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THE PROJECTION CONSTANT FOR THE TRACE CLASS

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We study the projection constant of the space of operators on n-dimensional Hilbert spaces with the trace norm $\mathscr{S}_1(n)$. We show an integral formula for the projection constant of $\mathscr{S}_1(n)$; namely, $\boldsymbol{\lambda}(\mathscr{S}_1(n)) = n\int_{\mathscr{U}_n}|\mathrm{tr}(U)|\,dU$, where the integration is with respect to the Haar probability measure on the group $\mathscr{U}_n$ of unitary operators. Using a probabilistic approach, we derive the limit formula $\lim_{n\to\infty}\boldsymbol{\lambda}(\mathscr{S}_1(n))/n = \sqrt{\pi}/2$.

Introduction

The projection constant is a fundamental concept in Banach spaces and their local theory. It has its origins in the study of complemented subspaces of Banach spaces. If X is a complemented subspace of a Banach space Y, then the relative projection constant of X in Y is defined by

$$\begin{aligned}\boldsymbol{\lambda}(X, Y) &= \inf\{\|P\| : P \in \mathscr{L}(Y, X),\ P|_X = \mathrm{Id}_X\}\\ &= \inf\{c > 0 : \forall\, T \in \mathscr{L}(X, Z)\ \exists \text{ an extension } \widetilde{T} \in \mathscr{L}(Y, Z) \text{ with } \|\widetilde{T}\| \le c\|T\|\},\end{aligned}$$

where Id_X denotes the identity operator on X and as usual $\mathscr{L}(U, V)$ denotes the Banach space of all bounded linear operators between the Banach spaces U and V with the uniform norm. In what follows $\mathscr{L}(U) := \mathscr{L}(U, U)$. We use here the convention that $\inf\varnothing = \infty$.

The (absolute) projection constant of X is given by

$$\boldsymbol{\lambda}(X) := \sup \boldsymbol{\lambda}(I(X), Y),$$

where the supremum is taken over all Banach spaces Y and isometric embeddings $I\colon X \to Y$.

It is well known that any Banach space X embeds isometrically into $\ell_\infty(\Gamma)$, where Γ is a nonempty set depending on X (and $\ell_\infty(\Gamma)$ as usual stands for the Banach space of all bounded scalar-valued functions on Γ), and then

$$\boldsymbol{\lambda}(X) = \boldsymbol{\lambda}(X, \ell_\infty(\Gamma)). \tag{1}$$

Thus finding $\boldsymbol{\lambda}(X)$ is equivalent to finding the norm of a minimal projection from $\ell_\infty(\Gamma)$ onto an isometric copy of X in $\ell_\infty(\Gamma)$. Note also the well-known fact that if X is a finite-dimensional Banach and X_1 is a subspace of some $C(K)$-space isometric to X, then $\boldsymbol{\lambda}(X) = \boldsymbol{\lambda}(X_1, C(K))$.

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Let us recall a few concrete cases relevant for our purposes — for an extensive treatment on all of this we refer to the excellent monographs [Diestel et al. 1995; Lindenstrauss and Tzafriri 1977; Pisier 1986; Tomczak-Jaegermann 1989; Wojtaszczyk 1991]. We use standard notation from (local) Banach space theory and note that, throughout the article, all Banach spaces are assumed to be complex. As usual $\mathscr{L}(X)$ denotes the Banach space of all (bounded) linear operators T on X together with the operator norm. For $1 \le p \le \infty$ and $n \in \mathbb{N}$, the symbol ℓ_p^n denotes the Banach space $\mathbb{C}^n$ equipped with the Minkowski norm $\|x\|_p = \big(\sum_{k=1}^n |x_k|^p\big)^{1/p}$ for $1 \le p < \infty$, and $\|x\|_\infty = \sup_{1\le k\le n} |x_k|$ for $p = \infty$.

A well-known simple application of the Hahn–Banach theorem shows that

$$\boldsymbol{\lambda}(\ell_\infty^n) = 1.$$

The exact values of $\boldsymbol{\lambda}(\ell_2^n)$ and $\boldsymbol{\lambda}(\ell_1^n)$ were computed by Grünbaum [1960] and Rutovitz [1965]: If $d\sigma$ stands for the normalised surface measure on the sphere $\mathbb{S}_n(\mathbb{C})$, then

$$\boldsymbol{\lambda}(\ell_2^n) = n \int_{\mathbb{S}_n} |x_1|\, d\sigma = \frac{\sqrt{\pi}}{2} \frac{n!}{\Gamma\big(n+\frac{1}{2}\big)}. \tag{2}$$

On the other hand, if dz denotes the normalised Lebesgue measure on the distinguished boundary $\mathbb{T}^n$ in $\mathbb{C}^n$ and J_0 is the zero Bessel function defined by $J_0(t) = \frac{1}{2\pi}\int_0^\infty \cos(t\cos\varphi)\, d\varphi$, then

$$\boldsymbol{\lambda}(\ell_1^n) = \int_{\mathbb{T}^n} \Big|\sum_{k=1}^n z_k\Big|\, dz = \int_0^\infty \frac{1 - J_0(t)^n}{t^2}\, dt. \tag{3}$$

Moreover, König, Schütt and Tomczak-Jagermann [König et al. 1999] proved that, for $1 \le p \le 2$,

$$\lim_{n\to\infty} \frac{\boldsymbol{\lambda}(\ell_p^n)}{\sqrt{n}} = \frac{\sqrt{\pi}}{2}. \tag{4}$$

Let us turn to the noncommutative analogs of these results. The operator analog of ℓ_∞^n is the Banach space $\mathscr{L}(\ell_2^n)$. By [Gordon and Lewis 1974, Theorem 5.6] it is known that

$$\boldsymbol{\lambda}(\mathscr{L}(\ell_2^n)) = \frac{\pi}{4} \frac{n!^2}{\Gamma\big(n+\frac{1}{2}\big)^2}.$$

The space of Hilbert–Schmidt operators $\mathscr{H}_2(n)$ on ℓ_2^n is a Hilbert space, and we may deduce from (2) that

$$\boldsymbol{\lambda}(\mathscr{H}_2(n)) = \frac{\sqrt{\pi}}{2} \frac{n^2!}{\Gamma\big(n^2+\frac{1}{2}\big)},$$

which in particular leads to the two limits

$$\lim_{n\to\infty} \frac{\boldsymbol{\lambda}(\mathscr{H}_2(n))}{n} = \frac{\sqrt{\pi}}{2} \quad \text{and} \quad \lim_{n\to\infty} \frac{\boldsymbol{\lambda}(\mathscr{L}(\ell_2^n))}{n} = \frac{\pi}{4}. \tag{5}$$

Finite dimensional Schatten classes form the building blocks of a variety of natural objects in noncommutative functional analysis. Recall that the singular numbers $(s_k(u))_{k=1}^n$ of $u \in \mathscr{L}(\ell_2^n)$ are given by the eigenvalues of $|u| = (u^*u)^{1/2}$, and that the Schatten p-class $\mathscr{S}_p(n)$, $1 \le p \le \infty$, by definition is the Banach space of all operators on ℓ_2^n endowed with the norm $\|u\|_p = \big(\sum_{k=1}^n |s_k(u)|^p\big)^{1/p}$ (for $p = \infty$

we here take the maximum over all $1 \le k \le n$). It is well known that the equalities $\mathscr{S}_\infty(n) = \mathscr{L}(\ell_2^n)$ and $\mathscr{S}_2(n) = \mathscr{H}_2(n)$ hold isometrically. We remark that the space $\mathscr{S}_1(n)$ is usually referred to as trace class.

For the noncommutative analog of (3) in the case of $\mathscr{S}_1(n)$, the best known estimate seems to be

$$\frac{n}{3} \le \boldsymbol{\lambda}(\mathscr{S}_1(n)) \le n. \tag{6}$$

The lower bound was proved by Gordon and Lewis [1974], while the upper bound is a consequence of the famous Kadets–Snobar inequality [Kadets and Snobar 1971].

As pointed out in (3), there is a useful integral formula for $\boldsymbol{\lambda}(\ell_1^n)$. Our main aim is to show a noncommutative analog for $\boldsymbol{\lambda}(\mathscr{S}_1(n))$ and to employ it to get the missing limit from (5). More precisely, we prove that

$$\boldsymbol{\lambda}(\mathscr{S}_1(n)) = n \int_{\mathscr{U}_n} |\mathrm{tr}(U)|\, dU,$$

where $\mathrm{tr}(U)$ denotes the trace of the matrix U and the integration is with respect to the Haar probability measure on the unitary group $\mathscr{U}_n$, and then we apply a probabilistic approach (within the so-called Weingarten calculus) to derive

$$\lim_{n\to\infty} \frac{\boldsymbol{\lambda}(\mathscr{S}_1(n))}{n} = \frac{\sqrt{\pi}}{2}.$$

We finish this introduction with a few words on the technique used. An important tool to calculate projection constants, and more generally to obtain minimal projections, is due to [Rudin 1962]; see also [Wojtaszczyk 1991, Chapter III.B]. This technique is sometimes called Rudin's averaging technique, and it for example may be used to prove (2) as well as (3).

Given an isometric subspace X of Y, we outline the main steps of the strategy to find the relative projection constant $\boldsymbol{\lambda}(X, Y)$. First, one selects a possible "natural candidate" $\boldsymbol{P} : Y \to X$ for a minimal projection. Then, one identifies a topological group G acting on $\mathscr{L}(Y)$ such that every $g \in G$ defines an operator T_g acting in a "compatible way" on Y. Next, it is shown that $\boldsymbol{P}$ is the unique projection commuting with all operators T_g, $g \in G$. Afterward, an arbitrary projection $\boldsymbol{Q} : Y \to X$ is considered, and all operators $T_g^{-1}\boldsymbol{Q}T_g$ are averaged with respect to the Haar measure on G. This average commutes with all operators T_g, $g \in G$, and must coincide with $\boldsymbol{P}$. A simple convexity argument is then employed to establish that $\boldsymbol{\lambda}(X, Y) = \|\boldsymbol{P} : Y \to X\|$, and this norm is subsequently analyzed to refine the formula for $\boldsymbol{\lambda}(X, Y)$.

