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ROTATING SPIRALS IN SEGREGATED REACTION-DIFFUSION SYSTEMS

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We give a complete characterization of the boundary traces φ_i $(i = 1, \dots, K)$ supporting spiraling waves, rotating with a given angular speed ω, which appear as singular limits of competition-diffusion systems of the type

$$\begin{cases} \partial_t u_i - \Delta u_i = \mu u_i - \beta u_i \sum_{j \neq i} a_{ij} u_j & \text{in } \Omega \times \mathbb{R}^+, \\ u_i = \varphi_i & \text{on } \partial\Omega \times \mathbb{R}^+, \\ u_i(\boldsymbol{x}, 0) = u_{i,0}(\boldsymbol{x}) & \text{for } \boldsymbol{x} \in \Omega, \end{cases}$$

as $\beta \to +\infty$. Here Ω is a rotationally invariant planar set, and $a_{ij} > 0$ for every i and j. We tackle also the homogeneous Dirichlet and Neumann boundary conditions, as well as entire solutions in the plane. As a byproduct of our analysis, we detect explicit families of eternal, entire solutions of the pure heat equation, parametrized by $\omega \in \mathbb{R}$, which reduce to homogeneous harmonic polynomials for $\omega = 0$.

1. Introduction

This paper deals with existence, uniqueness and qualitative properties of rotating spiraling waves arising in the singular limit of reaction-diffusion systems, when the interspecific competition rates become infinite. More precisely, we are concerned with the singular limits, as $\beta \to +\infty$, of the following model problem involving $K \geq 3$ species competing in the plane:

$$\begin{cases} \partial_t u_i - \Delta u_i = f_i(u_i) - \beta u_i \sum_{j \neq i} a_{ij} u_j & \text{in } \Omega \times \mathbb{R}^+, \\ u_i = \varphi_i & \text{on } \partial\Omega \times \mathbb{R}^+, \\ u_i(\boldsymbol{x}, 0) = u_{i,0}(\boldsymbol{x}) & \text{for } \boldsymbol{x} \in \Omega. \end{cases} \tag{1}$$

Here $\Omega \subset \mathbb{R}^2$ has a smooth boundary and $u_i = u_i(\boldsymbol{x}, t)$ represents the density of the i-th species $(1 \leq i \leq K)$, whose internal dynamic is described by the function f_i. The positive numbers βa_{ij} account for the interspecific competition rates, so that the interaction has a repulsive character. The boundary data φ_i are positive and segregated, i.e., $\varphi_i \varphi_j \equiv 0$ for $j \neq i$.

As already mentioned, we are concerned with the limit case of strong competition; that is, when the parameter β goes to $+\infty$ while the positive coefficients a_{ij} remain fixed. In this case it is known that the densities u_i *segregate*, in the sense that they converge uniformly to limit densities satisfying $u_i u_j \equiv 0$ for $j \neq i$; hence a pattern arises, and the common nodal set (where all densities vanish simultaneously) can be considered as a free boundary; see [Caffarelli et al. 2009; Conti et al. 2005a; 2005b; Wei and Weth 2008] for steady states and [Dancer et al. 2012a; 2012b; Dancer and Zhang 2002; Wang and Zhang 2010]

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for time-varying solutions. For such segregated limit profiles, the interface conditions are expressed by two systems of differential inequalities which play a fundamental role in our work:

$$\partial_t u_i - \Delta u_i \le f_i(u_i), \quad \partial_t \hat{u}_i - \Delta \hat{u}_i \ge \hat{f}_i(\hat{u}_i), \tag{2}$$

where the differential inequalities are understood in the variational sense, and

$$\hat{u}_i = u_i - \sum_{j\neq i} \frac{a_{ij}}{a_{ji}} u_j, \quad \hat{f}_i(\hat{u}_i) = f(u_i) - \sum_{j\neq i} \frac{a_{ij}}{a_{ji}} f(u_j). \tag{3}$$

These inequalities incorporate the transmission conditions at the free boundary, that is the closure of the interfaces $\partial\{u_i > 0\} \cap \partial\{u_j > 0\}$, which separate the supports of u_i and u_j at any fixed time t.

For planar stationary solutions, the structure of the free boundary has been the object of several papers. In the case of symmetric interactions ($a_{ij} = a_{ji}$ for every i and j), it is composed by a regular part, a collection of smooth curves, meeting at a locally finite number of (singular) clustering points, with definite tangents; see [Caffarelli et al. 2009; Conti et al. 2005a; 2006; Helffer et al. 2009]. On the other hand, the asymmetric case has been treated only more recently in [Terracini et al. 2019]: while the topological structure of the free boundary is analogous to the symmetric case (smooth curves meeting at isolated singular points), the geometric description differs strongly in a neighborhood of each singular point, where the nodal lines meet with logarithmic spiraling asymptotics.

Going back to time-dependent systems, rotating spiraling patterns have been detected numerically in the case of three competing populations in [Murakawa and Ninomiya 2011]. Driven by this phenomenology, in this paper we seek rotating spirals, that is rigidly rotating waves which are steady states of (2) in a reference frame spinning with frequency ω; such solutions satisfy $\partial_t u_i = \omega \partial_\theta u_i$ in a disk, subject to boundary conditions which are prescribed in the rotating frame, and exhibit spiraling interfaces near the origin. Hence, in comparison with the literature, our work tackles the segregation problem from a new perspective, that is the existence of limit segregated profiles satisfying additional qualitative properties or shadowing some given shapes. On the other hand, the literature on other aspects of segregation triggered by strong competition, starting from pioneering works by Dancer and Du [1995a; 1995b], is now very vast, and it is impossible to give a complete account of it here; besides the papers mentioned above, we mention a few more recent ones such as [Arakelyan and Bozorgnia 2017; Berestycki and Zilio 2018; 2019; Lanzara and Montefusco 2019; 2021; Verzini and Zilio 2014].

The rotating spiral shapes we investigate evoke some other typical examples of spatiotemporal patterns arising in reaction-diffusion systems in planar domains: the spiral waves. In the simplest case, spiral waves are stationary waves in a rotating frame, while modulated spiraling waves may emanate from rigidly rotating ones in some circumstances. Such waves arise in different models and appear in the literature about reaction-diffusion systems in contexts different from singular perturbation problems; see, e.g., [Sandstede et al. 1997; Sandstede and Scheel 2007; 2023]. As far as we know, this is the first study on spiraling rotating waves for segregated limit profiles of competition-diffusion systems. We also mention that spiraling interfaces arise in free boundary problems in entirely different contexts [Allen and Kriventsov 2020].

To construct eternal solutions of spiraling-type to the limit system (2), in this paper we deal with suitable classes of reactions f_i and boundary conditions. More precisely, let us consider identical, linear reactions in the unit ball (centered at $\mathbf{0}$):

$$\Omega = B, \quad f_i(u) = \mu u \text{ for some } \mu \in \mathbb{R}.$$

We insert into (2) the rotating wave ansatz

$$u_i(\boldsymbol{x}, t) = u_i(\mathcal{R}_{\omega t}\boldsymbol{x}),$$

where

$$\mathcal{R}_{\omega t} = \begin{pmatrix} \cos(\omega t) & -\sin(\omega t) \\ \sin(\omega t) & \cos(\omega t) \end{pmatrix}$$

is the rotation matrix of angular speed ω, and we obtain the stationary system of inequalities

$$\begin{cases} -\Delta u_i + \omega \boldsymbol{x}^{\perp} \cdot \nabla u_i \leq \mu u_i & \text{in } B, \\ -\Delta \hat{u}_i + \omega \boldsymbol{x}^{\perp} \cdot \nabla \hat{u}_i \geq \mu \hat{u}_i & \text{in } B, \\ u_i \cdot u_j = 0 & \text{for } i \neq j, \end{cases} \tag{4}$$

where $\boldsymbol{x}^{\perp} = \mathcal{R}_{\pi/2}\boldsymbol{x}$ and $\hat{u}_i$ is defined in (3). It is worth noting that, despite appearances, this system is strongly nonlinear and has to be tackled as a free boundary problem.

We are interested in solutions of (4) whose nodal set consists in smooth arcs, emanating from ∂B and spiraling towards $\mathbf{0}$, which is the unique singular point of the free boundary. In this way, each arc is a smooth interface between two adjacent densities, and the origin is the only point with higher multiplicity (see Figure 1). In this framework we provide a complete description of the nonhomogeneous Dirichlet problem associated with (4).

Let us consider a K-tuple $(\varphi_1, \dots, \varphi_K)$ of segregated boundary traces. Precisely, we assume that, for every $i = 1, \dots, K$,

$$\begin{cases} \varphi_i \in C^{0,1}(\partial B), \quad \varphi_i \geq 0, \\ \{\boldsymbol{x} : \varphi_i(\boldsymbol{x}) > 0\} \text{ are connected, nonempty and disjoint arcs,} \\ \bigcup_i \operatorname{supp} \varphi_i = \partial B. \end{cases} \tag{5}$$

Up to relabeling, we can assume that the traces φ_i are labeled in counterclockwise order.

In general, it is not reasonable to expect that any choice of the boundary data provides a solution of (4) with a unique singular point at $\mathbf{0}$. Indeed, we show that this happens exactly for an explicit subset having codimension $K-1$ in the space of traces. Let $s = (s_1, \dots, s_K) \in \mathbb{R}^K$, with $s_i > 0$ for all i, and let us consider the class of functions

$$\mathcal{S}_{\text{rot}} = \{U = (u_1, \dots, u_K) \in (H^1(B))^K : u_i \geq 0 \text{ satisfy (4)}, \ u_i = s_i \varphi_i \text{ on } \partial B\}. \tag{6}$$

To state our main result we introduce the parameter

$$\alpha = \frac{1}{2\pi} \ln\left(\frac{a_{12}}{a_{21}} \cdot \frac{a_{23}}{a_{32}} \cdots \frac{a_{K1}}{a_{1K}} \right), \tag{7}$$

which synthesizes the asymmetry of the coefficients a_{ij}; see [Terracini et al. 2019] for more details.

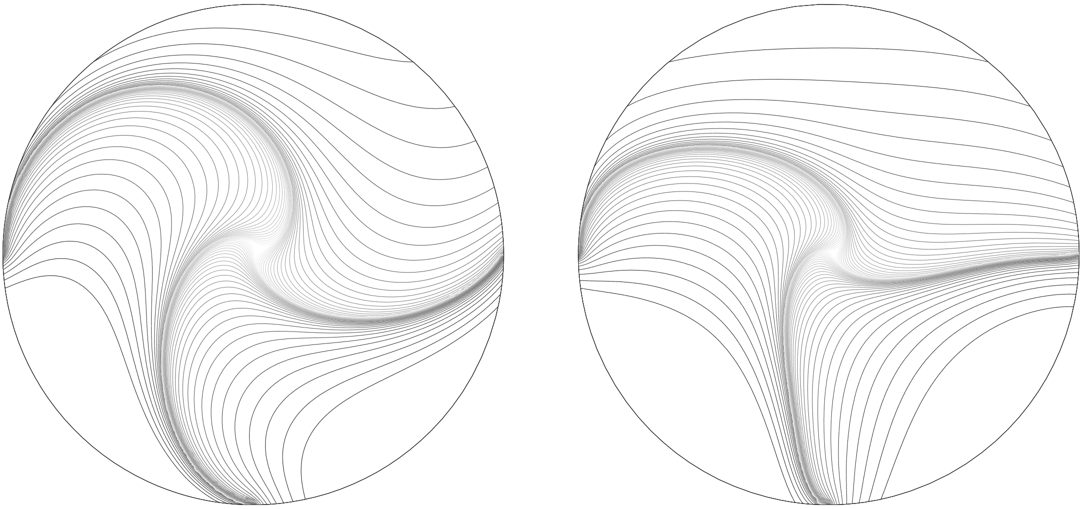

Figure 1. Contour lines of a numerical simulation (obtained in FreeFem++ [Hecht 2012]) in the case of $K = 3$ densities, with asymmetric competition such that $a_{12}/a_{21} = a_{23}/a_{32} = a_{31}/a_{13} = 10$, and reaction term $\mu = 0$. The angular velocity is $\omega = 3$ for the image on the left (counterclockwise spin) and $\omega = -3$ for the image on the right (clockwise spin). In both cases, we obtain a unique singular point at the center of the circle by choosing the same boundary conditions, which satisfy the necessary and sufficient conditions in Theorem 1.1; see (10). The rotation affects the shape of the spirals but not their asymptotic behavior close to the center. This is part of the content of Theorem 1.1.

Our main result is the following theorem.

Theorem 1.1. *Let $K \geq 3$, $a_{ij} > 0$ and $\omega \in \mathbb{R}$. Assume that $\mu < \pi^2$ and $(\varphi_1, \dots, \varphi_K)$ satisfies (5). There exists*

$$\bar{s} = (\bar{s}_1, \dots, \bar{s}_K) \in \mathbb{R}^K,$$

independent of μ and ω, with $\bar{s}_i > 0$ for all i, such that:

(1) *If $s = t\bar{s}$ for some $t > 0$, then $\mathcal{S}_{\mathrm{rot}}$ contains an element with a unique singular point at $\mathbf{0}$. Moreover, such an element is unique and, defining $\mathcal{U}$ as a suitable linear combination of its components, we have*

$$\mathcal{U}(r\cos\vartheta, r\sin\vartheta) = Ar^\gamma \cos\left(\frac{K}{2}\vartheta - \alpha \ln r\right) + o(r^\gamma) \quad \text{as } r \to 0, \tag{8}$$

where

$$\gamma = \frac{K}{2} + \frac{2\alpha^2}{K} \quad \text{and} \quad 0 < A_0 \leq A(\boldsymbol{x}) \leq A_1.$$

(2) *If $s \neq t\bar{s}$ for every $t > 0$, then $\mathcal{S}_{\mathrm{rot}}$ contains no element with a unique singular point at $\mathbf{0}$.*

Corollary 1.2. *Under the assumptions of the above theorem, if the problem is invariant under a rotation of* $2\pi/K$, *i.e.,*

$$\varphi_{i+1}(\boldsymbol{x}) = \varphi_1(\mathcal{R}_{2\pi i/K}\boldsymbol{x}) \quad \text{and} \quad \frac{a_{i(i+1)}}{a_{(i+1)i}} = \frac{a_{K1}}{a_{1K}} \tag{9}$$

for every i, *then*

$$\bar{s} = (1, 1, \dots, 1).$$

Remark 1.3. Notice that the asymptotic expansion (8) implies that the free boundary, near the singular point $\mathbf{0}$, is the union of K equidistributed logarithmic spirals, as long as $\alpha \neq 0$. On the other hand, in the case $\alpha = 0$, we obtain that the interfaces enter the origin with a definite angle. In particular, this holds true in the symmetric case $a_{ij} = a_{ji}$ for every $j \neq i$.

Remark 1.4. In this work, we normalize the radius of the disc, taking the slope of the reaction term μ at zero as a parameter. If we wish to work in a ball of radius R then we need $\mu < \pi^2/R^2$, as seen with a simple scaling.

Remark 1.5. A natural question concerns the dynamical stability of the solutions above. From this point of view, the study of the linearized problem of (1), due to the presence of the large parameter β, does not seem a viable path. This leaves open the problem of stability, for the moment, although numerical simulations for (1), with logistic reactions and β large, suggest stability for some specific angular velocity ω.

We shall adopt a constructive point of view, building the solution by superposition of fundamental elementary modes. The dependence of such building blocks on the parameter ω and μ shows the presence of resonances at exceptional values; see Section 6 for further details. As a byproduct of the analysis of resonances, we will prove the following results.

Theorem 1.6 (homogeneous boundary conditions). *Let* $K \geq 3$ *and* $a_{ij} > 0$. *If* (μ, ω) *belongs to a suitable discrete set then there exists a nontrivial element of* $\mathcal{S}_{\text{rot}}$ *with null traces. Analogous results hold for homogenous Neumann or Robin boundary conditions.*

Theorem 1.7 (entire solutions). *Let* $K \geq 3$ *and* $a_{ij} > 0$. *For almost every* (μ, ω), *there exists an entire solution of* (4) *in* $\mathbb{R}^2$.

In the above results, the conditions on (μ, ω) are explicit in terms of the zero set of suitable analytic functions in the complex plane. In both cases, the solutions are explicit in terms of trigonometric and Bessel's functions. This allows us to study the structure of the free boundary of the entire solutions far away from the origin. It turns out that, at least when $\omega \neq 0$, also at infinity the free boundary consists in equidistributed spirals, now of arithmetic type. We refer to Lemma 6.7 and Remark 6.8 for further details.

