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We derive estimates for linear potentials that hold away from thin subsets. And, inspired by the celebrated work of Huber (1957) and Cohn-Vossen (1935), we verify that, for a subset that is thin at a point, there is always a geodesic that reaches to the point and avoids the thin subset in general dimensions. As applications of these estimates on linear potentials, we consider the scalar curvature equations and improve the results of Schoen and Yau (1988, 1994) and Carron (2012) on the Hausdorff dimensions of singular sets which represent the ends of complete conformal metrics on domains in manifolds of dimension greater than 3. We also study Q -curvature equations in dimensions greater than 4 and obtain stronger results on the Hausdorff dimensions of the singular sets than those of Chang et al. (2004). More interestingly, our approach based on potential theory yields a significantly stronger finiteness theorem on the singular sets for Q -curvature equations in dimension 4 than those of Chang et al. (2000) and Carron and Herzlich (2002), which is a remarkable analogue of Huber's theorem.

1. Introduction

We employ linear potential theory to study scalar curvature equations and Q -curvature equations in conformal geometry. This is a continuation of our recent work on n -superharmonic functions (see [Bonini et al. 2018; 2019; Ma and Qing 2021; 2022]) inspired by Huber's theorem and related work on superharmonic functions in dimension 2 (see [Cohn-Vossen 1935; Huber 1957; Arsove and Huber 1973; Hayman and Kennedy 1976]).

Linear potential theory has always been a major subject in analysis and partial differential equations. We refer readers, for instance, to [Mizuta 1996; Adams and Hedberg 1996; Armitage and Gardiner 2001] for good introductions on potential theory. For clarity, the definitions of Riesz potentials and log potentials are given in Section 2. For our purpose, the kernel functions are not chosen for discussions on the boundary behavior of potentials and we focus on the outer capacity and thin subsets (please see Definitions 2.2 and 2.8 in Section 2). Also we set up some of the potential theory on Riemannian manifolds directly. The interesting result on Riesz potentials we obtain is:

Theorem 1.1. *Suppose that (M^n, g) is a complete Riemannian manifold and μ is a finite nonnegative Radon measure on a bounded domain $G \subset M^n$. Let S be a compact subset in G such that its Hausdorff*

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dimension is greater than d , where $d < n - \alpha$ and $\alpha \in (1, n)$. Then there is a point $p \in S$ and a subset E that is α -thin at p such that

$$\int_G \frac{1}{d(x, y)^{n-\alpha}} d\mu \leq \frac{C}{d(x, p)^{n-\alpha-d}} \quad (1-1)$$

for some constant C and all $x \in B_\delta(p) \setminus E$ for some small $\delta > 0$.

The proof of Theorem 1.1 uses a general decomposition result [Kpata 2019, Proposition 1.4] and multiscale analysis. We also give a proof of a slight extension of [Mizuta 1996, Theorem 6.3] for log potentials on manifolds, which is closely related to [Cohn-Vossen 1935; Huber 1957; Arsove and Huber 1973; Ma and Qing 2021; 2022] for us. What makes these estimates useful is the following key observation about thin subsets in general dimensions (see [Cohn-Vossen 1935; Huber 1957; Arsove and Huber 1973; Ma and Qing 2021; 2022]).

Theorem 1.2. *Let E be a subset in the Euclidean space \mathbb{R}^n and $p \in \mathbb{R}^n$. Suppose that E is α -thin at the point p for $\alpha \in (1, n]$. Then there is always a ray from p that avoids E at least within some small ball at p .*

The proof of Theorem 1.2 uses only the scaling property (Lemma 2.4), the contractive property (Lemma 2.5), and the calculation of $C^\alpha(S^{n-1}, B_2(0))$ (Lemma 2.6) for the outer capacity $C^\alpha(E, \Omega)$ defined in Definition 2.2 and α -thinness in Definition 2.8.

To better motivate our geometric applications, let us first recall the seminal theorem of Huber on surfaces. Huber [1957] showed that a complete open surface whose negative part of the Gaussian curvature is integrable is a closed surface with finitely many points removed. Huber's theorem uses the Gaussian curvature equation

$$-\Delta[\bar{g}]\phi + K[\bar{g}] = K[e^{2\phi}\bar{g}]e^{2\phi} \quad (1-2)$$

and the potential theory on superharmonic functions.

In conformal geometry, the scalar curvature equation

$$-\frac{4(n-1)}{n-2}\Delta[\bar{g}]u + R[\bar{g}]u = R[u^{\frac{4}{n-2}}\bar{g}]u^{\frac{n+2}{n-2}} \quad (1-3)$$

describes the conformal transformation of the scalar curvature in dimensions higher than 2. There have been many works on singular solutions after the seminal paper [Schoen and Yau 1988], where the singularities represent the ends of complete conformal metrics on domains in Riemannian manifolds (see, for instance, [Schoen and Yau 1994, Chapter VI; Carron 2012; Schoen 1988; Mazzeo and Smale 1991; Mazzeo and Pacard 1996]).

Theorem 1.3. *Let (M^n, \bar{g}) be a complete Riemannian manifold and S be a compact subset in M^n . And let D be a bounded open neighborhood of S . Suppose that $g = u^{4/(n-2)}\bar{g}$ is a conformal metric on $D \setminus S$ and is geodesically complete near S . Then the Hausdorff dimension satisfies*

$$\dim_{\mathcal{H}}(S) \leq \frac{n-2}{2} \quad (1-4)$$

provided $R^-[g] \in L^{2n/(n+2)}(D \setminus S, g) \cap L^p(D \setminus S, g)$ for some $p > \frac{n}{2}$, where $R^-[g]$ is the negative part of the scalar curvature of the metric g . Consequently, (1-4) holds when the scalar curvature $R[g]$ of the conformal metric g is nonnegative.

Theorem 1.3 is an improvement of [Schoen and Yau 1988, Theorem 2.7] and [Carron 2012, Theorem C]. Our approach is based on Theorems 1.1 and 1.2. Particularly, Theorem 1.3 covers domains in general manifolds, while others (see [Schoen and Yau 1988; Carron 2012]) are restricted to domains in round spheres. The use of auxiliary testing functions built from the level sets is the key analytic technique (see [Dolzmann et al. 1997; Bidaut-Véron 1989; Ma and Qing 2021; 2022]). We remark that, for our approach, the complement $M^n \setminus D$ is not relevant (see Theorem 1.3 in Section 3).

In conformal geometry, one considers the Paneitz operator

$$P_4 = \Delta^2 + \operatorname{div}(4A \cdot \nabla - (n - 2)J\nabla) + \frac{n - 4}{2} Q_4$$

and the associated Q -curvature

$$Q_4 = -\Delta J + \frac{n}{2} J^2 - 2|A|^2,$$

where $A = \frac{1}{n-2}(\operatorname{Ric} - Jg)$ is the Schouten curvature and $J = \frac{1}{2(n-1)}R$. The curvature Q_4 , under a conformal change of the metric, transforms by the Q -curvature equation:

$$P_4[\bar{g}]u = \frac{n - 4}{2} Q_4[u^{\frac{4}{n-4}} \bar{g}]u^{\frac{n+4}{n-4}} \quad \text{in dimensions } \geq 5, \tag{1-5}$$

$$P_4[\bar{g}]u + Q_4[\bar{g}] = Q_4[e^{2u} \bar{g}]e^{4u} \quad \text{in dimension 4.} \tag{1-6}$$

On Q -curvature equations in dimensions greater than 4, we have:

Theorem 1.4. *Let (M^n, \bar{g}) be a complete Riemannian manifold for $n \geq 5$ and S be a compact subset in M^n . And let D be a bounded open neighborhood of S . Suppose that $g = u^{4/(n-4)} \bar{g}$ is a conformal metric on $D \setminus S$ with nonnegative scalar curvature $R[g] \geq 0$ and is geodesically complete near S . And suppose also that*

$$Q_4^-[g] \in L^{\frac{2n}{n+4}}(D \setminus S, g),$$

where $Q_4^-[g]$ is the negative part of the Q -curvature of the metric g . Then

$$\dim_{\mathcal{H}}(S) \leq \frac{n - 4}{2}. \tag{1-7}$$

There have been a lot of works on the study of singular solutions to Q -curvature equations on manifolds of dimension greater than 4, notably [Qing and Raske 2006a; 2006b; Chang et al. 2004; González et al. 2012]. Theorem 1.4 is an improvement of [Chang et al. 2004, Theorem 1.2] in terms of curvature conditions and the coverage of domains in general manifolds. The preliminary estimates in Lemma 4.1 serve to facilitate the argument of treating the bi-Laplace as the iteration of the Laplace, which is an interesting alternative to the usual elliptic estimates of Q -curvature equations. Again, the complement $M^n \setminus D$ is not relevant for our approach (see Theorem 1.4 in Section 4).

On Q -curvature equations in dimension 4, there have been several attempts to establish results analogous to Huber's theorem on finiteness of singularities (see [Chang et al. 2000a; Carron and Herzlich 2002; Ma and Qing 2021; 2022]). Q -curvature in dimension 4 indeed plays a role similar to that of the Gaussian curvature in dimension 2 (please see (1-6) for instance). Our following result is a significant improvement of the finiteness result of [Chang et al. 2000a, Theorem 2] (see also [Chang et al. 2000b]). It covers domains in general manifolds and drops other additional curvature assumptions in [Chang et al. 2000a, Theorem 2]. The potential theory approach here, particularly Theorems 1.1 and 1.2, seems to be more effective. And the preliminary estimates in Lemma 4.4 are interesting for Q -curvature equations in dimension 4 too. Once again, the complement $M^n \setminus D$ is not relevant for our approach (see Theorem 1.4 in Section 4).

Theorem 1.5. *Let (M^4, \bar{g}) be a complete Riemannian manifold and S be a compact subset in M^n . And let D be a bounded open neighborhood of S . Suppose that $g = e^{2u} \bar{g}$ is a conformal metric on $D \setminus S$ with nonnegative scalar curvature $R[g] \geq 0$ and is geodesically complete near S . And suppose that*

$$\int_D Q_4^- [g] d\text{vol}[g] < \infty,$$

where $Q_4^- [g]$ is the negative part of the Q -curvature of the metric g . Then S consists of only finitely many points.

The organization of this paper is as follows: In Section 2 we define linear potentials and develop potential theory with the outer capacity and the notion of α -thinness. Then we prove Theorems 1.1 and 1.2. In Section 3 we build the framework to use potential theory developed in Section 2 to estimate the Hausdorff dimension of singular sets which correspond to the ends of complete conformal metrics on domains of manifolds. And we prove Theorem 1.3. In Section 4, based on the framework built in Section 3, we prepare some preliminary estimates and prove Theorems 1.4 and 1.5 for Q -curvature equations.

2. On linear potentials

The study of linear potentials has been extensive and full of great achievements. Readers are referred, for instance, to [Mizuta 1996; Adams and Hedberg 1996; Armitage and Gardiner 2001] for good introductions. In this section we will introduce the theory of linear potential to facilitate the discussion of some estimates of linear potentials inspired by the one in [Cohn-Vossen 1935; Huber 1957; Arsove and Huber 1973; Ma and Qing 2021; 2022]. The estimates provide us some alternative tools to study the problems on the Hausdorff dimensions of singularities of solutions to a class of geometric partial differential equations in conformal geometry (see [Schoen and Yau 1988; 1994; Chang et al. 2004; Carron and Herzlich 2002] for instance). We will introduce potential theory in a way that is brief, mostly self-contained, and suffices to serve our purpose.

2.1. Linear potential and the outer capacity in Euclidean spaces. For the purpose of relating potentials on Euclidean spaces to that on manifolds, we want to introduce potentials that are possibly confined to an open subset $\Omega \subseteq \mathbb{R}^n$ in the Euclidean space. We will use the definition of a Radon measure on locally compact Hausdorff spaces in [Royden and Fitzpatrick 2010, page 455].

Definition 2.1. Let $\Omega \subseteq \mathbb{R}^n$ be a bounded open subset in the Euclidean space \mathbb{R}^n . Then, for $x \in \Omega$, let

$$R_{\mu}^{\alpha, \Omega}(x) = \begin{cases} \int_{\Omega} \frac{1}{|x-y|^{n-\alpha}} d\mu(y) & \text{when } \alpha \in (1, n), \\ \int_{\Omega} \log \frac{D}{|x-y|} d\mu(y) & \text{when } \alpha = n \end{cases} \quad (2-1)$$

for a Radon measure μ on Ω , where D is the diameter of Ω .

For basic properties of the potential $R_{\mu}^{\alpha, \Omega}(x)$, readers are referred to [Mizuta 1996, Chapter 2]. Most facts, results, and arguments in that work that are relevant for the discussions in this paper hold with slight changes.

