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STOCHASTIC HOMOGENIZATION FOR VARIATIONAL SOLUTIONS OF HAMILTON–JACOBI EQUATIONS

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Let (Ω, μ) be a probability space endowed with an ergodic action τ of $(\mathbb{R}^n, +)$. Let $H(x, p; \omega) = H_\omega(x, p)$ be a smooth Hamiltonian on $T^*\mathbb{R}^n$ parametrized by $\omega \in \Omega$ and such that $H(a + x, p; \tau_a\omega) = H(x, p; \omega)$. We consider for an initial condition $f \in C^0(\mathbb{R}^n, \mathbb{R})$ the family of variational solutions of the stochastic Hamilton–Jacobi equations

$$\begin{cases} \dfrac{\partial u^\varepsilon}{\partial t}(t, x; \omega) + H\left(\dfrac{x}{\varepsilon}, \dfrac{\partial u^\varepsilon}{\partial x}(t, x; \omega)\right) = 0, \\ u^\varepsilon(0, x; \omega) = f(x). \end{cases}$$

Under some coercivity assumptions on p — but without any convexity assumption — we prove that for a.e. $\omega \in \Omega$ we have $C^0\!-\lim u^\varepsilon(t, x; \omega) = v(t, x)$, where v is the variational solution of the homogenized equation

$$\begin{cases} \dfrac{\partial v}{\partial t}(t, x) + \overline{H}\left(x, \dfrac{\partial v}{\partial x}(t, x)\right) = 0, \\ v(0, x) = f(x). \end{cases}$$

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1. Introduction

Let (Ω, μ) be a probability space endowed with an ergodic action τ of $(\mathbb{R}^n, +)$. This means that if $X \subset \Omega$ satisfies $\tau_a X \subset X$ for all $a \in \mathbb{R}^n$, then $\mu(X) = 0$ or 1.

Let $H(x, p; \omega) = H_\omega(x, p)$ be a smooth Hamiltonian on $T^*\mathbb{R}^n$ parametrized by $\omega \in \Omega$ and such that

$$H(a + x, p; \tau_a \omega) = H(x, p; \omega). \tag{Inv}$$

We shall specify later the assumptions satisfied by H. We now consider for an initial condition $f \in C^0(\mathbb{R}^n)$ the family of stochastic Hamilton–Jacobi equations

$$\begin{cases} \dfrac{\partial u^\varepsilon}{\partial t}(t, x; \omega) + H\Big(\dfrac{x}{\varepsilon}, \dfrac{\partial u^\varepsilon}{\partial x}(t, x; \omega); \omega\Big) = 0, \\ u^\varepsilon(0, x; \omega) = f(x). \end{cases} \tag{HJS$_\varepsilon$}$$

Fixing ω, we can consider different types of generalized solutions (there is generally no smooth solution) for this equation. The most interesting ones are either the viscosity solution of Crandall and Lions [1983] (see also [Barles 1994; Bardi and Capuzzo-Dolcetta 1997]), or the variational solutions defined in [Chaperon 1991; Viterbo 1996; 2006] (we also credit J. C. Sikorav [1989]), both requiring some assumptions on f and H that will be specified later. The problem of stochastic homogenization for the above equation is to determine whether, for μ-a.e. in ω, the sequence $u^\varepsilon(t, x; \omega)$ C^0-converges on compact sets to $\bar{u}(t, x)$, the solution of

$$\begin{cases} \dfrac{\partial v}{\partial t}(t, x) + \overline{H}\Big(\dfrac{\partial v}{\partial x}(t, x)\Big) = 0, \\ v(0, x) = f(x), \end{cases} \tag{HJH}$$

where $\overline{H}$ is to be determined (and in general cannot be defined explicitly). Note that $\overline{H}$ does *not* depend on ω by the ergodicity hypothesis. A classical case is the so-called (nonstochastic) periodic case, corresponding to the case where $\Omega = \mathbb{T}^n$ and τ_a is the translation on the torus. Then condition (Inv) means that there is a smooth function K on T^*T^n such that $H(x, p; \omega) = K(x - \omega, p)$. Then solving (HJS$_\varepsilon$) is equivalent to solving the (nonstochastic) equation

$$\frac{\partial u}{\partial t}(t, x) + K\Big(\frac{y}{\varepsilon}, \frac{\partial v^\varepsilon}{\partial y}(t, y)\Big) = 0$$

and in this case stochastic homogenization boils down[1] to deterministic homogenization for K. For viscosity solutions, homogenization in the periodic nonstochastic case has been settled in [Lions et al. 1988] in 1987, and for variational solutions in [Viterbo 2023] in 2014.

For the general stochastic case, this problem has been solved for viscosity solutions by Rezakhanlou and Tarver [2000] and Souganidis [1999], assuming H is convex in p. Beyond the quasiconvex case (i.e., functions having all their sublevels convex) and some very special cases (see for instance [Armstrong et al.

[1]Indeed, if $u^\varepsilon(t, x)$ is the solution (either viscosity or variational) of $\frac{\partial u^\varepsilon}{\partial t}(t, x) + K\big(\frac{x}{\varepsilon} - \omega, \frac{\partial u^\varepsilon}{\partial x}(t, x)\big) = 0$ then $v^\varepsilon(t, y) = u^\varepsilon(t, y + \varepsilon\omega)$ satisfies $\frac{\partial v^\varepsilon}{\partial t}(t, x) + K\big(\frac{y}{\varepsilon}, \frac{\partial v^\varepsilon}{\partial y}(t, y)\big) = 0$. Thus $u^\varepsilon(t, y) = v^\varepsilon(t, y - \varepsilon\omega)$, and convergence of v^ε to $\bar{v}$ as ε goes to 0 is equivalent to convergence of u^ε to $\bar{u} = \bar{v}$. See the proof of Corollary 1.7 for another method of reducing to the periodic case.

2015; Gao 2016]), nothing is known for viscosity solutions in the general (i.e., for H nonconvex in p) case, and counterexamples have been found, first by Ziliotto [2017] and then by Feldman and Souganidis [2017].

We settle here the case of variational solutions without any convexity assumption. Note that the construction of a variational solution relies on the choice of a field of coefficients for the homology theory we use, but once the field is chosen, the variational solution is uniquely defined.[2] We shall here fix once and for all a coefficient field (the reader can think of $\mathbb{Z}/2\mathbb{Z}$ or $\mathbb{R}$ for example). As in [Viterbo 2023], our results hold when H is either compactly supported or coercive in the p-direction. Note that fixing ω, if $V_t(H)f = u(t,x)$ is the variational solution operator[3] of the Hamilton–Jacobi equation, and $S_t(H)f$ is the viscosity semigroup, we know that for H convex in p we have $S_t(H) = V_t(H)$ [Zhukovskaya 1993; 1996]. Our result thus implies the stochastic homogenization for viscosity solutions in the convex case[4] as in [Rezakhanlou and Tarver 2000; Souganidis 1999]. In the general case it has been proved in [Wei 2013; 2014] (see also [Roos 2017, Theorem 1.19]) that

$$S_t(H) = \lim_{n\to+\infty} (V_{t/n}(H))^n.$$

Since there are counterexamples in the nonconvex case, stochastic homogenization of the viscosity solutions cannot hold in general.[5]

Of course, as in [Viterbo 2023], the equation (HJS$_\varepsilon$) is related to the Hamiltonian flow of $H\big(\frac{x}{\varepsilon}, p; \omega\big)$ given by

$$\varphi^t_{\varepsilon,\omega} = \rho_\varepsilon^{-1} \varphi_\omega^{t/\varepsilon} \rho_\varepsilon,$$

where φ^t_ω is the flow of $H(x,p;\omega)$ and $\rho_\varepsilon(x,p) = \big(\frac{x}{\varepsilon}, p\big)$.

We shall prove analogously to [Viterbo 2023] that, for almost all ω, we have

$$\varphi^t_{\varepsilon,\omega} \xrightarrow{\gamma_c} \bar{\varphi}^t_\omega,$$

but since we are on a noncompact base we have to redefine the γ-distance, which we shall denote by γ_c.

1.1. ***Statement of the main results.*** Our main result is:

Theorem 1.1 (Main Theorem). *Let $H(x,p;\omega)$ be a stochastic Hamiltonian on $T^*\mathbb{R}^n \times \Omega$, where (Ω,μ) is a probability space endowed with an action τ of $\mathbb{R}^n$. We assume the following conditions are satisfied*:

(1) *For all $a \in \mathbb{R}^n$, the map τ_a is measure-preserving and the action τ is ergodic for the measure μ (i.e., invariant sets have measure* 0 *or* 1).

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[2] See for example [Cardin and Viterbo 2008] and more explicitly [Wei 2014] and Appendix B in [Roos 2019].

[3] This means that it sends f to the variational solution of

$$\begin{cases} \frac{\partial u}{\partial t}(t,x) + H\big(x, \frac{\partial u}{\partial x}(t,x)\big) = 0, \\ u(0,x) = f(x). \end{cases} \tag{HJS}$$

Note that the operator is not a semigroup (since variational solutions do not have the Markov property).

[4] However in that case our method is much more complicated.

[5] Of course if in some cases we knew that $V_t(\varepsilon) = V_t(H_\varepsilon) = \overline{V}_t + R_t(\varepsilon)$, where $\|R_t(\varepsilon)\| \le Ct\varepsilon$, and $\overline{V}_t = V_t(\overline{H})$ is the homogenized operator, we would get that $\|(V_{t/n}(\varepsilon))^n - (\overline{V}_{t/n})^n\| \le Ct\varepsilon$. Hence, setting $\overline{S}_t = \lim_n (\overline{V}_{t/n})^n$, we would have $\|S^t(\varepsilon) - \overline{S}_t\| \le Ct\varepsilon$ and then $\lim_{\varepsilon\to 0} S^t_0(\varepsilon) = \overline{S}^t_0$.

(2) *We have, for all* $a \in \mathbb{R}^n$, $(x,p) \in T^*\mathbb{R}^n$ *and almost all* $\omega \in \Omega$, *the identity* $H(x+a,p,\tau_a\omega) = H(x,p,\omega)$.

(3) *The map* $(x,p) \mapsto H(x,p,\omega)$ *is* $C^{1,1}$ *for* μ*-almost all* ω.

(4) *For almost all* ω, H *is compactly supported in the* p*-direction, i.e., the set*

$$\{p \mid \exists x \in \mathbb{R}^n,\ H(x,p;\omega) \neq 0\}$$

is bounded.

(5) *There exists* C *such that for almost all* ω *and for all* $(x,p) \in T^*\mathbb{R}^n$ *we have* $\left|\frac{\partial H}{\partial p}(x,p;\omega)\right| \leq C$.

(6) *There exists* C *such that for almost all* ω *we have* $\sup_{(x,p)\in T^*\mathbb{R}^n} |H(x,p;\omega)| \leq C$.

Then if $\varphi_{\varepsilon,\omega}$ *is the flow of* $H_{\varepsilon,\omega}(x,p) = H\left(\frac{x}{\varepsilon},p;\omega\right)$ *there is a function* $\overline{H}$ *in* $C^0(\mathbb{R}^n,\mathbb{R})$ *such that*

$$\varphi^t_{\varepsilon,\omega} \xrightarrow{\gamma_c} \bar{\varphi}^t_\omega$$

for the topology γ_c *that will be defined in Section 4. Here* $\varphi^t_{\overline{H}}$ *denotes the flow of* $\overline{H}$ *in* $\widehat{\mathfrak{DHam}}(T^*\mathbb{R}^n)$, *the* γ_c*-completion of* $\mathfrak{DHam}(T^*\mathbb{R}^n)$. *As a consequence if* f *is uniformly continuous on* $\mathbb{R}^n$, *then a.s. in* $\omega \in \Omega$ *the variational solution* $u^\varepsilon(t,x;\omega)$ *of* (HJS$_\varepsilon$) *converges to the variational solution* $\bar{u}(t,x)$ *of* (HJH).

Let us try to give some intuition for the γ_c metric. The γ_c metric is a version, in the noncompact case, of the γ-metric first defined in [Viterbo 1992]. For a compact base, it is easier to describe it on Lagrangians. For example if L_k is the graph of df_k and f_k C^0-converges to a smooth function f_∞, then L_k converges to L_∞, the graph of df_∞. For this reason, the γ-metric is often called a C^{-1}-metric. However, as is quite natural, the γ-completion of the set of smooth Lagrangians contains more objects and in particular contains the graphs of continuous functions. For Hamiltonians maps, if φ_k is the time-one flow of the Hamiltonian H_k and $(H_k)_{k\geq 1}$ C^0-converges to H_∞, then φ_k γ-converges to φ_∞, the time-one flow of H_∞. Here again the time-one flow of a C^0 Hamiltonian is well-defined in the completion (see [Viterbo 1992; 2006; Humilière 2008]).

Remarks 1.2. (1) Existence and uniqueness of the variational solution for (HJS$_\varepsilon$) follows from [Cardin and Viterbo 2008, pp. 266–276] (since we are in the case of a noncompact base). The *bounded propagation speed* condition in [loc. cit.] is more general than the one in the present paper and is obviously satisfied in the fiberwise compactly supported case.

(2) By ergodicity, each of the conditions (4), (5), (6) either holds a.s. or fails a.s. Indeed, set

$$\Omega_c = \left\{\omega \in \Omega \mid \sup_{(x,p)\in T^*\mathbb{R}^n} |H(x,p;\omega)| \geq c\right\}.$$

This set is τ-invariant. If for some c this set has measure 0, then (6) holds; otherwise

$$\sup_{(x,p)\in T^*\mathbb{R}^n} |H(x,p;\omega)| = +\infty$$

for a.e. ω. Similarly, the set $\Omega'_R = \{\omega \in \Omega \mid \operatorname{supp}(H) \subset \mathbb{R}^n \times B(R)\}$ is also invariant by τ. It thus either has measure 1 for some R, and then the bound in (4) is independent from ω in a set of full measure, or it

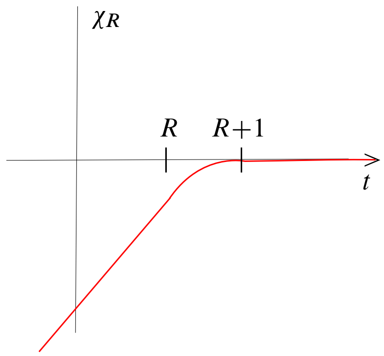

Figure 1. Graph of χ_R.

has measure 0 for all R and then, for a.e. ω, condition (4) is violated. In the first case, we shall say that the H_ω have *uniform fiber compact support*. This is assumption (4) of the Main Theorem.

(3) Let us compare our results to those of [Rezakhanlou and Tarver 2000; Souganidis 1999]. Note that if H is convex in p, then viscosity and variational solutions coincide. So consider a Hamiltonian H convex in p and uniformly coercive. In the ergodic case this implies that there exist functions $h_\pm(p)$ going to infinity such that $h_-(p) \le H(x, p; \omega) \le h_+(p)$ (this also follows from the assumptions in both [Rezakhanlou and Tarver 2000, (2.5)(ii) and (2.8), p. 280] and [Souganidis 1999, Condition 0.2]). Note that both authors assume $\lim_{|p|\to+\infty} h_\pm(p)/|p| = +\infty$, an assumption we do not require here.

Then we claim that the truncation $H_{\chi_R} = \chi_R(H)$, where χ_R is the function represented in Figure 1, satisfies assumption (5) of the Main Theorem (condition (4) is obvious) or equivalently, (2a) of the corollary. This is because

$$\frac{\partial H_{\chi_R}}{\partial p} = \chi_R'(H)\frac{\partial H}{\partial p},$$

so it is enough to prove that $\frac{\partial H}{\partial p}$ is bounded on a set $|p| \le C$. But if $\left|\frac{\partial H}{\partial p}(x_0, p_0)\right| \ge A$, we can find p_1 with $|p_1| \le 2C$ such that $p_0 - p_1$ is colinear with $\frac{\partial H}{\partial p}(x_0, p_0)$ and $|p_0 - p_1| = C$, so that

$$\sup_{|p|\le 2C} h_+(p) - \inf_{|p|\le 2C} h_-(p) \ge H(x, p_1) - H(x, p_0) \ge \left\langle \frac{\partial H}{\partial p}, p_0 - p_1 \right\rangle \ge C\left|\frac{\partial H}{\partial p}\right| = CA;$$

hence A is bounded.

The compactly supported case is usually not the most interesting in applications. However the above theorem implies

Corollary 1.3 (Main Corollary). *Let $H(x, p; \omega)$ be a stochastic Hamiltonian on $T^*\mathbb{R}^n \times \Omega$, where (Ω, μ) is a probability space endowed with an action τ of $\mathbb{R}^n$. We assume the following conditions are satisfied*:

(1a) *Conditions (1)–(3) as in the Main Theorem.*

(2a) *For all $(x, p; \omega)$ we have $\left|\frac{\partial H}{\partial p}(x, p; \omega)\right| \le h^1(|p|)$ for almost all ω for some continuous function $h^1 : \mathbb{R} \to \mathbb{R}$.*

(3a) *For almost all ω, H is coercive, that is $\lim_{|p|\to+\infty} |H(x, p; \omega)| = +\infty$ uniformly in x.*

If H satisfies the above assumptions and f is Lipschitz on $\mathbb{R}^n$, there is a coercive function $\overline{H}$ in $C^0(\mathbb{R}^n, \mathbb{R})$ such that a.e. in ω the variational solution $u^\varepsilon(t,x;\omega)$ of

$$\begin{cases} \dfrac{\partial u^\varepsilon}{\partial t}(t,x;\omega) + H\Big(\dfrac{x}{\varepsilon}, \dfrac{\partial u^\varepsilon}{\partial x}(t,x;\omega);\omega\Big) = 0, \\ u^\varepsilon(0,x;\omega) = f(x) \end{cases} \tag{HJS$_\varepsilon$}$$

converges to the variational solution $\bar{u}(t,x)$ of

$$\begin{cases} \dfrac{\partial v}{\partial t}(t,x) + \overline{H}\Big(\dfrac{\partial v}{\partial x}(t,x)\Big) = 0, \\ v(0,x) = f(x). \end{cases} \tag{HJH}$$

Remark 1.4. We shall reduce the case (3a) where H is coercive to the uniformly fiberwise compactly supported case by replacing H by $\chi_R(H)$, which is compactly supported where $\chi_R : \mathbb{R} \to \mathbb{R}$ is a function supported in $]-\infty, R+1]$ such that $\chi'(t) = 1$ for $t \le R$ (see [Cardin and Viterbo 2008, Appendix B]). Then $H_{\chi_R} = \chi_R(H)$ also satisfies $H_{\chi_R}(x+a, p; \tau_a\omega) = H_{\chi_R}(x, p; \omega)$.

Examples 1.5. (1) Let Ω be the space of C^1 functions on $\mathbb{R}^n$, $(\tau_a f)(x) = f(x+a)$ and μ be some measure on Ω invariant by τ_a and ergodic. Let V be a bounded function. Set $H(x,p;\omega) = \frac{1}{2}h(p) - V(\omega(x))$, where h is coercive. This satisfies the assumptions of the corollary and corresponds to a random potential, with probability μ.

(2) [Pelayo and Rezakhanlou 2018, Example 2.4(ii)] Let $H_0(q,p)$ be a Hamiltonian and $H(q,p;\omega) = \sum_{j\in\mathbb{Z}} H_0(q-q_j, p)$, where $\omega = (q_j)_{j\in\mathbb{Z}}$ is a stationary point process, that is, a probability on $\mathbb{R}^{\mathbb{Z}}$ invariant by translation. This makes sense provided H_0 decreases fast enough as q goes to infinity. Then H satisfies the assumption of the above corollary.

Remark 1.6. Here are a few comments:

(1) We could of course also state a convergence result in the coercive case for the sequence $\varphi_{\varepsilon,\omega}$; it is just that the statement of convergence would be a little more complicated to state.

(2) By ergodicity there exist $h_+(p) \in \mathbb{R} \cup \{+\infty\}$ and $h_-(p) \in \mathbb{R} \cup \{-\infty\}$ such that $\sup_{x\in\mathbb{R}^n} H(x,p;\omega) = h_+(p)$ a.e. in Ω and similarly $\inf_{x\in\mathbb{R}^n} H(x,p;\omega) = h_-(p)$ a.e. in Ω. Notice that (3a) implies that $h_\pm(p)$ is finite, and that $\lim_{|p|\to+\infty} h_\pm(p) = +\infty$. This condition is more or less explicit in both [Rezakhanlou and Tarver 2000, conditions (Aii)–(Aiii)] and [Souganidis 1999, Condition 0.2]. Similarly

$$h_+^1(p,\omega) = \sup_{x\in\mathbb{R}^n} \Big|\frac{\partial H}{\partial p}(x,p;\omega)\Big|$$

is invariant by τ, and hence independent from ω a.e. in Ω, and equal to $h_+^1(p)$, so (2a) and (5) either hold a.s. or do not hold a.s in Ω.

(3) Again by ergodicity, the coercivity is necessarily uniform: one has a function $f(r)$ such that $\lim_{r\to+\infty} f(r) = +\infty$ and for all $(x,p;\omega)$ we have $|H(x,p;\omega)| \ge f(|p|)$.

(4) Let us consider a Hamiltonian H convex in p and uniformly coercive. In the ergodic case this implies that there exist functions $h_\pm(p)$ going to infinity such that $h_-(p) \le H(x,p;\omega) \le h_+(p)$ (this also follows

from the assumptions in both [Rezakhanlou and Tarver 2000, (2.5)(ii) and (2.8), p. 280] and [Souganidis 1999, Condition 0.2]). Then we claim that its truncation $H_{\chi_R} = \chi_R(H)$ satisfies assumption (5) of the Main Theorem (condition (4) is obvious) or equivalently, (2a) of the corollary. This is because

$$\frac{\partial H_{\chi_R}}{\partial p} = \chi_R'(H)\frac{\partial H}{\partial p},$$

so it is enough to prove that $\frac{\partial H}{\partial p}$ is bounded on a set $|p| \le C$. But if $\left|\frac{\partial H}{\partial p}(x_0, p_0)\right| \ge A$, we can find p_1 with $|p_1| \le 2C$ such that $p_0 - p_1$ is colinear with $\frac{\partial H}{\partial p}(x_0, p_0)$ and $|p_0 - p_1| = C$, so that

$$\sup_{|p|\le 2C} h_+(p) - \inf_{|p|\le 2C} h_-(p) \ge H(x, p_1) - H(x, p_0) \ge \left\langle \frac{\partial H}{\partial p}, p_0 - p_1 \right\rangle \ge C\left|\frac{\partial H}{\partial p}\right| = CA;$$

hence A is bounded.

Our result can be easily extended, since we do not need the full action of $\mathbb{R}^n$. For example if we have an action of $\mathbb{Z}^n$ we get the following:

Corollary 1.7. *Take the same assumptions as in the Main Theorem except that we have an action of $\mathbb{Z}^n$ (instead of $\mathbb{R}^n$) on Ω, still denoted by τ, and the first two assumptions are replaced by*:

(1b) *For all $z \in \mathbb{Z}^n$, the map τ_z is measure-preserving and ergodic.*

(2b) *We have, for all $z \in \mathbb{Z}^n$, $(x, p) \in T^*\mathbb{R}^n$ and almost all $\omega \in \Omega$, the identity*

$$H(x + z, p, \tau_z\omega) = H(x, p, \omega),$$

while conditions (3)–(6) are unchanged. We then have the same conclusion as in the Main Theorem.

Finally, note that ergodicity of τ on Ω is not required, since we can use the ergodic decomposition theorem (see [Greschonig and Schmidt 2000]), which holds for Borel spaces[6] and obtain:

Corollary 1.8. *With the same assumptions as in the Main Theorem (resp. Corollary 1.7) except that the action τ is not supposed to be ergodic but we assume (Ω, μ) is a Borel space, we have the same conclusion, except that $\overline{H}(p; \omega)$ now depends on $\omega \in \Omega$ and is constant on each ergodic component of τ.*

1.2. ***Sketch of the proof of the Main Theorem.*** Our proof will require the following steps, starting from the uniformly fiber compactly supported case:

(1) On $\mathfrak{Ham}_{\rm fc}(T^*\mathbb{R}^n)$, the set of uniformly fiberwise compactly supported Hamiltonians on $T^*\mathbb{R}^n$, we define a metric γ_c (see Sections 3 and 4).

(2) We identify Ω to $\mathfrak{H}_\Omega$ the set of H_ω for $\omega \in \Omega$, and $\widehat{\mathfrak{H}}_\Omega$ its completion for γ_c. We then prove that ergodicity implies compactness of the metric space $(\widehat{\mathfrak{H}}_\Omega, \gamma_c)$ (see Sections 5 and 6). The action of $\mathbb{R}^n$ on $\mathfrak{H}_\Omega$ given by $(\tau_a H)(x, p; \omega) = H(x - a, p; \omega) = H(x, p; \tau_a\omega)$ extends to an action of a compact connected metric abelian group $\mathbb{A}_\Omega$ on $(\widehat{\mathfrak{H}}_\Omega, \gamma_c)$, and $\mathbb{R}^n$, through the action τ, is identified to a dense subgroup of $\mathbb{A}_\Omega$. Moreover we prove that for μ-almost all H in $\mathfrak{H}_\Omega$, the $\mathbb{A}_\Omega$ orbit of H is equal to $\widehat{\mathfrak{H}}_\Omega$.

[6] That is, isomorphic (as a measured space) to a complete separable metric space with a measure defined on its Borel algebra.