Remark 1.8. In the particular case $\alpha = \mu = 0$, we obtain that the entire solution found in Theorem 1.7 is related to the nodal components of a smooth rotating solution of the pure heat equation. Let $\omega > 0$, $k \geq 1$ be an integer, and let I_k denote the modified Bessel function of the first kind, with parameter k. We have that the function

$$U(re^{i\vartheta}, t) = \operatorname{Re}\big[e^{ik(\vartheta+\omega t)} I_k\big(\tfrac{1}{2}\sqrt{2\omega k}(1+i)r\big)\big]$$

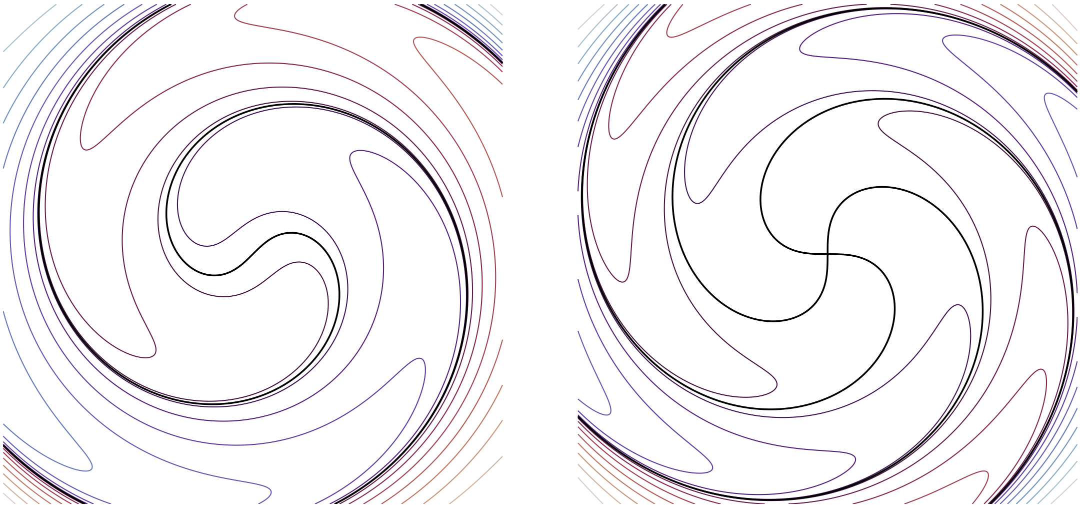

Figure 2. Contour lines of the rotating caloric functions in Remark 1.8. Here $\omega = 1$, and $k = 1$ and $k = 2$, respectively. In black the nodal lines: the appearance of arithmetic spirals for r large is rather clear in the picture.

is an entire, eternal rotating solution of the heat equation

$$U_t - \Delta U = 0 \quad \text{in } \mathbb{R}^2 \times \mathbb{R}$$

having $2k$ nodal regions, which coincide up to rotations that are a multiple of π/k. The equidistributed nodal lines admit a straight tangent as $r \to 0$, while they behave like arithmetic spirals of the equation $\vartheta = \sqrt{\omega/(2k)}r$ as $r \to +\infty$; see Figure 2. Notice that, as $\omega \to 0$, a suitable renormalization of U converges to the entire harmonic function $\operatorname{Re} z^k$.

Remark 1.9. Notice that, by separation of variables, one may treat boundary value problems for rotating solutions also on other rotationally invariant domains Ω, such as annuli or external domains. Of course, since in these cases $\mathbf{0} \notin \Omega$, this cannot provide spiraling solutions, at least in our sense.

Let us provide an explanation for our construction. When a smooth curve separates two densities of an element of $\mathcal{S}_{\text{rot}}$, at least locally, the gradients of the two densities are proportional across such an interface. Indeed, by definition of $\hat{u}_i$, the function $a_{21}u_1 - a_{12}u_2$ solves an elliptic equation in a neighborhood of the interface.

Let us assume, for concreteness, $K = 3$. In case the nodal structure of $(u_1, u_2, u_3) \in \mathcal{S}_{\text{rot}}$ is the required one, as depicted in Figure 1, then a suitable linear combination of the components u_i satisfies an equation on B, up to a curve. More precisely, let us define

$$\mathcal{U} = u_1 - \frac{a_{12}}{a_{21}}u_2 + \frac{a_{12}}{a_{21}} \cdot \frac{a_{23}}{a_{32}}u_3, \qquad \Gamma = \overline{\{u_1 > 0\}} \cap \overline{\{u_3 > 0\}}.$$

It is easy to check that

$$-\Delta \mathcal{U} + \omega \boldsymbol{x}^{\perp} \cdot \nabla \mathcal{U} = \mu \mathcal{U} \quad \text{in } B \setminus \Gamma,$$

while, if $\mathbf{0} \neq \boldsymbol{x}_0 \in \Gamma$ and α is defined as in (7),

$$\lim_{\substack{\boldsymbol{x}\to\boldsymbol{x}_0\\ u_3(\boldsymbol{x})>0}} \nabla\mathcal{U}(\boldsymbol{x}) = -e^{2\pi\alpha} \lim_{\substack{\boldsymbol{x}\to\boldsymbol{x}_0\\ u_1(\boldsymbol{x})>0}} \nabla\mathcal{U}(\boldsymbol{x}).$$

By composing with a conformal map between $B \setminus \{\mathbf{0}\}$ and its universal covering $\mathbb{R}\times(0,\infty)$, we can lift $\mathcal{U}$ to a solution of a linear equation in the half-plane (see (11) below) having a precise nodal structure. This connection is analyzed in Section 2.

To prove Theorem 1.1 we reverse the above argument: we start by solving the equation in the covering by separation of variables in Section 3; next, we show in Section 4 that, under suitable conditions, the solution has the appropriate nodal properties to be mapped back to the disk. In both these points, we have to deal with nonresonance/coerciveness conditions, leading to the assumption on μ. On the other hand, the existence of the vector $\bar{s}$ is equivalent to the validity of suitable compatibility conditions, expressed in terms of the Fourier coefficients of the boundary data. Specifically, when $K=3$, $\bar{s}$ is any componentwise positive solution of the system

$$\int_0^{2\pi} e^{-\alpha\vartheta}\Phi(\vartheta)\sin\left(\frac{\vartheta}{2}\right)dx = \int_0^{2\pi} e^{-\alpha\vartheta}\Phi(\vartheta)\cos\left(\frac{\vartheta}{2}\right)dx = 0, \tag{10}$$

where

$$\Phi = s_1\varphi_1 - s_2\frac{a_{12}}{a_{21}}\varphi_2 + s_3\frac{a_{12}}{a_{21}}\cdot\frac{a_{23}}{a_{32}}\varphi_3.$$

We analyze the general compatibility conditions in Section 5, concluding the proof of Theorem 1.1. Finally, Theorems 1.6 and 1.7 are proved in Section 6.

2. An equivalent problem in the half-plane

As we mentioned, the proof of Theorems 1.1, 1.6 and 1.7 is based on the connection between system (4) and an equation in the half-plane, seen as the universal covering of the punctured disk. In this section we analyze such a connection.

Let $\mu,\ \omega$ be real parameters and $v = v(x,y) \in C(\mathbb{R}\times[0,+\infty))$ be a classical solution of the equation

$$-\Delta v + \omega e^{-2y} v_x = e^{-2y}\mu v, \qquad x\in\mathbb{R}, \quad y>0. \tag{11}$$

In the following we assume that v satisfies the following properties:

(a) There exists $\sigma \neq 0$ such that

$$v(x+2\pi, y) = \sigma v(x,y) \tag{12}$$

for any $x \in \mathbb{R}$, $y \geq 0$.

(b) $v(x,y) = 0$ if and only if $(x,y) \in \bar{S}_i \cap \bar{S}_{i+1}$ for some $i \in \mathbb{Z}$, where the nonempty nodal regions S_i are open, connected, disjoint, unbounded and

$$\bar{S}_i \cap \{(x,0) : x\in\mathbb{R}\} = \{(x,0) : x_{i-1} \leq x \leq x_i\}, \qquad \bar{S}_i \cap \bar{S}_j \neq \varnothing \iff j-i = -1, 0, 1.$$

In particular, since v is analytic for $y>0$, we obtain that the set $\bar{S}_i \cap \bar{S}_{i+1}$ is actually a locally analytic curve which accumulates both at $(x_i, 0)$ and at $y=\infty$.

(c) $v|_{S_i} \in H^1(S_i)$ for every $i\in\mathbb{Z}$ (or, equivalently, their trivial extensions belong to $H^1(\mathbb{R}\times(0,+\infty))$).

We infer that $\bigcup_i \overline{S}_i = \mathbb{R}\times[0,+\infty)$, and that this covering is locally finite. Moreover, by (a), the nodal set of v is 2π-periodic in the x-direction. Up to a translation, we can assume that $x_0 = 0$, so that in particular $v(0,0) = 0$ and the number K of nodal components, up to periodicity, can be defined as

$$K = \#\{i : [x_{i-1}, x_i] \subset [0, 2\pi]\}, \text{ i.e., } S_{i+K} = S_i + (2\pi, 0), \quad \text{for all } i. \tag{13}$$

Notice that $\sigma > 0$ implies K even, while $\sigma < 0$ forces K odd.

Finally, we introduce the following conformal map between the half-plane and the punctured disk:

$$\mathcal{T}\colon \mathbb{R}\times(0,+\infty) \to B\setminus\{\mathbf{0}\}, \quad \mathcal{T}\colon (x,y) \mapsto \boldsymbol{x} = (e^{-y}\cos x, e^{-y}\sin x) \tag{14}$$

(for more details about this map, see Remarks 2.17 and 2.19 in [Terracini et al. 2019]).

The main result of this section is the following.

Proposition 2.1. *Let $v \in C(\mathbb{R}\times[0,+\infty))$ be a classical solution of (11) satisfying (a), (b) and (c), and let K be defined as in (13). Assume that the positive coefficients a_{ij} and the parameter α satisfy*

$$\prod_{i=1}^{K} \frac{a_{(i-1)i}}{a_{i(i-1)}} = (-1)^K \sigma \tag{15}$$

(understanding $a_{01} = a_{K1}$, $a_{10} = a_{1K}$).

For $i = 1, \dots, K$, let us define

$$u_i = (-1)^{i+1} l_i v|_{S_i} \circ \mathcal{T}, \quad \text{with } l_1 = 1, \;\; l_i = \frac{a_{i(i-1)}}{a_{(i-1)i}} \cdot l_{i-1} \tag{16}$$

(trivially extended in the whole B). Then $(u_1, \dots, u_K) \in \mathcal{S}_{\mathrm{rot}}$. Moreover, with respect to this K-tuple, the origin is the only point with higher multiplicity, with $m(\mathbf{0}) = K$.

Vice versa, if $(u_1, \dots, u_K) \in \mathcal{S}_{\mathrm{rot}}$ has the origin as its only singular point, then there exists v such that the first part of the proposition holds.

Remark 2.2. In the case that the asymptotic behavior of the nodal zones S_i is known as $y \to +\infty$, then by composition with $\mathcal{T}$ one can deduce the local description of the free boundary associated to $(u_1, \dots, u_K)$ near $\mathbf{0}$.

Proof. By condition (a) the functions u_i are well defined, by (b) they satisfy $u_i \cdot u_j \equiv 0$ as long as $j \neq i$, and by (c) they belong to $H^1(B)$ (recall that $\mathcal{T}$ is a conformal map). With direct computations one can check that

$$-\Delta u_i + \omega \boldsymbol{x}^\perp \cdot \nabla u_i = \mu u_i \quad \text{in } \omega_i := \{u_i > 0\}. \tag{17}$$

Analogously, using the definition of the coefficients l_i (see (16)), we have that

$$-\Delta\left(u_{i-1} - \frac{a_{(i-1)i}}{a_{i(i-1)}} u_i\right) + \omega \boldsymbol{x}^\perp \cdot \nabla\left(u_{i-1} - \frac{a_{(i-1)i}}{a_{i(i-1)}} u_i\right) = \mu\left(u_{i-1} - \frac{a_{(i-1)i}}{a_{i(i-1)}} u_i\right) \tag{18}$$

in the interior of $\overline{\omega}_{i-1} \cup \overline{\omega}_i$, $i = 1, \dots, K$ (in case $i = 1$ we keep the understanding $i - 1 = K$, and the validity of (18) follows by (15)). Notice that, when restricted to $\overline{\omega}_{i-1} \cup \overline{\omega}_i$, the function in (18) is a multiple of both $\hat{u}_{i-1}$ and $\hat{u}_i$.

We have to show the validity of the inequalities

$$\int_B \nabla u_i \cdot \nabla \varphi + [\omega \boldsymbol{x}^\perp \cdot \nabla u_i - \mu u_i]\varphi \le 0, \tag{19}$$

$$\int_B \nabla \hat{u}_i \cdot \nabla \varphi + [\omega \boldsymbol{x}^\perp \cdot \nabla \hat{u}_i - \mu \hat{u}_i]\varphi \ge 0 \tag{20}$$

for every Lipschitz, compactly supported, nonnegative φ.

First, let us consider any φ such that $\varphi \equiv 0$ in $B_\varepsilon(\mathbf{0})$. Then (19) follows by integration by parts, since

$$\int_B \nabla u_i \cdot \nabla \varphi + [\omega \boldsymbol{x}^\perp \cdot \nabla u_i - \mu u_i]\varphi = \int_{\omega_i \setminus B_\varepsilon} \nabla u_i \cdot \nabla \varphi + [\omega \boldsymbol{x}^\perp \cdot \nabla u_i - \mu u_i]\varphi = \int_{\partial \omega_i} \partial_\nu u_i \varphi \le 0,$$

where we used the regularity of $\partial\omega_i$ away from $\mathbf{0}$, the equation for u_i and the fact that $\partial_\nu u_i \le 0$ on $\partial\omega_i$. On the other hand, to prove (20), since $\varphi \equiv 0$ in $B_\varepsilon(\mathbf{0})$, we can use a partition of unity argument and assume that $\operatorname{supp}(\varphi)$ intersects at most two adjacent nodal regions. In case none of them is ω_i, then $\hat{u}_i = -c_1 u_j - c_2 u_{j+1}$, with $c_i > 0$, and (20) follows by applying (19) twice, with $i = j, j+1$; if $\operatorname{supp}(\varphi) \subset \overline{\omega}_{i-1} \cup \overline{\omega}_i \setminus B_\varepsilon$ then (18) yields

$$\int_B \nabla \hat{u}_i \cdot \nabla \varphi + [\omega \boldsymbol{x}^\perp \cdot \nabla \hat{u}_i - \mu u_i]\varphi = \int_{\overline{\omega}_{i-1} \cap \overline{\omega}_i \setminus B_\varepsilon} \nabla \hat{u}_i \cdot \nabla \varphi + [\omega \boldsymbol{x}^\perp \cdot \nabla \hat{u}_i - \mu \hat{u}_i]\varphi = 0,$$

and the same holds true if $\operatorname{supp}(\varphi) \subset \overline{\omega}_i \cup \overline{\omega}_{i+1} \setminus B_\varepsilon$.

Finally, let us consider any φ. We show how to prove (19); the proof of (20) is analogous. For any $\varepsilon > 0$ small, we define the function

$$\eta(\boldsymbol{x}) = \begin{cases} 0, & \boldsymbol{x} \in B_\varepsilon, \\ (|\boldsymbol{x}| - \varepsilon)/\varepsilon, & \boldsymbol{x} \in B_{2\varepsilon} \setminus B_\varepsilon, \\ 1, & \boldsymbol{x} \in B \setminus B_{2\varepsilon}. \end{cases}$$

Then $\varphi\eta = 0$ in B_ε, and by the previous part

$$\int_B (\nabla u_i \cdot \nabla \varphi)\eta + \int_B (\nabla u_i \cdot \nabla \eta)\varphi + \int_B [\omega \boldsymbol{x}^\perp \cdot \nabla u_i - \mu u_i]\eta\varphi \le 0.$$

Since φ is Lipschitz, we have

$$\left| \int_B (\nabla u_i \cdot \nabla \eta)\varphi \right| \le \frac{1}{\varepsilon} \int_{B_{2\varepsilon} \setminus B_\varepsilon} |\nabla u_i| \varphi \le \frac{1}{\varepsilon} \|u_i\|_{H^1(B_{2\varepsilon})} \|\varphi\|_{L^2(B_{2\varepsilon})} \le C \|u_i\|_{H^1(B_{2\varepsilon})} \|\varphi\|_{L^\infty}.$$

Thus we find the estimate

$$\int_B (\nabla u_i \cdot \nabla \varphi)\eta + \int_B [\omega \boldsymbol{x}^\perp \cdot \nabla u_i - \mu u_i]\eta\varphi \le C \|u_i\|_{H^1(B_{2\varepsilon})} \|\varphi\|_{L^\infty}.$$

Taking the limit as $\varepsilon \to 0$, since η converges monotonically to 1, we conclude that

$$\int_B \nabla u_i \cdot \nabla \varphi + [\omega \boldsymbol{x}^\perp \cdot \nabla u_i - \mu u_i]\varphi \le 0,$$

concluding the proof of the first assertion.

The second part follows by defining

$$v \circ \mathcal{T} = \sum_{i=1}^{K} \frac{(-1)^{i+1}}{l_i} u_i, \tag{21}$$

and then deriving v by a lifting argument. We refer to [Terracini et al. 2019, Section 2] for further details. □

3. Solutions in the half-plane

Let $\mu, \alpha, \omega \in \mathbb{R}$. Given the trace

$$\Phi\colon [0, 2\pi] \to \mathbb{R}, \qquad \Phi(0) = \Phi(2\pi) = 0,$$

we look for solutions v of the following problem in the half-plane:

$$\begin{cases} -\Delta v + \omega e^{-2y} v_x = e^{-2y} \mu v, & x \in \mathbb{R}, \quad y > 0, \\ v(x + 2\pi, y) = e^{2\pi\alpha} v(x, y), & x \in \mathbb{R}, \quad y \geq 0, \\ v(x, 0) = \Phi(x), & 0 \leq x \leq 2\pi. \end{cases} \tag{22}$$

Notice that we are considering (11) together with condition (12) in the case $\sigma = e^{2\pi\alpha} > 0$ (recall definition (7) and the relation (15)). As we noticed, this involves an even number of nodal zones in the period. One can easily modify our arguments to deal with an odd one, i.e., with $\sigma < 0$, for instance with the change of variables $(x, y) \mapsto \left(\frac{1}{2}x, \frac{1}{2}y\right)$, $\sigma \mapsto \sigma^2$. In a completely equivalent way, one can work with 2π-periodicity and take $\alpha = \frac{1}{2\pi} \ln|\sigma| + \frac{i}{2} \in \mathbb{C}$.

