Analysis & PDE

Volume 18 No. 3 2025

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Analysis & PDE
msp.org/apde

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Analysis & PDE (ISSN 1948-206X electronic, 2157-5045 printed) at Mathematical Sciences Publishers, 798 Evans Hall #3840, c/o University of California, Berkeley, CA 94720-3840, is published continuously online.

APDE peer review and production are managed by EditFlow® from MSP.

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http://msp.org/

ANALYSIS AND PDE
Vol. 18 (2025), No. 3, pp. 549–590
DOI: 10.2140/apde.2025.18.549

ROTATING SPIRALS IN SEGREGATED REACTION-DIFFUSION SYSTEMS

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We give a complete characterization of the boundary traces φ_i $(i = 1, \dots, K)$ supporting spiraling waves, rotating with a given angular speed ω, which appear as singular limits of competition-diffusion systems of the type

$$\begin{cases} \partial_t u_i - \Delta u_i = \mu u_i - \beta u_i \sum_{j \neq i} a_{ij} u_j & \text{in } \Omega \times \mathbb{R}^+, \\ u_i = \varphi_i & \text{on } \partial\Omega \times \mathbb{R}^+, \\ u_i(\boldsymbol{x}, 0) = u_{i,0}(\boldsymbol{x}) & \text{for } \boldsymbol{x} \in \Omega, \end{cases}$$

as $\beta \to +\infty$. Here Ω is a rotationally invariant planar set, and $a_{ij} > 0$ for every i and j. We tackle also the homogeneous Dirichlet and Neumann boundary conditions, as well as entire solutions in the plane. As a byproduct of our analysis, we detect explicit families of eternal, entire solutions of the pure heat equation, parametrized by $\omega \in \mathbb{R}$, which reduce to homogeneous harmonic polynomials for $\omega = 0$.

1. Introduction

This paper deals with existence, uniqueness and qualitative properties of rotating spiraling waves arising in the singular limit of reaction-diffusion systems, when the interspecific competition rates become infinite. More precisely, we are concerned with the singular limits, as $\beta \to +\infty$, of the following model problem involving $K \geq 3$ species competing in the plane:

$$\begin{cases} \partial_t u_i - \Delta u_i = f_i(u_i) - \beta u_i \sum_{j \neq i} a_{ij} u_j & \text{in } \Omega \times \mathbb{R}^+, \\ u_i = \varphi_i & \text{on } \partial\Omega \times \mathbb{R}^+, \\ u_i(\boldsymbol{x}, 0) = u_{i,0}(\boldsymbol{x}) & \text{for } \boldsymbol{x} \in \Omega. \end{cases} \tag{1}$$

Here $\Omega \subset \mathbb{R}^2$ has a smooth boundary and $u_i = u_i(\boldsymbol{x}, t)$ represents the density of the i-th species $(1 \leq i \leq K)$, whose internal dynamic is described by the function f_i. The positive numbers βa_{ij} account for the interspecific competition rates, so that the interaction has a repulsive character. The boundary data φ_i are positive and segregated, i.e., $\varphi_i \varphi_j \equiv 0$ for $j \neq i$.

As already mentioned, we are concerned with the limit case of strong competition; that is, when the parameter β goes to $+\infty$ while the positive coefficients a_{ij} remain fixed. In this case it is known that the densities u_i *segregate*, in the sense that they converge uniformly to limit densities satisfying $u_i u_j \equiv 0$ for $j \neq i$; hence a pattern arises, and the common nodal set (where all densities vanish simultaneously) can be considered as a free boundary; see [Caffarelli et al. 2009; Conti et al. 2005a; 2005b; Wei and Weth 2008] for steady states and [Dancer et al. 2012a; 2012b; Dancer and Zhang 2002; Wang and Zhang 2010]

MSC2020: primary 35B25, 35B36; secondary 35K51, 92D25.
Keywords: competition-diffusion systems, singular perturbation, free boundary problems, spiral waves.

for time-varying solutions. For such segregated limit profiles, the interface conditions are expressed by two systems of differential inequalities which play a fundamental role in our work:

$$\partial_t u_i - \Delta u_i \le f_i(u_i), \quad \partial_t \hat{u}_i - \Delta \hat{u}_i \ge \hat{f}_i(\hat{u}_i), \tag{2}$$

where the differential inequalities are understood in the variational sense, and

$$\hat{u}_i = u_i - \sum_{j \neq i} \frac{a_{ij}}{a_{ji}} u_j, \quad \hat{f}_i(\hat{u}_i) = f(u_i) - \sum_{j \neq i} \frac{a_{ij}}{a_{ji}} f(u_j). \tag{3}$$

These inequalities incorporate the transmission conditions at the free boundary, that is the closure of the interfaces $\partial\{u_i > 0\} \cap \partial\{u_j > 0\}$, which separate the supports of u_i and u_j at any fixed time t.

For planar stationary solutions, the structure of the free boundary has been the object of several papers. In the case of symmetric interactions ($a_{ij} = a_{ji}$ for every i and j), it is composed by a regular part, a collection of smooth curves, meeting at a locally finite number of (singular) clustering points, with definite tangents; see [Caffarelli et al. 2009; Conti et al. 2005a; 2006; Helffer et al. 2009]. On the other hand, the asymmetric case has been treated only more recently in [Terracini et al. 2019]: while the topological structure of the free boundary is analogous to the symmetric case (smooth curves meeting at isolated singular points), the geometric description differs strongly in a neighborhood of each singular point, where the nodal lines meet with logarithmic spiraling asymptotics.

Going back to time-dependent systems, rotating spiraling patterns have been detected numerically in the case of three competing populations in [Murakawa and Ninomiya 2011]. Driven by this phenomenology, in this paper we seek rotating spirals, that is rigidly rotating waves which are steady states of (2) in a reference frame spinning with frequency ω; such solutions satisfy $\partial_t u_i = \omega \partial_\theta u_i$ in a disk, subject to boundary conditions which are prescribed in the rotating frame, and exhibit spiraling interfaces near the origin. Hence, in comparison with the literature, our work tackles the segregation problem from a new perspective, that is the existence of limit segregated profiles satisfying additional qualitative properties or shadowing some given shapes. On the other hand, the literature on other aspects of segregation triggered by strong competition, starting from pioneering works by Dancer and Du [1995a; 1995b], is now very vast, and it is impossible to give a complete account of it here; besides the papers mentioned above, we mention a few more recent ones such as [Arakelyan and Bozorgnia 2017; Berestycki and Zilio 2018; 2019; Lanzara and Montefusco 2019; 2021; Verzini and Zilio 2014].

The rotating spiral shapes we investigate evoke some other typical examples of spatiotemporal patterns arising in reaction-diffusion systems in planar domains: the spiral waves. In the simplest case, spiral waves are stationary waves in a rotating frame, while modulated spiraling waves may emanate from rigidly rotating ones in some circumstances. Such waves arise in different models and appear in the literature about reaction-diffusion systems in contexts different from singular perturbation problems; see, e.g., [Sandstede et al. 1997; Sandstede and Scheel 2007; 2023]. As far as we know, this is the first study on spiraling rotating waves for segregated limit profiles of competition-diffusion systems. We also mention that spiraling interfaces arise in free boundary problems in entirely different contexts [Allen and Kriventsov 2020].

To construct eternal solutions of spiraling-type to the limit system (2), in this paper we deal with suitable classes of reactions f_i and boundary conditions. More precisely, let us consider identical, linear reactions in the unit ball (centered at $\mathbf{0}$):

$$\Omega = B, \quad f_i(u) = \mu u \text{ for some } \mu \in \mathbb{R}.$$

We insert into (2) the rotating wave ansatz

$$u_i(\boldsymbol{x}, t) = u_i(\mathcal{R}_{\omega t}\boldsymbol{x}),$$

where

$$\mathcal{R}_{\omega t} = \begin{pmatrix} \cos(\omega t) & -\sin(\omega t) \\ \sin(\omega t) & \cos(\omega t) \end{pmatrix}$$

is the rotation matrix of angular speed ω, and we obtain the stationary system of inequalities

$$\begin{cases} -\Delta u_i + \omega \boldsymbol{x}^{\perp} \cdot \nabla u_i \leq \mu u_i & \text{in } B, \\ -\Delta \hat{u}_i + \omega \boldsymbol{x}^{\perp} \cdot \nabla \hat{u}_i \geq \mu \hat{u}_i & \text{in } B, \\ u_i \cdot u_j = 0 & \text{for } i \neq j, \end{cases} \tag{4}$$

where $\boldsymbol{x}^{\perp} = \mathcal{R}_{\pi/2}\boldsymbol{x}$ and $\hat{u}_i$ is defined in (3). It is worth noting that, despite appearances, this system is strongly nonlinear and has to be tackled as a free boundary problem.

We are interested in solutions of (4) whose nodal set consists in smooth arcs, emanating from ∂B and spiraling towards $\mathbf{0}$, which is the unique singular point of the free boundary. In this way, each arc is a smooth interface between two adjacent densities, and the origin is the only point with higher multiplicity (see Figure 1). In this framework we provide a complete description of the nonhomogeneous Dirichlet problem associated with (4).

Let us consider a K-tuple $(\varphi_1, \dots, \varphi_K)$ of segregated boundary traces. Precisely, we assume that, for every $i = 1, \dots, K$,

$$\begin{cases} \varphi_i \in C^{0,1}(\partial B), \quad \varphi_i \geq 0, \\ \{\boldsymbol{x} : \varphi_i(\boldsymbol{x}) > 0\} \text{ are connected, nonempty and disjoint arcs,} \\ \bigcup_i \operatorname{supp} \varphi_i = \partial B. \end{cases} \tag{5}$$

Up to relabeling, we can assume that the traces φ_i are labeled in counterclockwise order.

In general, it is not reasonable to expect that any choice of the boundary data provides a solution of (4) with a unique singular point at $\mathbf{0}$. Indeed, we show that this happens exactly for an explicit subset having codimension $K-1$ in the space of traces. Let $s = (s_1, \dots, s_K) \in \mathbb{R}^K$, with $s_i > 0$ for all i, and let us consider the class of functions

$$\mathcal{S}_{\text{rot}} = \{U = (u_1, \dots, u_K) \in (H^1(B))^K : u_i \geq 0 \text{ satisfy (4)}, \ u_i = s_i \varphi_i \text{ on } \partial B\}. \tag{6}$$

To state our main result we introduce the parameter

$$\alpha = \frac{1}{2\pi} \ln\left(\frac{a_{12}}{a_{21}} \cdot \frac{a_{23}}{a_{32}} \cdots \frac{a_{K1}}{a_{1K}}\right), \tag{7}$$

which synthesizes the asymmetry of the coefficients a_{ij}; see [Terracini et al. 2019] for more details.

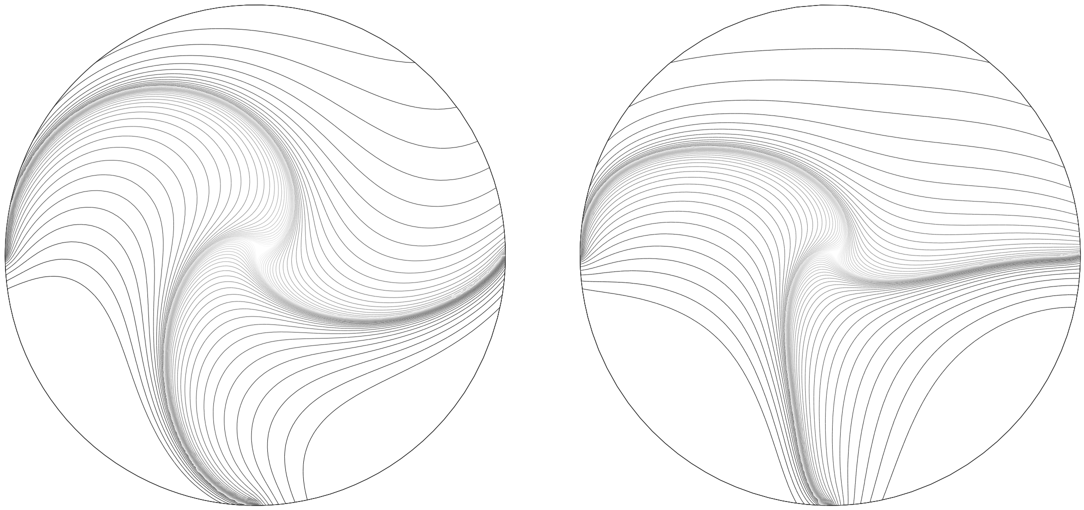

Figure 1. Contour lines of a numerical simulation (obtained in FreeFem++ [Hecht 2012]) in the case of $K = 3$ densities, with asymmetric competition such that $a_{12}/a_{21} = a_{23}/a_{32} = a_{31}/a_{13} = 10$, and reaction term $\mu = 0$. The angular velocity is $\omega = 3$ for the image on the left (counterclockwise spin) and $\omega = -3$ for the image on the right (clockwise spin). In both cases, we obtain a unique singular point at the center of the circle by choosing the same boundary conditions, which satisfy the necessary and sufficient conditions in Theorem 1.1; see (10). The rotation affects the shape of the spirals but not their asymptotic behavior close to the center. This is part of the content of Theorem 1.1.

Our main result is the following theorem.

Theorem 1.1. *Let $K \geq 3$, $a_{ij} > 0$ and $\omega \in \mathbb{R}$. Assume that $\mu < \pi^2$ and $(\varphi_1, \ldots, \varphi_K)$ satisfies* (5). *There exists*

$$\bar{s} = (\bar{s}_1, \ldots, \bar{s}_K) \in \mathbb{R}^K,$$

independent of μ and ω, with $\bar{s}_i > 0$ for all i, such that:

(1) *If $s = t\bar{s}$ for some $t > 0$, then $\mathcal{S}_{\mathrm{rot}}$ contains an element with a unique singular point at* $\mathbf{0}$. *Moreover, such an element is unique and, defining $\mathcal{U}$ as a suitable linear combination of its components, we have*

$$\mathcal{U}(r\cos\vartheta, r\sin\vartheta) = Ar^\gamma \cos\left(\frac{K}{2}\vartheta - \alpha \ln r\right) + o(r^\gamma) \quad \text{as } r \to 0, \tag{8}$$

where

$$\gamma = \frac{K}{2} + \frac{2\alpha^2}{K} \quad \text{and} \quad 0 < A_0 \leq A(\boldsymbol{x}) \leq A_1.$$

(2) *If $s \neq t\bar{s}$ for every $t > 0$, then $\mathcal{S}_{\mathrm{rot}}$ contains no element with a unique singular point at* $\mathbf{0}$.

Corollary 1.2. *Under the assumptions of the above theorem, if the problem is invariant under a rotation of* $2\pi/K$, *i.e.*,

$$\varphi_{i+1}(\boldsymbol{x}) = \varphi_1(\mathcal{R}_{2\pi i/K}\boldsymbol{x}) \quad \text{and} \quad \frac{a_{i(i+1)}}{a_{(i+1)i}} = \frac{a_{K1}}{a_{1K}} \tag{9}$$

for every i, *then*

$$\bar{s} = (1, 1, \dots, 1).$$

Remark 1.3. Notice that the asymptotic expansion (8) implies that the free boundary, near the singular point $\mathbf{0}$, is the union of K equidistributed logarithmic spirals, as long as $\alpha \neq 0$. On the other hand, in the case $\alpha = 0$, we obtain that the interfaces enter the origin with a definite angle. In particular, this holds true in the symmetric case $a_{ij} = a_{ji}$ for every $j \neq i$.

Remark 1.4. In this work, we normalize the radius of the disc, taking the slope of the reaction term μ at zero as a parameter. If we wish to work in a ball of radius R then we need $\mu < \pi^2/R^2$, as seen with a simple scaling.

Remark 1.5. A natural question concerns the dynamical stability of the solutions above. From this point of view, the study of the linearized problem of (1), due to the presence of the large parameter β, does not seem a viable path. This leaves open the problem of stability, for the moment, although numerical simulations for (1), with logistic reactions and β large, suggest stability for some specific angular velocity ω.

We shall adopt a constructive point of view, building the solution by superposition of fundamental elementary modes. The dependence of such building blocks on the parameter ω and μ shows the presence of resonances at exceptional values; see Section 6 for further details. As a byproduct of the analysis of resonances, we will prove the following results.

Theorem 1.6 (homogeneous boundary conditions). *Let* $K \geq 3$ *and* $a_{ij} > 0$. *If* (μ, ω) *belongs to a suitable discrete set then there exists a nontrivial element of* $\mathcal{S}_{\mathrm{rot}}$ *with null traces. Analogous results hold for homogenous Neumann or Robin boundary conditions.*

Theorem 1.7 (entire solutions). *Let* $K \geq 3$ *and* $a_{ij} > 0$. *For almost every* (μ, ω), *there exists an entire solution of* (4) *in* $\mathbb{R}^2$.

In the above results, the conditions on (μ, ω) are explicit in terms of the zero set of suitable analytic functions in the complex plane. In both cases, the solutions are explicit in terms of trigonometric and Bessel's functions. This allows us to study the structure of the free boundary of the entire solutions far away from the origin. It turns out that, at least when $\omega \neq 0$, also at infinity the free boundary consists in equidistributed spirals, now of arithmetic type. We refer to Lemma 6.7 and Remark 6.8 for further details.

Remark 1.8. In the particular case $\alpha = \mu = 0$, we obtain that the entire solution found in Theorem 1.7 is related to the nodal components of a smooth rotating solution of the pure heat equation. Let $\omega > 0$, $k \geq 1$ be an integer, and let I_k denote the modified Bessel function of the first kind, with parameter k. We have that the function

$$U(re^{i\vartheta}, t) = \mathrm{Re}\big[e^{ik(\vartheta + \omega t)} I_k\big(\tfrac{1}{2}\sqrt{2\omega k}(1 + i)r\big)\big]$$

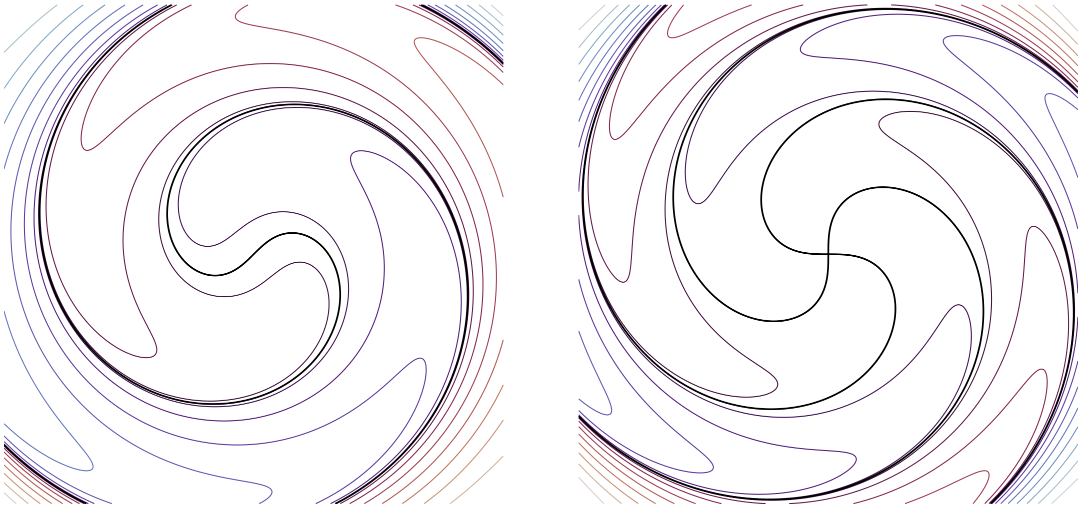

Figure 2. Contour lines of the rotating caloric functions in Remark 1.8. Here $\omega = 1$, and $k = 1$ and $k = 2$, respectively. In black the nodal lines: the appearance of arithmetic spirals for r large is rather clear in the picture.

is an entire, eternal rotating solution of the heat equation

$$U_t - \Delta U = 0 \quad \text{in } \mathbb{R}^2 \times \mathbb{R}$$

having $2k$ nodal regions, which coincide up to rotations that are a multiple of π/k. The equidistributed nodal lines admit a straight tangent as $r \to 0$, while they behave like arithmetic spirals of the equation $\vartheta = \sqrt{\omega/(2k)}r$ as $r \to +\infty$; see Figure 2. Notice that, as $\omega \to 0$, a suitable renormalization of U converges to the entire harmonic function $\operatorname{Re} z^k$.

Remark 1.9. Notice that, by separation of variables, one may treat boundary value problems for rotating solutions also on other rotationally invariant domains Ω, such as annuli or external domains. Of course, since in these cases $\mathbf{0} \notin \Omega$, this cannot provide spiraling solutions, at least in our sense.

Let us provide an explanation for our construction. When a smooth curve separates two densities of an element of $\mathcal{S}_{\text{rot}}$, at least locally, the gradients of the two densities are proportional across such an interface. Indeed, by definition of $\hat{u}_i$, the function $a_{21}u_1 - a_{12}u_2$ solves an elliptic equation in a neighborhood of the interface.

Let us assume, for concreteness, $K = 3$. In case the nodal structure of $(u_1, u_2, u_3) \in \mathcal{S}_{\text{rot}}$ is the required one, as depicted in Figure 1, then a suitable linear combination of the components u_i satisfies an equation on B, up to a curve. More precisely, let us define

$$\mathcal{U} = u_1 - \frac{a_{12}}{a_{21}}u_2 + \frac{a_{12}}{a_{21}} \cdot \frac{a_{23}}{a_{32}}u_3, \qquad \Gamma = \overline{\{u_1 > 0\}} \cap \overline{\{u_3 > 0\}}.$$

It is easy to check that

$$-\Delta\mathcal{U} + \omega\boldsymbol{x}^\perp \cdot \nabla\mathcal{U} = \mu\mathcal{U} \quad \text{in } B \setminus \Gamma,$$

while, if $\mathbf{0} \neq \boldsymbol{x}_0 \in \Gamma$ and α is defined as in (7),

$$\lim_{\substack{\boldsymbol{x}\to\boldsymbol{x}_0\\ u_3(\boldsymbol{x})>0}} \nabla\mathcal{U}(\boldsymbol{x}) = -e^{2\pi\alpha} \lim_{\substack{\boldsymbol{x}\to\boldsymbol{x}_0\\ u_1(\boldsymbol{x})>0}} \nabla\mathcal{U}(\boldsymbol{x}).$$

By composing with a conformal map between $B\setminus\{\mathbf{0}\}$ and its universal covering $\mathbb{R}\times(0,\infty)$, we can lift $\mathcal{U}$ to a solution of a linear equation in the half-plane (see (11) below) having a precise nodal structure. This connection is analyzed in Section 2.

To prove Theorem 1.1 we reverse the above argument: we start by solving the equation in the covering by separation of variables in Section 3; next, we show in Section 4 that, under suitable conditions, the solution has the appropriate nodal properties to be mapped back to the disk. In both these points, we have to deal with nonresonance/coerciveness conditions, leading to the assumption on μ. On the other hand, the existence of the vector $\bar{s}$ is equivalent to the validity of suitable compatibility conditions, expressed in terms of the Fourier coefficients of the boundary data. Specifically, when $K=3$, $\bar{s}$ is any componentwise positive solution of the system

$$\int_0^{2\pi} e^{-\alpha\vartheta}\Phi(\vartheta)\sin\left(\frac{\vartheta}{2}\right)dx = \int_0^{2\pi} e^{-\alpha\vartheta}\Phi(\vartheta)\cos\left(\frac{\vartheta}{2}\right)dx = 0, \tag{10}$$

where

$$\Phi = s_1\varphi_1 - s_2\frac{a_{12}}{a_{21}}\varphi_2 + s_3\frac{a_{12}}{a_{21}}\cdot\frac{a_{23}}{a_{32}}\varphi_3.$$

We analyze the general compatibility conditions in Section 5, concluding the proof of Theorem 1.1. Finally, Theorems 1.6 and 1.7 are proved in Section 6.

2. An equivalent problem in the half-plane

As we mentioned, the proof of Theorems 1.1, 1.6 and 1.7 is based on the connection between system (4) and an equation in the half-plane, seen as the universal covering of the punctured disk. In this section we analyze such a connection.

Let μ, ω be real parameters and $v = v(x,y) \in C(\mathbb{R}\times[0,+\infty))$ be a classical solution of the equation

$$-\Delta v + \omega e^{-2y} v_x = e^{-2y}\mu v, \quad x\in\mathbb{R},\ \ y>0. \tag{11}$$

In the following we assume that v satisfies the following properties:

(a) There exists $\sigma \neq 0$ such that

$$v(x+2\pi, y) = \sigma v(x,y) \tag{12}$$

for any $x\in\mathbb{R}$, $y\geq 0$.

(b) $v(x,y)=0$ if and only if $(x,y)\in \overline{S}_i\cap\overline{S}_{i+1}$ for some $i\in\mathbb{Z}$, where the nonempty nodal regions S_i are open, connected, disjoint, unbounded and

$$\overline{S}_i \cap \{(x,0) : x\in\mathbb{R}\} = \{(x,0) : x_{i-1}\leq x\leq x_i\}, \qquad \overline{S}_i\cap\overline{S}_j \neq \varnothing \quad\Longleftrightarrow\quad j-i=-1,0,1.$$

In particular, since v is analytic for $y>0$, we obtain that the set $\overline{S}_i\cap\overline{S}_{i+1}$ is actually a locally analytic curve which accumulates both at $(x_i,0)$ and at $y=\infty$.

(c) $v|_{S_i}\in H^1(S_i)$ for every $i\in\mathbb{Z}$ (or, equivalently, their trivial extensions belong to $H^1(\mathbb{R}\times(0,+\infty))$).

We infer that $\bigcup_i \bar{S}_i = \mathbb{R}\times[0,+\infty)$, and that this covering is locally finite. Moreover, by (a), the nodal set of v is 2π-periodic in the x-direction. Up to a translation, we can assume that $x_0 = 0$, so that in particular $v(0,0) = 0$ and the number K of nodal components, up to periodicity, can be defined as

$$K = \#\{i : [x_{i-1}, x_i] \subset [0, 2\pi]\}, \ \text{ i.e., } S_{i+K} = S_i + (2\pi, 0), \quad \text{for all } i. \tag{13}$$

Notice that $\sigma > 0$ implies K even, while $\sigma < 0$ forces K odd.

Finally, we introduce the following conformal map between the half-plane and the punctured disk:

$$\mathcal{T}\colon \mathbb{R}\times(0,+\infty) \to B\setminus\{\mathbf{0}\}, \quad \mathcal{T}\colon (x,y) \mapsto \boldsymbol{x} = (e^{-y}\cos x, e^{-y}\sin x) \tag{14}$$

(for more details about this map, see Remarks 2.17 and 2.19 in [Terracini et al. 2019]).

The main result of this section is the following.

Proposition 2.1. *Let $v \in C(\mathbb{R}\times[0,+\infty))$ be a classical solution of* (11) *satisfying* (*a*), (*b*) *and* (*c*), *and let K be defined as in* (13). *Assume that the positive coefficients a_{ij} and the parameter α satisfy*

$$\prod_{i=1}^{K} \frac{a_{(i-1)i}}{a_{i(i-1)}} = (-1)^K \sigma \tag{15}$$

(*understanding $a_{01} = a_{K1}$, $a_{10} = a_{1K}$*).

For $i = 1, \dots, K$, let us define

$$u_i = (-1)^{i+1} l_i v|_{S_i} \circ \mathcal{T}, \quad \text{with } l_1 = 1, \ \ l_i = \frac{a_{i(i-1)}}{a_{(i-1)i}} \cdot l_{i-1} \tag{16}$$

(*trivially extended in the whole B*). *Then $(u_1, \dots, u_K) \in \mathcal{S}_{\text{rot}}$. Moreover, with respect to this K-tuple, the origin is the only point with higher multiplicity, with $m(\mathbf{0}) = K$.*

Vice versa, if $(u_1, \dots, u_K) \in \mathcal{S}_{\text{rot}}$ has the origin as its only singular point, then there exists v such that the first part of the proposition holds.

Remark 2.2. In the case that the asymptotic behavior of the nodal zones S_i is known as $y \to +\infty$, then by composition with $\mathcal{T}$ one can deduce the local description of the free boundary associated to $(u_1, \dots, u_K)$ near $\mathbf{0}$.

Proof. By condition (a) the functions u_i are well defined, by (b) they satisfy $u_i \cdot u_j \equiv 0$ as long as $j \neq i$, and by (c) they belong to $H^1(B)$ (recall that $\mathcal{T}$ is a conformal map). With direct computations one can check that

$$-\Delta u_i + \omega \boldsymbol{x}^\perp \cdot \nabla u_i = \mu u_i \quad \text{in } \omega_i := \{u_i > 0\}. \tag{17}$$

Analogously, using the definition of the coefficients l_i (see (16)), we have that

$$-\Delta\left(u_{i-1} - \frac{a_{(i-1)i}}{a_{i(i-1)}} u_i\right) + \omega \boldsymbol{x}^\perp \cdot \nabla\left(u_{i-1} - \frac{a_{(i-1)i}}{a_{i(i-1)}} u_i\right) = \mu\left(u_{i-1} - \frac{a_{(i-1)i}}{a_{i(i-1)}} u_i\right) \tag{18}$$

in the interior of $\bar{\omega}_{i-1} \cup \bar{\omega}_i$, $i = 1, \dots, K$ (in case $i = 1$ we keep the understanding $i - 1 = K$, and the validity of (18) follows by (15)). Notice that, when restricted to $\bar{\omega}_{i-1} \cup \bar{\omega}_i$, the function in (18) is a multiple of both $\hat{u}_{i-1}$ and $\hat{u}_i$.

We have to show the validity of the inequalities

$$\int_B \nabla u_i \cdot \nabla \varphi + [\omega \boldsymbol{x}^\perp \cdot \nabla u_i - \mu u_i]\varphi \le 0, \tag{19}$$

$$\int_B \nabla \hat{u}_i \cdot \nabla \varphi + [\omega \boldsymbol{x}^\perp \cdot \nabla \hat{u}_i - \mu \hat{u}_i]\varphi \ge 0 \tag{20}$$

for every Lipschitz, compactly supported, nonnegative φ.

First, let us consider any φ such that $\varphi \equiv 0$ in $B_\varepsilon(\mathbf{0})$. Then (19) follows by integration by parts, since

$$\int_B \nabla u_i \cdot \nabla \varphi + [\omega \boldsymbol{x}^\perp \cdot \nabla u_i - \mu u_i]\varphi = \int_{\omega_i \setminus B_\varepsilon} \nabla u_i \cdot \nabla \varphi + [\omega \boldsymbol{x}^\perp \cdot \nabla u_i - \mu u_i]\varphi = \int_{\partial \omega_i} \partial_\nu u_i \varphi \le 0,$$

where we used the regularity of $\partial\omega_i$ away from $\mathbf{0}$, the equation for u_i and the fact that $\partial_\nu u_i \le 0$ on $\partial\omega_i$. On the other hand, to prove (20), since $\varphi \equiv 0$ in $B_\varepsilon(\mathbf{0})$, we can use a partition of unity argument and assume that $\mathrm{supp}(\varphi)$ intersects at most two adjacent nodal regions. In case none of them is ω_i, then $\hat{u}_i = -c_1 u_j - c_2 u_{j+1}$, with $c_i > 0$, and (20) follows by applying (19) twice, with $i = j, j+1$; if $\mathrm{supp}(\varphi) \subset \overline{\omega}_{i-1} \cup \overline{\omega}_i \setminus B_\varepsilon$ then (18) yields

$$\int_B \nabla \hat{u}_i \cdot \nabla \varphi + [\omega \boldsymbol{x}^\perp \cdot \nabla \hat{u}_i - \mu u_i]\varphi = \int_{\overline{\omega}_{i-1} \cap \overline{\omega}_i \setminus B_\varepsilon} \nabla \hat{u}_i \cdot \nabla \varphi + [\omega \boldsymbol{x}^\perp \cdot \nabla \hat{u}_i - \mu \hat{u}_i]\varphi = 0,$$

and the same holds true if $\mathrm{supp}(\varphi) \subset \overline{\omega}_i \cup \overline{\omega}_{i+1} \setminus B_\varepsilon$.

Finally, let us consider any φ. We show how to prove (19); the proof of (20) is analogous. For any $\varepsilon > 0$ small, we define the function

$$\eta(\boldsymbol{x}) = \begin{cases} 0, & \boldsymbol{x} \in B_\varepsilon, \\ (|\boldsymbol{x}| - \varepsilon)/\varepsilon, & \boldsymbol{x} \in B_{2\varepsilon} \setminus B_\varepsilon, \\ 1, & \boldsymbol{x} \in B \setminus B_{2\varepsilon}. \end{cases}$$

Then $\varphi\eta = 0$ in B_ε, and by the previous part

$$\int_B (\nabla u_i \cdot \nabla \varphi)\eta + \int_B (\nabla u_i \cdot \nabla \eta)\varphi + \int_B [\omega \boldsymbol{x}^\perp \cdot \nabla u_i - \mu u_i]\eta\varphi \le 0.$$

Since φ is Lipschitz, we have

$$\left| \int_B (\nabla u_i \cdot \nabla \eta)\varphi \right| \le \frac{1}{\varepsilon} \int_{B_{2\varepsilon} \setminus B_\varepsilon} |\nabla u_i| \varphi \le \frac{1}{\varepsilon} \|u_i\|_{H^1(B_{2\varepsilon})} \|\varphi\|_{L^2(B_{2\varepsilon})} \le C \|u_i\|_{H^1(B_{2\varepsilon})} \|\varphi\|_{L^\infty}.$$

Thus we find the estimate

$$\int_B (\nabla u_i \cdot \nabla \varphi)\eta + \int_B [\omega \boldsymbol{x}^\perp \cdot \nabla u_i - \mu u_i]\eta\varphi \le C \|u_i\|_{H^1(B_{2\varepsilon})} \|\varphi\|_{L^\infty}.$$

Taking the limit as $\varepsilon \to 0$, since η converges monotonically to 1, we conclude that

$$\int_B \nabla u_i \cdot \nabla \varphi + [\omega \boldsymbol{x}^\perp \cdot \nabla u_i - \mu u_i]\varphi \le 0,$$

concluding the proof of the first assertion.

The second part follows by defining

$$v \circ \mathcal{T} = \sum_{i=1}^{K} \frac{(-1)^{i+1}}{l_i} u_i, \tag{21}$$

and then deriving v by a lifting argument. We refer to [Terracini et al. 2019, Section 2] for further details. □

3. Solutions in the half-plane

Let $\mu, \alpha, \omega \in \mathbb{R}$. Given the trace

$$\Phi\colon [0, 2\pi] \to \mathbb{R}, \quad \Phi(0) = \Phi(2\pi) = 0,$$

we look for solutions v of the following problem in the half-plane:

$$\begin{cases} -\Delta v + \omega e^{-2y} v_x = e^{-2y} \mu v, & x \in \mathbb{R}, \ \ y > 0, \\ v(x + 2\pi, y) = e^{2\pi\alpha} v(x, y), & x \in \mathbb{R}, \ \ y \geq 0, \\ v(x, 0) = \Phi(x), & 0 \leq x \leq 2\pi. \end{cases} \tag{22}$$

Notice that we are considering (11) together with condition (12) in the case $\sigma = e^{2\pi\alpha} > 0$ (recall definition (7) and the relation (15)). As we noticed, this involves an even number of nodal zones in the period. One can easily modify our arguments to deal with an odd one, i.e., with $\sigma < 0$, for instance with the change of variables $(x, y) \mapsto \left(\frac{1}{2}x, \frac{1}{2}y\right)$, $\sigma \mapsto \sigma^2$. In a completely equivalent way, one can work with 2π-periodicity and take $\alpha = \frac{1}{2\pi} \ln|\sigma| + \frac{i}{2} \in \mathbb{C}$.

To solve (22), we first transform it into a periodic problem, and then use separation of variables to write the solution in Fourier series. To this aim, we notice that v solves (22) if and only if

$$w(x, y) := e^{-\alpha x} v(x, y)$$

solves

$$\begin{cases} -\Delta w + (\omega e^{-2y} - 2\alpha) w_x + [(\alpha\omega - \mu) e^{-2y} - \alpha^2] w = 0, & x \in \mathbb{R}, \ \ y > 0, \\ w(x + 2\pi, y) = w(x, y), & x \in \mathbb{R}, \ \ y \geq 0, \\ w(x, 0) = e^{-\alpha x} \Phi(x), & 0 \leq x \leq 2\pi. \end{cases} \tag{23}$$

Of course, if $\alpha = 0$ then v and w coincide. Either way, with a little abuse of notation, we can extend Φ to $\mathbb{R}$ in such a way that $e^{-\alpha x} \Phi(x)$ is 2π-periodic. At least formally we can expand w in Fourier series and write

$$w(x, y) = \sum_{k \in \mathbb{Z}} W_k(y) e^{ikx}.$$

Plugging this expression into (23), we obtain that the coefficients $W_k : \overline{\mathbb{R}}^+ \to \mathbb{C}$, $k \in \mathbb{Z}$, must solve the ordinary differential equation

$$W_k''(y) = [(k - i\alpha)^2 + (\omega\alpha - \mu + i\omega k) e^{-2y}] W_k(y), \quad y > 0. \tag{24}$$

We can solve boundary value problems associated with (24) by using the Fredholm alternative and the Lax–Milgram theorem, settled in complex Hilbert spaces. We are looking for solutions of (23) that change sign as $y \to +\infty$. As we will see in Lemma 3.9, this requires that the term corresponding to $k = 0$ in the expansion should not be present. For this reason we consider $k \neq 0$ from now on.

Lemma 3.1. *For any $k \in \mathbb{Z} \setminus \{0\}$, $\alpha \in \mathbb{R}$, there exists a sequence $\{\lambda_n\}_{n\in\mathbb{N}} \subset \mathbb{C}$, with $|\lambda_n| \to +\infty$ as $n \to +\infty$, such that the problem*

$$\begin{cases} X_k''(y) = [(k - i\alpha)^2 + (\omega\alpha - \mu + i\omega k)e^{-2y}]X_k(y), & y > 0, \\ X_k(0) = 1, \quad X_k \in H^1(\mathbb{R}^+; \mathbb{C}), \end{cases} \tag{25}$$

admits a unique solution if and only if

$$\omega\alpha - \mu + i\omega k \notin \{\lambda_n\}_{n\in\mathbb{N}}, \tag{26}$$

while no solution exists in the complementary case.

Proof. We shall consider the case $k \geq 1$, as the case $k \leq -1$ follows by the same arguments, up to the change of sign

$$(\alpha, \omega, \mu, k) \mapsto (-\alpha, -\omega, \mu, -k).$$

In particular, one can verify that $X_{-k}(y) = \overline{X_k(y)}$ for any $k \in \mathbb{Z}$ and $y \geq 0$ (in case one of them exists). We proceed through several steps.

Step 1. *Weak formulation of the problem.* Letting $X_k = U + U_0$, where $U_0 := e^{-(k-i\alpha)y}$, we are led to find, if it exists, a function $U \in H_0^1(\mathbb{R}^+; \mathbb{C})$, solution of

$$-U'' + [(k - i\alpha)^2 + (\omega\alpha - \mu + i\omega k)e^{-2y}]U = -(\omega\alpha - \mu + i\omega k)e^{-2y}e^{-(k-i\alpha)y}, \quad y > 0.$$

We settle the problem in the space

$$H = H_0^1(\mathbb{R}^+; \mathbb{C}), \quad \|u\|_H^2 = \int_0^\infty |U'|^2 + |U|^2.$$

To proceed, we introduce the sesquilinear forms a_R, a_I as

$$a_R(U, V) = \int_0^\infty U'\overline{V}' + [(k^2 - \alpha^2) + (\omega\alpha - \mu)e^{-2y}]U\overline{V}, \quad a_I(U, V) = \int_0^\infty (-2\alpha k + \omega k e^{-2y})U\overline{V},$$

and the antilinear form l as

$$l(V) = -(\omega\alpha - \mu + i\omega k)\int_0^\infty e^{-2y}U_0\overline{V} = -(\omega\alpha - \mu + i\omega k)\int_0^\infty e^{-(k+2-i\alpha)y}\,\overline{V}. \tag{27}$$

In this way, we are reduced to solve the following variational problem: finding $U \in H$ such that

$$a(U, V) = a_R(U, V) + i a_I(U, V) = l(V) \quad \text{for all } V \in H. \tag{28}$$

Notice that both a and l are continuous: indeed, since $|e^{-2y}| \leq 1$ for $y \geq 0$, it is easy to see that

$$|a(U, V)| \leq (k^2 + \alpha^2 + \sqrt{(\omega\alpha - \mu)^2 + (\omega k)^2})\|u\|_H\|v\|_H.$$

Similarly, for l we obtain

$$|l(V)| \le |(\omega\alpha - \mu + i\omega k)| \int_0^\infty e^{-(k+2)y}|V| \le \frac{\sqrt{(\omega\alpha-\mu)^2+(\omega k)^2}}{\sqrt{2(k+2)}} \left(\int_0^\infty |V|^2 \right)^{1/2}.$$

For future purposes we notice that, for every $U \in H$, both $a_R(U,U)$ and $a_I(U,U)$ are real numbers: indeed, $a_R(U,U)$ and $a_I(U,U)$ are, respectively, the real and imaginary part of $a(U,U)$. We can exploit the Cauchy–Schwarz inequality (for real two-dimensional vectors) to find that

$$\begin{aligned} |a(U,U)| &= \sup_{K\in\mathbb{R}} \frac{a_R(U,U)+Ka_I(U,U)}{\sqrt{1+K^2}} \ge \frac{k}{\sqrt{\alpha^2+k^2}}\Big(a_R(U,U) - \frac{\alpha}{k}a_I(U,U)\Big) \\ &= \frac{k}{\sqrt{\alpha^2+k^2}} \int_0^\infty [|U'|^2 + (k^2+\alpha^2)|U|^2] - \frac{k\mu}{\sqrt{\alpha^2+k^2}} \int_0^\infty e^{-2y}|U|^2. \end{aligned} \tag{29}$$

In order to prove existence and uniqueness of a solution U, we shall make use of the classical Fredholm alternative theorem. In particular, we shall find that (28) admits a unique solution $U \in H_0^1(\mathbb{R}^+;\mathbb{C})$ if and only if 0 is not an eigenvalue of a (more precisely, and equivalently, 0 is not an eigenvalue of the conjugate transpose sesquilinear form $a^\dagger$).

Step 2. *A related eigenvalue problem.* To proceed, we introduce the (adjoint) eigenvalue problem: finding $\lambda \in \mathbb{C}$ and $V \in H \setminus \{0\}$ such that

$$\int_0^\infty [U'\overline{V}' + (k-i\alpha)^2 U\overline{V}] + \lambda \int_0^\infty e^{-2y} U\overline{V} = 0 \quad \text{for all } U \in H.$$

Defining the weighted space

$$L = \left\{ U \in L^1_{\text{loc}}(\mathbb{R}^+;\mathbb{C}) : \|U\|_L^2 = \int_0^\infty e^{-2y}|U|^2 < +\infty \right\},$$

we have that $H \subset L = L^* \subset H^*$ is a Hilbert triplet, with H compactly embedded in L; see Lemma A.1. Then standard spectral theory (see, e.g., [Kato 1966, Chapter 3, Theorem 6.26]) yields the existence of a sequence of eigenvalues $\{\lambda_n\}_{n\in\mathbb{N}} \subset \mathbb{C}$, with $|\lambda_n| \to +\infty$, and it is straightforward to show that $V \neq 0$ satisfies

$$a(U,V) = 0 \text{ for all } U \in H \quad \Longleftrightarrow \quad \begin{array}{c} \omega\alpha - \mu + i\omega k = \lambda_n \\ \text{and } V = V_n \text{ is an associated eigenfunction.} \end{array} \tag{30}$$

Notice that each λ_n is a simple eigenvalue by uniqueness of the Cauchy problem for ODEs.

Step 3. *Application of the Babuška–Lax–Milgram theorem.* To conclude the invertible case, we show that, if $\omega\alpha - \mu + i\omega k \neq \lambda_n$ for every n, then there exists a unique solution to (28). To this aim, we apply a generalization of the Lax–Milgram theorem due to Babuška [1971, Theorem 2.1] (with $H_1 = H_2 = H$). After the previous steps, in order to apply such a result to (28), we only need to show that, if $\omega\alpha - \mu + i\omega k \neq \lambda_n$ for every n, then the following inf-sup conditions hold:

$$\inf_{\|V\|_H=1} \sup_{\|U\|_H=1} |a(U,V)| \ge C_2 > 0, \qquad \inf_{\|U\|_H=1} \sup_{\|V\|_H=1} |a(U,V)| \ge C_3 > 0$$

for suitable constants $C_2,\ C_3$. We prove the first inequality; the second is proved analogously. Assume by contradiction that the sequence $\{V_n\}_n$ satisfies

$$\|V_n\|_H = 1, \qquad |a(U,V_n)| \le \frac{1}{n}\|U\|_H \quad \text{for all } U \in H.$$

In particular, as $n \to +\infty,\ a(V_n, V_n) \to 0$. Moreover, up to subsequences, V_n converges to V_∞, both weakly in H and strongly in L (by compact embedding). Thus $a(U, V_\infty) = 0$ for every $U \in H$. Since $\omega\alpha - \mu + i\omega k \neq \lambda_n$ for every n and recalling (30), we deduce that $V_\infty \equiv 0$. Since $k^2 \ge 1$, (29) yields

$$o(1) = |a(V_n, V_n)| \ge \frac{k}{\sqrt{\alpha^2+k^2}}\|V_n\|_H^2 - \frac{k\mu}{\sqrt{\alpha^2+k^2}}\|V_n\|_L^2 = \frac{k}{\sqrt{\alpha^2+k^2}} + o(1)$$

as $n \to \infty$, a contradiction.

Step 4. *Nonexistence in the resonant case.* Finally, assume that $\omega\alpha - \mu + i\omega k = \lambda_n$ for some n, and let $V_n \not\equiv 0$ be an associated eigenfunction of the adjoint problem

$$a(U, V_n) = \int_0^\infty [U'\overline{V}'_n + (k-i\alpha)^2 U \overline{V}_n] + \lambda_n \int_0^\infty e^{-2y} U \overline{V}_n = 0 \quad \text{for all } U \in H.$$

This forces

$$-\overline{V}''_n + (k-i\alpha)^2 \overline{V}_n + \lambda_n e^{-2y} \overline{V}_n = 0 \quad \text{on } (0,\infty); \tag{31}$$

in particular, $V_n \in H^2(0,+\infty)$, and thus $V'_n(y) \to 0$ as $y \to +\infty$. Moreover, by uniqueness of the Cauchy problem, $V'_n(0) \neq 0$.

In the case we are considering, (28) can be rewritten as

$$a(U, V) = (-\lambda_n U_0, V)_L \quad \text{for all } V \in H,$$

where $U_0 = e^{-(k-i\alpha)y}$. By Fredholm's alternative, in this case (28) is solvable if and only if the compatibility condition

$$(-\lambda_n U_0, V_n)_L = 0$$

holds true. On the other hand, using (31), we have

$$(-\lambda_n U_0, V_n)_L = -\lambda_n \int_0^\infty e^{-2y} U_0 \overline{V}_n = U(0)\overline{V}'_n(0) + \int_0^\infty [U_0'\overline{V}'_n + (k-i\alpha)^2 U_0 \overline{V}_n] = \overline{V}'_n(0) \neq 0,$$

which concludes the proof. ☐

The resonance set in the previous lemma can be characterized in terms of the zero set of the following function Θ_ν, depending on the complex parameter ν:

$$\Theta_\nu(z) = \sum_{n=0}^\infty \frac{1}{n!\,\Gamma(n+1+\nu)} \left(\frac{z}{4}\right)^n. \tag{32}$$

Notice that, for any $\nu \in \mathbb{C},\ \Theta_\nu$ is analytic on $\mathbb{C}$ (recall that Γ has no zeros, but only simple poles at each nonpositive integer $-k$: in such a case, we understand $1/\Gamma(-k) = 0$). As a matter of fact, Θ_ν is related to I_ν, the modified Bessel function of the first kind, with parameter $\nu \in \mathbb{C}$, by the formula

$$I_\nu(z) = \left(\frac{z}{2}\right)^\nu \Theta_\nu(z^2) \tag{33}$$

(in turn, $I_\nu(z) = e^{-i\nu\pi/2} J_\nu(iz)$, where J_ν is the usual Bessel function of the first kind). Notice that, in the case $\nu \notin \mathbb{Z}$, I_ν is a multivalued function because of the complex exponentiation z^ν. Nonetheless, the zero set of (any determination of) I_ν coincides with the complex square root of the zero set of Θ_ν, with the exception of 0.

Lemma 3.2. *For any $k \in \mathbb{Z} \setminus \{0\}$, $\alpha \in \mathbb{R}$, let $\{\lambda_n\}_{n\in\mathbb{N}} \subset \mathbb{C}$ denote the sequence defined in Lemma 3.1. Then*

$$\{\lambda_n\}_{n\in\mathbb{N}} = \{z \in \mathbb{C} \setminus \{0\} : \Theta_{\mathrm{sign}(k)(k-i\alpha)}(z) = 0\},$$

where Θ_ν is defined in (32) *for every $\nu \in \mathbb{C}$.*

Moreover, whenever $\lambda := \omega\alpha - \mu + i\omega k \notin \{\lambda_n\}_{n\in\mathbb{N}}$, the unique solution of (25) *is*

$$X_k(y) = \frac{\Theta_\nu(\lambda e^{-2y})}{\Theta_\nu(\lambda)} e^{-\nu y}$$

($X_k(y) = e^{-\nu y}$ in the case $\lambda = 0$), where $\nu = \mathrm{sign}(k)(k - i\alpha)$ whenever $k \neq 0$.

Equivalently, we could write

$$X_k(y) = \frac{I_\nu(\sqrt{\lambda} e^{-y})}{I_\nu(\sqrt{\lambda})},$$

and such an identity is not ambiguous as long as we choose the same determinations both in the numerator and in the denominator.

Proof. Again, we treat the case $k \geq 1$; the case $k \leq -1$ follows with minor changes. With the above notation,

$$\nu = k - i\alpha, \quad \lambda = \omega\alpha - \mu + i\omega k,$$

the second-order linear ODE in (25) is written as

$$x''(y) = [\nu^2 + \lambda e^{-2y}] x(y). \tag{34}$$

We assume $\lambda \neq 0$; the complementary case is trivial. Let us consider the functions $x_{\pm\nu}(y)$ defined as

$$x_{\pm\nu}(y) = \Theta_{\pm\nu}(\lambda e^{-2y}) e^{\mp\nu y} = \sum_{n\geq 0} c_{\pm\nu,n} e^{(-2n\mp\nu)y}, \quad \text{where } c_{\pm\nu,n} = \frac{1}{n!\,\Gamma(n+1\pm\nu)}\left(\frac{\lambda}{4}\right)^n$$

(again, we understand $c_{\pm\nu,n} = 0$ whenever $-(n+1\pm\nu) \in \mathbb{N}$). We notice that $4n(n\pm\nu)c_{\pm\nu,n} = \lambda c_{\pm\nu,n-1}$. Then

$$\begin{aligned} x''_{\pm\nu}(y) &= \sum_{n\geq 0}(-2n \mp \nu)^2 c_{\pm\nu,n} e^{(-2n\mp\nu)y} = \sum_{n\geq 0} \nu^2 c_{\pm\nu,n} e^{(-2n\mp\nu)y} + \sum_{n\geq 0} 4n(n\pm\nu) c_{\pm\nu,n} e^{(-2n\mp\nu)y} \\ &= \nu^2 x_{\pm\nu}(y) + \lambda \sum_{n\geq 1} c_{\pm\nu,n-1} e^{(-2n\mp\nu)y} = [\nu^2 + \lambda e^{-2y}] x_{\pm\nu}(y); \end{aligned}$$

that is, both $x_{\pm\nu}$ solve the second-order linear ODE (34).

Let us first assume that $\alpha \neq 0$. Then $-(n+1\pm\nu) \notin \mathbb{N}$ for every n, and we obtain that $\Theta_{\pm\nu}(\lambda e^{-2y}) = 1/\Gamma(1\pm\nu) + o(1)$; that is,

$$x_{\pm\nu}(y) = \frac{1}{\Gamma(1\pm\nu)} e^{\mp\nu y} + o(e^{\mp\nu y}) \quad \text{as } y \to +\infty.$$

Then $x_{\pm\nu}$ are linearly independent, and any solution of (34) is of the form

$$x(y) = C_+ x_\nu(y) + C_- x_{-\nu}(y), \quad C_\pm \in \mathbb{C}.$$

Since $\nu = k - i\alpha$ and $k \geq 1$, we have that $x \in H^1(0,+\infty)$ if and only if $C_- = 0$. As a consequence, (25) is (uniquely) solvable if and only if $x_\nu(0) = \Theta_\nu(\lambda) \neq 0$, and the lemma follows.

On the other hand, let $\alpha = 0$ (and $\lambda \neq 0$). In this case $\nu = k \geq 1$, and

$$c_{-k,n+k} = \frac{1}{(n+k)!\,n!}\Big(\frac{\lambda}{4}\Big)^{n+k} = \Big(\frac{\lambda}{4}\Big)^k c_{k,n}$$

for every $n \geq 0$, therefore the functions $x_{\pm k}$ are no longer linearly independent. By differentiating (34) with respect to ν, one can easily see that a second independent solution of (34) can be obtained as

$$\tilde{x}_k = \Big[\Big(\frac{\lambda}{4}\Big)^k \frac{\partial x_\nu}{\partial \nu} - \frac{\partial x_{-\nu}}{\partial \nu}\Big]_{\nu=k},$$

mimicking the procedure that leads to the (modified) Bessel functions of the second kind. Since $\Gamma(n+1-k)$ has a simple pole at $n = 0$, we have

$$\lim_{\nu\to k} \frac{\partial c_{-\nu,0}}{\partial \nu} = (-1)^k (k-1)! \qquad \text{and} \qquad \tilde{x}_k(y) = (-1)^k (k-1)!\, e^{ky} + o(e^{ky}) \quad \text{as } y \to +\infty$$

(see [Erdélyi et al. 1953, Section 7.2.5, p. 9] for more details). Thus also in this case $\tilde{x}_k \notin H^1(0,+\infty)$, and the lemma follows. ☐

Corollary 3.3. *Let X_k denote the solution of* (25). *Then, for some $C \neq 0$,*

$$X_k(y) = Ce^{-\operatorname{sign}(k)(k-i\alpha)y} + O(e^{-(|k|+2)y}) \quad \text{as } y \to +\infty.$$

Remark 3.4. As a byproduct of the proof of Lemma 3.2, we have that the eigenvalues λ_n are all simple in $H_0^1(\mathbb{R}^+;\mathbb{C})$. Indeed, the general solution of the corresponding eigenequation is a two-dimensional vector space of complex-valued functions, but only a one-dimensional subspace consists of H^1 functions of the form

$$C\,\Theta_{\operatorname{sign}(k)(k-i\alpha)}(\lambda_n e^{-2y})e^{-\operatorname{sign}(k)(k-i\alpha)y}, \quad C \in \mathbb{C}.$$

In view of writing w as a series in terms of the solutions X_k, we need to estimate the asymptotic behaviors as $k \to \infty$ of their L^2 and H^1 norms.

Lemma 3.5. *Let α, μ, ω be fixed in such a way that* (26) *holds for every $k \neq 0$. Then X_k satisfies*

$$\Big(\int_0^\infty |X_k|^2\Big)^{1/2} \leq \frac{C}{\sqrt{|k|}}, \quad \Big(\int_0^\infty |X_k'|^2\Big)^{1/2} \leq C\sqrt{|k|} \quad \text{and} \quad \|X_k\|_{L^\infty(0,+\infty)} \leq \sqrt{2}C, \qquad (35)$$

where C depends only on α, μ, ω.

Proof. As usual, for concreteness, we assume $k \geq 1$. As in the proof of Lemma 3.1 we write $X_k = U + e^{-(k-i\alpha)y}$. In order to prove (35), we distinguish between two cases, corresponding to the instances k small and k large. Indeed, for any fixed $\bar{k}$, which we will choose later in terms of $\alpha,\ \mu,\ \omega$, the estimate (35) is true for $k < \bar{k}$ and a suitable constant C. Next, for $k \geq \bar{k}$, we estimate the norms of U using the identity

$$|a(U,U)| = |l(U)|.$$

Recalling (27), we have

$$|l(U)| \leq |\omega\alpha - \mu + i\omega k| \int_0^\infty |e^{-(k+2)y}||U| \leq \frac{\sqrt{(\omega\alpha-\mu)^2 + (\omega k)^2}}{\sqrt{2(k+2)}} \left(\int_0^\infty |U|^2\right)^{1/2}.$$

Using (29), we obtain

$$\frac{k}{\sqrt{k^2+\alpha^2}} \int_0^\infty [|U'|^2 + (k^2+\alpha^2-\mu^+)|U|^2] \leq \frac{\sqrt{(\omega\alpha-\mu)^2 + (\omega k)^2}}{\sqrt{2(k+2)}} \left(\int_0^\infty |U|^2\right)^{1/2}.$$

Then

$$\left(\int_0^\infty |U|^2\right)^{1/2} \leq \frac{\sqrt{k^2+\alpha^2}}{k(k^2+\alpha^2-\mu^+)} \cdot \frac{\sqrt{(\omega\alpha-\mu)^2 + (\omega k)^2}}{\sqrt{2(k+2)}} \leq \frac{|\omega|}{k^{3/2}},$$

whence

$$\left(\int_0^\infty |U'|^2\right)^{1/2} \leq \left[\frac{\sqrt{k^2+\alpha^2}}{k} \frac{\sqrt{(\omega\alpha-\mu)^2 + (\omega k)^2}}{\sqrt{2(k+2)}} \left(\int_0^\infty |U|^2\right)\right]^{1/2} \leq \frac{|\omega|^{3/2}}{k^{5/4}}$$

for $k \geq \bar{k}$ sufficiently large (depending on $\omega,\ \mu,\ \alpha$).

Coming back to $X_k = U + e^{-(k-i\alpha)y}$, we finally obtain

$$\left(\int_0^\infty |X_k|^2\right)^{1/2} \leq \left(\int_0^\infty |U|^2\right)^{1/2} + \left(\int_0^\infty e^{-2ky}\right)^{1/2} \leq \frac{|\omega|}{k^{3/2}} + \frac{1}{\sqrt{2k}} \leq \frac{1}{\sqrt{k}}$$

and

$$\left(\int_0^\infty |X_k'|^2\right)^{1/2} \leq \left(\int_0^\infty |U'|^2\right)^{1/2} + \left(\int_0^\infty |k-i\alpha|^2 e^{-2ky}\right)^{1/2} \leq \frac{|\omega|^{3/2}}{k^{5/4}} + \sqrt{\frac{k^2+\alpha^2}{2k}} \leq \sqrt{k}$$

for k sufficiently large (depending on $\omega,\ \mu,\ \alpha$), concluding the H^1 estimates. Finally, by Corollary 3.3, for any $y > 0$,

$$X_k(y)^2 = -\int_y^\infty 2X_k(t)X_k'(t)\,dt \leq 2\left(\int_0^\infty |X_k|^2\right)^{1/2}\left(\int_0^\infty |X_k'|^2\right)^{1/2} \leq 2C^2,$$

and the last estimate follows. □

Next we provide explicit sufficient conditions for the validity of condition (26).

Lemma 3.6. *A sufficient condition for* (26) *to hold true is*

$$\sup\left\{(j_{\tau,1})^2 - \frac{\omega}{2\alpha}\tau^2 : \tau > 0\right\} > \mu - \frac{\omega}{2\alpha}(k^2+\alpha^2), \tag{36}$$

where $j_{\tau,1}$ denotes the first (*positive*) *zero of the standard Bessel function of the first kind of order $\tau > 0$.*

This is the case, for instance, if

$$\text{either } \mu < (j_{0,1} + \sqrt{k^2+\alpha^2})^2, \quad \text{or } \frac{\omega}{\alpha} < 2. \tag{37}$$

In particular, for any choice of α, ω, μ, *if* $|k|$ *is sufficiently large then* (26) *holds.*

Proof. Using the notation introduced in the proof of Lemma 3.1, we are going to show that, under the present assumptions, the sesquilinear form a is coercive. By the first estimate in (29), this follows once we find $K \in \mathbb{R}$ such that the quadratic form (with real coefficients)

$$a_R(U,U) + a_I(U,U)K = \int_0^\infty |U'|^2 + (k^2 - \alpha^2 - 2\alpha k K)|U|^2 + ((\omega\alpha - \mu) + \omega k K)e^{-2y}|U|^2$$

is strictly positive. To this aim, it is not difficult to check that we have to ask that $k^2 - \alpha^2 - 2\alpha k K > 0$. For this reason, it is convenient to introduce the parameters $\tau > 0$ and $b = b(\tau)$ such that

$$K = \frac{k^2 - \alpha^2 - \tau^2}{2\alpha k}, \quad b = -((\omega\alpha - \mu) + \omega k K) = \mu + \frac{\omega}{2\alpha}(\tau^2 - (k^2 + \alpha^2)).$$

In this way, we are reduced to finding $\tau > 0$ such that the quadratic form

$$U \mapsto \int_0^\infty |U'|^2 + (\tau^2 - b e^{-2y})|U|^2$$

is strictly positive. This quadratic form can be studied by standard arguments; we postpone the details to Lemma A.2 in the Appendix. We obtain that it is coercive if and only if

$$b = \mu + \frac{\omega}{2\alpha}(\tau^2 - (k^2 + \alpha^2)) < (j_{\tau,1})^2,$$

and (36) follows. In order to make this condition more explicit, we exploit the fact that

$$j_{\tau,1} \geq j_{0,1} + \tau \quad \text{for every } \tau \geq 0$$

(see [McCann and Love 1982]). Therefore, a stronger condition than (36) is

$$\mu + \frac{\omega}{2\alpha}(\tau^2 - (k^2 + \alpha^2)) < (j_{0,1} + \tau)^2 \quad \text{for some } \tau > 0.$$

The conditions in (37) follow by taking either $\tau^2 = k^2 + \alpha^2$, or $\tau \to +\infty$, respectively. □

Corollary 3.7. *Let* α, μ, ω *be fixed, with*

$$\mu < (j_{0,1} + 1)^2. \tag{38}$$

Then (26) *holds true for every* $k \neq 0$.

We are ready to state and prove the main result of this section. For any $\Phi \in \mathrm{Lip}([0, 2\pi])$, we write the Fourier coefficients of $e^{-\alpha x}\Phi(x)$ as

$$\phi_k = \frac{1}{2\pi} \int_0^{2\pi} e^{-(ik+\alpha)x} \Phi(x)\, dx, \quad k \in \mathbb{Z}.$$

Proposition 3.8. *Let* α, μ, ω *be fixed and* $\Phi \in \mathrm{Lip}([0, 2\pi])$. *Let us assume that*

- $\mu < (j_{0,1} + 1)^2 \simeq 3.4^2$,
- $\Phi(0) = \Phi(2\pi) = 0$ *and* $\phi_0 = \int_0^{2\pi} e^{-\alpha x}\Phi(x)\, dx = 0$.

Then the functions

$$w(x,y)=\sum_{k\in\mathbb{Z}\setminus\{0\}}\phi_k X_k(y)e^{ikx}\quad\textit{and}\quad v(x,y)=e^{\alpha x}w(x,y),\tag{39}$$

where the functions X_k are as in Lemmas 3.1 and 3.2, satisfy:

(1) *$w\in H^1(\{(x,y)\in\mathbb{R}\times\mathbb{R}^+ : a<x+ly<b\})$ for any $l\in\mathbb{R}$ and $a<b$, and it solves* (23).

(2) *$v\in H^1(\{(x,y)\in\mathbb{R}\times\mathbb{R}^+ : a<x+ly<b\})$ for any l such that $l\alpha\geq 0$ and for every $a<b$, and it solves* (22).

(3) *Both v and w are analytic in $\mathbb{R}\times\mathbb{R}^+$ and $C^{0,\alpha}$ up to $y=0$ for every $\alpha<1$.*

Proof of Proposition 3.8. In view of Lemma 3.1, we have that all the terms in the series in (39) are smooth and satisfy the differential equations in (23). We now show that the series converges in H^1, ensuring that w also satisfies the corresponding equation. We start by observing that, by construction, the family $\{(x,y)\mapsto X_k(y)e^{ikx}\}_{k\in\mathbb{Z}\setminus\{0\}}$ is orthogonal in $H^1(S)$, $S=(0,2\pi)\times\mathbb{R}^+$, and, in particular, for any $k,h\in\mathbb{Z}\setminus\{0\}$ and $k\neq h$, we have

$$\int_S X_k(y)e^{ikx}\cdot\overline{(X_h(y)e^{ihx})}=0,\qquad \int_S X_k'(y)e^{ikx}\cdot\overline{(X_h'(y)e^{ihx})}=0$$

and, recalling (35),

$$\int_S |X_k(y)e^{ikx}|^2\leq\frac{C}{|k|},\qquad \int_S |X_k'(y)e^{ikx}|^2\leq C|k|,\qquad \int_S |X_k(y)(e^{ikx})'|^2\leq C|k|.$$

On the other hand, since $x\mapsto e^{-\alpha x}\Phi(x)$ can be extended to a 2π-periodic Lipschitz continuous function, it is an H^1-function on $\mathbb{S}^1$, and its Fourier coefficients ϕ_k satisfy

$$\sum_{k\in\mathbb{Z}}k^2|\phi_k|^2<+\infty$$

(recall that $\phi_0=0$). Combining the above inequalities, we infer

$$\Bigl\|\sum_{k\neq 0}W_k(y)e^{ikx}\Bigr\|^2_{H^1(S)}\leq C\sum_{k\geq 1}(|\phi_k|^2+|\phi_{-k}|^2)\Bigl(\frac{1}{|k|}+|k|\Bigr)<+\infty.$$

We conclude that the series defining w converges in $H^1(S)$, making w a weak solution of (23). Since w is periodic in the x-direction, we deduce that it belongs to $H^1((a,b)\times\mathbb{R}^+)$ for every $a<b$. Exploiting once again the periodicity in x of w, we can readily infer that $w\in H^1(\{(x,y)\in\mathbb{R}\times\mathbb{R}^+ : a<x+ly<b\})$ for any $l\in\mathbb{R}$ and $a<b$. Moreover, by elliptic regularity, w is analytic in $\mathbb{R}\times\mathbb{R}^+$ and Hölder continuous up to the boundary. Analogous conclusions for the function v can be drawn from the fact that $v(x,y)=e^{\alpha x}w(x,y)$, the only difference being that we need to exploit the assumption $l\alpha\geq 0$ in order to estimate the exponential factor. □

We conclude this section by showing that the Fourier expansions of the functions w and v can be exploited to give a description of their nodal sets for y large.

Lemma 3.9. *We consider again the assumptions of Proposition 3.8. Let $n \geq 1$ be the largest integer such that*

$$\phi_k = 0 \quad \text{for all } |k| < n.$$

Then there exists $y^* > 0$ and $2n$ disjoint simple curves $\Gamma_1, \dots, \Gamma_{2n}$ such that

$$\{(x, y) \in \mathbb{R} \times (y^*, +\infty) : w(x, y) = 0 (= v(x, y))\} = \bigcup_{\substack{j=1,\dots,2n \\ h \in \mathbb{Z}}} \Gamma_j + (2\pi h, 0). \tag{40}$$

The curves Γ_j are asymptotic to evenly spaced parallel lines: there exists $\beta \in \mathbb{R}$ such that

$$(x, y) \in \Gamma_j \iff \alpha y + nx = \beta + \pi j + o_y(1) \qquad \text{as } y \to +\infty.$$

Proof. By Lemma 3.5, we have that

$$\sup_{(x,y) \in \mathbb{R} \times \mathbb{R}^+} |w(x, y)| \leq \sup_{y > 0} \sum_{k \geq n} |\phi_k||X_k(y)| + |\phi_{-k}||X_{-k}(y)| \leq C \sum_{k \geq n} (|\phi_k| + |\phi_{-k}|) < +\infty,$$

which implies that the series converges also uniformly in $\mathbb{R} \times \mathbb{R}^+$. Moreover, we can extract the first term of the series and see that

$$|w(x, y) - \phi_n X_n(y) e^{inx} - \phi_{-n} X_{-n}(y) e^{-inx}| \leq C \sum_{k \geq n+1} (|\phi_k| + |\phi_{-k}|) e^{-ky} \leq C e^{-(n+1)y}$$

(see Corollary 3.3). This, in turn, implies that

$$w(x, y) = \phi_n X_n(y) e^{inx} + \phi_{-n} X_{-n}(y) e^{-inx} + O(e^{-(n+1)y}) \tag{41}$$

uniformly in $x \in \mathbb{R}$.

We claim that the nodal lines of the functions w (and of v) align asymptotically with those of the function

$$\begin{aligned}
(x, y) \mapsto A_n(x, y) &= \phi_n X_n(y) e^{inx} + \phi_{-n} X_{-n}(y) e^{-inx} \\
&= \phi_n C_n e^{-(n - i\alpha)y + inx} + \phi_{-n} C_{-n} e^{(-n - i\alpha)y - inx} + O(e^{-(n+2)y}) \\
&= e^{-ny} (a_n \cos(\alpha y + nx) + b_n \sin(\alpha y + nx) + O(e^{-2y})) \\
&= e^{-ny} (\sqrt{a_n^2 + b_n^2} \sin(\alpha y + nx - \beta) + O(e^{-2y})),
\end{aligned}$$

where the coefficients a_n, b_n and β are real numbers, $a_n^2 + b_n^2 \neq 0$ by assumption, and $\sin \beta = -a_n / \sqrt{a_n^2 + b_n^2}$. Indeed, recalling (41), we have that, as $y \to +\infty$,

$$e^{ny} w(x, y) = \sqrt{a_n^2 + b_n^2} \sin(\alpha y + nx - \beta) + O(e^{-y}).$$

Analogously, one can show that also the series of the derivatives converges uniformly in $x \in \mathbb{R}$ and that, as $y \to +\infty$,

$$e^{ny} w_x(x, y) = n \sqrt{a_n^2 + b_n^2} \cos(\alpha y + nx - \beta) + O(e^{-y}).$$

By the implicit function theorem, there exists $y^* > 0$ large enough that the nodal set of the function w in $\mathbb{R} \times (y^*, +\infty)$ is a countable union of graphs with respect to the y variable, each one asymptotic to

$$\alpha y + nx = \beta + h\pi \quad \text{for some } h \in \mathbb{Z}.$$

We choose $\Gamma_j,\ j = 1, \dots, 2n$, as $2n$ consecutive curves in this family of graphs by taking $h = j$. □

Remark 3.10. If the number of nodal zones for y small is different from $2n$, then the nodal lines of v must intersect. As a consequence, condition (b) in Section 2 fails for such a v, which cannot correspond to any element of $\mathcal{S}_{\mathrm{rot}}$ via Proposition 2.1.

4. Nodal sets in the half-plane

In this section, we study in detail the nodal structure of the function v constructed in Proposition 3.8. For this purpose, we let

$$\mathcal{N} = \{(x, y) \in \mathbb{R} \times \mathbb{R}_+ : v(x, y) = 0\}$$

be the nodal set of v, and we call a *nodal component* of v any connected component of $\mathbb{R} \times \mathbb{R}^+ \setminus \mathcal{N}$.

We state the main result of this section. Its assumptions should be compared to those of Proposition 3.8, in particular, we point out that they imply the existence of a unique solution v of (22). We recall that, for $\Phi \in \mathrm{Lip}([0, 2\pi])$, we write the Fourier coefficients of $e^{-\alpha x}\Phi(x)$ as

$$\phi_k = \frac{1}{2\pi} \int_0^{2\pi} e^{-(ik+\alpha)x} \Phi(x)\, dx, \quad k \in \mathbb{Z}.$$

Proposition 4.1. *Let $\alpha,\ \mu,\ \omega$ be fixed real numbers, $\Phi \in \mathrm{Lip}([0, 2\pi])$ and $n \geq 1$ be a given integer. Let us assume that*

- *the function Φ changes sign $2n$ times in $[0, 2\pi]$, more precisely, there exist*

$$x_1 = 0 < x_2 < \cdots < x_{2n+1} = 2\pi$$

such that

$$\{x \in (0, 2\pi) : \Phi(x) > 0\} = \bigcup_{k=0}^{n-1} (x_{2k+1}, x_{2k+2}) \quad \textit{and} \quad \{x \in (0, 2\pi) : \Phi(x) < 0\} = \bigcup_{k=0}^{n-1} (x_{2k+2}, x_{2k+3});$$

- *the coefficients of the equation satisfy $\mu < \pi^2$;*
- *we have the compatibility condition*

$$\sup\{|k| : \phi_k = 0\} = n - 1 \geq 0. \tag{42}$$

Moreover, let v denote the solution of (23), *whose existence is guaranteed by Proposition 3.8.*

Then there exist $2n$ connected, open sets $S_1, \dots, S_{2n} \subset \mathbb{R} \times \mathbb{R}^+$ such that

- *extending the definition of S_k, by periodicity, as $S_{k+2n} = S_k + (2\pi, 0),\ k \in \mathbb{Z}$, we have*

$$S_k \cap S_h = \varnothing \quad \textit{for every } k \neq h \qquad \textit{and} \qquad \overline{S}_k \cap \overline{S}_h \neq \varnothing \quad \Longleftrightarrow \quad k - h = -1, 0, 1;$$

- *any nodal component of* v *is one of the* S_k:

$$\mathbb{R}\times\mathbb{R}^+ \setminus \mathcal{N} = \bigcup_{k\in\mathbb{Z}} S_k;$$

- *each of them touches the* x*-axis in a single (connected) interval*:

$$\overline{S}_k \cap \{(x,0)\} = [x_k, x_{k+1}] \quad \text{for any } k = 1,\dots,2n;$$

- *they are asymptotic to a family of evenly spaced strips: there exists* $\beta\in\mathbb{R}$ *such that*

$$S_k \subset \{(x,y) : \beta + \pi k + o_y(1) < \alpha y + nx < \beta + \pi(k+1) + o_y(1)\} \quad \text{as } y\to+\infty.$$

The remaining part of this section is devoted to the proof of Proposition 4.1. We shall prove it in a series of intermediate steps. First we briefly investigate the local structure of the nodal set $\mathcal{N}$.

Lemma 4.2. *Under the above notation*,

- $\mathcal{C} = \{(x,y)\in\mathbb{R}\times\mathbb{R}_+ : v(x,y)=0, \nabla v(x,y)=0\}$ *is discrete in* $\mathbb{R}\times\mathbb{R}^+$;
- $\mathcal{N}\setminus\mathcal{C}$ *is the union of countably many analytic curves*;
- *If* $\Phi(\bar{x})\neq 0$ *and* $l\in\mathbb{R}$, *then the set*

$$\mathcal{N}\cap\{(x,y) : x+ly=\bar{x}\}$$

is discrete, and it does not accumulate at $\{y=0\}$.

We point out that, for the moment, it may still be that $\mathcal{C}$ accumulates at some point of the discrete set $\{(x,0) : \Phi(x)=0\}$.

Proof. We recall that v satisfies (22), and v is analytic in $\mathbb{R}\times\mathbb{R}^+$ and continuous up to the boundary $\{(x,y) : y=0\}$ (see Proposition 3.8). By well-known results of Hartman and Wintner [1953], the set $\mathcal{C}$ is discrete in $\mathbb{R}\times\mathbb{R}^+$.

As a consequence, by the analytic implicit function theorem, $\mathcal{N}\setminus\mathcal{C}$ is the disjoint union of countably many analytic curves which are either unbounded, accumulate at some point of $\{(x,0) : \Phi(x)=0\}$, or meet each other at points of $\mathcal{C}$.

Finally, let $\varphi : [0,+\infty)\to\mathbb{R}$ be defined as

$$\varphi(y) = v(\bar{x}-ly, y).$$

Then φ is real analytic for $y>0$, and continuous up to $y=0$ and $\varphi(0)\neq 0$. We deduce that its zero set is discrete. Since

$$\mathcal{N}\cap\{(x,y) : x+ly=\bar{x}\} \equiv \{(\bar{x}-ly, y) : \varphi(y)=0\},$$

the lemma follows. □

Let A be any nodal component of v. In the following, for any $h\in\mathbb{Z}$, we write

$$A_h = A - (2h\pi, 0).$$

Since v is 2π-periodic in x, A_h is itself a nodal component of v. As a consequence, either A and A_h coincide, or they are disjoint. We prove that this property is independent of $h\neq 0$.

Lemma 4.3. *Let A be any nodal component of v. Then*

- *either $A \equiv A_h$ for some $h \in \mathbb{Z}$, in which case $A \equiv A_k$ for every $k \in \mathbb{Z}$,*
- *or $A \cap A_h = \varnothing$ for some $h \in \mathbb{Z}$, in which case $A \cap A_k = \varnothing$ for every $k \neq 0$, and*

$$\sup_{y>0} |\{x : (x, y) \in A\}| \leq 2\pi,$$

where $|\cdot|$ denotes the one-dimensional Lebesgue measure.

Proof. We start by examining the first alternative. Let $(\bar{x}, \bar{y}) \in A \equiv A_h$, with $h \geq 1$, so that we also have $(\bar{x} + 2h\pi, \bar{y}) \in A$. By connectedness, there exists a curve $\gamma \subset A$ joining $(\bar{x}, \bar{y})$ and $(\bar{x} + 2h\pi, \bar{y})$. Since $2h\pi/(2\pi) = h \in \mathbb{N}$, by the universal chord theorem (see, e.g., [Oxtoby 1972]), there exists $(x_1, y_1), (x_2, y_2) \in \gamma$ such that $(x_2, y_2) = (x_1, y_1) + (2\pi, 0)$. Thus $A \cap A_1 \ni (x_2, y_2)$, which implies $A \equiv A_k$ for every $k \in \mathbb{Z}$.

Conversely, let us assume that $A \cap A_k = \varnothing$ for every $k \neq 0$. Then, for every $y > 0$,

$$\{x : (x, y) \in A\} = \bigcup_{k\in\mathbb{Z}} \{x \in [2k\pi, 2(k+1)\pi) : (x, y) \in A\} = \bigcup_{k\in\mathbb{Z}} \{x \in [0, 2\pi) : (x, y) \in A_k\},$$

and such a union is disjoint by assumption. We deduce that $|\{x : (x, y) \in A\}| \leq |[0, 2\pi)|$. □

To proceed, we need the following result, which is a consequence of a Poincaré-type inequality (see Lemma A.3).

Lemma 4.4. *Let A be any nodal component of v and assume that ($\mu < \pi^2$ and)*

$$\sup_{y>0} |\{x : (x, y) \in A\}| \leq 2\pi.$$

Then $v|_A \notin H_0^1(A)$.

Proof. By contradiction, let A be any nodal component of v and assume that $v|_A \in H_0^1(A)$ and

$$\sup_{y>0} |\{x : (x, y) \in A\}| \leq 2\pi.$$

We will show that this necessarily implies $\mu \geq \pi^2$.

By assumption, the function $v \in H^1(A)$ satisfies

$$\begin{cases} -\Delta v + \omega e^{-2y} v_x = e^{-2y} \mu v & \text{in } A, \\ v = 0 & \text{on } \partial A. \end{cases}$$

Multiplying by v and integrating by parts over A yields the identity

$$\int_A |\nabla v|^2 = \mu \int_A e^{-2y} v^2;$$

indeed,

$$\frac{\omega}{2} \int_A e^{-2y} (v^2)_x = 0$$

for every $v \in H_0^1(A)$ by density of the test functions.

We argue by Steiner symmetrization with respect to the y-axis; see, e.g., [Kawohl 1985]. We stress that the weight $(x, y) \mapsto e^{-2y}$ is independent of the x variable. Let $A^* \subset (-\pi, \pi) \times \mathbb{R}^+$ be defined as

$$A^* := \left\{(x, y) : y > 0, |x| < \tfrac{1}{2}|\{x : (x, y) \in A\}|\right\}$$

and $v^* \in H_0^1(A^*) \subset H_0^1((-\pi, \pi)\times\mathbb{R}^+)$ be the Steiner symmetrization of the function $v|_A$. By well-known properties of the Steiner symmetrization, we obtain

$$\int_{(-\pi,\pi)\times\mathbb{R}^+} |\nabla v^*|^2 \le \mu \int_{(-\pi,\pi)\times\mathbb{R}^+} e^{-2y}(v^*)^2.$$

Since v and v^* are not identically zero, by Lemma A.3, we obtain

$$\mu \ge (j_{1/2,1})^2 = \pi^2. \qquad \square$$

Lemma 4.5. *Let y^* be defined as in Lemma 3.9, and let A denote any nodal component of v such that $A \cap \{(x, y) : y > y^*\} \neq \varnothing$. Then*

$$\sup_{y>0} |\{x : (x, y) \in A\}| \le 2\pi.$$

Proof. Without loss of generality we can assume that $v > 0$ in A and, by Lemma 3.9, there exists a half-line $\ell := \{(x, y) : y \ge y^*, \alpha y + nx = q\}$ such that $\ell \subset A$. Let us assume by contradiction that $\sup_{y>0} |\{x : (x, y) \in A\}| > 2\pi$. By Lemma 4.3, we deduce that A is 2π-periodic in the x-direction, so that also $\ell + (2\pi, 0) \subset A$. By connectedness, we can find a simple curve γ such that

$$\gamma \subset A, \quad \gamma \cap \{(x, y) : y \ge y^*\} = \ell \cup \ell + (2\pi, 0) \quad \text{and} \quad \gamma \cap \{(x, y) : y \le y^*\} \text{ is compact.}$$

As a consequence, $\mathbb{R} \times \mathbb{R}^+ \setminus \gamma = O_0 \cup O_1$, where each O_i is open and connected and only one of them, say O_1, is such that

$$O_1 \supset \{(x, y^*) : x^* < x < x^* + 2\pi\} \neq \varnothing, \quad \text{where } \alpha y^* + nx^* = q.$$

Since $\gamma \cap \{y \le y^*\}$ is compact, we deduce that there exist q_1, q_2 and $y_0 > 0$ such that

$$O_1 \subset \{(x, y) : y \ge y_0, q_1 < \alpha y + nx < q_2\}. \tag{43}$$

Now, let $B \neq A$ be any other nodal component of v satisfying $B \subset O_1$ (B exists as v changes sign in O_1, by Lemma 3.9). Then B cannot be periodic in the x-direction, and hence, by Lemma 4.3, $\sup_{y>0} |\{x : (x, y) \in B\}| \le 2\pi$. By Proposition 3.8 and (43), we have that $v|_B \in H_0^1(B)$. Thus Lemma 4.4 applies, providing a contradiction since we are assuming $\mu < \pi^2$. $\square$

In the same spirit, we show the following.

Lemma 4.6. *Let y^* be defined as in Lemma 3.9, and let A denote any nodal component of v such that $A \cap \{(x, y) : y > y^*\} \neq \varnothing$. Then $A \cap \{(x, y) : y > y^*\}$ is connected.*

Proof. The proof follows the lines of that of Lemma 4.5. Assume by contradiction that $A \cap \{(x, y) : y > y^*\}$ contains at least two connected components, say A_1 and A_2. Then, by Lemma 3.9, we can find half-lines

$\ell_j := \{(x, y) : y \geq y^*, \alpha y + nx = q_j\} \subset A_j$ and a simple curve $\gamma \subset A$ which joins such half lines. Then $\mathbb{R} \times \mathbb{R}^+ \setminus \gamma$ is the disjoint union of O_0 and O_1, and one can find a contradiction as above. □

Motivated by Lemma 4.6, we introduce the following notation.

Definition 4.7. Let $y^* > 0$ and $\beta \in \mathbb{R}$ be fixed as in Lemma 3.9. We denote with $S_k,\ k \in \mathbb{Z}$, the nodal component of v asymptotic to

$$\{(x, y) : \beta + \pi k < \alpha y + nx < \beta + \pi(k+1)\} \quad \text{as } y \to +\infty.$$

By Lemma 4.6, we have that S_k and S_h are disjoint, as long as $h \neq k$. To conclude the proof of Proposition 4.1, we are left to show that the sets S_k exhaust the nodal components of v. At the moment we cannot be assured that each S_k intersects the x-axis. However, in such cases, the horizontal order is preserved.

Lemma 4.8. *Let $S_{k_1},\ S_{k_2}$ be two nodal components of v as in Definition 4.7, and let $k_1 < k_2$. If $\overline{S}_{k_i} \cap \{(x, 0)\} \neq \varnothing,\ i = 1, 2$, then*

$$(\hat{x}_i, 0) \in \overline{S}_{k_i} \quad \Longrightarrow \quad \hat{x}_1 < \hat{x}_2.$$

Proof. This follows by connectedness since the segments $S_k \cap \{(x, y^*)\}$ are ordered according to the index k. □

Lemma 4.9. *Let A denote any nodal component of v. There exist $q_- < q_+$ such that*

$$A \subset \{(x, y) : q_- < \alpha y + nx < q_+\}.$$

Proof. We only show that $A \subset \{(x, y) : \alpha y + nx < q_+\}$, for some q^+, because the other property follows by a similar argument. In the following, we fix x_0 such that $\Phi(x_0) \neq 0$, and we write

$$\ell := \{(x, y) : y > 0, \alpha y + n(x - x_0) = 0\}, \quad L^- := \{(x, y) : y > 0, \alpha y + n(x - x_0) < 0\}.$$

Moreover, by Lemma 3.9, we can assume that v does not vanish on $\ell \cap \{(x, y) : y \geq y^*\}$.

We have to show that, for some $h \in \mathbb{Z}$,

$$A_h := A - (2h\pi, 0) \subset L^-.$$

To start with, we observe that $A_h \cap L^- \neq \varnothing$ for every $h \geq \bar{h}$ sufficiently large (indeed A is not empty). Let us assume by contradiction that $A_h \setminus L^- \neq \varnothing$ for every $h \geq \bar{h}$ as well. By connectedness, we obtain that $I_h := \ell \cap A_h$ is nonempty, relatively open in ℓ, and with nonempty (relative) boundary $\partial I_h \subset \mathcal{N}$. Finally, by Lemmas 4.5 and 4.3, we have that $I_{h_1} \cap I_{h_2} = \varnothing$ for every $h_1 \neq h_2$. We deduce that the set

$$\bigcup_{h \geq \bar{h}} \partial I_h \subset (\mathcal{N} \cap \ell \cap \{y \leq y^*\}) \quad \text{is infinite.}$$

This contradicts the last part of Lemma 4.2. □

Lemma 4.10. *Let A denote any nodal component of v. Then $v|_A \in H^1(A)$.*

Proof. This follows by Lemma 4.9 and Proposition 3.8. □

Lemma 4.11. *Let S_k be a nodal component of v as in Definition 4.7. Then $v|_{\partial S_k} \not\equiv 0$. In particular,*

$$\{x : (x,0) \in \overline{S}_k\} \quad \textit{contains a nontrivial interval.}$$

Proof. The lemma follows by Lemmas 4.4, 4.5 and 4.10. □

Lemma 4.12. *Let A denote any nodal component of v. Then*

$$\sup_{y>0} |\{x : (x,y) \in A\}| \leq 2\pi.$$

Proof. Let A contradict the result; then $A \equiv A + (2\pi, 0)$ (Lemma 4.3) and $A \subset \{(x,y) : y < y^*\}$ (Lemma 4.5). As a consequence, there exists a simple curve $\gamma \subset A$, with $\gamma + (2\pi, 0) \equiv \gamma$. Then $\mathbb{R} \times \mathbb{R}^+ \setminus \gamma = O_0 \cup O_1$, where each O_i is open and connected and $O_1 \supset \{(x,y) : y \geq y^*\}$. Now, let A' be any nodal region of v intersecting $\{(x,y) : y \geq y^*\}$. Then $A \cap A' = \varnothing$. By Lemma 4.11 there exists $\gamma' \subset A'$ with one endpoint in O_1 and the other one in O_0, so that γ' intersects γ, a contradiction. □

Lemma 4.13. *Let A denote any nodal component of v. Then $v|_{\partial A} \not\equiv 0$. In particular,*

$$\{x : (x,0) \in \overline{A}\} \quad \textit{contains a nontrivial interval.}$$

Proof. The lemma follows by Lemmas 4.4, 4.12 and 4.10. □

We are ready to conclude the proof of the main result of the section.

End of the proof of Proposition 4.1. We are left to show that the sets S_k (Definition 4.7) exhaust the nodal components of the function v, so that, in particular, for each S_k, there exists two consecutive zeros of the function Φ, $x_j < x_{j+1} \in [0, 2\pi]$, and $h \in \mathbb{Z}$ such that

$$\overline{S}_k \cap \{(x,0)\} = [x_j, x_{j+1}] + (2h\pi, 0).$$

Let S_k be any connected component as in Definition 4.7; then, by Lemma 4.13 and continuity of the function v (see Proposition 3.8), there exist two consecutive zeros $x_j < x_{j+1}$ and $h \in \mathbb{Z}$ such that

$$[x_j, x_{j+1}] + (2h\pi, 0) \subset \overline{S}_k \cap \{(x,0)\}.$$

By periodicity in the x-direction, it follows that

$$[x_j, x_{j+1}] + (2(h+1)\pi, 0) \subset \overline{S}_{k+2n} \cap \{(x,0)\}.$$

Now, on the one hand, for $y \geq y^*$, we already know that the nodal set of v between S_k (included) and S_{k+2n} (excluded) is precisely given by the $2n$ sets $S_k, \dots, S_{k+2n-1}$. On the other hand, for $y = 0$, the nodal set of v between $(x_j + 2h\pi, 0)$ and $(x_j + 2(h+1)\pi, 0)$ consists in exactly $2n$ intervals. Once again, we appeal to Lemma 4.11 to infer that every $S_k, \dots, S_{k+2n-1}$ contains exactly one interval on $\{(x,0)\}$, and the intersections are ordered by Lemma 4.8. The remaining conclusions follow straightforwardly. □

5. End of the proof of Theorem 1.1

We give the proof in the case that $K = 2n$ is even. The odd case can be treated with minor changes; see the discussion at the beginning of Section 3.

In view of Proposition 2.1, the existence of an element of $\mathcal{S}_{\text{rot}}$, as defined in (6), with the required nodal properties is equivalent to the existence of a solution of (22) having trace

$$\Phi(x) = \sum_{m=1}^{K} \frac{(-1)^{m+1}}{l_m} s_m \varphi_m \tag{44}$$

(recall (16), (21)) and enjoying properties (b) and (c) in Section 2 (property (a) is already contained in (22)).

The existence of such functions is provided by Proposition 3.8, while properties (b) and (c) follow from Proposition 4.1 once Φ satisfies the compatibility conditions (42), i.e.,

$$\phi_k = \frac{1}{2\pi} \int_0^{2\pi} e^{-(ik+\alpha)x} \Phi(x)\, dx = 0, \quad |k| < n, \quad \text{and} \quad \phi_n \neq 0 \tag{45}$$

(or equivalently $\phi_{-n} = \bar{\phi}_n \neq 0$). Under the validity of these conditions, also the asymptotic expansion (8) follows from Proposition 4.1 and the definition of the map $\mathcal{T}$ (14); see also Remark 2.2. The details of these calculations are very similar to those in [Terracini et al. 2019, Proof of Theorem 1.5]

Writing $c_m = s_m / l_m$ in (44) and (45), and recalling also Remark 3.10, we obtain that Theorem 1.1 is equivalent to the following assertion: *there exists* $\bar{c} = (\bar{c}_1, \dots, \bar{c}_{2n})$, *with* $(-1)^{m+1} c_m > 0$, *such that*

$$\sum_{m=1}^{2n} \frac{1}{2\pi} \int_0^{2\pi} e^{-(ik+\alpha)x} c_m \varphi_m(x)\, dx = 0, \quad |k| < n,$$

and

$$\sum_{m=1}^{2n} \frac{1}{2\pi} \int_0^{2\pi} e^{-(in+\alpha)x} c_m \varphi_m(x)\, dx \neq 0$$

if and only if $c = t\bar{c}$.

To prove this last claim, let us define the matrix $A \in \mathbb{C}^{2n \times 2n}$,

$$A = (a_{km})_{\substack{k=-n+1,\dots,n \\ m=1,\dots,2n}} = \left(\frac{1}{2\pi} \int_0^{2\pi} e^{-(ik+\alpha)x} \varphi_m(x)\, dx \right)_{km} = \left(\frac{1}{2\pi} \int_0^{2\pi} e^{-(ik+\alpha)t_m} \varphi_m(t_m)\, dt_m \right)_{km}.$$

Observe that we have suitably renamed the dummy variables in each integral as, later, this will lead us to more manageable identities. We can write the set of compatibility conditions (45) as a system of linear equations,

$$A \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_{2n} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ \phi_n \end{pmatrix}. \tag{46}$$

To show our claim, we prove that the matrix A is invertible and that it is possible to choose $\phi_n \neq 0$ such that the solution vector is real and sign-alternating. First, exploiting the multilinearity of the determinant, we have

$$\det A = \frac{1}{(2\pi)^{2n}} \int_{[0,2\pi]^{2n}} \prod_{m=1}^{2n} e^{-\alpha t_m} \varphi_m(t_m) \cdot \det A',$$

where we have introduced the matrix

$$A' = \begin{pmatrix} e^{-i(-n+1)t_1} & e^{-i(-n+1)t_2} & \dots & e^{-i(-n+1)t_{2n}} \\ e^{-i(-n+2)t_1} & e^{-i(-n+2)t_2} & \dots & e^{-i(-n+2)t_{2n}} \\ \vdots & \vdots & \ddots & \vdots \\ e^{-int_1} & e^{-int_2} & \dots & e^{-int_{2n}} \end{pmatrix}.$$

Factoring out the coefficients of the first row, we recognize Vandermonde's determinant and compute

$$\begin{aligned} \det A' &= e^{-i(-n+1)\sum_{m=1}^{2n} t_m} \begin{vmatrix} 1 & \dots & 1 \\ e^{-it_1} & \dots & e^{-it_{2n}} \\ \vdots & \ddots & \vdots \\ e^{-(2n-1)it_1} & \dots & e^{-(2n-1)it_{2n}} \end{vmatrix} \\ &= e^{-i(-n+1)\sum_{m=1}^{2n} t_m} \prod_{1\le p<q\le 2n} (e^{-it_q} - e^{-it_p}) \\ &= e^{i(n-1)\sum_{m=1}^{2n} t_m} \prod_{1\le p<q\le 2n} (-1) e^{-\frac{1}{2}it_q - \frac{1}{2}it_p} (-e^{-\frac{1}{2}it_q + \frac{1}{2}it_p} + e^{-\frac{1}{2}it_p + \frac{1}{2}it_q}) \\ &= e^{i(n-1)\sum_{m=1}^{2n} t_m} (-1)^{\frac{2n(2n-1)}{2}} e^{-\frac{1}{2}i(2n-1)\sum_{m=1}^{2n} t_m} \prod_{1\le p<q\le 2n} (e^{-\frac{1}{2}it_p + \frac{1}{2}it_q} - e^{-\frac{1}{2}it_q + \frac{1}{2}it_p}) \\ &= (-1)^n (2i)^{\frac{2n(2n-1)}{2}} e^{-\frac{1}{2}i\sum_{m=1}^{2n} t_m} \prod_{1\le p<q\le 2n} \left(\frac{e^{\frac{1}{2}i(t_q-t_p)} - e^{-\frac{1}{2}i(t_q-t_p)}}{2i} \right) \\ &= (-1)^n (2i)^{n(2n-1)} e^{-\frac{1}{2}i\sum_{m=1}^{2n} t_m} \prod_{1\le p<q\le 2n} \sin\left(\frac{t_q-t_p}{2}\right). \end{aligned}$$

Thus we find

$$\det A = \frac{(-1)^n (2i)^{n(2n-1)}}{(2\pi)^{2n}} \int_{[0,2\pi]^{2n}} \underbrace{\prod_{m=1}^{2n} e^{-\alpha t_m} \varphi_m(t_m) \prod_{1\le p<q\le 2n} \sin\left(\frac{t_q-t_p}{2}\right)}_{\text{Mod}} \underbrace{e^{-\frac{1}{2}i\sum_{m=1}^{2n} t_m}}_{\text{Phase}}.$$

We show that the integral in the previous expression is always different from 0. We recall that, by assumption, the functions φ_m are supported on ordered intervals. More precisely, using the notation introduced in Proposition 4.1, we have

$$\{t \in [0, 2\pi] : \varphi_m(t) > 0\} = (x_m, x_{m+1}).$$

As a result, the integral can be restricted to the open and not empty set

$$\mathcal{O} = (x_1, x_2) \times (x_2, x_3) \times \cdots \times (x_{2n}, x_{2n+1}) \subset [0, 2\pi]^{2n}.$$

Moreover, for any choice $1 \le p < q \le 2n$, in $\mathcal{O}$ we have $0 < t_q - t_p < 2\pi$, and thus

$$0 < \frac{t_q - t_p}{2} < \pi \quad \Longrightarrow \quad \sin\left(\frac{t_q - t_p}{2}\right) > 0.$$

As it turns out, the factor denoted as Mod is strictly positive in $\mathcal{O}$. This function corresponds to the modulus of the integral function. On the other hand, the factor Phase is complex and of modulus 1. Let us investigate more closely the argument of Phase. We find

$$\sum_{m=1}^{2n} x_m < \sum_{m=1}^{2n} t_m < \sum_{m=1}^{2n} x_{m+1} = \sum_{m=1}^{2n} x_m + (x_{2n+1} - x_1) < \sum_{m=1}^{2n} x_m + 2\pi.$$

That is, letting $X = \sum_{m=1}^{2n} x_m$, for any $(t_1, \dots, t_{2n}) \in \mathcal{O}$,

$$0 < \frac{1}{2}\left(\sum_{m=1}^{2n} t_m - X\right) < \pi.$$

We can rewrite the determinant as

$$\det A = C\left[\int_{\mathcal{O}} \mathrm{Mod} \cdot \cos\frac{1}{2}\left(\sum_{m=1}^{2n} t_m - X\right) - i\int_{\mathcal{O}} \mathrm{Mod} \cdot \sin\frac{1}{2}\left(\sum_{m=1}^{2n} t_m - X\right)\right]$$

for some complex constant $C \in \mathbb{C} \setminus \{0\}$. By the previous discussion, the second integral is positive. It follows that the determinant of A is not zero, proving that the linear system (46) has a unique solution for any ϕ_n.

We now show that there exists $\phi_n \neq 0$ such that the solution vector is real and sign-alternating. By Cramer's rule, we have

$$c_l = (\det A)^{-1} \det A_l,$$

where A_l is the matrix obtained by replacing the l column of A with the right-hand side of system (46). Now, by the same considerations as before, we have

$$\det A_l = \frac{1}{(2\pi)^{2n}} \int_{[0,2\pi]^{2n}} \prod_{m=1, m\neq l}^{2n} e^{-\alpha t_m} \varphi_m(t_m) \cdot \det A_l',$$

where

$$A_l' = \begin{pmatrix} e^{-i(-n+1)t_1} & e^{-i(-n+1)t_2} & \dots & e^{-i(-n+1)t_{l-1}} & 0 & e^{-i(-n+1)t_{l+1}} & \dots & e^{-i(-n+1)t_{2n}} \\ e^{-i(-n+2)t_1} & e^{-i(-n+2)t_2} & \dots & e^{-i(-n+2)t_{l-1}} & 0 & e^{-i(-n+2)t_{l+1}} & \dots & e^{-i(-n+2)t_{2n}} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ e^{-int_1} & e^{-int_2} & \dots & e^{-int_{l-1}} & \phi_n & e^{-int_{l+1}} & \dots & e^{-int_{2n}} \end{pmatrix}.$$

Developing the determinant with respect to the l-th column, factoring out the first line and exploiting once more Vandermonde's determinant, we find

$$\begin{aligned}
\det A'_l &= (-1)^{l-1}\phi_n e^{-i(-n+1)\sum_{m=1,m\neq l}^{2n} t_m} \begin{vmatrix} 1 & \cdots & 1 \\ e^{-it_1} & \cdots & e^{-it_{2n}} \\ \vdots & \ddots & \vdots \\ e^{-(2n-2)it_1} & \cdots & e^{-(2n-2)it_{2n}} \end{vmatrix} \\
&= (-1)^{l-1}\phi_n e^{-i(-n+1)\sum_{m=1,m\neq l}^{2n} t_m} \prod_{\substack{1\le p<q\le 2n \\ p,q\neq l}} (e^{-it_q} - e^{-it_p}) \\
&= (-1)^{l-1}\phi_n e^{i(n-1)\sum_{m=1,m\neq l}^{2n} t_m} \prod_{\substack{1\le p<q\le 2n \\ p,q\neq l}} (-1) e^{-\frac{1}{2}it_q-\frac{1}{2}it_p}(-e^{-\frac{1}{2}it_q+\frac{1}{2}it_p} + e^{-\frac{1}{2}it_p+\frac{1}{2}it_q}) \\
&= (-1)^{l-1}\phi_n e^{i(n-1)\sum_{m=1,m\neq l}^{2n} t_m} (-1)^{\frac{(2n-1)(2n-2)}{2}} e^{-\frac{1}{2}i(2n-2)\sum_{m=1,m\neq l}^{2n} t_m} \\
&\qquad\qquad \times \prod_{\substack{1\le p<q\le 2n \\ p,q\neq l}} (e^{-\frac{1}{2}it_p+\frac{1}{2}it_q} - e^{-\frac{1}{2}it_q+\frac{1}{2}it_p}) \\
&= (-1)^{l+n-2}\phi_n (2i)^{\frac{(2n-1)(2n-2)}{2}} \prod_{\substack{1\le p<q\le 2n \\ p,q\neq l}} \left(\frac{e^{\frac{1}{2}i(t_q-t_p)} - e^{-\frac{1}{2}i(t_q-t_p)}}{2i}\right) \\
&= (-1)^{l+n}(2i)^{(2n-1)(n-1)}\phi_n \prod_{\substack{1\le p<q\le 2n \\ p,q\neq l}} \sin\left(\frac{t_q-t_p}{2}\right).
\end{aligned}$$

We obtain

$$\begin{aligned}
c_l &= \frac{(\det A)^{-1}(-1)^{l+n}(2i)^{(2n-1)(n-1)}\phi_n}{(2\pi)^{2n-1}} \int \prod_{m=1,m\neq l}^{2n} e^{-\alpha t_m}\varphi_m(t_m) \prod_{1\le p<q\le 2n,p,q\neq l} \sin\left(\frac{t_q-t_p}{2}\right) \\
&= (-1)^{l+1}\Gamma \int_{[0,2\pi]^{2n-1}} \prod_{m=1,m\neq l}^{2n} e^{-\alpha t_m}\varphi_m(t_m) \prod_{1\le p<q\le 2n,p,q\neq l} \sin\left(\frac{t_q-t_p}{2}\right),
\end{aligned}$$

where $\Gamma \in \mathbb{C}$. Reasoning as before, we see that the integral is always strictly positive. Thus c_l satisfies the condition $(-1)^{l+1}c_l > 0$ if and only if Γ is real and positive, $\Gamma = t > 0$. We obtain the solution

$$c_l = t(-1)^{l+1} \int_{[0,2\pi]^{2n-1}} \prod_{m=1,m\neq l}^{2n} e^{-\alpha t_m}\varphi_m(t_m) \prod_{1\le p<q\le 2n,p,q\neq l} \sin\left(\frac{t_q-t_p}{2}\right)$$

and

$$\phi_n = t(-1)^{n+1}\frac{2^{2n-2}}{\pi} \int_{[0,2\pi]^{2n}} \prod_{m=1}^{2n} e^{-\alpha t_m}\varphi_m(t_m) \prod_{1\le p<q\le 2n} \sin\left(\frac{t_q-t_p}{2}\right) e^{-\frac{1}{2}i\sum_{m=1}^{2n} t_m}.$$

Proof of Corollary 1.2. This follows by uniqueness of $\bar{s}$; indeed, notice that a rotation of $2\pi/K$ leaves the data unchanged, while the indexes of the densities are shifted by 1. By uniqueness, $\bar{s}_m = \bar{s}_{m-1}$ for every m. □

6. Single-mode special solutions

In the following we deal with the fundamental single-mode solutions that we constructed by separation of variables in Section 3. Theorems 1.6 and 1.7 will follow once again by Proposition 2.1.

6.1. *The homogeneous Dirichlet problem.* We now turn our attention to the homogeneous version of (22); that is, we look for conditions under which there exists a nonzero solution v of

$$\begin{cases} -\Delta v + \omega e^{-2y} v_x = e^{-2y}\mu v, & x \in \mathbb{R}, \quad y > 0, \\ v(x+2\pi, y) = e^{2\pi\alpha} v(x,y), & x \in \mathbb{R}, \quad y \geq 0, \\ v(x,0) = 0, & 0 \leq x \leq 2\pi, \end{cases} \tag{47}$$

with nodal set consisting of $2k$ strips (up to horizontal 2π-periodicity), $k \geq 1$, that connect the boundary $y = 0$ with $y \to +\infty$, as in the previous section. Clearly (47) may have nonzero solutions only for some specific choices of parameters (this is indeed the case according to Lemma 3.6). For this reason, in this section we consider the number $k \geq 1$ and the parameter $\alpha \in \mathbb{R}$ as givens of the problem, and we look for pairs of numbers $(\mu, \omega) \in \mathbb{R}^2$ such that a solution v as specified above exists.

The analysis that we have conducted in Section 3 can be exploited to give a direct solution to this problem. Indeed we have the following result.

Lemma 6.1. *For any $k \geq 1$, $\alpha \in \mathbb{R}$, there exists at least a value $\lambda \in \mathbb{C}$ satisfying*

$$\begin{cases} \Theta_{k-i\alpha}(\lambda) = 0, \\ \Theta_{k-i\alpha}(t\lambda) \neq 0 \quad \text{for all } t \in [0,1), \end{cases} \tag{48}$$

where Θ_ν is defined in (32) *for every $\nu \in \mathbb{C}$. For any such λ, the function*

$$v(x,y) = e^{\alpha x - ky} \operatorname{Re}(e^{i(kx+\alpha y)} \Theta_{k-i\alpha}(\lambda e^{-2y}))$$

is a solution of (47), *with*

$$\omega = \frac{\operatorname{Im}(\lambda)}{k}, \quad \mu = \alpha \frac{\operatorname{Im}(\lambda)}{k} - \operatorname{Re}(\lambda).$$

Moreover, there exists an analytic map $y \mapsto \zeta(y)$ such that

$$v(x,y) = 0 \iff x = \zeta(y) + \frac{h\pi}{k}, \quad h \in \mathbb{Z},$$

and

$$\zeta(y) = \frac{1}{k}(\beta - \alpha y) + o(1) \quad \text{for some } \beta \in \mathbb{R} \text{ and } y \to +\infty.$$

In particular, for any $y > 0$, $v(\cdot, y)$ has exactly $2k$ zeros in each period $x \in [0, 2\pi)$.

Proof. The result is a direct consequence of Lemma 3.2. We start by showing that, for any choice of parameters, there exists at least a value $\lambda \in \mathbb{C}$ satisfying (48). Indeed, $\Theta_{k-i\alpha}$ is a nonconstant analytic function with $\Theta_{k-i\alpha}(0) \neq 0$, and it suffices to consider a zero λ of $\Theta_{k-i\alpha}$ with the least absolute value in order to guarantee that $\Theta_{k-i\alpha}(t\lambda) \neq 0$ for any $t \in [0,1)$. Of course, many (if not all) the zeros of $\Theta_{k-i\alpha}$ may satisfy this assumption, but these constitute an at most countable discrete subset of $\mathbb{C}$.

Exploiting the fact that the coefficients of (47) are real, we find that the function

$$v(x,y) = e^{\alpha x}\,\mathrm{Re}(e^{ikx}D_k(y)) \tag{49}$$

is a solution of (47), where the function D_k solves

$$\begin{cases} D_k''(y) = [(k-i\alpha)^2 + (\omega\alpha - \mu + i\omega k)e^{-2y}]D_k(y), & y > 0, \\ D_k(0) = 0, \quad D_k(y) \to 0 & \text{as } y \to +\infty. \end{cases} \tag{50}$$

By Lemma 3.2, equation (50) is solved by any multiple of the function

$$y \mapsto e^{-(k-i\alpha)y}\Theta_{k-i\alpha}((\omega\alpha - \mu + i\omega k)e^{-2y}),$$

which in turns vanishes for $y \to +\infty$. The initial condition $D_k(0) = 0$ is satisfied since we chose $\lambda = \omega\alpha - \mu + i\omega k$ as a zero of the function $\Theta_{k-i\alpha}$ (observe that we are negating (26)).

To conclude, we need to study the nodal properties of the function v. From its expression we readily see, that for any fixed $y > 0$, the function $x \mapsto v(x,y)$ has exactly $2k$ evenly spaced zeros in $[0, 2\pi)$ since, by assumption, $\Theta_{k-i\alpha}(\lambda e^{-2y}) \neq 0$. From this we deduce also that the nodal lines of v can be described, up to translations, by a function $y \mapsto \zeta(y)$. We notice that ζ is continuous by the implicit function theorem, as

$$v(x,y) = 0 \quad \Longleftrightarrow \quad \mathrm{Re}(e^{ikx}D_k(y)) = 0$$

and, for such (x,y),

$$\frac{\partial}{\partial x}\mathrm{Re}(e^{ikx}D_k(y)) = ik\,\mathrm{Im}(e^{ikx}D_k(y)) \neq 0.$$

More explicitly, writing

$$D_k(y) = \rho(y)e^{i\vartheta(y)},$$

where $\rho(y) > 0$ for $y > 0$ and ϑ is an analytic lifting of the argument of D_k, we have that

$$e^{\alpha x}v(x,y) = \mathrm{Re}(e^{ikx}D_k(y)) = 0 \quad \Longleftrightarrow \quad x - \frac{h\pi}{k} = \frac{1}{k}(\beta - \vartheta(y)) =: \zeta(y).$$

Finally, the asymptotic behavior of ζ follows as in Lemma 3.9. □

We conclude with some additional remarks on the result.

Remark 6.2 (a question about uniqueness). If v is a solution of (47), then for any $A, \bar{x} \in \mathbb{R}$, the function $(x,y) \mapsto Av(x-\bar{x}, y)$ is again a solution. We may wonder whether this family of functions completely describes the set of solutions of (47) under some additional condition (for instance that, for any $x \in \mathbb{R}$, $v(x,y) \to 0$ as $y \to +\infty$). More precisely, fix ω, μ and α in such a way that (47) admits at least a solution. Is this solution unique (up to translation in x and multiplication by a real constant of course)? This seems to be a question of a nontrivial nature, and it is related to the position of the zeros of Bessel functions with different order. From the proof of Lemma 6.1, we can state the following: let $\alpha \in \mathbb{R}$ be such that, for any $k_1, k_2 \geq 1$ and $z_1, z_2 \in \mathbb{C}$, we have

$$\begin{cases} I_{k_1-i\alpha}(z_1) = I_{k_2-i\alpha}(z_2) = 0, \\ \mathrm{Re}(z_1^2) = \mathrm{Re}(z_2^2), \\ \mathrm{Im}(z_1^2)/k_1 = \mathrm{Im}(z_2^2)/k_2 \end{cases} \quad \Longrightarrow \quad k_1 = k_2.$$

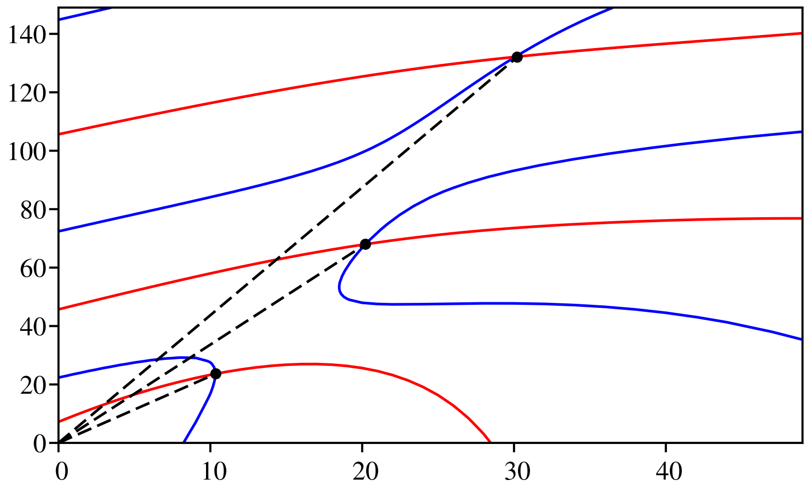

Figure 3. Numerical zeros of $\mathrm{Re}\,\Theta_{1-i}$ (blue) and $\mathrm{Im}\,\Theta_{1-i}$ (red). The three zeros located at $10.36+i23.66$, $20.22+i67.99$, $30.21+i132.04$ satisfy condition (48).

Then for this specific value of α, if (47) admits a solution, this solution is unique up to translation in x and multiplication by a real constant.

Remark 6.3 (the symmetric case $\alpha = 0$). If $\nu \in \mathbb{R}$ and $\nu \geq 1$, the zeros of the modified Bessel function I_ν are purely imaginary numbers (and are given by $ij_{\nu,l}$, where $j_{\nu,l}$ is the l-th zero of the Bessel function J_ν, with $l \in \mathbb{N}$). It follows that

$$\Theta_k(\lambda) = 0 \quad \Longrightarrow \quad \lambda = -t^2 \text{ for some } t > 0.$$

As a result, if $\alpha = 0$, then necessarily $\omega = 0$ (no rotation) and $\mu = j_{k,1}^2$. Since all the zeros belong to the same half-line emanating from the origin, the first nontrivial zero is also the only one that satisfies the assumptions of Lemma 6.1. We conclude that, in the case $\alpha = 0$, (47) has nonzero solutions only if $\mu = j_{k,1}^2$ and $\omega = 0$, and any solution (that converges to zero as $y \to +\infty$) is of the form

$$v(x, y) = (A\cos(kx) + B\sin(kx))J_k(j_{k,1}e^{-y})$$

for some $A, B \in \mathbb{R}$.

Remark 6.4 (the asymmetric case $\alpha \neq 0$). By Lemma 3.6, and in particular (37), we already know that, if $\alpha \neq 0$, for (47) to have a solution, it is necessary that

$$\mu \geq (j_{0,1} + \sqrt{k^2+\alpha^2})^2.$$

From numerical explorations (see, e.g., Figures 3 and 4), it seems that, if $\alpha \neq 0$, the zeros of the function $\Theta_{k-i\alpha}$ belong to different lines emanating from the origin. In contrast with the case $\alpha = 0$, it thus seems to be the case that, for $\alpha \neq 0$, (47) has infinitely many (but still countably many) solutions.

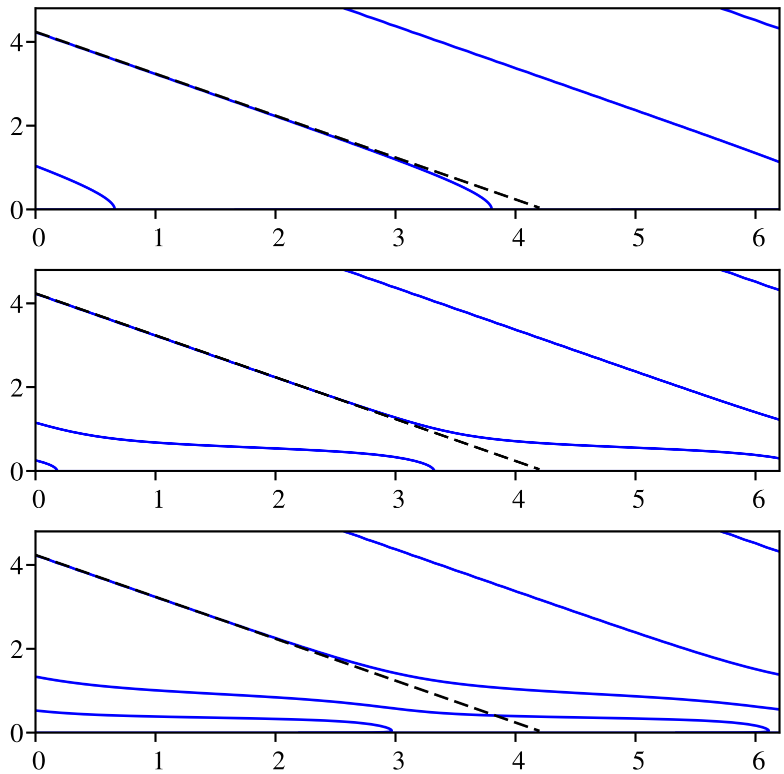

Figure 4. Nodal sets of the solutions corresponding to the three zeros in Figure 3.

6.2. ***The homogeneous Neumann/Robin problem.*** Let $\sigma \in \mathbb{R}$. We consider the problem

$$\begin{cases} -\Delta v + \omega e^{-2y} v_x = e^{-2y} \mu v, & x \in \mathbb{R}, \quad y > 0, \\ v(x + 2\pi, y) = e^{2\pi\alpha} v(x, y), & x \in \mathbb{R}, \quad y \geq 0, \\ \partial_y v(x, 0) + \sigma v(x, 0) = 0, & 0 \leq x \leq 2\pi, \end{cases} \tag{51}$$

which involves Robin ($\sigma \neq 0$) or Neumann ($\sigma = 0$) boundary conditions.

As in the previous section we can find single-mode solutions that exhibit a precise nodal behavior.

Lemma 6.5. *For any* $k \geq 1$, $\alpha \in \mathbb{R}$, *assume that there exists* $\lambda \in \mathbb{C}$ *satisfying*

$$\begin{cases} 2\lambda \Theta'_{k-i\alpha}(\lambda) + (k - i\alpha - \sigma)\Theta_{k-i\alpha}(\lambda) = 0, \\ \Theta_{k-i\alpha}(t\lambda) \neq 0 \quad \text{for all } t \in [0, 1). \end{cases}$$

Then we have

$$v(x, y) = e^{\alpha x - ky} \operatorname{Re}(e^{i(kx + \alpha y)} \Theta_{k-i\alpha}(\lambda e^{-2y}))$$

a solution of (51) *for the particular choice of parameters*

$$\omega = \frac{\operatorname{Im}(\lambda)}{k}, \quad \mu = \alpha \frac{\operatorname{Im}(\lambda)}{k} - \operatorname{Re}(\lambda).$$

Moreover, the nodal set of v *has the same properties as those described in Lemma 6.1.*

Proof. We already know that any function of the type

$$v(x, y) = e^{\alpha x} \operatorname{Re}(e^{ikx} N_k(y))$$

is a solution of the differential equation in (51) provided that

$$N_k''(y) = [(k - i\alpha)^2 + (\omega\alpha - \mu + i\omega k)e^{-2y}]N_k(y), \quad y > 0.$$

Once again we can appeal to Lemma 3.2 for an explicit expression for the function N_k. In order to impose the boundary condition at $y = 0$ we find

$$N_k'(y) = \Theta_{k-i\alpha}'(\lambda e^{-2y})(-2\lambda e^{-2y})e^{-(k-i\alpha)y} - \Theta_{k-i\alpha}(\lambda e^{-2y})(k - i\alpha)e^{-(k-i\alpha)y};$$

that is,

$$N_k'(0) = \Theta_{k-i\alpha}'(\lambda)(-2\lambda) - (k - i\alpha)\Theta_{k-i\alpha}(\lambda) = 0.$$

The rest of the proof follows easily. □

6.3. ***Entire solutions.*** Finally we consider the case of entire solutions; that is, we look for functions v that satisfy

$$\begin{cases} -\Delta v + \omega e^{-2y} v_x = e^{-2y}\mu v, \\ v(x + 2\pi, y) = e^{2\pi\alpha} v(x, y), \end{cases} \quad (x, y) \in \mathbb{R}^2, \tag{52}$$

vanish for $y \to +\infty$ and, as before, change sign exactly $2k$ times ($k \geq 1$) in each period of length 2π in the x-direction. Similar considerations as before lead us to the following result.

Lemma 6.6. *Let $k \geq 1$, $\alpha \in \mathbb{R}$. Consider any $\lambda \in \mathbb{C}$ such that*

$$\Theta_{k-i\alpha}(t\lambda) \neq 0 \quad \text{for all } t > 0. \tag{53}$$

Then the function

$$v(x, y) = e^{\alpha x - ky} \operatorname{Re}(e^{i(kx+\alpha y)}\Theta_{k-i\alpha}(\lambda e^{-2y})) \tag{54}$$

is a solution of (52) *for the particular choice of parameters*

$$\omega = \frac{\operatorname{Im}(\lambda)}{k}, \quad \mu = \alpha\frac{\operatorname{Im}(\lambda)}{k} - \operatorname{Re}(\lambda).$$

Once again, we point out that $\Theta_{k-i\alpha}$ is analytic and thus it has at most countably many zeros, meaning that, apart from a negligible set, any $\lambda \in \mathbb{C}$ gives rise to an entire solution.

In the case of entire solutions, it is interesting to study once again the shape of the nodal lines of the solutions, which now are defined also for $y < 0$.

Lemma 6.7. *Let v be the function* (54) *in Lemma 6.6. Then there exists an analytic function $y \mapsto \zeta(y)$, defined for any $y \in \mathbb{R}$, such that*

- *$v(x, y) = 0$ if and only if $x = \zeta(y) + h\pi/k$, $y \in \mathbb{R}$, $h \in \mathbb{Z}$, and consequently, in the regions $\{(x, y) : h\pi/k < x - \zeta(y) < (h + 1)\pi/k\}$, for any $h \in \mathbb{Z}$, v does not change sign*;

- *for $y \to +\infty$, ζ is asymptotic to a line: there exists $\beta \in \mathbb{R}$ such that*

$$\zeta(y) = \frac{1}{k}(\beta - \alpha y) + o(1) \quad \text{as } y \to +\infty;$$

- *for $y \to -\infty$, ζ is asymptotic to an exponential curve*

$$\zeta(y) = \gamma e^{-y} + O(1) \quad \text{as } y \to -\infty,$$

where

$$\gamma = \begin{cases} \frac{1}{k}\operatorname{sign}(\omega)\sqrt{\sqrt{\left(\frac{1}{2}(\omega\alpha - \mu)\right)^2 + \left(\frac{1}{2}\omega k\right)^2} - \frac{1}{2}(\omega\alpha - \mu)}, & \omega \neq 0, \\ 0, & \omega = 0,\ \mu < 0, \\ \frac{1}{k}\operatorname{sign}(\alpha)\sqrt{\mu}, & \omega = 0,\ \mu > 0, \end{cases}$$

unless $\omega = \mu = 0$, in which case

$$\zeta(y) = \frac{1}{k}(\beta - \alpha y), \quad y \in \mathbb{R}.$$

Proof. The first conclusions of the result follow from similar (and much simpler) considerations as in Proposition 4.1 and Lemma 6.1. We only study the asymptotic behavior of ζ as $y \to -\infty$. As we shall see, beyond the validity of (53), we need to distinguish three cases, according to the different expansions of the Bessel functions at infinity: (Case 1) $\omega = \mu = 0$; (Case 2) $\omega = 0,\ \mu > 0$; (Case 3) either $\omega = 0$ and $\mu < 0$, or $\omega \neq 0$.

Case 1. We start with the simplest case, that is $\omega = \mu = 0$. This is equivalent to assuming that $\lambda = 0$, whence (53) is automatically satisfied (recall that $\Theta_{k-i\alpha}(0) \neq 0$ for $k \geq 1$). Substituting in (52) we find that solutions are of the form

$$v(x, y) = e^{\alpha x - ky}\cos(kx + \alpha y).$$

In this case the nodal lines are described, up to translations, by the linear function

$$\zeta(y) = \frac{1}{k}\left(\frac{\pi}{2} - \alpha y\right), \quad y \in \mathbb{R},$$

and, in particular, the nodal set of v is a family of parallel straight lines.

Case 2. Next, we look at the case $\omega = 0$ and $\mu > 0$, which means $\lambda = -\mu < 0$. We have that $\sqrt{\lambda} = -i\sqrt{\mu}$, where we have chosen the determination of the square root with negative imaginary part. In this case, exploiting (54), (33) and the relation between the Bessel functions and their modified versions, we have

$$v(x, y) = e^{\alpha x}\left(\tfrac{1}{2}e^{ikx}J_\nu(\sqrt{\mu}e^{-y}) + \tfrac{1}{2}e^{-ikx}\overline{J_\nu(\sqrt{\mu}e^{-y})}\right)$$

(to be precise, we take the line $y \mapsto \sqrt{\lambda}e^{-y}$ as the path of monodromy for the determination of J_ν). In particular, from this expression we infer the necessary condition $\alpha \neq 0$: indeed, if $\nu = k \geq 1$, the Bessel function J_k has all of its zeros on the real line, and thus we are contradicting (53). We have that (see [Erdélyi et al. 1953, p. 85])

$$J_\nu(z) = \sqrt{\frac{2}{\pi z}}\left(\cos\left(z - \tfrac{\pi}{2}\nu - \tfrac{\pi}{4}\right) + O\left(\frac{1}{|z|}\right)\right) \quad \text{for } |z| \to +\infty \text{ with } |\arg z| < \pi.$$

As to what concerns us, we have that $z > 0$. Letting

$$w = \sqrt{\mu}e^{-y} - \tfrac{\pi}{2}\nu - \tfrac{\pi}{4} = \left(\sqrt{\mu}e^{-y} - \tfrac{\pi}{2}k - \tfrac{\pi}{4}\right) + i\tfrac{\pi}{2}\alpha,$$

we may simplify the expression for v and see that, for $y \to -\infty$, the following asymptotic expansion holds:

$$\sqrt{\tfrac{1}{2}\pi\sqrt{\mu}}e^{-\alpha x - \frac{1}{2}y}v(x, y) = \tfrac{1}{2}e^{ikx}\cos w + \tfrac{1}{2}e^{-ikx}\cos\overline{w} + O(e^{y}).$$

We point out that, in this peculiar case, the solution v decays for $y \to -\infty$ since $\operatorname{Im}(w)$ is bounded (constant). The last expression can be further simplified, since

$$\begin{aligned}
\tfrac{1}{2}e^{ikx}\cos w + \tfrac{1}{2}e^{-ikx}\cos\overline{w} &= \tfrac{1}{2}(\cos(kx) + i\sin(kx))\cos w + \tfrac{1}{2}(\cos(kx) - i\sin(kx))\cos\overline{w} \\
&= \tfrac{1}{2}\cos(kx)[\cos w + \cos\overline{w}] + \tfrac{1}{2}i\sin(kx)[\cos w - \cos\overline{w}] \\
&= \cos(kx)\cos(\operatorname{Re} w)\cosh(\operatorname{Im} w) + \sin(kx)\sin(\operatorname{Re} w)\sinh(\operatorname{Im} w).
\end{aligned}$$

In order to determine the asymptotic behavior of the nodal lines of v, we need to solve the equation

$$\cos(kx)\cos(\operatorname{Re} w)\cosh(\operatorname{Im} w) + \sin(kx)\sin(\operatorname{Re} w)\sinh(\operatorname{Im} w) = 0.$$

It seems that this equation cannot be solved explicitly, nevertheless we can describe its set of solutions with sufficient accuracy for our purpose. In order to simplify the notation, we introduce the real function

$$F(X, Y) = \cos(X)\cos(Y)\cosh(T) + \sin(X)\sin(Y), \sinh(T) \tag{55}$$

where we recall that the parameter $T = \operatorname{Im} w = \frac{\pi}{2}\alpha \neq 0$. In the plane $(X, Y) \in \mathbb{R}^2$, we want to describe the set $F(X, Y) = 0$. First of all, we point out that F is 2π-period both in X and in Y and enjoys the symmetries $F(X, Y) = F(Y, X)$, $F(-X, Y) = F(X, -Y)$, $F(X+\pi, Y) = F(X, Y+\pi) = -F(X, Y)$ and $F(-X, -Y) = F(X, Y)$ for any $(X, Y) \in \mathbb{R}^2$. In particular, we deduce that the equation $F(X, Y) = 0$ has infinitely many solutions and that, for any fixed $Y \in \mathbb{R}$ (resp. X), solutions of $F(X, Y) = 0$ are equally spaced and of the form $X = X_Y + h\pi$ for some given $X_Y \in \mathbb{R}$ and $h \in \mathbb{Z}$ (resp. $Y = Y_X + h\pi$, $Y_X \in \mathbb{R}$). We deduce that, for any given $Y \in [0, \pi)$, there exists a unique $X \in [0, \pi)$ such that $F(X, Y) = 0$, and vice versa.