Thus, if here $Y = \ell_\infty(\Gamma)$, then (1) shows that these steps may lead to a formula/estimate of $\boldsymbol{\lambda}(X)$. Let us see how our object of desire $\mathscr{S}_1(n)$ naturally embeds in some reasonable $\ell_\infty(\Gamma)$. It is well known that $\mathscr{L}(\ell_2^n)$ and $\mathscr{S}_1(n)$ are in trace duality; that is, the mapping

$$\mathscr{S}_1(n) \to \mathscr{L}(\ell_2^n)^*, \quad u \mapsto [v \mapsto \mathrm{tr}(uv)], \tag{7}$$

defines a linear and isometric bijection. To go one step further, we may compose this mapping with the restriction map $\mathscr{L}(\ell_2^n)^* \to C(\mathscr{U}_n)$, $u \mapsto u|_{\mathscr{U}_n}$, where $\mathscr{U}_n$ stands for the group of all unitary $n \times n$ matrices, and in fact this leads to an isometric embedding $\mathscr{S}_1(n) \hookrightarrow C(\mathscr{U}_n)$; see Proposition 2.11. So our aim in the following will be to analyze the relative projection constant $\boldsymbol{\lambda}(\mathscr{S}_1(n), C(\mathscr{U}_n))$.

In order to apply Rudin's averaging technique, we first need to develop what we call "unitary harmonics" on the unitary group $\mathscr{U}_n$, which (roughly speaking) are harmonic polynomials in finitely many "matrix variables" z and $\bar{z}$ from the unitary group $\mathscr{U}_n$. All this is deeply inspired by the classical theory of spherical harmonics (see, e.g., [Rudin 1980]), that is, the study of harmonic polynomials in finitely many complex variables z and $\bar{z}$ on the n-dimensional euclidean sphere $\mathbb{S}_n$. Unitary harmonics and their density in $C(\mathscr{U}_n)$ are described in Section 1.

In Section 2 we formulate and prove our main Theorem 2.1. And although this is the sole focus of this work, structuring the proof of Theorem 2.1 carefully shows that parts of it extend to a more abstract version given in Theorem 2.6.

1. Unitary harmonics and their density

We need to extend a few aspects of the theory of spherical harmonics on the sphere $\mathbb{S}_n$ (as developed for example in [Rudin 1980, Chapter 12] and [Atkinson and Han 2012, Chapter 2]) to what we call unitary harmonics on the unitary group $\mathscr{U}_n$.

1.1. ***Unitaries.*** Denote by M_n the space of all $n \times n$ matrices $Z = (z_{k\ell})$ with entries from $\mathbb{C}$. The group $\mathscr{U}_n$ of all unitary $n \times n$ matrices $U = (u_{ij})_{1 \le i,j \le n}$ endowed with the topology induced by $\mathscr{L}(\ell_2^n)$ forms a nonabelian compact group. It is unimodular, and we denote the integral, with respect to the Haar measure on $\mathscr{U}_n$, of a function $f \in L_2(\mathscr{U}_n)$ by

$$\int_{\mathscr{U}_n} f(U)\, dU.$$

Integrals of this type form the so-called Weingarten calculus, which is of outstanding importance in random matrix theory, mathematical physics, and the theory of quantum information; see, e.g., [Collins and Śniady 2006; Köstenberger 2021].

Basically, we will only need the precise values of two concrete integrals from the Weingarten calculus. The first is

$$\int_{\mathscr{U}_n} u_{i,j} \overline{u_{k,\ell}}\, dU = \frac{1}{n} \delta_{i,k} \delta_{j,\ell} \tag{8}$$

for all possible $1 \le i, j, k, \ell \le n$, and the second is

$$\int_{\mathscr{U}_n} |\operatorname{tr}(AU)|^2\, dU = \frac{1}{n} \operatorname{tr}(AA^*) \tag{9}$$

for every $A \in M_n$; see, e.g., [Cerezo et al. 2021, p. 16], [Collins and Śniady 2006], or [Zhang 2014, Corollary 3.6].

Every operator $T\colon M_n \to M_n$ that leaves $\mathscr{U}_n$ invariant (i.e., $T\mathscr{U}_n \subset \mathscr{U}_n$), defines a composition operator

$$C_T\colon L_2(\mathscr{U}_n) \to L_2(\mathscr{U}_n), \quad f \mapsto f \circ T.$$

There are in fact two such operators T (leaving $\mathscr{U}_n$ invariant) of special interest: the left and right multiplication operators L_V and R_V with respect to $V \in \mathscr{U}_n$ are given by

$$L_V(U) := VU \text{ and } R_V(U) := UV, \quad U \in M_n.$$

A subspace $S \subset L_2(\mathscr{U}_n)$ is said to be $\mathscr{U}_n$-invariant whenever it is invariant under all possible composition operators C_{L_V} and C_{R_V} with $V \in \mathscr{U}_n$.

For any closed subspace $S \subset L_2(\mathscr{U}_n)$, we denote by $\pi_S \colon L_2(\mathscr{U}_n) \to L_2(\mathscr{U}_n)$ the orthogonal projection on $L_2(\mathscr{U}_n)$ with range S.

1.2. ***Spherical harmonics.*** The symbol $\mathscr{P}(\mathbb{R}^N)$ stands for all polynomials $f : \mathbb{R}^N \to \mathbb{C}$ of the form

$$f(x) = \sum_{\alpha \in J} c_\alpha x^\alpha, \tag{10}$$

where $J \subset \mathbb{N}_0^N$ is finite and $(c_\alpha)_{\alpha \in J}$ are complex coefficients. Moreover, given $k \in \mathbb{N}_0$, we write $\mathscr{P}_k(\mathbb{R}^N)$ for all k-homogeneous polynomials f of this type; that is, f has a representation like that in (10) with $|\alpha| := \sum \alpha_i = k$ for each $\alpha \in J$.

Observe that in (10) one has $c_\alpha = \partial^\alpha f(0)/\alpha!$ for each $\alpha \in J$, which in particular shows the uniqueness of the coefficients for each $f \in \mathscr{P}(\mathbb{R}^N)$; in the following we often write $c_\alpha = c_\alpha(f)$. A particular consequence is that the linear space $\mathscr{P}(\mathbb{R}^N)$ carries a natural inner product given by

$$\langle f, g \rangle_{\mathscr{P}} := \sum_\alpha \alpha!\, c_\alpha(f)\overline{c_\alpha(g)}, \quad f, g \in \mathscr{P}(\mathbb{R}^N). \tag{11}$$

This scalar product has a useful reformulation. To see this, note first that every $f \in \mathscr{P}(\mathbb{R}^N)$ defines the differential operator $f(D) \colon \mathscr{P}(\mathbb{R}^N) \to \mathscr{P}(\mathbb{R}^N)$ given by

$$f(D)g := \sum_\alpha c_\alpha(f)\partial^\alpha g, \quad g \in \mathscr{P}(\mathbb{R}^N).$$

And then it is straightforward to verify for every $f, g \in \mathscr{P}(\mathbb{R}^N)$ the formula

$$\langle f, g \rangle_{\mathscr{P}} = [f(D)\bar{g}](0). \tag{12}$$

The polynomial $\boldsymbol{t} \in \mathscr{P}_2(\mathbb{R}^N)$ defined by

$$\boldsymbol{t}(x) := \|x\|_2^2, \quad x \in \mathbb{R}^N, \tag{13}$$

is of special importance, since then

$$\Delta := \boldsymbol{t}(D) = \sum_{j=1}^N \frac{\partial^2}{\partial x_j^2} \colon \mathscr{P}(\mathbb{R}^N) \to \mathscr{P}(\mathbb{R}^N)$$

is the Laplace operator. A polynomial $f \in \mathscr{P}(\mathbb{R}^N)$ is said to be harmonic whenever $\Delta f = 0$, and we write $\mathscr{H}(\mathbb{R}^N)$ for the subspace of all harmonic polynomials in $\mathscr{P}(\mathbb{R}^N)$ and $\mathscr{H}_k(\mathbb{R}^N)$ for all k-homogeneous, harmonic polynomials. For each $N \in \mathbb{N}$, one has

$$\mathscr{H}(\mathbb{R}^N) = \operatorname{span}_k \mathscr{H}_k(\mathbb{R}^N). \tag{14}$$

To see this, fix $f = \sum_{\alpha \in J} c_\alpha(f)x^\alpha \in \mathscr{H}(\mathbb{R}^N)$ with degree $d = \max_{\alpha \in J} |\alpha|$. For each $k \in \{0, 1, \ldots, d\}$, let $f_k = \sum_{|\alpha|=k} c_\alpha(f)x^\alpha$ be the k-homogeneous part of f. Since $f = \sum_{k=0}^d f_k$, it remains to show that each f_k is harmonic. Clearly, $\sum_{k=0}^d \Delta f_k = \Delta f = 0$. Since all f_k are supported on disjoint index sets of multi-indices, we conclude that $\Delta f_k = 0$ for each $0 \le k \le d$.

Much of what follows is based on the following well-known decomposition of $\mathscr{P}_k(\mathbb{R}^N)$ into harmonic subspaces; see, e.g., [Atkinson and Han 2012, Theorem 2.18]. For the sake of completeness we include a proof.

Proposition 1.1. *For each $k \in \mathbb{N}_0$ and $N \in \mathbb{N}$,*

$$\mathscr{P}_k(\mathbb{R}^N) = \mathscr{H}_k(\mathbb{R}^N) \oplus \boldsymbol{t} \cdot \mathscr{H}_{k-2}(\mathbb{R}^N) \oplus \boldsymbol{t}^2 \cdot \mathscr{H}_{k-4}(\mathbb{R}^N) \oplus \cdots,$$

where the orthogonal sum, taken with respect to the inner product from (12), *stops when the subscript reaches* 1 *or* 0*.*

Proof. Given $g \in \mathscr{P}_{k-2}(\mathbb{R}^N)$, we let $h(x) := \boldsymbol{t}(x)g(x)$ for all $x \in \mathbb{R}^N$. Since $\boldsymbol{t}(D) = \Delta$, this implies that $h(D) = \Delta \circ g(D) = g(D) \circ \Delta$. Clearly, if now $f \in \mathscr{P}_k(\mathbb{R}^N)$, then by (12)

$$\langle h, f\rangle_{\mathscr{P}} = [h(D)\bar{f}](0) = [g(D)(\Delta \bar{f})](0) = \langle g, \Delta f\rangle_{\mathscr{P}}.$$

Thus, the condition $f \perp \boldsymbol{t}g$ for every $g \in \mathscr{P}_{k-2}(\mathbb{R}^N)$ is equivalent to $\Delta f \perp g$ for every $g \in \mathscr{P}_{k-2}(\mathbb{R}^N)$, which is also equivalent to $f \in \mathscr{H}_k(\mathbb{R}^N)$. As a consequence, we get

$$\mathscr{P}_k(\mathbb{R}^N) = \mathscr{H}_k(\mathbb{R}^N) \oplus \boldsymbol{t} \cdot \mathscr{P}_{k-2}(\mathbb{R}^N).$$

The proof finishes by repeating this procedure for $\mathscr{H}_{k-2}(\mathbb{R}^N)$, $\mathscr{H}_{k-4}(\mathbb{R}^N)$, and so on. ☐

By $\mathbb{S}_N^{\mathbb{R}}$ we denote the sphere in the real Hilbert space $\ell_2^N(\mathbb{R})$. We write $\mathscr{P}(\mathbb{S}_N^{\mathbb{R}})$ for the linear space of all restrictions $f|_{\mathbb{S}_N^{\mathbb{R}}}$ of polynomials $f \in \mathscr{P}(\mathbb{R}^N)$ and $\mathscr{P}_k(\mathbb{S}_N^{\mathbb{R}})$ whenever we only consider restrictions of k-homogeneous polynomials.