To solve (22), we first transform it into a periodic problem, and then use separation of variables to write the solution in Fourier series. To this aim, we notice that v solves (22) if and only if

$$w(x, y) := e^{-\alpha x} v(x, y)$$

solves

$$\begin{cases} -\Delta w + (\omega e^{-2y} - 2\alpha) w_x + [(\alpha\omega - \mu) e^{-2y} - \alpha^2] w = 0, & x \in \mathbb{R}, \quad y > 0, \\ w(x + 2\pi, y) = w(x, y), & x \in \mathbb{R}, \quad y \geq 0, \\ w(x, 0) = e^{-\alpha x} \Phi(x), & 0 \leq x \leq 2\pi. \end{cases} \tag{23}$$

Of course, if $\alpha = 0$ then v and w coincide. Either way, with a little abuse of notation, we can extend Φ to $\mathbb{R}$ in such a way that $e^{-\alpha x}\Phi(x)$ is 2π-periodic. At least formally we can expand w in Fourier series and write

$$w(x, y) = \sum_{k \in \mathbb{Z}} W_k(y) e^{ikx}.$$

Plugging this expression into (23), we obtain that the coefficients $W_k : \overline{\mathbb{R}}^+ \to \mathbb{C}$, $k \in \mathbb{Z}$, must solve the ordinary differential equation

$$W_k''(y) = [(k - i\alpha)^2 + (\omega\alpha - \mu + i\omega k) e^{-2y}] W_k(y), \qquad y > 0. \tag{24}$$

We can solve boundary value problems associated with (24) by using the Fredholm alternative and the Lax–Milgram theorem, settled in complex Hilbert spaces. We are looking for solutions of (23) that change sign as $y \to +\infty$. As we will see in Lemma 3.9, this requires that the term corresponding to $k = 0$ in the expansion should not be present. For this reason we consider $k \neq 0$ from now on.

Lemma 3.1. *For any $k \in \mathbb{Z} \setminus \{0\}$, $\alpha \in \mathbb{R}$, there exists a sequence $\{\lambda_n\}_{n\in\mathbb{N}} \subset \mathbb{C}$, with $|\lambda_n| \to +\infty$ as $n \to +\infty$, such that the problem*

$$\begin{cases} X_k''(y) = [(k-i\alpha)^2 + (\omega\alpha - \mu + i\omega k)e^{-2y}]X_k(y), & y > 0, \\ X_k(0) = 1, \quad X_k \in H^1(\mathbb{R}^+;\mathbb{C}), \end{cases} \tag{25}$$

admits a unique solution if and only if

$$\omega\alpha - \mu + i\omega k \notin \{\lambda_n\}_{n\in\mathbb{N}}, \tag{26}$$

while no solution exists in the complementary case.

Proof. We shall consider the case $k \geq 1$, as the case $k \leq -1$ follows by the same arguments, up to the change of sign

$$(\alpha, \omega, \mu, k) \mapsto (-\alpha, -\omega, \mu, -k).$$

In particular, one can verify that $X_{-k}(y) = \overline{X_k(y)}$ for any $k \in \mathbb{Z}$ and $y \geq 0$ (in case one of them exists). We proceed through several steps.

Step 1. *Weak formulation of the problem.* Letting $X_k = U + U_0$, where $U_0 := e^{-(k-i\alpha)y}$, we are led to find, if it exists, a function $U \in H_0^1(\mathbb{R}^+;\mathbb{C})$, solution of

$$-U'' + [(k-i\alpha)^2 + (\omega\alpha - \mu + i\omega k)e^{-2y}]U = -(\omega\alpha - \mu + i\omega k)e^{-2y}e^{-(k-i\alpha)y}, \quad y > 0.$$

We settle the problem in the space

$$H = H_0^1(\mathbb{R}^+;\mathbb{C}), \quad \|u\|_H^2 = \int_0^\infty |U'|^2 + |U|^2.$$

To proceed, we introduce the sesquilinear forms a_R, a_I as

$$a_R(U,V) = \int_0^\infty U'\overline{V}' + [(k^2-\alpha^2) + (\omega\alpha-\mu)e^{-2y}]U\overline{V}, \quad a_I(U,V) = \int_0^\infty (-2\alpha k + \omega k e^{-2y})U\overline{V},$$

and the antilinear form l as

$$l(V) = -(\omega\alpha - \mu + i\omega k)\int_0^\infty e^{-2y}U_0\overline{V} = -(\omega\alpha - \mu + i\omega k)\int_0^\infty e^{-(k+2-i\alpha)y}\overline{V}. \tag{27}$$

In this way, we are reduced to solve the following variational problem: finding $U \in H$ such that

$$a(U,V) = a_R(U,V) + i a_I(U,V) = l(V) \quad \text{for all } V \in H. \tag{28}$$

Notice that both a and l are continuous: indeed, since $|e^{-2y}| \leq 1$ for $y \geq 0$, it is easy to see that

$$|a(U,V)| \leq (k^2 + \alpha^2 + \sqrt{(\omega\alpha-\mu)^2 + (\omega k)^2})\|u\|_H\|v\|_H.$$

Similarly, for l we obtain

$$|l(V)| \le |(\omega\alpha - \mu + i\omega k)| \int_0^\infty e^{-(k+2)y}|V| \le \frac{\sqrt{(\omega\alpha-\mu)^2+(\omega k)^2}}{\sqrt{2(k+2)}}\left(\int_0^\infty |V|^2\right)^{1/2}.$$

For future purposes we notice that, for every $U \in H$, both $a_R(U,U)$ and $a_I(U,U)$ are real numbers: indeed, $a_R(U,U)$ and $a_I(U,U)$ are, respectively, the real and imaginary part of $a(U,U)$. We can exploit the Cauchy–Schwarz inequality (for real two-dimensional vectors) to find that

$$\begin{aligned} |a(U,U)| &= \sup_{K\in\mathbb{R}} \frac{a_R(U,U)+Ka_I(U,U)}{\sqrt{1+K^2}} \ge \frac{k}{\sqrt{\alpha^2+k^2}}\Big(a_R(U,U) - \frac{\alpha}{k}a_I(U,U)\Big) \\ &= \frac{k}{\sqrt{\alpha^2+k^2}}\int_0^\infty [|U'|^2+(k^2+\alpha^2)|U|^2] - \frac{k\mu}{\sqrt{\alpha^2+k^2}}\int_0^\infty e^{-2y}|U|^2. \end{aligned} \tag{29}$$

In order to prove existence and uniqueness of a solution U, we shall make use of the classical Fredholm alternative theorem. In particular, we shall find that (28) admits a unique solution $U \in H_0^1(\mathbb{R}^+;\mathbb{C})$ if and only if 0 is not an eigenvalue of a (more precisely, and equivalently, 0 is not an eigenvalue of the conjugate transpose sesquilinear form $a^\dagger$).

Step 2. *A related eigenvalue problem.* To proceed, we introduce the (adjoint) eigenvalue problem: finding $\lambda \in \mathbb{C}$ and $V \in H \setminus \{0\}$ such that

$$\int_0^\infty [U'\overline{V}' + (k-i\alpha)^2 U\overline{V}] + \lambda\int_0^\infty e^{-2y}U\overline{V} = 0 \quad \text{for all } U \in H.$$

Defining the weighted space

$$L = \left\{U \in L^1_{\text{loc}}(\mathbb{R}^+;\mathbb{C}) : \|U\|_L^2 = \int_0^\infty e^{-2y}|U|^2 < +\infty\right\},$$

we have that $H \subset L = L^* \subset H^*$ is a Hilbert triplet, with H compactly embedded in L; see Lemma A.1. Then standard spectral theory (see, e.g., [Kato 1966, Chapter 3, Theorem 6.26]) yields the existence of a sequence of eigenvalues $\{\lambda_n\}_{n\in\mathbb{N}} \subset \mathbb{C}$, with $|\lambda_n| \to +\infty$, and it is straightforward to show that $V \neq 0$ satisfies

$$a(U,V) = 0 \text{ for all } U \in H \iff \begin{array}{c}\omega\alpha - \mu + i\omega k = \lambda_n \\ \text{and } V = V_n \text{ is an associated eigenfunction.}\end{array} \tag{30}$$

Notice that each λ_n is a simple eigenvalue by uniqueness of the Cauchy problem for ODEs.

Step 3. *Application of the Babuška–Lax–Milgram theorem.* To conclude the invertible case, we show that, if $\omega\alpha - \mu + i\omega k \neq \lambda_n$ for every n, then there exists a unique solution to (28). To this aim, we apply a generalization of the Lax–Milgram theorem due to Babuška [1971, Theorem 2.1] (with $H_1 = H_2 = H$). After the previous steps, in order to apply such a result to (28), we only need to show that, if $\omega\alpha - \mu + i\omega k \neq \lambda_n$ for every n, then the following inf-sup conditions hold:

$$\inf_{\|V\|_H=1}\sup_{\|U\|_H=1}|a(U,V)| \ge C_2 > 0, \qquad \inf_{\|U\|_H=1}\sup_{\|V\|_H=1}|a(U,V)| \ge C_3 > 0$$

for suitable constants C_2, C_3. We prove the first inequality; the second is proved analogously. Assume by contradiction that the sequence $\{V_n\}_n$ satisfies

$$\|V_n\|_H = 1, \qquad |a(U, V_n)| \le \frac{1}{n}\|U\|_H \quad \text{for all } U \in H.$$

In particular, as $n \to +\infty$, $a(V_n, V_n) \to 0$. Moreover, up to subsequences, V_n converges to V_∞, both weakly in H and strongly in L (by compact embedding). Thus $a(U, V_\infty) = 0$ for every $U \in H$. Since $\omega\alpha - \mu + i\omega k \ne \lambda_n$ for every n and recalling (30), we deduce that $V_\infty \equiv 0$. Since $k^2 \ge 1$, (29) yields

$$o(1) = |a(V_n, V_n)| \ge \frac{k}{\sqrt{\alpha^2+k^2}}\|V_n\|_H^2 - \frac{k\mu}{\sqrt{\alpha^2+k^2}}\|V_n\|_L^2 = \frac{k}{\sqrt{\alpha^2+k^2}} + o(1)$$

as $n \to \infty$, a contradiction.

Step 4. *Nonexistence in the resonant case.* Finally, assume that $\omega\alpha - \mu + i\omega k = \lambda_n$ for some n, and let $V_n \not\equiv 0$ be an associated eigenfunction of the adjoint problem

$$a(U, V_n) = \int_0^\infty [U'\overline{V}_n' + (k - i\alpha)^2 U\overline{V}_n] + \lambda_n \int_0^\infty e^{-2y} U\overline{V}_n = 0 \quad \text{for all } U \in H.$$

This forces

$$-\overline{V}_n'' + (k - i\alpha)^2 \overline{V}_n + \lambda_n e^{-2y}\overline{V}_n = 0 \quad \text{on } (0, \infty); \tag{31}$$

in particular, $V_n \in H^2(0, +\infty)$, and thus $V_n'(y) \to 0$ as $y \to +\infty$. Moreover, by uniqueness of the Cauchy problem, $V_n'(0) \ne 0$.

In the case we are considering, (28) can be rewritten as

$$a(U, V) = (-\lambda_n U_0, V)_L \quad \text{for all } V \in H,$$

where $U_0 = e^{-(k-i\alpha)y}$. By Fredholm's alternative, in this case (28) is solvable if and only if the compatibility condition

$$(-\lambda_n U_0, V_n)_L = 0$$

holds true. On the other hand, using (31), we have

$$(-\lambda_n U_0, V_n)_L = -\lambda_n \int_0^\infty e^{-2y} U_0 \overline{V}_n = U(0)\overline{V}_n'(0) + \int_0^\infty [U_0'\overline{V}_n' + (k - i\alpha)^2 U_0 \overline{V}_n] = \overline{V}_n'(0) \ne 0,$$

which concludes the proof. □

The resonance set in the previous lemma can be characterized in terms of the zero set of the following function Θ_ν, depending on the complex parameter ν:

$$\Theta_\nu(z) = \sum_{n=0}^{\infty} \frac{1}{n!\,\Gamma(n+1+\nu)} \left(\frac{z}{4}\right)^n. \tag{32}$$

Notice that, for any $\nu \in \mathbb{C}$, Θ_ν is analytic on $\mathbb{C}$ (recall that Γ has no zeros, but only simple poles at each nonpositive integer $-k$: in such a case, we understand $1/\Gamma(-k) = 0$). As a matter of fact, Θ_ν is related to I_ν, the modified Bessel function of the first kind, with parameter $\nu \in \mathbb{C}$, by the formula

$$I_\nu(z) = \left(\frac{z}{2}\right)^\nu \Theta_\nu(z^2) \tag{33}$$

(in turn, $I_\nu(z) = e^{-i\nu\pi/2} J_\nu(iz)$, where J_ν is the usual Bessel function of the first kind). Notice that, in the case $\nu \notin \mathbb{Z}$, I_ν is a multivalued function because of the complex exponentiation z^ν. Nonetheless, the zero set of (any determination of) I_ν coincides with the complex square root of the zero set of Θ_ν, with the exception of 0.

Lemma 3.2. *For any $k \in \mathbb{Z} \setminus \{0\}$, $\alpha \in \mathbb{R}$, let $\{\lambda_n\}_{n\in\mathbb{N}} \subset \mathbb{C}$ denote the sequence defined in Lemma 3.1. Then*

$$\{\lambda_n\}_{n\in\mathbb{N}} = \{z \in \mathbb{C} \setminus \{0\} : \Theta_{\mathrm{sign}(k)(k-i\alpha)}(z) = 0\},$$

where Θ_ν is defined in (32) for every $\nu \in \mathbb{C}$.

Moreover, whenever $\lambda := \omega\alpha - \mu + i\omega k \notin \{\lambda_n\}_{n\in\mathbb{N}}$, the unique solution of (25) is

$$X_k(y) = \frac{\Theta_\nu(\lambda e^{-2y})}{\Theta_\nu(\lambda)} e^{-\nu y}$$

($X_k(y) = e^{-\nu y}$ *in the case* $\lambda = 0$), *where* $\nu = \mathrm{sign}(k)(k - i\alpha)$ *whenever* $k \neq 0$.

Equivalently, we could write

$$X_k(y) = \frac{I_\nu(\sqrt{\lambda} e^{-y})}{I_\nu(\sqrt{\lambda})},$$

and such an identity is not ambiguous as long as we choose the same determinations both in the numerator and in the denominator.

Proof. Again, we treat the case $k \geq 1$; the case $k \leq -1$ follows with minor changes. With the above notation,

$$\nu = k - i\alpha, \quad \lambda = \omega\alpha - \mu + i\omega k,$$

the second-order linear ODE in (25) is written as

$$x''(y) = [\nu^2 + \lambda e^{-2y}]x(y). \tag{34}$$

We assume $\lambda \neq 0$; the complementary case is trivial. Let us consider the functions $x_{\pm\nu}(y)$ defined as

$$x_{\pm\nu}(y) = \Theta_{\pm\nu}(\lambda e^{-2y}) e^{\mp\nu y} = \sum_{n\geq 0} c_{\pm\nu,n} e^{(-2n\mp\nu)y}, \quad \text{where } c_{\pm\nu,n} = \frac{1}{n!\,\Gamma(n+1\pm\nu)}\left(\frac{\lambda}{4}\right)^n$$

(again, we understand $c_{\pm\nu,n} = 0$ whenever $-(n+1\pm\nu) \in \mathbb{N}$). We notice that $4n(n\pm\nu)c_{\pm\nu,n} = \lambda c_{\pm\nu,n-1}$. Then

$$\begin{aligned} x''_{\pm\nu}(y) &= \sum_{n\geq 0}(-2n\mp\nu)^2 c_{\pm\nu,n} e^{(-2n\mp\nu)y} = \sum_{n\geq 0} \nu^2 c_{\pm\nu,n} e^{(-2n\mp\nu)y} + \sum_{n\geq 0} 4n(n\pm\nu) c_{\pm\nu,n} e^{(-2n\mp\nu)y} \\ &= \nu^2 x_{\pm\nu}(y) + \lambda \sum_{n\geq 1} c_{\pm\nu,n-1} e^{(-2n\mp\nu)y} = [\nu^2 + \lambda e^{-2y}] x_{\pm\nu}(y); \end{aligned}$$

that is, both $x_{\pm\nu}$ solve the second-order linear ODE (34).

Let us first assume that $\alpha \neq 0$. Then $-(n+1\pm\nu) \notin \mathbb{N}$ for every n, and we obtain that $\Theta_{\pm\nu}(\lambda e^{-2y}) = 1/\Gamma(1\pm\nu) + o(1)$; that is,

$$x_{\pm\nu}(y) = \frac{1}{\Gamma(1\pm\nu)} e^{\mp\nu y} + o(e^{\mp\nu y}) \quad \text{as } y \to +\infty.$$

Then $x_{\pm\nu}$ are linearly independent, and any solution of (34) is of the form

$$x(y) = C_+ x_\nu(y) + C_- x_{-\nu}(y), \quad C_\pm \in \mathbb{C}.$$

Since $\nu = k - i\alpha$ and $k \geq 1$, we have that $x \in H^1(0,+\infty)$ if and only if $C_- = 0$. As a consequence, (25) is (uniquely) solvable if and only if $x_\nu(0) = \Theta_\nu(\lambda) \neq 0$, and the lemma follows.

