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TAREK M. ELGINDI AND KARIM R. SHIKH KHALIL

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STRONG ILL-POSEDNESS IN L^∞ FOR THE RIESZ TRANSFORM PROBLEM

TAREK M. ELGINDI AND KARIM R. SHIKH KHALIL

We prove strong ill-posedness in L^∞ for linear perturbations of the 2-dimensional Euler equations of the form

$$\partial_t \omega + u \cdot \nabla \omega = R(\omega),$$

where R is any nontrivial second-order Riesz transform. Namely, we prove that there exist smooth solutions that are initially small in L^∞ but become arbitrarily large in short time. Previous works in this direction relied on the strong ill-posedness of the linear problem, viewing the transport term perturbatively, which only led to mild growth. We derive a nonlinear model taking all of the leading-order effects into account to determine the precise pointwise growth of solutions for short time. Interestingly, the Euler transport term does counteract the linear growth so that the full nonlinear equation grows an order of magnitude less than the linear one. In particular, the (sharp) growth rate we establish is consistent with the global regularity of smooth solutions.

1. Introduction

The Euler equations for incompressible flow are a fundamental model in fluid dynamics that describe the motion of ideal fluids:

$$\begin{aligned} \partial_t u + u \cdot \nabla u + \nabla p &= 0, \\ \nabla \cdot u &= 0. \end{aligned} \tag{1-1}$$

In this equation, u is the velocity field and p is the pressure of an ideal fluid flowing in $\mathbb{R}^2$. A key difficulty in understanding the dynamics of 2-dimensional Euler flows is the nonlocality of the system due to the presence of the pressure term.

Defining the vorticity $\omega := \nabla^\perp \cdot u$, it is insightful to study the Euler equations in vorticity form:

$$\begin{aligned} \partial_t \omega + u \cdot \nabla \omega &= 0, \\ \nabla \cdot u &= 0, \\ u &= \nabla^\perp \Delta^{-1} \omega. \end{aligned} \tag{1-2}$$

Because the L^∞ norm of vorticity is conserved in the Euler equations in two dimensions, Yudovich [1963] proved that there is a unique global-in-time solution to the Euler equation corresponding to every initial bounded and decaying vorticity. See also [Wolibner 1933; Beale et al. 1984; Hölder 1933; Yudovich 1963; Kato 1967; Marchioro and Pulvirenti 1994; Majda and Bertozzi 2002]. This bound on the L^∞ norm is unfortunately unstable even to very mild perturbations of the equation [Constantin and Vicol 2012; Elgindi

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and Masmoudi 2020; Elgindi 2018]. To understand this phenomenon, we are interested in studying linear perturbations of the Euler equations in two dimensions as follows:

$$\begin{aligned} \partial_t u + u \cdot \nabla u + \nabla p &= \begin{pmatrix} 0 \\ u_1 \end{pmatrix}, \\ \nabla \cdot u &= 0. \end{aligned} \tag{1-3}$$

Equation (1-3) is a model for many problems in fluid dynamics that have a coupling with the Euler equations. For instance, similar types of equations appear in viscoelastic fluids, see [Constantin and Kliegl 2012; Elgindi and Rousset 2015; Lions and Masmoudi 2000; Chemin and Masmoudi 2001], and in magnetohydrodynamics, see [Boardman et al. 2020; Hmidi 2014; Cao and Wu 2011; Wu and Zhao 2023]. Further, they also appear when studying stochastic Euler equations; see [Glatt-Holtz and Vicol 2014].

Writing (1-3) in vorticity form, we get

$$\begin{aligned} \partial_t \omega + u \cdot \nabla \omega &= \partial_x u_1, \\ \nabla \cdot u &= 0, \\ u &= \nabla^{\perp} \Delta^{-1} \omega. \end{aligned} \tag{1-4}$$

We observe that the challenge of studying these equations is that the right-hand side of (1-4) can be written as the Riesz transform of vorticity $\partial_x u_1 = R(\omega)$, which is unbounded on L^∞. P. Constantin and V. Vicol [2012] considered these equations with weak dissipation, and they proved global well-posedness. However, without dissipation it is an open question whether these equations are globally well-posed. In this work, we are interested in the question of L^∞ ill/well-posedness of the Euler equations with Riesz forcing and the local rate of L^∞ growth. The first author and N. Masmoudi studied the Euler equations with Riesz forcing in [Elgindi and Masmoudi 2020], where they proved that it is mildly ill-posed. This means that there is a universal constant $c > 0$ such that, for all $\epsilon > 0$, there is $\omega_0 \in C^\infty$ for which the unique local solution to (1-4) satisfies

$$|\omega_0|_{L^\infty} \le \epsilon, \quad \text{but} \quad \sup_{t\in[0,\epsilon]} |\omega(t)|_{L^\infty} \ge c. \tag{1-5}$$

The authors in [Elgindi and Masmoudi 2020] conjectured that the Euler equation with Riesz forcing is actually strongly ill-posed in L^∞. Namely, that we can take c in (1-5) to be arbitrarily large. The goal of our work here is to show that indeed this is possible. To show this, we use the first author's Biot–Savart law decomposition [Elgindi 2021] to derive a leading-order system for the Euler equations with Riesz forcing. We then show that the leading-order system is strongly ill-posed in L^∞. Using this, we can show that the Euler equation with Riesz forcing is strongly ill-posed by estimating the error between the leading-order system and the Euler with Riesz forcing system on a specific time interval.

We should remark that the main application of the approach of [Elgindi and Masmoudi 2020] was to prove ill-posedness of the Euler equation in the integer C^k spaces, which was also proved independently by J. Bourgain and D. Li [2015]. Regarding the notion of mild ill-posedness in L^∞ for models related to the Euler with Riesz forcing system, see [Wu and Zhao 2023] about the 2-dimensional resistive MHD equations.

1.1. *Statement of the main result.*

Theorem 1. *For any $\alpha, \delta > 0$, there exists an initial data $\omega_0^{\alpha,\delta} \in C_c^\infty(\mathbb{R}^2)$ and $T(\alpha)$ such that the corresponding unique global solution, $\omega^{\alpha,\delta}$, to (1-4) is such that at $t = 0$ we have*

$$|\omega_0^{\alpha,\delta}|_{L^\infty} = \delta,$$

but for any $0 < t \leq T(\alpha)$ we have

$$|\omega^{\alpha,\delta}(t)|_{L^\infty} \geq |\omega_0|_{L^\infty} + c\log\Big(1 + \frac{c}{\alpha}t\Big),$$

where $T(\alpha) = c\alpha\log(c|\log(\alpha)|)$, and $c > 0$ is a constant independent of α that depends linearly on δ.

Remark 1.1. Note that at time $t = T(\alpha)$, we have

$$|\omega^{\alpha,\delta}|_{L^\infty} \geq c\log(c\log(c|\log(\alpha)|)),$$

which can be made arbitrarily large as $\alpha \to 0$. Fixing $\delta > 0$ small and then taking α sufficiently small thus gives strong ill-posedness for (1-4) in L^∞.

Remark 1.2. As we will discuss below, we in fact establish upper and lower bounds on the solutions we construct so that on the same time-interval we have

$$|\omega^{\alpha,\delta}(t)|_{L^\infty} \approx |\omega_0|_{L^\infty} + c\log\Big(1 + \frac{c}{\alpha}t\Big).$$

This should be contrasted with the linear problem where the upper and lower bounds for the same data come without the log:

$$|\omega_{\text{linear}}^{\alpha,\delta}(t)|_{L^\infty} \approx |\omega_0|_{L^\infty} + c\Big(1 + \frac{c}{\alpha}t\Big).$$

Remark 1.3. Our ill-posedness result applies to the equation

$$\partial_t\omega + u\cdot\nabla\omega = R(\omega),$$

where $R = R_{12} = \partial_{12}\Delta^{-1}$. Note that a direct consequence of the result gives strong ill-posedness when $R = R_{11}$ or $R = R_{22}$ even though these are dissipative on L^2. This can be seen just by noting that a linear change of coordinates can transform R_{12} to a constant multiple of $R_{11} - R_{22} = R_{11} - \mathrm{Id}$. The strong ill-posedness for the Euler equation with forcing by any second-order Riesz transform (other than the identity) follows. We further remark that the same strategy can be used to study the case of general Riesz transforms, though we do not undertake this here since the case of forcing by second-order Riesz transforms is the most relevant for applications we are aware of (such as the 3-dimensional Euler equations, the Boussinesq system, viscoelastic models, MHD, etc.).

1.2. *Comparison with the linear equation and the effect of transport.*

We now move to compare the result of this paper with the corresponding linear results and emphasize the regularizing effect of the nonlinearity in this problem. The ill-posedness result of [Elgindi and Masmoudi 2020] relies on viewing (1-4) as a perturbation of

$$\partial_t f = R(f). \tag{1-6}$$

For this simple linear equation, it is easy to show that L^∞ data can immediately develop a logarithmic singularity. Let us mention two ways to quantify this logarithmic singularity. One way is to study the growth of L^p norms as $p \to \infty$. For the linear equation (1-6), it is easy to show that the upper bound

$$|f(t)|_{L^p} \leq \exp(Ct) p |f_0|_{L^p}$$

is sharp in the sense that we can find localized L^∞ data for which the solution satisfies

$$|f(t)|_{L^p} \geq c(t) \cdot p.$$

This can be viewed as approximating L^∞ "from below". Similarly, the C^α bound for (1-6),

$$|f(t)|_{C^\alpha} \leq \frac{\exp(Ct)}{\alpha} |f_0|_{C^\alpha},$$

can also be shown to be sharp for short time in that we can find for each $\alpha > 0$ smooth and localized data with $|f_0|_{C^\alpha} = 1$ for which

$$|f(t)|_{L^\infty} \geq \frac{c(t)}{\alpha}.$$

The main result of [Elgindi and Masmoudi 2020] was that these upper and lower bounds remain unchanged in the presence of a transport term by a Lipschitz continuous velocity field. This is not directly applicable to our setting since the coupling between ω and u is such that u may not be Lipschitz even if ω is bounded. Interestingly, in [Elgindi 2018], it was shown that this growth could be significantly stronger in the presence of a merely bounded velocity field.

All of the above discussion leads us to understand that the nature of the well/ill-posedness of (1-4) will depend on the precise relationship between the velocity field and the linear forcing term in (1-4). In particular, for a natural class of data, we construct solutions to (1-4) satisfying

$$|\omega|_{L^\infty} \approx 1 + \log\Big(1 + \frac{t}{\alpha}\Big),$$

for short time, which is the best growth rate possible in this setting. This should be contrasted with the corresponding growth rate for the linear problem

$$|\omega_{\text{linear}}|_{L^\infty} \approx 1 + \frac{t}{\alpha}.$$

In particular, the nonlinear term in (1-4) actually tries to *prevent* L^∞ growth. Let us finally remark that the weak growth rate we found is consistent with the vorticity trying to develop a $\log\log$ singularity. It is curious that, in the Euler equation, vorticity with nearly $\log\log$ data is perfectly well-behaved and consistent with global regularity but with a triple exponential upper bound on gradients. Though establishing the global regularity rigorously remains a major open problem, this appears to be a sign that perhaps smooth solutions to (1-3) are globally regular.

1.3. ***A short discussion of the proof.*** The first step of the proof is to use the Biot–Savart law decomposition in [Elgindi 2021] to derive a leading-order model:

$$\partial_t \Omega + \frac{1}{2\alpha}(L_s(\Omega)\sin(2\theta) + L_c(\Omega)\cos(2\theta))\partial_\theta \Omega = \frac{1}{2\alpha} L_s(\Omega),$$

where the operators L_s and L_c are bounded linear operators on L^2 defined by

$$L_s(f)(R) = \frac{1}{\pi}\int_R^\infty\int_0^{2\pi} \frac{f(s,\theta)}{s}\sin(2\theta)\,d\theta\,ds \quad \text{and} \quad L_c(f)(R) = \frac{1}{\pi}\int_R^\infty\int_0^{2\pi} \frac{f(s,\theta)}{s}\cos(2\theta)\,d\theta\,ds.$$

Essentially all we do here is replace the velocity field by its most singular part. Upon inspecting this model, we observe that the forcing term on the right-hand side is purely radial, while the direction of transport is angular. Upon choosing a suitable unknown, we thus reduce the problem to solving a transport equation for some unknown f:

$$\partial_t f + \frac{1}{2\alpha} L_s(f)\sin(2\theta)\partial_\theta f = 0.$$

Surprisingly, this reduced equation propagates the usual "odd-odd" symmetry even though the original system does not. The leading-order model will then be strongly ill-posed if we can ensure that the solution of this transport equation satisfies that $\int_0^t L_s(f)$ can be arbitrarily large. One subtlety is that the growth of $L_s(f)$ enhances the transport effect, which in turn depletes the growth of $L_s(f)$. In fact, were the transport term to be stronger even by a log, the problem would *not* be strongly ill-posed. By a careful study of the characteristics of this equation, we obtain a closed nonlinear integrodifferential equation governing the evolution of $L_s(f)$ (see (3-4)). We study this nonlinear integrodifferential equation and establish upper and lower bounds on $L_s(f)$ proving strong ill-posedness for the leading-order equation; see Section 3 for more details. Finally, we close the argument by estimating the error incurred by approximating the dynamics with the leading-order model. An important idea here is to work on a time scale long enough to see the growth from the leading-order model but short enough to suppress any potential stronger nonlinear growth; see Section 6 for more details.