All restrictions of harmonic polynomials on $\mathbb{R}^N$ (so polynomials in $\mathscr{H}(\mathbb{R}^N)$) to the sphere $\mathbb{S}_N^{\mathbb{R}}$ are denoted by $\mathscr{H}(\mathbb{S}_N^{\mathbb{R}})$, and such polynomials are called spherical harmonics. Similarly, we denote by $\mathscr{H}_k(\mathbb{S}_N^{\mathbb{R}})$ the space collecting all k-homogeneous polynomials restricted to $\mathbb{S}_N^{\mathbb{R}}$. Endowed with the supremum norm taken on $\mathbb{S}_N^{\mathbb{R}}$, both spaces $\mathscr{H}(\mathbb{S}_N^{\mathbb{R}})$ and $\mathscr{H}_k(\mathbb{S}_N^{\mathbb{R}})$ form subspaces of $C(\mathbb{S}_N^{\mathbb{R}})$.

An important fact, not needed here, is that the spaces $\mathscr{H}_k(\mathbb{S}_N^{\mathbb{R}})$ are pairwise orthogonal in $L_2(\mathbb{S}_N^{\mathbb{R}})$; see, e.g., [Atkinson and Han 2012, Corollary 2.15].

1.3. ***Unitary harmonics.*** Going one step further, we extend the notion of spherical harmonics on the real sphere $\mathbb{S}_N^{\mathbb{R}}$ to what we call unitary harmonics on the unitary group $\mathscr{U}_n$.

Recall that M_n here stands for the space of all $n \times n$ matrices $Z = (z_{k\ell})$ with entries from $\mathbb{C}$. The subset of such matrices $\alpha = (\alpha_{k\ell})$ with entries from $\mathbb{N}_0$ is denoted by $M_n(\mathbb{N}_0)$. For $Z \in M_n$ and $\alpha = (\alpha_{k\ell}) \in M_n(\mathbb{N}_0)$, we define

$$Z^\alpha = \prod_{k,\ell=1}^n z_{k\ell}^{\alpha_{k\ell}}.$$

We identify M_n with $\mathbb{R}^{2n^2}$ in the canonical way through the bijective mapping

$$\boldsymbol{I}_n \colon M_n \longrightarrow \mathbb{R}^{2n^2}, \tag{15}$$

which assigns to every matrix $Z = (z_{k\ell})_{k\ell} = (x_{k\ell} + i y_{k\ell})_{k\ell} \in M_n$ the element

$$(x_{11}, y_{11}, \dots, x_{1n}, y_{1n}, x_{21}, y_{21}, \dots, x_{2n}, y_{2n}, \dots, x_{n1}, y_{n1}, \dots, x_{nn}, y_{nn}) \in \mathbb{R}^{2n^2}.$$

Then $\mathfrak{P}(M_n)$ denotes the linear space of all polynomials $f = g \circ \boldsymbol{I}_n$ with $g \in \mathscr{P}(\mathbb{R}^{2n^2})$. Hence, by definition, the mapping

$$\mathscr{P}(\mathbb{R}^{2n^2}) = \mathfrak{P}(M_n), \quad g \mapsto g \circ \boldsymbol{I}_n, \tag{16}$$

identifies both spaces as vector spaces.

We collect a couple of useful facts. Note first that if $f = g \circ \boldsymbol{I}_n \in \mathfrak{P}(M_n)$ with $g \in \mathscr{P}(\mathbb{R}^{2n^2})$, then

$$\Delta f = \sum_{i,j=1}^{n} \frac{\partial^2 f}{\partial z_{ij} \partial \bar{z}_{ij}} = \frac{1}{4} \sum_{i,j=1}^{n} \left(\frac{\partial^2 g}{\partial x_{ij}^2} + \frac{\partial^2 g}{\partial y_{ij}^2} \right);$$

a formula that follows directly from the definition of $\partial_{z_{ij}} = \frac{1}{2}(\partial_{x_{ij}} - i\partial_{y_{ij}})$ and $\partial_{\bar{z}_{ij}} = \frac{1}{2}(\partial_{x_{ij}} + i\partial_{y_{ij}})$.

Secondly, a function $f \colon M_n \to \mathbb{C}$ belongs to $\mathfrak{P}(M_n)$ if and only if it has a representation

$$f(Z) = \sum_{(\alpha,\beta) \in J} c_{(\alpha,\beta)} Z^\alpha \bar{Z}^\beta, \quad Z \in M_n, \tag{17}$$

where J is a finite index set in $M_n(\mathbb{N}_0) \times M_n(\mathbb{N}_0)$ and $c_{(\alpha,\beta)} \in \mathbb{C}$, $(\alpha, \beta) \in J$. Moreover, in this case this representation is unique.

Indeed, if f is given by (17), then $g = f \circ \boldsymbol{I}_n^{-1} \in \mathscr{P}(\mathbb{R}^{2n^2})$ and $f = g \circ \boldsymbol{I}_n \in \mathfrak{P}(M_n)$. Conversely, if $f = g \circ \boldsymbol{I}_n \in \mathfrak{P}(M_n)$ with $g = \sum_\alpha c_\alpha x^\alpha \in \mathscr{P}(\mathbb{R}^{2n^2})$, then the desired representation easily follows from the substitution $\operatorname{Re} z_{ij} = \frac{1}{2}(z_{ij} + \bar{z}_{ij})$ and $\operatorname{Im} z_{ij} = \frac{1}{2}(z_{ij} - \bar{z}_{ij})$. To see the uniqueness of the representation in (17), observe that if $f = 0$, then, given $(\alpha, \beta) \neq (0, 0)$, an application of the differential operator

$$\partial_{z_{11}}^{\alpha_{11}} \cdots \partial_{z_{1n}}^{\alpha_{1n}} \partial_{\bar{z}_{11}}^{\beta_{11}} \cdots \partial_{\bar{z}_{1n}}^{\beta_{1n}} \cdots \partial_{z_{n1}}^{\alpha_{n1}} \cdots \partial_{z_{nn}}^{\alpha_{nn}} \partial_{\bar{z}_{n1}}^{\beta_{n1}} \cdots \partial_{\bar{z}_{nn}}^{\beta_{nn}}$$

to f (and evaluating at zero), shows that $c_{(\alpha,\beta)} = 0$.

We again use the identification $g \mapsto g \circ \boldsymbol{I}_n$ from (15) to define the spaces

$$\mathfrak{P}_k(M_n) := \mathscr{P}_k(\mathbb{R}^{2n^2}), \quad \mathfrak{H}_k(M_n) := \mathscr{H}_k(\mathbb{R}^{2n^2}) \quad \text{and} \quad \mathfrak{H}(M_n) := \mathscr{H}(\mathbb{R}^{2n^2}). \tag{18}$$

The following lemma gives a simple description of the elements of $\mathfrak{P}_k(M_n)$.

Lemma 1.2. *Let $f \in \mathfrak{P}(M_n)$ and $k \in \mathbb{N}$. Then $f \in \mathfrak{P}_k(M_n)$ if and only if $f(\lambda Z) = \lambda^k f(Z)$ for all $\lambda \in \mathbb{R}$ and $Z \in M_n$.*

Proof. For $f \in \mathfrak{P}_k(M_n)$ there is $g \in \mathscr{P}_k(\mathbb{R}^{2n^2})$ such that $f = g \circ \boldsymbol{I}_n$. Clearly, $f(\lambda Z) = \lambda^k f(Z)$ for all $\lambda \in \mathbb{R}$ and $Z \in M_n$. Assume conversely that f is k-homogeneous in the meaning of the statement. Since $f \in \mathfrak{P}(M_n)$, there is a finite polynomial $g(x) = \sum_J c_\alpha(g) x^\alpha$, $x \in \mathbb{R}^{2n^2}$, such that $f = g \circ \boldsymbol{I}_n$. Since $g = f \circ \boldsymbol{I}_n^{-1}$, it follows that $g(\lambda x) = \lambda^k g(x)$ for all $\lambda \in \mathbb{R}$ and $x \in \mathbb{R}^{2n^2}$. But this by the uniqueness of the coefficients $c_\alpha(g)$ necessarily implies that $c_\alpha(g) \neq 0$ only if $|\alpha| = k$, so as desired $g \in \mathscr{P}_k(\mathbb{R}^{2n^2})$. ☐

Obviously,

$$\mathfrak{P}(M_n) = \operatorname{span}_k \mathfrak{P}_k(M_n) \tag{19}$$

(consider the polynomials on $\mathbb{R}^{2n^2}$ defining these spaces), and less trivially (as an immediate consequences of (14)) we have

$$\mathfrak{H}(M_n) = \operatorname{span}_k \mathfrak{H}_k(M_n). \tag{20}$$

The polynomial $\boldsymbol{t}_{M_n} \in \mathfrak{P}_2(M_n)$ given by

$$\boldsymbol{t}_{M_n}(Z) := \operatorname{tr}(ZZ^*), \quad Z \in M_n,$$

where $\operatorname{tr}\colon M_n \to \mathbb{C}$ denotes the trace, is of special importance. It is easily seen that under the identification from (16) the image of the polynomial $\boldsymbol{t} \in \mathscr{P}_2(\mathbb{R}^{2n^2})$ (see again (13)) is $\boldsymbol{t}_{M_n} \in \mathfrak{P}_2(M_n)$; that is,

$$\boldsymbol{t}_{M_n}(Z) = \boldsymbol{t}(\boldsymbol{I}_n Z), \quad Z \in M_n. \tag{21}$$

Recall again that $\mathscr{P}(\mathbb{R}^{2n^2})$ carries the natural inner product from (11), which then by the identification in (16) transfers to a natural inner product on $\mathfrak{P}(M_n)$; that is,

$$\langle f, g\rangle_{\mathfrak{P}} := \langle f \circ \boldsymbol{I}_n^{-1}, g \circ \boldsymbol{I}_n^{-1}\rangle_{\mathscr{P}}, \quad f, g \in \mathfrak{P}(M_n). \tag{22}$$

Using (18) and (21), we deduce from Proposition 1.1 its matrix analog, which is going to be of great value later on.