(3) In Section 7 we prove a regularization theorem showing that the action of $\mathbb{A}_\Omega$ on $\widehat{\mathfrak{H}}_\Omega$ can be approximated by an action of a finite-dimensional torus (note that $\mathbb{A}_\Omega$ is not in general a finite-dimensional torus, but is a projective limit of finite-dimensional tori).

(4) We prove in Section 8 that homogenization holds when $\mathbb{A}_\Omega$ is a finite-dimensional torus (the quasiperiodic case) and $\omega \mapsto H_\omega$ is continuous for the C^0-topology instead of the γ_c-topology.

(5) In Section 10 we conclude the proof in the fiberwise compact case, and in Section 11 for the coercive case and in Section 12 for the discrete case.

2. Notation and abbreviations

- Ω is a probability space with measure μ.
- a.s. or a.e. mean almost surely or almost everywhere in (Ω, μ).
- GFQI means "generating function quadratic at infinity".
- H^*, H_* are, respectively, cohomology and homology (either Čech or singular) with coefficients in some field $\mathbb{K}$.
- μ_N is the fundamental class in $H^d(N)$ (for a closed manifold) or $H^d(N, \partial N)$ (for a manifold with boundary) or $H_c^d(N)$ (for a noncompact manifold), where $d = \dim(N)$. When N is nonorientable, it is assumed that $\mathbb{K} = \mathbb{F}_2 = \mathbb{Z}/2\mathbb{Z}$.
- 1_N is the generator of $H^0(N)$.
- T^*N is the cotangent bundle of N with the standard symplectic form $\omega = d\lambda$, where $\lambda = p\,dq$.
- $\overline{T^*N}$ is the cotangent bundle of N with the opposite of the standard symplectic form $\omega = -d\lambda$, where $\lambda = p\,dq$.
- 0_N is the zero section of T^*N.
- $\mathfrak{Ham}_{\mathrm{fc}}(T^*N)$ is the set of smooth uniformly fiberwise compactly supported[7] autonomous Hamiltonians.
- $\mathfrak{Ham}_{\mathrm{fc}}([0,1] \times T^*N)$ is the set of smooth uniformly fiberwise compactly supported time-dependent Hamiltonians.
- $C^0_{\mathrm{fc}}([0,1] \times T^*N)$ is set of continuous functions on $[0,1] \times T^*N$ (viewed as "continuous Hamiltonians") which are fiberwise compact.
- For a Hamiltonian H on T^*N, $X_H(t,z)$ is the Hamiltonian vector field associated to H, defined by $\omega(X_H(t,z)) = -d_z H(t,z)$.
- For a Hamiltonian H on T^*N, φ_H^t is the solution of $\frac{d}{dt}\varphi_H^t(z) = X_H(t, \varphi_H^t(z))$ such that $\varphi_H^0(z) = z$. We set $\varphi_H = \varphi_H^1$.
- $\mathfrak{DHam}_{\mathrm{fc}}(T^*N)$ is the image by $H \mapsto \varphi_H$ of $\mathfrak{Ham}_{\mathrm{fc}}([0,1] \times T^*N)$.
- FPS means "finite propagation speed" (see Definition 3.1).

[7]That is, the support is contained in $\mathbb{R}^n \times B(R)$ for some R.

- BPS means “bounded propagation speed” (see Definition 3.8).
- $\mathfrak{DHam}_{\mathrm{FP}}(T^*N)$ (resp. $\mathfrak{Ham}_{\mathrm{FP}}(T^*N)$ or $\mathfrak{Ham}_{\mathrm{FP}}([0,1]\times T^*N)$) is the set of elements in $\mathfrak{DHam}(T^*N)$ (resp. $\mathfrak{Ham}(T^*N)$ or $\mathfrak{Ham}([0,1]\times T^*N)$) having FPS.
- $\mathfrak{DHam}_{\mathrm{BP}}(T^*N)$ (resp. $\mathfrak{Ham}_{\mathrm{BP}}(T^*N)$ or $\mathfrak{Ham}_{\mathrm{BP}}([0,1]\times T^*N)$) is the set of elements in $\mathfrak{DHam}(T^*N)$ (resp. $\mathfrak{Ham}(T^*N)$ or $\mathfrak{Ham}([0,1]\times T^*N)$) having BPS.
- $\mathfrak{L}(T^*N)$ is the set of pairs (L, f_L), where L is the image of 0_N by some element $\varphi\in\mathfrak{DHam}_{\mathrm{FP}}(T^*N)$ and f_L is a primitive of $\lambda_{|L}$. We often just write L if f_L is implicit.
- γ_c is the uniform topology on $\mathfrak{L}(T^*N)$ (see Definition 4.17).
- $\widehat{\mathfrak{L}}(T^*N)$ is the completion for γ_c of $\mathfrak{L}(T^*N)$ (see Definition 4.17).
- $\widehat{\mathfrak{DHam}}_{\mathrm{FP}}(T^*N)$ (resp. $\widehat{\mathfrak{DHam}}_{\mathrm{BP}}(T^*N)$ or $\widehat{\mathfrak{DHam}}_{\mathrm{fc}}(T^*N)$) is the completion for γ_c of $\mathfrak{DHam}_{\mathrm{FP}}(T^*N)$ (resp. $\mathfrak{DHam}_{\mathrm{BP}}(T^*N)$ or $\mathfrak{DHam}_{\mathrm{fc}}(T^*N)$) (see Definition 4.24)
- G_f is the graph of df in T^*N.
- $\bar{L}$: For $L\in\mathfrak{L}(T^*N)$ we define $\bar{L}=\{(x,-p)\mid(x,p)\in L\}$, where $f_{\bar{L}}=-f_L$.

3. Noncompactly supported Hamiltonians

Let N be a noncompact manifold. We shall assume that N is homeomorphic to the interior of a compact manifold with smooth boundary.[8]

Definition 3.1. Let $\varphi\in\mathfrak{DHam}(T^*N)$. We say that φ has *finite propagation speed* (FPS for short) if, for each bounded set U, there is a bounded set V such that $\varphi(T^*U)\subset T^*V$. A subset in $\mathfrak{DHam}(T^*N)$ has *uniformly finite propagation speed* if each element has finite propagation speed, and moreover, given U, the set V can be chosen to be the same for all the elements in the subset. We write $\mathfrak{DHam}_{\mathrm{FP}}(T^*N)$ for the set of Hamiltonian maps with finite propagation speed. By abuse of language, we use the same terminology in $\mathfrak{Ham}(T^*N)$: H has *finite propagation speed* if φ_H has finite propagation speed, etc. We use the notation $\mathfrak{Ham}_{\mathrm{FP}}(T^*N)$ for this set.

Note that for instance if $\left|\frac{\partial H}{\partial p}(t,q,p)\right|\le C_U$ for all $(q,p)\in T^*U$ then H has FPS.

The following lemma will prove useful.

Lemma 3.2. *Let $U\subset V$ be relatively compact open sets in N such that for any compact set K in N there exists an isotopy of N sending K in V. Let $\varphi\in\mathfrak{DHam}(T^*N)$ be such that $\varphi(T^*U)\subset T^*V$. Then we can find a Hamiltonian isotopy $(\varphi^t)_{t\in[0,1]}$ from the identity to φ such that for all $t\in[0,1]$ we have $\varphi^t(T^*U)\subset T^*V$.*

Proof. Let ψ^t be an isotopy from id to $\psi^1=\varphi$. Let X be a vector field corresponding to the isotopy for a compact set containing the projection of $\bigcup_{t\in[0,1]}\psi^t(U)=K$ and pointing inwards on ∂V. Let ρ^t be the Hamiltonian vector field of $H(t,x,p)=\langle p, X(t,x)\rangle$ which projects on the flow of X. Possibly replacing ρ^t by a $\rho^{\alpha(t)}$, we may assume that for all $t\in[0,1]$ we have $\rho^t\circ\psi^t(T^*U)\subset T^*V$. Then

[8] We eventually only use the case $N=\mathbb{R}^n$. For this section we actually only need that there is an exhausting sequence of open bounded sets $(U_j)_{j\in\mathbb{N}}$ such that $U_j\subset U_{j+1}$ and, for j large enough, U_j is ambient isotopic to U_{j+1}.

$\rho^1\psi^1(T^*U) \subset T^*V$ and, since $\psi^1(T^*U) \subset T^*V$, the set of t such that $\rho^t\psi^1(T^*U) \subset T^*V$ is an interval — because X points inward on ∂V — it must contain [0, 1]; hence concatenating the Hamiltonian isotopy $t \mapsto \rho^t\psi^t$ with $t \mapsto \rho^{1-t}\psi^1$, we get a new Hamiltonian isotopy that we denote by φ^t such that $\varphi^t(T^*U) \subset T^*V$ for all $t \in [0, 1]$. □

Note that our hypothesis on N implies that we can find an exhausting sequence $(U_j)_{j\geq 1}$ of N satisfying the assumptions of Lemma 3.2.

We shall now prove that $\mathfrak{DHam}_{\mathrm{fc}}$, the set of Hamiltonians which are uniformly fiberwise compactly supported, is contained in $\mathfrak{Ham}_{\mathrm{FP}}$.

Proposition 3.3. *If $H \in \mathfrak{Ham}_{\mathrm{fc}}(T^*N)$ is uniformly fiberwise compactly supported, then H has FPS.*

Proof. Indeed, if for some C, φ is the identity outside of $DT^*_C(N) = \{(q, p) \mid |p| \leq C\}$, then $\varphi(T^*U) \subset T^*U \cup \varphi(T^*U \cap T^*_C N)$, but since $T^*U \cap T^*_C N$ is compact, its image is contained in some T^*V for V bounded, and we get $\varphi(T^*U) \subset T^*(U \cup V)$. □

The usefulness of this notion will be clear on several occasions. Remember that a generating function quadratic at infinity for (L, f_L), where L is a smooth Lagrangian and f_L a function such that of $df_L = \lambda_{|L}$, is a smooth function $S : E = N \times F \to \mathbb{R}$, where F is a finite-dimensional vector space,[9] such that

(1) $S(x, \xi)$ coincides with a nondegenerate quadratic form Q on the vector space F for ξ large enough,

(2) $(x, \xi) \mapsto \frac{\partial S}{\partial \xi}(x, \xi)$ is transverse to 0,

(3) setting $\Sigma_S = \left\{(x, \xi) \mid \frac{\partial S}{\partial \xi}(x, \xi)\right\}$ the image of this submanifold by $i_S : (x, \xi) \mapsto \frac{\partial S}{\partial x}(x, \xi)$ has image L,

(4) $f_L \circ i_S = S$.

Let S_1, S_2 be two GFQI. They are said to be equivalent if they are fiberwise diffeomorphic after stabilization, that is, there are two nondegenerate quadratic forms q_1, q_2 such that if

$$\widetilde{S}_j(x, \xi_j, \eta_j) = S_j(x, \xi_j) + q_j(\eta_j),$$

there is a fiber-preserving diffeomorphism

$$(x, \xi_1, \eta_1) \to (x, \xi_2(x, \xi_1, \eta_1), \eta_2(x, \xi_1, \eta_1))$$

such that

$$S_2(x, \xi_2(x, \xi_1, \eta_1), \eta_2(x, \xi_1, \eta_1)) = S_1(x, \xi_1, \eta_1).$$

We shall say that S_1, S_2 are equivalent over U if the fiber-preserving diffeomorphism is defined for $x \in U$. Note that the customary "addition of a constant" for the equivalence of generating functions is not needed here, since generating functions are normalized so that $S_{|\Sigma_S} = f_L \circ i_S$.

We cannot expect a noncompact Lagrangian to have a GFQI in this sense, since the number of variables required could go to infinity. We can either assume F is a Hilbert space, but then positive and

[9] All this discussion also works if we replace $N \times F$ by a general finite-dimensional vector bundle. Then we must replace in the sequel the Künneth isomorphism by the Thom isomorphism.

negative eigenspaces will generally be infinite-dimensional so that $H^*(S^b, S^a) = 0$, which is a notorious drawback.[10] Here we have:

Definition 3.4. We say that a Lagrangian $L \subset T^*N$ has a GFQI if, for each bounded set U, there is a GFQI defined over $U \times F$ (where F depends on U), S_U, and a set $V \supset U$ such that the S_W are all equivalent over U for $W \supset V$. Two GFQI are equivalent if they are equivalent over each bounded set.

Theorem 3.5. *Let φ be an element in $\mathfrak{DHam}_{\mathrm{FP}}(T^*N)$. Then $\varphi(0_N)$ has a GFQI. Moreover such a GFQI is unique up to equivalence.*

Proof. See Appendix A. □

Remarks 3.6. Notice that

(1) If φ does not have FPS, $\varphi(0_N)$ does not even need to have surjective projection on N: For example take on $T^*\mathbb{R}$ the Hamiltonian $\frac{\pi}{4}(x^2 + p^2)$. Then $\varphi(0_{\mathbb{R}}) = \{0\} \times \mathbb{R}$!

(2) Using Lemma 3.2 we may assume we have a sequence U_ν of domains such that for all $t \in [0, 1]$ we have $\varphi^t(T^*U_\nu) \subset T^*U_{\nu+1}$. We let $S_\nu = S_{U_\nu}$ and notice that we may assume that the restriction of S_μ over U_ν is exactly $S_\nu \oplus q_{\nu,\mu}$ by composing S_μ with an extension of the fiber-preserving diffeomorphism realizing the equivalence.[11] We shall always make this assumption in the sequel.

(3) We will use the expression "S is a GFQI for L" meaning "there is a sequence $(S_\nu)_{\nu\geq 1}$ of GFQI for L over U_ν" to avoid cumbersome indexes. Most of the time this means we consider S_ν for ν large enough.

Definition 3.7. We denote by $\mathfrak{L}(T^*N)$ the set of Lagrangians of the type $\varphi(0_N)$, where $\varphi \in \mathfrak{DHam}_{\mathrm{FP}}(T^*N)$.

On a Riemannian manifold, there is a more precise notion than FPS.

Definition 3.8. Let N be a manifold with a distance d and $\varphi \in \mathfrak{DHam}(T^*N)$. We say that φ has *bounded propagation speed* (BPS for short) if there is a constant r_0 such that for any ball $B(x_0, r)$ we have $\varphi(T^*B(x_0, r)) \subset T^*B(x_0, r + r_0)$. A subset in $\mathfrak{DHam}(T^*N)$ has *uniformly bounded propagation speed* if each element has bounded propagation speed, and moreover the constant r_0 can be chosen to be the same for all the elements in the subset. We write $\mathfrak{DHam}_{\mathrm{BP}}(T^*N)$ for the set of Hamiltonians maps with bounded propagation speed. By abuse of language, we use the same terminology in $\mathfrak{Ham}(T^*N)$: H has *bounded propagation speed* if φ_H has bounded propagation speed.

Example 3.9. If $\left|\frac{\partial H}{\partial p}(t, q, p)\right| \leq C$ for all $(q, p) \in T^*\mathbb{R}^n$ then H has BPS. In particular assumption (5) implies BPS.

Remark 3.10. (1) Of course bounded propagation speed implies finite propagation speed.

(2) Our definition of finite propagation speed does not exactly coincide with the terminology of [Cardin and Viterbo 2008, Definition B.5, p. 271]. Our definition is more involved and the notion of finite propagation speed defined there is weaker than the present one, but would still be sufficient to prove our theorems. However this would have made an already long paper even longer.

[10] That we could avoid by using Floer homology everywhere, but would make reading this paper even harder for the Hamilton–Jacobi community!

[11] The existence of the extension follows from the fact that we may assume that, for μ, ν large enough, the inclusion $U_\nu \subset U_\mu$ is a homotopy equivalence.

4. Spectral invariants in cotangent bundles of noncompact manifolds

The goal of this section is to define and state the main properties of the metric γ that occurs in the statement of the Main Theorem. This has been done in [Viterbo 1992] in the case of a compact base; the present situation, for a noncompact base, is unfortunately slightly more involved. Even though we work on a general noncompact manifold, the reader can assume that $N = \mathbb{R}^n$. The general case will turn out to be useful for future applications, and the only extra difficulty is visual.

4.1. ***The case of Lagrangians.*** Let L be an exact Lagrangian in T^*N with N not necessarily compact (but assumed, for simplicity, to be connected). We assume a primitive of $\lambda_{|L}$, f_L, is given.[12]

We shall assume that L has a unique GFQI, S, such[13] that $f_L = S$ on L $\big($through the identification $i_S(x,\xi) = \big(x, \frac{\partial S}{\partial \xi}(x,\xi)\big)\big)$. For example according to Theorem 3.5, this is the case if $L = \varphi_H(0_N)$ with $\varphi \in \mathfrak{DHam}_{\mathrm{FP}}(T^*N)$. Note that in general, S_U, Q, F depend on U.

We denote by T_F the generator of $H^i(D(F^-), S(F^-))$, where F^- is the negative eigenspace of Q, $i = \dim(F^-)$ and $D(F^-), S(F^-)$ are respectively the disc and sphere in F^-, so that $\alpha \mapsto \alpha \otimes T_F$ is an isomorphism (the Künneth isomorphism) from $H^*(U)$ to

$$H^{*+i}(U \times D(F^-), U \times S(F^-)) = H^*(U) \otimes H^*(D(F^-), S(F^-))$$

for $U \subset N$. By abuse of language we again denote by T_F its homological counterpart in $H_i(D(F^-), S(F^-))$. We shall later write T instead of T_F.

We denote by $S_U^t = \{(x,\xi) \in U \times F \mid S(x,\xi) \le t\}$ (we omit the subscript for $U = N$) and $S_U^{-\infty}$ (resp. $S_U^{+\infty}$) any of the S_U^{-c} (resp. S^c) for c large enough (by Morse's lemma they are all isotopic).

Classically we have a homotopy equivalence between $(S_U^{+\infty}, S_U^{-\infty})$ and $U \times (D(F^-), S(F^-))$. In the following definitions, we set $\mu_U \in H^n(U, \partial U)$, $1_U \in H^0(U)$ to be the generators of these cohomology groups.

Definitions 4.1. Let S be a GFQI for $L \in \mathfrak{L}(T^*N)$ and U a bounded open set with smooth boundary. We define:

(1) For $\alpha \in H^*(U)$,

$$c(\alpha, S) = \inf\{t \mid T \otimes \alpha \neq 0 \text{ in } H^*(S_{|U}^t, S_{|U}^{-\infty})\}.$$

(2) For $a \in H_*(U, \partial U)$,

$$c(a, S) = \inf\{t \mid T \otimes a \text{ is in the image of } H_*(S_{|U}^t, S_{|U}^{-\infty} \cup S_{|\partial U}^t)\}.$$

(3) For $\alpha \in H_c^*(U) = H^*(U, \partial U)$,

$$c(\alpha, S) = \inf\{t \mid T \otimes \alpha \neq 0 \text{ in } H^*(S_{|U}^t, S_{|U}^{-\infty} \cup S_{|\partial U}^t)\}.$$

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[12] Even though we write L, we always mean the pair (L, f_L).

[13] Remember by Remarks 3.6(3) that this means there is a sequence S_ν of GFQI over U_ν such that, for $\nu \le \mu$, the function S_μ restricts to the stabilization of S_ν over U_ν.

(4) For $a \in H_*(U)$,

$$c(a,S) = \inf\{t \mid T \otimes a \text{ is in the image of } H_*(S^t_{|U}, S^{-\infty}_{|U})\}.$$

(5) For L_1, L_2 in $\mathfrak{L}(T^*N)$, having unique GFQI, S_1, S_2, we set $(S_1 \ominus S_2)(x;\xi,\eta) = S_1(x;\xi) - S_2(x;\eta)$ and, for $\alpha \in H^*(U)$ or $H^*(U, \partial U)$,

$$c(\alpha, L_1, L_2) = c(\alpha, (S_1 \ominus S_2))$$

and $c(\alpha, L) = c(\alpha, 0_N, L)$.

(6) We set $\gamma_U(L_1, L_2) = c(\mu_U, L_1, L_2) - c(1_U, L_1, L_2)$ and $\gamma_U(L) = \gamma_U(0_N, L)$.

(7) We write $L_2 \preceq_U L_1$ if $c(1_U, L_1, L_2) = 0$. If this holds for all bounded sets U, we write $L_2 \preceq L_1$.

(8) We set $GH^*(L_1, L_2; a, b) = H^{*-i}((S_1 \ominus S_2)^b, (S_1 \ominus S_2)^a)$.

Remark 4.2. We notice that

(1) As we said, S is shorthand for S_ν defined on U_ν. As long as $U \subset U_\nu$, it is easy to see that for $\alpha \in H^*(U)$ (resp. $H^*(U, \partial U)$) the $c(\alpha, S_\nu)$ do not depend on ν.

(2) The function $(S_1 \ominus S_2)$ is not quadratic at infinity, but a standard trick allows us to deform it to a function quadratic at infinity (see [Viterbo 2006, Proposition 1.6]). The GH^* functor is called generating function homology (see [Traynor 1994]) and coincides with Floer homology[14] that we shall not introduce here.

(3) Note that if S has no fiber variables, $c(1_U, S) = \inf_{x \in U} S(x)$ and $c(\mu_U, S) = \sup_{x \in U} S(x)$.

It is often convenient to express the cohomological critical values in terms of their homology counterparts. Note that $H^*(U)$ is dual to $H_{n-*}(U, \partial U)$ and $H^*(U, \partial U)$ is dual to $H_{n-*}(U)$ by Lefschetz duality (see [Hatcher 2002, p. 254]). We have a fundamental class $\mu_U \in H^n(U, \partial U)$ dual to $[pt_U] \in H_0(U)$ and $1_U \in H^0(U)$ dual to $[U] \in H_n(U, \partial U)$. The following lemma will be useful.

Lemma 4.3. *We have for S a GFQI*:

(1) $c(1_U, S) = c([pt_U], S)$.

(2) $c(\mu_U, S) = c([U], S)$.

We also have the duality identity

$$c(1_U, \bar{L}) = -c(\mu_U, L).$$

Proof. The first two properties follow from Proposition B.3 in [Viterbo 2023]. The duality identity is a consequence of the identity $c(1_U, -S) = -c(\mu_U, S)$. Both are easily adapted from the case $U = N$ closed to the present situation. This follows from the following argument (see [Viterbo 1992, Proposition 2.7, p. 692]). First notice that $(-S)^t = E \setminus S^{-t}$, so we look for the smallest t such that $1_U \neq 0$ in $H^*(E_{|U} \setminus S^{-t}_{|U}, E_{|U} \setminus S^{-\infty}_{|U})$. We then apply Alexander duality (see [Spanier 1966, Theorem 10, p. 342]), which claims that for any closed pair (A, B) contained in an orientable manifold X we have an isomorphism

$$H_k(X - B, X - A) \simeq H_c^{d-k}(B, A).$$

[14] See [Viterbo 2003] (or [Milinković and Oh 1997]) for the equivalence of the two homologies.

Note that $H_c^{d-k}(B, A)$ is invariant by proper homotopy equivalence, so if there is a proper retraction of the pair (A, B) to a compact pair (A', B'), then

$$H_c^{d-k}(B, A) \simeq H_c^{d-k}(B', A') \simeq H^{d-k}(B', A') \simeq H^{d-k}(B, A).$$

In particular this is always the case for pairs (S^b, S^a), where S is a GFQI. We then get the following diagram, where vertical maps correspond to long exact sequences of triples, and horizontal to Alexander isomorphisms (omitting the subscript U):

$$\begin{array}{ccccc}
H_d(S^{-t}, S^{-\infty}) & \xrightarrow{\simeq} & H^{n+k-d}(E \setminus S^{-\infty}, E \setminus S^{-t}) & = & H^{n+k-d}((-S)^t, (-S)^{-\infty}) \\
\downarrow & & \downarrow & & \downarrow \\
H_d(S^{+\infty}, S^{-\infty}) & \xrightarrow{\simeq} & H^{n+k-d}(E \setminus S^{-\infty}, E \setminus S^{+\infty}) & = & H^{n+k-d}((-S)^{+\infty}, (-S)^{-\infty}) \\
\downarrow & & \downarrow & & \downarrow \\
H_d(S^{+\infty}, S^{-t}) & \xrightarrow{\simeq} & H^{n+k-d}(E \setminus S^{-t}, E \setminus S^{\infty}) & = & H^{n+k-d}((-S)^t, (-S)^{-\infty})
\end{array}$$

Using the universal coefficient theorem (recall, our coefficient ring is a field) we see that $H_*(S^{+\infty}_{|U}, S^{-\infty}_{|U})$ is a vector space dual to $H^*(S^{+\infty}_{|U}, S^{-\infty}_{|U})$. By abuse of language, we denote by 1_U the element $pt_U \in H_*(S^{+\infty}_{|U}, S^{-\infty}_{|U})$ sent to $1_U \in H^*(S^{+\infty}_{|U}, S^{-\infty}_{|U})$, and we see that $c(1_U, S)$ is the same whether we consider 1_U in homology or cohomology. On the other hand the second line of the diagram sends $T \otimes 1_U$ to $T \otimes \mu_U$, since in this case Alexander duality corresponds to Poincaré duality. Now saying that 1_U is in the image of $H_*(S^{-t}, S^{+\infty})$ is equivalent to saying that μ_U is in the image of $H^*((-S)^t, (-S)^{-\infty})$. In other words, $-t \geq c(1_U, S)$ is equivalent to $t \geq c(\mu_U, -S)$ and this means $c(1_U, S) = -c(\mu_U, -S)$. □

Definition 4.4. Let U be a bounded domain with smooth boundary, ∂U. We say that the sequence of smooth functions $(f_k)_{k\geq 1}$ in $C^\infty(N)$ *defines* U if

(1) there is a decreasing family F_k of closed subset of N such that $\bigcap_k F_k = \overline{U}$,

(2) $f_k = 0$ on F_k,

(3) f_k is a decreasing sequence converging to $-\infty$ on $N \setminus U$.