On the other hand, let $\alpha = 0$ (and $\lambda \neq 0$). In this case $\nu = k \geq 1$, and

$$c_{-k,n+k} = \frac{1}{(n+k)!\,n!}\left(\frac{\lambda}{4}\right)^{n+k} = \left(\frac{\lambda}{4}\right)^k c_{k,n}$$

for every $n \geq 0$, therefore the functions $x_{\pm k}$ are no longer linearly independent. By differentiating (34) with respect to ν, one can easily see that a second independent solution of (34) can be obtained as

$$\tilde{x}_k = \left[\left(\frac{\lambda}{4}\right)^k \frac{\partial x_\nu}{\partial \nu} - \frac{\partial x_{-\nu}}{\partial \nu}\right]_{\nu=k},$$

mimicking the procedure that leads to the (modified) Bessel functions of the second kind. Since $\Gamma(n+1-k)$ has a simple pole at $n = 0$, we have

$$\lim_{\nu\to k} \frac{\partial c_{-\nu,0}}{\partial \nu} = (-1)^k (k-1)! \qquad \text{and} \qquad \tilde{x}_k(y) = (-1)^k (k-1)!\, e^{ky} + o(e^{ky}) \quad \text{as } y \to +\infty$$

(see [Erdélyi et al. 1953, Section 7.2.5, p. 9] for more details). Thus also in this case $\tilde{x}_k \notin H^1(0,+\infty)$, and the lemma follows. □

Corollary 3.3. *Let X_k denote the solution of (25). Then, for some $C \neq 0$,*

$$X_k(y) = C e^{-\operatorname{sign}(k)(k-i\alpha)y} + O(e^{-(|k|+2)y}) \quad \textit{as } y \to +\infty.$$

Remark 3.4. As a byproduct of the proof of Lemma 3.2, we have that the eigenvalues λ_n are all simple in $H^1_0(\mathbb{R}^+;\mathbb{C})$. Indeed, the general solution of the corresponding eigenequation is a two-dimensional vector space of complex-valued functions, but only a one-dimensional subspace consists of H^1 functions of the form

$$C\,\Theta_{\operatorname{sign}(k)(k-i\alpha)}(\lambda_n e^{-2y}) e^{-\operatorname{sign}(k)(k-i\alpha)y}, \quad C \in \mathbb{C}.$$

In view of writing w as a series in terms of the solutions X_k, we need to estimate the asymptotic behaviors as $k \to \infty$ of their L^2 and H^1 norms.

Lemma 3.5. *Let α, μ, ω be fixed in such a way that (26) holds for every $k \neq 0$. Then X_k satisfies*

$$\left(\int_0^\infty |X_k|^2\right)^{1/2} \leq \frac{C}{\sqrt{|k|}}, \quad \left(\int_0^\infty |X_k'|^2\right)^{1/2} \leq C\sqrt{|k|} \quad \textit{and} \quad \|X_k\|_{L^\infty(0,+\infty)} \leq \sqrt{2}C, \tag{35}$$

where C depends only on α, μ, ω.

Proof. As usual, for concreteness, we assume $k \geq 1$. As in the proof of Lemma 3.1 we write $X_k = U + e^{-(k-i\alpha)y}$. In order to prove (35), we distinguish between two cases, corresponding to the instances k small and k large. Indeed, for any fixed $\bar{k}$, which we will choose later in terms of $\alpha,\ \mu,\ \omega$, the estimate (35) is true for $k < \bar{k}$ and a suitable constant C. Next, for $k \geq \bar{k}$, we estimate the norms of U using the identity

$$|a(U,U)| = |l(U)|.$$

Recalling (27), we have

$$|l(U)| \leq |\omega\alpha - \mu + i\omega k| \int_0^\infty |e^{-(k+2)y}||U| \leq \frac{\sqrt{(\omega\alpha-\mu)^2+(\omega k)^2}}{\sqrt{2(k+2)}} \left(\int_0^\infty |U|^2\right)^{1/2}.$$

Using (29), we obtain

$$\frac{k}{\sqrt{k^2+\alpha^2}} \int_0^\infty [|U'|^2 + (k^2+\alpha^2-\mu^+)|U|^2] \leq \frac{\sqrt{(\omega\alpha-\mu)^2+(\omega k)^2}}{\sqrt{2(k+2)}} \left(\int_0^\infty |U|^2\right)^{1/2}.$$

Then

$$\left(\int_0^\infty |U|^2\right)^{1/2} \leq \frac{\sqrt{k^2+\alpha^2}}{k(k^2+\alpha^2-\mu^+)} \cdot \frac{\sqrt{(\omega\alpha-\mu)^2+(\omega k)^2}}{\sqrt{2(k+2)}} \leq \frac{|\omega|}{k^{3/2}},$$

whence

$$\left(\int_0^\infty |U'|^2\right)^{1/2} \leq \left[\frac{\sqrt{k^2+\alpha^2}}{k} \frac{\sqrt{(\omega\alpha-\mu)^2+(\omega k)^2}}{\sqrt{2(k+2)}} \left(\int_0^\infty |U|^2\right)\right]^{1/2} \leq \frac{|\omega|^{3/2}}{k^{5/4}}$$

for $k \geq \bar{k}$ sufficiently large (depending on $\omega,\ \mu,\ \alpha$).

Coming back to $X_k = U + e^{-(k-i\alpha)y}$, we finally obtain

$$\left(\int_0^\infty |X_k|^2\right)^{1/2} \leq \left(\int_0^\infty |U|^2\right)^{1/2} + \left(\int_0^\infty e^{-2ky}\right)^{1/2} \leq \frac{|\omega|}{k^{3/2}} + \frac{1}{\sqrt{2k}} \leq \frac{1}{\sqrt{k}}$$

and

$$\left(\int_0^\infty |X_k'|^2\right)^{1/2} \leq \left(\int_0^\infty |U'|^2\right)^{1/2} + \left(\int_0^\infty |k-i\alpha|^2 e^{-2ky}\right)^{1/2} \leq \frac{|\omega|^{3/2}}{k^{5/4}} + \sqrt{\frac{k^2+\alpha^2}{2k}} \leq \sqrt{k}$$

for k sufficiently large (depending on $\omega,\ \mu,\ \alpha$), concluding the H^1 estimates. Finally, by Corollary 3.3, for any $y > 0$,

$$X_k(y)^2 = -\int_y^\infty 2X_k(t)X_k'(t)\,dt \leq 2\left(\int_0^\infty |X_k|^2\right)^{1/2}\left(\int_0^\infty |X_k'|^2\right)^{1/2} \leq 2C^2,$$

and the last estimate follows. □

Next we provide explicit sufficient conditions for the validity of condition (26).

Lemma 3.6. *A sufficient condition for* (26) *to hold true is*

$$\sup\left\{(j_{\tau,1})^2 - \frac{\omega}{2\alpha}\tau^2 : \tau > 0\right\} > \mu - \frac{\omega}{2\alpha}(k^2+\alpha^2), \tag{36}$$

where $j_{\tau,1}$ denotes the first (positive) zero of the standard Bessel function of the first kind of order $\tau > 0$.

This is the case, for instance, if

$$\text{either } \mu < (j_{0,1} + \sqrt{k^2+\alpha^2})^2, \quad \text{or } \frac{\omega}{\alpha} < 2. \tag{37}$$

In particular, for any choice of α, ω, μ, *if* $|k|$ *is sufficiently large then* (26) *holds.*

Proof. Using the notation introduced in the proof of Lemma 3.1, we are going to show that, under the present assumptions, the sesquilinear form a is coercive. By the first estimate in (29), this follows once we find $K \in \mathbb{R}$ such that the quadratic form (with real coefficients)

$$a_R(U,U) + a_I(U,U)K = \int_0^\infty |U'|^2 + (k^2 - \alpha^2 - 2\alpha k K)|U|^2 + ((\omega\alpha - \mu) + \omega k K)e^{-2y}|U|^2$$

is strictly positive. To this aim, it is not difficult to check that we have to ask that $k^2 - \alpha^2 - 2\alpha k K > 0$. For this reason, it is convenient to introduce the parameters $\tau > 0$ and $b = b(\tau)$ such that

$$K = \frac{k^2 - \alpha^2 - \tau^2}{2\alpha k}, \quad b = -((\omega\alpha - \mu) + \omega k K) = \mu + \frac{\omega}{2\alpha}(\tau^2 - (k^2 + \alpha^2)).$$

In this way, we are reduced to finding $\tau > 0$ such that the quadratic form

$$U \mapsto \int_0^\infty |U'|^2 + (\tau^2 - b e^{-2y})|U|^2$$

is strictly positive. This quadratic form can be studied by standard arguments; we postpone the details to Lemma A.2 in the Appendix. We obtain that it is coercive if and only if

$$b = \mu + \frac{\omega}{2\alpha}(\tau^2 - (k^2 + \alpha^2)) < (j_{\tau,1})^2,$$

and (36) follows. In order to make this condition more explicit, we exploit the fact that

$$j_{\tau,1} \geq j_{0,1} + \tau \quad \text{for every } \tau \geq 0$$

(see [McCann and Love 1982]). Therefore, a stronger condition than (36) is

$$\mu + \frac{\omega}{2\alpha}(\tau^2 - (k^2 + \alpha^2)) < (j_{0,1} + \tau)^2 \quad \text{for some } \tau > 0.$$

The conditions in (37) follow by taking either $\tau^2 = k^2 + \alpha^2$, or $\tau \to +\infty$, respectively. □

Corollary 3.7. *Let* α, μ, ω *be fixed, with*

$$\mu < (j_{0,1} + 1)^2. \tag{38}$$

Then (26) *holds true for every* $k \neq 0$.

We are ready to state and prove the main result of this section. For any $\Phi \in \mathrm{Lip}([0, 2\pi])$, we write the Fourier coefficients of $e^{-\alpha x}\Phi(x)$ as

$$\phi_k = \frac{1}{2\pi}\int_0^{2\pi} e^{-(ik+\alpha)x}\Phi(x)\,dx, \quad k \in \mathbb{Z}.$$

Proposition 3.8. *Let* α, μ, ω *be fixed and* $\Phi \in \mathrm{Lip}([0, 2\pi])$. *Let us assume that*

- $\mu < (j_{0,1} + 1)^2 \simeq 3.4^2$,
- $\Phi(0) = \Phi(2\pi) = 0$ *and* $\phi_0 = \int_0^{2\pi} e^{-\alpha x}\Phi(x)\,dx = 0$.

Then the functions

$$w(x,y)=\sum_{k\in\mathbb{Z}\setminus\{0\}}\phi_k X_k(y)e^{ikx}\quad\text{and}\quad v(x,y)=e^{\alpha x}w(x,y),\tag{39}$$

where the functions X_k are as in Lemmas 3.1 and 3.2, satisfy:

(1) *$w\in H^1(\{(x,y)\in\mathbb{R}\times\mathbb{R}^+ : a<x+ly<b\})$ for any $l\in\mathbb{R}$ and $a<b$, and it solves (23).*

(2) *$v\in H^1(\{(x,y)\in\mathbb{R}\times\mathbb{R}^+ : a<x+ly<b\})$ for any l such that $l\alpha\geq 0$ and for every $a<b$, and it solves (22).*

(3) *Both v and w are analytic in $\mathbb{R}\times\mathbb{R}^+$ and $C^{0,\alpha}$ up to $y=0$ for every $\alpha<1$.*

Proof of Proposition 3.8. In view of Lemma 3.1, we have that all the terms in the series in (39) are smooth and satisfy the differential equations in (23). We now show that the series converges in H^1, ensuring that w also satisfies the corresponding equation. We start by observing that, by construction, the family $\{(x,y)\mapsto X_k(y)e^{ikx}\}_{k\in\mathbb{Z}\setminus\{0\}}$ is orthogonal in $H^1(S)$, $S=(0,2\pi)\times\mathbb{R}^+$, and, in particular, for any $k,h\in\mathbb{Z}\setminus\{0\}$ and $k\neq h$, we have

$$\int_S X_k(y)e^{ikx}\cdot\overline{(X_h(y)e^{ihx})}=0,\qquad \int_S X_k'(y)e^{ikx}\cdot\overline{(X_h'(y)e^{ihx})}=0$$

and, recalling (35),

$$\int_S |X_k(y)e^{ikx}|^2\leq\frac{C}{|k|},\qquad\int_S|X_k'(y)e^{ikx}|^2\leq C|k|,\qquad\int_S|X_k(y)(e^{ikx})'|^2\leq C|k|.$$

On the other hand, since $x\mapsto e^{-\alpha x}\Phi(x)$ can be extended to a 2π-periodic Lipschitz continuous function, it is an H^1-function on $\mathbb{S}^1$, and its Fourier coefficients ϕ_k satisfy

$$\sum_{k\in\mathbb{Z}}k^2|\phi_k|^2<+\infty$$

(recall that $\phi_0=0$). Combining the above inequalities, we infer

$$\Big\|\sum_{k\neq 0}W_k(y)e^{ikx}\Big\|^2_{H^1(S)}\leq C\sum_{k\geq 1}(|\phi_k|^2+|\phi_{-k}|^2)\left(\frac{1}{|k|}+|k|\right)<+\infty.$$

We conclude that the series defining w converges in $H^1(S)$, making w a weak solution of (23). Since w is periodic in the x-direction, we deduce that it belongs to $H^1((a,b)\times\mathbb{R}^+)$ for every $a<b$. Exploiting once again the periodicity in x of w, we can readily infer that $w\in H^1(\{(x,y)\in\mathbb{R}\times\mathbb{R}^+ : a<x+ly<b\})$ for any $l\in\mathbb{R}$ and $a<b$. Moreover, by elliptic regularity, w is analytic in $\mathbb{R}\times\mathbb{R}^+$ and Hölder continuous up to the boundary. Analogous conclusions for the function v can be drawn from the fact that $v(x,y)=e^{\alpha x}w(x,y)$, the only difference being that we need to exploit the assumption $l\alpha\geq 0$ in order to estimate the exponential factor. □

We conclude this section by showing that the Fourier expansions of the functions w and v can be exploited to give a description of their nodal sets for y large.

Lemma 3.9. *We consider again the assumptions of Proposition 3.8. Let $n \geq 1$ be the largest integer such that*

$$\phi_k = 0 \quad \textit{for all } |k| < n.$$

Then there exists $y^* > 0$ and $2n$ disjoint simple curves $\Gamma_1, \dots, \Gamma_{2n}$ such that

$$\{(x, y) \in \mathbb{R} \times (y^*, +\infty) : w(x, y) = 0 (= v(x, y))\} = \bigcup_{\substack{j=1,\dots,2n \\ h \in \mathbb{Z}}} \Gamma_j + (2\pi h, 0). \tag{40}$$

The curves Γ_j are asymptotic to evenly spaced parallel lines: there exists $\beta \in \mathbb{R}$ such that

$$(x, y) \in \Gamma_j \quad \iff \quad \alpha y + nx = \beta + \pi j + o_y(1) \qquad \textit{as } y \to +\infty.$$

Proof. By Lemma 3.5, we have that

$$\sup_{(x,y) \in \mathbb{R} \times \mathbb{R}^+} |w(x, y)| \leq \sup_{y > 0} \sum_{k \geq n} |\phi_k||X_k(y)| + |\phi_{-k}||X_{-k}(y)| \leq C \sum_{k \geq n} (|\phi_k| + |\phi_{-k}|) < +\infty,$$

which implies that the series converges also uniformly in $\mathbb{R} \times \mathbb{R}^+$. Moreover, we can extract the first term of the series and see that

$$|w(x, y) - \phi_n X_n(y) e^{inx} - \phi_{-n} X_{-n}(y) e^{-inx}| \leq C \sum_{k \geq n+1} (|\phi_k| + |\phi_{-k}|) e^{-ky} \leq C e^{-(n+1)y}$$

(see Corollary 3.3). This, in turn, implies that

$$w(x, y) = \phi_n X_n(y) e^{inx} + \phi_{-n} X_{-n}(y) e^{-inx} + O(e^{-(n+1)y}) \tag{41}$$

uniformly in $x \in \mathbb{R}$.

We claim that the nodal lines of the functions w (and of v) align asymptotically with those of the function

$$\begin{aligned}(x, y) \mapsto A_n(x, y) &= \phi_n X_n(y) e^{inx} + \phi_{-n} X_{-n}(y) e^{-inx} \\ &= \phi_n C_n e^{-(n-i\alpha)y + inx} + \phi_{-n} C_{-n} e^{(-n-i\alpha)y - inx} + O(e^{-(n+2)y}) \\ &= e^{-ny}(a_n \cos(\alpha y + nx) + b_n \sin(\alpha y + nx) + O(e^{-2y})) \\ &= e^{-ny}(\sqrt{a_n^2 + b_n^2} \sin(\alpha y + nx - \beta) + O(e^{-2y})),\end{aligned}$$

where the coefficients a_n, b_n and β are real numbers, $a_n^2 + b_n^2 \neq 0$ by assumption, and $\sin \beta = -a_n / \sqrt{a_n^2 + b_n^2}$. Indeed, recalling (41), we have that, as $y \to +\infty$,

$$e^{ny} w(x, y) = \sqrt{a_n^2 + b_n^2} \sin(\alpha y + nx - \beta) + O(e^{-y}).$$

Analogously, one can show that also the series of the derivatives converges uniformly in $x \in \mathbb{R}$ and that, as $y \to +\infty$,

$$e^{ny} w_x(x, y) = n \sqrt{a_n^2 + b_n^2} \cos(\alpha y + nx - \beta) + O(e^{-y}).$$

By the implicit function theorem, there exists $y^* > 0$ large enough that the nodal set of the function w in $\mathbb{R} \times (y^*, +\infty)$ is a countable union of graphs with respect to the y variable, each one asymptotic to

$$\alpha y + nx = \beta + h\pi \quad \text{for some } h \in \mathbb{Z}.$$

We choose Γ_j, $j = 1, \dots, 2n$, as $2n$ consecutive curves in this family of graphs by taking $h = j$. □

Remark 3.10. If the number of nodal zones for y small is different from $2n$, then the nodal lines of v must intersect. As a consequence, condition (b) in Section 2 fails for such a v, which cannot correspond to any element of $\mathcal{S}_{\text{rot}}$ via Proposition 2.1.

