

$$\mathrm{curl}_{\min} = \mathrm{curl}\,|_{H_0(\mathrm{curl},\Omega)} : H_0(\mathrm{curl}, \Omega) \to L^2(\Omega, \mathbb{C}^3)$$

is a closed densely defined operator. It coincides with the closure of the operator curl on the space of compactly supported smooth vector fields on Ω [Kirsch and Hettlich 2015, Theorem 5.25] and therefore equals the minimal closed extension of curl.

Its adjoint is the maximal extension, i.e., the closed operator

$$\mathrm{curl}_{\max} : H(\mathrm{curl}, \Omega) \to L^2(\Omega, \mathbb{C}^3). \tag{6}$$

For any closed densely defined operator A, the operator A^*A is automatically self-adjoint. If in addition $\mathrm{rg}(A) \subset \ker(A)$, then $A^*A + AA^*$ is self-adjoint if it is densely defined; see for example [Strohmaier and Waters 2022, Section 2]. It follows that $\mathrm{curl}_{\max}\mathrm{curl}_{\min}$ with domain

$$\{f \in H_0(\mathrm{curl}, \Omega) \mid \mathrm{curl}\, f \in H(\mathrm{curl}, \Omega)\}$$

is a nonnegative self-adjoint operator. Similarly, $\mathrm{div}_{\max} : H(\mathrm{div}, \Omega) \to L^2(\Omega)$ is a closed operator with adjoint $-\mathrm{grad}_{\min} : H^1_0(\Omega) \to L^2(\Omega)$. Therefore, the operator $-\mathrm{grad}_{\min}\mathrm{div}_{\max}$ is a nonnegative self-adjoint operator with domain

$$\{f \in H(\mathrm{div}, \Omega) \mid \mathrm{div}\, f \in H_0(\Omega)\}.$$

Their sum $\Delta_{\Omega,\mathrm{rel}} = \mathrm{curl}_{\max}\mathrm{curl}_{\min} - \mathrm{grad}_{\min}\mathrm{div}_{\max}$ is again self-adjoint and has domain

$$\{f \in H(\mathrm{div}, \Omega) \cap H_0(\mathrm{curl}, \Omega) \mid \mathrm{div}\, f \in H_0(\Omega),\ \mathrm{curl}\, f \in H(\mathrm{curl}, \Omega)\},$$

and on this domain $-\Delta_{\Omega,\mathrm{rel}}$ is given by $\mathrm{curl}\,\mathrm{curl} - \mathrm{grad}\,\mathrm{div}$. The implied boundary conditions of this operator are the so-called relative boundary conditions

$$\gamma_{t,+}(f) = 0, \quad \mathrm{div}\, f|_{\partial\Omega} = 0.$$

In the case of smooth boundary, the form domain of the interior relative Laplace operator is contained in $H^1(\Omega, \mathbb{C}^3)$. In the more general Lipschitz case this is no longer true, but it is known that the form domain is contained in $H^{1/2}(\Omega, \mathbb{C}^3)$; see [Costabel 1990, Theorem 2] and also [Mitrea and Mitrea 2002]. This is compactly embedded in $L^2(\Omega, \mathbb{C}^3)$, and therefore the interior relative Laplace operator has purely discrete spectrum. We have the classical Hodge–Helmholtz decomposition

$$L^2(\Omega) = \mathcal{H}^1(\Omega) \oplus \overline{\mathrm{rg}(\mathrm{grad}_{\min})} \oplus \overline{\mathrm{rg}(\mathrm{curl}_{\max})}$$

into an orthogonal direct sum. Here $\mathcal{H}^1(\Omega) = \ker(\Delta_{\Omega,\mathrm{rel}})$ is the finite-dimensional space of harmonic vector fields satisfying the relative boundary conditions. We will see in Section 3.3 that in fact the assumption that M is connected implies that $\mathcal{H}^1(\Omega) = \{0\}$.

We now describe the spectrum of the relative Laplace operator. On Ω we can choose an orthonormal basis (v_j) of Dirichlet eigenfunctions v_j in the domain of the Dirichlet Laplacian with eigenvalues λ_j^2:

$$-\Delta v_j = \lambda_{D,j}^2 v_j, \quad \lambda_{D,j} > 0, \quad v_j \in \{v \in H_0^1(\Omega, \mathbb{C}^3) \mid \nabla v \in H(\mathrm{div}, \Omega)\}.$$

We have $\lambda_j \to \infty$, and we arrange the eigenfunctions such that $\lambda_j \nearrow \infty$. Then $(1/\lambda_j)\nabla v_j$ form an orthonormal basis of eigenfunctions in $\overline{\mathrm{rg}(\mathrm{grad}_{\min})}$ of $-\Delta_\Omega$ with eigenvalues λ_j^2. One has the usual Weyl law for Lipschitz domains which can easily be inferred from the Weyl law for smooth domains using domain monotonicity and an approximation by smooth domains:

$$\lambda_{D,k} \sim \left(\frac{6\pi^2}{\mathrm{Vol}(\Omega)}\right)^{1/3} k^{1/3}, \quad k \to \infty.$$

The space $\overline{\mathrm{rg}(\mathrm{curl}_{\max})}$ on the other hand is the closure of the subspace spanned by ϕ_j, where (ϕ_j) is an orthonormal basis in $\ker(\mathrm{div}_{\max}) \subset L^2(\Omega, \mathbb{C}^3)$ satisfying the eigenvalue equation

$$-\Delta_{\Omega,\mathrm{rel}}\phi_j = \mu_j^2 \phi_j, \quad \mathrm{div}\,\phi_j = 0,$$

with boundary condition $\gamma_t(\phi_j) = 0$. Therefore 0 is not an eigenvalue. The numbers $\mu_j > 0$ are the Maxwell eigenvalues, and we again assume these are arranged such that $\mu_j \nearrow \infty$. The Maxwell eigenvalues are known to satisfy a Weyl law (see [Birman and Solomyak 1987] for Lipschitz domains, but also [Filonov 2013] and references for a general statement in arbitrary dimension):

$$\mu_k \sim \left(\frac{3\pi^2}{\mathrm{Vol}(\Omega)}\right)^{1/3} k^{1/3}, \quad k \to \infty.$$

The family $(\phi_j)_{\mu_j > 0}$ then forms an orthonormal basis in $\overline{\mathrm{rg}(\mathrm{curl}_{\max})}$ consisting of eigenfunctions of $-\Delta_{\Omega,\mathrm{rel}}$ with nonzero eigenvalues μ_j^2. Summarising, there is an orthonormal basis of eigenfunctions $\Delta_{\Omega,\mathrm{rel}}$ of the form

$$\left\{\frac{1}{\lambda_j}\,\mathrm{grad}\, v_j \;\middle|\; j \in \mathbb{N}\right\} \cup \{\phi_j \mid \mu_j > 0\},$$

where v_j are the Dirichlet eigenfunctions and ϕ_j the Maxwell eigenfunctions with Maxwell eigenvalues μ_j.

3.2. ***The absolute Laplacian.*** It will also be convenient to consider another operator $\Delta_{\Omega,\mathrm{abs}}$, which is defined by

$$-\Delta_{\Omega,\mathrm{abs}} = \mathrm{curl}_{\min}\,\mathrm{curl}_{\max} - \mathrm{grad}_{\max}\,\mathrm{div}_{\min},$$

with domain

$$\{f \in H_0(\mathrm{div}, \Omega) \cap H(\mathrm{curl}, \Omega) \mid \mathrm{div}\, f \in H^1(\Omega),\ \mathrm{curl}\, f \in H_0(\mathrm{curl}, \Omega)\}.$$

Again, it is known that the form domain is contained in $H^{1/2}(\Omega, \mathbb{C}^3)$ [Costabel 1990, Theorem 2] and the domain is therefore compactly embedded into $L^2(\Omega, \mathbb{C}^3)$. In the same way as for the relative Laplacian, there is an explicit description of the spectrum which we now give. Let (u_j) be an orthonormal basis consisting of eigenfunctions of the Neumann Laplacian with eigenvalues $\lambda_{N,j}$. Hence

$$-\Delta u_j = \lambda_{N,j}^2 u_j, \quad \partial_\nu u_j|_{\partial\Omega} = 0, \quad u_j \in \{u \in H^1(\Omega) \mid \nabla u \in H_0^1(\mathrm{div}, \Omega)\}.$$

Then the functions $(1/\lambda_{N,j})\nabla u_j$ form an orthonormal set consisting of eigenfunctions of $\Delta_{\Omega,\mathrm{abs}}$.

We can construct another orthogonal set (ψ_j) from the Maxwell eigenfunctions ϕ_j of the relative Laplace operator by defining

$$\psi_j = \frac{1}{\mu_j}\operatorname{curl}\phi_j.$$

Since the spectrum is discrete, standard Hodge theory applies for the absolute Laplacian, and we obtain an orthogonal decomposition

$$L^2(\Omega,\mathbb{C}^3) = \mathcal{H}^1_{\mathrm{abs}}(\Omega) \oplus \overline{\operatorname{span}\left\{\frac{1}{\lambda_{N,j}}\operatorname{grad} u_j\right\}} \oplus \overline{\operatorname{span}\{\psi_j\}},$$

where $\mathcal{H}^1_{\mathrm{abs}}(\Omega) = \ker\Delta_{\Omega,\mathrm{abs}}$. Unlike in the case of the relative Laplace operator, this space is in general nontrivial. We will in the following choose an orthonormal basis $(\psi_{0,k})_k$, where $1 \le k \le \dim(\mathcal{H}^1_{\mathrm{abs}}(\Omega))$. Therefore an orthonormal basis in $L^2(\Omega,\mathbb{C}^3)$ consisting of eigenfunctions of the absolute Laplacian is

$$\{\psi_{0,k} \mid 1 \le k \le \dim(H^1_{\mathrm{abs}}(\Omega))\} \cup \left\{\frac{1}{\lambda_{N,j}}\nabla u_j \;\Big|\; j \in \mathbb{N}\right\} \cup \{\psi_j \mid j \in \mathbb{N}\}.$$

3.3. ***Relation to singular and de Rham cohomology groups.*** Since Ω is an oriented smooth manifold, we have, by de Rham's theorem, a natural isomorphism identifying $H^p_{\mathrm{dR}}(\Omega,\mathbb{C})$ with $H^p_{\mathrm{sing}}(\Omega,\mathbb{C}) = H^p_{\mathrm{sing}}(\Omega,\mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{C}$. Hodge theory is also applicable for Lipschitz domains in the sense that the natural map from $\ker\Delta_{\Omega,\mathrm{abs}}$ to the first de Rham cohomology group $H^1_{\mathrm{dR}}(\Omega,\mathbb{C})$ is an isomorphism. This can for example be inferred from the statement of [Mitrea et al. 2001, Theorems 11.1 and 11.2] together with the universal coefficient theorem and de Rham's theorem. This theorem also applies to the absolute Laplacian on 2-forms as defined in [Mitrea et al. 2001]. Since this operator is obtained by conjugation of the relative Laplacian on 1-forms with the Hodge star operator $*$, we therefore have that $*\ker\Delta_{\Omega,\mathrm{rel}}$ is isomorphic to $H^2_{\mathrm{dR}}(\Omega,\mathbb{C})$. Because the inner product is nondegenerate on these spaces, we have the following nondegenerate dual pairing

$$\ker\Delta_{\Omega,\mathrm{rel}} \times (*\ker\Delta_{\Omega,\mathrm{rel}}) \to \mathbb{C}, \quad (f_1, f_2) \mapsto \int_\Omega f_1 \wedge f_2.$$

We also have, as a consequence of Poincaré duality, the nondegenerate dual pairing

$$H^1_{\mathrm{c,dR}}(\Omega,\mathbb{C}) \times H^2_{\mathrm{dR}}(\Omega,\mathbb{C}) \to \mathbb{C}, \quad (f_1, f_2) \mapsto \int_\Omega f_1 \wedge f_2.$$

This establishes an isomorphism $\ker\Delta_{\Omega,\mathrm{rel}} \to H^1_{\mathrm{c,dR}}(\Omega,\mathbb{C})$, which relates the harmonic forms to the de Rham cohomology groups with compact support. Since elements in $\ker\Delta_{\Omega,\mathrm{rel}}$ are not compactly supported, this map is defined indirectly by duality.

Our assumptions imply that in fact $H^1_{\mathrm{c,dR}}(\Omega,\mathbb{C})$ is trivial and therefore $\ker\Delta_{\Omega,\mathrm{rel}} = \{0\}$. This reflects the observation that a domain with connected exterior cannot have homologically nontrivial 2-cycles (inclusions).

Lemma 3.1. *Let U be an open C^0-domain with compact closure in $\mathbb{R}^d$ with $d \ge 2$ such that $\mathbb{R}^d \setminus \overline{U}$ is connected. Then $H^1_{\mathrm{c,dR}}(U) = \{0\}$.*

Proof. Let α be a smooth closed 1-form with compact support in U. By the Poincaré lemma, there is a smooth function $f : \mathbb{R}^d \to \mathbb{R}$ with $\alpha = \mathrm{d}f$. Since f is locally constant in the complement of the support of α, it must be constant in $\mathbb{R}^d \setminus \overline{U}$, as this set was assumed to be connected. By continuity, f is constant in $\mathbb{R}^d \setminus \overline{U}$ and, since locally constant, it is constant in a neighbourhood of $\mathbb{R}^d \setminus \overline{U}$. It follows that $f - c$ is compactly supported in U. Since $\alpha = \mathrm{d}(f - c)$, the class α vanishes $H^1_{\mathrm{c,dR}}(U)$, and therefore $H^1_{\mathrm{c,dR}}(U) = \{0\}$. □

4. Laplace operators on the exterior domain

As in the interior case, the operator

$$\mathrm{curl}_{\min} = \mathrm{curl}\,|_{H_0(\mathrm{curl},M)} : H_0(\mathrm{curl}, M) \to L^2(M, \mathbb{C}^3)$$

is a closed densely defined operator with adjoint

$$\mathrm{curl}_{\max} : H(\mathrm{curl}, M) \to L^2(M, \mathbb{C}^3).$$

It follows that $\mathrm{curl}_{\max}\,\mathrm{curl}_{\min}$ with domain

$$\{f \in H_0(\mathrm{curl}, M) \mid \mathrm{curl}\, f \in H(\mathrm{curl}, M)\}$$

is a nonnegative self-adjoint operator. Similarly, $\mathrm{div}_{\max} : H(\mathrm{div}, M) \to L^2(M)$ is a closed operator with adjoint $-\mathrm{grad}_{\min} : H^1_0(M) \to L^2(M)$. Therefore, the operator $-\mathrm{grad}_{\min}\,\mathrm{div}_{\max}$ is a nonnegative self-adjoint operator with domain

$$\{f \in H(\mathrm{div}, M) \mid \mathrm{div}\, f \in H_0(M)\}.$$

Their sum $-\Delta_{M,\mathrm{rel}} = \mathrm{curl}_{\max}\,\mathrm{curl}_{\min} - \mathrm{grad}_{\min}\,\mathrm{div}_{\max}$ then has domain

$$\{f \in H(\mathrm{div}, M) \cap H_0(\mathrm{curl}, M) \mid \mathrm{div}\, f \in H_0(M),\ \mathrm{curl}\, f \in H(\mathrm{curl}, M)\}.$$

The implied boundary conditions are the exterior relative boundary conditions

$$\gamma_{t,-}(f) = 0, \quad \mathrm{div}\, f|_{\partial\Omega} = 0.$$

The spectrum of the operator $\Delta_{M,\mathrm{rel}}$ consists of a finite multiplicity eigenvalue at 0 and a purely absolutely continuous part. This is the consequence of the finite-type meromorphic continuation of the resolvent and Rellich's theorem. We have described this in detail in [Strohmaier and Waters 2020] for smooth domains, but this part of the paper carries over to Lipschitz domains without change; see [Strohmaier and Waters 2022] for a discussion of this point. The absolutely continuous part of the spectrum can be described well by stationary scattering theory. For each $\Phi \in C^\infty(\mathbb{S}^2, \mathbb{C}^3)$ and $\lambda > 0$, there exists a unique generalised eigenfunction $E_\lambda(\Phi) \in C^\infty(M, \mathbb{C}^3)$ satisfying the boundary conditions of $\Delta_{M,\mathrm{rel}}$ near $\partial\Omega$ such that

$$(-\Delta - \lambda^2) E_\lambda(\Phi) = 0, \tag{7}$$

$$E_\lambda(\Phi) = \frac{\mathrm{e}^{-\mathrm{i}\lambda r}}{r}\Phi - \frac{\mathrm{e}^{\mathrm{i}\lambda r}}{r}\Psi_\lambda(\Phi) + O\left(\frac{1}{r^2}\right) \quad \text{for } r \to \infty \tag{8}$$

uniformly in the angular variables on the sphere for some $\Psi_\lambda(\Phi) \in C^\infty(\mathbb{S}^2, \mathbb{C}^3)$. The expansion (8) may be differentiated; see Proposition 2.6 and Appendix E in [Strohmaier and Waters 2020] for a justification. Here, satisfying the boundary conditions near $\partial\Omega$ means that $\chi E_\lambda(\Phi) \in \operatorname{dom}(\Delta_M)$ for any compactly supported smooth χ on M such that $\chi = 1$ near $\partial\Omega$.

The above implicitly defines the *scattering matrix* as a map $\widetilde{S}_\lambda : C^\infty(\mathbb{S}^2, \mathbb{C}^3) \to C^\infty(\mathbb{S}^2, \mathbb{C}^3)$ by $\Psi_\lambda(\Phi) = \tau \widetilde{S}_\lambda \Phi$, where $\tau : C^\infty(\mathbb{S}^2; \mathbb{C}^3) \to C^\infty(\mathbb{S}^2; \mathbb{C}^3)$ is the pullback of the antipodal map. It extends continuously as $\widetilde{S}_\lambda : L^2(\mathbb{S}^2, \mathbb{C}^3) \to L^2(\mathbb{S}^2, \mathbb{C}^3)$. The map $\tilde{A}_\lambda = \widetilde{S}_\lambda - \mathrm{id}$ is called the scattering amplitude. We have the equations

$$\operatorname{curl}\operatorname{curl} E_\lambda(\Phi) = \lambda^2 E_\lambda(\boldsymbol{r} \times \Phi \times \boldsymbol{r}), \quad \operatorname{div} E_\lambda(\Phi) = -\mathrm{i}\lambda E^0_\lambda(\boldsymbol{r} \cdot \Phi),$$

where $\boldsymbol{r}$ is the radius vector, i.e., the outward-pointing unit vector on the sphere. Here $E^0_\lambda(\boldsymbol{r} \cdot \Phi)$ is the generalised eigenfunction for the exterior Dirichlet problem on scalar-valued functions defined in an analogous way; see Proposition 4.7 in [Strohmaier and Waters 2022]. In particular this means that in the case that Φ is purely tangential, $\boldsymbol{r} \cdot \Phi = 0$, the generalised eigenfunction is a solution of the stationary Maxwell equation

$$\operatorname{curl}\operatorname{curl} E_\lambda(\Phi) = \lambda^2 E_\lambda(\Phi),$$
$$\operatorname{div} E_\lambda(\Phi) = 0$$

that satisfies the boundary conditions near $\partial\Omega$. These equations also imply that the scattering matrix is of the form

$$\widetilde{S}_\lambda = \begin{pmatrix} S^D_\lambda & 0 \\ 0 & S_\lambda \end{pmatrix}$$

if $L^2(\mathbb{S}^2, \mathbb{C}^3)$ is decomposed into $L^2(\mathbb{S}^2)\boldsymbol{r} \oplus L^2_{\tan}(\mathbb{S}^2, \mathbb{C}^3)$. Here $L^2_{\tan}(\mathbb{S}^2, \mathbb{C}^3)$ is the space of tangential square-integrable vector fields on the sphere. The operator S^D_λ is the scattering operator for scalar-valued functions with Dirichlet conditions imposed on $\partial\Omega$, and S_λ is the Maxwell scattering operator, describing the scattering of electromagnetic waves. Note that we have the weak Hodge–Helmholtz decomposition

$$L^2(M) = \mathcal{H}^1_{\mathrm{rel}}(M) \oplus \overline{\operatorname{rg}(\operatorname{grad}_{\min})} \oplus \overline{\operatorname{rg}(\operatorname{curl}_{\max})}, \tag{9}$$

which holds very generally in the abstract context of Hilbert complexes [Brüning and Lesch 1992]. The first summand is the discrete spectral subspace, and the splitting of its orthogonal complement into the last two subspaces corresponds to the above decomposition of the scattering matrix.

4.1. ***The exterior absolute Laplacian.*** In the same way as for the interior problem, there is also an exterior absolute Laplacian $\Delta_{M,\mathrm{abs}}$ defined by

$$-\Delta_{M,\mathrm{abs}} = \operatorname{curl}_{\min}\operatorname{curl}_{\max} - \operatorname{grad}_{\max}\operatorname{div}_{\min}.$$

The spectrum of $\Delta_{M,\mathrm{abs}}$ consists of a finite multiplicity eigenvalue at 0 and an absolutely continuous part. The absolutely continuous part is described by generalised eigenfunctions $E_{\mathrm{abs},\lambda}(\Phi)$ which are related to the generalised eigenfunctions $E_\lambda(\Phi)$ of the relative Laplacian by

$$E_{\mathrm{abs},\lambda}(\boldsymbol{r} \times \Phi) = -\frac{\mathrm{i}}{\lambda}\operatorname{curl} E_\lambda(\Phi). \tag{10}$$

One checks easily that

$$(-\Delta_{M,\mathrm{rel}}-\lambda^2)^{-1}\,\mathrm{curl}=\mathrm{curl}(-\Delta_{M,\mathrm{abs}}-\lambda^2)^{-1}$$

on the dense set of compactly supported smooth functions and, appropriately interpreted, extends by continuity to a larger space. This will allow us to reduce to statements about the absolute Laplace operator to statements about the relative Laplace operator. For the purposes of this paper, it will therefore not be necessary to introduce separate notation for the spectral decomposition. For example, the scattering matrix

$$\widetilde{S}_{\mathrm{abs},\lambda}=\begin{pmatrix} S_\lambda^N & 0\\ 0 & S_{\mathrm{abs},\lambda}\end{pmatrix}$$

for the absolute Laplacian is defined by the expansion of $E_{\mathrm{abs},\lambda}(\Phi)$. Here S_λ^N is the scattering matrix for the Neumann Laplace operator on M acting on functions. We then have the equation

$$S_{\mathrm{abs},\lambda}(g)=\boldsymbol{r}\times S_\lambda(g\times\boldsymbol{r}) \tag{11}$$

for $g\in L^2_{\tan}(\mathbb{S}^2,\mathbb{C}^3)$. This follows by applying curl to the expansion (8), the uniqueness of the generalised eigenfunctions, and (10).