Proposition 1.3. *For all $k \in \mathbb{N}_0$ and $n \in \mathbb{N}$,*

$$\mathfrak{P}_k(M_n) = \mathfrak{H}_k(M_n) \oplus \boldsymbol{t}_{M_n} \cdot \mathfrak{H}_{k-2}(M_n) \oplus \boldsymbol{t}_{M_n}^2 \cdot \mathfrak{H}_{k-4}(M_n) \oplus \cdots,$$

where the last term of the orthogonal sum is the span of $\boldsymbol{t}_{M_n}^{k/2}$ for even k and $\boldsymbol{t}_{M_n}^{(k-1)/2} \cdot \mathfrak{H}_1(M_n)$ for odd k.

We need two more lemmas.

Lemma 1.4. *Let $f \in \mathfrak{H}(M_n)$ and $U \in \mathscr{U}_n$. Then $f \circ L_U \in \mathfrak{H}(M_n)$. Moreover, if $f \in \mathfrak{H}_k(M_n)$, then also $f \circ L_U \in \mathfrak{H}_k(M_n)$.*

Proof. Recall the well-known fact that, for every harmonic function $F\colon \mathbb{C}^{n^2} \to \mathbb{C}$ and every $W \in \mathscr{U}_{n^2}$, we have $\triangle(F \circ \Phi_W) = \triangle F \circ \Phi_W$, where $\Phi_W z = Wz$ for $z \in \mathbb{C}^{n^2}$. Now identify M_n and $\mathbb{C}^{n^2}$ in the natural way by

$$\boldsymbol{J}_n(Z) = (z_{11}, \ldots, z_{1n}, z_{21}, \ldots, z_{2n}, \ldots, z_{n1}, \ldots, z_{nn}), \quad Z \in M_n, \tag{23}$$

and define $g = f \circ \boldsymbol{J}_n^{-1} : \mathbb{C}^{n^2} \to \mathbb{C}$. Then obviously $\triangle g = 0$, and moreover a simple calculation shows

$$f \circ L_U = g \circ \Phi_{U \otimes \mathrm{id}_{\mathbb{C}^n}} \circ \boldsymbol{J}_n.$$

Since $U \otimes \mathrm{id}_{\mathbb{C}^n} \in \mathscr{U}_{n^2}$ is unitary, it follows that

$$\triangle(f \circ L_U) = \triangle(g \circ \Phi_{U \otimes \mathrm{id}_{\mathbb{C}^n}}) = \triangle g \circ \Phi_{U \otimes \mathrm{id}_{\mathbb{C}^n}} = 0.$$

For the second statement, note that if $f \in \mathfrak{H}_k(M_n)$, then by the first statement $f \circ L_U \in \mathfrak{H}(M_n)$. But $f \circ L_U(\lambda Z) = \lambda f \circ L_U(Z)$ for all $Z \in M_n$ and $\lambda \in \mathbb{R}$, and hence the claim follows from Lemma 1.2. $\square$

For $p, q \in \mathbb{N}_0$, let $\mathfrak{H}_{(p,q)}(M_n) \subset \mathfrak{H}(M_n)$ be the subspace of all harmonic polynomials which are p-homogeneous in $Z = (z_{ij})$ and q-homogeneous in $\bar{Z} = (\bar{z}_{ij})$; that is, all polynomials $f \in \mathfrak{H}(M_n)$ of the form

$$f(Z) = \sum_{|\alpha|=p,|\beta|=q} c_{(\alpha,\beta)} Z^\alpha \bar{Z}^\beta, \quad Z \in M_n. \tag{24}$$

By Lemma 1.2, we immediately see that

$$\mathfrak{H}_{(p,q)}(M_n) \subset \mathfrak{H}_{p+q}(M_n). \tag{25}$$

The following result is crucial for our purpose; see also Lemma 1.7.

Lemma 1.5. *For all $f \in \mathfrak{H}_{(p,q)}(M_n)$ and $U \in \mathscr{U}_n$, one has*

$$f \circ L_U \in \mathfrak{H}_{(p,q)}(M_n) \quad \text{and} \quad f \circ R_U \in \mathfrak{H}_{(p,q)}(M_n).$$

Moreover,

$$\mathfrak{H}(M_n) = \operatorname{span}_{p,q} \mathfrak{H}_{(p,q)}(M_n). \tag{26}$$

Proof. Taking for f a representation as in (24), we have

$$f \circ L_U(Z) = \sum_{|\alpha|=p,|\beta|=q} c_{(\alpha,\beta)} (UZ)^\alpha (\overline{UZ})^\beta, \quad Z \in M_n.$$

Now, for each $1 \le i, j \le n$, we use the multinomial formula for $\left(\sum_\ell u_{i\ell} z_{\ell j}\right)^{\alpha_{ij}}$ to get

$$(UZ)^\alpha (\overline{UZ})^\beta = \sum_{|\gamma|\le p,|\zeta|\le q} d_{(\gamma,\delta)} Z^\gamma \bar{Z}^\zeta, \quad Z = (z_{k\ell})_{k,\ell} \in M_n.$$

Combining, we conclude that $f \circ L_U$ has a representation

$$f \circ L_U(Z) = \sum_{|\eta|\le p,|\sigma|\le q} e_{(\eta,\sigma)} Z^\eta \bar{Z}^\sigma, \quad Z \in M_n.$$

On the other hand, by (25) and Lemma 1.4, it follows that $f \circ L_U \in \mathfrak{H}_{p+q}(M_n)$, and hence, for all $\lambda \in \mathbb{R}$ and $Z \in M_n$, one has

$$\sum_{|\eta|\le p,|\sigma|\le q} e_{(\eta,\sigma)} \lambda^{|\eta|+|\sigma|} Z^\eta \bar{Z}^\sigma = (f \circ L_U)(\lambda Z) = \lambda^{p+q} (f \circ L_U)(Z) = \sum_{|\eta|\le p,|\sigma|\le q} c_{(\eta,\sigma)} \lambda^{p+q} Z^\eta \bar{Z}^\sigma.$$

Inserting $Z = \mathrm{id} \in M_n$ shows that $e_{(\eta,\sigma)} \neq 0$ only if $|\eta| + |\sigma| = p + q$, and since $|\eta| \le p$ and $|\sigma| \le q$, this is only possible whenever $|\eta| = p$ and $|\sigma| = q$. This as desired proves $f \circ L_U \in \mathfrak{H}_{(p,q)}(M_n)$.

The equality (26) follows from (20) since it may easily be seen that $\mathfrak{H}_k(M_n) = \operatorname{span}_{p+q=k} \mathfrak{H}_{(p,q)}(M_n)$; see also, e.g., [Rudin 1980, Proposition 12.2.2].

In order to prove that $f \circ R_U \in \mathfrak{H}_{(p,q)}(M_n)$, define $f^*(Z) = f(Z^*)$ for $Z \in M_n$. Since the mapping $\ell_2^{n^2} \to \ell_2^{n^2}$, $Z \mapsto Z^*$, is unitary (it is an isometry), the function f^* is harmonic. Now, looking at the representation of f as in (24), we see that $f^* \in \mathfrak{H}_{(q,p)}(M_n)$. This, by what is already proved, gives that $f^* \circ L_{U^*} \in \mathfrak{H}_{(q,p)}(M_n)$. But, for $Z \in M_n$,

$$f \circ R_U(Z) = f(ZU) = f((U^*Z^*)^*) = f^*(U^*Z^*) = f^* \circ L_{U^*}(Z^*) = (f^* \circ L_{U^*})^*(Z),$$

and hence $f \circ R_U = (f^* \circ L_{U^*})^* \in \mathfrak{H}_{(p,q)}(M_n)$. $\square$

1.4. ***Unitarily invariant subspaces of $C(\mathscr{U}_n)$.*** By $\mathfrak{P}(\mathscr{U}_n)$ and $\mathfrak{P}_k(\mathscr{U}_n)$ we denote the linear space of all restrictions $f|_{\mathscr{U}_n} : \mathscr{U}_n \to \mathbb{C}$ of polynomials $f \in \mathfrak{P}(M_n)$ and $f \in \mathfrak{P}_k(M_n)$, respectively.

Similarly, for all restrictions to $\mathscr{U}_n$ of harmonic polynomials from $\mathfrak{H}(M_n)$ and $\mathfrak{H}_k(M_n)$, we write $\mathfrak{H}(\mathscr{U}_n)$ and $\mathfrak{H}_k(\mathscr{U}_n)$, respectively, and the elements therein we address as unitary harmonics. All these constitute important subspaces of $C(\mathscr{U}_n)$.

Lemma 1.6. *For each k,*

$$\mathfrak{P}_k(\mathscr{U}_n) = \operatorname{span}_{\ell \le k} \mathfrak{H}_\ell(\mathscr{U}_n) \tag{27}$$

and

$$\mathfrak{P}(\mathscr{U}_n) = \operatorname{span}_k \mathfrak{P}_k(\mathscr{U}_n) = \operatorname{span}_\ell \mathfrak{H}_\ell(\mathscr{U}_n) = \mathfrak{H}(\mathscr{U}_n). \tag{28}$$

Proof. Proposition 1.3 and the fact that the function $\boldsymbol{t}_{M_n} = n$ on $\mathscr{U}_n$ imply (27). To prove (28), note that the first equality is a consequence of (19), the second of (27), and the last of (20). □

Moreover, for $p, q \in \mathbb{N}_0$, we write $\mathfrak{H}_{(p,q)}(\mathscr{U}_n)$ for all restrictions to $\mathscr{U}_n$ of functions in $\mathfrak{H}_{(p,q)}(M_n)$. Observe that a function $f : \mathscr{U}_n \to \mathbb{C}$ belongs to $\mathfrak{H}_{(p,q)}(\mathscr{U}_n)$ if and only if it has on $\mathscr{U}_n$ a representation like in (24). All needed information on these subspaces of $C(\mathscr{U}_n)$ is included in the following lemma, which is an immediate consequence of Lemma 1.5.

Lemma 1.7. *Each space $\mathfrak{H}_{(p,q)}(\mathscr{U}_n)$ is a $\mathscr{U}_n$-invariant subspace of $C(\mathscr{U}_n)$; that is, for all $f \in \mathfrak{H}_{(p,q)}(\mathscr{U}_n)$ and $U \in \mathscr{U}_n$, we have $f \circ L_U$, $f \circ R_U \in \mathfrak{H}_{(p,q)}(\mathscr{U}_n)$. Moreover,*

$$\mathfrak{H}(\mathscr{U}_n) = \operatorname{span}_{p,q} \mathfrak{H}_{(p,q)}(\mathscr{U}_n). \tag{29}$$

We finish with the following density result as it is crucial for our purposes.