Next, let $(X_0, Y_0) \in \mathbb{R}^2$ such that $F(X_0, Y_0) = 0$. By the implicit function theorem, the nodal set of F is described locally at (X_0, Y_0) by a function $X = Z(Y)$ if $\partial_X F(X_0, Y_0) \neq 0$. Arguing by contradiction, we have the system

$$\begin{cases}\cos(X_0)\cos(Y_0)\cosh(T) + \sin(X_0)\sin(Y_0)\sinh(T) = 0, \\ \cos(X_0)\sin(Y_0)\sinh(T) - \sin(X_0)\cos(Y_0)\cosh(T) = 0,\end{cases}$$

which has a solution if and only if

$$\cos^2(Y_0)\cosh^2(T) + \sin^2(Y_0)\sinh^2(T) = 0.$$

But this is impossible since $\cosh^2(T) \neq 0$ and $\sinh^2(T) \neq 0$ (recall that $T \neq 0$). Thus $\partial_X F(X_0, Y_0) \neq 0$ at any zero of F. Observe that we can perform similar computations exchanging variables and show that the function Z is a bijection (and thus monotone). By periodicity, we can assume that $Z(0) = \frac{\pi}{2}$. We can determine the sense of monotonicity of Z by computing $Z'(Y)$ for the zero $(X, Y) = \left(\frac{\pi}{2}, 0\right)$. We find

$$Z'(0) = -\frac{\partial_Y F\left(\frac{\pi}{2}, 0\right)}{\partial_X F\left(\frac{\pi}{2}, 0\right)} = \tanh(T) = \tanh\left(\tfrac{\pi}{2}\alpha\right).$$

Bringing together the previous conclusions, we infer that

$$0 \le Z(Y) - \operatorname{sign}(\alpha) Y < \pi \quad \text{for all } Y \in \mathbb{R}.$$

Going back to the original variable, we find the asymptotic behavior

$$\zeta(y) = \frac{1}{k} \operatorname{sign}(\alpha) \sqrt{\mu} e^{-y} + O(1) \quad \text{as } y \to -\infty.$$

Case 3. We conclude with the third and last case, that is $\lambda = \omega\alpha - \mu + i\omega k \in \mathbb{C} \setminus \mathbb{R}_-$ together with (53). We recall that the modified Bessel function I_ν satisfies (see [Erdélyi et al. 1953, p. 86])

$$I_\nu(z) = \frac{e^z}{\sqrt{2\pi z}}\left(1 + O\left(\frac{1}{|z|}\right)\right) \quad \text{for } |z| \to +\infty \text{ with } |\arg z| < \tfrac{\pi}{2} - \delta.$$

By (33), the entire function in (54) is equal to

$$v(x, y) = e^{\alpha x} \operatorname{Re}(e^{ikx} I_\nu(\sqrt{\lambda} e^{-y})),$$

where we choose as determination of the square root of λ the one with strictly positive real part (recall that $\lambda \in \mathbb{C} \setminus \mathbb{R}_-$). Then $|\arg \sqrt{\lambda}| < \frac{\pi}{2} - \delta$ for some $\delta > 0$. We find

$$\begin{aligned} v(x, y) &= e^{\alpha x} \operatorname{Re}\left(e^{ikx} \frac{e^{\sqrt{\lambda} e^{-y}}}{\sqrt{2\pi\sqrt{\lambda} e^{-y}}}(1 + O(e^y))\right) = e^{\alpha x} \operatorname{Re}(C_\lambda e^{ikx + \frac{1}{2}y + \sqrt{\lambda} e^{-y}}(1 + O(e^y))) \\ &= e^{\alpha x + \frac{1}{2}y + \operatorname{Re}(\sqrt{\lambda}) e^{-y}} \operatorname{Re}(C_\lambda e^{ikx + i \operatorname{Im}\sqrt{\lambda} e^{-y} + iO(e^y)} |1 + O(e^y)|) = 0, \end{aligned}$$

which in turns gives the asymptotic equation, as $y \to -\infty$,

$$kx + \operatorname{Im}(\sqrt{\lambda}) e^{-y} + O(e^y) = \beta,$$

where $\beta \in \mathbb{R}$ and

$$\operatorname{Im}(\sqrt{\lambda}) = \operatorname{sign}(\omega k) \sqrt{\sqrt{\left(\tfrac{1}{2}(\omega\alpha - \mu)\right)^2 + \left(\tfrac{1}{2}\omega k\right)^2} - \tfrac{1}{2}(\omega\alpha - \mu)}$$

(with $\operatorname{Im}(\sqrt{\lambda}) = 0$ in case $\omega = 0$). Notice that the sign above agrees with the fact that the nodal lines of the solution v are spanned by monotone functions; see the proof of Lemma 6.1. □

Remark 6.8. In view of the results of Section 2, we have that any solution constructed in this section corresponds to an element of the corresponding class $\mathcal{S}_{\mathrm{rot}}$. In particular, if $\alpha = 0$, we obtain (positive and negative parts of) smooth rotating solutions of the heat equation, with or without reaction term. Moreover, Lemma 6.7 provides a description of their nodal lines, which behave like arithmetic spirals of the equation $\vartheta = \gamma r$ as $r \to +\infty$, as we claimed in Remark 1.8.

Appendix: Weighted embeddings and Poincaré inequalities

In this appendix, we give the proof of some results cited in the paper for the sake of completeness. We start with a very classical compact embedding result.

Lemma A.1. *The functional space $H_0^1(\mathbb{R}^+;\mathbb{C})$ embeds compactly in*

$$L=\left\{U\in L^1_{\mathrm{loc}}(\mathbb{R}^+;\mathbb{C}):\|U\|_L^2=\int_{y>0}e^{-2y}|U|^2<+\infty\right\}.$$

Proof. Let $\{u_n\}_{n\in\mathbb{N}}\subset H_0^1(\mathbb{R}^+;\mathbb{C})$ be a weakly converging sequence, and let u be its limit. Since the embedding of H_0^1 in L is clearly continuous, $u_n\rightharpoonup u$ in L, and in order to show that $u_n\to u$ in L we just need to prove the convergence of the norms. Let

$$d_n=\left|\int_{y>0}e^{-2y}u_n^2-\int_{y>0}e^{-2y}u^2\right|.$$

Observe that $\{d_n\}_n$ is a positive sequence. We have that

$$\begin{aligned}d_n&\le\int_{y>0}e^{-2y}|u_n^2-u^2|=\int_0^T e^{-2y}|u_n^2-u^2|+\int_T^\infty e^{-2y}|u_n^2-u^2|\\&\le\int_0^T e^{-2y}|u_n^2-u^2|+e^{-2T}(\|u_n\|_{L^2}^2+\|u\|_{L^2}^2)\le\int_0^T e^{-2y}|u_n^2-u^2|+2Ce^{-2T}\end{aligned}$$

for any $T>0$. Since $H^1(0,T)$ is compactly embedded in $L^2(0,T)$, we conclude that there exists $\{\varepsilon_{n,T}\}_n$ such that $\varepsilon_{n,T}\to 0$ and

$$d_n\le\varepsilon_{n,T}+2Ce^{-2T}.$$

To conclude, for any given $\delta>0$, we can find $T>0$ such that $Ce^{-2T}<\frac{1}{2}\delta$ and subsequently $\bar{n}$ such that $\varepsilon_{n,T}\le\frac{1}{2}\delta$ for any $n\ge\bar{n}$. This implies that, for any $n\ge\bar{n}$, we have that $0\le d_n\le\delta$; that is,

$$\lim_{n\to+\infty}d_n=0\quad\Longrightarrow\quad\int_{y>0}e^{-2y}u^2=\lim_{n\to+\infty}\int_{y>0}e^{-2y}u_n^2,$$

and thus we conclude the strong convergence of the sequence $\{u_n\}_{n\in\mathbb{N}}$. □

Exploiting this compact embedding, we can show the following weighted Poincaré inequality.

Lemma A.2. *Let $a>0$ and $b\in\mathbb{R}$, then*

$$\int_{y>0}|u'|^2+(a^2-be^{-2y})u^2\ge 0$$

for any $u\in H_0^1(\mathbb{R}^+)$ as long as

$$b\le(j_{a,1})^2,$$

where $j_{a,1}$ is the first (positive) zero of the Bessel function of the first kind of order a.

Proof. The statement is equivalent to proving that

$$(j_{a,1})^2 = \inf_{u \in H_0^1(\mathbb{R}^+)} \left\{ \int_{y>0} |u'|^2 + a^2 u^2 : \int_{y>0} e^{-2y} u^2 = 1 \right\}. \tag{56}$$

The existence of a minimizer $u \in H_0^1(\mathbb{R}^+)$ follows directly from the embedding in Lemma A.1. As the functional and the constraint are even, we can assume that the minimizer u is positive. Standard regularity results imply that the function u is also smooth and strictly positive in $\mathbb{R}^+$. Let $\lambda \geq 0$ be the minimum of (56). We have that $u \in H_0^1(\mathbb{R}^+)$ is a solution of

$$\begin{cases} -u'' + (a^2 - \lambda e^{-2y})u = 0, \\ u(0) = 0, \quad u(y) > 0 \quad \text{for } y > 0. \end{cases}$$

We argue as in Lemma 3.2. We look for a solution defined by the series

$$u(y) = \sum_{n \geq 0} c_n e^{-(2n+a)y}, \quad \text{where } c_n \in \mathbb{R} \text{ for } n \in \mathbb{N}.$$

We first make some formal computations, plugging this expression directly into the equation. We find that the coefficients c_n must satisfy the following recursive relation for $n \geq 1$:

$$c_n(2n+a)^2 = c_n a^2 - c_{n-1}\lambda,$$

which is satisfied for instance by letting

$$c_n = \frac{(-1)^n}{n!\,\Gamma(n+1+a)} \left(\frac{\sqrt{\lambda}}{2} \right)^{2n+a} \quad \text{for all } n \in \mathbb{N},$$

thus leading us to the solution

$$u(y) = \sum_{n \in \mathbb{N}} \frac{(-1)^n}{n!\,\Gamma(n+1+a)} \left(\frac{\sqrt{\lambda}}{2} e^{-y} \right)^{2n+a} = J_a(\sqrt{\lambda} e^{-y}).$$

We recall that, if $a > 0$, then $J_a(0) = 0$. This gives that, for any $a > 0$,

$$\lim_{y \to +\infty} u(y) = 0.$$

One can easily check that the series does converge in $H^1(\mathbb{R}^+)$ to its sum u. We only need to ensure that

$$u(0) = 0 \quad \text{and} \quad u(y) > 0 \text{ for any } y > 0.$$

In terms of the function J_a, these conditions together mean that $\sqrt{\lambda}$ has to be the first (positive) zero for J_a; that is,

$$\sqrt{\lambda} = j_{a,1} \quad \Longleftrightarrow \quad \lambda = (j_{a,1})^2. \qquad \square$$

We can also show a similar Poincaré inequality for semi-infinite rectangles.

Lemma A.3. *For any $a > 0$ and $b \in \mathbb{R}$, we consider the semi-infinite rectangle*

$$Q_{a,b} = \left(-\tfrac{1}{2}a, \tfrac{1}{2}a\right) \times (b, +\infty)$$

and the corresponding functional space

$$H_0^1(Q_{a,b}) = \{u \in H^1(Q_{a,b}) : u = 0 \text{ on } \partial Q_{a,b}\}.$$

We have

$$\inf_{u \in H_0^1(Q_{a,b})} \left\{ \int_{Q_{a,b}} |\nabla u|^2 : \int_{Q_{a,b}} e^{-2y} u^2 = 1 \right\} = e^{2b} (j_{\pi/a,1})^2.$$

Proof. By the same compactness argument of Lemma A.1, we can show that the infimum is attained by a function $u \in H_0^1(Q_{a,b})$ which, by standard results, is also positive and smooth in $Q_{a,b}$. Up to a translation in y, the function u is then a positive solution of

$$\begin{cases} -\Delta u = \lambda e^{-2b} e^{-2y} u & \text{in } Q_{a,0}, \\ u = 0 & \text{on } \partial Q_{a,0}, \end{cases}$$

for some $\lambda \geq 0$. By separation of variables we can easily show that u is of the form

$$u(x, y) = \cos\left(\frac{\pi}{a} x\right) v(y),$$

where the new unknown function $v \in H_0^1(\mathbb{R}^+)$ solves

$$\begin{cases} -v'' + \left(\frac{\pi^2}{a^2} - \lambda e^{-2b} e^{-2y}\right) v = 0, \\ v(0) = 0, \quad v(y) > 0 \quad \text{for } y > 0. \end{cases}$$

By Lemma A.2, we conclude that

$$\lambda e^{-2b} = (j_{\pi/a,1})^2. \qquad \square$$

Acknowledgements

Verzini acknowledges support from the project Vain-Hopes within the program VALERE - Università degli Studi della Campania “Luigi Vanvitelli” and by the Portuguese government through FCT/Portugal under the project PTDC/MAT-PUR/1788/2020.

Zilio acknowledges support from the ANR via the project Indyana under grant agreement ANR-21-CE40-0008 and the project SHAPO under grant agreement ANR-18-CE40-0013.

This work was partially supported by the INdAM - GNAMPA group.

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Received 21 Feb 2022. Revised 1 Sep 2023. Accepted 21 Nov 2023.

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mathematical sciences publishers msp

ANALYSIS AND PDE
Vol. 18 (2025), No. 3, pp. 591–628
DOI: 10.2140/apde.2025.18.591

STAHL–TOTIK REGULARITY FOR CONTINUUM SCHRÖDINGER OPERATORS

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We develop a theory of regularity for continuum Schrödinger operators based on the Martin compactification of the complement of the essential spectrum. This theory is inspired by Stahl–Totik regularity for orthogonal polynomials, but requires a different approach, since Stahl–Totik regularity is formulated in terms of the potential-theoretic Green's function with a pole at ∞, logarithmic capacity, and the equilibrium measure, notions which do not extend to unbounded spectra. For any half-line Schrödinger operator with a bounded potential (in a locally L^1 sense), we prove that its essential spectrum obeys the Akhiezer–Levin condition, and moreover, that the Martin function at ∞ obeys the two-term asymptotic expansion $\sqrt{-z}+a/(2\sqrt{-z})+o(1/\sqrt{-z})$ as $z\to-\infty$. The constant a in that expansion has not appeared in the literature before; we show that it can be used to measure the size of the spectrum in a potential-theoretic sense and that it should be thought of as a renormalized Robin constant suited for semibounded sets. We prove that it enters a universal inequality $a\le\liminf_{x\to\infty}(1/x)\int_0^x V(t)\,\mathrm{d}t$, which leads to a notion of regularity, with connections to the root asymptotics of Dirichlet solutions and zero counting measures. We also present applications to decaying and ergodic potentials.

1. Introduction

The goal of this paper is to develop a theory of Stahl–Totik regularity suitable for continuum Schrödinger operators; it is natural for this topic to work in the half-line setting, so our Schrödinger operators are unbounded self-adjoint operators on $L^2((0,\infty))$, corresponding formally to

$$L_V=-\frac{\mathrm{d}^2}{\mathrm{d}x^2}+V.$$

The potential V will always be real-valued and assumed to be uniformly locally integrable, i.e.,

$$\sup_{x\ge 0}\int_x^{x+1}|V(t)|\,\mathrm{d}t<\infty \tag{1-1}$$

(in particular, 0 is a regular endpoint and $+\infty$ is a limit point endpoint in the sense of Weyl). We set the Dirichlet boundary condition at 0, so the domain of the operator is

$$D(L_V)=\{f\in L^2((0,\infty))\mid f\in W^{2,1}_{\mathrm{loc}}([0,\infty)),\,-f''+Vf\in L^2((0,\infty)),\,f(0)=0\},$$

where $W^{2,1}_{\mathrm{loc}}([0,\infty))$ denotes the set of functions such that $f\in W^{2,1}([0,x])$ for all $x<\infty$, i.e., $f''\in L^1([0,x])$ for all $x<\infty$.

Eichinger was supported by Austrian Science Fund FWF, project no. J 4138-N32 and project no. P 33885. Lukić was supported in part by NSF grant DMS–1700179.
MSC2020: primary 34L40; secondary 31C35, 35J10.
Keywords: Schrödinger operators, Stahl–Totik regularity.

The connection of orthogonal polynomials and potential theory goes back at least to [Faber 1920; Szegő 1924]. For further references on the subject we refer to [Simon 2007; Stahl and Totik 1992]. Building on the important work [Ullman 1972], Stahl and Totik developed a comprehensive theory for orthogonal polynomials for arbitrary measures with compact support in $\mathbb{C}$. It is shown that the asymptotic behavior of the orthogonal polynomials is intimately related with so-called Stahl–Totik regularity of the measure. Regularity of the measure is then used as a reference behavior in the description of many phenomena; in spectral theory, it has important consequences through the special cases of measures supported on the real line or unit circle. For instance, on the real line, the theory provides a universal inequality between the Jacobi coefficients of a compactly supported measure and the logarithmic capacity of its topological support E, and the measure is defined to be Stahl–Totik-regular if equality holds. The corresponding Jacobi matrix is then also said to be regular. This motivates the search for a similar theory for Schrödinger operators, as discussed in [Simon 2007, Section 9]. However, Stahl–Totik regularity is built on potential-theoretic notions, such as Green's functions on the domain $\Omega = \widehat{\mathbb{C}} \setminus \mathsf{E}$ with the pole at ∞, logarithmic capacity, and equilibrium measures — objects which are undefined for unbounded sets E, and therefore not applicable to continuum Schrödinger operators. For this reason, even the correct objects and extremal principles were not identified until now.

In this paper, we develop the corresponding theory for Schrödinger operators. Martin functions [1941] (see also [Armitage and Gardiner 2001]) serve as the counterpart of Green's functions, corresponding to boundary points $z_0 \in \partial\Omega$ instead of internal points $z_0 \in \Omega$; but whereas the Green's function is defined with an explicit logarithmic singularity at z_0, the existence and behavior of Martin functions is more varied. If $\mathsf{E} \subset \mathbb{R}$ is a closed unbounded set, ∞ is a boundary point of the Denjoy domain $\Omega = \mathbb{C} \setminus \mathsf{E}$. If this domain is Greenian, associated to the boundary point ∞ is a cone of dimension 1 or 2 of positive harmonic functions in Ω which are bounded on bounded sets and vanish at every Dirichlet-regular point of E. The cone is spanned by the minimal Martin functions at ∞ [Akhiezer and Levin 1960; Ancona 1979; Benedicks 1980; Gardiner and Sjödin 2009]. Moreover, if $\inf \mathsf{E} > -\infty$, the cone is of dimension 1, and the Martin function at ∞ is determined uniquely up to normalization; we denote it by M_{E} and simply call it the Martin function from now on.

The Akhiezer–Levin condition for semibounded sets (sets with $\inf \mathsf{E} > -\infty$) is

$$\lim_{z \to -\infty} \frac{M_{\mathsf{E}}(z)}{\sqrt{-z}} > 0 \tag{1-2}$$

(by general principles, the limit exists with a value in $[0, \infty)$). This is the semibounded version of a condition originally considered in [Akhiezer and Levin 1960] for arbitrary $\mathsf{E} \subset \mathbb{R}$; see also [Yuditskii 2020, Remark 1.13]. For sets obeying (1-2), we will normalize the Martin function so that the limit in (1-2) is equal to 1.

For a potential bounded in the sense (1-1), the spectrum $\sigma(L_V)$ is a closed subset of $\mathbb{R}$ bounded below but not above, so the above definitions are applicable. It will be noted that isolated points of the set don't affect the Martin function, so we can equally well use $\mathsf{E} = \sigma_{\mathrm{ess}}(L_V)$ in what follows (more generally, $M_{\mathsf{E}_1} = M_{\mathsf{E}_2}$ if the symmetric difference of E_1 and E_2 is a polar set).

In spectral theory, Martin functions first appear implicitly, in the classical work [Marchenko and Ostrovskii 1975] classifying the spectra of periodic Schrödinger operators. In this work, the discriminant of a 1-periodic operator is expressed in the form $\Delta(z) = 2\cos(\Theta(z))$, and it can be recognized that $\operatorname{Im}\Theta(z)$ is the Martin function at ∞ for the periodic spectrum. The explicit use of Martin functions in spectral theory starts with works of Yuditskii and coauthors [Sodin and Yuditskii 1995; Damanik and Yuditskii 2016; Eichinger et al. 2019], through inverse spectral-theoretic studies associated to Dirichlet-regular spectra obeying a Widom condition and finite gap length conditions.

In contrast to the previous works, our first theorem is a set of universal properties of the spectra of Schrödinger operators obeying (1-1); note that a boundedness condition such as (1-1) is essential for the following theory, since potentials going to $-\infty$ or $+\infty$ can give spectrum equal to $\mathbb{R}$ or spectrum which is a polar set.

Theorem 1.1. *For any potential V obeying* (1-1) *and $\mathsf{E} = \sigma_{\mathrm{ess}}(L_V)$, the domain $\Omega = \mathbb{C} \setminus \mathsf{E}$ is Greenian, ∞ is a Dirichlet-regular point for Ω, Ω obeys the Akhiezer–Levin condition, and there exists $a_{\mathsf{E}} \in \mathbb{R}$ such that the Martin function has the asymptotic behavior*

$$M_{\mathsf{E}}(z) = \operatorname{Re}\left(\sqrt{-z} + \frac{a_{\mathsf{E}}}{2\sqrt{-z}}\right) + o\left(\frac{1}{\sqrt{|z|}}\right) \tag{1-3}$$

as $z \to \infty$, $\arg z \in [\delta, 2\pi - \delta]$ for any $\delta > 0$.

Each of the conclusions of this theorem is strictly stronger than the previous; we will point out examples in Section 2. In particular, the second term of the expansion (1-3) is not an automatic property of Akhiezer–Levin sets, but rather an added feature corresponding to spectra of Schrödinger operators. It should be emphasized that spectra of Schrödinger operators with bounded potentials can be very thin in the sense that they can even have zero Hausdorff dimension [Damanik et al. 2017a] and zero lower box counting dimension [Damanik et al. 2019], while our result is a universal "thickness" result in the perspective of the Martin function.

In the references given above, the Martin function was used in spectral theory as a positive harmonic function in Ω that vanishes on the boundary. In fact, Martin theory provides a whole kernel $M(z,x)$ on $\Omega \times (\widehat{\Omega} \setminus \{z_*\})$, where $\widehat{\Omega}$ denotes the Martin compactification of Ω and $z_* \in \Omega$ is a normalization point. If $\partial_1^M \Omega$ denotes the so-called minimal Martin boundary of Ω, then for every positive harmonic function h on Ω there exists a unique finite measure ν such that

$$h(z) = \int_{\partial_1^M \Omega} M(z,x)\,\mathrm{d}\nu(x).$$

We will provide more details and precise definitions in Section 2. It is new to combine this theory with the spectral theory of unbounded self-adjoint operators and this was crucial for the proof of Theorem 1.1.

It is crucial that Theorem 1.1 associates to the essential spectrum E the real-valued constant a_{E}, which will serve as a substitute for the Robin constant from potential theory. Expansions of the form (1-3) have previously appeared in the spectral theory literature [Marchenko and Ostrovskii 1975] only under strong a priori assumptions on the spectrum. Namely, the set E is closed so it can be written in the form

$$\mathsf{E} = [b_0, \infty) \setminus \bigcup_{j=1}^{N} (a_j, b_j), \tag{1-4}$$

where j indexes the "gaps", i.e., connected components of $[b_0, \infty) \setminus \mathsf{E}$, and N is finite or ∞. If $\sum_j (b_j - a_j) < \infty$, the Martin function has an expansion (1-3) with $a_{\mathsf{E}} = b_0 + \sum_j (a_j + b_j - 2c_j)$, where c_j denotes the (unique) location of the maximum of the restriction of M_{E} to the interval $[a_j, b_j]$ (see Lemma 6.2) by harmonic/complex-theoretic arguments. Instead, our Theorem 1.1 applies even when the spectrum E is very thin and this is not a purely complex-theoretic result; its proof is a combination of spectral-theoretic arguments and the theory of the Martin boundary of Denjoy domains.

The renormalized Robin constant a_{E} obeys a decreasing property on the spectra of Schrödinger operators, so it should be interpreted as an inverse measure of the size of E. For instance, our next result is a universal inequality involving a_{E}, which should be seen as a *lower* bound on the size of the essential spectrum:

Theorem 1.2. *If V is a potential obeying* (1-1) *and* $\mathsf{E} = \sigma_{\text{ess}}(L_V)$, *then*

$$a_{\mathsf{E}} \le \liminf_{x\to\infty} \frac{1}{x} \int_0^x V(t)\,\mathrm{d}t. \tag{1-5}$$

The perspective on a_{E} as an inverse measure of the size of E will be most explicitly illustrated later, in the proof of Theorem 1.12, which will use the argument that if $\mathsf{E} \subset [0, \infty)$ and $a_{\mathsf{E}} \le a_{[0,\infty)}$, then $\mathsf{E} = [0, \infty)$. This kind of argument wasn't available before in this generality, because there was no known quantity with the correct properties: any quantity based on Lebesgue measure or dimension would sometimes give infinite or trivial values.

For any $z \in \mathbb{C}$, the Dirichlet eigensolution is the solution of the initial value problem

$$-\partial_x^2 u(x, z) + V(x)u(x, z) = zu(x, z), \quad u(0, z) = 0, \ (\partial_x u)(0, z) = 1.$$

Our next result is that the Martin function provides a universal lower bound on the growth rate of the Dirichlet solution.

Theorem 1.3. *If V is a potential obeying* (1-1) *and* $\mathsf{E} = \sigma_{\text{ess}}(L_V)$, *then*

$$M_{\mathsf{E}}(z) \le \liminf_{x\to\infty} \frac{1}{x} \log|u(x, z)| \quad \forall z \in \mathbb{C} \setminus [\min \mathsf{E}, \infty).$$

Exclusion of $[\min \mathsf{E}, \infty)$ in Theorem 1.3 is necessary because, for $z \in (\min \mathsf{E}, \infty)$, by Sturm oscillation theory [Simon 2005], the Dirichlet solution has infinitely many zeros.

Definition 1.4. The potential V is regular if

$$a_{\mathsf{E}} = \lim_{x\to\infty} \frac{1}{x} \int_0^x V(t)\,\mathrm{d}t. \tag{1-6}$$

Of course, due to (1-5), this is equivalent to requiring

$$a_{\mathsf{E}} \ge \limsup_{x\to\infty} \frac{1}{x} \int_0^x V(t)\,\mathrm{d}t.$$

In our next theorem, we will characterize regularity in terms of root asymptotics for the Dirichlet eigensolutions. We say that a property holds a.e. on E with respect to harmonic measure if it holds away from a set $A \subset \mathsf{E}$ such that $\omega_{\mathsf{E}}(A, z_0) = 0$, where $\omega_{\mathsf{E}}(\,\cdot\,, z_0)$ denotes the harmonic measure of Ω evaluated at some $z_0 \in \Omega$. This condition is independent of the choice of $z_0 \in \Omega$ since the harmonic measures are mutually absolutely continuous.

Theorem 1.5. *If V is a potential obeying* (1-1) *and* $\mathsf{E} = \sigma_{\mathrm{ess}}(L_V)$, *the following are equivalent*:

(i) *V is regular.*

(ii) *For every Dirichlet-regular* $z \in \mathsf{E}$, $\limsup_{x\to\infty} \frac{1}{x}\log|u(x,z)| \le 0$.

(iii) *For a.e.* $z \in \mathsf{E}$ *with respect to harmonic measure*, $\limsup_{x\to\infty} \frac{1}{x}\log|u(x,z)| \le 0$.

(iv) *There exists* $z \in \mathbb{C}_+$ *such that* $\limsup_{x\to\infty} \frac{1}{x}\log|u(x,z)| \le M_{\mathsf{E}}(z)$.

(v) *For all* $z \in \mathbb{C}$, $\limsup_{x\to\infty} \frac{1}{x}\log|u(x,z)| \le M_{\mathsf{E}}(z)$.

(vi) $\lim_{x\to\infty} \frac{1}{x}\log|u(x,z)| = M_{\mathsf{E}}(z)$ *uniformly on compact subsets of* $\mathbb{C} \setminus [\min \mathsf{E}, \infty)$.

Since (v) or (vi) trivially imply (iv), part (iv) is of interest as a criterion for establishing regularity of V, whereas (v), (vi) are of interest as consequences of regularity. Similarly, (ii) implies (iii), so (ii) is of interest as a consequence of regularity and (iii) as a condition for regularity. Instead of conditions (ii) and (iii), it would be customary to state the single condition that the inequality holds quasi-everywhere; this is between our conditions since the set of Dirichlet-irregular points is polar and polar sets have harmonic measure 0. The benefit of (ii) is that it can be used pointwise (in particular, for a Dirichlet-regular set E, the inequality holds everywhere on E). More importantly, the benefit of (iii) is that the characterization in terms of harmonic measure will be essential for our proof of Theorem 1.8 below.

There are no previous results on Stahl–Totik regularity for continuum Schrödinger operators, even in special cases. This topic was previously considered by Simon [2007, Section 9], who formulated several conjectures. The first is that for semibounded spectra that are "close" to $[0,\infty)$ (e.g., $[0,\infty) \setminus \mathsf{E}$ of finite Lebesgue measure) there should be a version of equilibrium measure ν_{E} and equilibrium potential Φ_{E}, characterized by several properties including a normalization $\Phi_{\mathsf{E}}(z) \sim \mathrm{Re}(\sqrt{-z})(1+o(1))$ as $z \to -\infty$. It was suggested that regularity for continuum Schrödinger operators can be defined by the condition $\limsup_{x\to\infty} \frac{1}{x}\log|u(x,z)| = \Phi_{\mathsf{E}}(z)$, and that this would have equivalent characterizations similar to the orthogonal polynomial case. Our work does not use a finite Lebesgue measure assumption for $[0,\infty) \setminus \mathsf{E}$, so it solves these conjectures in a far greater generality than they were even previously conjectured. Moreover, our work provides the correct potential-theoretic interpretation for the function Φ_{E} (now understood as the Martin function M_{E}), and that interpretation is crucial in the proofs.

Simon also conjectured that the asymptotics $\Phi_{\mathsf{E}}(z) = \mathrm{Re}(\sqrt{-z})(1+o(1))$ should improve to the asymptotic behavior $\mathrm{Re}\sqrt{-z} + o(1)$; this is motivated by the asymptotic behavior $\sqrt{-z} + o(1)$ of m-functions, proved in [Atkinson 1981]. While that asymptotic statement for individual m-functions cannot be improved for locally integrable potentials, we discover that due to averaging effects, the asymptotic behavior of our quantities improves even more, to the form (1-3). This discovery of (1-3) has enabled us to introduce the constant a_{E}, which was not previously conjectured, and to use it for the robust general definition of regularity given above.

We also define the correct "equilibrium measure" which will be related to a deterministic density of states. The Martin function can be extended to a subharmonic function on $\mathbb{C}$, so it has a Riesz measure, given by

$$\rho_{\mathsf{E}} = \frac{1}{2\pi}\Delta M_{\mathsf{E}},$$

which we will call the Martin measure of the set E. Conversely, the Martin function has a Hadamard representation of the form

$$M_{\mathsf{E}}(z) = M_{\mathsf{E}}(z_*) + \int_{\mathsf{E}} \log\left|1 - \frac{z - z_*}{t - z_*}\right| d\rho_{\mathsf{E}}(t),$$

where $z_* < \min \mathsf{E}$ is an arbitrary normalization point. The Martin measure will serve the same role in this theory that the logarithmic equilibrium measure serves for orthogonal polynomials. However, ρ_{E} is not defined with respect to any extremal property (and it is not even a finite measure), so different proofs will be needed in the current setting.

For any $x > 0$, let ρ_x denote the zero counting measure for $u(x, z)$ divided by x,

$$\rho_x = \frac{1}{x} \sum_{z:u(x,z)=0} \delta_z. \tag{1-7}$$

Note that ρ_x is the Riesz measure of $\frac{1}{x} \log|u(x, z)|$. The limit of ρ_x as $x \to \infty$, when it exists, is interpreted as a deterministic density of states associated to V. The convergence of measures will be understood in the weak-$*$ sense, i.e., when integrated against continuous functions with compact support. The Martin measure and the zero counting measures are related by the following pair of results:

Theorem 1.6. *Assume V is regular. Then ρ_x converges to ρ_{E} as $x \to \infty$, in the weak-$*$ sense.*

The following is a continuum analog of a result of [Stahl and Totik 1992]:

Theorem 1.7. *Assume that V obeys* (1-1) *and let μ be a maximal spectral measure for L_V. Suppose that ρ_x converges to ρ_{E} as $x \to \infty$ in the weak-$*$ sense. Then, either V is regular, or there exists a polar Borel set X such that $\mu(\mathbb{R} \setminus X) = 0$.*

Of course, the statement $\mu(\mathbb{R} \setminus X) = 0$ can be restated in the language of the Borel functional calculus as $\chi_{\mathbb{R}\setminus X}(L_V) = 0$.

So far, we have seen that regularity of V can be established from the root asymptotics of Dirichlet solutions. The next theorem shows that it can be established from spectral properties of the operator. It is the continuum counterpart of a theorem of [Widom 1967].

Theorem 1.8. *Let μ be a maximal spectral measure for L_V. If $\omega_{\mathsf{E}}(\,\cdot\,, z_0)$ for some $z_0 \in \mathbb{C} \setminus \mathsf{E}$ is absolutely continuous with respect to μ, then V is regular.*

This theory leads to several new results even for the special case of half-line essential spectrum $[0, \infty)$; we present those as our first applications. If V is a decaying potential in the sense

$$\lim_{x\to\infty} \int_x^{x+1} |V(t)|\, dt = 0 \tag{1-8}$$

then $\mathsf{E} = \sigma_{\mathrm{ess}}(L_V) = [0, \infty)$ by [Blumenthal 1898; Weyl 1909]. It follows that $M_{\mathsf{E}}(z) = \operatorname{Re} \sqrt{-z}$. In particular, $a_{\mathsf{E}} = 0$, so immediately from the definition:

Corollary 1.9. *If V is a decaying potential in the sense* (1-8), *then V is regular with $\sigma_{\mathrm{ess}}(L_V) = [0, \infty)$.*

Since harmonic measure for $\mathsf{E} = [0, \infty)$ is mutually absolutely continuous with $\chi_{(0,\infty)}(x)\,\mathrm{d}x$, the following is an immediate consequence of Theorem 1.8:

Corollary 1.10. *Assume that V obeys* (1-1) *and denote by μ a maximal spectral measure for L_V. Denote by $\mathrm{d}\mu = f\,\mathrm{d}x + \mathrm{d}\mu_s$ the Radon–Nikodym decomposition of μ with respect to Lebesgue measure. If $\sigma_{\mathrm{ess}}(L_V) = [0, \infty)$ and $f(x) > 0$ for Lebesgue-a.e. $x > 0$, then V is regular.*

More generally, a version of Corollary 1.10 holds, whenever the harmonic measure for the domain $\mathbb{C} \setminus \mathsf{E}$ is absolutely continuous with respect to the Lebesgue measure $\chi_{\mathsf{E}}(x)\,\mathrm{d}x$. In particular, it holds for finite gap sets (i.e., when N is finite in (1-4)) and regular Parreau–Widom sets. If E is Dirichlet-regular, the Green's function $G_{\mathsf{E}}(z, z_0)$, for $z_0 < \min \mathsf{E}$, has exactly one critical point $c_j \in (a_j, b_j)$ in each gap. If, in addition, the critical values of $G_{\mathsf{E}}(z, z_0)$ are summable, i.e.,

$$\sum_{j=1}^{\infty} G_{\mathsf{E}}(c_j, z_0) < \infty,$$

we call E a regular Parreau–Widom set. In fact, the harmonic measure for the domain $\mathbb{C} \setminus \mathsf{E}$ is absolutely continuous with respect to the Lebesgue measure if and only if E satisfies a certain sector condition [Eremenko and Yuditskii 2012, Theorem 4]. We will describe this generalization in Section 6.

Sparse potentials are not covered by Corollary 1.9 or Corollary 1.10, but nonetheless provide additional examples of regular potentials:

Example 1.11. Let $W \in L^1((0, \infty))$ be compactly supported, $W \geq 0$, let $x_n \geq 0$ be an increasing sequence such that $x_{n+1} - x_n \to \infty$ as $n \to \infty$ and $V(x) = \sum_n W(x - x_n)$. Then V is regular with $\sigma_{\mathrm{ess}}(L_V) = [0, \infty)$.

The sparse potentials from Example 1.11 are not decaying in the sense (1-8), so Corollary 1.9 does not have a converse; sparse potentials have purely singular spectrum by [Pearson 1978; Last and Simon 1999], so Corollary 1.10 does not have a converse.

However, we prove that Corollary 1.9 has the following partial converse; we have already described Theorem 1.1 as a universal thickness result about the spectrum, and the following result similarly guarantees presence of essential spectrum.

Theorem 1.12. *Assume that V obeys* (1-1) *and that $\sigma_{\mathrm{ess}}(L_V) \subset [0, \infty)$. Then*:

(a) $\liminf_{x\to\infty} \frac{1}{x} \int_0^x V(t)\,\mathrm{d}t \geq 0$.

(b) *If* $\liminf_{x\to\infty} \frac{1}{x} \int_0^x V(t)\,\mathrm{d}t \leq 0$, *then* $\sigma_{\mathrm{ess}}(L_V) = [0, \infty)$.

(c) *If* $\limsup_{x\to\infty} \frac{1}{x} \int_0^x V(t)\,\mathrm{d}t \leq 0$, *then* $\sigma_{\mathrm{ess}}(L_V) = [0, \infty)$ *and V is regular.*

Part (a) can also be established by other means, but we include it for completeness. Parts (b) and (c) generalize known results giving sufficient conditions for $\sigma_{\mathrm{ess}}(L_V) = [0, \infty)$. In particular, Damanik and Remling [2007, Theorem 1.2] showed that $\sigma_{\mathrm{ess}}(L_{\pm V}) \subset [0, \infty)$ implies $\sigma_{\mathrm{ess}}(L_V) = [0, \infty)$. Part (b) of our theorem is a strict generalization of that result; strict because it applies, e.g., to the sparse potentials of Example 1.11 where [loc. cit.] does not (for a positive sparse potential V, $\min \sigma_{\mathrm{ess}}(L_{-V}) < 0$), and a generalization because $\sigma_{\mathrm{ess}}(L_{-V}) \subset [0, \infty)$ implies $\limsup_{x\to\infty} \frac{1}{x} \int_0^x V(t)\,\mathrm{d}t \leq 0$ (by (a) applied to $-V$), so our parts (b), (c) also apply to the potentials in [Damanik and Remling 2007]. In particular, $\sigma_{\mathrm{ess}}(L_{\pm V}) \subset [0, \infty)$ implies that V is regular and $\sigma_{\mathrm{ess}}(L_V) = [0, \infty)$.

In the theory of Jacobi matrices, a result of [Simon 2009] shows that a regular Jacobi matrix with essential spectrum $[-2, 2]$ obeys a Cesàro–Nevai condition. The analog for Schrödinger operators is false — the continuum setting allows rapid oscillations which can break any Cesàro-type decay in an L^1 sense:

Example 1.13. The potential defined piecewise by $V(x) = (-1)^{\lfloor 2n(x-n) \rfloor}$ on $x \in [n-1, n)$ for an integer n is regular with $\sigma_{\mathrm{ess}}(L_V) = [0, \infty)$, but $\frac{1}{x}\int_0^x |V(t)|\,\mathrm{d}t \not\to 0$ as $x \to \infty$.

All objects considered above are deterministic (defined only in terms of a single half-line potential V), but for ergodic families of Schrödinger operators, they can be recognized almost surely as ergodic notions such as the Lyapunov exponent and the ergodic density of states, so our results can be interpreted in the ergodic setting. In the ergodic setting, it is natural to work with whole line potentials: let us consider a family $(V_\eta)_{\eta \in S}$ of real-valued potentials on $\mathbb{R}$ on a probability space S which is metrically transitive with respect to a group of measure-preserving transformations τ_y such that $V_{\tau_y \eta}(x) = V_\eta(x-y)$ and such that any measurable subset A of S which is invariant under all τ_y has probability 0 or 1. The group of transformations can be continuous (indexed by $y \in \mathbb{R}$) or discrete (indexed by $y \in \ell\mathbb{Z}$ for some $\ell > 0$); the former case includes almost periodic Schrödinger operators and the latter case includes many Anderson-type models studied in the literature [Kirsch 1985; Damanik et al. 2002], including those with a periodic background. We also assume that V_η almost surely obeys

$$\sup_{x \in \mathbb{R}} \int_x^{x+1} |V_\eta(t)|\,\mathrm{d}t < \infty; \tag{1-9}$$

in fact, much of the literature on ergodic Schrödinger operators is focused on bounded potentials. Let us denote by H_{V_η} the self-adjoint operators on $L^2(\mathbb{R})$ given by

$$D(H_{V_\eta}) = \{ f \in L^2(\mathbb{R}) \mid f \in W^{2,1}_{\mathrm{loc}}(\mathbb{R}),\ -f'' + V_\eta f \in L^2(\mathbb{R}) \}$$

and recall the basic properties of this ergodic family (see the textbooks [Carmona and Lacroix 1990; Pastur and Figotin 1992; Cycon et al. 1987] and the paper [Kirsch 1985] addressing some nuances for locally L^1 ergodic potentials with a discrete group of transformations). There is an almost sure spectrum $\mathsf{E} \subset \mathbb{R}$,

$$\mathsf{E} = \sigma(H_{V_\eta}) = \sigma_{\mathrm{ess}}(H_{V_\eta}) \quad \text{for a.e. } \eta \in S,$$

and the potentials V_η have an almost sure Birkhoff average $\mathbb{E}(V) \in \mathbb{R}$,

$$\mathbb{E}(V) = \lim_{x \to \infty} \frac{1}{x} \int_0^x V_\eta(t)\,\mathrm{d}t \quad \text{for a.e. } \eta \in S.$$

If L_{V_η} denotes the half-line operator corresponding to the restriction of V_η to $[0, \infty)$, then $\mathsf{E} = \sigma_{\mathrm{ess}}(L_{V_\eta})$ almost surely, so as a direct consequence of our deterministic results, E corresponds to a Martin function with an expansion (1-3), and

$$a_{\mathsf{E}} \le \mathbb{E}(V). \tag{1-10}$$

This inequality is new; several cases of the equality $a_{\mathsf{E}} = \mathbb{E}(V)$ are well known and among the most studied classes of ergodic Schrödinger operators (periodic, reflectionless almost periodic with finite gap length), and we can now interpret this through the fact that the corresponding potentials are regular.

In the ergodic setting, two central objects are the Lyapunov exponent $\gamma(z)$ and the density of states $d\rho$; both are almost sure ergodic averages of important spectral quantities. The transfer matrix $T_\eta(x, z)$ is the 2×2-matrix-valued solution of the initial value problem

$$(\partial_x T_\eta)(x, z) = \begin{pmatrix} 0 & V_\eta(x) - z \\ 1 & 0 \end{pmatrix} T_\eta(x, z), \quad T_\eta(0, z) = I,$$

and the corresponding Dirichlet solution is $u_\eta(x, z) = (T_\eta)_{2,1}(x, z)$. If $\rho_{\eta,x}$ denotes the measure corresponding to u_η as in (1-7), then

$$\gamma(z) = \lim_{x\to+\infty} \frac{1}{x} \log\|T_\eta(x, z)\| \quad \text{for a.e. } \eta \in S, \tag{1-11}$$

and

$$d\rho = \operatorname*{w-lim}_{x\to+\infty} d\rho_{\eta,x} \quad \text{for a.e. } \eta \in S.$$

Thus Theorem 1.5, specialized to the ergodic setting, immediately gives the following:

Corollary 1.14. *For any ergodic family of Schrödinger operators obeying* (1-9), *the following are equivalent*:

(i) $a_{\mathsf{E}} = \mathbb{E}(V)$.

(ii) *For every Dirichlet-regular* $z \in \mathsf{E}$, *we have* $\gamma(z) = 0$.

(iii) *For almost every* $z \in \mathsf{E}$ *with respect to harmonic measure*, *we have* $\gamma(z) = 0$.

(iv) *For all* $z \in \mathbb{C}_+$, *we have* $\gamma(z) \le M_{\mathsf{E}}(z)$.

(v) *For all* $z \in \mathbb{C} \setminus \mathsf{E}$, *we have* $\gamma(z) \le M_{\mathsf{E}}(z)$.

(vi) $\gamma(z) = M_{\mathsf{E}}(z)$ *for all* $z \in \mathbb{C} \setminus [\min \mathsf{E}, \infty)$.

We say that a family of ergodic Schrödinger operators is regular if one (and therefore all) of the statements of Corollary 1.14 holds. Although this notion is new, let us point out that it contains several of the most well-studied families of almost periodic Schrödinger operators known to have zero Lyapunov exponent on the spectrum, such as quasiperiodic operators at small coupling [Eliasson 1992; Damanik and Goldstein 2014; Damanik et al. 2016; 2017b; 2017c] and limit-periodic potentials superexponentially well-approximated by periodic operators [Chulaevsky 1981; Pastur and Tkachenko 1984; 1988; Fillman and Lukic 2017]. In fact, the question of when the Lyapunov exponent is zero or positive on E is one of the basic questions for an almost periodic family of operators and an important dichotomy in their study; this is especially well-studied in the setting of discrete Schrödinger operators; see, e.g., [Marx and Jitomirskaya 2017; Damanik 2017; Avila 2015]. In inverse spectral theory one considers reflectionless Schrödinger operators on Dirichlet-regular Widom spectra with the DCT property and associated solutions of the KdV equation [Damanik and Goldstein 2016; Egorova 1993; 1994; Sodin and Yuditskii 1995; Gesztesy and

Yuditskii 2006; Binder et al. 2018; Eichinger et al. 2019]; those operators have zero Lyapunov exponent on the spectrum so they are regular in the sense of this paper.

For a 1-periodic potential V, it is well known that the discriminant has an asymptotic expansion at ∞ whose coefficients are equal to averages of differential polynomials in V (under the appropriate regularity assumptions on V). The first of those equalities, rewritten for the Martin function, give the equality $a_{\mathsf{E}} = \int_0^1 V(x)\,\mathrm{d}x$. This can now be interpreted through the fact that periodic potentials are regular.

For an almost periodic potential V, Johnson and Moser [1982] introduced the spatial average of m-functions, whose real part is the Lyapunov exponent γ. Their construction relies heavily on almost periodicity through compactness of the hull, so their methods would not extend to our setting; Johnson and Moser [1982] noted as a consequence of their results, the spectrum of any almost periodic Schrödinger operator is not a polar set (i.e., Ω is Greenian), but further consequences of Theorem 1.1 were not previously known even in the almost periodic case.

The next theorem is a specialization of Theorems 1.6, 1.7 to the ergodic setting:

Theorem 1.15. *Let $(V_\eta)_{\eta\in S}$ be an ergodic family of Schrödinger operators obeying* (1-9). *If this ergodic family is regular, then its density of states ρ is equal to the Martin measure ρ_{E}. Conversely, if $\rho = \rho_{\mathsf{E}}$, then either the ergodic family is regular, or for a.e. η, the maximal spectral measure μ_η is supported on a polar set.*

Although positive Lyapunov exponents don't always correspond to localization, we can now prove that they always correspond to very thin spectral type. This is the analog of a Jacobi matrix result which has been described as the ultimate Pastur–Ishii theorem.

Theorem 1.16. *Let γ denote the Lyapunov exponent associated to the ergodic family $(V_\eta)_{\eta\in S}$ and let μ_η denote a maximal spectral measure for H_{V_η}. Let $Q \subset \mathbb{R}$ be the Borel set of $\lambda \in \mathbb{R}$ with $\gamma(\lambda) > 0$. Then for a.e. $\eta \in S$, there exists a polar set X_η such that $\mu_\eta(Q \setminus X_\eta) = 0$. In particular, the measure $\chi_Q\,\mathrm{d}\mu_\eta$ is of local Hausdorff dimension zero.*

It is known in great generality [Damanik et al. 2002] that one-dimensional random Schrödinger operators give rise to positive Lyapunov exponent throughout the spectrum. In particular, random Schrödinger operators provide examples of nonregular operators.

Throughout this paper, we follow the dominant literature by working with locally integrable potentials; we expect that the theory presented here can be extended to potentials which are in the negative Sobolev space $H^{-1}([0, x])$ for $x < \infty$, with an appropriate uniform bound replacing (1-1), and that it can be adapted to certain other classes of one-dimensional differential operators.

We expect that the notion of regularity introduced in this paper will pave the way to new kinds of results on Schrödinger operators which were previously beyond reach. For instance, regularity of measures is used as the standard reference behavior in the study of the local distribution of zeros of orthogonal polynomials, through so-called clock behavior and universality [Lubinsky 2009; Máté et al. 1991; Simon 2008]; we conjecture that similar results hold for regular Schrödinger operators. Without regularity, the only currently available Schrödinger result is inevitably more limited in scope to certain perturbations of periodic Schrödinger operators [Maltsev 2010]. Likewise, logarithmic capacity is used to formulate the generalization of the Shohat–Nevai theorem to measures whose essential supports are regular Parreau–Widom sets [Christiansen 2012]; the Schrödinger counterpart of this result couldn't even be formulated without the renormalized Robin constant a_{E}. We expect the theory in this paper to be an integral part of its eventual proof, and of the broader program of investigating sum rules for Schrödinger operators with regular Parreau–Widom essential spectra.

2. The Martin function and Akhiezer–Levin sets

In this section we consider in more detail the general Martin theory for Denjoy domains $\Omega = \mathbb{C} \setminus \mathsf{E}$ with $\min \mathsf{E} = b_0 > -\infty$. Clearly, we have in mind the application that E is the essential spectrum of some continuum Schrödinger operator, L_V, where V satisfies (1-1).

Recall that the capacity of a Borel set A is defined by

$$\operatorname{Cap}(A) = \sup\{\operatorname{Cap}(K) : K \text{ compact}, \ K \subset A\}$$

and we call a Borel set, A, polar, if $\operatorname{Cap}(A) = 0$. Moreover, a property holds quasi-everywhere on a set B if there exists a polar set A such that the property holds on $B \setminus A$. We start with a discussion of the Green's function $G_{\mathsf{E}}(z, z_0)$, $z_0 \in \Omega$. For standard references on potential theory see [Armitage and Gardiner 2001; Ransford 1995; Garnett and Marshall 2005]. If $z_0 \in \mathbb{R}$, then $G_{\mathsf{E}}(z, z_0)$ is symmetric, that is, $G_{\mathsf{E}}(\bar{z}, z_0) = G_{\mathsf{E}}(z, z_0)$. Let us fix $z_0 < b_0$. Then there exists a comb domain

$$\Pi_{z_0} = \{x + iy : 0 < x < \pi, \ y > s(x)\}, \tag{2-1}$$

where s is a positive upper semicontinuous function, bounded from above, and vanishes Lebesgue-a.e., and a conformal mapping $\theta_{z_0} : \mathbb{C}_+ \to \Pi_{z_0}$ such that

$$G_{\mathsf{E}}(z, z_0) = \operatorname{Im} \theta_{z_0}(z), \quad z \in \mathbb{C}_+. \tag{2-2}$$

(Such a representation was proved in [Eremenko and Yuditskii 2012] in the case that E is compact and $z_0 = \infty$; by a simple transformation $\lambda = 1/(z_0 - z)$ this yields a corresponding representation for the current setting). Note that $\theta_{z_0}(b_0) = i \limsup_{u \to 0} s(u)$ and $\theta_{z_0}(\infty) = i \limsup_{u \to \pi} s(u)$. Moreover, harmonic measure $\omega_{\mathsf{E}}(\,\cdot\,, z_0)$ corresponds to the pullback of the normalized (by π) Lebesgue measure on the base of the comb. The mapping can be extended by symmetry to $\mathbb{C} \setminus [b_0, \infty)$ such that (2-2) still holds there. In fact, any such function s leads to a Green's function of a certain domain.

The Martin kernel normalized at $z_* < b_0$ is defined on $\Omega \times (\Omega \setminus \{z_*\})$ by

$$M_{\mathsf{E}}(z, z_0) = \frac{G_{\mathsf{E}}(z, z_0)}{G_{\mathsf{E}}(z_*, z_0)}. \tag{2-3}$$

The *Martin compactification* $\widehat{\Omega}$ is the smallest metric compactification of Ω such that $M_{\mathsf{E}}(z, \cdot\,)$ can be continuously extended to the boundary $\partial^M \Omega = \widehat{\Omega} \setminus \Omega$ for each z. We will also write $M_{\mathsf{E}}(z, z_0)$ for the extended function. Note that by the Harnack principle the family $\{M_{\mathsf{E}}(z, z_0)\}$ is precompact in the space of positive harmonic functions equipped with uniform convergence on compacts. We call a positive harmonic function, M, *minimal* if any harmonic function, h, which satisfies $0 \le h \le M$, is a multiple of M, i.e., $h = cM,\ c \geq 0$. Finally, let $\partial_1^M \Omega \subset \partial^M \Omega$ denote the subset of the Martin boundary, which consists of minimal harmonic functions. In this case, for every positive harmonic function h, there exists a unique finite measure ν such that

$$h(z) = \int_{\partial_1^M \Omega} M_{\mathsf{E}}(z, x)\, \mathrm{d}\nu(x), \qquad h(z_*) = \nu(\partial_1^M \Omega). \tag{2-4}$$

In general $\partial_1^M \Omega$ can be quite abstract, but the situation is rather intuitive for Denjoy domains. In [Gardiner and Sjödin 2009, Theorem 6] it is shown that there exists a map $\pi : \partial_1^M \Omega \to \mathsf{E} \cup \{\infty\}$ such that for every $x \in \mathsf{E} \cup \{\infty\}$, $\#\pi^{-1}(\{x\})$ is either 1 or 2, depending on how "thin" $\mathbb{R} \cap \Omega$ is at x. To state this precisely we need some definitions. If A is a subset of the Martin boundary $\partial^M \Omega = \widehat{\Omega} \setminus \Omega$, then we say a property, P, holds near A if there is a Martin-neighborhood $A \subset W$ such that P holds on $W \cap \Omega$. Then, for $A \subset \widehat{\Omega}$ and a positive superharmonic function h on Ω we define the reduced function

$$R_h^A(x) = \inf\{u(x) : u \geq 0 \text{ is superharmonic, } h \leq u \text{ on } A \cap \Omega \text{ and } h \leq u \text{ near } A \cap \partial^M \Omega\} \tag{2-5}$$

and $\widehat{R}_h^A$ denotes its lower semicontinuous regularization. A set $A \subset \Omega$ is said to be minimally thin at $y \in \partial_1^M \Omega$ if

$$\widehat{R}^A_{M_{\mathsf{E}}(\,\cdot\,, y)} \neq M_{\mathsf{E}}(\,\cdot\,, y).$$

Then $\#\pi^{-1}(\{x\}) = 2$ if and only if there is $y \in \pi^{-1}(\{x\})$ such that $\Omega \cap \mathbb{R}$ is minimally thin at y. Informally, if E is sufficiently "dense" at x, then Ω locally splits into the two half-spaces $\mathbb{C}_+$ and $\mathbb{C}_-$ and we obtain a Martin function for each of them.

A reformulation of the above statement can be given in the following way. For $x \in \mathsf{E}$, let $\mathcal{P}_{\mathsf{E}}(x)$ denote the set of positive harmonic functions that are bounded outside every neighborhood of x and vanish quasi-everywhere on E. As in the proof of [Hirata 2007, Lemma 2.9] one can see that $\mathcal{P}_{\mathsf{E}}(x)$ is spanned by the Martin functions related to x. Hence, the above question is whether $\mathcal{P}_{\mathsf{E}}(x)$ is one- or two-dimensional. We will provide a simplified proof for the case that there is only one Martin function associated to x below. This question has attracted much interest and several conditions have been obtained [Ancona 1979; Benedicks 1980; Koosis 1988; Levin 1989c]. To note two extreme cases, if $x \in (a, b) \subset \mathsf{E}$, then $\mathcal{P}_{\mathsf{E}}(x)$ is two-dimensional, whereas if x is a endpoint of a gap of E, then $\mathcal{P}_{\mathsf{E}}(x)$ is one-dimensional, as discussed in [Gardiner and Sjödin 2009] after Theorem 6.

We are particularly interested in the Martin kernel related to ∞. Since E is semibounded, $\mathcal{P}_{\mathsf{E}} = \mathcal{P}_{\mathsf{E}}(\infty)$ is one-dimensional and we can talk about the Martin function $M_\infty(z) = M_{\mathsf{E}}(z, \infty)$ related to ∞, which is known to be symmetric, i.e., $M_\infty(\bar{z}) = M_\infty(z)$. Moreover, all limits with $z_n \to -\infty$ must lead to M_∞ and we have

$$M_\infty(z) = \lim_{z_0 \to -\infty} M(z, z_0) = \lim_{z_0 \to -\infty} \frac{\operatorname{Im} \theta_{z_0}(z)}{G_{\mathsf{E}}(z_*, z_0)}.$$

Note that M_∞ is not exactly M_{E} from the Introduction, because in the general situation we cannot use the normalization (1-2). For this reason, we keep the normalization at z_*, but once we have specified to sets where the limit in (1-2) is positive, we can pass to this normalization. Since $M_\infty(z)$ is positive and harmonic in Ω, setting $\lambda^2 = z - b_0$ it defines a positive harmonic function for $\lambda \in \mathbb{C}_+$ by

$$f(\lambda) = M_\infty(z).$$

Since f can be represented as

$$f(x+iy) = ay + \int \frac{y}{(x-t)^2+y^2}\, d\nu(t), \qquad \int \frac{d\nu(t)}{1+t^2} < \infty, \tag{2-6}$$

and

$$0 \le a = \lim_{y\to\infty} \frac{f(iy)}{y}, \tag{2-7}$$

we see that $M_\infty(z)$ can grow at most as $\sqrt{-z}$ as $z \to -\infty$. In case of two-sided unbounded sets, where the Martin function can grow at most linearly, Akhiezer and Levin showed that $\mathcal{P}_{\mathsf{E}}$ is two-dimensional whenever the Martin function admits the maximal possible growth. This explains why we call E an *Akhiezer–Levin set* if

$$\lim_{z\to-\infty} \frac{M_\infty(z)}{\sqrt{-z}} > 0. \tag{2-8}$$

Note that by (2-7) this limit indeed exists in $[0, \infty)$. Since in (2-6), the integral $\int y/((x-t)^2+y^2)\, d\nu(t)$ defines again a positive harmonic function it follows that

$$a \operatorname{Re} \sqrt{b_0 - z} \le M_\infty(z) \tag{2-9}$$

in Ω. The following theorem presents a list of equivalent characterizations of M_∞. We say that h vanishes continuously at a point $x \in \mathsf{E}$ if $\lim_{z\to x,\, z\in\Omega} h(z) = 0$. We call a subset of Ω bounded if it is bounded as a subset of $\mathbb{C}$.

Theorem 2.1. *Let $H_{+,b}(\Omega)$ denote the set of positive harmonic functions on Ω that are bounded on every bounded subset of Ω. Then, the following are equivalent*:

(i) *$h \in H_{+,b}(\Omega)$ and h vanishes continuously for every Dirichlet-regular point of* E.

(ii) *$h \in H_{+,b}(\Omega)$ and h vanishes continuously quasi-everywhere on* E.

(iii) *$h \in H_{+,b}(\Omega)$ and h vanishes continuously $\omega_{\mathsf{E}}(\,\cdot\,, z_0)$-a.e.*

(iv) *$h = cM_\infty$, where $c \ge 0$.*

Proof. Due to [Gardiner and Sjödin 2009, Remark 5, Theorem 6] (iv) ⇒ (i). Kellogg's theorem [Garnett and Marshall 2005, Corollary 6.4] yields (i) ⇒ (ii) and by [loc. cit., Theorem III.8.2] we get that (ii) ⇒ (iii). It remains to show that (iii) ⇒ (iv). Due to (2-4) there exists ν such that

$$h(z) = \int_{\partial_1^M \Omega} M_{\mathsf{E}}(z, x)\, d\nu(x).$$

Let $K \subset \partial^M \Omega \setminus \{M_\infty\}$ be compact. Then K has an open neighborhood U in $\widehat{\Omega}$ such that $U \cap \Omega$ is bounded. As in the proof of [Armitage and Gardiner 2001, Theorem 8.4.1]

$$R_h^K(z) = \int_K M_{\mathsf{E}}(x, z)\, d\nu(x).$$

Since $h \in H_{+,b}(\Omega)$, h is majorized by a constant in $U \cap \Omega$, so R_h^K is a bounded harmonic function in Ω which vanishes $\omega_{\mathsf{E}}(\,\cdot\,, z_0)$-a.e. on the boundary. By the maximum principle [Garnett and Marshall 2005, Theorem III.8.1] it follows that $R_h^K = 0$. In particular, $R_h^K(z_*) = \nu(K) = 0$. The claim follows. □

In a series of papers Levin [1989a; 1989b; 1989c], first systematically established the relation between extremal problems and comb mappings imposing Dirichlet regularity on the set E. Eremenko and Yuditskii [2012] provided a modern approach to it, giving a detailed proof for comb mappings for Green's functions as discussed above. It relies on the representation of Green's functions for a compact set E, as

$$G_E(z,\infty)=\int_E \log|z-t|\,\mathrm{d}\rho_E(t)+\gamma_E, \tag{2-10}$$

where $\operatorname{Cap}(E)=e^{-\gamma_E}$ and $\rho_E(X)=0$ for sets of zero capacity. It is also discussed that the proof carry over for Martin functions and the corresponding description is given. Since we were not able to find in our generality a reference for a representation of the type (2-10), which is certainly known to experts, for the readers convenience we survey the corresponding theory in the following.

Since M_∞ vanishes quasi-everywhere, we can extend M_∞ to a subharmonic function to all of $\mathbb{C}$ by

$$M_\infty(x)=\limsup_{\substack{z\to x\\ z\in\Omega}} M_\infty(z), \qquad x\in\mathsf{E}; \tag{2-11}$$

see [Armitage and Gardiner 2001, Theorem 5.2.1]. Hence, we obtain a subharmonic, symmetric function in $\mathbb{C}$, which is positive and harmonic in $\mathbb{C}_+$ and $\mathbb{C}_-$. For the following result we refer to [Levin 1989b, Lemma 2.3] and its corollary. It was initially proved for majorants of subharmonic functions, but it is mentioned that it extends to the version stated below:

Lemma 2.2. *Let v be a subharmonic, symmetric function in $\mathbb{C}$, which is positive and harmonic in $\mathbb{C}\setminus[b_0,\infty)$ for some $b_0\in\mathbb{R}$. Then*

$$v(z)=v(z_*)+\int_{b_0}^\infty \log\left|1-\frac{z-z_*}{t-z_*}\right|\mathrm{d}\nu(t), \qquad \int_{b_0}^\infty \frac{\mathrm{d}\nu(t)}{t-z_*}<\infty, \tag{2-12}$$

and for $y>0$

$$\frac{\partial v(x+iy)}{\partial y}=\int_{b_0}^\infty \frac{y}{(t-x)^2+y^2}\,\mathrm{d}\nu(t)>0. \tag{2-13}$$

Remark. Equation (2-12) is essentially the Hadamard representation for the subharmonic function v and ν is its *Riesz measure*. Usually the Hadamard representation would include a normalization term $(\operatorname{Re} z)/t$, which is not needed due to the convergence property of ν in (2-12).

Lemma 2.3. *Let Θ be such that $\operatorname{Im}\Theta=M_\infty$ for $z\in\mathbb{C}_+$ and ρ be the Riesz measure for M_∞. Then, the functions Θ and $i\Theta'$ are Herglotz functions and in particular*

$$i\Theta'(z)=\int_{\mathsf{E}}\frac{\mathrm{d}\rho(t)}{t-z}.$$

They can be analytically extended to $\mathbb{C}\setminus[b_0,\infty)$ and $\Theta'\neq 0$ there.

Proof. Applying Lemma 2.2 to M_∞ gives a representation of the form (2-12) in terms of the Riesz measure ρ supported on E and, in particular, $\int_{\mathsf{E}}\mathrm{d}\rho(t)/(t-z_*)<\infty$. Moreover,

$$i\Theta'(z)=c_0+\int_{\mathsf{E}}\frac{\mathrm{d}\rho(t)}{t-z} \tag{2-14}$$

for some $c_0 \in \mathbb{R}$, since the imaginary parts of the two sides are equal by (2-13). Since Θ is also a Herglotz function, for some measure μ supported on E,

$$i\Theta'(z) = i\int \frac{\mathrm{d}\mu(t)}{(t-z)^2}, \qquad \int \frac{\mathrm{d}\mu(t)}{1+t^2} < \infty. \tag{2-15}$$

Using monotone convergence and taking the limit as $z \to -\infty$ in (2-14) and (2-15) yields

$$\lim_{z\to-\infty} i\Theta'(z) = 0 = c_0.$$

Since $i\Theta'$ is Herglotz, $\Theta' \neq 0$ in $\mathbb{C}_+$ and $\mathbb{C}_-$. Moreover, since it is increasing on $(-\infty, b_0)$ and vanishes at $-\infty$ we obtain the final claim. □

The following lemma shows that, like the harmonic measure, ρ gives zero measure to polar sets. Of course, once we introduce the Martin measure ρ_{E}, it will be a scalar multiple of ρ, so the following claim will also hold for ρ_{E}.

Lemma 2.4. *Let $X \subset \mathbb{C}$ be a Borel polar set. Then $\rho(X) = 0$.*

Proof. By [Ransford 1995, Theorem 3.2.3] it suffices to show that for each $s > b_0$ we have

$$\int_{b_0}^{s}\int_{b_0}^{s} \log|x-t|\,\mathrm{d}\rho(x)\,\mathrm{d}\rho(t) > -\infty. \tag{2-16}$$

By means of the subharmonic extension (2-11), M_∞ is nonnegative on $\mathbb{C}$ and we get

$$0 \le \int_{b_0}^{s} M_\infty(x)\,\mathrm{d}\rho(x) = d + I_1 + I_2,$$

where

$$d = \rho(b_0, s)\left(1 - \int_{b_0}^{s} \log|t - z_*|\,\mathrm{d}\rho(t)\right),$$

$$I_1 = \int_{b_0}^{s}\int_{b_0}^{s} \log|x-t|\,\mathrm{d}\rho(t)\,\mathrm{d}\rho(x), \qquad I_2 = \int_{b_0}^{s}\int_{s}^{\infty} \log\left|1 - \frac{x-z_*}{t-z_*}\right|\mathrm{d}\rho(t)\,\mathrm{d}\rho(x).$$

Since $I_2 \le 0$, it follows that $-\infty < -d \le I_1$, i.e., we have (2-16). □

It was already encountered in [Levin 1989b, Lemma 2.4] that there is an explicit connection between ρ and the conformal map Θ defined in Lemma 2.3; see also [Eremenko and Yuditskii 2012]. Note that although in [Levin 1989b] Dirichlet regularity is assumed for the set E, the proof of the following lemma holds also in our setting. Namely, the Lebesgue measure on the base of the comb corresponds to the measure ρ on E. To be more precise, $\operatorname{Re}\Theta$ extends continuously to $\mathbb{R}$ and we have

$$\operatorname{Re}\Theta(b) - \operatorname{Re}\Theta(a) = \pi\rho((a,b)). \tag{2-17}$$

These are all the ingredients needed to describe the comb domains related to the conformal mapping Θ. There exists a positive upper semicontinuous function s on $(0, b)$, where $b \in (0, \infty]$, such that Θ maps $\mathbb{C}_+$ conformally onto

$$\Pi = \{x + iy : 0 < x < b,\ y > s(x)\}.$$

If $b < \infty$ then $\limsup_{x\to b} s(x) = \infty$. We will show in Corollary 2.8 that b being finite corresponds to ∞ being not Dirichlet-regular.

Example 2.5. In their classical work Marchenko and Ostrovskii [1975] studied the relation between spectra of 1-periodic L^2 potentials on the real line and corresponding data of the mapping Θ_{E}. They showed that E is the spectrum of a Schrödinger operator of this type if and only if the corresponding comb domain is of the form

$$\Pi_{\mathsf{E}} = \{x + iy : x > 0,\ y > 0\} \setminus \{k\pi + iy : k \in \mathbb{N},\ 0 \le y \le s_k\},$$

and the slit heights s_k satisfy $\sum_{k=1}^{\infty} k^2 s_k^2 < \infty$.

The next example demonstrates Akhiezer–Levin sets which don't have an expansion of the form (1-3).

Example 2.6. We will construct an explicit expression for the conformal map

$$\Theta : \mathbb{C}_+ \to \Pi = \mathbb{C}_+ \setminus \{n + iy : n \in \mathbb{Z}, 0 < y \le y_0\},$$

where $y_0 > 0$ is an arbitrary but fixed parameter. We will show that along the imaginary axis we have

$$\Theta(iy) = iy + ic(y_0) + o(1) \quad \text{as } y \to \infty,$$

where, $c(y_0)$ is a real constant that depends monotonically on y_0 and can attain in fact any real value. Note that Θ can be continuously extended to $\mathbb{R}$ and that $E := \Theta^{-1}(\mathbb{R})$ is symmetric, $E = -E = \{-x : x \in E\}$. Hence, again by defining $\widetilde{\Theta}(z) = \Theta(\lambda^2)$, the function $M(z) = \operatorname{Im} \widetilde{\Theta}(z)$ is an example for a Martin function of an Akhiezer–Levin set, which has a constant term in its asymptotic expansion. The Christoffel–Darboux transformation

$$f_1(w) = \frac{1}{\pi} \int_{-1}^{w} \frac{\mathrm{d}x}{\sqrt{1 - x^2}}$$

maps $\mathbb{C}_+$ onto $\Pi_1 = \{\vartheta = \xi + i\eta : \eta > 0,\ 0 < \xi < 1\}$. In particular $f_1(-1) = 0$ and $f_1(1) = 1$. We choose $\ell > 1$ so that $iy_0 = f_1(-\ell)$ and consider

$$f_2(w) = \frac{1}{\pi} \int_{-\ell}^{w} \frac{\mathrm{d}x}{\sqrt{\ell^2 - x^2}} = f_1\Big(\frac{w}{\ell}\Big).$$

Then $\Theta = f_1 \circ f_2^{-1}$ defines a conformal map $\Theta : \Pi_1 \to \Pi_1$ such that $\Theta(0) = iy_0$. By symmetry, we can extend Θ to a conformal map from $\Theta : \mathbb{C}_+ \to \Pi$. Calculations of f_1, f_2 along the imaginary axis give $\Theta(iy) = i \cosh^{-1}(\ell \cosh(y))$, so

$$\Theta(iy) = iy + i \log(\ell) + o(1) \quad \text{as } y \to \infty.$$

We emphasize that in order to show that the limit in (2-8) is always finite for the Martin function, it was only used that M_∞ represents a positive harmonic function in Ω. This shows that the same conclusion holds for any such function. In view of (2-4) this growth should also be reflected in the corresponding asymptotic behavior of M_∞, leading to the following criterion for E to be an Akhiezer–Levin set.

Lemma 2.7. *Assume that there exists a positive harmonic function in Ω such that*

$$\lim_{z\to-\infty}\frac{h(z)}{\sqrt{-z}}=1.$$

Then Ω is Greenian and E is an Akhiezer–Levin set. Moreover, in this case we have

$$M_{\mathsf{E}}(z)\le h(z) \tag{2-18}$$

for all $z\in\Omega$, where M_{E} is normalized by $\lim_{z\to-\infty}M_{\mathsf{E}}(z)/\sqrt{-z}=1$.

Proof. By Myrberg's theorem [Armitage and Gardiner 2001, Theorem 5.3.8] the existence of a nonconstant positive harmonic function on Ω implies that Ω is Greenian. Since h is a positive harmonic function in Ω, there exists a unique measure ν with $\nu(\partial_1^M\Omega)=h(z_*)$ such that

$$h(z)=\int_{\partial_1^M\Omega}M(z,x)\,\mathrm{d}\nu(x).$$

In particular, $\nu(\{\infty\})<\infty$. Recall that $\#\pi^{-1}(\{\infty\})=1$. Since $(-\infty,b_0)\subset\Omega$, the negative half-axis is clearly not minimally thin at ∞ so it follows by [Armitage and Gardiner 2001, Theorem 9.2.6] that

$$\liminf_{z\to-\infty}\frac{h(z)}{M_\infty(z)}\le\nu(\{\infty\})<\infty. \tag{2-19}$$

Let $\lambda^2=z-b_0$ and $g(\lambda)=h(z)$ and $f(\lambda)=M_\infty(z)$. Then f defines a positive harmonic function in $\mathbb{C}_+$ and

$$f(x+iy)=ay+\int\frac{y}{(x-t)^2+y^2}\,\mathrm{d}\mu(t),\qquad a=\lim_{y\to\infty}\frac{f(iy)}{y}.$$

Hence,

$$0<\limsup_{z\to-\infty}\frac{M_\infty(z)}{h(z)}=\limsup_{y\to\infty}\frac{f(iy)}{g(iy)}=\limsup_{y\to\infty}\frac{f(iy)}{y}=a.$$

Hence, E is an Akhiezer–Levin set. Due to [Armitage and Gardiner 2001, Theorem 9.3.3] we have

$$\nu(\{\infty\})=\inf_{z\in\Omega}\frac{h(z)}{M_\infty(z)}\le\frac{h(z)}{M_\infty(z)} \tag{2-20}$$

and the second claim follows. Finally, (2-20) shows that we actually have equality in (2-19) and it follows that $\nu(\{\infty\})$ corresponds to the normalization of M_∞ at ∞. □

Carleson and Totik [2004] showed that $\mathcal{P}_{\mathsf{E}}(x_0)$ being two-dimensional is equivalent to the fact that $G_{\mathsf{E}}(z,z_0)$ is Lipschitz continuous at x_0, where z_0 is some arbitrary interior point. As a corollary of the comb mapping representation for Θ, we show that E being an Akhiezer–Levin set implies continuity at infinity. Note that by the aforementioned equivalence, one cannot hope for Lipschitz continuity for semibounded sets, since in this case $\mathcal{P}_{\mathsf{E}}(\infty)$ is always one-dimensional. Alternatively, this could be seen from the fact that often, at a gap edge a, the Green's function has behavior $G_{\mathsf{E}}(z,z_0)\sim\sqrt{z-a}$ and thus is not Lipschitz continuous. Moreover, as discussed in [Volberg and Yuditskii 2016] the set $\mathsf{E}=\mathbb{R}_+\setminus\bigcup_{n\in\mathbb{Z}}r^n(a_1,b_1)$, where $0<a_1<b_1$ and $r>1$, provides an example of a set for which ∞ is Dirichlet-regular, but which is not an Akhiezer–Levin set. In this sense the following result is optimal.

Corollary 2.8. *Let* $\mathsf{E}\subset\mathbb{R}$ *be closed and semibounded and* Θ *be the corresponding comb-mapping. If* $\sup\{\operatorname{Re}\Theta(z) : z\in\mathbb{C}_+\}=\infty$, *then* ∞ *is a Dirichlet-regular point of* E. *This holds in particular if* E *is an Akhiezer–Levin set.*

Proof. We will assume that $\limsup_{z_0\to-\infty} G_{\mathsf{E}}(z_0, z_*) = \varepsilon > 0$ in order to obtain a contradiction. Note that $\sup\{\operatorname{Re}\theta_{z_0}(z) :\ z\in\mathbb{C}_+\}=\pi$, so for any $z\in\mathbb{C}_+$,

$$\lim_{z_0\to-\infty}\frac{\operatorname{Re}\theta_{z_0}(z)}{G_{\mathsf{E}}(z_*, z_0)}\le\liminf_{z_0\to-\infty}\frac{\pi}{G_{\mathsf{E}}(z_*, z_0)}.$$

Since

$$\Theta(z)=\lim_{z_0\to-\infty}\frac{\theta_{z_0}(z)}{G_{\mathsf{E}}(z_*, z_0)},$$

taking the supremum over $z\in\mathbb{C}_+$ gives $\sup\{\operatorname{Re}\Theta(z) :\ z\in\mathbb{C}_+\}\le\varepsilon^{-1}\pi<\infty$. Now, as already mentioned in [Eremenko and Yuditskii 2012], using upper semicontinuity of h it follows that vanishing of the radial limit of $G_{\mathsf{E}}(z_0, z_*)$ implies Dirichlet regularity. Let $\operatorname{Im}\theta_{z_*}=G_{\mathsf{E}}(z, z_*)$ and it will be more convenient to shift the mapping by $-\pi$. Then, $\lim_{z_0\to-\infty} G_{\mathsf{E}}(z_0, z_*)=0$ implies that $\limsup_{u\to0} h(u)=0$. Therefore, $(-\infty, z_*)$ is mapped by θ_{z_*} onto $i\mathbb{R}_+$ and we can extend θ_{z_*} by symmetry to $\mathbb{C}\setminus(\mathbb{R}\setminus(-\infty, z_*))$. In particular $i\mathbb{R}_+$ is an interior ray of the image, $\Pi_e=\Pi_{z_*}\cup i\mathbb{R}_+\cup\{-x+iy : x+iy\in\Pi_{z_*}\}$, of this extended map. Since $\limsup_{u\to0} h(u)=0$, we have Π_e is locally connected at 0 and hence θ_{z_*} can be continuously extended to 0, which implies that ∞ is a Dirichlet-regular point. This finishes the proof of the first claim.

In view of (2-17), $\sup\{\operatorname{Re}\Theta(z) : z\in\mathbb{C}_+\}<\infty$ means that ρ is finite. We show this implies that M_∞ can grow at most like $\rho(\mathbb{R})\log|z|$ and therefore E is not an Akhiezer–Levin set. Let's assume that $|z_*-b_0|>1$ and $z_*<0$. Then, using (2-12) we see that for $z<z_*$ we have

$$M_\infty(z)-\rho(\mathbb{R})\log|z| = M_\infty(z_*)+\int_{b_0}^{\infty}\log\left|\frac{1}{t-\lambda_*}\left(1-\frac{z_*}{z}\right)\right|\mathrm{d}\rho(t)\le M_\infty(z_*). \qquad\square$$

For Akhiezer–Levin sets one could also use the result of Carleson and Totik and the substitution $\lambda^2=z-b_0$ to see that G_{E} is Hölder continuous with exponent $\frac{1}{2}$ at ∞.

3. Asymptotic behavior of eigensolutions

We now turn our attention to the Schrödinger operator L_V and associated objects. Fundamental solutions at $z\in\mathbb{C}$ are defined as solutions $u(x, z)$, $v(x, z)$ of the initial value problems

$$-\partial_x^2 u+(V(x)-z)u=0,\quad u(0, z)=0,\quad (\partial_x u)(0, z)=1, \tag{3-1}$$

$$-\partial_x^2 v+(V(x)-z)v=0,\quad v(0, z)=1,\quad (\partial_x v)(0, z)=0. \tag{3-2}$$

The natural regularity class for the solutions is that of functions which are in $W^{2,1}([0, x])$ for every $x<\infty$, and the differential equations are interpreted as equality of L^1 functions, i.e., equality Lebesgue-a.e. It is useful to substitute

$$k=\sqrt{-z}$$

and view the initial value problems as perturbations by V of $-\partial_x^2 + k^2$. We will always assume that $\operatorname{Re} k \geq 0$; this can be done pointwise throughout $\mathbb{C}$, and later we will view k as a branch of the square root such that $\operatorname{Re} k > 0$ if $z \in \mathbb{C} \setminus [0, \infty)$. Note also that this makes $\operatorname{Im} k < 0$ if $z \in \mathbb{C}_+$. By choosing the branch $\sqrt{z} = ik$, we see that $\sqrt{z} \in \mathbb{C}_+$ if $z \in \mathbb{C} \setminus [0, \infty)$. In particular, $\operatorname{Im} \sqrt{z} = \operatorname{Re} k$.

The fundamental solutions for $V = 0$ are the functions

$$c(x, k) = \cosh(kx), \quad s(x, k) = \begin{cases} \sinh(kx)/k, & k \neq 0, \\ x, & k = 0. \end{cases}$$

By standard arguments, for general $V \in L^1([0, 1])$, the initial value problems (3-1), (3-2) are rewritten as integral equations, and by Volterra-type arguments, convergent series representations are then found for the fundamental solutions. With the notation $\Delta_n(x) = \{t \in \mathbb{R}^n \mid x \geq t_1 \geq t_2 \geq \cdots \geq t_n \geq 0\}$, the series representations for fundamental solutions and their first derivatives are

$$u(x, z) = s(x, k) + \sum_{n=1}^{\infty} \int_{\Delta_n(x)} s(x - t_1, k) \left(\prod_{j=1}^{n-1} V(t_j) s(t_j - t_{j+1}, k) \right) V(t_n) s(t_n, k) \, \mathrm{d}^n t, \tag{3-3}$$

$$v(x, z) = c(x, k) + \sum_{n=1}^{\infty} \int_{\Delta_n(x)} s(x - t_1, k) \left(\prod_{j=1}^{n-1} V(t_j) s(t_j - t_{j+1}, k) \right) V(t_n) c(t_n, k) \, \mathrm{d}^n t, \tag{3-4}$$

$$(\partial_x u)(x, z) = c(x, k) + \sum_{n=1}^{\infty} \int_{\Delta_n(x)} c(x - t_1, k) \left(\prod_{j=1}^{n-1} V(t_j) s(t_j - t_{j+1}, k) \right) V(t_n) s(t_n, k) \, \mathrm{d}^n t, \tag{3-5}$$

$$(\partial_x v)(x, z) = k^2 s(x, k) + \sum_{n=1}^{\infty} \int_{\Delta_n(x)} c(x - t_1, k) \left(\prod_{j=1}^{n-1} V(t_j) s(t_j - t_{j+1}, k) \right) V(t_n) c(t_n, k) \, \mathrm{d}^n t. \tag{3-6}$$

These expansions are derived, e.g., in [Pöschel and Trubowitz 1987] for $V \in L^2([0, x])$, but they hold for $V \in L^1([0, x])$ as well, due to the estimate

$$\left| \int_0^x \int_0^{t_1} \cdots \int_0^{t_{n-1}} e^{\operatorname{Re} k (x - t_1)} \left(\prod_{j=1}^{n} V(t_j) e^{\operatorname{Re} k (t_j - t_{j+1})} \right) V(t_n) e^{\operatorname{Re} k t_n} \, \mathrm{d}t_n \cdots \mathrm{d}t_2 \, \mathrm{d}t_1 \right| \leq \frac{1}{n!} \left(\int_0^x |V(s)| \, \mathrm{d}s \right)^n e^{\operatorname{Re} k x}, \tag{3-7}$$

which is proved by combining the exponentials and using permutations of t and symmetry, and the elementary estimates which follow directly from Euler's formula,

$$|c(x, k)| \leq e^{\operatorname{Re} k x}, \quad |s(x, k)| \leq |k|^{-1} e^{\operatorname{Re} k x}. \tag{3-8}$$

The same estimates which guarantee convergence provide exponential upper bounds on eigensolutions; these are often stated over a fixed interval, but we will need a kind of uniformity in x:

Lemma 3.1. *For all $z = -k^2 \in \mathbb{C}$ and $x > 0$,*

$$|u(x, -k^2)| \leq e^{(1 + \operatorname{Re} k) x + \int_0^x |V(t)| \, \mathrm{d}t}. \tag{3-9}$$

Proof. Using $|s(x,k)| = \left|\int_0^x c(t,k)\,\mathrm{d}t\right| \le xe^{\operatorname{Re} kx} \le e^{(1+\operatorname{Re} k)x}$ and then applying (3-7) to each term of (3-3) implies that

$$|u(x,-k^2)| \le e^{(1+\operatorname{Re} k)x} \sum_{n=0}^{\infty} \frac{1}{n!} \left(\int_0^x |V(t)| \right)^n . \qquad \square$$

Corollary 3.2. *If V obeys* (1-1), *for each $R > 0$ there exists C_R such that for all $|z| \le R$ and $x \ge 1$ we have $\frac{1}{x} \log|u(x,z)| \le C_R$.*

Proof. This is an immediate consequence of the previous lemma together with $\int_0^x |V(t)|\,\mathrm{d}t \le C(x+1) \le 2Cx$ for $x \ge 1$, where $C = \sup_{x \ge 0} \int_x^{x+1} |V(t)|\,\mathrm{d}t$. $\square$

We will need asymptotic statements about m-functions. Such statements are ubiquitous, especially for smooth potentials; we need an asymptotic expansion which doesn't assume any smoothness.

Lemma 3.3. *For fixed $x > 0$, as $z \to \infty$, $\arg z \in [\delta, 2\pi - \delta]$,*

$$-\frac{v(x,z)}{u(x,z)} = -k - \int_0^x V(t) e^{-2kt}\,\mathrm{d}t + \frac{1}{k} \int_0^x \int_0^{t_1} e^{-2kt_1}(1 - e^{-2kt_2}) V(t_1) V(t_2)\,\mathrm{d}t_2\,\mathrm{d}t_1 + O(|k|^{-2})$$

uniformly in V in bounded subsets of $L^1([0,x])$.

Proof. Assume that $\int_0^x |V(t)|\,\mathrm{d}t \le C$. Define

$$A_n = 2k^{n+1} e^{-kx} \int_{\Delta_n(x)} s(x - t_1, k) \left(\prod_{j=1}^{n-1} V(t_j) s(t_j - t_{j+1}, k) \right) V(t_n) s(t_n, k)\,\mathrm{d}^n t,$$

$$B_n = 2k^n e^{-kx} \int_{\Delta_n(x)} s(x - t_1, k) \left(\prod_{j=1}^{n-1} V(t_j) s(t_j - t_{j+1}, k) \right) V(t_n) c(t_n, k)\,\mathrm{d}^n t,$$

From (3-8) and (3-7) it follows that $|A_n|, |B_n| \le 2C^n/n!$. In the nontangential limit $z \to \infty$, $\arg z \in [\delta, 2\pi - \delta]$, we have the elementary estimates

$$\frac{s(x,k)}{e^{kx}/(2k)} = 1 - e^{-2kx} = 1 + O(|k|^{-3}), \qquad \frac{c(x,k)}{e^{kx}/2} = 1 + e^{-2kx} = 1 + O(|k|^{-3}),$$

so the series expansions for $u(x,z)$, $v(x,z)$ imply

$$u(x,z) = \frac{e^{kx}}{2k} \left(1 + \frac{A_1}{k} + \frac{A_2}{k^2} + O(|k|^{-3}) \right),$$

$$v(x,z) = \frac{e^{kx}}{2} \left(1 + \frac{B_1}{k} + \frac{B_2}{k^2} + O(|k|^{-3}) \right),$$

with the error $O(|k|^{-3})$ depending only on C and δ. Dividing,

$$-\frac{v(x,z)}{u(x,z)} = -k \left(1 + \frac{B_1 - A_1}{k} + \frac{B_2 - A_2 - A_1(B_1 - A_1)}{k^2} + O(|k|^{-3}) \right). \tag{3-10}$$

Moreover,

$$B_1 - A_1 = \int_0^x (1-e^{-2k(x-t)})V(t)e^{-2kt}\,\mathrm{d}t = \int_0^x V(t)e^{-2kt}\,\mathrm{d}t + O(e^{-2\operatorname{Re} kx}) \tag{3-11}$$

Multiplying by

$$A_1 = \frac{1}{2}\int_0^x (1-e^{-2k(x-s)})V(s)(1-e^{-2ks})\,\mathrm{d}s$$

gives a formula for $A_1(B_1-A_1)$ as a double integral over $[0,x]^2$, and using the substitution $t_1=\max\{s,t\}$, $t_2=\min\{s,t\}$ gives

$$A_1(B_1-A_1) = \frac{1}{2}\int_0^x\int_0^{t_1} (e^{-2kt_1}+e^{-2kt_2}-2e^{-2k(t_1+t_2)}-e^{-2k(x-t_1+t_2)})V(t_1)V(t_2)\,\mathrm{d}t_2\,\mathrm{d}t_1 + O(e^{-2\operatorname{Re} kx})$$

(some terms are grouped into the error $O(e^{-2\operatorname{Re} kx})$ since, e.g., $x-t_2+t_1\ge x$). Similarly,

$$\begin{aligned}B_2 - A_2 &= \frac{1}{2}\int_0^x\int_0^{t_1} (1-e^{-2k(x-t_1)})V(t_1)(1-e^{-2k(t_1-t_2)})V(t_2)e^{-2kt_2}\,\mathrm{d}t_2\,\mathrm{d}t_1\\ &= \frac{1}{2}\int_0^x\int_0^{t_1} (e^{-2kt_2}-e^{-2kt_1}-e^{-2k(x-t_1+t_2)})V(t_1)V(t_2)\,\mathrm{d}t_2\,\mathrm{d}t_1 + O(e^{-2\operatorname{Re} kx}).\end{aligned}$$

Substituting these formulas into (3-10) concludes the proof. ☐

Returning to the half-line setting from the Introduction, we recall that half-line potentials obeying the boundedness assumption (1-1) are in the limit point case at $+\infty$, i.e., for every $z\in\mathbb{C}\setminus\mathsf{E}$, the set of solutions of

$$-\partial_x^2\psi + V\psi = z\psi, \quad \psi\in L^2((0,\infty)),$$

is one-dimensional. Any such nontrivial solution is called the Weyl solution; it is uniquely determined up to normalization and we will not fix any particular normalization. We will use

$$m(x,z) = \frac{(\partial_x\psi)(x,z)}{\psi(x,z)}. \tag{3-12}$$

Proposition 3.4. *As* $z\to\infty$, $\arg z\in[\delta,\pi-\delta]$,

$$m(s,z) = -k - \int_0^1 V(s+t)e^{-2kt}\,\mathrm{d}t + \frac{1}{k}\int_0^1\int_0^{t_1} e^{-2kt_1}(1-e^{-2kt_2})V(s+t_1)V(s+t_2)\,\mathrm{d}t_2\,\mathrm{d}t_1 + O(|k|^{-2})$$

and the error is uniform in $s\in[0,\infty)$ *if* V *obeys* (1-1).

Proof. By an argument of [Atkinson 1981], for $\arg z\in[\delta,\pi-\delta]$, the Weyl circle at x has radius

$$r = \frac{2|k|^2}{|\operatorname{Im} k|}e^{-2x\operatorname{Re} k}(1+O(|k|^{-1})),$$

which decays exponentially as $z\to\infty$, $\arg z\in[\delta,\pi-\delta]$; the error term $O(|k|^{-1})$ is uniform for V in bounded subsets of $[0,x]$, since this term is derived by arguments like those in the proof of Lemma 3.3. Since $m_+(0,z)$ lies inside the Weyl circle and $-v(1,z)/u(1,z)$ lies on the circle, this radius allows us to estimate

$$\left| m(0,z) + \frac{v(1,z)}{u(1,z)}\right| \le \frac{4|k|^2}{|\operatorname{Im} k|}e^{-2\operatorname{Re} k}(1+O(|k|^{-1})).$$

In the nontangential limit as $\arg z \in [\delta, \pi - \delta]$, this error is $O(|k|^{-2})$, so the previous lemma implies

$$m(0,z) = -k - \int_0^1 V(t) e^{-2kt}\, \mathrm{d}t + \frac{1}{k}\int_0^1 \int_0^{t_1} e^{-2kt_1}(1 - e^{-2kt_2}) V(t_1) V(t_2)\, \mathrm{d}t_2\, \mathrm{d}t_1 + O(|k|^{-2}).$$

Applying this for an arbitrary $s \geq 0$ to the translated half-line potential $V_s(x) = V(x+s)$ on $[0,\infty)$ concludes the proof. □

For the half-line operator L_V, the Dirichlet solution can be interpreted as the Weyl solution corresponding to the endpoint 0. Therefore, the Atkinson argument can be applied also "in reverse", to produce uniform asymptotics on the logarithmic derivative of $u(x,z)$. To produce uniform asymptotics, we fix the interval length 1, as in the previous proof:

Corollary 3.5. *As $z \to \infty$, $\arg z \in [\delta, \pi - \delta]$, for all $s \geq 1$,*

$$-\frac{(\partial_x u)(s,z)}{u(s,z)} = -k - \int_0^1 V(s-t) e^{-2kt}\, \mathrm{d}t + \frac{1}{k}\int_0^1 \int_0^{t_1} e^{-2kt_1}(1 - e^{-2kt_2}) V(s-t_1) V(s-t_2)\, \mathrm{d}t_2\, \mathrm{d}t_1 + O(|k|^{-2})$$

and the error is uniform in $s \in [1,\infty)$ if V obeys (1-1).

To make some uniform statements for a family of Herglotz functions, we will use the Carathéodory inequality for the half-plane [Levin 1980, Proof of Theorem I.8]: for any Herglotz function f,

$$|f(z)| \leq |f(i)| + \operatorname{Im} f(i) \frac{2|z-i|}{|z+i| - |z-i|} \quad \forall z \in \mathbb{C}_+. \tag{3-13}$$

Lemma 3.6. *Fix a potential V which obeys* (1-1). *For each $z \in \mathbb{C}_+$,*

$$\sup_{x \geq 1} \left| \frac{(\partial_x u)(x,z)}{u(x,z)} \right| < \infty. \tag{3-14}$$

Proof. The ratio $-(\partial_x u)(x,z)/u(x,z)$ is a Herglotz function and obeys the nontangential asymptotics in Corollary 3.5. The error is uniform in $x \geq 1$ since V obeys (1-1). In particular, for $z = iy_0$ with some fixed $y_0 > 0$ large enough, Corollary 3.5 implies an upper bound independent of x and therefore (3-14). By rescaling by y_0 and using (3-13), the upper bound at iy_0 implies uniform upper bounds for z in compact subsets of $\mathbb{C}_+$. □

For $z \notin \sigma(L_V)$, ψ decays exponentially as $x \to \infty$. The Weyl solution ψ and the Dirichlet solution u are related by the Wronskian

$$W(\psi,u) = (\partial_x u)(x,z)\psi(x,z) - (\partial_x \psi)(x,z) u(x,z),$$

which is independent of x and nonzero, since u, ψ are linearly independent (otherwise they would give an eigenvalue of L_V). This strongly suggests that u should grow at the same rate at which ψ decays, but a proof based only on the Wronskian is difficult due to the derivative, especially if a pointwise statement is desired. We therefore use a different argument:

Lemma 3.7. *Fix a potential V which obeys* (1-1). *For each $z \in \mathbb{C}_+$, there exists C such that*, *for all* $x \in [1, \infty)$,

$$C^{-1} \le |u(x,z)\psi(x,z)| \le C.$$

Proof. We use the diagonal (spectral-theoretic) Green's function for L_V,

$$g(x,x;z) = \frac{u(x,z)\psi(x,z)}{W(\psi,u)}, \tag{3-15}$$

which can be written as

$$-\frac{1}{g(x,x;z)} = \frac{(\partial_x \psi)(x,z)}{\psi(x,z)} - \frac{(\partial_x u)(x,z)}{u(x,z)}. \tag{3-16}$$

Using the above asymptotics for m-functions gives a well-known asymptotic statement,

$$g(x,x;z) = \frac{1}{2\sqrt{-z}} + O(|z|^{-1}), \quad z \to \infty,\ \arg z \in [\delta, \pi - \delta],$$

and the proof given here shows that this asymptotic behavior is uniform in $x \in [1,\infty)$, since V obeys (1-1). In particular, for some fixed $z = iy$ with y large enough, this implies

$$\sup_{x \in [1,\infty)} |g(x,x;iy)| < \infty, \qquad \inf_{x \in [1,\infty)} |g(x,x;iy)| > 0.$$

Rescaling z by a factor y and applying (3-13) to the Herglotz functions $g(x,x;z)$ and $-1/g(x,x;z)$ implies uniform upper and lower bounds on compact subsets of $\mathbb{C}_+$.

For any $z \in \mathbb{C}_+$, the Wronskian is nonzero and independent of x, so by (3-15), uniform bounds in x for $g(x,x;z)$ imply uniform bounds in x (for each $z \in \mathbb{C}_+$) for $u(x,z)\psi(x,z)$. □

The growth rate of $u(x,z)$ can now be expressed in terms of averages of the m-functions:

Corollary 3.8. *For any $z \in \mathbb{C}_+$*,

$$\limsup_{x\to\infty} \left| \frac{1}{x} \log u(x,z) + \frac{1}{x} \int_0^x m(s,z)\, \mathrm{d}s \right| = 0. \tag{3-17}$$

Proof. This follows from Lemma 3.7 since $m(x,z)$ is the logarithmic derivative of $\psi(x,z)$. □

Expansions for $m(s,z)$ are often stated in terms of values of V and its derivatives at s, but such expansions assume some regularity of V, and the error terms in such expansions are usually not uniform in the appropriate local norm for V. By working directly with the expansion in Proposition 3.4, we can obtain uniform expansions for the averages without imposing any regularity on V.

Corollary 3.9. *If V obeys* (1-1),

$$\limsup_{x\to\infty} \left| \frac{1}{x} \int_0^x m(s,z)\, \mathrm{d}s + k + \frac{1}{2kx} \int_0^x V(s)\, \mathrm{d}s \right| = O(|k|^{-2}), \tag{3-18}$$

as $z = -k^2 \to \infty$, $\arg z \in [\delta, \pi-\delta]$ *for any* $\delta > 0$.

Proof. Due to the uniformity of the error in the asymptotic expansion from Proposition 3.4,

$$\frac{1}{x}\int_0^x m(s,z)\,\mathrm{d}s = -k - \frac{1}{x}\int_0^x\int_0^1 V(s+t)e^{-2kt}\,\mathrm{d}t\,\mathrm{d}s \\ + \frac{1}{kx}\int_0^x\int_0^1\int_0^{t_1} e^{-2kt_1}(1-e^{-2kt_2})V(s+t_1)V(s+t_2)\,\mathrm{d}t_2\,\mathrm{d}t_1\,\mathrm{d}s + O(|k|^{-2}),$$

with the error term independent of x. For the term linear in V, we use $p = s+t$ to rewrite the iterated integral as $\int_0^1\int_t^{x+t} V(p)e^{-2kt}\,\mathrm{d}p\,\mathrm{d}t$. Then we wish to note that

$$\frac{1}{x}\int_0^1\int_t^{x+t} V(p)e^{-2kt}\,\mathrm{d}p\,\mathrm{d}t = \frac{1}{x}\int_0^1\int_0^x V(p)e^{-2kt}\,\mathrm{d}p\,\mathrm{d}t + O(x^{-1}), \quad x\to\infty, \tag{3-19}$$

for any k. This is because the two iterated integrals describe similar regions in $\mathbb{R}^2$: the symmetric difference of the regions $\{(t,p) \mid 0\le t\le 1,\ t\le p\le x+t\}$ and $\{(t,p)\mid 0\le t\le 1,\ 0\le p\le x\}$ is contained in $[0,1]\times([0,1]\cup[x,x+1])$, and the double integral over that region is bounded uniformly in x due to (1-1). Now the integral in (3-19) separates and simplifies using $\int_0^1 e^{-2kt}\,\mathrm{d}t = \frac{1}{2k} + O(e^{-2\operatorname{Re}k})$. By analogous arguments, using $q = s+t_2$ to rewrite the quadratic term and comparing the regions $\{(t_1,t_2,q)\mid 0\le t_2\le t_1\le 1,\ t_2\le q\le x+t_2\}$ and $\{(t_1,t_2,q)\mid 0\le t_2\le t_1\le 1,\ 0\le q\le x\}$,

$$\frac{1}{kx}\int_0^x\int_0^1\int_0^{t_1} e^{-2kt_1}(1-e^{-2kt_2})V(s+t_1)V(s+t_2)\,\mathrm{d}t_2\,\mathrm{d}t_1\,\mathrm{d}s \\ = \frac{1}{kx}\int_0^1\int_0^{t_1}\int_0^x e^{-2kt_1}(1-e^{-2kt_2})V(q+t_1-t_2)V(q)\,\mathrm{d}q\,\mathrm{d}t_2\,\mathrm{d}t_1 + O(x^{-1}) \\ = \frac{1}{kx}\int_0^1\int_0^x h(u)V(q+u)V(q)\,\mathrm{d}q\,\mathrm{d}u + O(x^{-1})$$

as $x\to\infty$, for any k. For the last step we introduced $u = t_1 - t_2 \in [0,1]$ and

$$h(u) = \int_0^{1-u} e^{-2k(u+t_2)}(1-e^{-2kt_2})\,\mathrm{d}t_2.$$

For the remaining double integral, it is elementary to estimate that $h(u) = O(|k|^{-1})$ uniformly in $u\in[0,1]$ and that

$$\frac{1}{x}\int_0^1\int_0^x |V(q+u)V(q)|\,\mathrm{d}q\,\mathrm{d}u \le C^2,$$

where C denotes the sup in (1-1), so (3-18) follows. □

4. Regular measures for half-line Schrödinger operators

The main part of this section is devoted to the study of limits of the function

$$h(x,z) := \frac{1}{x}\log|u(x,z)| \tag{4-1}$$

as $x\to\infty$. Our first goal is to show that for $z\in\mathbb{C}_+$ we have that $\liminf_{x\to\infty} h(x,z)\ge 0$.

Lemma 4.1. *Fix $z\in\mathbb{C}_+$. Then*

$$\liminf_{x\to\infty}\frac{1}{x}\log|u(x,z)| \ge 0.$$

Proof. Note first of all that $u(x,z) \neq 0$ whenever $x > 0$, because the converse would correspond to a complex eigenvalue for the self-adjoint realization of L_V on $[0,x]$ with Dirichlet boundary conditions. The Weyl solution $\psi(x,z)$ is an eigensolution and is in $L^2((0,\infty))$; the condition (1-1) is sufficient to conclude that ψ decays pointwise [Lukic 2013, Theorem 1.1], i.e.,

$$\lim_{x\to\infty} \psi(x,z) = 0.$$

Combining with Lemma 3.7 shows that $|u(x,z)| \to \infty$ as $x \to \infty$, which completes the proof. □

Let $\mathsf{E} = \sigma_{\mathrm{ess}}(L_V)$ written in the form (1-4). That is $b_0 = \min \mathsf{E}$ and (a_j, b_j) denote the gaps of E.

Lemma 4.2. *For any $\varepsilon > 0$ there exists $x_0 > 0$ such that $u(x,z) \neq 0$ for $x > x_0$ and $z \le b_0 - \varepsilon$. Moreover, let $n_j(\varepsilon)$ denote the finite number of eigenvalues in $(a_j+\varepsilon, b_j-\varepsilon)$. Then, for any $x > 0$, $u(x,z)$ has at most $n_j(\varepsilon)+1$ zeros in $(a_j+\varepsilon, b_j-\varepsilon)$.*

Proof. Since L_V is semibounded there are at most finitely many eigenvalues below $b_0 - \varepsilon$. Hence, the first statement follows by Sturm oscillation theory.

As in the proof of Lemma 3.7, we use the spectral-theoretic Green's function $g(x,x;z)$. By the Weyl M-matrix representation for L_V centered at x, $g(x,x;\cdot)$ is analytic on $\mathbb{C} \setminus \sigma(L_V)$ and, since it is Herglotz, it is strictly increasing on intervals in $\mathbb{R} \setminus \sigma(L_V)$. In particular, every pole of $g(x,x;\cdot)$ is an eigenvalue of L_V, so it has at most $n_j(\varepsilon)$ poles in $(a_j+\varepsilon, b_j-\varepsilon)$. By (3-16), every zero of $u(x,z)$ is a pole of $-(\partial_x u)(x,z)/u(x,z)$ and a zero of $g(x,x;\cdot)$. Since zeros and poles of the Herglotz function $g(x,x;\cdot)$ strictly interlace on intervals in the domain of meromorphicity, it follows that $u(x,z)$ has at most $n_j(\varepsilon)+1$ zeros in $(a_j+\varepsilon, b_j-\varepsilon)$. □

We are now ready to study the existence of limit points for the family of functions $\mathcal{F} = \{h(x,z)\}_{x\in[1,\infty)}$. Since $u(x,\cdot)$ are entire functions, the functions $h(x,\cdot)$ are subharmonic in $\mathbb{C}$, and they can be viewed as elements of the space of distributions $\mathcal{D}'(\mathbb{C})$ with nonnegative distributional Laplacian.

Theorem 4.3. (a) *The family $\mathcal{F} = \{h(x,z)\}_{x\in[1,\infty)}$ is precompact in $\mathcal{D}'(\mathbb{C})$.*

(b) *For any sequence $(x_j)_{j=1}^{\infty}$ with $x_j \to \infty$ such that $h(x_j,\cdot)$ converges in $\mathcal{D}'(\mathbb{C})$, the limit $h = \lim_{j\to\infty} h(x_j,\cdot)$ is also a subharmonic function on $\mathbb{C}$, harmonic on $\mathbb{C}\setminus\mathsf{E}$, and $h(x_j,\cdot)$ also converge to h uniformly on compact subsets of $\mathbb{C}\setminus\mathsf{E}$.*

Proof. (a) By Corollary 3.2, $h(x,z)$ is uniformly bounded from above on compact subsets of $\mathbb{C}$. Moreover, Lemma 4.1 implies a pointwise lower bound at some arbitrary point $z_0 \in \mathbb{C}_+$. Hence, [Hörmander 1983, Theorem 4.1.9] shows that $\mathcal{F}$ is precompact in the topology of $\mathcal{D}'(\mathbb{C})$.

(b) On $\mathbb{C}_+$ and on $\mathbb{C}_-$, the functions $h(x,z)$ are harmonic and uniformly bounded above. Since they are also pointwise bounded below, they are uniformly bounded and uniformly equicontinuous on each compact subset of $\mathbb{C}_\pm$. Therefore, they are precompact in the topology of uniform convergence on compact subsets of $\mathbb{C}_\pm$. Since this convergence implies convergence in $L^1_{\mathrm{loc}}(\mathbb{C}_\pm)$, it follows that if the sequence $h(x_j,\cdot)$ converges in $\mathcal{D}'(\mathbb{C})$ to h, then it also converges to h uniformly on compact subsets of $\mathbb{C}_\pm$.

Next, we show that h has a harmonic extension through an arbitrary gap (a_m, b_m) of E. Fix $\varepsilon > 0$. By Lemma 4.2, there are at most $n_m(\varepsilon)+1$ zeros of $u(x_j,z)$ in $(a_m+\varepsilon, b_m-\varepsilon)$. Let p_j be the monic polynomial of degree at most $n_m(\varepsilon)+1$ which vanishes exactly at these zeros. Now consider

$$f_j(z) = \frac{1}{x_j}\log\left|\frac{u(x_j,z)}{p_j(z)}\right|,$$

which is harmonic on $\mathbb{C}_+\cup\mathbb{C}_-\cup(a_m+\varepsilon, b_m-\varepsilon)$. On the boundary of the rectangle $(a_m-1, b_m+1)\times(-1,1)$, p_j is uniformly bounded below by 1, so by the maximum principle, the analytic functions $u(x_j,z)/p_j(z)$ are also bounded above by e^{cx_j} in this rectangle for some constant c. Hence, $f_j(z)$ is locally uniformly bounded above on $R_m = (a_m+\varepsilon, b_m-\varepsilon)\times(-1,1)$. Since all zeros of p_j are in (a_m, b_m), there is still a pointwise lower bound for $z_0\in\mathbb{C}_+$. Hence, the functions f_j are harmonic on R_m and precompact in the topology of uniform convergence on compacts. For any $z\in R_m\setminus\mathbb{R}$,

$$\lim_{j\to\infty}(h_j(z)-f_j(z)) = \lim_{j\to\infty}\frac{1}{x_j}\log|p_j(z)| = 0$$

since $|\operatorname{Im} z|^{n_m(\varepsilon)+1}\le|p_j(z)|\le(b_m-a_m+1)^{n_m(\varepsilon)+1}$. Hence, any subsequential limit of the $f_j(z)$ is a harmonic function on R_m which agrees with h on $R_m\setminus\mathbb{R}$. It follows that f_j converge in R_m uniformly on compacts, so it provides a harmonic extension for h through $(a_m+\varepsilon, b_m-\varepsilon)$. Since $\varepsilon>0$ was arbitrary and the extensions must coincide on their common domain, we obtain an extension through (a_m, b_m) by letting $\varepsilon\to 0$. It follows from the weak identity principle for subharmonic functions [Ransford 1995, Theorem 2.7.5] that the harmonic extension coincides with h.

Consider a compact $K\subset\mathbb{C}\setminus[b_0,\infty)$. By possibly increasing K, assume that $K\not\subset\mathbb{R}$. Choose an open set U such that $K\subset U\subset\overline{U}\subset\mathbb{C}\setminus[b_0,\infty)$. By Lemma 4.2, for all sufficiently large j, we have $h_j(z)$ is harmonic in U. The functions h_j are uniformly bounded above and pointwise bounded below at $z_0\in K\cap(\mathbb{C}_+\cup\mathbb{C}_-)$, so they form a precompact sequence with respect to uniform convergence on K. As before, every limit is equal to h, so h_j converge to h uniformly on compacts. □

Collecting our results now yields that the limits define a positive harmonic function in $\Omega=\mathbb{C}\setminus\mathsf{E}$.

Theorem 4.4. *Let $x_j\to\infty$ be a sequence such that $h_j = h(x_j,\cdot\,)$ converge in $\mathcal{D}'(\mathbb{C})$. Then $h=\lim_{j\to\infty} h_j$ defines a positive harmonic function in Ω, the limit*

$$a = \lim_{j\to\infty}\frac{1}{x_j}\int_0^{x_j} V(x)\,\mathrm{d}x \tag{4-2}$$

exists, and h has the nontangential asymptotic behavior

$$h(z) = \operatorname{Re}\Big(k+\frac{a}{2k}\Big)+O(|k|^{-2}), \tag{4-3}$$

$z\to\infty$, $\delta\le\arg z\le 2\pi-\delta$ for any $\delta>0$.

Proof. Harmonicity of h was proved in Theorem 4.3 and positivity in $\mathbb{C}_+\cup\mathbb{C}_-$ follows from Lemma 4.1. That h is also positive in $\mathbb{R}\setminus\mathsf{E}$ follows by the maximum principle for harmonic functions, and by Corollary 3.8,

$$h(z) = -\lim_{j\to\infty}\frac{1}{x_j}\operatorname{Re}\int_0^{x_j} m(x,z)\,\mathrm{d}x. \tag{4-4}$$

Define $c = \min \sigma(L_V)$. By general spectral theory, $m(x, z)$ are analytic functions on $\mathbb{C} \setminus [c, \infty)$ and $m(x, z) < 0$ on $(-\infty, c)$. Since convergence of analytic functions follows from convergence of their real parts together with convergence at one point, from $\operatorname{Im} m(x, z) = 0$ for $z < c$ together with (4-4), it follows that the limit

$$w(z) = \lim_{j\to\infty} \frac{1}{x_j} \int_0^{x_j} m(x, z)\,\mathrm{d}x$$

converges uniformly on compact subsets of $\mathbb{C} \setminus [c, \infty)$. If a denotes some accumulation point of the sequence $(1/x_j) \int_0^{x_j} V(x)\,\mathrm{d}x$, applying Corollary 3.9 along the subsequence and using uniformity of the error term, it follows that

$$w(z) = -k - \frac{a}{2k} + O(|k|^{-2}) \tag{4-5}$$

nontangentially as $z \to \infty$, with $\arg z \in [\delta, \pi - \delta]$. This asymptotic behavior can only hold for one value of a, so it follows that the limit (4-2) exists.

We know that (4-5) holds as $z \to \infty$ with $\arg z \in [\delta, \pi - \delta]$ and, by symmetry, for $\arg z \in [\pi + \delta, 2\pi - \delta]$. It remains to extend this asymptotic behavior to a sector of the form $\arg z \in [\pi - \delta, \pi + \delta]$. Without loss of generality assume $c = 0$. Since $\operatorname{Re} w = -h \le 0$, the function $f(\lambda) = -i w(\lambda^2)$ is Herglotz, and obeys

$$f(\lambda) = \lambda - \frac{a}{2\lambda} + O(|\lambda|^{-2}), \quad |\lambda| \to \infty, \tag{4-6}$$

along the rays $\arg \lambda = \pi/2 - \delta/2$ and $\arg \lambda = \pi/2 + \delta/2$. In the sector $T = \{\lambda : \pi/2 - \delta/2 \le \arg \lambda \le \pi/2 + \delta/2\}$, the function $g(\lambda) = \lambda^2 (f(\lambda) - \lambda + a/(2\lambda))$ is analytic. It has a continuous extension to $\overline{T}$ with $g(0) = 0$, because $f(\lambda) = O(1/\lambda)$ as $\lambda \to 0$ nontangentially. By (4-6), g is bounded on the boundary of T. Finally, since f is Herglotz, f, g grow at most polynomially as $\lambda \to \infty$, $\lambda \in T$, so by Phragmén–Lindelöf, g is bounded in T. This implies that f has the asymptotic behavior (4-6) also in the sector T. Rewriting the conclusion for w and $h = -\operatorname{Re} w$ completes the proof. □

We need the following variant of the Herglotz representation:

Lemma 4.5. *Let f be a Herglotz function. Assume* $\operatorname{Im} f(iy) = O(y^{-1})$ *as $y \to \infty$. Then for some $\beta \in \mathbb{R}$*

$$f(\lambda) = \beta + \int_{\mathbb{R}} \frac{\mathrm{d}\mu(t)}{t - \lambda}, \quad \textit{with} \lim_{y\to\infty} y \operatorname{Im} f(iy) = \mu(\mathbb{R}) < \infty,$$

and

$$f(\lambda) = \beta - \frac{\mu(\mathbb{R})}{\lambda} + o(|\lambda|^{-1}), \tag{4-7}$$

$\lambda \to \infty$, $\delta \le \arg \lambda \le \pi - \delta$ *for any* $\delta > 0$.

Proof. Starting from the Herglotz representation, we can write $\operatorname{Im} f(iy) = ay + \int y/(t^2 + y^2)\,\mathrm{d}\mu(t)$, with $\lim_{y\to\infty} \operatorname{Im} f(iy)/y = a$. Hence, by our assumption, $a = 0$. Moreover, by monotone convergence

$$\lim_{y\to\infty} y \operatorname{Im} f(iy) = \lim_{y\to\infty} \int \frac{y^2}{t^2 + y^2}\,\mathrm{d}\mu(t) = \mu(\mathbb{R}).$$

By our assumption, this shows that $\mu(\mathbb{R}) < \infty$. We have

$$\lambda \int_{\mathbb{R}} \frac{\mathrm{d}\mu(t)}{t - \lambda} + \mu(\mathbb{R}) = \int_{\mathbb{R}} \frac{t}{t - \lambda}\,\mathrm{d}\mu(t) \to 0 \quad \text{as } \lambda \to \infty,$$

by dominated convergence since $|t/(t - \lambda)| \le 1/\sin \delta$. □

We are now ready to prove an asymptotic expansion (1-3) of higher order for M_{E}.

Proof of Theorem 1.1. By translation, we may assume that $0 = \min \mathsf{E}$. By precompactness of the family $\{h(x,z)\}_{x\ge 1}$, there is a sequence $x_n \to \infty$ for which the limit $h = \lim_{n\to\infty}(1/x_n)\log|u(x_n,\cdot)|$ is convergent in $\mathcal{D}'(\mathbb{C})$. By Theorem 4.4, h is a positive harmonic function in Ω and $h(z)/\sqrt{-z} \to 1$ as $z \to -\infty$, so by Lemma 2.7, Ω is Greenian, obeys the Akhiezer–Levin condition, and $h \ge M_{\mathsf{E}}$ in Ω. Using (2-9), we obtain for $z \in \Omega$

$$\operatorname{Re}\sqrt{-z} \le M_{\mathsf{E}}(z) \le h(z). \tag{4-8}$$

Hence, the difference $M_{\mathsf{E}}(-k^2) - \operatorname{Re} k$ defines a positive harmonic function in Ω and (4-3), (4-8) imply that $M_{\mathsf{E}}(-k^2) - \operatorname{Re} k = O(|k|^{-1})$. Set $z = \lambda^2$ and $v(\lambda) = M_{\mathsf{E}}(-k^2) - \operatorname{Re} k$. We thus obtain a positive harmonic function in $\mathbb{C}_+$ such that $v(iy) = O(y^{-1})$. By Lemma 4.5 there is a constant c such that

$$v(\lambda) = -\operatorname{Im}\Big(\frac{c}{\lambda}\Big) + o(|\lambda|^{-1})$$

as $\lambda \to \infty$ nontangentially in $\mathbb{C}_+$. Recalling that $\lambda = ik$, this shows that

$$M_{\mathsf{E}}(-k^2) - \operatorname{Re} k = \operatorname{Re}\Big(\frac{c}{k}\Big) + o(|k|^{-1}). \qquad \square$$

Proof of Theorem 1.2. Consider a sequence $x_n \to \infty$ such that

$$\lim_{n\to\infty} \frac{1}{x_n}\int_0^{x_n} V(t)\,\mathrm{d}t = \liminf_{x\to\infty} \frac{1}{x}\int_0^{x} V(t)\,\mathrm{d}t.$$

Due to Theorem 4.3, this sequence has a subsequence for which the limit $h = \lim_{j\to\infty}(1/x_{n_j})\log|u(x_{n_j},\cdot)|$ is convergent in $\mathcal{D}'(\mathbb{C})$. As in the proof of Theorem 1.1, we have $h \ge M_{\mathsf{E}}$ in Ω. Theorems 1.1 and 4.4 yield

$$a_{\mathsf{E}} = \lim_{k\to+\infty} 2k(M_{\mathsf{E}}(-k^2) - k) \le \lim_{k\to+\infty} 2k(h(-k^2) - k) = \lim_{j\to\infty} \frac{1}{x_{n_j}}\int_0^{x_{n_j}} V(s)\,\mathrm{d}s. \qquad \square$$

Proof of Theorem 1.3. Fix $z_0 \in \mathbb{C}\setminus[\min \mathsf{E}, \infty)$ and consider a sequence $x_n \to \infty$ such that

$$\lim_{n\to\infty} \frac{1}{x_n}\log|u(x_n, z_0)| = \liminf_{x\to\infty} \frac{1}{x}\log|u(x, z_0)|.$$

We can again pass to a subsequence such that $h = \lim_{j\to\infty}(1/x_{n_j})\log|u(x_{n_j},\cdot)|$ and $h \ge M_{\mathsf{E}}$ in Ω. In particular,

$$\liminf_{x\to\infty} \frac{1}{x}\log|u(x, z_0)| = h(z_0) \ge M_{\mathsf{E}}(z_0). \qquad \square$$

Proof of Theorem 1.5. By inclusions, we have (vi) $\Rightarrow$ (iv) and (v) $\Rightarrow$ (iv).

(iv) $\Rightarrow$ (vi): Consider any sequence $x_j \to \infty$ such that the limit $h = \lim_{j\to\infty} h(x_j,\cdot)$ converges. The limit h obeys $h \ge M_{\mathsf{E}}$ on $\mathbb{C}_+$ by Theorem 1.3 and obeys $h(z) \le M_{\mathsf{E}}(z)$ for some $z \in \mathbb{C}_+$. By the maximum principle, $h = M_{\mathsf{E}}$ on $\mathbb{C}_+$, and then on Ω by harmonic continuation. Thus, M_{E} is the only possible subsequential limit of $h(x,\cdot)$ as $x \to \infty$, so by precompactness, $\lim_{x\to\infty} h(x,z) = M_{\mathsf{E}}(z)$ uniformly on compact subsets of $\mathbb{C}\setminus[b_0,\infty)$.

(vi) ⇒ (v): Given (vi), we know that for any convergent sequence $h(x_n, z)$ the limit is M_{E}. For $z \in [b_0, \infty)$ we have by [Azarin 2009, Theorem 2.7.4.1] that

$$\limsup_{n\to\infty} h(x_n, z) \le (\limsup_{n\to\infty} h(x_n, z))\check{} = M_{\mathsf{E}}(z),$$

where $\check{f}$ denotes the upper semicontinuous regularization of f. The first inequality follows by the general fact that $f \le \check{f}$.

(v) ⇒ (ii): This follows from Theorem 2.1.

(ii) ⇒ (iii): Due to [Garnett and Marshall 2005, Corollary 6.4] the set of Dirichlet-irregular points is polar and thus, by [loc. cit., Theorem 8.2] it is of harmonic measure zero and the claim follows.

(iii) ⇒ (vi): Take a sequence $x_n \to \infty$ such that $\lim_{n\to\infty} h(x_n, z) = h(z)$ in $\mathcal{D}'(\mathbb{C})$ and uniformly on compact subsets of $\mathbb{C} \setminus [b_0, \infty)$. Due to the upper envelope theorem [Azarin 2009, Theorem 2.7.4.1], there is a polar set X_1 such that, for any $z \in \mathbb{C} \setminus X_1$,

$$\limsup_{n\to\infty} h(x_n, z) = h(z).$$

On the other hand, assuming (iii), there exists X_2 with $\omega_{\mathsf{E}}(X_2, z_0) = 0$ such that, for $t \in \mathsf{E} \setminus (X_1 \cup X_2)$ by upper semicontinuity

$$0 \le \liminf_{\substack{z\to t\\ z\in\Omega}} h(z) \le \limsup_{\substack{z\to t\\ z\in\Omega}} h(z) \le h(t) \le 0.$$

Since $\omega_{\mathsf{E}}(X_1 \cup X_2, z_0) = 0$, Theorem 2.1 gives $h = cM_{\mathsf{E}}$. Comparing the leading-order asymptotic behavior at ∞ shows that $c = 1$. Thus, M_{E} is the only possible subsequential limit of $h(x, \cdot\,)$ as $x \to \infty$, so by precompactness, $\lim_{x\to\infty} h(x, z) = M_{\mathsf{E}}(z)$ uniformly on compact subsets of $\mathbb{C} \setminus [b_0, \infty)$.

(vi) ⇒ (i): By Theorem 4.4, (vi) implies that $(1/x_j) \int_0^{x_j} V(t)\,\mathrm{d}t \to a_{\mathsf{E}}$ for every sequence $x_j \to \infty$, so (i) follows.

(i) ⇒ (vi): Take a sequence $x_n \to \infty$ such that $h = \lim_{n\to\infty} h(x_n, \cdot\,)$ converges in $\mathcal{D}'(\mathbb{C})$. Define $v(\lambda) = h(-k^2) - M(-k^2)$. Similarly to the proof of $\mathbb{C}_+$. Theorem 1.1, this yields a positive harmonic function in By Theorems 4.4 and 1.1, $v(iy) = o(y^{-1})$ as $y \to \infty$. By Lemma 4.5, $\lim_{y\to\infty} yv(iy) = 0$ implies that $v \equiv 0$. This shows that M_{E} is the only subsequential limit of $h(x, \cdot\,)$ as $x \to \infty$. By precompactness, (vi) follows. □

The functions $u(x, z)$ are entire functions of order $\frac{1}{2}$ and as such admit a product representation

$$u(x, z) = u(x, z_*) \prod_{j=1}^{\infty} \left(1 - \frac{z - z_*}{z_j - z_*}\right),$$

where the z_j depend on x and z_* is some normalization point. Then the Riesz measure, ρ_x, of the subharmonic function $\log |u(x, z)|$ is a rescaled zero counting measure of $u(x, z)$. That is,

$$\frac{1}{x} \log |u(x, z)| = \frac{1}{x} \log |u(x, z_*)| + \int \log \left| 1 - \frac{z - z_*}{t - z_*} \right| \mathrm{d}\rho_x(t),$$

where ρ_x is defined in (1-7).

Proof of Theorem 1.6. By Theorems 1.5 and 4.3, $h(x,\cdot) \to M_{\mathsf{E}}$ in $\mathcal{D}'(\mathbb{C})$ as $x \to \infty$. By the definition of the Riesz measure, for any $\phi \in C_c^\infty(\mathbb{C})$,

$$\lim_{x\to\infty} 2\pi \int \phi(z)\,\mathrm{d}\rho_x(z) = \lim_{x\to\infty} \int h(x,z)\Delta\phi(z)\,\mathrm{d}\lambda(z)$$
$$= \int M_{\mathsf{E}}(z)\Delta\phi(z)\,\mathrm{d}\lambda(z) = 2\pi\int \phi(z)\,\mathrm{d}\rho_{\mathsf{E}}(z),$$

where $\mathrm{d}\lambda$ denotes the Lebesgue measure on $\mathbb{C}$. The rest follows from density of $C_c^\infty(\mathbb{C})$ in $C_c(\mathbb{C})$. □

Proposition 4.6. *Let* $\mathrm{d}\mu$ *be the spectral measure of* L_V, *where* V *satisfies* (1-1) *and* $\sigma_{\mathrm{ess}}(L_V) = \mathsf{E}$. *Suppose that along a sequence* $x_n \to \infty$ *the Riesz measures* $\mathrm{d}\rho_{x_n}$ *converge to* ρ_{E} *in the weak-∗ sense. Then*, *either* $h(x_n, z)$ *converges to* $M_{\mathsf{E}}(z)$ *or there exists a polar Borel set* X *such that* $\mu(\mathbb{R}\setminus X) = 0$.

Proof. Assume that $h(x_n,\cdot)$ do not converge to M_{E} and consider a subsequence x_{n_j} such that $h(x_{n_j},\cdot) \to h$ in $\mathcal{D}'(\mathbb{C})$ with some limit h not equal to M_{E}. By the upper envelope theorem [Azarin 2009, Theorem 2.7.4.1] there is a polar set X_1 such that, for any $z \in \mathbb{C}\setminus X_1$,

$$\limsup_{j\to\infty} h(x_{n_j}, z) = h(z).$$

The subharmonic function h has some Riesz measure ρ and by the same arguments as in the proof of Theorem 1.6, $\rho_{x_{n_j}}$ converges to ρ in the weak-∗ sense. Hence, by uniqueness of the limits our assumption implies that $\rho = \rho_{\mathsf{E}}$ and, by Lemma 2.2 applied to h and M_{E},

$$h(z) = h(z_*) + \int \log\left|1 - \frac{z - z_*}{t - z_*}\right| \mathrm{d}\rho_{\mathsf{E}}(t) = d + M_{\mathsf{E}}(z),$$

where $d = h(z_*) - M_{\mathsf{E}}(z_*)$. Recall that M_{E} has a unique subharmonic extension to $\mathbb{C}$ which vanishes quasi-everywhere on E. Therefore, there is a polar set X_2 such that $h(z) = d$ for $z \in \mathsf{E}\setminus X_2$. Moreover, since $M_{\mathsf{E}} \le h$ on Ω we see that $d \ge 0$, and since h is not equal to M_{E}, $d > 0$. In particular,

$$\limsup_{j\to\infty} h(x_{n_j}, z) = d > 0 \quad \forall z \in \mathsf{E}\setminus(X_1 \cup X_2).$$

However, by Schnol's theorem [1954], for μ-a.e. $z \in \mathsf{E}$, the Dirichlet solution decays at most polynomially and, in particular,

$$\limsup_{j\to\infty} h(x_{n_j}, z) \le 0.$$

Thus $\mu(\mathsf{E}\setminus(X_1\cup X_2)) = 0$, which implies the claim with $X = X_1 \cup X_2$. □

In particular, Theorem 1.7 is now proved.

Proof of Theorem 1.8. By Schnol's theorem [1954] for μ-a.e. $z \in \mathsf{E}$

$$\limsup_{x\to\infty} h(x, z) \le 0. \tag{4-9}$$

Hence, by assumption, (4-9) holds $\omega_\Omega(\cdot, z_0)$-a.e. Therefore, V is regular by Theorem 1.5. □

5. Applications

Proof of Theorem 1.12. (a) Setting $\mathsf{E} = \sigma_{\text{ess}}(L_V)$, it follows from $\mathsf{E} \subset [0, \infty)$ that M_{E} is a positive harmonic function on $\mathbb{C} \setminus [0, \infty)$. Since the Martin function for the domain $\mathbb{C} \setminus [0, \infty)$ is $\operatorname{Re}\sqrt{-z}$, it follows from Lemma 2.7 that $M_{\mathsf{E}}(z) \ge \operatorname{Re}\sqrt{-z}$. Comparing this with the asymptotic expansion (1-3) as $z \to -\infty$ shows that $a_{\mathsf{E}} \ge 0$ so, by (1-5), $\liminf_{x\to\infty} \frac{1}{x}\int_0^x V(t)\,dt \ge 0$.

(b) As in (a), $a_{\mathsf{E}} \ge 0$. By (1-5) and $\liminf_{x\to\infty} \frac{1}{x}\int_0^x V(t)\,dt \le 0$, this implies that $a_{\mathsf{E}} = 0$. Moreover, $M_{\mathsf{E}}(z) - \operatorname{Re}\sqrt{-z} = o(\sqrt{|z|}^{\,-1})$ defines a positive harmonic function in $\mathbb{C} \setminus [0, \infty)$ so, by Lemma 2.7, $M_{\mathsf{E}}(z) = \operatorname{Re}\sqrt{-z}$. If E was a proper subset of $[0, \infty)$, since E is closed, there would exist a gap $(a, b) \subset [0, \infty) \setminus \mathsf{E}$, and on this gap M_{E} would be strictly positive, contradicting $M_{\mathsf{E}}(z) = \operatorname{Re}\sqrt{-z}$.

(c) Again by $a_{\mathsf{E}} \ge 0$ and (1-5), $\limsup_{x\to\infty} \frac{1}{x}\int_0^x V(t)\,dt \le 0$ implies that V is regular. □

We now turn to the construction of a potential which is regular for $\mathsf{E} = [0, \infty)$ but not decaying, even in the Cesàro sense. The potential will be constructed piecewise, so we begin by considering a 2δ-periodic potential defined by

$$W_\delta(x) = \begin{cases} 1, & x \in [0, \delta), \\ -1, & x \in [\delta, 2\delta). \end{cases}$$

Let us compute the discriminant $\Delta_\delta(z)$ and the smallest eigenvalue for the periodic problem,

$$\lambda_\delta = \min\{\lambda \in \mathbb{R} \mid \Delta_\delta(\lambda) = 2\}.$$

Lemma 5.1. $\lim_{\delta\downarrow 0} \lambda_\delta = 0.$

Proof. Since $|W_\delta| \le 1$ and λ_δ is the minimum of the periodic spectrum, by standard variational principles, $\lambda_\delta \in [-1, 1]$ for all $\delta > 0$. The transfer matrix corresponding to W_δ at energy $\lambda \in (-1, 1)$ is

$$T_\delta(\lambda) = \begin{pmatrix} \cosh(\delta\sqrt{1-\lambda}) & \sinh(\delta\sqrt{1-\lambda})/\sqrt{1-\lambda} \\ \sqrt{1-\lambda}\sinh(\delta\sqrt{1-\lambda}) & \cosh(\delta\sqrt{1-\lambda}) \end{pmatrix} \begin{pmatrix} \cos(\delta\sqrt{1+\lambda}) & \sin(\delta\sqrt{1+\lambda})/\sqrt{1+\lambda} \\ -\sqrt{1+\lambda}\sin(\delta\sqrt{1+\lambda}) & \cos(\delta\sqrt{1+\lambda}) \end{pmatrix}.$$

From this it is elementary to obtain the asymptotic behavior for the discriminant, $\Delta_\delta(\lambda) = \operatorname{tr} T_\delta(\lambda)$, in the form

$$\Delta_\delta(\lambda) = 2 - 4\lambda\delta^2 + O(\delta^3), \quad \delta \downarrow 0, \tag{5-1}$$

uniformly in $\lambda \in (-1, 0)$ (and then, by continuity, for $\lambda \in [-1, 0]$). From this, it follows that, for any $t < 0$, there exists $\delta_0 > 0$ such that $\delta \in (0, \delta_0)$ and $\lambda \in [-1, t)$ implies $\Delta_\delta(\lambda) > 2$ and therefore $\lambda_\delta \ge t$. It follows that $\liminf_{\delta\downarrow 0} \lambda_\delta \ge 0$.