Definition 2.2. Let E be a subset in Ω and Ω be a bounded open subset in \mathbb{R}^n . For $\alpha \in (1, n]$, we define a capacity by

$$C^{\alpha}(E, \Omega) = \inf\{\mu(\Omega) : \mu \geq 0 \text{ on } \Omega \text{ and } R_{\mu}^{\alpha, \Omega}(x) \geq 1 \text{ for all } x \in E\}. \quad (2-2)$$

Because of the choice of the kernel functions in Definition 2.1, the capacity $C^{\alpha}(E, \Omega)$ in Definition 2.2 is not intended to be the same as relative capacity where the kernel function is the Green's function for a so-called Greenian domain Ω . Similar to [Mizuta 1996, Theorem 4.1 in Chapter 2; Section 2.6], we have:

Lemma 2.3. *Let C^{α} be the capacity defined as in Definition 2.2 for $\alpha \in (1, n]$.*

(1) C^{α} is nondecreasing, that is,

$$C^{\alpha}(E_1, \Omega) \leq C^{\alpha}(E_2, \Omega)$$

when $E_1 \subseteq E_2 \subseteq \Omega \subseteq \mathbb{R}^n$.

(2) C^{α} is countably subadditive, that is,

$$C^{\alpha}\left(\bigcup_{i=1}^{\infty} E_i, \Omega\right) \leq \sum_{i=1}^{\infty} C^{\alpha}(E_i, \Omega)$$

for subsets $E_i \subseteq \Omega$.

(3) C^{α} is an outer capacity, that is,

$$C^{\alpha}(E, \Omega) = \inf\{C^{\alpha}(U, \Omega) : E \subseteq U \text{ and } U \subseteq \Omega \text{ open}\}.$$

The immediate and important property of the outer capacity C^{α} in Definition 2.2 is the scaling property (see [Armitage and Gardiner 2001, page 135]).

Lemma 2.4. *For a positive number λ , let*

$$A_{\lambda} = \{\lambda x : x \in A\}$$

for any subset A in \mathbb{R}^n . Then, for $\alpha \in (1, n]$,

$$C^{\alpha}(E_{\lambda}, \Omega_{\lambda}) = \lambda^{n-\alpha} C^{\alpha}(E, \Omega).$$

Proof. For a nonnegative Radon measure μ on Ω , we associate it with a nonnegative Radon measure

$$\mu^*(A_{\lambda}) = \mu(A)$$

on Ω_λ . Then

$$R_{\mu^*}^{\alpha, \Omega_\lambda}(\lambda x) = \lambda^{\alpha-n} R_\mu^{\alpha, \Omega}(x)$$

for $x \in \Omega$. Therefore

$$\begin{aligned} C^\alpha(E_\lambda, \Omega_\lambda) &= \inf\{\mu^*(\Omega_\lambda) : R_{\mu^*}^{\alpha, \Omega_\lambda}(\lambda x) \geq 1 \text{ for all } x \in E\} \\ &= \lambda^{n-\alpha} \inf\{\lambda^{\alpha-n} \mu(E) : R_{\lambda^{\alpha-n} \mu}^{\alpha, \Omega}(x) \geq 1 \text{ for all } x \in E\} \\ &= \lambda^{n-\alpha} C^\alpha(E, \Omega). \end{aligned} \quad \square$$

The next important property of the outer capacity C^α in Definition 2.2 is the contractive property (see [Mizuta 1996; Adams and Hedberg 1996; Armitage and Gardiner 2001]).

Lemma 2.5. *Suppose that*

$$\Phi : \Omega \rightarrow \Omega$$

is a contractive map, that is,

$$|\Phi(x) - \Phi(y)| \leq |x - y|$$

for all $x, y \in \Omega$. Then, for $\alpha \in (1, n]$,

$$C^\alpha(\Phi(E), \Omega) \leq C^\alpha(E, \Omega)$$

for any subset $E \subseteq \Omega$.

Proof. Let μ be a nonnegative Radon measure on Ω such that $R_\mu^{\alpha, \Omega}(x) \geq 1$ for all $x \in E$. Then let μ^* be a nonnegative Radon measure on Ω such that $\mu^*(A) = \mu(\Phi^{-1}(A))$ for any $A \subseteq \Omega$ and therefore

$$\int_\Omega f(\tilde{y}) d\mu^*(\tilde{y}) = \int_\Omega f \circ \Phi(y) d\mu(y).$$

Notice that

$$\begin{aligned} R_{\mu^*}^{\alpha, \Omega}(\Phi(x)) &= \int_\Omega \frac{1}{|\Phi(x) - \tilde{y}|^{n-\alpha}} d\mu^*(\tilde{y}) = \int_\Omega \frac{1}{|\Phi(x) - \Phi(y)|^{n-\alpha}} d\mu(y) \\ &\geq \int_\Omega \frac{1}{|x - y|^{n-\alpha}} d\mu(y) = R_\mu^{\alpha, \Omega}(x) \geq 1. \end{aligned}$$

Thus

$$\begin{aligned} C^\alpha(\Phi(E), \Omega) &= \inf\{v(\Omega) : v \geq 0 \text{ on } \Omega \text{ and } R_v^{\alpha, \Omega}(x) \geq 1 \text{ for all } x \in \Phi(E)\} \\ &\leq \inf\{\mu^*(\Omega) : \mu^* \text{ induced from } \mu \text{ and } R_{\mu^*}^{\alpha, \Omega}(\Phi(x)) \geq 1 \text{ for all } x \in E\} \\ &= \inf\{\mu(\Omega) : \mu \geq 0 \text{ on } \Omega \text{ and } R_\mu^{\alpha, \Omega}(x) \geq 1 \text{ for all } x \in E\} = C^\alpha(E, \Omega). \end{aligned}$$

The argument for $\alpha = n$ is similar and the proof is complete. □

Before we introduce the notion of thinness by C^α , for completeness, let us calculate the outer capacity $C^\alpha(S^{n-1}, B_2)$, where

$$B_2 = \{x \in \mathbb{R}^n : |x| < 2\} \quad \text{and} \quad S^{n-1} = \{x \in \mathbb{R}^n : |x| = 1\}.$$

Lemma 2.6 [Mizuta 1996, Example 5.4.3]. For $\alpha \in (1, n]$,

$$C^\alpha(S^{n-1}, B_2) = c(n, \alpha)$$

for some positive constant $c(n, \alpha)$.

Proof. It suffices to show that $C^\alpha(S^{n-1}, B_2)$ is finite and positive. Let σ be the volume measure for the unit sphere so that the total measure of S^{n-1} is 1. First we realize that the potential, for $\alpha \in (1, n]$ and $x \in S^{n-1}$, satisfies

$$R_\sigma^{\alpha, B_2}(x) \geq m$$

for some $m = m(n, \alpha) > 0$. Therefore $C^\alpha(S^{n-1}, B_2) \leq \frac{1}{m} < \infty$ by Definition 2.2. To see that $C^\alpha(S^{n-1}, B_2) > 0$ for any μ on B_2 , we use Lemma 2.7 below to pick up a point $p \in S^{n-1}$ such that (2-3) holds and calculate, for $\alpha \in (1, n)$,

$$\begin{aligned} R_\mu^{\alpha, B_2}(p) &= (n - \alpha) \int_0^\infty \mu\left(\left\{\frac{1}{r} - \frac{1}{3} > s\right\} \cap B_2\right) \frac{1}{\left(s + \frac{1}{3}\right)^{n-\alpha+1}} ds + \frac{1}{3^{n-\alpha}} \mu(B_2) \\ &= (n - \alpha) \int_0^3 \mu(B_r(p) \cap B_2) r^{\alpha-n-1} dr + \frac{1}{3^{n-\alpha}} \mu(B_2) \\ &\leq M(n, \alpha) \mu(B_2) \end{aligned}$$

for some $M(n, \alpha) > 0$ and $r = |x - p|$. For $\alpha = n$,

$$\begin{aligned} R_\mu^{\alpha, B_2}(p) &= \int_0^\infty \mu\left(\left\{\frac{3}{r} - 1 > s\right\} \cap B_2\right) \frac{1}{1+s} ds + \log \frac{4}{3} \mu(B_2) \\ &= \int_0^3 \mu(B_r(p) \cap B_2) \frac{1}{r} dr + \log \frac{4}{3} \mu(B_2) \\ &\leq M(n, n) \mu(B_2) \end{aligned}$$

for some $M(n, n) > 0$. In the above we used [Rudin 1987, Theorem 8.16]. This implies $C^\alpha(S^{n-1}, B_2) \geq 1/M(n, \alpha) > 0$ by Definition 2.2. \square

By the Vitali covering lemma, we prove the following fact used in the above.

Lemma 2.7. Let $n \geq 2$ and μ be a finite nonnegative Radon measure on $B_2 \subset \mathbb{R}^n$. Then there is a point $p \in S^{n-1}$ such that

$$\mu(B_r(p) \cap B_2) \leq c(n) \mu(B_2) r^{n-1} \quad \text{for all } r > 0, \tag{2-3}$$

for some dimensional constant $c = c(n)$.

Proof. For convenience, let $\mu(B_2) = 1$. Assume otherwise, for any $q \in S^{n-1}$, there is $r_q > 0$ such that

$$\mu(B_{r_q}(q) \cap B_2) \geq c(n) r_q^{n-1}.$$

Using the Vitali covering lemma, we have $\{q_1, q_2, \dots, q_k\} \subset S^{n-1}$ such that the balls in the collection

$$\{B_{r_{q_1}}(q_1), B_{r_{q_2}}(q_2), \dots, B_{r_{q_k}}(q_k)\}$$

are disjoint but the balls in the collection

$$\{B_{3r_{q_1}}(q_1), B_{3r_{q_2}}(q_2), \dots, B_{3r_{q_k}}(q_k)\}$$

cover the sphere S^{n-1} . Therefore, on one hand,

$$c(n) \sum_{i=1}^k r_{q_i}^{n-1} \leq \sum_{i=1}^k \mu(B_{r_{q_i}}(q_i) \cap B_2) \leq \mu(B_2) = 1.$$

On the other hand,

$$|S^{n-1}| \leq \sum_{i=1}^k |B_{3r_{q_i}}(q_i) \cap S^{n-1}| < |S^{n-1}| c(n) \sum_{i=1}^k r_{q_i}^{n-1},$$

when $c(n)$ is sufficiently large, where $|\cdot|$ stands for the Lebesgue measure on S^{n-1} . Therefore the lemma is proven by contradiction. \square

Now let us introduce the geometric definition of thinness. For notions of thinness in terms of the fine topology and Wiener criterion, readers are referred, for instance, to [Mizuta 1996; Adams and Hedberg 1996; Armitage and Gardiner 2001]. Let

$$\begin{aligned} \omega_i^\delta(p) &= \{x \in \mathbb{R}^n : |x - p| \in [2^{-i}\delta, 2^{-i+1}\delta]\}, \\ \Omega_i^\delta(p) &= \{x \in \mathbb{R}^n : |x - p| \in (2^{-i-1}\delta, 2^{-i+2}\delta)\}. \end{aligned}$$

Definition 2.8. Let E be a subset in the Euclidean space \mathbb{R}^n and $p \in \mathbb{R}^n$ be a point in \mathbb{R}^n . The subset E is said to be α -thin at the point p for $\alpha \in (1, n)$ if

$$\sum_{i \geq 1} \frac{C^\alpha(E \cap \omega_i^\delta(p), \Omega_i^\delta(p))}{C^\alpha(\partial B_{2^{-i}\delta}(p), B_{2^{-i+1}\delta}(p))} < \infty$$

for some small $\delta > 0$. The subset E is said to be n -thin at p if

$$\sum_{i \geq 1} i C^n(E \cap \omega_i^\delta(p), \Omega_i^\delta(p)) < \infty$$

for some small $\delta > 0$.

Combining Lemmas 2.3-2.6 with the above definition, we observe the following important property of α -thin sets, inspired by [Arsove and Huber 1973] (see also [Ma and Qing 2021; 2022]). We recall Theorem 1.2 from the Introduction for readers' convenience.

Theorem 1.2. *Let E be a subset in the Euclidean space \mathbb{R}^n and $p \in \mathbb{R}^n$ be a point. Suppose that E is α -thin at the point p for $\alpha \in (1, n]$. Then there is a ray from p that avoids E at least within some small ball at p .*

Proof. First of all, due to the translation invariance, we may simply assume p is the origin of the Euclidean space. Then, by the scaling property of the outer capacity C^α in Lemma 2.4, one notices that

$$\frac{C^\alpha(E \cap \omega_i^\delta, \Omega_i^\delta)}{C^\alpha(\partial B_{2^{-i}\delta}, B_{2^{-i+1}\delta})} = \frac{C^\alpha(S_i(E) \cap \omega_0^1, \Omega_0^1)}{C^\alpha(\partial B_1, B_2)},$$

where $S_i(v) = (2^i/\delta)v$ is the scaling map. Then we consider the projection

$$P(v) = \begin{cases} v/|v| & \text{when } v \in \mathbb{R}^n \text{ and } |v| \geq 1, \\ v & \text{when } v \in \mathbb{R}^n \text{ and } |v| < 1, \end{cases}$$

which is contractive. Therefore, in light of Lemma 2.5, we have

$$C^\alpha(P(S_i(E) \cap \omega_0^1), \Omega_0^1) \leq C^\alpha(S_i(E) \cap \omega_0^1, \Omega_0^1).$$

Next, using the countable subadditivity in Lemma 2.3, we have

$$C^\alpha\left(\bigcup_{i \geq k} P(S_i(E) \cap \omega_0^1), \Omega_0^1\right) \leq \sum_{i \geq k} C^\alpha(P(S_i(E) \cap \omega_0^1), \Omega_0^1).$$

Thus,

$$\begin{aligned} C^\alpha\left(\bigcup_{i \geq k} P(S_i(E) \cap \omega_0^1), \Omega_0^1\right) &\leq \sum_{i \geq k} C^\alpha(S_i(E) \cap \omega_0^1, \Omega_0^1) \\ &\leq C^\alpha(\partial B_1, B_2) \sum_{i \geq k} \frac{C^\alpha(S_i(E) \cap \omega_0^1, \Omega_0^1)}{C^\alpha(\partial B_1, B_2)} \\ &\leq C^\alpha(\partial B_1, B_2) \sum_{i \geq k} \frac{C^\alpha(E \cap \omega_i^\delta, \Omega_i^\delta)}{C^\alpha(\partial B_{2^{-i}\delta}, B_{2^{-i+1}\delta})}, \end{aligned}$$

which is arbitrarily small when k is appropriately large using Lemma 2.6 for $C^\alpha(\partial B_1, B_2)$. And then this implies that

$$\partial B_1 \setminus \bigcup_{i \geq k} P(S_i(E) \cap \omega_0^1) \neq \emptyset.$$

The argument for $\alpha = n$ is similar and easier. □

2.2. Linear potential on manifolds. On a given complete Riemannian manifold (M^n, g) , let $d(\cdot, \cdot)$ be the distance function associated with the given Riemannian metric g .