We say that $(f_k)_{k\geq 1}$ is a *standard defining sequence* if there is a function $r \in C^\infty(\mathbb{R})$ such that

(1) $r(t) = 0$ for $t \leq 0$,

(2) $r'(t) < 0$ for $0 < t < 1$,

(3) $r(t) = -1$ for $t \geq 1$,

and for some increasing sequence a_k converging to $+\infty$ we have

$$f_k(x) = a_k r_k(a_k \cdot d(x, U)).$$

Notice that given a sequence $(f_k)_{k\geq 1}$ defining U, we can find standard sequences $(g_k)_{k\geq 1}, (h_k)_{k\geq 1}$ such that $g_k \leq f_k \leq h_k$.

We define for a smooth function f the graph of its differential, $G_f = \{(x, df(x)) \mid x \in N\}$. This is an exact Lagrangian, with primitive f. If L is a Lagrangian with GFQI, S, we define $L + G_f$ to be the Lagrangian generated by $S + f$, where $S + f(x, \xi) = S(x, \xi) + f(x)$.

We notice that:

Lemma 4.5. *Let $(f_k)_{k\geq 1}$ be a sequence defining U, and V be any bounded open set such that $V \supset \bar{U}$. Then for $L_1, L_2 \in \mathfrak{L}(T^*N)$ we have*

$$c(1_U, L_1, L_2) = \lim_k c(1_V, L_1 - G_{f_k}, L_2) = \lim_k c(1_V, L_1, L_2 + G_{f_k}).$$

Proof. Let S_j be GFQI for L_j and $S = S_1 \ominus S_2$. We have $S^c_{|U} = \lim_k (S - f_k)^c_{|V}$; therefore for Čech cohomology, according to Theorem 5 in [Lee and Raymond 1968] we have

$$\lim_k H^*((S - f_k)^c_{|V}, (S - f_k)^b_{|V}) = H^*(S^c_{|U}, S^b_{|U})$$

and from the definition of $c(1_U, S)$ the proposition follows. □

Remark 4.6. One should be careful. We will often have to estimate $c(\mu_U, L_1, L_2)$ but it is not true that $c(\mu_U, L_1, L_2) = \lim_k c(\mu_N, L_1 - G_{f_k}, L_2)$. Indeed, if $L_1 = G_g$, $L_2 = 0_N$, then $c(\mu_U, L_1, L_2) = \sup_{x\in U} g(x) \neq \sup_{x\in N} g(x) - f_k(x)$. However it follows from Lemma 4.3 that

$$c(\mu_U, L_1, L_2) = -\lim_k c(1_N, L_2 + G_{f_k}, L_1).$$

Let U be an open set with smooth boundary and set $\nu(x) \in T^*_x U$ to be the exterior conormal to ∂U at $x \in \partial U$, i.e., $\nu(x) = 0$ on $T\partial U$ and $\langle \nu(x), n(x)\rangle = 1$, where $n(x)$ is the exterior unit normal to U at x. The conormal of U is then defined as

$$\nu^* U = \{(x, p) \in T^*N \mid x \in U, p = 0, \text{ or } x \in \partial U, p = c\nu(x), c \leq 0\}.$$

We now prove that the values of $c(\alpha, L)$ correspond to intersection points of L and ν^*U (or L and $\overline{\nu^*U}$).

Proposition 4.7 (representation theorem). *Let U be a bounded open set with smooth boundary and $(L_1, f_1), (L_2, f_2)$ be exact Lagrangians in T^*N. Then we have*:

(1) *For $\alpha \in H^*(U) \setminus \{0\}$, $c(\alpha; L_1, L_2)$ is given by $f_1(x_\alpha, p_{1,\alpha}) - f_2(x_\alpha, p_{2,\alpha})$, where $(x_\alpha, p_{1,\alpha}) \in L_\alpha$ and $(x_\alpha, p_{2,\alpha}) \in L_2$ and $(x_\alpha, p_{1,\alpha} - p_{2,\alpha}) \in \nu^*U$.*

(2) *The same holds for $\alpha \in H^*(U, \partial U) \setminus \{0\}$ but with ν^*U replaced by $\overline{\nu^*U}$.*

Proof. This is the representation theorem [Viterbo 1992, Proposition 2.4], using a standard defining sequence for U and the fact that $c(1_U; L_1, L_2) = \lim_k c(1_V; L_1 - G_{f_k}, L_2)$. Indeed, a converging sequence of points in G_{f_k} will converge to a point in ν^*U (remember f_k must also be bounded in the sequence!). Then the compactness of $L_1 \cap T^*U$ and $L_2 \cap T^*U$ implies the result. □

For $(f_k)_{k\geq 1}$ a defining sequence of U, we say ν^*U is the "limit" of the G_{f_k} for $k \geq 1$. We will formally write $c(\alpha, L, \nu^*U)$ for $c(\alpha_U, L)$.

Remarks 4.8. (1) The same will hold for $U \subset V$ and any $\alpha_V \in H^*(V)$ having restriction $\alpha_U \in H^*(U)$:

$$c(\alpha_U, L_1, L_2) = \lim_k c(\alpha_V, L_1 - G_{f_k}, L_2) = \lim_k c(\alpha_V, L_1, L_2 + G_{f_k}).$$

In particular, if M is a closed manifold containing N, we have

$$c(1_U, L_1, L_2) = \lim_k c(1_M, L_1 - G_{f_k}, L_2) = \lim_k c(1_M, L_1, L_2 + G_{f_k}) = c(1_M, L_1, L_2 + \nu^* U).$$

(2) Let $\overline{U} \subset V$. Then with obvious abuse of notation $c(1_V, \nu^* U) = -\infty$, $c(\mu_V, \nu^* U) = 0$ and of course $c(1_U, \nu^* V) = 0$, $c(\mu_U, \nu^* V) = +\infty$. This means that, for $(f_k)_{k \geq 1}$ and $(g_k)_{k \geq 1}$ defining U and V, we have $\lim_k c(1_M, G_{f_k}, G_{g_k}) = -\infty$ and $\lim_k c(\mu_M, G_{f_k}, G_{g_k}) = 0$.

We will now prove some of the properties of these invariants:

Proposition 4.9. *Let $\varphi \in \mathfrak{DHam}_{\mathrm{FP}}(T^*N)$ and $L = \varphi^1(0_N)$ be a Lagrangian submanifold. We have*

$$\gamma_U(L) := c(\mu_U, L) - c(1_U, L) \geq 0$$

and equality implies that $L \cap T^*U \supset 0_U$.

Proof. The proof follows from the triangle inequality (see [Viterbo 1992, Proposition 3.3, p. 693]) applied to the product

$$H^*(U) \otimes H_c^*(U) \to H_c^*(U).$$

Remember that the triangle inequality in [Viterbo 1992, Proposition 3.3] states that for two GFQI S_1, S_2 and two cohomology classes α, β, we have

$$c(\alpha \cup \beta, S_1 \oplus S_2) \geq c(\alpha, S_1) + c(\beta, S_2),$$

where $(S_1 \oplus S_2)(x; \xi, \eta) = S_1(x; \xi) + S_2(x; \eta)$. Here we apply it to S_1 a GFQI for L, and S_2 a nondegenerate quadratic form, that is, a GFQI for 0_N, $\alpha \in H^*(U), \beta \in H_c^*(U)$. We then have, since $c(\beta, 0_N) = 0$,

$$c(\alpha \cup \beta, L) \geq c(\alpha, L).$$

Thus we have $c(\mu_U, L) = c(1_U \cup \mu_U, L) \geq c(1_U, L)$ and equality implies that μ_U is nonzero in $K_c \simeq L \cap \overline{\nu^* U}$. But this implies $\pi(L \cap \nu^* U) \supset U$; hence L contains 0_U. Note that in general, contrary to the case where $N = U$ is compact, $L \cap T^*U$ may contain other connected components than 0_U. ☐

Proposition 4.10. *The following hold for $L_i \in \mathfrak{L}(T^*N)$:*

(1) *We have $c(\mu_U, L_1, L_2) = -c(1_U, L_2, L_1) = -c(1_U, \bar{L}_1, \bar{L}_2)$.*

(2) *For $U \subset V$ and L_1, L_2 Lagrangian submanifolds we have*
- $c(\mu_U, L_1, L_2) \leq c(\mu_V, L_1, L_2)$,
- $c(1_U, L_1, L_2) \geq c(1_V, L_1, L_2)$,
- $\gamma_U(L_1, L_2) \leq \gamma_V(L_1, L_2)$.

(3) *We have $\gamma_U(L_1, L_3) \leq \gamma_U(L_1, L_2) + \gamma_U(L_2, L_3)$.*

(4) *If* $\gamma_U(L_1, L_2) = 0$ *then* $L_1 \cap L_2$ *has a connected component with projection on* N *containing* U.

(5) *If* $L_1 \preceq L_2$ *then* $c(\alpha, L_1) \leq c(\alpha, L_2)$ *for all* $\alpha \neq 0$.

Proof. (1) The proof is the same as in Lemma 4.3, since $S_{L_1} \ominus S_{L_2} = -(S_{L_2} \ominus S_{L_1})$.

(2) If $U \subset N$ note that

$$c(1_U; L_1, L_2) = \lim_k c(1_N, L_1 - G_{f_k}, L_2).$$

Since we may choose defining sequences $(f_k)_{k\geq 1}, (g_k)_{k\geq 1}$ for U, V such that $f_k \leq g_k$, we have for S_1 a GFQI of L_1 that $S_1 - f_k \geq S_1 - g_k$, hence $c(1_N, L_1 - G_{f_k}) \geq c(1_N, L_1 - G_{g_k})$, and going to the limit, $c(1_U, L_1, L_2) \geq c(1_V, L_1, L_2)$. By the duality formula (1), we get $c(\mu_U; L_1, L_2) \leq c(\mu_V; L_1, L_2)$; hence $\gamma_U(L_1, L_2) \leq \gamma_V(L_1, L_2)$.

(3) We have

$$S_1 \ominus 2 \cdot f \ominus S_3 = (S_1 \ominus f) \ominus (S_3 \oplus f)$$

and $(S_1 \ominus f) \ominus S_2 = S_1 \ominus (f \oplus S_2)$. Now noting that if $(f_k)_{k\geq 1}$ defines U, then so does $(2 \cdot f_k)_{k\geq 1}$, we have

$$\begin{aligned}
\gamma_U(L_1, L_3) &= \lim_k \gamma_V(S_1 \ominus 2 \cdot f_k \ominus S_3) \\
&= \lim_k \gamma_V((S_1 \ominus f_k) \ominus (S_3 \oplus f_k)) \\
&\leq \lim_k \gamma_V(S_1 \ominus f_k \ominus S_2) + \lim_k \gamma_V(S_2 \ominus (f_k \oplus S_3)) \\
&= \gamma_U(L_1, L_2) + \gamma_U(L_2, L_3).
\end{aligned}$$

(4) This follows from Lusternik–Schnirelmann theory as in the proof of [Viterbo 1992, Proposition 2.2, p. 691] (see also Proposition 4.9).

(5) $L_1 \preceq L_2$ implies $c(\mu_U, L_1, L_2) = 0$ for all U. By the triangle inequality applied to $S_1 \oplus (-S_2)$ (where S_i is a GFQI for L_i) if $\beta \cup \alpha = \mu_U$, we have

$$0 = c(\mu_U, L_1, L_2) \geq c(\alpha, L_1, 0_N) + c(\beta, 0_N, L_2) \geq c(\alpha, L_1) - c(\alpha, L_2)$$

since $c(\beta, 0_N, L) = -c(\alpha, L, 0_N)$ according to the proof of Proposition B.3 in [Viterbo 2023]. □

We must now see what happens when we make a coordinates change in T^*N. We start with three lemmas.

Lemma 4.11. *Let* S *be a GFQI defined on* $E = Y \times F$ *and for* $f : X \to Y$ *a smooth map a map* $\tilde{f} : X \times F \to Y \times F$ *living over* f, *i.e., the diagram*

$$\begin{array}{ccc}
X \times F & \xrightarrow{\tilde{f}} & Y \times F \\
\downarrow & & \downarrow \\
X & \xrightarrow{f} & Y
\end{array}$$

is commutative. We then have, for $\alpha \in H^*(Y)$ *and* $(f)^*(\alpha) \in H^*(X)$,

$$c(\alpha, S) \leq c(f^*(\alpha), S \circ \tilde{f}).$$

Proof. Indeed, if $T \in H^*(D(F^-), S(F^-))$ is the Thom class for F^-, then $(\tilde{f})^*(T) = \tilde{T}$ is the Thom class for $\tilde{f}^*(F^-)$ and we have, denoting by $\pi, \tilde{\pi}$ the projections on Y and X,

$$(\tilde{f})^*(T \cup \pi^*(\alpha)) = \pi^*(f^*(\alpha)) \cup \tilde{T}.$$

Now if $c < c(\alpha, S)$ then $\pi^*(\alpha) \cup T$ vanishes in $H^*(S^c, S^{-\infty})$ and this implies that $(\tilde{f})^*(T \cup \pi^*(\alpha)) = \pi^*(f^*(\alpha)) \cup \tilde{T}$ vanishes in $H^*((S \circ \tilde{f})^c, (S \circ \tilde{f})^{-\infty})$, i.e., $c \le c(f^*(\alpha), S \circ \tilde{f})$. This implies the lemma. ☐

For the next lemma we use the notation $S_1 \boxtimes S_2$ to denote $S_1(x, y, \xi, \eta) = S_1(x; \xi) + S_2(y; \eta)$ (not to be confused with $S_1 \oplus S_2$) and $\alpha \otimes \beta$ to denote the class in $H^*(X \times Y)$ image of $\alpha \otimes \beta$ by Künneth's isomorphism.

Lemma 4.12. *We have*

$$c(1_U \otimes 1_U; L_1 \times L_2, \nu^* \Delta_N) = c(1_U; L_1, L_2).$$

Proof. Let $d^\varepsilon : N \times N \to \mathbb{R}$ be a smooth function vanishing on Δ_N and converging as ε goes to 0 to $-\infty \cdot (1 - \chi_{\Delta_N})$, where χ_{Δ_N} is the characteristic function of Δ_N. For example we can choose

$$d^\varepsilon(x, y) = -\frac{1}{\varepsilon} d(x, y).$$

Similarly define $d_U^\varepsilon(x, y) = d^\varepsilon(x, y) + f_U^\varepsilon(x) + f_U^\varepsilon(y)$, where f_U^ε converges to $-\infty(1 - \chi_U)$ as ε goes to 0.

Setting $[S_1 \boxtimes (-S_2)](x_1, x_2, \xi_1, \xi_2) = S_1(x_1, \xi_1) + S_2(x_2, \xi_2)$, and

$$[S_1 \oplus (-S_2)](x, \xi_1, \xi_2) = S_1(x, \xi_1) + S_2(x, \xi_2)$$

we may write

$$\begin{aligned} c(1_{U \times U}; L_1 \times L_2, \Delta_N) &= \lim_{\varepsilon \to 0} c(1_{N \times N}; (L_1 - G_{f_U^\varepsilon}) \times (L_2 - G_{f_U^\varepsilon}), \nu^* \Delta_N) \\ &= \lim_{\varepsilon \to 0} c(1_{N \times N}; (S_1 - f_U^\varepsilon) \boxtimes (-S_2 - f_U^\varepsilon), d^\varepsilon) \\ &= c(1_{N \times N}, [(S_1 - f_U^\varepsilon) \boxtimes (-S_2 - f_U^\varepsilon)] - d^\varepsilon). \end{aligned}$$

Now $\lim_{\varepsilon \to 0}(S_1 \boxtimes (-S_2) - d^\varepsilon)^c = (S_1 \oplus (-S_2))^c$ and if $\delta : \Delta_N \to N \times N$ is the diagonal map, $\delta^*(1_N \otimes 1_N) = 1_{\Delta_N}$, so from Lemma 4.11, we get

$$c(1_U \otimes 1_U; L_1 \times L_2, \nu^* \Delta_N) \le c(1_N, (S_1 - f_U^\varepsilon) \oplus (S_2 - f_U^\varepsilon)) \le c(1_U; L_1, L_2).$$

Conversely we notice that given c, for ε small enough, $(S_1 \boxtimes (-S_2) - d^\varepsilon)^c$ is contained in a neighborhood of Δ_N. Thus if $1_U \otimes 1_U$ does not vanish in

$$H_*([((S_1 - f_U^\varepsilon) \boxtimes (-S_2 - f_U^\varepsilon)) - d^\varepsilon]^c, [((S_1 - f_U^\varepsilon) \boxtimes (-S_2 - f_U^\varepsilon)) - d^\varepsilon]^{-\infty}),$$

i.e., $c \ge c(1_U \otimes 1_U; L_1 \times L_2, \nu^* \Delta_N)$, then its restriction to Δ_N, that is, 1_U does not vanish either, and $c \ge c(1_U; L_1, L_2)$, so

$$c(1_U \otimes 1_U; L_1 \times L_2, \nu^* \Delta_N) \ge c(1_U; L_1, L_2)$$

and we have equality. ☐

Lemma 4.13. *Let us consider a bounded open set with boundary $U \subset N$ and $\nu^*\Delta_U \subset T^*N \times \overline{T^*N}$, where Δ_U is the diagonal in U. Let φ^t be a Hamiltonian flow on T^*U such that $\varphi^1(T^*U) \subset T^*V$. We have*

$$(\varphi^1 \times \varphi^1)(\nu^*\Delta_U) \preceq \nu^*\Delta_V.$$

Proof. Let $(q, p, q, p') \in \nu^*\Delta_U$ and notice that unless $q \in \partial U$, we have $p = p'$. Then according to Lemma 3.2 we may assume $\varphi^t(T^*U) \subset T^*V$ for all $t \in [0, 1]$, so setting $(\varphi^t \times \varphi^t)(q, p, q, p') = (Q_t, P_t, Q'_t, P'_t)$ we know that when $(q, p, q, p') \in \nu^*\Delta_U \subset T^*\overline{U}$, we have $Q_t, Q'_t \notin \partial V$. So if $(Q_t, P_t, Q'_t, P'_t) \in \nu^*V$, we must have $Q_t = Q'_t$, $P_t = P'_t$, but then $p = p'$. In other words

$$(\varphi^t \times \varphi^t)(\nu^*\Delta_U) \cap \nu^*\Delta_V = (\varphi^t \times \varphi^t)(\Delta_{T^*U}) \cap \Delta_{T^*V} = (\varphi^t \times \varphi^t)(\Delta_{T^*U}).$$

So the intersection $(\varphi^t \times \varphi^t)(\nu^*\Delta_U) \cap \nu^*\Delta_V$ is constant and by a classical argument, this implies that as a function of t, $c(\alpha, (\varphi^t \times \varphi^t)(\nu^*\Delta_U), \nu^*\Delta_V)$ is constant. Since $\nu^*\Delta_U \preceq \nu^*\Delta_V$, we have for all t we have $(\varphi^t \times \varphi^t)(\nu^*\Delta_U) \preceq \nu^*\Delta_V$. ☐

Using Proposition 4.10(2), we may conclude that the limits in the following proposition are well-defined in $\mathbb{R} \cup \{\pm\infty\}$.

Definition 4.14. When U is an unbounded set we define $\mathscr{B}(U)$ to be the set of bounded subsets in U and

$$c(\mu_U, L_1, L_2) = \lim_{V \in \mathscr{B}(U)} c(\mu_V, L_1, L_2),$$
$$c(1_U, L_1, L_2) = \lim_{V \in \mathscr{B}(U)} c(1_V, L_1, L_2).$$

Remark 4.15. Symbolically we have for $\overline{U} \subset V$ that $\nu^*U + \nu^*V = \nu^*U$, meaning that if $(f_k)_{k\geq 1}$ defines U and $(g_k)_{k\geq 1}$ defines V then $(f_k + g_k)_{k\geq 1}$ defines U. More generally if $U \cap V \subset W$, we have $\nu^*U + \nu^*V \preceq \nu^*W$ where this means that if $(f_k)_{k\geq 1}$ defines U and $(g_k)_{k\geq 1}$ defines V, there is a sequence $(h_k)_{k\geq 1}$ defining W such that $f_k + g_k \leq h_k$.

Proposition 4.16. *We have for $\varphi \in \mathfrak{DHam}(T^*N)$ such that $\varphi(T^*U) \subset T^*V$ and $L_1, L_2 \in \mathfrak{L}(T^*N)$*

$$\gamma_U(\varphi(L_1), \varphi(L_2)) \leq \gamma_V(L_1, L_2).$$

Proof. We use Lemma 4.11 so we replace $c(1_U, \varphi(L_1), \varphi(L_2))$ by

$$c(1_U \otimes 1_U, \varphi(L_1) \times \varphi(L_2), \nu^*\Delta_N)$$

and this in turn equals

$$c(1_N \otimes 1_N, (\varphi \times \varphi)(L_1 \times L_2), \nu^*\Delta_N + \nu^*(U \times U)).$$

Using Remark 4.15 we have

$$\nu^*\Delta_N + \nu^*(U \times U) \preceq \nu^*(\Delta_N \cap (U \times U)) = \nu^*\Delta_U$$

and we get

$$\begin{aligned} c(1_N \otimes 1_N, (\varphi \times \varphi)(L_1 \times L_2), \nu^*\Delta_N + \nu^*(U \times U)) &\geq c(1_N \otimes 1_N, (\varphi \times \varphi)(L_1 \times L_2), \nu^*\Delta_U) \\ &= c(1_N \otimes 1_N, (L_1 \times L_2), (\varphi \times \varphi)^{-1}(\nu^*\Delta_U)) \end{aligned}$$

and using Lemma 4.13 we get that the last term is greater than

$$c(1_{N\times N}, L_1 \times L_2, \nu^*\Delta_V) = c(1_V, L_1, L_2).$$

We may thus conclude that

$$c(1_V, L_1, L_2) \le c(1_U, \varphi(L_1), \varphi(L_2)).$$

By duality, we get

$$c(\mu_V, L_1, L_2) \ge c(\mu_U, \varphi(L_1), \varphi(L_2))$$

and our result follows. □

Definition 4.17. A sequence $(L_k)_{k\ge 1} \in \mathfrak{L}(T^*N)$ γ_c*-converges* to $L \in \mathfrak{L}(T^*N)$ if for all bounded domains U the sequence $\gamma_U(L_k, L)$ converges to 0. We shall write $L_k \xrightarrow{\gamma_c} L$. The γ_c*-completion* of $\mathfrak{L}(T^*N)$ for γ_c is the set of equivalence classes of γ_c-Cauchy sequences $(L_k)_{k\ge 1}$ for the following relation: $(L_k)_{k\ge 1} \simeq (L'_k)_{k\ge 1}$ if for all bounded domains U the sequence $\gamma_U(L_k, L'_k)$ converges to 0. We denote this completion by $\widehat{\mathfrak{L}}(T^*N)$.

Remark 4.18. Of course we may take a cofinal sequence U_k of bounded open sets in N and define

$$d(L_1, L_2) = \sum_{j=1}^{+\infty} 2^{-j} \max\{1, \gamma_{U_j}(L_1, L_2)\}$$

and then take the completion with respect to this metric. It is easy to see that the completion coincides with the above, and hence does not depend on the choice of the sequence U_k (this is just rephrasing the fact that the γ_U define a uniform structure; see [Weil 1938] or [Bourbaki 2007, Chapter II]).

Example 4.19. Let f_k be a sequence of smooth functions. Then γ-convergence of the $L_k = \mathrm{gr}(df_k)$ is equivalent to uniform convergence on compact sets of the f_k.

We shall need the following proposition.

Proposition 4.20. *We have for* $L = \varphi_H^1(0_N) \in \mathfrak{L}(T^*N)$ *the inequalities*

$$c(\mu_U, L) \le \sup_{(q,p)\in T^*U} H(q,p),$$

$$c(1_U, L) \ge \inf_{(q,p)\in T^*U} H(q,p),$$

$$\gamma_U(L) \le \sup_{(q,p)\in T^*U} H(q,p) - \inf_{(q,p)\in T^*U} H(q,p) = \mathrm{osc}_{T^*U}(H) \le 2\|H\|_{C^0(T^*U)}.$$

Proof. Let $H(q,p) = h(q)$ and $L_h = \varphi_H(0_N)$. Then according to Remark 4.2(3) we have $c(\mu_U, L_h) \le \sup_{q\in U} h(q)$ and $c(1_U, L_h) \ge \inf_{q\in U} h(q)$ because $L_h = \{(q, dh(q)) \mid q \in N\}$.