5. The combined relative operators and the Birman–Krein formula

In the following, it will be convenient to combine the operators $\Delta_{M,\mathrm{rel}}$ and $\Delta_{\Omega,\mathrm{rel}}$ into a single operator acting on the Hilbert space $L^2(\mathbb{R}^3,\mathbb{C}^3)$. We have $L^2(\mathbb{R}^3,\mathbb{C}^3)=L^2(M,\mathbb{C}^3)\oplus L^2(\Omega,\mathbb{C}^3)$, and we define the operator $\Delta_{\mathrm{rel}}:=\Delta_{M,\mathrm{rel}}\oplus\Delta_{\Omega,\mathrm{rel}}$. In contrast to this, we also have the free Laplace operator Δ_{free} with domain $H^2(\mathbb{R}^3,\mathbb{C}^3)$. Following the paper [Hanisch et al. 2022], on the relative trace we also define the operator $\Delta_{j,\mathrm{rel}}$ for each boundary component Ω_j. This will correspond to the operator Δ_{rel} when all the other boundary components are absent, i.e., when $\Omega=\Omega_j$. As in [Hanisch et al. 2022], we would like to consider an analogue of the relative trace for the Laplace operator acting on divergence-free vector fields. In this section we assume that $f\in\mathcal{S}(\mathbb{R})$ is an even Schwartz function, but later on we will focus on another function class. We would like to compute the relative trace

$$\begin{aligned}\mathrm{tr}\Bigg(\mathrm{curl}\,\mathrm{curl}\Bigg(f((-\Delta_{\mathrm{rel}})^{1/2})-f((-\Delta_{\mathrm{free}})^{1/2})-\Bigg(\sum_{j=1}^N f((-\Delta_{j,\mathrm{rel}})^{1/2})-f((-\Delta_{\mathrm{free}})^{1/2})\Bigg)\Bigg)\Bigg)\\ =\mathrm{tr}\Bigg(\mathrm{curl}\,\mathrm{curl}\Bigg(f((-\Delta_{\mathrm{rel}})^{1/2})-\sum_{j=1}^N f((-\Delta_{j,\mathrm{rel}})^{1/2})+(N-1)f((-\Delta_{\mathrm{free}})^{1/2})\Bigg)\Bigg),\end{aligned}$$

which is the trace of the operator

$$D_{\mathrm{rel},f}=\mathrm{curl}\,\mathrm{curl}\Bigg(f((-\Delta_{\mathrm{rel}})^{1/2})-f((-\Delta_{\mathrm{free}})^{1/2})-\Bigg(\sum_{j=1}^N f((-\Delta_{j,\mathrm{rel}})^{1/2})-f((-\Delta_{\mathrm{free}})^{1/2})\Bigg)\Bigg).$$

We have the following Birman–Krein-type formula, proved recently in [Strohmaier and Waters 2022] and its simple consequence for the relative trace.

Theorem 5.1 [Strohmaier and Waters 2022, Theorem 1.5]. *Let $f \in C_0^\infty(\mathbb{R})$ be an even function. Then the operator*

$$\operatorname{curl}\operatorname{curl}(f((-\Delta_{\mathrm{rel}})^{1/2}) - f((-\Delta_{\mathrm{free}})^{1/2}))$$

extends to a trace-class operator on $L^2(\mathbb{R}^3, \mathbb{C}^3)$, and its trace equals

$$\operatorname{tr}(\operatorname{curl}\operatorname{curl}(f((-\Delta_{\mathrm{rel}})^{1/2}) - f((-\Delta_{\mathrm{free}})^{1/2}))) = \frac{1}{2\pi\mathrm{i}}\int_0^\infty \lambda^2 \operatorname{tr}(S_\lambda^{-1}(S_\lambda)')f(\lambda)\,\mathrm{d}\lambda + \sum_{j=1}^\infty f(\mu_j)\mu_j^2.$$

Moreover,

$$\operatorname{tr}(D_f) = -\int_0^\infty \xi_D(\lambda)(f(\lambda)\lambda^2)'\,\mathrm{d}\lambda,$$

where

$$\xi_D(\lambda) = \frac{1}{2\pi\mathrm{i}}\log\frac{\det S_\lambda}{\det(S_{1,\lambda})\cdots\det(S_{N,\lambda})}.$$

A similar statement holds for the absolute Laplacian. Using (11) and

$$\operatorname{curl} f((-\Delta_{\mathrm{rel}})^{1/2}) = f((-\Delta_{\mathrm{abs}})^{1/2})\operatorname{curl}$$

one obtains

$$\begin{aligned}\operatorname{tr}(\operatorname{curl}(f((-\Delta_{\mathrm{rel}})^{1/2}) - f((-\Delta_{\mathrm{free}})^{1/2}))\operatorname{curl}) &= \operatorname{tr}(\operatorname{curl}\operatorname{curl}(f((-\Delta_{\mathrm{abs}})^{1/2}) - f((-\Delta_{\mathrm{free}})^{1/2}))) \\ &= \operatorname{tr}(\operatorname{curl}\operatorname{curl}(f((-\Delta_{\mathrm{rel}})^{1/2}) - f((-\Delta_{\mathrm{free}})^{1/2}))) \\ &= \frac{1}{2\pi\mathrm{i}}\int_0^\infty \lambda^2 \operatorname{tr}(S_\lambda^{-1}(S_\lambda)')f(\lambda)\,\mathrm{d}\lambda + \sum_{j=1}^\infty f(\mu_j)\mu_j^2.\end{aligned}$$

The Birman–Krein formula can be proved for a slightly larger function class than the space of even Schwartz functions, but nondecaying functions are not admissible. The rest of the paper is devoted to dealing with exactly the trace-class properties of D_f when f is in a different function class that contains possibly growing functions.

6. Maxwell boundary layer operators

Maxwell boundary layer theory for Lipschitz domains is a well-developed subject in mathematics, and in this section we summarise the material that we are going to need. The distributional kernel of the resolvent of the operator $(-\Delta_{\mathrm{free}} - \lambda^2)^{-1}$ is called the Green's function and in dimension three is given explicitly by

$$G_{\lambda,\mathrm{free}}(x, y) = \frac{1}{4\pi}\frac{\mathrm{e}^{\mathrm{i}\lambda|x-y|}}{|x-y|}. \tag{12}$$

Note that this kernel is holomorphic at 0. As usual we define the single layer potential operator $\widetilde{\mathcal{S}}_\lambda : H^{-1/2}(\partial\Omega) \to H^1_{\mathrm{loc}}(\mathbb{R}^3)$ by

$$\widetilde{\mathcal{S}}_\lambda = (-\Delta_{\mathrm{free}} - \lambda^2)^{-1}\gamma^*.$$

This is defined for any $\lambda \in \mathbb{C}$ and a holomorphic family of operators. The single layer operator is defined by taking the trace $\mathcal{S}_\lambda = \gamma_+\widetilde{\mathcal{S}}_\lambda = \gamma_+(-\Delta_{\mathrm{free}} - \lambda^2)^{-1}\gamma^*$. The interior trace γ_+ and the exterior trace γ_-

coincide on the range of $\widetilde{\mathcal{S}}_\lambda$ and therefore we could also have used γ_- to define this operator. The operator $\mathcal{S}_\lambda$ is a holomorphic family of maps $H^{-1/2}(\partial\Omega) \to H^{1/2}(\partial\Omega)$. Both operators $\widetilde{\mathcal{S}}_\lambda$ and $\mathcal{S}_\lambda$ act componentwise on $H^{-1/2}(\partial\Omega, \mathbb{C}^3)$ and define maps to $H^1_{\mathrm{loc}}(\mathbb{R}^3, \mathbb{C}^3)$ and $H^{-1/2}(\partial\Omega, \mathbb{C}^3)$, respectively. We will distinguish this notationally from the map on functions.

We will also need the double layer operator $\mathcal{K}_\lambda$ and its transpose (complex conjugate-adjoint) $\mathcal{K}^t_\lambda$. The latter is given by

$$\mathcal{K}^t_\lambda u = \tfrac{1}{2}(\gamma_+ \nabla_\nu \mathcal{S}_\lambda u + \gamma_- \nabla_\nu \mathcal{S}_\lambda u)$$

and defines a continuous map $\mathcal{K}^t_\lambda : H^{-1/2}(\partial\Omega) \to H^{-1/2}(\partial\Omega)$. Its transpose $\mathcal{K}_\lambda$ therefore defines a continuous map $\mathcal{K}_\lambda : H^{1/2}(\partial\Omega) \to H^{1/2}(\partial\Omega)$. The following jump relations are characteristic:

$$\gamma_+ \mathcal{S}_\lambda u = \gamma_- \mathcal{S}_\lambda u, \quad \gamma_\pm \nabla_\nu \mathcal{S}_\lambda u = \left(\mp\tfrac{1}{2} + \mathcal{K}^t_\lambda\right) u.$$

We have the following representation formulae for divergence-free solutions $\phi \in H(\mathrm{curl}, M) \oplus H(\mathrm{curl}, \Omega)$ of the vector-valued Helmholtz equation

$$(-\Delta - \lambda^2)\phi = 0, \quad \operatorname{div} \phi = 0$$

by single layer potential operators:

$$\phi|_M = -\mathrm{curl}\, \widetilde{\mathcal{S}}_\lambda(\gamma_{t,-}\phi) + \nabla \widetilde{\mathcal{S}}_\lambda(\gamma_{\nu,-}\phi) - \widetilde{\mathcal{S}}_\lambda(\gamma_{t,-}\, \mathrm{curl}\, \phi) \tag{13}$$

and likewise

$$\phi|_\Omega = -\mathrm{curl}\, \widetilde{\mathcal{S}}_\lambda(\gamma_{t,+}\phi) + \nabla \widetilde{\mathcal{S}}_\lambda(\gamma_{\nu,+}\phi) - \widetilde{\mathcal{S}}_\lambda(\gamma_{t,+}\, \mathrm{curl}\, \phi); \tag{14}$$

see Corollary 3.3 in [Mitrea et al. 1997]

In Maxwell theory one defines additional layer potential operators as follows. Let L be the distribution defined by

$$L_\lambda(x, y) = \mathrm{curl}_x\, \mathrm{curl}_x\, G_{\lambda,\mathrm{free}}(x, y).$$

This is the kernel of the operator $(-\Delta_{\mathrm{free}} - \lambda^2)^{-1}\, \mathrm{curl}\, \mathrm{curl} = \mathrm{curl}\, \mathrm{curl}(-\Delta_{\mathrm{free}} - \lambda^2)^{-1}$. It is again holomorphic at $\lambda = 0$ as a kernel. The corresponding operator L_λ is related to the operator

$$(\lambda^2 + \mathrm{grad}\, \mathrm{div})(-\Delta_{\mathrm{free}} - \lambda^2)^{-1},$$

whose distributional integral kernel equals the so-called dyadic Green's function

$$K_\lambda(x, y) = (\lambda^2 + \mathrm{grad}_x\, \mathrm{div}_x) \frac{1}{4\pi} \frac{\mathrm{e}^{\mathrm{i}\lambda|x-y|}}{|x-y|},$$

which is more commonly used in computational electrodynamics. However, we also have the equality

$$L_\lambda(x, y) - K_\lambda(x, y) = \delta(x - y);$$

hence the kernels agree outside the diagonal. We define now the *Maxwell single layer potential operator* for $u \in H^{1/2}(\partial\Omega, \mathbb{C}^3) \cap L^2_{\tan}(\partial\Omega)$ as

$$u \mapsto \widetilde{\mathcal{L}}_\lambda u, \quad (\widetilde{\mathcal{L}}_\lambda u)(x) = \int_{\partial\Omega} L_\lambda(x, y) u(y)\, \mathrm{d}y = \int_{\partial\Omega} K_\lambda(x, y) u(y)\, \mathrm{d}y.$$

Therefore this can also be written as $\widetilde{\mathcal{L}}_\lambda u = \operatorname{curl}\operatorname{curl}\widetilde{\mathcal{S}}_\lambda u$. Similarly one defines the Maxwell magnetic layer potential operator $\widetilde{\mathcal{M}}_\lambda$ as $\widetilde{\mathcal{M}}_\lambda u = \operatorname{curl}\widetilde{\mathcal{S}}_\lambda u$. For all $\lambda \in \mathbb{C}$, these maps extend continuously to maps as follows:

$$\widetilde{\mathcal{L}}_\lambda : H^{-1/2}(\mathrm{Div}, \partial\Omega) \to H_{\mathrm{loc}}(\mathrm{curl}, M) \oplus H_{\mathrm{loc}}(\mathrm{curl}, \Omega),$$
$$\widetilde{\mathcal{M}}_\lambda : H^{-1/2}(\mathrm{Div}, \partial\Omega) \to H_{\mathrm{loc}}(\mathrm{curl}, M) \oplus H_{\mathrm{loc}}(\mathrm{curl}, \Omega).$$

It will be convenient to distinguish notationally between the exterior part $\widetilde{\mathcal{M}}_{-,\lambda}$ and the interior part $\widetilde{\mathcal{M}}_{+,\lambda}$ of $\widetilde{\mathcal{M}}_\lambda$. The boundedness of these maps is established in [Kirsch and Hettlich 2015] for $\operatorname{Im}\lambda \geq 0,\ \lambda \neq 0$, but these maps extend to holomorphic families on the entire complex plane as we will see later.

The *Maxwell single layer operator* $\mathcal{L}_\lambda$ is then defined for all $\lambda \in \mathbb{C}$ as a map

$$\mathcal{L}_\lambda : H^{-1/2}(\mathrm{Div}, \partial\Omega) \to H^{-1/2}(\mathrm{Div}, \partial\Omega), \quad u \mapsto \gamma_t \widetilde{\mathcal{L}}_\lambda$$

and is a holomorphic family of bounded operators on $H^{-1/2}(\mathrm{Div}, \partial\Omega)$ in λ. With respect to the above splitting, we then have

$$\widetilde{\mathcal{M}}_\lambda = \widetilde{\mathcal{M}}_{-,\lambda} \oplus \widetilde{\mathcal{M}}_{+,\lambda}.$$

One defines the *magnetic dipole operator* $\mathcal{M}_\lambda$ for all $\lambda \in \mathbb{C}$ by

$$\mathcal{M}_\lambda : H^{-1/2}(\mathrm{Div}, \partial\Omega) \to H^{-1/2}(\mathrm{Div}, \partial\Omega), \quad \mathcal{M}_\lambda = \tfrac{1}{2}(\gamma_t \widetilde{\mathcal{M}}_{-,\lambda} + \gamma_t \widetilde{\mathcal{M}}_{+,\lambda}).$$

By [Kirsch and Hettlich 2015, Theorem 5.52], this is a family of bounded operators on the space $H^{-1/2}(\mathrm{Div}, \partial\Omega)$ when $\operatorname{Im}\lambda > 0$. If $u = \widetilde{\mathcal{M}}a = \operatorname{curl}\widetilde{\mathcal{S}}_\lambda a$ then we have the jump conditions

$$\gamma_{t,\pm} u = \mp\tfrac{1}{2}a + \mathcal{M}_\lambda a, \quad \gamma_{t,\pm}\operatorname{curl} u = \mathcal{L}_\lambda a. \tag{15}$$

Moreover, the operator $\widetilde{\mathcal{L}}_\lambda a$ can be written as

$$\widetilde{\mathcal{L}}_\lambda a = \nabla\widetilde{\mathcal{S}}_\lambda \operatorname{Div} a + \lambda^2 \widetilde{\mathcal{S}}_\lambda a, \quad a \in H^{-1/2}(\mathrm{Div}, \partial\Omega). \tag{16}$$

We refer to [Kirsch and Hettlich 2015, Theorem 5.4] for both statements.

If $\operatorname{Im}\lambda \geq 0$ is nonzero then there exists a unique solution of the exterior boundary value problem for every $A \in H^{-1/2}(\mathrm{Div}, \partial\Omega)$, which satisfies the Silver–Müller radiation condition [Kirsch and Hettlich 2015, Theorem 5.64]. For the interior problem there exists a similar statement. If $\lambda \in \mathbb{C}\setminus\{0\}$ is not a Maxwell eigenvalue then there exists a unique solution of the interior boundary value problem for every $A \in H^{-1/2}(\mathrm{Div}, \partial\Omega)$. In both cases, if $\lambda \neq 0$ the solution can be written as a boundary layer potential of the form

$$E(x) = (\widetilde{\mathcal{L}}a)(x) = \operatorname{curl}^2\langle a, G_\lambda(x,\cdot)\rangle_{\partial\Omega}, \quad H(x) = \frac{\mathrm{i}\operatorname{curl} E}{-\lambda}, \quad x \notin \partial\Omega, \tag{17}$$

with the density $a \in H^{-1/2}(\mathrm{Div}, \partial\Omega)$, which satisfies $\mathcal{L}_\lambda a = A$; see again Theorem 5.60 in [Kirsch and Hettlich 2015].

The space of boundary data $(\gamma_t(E), \gamma_t(H))$ of solutions of Maxwell's equations is described by the Calderon projector. To describe this we first observe that, given $a, b \in H^{-1/2}(\mathrm{Div}, \partial\Omega)$, we obtain for any nonzero λ a solution of the interior Maxwell's equation $E, H \in H(\text{curl}, \Omega)$ given by

$$E = -\widetilde{\mathcal{M}}_\lambda a + \frac{1}{\mathrm{i}\lambda}\widetilde{\mathcal{L}}_\lambda b, \quad H = -\widetilde{\mathcal{M}}_\lambda b - \frac{1}{\mathrm{i}\lambda}\widetilde{\mathcal{L}}_\lambda a,$$

and therefore, using (15), the boundary data $(\gamma_t(E), \gamma_t(H))$ is described as

$$\begin{pmatrix}\gamma_t(E)\\ \gamma_t(H)\end{pmatrix} = \begin{pmatrix}\frac{1}{2} - \mathcal{M}_\lambda & \frac{1}{\mathrm{i}\lambda}\mathcal{L}_\lambda \\ -\frac{1}{\mathrm{i}\lambda}\mathcal{L}_\lambda & \frac{1}{2} - \mathcal{M}_\lambda\end{pmatrix}\begin{pmatrix}a\\ b\end{pmatrix}.$$

By the Stratton–Chu representation formula [Kirsch and Hettlich 2015, Theorem 5.49], we have that if (E, H) solves Maxwell's equations then E and H can be recovered from the boundary data as

$$E = -\widetilde{\mathcal{M}}_\lambda(\gamma_t E) + \frac{1}{\mathrm{i}\lambda}\widetilde{\mathcal{L}}_\lambda(\gamma_t H), \quad H = -\widetilde{\mathcal{M}}_\lambda(\gamma_t H) - \frac{1}{\mathrm{i}\lambda}\widetilde{\mathcal{L}}_\lambda(\gamma_t E).$$

Hence the operator

$$P_+ = \begin{pmatrix}\frac{1}{2} - \mathcal{M}_\lambda & \frac{1}{\mathrm{i}\lambda}\mathcal{L}_\lambda \\ -\frac{1}{\mathrm{i}\lambda}\mathcal{L}_\lambda & \frac{1}{2} - \mathcal{M}_\lambda\end{pmatrix}$$

acting on $H^{-1/2}(\text{Div}, \partial\Omega) \oplus H^{-1/2}(\text{Div}, \partial\Omega)$ is a projection onto the space of boundary data of solutions of Maxwell's equation in $H(\text{curl}, \Omega) \oplus H(\text{curl}, \Omega)$. This map is called the interior Calderon projector. In the same way, the exterior Calderon projector P_- acting on $H^{-1/2}(\text{Div}, \partial\Omega) \oplus H^{-1/2}(\text{Div}, \partial\Omega)$ is given by

$$P_- = \begin{pmatrix}\frac{1}{2} + \mathcal{M}_\lambda & -\frac{1}{\mathrm{i}\lambda}\mathcal{L}_\lambda \\ \frac{1}{\mathrm{i}\lambda}\mathcal{L}_\lambda & \frac{1}{2} + \mathcal{M}_\lambda\end{pmatrix}.$$

It projects onto the space of boundary data of solutions of Maxwell's equation in $H(\text{curl}, \Omega) \oplus H(\text{curl}, \Omega)$ when $\operatorname{Im}\lambda > 0$ and more generally solutions satisfying a radiation condition for nonzero real λ. As usual one has $P_+ + P_- = \text{id}$.

We now define the voltage-to-current mappings $\Lambda_\lambda^\pm : H^{-1/2}(\text{Div}, \partial\Omega) \to H^{-1/2}(\text{Div}, \partial\Omega)$ by

$$\Lambda_\lambda^\pm : \gamma_t(E) \to \gamma_t(H), \tag{18}$$

where E and H are solutions to the interior and exterior boundary value problem for the Maxwell system (1), respectively, whenever these solutions are unique. The graphs of $\Lambda_\lambda^\pm$ in $H^{-1/2}(\text{Div}, \partial\Omega) \oplus H^{-1/2}(\text{Div}, \partial\Omega)$ are therefore by definition the ranges of the Calderon projectors $P_\pm$. The voltage-to-current maps are henceforth the Maxwell analogues of the interior and exterior Helmholtz Dirichlet-to-Neumann maps.

The mapping Λ_λ^+ is well defined for any $\lambda \in \mathbb{C}$ which is not a Maxwell eigenvalue or 0. The mapping Λ_λ^- is well defined for all nonzero λ in the closed upper half-space. In this case these are bounded operators on $H^{-1/2}(\text{Div}, \partial\Omega)$. We will see later that these operators extend meromorphically to the complex plane. In anticipation of this we will not explicitly state the domains when dealing with algebraic identities. As a consequence of the symmetry $(E, H) \mapsto (H, -E)$ of the Maxwell system and the above relations, one obtains the formulae

$$(\Lambda_\lambda^\pm)^2 = -\text{id} \quad \text{and} \quad \mathcal{L}_\lambda = \mathrm{i}\lambda\Lambda_\lambda^\pm\big(\mp\tfrac{1}{2} + \mathcal{M}_\lambda\big) = -\mathrm{i}\lambda\big(\pm\tfrac{1}{2} + \mathcal{M}_\lambda\big)\Lambda_\lambda^\pm, \tag{19}$$

and as a consequence

$$-\mathrm{i}\lambda^{-1}\mathcal{L}_\lambda(\Lambda_\lambda^+ - \Lambda_\lambda^-) = \mathrm{id} \quad \text{and} \quad \mathcal{L}_\lambda^2 = -\lambda^2\big(-\tfrac{1}{2} + \mathcal{M}_\lambda\big)\big(\tfrac{1}{2} + \mathcal{M}_\lambda\big). \tag{20}$$

These are also manifestations of the Calderon projector being a projection mapping, i.e., $P_\pm^2 = P_\pm$. We refer to [Mitrea et al. 1997, Lemma 5.10] for these and more statements in the L^2-setting. Notice that we are using the opposite sign convention for $\widetilde{\mathcal{S}}_\lambda$ than in [Mitrea et al. 1997].

For later reference and completeness we also state the following identities.

