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A NEW APPROACH TO THE MEAN-FIELD LIMIT OF VLASOV–FOKKER–PLANCK EQUATIONS

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We introduce a novel approach to the mean-field limit of stochastic systems of interacting particles, leading to the first ever derivation of the mean-field limit to the Vlasov–Poisson–Fokker–Planck system for plasmas in dimension 2 together with a partial result in dimension 3. The method is broadly compatible with second-order systems that lead to kinetic equations and it relies on novel estimates on the BBGKY hierarchy. By taking advantage of the diffusion in velocity, those estimates bound weighted L^p norms of the marginals or observables of the system, uniformly in the number of particles. This allows us to qualitatively derive the mean-field limit for very singular interaction kernels between the particles, including repulsive Poisson interactions, together with quantitative estimates for a general kernel in L^2.

1. Introduction

The rigorous derivation of kinetic models such as the Vlasov–Poisson system from many-particle systems has been a long standing open question, ever since the introduction of the Vlasov–Poisson system in [Vlasov 1938; 1967]. While our understanding of the mean-field limit for singular interactions has made significant progress for first-order dynamics, the mean-field limit for second-order systems has remained frustratingly less understood. This article proposes a new approach that is broadly applicable to second-order systems with repulsive interactions and diffusion in velocity. In particular, this allows us to derive for the first time the Vlasov–Poisson–Fokker–Planck system in dimensions higher than 1 without any truncation or regularizing.

We more precisely consider the classical second-order Newton dynamics

$$\begin{aligned} \frac{d}{dt} X_i(t) &= V_i(t), \quad X_i(t=0) = X_i^0, \\ dV_i(t) &= \frac{1}{N} \sum_{j \neq i} K(X_i - X_j)\, dt + \sigma\, dW_i, \quad V_i(t=0) = V_i^0, \end{aligned} \tag{1}$$

where the W_i are N independent Wiener processes. For simplicity we take the positions X_i on the torus Π^d, while the velocities lie in $\mathbb{R}^d$. The kernel K models the pairwise interaction between particles and is taken to be *repulsive* throughout this paper, in the basic sense that it derives from a potential $K = -\nabla\phi$ that is even and positive, $\phi \geq 0$.

Remark 1. For simplicity, we write $\phi(0) = 0$ and $K(0) = 0$ even if ϕ and K are not continuous at 0. This simplifies the notation by allowing us to sum over all j in (1) since the term $j = i$ trivially vanishes.

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We naturally focus on singular kernels K with, as a main guiding example, the case of Coulombian interactions

$$K = \alpha \frac{x}{|x|^d} + K_0(x), \tag{2}$$

with $\alpha > 0$ and K_0 a smooth correction to periodize K. This corresponds, if $d \geq 3$, to the choice $\phi = \alpha(d-2)^{-1}|x|^{2-d} + \text{correction}$, or, if $d = 2$, the choice $\phi = -\alpha \ln|x| + \text{correction}$.

The Coulombian kernel (2) typically models electrostatic interactions between point charges, such as ions or electrons in a plasma, when the velocities are small enough with respect to the speed of light. In that setting, diffusion in (1) may for example represent collisions against a random background, such as the collision of the faster electrons against the background of ions. Such random collisions may also involve some friction in velocity, which we did not include in (1) but could be added to our method without difficulty. This makes (1) with (2) one of the most classical and important starting points for the modeling of plasmas; we refer in particular to the classical [Bogoliubov 1946].

Coulombian interactions are also a natural scaling in many models. The obvious counterpart to plasmas concerns the Newtonian dynamics of point masses through gravitational interactions. This consists in taking $\alpha < 0$ in (2) and leads to attractive interactions with a negative potential and for this reason cannot be handled with the method presented here.

The system (1) usually involves a very large number of particles, typically up to 10^{20}–10^{25} in plasmas for example. This makes the mean-field limit especially attractive. This is a kinetic, Vlasov–Fokker–Planck equation posed on the limiting one-particle density $f(t, x, v)$:

$$\partial_t f + v \cdot \nabla_x f + (K \star_x \rho) \cdot \nabla_v f = \frac{\sigma^2}{2} \Delta_v f, \quad \text{with } \rho = \int_{\mathbb{R}^d} f \, dv. \tag{3}$$

Well posedness for mean-field kinetic equations such as (3) is now reasonably well understood, including for singular Coulombian interactions such as (2) in dimension $d \leq 3$. For the nondiffusive case $\sigma = 0$, weak solutions were established in [Arsenev 1975], while classical solutions were obtained in dimension 2 in [Ukai and Okabe 1978]. The dimension 3 case is harder and obtaining classical solutions requires more difficult dispersive arguments and were only obtained later in [Lions and Perthame 1991; Pfaffelmoser 1992; Schaeffer 1991], see also the more recent [Gasser et al. 2000; Holding and Miot 2018; Loeper 2006; Pallard 2014]. In the case with diffusion $\sigma > 0$, we refer to [Victory 1991] for weak solutions, and to [Bouchut 1993; Degond 1986; Ono and Strauss 2000; Rein and Weckler 1992; Victory and O'Dwyer 1990] for classical solutions.

Of course the mean-field scaling is not the only possible scaling on systems such as (1). We mention in particular the likely even more critical Boltzmann–Grad limit, such as obtained in the classical [Lanford 1975] and the major results in [Bodineau et al. 2018; 2020; Gallagher et al. 2014; Pulvirenti and Simonella 2017; Pulvirenti et al. 2014]. We note as well that the derivation of macroscopic equations from mesoscopic systems such as (3) is another important and challenging question. For example the passage to the fluid macroscopic system from Vlasov–Poisson–Fokker–Planck has been approached in different low-field (parabolic) or high-field (hyperbolic) regimes depending on the space dimension; see for example [Carrillo et al. 2022; Goudon et al. 2005; Nieto et al. 2001; Poupaud and Soler 2000].

Mean-field limits have been rigorously derived for general systems, including second-order dynamics such as (1), in the case of Lipschitz interaction kernels K. We refer the reader to the classical works [McKean 1967; Sznitman 1991] in the stochastic case and [Braun and Hepp 1977; Dobrushsin 1979] for the deterministic case. Uniform-in-time propagation of chaos has also been obtained in the locally Lipschitz case, notably in a close to convex case in [Bolley et al. 2010] and more recently in a nonconvex setting in [Guillin et al. 2022].

There now exists a large literature on the question of the mean-field limits; see for example the survey in [Golse 2016; Jabin 2014; Jabin and Wang 2017]. However in the specific case of second-order systems such as (1) very little is known. In dimension $d = 1$, the Vlasov–Poisson–Fokker–Planck system was derived in [Guillin et al. 2023; Hauray and Salem 2019]. In dimensions $d \geq 2$, the only results for unbounded interaction kernels were obtained in [Hauray and Jabin 2007; 2015]. But those are valid only in the deterministic case $\sigma = 0$ and for only mildly singular kernels with

$$|K(x)| \lesssim |x|^{-\alpha} \quad \text{and} \quad |\nabla K| \lesssim |x|^{-\alpha-1} \qquad \text{for } \alpha < 1.$$

Jabin and Wang [2016] derived the mean-field limit with $K \in L^\infty$ and without extra derivative. Those cannot cover Coulombian interactions, even in dimension 2.