Theorem 1.8. *$\mathfrak{H}(\mathscr{U}_n)$ is dense in $C(\mathscr{U}_n)$. In particular, the span of the union of all $\mathfrak{H}_k(\mathscr{U}_n)$ as well as the span of the union of all $\mathfrak{H}_{(p,q)}(\mathscr{U}_n)$ are dense in $C(\mathscr{U}_n)$.*

Proof. Observe first that $\mathfrak{P}(\mathscr{U}_n)$ is a subalgebra of $C(\mathscr{U}_n)$, which is closed under conjugation, and that the collection of all coordinate functions e_{ij} separates the points of $\mathscr{U}_n$. Thus, by the Stone–Weierstrass theorem, $\mathfrak{P}(\mathscr{U}_n)$ is dense in $C(\mathscr{U}_n)$. The rest follows from (28) and (29). □

Remark 1.9. An important difference between spherical harmonics and unitary harmonics is that, for the case of the sphere, the corresponding spaces $\mathfrak{H}_{(p,q)}(\mathbb{S}_n^{\mathbb{C}})$ are mutually orthogonal in $L_2(\mathbb{S}_n^{\mathbb{C}})$; see [Rudin 1980, Theorem 12.2.3]. But for the subspaces $\mathfrak{H}_{(p,q)}(\mathscr{U}_n)$ of $L^2(\mathscr{U}_n)$ this is no longer true. To see an example, take $f \in \mathfrak{H}_{(1,0)}(\mathscr{U}_n)$ and $g \in \mathfrak{H}_{(2,1)}(\mathscr{U}_n)$ defined by $f(U) = u_{1,1}$ and $g(U) = \overline{u_{2,2}} u_{1,2} u_{2,1}$. Then (see, e.g., [Hiai and Petz 2000, Section 4.2])

$$\langle f, g \rangle_{L_2} = \int_{\mathscr{U}_n} u_{1,1} u_{2,2} \overline{u_{1,2} u_{2,1}} \, dU = -\frac{1}{(n-1)n(n+1)}. \tag{30}$$

On the other hand, using basic properties of the Haar measure on $\mathscr{U}_n$, it is not difficult to prove that

$$\mathfrak{H}_{(p,q)}(\mathscr{U}_n) \perp \mathfrak{H}_{(p',q')}(\mathscr{U}_n) \quad \text{whenever} \quad p+q = p'+q' \quad \text{and} \quad (p,q) \neq (p',q'); \tag{31}$$

see [Hewitt and Ross 1963, §29] or [Köstenberger 2021].

It is worth noting the following conclusion from (31) — not needed for our further purposes — which states that

$$\mathfrak{H}_k(\mathscr{U}_n) = \mathfrak{H}_{(k,0)}(\mathscr{U}_n) \oplus \mathfrak{H}_{(k-1,1)}(\mathscr{U}_n) \oplus \cdots \oplus \mathfrak{H}_{(0,k)}(\mathscr{U}_n),$$

where $\oplus$ indicates the orthogonal sum in $L_2(\mathscr{U}_n)$. We conclude with the observation that, in contrast to (30), we have $\langle f, g\rangle_{\mathfrak{P}} = 0$, so the euclidean structure, which $\mathfrak{H}_{(p,q)}(\mathscr{U}_n)$ inherits from $L_2(\mathscr{U}_n)$, is different from that induced by the inner product from (22).

2. Projection constants

As explained in the introduction, the main goal of this work is to prove the following result.

Theorem 2.1. *For each $n \in \mathbb{N}$,*

$$\boldsymbol{\lambda}(\mathscr{S}_1(n)) = \|\pi_{(1,0)} : C(\mathscr{U}_n) \to \mathscr{S}_1(n)\| = n \int_{\mathscr{U}_n} |\mathrm{tr}(V)|\, dV. \tag{32}$$

Moreover,

$$\lim_{n\to\infty} \frac{\boldsymbol{\lambda}(\mathscr{S}_1(n))}{n} = \frac{\sqrt{\pi}}{2}. \tag{33}$$

The proof of this theorem is presented in Section 2.5. It is based on preliminary results we prove in the following, which require some preliminary arguments.

2.1. ***Rudin's averaging technique.*** Given a topological group G and a Banach space Y, we say that G acts on Y (through T) whenever there is a mapping

$$T : G \to \mathscr{L}(Y), \quad g \mapsto T_g,$$

such that

$$T_e = I_Y, \quad T_{gh} = T_g T_h, \quad g, h \in G,$$

and all mappings

$$g \ni G \mapsto T_g(y) \in Y, \quad y \in Y,$$

are continuous. If in addition all operators T_g, $g \in G$, are isometries, then we say that G acts isometrically on Y. We say that $S \in \mathscr{L}(Y)$ commutes with the action T of G on Y whenever S commutes with all T_h, $h \in G$.

The following theorem was presented in [Rudin 1962]; see also [Wojtaszczyk 1991, Theorem III.B.13].

Theorem 2.2. *Let Y be a Banach space, X a complemented subspace of Y, and $\boldsymbol{Q} : Y \to Y$ a projection onto X. Suppose that G is a compact group with Haar measure* m *which acts on Y through T such that X is invariant under the action of G; that is, $T_g(X) \subset X$ for all $g \in G$. Then $\boldsymbol{P} : Y \to Y$ given by*

$$\boldsymbol{P}(y) := \int_G T_{g^{-1}} \boldsymbol{Q} T_g(y)\, d\mathrm{m}(g), \quad y \in Y, \tag{34}$$

is a projection onto X which commutes with the action of G on Y (meaning that $T_g \boldsymbol{P} = \boldsymbol{P} T_g$ for all $g \in G$) and satisfies

$$\|\boldsymbol{P}\| \leq \|\boldsymbol{Q}\| \sup_{g \in G} \|T_g\|^2.$$

Moreover, if there is a unique projection on Y onto X that commutes with the action of G on Y, and if G acts isometrically on Y, then $\boldsymbol{P}$ given in (34) *is minimal, i.e.,*

$$\boldsymbol{\lambda}(X, Y) = \|\boldsymbol{P}\|.$$

In order to be able to apply Rudin's technique, we need to endow $\mathscr{U}_n \times \mathscr{U}_n$ with a special group structure, which allows us to represent the resulting group in $\mathscr{L}(C(\mathscr{U}_n))$. To do so, consider on $\mathscr{U}_n \times \mathscr{U}_n$ the multiplication

$$(U_0, V_0) \cdot (U_1, V_1) := (U_1 U_0, V_0 V_1).$$

With this multiplication and endowed with the product topology, $\mathscr{U}_n \times \mathscr{U}_n$ turns into a compact topological group, and it may be seen easily that the Haar measure on $\mathscr{U}_n \times \mathscr{U}_n$ is given by the product measure of the Haar measure on $\mathscr{U}_n$ with itself.

Further, for any $(U, V) \in \mathscr{U}_n \times \mathscr{U}_n$ and any $f \in L_2(\mathscr{U}_n)$, we define

$$\rho_{(U,V)} f := (C_{L_U} \circ C_{R_V}) f = f \circ L_U \circ R_V,$$

which leads to an action of $\mathscr{U}_n \times \mathscr{U}_n$ on $C(\mathscr{U}_n)$ given by

$$\mathscr{U}_n \times \mathscr{U}_n \to \mathscr{L}(C(\mathscr{U}_n)), \quad (U, V) \mapsto [\rho_{(U,V)} : f \mapsto f \circ L_U \circ R_V]. \tag{35}$$

We say that a mapping $T : S_1 \to S_2$, where S_1 and S_2 both are $\mathscr{U}_n$-invariant subspaces of $L_2(\mathscr{U}_n)$, commutes with the action of $\mathscr{U}_n \times \mathscr{U}_n$ on $C(\mathscr{U}_n)$ whenever

$$(C_{L_U} \circ C_{R_V})(Tf) = T((C_{L_U} \circ C_{R_V}) f) \quad \text{for every } (U, V) \in \mathscr{U}_n \times \mathscr{U}_n \text{ and } f \in S_1.$$

2.2. ***Convolution.*** Recall from Section 1.1 that $\pi_S : L_2(\mathscr{U}_n) \to S$ denotes the orthogonal projection on $L_2(\mathscr{U}_n)$ onto a given closed subspace S. The following result shows that, under the assumption of $\mathscr{U}_n$-invariance of S, this projection is a convolution operator with respect to some kernel in S.

Theorem 2.3. *Let S be a $\mathscr{U}_n$-invariant subspace of $C(\mathscr{U}_n)$ which is closed in $L_2(\mathscr{U}_n)$. Then the following statements hold*:

(i) *There is a unique function* $\mathrm{t}_S \in S$ *such that, for all $f \in L_2(\mathscr{U}_n)$,*

$$\pi_S f = f * \mathrm{t}_S.$$

(ii) *π_S commutes with all L_U and R_U for $U \in \mathscr{U}_n$; that is, π_S commutes with the action of $\mathscr{U}_n \times \mathscr{U}_n$ on $C(\mathscr{U}_n)$.*

(iii) $\|\pi_S : C(\mathscr{U}_n) \to S\| = \int_{\mathscr{U}_n} |\mathrm{t}_S(V)| \, dV.$

The proof is an easy consequence of the following lemma.

Lemma 2.4. *Let S be a $\mathscr{U}_n$-invariant subspace of $C(\mathscr{U}_n)$ which is closed in $L_2(\mathscr{U}_n)$. Then, for every $U \in \mathscr{U}_n$, there exists a unique function $K_U^S \in S$ such that, for all $f \in L_2(\mathscr{U}_n)$,*

(i) $(\pi_S f)(U) = \langle f, K_U^S \rangle_{L_2} = \int_{\mathscr{U}_n} f(V)\overline{K_U^S(V)}\, dV$,

and moreover, for every choice of $U, V \in \mathscr{U}_n$, we have

(ii) $K_U^S(V) = \langle K_U^S, K_V^S \rangle_{L_2} = \overline{K_V^S(U)}$,

(iii) $K_U^S \circ L_{V^{-1}} = K_{VU}^S = K_V^S \circ R_{U^{-1}}$,

(iv) $K_V^S(V) = K_{\mathrm{Id}}^S(\mathrm{Id}) > 0$.

Proof. The claim from (i) is an immediate consequence of the Riesz representation theorem applied to the continuous linear functional $L_2(\mathscr{U}_n) \to \mathbb{C},\ f \mapsto (\pi_S f)(U)$.

(ii) $K_U^S(V) = \pi_S(K_U^S)(V) = \langle K_U^S, K_V^S \rangle_{L_2} = \overline{\langle K_V^S, K_U^S \rangle_{L_2}} = \overline{K_V^S(U)}$ for all $V \in \mathscr{U}_n$.