Lemma 6.1. *For $A \in H^{-1/2}(\mathrm{Div}, \partial\Omega)$ and $f \in H^{1/2}(\partial\Omega)$, we have*

$$\operatorname{div} \widetilde{\mathcal{S}}_\lambda A = \widetilde{\mathcal{S}}_\lambda \operatorname{Div} A, \tag{21}$$

$$\operatorname{curl} \widetilde{\mathcal{S}}_\lambda \nu f = -\widetilde{\mathcal{S}}_\lambda(\nu \times \nabla f), \tag{22}$$

$$\operatorname{Div} \mathcal{M}_\lambda A = -\lambda^2 \nu \cdot \mathcal{S}_\lambda A - \mathcal{K}_\lambda^{\mathrm{t}}(\operatorname{Div} A), \tag{23}$$

$$(\nu \times \nabla)\mathcal{K}_\lambda f = \lambda^2 \nu \times \mathcal{S}_\lambda(\nu f) + \mathcal{M}_\lambda(\nu \times \nabla f), \tag{24}$$

$$(\nu \times \nabla)\mathcal{K}_0 f = \mathcal{M}_0(\nu \times \nabla f). \tag{25}$$

These identities were for example proved in [Mitrea et al. 1997, Lemmas 4.2, 4.3, 4.4, and 5.11] in slightly different function spaces containing the image of $C_0^\infty(\mathbb{R}^3, \mathbb{C}^3)$ under the tangential restriction map γ_t. Since $C_0^\infty(\mathbb{R}^3, \mathbb{C}^3)$ is a dense subspace in $H(\mathrm{curl}, \mathbb{R}^3)$, the space $\gamma_t C_0^\infty(\mathbb{R}^3, \mathbb{C}^3)$ is dense in $H^{-1/2}(\mathrm{Div}, \partial\Omega)$. Hence these equations extend by continuity to the claimed larger space if we use the continuous mapping properties of the potential layer operators. We note here that the gradient ∇ defines a continuous map $H^{1/2}(\partial\Omega) \to H^{-1/2}(\mathrm{Curl}, \partial\Omega)$ and the map $\nu \times \nabla$ is continuous from $H^{1/2}(\partial\Omega) \to H^{-1/2}(\mathrm{Div}, \partial\Omega)$.

Lemma 6.2. *The map $\mathcal{S}_\lambda$ satisfies $\mathcal{S}_\lambda^* = \mathcal{S}_{\bar\lambda}$, where the adjoint is taken with respect to the L^2-induced dual pairing between $H^{1/2}(\partial\Omega)$ and $H^{-1/2}(\partial\Omega)$. In other words it is its own transpose: $\mathcal{S}_\lambda^{\mathrm{t}} = \mathcal{S}_\lambda$. We also have $(\mathcal{L}_\lambda(\nu \times))^{\mathrm{t}} = \mathcal{L}_\lambda(\nu \times)$, i.e., $\mathcal{L}_\lambda$ is symmetric with respect to the bilinear form induced by $\langle \cdot, \nu \times \cdot \rangle$.*

Proof. The symmetry of the operator $\mathcal{S}_\lambda$ with respect to the real inner product are classical and follow from the symmetry properties of the integral kernel. See for example Theorem 5.44 in [Kirsch and Hettlich 2015]. The statement about $\mathcal{L}_\lambda^{\mathrm{t}}$ is Lemma 5.6.1 in [Kirsch and Hettlich 2015]. □

The following lemma is implicit in [Kirsch and Hettlich 2015].

Lemma 6.3. *The operator $\pm\frac{1}{2} + \mathcal{M}_\lambda$ is for any $\operatorname{Im}\lambda > 0$ an isomorphism from $H^{-1/2}(\mathrm{Div}, \partial\Omega)$ to $H^{-1/2}(\mathrm{Div}, \partial\Omega)$.*

Proof. Assume that $\operatorname{Im}\lambda > 0$. It was shown in [Kirsch and Hettlich 2015, Theorem 5.52 (d)] that $\mathcal{L}_\lambda$ is invertible modulo compact operators and therefore is a Fredholm operator of index 0. Moreover, by [Kirsch and Hettlich 2015, Theorem 5.59], we know that $\mathcal{L}_\lambda$ is injective and hence invertible. Since $\Lambda_\lambda^\pm$ are invertible, it follows from (19) that $\pm\frac{1}{2} + \mathcal{M}_\lambda$ is also. As usual the inverse is continuous by the open mapping theorem. □

Invertibility of operators $\pm\frac{1}{2} + \mathcal{M}_\lambda$ on several other L^p-spaces has been shown in the works of M. Mitrea and D. Mitrea; see, for example, Theorem 4.1 in [Mitrea 1995].

Proposition 6.4. *The family* $\pm\frac{1}{2}+\mathcal{M}_\lambda$ *is a holomorphic family of Fredholm operators of index* 0 *from* $H^{-1/2}(\mathrm{Div}, \partial\Omega)$ *to* $H^{-1/2}(\mathrm{Div}, \partial\Omega)$. *The derivative* $\mathcal{M}'_\lambda = \frac{\mathrm{d}}{\mathrm{d}\lambda}\mathcal{M}_\lambda$ *is a continuous family of Hilbert–Schmidt operators on* $H^{-1/2}(\mathrm{Div}, \partial\Omega)$.

Proof. We will show that $\mathcal{M}_\lambda$ is complex-differentiable as a family of bounded operators $H^{-1/2}(\mathrm{Div}, \partial\Omega)$ and its derivative is compact. The first part of the theorem then follows from

$$\big(\pm\tfrac{1}{2}+\mathcal{M}_\lambda\big)-\big(\pm\tfrac{1}{2}+\mathcal{M}_\mathrm{i}\big)=\int_\mathrm{i}^\lambda \mathcal{M}'_\mu \,\mathrm{d}\mu$$

and the proposition above. We have used here that Fredholm operators are stable under compact perturbations; see, for example, Lemma 8.6 in [Shubin 1987]. It is therefore sufficient to show that $\mathcal{M}'_\lambda$ exists and is Hilbert–Schmidt. First choose a compactly supported smooth cut-off function χ supported in $(-2R, 2R)$ and which equals 1 on $[-R, R]$ for sufficiently large $R > 0$. The integral kernel of $\widetilde{\mathcal{M}}_{\pm,\lambda}$ is given by $\mathrm{curl}_x\, \mathrm{e}^{\mathrm{i}\lambda|x-y|}/(4\pi|x-y|)$. For x not far from $\partial\Omega$, we can replace this by

$$\chi(|x-y|)\,\mathrm{curl}_x \frac{\mathrm{e}^{\mathrm{i}\lambda|x-y|}}{4\pi|x-y|}.$$

Consider the Taylor expansion

$$\chi(|x-y|)\,\mathrm{curl}_x \frac{\mathrm{e}^{\mathrm{i}\lambda|x-y|}}{4\pi|x-y|}=\chi(|x-y|)\,\mathrm{curl}_x \frac{\mathrm{e}^{\mathrm{i}\mu|x-y|}}{4\pi|x-y|}+\chi(|x-y|)\,\mathrm{curl}_x \frac{\mathrm{e}^{\mathrm{i}\mu|x-y|}}{4\pi}(\lambda-\mu) \\ +\chi(|x-y|)T_\lambda(x-y)(\lambda-\mu)^2$$

with remainder term T_λ. This gives rise to an operator expansion

$$\mathcal{M}_{\pm,\lambda}=\mathcal{M}_{\pm,\mu}+A_\lambda(\lambda-\mu)+B_\lambda(\lambda-\mu)^2.$$

Here the operators A_λ and B_λ arise as compositions as

$$H^{-1/2}(\mathrm{Div}, \partial\Omega)\xrightarrow{\gamma_T^*} H^{-1}(U)\xrightarrow{K_A, K_B} H^1(\mathbb{R}^d)\longrightarrow H(\mathrm{curl}, M)\xrightarrow{\gamma_t} H^{-1/2}(\mathrm{Div}, \partial\Omega),$$

where K_A or K_B is the integral operator with kernel $\chi(|x-y|)\,\mathrm{curl}_x\, \mathrm{e}^{\mathrm{i}\mu|x-y|}/(4\pi)$ or $\chi(|x-y|)T_\lambda(x-y)$, respectively. Here U is a bounded open neighbourhood of $\partial\Omega$. It is now sufficient to show that the operators K_A and K_B are bounded as Hilbert–Schmidt operators. In view of Lemma A.3, we would like to bound the $H^2(\mathbb{R}^d\times\mathbb{R}^d)$-norm of the kernels. Taking two derivatives gives in both cases an integrable convolution kernel in $L^1(\mathbb{R}^d)$ and the $H^2(\mathbb{R}^d\times\mathbb{R}^d)$-norm is then, by Young's inequality, bounded by the L^1-norm of this kernel. □

Definition 6.5. The spaces $\mathcal{B}^{\pm}_{\partial\Omega}\subset H^{-1/2}(\mathrm{Div}, \partial\Omega)$ of interior/exterior boundary data of absolute harmonic forms are defined as

$$\mathcal{B}^{+}_{\partial\Omega}=\{\gamma_{t,+}(\phi)\mid\phi\in\mathcal{H}^1_{\mathrm{abs}}(\Omega)\}\quad\text{and}\quad\mathcal{B}^{-}_{\partial\Omega}=\{\gamma_{t,-}(\phi)\mid\phi\in\mathcal{H}^1_{\mathrm{abs}}(M)\}.$$

It is then obvious that $\mathcal{B}^{+}_{\partial\Omega}=\mathcal{B}^{+}_{\partial\Omega_1}\oplus\cdots\oplus\mathcal{B}^{+}_{\partial\Omega_N}$ with respect to the decomposition

$$H^{-1/2}(\mathrm{Div}, \partial\Omega)=H^{-1/2}(\mathrm{Div}, \partial\Omega_1)\oplus\cdots\oplus H^{-1/2}(\mathrm{Div}, \partial\Omega_N).$$

This is not true for the space $\mathcal{B}^-_{\partial\Omega}$. The spaces $\mathcal{B}^+_{\partial\Omega}$ are also known to be subspaces of $L^2(\partial\Omega, \mathbb{C}^3)$, see [Mitrea et al. 2001, Theorem 11.2], but this will not be needed.

The following was announced by D. Mitrea [2000] in the context of L^p-spaces, with p sufficiently close to 2. It is a reflection of general Hodge theory for Lipschitz domains, and we restate and prove this here for our choice of function spaces.

Proposition 6.6. *We have*

$$\mathcal{B}^{\pm}_{\partial\Omega} = \ker\big(\pm\tfrac{1}{2} + \mathcal{M}_0\big) \subset H^{-1/2}(\mathrm{Div}\, 0, \partial\Omega). \tag{26}$$

Proof. We will prove this only for $\mathcal{B}^+_{\partial\Omega}$ since the proof for $\mathcal{B}^-_{\partial\Omega}$, when supplemented by Lemma 3.1, is exactly the same. Suppose that $u \in \ker\big(\frac{1}{2} + \mathcal{M}_0\big)$, and define $\phi = -\widetilde{\mathcal{M}}_0 u$. Then ϕ is divergence-free and harmonic on M and on Ω. The jump relations (15) hold by analytic continuation for all $\lambda \in \mathbb{C}$, and they show that $\gamma_{t,-}\phi = 0$, $\gamma_{t,+}\phi = u$, and $\gamma_{\nu,+}\phi = \gamma_{\nu,-}\phi$. We first show that $q = \gamma_{\nu,+}\phi$ vanishes, thus establishing the inclusion $\phi|_\Omega \in \mathcal{H}^1_{\mathrm{abs}}(\Omega)$, $\gamma_{t,+}\phi = u$. The proof uses similar arguments as in [Verchota 1984] and reflects the mapping properties of the adjoint double layer operator.

On the exterior, ϕ is a harmonic vector field satisfying relative boundary conditions. The decay of $\mathrm{curl}(1/|x-y|)$ implies that ϕ is square-integrable. This shows that $\mathrm{curl}\,\phi$ must vanish in the exterior. From the representation (13) we obtain, using the jump relations and $\gamma_{t,-}\phi = 0$,

$$\phi|_M = \nabla \widetilde{\mathcal{S}}_0 q.$$

Taking the normal trace, one gets $q = \gamma_{\nu,-}\nabla\widetilde{\mathcal{S}}_0 q$. Taking the tangential trace, one obtains, from the jump relations,

$$\gamma_{t,-}\nabla\widetilde{\mathcal{S}}_0(\gamma_{\nu,-}\phi) = \nabla_{\partial\Omega}\mathcal{S}_0 q = 0.$$

This shows that $w = \mathcal{S}_0 q$ is locally constant (and in particular in $L^2(\partial\Omega)$). Using the divergence theorem on the interior of each of the components Ω_j one finds that $\int_{\partial\Omega_j} q = 0$. This gives $\langle \mathcal{S}_0 q, q\rangle_{L^2(\partial\Omega)} = 0$ and therefore

$$\langle \mathcal{S}_0 q, \nabla_\nu \widetilde{\mathcal{S}}_0 q\rangle_{L^2(\partial\Omega)} = 0.$$

Since this is the boundary term in the integration by parts formula for $\langle \nabla\widetilde{\mathcal{S}}_0 q, \nabla\widetilde{\mathcal{S}}_0 q\rangle = 0$, we can then imply that $\widetilde{\mathcal{S}}_0 q$ is constant. Since it decays we must have $\widetilde{\mathcal{S}}_0 q = 0$ and therefore $\mathcal{S}_0 q = 0$. By invertibility of the single layer operator, one obtains $q = 0$ as claimed.

We now show the inclusion in the other direction. Suppose that $u = \gamma_{t,+}(h)$, where $h \in \mathcal{H}^1_{\mathrm{abs}}(\Omega)$. This means in particular that h is divergence-free, curl-free, and $\gamma_{\nu,+}h = 0$. Taking the tangential trace in representation (14), we obtain

$$u = \big(\tfrac{1}{2} - \mathcal{M}_0\big)u,$$

and therefore $\big(\frac{1}{2} + \mathcal{M}_0\big)u = 0$ as claimed.

It finally remains to show that $\{\gamma_{t,+}(\phi) \mid \phi \in \mathcal{H}^1_{\mathrm{abs}}(\Omega)\} \subset H^{-1/2}(\mathrm{Div}\,0, \partial\Omega)$. This follows immediately from the fact that $\mathrm{curl}\,\phi = 0$ and $\mathrm{Div} \circ \gamma_{t,+} = -\gamma_{\nu,+} \circ \mathrm{curl}$. ☐

A similar but easier argument applies to other elements of the real line and gives the following.

Proposition 6.7. *If* $\lambda \in \mathbb{R} \setminus \{0\}$ *then* $\ker\big(\frac{1}{2} + \mathcal{M}_\lambda\big) = \{0\}$ *when* $|\lambda| \neq \mu_k$ *for all* $k \in \mathbb{N}$, *i.e.*, $|\lambda|$ *is not a Maxwell eigenvalue. Moreover*,

$$\ker\big(\tfrac{1}{2} + \mathcal{M}_{\mu_k}\big) = \{\gamma_{t,+}(u) \mid u \in V_{\mu_k}\}, \tag{27}$$

with V_{μ_k} *the eigenspace of* $\Delta_{\Omega,\mathrm{abs}}$ *for the eigenvalue* μ_k^2 *on the subspace of divergence-free vector fields.*

Proof. The proof is very similar to the proof of the previous proposition, and we therefore only give a brief sketch. As before let $u \in \ker\big(\frac{1}{2} + \mathcal{M}_{\mu_k}\big)$ and $\phi = -\widetilde{\mathcal{M}}_\lambda u$. Then $\phi|_M$ is a purely incoming or outgoing solution of the Helmholtz equation (see, e.g., [Strohmaier and Waters 2020, Appendix C] for details) satisfying relative boundary conditions. It therefore vanishes. By the jump relations (15), the function $\phi|_\Omega$ satisfies absolute boundary conditions, is divergence-free, and is a Maxwell eigenfunction with Maxwell eigenvalue μ_k. Moreover, again by the jump relation, $\gamma_{t,+}\phi = u$. This proves the inclusion in one direction. Conversely, assume that $u = \gamma_{t,+}\phi$, where ϕ is divergence-free, satisfies absolute boundary conditions, and $-\Delta\phi = \mu_k^2\phi$. Taking the tangential trace in representation (14), we obtain

$$u = \big(\tfrac{1}{2} - \mathcal{M}_{\mu_k}\big)u,$$

and therefore $\big(\frac{1}{2} + \mathcal{M}_{\mu_k}\big)u = 0$ as claimed. □

7. Estimates and low-energy expansions for the layer potential operators

For $0 < \epsilon < \frac{\pi}{2}$, define the sector $\mathfrak{D}_\epsilon$ in the upper half-plane by

$$\mathfrak{D}_\epsilon := \{z \in \mathbb{C} \mid \epsilon < \arg(z) < \pi - \epsilon\}.$$

The next proposition establishes properties of the single layer operator $\widetilde{\mathcal{S}}_\lambda$ and the operator $\widetilde{\mathcal{L}}_\lambda$.

Proposition 7.1. *For* $\epsilon \in \big(0, \frac{\pi}{2}\big)$ *and for all* $\lambda \in \mathfrak{D}_\epsilon$, *we have the following bounds*:

(1) *Let* $\Omega_0 \subset \mathbb{R}^d$ *be an open subset and assume* $\delta = \operatorname{dist}(\Omega_0, \partial\Omega) > 0$. *Let* $0 < \delta' < \delta$. *Assume that* $\varphi \in C_b^1(\mathbb{R}^3)$ *is bounded with bounded derivative and supported in* Ω_0. *For each* $\lambda \in \mathfrak{D}_\epsilon$, *the operators*

$$\begin{aligned}
\varphi\widetilde{\mathcal{L}}_\lambda &: H^{-1/2}(\mathrm{Div}, \partial\Omega) \to H(\mathrm{curl}, \mathbb{R}^3),\\
\varphi\widetilde{\mathcal{S}}_\lambda &: H^{-1/2}(\partial\Omega) \to H^1(\mathbb{R}^3),\\
\varphi\nabla\widetilde{\mathcal{S}}_\lambda &: H^{-1/2}(\partial\Omega) \to L^2(\mathbb{R}^3),\\
\varphi\widetilde{\mathcal{M}}_\lambda &: H^{-1/2}(\mathrm{Div}, \partial\Omega) \to H(\mathrm{div}, \mathbb{R}^3)
\end{aligned}$$

are Hilbert–Schmidt operators. There exists $C_{\delta',\epsilon} > 0$ *such that, for all* $\lambda \in \mathfrak{D}_\epsilon$, *we have the following bounds on the Hilbert–Schmidt norms between these spaces*:

$$\|\varphi\widetilde{\mathcal{L}}_\lambda\|_{\mathrm{HS}} \le C_{\delta',\epsilon}\mathrm{e}^{-\delta' \operatorname{Im}\lambda}, \tag{28}$$

$$\|\varphi\widetilde{\mathcal{S}}_\lambda\|_{\mathrm{HS}} \le |\lambda|^{-1/2}C_{\delta',\epsilon}\mathrm{e}^{-\delta' \operatorname{Im}\lambda}, \tag{29}$$

$$\|\varphi\nabla\widetilde{\mathcal{S}}_\lambda\|_{\mathrm{HS}} \le C_{\delta',\epsilon}\mathrm{e}^{-\delta' \operatorname{Im}\lambda}, \tag{30}$$

$$\|\varphi\widetilde{\mathcal{M}}_\lambda\|_{\mathrm{HS}} \le C_{\delta',\epsilon}\mathrm{e}^{-\delta' \operatorname{Im}\lambda}, \tag{31}$$

$$\|\varphi\widetilde{\mathcal{S}}_\lambda \,\mathrm{Div}\,\|_{\mathrm{HS}} \le C_{\delta',\epsilon}\mathrm{e}^{-\delta' \operatorname{Im}\lambda}. \tag{32}$$

(2) *For* $\lambda \in \mathfrak{D}_\epsilon$, *we have the operator norm bound*

$$\|\widetilde{\mathcal{L}}_\lambda\|_{H^{-1/2}(\mathrm{Div},\partial\Omega)\to H(\mathrm{curl},\mathbb{R}^3)} \le C_\epsilon(1+|\lambda|^2). \tag{33}$$

(3) *For* $\lambda \in \mathfrak{D}_\epsilon$, *we have the operator norm bound*

$$\|\widetilde{\mathcal{M}}_\lambda\|_{H^{-1/2}(\mathrm{Div},\partial\Omega)\to L^2(\mathbb{R}^3,\mathbb{C}^3)} \le C_\epsilon. \tag{34}$$

(4) *For* $\lambda \in \mathfrak{D}_\epsilon$, *we have the operator norm bounds*

$$\|\widetilde{\mathcal{S}}_\lambda\|_{H^{-1/2}(\partial\Omega)\to H^1(\mathbb{R}^3)} \le C_\epsilon|\lambda|^{-1/2}(1+|\lambda|^{1/2}), \tag{35}$$

$$\|\nabla\widetilde{\mathcal{S}}_\lambda\|_{H^{-1/2}(\partial\Omega)\to L^2(\mathbb{R}^3,\mathbb{C}^3)} \le C_\epsilon, \tag{36}$$

(5) *On the space of functions of mean zero,* $H_0^{-1/2}(\partial\Omega) = \{u \in H_0^{-1/2}(\partial\Omega) \mid \langle u, 1\rangle = 0\}$, *we have for* $\lambda \in \mathfrak{D}_\epsilon$ *the improved estimate*

$$\|\widetilde{\mathcal{S}}_\lambda|_{H_0^{-1/2}(\partial\Omega)}\|_{\mathrm{HS}} \le C_\epsilon. \tag{37}$$

Proof. The operator $\varphi\widetilde{\mathcal{L}}_\lambda$ can be written as $\varphi\,\mathrm{curl}\,\mathrm{curl}\,G_{\lambda,0}\gamma_T^*$. Similarly, we have $\varphi\widetilde{\mathcal{M}}\gamma_T^*$ and $\varphi\widetilde{\mathcal{S}}\gamma_T^*$. We choose a bounded open neighbourhood U of $\partial\Omega$ such that $\mathrm{dist}(\Omega_0, U) > \delta'$. Since γ_T^* continuously maps $H^{-1/2}(\partial\Omega)$ to $H^{-1}(U)$, we only need to show that the map $\mathrm{curl}\,\mathrm{curl}\,G_\lambda$ is a Hilbert–Schmidt operator from $H^{-1}(U)$ to $H^1(\Omega_0)$ and establish the corresponding bound on its Hilbert–Schmidt norm. By Lemma A.3, the Hilbert–Schmidt norm can be bounded by the $H^2(\Omega_0\times U)$-norm of the kernel of $\mathrm{curl}\,\mathrm{curl}\,G_{\lambda,0}$ on $\Omega_0\times U$. The corresponding bound has been established in Lemma A.1. The same argument works for $\varphi\widetilde{\mathcal{M}}_\lambda$ and $\varphi\widetilde{\mathcal{S}}_\lambda$. This concludes the proof of the estimates (28), (29), (31), (30), (32).