More is known for singular interaction kernels K that are smoothed or truncated at some N-dependent scale ε_N. In that truncated case, one can mention in particular [Ganguly and Victory 1989; Ganguly et al. 1991; Victory and Allen 1991; Wollman 2000] for the convergence of so-called particle methods. The recent works [Boers and Pickl 2016; Lazarovici 2016; Lazarovici and Pickl 2017] in the deterministic case and [Huang et al. 2020] in the stochastic case considerably extended the results for such truncated kernels and allowed for almost reaching the critical physical scale $\varepsilon_N \sim N^{-1/d}$. One can also mention [Carrillo et al. 2019] with polynomial cut-off. It is also possible to derive the Vlasov–Poisson system directly from many-particle quantum dynamics such as the Hartree equation, for which we briefly refer to [Golse and Paul 2019; Lafleche 2021; Saffirio 2020].

The mean-field limits for first-order systems with singular interactions appear to be more tractable. A classical example concerns the dynamics of point vortices or stochastic point vortices where the mean-field limit corresponds to the vorticity formulation of two-dimensional incompressible Euler or Navier–Stokes equations. The interaction between vortices obey the Biot–Savart law, which has the same singularity as the Coulombian kernel in dimension 2. In the deterministic case, the mean-field limit was classically obtained for example in [Goodman and Hou 1991; Goodman et al. 1990] or [Schochet 1995; 1996] for the two-dimensional Euler equation and extended remarkably to essentially any Riesz kernels in [Serfaty 2020]. In the stochastic case, we refer in particular to [Fournier et al. 2014; Jabin and Wang 2018; Osada 1987] for the limit to two-dimensional Navier–Stokes equations, to [Bresch et al. 2020; 2023] for singular attractive kernels, or to [Nguyen et al. 2022] for multiplicative noise. Uniform-in-time propagation of chaos was even recently obtained in [Guillin et al. 2024; Rosenzweig and Serfaty 2023].

One of the reasons second-order systems appear more difficult to handle stems from how the structure of the singularity interacts with the distribution of velocities. Because of the term $K(X_i - X_j)$, the singularity in pairwise interactions is typically localized on collisions $X_i = X_j$. For first-order systems this corresponds to a point singularity, while for second-order systems the presence of the additional velocity variables makes it into a plane. In that regard, we also note that the derivation of macroscopic systems directly from second-order dynamics is in fact better understood than the derivation of kinetic equations like (3). We refer to the derivation of incompressible Euler equations in [Han-Kwan and Iacobelli 2021], or to the derivation of monokinetic solutions to (3) (which are essentially equivalent to a macroscopic system) in [Serfaty 2020].

The main argument in our proof is a new quantitative estimate on the so-called marginals of the system through the BBGKY hierarchy. This leads to the propagation of some weighted L^p estimates on the marginals. It implies a weak propagation of chaos in the sense of [Sznitman 1991] but it applies more broadly to initial data that are not chaotic or not close to being chaotic.

Recently, new approaches have been introduced to bound marginals on systems with appropriate nondegenerate diffusion. Using relative entropy, Lacker [2023] was the first to derive quantitative estimates comparing the marginals to the limiting tensorized solution, thus deriving optimal rates for the propagation of chaos in $O(1/N)$, instead of $O(1/\sqrt{N})$ on the convergence of the marginals (as observed for smoother interactions in [Duerinckx 2021]). While formulated for first-order systems, the method also applies to second-order systems with diffusion in velocity, as observed by Lacker. The method takes advantage of the regularizing provided by the diffusion to avoid "losing" a derivative in the hierarchy estimates. The use of the relative entropy however imposes that the interaction kernel belongs to an exponential Orlicz space. In a different context of nonexchangeable systems, [Jabin et al. 2025] later used the propagation of L^2 norms on some equivalent of the marginals, again taking advantage of the diffusion but requiring that the interaction kernel K be in L^∞.

The present article focuses mostly on second-order singular systems, where our method combines this general idea with a specific choice of weights for the L^p norms that are propagated. Those weights are based on a total energy reduced to k particles when dealing with the marginal of order k. They allow us to take advantage of a further regularizing effect in the hierarchy to only require kernels K to be in some L^p with $p > 1$. The same idea to propagate L^p norms on the marginals also applies to first-order systems in confined domains, without then requiring weights.

A direct consequence of our approach is the first ever derivation of the mean-field limit for the repulsive Vlasov–Poisson–Fokker–Planck over a finite time interval. This applies to any chaotic initial data in dimension $d = 2$ and for initial data with more restrictive energy bound in any dimension $d \geq 3$. We are expecting to extend this derivation in a future work to any chaotic initial data in any dimension $d \geq 2$ by decomposing appropriately the initial data.

The paper is structured as follows: We start in Section 2 with the notation and main results. We first state our main result, Theorem 2, that proves the convergence to the Vlasov–Fokker–Planck equation as N tends to infinity followed with Theorem 3 proving quantitative estimates for singular kernels in L^2. We next introduce Proposition 5, which states the explicit propagation of weighted L^p bounds on the marginals. We in particular discuss more thoroughly the limitations and possible extensions of our approach after stating Proposition 5. Section 3 is devoted to the proof of Proposition 5 and Theorem 2 from the key technical contribution of the article around Lemma 9 and ends with the proof of Theorem 3.

2. Main results

2.1. *The new result.* We introduce the full N-particle joint law of the system f_N which satisfies the Liouville or forward Kolmogorov equation

$$\partial_t f_N + \sum_{i=1}^N v_i \cdot \nabla_{x_i} f_N + \sum_{i=1}^N \frac{1}{N} \sum_{j=1}^N K(x_i - x_j) \cdot \nabla_{v_i} f_N = \frac{\sigma^2}{2} \sum_i \Delta_{v_i} f_N, \tag{4}$$

which is a linear advection-diffusion equation. However the marginals $f_{k,N}$ of f_N will also play a critical role in the analysis. They correspond to the law of k among N particles and are represented through

$$f_{k,N}(t, x_1, v_1, \dots, x_k, v_k) = \int_{\Pi^{d(N-k)} \times \mathbb{R}^{d(N-k)}} f_N(t, x_1, v_1, \dots, x_N, v_N)\, dx_{k+1}\, dv_{k+1} \cdots dx_N\, dv_N. \tag{5}$$

The question of well-posedness for (4) can be delicate and is separate from the issue of the mean-field limit considered here. For this reason, we consider the notion of an entropy solution $f_N \in L^\infty(\mathbb{R}_+ \times \Pi^{dN} \times \mathbb{R}^{dN})$ to (4), fully described later in Section 2.4, to which we impose some Gaussian decay in velocity:

$$\sup_{t \le 1} \int_{\Pi^{dN} \times \mathbb{R}^{dN}} e^{\beta \sum_{i \le N} |v_i|^2} f_N\, dx_1\, dv_1 \cdots dx_N\, dv_N \le V^N \quad \text{for some } \beta > 0, \quad V > 0, \tag{6}$$

for which we refer to the short discussion in Section 2.4.

Our main result is the derivation of the mean-field limit for a broad class of singular kernels.