Now for general H, since for $H \le h(q) = \sup_{p\in T_q^*N} H(q,p)$ we have $H \le h$, we get $L \preceq L_h$, so $c(\mu_U, L) \le c(\mu, L_h) \le \sup_{q\in U} h(q) = \sup_{(q,p)\in T^*U} H(q,p)$ and we get the first inequality. The other two inequalities follow immediately from this one. □

4.2. *The case of Hamiltonians in $T^*\mathbb{R}^n$.* Let $H \in \mathfrak{Ham}_{\mathrm{fc}}([0,1]\times T^*\mathbb{R}^n)$ and φ_H^t be its flow. Let s_1, s_2 the symplectomorphisms

$$T^*\mathbb{R}^n \times \overline{T^*\mathbb{R}^n} \to T^*\Delta_{T^*\mathbb{R}^n}$$

defined respectively by

$$s_1(q,p,Q,P) = (q,P,p-P,Q-q),$$
$$s_2(q,p,Q,P) = (Q,p,p-P,Q-q).$$

Denoting by (x,y,X,Y) the coordinates in $T^*\Delta_{T^*\mathbb{R}^n}$, we have

$$s_i^*(dY\wedge dy + dX\wedge dx) = dp\wedge dq - dP\wedge dQ,$$

so the s_i are symplectic.

The graph of φ_H is $(\mathrm{id}\times\varphi_H)(\Delta_{T^*\mathbb{R}^n})$, and its image by s_1 is denoted by $\Gamma(\varphi_H)$, while its image by s_2 will be $\overline{\Gamma(\varphi_H^{-1})}$. Let S_H be a GFQI for $\Gamma(\varphi_H)$ which exists and is unique if $H \in \mathfrak{Ham}_{\mathrm{BP}}(T^*\mathbb{R}^n)$ by Theorem 3.5.

Definition 4.21. We set for W a domain contained in $\Delta_{T^*\mathbb{R}^n}$. Then

(1) $c_W^-(\varphi_H,\varphi_K) = c(1_W;\Gamma(\varphi_H),\Gamma(\varphi_H))$.
(2) $c_W^+(\varphi_H,\varphi_K) = c(\mu_W;\Gamma(\varphi_H),\Gamma(\varphi_H))$.
(3) $\gamma_W(\varphi_H,\varphi_K) = c_W^+(\varphi_H,\varphi_K) - c_W^-(\varphi_H,\varphi_K)$.
(4) $c_W^-(\varphi_K)$, $c_W^+(\varphi_K)$ and $\gamma_W(\varphi_K)$ are abbreviations for $c_W^-(\mathrm{id},\varphi_K)$, $c_W^+(\mathrm{id},\varphi_K)$ and $\gamma_W(\mathrm{id},\varphi_K)$ respectively.

Remark 4.22. In T^*N we may define for $U\subset N$ the number

$$\hat{\gamma}_U(\varphi_H) = \sup_{L\in\mathfrak{L}(T^*N)} \gamma_U(L,\varphi_H(L)),$$

which corresponds to (even though we do not claim it is equal to) $\gamma_{(U\times\mathbb{R}^n)}(\varphi_H)$.

Analogously to Proposition 4.16 we prove:

Proposition 4.23. *For $\varphi_1,\varphi_2 \in \mathfrak{DHam}_{\mathrm{BP}}(T^*\mathbb{R}^n)$ such that $\varphi_j(T^*U)\subset T^*V$ and $L\in\mathfrak{L}(T^*\mathbb{R}^n)$ we have*

$$\gamma_U(\varphi_1(L),\varphi_2(L)) \le \gamma_{V\times\mathbb{R}^n}(\varphi_1,\varphi_2).$$

Proof. We have

$$\begin{aligned} c(1_U,\varphi(L),L) &= c(1_U\otimes 1_U;\varphi(L)\times L,\nu^*\Delta_N)\\ &\ge c(1_U\otimes 1_U;(\varphi\times\mathrm{id})(L\times L),(\varphi\times\mathrm{id})(\nu^*\Delta_N)) + c(1_U\otimes 1_U;(\varphi\times\mathrm{id})(\nu^*\Delta_N),\nu^*\Delta_N). \end{aligned}$$

Equality follows from Lemma 4.12 and the inequality is the triangle inequality.

Now if $(\varphi\times\mathrm{id})T^*(U\times U)\subset T^*(V\times V)$, we have

$$c(1_U\otimes 1_U;(\varphi\times\mathrm{id})(L\times L),(\varphi\times\mathrm{id})(\nu^*\Delta_N)) \ge c(1_V\otimes 1_V;L\times L,\Delta_{T^*N}) = c(1_V;L,L) = 0.$$

As a result we have

$$c(1_U,\varphi(L),L)\geq c(1_U\otimes 1_U;(\varphi\times \mathrm{id})(\nu^*\Delta_N),\nu^*\Delta_N)=c(1_U\otimes 1_U;\Gamma(\varphi),\Gamma(\mathrm{id})).$$

We must now compare this last invariant with $c(1_W;\Gamma(\varphi))$. The map $s_1 : T^*\mathbb{R}^n\times\overline{T^*\mathbb{R}^n}\to T^*\Delta_{\mathbb{R}^{2n}}$ given by $s_1(q,p,Q,P)=(q,P,p-P,Q-q)$ sends $T^*(V\times V)$ into $T^*(V\times\mathbb{R}^n)$, so we have

$$c(1_V\otimes 1_V;\Gamma(\varphi),\Gamma(\mathrm{id}))\geq c(1_{V\times\mathbb{R}^n},\Gamma(\varphi)).$$

We may then conclude that

$$c(1_U,\varphi(L),L)\geq c(1_{V\times\mathbb{R}^n},\Gamma(\varphi))$$

and using the dual inequality we get our result. □

Let then $(H_\nu)_{\nu\geq 1}$ be a sequence of Hamiltonians in $\mathfrak{Ham}_{\mathrm{FP}}(T^*\mathbb{R}^n)$ and $\varphi_\nu=\varphi_{H_\nu}$.

Definition 4.24. The sequence $(\varphi_\nu)_{\nu\geq 1}$ γ_c*-converges* to φ if for all bounded domains W we have $\lim_\nu \gamma_W(\varphi_\nu,\varphi)=0$.

The γ_c*-completion* $\widehat{\mathfrak{DHam}}_{\mathrm{FP}}(T^*\mathbb{R}^n)$ is defined as the set of Cauchy sequences in $\mathfrak{DHam}_{\mathrm{FP}}(T^*\mathbb{R}^n)$ for the uniform structure defined by the γ_W, in other words the set of sequences which are Cauchy for each γ_W, modulo the equivalence relation $(\varphi_\nu)_{\nu\geq 1}\simeq(\psi_\nu)_{\nu\geq 1}$ if for all W we have $\lim_\nu\gamma_W(\varphi_\nu,\psi_\nu)=0$.

Similarly we define for $H\in\mathfrak{Ham}_{\mathrm{FP}}(T^*\mathbb{R}^n)$ the pseudometric

$$\gamma_W(H,K)=\sup_{t\in[0,1]}\gamma_W(\varphi_H^t,\varphi_K^t).$$

We then define analogously the γ_c-convergence of a sequence in $\mathfrak{Ham}_{\mathrm{FP}}(T^*\mathbb{R}^n)$ and its completion $\widehat{\mathfrak{Ham}}_{\mathrm{FP}}(T^*\mathbb{R}^n)$.

Note that the property of having FPS or being in $\mathfrak{H}am_{\mathrm{fc}}$ can be checked in the γ_c-completion.

Proposition 4.25. *There exist closed sets in* $\widehat{\mathfrak{DHam}}_{\mathrm{FP}}(T^*\mathbb{R}^n)$ *that intersect* $\mathfrak{DHam}(T^*\mathbb{R}^n)$ *on* $\mathfrak{DHam}_{\mathrm{FP}}(T^*\mathbb{R}^n)$, $\mathfrak{DHam}_{\mathrm{BP}}(T^*\mathbb{R}^n)$ *and* $\{\varphi\in\mathfrak{DHam}(T^*T^n)\mid \mathrm{supp}(\varphi)\subset\{|p|\leq r\}$ *respectively.*

Proof. Indeed $\varphi(T^*U)\subset T^*V$ is equivalent to

$$\Gamma(\varphi)\cap\{(x,p_x,y,p_y)\mid x\in U,\, y\notin V\}=\varnothing$$

and being supported in $|p|\leq r$ is equivalent to

$$\Gamma(\varphi)\cap\{(x,p_x,y,p_y)\mid |p_x|\geq r\}\subset\Gamma(\mathrm{id})$$

and both are closed conditions, which makes sense in the completion (see [Humilière 2008]). □

When dealing with fiberwise compactly supported Hamiltonians, we have:

Definition 4.26. We set for $\varphi\in\mathfrak{Ham}_{\mathrm{fc}}(T^*\mathbb{R}^n)$

$$\gamma_r(\varphi)=\gamma_{\mathbb{R}^n\times B^n(r)}(\varphi)=\lim_{R\to+\infty}\gamma_{B^n(R)\times B^n(r)}(\varphi),$$
$$\gamma_\infty(\varphi)=\lim_{r\to\infty}\gamma_r(\varphi)\in\mathbb{R}\cup\{+\infty\}.$$

Notice that convergence for γ_c and γ_∞ coincides on sequences supported in a fixed bounded set in the p-direction.

Proposition 4.27. *If $h_-(p) \le H(t,q,p) \le h_+(p)$, we have the inequality*

$$\gamma_r(\varphi_H) \le \sup_{|p|\le r} h_+(p) - \inf_{|p|\le r} h_-(p).$$

In particular if $a \le H(q,p) \le b$, we have $\gamma_\infty(\varphi_H) \le b-a$.

Proof. Indeed, $c_W^+(H) \le c_W^+(h_+)$, but $c^+_{\mathbb{R}^n\times B^n(r)}(h_+) = \sup_{|p|\le r} h_+(p)$. Indeed, the flow of $h(p)$ is $(q,p) \mapsto (q + t\,dh(p), p)$ and its graph is given by $(q,p,0,t\,dh(p))$, so a GFQI is $S(q,P) = h(P)$, and

$$c^+_{\mathbb{R}^n\times B^n(r)}(H) \le c^+_{\mathbb{R}^n\times B^n(r)}(h_+) = \sup_{|p|\le r} h_+(p).$$

Similarly $c^-_{\mathbb{R}^n\times B^n(r)}(h_-) = \inf_{|p|\le r} h_-(p)$ and $c^-_{\mathbb{R}^n\times B^n(r)}(H) \ge c^+_{\mathbb{R}^n\times B^n(r)}(h_-) = \inf_{|p|\le r} h_-(p)$.

By taking the difference of the above inequalities, we prove the proposition. □

Remark 4.28. The quantity $\gamma_\infty(\varphi)$ is finite for $\varphi \in \mathfrak{Ham}(T^*\mathbb{R}^n)$ such that $\|H\|_{C^0(T^*\mathbb{R}^n)} < +\infty$.

Our last results in this section will be:

Proposition 4.29. *We have the following, remembering that $\rho(x,p) = \left(\frac{x}{\varepsilon}, p\right)$:*

(1) *Assume ψ, ψ^{-1} send $W = U \times V$ into $W' = U' \times V'$, where $U, U' \subset \mathbb{R}^n$, $V, V' \subset (\mathbb{R}^n)^*$. Then we have*

$$\gamma_W(\psi^{-1}\circ\varphi\circ\psi) \le \gamma_{W'}(\varphi).$$

(2) $\gamma_r(\tau_{-a}\circ\varphi\circ\tau_a) = \gamma_r(\varphi)$.

(3) $\gamma_r(\rho_\varepsilon^{-1}\circ\varphi\circ\rho_\varepsilon) = \varepsilon\gamma_r(\varphi)$.

Proof. In $T^*(\mathbb{R}^n\times\mathbb{R}^n)$ we have that $\Gamma(\varphi)$ is the set of $(q,P,P-p,q-Q)$, where $(Q,P) = \varphi(q,p)$, while $\Gamma(\psi\circ\varphi\circ\psi^{-1})$ is obtained by applying $\psi\times\psi$ to (q,p,Q,P). In other words writing $(q',p') = \psi(q,p), (Q',P') = \psi(Q,P)$, $\Gamma(\psi\circ\varphi\circ\psi^{-1})$ is obtained as

$$\{(q',P',P'-p',q'-Q') \mid \varphi(q,p) = (Q,P)\}.$$

Now if $q \in U$ and $P \in V$, we have $q' \in U'$ and $P' \in V'$; hence $(\psi\times\psi)(T^*(U\times V)) \subset T^*(U'\times V')$, where $U\times V, U'\times V'$ are considered subsets of $\Delta_{T^*\mathbb{R}^n}$.

As a result, since $\psi\times\psi$ preserves the diagonal (that is the zero section in the new coordinates) we have, using Proposition 4.16,

$$\gamma_{U\times V}(\psi^{-1}\varphi\psi) = \gamma_{U\times V}((\psi\times\psi)\Gamma(\varphi), (\psi\times\psi)(\Delta)) \le \gamma_{U'\times V'}(\Gamma(\varphi),\Delta) = \gamma_{U'\times V'}(\varphi).$$

Statement (2) follows from first applying (1) to $\psi = \tau_a$ so that, setting $U_a = \bigcup_{t\in[-a,a]} \tau_t(U)$,

$$\gamma_{U\times B(r)}(\tau_{-a}\varphi\tau_a) \le \gamma_{U_a\times B(r)}(\varphi).$$

Hence taking the limit for $U \subset \mathbb{R}^n$ we get

$$\gamma_r(\tau_{-a}\varphi\tau_a) \le \gamma_r(\varphi)$$

and changing a to $-a$ we get equality. The last equality is rather obvious since ρ_ε is $\frac{1}{\varepsilon}$-conformal and $\rho_\varepsilon(U \times B_r) = \left(\frac{1}{\varepsilon} \cdot U\right) \times B_r$. □

Remark 4.30. One should be careful, in particular $\gamma_U(\varphi_1, \varphi_2)$ is *not* in general equal to $\gamma_U(\varphi_2^{-1} \circ \varphi_1) = \gamma_U(\varphi_2^{-1} \circ \varphi_1, \mathrm{id})$. We thus have a priori two types of convergence. We could say that φ_ν converges to φ if for all bounded sets U either the sequence $\gamma_U(\varphi_\nu, \varphi)$ goes to 0 or if $\gamma_U(\varphi_\nu \varphi^{-1})$ goes to 0. However if the φ_ν have uniformly bounded propagation speed, that is, $\varphi_\nu(T^* B_r) \subset T^* B_{r+r_0}$ for all ν and all r, then the two conditions are equivalent.

5. Compactness and ergodicity

Let $H : T^*\mathbb{R}^n \times \Omega \to \mathbb{R}$ be Hamiltonian satisfying properties (1)–(6). Then each $H_\omega = H(\cdot, \cdot, \omega)$ is in $\mathfrak{Ham}_{\mathrm{fc}}(T^*\mathbb{R}^n)$ and we identify Ω with its image in $\mathfrak{Ham}_{\mathrm{fc}}(T^*\mathbb{R}^n)$, denoted by $\mathfrak{H}_\Omega$. Its closure for the γ_c-topology in the completion $\widehat{\mathfrak{Ham}}_{\mathrm{FP}}(T^*\mathbb{R}^n)$ is denoted by $\widehat{\mathfrak{H}}_\Omega$. The action τ of $\mathbb{R}^n$ on Ω induces an action on $\mathfrak{H}_\Omega$ by

$$(\tau_a H)(x, p; \omega) = H(x + a, p; \omega) = H(x, p; \tau_{-a}\omega).$$

This action translates into $\varphi \mapsto \tau_{-a}\varphi\tau_a$ on $\mathfrak{DHam}_{\mathrm{fc}}(T^*\mathbb{R}^n)$.

We first want to prove:

Proposition 5.1. *The abelian group $\mathbb{R}^n$ acts continuously by isometries on $(\mathfrak{Ham}_{\mathrm{fc}}(T^*\mathbb{R}^n), \gamma_c)$ and $(\mathfrak{DHam}_{\mathrm{fc}}(T^*\mathbb{R}^n), \gamma_c)$ and hence on $(\widehat{\mathfrak{Ham}}_{\mathrm{fc}}(T^*\mathbb{R}^n), \gamma_c)$ and $(\widehat{\mathfrak{DHam}}_{\mathrm{fc}}(T^*\mathbb{R}^n), \gamma_c)$. Therefore the action τ of $\mathbb{R}^n$ on $\mathfrak{H}_\Omega$ is a continuous action by isometries for γ_c which extends to a continuous action by isometries on $\widehat{\mathfrak{H}}_\Omega$.*

Proof. That $\mathbb{R}^n$ acts by isometries follows from Proposition 4.29(2). It is enough according to a theorem by Chernoff and Marsden[15] to prove the separate continuity of the map $\mathbb{R}^n \times \mathfrak{Ham}_{\mathrm{fc}}(T^*\mathbb{R}^n) \to \mathfrak{Ham}_{\mathrm{fc}}(T^*\mathbb{R}^n)$ in each variable. In other words — since τ_a is an isometry, it is obviously continuous in the second variable — we must prove that, for all $H \in \mathfrak{Ham}_{\mathrm{fc}}(T^*\mathbb{R}^n)$, we have

$$\lim_{a \to 0} \gamma_c(H, \tau_a H) = 0,$$

i.e., we want to prove that for all $r > 0$, $\lim_{a\to 0} \gamma_r(\tau_a^{-1}\varphi^{-1}\tau_a, \varphi) = 0$. But

$$\Gamma(\varphi) = \{(q, P, p - P, Q - q) \mid \varphi(q, p) = (Q, P)\},$$

while

$$\Gamma(\tau_a^{-1}\varphi\tau_a) = \{(q - a, P, p - P, Q - q) \mid \varphi(q, p) = (Q, P)\},$$

so that $S(q, P; \xi)$ is a GFQI for $\Gamma(\varphi)$ and $(\tau_a S)(q, P, \xi) = S(q - a, P; \xi)$ is a GFQI for $\Gamma(\tau_a^{-1}\varphi\tau_a)$. Since critical points of $S(q, P, \xi)$ are contained in $|P| \le R$ and $a \mapsto S(q - a, P; \xi)$ is uniformly continuous on $|P| \le R$, we get that $c_W(\alpha, S \ominus \tau_a S)$ depends continuously on a, and for $a = 0$ is equal to 0 (since it is equal to $c_W(\varphi, \varphi) = 0$). □

[15]Which claims that, under our assumptions, a separately continuous action is jointly continuous. See [Chernoff and Marsden 1970, Theorem 1], extending a theorem of Ellis [1957].

Proposition 5.1 extends the action τ to a continuous action by isometries of $\widehat{\mathfrak{H}}_\Omega$. Since $\mathrm{Isom}(\mathfrak{H}_\Omega,\gamma)\subset \mathrm{Isom}(\widehat{\mathfrak{H}}_\Omega,\gamma)$, the map $\tau:\mathbb{R}^n\to \mathrm{Isom}(\mathfrak{H}_\Omega,\gamma)$ extends to a map, still denoted by τ, from $\mathbb{R}^n$ to $\mathrm{Isom}(\widehat{\mathfrak{H}}_\Omega,\gamma)$. Since this is obviously a group morphism, its closure in $\mathrm{Isom}(\widehat{\mathfrak{H}}_\Omega,\gamma)$ is an abelian connected and complete metric group.

Proposition 5.2. *Let us denote the closure of $\tau(\mathbb{R}^n)$ in $\mathrm{Isom}(\widehat{\mathfrak{H}}_\Omega,\gamma)$ by $\mathbb{A}_\Omega$. Then $\mathbb{A}_\Omega$ is an abelian, connected and complete metric group.*

The goal of this section is to prove that our assumptions on H imply that $\mathbb{A}_\Omega$ is compact. For this it is enough to prove that $\mathrm{Isom}(\widehat{\mathfrak{H}}_\Omega,\gamma_c)$ is compact, but this follows immediately by the Arzelà–Ascoli theorem if we prove that $(\widehat{\mathfrak{H}}_\Omega,\gamma_c)$ is compact. Because by assumption $(\widehat{\mathfrak{H}}_\Omega,\gamma_c)$ is complete, it is enough to show that it is totally bounded, that is, for any $\varepsilon>0$, $(\widehat{\mathfrak{H}}_\Omega,\gamma_c)$ can be covered by finitely many γ_c-balls of radius ε. Since $(\mathfrak{H}_\Omega,\gamma_c)$ is dense in $(\widehat{\mathfrak{H}}_\Omega,\gamma_c)$, it is enough to prove that $(\mathfrak{H}_\Omega,\gamma_c)$ is totally bounded. We shall prove slightly less but it will be good enough for our purposes:

Proposition 5.3. *Let $\hat{\mu}_\Omega$ be the push forward to $\widehat{\mathfrak{H}}_\Omega$ of the measure μ on Ω. Then the support of $\hat{\mu}_\Omega$ is totally bounded hence compact.*

This will follow from the following general result.

Proposition 5.4. *Let (X,μ) be a probability space endowed with a distance d such that (X,d) is separable.*[16] *Let G be a group acting ergodically on X by (measure-preserving) isometries. Then* $\mathrm{supp}(\mu)$ *is totally bounded.*

We shall first prove:

Lemma 5.5. *Let τ be a continuous ergodic action of a group G on a probability, separable metric space (X,μ,d). Then for μ-almost all points $x\in X$, the orbit $G\cdot x$ is dense in* $\mathrm{supp}(\mu)$.

Proof. This is an immediate consequence of Birkhoff's ergodic theorem, but we shall give a simpler (or at least easier) proof. Let Y be countable and dense in X and set

$$W=\bigcup_{\substack{y\in Y,r\in\mathbb{Q}_+^*\\ \mu(B(y,r))=0}} B(y,r).$$

If $\mu(B(x,r))=0$ for some $x\in X,\ r>0$ then $x\in W$. Indeed, we may assume r is rational, and choose $y\in Y$ such that $d(y,x)<\frac{r}{2}$. Then $x\in B\big(y,\frac{r}{2}\big)$ so $B\big(y,\frac{r}{2}\big)\subset B(x,r)$ and we get $\mu\big(B\big(y,\frac{r}{2}\big)\big)=0$. This argument implies that

$$W=\{x\in X\mid \exists U \text{ open } x\in U, \mu(U)=0\}$$

and W is τ invariant since τ preserves μ and the open sets. Now because W is a countable union of open sets of measure 0, it is open and has measure 0. We may then replace X by $X\setminus W$, so we are reduced to the situation where all balls have >0 measure, i.e., all open sets have positive measure.

[16] A separable topological space is a space having a countable dense subset.

Now let $(U_j)_{j\in\mathbb{N}}$ be a countable basis of open sets (since a separable metric space is second countable). Set $\tau_G A = \{\tau_g x \mid x \in A\}$; then the orbit of x misses U_j if and only if $\tau_G x \cap U_j = \varnothing$, i.e., $x \notin \tau_G(U_j)$. The points with nondense orbit must miss at least one $\tau_G(U_j)$ so they belong to

$$\bigcup_j (X \setminus \tau_G(U_j) = X \setminus \bigcap_j \tau_G(U_j),$$

but by ergodicity $\tau_G(U_j)$ being τ invariant has measure 1 (since it cannot be zero, as its measure is at least the measure of U_j that is positive by assumption). Therefore $\bigcap_j \tau_G(U_j)$ as a countable intersection of measure-1 sets has measure 1, and its complement has measure zero. ☐

We are now in a position to prove Proposition 5.4.

Proof of Proposition 5.4. By the lemma we may choose x such that $\tau_G x$ is a dense orbit in $\operatorname{supp}(\mu)$. We shall prove that $\tau_G(x)$ is totally bounded, arguing by contradiction.

Let $a_1, \dots, a_k, \dots \in G$ be a sequence in G such that:

- $\bigcup_{j=1}^k \overline{B}(\tau_{a_j} x, \varepsilon)$ does not cover $\tau_G x$, where $\overline{B}(x, r)$ is the closed ball of radius r.
- For all $i \neq j$ we have $B\big(\tau_{a_i} x, \frac{\varepsilon}{2}\big) \cap B\big(\tau_{a_j} x, \frac{\varepsilon}{2}\big) = \varnothing$.

We claim that if $\tau_G x$ cannot be covered by finitely many balls of size ε then we may construct such a sequence by induction. Indeed, assume $a_1, \dots, a_k$ have been constructed satisfying the above properties. Then by the first property we may find a_{k+1} such that $\tau_{a_{k+1}} x \notin \bigcup_{j=1}^k B(\tau_{a_j} x, \varepsilon)$ and this implies $B\big(\tau_{a_j} x, \frac{\varepsilon}{2}\big) \cap B\big(\tau_{a_{k+1}} x, \frac{\varepsilon}{2}\big) = \varnothing$. Hence $a_1, \dots, a_{k+1}$ satisfy both properties. But now we found infinitely many disjoint balls of radius $\frac{\varepsilon}{2}$ in $\tau_G x$. Since $\tau_{a_j} x \in \operatorname{supp}(\mu)$, we have $\mu(B\big(\tau_{a_j} x, \frac{\varepsilon}{2}\big)) > 0$ and since all the balls $B\big(\tau_a x, \frac{\varepsilon}{2}\big)$ are isometric, they have the same measure. But we cannot have infinitely many disjoint balls with the same positive measure, since the total measure of our space is 1. ☐

We may now conclude with:

Proof of Proposition 5.3. Here $G = \mathbb{R}^n$ and τ induces a measure-preserving ergodic action on $(\mathfrak{H}_\Omega, \gamma, \hat{\mu}_\Omega)$. This action is by isometries according to Proposition 5.1, so according to Proposition 5.3 the support of $\hat{\mu}_\Omega$ is totally bounded. ☐

Remark 5.6. As we pointed out already in [Viterbo 2023], there are not so many nontrivial examples of compact subset in $(\widehat{\mathfrak{Ham}}_{\rm fc}(T^*\mathbb{R}^n), \gamma)$ or $(\widehat{\mathfrak{DHam}}_{\rm fc}(T^*\mathbb{R}^n), \gamma)$, that is, sets that are not already compact for the C^0-topology (since γ is continuous for the C^0 topology on $\mathfrak{Ham}(T^*N)$ according to [Viterbo 1992]) and in $\mathfrak{DHam}(T^*N)$ according to [Seyfaddini 2012]). In [Viterbo 2023] we proved that in T^*T^n the sequence $(H_k)_{k\geq 1}$, where $H_k(q,p) = H(k\cdot q, p)$, is converging. Here we extend this to certain families of Hamiltonians on $T^*\mathbb{R}^n$.