4. Nodal sets in the half-plane

In this section, we study in detail the nodal structure of the function v constructed in Proposition 3.8. For this purpose, we let

$$\mathcal{N} = \{(x, y) \in \mathbb{R} \times \mathbb{R}_+ : v(x, y) = 0\}$$

be the nodal set of v, and we call a *nodal component* of v any connected component of $\mathbb{R} \times \mathbb{R}^+ \setminus \mathcal{N}$.

We state the main result of this section. Its assumptions should be compared to those of Proposition 3.8, in particular, we point out that they imply the existence of a unique solution v of (22). We recall that, for $\Phi \in \text{Lip}([0, 2\pi])$, we write the Fourier coefficients of $e^{-\alpha x}\Phi(x)$ as

$$\phi_k = \frac{1}{2\pi} \int_0^{2\pi} e^{-(ik+\alpha)x} \Phi(x)\, dx, \quad k \in \mathbb{Z}.$$

Proposition 4.1. *Let α, μ, ω be fixed real numbers, $\Phi \in \text{Lip}([0, 2\pi])$ and $n \geq 1$ be a given integer. Let us assume that*

- *the function Φ changes sign $2n$ times in $[0, 2\pi]$, more precisely, there exist*

$$x_1 = 0 < x_2 < \cdots < x_{2n+1} = 2\pi$$

such that

$$\{x \in (0, 2\pi) : \Phi(x) > 0\} = \bigcup_{k=0}^{n-1} (x_{2k+1}, x_{2k+2}) \quad \textit{and} \quad \{x \in (0, 2\pi) : \Phi(x) < 0\} = \bigcup_{k=0}^{n-1} (x_{2k+2}, x_{2k+3});$$

- *the coefficients of the equation satisfy $\mu < \pi^2$;*
- *we have the compatibility condition*

$$\sup\{|k| : \phi_k = 0\} = n - 1 \geq 0. \tag{42}$$

Moreover, let v denote the solution of (23), whose existence is guaranteed by Proposition 3.8.

Then there exist $2n$ connected, open sets $S_1, \dots, S_{2n} \subset \mathbb{R} \times \mathbb{R}^+$ such that

- *extending the definition of S_k, by periodicity, as $S_{k+2n} = S_k + (2\pi, 0)$, $k \in \mathbb{Z}$, we have*

$$S_k \cap S_h = \varnothing \quad \textit{for every } k \neq h \qquad \textit{and} \qquad \bar{S}_k \cap \bar{S}_h \neq \varnothing \iff k - h = -1, 0, 1;$$

- *any nodal component of* v *is one of the* S_k:
$$\mathbb{R}\times\mathbb{R}^+ \setminus \mathcal{N} = \bigcup_{k\in\mathbb{Z}} S_k;$$
- *each of them touches the* x*-axis in a single (connected) interval*:
$$\bar{S}_k \cap \{(x,0)\} = [x_k, x_{k+1}] \quad \textit{for any } k = 1,\dots,2n;$$
- *they are asymptotic to a family of evenly spaced strips*: *there exists* $\beta\in\mathbb{R}$ *such that*
$$S_k \subset \{(x,y) : \beta+\pi k + o_y(1) < \alpha y + nx < \beta + \pi(k+1) + o_y(1)\} \quad \textit{as } y\to+\infty.$$

The remaining part of this section is devoted to the proof of Proposition 4.1. We shall prove it in a series of intermediate steps. First we briefly investigate the local structure of the nodal set $\mathcal{N}$.

Lemma 4.2. *Under the above notation*,

- $\mathcal{C} = \{(x,y)\in\mathbb{R}\times\mathbb{R}_+ : v(x,y)=0, \nabla v(x,y)=0\}$ *is discrete in* $\mathbb{R}\times\mathbb{R}^+$;
- $\mathcal{N}\setminus\mathcal{C}$ *is the union of countably many analytic curves*;
- *If* $\Phi(\bar{x})\neq 0$ *and* $l\in\mathbb{R}$, *then the set*
$$\mathcal{N}\cap\{(x,y) : x+ly=\bar{x}\}$$
is discrete, *and it does not accumulate at* $\{y=0\}$.

We point out that, for the moment, it may still be that $\mathcal{C}$ accumulates at some point of the discrete set $\{(x,0) : \Phi(x)=0\}$.

Proof. We recall that v satisfies (22), and v is analytic in $\mathbb{R}\times\mathbb{R}^+$ and continuous up to the boundary $\{(x,y) : y=0\}$ (see Proposition 3.8). By well-known results of Hartman and Wintner [1953], the set $\mathcal{C}$ is discrete in $\mathbb{R}\times\mathbb{R}^+$.

As a consequence, by the analytic implicit function theorem, $\mathcal{N}\setminus\mathcal{C}$ is the disjoint union of countably many analytic curves which are either unbounded, accumulate at some point of $\{(x,0) : \Phi(x)=0\}$, or meet each other at points of $\mathcal{C}$.

Finally, let $\varphi : [0,+\infty)\to\mathbb{R}$ be defined as
$$\varphi(y) = v(\bar{x}-ly, y).$$
Then φ is real analytic for $y>0$, and continuous up to $y=0$ and $\varphi(0)\neq 0$. We deduce that its zero set is discrete. Since
$$\mathcal{N}\cap\{(x,y) : x+ly=\bar{x}\} \equiv \{(\bar{x}-ly, y) : \varphi(y)=0\},$$
the lemma follows. □

Let A be any nodal component of v. In the following, for any $h\in\mathbb{Z}$, we write
$$A_h = A-(2h\pi, 0).$$
Since v is 2π-periodic in x, A_h is itself a nodal component of v. As a consequence, either A and A_h coincide, or they are disjoint. We prove that this property is independent of $h\neq 0$.

Lemma 4.3. *Let A be any nodal component of v. Then*

- *either $A \equiv A_h$ for some $h \in \mathbb{Z}$, in which case $A \equiv A_k$ for every $k \in \mathbb{Z}$,*
- *or $A \cap A_h = \varnothing$ for some $h \in \mathbb{Z}$, in which case $A \cap A_k = \varnothing$ for every $k \neq 0$, and*

$$\sup_{y>0} |\{x : (x, y) \in A\}| \leq 2\pi,$$

where $|\cdot|$ denotes the one-dimensional Lebesgue measure.

Proof. We start by examining the first alternative. Let $(\bar{x}, \bar{y}) \in A \equiv A_h$, with $h \geq 1$, so that we also have $(\bar{x} + 2h\pi, \bar{y}) \in A$. By connectedness, there exists a curve $\gamma \subset A$ joining $(\bar{x}, \bar{y})$ and $(\bar{x} + 2h\pi, \bar{y})$. Since $2h\pi/(2\pi) = h \in \mathbb{N}$, by the universal chord theorem (see, e.g., [Oxtoby 1972]), there exists $(x_1, y_1), (x_2, y_2) \in \gamma$ such that $(x_2, y_2) = (x_1, y_1) + (2\pi, 0)$. Thus $A \cap A_1 \ni (x_2, y_2)$, which implies $A \equiv A_k$ for every $k \in \mathbb{Z}$.

Conversely, let us assume that $A \cap A_k = \varnothing$ for every $k \neq 0$. Then, for every $y > 0$,

$$\{x : (x, y) \in A\} = \bigcup_{k\in\mathbb{Z}} \{x \in [2k\pi, 2(k+1)\pi) : (x, y) \in A\} = \bigcup_{k\in\mathbb{Z}} \{x \in [0, 2\pi) : (x, y) \in A_k\},$$

and such a union is disjoint by assumption. We deduce that $|\{x : (x, y) \in A\}| \leq |[0, 2\pi)|$. □

To proceed, we need the following result, which is a consequence of a Poincaré-type inequality (see Lemma A.3).

Lemma 4.4. *Let A be any nodal component of v and assume that ($\mu < \pi^2$ and)*

$$\sup_{y>0} |\{x : (x, y) \in A\}| \leq 2\pi.$$

Then $v|_A \notin H^1_0(A)$.

Proof. By contradiction, let A be any nodal component of v and assume that $v|_A \in H^1_0(A)$ and

$$\sup_{y>0} |\{x : (x, y) \in A\}| \leq 2\pi.$$

We will show that this necessarily implies $\mu \geq \pi^2$.

By assumption, the function $v \in H^1(A)$ satisfies

$$\begin{cases} -\Delta v + \omega e^{-2y} v_x = e^{-2y} \mu v & \text{in } A, \\ v = 0 & \text{on } \partial A. \end{cases}$$

Multiplying by v and integrating by parts over A yields the identity

$$\int_A |\nabla v|^2 = \mu \int_A e^{-2y} v^2;$$

indeed,

$$\frac{\omega}{2} \int_A e^{-2y} (v^2)_x = 0$$

for every $v \in H^1_0(A)$ by density of the test functions.

We argue by Steiner symmetrization with respect to the y-axis; see, e.g., [Kawohl 1985]. We stress that the weight $(x, y) \mapsto e^{-2y}$ is independent of the x variable. Let $A^* \subset (-\pi, \pi) \times \mathbb{R}^+$ be defined as

$$A^* := \left\{(x, y) : y > 0,\ |x| < \tfrac{1}{2}|\{x : (x, y) \in A\}|\right\}$$

and $v^* \in H_0^1(A^*) \subset H_0^1((-\pi, \pi)\times\mathbb{R}^+)$ be the Steiner symmetrization of the function $v|_A$. By well-known properties of the Steiner symmetrization, we obtain

$$\int_{(-\pi,\pi)\times\mathbb{R}^+} |\nabla v^*|^2 \le \mu \int_{(-\pi,\pi)\times\mathbb{R}^+} e^{-2y}(v^*)^2.$$

Since v and v^* are not identically zero, by Lemma A.3, we obtain

$$\mu \ge (j_{1/2,1})^2 = \pi^2. \qquad \square$$

Lemma 4.5. *Let y^* be defined as in Lemma 3.9, and let A denote any nodal component of v such that $A \cap \{(x, y) : y > y^*\} \ne \varnothing$. Then*

$$\sup_{y>0} |\{x : (x, y) \in A\}| \le 2\pi.$$

Proof. Without loss of generality we can assume that $v > 0$ in A and, by Lemma 3.9, there exists a half-line $\ell := \{(x, y) : y \ge y^*, \alpha y + nx = q\}$ such that $\ell \subset A$. Let us assume by contradiction that $\sup_{y>0} |\{x : (x, y) \in A\}| > 2\pi$. By Lemma 4.3, we deduce that A is 2π-periodic in the x-direction, so that also $\ell + (2\pi, 0) \subset A$. By connectedness, we can find a simple curve γ such that

$$\gamma \subset A, \quad \gamma \cap \{(x, y) : y \ge y^*\} = \ell \cup \ell + (2\pi, 0) \quad \text{and} \quad \gamma \cap \{(x, y) : y \le y^*\} \text{ is compact.}$$

As a consequence, $\mathbb{R} \times \mathbb{R}^+ \setminus \gamma = O_0 \cup O_1$, where each O_i is open and connected and only one of them, say O_1, is such that

$$O_1 \supset \{(x, y^*) : x^* < x < x^* + 2\pi\} \ne \varnothing, \quad \text{where } \alpha y^* + nx^* = q.$$

Since $\gamma \cap \{y \le y^*\}$ is compact, we deduce that there exist q_1, q_2 and $y_0 > 0$ such that

$$O_1 \subset \{(x, y) : y \ge y_0, q_1 < \alpha y + nx < q_2\}. \tag{43}$$

Now, let $B \ne A$ be any other nodal component of v satisfying $B \subset O_1$ (B exists as v changes sign in O_1, by Lemma 3.9). Then B cannot be periodic in the x-direction, and hence, by Lemma 4.3, $\sup_{y>0} |\{x : (x, y) \in B\}| \le 2\pi$. By Proposition 3.8 and (43), we have that $v|_B \in H_0^1(B)$. Thus Lemma 4.4 applies, providing a contradiction since we are assuming $\mu < \pi^2$. $\square$

In the same spirit, we show the following.

Lemma 4.6. *Let y^* be defined as in Lemma 3.9, and let A denote any nodal component of v such that $A \cap \{(x, y) : y > y^*\} \ne \varnothing$. Then $A \cap \{(x, y) : y > y^*\}$ is connected.*

Proof. The proof follows the lines of that of Lemma 4.5. Assume by contradiction that $A \cap \{(x, y) : y > y^*\}$ contains at least two connected components, say A_1 and A_2. Then, by Lemma 3.9, we can find half-lines

$\ell_j := \{(x,y) : y \geq y^*, \alpha y + nx = q_j\} \subset A_j$ and a simple curve $\gamma \subset A$ which joins such half lines. Then $\mathbb{R} \times \mathbb{R}^+ \setminus \gamma$ is the disjoint union of O_0 and O_1, and one can find a contradiction as above. □

Motivated by Lemma 4.6, we introduce the following notation.

Definition 4.7. Let $y^* > 0$ and $\beta \in \mathbb{R}$ be fixed as in Lemma 3.9. We denote with S_k, $k \in \mathbb{Z}$, the nodal component of v asymptotic to

$$\{(x,y) : \beta + \pi k < \alpha y + nx < \beta + \pi(k+1)\} \quad \text{as } y \to +\infty.$$

By Lemma 4.6, we have that S_k and S_h are disjoint, as long as $h \neq k$. To conclude the proof of Proposition 4.1, we are left to show that the sets S_k exhaust the nodal components of v. At the moment we cannot be assured that each S_k intersects the x-axis. However, in such cases, the horizontal order is preserved.

Lemma 4.8. *Let S_{k_1}, S_{k_2} be two nodal components of v as in Definition 4.7, and let $k_1 < k_2$. If $\overline{S}_{k_i} \cap \{(x,0)\} \neq \varnothing$, $i = 1,2$, then*

$$(\hat{x}_i, 0) \in \overline{S}_{k_i} \quad \Longrightarrow \quad \hat{x}_1 < \hat{x}_2.$$

Proof. This follows by connectedness since the segments $S_k \cap \{(x,y^*)\}$ are ordered according to the index k. □

Lemma 4.9. *Let A denote any nodal component of v. There exist $q_- < q_+$ such that*

$$A \subset \{(x,y) : q_- < \alpha y + nx < q_+\}.$$

Proof. We only show that $A \subset \{(x,y) : \alpha y + nx < q_+\}$, for some q^+, because the other property follows by a similar argument. In the following, we fix x_0 such that $\Phi(x_0) \neq 0$, and we write

$$\ell := \{(x,y) : y > 0, \alpha y + n(x - x_0) = 0\}, \quad L^- := \{(x,y) : y > 0, \alpha y + n(x - x_0) < 0\}.$$

Moreover, by Lemma 3.9, we can assume that v does not vanish on $\ell \cap \{(x,y) : y \geq y^*\}$.

We have to show that, for some $h \in \mathbb{Z}$,

$$A_h := A - (2h\pi, 0) \subset L^-.$$

To start with, we observe that $A_h \cap L^- \neq \varnothing$ for every $h \geq \bar{h}$ sufficiently large (indeed A is not empty). Let us assume by contradiction that $A_h \setminus L^- \neq \varnothing$ for every $h \geq \bar{h}$ as well. By connectedness, we obtain that $I_h := \ell \cap A_h$ is nonempty, relatively open in ℓ, and with nonempty (relative) boundary $\partial I_h \subset \mathcal{N}$. Finally, by Lemmas 4.5 and 4.3, we have that $I_{h_1} \cap I_{h_2} = \varnothing$ for every $h_1 \neq h_2$. We deduce that the set

$$\bigcup_{h \geq \bar{h}} \partial I_h \subset (\mathcal{N} \cap \ell \cap \{y \leq y^*\}) \quad \text{is infinite.}$$

This contradicts the last part of Lemma 4.2. □

Lemma 4.10. *Let A denote any nodal component of v. Then $v|_A \in H^1(A)$.*

Proof. This follows by Lemma 4.9 and Proposition 3.8. □

Lemma 4.11. *Let S_k be a nodal component of v as in Definition 4.7. Then $v|_{\partial S_k} \not\equiv 0$. In particular,*

$$\{x : (x,0) \in \overline{S}_k\} \quad \textit{contains a nontrivial interval.}$$

Proof. The lemma follows by Lemmas 4.4, 4.5 and 4.10. □

Lemma 4.12. *Let A denote any nodal component of v. Then*

$$\sup_{y>0} |\{x : (x,y) \in A\}| \le 2\pi.$$

Proof. Let A contradict the result; then $A \equiv A + (2\pi, 0)$ (Lemma 4.3) and $A \subset \{(x,y) : y < y^*\}$ (Lemma 4.5). As a consequence, there exists a simple curve $\gamma \subset A$, with $\gamma + (2\pi, 0) \equiv \gamma$. Then $\mathbb{R} \times \mathbb{R}^+ \setminus \gamma = O_0 \cup O_1$, where each O_i is open and connected and $O_1 \supset \{(x,y) : y \ge y^*\}$. Now, let A' be any nodal region of v intersecting $\{(x,y) : y \ge y^*\}$. Then $A \cap A' = \varnothing$. By Lemma 4.11 there exists $\gamma' \subset A'$ with one endpoint in O_1 and the other one in O_0, so that γ' intersects γ, a contradiction. □

Lemma 4.13. *Let A denote any nodal component of v. Then $v|_{\partial A} \not\equiv 0$. In particular,*

$$\{x : (x,0) \in \overline{A}\} \quad \textit{contains a nontrivial interval.}$$

Proof. The lemma follows by Lemmas 4.4, 4.12 and 4.10. □

We are ready to conclude the proof of the main result of the section.