Theorem 2. *Assume that there exists some constant $\theta > 0$ such that the potential ϕ satisfies*

$$\int_\Pi e^{\theta \phi(x)}\, dx < +\infty \tag{7}$$

and that

$$K = -\nabla \phi \in L^p(\Pi^d) \quad \textit{for some } p > 1.$$

Let f be the unique smooth solution to the Vlasov equation (3) *with initial data $f^0 \in C^\infty(\Pi^d \times \mathbb{R}^d)$ such that $\int_{\Pi^d \times \mathbb{R}^d} f^0 e^{\beta |v|^2} < \infty$. Consider moreover an entropy solution f_N to* (4) (in the sense of Section 2.4) *satisfying* (6) *with initial data $f_N^0 \in L^\infty(\Pi^{dN} \times \mathbb{R}^{dN})$. Assume that $f_{k,N}^0$ converges weakly in L^1 to $(f^0)^{\otimes k}$ for each fixed k and that*

$$\| f_{k,N}^0 \|_{L^\infty(\Pi^{dN} \times \mathbb{R}^{dN})} \le M^k$$

for some $M > 0$ and for all $k \le N$. Then there exists T^* depending only on M, V, and $\|K\|_{L^p}$ such that the $f_{k,N}$, given by (5), *weakly converge to*

$$f_k = f^{\otimes k} \quad \textit{in } L^q_{\mathrm{loc}}([0, T^\star] \times \Pi^{kd} \times \mathbb{R}^{kd})$$

for any k and any $2 < q < \infty$, with $1/q + 1/p \le 1$.

Our estimates can also provide quantitative rates of convergence though we need to use a stronger assumption, namely $K \in L^2$.

Theorem 3. *Assume the same conditions and hypotheses of Theorem 2, with moreover $p = 2$. We also assume that there exists a constant C independent of N and $\varepsilon_N \to 0$ such that*

$$\int_{\Pi^{kd}\times\mathbb{R}^{kd}} |f^0_{k,N} - (f^0)^{\otimes k}|^2 e^{\lambda(0)e_k} \le C^k \varepsilon_N$$

for all k, with

$$e_k(x_1, v_1, \dots, x_k, v_k) = \sum_{i\le k}(1 + |v_i|^2) + \frac{1}{N}\sum_{i,j\le k}\phi(x_i - x_j) \tag{8}$$

and

$$\lambda(t) = \frac{1}{\Lambda(1+t)} \quad \textit{for a positive constant } \Lambda.$$

Then, there exists T^* such that $f_{k,N}$ converges strongly to f_k in $L^2_{\mathrm{loc}}([0, T^\star] \times \Pi^{kd} \times \mathbb{R}^{kd})$ for any k, and we have the quantitative estimate

$$\sup_{t\le T^\star} \int_{\Pi^{kd}\times\mathbb{R}^{kd}} |f_{N,k} - f^{\otimes k}|^2 e^{\lambda(t)e_k} \le \widetilde{C}^k \varepsilon_N$$

for some $\widetilde{C}$ independent of N.

In addition to the mean-field limit, Theorem 2 implies the weak propagation of chaos in the sense of the famous [Sznitman 1991], although with strong conditions on f^0_N. Theorem 2 also justifies for the first time the convergence to the Vlasov–Poisson–Fokker–Planck in two space dimensions. More precisely, we highlight the following result.

Corollary 4. *Let $d = 2$, and consider the Poisson kernel $K = -\nabla\phi$ with its associated potential $\phi(x) \simeq -\ln|x|$. Then, the convergence properties given by Theorem 2 hold true, leading to the Vlasov–Poisson–Fokker–Planck system.*

2.2. ***New stability estimates.*** Theorem 2 relies on a new approach to derive estimates on the BBGKY hierarchy solved by the marginals $f_{k,N}$, which is of significant interest in itself. In general, deriving bounds on either the BBGKY or limiting Vlasov hierarchy is complex. We refer for example to [Golse et al. 2013] for the Vlasov hierarchy, and to [Duerinckx and Saint-Raymond 2021] for the study of long-time corrections to mean-field limits. Bounds on the hierarchy are critical for the derivation of collisional models such as the Boltzmann equation, ever since [Lanford 1975]. Even a partial discussion of the challenges in the collisional setting would go well beyond the scope of this paper, and we simply refer again to [Bodineau et al. 2017; 2018; 2020; Gallagher et al. 2014; Kac 1956; Lanford 1975; Pulvirenti et al. 2014; Pulvirenti and Simonella 2017].

The main difficulty in handling the hierarchy consists in the term

$$\nabla_{v_i} \int_{\Pi^d\times\mathbb{R}^d} K(x_i - x_{k+1}) f_{k+1,N}\, dx_{k+1}\, dv_{k+1}, \tag{9}$$

as seen in (17), because this introduces the next-order marginal $f_{k+1,N}$ into the equation for $f_{k,N}$. When treated naively as a source term, it leads to a loss of one derivative on each equation of the hierarchy.

However, it was noticed first in [Lacker 2023] and then in [Jabin et al. 2025] that one may avoid this loss of derivative in the stochastic case for nondegenerate diffusion: any L^2 estimate then gains an additional H^1 dissipation which can be used to control the loss of one derivative. This idea still appears applicable in the present kinetic context: even though we only have diffusion in velocity, the derivative in (9) is also only on the velocity variable.

Both [Jabin et al. 2025] and [Lacker 2023] require high integrability on the kernel: $K \in L^\infty$ for [Jabin et al. 2025] and some sort of exponential Orlicz space of the type $\int e^{\lambda |K(x)|}\,dx < C$ for [Lacker 2023]. Lacker [2023] used quantitative relative entropy estimates to prove uniqueness on the BBGKY hierarchy, while [Jabin et al. 2025] proved uniqueness on a tree-indexed limiting hierarchy through L^2 bounds. Hence, in both cases, the corresponding bounds on the marginals was already known uniformly in N, and the challenge was to prove that the norm of the difference with the limit is small.

This leads to a first key difference with respect to the present approach and to the first critical new idea introduced in this paper. In essence, we note that the integral in (9) leads to a regularizing effect that has the same scaling as the convolution at the limit: one has by Hölder estimates that

$$\left\| \int_{\Pi^d} K(x_i - x_{k+1}) f(x_1, \dots, x_{k+1})\, dx_{k+1} \right\|_{L^q(\Pi^{dk})} \le \|K\|_{L^p(\Pi^d)} \|f\|_{L^q(\Pi^{d(k+1)})}, \tag{10}$$

provided that $1/p + 1/q \le 1$.

Taking advantage of (10) for singular $K \in L^p$ with p small naturally leads us to try to propagate L^q norms of the marginals $f_{k,N}$ for large exponents q; in opposition to [Jabin et al. 2025; Lacker 2023]. But it also leads to an additional major difficulty, due to the velocity variable in the unbounded space $\mathbb{R}^d$ in (9). In fact, trying to use (10) in (9) as is would force the use of a mixed norm $L^q_x L^1_v$ on the marginals. Unfortunately such mixed norms are notoriously ill-behaved on kinetic equations.

Instead, a more natural idea, from the point of view of kinetic equations, consists in using some moments or fast decay in velocity. Even if they are less usual for kinetic equations, the use of Gaussian moments is especially attractive in the current case because they are naturally tensorized. For example, one has the extension of (10)

$$\begin{aligned}\int_{\Pi^{dk}\times\mathbb{R}^{dk}} e^{|v_1|^2+\cdots+|v_k|^2} \left| \int_{\Pi^d\times\mathbb{R}^d} K(x_i - x_{k+1}) f_{k+1,N}\, dx_{k+1}\, dv_{k+1} \right|^q \\ \le C_d \|K\|^q_{L^p(\Pi^d)} \int_{\Pi^{d(k+1)}\times\mathbb{R}^{d(k+1)}} e^{|v_1|^2+\cdots+|v_{k+1}|^2} |f_{k+1,N}|^q,\end{aligned} \tag{11}$$

still provided $1/p + 1/q \le 1$.