(iii) Fix some $V \in \mathscr{U}_n$ and $f \in L_2(\mathscr{U}_n)$, and note first that $S^\perp$ is also $\mathscr{U}_n$-invariant. Then

$$(\mathrm{Id} - \pi_S)(f) \circ L_V \in S^\perp \quad \text{and} \quad f \circ L_V = \pi_S(f) \circ L_V + (\mathrm{Id} - \pi_S)(f) \circ L_V,$$

and hence

$$\pi_S(f \circ L_V) = \pi_S(\pi_S(f) \circ L_V) + \pi_S((\mathrm{Id} - \pi_S)(f) \circ L_V) = \pi_S(f) \circ L_V. \tag{36}$$

Then

$$\langle f, K_{VU}^S \rangle_{L_2} = \pi_S(f)(VU) = \pi_S(f) \circ L_V(U) = \pi_S(f \circ L_V)(U),$$

and thus, by (i),

$$\langle f, K_{VU}^S \rangle_{L_2} = \langle f \circ L_V, K_U^S \rangle_{L_2} = \langle C_{L_V} f, K_U^S \rangle_{L_2} = \langle f, C_{L_{V^{-1}}} K_U^S \rangle_{L_2} = \langle f, K_U^S \circ L_{V^{-1}} \rangle_{L_2}.$$

Since $f \in L_2(\mathscr{U}_n)$ was chosen arbitrarily, we obtain that $K_{VU}^S = K_U^S \circ L_{V^{-1}}$. The other identity follows similarly.

(iv) Let $V \in \mathscr{U}_n$. Then

$$K_V^S(V) = \langle K_V^S, K_V^S \rangle_{L_2} = \langle K_{\mathrm{Id}}^S \circ L_{V^{-1}}, K_V^S \rangle_{L_2} = \langle K_{\mathrm{Id}}^S, K_V^S \circ L_V \rangle_{L_2} = \langle K_{\mathrm{Id}}^S, K_{\mathrm{Id}}^S \rangle_{L_2} = K_{\mathrm{Id}}^S(\mathrm{Id}) > 0. \quad \square$$

It remains to prove Theorem 2.3. Defining

$$\mathrm{t}_S := K_{\mathrm{Id}}^S, \tag{37}$$

this proof is in fact a straightforward consequence of the preceding lemma. But before we do this, we collect two elementary properties of the kernel t_S.

Remark 2.5. Let S be a $\mathscr{U}_n$-invariant subspace of $C(\mathscr{U}_n)$ which is closed in $L_2(\mathscr{U}_n)$. Then $\mathrm{t}_S = K_{\mathrm{Id}}^S$ satisfies

- $\mathrm{t}_S(V^*) = \overline{\mathrm{t}_S(V)}$ for all $V \in \mathscr{U}_n$,
- $\mathrm{t}_S(V^*UV) = \mathrm{t}_S(U)$ for all $U, V \in \mathscr{U}_n$; that is, t_S is a so-called class function.

Indeed, for the first equality, note that

$$\mathrm{t}_S(V^*) = (K^S_{\mathrm{Id}} \circ L_{V^{-1}})(\mathrm{Id}) = K^S_V(\mathrm{Id}) = \overline{K^{\mathrm{Id}}_V(S)} = \overline{\mathrm{t}_S(V)},$$

and together with this we get

$$\begin{aligned}\mathrm{t}_S(V^{-1}UV) &= \overline{\mathrm{t}_S(V^{-1}U^*V)} = \overline{(K^S_{\mathrm{Id}} \circ L_{V^{-1}})(U^*V)} \\ &= \overline{K^S_V(U^*V)} = K^S_{U^*V}(V) = K^S_{\mathrm{Id}} \circ R_{V^{-1}U}(V) = \mathrm{t}_S(U).\end{aligned}$$

Proof of Theorem 2.3. By Lemma 2.4, for all $U \in \mathscr{U}_n$ and $f \in L_2(\mathscr{U}_n)$,

$$\begin{aligned}(\pi_S f)(U) &= \int_{\mathscr{U}_n} f(V)\overline{K^S_U(V)}\,dV = \int_{\mathscr{U}_n} f(V)K^S_V(U)\,dV \\ &= \int_{\mathscr{U}_n} f(V)K^S_{\mathrm{Id}}(UV^{-1})\,dV = \int_{\mathscr{U}_n} f(V)\mathrm{t}_S(UV^*)\,dV = (f * \mathrm{t}_S)(U),\end{aligned}$$

which proves (i). Statement (ii) was already shown in (36), and it remains to check (iii). Obviously, we have that

$$\|\pi_S : C(\mathscr{U}_n) \to S\| = \sup_{U\in\mathscr{U}_n} \int_{\mathscr{U}_n} |\mathrm{t}_S(UV^*)|\,dV,$$

and, for every $U \in \mathscr{U}_n$ by Remark 2.5,

$$\int_{\mathscr{U}_n} |\mathrm{t}_S(UV^*)|\,dV = \int_{\mathscr{U}_n} |\mathrm{t}_S(V^*)|\,dV = \int_{\mathscr{U}_n} |\mathrm{t}_S(V)|\,dV.$$

This completes the argument. □

2.3. ***Accessibility.*** Let S a be $\mathscr{U}_n$-invariant subspace of $C(\mathscr{U}_n)$ which is closed in $L_2(\mathscr{U}_n)$. Then S is called *accessible* if every projection Q on $C(\mathscr{U}_n)$ onto S which commutes with the action of $\mathscr{U}_n \times \mathscr{U}_n$ on $C(\mathscr{U}_n)$ equals $\pi_S|_{C(\mathscr{U}_n)}$.

Theorem 2.6. *Let S be a $\mathscr{U}_n$-invariant and accessible subspace of $C(\mathscr{U}_n)$ which is closed in $L_2(\mathscr{U}_n)$. Then*

$$\boldsymbol{\lambda}(S) = \|\pi_S : C(\mathscr{U}_n) \to S\| = \int_{\mathscr{U}_n} |\mathrm{t}_S(V)|\,dV.$$

Proof. The proof is an immediate consequence of Rudin's Theorem 2.2 and the assumptions on S, taking into account that we know (ii) and (iii) from Theorem 2.3 as well as (1). □

We say that a $\mathscr{U}_n$-invariant subspace S of $C(\mathscr{U}_n)$ which is closed in $L_2(\mathscr{U}_n)$ is *strongly accessible* whenever every $f \in S$ for which $f(VUV^*) = f(U)$ for all $U, V \in \mathscr{U}_n$ is a scalar multiple of t_S. In other words, every class function in S is a multiple of t_S.

As the name in the previous definition suggests, we have the following key result.

Proposition 2.7. *Let S be a $\mathscr{U}_n$-invariant subspace of $C(\mathscr{U}_n)$ which is closed in $L_2(\mathscr{U}_n)$. Then S is accessible whenever it is strongly accessible.*

The proof requires the next statement.

Lemma 2.8. *Let H and S be $\mathscr{U}_n$-invariant subspaces of $C(\mathscr{U}_n)$ which are both closed in $L_2(\mathscr{U}_n)$. Then, if S is strongly accessible, every operator $T : H \to S$ that commutes with the action of $\mathscr{U}_n \times \mathscr{U}_n$ on $C(\mathscr{U}_n)$ is a scalar multiple of $\pi_S|_H$.*

Moreover, if H is orthogonal to S and Q is a projection on $H \oplus S$ onto S that commutes with the action of $\mathscr{U}_n \times \mathscr{U}_n$ on $C(\mathscr{U}_n)$, then $Q = \pi_S|_{H\oplus S}$.

Proof. By the assumption on T and Lemma 2.4 (iii), for every $V \in \mathscr{U}_n$,

$$(C_{L_V} \circ C_{R_{V^{-1}}})(T\mathrm{t}_H) = T((C_{L_V} \circ C_{R_{V^{-1}}})\mathrm{t}_H) = Tt_H.$$

This implies that $(T\mathrm{t}_H)(V^*UV) = (T\mathrm{t}_H)(U)$ for all $U, V \in \mathscr{U}_n$. Since S is strongly accessible, we have that $T\mathrm{t}_H = \gamma\mathrm{t}_S$ for some $\gamma \in \mathbb{C}$. But from Theorem 2.3 we know that, for all $h \in H$,

$$h = \pi_H h = h * \mathrm{t}_H,$$

and hence

$$Th = h * T\mathrm{t}_H = \gamma h * \mathrm{t}_S = \gamma\pi_S h.$$

To see the second assertion, note that, by the first part of the lemma, we have $Q|_H = \gamma\pi_S|_H$ for some $\gamma \in \mathbb{C}$. But since by assumption $H \subset S^{\perp}$, this implies $Q|_H = 0 = \pi_S|_H$. On the other hand, since Q is a projection onto S, we see that $Q|_S = \mathrm{Id}_S = \pi_S|_S$, which finishes the proof. □

We are now ready to give the following.

Proof of Proposition 2.7. Let Q be a projection on $C(\mathscr{U}_n)$ onto S which commutes with the action of $\mathscr{U}_n \times \mathscr{U}_n$ on $C(\mathscr{U}_n)$. By Theorem 1.8, it suffices to show that, for each pair $(p, q) \in \mathbb{N}_0 \times \mathbb{N}_0$,

$$Q|_{\mathfrak{H}_{(p,q)}} = \pi_S|_{\mathfrak{H}_{(p,q)}}.$$

Given such a pair (p, q), we define the subspace

$$H := \{f - \pi_S f : f \in \mathfrak{H}_{(p,q)}\} \subset C(\mathscr{U}_n).$$

Then H is $\mathscr{U}_n$-invariant; indeed, by Theorem 2.3 (ii) and the fact that $\mathfrak{H}_{(p,q)}$ is $\mathscr{U}_n$-invariant (proved in Lemma 1.7), for every $f \in \mathfrak{H}_{(p,q)}$ and $U \in \mathscr{U}_n$, we have

$$(f - \pi_S f) \circ L_U = f \circ L_U - \pi_S f \circ L_U = f \circ L_U - \pi_S(f \circ L_U) \in H,$$

and the invariance under right multiplication follows similarly. Since $H \perp S$ and Q commutes with the action of $\mathscr{U}_n \times \mathscr{U}_n$ on $C(\mathscr{U}_n)$, Lemma 2.8 (the second part applied to the restriction of Q to $H \oplus S$) shows that

$$Q|_{H\oplus S} = \pi_S|_{H\oplus S},$$

so in particular $Q|_H = \pi_S|_H = 0$. But then, for every $f \in \mathfrak{H}_{(p,q)}(\mathscr{U}_n)$,

$$Q(f) = Q(f - \pi_S f) + Q(\pi_S f) = \pi_S f,$$

which completes the argument. □

2.4. ***The special case $S = \mathfrak{H}_{(1,0)}(\mathscr{U}_n)$.*** Recall from Section 1.4 the definition of the $\mathscr{U}_n$-invariant subspace $\mathfrak{H}_{(1,0)}(\mathscr{U}_n)$ of $C(\mathscr{U}_n)$ of all polynomials $f \in C(\mathscr{U}_n)$ of the form

$$f(U) = \sum_{1 \le i,j \le n} c_{i,j} u_{i,j},$$

where $U = (u_{i,j})_{1 \le i,j \le n} \in \mathscr{U}_n$.