Definition 2.9. Suppose that (M^n, g) is a complete Riemannian manifold and $U \subseteq M^n$ is a bounded open subset. For $\alpha \in (1, n]$, the linear potential on the Riemannian manifold (M^n, g) of order α for a Radon measure μ on U is given by

$$\mathcal{R}_\mu^{\alpha, U}(x) = \begin{cases} \int_U \frac{1}{d(x, y)^{n-\alpha}} d\mu(y) & \text{when } \alpha \in (1, n), \\ \int_\Omega \log \frac{D}{d(x, y)} d\mu(y) & \text{when } \alpha = n, \end{cases}$$

where D is the diameter of U .

From the discussion in the previous subsection, it is easily seen that one may generate an outer capacity $\mathcal{C}^\alpha(E, U)$ for any subset $E \subseteq U \subseteq M^n$ that behaves like the counterpart in Euclidean spaces. To use $R_\mu^{\alpha, \Omega}(x)$ and $C^\alpha(E, \Omega)$ on Euclidean spaces in the previous subsection to study $\mathcal{R}_\mu^{\alpha, U}(p)$ and $\mathcal{C}^\alpha(A, U)$ on manifolds, we first introduce the correspondence between Radon measures on the tangent space $T_p M^n$ at each point $p \in M^n$ and those on (M^n, g) . Suppose that (M^n, g) is a complete Riemannian manifold. Let $p \in M^n$ and U be a convex normal coordinate neighborhood at p on (M^n, g) , where the exponential map serves as the convex normal coordinate

$$\exp|_p : \Omega \rightarrow U.$$

The domain U is said to be convex if the unique geodesic joining any two points in U stays in U . Moreover, we may assume in the coordinate chart U the exponential map be uniformly bi-Lipschitz throughout this paper.

Then, for a Radon measure μ on $U \subseteq M^n$, one may introduce the Radon measure μ^* on $\Omega \subset T_p M^n$ such that, for a subset $E \subseteq \Omega$,

$$\mu^*(E) = \mu(\exp|_p E) \quad \text{and} \quad \int_{\Omega} f \circ \exp|_p d\mu^* = \int_U f d\mu.$$

It is then easily seen that the following equivalence between the linear potential $R_{\mu^*}^{\alpha, \Omega}$, the outer capacities $C^\alpha(\cdot, \Omega)$ and the corresponding $\mathcal{R}_\mu^{\alpha, U}, \mathcal{C}^\alpha(\cdot, U)$ holds. Namely:

Lemma 2.10. *Suppose that (M^n, g) is a complete Riemannian manifold and $p \in M^n$. Let*

$$\exp|_p : \Omega \rightarrow U$$

be the convex normal coordinate chart, where the exponential map is uniformly bi-Lipschitz. And let $\alpha \in (1, n]$. Then, for $A \subset U$ and $E = (\exp|_p)^{-1} A \subset \Omega$,

$$C^{-1} R_{\mu^*}^{\alpha, \Omega} \leq \mathcal{R}_\mu^{\alpha, U} \leq C R_{\mu^*}^{\alpha, \Omega},$$

$$C^{-1} C^\alpha(E, \Omega) \leq \mathcal{C}^\alpha(A, U) \leq C C^\alpha(E, \Omega)$$

for some constant $C = C(M^n, g, U, p)$. Consequently, a subset $A \subset U$ is α -thin at p if and only if $E = (\exp|_p)^{-1}(A) \subset \Omega$ is α -thin at the origin of $T_p M^n$.

Proof. The proof is straightforward based on the properties of the convex normal coordinate chart at a point in a complete Riemannian manifold, where the exponential map is bi-Lipschitz. □

2.3. Estimates of Riesz potentials. We now introduce our estimates of Riesz potentials on manifolds. We will recall some well-known estimates for Riesz potentials in Euclidean spaces [Mizuta 1996, Chapter 2].

Our estimates on Riesz potentials are designed to help understand the Hausdorff dimensions of singularities of solutions of partial differential equations on manifolds. Let us start with a general decomposition theorem for nonnegative Radon measures on a complete Riemannian manifold based on [Kpata 2019, Proposition 1.4], which is related to Lemma 2.7 and a broad generalization of the Lebesgue Differentiation Theorem in some way.

Lemma 2.11 [Kpata 2019, Proposition 1.4]. *Let μ be a nonnegative Radon measure on a complete Riemannian manifold (M^n, g) and let*

$$G_d^\infty = \left\{ x \in M^n : \limsup_{r \rightarrow 0} r^{-d} \mu(B_r(x)) = +\infty \right\}$$

for any $d \in [0, n]$. Then

$$\mathcal{H}_d(G_d^\infty) = 0,$$

where \mathcal{H}_d is the Hausdorff measure of dimension d .

Proof. Based on the general decomposition theorem [Kpata 2019, Proposition 1.4] on the Euclidean space and the correspondence of Radon measures in Lemma 2.10, this lemma is easily seen. Specifically, we first prove the statement for Radon measures supported in a convex normal coordinate chart used in Lemma 2.10. Then the lemma follows by using a countable covering for (M, g) by convex normal coordinate charts. \square

Now we are ready to state and prove one crucial analytic result in this paper on the behavior of the Riesz potentials. For readers' convenience, we recall Theorem 1.1 from the Introduction.

Theorem 1.1. *Suppose that (M^n, g) is a complete Riemannian manifold and μ is a finite Radon measure on a bounded domain $G \subset M^n$. Let S be a compact subset in G such that its Hausdorff dimension is greater than d . And let $\alpha \in (1, n)$ and $d < n - \alpha$. Then there is a point $p \in S$ and a subset E that is α -thin at p such that*

$$\int_G \frac{1}{d(x, y)^{n-\alpha}} d\mu \leq \frac{C}{d(x, p)^{n-\alpha-d}}$$

for some constant C and all $x \in B_\delta(p) \setminus E$ for some $\delta > 0$.

Proof. First, due to the assumption that the Hausdorff dimension of S is greater than d ,

$$\mathcal{H}_{d+\epsilon}(S) = \infty$$

for some small $\epsilon > 0$. Then, in light of Lemma 2.11, there is a point $p \in S$ such that

$$\limsup_{r \rightarrow 0} r^{-(d+\epsilon)} \mu(B_r(p)) \leq C < \infty.$$

That is to say

$$\mu(B_r(p)) \leq Cr^{d+\epsilon} \tag{2-4}$$

when r is appropriately small. Secondly, we may confine ourselves to a convex normal coordinate neighborhood U of p and we may work on the Euclidean space without loss of generality in light of the discussion in the previous subsection, particularly, Lemma 2.10, where $\exp|_p : \Omega \rightarrow U$ and $\exp|_p(0) = p$. For convenience, we will not differentiate μ and μ^* if no confusion rises. Therefore, for $x \in \omega_i^\delta \subset \Omega$ when δ is sufficiently small and i is appropriately large,

$$\begin{aligned} R_\mu^{\alpha, \Omega}(x) &= \int_\Omega \frac{1}{|x-y|^{n-\alpha}} d\mu \\ &= \int_{\Omega \setminus B_{2^{-i_0+2}\delta}} \frac{1}{|x-y|^{n-\alpha}} d\mu + \int_{B_{2^{-i_0+2}\delta} \setminus \Omega_i^\delta} \frac{1}{|x-y|^{n-\alpha}} d\mu + \int_{\Omega_i^\delta} \frac{1}{|x-y|^{n-\alpha}} d\mu, \end{aligned} \tag{2-5}$$

where $i_0 \leq i$ to be fixed. For the first term in the right-hand side of (2-5),

$$I = \int_{\Omega \setminus B_{2^{-i_0+2}\delta}} \frac{1}{|x-y|^{n-\alpha}} d\mu \leq \left(\frac{1}{2^{-i_0+2}\delta - 2^{-i+1}\delta} \right)^{n-\alpha} \mu(\Omega) \leq \left(\frac{1}{2^{-i_0+1}\delta} \right)^{n-\alpha} \mu(\Omega).$$

Recall that $2^{-i}\delta \leq |x| \leq 2^{-i+1}\delta$ for $x \in \omega_i$, we have

$$I \leq \mu(\Omega) \frac{(2^{-i+1}\delta)^{n-\alpha-d}}{(2^{-i+1}\delta)^{n-\alpha}} \frac{1}{|x|^{n-\alpha-d}} \leq C \frac{1}{|x|^{n-\alpha-d}}, \tag{2-6}$$

where $C = C(n, \alpha, d, \delta, i_0)$. For the second term in the right-hand side of (2-5),

$$\begin{aligned} \int_{B_{2^{-i_0+2}\delta} \setminus \Omega_i^\delta} \frac{1}{|x-y|^{n-\alpha}} d\mu &= \int_{B_{2^{-i_0+2}\delta} \setminus B_{2^{-i+2}\delta}} \frac{1}{|x-y|^{n-\alpha}} d\mu + \int_{B_{2^{-i-1}\delta}} \frac{1}{|x-y|^{n-\alpha}} d\mu \\ &\leq \int_{B_{2^{-i_0+2}\delta} \setminus B_{2^{-i+2}\delta}} \frac{1}{|x-y|^{n-\alpha}} d\mu + \left(\frac{1}{2^{-i-1}\delta}\right)^{n-\alpha} \mu(B_{2^{-i-1}\delta}) \\ &\leq \sum_{k=i_0}^{i-1} \int_{B_{2^{-k+2}\delta} \setminus B_{2^{-k+1}\delta}} \frac{1}{|x-y|^{n-\alpha}} d\mu + \left(\frac{1}{2^{-i-1}\delta}\right)^{n-\alpha} \mu(B_{2^{-i-1}\delta}) \\ &\leq \sum_{k=i_0}^{i-1} \left(\frac{1}{2^{-k}\delta}\right)^{n-\alpha} \mu(B_{2^{-k+2}\delta}) + \left(\frac{1}{2^{-i-1}\delta}\right)^{n-\alpha} \mu(B_{2^{-i-1}\delta}). \end{aligned}$$

Using (2-4) for $\epsilon = 0$, we continue from the above,

$$\begin{aligned} II &\leq C \left(4^d \sum_{k=i_0}^{i-1} \left(\frac{1}{2^{-k}\delta}\right)^{n-\alpha-d} + \left(\frac{1}{2^{-i-1}\delta}\right)^{n-\alpha-d} \right) \\ &\leq C \left(\frac{4^d}{1-2^{-(n-\alpha-d)}} \left(\frac{1}{2^{-i+1}\delta}\right)^{n-\alpha-d} + \left(\frac{1}{2^{-i-1}\delta}\right)^{n-\alpha-d} \right) \leq C \frac{1}{|x|^{n-\alpha-d}}, \end{aligned} \tag{2-7}$$

where $C = C(n, \alpha, d, \delta, i_0)$. To handle the third term in the right-hand side of (2-5), we let

$$E_i^\lambda = \left\{ x \in \omega_i^\delta : \int_{\Omega_i^\delta} \frac{1}{|x-y|^{n-\alpha}} d\mu \geq \lambda 2^{i(n-\alpha-d)} \right\},$$

where $\lambda > 0$ is fixed. By Definition 2.2, we know

$$C^\alpha(E_i^\lambda, \Omega_i^\delta) \leq \frac{\mu(\Omega_i^\delta)}{\lambda 2^{i(n-\alpha-d)}} \leq \frac{C (2^{-i+2}\delta)^{d+\epsilon}}{\lambda 2^{i(n-\alpha-d)}} = \frac{C 4^{d+\epsilon}}{\lambda} 2^{-i\epsilon} (2^{-i})^{n-\alpha},$$

where (2-4) for some $\epsilon > 0$ is used and $\Omega_i^\delta \subset B_{2^{-i+2}\delta}$. Now, from Lemma 2.6 and the scaling property, we know

$$C^\alpha(\partial B_{2^{-i}\delta}, B_{2^{-i+1}\delta}) = C(n, \alpha) (2^{-i}\delta)^{n-\alpha}$$

and

$$\sum_{i \geq i_0} \frac{C^\alpha(E_i^\lambda, \Omega_i^\delta)}{C^\alpha(\partial B_{2^{-i}\delta}, B_{2^{-i+1}\delta})} \leq \frac{C}{\lambda} \sum_{i \geq i_0} 2^{-\epsilon i} < \infty.$$

Thus, by Definition 2.8, the proof is completed. □

As a consequence of Theorems 1.2 and 1.1, we have:

Corollary 2.12. *Suppose that (M^n, g) is a complete Riemannian manifold and μ is a finite Radon measure on a bounded domain $G \subset M^n$. Let S be a compact subset in G such that its Hausdorff dimension is greater than d . And let $\alpha \in (1, n)$ and $d < n - \alpha$. Then there is a point $p \in S$ such that, for some constant C ,*

$$\int_G \frac{1}{d(x, y)^{n-\alpha}} d\mu \leq \frac{C}{d(x, p)^{n-\alpha-d}}$$

for all $x \in \text{Ray}_p \cap B(x, \delta)$, where Ray_p is a ray from p and $B(p, \delta)$ is the geodesic ball of radius $\delta > 0$.