End of the proof of Proposition 4.1. We are left to show that the sets S_k (Definition 4.7) exhaust the nodal components of the function v, so that, in particular, for each S_k, there exists two consecutive zeros of the function Φ, $x_j < x_{j+1} \in [0, 2\pi]$, and $h \in \mathbb{Z}$ such that

$$\overline{S}_k \cap \{(x,0)\} = [x_j, x_{j+1}] + (2h\pi, 0).$$

Let S_k be any connected component as in Definition 4.7; then, by Lemma 4.13 and continuity of the function v (see Proposition 3.8), there exist two consecutive zeros $x_j < x_{j+1}$ and $h \in \mathbb{Z}$ such that

$$[x_j, x_{j+1}] + (2h\pi, 0) \subset \overline{S}_k \cap \{(x,0)\}.$$

By periodicity in the x-direction, it follows that

$$[x_j, x_{j+1}] + (2(h+1)\pi, 0) \subset \overline{S}_{k+2n} \cap \{(x,0)\}.$$

Now, on the one hand, for $y \ge y^*$, we already know that the nodal set of v between S_k (included) and S_{k+2n} (excluded) is precisely given by the $2n$ sets $S_k, \dots, S_{k+2n-1}$. On the other hand, for $y = 0$, the nodal set of v between $(x_j + 2h\pi, 0)$ and $(x_j + 2(h+1)\pi, 0)$ consists in exactly $2n$ intervals. Once again, we appeal to Lemma 4.11 to infer that every $S_k, \dots, S_{k+2n-1}$ contains exactly one interval on $\{(x,0)\}$, and the intersections are ordered by Lemma 4.8. The remaining conclusions follow straightforwardly. □

5. End of the proof of Theorem 1.1

We give the proof in the case that $K = 2n$ is even. The odd case can be treated with minor changes; see the discussion at the beginning of Section 3.

In view of Proposition 2.1, the existence of an element of $\mathcal{S}_{\text{rot}}$, as defined in (6), with the required nodal properties is equivalent to the existence of a solution of (22) having trace

$$\Phi(x) = \sum_{m=1}^{K} \frac{(-1)^{m+1}}{l_m} s_m \varphi_m \tag{44}$$

(recall (16), (21)) and enjoying properties (b) and (c) in Section 2 (property (a) is already contained in (22)).

The existence of such functions is provided by Proposition 3.8, while properties (b) and (c) follow from Proposition 4.1 once Φ satisfies the compatibility conditions (42), i.e.,

$$\phi_k = \frac{1}{2\pi} \int_0^{2\pi} e^{-(ik+\alpha)x} \Phi(x)\, dx = 0, \quad |k| < n, \quad \text{and} \quad \phi_n \neq 0 \tag{45}$$

(or equivalently $\phi_{-n} = \bar{\phi}_n \neq 0$). Under the validity of these conditions, also the asymptotic expansion (8) follows from Proposition 4.1 and the definition of the map $\mathcal{T}$ (14); see also Remark 2.2. The details of these calculations are very similar to those in [Terracini et al. 2019, Proof of Theorem 1.5]

Writing $c_m = s_m / l_m$ in (44) and (45), and recalling also Remark 3.10, we obtain that Theorem 1.1 is equivalent to the following assertion: *there exists* $\bar{c} = (\bar{c}_1, \ldots, \bar{c}_{2n})$, *with* $(-1)^{m+1} c_m > 0$, *such that*

$$\sum_{m=1}^{2n} \frac{1}{2\pi} \int_0^{2\pi} e^{-(ik+\alpha)x} c_m \varphi_m(x)\, dx = 0, \quad |k| < n,$$

and

$$\sum_{m=1}^{2n} \frac{1}{2\pi} \int_0^{2\pi} e^{-(in+\alpha)x} c_m \varphi_m(x)\, dx \neq 0$$

if and only if $c = t\bar{c}$.

To prove this last claim, let us define the matrix $A \in \mathbb{C}^{2n \times 2n}$,

$$A = (a_{km})_{\substack{k=-n+1,\ldots,n \\ m=1,\ldots,2n}} = \left(\frac{1}{2\pi} \int_0^{2\pi} e^{-(ik+\alpha)x} \varphi_m(x)\, dx \right)_{km} = \left(\frac{1}{2\pi} \int_0^{2\pi} e^{-(ik+\alpha)t_m} \varphi_m(t_m)\, dt_m \right)_{km}.$$

Observe that we have suitably renamed the dummy variables in each integral as, later, this will lead us to more manageable identities. We can write the set of compatibility conditions (45) as a system of linear equations,

$$A \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_{2n} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ \phi_n \end{pmatrix}. \tag{46}$$

To show our claim, we prove that the matrix A is invertible and that it is possible to choose $\phi_n \neq 0$ such that the solution vector is real and sign-alternating. First, exploiting the multilinearity of the determinant, we have

$$\det A = \frac{1}{(2\pi)^{2n}} \int_{[0,2\pi]^{2n}} \prod_{m=1}^{2n} e^{-\alpha t_m} \varphi_m(t_m) \cdot \det A',$$

where we have introduced the matrix

$$A' = \begin{pmatrix} e^{-i(-n+1)t_1} & e^{-i(-n+1)t_2} & \dots & e^{-i(-n+1)t_{2n}} \\ e^{-i(-n+2)t_1} & e^{-i(-n+2)t_2} & \dots & e^{-i(-n+2)t_{2n}} \\ \vdots & \vdots & \ddots & \vdots \\ e^{-int_1} & e^{-int_2} & \dots & e^{-int_{2n}} \end{pmatrix}.$$

Factoring out the coefficients of the first row, we recognize Vandermonde's determinant and compute

$$\begin{aligned}
\det A' &= e^{-i(-n+1)\sum_{m=1}^{2n} t_m} \begin{vmatrix} 1 & \dots & 1 \\ e^{-it_1} & \dots & e^{-it_{2n}} \\ \vdots & \ddots & \vdots \\ e^{-(2n-1)it_1} & \dots & e^{-(2n-1)it_{2n}} \end{vmatrix} \\
&= e^{-i(-n+1)\sum_{m=1}^{2n} t_m} \prod_{1\le p<q\le 2n} (e^{-it_q} - e^{-it_p}) \\
&= e^{i(n-1)\sum_{m=1}^{2n} t_m} \prod_{1\le p<q\le 2n} (-1) e^{-\frac{1}{2}it_q - \frac{1}{2}it_p} (-e^{-\frac{1}{2}it_q + \frac{1}{2}it_p} + e^{-\frac{1}{2}it_p + \frac{1}{2}it_q}) \\
&= e^{i(n-1)\sum_{m=1}^{2n} t_m} (-1)^{\frac{2n(2n-1)}{2}} e^{-\frac{1}{2}i(2n-1)\sum_{m=1}^{2n} t_m} \prod_{1\le p<q\le 2n} (e^{-\frac{1}{2}it_p + \frac{1}{2}it_q} - e^{-\frac{1}{2}it_q + \frac{1}{2}it_p}) \\
&= (-1)^n (2i)^{\frac{2n(2n-1)}{2}} e^{-\frac{1}{2}i\sum_{m=1}^{2n} t_m} \prod_{1\le p<q\le 2n} \left(\frac{e^{\frac{1}{2}i(t_q-t_p)} - e^{-\frac{1}{2}i(t_q-t_p)}}{2i} \right) \\
&= (-1)^n (2i)^{n(2n-1)} e^{-\frac{1}{2}i\sum_{m=1}^{2n} t_m} \prod_{1\le p<q\le 2n} \sin\left(\frac{t_q-t_p}{2} \right).
\end{aligned}$$

Thus we find

$$\det A = \frac{(-1)^n (2i)^{n(2n-1)}}{(2\pi)^{2n}} \int_{[0,2\pi]^{2n}} \underbrace{\prod_{m=1}^{2n} e^{-\alpha t_m} \varphi_m(t_m) \prod_{1\le p<q\le 2n} \sin\left(\frac{t_q-t_p}{2} \right)}_{\text{Mod}} \underbrace{e^{-\frac{1}{2}i\sum_{m=1}^{2n} t_m}}_{\text{Phase}}.$$

We show that the integral in the previous expression is always different from 0. We recall that, by assumption, the functions φ_m are supported on ordered intervals. More precisely, using the notation introduced in Proposition 4.1, we have

$$\{t \in [0, 2\pi] : \varphi_m(t) > 0\} = (x_m, x_{m+1}).$$

As a result, the integral can be restricted to the open and not empty set

$$\mathcal{O} = (x_1, x_2) \times (x_2, x_3) \times \cdots \times (x_{2n}, x_{2n+1}) \subset [0, 2\pi]^{2n}.$$

Moreover, for any choice $1 \le p < q \le 2n$, in $\mathcal{O}$ we have $0 < t_q - t_p < 2\pi$, and thus

$$0 < \frac{t_q - t_p}{2} < \pi \quad \Longrightarrow \quad \sin\left(\frac{t_q - t_p}{2}\right) > 0.$$

As it turns out, the factor denoted as Mod is strictly positive in $\mathcal{O}$. This function corresponds to the modulus of the integral function. On the other hand, the factor Phase is complex and of modulus 1. Let us investigate more closely the argument of Phase. We find

$$\sum_{m=1}^{2n} x_m < \sum_{m=1}^{2n} t_m < \sum_{m=1}^{2n} x_{m+1} = \sum_{m=1}^{2n} x_m + (x_{2n+1} - x_1) < \sum_{m=1}^{2n} x_m + 2\pi.$$

That is, letting $X = \sum_{m=1}^{2n} x_m$, for any $(t_1, \dots, t_{2n}) \in \mathcal{O}$,

$$0 < \frac{1}{2}\left(\sum_{m=1}^{2n} t_m - X\right) < \pi.$$

We can rewrite the determinant as

$$\det A = C\left[\int_{\mathcal{O}} \mathrm{Mod} \cdot \cos \frac{1}{2}\left(\sum_{m=1}^{2n} t_m - X\right) - i \int_{\mathcal{O}} \mathrm{Mod} \cdot \sin \frac{1}{2}\left(\sum_{m=1}^{2n} t_m - X\right)\right]$$

for some complex constant $C \in \mathbb{C} \setminus \{0\}$. By the previous discussion, the second integral is positive. It follows that the determinant of A is not zero, proving that the linear system (46) has a unique solution for any ϕ_n.

We now show that there exists $\phi_n \neq 0$ such that the solution vector is real and sign-alternating. By Cramer's rule, we have

$$c_l = (\det A)^{-1} \det A_l,$$

where A_l is the matrix obtained by replacing the l column of A with the right-hand side of system (46). Now, by the same considerations as before, we have

$$\det A_l = \frac{1}{(2\pi)^{2n}} \int_{[0,2\pi]^{2n}} \prod_{m=1, m\neq l}^{2n} e^{-\alpha t_m} \varphi_m(t_m) \cdot \det A_l',$$

where

$$A_l' = \begin{pmatrix} e^{-i(-n+1)t_1} & e^{-i(-n+1)t_2} & \dots & e^{-i(-n+1)t_{l-1}} & 0 & e^{-i(-n+1)t_{l+1}} & \dots & e^{-i(-n+1)t_{2n}} \\ e^{-i(-n+2)t_1} & e^{-i(-n+2)t_2} & \dots & e^{-i(-n+2)t_{l-1}} & 0 & e^{-i(-n+2)t_{l+1}} & \dots & e^{-i(-n+2)t_{2n}} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ e^{-int_1} & e^{-int_2} & \dots & e^{-int_{l-1}} & \phi_n & e^{-int_{l+1}} & \dots & e^{-int_{2n}} \end{pmatrix}.$$

Developing the determinant with respect to the l-th column, factoring out the first line and exploiting once more Vandermonde's determinant, we find

$$\det A'_l = (-1)^{l-1}\phi_n e^{-i(-n+1)\sum_{m=1,m\neq l}^{2n} t_m} \begin{vmatrix} 1 & \cdots & 1 \\ e^{-it_1} & \cdots & e^{-it_{2n}} \\ \vdots & \ddots & \vdots \\ e^{-(2n-2)it_1} & \cdots & e^{-(2n-2)it_{2n}} \end{vmatrix}$$

$$= (-1)^{l-1}\phi_n e^{-i(-n+1)\sum_{m=1,m\neq l}^{2n} t_m} \prod_{\substack{1\le p<q\le 2n \\ p,q\neq l}} (e^{-it_q} - e^{-it_p})$$

$$= (-1)^{l-1}\phi_n e^{i(n-1)\sum_{m=1,m\neq l}^{2n} t_m} \prod_{\substack{1\le p<q\le 2n \\ p,q\neq l}} (-1)e^{-\frac{1}{2}it_q-\frac{1}{2}it_p}(-e^{-\frac{1}{2}it_q+\frac{1}{2}it_p} + e^{-\frac{1}{2}it_p+\frac{1}{2}it_q})$$

$$= (-1)^{l-1}\phi_n e^{i(n-1)\sum_{m=1,m\neq l}^{2n} t_m}(-1)^{\frac{(2n-1)(2n-2)}{2}} e^{-\frac{1}{2}i(2n-2)\sum_{m=1,m\neq l}^{2n} t_m} \times \prod_{\substack{1\le p<q\le 2n \\ p,q\neq l}} (e^{-\frac{1}{2}it_p+\frac{1}{2}it_q} - e^{-\frac{1}{2}it_q+\frac{1}{2}it_p})$$

$$= (-1)^{l+n-2}\phi_n(2i)^{\frac{(2n-1)(2n-2)}{2}} \prod_{\substack{1\le p<q\le 2n \\ p,q\neq l}} \left(\frac{e^{\frac{1}{2}i(t_q-t_p)} - e^{-\frac{1}{2}i(t_q-t_p)}}{2i}\right)$$

$$= (-1)^{l+n}(2i)^{(2n-1)(n-1)}\phi_n \prod_{\substack{1\le p<q\le 2n \\ p,q\neq l}} \sin\left(\frac{t_q-t_p}{2}\right).$$

We obtain

$$c_l = \frac{(\det A)^{-1}(-1)^{l+n}(2i)^{(2n-1)(n-1)}\phi_n}{(2\pi)^{2n-1}} \int \prod_{m=1,m\neq l}^{2n} e^{-\alpha t_m}\varphi_m(t_m) \prod_{1\le p<q\le 2n,p,q\neq l} \sin\left(\frac{t_q-t_p}{2}\right)$$

$$= (-1)^{l+1}\Gamma \int_{[0,2\pi]^{2n-1}} \prod_{m=1,m\neq l}^{2n} e^{-\alpha t_m}\varphi_m(t_m) \prod_{1\le p<q\le 2n,p,q\neq l} \sin\left(\frac{t_q-t_p}{2}\right),$$

where $\Gamma \in \mathbb{C}$. Reasoning as before, we see that the integral is always strictly positive. Thus c_l satisfies the condition $(-1)^{l+1}c_l > 0$ if and only if Γ is real and positive, $\Gamma = t > 0$. We obtain the solution

$$c_l = t(-1)^{l+1} \int_{[0,2\pi]^{2n-1}} \prod_{m=1,m\neq l}^{2n} e^{-\alpha t_m}\varphi_m(t_m) \prod_{1\le p<q\le 2n,p,q\neq l} \sin\left(\frac{t_q-t_p}{2}\right)$$

and

$$\phi_n = t(-1)^{n+1}\frac{2^{2n-2}}{\pi} \int_{[0,2\pi]^{2n}} \prod_{m=1}^{2n} e^{-\alpha t_m}\varphi_m(t_m) \prod_{1\le p<q\le 2n} \sin\left(\frac{t_q-t_p}{2}\right) e^{-\frac{1}{2}i\sum_{m=1}^{2n} t_m}.$$

Proof of Corollary 1.2. This follows by uniqueness of $\bar{s}$; indeed, notice that a rotation of $2\pi/K$ leaves the data unchanged, while the indexes of the densities are shifted by 1. By uniqueness, $\bar{s}_m = \bar{s}_{m-1}$ for every m. □

6. Single-mode special solutions

In the following we deal with the fundamental single-mode solutions that we constructed by separation of variables in Section 3. Theorems 1.6 and 1.7 will follow once again by Proposition 2.1.

6.1. *The homogeneous Dirichlet problem.* We now turn our attention to the homogeneous version of (22); that is, we look for conditions under which there exists a nonzero solution v of

$$\begin{cases} -\Delta v + \omega e^{-2y} v_x = e^{-2y} \mu v, & x \in \mathbb{R}, \quad y > 0, \\ v(x+2\pi, y) = e^{2\pi\alpha} v(x, y), & x \in \mathbb{R}, \quad y \geq 0, \\ v(x, 0) = 0, & 0 \leq x \leq 2\pi, \end{cases} \tag{47}$$

with nodal set consisting of $2k$ strips (up to horizontal 2π-periodicity), $k \geq 1$, that connect the boundary $y = 0$ with $y \to +\infty$, as in the previous section. Clearly (47) may have nonzero solutions only for some specific choices of parameters (this is indeed the case according to Lemma 3.6). For this reason, in this section we consider the number $k \geq 1$ and the parameter $\alpha \in \mathbb{R}$ as givens of the problem, and we look for pairs of numbers $(\mu, \omega) \in \mathbb{R}^2$ such that a solution v as specified above exists.

The analysis that we have conducted in Section 3 can be exploited to give a direct solution to this problem. Indeed we have the following result.