Meanwhile, $\Delta_\delta(0) = 2\cosh\delta\cos\delta = 2 - \delta^4/3 + o(\delta^4)$ as $\delta \to 0$ implies that $\limsup_{\delta\downarrow 0} \lambda_\delta \le 0$. □

Proof of Example 1.13. Consider the Dirichlet solution $u(x, t)$ corresponding to the given potential at some $t < 0$. There exists n_0 such that, for all $n \ge n_0$, $\lambda_{1/(2n)} > t$. At energies below the periodic spectrum, transfer matrices have strictly positive entries; applying this on intervals $[n, n+1]$ and since products of matrices with positive entries have positive entries, we conclude that $u(x, t)$ has at most one zero with $x > n_0 - 1$. Since zeros of an eigensolution are isolated, it follows that $u(\,\cdot\,, t)$ has finitely many zeros, so by Sturm oscillation theory, $\min \sigma_{\mathrm{ess}}(L_V) \ge t$. Since this holds for arbitrary $t < 0$, we conclude $\min \sigma_{\mathrm{ess}}(L_V) \ge 0$.

Conversely, since V obeys $\lim_{x\to\infty} \frac{1}{x}\int_0^x V(t)\,\mathrm{d}t = 0$, the statement is completed by Theorem 1.12. □

Proof of Example 1.11. For $x \in [x_n, x_{n+1}]$ we have

$$\frac{1}{x}\int_0^x V(t)\,\mathrm{d}t \le \int W(t)\,\mathrm{d}t\,\frac{n+1}{x_n}.$$

Since the condition on x_n implies that $x_n/n \to \infty$ we see that $\lim_{x\to\infty} \frac{1}{x}\int_0^x V(t)\,\mathrm{d}t = 0$. Since $V \ge 0$, we have $\sigma_{\mathrm{ess}}(L_V) \subset \sigma(L_V) \subset [0,\infty)$, so by Theorem 1.12, V is regular and $\sigma_{\mathrm{ess}}(L_V) = [0,\infty)$.

Let H_W be the whole-line operator with the potential $W(x)$. Since $W \ge 0$, we have $\sigma(H_W) \subset [0,\infty)$. Hence, we conclude that $\min \sigma(H_{-W}) < 0$, for otherwise [Damanik et al. 2005, Corollary 1] would imply that $W \equiv 0$. Now by [Last and Simon 2006, Theorem 7.1] it follows that $\sigma_{\mathrm{ess}}(H_{-V}) = \sigma(H_{-W})$ (where H_{-V} is the full-line operator with potential V extended to $\mathbb{R}_-$ by $V \equiv 0$). Since $\sigma_{\mathrm{ess}}(H_{-V}) = \sigma_{\mathrm{ess}}(L_0) \cup \sigma_{\mathrm{ess}}(L_{-V})$ this shows that $\min \sigma_{\mathrm{ess}}(L_{-V}) < 0$. □

Proof of Theorem 1.16. The Lyapunov exponent γ is harmonic in $\mathbb{C}_+ \cup \mathbb{C}_-$ and subharmonic in $\mathbb{C}$. By (1-11) for a.e. $\eta \in S$

$$\lim_{x\to\infty} \frac{1}{x}\log|u_\eta(x,z)| = \gamma(z)$$

converges pointwise in $\mathbb{C}_+ \cup \mathbb{C}_-$; by the weak identity principle for subharmonic functions and precompactness, convergence to γ is also in $\mathcal{D}'(\mathbb{C})$. By Schnol's theorem, for μ_η-a.e. z,

$$\limsup_{x\to\infty} \frac{1}{x}\log|u_\eta(x,z)| \le 0. \tag{5-2}$$

Fix a sequence $x_n \to \infty$. By the upper envelope theorem [Azarin 2009, Theorem 2.7.4.1] there is a polar set X_η such that, for any $z \in \mathbb{C} \setminus X_\eta$,

$$\limsup_{n\to\infty} \frac{1}{x_n}\log|u_\eta(x_n,z)| = \gamma(z).$$

On Q, $\gamma > 0$. Hence, since (5-2) holds for μ_η-a.e. z, we have $\mu_\eta(Q \setminus X_\eta) = 0$. □

6. Conformal maps

In view of Corollary 1.10 and the subsequent discussion, it is of great interest if the harmonic measure of the domain $\mathbb{C}\setminus\mathsf{E}$ is absolutely continuous with respect to the Lebesgue measure $\chi_{\mathsf{E}}(x)\,\mathrm{d}x$. Let $z_0 < \min E$ and $G_{\mathsf{E}}(z,z_0)$ be the Green's function of $\mathbb{C}\setminus\mathsf{E}$ with pole at z_0 and Π_{z_0} the associated comb domain, defined by the upper semicontinuous function s. We say that Π_{z_0} satisfies the sector condition if

$$S_{z_0}(x) = \sup_{y\in(0,\pi)} \frac{s_{z_0}(y)}{|x-y|}$$

is finite for Lebesgue-a.e. $x \in (0,\pi)$. Then, $\omega_{\mathsf{E}}(\,\cdot\,,z_0)$ is absolutely continuous with respect to the Lebesgue measure if and only if Π_{z_0} satisfies the sector condition.

The proceeding discussion holds for general semibounded sets E and does not assume that E is an Akhiezer–Levin set. Let M be the Martin function with pole at ∞, normalized at some internal point z_*,

ρ its Riesz measure and Π and Θ the corresponding comb and comb mapping. There is a similar characterization for absolute continuity of ρ. Let s be the upper semicontinuous function defining Π. Then ρ is absolutely continuous with respect to $\chi_{\mathsf{E}}(x)\,\mathrm{d}x$ if and only if the domain contains a Stolz angle at a.e. point at the base of the comb, i.e.,

$$S(x) = \limsup_{y\to x} \frac{s(y)}{|x-y|} \tag{6-1}$$

is finite for Lebesgue-a.e. $x \in (0, b)$.

Under various conditions on the set E, it is known that the conformal map $i\Theta'$ has a product representation. We now provide a general proof which does not assume Dirichlet regularity or any other additional assumptions.

Lemma 6.1. *Let E be a closed nonpolar set of the form* (1-4). *For each j there exists $c_j \in [a_j, b_j]$ such that M is strictly increasing on (a_j, c_j) and strictly decreasing on (c_j, b_j), and $\Theta'(z)$ is given on $z \in \mathbb{C}\setminus[b_0,\infty)$ by*

$$i\Theta'(z) = \frac{C}{\sqrt{b_0 - z}} e^{\int_{[b_0,\infty)\setminus\mathsf{E}} \xi(x) \frac{1+xz}{x-z} \frac{\mathrm{d}x}{1+x^2}} \tag{6-2}$$

where $\xi(x) = \frac{1}{2}$ for $x \in (a_j, c_j)$, $\xi(x) = -\frac{1}{2}$ for $x \in (c_j, b_j)$, $\xi(x) = 0$ for $x \notin [b_0,\infty)\setminus\mathsf{E}$, and $C > 0$ is a normalization constant.

Proof. For finite-gap sets, this is a reformulation of the Schwarz–Christoffel mapping. If E has infinitely many gaps, we consider them labeled by $j \in \mathbb{N}$ in an arbitrary way and define $\mathsf{E}_n = [b_0,\infty)\setminus\bigcup_{j=1}^n (a_j, b_j)$. Denote by M_n the Martin functions at ∞ corresponding to the sets E_n, normalized by $M_n(z_*) = 1$ for some fixed $z_* < b_0$. Since the functions M_n are all positive harmonic on $\mathbb{C}\setminus[b_0,\infty)$, for any $R > |b_0|$, by Harnack's principle they are uniformly bounded on the line segments parametrized by $-R+it$, $t+iR$, $t-iR$, with $t \in [-R, R]$. Since $M_n(x+iy)$ are increasing in $y > 0$ and symmetric, it follows that M_n are uniformly bounded above on the boundary of $(-R, R)\times(-R, R)$ for any R large enough. Since they are also nonnegative, they are a precompact sequence of subharmonic functions on $\mathbb{C}$. By the upper envelope theorem, for any subsequential limit $h = \lim_{k\to\infty} M_{n_k}$, quasi-everywhere on E, $h(z) = \lim_{k\to\infty} M_{n_k}(z) = 0$, so by Theorem 2.1, h is Martin function for the domain $\mathbb{C}\setminus\mathsf{E}$ with $h(z_*) = 1$. It follows that M_n converge to h in $\mathcal{D}'(\mathbb{C})$.

It follows that Θ_n converge to Θ since their real parts converge and their imaginary parts are zero on $(-\infty, b_0)$. In particular, the Herglotz functions $i\Theta_n'$ converge to $ci\Theta'$ uniformly on compact subsets of $\mathbb{C}_+$, so by interpreting this convergence in terms of their exponential Herglotz representations,

$$\lim_{n\to\infty} \int_{\mathbb{R}} g(x)\xi_n(x)\frac{\mathrm{d}x}{1+x^2} = \int_{\mathbb{R}} g(x)\xi(x)\frac{\mathrm{d}x}{1+x^2} \quad \forall g \in C(\mathbb{R}\cup\{\infty\}),$$

where ξ is determined by $\lim_{y\downarrow 0}\arg\Theta'(x+iy) = \pi\xi(x)$ Lebesgue-a.e. $x \in \mathbb{R}$. By using test functions g supported in (a_j, b_j), it follows that, for each j, the critical points $c_{j,n}$ must converge to a point $c_j \in [a_j, b_j]$. Then ξ_n converge pointwise to the function $\tilde{\xi}$ which is 1 on intervals (a_j, c_j), -1 on (c_j, b_j), and 0 on $[b_0, \infty)$, so by dominated convergence with dominating function $\|g\|_\infty (1/(1+x^2))\chi_{[b_0,\infty)\setminus \mathsf{E}}$,

$$\lim_{n\to\infty} \int_{\mathbb{R}} g(x)\xi_n(x)\frac{\mathrm{d}x}{1+x^2} = \int_{\mathbb{R}} g(x)\tilde{\xi}(x)\frac{\mathrm{d}x}{1+x^2} \quad \forall g \in C(\mathbb{R}\cup\{\infty\}).$$

Of course, this implies $\xi = \tilde{\xi}$, which implies (6-2). Finally, by separating the contribution from the gap (a_j, b_j) from the remainder of the integral, (6-2) can be extended into the gap (a_j, b_j) to show that $i\Theta' > 0$ on (a_j, c_j) and $i\Theta' < 0$ on (c_j, b_j). It follows that $M' > 0$ on (a_j, c_j) and $M' < 0$ on (c_j, b_j), so our construction of c_j as limits of $c_{j,n}$ satisfies the property in the lemma. □

As the final topic of this section, we describe a class of Akhiezer–Levin sets for which it can be seen by purely complex-theoretic arguments that the Martin function has the two-term expansion (1-3). While this is not as general as Theorem 1.1, within its scope of applicability, it provides a formula for a_{E} in terms of critical points of the Martin function.

Lemma 6.2. *Let* $\mathsf{E} \subset \mathbb{R}$ *be of the form* (1-4). *If* $\sum_{j=1}^{N}(b_j - a_j) < \infty$, *then* E *is an Akhiezer–Levin set*, *the Martin function obeys the two-term expansion* (1-3), *and*

$$a_{\mathsf{E}} = b_0 + \sum_{j=1}^{N}(a_j + b_j - 2c_j). \tag{6-3}$$

Proof. Finite gap length can be restated as $\int \chi_{[b_0,\infty)\setminus\mathsf{E}}(x)\mathrm{d}x < \infty$ and it implies that the exponent in (6-2) can be split into two separately integrable integrands, of which one is z-independent, to give

$$i\Theta'_{\mathsf{E}}(z) = \frac{C_{\mathsf{E}}}{\sqrt{b_0 - z}} e^{\int_{[b_0,\infty)\setminus\mathsf{E}} \xi(x)\frac{1}{x-z}\mathrm{d}x}.$$

For any $\delta > 0$, using finite gap length and dominated convergence,

$$\int_{[b_0,\infty)\setminus\mathsf{E}} \xi(x)\frac{1}{x-z}\,\mathrm{d}x = -\frac{1}{z}\int_{[b_0,\infty)\setminus\mathsf{E}} \xi(x)\,\mathrm{d}x + o(|z|^{-1}),$$

as $z \to \infty$, $\arg z \in [\delta, 2\pi - \delta]$. Evaluating the integral $\int_{[b_0,\infty)\setminus\mathsf{E}} \xi(x)\,\mathrm{d}x$ and substituting into $\Theta'(z)$,

$$i\Theta'_{\mathsf{E}}(z) = C_{\mathsf{E}}\left(\frac{1}{\sqrt{-z}} + \frac{1}{2}(b_0 + \sum_{j=1}^{N}(a_j + b_j - 2c_j))\frac{1}{\sqrt{-z}^{3}} + o(|z|^{-3/2})\right)$$

and integrating along rays shows that, as $z \to \infty$ with $\arg z \in [\delta, 2\pi - \delta]$,

$$i\Theta_{\mathsf{E}}(z) = C_{\mathsf{E}}\left(-2\sqrt{-z} + (b_0 + \sum_{j=1}^{N}(a_j + b_j - 2c_j))\frac{1}{\sqrt{-z}} + o(|z|^{-1/2})\right).$$

Taking imaginary parts gives a two-term expansion of M_{E}, which matches (1-3) with $C_{\mathsf{E}} = \frac{1}{2}$. Reading off the second term gives (6-3). □

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Received 13 Jun 2022. Revised 17 Sep 2023. Accepted 30 Nov 2023.

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ANALYSIS AND PDE
Vol. 18 (2025), No. 3, pp. 629–714
DOI: 10.2140/apde.2025.18.629

GLOBAL EXISTENCE AND MODIFIED SCATTERING FOR THE SOLUTIONS TO THE VLASOV–MAXWELL SYSTEM WITH A SMALL DISTRIBUTION FUNCTION

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The purpose of this paper is two-fold. In the first part, we provide a new proof of the global existence of the solutions to the Vlasov–Maxwell system with a small initial distribution function. Our approach relies on vector field methods, together with the Glassey–Strauss decomposition of the electromagnetic field, and does not require any support restriction on the initial data or smallness assumption on the Maxwell field. Contrary to previous works on Vlasov systems in dimension 3, we do not modify the linear commutators and avoid then many technical difficulties.

In the second part of this paper, we prove a modified scattering result for these solutions. More precisely, we obtain that the electromagnetic field has a radiation field along future null infinity and approaches, for large time, a smooth solution to the vacuum Maxwell equations. As for the Vlasov–Poisson system, in contrast, the distribution function converges to a new density function f_∞ along *modifications* of the characteristics of the free relativistic transport equation. In order to define these logarithmic corrections, we identify an effective asymptotic Lorentz force. By considering logarithmical modifications of the linear commutators, defined in terms of derivatives of the asymptotic Lorentz force, we finally prove higher-order regularity results for f_∞.

1. Introduction

The Vlasov–Maxwell system, which is used to describe the dynamics of collisionless plasma, can be written as

MSC2020: 35A01, 35B40, 35L99.
Keywords: Vlasov–Maxwell, scattering, asymptotic, stability.

$$\partial_t f + \hat{v} \cdot \nabla_x f + (E + \hat{v} \times B) \cdot \nabla_v f = 0, \tag{1}$$

$$\nabla_x \cdot E = \int_{\mathbb{R}^3_v} f \,\mathrm{d}v, \quad \partial_t E = \nabla_x \times B - \int_{\mathbb{R}^3_v} \hat{v} f \,\mathrm{d}v, \tag{2}$$

$$\nabla_x \cdot B = 0, \qquad \partial_t B = -\nabla_x \times E, \tag{3}$$

where

- $f : \mathbb{R}_{+,t} \times \mathbb{R}^3_x \times \mathbb{R}^3_v \to \mathbb{R}_+$ is the density distribution function of the particles,
- $\hat{v} = v/v^0$, with $v^0 := \sqrt{1+|v|^2}$, is the relativistic speed of a particle of momentum $v \in \mathbb{R}^3_v$,
- $\int_{\mathbb{R}^3_v} f \,\mathrm{d}v$ and $\int_{\mathbb{R}^3_v} \hat{v} f \,\mathrm{d}v$ are respectively the total charge density and the total current density,
- $E, B : \mathbb{R}_{+,t} \times \mathbb{R}^3_x \to \mathbb{R}^3$ are respectively the electric and the magnetic field.

For simplicity, we assume that the plasma is composed of one species of particles of charge $q = 1$ and mass $m = 1$. Our results can be extended without any additional difficulty to several families of particles of different charges and positive masses.[1] We refer to [Glassey 1996] for a detailed introduction to these equations.

The initial value problem for the Vlasov–Maxwell equations, together with a regular initial data set (f_0, E_0, B_0) composed of a function $f_0 : \mathbb{R}^3_x \times \mathbb{R}^3_v \to \mathbb{R}_+$ and two fields $E_0, B_0 : \mathbb{R}^3_x \to \mathbb{R}^3$ satisfying the constraint equations $\nabla_x \cdot E_0 = \int_v f_0 \,\mathrm{d}v$ and $\nabla_x \cdot B_0 = 0$, is well-posed [Wollman 1984]. On the other hand, the global existence problem for classical solutions to the Vlasov–Maxwell system is still open[2] and has only been addressed in some particular cases, such as under certain symmetry assumptions [Glassey and Schaeffer 1990; 1997; 1998; Luk and Strain 2016; Rein 1990; Wang 2022a]. For the general case, since the pioneering work [Glassey and Strauss 1986], several continuation criteria have been obtained [Glassey and Strauss 1987b; 1989; Klainerman and Staffilani 2002; Bouchut et al. 2003; Pallard 2005; 2015; Sospedra-Alfonso and Illner 2010; Luk and Strain 2014; Kunze 2015; Patel 2018].

1.1. ***Small data solutions to the Vlasov–Maxwell system.*** Much more is known for this particular perturbative regime, in which global existence holds and the solutions disperse. For small compactly supported initial data Glassey and Strauss [1987a] proved the optimal decay rate $\int_v f \,\mathrm{d}v \lesssim t^{-3}$ on the velocity average of the distribution function and obtained estimates for the electromagnetic field and its first-order derivatives. Shortly after, in the multispecies case, the smallness assumptions on the individual particle densities was relaxed by [Glassey and Schaeffer 1988]. Later, Schaeffer [2004] removed the support restriction on the velocity variable. However, his method leads to a loss on the estimate of $\int_v f \,\mathrm{d}v$.

It is only recently that all the compact support assumptions on the initial data were removed in two independent results [Bigorgne 2020a; Wang 2022b]. Both of these works are based on vector field methods and the latter used also Fourier analysis. These robust approaches allow for the derivation of sharp pointwise decay estimates on the solutions and their (high-order) derivatives. Moreover, in [Bigorgne 2020a], the initial decay hypothesis in v is optimal and improved estimates on certain *null* components of the electromagnetic field are derived. Finally, using the framework of Glassey and Strauss

[1] The case of massless particles requires in fact a different analysis [Bigorgne 2021b].

[2] In contrast, a global in time existence result for weak solutions was proved in [DiPerna and Lions 1989] and revisited in [Rein 2004].

and without any compact support restriction, Wei and Yang [2021] derived a global existence result which does not require the initial Maxwell field to be small.

In the first part of this article, we provide an alternative but shorter proof of the main results of [Bigorgne 2020a; Wang 2022b], without assuming any smallness assumption on the electromagnetic field. Compared to [Wei and Yang 2021], we require more regularity on the initial data but our method allows us to control the derivatives of the solutions, up to an arbitrary order N. This information is needed for the second part of the paper.

1.2. ***Modified scattering results for the Vlasov–Poisson system.*** Sharp decay estimates for the small data solutions to the Vlasov–Poisson system were first derived by [Bardos and Degond 1985] and then, with various improvements, by [Hwang et al. 2011; Smulevici 2016; Duan 2022; Schaeffer 2021] (for the relativistic cases, see [Glassey and Schaeffer 1985; Wang 2023; Bigorgne 2020b]). Modified scattering for these solutions was established in [Choi and Kwon 2016] and then in [Ionescu et al. 2022; Pankavich 2022], where more information was obtained on the asymptotic dynamics governing the modification of the linear characteristics. Furthermore, a scattering map has been constructed by [Flynn et al. 2023] and let us finally mention that similar results hold for perturbations of a point charge [Pausader and Widmayer 2021; Pausader et al. 2024].

In the second part of this paper, we investigate such problems in the context of the Vlasov–Maxwell equations. In particular, as in [Ionescu et al. 2022] for the Vlasov–Poisson system, we prove that

$$\int_{\mathbb{R}^3_x} f(t,x,v)\,\mathrm{d}x \to Q_\infty(v) \quad \text{as } t \to +\infty.$$

The scattering charge Q_∞ is deeply related to the leading-order term of the asymptotic expansion of both the charge density $\int_v f\,\mathrm{d}v$ and the current density. It allows us to define an asymptotic Lorentz force $v \mapsto \mathrm{Lor}(v)$, from which we deduce the modified scattering statement for f (see Theorem 1.1 and Remark 1.3 for more details). We also prove higher-order regularity properties for the limit distribution f_∞, which require a more thorough analysis. To our knowledge, there is no such regularity result for the Vlasov–Poisson system.

1.3. ***Vector field methods for relativistic transport equations.*** Our analysis of the asymptotic behavior of both the electromagnetic field and the distribution function relies on vector field methods (see Section 2.4 for an overview of the key ideas). This kind of technique was first developed by Klainerman [1985] in order to study solutions to nonlinear wave equations and then adapted in [Christodoulou and Klainerman 1990] to the Maxwell equations. It is only recently that the approach has been adapted to relativistic transport equations by Fajman, Joudioux and Smulevici [Fajman et al. 2017], leading in particular to a proof of the stability of Minkowski spacetime for both the massive and massless Einstein–Vlasov system [Fajman et al. 2021; Bigorgne et al. 2021] (see also [Lindblad and Taylor 2020; Taylor 2017] for alternative proofs). Our work [Bigorgne 2020a] concerning the small data solutions to the Vlasov–Maxwell system relies on such techniques as well. The method has also been successfully used to derive boundedness and decay estimates for the solutions to the massless Vlasov equation on a fixed Kerr black hole [Andersson et al. 2018; Bigorgne 2023]. Finally, even if it concerns the nonrelativistic setting, let us also mention that such approaches have been applied in the collisional regime [Chaturvedi 2021; 2022; Chaturvedi et al. 2023].

In order to deal with slowly decaying error terms, all the works on the small data solutions to massive relativistic Vlasov systems or the Vlasov–Poisson system [Fajman et al. 2021; Bigorgne 2020a; Smulevici 2016; Duan 2022], based on vector field methods, require dynamically modifying certain linear commutators of the Vlasov operator. One of the main novelties of this article consists in proving that the solutions are global without using these modified vector fields, which considerably simplifies the analysis. For this, even though certain quantities grow logarithmically in time, we are able to close the energy estimates by identifying several hierarchies in the commuted equations (see Section 2.8.2 for more details). It is then important to derive the optimal decay rate t^{-3} for $\int_v f \, \mathrm{d}v$ and its derivatives by a method allowing well-chosen weighted $W^{N,\infty}_{x,v}$ norms of the distribution function to grow slowly in time. We believe that this approach could be applied to other systems of equations, in particular for both the Einstein–Vlasov and the Vlasov–Poisson systems.

1.4. *The main result.* We present here a short version of our main result, stated in Theorems 2.10–2.11 below, where we also describe the behavior of the derivatives of the solutions.

Theorem 1.1. *Any solution (f, E, B) to the Vlasov–Maxwell system* (1)–(3) *arising from a small initial distribution function and smooth as well as sufficiently decaying initial data is global in time. Moreover*:

(1) *There exists a solution $(E^{\mathrm{vac}}, B^{\mathrm{vac}})$ to the vacuum Maxwell equations*[3] *approaching (E, B) as $t+|x| \to \infty$,*

$$\forall (t,x) \in \mathbb{R}_+ \times \mathbb{R}^3, \quad |E - E^{\mathrm{vac}}|(t,x) + |B - B^{\mathrm{vac}}|(t,x) \leq C_q (1+t+|x|)^{-1-q}, \quad \tfrac{1}{2} \leq q < 1.$$

(2) *The Lorentz force has a self-similar asymptotic profile $v \mapsto \mathrm{Lor}(v)$,*

$$\forall (t,x,v) \in \mathbb{R}_+ \times \mathbb{R}^3_x \times \mathbb{R}^3_v, \quad |t^2(E(t, x+t\hat{v}) + \hat{v} \times B(t, x+t\hat{v})) - \mathrm{Lor}(v)| \lesssim \langle x \rangle^2 |v^0|^8 \frac{\log^n(3+t)}{1+t},$$

where $\langle x \rangle := (1+|x|^2)^{1/2}$ and, say, $n = 70$. We have modified scattering to a new density function $f_\infty : \mathbb{R}^3_x \times \mathbb{R}^3_v \to \mathbb{R}_+$,

$$\forall t \geq 3, \quad \| f(t, X_{\mathscr{C}}(t, \cdot, \cdot), \cdot) - f_\infty \|_{L^1_{x,v} \cap L^\infty_{x,v}} \lesssim t^{-1} \log^n(t),$$

where the Cartesian components $X^k_{\mathscr{C}}$ of the modified spatial characteristics $X_{\mathscr{C}} \in \mathbb{R}^3_x$ are defined as

$$X^k_{\mathscr{C}}(t,x,v) := x^k + t\hat{v}^k - \frac{\log(t)}{v^0}(\mathrm{Lor}^k(v) - \hat{v} \cdot \mathrm{Lor}(v)\hat{v}^k), \quad 1 \leq k \leq 3.$$

Remark 1.2. No modification of the spatial characteristics is in fact required in the exterior of the light cone $\{|x| \geq t\}$ in order to prove such a result (see Section C.2). We already observed in [Bigorgne 2021a] that the small data solutions to the Vlasov–Maxwell system have better behavior in this region.

Similarly, no correction of the linear characteristics should in principle be necessary in order to prove a scattering statement in higher dimensions. This is consistent with the result of [Pankavich 2023]

[3] The vacuum Maxwell equations are given by (2)–(3) with $f = 0$.

concerning the Vlasov–Poisson system in dimension $d \geq 4$ and our study of the asymptotic behavior of the small data solutions of the Vlasov–Maxwell system in high dimensions [Bigorgne 2022].

The case of massless particles differs from the case of massive particles treated in this paper. Indeed, in view of [Bigorgne 2021b], we expect the small data solutions of the massless Vlasov–Maxwell system to satisfy linear scattering in dimension $d = 3$.

Remark 1.3. The behavior of the Lorentz force along the linear trajectories suggests that the characteristics of the Vlasov–Maxwell system satisfy, for $t \gg 1$,

$$\dot{X} = \widehat{V}, \quad \dot{V} \approx t^{-2}\,\mathrm{Lor}(V), \quad X(0) = x_0, \quad V(0) = v_0.$$

Hence, we can presume that V converges to v, so that

$$V(t) \approx v - \frac{1}{t}\,\mathrm{Lor}(v), \quad \dot{X}(t) \approx \hat{v} - \frac{1}{tv^0}\,\mathrm{Lor}(v) + \frac{\hat{v}\cdot\mathrm{Lor}(v)}{tv^0}\hat{v} + O(t^{-2}),$$

and we can then expect $X(t) \approx X_{\mathscr{C}}(t, x_0, v)$.

Moreover, we could in fact decompose $\mathrm{Lor}(v)$ as $E^\infty(v) + \hat{v} \times B^\infty(v)$ and observe that, as $v \to 0$,

$$X_{\mathscr{C}}(t, x, v) = x + tv - \log(t)E^\infty(v) + o(v).$$

In other words, for small velocities, the modified characteristics $X_{\mathscr{C}}$ of the Vlasov–Maxwell system approach the ones constructed in [Ionescu et al. 2022] for the Vlasov–Poisson system.

1.5. ***Structure of the paper.*** In Section 2 we introduce the notations and the tools used throughout this article. Then, we state our main results, Theorems 2.10–2.11, and present the key ideas of the proof. In Section 3, we set up the bootstrap assumptions and discuss their immediate consequences. Section 4 concerns the study of the distribution function. In particular, we prove that weighted $L^\infty_{x,v}$ norms of f and its derivatives grow at most logarithmically and we improve the bootstrap assumption on their velocity average. Then, in Section 5, we conclude the proof of the global existence of the small data solutions to (1)–(3) by exploiting the Glassey–Strauss decomposition of the electromagnetic field in order to improve the bounds on (E, B) and their derivatives. Next, in Section 6 we refine our estimates by proving that the particle current density and the electromagnetic field have a self-similar asymptotic profile. This allows us to define the modified trajectories along which the distribution function converges. Section 7 is devoted to the scattering results for the electromagnetic field. A crucial part of the proof consists in constructing a scattering map for the vacuum Maxwell equations. In Section 8, we relate the conserved total energy of the system to the ones of the scattering states. Finally, Appendices A and B contain two useful computations and Appendix C presents alternative expressions for the profile of F and the modified characteristics.

2. Preliminaries and detailed statement of the main result

2.1. ***Basic notations.*** In this paper we work on the 1+3-dimensional Minkowski spacetime $(\mathbb{R}^{1+3}, \eta)$. We will use two sets of coordinates, the Cartesian $(x^0 = t, x^1, x^2, x^3)$, in which $\eta = \mathrm{diag}(-1, 1, 1, 1)$, and null coordinates $(u, \underline{u}, \theta, \varphi)$, where

$$\underline{u} = t + r, \quad u = t - r, \quad r := |x| = \sqrt{|x^1|^2 + |x^2|^2 + |x^3|^2},$$

and $(\theta, \varphi) \in \,]0, \pi[\, \times \,]0, 2\pi[$ are spherical coordinates on the spheres of constant (t, r). These coordinates are defined globally on $\mathbb{R}^{1+3}$ apart from the usual degeneration of spherical coordinates and at $r = 0$. Sometimes, for a tensor field T defined on $\mathbb{R}_+ \times \mathbb{R}^3_x$, it will be convenient to write

$$T(u, \underline{u}, \omega) := T\left(\frac{\underline{u}+u}{2}, \frac{\underline{u}-u}{2}\omega\right), \quad \underline{u} \geq 0, \quad |u| \leq \underline{u}, \quad \omega \in \mathbb{S}^2.$$

We will work with the null frame $(L, \underline{L}, e_\theta, e_\varphi)$, where $L = 2\partial_u$, $\underline{L} = 2\partial_{\underline{u}}$ are null derivatives and (e_θ, e_φ) is the standard orthonormal basis on the spheres. More precisely,

$$L = \partial_t + \partial_r, \quad \underline{L} = \partial_t - \partial_r, \quad e_\theta = \frac{1}{r}\partial_\theta, \quad e_\varphi = \frac{1}{r\sin\theta}\partial_\varphi.$$

The Einstein summation convention will often be used; for instance $v^\mu \partial_{x^\mu} f = \sum_{\mu=0}^3 v^\mu \partial_{x^\mu} f$. The Latin indices goes from 1 to 3 and the Greek indices from 0 to 3. We will raise and lower indices using the Minkowski metric η, so that $x^i = x_i$ and $x^0 = -x_0$.

The four-momentum vector $\boldsymbol{v} = (v^\mu)_{0 \leq \mu \leq 3}$ is parametrized by $v = (v^i)_{1 \leq i \leq 3} \in \mathbb{R}^3_v$ and $v^0 = \sqrt{1+|v|^2}$ since the mass of the particles is equal to 1. Let $(v^L, v^{\underline{L}}, v^{e_1}, v^{e_2})$ be the null components of the momentum vector and $\not{v} = (v^{e_\theta}, v^{e_\varphi})$ its angular part, so that

$$\boldsymbol{v} = v^L L + v^{\underline{L}} \underline{L} + v^{e_\theta} e_\theta + v^{e_\varphi} e_\varphi, \quad v^L = \frac{v^0 + (x_i/r)v^i}{2}, \quad v^{\underline{L}} = \frac{v^0 - (x_i/r)v^i}{2}, \quad |\not{v}|^2 = |v^{e_\theta}|^2 + |v^{e_\varphi}|^2.$$

The relativistic speed $\hat{v} \in \mathbb{R}^3$ is given by $\hat{v}^i = v^i/v^0$ and, for convenience, we define the quantities

$$\hat{v}^0 := \frac{v^0}{v^0} = 1, \quad \hat{v}^L := \frac{v^L}{v^0}, \quad \hat{v}^{\underline{L}} := \frac{v^{\underline{L}}}{v^0}, \quad \hat{\not{v}} := \frac{\not{v}}{v^0}, \quad \hat{v}^{e_A} := \frac{v^{e_A}}{v^0}, \quad A \in \{\theta, \varphi\}.$$

Sometimes, we will write $(|v^0|^p g)(w)$ to denote $|w^0|^p g(w)$, where $w \in \mathbb{R}^3_v$ and $g : \mathbb{R}^3_v \to \mathbb{R}$.

In order to capture the good properties of certain geometric quantities associated to the solutions in the good null directions (L, e_θ, e_φ), we introduce the Faraday tensor $F_{\mu\nu}$, which is a 2-form, and the four-current density $J(f)_\mu$,

$$F = \begin{pmatrix} 0 & E^1 & E^2 & E^3 \\ -E^1 & 0 & -B^3 & B^2 \\ -E^2 & B^3 & 0 & -B^1 \\ -E^3 & -B^2 & B^1 & 0 \end{pmatrix}, \quad J(f) := \begin{pmatrix} -\int_{\mathbb{R}^3_v} f \, \mathrm{d}v \\ \int_{\mathbb{R}^3_v} (v_1/v^0) f \, \mathrm{d}v \\ \int_{\mathbb{R}^3_v} (v_2/v^0) f \, \mathrm{d}v \\ \int_{\mathbb{R}^3_v} (v_3/v^0) f \, \mathrm{d}v \end{pmatrix}. \tag{4}$$

The Cartesian components of F are then either equal to 0 or to a component of $\pm(E, B)$. We will in fact be more interested in its null decomposition $(\alpha(F), \underline{\alpha}(F), \rho(F), \sigma(F))$ defined, for $A \in \{\theta, \varphi\}$, as

$$\alpha(F)_{e_A} := F_{e_A L}, \quad \underline{\alpha}(F)_{e_A} := F_{e_A \underline{L}}, \quad \rho(F) := \tfrac{1}{2} F_{\underline{L} L}, \quad \sigma(F) := F_{e_\theta e_\varphi}. \tag{5}$$

In particular, $\rho(F) = E \cdot \partial_r$ and $-\sigma(F) = B \cdot \partial_r$ are the radial components of the electric field and the magnetic field. Moreover, the 1-forms $\alpha(F)$ and $\underline{\alpha}(F)$ are tangential to the 2-spheres and we will use the pointwise norms

$$|\alpha(F)|^2 := |\alpha(F)_{e_\theta}|^2 + |\alpha(F)_{e_\varphi}|^2, \quad |\underline{\alpha}(F)|^2 := |\underline{\alpha}(F)_{e_\theta}|^2 + |\underline{\alpha}(F)_{e_\varphi}|^2,$$
$$|F|^2 := \sum_{0 \leq \mu < \nu \leq 3} |F_{\mu\nu}|^2 = \tfrac{1}{2}|\alpha(F)|^2 + \tfrac{1}{2}|\underline{\alpha}(F)|^2 + |\rho(F)|^2 + |\sigma(F)|^2.$$

The Vlasov equation (1) can be rewritten as

$$\boldsymbol{T}_F(f) = 0, \quad \text{where } \boldsymbol{T}_F : f \mapsto \partial_t f + \hat{v} \cdot \nabla_x f + \hat{v}^\mu F_\mu{}^j \partial_{v^j} f, \tag{6}$$

and the Maxwell equations (2)–(3) take a concise form. The Gauss–Ampère law and the Gauss–Faraday law[4]

$$\nabla^\mu F_{\mu\nu} = J(f)_\nu, \quad \nabla^{\mu *}F_{\mu\nu} = 0, \tag{7}$$

where ${}^*F_{\mu\nu} = \frac{1}{2} F^{\lambda\sigma} \varepsilon_{\lambda\sigma\mu\nu}$ is the Hodge dual of F and ε is the Levi-Civita symbol. Here ∇ stands for the covariant derivative (or Levi-Civita connection), so that (7) holds in any coordinate system.

The operators ∇_x and ∇_v will denote the standard gradients in x and v respectively. For instance,

$$\nabla_x f = (\partial_{x^1} f, \partial_{x^2} f, \partial_{x^3} f), \quad \nabla_v f = (\partial_{v^1} f, \partial_{v^2} f, \partial_{v^3} f).$$

Given a 2-form G and $0 \leq \lambda \leq 3$, we will denote by $\nabla_{\partial_{x^\lambda}} G$ the covariant derivative of G according to ∂_{x^λ}, where $\partial_{x^0} = \partial_t$. For any multi-index $\kappa \in \{0, 1, 2, 3\}^p$, we define $\nabla^\kappa_{t,x} G := \nabla_{\partial_{x^{\kappa_1}}} \cdots \nabla_{\partial_{x^{\kappa_p}}} G$. In Cartesian coordinates, we then have

$$\nabla^\kappa_{t,x}(G)_{\mu\nu} = \partial^\kappa_{t,x}(G_{\mu\nu}), \quad 0 \leq \mu, \nu \leq 3.$$

Finally, for $x \in \mathbb{R}^3$ we will use the Japanese brackets $\langle x \rangle := (1 + |x|^2)^{1/2}$ and the notation $D_1 \lesssim D_2$ will stand for the statement that there exists $C > 0$ a positive constant independent of the solutions such as $D_1 \leq C D_2$.

2.2. ***Backward light cones and future null infinity.*** The scattering state for a smooth electromagnetic field F, which in our case is also called radiation field, will be a tensor field depending on the variables $(u, \omega) \in \mathbb{R} \times \mathbb{S}^2$. It will be obtained as the limit, when $\underline{u} \to +\infty$, of $rF(u, \underline{u}, \omega)$. For this reason, we introduce the backward light cones $\underline{C}_{\underline{u}}$ and give their induced volume form $\mathrm{d}\mu_{\underline{C}_{\underline{u}}}$ in accordance with the choice of the null vector field $\underline{L}$ as their normal vector. Let, for any $\underline{u} \geq 0$,

$$\underline{C}_{\underline{u}} := \{(t, x) \in \mathbb{R}_+ \times \mathbb{R}^3 \mid t + |x| = \underline{u}\}, \quad \mathrm{d}\mu_{\underline{C}_{\underline{u}}} = \tfrac{1}{2} r^2 \,\mathrm{d}u \,\mathrm{d}\mu_{\mathbb{S}^2},$$

where $\mathrm{d}\mu_{\mathbb{S}^2} = \sin(\theta)\,\mathrm{d}\theta\,\mathrm{d}\varphi$ is the volume form on $\mathbb{S}^2$.

Even if we will not need this formalism, we mention that the radiation field is in fact defined on a part of the conformal boundary of the Minkowski space, called future null infinity $\mathcal{I}^+$ and corresponding to the future end points of the null geodesics $t - |x| = u$. It can be viewed as $\underline{C}_{+\infty}$. More precisely,

$$(t, r, \omega) \mapsto (T(t, r) = \tan^{-1}(t + r) + \tan^{-1}(t - r), \quad R(t, r) = \tan^{-1}(t + r) - \tan^{-1}(t - r), \omega) \in \mathbb{R} \times \mathbb{S}^3$$

[4] Note that $\nabla^{\mu *}F_{\mu\nu} = 0$ is equivalent to $\nabla_{[\lambda} F_{\mu\nu]} := \nabla_\lambda F_{\mu\nu} + \nabla_\mu F_{\nu\lambda} + \nabla_\nu F_{\lambda\mu} = 0$.

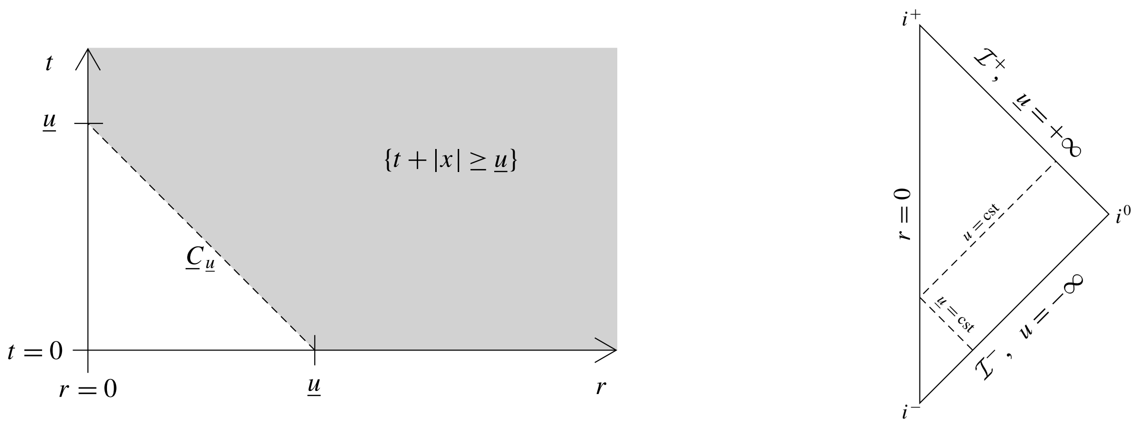

Figure 1. The set $\underline{C}_{\underline{u}}$ and the Penrose diagram of the Minkowski space.

is a conformal diffeomorphism between Minkowski spacetime and the interior of the triangle $0 \leq R \leq \pi$, $|T| = \pi - R$ of the space $\mathbb{R} \times \mathbb{S}^3$, equipped with the metric $-\mathrm{d}T^2 + \mathrm{d}R^2 + \sin^2(R)\, \mathrm{d}\mu_{\mathbb{S}^2}$. Then

$$\mathcal{I}^+ := \{(T, R, \omega) \in \mathbb{R} \times \mathbb{S}^3 \mid 0 < R < \pi,\ T = \pi - R\}.$$

Past null infinity $\mathcal{I}^-$ is defined similarly as $\{0 < R < \pi,\ T = R - \pi\}$ and can be viewed as $t - |x| = -\infty$. See Figure 1.

2.3. ***Charged electromagnetic field.*** For our global existence result, it will be sufficient to assume that the electromagnetic field satisfies $|F|(0, \cdot) \lesssim r^{-2}$, whereas our scattering result will require a slightly stronger initial decay hypothesis. However, if the plasma is not neutral, one cannot expect F to decay faster than r^{-2}. Indeed, if (f, F) is a sufficiently regular solution to (6)–(7) on $[0, T[$, we obtain from Gauss's law that the total charge

$$Q_F(t) := \lim_{r \to +\infty} \int_{\omega \in \mathbb{S}^2} \rho(F)(t, r\omega) r^2\, \mathrm{d}\mu_{\mathbb{S}^2} = \int_{x \in \mathbb{R}^3} \int_{v \in \mathbb{R}^3} f(t, x, v)\, \mathrm{d}v\, \mathrm{d}x, \qquad t \in [0, T[,$$

is a conserved quantity and that $|F| = o(r^{-2})$ implies $Q_F = 0$. In order to avoid such a restrictive assumption, we introduce the pure charge part $\overline{F}$ of F,

$$\overline{F}(t, x) := \frac{Q_F}{4\pi |x|^2} \frac{x_i}{|x|} dt \wedge dx^i, \qquad \rho(\overline{F})(t, x) = \frac{Q_F}{4\pi |x|^2}, \qquad \alpha(\overline{F}) = \underline{\alpha}(\overline{F}) = \sigma(\overline{F}) = 0, \tag{8}$$

which corresponds to the electromagnetic field generated by a point charge Q_F at $x = 0$. One can verify that $Q_{\overline{F}} = Q_F$, so that $F - \overline{F}$ is chargeless and it will then be consistent to assume that F has an asymptotic expansion of the form $F = \overline{F} + O(r^{-2-\delta})$, $\delta > 0$. In fact, $E = E^{\mathrm{df}} + E^{\mathrm{cf}}$ and $B = B^{\mathrm{df}} + B^{\mathrm{cf}}$ can be decomposed into their divergence-free and curl-free components. Then, $B^{\mathrm{cf}} = 0$ and $E^{\mathrm{cf}, i} = \overline{F}_{0i} + O(r^{-3})$ if $J(f)_0$ is sufficiently regular, so that the stronger initial decay assumption required for the scattering result concerns the divergence-free components of E and B.

2.4. ***Commutation vector fields.*** We will derive estimates on both the electromagnetic field and the distribution function using vector field methods. These kinds of approaches are usually based on

- a set of vector fields, which commute with the linear operator of the equation studied,
- energy inequalities, applied in order to prove boundedness for L^2 or L^1 norms of the solutions and their derivatives (for instance, see [Bigorgne 2020a, Section 4.1]),
- weighted Sobolev embeddings, such as [Fajman et al. 2017, Theorem 6], used to obtain decay estimates on the fields.

In this paper, in order to simplify the analysis, we will prove L^∞ estimates and then obtain pointwise decay estimates on the solutions in a different way (see Section 2.8 for more details). We now elaborate on the commutators for the Maxwell equations and the ones for the relativistic transport equation.

Definition 2.1. Let $\mathbb{K}$ be the set composed of the vector fields

$$\partial_t, \quad \partial_{x^i}, \quad \Omega_{0i} := t\,\partial_{x^i} + x^i\,\partial_t, \quad \Omega_{jk} := x^j\,\partial_{x^k} - x^k\,\partial_{x^j}, \quad S := t\,\partial_t + x^\ell\,\partial_{x^\ell} = t\,\partial_t + r\,\partial_r,$$

where $1 \leq i \leq 3$ and $1 \leq j < k \leq 3$. The translations ∂_{x^μ}, the Lorentz boosts Ω_{0i} and the rotations Ω_{jk} are Killing vector fields, so that they generate isometries of the Minkowski space. The scaling vector field S is merely conformal Killing.

We will use this set for differentiating the electromagnetic field geometrically. More precisely, for a 2-form F and a vector field $Z = Z^\mu\,\partial_{x^\mu}$, the Lie derivative $\mathcal{L}_Z(F)$ of F with respect to Z is given, in coordinates, by

$$\mathcal{L}_Z(F)_{\mu\nu} = Z(F_{\mu\nu}) + \partial_\mu(Z^\lambda)F_{\lambda\nu} + \partial_\nu(Z^\lambda)F_{\mu\lambda}.$$

Furthermore, if F is a smooth solution to the vacuum Maxwell equations $\nabla^\mu F_{\mu\nu} = \nabla^{\mu\,*}F_{\mu\nu} = 0$ and $Z \in \mathbb{K}$, then $\mathcal{L}_Z(F)$ is also a solution to the vacuum Maxwell equations, that is, $\nabla^\mu \mathcal{L}_Z(F)_{\mu\nu} = \nabla^{\mu\,*}\mathcal{L}_Z(F)_{\mu\nu} = 0$.

Definition 2.2. Let $\widehat{\mathbb{P}}_0$ be the set composed of

$$\partial_t, \quad \partial_{x^i}, \quad \widehat{\Omega}_{0i} := t\,\partial_{x^i} + x^i\,\partial_t + v^0\,\partial_{v^i}, \quad \widehat{\Omega}_{jk} := x^j\,\partial_{x^k} - x^k\,\partial_{x^j} + v^j\,\partial_{v^k} - v^k\,\partial_{v^j}, \quad S = t\,\partial_t + r\,\partial_r,$$

where $1 \leq i \leq 3$ and $1 \leq j < k \leq 3$. In fact, $\widehat{\partial}_{x^\mu} = \partial_{x^\mu}$, $\widehat{\Omega}_{0i}$ and $\widehat{\Omega}_{jk}$ are obtained as the complete lift, a classical operation in differential geometry,[5] of the Killing fields ∂_{x^μ}, Ω_{0i} and Ω_{jk}.

These vector fields have good commutation properties with the linear transport operator $\boldsymbol{T}_0 = \partial_t + \hat{v}\cdot\nabla_x$. Indeed, $[\boldsymbol{T}_0, S] = \boldsymbol{T}_0$ and $[v^0\boldsymbol{T}_0, \widehat{Z}] = 0$ for all $\widehat{Z} \in \widehat{\mathbb{P}}_0 \setminus \{S\}$.

In order to consider higher-order derivatives, we introduce an ordering on $\mathbb{K} = \{Z^i \mid 1 \leq i \leq 11\}$ and on $\widehat{\mathbb{P}}_0 = \{\widehat{Z}^i \mid 1 \leq i \leq 11\}$. It will be convenient to assume that $Z^{11} = \widehat{Z}^{11} = S$ and $\widehat{Z^i} = \widehat{Z}^i$ for any $1 \leq i \leq 10$. Moreover, for a multi-index $\beta \in [\![1, 11]\!]^p$ of length $|\beta| = p$, we denote by $\mathcal{L}_{Z^\beta}$ the

[5]We refer to [Fajman et al. 2017, Section 2G] for more details about the relations between the Vlasov operator on a Lorentzian manifold and the complete lift of its Killing vector fields.

Lie derivative $\mathcal{L}_{Z^{\beta_1}} \cdots \mathcal{L}_{Z^{\beta_p}}$ of order $|\beta|$. Similarly, we define $\widehat{Z}^\beta$ as $\widehat{Z}^{\beta_1} \cdots \widehat{Z}^{\beta_p}$. Note the equivalence between the pointwise norms

$$\sum_{|\gamma| \leq N} |\mathcal{L}_{Z^\gamma}(F)| \lesssim \sum_{|\beta| \leq N} \sum_{0 \leq \mu, \nu \leq 3} |Z^\beta(F_{\mu\nu})| \lesssim \sum_{|\gamma| \leq N} |\mathcal{L}_{Z^\gamma}(F)|. \tag{9}$$

Since $\mathcal{L}_{\partial_{x^\mu}}(F)$ and $\partial_{x^\mu} f$ have better behavior than the other derivatives, it will be crucial, in order to identify certain hierarchies in the commuted equations, to count the number of homogeneous vector fields composing Z^β or $\widehat{Z}^\beta$. We denote by β_H (respectively β_T) the number of homogeneous vector fields Ω_{0i}, Ω_{jk} and S (respectively translations ∂_{x^μ}) composing Z^β. Note that $\beta_H + \beta_T = |\beta|$ and that $\widehat{Z}^\beta$ is also composed of β_H homogenous vector fields and β_T translations. If $Z^\beta = \Omega_{01} \partial_t S$, we have $\beta_H = 2$ and $\beta_T = 1$.

The following geometric commutation formula, proved in [Bigorgne 2021b, Lemma 2.8], will be fundamental for us.

Lemma 2.3. *Let G be a 2-form and $g : [0, T[\times \mathbb{R}^3_x \times \mathbb{R}^3_v \to \mathbb{R}$ be a function, both of class C^1, such that*

$$\nabla^\mu G_{\mu\nu} = J(g)_\nu, \quad \nabla^{\mu *} G_{\mu\nu} = 0.$$

Let further $Z \in \mathbb{K} \setminus \{S\}$ be a Killing vector field and $\widehat{Z} \in \widehat{\mathbb{P}}_0 \setminus \{S\}$ be its complete lift. Then,

$$\begin{gathered}
\nabla^\mu \mathcal{L}_Z(G)_{\mu\nu} = J(\widehat{Z} g)_\nu, \quad \nabla^{\mu *} \mathcal{L}_Z(G)_{\mu\nu} = 0, \\
\nabla^\mu \mathcal{L}_S(G)_{\mu\nu} = J(Sg)_\nu + 3J(g)_\nu, \quad \nabla^{\mu *} \mathcal{L}_S(G)_{\mu\nu} = 0, \\
\widehat{Z}(v^\mu G_\mu{}^j \partial_{v^j} g) = v^\mu \mathcal{L}_Z(G)_\mu{}^j \partial_{v^j} g + v^\mu G_\mu{}^j \partial_{v^j} \widehat{Z} g, \\
S(v^\mu G_\mu{}^j \partial_{v^j} g) = v^\mu \mathcal{L}_S(G)_\mu{}^j \partial_{v^j} g - 2 v^\mu G_\mu{}^j \partial_{v^j} g + v^\mu G_\mu{}^j \partial_{v^j} Sg.
\end{gathered}$$

Iterating the above, we obtain that the structure of the Vlasov–Maxwell equations (6)–(7) is preserved by commutation.

Proposition 2.4. *Let (f, F) be a sufficiently regular solution to the Vlasov–Maxwell system. For any multi-index β, there exists $C^\beta_{\gamma,\kappa}, C^\beta_\xi \in \mathbb{Z}$ such that*

$$\begin{gathered}
\boldsymbol{T}_F(\widehat{Z}^\beta f) = \sum_{\substack{|\gamma| + |\kappa| \leq |\beta| \\ |\kappa| \leq |\beta| - 1}} C^\beta_{\gamma,\kappa} \hat{v}^\mu \mathcal{L}_{Z^\gamma}(F)_\mu{}^j \partial_{v^j} \widehat{Z}^\kappa f, \\
\nabla^\mu \mathcal{L}_{Z^\beta}(F)_{\mu\nu} = \sum_{|\xi| \leq |\beta|} C^\beta_\xi J(\widehat{Z}^\xi f), \quad \nabla^{\mu *} \mathcal{L}_{Z^\beta}(F)_{\mu\nu} = 0.
\end{gathered}$$

Moreover, the multi-indices $|\gamma| + |\kappa| \leq |\beta|$ satisfy $\gamma_H + \kappa_H \leq \beta_H$ and the equality $\kappa_H = \beta_H$ implies $\gamma_T \geq 1$.

Proof. For the condition on the multi-indices $|\gamma| + |\kappa| \leq |\beta|$, note from Lemma 2.3 that $\gamma_H + \kappa_H \leq \beta_H$ and $\gamma_T + \kappa_T = \beta_T$. Hence, if $\kappa_H = \beta_H$, we necessarily have $\kappa_T < \beta_H$ since $|\kappa| < |\beta|$. This implies $\gamma_T \geq 1$. □

2.5. ***Weights preserved along the linear flow.*** The set $\boldsymbol{k}_1$ of weight functions given by

$$z_{0i} := t\hat{v}^i - x^i, \quad z_{jk} := x^j \hat{v}^k - x^k \hat{v}^j, \quad 1 \leq i \leq 3, \ 1 \leq j < k \leq 3, \tag{10}$$

are conserved along any timelike straight line $t \mapsto (t, x + t\hat{v})$. They are obtained as $|v^0|^{-1}\eta(\boldsymbol{v}, K)$, where K is a Killing vector field[6] and they are then solutions to the relativistic transport equation, for all $z \in \boldsymbol{k}_1$, $\boldsymbol{T}_0(z) = 0$. As a consequence, if $\boldsymbol{T}_0(g) = 0$ then the same goes for zg, so that certain weighted norms of g are conserved. In our nonlinear setting these norms will grow logarithmically in time and will then provide useful decay properties on the Vlasov field. For convenience, we will rather work with

$$\boldsymbol{z} := \Big(1 + \sum_{z \in \boldsymbol{k}_1} z^2\Big)^{\frac{1}{2}}, \quad \boldsymbol{T}_0(\boldsymbol{z}) = \hat{v}^\mu \partial_{x^\mu}(\boldsymbol{z}) = 0. \tag{11}$$

In particular, as $z_{0i} \in \boldsymbol{k}_1$, one has

$$1 \leq \boldsymbol{z} \quad \text{and} \quad \forall (t, x, v) \in \mathbb{R}_+ \times \mathbb{R}^3_x \times \mathbb{R}^3_v, \quad \langle x \rangle \leq \boldsymbol{z}(t, x + t\hat{v}, v), \tag{12}$$

which will allow us to obtain space decay for $f(t, x + t\hat{v}, v)$, the particle density evaluated along the linear characteristics. Note also the following properties, which will be particularly useful for us in order to exploit the null structure of the system.

Lemma 2.5. *The four-momentum vector $\boldsymbol{v}$ has good null components, $v^{\underline{L}}$ and $\not{v}$. More precisely,*

$$\forall (t, x, v) \in \mathbb{R}_+ \times \mathbb{R}^3_x \times \mathbb{R}^3_v, \quad 0 < \hat{v}^{\underline{L}} \lesssim \frac{1 + |t - |x||}{1 + t + |x|} + \frac{\boldsymbol{z}}{1 + t + |x|}, \quad |\hat{\not{v}}| \lesssim \frac{\boldsymbol{z}}{1 + t + |x|}.$$

In certain circumstances, $v^{\underline{L}}$ will be the best component for exploiting decay in $t - r$. We will then use

$$|v^0|^{-2} + |\hat{\not{v}}|^2 \leq 4\hat{v}^{\underline{L}}.$$

Proof. The first two inequalities are proved in [Bigorgne 2020a, Lemma 2.4]; using

$$4v^0 v^{\underline{L}} \geq 4v^L v^{\underline{L}} = |v^0|^2 - \left|\frac{x^i}{r} v_i\right|^2 = 1 + |v|^2 - |v \cdot \partial_r|^2 = 1 + |v \cdot e_\theta|^2 + |v \cdot e_\varphi|^2 = 1 + |\not{v}|^2, \tag{13}$$

the last inequality follows. □

Since the particles are massive and then travel at a speed strictly lower than 1, the speed of light, Vlasov fields enjoy much better decay properties along null rays than along timelike geodesics $t \mapsto x + t\hat{v}$. After a long time, many of the particles should be located in the interior of the light cone. We will capture this property with the following inequality.

Lemma 2.6. *By losing powers of v^0 and $\boldsymbol{z}$, one can gain decay near the light cone $t = |x|$,*

$$\forall (t, x, v) \in \mathbb{R}_+ \times \mathbb{R}^3_x \times \mathbb{R}^3_v, \quad 1 \lesssim \frac{1 + |t - |x||}{1 + t + |x|}|v^0|^2 + \frac{|v^0|^2 \boldsymbol{z}}{1 + t + |x|}.$$

Moreover, in the exterior of the light cone, for $|x| \geq t$, one has $1 \lesssim (1 + t + |x|)^{-1}|v^0|^2\boldsymbol{z}$.

Proof. For the first inequality, note that (13) gives $1 \leq 4|v^0|^2\hat{v}^{\underline{L}}$ and apply Lemma 2.5. For the second one, we refer to [Bigorgne 2020a, Remark 2.5]. □

Recall from [Bigorgne 2020a, Lemma 3.2] that $\boldsymbol{z}$ enjoys good commutation properties with the vector fields of $\widehat{\mathbb{P}}_0$.

[6] On any smooth Lorentzian manifold (Y, g), if γ is a timelike geodesic and K a Killing vector field, then $g(\dot{\gamma}, K) = \text{constant}$.

Lemma 2.7. *For any $a \in \mathbb{R}$ and $\widehat{Z} \in \widehat{\mathbb{P}}_0$, we have $|\widehat{Z}(z^a)| \lesssim |a| z^a$.*

Finally, motivated by the fact that any regular solution to the linear relativistic transport equation $\boldsymbol{T}_0(h) = 0$ is constant along the timelike straight lines, $h(t, x + \hat{v}t, v) = h(0, x, v)$, it will sometimes be useful to work with $g(t, x, v) := f(t, x + t\hat{v}, v)$, in particular during the study of the asymptotic properties of $\int_v f \, \mathrm{d}v$ and its derivatives. The following result suggests that g enjoys strong space decay and that its v derivatives behave better than the ones of the distribution function f.

Lemma 2.8. *Let $f : [0, T[\times \mathbb{R}^3_x \times \mathbb{R}^3_v \to \mathbb{R}$ be a sufficiently regular function and $g(t, x, v) := f(t, x + t\hat{v}, v)$. Then the following properties hold*:

$$\langle x \rangle^a |g|(t, x, v) \leq |z^a f|(t, x + t\hat{v}, v), \quad v^0 |\nabla_v g|(t, x, v) \leq \sum_{\widehat{Z} \in \widehat{\mathbb{P}}_0} |z \widehat{Z} f|(t, x + t\hat{v}, v).$$

Proof. The first property follows from $z^2 \geq 1 + |z_{01}|^2 + |z_{02}|^2 + |z_{03}|^3$ and $|z_{0i}|(t, x + t\hat{v}, v) = |x^i|$. For the second one, we have, using the Einstein summation convention,

$$v^0 \partial_{v^j} g(t, x, v) = (v^0 \partial_{v^j} f)(t, x + t\hat{v}, v) + t \partial_{x^j} f(t, x + t\hat{v}, v) - t \hat{v}_j \hat{v}^i \partial_{x^i} f(t, x + t\hat{v}, v).$$

Then by $v^0 \partial_{v^j} = \widehat{\Omega}_{0j} - t \partial_{x^j} - x^j \partial_t$ and

$$x^j \partial_t + t \hat{v}^j \hat{v}^i \partial_{x^i} = (x^j - t\hat{v}^j)\partial_t + \hat{v}^j t \partial_t + \hat{v}^j (t\hat{v}^i - x^i)\partial_{x^i} + \hat{v}^j x^i \partial_{x^i} = -z_{0j} \partial_t + \hat{v}^j S + \sum_{1 \leq i \leq 3} \hat{v}^j z_{0i} \partial_{x^i}, \quad (14)$$

the result follows. □

2.6. *Inverse function of the relativistic speed.* In order to perform the change of variables $y = x - \hat{v}t$ for integrals on the domain $\mathbb{R}^3_v$, it will be useful to determine certain properties of the function $v \mapsto \hat{v}$.

Lemma 2.9. *We define, on the domain $\{y \in \mathbb{R}^3 \mid |y| < 1\}$, the operator $\check{\cdot}$ as*

$$y \mapsto \check{y} = \frac{y}{\sqrt{1 - |y|^2}}, \quad \text{so that} \quad \forall |y| < 1, \quad v \in \mathbb{R}^3_v, \quad \hat{\check{y}} = y, \quad \check{\hat{v}} = v.$$

Note also that $v^0 = (1 - |\hat{v}|^2)^{-1/2}$. Moreover, for any $(t, x) \in \mathbb{R}_+ \times \mathbb{R}^3$, the Jacobian determinant of the transformation $v \mapsto x - \hat{v}t$ is equal to $-t^3/|v^0|^5$.

Proof. The fact that $\check{\cdot}$ is the reciprocal function of $\hat{\cdot}$ can be obtained by direct computations. Let V be the column vector such that its transpose is $V^T = (v^1/v^0, v^2/v^0, v^3/v^0)$. Then the Jacobian determinant of the transformation $v \mapsto x - \hat{v}t$ is equal to

$$-\frac{t^3}{|v^0|^3} \det(\boldsymbol{I}_3 - V V^T) = -\frac{t^3}{|v^0|^3} \det\left(\operatorname{diag}\left(1, 1, 1 - \frac{|v|^2}{1 + |v|^2}\right)\right) = -\frac{t^3}{|v^0|^5}. \quad □$$

Let us also mention the inequality $2(1 - |\hat{v}|) \geq (1 - |\hat{v}|)(1 + |\hat{v}|) = |v^0|^{-2}$, which will be used several times throughout this paper.

2.7. *Complete version of the main result.* We are now ready to give a full and detailed version of Theorem 1.1. Recall the alternative geometric form (6)–(7) of the Vlasov–Maxwell equations (1)–(3).

Theorem 2.10. *Let $N \geq 3$ and (f_0, F_0) be an initial data set of class C^N for the Vlasov–Maxwell system. Consider further $\Lambda \geq \epsilon > 0$, two constants $(N_v, N_x) \in \mathbb{R}^2_+$ and assume that*

$$\sum_{|\gamma| \leq N+1} \sup_{x \in \mathbb{R}^3} \langle x \rangle^{2+|\gamma|} |\nabla_x^\gamma F_0|(x) \leq \Lambda, \qquad \sum_{|\beta|+|\kappa| \leq N} \sup_{(x,v) \in \mathbb{R}^6} \langle v \rangle^{N_v+|\kappa|} \langle x \rangle^{N_x+|\beta|} |\partial_v^\kappa \partial_x^\beta f_0|(x,v) \leq \epsilon.$$

If $N_v \geq 15$ and $N_x > 7$, there exist $D > 0$ and $\epsilon_0 > 0$, depending only on (N, N_v, N_x), such that, if $\bar{\epsilon} := \epsilon e^{D\Lambda} \leq \epsilon_0$, then the unique solution (f, F) to (1)–(3) arising from these data is global in time. Moreover:

- *The following pointwise estimates hold for the distribution function*:

$$\forall (t,x,v) \in \mathbb{R}_+ \times \mathbb{R}^3_v \times \mathbb{R}^3_x,\ \forall |\beta| \leq N, \quad |v^0|^{N_v-3} |z^{N_x-2} \widehat{Z}^\beta f|(t,x,v) \lesssim \bar{\epsilon} \log^{3N_x+3N}(3+t),$$
$$\forall |\kappa| \leq N, \qquad |v^0|^{N_v-3} |\partial_{t,x}^\kappa f|(t,x,v) \lesssim \bar{\epsilon}.$$

- *The electromagnetic field and its derivatives $\mathcal{L}_{Z^\gamma}(F)$, up to order $|\gamma| \leq N-1$, decay as,*

$$\forall (t,x) \in \mathbb{R}_+ \times \mathbb{R}^3, \quad |\mathcal{L}_{Z^\gamma} F|(t,x) \lesssim \Lambda (1+t+|x|)^{-1} (1+|t-|x||)^{-1}.$$

If $|\gamma| \leq N-2$, the good null components enjoy stronger decay properties near the light cone,

$$\forall (t,x,v) \in \mathbb{R}_+ \times \mathbb{R}^3, \quad |\alpha(\mathcal{L}_{Z^\gamma} F)|(t,x) + |\rho(\mathcal{L}_{Z^\gamma} F)|(t,x) + |\sigma(\mathcal{L}_{Z^\gamma} F)|(t,x) \lesssim \Lambda \frac{\log(3+t)}{(1+t+|x|)^2}.$$

Let us formulate two remarks.

(1) More estimates, such as $\int_v f \, \mathrm{d}v \lesssim t^{-3}$, are derived during the proof of Theorem 2.10.

(2) With our method, contrary to our previous work [Bigorgne 2020a], we cannot reach the optimal assumption $N_v = 3$. We list in Remark 3.3 below the precise parts of the proof where the control of higher spatial and momentum moments of f are required.

We now state our scattering result. For this, recall from (8) the definition of the pure charge part $\overline{F}$ of F.

Theorem 2.11. *Let $0 < \delta \leq 1$ and (f, F) be a smooth solution to the Vlasov–Maxwell system arising from initial data satisfying the assumptions of Theorem 2.10. Suppose further that the initial electromagnetic field has the asymptotic expansion*

$$\sum_{|\gamma| \leq N+1} \sup_{|x| \geq 1} \langle x \rangle^{2+\delta+|\gamma|} |\nabla_{t,x}^\gamma (F - \overline{F})|(0,x) \leq \Lambda. \tag{15}$$

Then, with $n := 7(N_x + N)$, we have the following properties.

- *The spatial average of f converges to a function $Q_\infty \in L^1(\mathbb{R}^3_v) \cap L^\infty(\mathbb{R}^3_v)$ of class C^{N-1},*

$$\forall t \in \mathbb{R}_+, \quad \left\| |v^0|^5 \left(\int_{\mathbb{R}^3_x} f(t,x,v) \, \mathrm{d}x - Q_\infty(v) \right) \right\|_{L^1_v \cap L^\infty_v} \lesssim \bar{\epsilon} \frac{\log^n(3+t)}{1+t}.$$

• *The four-current density* $J(\widehat{Z}^\beta f)_\mu = \int_v (v_\mu/v^0)\widehat{Z}^\beta f \,\mathrm{d}v$ *has the following self-similar asymptotic profile. For any* $|\beta| \leq N-1$ *and* $0 \leq \mu \leq 3$,

$$\forall t \in \mathbb{R}_+^*, \quad \sup_{|x|<t} \left| t^3 \int_{\mathbb{R}^3_v} \frac{v^\mu}{v^0} \widehat{Z}^\beta f(t,x,v)\,\mathrm{d}v - \frac{x^\mu}{t}(|v^0|^5 Q^\beta_\infty)\left(\frac{\check{x}}{t}\right)\right| \lesssim \bar{\epsilon}\,\frac{\log^n(3+t)}{t}, \quad x^0 = t,$$

where Q^β_∞ *can be computed in terms of* $\partial_v^\kappa Q_\infty$, $|\kappa| \leq |\beta|$. *Moreover,* $J(\widehat{Z}^\beta f)$ *decays much faster in the exterior of the light cone.*

• *The electromagnetic field and their derivatives up to order* $|\gamma| \leq N-1$ *have a self-similar asymptotic profile* $v \mapsto \mathcal{L}_{Z^\gamma}(F)^\infty(v)$,

$$\forall (t,x,v) \in \mathbb{R}_+ \times \mathbb{R}^3_x \times \mathbb{R}^3_v, \quad |t^2 \mathcal{L}_{Z^\gamma}(F)(t, x+\hat{v}t) - \mathcal{L}_{Z^\gamma}(F)^\infty(v)| \lesssim \Lambda \langle x\rangle^2 |v^0|^8 \frac{\log^n(3+t)}{(1+t)^\delta}.$$

F^∞ *is of class* C^{N-1} *and the components of* $\mathcal{L}_{Z^\gamma}(F)^\infty$ *can be computed in terms of* $\partial_v^\kappa F^\infty_{\mu\nu}$, $|\kappa| \leq |\gamma|$.

• *We have modified scattering to a state* $f_\infty \in L^1_{x,v} \cap L^\infty_{x,v}$ *of class* C^{N-2}. *For any* $|\kappa| + |\beta| \leq N-2$,

$$\forall t \geq 3, \quad \left\| |v^0|^{N_v - 10 + |\xi|} \langle x\rangle^{N_x - 4 - |\xi|} \left(\partial_v^\xi \partial_x^\kappa f(t, X_{\mathscr{C}}(t,x,v), v) - \partial_v^\xi \partial_x^\kappa f_\infty(x,v)\right)\right\|_{L^\infty_{x,v}} \lesssim \bar{\epsilon}\,\frac{\log^n(t)}{t^\delta},$$

where the corrections of the linear spatial characteristics are defined as

$$X^j_{\mathscr{C}}(t,x,v) := x^j + t\hat{v}^j - \frac{\log(t)}{v^0}\hat{v}^\mu\big(F^{\infty,j}_\mu(v) + \hat{v}^j F^\infty_{\mu 0}(v)\big), \quad 1 \leq j \leq 3. \tag{16}$$

• *The modified complete lifts, of the Lorentz boosts* $\widehat{\Omega}_{0k}$ *and the rotations* $\widehat{\Omega}_{jk}$, *and the modified scaling,*

$$\widehat{\Omega}^{\mathrm{mod}}_{\lambda k} := \widehat{\Omega}_{\lambda k} - \frac{\log(t)}{v^0}\hat{v}^\mu\big(\mathcal{L}_{\Omega_{\lambda k}}(F)^{\infty,j}_\mu(v) + \hat{v}^j \mathcal{L}_{\Omega_{\lambda k}}(F)^\infty_{\mu 0}(v)\big)\partial_{x^j}, \quad 0 \leq \lambda < k \leq 3,$$

$$S^{\mathrm{mod}} := S + \frac{\log(t)}{v^0}\hat{v}^\mu\big(F^{\infty,j}_\mu(v) + \hat{v}^j F^\infty_{\mu 0}(v)\big)\partial_{x^j},$$

satisfy the improved estimates $\|\widehat{\Omega}^{\mathrm{mod}}_{\lambda k} f(t,\cdot,\cdot)\|_{L^\infty_{x,v}} \lesssim \bar{\epsilon}$ *and* $\|S^{\mathrm{mod}} f(t,\cdot,\cdot)\|_{L^\infty_{x,v}} \lesssim \bar{\epsilon}$.

• *For any* $|\gamma| \leq N-3$, *there exists a scattering state* $\underline{\alpha}^{\mathcal{I}^+}_\gamma(u,\omega)$ *on* $\mathcal{I}^+$ *such that,*

$$\forall \underline{u} \geq 3, \quad \sup_{|u| \leq \underline{u},\, \omega \in \mathbb{S}^2} |r\underline{\alpha}(\mathcal{L}_{Z^\gamma}F)(u,\underline{u},\omega) - \underline{\alpha}^{\mathcal{I}^+}_\gamma(u,\omega)| \lesssim \Lambda \frac{\log(\underline{u})}{\underline{u}}.$$

Moreover, $\underline{\alpha}^{\mathcal{I}^+}$ *is of class* C^{N-3} *and* $\underline{\alpha}^{\mathcal{I}^+}_\gamma$ *can be expressed in terms of the derivatives of* $\underline{\alpha}^{\mathcal{I}^+}$.

• *The conserved energy of the system can be related to the ones of the scattering states. For all* $t \in \mathbb{R}_+$,

$$\int_{\mathbb{R}^3_x}\int_{\mathbb{R}^3_v} v^0 f(t,x,v)\,\mathrm{d}v\,\mathrm{d}x + \frac{1}{2}\int_{\mathbb{R}^3_x} |F|^2(t,x)\,\mathrm{d}x = \int_{\mathbb{R}^3_x}\int_{\mathbb{R}^3_v} v^0 f_\infty(x,v)\,\mathrm{d}v\,\mathrm{d}x + \frac{1}{4}\int_{\mathbb{R}^u}\int_{\mathbb{S}^2_\omega} |\underline{\alpha}^{\mathcal{I}^+}|^2(u,\omega)\,\mathrm{d}\mu_{\mathbb{S}^2}\,\mathrm{d}u.$$

• *If* $N \geq 10$, *there exists a solution* F^{vac} *of class* C^{N-5} *to the vacuum Maxwell equations* (19) *such that, for any* $\frac{1}{2} \leq q < 1$ *and* $|\gamma| \leq N-10$,

$$\forall (t,x) \in \mathbb{R}_+ \times \mathbb{R}^3, \quad |\mathcal{L}_{Z^\gamma}(F) - \mathcal{L}_{Z^\gamma}(F)^{\mathrm{vac}}|(t,x) \leq \Lambda C_q (1+t+|x|)^{-1-q}, \quad C_q > 0.$$

Remark 2.12. As suggested by the scattering result, we could improve the logarithmic powers in the $L^\infty_{x,v}$ estimates for f stated in Theorem 2.10. We could then prove that Theorem 2.11 holds for $n = 3N_x + 3N$. However, such a tiny improvement would require a relatively long and technical proof.

Remark 2.13. We emphasize two main differences with previous works on Vlasov systems in dimension 3 based on vector field methods [Fajman et al. 2021; Smulevici 2016; Bigorgne 2020a; Duan 2022].

(1) The logarithmic correction of the linear commutators $\widehat{\Omega}_{\lambda\nu}$ and S can be geometrically interpreted in terms of the asymptotic dynamic of the Lorentz force $\hat{v}^\mu F_{\mu k}$ and its derivatives (see also Remark 6.31).

(2) Our approach does not require modifying the linear commutators in order to prove the global existence of the solutions, so that we avoid many technical difficulties. In these previous works, the analysis of the Vlasov field relied on propagating $L^1_{x,v}$ bounds. The source term of the wave equations (or the Poisson equation) were estimated through weighted Sobolev embeddings as $t^3|Z^\beta \int_v f\,\mathrm{d}v| \le t^3 \int_v |\widehat{Z}^\beta f|\,\mathrm{d}v \lesssim \mathbb{E}(t)$, where $\mathbb{E}(t)$ is a certain $L^1_{x,v}$ norm. However, we know from Theorems 2.10–2.11 that, in general, $\|\widehat{Z} f\|_{L^1_{x,v}} \gtrsim \log(t)$ if $\widehat{Z} \neq \partial_{t,x}$. As a consequence, the optimal decay t^{-3} cannot be obtained in such a way without modifying the linear commutators.

Remark 2.14. The profile F^∞ of F can be explicitly expressed in terms of the limit of the spatial average Q_∞ (see Remark 6.17 and Appendix C.1). Moreover, the Maxwell field admits the decomposition $F = F^T + F^2$, where

$$\lim_{t\to+\infty} t^2 F(t, x + t\hat{v}) = \lim_{t\to+\infty} t^2 F^T(t, x + t\hat{v}) = F^\infty(v), \qquad \lim_{\underline{u}\to+\infty} r F^T(u, \underline{u}, \omega) = 0.$$

In other words, the part of the electromagnetic field which gives rise to F^∞ (respectively $\underline{\alpha}^{\mathcal{I}^+}$) has no impact on $\underline{\alpha}^{\mathcal{I}^+}$ (respectively F^∞).

2.8. ***Key ingredients of the proof.*** For the global existence result, our strategy relies on vector field methods and a continuity argument. The proof then essentially consists in improving bootstrap assumptions, which are pointwise decay estimates on the solutions and their derivatives. The scattering statements are then obtained by refining the analysis carried out during of the proof of Theorem 2.10 and by investigating further the asymptotic behavior of the electromagnetic field.

2.8.1. *The large Maxwell field.* The assumptions of Theorems 2.10–2.11 imply that, initially, the distribution function f is at most of size $\epsilon \ll 1$ and the electromagnetic field F is at most of size Λ. The goal of our bootstrap argument is to prove that these properties are preserved over time. Our proof allows for Λ to be large for the following reasons.

- Since the Maxwell equations are *linear*, we can expect $F(t, \cdot)$ and its derivatives to be at most of size $\Lambda + C\epsilon \sim \Lambda$, provided that ϵ is small enough. Here, the constant C possibly depends on Λ. Indeed, the data are bounded by Λ and we expect the source term $J(f)$ to remains of size ϵ.

- In contrast, the Vlasov equation is nonlinear and we can expect, at first glance, to bound $\|\partial^\kappa_{t,x} f(t, \cdot)\|_{L^\infty_{x,v}}$ by $\epsilon + D\Lambda\epsilon = C(\Lambda)\epsilon$.

In fact, since our argument will rely on Grönwall's inequality, $C(\Lambda)$ will rather be of the form $e^{D\Lambda}$. The difficulty, if Λ is large, is related to the logarithmic growth of quantities such as $\|\widehat{\Omega}_{01} f\|_{L^\infty_{x,v}}$. More precisely, certain error terms are at the threshold of time-integrability. Consequently a naive application of Grönwall's inequality would lead to $\|\widehat{\Omega}_{01} f\|_{L^\infty_{x,v}} \lesssim \epsilon(1+t)^{D\Lambda}$. We discuss how to circumvent this obstacle in the next section.

2.8.2. *Estimates for the Vlasov field.* In order to control sufficiently well the electromagnetic field and close our estimates, we would like to recover the linear decay for $\left|\int_v \widehat{Z}^\beta f(t,x,v)\,\mathrm{d}v\right| \lesssim t^{-3}$, with $|\beta| \leq N-1$, and similar quantities. This is done as follows:

- The main step consists in proving that $|v^0|^{N_v} z^{N_x} \widehat{Z}^\beta f$ grows slowly, and in fact logarithmically, in time.
- Then, by performing the standard change of variables $y = x - t\hat{v}$, we are able to reduce the problem to proving a uniform bound for the spatial averages $|v^0|^5 \int_y \widehat{Z}^\beta f(t,y,v)\,\mathrm{d}y$. This turns out to be a consequence of the first step as well but our argument requires a loss of regularity, which is why we do not attain the optimal decay t^{-3} for the top-order derivatives $|\beta| = N$.

Let us illustrate certain difficulties of the first step, which relies on Duhamel's formula, by considering the first-order derivatives. If $Z \in \mathbb{K} \setminus \{S\}$ is a Killing vector field, then

$$|\boldsymbol{T}_F(\widehat{Z} f)| = |\hat{v}^\mu \mathcal{L}_Z(F)_\mu{}^j \partial_{v^j} f| \lesssim \sum_{1 \leq j \leq 3} \frac{t+|x|}{v^0} |\hat{v}^\mu \mathcal{L}_Z(F)_\mu{}^j| |\partial_{t,x} f| + \text{better terms}. \tag{17}$$

Since $\mathcal{L}_Z(F)$ is supposed to decay as[7] $|\mathcal{L}_Z(F)| \lesssim \Lambda(1+t+|x|)^{-1}(1+|t-|x||)^{-1}$, there are two problems.

(1) The decay rate degenerates near the light cone $t = |x|$.

(2) Even far from the light cone, $|T_F(\widehat{Z} f)| \sim \Lambda t^{-1} |\partial_{t,x} f|$ is not integrable in time, preventing us from proving that $\|\widehat{Z} f\|_{L^\infty_{x,v}}$ grows slowly by a direct application of Grönwall's inequality if Λ is large.

We deal with the first issue by taking advantage of the null structure of the Lorentz force, which, roughly speaking, allows us to transform decay in $t-r$ into decay in $t+r$. More precisely, $\hat{v}^\mu \mathcal{L}_Z(F)_\mu{}^j$ can be decomposed as the sum of terms containing either a good null component α, ρ or σ of $\mathcal{L}_Z(F)$ or one of the good null components of $\hat{v}$. The first group enjoys improved decay estimates near the light cone, whereas the latter allows us to exploit the decay in $t-r$. We refer to Lemmas 4.1 and 4.4 for more details.

We circumvent the second problem by identifying hierarchies in the commuted equations. More precisely, if $Z = \partial_{x^\mu}$ is a translation, one can use that $|\mathcal{L}_{\partial_{x^\mu}}(F)| \lesssim t^{-1}(1+|t-|x||)^{-2}$ in order to prove that $T_F(\partial_{x^\mu} f)$ is in fact time-integrable. Then, one can observe that the system of the commuted Vlasov equations (17) is in some sense triangular and expect $\|\widehat{Z} f\|_{L^\infty_{x,v}}$ to grow at most logarithmically. A toy model for the system of the first-order commuted equations, once the null structure is well understood, is then

$$\boldsymbol{T}_F(g) = \Lambda(1+t)^{-2} g + \Lambda(1+t)^{-3} h, \quad \boldsymbol{T}_F(h) = \Lambda(1+t)^{-1} g + \Lambda(1+t)^{-2} h, \quad g \geq 0,\ h \geq 0,$$

where g is supposed to capture the behavior of $|\partial_{x^\mu} f|$, $0 \leq \mu \leq 3$, and h that of $|\widehat{Z} f|$, with $\widehat{Z}$ a homogeneous vector field such as $\widehat{\Omega}_{01}$. The source terms having h as a factor represent the strongly

[7] This pointwise decay estimate is consistent with the expected behavior of the source term of the Maxwell equations.

decaying error terms in (17). Using the Duhamel formula and applying Grönwall's inequality, we have, for $\mathbb{E}(t) := \|g(t,\cdot,\cdot)\|_{L^\infty_{x,v}} + \|h(t,\cdot,\cdot)\|_{L^\infty_{x,v}}$,

$$\mathbb{E}(t) \leq \mathbb{E}(0) + \int_{s=0}^{t} \frac{\Lambda}{1+s}\mathbb{E}(s)\,\mathrm{d}s, \qquad \mathbb{E}(t) \leq \mathbb{E}(0)(1+t)^{\Lambda}.$$

As mentioned earlier, without any smallness assumption on Λ, this estimate is not good enough to derive a satisfying decay estimate for $\int_v f\,\mathrm{d}v$. The idea then is to exploit that

$$\boldsymbol{T}_F(\log^{-1}(3+t)) \leq 0, \qquad \boldsymbol{T}_F(h\log^{-2}(3+t)) \leq \Lambda(1+t)^{-1}\log^{-2}(3+t)g + \Lambda(1+t)^{-2}h\log^{-2}(3+t).$$

By considering the hierarchized norm $\overline{\mathbb{E}}(t) := \|g(t,\cdot,\cdot)\|_{L^\infty_{x,v}} + \|h(t,\cdot,\cdot)\|_{L^\infty_{x,v}}\log^{-2}(3+t)$, we finally get

$$\overline{\mathbb{E}}(t) \leq \overline{\mathbb{E}}(0) + \int_{s=0}^{t} \frac{2\Lambda}{(1+s)\log^2(3+s)}\overline{\mathbb{E}}(s)\,\mathrm{d}s, \qquad \overline{\mathbb{E}}(t) \leq \overline{\mathbb{E}}(0)e^{2\Lambda}.$$

More generally, the hierarchies are determined by the number of homogeneous vector fields β_H composing $\widehat{Z}^\beta$ and the exponent of the weight $\boldsymbol{z}$.

A new difficulty arises for the higher-order derivatives since we do not have improved estimates at our disposal on the good null components of $\mathcal{L}_{Z^\gamma}(F)$ for $|\gamma| \geq N-1$. This time, we transform decay in $t-r$ into decay in $t+r$ by losing powers of $|v^0|^2\boldsymbol{z}$ through Lemma 2.6. For this, it is important to observe that, in the error terms, such a $\mathcal{L}_{Z^\gamma}(F)$ is always multiplied by a low-order derivative of f. We can then close the estimates by propagating weaker $L^\infty_{x,v}$ norms of $\widehat{Z}^\beta f$ when $|\beta| \geq N-1$.

Remark 2.15. Let us make some comparisons between the decay properties of the electromagnetic F and the ones of the electric field E associated to a solution to the Vlasov–Poisson system arising from small data.

- As $\|E(t,\cdot)\|_{L^\infty_x} \lesssim t^{-2}$ and $|F|(t,x) \lesssim t^{-1}(1+|t-|x||)^{-1}$, the electromagnetic field has a much weaker decay rate near the light cone $t=r$ than E.
- The difference is even more marked for their derivatives since $|\partial^\kappa_{t,x}E|(t,x) \lesssim t^{-2-|\kappa|}$, whereas we merely have $|\mathcal{L}_{\partial^\kappa_{t,x}}F|(t,x) \lesssim t^{-1}(1+|t-|x||)^{-1-|\kappa|}$. Thus, in order to exploit the extra decay provided by these derivatives of F, one has to take advantage of the null structure of the system or lose powers of $|v^0|^2\boldsymbol{z}$.

2.8.3. *Estimates for the electromagnetic field.* We control the Cartesian components of $\mathcal{L}_{Z^\gamma}(F)$ using the representation formula for the wave equation since, for instance, $\square F_{01} = -\int_v \partial_{x^1} f + \hat{v}^1\partial_t f\,\mathrm{d}v$. However, two difficulties arise for the higher-order derivatives:

(1) There is a loss of regularity. We need to control $\int_v \hat{v}^\mu \partial_{t,x}\widehat{Z}^\gamma f\,\mathrm{d}v$ in order to estimate $\mathcal{L}_{Z^\gamma}(F)$.

(2) With our method, we do not have the optimal decay rate for $\int_v \widehat{Z}^\gamma f\,\mathrm{d}v$, $|\gamma| = N$. Moreover, any logarithmic loss would prevent us from closing our estimates.

We treat the first problem by using the Glassey–Strauss decomposition [1986] of the electromagnetic field, presented in detail in Section 5.1. The idea is to express the derivatives ∂_{x^μ} in terms of derivatives tangential to backward light cones and $\boldsymbol{T}_0 = \partial_t + \hat{v}\cdot\nabla_x$, which is transverse to light cones. Exploiting then the Vlasov equation $\boldsymbol{T}_F(f) = 0$, we can perform integration by parts and save one derivative.

We deal with the second issue by estimating $\nabla_{t,x}\mathcal{L}_{Z^\xi}(F)$, for $|\xi| = N-1$, by the Glassey–Strauss decomposition of the derivatives of the electromagnetic field. Roughly speaking, it allows us to control the inhomogeneous part of $\nabla_{t,x}\mathcal{L}_{Z^\xi}(F)$ by $\int_v |v^0|^3|\widehat{Z}^\beta f|\,\mathrm{d}v$, where $|\beta| \leq N-1$ (see Proposition 5.7 and Corollary 5.8 for more details). However, with this process, we get a bad control of the other top-order derivatives near the light cone,

$$|\mathcal{L}_{ZZ^\xi}(F)|(t,x) \lesssim (1+t+|x|)|\nabla_{t,x}\mathcal{L}_{Z^\xi}F|(t,x)+|\mathcal{L}_{Z^\xi}F|(t,x) \lesssim (1+|t-r|)^{-2}\log(3+|t-r|), \quad |\xi| = N-1.$$

This forces us to lose a power more of $|v^0|^2 z$ for the estimates of the top-order derivatives of the Vlasov field f.

Once we proved that the solutions are global in time, we use null properties of the Maxwell equations (7) to derive the existence of a scattering state for F and its derivatives. We then address the problem of finding a solution F^{vac} to the vacuum Maxwell equations which approaches F by constructing a scattering map for these equations. For this, we make crucial use of the corresponding result for the homogeneous wave equation [Lindblad and Schlue 2023]. This is carried out in Section 7.

2.8.4. *Modified scattering result.* In the context of the Vlasov–Poisson system, except for the trivial solution, the distribution function does not converge along the linear characteristics [Choi and Ha 2011]. We then do not expect $f(t, x+t\hat{v}, v)$ to converge and the reason is related to the long-range effect of the Lorentz force (recall Remark 1.3). More precisely, isolating the leading-order term of the source term of the Maxwell equations,

$$\sup_{|x|<t}\left| t^3 \int_{\mathbb{R}^3_v} \frac{v^\mu}{v^0} f(t,x,v)\,\mathrm{d}v - \frac{x^\mu}{t}(|v^0|^5 Q_\infty)\left(\frac{\check{x}}{t}\right)\right| = O(t^{-\frac{\delta}{2}}), \qquad Q_\infty(v) := \lim_{t\to+\infty}\int_{\mathbb{R}^3_x} f(t,x,v)\,\mathrm{d}v,$$

where $x^0 = t$, we are able to prove $t^2 F(t, x+t\hat{v}) = F^\infty(v) + O(t^{-\delta/2})$. Consequently, the slow decay of the electromagnetic field along timelike trajectories implies that the right-hand side of

$$\partial_t(f(t,x+t\hat{v},v)) = \frac{t}{v^0}\hat{v}^\mu(F_\mu{}^j(t,x+t\hat{v})+\hat{v}^j F_{\mu 0}(t,x+t\hat{v}))\partial_{x^j} f(t,x+t\hat{v},v) + O(t^{-\frac{\delta}{2}})$$

should not be time-integrable, preventing $f(t, x+t\hat{v}, v)$ from converging. Instead, by considering the logarithmic corrections $X_{\mathscr{C}}$, given in (16), of the timelike straight lines, one can compensate for the worst term in the right-hand side of the previous identity and prove the modified scattering statement $f(t, X_{\mathscr{C}}, v) \to f_\infty(x, v)$.

Although the regularity of f_∞ according to x can be obtained in a similar fashion, the regularity in v requires a more thorough analysis. In fact, $v^0\partial_{v^i}(f(t, X_{\mathscr{C}}, v))$ can be expressed as terms such as $\widehat{\Omega}_{0i} f(t, X_{\mathscr{C}}, v)$ which, contrary to $\partial_{t,x} f(t, X_{\mathscr{C}}, v)$, does not converge. The reason is related to the weak decay of the error term $[\boldsymbol{T}_F, \widehat{\Omega}_{0i}] \sim t^{-1}$. As for the characteristics, the idea consists in considering a logarithmic correction of $\widehat{\Omega}_{0i}$, introduced and studied in Section 6.4, which has improved commutation properties with the Vlasov operator $\boldsymbol{T}_F$. As stated in Theorem 2.11, these corrections are given in terms of first-order derivatives of the effective electromagnetic field $F^\infty(v)$.

2.9. ***Null properties of electromagnetic fields.*** We recall here the classical results which will be used throughout this paper in order to study solutions to the Maxwell equations

$$\nabla^\mu F_{\mu\nu} = J_\nu, \quad \nabla^{\mu *}F_{\mu\nu} = 0, \tag{18}$$

where the source term $J = J_\mu \, dx^\mu$ is a sufficiently regular 1-form. In particular, solutions to the vacuum Maxwell equations will satisfy

$$\nabla^\mu F_{\mu\nu} = 0, \quad \nabla^{\mu *}F_{\mu\nu} = 0. \tag{19}$$

We point out that some of the estimates presented here could be refined in a general setting. For the purpose of performing energy estimates during the construction of the scattering map for (19), we recall the electromagnetic stress-energy tensor.

Definition 2.16. Let G be a 2-form of class C^1 such that $\nabla^\mu G_{\mu\nu} = J_\nu$ and $\nabla^{\mu *}G_{\mu\nu} = 0$. The energy-momentum tensor $\mathbb{T}[G]_{\mu\nu}$ is defined as

$$\mathbb{T}[G]_{\mu\nu} := G_{\mu\beta} G_\nu{}^\beta - \tfrac{1}{4}\eta_{\mu\nu} G_{\xi\lambda} G^{\xi\lambda}.$$

Moreover, we have

$$\nabla^\mu T[G]_{\mu\nu} = G_{\nu\lambda} J^\lambda, \quad T[G]_{LL} = |\alpha(G)|^2, \quad T[G]_{\underline{L}\underline{L}} = |\underline{\alpha}(G)|^2, \quad T[G]_{L\underline{L}} = |\rho(G)|^2 + |\sigma(G)|^2.$$

We now present inequalities relying on the relations

$$(t-r)\underline{L} = S - \frac{x^i}{r}\Omega_{0i}, \quad (t+r)L = S + \frac{x^i}{r}\Omega_{0i}, \quad re_\theta = -\cos(\varphi)\Omega_{13} - \sin(\varphi)\Omega_{23}, \quad re_\varphi = \Omega_{12}. \tag{20}$$

Lemma 2.17. *Let G be a sufficiently regular solution to the Maxwell equations* (18) *with a smooth source term J. Then*,

$$\forall |x| \geq \tfrac{1}{2}(1+t), \quad (|\nabla_{\underline{L}} \alpha(G)| + |\nabla_{\underline{L}} \rho(G)| + |\nabla_{\underline{L}} \sigma(G)|)(t,x) \lesssim |J|(t,x) + \sum_{|\gamma| \leq 1} \frac{|\mathcal{L}_{Z^\gamma}(G)|(t,x)}{1+t+|x|}$$

and,

$$\forall (t,x) \in \mathbb{R}_+ \times \mathbb{R}^3, \quad |\nabla_L(r\underline{\alpha}(G))|(t,x) \lesssim r|J|(t,x) + \sum_{|\gamma| \leq 1} |\rho(\mathcal{L}_{Z^\gamma} G)|(t,x) + |\sigma(\mathcal{L}_{Z^\gamma} G)|(t,x).$$

Remark 2.18. Compared to $Z \in \mathbb{K}$, $Z \neq \partial_{x^\mu}$, the derivatives tangential to the light cone (L, e_θ, e_φ) provide an extra decay in $t+r$, whereas $\underline{L}$ merely provides an additional decay in $t-r$. The second estimate then reflects that α, ρ and σ are the good null components. The last inequality provides an improved control of $\nabla_L(r\underline{\alpha})$ near the light cone and will be useful in order to prove the existence of scattering states.

Proof. Let us denote by $\not\nabla$ the intrinsic covariant differentiation on the spheres and by ζ any of the null components α, $\underline{\alpha}$, ρ or σ. Then, according to [Bigorgne 2021b, Lemma D.2], we have, for all $(t,x) \in \mathbb{R}_+ \times \mathbb{R}^3$,

$$(1+t+|x|)|\nabla_L \zeta(G)|(t,x) + (1+|x|)|\not\nabla \zeta(G)|(t,x) + (1+|t-|x||)|\nabla_{\underline{L}} \zeta(G)|(t,x) \lesssim \sum_{|\gamma| \leq 1} |\zeta(\mathcal{L}_{Z^\gamma} G)|(t,x).$$

We now express the Maxwell equations in null coordinates. According to [Christodoulou and Klainerman 1990, equations (M_1'')–(M_6'')], we have, for any $A \in \{\theta, \varphi\}$,

$$\nabla_{\underline{L}} \rho(G) - \frac{2}{r}\rho(G) - \not{\nabla}^{e_B} \underline{\alpha}(G)_{e_B} = J_{\underline{L}}, \quad \nabla_{\underline{L}} \alpha(G)_{e_A} - \frac{\alpha(G)_{e_A}}{r} + \not{\nabla}_{e_A} \rho(G) - \varepsilon^{AB} \not{\nabla}_{e_B} \sigma(G) = J_{e_A},$$
$$\nabla_{\underline{L}} \sigma(G) - \frac{2}{r}\sigma(G) + \varepsilon^{AB} \not{\nabla}_{e_A} \underline{\alpha}(G)_{e_B} = 0, \quad \nabla_{L} \underline{\alpha}(G)_{e_A} + \frac{\underline{\alpha}(G)_{e_A}}{r} - \not{\nabla}_{e_A} \rho(G) - \varepsilon^{AB} \not{\nabla}_{e_B} \sigma(G) = J_{e_A}.$$

This allows us to deduce the first estimate. For the last one, use the same arguments and remark further that $\nabla_L e_A = 0$ implies

$$|\nabla_L (r\underline{\alpha})| \lesssim \sum_{B \in \{\theta,\varphi\}} |\nabla_L (r\underline{\alpha})_{e_B}| = \sum_{B \in \{\theta,\varphi\}} |\nabla_L (r\underline{\alpha}_{e_B})| = \sum_{B \in \{\theta,\varphi\}} |r \nabla_L \underline{\alpha}(G)_{e_B} + \underline{\alpha}(G)_{e_B}|. \qquad \square$$

In the same spirit, we have the following identity which is proved in [Bigorgne 2020a, Proposition 3.7, equation (18)].

Lemma 2.19. *For any sufficiently regular* 2*-form* G *and any null component* $\zeta \in \{\alpha, \underline{\alpha}, \rho, \sigma\}$,

$$\forall (t,x) \in \mathbb{R}_+ \times \mathbb{R}^3, \quad |\zeta(\nabla_{t,x} G)|(t,x) \lesssim \sum_{|\gamma| \leq 1} \frac{|\zeta(\mathcal{L}_{Z^\gamma} G)|(t,x)}{1+|t-|x||} + \frac{|\mathcal{L}_{Z^\gamma}(G)|(t,x)}{1+t+|x|}.$$

We now illustrate how the previous lemmas can be used in order to obtain improved estimates for the good null components of the electromagnetic field.

Corollary 2.20. *Consider a* 2*-form* G *of class* C^1, *a solution to the Maxwell equations* (18) *with a continuous source term* J. *Assume that there exist two constants* $C[G] > 0$ *and* $q > 0$ *such that*,

$$\forall (t,x) \in \mathbb{R}_+ \times \mathbb{R}^3, \quad (1+t+|x|)|J|(t,x) + \sum_{|\gamma| \leq 1} |\mathcal{L}_{Z^\gamma}(G)|(t,x) \leq \frac{C[G]}{(1+t+|x|)(1+|t-|x||)^q}. \tag{21}$$

Then, *for all* $(t,x) \in \mathbb{R}_+ \times \mathbb{R}^3$,

$$(|\alpha(G)| + |\rho(G)| + |\sigma(G)|)(t,x) \lesssim C[G] \begin{cases} (1+t+|x|)^{-1-q} & \text{if } 0 < q < 1, \\ \log(3+t)(1+t+|x|)^{-2} & \text{if } q = 1, \\ (1+t+|x|)^{-2}(1+|t-|x||)^{-q+1} & \text{if } q > 1. \end{cases}$$

Moreover, *if* G *is merely defined on* $[0,T[\times \mathbb{R}^3$, $T > 0$, *we have the weaker estimate*, *for the case* $q > 1$,

$$\forall (t,x) \in [0,T[\times \mathbb{R}^3, \quad (|\alpha(G)| + |\rho(G)| + |\sigma(G)|)(t,x) \lesssim C[G](1+t+|x|)^{-2} \quad \text{if } q > 1.$$

Proof. Note first that the assumptions give $|G|(t,x) \lesssim (1+t+|x|)^{-1-q}$ if $1+t \geq 2|x|$ or $|x| \geq 2(1+t)$. We then fix $(t, r\omega) \in \mathbb{R}_+ \times \mathbb{R}^3$ such that $1+t \leq 2r \leq 4(1+t)$, $\omega \in \mathbb{S}^2$, and we denote by ζ any of the null components α, ρ or σ. Consider further

$$\phi(u, \underline{u}) := \zeta(G)\left(\frac{\underline{u}+u}{2}, \frac{\underline{u}-u}{2}\omega\right).$$

By Lemma 2.17 and (21), we have

$$|\nabla_{\partial_u}\phi|(u,\underline{u}) = \frac{1}{2}|\nabla_{\underline{L}}\zeta(G)|\left(\frac{\underline{u}+u}{2}, \frac{\underline{u}-u}{2}\omega\right) \lesssim \frac{C[G]}{(1+\underline{u})^2(1+|u|)^q}.$$

Now, note that, for $t-r \leq 0$,

$$\begin{aligned}|\zeta(G)|(t,r\omega) = |\phi|(t-r,t+r) &= |\phi(-t-r,t+r) + \int_{u=-t-r}^{-|t-r|} \nabla_{\partial_u}\phi(u,t+r)\,\mathrm{d}u| \\ &\lesssim |\zeta(G)|(0,t\omega+r\omega) + \frac{C[G]}{(1+t+r)^2}\int_{u=-t-r}^{-|t-r|}\frac{\mathrm{d}u}{(1+|u|)^q}.\end{aligned}$$

Similarly, if $t-r \geq 0$, we obtain by integrating between $u=t-r$ and $t+r$,

$$|\zeta(G)|(t,r\omega) \lesssim |\zeta(G)|(t+r,0) + \frac{C[G]}{(1+t+r)^2}\int_{u=|t-r|}^{t+r}\frac{\mathrm{d}u}{(1+|u|)^q}.$$

By (21),

$$|\zeta(G)|(t+r,0) + |\zeta(G)|(0,t\omega+r\omega) \lesssim C[G](1+t+r)^{-1-q}$$

and the first part of the result then follows from the computations of the integrals in the previous two estimates. For the case $q=1$, note that $\log(1+t+r) \leq 3\log(3+t)$ since $r \leq 2+2t$.

If G is merely defined on $[0,T[\times\mathbb{R}^3$ and $t<T$, then we cannot apply the previous computations in the case $t \geq r$. Instead, we integrate between $u=0$ and $t-r$ in order to get

$$|\zeta(G)|(t,r\omega) \lesssim |\zeta(G)|\left(\frac{t+r}{2}, \frac{t+r}{2}\omega\right) + \frac{C[G]}{(1+t+r)^2}\int_{u=0}^{|t-r|}\frac{\mathrm{d}u}{(1+|u|)^q}.$$

It remains to bound $|\zeta(G)|((t+r)/2,(t+r)\omega/2)$ by the estimate obtained in the region $t \leq r$ and to compute the integral in the three different cases. □

Finally, we prove pointwise decay estimates for a solution to the homogeneous wave equation. Since the Cartesian components $F_{\mu\nu}$ of a solution F to the vacuum Maxwell equations satisfy $\Box F_{\mu\nu}=0$, the next result will also allow us to estimate such electromagnetic fields.

Proposition 2.21. *Let ϕ be a C^2 solution to the free wave equation $\Box\phi=0$ such that*

$$\mathcal{E}^q[\phi] := \sup_{x\in\mathbb{R}^3}\langle x\rangle^q|\phi|(0,x) + \sup_{x\in\mathbb{R}^3}\langle x\rangle^{q+1}|\partial_{t,x}\phi|(0,x) < +\infty, \quad q \geq 2.$$

Then, there holds,

$$\forall (t,x)\in\mathbb{R}_+\times\mathbb{R}^3, \quad |\phi|(t,x) \lesssim \frac{\mathcal{E}^q[\phi]}{(1+t+|x|)(1+|t-|x||)^{q-1}}.$$

Proof. By Kirchhoff's formula we have

$$\phi(t,x) = \frac{1}{4\pi t^2}\int_{|y-x|=t}\phi(0,y)\,\mathrm{d}y + \frac{1}{4\pi t}\int_{|y-x|=t}\frac{y-x}{|y-x|}\cdot\nabla_y\phi(0,y) + \partial_t\phi(0,y)\,\mathrm{d}y.$$

We obtain the result by applying[8] [Wei and Yang 2021, Lemma 4.1], which gives that, for any $h \in C(\mathbb{R}^3)$ such that $|h|(x) \leq K_0(1+|x|)^{-p}$,

$$\int_{|y-x|=t} |h|(y)\,\mathrm{d}y \leq \begin{cases} 8\pi K_0 t^2 (1+t+|x|)^{-1}(1+|t-|x||)^{-p+1} & \text{if } 2 \leq p < 3, \\ 4\pi K_0 t (1+t+|x|)^{-1}(1+|t-|x||)^{-p+2} & \text{if } p \geq 3, \end{cases} \tag{22}$$

completing the proof. □

3. Strategy of the proof and the bootstrap assumptions

Let $N \geq 3$, $N_v \geq 15$, $N_x > 7$ and consider an initial data set (f_0, F_0) satisfying the hypotheses of Theorem 2.10. By a standard local well-posedness argument, there exists a unique maximal solution (f, F) to the Vlasov–Maxwell system arising from these data. Let $T_{\max} \in \mathbb{R}_+^* \cup \{+\infty\}$ such that the solution is defined on $[0, T_{\max}[$. By continuity, there exists a largest time $0 < T \leq T_{\max}$ and a constant $C_{\text{boot}} > 0$, independent of ϵ, such that the following bootstrap assumptions hold. For all $(t,x) \in [0,T[\times \mathbb{R}^3$,

$$\forall |\gamma| \leq N-1, \qquad |\mathcal{L}_{Z^\gamma}(F)|(t,x) \leq \frac{C_{\text{boot}}\Lambda}{(1+t+|x|)(1+|t-|x||)}, \tag{BA1}$$

$$\forall |\gamma| = N-1, \qquad |\nabla_{t,x}\mathcal{L}_{Z^\gamma}(F)|(t,x) \leq \frac{C_{\text{boot}}\Lambda \log(3+|t-|x||)}{(1+t+|x|)(1+|t-|x||)^2}, \tag{BA2}$$

$$\forall |\beta| \leq N-2, \quad \left| \int_{\mathbb{R}^3_v} \frac{v^\mu}{v^0} \widehat{Z}^\beta f(t,x,v)\,\mathrm{d}v \right| \leq \frac{C_{\text{boot}}\Lambda}{(1+t+|x|)^3}, \quad 0 \leq \mu \leq 3. \tag{BA3}$$

The goal consists in improving, for C_{boot} chosen large enough and if ϵ is small enough, (BA1)–(BA3). We stress that (BA3) will only be used for the proof of Proposition 3.1, where we improve the pointwise decay estimates for the good null components of the electromagnetic field.

3.1. ***Immediate consequences of the bootstrap assumptions.*** We start by improving, near the light cone, the estimates for the good null components of the electromagnetic field and its derivatives up to order $N-2$.

Proposition 3.1. *For any $|\gamma| \leq N-2$ and all $(t,x) \in [0,T[\times \mathbb{R}^3$, we have*

$$(|\alpha(\mathcal{L}_{Z^\gamma}F)| + |\rho(\mathcal{L}_{Z^\gamma}F)| + |\sigma(\mathcal{L}_{Z^\gamma}F)|)(t,x) \lesssim \frac{\Lambda \log(3+t)}{(1+t+|x|)^2(1+|t-|x||)^{\gamma_T}},$$

$$|\underline{\alpha}(\mathcal{L}_{Z^\gamma}F)|(t,x) \lesssim \frac{\Lambda}{(1+t+|x|)(1+|t-|x||)^{1+\gamma_T}},$$

where we recall that γ_T is number of translations ∂_{x^μ} composing Z^γ.

Proof. Consider $|\gamma| \leq N-2$ and recall from Proposition 2.4 that $\mathcal{L}_{Z^\gamma}F$ is solution to the Maxwell equations (18) with a source term which is a linear combination of $J(\widehat{Z}^\beta f)$, $|\beta| \leq N-2$, which are bounded by the bootstrap assumption (BA3). Hence, by applying Corollary 2.20 and using the bootstrap

[8] The case $2 < p < 3$, not considered by [Wei and Yang 2021], can be treated as the case $p = 2$ since $\int_b^a \lambda/(1+\lambda)^p\,\mathrm{d}\lambda \leq (1+b)^{p-2}\int_b^a \lambda/(1+\lambda)^2\,\mathrm{d}\lambda$.

assumption (BA1), we get

$$(|\alpha(\mathcal{L}_{Z^\gamma}F)|+|\rho(\mathcal{L}_{Z^\gamma}F)|+|\sigma(\mathcal{L}_{Z^\gamma}F)|)(t,x)\lesssim \Lambda\log(3+t)(1+t+|x|)^{-2},$$
$$|\underline{\alpha}(\mathcal{L}_{Z^\gamma}F)|(t,x)\lesssim |\mathcal{L}_{Z^\gamma}(F)|(t,x)\lesssim \Lambda(1+t+|x|)^{-1}(1+|t-|x||)^{-1}.$$

Now, note that, for any $0\leq\mu\leq 3$ and $Z\in\mathbb{K}$, we have $[Z,\partial_{x^\mu}]=0$ or $[Z,\partial_{x^\mu}]=\pm\partial_{x^\lambda}$ for a certain $0\leq\lambda\leq 3$. As a consequence, and since $\mathcal{L}_{\partial_{x^\mu}}=\nabla_{\partial_{x^\mu}}$, there exists constants $D^\gamma_{\kappa,\xi}\in\mathbb{N}$ such that

$$\mathcal{L}_{Z^\gamma}(F)=\sum_{|\kappa|=\gamma_T}\sum_{|\xi|\leq|\gamma|-\gamma_T}D^\gamma_{\kappa,\xi}\mathcal{L}_{\partial^\kappa_{t,x}Z^\xi}(F)=\sum_{|\kappa|=\gamma_T}\sum_{|\xi|\leq|\gamma|-\gamma_T}D^\gamma_{\kappa,\xi}\nabla^\kappa_{t,x}\mathcal{L}_{Z^\xi}(F). \tag{23}$$

The result then follows from γ_T applications of Lemma 2.19. □

In contrast, we have very bad control of the top-order derivatives near the light cone.

Proposition 3.2. *For any $|\gamma|=N$, there holds,*

$$\forall(t,x)\in[0,T[\times\mathbb{R}^3,\quad |\mathcal{L}_{Z^\gamma}F|(t,x)\lesssim\Lambda\frac{\log(3+|t-|x||)}{(1+|t-|x||)^{2+\gamma_T}}.$$

If $|\gamma|\leq N-1$, we have the better estimate,

$$\forall(t,x)\in[0,T[\times\mathbb{R}^3,\quad |\mathcal{L}_{Z^\gamma}F|(t,x)\lesssim\Lambda(1+t+|x|)^{-1}(1+|t-|x||)^{-1-\gamma_T}.$$

Proof. Let $|\gamma|=N$, $(t,x)\in[0,T[\times\mathbb{R}^3$ and note that $|\mathcal{L}_ZG|\lesssim(1+t+r)|\nabla_{t,x}G|+|G|$ for any $Z\in\mathbb{K}$ and any 2-form G. Consequently, we obtain from the bootstrap assumptions (BA1)–(BA2) that,

$$|\mathcal{L}_{Z^\gamma}F|(t,x)\lesssim(1+t+|x|)\frac{\Lambda\log(3+|t-|x||)}{(1+t+|x|)(1+|t-|x||)^2}+\frac{\Lambda}{(1+t+|x|)(1+|t-|x||)}\lesssim\Lambda\frac{\log(3+|t-|x||)}{(1+|t-|x||)^2}.$$

As previously, when $\gamma_T\geq 1$, the extra decay in $t-r$ is given by (23) and Lemma 2.19. The case $|\gamma|\leq N-1$ is easier and follows from (BA1), (23) and Lemma 2.19. □

3.2. ***Structure of the proof.*** The remainder of the paper is divided as follows.

(1) First, in Section 4, we prove that, for any $|\beta|\leq N$, an $L^\infty_{x,v}$ norm of $\widehat{Z}^\beta f$, weighted by powers of v^0 and z, grows at most logarithmically in time. Next, we control uniformly in time weighted space averages of $\widehat{Z}^\beta f$ for any $|\beta|\leq N-1$. This will allow us to prove, in Section 4.4, decay estimates for $\int_v\widehat{Z}^\beta f\,\mathrm{d}v$ and improve (BA3).

(2) Then, we introduce the Glassey–Strauss decomposition of the electromagnetic field in Section 5.1. It allows us to improve the bootstrap assumptions (BA1) and (BA2), respectively in Sections 5.3 and 5.4, thus implying the global existence of the solution (f,F).

(3) Finally, refining the estimates carried out during the previous steps, we prove our modified scattering result for the distribution function in Section 6. The scattering result for the electromagnetic field is treated in Section 7 and will require an additional step, the construction of a scattering map for the vacuum Maxwell equations.

Remark 3.3. If one is interested in relaxing the assumptions on N_v and N_x, though it would force us to either modify the proof or obtain weaker rate of convergences, we give here the precise results where losses in v^0 and z occur.

- Two powers of z are lost in order to close the $L^\infty_{x,v}$ estimates in Proposition 4.5; $5+\delta$ powers of z are required in order to apply Lemma 4.7 and prove boundedness for $\int_x f \,\mathrm{d}x$ and its derivatives.
- Three powers of v^0 are lost for closing the $L^\infty_{x,v}$ estimates, and eight for the pointwise decay estimates (see Lemma 4.12 and Proposition 4.13). Finally, the Glassey–Strauss decomposition of the derivative of the Maxwell field requires losing four powers of v^0, as suggested by Proposition 5.7 and Corollary 5.8.

Note that the various applications of Proposition 4.11 will not require controlling as many moments of f as the results mentioned here.

4. Estimates for the distribution function

4.1. ***Control of the Lorentz force.*** In view of the structure of the error terms for the commuted Vlasov equations, given by Proposition 2.4, it is important to obtain precise estimates of the Lorentz force and its derivatives by exploiting its null structure.

Lemma 4.1. *Let $|\gamma| \leq N-2$ and $j \in [\![1,3]\!]$. For all $(t,x,v) \in [0,T[\, \times \mathbb{R}^3_x \times \mathbb{R}^3_v$, we have*

$$\frac{1}{v^0}|\hat{v}^\mu \mathcal{L}_{Z^\gamma}(F)_\mu{}^j|(t,x) \lesssim \frac{\Lambda \log(3+t)}{(1+t+|x|)^2} + \frac{\Lambda \hat{v}^{\underline{L}}}{(1+t+|x|)(1+|t-|x||)}.$$

If $\gamma_T \geq 1$, then we have the improved estimate

$$\frac{1}{v^0}|\hat{v}^\mu \mathcal{L}_{Z^\gamma}(F)_\mu{}^j|(t,x) \lesssim \frac{\Lambda}{(1+t+|x|)^{\frac{5}{2}}} + \frac{\Lambda \hat{v}^{\underline{L}}}{(1+t+|x|)(1+|t-|x||)^2}.$$

Proof. Recall the definition of the null components of a 2-form (5) and expand $\hat{v}^\mu F_\mu{}^j$ according to the null frame $(\underline{L}, L, e_\theta, e_\varphi)$ in order to get

$$\begin{aligned}|\hat{v}^\mu F_\mu{}^j| &= |\hat{v}^L F_L{}^j + \hat{v}^{\underline{L}} F_{\underline{L}}{}^j + \hat{v}^{e_\theta} F_{e_\theta}{}^j + \hat{v}^{e_\varphi} F_{e_\varphi}{}^j| \\ &\lesssim \hat{v}^L(|\alpha(F)| + |\rho(F)|) + \hat{v}^{\underline{L}}(|\rho(F)| + |\underline{\alpha}(F)|) + |\hat{\slashed{v}}|(|\sigma(F)| + |\alpha(F)| + |\underline{\alpha}(F)|). \end{aligned} \tag{24}$$

Since $\hat{v}^L$, $\hat{v}^{\underline{L}}$, $|\hat{\slashed{v}}| \leq 1$ and $|\hat{\slashed{v}}| + |v^0|^{-1} \leq 2\sqrt{\hat{v}^{\underline{L}}}$ by Lemma 2.5, we obtain

$$\frac{1}{v^0}|\hat{v}^\mu F_\mu{}^j| \lesssim \sqrt{\hat{v}^{\underline{L}}}(|\alpha(F)| + |\rho(F)| + |\sigma(F)|) + \hat{v}^{\underline{L}}|\underline{\alpha}(F)|. \tag{25}$$

Note that the same applies to $\mathcal{L}_{Z^\gamma}(F)$, $|\gamma| \leq N-2$, so that the first estimate follows from Proposition 3.1. Assume now that $\gamma_T \geq 1$ and apply once again (25) to $\mathcal{L}_{Z^\gamma} F$ together with Proposition 3.1. We obtain

$$\begin{aligned}\frac{1}{v^0}|\hat{v}^\mu \mathcal{L}_{Z^\gamma}(F)_\mu{}^j|(t,x) &\lesssim \frac{\Lambda \log(3+t)\sqrt{\hat{v}^{\underline{L}}}}{(1+t+|x|)^2(1+|t-|x||)} + \frac{\Lambda \hat{v}^{\underline{L}}}{(1+t+|x|)(1+|t-|x||)^2} \\ &\lesssim \Lambda \frac{\log^2(3+t)}{(1+t+|x|)^3} + \frac{\Lambda \hat{v}^{\underline{L}}}{(1+t+|x|)(1+|t-|x||)^2},\end{aligned}$$

which implies the result. □

If $N-1 \leq |\gamma| \leq N$, we do not have improved estimates on the null components of the electromagnetic field. Moreover, if $|\gamma| = N$ and $\gamma_T = 0$, we have a very bad control of $\mathcal{L}_{Z^\gamma} F$ near the light cone. The idea then is to transform decay in $t-r$ into decay in $t+r$ at the cost of powers of z and v^0.

Lemma 4.2. *Consider $|\gamma| \leq N$ and $j \in [\![1, 3]\!]$. Then, for all $(t, x, v) \in \mathbb{R}_+ \times \mathbb{R}^3_x \times \mathbb{R}^3_v$,*

$$\frac{1}{v^0}|\hat{v}^\mu \mathcal{L}_{Z^\gamma}(F)_\mu{}^j|(t,x) \lesssim \frac{1}{v^0}|\mathcal{L}_{Z^\gamma} F|(t,x) \lesssim \Lambda \frac{\log(3+t+|x|)}{(1+t+|x|)^2}|v^0|^3 z^2(t,x,v),$$

and, if $\gamma_T \geq 1$,

$$\frac{1}{v^0}|\hat{v}^\mu \mathcal{L}_{Z^\gamma}(F)_\mu{}^j|(t,x) \lesssim \frac{1}{v^0}|\mathcal{L}_{Z^\gamma} F|(t,x) \lesssim \Lambda \frac{\log(3+t+|x|)}{(1+t+|x|)^3}|v^0|^3 z^2(t,x,v).$$

Proof. Recall from Lemma 2.6 that $(1+t+r)^2 \lesssim (1+|t-r|)^2|v^0|^4 z^2$. The first estimate then follows from Proposition 3.2 and the second one from (BA2) together with (23). □

Remark 4.3. If $|\gamma| \leq N-1$, we have $|\mathcal{L}_{Z^\gamma} F|(t,x) \lesssim \Lambda(1+t+|x|)^{-2}|v^0|^2 z(t,x,v)$. If $|\gamma| \leq N-2$, by combining Lemmas 2.5 and 4.1, we could even save a power of $|v^0|^3 z$ in the first estimate of the Lorentz force and then avoid any loss in v.

4.2. ***Pointwise bounds for f and its derivatives.*** As explained in Section 2.8.2, the main difficulties here are related to the weak decay rate of the electromagnetic field. We deal with them by exploiting several hierarchies in the commuted equations and by taking advantage of the null structure of the Lorentz force. Our approach, based on the method of characteristics, will require various applications of the following result.

Lemma 4.4. *Let $g : [0, T[\times \mathbb{R}^3_x \times \mathbb{R}^3_v \to \mathbb{R}_+$ and $h : [0, T[\times \mathbb{R}^3_x \times \mathbb{R}^3_v \to \mathbb{R}_+$ be two nonnegative sufficiently regular functions such that, for all $(t, x, v) \in [0, T[\times \mathbb{R}^3_x \times \mathbb{R}^3_v$,*

$$|\boldsymbol{T}_F(g)|(t,x,v) \leq \frac{C_g}{(1+t)\log^2(3+t)} g + \frac{C_g \hat{v}^{\underline{L}}}{(1+|t-|x||)\log^2(3+|t-|x||)} g + \frac{1}{(1+t)\log^2(3+t)} h$$

for some constant $C_g > 0$. Then,

$$\forall (t,x,v) \in [0,T[\times \mathbb{R}^3_x \times \mathbb{R}^3_v, \quad |g|(t,x,v) \leq (\|g(0,\cdot,\cdot)\|_{L^\infty_{x,v}} + 3\|h\|_{L^\infty_{t,x,v}})e^{6C_g}.$$

Proof. Fix, for all of this proof, $(x, v) \in \mathbb{R}^3_x \times \mathbb{R}^3_v$ and denote by $t \mapsto (X_t, V_t)$ the characteristic of the operator $\boldsymbol{T}_F = \partial_t + \hat{v}^i \partial_{x^i} + \hat{v}^\mu F_\mu{}^j \partial_{v^j}$ satisfying,

$$\forall\, 1 \leq j \leq 3, \quad \dot{X}^j_t = \widehat{V}^j_t, \quad \dot{V}^j_t = \widehat{V}^\mu_t F_\mu{}^j(t, X_t), \quad X_0 = x, \; V_0 = v.$$

According to the Duhamel formula, we have,

$$\forall t \in [0,T[, \quad g(t, X_t, V_t) = g(0,x,v) + \int_{s=0}^t \boldsymbol{T}_F(g)(s, X_s, V_s)\, \mathrm{d}s.$$

We are then lead to introduce the two functions

$$\psi_1(s) := (1+s)^{-1}\log^{-2}(3+s), \quad \psi_2(s) := \hat{v}^{\underline{L}}(X_s)(1+|s-|X_s||)^{-1}\log^{-2}(3+|s-|X_s||).$$

In view of the expression of $\boldsymbol{T}_F(g)$, we have, for all $t \in [0, T[$,

$$g(t, X_t, V_t) \leq \|g(0, \cdot, \cdot)\|_{L^\infty_{x,v}} + \|h\|_{L^\infty_{t,x,v}} \int_{s=0}^t \psi_1(s)\,\mathrm{d}s + \int_{s=0}^t C_g(\psi_1(s)+\psi_2(s))g(s, X_s, V_s)\,\mathrm{d}s.$$

Consequently, Grönwall's inequality and

$$\int_{s=0}^{+\infty} \psi_1(s)\,\mathrm{d}s = \int_{s=0}^{+\infty} \frac{\mathrm{d}s}{(1+s)\log^2(3+s)} \leq \int_{s=0}^{+\infty} \frac{3\,\mathrm{d}s}{(3+s)\log^2(3+s)} \leq \frac{3}{\log(3)} \leq 3$$

yield,

$$\forall t \in [0, T[, \qquad \sup_{0\leq s\leq t} g(s, X_s, V_s) \leq (\|g(0, \cdot, \cdot)\|_{L^\infty_{x,v}} + 3\|h\|_{L^\infty_{t,x,v}}) \exp\biggl(3C_g + C_g \int_{s=0}^t \psi_2(s)\,\mathrm{d}s\biggr).$$

It remains us to estimate the integral of ψ_2. For this, we will perform a change of variables reflecting that the Vlasov operator reads, in the coordinate system (u, x, v), where $u = t - |x|$,

$$\boldsymbol{T}_F = \partial_u - \hat{v}^i \frac{x_i}{|x|}\partial_u + \hat{v}^i \partial_{x^i} + \hat{v}^\mu F_\mu{}^j \partial_{v^j} = 2\hat{v}^{\underline{L}}\partial_u + \hat{v}^i \partial_{x^i} + \hat{v}^\mu F_\mu{}^j \partial_{v^j}.$$

As $\hat{v}^{\underline{L}} > 0$ by Lemma 2.5, we can then parametrize $t \mapsto (X_t, V_t)$ by the variable u. Hence, we perform the change of variables $\tilde{u}(s) = s - |X_s|$, so that $\tilde{u}'(s) = 2\widehat{V}^{\underline{L}}(X_s) > 0$ and

$$\int_{s=0}^t \psi_2(s)\,\mathrm{d}s = \int_{u=t-|x|}^{\tilde{u}(t)} \frac{\mathrm{d}u}{2(1+|u|)\log^2(3+|u|)} \leq \int_{u\in\mathbb{R}} \frac{\mathrm{d}u}{2(1+|u|)\log^2(3+|u|)} \leq 3. \qquad \square$$

We are now able to prove that quantities such as $z\widehat{Z}^\beta f$ are almost uniformly bounded in phase space. We recall that for a multi-index β, the number of homogeneous vector fields (respectively translations) composing $\widehat{Z}^\beta$ is denoted by β_H (respectively β_T).

Proposition 4.5. *There exists $D > 0$, depending only on (N, N_v, N_x), such that the following estimates hold. For all $(t, x, v) \in [0, T[\times \mathbb{R}^3_x \times \mathbb{R}^3_v$,*

$$\forall\, 0 \leq q \leq N_x,\ |\beta| \leq N-2, \qquad |v^0|^{N_v}|z^q \widehat{Z}^\beta f|(t, x, v) \lesssim \epsilon e^{D\Lambda} \log^{3q+3\beta_H}(3+t), \tag{26}$$

$$\forall\, 0 \leq q \leq N_x - 2,\ |\beta| \leq N, \quad |v^0|^{N_v-3}|z^q \widehat{Z}^\beta f|(t, x, v) \lesssim \epsilon e^{D\Lambda} \log^{3q+3\beta_H}(3+t). \tag{27}$$

Throughout this paper, it will be convenient to work with $\bar{\epsilon} := \epsilon e^{(D+1)\Lambda}$.