Since the operator norm is bounded in terms of the Hilbert–Schmidt norm and by the estimates (28), (29), (30), (31), it is sufficient to prove the estimates (33), (34), (35), and (36) for the operators $\chi\widetilde{\mathcal{L}}_\lambda$, $\chi\widetilde{\mathcal{S}}_\lambda$, $\chi\widetilde{\mathcal{M}}_\lambda$, $\chi\nabla\widetilde{\mathcal{S}}_\lambda$, where $\chi \in C_0^\infty(\mathbb{R}^3)$ is a compactly supported function that equals 1 near $\partial\Omega$. We write

$$\chi\widetilde{\mathcal{L}}_\lambda = \chi\nabla\widetilde{\mathcal{S}}_\lambda\mathrm{Div} + \lambda^2\chi\widetilde{\mathcal{S}}_\lambda.$$

The map γ_t^* is from $H^{-1/2}(\partial\Omega)$ to $H_c^{-1}(U)$, where U is an open neighbourhood of $\partial\Omega$. To prove all the bounds (33), (34), (35), and (36), it is therefore sufficient to show that the resolvent $(-\Delta_{\mathrm{free}}-\lambda^2)^{-1}$ is a bounded map from $H^{-1}_{\mathrm{comp}}(\mathbb{R}^3)$ to $H^1_{\mathrm{loc}}(\mathbb{R}^3)$ uniformly in λ for all $\lambda \in \mathfrak{D}_\epsilon$. This means that we need to show that the cut-off resolvent $\chi(-\Delta_{\mathrm{free}}-\lambda^2)^{-1}\chi$ is a uniformly bounded map from $H^{-1}(\mathbb{R}^3)$ to $H^1(\mathbb{R}^3)$ for all $\lambda \in \mathfrak{D}_\epsilon$. To see this, let $\eta \in C_0^\infty(\mathbb{R})$ be a function that is 1 near $[-R_1, R_1]$, where R_1 is the diameter of the support of χ. Let R be large enough that $\mathrm{supp}\,\eta \in (-R, R)$. This implies that $\chi(-\Delta-\lambda^2)^{-1}\chi = \chi R_{\eta,\lambda}\chi$, where $R_{\eta,\lambda}$ is the operator with integral kernel

$$\eta(|x-y|)\frac{1}{4\pi|x-y|}e^{i\lambda|x-y|} =: k_\lambda(x-y).$$

It is therefore sufficient to show that $R_{\eta,\lambda}$ is uniformly bounded for all $\lambda \in \mathfrak{D}_\epsilon$ as a map $H^s(\mathbb{R}^3)$ to $H^{s+2}(\mathbb{R}^3)$. Since this is a convolution operator, it commutes with the Laplace operator, and therefore it is sufficient to show that $R_{\eta,\lambda}$ is uniformly bounded as a map $L^2(\mathbb{R}^3)$ to $H^2(\mathbb{R}^3)$. We will show that $(-\Delta+1)R_{\eta,\lambda}$ is uniformly bounded as a map from $L^2(\mathbb{R}^3)$ to $L^2(\mathbb{R}^3)$. Using

$$(-\Delta+1)(-\Delta-\lambda^2)^{-1} = \mathrm{id} + (1+\lambda^2)(-\Delta-\lambda^2)^{-1},$$

one obtains that the integral kernel of $(-\Delta+1)R_{\eta,\lambda} - \mathrm{id}$ equals

$$(-(\Delta_x \eta(|x-y|)) + (1+\lambda^2)\eta(|x-y|))\frac{1}{4\pi|x-y|}\mathrm{e}^{\mathrm{i}\lambda|x-y|} - 2\nabla_x\eta(|x-y|)\nabla_x\frac{1}{4\pi|x-y|}\mathrm{e}^{\mathrm{i}\lambda|x-y|}.$$

This is a convolution operator, and we can use Young's inequality to estimate its operator norm. In particular, using spherical coordinates, the estimates

$$\int_0^R \frac{1}{4\pi r}|\mathrm{e}^{\mathrm{i}\lambda r}|r^2\,\mathrm{d}r \le C_R\frac{1}{1+|\mathrm{Im}\,\lambda|^2} \quad\text{and}\quad \int_0^R \frac{1}{4\pi r^2}|\mathrm{e}^{\mathrm{i}\lambda r}|r^2\,\mathrm{d}r \le C_R\frac{1}{1+|\mathrm{Im}\,\lambda|}$$

show that the convolution kernel is uniformly bounded in $L^1(\mathbb{R}^3)$ for $\lambda \in \mathfrak{D}_\epsilon$. Thus $R_{\eta,\lambda}$ is uniformly bounded as a map from $L^2(\mathbb{R}^3)$ to $H^2(\mathbb{R}^3)$ for $\lambda \in \mathfrak{D}_\epsilon$.

It remains to show the improved estimate (37). We again choose cutoffs χ, ψ as above, and we arrange them so that $\psi+\phi = 1$. Since the cut-off resolvent $\chi(-\Delta_{\mathrm{free}}-\lambda^2)^{-1}\chi$ is regular near 0 as a map from $H^{-1}(\mathbb{R}^3)$ to $H^1(\mathbb{R}^3)$, we know that $\chi\widetilde{\mathcal{S}}_\lambda : H^{-1/2}(\partial\Omega) \to H(\mathrm{curl}, \mathbb{R}^3)$ is regular near 0. It is therefore sufficient to establish the bound for $\phi\widetilde{\mathcal{S}}_\lambda$ as a map from $H_0^{-1/2}(\partial\Omega)$ to $H^1(\mathbb{R}^3)$. We argue similarly as above choosing an open neighbourhood U such that the support of ϕ has positive distance from U. For convenience, we will also assume that the support of ϕ is sufficiently separated from Ω; more precisely, we assume that the support of ϕ has positive distance to the convex hull of Ω. With $u \in H_0^{-1/2}(\partial\Omega)$, the distribution $\gamma^* u$ is in the space distributions $H_0^{-1}(U) = \{v \in H_c^{-1}(U) \mid \langle v, 1\rangle = 0\}$ of mean zero. We therefore only need to bound $\phi(-\Delta_{\mathrm{free}}-\lambda^2)^{-1}$ as a map from $H_0^{-1}(U)$ to $H^1(\mathbb{R}^n)$. This map is the restriction of the integral operator with smooth kernel

$$g(x, y) = \phi(x)\left(\frac{\mathrm{e}^{\mathrm{i}\lambda|x-y|}}{4\pi|x-y|} - \frac{\mathrm{e}^{\mathrm{i}\lambda|x-z|}}{4\pi|x-z|}\right)$$

to $H_0^{-1}(U)$, where z is any fixed point on $\partial\Omega$. One shows that this kernel is in the Sobolev space $H^2(\mathbb{R}^3 \times U)$ and is uniformly bounded in $\lambda \in \mathfrak{D}_\epsilon$. This kernel and its derivatives are easily bounded using the mean-value inequality

$$|\partial_x^\alpha g(x, y)| \le |y-z| \sup_{\tilde{y}\in K} \|\partial^\alpha \nabla_x g(x, \tilde{y})\| \le C \sup_{\tilde{y}\in K} \|\partial^\alpha \nabla_x g(x, \tilde{y})\|,$$

where K is the closure of the convex hull of $\partial\Omega$. The L^2-norm of this expression is uniformly bounded for all $\lambda \in \mathfrak{D}_\epsilon$ by the same estimate as in (88). This works essentially because, with repeated application of the product rule, the terms either have improved decay or have an extra λ-factor. □

The proof above can also be applied directly to $\chi\widetilde{\mathcal{L}}_\lambda$ in the entire complex plane to bound the operator norm, the norm of the derivative, and the norm of the remainder term. This gives the following result. We will not repeat the proof but simply state the result.

Lemma 7.2. *The families*

$$\widetilde{\mathcal{L}}_\lambda : H^{-1/2}(\mathrm{Div}, \partial\Omega) \to H_{\mathrm{loc}}(\mathrm{curl}, M) \oplus H_{\mathrm{loc}}(\mathrm{curl}, \Omega),$$
$$\mathcal{L}_\lambda : H^{-1/2}(\mathrm{Div}, \partial\Omega) \to H^{-1/2}(\mathrm{Div}, \partial\Omega)$$

are holomorphic families of bounded operators in the complex plane.

Lemma 7.3. *The families $\mathcal{L}_\lambda^{-1}$ and $\Lambda_\lambda^\pm$ are meromorphic in λ as families of bounded operators on $H^{-1/2}(\mathrm{Div}, \partial\Omega)$. The family Λ_λ^- has no poles in $\mathbb{R} \setminus \{0\}$ and in the upper half-plane.*

Proof. By Proposition 6.4 and Lemma 6.3, the operator $\left(\frac{1}{2} + \mathcal{M}_\lambda\right)$ is an analytic family of Fredholm operators which is invertible for $\operatorname{Im}\lambda > 0$. By the analytic Fredholm theorem, the inverse $\left(\frac{1}{2} + \mathcal{M}_\lambda\right)^{-1}$ is a meromorphic family of finite type, i.e., the negative Laurent coefficients are finite-rank operators. We have

$$\mathcal{L}_\lambda^{-2} = -\lambda^{-2}\left(\tfrac{1}{2} + \mathcal{M}_\lambda\right)^{-1}\left(\tfrac{1}{2} - \mathcal{M}_\lambda\right)^{-1},$$

which shows that $\mathcal{L}_\lambda^{-2}$ is meromorphic. Since $\mathcal{L}_\lambda$ is holomorphic, this shows that $\mathcal{L}_\lambda^{-1}$ is meromorphic. Finally $\Lambda^\pm$ is meromorphic by (19). Poles of Λ_λ^- are absent in the closed upper half-space because of the uniqueness of the exterior boundary value problem. Indeed, the most negative Laurent coefficient would give rise to an outgoing solution of the Helmholtz equation satisfying relative boundary conditions. But such an outgoing solution vanishes. ☐

Remark 7.4. The above cannot be easily concluded from analytic Fredholm theory since the operators $\mathcal{L}_\lambda$ and $\Lambda^\pm$ are not Fredholm operators. Indeed, the singular Laurent coefficients are not finite-rank operators.

We now aim to show a new formula for the voltage-to-current map in order to find bounds on $\mathcal{L}_\lambda^{-1}$ where it is well defined.

Theorem 7.5. *The interior voltage-to-current mapping Λ_λ^+ satisfies*

$$\mathrm{i}\Lambda_\lambda^+ = \frac{1}{\lambda} T + \lambda U_\lambda,$$

where T is a bounded operator on $H^{-1/2}(\mathrm{Div}, \partial\Omega)$ and U_λ is a meromorphic family of bounded operators on $H^{-1/2}(\mathrm{Div}, \partial\Omega)$ which is regular at $\lambda = 0$. We have explicitly

$$TA = \sum_{k=1}^{\beta_1} \langle A, \gamma_T \psi_{0,k}\rangle_{L^2(\partial\Omega)} \gamma_t(\psi_{0,k}) + \sum_{\lambda_{N,k}>0} \frac{1}{\lambda_{N,k}^2} \langle A, \gamma_T \nabla v_k\rangle_{L^2(\partial\Omega)} \gamma_t(\nabla v_k),$$
$$U_\lambda A = \sum_{k=1}^{\infty} \frac{1}{\lambda^2 - \mu_k^2} \langle A, \gamma_T \psi_k\rangle_{L^2(\partial\Omega)} \gamma_t(\psi_k)$$

for $A \in H^{-1/2}(\mathrm{Div}, \partial\Omega)$. Both sums converge in $H^{-1/2}(\mathrm{Div}, \partial\Omega)$. Here $\beta_1 = \dim \mathcal{H}^1_{\mathrm{abs}}(\Omega)$ is the first Betti number of the domain. We have $T^2 = 0$ and $TU_\lambda + U_\lambda T = \mathrm{id} - \lambda^2 U_\lambda$.

Proof. We start with an interior solution $E \in H(\text{curl}, \Omega)$ of the Maxwell system and assume $A = \gamma_t(E) \in H^{-1/2}(\text{Div}, \partial\Omega)$. First note that E satisfies $\text{div}\, E = 0$, but it is not in general in $\ker(\text{div}_0)$ because it may not satisfy the correct boundary conditions. We have

$$L^2(\Omega, \mathbb{C}^3) = \mathcal{H}^1_{\text{abs}}(\Omega) \oplus \{\psi_j \mid \mu_j > 0\} \oplus \{\nabla v_k | \lambda_{N,k}\}, \tag{38}$$

where v_k is an orthonormal basis of Neumann eigenfunctions on Ω. Define

$$\tilde{\psi}_k = \frac{1}{\lambda_{N,k}} \,\text{grad}\, v_k. \tag{39}$$

Now we can write

$$E = \sum \langle E, \psi_k \rangle \psi_k + \sum \langle E, \tilde{\psi}_k \rangle \tilde{\psi}_k, \tag{40}$$

which we need to show converges in $H(\text{curl}, \Omega)$. We have

$$\begin{aligned}
\langle E, \psi_k \rangle_{L^2(\Omega)} &= \frac{1}{\lambda^2 - \mu_k^2} (\langle -\Delta E, \psi_k \rangle_{L^2(\partial\Omega)} - \langle E, -\Delta \psi_k \rangle_{L^2(\partial\Omega)}) \\
&= \frac{1}{\lambda^2 - \mu_k^2} (\langle \gamma_t \,\text{curl}\, E, \gamma_T \psi_k \rangle_{L^2(\partial\Omega)} + \langle \gamma_t E, \gamma_T \,\text{curl}\, \psi_k \rangle_{L^2(\partial\Omega)}) \\
&= \frac{1}{\lambda^2 - \mu_k^2} \langle \gamma_t \,\text{curl}\, E, \gamma_T \psi_k \rangle_{L^2(\partial\Omega)} \\
&= \frac{\mathrm{i}\lambda}{\lambda^2 - \mu_k^2} \langle \gamma_t H, \gamma_T \psi_k \rangle_{L^2(\partial\Omega)} = \frac{\mathrm{i}\lambda}{\lambda^2 - \mu_k^2} \langle \Lambda_\lambda^+ A, \gamma_T \psi_k \rangle_{L^2(\partial\Omega)},
\end{aligned} \tag{41}$$

where we have used Stokes' theorem (4) as well as Maxwell system properties in (1) repeatedly. Since $E \in L^2(\Omega, \mathbb{C}^3)$, the sum $\sum \langle E, \psi_k \rangle \psi_k$ converges in $L^2(\Omega, \mathbb{C}^3)$. Let ϕ_k denote an orthonormal basis of eigenfunctions of Δ_{rel}. We now note that

$$\sum_{\lambda_k \neq 0} \langle E, \psi_k \rangle \,\text{curl}\, \psi_k = \sum_{\mu_k \neq 0} \langle E, \psi_k \rangle \,\text{curl}\, \frac{1}{\mu_k} \,\text{curl}\, \phi_k = \sum_{\lambda_k \neq 0} \langle E, \psi_k \rangle \mu_k \phi_k$$

converges in $L^2(\Omega, \mathbb{C}^3)$ whenever $(\langle E, \psi_k \rangle_{L^2(\Omega)} \mu_k)_k \in \ell^2$. The latter is true because

$$\mu_k \langle E, \psi_k \rangle_{L^2(\Omega)} = \langle E, \text{curl}\, \phi_k \rangle_{L^2(\Omega)} = \langle \text{curl}\, E, \phi_k \rangle_{L^2(\Omega)} \in \ell^2, \tag{42}$$

where we have used the fact $\text{curl}\, E \in L^2(\Omega, \mathbb{C}^3)$. Therefore

$$\sum_{k=1}^{\infty} \langle E, \psi_k \rangle \psi_k \tag{43}$$

converges in $H(\text{curl}, \Omega)$. For the second term, now we have

$$\begin{aligned}
\langle E, \tilde{\psi}_k \rangle_{L^2(\Omega)} &= \frac{\mathrm{i}}{\lambda} \Big\langle \text{curl}\, H, \frac{1}{\lambda_{N,k}} \,\text{grad}\, v_k \Big\rangle_{L^2(\Omega)} \\
&= \frac{\mathrm{i}}{\lambda_{N,k} \lambda} \langle \text{curl}\, H, \text{grad}\, v_k \rangle_{L^2(\Omega)} = \frac{\mathrm{i}}{\lambda_{N,k} \lambda} \langle \gamma_t H, \gamma_T \,\text{grad}\, v_k \rangle_{L^2(\partial\Omega)}.
\end{aligned} \tag{44}$$

This also gives that

$$\sum_{\lambda_k \neq 0} \langle E, \tilde{\psi}_k \rangle \tilde{\psi}_k \tag{45}$$

converges in $H(\mathrm{curl}, \Omega)$ as $\lambda_{N,k}^{-2}$ is summable. Therefore we have

$$E = \sum_{k=0}^{\infty} \frac{\mathrm{i}\lambda}{\lambda^2 - \mu_k^2} \langle \gamma_t H, \gamma_T \psi_k \rangle_{L^2(\partial\Omega)} \psi_k + \sum_{\lambda_{N,k} \neq 0} \frac{\mathrm{i}}{\lambda_k^2 \lambda} \langle \gamma_t H, \gamma_T \operatorname{grad} v_k \rangle_{L^2(\partial\Omega)} \operatorname{grad} v_k, \tag{46}$$

and this representation converges in $H(\mathrm{curl}, \Omega)$. As a result, we have convergence in $H^{-1/2}(\mathrm{Div}, \partial\Omega)$ of

$$\begin{aligned} A &= \nu \times E|_{\partial\Omega} \\ &= \sum_{\mu_k \geq 0} \frac{\mathrm{i}\lambda}{\lambda^2 - \mu_k^2} \langle \gamma_t H, \gamma_T \psi_k \rangle_{L^2(\partial\Omega)} \gamma_t(\psi_k) + \sum_{\lambda_{N,k} \neq 0} \frac{\mathrm{i}}{\lambda_{N,k}^2 \lambda} \langle \gamma_t H, \gamma_T \operatorname{grad} v_k \rangle_{L^2(\partial\Omega)} \gamma_t(\operatorname{grad} v_k). \end{aligned} \tag{47}$$

Then using the fact that $(\gamma_t(H)) = \Lambda_\lambda^+(\gamma_t(E)) = \Lambda_\lambda^+(A)$ and remarking that $(\Lambda^+)^2 = -\,\mathrm{id}$, we obtain the desired result. Expanding the formula $(\mathrm{i}\Lambda^+)^2 = \mathrm{id}$ also gives the claimed identities. □

We now aim to show operator bounds on the electric dipole map in order to find bounds on the large $|\lambda|$ behaviour of $\mathcal{L}_\lambda^{-1}$. Note that, for $\lambda \in \mathfrak{D}_\epsilon$, we have the estimate

$$\operatorname{Im} \lambda = |\operatorname{Im} \lambda| \leq |\lambda| \leq C_\epsilon \operatorname{Im} \lambda,$$

where $C_\epsilon := \sin(\epsilon)^{-1}$ is independent of $\lambda \in \mathfrak{D}_\epsilon$.

Theorem 7.6. *There exists a constant C such that, for all $\operatorname{Im} \lambda > 0$, we have the estimate*

$$\|\Lambda_\lambda^{\pm}\|_{H^{-1/2}(\mathrm{Div},\partial\Omega) \mapsto H^{-1/2}(\mathrm{Div},\partial\Omega)} \leq C \frac{1}{|\lambda|} \left(1 + \frac{|\lambda|(1 + |\lambda|^2)}{\operatorname{Im} \lambda} \right). \tag{48}$$

Proof. We first consider the case $\operatorname{Re} \lambda^2 < 0$, i.e., $|\operatorname{Im} \lambda| > |\operatorname{Re} \lambda|$. We have the integral identity

$$\langle v, \operatorname{curl} u \rangle_{L^2(M)} - \langle \operatorname{curl} v, u \rangle_{L^2(M)} = \langle \gamma_t v, \gamma_T u \rangle_{L^2(\partial\Omega)} \tag{49}$$

for $u, v \in H(\mathrm{curl}, M)$. Applying this integral identity with E and H gives

$$\begin{aligned} \mathrm{i}\lambda \langle \gamma_t H, \gamma_T E \rangle_{L^2(\partial\Omega)} &= \mathrm{i}\lambda (\langle H, \operatorname{curl} E \rangle_{L^2(M)} - \langle \operatorname{curl} H, E \rangle_{L^2(M)}) \\ &= \langle \operatorname{curl} E, \operatorname{curl} E \rangle - \lambda^2 \langle E, E \rangle. \end{aligned}$$

Taking the real part we obtain

$$|\operatorname{Re} \lambda^2| \langle E, E \rangle_{L^2(M)} \leq |\lambda| \cdot |\langle \gamma_t H, \gamma_T E \rangle_{L^2(\partial\Omega)}|.$$

The antisymmetric bilinear form $\langle \nu \times u, v \rangle_{L^2(\partial\Omega)}$ extends continuously to $H^{-1/2}(\mathrm{Div}, \partial\Omega)$ (see for example Lemma 5.61 in [Kirsch and Hettlich 2015]), and we therefore have

$$\|E\|_{L^2(M)}^2 \leq C_1 |\lambda| (-\operatorname{Re} \lambda^2)^{-1} \|\gamma_t E\|_{H^{-1/2}(\mathrm{Div},\partial\Omega)} \|\gamma_t(H)\|_{H^{-1/2}(\mathrm{Div},\partial\Omega)}. \tag{50}$$

Now we use the continuity of the tangential trace map and obtain

$$\begin{aligned}\|\gamma_t(\operatorname{curl} E)\|^2_{H^{-1/2}(\mathrm{Div},\partial\Omega)} &\le C_2\|(\operatorname{curl} E)\|^2_{H(\mathrm{curl},M)} = C_2(\|\operatorname{curl} E\|^2_{L^2(M)} + |\lambda|^4\|E\|^2_{L^2(M)})\\ &= C_2(|\lambda|^2\|E\|^2_{L^2(M)} + |\lambda|^4\|E\|^2_{L^2(M)} + \langle\gamma_t E, \gamma_T \operatorname{curl} E\rangle_{L^2(\partial\Omega)})\\ &\le C_3(|\lambda|^2(1+|\lambda|^2)\|E\|^2_{L^2(M)} + \|\gamma_t E\|_{H^{-1/2}(\mathrm{Div},\partial\Omega)}\|\gamma_t \operatorname{curl} E\|_{H^{-1/2}(\mathrm{Div},\partial\Omega)}).\end{aligned}$$

Choosing

$$a = C_3\|\gamma_t E\|_{H^{-1/2}(\mathrm{Div},\partial\Omega)} \quad \text{and} \quad b = \|\gamma_t \operatorname{curl} E\|_{H^{-1/2}(\mathrm{Div},\partial\Omega)}$$

and using the inequality $|ab| \le \frac{1}{2}(a^2+b^2)$, one obtains

$$\|\gamma_t(\operatorname{curl} E)\|^2_{H^{-1/2}(\mathrm{Div},\partial\Omega)} \le (2C_3|\lambda|^2(1+|\lambda|^2)\|E\|^2_{L^2(M)} + C_3^2\|\gamma_t E\|^2_{H^{-1/2}(\mathrm{Div},\partial\Omega)}).$$

Using (50), this further gives

$$\begin{aligned}&\|\gamma_t(\operatorname{curl} E)\|^2_{H^{-1/2}(\mathrm{Div},\partial\Omega)}\\ &\qquad\le C_4\biggl(\frac{|\lambda|^2(1+|\lambda|^2)}{-\operatorname{Re}\lambda^2}\|\gamma_t E\|_{H^{-1/2}(\mathrm{Div},\partial\Omega)}\|\gamma_t \operatorname{curl} E\|_{H^{-1/2}(\mathrm{Div},\partial\Omega)} + \|\gamma_t E\|^2_{H^{-1/2}(\mathrm{Div},\partial\Omega)}\biggr).\end{aligned}$$

The same trick as before with

$$a = C_4\frac{|\lambda|^2(1+|\lambda|^2)}{-\operatorname{Re}\lambda^2}\|\gamma_t E\|_{H^{-1/2}(\mathrm{Div},\partial\Omega)} \quad \text{and} \quad b = \|\gamma_t \operatorname{curl} E\|_{H^{-1/2}(\mathrm{Div},\partial\Omega)}$$

yields

$$\|\gamma_t(\operatorname{curl} E)\|^2_{H^{-1/2}(\mathrm{Div},\partial\Omega)} \le \biggl(C_4^2\frac{|\lambda|^4(1+|\lambda|^2)^2}{(-\operatorname{Re}\lambda^2)^2} + 2C_4\biggr)\|\gamma_t E\|^2_{H^{-1/2}(\mathrm{Div},\partial\Omega)},$$

which finally gives

$$\|\gamma_t(\operatorname{curl} E)\|_{H^{-1/2}(\mathrm{Div},\partial\Omega)} \le C\biggl(1 + \frac{|\lambda|^2(1+|\lambda|^2)}{-\operatorname{Re}\lambda^2}\biggr)\|\gamma_t E\|_{H^{-1/2}(\mathrm{Div},\partial\Omega)}. \tag{51}$$

Next consider the case $\operatorname{Im}\lambda^2 < 0$. The same proof with imaginary parts taken instead of real parts gives the estimate

$$\|\gamma_t(\operatorname{curl} E)\|_{H^{-1/2}(\mathrm{Div},\partial\Omega)} \le C\biggl(1 + \frac{|\lambda|^2(1+|\lambda|^2)}{-\operatorname{Im}\lambda^2}\biggr)\|\gamma_t E\|_{H^{-1/2}(\mathrm{Div},\partial\Omega)}. \tag{52}$$

These two estimates cover the upper half-space and are combined into

$$\|\gamma_t(\operatorname{curl} E)\|_{H^{-1/2}(\mathrm{Div},\partial\Omega)} \le C\biggl(1 + \frac{|\lambda|(1+|\lambda|^2)}{\operatorname{Im}\lambda}\biggr)\|\gamma_t E\|_{H^{-1/2}(\mathrm{Div},\partial\Omega)}, \tag{53}$$

which holds in the upper half-space except when $\operatorname{Im}\lambda < \operatorname{Re}\lambda$. The estimate holds in this region too as can be seen by replacing λ by $-\bar{\lambda}$, which is a symmetry operation of the Maxwell system that preserves the radiation condition. Hence, the estimate holds in the upper half-space. Since $\mathrm{i}\lambda H = \operatorname{curl} E$, this proves the claimed estimate. The same proof works for the interior with M replaced by Ω. ☐

Lemma 7.7. *The operator* $\left(\frac{1}{2}+\mathcal{M}_\lambda\right)^{-1}$ *is meromorphic of finite type, and we have near* 0 *the expansion*

$$\left(\tfrac{1}{2}+\mathcal{M}_\lambda\right)^{-1}=\frac{P}{\lambda^2}+\frac{B}{\lambda}+Q_\lambda, \tag{54}$$

where P *and* B *are finite-rank operators and* Q_λ *is analytic near* $\lambda=0$ *taking values in the bounded operators on* $H^{-1/2}(\mathrm{Div},\partial\Omega)$. *We also have*

$$\operatorname{image}(P)\cup\operatorname{image}(B)\subseteq\mathcal{B}_{\partial\Omega},\qquad P(\nu\times\nabla u)=B(\nu\times\nabla u)=0$$

for all $u\in H^{1/2}(\partial\Omega)$.