Lemma 6.1. *For any $k \geq 1$, $\alpha \in \mathbb{R}$, there exists at least a value $\lambda \in \mathbb{C}$ satisfying*

$$\begin{cases} \Theta_{k-i\alpha}(\lambda) = 0, \\ \Theta_{k-i\alpha}(t\lambda) \neq 0 \quad \text{for all } t \in [0, 1), \end{cases} \tag{48}$$

where Θ_ν is defined in (32) for every $\nu \in \mathbb{C}$. For any such λ, the function

$$v(x, y) = e^{\alpha x - ky} \operatorname{Re}(e^{i(kx+\alpha y)} \Theta_{k-i\alpha}(\lambda e^{-2y}))$$

is a solution of (47), with

$$\omega = \frac{\operatorname{Im}(\lambda)}{k}, \quad \mu = \alpha \frac{\operatorname{Im}(\lambda)}{k} - \operatorname{Re}(\lambda).$$

Moreover, there exists an analytic map $y \mapsto \zeta(y)$ such that

$$v(x, y) = 0 \quad \Longleftrightarrow \quad x = \zeta(y) + \frac{h\pi}{k}, \quad h \in \mathbb{Z},$$

and

$$\zeta(y) = \frac{1}{k}(\beta - \alpha y) + o(1) \quad \text{for some } \beta \in \mathbb{R} \text{ and } y \to +\infty.$$

In particular, for any $y > 0$, $v(\cdot, y)$ has exactly $2k$ zeros in each period $x \in [0, 2\pi)$.

Proof. The result is a direct consequence of Lemma 3.2. We start by showing that, for any choice of parameters, there exists at least a value $\lambda \in \mathbb{C}$ satisfying (48). Indeed, $\Theta_{k-i\alpha}$ is a nonconstant analytic function with $\Theta_{k-i\alpha}(0) \neq 0$, and it suffices to consider a zero λ of $\Theta_{k-i\alpha}$ with the least absolute value in order to guarantee that $\Theta_{k-i\alpha}(t\lambda) \neq 0$ for any $t \in [0, 1)$. Of course, many (if not all) the zeros of $\Theta_{k-i\alpha}$ may satisfy this assumption, but these constitute an at most countable discrete subset of $\mathbb{C}$.

Exploiting the fact that the coefficients of (47) are real, we find that the function

$$v(x, y) = e^{\alpha x} \operatorname{Re}(e^{ikx} D_k(y)) \tag{49}$$

is a solution of (47), where the function D_k solves

$$\begin{cases} D_k''(y) = [(k - i\alpha)^2 + (\omega\alpha - \mu + i\omega k)e^{-2y}] D_k(y), & y > 0, \\ D_k(0) = 0, \quad D_k(y) \to 0 & \text{as } y \to +\infty. \end{cases} \tag{50}$$

By Lemma 3.2, equation (50) is solved by any multiple of the function

$$y \mapsto e^{-(k-i\alpha)y} \Theta_{k-i\alpha}((\omega\alpha - \mu + i\omega k)e^{-2y}),$$

which in turns vanishes for $y \to +\infty$. The initial condition $D_k(0) = 0$ is satisfied since we chose $\lambda = \omega\alpha - \mu + i\omega k$ as a zero of the function $\Theta_{k-i\alpha}$ (observe that we are negating (26)).

To conclude, we need to study the nodal properties of the function v. From its expression we readily see, that for any fixed $y > 0$, the function $x \mapsto v(x, y)$ has exactly $2k$ evenly spaced zeros in $[0, 2\pi)$ since, by assumption, $\Theta_{k-i\alpha}(\lambda e^{-2y}) \neq 0$. From this we deduce also that the nodal lines of v can be described, up to translations, by a function $y \mapsto \zeta(y)$. We notice that ζ is continuous by the implicit function theorem, as

$$v(x, y) = 0 \quad \Longleftrightarrow \quad \operatorname{Re}(e^{ikx} D_k(y)) = 0$$

and, for such (x, y),

$$\frac{\partial}{\partial x} \operatorname{Re}(e^{ikx} D_k(y)) = ik \operatorname{Im}(e^{ikx} D_k(y)) \neq 0.$$

More explicitly, writing

$$D_k(y) = \rho(y) e^{i\vartheta(y)},$$

where $\rho(y) > 0$ for $y > 0$ and ϑ is an analytic lifting of the argument of D_k, we have that

$$e^{\alpha x} v(x, y) = \operatorname{Re}(e^{ikx} D_k(y)) = 0 \quad \Longleftrightarrow \quad x - \frac{h\pi}{k} = \frac{1}{k}(\beta - \vartheta(y)) =: \zeta(y).$$

Finally, the asymptotic behavior of ζ follows as in Lemma 3.9. □

We conclude with some additional remarks on the result.

Remark 6.2 (a question about uniqueness). If v is a solution of (47), then for any $A, \bar{x} \in \mathbb{R}$, the function $(x, y) \mapsto Av(x - \bar{x}, y)$ is again a solution. We may wonder whether this family of functions completely describes the set of solutions of (47) under some additional condition (for instance that, for any $x \in \mathbb{R}$, $v(x, y) \to 0$ as $y \to +\infty$). More precisely, fix ω, μ and α in such a way that (47) admits at least a solution. Is this solution unique (up to translation in x and multiplication by a real constant of course)? This seems to be a question of a nontrivial nature, and it is related to the position of the zeros of Bessel functions with different order. From the proof of Lemma 6.1, we can state the following: let $\alpha \in \mathbb{R}$ be such that, for any $k_1, k_2 \geq 1$ and $z_1, z_2 \in \mathbb{C}$, we have

$$\begin{cases} I_{k_1 - i\alpha}(z_1) = I_{k_2 - i\alpha}(z_2) = 0, \\ \operatorname{Re}(z_1^2) = \operatorname{Re}(z_2^2), \\ \operatorname{Im}(z_1^2)/k_1 = \operatorname{Im}(z_2^2)/k_2 \end{cases} \quad \Longrightarrow \quad k_1 = k_2.$$

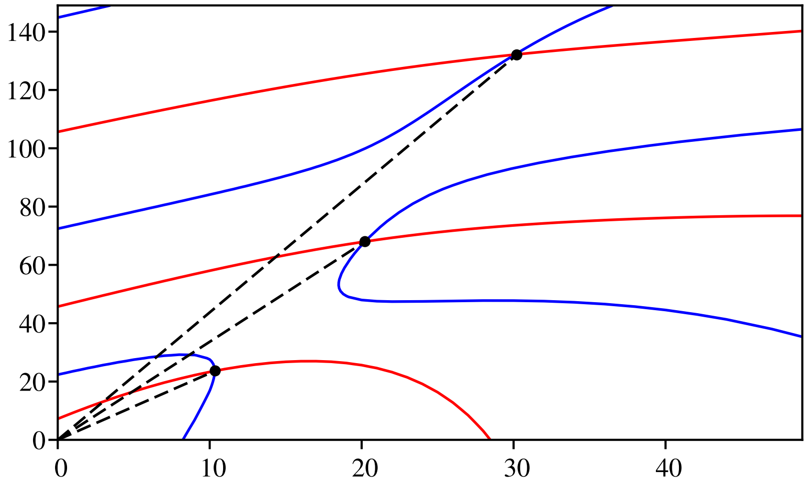

Figure 3. Numerical zeros of $\operatorname{Re}\Theta_{1-i}$ (blue) and $\operatorname{Im}\Theta_{1-i}$ (red). The three zeros located at $10.36 + i23.66$, $20.22 + i67.99$, $30.21 + i132.04$ satisfy condition (48).

Then for this specific value of α, if (47) admits a solution, this solution is unique up to translation in x and multiplication by a real constant.

Remark 6.3 (the symmetric case $\alpha = 0$). If $\nu \in \mathbb{R}$ and $\nu \geq 1$, the zeros of the modified Bessel function I_ν are purely imaginary numbers (and are given by $ij_{\nu,l}$, where $j_{\nu,l}$ is the l-th zero of the Bessel function J_ν, with $l \in \mathbb{N}$). It follows that

$$\Theta_k(\lambda) = 0 \quad \Longrightarrow \quad \lambda = -t^2 \text{ for some } t > 0.$$

As a result, if $\alpha = 0$, then necessarily $\omega = 0$ (no rotation) and $\mu = j_{k,1}^2$. Since all the zeros belong to the same half-line emanating from the origin, the first nontrivial zero is also the only one that satisfies the assumptions of Lemma 6.1. We conclude that, in the case $\alpha = 0$, (47) has nonzero solutions only if $\mu = j_{k,1}^2$ and $\omega = 0$, and any solution (that converges to zero as $y \to +\infty$) is of the form

$$v(x, y) = (A\cos(kx) + B\sin(kx))J_k(j_{k,1}e^{-y})$$

for some $A, B \in \mathbb{R}$.

Remark 6.4 (the asymmetric case $\alpha \neq 0$). By Lemma 3.6, and in particular (37), we already know that, if $\alpha \neq 0$, for (47) to have a solution, it is necessary that

$$\mu \geq (j_{0,1} + \sqrt{k^2 + \alpha^2})^2.$$

From numerical explorations (see, e.g., Figures 3 and 4), it seems that, if $\alpha \neq 0$, the zeros of the function $\Theta_{k-i\alpha}$ belong to different lines emanating from the origin. In contrast with the case $\alpha = 0$, it thus seems to be the case that, for $\alpha \neq 0$, (47) has infinitely many (but still countably many) solutions.

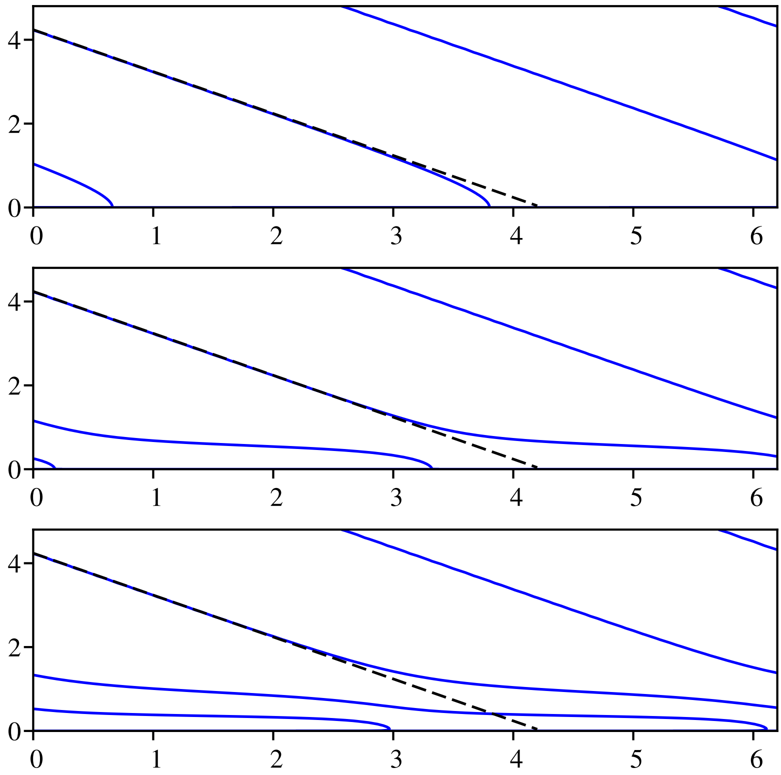

Figure 4. Nodal sets of the solutions corresponding to the three zeros in Figure 3.

6.2. ***The homogeneous Neumann/Robin problem.*** Let $\sigma \in \mathbb{R}$. We consider the problem

$$\begin{cases} -\Delta v + \omega e^{-2y} v_x = e^{-2y} \mu v, & x \in \mathbb{R}, \quad y > 0, \\ v(x + 2\pi, y) = e^{2\pi\alpha} v(x, y), & x \in \mathbb{R}, \quad y \geq 0, \\ \partial_y v(x, 0) + \sigma v(x, 0) = 0, & 0 \leq x \leq 2\pi, \end{cases} \tag{51}$$

which involves Robin ($\sigma \neq 0$) or Neumann ($\sigma = 0$) boundary conditions.

As in the previous section we can find single-mode solutions that exhibit a precise nodal behavior.

Lemma 6.5. *For any $k \geq 1$, $\alpha \in \mathbb{R}$, assume that there exists $\lambda \in \mathbb{C}$ satisfying*

$$\begin{cases} 2\lambda \Theta'_{k-i\alpha}(\lambda) + (k - i\alpha - \sigma)\Theta_{k-i\alpha}(\lambda) = 0, \\ \Theta_{k-i\alpha}(t\lambda) \neq 0 \quad \textit{for all } t \in [0, 1). \end{cases}$$

Then we have

$$v(x, y) = e^{\alpha x - ky} \operatorname{Re}(e^{i(kx+\alpha y)} \Theta_{k-i\alpha}(\lambda e^{-2y}))$$

a solution of (51) for the particular choice of parameters

$$\omega = \frac{\operatorname{Im}(\lambda)}{k}, \quad \mu = \alpha \frac{\operatorname{Im}(\lambda)}{k} - \operatorname{Re}(\lambda).$$

Moreover, the nodal set of v has the same properties as those described in Lemma 6.1.

Proof. We already know that any function of the type

$$v(x,y) = e^{\alpha x}\,\mathrm{Re}(e^{ikx}N_k(y))$$

is a solution of the differential equation in (51) provided that

$$N_k''(y) = [(k-i\alpha)^2 + (\omega\alpha - \mu + i\omega k)e^{-2y}]N_k(y), \quad y > 0.$$

Once again we can appeal to Lemma 3.2 for an explicit expression for the function N_k. In order to impose the boundary condition at $y = 0$ we find

$$N_k'(y) = \Theta_{k-i\alpha}'(\lambda e^{-2y})(-2\lambda e^{-2y})e^{-(k-i\alpha)y} - \Theta_{k-i\alpha}(\lambda e^{-2y})(k-i\alpha)e^{-(k-i\alpha)y};$$

that is,

$$N_k'(0) = \Theta_{k-i\alpha}'(\lambda)(-2\lambda) - (k-i\alpha)\Theta_{k-i\alpha}(\lambda) = 0.$$

The rest of the proof follows easily. □

6.3. ***Entire solutions.*** Finally we consider the case of entire solutions; that is, we look for functions v that satisfy

$$\begin{cases} -\Delta v + \omega e^{-2y} v_x = e^{-2y}\mu v, \\ v(x+2\pi, y) = e^{2\pi\alpha} v(x,y), \end{cases} \qquad (x,y) \in \mathbb{R}^2, \tag{52}$$

vanish for $y \to +\infty$ and, as before, change sign exactly $2k$ times ($k \ge 1$) in each period of length 2π in the x-direction. Similar considerations as before lead us to the following result.

Lemma 6.6. *Let $k \ge 1$, $\alpha \in \mathbb{R}$. Consider any $\lambda \in \mathbb{C}$ such that*

$$\Theta_{k-i\alpha}(t\lambda) \neq 0 \quad \textit{for all } t > 0. \tag{53}$$

Then the function

$$v(x,y) = e^{\alpha x - ky}\,\mathrm{Re}(e^{i(kx+\alpha y)}\Theta_{k-i\alpha}(\lambda e^{-2y})) \tag{54}$$

is a solution of (52) for the particular choice of parameters

$$\omega = \frac{\mathrm{Im}(\lambda)}{k}, \quad \mu = \alpha\frac{\mathrm{Im}(\lambda)}{k} - \mathrm{Re}(\lambda).$$

Once again, we point out that $\Theta_{k-i\alpha}$ is analytic and thus it has at most countably many zeros, meaning that, apart from a negligible set, any $\lambda \in \mathbb{C}$ gives rise to an entire solution.

In the case of entire solutions, it is interesting to study once again the shape of the nodal lines of the solutions, which now are defined also for $y < 0$.

Lemma 6.7. *Let v be the function (54) in Lemma 6.6. Then there exists an analytic function $y \mapsto \zeta(y)$, defined for any $y \in \mathbb{R}$, such that*

- *$v(x,y) = 0$ if and only if $x = \zeta(y) + h\pi/k$, $y \in \mathbb{R}$, $h \in \mathbb{Z}$, and consequently, in the regions $\{(x,y) : h\pi/k < x - \zeta(y) < (h+1)\pi/k\}$, for any $h \in \mathbb{Z}$, v does not change sign;*

- *for* $y \to +\infty$, ζ *is asymptotic to a line*: *there exists* $\beta \in \mathbb{R}$ *such that*

$$\zeta(y) = \frac{1}{k}(\beta - \alpha y) + o(1) \quad \text{as } y \to +\infty;$$

- *for* $y \to -\infty$, ζ *is asymptotic to an exponential curve*

$$\zeta(y) = \gamma e^{-y} + O(1) \quad \text{as } y \to -\infty,$$

where

$$\gamma = \begin{cases} \frac{1}{k}\operatorname{sign}(\omega)\sqrt{\sqrt{\left(\frac{1}{2}(\omega\alpha - \mu)\right)^2 + \left(\frac{1}{2}\omega k\right)^2} - \frac{1}{2}(\omega\alpha - \mu)}, & \omega \neq 0, \\ 0, & \omega = 0, \quad \mu < 0, \\ \frac{1}{k}\operatorname{sign}(\alpha)\sqrt{\mu}, & \omega = 0, \quad \mu > 0, \end{cases}$$

unless $\omega = \mu = 0$, *in which case*

$$\zeta(y) = \frac{1}{k}(\beta - \alpha y), \quad y \in \mathbb{R}.$$

Proof. The first conclusions of the result follow from similar (and much simpler) considerations as in Proposition 4.1 and Lemma 6.1. We only study the asymptotic behavior of ζ as $y \to -\infty$. As we shall see, beyond the validity of (53), we need to distinguish three cases, according to the different expansions of the Bessel functions at infinity: (Case 1) $\omega = \mu = 0$; (Case 2) $\omega = 0$, $\mu > 0$; (Case 3) either $\omega = 0$ and $\mu < 0$, or $\omega \neq 0$.

Case 1. We start with the simplest case, that is $\omega = \mu = 0$. This is equivalent to assuming that $\lambda = 0$, whence (53) is automatically satisfied (recall that $\Theta_{k-i\alpha}(0) \neq 0$ for $k \geq 1$). Substituting in (52) we find that solutions are of the form

$$v(x, y) = e^{\alpha x - ky}\cos(kx + \alpha y).$$

In this case the nodal lines are described, up to translations, by the linear function

$$\zeta(y) = \frac{1}{k}\left(\tfrac{\pi}{2} - \alpha y\right), \quad y \in \mathbb{R},$$

and, in particular, the nodal set of v is a family of parallel straight lines.