1.4. ***Organization.*** This paper is organized as follow: In Section 2, we derive a leading-order model for the Euler equations with Riesz forcing (1-4) based on the first author's Biot–Savart law approximation [Elgindi 2021]. Then, in Section 3, we obtain a pointwise estimate on the leading-order model which is the main ingredient in obtaining the strong ill-posedness result for the Euler with Riesz forcing system. In addition, in Section 3, we also obtain some estimates on the leading-order model in suitable norms which will be then used in estimating the remainder term in Section 6. After that, in Section 4 we will recall the Biot–Savart law decomposition obtained in [Elgindi 2021], and we will include a short sketch of the proof. In Section 5, we will obtain some embedding estimates which will also be used in Section 6 for the remainder term estimates. Then, in Section 6, we show that the remainder term remains small which will then allow us to prove the main result in Section 7.

1.5. ***Notation.*** In this paper, we will be working in a form polar coordinates introduced in [Elgindi 2021]. Let r be the radial variable,

$$r = \sqrt{x^2+y^2},$$

and since we will be working with functions of the variable r^α, where $0 < \alpha < 1$, we will use R to denote it:

$$R = r^\alpha.$$

We will use θ to denote the angle variable:

$$\theta = \arctan \frac{y}{x}.$$

We will use $|f|_{L^\infty}$ and $|f|_{L^2}$ to denote the usual L^∞ and L^2 norms, respectively. In addition, we will use f_t or f_τ to denote the time variable. Further, in this paper, following [Elgindi 2021], we will be working on $(R,\theta) \in [0,\infty) \times \left[0, \frac{\pi}{2}\right]$ where the L^2 norm will be with measure $dR\,d\theta$ and not $R\,dR\,d\theta$.

We define the weighted $\mathcal{H}^k\left([0,\infty) \times \left[0, \frac{\pi}{2}\right]\right)$ norm as

$$|f|_{\dot{\mathcal{H}}^m} = \sum_{i=0}^{m} |\partial_R^i \partial_\theta^{m-i} f|_{L^2} + \sum_{i=1}^{m} |R^i \partial_R^i \partial_\theta^{m-i} f|_{L^2}, \quad |f|_{\mathcal{H}^k} = \sum_{m=0}^{k} |f|_{\dot{\mathcal{H}}^m}.$$

We also define the $\mathcal{W}^{k,\infty}$ norm as

$$|f|_{\dot{\mathcal{W}}^{m,\infty}} = \sum_{i=0}^{m} |\partial_R^i \partial_\theta^{m-i} f|_{L^\infty} + \sum_{i=1}^{m} |R^i \partial_R^i \partial_\theta^{m-i} f|_{L^\infty}, \quad |f|_{\mathcal{W}^{k,\infty}} = \sum_{m=0}^{k} |f|_{\dot{\mathcal{W}}^{m,\infty}}.$$

Throughout this paper, we will use the notation

$$L(f)(R) = \int_R^\infty \frac{f(s)}{s}\,ds$$

to define operators, and by adding a subscript L_s or L_c we denote the projection onto $\sin(2\theta)$ and $\cos(2\theta)$ respectively. Namely,

$$L_s(f)(R) = \frac{1}{\pi} \int_R^\infty \int_0^{2\pi} \frac{f(s,\theta)}{s} \sin(2\theta)\,d\theta\,ds \quad \text{and} \quad L_c(f)(R) = \frac{1}{\pi} \int_R^\infty \int_0^{2\pi} \frac{f(s,\theta)}{s} \cos(2\theta)\,d\theta\,ds.$$

2. Leading-order model

In this section, we will derive a leading-order model for the Euler equation with Riesz forcing:

$$\begin{aligned} \partial_t \omega + u \cdot \nabla \omega &= \partial_x u_1, \\ \nabla \cdot u &= 0, \\ u &= \nabla^\perp \Delta^{-1} \omega. \end{aligned} \tag{2-1}$$

To do this, we follow [Elgindi 2021] and we write the equation in a form of polar coordinates. Namely, we set $r = \sqrt{x^2+y^2}$, $R = r^\alpha$, and $\theta = \arctan(y/x)$. We will the rewrite (2-1) in the new functions $\omega(x,y) = \Omega(R,\theta)$ and $\psi(x,y) = r^2 \Psi(R,\theta)$, with $u = \nabla^\perp \psi$, where $u_1 = -\partial_y \psi$ and $u_2 = \partial_x \psi$.

Equations of u in terms of Ψ:

$$\begin{aligned} u_1 &= -r(2\sin(\theta)\Psi + \alpha \sin(\theta) R\,\partial_R \Psi + \cos(\theta)\partial_\theta \Psi), \\ u_2 &= r(2\cos(\theta)\Psi + \alpha \cos(\theta) R\,\partial_R \Psi - \sin(\theta)\partial_\theta \Psi). \end{aligned}$$

Evolution equation for Ω:

$$\partial_t\Omega+(-\alpha R\partial_\theta\Psi)\partial_R\Omega+(2\Psi+\alpha R\partial_R\Psi)\partial_\theta\Omega$$
$$=\big(-2\alpha R\sin(\theta)\cos(\theta)-\alpha^2R\sin(\theta)\cos(\theta)\big)\partial_R\Psi+(-1+2\sin^2(\theta))\partial_\theta\Psi$$
$$+\big(-\alpha R\cos^2(\theta)+\alpha R\sin^2(\theta)\big)\partial_{R\theta}\Psi-(\alpha^2R^2\sin(\theta)\cos(\theta))\partial_{RR}\Psi+(\sin(\theta)\cos(\theta))\partial_{\theta\theta}\Psi.$$

The elliptic equation for $\Delta(r^2\Psi(R,\theta))=\Omega(R,\theta)$:

$$4\Psi+\alpha^2R^2\partial_{RR}\Psi+\partial_{\theta\theta}\Psi+(4\alpha+\alpha^2)R\partial_R\Psi=\Omega(R,\theta).$$

Now using the Biot–Savart decomposition of [Elgindi 2021], see Section 4 for more details, by defining the operators

$$L_s(\Omega)(R)=\frac{1}{\pi}\int_R^\infty\int_0^{2\pi}\frac{\Omega(s,\theta)}{s}\sin(2\theta)\,d\theta\,ds\quad\text{and}\quad L_c(\Omega)(R)=\frac{1}{\pi}\int_R^\infty\int_0^{2\pi}\frac{\Omega(s,\theta)}{s}\cos(2\theta)\,d\theta\,ds$$

we have

$$\Psi(R,\theta)=-\frac{1}{4\alpha}L_s(\Omega)\sin(2\theta)-\frac{1}{4\alpha}L_c(\Omega)\cos(2\theta)+\text{lower-order terms}.$$

Thus, if we ignore the α-terms in the evolution equation, we obtain

$$\partial_t\Omega+(2\Psi)\partial_\theta\Omega=(-1+2\sin^2(\theta))\partial_\theta\Psi+(\sin(\theta)\cos(\theta))\partial_{\theta\theta}\Psi. \tag{2-2}$$

Now we consider Ψ of the form

$$\Psi=-\frac{1}{4\alpha}L_s(\Omega)\sin(2\theta)-\frac{1}{4\alpha}L_c(\Omega)\cos(2\theta),$$

and plugging it into the evolution equation, we have

$$\partial_t\Omega-\Big(\frac{1}{2\alpha}L_s(\Omega)\sin(2\theta)+\frac{1}{2\alpha}L_c(\Omega)\cos(2\theta)\Big)\partial_\theta\Omega=-(\cos(2\theta))\Big(-\frac{1}{2\alpha}L_s(\Omega)\cos(2\theta)+\frac{1}{2\alpha}L_c(\Omega)\sin(2\theta)\Big)$$
$$+\Big(\frac{1}{2}\sin(2\theta)\Big)\Big(\frac{1}{\alpha}L_s(\Omega)\sin(2\theta)+\frac{1}{\alpha}L_c(\Omega)\cos(2\theta)\Big),$$

which simplifies to

$$\partial_t\Omega-\Big(\frac{1}{2\alpha}L_s(\Omega)\sin(2\theta)+\frac{1}{2\alpha}L_c(\Omega)\cos(2\theta)\Big)\partial_\theta\Omega=\frac{1}{2\alpha}L_s(\Omega).$$

In order to work with positive solutions and have the angular trajectories moving to the right, we make the change $\Omega\to-\Omega$ and get the final model

$$\partial_t\Omega+\Big(\frac{1}{2\alpha}L_s(\Omega)\sin(2\theta)+\frac{1}{2\alpha}L_c(\Omega)\cos(2\theta)\Big)\partial_\theta\Omega=\frac{1}{2\alpha}L_s(\Omega). \tag{2-3}$$

We now move to study the dynamics of solutions to (2-3).

Proposition 2.1. *Let Ω be a solution to the leading-order model*

$$\partial_t\Omega+\Big(\frac{1}{2\alpha}L_s(\Omega)\sin(2\theta)+\frac{1}{2\alpha}L_c(\Omega)\cos(2\theta)\Big)\partial_\theta\Omega=\frac{1}{2\alpha}L_s(\Omega), \tag{2-4}$$

with initial data of the form $\Omega|_{t=0} = f_0(R)\sin(2\theta)$. *Then we can write* Ω *as*

$$\Omega = f + \frac{1}{2\alpha}\int_0^t L_s(f_\tau)\,d\tau, \tag{2-5}$$

where f *satisfies the transport equation*

$$\partial_t f + \frac{1}{2\alpha}\sin(2\theta)L_s(f)\partial_\theta f = 0. \tag{2-6}$$

Proof. The right-hand side term of (2-4) is radial, and hence if we take the inner product with $\sin(2\theta)$ it will be zero. Now if write Ω as

$$\Omega_t(R,\theta) = f_t(R,\theta) + \frac{1}{2\alpha}\int_0^t L_s(\Omega_\tau)(R)\,d\tau,$$

and consider it to be a solution to (2-4), we obtain that f satisfies

$$\partial_t f_t + \Big(\frac{1}{2\alpha}L_s(f_t)\sin(2\theta) + \frac{1}{2\alpha}L_c(f_t)\cos(2\theta)\Big)\,\partial_\theta f_t = 0. \tag{2-7}$$

Here we used that $L_s(\Omega_\tau)(R)$ is a radial function. Notice that (2-7) is a transport equation that preserves odd symmetry. Now if we set

$$f_t^s = \int_0^{2\pi} f_t(R,\theta)\sin(2\theta)\,d\theta \quad \text{and} \quad \Omega_t^s = \int_0^{2\pi}\Omega_t(R,\theta)\sin(2\theta)\,d\theta,$$

we notice that f_t^s and Ω_t^s will satisfy the same equation. Thus, if we start with the same initial conditions $f_0 = \Omega_0$, then

$$f_t^s = \Omega_t^s \quad \text{for all } t.$$

Thus, we have $L_s(\Omega_t) = L_s(f_t)$, and hence

$$\Omega_t = f_t + \frac{1}{2\alpha}\int_0^t L_s(f_\tau)\,d\tau.$$

Now since the initial data which we are considering have odd symmetry, it suffices to consider the transport equation:

$$\partial_t f_t + \frac{1}{2\alpha}\sin(2\theta)L_s(f_t)\partial_\theta f_t = 0. \qquad \square$$

3. Leading-order model estimate

The purpose of this section is to obtain L^∞ estimates for the leading-order model, which is the main ingredient in obtaining the ill-posedness result for the Euler with Riesz forcing system. This will be done in Section 3.1 in three steps: Lemma 3.1, Lemma 3.2, and Proposition 3.3. Then in Section 3.2, we will obtain an estimate for the leading-order model which will be useful in remainder estimates in Section 6.