Proof. By the proof of Lemma 7.3, we know that $\left(\frac{1}{2}+\mathcal{M}_\lambda\right)^{-1}$ is a meromorphic family of finite type. The order of the singularity at 0 is at most 2 since, for $\lambda\in\mathfrak{D}_\epsilon$, $\lambda\neq 0$, we have

$$\left(\tfrac{1}{2}+\mathcal{M}_\lambda\right)^{-1}=-\Lambda_\lambda^{+}(\Lambda_\lambda^{+}-\Lambda_\lambda^{-}), \tag{55}$$

and the bound in Theorem 7.6 holds. Hence $\left(\frac{1}{2}+\mathcal{M}_\lambda\right)^{-1}$ has the claimed form:

$$\left(\tfrac{1}{2}+\mathcal{M}_\lambda\right)^{-1}=\frac{P}{\lambda^2}+\frac{B}{\lambda}+Q_\lambda,$$

with P and B of finite rank.

We must naturally have for these λ

$$\left(\tfrac{1}{2}+\mathcal{M}_\lambda\right)^{-1}\left(\tfrac{1}{2}+\mathcal{M}_\lambda\right)=\left(\tfrac{1}{2}+\mathcal{M}_\lambda\right)\left(\tfrac{1}{2}+\mathcal{M}_\lambda\right)^{-1}=\mathrm{id}. \tag{56}$$

Expanding $\left(\frac{1}{2}+\mathcal{M}_\lambda\right)$ around $\lambda=0$, we see that it has operator kernel

$$\frac{1}{2}+\frac{1}{4\pi}\gamma_{t,x}\gamma_{T,y}^{*}\operatorname{curl}\left(\frac{1}{|x-y|}\right)+O(\lambda^2) \tag{57}$$

since the first-order term in the expansion distributional kernel of the free Green's function is constant and therefore curl-free. Hence,

$$\left(\tfrac{1}{2}+\mathcal{M}_\lambda\right)=\tfrac{1}{2}+\mathcal{M}_0+O(\lambda^2)$$

near $\lambda=0$. Inserting this into (56) and comparing coefficients, one obtains

$$\left(\tfrac{1}{2}+\mathcal{M}_0\right)P=0,\quad \left(\tfrac{1}{2}+\mathcal{M}_0\right)B=0,\quad P\left(\tfrac{1}{2}+\mathcal{M}_0\right)=0,\quad B\left(\tfrac{1}{2}+\mathcal{M}_0\right)=0.$$

By Proposition 6.6, we therefore obtain $\operatorname{image}(P)$, $\operatorname{image}(B)\subseteq\mathcal{B}_{\partial\Omega}$ as claimed. It remains to show that

$$P(\nu\times\nabla u)=B(\nu\times\nabla u)=0.$$

To see this, it is sufficient to show that $\nu\times\nabla u$ is in the range of $\frac{1}{2}+\mathcal{M}_0$. To see this we use a classical result in potential layer theory, namely the invertibility of $\left(\frac{1}{2}+\mathcal{K}_0\right)$; see [Verchota 1984]. We then have by (25)

$$\nu\times\nabla u=\nu\times\nabla\left(\tfrac{1}{2}+\mathcal{K}_0\right)\left(\tfrac{1}{2}+\mathcal{K}_0\right)^{-1}u=\left(\tfrac{1}{2}+\mathcal{M}_0\right)\left(\nu\times\nabla\left(\tfrac{1}{2}+\mathcal{K}_0\right)^{-1}u\right). \qquad\square$$

Lemma 7.8. *The nonzero poles of* $\left(\frac{1}{2}+\mathcal{M}_\lambda\right)^{-1}$ *in the closed upper half-space are precisely the Maxwell eigenvalues of* Ω. *Near a Maxwell eigenvalue* $\mu=\mu_k$, *we have the expansion*

$$\left(\tfrac{1}{2}+\mathcal{M}_\lambda\right)^{-1}=\frac{P_\mu}{(\lambda-\mu)^2}+\frac{B_\mu}{\lambda-\mu}+Q_{\mu,\lambda}, \tag{58}$$

where P_μ *and* B_μ *are finite-rank operators with range in* $\ker\left(\frac{1}{2}+\mathcal{M}_\mu\right)^{-1}$ *and* $Q_{\mu,\lambda}$ *is holomorphic in* λ *near* μ.

Proof. The poles are precisely where $\left(\frac{1}{2}+\mathcal{M}_\lambda\right)$ is not injective. On the closed upper half-space, this means that the only poles are at 0 and at the Maxwell eigenvalues by Propositions 6.6 and 6.7. The statement now follows immediately from the formula

$$\left(\tfrac{1}{2}+\mathcal{M}_\lambda\right)^{-1}=-\Lambda^+_\lambda(\Lambda^+_\lambda-\Lambda^-_\lambda), \tag{59}$$

the expansion of Theorem 7.5, and the fact that Λ^- is holomorphic near $\mathbb{R}\setminus\{0\}$ by Lemma 7.3. ☐

Theorem 7.9. *For any* $\epsilon>0$, *we have there exists a constant* $C>0$ *such that*

$$\|\mathcal{L}_\lambda^{-1}\|_{H^{-1/2}(\mathrm{Div},\partial\Omega)\to H^{-1/2}(\mathrm{Div},\partial\Omega)}\le\frac{1+|\lambda|^2}{|\lambda|^2}C(1+|\lambda|^2),$$
$$\|\mathrm{Div}\circ(\mathcal{L}_\lambda^{-1})\circ(\nu\times\nabla)\|_{H^{-1/2}(\partial\Omega)\to H^{-1/2}(\partial\Omega)}\le C(1+|\lambda|^2)$$

for all λ *in the sector* $\mathfrak{D}_\epsilon$.

Proof. We use the identity, derived from (20),

$$\mathcal{L}_\lambda^{-1}=-\frac{\mathrm{i}(\Lambda^+_\lambda-\Lambda^-_\lambda)}{\lambda} \tag{60}$$

to reduce the analysis to that of $\Lambda^\pm_\lambda$. The bounds on the operator norm on the space $H^{-1/2}(\mathrm{Div},\partial\Omega)$ then follow immediately from Theorem 7.6. By (19), we have the identity

$$\mathcal{L}_\lambda^{-1}=\frac{1}{\mathrm{i}\lambda}\Lambda^+_\lambda\left(\tfrac{1}{2}+\mathcal{M}_\lambda\right)^{-1}.$$

Using Theorem 7.5, we obtain

$$\begin{aligned}\mathcal{L}_\lambda^{-1}&=-\left(\frac{1}{\lambda^2}T+U_\lambda\right)\left(\frac{1}{\lambda^2}P+\frac{1}{\lambda}B+Q_\lambda\right)\\&=-\frac{1}{\lambda^2}(TQ_\lambda+U_\lambda P)-\frac{1}{\lambda}U_\lambda B+U_\lambda Q_\lambda.\end{aligned} \tag{61}$$

We have used that $TP=TB=0$, which follows from Lemma 7.7 and Theorem 7.5. Since $\mathrm{Div}\circ T=0$, $P\circ(\nu\times\nabla)=0$, and $B\circ(\nu\times\nabla)=0$, we then obtain

$$\mathrm{Div}\circ(\mathcal{L}_\lambda^{-1})\circ(\nu\times\nabla)=\mathrm{Div}\circ U_\lambda Q_\lambda\circ(\nu\times\nabla),$$

which is regular at 0. ☐

8. Resolvent formulae and estimates

Proposition 8.1. *Assume that* $\operatorname{Im}\lambda > 0$. *For* $f \in C_0^\infty(\mathbb{R}^3, \mathbb{C}^3)$, *we have the following formulae for the difference of resolvents*:

$$((-\Delta_{\mathrm{rel}} - \lambda^2)^{-1} - (-\Delta_{\mathrm{free}} - \lambda^2)^{-1}) \operatorname{curl}\operatorname{curl} f = -\widetilde{\mathcal{L}}_\lambda(\mathcal{L}_\lambda)^{-1}(\nu \times)\widetilde{\mathcal{L}}_\lambda^{\mathfrak{t}} f, \tag{62}$$

$$((-\Delta_{\mathrm{rel}} - \lambda^2)^{-1} - (-\Delta_{\mathrm{free}} - \lambda^2)^{-1}) \operatorname{curl} f = -\widetilde{\mathcal{L}}_\lambda(\mathcal{L}_\lambda)^{-1}(\nu \times)\widetilde{\mathcal{M}}_\lambda^{\mathfrak{t}} f, \tag{63}$$

$$\operatorname{curl}((-\Delta_{\mathrm{rel}} - \lambda^2)^{-1} - (-\Delta_{\mathrm{free}} - \lambda^2)^{-1}) \operatorname{curl} f = -\lambda^2\widetilde{\mathcal{M}}_\lambda(\mathcal{L}_\lambda)^{-1}(\nu \times)\widetilde{\mathcal{M}}_\lambda^{\mathfrak{t}} f. \tag{64}$$

Here $\widetilde{\mathcal{L}}_\lambda^{\mathfrak{t}}$ *is the transpose operator to* $\widetilde{\mathcal{L}}_\lambda$ *obtained from the real* L^2*-inner product, i.e.,* $\widetilde{\mathcal{L}}_\lambda^{\mathfrak{t}} f = \overline{\widetilde{\mathcal{L}}_\lambda^{*}\overline{f}}$. *Similarly,* $\widetilde{\mathcal{M}}_\lambda^{\mathfrak{t}}$ *is the transpose of* $\widetilde{\mathcal{M}}_\lambda$.

Proof. We begin with the first formula. We know that $\widetilde{\mathcal{L}}_\lambda$ maps to functions satisfying the Helmholtz equation $(-\Delta - \lambda^2)v = 0$. Therefore we only need to show that, given $f \in C_0^\infty(\mathbb{R}^3, \mathbb{C}^3)$, the function

$$u = (-\Delta_{\mathrm{free}} - \lambda^2)^{-1}\operatorname{curl}\operatorname{curl} f - \widetilde{\mathcal{L}}_\lambda(\mathcal{L}_\lambda)^{-1}(\nu \times)\widetilde{\mathcal{L}}_\lambda^{\mathfrak{t}} f$$

satisfies relative boundary conditions. Since clearly $\operatorname{div} u = 0$ we only need to check that $\gamma_t u = 0$. One computes

$$\begin{aligned}\gamma_t u &= \gamma_t \operatorname{curl}\operatorname{curl}(-\Delta_{\mathrm{free}} - \lambda^2)^{-1} f - \mathcal{L}_\lambda(\mathcal{L}_\lambda)^{-1}(\nu \times)\widetilde{\mathcal{L}}_\lambda^{\mathfrak{t}} f \\ &= \gamma_t \operatorname{curl}\operatorname{curl}(-\Delta_{\mathrm{free}} - \lambda^2)^{-1} f - (\nu \times)\gamma_T \operatorname{curl}\operatorname{curl}(-\Delta_{\mathrm{free}} - \lambda^2)^{-1} f = 0,\end{aligned}$$

which gives the result.

Next consider the second formula. We again only need to check that $\gamma_{t,\pm}(u) = 0$, where

$$u = (-\Delta_{\mathrm{free}} - \lambda^2)^{-1}\operatorname{curl} f - \widetilde{\mathcal{L}}_\lambda(\mathcal{L}_\lambda)^{-1}(\nu \times)\widetilde{\mathcal{M}}_\lambda^{\mathfrak{t}} f.$$

The third formula follows from the second by applying the curl operator from the left and using $\operatorname{curl}\operatorname{curl}\operatorname{curl}\widetilde{\mathcal{S}}_\lambda = \lambda^2 \operatorname{curl}\widetilde{\mathcal{S}}_\lambda$. □

This can be used to show the following.

Theorem 8.2. *Let* $\epsilon > 0$, *and also suppose that* Ω_0 *is a smooth open set in* $\mathbb{R}^3$ *whose complement contains* $\overline{\Omega}$. *Let* $\delta = \operatorname{dist}(\partial\Omega, \Omega_0)$. *If* p *is the projection onto* $L^2(\Omega_0; \mathbb{C}^3)$ *in* $L^2(\mathbb{R}^3; \mathbb{C}^3)$ *then the operators*

$$p(-\Delta_{\mathrm{rel}} - \lambda^2)^{-1}\operatorname{curl}\operatorname{curl} p - p(-\Delta_{\mathrm{free}} - \lambda^2)^{-1}\operatorname{curl}\operatorname{curl} p,$$
$$p(-\Delta_{\mathrm{abs}} - \lambda^2)^{-1}\operatorname{curl}\operatorname{curl} p - p(-\Delta_{\mathrm{free}} - \lambda^2)^{-1}\operatorname{curl}\operatorname{curl} p,$$

are trace-class for all $\lambda \in \mathfrak{D}_\epsilon$ *as operators on* $L^2(\mathbb{R}^3; \mathbb{C}^3)$. *Moreover, for any* $\delta' \in (0, \delta)$, *their trace norms satisfy the bounds*

$$\| p(-\Delta_{\mathrm{rel}} - \lambda^2)^{-1}\operatorname{curl}\operatorname{curl} p - p(-\Delta_{\mathrm{free}} - \lambda^2)^{-1}\operatorname{curl}\operatorname{curl} p\|_1 \le C_{\delta',\epsilon} \mathrm{e}^{-\delta' \operatorname{Im}\lambda}, \tag{65}$$

$$\| p(-\Delta_{\mathrm{abs}} - \lambda^2)^{-1}\operatorname{curl}\operatorname{curl} p - p(-\Delta_{\mathrm{free}} - \lambda^2)^{-1}\operatorname{curl}\operatorname{curl} p\|_1 \le C_{\delta',\epsilon} \mathrm{e}^{-\delta' \operatorname{Im}\lambda} \tag{66}$$

for all $\lambda \in \mathfrak{D}_\epsilon$. *Moreover, both operators have integral kernels* $\kappa_{\mathrm{rel},\lambda}$, $\kappa_{\mathrm{abs},\lambda}$ *that are smooth on* $\Omega_0 \times \Omega_0$ *for all* $\lambda \in \mathfrak{D}_\epsilon$. *There exists* $C_{\Omega_0,\epsilon} > 0$ *depending on* Ω_0 *and* ϵ *such that*

$$\|k_{\mathrm{rel},\lambda}(x, x)\| + \|k_{\mathrm{abs},\lambda}(x, x)\| \le \left(C_{\Omega_0,\epsilon}\frac{\mathrm{e}^{-\operatorname{dist}(x,\partial\Omega)\operatorname{Im}\lambda}}{(\operatorname{dist}(x, \partial\Omega))^4}\right). \tag{67}$$

Proof. Given $\delta' \in (0, \delta)$, we choose a compactly supported smooth cut-off function χ which vanishes in Ω_0 such that the support of $\varphi = 1 - \chi$ has distance at least δ' from Ω. Then, since $\varphi p = p$, it is sufficient to show the estimates with p replaced by φ. From (62), we have

$$\varphi(-\Delta_{\text{rel}} - \lambda^2)^{-1}(\operatorname{curl}\operatorname{curl})\varphi - \varphi(-\Delta_{\text{free}} - \lambda^2)^{-1}(\operatorname{curl}\operatorname{curl})\varphi$$
$$= -\varphi\widetilde{\mathcal{L}}_\lambda \mathcal{L}_\lambda^{-1}(\nu\times)\widetilde{\mathcal{L}}_\lambda^{\mathrm{t}}\varphi = -(\varphi\widetilde{\mathcal{L}}_\lambda)\mathcal{L}_\lambda^{-1}(\nu\times)(\varphi\widetilde{\mathcal{L}}_\lambda)^{\mathrm{t}}. \quad (68)$$

The operator $\varphi\widetilde{\mathcal{L}}_\lambda$ is Hilbert–Schmidt by Proposition 7.1. Since $\mathcal{L}_\lambda^{-1}$ is bounded by Corollary 7.9 on the correct domains, this factorises the right-hand side of (68) into a product of the two Hilbert–Schmidt operators $(\varphi\widetilde{\mathcal{L}}_\lambda)$, $(\varphi\widetilde{\mathcal{L}}_\lambda)^{\mathrm{t}}$ and a bounded operator $\mathcal{L}_\lambda^{-1}(\nu\times)$. This shows it is trace-class; see for example [Shubin 1987, (A.3.4) and (A.3.2)]. We need to show the bound for the trace-norm. We now employ the more explicit description of $\varphi\widetilde{\mathcal{L}}_\lambda = \varphi(\nabla\widetilde{\mathcal{S}}_\lambda\mathrm{Div} + \lambda^2\widetilde{\mathcal{S}}_\lambda)$. This gives

$$\begin{aligned}
&(\varphi\widetilde{\mathcal{L}}_\lambda)\mathcal{L}_\lambda^{-1}(\nu\times)(\varphi\widetilde{\mathcal{L}}_\lambda)^{\mathrm{t}}\\
&\quad= (\varphi\nabla\widetilde{\mathcal{S}}_\lambda\mathrm{Div} + \lambda^2\varphi\widetilde{\mathcal{S}}_\lambda)\mathcal{L}_\lambda^{-1}((\nu\times)\nabla\widetilde{\mathcal{S}}_\lambda^{\mathrm{t}}\operatorname{div}\varphi + \lambda^2(\nu\times)(\varphi\widetilde{\mathcal{S}}_\lambda)^{\mathrm{t}})\\
&\quad= (\varphi\nabla\widetilde{\mathcal{S}}_\lambda\mathrm{Div} + \lambda^2\varphi\widetilde{\mathcal{S}}_\lambda)\mathcal{L}_\lambda^{-1}((\nu\times)\nabla\widetilde{\mathcal{S}}_\lambda^{\mathrm{t}}\operatorname{div}\varphi + \lambda^2(\nu\times)(\varphi\widetilde{\mathcal{S}}_\lambda)^{\mathrm{t}})\\
&\quad= \varphi\nabla\widetilde{\mathcal{S}}_\lambda\,\mathrm{Div}\,\mathcal{L}_\lambda^{-1}(\nu\times)\nabla\widetilde{\mathcal{S}}_\lambda^{\mathrm{t}}\operatorname{div}\varphi + \lambda^4\varphi\widetilde{\mathcal{S}}_\lambda\mathcal{L}_\lambda^{-1}(\nu\times)(\varphi\widetilde{\mathcal{M}}_\lambda)^{\mathrm{t}} + \lambda^2\varphi\nabla\widetilde{\mathcal{S}}_\lambda\,\mathrm{Div}\,\mathcal{L}_\lambda^{-1}(\nu\times)(\varphi\widetilde{\mathcal{S}}_\lambda)^{\mathrm{t}}\\
&\qquad\qquad + \lambda^2\varphi\widetilde{\mathcal{S}}_\lambda\mathcal{L}_\lambda^{-1}(\nu\times)\nabla\widetilde{\mathcal{S}}_\lambda^{\mathrm{t}}\operatorname{div}\varphi\\
&\quad= (\mathrm{I}) + (\mathrm{II}) + (\mathrm{III}) + (\mathrm{IV}).
\end{aligned} \quad (69)$$

We will show that the estimate holds for the individual terms. The trace-norm of (I) is bounded by $\|\varphi\nabla\widetilde{\mathcal{S}}_\lambda\|_{\mathrm{HS}}^2\|\mathrm{Div}\,\mathcal{L}_\lambda^{-1}(\nu\times)\nabla\|$ using the fact that the Hilbert–Schmidt norm is invariant under transposition. This is bounded by $C\mathrm{e}^{-\delta'\operatorname{Im}\lambda}$ in the sector by Theorem 7.9 and estimate (30) of Proposition 7.1.