Case 2. Next, we look at the case $\omega = 0$ and $\mu > 0$, which means $\lambda = -\mu < 0$. We have that $\sqrt{\lambda} = -i\sqrt{\mu}$, where we have chosen the determination of the square root with negative imaginary part. In this case, exploiting (54), (33) and the relation between the Bessel functions and their modified versions, we have

$$v(x, y) = e^{\alpha x}\left(\tfrac{1}{2}e^{ikx}J_\nu(\sqrt{\mu}e^{-y}) + \tfrac{1}{2}e^{-ikx}\overline{J_\nu(\sqrt{\mu}e^{-y})}\right)$$

(to be precise, we take the line $y \mapsto \sqrt{\lambda}e^{-y}$ as the path of monodromy for the determination of J_ν). In particular, from this expression we infer the necessary condition $\alpha \neq 0$: indeed, if $\nu = k \geq 1$, the Bessel function J_k has all of its zeros on the real line, and thus we are contradicting (53). We have that (see [Erdélyi et al. 1953, p. 85])

$$J_\nu(z) = \sqrt{\frac{2}{\pi z}}\left(\cos\left(z - \tfrac{\pi}{2}\nu - \tfrac{\pi}{4}\right) + O\left(\frac{1}{|z|}\right)\right) \quad \text{for } |z| \to +\infty \text{ with } |\arg z| < \pi.$$

As to what concerns us, we have that $z > 0$. Letting

$$w = \sqrt{\mu}e^{-y} - \tfrac{\pi}{2}\nu - \tfrac{\pi}{4} = \left(\sqrt{\mu}e^{-y} - \tfrac{\pi}{2}k - \tfrac{\pi}{4}\right) + i\tfrac{\pi}{2}\alpha,$$

we may simplify the expression for v and see that, for $y \to -\infty$, the following asymptotic expansion holds:

$$\sqrt{\tfrac{1}{2}\pi\sqrt{\mu}}e^{-\alpha x - \frac{1}{2}y}v(x, y) = \tfrac{1}{2}e^{ikx}\cos w + \tfrac{1}{2}e^{-ikx}\cos\overline{w} + O(e^{y}).$$

We point out that, in this peculiar case, the solution v decays for $y \to -\infty$ since $\operatorname{Im}(w)$ is bounded (constant). The last expression can be further simplified, since

$$\begin{aligned}\tfrac{1}{2}e^{ikx}\cos w + \tfrac{1}{2}e^{-ikx}\cos\overline{w} &= \tfrac{1}{2}(\cos(kx) + i\sin(kx))\cos w + \tfrac{1}{2}(\cos(kx) - i\sin(kx))\cos\overline{w} \\ &= \tfrac{1}{2}\cos(kx)[\cos w + \cos\overline{w}] + \tfrac{1}{2}i\sin(kx)[\cos w - \cos\overline{w}] \\ &= \cos(kx)\cos(\operatorname{Re} w)\cosh(\operatorname{Im} w) + \sin(kx)\sin(\operatorname{Re} w)\sinh(\operatorname{Im} w).\end{aligned}$$

In order to determine the asymptotic behavior of the nodal lines of v, we need to solve the equation

$$\cos(kx)\cos(\operatorname{Re} w)\cosh(\operatorname{Im} w) + \sin(kx)\sin(\operatorname{Re} w)\sinh(\operatorname{Im} w) = 0.$$

It seems that this equation cannot be solved explicitly, nevertheless we can describe its set of solutions with sufficient accuracy for our purpose. In order to simplify the notation, we introduce the real function

$$F(X, Y) = \cos(X)\cos(Y)\cosh(T) + \sin(X)\sin(Y), \sinh(T) \tag{55}$$

where we recall that the parameter $T = \operatorname{Im} w = \frac{\pi}{2}\alpha \neq 0$. In the plane $(X, Y) \in \mathbb{R}^2$, we want to describe the set $F(X, Y) = 0$. First of all, we point out that F is 2π-period both in X and in Y and enjoys the symmetries $F(X, Y) = F(Y, X)$, $F(-X, Y) = F(X, -Y)$, $F(X+\pi, Y) = F(X, Y+\pi) = -F(X, Y)$ and $F(-X, -Y) = F(X, Y)$ for any $(X, Y) \in \mathbb{R}^2$. In particular, we deduce that the equation $F(X, Y) = 0$ has infinitely many solutions and that, for any fixed $Y \in \mathbb{R}$ (resp. X), solutions of $F(X, Y) = 0$ are equally spaced and of the form $X = X_Y + h\pi$ for some given $X_Y \in \mathbb{R}$ and $h \in \mathbb{Z}$ (resp. $Y = Y_X + h\pi$, $Y_X \in \mathbb{R}$). We deduce that, for any given $Y \in [0, \pi)$, there exists a unique $X \in [0, \pi)$ such that $F(X, Y) = 0$, and vice versa.

Next, let $(X_0, Y_0) \in \mathbb{R}^2$ such that $F(X_0, Y_0) = 0$. By the implicit function theorem, the nodal set of F is described locally at (X_0, Y_0) by a function $X = Z(Y)$ if $\partial_X F(X_0, Y_0) \neq 0$. Arguing by contradiction, we have the system

$$\begin{cases}\cos(X_0)\cos(Y_0)\cosh(T) + \sin(X_0)\sin(Y_0)\sinh(T) = 0, \\ \cos(X_0)\sin(Y_0)\sinh(T) - \sin(X_0)\cos(Y_0)\cosh(T) = 0,\end{cases}$$

which has a solution if and only if

$$\cos^2(Y_0)\cosh^2(T) + \sin^2(Y_0)\sinh^2(T) = 0.$$

But this is impossible since $\cosh^2(T) \neq 0$ and $\sinh^2(T) \neq 0$ (recall that $T \neq 0$). Thus $\partial_X F(X_0, Y_0) \neq 0$ at any zero of F. Observe that we can perform similar computations exchanging variables and show that the function Z is a bijection (and thus monotone). By periodicity, we can assume that $Z(0) = \frac{\pi}{2}$. We can determine the sense of monotonicity of Z by computing $Z'(Y)$ for the zero $(X, Y) = \left(\frac{\pi}{2}, 0\right)$. We find

$$Z'(0) = -\frac{\partial_Y F\left(\frac{\pi}{2}, 0\right)}{\partial_X F\left(\frac{\pi}{2}, 0\right)} = \tanh(T) = \tanh\left(\tfrac{\pi}{2}\alpha\right).$$

Bringing together the previous conclusions, we infer that

$$0 \le Z(Y) - \operatorname{sign}(\alpha) Y < \pi \quad \text{for all } Y \in \mathbb{R}.$$

Going back to the original variable, we find the asymptotic behavior

$$\zeta(y) = \frac{1}{k} \operatorname{sign}(\alpha) \sqrt{\mu} e^{-y} + O(1) \quad \text{as } y \to -\infty.$$

Case 3. We conclude with the third and last case, that is $\lambda = \omega\alpha - \mu + i\omega k \in \mathbb{C} \setminus \mathbb{R}_-$ together with (53). We recall that the modified Bessel function I_ν satisfies (see [Erdélyi et al. 1953, p. 86])

$$I_\nu(z) = \frac{e^z}{\sqrt{2\pi z}}\left(1 + O\left(\frac{1}{|z|}\right)\right) \quad \text{for } |z| \to +\infty \text{ with } |\arg z| < \tfrac{\pi}{2} - \delta.$$

By (33), the entire function in (54) is equal to

$$v(x, y) = e^{\alpha x} \operatorname{Re}(e^{ikx} I_\nu(\sqrt{\lambda} e^{-y})),$$

where we choose as determination of the square root of λ the one with strictly positive real part (recall that $\lambda \in \mathbb{C} \setminus \mathbb{R}_-$). Then $|\arg \sqrt{\lambda}| < \frac{\pi}{2} - \delta$ for some $\delta > 0$. We find

$$\begin{aligned} v(x, y) &= e^{\alpha x} \operatorname{Re}\left(e^{ikx} \frac{e^{\sqrt{\lambda} e^{-y}}}{\sqrt{2\pi \sqrt{\lambda} e^{-y}}}(1 + O(e^y))\right) = e^{\alpha x} \operatorname{Re}(C_\lambda e^{ikx + \frac{1}{2} y + \sqrt{\lambda} e^{-y}}(1 + O(e^y))) \\ &= e^{\alpha x + \frac{1}{2} y + \operatorname{Re}(\sqrt{\lambda}) e^{-y}} \operatorname{Re}(C_\lambda e^{ikx + i \operatorname{Im} \sqrt{\lambda} e^{-y} + iO(e^y)} |1 + O(e^y)|) = 0, \end{aligned}$$

which in turns gives the asymptotic equation, as $y \to -\infty$,

$$kx + \operatorname{Im}(\sqrt{\lambda}) e^{-y} + O(e^y) = \beta,$$

where $\beta \in \mathbb{R}$ and

$$\operatorname{Im}(\sqrt{\lambda}) = \operatorname{sign}(\omega k) \sqrt{\sqrt{\left(\tfrac{1}{2}(\omega\alpha - \mu)\right)^2 + \left(\tfrac{1}{2}\omega k\right)^2} - \tfrac{1}{2}(\omega\alpha - \mu)}$$

(with $\operatorname{Im}(\sqrt{\lambda}) = 0$ in case $\omega = 0$). Notice that the sign above agrees with the fact that the nodal lines of the solution v are spanned by monotone functions; see the proof of Lemma 6.1. □

Remark 6.8. In view of the results of Section 2, we have that any solution constructed in this section corresponds to an element of the corresponding class $\mathcal{S}_{\text{rot}}$. In particular, if $\alpha = 0$, we obtain (positive and negative parts of) smooth rotating solutions of the heat equation, with or without reaction term. Moreover, Lemma 6.7 provides a description of their nodal lines, which behave like arithmetic spirals of the equation $\vartheta = \gamma r$ as $r \to +\infty$, as we claimed in Remark 1.8.

Appendix: Weighted embeddings and Poincaré inequalities

In this appendix, we give the proof of some results cited in the paper for the sake of completeness. We start with a very classical compact embedding result.

Lemma A.1. *The functional space $H_0^1(\mathbb{R}^+;\mathbb{C})$ embeds compactly in*

$$L = \left\{ U \in L^1_{\mathrm{loc}}(\mathbb{R}^+;\mathbb{C}) : \|U\|_L^2 = \int_{y>0} e^{-2y}|U|^2 < +\infty \right\}.$$

Proof. Let $\{u_n\}_{n\in\mathbb{N}} \subset H_0^1(\mathbb{R}^+;\mathbb{C})$ be a weakly converging sequence, and let u be its limit. Since the embedding of H_0^1 in L is clearly continuous, $u_n \rightharpoonup u$ in L, and in order to show that $u_n \to u$ in L we just need to prove the convergence of the norms. Let

$$d_n = \left| \int_{y>0} e^{-2y} u_n^2 - \int_{y>0} e^{-2y} u^2 \right|.$$

Observe that $\{d_n\}_n$ is a positive sequence. We have that

$$\begin{aligned} d_n &\le \int_{y>0} e^{-2y}|u_n^2 - u^2| = \int_0^T e^{-2y}|u_n^2 - u^2| + \int_T^\infty e^{-2y}|u_n^2 - u^2| \\ &\le \int_0^T e^{-2y}|u_n^2 - u^2| + e^{-2T}(\|u_n\|_{L^2}^2 + \|u\|_{L^2}^2) \le \int_0^T e^{-2y}|u_n^2 - u^2| + 2Ce^{-2T} \end{aligned}$$

for any $T > 0$. Since $H^1(0,T)$ is compactly embedded in $L^2(0,T)$, we conclude that there exists $\{\varepsilon_{n,T}\}_n$ such that $\varepsilon_{n,T} \to 0$ and

$$d_n \le \varepsilon_{n,T} + 2Ce^{-2T}.$$

To conclude, for any given $\delta > 0$, we can find $T > 0$ such that $Ce^{-2T} < \frac{1}{2}\delta$ and subsequently $\bar{n}$ such that $\varepsilon_{n,T} \le \frac{1}{2}\delta$ for any $n \ge \bar{n}$. This implies that, for any $n \ge \bar{n}$, we have that $0 \le d_n \le \delta$; that is,

$$\lim_{n\to+\infty} d_n = 0 \quad\Longrightarrow\quad \int_{y>0} e^{-2y}u^2 = \lim_{n\to+\infty} \int_{y>0} e^{-2y}u_n^2,$$

and thus we conclude the strong convergence of the sequence $\{u_n\}_{n\in\mathbb{N}}$. □

Exploiting this compact embedding, we can show the following weighted Poincaré inequality.

Lemma A.2. *Let $a > 0$ and $b \in \mathbb{R}$, then*

$$\int_{y>0} |u'|^2 + (a^2 - be^{-2y})u^2 \ge 0$$

for any $u \in H_0^1(\mathbb{R}^+)$ as long as

$$b \le (j_{a,1})^2,$$

where $j_{a,1}$ is the first (positive) zero of the Bessel function of the first kind of order a.

Proof. The statement is equivalent to proving that

$$(j_{a,1})^2 = \inf_{u \in H_0^1(\mathbb{R}^+)} \left\{ \int_{y>0} |u'|^2 + a^2 u^2 : \int_{y>0} e^{-2y} u^2 = 1 \right\}. \tag{56}$$

The existence of a minimizer $u \in H_0^1(\mathbb{R}^+)$ follows directly from the embedding in Lemma A.1. As the functional and the constraint are even, we can assume that the minimizer u is positive. Standard regularity results imply that the function u is also smooth and strictly positive in $\mathbb{R}^+$. Let $\lambda \geq 0$ be the minimum of (56). We have that $u \in H_0^1(\mathbb{R}^+)$ is a solution of

$$\begin{cases} -u'' + (a^2 - \lambda e^{-2y})u = 0, \\ u(0) = 0, \quad u(y) > 0 \quad \text{for } y > 0. \end{cases}$$

We argue as in Lemma 3.2. We look for a solution defined by the series

$$u(y) = \sum_{n \geq 0} c_n e^{-(2n+a)y}, \quad \text{where } c_n \in \mathbb{R} \text{ for } n \in \mathbb{N}.$$

We first make some formal computations, plugging this expression directly into the equation. We find that the coefficients c_n must satisfy the following recursive relation for $n \geq 1$:

$$c_n (2n+a)^2 = c_n a^2 - c_{n-1} \lambda,$$

which is satisfied for instance by letting

$$c_n = \frac{(-1)^n}{n!\,\Gamma(n+1+a)} \left(\frac{\sqrt{\lambda}}{2} \right)^{2n+a} \quad \text{for all } n \in \mathbb{N},$$

thus leading us to the solution

$$u(y) = \sum_{n \in \mathbb{N}} \frac{(-1)^n}{n!\,\Gamma(n+1+a)} \left(\frac{\sqrt{\lambda}}{2} e^{-y} \right)^{2n+a} = J_a(\sqrt{\lambda} e^{-y}).$$

We recall that, if $a > 0$, then $J_a(0) = 0$. This gives that, for any $a > 0$,

$$\lim_{y \to +\infty} u(y) = 0.$$

One can easily check that the series does converge in $H^1(\mathbb{R}^+)$ to its sum u. We only need to ensure that

$$u(0) = 0 \quad \text{and} \quad u(y) > 0 \text{ for any } y > 0.$$

In terms of the function J_a, these conditions together mean that $\sqrt{\lambda}$ has to be the first (positive) zero for J_a; that is,

$$\sqrt{\lambda} = j_{a,1} \iff \lambda = (j_{a,1})^2. \qquad \square$$

We can also show a similar Poincaré inequality for semi-infinite rectangles.

Lemma A.3. *For any $a > 0$ and $b \in \mathbb{R}$, we consider the semi-infinite rectangle*

$$Q_{a,b} = \left(-\tfrac{1}{2}a, \tfrac{1}{2}a\right) \times (b, +\infty)$$

and the corresponding functional space

$$H_0^1(Q_{a,b}) = \{u \in H^1(Q_{a,b}) : u = 0 \text{ on } \partial Q_{a,b}\}.$$

We have

$$\inf_{u \in H_0^1(Q_{a,b})} \left\{ \int_{Q_{a,b}} |\nabla u|^2 : \int_{Q_{a,b}} e^{-2y} u^2 = 1 \right\} = e^{2b} (j_{\pi/a,1})^2.$$

Proof. By the same compactness argument of Lemma A.1, we can show that the infimum is attained by a function $u \in H_0^1(Q_{a,b})$ which, by standard results, is also positive and smooth in $Q_{a,b}$. Up to a translation in y, the function u is then a positive solution of

$$\begin{cases} -\Delta u = \lambda e^{-2b} e^{-2y} u & \text{in } Q_{a,0}, \\ u = 0 & \text{on } \partial Q_{a,0}, \end{cases}$$

for some $\lambda \geq 0$. By separation of variables we can easily show that u is of the form

$$u(x, y) = \cos\left(\frac{\pi}{a} x\right) v(y),$$

where the new unknown function $v \in H_0^1(\mathbb{R}^+)$ solves

$$\begin{cases} -v'' + \left(\frac{\pi^2}{a^2} - \lambda e^{-2b} e^{-2y}\right) v = 0, \\ v(0) = 0, \quad v(y) > 0 \quad \text{for } y > 0. \end{cases}$$

By Lemma A.2, we conclude that

$$\lambda e^{-2b} = (j_{\pi/a,1})^2. \qquad \square$$

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