However, pure Gaussian moments in velocity do not seem to be naturally propagated at the discrete level of the hierarchy, even though they would trivially be propagated on the limiting mean-field equation at least for short time. This leads to the final critical idea of the paper, which is to incorporate the potential energy in the Gaussian: namely to consider $e^{\lambda(t)e_k}$ instead of a pure Gaussian with e_k defined by (8).

We observe that our definition of e_k uses $1 + |v_i|^2$ but could just as well be reduced to $|v_i|^2$ instead as (11) suggests. The extra constant in e_k allows us to normalize the weight of each marginal by a factor $e^{\lambda(0)k}$, which saves some extra numerical constants in the proof.

We also remark that the use of a dynamical weights argument has been recently developed in [Bresch et al. 2023] for first-order particle systems with singular kernels. We also note that Proposition 5, stated below, shows the propagation of weighted L^q bounds on the marginals, without requiring the initial data to be chaotic or close to chaotic as introduced in [Kac 1956]. It hence applies to a broader framework than just the mean-field limit.

Proposition 5. *Let us assume $K \in L^p(\Pi^d)$ for some $p > 1$ and define*

$$\lambda(t) = \frac{1}{\Lambda(1+t)} \quad \text{and} \quad L = \frac{C}{\lambda(1)^\theta} \|K\|_{L^p}^q$$

for positive constants Λ and C, θ depending only on q, d, and σ, and provided that $1/q + 1/p \le 1$. Consider a renormalized solution f_N to (4) *satisfying* (6) *with initial data $f_N^0 \in L^\infty(\Pi^{dN} \times \mathbb{R}^{dN})$ and satisfying*

$$\begin{aligned} \int_{\Pi^{kd}\times\mathbb{R}^{kd}} |f_{k,N}^0|^q e^{\lambda(0)e_k} &\le F_0^k, \\ \sup_{t\le 1} \int_{\Pi^{Nd}\times\mathbb{R}^{Nd}} |f_N|^q e^{\lambda(t)e_N} &\le F^N \end{aligned} \tag{12}$$

for some $F > 0$, $F_0 > 0$, and q such that $2 \le q < \infty$, with $1/q + 1/p \le 1$. Then, one has that

$$\sup_{t\le T} \int_{\Pi^{kd}\times\mathbb{R}^{kd}} |f_{k,N}|^q e^{\lambda(t)e_k} \le 2^k F_0^k + F^k 2^{2k-N-1}, \tag{13}$$

where T is given by

$$T = \min\left(1, \frac{1}{4L\max(F_0, F)}\right).$$

Proposition 5 shows that the corresponding L^q norm of a marginal at order k behaves like C^k for some constant C. This is the expected scaling for propagation of chaos and tensorized marginals $f_k = f^{\otimes k}$.

However, Proposition 5 also presents several intriguing features that we want to highlight.

- *Vlasov–Poisson–Fokker–Planck in higher dimensions.* Proposition 5 handles just as easily Coulombian interactions in any dimension d, and not only dimension $d = 2$ as Theorem 2. Therefore, Proposition 5 would imply some form of propagation of chaos for the Vlasov–Poisson–Fokker–Planck system in any dimension if we are able to consider initial N-particle laws f_N^0 which are f^0-chaotic as $N \to +\infty$ and whose marginals f^0 and associated solution f_n to the forward Kolmogorov equation satisfy (12). While there are examples of such initial data, take $f_N^0 = Z\exp(-e_N)$ for instance, they demand some sort of truncation or decay of the configurations with high energy. This is not satisfying because we cannot even take $f_N^0 = (f^0)^{\otimes N}$: Assumption (12) cannot hold in such a case as $e^{\lambda(0)e_k}$ is not integrable if K is the Poisson kernel in dimension $d > 2$. The issue is that by taking $f_N^0 = (f^0)^{\otimes N}$, we allow some configurations with high potential energy. And roughly speaking the existence time T in the proposition vanishes as the starting potential energy increases in that case.

- *Repulsive potentials.* Proposition 5 does require repulsive potentials $\phi \ge 0$ as this assumption is critical in the proof. The repulsive assumption on the potential only appears to be needed to handle the discrete many-particle system. The extension to nonrepulsive settings remains an open problem.

• *Extension to the stochastic case of mildly singular kernels*. A special case concerns mildly singular kernels K with $K \in L^p$ for some $p > 1$ such that $\phi \in L^\infty$. In that situation, by considering $\phi + \|\phi\|_{L^\infty}$ instead of ϕ, yielding the same interaction kernel K, we can always ensure that $\phi \geq 0$. For example this easily extends for the first time to the stochastic settings the results of [Hauray and Jabin 2007; 2015], which had been obtained only for deterministic second-order systems with

$$|K| \lesssim |x|^{-\alpha} \quad \text{for } \alpha < 1.$$

• *Convergence for finite times*. We finally emphasize that, like Theorem 2, Proposition 5 holds over a finite time interval, independent of N. This may initially appear puzzling since we are dealing with linear equations for any fixed N. However, because those estimates are essentially independent of N, they also extend to the nonlinear limiting Vlasov equation. Moreover Proposition 5 includes a propagation of Gaussian moments in velocity over the marginals from the term $e^{\lambda(t)e_k}$ and the definition (8) of e_k. The propagation for all times of such moments for Vlasov–Poisson is only known in dimension $d = 2$, see [Degond 1986; Ukai and Okabe 1978], and dimension $d = 3$, see [Bouchut 1993; Gasser et al. 2000; Holding and Miot 2018; Lions and Perthame 1991; Ono and Strauss 2000; Pallard 2014; Pfaffelmoser 1992; Rein and Weckler 1992; Schaeffer 1991; Victory and O'Dwyer 1990] as cited in the introduction; it also requires in dimension 3 the use of dispersion estimates that are not present in our proof. As we already noted, Proposition 5 is in fact valid in any dimension which naturally limits it to some given finite time interval.

2.3. ***The case of first-order systems.*** While we focus on second-order systems, we also emphasize that our method directly applies to first-order systems on bounded domains (in a much simpler manner in fact) and provides the mean-field limit under very weak assumptions on the kernel K again. Consider in that case

$$\frac{d}{dt} X_i(t) = \frac{1}{N} \sum_{j \neq i} K(X_i - X_j)\, dt + \sigma\, dW_i, \qquad X_i(t=0) = X_i^0, \tag{14}$$

fully on the torus Π^d. The mean-field limit is similar to (3):

$$\partial_t f + (K \star_x f) \cdot \nabla_x f = \frac{\sigma^2}{2} \Delta_x f. \tag{15}$$

Similarly, the joint law $f_N(t, x_1, \ldots, x_N)$ solves an appropriately modified Liouville equation

$$\partial_t f_N + \sum_{i=1}^N \frac{1}{N} \sum_{j=1}^N K(x_i - x_j) \cdot \nabla_{x_i} f_N = \frac{\sigma^2}{2} \sum_i \Delta_{x_i} f_N. \tag{16}$$

Because system (14) does not involve velocities, many technical difficulties in our proofs actually vanish. For example, we no longer need to add assumptions such as (6). We also do not need to require that K derives from a potential, and hence do not require assumptions like (7). We then have the following equivalent of Theorem 2.