We thus proved that $\mathbb{A}_\Omega$, the closure of $\mathbb{R}^n$ in $\operatorname{Isom}(\widehat{\mathfrak{H}}_\Omega, \gamma)$, is a compact, connected, metric abelian group.

We are thus in the following situation: we have an action — again denoted by τ — of the group $\mathbb{A}_\Omega$ acting by γ-isometries on the space $\widehat{\mathfrak{H}}_\Omega$ and preserving $\hat{\mu}_\Omega$. By compactness of $\mathbb{A}_\Omega$, we have that $\mathbb{A}_\Omega \cdot H$

is closed for all $H \in \widehat{\mathfrak{H}}_\Omega$. But since by Lemma 5.5 for almost all H, $\tau_{\mathbb{R}^n} H$ is dense, we conclude that for almost all H we have $\mathbb{A}_\Omega \cdot H = \widehat{\mathfrak{H}}_\Omega$.

Thus $\widehat{\mathfrak{H}}_\Omega \simeq \mathbb{A}_\Omega/\mathbb{K}_\Omega$, but $\mathbb{A}_\Omega/\mathbb{K}_\Omega$ is again a compact metric abelian group. Moreover the measure $\hat{\mu}_\Omega$ on $\widehat{\mathfrak{H}}_\Omega$ induces a measure on $\mathbb{A}_\Omega/\mathbb{K}_\Omega$, invariant by the action. It is therefore the Haar measure. To conclude, and writing from now on $\mathbb{A}_\Omega$ for $\mathbb{A}_\Omega/\mathbb{K}_\Omega$, we are reduced to the situation where:

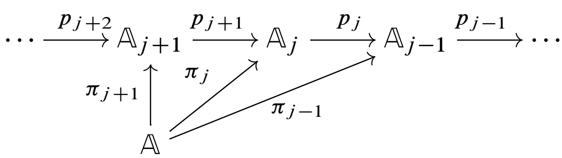
(1) $\Omega = \mathbb{A}_\Omega$.

(2) $\omega \to H_\omega \in \widehat{\mathfrak{Ham}}_{\mathrm{BP}}(T^*T^n)$ is continuous for the γ-topology.

(3) On the subgroup $\mathbb{R}^n$ in $\mathbb{A}_\Omega$ the action of $\mathbb{R}^n$ on Ω can be identified with the action by translation of $\mathbb{R}^n$ as a dense subgroup of $\mathbb{A}_\Omega$. The invariant measure on $\mathbb{A}_\Omega$ is the Haar measure and the action of $\mathbb{R}^n$ on $\mathbb{A}_\Omega$ is ergodic.

6. Some results on compact abelian metric groups

Let $\mathbb{A}$ be a compact metric abelian group having $\mathbb{R}^n$ as a dense subgroup (in particular $\mathbb{A}$ is connected). According to A. Weil [1965, p. 110] (see also [Hofmann and Morris 2013, Theorem 8.45]) $\mathbb{A}$ is the projective limit of finite-dimensional tori. In other words there are tori T^{n_j} and group morphisms $f_{j,i} : T^{n_j} \to T^{n_i}$ for $i < j$ integers such that $f_{k,j} \circ f_{j,i} = f_{k,i}$ and a map $f_{\infty,i} : \mathbb{A} \to T^{n_i}$ such that $\mathbb{A} = \varprojlim_j T^{n_j}$. We denote by $\mathbb{A}_j$ the image of $\mathbb{A}$ in T^{n_j}, which is clearly a connected compact subgroup of T^{n_j} and hence a subtorus, and we may replace T^{n_j} by $\mathbb{A}_j$. Setting by $p_j = f_{j+1,j}$ and $\pi_j = f_{\infty,j}$, we have the following sequence:



We set $\mathbb{K}_j = \mathrm{Ker}(\pi_j)$. We then have:

Lemma 6.1. *We have*

$$\lim_j \mathrm{diam}(\mathbb{K}_j) = 0.$$

Proof. The $\mathbb{K}_j$ are a decreasing sequence of closed — hence compact — subgroups such that $\bigcap_j \mathbb{K}_j = \{0\}$ by the definition of the projective limit. But this implies the lemma by an easy exercise (or [Rudin 1976, Theorem 3.10]). □

Now we need:

Lemma 6.2. *Let us consider the embeddings*

$$\pi_j^* : C^0(\mathbb{A}_j, \mathbb{R}) \to C^0(\mathbb{A}, \mathbb{R}), \quad f \mapsto f \circ \pi_j.$$

Then the union of the images of the π_j^* is dense in $C^0(\mathbb{A}, \mathbb{R})$.

Proof. Let $f \in C^0(\mathbb{A}, \mathbb{R})$. Then f is uniformly continuous by the Heine–Cantor theorem (see [Rudin 1976, Theorem 4.19]):

$$\forall \varepsilon > 0,\ \exists \eta > 0,\ \forall x, y \in \mathbb{A}, \quad d(x, y) < \eta \implies \delta(f(x), f(y)) < \varepsilon.$$

For j large enough we have $\operatorname{diam}(\mathbb{K}_j) < \eta$, so setting $f_j(x) = \min\{f(x+u) \mid u \in \mathbb{K}_j\}$, we see that by the compactness of $\mathbb{K}_j$ the function f_j is well-defined and continuous. Moreover $d(f(x), f_j(x)) < \varepsilon$ provided $\operatorname{diam}(\mathbb{K}_j) < \eta$. □

Now remember that we have a group morphism $\tau : \mathbb{R}^n \to \mathbb{A}$ with dense image. By the definition of a projective limit, the map τ is defined by a sequence of maps $\tau_j : \mathbb{R}^n \to \mathbb{A}_j$ such that $p_j \circ \tau_j = \tau_{j-1}$. Of course the density of $\tau(\mathbb{R}^n)$ implies the density of $\tau_j(\mathbb{R}^n)$ because the preimage by π_j of a proper closed subset is a proper closed subset (remember π_j is onto by assumption). Since the density of the image of τ is equivalent to the ergodicity of the action, we may conclude that τ is ergodic on $\mathbb{A}_j$.

We are now in the following situation: we have a subgroup $\mathbb{A}_\Omega$ in $\operatorname{Isom}(\widehat{\mathfrak{H}}_\omega, \gamma)$ and for almost every H (for the measure $\hat{\mu}_\Omega$) we have $\mathbb{A}_\Omega \cdot H = \widehat{\mathfrak{H}}_\omega$. Now $\mathbb{A}_\Omega \cdot H$ is approximated by $\mathbb{A}_j \cdot H$ for a finite-dimensional torus $\mathbb{A}_j$, and the action of $\mathbb{R}^n$ by τ yields a dense subgroup of $\mathbb{A}_j$. At the cost of an approximation, we have thus replaced H_ω for $\omega \in \mathbb{A}_\omega$ by the H_ω for $\omega \in \mathbb{A}_j$, that is, we have a continuous map $\mathbb{A}_j \to (\widehat{\mathfrak{Ham}}_{\mathrm{fc}}, \gamma)$ and $\mathbb{A}_j$ is a finite-dimensional torus.

7. Regularization of the Hamiltonians in $\widehat{\mathfrak{Ham}}_{\mathrm{fc}}$

Let $H \in \widehat{\mathfrak{Ham}}_{\mathrm{FP}}(T^*\mathbb{R}^n)$ and φ_H^t be its flow in $\widehat{\mathfrak{DHam}}_{\mathrm{FP}}(T^*\mathbb{R}^n)$. Let $S(q, p; \xi)$ be a GFQI for $\Gamma(\varphi_H)$, set $S_{(q,p)}(\xi) = S(q, p; \xi)$, and let $c(1_{(q,p)}, S) := c(1_{(q,p)}, S_{(q,p)})$ be the critical value corresponding to the unique cohomology class $1_{(q,p)} \in H^0(\{(q, p)\})$. The map $\varphi \mapsto c(1_{(q,p)}, \varphi)$ obviously extends to $\widehat{\mathfrak{DHam}}_{\mathrm{FP}}(T^*\mathbb{R}^n)$. We now set:

Definition 7.1. For $\eta > 0$ we set

$$H^\eta(q, p) = \frac{1}{\eta} c(1_{(q,p)}, \varphi_H^\eta) = \frac{1}{\eta} c(1_{(q,p)}, \Gamma(\varphi_H^\eta)).$$

This defines a map

$$\sigma_\eta : \widehat{\mathfrak{Ham}}_{\mathrm{fc}}(T^*\mathbb{R}^n) \to C^{0,1}_{\mathrm{fc}}(T^*\mathbb{R}^n),$$

where $C^{0,1}_{\mathrm{fc}}(T^*\mathbb{R}^n)$ is the set of Lipschitz functions with fiberwise compact support.

Our goal is to prove that σ_η is a regularizing operator. This is the content of:

Proposition 7.2. *We have for* $H \in \widehat{\mathfrak{Ham}}_{\mathrm{fc}}(T^*\mathbb{R}^n)$:

1. $\gamma_c - \lim_{\eta \to 0} \sigma_\eta(H) = H$.
2. *For each R there exists a constant C such that for H supported in $\mathbb{R}^n \times B(R)$ and such that $\varphi_H(T^*B(\rho)) \subset T^*B(\rho + r)$ we have $\sigma_\eta(H)$ is $\frac{C(R+r)}{\eta}$-Lipschitz.*
3. $\sigma_\eta : \widehat{\mathfrak{Ham}}_{\mathrm{fc}}(T^*\mathbb{R}^n) \to C^0_{\mathrm{fc}}(T^*\mathbb{R}^n, \mathbb{R})$ *is continuous for the γ-topology.*
4. $\sigma_\eta \circ \tau_a = \tau_a \circ \sigma_\eta$.

Remark 7.3. One should be careful: the γ_c-limit in (1) is of course not a C^0 limit, since H is not continuous in general — it is not even a function! But even if H is continuous, we do not claim this.

We need the following lemma, which we shall prove in Appendix C.

Lemma 7.4. *For η small enough we can find a GFQI for φ_K^η, $S_{K,\eta}$, such that*

$$\|S_{K,\eta}(q,p)-\eta K(q,p)\| \leq C\eta^2\|\nabla K\|^2_{C^0}.$$

Proof of Proposition 7.2.

(1) By density we can find $K \in C^\infty_{\mathrm{fc}}(T^*\mathbb{R}^n,\mathbb{R})$ such that $\gamma(H,K)\leq\varepsilon$. Now for $K \in C^\infty_{\mathrm{fc}}(T^*\mathbb{R}^n,\mathbb{R})$ we may find a GFQI, $S_{K,\eta}$ of φ_K^η such that

$$S_{K,\eta}(q,p)=\eta\cdot K(q,p)+o(\eta)$$

as η goes to zero so that $K^\eta(q,p)=\frac{1}{\eta}c(1_{(q,p)},S_{K,\eta})=K(q,p)+o(1)$.

Now the formula $c(1_{(q,p)},S_{K,\eta})=\eta K(q,p)+o(\eta)$ follows immediately from the lemma by applying on one hand the triangle inequality (see [Viterbo 1992, Proposition 3.3, p. 693])

$$|c(1_x,L)-c(1_x,L')|\leq\gamma(L,L')$$

and on the other hand Proposition 4.20,

$$\|K^\eta(q,p)-K(q,p)\|\leq\eta\cdot\|\nabla K\|^2_{C^0}.$$

Now for η small enough we have $\gamma(K^\eta,K)\leq\varepsilon$. Remember from Definitions 4.1 that for $H,K\in\widehat{\mathfrak{Ham}}(T^*\mathbb{R}^n)$, $H\preceq K$ means $c(1_W,\varphi_K,\varphi_H)=0$ for all W. The reduction inequality [Viterbo 1992, Proposition 5.1, p. 705] implies that $H^\eta(q,p)\leq K^\eta(q,p)$ for all $(q,p)\in T^*\mathbb{R}^n$.

Let $\zeta_R(p)$ be a function such that $0\leq\eta_R(p)\leq 1$, vanishing for $|p|\leq R$ and equal to 1 for $|p|2\geq R$. Now $\gamma(H,K)\leq\varepsilon$ implies that $K-\varepsilon\zeta_R\preceq H\preceq K+\varepsilon\zeta_R$ for R large enough: this follows from the formula $c(1_W,\varphi_{K+\varepsilon\zeta_R},\varphi_H)=c(1_W,\varphi_K,\varphi_H)+\varepsilon$ for W large enough because if S is a GFQI for φ_K then $S_\varepsilon(q,p;\xi)=S_0(q,p;\xi)+\varepsilon\zeta_R(p)$ is a GFQI for $\varphi_{K+\varepsilon\zeta_R}=\varphi_K\circ\varphi_{\eta_R}$ and $c(1_W,S_\varepsilon)=c(1_W,S_0)+\varepsilon$ for R and W large enough.

Now we have $K^\eta-\varepsilon\zeta_R\preceq H^\eta\preceq K^\eta+\varepsilon\zeta_R$ and for η small enough we get $\|K-K^\eta\|\leq\varepsilon$ so

$$K-2\varepsilon\preceq H^\eta\preceq K+2\varepsilon.$$

Thus

$$H-3\varepsilon\preceq K-2\varepsilon\preceq H^\eta\preceq K+2\varepsilon\preceq H+3\varepsilon;$$

hence $\gamma(H^\eta,H)\leq 3\varepsilon$.

(2) We have for $|q_1-q_2|+|p_1-p_2|\leq r$

$$c(1_{(q_1,p_1)}\varphi_H^\eta)-c(1_{(q_2,p_2)}\varphi_H^\eta)\leq C(r)$$

because for $L_{(q,p)}$ Hamiltonianly isotopic to the vertical and coinciding with $T^*_{(q,p)}\Delta_{\mathbb{R}^{2n}}$ in $\Delta_{\mathbb{R}^{2n}}\times B_r^{2n}$ we have

$$c(1_{(q,p)},\Gamma(\varphi_H^\eta))=c(\Gamma(\varphi_H^\eta),L_{(q,p)})$$

and

$$|c(\Gamma(\varphi_H^\eta), L) - c(\Gamma(\varphi_H^\eta), \psi(L))| \leq \gamma(L, \psi(L)) \leq \gamma(\psi).$$

As a result, there is a Hamiltonian map ψ with $\gamma(\psi) \leq C(r)$ such that

$$\psi(T^*_{(q_1,p_1)}\Delta_{\mathbb{R}^{2n}}) \cap (\Delta_{\mathbb{R}^{2n}} \times B_\rho^{2n}) = T^*_{(q_2,p_2)}\Delta_{\mathbb{R}^{2n}} \cap (\Delta_{\mathbb{R}^{2n}} \times B_\rho^{2n}),$$

where ρ is such that $\Gamma(\varphi_H^\eta) \subset \mathbb{R}^{2n} \times B_\rho^{2n}$. Since we assumed that H is supported in B_R we may assume $\rho = 2R$ and we have $C(r) = CR\cdot r$. Indeed if ψ_t is an isotopy such that ψ_1 sends (q_1, p_1) to (q_2, p_2), and Ψ_t its natural extension to a Hamiltonian isotopy $T^*(\Delta_{T^*\mathbb{R}})$, we truncate the Hamiltonian generating Ψ_t to $\mathbb{R}^{2n} \times B_\rho^{2n}$, where ρ is an upper bound for $|Q_H(q, p) - q| + |P_H(q, p) - p|$. Such an upper bound is given by $r + 2R$ (r for $|Q - q|$ and $2R$ for $|P - p|$). This proves the inequality.[17]

(3) We have

$$\|\sigma_\eta(H) - \sigma_\eta(K)\|_{C^0} \leq \frac{1}{\eta} \sup_{(q,p)} c(1_{(q,p)}, \varphi^\eta(\psi^\eta)^{-1}) \leq \frac{1}{\eta}\gamma(\varphi_H^\eta, \varphi_K^\eta) \leq \frac{1}{\eta}\gamma(H, K),$$

where the first inequality is just the triangle inequality (see [Viterbo 1992, Proposition 3.3, p. 693]) and the second inequality follows by the reduction inequality in [loc. cit., Proposition 5.1, p. 705].

(4) We have $\sigma_\eta(H_\omega)(x + a, p) = \frac{1}{\eta}c(1_{x+a,p}, \varphi^\eta_{H_\omega}) = c(1, S^\omega(x + a, P; \xi))$ but $S^\omega(x + a, P; \xi)$ is the generating function corresponding to $\tau_a H_\omega$, i.e., $\Gamma(\tau_{-a}\varphi^\eta_{H_\omega}\tau_a)$ is the set of $(q + a, P, P - p, Q - q)$, where $\varphi^\eta_{H_\omega}(q, p) = (Q, P)$. So we have $\Gamma(\tau_{-a}\varphi^\eta_{H_\omega}\tau_a) = \tau_a(\Gamma(\varphi^\eta_{H_\omega}))$ and

$$S_{H_{\tau_{-a}\omega}}(x, P, \tau_{-a}\xi) = S_{\tau_a H_\omega}(x, P; \xi) = S_{H_\omega}(x + a, P; \xi).$$

We thus proved that

$$\tau_a\sigma_\eta(H_\omega)(x, p) = \sigma_\eta(H_\omega)(x + a, p) = \sigma_\eta(\tau_a H_\omega)(x, p) = \sigma_\eta(H_{\tau_{-a}\omega})(x, p) = \sigma_\eta(\tau_a H_\omega)(x, p). \quad \square$$

We are now in the following situation: we started from a continuous map

$$H : \mathbb{A}_j \to (\widehat{\mathfrak{Ham}}(T^*\mathbb{R}^n), \gamma)$$

and have constructed a map

$$H^\eta : \mathbb{A}_j \to (C^0_{\text{fc}}(T^*\mathbb{R}^n), d_{C_0})$$

which is continuous and satisfies $\tau_a H^\eta = H^\eta$. Note that we may replace if needed C^0_{fc} by C^k_{fc} by applying convolution since $\tau_a(H \star \chi) = (\tau_a H) \star \chi = H \star \chi$ (and of course, since $\|H \star \chi - H\| \to 0$ as $\chi \to \delta_0$, we also have γ_c-convergence).

Let us summarize our findings combining the results of Proposition 7.2 and the conclusions of Sections 5 and 6:

Corollary 7.5. *Let $H : T^*\mathbb{R}^n \times \Omega \to \mathbb{R}$ satisfy assumptions (1)–(6) of the Main Theorem. Define $\pi_d : \Omega \to \mathbb{A}_d = T^d$ be the projection defined in Section 6. Then, given $\varepsilon > 0$, there exist $d \in \mathbb{N}$ and $H^\varepsilon : T^*\mathbb{R}^n \times T^d \to \mathbb{R}$ such that:*

[17] We also can take $R \simeq \eta\|H\|_{C^{0,1}}$, and then $C(r) \simeq Cr\eta\|H\|_{C^{0,1}}$ but this requires H to be Lipschitz. But this proves that the map σ_η does increase the Lipschitz norm by a bounded multiplicative constant only.

(1) *$\omega \mapsto H_\omega^\varepsilon$ is continuous from T^d to $C^\infty_{\mathrm{fc}}(T^*\mathbb{R}^n, \mathbb{R})$.*

(2) *$\gamma(H_\omega, H^\varepsilon_{\pi_d(\omega)}) \leq \varepsilon$ for all $\omega \in \Omega$.*

(3) *The Hamiltonians $H^\varepsilon_\omega, H^\varepsilon_{\pi_d(\omega)}$ satisfy assumptions (1)–(6).*

Proof. From Section 5 we get H from $\mathbb{A}_\Omega$ to $\widehat{\mathfrak{Ham}}_c(T^*\mathbb{R}^n)$. From Section 6 we can approximate H by a map from T^d to $\widehat{\mathfrak{Ham}}_c(T^*\mathbb{R}^n)$ and from the present section, we have an approximating map to $C^\infty_{\mathrm{fc}}(T^*\mathbb{R}^n, \mathbb{R})$. □

8. Homogenization in the almost periodic case

We assume in this section that we have a map $(q, p; \omega) \mapsto H(q, p; \omega) = H_\omega(q, p)$ such that:

(1) $\omega \in \Omega = T^d$.

(2) The map $\omega \mapsto H_\omega$ is continuous for the C^∞_{fc} topology. In particular the H_ω have uniformly fiberwise compact support and the H_ω are uniformly BPS by Proposition 3.3.

We set φ^t_ω to be the time t flow for H_ω and $\varphi_{\varepsilon,\omega} = \rho_\varepsilon^{-1} \varphi_\omega^{1/\varepsilon} \rho_\varepsilon$. By the compactness of Ω we also have a map $\omega \mapsto S_\omega(q, p; \xi)$ of GFQI for $\varphi_\omega = \varphi^1_\omega$, with ξ living in a vector space independent from ω: indeed its dimension is bounded by $2nN$ such that $\varphi_\omega^{1/N}$ is in a given neighborhood of id for all $\omega \in \Omega$ (see Appendix A for the number of fiber variables needed for a GFQI).

As we are going to use a number of results from [Viterbo 2023]. We will assume in the sequel that $\varepsilon = \frac{1}{k}$ and write ρ_k for $\rho_{1/k}$, h_k for $h_{1/k}$ and so on.

Definition 8.1. We set

$$h^\omega_{k,U}(p) = \lim_{V \ni p} c(\mu_{U \times V}, \varphi_{k,\omega})$$

and

$$h^\omega_k = \lim_{U \in \mathbb{R}^n} h^\omega_{k,U}.$$

Proposition 8.2. *The sequence h^ω_k is equicontinuous and equibounded. All its converging subsequences have the same limit $h_\omega(p)$, which is in fact independent from ω and denoted by $\overline{H}(p)$. We denote by $\varphi^t_{\overline{H}}$ the flow of $\overline{H}$ in $\widehat{\mathfrak{DHam}}_{\mathrm{fc}}(T^*\mathbb{R}^n)$ which belongs to $\widehat{\mathfrak{DHam}}_{\mathrm{FP}}(T^*\mathbb{R}^n)$.*

Proof. Let us start to examine what happens for fixed ω. For typographical reasons, the ω parameter will be omitted in the notation, but of course, everything depends on $\omega \in \Omega$, and the ω subscript will be reinstated when we prove that h_ω does not depend on ω.

Set $\varphi_k(q, p) = (Q_k(q, p), P_k(q, p))$ and $Q = Q_1$, $P = P_1$. By the definition of S_k we have

$$\frac{\partial S_k}{\partial \xi}(q, P_k(q, p); \xi) = 0 \quad \text{and} \quad \frac{\partial S_k}{\partial p}(q, P_k(q, p); \xi) = Q_k(q, p) - q.$$

By assumption we have

$$h_{k,U}(p) = S_k(q(p), p; \xi(p)),$$

where $(q(p), p; \xi(p))$ satisfies

$$\frac{\partial S_k}{\partial \xi}(q(p), p; \xi) = 0$$

and

$$\frac{\partial S_k}{\partial q}(q(p), p; \xi) = \begin{cases} 0 & \text{if } q \in U, \\ \lambda \cdot \nu_U(q) & \text{if } q \in \partial U \text{ and } \nu_U(q) \text{ is the exterior normal.} \end{cases}$$

Now as p varies, we can choose $p \mapsto (q(p), \xi(p))$ to be piecewise smooth, so that for p in the smooth locus

$$dh_{k,U}(p) = \frac{\partial S_k}{\partial p}(q(p), p; \xi(p)) + \frac{\partial S_k}{\partial q}(q(p), p; \xi(p)) \cdot \frac{\partial q}{\partial p} + \frac{\partial S_k}{\partial \xi}(q(p), p; \xi(p)) \cdot \frac{\partial \xi}{\partial p}.$$

Then we have

$$\frac{\partial S_k}{\partial \xi}(q, P_k(q, p); \xi) = 0 \quad \text{and} \quad \frac{\partial S_k}{\partial p}(q, P_k(q, p); \xi) = Q_k(q, p) - q.$$

But

$$h_{k,U}(p) = S_k(q(p), p; \xi(p)),$$

where

$$\frac{\partial S_k}{\partial \xi}(q(p), p; \xi) = 0$$

and

$$\frac{\partial S_k}{\partial q}(q(p), p; \xi) = \begin{cases} 0 & \text{if } q \in U, \\ \lambda \cdot \nu_U(q) & \text{if } q \in \partial U \text{ and } \nu_U(q) \text{ is the exterior normal.} \end{cases}$$

But if $q \in \partial U$, then $\frac{\partial q}{\partial p} \in T(\partial U)$, so that the term $\frac{\partial S_k}{\partial q}(q(p), p; \xi(p)) \cdot \frac{\partial q}{\partial p}$ also vanishes. We thus proved that where $h_{k,U}$ is smooth, we have

$$dh_k(p) = \frac{\partial S_k}{\partial p}(q(p), p; \xi(p)) = Q_k(q(p), p) - q(p) = \frac{1}{k}(Q(kq, p) - kq).$$

The assumption of finite propagation speed implies that this last quantity is uniformly bounded, so $|dh_{k,U}(p)|$ is uniformly bounded (independently from k, U).