2.4. Estimates of the log potential. First, as stated in [Mizuta 1996, Theorem 6.3], for the log potential $U_n\mu(x)$ on Euclidean spaces defined on page 82 of that work,

$$\lim_{x \rightarrow p \text{ and } x \in \Omega \setminus E} \frac{U_n\mu(x)}{\log(1/|x - p|)} = \mu(\{p\}).$$

The following is our version of [Mizuta 1996, Theorem 6.3] on manifolds. For us it is a generalization of [Arsove and Huber 1973, Theorem 1.3] in higher dimensions and linear versions of such behaviors for n -superharmonic functions (see [Huber 1957; Bonini et al. 2018; 2019; Ma and Qing 2021; 2022]). For convenience, we present a brief but full proof based on the potential theory developed in previous subsections in this paper.

Theorem 2.13. *Suppose (M^n, g) is a complete Riemannian manifold. Let μ be a finite Radon measure on a bounded domain $G \subset M^n$. Then, for all $p \in G$, there is a subset A that is n -thin at p and*

$$\lim_{x \rightarrow p \text{ and } x \in M^n \setminus A} \frac{\int_G \log(1/d(x, y)) d\mu(y)}{\log(1/d(x, p))} = \mu(\{p\}).$$

Proof. Let

$$\exp|_p : \Omega \rightarrow U$$

be a convex normal coordinate at $p \in M^n$. Clearly, it suffices to show that there is a subset A in U , which is n -thin at p , such that

$$\lim_{x \rightarrow p \text{ and } x \in U \setminus A} \frac{\mathcal{R}_\mu^{n,U}(x)}{\log(1/d(x, p))} = \mu(\{p\}). \tag{2-8}$$

Therefore, for $x \in \omega_i^\delta(p)$, we write

$$\begin{aligned} \mathcal{R}_\mu^{n,U}(x) &= \int_U \log \frac{D}{d(x, y)} d\mu(y) \\ &= \int_{U \setminus B_{2^{-i_0+2}\delta}} \log \frac{D}{d(x, y)} d\mu + \int_{B_{2^{-i_0+2}\delta} \setminus \Omega_i^\delta} \log \frac{D}{d(x, y)} d\mu + \int_{\Omega_i^\delta} \log \frac{D}{d(x, y)} d\mu. \end{aligned} \tag{2-9}$$

Here we omit the center p for each ball or annulus for simplicity. For the first term in the right-hand side of (2-9),

$$I = \int_{U \setminus B_{2^{-i_0+2}\delta}} \log \frac{D}{d(x, y)} d\mu \leq \mu(U) \log \frac{D}{2^{-i_0+1}\delta} = o(1) \log \frac{1}{d(x, p)} \quad \text{as } x \rightarrow p. \tag{2-10}$$

For the second term in the right-hand side of (2-9),

$$\begin{aligned} \int_{B_{2^{-i_0+2\delta}} \setminus \Omega_i^\delta} \log \frac{D}{d(x, y)} d\mu(y) &= \int_{B_{2^{-i_0+2\delta}} \setminus B_{2^{-i+1\delta}}} \log \frac{D}{d(x, y)} d\mu + \int_{B_{2^{-i-2\delta}}} \log \frac{D}{d(x, y)} d\mu \\ &\leq C \left[\sum_{k=i_0}^i k\mu(B_{2^{-k+2\delta}} \setminus B_{2^{-k+1\delta}}) \right] + \mu(B_{2^{-i-2\delta}}) \log \frac{D}{2^{-i-2\delta}}. \end{aligned}$$

Due to the regularity of Radon measures and $d(x, p) \in [2^{-i-1}\delta, 2^{-i}\delta]$, we know

$$\mu(B_{2^{-i-2\delta}}) \log \frac{D}{2^{-i-2\delta}} = \mu(\{p\}) \log \frac{1}{d(x, p)} + o\left(\log \frac{1}{d(x, p)}\right) \quad \text{as } x \rightarrow p \tag{2-11}$$

and

$$\sum_{k=i_0}^i k\mu(B_{2^{-k+2\delta}} \setminus B_{2^{-k+1\delta}}) = o(1)i = o(1) \log \frac{1}{d(x, p)} \tag{2-12}$$

as $i \rightarrow \infty$ or equivalently $x \rightarrow p$. To see (2-12), for any $\epsilon > 0$, we first find k_0 such that

$$\mu(B_{2^{-l+2\delta}} \setminus B_{2^{-m+1\delta}}) \leq \frac{\epsilon}{2}$$

for all $m \geq l \geq k_0$ due to the regularity of μ . Next, we find N such that

$$\frac{\sum_{k=i_0}^{k_0} k\mu(B_{2^{-k+2\delta}} \setminus B_{2^{-k+1\delta}})}{i} \leq \frac{\epsilon}{2}$$

for all $i \geq N$. Together, this gives

$$\begin{aligned} \frac{\sum_{k=i_0}^i k\mu(B_{2^{-k+2\delta}} \setminus B_{2^{-k+1\delta}})}{i} &= \frac{\sum_{k=i_0}^{k_0} k\mu(B_{2^{-k+2\delta}} \setminus B_{2^{-k+1\delta}})}{i} + \frac{\sum_{k=k_0+1}^i k\mu(B_{2^{-k+2\delta}} \setminus B_{2^{-k+1\delta}})}{i} \\ &= \frac{\sum_{k=i_0}^{k_0} k\mu(B_{2^{-k+2\delta}} \setminus B_{2^{-k+1\delta}})}{i} + \sum_{k=k_0+1}^i \mu(B_{2^{-k+2\delta}} \setminus B_{2^{-k+1\delta}}) \leq \epsilon \end{aligned}$$

for all $i \geq N$. Thus we conclude that

$$II = (\mu(\{p\}) + o(1)) \log \frac{1}{d(x, p)} \quad \text{as } x \rightarrow p. \tag{2-13}$$

To handle the third term in the right side of (2-9), for $\lambda_i > 0$ to be determined, we consider

$$A^{\lambda_i} = \left\{ x \in \omega_i^\delta : \int_{\Omega_i^\delta} \log \frac{D_i}{d(x, y)} d\mu \geq i\lambda_i \right\},$$

where D_i is the diameter of Ω_i^δ . By Definition 2.2,

$$\mathcal{C}^n(A^{\lambda_i}, \Omega_i^\delta) \leq \frac{\mu(\Omega_i^\delta)}{i\lambda_i}.$$

In light of Definition 2.8, we consider

$$\sum_{i \geq i_0} i \mathcal{C}^n(A^{\lambda_i}, \Omega_i^\delta) \leq \sum_{i \geq i_0} \frac{\mu(\Omega_i^\delta)}{\lambda_i}$$

and pick up $\lambda_i \rightarrow 0$ as $i \rightarrow \infty$ by the classic Paul du Bois-Reymond theorem [1873] (see [Bromwich 1908, (5), page 40]) for infinite series such that $\sum_{i \geq i_0} \mu(\Omega_i^\delta)/\lambda_i$ converges when $\sum_{i \geq i_0} \mu(\Omega_i^\delta)$ converges. This is to say that the third term in the right side of (2-9) satisfies

$$\begin{aligned} III &= \int_{\Omega_i^\delta} \log \frac{D}{d(x, y)} d\mu(y) = \int_{\Omega_i^\delta} \log \frac{D_i}{d(x, y)} d\mu(y) + \log \frac{D}{D_i} \mu(\Omega_i^\delta) \\ &\leq \left(\lambda_i + \left(1 + \frac{1}{i} \log \frac{1}{\delta} \right) \mu(\Omega_i^\delta) \right) \log \frac{D}{d(x, p)} \\ &= o(1) \log \frac{1}{d(x, p)} \quad \text{as } x \in \omega_i^\delta \setminus E^{\lambda_i} \text{ and } x \rightarrow p. \end{aligned} \tag{2-14}$$

Finally, if let $A = \bigcup_i A^{\lambda_i}$, we have

$$\lim_{x \rightarrow p \text{ and } x \in U \setminus A} \frac{\mathcal{R}_\mu^{n,U}(x)}{\log(1/d(x, p))} = \mu(\{p\}),$$

where A is n -thin at p . □

3. On scalar curvature equations

We now focus on the scalar curvature equations for conformal deformation of metrics. Let (M^n, \bar{g}) be a compact Riemannian manifold for $n \geq 3$. Let $R_{ijkl}[\bar{g}]$ be the Riemann curvature tensor, $R_{ij}[\bar{g}] = R_{ijkl}\bar{g}^{kl}$ be the Ricci curvature tensor, and $R[\bar{g}] = R_{ij}\bar{g}^{ij}$ be the scalar curvature. The scalar curvature equation in conformal geometry is

$$-\frac{4(n-1)}{n-2} \Delta[\bar{g}]u + R[\bar{g}]u = R[u^{\frac{4}{n-2}}\bar{g}]u^{\frac{n+2}{n-2}} \tag{3-1}$$

for a positive function u . The scalar curvature equation describes how the scalar curvature transforms under conformal change of metrics. In this section we want to use the estimates for the Newton potential in the previous section to study the Hausdorff dimensions of the singularities of solutions u to the scalar equations which represent the ends of a complete conformal metric $u^{4/(n-2)}\bar{g}$.

We remark here that all of the results in this section hold if we assume S is compact, $D \subset M^n$ is a bounded domain that contains S , and (M^n, \bar{g}) is just complete, because the possible noncompact part $M^n \setminus \bar{D}$ is not relevant for the purpose here.

3.1. Preliminaries. Let us start with [Ma and Qing 2022, Lemma 3.1], which is a slight improvement of [Chang et al. 2004, Proposition 8.1].

Lemma 3.1 [Ma and Qing 2022, Lemma 3.1]. *Let (M^n, \bar{g}) be a compact Riemannian manifold and S be a closed subset in M^n . And let D be an open neighborhood of S . Suppose that $g = u^{4/(n-2)}\bar{g}$ is a conformal metric on $D \setminus S$ and is geodesically complete near S . Then*

$$u(x) \rightarrow +\infty \quad \text{as } x \rightarrow S$$

if $R^-[g] \in L^p(D \setminus S, g)$ for some $p > n/2$, where $R^-[g] = \max\{-R[g], 0\}$ stands for the negative part of the scalar curvature $R[g]$ and $L^p(D \setminus S, g)$ is the L^p space with respect to the metric g .

For a preliminary estimate on the Hausdorff dimension of S , we follow the proof of [Ma and Qing 2022, Theorem 3.1] and get:

Proposition 3.2. *Let (M^n, \bar{g}) be a compact Riemannian manifold and S be a closed subset in M^n . And let D be an open neighborhood of S where the scalar curvature $R[\bar{g}]$ is nonpositive. Suppose that $g = u^{4/(n-2)}\bar{g}$ is a conformal metric on $D \setminus S$ and is geodesically complete near S . Then the Newton capacity of S is zero and therefore the Hausdorff dimension satisfies*

$$\dim_{\mathcal{H}}(S) \leq n - 2,$$

provided that

$$R^-[g] \in L^{\frac{2n}{n+2}}(D \setminus S, g) \cap L^p(D \setminus S, g)$$

for some $p > \frac{n}{2}$.