Theorem 6. *Assume that*

$$K \in L^p(\Pi^d) \text{ for some } p > 1, \quad (\operatorname{div} K)_- \in L^\infty(\Pi^d),$$

where x_- denotes the negative part of x. Let f be the unique smooth solution to the Vlasov equation (15) *with initial data $f^0 \in C^\infty(\Pi^d)$. Consider moreover an entropy solution f_N to* (16) (still in the sense of Section 2.4) *with initial data $f_N^0 \in L^\infty(\Pi^{dN})$. Assume that $f_{k,N}^0$ converges weakly in L^1 to $(f^0)^{\otimes k}$ for each fixed k and that*

$$\|f_{k,N}^0\|_{L^\infty(\Pi^{dN})} \le M^k$$

for some $M > 0$ and for all $k \le N$. Then there exists T^* depending only on M, $\|K\|_{L^p}$, and $\|(\operatorname{div} K)_-\|_{L^\infty}$ such that the $f_{k,N}$, given by (5), *weakly converge to $f_k = f^{\otimes k}$ in $L^q_{\text{loc}}([0, T^\star] \times \Pi^{kd})$ for any k and any $2 < q < \infty$, with $1/q + 1/p \le 1$.*

Because it is not our main focus, we do not give a distinct proof of Theorem 6.

As mentioned above, there exists now a large literature for the mean-field limit of first-order systems in the stochastic case, with much recent progress for singular kernels. We refer for example to the derivation of two-dimensional Navier–Stokes equations from a system of many vortices in [Fournier et al. 2014; Jabin and Wang 2018; Osada 1987]. The derivation of the two-dimensional Keller–Segel system, corresponding to attractive Coulombian potentials, was recently obtained in [Bresch et al. 2020; Tardy 2024]; see also [Fournier and Tardy 2024] for a precise description of the collisions leading to the blow-up. We also cite [Lacker 2023] which only requires the kernel to be in an Orlicz space similar to Exp, together with [Lacker and Le Flem 2023] which obtains global-in-time regularity for Lipschitz kernels with a smallness assumption on $\operatorname{div} K$.

All those results require stronger assumptions on the kernel K than just $K \in L^p$ with $p > 1$ as here. A similar scaling was however obtained in [Serfaty 2020] on first-order systems with no diffusion. The breakthrough method in that seminal paper is based on a modulated energy between the empirical measure and the limit and it applies to Riesz kernels where $K \sim 1/|x|^\alpha$ with $\alpha < d$ (corresponding to $K \in L^p$ with $p > 1$), with either a repulsive gradient flow or Hamiltonian interactions, or alternatively where $K * f \in W^{1,\infty}$. Uniform-in-time propagation of chaos was later obtained in [Rosenzweig and Serfaty 2023] including diffusion with the restriction $\alpha < d - 1$ using the modulated energy method and some relaxation rates properties. This was recently improved in [Chodron de Courcel et al. 2023] to again $\alpha < d$ combining precise relaxation rates with the new modulated free energy introduced in [Bresch et al. 2020]. One obvious advantage of our method here is that it allows for a much more general form of interaction, with singularities far away from the origin. On the other hand, Theorem 6 does require a nonvanishing diffusion and is again only valid for a finite time, instead of the much stronger uniform-in-time estimates above.

Contrary to the case of second-order systems, this short-time limitation appears less fundamental as many limiting systems do not blow up, with the obvious exception of attractive interactions such as Keller–Segel. We conjecture that the present method could lead to large-time results by taking advantage of the full nondegenerate diffusion for first-order systems.

2.4. ***Our notion of entropy solution for the hierarchy*: *the well-posedness of* (4).**

The definition. Being nonlinear, our estimates cannot be performed on any weak solutions. Moreover, the concept of a solution for f_N is carried over the marginals $f_{k,N}$ and not just the joint law f_N, so we also need an appropriate notion of entropy solutions on those marginals.

The hierarchy for the marginals from the Liouville equation. From (4), the $f_{k,N}$ solve the so-called BBGKY hierarchy

$$\partial_t f_{k,N} + \sum_{i=1}^{k} v_i \cdot \nabla_{x_i} f_{k,N} + \sum_{i\le k} \frac{1}{N} \sum_{j\le k} K(x_i - x_j) \cdot \nabla_{v_i} f_{k,N} + \frac{N-k}{N} \sum_{i\le k} \nabla_{v_i} \cdot \int_{\Pi^d\times\mathbb{R}^d} f_{k+1,N} K(x_i - x_{k+1})\, dx_{k+1}\, dv_{k+1} = \frac{\sigma^2}{2} \sum_{i\le k} \Delta_{v_i} f_{k,N}. \quad (17)$$

If f_N belongs to L^∞ and satisfies (6), then all marginals $f_{k,N}$ belong to $L^\infty_t L^q_{x,v}$ for every $q < \infty$ with similar Gaussian decay. For simplicity, we denote here abstractly by $L^q_{x,v}$ any space $L^q(\Pi^{kd} \times \mathbb{R}^{kd})$ when there is no confusion about the dimension k, as in our case. We also denote by $L^q_{\lambda e_k}$ the weighted L^q space

$$\|f\|^q_{L^q_{\lambda e_k}} = \int_{\Pi^{kd}\times\mathbb{R}^{kd}} |f|^q e^{\lambda e_k}.$$

Since $K \in L^p$ for some $p > 1$, by using a direct Hölder inequality, those bounds on the $f_{k,N}$ imply that

$$\int_{\Pi^d\times\mathbb{R}^d} f_{k+1,N} K(x_i - x_{k+1})\, dx_{k+1}\, dv_{k+1} \in L^\infty_t L^q_{x,v}$$

for all $q < \infty$. This allows us to immediately and rigorously derive (17) from (4).

Definition of entropy solutions. We write the advection component of (17) as

$$L_k = \sum_{i\le k} v_i \cdot \nabla_{x_i} + \frac{1}{N} \sum_{i,j\le k} K(x_i - x_j) \cdot \nabla_{v_i}. \quad (18)$$

The argument above implies that the only difficulties to propagate our estimates in (17) stem from L_k. Consequently we define our entropy solution as follows: a function $f_N \in L^\infty([0,1] \times \Pi^{dN} \times \mathbb{R}^{dN})$ satisfying (6) is an entropy solution if and only if all marginals $f_{k,N}$ for $1 \le k \le N$, as defined by (5), satisfy

$$\int_0^T \int_{\Pi^{dk}\times\mathbb{R}^{dk}} e^{\lambda e_k} |f_{k,N}|^{q-1} \operatorname{sign}(f_{k,N}) L_k f_{k,N}\, dx_1\, dv_1 \cdots dx_k\, dv_k\, dt \ge 0 \quad (19)$$

for any $T \in [0,1]$, any $1 < q < \infty$, and any $\lambda < \lambda_0$. Inequality (19) is still somewhat formal and should be understood in the following rigorous sense: for some smooth convolution kernel K_ε, one has that

$$\liminf_{\varepsilon\to 0} \int_0^T \int_{\Pi^{dk}\times\mathbb{R}^{dk}} e^{\lambda e_k} |K_\varepsilon^{\otimes k} \star f_{k,N}|^{q-1} \operatorname{sign}(K_\varepsilon^{\otimes k} \star f_{k,N}) K_\varepsilon^{\otimes k} \star (L_k f_{k,N})\, dx_1\, dv_1 \cdots dx_k\, dv_k\, dt \ge 0, \quad (20)$$

where we define

$$K_\varepsilon^{\otimes k}\star g=\int_{\Pi^{dk}\times\mathbb{R}^{dk}} K_\varepsilon(x_1-y_1,v_1-w_1)\cdots K_\varepsilon(x_k-y_k,v_k-w_k)g(y_1,w_1,\dots,y_k,w_k)\,dy_1\,dw_1\cdots dy_k\,dw_k,$$

with $K_\varepsilon\to\delta$ when $\varepsilon\to 0$. However, it is usually more delicate to determine whether any weak solution f_N in L^∞ and with the bound (6) is an entropy solution according to our definition. For linear advection-diffusion equations such as (4), this is usually approached through the notion of renormalized solutions as introduced in [DiPerna and Lions 1989]. In that context, (20) is obviously similar to the classical commutator estimate at the basis of many methods for renormalized solutions.