In Theorem 2.3 we showed that the orthogonal projection $\pi_{(1,0)} = \pi_{\mathfrak{H}_{(1,0)}(\mathscr{U}_n)}$ on $L_2(\mathscr{U}_n)$ onto $\mathfrak{H}_{(1,0)}(\mathscr{U}_n)$ is a convolution operator with respect to the kernel $t_{(1,0)} = t_{\mathfrak{H}_{(1,0)}(\mathscr{U}_n)}$. We need an alternative description of this projection in terms of the canonical orthonormal basis of $\mathfrak{H}_{(1,0)}(\mathscr{U}_n)$.

By (8), the collection of all normalised functions $\sqrt{n} e_{ij}$, $1 \le i, j \le n$, forms an orthonormal system in $L_2(\mathscr{U}_n)$, and hence an orthonormal basis of $\mathfrak{H}_{(1,0)}(\mathscr{U}_n)$ considered as a subspace of $L_2(\mathscr{U}_n)$. Consequently, for each $f \in L_2(\mathscr{U}_n)$,

$$\pi_{(1,0)}(f) = \sum_{1 \le i,j \le n} \langle f, \sqrt{n} e_{ij} \rangle_{L_2} \sqrt{n} e_{ij} = n \sum_{1 \le i,j \le n} \langle f, e_{ij} \rangle_{L_2} e_{ij}, \tag{38}$$

where $e_{ij} \in \mathfrak{H}_{(1,0)}(\mathscr{U}_n)$ is defined by $e_{ij}(U) = u_{i,j}$ for $U \in \mathscr{U}_n$.

Comparing the two representations of $\pi_{(1,0)}$ we now have leads to the following.

Proposition 2.9. *For each $n \in \mathbb{N}$, we have $\mathrm{t}_{(1,0)} = n\,\mathrm{tr}$, and moreover*

$$\pi_{(1,0)} f = n(f * \mathrm{tr}), \quad f \in L_2(\mathscr{U}_n),$$

and

$$\|\pi_{(1,0)} : C(\mathscr{U}_n) \to \mathfrak{H}_{(1,0)}(\mathscr{U}_n)\| = n \int_{\mathscr{U}_n} |\mathrm{tr}(V)|\, dV.$$

Proof. To check the equality $\mathrm{t}_{(1,0)} = n\,\mathrm{tr}$, recall that, by Lemma 2.4 (i) and the definition of $\mathrm{t}_{(1,0)}$ from (37), for all $f \in L_2(\mathscr{U}_n)$, one gets

$$(\pi_{(1,0)} f)(\mathrm{Id}) = \langle f, \mathrm{t}_{(1,0)} \rangle_{L_2}.$$

On the other hand, by (38), for all $f \in L_2(\mathscr{U}_n)$,

$$(\pi_{(1,0)} f)(\mathrm{Id}) = n \sum_{i,j} \langle f, e_{ij} \rangle_{L_2} e_{ij}(\mathrm{Id}) = n \sum_i \langle f, e_{ii} \rangle_{L_2} = n \langle f, \mathrm{tr} \rangle_{L_2},$$

which together with the preceding equality is what we were looking for. Deducing the second and third claim is then immediate from Theorem 2.3 (iii). □

Proposition 2.10. *$\mathfrak{H}_{(1,0)}(\mathscr{U}_n)$ is a strongly accessible $\mathscr{U}_n$-invariant subspace of $C(\mathscr{U}_n)$.*

Proof. Take $f = \sum_{1 \le i,j \le n} c_{i,j} e_{i,j} \in \mathfrak{H}_{(1,0)}(\mathscr{U}_n)$ such that $f(V^{-1}UV) = f(U)$ for every $U, V \in \mathscr{U}_n$. Clearly, f can be considered as a linear functional on M_n. This implies that there exists $A \in M_n$ such that $f(U) = \mathrm{tr}(AU)$ for all $U \in M_n$. Then, from the assumption on f, it follows that, for all $U, V \in \mathscr{U}_n$,

$$\mathrm{tr}(AU) = f(U) = f(V^{-1}UV) = \mathrm{tr}(AV^{-1}UV) = \mathrm{tr}(VAV^{-1}U).$$

Combining this with the fact that any matrix in M_n is a linear combination of unitary matrices, we deduce that $A = VAV^{-1}$ for every $V \in \mathscr{U}_n$, and so A commutes with all matrices in M_n. This implies that $A = \gamma \mathrm{Id}$ for some $\gamma \in \mathbb{C}$, and hence as desired $f = \gamma\,\mathrm{tr}$. □

A comment is in order: if $p+q > 1$, then $g_1(A) := \mathrm{tr}(A^p(A^*)^q)$ and $g_2(A) := \mathrm{tr}(A)^p\,\mathrm{tr}(A^*)^q$ are different class functions. Thus, in this case, $\mathfrak{H}_{(p,q)}(\mathscr{U}_n)$ is not strongly accessible.

The following result identifies $\mathfrak{H}_{(1,0)}(\mathscr{U}_n)$ with the trace class $\mathscr{S}_1(n)$.

Proposition 2.11. *The space $\mathfrak{H}_{(1,0)}(\mathscr{U}_n)$ is isometrically isomorphic to $\mathscr{S}_1(n)$. More precisely,*

$$\mathscr{S}_1(n) \to \mathfrak{H}_{(1,0)}(\mathscr{U}_n), \quad A \mapsto [f : U \mapsto \mathrm{tr}(AU)], \tag{39}$$

is a surjective isometry.

Proof. Obviously, the mapping in (39) is a linear bijection. Indeed, as a linear space $\mathscr{S}_1(n)$ equals M_n, and $\mathfrak{H}_{(1,0)}(\mathscr{U}_n)$ equals the algebraic dual $M_n^\times$ of M_n. Moreover, it is well known that the mapping $A \mapsto [f : U \mapsto \mathrm{tr}(AU)]$ identifies M_n and $M_n^\times$. So it remains to prove that the mapping in (39) is isometric. To prove this, we use a result of Nelson [1961] (see also [Harris 1997, Theorem 1]) showing that, for any complex-valued function f which is continuous on the closed and analytic on the open unit ball of $\mathscr{L}(\ell_2^n)$, we have $\sup_{\|T\|\le 1} |f(T)| = \sup_{U\in\mathscr{U}_n} |f(U)|$. But then, by (7), for every $A \in \mathscr{S}_1(n)$,

$$\|A\|_1 = \sup_{\|T\|\le 1} |\mathrm{tr}(AT)| = \sup_{U\in\mathscr{U}_n} |\mathrm{tr}(AU)|,$$

completing the argument. □

2.5. *Proof of the main result.* We begin with the following presentation.

Proof of the integral formula from (32). We use the identification from Proposition 2.11 and combine it with Proposition 2.10 and Theorem 2.6. Then Proposition 2.9 completes the argument. □

Now we deal with the limit formula from (33). For this we need to recall some well-known results from probability theory; for more on this see [Billingsley 1999]. We are going to use that, given any sequence (Y_n) of random variables which converges in distribution to the random variable Y and any continuous real-valued function f, the sequence $(f(Y_n))$ converges in distribution to $f(Y)$. Recall also that a sequence $(Y_n)_n$ of random variables is said to be uniformly integrable whenever

$$\lim_{a\to\infty} \sup_{n\ge 1} \int_{|Y_n|\ge a} |Y_n|\, dP = 0.$$

Uniform integrability will be useful for us due to the fact (see for example [Billingsley 1999, Theorem 3.5]) that if $(Y_n)_n$ is a uniformly integrable sequence of random variables and $Y_n \xrightarrow{D} Y$, then Y is integrable and

$$\mathbb{E}(Y_n) \to \mathbb{E}(Y). \tag{40}$$

To check uniform integrability we cite a standard criterion.

Remark 2.12. If $\sup_n \mathbb{E}(|Y_n|^{1+\varepsilon}) \le C$ for some $\varepsilon, C > 0$, then $(Y_n)_n$ is uniformly integrable; indeed, this is a consequence of

$$\lim_{a\to\infty} \sup_{n\ge 1} \int_{|Y_n|\ge a} |Y_n|\, dP \le \lim_{a\to\infty} \frac{1}{a^{\varepsilon}} C.$$

We are now ready to provide the following.

Proof of the limit formula from (33). Consider the sequence $(\mathrm{tr}(U(n)))$ of random variables on $\mathscr{U}_n$, where $U(n)$ is a unitary matrix uniformly Haar distributed. Then, by [Johansson 1997, Corollary 2.4] (see also [Diaconis and Shahshahani 1994] or [Pastur and Shcherbina 2011, Problem 8.5.5]), the previous sequence converges in distribution to the standard Gaussian complex random variable $\boldsymbol{\gamma}$. Indeed, the random variables $\sqrt{2}\,\mathrm{Re}[\mathrm{tr}(U(n))]$ and $\sqrt{2}\,\mathrm{Im}[\mathrm{tr}(U(n))]$ converge in distribution to a standard real Gaussian random variable.

Thus, the sequence $(\sqrt{2}|\mathrm{tr}(U(n))|)$ of random variables on $\mathscr{U}_n$ converges in distribution to a Rayleigh random variable. Moreover, since, as mentioned in (9), for each n,

$$\mathbb{E}(|\mathrm{tr}(U(n))|^2) = \int_{\mathscr{U}_n} |\mathrm{tr}(V)|^2\, dV = 1,$$

the sequence of random variables $\mathrm{tr}(U(n))$ by Remark 2.12 is uniformly integrable. Consequently, we deduce from (40) that $(\mathbb{E}(\sqrt{2}|\mathrm{tr}(U(n))|))$ converges to the expectation of a Rayleigh random variable. That is,

$$\lim_{n\to\infty} \mathbb{E}(\sqrt{2}|\mathrm{tr}(U(n))|) \to \sqrt{\frac{\pi}{2}}.$$

Using (32), we arrive at

$$\lim_{n\to\infty} \frac{1}{n}\boldsymbol{\lambda}(\mathscr{S}_1(n)) = \frac{1}{\sqrt{2}} \lim_{n\to\infty} \mathbb{E}(\sqrt{2}|\mathrm{tr}(U(n))|) = \frac{\sqrt{\pi}}{2},$$

which completes the proof. □

2.6. *Other examples.* In this final subsection, we give some other examples where the theory developed to reach our main objective (Theorem 2.1) could be applied.