Proof. Recall the scalar curvature equation

$$-\frac{4(n-1)}{n-2} \Delta u = -Ru + R^+[g]u^{\frac{n+2}{n-2}} - R^-[g]u^{\frac{n+2}{n-2}} \quad \text{in } D \setminus S, \tag{3-2}$$

where

$$\begin{aligned} \int_{D \setminus S} R^-[g]u^{\frac{n+2}{n-2}} d\text{vol}[\bar{g}] &\leq \left(\int_{D \setminus S} (R^-[g])^{\frac{2n}{n+2}} u^{\frac{2n}{n-2}} d\text{vol}[\bar{g}] \right)^{\frac{n+2}{2n}} \text{vol}(D)^{\frac{n-2}{2n}} \\ &\leq \left(\int_{D \setminus S} (R^-[g])^{\frac{2n}{n+2}} d\text{vol}[g] \right)^{\frac{n+2}{2n}} \text{vol}(D)^{\frac{n-2}{2n}} < \infty. \end{aligned} \tag{3-3}$$

Here, and from now on, all geometric quantities are under the background metric \bar{g} unless indicated otherwise. And, in light of Lemma 3.1, we know

$$u(x) \rightarrow +\infty \quad \text{as } x \rightarrow S.$$

As in the proof of [Ma and Qing 2022, Theorem 3.1] (adopted from [Bidaut-Véron 1989, Lemma 1.2]), we use the following test functions. First we let

$$u_{\alpha,\beta} = \begin{cases} \beta, & u \geq \alpha + \beta, \\ u - \alpha, & u < \alpha + \beta, \end{cases} \quad \text{and} \quad \phi_{\alpha,\beta} = u_{\alpha,\beta} - \beta + \beta(1 - \eta),$$

where $\eta \in C_c^\infty(\Sigma_\alpha)$ is a fixed cut-off function that is equal to 1 in a neighborhood of S and $\Sigma_\alpha = \{x \in D : u(x) > \alpha\}$. Notice that, for β sufficiently large,

$$u_{\alpha,\beta} \in (0, \beta] \quad \text{in } \Sigma_\alpha \quad \text{and} \quad \phi_{\alpha,\beta} = 0 \quad \text{on } \{x \in D : u(x) = \alpha\} \cup \{x \in D : u \geq \alpha + \beta\}$$

and

$$\nabla \phi_{\alpha,\beta} = \nabla u_{\alpha,\beta} + \beta \nabla \eta \quad \text{and} \quad \nabla u = \nabla u_{\alpha,\beta} \quad \text{when } \nabla u_{\alpha,\beta} \neq 0.$$

We then multiply $\phi_{\alpha,\beta}$ to (3-2) and get

$$\frac{4(n-1)}{n-2} \int_{\Sigma_\alpha} \nabla u \cdot \nabla \phi_{\alpha,\beta} d\text{vol}[\bar{g}] = \int_{\Sigma_\alpha} (-Ru + R[g]u^{\frac{n+2}{n-2}}) \phi_{\alpha,\beta} d\text{vol}[\bar{g}].$$

Therefore

$$\begin{aligned} \frac{4(n-1)}{n-2} \int_{\Sigma_\alpha} |\nabla u_{\alpha,\beta}|^2 d\text{vol}[\bar{g}] &= \beta \int_{\Sigma_\alpha} \left(\frac{n-2}{4(n-1)} \nabla u \cdot \nabla \eta + (-Ru + R[g]u^{\frac{n+2}{n-2}})(1-\eta) \right) d\text{vol}[\bar{g}] \\ &\quad - \int_{\Sigma_\alpha} (-Ru + R^+[g]u^{\frac{n+2}{n-2}})(\beta - u_{\alpha,\beta}) d\text{vol}[\bar{g}] \\ &\quad + \int_{\Sigma_\alpha} R^-[g]u^{\frac{n+2}{n-2}}(\beta - u_{\alpha,\beta}) d\text{vol}[\bar{g}] \leq C\beta, \end{aligned} \tag{3-4}$$

where C depends on α and η but does not depend on β , due the support of $1 - \eta$ and (3-3). That is,

$$\int_{\Sigma_\alpha} |\nabla \frac{u_{\alpha,\beta}}{\beta}|^2 d\text{vol}[\bar{g}] \leq \frac{C}{\beta} \rightarrow 0$$

as $\beta \rightarrow \infty$, where $u_{\alpha,\beta}/\beta$ is a function that is identically 1 in a neighborhood of S . This implies the Newton capacity $\text{Cap}_2(S, D)$ of S is zero. Consequently, we know S is of Hausdorff dimension not greater than $n - 2$ (see [Adams and Meyers 1972; Schoen and Yau 1994, Theorem 2.10 in Chapter VI]). \square

3.2. $-\Delta u$ is a Radon measure on D . In order to use the estimates of potentials in the previous section, we need the following lemma (see [Ma and Qing 2022, Lemma 3.2–3.4]).

Lemma 3.3. *Let (M^n, \bar{g}) be a compact Riemannian manifold and S be a closed subset in M^n . And let D be an open neighborhood of S where the scalar curvature $R[\bar{g}]$ is nonpositive. Suppose that $g = u^{4/(n-2)}\bar{g}$ is a conformal metric on $D \setminus S$ and is geodesically complete near S . Then $-\Delta u$ is a Radon measure on D and $-\Delta u|_S \geq 0$, provided that*

$$R^-[g] \in L^{\frac{2n}{n+2}}(D \setminus S, g) \cap L^p(D \setminus S, g)$$

for some $p > \frac{n}{2}$.

Proof. Again, recall the scalar curvature equation

$$-\frac{4(n-1)}{n-2} \Delta u = -Ru + R^+[g]u^{\frac{n+2}{n-2}} - R^-[g]u^{\frac{n+2}{n-2}} = f \quad \text{in } D \setminus S, \tag{3-5}$$

where

$$\int_D R^-[g]u^{\frac{n+2}{n-2}} d\text{vol}[\bar{g}] < \infty.$$

And, in light of Lemma 3.1, we know

$$u(x) \rightarrow \infty \quad \text{as } x \rightarrow S.$$

Then we claim the right-hand side f of (3-5) is in $L^1(D)$. To prove this claim, we follow the argument in the proof of [Ma and Qing 2022, Theorem 3.2] (stated as Lemma 3.2 there). Let

$$\alpha_s(t) = \begin{cases} t, & t \leq s, \\ \text{increasing}, & t \in [s, 10s], \\ 2s, & t \geq 10s \end{cases}$$

(this function was used in [Dolzmann et al. 1997]). Notice that one may require $\alpha'_s \in [0, 1]$ and $\alpha''_s \leq 0$. We calculate

$$-\Delta\alpha_s(u) = -\alpha''(u)|\nabla u|^2 + \alpha'_s(u)(-\Delta u)$$

and, for $s > \max\{u(x) : x \in \partial D\}$,

$$\int_{\partial D} \frac{\partial u}{\partial \nu} d\sigma = \int_D \Delta\alpha_s(u) d\text{vol}[\bar{g}] = \int_D \left(-\alpha''(u)|\nabla u|^2 + \alpha'_s(u) \frac{n-2}{4(n-1)} f\right) d\text{vol}[\bar{g}].$$

Hence

$$\int_D \left(-\alpha''_s(u)|\nabla u|^2 + \alpha'_s(u) \frac{n-2}{4(n-1)} f^+\right) d\text{vol}[\bar{g}] \leq \int_{\partial D} \frac{\partial u}{\partial \nu} d\sigma + \frac{n-2}{4(n-1)} \int_D R^-[g] u^{\frac{n+2}{n-2}} d\text{vol}[\bar{g}]$$

and

$$\int_D |\Delta\alpha_s(u)| d\text{vol}[\bar{g}] = \int_D \left(-\alpha''(u)|\nabla u|^2 + \alpha'_s(u) \frac{n-2}{4(n-1)} (f^+ + f^-)\right) d\text{vol}[\bar{g}].$$

By Fatou's lemma, as $s \rightarrow \infty$, we have

$$\int_D f^+ d\text{vol}[\bar{g}] \leq \frac{4(n-1)}{n-2} \int_{\partial D} \frac{\partial u}{\partial \nu} d\sigma + \int_D R^-[g] u^{\frac{n+2}{n-2}} d\text{vol}[\bar{g}].$$

So the claim is proven. Moreover,

$$\int_D |\Delta\alpha_s(u)| d\text{vol}[\bar{g}] \leq \int_{\partial D} \frac{\partial u}{\partial \nu} d\sigma + \frac{n-2}{2(n-1)} \int_D R^-[g] u^{\frac{n+2}{n-2}} d\text{vol}[\bar{g}].$$

Consequently, for $\phi \in C_c^\infty(D)$,

$$\begin{aligned} |-\Delta\alpha_s(u)(\phi)| &= \left| \int_D (-\Delta\alpha_s(u))\phi d\text{vol}[\bar{g}] \right| \\ &\leq \int_D |\Delta\alpha_s(u)| d\text{vol}[\bar{g}] \|\phi\|_{C^0(D)} \\ &\leq \left(\int_{\partial D} \frac{\partial u}{\partial \nu} d\sigma + \frac{n-2}{2(n-1)} \int_{D \setminus S} R^-[g] u^{\frac{n+2}{n-2}} d\text{vol}[\bar{g}] \right) \|\phi\|_{C^0(D)} \end{aligned}$$

for any s larger. Before we show $-\Delta u$ is a Radon measure, let us state and prove a lemma which is useful for the proof now and later in the following sections.

Lemma 3.4. *Let (M^n, \bar{g}) be a compact Riemannian manifold and S be a closed subset in M^n . And let D be an open neighborhood of S where the scalar curvature $R[\bar{g}]$ is nonpositive. Suppose that $g = u^{4/(n-2)}\bar{g}$ is a conformal metric on $D \setminus S$ and is geodesically complete near S . Then*

$$\nabla u \in L^p(D) \quad \text{and} \quad u \in L^q(D) \tag{3-6}$$

for $p \in [1, \frac{n}{n-1})$ and $q \in [1, \frac{n}{n-2})$, provided that

$$R^-[g] \in L^{\frac{2n}{n+2}}(D \setminus S, g) \cap L^p(D \setminus S, g)$$

for some $p > \frac{n}{2}$.

Proof. In fact, we continue from the above, for $\phi \in C_c^\infty(D)$,

$$\begin{aligned} \left| \int_D \nabla \alpha_s(u) \cdot \nabla \phi \, d\text{vol}[\bar{g}] \right| &= \left| \int_D (-\Delta[\bar{g}]\alpha_s(u)\phi) \, d\text{vol}[\bar{g}] \right| \\ &\leq \left(\int_{\partial D} \frac{\partial u}{\partial \nu} \, d\sigma + \frac{n-2}{2(n-1)} \int_{D \setminus S} R^-[g]u^{\frac{n+2}{n-2}} \, d\text{vol}[\bar{g}] \right) \|\phi\|_{C^0(D)} \\ &\leq C \left(\int_{\partial D} \frac{\partial u}{\partial \nu} \, d\sigma + \frac{n-2}{2(n-1)} \int_{D \setminus S} R^-[g]u^{\frac{n+2}{n-2}} \, d\text{vol}[\bar{g}] \right) \|\nabla \phi\|_{L^\lambda(D)} \end{aligned} \quad (3-7)$$

for any $\lambda > n$ due to the Sobolev embedding theorem. Therefore, for any s appropriately large,

$$\|\nabla \alpha_s(u)\|_{L^p(D)} \leq C \quad \text{and} \quad \|\alpha_s(u)\|_{L^q(D)} \leq C$$

for some constant C and $p = \lambda' \in (1, \frac{n}{n-1})$ and $q \in [1, \frac{n}{n-2})$, where C is independent of s . Therefore we first have, by Fatou's lemma,

$$\|u\|_{L^q(D)} \leq C$$

for some C and $q \in [1, \frac{n}{n-2})$. Moreover, we calculate

$$\begin{aligned} |\nabla u(\phi)| &= \left| \int_D u \nabla \phi \, d\text{vol}[\bar{g}] \right| = \left| \lim_{s \rightarrow \infty} \int_D \alpha_s(u) \nabla \phi \, d\text{vol}[\bar{g}] \right| = \left| \lim_{s \rightarrow \infty} \int_D \alpha'_s(u) \nabla u \phi \, d\text{vol}[\bar{g}] \right| \\ &\leq \limsup_{s \rightarrow \infty} \|\alpha'_s(u) \nabla u\|_{L^p(D)} \|\phi\|_{L^\lambda} \leq C \|\phi\|_{L^\lambda}. \end{aligned} \quad (3-8)$$

This implies

$$\nabla u \in L^p(D) \quad \text{and} \quad u \in L^q(D) \quad (3-9)$$

for $p \in [1, \frac{n}{n-1})$ and $q \in [1, \frac{n}{n-2})$, completing the proof of Lemma 3.4. □

Back to the proof of Lemma 3.3,

$$(-\Delta u)(\phi) = \int_D \nabla u \cdot \nabla \phi \, d\text{vol}[\bar{g}] = \lim_{s \rightarrow \infty} \int_D \alpha'_s(u) \nabla u \cdot \nabla \phi \, d\text{vol}[\bar{g}] = \lim_{s \rightarrow \infty} (-\Delta \alpha_s(u))(\phi), \quad (3-10)$$

where the dominated convergence theorem is applied due to $\nabla u \in L^1(D)$. Thus, for $\phi \in C_c^\infty(D)$,

$$|(-\Delta u)(\phi)| \leq \left(\int_{\partial D} \frac{\partial u}{\partial \nu} \, d\sigma + \frac{n-2}{2(n-1)} \int_{D \setminus S} R^-[g]u^{\frac{n+2}{n-2}} \, d\text{vol}[\bar{g}] \right) \|\phi\|_{C^0(D)},$$

which implies that $-\Delta u$ is a Radon measure on D . To show that $-\Delta u|_S \geq 0$, we calculate, for a nonnegative function $\phi \in C_c^\infty(D)$,

$$\begin{aligned} (-\Delta u)(\phi) &= \int_D \nabla u \cdot \nabla \phi \, d\text{vol}[\bar{g}] = \lim_{s \rightarrow \infty} \int_D \nabla \alpha_s(u) \cdot \nabla \phi \, d\text{vol}[\bar{g}] = \lim_{s \rightarrow \infty} \int_D (-\Delta \alpha_s(u))\phi \, d\text{vol}[\bar{g}] \\ &= \lim_{s \rightarrow \infty} \int_D \left[\alpha'_s(u) \frac{n-2}{4(n-1)} (-Ru + R[g]u^{\frac{n+2}{n-2}}) - \alpha''_s(u) |\nabla u|^2 \right] \phi \, d\text{vol}[\bar{g}] \\ &\geq - \left\{ \frac{n-2}{4(n-1)} \int_{\text{supp } \phi \setminus S} |-Ru + R[g]u^{\frac{n+2}{n-2}}| \, d\text{vol}[\bar{g}] \right\} \|\phi\|_{C^0(D)} \rightarrow 0 \end{aligned}$$

as $\int_{\text{supp } \phi \setminus S} d\text{vol}[\bar{g}] \rightarrow 0$ and $\|\phi\|_{C^0(D)} = 1$, which implies $-\Delta u|_S \geq 0$. □

3.3. Main result on the Hausdorff dimensions. Now we are ready to state and prove our result on the Hausdorff dimension of the singular set S , which is a significant improvement of Proposition 3.2. For the readers' convenience, we recall Theorem 1.3 from the Introduction.