Remark 7. (1) We first remark that (19) is automatically satisfied if we have classical solutions. Indeed, L_k is an antisymmetric operator, so we expect it to propagate L^q norms such that, if all terms are smooth, we have

$$|f_{k,N}|^{q-1}\mathrm{sign}(f_{k,N})L_k f_{k,N}=L_k|f_{k,N}|^q.$$

(2) We immediately observe that the reduced energy e_k is formally invariant under the advection component of (17):

$$L_k e_k=\frac{2}{N}\sum_{i,j\le k} v_i\cdot\nabla_{x_i}\phi(x_i-x_j)+\frac{2}{N}\sum_{i,j\le k} K(x_i-x_j)\cdot v_i=0$$

since $K=-\nabla_x\phi$. In the same way, we have $L_k\Phi(e_k)=0$ for any locally Lipschitz function Φ.

(3) If K is smooth and f_N is a classical solution to (4), we would hence immediately have equality in (19). With K only in L^p, it would be straightforward to obtain one entropy solution in the sense defined above, through passing to the limit in a sequence of solutions for a smoother kernel K.

Remark 8. There exists an extensive literature on renormalized solutions with a comparably large variety of potential assumptions that one may consider. While we cannot do justice to this question in this short discussion, we briefly mention for instance [Hauray 2004] that studies the specific case of the Liouville equation (4) for second-order systems without diffusion. In the present setting of a constant nonvanishing diffusion, we also refer to [Bogachev et al. 2015; Le Bris and Lions 2008; 2019] that provide broad results of well-posedness for velocity fields in L^p.

We in particular note that renormalized solutions apply to the case $K\in L^p$ with $p>2$ and f_N in L^∞ with $\nabla_{v_i} f_N^{q/2}\in L^2$ for any $q<\infty$ and satisfying the extension of (6)

$$\sup_{t\le 1}\int_{\Pi^{dN}\times\mathbb{R}^{dN}} e^{\lambda_0 e_k} f_N\,dx_1\,dv_1\cdots dx_N\,dv_N<\infty.$$

The latter estimates are natural for the Liouville equation (4), as demonstrated by Lemma 9 for the case $k=N$ in Section 3. In that situation, all marginals $f_{k,N}$ belong to $L_t^\infty L_{x,v}^q$ for every $q<\infty$ with similar exponential decay in e_k and with as well $\nabla_{v_i} f_{k,N}^{q/2}\in L_{t,x,v}^r$ for any $r<2$. This regularity easily allows us to prove that (20) holds for $\lambda<\lambda_0$.

We also mention that so-called mild solutions can also offer a natural way to prove (20). We simply refer to [Bouchut 1993; Carrillo and Soler 1997] for such formulations through the Fokker–Planck kernel in the whole space, or to [Clark 1993] or [Degond 1986; Victory and O'Dwyer 1990] for periodic conditions.

Strong solutions up to the first collision. We also emphasize that, in the case of repulsive kernels smooth out of the origin but with singular potentials $\lim_{x\to 0}\phi(x) = +\infty$, a straightforward bound on the energy of the system can easily lead to strong solutions on the many-particle system (1), bypassing the need for entropy or renormalized solutions.

Very roughly, if $K \in C^\infty(\Pi^d \setminus \{0\})$, then up to the conditional time of first collision in (1), we may write

$$d\bigg(\sum_{i=1}^N |V_i|^2 + \frac{1}{N}\sum_{i\neq j}\phi(X_i - X_j)\bigg) = \sigma^2\,dt + \sum_{i=1}^N 2\sigma V_i\cdot dW_i.$$

This implies that, with probability 1, the total energy remains finite if it was so initially. Because $\lim_{x\to 0}\phi(x) = +\infty$, it also implies that collisions almost surely never happen. This argument would in particular apply to the Coulombian case in any dimension $d \geq 2$.

To conclude this discussion of the well-posedness of (4) or (1) for a fixed N, we emphasize the estimates that we described here cannot easily be made uniform in N. The previous discussion of the energy bound on the system (1) for the Coulombian interaction in dimension $d = 2$ is an excellent illustration: if we have the bound

$$\sum_{i=1}^N |V_i|^2 + \frac{1}{N}\sum_{i\neq j}\phi(X_i - X_j) \leq E$$

with some large probability on some time interval and for $\phi(x) = -\log|x|$, then this only proves that, for any $i \neq j$,

$$|X_i - X_j| \geq e^{-NE},$$

which is indeed finite for any fixed N but is completely unhelpful when considering the limit $N \to \infty$.

Hence the present discussion remains focused on renormalized solutions for a fixed N. Quantitative approaches to renormalized solutions have for example been introduced in [Crippa and De Lellis 2008], which are based on the propagation of a sort of log-derivative on the characteristics; see also for example the discussion on Eulerian variants in [Bresch and Jabin 2018]. This leads to an interesting and so far mostly fully open question as to whether it would be possible to obtain quantitative bounds that would combine the limit $N \to \infty$ with some regularity estimates on the solution for a fixed N.

3. Proof of the main results

3.1. ***The BBGKY and Vlasov hierarchies.*** Using (3), the tensorized limits $f_k = \bar{f}^{\otimes k}$ satisfy the Vlasov hierarchy

$$\partial_t f_k + \sum_{i=1}^k v_i\cdot\nabla_{x_i} f_k + \sum_{i=1}^k \bigg(K \star \int_{\mathbb{R}^d} f\,dv\bigg)\cdot\nabla_{v_i} f_k = \frac{\sigma^2}{2}\sum_{i=1}^k \Delta_{v_i} f_k. \tag{21}$$

To avoid repeating the analysis working on (17) or (21), we introduce the generalized hierarchy equation

$$\partial_t F_{k,N} + \sum_{i=1}^{k} v_i \cdot \nabla_{x_i} F_{k,N} + \sum_{i\le k} \frac{\gamma}{N} \sum_{j\le k} K(x_i - x_j)\cdot \nabla_{v_i} F_{k,N} \\ + \frac{N-\gamma k}{N} \sum_{i\le k} \nabla_{v_i} \cdot \int_{\Pi^d\times\mathbb{R}^d} F_{k+1,N} K(x_i - x_{k+1})\, dx_{k+1}\, dv_{k+1} = \frac{\sigma^2}{2} \sum_{i\le k} \Delta_{v_i} F_{k,N} + R_{k,N}. \quad (22)$$

Note that (22) is exactly (21) for $\gamma = 0,\ R_{k,N} = 0$ and exactly (17) for $\gamma = 1,\ R_{k,N} = 0$. In the same spirit we define

$$e_{k,\gamma} = \sum_{i\le k} (1 + |v_i|^2) + \frac{\gamma}{N} \sum_{i,j\le k} \phi(x_i - x_j),$$

$$L_{k,\gamma} = \sum_{i\le k} v_i \cdot \nabla_{x_i} + \frac{\gamma}{N} \sum_{i,j\le k} K(x_i - x_j)\cdot \nabla_{v_i}$$

and observe that we of course still have $L_{k,\gamma} e_{k,\gamma} = 0$.