From this we conclude that the sequence h_k is equicontinuous. Equiboundedness follows from Definition 4.8 in [Viterbo 2023] (or Proposition 9.1 of the current paper), which states that a GFQI S_k of φ_k is given by

$$S_k(q, p; \zeta) = \frac{1}{k}\left[S(kq, p_1) + \sum_{j=2}^{k-1} S(kq_j, p_j) + S((kq_k, p)\right] + B_k(q, p; \zeta),$$

where $S(q, p; \zeta) = S_1(q, p; \zeta)$ is a GFQI for $\varphi = \varphi_1$, $\zeta = (p_1, q_2, \dots, p_{k-1}, q_k)$ and B_k is a nondegenerate quadratic form. As a result $|S_k - B_k| \le C$, where C is a bound for $|S(q, p; \zeta) - B_1(q, p; \zeta)|$.

This implies that $|h_k(p)| \le C$ and since all these estimates are uniform in ω, this implies (uniform) equiboundedness.

We may thus apply the Arzelà–Ascoli theorem, and conclude that h_k^ω has a converging subsequence. Proving that the limit is unique follows as in [Viterbo 2023, Lemma 4.11 and Proposition 4.12].

Finally we prove that $h_\omega(p)$ is independent from ω, using the commutation of τ_a and ρ_k. We have

$$\begin{aligned} h_{k,\tau_a\omega}(p) &= \lim_{U\subset\mathbb{R}^n} c(\mu_U\otimes 1_p, \Gamma(\varphi_{k,\tau_a\omega})) \\ &= \lim_{U\subset\mathbb{R}^n} c(\mu_U\otimes 1_p, \Gamma(\tau_a^{-1}\varphi_{k,\omega}\tau_a)) \\ &= \lim_{U\subset\mathbb{R}^n} c(\mu_{\tau_a U}\otimes 1_p, \Gamma(\varphi_{k,\omega})) = h_{k,\omega}(p). \end{aligned}$$

Since $\omega\mapsto\varphi_{k,\omega}$ is γ-continuous, we infer that $\omega\mapsto h_{k,\omega}(p)$ is continuous and we just proved that it is τ-invariant. Ergodicity then implies that it is constant in ω. □

We define

$$\widehat{\mathfrak{Ham}}_{fc,BP}(T^*\mathbb{R}^n) = \widehat{\mathfrak{Ham}}_{\mathrm{BP}}(T^*\mathbb{R}^n)\cap\widehat{\mathfrak{Ham}}_{\mathrm{fc}}(T^*\mathbb{R}^n).$$

From now on we write $\bar\varphi^t$ instead of $\varphi^t_{\overline{H}}$ for typographical reasons.

The next proposition is the analog of Proposition 4.15 in [Viterbo 2023].

Proposition 8.3. *Let $\alpha\in\widehat{\mathfrak{DHam}}_{fc,BP}(T^*\mathbb{R}^n)$. There exists a sequence k_ν such that*

$$\lim_{\nu\to+\infty}\lim_{U\subset\mathbb{R}^n} c(\mu_U,\varphi_{k_\nu,\omega}\alpha)\le\lim_{U\subset\mathbb{R}^n} c(\mu_U,\bar\varphi\alpha).$$

Proof. The proof is identical to the proof of Proposition 4.15 in Section 4 of [Viterbo 2023] and can be found in Appendix D. □

The next proposition is the analog of Proposition 6.2 in [Viterbo 2023], but requires an adaptation. It will be proved in Section 9.

Proposition 8.4. *For each $\varepsilon>0$ there exists K such that, for all $k\ge K$ and U large enough, we have*

$$c(\mu_U\otimes 1_p,\varphi_{k,\omega})\le c(1_U\otimes 1_p,\varphi_{k,\omega})+\varepsilon.$$

This implies:

Corollary 8.5. *We have $\overline{\varphi^{-1}}=(\bar\varphi)^{-1}$, or equivalently $\overline{H}_{\varphi^{-1}}=-\overline{H}_\varphi$.*

Now putting together Proposition 8.3 and Corollary 8.5 we get:

Proposition 8.6. *For almost all $\omega\in\Omega$, the sequence $\varphi_{k,\omega}$ γ_∞-converges to $\bar\varphi$.*

Proof assuming Corollary 8.5 and Proposition 8.3. Let us prove the above proposition as a consequence of Corollary 8.5 and Proposition 8.3. Indeed Proposition 8.3 implies

$$\lim_{k\to+\infty}\lim_U c(\mu_U,\varphi_{k,\omega}\bar\varphi^{-1})\le\lim_U c(\mu_U,\mathrm{id})=0.$$

Applying the same inequality for φ^{-1} instead of φ and using the corollary, we get

$$\lim_{k\to+\infty}\lim_U c(\mu_U,\varphi_{k,\omega}^{-1}\bar\varphi)\le\lim_U c(\mu_U,\mathrm{id})=0$$

and this implies

$$\lim_{k\to+\infty}\lim_U\gamma(\mu_U,\varphi_{k,\omega}^{-1}\bar\varphi)=0,$$

which proves our claim. □

Proof of Corollary 8.5 assuming Proposition 8.4. Set

$$h^+_{k,\omega}(\varphi;p) = \lim_{U\subset\mathbb{R}^n} c(\mu_U\otimes 1(p),\varphi_{k,\omega}),$$
$$h^-_{k,\omega}(\varphi;p) = \lim_{U\subset\mathbb{R}^n} c(1_U\otimes 1(p),\varphi_{k,\omega})$$

so that $h^-_{k,\omega}(\varphi;p)\le h^+_{k,\omega}(\varphi;p)$. Set $\sigma_{p_0}(q,p)=(q,p+p_0)$. If $S(q,p;\xi)$ is a GFQI for φ, then $S_p(x;\xi)=S(x,p;\xi)$ is a GFQI for $\sigma_p(0_{\mathbb{R}^n})-\varphi(\sigma_p(0_{\mathbb{R}^n}))$. If we assume φ has FPS we have from Proposition 4.16

$$c(\mu_U,\sigma_p(0_{\mathbb{R}^n})-\varphi(\sigma_p(0_{\mathbb{R}^n})))\le c(\mu_V,\sigma_{-p}\varphi^{-1}(\sigma_p(0_{\mathbb{R}^n})))$$

for V such that $\varphi(T^*U)\subset T^*V$. Taking the limit for $U\subset\mathbb{R}^n$ we get

$$\lim_{U\subset\mathbb{R}^n} c(\mu_U,S_p)=\lim_{U\subset\mathbb{R}^n} c(\mu_U,\sigma_{-p}\varphi^{-1}\sigma_p(0_{\mathbb{R}^n}))$$

and the same holds for 1_U instead of μ_U. Now we may write (again omitting the ω) using first Proposition 4.10(1) and then FPS of φ

$$\begin{aligned} h^+_k(\varphi^{-1};p) &= \lim_{U\subset\mathbb{R}^n} c(\mu_U\otimes 1(p),\sigma_{-p}\varphi_k\sigma_p(0_{\mathbb{R}^n}))\\ &= -\lim_{U\subset\mathbb{R}^n} c(1_U\otimes 1(p),0_{\mathbb{R}^n}-\sigma_{-p}\varphi_k\sigma_p(0_{\mathbb{R}^n}))\\ &\le -\lim_{V\subset\mathbb{R}^n} c(1_V\otimes 1(p),\sigma_{-p}\varphi_k^{-1}\sigma_p(0_{\mathbb{R}^n}))=-h^-_k(\varphi;p).\end{aligned}$$

As a result

$$h^+_k(\varphi^{-1};p)+h^-_k(\varphi;p)\le 0 \tag{a}$$

and as k goes to $+\infty$, Proposition 8.4 implies

$$h^+_k(\varphi^{-1};p)-h^-_k(\varphi;p)\le\varepsilon$$

and we get

$$h^+_k(\varphi^{-1};p)+h^+_k(\varphi;p)\le\varepsilon. \tag{b}$$

On the other hand, we have using again Proposition 4.10(1)

$$-c(1_U,\sigma_{-p}\varphi_k\sigma_p(0_{\mathbb{R}^n}))\le -c(1_V,0_{\mathbb{R}^n},\sigma_{-p}\varphi_k^{-1}\sigma_p(0_{\mathbb{R}^n}))=c(\mu_V,\sigma_{-p}\varphi_k^{-1}\sigma_p(0_{\mathbb{R}^n})),$$

so

$$-h^-_k(\varphi;p)\le h^+_k(\varphi;p),$$

and using (a) we get

$$h^+_k(\varphi;p)+h^-_k(\varphi;p)=0. \tag{c}$$

Using again Proposition 8.4 we get for k large enough

$$h^-_k(\varphi^{-1};p)+h^-_k(\varphi;p)\ge-\varepsilon. \tag{d}$$

Adding (b) and (d) we get

$$[h^+_k(\varphi^{-1};p)-h^-_k(\varphi^{-1};p)]+[h^+_k(\varphi;p)-h^-_k(\varphi;p)]\le 2\varepsilon. \tag{e}$$

Since $\overline{H}_{\varphi^{-1}} = \lim_k h_k^+(\varphi^{-1}; p)$, inequality (b) implies

$$\overline{H}_{\varphi^{-1}} + \overline{H}_{\varphi} \le 0.$$

Using (d) and (e) we get

$$\overline{H}_{\varphi^{-1}} + \overline{H}_{\varphi} \ge 0$$

so we may conclude

$$\overline{H}_{\varphi^{-1}} + \overline{H}_{\varphi} = 0. \qquad \square$$

9. Proof of Proposition 8.4

We shall interchangeably use the notation $S_\omega(q, p; \xi)$ and $S(q, p; \xi; \omega)$ for the GFQI of φ_ω. We shall make repeated use of the iteration formula (see [Viterbo 2023, Lemma 4.5]), defining the GFQI $S_{k,\omega}$ for $\varphi_{k,\omega}$ in terms of the GFQI S_ω of φ_ω.

Proposition 9.1 (iteration formula). *Let S_ω be a GFQI for φ_ω. Then the following formula defines a GFQI for $\varphi_{k,\omega}$:*

$$S_{k,\omega}(x, y; \zeta, \xi) = \frac{1}{k}\left[S_\omega(kx, p_1; \xi_1) + \sum_{j=2}^{k-1} S_\omega(kq_j, p_j; \xi_j) + S_\omega(kq_k, y; \xi_k)\right] + B_k(x, y; \zeta),$$

where $\zeta = (p_1, q_2, \dots, p_{k-1}, q_k)$, $q_1 = x$, $p_k = y$, $\xi = (\xi_1, \dots, \xi_k)$ and

$$B_k(x, y; \zeta) = \langle p_1, q_2 - x\rangle + \sum_{j=2}^{k-1} \langle p_j, q_{j+1} - q_j\rangle + \langle y, x - q_k\rangle.$$

We shall set $F_{k,\omega} = S_{k,\omega} - B_k$.

The action of $\mathbb{R}^n$ is given by

$$\tau_a^{(k)}(x, y; \xi, \zeta; \omega) = \Big(x + \frac{a}{k}, y; \xi; \tau_{a/k}\zeta; \tau_a\omega\Big).$$

Remark 9.2. We will mostly use this formula when $S(q, p, \xi) = S(q, p)$, i.e., we have no fiber variables for S.

Lemma 9.3. *Assume $\omega \mapsto \varphi_\omega$ for $\omega \in \Omega = T^d$ to be continuous. Then we may choose $\omega \mapsto S_\omega(q, p; \xi)$ to be continuous and such that*

$$S(q + a, p; \tau_a\xi; \tau_a\omega) = S(q, p; \xi; \omega).$$

Proof. It is enough to prove this assuming φ_ω is C^1 small, that is, for $\varphi_\omega^{1/N}$ with N large enough, and then use iteration formula. But then the graph of φ_ω is the graph of a generating function with no fiber variable, which obviously depends continuously on ω and satisfies the above formula. $\square$

Now remember that τ_a is given on $\Omega = T^d$ by $\tau_a(\omega) = \omega + A\cdot a$, where $A : \mathbb{R}^n \to \mathbb{R}^d$ is a linear injective map with dense image in T^d. Consider triples α, β, γ, with $\alpha \in H^*(T^d)$, $\beta \in H^*(U)$ or $H^*(U, \partial U)$, $\gamma \in H^*(V)$ or $H^*(V, \partial V)$. We may then define[18] $c(\alpha \otimes \beta \otimes \gamma, S)$, and we have:

[18] Caveat: the cohomology class α corresponds to the last variable, ω!

Lemma 9.4. *We have the inequalities*

$$c(\mu_U\otimes 1(p);S_\omega)\le c(\mu_{T^d}\otimes\mu_U\otimes 1(p);S),$$
$$c(1_{T^d}\otimes\mu_U\otimes 1(p);S)\le c(1_U\otimes 1(p);S_\omega).$$

Proof. This is the reduction inequality (see [Viterbo 1992, Proposition 5.1, p. 705]). □

We now compare spectral invariants of S with those of S^0, where we define $S^0(p;\xi;\omega)=S(0,p;\xi;\omega)$.

Lemma 9.5. *We have*

$$\lim_{U\subset\mathbb{R}^n} c(\mu_{T^d}\otimes\mu_U\otimes 1(p);S)=c(\mu_{T^d}\otimes 1(0)\otimes 1(p);S)=c(\mu_{T^d}\otimes 1(p);S^0),$$
$$\lim_{U\subset\mathbb{R}^n} c(1_{T^d}\otimes\mu_U\otimes 1(p);S)=c(1_{T^d}\otimes 1(0)\otimes 1(p);S)=c(1_{T^d}\otimes 1(p);S^0).$$

Remarks 9.6. (1) The point of replacing S by S^0 is to avoid the complications related to the noncompactness of $x\in\mathbb{R}^n$. Our proofs could be adapted to work directly with S, but proving that the cycles we construct are in the right homology class is slightly more involved.

(2) This is an extension to GFQI of the following obvious identity for continuous functions $f:\mathbb{R}^n\times T^d\to\mathbb{R}$ such that $f(x+a,\tau_a\omega)=f(x,\omega)$: for any $x_0\in\mathbb{R}^n$ we have

$$\sup_{(x,\omega)\in\mathbb{R}^n\times T^d} f(x,\omega)=\sup_{\omega\in T^d} f(x_0,\omega).$$

Moreover if the action of τ has dense orbits, this is also equal to $\sup_{x\in\mathbb{R}^n} f(x,\omega_0)$ for any $\omega_0\in\Omega$. The analog of this last statement will be our main result.

Proof. Clearly if $0\in U$, we have

$$c(\mu_{T^d}\otimes\mu_U\otimes 1(p);S)\ge c(\mu_{T^d}\otimes 1(0)\otimes 1(p);S)$$

and we need to prove the reverse inequality. Let C be a cycle representing $\mu_{T^d}\otimes 1(p)\in H_*((S_p^0)^c,S_p^0)^{-\infty})$ with $c\le c(\alpha\otimes 1(0)\otimes 1(p),S)+\varepsilon$ and set

$$\widetilde{C}_U=\{(x,p,\tau_x\xi;\tau_x\omega)\mid(0,p;\xi;\omega)\in C,x\in U\}.$$

Then $\widetilde{C}_U\subset S_p^c$ and clearly $[\widetilde{C}_U]=\mu_{T^d}\otimes\mu_U\otimes 1(p)$. The above is in fact an abuse of language for $f_*(\mu_U\otimes[C])$, where

$$f:U\times((S_p^0)^c,(S_p^0)^{-\infty})\to((S_p)^c,(S_p)^{-\infty})$$

is defined by $f(x;(0,p,\xi,\omega))=(x,p,\tau_x\xi,\tau_x\omega)$.

Thus

$$c(\mu_{T^d}\otimes\mu_U\otimes 1(p),S)\le S(\widetilde{C}_U)=S^0(C)$$

because $S(x,p,\tau_x\xi,\tau_x\omega)=S(0,p;\xi;\omega)$ and $S^0(C)\le c$.

This implies

$$c(\mu_{T^D}\otimes\mu_U\otimes 1(p);S)\le c(\mu_{T^D}\otimes 1(0)\otimes 1(p);S)$$

and proves the first equality. The second one is the dual of the first one, since $\mu_{T^d}\otimes\mu_U$ is dual to $1_{T^d}\otimes 1(U)$. □

Our Proposition 8.4 then follows from:

Proposition 9.7. *For each $\varepsilon > 0$ there exists K such that for $k \geq K$*

$$c(\mu_{T^d} \otimes 1(p), S_k^0) \leq c(1_{T^d} \otimes 1(p), S_k^0) + \varepsilon.$$

Remark 9.8. The idea behind the proof is that as we homogenize, the difference between the largest and smallest spectral invariants goes to zero. The proof is a Hamiltonian version of the following ancient result [Acerbi and Buttazzo 1983] that states that if we replace a metric g by a rescaled version g_k, so that the distance $d(x, y)$ becomes $d_k(x, y) = \frac{1}{k} d(k \cdot x, k \cdot y)$, then $\lim_{k\to\infty} d_k(x, y) = \bar{d}(x, y)$ is the distance associated to a flat Finsler metric, g_∞. In particular on a 2-torus for each homotopy class α of loops, α, there are two "spectral values" associated to the geodesic problem $l_1(g, \alpha) \leq l_2(g, \alpha)$, where $l_1(g, \alpha)$ is the shortest geodesic in the homotopy class α, while $l_2(g, \alpha)$ is the "second shortest", i.e., given by the Birkhoff minmax procedure:

$$l_2(g, \alpha) = \inf\Big\{c \;\Big|\; \exists \gamma_s \in C_\alpha^\infty(S^1, T^2),\ s \in S^1,\ \int_{S^1} |\dot{\gamma}_s(t)|\, dt \leq c,\ [s \mapsto \gamma_s(0)] \in \beta \neq \alpha\Big\}.$$

One then checks that $\lim_{k\to+\infty} l_1(g_k, \alpha) = \lim_{k\to\infty} l_2(g_k, \alpha) = l_1(g_\infty, \alpha) = l_2(g_\infty, \alpha)$. Our proof is the analog of the proof of the inequality $l_2(g_k, \alpha) \leq l_1(g_\infty, \alpha) + \varepsilon$ for k large enough, which obviously implies $\lim_{k\to+\infty} l_1(g_k, \alpha) = \lim_{k\to\infty} l_2(g_k, \alpha)$.

Proof. The proof will take up the rest of the section. We rewrite the iteration formula

$$S_{k,\omega}(x, y; \zeta; \omega) = \frac{1}{k}\left[S_\omega(kx, p_1) + \sum_{j=2}^{k-1} S_\omega(kq_j, p_j) + S_\omega(kq_k, y)\right] + B_k(x, y; \zeta),$$

where $\zeta = (p_1, q_2, \ldots, p_{k-1}, q_k),\ q_1 = x,\ p_k = y$ and

$$B_k(x, y; \zeta) = \langle p_1, q_2 - x\rangle + \sum_{j=2}^{k-1} \langle p_j, q_{j+1} - q_j\rangle + \langle y, x - q_k\rangle$$

and $F_{k,\omega} = S_{k,\omega} - B_k$. The action of $\mathbb{R}^n$ is given by

$$\tau_a^{(k)}(x, y; \zeta; \omega) = \Big(x + \frac{a}{k}, y; \tau_{a/k}\zeta; \tau_a\omega\Big)$$

and now $S_{k,\omega}$ is $\tau_a^{(k)}$-invariant, i.e.,

$$S_k\Big(x + \frac{a}{k}, y; \tau_{a/k}\zeta; \tau_a\omega\Big) = S(x, y; \zeta; \omega).$$

Let $a \in \mathbb{R}^n$ such that for some $\nu \in \mathbb{Z}^d$ we have $|A \cdot a - \nu| \leq \delta$ (that is, $d_{T^d}(\tau_a(0), 0) \leq \delta$, where d_{T^d} is the distance on the torus). Then for some constant depending on H and provided δ is small enough

$$\forall t \in [0, 1],\ \forall (q, p; \xi; \omega) \in \mathbb{R}^n \times \mathbb{R}^n \times E \times \Omega, \quad |S(kq + ta, p; \xi; \omega) - S(kq, p; \xi; \omega)| \leq C \qquad (\star)$$

and

$$\forall (q, p; \xi; \omega) \in \mathbb{R}^n \times \mathbb{R}^n \times E \times \Omega,$$
$$|S(kq + a, p; \xi; \omega) - S(kq, p; \xi; \omega)| = |S(kq, p; \xi; \tau_{-a}\omega) - S(kq, p; \xi; \omega)| \leq \varepsilon. \qquad (\star\star)$$

Indeed the first inequality holds because

$$|S(q+a,p;\xi;\omega)-S(q,p;\xi;\omega)| = |S(q,p;\xi;\tau_{-a}\omega)-S(q,p;\xi;\omega)| \le \sup_{\omega,\omega'} |S(q,p;\xi;\omega)-S(q,p;\xi;\omega')|.$$

This follows by using the iteration formula. In this case we may assume $|S(q,p;\omega)-S(q,p;\omega')| \le \gamma(\varphi_\omega,\varphi_{\omega'})$. The second inequality follows from the fact that $d_{T^d}(\tau_a\omega,\omega)\le\delta$ and by the continuity of S.

Now let γ be the path in $\mathbb{R}^n$ defined by $\gamma(t)=t\cdot a$ for $0\le t\le 1$. Set $\tilde{\gamma}^{(k)}$ to be the path in $(\mathbb{R}^n)^k$ defined as the concatenation of the k paths

$$\begin{aligned}
&t\mapsto(\gamma(t),0,\dots,0) && \text{for } t\in\Big[0,\frac{1}{k}\Big],\\
&t\mapsto\Big(\gamma\Big(\frac{1}{k}\Big),\gamma\Big(t-\frac{1}{k}\Big),\dots,0\Big) && \text{for } t\in\Big[\frac{1}{k},\frac{2}{k}\Big],\\
&\vdots && \vdots\\
&t\mapsto\Big(\gamma\Big(\frac{1}{k}\Big),\gamma\Big(\frac{1}{k}\Big),\dots,\gamma\Big(\frac{1}{k}\Big),\gamma\Big(t-\frac{k-1}{k}\Big)\Big) && \text{for } t\in\Big[\frac{k-1}{k},1\Big].
\end{aligned}\tag{9-1}$$

The path $\tilde{\gamma}^{(k)}$ connects $\tilde{\gamma}^{(k)}(0)=(0,\dots,0)$ to $\tilde{\gamma}^{(k)}(1)=\big(\frac{a}{k},\frac{a}{k},\dots,\frac{a}{k}\big)$ through the points

$$\tilde{\gamma}^{(k)}\Big(\frac{1}{k}\Big)=\Big(\frac{a}{k},0,\dots,0\Big),\quad \tilde{\gamma}^{(k)}\Big(\frac{2}{k}\Big)=\Big(\frac{a}{k},\frac{a}{k},0,\dots,0\Big),\quad \dots,\quad \tilde{\gamma}^{(k)}\Big(\frac{k-1}{k}\Big)=\Big(\frac{a}{k},\frac{a}{k},\dots,\frac{a}{k},0\Big).$$

We shall omit the superscript k and set $\tilde{\gamma}^{(k)}(t)=\tilde{\gamma}(t)=(\gamma_1(t),\dots,\gamma_k(t))=(\gamma_1(t),\bar{\gamma}(t))$. We then set $\tau_{\bar{\gamma}(t)}\zeta=\tau_{\bar{\gamma}(t)}(p_1,q_2,\dots,p_{k-1},q_k)=(p_1,q_2+\gamma_2(t),\dots,p_{k-1},q_k+\gamma_k(t))$ and $\tau_{\tilde{\gamma}(t)}(x,y;\zeta)=(x+\gamma_1(t),y;\tau_{\bar{\gamma}(t)}\zeta)$. Now from $(\star)$ and $(\star\star)$ and the formula

$$F_k(x,y;\xi;\zeta;\omega)=\frac{1}{k}\Big[S_\omega(kx,p_1)+\sum_{j=2}^{k-1}S_\omega(kq_j,p_j)+S_\omega(kq_k,y)\Big],$$

we infer that on $\big[\frac{l}{k},\frac{l+1}{k}\big]$ for $1\le l\le k$

$$\begin{aligned}
F_k(\tau_{\tilde{\gamma}(t)}(x,y;\zeta;\omega)) = F_{k,\omega}(x,y;\zeta;\omega)+\frac{1}{k}\Big[&S(kx+a,p_1;\zeta;\omega)-S(kx,p_1;\omega)\\
&+\sum_{k=2}^{l}(S(kq_j+a,p_j;\omega)-S(kq_j,p_j;\omega))\\
&+S\Big(kq_{l+1}+\Big(t-\frac{l}{k}\Big)a,p_{l+1};\omega\Big)-S(kq_{l+1},p_{l+1};\omega)\Big],
\end{aligned}$$

so we get

$$|F_k(\tau_{\tilde{\gamma}(t)}(x,y;\zeta);\omega)-F_k(x,y;\zeta;\omega)| \le \frac{\varepsilon l}{k}+\frac{C}{k}\le\frac{C}{k}+\varepsilon. \tag{9-2}$$

We now want to estimate the variation of B_k on $\tau_{\tilde{\gamma}(t)}(x,y;\zeta)$ as t varies from 0 to 1. Note that the choice of this path is crucial to our argument: by changing coordinates one at the time, we achieve an increase of S by $O\big(\frac{1}{k}\big)$ instead of $O(1)$.