Proof. For simplicity, we assume here that $N \geq 4$ and we sketch the proof of the case $N = 3$ in Remark 4.6 below. Note further that, by interpolation, it suffices to deal with the cases $q \in \{0, N_x\}$ for (26) and $q \in \{0, N_x - 2\}$ for (27). Motivated by the analysis of the toy model carried out in Section 2.8.2, we introduce the following hierarchized norms in order to deal with nonintegrable error terms and still obtain satisfying estimates if the electromagnetic field is large. Consider, for $(N_0, p, q) = (N-2, N_v, N_x)$ or $(N, N_v - 3, N_x - 2)$,

$$\mathbb{E}^{p,q}_{N_0}[f](t, x, v) := \sum_{|\beta|\leq N_0} \frac{|v^0|^p|\widehat{Z}^\beta f|(t, x, v)}{\log^{3\beta_H}(3+t)} + \frac{|v^0|^p|z^q \widehat{Z}^\beta f|(t, x, v)}{\log^{3q+3\beta_H}(3+t)}$$

and let us prove that, for all $(t,x,v) \in [0,T[\times \mathbb{R}^3_x \times \mathbb{R}^3_v$,

$$\boldsymbol{T}_F(\mathbb{E}^{N_v,N_x}_{N-2}[f])(t,x,v) \lesssim \frac{\Lambda \mathbb{E}[f]^{N_v,N_x}_{N-2}(t,x,v)}{(1+t)\log^2(3+t)} + \frac{\Lambda \hat{v}^{\underline{L}}(x)\mathbb{E}[f]^{N_v,N_x}_{N-2}(t,x,v)}{(1+|t-|x||)\log^2(3+|t-|x||)}, \tag{28}$$

$$\begin{aligned}\boldsymbol{T}_F(\mathbb{E}^{N_v-3,N_x-2}_{N}[f])(t,x,v) \lesssim{}& \frac{\Lambda \mathbb{E}^{N_v-3,N_x-2}_{N}[f](t,x,v)}{(1+t)\log^2(3+t)} \\ &+ \frac{\Lambda \hat{v}^{\underline{L}}(x)\mathbb{E}^{N_v-3,N_x-2}_{N}[f]}{(1+|t-|x||)\log^2(3+|t-|x||)} + \frac{\Lambda \mathbb{E}^{N_v,N_x}_{N-2}[f](t,x,v)}{(1+t)\log^2(3+t)}. \end{aligned}\tag{29}$$

We are able to apply $\boldsymbol{T}_F$ to these energy norms since $\boldsymbol{T}_F(|h|) = \boldsymbol{T}_F(h)(h/|h|)$ almost everywhere for any $h \in W^{1,1}_{\mathrm{loc}}$. The result would then follow from two applications of Lemma 4.4. Fix now $(t,x,v) \in [0,T[\times \mathbb{R}^3_x \times \mathbb{R}^3_v$ as well as either $|\beta| \le N-2,\ p = N_v$ and $a \in \{0, N_x\}$ or $|\beta| \le N,\ p = N_v - 3$ and $a \in \{0, N_x - 2\}$. Note then, since $\boldsymbol{T}_F(\log^{-1}(3+t)) < 0$, that

$$\begin{aligned}\boldsymbol{T}_F\left(\frac{|v^0|^p \boldsymbol{z}^a |\widehat{Z}^\beta f|}{\log^{3a+3\beta_H}(3+t)}\right) \le{}& p\boldsymbol{T}_F(v^0)\frac{|v^0|^{p-1}\boldsymbol{z}^a|\widehat{Z}^\beta f|}{\log^{3a+3\beta_H}(3+t)} \\ &+ a\boldsymbol{T}_F(\boldsymbol{z})\frac{|v^0|^p \boldsymbol{z}^{a-1}|\widehat{Z}^\beta f|}{\log^{3a+3\beta_H}(3+t)} + \boldsymbol{T}_F(\widehat{Z}^\beta f)\frac{\widehat{Z}^\beta f}{|\widehat{Z}^\beta f|}\frac{|v^0|^p \boldsymbol{z}^a}{\log^{3a+3\beta_H}(3+t)}. \end{aligned}\tag{30}$$

It is important to note that the second term on the right-hand side vanishes if $a = 0$. We start by dealing with the first two terms on the right-hand side since the last one requires a more thorough analysis. As $|\nabla_v v^0| \le 1$, we obtain, by applying Lemma 4.1,

$$\frac{1}{v^0}|\boldsymbol{T}_F(v^0)|(t,x,v) = \frac{1}{v^0}|\hat{v}^\mu F_\mu{}^j \partial_{v^j}(v^0)|(t,x) \lesssim \frac{\Lambda \log(3+t)}{(1+t+|x|)^2} + \frac{\Lambda \hat{v}^{\underline{L}}}{(1+t+|x|)(1+|t-|x||)}, \tag{31}$$

so that

$$|\boldsymbol{T}_F(v^0)|\frac{|v^0|^{p-1}|\boldsymbol{z}^a\widehat{Z}^\beta f|(t,x,v)}{\log^{3a+3\beta_H}(3+t)} \lesssim \left(\frac{\Lambda}{(1+t)^{\frac{3}{2}}} + \frac{\Lambda \hat{v}^{\underline{L}}}{(1+t)(1+|t-|x||)}\right)\frac{|v^0|^p|\boldsymbol{z}^a\widehat{Z}^\beta f|(t,x,v)}{\log^{3a+3\beta_H}(3+t)}. \tag{32}$$

Next, recall from (11) the identity $\hat{v}^\mu \partial_{x^\mu}(\boldsymbol{z}) = 0$ and note that $|\nabla_v \boldsymbol{z}| \lesssim (t+r)/v^0$. We get, using Lemma 4.1,

$$|\boldsymbol{T}_F(\boldsymbol{z})|(t,x,v) \lesssim \sum_{1\le j\le 3}\frac{t+|x|}{v^0}|\hat{v}^\mu F_\mu{}^j(t,x)| \lesssim \frac{\Lambda \log(3+t)}{1+t+|x|} + \frac{\Lambda \hat{v}^{\underline{L}}}{1+|t-|x||}.$$

Using Young inequality for products, we obtain, if $a \neq 0$,

$$\frac{\boldsymbol{z}^{a-1}}{\log^{3a}(3+t)} \le \frac{a-1}{a\log^3(3+t)}\frac{\boldsymbol{z}^a}{\log^{3a}(3+t)} + \frac{1}{a\log^3(3+t)}.$$

We then deduce that

$$\begin{aligned} a|\boldsymbol{T}_F(\boldsymbol{z})|&\frac{|v^0|^p \boldsymbol{z}^{a-1}|\widehat{Z}^\beta f|}{\log^{3a+3\beta_H}(3+t)} \\ &\lesssim \left(\frac{\Lambda}{(1+t)\log^2(3+t)} + \frac{\Lambda \hat{v}^{\underline{L}}}{(1+|t-|x||)\log^3(3+t)}\right)\left(\frac{|v^0|^p|\widehat{Z}^\beta f|}{\log^{3\beta_H}(3+t)} + \frac{|v^0|^p \boldsymbol{z}^a|\widehat{Z}^\beta f|}{\log^{3a+3\beta_H}(3+t)}\right). \end{aligned}\tag{33}$$

We now focus on the last term in (30). The first step consists in applying the commutation formula of Proposition 2.4 and noting that $v^0\partial_{v^i} = \widehat{\Omega}_{0i} - t\partial_{x^i} - x^i\partial_t$. We can then bound

$$|\boldsymbol{T}_F(\widehat{Z}^\beta f)||v^0|^p z^a \log^{-3a-3\beta_H}(3+t)$$

by a linear combination of terms of the following form. The good ones, which are strongly decaying and can then be easily handled,

$$\mathcal{G}^{p,a}_{\gamma,\kappa} := \frac{1}{v^0}|\hat{v}^\mu \mathcal{L}_{Z^\gamma}(F)_\mu{}^j| \frac{|v^0|^p z^a |\widehat{\Omega}_{0j}\widehat{Z}^\kappa f|}{\log^{3a+3\beta_H}(3+t)}, \quad |\gamma|+|\kappa| \le |\beta|,\ |\kappa| \le |\beta|-1, \tag{34}$$

and the bad ones,

$$\mathcal{B}^{p,a}_{\gamma,\kappa} := (t+r)\sup_{1\le j\le 3}\frac{1}{v^0}|\hat{v}^\mu \mathcal{L}_{Z^\gamma}(F)_\mu{}^j| \frac{|v^0|^p z^a |\partial_{t,x}\widehat{Z}^\kappa f|}{\log^{3a+3\beta_H}(3+t)}, \quad \begin{cases}\gamma_H+\kappa_H \le \beta_H, \\ \kappa_H = \beta_H \implies \gamma_T \ge 1,\end{cases} \tag{35}$$

where, again, $|\gamma|+|\kappa| \le |\beta|$ and $|\kappa| \le |\beta|-1$. We emphasize that $\widehat{Z}^\xi := \partial_{t,x}\widehat{Z}^\kappa$ is composed of the same number of homogeneous vector fields as $\widehat{Z}^\kappa$, so that $\xi_H = \kappa_H$. In contrast, $\widehat{Z}^\zeta := \widehat{\Omega}_{0j}\widehat{Z}^\kappa$ satisfies $\zeta_H = \kappa_H + 1$. Moreover, $\widehat{\Omega}_{0j}\widehat{Z}^\kappa$ and $\partial_{t,x}\widehat{Z}^\kappa$ are of order at most $|\beta|$.

Consider first the case $|\beta| \le N-2$, so that $p = N_v$ and $a \in \{0, N_x\}$, and fix two multi-indices $|\gamma| \le |\beta|$, $|\kappa| \le |\beta|-1$. Then, according to Lemma 4.1, we have

$$\begin{aligned}\mathcal{G}^{N_v,a}_{\gamma,\kappa} &\lesssim \Lambda\left(\frac{\log(3+t)}{(1+t+|x|)^2} + \frac{\hat{v}^{\underline{L}}}{(1+t+|x|)(1+|t-|x||)}\right)\frac{|v^0|^{N_v}|z^a\widehat{\Omega}_{0j}\widehat{Z}^\kappa f|(t,x,v)}{\log^{3a+3\beta_H}(3+t)} \\ &\lesssim \left(\frac{\Lambda}{(1+t)^{\frac{3}{2}}} + \frac{\Lambda\hat{v}^{\underline{L}}}{(1+t)^{\frac{1}{2}}(1+|t-|x||)}\right)\frac{|v^0|^{N_v}|z^a\widehat{\Omega}_{0j}\widehat{Z}^\kappa f|(t,x,v)}{\log^{3a+3(\kappa_H+1)}(3+t)}.\end{aligned} \tag{36}$$

We now focus on $\mathcal{B}^{N_v,a}_{\gamma,\kappa}$ and we start by treating the case $\kappa_H = \beta_H$ and $\gamma_T \ge 1$. Applying once again Lemma 4.1, we get

$$\begin{aligned}\mathcal{B}^{N_v,a}_{\gamma,\kappa} &\lesssim (t+|x|)\left(\frac{\Lambda}{(1+t+|x|)^{\frac{5}{2}}} + \frac{\Lambda\hat{v}^{\underline{L}}}{(1+t+|x|)(1+|t-|x||)^2}\right)\frac{|v^0|^{N_v}|z^a\partial_{t,x}\widehat{Z}^\kappa f|}{\log^{3a+3\beta_H}(3+t)} \\ &\le \left(\frac{\Lambda}{(1+t)^{\frac{3}{2}}} + \frac{\Lambda\hat{v}^{\underline{L}}}{(1+|t-|x||)^2}\right)\frac{|v^0|^{N_v}|z^a\partial_{t,x}\widehat{Z}^\kappa f|}{\log^{3a+3\kappa_H}(3+t)}.\end{aligned} \tag{37}$$

Otherwise $\kappa_H \le \beta_H - 1$, so necessarily $\beta_H \ge 1$, and

$$\begin{aligned}\mathcal{B}^{N_v,a}_{\gamma,\kappa} &\lesssim (t+|x|)\left(\frac{\Lambda\log(3+t)}{(1+t+|x|)^2} + \frac{\Lambda\hat{v}^{\underline{L}}}{(1+t+|x|)(1+|t-|x||)}\right)\frac{|v^0|^{N_v}|z^a\partial_{t,x}\widehat{Z}^\kappa f|}{\log^{3a+3\beta_H}(3+t)} \\ &\le \left(\frac{\Lambda}{(1+t)\log^2(3+t)} + \frac{\Lambda\hat{v}^{\underline{L}}}{(1+|t-|x||)\log^3(3+t)}\right)\frac{|v^0|^{N_v}|z^a\partial_{t,x}\widehat{Z}^\kappa f|}{\log^{3a+3\kappa_H}(3+t)}.\end{aligned} \tag{38}$$

We obtain from (30)–(33) and (36)–(38),

$$\boldsymbol{T}_F\left(\frac{|v^0|^{N_v}z^a|\widehat{Z}^\beta f|}{\log^{3a+3\beta_H}(3+t)}\right) \lesssim \frac{\Lambda\mathbb{E}^{N_v,N_x}_{N-2}[f](t,x,v)}{(1+t)\log^2(3+t)} + \frac{\Lambda\hat{v}^{\underline{L}}(x)\mathbb{E}^{N_v,N_x}_{N-2}[f](t,x,v)}{(1+|t-|x||)\log^2(3+t)} + \frac{\Lambda\hat{v}^{\underline{L}}(x)\mathbb{E}^{N_v,N_x}_{N-2}[f](t,x,v)}{(1+|t-|x||)^2}.$$

As $|t-r| \gtrsim t$ for $r \geq 2t$ and $t \geq |t-r|$ otherwise, we have

$$(1+|t-r|)^{-1}\log^{-2}(3+t) \lesssim (1+t)^{-1}\log^{-2}(3+t) + (1+|t-r|)^{-1}\log^{-2}(3+|t-r|) \tag{39}$$

and we then deduce that (28) holds. Lemma 4.4 then implies (26).

Assume now that $N-1 \leq |\beta| \leq N$, $p = N_v - 3$ and $a \in \{0, N_x - 2\}$. We fix two multi-indices γ, κ verifying $|\gamma| + |\kappa| \leq |\beta|$, $|\kappa| \leq |\beta| - 1$ and we consider two cases.

Case 1: $|\gamma| \leq N-2$. The Lorentz force can still be estimated using Lemma 4.1. One can then follow the analysis carried out in (36)–(39) and obtain

$$\mathcal{G}^{N_v-3,a}_{\gamma,\kappa},\ \mathcal{B}^{N_v-3,a}_{\gamma,\kappa} \lesssim \frac{\Lambda \mathbb{E}^{N_v-3,N_x-2}_N[f](t,x,v)}{(1+t)\log^2(3+t)} + \frac{\Lambda \hat{v}^{\underline{L}}(x)\mathbb{E}^{N_v-3,N_x-2}_N[f](t,x,v)}{(1+|t-|x||)\log^2(3+|t-|x||)}, \tag{40}$$

where the term $\mathcal{B}^{N_v-3,a}_{\gamma,\kappa}$ is of course merely defined when γ_T and κ_H satisfy the condition given in (35).

Case 2: $N-1 \leq |\gamma| \leq N$. Then, as $N \geq 4$, we have $|\kappa| \leq 1$ so that we will be able to control the terms (34)–(35) using (26). In particular, we are allowed to lose two powers of $|v^0|^2 z$ in the upcoming estimates in order to deal with the weak decay rate of $\mathcal{L}_{Z^\gamma}F$ near the light cone. More precisely, using first Lemma 4.2 and then $a+2 \leq N_x$,

$$\begin{aligned}\mathcal{G}^{N_v-3,a}_{\gamma,\kappa} &\lesssim \frac{\Lambda\log(3+t+|x|)}{(1+t+|x|)^2}|v^0|^3 z^2(t,x,v)\frac{|v^0|^{N_v-3}|z^a\widehat{\Omega}_{0j}\widehat{Z}^\kappa f|(t,x,v)}{\log^{3a+3\beta_H}(3+t)} \\ &\lesssim \frac{\Lambda}{(1+t)^{\frac{3}{2}}}\frac{|v^0|^{N_v}|z^{N_x}\widehat{\Omega}_{0j}\widehat{Z}^\kappa f|(t,x,v)}{\log^{3N_x+3(\kappa_H+1)}(3+t)}.\end{aligned}$$

Next, consider the terms (35) and assume first that $\gamma_T \geq 1$. In that case, $\mathcal{B}^{N_v-3,a}_{\gamma,\kappa}$ can be easily handled since it is strongly decaying. Indeed, using again Lemma 4.2, we get

$$\begin{aligned}\mathcal{B}^{N_v-3,a}_{\gamma,\kappa} &\lesssim (t+|x|)\frac{\Lambda\log(3+t+|x|)}{(1+t+|x|)^3}|v^0|^3 z^2(t,x,v)\frac{|v^0|^{N_v-3}|z^a\partial_{t,x}\widehat{Z}^\kappa f|(t,x,v)}{\log^{3a+3\beta_H}(3+t)} \\ &\lesssim \frac{\Lambda}{(1+t)^{\frac{3}{2}}}\frac{|v^0|^{N_v}|z^{N_x}\partial_{t,x}\widehat{Z}^\kappa f|(t,x,v)}{\log^{3N_x+3\kappa_H}(3+t)}.\end{aligned}$$

Finally, if $\gamma_T = 0$, we necessarily have $\gamma_H = |\gamma| \geq N-1 \geq 3$. Since $\beta_H \geq \gamma_H + \kappa_H$, we have $\kappa_H \leq \beta_H - 3$, so that $3a + 3\beta_H \geq 3(a+2) + 3\kappa_H + 3$. Thus, Lemma 4.2 yields

$$\begin{aligned}\mathcal{B}^{N_v-3,a}_{\gamma,\kappa} &\lesssim (t+|x|)\frac{\Lambda\log(3+t+|x|)}{(1+t+|x|)^2}|v^0|^3 z^2(t,x,v)\frac{|v^0|^{N_v-3}|z^a\partial_{t,x}\widehat{Z}^\kappa f|(t,x,v)}{\log^{3a+3\beta_H}(3+t)} \\ &\lesssim \frac{\Lambda}{(1+t)\log^2(3+t)}\frac{|v^0|^{N_v}|z^{a+2}\partial_{t,x}\widehat{Z}^\kappa f|(t,x,v)}{\log^{3(a+2)+3\kappa_H}(3+t)}.\end{aligned}$$

We then deduce that, in this case,

$$\mathcal{G}^{N_v-3,a}_{\gamma,\kappa},\ \mathcal{B}^{N_v-3,a}_{\gamma,\kappa} \lesssim \frac{\Lambda}{(1+t)\log^2(3+t)}\mathbb{E}^{N_v,N_x}_{N-2}[f](t,x,v).$$

The estimate (29) ensues from (40) and this last inequality. To conclude the proof, it then remains to apply again the previous Lemma 4.4. ☐

Remark 4.6. If $N = 3$, the proof of Proposition 4.5 requires an additional step. Once the estimate for $\mathbb{E}^{N_v,N_x}_{N-2}[f]$ is proved, we need to control the intermediary norm $\mathbb{E}^{N_v-1,N_x-1}_{N-1}[f]$. For this, compared to the treatment of $\mathbb{E}^{N_v-3,N_x-2}_{N}[f]$ carried out during the proof of Proposition 4.5, there are two differences.

- First, we can exploit the much stronger decay estimate satisfied by the derivatives of order $N-1$ of the electromagnetic field than that on its top-order ones (see Proposition 3.2). This explains why we can propagate higher moments for the derivatives of order $N-1$ of f than for the top-order ones.
- Moreover, for controlling sufficiently well $\mathcal{B}^{N_v-1,0}_{\gamma,\kappa}$ and $\mathcal{B}^{N_v-1,N_x-1}_{\gamma,\kappa}$ in the case $\beta_H = \kappa_H$, we can prove, through a direct application of Lemma 2.17, that the good null components of $\mathcal{L}_{Z^\gamma}(F)$ still satisfy improved estimates when $|\gamma| = N-1$ and $\gamma_T \geq 1$.

Finally, in order to bound uniformly in time $\mathbb{E}^{N_v-3,N_x-2}_{N}[f]$, the analysis of the terms (34)–(35) is slightly more technical. It is necessary to consider three cases ($|\gamma| \leq N-2$, $|\gamma| = N-1$ as well as $|\gamma| = N$) and to use the estimates on the first two energy norms.

4.3. ***Uniform boundedness of the spatial averages.*** We start by a preparatory result, which will also be useful later in Section 6. Recall the constant $\bar{\epsilon} := \epsilon e^{(D+1)\Lambda}$ introduced in Proposition 4.5.

Lemma 4.7. *For any $|\beta| \leq N-1$, we have,*

$$\forall (t,v) \in [0,T[\times \mathbb{R}^3_v, \quad |v^0|^{N_v-6} \left| \partial_t \int_{\mathbb{R}^3_x} \widehat{Z}^\beta f(t,x,v)\,\mathrm{d}x \right| \lesssim \bar{\epsilon} \frac{\log^{3N_x+3N}(3+t)}{(1+t)^2}.$$

Proof. Fix $|\beta| \leq N-1$, $t \in [0,T[$ and $v \in \mathbb{R}^3_v$. Integrating the commutation formula of Proposition 2.4 for $\widehat{Z}^\beta f$ and performing integration by parts in x gives

$$\partial_t \int_{\mathbb{R}^3_x} \widehat{Z}^\beta f(t,x,v)\,\mathrm{d}x = -\int_{\mathbb{R}^3_x} \hat{v}^\mu F_\mu{}^j \partial_{v^j} \widehat{Z}^\beta f(t,x,v)\,\mathrm{d}x + \sum_{|\gamma|+|\kappa| \leq |\beta|} C^\beta_{\gamma,\kappa} \int_{\mathbb{R}^3_x} \hat{v}^\mu \mathcal{L}_{Z^\gamma}(F)_\mu{}^j \partial_{v^j} \widehat{Z}^\kappa f(t,x,v)\,\mathrm{d}x.$$

Now, we write

$$v^0 \partial_{v^j} = \widehat{\Omega}_{0j} - x^j \partial_t - t\partial_{x^j} = \widehat{\Omega}_{0j} - (x^j - \hat{v}^j t)\partial_t - v^j S + v^j x^i \partial_{x^i} - t\partial_{x^j}, \quad |x^j - \hat{v}^j t| \leq z,$$

so that, integrating once again by parts,

$$\left| \partial_t \int_{\mathbb{R}^3_x} \widehat{Z}^\beta f(t,x,v)\,\mathrm{d}x \right| \lesssim \sum_{\substack{|\gamma|+|\kappa| \leq |\beta|+1 \\ |\gamma| \leq |\beta|}} \sup_{1 \leq j \leq 3} \int_{\mathbb{R}^3_x} \frac{1}{v^0} |\hat{v}^\mu \mathcal{L}_{Z^\gamma}(F)_\mu{}^j(t,x)| |z\widehat{Z}^\kappa f|(t,x,v)\,\mathrm{d}x + \int_{\mathbb{R}^3_x} \frac{t+|x|}{v^0} |\hat{v}^\mu \nabla_{t,x} \mathcal{L}_{Z^\gamma}(F)_\mu{}^j(t,x)| |\widehat{Z}^\kappa f|(t,x,v)\,\mathrm{d}x.$$

According to the bootstrap assumptions (BA1)–(BA2) and Lemma 2.6, we have

$$|\mathcal{L}_{Z^\gamma}(F)_\mu{}^j(t,x)| \lesssim \Lambda (1+t+|x|)^{-2} |v^0|^2 z,$$
$$|\nabla_{t,x} \mathcal{L}_{Z^\gamma}(F)_\mu{}^j(t,x)| \lesssim \Lambda \log(3+t+|x|)(1+t+|x|)^{-3} |v^0|^4 z^2,$$

so that

$$\left|\partial_t \int_{\mathbb{R}^3_x} \widehat{Z}^\beta f(t,x,v)\,\mathrm{d}x\right| \lesssim \Lambda \sum_{|\kappa|\leq|\beta|+1} \int_{\mathbb{R}^3_x} \frac{\log(3+t+|x|)}{(1+t+|x|)^2} |v^0|^3 |z^2 \widehat{Z}^\kappa f|(t,x,v)\,\mathrm{d}x$$
$$\leq \Lambda \sup_{|\kappa|\leq|\beta|+1} \sup_{x\in\mathbb{R}^3} \left(\frac{\log(3+t+|x|)}{(1+t+|x|)^2} |v^0|^3 |z^{N_x-2} \widehat{Z}^\kappa f|(t,x,v)\right) \int_{\mathbb{R}^3_x} \frac{\mathrm{d}x}{z^{N_x-4}(t,x,v)}.$$

Note then that, in view of (11) and $N_x > 7$,

$$\int_{\mathbb{R}^3_x} \frac{\mathrm{d}x}{z^{N_x-4}(t,x,v)} \leq \int_{\mathbb{R}^3_x} \frac{\mathrm{d}x}{(1+|x-\hat{v}t|)^{N_x-4}} = \int_{y\in\mathbb{R}^3} \frac{\mathrm{d}y}{(1+|y|)^{N_x-4}} < +\infty.$$

Then, multiply both sides of the inequality by $|v^0|^{N_v-6}$ and bound above the right-hand side by applying Proposition 4.5. It remains to use $\Lambda\epsilon e^{D\Lambda} \leq \epsilon e^{(D+1)\Lambda} = \bar{\epsilon}$. □

Remark 4.8. If $|\beta| \leq N-3$, by using the estimates of the Lorentz force provided by Lemma 4.1, we can even prove $|v^0|^{N_v} \left|\partial_t \int_x \widehat{Z}^\beta f\,\mathrm{d}v\right| \lesssim \bar{\epsilon}(1+t)^{-2}\log^{-3N_x-3N}(3+t)$.

Note now that $\left|\int_x \widehat{Z}^\beta f(0,x,v)\,\mathrm{d}x\right| \leq 2\sup_x |z^4 \widehat{Z}^\beta f|(0,x,v) \leq 2\epsilon$. Hence, by integrating in time the inequality of the previous Lemma 4.7, we obtain, for any $|\beta| \leq N-1$,

$$\forall (t,v) \in [0,T[\times\mathbb{R}^3_v, \quad |v^0|^{N_v-6}\left|\int_{\mathbb{R}^3_x} \widehat{Z}^\beta f(t,x,v)\,\mathrm{d}x\right| \lesssim \epsilon + \bar{\epsilon}\int_{\tau=0}^t \frac{\log^{3N_x+3N}(3+\tau)}{(1+\tau)^2}\,\mathrm{d}\tau \lesssim \bar{\epsilon}.$$

It directly implies the following result.

Corollary 4.9. *Let $|\beta| \leq N-1$ and $\psi : \mathbb{S}^2_\omega \times \mathbb{R}^3_v \to \mathbb{R}$ be a function such that $\|\psi(\cdot,v)\|_{L^\infty_\omega} \lesssim |v^0|^{N_v-6}$. Then, for any $\omega \in \mathbb{S}^2$,*

$$\forall (t,v) \in [0,T[\times\mathbb{R}^3_v, \quad \left|\psi(\omega,v)\int_{\mathbb{R}^3_x} \widehat{Z}^\beta f(t,x,v)\,\mathrm{d}x\right| \lesssim \bar{\epsilon}.$$

We allowed the function ψ to depend on a parameter $\omega \in \mathbb{S}^2$ in order to prove optimal decay estimates on certain elements of the Glassey–Strauss decomposition of the electromagnetic field, defined as integral kernels.

4.4. ***Pointwise decay estimates for velocity averages.*** We prove here that the decay rate of $\int_v \widehat{Z}^\beta f\,\mathrm{d}v$, for $|\beta| \leq N-1$, coincides with the one of the linear setting. In particular, we improve the bootstrap assumption (BA3). The starting point consists of performing the change of variables $y = x - t\hat{v}$. For this, recall Lemma 2.9 and that $y \mapsto \check{y}$ is the inverse function of $v \mapsto \hat{v}$.

Lemma 4.10. *Let $g : [0,T[\times\mathbb{R}^3_x\times\mathbb{R}^3_v \to \mathbb{R}$ be a sufficiently regular function. Then,*

$$\forall (t,x) \in [0,T[\times\mathbb{R}^3, \quad t^3\int_{\mathbb{R}^3_v} g(t,x-\hat{v}t,v)\,\mathrm{d}v = \int_{|y-x|<t} (|v^0|^5 g)\left(t,y,\widecheck{\frac{x-y}{t}}\right)\mathrm{d}y.$$

This change of variables is motivated by the linear case. Any regular solution to the relativistic transport equation $\partial_t h + \hat{v}\cdot\nabla_x h = 0$ is constant along the timelike straight lines, $h(t,x+\hat{v}t,v) = h(0,x,v)$. The previous lemma, applied for $g(t,x,v) = h(0,x,v)$, then leads to $\int_v h(t,x,v)\,\mathrm{d}v \lesssim t^{-3}$.

As a first step, we control $\int_v |\widehat{Z}^\beta f|\,\mathrm{d}v$ for any $|\beta| \leq N$, which has a slightly slower decay rate than in the linear case in the interior of the light cone. These estimates will also be useful on their own.

Proposition 4.11. *Let $|\beta| \leq N$ and $0 \leq a \leq N_x - 6$. Then, the following properties hold.*

- *Almost optimal pointwise decay estimate,*

$$\forall (t,x) \in [0,T[\times \mathbb{R}^3_x, \qquad \int_{\mathbb{R}^3_v} |v^0|^{N_v-8} |z^a \widehat{Z}^\beta f|(t,x,v)\,\mathrm{d}v \lesssim \bar{\epsilon} \frac{\log^{3N_x+3N}(3+t)}{(1+t)^3}.$$

- *Improved decay estimates near and in the exterior of the light cone,*

$$\forall |x| \leq t < T, \qquad \int_{\mathbb{R}^3_v} |v^0|^{N_v-8-2a} |\widehat{Z}^\beta f|(t,x,v)\,\mathrm{d}v \lesssim \bar{\epsilon} \log^{3N_x+3N}(3+t) \frac{(1+t-|x|)^a}{(1+t)^{3+a}},$$

$$\forall\, t < \sup(|x|,T), \quad \int_{\mathbb{R}^3_v} |v^0|^{N_v-8-2a} |\widehat{Z}^\beta f|(t,x,v)\,\mathrm{d}v \lesssim \bar{\epsilon} \frac{\log^{3N_x+3N}(3+t)}{(1+t+|x|)^{3+a}}.$$

Proof. Fix $|\beta| \leq N$, $(t,x) \in [0,T[\times \mathbb{R}^3_x$ and $0 \leq a \leq N_x - 6$. If $t \leq 1$, we have by Proposition 4.5,

$$\int_{\mathbb{R}^3_v} |v^0|^{N_v-7} |z^a \widehat{Z}^\beta f|(t,x,v)\,\mathrm{d}v \lesssim \sup_{v \in \mathbb{R}^3} |v^0|^{N_v-3} |z^{N_x-6} \widehat{Z}^\beta f|(t,x,v) \int_{\mathbb{R}^3_w} \frac{\mathrm{d}w}{\langle w \rangle^4} \lesssim \bar{\epsilon}.$$

Assume now, unless $T \leq 1$, that $t \geq 1$ and introduce the function $g(t,x,v) := |v^0|^{N_v-8} |z^a \widehat{Z}^\beta f|(t, x+t\hat{v}, v)$. Applying the previous Lemma 4.10 to g, we get

$$\begin{aligned} t^3 \int_{\mathbb{R}^3_v} |v^0|^{N_v-8} |z^a \widehat{Z}^\beta f|(t,x,v)\,\mathrm{d}v &\leq \int_{|y-x|<t} \sup_{v \in \mathbb{R}^3} |v^0|^5 g(t,y,v)\,\mathrm{d}y \\ &\leq \sup_{(y,v) \in \mathbb{R}^3 \times \mathbb{R}^3} |v^0|^5 \langle y \rangle^4 g(t,y,v) \int_{\mathbb{R}^3_y} \frac{\mathrm{d}y}{\langle y \rangle^4}. \end{aligned}$$

Using now Lemma 2.8 and then Proposition 4.5, we obtain

$$t^3 \int_{\mathbb{R}^3_v} |v^0|^{N_v-8} |z^a \widehat{Z}^\beta f|(t,x,v)\,\mathrm{d}v \leq \sup_{(y,v) \in \mathbb{R}^3 \times \mathbb{R}^3} |v^0|^{N_v-3} |z^{a+4} \widehat{Z}^\beta f|(t,y,v) \lesssim \bar{\epsilon} \log^{3a+12+3N}(3+t).$$

This concludes the proof of the first estimate, which, together with Lemma 2.6, implies the second one as well as the last one in the region $t < |x| \leq 2t$. If $|x| \geq 2t$, note that $z \gtrsim 1 + |x - t\hat{v}| \gtrsim 1 + t + |x|$, so that

$$\int_{\mathbb{R}^3_v} |v^0|^{N_v-7} |\widehat{Z}^\beta f|(t,x,v)\,\mathrm{d}v \lesssim (1+t+|x|)^{-N_x+2} \sup_{(y,v) \in \mathbb{R}^3 \times \mathbb{R}^3} |v^0|^{N_v-3} |z^{N_x-2} \widehat{Z}^\beta f|(t,y,v) \int_{\mathbb{R}^3_w} \frac{\mathrm{d}w}{\langle w \rangle^4}.$$

It remains to apply Proposition 4.5. □

Our goal now is to remove the logarithmic loss of the estimate of $\int_v \widehat{Z}^\beta f\,\mathrm{d}v$ provided by Proposition 4.11. Since our analysis will rely on the following result, we will not be able to deal with top-order derivatives. We recall that $N_x - 3 > 4$.

Lemma 4.12. *Let $g : [0,T[\times \mathbb{R}^3_x \times \mathbb{R}^3_v \to \mathbb{R}$ be a sufficiently regular function. Then, for all $|x| < t < T$,*

$$\left| t^3 \int_{\mathbb{R}^3_v} g(t, x - \hat{v}t, v)\,\mathrm{d}v - \int_{y \in \mathbb{R}} (|v^0|^5 g)\left(t, y, \frac{\check{x}}{t}\right) \mathrm{d}y \right| \lesssim \frac{1}{t} \sup_{(y,v) \in \mathbb{R}^3 \times \mathbb{R}^3} \langle y \rangle^{N_x-3} (|v^0|^7 |g| + |v^0|^8 |\nabla_v g|)(t,y,v).$$

Proof. According to Lemma 4.10, we have

$$t^3 \int_{\mathbb{R}^3_v} g(t, x-\hat{v}t, v)\,\mathrm{d}v - \int_{y\in\mathbb{R}} g\Big(t, y, \frac{\check{x}}{t}\Big)\,\mathrm{d}v = \mathcal{I}_1 + \mathcal{I}_2,$$

where

$$\mathcal{I}_1 := \int_{|x-y|<t} (|v^0|^5 g)\Big(t, y, \widecheck{\frac{x-y}{t}}\Big)\,\mathrm{d}y - \int_{|x-y|<t} (|v^0|^5 g)\Big(t, y, \frac{\check{x}}{t}\Big)\,\mathrm{d}y,$$

$$\mathcal{I}_2 := -\int_{|x-y|\geq t} (|v^0|^5 g)\Big(t, y, \frac{\check{x}}{t}\Big)\,\mathrm{d}y.$$

Since, by Lemma 2.9, we have $|\nabla_y \check{y}| \lesssim \sqrt{1-|y|^2}^{\,-3} = \langle \check{y}\rangle^3 = |v^0|^3(\check{y})$, the mean value theorem gives us

$$\left|(|v^0|^5 g)\Big(t, y, \widecheck{\frac{x-y}{t}}\Big) - (|v^0|^5 g)\Big(t, y, \frac{\check{x}}{t}\Big)\right| \lesssim \frac{|y|}{t} \sup_{v\in\mathbb{R}^3} |v^0|^7 |g|(t, y, v) + |v^0|^8 |\nabla_v g|(t, y, v).$$

Consequently,

$$|\mathcal{I}_1| \lesssim \frac{1}{t} \sup_{(y,v)\in\mathbb{R}^3\times\mathbb{R}^3} \langle y\rangle^{N_x-3}(|v^0|^7|g|(t, y, v) + |v^0|^8|\nabla_v g|(t, y, v)) \int_{|x-y|<t} \frac{\mathrm{d}y}{\langle y\rangle^{N_x-4}}, \quad N_x - 4 > 3.$$

In order to bound $\mathcal{I}_2$ recall that $|x| < t$ and note that, for $v = \widecheck{x/t}$ and any $y \in \mathbb{R}$ such that $|y-x| \geq t$,

$$1 = |v^0|^2\Big(1 - \frac{|x|^2}{t^2}\Big) \leq |v^0|^2 \frac{|y|(t+|x|)}{t^2} \leq 2\frac{|y|\,|v^0|^2}{t}.$$

We then finally deduce that

$$|\mathcal{I}_2| \leq \frac{2}{t} \int_{|y-x|\geq t} (|v^0|^7 g)\Big(t, y, \frac{\check{x}}{t}\Big) \frac{\langle y\rangle^{N_x-3}}{\langle y\rangle^{N_x-4}}\,\mathrm{d}y \leq \frac{4}{t} \sup_{(y,v)\in\mathbb{R}^3\times\mathbb{R}^3} |v^0|^7 \langle y\rangle^{N_x-3}|g|(t, y, v). \qquad \square$$

We are able to prove that the decay of quantities such as $\int_v \widehat{Z}^\beta f\,\mathrm{d}v$ is optimal. We state a general result since we will later have to deal with integral kernels.

Proposition 4.13. *Let $|\beta| \leq N-1$ and $\Psi : \mathbb{S}^2_\omega \times \mathbb{R}^3_v \to \mathbb{R}$ be a sufficiently regular function such that $\|\Psi(\cdot, v)\|_{L^\infty_\omega} + \|v^0\nabla_v\Psi(\cdot, v)\|_{L^\infty_\omega} \lesssim |v^0|^{N_v-11}$. Then, for any $\omega \in \mathbb{S}^2$,*

$$\forall (t, x) \in [0, T[\times\mathbb{R}^3_x, \qquad \left|\int_{\mathbb{R}^3_v} \Psi(\omega, v)\widehat{Z}^\beta f(t, x, v)\,\mathrm{d}v\right| \lesssim \frac{\bar{\epsilon}}{(1+t+|x|)^3}.$$

Proof. Assume first that $|x| \leq t \leq 1$ or $|x| \geq t$. Then, as $|\Psi|(\cdot, v) \lesssim |v^0|^{N_v-9}$, it suffices to use the first or the third estimate of Proposition 4.11, applied for $a = \frac{1}{2}$. Otherwise $t > \max(1, |x|)$ and we introduce, for any $\omega \in \mathbb{S}^2$, $g_\omega(t, x, v) = \Psi(\omega, v)\widehat{Z}^\beta f(t, x + t\hat{v}, v)$. Using first Lemma 2.8 and then Proposition 4.5, we have

$$\begin{aligned}
&\sup_{(y,v)\in\mathbb{R}^3\times\mathbb{R}^3} \langle y\rangle^{N_x-3}(|v^0|^7|g_\omega| + |v^0|^8|\nabla_v g_\omega|)(t, y, v) \\
&\quad\lesssim \sup_{(y,v)\in\mathbb{R}^3\times\mathbb{R}^3} |\nabla_v\Psi|(\omega, v)|v^0|^8 |z^{N_x-3}\widehat{Z}^\beta f|(t, y, v) + \sum_{|\kappa|\leq 1} |\Psi|(\omega, v)|v^0|^7 |z^{N_x-2}\widehat{Z}^\kappa\widehat{Z}^\beta f|(t, y, v) \\
&\quad\lesssim \sum_{|\xi|\leq N} \sup_{(y,v)\in\mathbb{R}^3\times\mathbb{R}^3} |v^0|^{N_v-3}|z^{N_x-2}\widehat{Z}^\xi f|(t, y, v) \lesssim \bar{\epsilon}\log^{3N_x+3N}(3+t).
\end{aligned} \tag{41}$$

Now, apply Lemma 4.12 to g_ω in order to get,

$$\forall \omega \in \mathbb{S}^2, \quad t^3 \left| \int_{\mathbb{R}^3_v} \Psi(\omega, v) \widehat{Z}^\beta f(t, x, v) \, \mathrm{d}v \right| \lesssim \left| \int_{\mathbb{R}^3_y} (|v^0|^5 g_\omega)\Big(t, y, \frac{\check{x}}{t}\Big) \, \mathrm{d}y \right| + \bar{\epsilon} \frac{\log^{3N_x+3N}(3+t)}{t}.$$

As $t \geq 1$, it remains to bound by $\bar{\epsilon}$ the first term on the right-hand side. For this, perform the change of variables $z = y - t\hat{v}$ and apply Corollary 4.9 with $\psi(\omega, v) = |v^0|^5 \Psi(\omega, v)$. □

The next result is a direct application of the previous proposition to $\Psi(\omega, v) = v^\mu / v^0$ for any $0 \leq \mu \leq 3$.

Corollary 4.14. *For any $|\beta| \leq N-1$, the decay of the current density $J(\widehat{Z}^\beta f)$ is optimal. There exists a constant $C > 0$ independent of ϵ such that,*

$$\forall (t, x) \in [0, T[\times \mathbb{R}^3_x, \qquad \left| \int_{\mathbb{R}^3_v} \frac{v^\mu}{v^0} \widehat{Z}^\beta f(t, x, v) \, \mathrm{d}v \right| \leq \frac{C\bar{\epsilon}}{(1+t+|x|)^3}, \qquad 0 \leq \mu \leq 3.$$

If ϵ satisfies $C\bar{\epsilon} = C\epsilon e^{(D+1)\Lambda} < C_{\mathrm{boot}}\Lambda$, it improves the bootstrap assumption (BA3).

4.5. ***Improved estimates for derivatives of velocity averages.*** In the linear case, derivatives of averages in v, such as $\partial_{t,x} \int_v f \, \mathrm{d}v$, enjoy stronger decay properties. Our study of the top-order derivatives of the electromagnetic field will require the following improved estimates.

Proposition 4.15. *Let $|\beta| \leq N-1$, $\mu \in [\![0, 3]\!]$ and $\Phi : \mathbb{S}^2 \times \mathbb{R}^3_v \to \mathbb{R}$ be a sufficiently regular function such that $\|\Phi(\,\cdot\,, v)\|_{L^\infty_\omega} + \|v^0 \nabla_v \Phi(\,\cdot\,, v)\|_{L^\infty_\omega} \lesssim |v^0|^{N_v - 10}$. Then, for any $\omega \in \mathbb{S}^2$,*

$$\forall (t, x) \in [0, T[\times \mathbb{R}^3_x, \qquad \left| \int_{\mathbb{R}^3_v} \Phi(\omega, v) \partial_{x^\mu} \widehat{Z}^\beta f(t, x, v) \, \mathrm{d}v \right| \lesssim \bar{\epsilon} \frac{\log^{3N_x+3N}(3+t)}{(1+t+|x|)^4}.$$

Proof. Let $(t, x) \in [0, T[\times \mathbb{R}^3_x$ and note that, if $|x| \geq t-1$, the result is given by Proposition 4.11, applied for $a = 1$. We then consider the case $t - |x| \geq 1$. Using (20) together with $t\Omega_{ij} = (x^i/r)\Omega_{0j} - (x^j/r)\Omega_{0i}$, one has

$$\mathcal{I}^\beta_{t,x} := |t - |x|| \left| \int_{\mathbb{R}^3_v} \Phi(\omega, v) \partial_{x^\mu} \widehat{Z}^\beta f(t, x, v) \, \mathrm{d}v \right| \leq \sum_{Z \in \mathbb{K}} \left| \int_{\mathbb{R}^3_v} \Phi(\omega, v) Z \widehat{Z}^\beta f(t, x, v) \, \mathrm{d}v \right|.$$

Fix now $Z \in \mathbb{K}$. If Z is a translation ∂_{x^μ} or if $Z = S$, then $Z \in \widehat{\mathbb{P}}_0$. Otherwise, either $Z = \Omega_{ij}$ is a rotation and $Z = \widehat{Z} - v^i \partial_{v^j} + v^j \partial_{v^i}$ or $Z = \Omega_{0k}$ is a Lorentz boost and $Z = \widehat{Z} - v^0 \partial_{v^k}$, so that

$$\mathcal{I}^\beta_{t,x} \leq \sum_{\widehat{Z} \in \widehat{\mathbb{P}}_0} \left| \int_{\mathbb{R}^3_v} \Phi(\omega, v) \widehat{Z} \widehat{Z}^\beta f(t, x, v) \, \mathrm{d}v \right| + \sum_{\lambda=0}^{3} \sum_{k=1}^{3} \left| \int_{\mathbb{R}^3_v} \Phi(\omega, v) v^\lambda \partial_{v^k} \widehat{Z}^\beta f(t, x, v) \, \mathrm{d}v \right|.$$

Integration by parts and $|\partial_{v^k}(\Phi(\omega, v) v^\lambda)| \leq v^0 |\nabla_v \Phi|(\omega, v) + |\Phi|(\omega, v) \lesssim |v^0|^{N_v - 10}$ yield

$$\left| \int_{\mathbb{R}^3_v} \Phi(\omega, v) \partial_{x^\mu} \widehat{Z}^\beta f(t, x, v) \, \mathrm{d}v \right| \lesssim \frac{1}{|t - |x||} \sum_{|\kappa| \leq 1} \int_{\mathbb{R}^3_v} |v^0|^{N_v - 10} |\widehat{Z}^\kappa \widehat{Z}^\beta f|(t, x, v) \, \mathrm{d}v.$$

As $t - |x| \geq 1$, it remains to apply once again Proposition 4.11 for $a = 1$. □

5. Improvement of the bootstrap assumptions on the electromagnetic field

We are now able to prove pointwise decay estimates for the Maxwell field and its derivatives. We improve first (BA1), whereas the case of the top-order derivatives (BA2) will require a different strategy since we did not recover the linear decay t^{-3} for $\int_v \widehat{Z}^\beta f(t,x,v)\,\mathrm{d}v,\ |\beta|=N$.

5.1. ***The Glassey–Strauss decomposition of the electromagnetic field.*** We separate F as well as its derivatives $\mathcal{L}_{Z^\gamma}(F)$ into three parts according to the Glassey–Strauss decomposition. For this, recall from (4) the relation between the electric field E, the magnetic field B and the Faraday tensor F. We have $E^i = F_{0i},\ B^1 = F_{32},\ B^2 = F_{13}$ and $B^3 = F_{21}$. To simplify the statement of the decomposition, we introduce a weight tensor field.

Definition 5.1. Let $\boldsymbol{w}_{\mu\nu}(\omega,v)$ be the antisymmetric tensor defined for all $(\omega,v)\in\mathbb{S}^2\times\mathbb{R}^3_v$ by

$$\boldsymbol{w}_{0i}(\omega,v) = -\boldsymbol{w}_{i0}(\omega,v) := \omega_i + \hat{v}_i, \quad \boldsymbol{w}_{jk}(\omega,v) := \omega_j\hat{v}_k - \omega_k\hat{v}_j, \quad 1\le i,j,k\le 3,$$

where $\omega_i := x_i/|x|$ if $x\in\mathbb{R}^3$ satisfies $\omega = x/|x|$. We further define

$$\mathcal{W}_{\mu\nu}(\omega,v) := \frac{\boldsymbol{w}_{\mu\nu}(\omega,v)}{1+\omega\cdot\hat{v}}.$$

Remark 5.2. Since $\boldsymbol{w}$ is antisymmetric, $\boldsymbol{w}_{\mu\mu}=0$ for any $\mu\in[\![0,3]\!]$. Note also that $1+\omega\cdot\hat{v} = 2v^L > 0$.

We can now prove an adaptation of [Glassey and Strauss 1986, Theorem 3]. The key idea of their proof consists in rewriting the standard derivatives $\partial_{t,x}$ as combinations of derivatives tangential to a backward light cone, which naturally appears in the representation formula for solutions to wave equations, and $\boldsymbol{T}_0 := \partial_t + \hat{v}\cdot\nabla_x$, the free relativistic transport operator which is transverse to light cones. To avoid any confusion with the scaling vector field, we do not keep the notation S, used by Glassey and Strauss, in order to denote $\boldsymbol{T}_0$.

Proposition 5.3. *Let $|\gamma|\le N-1$. Then, there exist $C^\gamma_\beta,\ N^\gamma_{\xi,\kappa}\in\mathbb{N}$ such that*

$$4\pi\mathcal{L}_{Z^\gamma}(F) = \mathcal{L}_{Z^\gamma}(F)^{\mathrm{data}} + \mathcal{L}_{Z^\gamma}(F)^T + \mathcal{L}_{Z^\gamma}(F)^S,$$

where, for any $0\le\mu,\nu\le 3$ and with $\omega = (y-x)/|y-x|$ in the following integrals:

- *$\mathcal{L}_{Z^\gamma}(F)^{\mathrm{data}}_{\mu\nu}$ can be explicitly computed in terms of the initial data. More precisely,*

$$\begin{aligned}\mathcal{L}_{Z^\gamma}(F)^{\mathrm{data}}_{\mu\nu}&(t,x)\\ &= 4\pi\mathcal{L}_{Z^\gamma}(F)^{\mathrm{hom}}_{\mu\nu}(t,x) - \sum_{|\beta|\le|\gamma|}\frac{C^\gamma_\beta}{t}\int_{|y-x|=t}\int_{\mathbb{R}^3_v}(\mathcal{W}_{\mu\nu}(\omega,v) - \delta^0_\mu\hat{v}^\nu + \delta^0_\nu\hat{v}^\mu)\widehat{Z}^\beta f(0,y,v)\,\mathrm{d}v\,\mathrm{d}y\end{aligned}$$

and $\mathcal{L}_{Z^\gamma}(F)^{\mathrm{hom}}_{\mu\nu}$ is the unique solution to the homogeneous wave equation $\square\mathcal{L}_{Z^\gamma}(F)^{\mathrm{hom}}_{\mu\nu}=0$ which initially verifies $\mathcal{L}_{Z^\gamma}(F)^{\mathrm{hom}}_{\mu\nu}(0,\cdot) = \mathcal{L}_{Z^\gamma}(F)_{\mu\nu}(0,\cdot)$ and $\partial_t\mathcal{L}_{Z^\gamma}(F)^{\mathrm{hom}}_{\mu\nu}(0,\cdot) = \partial_t\mathcal{L}_{Z^\gamma}(F)_{\mu\nu}(0,\cdot)$.

- *The 2-form $\mathcal{L}_{Z^\gamma}(F)^T$ is given by*

$$\mathcal{L}_{Z^\gamma}(F)^T_{\mu\nu}(t,x) := -\sum_{|\beta|\le|\gamma|} C^\gamma_\beta[\widehat{Z}^\beta f]^T_{\mu\nu}(t,x),$$

where the field $[\widehat{Z}^\beta f]^T$ *generated by* $\widehat{Z}^\beta f$ *is*

$$[\widehat{Z}^\beta f]^T_{\mu\nu}(t,x) := \int_{|y-x|\le t}\int_{\mathbb{R}^3_v} \frac{\mathcal{W}_{\mu\nu}(\omega,v)}{|v^0|^2(1+\omega\cdot\hat{v})}\widehat{Z}^\beta f(t-|y-x|,y,v)\frac{\mathrm{d}v\,\mathrm{d}y}{|y-x|^2}.$$

- *The 2-form* $\mathcal{L}_{Z^\gamma}(F)^S$ *is defined by*

$$\mathcal{L}_{Z^\gamma}(F)^S_{\mu\nu}(t,x) := \sum_{|\xi|+|\kappa|\le|\gamma|} N^\gamma_{\xi,\kappa}\int_{|y-x|\le t}\int_{\mathbb{R}^3_v}(\widehat{Z}^\kappa f\,\hat{v}^\lambda\mathcal{L}_{Z^\xi}(F)_\lambda{}^j)(t-|y-x|,y,v)\partial_{v^j}\mathcal{W}_{\mu\nu}(\omega,v)\frac{\mathrm{d}v\,\mathrm{d}y}{|y-x|}.$$

Proof. Fix $|\gamma|\le N-1$ and apply Proposition 2.4 in order to rewrite the Maxwell equations satisfied by $\mathcal{L}_{Z^\gamma}(F)$ as

$$\nabla^\mu\mathcal{L}_{Z^\gamma}(F)_{\mu\nu} = \int_{\mathbb{R}^3_v}\frac{v_\nu}{v^0}f_\gamma(t,x,v)\,\mathrm{d}v,\quad \nabla^{\mu}{}^*\mathcal{L}_{Z^\gamma}(F)_{\mu\nu}=0,\quad \nu\in[\![0,3]\!],\quad f_\gamma := \sum_{|\beta|\le|\gamma|}C^\gamma_\beta\widehat{Z}^\beta f, \tag{42}$$

with $C^\gamma_\beta\in\mathbb{N}$. Introduce further the electric E_γ and magnetic B_γ parts of $\mathcal{L}_{Z^\gamma}(F)$,

$$E^i_\gamma := \mathcal{L}_{Z^\gamma}(F)_{0i},\quad i\in[\![1,3]\!],\quad B^1_\gamma=\mathcal{L}_{Z^\gamma}(F)_{32},\quad B^2_\gamma=\mathcal{L}_{Z^\gamma}(F)_{13},\quad B^3_\gamma=\mathcal{L}_{Z^\gamma}(F)_{21}. \tag{43}$$

In such a way, our framework exactly corresponds to the one of Glassey and Strauss. More precisely, one can compute the source terms of the wave equations satisfied by the components of E_γ and B_γ. For any $0\le\mu,\nu\le 3$, we have

$$\Box\mathcal{L}_{Z^\gamma}(F)_{\mu\nu} = \int_{\mathbb{R}^3_v}\hat{v}_\mu\partial_{x^\nu}f_\gamma-\hat{v}_\nu\partial_{x^\mu}f_\gamma\,\mathrm{d}v,\quad\text{so, for instance,}\quad \Box E^i_\gamma = -\int_{\mathbb{R}^3_v}\partial_{x^i}f_\gamma+\hat{v}_i\partial_t f_\gamma\,\mathrm{d}v.$$

Applying [Glassey and Strauss 1986, Theorem 3] to $(f_\gamma,E_\gamma,B_\gamma)$ provides us, for any $0\le\mu,\nu\le 3$,

$$4\pi\mathcal{L}_{Z^\gamma}(F)_{\mu\nu} = \mathcal{L}_{Z^\gamma}(F)^{\mathrm{data}}_{\mu\nu}+\mathcal{L}_{Z^\gamma}(F)^T_{\mu\nu}-\int_{|y-x|\le t}\int_{\mathbb{R}^3_v}\mathcal{W}_{\mu\nu}(\omega,v)(\boldsymbol{T}_0 f_\gamma)(t-|y-x|,y,v)\frac{\mathrm{d}v\,\mathrm{d}y}{|y-x|},$$

where we recall that $\boldsymbol{T}_0=\hat{v}^\lambda\partial_{x^\lambda}$. Note that the constants C^γ_β in the definitions of $\mathcal{L}_{Z^\gamma}(F)^{\mathrm{data}}$, $\mathcal{L}_{Z^\gamma}(F)^T$ and f_γ are the same. Applying the commutation formula of Proposition 2.4 for any $|\beta|\le|\gamma|$ yields

$$\boldsymbol{T}_0 f_\gamma = -\sum_{|\beta|\le|\gamma|}C^\gamma_\beta\hat{v}^\mu F_\mu{}^j\partial_{v^j}\widehat{Z}^\beta f + C^\gamma_\beta\sum_{|\xi|+|\kappa|\le|\beta|}C^\beta_{\xi,\kappa}\hat{v}^\mu\mathcal{L}_{Z^\xi}(F)_\mu{}^j\partial_{v^j}\widehat{Z}^\kappa f, \tag{44}$$

with $C^\beta_{\xi,\kappa}\in\mathbb{N}$. It remains to integrate by parts in v and to recall that $\nabla_{v^j}\cdot\hat{v}^\mu\mathcal{L}_{Z^\xi}(F)_\mu{}^j=\mathcal{L}_{Z^\xi}(F)_j{}^j=0$. □

It will then be important to estimate the kernels introduced in the previous proposition.

Lemma 5.4. *For all* $(\omega,v)\in\mathbb{S}^2\times\mathbb{R}^3_v$, *we have* $|\omega+\hat{v}|^2$, $|\omega\wedge\hat{v}|^2\le 2(1+\omega\cdot\hat{v})$ *and* $(1+\omega\cdot\hat{v})^{-1}\le 2|v^0|^2$.

Proof. For the second inequality, simply note that

$$2|v^0|^2(1+\omega\cdot\hat{v})\ge 2|v^0|^2(1-|\hat{v}|)=2v^0(v^0-|v|)\ge(v^0+|v|)(v^0-|v|)=|v^0|^2-|v|^2=1.$$

For the first ones, since $|\omega|=1$ and $|\hat{v}|\le 1$,

$$|\omega+\hat{v}|^2=|\omega|^2+|\hat{v}|^2+2\omega\cdot\hat{v}\le 2(1+\omega\cdot\hat{v}),$$
$$|\omega\wedge\hat{v}|^2=|\omega|^2|\hat{v}|^2-|\omega\cdot\hat{v}|^2\le(1+\omega\cdot\hat{v})(1-\omega\cdot\hat{v})\le 2(1+\omega\cdot\hat{v}).$$
□

Corollary 5.5. *For any* $0 \leq \mu, \nu \leq 3$ *and all* $(\omega, v) \in \mathbb{S}^2 \times \mathbb{R}^3_v$, *there holds*

$$|\mathcal{W}_{\mu\nu}|(\omega, v) \leq 2v^0, \qquad \frac{|\mathcal{W}_{\mu\nu}|(\omega, v)}{|v^0|^2(1+\omega\cdot\hat{v})} \leq 4v^0, \quad |\nabla_v \mathcal{W}_{\mu\nu}|(\omega, v) \leq 6v^0.$$

We have similar bounds for their first-order derivatives,

$$|\nabla_v \mathcal{W}_{\mu\nu}|(\omega, v) \lesssim v^0, \quad \left|\nabla_v\left(\frac{\mathcal{W}_{\mu\nu}(\omega, v)}{|v^0|^2(1+\omega\cdot\hat{v})}\right)\right| \lesssim v^0, \quad |\nabla_v \nabla_v \mathcal{W}_{\mu\nu}|(\omega, v) \lesssim v^0.$$

Proof. The first two inequalities are a direct consequence of the previous lemma. The other ones ensue from straightforward computations carried out in Lemma A.2. □

Remark 5.6. These bounds are sharp. Let us focus for instance on the first one, $|\mathcal{W}_{\mu\nu}|(\omega, v) \leq 2v^0$. For this, consider, for any $v \in \mathbb{R}^3_v$, the function $\phi_v : \omega \mapsto 1+\omega\cdot\hat{v}$ defined on $\mathbb{S}^2$. Then,

$$\min_{\omega\in\mathbb{S}^2} \phi_v(\omega) = \frac{v^0-|v|}{v^0} = \frac{1}{v^0(v^0+|v|)} \leq \frac{1}{|v^0|^2}, \quad \max_{\omega\in\mathbb{S}^2} \phi_v(\omega) = \frac{v^0+|v|}{v^0} \geq 1.$$

By continuity, there exists $\omega_v \in \mathbb{S}^2$ such that $1+\omega_v\cdot\hat{v} = |v^0|^{-2}$. Then, using $|\omega+\hat{v}|^2 = 2(1+\omega\cdot\hat{v}) - |v^0|^{-2}$, we have

$$\sum_{1\leq i\leq 3} |\mathcal{W}_{0i}|^2(\omega_v, v) = \frac{|\omega_v+\hat{v}|^2}{|1+\omega_v\cdot\hat{v}|^2} = \frac{1}{1+\omega_v\cdot\hat{v}}\left(2-\frac{1}{|v^0|^2(1+\omega_v\cdot\hat{v})}\right) = v^0.$$

In order to improve the bootstrap assumption (BA2), we will need to use the Glassey–Strauss decomposition of the spatial derivatives of the electromagnetic field. A similar result holds for the time derivative but we will estimate it by exploiting the Maxwell equations. For instance, one can check that (2)–(3) imply $|\nabla_{\partial_t} F| \lesssim \sum_{1\leq k\leq 3} |\nabla_{\partial_{x^k}} F| + |J(f)|$. We lighten the notations by denoting the Lorentz force as

$$K^j := \hat{v}^\mu F_\mu{}^j, \quad K^j_\xi := \hat{v}^\mu \mathcal{L}_{Z^\xi}(F)_\mu{}^j, \quad 1 \leq j \leq 3, \ 1 \leq |\xi| \leq N. \tag{45}$$

Proposition 5.7. *Let* $|\gamma| = N-1$ *and* $1 \leq k \leq 3$. *Then,* $\nabla_{\partial_{x^k}} \mathcal{L}_{Z^\gamma}(F)$ *can be written as*

$$4\pi \nabla_{\partial_{x^k}} \mathcal{L}_{Z^\gamma}(F) = A^{\mathrm{data}}_{\gamma,k} + A^{\mathrm{ver}}_{\gamma,k} + A^{TT}_{\gamma,k} + A^{TS}_{\gamma,k} + A^{SS}_{\gamma,k},$$

where the five 2*-forms satisfy the following properties. We fix* $0 \leq \mu, \nu \leq 3$ *and we use again the notation* $\omega = (y-x)/|y-x|$ *in the integrals written below. Moreover, we give the definition of the kernels at the end of the statement.*[9]

- $A^{\mathrm{data}}_{\gamma,k}$ *can be explicitly computed in terms of the initial data,*

$$A^{\mathrm{data}}_{\gamma,k,\mu\nu}(t,x) = 4\pi\,\partial_{x^k}\mathcal{L}_{Z^\gamma}(F)^{\mathrm{data}}_{\mu\nu}(t,x) - \sum_{|\beta|\leq N-1} C^\gamma_\beta \frac{1}{t^2}\int_{|y-x|=t}\int_{\mathbb{R}^3_v} \mathcal{D}^k_{\mu\nu}(\omega, v)\widehat{Z}^\beta f(0,y,v)\,\mathrm{d}v\,\mathrm{d}y$$
$$- \sum_{|\beta|\leq N-1} C^\gamma_\beta \frac{1}{t}\int_{|y-x|=t}\int_{\mathbb{R}^3_v} \mathcal{C}^k_{\mu\nu}(\omega, v)\boldsymbol{T}_0\widehat{Z}^\beta f(0,y,v)\,\mathrm{d}v\,\mathrm{d}y.$$

[9]We point out that we are only interested in the qualitative properties of these kernels.

- $A^{\mathrm{ver}}_{\gamma,k}$ *is the vertex term,*

$$A^{\mathrm{ver}}_{\gamma,k,\mu\nu}(t,x) := \sum_{|\beta|\leq N-1} C^{\gamma}_{\beta} \int_{\sigma\in\mathbb{S}^2}\int_{\mathbb{R}^3_v} \mathcal{D}^k_{\mu\nu}(\sigma,v)\widehat{Z}^{\beta} f(t,x,v)\,\mathrm{d}v\,\mathrm{d}\mu_{\mathbb{S}^2}.$$

- $A^{TT}_{\gamma,k}$ *is the most singular term,*

$$A^{TT}_{\gamma,k,\mu\nu}(t,x) := \sum_{|\beta|\leq N-1} C^{\gamma}_{\beta} \int_{|y-x|\leq t}\int_{\mathbb{R}^3_v} \mathcal{A}^k_{\mu\nu}(\omega,v)\widehat{Z}^{\beta} f(t-|y-x|,y,v)\frac{\mathrm{d}v\,\mathrm{d}y}{|y-x|^3}$$

and the crucial identity $\int_{|\sigma|=1}\mathcal{A}^k_{\mu\nu}(\sigma,\hat{v})\,\mathrm{d}\mu_{\mathbb{S}^2}=0$ *holds for all* $v\in\mathbb{R}^3_v$.

- $A^{T,S}_{\gamma,k}$ *is given by*

$$A^{T,S}_{\gamma,k,\mu\nu}(t,x) := \sum_{|\xi|+|\kappa|\leq N-1} N^{\gamma}_{\xi,\kappa} \int_{|y-x|\leq t}\int_{\mathbb{R}^3_v} \nabla_v\mathcal{B}^k_{\mu\nu}(\omega,v)\cdot(\widehat{Z}^{\kappa} f K_{\xi})(t-|y-x|,y,v)\frac{\mathrm{d}v\,\mathrm{d}y}{|y-x|^2}.$$

- $A^{SS}_{\gamma,k}$ *is the sum of the four following quantities, where* $N^{\gamma}_{\xi,\zeta,\kappa}\in\mathbb{N}$,

$$A^{SS,I}_{\gamma,k,\mu\nu} := \sum_{|\xi|+|\zeta|+|\kappa|\leq N-1} N^{\gamma}_{\xi,\zeta,\kappa} \int_{|y-x|\leq t}\int_{\mathbb{R}^3_v} [\nabla_v(\nabla_v\mathcal{C}^k_{\mu\nu}(\omega,\cdot)\cdot K_{\xi})\cdot K_{\zeta}\widehat{Z}^{\kappa} f](t-|y-x|,y,v)\frac{\mathrm{d}v\,\mathrm{d}y}{|y-x|},$$

$$A^{SS,II}_{\gamma,k,\mu\nu} := \sum_{|\xi|+|\kappa|\leq N-1} N^{\gamma}_{\xi,\kappa} \int_{|y-x|\leq t}\int_{\mathbb{R}^3_v} \nabla_v\mathcal{C}^k_{\mu\nu}(\omega,v)\cdot(\boldsymbol{T}_0(K_{\xi})\widehat{Z}^{\kappa} f)(t-|y-x|,y,v)\frac{\mathrm{d}v\,\mathrm{d}y}{|y-x|},$$

$$A^{SS,III}_{\gamma,k,\mu\nu} := \sum_{|\xi|+|\kappa|\leq N-1} N^{\gamma}_{\xi,\kappa} \int_{|y-x|\leq t}\int_{\mathbb{R}^3_v} \mathcal{C}^k_{\mu\nu}(\omega,v)\frac{\delta^n_j-\hat{v}_j\hat{v}^n}{v^0}(\partial_{x^n}(K^j_{\xi})\widehat{Z}^{\kappa} f)(t-|y-x|,y,v)\frac{\mathrm{d}v\,\mathrm{d}y}{|y-x|},$$

$$A^{SS,IV}_{\gamma,k,\mu\nu} := \sum_{|\xi|+|\kappa|\leq N-1} N^{\gamma}_{\xi,\kappa} \int_{|y-x|\leq t}\int_{\mathbb{R}^3_v} \mathcal{C}^k_{\mu\nu}(\omega,v)\frac{\delta^n_j-\hat{v}_j\hat{v}^n}{v^0}(K^j_{\xi}\partial_{x^n}\widehat{Z}^{\kappa} f)(t-|y-x|,y,v)\frac{\mathrm{d}v\,\mathrm{d}y}{|y-x|}.$$

- *The kernels are smooth functions of* $(\omega,v)\in\mathbb{S}^2\times\mathbb{R}^3_v$ *given by*

$$\mathcal{A}^k_{\mu\nu}(\omega,v) := -3\frac{\boldsymbol{w}_{\mu\nu}(\omega,v)\omega_k}{|v^0|^4(1+\omega\cdot\hat{v})^4} - 3\frac{\boldsymbol{w}_{\mu\nu}(\omega,v)\hat{v}_k}{|v^0|^2(1+\omega\cdot\hat{v})^3} + \frac{\delta_{k\mu}\hat{v}_\nu-\delta_{k\nu}\hat{v}_\mu}{|v^0|^2(1+\omega\cdot\hat{v})^2},$$
$$\mathcal{B}^k_{\mu\nu}(\omega,v) := 3\frac{\boldsymbol{w}_{\mu\nu}(\omega,v)\omega_k}{|v^0|^2(1+\omega\cdot\hat{v})^3} - 2\frac{\boldsymbol{w}_{\mu\nu}(\omega,v)\hat{v}_k}{(1+\omega\cdot\hat{v})^2} - \frac{\delta_{k\mu}\hat{v}_\nu-\delta_{k\nu}\hat{v}_\mu}{1+\omega\cdot\hat{v}},$$
$$\mathcal{C}^k_{\mu\nu}(\omega,v) := \frac{\omega_k\boldsymbol{w}_{\mu\nu}(\omega,v)}{(1+\omega\cdot\hat{v})^2}, \quad \mathcal{D}^k_{\mu\nu}(\omega,v) := \frac{\omega_k\boldsymbol{w}_{\mu\nu}(\omega,v)}{|v^0|^2(1+\omega\cdot\hat{v})^3}.$$

Proof. Let $k\in[\![1,3]\!]$, $|\gamma|=N-1$ and recall from (42) the definition of f_γ and that $\mathcal{L}_{Z^\gamma}(F)$ solves the Maxwell equations with source term $J(f_\gamma)$. Recall further the electric and magnetic parts (E_γ,B_γ) of $\mathcal{L}_{Z^\gamma}(F)$, introduced in (43). In the same spirit as in the proof of Proposition 5.3, we apply[10] [Glassey 1996, Theorem 5.4.1] to $(f_\gamma,E_\gamma,B_\gamma)$. This yields

$$\nabla_{\partial_{x^k}}\mathcal{L}_{Z^\gamma}(F)_{\mu\nu} = A^{\mathrm{data}}_{\gamma,k,\mu\nu} + A^{\mathrm{ver}}_{\gamma,k,\mu\nu} + A^{TT}_{\gamma,k,\mu\nu} + \tilde{A}^{T,S}_{\gamma,k,\mu\nu} + \tilde{A}^{SS}_{\gamma,k,\mu\nu},$$

[10] See also the original version of the result, [Glassey and Strauss 1986, Theorem 4].

where

$$\tilde{A}^{T,S}_{\gamma,k,\mu\nu} := \int_{|y-x|\le t}\int_{\mathbb{R}^3_v} \mathcal{B}^k_{\mu\nu}(\omega, v)(\boldsymbol{T}_0 f_\gamma)(t-|y-x|, y, v)\frac{\mathrm{d}v\,\mathrm{d}y}{|y-x|^2},$$
$$\tilde{A}^{SS}_{\gamma,k,\mu\nu} := -\int_{|y-x|\le t}\int_{\mathbb{R}^3_v} \mathcal{C}^k_{\mu\nu}(\omega, v)(\boldsymbol{T}_0\boldsymbol{T}_0 f_\gamma)(t-|y-x|, y, v)\frac{\mathrm{d}v\,\mathrm{d}y}{|y-x|},$$

as well as $\int_{|\sigma|=1}\mathcal{A}^k_{\mu\nu}(\sigma,\hat{v})\,\mathrm{d}\mu_{\mathbb{S}^2}=0$. One can then prove that $\tilde{A}^{T,S}_{\gamma,k,\mu\nu}=A^{T,S}_{\gamma,k,\mu\nu}$ by rewriting $\boldsymbol{T}_0 f_\gamma$ using the (commuted) Vlasov equation. More precisely, we use (44) and we then integrate by parts in v. It remains to deal with $\tilde{A}^{SS}_{\gamma,k,\mu\nu}$ and we recall for this that $\nabla_v\cdot K_\xi=\nabla_{v^j}\cdot\hat{v}^\mu\mathcal{L}_{Z^\xi}(F)_\mu{}^j=0$. Hence, using again (44), we get that there exists $N^\gamma_{\xi,\kappa}\in\mathbb{N}$ such that

$$\boldsymbol{T}_0\boldsymbol{T}_0(f_\gamma)=\sum_{|\xi|+|\kappa|\le|\gamma|}N^\gamma_{\xi,\kappa}\boldsymbol{T}_0\partial_{v^j}(K^j_\xi\widehat{Z}^\kappa f).$$

Now, we write $\boldsymbol{T}_0\partial_{v^j}=\partial_{v^j}\boldsymbol{T}_0-\partial_{v^j}(\hat{v}^n)\partial_{x^n}$ and we apply the commutation formula of Proposition 2.4 to $\widehat{Z}^\kappa f$. We get

$$\boldsymbol{T}_0\partial_{v^j}(\hat{v}^\lambda\mathcal{L}_{Z^\xi}(F)_\lambda{}^j\widehat{Z}^\kappa f)=\partial_{v^j}(\boldsymbol{T}_0(K^j_\xi)\widehat{Z}^\kappa f)+\partial_{v^j}(K^j_\xi\boldsymbol{T}_0(\widehat{Z}^\kappa f))-\frac{\delta^n_j-\hat{v}_j\hat{v}^n}{v^0}(\partial_{x^n}(K^j_\xi)\widehat{Z}^\kappa f+K^j_\xi\partial_{x^n}\widehat{Z}^\kappa f),$$

so that, by integration by parts in v for the quantities related to the two first terms on the right-hand side of the previous equality,

$$\tilde{A}^{SS}_{\gamma,k,\mu\nu}=A^{SS,II}_{\gamma,k,\mu\nu}+A^{SS,III}_{\gamma,k,\mu\nu}+A^{SS,IV}_{\gamma,k,\mu\nu} + \sum_{|\xi|+|\kappa|\le|\gamma|}N^\gamma_{\xi,\kappa}\int_{|y-x|\le t}\int_{\mathbb{R}^3_v}\nabla_{v^j}(\mathcal{C}^k_{\mu\nu}(\omega,v))(K^j_\xi\boldsymbol{T}_0(\widehat{Z}^\kappa f))(\tau_y,y,v)\frac{\mathrm{d}v\,\mathrm{d}y}{|y-x|},$$

where $\tau_y:=t-|y-x|$. Finally, we deal with the last term by applying first the commutation relation of Proposition 2.4, giving that $\boldsymbol{T}_0(\widehat{Z}^\kappa f)=-K\cdot\nabla_v\widehat{Z}^\kappa f+C^\kappa_{\zeta,\beta}K_\zeta\cdot\nabla_v\widehat{Z}^\beta f$, and then by integrating by parts in v. □

These kernels and their derivatives can be estimated by a direct application of Lemmas 5.4 and A.2.

Corollary 5.8. *For any* $1\le k, j, n\le 3$ *and for all* $v\in\mathbb{R}^3_v$, *we have*

$$(|\mathcal{A}^k|+|\nabla_v\mathcal{A}^k|+|\nabla_v\mathcal{B}^k|+|\mathcal{C}^k|+|\nabla_v\mathcal{C}^k|+|\nabla_v\nabla_v\mathcal{C}^k|+|\mathcal{D}^k|+|\nabla_v\mathcal{D}^k|)(\,\cdot\,,v)\lesssim|v^0|^3.$$

5.2. *Three integral bounds.* The estimate of most of the terms listed in Propositions 5.3 and 5.7 will in fact be reduced to the analysis of three different integrals. We will deal with all of them by applying a particular case of [Glassey 1996, Lemma 6.5.2].

Lemma 5.9. *Let* $p\in\mathbb{R}$ *and* $g:\mathbb{R}^2_+\to\mathbb{R}_+$ *be a continuous function. Then, for all* $(t,x)\in[0,T[\times\mathbb{R}^3\setminus\{0\}$,

$$\int_{|y-x|\le t}g(t-|y-x|,|y|)\frac{\mathrm{d}y}{|y-x|^p}=\frac{2\pi}{|x|}\int_{\tau=0}^{t}\int_{\lambda=||x|-t+\tau|}^{|x|+t-\tau}g(\tau,\lambda)\lambda\,\mathrm{d}\lambda\frac{\mathrm{d}\tau}{(t-\tau)^{p-1}}.$$

The following result will be useful for controlling $\mathcal{L}_{Z^\gamma}(F)^S$ and $A^{SS}_{\gamma,k}$.

Lemma 5.10. *For any $b \geq 4$ and for all $(t, x) \in \mathbb{R}_+ \times \mathbb{R}$, there holds*

$$\boldsymbol{Y}^{p=1}_{b,1}(t, x) := \int_{|y-x|\leq t} \frac{1}{(1+t-|y-x|+|y|)^b(1+|t-|y-x|-|y||)} \frac{\mathrm{d}y}{|y-x|} \lesssim \frac{\log(3+|t-|x||)}{(1+t+|x|)(1+|t-|x||)^{b-2}}.$$

Proof. Note first that, on the domain of integration,

$$t-|y-x|+|y| \geq t-|y|-|x|+|y| = t-|x|, \quad t-|y-x|+|y| \geq |y| \geq |x|-|y-x| \geq |x|-t, \tag{46}$$

so that $\boldsymbol{Y}^{p=1}_{b,1}(t, x) \leq (1+|t-|x||)^{-b+4}\boldsymbol{Y}^{p=4}_{4,1}(t, x)$ and it suffices to treat the case $b = 4$. By continuity, we can assume further that $x \neq 0$. According to Lemma 5.9,

$$\boldsymbol{Y}^{p=1}_{4,1}(t, x) \leq \frac{2\pi}{|x|} \int_{\tau=0}^{t} \int_{\lambda=||x|-t+\tau|}^{|x|+t-\tau} \frac{\mathrm{d}\lambda\, \mathrm{d}\tau}{(1+\tau+\lambda)^3(1+|\tau-\lambda|)}.$$

We perform the change of variables $\underline{u} = \tau+\lambda$ and $u = \tau-\lambda$. Then, on the domain of integration $||x|-t| \leq \underline{u} \leq t+|x|$ and $u \leq ||x|-t|$. Moreover, $u \geq -\underline{u}$ since $2\tau \geq 0$. Consequently,

$$\boldsymbol{Y}^{p=1}_{4,1}(t, x) \leq \frac{\pi}{|x|} \int_{\underline{u}=||x|-t|}^{t+|x|} \int_{u=-\underline{u}}^{||x|-t|} \frac{\mathrm{d}u}{1+|u|} \frac{\mathrm{d}\underline{u}}{(1+\underline{u})^3} \leq \frac{2\pi}{|x|} \int_{\underline{u}=||x|-t|}^{t+|x|} \frac{\log(3+\underline{u})}{(1+\underline{u})^3}\, \mathrm{d}\underline{u}.$$

Now, note that

$$\boldsymbol{Y}^{p=1}_{4,1}(t, x) \lesssim \frac{2\pi \log(3+|t-|x||)}{(1+|t-|x||)|x|} \int_{\underline{u}=||x|-t|}^{t+|x|} \frac{\mathrm{d}\underline{u}}{(1+\underline{u})^2} = \frac{2\pi \log(3+|t-|x||)}{(1+t+|x|)(1+|t-|x||)^2} \frac{t+|x|-|t-|x||}{|x|}$$

and it remains to note that the last factor on the right-hand side is bounded by $2\min(t, |x|)/|x| \leq 2$. □

We will apply the next lemma in order to deal with $\mathcal{L}_{Z^\gamma}(F)^T$ and $A^{T,S}_{\gamma,k}$.

Lemma 5.11. *Let, for any $b \geq 3$ and all $(t, x) \in \mathbb{R}_+ \times \mathbb{R}^3$,*

$$\boldsymbol{Y}^{p=2}_{b}(t, x) := \int_{|y-x|\leq t} (1+t-|y-x|+|y|)^{-b} \frac{\mathrm{d}y}{|y-x|^2}.$$

Then, the following range of estimates holds. For any $0 < \delta \leq 1$,

$$\boldsymbol{Y}^{p=2}_{b}(t, x) \lesssim \delta^{-1}(1+t+|x|)^{-2+\delta}(1+|t-|x||)^{-b-\delta+3},$$
$$\boldsymbol{Y}^{p=2}_{b}(t, x) \lesssim (1+t+|x|)^{-2}(1+|t-|x||)^{-b+3}\log(1+t).$$

Proof. In view of (46), we have $\boldsymbol{Y}^{p=2}_{b}(t, x) \leq (1+|t-|x||)^{-b+3}\boldsymbol{Y}^{p=2}_{3}(t, x)$ and it suffices to treat the case $b = 3$. Then note that

$$\boldsymbol{Y}^{p=2}_{3}(t, x) = \boldsymbol{K}_{[0,\frac{t}{2}]} + \boldsymbol{K}_{[\frac{t}{2},t]}, \qquad \boldsymbol{K}_I := \int_{|y-x|\in I} (1+t-|y-x|+|y|)^{-3} \frac{\mathrm{d}y}{|y-x|^2}.$$

On the domain of integration of $\boldsymbol{K}_{[0,t/2]}$, we have $t-|y-x|+|y| \gtrsim t+|x|$. Indeed, $t-|y-x| \geq t/2$ and $|y| \geq |x|-t$ (which controls $|x|/2$ if $|x| \geq 2t$). Consequently,

$$\boldsymbol{K}_{[0,\frac{t}{2}]} \lesssim (1+t+|x|)^{-3} \int_{r=0}^{\frac{t}{2}} \mathrm{d}r \leq \tfrac{1}{2}(1+t+|x|)^{-2}. \tag{47}$$

Applying Lemma 5.9, we have

$$\boldsymbol{K}_{[\frac{t}{2},t]} \leq \frac{2\pi}{|x|} \int_{\tau=0}^{\frac{t}{2}} \int_{\lambda=||x|-t+\tau|}^{|x|+t-\tau} \frac{\mathrm{d}\lambda\,\mathrm{d}\tau}{(1+\tau+\lambda)^2(t-\tau)}.$$

Now, observe that, for all $0 \leq \tau \leq t/2$,

$$\begin{aligned}\frac{1}{|x|(t-\tau)} \int_{\lambda=||x|-t+\tau|}^{|x|+t-\tau} \frac{\mathrm{d}\lambda}{(1+\tau+\lambda)^2} &= \frac{2\min(|x|,t-\tau)}{|x|(t-\tau)(1+t+|x|)(1+\tau+||x|-t+\tau|)} \\ &\leq \frac{8}{\max(|x|,t)(1+t+|x|)(1+\tau+|t-|x||)}.\end{aligned} \tag{48}$$

Let $0 \leq \delta \leq 1$ and write $(1+\tau+|t-|x||) \geq (1+\tau)^{1-\delta}(1+|t-|x||)^{\delta}$. It remains to integrate in τ in order to derive the expected range of estimates for $\boldsymbol{K}_{[t/2,t]}$. □

Finally, a part of our analysis of $A^{TT}_{\gamma,k}$ will rely on the following estimate.

Lemma 5.12. *For all* $(t,x) \in [1,+\infty[\times \mathbb{R}^3$, *we have*

$$\boldsymbol{Y}_3^{p=3}(t,x) := \int_{1 \leq |y-x| \leq t} (1+t-|y-x|+|y|)^{-3} \frac{\mathrm{d}y}{|y-x|^3} \lesssim \frac{\log(t)}{(1+t+|x|)^3}.$$

Proof. The inequality can be easily proved if $t \leq 2$ so we assume $t \geq 2$. Start by writing

$$\boldsymbol{Y}_3^{p=3}(t,x) = \overline{\boldsymbol{K}}_{[1,\frac{t}{2}]} + \overline{\boldsymbol{K}}_{[\frac{t}{2},t]}, \qquad \overline{\boldsymbol{K}}_I := \int_{|y-x| \in I} (1+t-|y-x|+|y|)^{-3} \frac{\mathrm{d}y}{|y-x|^3}.$$

Following (47), we have

$$\overline{\boldsymbol{K}}_{[1,\frac{t}{2}]} \lesssim (1+t+|x|)^{-3} \int_{r=1}^{\frac{t}{2}} \frac{\mathrm{d}r}{r} \leq \log\left(\frac{t}{2}\right)(1+t+|x|)^{-3}.$$

Next, we apply Lemma 5.9 to get

$$\overline{\boldsymbol{K}}_{[\frac{t}{2},t]} \leq \frac{2\pi}{|x|} \int_{\tau=0}^{\frac{t}{2}} \int_{\lambda=||x|-t+\tau|}^{|x|+t-\tau} \frac{\mathrm{d}\lambda\,\mathrm{d}\tau}{(1+\tau+\lambda)^2(t-\tau)^2}.$$

If $2t \geq |x|$, we use (48) and $t-\tau \geq t/2$ in order to derive $\overline{\boldsymbol{K}}_{[t/2,t]} \lesssim t^{-2}(1+t+|x|)^{-1}\log(1+t/2)$, which implies the result. Otherwise, $2t \leq |x|$ and we have, for all $0 \leq \tau \leq t/2$,

$$\begin{aligned}\frac{1}{|x|(t-\tau)^2} \int_{\lambda=||x|-t+\tau|}^{|x|+t-\tau} \frac{\mathrm{d}\lambda\,\mathrm{d}\tau}{(1+\tau+\lambda)^2} &= \frac{2\min(|x|,t-\tau)}{|x|(t-\tau)^2(1+t+|x|)(1+\tau+||x|-t+\tau|)} \\ &\leq \frac{2}{|x|(1+t+|x|)(1+|x|-t)(t-\tau)}.\end{aligned} \tag{49}$$

We get, as $2 \leq 2t \leq |x|$,

$$\boldsymbol{K}_{[\frac{t}{2},t]} \leq 4\pi\log(2)|x|^{-1}(1+|x|/2)^{-1}(1+t+|x|)^{-1} \lesssim (1+t+|x|)^{-3}.$$ □

5.3. ***The derivatives of order up to $N-1$.*** In this subsection, we prove pointwise decay estimates for each of the elements of the decomposition of $\mathcal{L}_{Z^\gamma}(F)$ provided by Proposition 5.3. We start by dealing with $\mathcal{L}_{Z^\gamma}(F)^{\mathrm{data}}$, which is defined on $\mathbb{R}_+ \times \mathbb{R}^3$.

Proposition 5.13. *There exists $C_{\mathrm{data}} > 0$, depending only on N, such that,*

$$\forall |\gamma| \leq N-1,\ \forall (t,x) \in \mathbb{R}_+ \times \mathbb{R}^3, \quad |\mathcal{L}_{Z^\gamma}(F)^{\mathrm{data}}|(t,x) \leq \Lambda C_{\mathrm{data}} (1+t+|x|)^{-1}(1+|t-|x||)^{-1}.$$

Proof. In view of the assumptions on the initial data (see Theorem 2.10) and applying Corollary 5.5 in order to estimate $\mathcal{W}_{\mu\nu}$, we have, for any $|\beta| \leq N-1$, $\omega \in \mathbb{S}^2$ and $0 \leq \mu, \nu \leq 3$,

$$\forall y \in \mathbb{R}^3, \quad |\mathcal{L}_{Z^\beta}(F)|(0,y) + \langle y \rangle |\nabla_{t,x} \mathcal{L}_{Z^\beta}(F)|(0,y) \lesssim \sum_{|\kappa| \leq |\beta|+1} \langle y \rangle^{|\kappa|} |\nabla^\kappa_{t,x} F|(0,y) \lesssim \Lambda \langle y \rangle^{-2},$$

$$\left| \int_{\mathbb{R}^3_v} (\mathcal{W}_{\mu\nu}(\omega,\hat{v}) - \delta^0_\mu \hat{v}^\nu + \delta^0_\nu \hat{v}^\mu) \widehat{Z}^\beta f(0,y,v)\, \mathrm{d}v \right| \leq 3 \int_{\mathbb{R}^3_v} |v^0|^{N_v} |\widehat{Z}^\beta f(0,y,v)| \frac{\mathrm{d}v}{\langle v \rangle^{N_v-1}} \lesssim \epsilon \langle y \rangle^{-N_x}.$$

The estimates, at $t=0$, for the time derivatives of the solutions are obtained by using that (1)–(3) are initially verified. Using (22) for $p = N_x \geq 3$, we then deduce that,

$$\forall (t,x) \in \mathbb{R}_+ \times \mathbb{R}^3, \quad |\mathcal{L}_{Z^\gamma}(F)^{\mathrm{data}} - \mathcal{L}_{Z^\gamma}(F)^{\mathrm{hom}}|(t,x) \lesssim \epsilon (1+t+|x|)^{-1}(1+|t-|x||)^{-N_x+1} \tag{50}$$

and it remains to use $\epsilon \leq \Lambda$ and to apply Proposition 2.21 to $\mathcal{L}_{Z^\gamma}(F)^{\mathrm{hom}}_{\mu\nu}$. □

Next, we consider $\mathcal{L}_{Z^\gamma}(F)^S$, which is strongly decaying far from the light cone.

Proposition 5.14. *For any $|\gamma| \leq N-1$, there holds,*

$$\forall (t,x) \in [0,T[\times \mathbb{R}^3, \quad |\mathcal{L}_{Z^\gamma}(F)^S|(t,x) \lesssim \bar{\epsilon} \Lambda \frac{\log(3+|t-|x||)}{(1+t+|x|)(1+|t-|x||)^2}.$$

Proof. Fix $0 \leq \mu, \nu \leq 3$ and recall from Proposition 5.3 the definition of $\mathcal{L}_{Z^\gamma}(F)^S$. We have, with $\omega = (y-x)/|y-x|$,

$$|\mathcal{L}_{Z^\gamma}(F)^S_{\mu\nu}|(t,x)$$
$$\lesssim \sum_{|\xi|+|\kappa| \leq |\gamma|} \int_{|y-x| \leq t} |\mathcal{L}_{Z^\xi}(F)_\lambda{}^j|(t-|y-x|,y) \left| \int_{\mathbb{R}^3_v} \hat{v}^\lambda \partial_{v^j} \mathcal{W}_{\mu\nu}(\omega,v) \widehat{Z}^\kappa f(t-|y-x|,y,v)\, \mathrm{d}v \right| \frac{\mathrm{d}y}{|y-x|}.$$

Fix now $|\xi|+|\kappa| \leq N-1$, $j \in [\![1,3]\!]$ and $\lambda \in [\![0,3]\!]$. In view of Corollary 5.5, $\Psi(\omega,v) := \hat{v}^\lambda \partial_{v^j} \mathcal{W}_{\mu\nu}(\omega,v)$ satisfies $|\Psi|(\,\cdot\,,v) + |v^0 \nabla_v \Psi|(\,\cdot\,,v) \lesssim |v^0|^2 \leq |v^0|^{N_v-11}$. As $|\kappa| \leq N-1$, Proposition 4.13 then gives us,

$$\forall (\sigma,\tau,y) \in \mathbb{S}^2 \times [0,T[\times \mathbb{R}^3, \quad \left| \int_{\mathbb{R}^3_v} \hat{v}^\lambda \partial_{v^j} \mathcal{W}_{\mu\nu}(\sigma,v) \widehat{Z}^\kappa f(\tau,y,v)\, \mathrm{d}v \right| \lesssim \frac{\bar{\epsilon}}{(1+\tau+|y|)^3}.$$

Applying this last inequality for $(\sigma,\tau) = (\omega, t-|y-x|)$ and estimating the electromagnetic field using the bootstrap assumption (BA1), we get

$$|\mathcal{L}_{Z^\gamma}(F)^S|(t,x) \lesssim \int_{|y-x| \leq t} \frac{\bar{\epsilon}\Lambda}{(1+t-|y-x|+|y|)^4(1+|t-|y-x|-|y||)} \frac{\mathrm{d}y}{|y-x|} = \bar{\epsilon}\Lambda \boldsymbol{Y}^{p=1}_{4,1}(t,x).$$

The result then follows from Lemma 5.10. □

We finally deal with $\mathcal{L}_{Z^\gamma}(F)^T$, which actually enjoys stronger decay properties than $\mathcal{L}_{Z^\gamma}(F)^S$ for $t \sim |x|$ (see Remark 5.16 below).

Proposition 5.15. *For any $|\gamma| \leq N-1$ and all $(t,x) \in [0,T[\times \mathbb{R}^3$, we have*

$$|\mathcal{L}_{Z^\gamma}(F)^T|(t,x) \lesssim \bar{\epsilon}(1+t+|x|)^{-\frac{7}{4}}(1+|t-|x||)^{-\frac{1}{4}}.$$

Proof. In view of the definition of $\mathcal{L}_{Z^\gamma}(F)^T$, introduced in Proposition 5.3, we have

$$|\mathcal{L}_{Z^\gamma}(F)^T|(t,x) \lesssim \sum_{0\leq \mu,\nu\leq 3}\sum_{|\beta|\leq|\gamma|}\int_{|y-x|\leq t}\left|\int_{\mathbb{R}^3_v}\frac{\mathcal{W}_{\mu\nu}(\omega,v)}{|v^0|^2(1+\omega\cdot\hat{v})}\widehat{Z}^\beta f(t-|y-x|,y,v)\,\mathrm{d}v\right|\frac{\mathrm{d}y}{|y-x|^2},\quad \omega=\frac{y-x}{|y-x|}.$$

Fix $0\leq\mu,\nu\leq 3$, $|\beta|\leq|\gamma|$ and recall from Corollary 5.5 that

$$\Psi(\sigma,v) := \frac{\mathcal{W}_{\mu\nu}(\sigma,v)}{|v^0|^2(1+\omega\cdot\hat{v})}$$

satisfies $|\Psi|(\,\cdot\,,v)+|\nabla_v\Psi|(\,\cdot\,,v)\lesssim v^0$. We then obtain from Proposition 4.13 that,

$$\forall \sigma\in\mathbb{S}^2,\ \forall(\tau,z)\in[0,T[\times\mathbb{R}^3,\qquad \left|\int_{\mathbb{R}^3_v}\frac{\mathcal{W}_{\mu\nu}(\sigma,v)}{|v^0|^2(1+\sigma\cdot\hat{v})}\widehat{Z}^\beta f(\tau,z,v)\,\mathrm{d}v\right|\lesssim\frac{\bar{\epsilon}}{(1+\tau+|z|)^3}.$$

Applying this estimate for $\sigma=\omega$, $\tau=t-|y-x|$ and $z=y$, we get from Lemma 5.11 that

$$|\mathcal{L}_{Z^\gamma}(F)^T|(t,x)\lesssim\bar{\epsilon}\boldsymbol{Y}_3^{p=2}(t,x)\lesssim\bar{\epsilon}(1+t+|x|)^{-\frac{7}{4}}(1+|t-|x||)^{-\frac{1}{4}}.\qquad\square$$

Remark 5.16. In fact, Lemma 5.11 also provides $|\mathcal{L}_{Z^\gamma}(F)^T|(t,x)\lesssim\bar{\epsilon}(1+t+|x|)^{-2}\log(1+t)$. Moreover, the estimate could be significantly improved in the exterior of the light cone, where $|x|\geq t$.

If the constant C_{boot} is chosen such that $C_{\mathrm{boot}}\geq 2C_{\mathrm{data}}$ and if ϵ is small enough, Propositions 5.13, 5.14 and 5.15 allow us to improve the bootstrap assumption (BA1).

5.4. *The top-order derivatives.* In this section, we estimate all the terms listed in Proposition 5.7 in order to improve the bootstrap assumption (BA2). We start by dealing with the ones depending explicitly on the data.

Proposition 5.17. *There exists a constant $\overline{C}_{\mathrm{data}}$, depending only on N, such that, for any $k\in[\![1,3]\!]$ and $|\gamma|=N-1$,*

$$\forall(t,x)\in[0,T[\times\mathbb{R}^3,\quad |A^{\mathrm{data}}_{\gamma,k}|(t,x)\leq\Lambda\overline{C}_{\mathrm{data}}(1+t+|x|)^{-1}(1+|t-|x||)^{-2}.$$

Proof. Recall from Propositions 5.3 and 5.7 the expression of $A^{\mathrm{data}}_{\gamma,k}$ and from Corollaries 5.5 and 5.8 the bounds on the kernels. Hence, for $(t,x)\in[0,T[\times\mathbb{R}^3$,

$$|A^{\mathrm{data}}_{\gamma,k}|(t,x)\lesssim|\nabla_{\partial_{x^k}}\mathcal{L}_{Z^\gamma}(F)^{\mathrm{hom}}|(t,x)+\sum_{|\beta|\leq|\gamma|+1}\min(t^{-1},t^{-2})\int_{|y-x|=t}\int_{\mathbb{R}^3_v}|v^0|^3|\widehat{Z}^\beta f|(0,y,v)\,\mathrm{d}v\,\mathrm{d}y.$$

As $[\partial_{x^\mu}, Z] = 0$ or $[\partial_{x^\mu}, Z] = \pm\partial_{x^\nu}$ for any $Z \in \mathbb{K}$, by the equivalence of the pointwise norms (9) and in view of the smallness assumptions on the initial data, there holds

$$|\nabla_{\partial_{x^k}} \mathcal{L}_{Z^\gamma}(F)^{\mathrm{hom}}|(0, y) = |\nabla_{\partial_{x^k}} \mathcal{L}_{Z^\gamma}(F)|(0, y) \lesssim \sum_{1 \leq |\kappa| \leq N} \langle y \rangle^{|\kappa|-1} |\nabla^\kappa_{t,x} F|(0, y) \lesssim \Lambda \langle y \rangle^{-3},$$

$$|\nabla_{t,x}\nabla_{\partial_{x^k}} \mathcal{L}_{Z^\gamma}(F)^{\mathrm{hom}}|(0, y) = |\nabla_{t,x}\nabla_{\partial_{x^k}} \mathcal{L}_{Z^\gamma}(F)|(0, y) \lesssim \sum_{2 \leq |\kappa| \leq N+1} \langle y \rangle^{|\kappa|-2} |\nabla^\kappa_{t,x} F|(0, y) \lesssim \Lambda \langle y \rangle^{-4}.$$

As $\nabla_{\partial_{x^k}} \mathcal{L}_{Z^\gamma}(F)^{\mathrm{hom}}_{\mu\nu}$ is solution to the homogeneous wave equation, Proposition 2.21 gives

$$|\nabla_{\partial_{x^k}} \mathcal{L}_{Z^\gamma}(F)^{\mathrm{hom}}|(t, x) \lesssim \Lambda(1+t+|x|)^{-1}(1+|t-|x||)^{-2}.$$

Since $|v^0|^{-N_v+3} \in L^1(\mathbb{R}^3_v)$, we have, for any $|\beta| \leq N$,

$$\int_{\mathbb{R}^3_v} |v^0|^3 |\widehat{Z}^\beta f|(0, y, v)\, \mathrm{d}v \lesssim \langle y \rangle^{-N_x} \sup_{|\kappa|+|\xi| \leq N} \sup_{(x,v) \in \mathbb{R}^6} |v^0|^{N_v+|\xi|} \langle x \rangle^{N_x+|\kappa|} |\partial^\kappa_v \partial^\xi_x f|(0, x, v) \lesssim \epsilon \langle y \rangle^{-N_x}.$$

Consequently, as $N_x \geq 5$, we have

$$|A^{\mathrm{data}}_{\gamma,k}|(t, x) \lesssim \Lambda(1+t+|x|)^{-1}(1+|t-|x||)^{-2} + \epsilon \min(t^{-1}, t^{-2}) \mathcal{Q}_{t,x}, \qquad \mathcal{Q}_{t,x} := \int_{|y-x|=t} \frac{\mathrm{d}\mu_{\mathbb{S}^2}}{\langle y \rangle^5}.$$

As $\epsilon \leq \Lambda$, it remains to prove $\min(t^{-1}, t^{-2})\mathcal{Q}_{t,x} \lesssim (1+t+|x|)^{-1}(1+|t-|x||)^{-2}$ and, for this, we consider two cases.

- If $t \leq 1$, then $|y| \geq |x| - 1$ on the domain of integration and $\mathcal{Q}_{t,x} \lesssim 4\pi t^2 \langle x \rangle^{-5}$. It remains to note that $\langle x \rangle \geq 1 + t + |x| \geq 1 + |t - |x||$ and $t^{-1} \leq t^{-2}$ in the region considered.
- Otherwise $t \geq 1$ and we have $\mathcal{Q}_{t,x} \lesssim t(1+t+|x|)^{-1}(1+|t-|x||)^{-2}$ according to the estimate (22). The result follows from $t^{-2} \leq t^{-1}$ in the domain treated here. □

Next, we consider the vertex term.

Proposition 5.18. *Let $k \in [\![1, 3]\!]$ and $|\gamma| = N - 1$. We have,*

$$\forall (t, x) \in [0, T[\, \times \mathbb{R}^3, \qquad |A^{\mathrm{ver}}_{\gamma,k}|(t, x) \lesssim \bar{\epsilon}(1+t+|x|)^{-3}.$$

Proof. Fix $0 \leq \mu, \nu \leq 3$, $(t, x) \in [0, T[\, \times \mathbb{R}^3$ and recall $N_v \geq 15$, so that Corollary 5.8 implies $|\mathcal{D}^k_{\mu\nu}|(\omega, v) + |v^0 \nabla_v \mathcal{D}^k_{\mu\nu}|(\omega, v) \lesssim |v^0|^{N_v-11}$. Proposition 4.13, applied for $\Psi = \mathcal{D}^k_{\mu\nu}$ and to any $|\beta| \leq N-1$, then yields

$$|A^{\mathrm{ver}}_{\gamma,k,\mu\nu}|(t, x) \lesssim \sum_{|\beta| \leq |\gamma|} \int_{\sigma \in \mathbb{S}^2} \left| \int_{\mathbb{R}^3_v} \mathcal{D}^k_{\mu\nu}(\sigma, v) \widehat{Z}^\beta f(t, x, v)\, \mathrm{d}v \right| \mathrm{d}\mu_{\mathbb{S}^2}$$
$$\lesssim \frac{\bar{\epsilon}}{(1+t+|x|)^3} \int_{\sigma \in \mathbb{S}^2} \mathrm{d}\mu_{\mathbb{S}^2} = \frac{4\pi \bar{\epsilon}}{(1+t+|x|)^3}. \qquad \square$$

We now estimate $A^{T,S}_{\gamma,k}$. Note that the next result could be easily improved but it is more than enough for the purpose of improving the bootstrap assumption (BA2).

Proposition 5.19. *For any* $k \in [\![1,3]\!]$ *and* $|\gamma| = N-1$, *there holds*,

$$\forall (t,x) \in [0,T[\times \mathbb{R}^3, \quad |A^{T,S}_{\gamma,k}|(t,x) \lesssim \bar{\epsilon}\Lambda(1+t+|x|)^{-\frac{3}{2}}(1+|t-|x||)^{-2}.$$

Proof. Let $0 \le \mu,\nu \le 3$, $(t,x) \in [0,T[\times \mathbb{R}^3$ and recall that $K^j_\xi := \hat{v}^\lambda \mathcal{L}_{Z^\xi}(F)_\lambda{}^j$. Consequently, $|A^{T,S}_{\gamma,k,\mu\nu}|(t,x)$ is bounded by a linear combination of terms of the form

$$\mathcal{Q}^{\xi,\kappa}_{t,x} := \int_{|y-x|\le t} |\mathcal{L}_{Z^\xi}(F)_\lambda{}^j|(t-|y-x|,y) \int_{\mathbb{R}^3_v} |\partial_{v^j}\mathcal{B}^k_{\mu\nu}(\omega,v)||\widehat{Z}^\kappa f(t-|y-x|,y,v)|\,\mathrm{d}v \frac{\mathrm{d}y}{|y-x|^2},$$

with $|\xi|+|\kappa| \le N-1$ and where we recall that $\omega = (y-x)/|y-x|$. Since $|\partial_{v^j}\mathcal{B}^k_{\mu\nu}(\omega,v)| \lesssim |v^0|^3$ by Corollary 5.8 and $N_v \ge 13$, Proposition 4.11, applied for $a=1$, provides

$$\int_{\mathbb{R}^3_v} |\partial_{v^j}\mathcal{B}^k_{\mu\nu}(\omega,v)||\widehat{Z}^\kappa f(t-|y-x|,y,v)|\,\mathrm{d}v \lesssim \frac{\bar{\epsilon}(1+|t-|y-x|-|y||)}{(1+t-|y-x|+|y|)^{3+\frac{1}{2}}}.$$

Moreover, as $|\xi| \le N-1$, the bootstrap assumption (BA1) gives

$$|\mathcal{L}_{Z^\xi}(F)_\lambda{}^j|(\tau,y) \lesssim \Lambda(1+t-|y-x|+|y|)^{-1}(1+|t-|y-x|-|y||)^{-1}.$$

Consequently, the last two estimates yield

$$\mathcal{Q}^{\xi,\kappa}_{t,x} \lesssim \bar{\epsilon}\Lambda \int_{|y-x|\le t} (1+t-|y-x|)^{-4-\frac{1}{2}} \frac{\mathrm{d}y}{|y-x|^2} = \bar{\epsilon}\Lambda \boldsymbol{Y}^{p=2}_{4+\frac{1}{2}}(t,x)$$

and the result follows from Lemma 5.11. □

We pursue with the analysis of $A^{SS}_{\gamma,k}$. As for the previous term, the estimate could be improved.

Proposition 5.20. *We have*, *for any* $k \in [\![1,3]\!]$ *and* $|\gamma| = N-1$,

$$\forall (t,x) \in [0,T[\times \mathbb{R}^3, \quad |A^{SS}_{\gamma,k}|(t,x) \lesssim \bar{\epsilon}\Lambda\langle\Lambda\rangle(1+t+|x|)^{-1}(1+|t-|x||)^{-\frac{5}{2}}.$$

Proof. We fix $(t,x) \in [0,T[\times \mathbb{R}^3$ and we recall that $K^j_\xi := \hat{v}^\lambda \mathcal{L}_{Z^\xi}(F)_\lambda{}^j$. Recall further from Proposition 5.7 that $A^{SS}_{\gamma,k}$ can be decomposed as the sum of four terms. Bounding the kernel in $A^{SS,I}_{\gamma,k}$ by Corollary 5.8 and estimating the derivatives of the electromagnetic field using (BA1), we have

$$|A^{SS,I}_{\gamma,k}|(t,x) \lesssim \sum_{|\kappa|\le N-1} \int_{|y-x|\le t} \frac{\Lambda^2}{(1+\tau+|y|)^2(1+|\tau-|y||)^2} \int_{\mathbb{R}^3_v} |v^0|^3|\widehat{Z}^\kappa f|(\tau,y,v)\,\mathrm{d}v \frac{\mathrm{d}y}{|y-x|}, \quad \tau := t-|y-x|.$$

For the next two terms, recall that $\boldsymbol{T}_0 = \hat{v}^\lambda \partial_{x^\lambda}$ and the expression of K_ξ. Recall further from Corollary 5.8 that the integral kernels are bounded by $|v^0|^3$. Consequently, we can bound $|A^{SS,II}_{\gamma,k}|(t,x) + |A^{SS,III}_{\gamma,k}|(t,x)$ by a linear combination of terms of the form

$$\mathcal{R}^{\xi,\kappa}_{t,x} := \int_{|y-x|\le t} |\nabla_{t,x}\mathcal{L}_{Z^\xi}(F)|(t-|y-x|,y) \int_{\mathbb{R}^3_v} |v^0|^3|\widehat{Z}^\kappa f(t-|y-x|,y,v)|\,\mathrm{d}v \frac{\mathrm{d}y}{|y-x|},$$

where $|\xi| + |\kappa| \leq N-1$. We estimate the electromagnetic field through (BA2) if $|\xi| = N-1$ or by Proposition 3.2 if $|\xi| \leq N-2$. This leads to the bound

$$|A^{SS,II}_{\gamma,k}|(t,x) + |A^{SS,III}_{\gamma,k}|(t,x) \lesssim \sum_{|\kappa|\leq N-1} \int_{|y-x|\leq t} \frac{\Lambda \log(3+|\tau-|y||)}{(1+\tau+|y|)(1+|\tau-|y||)^2} \int_{\mathbb{R}^3_v} |v^0|^3 |\widehat{Z}^\kappa f|(\tau,y,v)\,\mathrm{d}v \frac{\mathrm{d}y}{|y-x|},$$

where $\tau = t-|y-x|$. Controlling the velocity average through the improved estimates of Proposition 4.11 yields, as $N_v \geq 13$,

$$|A^{SS,I}_{\gamma,k}|(t,x) + |A^{SS,II}_{\gamma,k}|(t,x) + |A^{SS,III}_{\gamma,k}|(t,x) \lesssim \bar{\epsilon}\Lambda\langle\Lambda\rangle \int_{|y-x|\leq t} \frac{\log^{3N_x+3N+1}(3+t-|y-x|+|y|)}{(1+t-|y-x|+|y|)^5(1+|t-|y-x|-|y||)} \frac{\mathrm{d}y}{|y-x|} \lesssim \bar{\epsilon}\Lambda\langle\Lambda\rangle \boldsymbol{Y}^{p=1}_{4+\frac{3}{4},1}(t,x).$$

Finally, we can bound similarly $|A^{SS,IV}_{\gamma,k}|(t,x)$ by a linear combination of terms of the form

$$\overline{\mathcal{R}}^{\xi,\kappa}_{t,x} := \int_{|y-x|\leq t} |\mathcal{L}_{Z^\xi}(F)|(t-|y-x|,y) \left| \int_{\mathbb{R}^3_v} \mathcal{V}(\omega,v)\partial_{x^n}\widehat{Z}^\kappa f(t-|y-x|,y,v)\,\mathrm{d}v \right| \frac{\mathrm{d}y}{|y-x|},$$

where $|\xi|+|\kappa| \leq N-1$, $1 \leq n \leq 3$ and $\mathcal{V}(\omega,v)$ is of the form $\mathcal{C}^k_{\mu\nu}(\omega,v)\hat{v}^\lambda|v^0|^{-1}$. We get from Corollary 5.8 that $|\mathcal{V}(\omega,v)| + |v^0\nabla_v\mathcal{V}(\omega,v)| \lesssim |v^0|^3$, so that Proposition 4.15 gives,

$$\forall(\tau,y,\sigma) \in [0,T[\times\mathbb{R}^3\times\mathbb{S}^2, \quad \left| \int_{\mathbb{R}^3_v} \mathcal{V}(\sigma,v)\partial_{x^n}\widehat{Z}^\kappa f(\tau,y,v)\,\mathrm{d}v \right| \lesssim \bar{\epsilon}\frac{\log^{3N_x+3N}(3+\tau)}{(1+\tau+|y|)^4}.$$

Applying it to $\sigma = \omega$ and $\tau = t-|y-x|$ and estimating the electromagnetic field using (BA1), we get

$$\overline{\mathcal{R}}^{\xi,\kappa}_{t,x} \lesssim \bar{\epsilon}\Lambda \int_{|y-x|\leq t} \frac{\log^{3N_x+3N}(3+t-|y-x|)}{(1+t-|y-x|+|y|)^5(1+|t-|y-x|-|y||)} \frac{\mathrm{d}y}{|y-x|} \lesssim \bar{\epsilon}\Lambda \boldsymbol{Y}^{p=1}_{4+\frac{3}{4},1}(t,x).$$

Consequently, $|A^{SS}_{\gamma,k}|(t,x) \lesssim \bar{\epsilon}\Lambda\langle\Lambda\rangle\boldsymbol{Y}^{p=1}_{4+3/4,1}(t,x)$, so that the result follows from Lemma 5.10. □

Finally, we deal with the most problematic term, the one with an integral kernel carrying the nonintegrable weight $|y-x|^{-3}$.

Proposition 5.21. *Let $k \in [\![1,3]\!]$ and $|\gamma| = N-1$. Then,*

$$\forall(t,x) \in [0,T[\times\mathbb{R}^3, \quad |A^{TT}_{\gamma,k}|(t,x) \lesssim \bar{\epsilon}\frac{\log(3+t)}{(1+t+|x|)^3}.$$

Proof. Let $0 \leq \mu,\nu \leq 3$, $|\beta| \leq N-1$ and

$$G^\beta_\sigma(\tau,y) := \int_{\mathbb{R}^3} \mathcal{A}^k_{\mu\nu}(\sigma,v)\widehat{Z}^\beta f(\tau,y,v)\,\mathrm{d}v, \quad (\sigma,\tau,y) \in \mathbb{S}^2\times[0,T[\times\mathbb{R}^3.$$

Recall from Corollary 5.8 the bound on the kernel $\mathcal{A}^k_{\mu\nu}$ and apply Proposition 4.13 for $\Psi = \mathcal{A}^k_{\mu\nu}$. We obtain,

$$\forall(\sigma,\tau,y) \in \mathbb{S}^2\times[0,T[\times\mathbb{R}^3, \quad |G^\beta_\sigma|(\tau,y) \lesssim \bar{\epsilon}(1+\tau+|y|)^{-3},$$

which, applied for $(\sigma,\tau)=(\omega,t-|y-x|)$, yields

$$|A^{TT}_{\gamma,k,\mu\nu}|(t,x)\lesssim\bar\epsilon\boldsymbol{Y}^{p=3}_3(t,x)+\sum_{|\beta|\le N-1}\mathcal{U}^\beta_{t,x},\quad \mathcal{U}^\beta_{t,x}:=\left|\int_{|y-x|\le 1}\int_{\mathbb{R}^3_v}\mathcal{A}^k_{\mu\nu}(\omega,v)\widehat{Z}^\beta f(t-|y-x|,y,v)\frac{\mathrm{d}v\,\mathrm{d}y}{|y-x|^3}\right|.$$

Fix $|\beta|\le N-1$ and recall from Proposition 5.7 that the average of $\sigma\mapsto\mathcal{A}^k_{\mu\nu}(\sigma,\cdot)$ on $\mathbb{S}^2$ vanishes. Hence,

$$\begin{aligned}\mathcal{U}^\beta_{t,x}&=\left|\int_{|y-x|\le1}\int_{\mathbb{R}^3_v}\mathcal{A}^k_{\mu\nu}(\omega,v)(\widehat{Z}^\beta f(t-|y-x|,y,v)-\widehat{Z}^\beta f(t-|y-x|,x,v))\frac{\mathrm{d}v\,\mathrm{d}y}{|y-x|^3}\right|\\&\le\int_{|y-x|\le1}|G^\beta_\omega(t-|y-x|,y)-G^\beta_\omega(t-|y-x|,x)|\frac{\mathrm{d}y}{|y-x|^3}.\end{aligned}$$

For any $(\sigma,\tau)\in\mathbb{S}^2\times[0,T[$, we apply the mean value theorem to $s\mapsto G^\beta_\sigma(\tau,x+s(y-x))$ on the interval $[0,1]$. Then, there exists $x_{\sigma,\tau}$ in the segment $[x,y]\subset\mathbb{R}^3$ such that

$$G^\beta_\sigma(\tau,y)-G^\beta_\sigma(\tau,x)=\omega\cdot\nabla_x G^\beta_\sigma(\tau,x_{\sigma,\tau})|y-x|,\quad \omega=\frac{y-x}{|y-x|}.$$

Apply now Proposition 4.15 for $\Phi=\mathcal{A}_{\mu\nu}$ in order to get, for any $1\le i\le 3$,

$$\begin{aligned}\forall(\sigma,\tau,z)\in\mathbb{S}^2\times[0,T[\times\mathbb{R}^3,\quad |\partial_{x^i}G^\beta_\sigma|(\tau,z)&=\left|\int_{\mathbb{R}^3}\mathcal{A}^k_{\mu\nu}(\sigma,v)\partial_{x^i}\widehat{Z}^\beta f(\tau,z,v)\,\mathrm{d}v\right|\\&\lesssim\bar\epsilon\frac{\log^{3N_x+3N}(3+\tau)}{(1+\tau+|z|)^4}.\end{aligned}$$

Applying the last two identities for $\sigma=\omega$, $\tau=t-|y-x|$ and $z=x_{\sigma,\tau}$ yields

$$\mathcal{U}^\beta_{t,x}\lesssim\int_{|y-x|\le1}\frac{\bar\epsilon}{(1+t-|y-x|+|x_{\omega,t-|y-x|}|)^3}\frac{\mathrm{d}y}{|y-x|^2}.$$

As $|y-x|\le1$ and $x_{\omega,t-|y-x|}\in[x,y]$, we have $1+t-|y-x|\ge\frac12(1+t)$ and $|x_{\omega,t-|y-x|}|\ge|x|-1$, so that

$$|A^{TT}_{\gamma,k,\mu\nu}|(t,x)\lesssim\bar\epsilon\boldsymbol{Y}^{p=3}_3(t,x)+\bar\epsilon(1+t+|x|)^{-3}.$$

We conclude the proof by applying Lemma 5.12. □

As in the previous subsection, if C_{boot} is chosen such that $C_{\mathrm{boot}}\ge 2\overline{C}_{\mathrm{data}}$ and if ϵ is small enough, we can improve the bootstrap assumption (BA2) for the spatial derivatives $\nabla_{\partial_{x^k}}\mathcal{L}_{Z^\gamma}(F)$, with $1\le k\le3$ and $|\gamma|=N-1$, by applying Propositions 5.17–5.21. The time derivative can then be controlled using

$$|\nabla_{\partial_t}\mathcal{L}_{Z^\gamma}(F)|\lesssim\sum_{1\le k\le3}|\nabla_{\partial_{x^k}}\mathcal{L}_{Z^\gamma}(F)|+\sum_{|\beta|\le|\gamma|}|J(\widehat{Z}^\beta f)|,$$

which follows from the commuted Maxwell equations (see Proposition 2.4). We stress, for the smallness condition on ϵ, that $\bar\epsilon\langle\Lambda\rangle^2\le 2\epsilon e^{(D+3)\Lambda}$.

6. Modified scattering for the distribution function

In this section, we determine the precise asymptotic behavior of the particle density and its derivatives under the additional assumption (15) on the initial electromagnetic field. In particular, we determine the self-similar profile of the current density $J(f)$ as well as the one of the Maxwell field F and we define modified trajectories along which f converges to a new smooth density function.

6.1. ***Convergence of the spatial averages.*** Since the solution (f, F) is global in time, all the statements of Sections 3–5 hold true for $T=+\infty$. We can then deduce that $\int_x \widehat{Z}^\beta f \, \mathrm{d}x$ converges to a function defined on $\mathbb{R}^3_v$.

Proposition 6.1. *Let $|\beta| \leq N-1$. There exists a continuous function $Q^\beta_\infty \in L^1_v \cap L^\infty_v$ such that,*

$$\forall t \in \mathbb{R}_+, \qquad \left\| |v^0|^{N_v-6}\left(Q^\beta_\infty - \int_{\mathbb{R}^3_x} \widehat{Z}^\beta f(t,x,\cdot)\,\mathrm{d}x \right) \right\|_{L^\infty(\mathbb{R}^3_v)} \lesssim \bar{\epsilon}\,\frac{\log^{3N_x+3N}(3+t)}{1+t}.$$

Remark 6.2. This estimate directly implies that $|v^0|^{N_v-10}\int_{\mathbb{R}^3_x} \widehat{Z}^\beta f(t,x,\cdot)\,\mathrm{d}x \to |v^0|^{N_v-10} Q^\beta_\infty$ in $L^1(\mathbb{R}^3_v)$, as $t \to +\infty$, with the same rate for convergence.

Proof. Let $v \in \mathbb{R}^3_v$ and apply Lemma 4.7 in order to get, for all $0 \leq t \leq s$,

$$|v^0|^{N_v-6}\left| \int_{\mathbb{R}^3_x} \widehat{Z}^\beta f(s,x,v)\,\mathrm{d}x - \int_{\mathbb{R}^3_x} \widehat{Z}^\beta f(t,x,v)\,\mathrm{d}x \right| \lesssim \bar{\epsilon} \int_{\tau=t}^{s} \frac{\log^{3N_x+3N}(3+\tau)}{(1+\tau)^2}\,\mathrm{d}\tau \leq \bar{\epsilon}\,\frac{\log^{3N_x+3N}(3+t)}{1+t}.$$

Consequently, there exists $Q^\beta_\infty \in L^\infty_v$ such that $\int_{\mathbb{R}^3_x} \widehat{Z}^\beta f(s,x,v)\,\mathrm{d}x \to Q^\beta_\infty$ in L^∞_v as $s \to +\infty$. Moreover, letting $s \to +\infty$ in the previous estimate provides the rate of convergence stated in the proposition. It implies $|v^0|^{N_v-6} Q^\beta_\infty \in L^\infty_v$ and then, as $N_v > 9$, $Q^\beta_\infty \in L^1_v$. □

It turns out that these functions are differentiable for $|\beta| \leq N-2$ and that $\partial_{v^i} Q^\beta_\infty$ can be related to other such functions Q^κ_∞. For this reason, if $\widehat{Z}^\kappa = \widehat{\Omega}_{0i}\widehat{Z}^\beta$, we will use $Q^{\widehat{\Omega}_{0i}\beta}_\infty$ in order to denote Q^κ_∞.

Proposition 6.3. *For any $|\beta| \leq N-2$, $Q^\beta_\infty \in C^{N-1-|\beta|}(\mathbb{R}^3_v)$ and its derivatives can be obtained by iterating the relations*

$$v^0 \partial_{v^i} Q^\beta_\infty = Q^{\widehat{\Omega}_{0i}\beta}_\infty - \hat{v}^i Q^\beta_\infty, \qquad 1 \leq i \leq 3. \tag{51}$$

Proof. Let $(t, v) \in \mathbb{R}_+ \times \mathbb{R}^3_v$ and note that

$$v^0 \partial_{v^i} \int_{\mathbb{R}^3_x} \widehat{Z}^\beta f(t,x,v)\,\mathrm{d}x = \int_{\mathbb{R}^3_x} \widehat{\Omega}_{0i}\widehat{Z}^\beta f(t,x,v)\,\mathrm{d}x - t\int_{\mathbb{R}^3_x} \partial_{x^i}\widehat{Z}^\beta f(t,x,v)\,\mathrm{d}x - \int_{\mathbb{R}^3_x} x^i \partial_t \widehat{Z}^\beta f(t,x,v)\,\mathrm{d}x.$$

Writing $\partial_t = -\hat{v}\cdot\nabla_x - \hat{v}^\mu F_\mu{}^j \partial_{v^j} + \boldsymbol{T}_F$, we get by performing integration by parts,

$$\begin{aligned} v^0 \partial_{v^i} &\int_{\mathbb{R}^3_x} \widehat{Z}^\beta f(t,x,v)\,\mathrm{d}x \\ &= \int_{\mathbb{R}^3_x} \widehat{\Omega}_{0i}\widehat{Z}^\beta f(t,x,v) - \hat{v}^i \widehat{Z}^\beta f(t,x,v)\,\mathrm{d}x + \int_{\mathbb{R}^3_x} x^i (\hat{v}^\mu F_\mu{}^j \partial_{v^j} - \boldsymbol{T}_F)(\widehat{Z}^\beta f)(t,x,v)\,\mathrm{d}x. \end{aligned}$$

According to Proposition 6.1, the first term on the right-hand side converges to $Q^{\widehat{\Omega}_{0i}\beta}_\infty - \hat{v}^i Q^\beta_\infty$, as $t \to +\infty$ and in $L^\infty(\mathbb{R}^3_v)$. Following the proof of Lemma 4.7 and then using Proposition 4.5, one can prove

$$\left|\int_{\mathbb{R}^3_x} x^i(\hat{v}^\mu F_\mu{}^j \partial_{v^j} - \boldsymbol{T}_F)(\widehat{Z}^\beta f)(t,x,v)\,\mathrm{d}x\right| \lesssim \Lambda \frac{\log(3+t)}{1+t} \sup_{|\kappa|\le|\beta|+1} \sup_{x\in\mathbb{R}^3} ||v^0|^3 z^{N_x-2}\widehat{Z}^\kappa f|(t,x,v)$$
$$\lesssim \bar{\epsilon}\frac{\log^{3N_x+3N}(3+t)}{1+t}.$$

We then deduce (51) and, by a direct induction, $Q^\beta_\infty \in C^{N-1-|\beta|}(\mathbb{R}^3_v)$. □

Let us mention that any Q^β_∞ can be written as a combination of Q_∞ and Q^κ_∞, where $\widehat{Z}^\kappa$ is only composed of complete lifts of Lorentz boosts $\widehat{\Omega}_{0i}$.

Proposition 6.4. *Let* $|\beta| \le N-1$. *Then*:

- *If* $\beta_T \ge 1$, *which means that* $\widehat{Z}^\beta$ *is composed of at least one translation, we have* $Q^\beta_\infty = 0$.
- *Otherwise there exists* $n+|\kappa| \le |\beta|$ *such that* $\widehat{Z}^\beta = S^n\widehat{Z}^\kappa$ *and* $Q^\beta_\infty = (-3)^n Q^\kappa_\infty$.
- *Moreover, if* $\widehat{Z}^\beta = \widehat{\Omega}_{jk}\widehat{Z}^\kappa$, $1\le j<k\le 3$, *then* $Q^\beta_\infty = \hat{v}^j Q^{\widehat{\Omega}_{0k}\kappa} - \hat{v}^k Q^{\widehat{\Omega}_{0j}\kappa}$.

Proof. Assume first that $\beta_T \ge 1$. Since $[\widehat{Z}, \partial_{x^\mu}] = 0$ or $\pm\partial_{x^\nu}$ for any $0 \le \mu \le 3$ and $\widehat{Z} \in \widehat{\mathbb{P}}_0$, it suffices to consider the case $\widehat{Z}^\beta = \partial_{x^\mu}\widehat{Z}^\xi$. Then, by either applying Lemma 4.7 or by performing integration by parts,

$$\left|\int_{\mathbb{R}^3_x} \partial_t \widehat{Z}^\xi f(t,x,v)\,\mathrm{d}x\right| \lesssim \bar{\epsilon}(1+t)^{-\frac{3}{2}} \to 0, \qquad \int_{\mathbb{R}^3_x} \partial_{x^i}\widehat{Z}^\xi f(t,x,v)\,\mathrm{d}x = 0, \quad 1\le i\le 3.$$

Otherwise $\beta_T = 0$ and since S commutes with $\widehat{\Omega}_{jk}$ and $\widehat{\Omega}_{0i}$, there exists $n+|\kappa|\le|\beta|$ such that $\widehat{Z}^\beta = S^n\widehat{Z}^\kappa$. The result follows from an easy induction and the following properties, which hold for any $|\xi| \le N-2$:

$$\left|\int_{\mathbb{R}^3_x} t\partial_t \widehat{Z}^\xi f(t,x,v)\,\mathrm{d}x\right| \lesssim \bar{\epsilon}(1+t)^{-\frac{1}{2}} \to 0, \quad \int_{\mathbb{R}^3_x} x_i\partial_{x^i}\widehat{Z}^\xi f(t,x,v)\,\mathrm{d}x = -\int_{\mathbb{R}^3_x}\widehat{Z}^\xi f(t,x,v)\,\mathrm{d}x, \quad 1\le i\le 3.$$

Finally, if $\widehat{Z}^\beta = \widehat{\Omega}_{jk}\widehat{Z}^\kappa$, note that by integration by parts,

$$\int_x \widehat{Z}^\beta f\,\mathrm{d}x = \hat{v}^j\int_x v^0\partial_{v^k}\widehat{Z}^\kappa f\,\mathrm{d}x - \hat{v}^k\int_x v^0\partial_{v^j}\widehat{Z}^\kappa f\,\mathrm{d}x$$

and it remains to apply Proposition 6.3. □

We are now able to establish the precise behavior of $J(f)$ in the interior of the light cone. In other words, we improve Corollary 4.14. No such result holds for the exterior region since the decay can be arbitrarily fast (we refer for this to the third estimate of Proposition 4.11). Recall the notation $x^0 = t$.

Proposition 6.5. *For any* $|\beta| \le N-1$, *the components of the electric current density* $J(\widehat{Z}^\beta f)$, *that is*, $J^\mu(\widehat{Z}^\beta f) = \int_{\mathbb{R}^3_v}(v^\mu/v^0)\widehat{Z}^\beta f\,\mathrm{d}v$, *satisfy*,

$$\forall |x|<t, \qquad \left|t^3 J^\mu(\widehat{Z}^\beta f)(t,x) - \frac{x^\mu}{t}(|v^0|^5 Q^\beta_\infty)\left(\frac{\check{x}}{t}\right)\right| \lesssim \bar{\epsilon}\frac{\log^{3N_x+3N}(3+t)}{t}, \qquad \mu \in [\![0,3]\!].$$

Proof. Let $|\beta|\le N-1$, $0\le\mu\le 3$ and $|x|<t$. Apply Lemma 4.12 and the estimate (41) to $g(t,x,v) := \hat{v}^\mu\widehat{Z}^\beta f(t, x+t\hat{v}, v)$. Since the spatial average of $|v^0|^5 g$ is equal to the one of $\hat{v}^\mu|v^0|^5\widehat{Z}^\beta f$, we get

$$\left|t^3\int_{\mathbb{R}^3_v}\frac{v^\mu}{v^0}\widehat{Z}^\beta f(t,x,v)\,\mathrm{d}v - \int_{\mathbb{R}^3_y}\left(\frac{v^\mu}{v^0}|v^0|^5\widehat{Z}^\beta f\right)\left(t,y,\frac{\check{x}}{t}\right)\mathrm{d}y\right| \lesssim \bar{\epsilon}\frac{\log^{3N_x+3N}(3+t)}{t}. \tag{52}$$

As $N_v - 6 \geq 5$, we obtain from Proposition 6.1 that,

$$\forall v \in \mathbb{R}^3_v, \quad \left| \frac{v^\mu}{v^0} |v^0|^5 Q^\beta_\infty(v) - \frac{v^\mu}{v^0} |v^0|^5 \int_{\mathbb{R}^3_y} \widehat{Z}^\beta f(t, y, v)\, \mathrm{d}y \right| \lesssim \bar{\epsilon} \frac{\log^{3N_x+3N}(3+t)}{1+t}.$$

The result follows from (52) and the last estimate, applied for $v = \widecheck{x/t}$. □

6.2. ***Self-similar asymptotic profile of the electromagnetic field.*** To identify the profile of F, we will see that Q_∞ generates an effective electromagnetic field. For this, we study F^T since it is the element of the Glassey–Strauss decomposition of F with the slower decay rate along timelike geodesics $t \mapsto (t, x + t\hat{v})$. If the plasma is not neutral, $Q_F \neq 0$, we will also have to improve the estimate for F^{data}.

6.2.1. *Behavior of $\mathcal{L}_{Z^\gamma}(F)^T$ along timelike straight lines.* It will be convenient to lighten the notations by denoting the kernel in the integral defining F^T, which was bounded in Corollary 5.5, as

$$\mathcal{W}^T(\omega, v) := \frac{\mathcal{W}(\omega, v)}{|v^0|^2(1+\omega\cdot\hat{v})}, \quad |\mathcal{W}^T|(\,\cdot\,, v) + |\nabla_v \mathcal{W}^T|(\,\cdot\,, v) \lesssim v^0. \tag{53}$$

Definition 6.6. Let, for any $|\beta| \leq N-1$, $[\widehat{Z}^\beta f]^\infty(v)$ be the 2-form defined as,

$$\forall v \in \mathbb{R}^3_v, \quad [\widehat{Z}^\beta f]^\infty(v) := \int_{\substack{|z|\leq 1 \\ |z+\hat{v}|<1-|z|}} \mathcal{W}^T\left(\frac{z}{|z|}, \widecheck{\frac{z+\hat{v}}{1-|z|}}\right)(|v^0|^5 Q^\beta_\infty)\left(\widecheck{\frac{z+\hat{v}}{1-|z|}}\right) \frac{\mathrm{d}z}{|z|^2(1-|z|)^3}.$$

Remark 6.7. We recall our convention $(|v^0|^5 Q^\beta_\infty)(w) := |w^0|^5 Q^\beta_\infty(w)$ for any $w \in \mathbb{R}^3_v$.

Remark 6.8. It is crucial to observe that the domain of integration is included in $\{0 \leq |z| \leq (1+|\hat{v}|)/2\}$. Indeed, if $|z| \geq (1+|\hat{v}|)/2$, we have

$$|z+\hat{v}| \geq |z| - 1 + 1 - |\hat{v}| \geq \frac{1-|\hat{v}|}{2} \geq 1 - |z|.$$

Consequently,

$$|z| \leq 1,\ |z+\hat{v}| < 1-|z| \quad \Longrightarrow \quad \frac{1}{4|v^0|^2} \leq \frac{1-|\hat{v}|}{2} \leq 1-|z| \leq 1.$$

In order to transform decay in $|t-r|$ into decay in t along timelike trajectories, we will use the next property.

Lemma 6.9. *Let $(x, v) \in \mathbb{R}^3_x \times \mathbb{R}^3_v$. Then,*

$$\forall\, 1 \leq t \leq 4\langle x\rangle|v^0|^2, \quad 1 \leq 4\frac{\langle x\rangle|v^0|^2}{t}, \qquad \forall t \geq 4\langle x\rangle|v^0|^2, \quad t - |x+t\hat{v}| \geq \frac{t}{4|v^0|^2}.$$

Proof. It suffices to observe that,

$$\forall t \geq 4|x||v^0|^2, \quad t \geq \frac{2|x|}{1-|\hat{v}|}, \quad \text{so that} \quad t - |x+t\hat{v}| \geq t - \frac{1-|\hat{v}|}{2}t - |\hat{v}|t = t - \frac{1+|\hat{v}|}{2}t \geq \frac{t}{4|v^0|^2}. \ \square$$

We have the following convergence result.