Theorem 1.3. *Let (M^n, \bar{g}) be a compact Riemannian manifold and S be a closed subset in M^n . And let D be an open neighborhood of S . Suppose that $g = u^{4/(n-2)}\bar{g}$ is a conformal metric on $D \setminus S$ and is geodesically complete near S . Then the Hausdorff dimension satisfies*

$$\dim_{\mathcal{H}}(S) \leq \frac{n-2}{2} \tag{3-11}$$

provided $R^-[g] \in L^{2n/(n+2)}(D \setminus S, g) \cap L^p(D \setminus S, g)$ for some $p > \frac{n}{2}$. Consequently, (3-11) holds when the scalar curvature $R[g]$ of the conformal metric g is nonnegative.

Proof. The outline of the proof is as follows: We first show that one may assume the scalar curvature $R[\bar{g}]$ is nonpositive without loss of generality for our purpose. Then we use the Green's function to construct the integral representation of the solution to the Laplace equation. Finally we apply Lemma 3.3, Theorem 1.1, and the geodesic completeness to complete the proof.

Step I: In this step, we find a conformal change $\bar{h} = v^{4/(n-2)}\bar{g}$ such that the scalar curvature $R[\bar{h}]$ is nonpositive (or even negative) in D , based on the similar idea used in the proof of [Ma and Qing 2022, Lemma 3.1]. This is trivial if the Yamabe constant of (M^n, \bar{g}) is nonpositive. Otherwise, take a point $p \in M^n \setminus D$ and consider a connected sum of M^n with another compact Riemannian manifold (M_1^n, \bar{g}_1) with very negative Yamabe constant in such way that the conformal structure on the connected sum $M^n \sharp M_1^n$ is unchanged in $D \subset M^n \sharp M_1^n$. Then, by [Gil-Medrano 1986, Theorem 5], the Yamabe constant of such a connected sum is negative. Therefore one easily finds a conformal metric $\bar{h} = v^{4/(n-2)}\bar{g}$ whose scalar curvature is negative in D , where $v \in C^\infty(\bar{D})$ and

$$C^{-1} \leq v \leq C \quad \text{in } \bar{D} \tag{3-12}$$

for some positive constant C . In any case, we have $g = u^{4/(n-2)}\bar{g} = (\frac{u}{v})^{4/(n-2)}\bar{h}$ and the scalar curvature $R[\bar{h}]$ is nonpositive. In conclusion, due to (3-12), we may simply assume $R[\bar{g}]$ is nonpositive (or even negative) in D without loss of any generality for the purpose of obtaining the growth estimate like the one given in Theorem 1.1.

Step II: In this step, we use the Green's function to construct the integral representation of the solution u . In light of Lemma 3.3, we may write

$$-\Delta u = \mu \quad \text{in } D$$

for a Radon measure μ on D . Let $G(x, y)$ be the Green's function on D given by [Aubin 1982, Theorem 4.17]. Then

$$u = \int_D G(x, y) d\mu(y) + h$$

for a smooth function h that is harmonic in D . By [Aubin 1982, Theorem 4.17(c)], we have

$$0 < G(x, y) \leq \frac{C}{d(x, y)^{n-2}}$$

for some constant C and $x, y \in D$. We therefore arrive at, for $x \in D$,

$$u(x) \leq \int_D G(x, y) d\mu^+ + h(x) \leq C \mathcal{R}_{\mu^+}^{2,D}(x) + h(x). \tag{3-13}$$

Step III: Assume otherwise that $\dim_{\neq}(S) = d > \frac{n-2}{2}$. From Corollary 2.12, there is a point $p \in S$ such that

$$\mathcal{R}_{\mu^+}^{2,D}(x) \leq \frac{C}{d(x, p)^{n-2-d}}$$

at least for x along a short geodesic ray γ from p , which implies

$$u(x)^{\frac{2}{n-2}} \leq \frac{C}{d(x, p)^{\frac{2(n-2-d)}{n-2}}} \tag{3-14}$$

at least for x along a short geodesic ray γ from p , where

$$\frac{2(n-2-d)}{n-2} = 2 - \frac{2d}{n-2} < 1$$

when $d > \frac{n-2}{2}$. Now the length of the curve γ with respect to the conformal metric $g = u^{4/(n-2)}\bar{g}$ is

$$L(\gamma, g) \leq C \int_0^{l_0} \frac{1}{s^{\frac{2(n-2-d)}{n-2}}} ds < \infty$$

when $d > \frac{n-2}{2}$, which contradicts the geodesic completeness of the conformal metric $g = u^{4/(n-2)}\bar{g}$. \square

The study of singular solutions to the scalar curvature equations started from the seminal paper [Schoen and Yau 1988] (see also [Schoen and Yau 1994, Chapter VI; Carron 2012; Schoen 1988; Mazzeo and Smale 1991; Mazzeo and Pacard 1996]) on domains of the sphere. Theorem 1.3 here can be considered as a necessary condition for the existences of singular solutions on domains in general Riemannian manifolds and compared with [Schoen and Yau 1988, Theorem 2.7; Carron 2012, Theorem C], which stated the similar result for domains in the round sphere S^n and slightly stronger curvature assumptions. Clearly [Schoen and Yau 1988, Proposition 2.4] and the quantity $d(M)$ there are not of local nature, while our approach here is very much local in nature.

4. On Q -curvature equations

We will use linear potential theory developed in Section 2 to study Q -curvature equations and prove our results on the Hausdorff dimensions of the singular sets of positive solutions of Q -curvature equations which correspond to ends of complete conformal metrics on domains of a compact Riemannian manifold.

Again we remark here that all of the results in this section hold if we assume S is compact, $D \subset M^n$ is a bounded domain that contains S , and (M^n, \bar{g}) is just complete. Because the possible noncompact part $M^n \setminus \bar{D}$ is not relevant for the purpose here.

4.1. Q -curvature equations in dimensions greater than 4. We now focus on (1-5) in dimensions greater than 4. We will always assume that the scalar curvature of the conformal metric $g = u^{4/(n-4)}\bar{g}$ is nonnegative. We will first prove some preliminary estimates based on discussions in the previous section. Our strategy is to consider the bi-Laplace operator as the composition of the Laplace operators. Let us write the scalar curvature equation and its consequence:

$$-\Delta u^{\frac{n-2}{n-4}} + \frac{n-2}{4(n-1)}Ru^{\frac{n-2}{n-4}} = \frac{n-2}{4(n-1)}R[g]u^{\frac{n+2}{n-4}} \quad \text{in } D \setminus S, \tag{4-1}$$

$$-\Delta u = \frac{2}{n-4} \frac{|\nabla u|^2}{u} + \frac{n-4}{4(n-1)}(-Ru + R[g]u^{\frac{n}{n-4}}) \quad \text{in } D \setminus S. \tag{4-2}$$

Here, and from now on, all geometric quantities are under the background metric \bar{g} unless indicated otherwise.

Lemma 4.1. *Let (M^n, \bar{g}) be a compact Riemannian manifold for $n \geq 5$ and S be a closed subset in M^n . And let D be an open neighborhood of S where the scalar curvature satisfies $R \leq -c_0 < 0$. Suppose that $g = u^{4/(n-4)}\bar{g}$ is a conformal metric on $D \setminus S$ with nonnegative scalar curvature $R[g] \geq 0$ and is geodesically complete near S . And suppose also that*

$$Q_4^-[g] \in L^{\frac{2n}{n+4}}(D \setminus S, g).$$

Then

$$\begin{aligned} &\text{as a function on } D \setminus S, \quad -\Delta u \rightarrow +\infty \quad \text{as } x \rightarrow S, \\ &\text{as a Radon measure on } D, \quad \Delta u|_S = 0, \\ &\text{in fact, } \Delta u \in L^p(D) \quad \text{for any } p \in [1, \frac{n}{n-2}). \end{aligned} \tag{4-3}$$

Proof. First, using Lemma 3.4 for $u^{(n-2)/(n-4)}$, we know that

$$u \in L^p(D) \quad \text{for } p \in [1, \frac{n}{n-4}). \tag{4-4}$$

Also, from Lemma 3.1, for $u^{(n-2)/(n-4)}$,

$$u(x) \rightarrow +\infty \quad \text{as } x \rightarrow S, \tag{4-5}$$

which implies, by (4-2),

$$-\Delta u \rightarrow +\infty \quad \text{as } x \rightarrow S.$$

To prove $-\Delta u$ is an integrable function in distributional sense, we first realize $-\Delta u$ is a Radon measure on D following (4-2) and Lemma 3.3. And, as a side product, we also have

$$\int_D \left[\frac{2}{n-4} \frac{|\nabla u|^2}{u} + \frac{n-4}{4(n-1)}(-Ru + R[g]u^{\frac{n}{n-4}}) \right] d\text{vol}[\bar{g}] < \infty.$$

In fact, from (4-1) and Lemma 3.3, we also know $-\Delta u^{(n-2)/(n-4)}$ is a Radon measure on D . To use this fact we calculate

$$-\Delta \alpha_s(u) = -\Delta(\alpha_s(u)^{\frac{n-2}{n-4}})^{\frac{n-4}{n-2}} = \frac{n-4}{n-2} \alpha_s(u)^{-\frac{2}{n-4}} (-\Delta \alpha_s(u)^{\frac{n-2}{n-4}}) + \frac{2}{n-4} \frac{|\nabla \alpha_s(u)|^2}{\alpha_s(u)}.$$

To prove $-\Delta u|_S = 0$, we consider

$$(-\Delta u)(\phi) = \int_D \nabla u \cdot \nabla \phi \, d\text{vol}[\bar{g}],$$

where ∇u is integrable in the distributional sense directly from (3-8) and (3-9). Therefore

$$\int_D \nabla u \cdot \nabla \phi \, d\text{vol}[\bar{g}] = \lim_{s \rightarrow \infty} \int_D \alpha'_s(u) \nabla u \cdot \nabla \phi \, d\text{vol}[\bar{g}] = \lim_{s \rightarrow \infty} \int_D (-\Delta \alpha_s(u)) \phi \, d\text{vol}[\bar{g}]$$

and

$$\begin{aligned} (-\Delta u)(\phi) &= \frac{n-4}{n-2} \lim_{s \rightarrow \infty} \int_D \alpha_s(u)^{-\frac{2}{n-4}} (-\Delta \alpha_s(u)^{\frac{n-2}{n-4}}) \phi \, d\text{vol}[\bar{g}] + \frac{2}{n-4} \lim_{s \rightarrow \infty} \int_D \frac{|\nabla \alpha_s(u)|^2}{\alpha_s(u)} \phi \, d\text{vol}[\bar{g}] \\ &= \frac{n-4}{n-2} u^{-\frac{2}{n-4}} (-\Delta u^{\frac{n-2}{n-4}})(\phi) + \frac{2}{n-4} \int_D \frac{|\nabla u|^2}{u} \phi \, d\text{vol}[\bar{g}] \rightarrow 0 \end{aligned}$$

as $\int_{\text{supp} \phi \setminus S} d\text{vol}[\bar{g}] \rightarrow 0$ and $\|\phi\|_{C^0(D)} \leq 1$. The proof will be complete after the following L^p estimate. To get the L^p estimate, we first calculate

$$\begin{aligned} \int_{D \setminus S} Q_4^- [g] u^{\frac{n+4}{n-4}} \, d\text{vol}[\bar{g}] &= \left(\int_{D \setminus S} (Q_4^- [g])^{\frac{2n}{n+4}} u^{\frac{2n}{n-4}} \, d\text{vol}[\bar{g}] \right)^{\frac{n+4}{2n}} \text{vol}(D)^{\frac{2n}{n-4}} \\ &= \left(\int_{D \setminus S} (Q_4^- [g])^{\frac{2n}{n+4}} \, d\text{vol}[g] \right)^{\frac{n+4}{2n}} \text{vol}(D)^{\frac{2n}{n-4}} < \infty. \end{aligned} \tag{4-6}$$