The main technical contribution of this section and of the paper is Lemma 9 stated in Section 3.2, which provides estimates for the solutions to (17). We will then use the uniform bound on the k-marginals $f_{k,N}$ for the proof of Proposition 5. Proposition 5 allows passing to the limit in the hierarchy (17), and a final use of Lemma 9 leads to proving uniqueness of the limiting hierarchy (21) to conclude the result of Theorem 2.

3.2. ***The key technical lemma.*** We first present the key technical lemma which links the k-marginal L^q_w control to the $(k+1)$-marginal L^q_w estimate control.

Lemma 9. *Assume that $K \in L^p(\Pi^d)$ for some $p > 1$. There exist some constants Λ, C, and θ depending only on q, d, and σ such that*

$$\|F_{k,N}\|^q_{L^q_{\lambda(t)e_k}} \le \|F_{k,N}(t=0)\|^q_{L^q_{\lambda(0)e_k}} + q \int_0^t \int |F_{k,N}|^{q-1} \operatorname{sign}(F_{k,N}) R_{k,N} e^{\lambda(s)e_{k,\gamma}}\, ds \\ + k\frac{N-\gamma k}{N} \frac{C}{\lambda^\theta(t)} \|K\|^q_{L^p} \int_0^t \|F_{k+1,N}(s)\|^q_{L^q_{\lambda(s)e_{k+1}}}\, ds$$

for any entropy solution $F_{k,N}$ to (22) *(in the sense of Section 2.4) and satisfying* (6) *with $F_{k,N} \in L^q_{\lambda(t)e_{k,\gamma}}$ and for any $2 \le q < \infty$ such that $1/q + 1/p \le 1$, with $\lambda(t)$ defined by $\lambda(t) = (\Lambda(1+t))^{-1}$.*

Proof. To be made fully rigorous, many calculations in this proof should involve a convolution kernel K_ε, estimating

$$\frac{d}{dt} \int |K_\varepsilon^{\otimes k} \star F_{k,N}|^q e^{\lambda(t)e_{k,\gamma}},$$

and passing to the limit in $\varepsilon \to 0$ while using appropriately the entropy condition (20). For simplicity, however, we will only present the corresponding formal calculations.

We hence calculate in a straightforward manner

$$\frac{d}{dt} \int |F_{k,N}|^q e^{\lambda(t)e_{k,\gamma}} = q \int |F_{k,N}|^{q-1} \operatorname{sign}(F_{k,N}) \partial_t F_{k,N} e^{\lambda(t)e_{k,\gamma}} + \lambda'(t) \int e_{k,\gamma} |F_{k,N}|^q e^{\lambda(t)e_{k,\gamma}}.$$

Inserting now in this identity the definition of $\lambda(t)$ and the (17), we find

$$\begin{aligned}
&\frac{d}{dt}\int |F_{k,N}|^q e^{\lambda(t)e_{k,\gamma}}\\
&= -q\int |F_{k,N}|^{q-1}\operatorname{sign}(F_{k,N})(L_{k,\gamma}F_{k,N})e^{\lambda(t)e_{k,\gamma}} + q\frac{\sigma^2}{2}\int |F_{k,N}|^{q-1}\operatorname{sign}(F_{k,N})\Big(\sum_{i\le k}\Delta_{v_i}F_{k,N}\Big)e^{\lambda(t)e_{k,\gamma}}\\
&\qquad - q\frac{N-\gamma k}{N}\sum_{i\le k}\int |F_{k,N}|^{q-1}\operatorname{sign}(F_{k,N})\nabla_{v_i}\cdot\int K(x_i-x_{k+1})F_{k+1,N}\,dx_{k+1}\,dv_{k+1}e^{\lambda(t)e_{k,\gamma}}\\
&\qquad - \Lambda\lambda^2(t)\int e_{k,\gamma}|F_{k,N}|^q e^{\lambda(t)e_{k,\gamma}} + q\int |F_{k,N}|^{q-1}\operatorname{sign}(F_{k,N})R_{k,N}e^{\lambda(t)e_{k,\gamma}}.
\end{aligned}$$

Note that

$$q|F_{k,N}|^{q-1}\operatorname{sign}(F_{k,N})(L_{k,\gamma}F_{k,N}) = L_{k,\gamma}|F_{k,N}|^q,$$

so that by integration by parts, we formally have

$$q\int |F_{k,N}|^{q-1}\operatorname{sign}(F_{k,N})(L_{k,\gamma}F_{k,N})e^{\lambda(t)e_{k,\gamma}} = -\int |F_{k,N}|^q L_{k,\gamma}e^{\lambda(t)e_{k,\gamma}} = 0.$$

On the other hand, again by integration by parts,

$$\begin{aligned}
&q\frac{\sigma^2}{2}\int |F_{k,N}|^{q-1}\operatorname{sign}(F_{k,N})\Big(\sum_{i\le k}\Delta_{v_i}F_{k,N}\Big)e^{\lambda(t)e_{k,\gamma}}\\
&= -q(q-1)\sum_{i\le k}\frac{\sigma^2}{2}\int |F_{k,N}|^{q-2}|\nabla_{v_i}F_{k,N}|^2 e^{\lambda(t)e_{k,\gamma}}\\
&\qquad - 2q\lambda(t)\sum_{i\le k}\frac{\sigma^2}{2}\int |F_{k,N}|^{q-1}\operatorname{sign}(F_{k,N})v_i\cdot\nabla_{v_i}F_{k,N}e^{\lambda(t)e_{k,\gamma}}.
\end{aligned}$$

By the Cauchy–Schwartz inequality, since $q \ge 2$, we obtain

$$\begin{aligned}
&q\frac{\sigma^2}{2}\int |F_{k,N}|^{q-1}\operatorname{sign}(F_{k,N})\Big(\sum_{i\le k}\Delta_{v_i}F_{k,N}\Big)e^{\lambda(t)e_{k,\gamma}}\\
&\qquad \le -q(q-1)\sum_{i\le k}\frac{\sigma^2}{4}\int |F_{k,N}|^{q-2}|\nabla_{v_i}F_{k,N}|^2 e^{\lambda(t)e_{k,\gamma}} + \frac{q}{q-1}\lambda^2\frac{\sigma^2}{2}\int |F_{k,N}|^q\sum_{i\le k}|v_i|^2 e^{\lambda(t)e_{k,\gamma}}.
\end{aligned}$$

Note that, since $\phi \ge 0$, we have $\sum_{i\le k}|v_i|^2 \le e_k$ and, therefore, combining all our estimates so far, we deduce that

$$\begin{aligned}
&\frac{d}{dt}\int |F_{k,N}|^q e^{\lambda(t)e_{k,\gamma}}\\
&\le -q(q-1)\sum_{i\le k}\frac{\sigma^2}{4}\int |F_{k,N}|^{q-2}|\nabla_{v_i}F_{k,N}|^2 e^{\lambda(t)e_{k,\gamma}}\\
&\qquad - q\frac{N-\gamma k}{N}\sum_{i\le k}\int |F_{k,N}|^{q-1}\operatorname{sign}(F_{k,N})\nabla_{v_i}\cdot\int K(x_i-x_{k+1})F_{k+1,N}\,dx_{k+1}\,dv_{k+1}e^{\lambda(t)e_{k,\gamma}}\\
&\qquad - \frac{\Lambda}{2}\lambda^2(t)\int e_{k,\gamma}|F_{k,N}|^q e^{\lambda(t)e_{k,\gamma}} + q\int |F_{k,N}|^{q-1}\operatorname{sign}(F_{k,N})R_{k,N}e^{\lambda(t)e_{k,\gamma}},
\end{aligned}$$

provided that $\Lambda \ge q\sigma^2/(q-1)$.