Proposition 6.10. *Let $|\beta| \leq N-1$ and $(x, v) \in \mathbb{R}^3_x \times \mathbb{R}^3_v$. For all $t \geq 1$, there holds*

$$|t^2[\widehat{Z}^\beta f]^T(t, x+\hat{v}t) - [\widehat{Z}^\beta f]^\infty(v)| \lesssim \bar{\epsilon}\langle x\rangle^2|v^0|^8 \frac{\log^{3N_x+3N+1}(3+t)}{t}.$$

Proof. Fix $|\beta| \leq N-1$, $(t,x,v) \in [1,+\infty[\times \mathbb{R}^3_x \times \mathbb{R}^3_v$ and recall from Proposition 5.3 the definition of $[\widehat{Z}^\beta f]^T$. Next, we split the domain of integration of $[\widehat{Z}^\beta f]^T$ into two parts,

$$t^2[\widehat{Z}^\beta f]^T(t,x+\hat{v}t) = t^2 \int_{\substack{|y-x-t\hat{v}|\leq t \\ |y-x|\geq t-|y-x-t\hat{v}|}} \int_{\mathbb{R}^3_w} \mathcal{W}^T\Big(\frac{y-x}{|y-x|},w\Big)\widehat{Z}^\beta f(t-|y-x-t\hat{v}|,y,w)\frac{\mathrm{d}w\,\mathrm{d}y}{|y-x-t\hat{v}|^2}+\mathcal{J},$$

$$\mathcal{J} := \int_{\substack{|z|\leq 1 \\ |z+\hat{v}|<1-|z|}} \int_{\mathbb{R}^3_w} \mathcal{W}^T\Big(\frac{z}{|z|},w\Big)\widehat{Z}^\beta f(t(1-|z|),x+tz+t\hat{v},w)\mathrm{d}w\frac{t^3\,\mathrm{d}z}{|z|^2},$$

where we performed the change of variables $z=(y-x-t\hat{v})/t$ in order to obtain the second integral $\mathcal{J}$. As we shall see below, this splitting is useful in order to identify and isolate the asymptotic profile. We start by controlling the first term. For this, note that (53), $N_v \geq 10$ and the last two estimates of Proposition 4.11, applied for $a=1$, yield, for all $(\omega,\tau,y) \in \mathbb{S}^2 \times \mathbb{R}_+ \times \mathbb{R}^3$,

$$\bigg|\int_{\mathbb{R}^3_w} \mathcal{W}^T(\omega,w)Z^\beta f(\tau,y,w)\,\mathrm{d}w\bigg| \lesssim \int_{\mathbb{R}^3_w} w^0|Z^\beta f|(\tau,y,w)\,\mathrm{d}w \lesssim \bar{\epsilon}\log^{3N_x+3N}(3+\tau)\frac{1+\max(\tau-|y|,0)}{(1+\tau+|y|)^4}.$$

Note now that $|y-x| \geq t-|y-x-t\hat{v}|$ implies

$$t-|y-x-t\hat{v}|-|y| \leq t-|y-x-t\hat{v}|-|y-x|+|x| \leq |x|.$$

Hence, applying first the previous estimate for $\tau = t-|y-x-t\hat{v}|$ and then (46), we get

$$\begin{aligned}|t^2[\widehat{Z}^\beta f]^T(t,x+\hat{v}t)-\mathcal{J}| &\lesssim \bar{\epsilon}(1+|x|)t^2\int_{\substack{|y-x-t\hat{v}|\leq t \\ |y-x|\geq t-|y-x-t\hat{v}|}} \frac{\log^{3N_x+3N}(3+t-|y-x-t\hat{v}|)}{(1+t-|y-x-t\hat{v}|+|y|)^4}\frac{\mathrm{d}y}{|y-x-t\hat{v}|^2} \\ &\lesssim \bar{\epsilon}\langle x\rangle \frac{\log^{3N_x+3N}(3+|t-|x+t\hat{v}||)}{1+|t-|x+t\hat{v}||}t^2\boldsymbol{Y}_3^{p=2}(t,x+t\hat{v}).\end{aligned}$$

According to Lemma 5.11, $t^2\boldsymbol{Y}_3^{p=2}(t,x+t\hat{v}) \lesssim \log(1+t)$. By applying Lemma 6.9, we then deduce

$$|t^2[\widehat{Z}^\beta f]^T(t,x+\hat{v}t)-\mathcal{J}| \lesssim \bar{\epsilon}\langle x\rangle\log(1+t)\bigg(\frac{\langle x\rangle|v^0|^2}{1+t}+|v^0|^2\frac{\log^{3N_x+3N}(3+t)}{1+t}\bigg),$$

so that it remains for us to compare $\mathcal{J}$ with $[\widehat{Z}^\beta f]^\infty(v)$. As in Section 4.4, it is convenient to change the reference frame and work with $g^\beta(\tau,y,w) := \widehat{Z}^\beta f(\tau,y+\tau\hat{w},w)$. In view of Lemma 2.9, the change of variables $y = x+tz+\hat{v}t-\hat{w}t(1-|z|)$, for z fixed, leads to

$$\mathcal{J} = \int_{\substack{|z|\leq 1 \\ |z+\hat{v}|<1-|z|}} \int_{|x-y+tz+\hat{v}t|<t(1-|z|)} \mathcal{W}^T\Big(\frac{z}{|z|},w\Big)(|v^0|^5g^\beta)(t(1-|z|),y,w)\frac{\mathrm{d}y\,\mathrm{d}z}{|z|^2(1-|z|)^3},$$

where we used w to denote the following function of (y,z):

$$w = \overbrace{\frac{x-y+tz+t\hat{v}}{t(1-|z|)}} \quad\Longleftrightarrow\quad \hat{w} = \frac{x-y+tz+t\hat{v}}{t(1-|z|)}.$$

By the triangular inequality, we have $|\mathcal{J}-[\widehat{Z}^\beta f]^\infty| \leq \mathcal{J}_1+\mathcal{J}_2+\mathcal{J}_3$, where

$$\mathcal{J}_1 := \int_{\substack{|z|\leq 1\\ |z+\hat{v}|<1-|z|}} \int_{|x-y+tz+t\hat{v}|<t(1-|z|)} |\Delta_1^\beta| \frac{\mathrm{d}y\,\mathrm{d}z}{|z|^2(1-|z|)^3},$$

$$\Delta_1^\beta := \mathcal{W}^T\left(\frac{z}{|z|}, w\right)(|v^0|^5 g^\beta)(t(1-|z|), y, w) - \mathcal{W}^T\left(\frac{z}{|z|}, \frac{\widecheck{z+\hat{v}}}{1-|z|}\right)(|v^0|^5 g^\beta)\left(t(1-|z|), y, \frac{\widecheck{z+\hat{v}}}{1-|z|}\right),$$

$$\mathcal{J}_2 := \left|\int_{\substack{|z|\leq 1\\ |z+\hat{v}|<1-|z|}} \int_{|x-y+tz+t\hat{v}|\geq t(1-|z|)} \mathcal{W}^T\left(\frac{z}{|z|}, \frac{\widecheck{z+\hat{v}}}{1-|z|}\right)(|v^0|^5 g^\beta)\left(t(1-|z|), y, \frac{\widecheck{z+\hat{v}}}{1-|z|}\right) \frac{\mathrm{d}y\,\mathrm{d}z}{|z|^2(1-|z|)^3}\right|,$$

$$\mathcal{J}_3 := \int_{\substack{|z|\leq 1\\ |z+\hat{v}|<1-|z|}} |\Delta_3^\beta| \frac{\mathrm{d}z}{|z|^2(1-|z|)^3},$$

$$\Delta_3^\beta := \mathcal{W}^T\left(\frac{z}{|z|}, \frac{\widecheck{z+\hat{v}}}{1-|z|}\right)\left[\int_{\mathbb{R}^3_y} (|v^0|^5 \widehat{Z}^\beta f)\left(t(1-|z|), y, \frac{\widecheck{z+\hat{v}}}{1-|z|}\right)\mathrm{d}y - (|v^0|^5 Q^\beta_\infty)\left(\frac{\widecheck{z+\hat{v}}}{1-|z|}\right)\right],$$

where, for Δ_3^β, we used that the spatial average of g^β is equal to the one of $\widehat{Z}^\beta f$. In view of Remark 6.8, we will be able to transform time decay for the integrands of $\mathcal{J}_i$ into decay in t, at the cost of powers of v^0. In particular, Remark 6.8 and $N_x > 7$ imply the following inequality that we will use several times:

$$\int_{\substack{|z|\leq 1\\ |z+\hat{v}|<1-|z|}} \int_{\mathbb{R}^3_y} \frac{\mathrm{d}y}{\langle y\rangle^{N_x-4}} \frac{\mathrm{d}z}{|z|^2(1-|z|)^n} \lesssim \int_{\substack{|z|\leq 1\\ |z+\hat{v}|<1-|z|}} \frac{\mathrm{d}z}{|z|^2(1-|z|)^n} \leq 2^{2n+2}\pi|v^0|^{2n}, \quad n\in\mathbb{N}. \tag{54}$$

We start by dealing with $\mathcal{J}_1$. Since $|\nabla_V \check{V}| \lesssim (1-|V|^2)^{-3/2} = |\check{V}^0|^3$ for all $|V|<1$ by Lemma 2.9 and in view of the bounds (53) on $\mathcal{W}^T$, the mean value theorem yields

$$\begin{aligned}|\Delta_1^\beta| &\leq \frac{|x-y|}{t(1-|z|)} \sup_{V\in\mathbb{R}^3} |V^0|^9(|g^\beta|+|\nabla_v g^\beta|)(t(1-|z|), y, V)\\ &\leq \frac{1+|x|}{t(1-|z|)\langle y\rangle^{N_x-4}} \sup_{(X,V)\in\mathbb{R}^6} |V^0|^9\langle X\rangle^{N_x-3}(|g^\beta|+|\nabla_v g^\beta|)(t(1-|z|), X, V).\end{aligned}$$

By applying Lemma 2.8 and then the estimates of Proposition 4.5, we obtain

$$|\Delta_1^\beta| \leq \frac{\langle x\rangle}{t(1-|z|)\langle y\rangle^{N_x-4}} \sum_{|\kappa|\leq N} \sup_{(X,V)\in\mathbb{R}^6} |V^0|^9|z^{N_x-2}\widehat{Z}^\kappa f|(t(1-|z|), X, V) \lesssim \frac{\bar{\epsilon}\langle x\rangle \log^{3N_x+3N}(3+t)}{t(1-|z|)\langle y\rangle^{N_x-4}},$$

where we used $N_v \geq 12$ and $|\beta|+1\leq N$. We then deduce from (54) that

$$\mathcal{J}_1 \lesssim \bar{\epsilon}\langle x\rangle|v^0|^8 \frac{\log^{3N_x+3N}(3+t)}{t}.$$

Next, we control Δ_3^β using $|\mathcal{W}^T|(\,\cdot\,, V) \lesssim V^0$, $N_v \geq 12$ and Proposition 6.1. This allows us to bound $\mathcal{J}_3$ through (54),

$$\Delta_3^\beta \lesssim \bar{\epsilon}\frac{\log^{3N_x+3N}(3+t)}{(1+t)(1-|z|)}, \quad \mathcal{J}_3 \lesssim \bar{\epsilon}|v^0|^8 \frac{\log^{3N_x+3N}(3+t)}{1+t}.$$

Finally, note that on the domain of integration of $\mathcal{J}_2$, we have, for $\hat{w} = (z+\hat{v})/(1-|z|)$,

$$1 = |w^0|^2\left(1 - \frac{|z+\hat{v}|^2}{(1-|z|)^2}\right) = |w^0|^2 \frac{(1-|z|+|z+\hat{v}|)(1-|z|-|z+\hat{v}|)}{(1-|z|)^2} \leq |w^0|^2 \frac{2|x-y|}{(1-|z|)t}.$$

Since $|\mathcal{W}^T|(\,\cdot\,, w) \lesssim w^0$, we get

$$\mathcal{J}_2 := \frac{\langle x\rangle}{t} \sup_{\tau \leq t} \sup_{(y,w)\in\mathbb{R}^6} |w^0|^8 \langle y\rangle^{N_x-3} |g^\beta|(\tau, y, w) \int_{\substack{|z|\leq 1 \\ |z+\hat{v}|<1-|z|}} \int_{\mathbb{R}^3_y} \frac{\mathrm{d}y}{\langle y\rangle^{N_x-4}} \frac{\mathrm{d}z}{|z|^2(1-|z|)^4}.$$

Using once again Lemma 2.8 together with Proposition 4.5, we get, in view of (54),

$$\mathcal{J}_2 \lesssim \bar{\epsilon}\langle x\rangle t^{-1} \log^{3N_x+3N}(3+t)|v^0|^8. \qquad \square$$

This directly provides us the asymptotic profile of $\mathcal{L}_{Z^\gamma}(F)^T = -\sum_{|\beta|\leq|\gamma|} C^\gamma_\beta [\widehat{Z}^\beta f]^T$.

Corollary 6.11. *Let $|\gamma| \leq N-1$ and $\mathcal{L}_{Z^\gamma}(F)^\infty := -\sum_{|\beta|\leq|\gamma|} C^\gamma_\beta [\widehat{Z}^\beta f]^\infty$. Then,*

$\forall (t, x, v) \in [1, +\infty[\times \mathbb{R}^3_x \times \mathbb{R}^3_v$,

$$|t^2 \mathcal{L}_{Z^\gamma}(F)^T(t, x+\hat{v}t) - \mathcal{L}_{Z^\gamma}(F)^\infty(v)| \lesssim \bar{\epsilon}\langle x\rangle^2 |v^0|^8 \frac{\log^{3N_x+3N+1}(3+t)}{t}.$$

Moreover, if Z^γ contains a translation ∂_{x^μ} or the scaling vector field S, then $\mathcal{L}_{Z^\gamma}(F)^\infty = 0$.

Proof. We only have to focus on the second part of the statement. Recall from the proof of Proposition 6.4 that we can reduce the analysis to the cases $Z^\gamma = \partial_{x^\lambda} Z^\kappa$ if $\gamma_T \geq 1$, and $Z^\gamma = SZ^\kappa$ otherwise. Recall further from the commutation formula of Lemma 2.3 and Proposition 2.4 that

$$\nabla^\mu \mathcal{L}_{\partial_{x^\lambda} Z^\kappa}(F)_{\mu\nu} = \sum_{|\xi|\leq|\kappa|} C^\kappa_\xi J(\partial_{x^\lambda} \widehat{Z}^\xi f)_\nu, \quad \nabla^\mu \mathcal{L}_{SZ^\kappa}(F)_{\mu\nu} = \sum_{|\xi|\leq|\kappa|} C^\kappa_\xi J(S\widehat{Z}^\xi f)_\nu + 3C^\kappa_\xi J(\widehat{Z}^\xi f)_\nu.$$

It remains to recall from Proposition 6.4 that $Q^{\partial_{x^\lambda}\xi}_\infty = 0$ and $Q^{S\xi}_\infty = -3Q^\xi$, so that $\mathcal{L}_{Z^\gamma}(F)^\infty = 0$. $\square$

6.2.2. *Behavior of $\mathcal{L}_{Z^\gamma}(F)^{\mathrm{data}}$ along timelike straight lines.* Recall from Proposition 5.3 and (50) that F^{data} is the sum of F^{hom}, which verifies $\square F^{\mathrm{hom}}_{\mu\nu} = 0$, and a term which is strongly decaying in the interior of the light cone. If $Q_F \neq 0$, F decays initially as r^{-2} and one cannot expect to prove strong decay estimates for F^{hom} through Proposition 2.21. For this reason, we need to analyse in detail the homogeneous part F^{hom}. It turns out that it decays faster in the interior of the light cone and then along timelike straight lines, so that it will not contribute to the asymptotic Lorentz force.

In order to improve the naive estimate of Proposition 5.13, one can note that the leading-order term $\overline{F}(0, x) = Q_F x_i/(4\pi|x|^3)\, \mathrm{d}t \wedge \mathrm{d}x^i$ of the asymptotic expansion of $F^{\mathrm{hom}}(0, \cdot)$ corresponds to the static electromagnetic field generated by a point charge Q_F located at $x = 0$. It is derived from the potential $A = Q(4\pi r)^{-1}\, \mathrm{d}t$ which satisfies the Lorenz gauge, and then $\square A_\mu = 0$ on $\mathbb{R} \times (\mathbb{R}^3 \setminus \{0\})$. To deal with our evolution problem and the singularity of the Newton potential, we introduce

$$\tilde{A}(t, x) := \chi(|x|-t)A(t, x) = \frac{Q_F}{4\pi|x|}\chi(|x|-t)\mathrm{d}t, \quad \chi \in C^\infty(\mathbb{R}, [0, 1]), \quad \chi|_{]-\infty,\frac{1}{2}]} = 0, \quad \chi|_{[1,+\infty[} = 1.$$

Then, $\tilde{A}$ is smooth on $\mathbb{R}_+ \times \mathbb{R}^3$ and $\Box \tilde{A}_\mu = 0$. It motivates the introduction of

$$\widetilde{F}(t,x) := \mathrm{d}\tilde{A}(t,x) = \frac{Q_F x_i}{4\pi |x|^3} \chi(|x|-t)\,\mathrm{d}t \wedge \mathrm{d}x^i - \frac{Q_F x_i}{4\pi |x|^2} \chi'(|x|-t)\,\mathrm{d}t \wedge \mathrm{d}x^i$$
$$= \chi(|x|-t)\bar{F}(t,x) - \frac{Q_F x_i}{4\pi |x|^2} \chi'(|x|-t)\,\mathrm{d}t \wedge \mathrm{d}x^i,$$

which, in view of $[\Box, \partial_{x^\lambda}] = 0$ and $\Box \tilde{A}_\lambda = 0$, satisfies $\Box \widetilde{F}_{\mu\nu} = 0$. Since,

- for any $\Gamma \in \mathbb{K} \setminus \{S\}$, $[\Box, \Gamma] = 0$ and $[\Box, S] = 2\Box$,
- for any $Z = Z^\lambda \partial_{x^\lambda} \in \mathbb{K}$ and any 2-form H, we have $\mathcal{L}_Z(H)_{\mu\nu} = Z(H_{\mu\nu}) + \partial_{x^\mu}(Z^\lambda) H_{\lambda\nu} + \partial_{x^\nu}(Z^\lambda) H_{\mu\lambda}$,

we then have $\Box \mathcal{L}_{Z^\gamma}(\widetilde{F})_{\mu\nu} = 0$ for any $|\gamma| \le N-1$. The key idea will then be to consider $\mathcal{L}_{Z^\gamma}(F)^{\mathrm{hom}} - \mathcal{L}_{Z^\gamma}(\widetilde{F})$. More precisely, the following estimates hold.

Proposition 6.12. *For any* $|\gamma| \le N-1$, *we have*,

$$\forall (t,x) \in \mathbb{R}_+ \times \mathbb{R}^3, \quad |\mathcal{L}_{Z^\gamma}(F)^{\mathrm{data}}(t,x) - \mathcal{L}_{Z^\gamma}(\widetilde{F})(t,x)| \lesssim \Lambda(1+t+|x|)^{-1}(1+|t-|x||)^{-1-\delta}.$$

Remark 6.13. We will not use it here, but we have

$$|\mathcal{L}_{Z^\gamma}(\widetilde{F}) - \chi(|x|-t)\mathcal{L}_{Z^\gamma}(\bar{F})(t,x)| \lesssim Q_F(1+t)^{-1} \mathbb{1}_{0 \le |x|-t \le 1}.$$

Moreover, $\mathcal{L}_{\partial_t}(\bar{F}) = \mathcal{L}_{\Omega_{jk}}(\bar{F}) = \mathcal{L}_S(\bar{F}) = 0$ for all $1 \le j < k \le 3$. We refer to [Bigorgne 2020a, Section 5] for more information concerning $\bar{F}$.

This result implies that the leading-order term of $\mathcal{L}_{Z^\gamma}(F)^{\mathrm{data}}(t,x)$ is supported in the exterior of the light cone. Before proving it, let us investigate its direct consequence for the behavior of F^{data} along timelike trajectories.

Proposition 6.14. *For any* $|\gamma| \le N-1$, *we have*,

$$\forall (t,x,v) \in [1,+\infty[\times \mathbb{R}^3_x \times \mathbb{R}^3_v, \quad |t^2 \mathcal{L}_{Z^\gamma}(F)^{\mathrm{data}}(t, x+t\hat{v})| \lesssim \Lambda \langle x \rangle^2 |v^0|^4 t^{-\delta}.$$

Proof. Let $(t,x,v) \in [1,+\infty[\times \mathbb{R}^3_x \times \mathbb{R}^3_v$. If $t \le 4\langle x \rangle |v^0|^2$, it suffices to apply Proposition 5.13, providing

$$|\mathcal{L}_{Z^\gamma}(F)^{\mathrm{data}}(t, x+t\hat{v})| \lesssim \Lambda t^{-1} \le 16\Lambda \langle x \rangle^2 |v^0|^4 t^{-3}.$$

Otherwise, according to Lemma 6.9, we have $t - |x+t\hat{v}| \ge t/(4|v^0|^2)$, so that $\chi^{(n)}(|x+t\hat{v}|-t) = 0$ for all $n \in \mathbb{N}$. Consequently, we get from Proposition 6.12 that

$$|\mathcal{L}_{Z^\gamma}(F)^{\mathrm{data}}(t, x+t\hat{v})| \lesssim \Lambda t^{-1}(1+|t-|x+t\hat{v}||)^{-1-\delta} \le 16\Lambda |v^0|^4 t^{-3}. \qquad \Box$$

The first step of the proof of Proposition 6.12 consists in controlling the initial data for $\mathcal{L}_{Z^\gamma}(F)^{\mathrm{hom}}$.

Lemma 6.15. *The assumption* (15) *on the initial electromagnetic field* $F(0,\cdot)$ *implies*,

$$\forall |\gamma| \le N-1, \quad \sup_{|\kappa| \le 1} \sup_{|x| \ge 1} \langle x \rangle^{2+\delta+|\kappa|} |\nabla^\kappa_{t,x} \mathcal{L}_{Z^\gamma}(F)^{\mathrm{hom}} - \nabla^\kappa_{t,x} \mathcal{L}_{Z^\gamma}(\bar{F})|(0,x) \lesssim \Lambda. \tag{55}$$

Note that $\nabla^\kappa_{t,x} \mathcal{L}_{Z^\gamma}(\bar{F})(0,x) = \nabla^\kappa_{t,x} \mathcal{L}_{Z^\gamma}(\widetilde{F})(0,x)$ *for all* $|x| \ge 1$ *since* $\chi = 1$ *on* $[1,+\infty[$.

Proof. As $\bar{F}$ is defined on $\mathbb{R} \times \mathbb{R}^3 \setminus \{0\}$, $\mathcal{L}_{Z^\gamma}(\bar{F})$ is well-defined for $|x| \geq 1$. We point out that $\nabla_t \mathcal{L}_{Z^\gamma}(\bar{F})(0,\cdot)$ does not necessarily vanish (consider for instance the case $Z^\gamma = \Omega_{01}$). Moreover, $\mathcal{L}_{Z^\gamma}(F)^{\mathrm{hom}}(0,\cdot) = \mathcal{L}_{Z^\gamma}(F)(0,\cdot)$ by definition. Hence, the left-hand side of (55) is bounded by

$$\begin{aligned}\sup_{|\kappa|\leq 1}\sup_{|x|\geq 1}\langle x\rangle^{2+\delta+|\kappa|}|\nabla^{\kappa}_{t,x}\mathcal{L}_{Z^\gamma}(F-\bar{F})|(0,x) &\lesssim \sup_{|\xi|\leq|\gamma|+1}\sup_{|x|\geq 1}\langle x\rangle^{2+\delta+|\xi|}|\nabla^{\xi}_{t,x}(F-\bar{F})|(0,x)\\ &\leq \Lambda + \sup_{|\beta|\leq|\gamma|}\sup_{|x|\geq 1}\langle x\rangle^{2+\delta+n+|\beta|}|\nabla_{\partial_t}\nabla^{\beta}_{t,x}F|(0,x), \qquad (56)\end{aligned}$$

where, in the last step, we used the assumption (15) and that $\bar{F}$ is independent of t. Now, remark that if $n \geq 1$, the Maxwell equations implies

$$\partial_t(\partial_t^{n-1}\partial_x^\beta B) = -\partial_t^{n-1}\partial_x^\beta(\nabla_x \times E), \quad \partial_t(\partial_t^{n-1}\partial_x^\beta E) = \partial_t^{n-1}\partial_x^\beta(\nabla_x \times B) - \int_{\mathbb{R}^3_v}\hat{v}\,\partial_t^{n-1}\partial_x^\beta f\,\mathrm{d}v.$$

Let $\bar{E}$ and $\bar{B}$ be the electric and magnetic field associated to $\bar{F}$ according to (4), so that $\bar{E}^i = x^i Q_F/(4\pi r^3)$ and $\bar{B}=0$. As $\nabla_x \times \bar{E} = \nabla_x \times \bar{B} = 0$, we can bound (56) by Λ by performing an induction and using (15) as well as the initial assumptions on f. ☐

We are now able to prove Proposition 6.12 and conclude this subsection. As $\epsilon \leq \Lambda$, (50) implies,

$$\forall (t,x) \in \mathbb{R}_+ \times \mathbb{R}^3, \quad |\mathcal{L}_{Z^\gamma}(F)^{\mathrm{data}} - \mathcal{L}_{Z^\gamma}(F)^{\mathrm{hom}}|(t,x) \lesssim \Lambda(1+t+|x|)^{-1}(1+|t-|x||)^{-1-\delta}.$$

Finally, $\square\mathcal{L}_{Z^\gamma}(F)^{\mathrm{hom}}_{\mu\nu} - \square\mathcal{L}_{Z^\gamma}(\widetilde{F})_{\mu\nu} = 0$, the decay assumptions on the initial data given by Lemma 6.15 and Proposition 2.21 yield,

$$\forall (t,x) \in \mathbb{R}_+ \times \mathbb{R}^3, \quad |\mathcal{L}_{Z^\gamma}(F)^{\mathrm{hom}} - \mathcal{L}_{Z^\gamma}(\widetilde{F})|(t,x) \lesssim \Lambda(1+t+|x|)^{-1}(1+|t-|x||)^{-1-\delta}.$$

6.2.3. *Self-similar asymptotic profile of $\mathcal{L}_{Z^\gamma}(F)$.* We are now able to study the full Maxwell field.

Corollary 6.16. *For any $|\gamma| \leq N-1$, there exists a 2-form $\mathcal{L}_{Z^\gamma}(F)^\infty$, independent of t, such that,*

$$\forall (t,x,v) \in [1,\infty[\times\mathbb{R}^3_x\times\mathbb{R}^3_v, \quad |t^2\mathcal{L}_{Z^\gamma}(F)(t,x+\hat{v}t) - \mathcal{L}_{Z^\gamma}(F)^\infty(v)| \lesssim \Lambda\langle x\rangle^2|v^0|^8\frac{\log^{3N_x+3N+1}(3+t)}{t^\delta}.$$

Moreover, for any $\eta > 0$, there exists $C_\eta > 0$ such that,

$$\forall (t,x) \in [1,+\infty[\times\mathbb{R}^3_x,\ \frac{|x|}{t} \leq 1-\eta, \quad \left|t^2\mathcal{L}_{Z^\gamma}(F)(t,x) - \mathcal{L}_{Z^\gamma}(F)^\infty\left(\frac{\check{x}}{t}\right)\right| \lesssim \Lambda C_\eta\frac{\log^{3N_x+3N+1}(3+t)}{t^\delta}.$$

Remark 6.17. For the most important case, $|\gamma| = 0$, we have $4\pi F^\infty = -[f]^\infty$, where $[f]^\infty$ is explicitly written in Definition 6.6.

Proof. Fix $|\gamma| \leq N-1$ and $(t,x,v) \in [1,\infty[\times\mathbb{R}^3_x\times\mathbb{R}^3_v$. Applying Proposition 5.14 and Lemma 6.9, we have

$$t^2|\mathcal{L}_{Z^\gamma}(F)^S|(t,x+\hat{v}t) \lesssim \bar{\epsilon}\Lambda\frac{t\log(3+|t-|x-t\hat{v}||)}{(1+|t-|x-t\hat{v}||)^2} \lesssim \Lambda\left(\frac{\langle x\rangle^2|v^0|^4}{t} + |v^0|^4\frac{\log(3+t)}{t}\right).$$

We then get the first part of the statement using the Glassey–Strauss decomposition given by Proposition 5.3, Corollary 6.11, where $\mathcal{L}_{Z^\gamma}(F)^\infty$ is introduced, and Proposition 6.14. For the second part, it suffices to apply the first estimate, with a slight abuse of notation, for $x=0$ and $\hat{v}=x/t$. □

We deduce from the previous result a uniform bound on $\mathcal{L}_{Z^\gamma}(F)^\infty$. Moreover, it turns out that this quantity vanishes in certain cases, providing improved estimates for $\mathcal{L}_{Z^\gamma}(F)$.

Proposition 6.18. *For any $|\gamma|\leq N-1$, we have $|\mathcal{L}_{Z^\gamma}(F)^\infty|(v)\lesssim \bar{\epsilon}\sqrt{v^0}$. Moreover, if $|\gamma|\geq 1$ and Z^γ contains a translation ∂_{x^μ} or the scaling vector field S, then $\mathcal{L}_{Z^\gamma}(F)^\infty=0$.*

Proof. According to Proposition 5.15, $t^2|\mathcal{L}_{Z^\gamma}(F)^T|(t,t\hat{v})\lesssim \bar{\epsilon}(1-|\hat{v}|)^{-1/4}\leq 2\bar{\epsilon}\sqrt{v^0}$. All the properties then follow from Corollary 6.11. □

Finally, we investigate the regularity of $\mathcal{L}_{Z^\gamma}(F)^\infty$.

Proposition 6.19. *For any $|\gamma|\leq N-2$ and $0\leq\mu,\nu\leq 3$, $\mathcal{L}_{Z^\gamma}(F)^\infty_{\mu\nu}$ is of class $C^{N-1-|\gamma|}$. Moreover, for any $1\leq k\leq 3$, we have*

$$v^0\partial_{v^k}\mathcal{L}_{Z^\gamma}(F)^\infty_{\mu\nu}=\mathcal{L}_{\Omega_{0k}Z^\gamma}(F)^\infty_{\mu\nu}+2\hat{v}^k\mathcal{L}_{Z^\gamma}(F)^\infty_{\mu\nu}-\delta^0_\mu\mathcal{L}_{Z^\gamma}(F)^\infty_{k\nu}-\delta^k_\mu\mathcal{L}_{Z^\gamma}(F)^\infty_{0\nu}-\delta^0_\nu\mathcal{L}_{Z^\gamma}(F)^\infty_{\mu k}-\delta^k_\nu\mathcal{L}_{Z^\gamma}(F)^\infty_{\mu 0}.$$

The angular derivatives satisfy

$$(v^j\partial_{v^k}-v^k\partial_{v^j})\mathcal{L}_{Z^\gamma}(F)^\infty_{\mu\nu}=\mathcal{L}_{\Omega_{jk}Z^\gamma}(F)^\infty_{\mu\nu}-\delta^j_\mu\mathcal{L}_{Z^\gamma}(F)^\infty_{k\nu}+\delta^k_\mu\mathcal{L}_{Z^\gamma}(F)^\infty_{j\nu}-\delta^j_\nu\mathcal{L}_{Z^\gamma}(F)^\infty_{\mu k}+\delta^k_\nu\mathcal{L}_{Z^\gamma}(F)^\infty_{\mu j}.$$

Proof. In order to lighten the notations, we introduce $X:=x+t\hat{v}$. Then, we compute

$$\begin{aligned}
&v^0\partial_{v^k}(\mathcal{L}_{Z^\gamma}(F)_{\mu\nu}(t,X))\\
&\quad=t(\delta^i_k-\hat{v}^k\hat{v}^i)\partial_{x^i}(\mathcal{L}_{Z^\gamma}(F)_{\mu\nu})(t,X)\\
&\quad=(\Omega_{0k}\mathcal{L}_{Z^\gamma}(F)_{\mu\nu})(t,X)-X^k\partial_t(\mathcal{L}_{Z^\gamma}(F)_{\mu\nu})(t,X)+\hat{v}^k(x^i-X^i)\partial_{x^i}(\mathcal{L}_{Z^\gamma}(F)_{\mu\nu})(t,X)\\
&\quad=(\Omega_{0k}\mathcal{L}_{Z^\gamma}(F)_{\mu\nu})(t,X)-\hat{v}^k(S\mathcal{L}_{Z^\gamma}(F)_{\mu\nu})(t,X)-x^k\partial_t(\mathcal{L}_{Z^\gamma}(F)_{\mu\nu})(t,X)+\hat{v}^kx^i\partial_{x^i}(\mathcal{L}_{Z^\gamma}(F)_{\mu\nu})(t,X).
\end{aligned}$$

One can already notice that the last two terms enjoy strong decay properties. More precisely, since Lemma 6.9 implies $1+|t-|X||\gtrsim(1+t)/(\langle x\rangle|v^0|^2)$, we have from Proposition 3.2

$$t^2|-x^k\partial_t(\mathcal{L}_{Z^\gamma}(F)_{\mu\nu})(t,X)+\hat{v}^kx^i\partial_{x^i}(\mathcal{L}_{Z^\gamma}(F)_{\mu\nu})(t,X)|\lesssim\frac{\Lambda\langle x\rangle^3|v^0|^4}{1+t}.$$

The result then follows from

$$\mathcal{L}_{SZ^\gamma}(F)_{\mu\nu}=S(\mathcal{L}_{Z^\gamma}(F)_{\mu\nu})+2\mathcal{L}_{Z^\gamma}(F)_{\mu\nu},\quad \mathcal{L}_{SZ^\gamma}(F)^\infty=0,\tag{57}$$

$$\mathcal{L}_{\Omega_{0k}Z^\gamma}(F)_{\mu\nu}=\Omega_{0k}(\mathcal{L}_{Z^\gamma}(F)_{\mu\nu})+\delta^0_\mu\mathcal{L}_{Z^\gamma}(F)_{k\nu}+\delta^k_\mu\mathcal{L}_{Z^\gamma}(F)_{0\nu}+\delta^0_\nu\mathcal{L}_{Z^\gamma}(F)_{\mu k}+\delta^k_\nu\mathcal{L}_{Z^\gamma}(F)_{\mu 0}$$

and Corollary 6.16, which give us

$$|t^2v^0\partial_{v^k}\mathcal{L}_{Z^\gamma}(F)_{\mu\nu}(t,x+t\hat{v})-v^0\partial_{v^k}\mathcal{L}_{Z^\gamma}(F)^\infty_{\mu\nu}(v)|\lesssim\Lambda\langle x\rangle^3|v^0|^8\frac{\log^{1+3N_x+3N}(3+t)}{(1+t)^\delta},$$

where $v^0\partial_{v^k}\mathcal{L}_{Z^\gamma}(F)^\infty_{\mu\nu}(v)$ is given in the statement of the proposition. To get the expression of the angular derivatives, notice that

$$(v^j\partial_{v^k}-v^k\partial_{v^j})(\mathcal{L}_{Z^\gamma}(F)_{\mu\nu}(t,X))=(\Omega_{jk}\mathcal{L}_{Z^\gamma}(F)_{\mu\nu})(t,X)-(x^j\partial_{x^k}-x^k\partial_{x^j})(\mathcal{L}_{Z^\gamma}(F)_{\mu\nu})(t,X),$$

and

$$\mathcal{L}_{\Omega_{jk}Z^\gamma}(F)_{\mu\nu} = \Omega_{jk}(\mathcal{L}_{Z^\gamma}(F)_{\mu\nu}) + \delta^j_\mu \mathcal{L}_{Z^\gamma}(F)_{k\nu} - \delta^k_\mu \mathcal{L}_{Z^\gamma}(F)_{j\nu} + \delta^j_\nu \mathcal{L}_{Z^\gamma}(F)_{\mu k} - \delta^k_\nu \mathcal{L}_{Z^\gamma}(F)_{\mu j}$$

and apply the same arguments. The $C^{N-1-|\gamma|}$ regularity is obtained by an induction. □

For later use, we prove that the structure of the asymptotic Lorentz force is preserved by differentiation.

Corollary 6.20. *Let* $0 \le \nu \le 3$ *and define*

$$\Delta_{Z^\gamma,\nu}(t,x,v) := t^2 \frac{\widehat{v}^\mu}{v^0} \mathcal{L}_{Z^\gamma}(F)_{\mu\nu}(t,x) - \frac{\widehat{v}^\mu}{v^0} \mathcal{L}_{Z^\gamma}(F)^\infty_{\mu\nu}(v), \quad |\gamma| \le N-1.$$

For any $|\gamma| \le N-2$, *there holds*

$$\begin{aligned}
&S(\Delta_{Z^\gamma,\nu}) = \Delta_{SZ^\gamma,\nu}, \\
&\widehat{\Omega}_{jk}(\Delta_{Z^\gamma,\nu}) = \Delta_{\Omega_{jk}Z^\gamma,\nu} - \delta^j_\nu \Delta_{Z^\gamma,k} + \delta^k_\nu \Delta_{Z^\gamma,j}, && 1 \le j < k \le 3, \\
&\widehat{\Omega}_{0i}(\Delta_{Z^\gamma,\nu}) = \Delta_{\Omega_{0i}Z^\gamma,\nu} - \delta^0_\nu \Delta_{Z^\gamma,i} - \delta^i_\nu \Delta_{Z^\gamma,0} + 2\frac{t}{v^0}(x^i - t\widehat{v}^i)\widehat{v}^\mu \mathcal{L}_{Z^\gamma}(F)_{\mu\nu}(t,x), && 1 \le i \le 3.
\end{aligned}$$

Proof. The first identity follows from $S(t^2) = 2t^2$ and (57). For the other ones, start by noticing that, according to Proposition 6.19 and for $1 \le i \le 3$,

$$\begin{aligned}
\widehat{\Omega}_{0i}\left(\frac{\widehat{v}^\mu}{v^0}\mathcal{L}_{Z^\gamma}(F)^\infty_{\mu\nu}(v)\right) &= v^0 \partial_{v^i}\left(\frac{\widehat{v}^\mu}{v^0}\mathcal{L}_{Z^\gamma}(F)^\infty_{\mu\nu}(v)\right) \\
&= \frac{\widehat{v}^\mu}{v^0}\mathcal{L}_{\Omega_{0i}Z^\gamma}(F)^\infty_{\mu\nu}(v) - \delta^0_\nu \frac{\widehat{v}^\mu}{v^0}\mathcal{L}_{Z^\gamma}(F)^\infty_{\mu i} - \delta^i_\nu \frac{\widehat{v}^\mu}{v^0}\mathcal{L}_{Z^\gamma}(F)^\infty_{\mu 0}. \qquad (58)
\end{aligned}$$

Similarly, for $1 \le j < k \le 3$,

$$\widehat{\Omega}_{jk}\left(\frac{\widehat{v}^\mu}{v^0}\mathcal{L}_{Z^\gamma}(F)^\infty_{\mu\nu}(v)\right) = \frac{\widehat{v}^\mu}{v^0}\mathcal{L}_{\Omega_{jk}Z^\gamma}(F)^\infty_{\mu\nu}(v) - \delta^j_\nu \frac{\widehat{v}^\mu}{v^0}\mathcal{L}_{Z^\gamma}(F)^\infty_{\mu k} + \delta^k_\nu \frac{\widehat{v}^\mu}{v^0}\mathcal{L}_{Z^\gamma}(F)^\infty_{\mu j}. \qquad (59)$$

Recall that we denote by $\boldsymbol{v}$ the 4-vector $(v^\mu)_{0\le\mu\le4}$, so that

$$\begin{aligned}
\widehat{Z}\left(t^2\frac{\widehat{v}^\mu}{v^0}\mathcal{L}_{Z^\gamma}(F)_{\mu\nu}\right) = \widehat{Z}\left(\frac{t^2}{|v^0|^2}\right)v^\mu \mathcal{L}_{Z^\gamma}(F)_{\mu\nu} + \frac{t^2}{|v^0|^2}\mathcal{L}_{ZZ^\gamma}(F)(\boldsymbol{v},\partial_{x^\nu}) + \frac{t^2}{|v^0|^2}\mathcal{L}_{Z^\gamma}(F)(\boldsymbol{v},[Z,\partial_{x^\nu}]) \quad (60) \\
+ \frac{t^2}{|v^0|^2}\mathcal{L}_{Z^\gamma}(F)([Z,\boldsymbol{v}],\partial_{x^\nu}) + \frac{t^2}{|v^0|^2}\widehat{Z}(v^\mu)\mathcal{L}_{Z^\gamma}(F)_{\mu\nu}. \quad (61)
\end{aligned}$$

- If $Z = \Omega_{0i}$, we have $[Z,\boldsymbol{v}] = -v^i\partial_t - v^0\partial_{x^i}$ and $\widehat{Z}(v^\mu) = \delta^0_\mu v^i + \delta^i_\mu v^0$, so that the sum of two terms in (61) vanishes. It remains to remark that $[Z,\partial_{x^\nu}] = -\delta^i_\nu\partial_t - \delta^0_\nu\partial_i$, $\widehat{Z}(t^2/|v^0|^2) = 2t(x^i - t\widehat{v}^i)/|v^0|^2$ and to combine (58) with (60).

- If $Z = \Omega_{jk}$, there holds $[Z,\boldsymbol{v}] = -v^j\partial_{x^k} + v^k\partial_{x^j}$ and $\widehat{Z}(v^\mu) = \delta^k_\mu v^j - \delta^j_\mu v^k$, so that the sum of the two terms in (61) vanishes once again. The result then ensues from $\widehat{Z}(t^2/|v^0|^2) = 0$, $[Z,\partial_{x^\nu}] = -\delta^j_\nu\partial_{x^k} + \delta^k_\nu\partial_{x^j}$, (59) and (60). □

6.3. ***Convergence of the distribution function along modified characteristics.*** Motivated by the discussion in Section 2.8.4 and by Corollary 6.16, we modify the linear spatial characteristics $t \mapsto x + t\widehat{v}$ as follows.

Definition 6.21. For $(x, v) \in \mathbb{R}^3_x \times \mathbb{R}^3_v$, let $X_{\mathscr{C}}(\,\cdot\,, x, v) : t \mapsto x + t\hat{v} + \mathscr{C}(t, v)$ be the trajectory[11]

$$X^i_{\mathscr{C}}(t, x, v) := x^i + t\hat{v}^i - \log(t)\hat{v}^\mu F^{\infty, j}_{\mu}(v)\frac{\delta^i_j - \hat{v}_j\hat{v}^i}{v^0}$$
$$= x^i + t\hat{v}^i - \frac{\log(t)}{v^0}(\hat{v}^\mu F^\infty_{\mu i}(v) + \hat{v}^i\hat{v}^\mu F^\infty_{\mu 0}(v)), \quad t \in \mathbb{R}^*_+,\ i \in [\![1, 3]\!]. \tag{62}$$

For simplicity, we will often write $X_{\mathscr{C}}$ instead of $X_{\mathscr{C}}(t, x, v)$. By Proposition 6.18, the components $\mathscr{C}^i$ of the correction term $\mathscr{C}$ satisfy,

$$\forall t > 0, \quad |\mathscr{C}^i|(t, v) \lesssim \bar{\epsilon}|v^0|^{-\frac{1}{2}}\log(t), \quad i \in [\![1, 3]\!]. \tag{63}$$

We now bound the time derivative of a function evaluated along the modified characteristics.

Proposition 6.22. *Let $\boldsymbol{f} : \mathbb{R}_+ \times \mathbb{R}^3_x \times \mathbb{R}^3_v \to \mathbb{R}$ be a sufficiently regular function and introduce $h(t, x, v) := \boldsymbol{f}(t, X_{\mathscr{C}}(t, x, v), v)$. Then, for all $(t, x, v) \in [1, +\infty[\times \mathbb{R}^3_x \times \mathbb{R}^3_v$,*

$$|\partial_t h|(t, x, v) \le |\boldsymbol{T}_F(\boldsymbol{f})|(t, X_{\mathscr{C}}, v) + \Lambda\frac{\log^{3+3N_x+3N}(3+t)}{(1+t)^{1+\delta}}\sum_{\widehat{Z}\in\widehat{\mathbb{P}}_0}||v^0|^7 z^2\widehat{Z}\boldsymbol{f}|(t, X_{\mathscr{C}}, v).$$

Proof. We have, for all $(t, x, v) \in [1, +\infty[\times \mathbb{R}^3_x \times \mathbb{R}^3_v$,

$$\partial_t h(t, x, v) = (\partial_t \boldsymbol{f} + \hat{v}^i\partial_{x^i}\boldsymbol{f})(t, X_{\mathscr{C}}, v) + \partial_t\mathscr{C}^i(t, v)\partial_{x^i}\boldsymbol{f}(t, X_{\mathscr{C}}, v)$$
$$= \boldsymbol{T}_F(\boldsymbol{f})(t, X_{\mathscr{C}}) - \hat{v}^\mu {F_\mu}^j(t, X_{\mathscr{C}})\partial_{v^j}\boldsymbol{f}(t, X_{\mathscr{C}}, v) + \partial_t\mathscr{C}^i(t, v)\partial_{x^i}\boldsymbol{f}(t, X_{\mathscr{C}}, v). \tag{64}$$

Recall from (14) the relation

$$v^0\partial_{v^j} = -t(\partial_{x^j} - \hat{v}^j\hat{v}^i\partial_{x^i}) + \widehat{\Omega}_{0j} + z_{0j}\partial_t - \hat{v}^j S - \sum_{1\le i\le 3}\hat{v}^j z_{0i}\partial_{x^i}, \quad 1 \le j \le 3, \tag{65}$$

in order to rewrite $\partial_{v^j}\boldsymbol{f}(t, X_{\mathscr{C}}, v)$. As $v^0\partial_t\mathscr{C}^i(t, v) = -(1/t)\hat{v}^\mu F^{\infty, j}_\mu(v)(\delta^i_j - \hat{v}_j\hat{v}^i)$, we get

$$|\partial_t h|(t, x, v) \le |\boldsymbol{T}_F(\boldsymbol{f})|(t, X_{\mathscr{C}}, v) + \sum_{1\le j\le 3}\sum_{\widehat{Z}\in\widehat{\mathbb{P}}_0}|\hat{v}^\mu {F_\mu}^j|(t, X_{\mathscr{C}})\left|\frac{z}{v^0}\widehat{Z}\boldsymbol{f}\right|(t, X_{\mathscr{C}}, v)$$
$$+ \frac{1}{tv^0}|t^2F(t, X_{\mathscr{C}}) - F^\infty(v)||\partial_{t,x}\boldsymbol{f}|(t, X_{\mathscr{C}}, v).$$

We deal with the second term on the right-hand side of the previous inequality by controlling the Lorentz force through Remark 4.3, so that $|\hat{v}^\mu {F_\mu}^j|(t, X_{\mathscr{C}}) \lesssim \Lambda(1+t)^{-2}|v^0|^2 z(t, X_{\mathscr{C}}, v)$. Next, by Corollary 6.16 and the mean value theorem,

$$|t^2F(t, X_{\mathscr{C}}) - F^\infty(v)| \le |t^2F(t, x+t\hat{v}) - F^\infty(v)| + t^2|F(t, X_{\mathscr{C}}) - F(t, x+t\hat{v})|$$
$$\lesssim \Lambda\langle x\rangle^2|v^0|^8\frac{\log^{3N_x+3N+1}(3+t)}{(1+t)^\delta} + t^2|\mathscr{C}(t, v)|\sup_{|y-X_{\mathscr{C}}|\le|\mathscr{C}|(t,v)}|\nabla_{t,x}F|(t, y).$$

[11] Recall that F^∞ is a 2-form, so that $\hat{v}^\mu\hat{v}^\nu F^\infty_{\mu\nu} = 0$.

In view of the estimate of $\nabla_{t,x} F$ given by Lemma 4.2 and the bound (63) on $\mathscr{C}$, we have

$$t^2|\mathscr{C}(t,v)| \sup_{|y-x|\leq|\mathscr{C}|(t,v)} |\nabla_{t,x}F|(t,y) \lesssim \frac{\Lambda}{\sqrt{v^0}} t^2 \log(3+t) \frac{\log(3+t)}{(1+t)^3} |v^0|^4 \sup_{|y-X_{\mathscr{C}}|\leq|C|(t,v)} z^2(t,y,v).$$

Since $|\nabla_x z| \lesssim 1$, the mean value theorem yields

$$z(t, x+t\hat{v}, v) \leq \sup_{|y-X_{\mathscr{C}}|\leq|\mathscr{C}|(t,v)} z(t,y,v) \leq z(t, X_{\mathscr{C}}, v) + \frac{\bar{\epsilon}}{\sqrt{v^0}} \log(3+t) \lesssim \log(3+t) z(t, X_{\mathscr{C}}, v). \quad (66)$$

Consequently, as $\langle x \rangle \leq z(t, x+t\hat{v}, v)$, we have

$$|t^2 F(t, X_{\mathscr{C}}) - F^{\infty}(v)| \lesssim \Lambda (1+t)^{-\delta} \log^{3N_x+3N+3}(3+t) |v^0|^8 z^2(t, X_{\mathscr{C}}, v).$$

We then deduce the result from the previous estimates. □

By applying this result to f, we obtain that there exists $f_\infty \in L^\infty_{x,v}$ such that $f(t, X_{\mathscr{C}}, v) \to f_\infty(x,v)$ as $t \to 0$ (see Proposition 6.34 for more details). Applying it again to $\partial^\kappa_x f$ we could easily deduce that f_∞ is smooth with respect to the spatial variables. However, obtaining the regularity in the velocity variables requires a more thorough analysis. Indeed, $\partial_{v^i}(f(t, X_{\mathscr{C}}, v))$ is deeply related to $\widehat{\Omega}_{0i} f(t, X_{\mathscr{C}}, v)$, which does not converge.

6.4. ***Modified commutators.*** Let $\widehat{Z} \in \widehat{\mathbb{P}}_0 \setminus \{\partial_t, \partial_{x^1}, \partial_{x^2}, \partial_{x^3}\}$ be a homogeneous vector field. Contrary to the case of the translations, the error term $[\boldsymbol{T}_F, \widehat{Z}](f)$ does not decay sufficiently fast in order to prove a convergence result for $\widehat{Z} f$, even along the modified characteristics. Indeed, recall from Lemma 2.3 that

$$\boldsymbol{T}_F(\widehat{Z} f) = -\hat{v}^\mu \mathcal{L}_Z(F)_\mu{}^j \partial_{v^j} f + \delta^S_{\widehat{Z}} \hat{v}^\mu F_\mu{}^j \partial_{v^j} f$$

and let us identify the terms with the slowest decay rate. Rewriting ∂_{v^j} by using (65) and estimating the electromagnetic field through Remark 4.3, we have

$$\left| \boldsymbol{T}_F(\widehat{Z} f) - \frac{t}{v^0} (\hat{v}^\mu \mathcal{L}_Z(F)_\mu{}^j - \delta^S_{\widehat{Z}} \hat{v}^\mu F_\mu{}^j)(\delta^i_j - \hat{v}_j \hat{v}^i) \partial_{x^i} f \right| \lesssim \Lambda (1+t)^{-2} \sum_{\widehat{\Gamma} \in \widehat{\mathbb{P}}_0} v^0 |z^2 \widehat{\Gamma} f|. \quad (67)$$

In view of Proposition 4.5, the right-hand side is bounded by $\bar{\epsilon}(1+t)^{-2} \log^9(3+t)$ and then belongs to $L^1_t L^\infty_{x,v}$. On the other hand, if $\mathcal{L}_Z(F)^\infty$ and F^∞ does not vanish, the decay rate of $t|\mathcal{L}_Z F| + t|F| \lesssim t^{-1}$ along timelike trajectories is at the threshold of time-integrability. For this reason, we modify the linear commutator $\widehat{Z}$ in a way that is similar to how we modify the spatial characteristics. More precisely, motivated by Corollary 6.16 and (67), we introduce the following vector fields.

Definition 6.23. For any $\widehat{Z} \in \widehat{\mathbb{P}}_0 \setminus \{\partial_t, \partial_{x^1}, \partial_{x^2}, \partial_{x^3}, S\}$, we define $\widehat{Z}^{\mathrm{mod}}$ and S^{mod} as

$$\widehat{Z}^{\mathrm{mod}} := \widehat{Z} - \log(t) \hat{v}^\mu \mathcal{L}_Z(F)^{\infty, j}_\mu (v) \frac{\delta^i_j - \hat{v}_j \hat{v}^i}{v^0} \partial_{x^i}, \qquad S^{\mathrm{mod}} := S + \log(t) \hat{v}^\mu F^{\infty, j}_\mu (v) \frac{\delta^i_j - \hat{v}_j \hat{v}^i}{v^0} \partial_{x^i}.$$

We further define the correction coefficients $\mathscr{C}^i_S(t,v) = -\mathscr{C}^i(t,v)$ and

$$\mathscr{C}^i_{\widehat{Z}}(t,v) = -\log(t) \hat{v}^\mu \mathcal{L}_Z(F)^{\infty, j}_\mu (v) \frac{\delta^i_j - \hat{v}_j \hat{v}^i}{v^0} = -\frac{\log(t)}{v^0} \big(\hat{v}^\mu \mathcal{L}_Z(F)^{\infty, i}_\mu (v) + \hat{v}^i \hat{v}^\mu \mathcal{L}_Z(F)^\infty_{\mu 0}(v) \big),$$

so that $S^{\mathrm{mod}} = S + \mathscr{C}^i_S(t,v) \partial_{x^i}$ and $\widehat{Z}^{\mathrm{mod}} = \widehat{Z} + \mathscr{C}^i_{\widehat{Z}}(t,v) \partial_{x^i}$.

Remark 6.24. Recall that $t|\mathcal{L}_S(F)| \lesssim (1+t)^{-1-\delta}$ in domains of the form $\{t \geq (1-\delta)r\}$ since $\mathcal{L}_S(F)^\infty = 0$. This is why we do not need to compensate the term related to $\mathcal{L}_S(F)$ in (67).

We have the improved commutation relations.

Proposition 6.25. *Let $Z \in \mathbb{K}$ be a rotational vector field Ω_{jk} or a Lorentz boost Ω_{0i}. Then, for $t > 0$,*

$$[\boldsymbol{T}_F, \widehat{Z}^{\mathrm{mod}}] = \frac{1}{t}\left(t^2 \hat{v}^\mu (\mathcal{L}_Z(F)_\mu{}^j - \mathcal{L}_Z(F)_\mu^{\infty,j}) \frac{\delta^i_j - \hat{v}_j \hat{v}^i}{v^0}\right)\partial_{x^i}$$
$$- \frac{\hat{v}^\mu}{v^0}\mathcal{L}_Z(F)_\mu{}^j\left(\widehat{\Omega}_{0j} + z_{0j}\partial_t - \hat{v}^j S - \sum_{1 \leq i \leq 3} \hat{v}^j z_{0i}\partial_{x^i}\right) - \mathscr{C}^i_{\widehat{Z}}\hat{v}^\mu \mathcal{L}_{\partial_{x^i}}(F)_\mu{}^j \partial_{v^j} + \hat{v}^\mu F_\mu{}^j \partial_{v^j}\mathscr{C}^i_{\widehat{Z}}\partial_{x^i}.$$

For the scaling vector field, we have

$$[\boldsymbol{T}_F, S^{\mathrm{mod}}] = -\frac{1}{t}\left(t^2 \hat{v}^\mu (F_\mu{}^j - F_\mu^{\infty,j}) \frac{\delta^i_j - \hat{v}_j \hat{v}^i}{v^0}\right)\partial_{x^i} + \frac{1}{t}\left(t^2 \hat{v}^\mu (\mathcal{L}_S(F)_\mu{}^j - \mathcal{L}_S(F)_\mu^{\infty,j}) \frac{\delta^i_j - \hat{v}_j \hat{v}^i}{v^0}\right)\partial_{x^i}$$
$$+ \frac{\hat{v}^\mu}{v^0}(F_\mu{}^j - \mathcal{L}_S(F)_\mu{}^j)\left(\widehat{\Omega}_{0j} + z_{0j}\partial_t - \hat{v}^j S - \sum_{1 \leq i \leq 3} \hat{v}^j z_{0i}\partial_{x^i}\right)$$
$$- \mathscr{C}^i_S \hat{v}^\mu \mathcal{L}_{\partial_{x^i}}(F)_\mu{}^j \partial_{v^j} + \hat{v}^\mu F_\mu{}^j \partial_{v^j}\mathscr{C}^i_S \partial_{x^i}.$$

Proof. Consider first the case $Z = \Omega_{jk}$ or $Z = \Omega_{0i}$. In view of the commutation relation of Lemma 2.3,

$$[\boldsymbol{T}_F, \widehat{Z}^{\mathrm{mod}}] = \boldsymbol{T}_F(\mathscr{C}^i_{\widehat{Z}})\partial_{x^i} + [\boldsymbol{T}_F, \widehat{Z}] + \mathscr{C}^i_{\widehat{Z}}[\boldsymbol{T}_F, \partial_{x^i}] = \boldsymbol{T}_F(\mathscr{C}^i_{\widehat{Z}})\partial_{x^i} - \hat{v}^\mu \mathcal{L}_Z(F)_\mu{}^j \partial_{v^j} - \mathscr{C}^i_{\widehat{Z}}\hat{v}^\mu \mathcal{L}_{\partial_{x^i}}(F)_\mu{}^j \partial_{v^j}.$$

It then suffices to use (65) in order to rewrite ∂_{v^j} in the second term and to compute

$$\boldsymbol{T}_F(\mathscr{C}^i_{\widehat{Z}}) = -\frac{1}{t}\hat{v}^\mu \mathcal{L}_Z(F)_\mu^{\infty,j}(v)\frac{\delta^i_j - \hat{v}_j \hat{v}^i}{v^0} + \hat{v}^\mu F_\mu{}^j \partial_{v^j}\mathscr{C}^i_{\widehat{Z}}.$$

The case of the scaling S can be treated similarly since $\mathcal{L}_S(F)^\infty = 0$ according to Proposition 6.18. □

Apart from the term involving $\mathcal{L}_S(F)$, already discussed in Remark 6.24, it is clear that any of the error terms decay almost as $t^{-1-\delta}$ for, say, $|x| < t/2$. At this point, we could then prove that f_∞ is C^1 in v. However, since we would like to show $f_\infty \in C^{N-2}(\mathbb{R}^3_x \times \mathbb{R}^3_v)$, we need to state a higher-order commutator formula for the modified vector fields. For this purpose, we introduce the set

$$\widehat{\mathbb{P}}_0^{\mathrm{mod}} := \{\partial_t, \partial_{x^i}, \widehat{\Omega}^{\mathrm{mod}}_{0i}, \widehat{\Omega}^{\mathrm{mod}}_{jk}, S^{\mathrm{mod}} \mid 1 \leq i \leq 3,\ 1 \leq j < k \leq 3\},$$

and we consider an ordering on it, so that $\widehat{\mathbb{P}}_0^{\mathrm{mod}} = \{\widehat{Z}^{\mathrm{mod},i} \mid 1 \leq i \leq 11\}$. Given a multi-index $\beta \in [\![1, 11]\!]^p$, we will then denote $\widehat{Z}^{\mathrm{mod},\beta_1} \dots \widehat{Z}^{\mathrm{mod},\beta_p}$ by $\widehat{Z}^{\mathrm{mod},\beta}$. We will further denote by β_H (respectively β_T) the number of modified vector fields (respectively translations) composing $\widehat{Z}^{\mathrm{mod},\beta}$, so that $|\beta| = \beta_H + \beta_T$. Furthermore, we will use the schematic notation $P_{p,q}(\mathscr{C})$ in order to denote any quantity of the form

$$\prod_{1 \leq k \leq p} \widehat{Z}^{\xi_k}(\mathscr{C}^{i_k}_{\widehat{Z}^k}), \quad (p,q) \in \mathbb{N}^2, \quad 1 \leq i_k \leq 3, \quad \widehat{Z}^k \in \widehat{\mathbb{P}}_0, \quad \sum_{1 \leq k \leq p} |\xi_k| = q, \quad q_T := \sum_{1 \leq k \leq q} \xi_{k,T}, \quad q_H := q - q_T,$$

where $q_T \geq 1$ when at least one translation ∂_{x^μ} is applied to at least one of the correction coefficients. By convention, we set $P_{0,0}(\mathscr{C}) = 1$ for $p = q = 0$. We recall from (10) the weights $z_{\lambda k} \in \boldsymbol{k}_1$, $0 \leq \lambda < k \leq 3$, which commute with the linear transport operator $\boldsymbol{T}_0$.

Proposition 6.26. *Let $\widehat{Z}^{\mathrm{mod},\beta} \in \widehat{\mathbb{P}}_0^{|\beta|}$. Then, $[\boldsymbol{T}_F, \widehat{Z}^{\mathrm{mod},\beta}]$ can be written as a linear combination of the following types of terms*:

$$\frac{1}{v^0 t} R\Big(\frac{1}{t}, \hat{v}, z\Big) P_{p,q}(\mathscr{C}) \big(t^2 \hat{v}^\mu \mathcal{L}_{Z^\gamma}(F)_{\mu\nu} - \hat{v}^\mu \mathcal{L}_{Z^\gamma}(F)^\infty_{\mu\nu}(v)\big) \widehat{Z}^\kappa, \tag{T-1}$$

$$\frac{1}{v^0} R\Big(\frac{1}{t}, \hat{v}, z\Big) P_{p,q}(\mathscr{C}) \mathcal{L}_{Z^\gamma}(F)_{\lambda\nu} \widehat{Z}^\kappa, \tag{T-2}$$

$$\frac{x^\alpha}{v^0} R\Big(\frac{1}{t}, \hat{v}, z\Big) P_{p,q}(\mathscr{C}) \mathcal{L}_{Z^\gamma}(F)_{\lambda\nu} \widehat{Z}^\kappa, \quad q_T + \gamma_T \geq 1, \tag{T-3}$$

where R is a polynomial in $1/t$, $\hat{v} = (\hat{v}^i)_{1 \leq i \leq 3}$ and $z = (z_{\mu k})_{0 \leq \mu < k \leq 3}$, of degree $\deg_z R$ in z, and

$$q_H + \deg_z R \leq \beta_H, \quad p \leq \beta_H, \quad q + |\gamma| + |\kappa| \leq |\beta| + 1, \quad q, |\gamma|, |\kappa| \leq |\beta|, \quad 0 \leq \alpha, \lambda, \nu \leq 3.$$

Remark 6.27. In fact, we could prove that, as for the first-order commutation formula, most of the error terms satisfy a form of null condition. Since this property is not crucial for our purpose, we chose to demonstrate a result requiring a much simpler analysis.

Proof. Note first that the result holds for any $|\beta| = 1$. One can see it by applying either Lemma 2.3, for the translation, or Proposition 6.25 and by rewriting all the v derivatives as $v^0 \partial_{v^j} = \widehat{\Omega}_{0j} - t\partial_{x^j} - x^j \partial_t$. Let $n \geq 1$ such that the proposition holds for any $|\beta| = n$ and consider a multi-index $|\beta_0| = n+1$. Consider further $|\beta| = n$ as well as $\widehat{Z}^{\mathrm{mod}} \in \widehat{\mathbb{P}}_0^{\mathrm{mod}}$ such that $\widehat{Z}^{\mathrm{mod},\beta_0} = \widehat{Z}^{\mathrm{mod}} \widehat{Z}^{\mathrm{mod},\beta}$ and note

$$[\boldsymbol{T}_F, \widehat{Z}^{\mathrm{mod},\beta_0}] = [\boldsymbol{T}_F, \widehat{Z}^{\mathrm{mod}}] \widehat{Z}^{\mathrm{mod},\beta} + \widehat{Z}^{\mathrm{mod}} [\boldsymbol{T}_F, \widehat{Z}^{\mathrm{mod},\beta}]. \tag{68}$$

We can deal with the first term on the right-hand side by applying the result for first-order operators and by noticing that $\widehat{Z}^\xi \widehat{Z}^{\mathrm{mod},\beta}$, for $|\xi| \leq 1$, can be written as a linear combination of terms of the form

$$P_{p,q}(\mathscr{C}) \widehat{Z}^\zeta, \quad p \leq \beta_H, \quad q_H \leq \beta_H + \xi_H - 1, \quad q \leq |\beta| + |\xi| - 1, \quad q + |\zeta| \leq |\beta| + |\xi|. \tag{69}$$

For the second term, we apply the induction hypothesis, so that $[\boldsymbol{T}_F, \widehat{Z}^{\mathrm{mod},\beta}]$ can be written as a linear combination of terms of the form (T-1)–(T-3). In order to deal with them, we will use the following properties:

- $\partial_t(t) = 1$, $\widehat{\Omega}_{0j}^{\mathrm{mod}}(t) = x^j = -z_{0j} - t\hat{v}^j$, $S^{\mathrm{mod}}(t) = t$ and $\widehat{Z}^{\mathrm{mod}}(t) = 0$ otherwise.
- If $\widehat{Z}^{\mathrm{mod}} = \partial_{x^\mu}$, then $\widehat{Z}^{\mathrm{mod}}(x^k) = \delta^k_\mu$. Otherwise, there exists $0 \leq \lambda \leq 3$ such that $\widehat{Z}^{\mathrm{mod}}(x^k) = \pm x^\lambda + \mathscr{C}^k_{\widehat{Z}}$.
- $\widehat{\Omega}_{0j}^{\mathrm{mod}}(v^0) = v^j$ for any $1 \leq j \leq 3$ and $\widehat{Z}^{\mathrm{mod}}(v^0) = 0$ otherwise.
- There exist four polynomials $R_0, \dots, R_3$ such that

$$\widehat{Z}^{\mathrm{mod}}(R(1/t, \hat{v}, z)) = R_0(1/t, \hat{v}, z) + \mathscr{C}^i_{\widehat{Z}} R_i(1/t, \hat{v}, z), \quad \deg_z R_0 \leq \deg_z R + 1, \ \deg_z R_i \leq \deg_z R,$$

where we set $\mathscr{C}^i_{\partial_{x^\mu}} := 0$. Moreover, if $\widehat{Z}^{\mathrm{mod}} \neq \widehat{\Omega}^{\mathrm{mod}}_{0j}$, then $\deg_z R_0 \leq \deg_z R$. This can be obtained by the first property and [Bigorgne 2020a, Lemma 3.2], giving,

$$\forall\, \widehat{\Gamma} \in \widehat{\mathbb{P}}_0, \ \forall 1 \leq i \leq 3, \ \forall z \in \boldsymbol{k}_1, \quad \widehat{\Gamma}(v^0 z) \in \{0\} \cup \boldsymbol{k}_1, \quad \partial_{x^i}(z) \in \{0, 1, \hat{v}^k \mid 1 \leq k \leq 3\}.$$

• If $\widehat{Z}^{\text{mod}} = \partial_{x^\mu}$, we schematically have $\widehat{Z}^{\text{mod}}(P_{p,q}(\mathscr{C})) = P^0_{p,q^0}(\mathscr{C})$, with $q^0 = q+1$ and $q^0_H = q_H$. Otherwise, $\widehat{Z}^{\text{mod}}(P_{p,q}(\mathscr{C})) = P^1_{p,q^1}(\mathscr{C}) + P^2_{p+1,q^2}(\mathscr{C})$, with $q^1 = q^2 = q+1$, $q^1_H = q_H + 1$ and $q^2_H = q_H$.
• $\widehat{Z}^{\text{mod}}\widehat{Z}^\kappa = \widehat{Z}\widehat{Z}^\kappa + \mathscr{C}^i_{\widehat{Z}}\partial_{x^i}\widehat{Z}^\kappa$ and $\widehat{Z}^{\text{mod}}\mathcal{L}_{Z^\gamma}(F)_{\lambda\nu}$ can be written as a linear combination of

$$\mathcal{L}_{ZZ^\gamma}(F)_{\lambda\nu}, \quad \mathscr{C}^i_{\widehat{Z}}\mathcal{L}_{\partial_{x^i}Z^\gamma}(F)_{\lambda\nu}, \quad \mathcal{L}_{Z^\gamma}(F)_{\mu\xi}, \quad 0 \le \mu, \xi \le 3.$$

Hence, we obtain by applying $\widehat{Z}^{\text{mod}}$ to any quantity of the form (T-1), (T-2) or (T-3) (corresponding to $|\beta| = n$), a combination of terms of the form (T-1)–(T-3) (corresponding to $|\beta_0| = n+1$), as well as

$$\mathcal{T}[\widehat{Z}^{\text{mod}}] = \frac{1}{t}R\left(\frac{1}{t}, \hat{v}, z\right)P_{p,q}(\mathscr{C})\widehat{Z}^{\text{mod}}\left(t^2\frac{\hat{v}^\mu}{v^0}\mathcal{L}_{Z^\gamma}(F)_{\mu\nu} - \frac{\hat{v}^\mu}{v^0}\mathcal{L}_{Z^\gamma}(F)^\infty_{\mu\nu}(v)\right)\widehat{Z}^\kappa,$$

where $0 \le \nu \le 3$, $q + |\gamma| + |\kappa| \le |\beta| + 1$, $\max(q, |\gamma|, |\kappa|) \le |\beta|$, $p \le \beta_H$ and $q_H + \deg_z R \le \beta_H$. Assume first that $\widehat{Z}^{\text{mod}}$ is a translation ∂_{x^λ}. Then,

$$\mathcal{T}[\partial_{x^\lambda}] = \frac{2\delta^0_\lambda}{v^0}R(1/t, \hat{v}, z)P_{p,q}(\mathscr{C})\hat{v}^\mu\mathcal{L}_{Z^\gamma}(F)_{\mu\nu}\widehat{Z}^\kappa + \frac{t}{v^0}R(1/t, \hat{v}, z)P_{p,q}(\mathscr{C})\hat{v}^\mu\mathcal{L}_{\partial_{x^\lambda}Z^\gamma}(F)_{\mu\nu}\widehat{Z}^\kappa$$

is the sum of a term of type (T-2) and a term of type (T-3). Otherwise, $\widehat{Z}^{\text{mod}} = \widehat{Z} + \mathscr{C}^i_{\widehat{Z}}\partial_{x^i}$ and, following the previous computations, we have

$$\mathcal{T}[\widehat{Z}^{\text{mod}}] = \mathcal{T}[\widehat{Z}] + \mathscr{C}^i_{\widehat{Z}}\mathcal{T}[\partial_{x^i}] = \mathcal{T}[\widehat{Z}] + \frac{t}{v^0}R(1/t, \hat{v}, z)P_{p,q}(\mathscr{C})\mathscr{C}^i_{\widehat{Z}}\hat{v}^\mu\mathcal{L}_{\partial_{x^i}Z^\gamma}(F)_{\mu\nu}\widehat{Z}^\kappa,$$

where the last three terms are of type (T-3). According to Corollary 6.20, $\mathcal{T}[\widehat{Z}]$ is a combination of terms of type (T-1) and, in the case $\widehat{Z} = \widehat{\Omega}_{0j}$, (T-2). □

We now control these error terms and then prove a uniform boundedness statement for $\widehat{Z}^{\text{mod},\beta}f$. Because of regularity issues on the coefficients $\mathscr{C}_{\widehat{Z}}$, which are of class C^{N-2}, we are not able to deal with the multi-indices $|\beta| \ge N-1$.

Proposition 6.28. *Let $|\beta| \le N-2$. For all $(t, x, v) \in [3, +\infty[\times \mathbb{R}^3_x \times \mathbb{R}^3_v$, there holds*

$$|\boldsymbol{T}_F(\widehat{Z}^{\text{mod},\beta}f)|(t, x, v) \lesssim \Lambda\frac{\log^{3N_x+4N}(t)}{t^{1+\delta}}\sum_{|\kappa|\le|\beta|}|v^0|^7|z^{2+\beta_H}\widehat{Z}^\kappa f|(t, x, v).$$

Moreover,

$$|v^0|^{N_v-7}|\boldsymbol{z}^{N_x-2-\beta_H}\boldsymbol{T}_F(\widehat{Z}^{\text{mod},\beta}f)|(t, x, v) \lesssim \bar{\epsilon}\frac{\log^{6N_x+7N}(t)}{t^{1+\delta}}.$$

Proof. Fix $(t, x, v) \in [3, +\infty[\times \mathbb{R}^3_x \times \mathbb{R}^3_v$ and let us prove first the following property. Consider $P_{p,q}(\mathscr{C})$ and $R(1/t, \hat{v}, z)$ a polynomial such that $p \le \beta_H$, $q \le |\beta|$ and $q_H + \deg_z R \le \beta_H$. Then,

$$|R(1/t, \hat{v}, z)||P_{p,q}(\mathscr{C})|(t, x, v) \lesssim \frac{\log^{N-2}(t)}{t^{q_T}}z^{\beta_H}(t, x, v). \tag{70}$$

For this, remark first that, for $|\xi| \le N-2$, $i \in [\![1, 3]\!]$ and $\widehat{Z} \in \widehat{\mathbb{P}}_0 \setminus \{\partial_t, \partial_{x^1}, \partial_{x^2}, \partial_{x^3}, S\}$,

$$|\widehat{Z}^\xi(\mathscr{C}^i_{\widehat{Z}})|(t, x, v) \le \sum_{\substack{|\gamma|+|\kappa|\le|\xi| \\ \gamma_T=\xi_T}}\mathcal{I}_{\gamma,\kappa}, \quad \mathcal{I}_{\gamma,\kappa} := \sum_{0\le\nu\le3}|\widehat{Z}^\gamma\log(t)|\left|\widehat{Z}^\kappa\left(\frac{\hat{v}^\mu}{v^0}\mathcal{L}_Z(F)^\infty_{\mu\nu}\right)\right|(v).$$

Note that the case $\widehat{Z} = S$ leads to a similar estimate.

• We have $|\widehat{Z}^\gamma \log(t)| \lesssim t^{-\xi_T} z^{\xi_H}(t,x,v)\log(t)$. Indeed, $|\widehat{Z}^\gamma \log(t)| \leq |t^{-\gamma_T} P_{\gamma_H}(x/t)\log(t)|$, where P_{γ_H} is a polynomial of degree at most $\gamma_H \leq \xi_H$, and $\gamma_T = \xi_T$. Finally, recall that $|x|/t \leq |x-t\hat{v}|/t+1 \leq 2z(t,x,v)$.

• To deal with the last factor in $\mathcal{I}_{\gamma,\kappa}$, note first that $|\kappa|+1 \leq N-1$ and that this quantity vanishes if κ is composed of at least a translation or the scaling vector field S according to Proposition 6.18. Then, using first the relations (58)–(59) and then Proposition 6.18, we get

$$\left|\widehat{Z}^\kappa\left(\frac{\hat{v}^\mu}{v^0}\mathcal{L}_Z(F)^\infty_{\mu\nu}\right)\right|(v) \lesssim \sum_{|\zeta|\leq|\kappa|+1}\left|\frac{\hat{v}^\mu}{v^0}\mathcal{L}_{Z^\zeta}(F)^\infty_{\mu\nu}\right|(v) \lesssim \bar{\epsilon}. \tag{71}$$

We then deduce that

$$|R(1/t,\hat{v},z)|\,|P_{p,q}(\mathscr{C})|(t,x,v) \lesssim z^{\deg_z R}(t,x,v)t^{-q_T}z^{q_H}(t,x,v)\log^p(t)\bar{\epsilon}^p,$$

which implies (70).

Apply Proposition 6.26 in order to reduce the analysis to the treatment of terms of type (T-1), (T-2) and (T-3). By Corollary 6.16 and (70), we can bound any term of type (T-1) by

$$\Lambda\frac{|v^0|^8\log^{3N_x+4N}(t)}{v^0t^{1+\delta}}\langle x-t\hat{v}\rangle^2|z^{\beta_H}\widehat{Z}^\kappa f|(t,x,v) \lesssim \Lambda\frac{\log^{3N_x+4N}(t)}{t^{1+\delta}}|v^0|^7\,|z^{2+\beta_H}\widehat{Z}^\kappa f|(t,x,v),$$

since $\langle x-t\hat{v}\rangle \leq z(t,x,v)$ and where $|\kappa|\leq N-2$. We deal with the ones of type (T-2) by using (BA1), (70) and Lemma 2.6. There are bounded above by

$$\frac{\Lambda\log^{N-2}(t)}{(t+|x|)(1+|t-|x||)v^0}\frac{(1+|t-|x||)|v^0|^2z}{t+|x|}|z^{\beta_H}\widehat{Z}^\kappa f|(t,x,v) \lesssim \Lambda\frac{\log^{N-2}(t)}{t^2}v^0|z^{1+\beta_H}\widehat{Z}^\kappa f|(t,x,v).$$

Finally, let $\mathcal{T}_3$ be a term of type (T-3). Using first (70) together with Proposition 3.2 and then Lemma 2.6,

$$\mathcal{T}_3 \lesssim \frac{\Lambda\log^{N-2}(t)}{v^0t^{q_T}(1+|t-|x||)^{1+\gamma_T}}|z^{\beta_H}\widehat{Z}^\kappa f|(t,x,v) \lesssim \Lambda\frac{\log^N(t)}{t^2}|v^0|^3|z^{2+\beta_H}\widehat{Z}^\kappa f|(t,x,v).$$

We deduce from that the first estimate of the statement, which, through an application of Proposition 4.5, implies the second one. □

Corollary 6.29. *Let $|\beta|\leq N-2$. If $\beta_H \leq N_x-2$, there exists $\overline{D}>0$ such that,*

$$\forall t\geq 3,\quad \||v^0|^{N_v-7}\widehat{Z}^{\mathrm{mod},\beta}f(t,\cdot,\cdot)\|_{L^\infty_{x,v}} \lesssim \epsilon e^{\overline{D}\Lambda}. \tag{72}$$

Proof. Note first that we can obtain, by a much simpler analysis than in the proof of Proposition 4.5, that $\||v^0|^{N_v}z^{N_x}\widehat{Z}^\beta f(3,\cdot,\cdot)\|_{L^\infty_{x,v}} \lesssim \bar{\epsilon}$ for all $|\beta|\leq N$. Consequently, using (69) and (71), we get,

$$\forall|\beta|\leq N-1,\quad \||v^0|^{N_v}z^{N_x}\widehat{Z}^{\mathrm{mod},\beta}f(3,\cdot,\cdot)\|_{L^\infty_{x,v}} \lesssim \sum_{|\kappa|\leq|\beta|}\||v^0|^{N_v}z^{N_x}\widehat{Z}^\kappa f(3,\cdot,\cdot)\|_{L^\infty_{x,v}} \lesssim \bar{\epsilon}.$$

Hence, it suffices to prove, according to Lemma 4.4, that

$$|\boldsymbol{T}_F(|v^0|^{N_v-7}\widehat{Z}^{\mathrm{mod},\beta}f)|(t,x,v) \lesssim \left(\frac{\Lambda|v^0|^{N_v-7}|\widehat{Z}^{\mathrm{mod},\beta}f|}{(1+t)^{\frac{3}{2}}}+\frac{\Lambda\hat{v}^{\underline{L}}|v^0|^{N_v-7}|\widehat{Z}^{\mathrm{mod},\beta}f|}{(1+|t-|x||)^2}\right)+\frac{\bar{\epsilon}}{(1+t)\log^2(3+t)}$$

for all $(t,x,v) \in [3,T[\times \mathbb{R}^3_x \times \mathbb{R}^3_v$ and any $|\beta| \leq N-2$. For this, we bound $\boldsymbol{T}_F(v^0)$ using (31) and we apply the previous Proposition 6.28 in order to control $\boldsymbol{T}_F(\widehat{Z}^{\mathrm{mod},\beta} f)$. $\square$

6.5. *Regularity of the asymptotic state.* In order to prove that f_∞ is differentiable with respect to v, we will need to compute the first-order v-derivatives of the correction terms in the modified spatial characteristics and to bound their higher-order derivatives.

Lemma 6.30. *Let $(i,k) \in [\![1,3]\!]^2$. Then, for all $(t,x,v) \in [3,+\infty[\times \mathbb{R}^3_x \times \mathbb{R}^3_v$,*

$$v^0 \partial_{v^k} \mathscr{C}^i(t,v) = \mathscr{C}^i_{\Omega_{0k}}(t,v) - \hat{v}^i \mathscr{C}^k(t,v).$$

More generally, for any multi-index $|\kappa| \leq N-1$,

$$|v^0|^{|\kappa|} |\partial_v^\kappa \mathscr{C}^i|(t,v) \lesssim \bar{\epsilon} |v^0|^{-\frac{1}{2}} \log(t).$$

Proof. According to (58), we have, for any $\nu \in [\![0,3]\!]$,

$$v^0 \partial_{v^k}\left(\frac{\hat{v}^\mu}{v^0} F^\infty_{\mu\nu}\right) = \frac{\hat{v}^\mu}{v^0} \mathcal{L}_{\Omega_{0k}}(F)^\infty_{\mu\nu} - \delta^0_\nu \frac{\hat{v}^\mu}{v^0} F^\infty_{\mu k} - \delta^k_\nu \frac{\hat{v}^\mu}{v^0} F^\infty_{\mu 0}.$$

This implies in particular that

$$v^0 \partial_{v^k}\left(\frac{\hat{v}^i \hat{v}^\mu}{v^0} F^\infty_{\mu 0} + \frac{\hat{v}^\mu}{v^0} F^\infty_{\mu i}\right) = \frac{\hat{v}^i \hat{v}^\mu}{v^0} \mathcal{L}_{\Omega_{0k}}(F)^\infty_{\mu 0} + \frac{\hat{v}^\mu}{v^0} \mathcal{L}_{\Omega_{0k}}(F)^\infty_{\mu i} - \hat{v}^i \left(\frac{\hat{v}^k \hat{v}^\mu}{v^0} F^\infty_{\mu 0} + \frac{\hat{v}^\mu}{v^0} F^\infty_{\mu k}\right).$$

In view of the definition of the correction coefficients (see Definitions 6.21 and 6.23), we deduce from this last equality the first part of the statement. The second part follows from a direct induction as well as Propositions 6.18–6.19. $\square$

Remark 6.31. Similarly, we could prove using (59) that $\Omega^v_{jk} \mathscr{C}^i(t,v) = \mathscr{C}^i_{\Omega_{jk}}(t,v) - \delta^i_j \mathscr{C}^k(t,v) + \delta^i_k \mathscr{C}^j(t,v)$, where $\Omega^v_{jk} := v^j \partial_{v^k} - v^k \partial_{v^j}$. Consequently, the following quantities, related to the asymptotic Lorentz force,

$$\Gamma(v) := \frac{\hat{v}^\mu}{v^0}(F^\infty_{\mu i}(v) + \hat{v}_i F^\infty_{\mu 0}(v)) \mathrm{d}v^i, \quad \Gamma_Z(v) := \frac{\hat{v}^\mu}{v^0}(\mathcal{L}_Z(F)^\infty_{\mu i}(v) + \hat{v}_i \mathcal{L}_Z(F)^\infty_{\mu 0}(v)) \mathrm{d}v^i,$$

satisfy $\mathcal{L}_{v^0 \partial_{v^k}}(\Gamma) = \Gamma_{\Omega_{0k}}$ and $\mathcal{L}_{\Omega^v_{jk}}(\Gamma) = \Gamma_{\Omega_{jk}}$.

We now perform a computation, which holds for any sufficiently regular function $\boldsymbol{f}$. In particular, we will apply it to $\boldsymbol{f} = \partial^\kappa_{t,x} f$. We have

$$v^0 \partial_{v^k}(\boldsymbol{f}(t, X_{\mathscr{C}}, v)) = t \partial_{x^k} \boldsymbol{f}(t, X_{\mathscr{C}}, v) - t \hat{v}^k \hat{v}^i \partial_{x^i} \boldsymbol{f}(t, X_{\mathscr{C}}, v) + v^0 \partial_{v^k} \boldsymbol{f}(t, X_{\mathscr{C}}, v) + v^0 \partial_{v^k} \mathscr{C}^i(t,v) \partial_{x^i} \boldsymbol{f}(t, X_{\mathscr{C}}, v).$$

Then, we use (65) in order to rewrite the third term on the right-hand side. We get

$$v^0 \partial_{v^k}(\boldsymbol{f}(t, X_{\mathscr{C}}, v)) = \left(\widehat{\Omega}_{0k} \boldsymbol{f} + z_{0k} \partial_t \boldsymbol{f} - \hat{v}^k S \boldsymbol{f} - \hat{v}^k \sum_{1 \leq i \leq 3} z_{0i} \partial_{x^i} \boldsymbol{f}\right)(t, X_{\mathscr{C}}, v) + v^0 \partial_{v^k} \mathscr{C}^i(t,v) \partial_{x^i} \boldsymbol{f}(t, X_{\mathscr{C}}, v).$$

Hence, as $z_{0i}(t, X_{\mathscr{C}}, v) = -x^i - \mathscr{C}^i(t, v)$,

$$\begin{aligned} v^0 \partial_{v^k}(\boldsymbol{f}(t, X_{\mathscr{C}}, v)) = (\widehat{\Omega}_{0k}\boldsymbol{f})(t, X_{\mathscr{C}}, v) - x^k(\partial_t \boldsymbol{f})(t, X_{\mathscr{C}}, v) - \frac{\mathscr{C}^k(t, v)}{t}(S\boldsymbol{f})(t, X_{\mathscr{C}}, v) \\ + \frac{\mathscr{C}^k(t, v)}{t} X^i_{\mathscr{C}} \partial_{x^i} \boldsymbol{f}(t, X_{\mathscr{C}}, v) - \hat{v}^k(S\boldsymbol{f})(t, X_{\mathscr{C}}, v) + \hat{v}^k \mathscr{C}^i(t, v)\partial_{x^i}\boldsymbol{f}(t, X_{\mathscr{C}}, v) \\ + \hat{v}^k x^i \partial_{x^i} \boldsymbol{f}(t, X_{\mathscr{C}}, v) + v^0 \partial_{v^k} \mathscr{C}^i(t, v) \partial_{x^i} \boldsymbol{f}(t, X_{\mathscr{C}}, v). \end{aligned}$$

Now, according to Lemma 6.30,

$$\widehat{\Omega}_{0k} + v^0 \partial_{v^k} \mathscr{C}^i(t, v)\partial_{x^i} = \widehat{\Omega}_{0k} + \mathscr{C}^i_{\Omega_{0k}}(t, v)\partial_{x^i} - \mathscr{C}^k(t, v)\hat{v}^i \partial_{x^i} = \widehat{\Omega}^{\mathrm{mod}}_{0k} - \mathscr{C}^k(t, v)\hat{v}^i \partial_{x^i},$$

and, in view of the relations $S^{\mathrm{mod}} = S - \mathscr{C}^i(t, v)\partial_{x^i}$ and $X^i_{\mathscr{C}} = x^i + t\hat{v}^i + \mathscr{C}^i(t, v)$,

$$\begin{aligned} v^0 \partial_{v^k}(\boldsymbol{f}(t, X_{\mathscr{C}}, v)) = (\widehat{\Omega}^{\mathrm{mod}}_{0k}\boldsymbol{f})(t, X_{\mathscr{C}}, v) - \left(\hat{v}^k + \frac{\mathscr{C}^k(t, v)}{t}\right)(S^{\mathrm{mod}}\boldsymbol{f})(t, X_{\mathscr{C}}, v) \\ - x^k(\partial_t \boldsymbol{f})(t, X_{\mathscr{C}}, v) + \left(\hat{v}^k + \frac{\mathscr{C}^k(t, v)}{t}\right) x^i \partial_{x^i} \boldsymbol{f}(t, X_{\mathscr{C}}, v). \end{aligned} \tag{73}$$

Iterating this process to the functions $\boldsymbol{f} = \partial^\kappa_{t,x} f$ yields the following result.

Proposition 6.32. *Let $|\kappa| + |\xi| \leq N - 2$. Then, there exist functions $P^{\kappa,\xi}_\beta$ such that,*

$$\forall (t,x,v) \in [3,+\infty[\times \mathbb{R}^3_x \times \mathbb{R}^3_v, \quad |v^0|^{|\xi|} \partial^\xi_v ((\partial^\kappa_{t,x} f)(t, X_{\mathscr{C}}, v)) = \sum_{|\beta| \leq |\kappa|+|\xi|} P^{\kappa,\xi}_\beta(t,x,v) \widehat{Z}^{\mathrm{mod},\beta} f(t, X_{\mathscr{C}}, v)$$

and $P^{\kappa,\xi}_\beta(t,x,v)$ is a linear combination of terms of the form $P(x,\hat{v})M(\mathscr{C})$, where P is a polynomial and

$$M(\mathscr{C}) = \prod_{k=1}^d \frac{1}{t} |v^0|^{|\xi_k|} \partial^{\xi_k}_v \mathscr{C}^{i_k}(t, v), \quad d + \sum_{1 \leq k \leq d} |\xi_k| \leq |\xi|, \quad |\beta| + \sum_{1 \leq k \leq d} |\xi_k| \leq |\xi|, \quad \deg_x(P) + \beta_H \leq |\xi|.$$

The value $d = 0$ is allowed, in which case we set $M(\mathscr{C}) = 1$.

In order to prove, through Proposition 6.22, that the functions considered in the previous statement converge, as $t \to +\infty$, we will be lead to estimate these polynomials and their time derivative.

Lemma 6.33. *Let $|\kappa| + |\xi| \leq N - 2$ and $|\beta| \leq |\kappa| + |\xi|$. Then, for all $(t, x, v) \in [3, +\infty[\times \mathbb{R}^3_x \times \mathbb{R}^3_v$,*

$$|P^{\kappa,\xi}_\beta|(t, x, v) \lesssim \langle x \rangle^{|\xi| - \beta_H}, \quad |\partial_t P^{\kappa,\xi}_\beta|(t, x, v) \lesssim \bar{\epsilon} \langle x \rangle^{|\xi| - \beta_H} \frac{\log(t)}{t^2}.$$

Proof. It is enough to bound terms of the form $P(x, \hat{v})M(\mathscr{C})$ satisfying the conditions given in Proposition 6.32. The first factor satisfies $|P(x, \hat{v})| \lesssim \langle x \rangle^{\deg_x P} \leq \langle x \rangle^{|\xi| - \beta_H}$ and does not depend on t. In view of Lemma 6.30, we have $|M(\mathscr{C})| \lesssim \bar{\epsilon}^d \log^d(t) t^{-d}$, which implies the first estimate. The second one can be obtained similarly. Either $|\partial_t M(\mathscr{C})| = 0$ or $d \geq 1$ and $|\partial_t M(\mathscr{C})| \lesssim \bar{\epsilon}^d \log^d(t) t^{-d-1}$ by Lemma 6.30. □

We are now able to prove the main result of this paper. For this, let us introduce

$$h : (t, x, v) \mapsto f(t, x + t\hat{v} + \mathscr{C}(t, v), v), \quad h^{\xi,\kappa} := |v^0|^{|\xi|} \partial^\xi_v \partial^\kappa_x h(t, x, v) = |v^0|^{|\xi|} \partial^\xi_v \big(\partial^\kappa_x f(t, X_{\mathscr{C}}(t, x, v), v)\big).$$

Proposition 6.34. *There exists a function* $f_\infty \in C^{N-2}(\mathbb{R}^3_x \times \mathbb{R}^3_v, \mathbb{R}_+)$ *such that, for any* $|\kappa| + |\xi| \leq N-2$,

$$\forall t \geq 3, \quad \| |v^0|^{N_v-10+|\xi|} \langle x \rangle^{N_x-4-|\xi|} (\partial_v^\xi \partial_x^\kappa h(t,\cdot,\cdot) - \partial_v^\xi \partial_x^\kappa f_\infty) \|_{L^\infty_{x,v}} \lesssim \bar{\epsilon} \frac{\log^{7(N_x+N)}(t)}{t^\delta}.$$

In particular, as $N_v > 13$ *and if* $N_x > 7 + |\xi|$, *we have* $\partial_v^\xi \partial_x^\kappa f_\infty \in L^1_{x,v}$.

Proof. Fix $t \geq 3$ and $(x,v) \in \mathbb{R}^3_x \times \mathbb{R}^3_v$. Applying the previous Proposition 6.32 and Lemma 6.33, we get

$$|\partial_t h^{\xi,\kappa}|(t,x,v) \lesssim \sum_{\substack{|\beta| \leq N-2 \\ \beta_H \leq |\xi|}} \langle x \rangle^{|\xi|-\beta_H} |\partial_t \widehat{Z}^{\mathrm{mod},\beta} f|(t, X_{\mathscr{C}}, v) + \bar{\epsilon} \frac{\log(t)}{t^2} \langle x \rangle^{|\xi|-\beta_H} |\widehat{Z}^{\mathrm{mod},\beta} f|(t, X_{\mathscr{C}}, v).$$

Next, we recall from (66) the inequality $\langle x \rangle \lesssim \log(t) z(t, X_{\mathscr{C}}, v)$ and note, using the same arguments, that $z(t, X_{\mathscr{C}}, v) \lesssim \log(t)\langle x \rangle$ holds as well. Bounding $\partial_t \widehat{Z}^{\mathrm{mod},\beta} f$ by Proposition 6.22, we then get

$$\begin{aligned} |v^0|^{N_v-10} \langle x \rangle^{N_x-4-|\xi|} |\partial_t h^{\kappa,\xi}|(t,x,v) \lesssim & \sum_{\substack{|\beta| \leq N-2 \\ \beta_H \leq |\xi|}} \log^{N_x}(t) |v^0|^{N_v-10} |z^{N_x-4-\beta_H} \boldsymbol{T}_F(\widehat{Z}^{\mathrm{mod},\beta} f)|(t, X_{\mathscr{C}}, v) \\ & + \Lambda \frac{\log^{4N_x+3N}(t)}{t^{1+\delta}} \sum_{|\gamma| \leq 1} |v^0|^{N_v-3} |z^{N_x-2-\beta_H} \widehat{Z}^\gamma \widehat{Z}^{\mathrm{mod},\beta} f|(t, X_{\mathscr{C}}, v). \end{aligned}$$

We control the first term on the right-hand side by Proposition 6.28 and we claim that the second one is bounded by

$$\Lambda \frac{\log^{4N_x+4N}(t)}{t^{1+\delta}} \sum_{|\kappa| \leq N-1} |v^0|^{N_v-3} |z^{N_x-2} \widehat{Z}^\kappa f|(t, X_{\mathscr{C}}, v).$$

Indeed, we rewrite the modified vector fields using (69) and we control $P_{p,q}(\mathscr{C})$ by (70). We then deduce from Proposition 4.5 that

$$|v^0|^{N_v-10} \langle x \rangle^{N_x-4-|\xi|} |\partial_t h^{\kappa,\xi}|(t,x,v) \lesssim \bar{\epsilon} \frac{\log^{7N_x+7N}(t)}{t^{1+\delta}}.$$

We obtain from that,

$$\forall\, 3 \leq t \leq \tau, \quad \left| |v^0|^{N_v-10} \langle x \rangle^{N_x-4-|\xi|} (h^{\kappa,\xi}(\tau,x,v) - h^{\kappa,\xi}(t,x,v)) \right| \lesssim \bar{\epsilon} \frac{\log^{7(N_x+N)}(t)}{t^\delta}. \tag{74}$$

Consequently, there exists $f_\infty^{\kappa,\xi} \in L^\infty_{x,v}$ such that $h^{\kappa,\xi}(t,\cdot,\cdot) \to f_\infty^{\kappa,\xi}$ as $t \to +\infty$, uniformly on any compact subset of $\mathbb{R}^3_x \times \mathbb{R}^3_v$. By uniqueness of the limit in $\mathcal{D}'(\mathbb{R}^3_x \times \mathbb{R}^3_v)$ and by continuity of the distributional partial derivatives, we get $f_\infty^{\kappa,\xi} = |v^0|^{|\xi|} \partial_v^\xi \partial_x^\kappa f_\infty$. Letting $\tau \to +\infty$ in (74) yields the stated rate of convergence and concludes the proof. □

Remark 6.35. We can improve the result for f_∞. Propositions 4.5 and 6.22 give,

$$\forall t \geq 3, \quad \left\| |v^0|^{N_v-7} \langle x \rangle^{N_x-2} (f(t, X_{\mathscr{C}}(t,\cdot,\cdot),\cdot) - f_\infty) \right\|_{L^\infty_{x,v}} \lesssim \bar{\epsilon} \frac{\log^{12+3N_x+3N}(t)}{t^\delta}.$$

Moreover, we could prove that f_∞ is of class C^{N-1} according to the spatial variable x.

Remark 6.36. We could prove that $\partial_v^\xi(\partial_t^n \partial_x^\kappa f(t, X_{\mathscr{C}}, v)) \to \partial_v^\xi (-\hat{v} \cdot \nabla_x)^n \partial_x^\kappa f_\infty$. The idea consists in rewriting the time derivatives using $\partial_t = -\hat{v} \cdot \nabla_x + \boldsymbol{T}_F - \hat{v}^\mu F_\mu{}^j \partial_{v^j}$.

7. Scattering result for the electromagnetic field

In this section, we start by defining the scattering state of a sufficiently regular Maxwell field. Then, we construct a scattering map for the vacuum Maxwell equations. Finally, we apply these results together with the estimates derived in Section 3.1 in order to prove that the electromagnetic field F scatters, in the sense that it is approached by a solution to the homogeneous Maxwell equations.

Since the asymptotic states will be functions of the variables (u, θ, φ), defined on future null infinity $\mathcal{I}^+$ introduced in Section 2.2, it will be convenient to work in null coordinates. For a function $\psi(t, x)$, in order to simplify the presentation, we will write $\psi(u, \underline{u}, \omega)$ to denote $\psi((\underline{u}+u)/2, (\underline{u}-u)\omega/2)$, where $(u, \underline{u}, \omega)$ are the null coordinates such that $x = r\omega$, $\underline{u} = t + r$ and $u = t - r$.

The scattering state of a smooth electromagnetic field G will give the leading-order term in the asymptotic expansion of rG, as $\underline{u} \to +\infty$. This motivates the introduction of the following terminology.

Definition 7.1. Let $\phi : \mathbb{R}_+ \times \mathbb{R}^3 \to \mathbb{R}$ be a function such that the limit

$$\Phi(u, \omega) := \lim_{r \to +\infty} r\phi(u + r, r\omega) = \lim_{\underline{u} \to +\infty} (r\phi)(u, \underline{u}, \omega), \qquad \Phi(u, \omega) < +\infty,$$

exists and is finite for all $(u, \omega) \in \mathbb{R}_u \times \mathbb{S}^2$. Then, we say that the function Φ, defined on $\mathbb{R}_u \times \mathbb{S}^2$, is the radiation field $\mathscr{R}(\phi)$ of ϕ along future null infinity $\mathcal{I}^+$.

Definition 7.2. Similarly, consider β, a 1-form on $\mathbb{R}_+ \times \mathbb{R}^3$ tangential to the 2-spheres[12] such that β_{e_θ} and β_{e_φ} have a radiation field $\beta^{\mathcal{I}^+}_{e_\theta}$ and $\beta^{\mathcal{I}^+}_{e_\varphi}$. Then, $\beta^{\mathcal{I}^+}$, defined on $\mathbb{R}_u \times \mathbb{S}^2$ as the 1-form $\beta^{\mathcal{I}^+}_{e_\theta} \mathrm{d}\theta + \beta^{\mathcal{I}^+}_{e_\varphi} \mathrm{d}\varphi$ tangential to the 2-spheres, is called the radiation field of β along $\mathcal{I}^+$.

If $\beta^{\mathcal{I}^+}$ is of class C^1, we define

$$\nabla_{\partial_u}(\beta) := \partial_u(\beta^{\mathcal{I}^+}_{e_\theta})\mathrm{d}\theta + \partial_u(\beta^{\mathcal{I}^+}_{e_\varphi})\mathrm{d}\varphi, \quad \not{\nabla}_{e_\theta}(\beta)(u,\cdot,\cdot) := \not{\nabla}_{e_\theta}(\beta(u,\cdot,\cdot)), \quad \not{\nabla}_{e_\varphi}(\beta)(u,\cdot,\cdot) := \not{\nabla}_{e_\varphi}(\beta(u,\cdot,\cdot)),$$

where $\not{\nabla}$ denotes the covariant derivative on $\mathbb{S}^2$.

We already know from Corollary 2.20 that, given a sufficiently decaying electromagnetic field G, the radiation field of the good null components $\alpha(G)$, $\rho(G)$ and $\sigma(G)$ exist and vanish. Concerning the component $\underline{\alpha}(G)$, we have the following result.

Proposition 7.3. *Let G be a C^1 solution to the Maxwell equations* (18) *with a continuous source term J. Assume that there exist three constants $C[G] > 0$, $p \in \mathbb{N}$ and $q > 0$ such that, for all $(t, x) \in \mathbb{R}_+ \times \mathbb{R}^3$,*

$$r|J|(t, x) + \sum_{|\gamma| \le 1} |\rho(\mathcal{L}_{Z^\gamma} G)|(t, x) + |\sigma(\mathcal{L}_{Z^\gamma} G)|(t, x) \le \frac{C[G] \log^p(3 + t + |x|)}{(1 + t + |x|)^{1+q}}. \tag{75}$$

Then, $\underline{\alpha}(G)$ has a radiation field along $\mathcal{I}^+$. For any $B \in \{\theta, \varphi\}$ and for all $(u, \omega) \in \mathbb{R}_u \times \mathbb{S}^2$, the limit

$$\underline{\alpha}^{\mathcal{I}^+}_{e_B}(u, \omega) := \lim_{r \to +\infty} r\underline{\alpha}(G)_{e_B}(r + u, r\omega) = \lim_{\underline{u} \to +\infty} r\underline{\alpha}(G)_{e_B}(u, \underline{u}, \omega)$$

[12] More generally, we could consider tensor fields tangential to the cones $\underline{C}_{\underline{u}}$.

exists and is finite. Moreover,

$$\forall (t,x)\in \mathbb{R}_+\times\mathbb{R}^3, \qquad \left| r\underline{\alpha}(G)_{e_B}(t,x)-\underline{\alpha}^{\mathcal{I}^+}_{e_B}\left(t-|x|,\frac{x}{|x|}\right)\right| \lesssim C[G]\frac{\log^p(3+t+|x|)}{(1+t+|x|)^q}.$$

Consequently, $\underline{\alpha}^{\mathcal{I}^+}$ *is a continuous tensor field*, *defined on* $\mathbb{R}_u\times\mathbb{S}^2$ *and tangential to the* 2-*spheres*.

Proof. The last inequality of Lemma 2.17, together with (75), provides,

$$\forall (t,x)\in \mathbb{R}_+\times\mathbb{R}^3, \quad |\nabla_L(r\underline{\alpha}(G))|(t,x)\lesssim \log^p(3+t+|x|)(1+t+|x|)^{-1-q}. \tag{76}$$

Using the null coordinates $\underline{u}=t+r$ and $u=t-r$, where $x=r\omega$, we get, as $L=2\partial_{\underline{u}}$ and $\nabla_L e_B=0$,

$$\forall\, 0\le \underline{u}\le \underline{z}, \quad |r\underline{\alpha}(F)(u,\underline{z},\omega)-r\underline{\alpha}(F)(u,\underline{u},\omega)|\lesssim \int_{s=\underline{u}}^{\underline{z}}\frac{\log^p(3+s)\,\mathrm{d}s}{(1+s)^{1+q}}\lesssim\frac{\log^p(3+\underline{u})}{(1+\underline{u})^q},$$

implying the existence of $\underline{\alpha}^{\mathcal{I}^+}_{e_B}$, for any $B\in\{\theta,\varphi\}$, and the rate of convergence given in the statement. □

If the electromagnetic field is sufficiently regular, we can relate the radiation fields of the derivatives of G to the ones of $\underline{\alpha}^{\mathcal{I}^+}$. For this, we will use the bounded functions $\omega_i:=x^i/|x|$ and $\omega_i^A:=\langle\partial_{x^i},e_A\rangle$, where $1\le i\le 3$ and $A\in\{\theta,\varphi\}$, which depend only on $\omega\in\mathbb{S}^2$ and which are given explicitly in Appendix B.

Proposition 7.4. *Suppose that G satisfies*, *in addition to the hypotheses of the previous Proposition 7.3*, *the inequality* $|rG|(t,x)\le C[G]$. *Then*, *for any* $Z\in\mathbb{K}$,

$$\exists\, \underline{\alpha}^{\mathcal{I}^+}_Z\in\mathcal{D}'(\mathbb{R}_u\times\mathbb{S}^2), \quad r\underline{\alpha}(\mathcal{L}_ZG)(\cdot,\underline{u},\cdot)\xrightarrow[\underline{u}\to+\infty]{}\underline{\alpha}^{\mathcal{I}^+}_Z \quad \text{in } \mathcal{D}'(\mathbb{R}_u\times\mathbb{S}^2).$$

Moreover, *for any* $1\le i\le 3$ *and* $1\le j<k\le 3$,

$$\underline{\alpha}^{\mathcal{I}^+}_{\partial_t}=\nabla_u\underline{\alpha}^{\mathcal{I}^+}, \qquad \underline{\alpha}^{\mathcal{I}^+}_{\partial_{x^i}}=-\omega_i\nabla_u\underline{\alpha}^{\mathcal{I}^+}, \quad \underline{\alpha}^{\mathcal{I}^+}_S=u\nabla_u\underline{\alpha}^{\mathcal{I}^+}+\underline{\alpha}^{\mathcal{I}^+},$$
$$\underline{\alpha}^{\mathcal{I}^+}_{\Omega_{jk}}=\mathcal{L}_{\Omega_{jk}}(\underline{\alpha}^{\mathcal{I}^+}), \quad \underline{\alpha}^{\mathcal{I}^+}_{\Omega_{0i}}=-\omega_i u\nabla_u\underline{\alpha}^{\mathcal{I}^+}-2\omega_i\underline{\alpha}^{\mathcal{I}^+}+\omega_i^{e_A}\nabla\!\!\!/_{e_A}\underline{\alpha}^{\mathcal{I}^+}.$$

This result is proved in Appendix B.

7.1. ***Scattering map for the vacuum Maxwell equations.*** Before starting the construction of the forward map for the homogeneous Maxwell equations, we introduce two functional spaces adapted to our problem. The first one contains the initial electromagnetic fields which are in L^2 and the second one contains the scattering states which belong to L^2. For a smooth solution F to (19), this state will be the radiation field of $\underline{\alpha}(F)$. Note that the electromagnetic fields considered in this subsection will be denoted by F. Since, we will only consider solutions to the homogeneous Maxwell equations here, there is no risk of confusion with the electromagnetic field of the plasma considered in the remainder of the article.

Definition 7.5. Let $\mathcal{E}_{\{t=0\}}$ be the set containing all the 2-form on $\mathbb{R}^{1+3}$ which does not depend on t and which is in $L^2(\mathbb{R}^3)$. Equipped with the norm

$$\|F_0\|^2_{\mathcal{E}_{\{t=0\}}}:=\int_{\mathbb{R}^3_x}\big(|\alpha(F_0)|^2+|\underline{\alpha}(F_0)|^2+2|\rho(F_0)|^2+2|\sigma(F_0)|^2\big)(x)\,\mathrm{d}x,$$

$\mathcal{E}_{\{t=0\}}$ is a Hilbert space.

We define $\mathcal{E}_{\mathcal{I}^+}$ as the set of the 1-forms on $\mathbb{R}_u \times \mathbb{S}^2$ which are tangential to the 2-spheres and in L^2. For

$$\|\underline{\alpha}^{\mathcal{I}^+}\|^2_{\mathcal{I}^+} := \int_{\mathbb{R}_u}\int_{\mathbb{S}^2_\omega} |\underline{\alpha}^{\mathcal{I}^+}|^2(u,\omega)\,\mathrm{d}\mu_{\mathbb{S}^2}\,\mathrm{d}u,$$

$(\mathcal{E}_{\mathcal{I}^+}, \|\cdot\|_{\mathcal{I}^+})$ is a Hilbert space.

We now state the two main results of this section.

Theorem 7.6. *The linear map*

$$\mathscr{F}^+ : \mathcal{E}_{\{t=0\}} \cap C^\infty_c \to \mathcal{E}_{\mathcal{I}^+}, \qquad F_0 \mapsto \lim_{\underline{u}\to+\infty} r\underline{\alpha}(F)(u,\underline{u},\omega),$$

where F is the unique solution to the vacuum Maxwell equations (19) *such that $F(0,\cdot) = F_0$, is well-defined and preserves the norm $\|F_0\|_{\mathcal{E}_{\{t=0\}}} = \|\mathscr{F}^+(F_0)\|_{\mathcal{I}^+}$.*

Moreover, this forward map can be uniquely extended in a bijective isometry $\mathscr{F}^+ : \mathcal{E}_{\{t=0\}} \to \mathcal{E}_{\mathcal{I}^+}$.

Remark 7.7. When $F_0 \notin C^\infty_c$ but is still sufficiently regular, $\mathscr{F}^+(F_0)$ is also given by the formula written in Theorem 7.6. Otherwise, $\mathscr{F}^+(F_0)$ can still be interpreted, in a weak sense, as the radiation field of $\underline{\alpha}(F)$, with F the solution to (19) arising from the data F_0 (see Lemma 7.9 below).

The proof will in particular rely on the following result, which is also important in itself. It provides precise estimates for solutions arising from the preimage by $\mathscr{F}^+$ of smooth elements of $\mathcal{E}_{\mathcal{I}^+}$.

Proposition 7.8. *Let $0 < a < \frac{1}{2}$, $N \in \mathbb{N}$ and $\underline{\alpha}^{\mathcal{I}^+} \in \mathcal{E}_{\mathcal{I}^+}$ be a sufficiently regular scattering state. Then, the unique solution F to the vacuum Maxwell equations* (19) *satisfying $\mathscr{F}^+(F) = \underline{\alpha}^{\mathcal{I}^+}$ satisfies, for any $0 \leq q - \frac{1}{2} < a$,*

$$\sum_{|\gamma|\leq N} \|\langle t-r\rangle^{q-\frac{1}{2}} |\mathcal{L}_{Z^\gamma} F|(t,\cdot)\|^2_{L^2_x} \lesssim C[\underline{\alpha}^{\mathcal{I}^+}] := \sum_{n_1+n_2+n_3\leq N+3} \int_{\mathbb{R}_u}\int_{\mathbb{S}^2_\omega} \langle u\rangle^{2a+2n_1} |\nabla_u^{n_1} \not\nabla_{e_\theta}^{n_2} \not\nabla_{e_\varphi}^{n_3} \underline{\alpha}^{\mathcal{I}^+}|^2(u,\omega)\,\mathrm{d}\mu_{\mathbb{S}^2}\,\mathrm{d}u$$

for all $t \in \mathbb{R}_+$. In particular, if $N \geq 4$, we have, for any $|\gamma| \leq N-3$ and $|\xi| \leq N-4$,

$$\forall (t,x) \in \mathbb{R}_+ \times \mathbb{R}^3, \quad \big(|\alpha(\mathcal{L}_{Z^\gamma} F)| + |\rho(\mathcal{L}_{Z^\gamma} F)| + |\sigma(\mathcal{L}_{Z^\gamma} F)|\big)(t,x) \leq \frac{C}{(1+t+|x|)^{1+q}},$$

$$\left| r\underline{\alpha}(\mathcal{L}_{Z^\xi} F)(t,x) - \mathscr{F}^+(\mathcal{L}_{Z^\xi} F(0,\cdot))\left(t-|x|, \frac{x}{|x|}\right)\right| \leq \frac{C}{(1+t+|x|)^q},$$

where the constant C depends only on $C[\underline{\alpha}^{\mathcal{I}^+}]$ and q.

We start by proving that $\mathscr{F}^+$ is well-defined for sufficiently regular electromagnetic field, including those arising from smooth compactly supported data.

Lemma 7.9. *The linear map $\mathscr{F}^+$ introduced in Theorem 7.6 is well-defined and extends in an injective isometry from $\mathcal{E}_{\{t=0\}}$ to $\mathcal{E}_{\mathcal{I}^+}$. Moreover, if F is a solution to the free Maxwell equations* (19) *such that*

$$C_F := \sum_{|\gamma|\leq 4} \|\mathcal{L}_{Z^\gamma} F(0,\cdot)\|_{\{t=0\}} < +\infty, \tag{77}$$

then, $\underline{\alpha}(F)$ has a continuous radiation field $\mathscr{F}^+(F(0,\cdot))$ and, for all $(t,x) \in \mathbb{R}_+ \times \mathbb{R}^3$,

$$(|\alpha(F)| + |\rho(F)| + |\sigma(F)|)(t,x) \lesssim C_F(1+t+|x|)^{-\frac{3}{2}}, \tag{78}$$

$$\left| r\underline{\alpha}(F)(t,x) - \mathscr{F}^+(F(0,\cdot))\left(t-|x|, \frac{x}{|x|}\right)\right| \lesssim C_F(1+t+|x|)^{-\frac{1}{2}}. \tag{79}$$

This implies that the radiation fields of $\alpha(F)$, $\rho(F)$ and $\sigma(F)$ vanish.

Finally, if F is a mildly regular solution to (19) *such that $F(0,\cdot) \in \mathcal{E}_{\{t=0\}}$, then $r\underline{\alpha}(F)$ converges to $\mathscr{F}^+(F(0,\cdot))$, as $\underline{u} \to +\infty$, in the space of distributions $\mathcal{D}'(\mathbb{R}_u \times \mathbb{S}^2)$.*

Proof. Recall from Definition 2.16 the energy momentum tensor $\mathbb{T}[F]_{\mu\nu}$, its principal null components and that $\nabla^\mu \mathbb{T}[F]_{\mu 0} = 0$. For any $t > 0$, the divergence theorem, applied to $\mathbb{T}[F]_{\mu 0}$ in the domain $\{(s,x) \in \mathbb{R}^{1+3} \mid 0 \leq s \leq t\}$, gives

$$\begin{aligned}\|F(0,\cdot)\|_{\{t=0\}} &= 4\int_{\mathbb{R}^3_x} \mathbb{T}[F]_{00}(0,x)\,\mathrm{d}x = 4\int_{\mathbb{R}^3_x} \mathbb{T}[F]_{00}(t,x)\,\mathrm{d}x \\ &= 2\sum_{0\leq\mu,\nu\leq 3}\int_{\mathbb{R}^3_x} |F_{\mu\nu}|^2(t,x)\,\mathrm{d}x = 2\|F(t,\cdot)\|_{L^2_x}.\end{aligned}$$

This also applies to $\mathcal{L}_{Z^\gamma}(F)$, for any $|\gamma| \leq 4$, since it is a solution to the free Maxwell equations (19) as well. In view of the equivalence of the pointwise norms (9), the standard Klainerman–Sobolev inequality (see for instance Theorem 1.3 of [Sogge 1995, Chapter II]) yields, for any $|\gamma| \leq 2$,

$$\begin{aligned}\forall (t,x) \in \mathbb{R}_+ \times \mathbb{R}^3_x, \quad |\mathcal{L}_{Z^\gamma} F|(t,x) &\lesssim \sum_{|\beta| \leq 2+|\gamma|}\sum_{0\leq\mu,\nu\leq 3} |Z^\beta(F_{\mu\nu})|(t,x) \\ &\lesssim \frac{C_F}{(1+t+|x|)(1+|t-|x||)^{\frac{1}{2}}}.\end{aligned} \tag{80}$$

Applying Corollary 2.20 to $\mathcal{L}_{Z^\xi} F$, for any $|\xi| \leq 1$ and $q = \frac{1}{2}$, gives,

$$\forall |\xi| \leq 1,\ \forall (t,x) \in \mathbb{R}_+ \times \mathbb{R}^3, \quad \big(|\alpha(\mathcal{L}_{Z^\xi} F)| + |\rho(\mathcal{L}_{Z^\xi} F)| + |\sigma(\mathcal{L}_{Z^\xi} F)|\big)(t,x) \leq C_F(1+t+|x|)^{-\frac{3}{2}}.$$

The existence of the radiation field $\underline{\alpha}^{\mathcal{I}^+}$ of $\underline{\alpha}(F)$ and the rate of convergence given in the statement then follows from Proposition 7.3. Since the convergence is uniform in (u,ω), $\underline{\alpha}^{\mathcal{I}^+}$ is continuous on $\mathbb{R}_u \times \mathbb{S}^2$.

Before defining $\mathscr{F}^+$, we need to bound the L^2 norm of the radiation field. For this, we prove conservation laws which hold for any mildly regular solution G to the free Maxwell equations (19).

Fix $\underline{u} \geq 0$ and apply the divergence theorem to $\mathbb{T}[G]_{\mu 0}$, in the domain $\{t + |x| \leq \underline{u}\}$, in order to get

$$\begin{aligned}\int_{\underline{C}_{\underline{u}}} \mathbb{T}[G]_{\underline{L}0}\,\mathrm{d}\mu_{\underline{C}_{\underline{u}}} &= \int_{|x|\leq \underline{u}} \mathbb{T}[G]_{00}(0,x)\,\mathrm{d}x \\ &= \frac{1}{4}\int_{|x|\leq\underline{u}} \big(|\alpha(G)|^2 + |\underline{\alpha}(G)|^2 + 2|\rho(F)|^2 + 2|\sigma(G)|^2\big)(0,x)\,\mathrm{d}x,\end{aligned} \tag{81}$$

where

$$\int_{\underline{C}_{\underline{u}}} \mathbb{T}[G]_{\underline{L}0}\,\mathrm{d}\mu_{\underline{C}_{\underline{u}}} = \frac{1}{4}\int_{|u|\leq\underline{u}}\int_{\mathbb{S}^2_\omega} (|\underline{\alpha}(G)|^2 + |\rho(G)|^2 + |\sigma(G)|^2)(u,\underline{u},\omega) r^2\,\mathrm{d}\mu_{\mathbb{S}^2}\,\mathrm{d}u. \tag{82}$$

Assume now that $F_{\mu\nu}(0,\cdot) \in C_c^\infty(\mathbb{R}^3_x)$ for all $0 \leq \mu, \nu \leq 3$ and let us apply the previous equality to F. On the one hand, the right-hand side of (81) converges to $\frac{1}{4}\|F(0,\cdot)\|^2_{\{t=0\}}$ as $\underline{u} \to +\infty$. On the other hand, we know from the Huygens–Fresnel principle that there exists $U > 0$ such that $F(t,x) = 0$ for all $|t-|x|| = |u| \geq U$. This implies that the domain of integration of the integrals in (82) is in fact included in $\{|u| \leq U\}$ for all $\underline{u} \geq 0$. The triangular inequality in L^2 together with the estimates (78)–(79) then leads to

$$\int_{\underline{C}_{\underline{u}}} \mathbb{T}[F]_{\underline{L}0}\,\mathrm{d}\mu_{\underline{C}_{\underline{u}}} \xrightarrow[\underline{u}\to+\infty]{} \frac{1}{4}\int_{|u|\leq U}\int_{\mathbb{S}^2_\omega} |\mathscr{F}^+(F(0,\cdot))|^2\,\mathrm{d}\mu_{\mathbb{S}^2}\,\mathrm{d}u = \frac{1}{4}\|\underline{\alpha}^{\mathcal{I}^+}\|^2_{\mathcal{I}^+}.$$

We can then define $\mathscr{F}^+ : \mathcal{E}_{\{t=0\}} \cap C_c^\infty \to \mathcal{E}_{\mathcal{I}^+}$, with $\mathscr{F}^+(F(0,\cdot)) := \underline{\alpha}^{\mathcal{I}^+}$, and extend it to an injective isometry from $\mathcal{E}_{\{t=0\}}$ to $\mathcal{E}_{\mathcal{I}^+}$.

Consider now a, say, C^1 solution F to (19) such that $F(0,\cdot) \in \mathcal{E}_{\{t=0\}}$. Fix $\psi \in C_c^\infty(\mathbb{R}_u \times \mathbb{S}^2)$ and $R > 0$ satisfying $\mathrm{supp}(\psi) \subset [-R, R] \times \mathbb{S}^2$. Let further $(F_n)_{n\geq 0}$ be a sequence of smooth solutions to the vacuum Maxwell equations such that $F_n(0,\cdot)$ is compactly supported for any $n \in \mathbb{N}$ and $F_n(0,\cdot) \to F(0,\cdot)$ in $\mathcal{E}_{\{t=0\}}$. Fix $A \in \{\theta, \varphi\}$ and start by observing that

$$\begin{aligned}|(r\underline{\alpha}(F)_{e_A} - \mathscr{F}^+(F(0,\cdot))_{e_A})\psi| \\ \lesssim \big(|r\underline{\alpha}(F) - r\underline{\alpha}(F_n)| + |r\underline{\alpha}(F_n) - \mathscr{F}^+(F_n(0,\cdot))| + |\mathscr{F}^+((F_n - F)(0,\cdot))|\big)\mathbb{1}_{|u|\leq R}.\end{aligned}$$

Then, in order to prove $r\underline{\alpha}_{e_A} \to \mathscr{F}^+(F(0,\cdot))_{e_A}$ in $\mathcal{D}'(\mathbb{R}_u \times \mathbb{S}^2)$, as $\underline{u} \to +\infty$, it suffices to prove that the integral on $\mathbb{R}_u \times \mathbb{S}^2$ of each of the three terms on the right-hand side converges to 0 as $\underline{u} \to +\infty$. For this, consider $\epsilon > 0$ and start by noticing that the energy equality (81)–(82), applied to $F - F_n$, gives,

$$\forall n \geq 0,\ \forall \underline{u} \geq 0, \qquad \int_{\mathbb{R}_u}\int_{\mathbb{S}^2_\omega} |r\underline{\alpha}(F) - r\underline{\alpha}(F_n)|^2(u,\underline{u},\omega)\,\mathrm{d}\mu_{\mathbb{S}^2}\,\mathrm{d}u \leq \|F(0,\cdot) - F_n(0,\cdot)\|^2_{\{t=0\}}.$$

According to (79), applied to F_n, there exists a constant C_n, such that,

$$\forall n \in \mathbb{N},\ \forall \underline{u} \geq 0, \qquad \int_{|u|\leq R}\int_{\mathbb{S}^2_\omega} |r\underline{\alpha}(F_n)(u,\underline{u},\omega) - \mathscr{F}^+(F_n(0,\cdot))(u,\omega)|\,\mathrm{d}\mu_{\mathbb{S}^2}\,\mathrm{d}u \leq \frac{C_n}{(1+\underline{u})^{\frac{1}{2}}}.$$

Moreover, since $\mathscr{F}^+$ is an isometry, we have $\|\mathscr{F}^+(F_n(0,\cdot)) - \mathscr{F}^+(F(0,\cdot))\|_{\mathcal{I}^+} = \|F(0,\cdot) - F_n(0,\cdot)\|_{\{t=0\}}$. The last four estimates, together with the Cauchy–Schwarz inequality in $L^2([-R,R]\times\mathbb{S}^2)$, yields

$$\left|\int_{\mathbb{R}_u}\int_{\mathbb{S}^2_\omega} \big(r\underline{\alpha}(F)_{e_A}(u,\underline{u},\omega) - \mathscr{F}^+(F)_{e_A}(u,\omega)\big)\psi(u,\omega)\,\mathrm{d}\mu_{\mathbb{S}^2}\,\mathrm{d}u\right| \lesssim \|F(0,\cdot) - F_n(0,\cdot)\|_{\{t=0\}} + \frac{C_n}{(1+\underline{u})^{\frac{1}{2}}}$$

for all $n \in \mathbb{N}$ and $\underline{u} \geq 0$. For a sufficiently large n and $\underline{U}$, which depends on n, we can bound the right-hand side by ϵ for all $\underline{u} \geq \underline{U}$. This concludes the proof of the last part of the lemma.

It remains to show that for any $F(0,\cdot)$ satisfying (77), we have $\mathscr{F}^+(F(0,\cdot)) = \underline{\alpha}^{\mathcal{I}^+}$. For this, it suffices to recall that we proved $r\underline{\alpha}(F) \to \underline{\alpha}^{\mathcal{I}^+}$ in $L^\infty_{u,\omega}$. □

Remark 7.10. In fact, assuming more decay on the initial data, we could prove using the equations (M_2''), (M_5'') and (M_6'') of [Christodoulou and Klainerman 1990] that $|\alpha(F)| = O(\underline{u}^{-2-\delta})$ and that $r^2\rho(F)$ as well as $r^2\sigma(F)$ converge as $\underline{u} \to +\infty$.

To conclude the proof of Theorem 7.6, it remains us to show that $\mathscr{F}^+$ is surjective. For this, it suffices to prove Proposition 7.8, which in particular implies that any smooth and compactly supported $\underline{\alpha}^{\mathcal{I}^+}$ has a preimage by $\mathscr{F}^+$. For this, we will make crucial use of [Lindblad and Schlue 2023, Theorem 1.1], which is a similar result for solutions to the homogeneous wave equation, and exploit that $\Box F_{\mu\nu} = 0$ for any Cartesian component $F_{\mu\nu}$.

Lemma 7.11. *Let $\Phi \in C(\mathbb{R}_u \times \mathbb{S}^2)$ be a sufficiently regular function, $0 < a < \frac{1}{2}$ and $N \in \mathbb{N}$. Then, there exists a unique solution to wave equation $\Box\phi = 0$ on $\mathbb{R}_+ \times \mathbb{R}^3$ satisfying, for any $0 < \delta \leq a$ and all $t \in \mathbb{R}_+$,*

$$\sum_{|\gamma|\leq N} \|\langle t-r\rangle^{a-\delta} Z^\gamma \phi(t,\cdot)\|^2_{L^2(\mathbb{R}^3_x)} \lesssim \sum_{|k|+|\beta|\leq N+3} \int_{u=-\infty}^{+\infty} \int_{\omega\in\mathbb{S}^2} |(\langle u\rangle\partial_u)^k \partial_\omega^\beta \Phi(u,\omega)|^2 \langle u\rangle^{2a} \, \mathrm{d}\mu_{\mathbb{S}^2} \, \mathrm{d}u$$

and such that Φ is the radiation field $\mathscr{R}(\phi)$ of ϕ along $\mathcal{I}^+$.

We will also require standard estimates for smooth solutions to the wave equation.

Lemma 7.12. *Let ϕ be a smooth solution to the wave equation $\Box\phi = 0$ such that $\|Z^\gamma\phi(0,\cdot)\|_{L^2_x} < +\infty$ for any $|\gamma| \leq 5$. Then, for any $|\beta| \leq 1$, the radiation field $\mathscr{R}(\partial^\beta_{t,x}\phi)$ of $\partial^\beta_{t,x}\phi$ is well-defined and*

$$\forall \underline{u} \geq 1,\ \forall (u,\omega) \in [-\underline{u},\underline{u}] \times \mathbb{S}^2, \quad |r\partial^\beta_{t,x}\phi(u,\underline{u},\omega) - \mathscr{R}(\partial^\beta_{t,x}\phi)(u,\omega)| \lesssim \underline{u}^{-\frac{1}{2}}.$$

Moreover, $\mathscr{R}(\partial_t\phi) = \partial_u\mathscr{R}(\phi)$ and $\mathscr{R}(\partial_{x^i}\phi) = -(x^i/|x|)\partial_u\mathscr{R}(\phi)$ for all $i \in [\![1,3]\!]$.