Then we continue to use notation in the proof of Proposition 3.2 and let

$$\alpha = \max\{u(x) : x \in \partial D\}$$

and $\alpha < \beta$. And recall

$$u_{\alpha,\beta} = \begin{cases} \beta, & x \in \Sigma_{\alpha+\beta}, \\ u(x) - \alpha, & x \in \Sigma_\alpha \setminus \Sigma_{\alpha+\beta}, \end{cases}$$

and

$$\phi_{\alpha,\beta} = \begin{cases} u_{\alpha,\beta} - \beta\eta = u - (\alpha + \beta) + \beta(1 - \eta) & \text{in } \Sigma_\alpha \setminus \Sigma_{\alpha+\beta}, \\ 0 & \text{on } \partial \Sigma_\alpha, \\ 0 & \text{on } \partial \Sigma_{\alpha+\beta}, \end{cases}$$

where η is a fixed cut-off function in $C_c^\infty(\Sigma_\alpha)$ and is identically 1 in a neighborhood of S , and β is arbitrarily large. We now first multiply the Q -curvature equation (1-5) by $1 - \eta$, integrate over D , apply integration by parts multiple times, and get

$$\int_D (1 - \eta) Q_4^+ u^{\frac{n+4}{n-4}} \, d\text{vol}[\bar{g}] \leq \int_D Q_4^- u^{\frac{n+4}{n-4}} \, d\text{vol}[\bar{g}] + C \tag{4-7}$$

for some constant C depending on the cut-off function η , u at ∂D , and $\|u\|_{L^1(D)}$. We then multiply both sides of the Q -curvature equation (1-5) by $\phi_{\alpha,\beta}$, integrate over $\Sigma_\alpha \setminus \Sigma_{\alpha+\beta}$, and get

$$\begin{aligned}
 & \int_{\Sigma_\alpha \setminus \Sigma_{\alpha+\beta}} \Delta u \Delta \phi_{\alpha,\beta} \, d\text{vol}[\bar{g}] - \int_{\partial \Sigma_\alpha} \Delta u \frac{\partial u}{\partial \nu} \, d\sigma - \int_{\partial \Sigma_{\alpha+\beta}} \frac{\partial u}{\partial \nu} \Delta u \, d\sigma \\
 & - \int_{\Sigma_\alpha \setminus \Sigma_{\alpha+\beta}} (4A(\nabla u, \nabla \phi_{\alpha,\beta}) - (n-2)J \nabla u \cdot \nabla \phi_{\alpha,\beta}) \, d\text{vol}[\bar{g}] + \frac{n-4}{2} \int_{\Sigma_\alpha \setminus \Sigma_{\alpha+\beta}} Q_4 u \phi_{\alpha,\beta} \, d\text{vol}[\bar{g}] \\
 & = \frac{n-4}{2} \int_{\Sigma_\alpha \setminus \Sigma_{\alpha+\beta}} Q_4 [g] u^{\frac{n+4}{n-4}} \phi_{\alpha,\beta} \, d\text{vol}[\bar{g}], \tag{4-8}
 \end{aligned}$$

where ν is the outward normal direction at the boundary and the boundary term $\int_{\partial \Sigma_{\alpha+\beta}} \frac{\partial u}{\partial \nu} (-\Delta u) \, d\sigma$ is nonnegative due to (4-2) and $\frac{\partial u}{\partial \nu} |_{\partial \Sigma_{\alpha+\beta}} = |\nabla u|$. Therefore,

$$\begin{aligned}
 & \int_{\Sigma_\alpha \setminus \Sigma_{\alpha+\beta}} (\Delta u)^2 \, d\text{vol}[\bar{g}] + \beta \int_{\Sigma_\alpha \setminus \Sigma_{\alpha+\beta}} (\Delta u)(\Delta(1-\eta)) \, d\text{vol}[\bar{g}] \\
 & \leq - \int_{\partial \Sigma_\alpha} (-\Delta u) \frac{\partial u}{\partial \nu} \, d\sigma + C \int_{\Sigma_\alpha \setminus \Sigma_{\alpha+\beta}} |\nabla u|^2 \, d\text{vol}[\bar{g}] \\
 & - \beta \int_{\Sigma_\alpha} (4A(\nabla u, \nabla(1-\eta)) - (n-2)J \nabla u \cdot \nabla(1-\eta)) \, d\text{vol}[\bar{g}] \\
 & + C\beta \int_D u \, d\text{vol}[\bar{g}] + C\beta \int_D Q_4^- [g] u^{\frac{n+4}{n-4}} \, d\text{vol}[\bar{g}], \tag{4-9}
 \end{aligned}$$

where we use (4-7) and $|\phi| \leq \beta$ in $\Sigma_\alpha \setminus \Sigma_{\alpha+\beta}$. After applying integration by parts, we get

$$\int_{\Sigma_\alpha \setminus \Sigma_{\alpha+\beta}} |\Delta u|^2 \, d\text{vol}[\bar{g}] \leq C \int_{\Sigma_\alpha \setminus \Sigma_{\alpha+\beta}} |\nabla u|^2 \, d\text{vol}[\bar{g}] + C\beta \tag{4-10}$$

for some constant C depending on the cut-off function η , u at $\partial \Sigma_\alpha$, and $\|u\|_{L^1(D)}$, because

$$\int_{\Sigma_\alpha \setminus \Sigma_{\alpha+\beta}} \Delta u \Delta \eta \, d\text{vol}[\bar{g}] = \int_{\Sigma_\alpha \setminus \Sigma_{\alpha+\beta}} u \Delta^2 \eta \, d\text{vol}[\bar{g}]$$

and similarly we may unload all derivatives from u by integration by parts for the other terms in the above (4-9). Now, to get an a priori estimate, we calculate

$$\begin{aligned}
 \int_{\Sigma_\alpha \setminus \Sigma_{\alpha+\beta}} |\nabla u|^2 \, d\text{vol}[\bar{g}] & \leq \frac{1}{(n-4)C} \int_{\Sigma_\alpha \setminus \Sigma_{\alpha+\beta}} \frac{|\nabla u|^4}{u^2} \, d\text{vol}[\bar{g}] + \frac{(n-4)C}{4} \int_{\Sigma_\alpha \setminus \Sigma_{\alpha+\beta}} u^2 \, d\text{vol}[\bar{g}] \\
 & \leq \frac{1}{2C} \int_{\Sigma_\alpha \setminus \Sigma_{\alpha+\beta}} |\Delta u|^2 \, d\text{vol}[\bar{g}] + \frac{(n-4)C}{4} (\alpha + \beta) \int_D u \, d\text{vol}[\bar{g}],
 \end{aligned}$$

due to (4-2), which implies, from (4-10),

$$\int_{\Sigma_\alpha \setminus \Sigma_{\alpha+\beta}} |\Delta u|^2 \, d\text{vol}[\bar{g}] \leq C(\alpha + \beta). \tag{4-11}$$

We claim that (4-11) implies

$$\Delta u \in L^p(D) \tag{4-12}$$

for all $p \in [1, \frac{n}{n-2})$. To prove (4-12), we first derive from (4-11)

$$2^{-i} \int_{\Sigma_{2^{i-1}} \setminus \Sigma_{2^i}} |\Delta u|^2 d\text{vol}[\bar{g}] \leq C$$

for $i \geq i_0$ large, which implies

$$\int_{\Sigma_{2^{i-1}} \setminus \Sigma_{2^i}} \frac{|\Delta u|^2}{u} d\text{vol}[\bar{g}] \leq 2C$$

and, for $s > 0$ appropriately small for any $p \in [1, \frac{n}{n-2})$,

$$\int_{\Sigma_{2^{i_0-1}} \setminus S} \frac{|\Delta u|^2}{u^{1+s}} d\text{vol}[\bar{g}] = \sum_{i=i_0}^{\infty} \int_{\Sigma_{2^{i-1}} \setminus \Sigma_{2^i}} \frac{|\Delta u|^2}{u^{1+s}} d\text{vol}[\bar{g}] \leq \sum_{i=i_0}^{\infty} 2^{s(-i+1)} \int_{\Sigma_{2^{i-1}} \setminus \Sigma_{2^i}} \frac{|\Delta u|^2}{u} d\text{vol}[\bar{g}] < \infty.$$

Thus

$$\int_{D \setminus S} |\Delta u|^p d\text{vol}[\bar{g}] \leq \left(\int_{D \setminus S} \frac{|\Delta u|^2}{u^{1+s}} d\text{vol}[\bar{g}] \right)^{\frac{p}{2}} \left(\int_{D \setminus S} u^{\frac{(1+s)p}{2-p}} d\text{vol}[\bar{g}] \right)^{1-\frac{p}{2}} < \infty,$$

where

$$\frac{(1+s)p}{2-p} < \frac{n}{n-4}. \quad \square$$

Corollary 4.2. *Under the same assumptions as in Lemma 4.1 we have*

$$\dim_{\mathcal{H}}(S) \leq n - 4.$$

Proof. Consequently from (4-2) and (4-11), we have

$$\begin{aligned} \int_{\Sigma_{\alpha} \setminus \Sigma_{\alpha+\beta}} \left| \nabla \frac{u_{\alpha,\beta}}{\beta} \right|^4 d\text{vol}[\bar{g}] &\leq \frac{(\alpha + \beta)^2}{\beta^4} \int_{\Sigma_{\alpha} \setminus \Sigma_{\alpha+\beta}} \frac{|\nabla u|^4}{u^2} d\text{vol}[\bar{g}] \\ &\leq \frac{(\alpha + \beta)^2}{\beta^4} \int_{\Sigma_{\alpha} \setminus \Sigma_{\alpha+\beta}} |\Delta u|^2 d\text{vol}[\bar{g}] \leq C \frac{(\alpha + \beta)^3}{\beta^4} \end{aligned}$$

for some α appropriately large and $\beta \rightarrow \infty$, which leads to $\text{Cap}_4(S) = 0$ and completes the proof similar to the proof of Proposition 3.2 (see [Adams and Meyers 1972; Schoen and Yau 1994, Theorem 2.10 in Chapter VI]). \square

Lemma 4.3. *Let (M^n, \bar{g}) be a compact Riemannian manifold for $n \geq 5$ and S be a closed subset in M^n . And let D be an open neighborhood of S where the scalar curvature $R[\bar{g}] \leq -c_0 < 0$. Suppose that $g = u^{4/(n-4)} \bar{g}$ is a conformal metric on $D \setminus S$ with nonnegative scalar curvature $R[g] \geq 0$ and is geodesically complete near S . And suppose also that*

$$Q_4^- [g] \in L^{\frac{2n}{n+4}}(D \setminus S, g).$$

Then $\Delta^2 u$ is a Radon measure on D and $\Delta^2 u|_S \geq 0$.

Proof. Let $v = -\Delta u$. We will follow the proof of Lemma 3.3 to show that $-\Delta v$ is a Radon measure on D using Lemma 4.1. We continue to use the notation from the proof of Lemma 3.3. We calculate

$$-\Delta \alpha_s(v) = \alpha'(v)(-\Delta v) - \alpha''(v)|\nabla v|^2,$$

where, by the Q -curvature equation (1-5), we have

$$-\Delta v = -\operatorname{div}(4A(\nabla u) - (n - 2)J\nabla u) - \frac{n - 4}{2}Q_4u + Q_4[g]u^{\frac{n+4}{n-4}} \quad \text{in } D \setminus S$$

and

$$-\Delta\alpha_s(v) = -\alpha_s''(v)|\nabla v|^2 + \alpha_s'(v)(-\operatorname{div}(4A(\nabla u) - (n - 2)J\nabla u) - \frac{n - 4}{2}Q_4u + Q_4[g]u^{\frac{n+4}{n-4}})$$

in D . In light of Lemma 4.1, terms in the right-hand side of the above equation are all integrable except $-\alpha_s''(v)|\nabla v|^2 + Q_4^+[g]u^{(n+4)/(n-4)}$. Therefore the argument in the proof of Lemma 3.3 works from this point and completes the proof. \square

We now are ready to state and prove our main results for Q -curvature equations in dimensions greater than 4. For this, we recall Theorem 1.4 from the Introduction.