3.1. *Pointwise leading-order model estimate.*

Lemma 3.1. *Let* f *be a solution to the transport equation*

$$\partial_t f + \frac{1}{2\alpha}\sin(2\theta)L_s(f)\partial_\theta f = 0, \tag{3-1}$$

with initial data $f|_{t=0} = f_0(R)\sin(2\theta)$. *Then we have the following estimate on the operator* $L_s(f)$:

$$c_1 \int_R^\infty \frac{f_0(s)}{s} \exp\Bigl(-\frac{1}{\alpha}\int_0^t L_s(f_\tau)(s)\,d\tau\Bigr)\,ds \le L_s(f_t)(R) \le c_2 \int_R^\infty \frac{f_0(s)}{s} \exp\Bigl(-\frac{1}{\alpha}\int_0^t L_s(f_\tau)(s)\,d\tau\Bigr)\,ds, \quad \text{(3-2)}$$

where c_1 *and* c_2 *are independent of* α.

Proof. To prove this, we consider the following variable change. For $\theta \in \bigl[0, \frac{\pi}{2}\bigr)$, let γ be defined as

$$\gamma := \tan(\theta) \implies \frac{d\gamma}{d\theta} = \sec^2(\theta), \quad \text{and} \quad \sin(2\theta) = \frac{2\gamma}{1+\gamma^2}.$$

Applying the chain rule, we rewrite (3-1) in the (R, γ)-variables

$$\partial_t f_t + \frac{1}{\alpha}\gamma L_s(f_t)(R)\,\partial_\gamma f = 0, \quad \text{(3-3)}$$

with initial data

$$f|_{t=0} = f_0(R)\sin(2\theta) = f_0(R)\frac{2\gamma}{1+\gamma^2}.$$

Let $\phi_t(\gamma)$ be the flow map associated with (3-3), so we have

$$\frac{d\phi_t(\gamma)}{dt} = \frac{1}{\alpha}\phi_t(\gamma)L_s(f_t) \implies \phi_t(\gamma) = \gamma\exp\Bigl(\frac{1}{\alpha}\int_0^t L_s(f_\tau)\,d\tau\Bigr).$$

Thus,

$$\phi_t^{-1}(\gamma) = \gamma\exp\Bigl(-\frac{1}{\alpha}\int_0^t L_s(f_\tau)\,d\tau\Bigr).$$

Hence, we now write the solution to (3-3) as

$$f_t(R,\gamma) = f_0(R, \phi_t^{-1}(\gamma)) = f_0(R)\frac{2\phi_t^{-1}(\gamma)}{1+\phi_t^{-1}(\gamma)^2} = f_0(R)\frac{2\gamma\exp\bigl(-(1/\alpha)\int_0^t L_s(f_\tau)\,d\tau\bigr)}{1+\gamma^2\exp\bigl(-(2/\alpha)\int_0^t L_s(f_\tau)\,d\tau\bigr)}.$$

Now we consider the operator L_s in the $(R,\gamma) \in [0,\infty) \times \bigl[0, \frac{\pi}{2}\bigr)$-variables:

$$L_s(f_t)(R) = \frac{1}{\pi}\int_R^\infty \frac{1}{s}\int_0^\infty f_t(s,\gamma)\frac{2\gamma}{(1+\gamma^2)^2}\,d\gamma\,ds.$$

Plugging in the expression for f_t, we have

$$L_s(f_t)(R) = \frac{1}{\pi}\int_R^\infty \frac{1}{s}\int_0^\infty f_0(s)\frac{\exp\bigl(-(1/\alpha)\int_0^t L_s(f_\tau)(s)\,d\tau\bigr)}{1+\gamma^2\exp\bigl(-(2/\alpha)\int_0^t L_s(f_\tau)(s)\,d\tau\bigr)}\frac{4\gamma^2}{(1+\gamma^2)^2}\,d\gamma\,ds. \quad \text{(3-4)}$$

Now since $0 \le \exp\bigl(-(2/\alpha)\int_0^t L_s(f_\tau)(s)\,d\tau\bigr) \le 1$, we have an upper and a lower bound on the operator on $L_s(f_t)(R)$ with constants c_1, c_2 independent of α (in fact, these constants can be explicitly computed). Namely,

$$c_1\int_R^\infty \frac{f_0(s)}{s}\exp\Bigl(-\frac{1}{\alpha}\int_0^t L_s(f_\tau)(s)\,d\tau\Bigr)\,ds \le L_s(f_t)(R) \le c_2\int_R^\infty \frac{f_0(s)}{s}\exp\Bigl(-\frac{1}{\alpha}\int_0^t L_s(f_\tau)(s)\,d\tau\Bigr)\,ds.$$

Thus, we have our desired inequalities. □

Lemma 3.2. *Define the operator*

$$\hat{L}(f_t)(R) := \int_R^\infty \frac{f_0(s)}{s} \exp\left(-\frac{1}{\alpha}\int_0^t \hat{L}(f_s)(s)\,d\tau\right) ds. \tag{3-5}$$

Then we have

$$\int_0^t \hat{L}(f_\tau)(R)\,d\tau = 2\alpha \log\left(1 + \frac{t}{2\alpha} L(f_0)(R)\right),$$

where $L(f_0)(R) = \int_R^\infty f_0(s)/s\,ds$.

Proof. We introduce $g_t(R) := \exp\bigl(-(1/\alpha)\int_0^t \hat{L}(f_\tau)(R)\,d\tau\bigr)$ and $K(R) := f_0(R)/R$. Then the operator $\hat{L}$ can be rewritten as

$$\hat{L}(f_t)(R) = \int_R^\infty K(s) g_t(s)\,ds. \tag{3-6}$$

Now taking the time derivative of (3-6), and using that $\partial_t g_t(R) = -(1/\alpha) g_t(R) \int_R^\infty K(s) g_t(s)\,ds$, we can obtain

$$\partial_t \hat{L}(f_t) = -\frac{1}{2\alpha}(\hat{L}(f_t))^2,$$

which can be solved explicitly:

$$\hat{L}(f_t)(R) = \frac{L(f_0)(R)}{1 + (t/(2\alpha))L(f_0)(R)}. \tag{3-7}$$

Then it follows that

$$\int_0^t \hat{L}(f_t)(R)\,d\tau = 2\alpha \log\left(1 + \frac{t}{2\alpha} L(f_0)(R)\right). \qquad \square$$

Proposition 3.3. *Let f be a solution to the transport equation*

$$\partial_t f + \frac{1}{2\alpha}\sin(2\theta) L_s(f)\partial_\theta f = 0, \tag{3-8}$$

with initial data $f|_{t=0} = f_0(R)\sin(2\theta)$. Then we have the following estimate on the operator $L_s(f)$:

$$\frac{2\alpha}{c_1}\log\left(1 + \frac{c_1}{2\alpha} t L(f_0)(R)\right) \geq \int_0^t L_s(f_\tau)(R) \geq \frac{2\alpha}{c_2}\log\left(1 + \frac{c_2}{2\alpha} t L(f_0)(R)\right), \tag{3-9}$$

where c_1 and c_2 are independent of α.

Proof. In the section, we will use the bounds in (3-2), namely

$$c_1 \int_R^\infty \frac{f_0(s)}{s} \exp\left(-\frac{1}{\alpha}\int_0^t L_s(f_\tau)(s)\,d\tau\right) ds \\ \leq L_s(f_t)(R) \leq c_2 \int_R^\infty \frac{f_0(s)}{s}\exp\left(-\frac{1}{\alpha}\int_0^t L_s(f_\tau)(s)\,d\tau\right) ds, \tag{3-10}$$

to obtain and upper and lower estimate on $\int_0^t L_s(f)$. As before we set

$$g_t(R) = \exp\left(-\frac{1}{\alpha}\int_0^t L_s(f_\tau)(R)\,d\tau\right) \quad \text{and} \quad K(R) = \frac{f_0(R)}{R}.$$

Using (3-10), we can obtain that

$$-\frac{c_1}{2\alpha}\left(\int_R^\infty g_t(s)K(s)\,ds\right)^2 \geq \partial_t \int_R^\infty g_t(s)K(s)\,ds \geq -\frac{c_2}{2\alpha}\left(\int_R^\infty g_t(s)K(s)\,du\right)^2. \tag{3-11}$$

Similar to Lemma 3.2, we define

$$L_s(f_t)(R) := \int_R^\infty g_t(s)K(s)\,ds.$$

Now from (3-11), we have

$$-\frac{c_1}{2\alpha}(L_s(f_t)(R))^2 \geq \partial_t L_s(f_t)(R) \geq -\frac{c_2}{2\alpha}(L_s(f_t)(R))^2.$$

Thus,

$$\frac{L(f_0)(R)}{1+(c_1/(2\alpha))tL(f_0)(R)} \geq L_s(f_t)(R) \geq \frac{L(f_0)(R)}{1+(c_2/(2\alpha))tL(f_0)(R)}, \tag{3-12}$$

which will give us that

$$\frac{2\alpha}{c_1}\log\left(1+\frac{c_1}{2\alpha}tL(f_0)(R)\right) \geq \int_0^t L_s(f_\tau)(R) \geq \frac{2\alpha}{c_2}\log\left(1+\frac{c_2}{2\alpha}tL(f_0)(R)\right),$$

and this completes the proof. □

3.2. *Estimate for the leading-order model in $\mathcal{W}^{k,\infty}$ and $\mathcal{H}^k$ norms.* The purpose of this subsection is to obtain some estimate on the leading-order model in $\mathcal{W}^{k,\infty}$ and $\mathcal{H}^k$ norms. These will be used to estimate the size of the remainder term in Section 6. First we will obtain estimates on Ψ_2 in Lemma 3.4. Then in Lemma 3.5, we will obtain estimates on Ω_2.

Lemma 3.4. *Let Ω_2 be a solution to the leading-order model*:

$$\partial_t\Omega_2 + \left(\frac{1}{2\alpha}L_s(\Omega_2)\sin(2\theta) + \frac{1}{2\alpha}L_c(\Omega_2)\cos(2\theta)\right)\partial_\theta\Omega_2 = \frac{1}{2\alpha}L_s(\Omega_2),$$

with initial data $\Omega_2|_{t=0} = f_0(R)\sin(2\theta)$, where $f_0(R)$ is smooth and compactly supported. Consider

$$\Psi_2 = \frac{1}{4\alpha}L_s(\Omega_2)\sin(2\theta) + \frac{1}{4\alpha}L_c(\Omega_2)\cos(2\theta).$$

Then, we have the following estimates on Ψ_2:

$$|\Psi_2|_{\mathcal{W}^{k+1,\infty}} \leq \frac{c_k}{\alpha}, \qquad |\Psi_2|_{\mathcal{H}^{k+1}} \leq \frac{c_k}{\alpha}, \tag{3-13}$$

where c_k depends on the initial conditions and is independent of α.

Proof. Recall that from Proposition 2.1, we can write Ω_2 as

$$\Omega_2 = f + \frac{1}{2\alpha}\int_0^t L_s(f_\tau)\,d\tau,$$

and since the initial data is odd in θ, we have

$$\Psi_2 = \frac{1}{4\alpha}L_s(\Omega_t)\sin(2\theta) = \frac{1}{4\alpha}L_s(f_t)\sin(2\theta).$$

To estimate the size of Ψ_2, from (3-4), we have

$$L_s(f_t)(R)=\int_R^\infty \frac{1}{s}\int_0^\infty f_0(s)\frac{\exp\bigl(-(1/\alpha)\int_0^t L_s(f_\tau)(s)\,d\tau\bigr)}{1+\gamma^2\exp\bigl(-(2/\alpha)\int_0^t L_s(f_\tau)(s)\,d\tau\bigr)}\frac{4\gamma^2}{(1+\gamma^2)^2}\,d\gamma\,ds.$$

Using (3-2), we have

$$|\Psi_2|_{L^\infty}\le\frac{c}{\alpha}\int_R^\infty\frac{f_0(s)}{s}\,ds\le\frac{c_0}{\alpha}.$$

For $\partial_\theta\Psi_2$, it is clear that we have

$$|\partial_\theta\Psi_2|_{L^\infty}\le\frac{c_0}{\alpha},$$

where, similarly, c_0 depends on the initial condition.