Proof. The first part of the result is classical. Indeed, since $\Box Z^\gamma\phi = 0$ for any $|\gamma| \leq 4$, we obtain by applying the standard Klainerman–Sobolev inequality and then an energy inequality (for a proof, see for instance Theorem 1.3 and Lemma 3.5 of [Sogge 1995, Chapter II]), that, for all $|\gamma| \leq 2$ and $(t,x) \in \mathbb{R}_+ \times \mathbb{R}^3$,

$$(1+t+|x|)(1+|t-|x||)^{\frac{1}{2}}|Z^\gamma\phi|(t,x) \lesssim \sum_{|\beta|\leq|\gamma|+2} \|Z^\beta\phi(t,\cdot)\|_{L^2_x} \lesssim \sum_{|\beta|\leq 4} \|Z^\beta\phi(0,\cdot)\|_{L^2_x}. \quad (83)$$

Now we claim that,

$$\forall (t,x) \in \mathbb{R}_+ \times \mathbb{R}^3, \quad |L(r\phi)|(t,x) \lesssim (1+t+|x|)^{-\frac{3}{2}}.$$

Indeed, if $|x| = r \leq (1+t)/2$, we have $1+t+r \lesssim 1+|t-r|$. Moreover, (20) leads to $|L(r\phi)| \leq \sum_{|\beta|\leq 1} |Z^\beta\phi|$, so that the claim is implied by (83). Otherwise, $|x| \gtrsim 1+t+|x| = 1+\underline{u}$ and, by writing the d'Alembertian in spherical coordinates, we obtain from $\Box\phi = 0$ that

$$0 = -L\underline{L}\phi + \frac{2}{r}\frac{L-\underline{L}}{2}\phi + \sum_{1\leq i<j\leq 3} \frac{\Omega_{ij}\Omega_{ij}\phi}{r^2}, \quad \text{leading to} \quad \underline{L}(L(r\phi)) = \sum_{1\leq i<j\leq 3} \frac{\Omega_{ij}\Omega_{ij}\phi}{r}. \quad (84)$$

In order to integrate along a null straight line $t+r = \underline{u}$, it will be convenient to work with the null coordinate system. We then write $x = |x|\omega$, with $\omega \in \mathbb{S}^2$. As $\underline{L} = 2\partial_u$ and in view of (83)–(84), we have

$$|L(r\phi)|(t,x) = |L(r\phi)|(t-|x|,\underline{u},\omega) \leq |L(r\phi)|(-t-|x|,\underline{u},\omega) + \frac{1}{2}\int_{u=-t-|x|}^{t-|x|} |\underline{L}(L(r\phi))|(u,\underline{u},\omega)\,\mathrm{d}u$$

$$\lesssim |L(r\phi)|(0,(t+|x|)\omega) + \int_{u=-t-|x|}^{t-|x|} \frac{\mathrm{d}u}{(1+\underline{u})^2(1+|u|)^{\frac{1}{2}}} \lesssim (1+\underline{u})^{-\frac{3}{2}},$$

which concludes the proof of the claim. As $L = 2\partial_{\underline{u}}$, we directly deduce from it that,

$$\forall \underline{z} \geq \underline{u} \geq 0,\ \forall |u| \leq \underline{u},\ \forall \omega \in \mathbb{S}^2, \quad |r\phi(u,\underline{z},\omega) - r\phi(u,\underline{u},\omega)| \lesssim \int_{s=\underline{u}}^{\underline{z}} |L(r\phi)|(u,s,\omega)\,\mathrm{d}s \lesssim (1+\underline{u})^{-\frac{1}{2}}.$$

This implies the existence of the radiation field $\mathscr{R}(\phi)$ of ϕ as well as the rate of convergence given in the statement of the lemma. Since $\Box \partial_{x^\mu}\phi = 0$ and $\|Z^\gamma \partial_{x^\mu}\phi(0,\cdot)\|_{L^2_x} < +\infty$ for any $|\gamma| \leq 4$, the same applies to $\partial_{x^\mu}\phi$. Now, note that

$$2r\partial_t\phi = rL\phi + r\underline{L}\phi, \quad 2r\partial_{x^i}\phi = \frac{x^i}{|x|} rL\phi - \frac{x^i}{|x|} r\underline{L}\phi + 2\langle \partial_{x^i}, e_\theta\rangle r e_\theta\phi + 2\langle \partial_{x^i}, e_\varphi\rangle r e_\varphi\phi, \quad 1 \leq i \leq 3.$$

Combining (83) with (20) yields $r|L\phi| + r|e_\theta\phi| + r|e_\varphi| + |\phi| \lesssim \underline{u}^{-1}$ so that

$$\text{there exists } \phi^{\underline{L}}_\infty \in L^\infty(\mathbb{R}_u \times \mathbb{S}^2_\omega) \quad \text{such that} \quad \underline{L}(r\phi) \xrightarrow[\underline{u}\to+\infty]{L^\infty_{u,\omega}} \phi^{\underline{L}}_\infty, \quad \phi^{\underline{L}}_\infty = 2\mathscr{R}(\partial_t\phi), \quad \frac{x^i}{|x|}\phi^{\underline{L}}_\infty = -2\mathscr{R}(\partial_{x^i}\phi).$$

It remains to use that $\underline{L}(r\phi)(\cdot,\underline{u},\cdot) \rightharpoonup 2\partial_u \mathscr{R}(\phi)$ in $\mathcal{D}'(\mathbb{R}_u \times \mathbb{S}^2)$ since $r\phi(\cdot,\underline{u},\cdot)$ converges to $\mathscr{R}(\phi)$ in $L^\infty_{u,\omega}$. □

We are now ready for the last part of this subsection.

Proof of Proposition 7.8. Fix $0 \leq q - \frac{1}{2} < a < \frac{1}{2}$, $N \in \mathbb{N}$ and $\underline{\alpha}^{\mathcal{I}^+} \in \mathcal{E}_{\mathcal{I}^+}$ such that the norm $C[\underline{\alpha}^{\mathcal{I}^+}]$ is finite. Recall that any sufficiently regular solution F to the vacuum Maxwell equations (19) satisfies $\Box F_{\mu\nu} = 0$ for any $0 \leq \mu, \nu \leq 3$. The first step consists in constructing each Cartesian component $F_{\mu\nu}$ of the electromagnetic field by applying Lemma 7.11 to well-chosen radiation fields. This will define a 2-form F which will verify the stated estimate. Then, we will prove that F is indeed a solution to the Maxwell equations and, finally, we will derive the pointwise decay estimates.

Assume first that $N \geq 5$ and let us start by identifying the expected radiation field of $F_{\mu\nu}$. For this, assume that F exists and recall the transfer matrix between the Cartesian and the null frame

$$\partial_t = \tfrac{1}{2}L + \tfrac{1}{2}\underline{L}, \quad \partial_{x^i} = \tfrac{1}{2}\omega_i L - \tfrac{1}{2}\omega_i \underline{L} + \omega_i^{e_\theta} e_\theta + \omega_i^{e_\varphi} e_\varphi, \quad 1 \leq i \leq 3,$$

where ω_i and $\omega_i^{e_A}$ are bounded functions of the spherical variables and are given explicitly in Appendix B. For convenience, we set $\omega_0 := -1$ and $\omega_0^{e_A} := 0$. Consequently, for any $0 \leq \mu, \nu \leq 3$, there exist smooth functions of $\omega \in \mathbb{S}^2$, $g^{\alpha,\theta}_{\mu\nu}$, $g^{\alpha,\varphi}_{\mu\nu}$, $g^\rho_{\mu\nu}$ and $g^\sigma_{\mu\nu}$, such that

$$rF_{\mu\nu} = -\tfrac{1}{2}(\omega_\mu^{e_A}\omega_\nu - \omega_\mu\omega_\nu^{e_A}) r\underline{\alpha}(F)_{e_A} + g^{\alpha,A}_{\mu\nu} r\alpha(F)_{e_A} + g^\rho_{\mu\nu} r\rho(F) + g^\sigma_{\mu\nu} r\sigma(F).$$

We then obtain by (78)–(79) that

$$\mathscr{R}(F_{\mu\nu}) = -\tfrac{1}{2}(\omega_\mu^{e_A}\omega_\nu - \omega_\mu\omega_\nu^{e_A})\underline{\alpha}^{\mathcal{I}^+}_{e_A}, \quad 0 \leq \mu, \nu \leq 3. \tag{85}$$

According to Lemma 7.11, we can indeed define a 2-form F satisfying (85) as well as $\Box F_{\mu\nu} = 0$ and, for all $t \in \mathbb{R}_+$,

$$\sum_{|\gamma|\leq N} \|\langle t-r\rangle^{q-\frac{1}{2}} |\mathcal{L}_{Z^\gamma} F|(t,\cdot)\|_{L^2_x} \lesssim \sum_{|\gamma|\leq N}\ \sum_{0\leq\mu,\nu\leq 3} \|\langle t-r\rangle^{q-\frac{1}{2}} Z^\gamma(F_{\mu\nu})(t,\cdot)\|_{L^2_x} \lesssim C[\underline{\alpha}^{\mathcal{I}^+}]. \tag{86}$$

The remainder of the proof of the case $N \geq 5$ essentially consists in performing linear algebra computations. In order to lighten the notations we temporarily denote ∂_{x^λ} by ∂_λ. Our goal now is to prove that F is a solution to the vacuum Maxwell equations (19), which read in Cartesian coordinates

$$\partial^\mu F_{\mu\nu} = 0, \quad \partial^{\mu *}F_{\mu\nu} = \partial_{[\lambda} F_{\mu\nu]} := \partial_\lambda F_{\mu\nu} + \partial_\mu F_{\nu\lambda} + \partial_\nu F_{\lambda\mu} = 0. \tag{87}$$

For a proof of the second identity, see for instance [Bigorgne 2021b, Lemma 2.2]. Since $\Box \partial^\mu F_{\mu\nu} = 0$ and $\Box \partial^{\mu *}F_{\mu\nu} = 0$, (87) would be implied, according to Lemma 7.11, by

$$\mathscr{R}(\partial^\mu F_{\mu\nu}) = 0, \quad \mathscr{R}(\partial^{\mu *}F_{\mu\nu}) = 0, \quad 0 \leq \nu \leq 3.$$

We compute, using Lemma 7.12, that, for any $0 \leq \lambda \leq 3$,

$$\mathscr{R}(\partial_\lambda F_{\mu\nu}) = -\omega_\lambda \partial_u \mathscr{R}(F_{\mu\nu}) = \tfrac{1}{2}\omega_\lambda(\omega_\mu^{e_A}\omega_\nu - \omega_\mu \omega_\nu^{e_A})\partial_u \underline{\alpha}_{e_A}^{\mathcal{I}^+}, \quad 0 \leq \mu, \nu \leq 3.$$

This implies in particular that $\mathscr{R}(\partial_{[\lambda} F_{\mu\nu]}) = 0$. Furthermore, as $\partial^\mu = \eta^{\mu\lambda}\partial_\lambda$, we have

$$\mathscr{R}(\partial^\mu F_{\mu\nu}) = \tfrac{1}{2}\eta^{\mu\lambda}\omega_\lambda(\omega_\mu^{e_A}\omega_\nu - \omega_\mu\omega_\nu^{e_A})\partial_u \underline{\alpha}_{e_A}^{\mathcal{I}^+} = \tfrac{1}{2}(\eta(e_A, L)\omega_\nu - \eta(L, L)\omega_\nu^{e_A})\partial_u \underline{\alpha}_{e_A}^{\mathcal{I}^+} = 0.$$

We then deduce that F is a smooth solution to the vacuum Maxwell equations. Finally, since the Cartesian components of $\underline{L} = \eta_{\underline{L}}{}^\mu \partial_\mu$ and $e_A = \eta_{e_A}{}^\mu \partial_\mu$ are bounded functions of $\omega \in \mathbb{S}^2$, we obtain from (85) and Lemmas 7.9, 7.12 that

$$\begin{aligned}\mathscr{F}^+(F(0,\cdot))_{e_A} &= \lim_{\underline{u}\to\infty} r\underline{\alpha}(F)_{e_A}(\cdot, \underline{u}, \cdot) \\ &= \eta_{e_A}{}^\mu \eta_{\underline{L}}{}^\nu \lim_{\underline{u}\to\infty} rF_{\mu\nu}(\cdot, \underline{u}, \cdot) = \eta_{e_A}{}^\mu \eta_{\underline{L}}{}^\nu \mathscr{R}(F_{\mu\nu}) = \underline{\alpha}_{e_A}^{\mathcal{I}^+}, \quad A \in \{\theta, \varphi\}.\end{aligned}$$

This concludes the proof of the first part of the proposition for the case $N \geq 5$. Consider now the case $N = 0$ and define similarly $F_{\mu\nu}$, through Lemma 7.11, as the unique solution to $\Box F_{\mu\nu} = 0$ such that (85) holds. This directly provides the estimate (86); let us prove that F is a weak solution to (19). For this, consider a sequence $(\underline{\alpha}_n^{I^+}) \in \mathcal{E}_{\mathcal{I}^+}^{\mathbb{N}}$ of smooth and compactly supported scattering states such that $C[\underline{\alpha}^{\mathcal{I}^+} - \underline{\alpha}_n^{I^+}] \to 0$ as $n \to +\infty$. Then, denote by F_n the unique smooth solution to the vacuum Maxwell equations such that $\mathscr{F}^+(F_n(0,\cdot)) = \underline{\alpha}_n^{\mathcal{I}^+}$. Applying once again Lemma 7.11 to $\mathscr{R}(F_{\mu\nu} - F_{n,\mu\nu})$ yields

$$\sup_{t\in\mathbb{R}_+} \|F(t,\cdot) - F_n(t,\cdot)\|_{L^2_x} \lesssim C[\underline{\alpha}^{I^+} - \underline{\alpha}_n^{I^+}]. \tag{88}$$

Fix $\psi \in C_c^\infty(\mathbb{R}_+ \times \mathbb{R}^3_x)$ and T_ψ such that $\psi(t,\cdot) = 0$ for all $t \geq T_\psi$. Note, since F_n is a classical and then a weak solution to (19), that for any $0 \leq \nu \leq 3$ and $n \in \mathbb{N}$,

$$\begin{aligned}&\left| \int_{\mathbb{R}_+\times\mathbb{R}^3_x} F_{\mu\nu}(t,x)\partial^\mu\psi(t,x)\,\mathrm{d}x\,\mathrm{d}t + \int_{\mathbb{R}^3_x} F_{\mu\nu}(0,x)\psi(0,x)\,\mathrm{d}x \right| \\ &\qquad = \left| \int_{\mathbb{R}_+\times\mathbb{R}^3_x} (F-F_n)_{\mu\nu}(t,x)\partial^\mu\psi(t,x)\,\mathrm{d}x\,\mathrm{d}t + \int_{\mathbb{R}^3_x} (F-F_n)_{\mu\nu}(0,x)\psi(0,x)\mathrm{d}x \right| \\ &\qquad \lesssim (1+T_\psi) \sup_{t\in\mathbb{R}_+} \|(F-F_n)(t,\cdot)\|_{L^2_x}.\end{aligned} \tag{89}$$

By (88), the right-hand side converges to 0 as $n \to +\infty$ whereas the left-hand side does not depend on n. This implies that (89) vanishes. The same applies to ${}^*F$, so that F is a weak solution to the vacuum Maxwell equations (19). Finally, by continuity of $\mathscr{F}^+$ and (88), $\mathscr{F}^+(F(0,\cdot)) = \underline{\alpha}^{\mathcal{I}^+}$.

We now focus on the second part of Proposition 7.8, which merely concerns the cases $N \geq 4$. We apply [Lindblad and Schlue 2023, Lemma 3.3], a weighted version of the standard Klainerman–Sobolev inequality, to $Z^\beta(F_{\mu\nu})$. Using (9), we obtain, for any $|\gamma| \leq N-2$ and all $(t,x) \in \mathbb{R}_+ \times \mathbb{R}^3$,

$$|\mathcal{L}_{Z^\gamma}(F)|(t,x) \lesssim \sum_{|\beta|\leq N-2} \sum_{0\leq \mu,\nu\leq 3} |Z^\beta(F_{\mu\nu})|(t,x) \lesssim \sum_{|\beta|\leq N} \frac{\|\langle t-r\rangle^{q-\frac{1}{2}} |\mathcal{L}_{Z^\beta}(F)|(t,\cdot)\|_{L^2_x}}{(1+t+|x|)(1+|t-|x||)^q}. \tag{90}$$

The numerator in the right-hand side is bounded by $C[\underline{\alpha}^{\mathcal{I}^+}]$. Recall now that $\mathcal{L}_{Z^\gamma}(F)$ is a solution to the vacuum Maxwell equations as well. To conclude the proof, it then suffices to use the previous estimate and to apply Corollary 2.20 to $\mathcal{L}_{Z^\gamma}(F)$ for any $|\gamma| \leq N-3$, as well as Proposition 7.3, to $\mathcal{L}_{Z^\xi}(F)$ for any $|\xi| \leq N-4$. □

Remark 7.13. A statement similar to Theorem 7.6 holds for scattering toward past null infinity $\mathcal{I}^- \cong \mathbb{R}_{\underline{u}} \times \mathbb{S}^2$. One can construct the past forward evolution bijective isometry $\mathscr{F}^- : \mathcal{E}_{\{t=0\}} \to \mathcal{E}_{\mathcal{I}^-}$, where, if $F(0,\cdot) \in \mathcal{E}_{\{t=0\}} \cap C_c^\infty$, $\mathscr{F}^-(F)(\underline{u},\omega) := \lim_{u\to-\infty} r\alpha(F)(u,\underline{u},\omega)$ and $\|\cdot\|_{\mathcal{I}^-} := \|\cdot\|_{L^2(\mathbb{R}_{\underline{u}}\times\mathbb{S}^2)}$. The scattering map $\mathscr{S} = (\mathscr{F}^-)^{-1} \circ \mathscr{F}^+$ then defines a unitary isomorphism of Hilbert spaces.

Finally, we state a direct consequence of Theorem 7.6, Proposition 7.8 and the commutation properties of the vacuum Maxwell equations with $\mathcal{L}_Z,\ Z \in \mathbb{K}$.

Definition 7.14. Let $N \geq 0$ and $\mathcal{E}^N_{\{t=0\}} \subset \mathcal{E}_{\{t=0\}}$ be the set of the 2-forms on $\mathbb{R}^{1+3}$ independent of t verifying

$$\|F_0\|^2_{\mathcal{E}^N_{\{t=0\}}} := \sum_{|\gamma|\leq N} \|\mathcal{L}_{Z^\gamma}(F_0)(0,\cdot)\|^2_{\mathcal{E}_{\{t=0\}}} < +\infty.$$

Consider $\mathcal{E}^N_{\mathcal{I}^+} \subset \mathcal{E}_{\mathcal{I}^+}$, the set of the 1-forms on $\mathbb{R}_u \times \mathbb{S}^2$ which are tangential to the 2-spheres and such that

$$\|\underline{\alpha}^{\mathcal{I}^+}\|^2_{N,\mathcal{I}^+} := \sum_{|\gamma|\leq N} \|\underline{\alpha}^{\mathcal{I}^+}_{Z^\gamma}\|^2_{\mathcal{I}^+} < +\infty,$$

where $\underline{\alpha}^{\mathcal{I}^+}_{Z^\gamma}$ is defined recursively from $\underline{\alpha}^{\mathcal{I}^+}$ through Proposition 7.4. Then, $(\mathcal{E}^N_{\{t=0\}}, \|\cdot\|^2_{\mathcal{E}^N_{\{t=0\}}})$ and $(\mathcal{E}^N_{\mathcal{I}^+}, \|\cdot\|_{N,\mathcal{I}^+})$ are Hilbert spaces.

Corollary 7.15. *For any $N \geq 0$, the restriction of $\mathscr{F}^+$ to $\mathcal{E}^N_{\{t=0\}}$ is a bijective isometry from $\mathcal{E}^N_{\{t=0\}}$ to $\mathcal{E}^N_{\mathcal{I}^+}$.*

7.2. *Existence of an asymptotic state for F and its derivatives.* In order to avoid any confusion, we make precise that, as in Sections 3–6, F denotes the electromagnetic field of our solution to the Vlasov–Maxwell system (f, F). The following statement can be easily deduced from previous results.

Proposition 7.16. *For any $|\gamma| \leq N-3$, $\underline{\alpha}(\mathcal{L}_{Z^\gamma}F)$ has a continuous radiation field $\underline{\alpha}^{\mathcal{I}^+}_\gamma$. Moreover, for any $0 \leq \eta < 1$, we have the rate of convergence,*

$$\forall \underline{u} \in \mathbb{R}_+,\ |u| \leq \underline{u},\ \omega \in \mathbb{S}^2, \quad \left|\langle u\rangle^\eta \big(r\underline{\alpha}(\mathcal{L}_{Z^\gamma}F)(\underline{u},u,\omega) - \underline{\alpha}^{\mathcal{I}^+}_\gamma(u,\omega)\big)\right| \lesssim \Lambda \frac{\log(3+\underline{u})}{(1+\underline{u})^{1-\eta}}.$$

If $|\gamma| = 0$, we simply denote the radiation field of F by $\underline{\alpha}^{\mathcal{I}^+}$.

Proof. Recall from Proposition 2.4 the form of the source term in the commuted Maxwell equations. Hence, according to the estimates of Proposition 3.1 and Corollary 4.14, $\mathcal{L}_{Z^\gamma} F$ satisfies the hypotheses of Proposition 7.3. □

It turns out that our decomposition of F allows us to improve the estimate on the radiation field.

Proposition 7.17. *For any* $|\gamma| \leq N-3$, *we have*,

$$\forall (u, \omega) \in \mathbb{R} \times \mathbb{S}^2, \quad |\underline{\alpha}_\gamma^{\mathcal{I}^+}|(u, \omega) \lesssim \begin{cases} \Lambda \langle u \rangle^{-1-\delta} & \text{if } 0 < \delta < 1, \\ \Lambda \langle u \rangle^{-2} \log(1+\langle u \rangle) & \text{if } \delta = 1. \end{cases}$$

Proof. Recall the decomposition

$$r\mathcal{L}_{Z^\gamma} F = r\mathcal{L}_{Z^\gamma}(F)^S + r(\mathcal{L}_{Z^\gamma}(F)^{\mathrm{data}}(t,x) - \mathcal{L}_{Z^\gamma}(\widetilde{F})) + r\mathcal{L}_{Z^\gamma}(\widetilde{F}) + r\mathcal{L}_{Z^\gamma}(F)^T.$$

Then, we use that $u = t - r$ as well as:

- The first term is bounded by $\Lambda \langle t-r \rangle^{-2} \log(1+\langle t-r \rangle)$ according to Proposition 5.14.
- By Proposition 6.12, the second one is controlled by $\Lambda \langle t-r \rangle^{-1-\delta}$.
- By Remark 6.13, the third term is bounded by $\epsilon(1+t+r)^{-1} + \epsilon \mathbb{1}_{|t-r|\leq 1}$ and $u = t-r$.
- Finally, the last one is bounded above by $\bar{\epsilon}(1+t+r)^{-3/4}$ according to Proposition 5.15. □

The last goal of this section consists in proving, if N is large enough, that F can be approached by a solution to the vacuum Maxwell equations through an application of Proposition 7.8, which requires us to control $\underline{\alpha}^{\mathcal{I}^+}$ and its derivatives up to order at least 3. Note then that by iterating Proposition 7.4, we get that $\underline{\alpha}_\gamma^{\mathcal{I}^+}$ can be computed in terms of derivatives of $\underline{\alpha}^{\mathcal{I}^+}$. Conversely, for any $0 \leq a < \frac{1}{2}$, we have

$$\sum_{n_u+n_\theta+n_\varphi \leq N-3} \int_{\mathbb{R}_u} \int_{\mathbb{S}^2} \langle u \rangle^{2a+2n_u} |\nabla_u^{n_u} \nabla\!\!\!\!/_{e_\theta}^{n_\theta} \nabla\!\!\!\!/_{e_\varphi}^{n_\varphi} \underline{\alpha}^{\mathcal{I}^+}|^2(u,\omega)\, \mathrm{d}\mu_{\mathbb{S}^2}\, \mathrm{d}u \lesssim \sum_{|\gamma| \leq N-3} \int_{\mathbb{R}_u} \int_{\mathbb{S}^2} \langle u \rangle^{2a} |\underline{\alpha}_\gamma^{\mathcal{I}^+}|^2(u,\omega)\, \mathrm{d}\mu_{\mathbb{S}^2}\, \mathrm{d}u.$$

Applying Proposition 7.16 for $\eta = (3+2a)/4$ then yields

$$\sum_{n_u+n_\theta+n_\varphi \leq N-3} \int_{\mathbb{R}_u} \int_{\mathbb{S}^2} \langle u \rangle^{2a+2n_u} |\nabla_u^{n_u} \nabla\!\!\!\!/_{e_\theta}^{n_\theta} \nabla\!\!\!\!/_{e_\varphi}^{n_\varphi} \underline{\alpha}^{\mathcal{I}^+}|^2(u,\omega)\, \mathrm{d}\mu_{\mathbb{S}^2}\, \mathrm{d}u \lesssim \Lambda \int_{\mathbb{R}_u} \langle u \rangle^{a-\frac{3}{2}}\, \mathrm{d}u \lesssim \frac{\Lambda}{1-2a}. \tag{91}$$

We are now ready to prove the following result.

Proposition 7.18. *If* $N \geq 10$, *there exists a solution* F^{vac} *of class* C^{N-8} *to the vacuum Maxwell equations* (19) *such that*, *for any* $\frac{1}{2} \leq q < 1$ *and* $|\gamma| \leq N-10$,

$$\forall (t,x) \in \mathbb{R}_+ \times \mathbb{R}^3, \quad r|\mathcal{L}_{Z^\gamma}(F) - \mathcal{L}_{Z^\gamma}(F^{\mathrm{vac}})|(t,x) \leq \Lambda C_q (1+t+|x|)^{-q},$$

where the constant $C_q > 0$ *depends on* q.

Proof. We fix $0 \leq q - \frac{1}{2} < a < \frac{1}{2}$. Since (91) holds, we get from Proposition 7.8 that there exists a solution F^{vac} of class C^{N-8} to the vacuum Maxwell equations satisfying, for any $|\gamma| \leq N-9$ and $|\xi| \leq N-10$,

$$\forall (t,x) \in \mathbb{R}_+ \times \mathbb{R}^3, \quad \big(|\alpha(\mathcal{L}_{Z^\gamma} F^{\mathrm{vac}})| + |\rho(\mathcal{L}_{Z^\gamma} F^{\mathrm{vac}})| + |\sigma(\mathcal{L}_{Z^\gamma} F^{\mathrm{vac}})|\big)(t,x) \lesssim \frac{\Lambda}{(1+t+|x|)^{1+q}}, \tag{92}$$

$$\left| r\underline{\alpha}(\mathcal{L}_{Z^\xi} F^{\mathrm{vac}})(t,x) - \mathscr{F}^+(\mathcal{L}_{Z^\xi} F^{\mathrm{vac}}(0,\cdot))\left(t-|x|, \frac{x}{|x|}\right)\right| \lesssim \frac{\Lambda}{(1+t+|x|)^q} \tag{93}$$

and $\mathscr{F}^+(F^{\mathrm{vac}}(0,\cdot)) = \underline{\alpha}^{\mathcal{I}^+}$. Together with Proposition 3.1 and Corollary 4.14, these estimates imply that $\mathcal{L}_{Z^\gamma}(F - F^{\mathrm{vac}})$ satisfies the assumptions of Proposition 7.4 for any $|\gamma| \leq N-10$. We then deduce, by a straightforward induction, that $\underline{\alpha}^{\mathcal{I}^+}_\gamma = \mathscr{F}^+(\mathcal{L}_{Z^\gamma} F^{\mathrm{vac}}(0,\cdot))$. Combining (93) with Proposition 7.16 then yields,

$$\forall (t,x) \in \mathbb{R}_+ \times \mathbb{R}^3, \quad r|\underline{\alpha}(\mathcal{L}_{Z^\gamma} F) - \underline{\alpha}(\mathcal{L}_{Z^\gamma} F^{\mathrm{vac}})|(t,x) \lesssim \Lambda(1+t+|x|)^{-q}, \quad |\gamma| \leq N-10.$$

On the other hand, Proposition 3.1 and (92) give, for any null component $\zeta \in \{\alpha, \rho, \sigma\}$,

$$\forall (t,x) \in \mathbb{R}_+ \times \mathbb{R}^3, \quad r|\zeta(\mathcal{L}_{Z^\gamma} F) - \zeta(\mathcal{L}_{Z^\gamma} F^{\mathrm{vac}})|(t,x) \lesssim \Lambda(1+t+|x|)^{-q}, \quad |\gamma| \leq N-9,$$

which concludes the proof. □

Remark 7.19. According to Corollary 7.15 and Lemma 7.9, F^{vac} is in fact of class C^{N-5}. Moreover, if $N \geq 7$, then the statement of Proposition 7.18 still holds for any $|\gamma| \leq N-7$ and the particular value $q = \frac{1}{2}$.

8. Conservation of the total energy of the system

Since (f, F) is a solution to the Vlasov–Maxwell system, the energy momentum tensor $\mathbb{T}[f, F]$, defined as

$$\mathbb{T}[f,F]_{\mu\nu} := \mathbb{T}[f]_{\mu\nu} + \mathbb{T}[F]_{\mu\nu}, \quad \mathbb{T}[f]_{\mu\nu} := \int_{\mathbb{R}^3_v} f v_\mu v_\nu \frac{\mathrm{d}v}{v^0}, \quad \mathbb{T}[F]_{\mu\nu} := F_{\mu\beta} F_\nu{}^\beta - \tfrac{1}{4}\eta_{\mu\nu} F_{\xi\lambda} F^{\xi\lambda},$$

is divergence free. It provides the conservation of the total energy of the system

$$\mathbb{E}_t := \int_{\mathbb{R}^3_x}\int_{\mathbb{R}^3_v} v^0 f(t,x,v)\,\mathrm{d}v\,\mathrm{d}x + \frac{1}{2}\int_{\mathbb{R}^3_x} |F|^2(t,x)\,\mathrm{d}x = \mathbb{E}_0, \quad |F|^2 = \sum_{0\leq\mu<\nu\leq 3} |F_{\mu\nu}|^2 = |E|^2 + |B|^2.$$

We would like to relate $\mathbb{E}_0$ to the energy of the scattering states f_∞ and $\underline{\alpha}^{\mathcal{I}^+}$. More precisely, the goal of this section is to prove

$$\mathbb{E}_\infty := \int_{\mathbb{R}^3_x}\int_{\mathbb{R}^3_v} v^0 f_\infty(x,v)\,\mathrm{d}v\,\mathrm{d}x + \frac{1}{4}\int_{\mathbb{R}_u}\int_{\mathbb{S}^2_\omega} |\underline{\alpha}^{\mathcal{I}^+}|^2(u,\omega)\,\mathrm{d}\mu_{\mathbb{S}^2}\,\mathrm{d}u = \mathbb{E}_0. \tag{94}$$

Note that $\mathbb{E}_\infty < +\infty$ according to Remark 6.35 and Proposition 7.16. The statement (94) is a consequence of $\mathbb{E}_t = \mathbb{E}_0$ and the following two propositions.

Proposition 8.1. *There holds*

$$\lim_{t\to+\infty} \int_{\mathbb{R}^3_x}\int_{\mathbb{R}^3_v} v^0 f(t,x,v)\,\mathrm{d}v\,\mathrm{d}x = \int_{\mathbb{R}^3_x}\int_{\mathbb{R}^3_v} v^0 f_\infty(x,v)\,\mathrm{d}v\,\mathrm{d}x.$$

Proof. Let $t \geq 3$ and perform the change of variables

$$x^j = y^j + \hat{v}^j t - \log(t)\hat{v}^\mu(F^\infty_{\mu j}(v) + \hat{v}^j F^\infty_{\mu 0}(v))$$

to get

$$\int_{\mathbb{R}^3_x}\int_{\mathbb{R}^3_v} v^0 f(t,x,v)\,\mathrm{d}v\,\mathrm{d}x = \int_{\mathbb{R}^3_y}\int_{\mathbb{R}^3_v} v^0 f(t, X_{\mathscr{C}}(t,y,v), v)\,\mathrm{d}v\,\mathrm{d}y.$$

We then deduce that

$$\left|\int_{\mathbb{R}^3_x}\int_{\mathbb{R}^3_v} v^0 f(t,x,v)\,\mathrm{d}v\,\mathrm{d}x - \int_{\mathbb{R}^3_x}\int_{\mathbb{R}^3_v} v^0 f_\infty(x,v)\,\mathrm{d}v\,\mathrm{d}x\right| \leq \sup_{(x,v)\in\mathbb{R}^6} \langle x\rangle^{\frac{7}{2}}|v^0|^5|f(t, X_{\mathscr{C}}(t,x,v),v) - f_\infty(x,v)|,$$

which, in view of $N_v \geq 12$, $N_x \geq \frac{11}{2}$ and Remark 6.35, implies the result. □

Proposition 8.2. *We have*

$$\lim_{t\to+\infty} \frac{1}{2}\int_{\mathbb{R}^3_x} |F|^2(t,x)\,\mathrm{d}x = \frac{1}{4}\int_{\mathbb{R}_u}\int_{\mathbb{S}^2_\omega} |\underline{\alpha}^{\mathcal{I}^+}|^2(u,\omega)\,\mathrm{d}\mu_{\mathbb{S}^2}\,\mathrm{d}u.$$

Proof. Consider $\underline{u} \geq \tau \geq 3$ and introduce the domain $\mathcal{D}^\tau_{\underline{u}} = \{t+|x| \leq \underline{u},\ t\geq\tau\}$, which is bounded by the truncated backward light cone $\underline{C}^\tau_{\underline{u}} := \{t+|x| = \underline{u},\ t\geq\tau\}$ and $\{t=\tau\}\cap\{|x|\leq \underline{u}-\tau\}$. In the same spirit as (81), the divergence theorem, applied to $\mathbb{T}[F]_{\mu 0}$ in $\mathcal{D}^\tau_{\underline{u}}$, yields

$$\int_{\underline{C}^\tau_{\underline{u}}} \mathbb{T}[F]_{\underline{L}0}\mathrm{d}\mu_{\underline{C}_{\underline{u}}} = \int_{|x|\leq\underline{u}-\tau} \mathbb{T}[F]_{00}(\tau,x)\,\mathrm{d}x + \int_{(t,x)\in\mathcal{D}^\tau_{\underline{u}}} F_{0\lambda}J(f)^\lambda\,\mathrm{d}x\,\mathrm{d}t. \tag{95}$$

First, we have

$$\lim_{\underline{u}\to+\infty}\int_{|x|\leq\underline{u}-\tau}\mathbb{T}[F]_{00}(\tau,x)\,\mathrm{d}x = \lim_{\underline{u}\to+\infty}\frac{1}{2}\int_{|x|\leq\underline{u}-\tau}|F|^2(\tau,x)\,\mathrm{d}x = \frac{1}{2}\int_{\mathbb{R}^3_x}|F|^2(\tau,x)\,\mathrm{d}x.$$

Next, since $|F|(t,x) \lesssim (1+t+|x|)^{-1}(1+|t-|x||)^{-1}$ by (BA1) and $|J(f)| \lesssim (1+t+|x|)^{-3}$ by Corollary 4.14,

$$\int_{(t,x)\in\mathcal{D}^\tau_{\underline{u}}} F_{0\lambda}J(f)^\lambda\,\mathrm{d}x\,\mathrm{d}t \lesssim \int_{t=\tau}^{+\infty}\int_{r=0}^{+\infty}\frac{r^2\,\mathrm{d}r\,\mathrm{d}t}{(1+t+r)^4(1+|t-r|)} \lesssim \int_{t=\tau}^{+\infty}\int_{r=0}^{+\infty}\frac{\mathrm{d}r\,\mathrm{d}t}{(1+t)^{\frac{3}{2}}(1+|t-r|)^{\frac{3}{2}}} \lesssim \tau^{-\frac{1}{2}}.$$

Recall from Definition 2.16 the value of the null components of $\mathbb{T}[F]$. As

$$|\rho|(t,x)+|\sigma|(t,x) \lesssim (1+t+|x|)^{-\frac{7}{4}}$$

by Proposition 3.1 and in view of Proposition 7.16, applied for $\eta > \frac{1}{2}$,

$$\begin{aligned}\int_{\underline{C}^\tau_{\underline{u}}}\mathbb{T}[F]_{\underline{L}0}\,\mathrm{d}\mu_{\underline{C}_{\underline{u}}} &= \frac{1}{4}\int_{2\tau-\underline{u}\leq u\leq\underline{u}}\int_{\mathbb{S}^2_\omega}(|\underline{\alpha}(F)|^2+|\rho(F)|^2+|\sigma(F)|^2)(u,\underline{u},\omega)r^2\,\mathrm{d}\mu_{\mathbb{S}^2}\,\mathrm{d}u\\ &= \frac{1}{4}\int_{2\tau-\underline{u}\leq u\leq\underline{u}}\int_{\mathbb{S}^2_\omega} r^2|\underline{\alpha}(F)|^2(u,\underline{u},\omega)\,\mathrm{d}\mu_{\mathbb{S}^2}\,\mathrm{d}u + O(\underline{u}^{-\frac{1}{2}})\\ &\xrightarrow[\underline{u}\to+\infty]{} \frac{1}{4}\int_{\mathbb{R}_u}\int_{\mathbb{S}^2_\omega}|\underline{\alpha}^{\mathcal{I}^+}|^2(u,\omega)\,\mathrm{d}\mu_{\mathbb{S}^2}\,\mathrm{d}u.\end{aligned}$$

Letting $\underline{u}\to+\infty$ and then $\tau\to+\infty$ in (95) yields the result. □

Appendix A: Estimates for the gradients of the kernels

In order to estimate the kernels and their derivatives in the integrals of Propositions 5.3 and 5.7, we introduce the following class of terms.

Definition A.1. Let $(p, q, d, d_{\boldsymbol{w}}) \in \mathbb{N}^4$. We define $\boldsymbol{S}^{d,d_{\boldsymbol{w}}}_{p,q}$ as the set of the functions $\mathcal{G} : \mathbb{S}^2 \times \mathbb{R}^3 \to \mathbb{R}$ of the form

$$\mathcal{G}(\omega, v) = \frac{P(\hat{v}, \omega) Q(\boldsymbol{w}(\omega, v))}{|v^0|^p (1 + \omega \cdot \hat{v})^q}, \tag{96}$$

where P is a monomial of degree d in $(\hat{v}^1, \hat{v}^2, \hat{v}^3, \omega_1, \omega_2, \omega_3)$ and Q is a monomial of degree $d_{\boldsymbol{w}}$ in $\boldsymbol{w}_{\mu\nu}(\omega, v)$, where $0 \leq \mu < \nu \leq 3$.

All the kernels considered in this paper can be written as linear combination of such terms, with $d_{\boldsymbol{w}} \in [\![0, 3]\!]$. Moreover, if $2q \geq d_{\boldsymbol{w}}$, by a direct application of Lemma 5.4, one can bound $\mathcal{G}(\omega, v)$ in (96) by $|v^0|^{2q-d_{\boldsymbol{w}}-p}$. The estimates of Corollaries 5.5 and 5.8 of the derivatives of the kernels then follows from the next result.

Lemma A.2. *Let $(p, q, d, d_{\boldsymbol{w}}) \in \mathbb{N}^4$ and consider $\mathcal{G} \in \boldsymbol{S}^{d,d_{\boldsymbol{w}}}_{p,q}$. Then, for any multi-index γ, $\partial_v^\gamma \mathcal{G}(\omega, v)$ can be written as linear combination of terms belonging to certain $\boldsymbol{S}^{d_0,d_{\boldsymbol{w},0}}_{p_0,q_0}$, where*

$$(p_0, q_0, d_0, d_{\boldsymbol{w},0}) \in \mathbb{N}^4, \quad 2q_0 - d_{\boldsymbol{w},0} - p_0 \leq 2q - d_{\boldsymbol{w}} - p, \quad q - d_{\boldsymbol{w}} \leq q_0 - d_{\boldsymbol{w},0}.$$

This implies $|\partial_v^\gamma \mathcal{G}|(\cdot, v) \lesssim |v^0|^{2q-d_{\boldsymbol{w}}-p}$ if $2q \geq d_{\boldsymbol{w}}$.

Proof. This follows from a straightforward induction and the following relations. For any $(i, j, k) \in [\![1, 3]\!]^3$,

$$\partial_{v^j} \hat{v}^i = \frac{\delta_i^j - \hat{v}^i \hat{v}^j}{v^0}, \quad \partial_{v^j} \omega^i = 0, \quad \partial_{v^j} |v^0|^{-p} = -p \frac{\hat{v}^j}{|v^0|^{p+1}},$$

$$\partial_{v^j} \boldsymbol{w}_{0i}(\omega, v) = \frac{\delta_i^j - \hat{v}^i \hat{v}^j}{v^0}, \quad \partial_{v^j} \boldsymbol{w}_{ik}(\omega, v) = \omega^i \frac{\delta_k^j - \hat{v}^k \hat{v}^j}{v^0} - \omega^k \frac{\delta_i^j - \hat{v}^i \hat{v}^j}{v^0},$$

$$\partial_{v^j} \left(\frac{1}{1 + \omega \cdot \hat{v}} \right) = \frac{\hat{v}^j}{v^0 (1 + \omega \cdot \hat{v})} - \frac{\boldsymbol{w}_{0j}(\omega, v)}{v^0 (1 + \omega \cdot \hat{v})^2}. \qquad \square$$

Appendix B: The radiation field of the derivatives of the Maxwell field

We fix, for all of this section, a C^1 solution G to the Maxwell equations (18) with a continuous source term J. We assume that there exist $C[G] > 0$ and $q > 0$ such that, for all $(t, x) \in \mathbb{R}_+ \times \mathbb{R}^3$,

$$|rG|(t, x) \leq C[G], \quad r|J|(t, x) + \sum_{|\gamma| \leq 1} |\rho(\mathcal{L}_{Z^\gamma} G)|(t, x) + |\sigma(\mathcal{L}_{Z^\gamma} G)|(t, x) \leq \frac{C[G]}{(1 + t + |x|)^{1+q}}.$$

As a consequence, G verifies the hypotheses (75) of Proposition 7.3 and then has a radiation field $\underline{\alpha}^{\mathcal{I}^+}$. The purpose of this section is to prove that, for any $Z \in \mathbb{K}$, $\mathcal{L}_Z G$ has a radiation field $\underline{\alpha}^{\mathcal{I}^+}_Z$ which can be expressed in terms of the derivatives of $\underline{\alpha}^{\mathcal{I}^+}$. For this, we will use the following bounded functions depending only on the spherical variables:

$$\omega_i := \langle \partial_{x^i}, \partial_r \rangle = \frac{x^i}{|x|}, \quad \omega_i^{e_A} := \langle \partial_{x^i}, e_A \rangle, \quad 1 \le i \le 3, \quad A \in \{\theta, \varphi\},$$
$$\omega_1^{e_\theta} = \cos(\varphi)\cos(\theta), \quad \omega_2^{e_\theta} = \sin(\varphi)\cos(\theta), \quad \omega_3^{e_\theta} = -\sin(\theta),$$
$$\omega_1^{e_\varphi} = -\sin(\varphi), \quad \omega_2^{e_\varphi} = \cos(\varphi), \quad \omega_3^{e_\varphi} = 0,$$

and we will work in the space of distributions $\mathcal{D}'(\mathbb{R}_u \times \mathbb{S}^2)$. For simplicity, we will simply write $\psi \rightharpoonup \psi^{\mathcal{I}^+}$ if the weak convergence

$$\psi(u, \underline{u}, \omega) \xrightharpoonup[\underline{u} \to +\infty]{} \psi^{\mathcal{I}^+}(u, \omega) \quad \text{in } \mathcal{D}'(\mathbb{R}_u \times \mathbb{S}^2)$$

holds. In particular, the following convergences will be crucial for us.

Lemma B.1. *For any* $1 \le i \le 3$ *and* $B \in \{\theta, \varphi\}$,

$$|G| \rightharpoonup 0, \quad \tfrac{1}{2} r \underline{L}(\underline{\alpha}(G)_{e_B}) \rightharpoonup \partial_u(\underline{\alpha}^{\mathcal{I}^+}_{e_B}), \quad r^2 L(\underline{\alpha}(G)_{e_B}) \rightharpoonup -\underline{\alpha}^{\mathcal{I}^+}_{e_B},$$
$$r^2 \omega_i^A e_A(\underline{\alpha}(G)_{e_B}) \rightharpoonup \omega_i^A e_A(\underline{\alpha}^{\mathcal{I}^+}_{e_B}), \quad r\rho(G) \rightharpoonup 0, \quad r\sigma(G) \rightharpoonup 0.$$

Since $2r = \underline{u} - u$, *we also have*

$$rL(\underline{\alpha}(G)_{e_B}) \rightharpoonup 0, \quad r\omega_i^A e_A(\underline{\alpha}(G)_{e_B}) \rightharpoonup 0, \quad \rho(G) \rightharpoonup 0, \quad \sigma(G) \rightharpoonup 0.$$

Proof. The first weak convergence follows from $2|G|(u, \underline{u}, \omega) \le C[G](\underline{u} - u)^{-1}$, so that $|G|(\cdot, \underline{u}, \cdot) \to 0$ uniformly on any compact subset of $\mathbb{R}_u \times \mathbb{S}^2$. The others are a direct consequence of the strong uniform convergence $r\underline{\alpha}(G)(u, \underline{u}, \omega) \to \underline{\alpha}^{\mathcal{I}^+}(u, \omega)$ as $\underline{u} \to +\infty$, which is given by Proposition 7.3 since G satisfies (75).

- For the second one, use $r\underline{L} = \underline{L}r + 1$, $\underline{L} = 2\partial_u$ and that $\underline{\alpha}(F)_{e_B}(\cdot, \underline{u}, \cdot) \to 0$ uniformly on compact subsets of $\mathbb{R}_u \times \mathbb{S}^2$.
- The third one is in fact a strong and uniform convergence. Indeed, $r^2 L(\underline{\alpha}(G)_{e_B}) = rL(r\underline{\alpha}(G)_{e_B}) - r\underline{\alpha}(G)_{e_B}$ and according to (76), $r|L(r\underline{\alpha}(G)_{e_B})| \lesssim \underline{u}^{-q}$.
- Next, fix $(t, r) \in \mathbb{R}^2_+$, $\psi \in C_c^\infty(\mathbb{R}_u \times \mathbb{S}^2)$ and denote by $\vec{v}_i$ the vector field $\omega_i^{e_A} e_A$, which is the projection on the 2-spheres of ∂_{x^i}. Since $(re_\theta, re_\varphi) = (\partial_\theta, \partial_\varphi / \sin(\theta))$, we have

$$\omega_i^A r^2 (e_A(\underline{\alpha}(G)_{e_B}))(t, r\omega)\psi(u, \omega) = r\psi(u, \omega)\vec{v}_i \cdot \nabla\!\!\!\!/ (\underline{\alpha}(G)_{e_B}(t, r\omega)),$$
$$\omega_i^A e_A(\underline{\alpha}^{\mathcal{I}^+}_{e_B})(u, \omega)\psi(u, \omega) = \psi(u, \omega)\vec{v}_i \cdot \nabla\!\!\!\!/ (\underline{\alpha}^{\mathcal{I}^+}_{e_B})(u, \omega),$$

so that it suffices to apply the divergence theorem on $\mathbb{S}^2$ and to use $r\underline{\alpha}(G)_{e_B} \rightharpoonup \underline{\alpha}^{\mathcal{I}^+}_{e_B}$.
- Finally, the last two follow from $r|\rho(G)| + r|\sigma(G)| \lesssim \underline{u}^{-q}$. □

We now prove a result which directly implies Proposition 7.4. We consider a more general setting since it does not complicate the proof and so we will be able to apply these properties in different contexts. For this, given a strictly increasing and unbounded sequence of advanced times $s = (\underline{u}_n)_{n \ge 0}$, we will write $\psi \rightharpoonup_s \psi^{\mathcal{I}^+}$ if the following weak convergence holds:

$$\psi(u, \underline{u}_n, \omega) \xrightharpoonup[n \to +\infty]{} \psi^{\mathcal{I}^+}(u, \omega) \quad \text{in } \mathcal{D}'(\mathbb{R}_u \times \mathbb{S}^2).$$

Proposition B.2. *Consider G an $H^1_{\mathrm{loc}}(\mathbb{R}_+ \times \mathbb{R}^3)$ 2-form and $\underline{\alpha}^{\mathcal{I}^+}$ an $L^2_{\mathrm{loc}}(\mathbb{R}_u \times \mathbb{S}^2_\omega)$ 2-form tangential to the spheres. Assume that there exists a strictly increasing and unbounded sequence of advanced times $\mathbf{s} = (\underline{u}_n)_{n\geq 0}$ such that*

- $r\underline{\alpha}(G) \rightharpoonup_{\mathbf{s}} \underline{\alpha}^{\mathcal{I}^+}$,
- *all the weak convergences of Lemma B.1 hold, at least for the sequence $\mathbf{s} \subset \mathbb{R}_{+,\underline{u}}$.*

Then, for any $Z \in \mathbb{K}$, there exists $\underline{\alpha}^{\mathcal{I}^+}_Z \in L^2_{\mathrm{loc}}(\mathbb{R}_u \times \mathbb{S}^2_\omega)$, a 2-form tangential to the spheres, which satisfies $r\underline{\alpha}(\mathcal{L}_Z G) \rightharpoonup_{\mathbf{s}} \underline{\alpha}^{\mathcal{I}^+}_Z$. Moreover, for any $1 \leq i \leq 3$ and $1 \leq j < k \leq 3$,

$$\underline{\alpha}^{\mathcal{I}^+}_{\partial_t} = \nabla_u \underline{\alpha}^{\mathcal{I}^+}, \quad \underline{\alpha}^{\mathcal{I}^+}_{\partial_{x^i}} = -\omega_i \nabla_u \underline{\alpha}^{\mathcal{I}^+}, \quad \underline{\alpha}^{\mathcal{I}^+}_S = u\nabla_u \underline{\alpha}^{\mathcal{I}^+} + \underline{\alpha}^{\mathcal{I}^+},$$
$$\underline{\alpha}^{\mathcal{I}^+}_{\Omega_{jk}} = \mathcal{L}_{\Omega_{jk}}(\underline{\alpha}^{\mathcal{I}^+}), \quad \underline{\alpha}^{\mathcal{I}^+}_{\Omega_{0i}} = -\omega_i u \nabla_u \underline{\alpha}^{\mathcal{I}^+} - 2\omega_i \underline{\alpha}^{\mathcal{I}^+} + \omega_i^{e_A} \nabla\!\!\!/_{e_A} \underline{\alpha}^{\mathcal{I}^+}.$$

Proof. In order to avoid technical difficulties related to the degeneracies of the spherical coordinate system, we will in fact prove weak convergences in

$$\mathcal{D}'(\mathbb{R}_u \times K), \quad K := \left\{\omega \in \mathbb{S}^2 \mid \sin\theta > \tfrac{1}{8}\right\}.$$

The convergences in the full space $\mathcal{D}'(\mathbb{R}_u \times \mathbb{S}^2)$ can then be obtained by applying the upcoming results to another well-chosen spherical coordinate system.

We fix, for all of this proof, $B \in \{\theta, \varphi\}$, $i \in [\![1, 3]\!]$ and we recall that, for any $Z \in \mathbb{K}$,

$$r\underline{\alpha}(\mathcal{L}_Z G)_{e_B} = rZ(\underline{\alpha}(G)_{e_B}) - rG([Z, e_B], \underline{L}) - rG(e_B, [Z, \underline{L}]).$$

Then, we have

$$r\underline{\alpha}(\mathcal{L}_{\partial_t} G)_{e_B} = \frac{r}{2}\underline{L}(\underline{\alpha}(G)_{e_B}) + \frac{r}{2}L(\underline{\alpha}(G)_{e_B}) \rightharpoonup_{\mathbf{s}} \partial_u(\underline{\alpha}^{\mathcal{I}^+}_{e_B}).$$

For the spatial translation $\partial_{x^i} = -\frac{1}{2}\omega_i \underline{L} + \frac{1}{2}\omega_i L + \omega_i^A e_A$, we use that

$$[\partial_{x^i}, \underline{L}] = -\frac{\omega_i^{e_A}}{r} e_A$$

and $[\partial_{x^i}, e_A] = \partial_{x^i}(\omega_j^{e_A})\partial_{x^j}$, with $\partial_{x^i}(\omega_j^{e_A}) \lesssim r^{-1}$ on K. We get

$$\begin{aligned} &r\underline{\alpha}(\mathcal{L}_{\partial_{x^i}} G)_{e_B} \\ &\quad = -\frac{\omega_i r}{2}\underline{L}(\underline{\alpha}(G)_{e_B}) + \frac{\omega_i r}{2} L(\underline{\alpha}(G)_{e_B}) + r\omega_i^A e_A(\underline{\alpha}(G)_{e_B}) - r\partial_{x^i}(\omega_j^{e_A})G(e_B, \partial_{x^j}) + \omega_i^{e_A} G(e_B, e_A) \\ &\quad \rightharpoonup_{\mathbf{s}} -\omega_i \partial_u(\underline{\alpha}^{\mathcal{I}^+}_{e_B}). \end{aligned}$$

For the scaling, recall that $[S, \underline{L}] = -\underline{L}$ and $[S, e_B] = -e_B$. As $2S = u\underline{L} + (u + 2r)L$, we have

$$r\underline{\alpha}(\mathcal{L}_S G)_{e_B} = \frac{u}{2} r\underline{L}(\underline{\alpha}(G)_{e_A}) + \frac{u+2r}{2} rL(\underline{\alpha}(G)_{e_A}) + 2rG(e_B, \underline{L}) \rightharpoonup_{\mathbf{s}} u\partial_u(\underline{\alpha}^{\mathcal{I}^+}_{e_B}) + \underline{\alpha}^{\mathcal{I}^+}_{e_B}.$$

Next, for the Lorentz boost Ω_{0i}, we use

$$\Omega_{0i} = \frac{\omega_i}{2}(\underline{u}L - u\underline{L}) + t\omega_i^{e_A} e_A, \quad [\Omega_{0i}, e_B] = \frac{\omega_i^{e_B}}{2r}(u\underline{L} - \underline{u}L) + \frac{t}{r}\omega_i^{e_A} \Gamma\!\!\!/^{D}_{AB} e_D, \quad [\Omega_{0i}, \underline{L}] = \omega_i \underline{L} - \frac{u}{r}\omega_i^{e_A} e_A,$$

where $\not{\Gamma}^D_{AB}$ are the Christoffel symbols of $\mathbb{S}^2$ in the nonholonomic basis (e_θ, e_φ). In particular, $\not{\Gamma}^D_{AB}$ is bounded on K. As $\underline{u} = u + 2r$ and $t = u + r$, we obtain

$$r\underline{\alpha}(\mathcal{L}_{\Omega_{0i}} G)_{e_B} = -\frac{\omega_i u}{2} r \underline{L}(\underline{\alpha}(G)_{e_A}) + \frac{\omega_i(u+2r)}{2} r L(\underline{\alpha}(G)_{e_B}) + \omega_i^{e_A}(u+r) r e_A(\underline{\alpha}(G)_{e_B}) - \frac{\omega_i^{e_B}}{2} u G(\underline{L}, \underline{L})$$
$$+ \frac{\omega_i^{e_B}}{2} \underline{u} G(L, \underline{L}) - (u+r)\omega_i^{e_A} \not{\Gamma}^D_{AB} G(e_D, \underline{L}) - \omega_i r G(e_B, \underline{L}) + \underline{u}\omega_i^{e_A} G(e_B, e_A).$$

Since $G(\underline{L}, \underline{L}) = 0$ and $\underline{u}(|G(L, \underline{L})| + |G(e_A, e_B)|) = (u+2r)(2|\rho(G)| + |\sigma(G)|) \rightharpoonup_s 0$, we get

$$\begin{aligned} r\underline{\alpha}(\mathcal{L}_{\Omega_{0i}} G)_{e_B} &\rightharpoonup_s -\omega_i u \partial_u(\underline{\alpha}^{\mathcal{I}^+}_{e_B}) - \omega_i \underline{\alpha}^{\mathcal{I}^+}_{e_B} + \omega_i^{e_A} e_A(\underline{\alpha}^{\mathcal{I}^+}_{e_B}) - 0 + 0 - \omega_i^{e_A} \not{\Gamma}^D_{AB} \underline{\alpha}^{\mathcal{I}^+}_{e_D} - \omega_i \underline{\alpha}^{\mathcal{I}^+}_{e_B} + 0 \\ &= -\omega_i u \partial_u(\underline{\alpha}^{\mathcal{I}^+}_{e_B}) - 2\omega_i \underline{\alpha}^{\mathcal{I}^+}_{e_B} + \omega_i^{e_A} \not{\nabla}_{e_A}(\underline{\alpha}^{\mathcal{I}^+})_{e_B}. \end{aligned}$$

Finally, we recall the expression of the rotations in the spherical coordinate system (t, r, θ, φ),

$$\Omega_{12} = \partial_\varphi, \quad \Omega_{13} = \cos(\varphi)\partial_\theta - \cot(\theta)\sin(\varphi)\partial_\varphi, \quad \Omega_{23} = -\sin(\varphi)\partial_\theta - \cot(\theta)\cos(\varphi)\partial_\varphi.$$

In particular, these vector fields, tangential to the spheres, are well-defined on $\mathcal{I}^+ \simeq \mathbb{R}_u \times \mathbb{S}^2$. Fix now $(j, k) \in [\![1, 3]\!]^2$ and write $\Omega_{jk} = \Omega^\theta_{jk}\partial_\theta + \Omega^\varphi_{jk}\partial_\varphi$. Note, using first $[\Omega_{jk}, \underline{L}] = 0$ and then the expression of the Lie derivative in the spherical coordinate system, that

$$\underline{\alpha}(\mathcal{L}_{\Omega_{jk}} G)_{\partial_B} = \mathcal{L}_{\Omega_{jk}}(\underline{\alpha}(G))_{\partial_B} = \Omega_{jk}(\underline{\alpha}(G)_{\partial_B}) + \partial_B(\Omega^A_{jk})\underline{\alpha}(G)_{\partial_A}.$$

Recall now that $(re_\theta, re_\varphi) = (\partial_\theta, \partial_\varphi/\sin(\theta))$ on $\mathbb{R}_+ \times \mathbb{R}^3$ and $(e_\theta, e_\varphi) = (\partial_\theta, \partial_\varphi/\sin(\theta))$ on $\mathbb{R}_u \times \mathbb{S}^2$. Hence, using $r\underline{\alpha}(G)_{e_A} \rightharpoonup_s \underline{\alpha}^{\mathcal{I}^+}_{e_A}$ and since any of the quantities considered is smooth and bounded on K,

$$\begin{aligned} r\underline{\alpha}(\mathcal{L}_{\Omega_{jk}} G)_{e_\theta} &= \Omega^\theta_{jk}\partial_\theta(r\underline{\alpha}(G)_{e_\theta}) + \Omega^\varphi_{jk}\partial_\varphi(r\underline{\alpha}(G)_{e_\theta}) + \partial_\theta(\Omega^\theta_{jk}) r\underline{\alpha}(G)_{e_\theta} + \sin(\theta)\partial_\theta(\Omega^\varphi_{jk}) r\underline{\alpha}(G)_{e_\varphi} \\ &\rightharpoonup_s \Omega_{jk}(\underline{\alpha}^{\mathcal{I}^+}_{e_\theta}) + \partial_\theta(\Omega^\theta_{jk})\underline{\alpha}^{\mathcal{I}^+}_{e_\theta} + \sin(\theta)\partial_\theta(\Omega^\varphi_{jk})\underline{\alpha}^{\mathcal{I}^+}_{e_\varphi} \\ &= \Omega_{jk}(\underline{\alpha}^{\mathcal{I}^+}_{\partial_\theta}) + \partial_\theta(\Omega^A_{jk})\underline{\alpha}^{\mathcal{I}^+}_{\partial_A} = \mathcal{L}_{\Omega_{kl}}(\underline{\alpha}^{\mathcal{I}^+})_{e_\theta}. \end{aligned}$$

Similarly, we get

$$\begin{aligned} r\underline{\alpha}(\mathcal{L}_{\Omega_{jk}} G)_{e_\varphi} &= \Omega_{jk}(r\underline{\alpha}(G)_{e_\varphi}) - \Omega_{kl}\Big(\frac{1}{\sin\theta}\Big) r\underline{\alpha}(G)_{e_\varphi} + \frac{1}{\sin\theta}\partial_\varphi(\Omega^\theta_{jk}) r\underline{\alpha}(G)_{e_\theta} + \partial_\varphi(\Omega^\varphi_{jk}) r\underline{\alpha}(G)_{e_\varphi} \\ &\rightharpoonup_s \Omega_{jk}(\underline{\alpha}^{\mathcal{I}^+}_{e_\varphi}) - \Omega_{kl}\Big(\frac{1}{\sin\theta}\Big)\underline{\alpha}^{\mathcal{I}^+}_{e_\varphi} + \frac{1}{\sin\theta}\partial_\varphi(\Omega^\theta_{jk})\underline{\alpha}^{\mathcal{I}^+}_{e_\theta} + \partial_\varphi(\Omega^\varphi_{jk})\underline{\alpha}^{\mathcal{I}^+}_{e_\varphi} = \mathcal{L}_{\Omega_{kl}}(\underline{\alpha}^{\mathcal{I}^+})_{e_\varphi}. \qquad \square \end{aligned}$$

Appendix C: Remarks on F^∞ and the modified characteristics

C.1. ***Alternative expression for F^∞.*** We could define F^∞ in a slightly different way. However, contrary to what we did in Section 6.2, we could not define in such a way $\mathcal{L}_{Z^\gamma}(F)^\infty$ for the derivatives of order $|\gamma| = N - 1$. Using the representation formula for the wave equation satisfied by $F_{\mu\nu}$,

$$F_{\mu\nu} = F^{\mathrm{hom}}_{\mu\nu} + [f]^{\mathrm{inh}}_{\mu\nu}, \quad [f]^{\mathrm{inh}}_{\mu\nu}(t, x) := \frac{1}{4\pi}\int_{|y-x|\le t}\int_{\mathbb{R}^3_v} (\hat{v}_\mu \partial_{x^\nu} f - \hat{v}_\nu \partial_{x^\mu} f)(t - |y-x|, y, v)\,\mathrm{d}v \frac{\mathrm{d}y}{|y-x|}.$$

In order to investigate the asymptotic behavior of $[f]^{\mathrm{inh}}$, it is important to determine the asymptotic profile of the source term of the wave equation. In particular, we need to obtain a better estimate than the one given by Proposition 4.15 which does not provide the expected time decay t^{-4}. The starting point consists in observing that a kind of null condition holds,

$$t(\partial_{x^i}+\hat{v}_i\partial_t)=\Omega_{0i}+z_{0i}\partial_t=\widehat{\Omega}_{0i}-v^0\partial_{v^i}+\partial_t z_{0i}-\hat{v}_i=\widehat{\Omega}_{0i}-\partial_{v^i}v^0+\partial_t z_{0i},\quad 1\le i\le 3,$$
$$t(\hat{v}_j\partial_{x^k}-\hat{v}_k\partial_{x^j})=\hat{v}_j\widehat{\Omega}_{0k}-\hat{v}_k\widehat{\Omega}_{0j}-\partial_t z_{jk}-\hat{v}_j\partial_{v^k}v^0+\hat{v}_k\partial_{v^j}v^0,\qquad 1\le j<k\le 3.$$

Hence, using the convention $\widehat{\Omega}_{00}=0$ and performing integration by parts, we obtain, for any $0\le\mu<\nu\le 3$,

$$\int_{\mathbb{R}^3_v}\hat{v}_\mu\partial_{x^\nu}f-\hat{v}_\nu\partial_{x^\mu}f\,\mathrm{d}v=\frac{1}{t}\int_{\mathbb{R}^3_v}(\hat{v}_\mu\widehat{\Omega}_{0\nu}f-\hat{v}_\nu\widehat{\Omega}_{0\mu}f)\,\mathrm{d}v-\frac{1}{t}\int_{\mathbb{R}^3_v}\partial_t(z_{\mu\nu}f)\,\mathrm{d}v.$$

The leading-order term of its asymptotic expansion is the first term on the right-hand side. Its behavior can be easily obtained from Proposition 6.5. Following the proof of Proposition 4.15, one could prove that last term almost decay as t^{-5}. It will then be convenient to use the notation $Q_\infty^{\widehat{\Omega}_{0\ell}}$ in order to denote Q_∞^κ, where $\widehat{Z}^\kappa=\widehat{\Omega}_{0\ell}$, and to set $Q_\infty^{\widehat{\Omega}_{00}}:=0$. Following the proof Proposition 6.10, we could obtain

$$\lim_{t\to+\infty}t^2[f]^{\mathrm{inh}}_{\mu\nu}(t,x+t\hat{v}):=\frac{1}{4\pi}\int_{\substack{|z|\le 1\\|z+\hat{v}|<1-|z|}}(|v^0|^5(\hat{v}_\mu Q_\infty^{\widehat{\Omega}_{0\nu}}-\hat{v}_\nu Q_\infty^{\widehat{\Omega}_{0\mu}}))\biggl(\frac{\widecheck{z+\hat{v}}}{1-|z|}\biggr)\frac{\mathrm{d}z}{|z|(1-|z|)^4},$$

which is necessarily equal to F^∞.

C.2. ***The support of the corrections of the linear characteristics and commutators.*** We could obtain similar results by modifying the trajectories and the homogeneous vector fields only inside the light cone. More precisely, we could consider, for $\widehat{Z}\in\widehat{\mathbb{P}}_0\setminus\{\partial_t,\partial_{x^1},\partial_{x^2},\partial_{x^3}\}$,

$$\widetilde{X}_{\mathscr{C}}(t,x,v):=x+t\hat{v}+\mathscr{C}(t,v)\chi(t-|x-t\hat{v}|),\qquad \widetilde{Z}^{\mathrm{mod}}:=\widehat{Z}+\mathscr{C}^i_{\widehat{Z}}\chi(t-r)\partial_{x^i},$$

where χ is a cutoff function satisfying $\chi(s)=0$ for $s\le 1$ and $\chi(s)=1$ for $s\ge 2$. It is not surprising that all the results proved for $X_{\mathscr{C}}$ and $\widehat{Z}^{\mathrm{mod}}$ hold as well with these corrections since the Vlasov field enjoys strong decay properties in the exterior of the light cone (see Lemma 2.6). We could even avoid the loss of the weight z^{β_H} in Proposition 6.28 and Corollary 6.29. Indeed, these weights come from the identity $x^i/t=(x^i-t\hat{v}^i)/t+\hat{v}^i$ that we performed during the proof of Proposition 6.26. On the support of χ, we can simply use that $|x|/t\le 1$. However, we could not save any $\langle x\rangle$ weight in the analogue version of the scattering statement of Proposition 6.34 since we would have to lose a power of z^{β_H} in order to estimate $|v^0|^{|\xi}\partial_v^\xi(\chi(t-|x+t\hat{v}|))$.

Acknowledgements

I would like to thank Jacques Smulevici for suggesting this problem to me. This work was conducted within the France 2030 framework programme, the Centre Henri Lebesgue ANR-11-LABX-0020-01.

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Received 26 Aug 2022. Revised 31 Aug 2023. Accepted 21 Nov 2023.

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ANALYSIS AND PDE
Vol. 18 (2025), No. 3, pp. 715–741
DOI: 10.2140/apde.2025.18.715

STRONG ILL-POSEDNESS IN L^∞ FOR THE RIESZ TRANSFORM PROBLEM

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We prove strong ill-posedness in L^∞ for linear perturbations of the 2-dimensional Euler equations of the form

$$\partial_t \omega + u \cdot \nabla \omega = R(\omega),$$

where R is any nontrivial second-order Riesz transform. Namely, we prove that there exist smooth solutions that are initially small in L^∞ but become arbitrarily large in short time. Previous works in this direction relied on the strong ill-posedness of the linear problem, viewing the transport term perturbatively, which only led to mild growth. We derive a nonlinear model taking all of the leading-order effects into account to determine the precise pointwise growth of solutions for short time. Interestingly, the Euler transport term does counteract the linear growth so that the full nonlinear equation grows an order of magnitude less than the linear one. In particular, the (sharp) growth rate we establish is consistent with the global regularity of smooth solutions.

1. Introduction

The Euler equations for incompressible flow are a fundamental model in fluid dynamics that describe the motion of ideal fluids:

$$\begin{aligned} \partial_t u + u \cdot \nabla u + \nabla p &= 0, \\ \nabla \cdot u &= 0. \end{aligned} \tag{1-1}$$

In this equation, u is the velocity field and p is the pressure of an ideal fluid flowing in $\mathbb{R}^2$. A key difficulty in understanding the dynamics of 2-dimensional Euler flows is the nonlocality of the system due to the presence of the pressure term.

Defining the vorticity $\omega := \nabla^\perp \cdot u$, it is insightful to study the Euler equations in vorticity form:

$$\begin{aligned} \partial_t \omega + u \cdot \nabla \omega &= 0, \\ \nabla \cdot u &= 0, \\ u &= \nabla^\perp \Delta^{-1} \omega. \end{aligned} \tag{1-2}$$

Because the L^∞ norm of vorticity is conserved in the Euler equations in two dimensions, Yudovich [1963] proved that there is a unique global-in-time solution to the Euler equation corresponding to every initial bounded and decaying vorticity. See also [Wolibner 1933; Beale et al. 1984; Hölder 1933; Yudovich 1963; Kato 1967; Marchioro and Pulvirenti 1994; Majda and Bertozzi 2002]. This bound on the L^∞ norm is unfortunately unstable even to very mild perturbations of the equation [Constantin and Vicol 2012; Elgindi

MSC2020: 35Q31, 35Q35.
Keywords: Euler equations, ill-posedness, Euler–Riesz equations.

and Masmoudi 2020; Elgindi 2018]. To understand this phenomenon, we are interested in studying linear perturbations of the Euler equations in two dimensions as follows:

$$\begin{aligned} \partial_t u + u \cdot \nabla u + \nabla p &= \begin{pmatrix} 0 \\ u_1 \end{pmatrix}, \\ \nabla \cdot u &= 0. \end{aligned} \tag{1-3}$$

Equation (1-3) is a model for many problems in fluid dynamics that have a coupling with the Euler equations. For instance, similar types of equations appear in viscoelastic fluids, see [Constantin and Kliegl 2012; Elgindi and Rousset 2015; Lions and Masmoudi 2000; Chemin and Masmoudi 2001], and in magnetohydrodynamics, see [Boardman et al. 2020; Hmidi 2014; Cao and Wu 2011; Wu and Zhao 2023]. Further, they also appear when studying stochastic Euler equations; see [Glatt-Holtz and Vicol 2014].

Writing (1-3) in vorticity form, we get

$$\begin{aligned} \partial_t \omega + u \cdot \nabla \omega &= \partial_x u_1, \\ \nabla \cdot u &= 0, \\ u &= \nabla^{\perp} \Delta^{-1} \omega. \end{aligned} \tag{1-4}$$

We observe that the challenge of studying these equations is that the right-hand side of (1-4) can be written as the Riesz transform of vorticity $\partial_x u_1 = R(\omega)$, which is unbounded on L^∞. P. Constantin and V. Vicol [2012] considered these equations with weak dissipation, and they proved global well-posedness. However, without dissipation it is an open question whether these equations are globally well-posed. In this work, we are interested in the question of L^∞ ill/well-posedness of the Euler equations with Riesz forcing and the local rate of L^∞ growth. The first author and N. Masmoudi studied the Euler equations with Riesz forcing in [Elgindi and Masmoudi 2020], where they proved that it is mildly ill-posed. This means that there is a universal constant $c > 0$ such that, for all $\epsilon > 0$, there is $\omega_0 \in C^\infty$ for which the unique local solution to (1-4) satisfies

$$|\omega_0|_{L^\infty} \leq \epsilon, \quad \text{but} \quad \sup_{t\in[0,\epsilon]} |\omega(t)|_{L^\infty} \geq c. \tag{1-5}$$

The authors in [Elgindi and Masmoudi 2020] conjectured that the Euler equation with Riesz forcing is actually strongly ill-posed in L^∞. Namely, that we can take c in (1-5) to be arbitrarily large. The goal of our work here is to show that indeed this is possible. To show this, we use the first author's Biot–Savart law decomposition [Elgindi 2021] to derive a leading-order system for the Euler equations with Riesz forcing. We then show that the leading-order system is strongly ill-posed in L^∞. Using this, we can show that the Euler equation with Riesz forcing is strongly ill-posed by estimating the error between the leading-order system and the Euler with Riesz forcing system on a specific time interval.

We should remark that the main application of the approach of [Elgindi and Masmoudi 2020] was to prove ill-posedness of the Euler equation in the integer C^k spaces, which was also proved independently by J. Bourgain and D. Li [2015]. Regarding the notion of mild ill-posedness in L^∞ for models related to the Euler with Riesz forcing system, see [Wu and Zhao 2023] about the 2-dimensional resistive MHD equations.

1.1. ***Statement of the main result.***

Theorem 1. *For any $\alpha, \delta > 0$, there exists an initial data $\omega_0^{\alpha,\delta} \in C_c^\infty(\mathbb{R}^2)$ and $T(\alpha)$ such that the corresponding unique global solution, $\omega^{\alpha,\delta}$, to* (1-4) *is such that at $t = 0$ we have*

$$|\omega_0^{\alpha,\delta}|_{L^\infty} = \delta,$$

but for any $0 < t \le T(\alpha)$ we have

$$|\omega^{\alpha,\delta}(t)|_{L^\infty} \ge |\omega_0|_{L^\infty} + c\log\Big(1 + \frac{c}{\alpha}t\Big),$$

where $T(\alpha) = c\alpha\log(c|\log(\alpha)|)$, and $c > 0$ is a constant independent of α that depends linearly on δ.

Remark 1.1. Note that at time $t = T(\alpha)$, we have

$$|\omega^{\alpha,\delta}|_{L^\infty} \ge c\log(c\log(c|\log(\alpha)|)),$$

which can be made arbitrarily large as $\alpha \to 0$. Fixing $\delta > 0$ small and then taking α sufficiently small thus gives strong ill-posedness for (1-4) in L^∞.

Remark 1.2. As we will discuss below, we in fact establish upper and lower bounds on the solutions we construct so that on the same time-interval we have

$$|\omega^{\alpha,\delta}(t)|_{L^\infty} \approx |\omega_0|_{L^\infty} + c\log\Big(1 + \frac{c}{\alpha}t\Big).$$

This should be contrasted with the linear problem where the upper and lower bounds for the same data come without the log:

$$|\omega_{\text{linear}}^{\alpha,\delta}(t)|_{L^\infty} \approx |\omega_0|_{L^\infty} + c\Big(1 + \frac{c}{\alpha}t\Big).$$

Remark 1.3. Our ill-posedness result applies to the equation

$$\partial_t\omega + u \cdot \nabla\omega = R(\omega),$$

where $R = R_{12} = \partial_{12}\Delta^{-1}$. Note that a direct consequence of the result gives strong ill-posedness when $R = R_{11}$ or $R = R_{22}$ even though these are dissipative on L^2. This can be seen just by noting that a linear change of coordinates can transform R_{12} to a constant multiple of $R_{11} - R_{22} = R_{11} - \text{Id}$. The strong ill-posedness for the Euler equation with forcing by any second-order Riesz transform (other than the identity) follows. We further remark that the same strategy can be used to study the case of general Riesz transforms, though we do not undertake this here since the case of forcing by second-order Riesz transforms is the most relevant for applications we are aware of (such as the 3-dimensional Euler equations, the Boussinesq system, viscoelastic models, MHD, etc.).

1.2. ***Comparison with the linear equation and the effect of transport.*** We now move to compare the result of this paper with the corresponding linear results and emphasize the regularizing effect of the nonlinearity in this problem. The ill-posedness result of [Elgindi and Masmoudi 2020] relies on viewing (1-4) as a perturbation of

$$\partial_t f = R(f). \tag{1-6}$$

For this simple linear equation, it is easy to show that L^∞ data can immediately develop a logarithmic singularity. Let us mention two ways to quantify this logarithmic singularity. One way is to study the growth of L^p norms as $p \to \infty$. For the linear equation (1-6), it is easy to show that the upper bound

$$|f(t)|_{L^p} \le \exp(Ct)p|f_0|_{L^p}$$

is sharp in the sense that we can find localized L^∞ data for which the solution satisfies

$$|f(t)|_{L^p} \ge c(t) \cdot p.$$

This can be viewed as approximating L^∞ "from below". Similarly, the C^α bound for (1-6),

$$|f(t)|_{C^\alpha} \le \frac{\exp(Ct)}{\alpha}|f_0|_{C^\alpha},$$

can also be shown to be sharp for short time in that we can find for each $\alpha > 0$ smooth and localized data with $|f_0|_{C^\alpha} = 1$ for which

$$|f(t)|_{L^\infty} \ge \frac{c(t)}{\alpha}.$$

The main result of [Elgindi and Masmoudi 2020] was that these upper and lower bounds remain unchanged in the presence of a transport term by a Lipschitz continuous velocity field. This is not directly applicable to our setting since the coupling between ω and u is such that u may not be Lipschitz even if ω is bounded. Interestingly, in [Elgindi 2018], it was shown that this growth could be significantly stronger in the presence of a merely bounded velocity field.

All of the above discussion leads us to understand that the nature of the well/ill-posedness of (1-4) will depend on the precise relationship between the velocity field and the linear forcing term in (1-4). In particular, for a natural class of data, we construct solutions to (1-4) satisfying

$$|\omega|_{L^\infty} \approx 1 + \log\left(1 + \frac{t}{\alpha}\right),$$

for short time, which is the best growth rate possible in this setting. This should be contrasted with the corresponding growth rate for the linear problem

$$|\omega_{\text{linear}}|_{L^\infty} \approx 1 + \frac{t}{\alpha}.$$

In particular, the nonlinear term in (1-4) actually tries to *prevent* L^∞ growth. Let us finally remark that the weak growth rate we found is consistent with the vorticity trying to develop a $\log\log$ singularity. It is curious that, in the Euler equation, vorticity with nearly $\log\log$ data is perfectly well-behaved and consistent with global regularity but with a triple exponential upper bound on gradients. Though establishing the global regularity rigorously remains a major open problem, this appears to be a sign that perhaps smooth solutions to (1-3) are globally regular.

1.3. ***A short discussion of the proof.*** The first step of the proof is to use the Biot–Savart law decomposition in [Elgindi 2021] to derive a leading-order model:

$$\partial_t \Omega + \frac{1}{2\alpha}(L_s(\Omega)\sin(2\theta) + L_c(\Omega)\cos(2\theta))\partial_\theta \Omega = \frac{1}{2\alpha}L_s(\Omega),$$

where the operators L_s and L_c are bounded linear operators on L^2 defined by

$$L_s(f)(R) = \frac{1}{\pi}\int_R^\infty\int_0^{2\pi} \frac{f(s,\theta)}{s}\sin(2\theta)\,d\theta\,ds \quad \text{and} \quad L_c(f)(R) = \frac{1}{\pi}\int_R^\infty\int_0^{2\pi} \frac{f(s,\theta)}{s}\cos(2\theta)\,d\theta\,ds.$$

Essentially all we do here is replace the velocity field by its most singular part. Upon inspecting this model, we observe that the forcing term on the right-hand side is purely radial, while the direction of transport is angular. Upon choosing a suitable unknown, we thus reduce the problem to solving a transport equation for some unknown f:

$$\partial_t f + \frac{1}{2\alpha}L_s(f)\sin(2\theta)\partial_\theta f = 0.$$

Surprisingly, this reduced equation propagates the usual "odd-odd" symmetry even though the original system does not. The leading-order model will then be strongly ill-posed if we can ensure that the solution of this transport equation satisfies that $\int_0^t L_s(f)$ can be arbitrarily large. One subtlety is that the growth of $L_s(f)$ enhances the transport effect, which in turn depletes the growth of $L_s(f)$. In fact, were the transport term to be stronger even by a log, the problem would *not* be strongly ill-posed. By a careful study of the characteristics of this equation, we obtain a closed nonlinear integrodifferential equation governing the evolution of $L_s(f)$ (see (3-4)). We study this nonlinear integrodifferential equation and establish upper and lower bounds on $L_s(f)$ proving strong ill-posedness for the leading-order equation; see Section 3 for more details. Finally, we close the argument by estimating the error incurred by approximating the dynamics with the leading-order model. An important idea here is to work on a time scale long enough to see the growth from the leading-order model but short enough to suppress any potential stronger nonlinear growth; see Section 6 for more details.

1.4. ***Organization.*** This paper is organized as follow: In Section 2, we derive a leading-order model for the Euler equations with Riesz forcing (1-4) based on the first author's Biot–Savart law approximation [Elgindi 2021]. Then, in Section 3, we obtain a pointwise estimate on the leading-order model which is the main ingredient in obtaining the strong ill-posedness result for the Euler with Riesz forcing system. In addition, in Section 3, we also obtain some estimates on the leading-order model in suitable norms which will be then used in estimating the remainder term in Section 6. After that, in Section 4 we will recall the Biot–Savart law decomposition obtained in [Elgindi 2021], and we will include a short sketch of the proof. In Section 5, we will obtain some embedding estimates which will also be used in Section 6 for the remainder term estimates. Then, in Section 6, we show that the remainder term remains small which will then allow us to prove the main result in Section 7.

1.5. ***Notation.*** In this paper, we will be working in a form polar coordinates introduced in [Elgindi 2021]. Let r be the radial variable,

$$r = \sqrt{x^2+y^2},$$

and since we will be working with functions of the variable r^α, where $0 < \alpha < 1$, we will use R to denote it:

$$R = r^\alpha.$$

We will use θ to denote the angle variable:

$$\theta = \arctan \frac{y}{x}.$$

We will use $|f|_{L^\infty}$ and $|f|_{L^2}$ to denote the usual L^∞ and L^2 norms, respectively. In addition, we will use f_t or f_τ to denote the time variable. Further, in this paper, following [Elgindi 2021], we will be working on $(R, \theta) \in [0, \infty) \times \left[0, \frac{\pi}{2}\right]$ where the L^2 norm will be with measure $dR\, d\theta$ and not $R\, dR\, d\theta$.

We define the weighted $\mathcal{H}^k\left([0, \infty) \times \left[0, \frac{\pi}{2}\right]\right)$ norm as

$$|f|_{\dot{\mathcal{H}}^m} = \sum_{i=0}^{m} |\partial_R^i \partial_\theta^{m-i} f|_{L^2} + \sum_{i=1}^{m} |R^i \partial_R^i \partial_\theta^{m-i} f|_{L^2}, \quad |f|_{\mathcal{H}^k} = \sum_{m=0}^{k} |f|_{\dot{\mathcal{H}}^m}.$$

We also define the $\mathcal{W}^{k,\infty}$ norm as

$$|f|_{\dot{\mathcal{W}}^{m,\infty}} = \sum_{i=0}^{m} |\partial_R^i \partial_\theta^{m-i} f|_{L^\infty} + \sum_{i=1}^{m} |R^i \partial_R^i \partial_\theta^{m-i} f|_{L^\infty}, \quad |f|_{\mathcal{W}^{k,\infty}} = \sum_{m=0}^{k} |f|_{\dot{\mathcal{W}}^{m,\infty}}.$$

Throughout this paper, we will use the notation

$$L(f)(R) = \int_R^\infty \frac{f(s)}{s}\, ds$$

to define operators, and by adding a subscript L_s or L_c we denote the projection onto $\sin(2\theta)$ and $\cos(2\theta)$ respectively. Namely,

$$L_s(f)(R) = \frac{1}{\pi} \int_R^\infty \int_0^{2\pi} \frac{f(s, \theta)}{s} \sin(2\theta)\, d\theta\, ds \quad \text{and} \quad L_c(f)(R) = \frac{1}{\pi} \int_R^\infty \int_0^{2\pi} \frac{f(s, \theta)}{s} \cos(2\theta)\, d\theta\, ds.$$

2. Leading-order model

In this section, we will derive a leading-order model for the Euler equation with Riesz forcing:

$$\begin{aligned} \partial_t \omega + u \cdot \nabla \omega &= \partial_x u_1, \\ \nabla \cdot u &= 0, \\ u &= \nabla^\perp \Delta^{-1} \omega. \end{aligned} \tag{2-1}$$

To do this, we follow [Elgindi 2021] and we write the equation in a form of polar coordinates. Namely, we set $r = \sqrt{x^2 + y^2}$, $R = r^\alpha$, and $\theta = \arctan(y/x)$. We will the rewrite (2-1) in the new functions $\omega(x, y) = \Omega(R, \theta)$ and $\psi(x, y) = r^2 \Psi(R, \theta)$, with $u = \nabla^\perp \psi$, where $u_1 = -\partial_y \psi$ and $u_2 = \partial_x \psi$.

<u>Equations of u in terms of Ψ</u>:

$$u_1 = -r(2\sin(\theta)\Psi + \alpha \sin(\theta) R\, \partial_R \Psi + \cos(\theta) \partial_\theta \Psi),$$
$$u_2 = r(2\cos(\theta)\Psi + \alpha \cos(\theta) R\, \partial_R \Psi - \sin(\theta) \partial_\theta \Psi).$$

Evolution equation for Ω:

$$\begin{aligned}\partial_t\Omega+(-\alpha R\partial_\theta\Psi)\partial_R\Omega+(2\Psi+\alpha R\partial_R\Psi)\partial_\theta\Omega\\ &=\big(-2\alpha R\sin(\theta)\cos(\theta)-\alpha^2R\sin(\theta)\cos(\theta)\big)\partial_R\Psi+(-1+2\sin^2(\theta))\partial_\theta\Psi\\ &\quad+\big(-\alpha R\cos^2(\theta)+\alpha R\sin^2(\theta)\big)\partial_{R\theta}\Psi-(\alpha^2R^2\sin(\theta)\cos(\theta))\partial_{RR}\Psi+(\sin(\theta)\cos(\theta))\partial_{\theta\theta}\Psi.\end{aligned}$$

The elliptic equation for $\Delta(r^2\Psi(R,\theta))=\Omega(R,\theta)$:

$$4\Psi+\alpha^2R^2\partial_{RR}\Psi+\partial_{\theta\theta}\Psi+(4\alpha+\alpha^2)R\partial_R\Psi=\Omega(R,\theta).$$

Now using the Biot–Savart decomposition of [Elgindi 2021], see Section 4 for more details, by defining the operators

$$L_s(\Omega)(R)=\frac{1}{\pi}\int_R^\infty\int_0^{2\pi}\frac{\Omega(s,\theta)}{s}\sin(2\theta)\,d\theta\,ds\quad\text{and}\quad L_c(\Omega)(R)=\frac{1}{\pi}\int_R^\infty\int_0^{2\pi}\frac{\Omega(s,\theta)}{s}\cos(2\theta)\,d\theta\,ds$$

we have

$$\Psi(R,\theta)=-\frac{1}{4\alpha}L_s(\Omega)\sin(2\theta)-\frac{1}{4\alpha}L_c(\Omega)\cos(2\theta)+\text{lower-order terms.}$$

Thus, if we ignore the α-terms in the evolution equation, we obtain

$$\partial_t\Omega+(2\Psi)\partial_\theta\Omega=(-1+2\sin^2(\theta))\partial_\theta\Psi+(\sin(\theta)\cos(\theta))\partial_{\theta\theta}\Psi. \tag{2-2}$$

Now we consider Ψ of the form

$$\Psi=-\frac{1}{4\alpha}L_s(\Omega)\sin(2\theta)-\frac{1}{4\alpha}L_c(\Omega)\cos(2\theta),$$

and plugging it into the evolution equation, we have

$$\begin{aligned}\partial_t\Omega-\Big(\frac{1}{2\alpha}L_s(\Omega)\sin(2\theta)+\frac{1}{2\alpha}L_c(\Omega)\cos(2\theta)\Big)\partial_\theta\Omega&=-(\cos(2\theta))\Big(-\frac{1}{2\alpha}L_s(\Omega)\cos(2\theta)+\frac{1}{2\alpha}L_c(\Omega)\sin(2\theta)\Big)\\&\quad+\Big(\frac{1}{2}\sin(2\theta)\Big)\Big(\frac{1}{\alpha}L_s(\Omega)\sin(2\theta)+\frac{1}{\alpha}L_c(\Omega)\cos(2\theta)\Big),\end{aligned}$$

which simplifies to

$$\partial_t\Omega-\Big(\frac{1}{2\alpha}L_s(\Omega)\sin(2\theta)+\frac{1}{2\alpha}L_c(\Omega)\cos(2\theta)\Big)\partial_\theta\Omega=\frac{1}{2\alpha}L_s(\Omega).$$

In order to work with positive solutions and have the angular trajectories moving to the right, we make the change $\Omega\to-\Omega$ and get the final model

$$\partial_t\Omega+\Big(\frac{1}{2\alpha}L_s(\Omega)\sin(2\theta)+\frac{1}{2\alpha}L_c(\Omega)\cos(2\theta)\Big)\partial_\theta\Omega=\frac{1}{2\alpha}L_s(\Omega). \tag{2-3}$$

We now move to study the dynamics of solutions to (2-3).

Proposition 2.1. *Let Ω be a solution to the leading-order model*

$$\partial_t\Omega+\Big(\frac{1}{2\alpha}L_s(\Omega)\sin(2\theta)+\frac{1}{2\alpha}L_c(\Omega)\cos(2\theta)\Big)\partial_\theta\Omega=\frac{1}{2\alpha}L_s(\Omega), \tag{2-4}$$

with initial data of the form $\Omega|_{t=0} = f_0(R)\sin(2\theta)$. *Then we can write* Ω *as*

$$\Omega = f + \frac{1}{2\alpha}\int_0^t L_s(f_\tau)\,d\tau, \tag{2-5}$$

where f *satisfies the transport equation*

$$\partial_t f + \frac{1}{2\alpha}\sin(2\theta)L_s(f)\partial_\theta f = 0. \tag{2-6}$$

Proof. The right-hand side term of (2-4) is radial, and hence if we take the inner product with $\sin(2\theta)$ it will be zero. Now if write Ω as

$$\Omega_t(R,\theta) = f_t(R,\theta) + \frac{1}{2\alpha}\int_0^t L_s(\Omega_\tau)(R)\,d\tau,$$

and consider it to be a solution to (2-4), we obtain that f satisfies

$$\partial_t f_t + \Big(\frac{1}{2\alpha}L_s(f_t)\sin(2\theta) + \frac{1}{2\alpha}L_c(f_t)\cos(2\theta)\Big)\,\partial_\theta f_t = 0. \tag{2-7}$$

Here we used that $L_s(\Omega_\tau)(R)$ is a radial function. Notice that (2-7) is a transport equation that preserves odd symmetry. Now if we set

$$f_t^s = \int_0^{2\pi} f_t(R,\theta)\sin(2\theta)\,d\theta \quad \text{and} \quad \Omega_t^s = \int_0^{2\pi} \Omega_t(R,\theta)\sin(2\theta)\,d\theta,$$

we notice that f_t^s and Ω_t^s will satisfy the same equation. Thus, if we start with the same initial conditions $f_0 = \Omega_0$, then

$$f_t^s = \Omega_t^s \quad \text{for all } t.$$

Thus, we have $L_s(\Omega_t) = L_s(f_t)$, and hence

$$\Omega_t = f_t + \frac{1}{2\alpha}\int_0^t L_s(f_\tau)\,d\tau.$$

Now since the initial data which we are considering have odd symmetry, it suffices to consider the transport equation:

$$\partial_t f_t + \frac{1}{2\alpha}\sin(2\theta)L_s(f_t)\partial_\theta f_t = 0. \qquad \square$$

3. Leading-order model estimate

The purpose of this section is to obtain L^∞ estimates for the leading-order model, which is the main ingredient in obtaining the ill-posedness result for the Euler with Riesz forcing system. This will be done in Section 3.1 in three steps: Lemma 3.1, Lemma 3.2, and Proposition 3.3. Then in Section 3.2, we will obtain an estimate for the leading-order model which will be useful in remainder estimates in Section 6.

3.1. *Pointwise leading-order model estimate.*

Lemma 3.1. *Let* f *be a solution to the transport equation*

$$\partial_t f + \frac{1}{2\alpha}\sin(2\theta)L_s(f)\partial_\theta f = 0, \tag{3-1}$$

with initial data $f|_{t=0} = f_0(R)\sin(2\theta)$. *Then we have the following estimate on the operator* $L_s(f)$:

$$c_1 \int_R^\infty \frac{f_0(s)}{s} \exp\Big(-\frac{1}{\alpha}\int_0^t L_s(f_\tau)(s)\,d\tau\Big)\,ds \le L_s(f_t)(R) \le c_2 \int_R^\infty \frac{f_0(s)}{s} \exp\Big(-\frac{1}{\alpha}\int_0^t L_s(f_\tau)(s)\,d\tau\Big)\,ds, \tag{3-2}$$

where c_1 *and* c_2 *are independent of* α.

Proof. To prove this, we consider the following variable change. For $\theta \in \big[0, \frac{\pi}{2}\big)$, let γ be defined as

$$\gamma := \tan(\theta) \implies \frac{d\gamma}{d\theta} = \sec^2(\theta), \quad \text{and} \quad \sin(2\theta) = \frac{2\gamma}{1+\gamma^2}.$$

Applying the chain rule, we rewrite (3-1) in the (R, γ)-variables

$$\partial_t f_t + \frac{1}{\alpha}\gamma L_s(f_t)(R)\,\partial_\gamma f = 0, \tag{3-3}$$

with initial data

$$f|_{t=0} = f_0(R)\sin(2\theta) = f_0(R)\frac{2\gamma}{1+\gamma^2}.$$

Let $\phi_t(\gamma)$ be the flow map associated with (3-3), so we have

$$\frac{d\phi_t(\gamma)}{dt} = \frac{1}{\alpha}\phi_t(\gamma)L_s(f_t) \implies \phi_t(\gamma) = \gamma \exp\Big(\frac{1}{\alpha}\int_0^t L_s(f_\tau)\,d\tau\Big).$$

Thus,

$$\phi_t^{-1}(\gamma) = \gamma \exp\Big(-\frac{1}{\alpha}\int_0^t L_s(f_\tau)\,d\tau\Big).$$

Hence, we now write the solution to (3-3) as

$$f_t(R,\gamma) = f_0(R, \phi_t^{-1}(\gamma)) = f_0(R)\frac{2\phi_t^{-1}(\gamma)}{1+\phi_t^{-1}(\gamma)^2} = f_0(R)\frac{2\gamma\exp\big(-(1/\alpha)\int_0^t L_s(f_\tau)\,d\tau\big)}{1+\gamma^2\exp\big(-(2/\alpha)\int_0^t L_s(f_\tau)\,d\tau\big)}.$$

Now we consider the operator L_s in the $(R,\gamma) \in [0,\infty) \times \big[0, \frac{\pi}{2}\big)$-variables:

$$L_s(f_t)(R) = \frac{1}{\pi}\int_R^\infty \frac{1}{s}\int_0^\infty f_t(s,\gamma)\frac{2\gamma}{(1+\gamma^2)^2}\,d\gamma\,ds.$$

Plugging in the expression for f_t, we have

$$L_s(f_t)(R) = \frac{1}{\pi}\int_R^\infty \frac{1}{s}\int_0^\infty f_0(s)\frac{\exp\big(-(1/\alpha)\int_0^t L_s(f_\tau)(s)\,d\tau\big)}{1+\gamma^2\exp\big(-(2/\alpha)\int_0^t L_s(f_\tau)(s)\,d\tau\big)}\frac{4\gamma^2}{(1+\gamma^2)^2}\,d\gamma\,ds. \tag{3-4}$$

Now since $0 \le \exp\big(-(2/\alpha)\int_0^t L_s(f_\tau)(s)\,d\tau\big) \le 1$, we have an upper and a lower bound on the operator on $L_s(f_t)(R)$ with constants c_1, c_2 independent of α (in fact, these constants can be explicitly computed). Namely,

$$c_1 \int_R^\infty \frac{f_0(s)}{s} \exp\Big(-\frac{1}{\alpha}\int_0^t L_s(f_\tau)(s)\,d\tau\Big)\,ds \le L_s(f_t)(R) \le c_2 \int_R^\infty \frac{f_0(s)}{s} \exp\Big(-\frac{1}{\alpha}\int_0^t L_s(f_\tau)(s)\,d\tau\Big)\,ds.$$

Thus, we have our desired inequalities. □

Lemma 3.2. *Define the operator*

$$\hat{L}(f_t)(R) := \int_R^\infty \frac{f_0(s)}{s} \exp\Bigl(-\frac{1}{\alpha}\int_0^t \hat{L}(f_s)(s)\,d\tau\Bigr)\,ds. \tag{3-5}$$

Then we have

$$\int_0^t \hat{L}(f_\tau)(R)\,d\tau = 2\alpha \log\Bigl(1+\frac{t}{2\alpha}L(f_0)(R)\Bigr),$$

where $L(f_0)(R) = \int_R^\infty f_0(s)/s\,ds$.

Proof. We introduce $g_t(R) := \exp\bigl(-(1/\alpha)\int_0^t \hat{L}(f_\tau)(R)\,d\tau\bigr)$ and $K(R) := f_0(R)/R$. Then the operator $\hat{L}$ can be rewritten as

$$\hat{L}(f_t)(R) = \int_R^\infty K(s)g_t(s)\,ds. \tag{3-6}$$

Now taking the time derivative of (3-6), and using that $\partial_t g_t(R) = -(1/\alpha)g_t(R)\int_R^\infty K(s)g_t(s)\,ds$, we can obtain

$$\partial_t \hat{L}(f_t) = -\frac{1}{2\alpha}(\hat{L}(f_t))^2,$$

which can be solved explicitly:

$$\hat{L}(f_t)(R) = \frac{L(f_0)(R)}{1+(t/(2\alpha))L(f_0)(R)}. \tag{3-7}$$

Then it follows that

$$\int_0^t \hat{L}(f_t)(R)\,d\tau = 2\alpha \log\Bigl(1+\frac{t}{2\alpha}L(f_0)(R)\Bigr). \qquad \square$$

Proposition 3.3. *Let f be a solution to the transport equation*

$$\partial_t f + \frac{1}{2\alpha}\sin(2\theta)L_s(f)\partial_\theta f = 0, \tag{3-8}$$

with initial data $f|_{t=0} = f_0(R)\sin(2\theta)$. Then we have the following estimate on the operator $L_s(f)$:

$$\frac{2\alpha}{c_1}\log\Bigl(1+\frac{c_1}{2\alpha}tL(f_0)(R)\Bigr) \ge \int_0^t L_s(f_\tau)(R) \ge \frac{2\alpha}{c_2}\log\Bigl(1+\frac{c_2}{2\alpha}tL(f_0)(R)\Bigr), \tag{3-9}$$

where c_1 and c_2 are independent of α.

Proof. In the section, we will use the bounds in (3-2), namely

$$c_1\int_R^\infty \frac{f_0(s)}{s}\exp\Bigl(-\frac{1}{\alpha}\int_0^t L_s(f_\tau)(s)\,d\tau\Bigr)\,ds \\ \le L_s(f_t)(R) \le c_2\int_R^\infty \frac{f_0(s)}{s}\exp\Bigl(-\frac{1}{\alpha}\int_0^t L_s(f_\tau)(s)\,d\tau\Bigr)\,ds, \tag{3-10}$$

to obtain and upper and lower estimate on $\int_0^t L_s(f)$. As before we set

$$g_t(R) = \exp\Bigl(-\frac{1}{\alpha}\int_0^t L_s(f_\tau)(R)\,d\tau\Bigr) \quad \text{and} \quad K(R) = \frac{f_0(R)}{R}.$$

Using (3-10), we can obtain that

$$-\frac{c_1}{2\alpha}\left(\int_R^\infty g_t(s)K(s)\,ds\right)^2 \geq \partial_t \int_R^\infty g_t(s)K(s)\,ds \geq -\frac{c_2}{2\alpha}\left(\int_R^\infty g_t(s)K(s)\,du\right)^2. \tag{3-11}$$

Similar to Lemma 3.2, we define

$$L_s(f_t)(R) := \int_R^\infty g_t(s)K(s)\,ds.$$

Now from (3-11), we have

$$-\frac{c_1}{2\alpha}(L_s(f_t)(R))^2 \geq \partial_t L_s(f_t)(R) \geq -\frac{c_2}{2\alpha}(L_s(f_t)(R))^2.$$

Thus,

$$\frac{L(f_0)(R)}{1+(c_1/(2\alpha))tL(f_0)(R)} \geq L_s(f_t)(R) \geq \frac{L(f_0)(R)}{1+(c_2/(2\alpha))tL(f_0)(R)}, \tag{3-12}$$

which will give us that

$$\frac{2\alpha}{c_1}\log\left(1+\frac{c_1}{2\alpha}tL(f_0)(R)\right) \geq \int_0^t L_s(f_\tau)(R) \geq \frac{2\alpha}{c_2}\log\left(1+\frac{c_2}{2\alpha}tL(f_0)(R)\right),$$

and this completes the proof. □

3.2. ***Estimate for the leading-order model in $\mathcal{W}^{k,\infty}$ and $\mathcal{H}^k$ norms.*** The purpose of this subsection is to obtain some estimate on the leading-order model in $\mathcal{W}^{k,\infty}$ and $\mathcal{H}^k$ norms. These will be used to estimate the size of the remainder term in Section 6. First we will obtain estimates on Ψ_2 in Lemma 3.4. Then in Lemma 3.5, we will obtain estimates on Ω_2.

Lemma 3.4. *Let Ω_2 be a solution to the leading-order model*:

$$\partial_t\Omega_2 + \left(\frac{1}{2\alpha}L_s(\Omega_2)\sin(2\theta) + \frac{1}{2\alpha}L_c(\Omega_2)\cos(2\theta)\right)\partial_\theta\Omega_2 = \frac{1}{2\alpha}L_s(\Omega_2),$$

with initial data $\Omega_2|_{t=0} = f_0(R)\sin(2\theta)$, where $f_0(R)$ is smooth and compactly supported. Consider

$$\Psi_2 = \frac{1}{4\alpha}L_s(\Omega_2)\sin(2\theta) + \frac{1}{4\alpha}L_c(\Omega_2)\cos(2\theta).$$

Then, we have the following estimates on Ψ_2:

$$|\Psi_2|_{\mathcal{W}^{k+1,\infty}} \leq \frac{c_k}{\alpha}, \quad |\Psi_2|_{\mathcal{H}^{k+1}} \leq \frac{c_k}{\alpha}, \tag{3-13}$$

where c_k depends on the initial conditions and is independent of α.

Proof. Recall that from Proposition 2.1, we can write Ω_2 as

$$\Omega_2 = f + \frac{1}{2\alpha}\int_0^t L_s(f_\tau)\,d\tau,$$

and since the initial data is odd in θ, we have

$$\Psi_2 = \frac{1}{4\alpha}L_s(\Omega_t)\sin(2\theta) = \frac{1}{4\alpha}L_s(f_t)\sin(2\theta).$$

To estimate the size of Ψ_2, from (3-4), we have

$$L_s(f_t)(R)=\int_R^\infty \frac{1}{s}\int_0^\infty f_0(s)\frac{\exp\bigl(-(1/\alpha)\int_0^t L_s(f_\tau)(s)\,d\tau\bigr)}{1+\gamma^2\exp\bigl(-(2/\alpha)\int_0^t L_s(f_\tau)(s)\,d\tau\bigr)}\frac{4\gamma^2}{(1+\gamma^2)^2}\,d\gamma\,ds.$$

Using (3-2), we have

$$|\Psi_2|_{L^\infty}\le \frac{c}{\alpha}\int_R^\infty \frac{f_0(s)}{s}\,ds\le\frac{c_0}{\alpha}.$$

For $\partial_\theta\Psi_2$, it is clear that we have

$$|\partial_\theta\Psi_2|_{L^\infty}\le\frac{c_0}{\alpha},$$

where, similarly, c_0 depends on the initial condition.

Now for $\partial_R\Psi_2$, we have

$$\partial_R\Psi_2=\frac{1}{4\alpha}\partial_R L_s(f_t)\sin(2\theta).$$

Thus,

$$\partial_R L_s(f_t)(R)=-\frac{1}{R}\int_0^\infty f_0(R)\frac{\exp\bigl(-(1/\alpha)\int_0^t L_s(f_\tau)(R)\,d\tau\bigr)}{1+\gamma^2\exp\bigl(-(2/\alpha)\int_0^t L_s(f_\tau)(R)\,d\tau\bigr)}\frac{4\gamma^2}{(1+\gamma^2)^2}\,d\gamma,$$

and similarly, we have

$$|\partial_R\Psi_2|_{L^\infty}\le\frac{c}{\alpha}.$$

Now the estimate on $R\,\partial_R\Psi_2$ follows from the estimate on $\partial_R\Psi_2$ and the fact that the initial data have compact support. Thus,

$$|R\,\partial_R\Psi_2|_{L^\infty}\le\frac{c}{\alpha}.$$

For higher-order derivatives, we can obtain the estimate following the same steps. Hence, we have

$$|\Psi|_{\mathcal{W}^{k+1,\infty}}\le\frac{c_k}{\alpha}.$$

The $\mathcal{H}^k$ estimates also follow using the same steps:

$$|\Psi|_{\mathcal{H}^{k+1}}\le\frac{c_k}{\alpha}.\qquad\square$$

In the following lemma, we will obtain the $\mathcal{H}^k$ estimates on Ω_2. Here we will use Lemma 3.5 and transport estimates.

Lemma 3.5. *Let Ω_2 be a solution to the leading-order model*

$$\partial_t\Omega_2+\Bigl(\frac{1}{2\alpha}L_s(\Omega_2)\sin(2\theta)+\frac{1}{2\alpha}L_c(\Omega_2)\cos(2\theta)\Bigr)\partial_\theta\Omega_2=\frac{1}{2\alpha}L_s(\Omega_2),$$

with initial data $\Omega_2|_{t=0}=f_0(R)\sin(2\theta)$, where $f_0(R)$ is smooth and compactly supported. Then, we have the following estimates on Ω_2:

$$|\Omega_2|_{\mathcal{H}^k}\le c_k e^{(c_k/\alpha)t},\tag{3-14}$$

where c_k depends on the initial conditions and is independent of α.

Proof. Recall that from Proposition 2.1 we can write Ω_2 as

$$\Omega_2=f+\frac{1}{2\alpha}\int_0^t L_s(f_\tau)\,d\tau,$$

where f satisfies the transport equation

$$\partial_t f_t + 2\Psi_2\, \partial_\theta f_t = 0.$$

When we consider the derivatives of Ω_2, the transport term f dominates the radial term $(1/(2\alpha))\int_0^t L_s(f)\, d\tau$. Thus, it suffices to consider the $\mathcal{H}^k$ estimates on f which will follow from the standard L^2 estimate for the transport equation. Thus, we have

$$\partial_t f_t + 2\Psi_2 \partial_\theta f_t = 0 \quad \Longrightarrow \quad \partial_t \partial_\theta f_t + 2\partial_\theta \Psi_2 \partial_\theta f_t + 2\Psi_2 \partial_{\theta\theta} f_t = 0.$$

Hence,

$$|\partial_\theta f_t|_{L^2} \le |\partial_\theta f_0|_{L^2} e^{\int_0^t |\partial_\theta \Psi_2|_{L^\infty}}.$$

From (3-13) we have $|\partial_\theta \Psi_2|_{L^\infty} \le c_0/\alpha$. Thus, applying the Gronwall inequality, we have

$$|\partial_\theta f_t|_{L^2} \le |\partial_\theta f_0|_{L^2} e^{(c_0/\alpha)t}. \tag{3-15}$$

To obtain $\mathcal{H}^k$ estimates, we need to estimate terms of the form $R^k \partial_R^k$. We will show how to obtain the $R\partial_R$ estimate, and for general k, it will follow similarly. Thus, similar to L^2 estimate for the $\partial_\theta f$ case, since

$$\partial_t f_t + 2\Psi_2 \partial_\theta f_t = 0,$$

we have

$$\partial_t \partial_R f_t + 2\partial_R \Psi_2 \partial_\theta f_t + 2\Psi_2 \partial_{R\theta} f_t = 0,$$

and thus,

$$\partial_t |R\partial_R f_t|_{L^2} \le 2|R\partial_R \Psi_2|_{L^\infty} |\partial_\theta f|_{L^2} + |\partial_\theta \Psi_2|_{L^\infty} |R\partial_R f_t|_{L^2}.$$

Now from (3-13), (3-15), and applying the Gronwall inequality we have

$$|R\partial_R f_t|_{L^2} \le (|R\partial_R f_0|_{L^2} + |\partial_\theta f_0|_{L^2} e^{(c_0/\alpha)t}) e^{(c_0/\alpha)t}.$$

Hence,

$$|f(t)|_{\mathcal{H}^1} \le |f_0|_{\mathcal{H}^1} e^{(c_1/\alpha)t},$$

which implies that

$$|\Omega_2(t)|_{\mathcal{H}^1} \le |\Omega_2(0)|_{\mathcal{H}^1} e^{(c_1/\alpha)t}.$$

Similarly, using (3-13), the transport estimate, and following the same steps as above, we can obtain the general $\mathcal{H}^k$ estimates. Hence

$$|\Omega_2|_{\mathcal{H}^k} \le |\Omega_2(0)|_{\mathcal{H}^k} e^{(c_k/\alpha)t}. \qquad \square$$

4. Elliptic estimate

The purpose of this section is to recall the Biot–Savart law decomposition of [Elgindi 2021], which is used here to derive the leading-order model. In this section, we highlight the main ideas in the proof, and for more details, see [Elgindi 2021; Drivas and Elgindi 2023]. We remark that this is also related to the Key Lemma of A. Kiselev and V. Šverák [2014]; see also [Elgindi 2016; Elgindi and Jeong 2023] for generalizations.

Proposition 4.1 [Elgindi 2021]. *Given $\Omega \in H^k$ such that for every R we have*

$$\int_0^{2\pi} \Omega(R,\theta)\sin(n\theta)\,d\theta = \int_0^{2\pi} \Omega(R,\theta)\cos(n\theta)\,d\theta = 0$$

for $n = 0, 1, 2$, the unique solution to

$$4\Psi + \partial_{\theta\theta}\Psi + \alpha^2 R^2 \partial_{RR}\Psi + (4\alpha+\alpha^2)R\partial_R\Psi = \Omega(R,\theta)$$

satisfies

$$|\partial_{\theta\theta}\Psi|_{H^k} + \alpha|R\partial_{R\theta}\Psi|_{H^k} + \alpha^2|R^2\partial_{RR}\Psi|_{H^k} \le C_k|\Omega|_{H^k}, \tag{4-1}$$

*where C_k is **independent** of α. In addition, we have the weights estimate*

$$|\partial_{\theta\theta}D_R^k(\Psi)|_{L^2} + \alpha|R\partial_{R\theta}D_R^k(\Psi)|_{L^2} + \alpha^2|R^2\partial_{RR}D_R^k(\Psi)|_{L^2} \le C_k|D_R^k(\Omega)|_{L^2}, \tag{4-2}$$

*where C_k is **independent** of α. Recall that $D_R = R\partial_R$.*

Proof. First, we will show how to obtain (4-1). Since Ω is orthogonal to $\sin(n\theta)$ and $\cos(n\theta)$ for $n=0, 1, 2$, Ψ must also be orthogonal to $\sin(n\theta)$ and $\cos(n\theta)$ for $n = 0, 1, 2$. Consider the elliptic equation, and we consider the L^2 estimate

$$4\Psi + \partial_{\theta\theta}\Psi + \alpha^2 R^2 \partial_{RR}\Psi + (4\alpha+\alpha^2)R\partial_R\Psi = \Omega(R,\theta).$$

Taking the inner product with $\partial_{\theta\theta}\Psi$ and integrating by parts, we obtain

$$-4|\partial_\theta\Psi|_{L^2}^2 + |\partial_{\theta\theta}\Psi|_{L^2}^2 - \alpha^2|\partial_\theta\Psi|_{L^2}^2 + \alpha^2|R\partial_{R\theta}\Psi|_{L^2}^2 + \tfrac{1}{2}(4\alpha+\alpha^2)|\partial_\theta\Psi|_{L^2}^2 \le |\Omega|_{L^2}|\partial_{\theta\theta}\Psi|_{L^2}.$$

Now by assumption, we have

$$\Psi(R,\theta) = \sum_{n\ge 3}\Psi_n(R)e^{in\theta},$$

and hence

$$|\partial_\theta\Psi|_{L^2}^2 \le \tfrac{1}{9}|\partial_{\theta\theta}\Psi|_{L^2}^2.$$

Using the above inequality, we can show that

$$\tfrac{5}{9}|\partial_{\theta\theta}\Psi|_{L^2}^2 + \alpha^2|R\partial_{R\theta}\Psi|_{L^2}^2 + \tfrac{1}{2}(4\alpha-\alpha^2)|\partial_\theta\Psi|_{L^2}^2 \le |\Omega|_{L^2}|\partial_{\theta\theta}\Psi|_{L^2},$$

and thus we have

$$|\partial_{\theta\theta}\Psi|_{L^2} \le C_0|\Omega|_{L^2},$$

where C_0 is independent of α. The estimate for the $R^2\partial_{RR}\Psi$-term will follow similarly. We can also obtain the H^k estimates by following the same strategy. To obtain the (4-2) estimates, recall that $D_R = R\partial_R$ and we notice that we can write the elliptic equation in the form

$$4\Psi + \partial_{\theta\theta}\Psi + \alpha^2 D_R^2(\Psi) + 4\alpha\, D_R(\Psi) = \Omega(R,\theta).$$

From this, we observe that the D_R operator commutes with the elliptic equation, and hence (4-2) estimates will follow from (4-1). ☐

Theorem 2 [Elgindi 2021]. *Given $\Omega \in H^k$, where Ω has the form of*

$$\Omega(R,\theta) = f(R)\sin(2\theta) \quad \big(\Omega(R,\theta) = f(R)\cos(2\theta)\big),$$

the unique solution to

$$4\Psi + \partial_{\theta\theta}\Psi + \alpha^2 R^2 \partial_{RR}\Psi + (4\alpha+\alpha^2)R\partial_R\Psi = \Omega(R,\theta)$$

is

$$\Psi = -\frac{1}{4\alpha}L(f)(R)\sin(2\theta) + \mathcal{R}(f) \quad \Big(\Psi = -\frac{1}{4\alpha}L(f)(R)\cos(2\theta) + \mathcal{R}(f)\Big),$$

where

$$L(f)(R) = \int_R^\infty \frac{f(s)}{s}\,ds$$

and

$$|\mathcal{R}(f)|_{H^k} \le c|f|_{H^k},$$

where c is independent of α.

Proof. Consider the case where $\Omega(R,\theta) = f(R)\sin(2\theta)$; the case where $\Omega(R,\theta) = f(R)\cos(2\theta)$ can be handled similarly. In this case $\Psi(R,\theta)$ will be of the form $\Psi(R,\theta) = \Psi_2(R)\sin(2\theta)$, where $\Psi_2(R)$ will satisfy the ODE

$$\alpha^2 R^2 \partial_{RR}\Psi_2 + (4\alpha+\alpha^2)R\partial_R\Psi_2 = f(R).$$

We can solve the ODE, see Theorem 4.24 in [Drivas and Elgindi 2023], and obtain

$$\partial_R\Psi_2(R) = \frac{1}{\alpha^2}\frac{1}{R^{4/\alpha+1}}\int_0^R \frac{f(s)}{s^{1-4/\alpha}}\,ds.$$

Now using that $\Psi_2(R) \to 0$ as $R \to \infty$, we obtain

$$\Psi_2(R) = -\frac{1}{\alpha^2}\int_R^\infty \frac{1}{\rho^{4/\alpha+1}}\int_0^\rho \frac{f(s)}{s^{1-4/\alpha}}\,ds\,d\rho.$$

We notice that we can write the above as

$$\Psi_2(R) = -\frac{1}{\alpha^2}\int_R^\infty \frac{1}{\rho^{4/\alpha+1}}\int_0^\rho \frac{f(s)}{s^{1-4/\alpha}}\,ds\,d\rho = \frac{1}{4\alpha}\int_R^\infty \partial_\rho\Big(\frac{1}{\rho^{4/\alpha}}\Big)\int_0^\rho \frac{f(s)}{s^{1-4/\alpha}}\,ds\,d\rho.$$

Thus, by integrating by parts, it follows that

$$\Psi_2(R) = -\frac{1}{4\alpha}\int_R^\infty \frac{f(s)}{s}\,ds - \frac{1}{4\alpha}\frac{1}{R^{4/\alpha}}\int_0^R \frac{f(s)}{s^{1-4/\alpha}}\,ds := -\frac{1}{4\alpha}L(f)(R) + \mathcal{R}(f).$$

Using Hardy-type inequality, see Lemma 4.25 in [Drivas and Elgindi 2023], one can show that

$$|\mathcal{R}(f)|_{L^2} \le c|f|_{L^2},$$

where c is independent of α. □

5. Embedding estimate in terms of the $\mathcal{H}^k$ norm

In this section we consider some embedding estimate in the $\mathcal{H}^k$ norm which will be used in Section 6. These estimates will be used various times as we estimate the remainder term. Recall that the $\mathcal{H}^k$ norm is defined as

$$|f|_{\dot{\mathcal{H}}^m} = \sum_{i=0}^{m} |\partial_R^i \partial_\theta^{m-i} f|_{L^2} + \sum_{i=1}^{m} |R^i \partial_R^i \partial_\theta^{m-i} f|_{L^2}, \quad |f|_{\mathcal{H}^k} = \sum_{m=0}^{k} |f|_{\dot{\mathcal{H}}^m}.$$

Lemma 5.1. *Let $f \in \mathcal{H}^N$, where $N \in \mathbb{N}$. Then we have*

$$|\partial_R^k \partial_\theta^m f|_{L^\infty} \le c_{k,m} |f|_{\mathcal{H}^{k+m+2}}, \tag{5-1}$$

$$|R^k \partial_R^k \partial_\theta^m f|_{L^\infty} \le c_{k,m} |f|_{\mathcal{H}^{k+m+2}} \tag{5-2}$$

for any $k+m+2 \le N$.

Proof. We will show how to obtain inequality (5-2), since inequality (5-1) follows from standard Sobolev embedding. To show that

$$|R^k \partial_R^k \partial_\theta^m f|_{L^\infty} \le c_{k,m} |f|_{\mathcal{H}^{k+m+2}},$$

for any $k+m+2 \le N$, we apply Sobolev embedding to obtain

$$|R^k \partial_R^k \partial_\theta^m f|_{L^\infty} \le c_{k,m} |R^k \partial_R^k \partial_\theta^m f|_{H^2_{R,\theta}},$$

where $H^2_{R,\theta}$ is the standard H^2 norm in R and θ. When considering the second derivative terms of $R^k \partial_R^k \partial_\theta^m f$, for the angular derivatives term, we have $|R^k \partial_R^k \partial_\theta^{m+2} f|_{L^2} \le |f|_{\mathcal{H}^{k+m+2}}$. Now for the radial derivatives, we have three cases. Considering the case when the two radial derivatives land on $\partial_R^k \partial_\theta^m f$, we have

$$|R^k \partial_R^{k+2} \partial_\theta^m f|_{L^2} \le |R^{k+2} \partial_R^{k+2} \partial_\theta^m f|_{L^2} + |\partial_R^{k+2} \partial_\theta^m f| \le |f|_{\mathcal{H}^{k+m+2}},$$

where the last inequality follows from the definition of the $\mathcal{H}^N$ norm. The other two cases follow in a similar way. □

We will also need some embedding estimates for the stream function Ψ in terms of Ω.

Lemma 5.2. *Let $\Omega \in \mathcal{H}^N$, where $N \in \mathbb{N}$, satisfy the same conditions as in Proposition 4.1. Then for the solution Ψ of*

$$4\Psi + \partial_{\theta\theta}\Psi + \alpha^2 R^2 \partial_{RR}\Psi + (4\alpha + \alpha^2) R \partial_R \Psi = \Omega(R, \theta),$$

we have

$$|\partial_R^k \partial_\theta^m \Psi|_{L^\infty} \le c_{k,m} |\Omega|_{\mathcal{H}^{k+m+1}} \tag{5-3}$$

for $k, m \in \mathbb{N}$ with $k+m+1 \le N$.

Proof. As in Lemma 5.1, applying the Sobolev embedding, we have

$$|\partial_R^k \partial_\theta^m \Psi|_{L^\infty} \le c_{k,m} |\partial_R^k \partial_\theta^m \Psi|_{H^2_{R,\theta}}.$$

From the elliptic estimates in Proposition 4.1, for any $i, n \in \mathbb{N}$, we have

$$|\partial_R^i \partial_\theta^n \Psi|_{L^2} \le c_{i,n} |\Omega|_{\mathcal{H}^{i+n-1}}. \tag{5-4}$$

Thus, to bound $|\partial_R^k \partial_\theta^m \Psi|_{H^2_{R,\theta}}$, we take Ω to be in $\mathcal{H}^{k+m+1}$. Hence, we have

$$|\partial_R^k \partial_\theta^m \Psi|_{L^\infty} \le c_{k,m} |\Omega|_{\mathcal{H}^{k+m+1}}, \tag{5-5}$$

completing the proof. □

Lemma 5.3. *Let $\Omega \in \mathcal{H}^N$, where $N \in \mathbb{N}$, satisfying the same conditions as in Proposition 4.1. Then for the solution Ψ of*

$$4\Psi + \partial_{\theta\theta}\Psi + \alpha^2 R^2 \partial_{RR}\Psi + (4\alpha + \alpha^2) R \partial_R \Psi = \Omega(R,\theta),$$

we have

$$|R^k \partial_R^k \partial_\theta^m \Psi|_{L^\infty} \le c_{k,m} |\Omega|_{\mathcal{H}^{k+m+1}} \tag{5-6}$$

for $k, m \in \mathbb{N}$ with $k + m + 1 \le N$.

Proof. As in Lemma 5.1, applying the Sobolev embedding, we have

$$|R^k \partial_R^k \partial_\theta^m \Psi|_{L^\infty} \le c_{k,m} |R^k \partial_R^k \partial_\theta^m \Psi|_{H^2_{R,\theta}}.$$

From the elliptic estimates in Proposition 4.1, for any $i, n \in \mathbb{N}$, we have

$$|\partial_R^i \partial_\theta^n \Psi|_{L^2} \le c_{i,n} |\partial_R^i \partial_\theta^{n-1} \Omega|_{L^2} \le c_{i,n} |\Omega|_{\mathcal{H}^{i+n-1}} \tag{5-7}$$

and

$$|R^i \partial_R^i \partial_\theta^n \Psi|_{L^2} \le c_{i,n} |\Omega|_{\mathcal{H}^{i+n-1}}. \tag{5-8}$$

Thus, if we look at the second derivative terms of $R^k \partial_R^k \partial_\theta^m \Psi$, we can use the above inequalities to obtain the desired estimate. For the angular derivative term, we have $|R^k \partial_R^k \partial_\theta^{m+2} \Psi|_{L^2} \le c_{k,m} |\Omega|_{\mathcal{H}^{k+m+1}}$. When considering the radial derivative terms, we have three terms. For the $R^k \partial_R^{k+2} \partial_\theta^m \Psi$-term, applying (5-7) and (5-8), we have

$$|R^k \partial_R^{k+2} \partial_\theta^m \Psi|_{L^2} \le |R^{k+2} \partial_R^{k+2} \partial_\theta^m \Psi|_{L^2} + |\partial_R^{k+2} \partial_\theta^m \Psi| \le c_{k,m} |\Omega|_{\mathcal{H}^{k+m+1}}.$$

The other terms can be handled in similar way. Hence, we have our desired result. □

6. Reminder estimate

In this section, we obtain an error estimate on the remaining terms in the Euler with Riesz forcing. Recall that Ω satisfies the evolution equation

$$\begin{aligned}\partial_t \Omega &+ (-\alpha R \partial_\theta \Psi)\, \partial_R \Omega + (2\Psi + \alpha R \partial_R \Psi)\, \partial_\theta \Omega \\ &= \big(2\alpha R \sin(\theta)\cos(\theta) + \alpha^2 R \sin(\theta)\cos(\theta)\big)\, \partial_R \Psi + (1 - 2\sin^2(\theta))\, \partial_\theta \Psi \\ &\quad + \big(\alpha R \cos^2(\theta) + \alpha R \sin^2(\theta)\big)\, \partial_{R\theta}\Psi + (\alpha^2 R^2 \sin(\theta)\cos(\theta))\, \partial_{RR}\Psi - (\sin(\theta)\cos(\theta))\, \partial_{\theta\theta}\Psi,\end{aligned} \tag{6-1}$$

and the elliptic equation is

$$4\Psi + \alpha^2 R^2 \partial_{RR}\Psi + \partial_{\theta\theta}\Psi + (4\alpha + \alpha^2) R \partial_R \Psi = \Omega(R,\theta). \tag{6-2}$$

From Section 2, the leading-order model for the Euler with Riesz forcing equation satisfies

$$\partial_t \Omega_2 + (2\Psi_2)\partial_\theta \Omega_2 = (-1 + 2\sin^2(\theta))\partial_\theta \Psi_2 + (\sin(\theta)\cos(\theta))\, \partial_{\theta\theta}\Psi_2, \tag{6-3}$$

where

$$\Psi_2(R,\theta)=\frac{1}{4\alpha}L_s(\Omega_2)\sin(2\theta)+\frac{1}{4\alpha}L_c(\Omega_2)\cos(2\theta). \tag{6-4}$$

Now set $\Omega_r := \Omega-\Omega_2$ to be the remainder term for the vorticity, and similarly set $\Psi_r := \Psi-\Psi_2$ to be the remainder term for the stream function. Thus, we have that the remainder, Ω_r, satisfies the evolution equation

$$\begin{aligned}\partial_t\Omega_r+(-\alpha R(\partial_\theta\Psi_2+\partial_\theta\Psi_r))(\partial_R\Omega_2+\partial_R\Omega_r)+(2\Psi_2\partial_\theta\Omega_r+2\Psi_r\partial_\theta\Omega_2+2\Psi_r\partial_\theta\Omega_r)&\\ +(\alpha R(\partial_R\Psi_2+\partial_R\Psi_r))(\partial_\theta\Omega_2+\partial_\theta\Omega_r)&\\ =\big(2\alpha R\sin(\theta)\cos(\theta)+\alpha^2R\sin(\theta)\cos(\theta)\big)(\partial_R\Psi_2+\partial_R\Psi_r)&\\ +(1-2\sin^2(\theta))\partial_\theta\Psi_r+\alpha(R\cos^2(\theta)-R\sin^2(\theta))(\partial_{R\theta}\Psi_2+\partial_{R\theta}\Psi_r)&\\ +\alpha^2(R^2\sin(\theta)\cos(\theta))(\partial_{RR}\Psi_2+\partial_{RR}\Psi_r)-(\sin(\theta)\cos(\theta))\partial_{\theta\theta}\Psi_r.&\end{aligned} \tag{6-5}$$

The goal of this section is to show that Ω_r remains small. Namely, using energy methods, for some time T, we show that

$$\sup_{t\le T}|\Omega_r(t)|_{L^\infty}\le C\alpha^{1/2}$$

for some constant C independent of α.

Lemma 6.1. *Let $\Omega_r=\Omega-\Omega_2$ satisfy* (6-5) *with Ω and Ω_2 satisfying* (6-1) *and* (6-3), *respectively. Let $\Psi_r=\Psi-\Psi_2$ with Ψ and Ψ_2 satisfying* (6-2) *and* (6-4), *respectively. Then we have the estimates*

$$|\partial_R^k\partial_\theta^m\Psi_r|_{L^2}\le\frac{c_{k,m}}{\alpha}|\Omega_r|_{\mathcal{H}^{k+m-1}}\quad\textit{and}\quad|R^k\partial_R^k\partial_\theta^m\Psi_r|_{L^2}\le\frac{c_{k,m}}{\alpha}|\Omega_r|_{\mathcal{H}^{k+m-1}} \tag{6-6}$$

for $k,m\in\mathbb{N}$.

Proof. Recall that by the Biot–Savart law decomposition [Elgindi 2021] (see Section 4 for more details), we have the following decomposition for the elliptic equation (6-2):

$$\Psi(R,\theta)=\frac{1}{4\alpha}L_s(\Omega)\sin(2\theta)+\frac{1}{4\alpha}L_c(\Omega)\cos(2\theta)+\mathcal{R}(\Omega),$$

with $\mathcal{R}(\Omega)$ bounded on $\mathcal{H}^N$ with a constant independent of α. This follows from the elliptic estimates in Proposition 4.1 and Theorem 2 in Section 4. Now since we defined $\Omega_r=\Omega-\Omega_2$ and $\Psi_r=\Psi-\Psi_2$, with Ω_2, and Ψ_2 satisfying (6-3), and (6-4), respectively, we have the following decomposition for Ψ_r:

$$\Psi_r(R,\theta)=\frac{1}{4\alpha}L_s(\Omega_r)\sin(2\theta)+\frac{1}{4\alpha}L_c(\Omega_r)\cos(2\theta)+\mathcal{R}(\Omega_r)+\mathcal{R}(\Omega_2). \tag{6-7}$$

Hence, this gives the estimates

$$|\partial_R^k\partial_\theta^m\Psi_r|_{L^2}\le\frac{c_{k,m}}{\alpha}|\Omega_r|_{\mathcal{H}^{k+m-1}}\quad\text{and}\quad|R^k\partial_R^k\partial_\theta^m\Psi_r|_{L^2}\le\frac{c_{k,m}}{\alpha}|\Omega_r|_{\mathcal{H}^{k+m-1}}. \qquad\square$$

We define the following terms to shorten the notation:

$$\begin{aligned}I_1&=-\alpha R(\partial_\theta\Psi_2+\partial_\theta\Psi_r)(\partial_R\Omega_2+\partial_R\Omega_r),\\ I_2&=(2\Psi_2\partial_\theta\Omega_r+2\Psi_r\partial_\theta\Omega_2+2\Psi_r\partial_\theta\Omega_r),\\ I_3&=\alpha R(\partial_R\Psi_2+\partial_R\Psi_r)(\partial_\theta\Omega_2+\partial_\theta\Omega_r),\\ I_4&=2\alpha(1-\alpha)R\sin(\theta)\cos(\theta)(\partial_R\Psi_2+\partial_R\Psi_r),\end{aligned}$$

$$
\begin{aligned}
I_5 &= (1-2\sin^2(\theta))\partial_\theta \Psi_r,\\
I_6 &= \alpha(R\cos^2(\theta) - R\sin^2(\theta))(\partial_{R\theta}\Psi_2 + \partial_{R\theta}\Psi_r),\\
I_7 &= \alpha^2(R^2\sin(\theta)\cos(\theta))(\partial_{RR}\Psi_2 + \partial_{RR}\Psi_r),\\
I_8 &= -(\sin(\theta)\cos(\theta))\partial_{\theta\theta}\Psi_r.
\end{aligned}
$$

Now we have the error estimate proposition.

Proposition 6.2. *Let* $\Omega_r = \Omega - \Omega_2$ *satisfy* (6-5) *with* $\Omega_r|_{t=0} = 0$. *Then*

$$\sup_{0\le t<T} |\Omega_r(t)|_{L^\infty} \le c_N \alpha^{1/2},$$

where $T = c\alpha\log(c|\log(\alpha)|)$ *and* c *is a small constant independent of* α.

Proof. We will use ∂^N to refer to any mixed derivatives in R and θ of order N (not excluding pure R- and θ-derivatives). From the definition of the $\mathcal{H}^N$ norm, to obtain the $\mathcal{H}^N$ estimate we will take the following inner product with each I_i-term:

$$\langle \partial^N I_i, \partial^N \Omega_r\rangle \quad\text{and}\quad \langle R^k\partial_R^k\partial_\theta^{N-k} I_i, R^k\partial_R^k\partial_\theta^{N-k}\Omega_r\rangle$$

for $0\le k\le N$ and $1\le i\le 8$.

Estimate on I_1 and I_3: Here we will estimate I_1 and I_3. The estimate of I_3 is very similar to I_1, and so we will just show how to obtain the estimate on I_1.