Theorem 1.4. *Let (M^n, \bar{g}) be a compact Riemannian manifold for $n \geq 5$ and S be a closed subset in M^n . And let D be an open neighborhood of S . Suppose that $g = u^{4/(n-4)}\bar{g}$ is a conformal metric on $D \setminus S$ with nonnegative scalar curvature $R[g] \geq 0$ and is geodesically complete near S . And suppose also that*

$$Q_4^-[g] \in L^{\frac{2n}{n+4}}(D \setminus S, g).$$

Then

$$\dim_{\mathcal{H}}(S) \leq \frac{n - 4}{2}.$$

Proof. In light of Step I in the proof of Theorem 1.3, we may assume the scalar curvature $R \leq -c_0 < 0$ for some c_0 without loss of any generality. Then we use Lemmas 4.1 and 4.3 and conclude that

$$\Delta^2 u = \mu$$

for a Radon measure μ on D . We use [Aubin 1982, Theorem 4.7] first to write

$$-\Delta u = \int_D G(x, y) d\mu + h(x)$$

for some harmonic function $h(x)$, where $G(x, y)$ is the Green's function for $-\Delta$. Then we have

$$u(x) = \int_D G(x, z) \int_D G(z, y) d\mu(y) d\operatorname{vol}[\bar{g}](z) + b(x),$$

where $b(x)$ is biharmonic, where

$$\int_D G(x, z) \int_D G(z, y) d\mu(y) d\operatorname{vol}[\bar{g}](z) = \int_D \left(\int_D G(x, z)G(z, y) d\operatorname{vol}[\bar{g}](z) \right) d\mu(y)$$

and

$$0 < \int_D G(x, z)G(z, y) d\operatorname{vol}[\bar{g}](z) \leq \frac{C}{d(x, y)^{n-4}}$$

for a constant C and $n \geq 5$ due to [Aubin 1982, Proposition 4.12], which can be easily proven to be available for bounded domains in Riemannian manifolds. Hence

$$u(x) \leq \mathcal{R}_{\mu^+}^{4,D}(x) + b(x).$$

From now on, using the same argument of the proof of Theorem 1.3, based on Theorem 1.1 for $\alpha = 4$ and $n \geq 5$, we conclude

$$\dim \mathcal{H}(S) \leq \frac{n-4}{2}$$

and finish the proof. □

There have been a lot of works on the study of singular solutions to Q -curvature equations on manifolds of dimension greater than 4, notably [Qing and Raske 2006a; 2006b; Chang et al. 2004; González et al. 2012], for example. Theorem 1.4, for instance, is an improvement of [Chang et al. 2004, Theorem 1.2] in terms of curvature conditions. And the approach here is different from [Chang et al. 2004].

4.2. Q -curvature equations in dimension 4. We will now study the Q -curvature equation (1-6). Our approach here in principle is similar to that in the previous subsection but different in calculations and details. We will always assume that the scalar curvature of the conformal metric $g = e^{2u} \bar{g}$ is nonnegative. We will first derive some preliminary estimates from the scalar curvature equation for $w = e^u$ and the Q -curvature equation (1-6) for u . Let us write the scalar curvature equation for e^u

$$-\Delta e^u = \frac{1}{6}(-R e^u + R[g] e^{3u}) \quad \text{in } D \setminus S \tag{4-13}$$

and consequently,

$$-\Delta u = |\nabla u|^2 + \frac{1}{6}(-R + R[g] e^{2u}) \quad \text{in } D \setminus S. \tag{4-14}$$

Lemma 4.4. *Let (M^4, \bar{g}) be a compact Riemannian manifold and S be a closed subset in M^n . And let D be an open neighborhood of S where the scalar curvature $R \leq 0$. Suppose that $g = e^{2u} \bar{g}$ is a conformal metric on $D \setminus S$ with nonnegative scalar curvature $R[g] \geq 0$ and is geodesically complete near S . And suppose also that*

$$Q_4^-[g] \in L^1(D \setminus S, g).$$

Then

$$\begin{aligned} &\text{as a Radon measure, } -\Delta u|_S = 0, \\ &\text{in fact, } \Delta u \in L^p(D) \text{ for any } p \in [1, \frac{4}{3}). \end{aligned} \tag{4-15}$$

Proof. First, by Lemma 3.1 for e^u , we have

$$u(x) \rightarrow \infty \quad \text{as } x \rightarrow S.$$

Then, by the proof of Lemma 3.3 and (4-14), we know that

- $-\Delta u$ is a Radon measure on D ,
- $\nabla u \in L^p(D)$ for any $p \in [1, \frac{4}{3})$ and $u \in L^p(D)$ for any $p \in [1, 2)$,
- $|\nabla u|^2 + \frac{1}{6}(-R + R[g] e^{2u}) \in L^1(D)$.

Using the same argument as in the proof of Lemma 4.1 we can prove that $-\Delta u|_S = 0$ as a Radon measure. Also, for the L^p estimate, following the proof of Lemma 4.1, we multiply both sides of (1-6) by $1 - \eta$ and get

$$\int_D (1 - \eta) Q_4^+ e^{4u} d\text{vol}[\bar{g}] \leq \int_D Q_4^-[g] d\text{vol}[g] + C.$$

Next we multiply both sides of (1-6) by $\phi_{\alpha,\beta}$ and integrate,

$$\begin{aligned} & \int_{\Sigma_\alpha \setminus \Sigma_{\alpha+\beta}} \Delta u \Delta \phi_{\alpha,\beta} d\text{vol}[\bar{g}] - \int_{\partial \Sigma_\alpha} \Delta u \frac{\partial \phi_{\alpha,\beta}}{\partial \nu} d\sigma - \int_{\partial \Sigma_{\alpha+\beta}} \Delta u \frac{\partial \phi_{\alpha,\beta}}{\partial \nu} d\sigma \\ & \quad - \int_{\Sigma_\alpha \setminus \Sigma_{\alpha+\beta}} (4A(\nabla u, \nabla \phi_\alpha) - 2J \nabla u \cdot \nabla \phi_{\alpha,\beta}) d\text{vol}[\bar{g}] + \int_{\Sigma_\alpha \setminus \Sigma_{\alpha+\beta}} Q_4 \phi_{\alpha,\beta} d\text{vol}[\bar{g}] \\ & = \int_{\Sigma_\alpha \setminus \Sigma_{\alpha+\beta}} Q_4[g] e^{4u} \phi_{\alpha,\beta} d\text{vol}[\bar{g}], \end{aligned}$$

and, again, the boundary term at $\partial \Sigma_{\alpha+\beta}$ is with the sign in our favor, thanks to (4-14) and $\frac{\partial u}{\partial \nu}|_{\Sigma_{\alpha+\beta}} = |\nabla u|$ for the outward normal ν of $\Sigma_\alpha \setminus \Sigma_{\alpha+\beta}$. Similar to the estimates in the proof of Lemma 4.1, we get

$$\int_{\Sigma_\alpha \setminus \Sigma_{\alpha+\beta}} |\Delta u|^2 d\text{vol}[\bar{g}] \leq C \int_{\Sigma_\alpha \setminus \Sigma_{\alpha+\beta}} |\nabla u|^2 d\text{vol}[\bar{g}] + C\beta.$$

And we handle $\int_{\Sigma_\alpha \setminus \Sigma_{\alpha+\beta}} |\nabla u|^2 d\text{vol}[\bar{g}]$ much as before,

$$\int_{\Sigma_\alpha \setminus \Sigma_{\alpha+\beta}} |\nabla u|^2 d\text{vol}[\bar{g}] \leq \frac{1}{2C} \int_{\Sigma_\alpha \setminus \Sigma_{\alpha+\beta}} |\nabla u|^4 d\text{vol}[\bar{g}] + C \leq \frac{1}{2C} \int_{\Sigma_\alpha \setminus \Sigma_{\alpha+\beta}} |\Delta u|^2 d\text{vol}[\bar{g}] + C$$

due to (4-14). Therefore

$$\int_{\Sigma_\alpha \setminus \Sigma_{\alpha+\beta}} |\Delta u|^2 d\text{vol}[\bar{g}] \leq C\beta. \quad (4-16)$$

Now, using the same idea as in the proof of Lemma 4.1, we rewrite (4-16) as

$$\int_{\Sigma_{2i-1} \setminus \Sigma_{2i}} \frac{|\Delta u|^2}{u} d\text{vol}[\bar{g}] \leq C$$

and, for $s > 0$ appropriately small for any $p \in [1, \frac{4}{3})$, we derive

$$\int_{D \setminus S} \frac{|\Delta u|^2}{u^{1+s}} d\text{vol}[\bar{g}] \leq C,$$

which implies

$$\int_{D \setminus S} |\Delta u|^p d\text{vol}[\bar{g}] \leq \left(\int_{D \setminus S} \frac{|\Delta u|^2}{u^{1+s}} d\text{vol}[\bar{g}] \right)^{\frac{p}{2}} \left(\int_{D \setminus S} u^{\frac{(1+s)p}{2-p}} d\text{vol}[\bar{g}] \right)^{1-\frac{p}{2}}$$

when

$$\frac{(1+s)p}{2-p} < 2. \quad \square$$

Corollary 4.5. *Under the assumptions of Lemma 4.4, we know the singular set S is of zero Hausdorff dimension.*

Proof. From (4-14) and (4-16) in the above we have

$$\int_{\Sigma_\alpha \setminus \Sigma_{\alpha+\beta}} \left| \nabla \frac{u_{\alpha,\beta}}{\beta} \right|^4 d\text{vol}[\bar{g}] \leq C\beta^{-3}$$

for some α appropriately large and $\beta \rightarrow \infty$, which leads to $\text{Cap}_4(S, D) = 0$ and completes the proof as in Proposition 3.2 (see [Adams and Meyers 1972; Schoen and Yau 1994, Theorem 2.10 in Chapter VI]). \square

What follows is to go beyond that S is of zero Hausdorff dimension. We now are ready to state and prove our main result on the finiteness of singularities for the Q -curvature equation in dimension 4. This is inspired by [Cohn-Vossen 1935; Huber 1957; Arsove and Huber 1973; Ma and Qing 2021; 2022]. We recall Theorem 1.5 from the Introduction.

Theorem 1.5. *Let (M^4, \bar{g}) be a compact Riemannian manifold and S be a closed subset in M^n . And let D be an open neighborhood of S . Suppose that $g = e^{2u}\bar{g}$ is a conformal metric on $D \setminus S$ with nonnegative scalar curvature $R[g] \geq 0$ and is geodesically complete near S . And suppose that*

$$\int_D Q_4^-[g] d\text{vol}[g] < \infty.$$

Then S consists of only finitely many points.

Proof. As before, we use the argument in Step I on Theorem 1.3 to assume that the scalar curvature of the background metric \bar{g} is less than a negative number, i.e., $R \leq -c_0 < 0$, without loss of any generality for our purpose. Let

$$v = -\Delta u + u$$

and claim $-\Delta v$ is a Radon measure on D with $-\Delta v|_S \geq 0$. Let us start with

$$-\Delta v = \Delta^2 u - \Delta u = -\text{div}(4A(\nabla u) - 2J\nabla u) - Q_4 + Q_4[g]e^{4u} - \Delta u. \tag{4-17}$$

By Lemma 4.4 and (4-14), we know

- $v(x) \rightarrow \infty$ as $x \rightarrow S$,
- all terms in the right side of (4-17) except $Q_4^+[g]e^{4u}$ are integrable.

Therefore, following the same argument as in the proof of Lemma 4.3, the claim is proven. Obviously, the same conclusion holds for $\Delta^2 u = -\Delta v + \Delta u$ from $-\Delta v$ and what we know about Δu in Lemma 4.4. Thus we let

$$\Delta^2 u = \mu$$

for a Radon measure on D with $\Delta^2 u|_S \geq 0$. Like in the proof of Theorem 1.4, we first write

$$-\Delta u(x) = \int_D G(x, y) d\mu(y) + h(x)$$

by [Aubin 1982, Theorem 4.17], where $h(x)$ is a harmonic function. Then we write

$$u(x) = \int_D G(x, z) \int_D G(z, y) d\mu(y) d\text{vol}[\bar{g}](z) + b(x),$$

where $b(x)$ is a biharmonic function and, due to [Aubin 1982, Proposition 4.12],

$$\int_D G(x, z)G(z, y) d\text{vol}[\bar{g}](z) \leq C \left(1 + \log \frac{1}{d(x, y)}\right)$$

for some constant C in dimension 4, where [Aubin 1982, Proposition 4.12] can be easily made available on bounded domains in manifolds. Therefore

$$u(x) \leq C \mathcal{R}_{\mu^+}^{A,D}(x) + b(x).$$

Applying Theorem 2.13, we have

$$\lim_{x \rightarrow p \text{ and } x \notin E} \frac{u(x)}{\log(1/d(x, p))} \leq C\mu^+(\{p\}) = C\mu(\{p\}),$$

where E is a subset that is n -thin at p . Next, in light of Corollary 2.12, we conclude that $\mu(\{p\}) \geq \frac{1}{C}$ for each $p \in S$ since otherwise $u(x) \leq m \log(1/d(x, p))$ for some $m < 1$, which violates the completeness of the metric g near S , because the \bar{g} -geodesic mentioned in Theorem 1.2, which avoids E , would have finite length with respect to the metric g . So we conclude that S can only have finitely many points. \square

Theorem 1.4 is a significant improvement of [Chang et al. 2000a, Theorem 2] (please see also [Carron and Herzlich 2002; Chang et al. 2000b; Ma and Qing 2022]).

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