The first result shows that examples of accessible $\mathscr{U}_n$-invariant subspaces come in pairs. To see this we define the linear and isometric bijection

$$\phi : C(\mathscr{U}_n) \to C(\mathscr{U}_n), \quad f \mapsto [U \mapsto f(U^*)].$$

For any subspace S in $C(\mathscr{U}_n)$, we write $S_* := \phi S$. As a first example we mention that, isometrically,

$$(\mathfrak{H}_{(1,0)}(\mathscr{U}_n))_* = \phi(\mathfrak{H}_{(1,0)}(\mathscr{U}_n)) = \mathfrak{H}_{(0,1)}(\mathscr{U}_n).$$

Proposition 2.13. *Let S be a $\mathscr{U}_n$-invariant subspace of $C(\mathscr{U}_n)$ which is closed in $L_2(\mathscr{U}_n)$. Then S_* is $\mathscr{U}_n$-invariant and $\mathrm{t}_{S_*} = \overline{\mathrm{t}_S}$. Moreover, S is strongly accessible* (*resp. accessible*) *if and only if S_* is strongly accessible* (*resp. accessible*).

Proof. Obviously, S_* is $\mathscr{U}_n$-invariant. In order to show that $\mathrm{t}_{S_*} = \overline{\mathrm{t}_S}$ note first that $\pi_{S_*} = \phi \circ \pi_S \circ \phi$. Then, for every $f \in L_2(\mathscr{U}_n)$ and $U \in \mathscr{U}_n$, it follows by Theorem 2.3 and Remark 2.5 that

$$\begin{aligned}(\pi_{S_*} f)(U) &= ((\pi_S \phi f))(U^*) \\ &= (\phi f * \mathrm{t}_S)(U^*) = \int_{\mathscr{U}_n} f(V^*)\mathrm{t}_S(U^*V^*)\,dV \\ &= \int_{\mathscr{U}_n} f(V^*)\overline{\mathrm{t}_S}(VU)\,dV = \int_{\mathscr{U}_n} f(V)\overline{\mathrm{t}_S}(V^*U)\,dV \\ &= \int_{\mathscr{U}_n} f(V)\overline{\mathrm{t}_S}(UV^*)\,dV = (f * \overline{\mathrm{t}_S})(U),\end{aligned}$$

which by the uniqueness of t_{S_*} leads to the claim. Let us turn to the “moreover part”. It is immediate that strong accessibility of S is equivalent to strong accessibility of S_*. So let us assume that S is accessible and show that then S_* is accessible. Take any projection $Q : C(\mathscr{U}_n) \to S_*$ which commutes with the action of $\mathscr{U}_n \times \mathscr{U}_n$ on $C(\mathscr{U}_n)$. Since $\phi \circ C_{L_V} = R_{V^*} \circ \phi$ and $\phi \circ C_{R_V} = L_{V^*} \circ \phi$ for all $V \in \mathscr{U}_n$, the projection $\phi \circ Q \circ \phi$ onto S commutes with the action of $\mathscr{U}_n \times \mathscr{U}_n$ on $C(\mathscr{U}_n)$, and hence by assumption $\phi \circ Q \circ \phi = \pi_S$. But then clearly $Q = \phi \circ \pi_S \circ \phi = \pi_{S_*}$, which is the desired conclusion. □

Note that, in particular, $\mathfrak{H}_{(0,1)}(\mathscr{U}_n)$ is $\mathscr{U}_n$-invariant and accessible, and $\mathrm{t}_{(0,1)} = \overline{\mathrm{tr}}$; therefore, by Theorem 2.6,

$$\boldsymbol{\lambda}(\mathfrak{H}_{(0,1)}(\mathscr{U}_n)) = \|\pi_{(0,1)} : C(\mathscr{U}_n) \to \mathfrak{H}_{(0,1)}(\mathscr{U}_n)\| = n \int_{\mathscr{U}_n} |\mathrm{tr}(V)|\,dV.$$

Also,

$$\lim_{n\to\infty} \frac{\boldsymbol{\lambda}(\mathfrak{H}_{(0,1)}(\mathscr{U}_n))}{n} = \frac{\sqrt{\pi}}{2}.$$

Of course, this is also a simple consequence of Theorem 2.1 using that ϕ identifies $\mathfrak{H}_{(0,1)}(\mathscr{U}_n)$ and $\mathfrak{H}_{(1,0)}(\mathscr{U}_n)$ isometrically.

We continue with another simple stability property of accessible subspaces.

Proposition 2.14. *Let S_1 and S_2 be accessible, $\mathscr{U}_n$-invariant subspaces of $C(\mathscr{U}_n)$ which in $L_2(\mathscr{U}_n)$ are closed and orthogonal. Then $S_1 \oplus S_2$ is accessible and $\mathscr{U}_n$-invariant, and moreover $t_{S_1 \oplus S_2} = \mathrm{t}_{S_1} + \mathrm{t}_{S_2}$. Consequently,*

$$\boldsymbol{\lambda}(S_1 \oplus S_2) = \|\pi_{S_1} + \pi_{S_2} : C(\mathscr{U}_n) \to S_1 \oplus S_2\| = \int_{\mathscr{U}_n} |\mathrm{t}_{S_1}(V) + \mathrm{t}_{S_2}(V)|\,dV. \tag{41}$$

Proof. That $S_1 \oplus S_2$ is $\mathscr{U}_n$-invariant is straightforward. Note that $\pi_{S_1 \oplus S_2} = \pi_{S_1} + \pi_{S_2}$ is the orthogonal projection on $L_2(\mathscr{U}_n)$ onto $S_1 \oplus S_2$. Then, by Theorem 2.3, for all $f \in L_2(\mathscr{U}_n)$, we have

$$\pi_{S_1 \oplus S_2} f = \pi_{S_1} f + \pi_{S_2} f = f * \mathrm{t}_{S_1} + f * \mathrm{t}_{S_2} = f * (\mathrm{t}_{S_1} + \mathrm{t}_{S_2}).$$

Hence by the uniqueness of $t_{S_1 \oplus S_2}$, we get

$$\mathrm{t}_{S_1 \oplus S_2} = \mathrm{t}_{S_1} + \mathrm{t}_{S_2}.$$

Let us now show that $S_1 \oplus S_2$ is accessible. So let Q be a projection on $C(\mathscr{U}_n)$ onto $S_1 \oplus S_2$ which commutes with the action of $\mathscr{U}_n \times \mathscr{U}_n$ on $C(\mathscr{U}_n)$. We claim that $Q = \pi_{S_1 \oplus S_2}$. Indeed, consider the two projections

$$Q_{S_1} = \pi_{S_1} \circ Q \quad \text{and} \quad Q_{S_2} = \pi_{S_2} \circ Q$$

on $C(\mathscr{U}_n)$ onto S_1 and S_2, respectively. Since π_{S_1} and π_{S_2} both commute with the action of $\mathscr{U}_n \times \mathscr{U}_n$ on $C(\mathscr{U}_n)$, we have that Q_{S_1} and Q_{S_2} also do. Then, by the accessibility of S_1 and S_2, we see that

$$Q_{S_1} = \pi_{S_1} \quad \text{and} \quad Q_{S_2} = \pi_{S_2},$$

and hence, for all $f \in C(\mathscr{U}_n)$, as desired,

$$Qf = \pi_{S_1}(Qf) + \pi_{S_2}(Qf) = \pi_{S_1} f + \pi_{S_2} f = \pi_{S_1 \oplus S_2} f.$$

To conclude the proof just note that (41) is then a direct consequence of Theorem 2.6. □

Combining the previous two propositions we obtain the following.

Corollary 2.15. *For each $n \in \mathbb{N}$,*

$$\begin{aligned}\boldsymbol{\lambda}(\mathfrak{H}_{(1,0)}(\mathscr{U}_n) \oplus \mathfrak{H}_{(0,1)}(\mathscr{U}_n)) &= \|\pi_{(1,0)} \oplus \pi_{(0,1)} : C(\mathscr{U}_n) \to \mathfrak{H}_{(1,0)}(\mathscr{U}_n) \oplus \mathfrak{H}_{(0,1)}(\mathscr{U}_n)\| \\ &= 2n \int_{\mathscr{U}_n} |\mathrm{Re}(\mathrm{tr}(V))| \, dV.\end{aligned}$$

Moreover,

$$\lim_{n\to\infty} \frac{\boldsymbol{\lambda}(\mathfrak{H}_{(1,0)}(\mathscr{U}_n) \oplus \mathfrak{H}_{(0,1)}(\mathscr{U}_n))}{\sqrt{2}n} = \sqrt{\frac{2}{\pi}}. \tag{42}$$

Before giving a proof of this, we mention that the denominator of the fraction above (so $\sqrt{2}n$) is exactly the square root of the dimension of the sum space $\mathfrak{H}_{(1,0)}(\mathscr{U}_n) \oplus \mathfrak{H}_{(0,1)}(\mathscr{U}_n)$.

Proof of Corollary 2.15. We only have to prove (42) since the integral formula for the projection constant follows directly from (41) and Proposition 2.9.

We repeat an argument similar to the proof of (33). We know, by [Johansson 1997, Corollary 2.4], that the sequence $(\sqrt{2}\,\mathrm{Re}[\mathrm{tr}(U(n))])$ of random variables converges in distribution to a standard real Gaussian random variable g. In particular, $(\sqrt{2}|\mathrm{Re}[\mathrm{tr}(U(n))]|)$ converges in distribution to $|g|$. Note that the sequence $(\sqrt{2}|\mathrm{Re}[\mathrm{tr}(U(n))]|)$ is uniformly integrable. Indeed,

$$\mathbb{E}(|\mathrm{Re}[\mathrm{tr}(U(n))]|^2) \le \mathbb{E}(|\mathrm{tr}(U(n))|^2) = 1;$$

see again Remark 2.12. Thus, by (40),

$$\begin{aligned}\lim_{n\to\infty} \frac{\boldsymbol{\lambda}(\mathfrak{H}_{(1,0)}(\mathscr{U}_n) \oplus \mathfrak{H}_{(0,1)}(\mathscr{U}_n))}{\sqrt{2}n} &= \lim_{n\to\infty} \mathbb{E}(\sqrt{2}|\mathrm{Re}[\mathrm{tr}(U(n))]|) \\ &= \mathbb{E}|g| = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} |x| e^{-x^2/2} \, dx = \sqrt{\frac{2}{\pi}}.\end{aligned}$$

□

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