Estimate on I_1: We can write I_1 as

$$
\begin{aligned}
I_1 &= -\alpha R(\partial_\theta\Psi_2 + \partial_\theta\Psi_r)(\partial_R\Omega_2 + \partial_R\Omega_r)\\
&= -\alpha(\partial_\theta\Psi_2)R(\partial_R\Omega_2) - \alpha(\partial_\theta\Psi_2)R(\partial_R\Omega_r) - \alpha(\partial_\theta\Psi_r)R(\partial_R\Omega_2) - \alpha(\partial_\theta\Psi_r)R(\partial_R\Omega_r)\\
&= I_{1,1} + I_{1,2} + I_{1,3} + I_{1,4},
\end{aligned}
$$

and we will estimate each term separately.

- $I_{1,1} = -\alpha\partial_\theta\Psi_2\, R\partial_R\Omega_2$. Here we have

$$\langle \partial^N(\alpha\partial_\theta\Psi_2\, R\partial_R\Omega_2), \partial^N\Omega_r\rangle = \sum_{i=0}^N c_{i,N}\int \partial^i(\alpha\partial_\theta\Psi_2)\partial^{N-i}(R\partial_R\Omega_2)\,\partial^N\Omega_r.$$

Now from Lemmas 3.4 and 3.5, we know that

$$|\Psi_2|_{\mathcal{W}^{k+1,\infty}} \le \frac{c_k}{\alpha} \quad\text{and}\quad |\Omega_2|_{\mathcal{H}^k} \le |\Omega_2(0)|_{\mathcal{H}^k} e^{(c_k/\alpha)t}.$$

Thus, we have

$$
\begin{aligned}
\sum_{i=0}^N \int \alpha\partial^i(\partial_\theta\Psi_2)\partial^{N-i}(R\partial_R\Omega_2)\,\partial^N\Omega_r &\le c_N\sum_{i=0}^N \alpha|\partial^i\partial_\theta\Psi_2|_{L^\infty}\,|\partial^{N-i}(R\partial_R\Omega_2)|_{L^2}\,|\partial^N\Omega_r|_{L^2}\\
&\le c_N\alpha|\Psi_2|_{\mathcal{W}^{N+1,\infty}}\,|\Omega_2|_{\mathcal{H}^{N+1}}\,|\Omega_r|_{\mathcal{H}^N}\\
&\le \alpha\frac{c_N}{\alpha}e^{(c_N/\alpha)t}|\Omega_r|_{\mathcal{H}^N} \le c_N e^{(c_N/\alpha)t}|\Omega_r|_{\mathcal{H}^N},
\end{aligned}
$$

and similarly we have

$$\langle \partial_R^k \partial_\theta^{N-k}(\alpha \partial_\theta \Psi_2\, R\partial_R \Omega_2),\, R^{2k}\partial_R^k \partial_\theta^{N-k}\Omega_r\rangle = c_{i,m,N}\int \sum_{i+m=0}^{N} \partial_R^i \partial_\theta^m(\alpha\partial_\theta\Psi_2)\, \partial_R^{k-i}\partial_\theta^{N-k-m}(R\partial_R\Omega_2)\, R^{2k}\partial_R^k\partial_\theta^{N-k}\Omega_r.$$

From the definition of the $\mathcal{W}^{N+1,\infty}$ norm, we have for $i+m\le N$,

$$|R^i\partial_R^i\partial_\theta^{m+1}\Psi_2|_{L^\infty} \le |\Psi_2|_{\mathcal{W}^{N+1,\infty}}.$$

Again, applying Lemmas 3.4 and 3.5, we obtain

$$\begin{aligned}\sum_{i+m=0}^{N}\int R^i\partial_R^i\partial_\theta^m(\alpha\partial_\theta\Psi_2)\, R^{k-i}\,\partial_R^{k-i}\partial_\theta^{N-k-m}(R\partial_R\Omega_2)\, R^k\partial_R^k\partial_\theta^{N-k}\Omega_r \\ \le c_N\sum_{i+m=0}^{N}\alpha|R^i\partial_R^i\partial_\theta^{m+1}\Psi_2|_{L^\infty}\, |R^{k-i}\,\partial_R^{k-i}\partial_\theta^{N-k-m}(R\partial_R\Omega_2)|_{L^2}\, |R^k\partial_R^k\partial_\theta^{N-k}\Omega_r|_{L^2} \\ \le c_N\alpha|\Psi_2|_{\mathcal{W}^{N+1,\infty}}\,|\Omega_2|_{\mathcal{H}^{N+1}}\,|\Omega_r|_{\mathcal{H}^N} \le \alpha\frac{c_N}{\alpha}e^{(c_N/\alpha)t}|\Omega_r|_{\mathcal{H}^N} \le c_N e^{(c_N/\alpha)t}|\Omega_r|_{\mathcal{H}^N}.\end{aligned}$$

Thus, we have

$$\langle I_{1,1},\,\Omega_r\rangle_{\mathcal{H}^N} \le c_N e^{(c_N/\alpha)t}|\Omega_r|_{\mathcal{H}^N}. \tag{6-8}$$

- $I_{1,2} = -\alpha\partial_\theta\Psi_2\, R\partial_R\Omega_r$. Here we have

$$\langle \partial^N(\alpha\partial_\theta\Psi_2\, R\partial_R\Omega_r),\, \partial^N\Omega_r\rangle = \sum_{i=0}^{N} c_{i,N}\int \partial^i(\alpha\partial_\theta\Psi_2)\partial^{N-i}(R\partial_R\Omega_r)\partial^N\Omega_r.$$

To obtain this estimate, we again apply Lemma 3.4. Namely, that $|\Psi_2|_{\mathcal{W}^{k+1,\infty}} \le c_k/\alpha$. When $i=0$, we integrate by parts and obtain

$$\int (\alpha\partial_\theta\Psi_2)\partial^N(R\partial_R\Omega_r)\,\partial^N\Omega_r \le c|\Psi_2|_{\mathcal{W}^{2,\infty}}|\Omega_r|^2_{\mathcal{H}^N} \le \frac{c_N}{\alpha}|\Omega_r|^2_{\mathcal{H}^N}.$$

For $1\le i\le N$ we have

$$\begin{aligned}\sum_{i=1}^{N}\int \alpha\partial^i(\partial_\theta\Psi_2)\partial^{N-i}(R\partial_R\Omega_r)\,\partial^N\Omega_r \le c_N\sum_{i=1}^{N}\alpha|\partial^i\partial_\theta\Psi_2|_{L^\infty}\,|\partial^{N-i}(R\partial_R\Omega_r)|_{L^2}\,|\partial^N\Omega_r|_{L^2} \\ \le c_N\alpha|\Psi_2|_{\mathcal{W}^{N+1,\infty}}\,|\Omega_r|_{\mathcal{H}^N}\,|\Omega_r|_{\mathcal{H}^N} \le \alpha\frac{c_N}{\alpha}|\Omega_r|^2_{\mathcal{H}^N} \le c_N|\Omega_r|^2_{\mathcal{H}^N}.\end{aligned}$$

Similarly, now for the $R^k\partial_R^k\partial_\theta^{N-k}$-terms we have

$$\langle R^k\partial_R^k\partial_\theta^{N-k}(\alpha\partial_\theta\Psi_2\, R\partial_R\Omega_r),\, R^k\partial_R^k\partial_\theta^{N-k}\Omega_r\rangle = c_{i,m,N}\int\sum_{i+m=0}^{N} R^k\partial_R^i\partial_\theta^m(\alpha\partial_\theta\Psi_2)\,\partial_R^{k-i}\partial_\theta^{N-k-m}(R\partial_R\Omega_r)\, R^k\partial_R^k\partial_\theta^{N-k}\Omega_r.$$

We again use $|\Psi_2|_{\mathcal{W}^{k+1,\infty}} \le c_k/\alpha$. Hence, we have

$$\sum_{i+m=0}^{N} \int R^i \partial_R^i \partial_\theta^m (\alpha \partial_\theta \Psi_2)\, R^{k-i}\, \partial_R^{k-i} \partial_\theta^{N-k-m} (R\partial_R \Omega_r)\, R^k \partial_R^k \partial_\theta^{N-k} \Omega_r$$
$$\le c_N \sum_{i+m=0}^{N} \alpha |R^i \partial_R^i \partial_\theta^{m+1} \Psi_2|_{L^\infty} \, |R^{k-i}\, \partial_R^{k-i} \partial_\theta^{N-k-m} (R\partial_R \Omega_r)|_{L^2} \, |R^k \partial_R^k \partial_\theta^{N-k} \Omega_r|_{L^2}$$
$$\le c_N \alpha |\Psi_2|_{\mathcal{W}^{N+1,\infty}} \, |\Omega_r|_{\mathcal{H}^N} \, |\Omega_r|_{\mathcal{H}^N} \le \alpha \frac{c_N}{\alpha} |\Omega_r|_{\mathcal{H}^N}^2 \le c_N |\Omega_r|_{\mathcal{H}^N}^2.$$

Thus, we have

$$\langle I_{1,2}, \Omega_r \rangle_{\mathcal{H}^N} \le c_N |\Omega_r|_{\mathcal{H}^N}^2. \tag{6-9}$$

• $I_{1,3} = -\alpha(\partial_\theta \Psi_r) R\partial_R \Omega_2$. To obtain the estimate on $I_{1,3}$, we will use Lemma 3.5, which will give us the estimate on Ω_2. In addition, to bound the $\partial_\theta \Psi_r$-term, we will use the decomposition of Ψ_r (6-7) and estimate (6-6) combined with the elliptic estimates from Proposition 4.1 and embedding estimates from Lemma 5.2. Now we have

$$\langle \partial^N (\alpha \partial_\theta \Psi_r\, R\partial_R \Omega_2),\, \partial^N \Omega_r \rangle = \sum_{i=0}^{N} c_{i,N} \int \partial^i (\alpha \partial_\theta \Psi_r) \partial^{N-i} (R\partial_R \Omega_2)\, \partial^N \Omega_r.$$

When $0 \le i \le N/2$, we will use the embedding from Lemma 5.1. Namely that

$$|\partial^i \partial_\theta \Psi_r|_{L^\infty} \le c_i |\partial_\theta \Psi_r|_{\mathcal{H}^{i+2}}.$$

Then, applying Lemma 6.1, we have

$$|\partial_\theta \Psi_r|_{\mathcal{H}^{i+2}} \le \frac{c_i}{\alpha} |\Omega_r|_{\mathcal{H}^{i+2}}.$$

Thus,

$$\sum_{i=0}^{N/2} \int \partial^i (\alpha \partial_\theta \Psi_r) \partial^{N-i} (R\partial_R \Omega_2)\, \partial^N \Omega_r \le \sum_{i=0}^{N/2} \alpha |\partial^i \partial_\theta \Psi_r|_{L^\infty} \, |\partial^{N-i} (R\partial_R \Omega_2)|_{L^2} |\partial^N \Omega_r|_{L^2}$$
$$\le \sum_{i=0}^{N/2} \alpha \frac{c_i}{\alpha} |\Omega_r|_{\mathcal{H}^{i+2}} |\Omega_2|_{\mathcal{H}^{N+1}} |\Omega_r|_{\mathcal{H}^N}$$
$$\le |\Omega_r|_{\mathcal{H}^{N/2+2}} |\Omega_2|_{\mathcal{H}^{N+1}} |\Omega_r|_{\mathcal{H}^N} \le c_N e^{c_N/\alpha} |\Omega_r|_{\mathcal{H}^N}^2.$$

Here we used Lemma 3.5 for the $|\Omega_2|_{\mathcal{H}^{N+1}}$-term.

When $N/2 \le i \le N$, we will use Lemma 6.1. Namely,

$$|\partial^i \partial_\theta \Psi_r|_{L^2} \le \frac{c_i}{\alpha} |\Omega_r|_{\mathcal{H}^i}.$$

Thus, we have

$$\sum_{i=N/2}^{N} \int \partial^i (\alpha \partial_\theta \Psi_r) \partial^{N-i} (R\partial_R \Omega_2)\, \partial^N \Omega_r \le \sum_{i=N/2}^{N} \alpha |\partial^i \partial_\theta \Psi_r|_{L^2} \, |R\partial_R \Omega_2|_{\mathcal{W}^{N-i,\infty}} |\partial^N \Omega_r|_{L^2}$$
$$\le \sum_{i=N/2}^{N} \alpha \frac{c_i}{\alpha} |\Omega_r|_{\mathcal{H}^i} |\Omega_2|_{\mathcal{W}^{N/2,\infty}} |\Omega_r|_{\mathcal{H}^N}$$
$$\le c_N |\Omega_r|_{\mathcal{H}^N} |\Omega_2|_{\mathcal{H}^N} |\Omega_r|_{\mathcal{H}^N} \le c_N e^{(c_N/\alpha)t} |\Omega_r|_{\mathcal{H}^N}^2.$$

Similarly, to estimate the inner product

$$\langle \partial_R^k \partial_\theta^{N-k}(\alpha(\partial_\theta \Psi_r) R\partial_R \Omega_2),\, R^{2k}\partial_R^k\partial_\theta^{N-k}\Omega_r\rangle \le c_N e^{(c_N/\alpha)t}|\Omega_r|^2_{\mathcal{H}^N},$$

we will use the weighted embedding estimates from Lemma 5.1 combined with Lemma 6.1. Following the same steps as we did in the previous inner product, we obtain

$$\langle I_{1,3}, \Omega_r\rangle_{\mathcal{H}^N} \le c_N e^{(c_N/\alpha)t}|\Omega_r|^2_{\mathcal{H}^N}. \tag{6-10}$$

- $I_{1,4} = -\alpha(\partial_\theta\Psi_r)R\partial_R\Omega_r$. To obtain the estimate on $I_{1,4}$, we will use Lemma 6.1 and the embedding estimate from Lemma 5.1 to handle the $\partial_\theta\Psi_r$-term. To handle the $R\partial_R\Omega_r$-term, we will use embedding estimates from Lemma 5.1 and follow the same steps as we did in the previous inner product. We will only show how to obtain the estimate on the term

$$\begin{aligned}\langle \partial_R^k\partial_\theta^{N-k}(\alpha\partial_\theta\Psi_r\, R\partial_R\Omega_r),\, R^{2k}\partial_R^k\partial_\theta^{N-k}\Omega_r\rangle \\ = c_{i,m,N}\int\sum_{i+m=0}^{N}\partial_R^i\partial_\theta^m(\alpha\partial_\theta\Psi_r)\,\partial_R^{k-i}\partial_\theta^{N-k-m}(R\partial_R\Omega_r)\,R^{2k}\partial_R^k\partial_\theta^{N-k}\Omega_r.\end{aligned}$$

For the other inner product, the idea is the same. To start the estimate, first we consider the case when $i=m=0$. We integrate by parts and use the embedding estimates in Lemmas 5.1 and 6.1 to estimate the $\partial_\theta\Psi_r$-term. We have

$$\begin{aligned}&\int \alpha\partial_\theta\Psi_r(R^{k+1}\partial_R^{k+1}\partial_\theta^{N-k}\Omega_r + R^k\partial_R^k\partial_\theta^{N-k}\Omega_r)\,R^k\partial_R^k\partial_\theta^{N-k}\Omega_r\\ &\qquad\le \alpha|R\partial_{R\theta}\Psi_r|_{L^\infty}|R^k\partial_R^k\partial_\theta^{N-k}\Omega_r|^2_{L^2} + \alpha|\partial_\theta\Psi_r|_{L^\infty}|R^k\partial_R^k\partial_\theta^{N-k}\Omega_r|^2_{L^2}\\ &\qquad\le c_N(|\Omega_r|_{\mathcal{H}^3}|\Omega_r|^2_{\mathcal{H}^N} + |\Omega_r|_{\mathcal{H}^2}|\Omega_r|^2_{\mathcal{H}^N})\\ &\qquad\le c_N|\Omega_r|^3_{\mathcal{H}^N}.\end{aligned}$$

Now when $1\le i+m\le N/2$, we will again use Lemmas 5.1 and 6.1 and the definition of the $\mathcal{H}^k$ norm to obtain

$$\begin{aligned}&\sum_{i+m\ge1}^{N/2} R^i\partial_R^i\partial_\theta^m(\alpha\partial_\theta\Psi_r)\big(R^{k+1-i}\partial_R^{k+1-i}\partial_\theta^{N-k-m}\Omega_r + R^{k-i}\partial_R^{k-i}\partial_\theta^{N-k-m}\Omega_r\big)\,R^k\partial_R^k\partial_\theta^{N-k}\Omega_r\\ &\quad\le \sum_{i+m\ge1}^{N/2}\alpha|R^i\,\partial_R^i\partial_\theta^{m+1}\Psi_r|_{L^\infty}|R^{k+1-i}\partial_R^{k+1-i}\partial_\theta^{N-k-m}\Omega_r|_{L^2}|R^k\partial_R^k\partial_\theta^{N-k}\Omega_r|_{L^2}\\ &\qquad\qquad + \sum_{i+m\ge1}^{N/2}\alpha|R^i\,\partial_R^i\partial_\theta^{m+1}\Psi_r|_{L^\infty}|R^{k-i}\partial_R^{k-i}\partial_\theta^{N-k-m}\Omega_r|_{L^2}|R^k\partial_R^k\partial_\theta^{N-k}\Omega_r|_{L^2}\\ &\quad\le c_N\sum_{i+m\ge1}^{N/2}|\Omega_r|_{\mathcal{H}^{i+m+2}}(|\Omega_r|_{\mathcal{H}^N} + |\Omega_r|_{\mathcal{H}^{N-1}})|\Omega_r|_{\mathcal{H}^N}\\ &\quad\le c_N|\Omega_r|_{\mathcal{H}^{N/2+3}}(|\Omega_r|_{\mathcal{H}^N} + |\Omega_r|_{\mathcal{H}^{N-1}})|\Omega_r|_{\mathcal{H}^N}\\ &\quad\le c_N|\Omega_r|^3_{\mathcal{H}^N}.\end{aligned}$$

Now for the case when $N/2 \le i+m \le N$, we will use Lemmas 5.1 and 6.1 to obtain

$$\begin{aligned}
&\sum_{i+m\ge N/2}^{N} R^i\partial_R^i\partial_\theta^m(\alpha\partial_\theta\Psi_r)\big(R^{k+1-i}\partial_R^{k+1-i}\partial_\theta^{N-k-m}\Omega_r + R^{k-i}\partial_R^{k-i}\partial_\theta^{N-k-m}\Omega_r\big)\, R^k\partial_R^k\partial_\theta^{N-k}\Omega_r \\
&\quad\le \sum_{i+m\ge N/2}^{N} \alpha|R^i\,\partial_R^i\partial_\theta^{m+1}\Psi_r|_{L^2}\left(|R^{k+1-i}\,\partial_R^{k+1-i}\,\partial_\theta^{N-k-m}\Omega_r|_{L^\infty}\right)|R^k\partial_R^k\partial_\theta^{N-k}\Omega_r|_{L^2} \\
&\qquad\qquad + \sum_{i+m\ge N/2}^{N} \alpha|R^i\,\partial_R^i\partial_\theta^{m+1}\Psi_r|_{L^2}\big(|R^{k-i}\,\partial_R^{k-i}\,\partial_\theta^{N-k-m}\Omega_r|_{L^\infty}\big)|R^k\partial_R^k\partial_\theta^{N-k}\Omega_r|_{L^2} \\
&\quad\le \sum_{i+m\ge N/2}^{N} |\Omega_r|_{\mathcal{H}^{i+m}}\left(|\Omega_r|_{\mathcal{H}^{N-(i+m)+3}} + |\Omega_r|_{\mathcal{H}^{N-(i+m)+2}}\right)|\Omega_r|_{\mathcal{H}^N} \\
&\quad\le c_N|\Omega_r|_{\mathcal{H}^N}|\Omega_r|_{\mathcal{H}^{N/2+3}}|\Omega_r|_{\mathcal{H}^N} \le c_N|\Omega_r|^3_{\mathcal{H}^N},
\end{aligned}$$

and thus, we have

$$\langle I_{1,4},\Omega_r\rangle_{\mathcal{H}^N} \le c_N|\Omega_r|^3_{\mathcal{H}^N}. \tag{6-11}$$

Thus, we have the following estimate on the I_1-term:

$$\langle I_1,\Omega_r\rangle_{\mathcal{H}^N} \le c_N e^{(c_N/\alpha)t}|\Omega_r|_{\mathcal{H}^N} + c_N e^{(c_N/\alpha)t}|\Omega_r|^2_{\mathcal{H}^N} + c_N|\Omega_r|^3_{\mathcal{H}^N}. \tag{6-12}$$

Estimate on I_3: The estimate on I_3 follows similarly to I_1, so we skip the details for this case. One can obtain

$$\langle I_3,\Omega_r\rangle_{\mathcal{H}^N} \le c_N e^{(c_N/\alpha)t}|\Omega_r|_{\mathcal{H}^N} + c_N e^{(c_N/\alpha)t}|\Omega_r|^2_{\mathcal{H}^N} + c_N|\Omega_r|^3_{\mathcal{H}^N}. \tag{6-13}$$

Estimate on I_2: Here we have

$$I_2 = (2\Psi_2\partial_\theta\Omega_r + 2\Psi_r\partial_\theta\Omega_2 + 2\Psi_r\partial_\theta\Omega_r) = I_{2,1} + I_{2,2} + I_{2,3}.$$

- $I_{2,1} = 2\Psi_2\partial_\theta\Omega_r$. To estimate $I_{2,1}$, we follow the same steps as in the I_1-term. Using Lemma 3.4, namely that $|\Psi_2|_{\mathcal{W}^{N,\infty}} \le c_N/\alpha$, we have

$$\langle I_{2,1},\Omega_r\rangle_{\mathcal{H}^N} \le \frac{c_N}{\alpha}|\Omega_r|^2_{\mathcal{H}^N}. \tag{6-14}$$

- $I_{2,2} = 2\Psi_r\partial_\theta\Omega_2$. Similarly, to estimate $I_{2,2}$ we also follow the same steps as we did in I_1. More specifically, to handle the Ψ_r-term, we will follow similar steps as for the terms $I_{1,3}$ and $I_{1,4}$. Namely, we will apply embedding estimates and Lemma 6.1 to estimate the Ψ_r-term. To estimate Ω_2, we use Lemma 3.5 to obtain that $|\Omega_2|_{\mathcal{H}^k} \le |\Omega_2(0)|_{\mathcal{H}^k} e^{(c_k/\alpha)t}$. Thus we have

$$\langle I_{2,2},\Omega_r\rangle_{\mathcal{H}^N} \le \frac{c_N}{\alpha}e^{(c_N/\alpha)t}|\Omega_r|^2_{\mathcal{H}^N}. \tag{6-15}$$

- $I_{2,3} = 2\Psi_r\partial_\theta\Omega_r$. This term $I_{2,3}$ can be estimated similarly to the $I_{1,4}$-term by using embedding and Lemma 6.1. Hence, we obtain

$$\langle I_{2,3},\Omega_r\rangle_{\mathcal{H}^N} \le \frac{c_N}{\alpha}|\Omega_r|^3_{\mathcal{H}^N}. \tag{6-16}$$

Thus we have

$$\langle I_2, \Omega_r\rangle_{\mathcal{H}^N} \le \frac{c_N}{\alpha}|\Omega_r|^2_{\mathcal{H}^N} + \frac{c_N}{\alpha}e^{(c_N/\alpha)t}|\Omega_r|^2_{\mathcal{H}^N} + \frac{c_N}{\alpha}|\Omega_r|^3_{\mathcal{H}^N} \le \frac{c_N}{\alpha}e^{(c_N/\alpha)t}|\Omega_r|^2_{\mathcal{H}^N} + \frac{c_N}{\alpha}|\Omega_r|^3_{\mathcal{H}^N}. \quad (6\text{-}17)$$

Estimates on I_4, I_5, I_6, I_7, and I_8: We can write I_4 as

$$\begin{aligned} I_4 &= 2\alpha R\sin(\theta)\cos(\theta) + \alpha^2 R\sin(\theta)\cos(\theta))(\partial_R\Psi_2 + \partial_R\Psi_r) \\ &= \alpha(2+\alpha)\sin(\theta)\cos(\theta)\, R\partial_R\Psi_2 + \alpha(2+\alpha)\sin(\theta)\cos(\theta)\, R\partial_R\Psi_r = I_{4,1} + I_{4,2}. \end{aligned}$$

Recall that

$$I_5 = (1 - 2\sin^2(\theta))\partial_\theta\Psi_r.$$

We can also rewrite and I_6 and I_7 as

$$\begin{aligned} I_6 &= \alpha(\cos^2(\theta) - \sin^2(\theta))R(\partial_{R\theta}\Psi_2 + \partial_{R\theta}\Psi_r) \\ &= \alpha(\cos^2(\theta) - \sin^2(\theta))R\partial_{R\theta}\Psi_2 + \alpha(\cos^2(\theta) - \sin^2(\theta))R\partial_{R\theta}\Psi_r = I_{6,1} + I_{6,2} \end{aligned}$$

and

$$\begin{aligned} I_7 &= \alpha^2(\sin(\theta)\cos(\theta))R^2(\partial_{RR}\Psi_2 + \partial_{RR}\Psi_r) \\ &= \alpha^2(\sin(\theta)\cos(\theta))R^2\,\partial_{RR}\Psi_2 + \alpha^2(\sin(\theta)\cos(\theta))R^2\,\partial_{RR}\Psi_r = I_{7,1} + I_{7,2}. \end{aligned}$$

Recall that

$$I_8 = -\sin(\theta)\cos(\theta)\,\partial_{\theta\theta}\Psi_r.$$

Now for $i = 4$, 6, and 7, using Lemma 3.4, namely that $|\Psi|_{\mathcal{H}^{k+1}} \le c_k/\alpha$, we have the estimate

$$\langle I_{i,1}, \Omega_r\rangle_{\mathcal{H}^N} \le c_N|\Omega_r|_{\mathcal{H}^N} \quad \text{for } i = 4, 6, 7. \quad (6\text{-}18)$$

Using Lemma 6.1, we obtain

$$\langle I_{i,2}, \Omega_r\rangle_{\mathcal{H}^N} \le \alpha\frac{c_N}{\alpha}|\Omega_r|^2_{\mathcal{H}^N} = c_N|\Omega_r|^2_{\mathcal{H}^N} \quad \text{for } i = 4, 6, 7 \quad (6\text{-}19)$$

and

$$\langle I_i, \Omega_r\rangle_{\mathcal{H}^N} \le \frac{c_N}{\alpha}|\Omega_r|^2_{\mathcal{H}^N} \quad \text{for } i = 5, 8. \quad (6\text{-}20)$$

Hence, from (6-18), (6-19), (6-20), we have

$$\langle I_i, \Omega_r\rangle_{\mathcal{H}^N} \le c_N|\Omega_r|_{\mathcal{H}^N} + \frac{c_N}{\alpha}|\Omega_r|^2_{\mathcal{H}^N} \quad \text{for } i = 4, 5, \dots, 8. \quad (6\text{-}21)$$

Total remainder estimate: Here we obtain the total error estimate. From our previous work we have

$$\frac{d}{dt}|\Omega_r|^2_{\mathcal{H}^N} = \langle\partial_t\Omega_r, \Omega_r\rangle_{\mathcal{H}^N} \le \sum_{i=1}^{8}|\langle I_i, \Omega_r\rangle_{\mathcal{H}^N}|,$$

and thus from (6-12), (6-13), (6-17), and (6-21), we have

$$\frac{d}{dt}|\Omega_r|^2_{\mathcal{H}^N} \le c_N e^{(c_N/\alpha)t}|\Omega_r|_{\mathcal{H}^N} + \frac{c_N}{\alpha}e^{(c_N/\alpha)t}|\Omega_r|^2_{\mathcal{H}^N} + \frac{c_N}{\alpha}|\Omega_r|^3_{\mathcal{H}^N},$$

and hence

$$\frac{d}{dt}|\Omega_r|_{\mathcal{H}^N} \le c_N e^{(c_N/\alpha)t} + \frac{c_N}{\alpha}e^{(c_N/\alpha)t}|\Omega_r|_{\mathcal{H}^N} + \frac{c_N}{\alpha}|\Omega_r|^2_{\mathcal{H}^N}. \quad (6\text{-}22)$$

Now since we have $\Omega_r|_{t=0} = 0$, we will use bootstrap argument to close the remainder estimate. We will assume that $|\Omega_r|_{\mathcal{H}^N} \le 2c_N\alpha^{1/2}$ for time $0 < t \le T$, and then show that $|\Omega_r(t)|_{\mathcal{H}^N} \le c_N\alpha^{1/2}$, and this will give the remainder estimate. Let us assume that

$$|\Omega_r|_{\mathcal{H}^N} \le 2c_N\alpha^{1/2}.$$

Then from (6-22) we have

$$\frac{d}{dt}|\Omega_r|_{\mathcal{H}^N} \le c_N e^{(c_N/\alpha)t} + \frac{c_N}{\alpha} e^{(c_N/\alpha)t}|\Omega_r|_{\mathcal{H}^N} + 4c_N^3,$$

and thus

$$|\Omega_r|_{\mathcal{H}^N} \le \left(\int_0^t c_N e^{(c_N/\alpha)\tau} + 4c_N^3\,d\tau\right)\exp\left(\int_0^t \frac{c_N}{\alpha} e^{(c_N/\alpha)\tau}\,d\tau\right) \le (\alpha c_N e^{(c_N/\alpha)t} + 4c_N^3 t)\exp(c_N e^{(c_N/\alpha)t}).$$

Hence, if we choose our time scale $0 < t \le T(\alpha) = c_1\alpha\log(c_2|\log(\alpha)|)$ for c_1 and c_2 small constants, for example, take $c_1 = 1/c_N$, and $c_2 = 1/(4c_N)$, we have

$$|\Omega_r|_{\mathcal{H}^N} \le c_N\alpha^{1/2},$$

which completes the bootstrap argument and gives the proof of Proposition 6.2. □

7. Main result

We now recall and prove the main theorem of this work.

Theorem 3. *For any $\alpha, \delta > 0$, there exists initial data $\omega_0^{\alpha,\delta} \in C_c^\infty(\mathbb{R}^2)$ and $T(\alpha)$ such that the corresponding unique global solution, $\omega^{\alpha,\delta}$, to* (1-4) *is such that at $t = 0$ we have*

$$|\omega_0^{\alpha,\delta}|_{L^\infty} = \delta,$$

but for any $0 < t \le T(\alpha)$ we have

$$|\omega^{\alpha,\delta}(t)|_{L^\infty} \ge |\omega_0|_{L^\infty} + c\log\left(1 + \frac{c}{\alpha}t\right),$$

where $T(\alpha) = c\alpha\log(c|\log(\alpha)|)$, and $c > 0$ is a constant independent of α that depends linearly on δ.

Proof. Consider the initial data of the form

$$\omega_0 = \Omega|_{t=0} = f_0(R)\sin(2\theta),$$

where $f_0(R)$, with $R = r^\alpha$, is a nonnegative compactly supported smooth function which is zero on $\left[0, \frac{1}{2}\right] \cup [1, \infty)$ and positive outside. We know that we can write $\Omega = \Omega_2 + \Omega_r$, and from the form of the initial data, we have $\Omega_r|_{t=0} = 0$ and thus from Proposition 6.2 we have

$$|\Omega_r(t)|_{L^\infty} \le c_N\alpha^{1/2}$$

for $0 \le t \le T(\alpha) = c\alpha\log(c|\log(\alpha)|)$, where recall that c is a small constant independent of α. Recall also that we can write Ω_2 as

$$\Omega_2 = f + \frac{1}{2\alpha}\int_0^t L_s(f_\tau)\,d\tau,$$

and thus from Proposition 3.3, we obtain

$$\Omega_2 = f + \frac{1}{2\alpha}\int_0^t L_s(f_\tau)\,d\tau \geq f + c_0 \log\Big(1 + \frac{c_0}{\alpha} t\Big)$$

for some c_0 independent of α and thus we have our desired result. □

Acknowledgements

Both authors were partially supported by the NSF grants DMS 2043024 and DMS 2124748. Elgindi was partially supported by an Alfred P. Sloan Fellowship. The authors are grateful to Prof. Jiefeng Zhao and Prof. Roberta Bianchini and would like to thank them for bringing their attention to an issue in the remainder estimate in Section 6 of the previous version of the paper. The authors would also like to thank the referees for their feedback and suggestions that improved the quality of the paper.

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Received 4 Oct 2022. Revised 1 Nov 2023. Accepted 21 Dec 2023.

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ANALYSIS AND PDE
Vol. 18 (2025), No. 3, pp. 743–772
DOI: 10.2140/apde.2025.18.743

FRACTAL UNCERTAINTY FOR DISCRETE TWO-DIMENSIONAL CANTOR SETS

ALEX COHEN

We prove that a self-similar Cantor set in $\mathbb{Z}_N \times \mathbb{Z}_N$ has a fractal uncertainty principle if and only if it does not contain a pair of orthogonal lines. The key ingredient in our proof is a quantitative form of Lang's conjecture in number theory due to Ruppert and to Beukers and Smyth. Our theorem answers a question of Dyatlov and has applications to open quantum maps.

1. Introduction

1.1. ***One-dimensional fractal uncertainty.*** The Bourgain–Dyatlov [2018] fractal uncertainty principle (FUP) says, in a precise quantitative sense, that a function $f : \mathbb{R} \to \mathbb{C}$ cannot simultaneously have large L^2 mass on a fractal set in physical space and large L^2 mass on a fractal set in Fourier space. This theorem and its variants have many applications to quantum chaos; see the survey article [Dyatlov 2019]. The proof of FUP in [Bourgain and Dyatlov 2018] is quite tricky, but the analogous result in the discrete setting has similar ingredients and is much simpler.

We begin with some notation. In this paper $\mathbb{Z}_N = \mathbb{Z}/N\mathbb{Z}$ refers to the integers mod N. We use the unitary discrete Fourier transform $\mathcal{F} : \ell^2(\mathbb{Z}_N) \to \ell^2(\mathbb{Z}_N)$, given by

$$\mathcal{F}f(\xi) = \hat{f}(\xi) = \frac{1}{\sqrt{N}} \sum_{x \in \mathbb{Z}_N} f(x) e^{-\frac{2\pi i}{N}\xi x},$$

$$\mathcal{F}^{-1}f(x) = f^{\vee}(x) = \frac{1}{\sqrt{N}} \sum_{\xi \in \mathbb{Z}_N} f(\xi) e^{\frac{2\pi i}{N}\xi x}.$$

We will also use the one-dimensional and two-dimensional tori $\mathbb{T}$, $\mathbb{T}^2$, which are identified as sets with $[0, 1)$ and $[0, 1) \times [0, 1)$.

Let us restrict our attention to self-similar Cantor sets (when we say Cantor set we always mean self-similar). Fix a base M and an alphabet $\mathcal{A} \subsetneq \mathbb{Z}_M$. Then let

$$\mathcal{X}_k = \{a_0 + a_1 M + \cdots + a_{k-1} M^{k-1} : a_j \in \mathcal{A}\} \subset \mathbb{Z}_{M^k}$$

be the k-th iterate. It will be convenient to let $N = M^k$, so $\mathcal{X}_k \subset \mathbb{Z}_N$. We say $\mathcal{A}$ has dimension $\delta_A = \log_M |\mathcal{A}|$, so $M^{k\delta} = |\mathcal{X}_k|$ for all k. Similarly, let $\mathcal{Y}_k$ be the Cantor iterates associated with the alphabet $\mathcal{B}$. Dyatlov and Jin [2017] proved the following fractal uncertainty principle for discrete one-dimensional Cantor sets. They were motivated by applications to open quantum maps; see Section 1.4 for more discussion.

Supported by a Fannie and John Hertz Foundation fellowship and an NSF GRFP fellowship. Partial support from NSF CAREER grant DMS-1749858 is acknowledged.
MSC2020: 28A80, 43A32, 81Q12.
Keywords: fractal uncertainty principle, quantum chaos, Lang conjecture.

Theorem 1 (one-dimensional FUP [Dyatlov and Jin 2017, Theorem 2]). *For all alphabets $\mathcal{A}, \mathcal{B} \subsetneq \mathbb{Z}_M$, the estimate*

$$\|1_{\mathcal{Y}_k} \mathcal{F}\, 1_{\mathcal{X}_k}\|_{2\to 2} \lesssim M^{-k\beta} \tag{1}$$

holds for some $\beta > 0$.

Because of self-similarity, Cantor sets enjoy the submultiplicativity estimate (see Section 5.1 for a proof)

$$\|1_{\mathcal{Y}_{r+k}} \mathcal{F}\, 1_{\mathcal{X}_{r+k}}\|_{2\to 2} \le \|1_{\mathcal{Y}_r} \mathcal{F}\, 1_{\mathcal{X}_r}\|_{2\to 2} \|1_{\mathcal{Y}_k} \mathcal{F}\, 1_{\mathcal{X}_k}\|_{2\to 2}, \tag{2}$$

which reduces (1) to the problem of proving that for *some* $k > 0$

$$\|1_{\mathcal{Y}_k} \mathcal{F}\, 1_{\mathcal{X}_k}\|_{2\to 2} < 1.$$

This estimate holds if there is no nonzero function f with $\operatorname{supp} f \subset \mathcal{X}_k$ and $\operatorname{supp} \hat{f} \subset \mathcal{Y}_k$. To recap, proving a one-dimensional FUP reduces to showing that, for some k, there is no function f with $\operatorname{supp} f \subset \mathcal{X}_k$ and $\operatorname{supp} \hat{f} \subset \mathcal{Y}_k$.

In the general case of arbitrary porous sets (not necessarily Cantor sets), submultiplicativity is replaced by an induction-on-scales argument which allows one to find significant L^2 mass of $\hat{f}$ in the gaps of $\mathcal{Y}_k$ at every scale.

1.2. ***Two-dimensional fractal uncertainty.*** In two dimensions, Cantor sets are determined by an alphabet $\mathcal{A} \subsetneq \mathbb{Z}_M^2$. We set

$$\mathcal{X}_k = \{(a_0 + \cdots + a_k M^{k-1}, b_0 + \cdots + b_k M^{k-1}) : (a_j, b_j) \in \mathcal{A}\} \subset \mathbb{Z}_N^2, \tag{3}$$

where $N := M^k$. We have $|\mathcal{A}| = M^\delta$ with $0 < \delta < 2$, and $|\mathcal{X}_k| = M^{k\delta}$. The unitary Fourier transform in two dimensions is given by

$$\mathcal{F} f(\xi, \eta) = \hat{f}(\xi, \eta) = \frac{1}{N} \sum_{(x,y)\in\mathbb{Z}_N^2} f(x, y) e^{-\frac{2\pi i}{N}(x\xi + y\eta)}$$
$$\hat{f}(\boldsymbol{\xi}) = \frac{1}{N} \sum_{\boldsymbol{x}\in\mathbb{Z}_N^2} f(\boldsymbol{x}) e^{-\frac{2\pi i}{N}\boldsymbol{x}\cdot\boldsymbol{\xi}} \quad \text{in vector notation.}$$

Submultiplicativity (2) holds in two dimensions as well (see Section 5.1), so proving a two-dimensional FUP reduces to showing that, for some k, there is no nonzero f with $\operatorname{supp} f \subset \mathcal{X}_k$ and $\operatorname{supp} \hat{f} \subset \mathcal{Y}_k$.

Unfortunately, this claim is not true in general. Indeed,

$$f(x, y) = N^{-\frac{1}{2}} 1_{y=0} \quad \text{has} \quad \hat{f} = N^{-\frac{1}{2}} 1_{x=0}$$

and fractal sets can contain vertical and horizontal lines. We show that the fractal sets generated by the alphabets $\mathcal{A}, \mathcal{B}$ containing a pair of orthogonal lines are the only obstruction to a two-dimensional FUP. For $\mathcal{A} \subset \mathbb{Z}_M^2$ an alphabet, let

$$\boldsymbol{A} = \overline{\{(x, y) \in \mathbb{T}^2 : (\lfloor Mx \rfloor, \lfloor My \rfloor) \in \mathcal{A}\}}.$$

This is a closed drawing of $\mathcal{A}$ in $\mathbb{T}^2$, and we draw the Cantor iterate $\mathcal{X}_k$ as

$$\boldsymbol{X}_k = \overline{\{(x, y) \in \mathbb{T}^2 : (\lfloor M^k x \rfloor, \lfloor M^k y \rfloor) \in \mathcal{X}_k\}} \subset \mathbb{T}^2. \tag{4}$$

We write $\boldsymbol{X} = \bigcap_k \boldsymbol{X}_k \subset \mathbb{T}^2$ as the limiting Cantor set, so

$$\boldsymbol{A} = \boldsymbol{X}_0 \supset \boldsymbol{X}_1 \supset \boldsymbol{X}_2 \supset \cdots \supset \boldsymbol{X},$$
$$\boldsymbol{X} = \{(0.a_0a_1\ldots, 0.b_0b_1\ldots) : (a_j, b_j) \in \mathcal{A} \text{ for all } j \geq 0\} \quad \text{in base } M.$$

Note that if $x \in \mathbb{T}$ is of the form a/M^k then there are two possible decimal expansions — the point $(x, y) \in \mathbb{T}$ is in $\boldsymbol{X}$ if *some* decimal expansion has all digits in the alphabet. For $\mathcal{B}$ a second alphabet we write $\boldsymbol{B} \subset \mathbb{T}^2$ as the drawing of $\mathcal{B}$ and $\boldsymbol{Y} \subset \mathbb{T}^2$ as the limiting Cantor set for $\mathcal{B}$. We need these closed sets to state the condition of our main theorem.

Theorem 2 (two-dimensional FUP). *Suppose $\mathcal{A}, \mathcal{B}$ are alphabets. Then either*

$$\mathbb{R}\boldsymbol{v} + \boldsymbol{p} \subset \boldsymbol{X} \quad \text{and} \quad \mathbb{R}\boldsymbol{v}^\perp + \boldsymbol{q} \subset \boldsymbol{Y} \tag{5}$$

for some $\boldsymbol{v} = (a, b) \in \mathbb{R}^2 - \{0\}$, $\boldsymbol{p}, \boldsymbol{q} \in \mathbb{T}^2$, or if not then $\mathcal{X}_k$, $\mathcal{Y}_k$ satisfy

$$\|1_{\mathcal{Y}_k} \mathcal{F}\, 1_{\mathcal{X}_k}\|_{2\to 2} \lesssim M^{-k\beta} \tag{6}$$

for some $\beta > 0$.

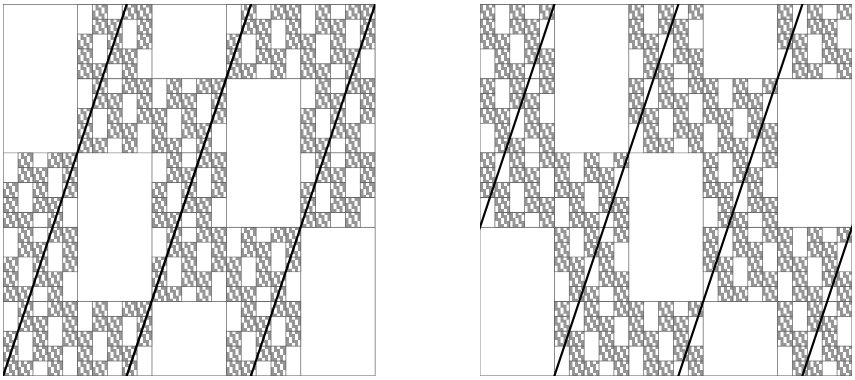
In particular, if $\boldsymbol{X}$ does not contain any line then it has an FUP. We note that in this theorem, (a, b) can be taken to be integers. Otherwise a/b is irrational and the coset $\mathbb{R}\boldsymbol{v} + \boldsymbol{p}$ is dense in $\mathbb{T}^2$, so it cannot lie entirely in the closed set $\boldsymbol{X} \subsetneq \mathbb{T}^2$. The main outside ingredient we use is Theorem 19 due to [Ruppert 1993, Corollary 5] and [Beukers and Smyth 2002, Theorem 4]; see Section 4.

In Section 5.2 we show that this theorem is sharp: if $\boldsymbol{X}, \boldsymbol{Y}$ contain a pair of orthogonal lines, FUP will fail. Notice that the condition of the theorem depends on the limiting Cantor sets $\boldsymbol{X}, \boldsymbol{Y} \subset \mathbb{T}^2$, and it is not immediately clear when alphabets $\mathcal{A}, \mathcal{B}$ generate Cantor sets satisfying this orthogonal line condition. The following proposition reduces this question to a finite combinatorial problem.

Proposition 3. *A line $\mathbb{R}\boldsymbol{v} + \boldsymbol{p}$ lies on $\boldsymbol{X}$ if and only if $\mathbb{R}\boldsymbol{v} + M^k\boldsymbol{p}$ lies on $\boldsymbol{A}$ for all $k \geq 0$. Additionally, suppose $(a, b) \in \mathbb{Z}^2 - \{0\}$ is given, a, b coprime. In order for there to be some $\boldsymbol{p}$ with $\mathbb{R}\boldsymbol{v} + \boldsymbol{p} \subset \boldsymbol{X}$, we must have* $\max(|a|, |b|) \leq M$.

Proposition 3 leaves open a natural algorithmic question. Given an alphabet $\mathcal{A}$ and vector $\boldsymbol{v} \in \mathbb{Z}^2 - \{0\}$, does there exist a point $\boldsymbol{p} \in \mathbb{T}^2$ such that $\mathbb{R}\boldsymbol{v} + \boldsymbol{p} \subset \boldsymbol{X}$? An efficient algorithm for this problem would lead to an efficient algorithm for testing when two alphabets $\mathcal{A}, \mathcal{B}$ satisfy the conditions of Theorem 2. For the proof and more discussion see Section 5.3.

Remark 4. Theorem 2 refines Conjecture 6.7 from [Dyatlov 2019]. That conjecture recognizes the potential obstruction of $\boldsymbol{X}, \boldsymbol{Y}$ containing a pair of vertical/horizontal or diagonal/antidiagonal lines (the case $\max(|a|, |b|) \leq 1$ in Proposition 3), but does account for lines with other slopes, which may occur in practice. See Figure 1.



(a) All iterates contain a skewed line.

(b) The first iterate contains a skewed line, but further iterates do not.

Figure 1. Cantor sets can contain lines that aren't horizontal, vertical, or diagonal, but they are less stable.

Theorem 2 is only interesting when $\frac{1}{2}(\delta_A + \delta_B) \geq 1$. Indeed, equation (6.8) from [Dyatlov 2019] says that (6) always holds with $\beta = \max\bigl(0, 1 - \frac{1}{2}(\delta_A + \delta_B)\bigr)$. Combining Theorem 2 with Proposition 6.8 from [Dyatlov 2019], we can classify exactly which discrete two-dimensional Cantor sets exhibit a fractal uncertainty principle.

Corollary 5. *Let $\mathcal{A}$, $\mathcal{B}$ be a pair of alphabets. Equation* (6) *holds for some* $\beta > \max\bigl(0, 1 - \frac{1}{2}(\delta_A + \delta_B)\bigr)$ *if and only if*

- $\delta_A + \delta_B \geq 2$ *and the orthogonal line condition from Theorem 2 holds*,
- $\delta_A + \delta_B \leq 2$ *and for some* $\boldsymbol{j}, \boldsymbol{j}' \in \mathcal{A}$, $\boldsymbol{k}, \boldsymbol{k}' \in \mathcal{B}$,

$$\langle \boldsymbol{j} - \boldsymbol{j}', \boldsymbol{k} - \boldsymbol{k}' \rangle \neq 0 \quad \textit{as an inner product in } \mathbb{Z}.$$

The second condition above is a different sort of orthogonal line condition from the first. Although it is not initially obvious, the two conditions are the same when $\delta_A + \delta_B = 2$. Indeed, this must be the case, because both conditions are if and only if statements. If $\delta_A + \delta_B = 2$ and $\mathcal{A}$, $\mathcal{B}$ do not obey an FUP, then $\delta_A = \delta_B = 1$ and

$$\mathcal{A} = \{(x_0, t) : t \in \mathbb{Z}_M\} \quad \text{and} \quad \mathcal{B} = \{(t, y_0) : t \in \mathbb{Z}_M\} \quad \text{for some } x_0, y_0 \in \mathbb{Z}_M$$

or

$$\mathcal{A} = \{(t, t) : t \in \mathbb{Z}_M\} \quad \text{and} \quad \mathcal{B} = \{(t, M - 1 - t) : t \in \mathbb{Z}_M\},$$

or the reverse of these. Indeed, if $\delta_A < 1$ then $\boldsymbol{X}$ is less than one-dimensional and it cannot contain any line, so Theorem 2 applies. If $\delta_A = \delta_B = 1$ then $|\mathcal{A}| = |\mathcal{B}| = N$, and $\mathcal{A} - \mathcal{A}$, $\mathcal{B} - \mathcal{B}$ must both lie on one-dimensional cosets as subsets of $\mathbb{Z}^2$. This can only be true in one of the two cases listed above.

1.3. ***Sketch of the argument.*** Suppose $f : \mathbb{Z}_N^2 \to \mathbb{C}$ has $\operatorname{supp} f = S$, $\operatorname{supp} \hat{f} = T$. Our argument shows that if S avoids lines in a robust sense, then $|T| \gtrsim N^2$. Proposition 15 is a realization of this heuristic.

We start by writing functions on $\mathbb{Z}_N^2$ with Fourier support in $[0, D]^2$ as a trigonometric polynomial on $\mathbb{T}^2 \subset \mathbb{C}^2$ with degree $\le D$. We gain two things from using polynomials: unique factorization and Bezout's theorem on the intersection of zero loci. The heart of the argument is constructing a trigonometric polynomial

$$h(x, y) = \sum_{0\le k,l\le D} a_{kl} z^k w^l, \quad z = e^{\frac{2\pi i}{N}x},\ w = e^{\frac{2\pi i}{N}y},\ D \lesssim \sqrt{|T|}, \tag{7}$$

which vanishes on all of T except one line (and does not vanish on all of T). Then $h\hat{f}$ is nonzero and supported along a line, so $(h\hat{f})^\vee$ has constant magnitude along dual lines. We have $(h\hat{f})^\vee = h^\vee * f$, so

$$\operatorname{supp}(h\hat{f})^\vee \subset S - [0, D] \times [0, D].$$

Thus $S - [0, D] \times [0, D]$ contains *some* dual line, and combining this fact with the structural condition on S implies $D \gtrsim N$. Thus $|T| \gtrsim N^2$. Because we end up analyzing the function $h\hat{f}$, h is called a *multiplier*.

It is useful to consider a hypothetical scenario: what if T is the vanishing set of some low-degree trigonometric polynomial in $\mathbb{Z}_N^2$, e.g.,

$$T = \{(x, y) \in \mathbb{Z}_N^2 : z^2 + 4zw + w = 1\}, \quad z = e^{\frac{2\pi i}{N}x},\ w = e^{\frac{2\pi i}{N}y}?$$

Bezout's inequality (Theorem 24) states that any trigonometric polynomial h can only vanish on at most $4D$ points of T, or it must vanish on all of T. So any multiplier as in (7) would have degree $\sim |T| \gg \sqrt{|T|}$, obstructing our strategy if $|T|$ is large.

Luckily, Theorem 19 from [Ruppert 1993, Corollary 5] and [Beukers and Smyth 2002, Theorem 4.1] excludes this possibility. They prove that the vanishing set of a degree-D trigonometric polynomial in $\mathbb{Z}_N^2$ either has order $\le 22D^2$ or contains a line. Concretely, with T defined as above, $|T| \le 88$ for all N. This theorem gives a sharp quantitative form to Lang's conjecture, which is a qualitative statement about cyclotomic roots of polynomials in $\mathbb{C}^n$ — see Section 4 for more details. Lemma 11 encapsulates this number-theoretic input as it applies to our result.

1.4. ***An application to quantum chaos.*** Dyatlov and Jin [2017] initially introduced Theorem 1 to prove results in quantum chaos. In particular, they used Theorem 1 to prove a class of one-dimensional *quantum open baker's maps*, a discrete model for open quantum maps, always have a spectral gap. Adapting their pipeline we can use our Theorem 2 to prove a large class of two-dimensional quantum open baker's maps have a spectral gap.

One-dimensional baker's maps. First we will review the one-dimensional situation as discussed in [Dyatlov and Jin 2017]. The quantum open baker's maps in consideration are parametrized by triples

$$(M, \mathcal{A}, \chi), \quad M \in \mathbb{Z}_{>0},\ \mathcal{A} \subsetneq \mathbb{Z}_M,\ \chi \in C_0^\infty((0, 1); [0, 1]).$$

Here M is the base, $\mathcal{A}$ is the alphabet, and χ is the cutoff function. For any $N \ge 1$, let $\chi_N \in \ell^2(\mathbb{Z}_N)$ be given by $\chi_N(x) = \chi(x/N)$. For each $k \ge 1$ the corresponding quantum open baker's map is the operator on $\ell^2(\mathbb{Z}_N)$, $N = M^k$, given by

$$B_N = \mathcal{F}_N^* \begin{pmatrix} \chi_{N/M}\, \mathcal{F}_{N/M}\, \chi_{N/M} & & \\ & \ddots & \\ & & \chi_{N/M}\, \mathcal{F}_{N/M}\, \chi_{N/M} \end{pmatrix} I_{\mathcal{A},M}.$$

Here $I_{\mathcal{A},M}$ is the $N \times N$ diagonal matrix with k-th diagonal entry equal to 1 if $\left\lfloor \frac{\ell}{N/M} \right\rfloor \in \mathcal{A}$ and 0 otherwise, and $\chi_{N/M}\, \mathcal{F}_{N/M}\, \chi_{N/M}$ is an $(N/M) \times (N/M)$ block matrix given by the corresponding operator on $\ell^2(\mathbb{Z}_{N/M})$. It is convenient to introduce the projection operator

$$\Pi_a : \ell^2(\mathbb{Z}_N) \to \ell^2(\mathbb{Z}_{N/M}), \quad a \in \mathbb{Z}_M, \qquad \Pi_a u(j) = u\left(j + a\frac{N}{M}\right).$$

Then

$$B_N = \sum_{a \in \mathcal{A}} B_N^a, \quad B_N^a := \mathcal{F}_N^*\, \Pi_a^*\, \chi_{N/M}\, \mathcal{F}_{N/M}\, \chi_{N/M}\, \Pi_a.$$

Let $\mathcal{X}_k \subset \mathbb{Z}_{M^k}$ denote the Cantor iterates of $\mathcal{A}$ as before. The following proposition relates the fractal uncertainty principle to spectral gaps for B_N.

Proposition 6 [Dyatlov and Jin 2017, Proposition 2.6]. *Suppose*

$$\|1_{\mathcal{X}_k} \mathcal{F}\, 1_{\mathcal{X}_k}\|_{2\to 2} \le C_\beta M^{-k\beta} \quad \textit{for all } k. \tag{8}$$

Then

$$\limsup_{N\to\infty} \max\{|\lambda| : \lambda \in \operatorname{Sp}(B_N)\} \le M^{-\beta}, \tag{9}$$

where $\operatorname{Sp}(B_N)$ *is the spectrum.*

Combining Proposition 6 with Theorem 1, Dyatlov and Jin obtain a spectral gap for our quantum open bakers maps.

Theorem 7 [Dyatlov and Jin 2017, Theorem 1]. *There exists* $\beta = \beta(M, \mathcal{A}) > 0$ *such that*

$$\limsup_{N\to\infty} \max\{|\lambda| : \lambda \in \operatorname{Sp}(B_N)\} \le M^{-\beta},$$

where $\operatorname{Sp}(B_N)$ *is the spectrum.*

It is not hard to show that (8) always holds with $\beta = \max\left(0, \frac{1}{2} - \delta\right)$, δ the fractal dimension, so this theorem is only interesting when $\delta \ge \frac{1}{2}$. A different argument for $\delta < \frac{1}{2}$ shows that in Theorem 1 we can take $\beta > \max\left(0, \frac{1}{2} - \delta\right)$ for all δ, giving an improved spectral gap for all fractal dimensions in Theorem 7.

Two-dimensional baker's maps. A two-dimensional quantum open baker's map is parametrized by a triple

$$(M, \mathcal{A}, \chi), \quad M \in \mathbb{Z}_{>0},\ \mathcal{A} \subsetneq (\mathbb{Z}_M)^2,\ \chi \in C_0^\infty((0,1)^2; [0,1]).$$

We will define baker's maps $B_N : \ell^2(\mathbb{Z}_N^2) \to \ell^2(\mathbb{Z}_N^2)$, $N = M^k$. As before, define

$$\Pi_{\boldsymbol{a}} : \ell^2(\mathbb{Z}_N^2) \to \ell^2(\mathbb{Z}_{N/M}^2), \quad \boldsymbol{a} = (a_1, a_2) \in (\mathbb{Z}_M)^2, \qquad \Pi_{\boldsymbol{a}} u(\boldsymbol{j}) = u\left(\boldsymbol{j} + \boldsymbol{a}\frac{N}{M}\right).$$

Then set

$$B_N = \sum_{\boldsymbol{a}\in\mathcal{A}} B_N^{\boldsymbol{a}}, \qquad B_N^{\boldsymbol{a}} := \mathcal{F}_N^* \, \Pi_{\boldsymbol{a}}^* \, \chi_{N/M} \, \mathcal{F}_{N/M} \, \chi_{N/M} \, \Pi_{\boldsymbol{a}},$$

where $\mathcal{F}_N$ denotes the unitary Fourier transform on $\ell^2(\mathbb{Z}_N^2)$ and $\chi_N(\boldsymbol{j}) = \chi(\boldsymbol{j}/N)$. In Section 5.4 we sketch the proof that Proposition 6 holds for two-dimensional bakers maps as well, leading to the following.

Theorem 8. *Suppose $\mathcal{A} \subsetneq \mathbb{Z}_M^2$ is an alphabet such that $\boldsymbol{X}$, the Cantor set generated by $\mathcal{A}$, does not contain a pair of orthogonal lines as in Theorem 2. Then there is some $\beta = \beta(M, \mathcal{A}) > 0$ so that*

$$\limsup_{N\to\infty} \max\{|\lambda| : \lambda \in \mathrm{Sp}(B_N)\} \le M^{-\beta}.$$

Just as Theorem 7 is only interesting for $\delta \ge \frac{1}{2}$, Theorem 8 is only interesting for $\delta \ge 1$, because we can always take $\beta = \max(0, 1-\delta)$ in (8).

1.5. ***Organization.*** In Section 2 we give a new proof of a one-dimensional FUP (Theorem 1) as a warmup for our two-dimensional argument. In Section 3 we prove Theorem 2, up to the proof of the main Lemma 11, which we defer to Section 4. In Section 5 we supply proofs of several earlier claims which are not directly relevant to Theorem 2. In particular, we show the condition of Theorem 2 is sharp, prove Proposition 3 regarding lines in Cantor sets, and sketch the two-dimensional proof of Proposition 6 regarding the application of FUP to quantum baker's maps. In Appendix A we give a sketch of Ruppert and Beukers–Smyth's Theorem 19, which is the essential ingredient to our Lemma 11. Finally, in Appendix B, we compare Theorem 2 to a more recent higher-dimensional FUP the author [Cohen 2023] proved in $\mathbb{R}^d$. The more recent result can be used to prove an FUP for discrete Cantor sets in any dimension that avoid all lines, but cannot recover the precise orthogonal line condition proved in two dimensions in the present paper.

2. The one-dimensional argument

Our starting point is the following simple argument which can be used to establish a one-dimensional FUP.

Proposition 9. *Let $I = [a, b)$ be an interval, and suppose $f : \mathbb{Z}_N \to \mathbb{C}$ is nonzero and has $\hat{f}|_I = 0$. Then $|\mathrm{supp}\, f| > |I| = b - a$.*

Proof. Suppose $|\mathrm{supp}\, f| = k$. Let $S = \mathrm{supp}\, f = \{x_1, \ldots, x_k\}$. Let $F(z)$ be the polynomial

$$F(z) = (z - e^{\frac{2\pi i}{N}x_1}) \cdots (z - e^{\frac{2\pi i}{N}x_{k-1}}) = \sum_{j=0}^{k-1} a_j z^j.$$

Let $h : \mathbb{Z}_N \to \mathbb{C}$ be defined by

$$h(x) = \frac{1}{\sqrt{N}} F(e^{\frac{2\pi i}{N}x}), \qquad \hat{h}(j) = \begin{cases} a_j, & 0 \le j \le k-1, \\ 0, & \text{else.} \end{cases}$$

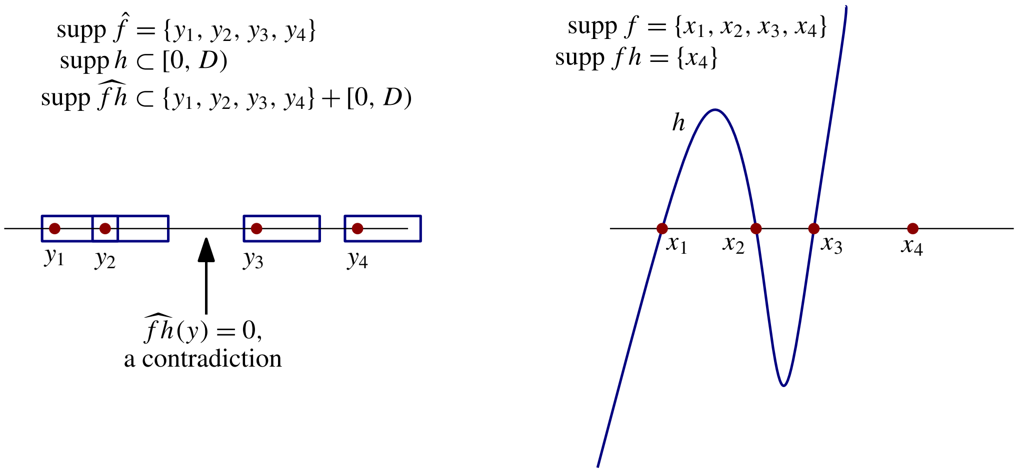


Figure 2. Diagram of the one-dimensional argument.

Then h vanishes on all of S except for x_k (and h is nonzero at x_k). Thus $hf = c\delta_{x_k}$, $c \neq 0$. So $\widehat{hf}$ has full Fourier support. But

$$\widehat{hf}(b-1) = (\hat{h} * \hat{f})(b-1) = \sum_{j=0}^{k-1} \hat{h}(j)\hat{f}(b-1-j).$$

If $k \leq |I|$ we have $\widehat{hf}(b-1) = 0$ leading to a contradiction. Thus $|\operatorname{supp} f| > |I|$. □

See Figure 2 for a visualization.

Remark 10. This proof shares some similarities with Bourgain and Dyatlov's proof of a one-dimensional FUP for general fractal sets. They constructed a function ψ with compact Fourier support and which decays quickly on a fractal set. They multiply by this function to discover that a function supported on a fractal set must have substantial Fourier mass in a union of intervals. In the discrete setting, things are much simpler: we may construct a multiplier that vanishes on all but one element of the fractal set, and then multiply by this function to discover some Fourier mass in every gap.

3. The two-dimensional argument

We first state our main lemma, then derive Theorem 2 from this lemma, and finally discuss the proof of the lemma. For $A \subset \mathbb{Z}_N^2$, let

$$\mathsf{N}_R(A) = A + [0, R) \times [0, R) = \operatorname{supp}(1_{[0,R)\times[0,R)} * 1_A)$$

be the R-neighborhood of A. A line $\ell \subset \mathbb{Z}_N^2$ is a coset of the form

$$\ell = \{(x, y) \in \mathbb{Z}_N^2 : ax + by = c\}.$$

The coefficients (a, b, c) are only determined up to multiplication by $\mathbb{Z}_N^\times$. We say ℓ is *irreducible* if a, b are coprime over $\mathbb{Z}_N$, and $\|\ell\| = R$ is the minimal number so that we can write

$$\ell = \{(x, y) \in \mathbb{Z}_N^2 : ax + by = c\}, \quad |a|, |b| \leq R. \tag{10}$$

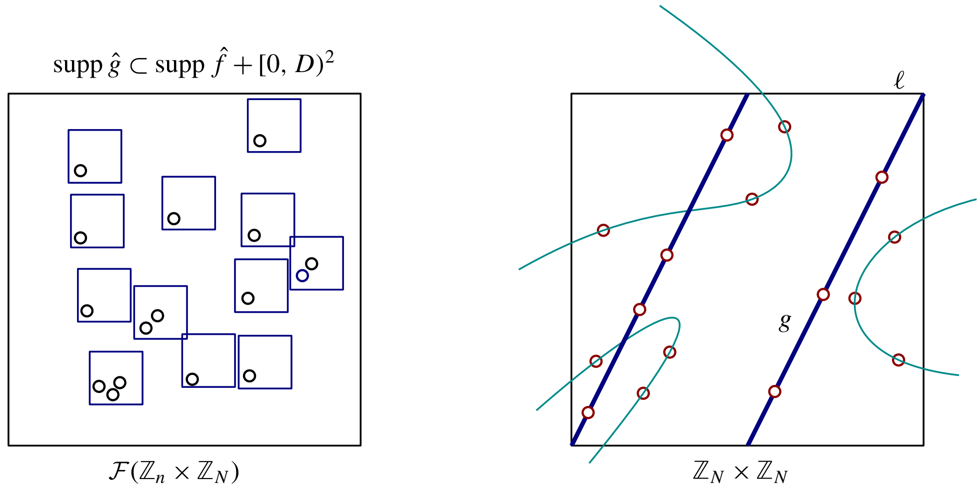


Figure 3. Visualization of Lemma 11.

Lemma 11. *Let* $f : \mathbb{Z}_N^2 \to \mathbb{C}$ *be a nonzero function with* supp $f = S$. *Let* $R = \lfloor 200|S|^{1/2} \rfloor$. *There is an irreducible line* ℓ *with* $\|\ell\| \le R$ *and a nonzero function* g *with* supp $g \subset S \cap \ell$ *and* supp $\hat{g} \subset \mathsf{N}_R(\text{supp } \hat{f})$.

This lemma is analogous to the proof of Proposition 9, except we can only localize the support of f to a line ℓ rather than to a single point. See Figure 3. Before showing how to derive Theorem 2 using this lemma we discuss discretizations of sets in $\mathbb{T}^2$, lines in $\mathbb{T}^2$, and lines in $\mathbb{Z}_N^2$.

3.1. ***Discretization of fractal sets.*** It will be more convenient to state our main results for discretizations of general fractal sets in $\mathbb{T}^2$ and then specialize to Cantor sets later. Let $\boldsymbol{X} \subset \mathbb{T}^2$ be closed. For $0 < r < 1$, let $\mathsf{N}_r(\boldsymbol{X}) = \boldsymbol{X} + [-r, r] \times [-r, r]$ be the r-neighborhood.[1] Let

$$X_N = \left\{ (x, y) \in \mathbb{Z}_N^2 : \left[\frac{x}{N}, \frac{x+1}{N}\right] \times \left[\frac{y}{N}, \frac{y+1}{N}\right] \cap \boldsymbol{X} \neq \varnothing \right\}$$
$$\subset \left\{ (x, y) \in \mathbb{Z}_N^2 : \left(\frac{x}{N}, \frac{y}{N}\right) \in \mathsf{N}_{1/N}(\boldsymbol{X}) \right\}$$

be a discretization of $\boldsymbol{X}$ to $\mathbb{Z}_N^2$. If $\boldsymbol{X}$ is the limiting Cantor set for an alphabet $\mathcal{A}$, then $X_{M^k} \subset \mathbb{Z}_{M^k}^2$ is just slightly larger than the k-th Cantor iterate $\mathcal{X}_k$ of $\mathcal{A}$ (due to endpoint considerations). Likewise, the drawing $\boldsymbol{X}_k$ (4) of the k-th iterate in $\mathbb{T}^2$ is slightly smaller than $\mathsf{N}_{M^{-k}}(\boldsymbol{X})$. If R is an integer and $\mathsf{N}_R(X_N) = X_N + [0, R) \times [0, R)$, then

$$\mathsf{N}_R(X_N) \subset \left\{ (x, y) \in \mathbb{Z}_N^2 : \left(\frac{x}{N}, \frac{y}{N}\right) \in \mathsf{N}_{R/N}(\boldsymbol{X}) \right\}, \tag{11}$$

where $\mathsf{N}_{R/N}(\boldsymbol{X}) \subset \mathbb{T}^2$. In what follows R will be $\sim N^\beta$, $\beta < 1$, so $R/N \sim N^{\beta-1}$, and $\mathsf{N}_{R/N}(\boldsymbol{X})$ will look like a very small neighborhood of $\boldsymbol{X}$ in $\mathbb{T}$.

[1]Our convention is that $\mathsf{N}_r(\boldsymbol{X}) = \boldsymbol{X} + [-r, r] \times [-r, r]$ denotes a neighborhood in $\mathbb{T}^2$, and $\mathsf{N}_R(A) = A + [0, R) \times [0, R)$ denotes an "upper right" neighborhood in $\mathbb{Z}_N^2$. We take the full neighborhood in $\mathbb{T}^2$ rather than just the upper right neighborhood for technical reasons — this convention makes (11) true, and otherwise it would be more complicated to state.

3.2. ***Some useful lemmas on lines.***

Lemma 12. *Let*

$$\ell = \{(x, y) \in \mathbb{Z}_N^2 : ax + by = c\}, \quad a, b, c \in \mathbb{Z}_N,$$

be an irreducible line, i.e., a, b are coprime as elements of $\mathbb{Z}_N$. Then $\ell = \mathbb{Z}(-b, a) + \boldsymbol{p}$, where $\boldsymbol{p} \in \ell$ is arbitrary. We have $|\ell| = N$. Also, a, b can be taken as coprime integers.

Proof. Pick s, t so that $sa + tb = 1 \pmod N$. We have $(cs, ct) \in \ell$. Suppose $ax + by = 0$. We claim $(x, y) = (-b, a) \cdot (-tx + sy)$. Indeed,

$$-b(-tx + sy) = tbx - sby = tbx + sax = x \pmod N,$$
$$a(-tx + sy) = -atx + asy = tby + say = y \pmod N$$

as needed. This shows that for $(x, y) \in \ell$, $(x, y) - \boldsymbol{p} \in (-b, a)\mathbb{Z}$.

To see $|\ell| = N$, notice $(-nb, na) + \boldsymbol{p} = (-mb, ma) + \boldsymbol{p} \pmod N$ if and only if $(-(n-m)b, (n-m)a) = 0 \pmod N$ if and only if $n = m \pmod N$, using that a, b are coprime.

Finally, suppose a and b are not coprime integers, but $a = \alpha a'$, $b = \alpha b'$, where a', b' are coprime integers. Then because a, b are coprime mod N, α, N are coprime, so

$$ax + by = c \iff \alpha(a'x + b'y) = c \iff a'x + b'y = \alpha^{-1}c,$$

where the equalities above are mod N. □

We will need a uniformity result for lines through closed sets $\boldsymbol{X} \subset \mathbb{T}^2$. In what follows

$$d(\boldsymbol{p}, \boldsymbol{q}) = \max(|p_1 - q_1|_{\mathbb{T}}, |p_2 - q_2|_{\mathbb{T}}), \quad |x|_{\mathbb{T}} = \min_{n \in \mathbb{Z}} |x - n|_{\mathbb{R}}, \tag{12}$$

is the ℓ^∞ distance on $\mathbb{T}^2$. First we need a lemma.

Lemma 13. *Let $\boldsymbol{v} = (a, b)$ with a, b coprime integers. Every coset $\ell = \mathbb{R}\boldsymbol{v} + \boldsymbol{p}$ is quantitatively dense in $\mathbb{T}^2$, in the sense that, for every $\boldsymbol{q} \in \mathbb{T}^2$, we have $d(\boldsymbol{q}, \ell) \leq 1/\max(|a|, |b|)$.*

In the following proof we let $\frac{1}{b}\mathbb{Z} = \left\{\frac{n}{b} : n \in \mathbb{Z}\right\}$.

Proof. For every $y_0 \in \mathbb{T}$, $(\mathbb{R}\boldsymbol{v} + \boldsymbol{p}) \cap \{y = y_0\}$ is a coset of $\frac{1}{b}\mathbb{Z}$, and, for every $x_0 \in \mathbb{T}$, $(\mathbb{R}\boldsymbol{v} + \boldsymbol{p}) \cap \{x = x_0\}$ is a coset of $\frac{1}{a}\mathbb{Z}$. Thus

$$d((x_0, y_0), \ell) \leq d((x_0, y_0), \ell \cap \{y = y_0\}) \leq \frac{1}{|a|},$$
$$d((x_0, y_0), \ell) \leq d((x_0, y_0), \ell \cap \{x = x_0\}) \leq \frac{1}{|b|},$$

giving the result. □

Lemma 14. *Suppose $\boldsymbol{X} \subsetneq \mathbb{T}^2$ is closed. There is a constant $c_{\boldsymbol{X}} > 0$ such that, for every direction $\boldsymbol{v} \in \mathbb{R}^2 - \{0\}$, either some coset $\mathbb{R}\boldsymbol{v} + \boldsymbol{p}$ lies entirely on $\boldsymbol{X}$, or*

$$\sup_{x \in \mathbb{R}\boldsymbol{v} + \boldsymbol{p}} d(x, \boldsymbol{X}) \geq c_{\boldsymbol{X}} \tag{13}$$

for every $\boldsymbol{p}$. Moreover, there is some $C_{\boldsymbol{X}} > 0$ so that if a, b are coprime integers with $\max(|a|, |b|) > C_{\boldsymbol{X}}$, then (13) holds for $\boldsymbol{v} = (a, b)$.

Proof. Because $\boldsymbol{X}$ is a closed proper subset of $\mathbb{T}^2$, it is not dense, and there is some $x_0 \in \mathbb{T}^2$ with $d(x_0, \boldsymbol{X}) \geq 2c_0$. If $\boldsymbol{v} = (\alpha, \beta)$ with α/β or β/α irrational, then $\mathbb{R}\boldsymbol{v} + \boldsymbol{p}$ is dense and has points coming arbitrarily close to x_0. Thus

$$\sup_{x \in \mathbb{R}\boldsymbol{v}+\boldsymbol{p}} d(x, \boldsymbol{X}) \geq 2c_0.$$

Otherwise, let $\boldsymbol{v} = (a, b)$ with a, b coprime integers. By Lemma 13,

$$\inf_{x \in \mathbb{R}\boldsymbol{v}+\boldsymbol{p}} d(x, x_0) \leq 1/\max(|a|, |b|),$$
$$\sup_{x \in \mathbb{R}\boldsymbol{v}+\boldsymbol{p}} d(x, \boldsymbol{X}) \geq 2c_0 - 1/\max(|a|, |b|).$$

Hence if $\max(|a|, |b|) > 1/c_0$, then $\sup_{x \in \mathbb{R}\boldsymbol{v}+\boldsymbol{p}} d(x, \boldsymbol{X}) \geq c_0$. For each pair of coprime integers a, b with $\max(|a|, |b|) \leq 1/c_0$, either some coset $\mathbb{R}(a, b) + \boldsymbol{p}$ lies on $\boldsymbol{X}$, or there is a c_1 so

$$\sup_{x \in \mathbb{R}(a,b)=\boldsymbol{p}} d(x, \boldsymbol{X}) \geq c_1 \quad \text{for all } \boldsymbol{p} \in \mathbb{T}^2.$$

There are finitely many such choices of (a, b), so c_1 can be chosen uniformly in all of them. We take $c_{\boldsymbol{X}} = \min(c_0, c_1)$ in (13). □

3.3. *Proof of Theorem 2 assuming Lemma 11.* Before proving Theorem 2, we prove the following simpler proposition, which applies when one of the fractal sets $\boldsymbol{X}, \boldsymbol{Y}$ avoids all lines.

Proposition 15. *Suppose $\boldsymbol{X} \subsetneq \mathbb{T}^2$ is closed and does not contain any closed cosets $\mathbb{R}\boldsymbol{v} + \boldsymbol{p} \subset \mathbb{T}^2$. By Lemma 14, there is some $c_{\boldsymbol{X}} > 0$ so that*

$$\sup_{x \in \ell} d(x, \boldsymbol{X}) \geq c_{\boldsymbol{X}}, \quad \ell = \mathbb{R}\boldsymbol{v} + \boldsymbol{p} \textit{ arbitrary}.$$

If $f : \mathbb{Z}_N^2 \to \mathbb{C}$ is nonzero and has $\operatorname{supp} \hat{f} \subset X_N$, then

$$|\operatorname{supp} f| \geq \frac{c_{\boldsymbol{X}}^2}{400^2} N^2. \tag{14}$$

Proof. Suppose $\operatorname{supp} f = S$, $\operatorname{supp} \hat{f} \subset X_N$. Apply Lemma 11 to f. We obtain an $R \leq 200|S|^{1/2}$, a line

$$\ell = \{(x, y) : ax + by = c\}, \quad a, b \text{ coprime}, \quad \max(|a|, |b|) \leq R,$$

and a nonzero g supported on ℓ with $\operatorname{supp} \hat{g} \subset \mathsf{N}_R(X_N)$. We claim $R/N \geq c_{\boldsymbol{X}}/2$, which would imply (14).

Suppose $R/N < c_{\boldsymbol{X}}/2$. We show $g = 0$. Set $\boldsymbol{v} = (a, b)$ and $\boldsymbol{v}^\perp = (-b, a)$. Because g is supported on ℓ, $\hat{g}$ has constant magnitude on dual lines $\mathbb{Z}\boldsymbol{v} + \boldsymbol{p}$. Indeed,

$$\hat{g}(\boldsymbol{\xi}) = \frac{1}{N} \sum_{\boldsymbol{v}\cdot\boldsymbol{x}=c \pmod N} g(\boldsymbol{x}) e^{\frac{2\pi i}{N} \boldsymbol{\xi}\cdot\boldsymbol{x}},$$
$$\hat{g}(\boldsymbol{\xi} + n\boldsymbol{v}) = \frac{1}{N} \sum_{\boldsymbol{v}\cdot\boldsymbol{x}=c \pmod N} g(\boldsymbol{x}) e^{\frac{2\pi i}{N} n\boldsymbol{v}\cdot\boldsymbol{x}} e^{\frac{2\pi i}{N} \boldsymbol{\xi}\cdot\boldsymbol{x}} = e^{\frac{2\pi i}{N} nc} \hat{g}(\boldsymbol{\xi}).$$

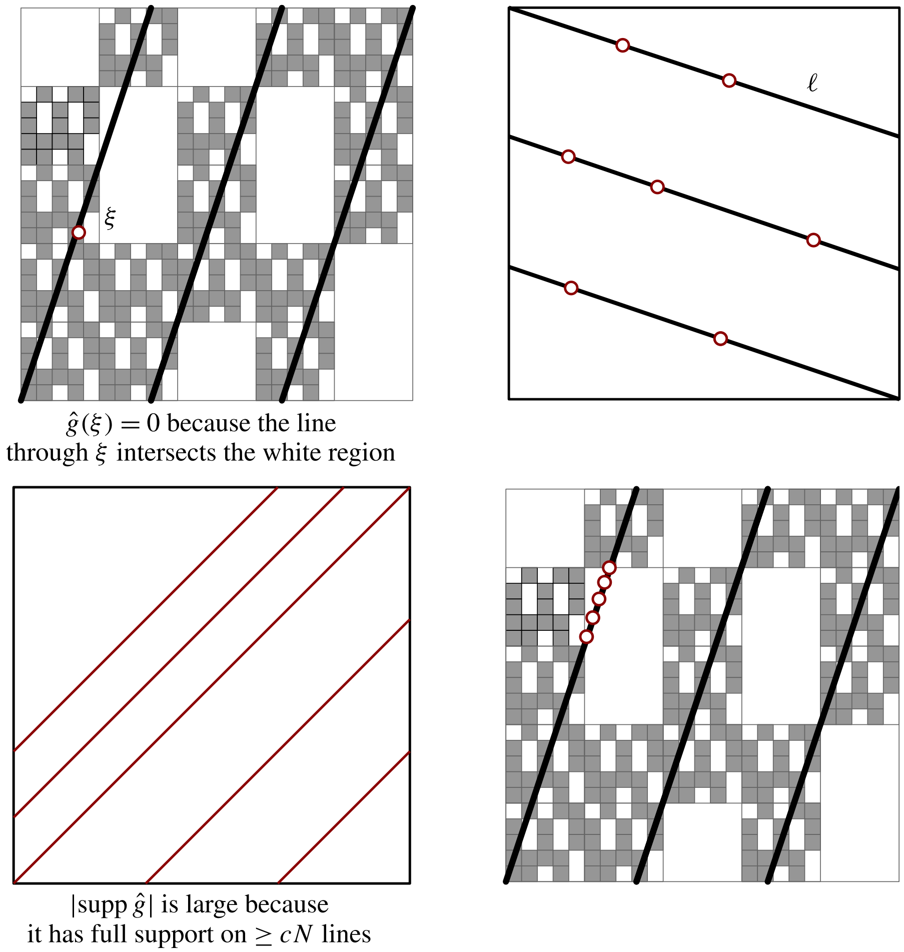


Figure 4. The two cases in Proposition 16 obtain contradictions in different ways.

Let $\boldsymbol{\xi} \in \mathbb{Z}_N^2$ be arbitrary. Let $t \in \mathbb{R}$ be such that $d(t\boldsymbol{v}/N + \boldsymbol{\xi}/N, \boldsymbol{X}) \geq c_{\boldsymbol{X}}$. Let n be the nearest integer to t. Then

$$d\left(\frac{n\boldsymbol{v}}{N} + \frac{\boldsymbol{\xi}}{N}, \boldsymbol{X}\right) \geq c_{\boldsymbol{X}} - \max\left(\frac{|a|}{N}, \frac{|b|}{N}\right) \geq \frac{c_{\boldsymbol{X}}}{2}.$$

By (11), since $R/N < c_{\boldsymbol{X}}/2$, we have $n\boldsymbol{v} + \boldsymbol{\xi} \notin \mathrm{N}_R(X_N)$, so $\hat g(n\boldsymbol{v} + \boldsymbol{\xi}) = 0$ by hypothesis. Thus $\hat g(\boldsymbol{\xi}) = 0$ as well. Since $\boldsymbol{\xi} \in \mathbb{Z}_N^2$ was arbitrary, $g = 0$. □

Now we prove a more general proposition applying to measure-zero sets $\boldsymbol{X}, \boldsymbol{Y}$ which don't contain a pair of orthogonal lines. Theorem 2 follows directly from this proposition by submultiplicativity.

Proposition 16. *Suppose $\boldsymbol{X}, \boldsymbol{Y} \subset \mathbb{T}^2$ are closed and have Lebesgue measure zero. Suppose that, for every direction $\boldsymbol{v} = (a, b) \in \mathbb{R}^2 - \{0\}$, $\boldsymbol{v}^\perp = (-b, a)$, either $\boldsymbol{X}$ contains no coset $\mathbb{R}\boldsymbol{v} + \boldsymbol{p}$ or $\boldsymbol{Y}$ contains no coset $\mathbb{R}\boldsymbol{v}^\perp + \boldsymbol{p}$. Then for large enough N, there is no nonzero $f : \mathbb{Z}_N^2 \to \mathbb{C}$ with* $\text{supp}\, f \subset X_N$ *and* $\text{supp}\, \hat f \subset Y_N$.

The proof involves two cases; see Figure 4.

Proof. First notice that by continuity of measure,

$$\lim_{r\to 0} |\mathsf{N}_r(\boldsymbol{X})| = \lim_{r\to 0} |\mathsf{N}_r(\boldsymbol{Y})| = 0, \tag{15}$$

where $|\cdot|$ denotes the Lebesgue measure. It follows that

$$|X_N|, |Y_N| = o(N^2) \tag{16}$$

as $N \to \infty$.

Using the hypothesis and Lemma 14, there is some $c > 0$ such that, for every coprime a, b, either

$$\sup_{y\in\mathbb{R}(a,b)+\boldsymbol{p}} d(y, \boldsymbol{Y}) \geq c \quad \text{for all } \boldsymbol{p} \tag{17}$$

or

$$\sup_{x\in\mathbb{R}(-b,a)+\boldsymbol{p}} d(x, \boldsymbol{X}) \geq c \quad \text{for all } \boldsymbol{p}. \tag{18}$$

There is also some $C > 0$ so that if $\max(|a|, |b|) > C$, then (17) and (18) both hold.

Suppose $\operatorname{supp} f = S \subset X_N$ and $\operatorname{supp} \hat{f} \subset Y_N$. Apply Lemma 11 to f to obtain an $R \leq o(N)$, a line

$$\ell = \{(x, y) : ax + by = c\}, \quad a, b \text{ coprime}, \ \max(|a|, |b|) \leq R,$$

and a nonzero g supported on $\ell \cap X_N$ with $\operatorname{supp} \hat{g} \subset \mathsf{N}_R(Y_N)$. Let $\boldsymbol{v} = (a, b)$, $\boldsymbol{v}^\perp = (-b, a)$.

<u>Case 1</u>: Suppose (17) holds. Then we are in the same position as Proposition 15, and for N large enough we conclude $g = 0$, which is a contradiction.

<u>Case 2</u>: Suppose (17) does not hold. Then (18) holds and $\max(|a|, |b|) \leq C$. Choose $\boldsymbol{p} = (p_1, p_2) \in \ell$, so $\ell = \mathbb{Z}\boldsymbol{v}^\perp + \boldsymbol{p}$. Write $g(n\boldsymbol{v}^\perp + \boldsymbol{p}) = \tilde{g}(n)$. Then

$$\hat{g}(\boldsymbol{\xi}) = \frac{1}{N} \sum_{n\in\mathbb{Z}_N} \tilde{g}(n) e^{-\frac{2\pi i}{N}\boldsymbol{\xi}\cdot(n\boldsymbol{v}^\perp+\boldsymbol{p})} = e^{-\frac{2\pi i}{N}\boldsymbol{\xi}\cdot\boldsymbol{p}} N^{-1} \sum_{n\in\mathbb{Z}_N} \tilde{g}(n) e^{-\frac{2\pi i}{N} n\boldsymbol{\xi}\cdot\boldsymbol{v}^\perp}.$$

Notice in particular that $\hat{g}$ only depends on $\boldsymbol{\xi}\cdot\boldsymbol{v}^\perp$. By Lemma 12, for every $d \in \mathbb{Z}_N$ there are N solutions in $\boldsymbol{\xi}$ to $\boldsymbol{\xi}\cdot\boldsymbol{v}^\perp = d$. So we may write

$$\hat{g}(\boldsymbol{\xi}) = \frac{1}{\sqrt{N}} e^{-\frac{2\pi i}{N}\boldsymbol{\xi}\cdot\boldsymbol{p}} \hat{\tilde{g}}(\boldsymbol{\xi}\cdot\boldsymbol{v}^\perp) = \frac{1}{N} \sum_{n\in\mathbb{Z}_N} \tilde{g}(n) e^{-\frac{2\pi i}{N} n\boldsymbol{\xi}\cdot\boldsymbol{v}^\perp},$$

$$|\hat{g}(\boldsymbol{\xi})| = \frac{1}{\sqrt{N}} |\hat{\tilde{g}}(\boldsymbol{\xi}\cdot\boldsymbol{v}^\perp)|,$$

Thus $|\operatorname{supp} \hat{g}| = N|\operatorname{supp} \hat{\tilde{g}}|$.

Choose $t \in \mathbb{R}$ so that $d(t\boldsymbol{v}^\perp/N + \boldsymbol{p}/N, \boldsymbol{X}) \geq c$. Then

$$d\left(\frac{s\boldsymbol{v}^\perp}{N} + \frac{\boldsymbol{p}}{N}, \boldsymbol{X}\right) \geq c - |s-t|\frac{C}{N} \geq \frac{c}{2} \quad \text{for } |s-t| \leq \frac{c}{2C}N.$$

If s is an integer satisfying the above and $N > 100/c$, we conclude that $s\boldsymbol{v}^\perp + \boldsymbol{p} \notin X_N$.

Let $I = [t - (c/2C)N, t + (c/2C)N] \cap \mathbb{Z}$. Then $|I| \geq (c/C)N$ and $\tilde{g}|_I = 0$. By Proposition 9,

$$|\operatorname{supp} \hat{\tilde{g}}| = N|\hat{\tilde{g}}| \geq \frac{c}{C}N^2.$$

On the other hand, $|Y_N| \leq o(N^2)$, leading to a contradiction for large enough N. □

Remark 17. Although Proposition 16 applies to arbitrary fractal sets, Theorem 2 only applies to Cantor sets, because we need submultiplicativity in order to prove exponential decay (6).

Remark 18. Let us note a quantitative difference between Cases 1 and 2 above. In Case 1 the coefficients a, b determining the line ℓ should have size $o(N)$ in order to obtain a contradiction. In Case 2, the coefficients a, b must have size $O(1)$. Lemma 11 only gives $a, b = o(N)$, so we argue that if $\max(|a|, |b|) > C$, we can use Case 1, and Case 2 only arises when $\max(|a|, |b|) \leq C$.

Now we can conclude Theorem 2.

Proof of Theorem 2. Suppose $\mathcal{A}$ and $\mathcal{B}$ are alphabets satisfying the condition of Theorem 2. The Cantor sets they generate, $\boldsymbol{X}$ and $\boldsymbol{Y}$, satisfy the conditions of Proposition 16. Indeed, $\boldsymbol{X}, \boldsymbol{Y}$ have dimension < 2, so certainly $|\boldsymbol{X}| = |\boldsymbol{Y}| = 0$.

Let $\mathcal{X}_k, \mathcal{Y}_k \subset \mathbb{Z}_N^2,\ N = M^k$, be the k-th Cantor iterates. Then $\mathcal{X}_k \subset X_N,\ \mathcal{Y}_k \subset Y_N$, where X_N and Y_N are obtained by discretizing $\boldsymbol{X}, \boldsymbol{Y}$ as in Section 3.1. By Proposition 16, for N large enough there is no $f : \mathbb{Z}_N^2 \to \mathbb{C}$ with $\operatorname{supp} f \subset X_N$ and $\operatorname{supp} \hat{f} \subset Y_N$. Thus for k large enough, there is no f with $\operatorname{supp} f \subset \mathcal{X}_k$ and $\operatorname{supp} \hat{f} \subset Y_k$. For this k,

$$\|1_{\mathcal{Y}_k} \mathcal{F}\, 1_{\mathcal{X}_k}\|_{2\to 2} < 1$$

and so by submultiplicativity we conclude

$$\|1_{\mathcal{Y}_k} \mathcal{F}\, 1_{\mathcal{X}_k}\|_{2\to 2} \lesssim M^{-k\beta}$$

for some $\beta > 0$. ☐

4. Proof of the main lemma

Lang [1965] conjectured that if C is an irreducible algebraic curve in $\mathbb{C}^{\times n}$ with infinitely many cyclotomic points — that is, points $(z_1, \ldots, z_n) \in C$ all of which are roots of unity — then C is a translate of a subgroup of $\mathbb{C}^{\times n}$ by a root of unity [Granville and Rudnick 2007].

The key ingredient in proving Lemma 11 is the following theorem from [Ruppert 1993, Corollary 5] and [Beukers and Smyth 2002, Theorem 4.1], which can be viewed as a sharp quantitative form of Lang's conjecture in two dimensions.

Theorem 19 [Ruppert 1993; Beukers and Smyth 2002]. *Let*

$$F(z, w) = \sum_{0\leq k,l\leq D} a_{kl} z^k w^l \tag{19}$$

be a polynomial in $\mathbb{C}[z, w]$ *with degree at most* D *in* z, w *separately. Then* F *has either at most* $22D^2$ *cyclotomic points, or infinitely many. In the latter case* F *has an irreducible factor*

$$z^a w^b - \zeta \quad \text{or} \quad z^a - \zeta w^b \tag{20}$$

for some root of unity ζ *and coprime integers* a, b.

We note that $z^a w^b - \zeta$ or $z^a - \zeta w^b$ is only irreducible if a and b are coprime integers, which is why that is part of the conclusion. In their paper Beukers Smyth actually proved significantly more; they gave an algorithm to compute this factor. The approach is to find seven polynomials $F_1, \dots, F_7$ so that every cyclotomic root of F is also a root of some F_j, and then apply Bezout's inequality to bound their pairwise intersection; see Appendix A for a sketch. In what follows

$$\deg F = \max_{a_{kl}\neq 0} \max(|k|, |l|), \qquad F(z, w) = \sum_{k,l} a_{kl} z^k w^l \tag{21}$$

so that (19) is the general form of a polynomial with degree $\leq D$.

Recall that we can embed $\mathbb{T}^2 \to \mathbb{C}^{\times 2}$ via

$$(x, y) \mapsto (e^{2\pi i x}, e^{2\pi i y}).$$

The cyclotomic points in $\mathbb{C}^{\times 2}$ are precisely the image of $(\mathbb{Q}/\mathbb{Z})^2$. For $F(z, w)$ a polynomial, we let

$$\begin{aligned} \mathsf{Z}(F) &= \{(x, y) \in \mathbb{T}^2 : F(e^{2\pi i x}, e^{2\pi i y}) = 0\}, \\ \mathsf{Z}_N(F) &= \{(x, y) \in \mathbb{Z}_N^2 : F(e^{\frac{2\pi i}{N} x}, e^{\frac{2\pi i}{N} y}) = 0\}. \end{aligned} \tag{22}$$

If we view $\mathbb{Z}_N^2$ as the subgroup of $\mathbb{T}^2$ given by

$$\mathbb{Z}_N^2 \cong \mathbb{T}_N^2 = \left\{ \left(\frac{x}{N}, \frac{y}{N} \right) \in \mathbb{T}^2 | x, y \in \mathbb{Z} \right\}$$

then $\mathsf{Z}_N(F) = \mathsf{Z}(F) \cap \mathbb{T}_N^2$. We say that a polynomial F of the form (20) *cuts out a line* because

$$\mathsf{Z}(F) = \{(x, y) \in \mathbb{T}^2 : ax + by = c\} \quad \text{or} \quad \mathsf{Z}(F) = \{(x, y) \in \mathbb{T}^2 : ax - by = c\},$$

with $a, b \geq 0$ integers and $c \in \mathbb{Q}$. If $c = c'/N,\ c' \in \mathbb{Z}$, then we say ℓ cuts out a line in $\mathbb{Z}_N^2$. Conversely, suppose

$$\ell = \{(x, y) \in \mathbb{Z}_N^2 : ax + by = c \pmod N\}$$

is an irreducible line. By Lemma 12, a, b can be taken as coprime integers. Then

$$\ell = \mathsf{Z}_N(P_\ell), \qquad P_\ell(z, w) = \begin{cases} z^a w^b - e^{\frac{2\pi i c}{N}}, & a, b \geq 0, \\ z^a - e^{\frac{2\pi i c}{N}} w^{|b|}, & a \geq 0,\ b < 0, \end{cases}$$

and P_ℓ is an irreducible polynomial with $\deg P_\ell \leq 2\|\ell\|$. Theorem 19 is related to Lemma 11 because functions $g : \mathbb{Z}_N^2 \to \mathbb{C}$ with $\operatorname{supp} \hat{g} \subset [0, D] \times [0, D]$ have values given by polynomials at cyclotomic points:

$$g(x, y) = \frac{1}{N} \sum_{0 \leq k, l \leq D} \hat{g}(k, l) z^k w^l, \qquad z = e^{\frac{2\pi i}{N} x}, w = e^{\frac{2\pi i}{N} y}.$$

Lemma 11 is a quick consequence of the following. We don't try to optimize the constant 200.

Lemma 20. *Let $S \subset \mathbb{Z}_N^2$ be an arbitrary nonempty set. Then there is a polynomial F^* with* $\deg F^* < 200|S|^{1/2} - 1$ *so that $S - \mathsf{Z}_N(F^*)$ is nonempty and lies on an irreducible line ℓ with $\|\ell\| \leq 200|S|^{1/2}$.*

We prove the slightly awkward bound $\deg F^* < 200|S|^{1/2} - 1$ in order to make the proof and application of Lemma 11 cleaner. Before proving this lemma, it is helpful to consider how it could fail to be true. Consider a quadratic polynomial

$$G(z, w) = a + bz + cw + fzw + dz^2 + ew^2$$

which does not cut out a line (e.g., G is not of the form $z = w^2$). Theorem 19 says that $|\mathsf{Z}_N(G)| \le 44$ for all N (the quadratic polynomial G cannot pass through many cyclotomic points). Ignoring this fact for a moment, it turns out that if for some G, N we have $|\mathsf{Z}_N(G)| > 1200^2$, then Lemma 20 would fail.

Let $S = \mathsf{Z}_N(G)$. Suppose F^* is a polynomial of degree $\le 200|S|^{1/2}$ such that $S - \mathsf{Z}_N(F^*)$ is nonempty and lies on a line ℓ with $\|\ell\| \le 200|S|^{1/2}$. The polynomial G cannot be a component of F^*, because that would mean $S \subset \mathsf{Z}_N(F^*)$. So by Bezout's inequality (Theorem 24),

$$|\mathsf{Z}_N(F^*) \cap S| \le 2 \deg F^* \le 400|S|^{\frac{1}{2}}.$$

If ℓ is a line with $\|\ell\| \le 200|S|^{1/2}$, then

$$|\ell \cap S| = |\mathsf{Z}_N(P_\ell) \cap S| \le 2 \deg P_\ell \le 4\|\ell\| \le 800|S|^{\frac{1}{2}}$$

again by Bezout's inequality. Thus if $S - \mathsf{Z}_N(F^*)$ lies on such a line ℓ,

$$|S| - 800|S|^{\frac{1}{2}} \le 400|S|^{\frac{1}{2}} \quad \Longrightarrow \quad |S| \le 1200^2$$

as claimed. Before proving Lemma 20 we need another lemma.

Lemma 21. *For every nonempty set $S \subset \mathbb{C}^2$, $D = \lfloor |S|^{1/2} \rfloor$, there is a nonzero polynomial $F(z, w) = \sum_{0 \le k,l \le D} a_{kl} z^k w^l$ vanishing on S.*

Proof. Consider the linear map taking

$$(a_{kl})_{0 \le k,l \le D} \mapsto \left(\sum_{kl} a_{kl} z^k w^l : (z, w) \in S \right).$$

If $(D+1)^2 > |S|$ then by rank nullity this has a nontrivial kernel, which is our desired polynomial F. Thus we may take $D = \lfloor |S|^{1/2} \rfloor$. □

The proof of Lemma 20 involves four cases; see Figure 5.

Proof of Lemma 20. We give a recursive algorithm to find our polynomial F^*. Mathematically this is phrased as induction on the size of S. For ease of presentation we prove we can take $\deg F^* \le 200|S|^{1/2}$, but the same argument can be optimized to give $\deg F^* \le 198|S|^{1/2}$, yielding the claim $\deg F^* < 200|S|^{1/2} - 1$.

Let F be a polynomial of minimal degree D with $S \subset \mathsf{Z}_N(F)$. We have $D \le |S|^{1/2}$ by Lemma 21. If there are several such polynomials, choose one with the minimal number of irreducible factors.

Case 1: F cuts out a line ℓ. In this case

$$\mathsf{Z}_N(F) = \left\{ (x, y) \in \mathbb{Z}_N^2 : \frac{ax}{N} + \frac{by}{N} = c \right\}, \quad c \in \mathbb{Q}.$$

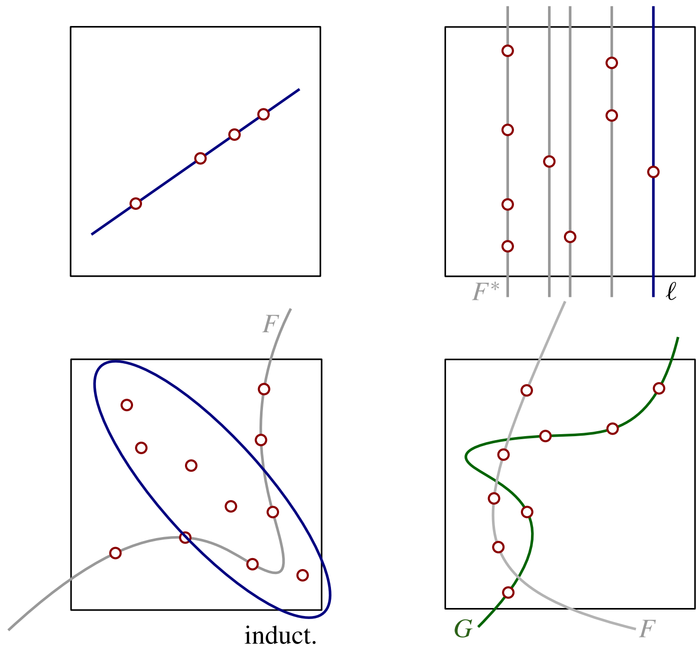


Figure 5. Cases 1–4 (left to right, top to bottom) in the proof of Lemma 20.

Because S is nonempty there is some $(x_0, y_0) \in S$ with $c = (ax_0 + by_0)/N$. So F cuts out a line ℓ in $\mathbb{Z}_N^2$, and

$$\mathsf{Z}_N(F) = \ell = \{(x, y) \in \mathbb{Z}_N^2 : ax + by = ax_0 + by_0\}, \quad \|\ell\| \le \deg F \le |S|^{\frac{1}{2}}.$$

Thus we are already done — we may take $F^* = 1$, and S already lies on a desired line ℓ.

Case 2: $|S| \le 200$. Let $S = \{(x_k, y_k) \in \mathbb{Z}_N^2\}$, and $\{x_1, \dots, x_m\}$ be the distinct x-coordinates appearing in S. If $m = 1$, we are in Case 1. Otherwise, set

$$F^* = (z - e^{2\pi i x_1/N}) \cdots (z - e^{2\pi i x_{m-1}/N}).$$

Then $\deg F^* < 200 \le 200|S|^{1/2}$, and

$$S - \mathsf{Z}_N(F^*) \subset \{x = x_m\}$$

lies on a line.

Case 3: F is irreducible but does not cut out a line. In this case, $|S| \le 22D^2$ by Theorem 19. Because $|S| \ge 200$, we have $D \ge 4$. Choose a curve G of degree $D - 1$ passing through at least

$$(D-1)^2 \ge \frac{1}{22}|S|\Big(1 - \frac{1}{D}\Big)^2 \ge \frac{|S|}{40}$$

points of S. Let $A = S \cap \mathsf{Z}_N(G)$. Notice $S - A$ is nonempty by the minimality of D. Now apply the inductive hypothesis to find a polynomial H passing through all but one line of $S - A$ with $\deg H \le 200|S - A|^{1/2}$, and set $F^* = GH$. We have

$$\deg G \le 200\sqrt{|S|\big(1 - \tfrac{1}{40}\big)} + D - 1 \le |S|^{\frac{1}{2}}(198 + 1) \le 200|S|^{\frac{1}{2}}$$

as needed.

Case 4: F is reducible. Let $F = GH$, where neither G nor H are scalars and $|\mathsf{Z}_N(G)| \le |\mathsf{Z}_N(H)|$. Let $T = \mathsf{Z}_N(G) - \mathsf{Z}_N(H)$. Because $\deg H \le \deg F$ and H has fewer irreducible factors than F, $S \not\subset \mathsf{Z}_N(H)$, so T is nonempty. Using the inductive hypothesis we may find a polynomial P passing through all but one line of the set $T = \mathsf{Z}_N(G) - \mathsf{Z}_N(H)$ (notice T is nonempty by the minimality of the number of irreducible factors). We have $|T| \le |S|/2$. Set $F^* = HP$. We have

$$\deg F^* \le 200|T|^{\frac{1}{2}} + \deg H \le 200\big(\tfrac{1}{2}|S|\big)^{\frac{1}{2}} + |S|^{\frac{1}{2}} \le 143|S|^{\frac{1}{2}}$$

as needed. □

Now we prove Lemma 11.

Proof of Lemma 11. Let $f : \mathbb{Z}_N^2 \to \mathbb{C}$ have $\operatorname{supp} f = S$. Let $R = \lfloor 200|S|^{1/2} \rfloor$. By Lemma 20 let

$$F^* = \sum_{0 \le k,l < R} a_{k,l} z^k w^l \in \mathbb{C}[z, w], \quad \ell = \{(x, y) \in \mathbb{Z}_N^2 : ax + by = c\},\ \max(|a|, |b|) \le R$$

be such that

$$A = S - \mathsf{Z}_N(F^*)$$

is nonempty and lies on ℓ. Let $h : \mathbb{Z}_N^2 \to \mathbb{C}$ be defined by

$$h(x, y) = \frac{1}{N} F(e^{\frac{2\pi i}{N} x}, e^{\frac{2\pi i}{N} y}), \quad \hat{h}(k, l) = \begin{cases} a_{k,l}, & 0 \le k, l < R, \\ 0, & \text{else.} \end{cases}$$

Thus hf is nonzero and supported in ℓ. Also,

$$\operatorname{supp} \widehat{hf} = \operatorname{supp} \hat{h} * \hat{f} \subset \operatorname{supp}(1_{[0,R)^2} * 1_{\operatorname{supp} \hat{f}}) = \mathsf{N}_R(\operatorname{supp} \hat{f}).$$

Setting $g := hf$ we are done. □

Remark 22. In order to obtain Theorem 2, it would suffice to replace Theorem 19 with a quantitatively weaker version which says that for $F(z, w)$ a degree-D irreducible polynomial not cutting out a line,

$$\#\{(\zeta_1, \zeta_2) : F(\zeta_1, \zeta_2) = 0\} \lesssim_\varepsilon D^{2+\varepsilon}, \quad \zeta_1, \zeta_2 \text{ cyclotomic},$$

for all $\varepsilon > 0$.

5. Loose ends

5.1. ***Submultiplicativity.*** We prove the submultiplicativity estimate (2) in two dimensions.

Proof. We first recall how Dyatlov [2019, Lemma 4.6] proved submultiplicativity for discrete one-dimensional cantor sets. The Fourier transform $\mathcal{F} : \mathbb{Z}_{N_1 N_2} \to \mathbb{Z}_{N_1 N_2}$ can be realized as follows. We realize $L^2(\mathbb{Z}_{N_1 N_2})$ as $L^2(\mathrm{Mat}_{N_1 \times N_2})$. In this basis,

$$\mathcal{F} = \mathcal{F}_{\mathrm{col}} D \mathcal{F}_{\mathrm{row}},$$

where

$$(\mathcal{F}_{\mathrm{row}} U)_{pb} = \frac{1}{\sqrt{N_2}} \sum_{q=0}^{N_2} e^{-\frac{2\pi i}{N_2} bq} U_{pq} \qquad \text{applies the Fourier transform to each row,}$$

$$(\mathcal{F}_{\text{col}}U)_{pb} = \frac{1}{\sqrt{N_1}}\sum_{q=0}^{N_1} e^{-\frac{2\pi i}{N_1}pq}U_{qb} \quad \text{applies the Fourier transform to each column,}$$

$$(DU)_{pb} = e^{-\frac{2\pi i}{N}pb}U_{pb} \quad \text{applies a phase shift to each entry.}$$

Abstractly, $L^2(\mathbb{Z}_{N_1N_2}) = L^2(\mathbb{Z}_{N_1}) \otimes L^2(\mathbb{Z}_{N_2})$. But

$$\mathcal{F}_{\mathbb{Z}_{N_1}} \otimes \mathcal{F}_{\mathbb{Z}_{N_2}} = \mathcal{F}_{\mathbb{Z}_{N_1}\times\mathbb{Z}_{N_2}} \neq \mathcal{F}_{\mathbb{Z}_{N_1N_2}}.$$

The phase shift operator D corrects this issue. We can write

$$\begin{gathered}\mathcal{F}_{\text{row}} = \text{Id} \otimes \mathcal{F}_{\mathbb{Z}_{N_2}}, \quad \mathcal{F}_{\text{col}} = \mathcal{F}_{\mathbb{Z}_{N_1}} \otimes \text{Id}, \\ \mathcal{F} = (\mathcal{F}_{\mathbb{Z}_{N_1}} \otimes \text{Id}) \circ D \circ (\text{Id} \otimes \mathcal{F}_{\mathbb{Z}_{N_2}}).\end{gathered} \tag{23}$$

In the notation of tensor products, if $N_1 = M^k$, $N_2 = M^r$, then

$$1_{\mathcal{X}_{k+r}} = 1_{\mathcal{X}_k} \otimes 1_{\mathcal{X}_r}, \quad 1_{\mathcal{Y}_{k+r}} = 1_{\mathcal{Y}_k} \otimes 1_{\mathcal{Y}_r}.$$

Because $1_{\mathcal{X}_{k+r}}$, $1_{\mathcal{Y}_{k+r}}$ commute with D,

$$\begin{aligned}1_{\mathcal{Y}_{k+r}}\mathcal{F}_{k+r}1_{\mathcal{X}_{k+r}} &= (1_{\mathcal{Y}_k} \otimes 1_{\mathcal{Y}_r}) \circ (\mathcal{F}_{\mathbb{Z}_{N_1}} \otimes \text{Id}) \circ D \circ (\text{Id} \otimes \mathcal{F}_{\mathbb{Z}_{N_2}}) \circ (1_{\mathcal{X}_k} \otimes 1_{\mathcal{X}_r}) \\ &= (1_{\mathcal{Y}_k}\mathcal{F}_{\mathbb{Z}_{N_1}} \otimes 1_{\mathcal{Y}_r}) \circ D \circ (1_{\mathcal{X}_k} \otimes \mathcal{F}_{\mathbb{Z}_{N_2}}1_{\mathcal{X}_r}) \\ &= (1_{\mathcal{Y}_k}\mathcal{F}_{\mathbb{Z}_{N_1}}1_{\mathcal{X}_k} \otimes 1_{\mathcal{Y}_r}) \circ D \circ (1_{\mathcal{X}_k} \otimes 1_{\mathcal{Y}_r}\mathcal{F}_{\mathbb{Z}_{N_2}}1_{\mathcal{X}_r}).\end{aligned}$$

It follows from the above that

$$\|1_{\mathcal{Y}_{k+r}}\mathcal{F}_{k+r}1_{\mathcal{X}_{k+r}}\|_{2\to 2} \le \|1_{\mathcal{Y}_k}\mathcal{F}_{\mathbb{Z}_{N_1}}1_{\mathcal{X}_k}\|_{2\to 2}\|1_{\mathcal{Y}_r}\mathcal{F}_{\mathbb{Z}_{N_2}}1_{\mathcal{X}_r}\|_{2\to 2}$$

as desired. Written in this way, it is easy to see that the submultiplicativity estimate extends to two dimensions. We have the equation

$$\mathcal{F}_{\mathbb{Z}^2_{N_1N_2}} = (\mathcal{F}_{\mathbb{Z}^2_{N_1}} \otimes \text{Id}) \circ D \circ (\text{Id} \otimes \mathcal{F}_{\mathbb{Z}^2_{N_2}}),$$

where D is a multiplication operator (indeed, this can be seen from writing $\mathbb{Z}^2_{N_1N_2}$ as a product of two copies of $\mathbb{Z}_{N_1N_2}$ and tensoring (23) with itself) and the rest of the proof goes through verbatim. □

5.2. ***Theorem 2 is sharp.*** Suppose $\mathcal{A}$, $\mathcal{B}$ are alphabets generating fractal sets $\boldsymbol{X}, \boldsymbol{Y} \subset \mathbb{T}^2$ with

$$\begin{gathered}\mathbb{R}\boldsymbol{v} + \boldsymbol{p} \subset \boldsymbol{X} \quad \text{and} \quad \mathbb{R}\boldsymbol{v}^\perp + \boldsymbol{q} \subset \boldsymbol{Y}, \\ \boldsymbol{v} = (a, b), \ \boldsymbol{v}^\perp = (-b, a), \quad a \text{ and } b \text{ coprime integers.}\end{gathered}$$

We show $\mathcal{A}$, $\mathcal{B}$ do not obey an FUP. This amounts to showing that, for infinitely many k (in fact, for all k), there exists $f : \mathbb{Z}_N^2 \to \mathbb{C}$ with

$$\text{supp } f \subset \mathcal{X}_k, \quad \text{supp } \hat{f} \subset \mathcal{Y}_k,$$

where $\mathcal{X}_k$, $\mathcal{Y}_k$ are defined in (3).

Case 1: $a = 0$ or $b = 0$. Assume $(a, b) = (0, 1)$. Then $\boldsymbol{X}$ contains a vertical line and $\boldsymbol{Y}$ contains a horizontal line. It follows that $\mathcal{A}$ contains some vertical line $\{x = x_0\}$, $x_0 \in \mathbb{Z}_M$, and $\mathcal{B}$ contains a horizontal line $\{y = y_0\}$. Let

$$x^{(k)} = x_0 + Mx_0 + \cdots + M^{k-1}x_0, \quad \{(x^{(k)}, y) : y \in \mathbb{Z}_N\} \subset \mathcal{X}_k,$$
$$y^{(k)} = y_0 + My_0 + \cdots + M^{k-1}y_0, \quad \{(x, y^{(k)}) : y \in \mathbb{Z}_N\} \subset \mathcal{Y}_k.$$

We have

$$\mathcal{F}N^{-\frac{1}{2}}1_{x=0}(\xi, \eta) = N^{-\frac{3}{2}} \sum_{y \in \mathbb{Z}_N} e^{-\frac{2\pi i}{N} y\eta} = N^{-\frac{1}{2}}1_{\eta=0},$$

so

$$f = N^{-\frac{1}{2}} e^{2\pi i y^{(k)}} 1_{x=x^{(k)}}$$

satisfies

$$\operatorname{supp} f = \{x = x^{(k)}\} \subset \mathcal{X}_k, \quad \operatorname{supp} \hat{f} = \{y = y^{(k)}\} \subset \mathcal{Y}_k$$

as needed.

Case 2: $a, b \neq 0$. In this case we claim

$$\begin{aligned} \mathcal{X}_k &= \left\{(x, y) \in \mathbb{Z}_N^2 : \left(\frac{x}{M}, \frac{x+1}{M}\right) \times \left(\frac{y}{M}, \frac{y+1}{M}\right) \cap \boldsymbol{X} \neq \varnothing\right\}, \\ \mathcal{Y}_k &= \left\{(x, y) \in \mathbb{Z}_N^2 : \left(\frac{x}{M}, \frac{x+1}{M}\right) \times \left(\frac{y}{M}, \frac{y+1}{M}\right) \cap \boldsymbol{Y} \neq \varnothing\right\}. \end{aligned} \tag{24}$$

It is clear that if $(x/M, (x+1)/M) \times (y/M, (y+1)/M) \cap \boldsymbol{X} \neq \varnothing$ then $(x, y) \in \mathcal{X}_k$. For the other direction, we first note that $(0, 1)^2 \cap \boldsymbol{X} \neq \varnothing$ — the only way for this to fail is if $\mathcal{A}$ lies on one of the horizontal or vertical lines $x = 0$, $x = M - 1$, $y = 0$, $y = M - 1$ in which case we are back in Case 1. Now if $(x, y) \in \mathcal{X}_k$, then $(x, x + 1) \times (y, y + 1) \cap \boldsymbol{X} \neq \varnothing$ by the self-similarity of $\boldsymbol{X}$.

Now, assume without loss of generality that a, b are coprime. We will show that, for all k, there exist $\boldsymbol{p}^{(k)}, \boldsymbol{q}^{(k)} \in \mathbb{Z}_{M^k}^2$ so that

$$\mathbb{Z}\boldsymbol{v} + \boldsymbol{p}^{(k)} \subset \mathcal{X}_k \quad \text{and} \quad \mathbb{Z}\boldsymbol{v}^{\perp} + \boldsymbol{q}^{(k)} \subset \mathcal{Y}_k. \tag{25}$$

We show it just for $\mathcal{X}_k$. By (24), we would like to choose $\boldsymbol{p}^{(k)} = (p_1^{(k)}, p_2^{(k)})$ so that, for all $t \in \mathbb{Z}$,

$$p_1^{(k)} + ta < M^k p_1 + (t + \varepsilon)a < p_1^{(k)} + ta + 1,$$
$$p_2^{(k)} + tb < M^k p_2 + (t + \varepsilon)b < p_2^{(k)} + tb + 1$$

for some small ε. Rearranging, this amounts to

$$\begin{aligned} 0 &< (M^k p_1 - p_1^{(k)}) + \varepsilon a < 1, \\ 0 &< (M^k p_2 - p_2^{(k)}) + \varepsilon b < 1. \end{aligned} \tag{26}$$

To make this true, we select $p_1^{(k)}, p_2^{(k)}$ to be integers so that

$$M^k p_1 - p_1^{(k)} \in \begin{cases} [0, 1) & \text{if } a > 0, \\ (0, 1] & \text{if } a < 0, \end{cases} \qquad M^k p_2 - p_2^{(k)} \in \begin{cases} [0, 1) & \text{if } b > 0, \\ (0, 1] & \text{if } b < 0. \end{cases}$$

In each of these cases (26) will hold, which yields (25). Now, we have

$$\mathcal{F}1_{\mathbb{Z}\boldsymbol{v}}(\boldsymbol{\xi}) = \frac{1}{\sqrt{N}} \sum_{t\in\mathbb{Z}_N} e^{-\frac{2\pi i}{N} t\boldsymbol{v}\cdot\boldsymbol{\xi}} = 1_{\boldsymbol{v}\cdot\boldsymbol{\xi}=0} = 1_{\mathbb{Z}\boldsymbol{v}^\perp}$$

by Lemma 12. Thus with $T_{\boldsymbol{a}} f = f(\cdot - \boldsymbol{a})$ the translation operator, we see that

$$f = \mathcal{F}^{-1} T_{\boldsymbol{q}^{(k)}} \mathcal{F} T_{\boldsymbol{p}^{(k)}} 1_{\mathbb{Z}\boldsymbol{v}}$$

satisfies

$$\operatorname{supp} f \subset \mathbb{Z}\boldsymbol{v} + \boldsymbol{p}^{(k)} \subset \mathcal{X}_k, \quad \operatorname{supp} \hat{f} \subset \mathbb{Z}\boldsymbol{v}^\perp + \boldsymbol{q}^{(k)} \subset \mathcal{Y}_k,$$

contradicting a fractal uncertainty principle.

5.3. *Proof of Proposition 3.* Let $\mathcal{A} \subset \mathbb{Z}_M^2$ be an alphabet and $\boldsymbol{X} \subset \mathbb{T}^2$ the Cantor set it generates. Let $\boldsymbol{A} \subset \mathbb{T}^2$ be the drawing of $\mathcal{A}$.

First we show that a line $\mathbb{R}\boldsymbol{v} + \boldsymbol{p}$ lies on $\boldsymbol{X}$ if and only if $\mathbb{R}\boldsymbol{v} + M^k\boldsymbol{p}$ lies on $\boldsymbol{A}$ for all $k \geq 0$. Recall that

$$\boldsymbol{x} \in \boldsymbol{X} \quad \text{if and only if} \quad M^k\boldsymbol{x} \in \boldsymbol{A} \text{ for all } k \geq 0.$$

Now, suppose $\mathbb{R}\boldsymbol{v} + \boldsymbol{p} \subset \boldsymbol{X}$. Then

$$t\boldsymbol{v} + \boldsymbol{p} \in \boldsymbol{X} \quad \Longrightarrow \quad (M^k t)\boldsymbol{v} + M^k\boldsymbol{p} \in \boldsymbol{X},$$

so rescaling, $\mathbb{R}\boldsymbol{v} + M^k\boldsymbol{p} \subset \boldsymbol{X} \subset \boldsymbol{A}$. In the reverse direction, suppose $\mathbb{R}\boldsymbol{v} + M^k\boldsymbol{p} \subset \boldsymbol{A}$ for all k. Then

$$M^k(t\boldsymbol{v} + \boldsymbol{p}) \in \boldsymbol{A} \text{ for all } k \quad \Longrightarrow \quad t\boldsymbol{v} + \boldsymbol{p} \in \boldsymbol{X}$$

as needed. Also, by Lemma 13, if $\boldsymbol{v} = (a, b)$ and $\max(|a|, |b|) > M$ then $\mathbb{R}\boldsymbol{v} + \boldsymbol{p} \not\subset \boldsymbol{A}$ for any $\boldsymbol{p}$.

More discussion on a procedure for checking lines. Suppose $\ell = \mathbb{R}\boldsymbol{v} + \boldsymbol{p}$. If $\boldsymbol{v}$ is a multiple of $(1, 0)$ then ℓ is a horizontal line, and $\boldsymbol{X}$ can only contain a horizontal line if $\mathcal{A}$ does.

Otherwise, let $\boldsymbol{v} = (a, b)$ with a, b coprime integers, $b \neq 0$. Assume a, b are fixed and $\max(|a|, |b|) \leq M$. There is some $\boldsymbol{p}' = (p_0, 0) \in \ell$, so $\ell = \mathbb{R}\boldsymbol{v} + (p_0, 0)$. We will turn the question around and consider the closed set

$$\boldsymbol{S_v} = \{s \in \mathbb{T} : \mathbb{R}\boldsymbol{v} + (s, 0) \subset \boldsymbol{A}\}.$$

The only possible boundary points are those for which $\mathbb{R}\boldsymbol{v} + (s, 0)$ intersects a point of the form $(j/M, k/M) \in \mathbb{T}^2$. If $t\boldsymbol{v} + (s, 0) = (j/M, k/M)$ then we can compute s as

$$s = \frac{jb - ka + Mr}{Mb} \quad \text{for some } r \in \mathbb{Z}, \qquad \text{so } s = \frac{c}{Mb} \quad \text{for some } c \in \mathbb{Z}.$$

Now we can write $\boldsymbol{S_v}$ as a union of intervals,

$$S_{\boldsymbol{v}} = \left\{ s_j \in \{0, \dots, Mb - 1\} : \left[\frac{s_j}{Mb}, \frac{s_j + 1}{Mb}\right] \subset \boldsymbol{S_v} \right\},$$

$$\boldsymbol{S_v} = \bigcup_{s_j \in S_{\boldsymbol{v}}} \left[\frac{s_j}{Mb}, \frac{s_j + 1}{Mb}\right].$$

Given the alphabet $\boldsymbol{A}$ and $\boldsymbol{v} = (a, b)$, one can efficiently compute the finite set $S_{\boldsymbol{v}} \subset \mathbb{Z}_{Mb}$. It is then a combinatorial question whether or not there exists $x \in \mathbb{T}$ so that $M^k x \in \boldsymbol{S_v}$ for all k. It would be interesting to find an algorithm to answer this question.

5.4. ***FUP implies spectral gap for bakers maps.*** We would like to show that the results in [Dyatlov and Jin 2017, §2] hold for two-dimensional bakers maps, in particular Proposition 6 ([loc. cit., Proposition 2.6]). We prove here that [loc. cit., Proposition 2.3] holds in two dimensions. The deduction of Proposition 2.4 from Proposition 2.3 is the same in two dimensions, and the proofs of Proposition 2.5 and 2.6 go through verbatim.

In what follows we use the ℓ^∞ distance on $\mathbb{T}^2$ as in (12). Let Φ be the expanding map

$$\Phi = \Phi_{M,\mathcal{A}} : \bigsqcup_{\boldsymbol{a}\in\mathcal{A}} \left(\frac{a_1}{M}, \frac{a_1+1}{M}\right) \times \left(\frac{a_2}{M}, \frac{a_2+1}{M}\right) \to (0, 1)^2,$$

$$\Phi(x, y) = (Mx - a_1, My - a_2), \quad (x, y) \in \left(\frac{a_1}{M}, \frac{a_1+1}{M}\right) \times \left(\frac{a_2}{M}, \frac{a_2+1}{M}\right).$$

For each $\varphi : \mathbb{T}^2 \to \mathbb{R}$ define

$$\varphi_N \in \ell^2(\mathbb{Z}_N^2), \quad \varphi_N(\boldsymbol{j}) = \varphi(\boldsymbol{j}/N).$$

The function φ_N defines a multiplication operator as well as a Fourier multiplier $\varphi_N^{\mathcal{F}} = \mathcal{F}_N^* \varphi_N \mathcal{F}_N$.

Proposition 23 (propagation of singularities). *Assume that* $\varphi, \psi : \mathbb{T}^2 \to [0, 1]$ *and*, *for some* $c > 0$, $0 \le \rho < 1$,

$$d(\Phi(\operatorname{supp}\psi \cap \Phi^{-1}(\operatorname{supp}\chi)), \operatorname{supp}\varphi) \ge cN^{-\rho}. \tag{27}$$

Then

$$\|\varphi_N B_N \psi_N\|_{2\to 2} = \mathcal{O}(N^{-\infty}), \tag{28}$$

$$\|\varphi_N^{\mathcal{F}} B_N \psi_N^{\mathcal{F}}\|_{2\to 2} = \mathcal{O}(N^{-\infty}), \tag{29}$$

where $\mathcal{O}(N^{-\infty})$ *means decay faster than any polynomial*, *with constants depending only on* c, ρ, χ. *In particular*, *these hold when*

$$d(\operatorname{supp}\psi, \Phi^{-1}(\operatorname{supp}\varphi)) \ge cN^{-\rho}. \tag{30}$$

The proof is almost identical to that in [Dyatlov and Jin 2017].

Proof. We have

$$\varphi_N B_N \psi_N u(\boldsymbol{j}) = \sum_{\boldsymbol{a}\in\mathcal{A}} \sum_{\substack{0\le k_1,k_2\le N/M-1 \\ \boldsymbol{k}=(k_1,k_2)}} A^{\boldsymbol{a}}_{\boldsymbol{jk}} u\left(\boldsymbol{k} + \boldsymbol{a}\frac{N}{M}\right),$$

where

$$A^{\boldsymbol{a}}_{\boldsymbol{jk}} = \frac{M}{N^2} \varphi\left(\frac{\boldsymbol{j}}{N}\right) \exp\left(\frac{2\pi i}{M} \boldsymbol{a}\cdot\boldsymbol{j}\right) \chi\left(\boldsymbol{k}\frac{M}{N}\right) \psi\left(\frac{\boldsymbol{k}}{N} + \frac{\boldsymbol{a}}{M}\right) \tilde{A}_{\boldsymbol{jk}},$$

and

$$\tilde{A}_{\boldsymbol{jk}} = \sum_{\substack{0\le m_1,m_2\le N/M-1 \\ \boldsymbol{m}=(m_1,m_2)}} \exp\left(\frac{2\pi i}{N} \boldsymbol{m}\cdot(\boldsymbol{j} - \boldsymbol{k}M)\right) \chi\left(\boldsymbol{m}\frac{M}{N}\right).$$

We can write

$$\tilde{A}_{\boldsymbol{jk}} = \sum_{\boldsymbol{m}\in\mathbb{Z}_N^2} \exp\left(\frac{2\pi i}{N}\boldsymbol{b}\cdot\boldsymbol{m}\right)\chi_1\left(\frac{\boldsymbol{m}}{N}\right), \quad \boldsymbol{b}L = \boldsymbol{j} - \boldsymbol{k}M, \quad \chi_1(\boldsymbol{x}) = \chi(M\boldsymbol{x}).$$

We have $A^{\boldsymbol{a}}_{\boldsymbol{jk}} = 0$ unless

$$\frac{\boldsymbol{j}}{N} \in \operatorname{supp}\varphi, \quad \frac{\boldsymbol{k}}{N} + \frac{\boldsymbol{a}}{M} \in \operatorname{supp}\psi, \quad \boldsymbol{k}\frac{M}{N} = \Phi\left(\frac{\boldsymbol{k}}{N} + \frac{\boldsymbol{a}}{M}\right) \in \operatorname{supp}\chi.$$

It follows that $d(\boldsymbol{b}/N, 0) \geq cN^{-\rho}$, so by the method of nonstationary phase [Dyatlov and Jin 2017, Lemma 2.2], we see $\max_{\boldsymbol{a},\boldsymbol{j},\boldsymbol{k}} |A^{\boldsymbol{a}}_{\boldsymbol{jk}}| = \mathcal{O}(N^{-\infty})$ and (28) follows. Equation (29) is a consequence, as

$$\psi_N^{\mathcal{F}} B_N \varphi_N^{\mathcal{F}} = \mathcal{F}_N^* \overline{(\varphi_N B_N \psi_N)}^* \mathcal{F}_N$$

and the Fourier transform is unitary. □

Appendix A: Sketch of the proof of Theorem 19

In this section we will try to illustrate the main ideas of Beukers and Smyth's proof of Theorem 19 as directly as possible. In what follows the degree of a polynomial is

$$\deg F = \max_{a_{kl}\neq 0}(k+l), \quad F(z, w) = \sum_{k,l} a_{kl} z^k w^l \in \mathbb{C}[z, w],$$

which is different from Section 4. We will use the notation $Z_N(F) \subset \mathbb{Z}_N^2$ as in (22).

A.1. ***Bezout's inequality.*** We first state Bezout's theorem.

Theorem 24. *Let $F, G \in \mathbb{C}[z, w]$ be coprime irreducible polynomials with degrees D, E which are not multiples of each other,*

$$F = \sum_{0\leq k+l\leq D} a_{kl} z^k w^l, \quad G = \sum_{0\leq k+l\leq E} b_{kl} z^k w^l.$$

Then

$$|\{(z, w) \in \mathbb{C}^2 : F(z, w) = G(z, w) = 0\}| \leq DE.$$

If intersections are taken in $\mathbb{CP}^2$ and counted with multiplicity, then this is an equality.

We denote by $\mathsf{V}(F), \mathsf{V}(G) \subset \mathbb{C}^2$ the zero sets of F and G. Then Bezout's inequality can be written

$$|\mathsf{V}(F) \cap \mathsf{V}(G)| \leq DE. \tag{31}$$

A.2. ***Setup for Theorem 19.*** To prove Theorem 19, it is more convenient to work with Laurent polynomials $F \in \mathbb{C}[z, w, z^{-1}, w^{-1}]$. Like polynomials in two variables, Laurent polynomials in two variables also enjoy unique factorization up to units and satisfy a version of Bezout's inequality. From this perspective, the factors $z^a - \zeta w^b$ can be written as $z^a w^{-b} - \zeta$, so we can just look for factors of the form $z^a w^b - \zeta$.