The trace-norm of term (II) is bounded by $|\lambda|^4\|\varphi\widetilde{\mathcal{S}}_\lambda\|_{\mathrm{HS}}^2\|\mathcal{L}_\lambda^{-1}\|$. This is again bounded by $C\mathrm{e}^{-\delta'\operatorname{Im}\lambda}$ by Theorem 7.9 and (29) of Proposition 7.1. Expression (III) is the transpose of (IV) as one computes easily from Lemma 6.2. It is therefore sufficient to bound the trace-norm of (IV). We have

$$\begin{aligned}
(\mathrm{IV}) &= \lambda^2\varphi\widetilde{\mathcal{S}}_\lambda\Big(-\frac{1}{\lambda^2}(TQ_\lambda + U_\lambda P) - \frac{1}{\lambda}U_\lambda B + U_\lambda Q_\lambda\Big)(\nu\times)\nabla(\varphi\widetilde{\mathcal{S}}_\lambda)^{\mathrm{t}}\\
&= \lambda^2(\varphi\widetilde{\mathcal{S}}_\lambda)\Big(\frac{1}{\lambda^2}TQ_\lambda + U_\lambda Q_\lambda\Big)(\nu\times\nabla)(\varphi\widetilde{\mathcal{S}}_\lambda)^{\mathrm{t}},
\end{aligned} \quad (70)$$

where we have used Lemma 7.7, the expansion (61), and the fact that $P(\nu\times\nabla) = 0$ and $B(\nu\times\nabla) = 0$. The range of T consists of distributions in $H_0^{-1/2}(\partial\Omega, \mathbb{C}^3) \cap H^{-1/2}(\mathrm{Div}, \partial\Omega)$. To see this, note that the range of T consists, by Theorem 7.5, of limits in $H^{-1/2}(\mathrm{Div}, \partial\Omega)$ of boundary values of curl-free vector fields. Applying the integration by parts formula (4) with $\phi \in \mathrm{rg}(T)$ and E a constant unit vector field, noting that $\operatorname{curl}\phi = \operatorname{curl}E = 0$, one obtains that $\langle\gamma_t\phi, \gamma E\rangle_{L^2(\partial\Omega,\mathbb{C}^3)} = \langle\gamma_t\phi, \gamma_T E\rangle_{L^2(\partial\Omega,\mathbb{C}^3)} = 0$ as claimed. It follows that the trace-norm of (III) and (IV) are bounded by $C\mathrm{e}^{-\delta'\operatorname{Im}\lambda}$ by Theorem 7.9 and by the estimates (30), (29).

Next we use (64) to obtain

$$\varphi(-\Delta_{\text{abs}} - \lambda^2)^{-1}(\operatorname{curl}\operatorname{curl})\varphi - \varphi(-\Delta_{\text{free}} - \lambda^2)^{-1}(\operatorname{curl}\operatorname{curl})\varphi$$
$$= -\lambda^2\varphi\widetilde{\mathcal{M}}_\lambda\mathcal{L}_\lambda^{-1}(\nu\times)\widetilde{\mathcal{M}}_\lambda^{\mathrm{t}}\varphi = -\lambda^2(\varphi\widetilde{\mathcal{M}}_\lambda)\mathcal{L}_\lambda^{-1}(\nu\times)(\varphi\widetilde{\mathcal{M}}_\lambda)^{\mathrm{t}}. \quad (71)$$

The operators $\varphi\widetilde{\mathcal{M}}_\lambda$, $(\varphi\widetilde{\mathcal{M}}_\lambda)^t$ are Hilbert–Schmidt, and their Hilbert–Schmidt norms are bounded by $e^{-\delta' \operatorname{Im}\lambda}$ by Proposition 7.1 (31). This gives the claimed estimate for the trace-norm since the operator $\lambda^2\mathcal{L}_\lambda^{-1}$ is polynomially bounded in any sector by Theorem 7.9.

It remains to show the estimate on the diagonal of the integral kernel. This is done the same way using the pointwise estimate

$$\|\partial_x^\alpha \widetilde{\mathcal{S}}_\lambda(x,\cdot)\|_{H^{-1}(\partial\Omega)} \le C\frac{1}{(\operatorname{dist}(x,\partial\Omega))^{1+|\alpha|}} e^{-\operatorname{Im}\lambda \operatorname{dist}(x,\partial\Omega)/2},$$

which is easily obtained directly from the integral kernel, noting that differentiation in the x or y-variable gives a linear combination of terms that are bounded by

$$\frac{\lambda^k(\operatorname{dist}(x,\partial\Omega))^k}{(\operatorname{dist}(x,\partial\Omega))^{1+|\alpha|}} e^{-\operatorname{Im}\lambda \operatorname{dist}(x,\partial\Omega)} \le C_{k,\epsilon}\frac{1}{(\operatorname{dist}(x,\partial\Omega))^{1+|\alpha|}} e^{-\operatorname{Im}\lambda \operatorname{dist}(x,\partial\Omega)/2},$$

with $0 \le k \le \alpha$. One now applies this estimate to each of the four terms (I), (II), (III), (IV) and observes that every factor of λ can be absorbed using the bound

$$|\lambda|^k e^{-\operatorname{Im}\lambda \operatorname{dist}(x,\partial\Omega)} = C_{k,\epsilon}\frac{1}{\operatorname{dist}(x,\partial\Omega)^k} e^{-\operatorname{Im}\lambda \operatorname{dist}(x,\partial\Omega)/2}.$$

This gives the first claimed estimate. The second estimate follows the same way, since the above implies

$$\|\widetilde{\mathcal{M}}_\lambda(x,\cdot)\|_{H^{-1}(\partial\Omega)} \le C\frac{1}{(\operatorname{dist}(x,\partial\Omega))^2} e^{-\operatorname{Im}\lambda \operatorname{dist}(x,\partial\Omega)/2}. \qquad \square$$

9. The function Ξ

Recall that the boundary $\partial\Omega$ consists of N connected components $\partial\Omega_j$. To keep the discussion meaningful, we will assume throughout this section that $N \ge 2$. This gives a natural decomposition

$$H^{-1/2}(\operatorname{Div},\partial\Omega) = \bigoplus_{j=1}^N H^{-1/2}(\operatorname{Div},\partial\Omega_j).$$

Let q_j be the orthogonal projection $H^{-1/2}(\operatorname{Div},\partial\Omega) \to H^{-1/2}(\operatorname{Div},\partial\Omega_j)$ and $\mathcal{L}_{j,\lambda} = q_j\mathcal{L}_\lambda q_j$. We then can write

$$\mathcal{L}_\lambda = \sum_{j=1}^N \mathcal{L}_{j,\lambda} + \sum_{j\neq k} q_j\mathcal{L}_\lambda q_k = \mathcal{L}_{D,\lambda} + \mathcal{T}_\lambda. \tag{72}$$

We remark that $\mathcal{L}_{j,\lambda}$, which is regarded as a map from $H^{-1/2}(\operatorname{Div},\partial\Omega) \to H^{-1/2}(\operatorname{Div},\partial\Omega)$, is independent of the other components. The sum $\mathcal{L}_{D,\lambda}$ describes the diagonal part of the operator $\mathcal{L}$ with respect to the decomposition above.

We have a similar decomposition for the operator

$$\mathcal{M}_\lambda = \mathcal{M}_{D,\lambda} + \mathcal{J}_\lambda.$$

We set

$$\delta = \min_{j\neq k} \operatorname{dist}(\partial\Omega_j,\partial\Omega_k) > 0. \tag{73}$$

Then we have the following proposition.

Proposition 9.1. *The families $\mathcal{T}_\lambda, \mathcal{J}_\lambda : H^{-1/2}(\mathrm{Div}, \partial\Omega) \to H^{-1/2}(\mathrm{Div}, \partial\Omega)$ are holomorphic families of trace-class operators in the complex plane. For any $\epsilon > 0$ and any $\delta' \in (0, \delta)$, the following estimates for their trace-norms $\|\cdot\|_1$ hold*:

$$\|\mathcal{T}_\lambda\|_1 \le C_{\delta',\epsilon} e^{-\delta' \operatorname{Im}\lambda}, \qquad \|\mathcal{J}_\lambda\|_1 \le C_{\delta',\epsilon} e^{-\delta' \operatorname{Im}\lambda}, \tag{74}$$

$$\Big\| \frac{d}{d\lambda} \mathcal{T}_\lambda \Big\|_1 \le C_{\delta',\epsilon} e^{-\delta' \operatorname{Im}\lambda}, \quad \Big\| \frac{d}{d\lambda} \mathcal{J}_\lambda \Big\|_1 \le C_{\delta',\epsilon} e^{-\delta' \operatorname{Im}\lambda} \tag{75}$$

for all λ in the sector $\mathfrak{D}_\epsilon$. We also have

$$\|\mathcal{T}_\lambda|_{H^{-1/2}(\mathrm{Div}\,0,\partial\Omega)}\|_1 \le C_{\delta',\epsilon} |\lambda|^2 e^{-\delta' \operatorname{Im}\lambda}, \tag{76}$$

$$\Big\| \frac{d}{d\lambda} \mathcal{T}_\lambda|_{H^{-1/2}(\mathrm{Div}\,0,\partial\Omega)} \Big\|_1 \le C_{\delta',\epsilon} |\lambda| e^{-\delta' \operatorname{Im}\lambda}. \tag{77}$$

Proof. We will prove this estimate only for $\mathcal{T}_\lambda$ as the estimate for $\mathcal{J}_\lambda$ is proved in the same way. It is sufficient to show this for the individual terms $q_j \mathcal{L}_\lambda q_k$ with $j \neq k$. We choose an open bounded neighbourhood U of $\partial\Omega_j$ and an open bounded neighbourhood V of $\partial\Omega_k$ such that $\mathrm{dist}(U, V) > \delta'$. The first two estimates are implied by Lemma A.4 by observing that the operator is the composition

$$H^{-1/2}(\mathrm{Div}, \partial\Omega) \to H^{-1}(V) \to H^1(U) \to H^{-1/2}(\mathrm{Div}, \partial\Omega),$$

and the map $H^{-1}(V) \to H^1(U)$ has smooth integral kernel

$$\chi(x, y) \operatorname{curl} \operatorname{curl}_x \frac{e^{i\lambda|x-y|}}{4\pi|x-y|}$$

for a suitable cut-off function that is compactly supported in $U \times V$. The same argument applies to the λ-derivative.

To show the bounds on the restriction to $H^{-1/2}(\mathrm{Div}\,0, \partial\Omega)$, one uses that $\mathcal{L}_\lambda = \gamma_t \nabla \mathcal{S}_\lambda \mathrm{Div} + \lambda^2 \mathcal{S}_\lambda$. To bound the trace-norm of $\lambda^2 q_j \mathcal{S}_\lambda q_k$, one uses exactly the same argument as above applied to the kernel

$$\lambda^2 \chi(x, y) \frac{e^{i\lambda|x-y|}}{4\pi|x-y|}$$

and its λ-derivative. ☐

Proposition 9.2. *Fix $\epsilon > 0$. Then $(\mathcal{L}_\lambda \mathcal{L}_{D,\lambda}^{-1} - \mathrm{id}) : H^{-1/2}(\mathrm{Div}, \partial\Omega) \to H^{-1/2}(\mathrm{Div}, \partial\Omega)$ is a meromorphic family of trace-class operators with no poles in the closed upper half-plane. In the sector, we have, for any $\delta' \in (0, \delta)$, the estimate*

$$\|\mathcal{L}_{D,\lambda}^{-1} \mathcal{L}_\lambda - \mathrm{id}\|_1 = \|\mathcal{L}_\lambda \mathcal{L}_{D,\lambda}^{-1} - \mathrm{id}\|_1 \le C_{\delta',\epsilon} e^{-\delta' \operatorname{Im}\lambda}. \tag{78}$$

Proof. We use (19) and obtain

$$\mathcal{L}_\lambda \mathcal{L}_{D,\lambda}^{-1} - \mathrm{id} = \big(\tfrac{1}{2} + \mathcal{M}_\lambda\big)\big(\tfrac{1}{2} + \mathcal{M}_{D,\lambda}\big)^{-1} - \mathrm{id},$$

bearing in mind that $\Lambda_\lambda^+ = \Lambda_{D,\lambda}^+$. With

$$\big(\tfrac{1}{2} + \mathcal{M}_{D,\lambda}\big)^{-1} = \frac{1}{\lambda^2} P_D + \frac{1}{\lambda} B_D + Q_\lambda,$$

we remark that

$$\left(\tfrac{1}{2}+\mathcal{M}_{D,0}\right)P_D=\left(\tfrac{1}{2}+\mathcal{M}_{D,0}\right)B_D=0,$$

but then also

$$\left(\tfrac{1}{2}+\mathcal{M}_{0}\right)P_D=\left(\tfrac{1}{2}+\mathcal{M}_{0}\right)B_D=0$$

because, according to Proposition 6.6, we know that the kernels of $\left(\tfrac{1}{2}+\mathcal{M}_0\right)$ and $\left(\tfrac{1}{2}+\mathcal{M}_{D,0}\right)$ coincide. We have used here, as in the proof of Lemma 7.7, that the first-order terms in the expansion of $\mathcal{M}_\lambda$ vanish at $\lambda=0$, i.e., $\left(\tfrac{\mathrm{d}}{\mathrm{d}\lambda}\mathcal{M}_\lambda\right)\big|_{\lambda=0}=\left(\tfrac{\mathrm{d}}{\mathrm{d}\lambda}\mathcal{M}_{D,\lambda}\right)\big|_{\lambda=0}=0$. Using the abbreviation $\mathcal{J}_\lambda=\mathcal{M}_\lambda-\mathcal{M}_{D,\lambda}$, this implies $\mathcal{J}_0P_D=\mathcal{J}_0B_D=0$. Moreover, $\mathcal{J}_\lambda$ is trace-class. This shows that

$$\left(\tfrac{1}{2}+\mathcal{M}_\lambda\right)\left(\tfrac{1}{2}+\mathcal{M}_{D,\lambda}\right)^{-1}-\mathrm{id}$$

is a meromorphic family of trace-class operators and 0 is not a pole. Interior Maxwell eigenvalues are not poles by the same argument, since the kernel of $\left(\tfrac{1}{2}+\mathcal{M}_\mu\right)$ coincides with the kernel of $\left(\tfrac{1}{2}+\mathcal{M}_{D,\mu}\right)$ and by the expansion of Lemma 7.8.

Moreover, $\left(\tfrac{1}{2}+\mathcal{M}_{D,\lambda}\right)$ is invertible for all the other points in the closed upper half-space, and hence there are no poles there. To show the estimate in the sector, we note that

$$\mathcal{L}_\lambda\mathcal{L}_{D,\lambda}^{-1}-\mathrm{id}=\mathcal{T}_\lambda\mathcal{L}_{D,\lambda}^{-1}. \tag{79}$$

Then the bound for large $|\lambda|$ is a result of Theorem 7.9 and Proposition 9.1. □

Proposition 9.3. *The Fredholm determinant* $\det(\mathcal{L}_\lambda\mathcal{L}_{D,\lambda}^{-1})$ *in the space* $H^{-1/2}(\mathrm{Div},\partial\Omega)$ *is well defined and holomorphic in a neighbourhood of the closed upper half-space. For any* $\epsilon>0$ *and* $\delta'\in(0,\delta)$, *we have the bound*

$$|\det(\mathcal{L}_\lambda\mathcal{L}_{D,\lambda}^{-1})-1|\le C_{\delta',\epsilon}\mathrm{e}^{-\delta'\operatorname{Im}\lambda} \tag{80}$$

for all λ *in the sector* $\mathfrak{D}_\epsilon$. *Moreover,* $\det(\mathcal{L}_\lambda\mathcal{L}_{D,\lambda}^{-1})$ *is nonzero in the closed upper half-space.*

Proof. The trace of $(\mathcal{L}_\lambda\mathcal{L}_{D,\lambda}^{-1}-\mathrm{id})$ is bounded by Proposition 9.2. Using the bound

$$|\det(1+A)-1|\le\|A\|_1\mathrm{e}^{1+\|A\|_1}$$

for the Fredholm determinant (see for example [Simon 1977, (3.7)]), one obtains

$$|\Xi(\lambda)|\le|\log\det(\mathcal{L}_\lambda\mathcal{L}_{D,\lambda}^{-1})|\le C_{\delta',\epsilon}\mathrm{e}^{-\delta'\operatorname{Im}\lambda}. \tag{81}$$

By analyticity of $(\mathcal{L}_\lambda\mathcal{L}_{D,\lambda}^{-1}-\mathrm{id})=\mathcal{J}_\lambda\left(\tfrac{1}{2}+\mathcal{M}_{D,\lambda}\right)^{-1}$ as a family of trace-class operators in the upper half-space and near 0, the determinant also depends analytically on λ (e.g., [Simon 1977, Theorem 3.3]). By invertibility of the operator in the closed upper half-space, the determinant never vanishes [Simon 1977, Theorem 3.9], and therefore log det is analytic in the union of the upper half-space and a neighbourhood of 0. □

Since the determinant does not vanish near the closed upper half-space, we can choose a simply connected open neighbourhood $\mathcal{U}$ of the closed upper half-space, and it then defines a holomorphic function $\mathcal{U}\to\mathbb{C}\setminus\{0\}$ which we can lift to a holomorphic function on the logarithmic cover of the complex plane, where we choose the branch cut to be the negative real line $(-\infty,0)$. Composition with log is then well defined, and we write $\log\det(\mathcal{L}_\lambda \mathcal{L}_{D,\lambda}^{-1})$ to mean this composition. This means that this function and the branch of the logarithm is fixed by requiring this to be a holomorphic function that decays exponentially fast along the positive imaginary axis.

Definition 9.4. The function Ξ is defined in a sufficiently small simply connected open neighbourhood of the closed upper half-space by

$$\Xi(\lambda) = \log\det(\mathcal{L}_\lambda \mathcal{L}_{D,\lambda}^{-1}),$$

where the branch of the logarithm is chosen as explained above.

Theorem 9.5. *The function $\Xi(\lambda)$ is holomorphic near the closed upper half-space and, for any $\epsilon > 0$ and $\delta' \in (0, \delta)$, we have the bounds*

$$|\Xi(\lambda)| \le C_{\delta',\epsilon} \mathrm{e}^{-\delta' \operatorname{Im}\lambda}, \quad |\Xi'(\lambda)| \le C_{\delta',\epsilon} \mathrm{e}^{-\delta' \operatorname{Im}\lambda} \tag{82}$$

for λ in the sector $\mathfrak{D}_\epsilon$.

Proof. The first bound is a direct consequence of the proposition above. The second bound is a direct consequence of the maximum modulus principle. ☐

10. Relative trace formula

We consider the two Maxwell resolvent differences

$$R_{D,\mathrm{rel},\lambda} = \left(((-\Delta_{\mathrm{rel}} - \lambda^2)^{-1} - (-\Delta_{\mathrm{free}} - \lambda^2)^{-1}) - \sum_{j=1}^{N} ((-\Delta_{\mathrm{rel},j} - \lambda^2)^{-1} - (-\Delta_{\mathrm{free}} - \lambda^2)^{-1}) \right) \operatorname{curl}\operatorname{curl},$$

$$R_{D,\mathrm{abs},\lambda} = \left(((-\Delta_{\mathrm{abs}} - \lambda^2)^{-1} - (-\Delta_{\mathrm{free}} - \lambda^2)^{-1}) - \sum_{j=1}^{N} ((-\Delta_{\mathrm{abs},j} - \lambda^2)^{-1} - (-\Delta_{\mathrm{free}} - \lambda^2)^{-1}) \right) \operatorname{curl}\operatorname{curl}.$$

Using (62) and (64), we conclude

$$((-\Delta_{\mathrm{rel},j} - \lambda^2)^{-1} - (-\Delta_{\mathrm{free}} - \lambda^2)^{-1}) \operatorname{curl}\operatorname{curl} = -\widetilde{\mathcal{L}}_\lambda \mathcal{L}_{j,\lambda}^{-1} (\nu \times) \widetilde{\mathcal{L}}_\lambda^{\mathfrak{t}},$$
$$((-\Delta_{\mathrm{abs},j} - \lambda^2)^{-1} - (-\Delta_{\mathrm{free}} - \lambda^2)^{-1}) \operatorname{curl}\operatorname{curl} = -\lambda^2 \widetilde{\mathcal{M}}_\lambda \mathcal{L}_{j,\lambda}^{-1} (\nu \times) \widetilde{\mathcal{M}}_\lambda^{\mathfrak{t}},$$

and hence

$$R_{D,\mathrm{rel},\lambda} = -\widetilde{\mathcal{L}}_\lambda \mathcal{L}_\lambda^{-1} (\nu \times) \widetilde{\mathcal{L}}_\lambda^{\mathfrak{t}} + \widetilde{\mathcal{L}}_\lambda \mathcal{L}_{D,\lambda}^{-1} (\nu \times) \widetilde{\mathcal{L}}_\lambda^{\mathfrak{t}},$$
$$R_{D,\mathrm{abs},\lambda} = -\widetilde{\mathcal{M}}_\lambda \mathcal{L}_\lambda^{-1} (\nu \times) \widetilde{\mathcal{M}}_\lambda^{\mathfrak{t}} + \widetilde{\mathcal{M}}_\lambda \mathcal{L}_{D,\lambda}^{-1} (\nu \times) \widetilde{\mathcal{M}}_\lambda^{\mathfrak{t}}.$$

We have the following improvement of Theorem 8.2 in the relative setting.