Now for $\partial_R\Psi_2$, we have

$$\partial_R\Psi_2=\frac{1}{4\alpha}\partial_R L_s(f_t)\sin(2\theta).$$

Thus,

$$\partial_R L_s(f_t)(R)=-\frac{1}{R}\int_0^\infty f_0(R)\frac{\exp\bigl(-(1/\alpha)\int_0^t L_s(f_\tau)(R)\,d\tau\bigr)}{1+\gamma^2\exp\bigl(-(2/\alpha)\int_0^t L_s(f_\tau)(R)\,d\tau\bigr)}\frac{4\gamma^2}{(1+\gamma^2)^2}\,d\gamma,$$

and similarly, we have

$$|\partial_R\Psi_2|_{L^\infty}\le\frac{c}{\alpha}.$$

Now the estimate on $R\,\partial_R\Psi_2$ follows from the estimate on $\partial_R\Psi_2$ and the fact that the initial data have compact support. Thus,

$$|R\,\partial_R\Psi_2|_{L^\infty}\le\frac{c}{\alpha}.$$

For higher-order derivatives, we can obtain the estimate following the same steps. Hence, we have

$$|\Psi|_{\mathcal{W}^{k+1,\infty}}\le\frac{c_k}{\alpha}.$$

The $\mathcal{H}^k$ estimates also follow using the same steps:

$$|\Psi|_{\mathcal{H}^{k+1}}\le\frac{c_k}{\alpha}.\qquad\square$$

In the following lemma, we will obtain the $\mathcal{H}^k$ estimates on Ω_2. Here we will use Lemma 3.4 and transport estimates.

Lemma 3.5. *Let Ω_2 be a solution to the leading-order model*

$$\partial_t\Omega_2+\Bigl(\frac{1}{2\alpha}L_s(\Omega_2)\sin(2\theta)+\frac{1}{2\alpha}L_c(\Omega_2)\cos(2\theta)\Bigr)\partial_\theta\Omega_2=\frac{1}{2\alpha}L_s(\Omega_2),$$

with initial data $\Omega_2|_{t=0}=f_0(R)\sin(2\theta)$, where $f_0(R)$ is smooth and compactly supported. Then, we have the following estimates on Ω_2:

$$|\Omega_2|_{\mathcal{H}^k}\le c_k e^{(c_k/\alpha)t},\tag{3-14}$$

where c_k depends on the initial conditions and is independent of α.

Proof. Recall that from Proposition 2.1 we can write Ω_2 as

$$\Omega_2=f+\frac{1}{2\alpha}\int_0^t L_s(f_\tau)\,d\tau,$$

where f satisfies the transport equation

$$\partial_t f_t + 2\Psi_2\, \partial_\theta f_t = 0.$$

When we consider the derivatives of Ω_2, the transport term f dominates the radial term $(1/(2\alpha))\int_0^t L_s(f)\, d\tau$. Thus, it suffices to consider the $\mathcal{H}^k$ estimates on f which will follow from the standard L^2 estimate for the transport equation. Thus, we have

$$\partial_t f_t + 2\Psi_2 \partial_\theta f_t = 0 \quad \Longrightarrow \quad \partial_t \partial_\theta f_t + 2\partial_\theta \Psi_2 \partial_\theta f_t + 2\Psi_2 \partial_{\theta\theta} f_t = 0.$$

Hence,

$$|\partial_\theta f_t|_{L^2} \leq |\partial_\theta f_0|_{L^2} e^{\int_0^t |\partial_\theta \Psi_2|_{L^\infty}}.$$

From (3-13) we have $|\partial_\theta \Psi_2|_{L^\infty} \leq c_0/\alpha$. Thus, applying the Gronwall inequality, we have

$$|\partial_\theta f_t|_{L^2} \leq |\partial_\theta f_0|_{L^2} e^{(c_0/\alpha)t}. \tag{3-15}$$

To obtain $\mathcal{H}^k$ estimates, we need to estimate terms of the form $R^k \partial_R^k$. We will show how to obtain the $R\partial_R$ estimate, and for general k, it will follow similarly. Thus, similar to L^2 estimate for the $\partial_\theta f$ case, since

$$\partial_t f_t + 2\Psi_2 \partial_\theta f_t = 0,$$

we have

$$\partial_t \partial_R f_t + 2\partial_R \Psi_2 \partial_\theta f_t + 2\Psi_2 \partial_{R\theta} f_t = 0,$$

and thus,

$$\partial_t |R\partial_R f_t|_{L^2} \leq 2|R\partial_R \Psi_2|_{L^\infty} |\partial_\theta f|_{L^2} + |\partial_\theta \Psi_2|_{L^\infty} |R\partial_R f_t|_{L^2}.$$

Now from (3-13), (3-15), and applying the Gronwall inequality we have

$$|R\partial_R f_t|_{L^2} \leq (|R\partial_R f_0|_{L^2} + |\partial_\theta f_0|_{L^2} e^{(c_0/\alpha)t}) e^{(c_0/\alpha)t}.$$

Hence,

$$|f(t)|_{\mathcal{H}^1} \leq |f_0|_{\mathcal{H}^1} e^{(c_1/\alpha)t},$$

which implies that

$$|\Omega_2(t)|_{\mathcal{H}^1} \leq |\Omega_2(0)|_{\mathcal{H}^1} e^{(c_1/\alpha)t}.$$

Similarly, using (3-13), the transport estimate, and following the same steps as above, we can obtain the general $\mathcal{H}^k$ estimates. Hence

$$|\Omega_2|_{\mathcal{H}^k} \leq |\Omega_2(0)|_{\mathcal{H}^k} e^{(c_k/\alpha)t}.$$ □

4. Elliptic estimate

The purpose of this section is to recall the Biot–Savart law decomposition of [Elgindi 2021], which is used here to derive the leading-order model. In this section, we highlight the main ideas in the proof, and for more details, see [Elgindi 2021; Drivas and Elgindi 2023]. We remark that this is also related to the Key Lemma of A. Kiselev and V. Šverák [2014]; see also [Elgindi 2016; Elgindi and Jeong 2023] for generalizations.

Proposition 4.1 [Elgindi 2021]. *Given $\Omega \in H^k$ such that for every R we have*

$$\int_0^{2\pi} \Omega(R,\theta)\sin(n\theta)\,d\theta = \int_0^{2\pi} \Omega(R,\theta)\cos(n\theta)\,d\theta = 0$$

for $n = 0, 1, 2$, the unique solution to

$$4\Psi + \partial_{\theta\theta}\Psi + \alpha^2 R^2 \partial_{RR}\Psi + (4\alpha + \alpha^2) R\partial_R \Psi = \Omega(R,\theta)$$

satisfies

$$|\partial_{\theta\theta}\Psi|_{H^k} + \alpha|R\partial_{R\theta}\Psi|_{H^k} + \alpha^2|R^2\partial_{RR}\Psi|_{H^k} \le C_k|\Omega|_{H^k}, \tag{4-1}$$

where C_k is ***independent*** *of α. In addition, we have the weights estimate*

$$|\partial_{\theta\theta} D_R^k(\Psi)|_{L^2} + \alpha|R\partial_{R\theta} D_R^k(\Psi)|_{L^2} + \alpha^2|R^2\partial_{RR} D_R^k(\Psi)|_{L^2} \le C_k|D_R^k(\Omega)|_{L^2}, \tag{4-2}$$

where C_k is ***independent*** *of α. Recall that $D_R = R\partial_R$.*

Proof. First, we will show how to obtain (4-1). Since Ω is orthogonal to $\sin(n\theta)$ and $\cos(n\theta)$ for $n = 0, 1, 2$, Ψ must also be orthogonal to $\sin(n\theta)$ and $\cos(n\theta)$ for $n = 0, 1, 2$. Consider the elliptic equation, and we consider the L^2 estimate

$$4\Psi + \partial_{\theta\theta}\Psi + \alpha^2 R^2 \partial_{RR}\Psi + (4\alpha + \alpha^2) R\partial_R \Psi = \Omega(R,\theta).$$

Taking the inner product with $\partial_{\theta\theta}\Psi$ and integrating by parts, we obtain

$$-4|\partial_\theta\Psi|_{L^2}^2 + |\partial_{\theta\theta}\Psi|_{L^2}^2 - \alpha^2|\partial_\theta\Psi|_{L^2}^2 + \alpha^2|R\partial_{R\theta}\Psi|_{L^2}^2 + \tfrac{1}{2}(4\alpha+\alpha^2)|\partial_\theta\Psi|_{L^2}^2 \le |\Omega|_{L^2}|\partial_{\theta\theta}\Psi|_{L^2}.$$

Now by assumption, we have

$$\Psi(R,\theta) = \sum_{n\ge 3} \Psi_n(R) e^{in\theta},$$

and hence

$$|\partial_\theta\Psi|_{L^2}^2 \le \tfrac{1}{9}|\partial_{\theta\theta}\Psi|_{L^2}^2.$$

Using the above inequality, we can show that

$$\tfrac{5}{9}|\partial_{\theta\theta}\Psi|_{L^2}^2 + \alpha^2|R\partial_{R\theta}\Psi|_{L^2}^2 + \tfrac{1}{2}(4\alpha - \alpha^2)|\partial_\theta\Psi|_{L^2}^2 \le |\Omega|_{L^2}|\partial_{\theta\theta}\Psi|_{L^2},$$

and thus we have

$$|\partial_{\theta\theta}\Psi|_{L^2} \le C_0|\Omega|_{L^2},$$

where C_0 is independent of α. The estimate for the $R^2\partial_{RR}\Psi$-term will follow similarly. We can also obtain the H^k estimates by following the same strategy. To obtain the (4-2) estimates, recall that $D_R = R\partial_R$ and we notice that we can write the elliptic equation in the form

$$4\Psi + \partial_{\theta\theta}\Psi + \alpha^2 D_R^2(\Psi) + 4\alpha\, D_R(\Psi) = \Omega(R,\theta).$$

From this, we observe that the D_R operator commutes with the elliptic equation, and hence (4-2) estimates will follow from (4-1). ☐

Theorem 2 [Elgindi 2021]. *Given $\Omega \in H^k$, where Ω has the form of*

$$\Omega(R,\theta) = f(R)\sin(2\theta) \quad \big(\Omega(R,\theta) = f(R)\cos(2\theta)\big),$$

the unique solution to

$$4\Psi + \partial_{\theta\theta}\Psi + \alpha^2 R^2 \partial_{RR}\Psi + (4\alpha + \alpha^2) R\partial_R \Psi = \Omega(R,\theta)$$

is

$$\Psi = -\frac{1}{4\alpha} L(f)(R)\sin(2\theta) + \mathcal{R}(f) \quad \Big(\Psi = -\frac{1}{4\alpha} L(f)(R)\cos(2\theta) + \mathcal{R}(f)\Big),$$

where

$$L(f)(R) = \int_R^\infty \frac{f(s)}{s}\, ds$$

and

$$|\mathcal{R}(f)|_{H^k} \le c|f|_{H^k},$$

where c is independent of α.