Beukers and Smyth make the following reduction. For $F = \sum_{kl} a_{kl} z^k w^l \in \mathbb{C}[z, w, z^{-1}, w^{-1}]$ a Laurent polynomial, let $\mathcal{L}(F)$ be the sublattice of $\mathbb{Z}^2$ generated by

$$\{(k, l) - (k', l') : a_{kl}, a_{k',l'} \neq 0\}.$$

Notice that if $F = z^a w^b - \zeta$, then $\mathcal{L}(F) = \mathbb{Z}(a, b)$ has rank 1. If $F = z^a - \zeta w^b$, then $\mathcal{L}(F) = \mathbb{Z}(a, -b)$. More generally, if $\mathcal{L}(F)$ has rank 1 then F can be written as a function of $z^a w^b$ and one can reduce to the one variable case. If $\mathcal{L}(F)$ has rank 2 but is not all of $\mathbb{Z}^2$, one can change variables within the class of Laurent polynomials to reduce to the case where $\mathcal{L}(F) = \mathbb{Z}^2$. Rather than fully explain this, we will just prove Theorem 19 in the case where F is a genuine polynomial and $\mathcal{L}(F) = \mathbb{Z}^2$.

Here is part of Lemma 1 from [Beukers and Smyth 2002].

Lemma 25. *If ζ is a root of unity, then it is Galois conjugate to exactly one of $-\zeta, \zeta^2, -\zeta^2$.*

Now we partially prove a lemma covering the relevant portions of [Beukers and Smyth 2002, §3]. We follow them directly.

Lemma 26. *Let $F \in \mathbb{C}[z, w]$ be an irreducible polynomial with $\mathcal{L}(F) = \mathbb{Z}^2$. Then there are seven other polynomials $F_1, \dots, F_7$ none of which have F as a component, and such that if (z, w) is a cyclotomic point ($z^N = w^N = 1$ for some N) with $F(z, w) = 0$, then $F_j(z, w) = 0$ for some $1 \leq j \leq 7$. We may take*

$$\deg F_1 = \deg F_2 = \deg F_3 = \deg F,$$
$$\deg F_4 = \deg F_5 = \deg F_6 = \deg F_7 = 2 \deg F.$$

It follows directly from Bezout's inequality (31) that

$$\mathsf{Z}_N(F) \subset \bigcup_{j=1}^{7} \mathsf{Z}_N(F) \cap \mathsf{Z}_N(F_j) \quad \text{for all } N,$$
$$|\mathsf{Z}_N(F)| \leq 3D^2 + 8D^2 = 11D^2 \qquad \text{for all } N.$$

Remark 27. In Theorem 19 we state the bound $22D^2$ rather than $11D^2$ because we allow terms of the form $z^D w^D$, which has degree $2D$. The bound is $22D^2$ rather than $11(2D)^2 = 44D^2$ because the *Newton polytope* of F has volume $\leq D^2$, so [Beukers and Smyth 2002, Theorem 4.1] gives the sharper bound of $22D^2$.

A.3. ***Proof sketch of some special cases of Lemma 26.*** In the proof we split into cases depending on whether or not F can be defined over an abelian extension of $\mathbb{Q}$. The hardest case is when F is defined in some nontrivial abelian extension of $\mathbb{Q}$ — there are a few subcases involved. We prove Lemma 26 in the two easier cases where F has coefficients in $\mathbb{Q}$, and where F is not defined over any abelian extension.

First, multiply F by a constant so one of its coefficients is rational.

Case 1: $F \in \mathbb{Q}[z, w]$. We take

$$F_1 = F(-z, w), \quad F_2 = F(z, -w), \quad F_3 = F(-z, -w),$$
$$F_4 = F(z^2, w^2), \quad F_5(-z^2, w^2), \quad F_6(z^2, -w^2), \quad F_7(-z^2, -w^2).$$

We must show that if $F(z, w) = 0$ is a cyclotomic point, $F_j(z, w) = 0$ for some j. Let ζ be a root of unity and $z = \zeta^a,\ w = \zeta^b,\ a, b$ coprime. Then $f(\zeta) = F(\zeta^a, \zeta^b)$ is a polynomial in ζ with rational coefficients.

Thus every conjugate of ζ is also a root of f. By Lemma 25 exactly one of $\{-\zeta, \zeta^2, -\zeta^2\}$ is conjugate to ζ, so one of

$$(-\zeta^a, \zeta^b), \quad (\zeta^a, -\zeta^b), \quad (-\zeta^a, -\zeta^b), \quad (\zeta^{2a}, \zeta^{2b}), \quad (-\zeta^{2a}, \zeta^{2b}), \quad (\zeta^{2a}, -\zeta^{2b}), \quad (-\zeta^{2a}, -\zeta^{2b})$$

is also a zero of F as needed. It remains to show that F is not a component of any F_j. Because they arise from a linear change of variables of F, we know F_1, F_2, F_3 are irreducible. If F_1 is a linear multiple of F, then all nonzero a_{kl} must have the same parity for k. Thus $\mathcal{L}(F)$ would span a proper sublattice of $\mathbb{Z}^2$ contradicting our assumption. Similar arguments show that F_2 and F_3 are not linear multiples of F, and because they have the same degree, F is not a component. If F were a component of F_4 then $F(z^2, w^2) = F(z, w)G(z, w)$, so $F(z^2, w^2) = F_1(z, w)G(-z, w)$, and F_1 is a component of F_4 as well. An analogous argument shows F_2, F_3 are components as well. This would imply that $\deg F_4 \geq \deg F F_1 F_2 F_3 \geq 4D$ using the fact that F_1, F_2, F_3, F_4 are all distinct irreducibles. But $\deg F_4 = 2D$, a contradiction. A similar argument shows F is not a factor of F_5, F_6, F_7.

Case 2: The coefficients of F do not lie in any abelian extension of $\mathbb{Q}$. This case is easier. Let $\sigma \in \mathrm{Gal}(\mathbb{C}/\mathbb{Q}^{ab})$ be an automorphism of $\mathbb{C}$ which fixes $\mathbb{Q}^{ab}$ and does not fix the coefficients of F. Here $\mathbb{Q}^{ab}$ is the maximal abelian extension of $\mathbb{Q}$, which is the composite of all the cyclotomic extensions $\mathbb{Q}[e^{2\pi i/N}]$. Let

$$F^\sigma = \sum_{kl} \sigma(a_{kl}) z^k w^l.$$

For z, w a cyclotomic root of F, we have $\sigma(z) = z$ and $\sigma(w) = w$, so

$$F^\sigma(z, w) = \sigma(F(z, w)) = 0.$$

Thus the cyclotomic points of F are contained in $V(F) \cap V(F^\sigma)$. But F^σ is not a multiple of F, because some coefficient of F (the rational one) is fixed by σ and another must be different. Thus $V(F) \cap V(F^\sigma) \leq D^2$.

Appendix B: Higher dimensions and continuous FUP

B.1. *Results from a new method.* It seems difficult to use the ideas in the present paper to prove a discrete FUP in $d \geq 3$ dimensions. We would need a higher-dimensional analogue of Theorem 19 with very strong bounds that are currently unavailable.

However, after this work was completed the author [Cohen 2023] proved a fractal uncertainty principle for sets $\boldsymbol{X} \subset \mathbb{R}^d$ that avoid lines in a quantitative sense called *line porosity*. The core of the latter paper involves constructing plurisubharmonic functions, and the methods are completely different from those used here — there is no arithmetic input. Using the new work we can prove the following higher-dimensional result for discrete Cantor sets.

Theorem 28. *Suppose $\mathcal{A}, \mathcal{B} \subsetneq \mathbb{Z}_M^d$ are alphabets with drawings $\boldsymbol{X}, \boldsymbol{Y} \subset \mathbb{T}^d$. If $\boldsymbol{Y}$ does not contain any lines, then $\mathcal{X}_k, \mathcal{Y}_k$ satisfy*

$$\|1_{\mathcal{Y}_k} \mathcal{F} 1_{\mathcal{X}_k}\|_{2\to 2} \lesssim M^{-k\beta}$$

for some $\beta > 0$.

The more recent work has a few advantages. We don't need self-similarity, the result applies in any dimension, and most importantly, we move from the model setting $\mathbb{Z}_N^d$ to the physically relevant domain $\mathbb{R}^d$.

On the other hand, Theorem 2 gives a precise condition involving pairs of orthogonal lines which is currently unavailable in the continuous setting: Theorem 28 needs one of the Cantor sets to avoid all lines. It is an interesting challenge to improve the main result of [Cohen 2023] so the condition involves pairs of orthogonal subspaces.

B.2. ***Statement of higher-dimensional continuous FUP.*** For $\boldsymbol{x} \in \mathbb{R}^d$, let $B_R(\boldsymbol{x})$ be the radius-R ball about $\boldsymbol{x}$.

Definition 29. Let $\nu \le \frac{1}{3}$.

- A set $\boldsymbol{X} \subset \mathbb{R}^d$ is *ν-porous on balls* from scales α_0 to α_1 if for every ball B of diameter $\alpha_0 < R < \alpha_1$ there is some $\boldsymbol{x} \in B$ such that $B_{\nu R}(\boldsymbol{x}) \cap \boldsymbol{X} = \varnothing$.
- A set $\boldsymbol{X}$ is *ν-porous on lines from scales α_0 to α_1* if for all line segments τ with length $\alpha_0 < R < \alpha_1$ there is some $\boldsymbol{x} \in \tau$ such that $B_{\nu R}(\boldsymbol{x}) \cap \boldsymbol{X} = \varnothing$.

We are ready to state the main theorem of [Cohen 2023].

Theorem 30. *Let $\nu > 0$ and assume that*

- *$\boldsymbol{X} \subset [-1, 1]^d$ is ν-porous on balls from scales h to 1, and*
- *$\boldsymbol{Y} \subset [-h^{-1}, h^{-1}]^d$ is ν-porous on lines from scales 1 to h^{-1}.*

Then there exist $\beta, C > 0$ depending only on ν and d such that for all $f \in L^2(\mathbb{R}^d)$

$$\operatorname{supp} \hat{f} \subset \boldsymbol{Y} \quad \Longrightarrow \quad \|f 1_{\boldsymbol{X}}\|_2 \le C h^{\beta} \|f\|_2. \tag{32}$$

To prove Theorem 28 we first show that the drawing of a Cantor set avoiding lines is porous on lines, and then prove a discrete FUP using continuous FUP.

B.3. ***Line porosity for self-similar Cantor sets.*** In this section $\boldsymbol{x} \in [0, 1]^d$ denotes a point in $\mathbb{R}^d$ and $\bar{\boldsymbol{x}} \in \mathbb{T}^d$ denotes the image in the torus. It is similar for sets $\boldsymbol{Y} \subset [0, 1]^d$ and $\overline{\boldsymbol{Y}} \subset \mathbb{T}^d$.

Definition 31. Let $\overline{\boldsymbol{X}} \subsetneq \mathbb{T}^d$ be a closed set. We say $\overline{\boldsymbol{X}}$ is a self-similar Cantor set at level M if $M \cdot \overline{\boldsymbol{X}} = \overline{\boldsymbol{X}}$, where

$$M \cdot \overline{\boldsymbol{X}} = \{M \bar{\boldsymbol{x}} : \bar{\boldsymbol{x}} \in \overline{\boldsymbol{X}}\}.$$

In particular, if $\mathcal{X}_k$ is a sequence of Cantor sets in $\mathbb{Z}_{M^k}$, then the drawing $\overline{\boldsymbol{X}} \subset \mathbb{T}^d$ is a self-similar Cantor set.

We first prove that if a Cantor set does not contain any lines, then it also does not contain any line segments. By a *line in* $\mathbb{T}^d$, we mean an irreducible one-dimensional closed coset. By a line segment $\bar{\tau} \subset \mathbb{T}^d$, we mean the image of a line segment in $\mathbb{R}^d$.

Lemma 32. *Let $\overline{\boldsymbol{Y}} \subset \mathbb{T}^d$ be a self-similar Cantor set which contains no lines. Then $\overline{\boldsymbol{Y}}$ also does not contain any line segments $\bar{\tau}$.*

Proof. Suppose by way of contradiction that $\bar{\tau} \subset \mathbb{T}^d$ is a line segment with $\bar{\tau} \subset \overline{\boldsymbol{Y}}$. Let $\bar{\tau}$ point in direction $\hat{\boldsymbol{v}} \in S^{d-1}$, and let

$$C_{\hat{\boldsymbol{v}}} = \mathrm{cl}\{t\bar{\hat{\boldsymbol{v}}} : t \in \mathbb{R}\} \subset \mathbb{T}^d$$

be the closure of the geodesic based at the origin and pointing in direction $\hat{\boldsymbol{v}}$. The set $C_{\hat{\boldsymbol{v}}}$ is a closed subgroup which contains at least one torus line. Choose $\boldsymbol{x}_0$ in the interior of $\bar{\tau}$. Select a subsequence $\{k_j\}_{j\geq 0}$ such that $M^{k_j}\bar{\boldsymbol{x}}_0 \to \bar{\boldsymbol{x}}_0' \in \mathbb{T}^d$. For any $t \in \mathbb{R}$,

$$M^{k_j}(\bar{\boldsymbol{x}}_0 + M^{-k_j}t\bar{\hat{\boldsymbol{v}}}) \to \bar{\boldsymbol{x}}_0' + t\bar{\hat{\boldsymbol{v}}} \in \bar{\boldsymbol{x}}_0' + C_{\hat{\boldsymbol{v}}}.$$

For large enough j, $M^{k_j}(\bar{\boldsymbol{x}}_0 + M^{-k_j}t\bar{\hat{\boldsymbol{v}}}) \in \overline{\boldsymbol{Y}}$, and because $\overline{\boldsymbol{Y}}$ is closed, we see $\bar{\boldsymbol{x}}_0' + C_{\hat{\boldsymbol{v}}} \subset \overline{\boldsymbol{Y}}$ contradicting our assumption. □

We prove if a Cantor set does not contain lines then it is porous on lines.

Lemma 33. *Suppose that $\overline{\boldsymbol{Y}} \subset \mathbb{T}^d$ is a self-similar Cantor set which does not contain any lines. Then for some $\nu > 0$, $\boldsymbol{Y} \subset [0, 1]^d$ is ν-porous on lines from scales* 0 *to* 1.

Proof. Let $\overline{\boldsymbol{Y}} \subset \mathbb{T}^d$ be a Cantor set which does not contain any lines. We show by a compactness argument that, for some $c_0 > 0$, every line segment $\bar{\tau}$ with length 1 has some $\bar{\boldsymbol{x}} \in \bar{\tau}$ such that $d(\bar{\boldsymbol{x}}, \overline{\boldsymbol{Y}}) \geq c_0$. Suppose by way of contradiction that this is not the case. Then there is a sequence $\bar{\tau}_j$ of unit line segments such that $\max_{\bar{\boldsymbol{x}}\in\bar{\tau}_j} d(\bar{\boldsymbol{x}}, \overline{\boldsymbol{Y}}) \leq c_j$, where $c_j \to 0$. The space of unit line segments in $\mathbb{T}^d$ is compact, so there is some line segment $\bar{\tau}$ which is a limit of these, and it follows that $\bar{\tau} \subset \overline{\boldsymbol{Y}}$ contradicting Lemma 32.

Now let $\tau \subset \mathbb{R}^d$ be a line segment of length $0 < R < 1$. We would like to show there is some $\boldsymbol{x} \in \tau$ such that $d(\boldsymbol{x}, \boldsymbol{Y}) \geq \nu R$. The torus metric is stronger than the ambient $\mathbb{R}^d$ metric, so it suffices to show that there is some $\bar{\boldsymbol{x}} \in \bar{\boldsymbol{\tau}}$ such that $d(\bar{\boldsymbol{x}}, \bar{\boldsymbol{\tau}}) \geq \nu R$. Let $j \geq 0$ be the smallest integer so that $M^j R \geq 1$. Because $M^j \cdot \bar{\tau}$ is a line segment with length ≥ 1, there is some $\bar{\boldsymbol{x}} \in \bar{\tau}$ such that $d(M^j\bar{\boldsymbol{x}}, \boldsymbol{Y}) \geq c_0$. So by self-similarity $d(\bar{\boldsymbol{x}}, \boldsymbol{Y}) \geq M^{-j}c_0 \geq (c_0/M)R$ and $\boldsymbol{Y}$ is ν-porous on lines from scales 0 to 1 with $\nu = c_0/M$. □

B.4. ***Proof of Theorem 28.*** We roughly follow the argument in [Dyatlov and Jin 2018, Proposition 5.8]. We state a general proposition which allows us to prove discrete fractal uncertainty from continuous fractal uncertainty. We will need the locally constant property from Fourier analysis, which we explain in a certain form now. Construct a $w \in C^\infty(\mathbb{R}^d)$ by setting $\widehat{w}$ to be a smooth bump function with $\widehat{w} = 1$ on B_1 and $\operatorname{supp} \widehat{w} \subset B_2$. Then

$$|w(\boldsymbol{x})| \lesssim_{m,d} \langle \boldsymbol{x} \rangle^{-m} \quad \text{for all } m \geq 0.$$

Moreover, if $f \in L^2(\mathbb{R}^d)$ is a function with $\operatorname{supp} \hat{f} \subset B_N$ then

$$f = f * w_N, \quad w_N(\boldsymbol{x}) = N^d w(N\boldsymbol{x}).$$

In particular we have the pointwise bound

$$|f(\boldsymbol{x})| \lesssim N^{d/2} \|f(\cdot)w(N(\cdot - \boldsymbol{x}))\|_2. \tag{33}$$

Let $\mathcal{X}, \mathcal{Y} \subset \mathbb{Z}_N^d$ be sets. Let

$$X = N^{-1} \cdot \{\boldsymbol{x} \in \{0, \dots, N-1\}^d : \bar{\boldsymbol{x}} \in \mathcal{X}\},\\ Y = \{\boldsymbol{y} \in \{0, \dots, N-1\}^d : \bar{\boldsymbol{y}} \in \mathcal{Y}\}. \tag{34}$$

Here is the main proposition connecting discrete and continuous FUP.

Proposition 34. *Let $\mathcal{X}, \mathcal{Y} \subset \mathbb{Z}_N^d$ and $X, Y \subset \mathbb{R}^d$ be as above. For any $\frac{10}{N} < r < \frac{1}{10}$ and $m > 0$ we have*

$$\|1_{\mathcal{X}}\, \mathcal{F}\, 1_{\mathcal{Y}}\|_{2\to 2} \lesssim_{d,m} \|1_{X+B_r}\, \mathcal{F}\, 1_{Y+B_{1/4}}\|_{2\to 2} + (Nr)^{-m}. \tag{35}$$

Proof. Let $u \in L^2(\mathbb{Z}_N^d)$ have $\operatorname{supp} \hat{u} \subset \mathcal{Y}_k$. We will construct an auxiliary function $f \in L^2(\mathbb{R}^d)$ based on u.

Let $\chi \in C_0^\infty(\mathbb{R}^d)$ be a bump function supported in $B_{1/4}$. We can design χ so that

$$|\chi^\vee(\boldsymbol{x})| \geq 1 \quad \text{for } \boldsymbol{x} \in [-10, 10]^d, \tag{36}$$

$$\|\chi\|_2 \leq C_d. \tag{37}$$

Let f be given by

$$\hat{f}(\xi) = \sum_{\boldsymbol{\xi}' \in \{0,\dots,N-1\}^d} \hat{u}(\xi' \chi(\xi - \xi').$$

We have

$$\|f\|_2^2 = \|u\|_2^2 \|\chi\|_2^2 \lesssim \|u\|_2^2.$$

Notice that, for $\boldsymbol{x} \in X$,

$$f(\boldsymbol{x}) = N^{d/2}\, \chi^\vee(\boldsymbol{x})\, u(\overline{N\boldsymbol{x}}),$$

so

$$\|u\, 1_{\mathcal{X}_k}\|_{L^2(\mathbb{Z}_N^d)}^2 \lesssim N^{-d} \sum_{\boldsymbol{x} \in X} |f(\boldsymbol{x})|^2.$$

If we let

$$\tilde{w}(\boldsymbol{x}) = \left(\sum_{\boldsymbol{x}' \in X} |w(N(\boldsymbol{x} - \boldsymbol{x}'))|^2 \right)^{\frac{1}{2}},$$

by (33), $|f(\boldsymbol{x})|^2 \lesssim N^d \|w(N(x - x')) f\|_2^2$, so summing over $\boldsymbol{x} \in X$ we find

$$\sum_{\boldsymbol{x} \in X} |f(\boldsymbol{x})|^2 \lesssim N^d \|f\, \tilde{w}\|_2^2,$$

$$\|u 1_{\mathcal{X}_k}\|_2^2 \lesssim \|f \tilde{w}\|_2^2.$$

Using the fact that X is an N^{-1}-separated set,

$$|\tilde{w}(\boldsymbol{x})|^2 = \sum_{\boldsymbol{x}' \in X} |w(N(\boldsymbol{x} - \boldsymbol{x}'))|^2 \lesssim_m \sum_{\boldsymbol{x}' \in X} (1 + N|\boldsymbol{x} - \boldsymbol{x}'|)^{-m}$$
$$\lesssim \sum_{N^{-1} \leq 2^j \leq 10} (1 + N\, 2^j)^{-m}\, |X \cap B_{2^j}(\boldsymbol{x})| \lesssim \sum_{2^j \geq \max(N^{-1}, d(\boldsymbol{x}, X))} (N2^j)^{d-m} \lesssim (1 + N\, d(\boldsymbol{x}, X))^{d-m}$$

for m large enough. Thus for any $r > N^{-1}$ and $m \geq 0$,

$$|\tilde{w}(\boldsymbol{x})| \lesssim_{m,d} 1_{X+B_r}(\boldsymbol{x}) + (Nr)^{-m}.$$

Because $\operatorname{supp}\hat{f} \subset Y + B_{1/4}$,

$$\begin{aligned}\|u 1_{\mathcal{X}_k}\|_2 &\lesssim \|f\, 1_{X+B_r}\|_2 + (Nr)^{-m}\|f\|_2 \\ &\lesssim (\|1_{X+B_r}\,\mathcal{F}\,1_{Y+B_{1/4}}\|_{2\to 2} + (Nr)^{-m})\|u\|_2\end{aligned}$$

giving (35). □

Now we prove the FUP for arithmetic Cantor sets that avoid lines.

Proof of Theorem 28. Let $\mathcal{X}_k$ and $\mathcal{Y}_k$ be a sequence of Cantor iterates such that the drawing $\overline{\boldsymbol{Y}} \subset \mathbb{T}^d$ does not contain any lines. Let $N = M^k$. Let $X_k \subset [0, 1]^d$ and $Y_k \subset [0, N]^d$ be the corresponding point sets as in (34). By choosing $r = N^{\varepsilon-1}$ in Proposition 34, we have for any $\varepsilon > 0$ the estimate

$$\|1_{\mathcal{X}_k}\,\mathcal{F}\,1_{\mathcal{Y}_k}\|_{2\to 2} \lesssim \|1_{X_k+B_r}\,\mathcal{F}\,1_{Y_k+B_{1/4}}\|_{2\to 2} + N^{-\varepsilon}.$$

Letting $\boldsymbol{X}, \boldsymbol{Y} \subset [0, 1]^d$ be the drawings of these Cantor sets, we have

$$X_k \subset \boldsymbol{X} + [-N^{-1}, N^{-1}]^d, \qquad Y_k \subset N\cdot\boldsymbol{Y} + [-1, 1]. \tag{38}$$

Thus

- The set $\boldsymbol{X}$ is ν-porous on balls from scales 0 to 1. So $X_k + B_{N^{\varepsilon-1}}$ is ν-porous on balls from scales $2N^{\varepsilon-1}$ to 1.
- By Lemma 33, the set $\boldsymbol{Y}$ is ν-porous on lines from scales 0 to 1. So the set $Y_k + B_{1/4}$ is ν-porous on lines from scales $\frac{1}{4}\sqrt{d}$ to N.

In the above, the value of ν changes from line to line. Split up $[-N-1, N+1]^d$ into a disjoint union of $\lesssim N^{\varepsilon d}$ many cubes $Q \in \mathcal{Q}$ that have side length $N^{1-\varepsilon}$. By Theorem 30, there is $\beta = \beta(\nu, d) > 0$ so that

$$\|1_{X_k+B_r}\,\mathcal{F}\,1_{(Y_k+B_{1/4})\cap Q}\|_{2\to 2} \lesssim N^{-(1-\varepsilon)\beta}.$$

Summing this over all the boxes $Q \in \mathcal{Q}$, we have

$$\|1_{X_k+B_r}\,\mathcal{F}\,1_{Y_k+B_{1/4}}\|_{2\to 2} \lesssim N^{-\beta+\varepsilon(d+\beta)}.$$

Choose $\varepsilon > 0$ small enough that the exponent is negative and apply Proposition 34 to obtain

$$\|1_{\mathcal{X}_k}\,\mathcal{F}\,1_{\mathcal{Y}_k}\|_{2\to 2} \le CN^{-\beta'}$$

for some $\beta' > 0$. □

Acknowledgements

Thanks to Larry Guth and Peter Sarnak for helpful discussions. Many thanks to Semyon Dyatlov for detailed and helpful comments and for useful conversations along the way.

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Received 21 Oct 2022. Revised 28 Sep 2023. Accepted 21 Nov 2023.

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ANALYSIS AND PDE
Vol. 18 (2025), No. 3, pp. 773–803
DOI: 10.2140/apde.2025.18.773

LINEAR POTENTIALS AND APPLICATIONS IN CONFORMAL GEOMETRY

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We derive estimates for linear potentials that hold away from thin subsets. And, inspired by the celebrated work of Huber (1957) and Cohn-Vossen (1935), we verify that, for a subset that is thin at a point, there is always a geodesic that reaches to the point and avoids the thin subset in general dimensions. As applications of these estimates on linear potentials, we consider the scalar curvature equations and improve the results of Schoen and Yau (1988, 1994) and Carron (2012) on the Hausdorff dimensions of singular sets which represent the ends of complete conformal metrics on domains in manifolds of dimension greater than 3. We also study Q-curvature equations in dimensions greater than 4 and obtain stronger results on the Hausdorff dimensions of the singular sets than those of Chang et al. (2004). More interestingly, our approach based on potential theory yields a significantly stronger finiteness theorem on the singular sets for Q-curvature equations in dimension 4 than those of Chang et al. (2000) and Carron and Herzlich (2002), which is a remarkable analogue of Huber's theorem.

1. Introduction

We employ linear potential theory to study scalar curvature equations and Q-curvature equations in conformal geometry. This is a continuation of our recent work on n-superharmonic functions (see [Bonini et al. 2018; 2019; Ma and Qing 2021; 2022]) inspired by Huber's theorem and related work on superharmonic functions in dimension 2 (see [Cohn-Vossen 1935; Huber 1957; Arsove and Huber 1973; Hayman and Kennedy 1976]).

Linear potential theory has always been a major subject in analysis and partial differential equations. We refer readers, for instance, to [Mizuta 1996; Adams and Hedberg 1996; Armitage and Gardiner 2001] for good introductions on potential theory. For clarity, the definitions of Riesz potentials and log potentials are given in Section 2. For our purpose, the kernel functions are not chosen for discussions on the boundary behavior of potentials and we focus on the outer capacity and thin subsets (please see Definitions 2.2 and 2.8 in Section 2). Also we set up some of the potential theory on Riemannian manifolds directly. The interesting result on Riesz potentials we obtain is:

Theorem 1.1. *Suppose that (M^n, g) is a complete Riemannian manifold and μ is a finite nonnegative Radon measure on a bounded domain $G \subset M^n$. Let S be a compact subset in G such that its Hausdorff*

Ma is supported by the Natural Science Foundation of Tianjin, no. 22JCJQJC00130, Fundamental Research Funds for the Central Universities and Natural Science Foundation of Fujian Province, China (2022J02050). Ma also thanks the support of CIM Visiting Scholars Program of Nankai University. Qing is supported by a Simons Foundation Collaboration Grant 2022–2027.
MSC2020: 53C21, 31B35, 31B05, 31B15.
Keywords: Riesz potentials, log potentials, outer capacities, α-thinness, scalar curvature equations, Q-curvature equations, Hausdorff dimensions.

dimension is greater than d, where $d < n-\alpha$ and $\alpha \in (1, n)$. Then there is a point $p \in S$ and a subset E that is α-thin at p such that

$$\int_G \frac{1}{d(x,y)^{n-\alpha}}\, d\mu \le \frac{C}{d(x,p)^{n-\alpha-d}} \tag{1-1}$$

for some constant C and all $x \in B_\delta(p) \setminus E$ for some small $\delta > 0$.

The proof of Theorem 1.1 uses a general decomposition result [Kpata 2019, Proposition 1.4] and multiscale analysis. We also give a proof of a slight extension of [Mizuta 1996, Theorem 6.3] for log potentials on manifolds, which is closely related to [Cohn-Vossen 1935; Huber 1957; Arsove and Huber 1973; Ma and Qing 2021; 2022] for us. What makes these estimates useful is the following key observation about thin subsets in general dimensions (see [Cohn-Vossen 1935; Huber 1957; Arsove and Huber 1973; Ma and Qing 2021; 2022]).

Theorem 1.2. *Let E be a subset in the Euclidean space $\mathbb{R}^n$ and $p \in \mathbb{R}^n$. Suppose that E is α-thin at the point p for $\alpha \in (1, n]$. Then there is always a ray from p that avoids E at least within some small ball at p.*

The proof of Theorem 1.2 uses only the scaling property (Lemma 2.4), the contractive property (Lemma 2.5), and the calculation of $C^\alpha(S^{n-1}, B_2(0))$ (Lemma 2.6) for the outer capacity $C^\alpha(E, \Omega)$ defined in Definition 2.2 and α-thinness in Definition 2.8.

To better motivate our geometric applications, let us first recall the seminal theorem of Huber on surfaces. Huber [1957] showed that a complete open surface whose negative part of the Gaussian curvature is integrable is a closed surface with finitely many points removed. Huber's theorem uses the Gaussian curvature equation

$$-\Delta[\bar{g}]\phi + K[\bar{g}] = K[e^{2\phi}\bar{g}]e^{2\phi} \tag{1-2}$$

and the potential theory on superharmonic functions.

In conformal geometry, the scalar curvature equation

$$-\frac{4(n-1)}{n-2}\Delta[\bar{g}]u + R[\bar{g}]u = R[u^{\frac{4}{n-2}}\bar{g}]u^{\frac{n+2}{n-2}} \tag{1-3}$$

describes the conformal transformation of the scalar curvature in dimensions higher than 2. There have been many works on singular solutions after the seminal paper [Schoen and Yau 1988], where the singularities represent the ends of complete conformal metrics on domains in Riemannian manifolds (see, for instance, [Schoen and Yau 1994, Chapter VI; Carron 2012; Schoen 1988; Mazzeo and Smale 1991; Mazzeo and Pacard 1996]).

Theorem 1.3. *Let $(M^n, \bar{g})$ be a complete Riemannian manifold and S be a compact subset in M^n. And let D be a bounded open neighborhood of S. Suppose that $g = u^{4/(n-2)}\bar{g}$ is a conformal metric on $D \setminus S$ and is geodesically complete near S. Then the Hausdorff dimension satisfies*

$$\dim_{\mathscr{H}}(S) \le \frac{n-2}{2} \tag{1-4}$$

provided $R^-[g] \in L^{2n/(n+2)}(D \setminus S, g) \cap L^p(D \setminus S, g)$ for some $p > \frac{n}{2}$, where $R^-[g]$ is the negative part of the scalar curvature of the metric g. Consequently, (1-4) *holds when the scalar curvature $R[g]$ of the conformal metric g is nonnegative.*

Theorem 1.3 is an improvement of [Schoen and Yau 1988, Theorem 2.7] and [Carron 2012, Theorem C]. Our approach is based on Theorems 1.1 and 1.2. Particularly, Theorem 1.3 covers domains in general manifolds, while others (see [Schoen and Yau 1988; Carron 2012]) are restricted to domains in round spheres. The use of auxiliary testing functions built from the level sets is the key analytic technique (see [Dolzmann et al. 1997; Bidaut-Véron 1989; Ma and Qing 2021; 2022]). We remark that, for our approach, the complement $M^n \setminus D$ is not relevant (see Theorem 1.3 in Section 3).

In conformal geometry, one considers the Paneitz operator

$$P_4 = \Delta^2 + \operatorname{div}(4A \cdot \nabla - (n-2)J\nabla) + \frac{n-4}{2} Q_4$$

and the associated Q-curvature

$$Q_4 = -\Delta J + \frac{n}{2} J^2 - 2|A|^2,$$

where $A = \frac{1}{n-2}(\mathrm{Ric} - Jg)$ is the Schouten curvature and $J = \frac{1}{2(n-1)} R$. The curvature Q_4, under a conformal change of the metric, transforms by the Q-curvature equation:

$$P_4[\bar{g}]u = \frac{n-4}{2} Q_4[u^{\frac{4}{n-4}} \bar{g}] u^{\frac{n+4}{n-4}} \quad \text{in dimensions} \geq 5, \tag{1-5}$$

$$P_4[\bar{g}]u + Q_4[\bar{g}] = Q_4[e^{2u} \bar{g}] e^{4u} \quad \text{in dimension 4.} \tag{1-6}$$

On Q-curvature equations in dimensions greater than 4, we have:

Theorem 1.4. *Let $(M^n, \bar{g})$ be a complete Riemannian manifold for $n \geq 5$ and S be a compact subset in M^n. And let D be a bounded open neighborhood of S. Suppose that $g = u^{4/(n-4)} \bar{g}$ is a conformal metric on $D \setminus S$ with nonnegative scalar curvature $R[g] \geq 0$ and is geodesically complete near S. And suppose also that*

$$Q_4^-[g] \in L^{\frac{2n}{n+4}}(D \setminus S, g),$$

where $Q_4^-[g]$ is the negative part of the Q-curvature of the metric g. Then

$$\dim_{\mathscr{H}}(S) \leq \frac{n-4}{2}. \tag{1-7}$$

There have been a lot of works on the study of singular solutions to Q-curvature equations on manifolds of dimension greater than 4, notably [Qing and Raske 2006a; 2006b; Chang et al. 2004; González et al. 2012]. Theorem 1.4 is an improvement of [Chang et al. 2004, Theorem 1.2] in terms of curvature conditions and the coverage of domains in general manifolds. The preliminary estimates in Lemma 4.1 serve to facilitate the argument of treating the bi-Laplace as the iteration of the Laplace, which is an interesting alternative to the usual elliptic estimates of Q-curvature equations. Again, the complement $M^n \setminus D$ is not relevant for our approach (see Theorem 1.4 in Section 4).

On Q-curvature equations in dimension 4, there have been several attempts to establish results analogous to Huber's theorem on finiteness of singularities (see [Chang et al. 2000a; Carron and Herzlich 2002; Ma and Qing 2021; 2022]). Q-curvature in dimension 4 indeed plays a role similar to that of the Gaussian curvature in dimension 2 (please see (1-6) for instance). Our following result is a significant improvement of the finiteness result of [Chang et al. 2000a, Theorem 2] (see also [Chang et al. 2000b]). It covers domains in general manifolds and drops other additional curvature assumptions in [Chang et al. 2000a, Theorem 2]. The potential theory approach here, particularly Theorems 1.1 and 1.2, seems to be more effective. And the preliminary estimates in Lemma 4.4 are interesting for Q-curvature equations in dimension 4 too. Once again, the complement $M^n \setminus D$ is not relevant for our approach (see Theorem 1.4 in Section 4).

Theorem 1.5. *Let $(M^4, \bar{g})$ be a complete Riemannian manifold and S be a compact subset in M^n. And let D be a bounded open neighborhood of S. Suppose that $g = e^{2u}\bar{g}$ is a conformal metric on $D \setminus S$ with nonnegative scalar curvature $R[g] \geq 0$ and is geodesically complete near S. And suppose that*

$$\int_D Q_4^-[g]\, d\,\mathrm{vol}[g] < \infty,$$

where $Q_4^-[g]$ is the negative part of the Q-curvature of the metric g. Then S consists of only finitely many points.

The organization of this paper is as follows: In Section 2 we define linear potentials and develop potential theory with the outer capacity and the notion of α-thinness. Then we prove Theorems 1.1 and 1.2. In Section 3 we build the framework to use potential theory developed in Section 2 to estimate the Hausdorff dimension of singular sets which correspond to the ends of complete conformal metrics on domains of manifolds. And we prove Theorem 1.3. In Section 4, based on the framework built in Section 3, we prepare some preliminary estimates and prove Theorems 1.4 and 1.5 for Q-curvature equations.

2. On linear potentials

The study of linear potentials has been extensive and full of great achievements. Readers are referred, for instance, to [Mizuta 1996; Adams and Hedberg 1996; Armitage and Gardiner 2001] for good introductions. In this section we will introduce the theory of linear potential to facilitate the discussion of some estimates of linear potentials inspired by the one in [Cohn-Vossen 1935; Huber 1957; Arsove and Huber 1973; Ma and Qing 2021; 2022]. The estimates provide us some alternative tools to study the problems on the Hausdorff dimensions of singularities of solutions to a class of geometric partial differential equations in conformal geometry (see [Schoen and Yau 1988; 1994; Chang et al. 2004; Carron and Herzlich 2002] for instance). We will introduce potential theory in a way that is brief, mostly self-contained, and suffices to serve our purpose.

2.1. ***Linear potential and the outer capacity in Euclidean spaces.*** For the purpose of relating potentials on Euclidean spaces to that on manifolds, we want to introduce potentials that are possibly confined to an open subset $\Omega \subseteq \mathbb{R}^n$ in the Euclidean space. We will use the definition of a Radon measure on locally compact Hausdorff spaces in [Royden and Fitzpatrick 2010, page 455].

Definition 2.1. Let $\Omega \subseteq \mathbb{R}^n$ be a bounded open subset in the Euclidean space $\mathbb{R}^n$. Then, for $x \in \Omega$, let

$$R_\mu^{\alpha,\Omega}(x) = \begin{cases} \displaystyle\int_\Omega \frac{1}{|x-y|^{n-\alpha}}\, d\mu(y) & \text{when } \alpha \in (1,n), \\ \displaystyle\int_\Omega \log \frac{D}{|x-y|}\, d\mu(y) & \text{when } \alpha = n \end{cases} \tag{2-1}$$

for a Radon measure μ on Ω, where D is the diameter of Ω.

For basic properties of the potential $R_\mu^{\alpha,\Omega}(x)$, readers are referred to [Mizuta 1996, Chapter 2]. Most facts, results, and arguments in that work that are relevant for the discussions in this paper hold with slight changes.

Definition 2.2. Let E be a subset in Ω and Ω be a bounded open subset in $\mathbb{R}^n$. For $\alpha \in (1,n]$, we define a capacity by

$$C^\alpha(E,\Omega) = \inf\{\mu(\Omega) : \mu \geq 0 \text{ on } \Omega \text{ and } R_\mu^{\alpha,\Omega}(x) \geq 1 \text{ for all } x \in E\}. \tag{2-2}$$

Because of the choice of the kernel functions in Definition 2.1, the capacity $C^\alpha(E,\Omega)$ in Definition 2.2 is not intended to be the same as relative capacity where the kernel function is the Green's function for a so-called Greenian domain Ω. Similar to [Mizuta 1996, Theorem 4.1 in Chapter 2; Section 2.6], we have:

Lemma 2.3. *Let C^α be the capacity defined as in Definition 2.2 for $\alpha \in (1,n]$.*

(1) *C^α is nondecreasing, that is,*

$$C^\alpha(E_1,\Omega) \leq C^\alpha(E_2,\Omega)$$

when $E_1 \subseteq E_2 \subseteq \Omega \subseteq \mathbb{R}^n$.

(2) *C^α is countably subadditive, that is,*

$$C^\alpha\Big(\bigcup_{i=1}^\infty E_i,\Omega\Big) \leq \sum_{i=1}^\infty C^\alpha(E_i,\Omega)$$

for subsets $E_i \subseteq \Omega$.

(3) *C^α is an outer capacity, that is,*

$$C^\alpha(E,\Omega) = \inf\{C^\alpha(U,\Omega) : E \subseteq U \text{ and } U \subseteq \Omega \text{ open}\}.$$

The immediate and important property of the outer capacity C^α in Definition 2.2 is the scaling property (see [Armitage and Gardiner 2001, page 135]).

Lemma 2.4. *For a positive number λ, let*

$$A_\lambda = \{\lambda x : x \in A\}$$

for any subset A in $\mathbb{R}^n$. Then, for $\alpha \in (1,n]$,

$$C^\alpha(E_\lambda,\Omega_\lambda) = \lambda^{n-\alpha} C^\alpha(E,\Omega).$$

Proof. For a nonnegative Radon measure μ on Ω, we associate it with a nonnegative Radon measure

$$\mu^*(A_\lambda) = \mu(A)$$

on Ω_λ. Then

$$R^{\alpha,\Omega_\lambda}_{\mu^*}(\lambda x) = \lambda^{\alpha-n} R^{\alpha,\Omega}_{\mu}(x)$$

for $x \in \Omega$. Therefore

$$\begin{aligned} C^\alpha(E_\lambda, \Omega_\lambda) &= \inf\{\mu^*(\Omega_\lambda) : R^{\alpha,\Omega_\lambda}_{\mu^*}(\lambda x) \geq 1 \text{ for all } x \in E\} \\ &= \lambda^{n-\alpha} \inf\{\lambda^{\alpha-n}\mu(E) : R^{\alpha,\Omega}_{\lambda^{\alpha-n}\mu}(x) \geq 1 \text{ for all } x \in E\} \\ &= \lambda^{n-\alpha} C^\alpha(E, \Omega). \end{aligned}$$

□

The next important property of the outer capacity C^α in Definition 2.2 is the contractive property (see [Mizuta 1996; Adams and Hedberg 1996; Armitage and Gardiner 2001]).

Lemma 2.5. *Suppose that*

$$\Phi : \Omega \to \Omega$$

is a contractive map, that is,

$$|\Phi(x) - \Phi(y)| \leq |x - y|$$

for all $x, y \in \Omega$. Then, for $\alpha \in (1, n]$,

$$C^\alpha(\Phi(E), \Omega) \leq C^\alpha(E, \Omega)$$

for any subset $E \subseteq \Omega$.

Proof. Let μ be a nonnegative Radon measure on Ω such that $R^{\alpha,\Omega}_{\mu}(x) \geq 1$ for all $x \in E$. Then let μ^* be a nonnegative Radon measure on Ω such that $\mu^*(A) = \mu(\Phi^{-1}(A))$ for any $A \subseteq \Omega$ and therefore

$$\int_\Omega f(\tilde{y})\, d\mu^*(\tilde{y}) = \int_\Omega f \circ \Phi(y)\, d\mu(y).$$

Notice that

$$\begin{aligned} R^{\alpha,\Omega}_{\mu^*}(\Phi(x)) &= \int_\Omega \frac{1}{|\Phi(x) - \tilde{y}|^{n-\alpha}}\, d\mu^*(\tilde{y}) = \int_\Omega \frac{1}{|\Phi(x) - \Phi(y)|^{n-\alpha}}\, d\mu(y) \\ &\geq \int_\Omega \frac{1}{|x - y|^{n-\alpha}}\, d\mu(y) = R^{\alpha,\Omega}_{\mu}(x) \geq 1. \end{aligned}$$

Thus

$$\begin{aligned} C^\alpha(\Phi(E), \Omega) &= \inf\{\nu(\Omega) : \nu \geq 0 \text{ on } \Omega \text{ and } R^{\alpha,\Omega}_{\nu}(x) \geq 1 \text{ for all } x \in \Phi(E)\} \\ &\leq \inf\{\mu^*(\Omega) : \mu^* \text{ induced from } \mu \text{ and } R^{\alpha,\Omega}_{\mu^*}(\Phi(x)) \geq 1 \text{ for all } x \in E\} \\ &= \inf\{\mu(\Omega) : \mu \geq 0 \text{ on } \Omega \text{ and } R^{\alpha,\Omega}_{\mu}(x) \geq 1 \text{ for all } x \in E\} = C^\alpha(E, \Omega). \end{aligned}$$

The argument for $\alpha = n$ is similar and the proof is complete. □

Before we introduce the notion of thinness by C^α, for completeness, let us calculate the outer capacity $C^\alpha(S^{n-1}, B_2)$, where

$$B_2 = \{x \in \mathbb{R}^n : |x| < 2\} \quad \text{and} \quad S^{n-1} = \{x \in \mathbb{R}^n : |x| = 1\}.$$

Lemma 2.6 [Mizuta 1996, Example 5.4.3]. *For $\alpha \in (1, n]$,*

$$C^\alpha(S^{n-1}, B_2) = c(n, \alpha)$$

for some positive constant $c(n, \alpha)$.

Proof. It suffices to show that $C^\alpha(S^{n-1}, B_2)$ is finite and positive. Let σ be the volume measure for the unit sphere so that the total measure of S^{n-1} is 1. First we realize that the potential, for $\alpha \in (1, n]$ and $x \in S^{n-1}$, satisfies

$$R_\sigma^{\alpha, B_2}(x) \geq m$$

for some $m = m(n, \alpha) > 0$. Therefore $C^\alpha(S^{n-1}, B_2) \leq \frac{1}{m} < \infty$ by Definition 2.2. To see that $C^\alpha(S^{n-1}, B_2) > 0$ for any μ on B_2, we use Lemma 2.7 below to pick up a point $p \in S^{n-1}$ such that (2-3) holds and calculate, for $\alpha \in (1, n)$,

$$\begin{aligned} R_\mu^{\alpha, B_2}(p) &= (n-\alpha) \int_0^\infty \mu\Big(\Big\{\frac{1}{r} - \frac{1}{3} > s\Big\} \cap B_2\Big) \frac{1}{\big(s + \frac{1}{3}\big)^{n-\alpha+1}} ds + \frac{1}{3^{n-\alpha}} \mu(B_2) \\ &= (n-\alpha) \int_0^3 \mu(B_r(p) \cap B_2) r^{\alpha-n-1} dr + \frac{1}{3^{n-\alpha}} \mu(B_2) \\ &\leq M(n, \alpha) \mu(B_2) \end{aligned}$$

for some $M(n, \alpha) > 0$ and $r = |x - p|$. For $\alpha = n$,

$$\begin{aligned} R_\mu^{\alpha, B_2}(p) &= \int_0^\infty \mu\Big(\Big\{\frac{3}{r} - 1 > s\Big\} \cap B_2\Big) \frac{1}{1+s} ds + \log \frac{4}{3} \mu(B_2) \\ &= \int_0^3 \mu(B_r(p) \cap B_2) \frac{1}{r} dr + \log \frac{4}{3}\, \mu(B_2) \\ &\leq M(n, n) \mu(B_2) \end{aligned}$$

for some $M(n, n) > 0$. In the above we used [Rudin 1987, Theorem 8.16]. This implies $C^\alpha(S^{n-1}, B_2) \geq 1/M(n, \alpha) > 0$ by Definition 2.2. □

By the Vitali covering lemma, we prove the following fact used in the above.

Lemma 2.7. *Let $n \geq 2$ and μ be a finite nonnegative Radon measure on $B_2 \subset \mathbb{R}^n$. Then there is a point $p \in S^{n-1}$ such that*

$$\mu(B_r(p) \cap B_2) \leq c(n) \mu(B_2) r^{n-1} \quad \textit{for all } r > 0, \tag{2-3}$$

for some dimensional constant $c = c(n)$.

Proof. For convenience, let $\mu(B_2) = 1$. Assume otherwise, for any $q \in S^{n-1}$, there is $r_q > 0$ such that

$$\mu(B_{r_q}(q) \cap B_2) \geq c(n) r_q^{n-1}.$$

Using the Vitali covering lemma, we have $\{q_1, q_2, \ldots, q_k\} \subset S^{n-1}$ such that the balls in the collection

$$\{B_{r_{q_1}}(q_1), B_{r_{q_2}}(q_2), \ldots, B_{r_{q_k}}(q_k)\}$$

are disjoint but the balls in the collection

$$\{B_{3r_{q_1}}(q_1), B_{3r_{q_2}}(q_2), \ldots, B_{3r_{q_k}}(q_k)\}$$

cover the sphere S^{n-1}. Therefore, on one hand,

$$c(n)\sum_{i=1}^{k} r_{q_i}^{n-1} \le \sum_{i=1}^{k} \mu(B_{r_{q_i}}(q_i)\cap B_2) \le \mu(B_2) = 1.$$

On the other hand,

$$|S^{n-1}| \le \sum_{i=1}^{k} |B_{3r_{q_i}}(q_i)\cap S^{n-1}| < |S^{n-1}| c(n)\sum_{i=1}^{k} r_{q_i}^{n-1},$$

when $c(n)$ is sufficiently large, where $|\cdot|$ stands for the Lebesgue measure on S^{n-1}. Therefore the lemma is proven by contradiction. □

Now let us introduce the geometric definition of thinness. For notions of thinness in terms of the fine topology and Wiener criterion, readers are referred, for instance, to [Mizuta 1996; Adams and Hedberg 1996; Armitage and Gardiner 2001]. Let

$$\omega_i^\delta(p) = \{x\in\mathbb{R}^n : |x-p| \in [2^{-i}\delta, 2^{-i+1}\delta]\},$$
$$\Omega_i^\delta(p) = \{x\in\mathbb{R}^n : |x-p| \in (2^{-i-1}\delta, 2^{-i+2}\delta)\}.$$

Definition 2.8. Let E be a subset in the Euclidean space $\mathbb{R}^n$ and $p\in\mathbb{R}^n$ be a point in $\mathbb{R}^n$. The subset E is said to be α-thin at the point p for $\alpha\in(1,n)$ if

$$\sum_{i\ge 1} \frac{C^\alpha(E\cap\omega_i^\delta(p), \Omega_i^\delta(p))}{C^\alpha(\partial B_{2^{-i}\delta}(p), B_{2^{-i+1}\delta}(p))} < \infty$$

for some small $\delta > 0$. The subset E is said to be n-thin at p if

$$\sum_{i\ge 1} i C^n(E\cap\omega_i^\delta(p), \Omega_i^\delta(p)) < \infty$$

for some small $\delta > 0$.

Combining Lemmas 2.3-2.6 with the above definition, we observe the following important property of α-thin sets, inspired by [Arsove and Huber 1973] (see also [Ma and Qing 2021; 2022]). We recall Theorem 1.2 from the Introduction for readers' convenience.

Theorem 1.2. *Let E be a subset in the Euclidean space $\mathbb{R}^n$ and $p\in\mathbb{R}^n$ be a point. Suppose that E is α-thin at the point p for $\alpha\in(1,n]$. Then there is a ray from p that avoids E at least within some small ball at p.*

Proof. First of all, due to the translation invariance, we may simply assume p is the origin of the Euclidean space. Then, by the scaling property of the outer capacity C^α in Lemma 2.4, one notices that

$$\frac{C^\alpha(E\cap\omega_i^\delta, \Omega_i^\delta)}{C^\alpha(\partial B_{2^{-i}\delta}, B_{2^{-i+1}\delta})} = \frac{C^\alpha(S_i(E)\cap\omega_0^1, \Omega_0^1)}{C^\alpha(\partial B_1, B_2)},$$

where $S_i(v) = (2^i/\delta)v$ is the scaling map. Then we consider the projection

$$P(v) = \begin{cases} v/|v| & \text{when } v\in\mathbb{R}^n \text{ and } |v|\ge 1,\\ v & \text{when } v\in\mathbb{R}^n \text{ and } |v| < 1,\end{cases}$$

which is contractive. Therefore, in light of Lemma 2.5, we have

$$C^\alpha(P(S_i(E)\cap\omega_0^1),\Omega_0^1)\le C^\alpha(S_i(E)\cap\omega_0^1,\Omega_0^1).$$

Next, using the countable subadditivity in Lemma 2.3, we have

$$C^\alpha\biggl(\bigcup_{i\ge k}P(S_i(E)\cap\omega_0^1),\Omega_0^1\biggr)\le\sum_{i\ge k}C^\alpha(P(S_i(E)\cap\omega_0^1),\Omega_0^1).$$

Thus,

$$\begin{aligned}C^\alpha\biggl(\bigcup_{i\ge k}P(S_i(E)\cap\omega_0^1),\Omega_0^1\biggr)&\le\sum_{i\ge k}C^\alpha(S_i(E)\cap\omega_0^1,\Omega_0^1)\\&\le C^\alpha(\partial B_1,B_2)\sum_{i\ge k}\frac{C^\alpha(S_i(E)\cap\omega_0^1,\Omega_0^1)}{C^\alpha(\partial B_1,B_2)}\\&\le C^\alpha(\partial B_1,B_2)\sum_{i\ge k}\frac{C^\alpha(E\cap\omega_i^\delta,\Omega_i^\delta)}{C^\alpha(\partial B_{2^{-i}\delta},B_{2^{-i+1}\delta})},\end{aligned}$$

which is arbitrarily small when k is appropriately large using Lemma 2.6 for $C^\alpha(\partial B_1,B_2)$. And then this implies that

$$\partial B_1\setminus\bigcup_{i\ge k}P(S_i(E)\cap\omega_0^1)\ne\varnothing.$$

The argument for $\alpha=n$ is similar and easier. □

2.2. ***Linear potential on manifolds.*** On a given complete Riemannian manifold (M^n,g), let $d(\cdot,\cdot)$ be the distance function associated with the given Riemannian metric g.

Definition 2.9. Suppose that (M^n,g) is a complete Riemannian manifold and $U\subseteq M^n$ is a bounded open subset. For $\alpha\in(1,n]$, the linear potential on the Riemannian manifold (M^n,g) of order α for a Radon measure μ on U is given by

$$\mathscr{R}_\mu^{\alpha,U}(x)=\begin{cases}\displaystyle\int_U\frac{1}{d(x,y)^{n-\alpha}}\,d\mu(y) & \text{when } \alpha\in(1,n),\\ \displaystyle\int_\Omega\log\frac{D}{d(x,y)}\,d\mu(y) & \text{when } \alpha=n,\end{cases}$$

where D is the diameter of U.

From the discussion in the previous subsection, it is easily seen that one may generate an outer capacity $\mathscr{C}^\alpha(E,U)$ for any subset $E\subseteq U\subseteq M^n$ that behaves like the counterpart in Euclidean spaces. To use $R_\mu^{\alpha,\Omega}(x)$ and $C^\alpha(E,\Omega)$ on Euclidean spaces in the previous subsection to study $\mathscr{R}_\mu^{\alpha,U}(p)$ and $\mathscr{C}^\alpha(A,U)$ on manifolds, we first introduce the correspondence between Radon measures on the tangent space T_pM^n at each point $p\in M^n$ and those on (M^n,g). Suppose that (M^n,g) is a complete Riemannian manifold. Let $p\in M^n$ and U be a convex normal coordinate neighborhood at p on (M^n,g), where the exponential map serves as the convex normal coordinate

$$\exp|_p:\Omega\to U.$$

The domain U is said to be convex if the unique geodesic joining any two points in U stays in U. Moreover, we may assume in the coordinate chart U the exponential map be uniformly bi-Lipschitz throughout this paper.

Then, for a Radon measure μ on $U \subseteq M^n$, one may introduce the Radon measure μ^* on $\Omega \subset T_p M^n$ such that, for a subset $E \subseteq \Omega$,

$$\mu^*(E) = \mu(\exp|_p E) \quad \text{and} \quad \int_\Omega f \circ \exp|_p \, d\mu^* = \int_U f \, d\mu.$$

It is then easily seen that the following equivalence between the linear potential $R^{\alpha,\Omega}_{\mu^*}$, the outer capacities $C^\alpha(\,\cdot\,,\Omega)$ and the corresponding $\mathscr{R}^{\alpha,U}_\mu$, $\mathscr{C}^\alpha(\,\cdot\,,U)$ holds. Namely:

Lemma 2.10. *Suppose that (M^n, g) is a complete Riemannian manifold and $p \in M^n$. Let*

$$\exp|_p : \Omega \to U$$

be the convex normal coordinate chart, where the exponential map is uniformly bi-Lipschitz. And let $\alpha \in (1, n]$. Then, for $A \subset U$ and $E = (\exp|_p)^{-1} A \subset \Omega$,

$$C^{-1} R^{\alpha,\Omega}_{\mu^*} \le \mathscr{R}^{\alpha,U}_\mu \le C R^{\alpha,\Omega}_{\mu^*},$$
$$C^{-1} C^\alpha(E,\Omega) \le \mathscr{C}^\alpha(A,U) \le C C^\alpha(E,\Omega)$$

for some constant $C = C(M^n, g, U, p)$. Consequently, a subset $A \subset U$ is α-thin at p if and only if $E = (\exp|_p)^{-1}(A) \subset \Omega$ is α-thin at the origin of $T_p M^n$.

Proof. The proof is straightforward based on the properties of the convex normal coordinate chart at a point in a complete Riemannian manifold, where the exponential map is bi-Lipschitz. □

2.3. *Estimates of Riesz potentials.* We now introduce our estimates of Riesz potentials on manifolds. We will recall some well-known estimates for Riesz potentials in Euclidean spaces [Mizuta 1996, Chapter 2].

Our estimates on Riesz potentials are designed to help understand the Hausdorff dimensions of singularities of solutions of partial differential equations on manifolds. Let us start with a general decomposition theorem for nonnegative Radon measures on a complete Riemannian manifold based on [Kpata 2019, Proposition 1.4], which is related to Lemma 2.7 and a broad generalization of the Lebesgue Differentiation Theorem in some way.

Lemma 2.11 [Kpata 2019, Proposition 1.4]. *Let μ be a nonnegative Radon measure on a complete Riemannian manifold (M^n, g) and let*

$$G^\infty_d = \Big\{ x \in M^n : \limsup_{r \to 0} r^{-d} \mu(B_r(x)) = +\infty \Big\}$$

for any $d \in [0, n]$. Then

$$\mathscr{H}_d(G^\infty_d) = 0,$$

where $\mathscr{H}_d$ is the Hausdorff measure of dimension d.

Proof. Based on the general decomposition theorem [Kpata 2019, Proposition 1.4] on the Euclidean space and the correspondence of Radon measures in Lemma 2.10, this lemma is easily seen. Specifically, we first prove the statement for Radon measures supported in a convex normal coordinate chart used in Lemma 2.10. Then the lemma follows by using a countable covering for (M, g) by convex normal coordinate charts. □

Now we are ready to state and prove one crucial analytic result in this paper on the behavior of the Riesz potentials. For readers' convenience, we recall Theorem 1.1 from the Introduction.

Theorem 1.1. *Suppose that (M^n, g) is a complete Riemannian manifold and μ is a finite Radon measure on a bounded domain $G \subset M^n$. Let S be a compact subset in G such that its Hausdorff dimension is greater than d. And let $\alpha \in (1, n)$ and $d < n - \alpha$. Then there is a point $p \in S$ and a subset E that is α-thin at p such that*

$$\int_G \frac{1}{d(x, y)^{n-\alpha}}\, d\mu \le \frac{C}{d(x, p)^{n-\alpha-d}}$$

for some constant C and all $x \in B_\delta(p) \setminus E$ for some $\delta > 0$.

Proof. First, due to the assumption that the Hausdorff dimension of S is greater than d,

$$\mathscr{H}_{d+\epsilon}(S) = \infty$$

for some small $\epsilon > 0$. Then, in light of Lemma 2.11, there is a point $p \in S$ such that

$$\limsup_{r \to 0} r^{-(d+\epsilon)} \mu(B_r(p)) \le C < \infty.$$

That is to say

$$\mu(B_r(p)) \le C r^{d+\epsilon} \tag{2-4}$$

when r is appropriately small. Secondly, we may confine ourselves to a convex normal coordinate neighborhood U of p and we may work on the Euclidean space without loss of generality in light of the discussion in the previous subsection, particularly, Lemma 2.10, where $\exp|_p : \Omega \to U$ and $\exp|_p(0) = p$. For convenience, we will not differentiate μ and μ^* if no confusion rises. Therefore, for $x \in \omega_i^\delta \subset \Omega$ when δ is sufficiently small and i is appropriately large,

$$\begin{aligned} R_\mu^{\alpha,\Omega}(x) &= \int_\Omega \frac{1}{|x-y|^{n-\alpha}}\, d\mu \\ &= \int_{\Omega \setminus B_{2^{-i_0+2}\delta}} \frac{1}{|x-y|^{n-\alpha}}\, d\mu + \int_{B_{2^{-i_0+2}\delta} \setminus \Omega_i^\delta} \frac{1}{|x-y|^{n-\alpha}}\, d\mu + \int_{\Omega_i^\delta} \frac{1}{|x-y|^{n-\alpha}}\, d\mu, \end{aligned} \tag{2-5}$$

where $i_0 \le i$ to be fixed. For the first term in the right-hand side of (2-5),

$$I = \int_{\Omega \setminus B_{2^{-i_0+2}\delta}} \frac{1}{|x-y|^{n-\alpha}}\, d\mu \le \left(\frac{1}{2^{-i_0+2}\delta - 2^{-i+1}\delta} \right)^{n-\alpha} \mu(\Omega) \le \left(\frac{1}{2^{-i_0+1}\delta} \right)^{n-\alpha} \mu(\Omega).$$

Recall that $2^{-i}\delta \le |x| \le 2^{-i+1}\delta$ for $x \in \omega_i$, we have

$$I \le \mu(\Omega)\frac{(2^{-i+1}\delta)^{n-\alpha-d}}{(2^{-i_0+1}\delta)^{n-\alpha}}\frac{1}{|x|^{n-\alpha-d}} \le C\frac{1}{|x|^{n-\alpha-d}}, \tag{2-6}$$

where $C = C(n, \alpha, d, \delta, i_0)$. For the second term in the right-hand side of (2-5),

$$\begin{aligned}\int_{B_{2^{-i_0+2}\delta}\setminus\Omega_i^\delta}\frac{1}{|x-y|^{n-\alpha}}\,d\mu &= \int_{B_{2^{-i_0+2}\delta}\setminus B_{2^{-i+2}\delta}}\frac{1}{|x-y|^{n-\alpha}}\,d\mu + \int_{B_{2^{-i-1}\delta}}\frac{1}{|x-y|^{n-\alpha}}\,d\mu \\ &\le \int_{B_{2^{-i_0+2}\delta}\setminus B_{2^{-i+2}\delta}}\frac{1}{|x-y|^{n-\alpha}}\,d\mu + \left(\frac{1}{2^{-i-1}\delta}\right)^{n-\alpha}\mu(B_{2^{-i-1}\delta}) \\ &\le \sum_{k=i_0}^{i-1}\int_{B_{2^{-k+2}\delta}\setminus B_{2^{-k+1}\delta}}\frac{1}{|x-y|^{n-\alpha}}\,d\mu + \left(\frac{1}{2^{-i-1}\delta}\right)^{n-\alpha}\mu(B_{2^{-i-1}\delta}) \\ &\le \sum_{k=i_0}^{i-1}\left(\frac{1}{2^{-k}\delta}\right)^{n-\alpha}\mu(B_{2^{-k+2}\delta}) + \left(\frac{1}{2^{-i-1}\delta}\right)^{n-\alpha}\mu(B_{2^{-i-1}\delta}).\end{aligned}$$

Using (2-4) for $\epsilon = 0$, we continue from the above,

$$\begin{aligned}II &\le C\left(4^d\sum_{k=i_0}^{i-1}\left(\frac{1}{2^{-k}\delta}\right)^{n-\alpha-d} + \left(\frac{1}{2^{-i-1}\delta}\right)^{n-\alpha-d}\right) \\ &\le C\left(\frac{4^d}{1-2^{-(n-\alpha-d)}}\left(\frac{1}{2^{-i+1}\delta}\right)^{n-\alpha-d} + \left(\frac{1}{2^{-i-1}\delta}\right)^{n-\alpha-d}\right) \le C\frac{1}{|x|^{n-\alpha-d}},\end{aligned} \tag{2-7}$$

where $C = C(n, \alpha, d, \delta, i_0)$. To handle the third term in the right-hand side of (2-5), we let

$$E_i^\lambda = \left\{x \in \omega_i^\delta : \int_{\Omega_i^\delta}\frac{1}{|x-y|^{n-\alpha}}\,d\mu \ge \lambda 2^{i(n-\alpha-d)}\right\},$$

where $\lambda > 0$ is fixed. By Definition 2.2, we know

$$C^\alpha(E_i^\lambda, \Omega_i^\delta) \le \frac{\mu(\Omega_i^\delta)}{\lambda 2^{i(n-\alpha-d)}} \le \frac{C}{\lambda}\frac{(2^{-i+2}\delta)^{d+\epsilon}}{2^{i(n-\alpha-d)}} = \frac{C\,4^{d+\epsilon}}{\lambda}2^{-i\epsilon}(2^{-i})^{n-\alpha},$$

where (2-4) for some $\epsilon > 0$ is used and $\Omega_i^\delta \subset B_{2^{-i+2}\delta}$. Now, from Lemma 2.6 and the scaling property, we know

$$C^\alpha(\partial B_{2^{-i}\delta}, B_{2^{-i+1}\delta}) = C(n, \alpha)(2^{-i}\delta)^{n-\alpha}$$

and

$$\sum_{i \ge i_0}\frac{C^\alpha(E_i^\lambda, \Omega_i^\delta)}{C^\alpha(\partial B_{2^{-i}\delta}, B_{2^{-i+1}\delta})} \le \frac{C}{\lambda}\sum_{i \ge i_0}2^{-\epsilon i} < \infty.$$

Thus, by Definition 2.8, the proof is completed. □

As a consequence of Theorems 1.2 and 1.1, we have:

Corollary 2.12. *Suppose that (M^n, g) is a complete Riemannian manifold and μ is a finite Radon measure on a bounded domain $G \subset M^n$. Let S be a compact subset in G such that its Hausdorff dimension is greater than d. And let $\alpha \in (1, n)$ and $d < n - \alpha$. Then there is a point $p \in S$ such that, for some constant C,*

$$\int_G \frac{1}{d(x,y)^{n-\alpha}}\, d\mu \le \frac{C}{d(x,p)^{n-\alpha-d}}$$

for all $x \in \mathrm{Ray}_p \cap B(x,\delta)$, where Ray_p is a ray from p and $B(p,\delta)$ is the geodesic ball of radius $\delta > 0$.

2.4. ***Estimates of the log potential.*** First, as stated in [Mizuta 1996, Theorem 6.3], for the log potential $U_n\mu(x)$ on Euclidean spaces defined on page 82 of that work,

$$\lim_{x\to p \text{ and } x\in\Omega\setminus E} \frac{U_n\mu(x)}{\log(1/|x-p|)} = \mu(\{p\}).$$

The following is our version of [Mizuta 1996, Theorem 6.3] on manifolds. For us it is a generalization of [Arsove and Huber 1973, Theorem 1.3] in higher dimensions and linear versions of such behaviors for n-superharmonic functions (see [Huber 1957; Bonini et al. 2018; 2019; Ma and Qing 2021; 2022]). For convenience, we present a brief but full proof based on the potential theory developed in previous subsections in this paper.

Theorem 2.13. *Suppose (M^n, g) is a complete Riemannian manifold. Let μ be a finite Radon measure on a bounded domain $G \subset M^n$. Then, for all $p \in G$, there is a subset A that is n-thin at p and*

$$\lim_{x\to p \text{ and } x\in M^n\setminus A} \frac{\int_G \log(1/d(x,y))\, d\mu(y)}{\log(1/d(x,p))} = \mu(\{p\}).$$

Proof. Let

$$\exp|_p : \Omega \to U$$

be a convex normal coordinate at $p \in M^n$. Clearly, it suffices to show that there is a subset A in U, which is n-thin at p, such that

$$\lim_{x\to p \text{ and } x\in U\setminus A} \frac{\mathscr{R}^{n,U}_\mu(x)}{\log(1/d(x,p))} = \mu(\{p\}). \tag{2-8}$$

Therefore, for $x \in \omega_i^\delta(p)$, we write

$$\begin{aligned}\mathscr{R}^{n,U}_\mu(x) &= \int_U \log\frac{D}{d(x,y)}\, d\mu(y)\\ &= \int_{U\setminus B_{2^{-i_0+2}\delta}} \log\frac{D}{d(x,y)}\, d\mu + \int_{B_{2^{-i_0+2}\delta}\setminus\Omega_i^\delta} \log\frac{D}{d(x,y)}\, d\mu + \int_{\Omega_i^\delta} \log\frac{D}{d(x,y)}\, d\mu.\end{aligned} \tag{2-9}$$

Here we omit the center p for each ball or annulus for simplicity. For the first term in the right-hand side of (2-9),

$$I = \int_{U\setminus B_{2^{-i_0+2}\delta}} \log\frac{D}{d(x,y)}\, d\mu \le \mu(U)\log\frac{D}{2^{-i_0+1}\delta} = o(1)\log\frac{1}{d(x,p)} \quad \text{as } x\to p. \tag{2-10}$$

For the second term in the right-hand side of (2-9),

$$\int_{B_{2^{-i_0+2}\delta}\setminus\Omega_i^\delta}\log\frac{D}{d(x,y)}\,d\mu(y)=\int_{B_{2^{-i_0+2}\delta}\setminus B_{2^{-i+1}\delta}}\log\frac{D}{d(x,y)}\,d\mu+\int_{B_{2^{-i-2}\delta}}\log\frac{D}{d(x,y)}\,d\mu$$
$$\le C\left[\sum_{k=i_0}^{i}k\mu(B_{2^{-k+2}\delta}\setminus B_{2^{-k+1}\delta})\right]+\mu(B_{2^{-i-2}\delta})\log\frac{D}{2^{-i-2}\delta}.$$

Due to the regularity of Radon measures and $d(x,p)\in[2^{-i-1}\delta,2^{-i}\delta]$, we know

$$\mu(B_{2^{-i-2}\delta})\log\frac{D}{2^{-i-2}\delta}=\mu(\{p\})\log\frac{1}{d(x,p)}+o\left(\log\frac{1}{d(x,p)}\right)\quad\text{as } x\to p \tag{2-11}$$

and

$$\sum_{k=i_0}^{i}k\mu(B_{2^{-k+2}\delta}\setminus B_{2^{-k+1}\delta})=o(1)i=o(1)\log\frac{1}{d(x,p)} \tag{2-12}$$

as $i\to\infty$ or equivalently $x\to p$. To see (2-12), for any $\epsilon>0$, we first find k_0 such that

$$\mu(B_{2^{-l+2}\delta}\setminus B_{2^{-m+1}\delta})\le\frac{\epsilon}{2}$$

for all $m\ge l\ge k_0$ due to the regularity of μ. Next, we find N such that

$$\frac{\sum_{k=i_0}^{k_0}k\mu(B_{2^{-k+2}\delta}\setminus B_{2^{-k+1}\delta})}{i}\le\frac{\epsilon}{2}$$

for all $i\ge N$. Together, this gives

$$\frac{\sum_{k=i_0}^{i}k\mu(B_{2^{-k+2}\delta}\setminus B_{2^{-k+1}\delta})}{i}=\frac{\sum_{k=i_0}^{k_0}k\mu(B_{2^{-k+2}\delta}\setminus B_{2^{-k+1}\delta})}{i}+\frac{\sum_{k=k_0+1}^{i}k\mu(B_{2^{-k+2}\delta}\setminus B_{2^{-k+1}\delta})}{i}$$
$$=\frac{\sum_{k=i_0}^{k_0}k\mu(B_{2^{-k+2}\delta}\setminus B_{2^{-k+1}\delta})}{i}+\sum_{k=k_0+1}^{i}\mu(B_{2^{-k+2}\delta}\setminus B_{2^{-k+1}\delta})\le\epsilon$$

for all $i\ge N$. Thus we conclude that

$$II=(\mu(\{p\})+o(1))\log\frac{1}{d(x,p)}\quad\text{as } x\to p. \tag{2-13}$$

To handle the third term in the right side of (2-9), for $\lambda_i>0$ to be determined, we consider

$$A^{\lambda_i}=\left\{x\in\omega_i^\delta:\int_{\Omega_i^\delta}\log\frac{D_i}{d(x,y)}\,d\mu\ge i\lambda_i\right\},$$

where D_i is the diameter of Ω_δ^i. By Definition 2.2,

$$\mathscr{C}^n(A^{\lambda_i},\Omega_i^\delta)\le\frac{\mu(\Omega_i^\delta)}{i\lambda_i}.$$

In light of Definition 2.8, we consider

$$\sum_{i\ge i_0}i\mathscr{C}^n(A^{\lambda_i},\Omega_i^\delta)\le\sum_{i\ge i_0}\frac{\mu(\Omega_i^\delta)}{\lambda_i}$$

and pick up $\lambda_i \to 0$ as $i \to \infty$ by the classic Paul du Bois-Reymond theorem [1873] (see [Bromwich 1908, (5), page 40]) for infinite series such that $\sum_{i\ge i_0} \mu(\Omega_i^\delta)/\lambda_i$ converges when $\sum_{i\ge i_0} \mu(\Omega_i^\delta)$ converges. This is to say that the third term in the right side of (2-9) satisfies

$$\begin{aligned} III &= \int_{\Omega_i^\delta} \log \frac{D}{d(x,y)}\, d\mu(y) = \int_{\Omega_i^\delta} \log \frac{D_i}{d(x,y)}\, d\mu(y) + \log \frac{D}{D_i}\mu(\Omega_\delta^i) \\ &\le \left(\lambda_i + \left(1 + \frac{1}{i}\log\frac{1}{\delta}\right)\mu(\Omega_\delta^i)\right)\log\frac{D}{d(x,p)} \\ &= o(1)\log\frac{1}{d(x,p)} \quad \text{as } x \in \omega_i^\delta \setminus E^{\lambda_i} \text{ and } x \to p. \end{aligned} \tag{2-14}$$

Finally, if let $A = \bigcup_i A^{\lambda_i}$, we have

$$\lim_{x\to p \text{ and } x\in U\setminus A} \frac{\mathcal{R}_\mu^{n,U}(x)}{\log(1/d(x,p))} = \mu(\{p\}),$$

where A is n-thin at p. □

3. On scalar curvature equations

We now focus on the scalar curvature equations for conformal deformation of metrics. Let $(M^n, \bar g)$ be a compact Riemannian manifold for $n \ge 3$. Let $R_{ijkl}[\bar g]$ be the Riemann curvature tensor, $R_{ij}[\bar g] = R_{ijkl}\bar g^{kl}$ be the Ricci curvature tensor, and $R[\bar g] = R_{ij}\bar g^{ij}$ be the scalar curvature. The scalar curvature equation in conformal geometry is

$$-\frac{4(n-1)}{n-2}\Delta[\bar g]u + R[\bar g]u = R[u^{\frac{4}{n-2}}\bar g]u^{\frac{n+2}{n-2}} \tag{3-1}$$

for a positive function u. The scalar curvature equation describes how the scalar curvature transforms under conformal change of metrics. In this section we want to use the estimates for the Newton potential in the previous section to study the Hausdorff dimensions of the singularities of solutions u to the scalar equations which represent the ends of a complete conformal metric $u^{4/(n-2)}\bar g$.

We remark here that all of the results in this section hold if we assume S is compact, $D \subset M^n$ is a bounded domain that contains S, and $(M^n, \bar g)$ is just complete, because the possible noncompact part $M^n \setminus \bar D$ is not relevant for the purpose here.

3.1. ***Preliminaries.*** Let us start with [Ma and Qing 2022, Lemma 3.1], which is a slight improvement of [Chang et al. 2004, Proposition 8.1].

Lemma 3.1 [Ma and Qing 2022, Lemma 3.1]. *Let $(M^n, \bar g)$ be a compact Riemannian manifold and S be a closed subset in M^n. And let D be an open neighborhood of S. Suppose that $g = u^{4/(n-2)}\bar g$ is a conformal metric on $D \setminus S$ and is geodesically complete near S. Then*

$$u(x) \to +\infty \quad \text{as } x \to S$$

if $R^-[g] \in L^p(D\setminus S, g)$ for some $p > n/2$, where $R^-[g] = \max\{-R[g], 0\}$ stands for the negative part of the scalar curvature $R[g]$ and $L^p(D\setminus S, g)$ is the L^p space with respect to the metric g.

For a preliminary estimate on the Hausdorff dimension of S, we follow the proof of [Ma and Qing 2022, Theorem 3.1] and get:

Proposition 3.2. *Let $(M^n, \bar{g})$ be a compact Riemannian manifold and S be a closed subset in M^n. And let D be an open neighborhood of S where the scalar curvature $R[\bar{g}]$ is nonpositive. Suppose that $g = u^{4/(n-2)}\bar{g}$ is a conformal metric on $D \setminus S$ and is geodesically complete near S. Then the Newton capacity of S is zero and therefore the Hausdorff dimension satisfies*

$$\dim_{\mathscr{H}}(S) \le n-2,$$

provided that

$$R^-[g] \in L^{\frac{2n}{n+2}}(D \setminus S, g) \cap L^p(D \setminus S, g)$$

for some $p > \frac{n}{2}$.

Proof. Recall the scalar curvature equation

$$-\frac{4(n-1)}{n-2}\Delta u = -Ru + R^+[g]u^{\frac{n+2}{n-2}} - R^-[g]u^{\frac{n+2}{n-2}} \quad \text{in } D \setminus S, \tag{3-2}$$

where

$$\begin{aligned}\int_{D\setminus S} R^-[g]u^{\frac{n+2}{n-2}}\, d\mathrm{vol}[\bar{g}] &\le \left(\int_{D\setminus S} (R^-[g])^{\frac{2n}{n+2}} u^{\frac{2n}{n-2}}\, d\mathrm{vol}[\bar{g}]\right)^{\frac{n+2}{2n}} \mathrm{vol}(D)^{\frac{n-2}{2n}} \\ &\le \left(\int_{D\setminus S} (R^-[g])^{\frac{2n}{n+2}}\, d\mathrm{vol}[g]\right)^{\frac{n+2}{2n}} \mathrm{vol}(D)^{\frac{n-2}{2n}} < \infty. \end{aligned} \tag{3-3}$$

Here, and from now on, all geometric quantities are under the background metric $\bar{g}$ unless indicated otherwise. And, in light of Lemma 3.1, we know

$$u(x) \to +\infty \quad \text{as } x \to S.$$

As in the proof of [Ma and Qing 2022, Theorem 3.1] (adopted from [Bidaut-Véron 1989, Lemma 1.2]), we use the following test functions. First we let

$$u_{\alpha,\beta} = \begin{cases} \beta, & u \ge \alpha+\beta, \\ u-\alpha, & u < \alpha+\beta, \end{cases} \qquad \text{and} \qquad \phi_{\alpha,\beta} = u_{\alpha,\beta} - \beta + \beta(1-\eta),$$

where $\eta \in C_c^\infty(\Sigma_\alpha)$ is a fixed cut-off function that is equal to 1 in a neighborhood of S and $\Sigma_\alpha = \{x \in D : u(x) > \alpha\}$. Notice that, for β sufficiently large,

$$u_{\alpha,\beta} \in (0, \beta] \quad \text{in } \Sigma_\alpha \qquad \text{and} \qquad \phi_{\alpha,\beta} = 0 \quad \text{on } \{x \in D : u(x) = \alpha\} \cup \{x \in D : u \ge \alpha+\beta\}$$

and

$$\nabla\phi_{\alpha,\beta} = \nabla u_{\alpha,\beta} + \beta\nabla\eta \quad \text{and} \quad \nabla u = \nabla u_{\alpha,\beta} \quad \text{when } \nabla u_{\alpha,\beta} \ne 0.$$

We then multiply $\phi_{\alpha,\beta}$ to (3-2) and get

$$\frac{4(n-1)}{n-2}\int_{\Sigma_\alpha} \nabla u \cdot \nabla\phi_{\alpha,\beta}\, d\mathrm{vol}[\bar{g}] = \int_{\Sigma_\alpha} (-Ru + R[g]u^{\frac{n+2}{n-2}})\phi_{\alpha,\beta}\, d\mathrm{vol}[\bar{g}].$$

Therefore

$$\begin{aligned}\frac{4(n-1)}{n-2}\int_{\Sigma_\alpha}|\nabla u_{\alpha,\beta}|^2\,d\mathrm{vol}[\bar g] &= \beta\int_{\Sigma_\alpha}\left(\frac{n-2}{4(n-1)}\nabla u\cdot\nabla\eta + (-Ru + R[g]u^{\frac{n+2}{n-2}})(1-\eta)\right)d\mathrm{vol}[\bar g]\\ &\quad -\int_{\Sigma_\alpha}(-Ru + R^+[g]u^{\frac{n+2}{n-2}})(\beta - u_{\alpha,\beta})\,d\mathrm{vol}[\bar g]\\ &\quad +\int_{\Sigma_\alpha}R^-[g]u^{\frac{n+2}{n-2}}(\beta-u_{\alpha,\beta})\,d\mathrm{vol}[\bar g] \le C\beta, \end{aligned}\tag{3-4}$$

where C depends on α and η but does not depend on β, due the support of $1-\eta$ and (3-3). That is,

$$\int_{\Sigma_\alpha}|\nabla\frac{u_{\alpha,\beta}}{\beta}|^2\,d\mathrm{vol}[\bar g] \le \frac{C}{\beta}\to 0$$

as $\beta\to\infty$, where $u_{\alpha,\beta}/\beta$ is a function that is identically 1 in a neighborhood of S. This implies the Newton capacity $\mathrm{Cap}_2(S,D)$ of S is zero. Consequently, we know S is of Hausdorff dimension not greater than $n-2$ (see [Adams and Meyers 1972; Schoen and Yau 1994, Theorem 2.10 in Chapter VI]). □

3.2. $-\Delta u$ *is a Radon measure on* D. In order to use the estimates of potentials in the previous section, we need the following lemma (see [Ma and Qing 2022, Lemma 3.2–3.4]).

Lemma 3.3. *Let $(M^n,\bar g)$ be a compact Riemannian manifold and S be a closed subset in M^n. And let D be an open neighborhood of S where the scalar curvature $R[\bar g]$ is nonpositive. Suppose that $g=u^{4/(n-2)}\bar g$ is a conformal metric on $D\setminus S$ and is geodesically complete near S. Then $-\Delta u$ is a Radon measure on D and $-\Delta u|_S\ge 0$, provided that*

$$R^-[g]\in L^{\frac{2n}{n+2}}(D\setminus S,g)\cap L^p(D\setminus S,g)$$

for some $p>\frac{n}{2}$.

Proof. Again, recall the scalar curvature equation

$$-\frac{4(n-1)}{n-2}\Delta u = -Ru + R^+[g]u^{\frac{n+2}{n-2}} - R^-[g]u^{\frac{n+2}{n-2}} = f \quad \text{in } D\setminus S,\tag{3-5}$$

where

$$\int_D R^-[g]u^{\frac{n+2}{n-2}}\,d\mathrm{vol}[\bar g]<\infty.$$

And, in light of Lemma 3.1, we know

$$u(x)\to\infty \quad \text{as } x\to S.$$

Then we claim the right-hand side f of (3-5) is in $L^1(D)$. To prove this claim, we follow the argument in the proof of [Ma and Qing 2022, Theorem 3.2] (stated as Lemma 3.2 there). Let

$$\alpha_s(t)=\begin{cases} t, & t\le s,\\ \text{increasing}, & t\in[s,10s],\\ 2s, & t\ge 10s\end{cases}$$

(this function was used in [Dolzmann et al. 1997]). Notice that one may require $\alpha_s' \in [0,1]$ and $\alpha_s'' \le 0$. We calculate

$$-\Delta\alpha_s(u) = -\alpha''(u)|\nabla u|^2 + \alpha_s'(u)(-\Delta u)$$

and, for $s > \max\{u(x) : x \in \partial D\}$,

$$\int_{\partial D} \frac{\partial u}{\partial \nu}\, d\sigma = \int_D \Delta\alpha_s(u)\, d\mathrm{vol}[\bar{g}] = \int_D \left(-\alpha''(u)|\nabla u|^2 + \alpha'(u)\frac{n-2}{4(n-1)} f\right) d\mathrm{vol}[\bar{g}].$$

Hence

$$\int_D \left(-\alpha_s''(u)|\nabla u|^2 + \alpha_s'(u)\frac{n-2}{4(n-1)} f^+\right) d\mathrm{vol}[\bar{g}] \le \int_{\partial D} \frac{\partial u}{\partial \nu}\, d\sigma + \frac{n-2}{4(n-1)} \int_D R^-[g] u^{\frac{n+2}{n-2}}\, d\mathrm{vol}[\bar{g}]$$

and

$$\int_D |\Delta\alpha_s(u)|\, d\mathrm{vol}[\bar{g}] = \int_D \left(-\alpha''(u)|\nabla u|^2 + \alpha'(u)\frac{n-2}{4(n-1)}(f^+ + f^-)\right) d\mathrm{vol}[\bar{g}].$$

By Fatou's lemma, as $s \to \infty$, we have

$$\int_D f^+\, d\mathrm{vol}[\bar{g}] \le \frac{4(n-1)}{n-2} \int_{\partial D} \frac{\partial u}{\partial \nu}\, d\sigma + \int_D R^-[g] u^{\frac{n+2}{n-2}}\, d\mathrm{vol}[\bar{g}].$$

So the claim is proven. Moreover,

$$\int_D |\Delta\alpha_s(u)|\, d\mathrm{vol}[\bar{g}] \le \int_{\partial D} \frac{\partial u}{\partial \nu}\, d\sigma + \frac{n-2}{2(n-1)} \int_D R^-[g] u^{\frac{n+2}{n-2}}\, d\mathrm{vol}[\bar{g}].$$

Consequently, for $\phi \in C_c^\infty(D)$,

$$\begin{aligned}
|-\Delta\alpha_s(u)(\phi)| &= \left| \int_D (-\Delta\alpha_s(u))\phi\, d\mathrm{vol}[\bar{g}] \right| \\
&\le \int_D |\Delta\alpha_s(u)|\, d\mathrm{vol}[\bar{g}] \|\phi\|_{C^0(D)} \\
&\le \left(\int_{\partial D} \frac{\partial u}{\partial \nu}\, d\sigma + \frac{n-2}{2(n-1)} \int_{D\setminus S} R^-[g] u^{\frac{n+2}{n-2}}\, d\mathrm{vol}[\bar{g}] \right) \|\phi\|_{C^0(D)}
\end{aligned}$$

for any s larger. Before we show $-\Delta u$ is a Radon measure, let us state and prove a lemma which is useful for the proof now and later in the following sections.

Lemma 3.4. *Let $(M^n, \bar{g})$ be a compact Riemannian manifold and S be a closed subset in M^n. And let D be an open neighborhood of S where the scalar curvature $R[\bar{g}]$ is nonpositive. Suppose that $g = u^{4/(n-2)}\bar{g}$ is a conformal metric on $D \setminus S$ and is geodesically complete near S. Then*

$$\nabla u \in L^p(D) \quad \text{and} \quad u \in L^q(D) \tag{3-6}$$

for $p \in \left[1, \frac{n}{n-1}\right)$ and $q \in \left[1, \frac{n}{n-2}\right)$, provided that

$$R^-[g] \in L^{\frac{2n}{n+2}}(D \setminus S, g) \cap L^p(D \setminus S, g)$$

for some $p > \frac{n}{2}$.

Proof. In fact, we continue from the above, for $\phi \in C_c^\infty(D)$,

$$\begin{aligned}\left|\int_D \nabla\alpha_s(u)\cdot\nabla\phi \, d\mathrm{vol}[\bar g]\right| &= \left|\int_D (-\Delta[\bar g]\alpha_s(u)\phi)\, d\mathrm{vol}[\bar g]\right| \\ &\le \left(\int_{\partial D}\frac{\partial u}{\partial \nu}\, d\sigma + \frac{n-2}{2(n-1)}\int_{D\setminus S} R^-[g]u^{\frac{n+2}{n-2}}\, d\mathrm{vol}[\bar g]\right)\|\phi\|_{C^0(D)} \\ &\le C\left(\int_{\partial D}\frac{\partial u}{\partial \nu}\, d\sigma + \frac{n-2}{2(n-1)}\int_{D\setminus S} R^-[g]u^{\frac{n+2}{n-2}}\, d\mathrm{vol}[\bar g]\right)\|\nabla\phi\|_{L^\lambda(D)} \quad (3\text{-}7)\end{aligned}$$

for any $\lambda > n$ due to the Sobolev embedding theorem. Therefore, for any s appropriately large,

$$\|\nabla\alpha_s(u)\|_{L^p(D)} \le C \quad \text{and} \quad \|\alpha_s(u)\|_{L^q(D)} \le C$$

for some constant C and $p = \lambda' \in \left(1, \frac{n}{n-1}\right)$ and $q \in \left[1, \frac{n}{n-2}\right)$, where C is independent of s. Therefore we first have, by Fatou's lemma,

$$\|u\|_{L^q(D)} \le C$$

for some C and $q \in \left[1, \frac{n}{n-2}\right)$. Moreover, we calculate

$$\begin{aligned}|\nabla u(\phi)| &= \left|\int_D u\nabla\phi\, d\mathrm{vol}[\bar g]\right| = \left|\lim_{s\to\infty}\int_D \alpha_s(u)\nabla\phi\, d\mathrm{vol}[\bar g]\right| = \left|\lim_{s\to\infty}\int_D \alpha_s'(u)\nabla u\phi\, d\mathrm{vol}[\bar g]\right| \\ &\le \limsup_{s\to\infty}\|\alpha_s'(u)\nabla u\|_{L^p(D)}\|\phi\|_{L^\lambda} \le C\|\phi\|_{L^\lambda}. \quad (3\text{-}8)\end{aligned}$$

This implies

$$\nabla u \in L^p(D) \quad \text{and} \quad u \in L^q(D) \quad (3\text{-}9)$$

for $p \in \left[1, \frac{n}{n-1}\right)$ and $q \in \left[1, \frac{n}{n-2}\right)$, completing the proof of Lemma 3.4. □

Back to the proof of Lemma 3.3,

$$(-\Delta u)(\phi) = \int_D \nabla u\cdot\nabla\phi\, d\mathrm{vol}[\bar g] = \lim_{s\to\infty}\int_D \alpha_s'(u)\nabla u\cdot\nabla\phi\ d\mathrm{vol}[\bar g] = \lim_{s\to\infty}(-\Delta\alpha_s(u))(\phi)), \quad (3\text{-}10)$$

where the dominated convergence theorem is applied due to $\nabla u \in L^1(D)$. Thus, for $\phi \in C_c^\infty(D)$,

$$|(-\Delta u)(\phi)| \le \left(\int_{\partial D}\frac{\partial u}{\partial \nu}\, d\sigma + \frac{n-2}{2(n-1)}\int_{D\setminus S} R^-[g]u^{\frac{n+2}{n-2}}\, d\mathrm{vol}[\bar g]\right)\|\phi\|_{C^0(D)},$$

which implies that $-\Delta u$ is a Radon measure on D. To show that $-\Delta u|_S \ge 0$, we calculate, for a nonnegative function $\phi \in C_c^\infty(D)$,

$$\begin{aligned}(-\Delta u)(\phi) &= \int_D \nabla u\cdot\nabla\phi\, d\mathrm{vol}[\bar g] = \lim_{s\to\infty}\int_D \nabla\alpha_s(u)\cdot\nabla\phi\, d\mathrm{vol}[\bar g] = \lim_{s\to\infty}\int_D (-\Delta\alpha_s(u))\phi\, d\mathrm{vol}[\bar g] \\ &= \lim_{s\to\infty}\int_D\left[\alpha_s'(u)\frac{n-2}{4(n-1)}(-Ru + R[g]u^{\frac{n+2}{n-2}}) - \alpha_s''(u)|\nabla u|^2\right]\phi\, d\mathrm{vol}[\bar g] \\ &\ge -\left\{\frac{n-2}{4(n-1)}\int_{\operatorname{supp}\phi\setminus S}|-Ru + R[g]u^{\frac{n+2}{n-2}}|\, d\mathrm{vol}[\bar g]\right\}\|\phi\|_{C^0(D)} \to 0\end{aligned}$$

as $\int_{\operatorname{supp}\phi\setminus S} d\mathrm{vol}[\bar g] \to 0$ and $\|\phi\|_{C^0(D)} = 1$, which implies $-\Delta u|_S \ge 0$. □

3.3. ***Main result on the Hausdorff dimensions.*** Now we are ready to state and prove our result on the Hausdorff dimension of the singular set S, which is a significant improvement of Proposition 3.2. For the readers' convenience, we recall Theorem 1.3 from the Introduction.

Theorem 1.3. *Let $(M^n, \bar{g})$ be a compact Riemannian manifold and S be a closed subset in M^n. And let D be an open neighborhood of S. Suppose that $g = u^{4/(n-2)}\bar{g}$ is a conformal metric on $D \setminus S$ and is geodesically complete near S. Then the Hausdorff dimension satisfies*

$$\dim_{\mathscr{H}}(S) \leq \frac{n-2}{2} \tag{3-11}$$

provided $R^-[g] \in L^{2n/(n+2)}(D \setminus S, g) \cap L^p(D \setminus S, g)$ for some $p > \frac{n}{2}$. Consequently, (3-11) *holds when the scalar curvature $R[g]$ of the conformal metric g is nonnegative.*

Proof. The outline of the proof is as follows: We first show that one may assume the scalar curvature $R[\bar{g}]$ is nonpositive without loss of generality for our purpose. Then we use the Green's function to construct the integral representation of the solution to the Laplace equation. Finally we apply Lemma 3.3, Theorem 1.1, and the geodesic completeness to complete the proof.

Step I: In this step, we find a conformal change $\bar{h} = v^{4/(n-2)}\bar{g}$ such that the scalar curvature $R[\bar{h}]$ is nonpositive (or even negative) in D, based on the similar idea used in the proof of [Ma and Qing 2022, Lemma 3.1]. This is trivial if the Yamabe constant of $(M^n, \bar{g})$ is nonpositive. Otherwise, take a point $p \in M^n \setminus D$ and consider a connected sum of M^n with another compact Riemannian manifold $(M_1^n, \bar{g}_1)$ with very negative Yamabe constant in such way that the conformal structure on the connected sum $M^n \sharp M_1^n$ is unchanged in $D \subset M^n \sharp M_1^n$. Then, by [Gil-Medrano 1986, Theorem 5], the Yamabe constant of such a connected sum is negative. Therefore one easily finds a conformal metric $\bar{h} = v^{4/(n-2)}\bar{g}$ whose scalar curvature is negative in D, where $v \in C^\infty(\bar{D})$ and

$$C^{-1} \leq v \leq C \quad \text{in } \bar{D} \tag{3-12}$$

for some positive constant C. In any case, we have $g = u^{4/(n-2)}\bar{g} = \left(\frac{u}{v}\right)^{4/(n-2)}\bar{h}$ and the scalar curvature $R[\bar{h}]$ is nonpositive. In conclusion, due to (3-12), we may simply assume $R[\bar{g}]$ is nonpositive (or even negative) in D without loss of any generality for the purpose of obtaining the growth estimate like the one given in Theorem 1.1.

Step II: In this step, we use the Green's function to construct the integral representation of the solution u. In light of Lemma 3.3, we may write

$$-\Delta u = \mu \quad \text{in } D$$

for a Radon measure μ on D. Let $G(x, y)$ be the Green's function on D given by [Aubin 1982, Theorem 4.17]. Then

$$u = \int_D G(x, y)\, d\mu(y) + h$$

for a smooth function h that is harmonic in D. By [Aubin 1982, Theorem 4.17(c)], we have

$$0 < G(x, y) \leq \frac{C}{d(x, y)^{n-2}}$$

for some constant C and $x, y \in D$. We therefore arrive at, for $x \in D$,

$$u(x) \le \int_D G(x,y)\, d\mu^+ + h(x) \le C\mathscr{R}^{2,D}_{\mu^+}(x) + h(x). \tag{3-13}$$

Step III: Assume otherwise that $\dim_{\mathscr{H}}(S) = d > \frac{n-2}{2}$. From Corollary 2.12, there is a point $p \in S$ such that

$$\mathscr{R}^{2,D}_{\mu^+}(x) \le \frac{C}{d(x,p)^{n-2-d}}$$

at least for x along a short geodesic ray γ from p, which implies

$$u(x)^{\frac{2}{n-2}} \le \frac{C}{d(x,p)^{\frac{2(n-2-d)}{n-2}}} \tag{3-14}$$

at least for x along a short geodesic ray γ from p, where

$$\frac{2(n-2-d)}{n-2} = 2 - \frac{2d}{n-2} < 1$$

when $d > \frac{n-2}{2}$. Now the length of the curve γ with respect to the conformal metric $g = u^{4/(n-2)}\bar{g}$ is

$$L(\gamma, g) \le C \int_0^{l_0} \frac{1}{s^{\frac{2(n-2-d)}{n-2}}}\, ds < \infty$$

when $d > \frac{n-2}{2}$, which contradicts the geodesic completeness of the conformal metric $g = u^{4/(n-2)}\bar{g}$. □

The study of singular solutions to the scalar curvature equations started from the seminal paper [Schoen and Yau 1988] (see also [Schoen and Yau 1994, Chapter VI; Carron 2012; Schoen 1988; Mazzeo and Smale 1991; Mazzeo and Pacard 1996]) on domains of the sphere. Theorem 1.3 here can be considered as a necessary condition for the existences of singular solutions on domains in general Riemannian manifolds and compared with [Schoen and Yau 1988, Theorem 2.7; Carron 2012, Theorem C], which stated the similar result for domains in the round sphere S^n and slightly stronger curvature assumptions. Clearly [Schoen and Yau 1988, Proposition 2.4] and the quantity $d(M)$ there are not of local nature, while our approach here is very much local in nature.

4. On Q-curvature equations

We will use linear potential theory developed in Section 2 to study Q-curvature equations and prove our results on the Hausdorff dimensions of the singular sets of positive solutions of Q-curvature equations which correspond to ends of complete conformal metrics on domains of a compact Riemannian manifold.

Again we remark here that all of the results in this section hold if we assume S is compact, $D \subset M^n$ is a bounded domain that contains S, and $(M^n, \bar{g})$ is just complete. Because the possible noncompact part $M^n \setminus \overline{D}$ is not relevant for the purpose here.

4.1. ***Q-curvature equations in dimensions greater than* 4.** We now focus on (1-5) in dimensions greater than 4. We will always assume that the scalar curvature of the conformal metric $g = u^{4/(n-4)}\bar{g}$ is nonnegative. We will first prove some preliminary estimates based on discussions in the previous section. Our strategy is to consider the bi-Laplace operator as the composition of the Laplace operators. Let us write the scalar curvature equation and its consequence:

$$-\Delta u^{\frac{n-2}{n-4}} + \frac{n-2}{4(n-1)} R u^{\frac{n-2}{n-4}} = \frac{n-2}{4(n-1)} R[g] u^{\frac{n+2}{n-4}} \quad \text{in } D \setminus S, \tag{4-1}$$

$$-\Delta u = \frac{2}{n-4}\frac{|\nabla u|^2}{u} + \frac{n-4}{4(n-1)}(-Ru + R[g]u^{\frac{n}{n-4}}) \quad \text{in } D \setminus S. \tag{4-2}$$

Here, and from now on, all geometric quantities are under the background metric $\bar{g}$ unless indicated otherwise.

Lemma 4.1. *Let $(M^n, \bar{g})$ be a compact Riemannian manifold for $n \geq 5$ and S be a closed subset in M^n. And let D be an open neighborhood of S where the scalar curvature satisfies $R \leq -c_0 < 0$. Suppose that $g = u^{4/(n-4)}\bar{g}$ is a conformal metric on $D \setminus S$ with nonnegative scalar curvature $R[g] \geq 0$ and is geodesically complete near S. And suppose also that*

$$Q_4^-[g] \in L^{\frac{2n}{n+4}}(D \setminus S, g).$$

Then

$$\begin{gathered} \text{as a function on } D \setminus S, \quad -\Delta u \to +\infty \quad \text{as } x \to S, \\ \text{as a Radon measure on } D, \quad \Delta u|_S = 0, \\ \text{in fact}, \quad \Delta u \in L^p(D) \quad \text{for any } p \in \left[1, \tfrac{n}{n-2}\right). \end{gathered} \tag{4-3}$$

Proof. First, using Lemma 3.4 for $u^{(n-2)/(n-4)}$, we know that

$$u \in L^p(D) \quad \text{for } p \in \left[1, \tfrac{n}{n-4}\right). \tag{4-4}$$

Also, from Lemma 3.1, for $u^{(n-2)/(n-4)}$,

$$u(x) \to +\infty \quad \text{as } x \to S, \tag{4-5}$$

which implies, by (4-2),

$$-\Delta u \to +\infty \quad \text{as } x \to S.$$

To prove $-\Delta u$ is an integrable function in distributional sense, we first realize $-\Delta u$ is a Radon measure on D following (4-2) and Lemma 3.3. And, as a side product, we also have

$$\int_D \left[\frac{2}{n-4}\frac{|\nabla u|^2}{u} + \frac{n-4}{4(n-1)}(-Ru + R[g]u^{\frac{n}{n-4}})\right] d\,\mathrm{vol}[\bar{g}] < \infty.$$

In fact, from (4-1) and Lemma 3.3, we also know $-\Delta u^{(n-2)/(n-4)}$ is a Radon measure on D. To use this fact we calculate

$$-\Delta \alpha_s(u) = -\Delta(\alpha_s(u)^{\frac{n-2}{n-4}})^{\frac{n-4}{n-2}} = \frac{n-4}{n-2}\alpha_s(u)^{-\frac{2}{n-4}}(-\Delta \alpha_s(u)^{\frac{n-2}{n-4}}) + \frac{2}{n-4}\frac{|\nabla \alpha_s(u)|^2}{\alpha_s(u)}.$$

To prove $-\Delta u|_S = 0$, we consider

$$(-\Delta u)(\phi) = \int_D \nabla u \cdot \nabla \phi \, d\mathrm{vol}[\bar{g}],$$

where ∇u is integrable in the distributional sense directly from (3-8) and (3-9). Therefore

$$\int_D \nabla u \cdot \nabla \phi \, d\mathrm{vol}[\bar{g}] = \lim_{s\to\infty} \int_D \alpha_s'(u) \nabla u \cdot \nabla \phi \, d\mathrm{vol}[\bar{g}] = \lim_{s\to\infty} \int_D (-\Delta \alpha_s(u))\phi \, d\mathrm{vol}[\bar{g}]$$

and

$$\begin{aligned}(-\Delta u)(\phi) &= \frac{n-4}{n-2} \lim_{s\to\infty} \int_D \alpha_s(u)^{-\frac{2}{n-4}} (-\Delta \alpha_s(u)^{\frac{n-2}{n-4}})\phi \, d\mathrm{vol}[\bar{g}] + \frac{2}{n-4} \lim_{s\to\infty} \int_D \frac{|\nabla \alpha_s(u)|^2}{\alpha_s(u)} \phi \, d\mathrm{vol}[\bar{g}] \\ &= \frac{n-4}{n-2} u^{-\frac{2}{n-4}} (-\Delta u^{\frac{n-2}{n-4}})(\phi) + \frac{2}{n-4} \int_D \frac{|\nabla u|^2}{u} \phi \, d\mathrm{vol}[\bar{g}] \to 0\end{aligned}$$

as $\int_{\mathrm{supp}\,\phi \setminus S} d\mathrm{vol}[\bar{g}] \to 0$ and $\|\phi\|_{C^0(D)} \le 1$. The proof will be complete after the following L^p estimate. To get the L^p estimate, we first calculate

$$\begin{aligned}\int_{D\setminus S} Q_4^-[g] u^{\frac{n+4}{n-4}} \, d\mathrm{vol}[\bar{g}] &= \left(\int_{D\setminus S} (Q_4^-[g])^{\frac{2n}{n+4}} u^{\frac{2n}{n-4}} \, d\mathrm{vol}[\bar{g}] \right)^{\frac{n+4}{2n}} \mathrm{vol}(D)^{\frac{2n}{n-4}} \\ &= \left(\int_{D\setminus S} (Q_4^-[g])^{\frac{2n}{n+4}} \, d\mathrm{vol}[g] \right)^{\frac{n+4}{2n}} \mathrm{vol}(D)^{\frac{2n}{n-4}} < \infty. \end{aligned} \tag{4-6}$$

Then we continue to use notation in the proof of Proposition 3.2 and let

$$\alpha = \max\{u(x) : x \in \partial D\}$$

and $\alpha < \beta$. And recall

$$u_{\alpha,\beta} = \begin{cases} \beta, & x \in \Sigma_{\alpha+\beta}, \\ u(x) - \alpha, & x \in \Sigma_\alpha \setminus \Sigma_{\alpha+\beta}, \end{cases}$$

and

$$\phi_{\alpha,\beta} = \begin{cases} u_{\alpha,\beta} - \beta\eta = u - (\alpha+\beta) + \beta(1-\eta) & \text{in } \Sigma_\alpha \setminus \Sigma_{\alpha+\beta}, \\ 0 & \text{on } \partial\Sigma_\alpha, \\ 0 & \text{on } \partial\Sigma_{\alpha+\beta}, \end{cases}$$

where η is a fixed cut-off function in $C_c^\infty(\Sigma_\alpha)$ and is identically 1 in a neighborhood of S, and β is arbitrarily large. We now first multiply the Q-curvature equation (1-5) by $1-\eta$, integrate over D, apply integration by parts multiple times, and get

$$\int_D (1-\eta) Q_4^+ u^{\frac{n+4}{n-4}} \, d\mathrm{vol}[\bar{g}] \le \int_D Q_4^- u^{\frac{n+4}{n-4}} \, d\mathrm{vol}[\bar{g}] + C \tag{4-7}$$

for some constant C depending on the cut-off function η, u at ∂D, and $\|u\|_{L^1(D)}$. We then multiply both sides of the Q-curvature equation (1-5) by $\phi_{\alpha,\beta}$, integrate over $\Sigma_\alpha \setminus \Sigma_{\alpha+\beta}$, and get

$$\int_{\Sigma_\alpha\setminus\Sigma_{\alpha+\beta}} \Delta u \Delta\phi_{\alpha,\beta}\, d\mathrm{vol}[\bar g] - \int_{\partial\Sigma_\alpha} \Delta u \frac{\partial u}{\partial\nu}\, d\sigma - \int_{\partial\Sigma_{\alpha+\beta}} \frac{\partial u}{\partial \nu}\Delta u\, d\sigma$$
$$- \int_{\Sigma_\alpha\setminus\Sigma_{\alpha+\beta}} (4A(\nabla u, \nabla\phi_{\alpha,\beta}) - (n-2)J\nabla u\cdot\nabla\phi_{\alpha,\beta})\, d\mathrm{vol}[\bar g] + \frac{n-4}{2}\int_{\Sigma_\alpha\setminus\Sigma_{\alpha+\beta}} Q_4 u\phi_{\alpha,\beta}\, d\mathrm{vol}[\bar g]$$
$$= \frac{n-4}{2}\int_{\Sigma_\alpha\setminus\Sigma_{\alpha+\beta}} Q_4[g]u^{\frac{n+4}{n-4}}\phi_{\alpha,\beta}\, d\mathrm{vol}[\bar g], \tag{4-8}$$

where ν is the outward normal direction at the boundary and the boundary term $\int_{\partial\Sigma_{\alpha+\beta}} \frac{\partial u}{\partial\nu}(-\Delta u)\, d\sigma$ is nonnegative due to (4-2) and $\frac{\partial u}{\partial \nu}|_{\partial\Sigma_{\alpha+\beta}} = |\nabla u|$. Therefore,

$$\int_{\Sigma_\alpha\setminus\Sigma_{\alpha+\beta}} (\Delta u)^2\, d\mathrm{vol}[\bar g] + \beta\int_{\Sigma_\alpha\setminus\Sigma_{\alpha+\beta}} (\Delta u)(\Delta(1-\eta))\, d\mathrm{vol}[\bar g]$$
$$\le -\int_{\partial\Sigma_\alpha} (-\Delta u)\frac{\partial u}{\partial\nu}\, d\sigma + C\int_{\Sigma_\alpha\setminus\Sigma_{\alpha+\beta}} |\nabla u|^2\, d\mathrm{vol}[\bar g]$$
$$-\beta\int_{\Sigma_\alpha}\big(4A(\nabla u, \nabla(1-\eta)) - (n-2)J\nabla u\cdot\nabla(1-\eta)\big)\, d\mathrm{vol}[\bar g]$$
$$+ C\beta\int_D u\, d\mathrm{vol}[\bar g] + C\beta\int_D Q_4^-[g]u^{\frac{n+4}{n-4}}\, d\mathrm{vol}[\bar g], \tag{4-9}$$

where we use (4-7) and $|\phi| \le \beta$ in $\Sigma_\alpha \setminus \Sigma_{\alpha+\beta}$. After applying integration by parts, we get

$$\int_{\Sigma_\alpha\setminus\Sigma_{\alpha+\beta}} |\Delta u|^2\, d\mathrm{vol}[\bar g] \le C\int_{\Sigma_\alpha\setminus\Sigma_{\alpha+\beta}} |\nabla u|^2\, d\mathrm{vol}[\bar g] + C\beta \tag{4-10}$$

for some constant C depending on the cut-off function η, u at $\partial\Sigma_\alpha$, and $\|u\|_{L^1(D)}$, because

$$\int_{\Sigma_\alpha\setminus\Sigma_{\alpha+\beta}} \Delta u\Delta\eta\, d\mathrm{vol}[\bar g] = \int_{\Sigma_\alpha\setminus\Sigma_{\alpha+\beta}} u\Delta^2\eta\, d\mathrm{vol}[\bar g]$$

and similarly we may unload all derivatives from u by integration by parts for the other terms in the above (4-9). Now, to get an a priori estimate, we calculate

$$\int_{\Sigma_\alpha\setminus\Sigma_{\alpha+\beta}} |\nabla u|^2\, d\mathrm{vol}[\bar g] \le \frac{1}{(n-4)C}\int_{\Sigma_\alpha\setminus\Sigma_{\alpha+\beta}} \frac{|\nabla u|^4}{u^2}\, d\mathrm{vol}[\bar g] + \frac{(n-4)C}{4}\int_{\Sigma_\alpha\setminus\Sigma_{\alpha+\beta}} u^2\, d\mathrm{vol}[\bar g]$$
$$\le \frac{1}{2C}\int_{\Sigma_\alpha\setminus\Sigma_{\alpha+\beta}} |\Delta u|^2\, d\mathrm{vol}[\bar g] + \frac{(n-4)C}{4}(\alpha+\beta)\int_D u\, d\mathrm{vol}[\bar g],$$

due to (4-2), which implies, from (4-10),

$$\int_{\Sigma_\alpha\setminus\Sigma_{\alpha+\beta}} |\Delta u|^2\, d\mathrm{vol}[\bar g] \le C(\alpha+\beta). \tag{4-11}$$

We claim that (4-11) implies

$$\Delta u \in L^p(D) \tag{4-12}$$

for all $p \in \left[1, \frac{n}{n-2}\right)$. To prove (4-12), we first derive from (4-11)

$$2^{-i} \int_{\Sigma_{2^{i-1}} \setminus \Sigma_{2^i}} |\Delta u|^2 \, d\mathrm{vol}[\bar{g}] \le C$$

for $i \ge i_0$ large, which implies

$$\int_{\Sigma_{2^{i-1}} \setminus \Sigma_{2^i}} \frac{|\Delta u|^2}{u} \, d\mathrm{vol}[\bar{g}] \le 2C$$

and, for $s > 0$ appropriately small for any $p \in \left[1, \frac{n}{n-2}\right)$,

$$\int_{\Sigma_{2^{i_0-1}} \setminus S} \frac{|\Delta u|^2}{u^{1+s}} \, d\mathrm{vol}[\bar{g}] = \sum_{i=i_0}^{\infty} \int_{\Sigma_{2^{i-1}} \setminus \Sigma_{2^i}} \frac{|\Delta u|^2}{u^{1+s}} \, d\mathrm{vol}[\bar{g}] \le \sum_{i=i_0}^{\infty} 2^{s(-i+1)} \int_{\Sigma_{2^{i-1}} \setminus \Sigma_{2^i}} \frac{|\Delta u|^2}{u} \, d\mathrm{vol}[\bar{g}] < \infty.$$

Thus

$$\int_{D \setminus S} |\Delta u|^p \, d\mathrm{vol}[\bar{g}] \le \left(\int_{D \setminus S} \frac{|\Delta u|^2}{u^{1+s}} \, d\mathrm{vol}[\bar{g}] \right)^{\frac{p}{2}} \left(\int_{D \setminus S} u^{\frac{(1+s)p}{2-p}} \, d\mathrm{vol}[\bar{g}] \right)^{1-\frac{p}{2}} < \infty,$$

where

$$\frac{(1+s)p}{2-p} < \frac{n}{n-4}. \qquad \square$$

Corollary 4.2. *Under the same assumptions as in Lemma 4.1 we have*

$$\dim_{\mathscr{H}}(S) \le n - 4.$$

Proof. Consequently from (4-2) and (4-11), we have

$$\begin{aligned} \int_{\Sigma_\alpha \setminus \Sigma_{\alpha+\beta}} \left|\nabla \frac{u_{\alpha,\beta}}{\beta}\right|^4 d\mathrm{vol}[\bar{g}] &\le \frac{(\alpha+\beta)^2}{\beta^4} \int_{\Sigma_\alpha \setminus \Sigma_{\alpha+\beta}} \frac{|\nabla u|^4}{u^2} \, d\mathrm{vol}[\bar{g}] \\ &\le \frac{(\alpha+\beta)^2}{\beta^4} \int_{\Sigma_\alpha \setminus \Sigma_{\alpha+\beta}} |\Delta u|^2 \, d\mathrm{vol}[\bar{g}] \le C \frac{(\alpha+\beta)^3}{\beta^4} \end{aligned}$$

for some α appropriately large and $\beta \to \infty$, which leads to $\mathrm{Cap}_4(S) = 0$ and completes the proof similar to the proof of Proposition 3.2 (see [Adams and Meyers 1972; Schoen and Yau 1994, Theorem 2.10 in Chapter VI]). $\square$

Lemma 4.3. *Let $(M^n, \bar{g})$ be a compact Riemannian manifold for $n \ge 5$ and S be a closed subset in M^n. And let D be an open neighborhood of S where the scalar curvature $R[\bar{g}] \le -c_0 < 0$. Suppose that $g = u^{4/(n-4)} \bar{g}$ is a conformal metric on $D \setminus S$ with nonnegative scalar curvature $R[g] \ge 0$ and is geodesically complete near S. And suppose also that*

$$Q_4^-[g] \in L^{\frac{2n}{n+4}}(D \setminus S, g).$$

Then $\Delta^2 u$ is a Radon measure on D and $\Delta^2 u|_S \ge 0$.

Proof. Let $v = -\Delta u$. We will follow the proof of Lemma 3.3 to show that $-\Delta v$ is a Radon measure on D using Lemma 4.1. We continue to use the notation from the proof of Lemma 3.3. We calculate

$$-\Delta \alpha_s(v) = \alpha'(v)(-\Delta v) - \alpha''(v) |\nabla v|^2,$$

where, by the Q-curvature equation (1-5), we have

$$-\Delta v = -\operatorname{div}(4A(\nabla u) - (n-2)J\nabla u) - \frac{n-4}{2}Q_4 u + Q_4[g]u^{\frac{n+4}{n-4}} \quad \text{in } D\setminus S$$

and

$$-\Delta\alpha_s(v) = -\alpha_s''(v)|\nabla v|^2 + \alpha'(v)(-\operatorname{div}(4A(\nabla u) - (n-2)J\nabla u) - \frac{n-4}{2}Q_4 u + Q_4[g]u^{\frac{n+4}{n-4}})$$

in D. In light of Lemma 4.1, terms in the right-hand side of the above equation are all integrable except $-\alpha''(v)|\nabla v|^2 + Q_4^+[g]u^{(n+4)/(n-4)}$. Therefore the argument in the proof of Lemma 3.3 works from this point and completes the proof. □

We now are ready to state and prove our main results for Q-curvature equations in dimensions greater than 4. For this, we recall Theorem 1.4 from the Introduction.

Theorem 1.4. *Let $(M^n, \bar{g})$ be a compact Riemannian manifold for $n \geq 5$ and S be a closed subset in M^n. And let D be an open neighborhood of S. Suppose that $g = u^{4/(n-4)}\bar{g}$ is a conformal metric on $D\setminus S$ with nonnegative scalar curvature $R[g] \geq 0$ and is geodesically complete near S. And suppose also that*

$$Q_4^-[g] \in L^{\frac{2n}{n+4}}(D\setminus S, g).$$

Then

$$\dim_{\mathscr{H}}(S) \leq \frac{n-4}{2}.$$

Proof. In light of Step I in the proof of Theorem 1.3, we may assume the scalar curvature $R \leq -c_0 < 0$ for some c_0 without loss of any generality. Then we use Lemmas 4.1 and 4.3 and conclude that

$$\Delta^2 u = \mu$$

for a Radon measure μ on D. We use [Aubin 1982, Theorem 4.7] first to write

$$-\Delta u = \int_D G(x,y)\,d\mu + h(x)$$

for some harmonic function $h(x)$, where $G(x,y)$ is the Green's function for $-\Delta$. Then we have

$$u(x) = \int_D G(x,z)\int_D G(z,y)\,d\mu(y)\,d\mathrm{vol}[\bar{g}](z) + b(x),$$

where $b(x)$ is biharmonic, where

$$\int_D G(x,z)\int_D G(z,y)\,d\mu(y)\,d\mathrm{vol}[\bar{g}](z) = \int_D (\int_D G(x,z)G(z,y)\,d\mathrm{vol}[\bar{g}](z))\,d\mu(y)$$

and

$$0 < \int_D G(x,z)G(z,y)\,d\mathrm{vol}[\bar{g}](z) \leq \frac{C}{d(x,y)^{n-4}}$$

for a constant C and $n \geq 5$ due to [Aubin 1982, Proposition 4.12], which can be easily proven to be available for bounded domains in Riemannian manifolds. Hence

$$u(x) \leq \mathscr{R}^{4,D}_{\mu^+}(x) + b(x).$$

From now on, using the same argument of the proof of Theorem 1.3, based on Theorem 1.1 for $\alpha = 4$ and $n \geq 5$, we conclude

$$\dim_{\mathscr{H}}(S) \leq \frac{n-4}{2}$$

and finish the proof. □

There have been a lot of works on the study of singular solutions to Q-curvature equations on manifolds of dimension greater than 4, notably [Qing and Raske 2006a; 2006b; Chang et al. 2004; González et al. 2012], for example. Theorem 1.4, for instance, is an improvement of [Chang et al. 2004, Theorem 1.2] in terms of curvature conditions. And the approach here is different from [Chang et al. 2004].

4.2. *Q-curvature equations in dimension* 4. We will now study the Q-curvature equation (1-6). Our approach here in principle is similar to that in the previous subsection but different in calculations and details. We will always assume that the scalar curvature of the conformal metric $g = e^{2u}\bar{g}$ is nonnegative. We will first derive some preliminary estimates from the scalar curvature equation for $w = e^u$ and the Q-curvature equation (1-6) for u. Let us write the scalar curvature equation for e^u

$$-\Delta e^u = \tfrac{1}{6}(-Re^u + R[g]e^{3u}) \quad \text{in } D \setminus S \tag{4-13}$$

and consequently,

$$-\Delta u = |\nabla u|^2 + \tfrac{1}{6}(-R + R[g]e^{2u}) \quad \text{in } D \setminus S. \tag{4-14}$$

Lemma 4.4. *Let $(M^4, \bar{g})$ be a compact Riemannian manifold and S be a closed subset in M^n. And let D be an open neighborhood of S where the scalar curvature $R \leq 0$. Suppose that $g = e^{2u}\bar{g}$ is a conformal metric on $D \setminus S$ with nonnegative scalar curvature $R[g] \geq 0$ and is geodesically complete near S. And suppose also that*

$$Q_4^-[g] \in L^1(D \setminus S, g).$$

Then

$$\begin{aligned} &\textit{as a Radon measure, } -\Delta u|_S = 0,\\ &\textit{in fact, } \Delta u \in L^p(D) \textit{ for any } p \in \left[1, \tfrac{4}{3}\right). \end{aligned} \tag{4-15}$$

Proof. First, by Lemma 3.1 for e^u, we have

$$u(x) \to \infty \quad \text{as } x \to S.$$

Then, by the proof of Lemma 3.3 and (4-14), we know that

- $-\Delta u$ is a Radon measure on D,
- $\nabla u \in L^p(D)$ for any $p \in \left[1, \tfrac{4}{3}\right)$ and $u \in L^p(D)$ for any $p \in [1, 2)$,
- $|\nabla u|^2 + \tfrac{1}{6}(-R + R[g]e^{2u}) \in L^1(D)$.

Using the same argument as in the proof of Lemma 4.1 we can prove that $-\Delta u|_S = 0$ as a Radon measure. Also, for the L^p estimate, following the proof of Lemma 4.1, we multiply both sides of (1-6) by $1 - \eta$ and get

$$\int_D (1-\eta) Q_4^+ e^{4u} \, d\text{vol}[\bar{g}] \leq \int_D Q_4^-[g] \, d\text{vol}[g] + C.$$

Next we multiply both sides of (1-6) by $\phi_{\alpha,\beta}$ and integrate,

$$\int_{\Sigma_\alpha\setminus\Sigma_{\alpha+\beta}} \Delta u\Delta\phi_{\alpha,\beta}\, d\mathrm{vol}[\bar g] - \int_{\partial\Sigma_\alpha} \Delta u \frac{\partial\phi_{\alpha,\beta}}{\partial\nu}\, d\sigma - \int_{\partial\Sigma_{\alpha+\beta}} \Delta u \frac{\partial\phi_{\alpha,\beta}}{\partial\nu}\, d\sigma \\ - \int_{\Sigma_\alpha\setminus\Sigma_{\alpha+\beta}} (4A(\nabla u, \nabla\phi_\alpha) - 2J\nabla u\cdot\nabla\phi_{\alpha,\beta}\, d\mathrm{vol}[\bar g] + \int_{\Sigma_\alpha\setminus\Sigma_{\alpha+\beta}} Q_4\phi_{\alpha,\beta}\, d\mathrm{vol}[\bar g] \\ = \int_{\Sigma_\alpha\setminus\Sigma_{\alpha+\beta}} Q_4[g]e^{4u}\phi_{\alpha,\beta}\, d\mathrm{vol}[\bar g],$$

and, again, the boundary term at $\partial\Sigma_{\alpha+\beta}$ is with the sign in our favor, thanks to (4-14) and $\frac{\partial u}{\partial\nu}|_{\Sigma_{\alpha+\beta}} = |\nabla u|$ for the outward normal ν of $\Sigma_\alpha \setminus \Sigma_{\alpha+\beta}$. Similar to the estimates in the proof of Lemma 4.1, we get

$$\int_{\Sigma_\alpha\setminus\Sigma_{\alpha+\beta}} |\Delta u|^2\, d\mathrm{vol}[\bar g] \le C\int_{\Sigma_\alpha\setminus\Sigma_{\alpha+\beta}} |\nabla u|^2\, d\mathrm{vol}[\bar g] + C\beta.$$

And we handle $\int_{\Sigma_\alpha\setminus\Sigma_{\alpha+\beta}} |\nabla u|^2\, d\mathrm{vol}[\bar g]$ much as before,

$$\int_{\Sigma_\alpha\setminus\Sigma_{\alpha+\beta}} |\nabla u|^2\, d\mathrm{vol}[\bar g] \le \frac{1}{2C}\int_{\Sigma_\alpha\setminus\Sigma_{\alpha+\beta}} |\nabla u|^4\, d\mathrm{vol}[\bar g] + C \le \frac{1}{2C}\int_{\Sigma_\alpha\setminus\Sigma_{\alpha+\beta}} |\Delta u|^2\, d\mathrm{vol}[\bar g] + C$$

due to (4-14). Therefore

$$\int_{\Sigma_\alpha\setminus\Sigma_{\alpha+\beta}} |\Delta u|^2\, d\mathrm{vol}[\bar g] \le C\beta. \tag{4-16}$$

Now, using the same idea as in the proof of Lemma 4.1, we rewrite (4-16) as

$$\int_{\Sigma_{2i-1}\setminus\Sigma_{2i}} \frac{|\Delta u|^2}{u}\, d\mathrm{vol}[\bar g] \le C$$

and, for $s > 0$ appropriately small for any $p \in \left[1, \frac{4}{3}\right)$, we derive

$$\int_{D\setminus S} \frac{|\Delta u|^2}{u^{1+s}}\, d\mathrm{vol}[\bar g] \le C,$$

which implies

$$\int_{D\setminus S} |\Delta u|^p\, d\mathrm{vol}[\bar g] \le \left(\int_{D\setminus S} \frac{|\Delta u|^2}{u^{1+s}}\, d\mathrm{vol}[\bar g]\right)^{\frac{p}{2}} \left(\int_{D\setminus S} u^{\frac{(1+s)p}{2-p}}\, d\mathrm{vol}[\bar g]\right)^{1-\frac{p}{2}}$$

when

$$\frac{(1+s)p}{2-p} < 2. \qquad \square$$

Corollary 4.5. *Under the assumptions of Lemma 4.4, we know the singular set S is of zero Hausdorff dimension.*

Proof. From (4-14) and (4-16) in the above we have

$$\int_{\Sigma_\alpha\setminus\Sigma_{\alpha+\beta}} |\nabla \frac{u_{\alpha,\beta}}{\beta}|^4\, d\mathrm{vol}[\bar g] \le C\beta^{-3}$$

for some α appropriately large and $\beta \to \infty$, which leads to $\mathrm{Cap}_4(S, D) = 0$ and completes the proof as in Proposition 3.2 (see [Adams and Meyers 1972; Schoen and Yau 1994, Theorem 2.10 in Chapter VI]). ☐

What follows is to go beyond that S is of zero Hausdorff dimension. We now are ready to state and prove our main result on the finiteness of singularities for the Q-curvature equation in dimension 4. This is inspired by [Cohn-Vossen 1935; Huber 1957; Arsove and Huber 1973; Ma and Qing 2021; 2022]. We recall Theorem 1.5 from the Introduction.

Theorem 1.5. *Let $(M^4, \bar{g})$ be a compact Riemannian manifold and S be a closed subset in M^n. And let D be an open neighborhood of S. Suppose that $g = e^{2u}\bar{g}$ is a conformal metric on $D \setminus S$ with nonnegative scalar curvature $R[g] \geq 0$ and is geodesically complete near S. And suppose that*

$$\int_D Q_4^-[g]\, d\mathrm{vol}[g] < \infty.$$

Then S consists of only finitely many points.

Proof. As before, we use the argument in Step I on Theorem 1.3 to assume that the scalar curvature of the background metric $\bar{g}$ is less than a negative number, i.e., $R \leq -c_0 < 0$, without loss of any generality for our purpose. Let

$$v = -\Delta u + u$$

and claim $-\Delta v$ is a Radon measure on D with $-\Delta v|_S \geq 0$. Let us start with

$$-\Delta v = \Delta^2 u - \Delta u = -\operatorname{div}(4A(\nabla u) - 2J\nabla u) - Q_4 + Q_4[g]e^{4u} - \Delta u. \tag{4-17}$$

By Lemma 4.4 and (4-14), we know

- $v(x) \to \infty$ as $x \to S$,
- all terms in the right side of (4-17) except $Q_4^+[g]e^{4u}$ are integrable.

Therefore, following the same argument as in the proof of Lemma 4.3, the claim is proven. Obviously, the same conclusion holds for $\Delta^2 u = -\Delta v + \Delta u$ from $-\Delta v$ and what we know about Δu in Lemma 4.4. Thus we let

$$\Delta^2 u = \mu$$

for a Radon measure on D with $\Delta^2 u|_S \geq 0$. Like in the proof of Theorem 1.4, we first write

$$-\Delta u(x) = \int_D G(x, y)\, d\mu(y) + h(x)$$

by [Aubin 1982, Theorem 4.17], where $h(x)$ is a harmonic function. Then we write

$$u(x) = \int_D G(x, z) \int_D G(z, y)\, d\mu(y)\, d\mathrm{vol}[\bar{g}](z) + b(x),$$

where $b(x)$ is a biharmonic function and, due to [Aubin 1982, Proposition 4.12],

$$\int_D G(x, z)G(z, y)\, d\mathrm{vol}[\bar{g}](z) \leq C\left(1 + \log \frac{1}{d(x, y)}\right)$$

for some constant C in dimension 4, where [Aubin 1982, Proposition 4.12] can be easily made available on bounded domains in manifolds. Therefore

$$u(x) \leq C\mathscr{R}^{4,D}_{\mu^+}(x) + b(x).$$

Applying Theorem 2.13, we have

$$\lim_{x\to p \text{ and } x\notin E} \frac{u(x)}{\log(1/d(x,p))} \leq C\mu^+(\{p\}) = C\mu(\{p\}),$$

where E is a subset that is n-thin at p. Next, in light of Corollary 2.12, we conclude that $\mu(\{p\}) \geq \frac{1}{C}$ for each $p \in S$ since otherwise $u(x) \leq m \log(1/d(x,p))$ for some $m < 1$, which violates the completeness of the metric g near S, because the $\bar{g}$-geodesic mentioned in Theorem 1.2, which avoids E, would have finite length with respect to the metric g. So we conclude that S can only have finitely many points. □

Theorem 1.4 is a significant improvement of [Chang et al. 2000a, Theorem 2] (please see also [Carron and Herzlich 2002; Chang et al. 2000b; Ma and Qing 2022]).

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Received 14 Nov 2022. Revised 9 Aug 2023. Accepted 21 Dec 2023.

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ANALYSIS & PDE

Volume 18 No. 3 2025