Proposition 10.1. *Let $\epsilon > 0$, and let $\delta' > 0$ be smaller than $\delta = \operatorname{dist}(\partial\Omega_j, \partial\Omega_k)$. Then the operators $R_{D,\mathrm{rel},\lambda}, R_{D,\mathrm{abs},\lambda} : L^2(\mathbb{R}^3, \mathbb{C}^3) \to L^2(\mathbb{R}^3, \mathbb{C}^3)$ are trace-class for all $\lambda \in \mathfrak{D}_\epsilon$, and their trace norm can be estimated by*

$$\|R_{D,\mathrm{rel},\lambda}\|_1 + \|R_{D,\mathrm{abs},\lambda}\|_1 \le C_{\delta',\epsilon} \mathrm{e}^{-\delta' \operatorname{Im}\lambda}, \quad \lambda \in \mathfrak{D}_\epsilon.$$

Proof. First note that

$$(\mathcal{L}_\lambda^{-1} - \mathcal{L}_{\lambda,D}^{-1}) = -(\mathcal{L}_\lambda^{-1} \mathcal{T}_\lambda \mathcal{L}_{\lambda,D}^{-1})$$

is a meromorphic family of trace-class operators $H^{-1/2}(\mathrm{Div}, \partial\Omega) \to H^{-1/2}(\mathrm{Div}, \partial\Omega)$ in the complex plane. For $|\lambda| > 1$, the bound then follows from Proposition 9.1 and the bounds in Theorem 7.9. In particular, the expansion

$$(\mathcal{L}_\lambda^{-1} - \mathcal{L}_{\lambda,D}^{-1}) = \frac{1}{\lambda^2} L_2 + \frac{1}{\lambda} L_1 + L_{0,\lambda}$$

resulting from (61) is in terms of trace-class operators $L_2,\ L_1$ and the holomorphic family of trace-class operators $L_{0,\lambda}$. Specifically,

$$\begin{aligned} L_2 &= -T(Q^{(0)} - Q_D^{(0)}) - (U^{(0)} P - U_D^{(0)} P_D) = T W_2 + V_2, \\ L_1 &= -T(Q^{(1)} - Q_D^{(1)}) - (U^{(0)} B - U_D^{(0)} B_D) - (U^{(1)} P - U_D^{(1)} P_D) = T W_1 + V_1, \end{aligned}$$

where $Q^{(0)}$ and $Q^{(1)}$ are the expansion coefficients of

$$Q_\lambda = Q^{(0)} + Q^{(1)}\lambda + O(|\lambda|^2)$$

near $\lambda = 0$. The same notation is used for the expansion coefficients of $Q_{D,\lambda},\ U_\lambda,\ U_{D,\lambda}$. Since the operator

$$\left(\tfrac{1}{2} + \mathcal{M}_\lambda\right)^{-1} - \left(\tfrac{1}{2} + \mathcal{M}_{D,\lambda}\right)^{-1} = -\left(\tfrac{1}{2} + \mathcal{M}_\lambda\right)^{-1} \mathcal{J}_\lambda \left(\tfrac{1}{2} + \mathcal{M}_{D,\lambda}\right)^{-1}$$

is a meromorphic family of trace-class operators, we know that the expansion coefficients $W_2 = Q^{(0)} - Q_D^{(0)}$ and $W_1 = Q^{(1)} - Q_D^{(1)}$ are trace-class. We also record that $V_2(\nu \times \nabla) = 0$ and $V_1(\nu \times \nabla) = 0$ and recall that $\mathrm{Div} \circ T = 0$. Now we are ready to estimate the resolvent differences. We first focus on $R_{D,\mathrm{rel},\lambda}$. We have

$$R_{D,\mathrm{rel},\lambda} = -\widetilde{\mathcal{L}}_\lambda (\mathcal{L}_\lambda^{-1} - \mathcal{L}_{D,\lambda}^{-1})(\nu \times)\widetilde{\mathcal{L}}_\lambda^{\mathtt{t}} = -\widetilde{\mathcal{L}}_\lambda \left(\frac{1}{\lambda^2}(T W_2 + V_2) + \frac{1}{\lambda}(T W_1 + V_1) + L_{0,\lambda} \right)(\nu \times)\widetilde{\mathcal{L}}_\lambda^{\mathtt{t}}.$$

We expand this further using $\widetilde{\mathcal{L}}_\lambda = \nabla \widetilde{\mathcal{S}}_\lambda \mathrm{Div} + \lambda^2 \widetilde{\mathcal{S}}_\lambda$ to obtain that, modulo terms that have bounded trace-norm near $\lambda = 0$, the operator $R_{D,\lambda}$ equals

$$\begin{aligned} &(\nabla \widetilde{\mathcal{S}}_\lambda \mathrm{Div} + \lambda^2 \widetilde{\mathcal{S}}_\lambda)\left(\frac{1}{\lambda^2}(T W_2 + V_2) + \frac{1}{\lambda}(T W_1 + V_1) \right)((\nu \times)\nabla \widetilde{\mathcal{S}}_\lambda^{\mathtt{t}} \mathrm{div} + \lambda^2 (\nu \times)(\widetilde{\mathcal{S}}_\lambda)^{\mathtt{t}}) \\ &\quad = \nabla \widetilde{\mathcal{S}}_\lambda \, \mathrm{Div}((T W_2 + V_2) + \lambda(T W_1 + V_1))(\nu \times)(\widetilde{\mathcal{S}}_\lambda)^{\mathtt{t}} + \widetilde{\mathcal{S}}_\lambda((T W_2 + V_2) + \lambda(T W_1 + V_1))(\nu \times)\nabla \widetilde{\mathcal{S}}_\lambda^{\mathtt{t}} \, \mathrm{div} \\ &\qquad\qquad + \lambda^4 \widetilde{\mathcal{S}}_\lambda \left(\frac{1}{\lambda^2}(T W_2 + V_2) + \frac{1}{\lambda}(T W_1 + V_1) \right)(\nu \times)(\widetilde{\mathcal{S}}_\lambda)^{\mathtt{t}} \\ &\quad = (\mathrm{I}) + (\mathrm{II}) + (\mathrm{III}). \end{aligned}$$

Since $\mathcal{L}_{D,\lambda}$ and $\mathcal{L}_\lambda$ are self-adjoint with respect to the antisymmetric bilinear and since

$$\mathcal{L}_\lambda^{-1} - \mathcal{L}_{D,\lambda}^{-1} = \left(\frac{1}{\lambda^2}(T W_2 + V_2) + \frac{1}{\lambda}(T W_1 + V_1) + L_{0,\lambda} \right),$$

one obtains

$$\left(\left(\frac{1}{\lambda^2}(T W_2 + V_2) + \frac{1}{\lambda}(T W_1 + V_1) \right)(\nu \times) \right)^{\mathtt{t}} = \left(\frac{1}{\lambda^2}(T W_2 + V_2) + \frac{1}{\lambda}(T W_1 + V_1) \right)(\nu \times),$$

and therefore (II) is the transpose of (I). (III) has bounded trace-norm near $\lambda = 0$. Finally

$$(\mathrm{II}) = \widetilde{\mathcal{S}}_\lambda((T W_2 + V_2) + \lambda(T W_1 + V_1))(\nu \times)\nabla \widetilde{\mathcal{S}}_\lambda^{\mathtt{t}} \, \mathrm{div} = \widetilde{\mathcal{S}}_\lambda((T W_2) + \lambda(T W_1))(\nu \times)\nabla \widetilde{\mathcal{S}}_\lambda^{\mathtt{t}} \, \mathrm{div}$$

has bounded trace-norm near $\lambda = 0$ as $\widetilde{\mathcal{S}}_\lambda T$ and $\nabla \widetilde{\mathcal{S}}_\lambda^{\mathtt{t}}$ div have bounded operator norm, and W_2 and W_1 are trace-class. Finally we consider $R_{D,\mathrm{abs},\lambda}$. We compute as above

$$\begin{aligned} R_{D,\mathrm{abs},\lambda} &= -\lambda^2 \widetilde{\mathcal{M}}_\lambda (\mathcal{L}_\lambda^{-1} - \mathcal{L}_{D,\lambda}^{-1})(\nu \times) \widetilde{\mathcal{M}}_\lambda^{\mathtt{t}} \\ &= -\lambda^2 \widetilde{\mathcal{M}}_\lambda \left(\frac{1}{\lambda^2}(TW_2 + V_2) + \frac{1}{\lambda}(TW_1 + V_1) + L_{0,\lambda} \right)(\nu \times) \widetilde{\mathcal{M}}_\lambda^{\mathtt{t}} \\ &= -\widetilde{\mathcal{M}}_\lambda ((TW_2 + V_2) + \lambda(TW_1 + V_1) + \lambda^2 L_{0,\lambda})(\nu \times) \widetilde{\mathcal{M}}_\lambda^{\mathtt{t}}, \end{aligned}$$

whose trace-norm is bounded near 0 since $\widetilde{\mathcal{M}}_\lambda$ is uniformly bounded. □

Lemma 10.2. *We have*

$$\operatorname{tr}(R_{D,\mathrm{rel},\lambda}) = \operatorname{tr}(R_{D,\mathrm{abs},\lambda}) = -\frac{\lambda}{2} \Xi'(\lambda).$$

Proof. One has

$$\begin{aligned} (\nu \times) \widetilde{\mathcal{L}}_\lambda^{\mathtt{t}} \widetilde{\mathcal{L}}_\lambda &= \gamma_t \operatorname{curl}\operatorname{curl}\operatorname{curl}\operatorname{curl}(-\Delta_{\mathrm{free}} - \lambda^2)^{-2} \gamma_T^{\mathtt{t}} = \gamma_t \operatorname{curl}\operatorname{curl}(-\Delta_{\mathrm{free}})(-\Delta_{\mathrm{free}} - \lambda^2)^{-2} \gamma_T^{\mathtt{t}} \\ &= \gamma_t \operatorname{curl}\operatorname{curl}(-\Delta_{\mathrm{free}} - \lambda^2)^{-1} \gamma_T^{\mathtt{t}} + \gamma_t \operatorname{curl}\operatorname{curl} \lambda^2 (-\Delta_{\mathrm{free}} - \lambda^2)^{-2} \gamma_T^{\mathtt{t}} = \mathcal{L}_\lambda + \frac{\lambda}{2} \frac{\mathrm{d}}{\mathrm{d}\lambda} \mathcal{L}_\lambda. \end{aligned}$$

Similarly, we also have

$$\lambda^2 (\nu \times) \widetilde{\mathcal{M}}_\lambda^{\mathtt{t}} \widetilde{\mathcal{M}}_\lambda = \lambda^2 \gamma_t \operatorname{curl}\operatorname{curl}(-\Delta_{\mathrm{free}} - \lambda^2)^{-2} \gamma_T^{\mathtt{t}} = \frac{\lambda}{2} \frac{\mathrm{d}}{\mathrm{d}\lambda} \mathcal{L}_\lambda.$$

Using invariance of the trace in $H^{-1/2}(\mathrm{Div}, \partial\Omega)$ under cyclic permutations, we get

$$\begin{aligned} \operatorname{tr}(R_{D,\mathrm{rel},\lambda}) &= -\operatorname{tr}(-\widetilde{\mathcal{L}}_\lambda (\mathcal{L}_\lambda^{-1} - \mathcal{L}_{D,\lambda}^{-1})(\nu \times) \widetilde{\mathcal{L}}_\lambda^{\mathtt{t}}) = -\operatorname{tr}\Big(\Big(\mathcal{L}_\lambda + \frac{\lambda}{2} \frac{\mathrm{d}}{\mathrm{d}\lambda} \mathcal{L}_\lambda\Big)(\mathcal{L}_\lambda^{-1} - \mathcal{L}_{D,\lambda}^{-1})\Big) \\ &= -\operatorname{tr}(\mathrm{id} - \mathcal{L}_\lambda \mathcal{L}_{D,\lambda}^{-1}) - \frac{\lambda}{2} \frac{\mathrm{d}}{\mathrm{d}\lambda} \log\det(\mathcal{L}_\lambda \mathcal{L}_{D,\lambda}^{-1}) = -\frac{\lambda}{2} \frac{\mathrm{d}}{\mathrm{d}\lambda} \log\det(\mathcal{L}_\lambda \mathcal{L}_{D,\lambda}^{-1}). \end{aligned}$$

Here we have used that $\operatorname{tr}\big(\mathcal{L}_{D,\lambda}^{-1} \frac{\mathrm{d}}{\mathrm{d}\lambda}(\mathcal{T}_\lambda)\big) = 0$ and $\operatorname{tr}(\mathcal{L}_{D,\lambda}^{-1} \mathcal{T}_\lambda) = 0$. Indeed this follows as

$$\operatorname{Tr}\Big(\mathcal{L}_{\lambda,D}^{-1}\Big(\frac{\mathrm{d}}{\mathrm{d}\lambda}\mathcal{T}_\lambda\Big)\Big) = \sum_{j \neq k} \operatorname{Tr}\Big(\mathcal{L}_{\lambda,D}^{-1}\Big(q_j \Big(\frac{\mathrm{d}}{\mathrm{d}\lambda}\mathcal{L}_\lambda\Big) q_k\Big)\Big) = \sum_{j \neq k} \operatorname{Tr}\Big(\Big(q_j \mathcal{L}_{\lambda,D}^{-1} \frac{\mathrm{d}}{\mathrm{d}\lambda}\mathcal{L}_\lambda\Big) q_k\Big) = 0.$$

We have also used the fact that, for a holomorphic family of trace-class operators $A(\lambda)$, we have that $\log\det(\mathrm{id} + A(\lambda))$ is holomorphic and we have the identity

$$\frac{\mathrm{d}}{\mathrm{d}\lambda} \log\det(\mathrm{id} + A(\lambda)) = \operatorname{tr}\Big((\mathrm{id} + A(\lambda))^{-1} \frac{\mathrm{d}}{\mathrm{d}\lambda} A(\lambda)\Big),$$

so that

$$\frac{\mathrm{d}}{\mathrm{d}\lambda} \log\det(\mathcal{L}_\lambda \mathcal{L}_{D,\lambda}^{-1}) = \operatorname{tr}\Big(\mathcal{L}_\lambda^{-1} \frac{\mathrm{d}}{\mathrm{d}\lambda}(\mathcal{L}_\lambda) - \mathcal{L}_{D,\lambda}^{-1} \frac{\mathrm{d}}{\mathrm{d}\lambda}(\mathcal{L}_{D,\lambda})\Big).$$

In the same way,

$$\begin{aligned} \operatorname{tr}(R_{D,\mathrm{abs},\lambda}) &= -\operatorname{tr}(\lambda^2 \widetilde{\mathcal{M}}_\lambda (\mathcal{L}_\lambda^{-1} - \mathcal{L}_{D,\lambda}^{-1})(\nu \times) \widetilde{\mathcal{M}}_\lambda^{\mathtt{t}}) = -\operatorname{tr}\Big(\Big(\frac{\lambda}{2} \frac{\mathrm{d}}{\mathrm{d}\lambda} \mathcal{L}_\lambda\Big)(\mathcal{L}_\lambda^{-1} - \mathcal{L}_{D,\lambda}^{-1})\Big) \\ &= -\frac{\lambda}{2} \frac{\mathrm{d}}{\mathrm{d}\lambda} \log\det(\mathcal{L}_\lambda \mathcal{L}_{D,\lambda}^{-1}) \end{aligned}$$

□

11. Proof of the main theorems

Proof of Theorem 1.1. This theorem is the combination of Proposition 9.3 and Theorem 9.5 in Section 9. □

Proof of Theorem 1.3. We set $f(z) = z^{-2}g(z^2)$, where $g \in \mathcal{P}_\epsilon$. By the decay properties of Ξ, it is sufficient to show equality for small ϵ, so we assume $\epsilon < \frac{\pi}{4}$. Then the function e^{-1/nz^2} is holomorphic in the sector $\mathfrak{S}_\epsilon$ and decays faster than exponentially. The function $g_n(z) = e^{-1/nz^2} g(z)$ is therefore an admissible function for the Riesz–Dunford functional calculus, and we therefore have

$$f_n((-\Delta_{\mathrm{rel}})^{1/2})\,\mathrm{curl}\,\mathrm{curl} = f_n((\delta\,\mathrm{d})^{1/2})\delta\,\mathrm{d} = g_n(\delta\,\mathrm{d}) = -\frac{1}{2\pi\mathrm{i}}\int_{\Gamma_\epsilon}(\delta\,\mathrm{d}-z)^{-1}g_n(z)\,\mathrm{d}z$$

and similarly for the other terms appearing in $R_{D,\mathrm{rel},\lambda}$. The integral converges despite the pole of order 1 at 0 since $g \in \mathcal{P}_\epsilon$ implies that $g_n(z) = O(|z|^\alpha)$ for some $\alpha > 0$ near $z = 0$. Here $f_n(z) = z^{-2}g_n(z^2)$. If $h \in C_0^\infty(X, \mathbb{C}^3)$ then we have convergence of $g_n(\delta\,\mathrm{d})$ to $g(\delta\,\mathrm{d})$ in L^2. Indeed, by our definition of the function class $\mathcal{P}_\epsilon$, the function g is polynomially bounded on the real line and therefore h is in the domain of the operator $g(\delta\,\mathrm{d})$. Consequently the function g is square-integrable with respect to the measure $\langle \mathrm{d}E_\lambda h, h\rangle$, where $\mathrm{d}E_\lambda$ is the spectral measure of $\delta\,\mathrm{d}$. Then we have

$$\|(g(\delta\,\mathrm{d}) - g_n(\delta\,\mathrm{d}))h\|_{L^2} = \int_{\mathbb{R}}(1-e^{-x^2/n})^2|g|^2(x)\,\langle \mathrm{d}E_\lambda h, h\rangle,$$

which tends to 0 as $n \to \infty$ by the dominated convergence theorem.

We note now that

$$(-\Delta_{\mathrm{rel}} - z)^{-1}\delta\,\mathrm{d} = (\delta\,\mathrm{d}-z)^{-1}\delta\,\mathrm{d} = \mathrm{id} + z(\delta\,\mathrm{d}-z)^{-1},$$

and again this formula applies to the other terms in $R_{D,\mathrm{rel},\lambda}$. This gives

$$D_{\mathrm{rel},f_n} = -\frac{1}{2\pi\mathrm{i}}\int_{\Gamma_\epsilon}R_{D,\mathrm{rel}}(\sqrt{z})\frac{1}{z}g_n(z)\,\mathrm{d}z = -\frac{1}{\mathrm{i}\pi}\int_{\widetilde{\Gamma}_{\epsilon/2}}R_{D,\mathrm{rel}}(\lambda)\lambda f_n(\lambda)\,\mathrm{d}\lambda.$$

Moreover, $D_{\mathrm{rel},f_n}h$ converges in L^2 to $D_{\mathrm{rel},f}h$ for any $h \in C_0^\infty(X, \mathbb{C}^3)$. By the decay properties of $R_{D,\mathrm{rel}}(\lambda)$, Proposition 10.1, the integral converges in the Banach space of trace-class operators, and the sequence D_{f_n} is Cauchy in the Banach space of trace-class operators. We conclude that $D_{\mathrm{rel},f}$ is trace-class.

To compute the trace, we can again use the convergence of the integral in the space of trace-class operators and therefore, using Lemma 10.2, we obtain

$$D_{\mathrm{rel},f} = -\frac{1}{\mathrm{i}\pi}\int_{\widetilde{\Gamma}_{\epsilon/2}}R_{\mathrm{rel},D}(\lambda)\lambda f(\lambda)\,\mathrm{d}\lambda = \frac{1}{2\pi\mathrm{i}}\int_{\widetilde{\Gamma}_{\epsilon/2}}(\Xi'(\lambda))\lambda^2 f(\lambda)\,\mathrm{d}\lambda.$$

Integration by parts and the decay of Ξ, Theorem 9.5, then completes the proof for $D_{\mathrm{rel},f}$. The proof for $D_{\mathrm{abs},f}$ is exactly the same. □

Proof of Theorem 1.4. We first establish the smoothness away from the objects. To see this we again use the Riesz–Dunford functional calculus. Let $\kappa_\lambda(x, y)$ be the integral kernel of the difference

$$(-\Delta_{\mathrm{rel}} - \lambda^2)^{-1}\mathrm{curl}\,\mathrm{curl} - (-\Delta_{\mathrm{free}} - \lambda^2)^{-1}\,\mathrm{curl}\,\mathrm{curl}.$$

Let U be an open neighbourhood of $\partial\Omega$ such that $\operatorname{dist}(U, \Omega_0) > \delta' \in (0, \delta)$. Then, on $\Omega_0 \times \Omega_0$, the integral kernel of $\kappa_\lambda(x, y)$ satisfies the estimate

$$\|\kappa_\lambda(x, y)\|_{C^k(K)} \leq C_{k,K} e^{-\delta' \operatorname{Im} \lambda}$$

for any compact subset $K \subset \Omega_0 \times \Omega_0$. This can be seen directly from (62), as Lemma A.1 implies that the integral kernel of $\widetilde{\mathcal{L}}_\lambda$ is smooth and C^∞-seminorms satisfy an exponential decay estimate on $\Omega_0 \times U$, whereas the norm of $\mathcal{L}_\lambda^{-1}$ is polynomially bounded by Theorem 7.9. By the same argument as in the proof of Theorem 1.3 above, the integral

$$2 \int_{\widetilde{\Gamma}_{\epsilon/2}} \kappa_\lambda(x, y) \lambda f(\lambda)\, d\lambda$$

then converges in $C^\infty(\Omega_0 \times \Omega_0)$ to the integral kernel of B_f restricted to $\Omega_0 \times \Omega_0$. Hence this kernel is smooth on $\Omega_0 \times \Omega_0$. It remains to show the decay estimate. For large $|x|$, we have by (67) the estimate

$$\|\kappa_\lambda(x, x)\| \leq \frac{C}{|x|^4} e^{-\delta' |x| \operatorname{Im} \lambda}.$$

Then, using functional calculus as before, we have the representation

$$\kappa(x, x) = \frac{i}{\pi} \int_{\widetilde{\Gamma}_{\epsilon/2}} \kappa_\lambda(x, x) \lambda f(\lambda)\, d\lambda,$$

which gives the estimate

$$\|\kappa(x, x)\| \leq \int_1^\infty \frac{C}{|x|^4} \lambda e^{-\delta_1 \lambda |x|}\, d\lambda + \int_0^1 \frac{C}{|x|^4} \lambda e^{-\delta_1 \lambda |x|} \lambda^a\, d\lambda \leq \frac{C_1}{|x|^{6+a}}.$$

This shows that $\kappa(x, x)$ is integrable and by Mercer's theorem the integral of $\operatorname{tr}(\kappa(x, x))$ is equal to the trace, as claimed. □

Proof of Theorem 1.5. Define the relative spectral shift function

$$\xi_D(\lambda) = \frac{1}{2\pi i} \log \frac{\det S_\lambda}{\det(S_{1,\lambda}) \cdots \det(S_{N,\lambda})}.$$

By the Birman–Krein formula we have

$$\operatorname{tr} D_{\mathrm{rel}, f} = -\int_0^\infty \xi_D(\lambda) \frac{d}{d\lambda} (\lambda^2 f(\lambda))\, d\lambda$$

for any even Schwartz function f.

Recall that Ξ' has a meromorphic extension to the complex plane and is holomorphic on the real line. Now assume that f is a compactly supported even test function, and let $\tilde{f}$ be a compactly supported almost analytic extension; see for example [Davies 1995, p. 169-170]. Let $dm(z) = dx\, dy$ be the Lebesgue measure on $\mathbb{C}$. By the Helffer–Sjöstrand formula [Davies 1995; Helffer and Sjöstrand 1989] combined with the substitution $z \mapsto z^2$, we have

$$\operatorname{curl} \operatorname{curl} f(\Delta_{\mathrm{rel}}^{1/2}) = \frac{2}{\pi} \operatorname{curl} \operatorname{curl} \int_{\operatorname{Im} z > 0} z \frac{\partial \tilde{f}}{\partial \bar{z}} (\Delta_{\mathrm{rel}} - z^2)^{-1}\, dm(z).$$

Therefore

$$D_{\mathrm{rel},f} = \frac{2}{\pi}\int_{\operatorname{Im} z>0} z\frac{\partial \tilde{f}}{\partial \bar{z}} R_{D,\mathrm{rel}}(z)\,\mathrm{d}m(z),$$

and hence, by Lemma 10.2, we have

$$\operatorname{Tr}(D_{\mathrm{rel},f}) = -\frac{1}{\pi}\int_{\operatorname{Im} z>0} z^2\frac{\partial \tilde{f}}{\partial \bar{z}}\Xi'(z)\,\mathrm{d}m(z).$$

Using Stokes' theorem in the form of [Hörmander 2003, p. 62-63], we therefore obtain

$$\operatorname{Tr}(D_f) = \frac{\mathrm{i}}{2\pi}\int_{\mathbb{R}}(\Xi'(x)+\Xi'(-x))x^2 f(x)\,\mathrm{d}x.$$

Comparing this with the Birman–Krein formula in Theorem 5.1 gives $\frac{\mathrm{i}}{2\pi}(\Xi'(x)+\Xi'(-x)) = \xi_D'(x)$. Since both functions are meromorphic, we have that this identity holds everywhere. We conclude that $\frac{\mathrm{i}}{2\pi}(\Xi(\lambda)-\Xi(-\lambda))-\xi_D(\lambda)$ is constant. Clearly, $(\Xi(\lambda)-\Xi(-\lambda))$ vanishes at 0, so the statement follows if we can show that $\xi_D(0)=0$. The estimate [Strohmaier and Waters 2020, Theorem 1.10] shows that $S_0 = S_{1,0} = \cdots = S_{N,0} = \mathrm{id}$, which then indeed implies $\xi_D(0)=0$. The paper [Strohmaier and Waters 2020] assumes the boundary of Ω to be smooth, but the section on the expansions in this paper carry over unmodified to the Lipschitz case (see also the remarks in [Strohmaier and Waters 2022] where this is made explicit). □

Appendix

A.1. *Norm estimates.* In the following we assume that Ω and M are as in the main body of the text. Recall that the integral kernel of the free resolvent is given by (12). We will subsequently prove norm and pointwise estimates for $G_{\lambda,0}$ and its derivatives, which are used in the main body of the text.