Proof. Consider the case where $\Omega(R,\theta) = f(R)\sin(2\theta)$; the case where $\Omega(R,\theta) = f(R)\cos(2\theta)$ can be handled similarly. In this case $\Psi(R,\theta)$ will be of the form $\Psi(R,\theta) = \Psi_2(R)\sin(2\theta)$, where $\Psi_2(R)$ will satisfy the ODE

$$\alpha^2 R^2 \partial_{RR}\Psi_2 + (4\alpha + \alpha^2) R\partial_R \Psi_2 = f(R).$$

We can solve the ODE, see Theorem 4.24 in [Drivas and Elgindi 2023], and obtain

$$\partial_R \Psi_2(R) = \frac{1}{\alpha^2}\frac{1}{R^{4/\alpha+1}} \int_0^R \frac{f(s)}{s^{1-4/\alpha}}\, ds.$$

Now using that $\Psi_2(R) \to 0$ as $R \to \infty$, we obtain

$$\Psi_2(R) = -\frac{1}{\alpha^2}\int_R^\infty \frac{1}{\rho^{4/\alpha+1}} \int_0^\rho \frac{f(s)}{s^{1-4/\alpha}}\, ds\, d\rho.$$

We notice that we can write the above as

$$\Psi_2(R) = -\frac{1}{\alpha^2}\int_R^\infty \frac{1}{\rho^{4/\alpha+1}} \int_0^\rho \frac{f(s)}{s^{1-4/\alpha}}\, ds\, d\rho = \frac{1}{4\alpha}\int_R^\infty \partial_\rho\Big(\frac{1}{\rho^{4/\alpha}}\Big) \int_0^\rho \frac{f(s)}{s^{1-4/\alpha}}\, ds\, d\rho.$$

Thus, by integrating by parts, it follows that

$$\Psi_2(R) = -\frac{1}{4\alpha}\int_R^\infty \frac{f(s)}{s}\, ds - \frac{1}{4\alpha}\frac{1}{R^{4/\alpha}} \int_0^R \frac{f(s)}{s^{1-4/\alpha}}\, ds := -\frac{1}{4\alpha} L(f)(R) + \mathcal{R}(f).$$

Using Hardy-type inequality, see Lemma 4.25 in [Drivas and Elgindi 2023], one can show that

$$|\mathcal{R}(f)|_{L^2} \le c|f|_{L^2},$$

where c is independent of α. ☐

5. Embedding estimate in terms of the $\mathcal{H}^k$ norm

In this section we consider some embedding estimate in the $\mathcal{H}^k$ norm which will be used in Section 6. These estimates will be used various times as we estimate the remainder term. Recall that the $\mathcal{H}^k$ norm is defined as

$$|f|_{\dot{\mathcal{H}}^m} = \sum_{i=0}^{m} |\partial_R^i \partial_\theta^{m-i} f|_{L^2} + \sum_{i=1}^{m} |R^i \partial_R^i \partial_\theta^{m-i} f|_{L^2}, \quad |f|_{\mathcal{H}^k} = \sum_{m=0}^{k} |f|_{\dot{\mathcal{H}}^m}.$$

Lemma 5.1. *Let $f \in \mathcal{H}^N$, where $N \in \mathbb{N}$. Then we have*

$$|\partial_R^k \partial_\theta^m f|_{L^\infty} \le c_{k,m} |f|_{\mathcal{H}^{k+m+2}}, \tag{5-1}$$

$$|R^k \partial_R^k \partial_\theta^m f|_{L^\infty} \le c_{k,m} |f|_{\mathcal{H}^{k+m+2}} \tag{5-2}$$

for any $k+m+2 \le N$.

Proof. We will show how to obtain inequality (5-2), since inequality (5-1) follows from standard Sobolev embedding. To show that

$$|R^k \partial_R^k \partial_\theta^m f|_{L^\infty} \le c_{k,m} |f|_{\mathcal{H}^{k+m+2}},$$

for any $k+m+2 \le N$, we apply Sobolev embedding to obtain

$$|R^k \partial_R^k \partial_\theta^m f|_{L^\infty} \le c_{k,m} |R^k \partial_R^k \partial_\theta^m f|_{H^2_{R,\theta}},$$

where $H^2_{R,\theta}$ is the standard H^2 norm in R and θ. When considering the second derivative terms of $R^k \partial_R^k \partial_\theta^m f$, for the angular derivatives term, we have $|R^k \partial_R^k \partial_\theta^{m+2} f|_{L^2} \le |f|_{\mathcal{H}^{k+m+2}}$. Now for the radial derivatives, we have three cases. Considering the case when the two radial derivatives land on $\partial_R^k \partial_\theta^m f$, we have

$$|R^k \partial_R^{k+2} \partial_\theta^m f|_{L^2} \le |R^{k+2} \partial_R^{k+2} \partial_\theta^m f|_{L^2} + |\partial_R^{k+2} \partial_\theta^m f| \le |f|_{\mathcal{H}^{k+m+2}},$$

where the last inequality follows from the definition of the $\mathcal{H}^N$ norm. The other two cases follow in a similar way. □

We will also need some embedding estimates for the stream function Ψ in terms of Ω.

Lemma 5.2. *Let $\Omega \in \mathcal{H}^N$, where $N \in \mathbb{N}$, satisfy the same conditions as in Proposition 4.1. Then for the solution Ψ of*

$$4\Psi + \partial_{\theta\theta}\Psi + \alpha^2 R^2 \partial_{RR}\Psi + (4\alpha + \alpha^2) R \partial_R \Psi = \Omega(R,\theta),$$

we have

$$|\partial_R^k \partial_\theta^m \Psi|_{L^\infty} \le c_{k,m} |\Omega|_{\mathcal{H}^{k+m+1}} \tag{5-3}$$

for $k, m \in \mathbb{N}$ with $k+m+1 \le N$.

Proof. As in Lemma 5.1, applying the Sobolev embedding, we have

$$|\partial_R^k \partial_\theta^m \Psi|_{L^\infty} \le c_{k,m} |\partial_R^k \partial_\theta^m \Psi|_{H^2_{R,\theta}}.$$

From the elliptic estimates in Proposition 4.1, for any $i, n \in \mathbb{N}$, we have

$$|\partial_R^i \partial_\theta^n \Psi|_{L^2} \le c_{i,n} |\Omega|_{\mathcal{H}^{i+n-1}}. \tag{5-4}$$

Thus, to bound $|\partial_R^k\partial_\theta^m\Psi|_{H^2_{R,\theta}}$, we take Ω to be in $\mathcal{H}^{k+m+1}$. Hence, we have

$$|\partial_R^k\partial_\theta^m\Psi|_{L^\infty} \le c_{k,m}|\Omega|_{\mathcal{H}^{k+m+1}}, \tag{5-5}$$

completing the proof. □

Lemma 5.3. *Let $\Omega \in \mathcal{H}^N$, where $N \in \mathbb{N}$, satisfying the same conditions as in Proposition 4.1. Then for the solution Ψ of*

$$4\Psi + \partial_{\theta\theta}\Psi + \alpha^2R^2\partial_{RR}\Psi + (4\alpha+\alpha^2)R\partial_R\Psi = \Omega(R,\theta),$$

we have

$$|R^k\partial_R^k\partial_\theta^m\Psi|_{L^\infty} \le c_{k,m}|\Omega|_{\mathcal{H}^{k+m+1}} \tag{5-6}$$

for $k, m \in \mathbb{N}$ with $k+m+1 \le N$.

Proof. As in Lemma 5.1, applying the Sobolev embedding, we have

$$|R^k\partial_R^k\partial_\theta^m\Psi|_{L^\infty} \le c_{k,m}|R^k\partial_R^k\partial_\theta^m\Psi|_{H^2_{R,\theta}}.$$

From the elliptic estimates in Proposition 4.1, for any $i, n \in \mathbb{N}$, we have

$$|\partial_R^i\partial_\theta^n\Psi|_{L^2} \le c_{i,n}|\partial_R^i\partial_\theta^{n-1}\Omega|_{L^2} \le c_{i,n}|\Omega|_{\mathcal{H}^{i+n-1}} \tag{5-7}$$

and

$$|R^i\partial_R^i\partial_\theta^n\Psi|_{L^2} \le c_{i,n}|\Omega|_{\mathcal{H}^{i+n-1}}. \tag{5-8}$$

Thus, if we look at the second derivative terms of $R^k\partial_R^k\partial_\theta^m\Psi$, we can use the above inequalities to obtain the desired estimate. For the angular derivative term, we have $|R^k\partial_R^k\partial_\theta^{m+2}\Psi|_{L^2} \le c_{k,m}|\Omega|_{\mathcal{H}^{k+m+1}}$. When considering the radial derivative terms, we have three terms. For the $R^k\partial_R^{k+2}\partial_\theta^m\Psi$-term, applying (5-7) and (5-8), we have

$$|R^k\partial_R^{k+2}\partial_\theta^m\Psi|_{L^2} \le |R^{k+2}\partial_R^{k+2}\partial_\theta^m\Psi|_{L^2} + |\partial_R^{k+2}\partial_\theta^m\Psi| \le c_{k,m}|\Omega|_{\mathcal{H}^{k+m+1}}.$$

The other terms can be handled in similar way. Hence, we have our desired result. □

6. Reminder estimate

In this section, we obtain an error estimate on the remaining terms in the Euler with Riesz forcing. Recall that Ω satisfies the evolution equation

$$\begin{aligned}\partial_t\Omega &+ (-\alpha R\partial_\theta\Psi)\,\partial_R\Omega + (2\Psi+\alpha R\partial_R\Psi)\,\partial_\theta\Omega \\ &= \big(2\alpha R\sin(\theta)\cos(\theta) + \alpha^2R\sin(\theta)\cos(\theta)\big)\,\partial_R\Psi + (1-2\sin^2(\theta))\,\partial_\theta\Psi \\ &\quad + \big(\alpha R\cos^2(\theta) + \alpha R\sin^2(\theta)\big)\,\partial_{R\theta}\Psi + (\alpha^2R^2\sin(\theta)\cos(\theta))\,\partial_{RR}\Psi - (\sin(\theta)\cos(\theta))\,\partial_{\theta\theta}\Psi,\end{aligned} \tag{6-1}$$

and the elliptic equation is

$$4\Psi + \alpha^2R^2\partial_{RR}\Psi + \partial_{\theta\theta}\Psi + (4\alpha+\alpha^2)R\partial_R\Psi = \Omega(R,\theta). \tag{6-2}$$

From Section 2, the leading-order model for the Euler with Riesz forcing equation satisfies

$$\partial_t\Omega_2 + (2\Psi_2)\partial_\theta\Omega_2 = (-1+2\sin^2(\theta))\partial_\theta\Psi_2 + (\sin(\theta)\cos(\theta))\,\partial_{\theta\theta}\Psi_2, \tag{6-3}$$

where

$$\Psi_2(R,\theta) = \frac{1}{4\alpha} L_s(\Omega_2)\sin(2\theta) + \frac{1}{4\alpha} L_c(\Omega_2)\cos(2\theta). \tag{6-4}$$

Now set $\Omega_r := \Omega - \Omega_2$ to be the remainder term for the vorticity, and similarly set $\Psi_r := \Psi - \Psi_2$ to be the remainder term for the stream function. Thus, we have that the remainder, Ω_r, satisfies the evolution equation

$$\begin{aligned}
&\partial_t \Omega_r + (-\alpha R(\partial_\theta \Psi_2 + \partial_\theta \Psi_r))(\partial_R \Omega_2 + \partial_R \Omega_r) + (2\Psi_2 \partial_\theta \Omega_r + 2\Psi_r \partial_\theta \Omega_2 + 2\Psi_r \partial_\theta \Omega_r) \\
&\qquad\qquad + (\alpha R(\partial_R \Psi_2 + \partial_R \Psi_r))(\partial_\theta \Omega_2 + \partial_\theta \Omega_r) \\
&= \big(2\alpha R\sin(\theta)\cos(\theta) + \alpha^2 R\sin(\theta)\cos(\theta)\big)(\partial_R \Psi_2 + \partial_R \Psi_r) \\
&\qquad + (1 - 2\sin^2(\theta))\partial_\theta \Psi_r + \alpha(R\cos^2(\theta) - R\sin^2(\theta))(\partial_{R\theta}\Psi_2 + \partial_{R\theta}\Psi_r) \\
&\qquad\qquad + \alpha^2(R^2\sin(\theta)\cos(\theta))(\partial_{RR}\Psi_2 + \partial_{RR}\Psi_r) - (\sin(\theta)\cos(\theta))\partial_{\theta\theta}\Psi_r.
\end{aligned} \tag{6-5}$$

The goal of this section is to show that Ω_r remains small. Namely, using energy methods, for some time T, we show that

$$\sup_{t\le T} |\Omega_r(t)|_{L^\infty} \le C\alpha^{1/2}$$

for some constant C independent of α.

Lemma 6.1. *Let $\Omega_r = \Omega - \Omega_2$ satisfy (6-5) with Ω and Ω_2 satisfying (6-1) and (6-3), respectively. Let $\Psi_r = \Psi - \Psi_2$ with Ψ and Ψ_2 satisfying (6-2) and (6-4), respectively. Then we have the estimates*

$$|\partial_R^k \partial_\theta^m \Psi_r|_{L^2} \le \frac{c_{k,m}}{\alpha} |\Omega_r|_{\mathcal{H}^{k+m-1}} \quad \textit{and} \quad |R^k \partial_R^k \partial_\theta^m \Psi_r|_{L^2} \le \frac{c_{k,m}}{\alpha} |\Omega_r|_{\mathcal{H}^{k+m-1}} \tag{6-6}$$

for $k, m \in \mathbb{N}$.