Lemma 9.9. *We have*

$$|B_k(\tau_{\tilde{\gamma}(t)}(x,y;\zeta))-B_k(x,y;\zeta)|\le(|p_{l+2}-p_{l+1}|+|p_{l+1}-p_l|)\frac{|a|}{k}.$$

Proof. Indeed,

$$B_k(x,y;\zeta)=\langle p_1,q_2-x\rangle+\sum_{j=2}^{k-1}\langle p_j,q_{j+1}-q_j\rangle+\langle y,x-q_k\rangle$$

and replacing $(x,q_2,\dots,q_l)$ by $\left(x+\frac{a}{k},q_2+\frac{a}{k},\dots,q_l+\frac{a}{k}\right)$ then q_{l+1} by $q_{l+1}+\left(t-\frac{l}{k}\right)a$, with $t\in\left[\frac{l}{k},\frac{l+1}{k}\right]$, and leaving $q_{l+2},\dots,q_k$ unchanged, we get for $t\in\left[\frac{l}{k},\frac{l+1}{k}\right]$

$$B_k(\tau_{\tilde{\gamma}(t)}(x,y;\zeta))=B_k(x,y;\zeta)+\left\langle p_{l+1}-p_l,\left(t-\frac{l}{k}\right)a-\frac{a}{k}\right\rangle-\left\langle p_{l+2}-p_{l+1},\left(t-\frac{l}{k}\right)a\right\rangle$$

and this proves the lemma since for t in $\left[\frac{l}{k},\frac{l+1}{k}\right]$, $\left|\left(t-\frac{l}{k}\right)a-\frac{a}{k}\right|$ and $\left|\left(t-\frac{l}{k}\right)a-\frac{a}{k}\right|$ are bounded by $\frac{|a|}{k}$. □

We must then bound the quantity $(|p_{l+2}-p_{l+1}|+|p_{l+1}-p_l|)\frac{|a|}{k}$ and we shall modify the cycle C representing the class in $H_k(S_k^c,S_k^{-\infty})$ so that the $|p_l|$ remain bounded. This follows from the lemma below.

Lemma 9.10 [Viterbo 2023, Lemma 6.5]. *There exist constants K,M such that, given a cycle $C\subset S_k^c$ representing a class $[C]\in H_*(S_k^c,S_k^{-\infty})$, we have a cycle $\widetilde{C}\subset S_k^c$ such that $[\widetilde{C}]=[C]$ in $H_*(S_k^c,S_k^{-\infty})$ and*

(1) $\widetilde{C}\subset S_k^{-4K}\cup(\{(x,y;\zeta;\omega)\mid \max_j|p_j|\le M\}\cap S_k^c)$,

(2) $\widetilde{C}\cap S_k^{-3K}\subset\{(x,y;\zeta;\omega)\mid \zeta\in E_k^-\}$, *where E_k^- is the negative eigenspace of B_k.*

The lemma means that we can deform C so that below a certain level of S_k it coincides with the negative bundle of B_k.

Proof. This is as in [Viterbo 2023, Lemma 6.5]. Let Z be the vector field defined by

$$\dot{q}_j=\chi(|p_j|)(p_{j+1}-p_j)=Z_{q_j}(q,p),\quad \dot{p}_j=0=Z_{p_j}(q,p),$$

where $\chi(r)$ vanishes for $r\le 1$. Denoting by ψ^s its flow, we have

$$\begin{aligned}\frac{d}{ds}S_k(\psi^s(q,p))&=dS_k(q,p)\cdot Z(q,p)=\left\langle\frac{\partial}{\partial q}S_k(q,p),Z_q(q,p)\right\rangle\\&=-\sum_{j=1}^{k}\chi(|p_j|)\left\langle\frac{d}{dq_j}S_k(q,p),p_{j+1}-2p_j+p_{j-1}\right\rangle\\&=-\sum_{j=1}^{k}\chi(|p_j|)|p_{j+1}-p_j|^2+\left\langle\frac{d}{dq}S_k(k\cdot q_j,p_j),p_{j+1}-p_j\right\rangle\\&=-\sum_{j=1}^{k}\chi(|p_j|)|p_{j+1}-p_j|^2;\end{aligned}$$

the last equality holds because S vanishes on the support of $\chi(|p_j|)$.

Now given $y=p_k$, if $\sup_j|p_j|\ge M$, we have that $\sum_{j=1}^{k}\chi(|p_j|)|p_{j+1}-p_j|^2$ is bounded from below by some positive quantity c_k (which is $O\left(\frac{1}{k}\right)$ but it does not matter). Thus, outside the region $\{(q,p)\mid |p_j|\le M\}$, the vector field Z is a pseudogradient vector field for F_k. Since Z is complete, its flow ψ^s has the following properties:

(1) It preserves the p_j.

(2) Outside $\{(q,p)\mid |p_j|\le M\}$, we have $\frac{d}{ds}S_k(\psi^s(q,p))\le -c_k$.

As a result if $F_k(q,p) \le c$, we have

$$\psi^{(c+4K)/c_k}(q,p) \in (\{(q,p) \mid |p_j| \le M\} \cup F_k^{-4K}) \cap F_k^c.$$

Thus $\widetilde{C}_1 = \psi^{(c+4K)/c_k}(C)$ satisfies (1).

Now to satisfy (2), we use a "cut and paste" as in [Viterbo 2023, Lemma 6.5]. □

Using Lemmas 9.9 and 9.10 and the inequality (9-2) we obtain the following:

Proposition 9.11. *Given a class a in $H_*(S_k^c, S_k^{-\infty})$, we can find a cycle C representing a and constants M_1, M_2 such that*

$$S_k(\tau_{\bar{\gamma}(t)}(C)) \le S_k(C) + \varepsilon + \frac{M_1}{k} + \frac{2M_2|a|}{k}.$$

Now let $a \in H_*(T^d)$ be represented by a map $u : C \to T^d$ and $b \in H_1(T^d)$ be represented by a map $v : S^1 \to T^d$. Then the Pontryagin product $a \cdot b$ is represented by $u \cdot v : S^1 \times C \to T^d$ given by $u \cdot v(z,\theta) = u(z) + v(\theta)$.

To conclude the proof of Proposition 9.7 (and as a consequence of Proposition 8.4) we need:

Lemma 9.12. *Let $\nu \in \mathbb{Z}^d$ be such that $|A \cdot a - \nu| \le \delta$, and let β_ν be the class in $H_1(T^d)$ of the loop $t \mapsto t \cdot \nu$ (for $t \in [0,1]$). Then given $\varepsilon > 0$, we have, for k large enough,*

$$c(\alpha \cdot \beta_\nu \otimes 1(p), S_k^0) \le c(\alpha \otimes 1(p), S_k^0) + \varepsilon.$$

Proof. Let C be a cycle representing a class in $H_*((S_k^0)^c, (S_k^0)^{-\infty})$. We may assume C satisfies properties (1) and (2). We are going to construct a cycle in the class of $\alpha \cdot \beta$ made of three pieces. First set

$$C_1 = \bigcup_{t \in [0,1]} C_1(t),$$

where

$$C_1(t) = \{(0, p; \tau_{-\gamma_1(t)}\tau_{\bar{\gamma}(t)}\zeta; \tau_{k\gamma_1(t)}\omega) \mid (0, p; \zeta; \omega) \in C\}.$$

According to Proposition 9.11 since

$$\begin{aligned} S_k(0, p; \tau_{-\gamma_1(t)}\tau_{\bar{\gamma}(t)}\zeta; \tau_{k\gamma_1(t)}\omega) &= S_k(\gamma_1(t), p; \tau_{\bar{\gamma}(t)}\zeta; \omega) \\ &= S_k(\tau_{\gamma(t)}(0, p; \zeta); \omega) \le S_k(C) + \varepsilon + \frac{M_1 + 2M_2|a|}{k}, \end{aligned}$$

as a result we have for each $t \in [0,1]$

$$S_k^0(C_1(t)) \le S_k^0(C) + \varepsilon + \frac{M_1 + 2M_2|a|}{k};$$

hence

$$S_k^0(C_1) \le S_k^0(C) + \varepsilon + \frac{M_1 + 2M_2|a|}{k}.$$

Note that

$$C_1(0) = C,$$

$$C_1(1) = \{(0, p; \zeta; \tau_a\omega) \mid (0, p; \zeta; \omega) \in C\}.$$

Now for $u \in [0,1]$ define the path $\eta(u) = (1-u)A\cdot a + u\nu$ so that $\eta(0)+\omega = \tau_a\omega$. Set

$$C_1(1+u) = \{(0,p;\zeta;\omega+\eta(u)) \mid (0,p;\zeta;\omega) \in C\}.$$

Now $C_1(2) = C$ and since $|\eta(u) - A\cdot a| \le \delta$, we have

$$S_k^0(C_1(1+u) \le S_k^0(C_1(1)) + \varepsilon$$

so that the cycle[19]

$$\widehat{C} = \bigcup_{s\in[0,2]} C_1(s)$$

satisfies for k large enough

$$S_k^0(\widehat{C}) \le S_k^0(C) + 2\varepsilon.$$

Moreover we claim that the cycle $\widehat{C}$ defines a cycle in the homology class of $\alpha\cdot\beta_\nu$. Indeed the lift of the variable ω starting from ω_0 is given

(1) for $s \in [0,1]$ by the path $s \mapsto \omega_0 + sA\cdot a$,

(2) for $s \in [1,2]$ by $s \mapsto \omega_0 + (2-s)A\cdot a + (s-1)\nu$,

and since it joins ω_0 to $\omega+\nu$, it belongs to the class β_ν. As a result $[\widehat{C}] = \alpha\cdot\beta \in H_*((S_k^0)^{+\infty},(S_k^0)^{-\infty})$ and this proves the lemma. □

We shall also need:

Lemma 9.13. *Let $\varepsilon > 0$ and $A : \mathbb{R}^n \to \mathbb{R}^d$ be a linear map such that $A(\mathbb{R}^n)$ has dense projection on T^d. Then there are integral vectors $\nu_1,\dots,\nu_d$ in $\mathbb{Z}^d$ forming a basis of $\mathbb{R}^d$ such that there exist vectors $a_1,\dots,a_d$ in $\mathbb{R}^n$ such that*

$$|A\cdot a_j - \nu_j| \le \varepsilon.$$

Proof. See Appendix B. □

Proof of Propositions 9.7 and 8.4. Let $\alpha_j \in H_1(T^d)$ be the homotopy class of the path $t \mapsto t\cdot\nu_j$, where ν_j is a basis of $\mathbb{R}^d$ given by Lemma 9.13. Then $\alpha_1\cdot\alpha_2\cdot\dots\cdot\alpha_d = c_d\mu_{T^d}$ for some $c_d \ne 0$. since $c(c_d\mu_{T^d}, f) = c(\mu_{T^d}, f)$ we obtain by repeated applications of Lemma 9.12 that $c(\mu_{T^d}\otimes 1(p), S_k^0) \le c(1_{T^d}\otimes 1(p), S_k^0) + \varepsilon$ and this proves Proposition 9.7 and hence Proposition 8.4. □

10. Proof of the Main Theorem

We first prove that under assumptions (1)–(6) we have $\lim_{\varepsilon\to 0}\varphi^t_{\varepsilon,\omega} = \varphi^t_{\overline{H}}$ for almost all $\omega \in \Omega$. We start from H satisfying (1)–(6), then, using the results of Section 5, we get a map $H : \mathbb{A}_\Omega \to \widehat{\mathfrak{Ham}}(T^*T^n)$ such that $\mathbb{A}_\Omega$ is a compact connected metric abelian group. According to Section 6, $\mathbb{A}_\Omega$ is the projective limit of finite-dimensional tori, $\mathbb{A}_j$, on which τ_a is given by $\tau_a\omega = \omega + A_j\cdot a$, where the projection of $A_j(\mathbb{R}^n)$ is dense in $\mathbb{A}_j$ and $\omega \mapsto H(\dots,\cdot;\omega)$ is continuous from $\mathbb{A}_j$ to $C^\infty_{\mathrm{fc}}(T^*\mathbb{R}^n,\mathbb{R})$ and satisfies (1)–(6).

[19] Similarly to the proof of Lemma 9.5, this is an abuse of language for $f_*(\mathbb{R}/2\mathbb{Z}\times C)$, where

$$f(t,(0,p;\zeta;\omega)) = \begin{cases}(0,p;\tau_{-\gamma_1(t)}\tau_{\bar\gamma(t)}\zeta;\tau_{k\gamma_1(t)}\omega) & \text{for } 0\le t\le 1,\\ (0,p;\zeta;\omega+\eta(t-1)) & \text{for } 1\le t\le 2.\end{cases}$$

By Corollary 7.5 we find H^η in $C^\infty(T^*\mathbb{R}^n \times \mathbb{A}_j, \mathbb{R})$ such that

$$\gamma_\infty(H^\eta_{\pi_j(\omega)}, H_\omega) \le \eta,$$

where $\pi_j : \mathbb{A}_\Omega \to \mathbb{A}_j$ is the projection map. According to Proposition 8.6, we know that

$$\gamma_c - \lim_k H^\eta_{k,\pi_j(\omega)} = \overline{H}^\eta_{\pi_j(\omega)}$$

and since, for all k, ω, we have $\gamma_\infty(H^\eta_{k,\pi_j(\omega)}, H_{k,\omega}) \le \eta$, we infer for k large enough

$$\gamma_\infty(\overline{H}^\eta_{\pi(\omega)}, H_{k,\omega}) \le 2\eta.$$

Now consider a sequence η_ν converging to 0 so that $H^{\eta_\nu}_{\pi_\nu(\omega)}$ is a γ_c-Cauchy sequence, γ_c-converging to H_ω uniformly in ω. Then $\overline{H}^{\eta_\nu}_{\pi_\nu(\omega)}$ is also a Cauchy sequence, so it converges to some $\overline{H} \in \widehat{\mathfrak{Ham}}(T^*T^n)$. But then $(H_{k,\omega})_{k\ge 1}$ converges a.s. in ω to $\overline{H}$.

For the second part of the Main Theorem, we must go from γ-convergence of the flow to γ-convergence of the solution of the corresponding Hamilton–Jacobi equation. In the case of a compact base this is achieved in [Viterbo 2006], and the extension to a noncompact base was spelled out in [Cardin and Viterbo 2008, pp. 266–276].

For $L \in \mathfrak{L}(T^*N)$ we define $u_L(x) = c(1_x, L)$. Our first claim is that γ-convergence for L implies C^0-convergence of the u_L uniformly on compact sets.

Lemma 10.1. *Let U be bounded domain in N. If $(L_\nu)_{\nu\ge 1}$ is a Cauchy sequence for γ_U, then the sequence u_{L_ν} is a Cauchy sequence for the topology of uniform convergence on U. As a result if $(L_\nu)_{\nu\ge 1}$ γ-converges to $L \in \widehat{\mathfrak{L}}(T^*N)$ then the sequence u_{L_ν} converges uniformly on compact sets to u_L.*

Proof. This is an immediate consequence of the reduction inequality [Viterbo 1992, Proposition 5.1, p. 705], which implies that, for any $x \in U$,

$$|c(1_x, L) - c(1_x, L')| \le \gamma_U(L, L'). \qquad \square$$

Proposition 10.2. *Let $(\varphi_\nu)_{\nu\ge 1}$ be a sequence in $\widehat{\mathfrak{DHam}}_{c,\mathrm{FP}}$ γ_c-converging to $\varphi_\infty \in \widehat{\mathfrak{DHam}}_{c,\mathrm{FP}}$. Then for any $L \in \mathfrak{L}(T^*\mathbb{R}^n)$ (or in $\widehat{\mathfrak{L}}(T^*\mathbb{R}^n)$) the sequence $\varphi_\nu(L)$ γ_c-converges to $\varphi_\infty(L)$.*

Proof. Indeed, we proved in Proposition 4.23 that $\gamma_U(\psi_1(L), \psi_2(L)) \le \gamma_{V\times B^n(r)}(\psi_1, \psi_2)$ provided $\psi_j^{\pm 1}$ sends T^*U to T^*V and $L \subset \mathbb{R}^n \times B(r)$. In our case, we get that for $L \subset \mathbb{R}^n \times B^n(r)$

$$\gamma_U(\varphi_\nu(L), \varphi_\infty(L)) \le \gamma_{V\times B^n(r)}(\varphi_\nu, \varphi_\infty)$$

and since the right-hand side converges to 0, so does the left-hand side. $\square$

We may now conclude our proof. Since a.s. in ω, $\varphi^t_{k,\omega}$ γ_∞-converges to $\bar\varphi^t$ and is uniformly FPS for bounded t, we have by Proposition 10.2

$$\varphi^t_{k,\omega}(L_f) \underset{\gamma_\infty}{\to} \bar\varphi^t(L_f))$$

a.s. in ω. Then applying Lemma 10.1 to the sequence $(\varphi^t_{k,\omega})_{k\ge 1}$, this implies uniform convergence on compact sets of the sequence $(u_{k,\omega})_{k\ge 1}$ to its limit $\bar u$. This concludes the proof of our Main Theorem.

11. The coercive case

We now assume H satisfies assumptions (1a)–(3a) of Corollary 1.3. Let χ_A be a truncation function, that is, an increasing function such that $0 \le \chi_A'(t) \le 1$ and $\chi_A(t) = t - \frac{3}{2}A$ for $t \le A$ and $\chi_A(t) = 0$ for $t \ge 2A$. We set $H_A(x, p; \omega) = \chi_A(H(x, p, \omega))$. Then coercivity implies[20] that H_A has a.s. in $\omega \in \Omega$ the same flow as H in $U_R = \{(x, p) \mid |p| \le r(A)\}$ where $\lim_{A\to+\infty} r(A) = +\infty$. We apply the Main Theorem to H_A and obtain a homogenized Hamiltonian $\overline{H}_A$. We claim now that for $B \ge A$ we have $\overline{H}_A = \overline{H}_B$ on U_R. This follows the same proof as Section 11 in [Viterbo 2023]. Because f is Lipschitz, it can be approximated by functions f_k which have a bounded differential, so the image of the graph of df_k remains in some domain bounded in the p-direction. Therefore for A large enough, $\varphi^t_{H_A}(G_{f_k}) = \varphi_A(G_{f_k})$ for all k and all t. Therefore homogenization for H_A yields homogenization for H.

12. The discrete case (Proof of Corollary 1.7)

If we have a $\mathbb{Z}^n$ action on Ω, and its standard action on $\mathbb{R}^n$ we construct an $\mathbb{R}^n$ action on $\widetilde{\Omega} = \Omega \times \mathbb{R}^n / \simeq$, where

$$(\omega, t_1, \dots, t_n) \simeq (T_{-z}\omega, z_1 + t_1, \dots, z_n + t_n),$$

where $z = (z_1, \dots, z_n) \in \mathbb{Z}^n$. Then $\mathbb{R}^n$ acts on $\widetilde{\Omega}$ by translation, i.e.,

$$\widetilde{T}_a(\omega, t_1, \dots, t_n) = (\omega, t_1 + a_1, \dots, t_n + a_n).$$

Notice that if $z \in \mathbb{Z}^n$, we have $\widetilde{T}_z(\omega, t_1, \dots, t_n) = (T_z\omega, t_1, \dots, t_n)$.

Now it is easy to see that T is ergodic for the measure μ on Ω if and only if $\widetilde{T}$ is ergodic for the measure $\mu \times \lambda$ (where λ is the Lebesgue measure on $[0, 1[^n$), since any $\widetilde{T}$-invariant set will be of the form $U \times [0, 1[^n$ with U a T-invariant set. Then if H satisfies $H(x + z, p, T_z\omega) = H(x, p; \omega)$, we can consider $K(x, p, [\omega, t]) = H(x - t, p; \omega)$ and this satisfies

$$K(x + a, p, \widetilde{T}_a[\omega, t]) = K(x, p, [\omega, t])$$

for all $a \in \mathbb{R}^n$, and we can apply the stochastic homogenization from the Main Theorem.

13. Extending the Main Theorem

Note that one should be able to extend our methods to the case where we have a Hamiltonian satisfying the assumptions of the Main Theorem, but:

(1) We have a time-dependent Hamiltonian, $H(t, x, p; \omega)$, and an action of $\mathbb{R} \times \mathbb{R}^n$ such that $H(t + s, x + a, p; \tau(s, a)\omega)$ and consider the sequence $H\big(\frac{t}{\varepsilon}, \frac{x}{\varepsilon}, \omega\big)$ This has been reduced to our case in the nonstochastic situation in [Viterbo 2023, Section 11.2. The nonautonomous case].

(2) We consider partial homogenization. For example if $X = N \times \mathbb{R}^k$, then we should be able to apply the above propositions as in [loc. cit.].

[20] See Remark 1.6, since $h_-(p) \le H(x, p; \omega) \le h_+(p)$ a.s. in ω, where $\lim_{|p|\to+\infty} h_\pm(p) = +\infty$.

(3) We consider the homogenization $H_\varepsilon\big(x,\frac{p}{\varepsilon};\omega\big)$ as ε goes to 0. This has been reduced to our case in the nonstochastic situation in [Viterbo 2023, Section 12. Homogenization in the p variable].

(4) We have a $\mathbb{Z}^n$ action on a manifold X such that the quotient $X/\mathbb{Z}^n$ is compact and the Hamiltonian satisfies $H(T_zx,T_z^*p,T_z\omega)=H(x,p;\omega)$, where T is the action of $\mathbb{Z}^n$ on X and we consider again the sequence $H_\varepsilon\big(x,\frac{p}{\varepsilon},\omega\big)$ as ε goes to 0.

The proof in this last case should be the same as the Main Theorem. We just need to replace $\gamma(\varphi)$ (which is not defined on T^*X) with $\hat{\gamma}(\varphi)$ and we shall get an embedding of $\mathbb{Z}^n$ into $\operatorname{Isom}(\widehat{\mathfrak{H}}_\Omega,\gamma)$. According to [Weil 1965], the closure of the image of $\mathbb{Z}^n$ is the product of an abelian compact connected metric group, A_Ω^0, and a totally disconnected compact metric abelian group D_Ω. Since we have a morphism $c:\mathbb{Z}^n\to D_\Omega$ and the kernel L must be a cocompact free abelian group, hence a lattice, so L is isomorphic to $\mathbb{Z}^n$ and in suitable integral coordinates, we see that $L=a_1\mathbb{Z}\oplus a_2\mathbb{Z}\oplus\cdots\oplus a_n\mathbb{Z}$, so $D_\Omega=\mathbb{Z}^n/L\simeq\mathbb{Z}/a_1\mathbb{Z}\oplus\mathbb{Z}/a_1\mathbb{Z}\oplus\cdots\oplus\mathbb{Z}/a_n\mathbb{Z}$. Replacing $\mathbb{Z}^n$ by L, we can reduce ourselves to the case of a compact connected abelian group so we get $\overline{K}(p,\omega)$, where $\overline{K}(p,\cdot)$ is constant on the ergodic components of the action of L and the ergodic components are interchanged by an element of D_Ω; thus we get that $\overline{K}(p,\cdot)$ is indeed constant a.e.

It would be also interesting to see what can be done in the framework of more general groups, as explained in [Sorrentino 2019] (see also [Contreras et al. 2015]). In this setting a discrete group G is a quotient of the $\pi_1(M)$, where M is a compact manifold, and we see a Hamiltonian on M as a G-invariant one on $\widetilde{M}$ a cover of M. Then Sorrentino considers the Hamiltonian $H\big(x,\frac{1}{\varepsilon}p\big)$ as ε goes to zero, and proves that it converges in some weak sense (we would say in the γ topology) to a Hamiltonian defined on G_∞ a graded Lie group associated to G (at least if G is nilpotent).

Appendix A: Generating functions for noncompact Lagrangians: Proof of Theorem 3.5

The goal of this section is to prove Theorem 3.5 that is:

Theorem 3.5. *Let φ be an element in $\mathfrak{DHam}_{\mathrm{FP}}(T^*N)$. Then $\varphi(0_N)$ has a GFQI. Moreover such a GFQI is unique.*

First we claim that the fibration theorem of Théret [1999, Theorem 4.2] goes through. Here $\mathscr{F}$ is the set of sequences of GFQI $(S_\nu)_{\nu\geq 1}$ satisfying the above property and $\mathscr{L}=\mathfrak{L}(T^*\mathbb{R}^n)$ and we have:

Proposition A.1. *The projection $\pi:\mathscr{F}\to\mathscr{L}$ is a Serre fibration up to equivalence.*

The proof is the same as Theorem 4.2 in [Théret 1999]. We may reduce ourselves to the case of a single parameter (as in [loc. cit.]). The proof is then based on Sikorav's existence theorem, which uses only the fact that, for t small enough, if L has a GFQI over U_ν then so does $\varphi^t(L)$. Note that we may always assume that $\varphi^t(T^*U_\nu)\subset T^*U_{\nu+1}$ and by truncating φ^t beyond $T^*U_{\nu+1}$, we are reduced to the compact situation.