Lemma A.1. *Let $\Omega_0 \subset M$ be an open set with $\operatorname{dist}(\Omega_0,\partial\Omega)=\delta>0$, and choose $\epsilon\in(0,\pi]$. Let U be a bounded open neighbourhood of the boundary $\partial\Omega$ such that $\operatorname{dist}(\Omega_0,U)>0$, and fix $\delta'>0$ such that $\delta'<\operatorname{dist}(\Omega_0,U)\le\delta$. Then, for any $k\in\mathbb{N}_0$, there exists $C_{k,\delta',\epsilon}>0>0$ such that*

$$\|G_{\lambda,0}\|^2_{H^k(\Omega_0\times U)} \le C_{k,\delta',\epsilon}\frac{(1+\operatorname{Im}\lambda)\mathrm{e}^{-2\delta'\operatorname{Im}\lambda}}{\operatorname{Im}\lambda}, \tag{83}$$

$$\|\nabla_x G_{\lambda,0}\|^2_{H^k(\Omega_0\times U)} \le C_{k,\delta',\epsilon}\mathrm{e}^{-2\delta'\operatorname{Im}\lambda} \tag{84}$$

for all $\lambda\in\mathfrak{D}_\epsilon$. Here ∇_x denotes differentiation in the first variable, i.e., $(\nabla_x G_\lambda)(x,y)=\nabla_x G_\lambda(x,y)$.

Proof. Let us set $\lambda=\theta|\lambda|$ and note that $\operatorname{Im}\theta\ge\sin(\epsilon)>0$. Since the kernel $G_{\lambda,0}$ satisfies the Helmholtz equation in both variables away from the diagonal, we have $((-\Delta_x)^k+(-\Delta_y)^k)G_{\lambda,0}(x,y) = 2\lambda^{2k}G_{\lambda,0}(x,y)$. We then change variables, so that $r:=|x-y|\ge\delta_0$. By homogeneity, all of the integration will be carried out in this variable, with the angular variables only contributing a constant. Substituting $s:=\operatorname{Im}\lambda r$ into the formula for the Green's function implies, for all $k\in\mathbb{N}$, that

$$\|\Delta^k G_{\lambda,0}\|^2_{L^2(\Omega_0'\times U')} \le C_k(\operatorname{Im}\lambda)^{4k}\int_{\delta_0}^{\infty}|G_{\lambda,0}(r)|^2 r^2\,\mathrm{d}r \le C_k(\operatorname{Im}\lambda)^{4k}\int_{\delta_0}^{\infty}\mathrm{e}^{-2\operatorname{Im}\lambda r}\,\mathrm{d}r. \tag{85}$$

Here we have enlarged the domains slightly, so that $\Omega_0' \times U'$ has positive distance from $\Omega_0 \times U$ and $\mathrm{dist}(\Omega_0', U') > \delta'$. This allows us to estimate the Sobolev norms using Lemma A.5. We then have

$$\int_{\delta_0}^{\infty} \mathrm{e}^{-2\,\mathrm{Im}\,\lambda r}\,\mathrm{d}r = \frac{\mathrm{e}^{-2\delta_0\,\mathrm{Im}\,\lambda}}{-2\,\mathrm{Im}\,\lambda}. \tag{86}$$

Let $C_{\delta',\epsilon,k}$ denote a generic constant depending on $\delta',\ \epsilon,\ k$. Using (85) and interpolation, we can conclude, for all $k \geq 0$, that we have

$$\|G_{\lambda,0}\|^2_{H^k(\Omega_0\times U)} \leq C_{\delta',\epsilon,k}\frac{(1+\mathrm{Im}\,\lambda)\mathrm{e}^{-2\delta'\,\mathrm{Im}\,\lambda}}{\mathrm{Im}\,\lambda}. \tag{87}$$

The second inequality follows by replacing $G_{\lambda,0}$ by $\nabla_x G_{\lambda,0}$ in (85). We then have

$$\begin{aligned}\|\Delta_x^k\nabla_x G_{\lambda,0}\|^2_{L^2(\Omega_0'\times U')} &\leq C_k(\mathrm{Im}\,\lambda)^{4k}\int_{\delta_0}^{\infty}|\nabla_x G_{\lambda,0}(r)|^2 r^2\,\mathrm{d}r\\ &\leq C_k(\mathrm{Im}\,\lambda)^{4k}\int_{\delta_0}^{\infty}\Big(|\mathrm{Im}\,\lambda|^2+\frac{1}{r^2}\Big)\mathrm{e}^{-2\,\mathrm{Im}\,\lambda r}\,\mathrm{d}r \leq C_k\mathrm{e}^{-2\delta'\,\mathrm{Im}\,\lambda}.\end{aligned} \tag{88}$$

The proof is complete. □

We now combine these estimates to get an estimate on the Maxwell layer potential operator.

Lemma A.2. *Let $\Omega_0 \subset M$ be an open set with $\mathrm{dist}(\Omega_0, \Omega) = \delta > 0$ and $\lambda \in \mathfrak{D}_\epsilon$. Then, for any $0 < \delta' < \delta$, there exists $C_{\delta',\epsilon} > 0$ such that*

$$\|\widetilde{\mathcal{L}}_\lambda\|^2_{H^{-1/2}(\mathrm{Div},\partial\Omega)\to H(\mathrm{curl},\Omega_0)} \leq C_{\delta',\epsilon}\mathrm{e}^{-2\delta'\,\mathrm{Im}\,\lambda} \tag{89}$$

and

$$\|\widetilde{\mathcal{L}}_\lambda\|^2_{H^{-1/2}(\mathrm{Div}\,0,\partial\Omega)\to H(\mathrm{curl},\Omega_0)} \leq C_{\delta',\epsilon}|\mathrm{Im}\,\lambda|^3\mathrm{e}^{-2\delta'\,\mathrm{Im}\,\lambda}. \tag{90}$$

Proof. We choose as in Lemma A.1 a bounded open neighbourhood of $\partial\Omega$. For $a \in H^{-1/2}(\partial\Omega)$, the distribution $\gamma_t^*(a)$ is, by duality, in $H_{\mathrm{c}}^{-1}(U)$. The first inequality then follows by using Lemma A.1 and bearing in mind that integration defines a continuous map

$$H^k(\Omega_0\times U)\times H_{\mathrm{c}}^{-s}(U) \to H^{k-s}(\Omega_0)$$

for k large enough. The second inequality follows from the identity (16), namely that we can write

$$\widetilde{\mathcal{L}}_\lambda a = \nabla\widetilde{\mathcal{S}}_\lambda\,\mathrm{Div}\,a + \lambda^2\widetilde{\mathcal{S}}_\lambda a, \quad a \in H^{-1/2}(\mathrm{Div},\partial\Omega), \tag{91}$$

and again using Lemma A.1 in the same way as above. □

Lemma A.3. *Let $k \in H^2(\mathbb{R}^d \times \mathbb{R}^d)$. Then k is the integral kernel of a Hilbert–Schmidt operator*

$$K : H^{-1}(\mathbb{R}^d) \to H^1(\mathbb{R}^d),$$

with Hilbert–Schmidt norm bounded by $\|k\|_{H^2(\mathbb{R}^d\times\mathbb{R}^d)}$.

Proof. Let K be the integral operator with kernel k. Since $(-\Delta+1)^{1/2}$ is an isometry from $L^2(\mathbb{R}^d)$ to $H^{-1}(\mathbb{R}^d)$ and from $H^1(\mathbb{R}^d)$ to $L^2(\mathbb{R}^d)$, it suffices to show that $(-\Delta+1)^{1/2}K(-\Delta+1)^{1/2}$ is Hilbert–Schmidt from $L^2(\mathbb{R}^d)$ to $L^2(\mathbb{R}^d)$ and bound its Hilbert–Schmidt norm. This is equivalent to the distributional integral kernel of $(-\Delta+1)^{1/2}K(-\Delta+1)^{1/2}$ being in $L^2(\mathbb{R}^d\times\mathbb{R}^d)$; see for example [Shubin 1987]. The Hilbert–Schmidt norm is equal to the L^2-norm of the kernel. The Fourier transform is given by $(\xi^2+1)^{1/2}(\eta^2+1)^{1/2}\hat{k}(\xi,\eta)$ and this is in L^2 with the L^2-norm bounded by $\|k\|_{H^2(\mathbb{R}^d\times\mathbb{R}^d)}$ thanks to the inequality

$$\frac{(\xi^2+1)^{1/2}(\eta^2+1)^{1/2}}{\xi^2+\eta^2+1}\le 1. \qquad \square$$

Lemma A.4. *Let $k\in H^4_{\mathrm{c}}(\mathbb{R}^3\times\mathbb{R}^3)$ be supported in a compact set $Q\times Q\subset\mathbb{R}^3\times\mathbb{R}^3$. Then k is the integral kernel of a nuclear operator*

$$K: H^{-1}(\mathbb{R}^3)\to H^1(\mathbb{R}^3),$$

with trace norm bounded by $C_Q\|k\|_{H^4(\mathbb{R}^d\times\mathbb{R}^d)}$.

Proof. Since k is compactly supported in Q, we can assume without loss of generality that Q is a subset of a torus $\mathbb{T}^n$ by imposing periodic boundary conditions on a sufficiently large rectangle and remarking that the Sobolev norms on the torus restricted to a neighbourhood of Q are then equivalent to those of $\mathbb{R}^d$ restricted to that neighbourhood. We can therefore assume without loss of generality that we are on a compact manifold Y. We can then write K as $K=(-\Delta_Y+1)^{-1}(-\Delta_Y+1)K$. The operator $(-\Delta_Y+1)^{-1}$ is Hilbert–Schmidt from $H^1(Y)$ to $H^1(Y)$, as for example can be seen from Weyl's law. The operator $(-\Delta_Y+1)K$ is Hilbert–Schmidt by Lemma A.3. Since we have written the operator as a product of two Hilbert–Schmidt operators, it is nuclear and the corresponding estimate for the nuclear norm follows by estimating in terms of the Hilbert–Schmidt norms. $\square$

Lemma A.5. *Suppose that $\Omega\subset\mathbb{R}^d$ is an open subset, and assume that $\Omega'\subset\mathbb{R}^d$ is a larger subset such that $\overline{\Omega}\subset\Omega'$ and $\operatorname{dist}(\partial\Omega,\partial\Omega')>0$. Let $N\in\mathbb{N}$. Then, for any $f\in L^2(\Omega')$ with $(-\Delta)^k f\in L^2(\Omega')$ for all $k=0,1,\dots,N$, we have $f|_\Omega\in H^{2N}(\Omega)$, and there exists a constant $C_{N,\Omega',\Omega}>0$, independent of f, such that $\|f|_\Omega\|_{H^{2N}(\Omega)}\le C_{N,\Omega',\Omega}\sum_{k\le N}\|(-\Delta)^k f\|_{L^2(\Omega')}$.*

Proof. This is the usual proof of interior regularity applied to the possibly noncompact domain Ω'. We will show that $f\in H^s(\Omega')$, $\Delta f\in H^s(\Omega')$ implies $f\in H^{s+2}(\Omega)$ with the corresponding norm-estimates. The result then follows from this statement by iterating using a sequence of intermediate domains $\Omega\subset\Omega_1\subset\cdots\subset\Omega_{N-1}\subset\Omega'$. We will choose U such that $\Omega\subset U\subset\Omega'$ while we still have $\operatorname{dist}(\partial U,\partial\Omega')>0$, $\operatorname{dist}(\partial U,\partial\Omega)>0$. We can choose a regularised distance function and construct a function $\chi\in C^\infty_b(\mathbb{R}^d)$ which is compactly supported in Ω' which equals 1 in a neighbourhood of U. Then, if $f\in H^s(\Omega')$ and $\Delta f\in H^s(\Omega')$, we have

$$(1-\Delta)(\chi f)=(\chi-\Delta(\chi))f-\chi\Delta f-2(\nabla\chi)\nabla f.$$

From this we see that $(-\Delta+1)(\chi f)\in H^{s-1}(\mathbb{R}^d)$ and therefore $\chi f\in H^{s+1}(\mathbb{R}^d)$. Hence the restriction of f to U is in $H^{s+1}(\Omega_1)$. Now we choose another cut-off function η in $C^\infty_b(\mathbb{R}^d)$ supported in U that equals 1 near Ω. Then $(\nabla\eta)\nabla f$ is in $H^s(U)$, and we now conclude in the same way that $f|_\Omega\in H^{s+2}(\Omega)$. $\square$

References

[Balian and Duplantier 1978] R. Balian and B. Duplantier, "Electromagnetic waves near perfect conductors, II: Casimir effect", *Ann. Physics* **112**:1 (1978), 165–208. MR Zbl

[Bimonte and Emig 2021] G. Bimonte and T. Emig, "Unifying theory for Casimir forces: bulk and surface formulations", *Universe* **7**:7 (2021), art. id. 225. Zbl

[Birman and Solomyak 1987] M. S. Birman and M. Z. Solomyak, "Weyl asymptotics of the spectrum of the Maxwell operator for domains with a Lipschitz boundary", *Vestnik Leningrad. Univ. Mat. Mekh. Astronom.* (1987), 23–28. In Russian. MR Zbl

[Bordag et al. 2009] M. Bordag, G. L. Klimchitskaya, U. Mohideen, and V. M. Mostepanenko, *Advances in the Casimir effect*, Oxford University Press, 2009. Zbl

[Brüning and Lesch 1992] J. Brüning and M. Lesch, "Hilbert complexes", *J. Funct. Anal.* **108**:1 (1992), 88–132. MR Zbl

[Buffa and Hiptmair 2003] A. Buffa and R. Hiptmair, "Galerkin boundary element methods for electromagnetic scattering", pp. 83–124 in *Topics in computational wave propagation*, edited by M. Ainsworth et al., Lect. Notes Comput. Sci. Eng. **31**, Springer, Berlin, 2003. MR Zbl

[Buffa et al. 2002] A. Buffa, M. Costabel, and D. Sheen, "On traces for $\mathbf{H}(\mathbf{curl}, \Omega)$ in Lipschitz domains", *J. Math. Anal. Appl.* **276**:2 (2002), 845–867. MR Zbl

[Buslaev and Merkur' ev 1969] V. S. Buslaev and S. P. Merkur' ev, "Trace equation for a three-particle system", *Dokl. Akad. Nauk SSSR* (1969), 1055–1057. In Russian; translated in *Soviet Physics Dokl.* **14** (1969), 1055–1057. MR Zbl

[Candelas 1982] P. Candelas, "Vacuum energy in the presence of dielectric and conducting surfaces", *Ann. Physics* **143**:2 (1982), 241–295. MR

[Carron 2002] G. Carron, "Déterminant relatif et la fonction Xi", *Amer. J. Math.* **124**:2 (2002), 307–352. MR Zbl

[Casimir 1948] H. B. G. Casimir, "On the attraction between two perfectly conducting plates", *Proc. Akad. Wet. Amsterdam* **51** (1948), 793–795. Zbl

[Claeys and Hiptmair 2019] X. Claeys and R. Hiptmair, "First-kind boundary integral equations for the Hodge–Helmholtz operator", *SIAM J. Math. Anal.* **51**:1 (2019), 197–227. MR Zbl

[Costabel 1988] M. Costabel, "Boundary integral operators on Lipschitz domains: elementary results", *SIAM J. Math. Anal.* **19**:3 (1988), 613–626. MR Zbl

[Costabel 1990] M. Costabel, "A remark on the regularity of solutions of Maxwell's equations on Lipschitz domains", *Math. Methods Appl. Sci.* **12**:4 (1990), 365–368. MR Zbl

[Davies 1995] E. B. Davies, "The functional calculus", *J. London Math. Soc.* (2) **52**:1 (1995), 166–176. MR Zbl

[Emig et al. 2007] T. Emig, N. Graham, R. L. Jaffe, and M. Kardar, "Casimir forces between arbitrary compact objects", *Phys. Rev. Lett.* **99** (2007), art. id. 170403. Zbl

[Fang and Strohmaier 2022] Y.-L. Fang and A. Strohmaier, "A mathematical analysis of Casimir interactions, I: The scalar field", *Ann. Henri Poincaré* **23**:4 (2022), 1399–1449. MR Zbl

[Filonov 2013] N. Filonov, "Weyl asymptotics of the spectrum of the Maxwell operator in Lipschitz domains of arbitrary dimension", *Algebra i Analiz* **25**:1 (2013), 170–215. In Russian; translated in *St. Petersburg Math. J.* **25**:1 (2014), 117–149. MR Zbl

[Gesztesy and Simon 1996] F. Gesztesy and B. Simon, "The xi function", *Acta Math.* **176**:1 (1996), 49–71. MR Zbl

[Gol'dshtein et al. 2011] V. Gol'dshtein, I. Mitrea, and M. Mitrea, "Hodge decompositions with mixed boundary conditions and applications to partial differential equations on Lipschitz manifolds", *J. Math. Sci.* **172**:3 (2011), 347–400. MR Zbl

[Hanisch et al. 2022] F. Hanisch, A. Strohmaier, and A. Waters, "A relative trace formula for obstacle scattering", *Duke Math. J.* **171**:11 (2022), 2233–2274. MR Zbl

[Helffer and Sjöstrand 1989] B. Helffer and J. Sjöstrand, "Équation de Schrödinger avec champ magnétique et équation de Harper", pp. 118–197 in *Schrödinger operators* (Sønderborg, Denmark, 1988), edited by H. Holden and A. Jensen, Lecture Notes in Phys. **345**, Springer, 1989. MR Zbl

[Hörmander 2003] L. Hörmander, *The analysis of linear partial differential operators, I: Distribution theory and Fourier analysis*, 2nd ed., Springer, 2003. MR Zbl

[Johnson 2011] S. G. Johnson, "Numerical methods for computing Casimir interactions", pp. 175–218 in *Casimir physics*, edited by D. Dalvit et al., Springer, 2011. Zbl

[Kay 1979] B. S. Kay, "Casimir effect in quantum field theory", *Phys. Rev. D* **20** (1979), 3052–3062. Zbl

[Kenneth and Klich 2006] O. Kenneth and I. Klich, "Opposites attract: a theorem about the Casimir force", *Phys. Rev. Lett.* **97** (2006), art. id. 160401. Zbl

[Kenneth and Klich 2008] O. Kenneth and I. Klich, "Casimir forces in a T-operator approach", *Phys. Rev. B* **78** (2008), art. id. 014103. Zbl

[Kirsch and Hettlich 2015] A. Kirsch and F. Hettlich, *The mathematical theory of time-harmonic Maxwell's equations*: *expansion-, integral-, and variational methods*, Applied Mathematical Sciences **190**, Springer, 2015. MR Zbl

[Kirsten 2002] K. Kirsten, *Spectral functions in mathematics and physics*, Chapman & Hall/CRC, 2002. MR Zbl

[Mitrea 1995] M. Mitrea, "The method of layer potentials in electromagnetic scattering theory on nonsmooth domains", *Duke Math. J.* **77**:1 (1995), 111–133. MR Zbl

[Mitrea 2000] D. Mitrea, "Boundary value problems for harmonic vector fields on nonsmooth domains", pp. 234–239 in *Integral methods in science and engineering* (Houghton, MI, 1998), edited by B. Bertram et al., Chapman & Hall/CRC Res. Notes Math. **418**, Chapman & Hall/CRC, Boca Raton, FL, 2000. MR Zbl

[Mitrea and Mitrea 2002] D. Mitrea and M. Mitrea, "Finite energy solutions of Maxwell's equations and constructive Hodge decompositions on nonsmooth Riemannian manifolds", *J. Funct. Anal.* **190**:2 (2002), 339–417. MR Zbl

[Mitrea et al. 1997] D. Mitrea, M. Mitrea, and J. Pipher, "Vector potential theory on nonsmooth domains in $\mathbb{R}^3$ and applications to electromagnetic scattering", *J. Fourier Anal. Appl.* **3**:2 (1997), 131–192. MR Zbl

[Mitrea et al. 2001] D. Mitrea, M. Mitrea, and M. Taylor, *Layer potentials, the Hodge Laplacian, and global boundary problems in nonsmooth Riemannian manifolds*, Mem. Amer. Math. Soc. **713**, Amer. Math. Soc., Providence, RI, 2001. MR Zbl

[Renne 1971] M. Renne, "Microscopic theory of retarded Van der Waals forces between macroscopic dielectric bodies", *Physica* **56**:1 (1971), 125–137. Zbl

[Shubin 1987] M. A. Shubin, *Pseudodifferential operators and spectral theory*, Springer, 1987. MR Zbl

[Simon 1977] B. Simon, "Notes on infinite determinants of Hilbert space operators", *Advances in Math.* **24**:3 (1977), 244–273. MR Zbl

[Strohmaier 2021] A. Strohmaier, "The classical and quantum photon field for non-compact manifolds with boundary and in possibly inhomogeneous media", *Comm. Math. Phys.* **387**:3 (2021), 1441–1489. MR Zbl

[Strohmaier and Waters 2020] A. Strohmaier and A. Waters, "Geometric and obstacle scattering at low energy", *Comm. Partial Differential Equations* **45**:11 (2020), 1451–1511. MR Zbl

[Strohmaier and Waters 2022] A. Strohmaier and A. Waters, "The Birman–Krein formula for differential forms and electromagnetic scattering", *Bull. Sci. Math.* **179** (2022), art. id. 103166. MR Zbl

[Vasy and Wang 2002] A. Vasy and X. P. Wang, "Smoothness and high energy asymptotics of the spectral shift function in many-body scattering", *Comm. Partial Differential Equations* **27**:11-12 (2002), 2139–2186. MR Zbl

[Verchota 1984] G. Verchota, "Layer potentials and regularity for the Dirichlet problem for Laplace's equation in Lipschitz domains", *J. Funct. Anal.* **59**:3 (1984), 572–611. MR Zbl

Received 9 Sep 2022. Revised 28 Jul 2023. Accepted 21 Nov 2023.

ALEXANDER STROHMAIER: a.strohmaier@math.uni-hannover.de
Institute of Analysis, Leibniz University Hannover, Hannover, Germany

ALDEN WATERS: alden.waters@math.uni-hannover.de
Institute of Analysis, Leibniz University Hannover, Hannover, Germany

Analysis & PDE
msp.org/apde

PRODUCTION

production@msp.org

Silvio Levy, Scientific Editor

Cover image: Eric J. Heller: “Linear Ramp”

See inside back cover or msp.org/apde for submission instructions.

The subscription price for 2025 is US $475/year for the electronic version, and $735/year (+$70, if shipping outside the US) for print and electronic. Subscriptions, requests for back issues from the last three years and changes of subscriber address should be sent to MSP.

Analysis & PDE (ISSN 1948-206X electronic, 2157-5045 printed) at Mathematical Sciences Publishers, 798 Evans Hall #3840, c/o University of California, Berkeley, CA 94720-3840, is published continuously online.

APDE peer review and production are managed by EditFlow® from MSP.

PUBLISHED BY

mathematical sciences publishers

nonprofit scientific publishing

http://msp.org/

Analysis & PDE

Volume 18 No. 2 2025