Proof. Recall that by the Biot–Savart law decomposition [Elgindi 2021] (see Section 4 for more details), we have the following decomposition for the elliptic equation (6-2):

$$\Psi(R,\theta) = \frac{1}{4\alpha} L_s(\Omega)\sin(2\theta) + \frac{1}{4\alpha} L_c(\Omega)\cos(2\theta) + \mathcal{R}(\Omega),$$

with $\mathcal{R}(\Omega)$ bounded on $\mathcal{H}^N$ with a constant independent of α. This follows from the elliptic estimates in Proposition 4.1 and Theorem 2 in Section 4. Now since we defined $\Omega_r = \Omega - \Omega_2$ and $\Psi_r = \Psi - \Psi_2$, with Ω_2, and Ψ_2 satisfying (6-3), and (6-4), respectively, we have the following decomposition for Ψ_r:

$$\Psi_r(R,\theta) = \frac{1}{4\alpha} L_s(\Omega_r)\sin(2\theta) + \frac{1}{4\alpha} L_c(\Omega_r)\cos(2\theta) + \mathcal{R}(\Omega_r) + \mathcal{R}(\Omega_2). \tag{6-7}$$

Hence, this gives the estimates

$$|\partial_R^k \partial_\theta^m \Psi_r|_{L^2} \le \frac{c_{k,m}}{\alpha} |\Omega_r|_{\mathcal{H}^{k+m-1}} \quad \text{and} \quad |R^k \partial_R^k \partial_\theta^m \Psi_r|_{L^2} \le \frac{c_{k,m}}{\alpha} |\Omega_r|_{\mathcal{H}^{k+m-1}}. \qquad \square$$

We define the following terms to shorten the notation:

$$\begin{aligned}
I_1 &= -\alpha R(\partial_\theta \Psi_2 + \partial_\theta \Psi_r)(\partial_R \Omega_2 + \partial_R \Omega_r), \\
I_2 &= (2\Psi_2 \partial_\theta \Omega_r + 2\Psi_r \partial_\theta \Omega_2 + 2\Psi_r \partial_\theta \Omega_r), \\
I_3 &= \alpha R(\partial_R \Psi_2 + \partial_R \Psi_r)(\partial_\theta \Omega_2 + \partial_\theta \Omega_r), \\
I_4 &= 2\alpha(1-\alpha)R\sin(\theta)\cos(\theta)(\partial_R \Psi_2 + \partial_R \Psi_r),
\end{aligned}$$

$$
\begin{aligned}
I_5 &= (1-2\sin^2(\theta))\partial_\theta \Psi_r,\\
I_6 &= \alpha(R\cos^2(\theta) - R\sin^2(\theta))(\partial_{R\theta}\Psi_2 + \partial_{R\theta}\Psi_r),\\
I_7 &= \alpha^2(R^2\sin(\theta)\cos(\theta))(\partial_{RR}\Psi_2 + \partial_{RR}\Psi_r),\\
I_8 &= -(\sin(\theta)\cos(\theta))\partial_{\theta\theta}\Psi_r.
\end{aligned}
$$

Now we have the error estimate proposition.

Proposition 6.2. *Let $\Omega_r = \Omega - \Omega_2$ satisfy (6-5) with $\Omega_r|_{t=0} = 0$. Then*

$$\sup_{0\le t<T} |\Omega_r(t)|_{L^\infty} \le c_N \alpha^{1/2},$$

where $T = c\alpha \log(c|\log(\alpha)|)$ and c is a small constant independent of α.

Proof. We will use ∂^N to refer to any mixed derivatives in R and θ of order N (not excluding pure R- and θ-derivatives). From the definition of the $\mathcal{H}^N$ norm, to obtain the $\mathcal{H}^N$ estimate we will take the following inner product with each I_i-term:

$$\langle \partial^N I_i, \partial^N \Omega_r\rangle \quad\text{and}\quad \langle R^k\partial_R^k\partial_\theta^{N-k} I_i, R^k\partial_R^k\partial_\theta^{N-k}\Omega_r\rangle$$

for $0\le k\le N$ and $1\le i\le 8$.

Estimate on I_1 and I_3: Here we will estimate I_1 and I_3. The estimate of I_3 is very similar to I_1, and so we will just show how to obtain the estimate on I_1.

Estimate on I_1: We can write I_1 as

$$
\begin{aligned}
I_1 &= -\alpha R(\partial_\theta\Psi_2 + \partial_\theta\Psi_r)(\partial_R\Omega_2 + \partial_R\Omega_r)\\
&= -\alpha(\partial_\theta\Psi_2)R(\partial_R\Omega_2) - \alpha(\partial_\theta\Psi_2)R(\partial_R\Omega_r) - \alpha(\partial_\theta\Psi_r)R(\partial_R\Omega_2) - \alpha(\partial_\theta\Psi_r)R(\partial_R\Omega_r)\\
&= I_{1,1} + I_{1,2} + I_{1,3} + I_{1,4},
\end{aligned}
$$

and we will estimate each term separately.

- $I_{1,1} = -\alpha\partial_\theta\Psi_2\, R\partial_R\Omega_2$. Here we have

$$\langle \partial^N(\alpha\partial_\theta\Psi_2\, R\partial_R\Omega_2), \partial^N\Omega_r\rangle = \sum_{i=0}^N c_{i,N}\int \partial^i(\alpha\partial_\theta\Psi_2)\partial^{N-i}(R\partial_R\Omega_2)\,\partial^N\Omega_r.$$

Now from Lemmas 3.4 and 3.5, we know that

$$|\Psi_2|_{\mathcal{W}^{k+1,\infty}} \le \frac{c_k}{\alpha} \quad\text{and}\quad |\Omega_2|_{\mathcal{H}^k} \le |\Omega_2(0)|_{\mathcal{H}^k} e^{(c_k/\alpha)t}.$$

Thus, we have

$$
\begin{aligned}
\sum_{i=0}^N \int \alpha\partial^i(\partial_\theta\Psi_2)\partial^{N-i}(R\partial_R\Omega_2)\,\partial^N\Omega_r &\le c_N\sum_{i=0}^N \alpha|\partial^i\partial_\theta\Psi_2|_{L^\infty}\,|\partial^{N-i}(R\partial_R\Omega_2)|_{L^2}\,|\partial^N\Omega_r|_{L^2}\\
&\le c_N\alpha|\Psi_2|_{\mathcal{W}^{N+1,\infty}}\,|\Omega_2|_{\mathcal{H}^{N+1}}\,|\Omega_r|_{\mathcal{H}^N}\\
&\le \alpha\frac{c_N}{\alpha}e^{(c_N/\alpha)t}|\Omega_r|_{\mathcal{H}^N} \le c_N e^{(c_N/\alpha)t}|\Omega_r|_{\mathcal{H}^N},
\end{aligned}
$$

and similarly we have

$$\langle \partial_R^k \partial_\theta^{N-k}(\alpha\partial_\theta\Psi_2\, R\partial_R\Omega_2),\, R^{2k}\partial_R^k\partial_\theta^{N-k}\Omega_r\rangle = c_{i,m,N}\int \sum_{i+m=0}^{N} \partial_R^i\partial_\theta^m(\alpha\partial_\theta\Psi_2)\,\partial_R^{k-i}\partial_\theta^{N-k-m}(R\partial_R\Omega_2)\, R^{2k}\partial_R^k\partial_\theta^{N-k}\Omega_r.$$

From the definition of the $\mathcal{W}^{N+1,\infty}$ norm, we have for $i+m\le N$,

$$|R^i\partial_R^i\partial_\theta^{m+1}\Psi_2|_{L^\infty} \le |\Psi_2|_{\mathcal{W}^{N+1,\infty}}.$$

Again, applying Lemmas 3.4 and 3.5, we obtain

$$\begin{aligned}\sum_{i+m=0}^{N}\int R^i\partial_R^i\partial_\theta^m(\alpha\partial_\theta\Psi_2)\, R^{k-i}\,\partial_R^{k-i}\partial_\theta^{N-k-m}(R\partial_R\Omega_2)\, R^k\partial_R^k\partial_\theta^{N-k}\Omega_r \\ \le c_N\sum_{i+m=0}^{N}\alpha|R^i\partial_R^i\partial_\theta^{m+1}\Psi_2|_{L^\infty}\,|R^{k-i}\,\partial_R^{k-i}\partial_\theta^{N-k-m}(R\partial_R\Omega_2)|_{L^2}\,|R^k\partial_R^k\partial_\theta^{N-k}\Omega_r|_{L^2} \\ \le c_N\alpha|\Psi_2|_{\mathcal{W}^{N+1,\infty}}\,|\Omega_2|_{\mathcal{H}^{N+1}}\,|\Omega_r|_{\mathcal{H}^N} \le \alpha\frac{c_N}{\alpha}e^{(c_N/\alpha)t}|\Omega_r|_{\mathcal{H}^N} \le c_N e^{(c_N/\alpha)t}|\Omega_r|_{\mathcal{H}^N}.\end{aligned}$$

Thus, we have

$$\langle I_{1,1},\,\Omega_r\rangle_{\mathcal{H}^N} \le c_N e^{(c_N/\alpha)t}|\Omega_r|_{\mathcal{H}^N}. \tag{6-8}$$

- $I_{1,2} = -\alpha\partial_\theta\Psi_2\, R\partial_R\Omega_r$. Here we have

$$\langle \partial^N(\alpha\partial_\theta\Psi_2\, R\partial_R\Omega_r),\,\partial^N\Omega_r\rangle = \sum_{i=0}^{N} c_{i,N}\int \partial^i(\alpha\partial_\theta\Psi_2)\partial^{N-i}(R\partial_R\Omega_r)\partial^N\Omega_r.$$

To obtain this estimate, we again apply Lemma 3.4. Namely, that $|\Psi_2|_{\mathcal{W}^{k+1,\infty}} \le c_k/\alpha$. When $i=0$, we integrate by parts and obtain

$$\int (\alpha\partial_\theta\Psi_2)\partial^N(R\partial_R\Omega_r)\,\partial^N\Omega_r \le c|\Psi_2|_{\mathcal{W}^{2,\infty}}|\Omega_r|^2_{\mathcal{H}^N} \le \frac{c_N}{\alpha}|\Omega_r|^2_{\mathcal{H}^N}.$$

For $1\le i\le N$ we have

$$\begin{aligned}\sum_{i=1}^{N}\int \alpha\partial^i(\partial_\theta\Psi_2)\partial^{N-i}(R\partial_R\Omega_r)\,\partial^N\Omega_r \le c_N\sum_{i=1}^{N}\alpha|\partial^i\partial_\theta\Psi_2|_{L^\infty}\,|\partial^{N-i}(R\partial_R\Omega_r)|_{L^2}\,|\partial^N\Omega_r|_{L^2} \\ \le c_N\alpha|\Psi_2|_{\mathcal{W}^{N+1,\infty}}\,|\Omega_r|_{\mathcal{H}^N}\,|\Omega_r|_{\mathcal{H}^N} \le \alpha\frac{c_N}{\alpha}|\Omega_r|^2_{\mathcal{H}^N} \le c_N|\Omega_r|^2_{\mathcal{H}^N}.\end{aligned}$$

Similarly, now for the $R^k\partial_R^k\partial_\theta^{N-k}$-terms we have

$$\langle R^k\partial_R^k\partial_\theta^{N-k}(\alpha\partial_\theta\Psi_2\, R\partial_R\Omega_r),\, R^k\partial_R^k\partial_\theta^{N-k}\Omega_r\rangle = c_{i,m,N}\int\sum_{i+m=0}^{N} R^k\partial_R^i\partial_\theta^m(\alpha\partial_\theta\Psi_2)\,\partial_R^{k-i}\partial_\theta^{N-k-m}(R\partial_R\Omega_r)\, R^k\partial_R^k\partial_\theta^{N-k}\Omega_r.$$