Proof of Theorem 3.5. Using Lemma 3.2 we may assume we have a sequence U_ν of domains such that $\varphi^t(T^*U_\nu)\subset T^*U_{\nu+1}$. Applying a sequence of cut-offs to the Hamiltonian defining φ we can then find a sequence L_ν of Lagrangians of the type $\varphi_\nu^1(0_N)$, where

(1) $\varphi_\nu^t(T^*U_\nu) \subset T^*U_{\nu+1}$ for all $t \in [0,1]$,

(2) φ_ν^t has compact support in $T^*U_{\nu+1}$,

(3) setting $\varphi_\nu^t(0_N) = L_\nu(t)$, we have for $\mu \geq \nu$

$$L_\nu(t) \cap T^*U_\nu = L_\mu(t) \cap T^*U_\nu = \varphi^t(L) \cap T^*U_\nu.$$

Then each $L_\nu(t)$ has a GFQI, $S_\nu(t) : N \times E_\nu \to \mathbb{R}$ and we claim that, for $\mu \geq \nu$, $S_\nu(t)$ and $S_\mu(t)$ are equivalent over U_ν. Indeed, we have a deformation from L_ν to L_μ that is the identity on T^*U_ν. If we denote by S_s a GFQI covering this deformation (the existence of which follows from [Théret 1999], since we are again in the compactly supported case), then S_s generates a Lagrangian L_s that is constant over T^*U_ν. Then using [loc. cit., Lemma 5.3] we can assume, after applying a fiber-preserving diffeomorphism, that $\Sigma_s \cap (U \times F) = \Sigma_0 \cap (U \times F)$, where

$$\Sigma_s = \left\{ (x,\xi) \;\middle|\; \frac{\partial S_s}{\partial \xi}(x,\xi) = 0 \right\}.$$

But then as in [loc. cit., p. 259], using Hadamard's lemma we prove that there is a fiber-preserving diffeomorphism such that $S_1(x, \xi(x,\eta)) = S_0(x,\eta)$.

So may now assume that the restriction of S_μ over U_ν is exactly $S_\nu \oplus q_{\nu,\mu}$ by composing S_μ with an extension of the fiber-preserving diffeomorphism realizing the equivalence.[21] □

Appendix B: Proof of Lemma 9.13

Lemma 9.13. *Let $\varepsilon > 0$ and $A : \mathbb{R}^n \to \mathbb{R}^d$ be a linear map such that $A(\mathbb{R}^n)$ has dense projection on T^d. Then there are integral vectors $v_1, \dots, v_d$ in $\mathbb{Z}^d$ forming a basis of $\mathbb{R}^d$ such that there exist vectors $a_1, \dots, a_d$ in $\mathbb{R}^n$ such that*

$$|A \cdot a_j - v_j| \leq \varepsilon.$$

Remark B.1. We do not claim the basis is an integral basis, i.e., it does not necessarily have determinant 1.

Proof suggested by the referee. We know that $A(\mathbb{R}^n) + \mathbb{Z}^d$ is dense in $\mathbb{R}^d$, so we may find $b_1, \dots, b_d \in \mathbb{R}^n$, $w_1, \dots, w_n \in \mathbb{Z}^d$ such that

$$\left| Ab_i - w_i - \frac{e_i}{2} \right| \leq \frac{\varepsilon}{2}.$$

Then $a_i = 2b_i$, $v_i = e_i + 2w_i$ satisfy $|Aa_i - v_i| \leq \varepsilon$, and since $\det(v_i)$ is odd, $(v_1, \dots, v_d)$ is a basis of $\mathbb{R}^d$. □

Appendix C: Approximation of generating functions and symplectic integrators

Our goal is to prove Lemma 7.4. It is a consequence of the more precise result:

Lemma C.1. *Let φ_H^t have $S_t(q,p)$ as generating function. We have*

$$\|S_t(q,p) - tH(q,p)\|_{C^0} \leq \frac{t^2}{2} \left\| \frac{\partial H}{\partial q} \right\|_{C^0} \cdot \left\| \frac{\partial H}{\partial p} \right\|_{C^0}.$$

[21] The existence of the extension follows from the fact that we may assume that for μ, ν large enough, the inclusion $U_\nu \subset U_\mu$ is a homotopy equivalence.

Proof. In the sequel, $\|\cdot\|$ denotes the C^0 norm. Note that S_t has no fiber variable. It is a classical fact [Hamilton 1834; 1835; Jacobi 2009] (see also [Arnold 1978]) that S_t satisfies the Hamilton–Jacobi equation

$$\begin{cases} \dfrac{\partial}{\partial t} S_t(q,p) = H\Big(q + \dfrac{\partial S_t}{\partial p}(q,p), p\Big), \\ S_0(q,p) = 0. \end{cases}$$

Indeed, setting $\varphi_H^t(q,p) = (Q_t(q,p), P_t(q,p))$, the Lagrangian submanifold

$$\Lambda(\varphi) = \{(t, -H(t, Q_t(q,p), P_t(q,p)), q, p, Q_t(q,p), P_t(q,p)) \mid t \in \mathbb{R}, (q,p) \in T^*N\}$$

in $T^*\mathbb{R} \times T^*N \times \overline{T^*N}$ is contained in

$$\{(t, \tau, q, p, Q, P) \mid \tau + H(Q,P) = 0\}$$

since $\Lambda(\varphi)$ is easily seen to be invariant by the flow of the Hamiltonian $K(t,\tau,q,p,Q,P) = \tau + H(Q,P)$, which is given by

$$(t,\tau,q,p,Q,P) \to (t+s, \tau, q, p, Q_s(Q,P), P_s(Q,P)).$$

Since $Q_t = q + \frac{\partial S_t}{\partial q}$, the equation follows.

Now set $S_t(q,p) = t \cdot H(q,p) + R_t(q,p)$ and replace in the equation, using

$$|H(q+\xi, p) - H(q,p)| \le |\xi| \cdot \Big\| \frac{\partial H}{\partial q} \Big\|_{C^0},$$

$$\frac{\partial R_t}{\partial t}(q,p) \le \Big\| \frac{\partial H}{\partial q} \Big\|_{C^0} \Big| \frac{\partial S_t}{\partial p} \Big| \le t \cdot \Big\| \frac{\partial H}{\partial q} \Big\|_{C^0} \cdot \Big\| \frac{\partial H}{\partial p} \Big\|_{C^0} + \Big\| \frac{\partial H}{\partial q} \Big\|_{C^0} \cdot \Big| \frac{\partial R_t}{\partial p}(q,p) \Big|$$

and $R_0(q,p) = 0$. Now the relation

$$\partial_t R_t(q,p) \le tA + B \Big| \frac{\partial R_t}{\partial p} \Big|$$

implies by monotonicity of the solutions of the Hamilton–Jacobi equations[22] that R_t is bounded by the solution u_t of $\partial_t u = tA + B|\nabla_x u|$, that is, $u(t,x) = \frac{1}{2}t^2 A$, so

$$R_t(q,p) \le \frac{t^2}{2} \Big\| \frac{\partial H}{\partial q} \Big\|_{C^0} \cdot \Big\| \frac{\partial H}{\partial p} \Big\|_{C^0}.$$

The same argument gives an estimate from below. □

Appendix D: Proof of Proposition 8.3

Proposition 8.3. *Given any α, there exists a sequence $(\ell_\nu)_{\nu \ge 1}$ such that for almost all $\omega \in \Omega$*

$$\lim_{\nu \to \infty} \lim_{U \subset \mathbb{R}^n} c(\mu_U, \varphi_{\ell_\nu, \omega} \alpha) \le \lim_{U \subset \mathbb{R}^n} c(\mu_U, \bar{\varphi}\alpha).$$

The proof is essentially the same as in Section 5 of [Viterbo 2023]. We reproduce it here adapted to our situation and notation but notice that ω just appears as a parameter and so does not change the proof of Proposition 8.3. In particular the cycles we construct in the proof do not need to depend continuously on ω. We first need the next lemma. We define a cycle with closed support in X to be a cycle for the

[22]That is, $H \le K$ implies that the solutions v, w of $\partial_t u = H(x, D_x u)$ corresponding to the same initial condition satisfy $u \le v$.

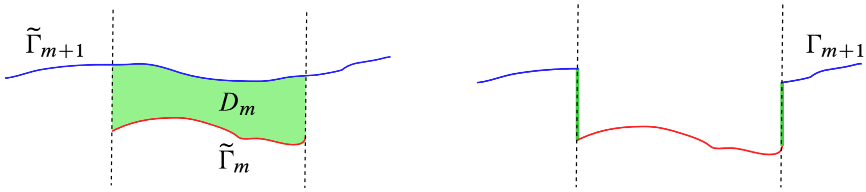


Figure 2. $\widetilde{\Gamma}_m$ in red, $\widetilde{\Gamma}_{m+1}$ in blue and D_m in green on the left and Γ_{m+1} on the right.

singular homology with locally finite support. These are the cycles of Borel–Moore homology (i.e., homology with closed supports) of X. Chains are infinite sums $\sum_\sigma a_\sigma \sigma$ of singular simplices such that there are only finitely many simplices with $a_\sigma \neq 0$ touching any compact set. As a result it is clear what it means for such a chain to be a cycle. For such homology, admissible maps are the proper maps, i.e., only a proper map $f : X \to Y$ will induce a map f_* between the corresponding Borel–Moore homology groups. Any proper submanifold without boundary represents a cycle in Borel–Moore homology, while in ordinary homology, this is the case only for compact submanifolds.

Lemma D.1. *Let S be a GFQI defined on E and $c = \lim_{U \subset N} c(\mu_U, S)$. There exists a closed cycle Γ such that $\Gamma_U = \Gamma \cap \pi^{-1}(U)$ satisfies $[\Gamma_U] = \mu_U$ in $H_*(S_U^{+\infty}, S^{-\infty})$ and $S(\Gamma_U) \leq c(\mu_U, S) + \varepsilon$ for U belonging to a sequence of exhausting open sets with smooth boundary.*

Remark D.2. Note that $[\Gamma_U]$ is assumed to be an ordinary cycle, so that its class in $H_*(S_U^{+\infty}, S_U^{-\infty})$ is well-defined.

Proof. Consider an increasing sequence U_n of open sets with smooth boundary such that $N = \bigcup_n U_n$. Notice that there is a restriction map for $U \subset V$ sending $H_*(V, \partial V) \to H_*(U, \partial U)$. It induces a map that we denote by $\rho_{U,V}$,

$$H_*(S_V^t, S_V^{-\infty} \cup E_{|\partial V}) \to H_*(S_U^t, S_U^{-\infty} \cup E_{|\partial U}),$$

and a diagram

$$\begin{array}{ccc} H_*(S_V^{+\infty}, S_V^{-\infty} \cup E_{|\partial V}) & \xrightarrow{\rho_{U,V}} & H_*(S_U^{+\infty}, S_U^{-\infty} \cup E_{|\partial U}) \\ \uparrow & & \uparrow \\ H_*(S_V^{c+\varepsilon}, S_V^{-\infty} \cup E_{|\partial V}) & \xrightarrow{\rho_{U,V}} & H_*(S_U^{c+\varepsilon}, S_U^{-\infty} \cup E_{|\partial U}) \end{array}$$

Now the upper horizontal map sends μ_V to μ_U, so applying this to the sequence U_n, we get a sequence $\widetilde{\Gamma}_n \in H_*(S_{U_n}^{c+\varepsilon}, S_{U_n}^{-\infty} \cup E_{|\partial U_n})$ with image μ_{U_n}, and we have a sequence such that $\rho_{U_n,U_m}[\widetilde{\Gamma}_n] = [\widetilde{\Gamma}_n \cap \pi^{-1}(U_m)]$ is constant for $n \geq m$. Then we may glue the $\widetilde{\Gamma}_n$ as follows: since $[\widetilde{\Gamma}_m] = [\widetilde{\Gamma}_{m+1} \cap \pi^{-1}(U_m)]$ in $H_*(S_{U_n}^{c+\varepsilon}, S_{U_n}^{-\infty} \cup E_{|\partial U_n})$, we have D_m such that $\partial D_m \cap \pi^{-1}(U_m) = \widetilde{\Gamma}_m - \widetilde{\Gamma}_{m+1} \cap \pi^{-1}(U_m)$ and we can assume $D_m \subset \pi^{-1}(\overline{U}_m)$. This is illustrated in Figure 2. Now we may consider the cycle

$$\Gamma_m = \widetilde{\Gamma}_m \cup (\partial D_m \cap \pi^{-1}(\overline{U}_m)) \cup \widetilde{\Gamma}_{m+1} \cap \pi^{-1}(U_{m+1} \setminus U_m).$$

We easily check that

(1) $\Gamma_m \cap \pi^{-1}(U_m) = \widetilde{\Gamma}_m$,

(2) $\Gamma_m \cap \pi^{-1}(U_{m+1} \setminus U_m) = (\widetilde{\Gamma}_{m+1} \cap \pi^{-1}(U_{m+1}) \cup (\partial D_m \cap \pi^{-1}(\partial U_m))),$

(3) $\partial\Gamma_m \subset E_{\partial U_{m+1}}$,

(4) $\Gamma_m \subset S^{c+\varepsilon}$.

By induction we can build a sequence Γ_m and we have $\Gamma_n \cap \pi^{-1}(U_m) = \Gamma_m \cap \pi^{-1}(U_m)$ for $n > m$. Therefore $\bigcup_n \Gamma_n$ is stationary over any compact set and defines a closed cycle Γ such that $S(\Gamma) \le c+\varepsilon$. □

Now the generating function for $\varphi_{k,\omega}$ is given by Proposition 9.1:

$$S_{k,\omega}(x,y;\xi,\zeta) = \frac{1}{k}\Bigg[S_\omega(kx,p_1,\zeta_1) + \sum_{j=2}^{k-1} S_\omega(kq_j,p_j,\zeta_j) + S_\omega(kq_k,y,\zeta_k)\Bigg] + B_k(x,y,\zeta),$$

where $\zeta = (\zeta_1,\dots,\zeta_k)$, $\xi = (p_1,q_2,\dots,q_{k-1},p_{k-1},q_k)$,

$$\tau_a\zeta = (p_1,q_2+a,\dots,q_{k-1}+a,p_{k-1},q_k+a)$$

and

$$B_k(x,y,\zeta) = \langle p_1, q_2 - x\rangle + \sum_{j=2}^{k-1}\langle p_j, q_{j+1}-q_j\rangle + \langle y, x-q_k\rangle.$$

Now let $F(q,P;\eta)$ be a GFQI for the graph of α. Then

$$G_k^\omega(u,v;x,y,\eta;\xi,\zeta) = S_k^\omega(u,y;\xi) + F(x,v;\eta) + \langle y-v, u-x\rangle$$

is a GFQI of $\varphi_k\alpha$. We set

$$\bar{G}_k^\omega(u,v;x,y,\eta) = h_k^\omega(y) + F(x,v;\eta) + \langle y-v, u-x\rangle.$$

We shall omit the subscripts a,χ for the moment, so in the sequel, $\bar{G}_{k,a,\chi}^\omega$ means $\bar{G}_{k,a,\chi}^\omega$ Here the variables u,v,x,y are in $\mathbb{R}^n$ and we denote by E_k the space of the $\theta = (\zeta,\xi)$, where $\xi \in E^k$, $\zeta \in (\mathbb{R}^{2n})^k$ and $\eta \in V$. By definition we have a cycle Γ_U^ω in $U \times \mathbb{R}_v^n \times \mathbb{R}_x^n \times \mathbb{R}_y^n \times E \times V$ relative to $(\bar{G}_k^\omega)^{-\infty} \cup \partial U \times \mathbb{R}_v^n \times \mathbb{R}_x^n \times \mathbb{R}_y^n \times E \times V$ and homologous (as a closed cycle) to $U \times \mathbb{R}_v^n \times \Delta_{x,y} \times E^- \times V^-$ (where Δ is the diagonal) such that

$$\bar{G}_k^\omega(\Gamma_U^\omega) \le c(\mu_U, \bar{G}_k^\omega) + \varepsilon = c(\mu_U, \bar{\varphi}_{k,U}^\omega\alpha) + \varepsilon,$$

where $\bar{\varphi}_{k,U}^\omega$ is the flow of $h_{k,U}^\omega(y)$.

Moreover according to Lemma D.1, we can assume there is a closed (i.e., Borel–Moore) cycle Γ^ω such that $\Gamma_U^\omega = \Gamma^\omega \cap \pi^{-1}(U)$ (at least for a cofinal sequence of U's).

Now let $C_U^\omega(y)$ be a cycle in the class of $U \times E_k^-$ in $H_*((S_{k,y}^\omega)^{+\infty}, (S_{k,y}^\omega)^{-\infty})$, depending continuously on y, such that[23]

$$S_k^\omega(y, C_U^\omega(y)) \le h_{k,U}^\omega(y) + a\chi(y) + \varepsilon.$$

As in [Viterbo 2023, Section 5, Lemma 5.1], this is possible provided χ is the characteristic function of Λ_δ, the complement of a disjoint union of sets of diameter less than δ. For example, we can take Λ_δ

[23]The notation is unfortunate since it does not respect the order of our variables. By $S_k(y, C_U(y))$ we mean the maximum of $S_k(x,y;\xi,\zeta)$, where $(x;\xi,\zeta) \in C(y)$

to be a neighborhood of $\Lambda(\delta)=\{(x_1,\dots,x_n)\mid \exists j,\ x_j\in\delta\mathbb{Z}\}$. Thus we set for $a\in\mathbb{R}_+$, $\chi\in C^\infty(\mathbb{R}^n)$

$$\bar{G}^{\omega}_{k,a,\chi}(u,v;x,y,\eta)=h^{\omega}_k(y)+F(x,v;\eta)+\langle y-v,u-x\rangle+a\chi(y).$$

We shall omit the subscripts a,χ for the moment, so in the sequel, $\bar{G}^{\omega}_{k,a,\chi}$ means $\bar{G}^{\omega}_{k,a,\chi}$.

We again invoke Lemma D.1 in order to obtain a (closed) cycle $C^{\omega}(y)$ such that for a cofinal sequence of U's we have $C^{\omega}_U(y)=C^{\omega}(y)\cap\pi^{-1}(U)$ and, like $C^{\omega}_U(y)$, the cycle $C(y)$ depends continuously on y.

We now construct a new (Borel–Moore) cycle, symbolically denoted by $\Gamma\times_Y C$ and defined as follows (everything depends on ω but for notational convenience we omit it):

$$\Gamma\times_Y C=\{(u,v;x,y,\theta,\eta)\mid (u,v,x,y,\eta)\in\Gamma,(u,\theta)\in C(y)\}.$$

We have

(1) $(\Gamma\times_Y C)_U$ is a Borel–Moore cycle homologous to $U\times\mathbb{R}^n_v\times\Delta_{x,y}\times E^-_k\times V^-$.

(2) $G^{\omega}_k((\Gamma\times C)_U)\le\bar{G}^{\omega}_{k,a,\chi}(\Gamma_U)+\varepsilon$.

Indeed for (1), it is a cycle by the continuity of $C(y)$ in y. That its homology class is the stated one follows from the fact that the homology class of $A\times_Y B$ only depends on the homology class of A,B and so $\Gamma_U\times_Y C^-_U$ is homologous to

$$\begin{aligned}(U\times\mathbb{R}^n_v\times\Delta_{x,y}\times V^-)\times_Y(U\times E^-_k)&=\{(u,v;x,y,\eta,\theta)\mid u\in U,\ x=y,\ \eta\in V^-,\ \theta\in E^-_k\}\\&=U\times\mathbb{R}^n_v\times\Delta_{x,y}\times V^-\times E^-_k.\end{aligned}$$

As for (2), we have $(\Gamma\times_Y C^-)_U=\Gamma_U\times_Y C^-_U$ and

$$G^{\omega}_k(\Gamma_U\times_Y C^-_U)\overset{\text{def}}{=}\sup\{S^{\omega}_k(u,y;\theta)+F(x,v;\eta)+\langle y-v,u-x\rangle\mid (u,v;x,y,\eta)\in\Gamma,\ (u;\theta)\in C(y)\}$$

but since $S_k(u,y;\theta)\le h^+_{k,U}(y)+a\chi(y)+\varepsilon$ for $(u;\theta)\in C(y)$, we have

$$\begin{aligned}&G^{\omega}_k(\Gamma_U\times_Y C^-_U)\\&\quad\le\sup\{F(x,v;\eta)+h^+_{k,U}(y)+a\chi(y)+\varepsilon+\langle y-v,u-x\rangle\mid(u,v;x,y,\eta)\in\Gamma^{\omega}_U,(u,\theta)\in C^{\omega}_U(y)\}\\&\quad\le\bar{G}^{\omega}_{k,a,\chi}(\Gamma_U)+\varepsilon.\end{aligned}$$

Now as in [Viterbo 2023, Section 5, p. 95], let us consider a collection of ℓ open sets Λ^j_δ for $1\le j\le\ell$ such that each of them is a translate of Λ_δ and any $n+1$ of them have empty intersection. We denote by χ_j $(1\le j\le\ell)$ the corresponding functions. We set $\bar{x}=(x_1,\dots,x_\ell)$, $\bar{y}=(y_1,\dots,y_\ell)$, $\bar{\theta}=(\theta_1,\dots,\theta_\ell)$ and define[24]

$$G_{k,\ell}(u,v,\bar{x},\bar{y},\bar{\theta},\eta)=F(x_1,v;\eta)+\frac{1}{\ell}\sum_{j=1}^{\ell}S_k(\ell x_j,y_j,\theta_j)+B_\ell(\bar{x},\bar{y})+\langle y_\ell-v,u-x_1\rangle$$

This is a GFQI for $\rho_\ell^{-1}\varphi_k^{\ell}\rho_\ell^{-1}\alpha=\rho_\ell^{-1}\rho_k^{-1}\varphi^{k\ell}\rho_k\rho_\ell\alpha=\varphi_{k\ell}\alpha$.

Let

$$\bar{G}_{k,\ell}(u,v;\bar{x},\bar{y},\eta)=F(x_1,v,\eta)+\frac{1}{\ell}\sum_{j=1}^{\ell}(h^+_k(y_j)+a\chi_j(y_j))+B_\ell(\bar{x},\bar{y})+\langle y_\ell-v,u-x_1\rangle.$$

[24]Here we omit ω from the notation, which would otherwise become unwieldy.

By definition there is a Borel–Moore cycle, $\Gamma_{k,\ell}$, such that

$$\overline{G}_{k,\ell}((\Gamma_{k,\ell})_U) \le c(\mu_U, \overline{G}_{k,\ell}) + \varepsilon,$$

and using $C_j(y_j)$ as before for $1 \le j \le \ell$ and setting

$$\Gamma_{k,\ell} \times_Y C^-[\ell] = \{(u, v; \bar{x}, \bar{y}, \bar{\theta}, \eta) \mid (u, v, x, y, \eta) \in \overline{\Gamma}, (\ell x_j, \xi_j) \in C_j^-(y_j)\},$$

we have

$$c(\mu_U, G_{k,\ell}) \le G_{k,\ell}((\Gamma_{k,\ell})_U \times_Y (C^-)_U[\ell]) \le \overline{G}_{k,\ell}((\Gamma_{k,\ell})_U) \le c(\mu_U, \overline{G}_{k,\ell}) + 2\varepsilon.$$

Finally we claim that

$$c(\mu_U, \overline{G}_{k,\ell}) \le c(\mu_U, \overline{G}_k) + \frac{(n+1)a}{\ell}.$$

Indeed, $\overline{G}_{k,\ell}$ is the generating function of $\psi_{k,\ell} = \rho_\ell^{-1} \psi_k^1 \circ \cdots \circ \psi_k^\ell \rho_\ell$, where ψ_k^j is the time-one flow of $h_k(y) + a\chi_j(y)$. But these flows commute, so $\psi_{k,\ell}$ is the time-one flow of

$$K_{k,\ell}(y) = \frac{1}{\ell} \sum_{j=1}^{\ell} (h_{k,U}(y) + a\chi_j(y))$$

and we have $|K_{k,\ell}(y) - h_k(y)| \le \frac{(1+n)a}{\ell} + \varepsilon$. Therefore

$$c(\mu_U, \overline{G}_{k,\ell}) \le c(\mu_U, \psi_{k,\ell}\alpha) \le c(\mu_U, \psi_k^1\alpha) + \frac{(1+n)a}{\ell} + \varepsilon \le c(\mu_U, \bar{\varphi}_k\alpha) + \frac{(1+n)a}{\ell} + \varepsilon.$$

Thus for ℓ large enough $c(\mu_U, \overline{G}_{k,\ell}) \le c(\mu_U, \psi_k^1\alpha) + 2\varepsilon$. Taking the limit as k goes to infinity, we get

$$c(\mu_U, \varphi_{k\ell}\alpha) = c(\mu_U, G_{k,\ell}) \le c(\mu_U, \bar{\varphi}_k\alpha) + 2\varepsilon \le c(\mu_U, \bar{\varphi}\alpha) + 3\varepsilon.$$

This concludes the proof of Proposition 8.3.

Acknowledgments and general remarks

I would like to acknowledge the hospitality of MSRI during the semester "Hamiltonian systems, from topology to applications through analysis" in the fall of 2018. The idea of this work came through attending talks and several discussions with Fraydoun Rezakhanlou, whom I want to thank here for generously sharing his ideas.

As the reader will check, and analogously to [Viterbo 2023], the methods used here are drawn from symplectic topology. This paper can be considered as part of a program to study symplectic topology in a random framework (or random phenomena having a symplectic structure) of which a foundational example is the random version of Poincaré–Birkhoff theorem from [Pelayo and Rezakhanlou 2018; 2025]. Last but not least, I would like to very warmly thank the referees for their in-depth reading of the manuscript and their many very thoughtful suggestions.

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CLAUDE VITERBO: claude.viterbo@universite-paris-saclay.fr
Laboratoire Mathématique d'Orsay, UMR 8628, Université de Paris-Saclay, CNRS, Orsay, France

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