We again use $|\Psi_2|_{\mathcal{W}^{k+1,\infty}} \le c_k/\alpha$. Hence, we have

$$\begin{aligned}\sum_{i+m=0}^{N} \int R^i \partial_R^i \partial_\theta^m (\alpha \partial_\theta \Psi_2)\, R^{k-i}\, \partial_R^{k-i} \partial_\theta^{N-k-m} (R\partial_R \Omega_r)\, R^k \partial_R^k \partial_\theta^{N-k} \Omega_r \\ \le c_N \sum_{i+m=0}^{N} \alpha |R^i \partial_R^i \partial_\theta^{m+1} \Psi_2|_{L^\infty}\, |R^{k-i}\, \partial_R^{k-i} \partial_\theta^{N-k-m} (R\partial_R \Omega_r)|_{L^2}\, |R^k \partial_R^k \partial_\theta^{N-k} \Omega_r|_{L^2} \\ \le c_N \alpha |\Psi_2|_{\mathcal{W}^{N+1,\infty}}\, |\Omega_r|_{\mathcal{H}^N}\, |\Omega_r|_{\mathcal{H}^N} \le \alpha \frac{c_N}{\alpha} |\Omega_r|^2_{\mathcal{H}^N} \le c_N |\Omega_r|^2_{\mathcal{H}^N}.\end{aligned}$$

Thus, we have

$$\langle I_{1,2}, \Omega_r \rangle_{\mathcal{H}^N} \le c_N |\Omega_r|^2_{\mathcal{H}^N}. \tag{6-9}$$

- $I_{1,3} = -\alpha(\partial_\theta \Psi_r) R\partial_R \Omega_2$. To obtain the estimate on $I_{1,3}$, we will use Lemma 3.5, which will give us the estimate on Ω_2. In addition, to bound the $\partial_\theta \Psi_r$-term, we will use the decomposition of Ψ_r (6-7) and estimate (6-6) combined with the elliptic estimates from Proposition 4.1 and embedding estimates from Lemma 5.2. Now we have

$$\langle \partial^N (\alpha \partial_\theta \Psi_r\, R\partial_R \Omega_2), \partial^N \Omega_r \rangle = \sum_{i=0}^{N} c_{i,N} \int \partial^i (\alpha \partial_\theta \Psi_r) \partial^{N-i} (R\partial_R \Omega_2)\, \partial^N \Omega_r.$$

When $0 \le i \le N/2$, we will use the embedding from Lemma 5.1. Namely that

$$|\partial^i \partial_\theta \Psi_r|_{L^\infty} \le c_i |\partial_\theta \Psi_r|_{\mathcal{H}^{i+2}}.$$

Then, applying Lemma 6.1, we have

$$|\partial_\theta \Psi_r|_{\mathcal{H}^{i+2}} \le \frac{c_i}{\alpha} |\Omega_r|_{\mathcal{H}^{i+2}}.$$

Thus,

$$\begin{aligned}\sum_{i=0}^{N/2} \int \partial^i (\alpha \partial_\theta \Psi_r) \partial^{N-i} (R\partial_R \Omega_2)\, \partial^N \Omega_r &\le \sum_{i=0}^{N/2} \alpha |\partial^i \partial_\theta \Psi_r|_{L^\infty}\, |\partial^{N-i} (R\partial_R \Omega_2)|_{L^2} |\partial^N \Omega_r|_{L^2} \\ &\le \sum_{i=0}^{N/2} \alpha \frac{c_i}{\alpha} |\Omega_r|_{\mathcal{H}^{i+2}} |\Omega_2|_{\mathcal{H}^{N+1}} |\Omega_r|_{\mathcal{H}^N} \\ &\le |\Omega_r|_{\mathcal{H}^{N/2+2}} |\Omega_2|_{\mathcal{H}^{N+1}} |\Omega_r|_{\mathcal{H}^N} \le c_N e^{c_N/\alpha} |\Omega_r|^2_{\mathcal{H}^N}.\end{aligned}$$

Here we used Lemma 3.5 for the $|\Omega_2|_{\mathcal{H}^{N+1}}$-term.

When $N/2 \le i \le N$, we will use Lemma 6.1. Namely,

$$|\partial^i \partial_\theta \Psi_r|_{L^2} \le \frac{c_i}{\alpha} |\Omega_r|_{\mathcal{H}^i}.$$

Thus, we have

$$\begin{aligned}\sum_{i=N/2}^{N} \int \partial^i (\alpha \partial_\theta \Psi_r) \partial^{N-i} (R\partial_R \Omega_2)\, \partial^N \Omega_r &\le \sum_{i=N/2}^{N} \alpha |\partial^i \partial_\theta \Psi_r|_{L^2}\, |R\partial_R \Omega_2|_{\mathcal{W}^{N-i,\infty}} |\partial^N \Omega_r|_{L^2} \\ &\le \sum_{i=N/2}^{N} \alpha \frac{c_i}{\alpha} |\Omega_r|_{\mathcal{H}^i} |\Omega_2|_{\mathcal{W}^{N/2,\infty}} |\Omega_r|_{\mathcal{H}^N} \\ &\le c_N |\Omega_r|_{\mathcal{H}^N} |\Omega_2|_{\mathcal{H}^N} |\Omega_r|_{\mathcal{H}^N} \le c_N e^{(c_N/\alpha)t} |\Omega_r|^2_{\mathcal{H}^N}.\end{aligned}$$

Similarly, to estimate the inner product

$$\langle \partial_R^k \partial_\theta^{N-k}(\alpha(\partial_\theta \Psi_r) R\partial_R \Omega_2),\, R^{2k}\partial_R^k \partial_\theta^{N-k}\Omega_r\rangle \le c_N e^{(c_N/\alpha)t}|\Omega_r|^2_{\mathcal{H}^N},$$

we will use the weighted embedding estimates from Lemma 5.1 combined with Lemma 6.1. Following the same steps as we did in the previous inner product, we obtain

$$\langle I_{1,3},\, \Omega_r\rangle_{\mathcal{H}^N} \le c_N e^{(c_N/\alpha)t}|\Omega_r|^2_{\mathcal{H}^N}. \tag{6-10}$$

- $I_{1,4} = -\alpha(\partial_\theta \Psi_r) R\partial_R \Omega_r$. To obtain the estimate on $I_{1,4}$, we will use Lemma 6.1 and the embedding estimate from Lemma 5.1 to handle the $\partial_\theta \Psi_r$-term. To handle the $R\partial_R\Omega_r$-term, we will use embedding estimates from Lemma 5.1 and follow the same steps as we did in the previous inner product. We will only show how to obtain the estimate on the term

$$\langle \partial_R^k \partial_\theta^{N-k}(\alpha \partial_\theta \Psi_r\, R\partial_R \Omega_r),\, R^{2k}\partial_R^k \partial_\theta^{N-k}\Omega_r\rangle = c_{i,m,N}\int \sum_{i+m=0}^{N} \partial_R^i\partial_\theta^m(\alpha\partial_\theta\Psi_r)\,\partial_R^{k-i}\partial_\theta^{N-k-m}(R\partial_R\Omega_r)\,R^{2k}\partial_R^k\partial_\theta^{N-k}\Omega_r.$$

For the other inner product, the idea is the same. To start the estimate, first we consider the case when $i = m = 0$. We integrate by parts and use the embedding estimates in Lemmas 5.1 and 6.1 to estimate the $\partial_\theta\Psi_r$-term. We have

$$\begin{aligned}\int \alpha\partial_\theta\Psi_r(R^{k+1}\partial_R^{k+1}\partial_\theta^{N-k}\Omega_r + R^k\partial_R^k\partial_\theta^{N-k}\Omega_r)\,R^k\partial_R^k\partial_\theta^{N-k}\Omega_r &\le \alpha|R\partial_{R\theta}\Psi_r|_{L^\infty}\,|R^k\partial_R^k\partial_\theta^{N-k}\Omega_r|^2_{L^2} + \alpha|\partial_\theta\Psi_r|_{L^\infty}\,|R^k\partial_R^k\partial_\theta^{N-k}\Omega_r|^2_{L^2}\\ &\le c_N(|\Omega_r|_{\mathcal{H}^3}\,|\Omega_r|^2_{\mathcal{H}^N} + |\Omega_r|_{\mathcal{H}^2}\,|\Omega_r|^2_{\mathcal{H}^N})\\ &\le c_N|\Omega_r|^3_{\mathcal{H}^N}.\end{aligned}$$

Now when $1 \le i+m \le N/2$, we will again use Lemmas 5.1 and 6.1 and the definition of the $\mathcal{H}^k$ norm to obtain

$$\begin{aligned}&\sum_{i+m\ge 1}^{N/2} R^i\partial_R^i\partial_\theta^m(\alpha\partial_\theta\Psi_r)\big(R^{k+1-i}\partial_R^{k+1-i}\partial_\theta^{N-k-m}\Omega_r + R^{k-i}\partial_R^{k-i}\partial_\theta^{N-k-m}\Omega_r\big)\,R^k\partial_R^k\partial_\theta^{N-k}\Omega_r\\ &\quad\le \sum_{i+m\ge 1}^{N/2}\alpha|R^i\,\partial_R^i\partial_\theta^{m+1}\Psi_r|_{L^\infty}\,|R^{k+1-i}\partial_R^{k+1-i}\partial_\theta^{N-k-m}\Omega_r|_{L^2}\,|R^k\partial_R^k\partial_\theta^{N-k}\Omega_r|_{L^2}\\ &\qquad\qquad + \sum_{i+m\ge 1}^{N/2}\alpha|R^i\,\partial_R^i\partial_\theta^{m+1}\Psi_r|_{L^\infty}\,|R^{k-i}\partial_R^{k-i}\partial_\theta^{N-k-m}\Omega_r|_{L^2}\,|R^k\partial_R^k\partial_\theta^{N-k}\Omega_r|_{L^2}\\ &\quad\le c_N\sum_{i+m\ge 1}^{N/2}|\Omega_r|_{\mathcal{H}^{i+m+2}}(|\Omega_r|_{\mathcal{H}^N} + |\Omega_r|_{\mathcal{H}^{N-1}})|\Omega_r|_{\mathcal{H}^N}\\ &\quad\le c_N|\Omega_r|_{\mathcal{H}^{N/2+3}}(|\Omega_r|_{\mathcal{H}^N} + |\Omega_r|_{\mathcal{H}^{N-1}})|\Omega_r|_{\mathcal{H}^N}\\ &\quad\le c_N|\Omega_r|^3_{\mathcal{H}^N}.\end{aligned}$$

Now for the case when $N/2 \le i+m \le N$, we will use Lemmas 5.1 and 6.1 to obtain

$$\sum_{i+m\ge N/2}^{N} R^i\partial_R^i\partial_\theta^m(\alpha\partial_\theta\Psi_r)\big(R^{k+1-i}\partial_R^{k+1-i}\partial_\theta^{N-k-m}\Omega_r + R^{k-i}\partial_R^{k-i}\partial_\theta^{N-k-m}\Omega_r\big)\, R^k\partial_R^k\partial_\theta^{N-k}\Omega_r$$
$$\le \sum_{i+m\ge N/2}^{N} \alpha|R^i\,\partial_R^i\partial_\theta^{m+1}\Psi_r|_{L^2}\,\big(|R^{k+1-i}\partial_R^{k+1-i}\partial_\theta^{N-k-m}\Omega_r|_{L^\infty}\big)\,|R^k\partial_R^k\partial_\theta^{N-k}\Omega_r|_{L^2}$$
$$+\sum_{i+m\ge N/2}^{N} \alpha|R^i\,\partial_R^i\partial_\theta^{m+1}\Psi_r|_{L^2}\big(|R^{k-i}\partial_R^{k-i}\partial_\theta^{N-k-m}\Omega_r|_{L^\infty}\big)\,|R^k\partial_R^k\partial_\theta^{N-k}\Omega_r|_{L^2}$$
$$\le \sum_{i+m\ge N/2}^{N} |\Omega_r|_{\mathcal{H}^{i+m}}\big(|\Omega_r|_{\mathcal{H}^{N-(i+m)+3}} + |\Omega_r|_{\mathcal{H}^{N-(i+m)+2}}\big)|\Omega_r|_{\mathcal{H}^N}$$
$$\le c_N|\Omega_r|_{\mathcal{H}^N}|\Omega_r|_{\mathcal{H}^{N/2+3}}|\Omega_r|_{\mathcal{H}^N} \le c_N|\Omega_r|^3_{\mathcal{H}^N},$$

and thus, we have

$$\langle I_{1,4},\,\Omega_r\rangle_{\mathcal{H}^N} \le c_N|\Omega_r|^3_{\mathcal{H}^N}. \tag{6-11}$$

Thus, we have the following estimate on the I_1-term:

$$\langle I_1,\,\Omega_r\rangle_{\mathcal{H}^N} \le c_N e^{(c_N/\alpha)t}|\Omega_r|_{\mathcal{H}^N} + c_N e^{(c_N/\alpha)t}|\Omega_r|^2_{\mathcal{H}^N} + c_N|\Omega_r|^3_{\mathcal{H}^N}. \tag{6-12}$$

<u>Estimate on I_3</u>: The estimate on I_3 follows similarly to I_1, so we skip the details for this case. One can obtain

$$\langle I_3,\,\Omega_r\rangle_{\mathcal{H}^N} \le c_N e^{(c_N/\alpha)t}|\Omega_r|_{\mathcal{H}^N} + c_N e^{(c_N/\alpha)t}|\Omega_r|^2_{\mathcal{H}^N} + c_N|\Omega_r|^3_{\mathcal{H}^N}. \tag{6-13}$$

<u>Estimate on I_2</u>: Here we have

$$I_2 = (2\Psi_2\partial_\theta\Omega_r + 2\Psi_r\partial_\theta\Omega_2 + 2\Psi_r\partial_\theta\Omega_r) = I_{2,1} + I_{2,2} + I_{2,3}.$$

- $I_{2,1} = 2\Psi_2\partial_\theta\Omega_r$. To estimate $I_{2,1}$, we follow the same steps as in the I_1-term. Using Lemma 3.4, namely that $|\Psi_2|_{\mathcal{W}^{N,\infty}} \le c_N/\alpha$, we have

$$\langle I_{2,1},\,\Omega_r\rangle_{\mathcal{H}^N} \le \frac{c_N}{\alpha}|\Omega_r|^2_{\mathcal{H}^N}. \tag{6-14}$$

- $I_{2,2} = 2\Psi_r\partial_\theta\Omega_2$. Similarly, to estimate $I_{2,2}$ we also follow the same steps as we did in I_1. More specifically, to handle the Ψ_r-term, we will follow similar steps as for the terms $I_{1,3}$ and $I_{1,4}$. Namely, we will apply embedding estimates and Lemma 6.1 to estimate the Ψ_r-term. To estimate Ω_2, we use Lemma 3.5 to obtain that $|\Omega_2|_{\mathcal{H}^k} \le |\Omega_2(0)|_{\mathcal{H}^k}e^{(c_k/\alpha)t}$. Thus we have

$$\langle I_{2,2},\,\Omega_r\rangle_{\mathcal{H}^N} \le \frac{c_N}{\alpha}e^{(c_N/\alpha)t}|\Omega_r|^2_{\mathcal{H}^N}. \tag{6-15}$$

- $I_{2,3} = 2\Psi_r\partial_\theta\Omega_r$. This term $I_{2,3}$ can be estimated similarly to the $I_{1,4}$-term by using embedding and Lemma 6.1. Hence, we obtain

$$\langle I_{2,3},\,\Omega_r\rangle_{\mathcal{H}^N} \le \frac{c_N}{\alpha}|\Omega_r|^3_{\mathcal{H}^N}. \tag{6-16}$$

Thus we have

$$\langle I_2, \Omega_r\rangle_{\mathcal{H}^N} \le \frac{c_N}{\alpha}|\Omega_r|^2_{\mathcal{H}^N} + \frac{c_N}{\alpha}e^{(c_N/\alpha)t}|\Omega_r|^2_{\mathcal{H}^N} + \frac{c_N}{\alpha}|\Omega_r|^3_{\mathcal{H}^N} \le \frac{c_N}{\alpha}e^{(c_N/\alpha)t}|\Omega_r|^2_{\mathcal{H}^N} + \frac{c_N}{\alpha}|\Omega_r|^3_{\mathcal{H}^N}. \quad (6\text{-}17)$$

Estimates on I_4, I_5, I_6, I_7, and I_8: We can write I_4 as

$$\begin{aligned} I_4 &= 2\alpha R\sin(\theta)\cos(\theta) + \alpha^2 R\sin(\theta)\cos(\theta))(\partial_R\Psi_2 + \partial_R\Psi_r) \\ &= \alpha(2+\alpha)\sin(\theta)\cos(\theta)\, R\partial_R\Psi_2 + \alpha(2+\alpha)\sin(\theta)\cos(\theta)\, R\partial_R\Psi_r = I_{4,1} + I_{4,2}. \end{aligned}$$

Recall that

$$I_5 = (1 - 2\sin^2(\theta))\partial_\theta\Psi_r.$$

We can also rewrite and I_6 and I_7 as

$$\begin{aligned} I_6 &= \alpha(\cos^2(\theta) - \sin^2(\theta))R(\partial_{R\theta}\Psi_2 + \partial_{R\theta}\Psi_r) \\ &= \alpha(\cos^2(\theta) - \sin^2(\theta))R\partial_{R\theta}\Psi_2 + \alpha(\cos^2(\theta) - \sin^2(\theta))R\partial_{R\theta}\Psi_r = I_{6,1} + I_{6,2} \end{aligned}$$

and

$$\begin{aligned} I_7 &= \alpha^2(\sin(\theta)\cos(\theta))R^2(\partial_{RR}\Psi_2 + \partial_{RR}\Psi_r) \\ &= \alpha^2(\sin(\theta)\cos(\theta))R^2\,\partial_{RR}\Psi_2 + \alpha^2(\sin(\theta)\cos(\theta))R^2\,\partial_{RR}\Psi_r = I_{7,1} + I_{7,2}. \end{aligned}$$

Recall that

$$I_8 = -\sin(\theta)\cos(\theta)\,\partial_{\theta\theta}\Psi_r.$$

Now for $i = 4$, 6, and 7, using Lemma 3.4, namely that $|\Psi|_{\mathcal{H}^{k+1}} \le c_k/\alpha$, we have the estimate

$$\langle I_{i,1}, \Omega_r\rangle_{\mathcal{H}^N} \le c_N|\Omega_r|_{\mathcal{H}^N} \quad \text{for } i = 4, 6, 7. \quad (6\text{-}18)$$

Using Lemma 6.1, we obtain

$$\langle I_{i,2}, \Omega_r\rangle_{\mathcal{H}^N} \le \alpha\frac{c_N}{\alpha}|\Omega_r|^2_{\mathcal{H}^N} = c_N|\Omega_r|^2_{\mathcal{H}^N} \quad \text{for } i = 4, 6, 7 \quad (6\text{-}19)$$

and

$$\langle I_i, \Omega_r\rangle_{\mathcal{H}^N} \le \frac{c_N}{\alpha}|\Omega_r|^2_{\mathcal{H}^N} \quad \text{for } i = 5, 8. \quad (6\text{-}20)$$

Hence, from (6-18), (6-19), (6-20), we have

$$\langle I_i, \Omega_r\rangle_{\mathcal{H}^N} \le c_N|\Omega_r|_{\mathcal{H}^N} + \frac{c_N}{\alpha}|\Omega_r|^2_{\mathcal{H}^N} \quad \text{for } i = 4, 5, \dots, 8. \quad (6\text{-}21)$$

Total remainder estimate: Here we obtain the total error estimate. From our previous work we have

$$\frac{d}{dt}|\Omega_r|^2_{\mathcal{H}^N} = \langle\partial_t\Omega_r, \Omega_r\rangle_{\mathcal{H}^N} \le \sum_{i=1}^{8}|\langle I_i, \Omega_r\rangle_{\mathcal{H}^N}|,$$

and thus from (6-12), (6-13), (6-17), and (6-21), we have

$$\frac{d}{dt}|\Omega_r|^2_{\mathcal{H}^N} \le c_N e^{(c_N/\alpha)t}|\Omega_r|_{\mathcal{H}^N} + \frac{c_N}{\alpha}e^{(c_N/\alpha)t}|\Omega_r|^2_{\mathcal{H}^N} + \frac{c_N}{\alpha}|\Omega_r|^3_{\mathcal{H}^N},$$

and hence

$$\frac{d}{dt}|\Omega_r|_{\mathcal{H}^N} \le c_N e^{(c_N/\alpha)t} + \frac{c_N}{\alpha}e^{(c_N/\alpha)t}|\Omega_r|_{\mathcal{H}^N} + \frac{c_N}{\alpha}|\Omega_r|^2_{\mathcal{H}^N}. \quad (6\text{-}22)$$

Now since we have $\Omega_r|_{t=0} = 0$, we will use bootstrap argument to close the remainder estimate. We will assume that $|\Omega_r|_{\mathcal{H}^N} \leq 2c_N\alpha^{1/2}$ for time $0 < t \leq T$, and then show that $|\Omega_r(t)|_{\mathcal{H}^N} \leq c_N\alpha^{1/2}$, and this will give the remainder estimate. Let us assume that

$$|\Omega_r|_{\mathcal{H}^N} \leq 2c_N\alpha^{1/2}.$$

Then from (6-22) we have

$$\frac{d}{dt}|\Omega_r|_{\mathcal{H}^N} \leq c_N e^{(c_N/\alpha)t} + \frac{c_N}{\alpha} e^{(c_N/\alpha)t}|\Omega_r|_{\mathcal{H}^N} + 4c_N^3,$$

and thus

$$|\Omega_r|_{\mathcal{H}^N} \leq \left(\int_0^t c_N e^{(c_N/\alpha)\tau} + 4c_N^3\,d\tau\right)\exp\left(\int_0^t \frac{c_N}{\alpha} e^{(c_N/\alpha)\tau}\,d\tau\right) \leq (\alpha c_N e^{(c_N/\alpha)t} + 4c_N^3 t)\exp(c_N e^{(c_N/\alpha)t}).$$

Hence, if we choose our time scale $0 < t \leq T(\alpha) = c_1\alpha\log(c_2|\log(\alpha)|)$ for c_1 and c_2 small constants, for example, take $c_1 = 1/c_N$, and $c_2 = 1/(4c_N)$, we have

$$|\Omega_r|_{\mathcal{H}^N} \leq c_N\alpha^{1/2},$$

which completes the bootstrap argument and gives the proof of Proposition 6.2. □

7. Main result

We now recall and prove the main theorem of this work.

Theorem 3. *For any $\alpha, \delta > 0$, there exists initial data $\omega_0^{\alpha,\delta} \in C_c^\infty(\mathbb{R}^2)$ and $T(\alpha)$ such that the corresponding unique global solution, $\omega^{\alpha,\delta}$, to (1-4) is such that at $t = 0$ we have*

$$|\omega_0^{\alpha,\delta}|_{L^\infty} = \delta,$$

but for any $0 < t \leq T(\alpha)$ we have

$$|\omega^{\alpha,\delta}(t)|_{L^\infty} \geq |\omega_0|_{L^\infty} + c\log\left(1 + \frac{c}{\alpha}t\right),$$

where $T(\alpha) = c\alpha\log(c|\log(\alpha)|)$, and $c > 0$ is a constant independent of α that depends linearly on δ.

Proof. Consider the initial data of the form

$$\omega_0 = \Omega|_{t=0} = f_0(R)\sin(2\theta),$$

where $f_0(R)$, with $R = r^\alpha$, is a nonnegative compactly supported smooth function which is zero on $\left[0, \frac{1}{2}\right] \cup [1, \infty)$ and positive outside. We know that we can write $\Omega = \Omega_2 + \Omega_r$, and from the form of the initial data, we have $\Omega_r|_{t=0} = 0$ and thus from Proposition 6.2 we have

$$|\Omega_r(t)|_{L^\infty} \leq c_N\alpha^{1/2}$$

for $0 \leq t \leq T(\alpha) = c\alpha\log(c|\log(\alpha)|)$, where recall that c is a small constant independent of α. Recall also that we can write Ω_2 as

$$\Omega_2 = f + \frac{1}{2\alpha}\int_0^t L_s(f_\tau)\,d\tau,$$

and thus from Proposition 3.3, we obtain

$$\Omega_2 = f + \frac{1}{2\alpha}\int_0^t L_s(f_\tau)\,d\tau \geq f + c_0\log\Big(1+\frac{c_0}{\alpha}t\Big)$$

for some c_0 independent of α and thus we have our desired result. □

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TAREK M. ELGINDI: tarek.elgindi@duke.edu
Department of Mathematics, Duke University, Durham, NC, United States

KARIM R. SHIKH KHALIL: karimridamoh.shikh.khalil@duke.edu
Department of Mathematics, Duke University, Durham, NC, United States

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