We integrate by parts the second term in the right-hand side to obtain

$$\sum_{i\le k}\int |F_{k,N}|^{q-1}\mathrm{sign}(F_{k,N})\nabla_{v_i}\cdot\int K(x_i-x_{k+1})F_{k+1,N}\,dx_{k+1}\,dv_{k+1}e^{\lambda(t)e_{k,\gamma}}=\mathrm{RH}_1+\mathrm{RH}_2,$$

with

$$\mathrm{RH}_1=-(q-1)\sum_{i\le k}\int |F_{k,N}|^{q-2}\nabla_{v_i}F_{k,N}\int K(x_i-x_{k+1})F_{k+1,N}\,dx_{k+1}\,dv_{k+1}e^{\lambda(t)e_{k,\gamma}}$$

and

$$\mathrm{RH}_2=-2\lambda(t)\sum_{i\le k}\int |F_{k,N}|^{q-1}\mathrm{sign}(F_{k,N})v_i\int K(x_i-x_{k+1})F_{k+1,N}\,dx_{k+1}\,dv_{k+1}e^{\lambda(t)e_{k,\gamma}}.$$

We perform a straightforward Cauchy–Schwartz inequality on both terms to find that

$$\mathrm{RH}_2\le\lambda^2(t)\sum_{i\le k}\int |F_{k,N}|^q|v_i|^2e^{\lambda(t)e_{k,\gamma}}+\sum_{i\le k}\int |F_{k,N}|^{q-2}\left|\int K(x_i-x_{k+1})F_{k+1,N}\,dx_{k+1}\,dv_{k+1}\right|^2e^{\lambda(t)e_{k,\gamma}},$$

and similarly

$$\mathrm{RH}_1\le\frac{\sigma^2}{4}\sum_{i\le k}\int |F_{k,N}|^{q-2}|\nabla_{v_i}F_{k,N}|^2e^{\lambda(t)e_{k,\gamma}}+\frac{(q-1)^2}{\sigma^2}\sum_{i\le k}\int |F_{k,N}|^{q-2}\left|\int K(x_i-x_{k+1})F_{k+1,N}\,dx_{k+1}\,dv_{k+1}\right|^2e^{\lambda(t)e_{k,\gamma}}.$$

Note that by Young estimates

$$\int |F_{k,N}|^{q-2}\left|\int K(x_i-x_{k+1})F_{k+1,N}\,dx_{k+1}\,dv_{k+1}\right|^2e^{\lambda(t)e_{k,\gamma}}\le\frac{q-2}{q}\lambda^2\int |F_{k,N}|^qe^{\lambda(t)e_{k,\gamma}}+\frac{2}{q\lambda^{q-2}}\int e^{\lambda(t)e_{k,\gamma}}\left|\int K(x_i-x_{k+1})F_{k+1,N}\,dx_{k+1}\,dv_{k+1}\right|^q.$$

Therefore, combining together all those terms, we obtain the further estimate

$$\begin{aligned}\sum_{i\le k}\int |F_{k,N}|^{q-1}\mathrm{sign}(F_{k,N})\nabla_{v_i}\cdot\int K(x_i-x_{k+1})F_{k+1,N}\,dx_{k+1}\,dv_{k+1}e^{\lambda(t)e_{k,\gamma}}\\ \le\frac{\sigma^2}{4}\sum_{i\le k}\int |F_{k,N}|^{q-2}|\nabla_{v_i}F_{k,N}|^2e^{\lambda(t)e_{k,\gamma}}+\lambda^2(t)\left(1+\frac{(q-2)(q-1)^2}{q\sigma^2}\right)\sum_{i\le k}\int |F_{k,N}|^q(1+|v_i|^2)e^{\lambda(t)e_{k,\gamma}}\\ +\frac{2}{q\lambda^{q-2}}\left(1+\frac{(q-1)^2}{\sigma^2}\right)\sum_{i\le k}\int e^{\lambda(t)e_{k,\gamma}}\left|\int K(x_i-x_{k+1})F_{k+1,N}\,dx_{k+1}\,dv_{k+1}\right|^q.\end{aligned}$$

Hence, provided that

$$\Lambda\ge 2q\left(1+\frac{(q-2)(q-1)^2}{q\sigma^2}\right),$$

we obtain

$$\frac{d}{dt}\int |F_{k,N}|^qe^{\lambda(t)e_{k,\gamma}}\le C_{q,\sigma,d}k\frac{N-\gamma k}{\lambda^{q-2}N}\int e^{\lambda(t)e_{k,\gamma}}\left|\int K(x_1-x_{k+1})F_{k+1,N}dx_{k+1}dv_{k+1}\right|^q.$$

At this point is where we take advantage of the specific structure of the hierarchy. Denoting by q^* the conjugate of q, namely such that $1/q^* + 1/q = 1$, we bound

$$\left|\int K(x_1 - x_{k+1}) F_{k+1,N}\, dx_{k+1}\, dv_{k+1}\right|^q \leq \left(\int |K(x_1 - x_{k+1})|^{q^*} e^{-(q^*/q)\lambda(t)|v_{k+1}|^2}\, dx_{k+1}\, dv_{k+1}\right)^{q/q^*} \int |F_{k+1,N}|^q e^{\lambda(t)|v_{k+1}|^2}\, dx_{k+1}\, dv_{k+1},$$

which implies

$$\left|\int K(x_1 - x_{k+1}) F_{k+1,N}\, dx_{k+1}\, dv_{k+1}\right|^q \leq \frac{C_{q,\sigma,d}}{\lambda^{qd/(2q^*)}(t)} \|K\|_{L^p}^q \int |F_{k+1,N}|^q e^{\lambda(t)|v_{k+1}|^2}\, dx_{k+1}\, dv_{k+1}$$

since $q \geq p^*$. Consequently

$$\int e^{\lambda(t)e_{k,\gamma}} \left|\int K(x_1 - x_{k+1}) F_{k+1,N}\, dx_{k+1}\, dv_{k+1}\right|^q \leq \frac{C_{q,\sigma,d}}{\lambda^{qd/(2q^*)}(t)} \|K\|_{L^p}^q \int |F_{k+1,N}|^q e^{\lambda(t)|v_{k+1}|^2 + \lambda(t)e_{k,\gamma}}\, dx_1\, dv_1 \cdots dx_{k+1}\, dv_{k+1}.$$

Note that

$$e_{k+1,\gamma} = e_{k,\gamma} + 1 + |v_{k+1}|^2 + \frac{2\gamma}{N} \sum_{i \leq k} \phi(x_i - x_{k+1}) \geq e_{k,\gamma} + 1 + |v_{k+1}|^2,$$

so that

$$\int e^{\lambda(t)e_k} \left|\int K(x_i - x_{k+1}) f_{k+1,N}\, dx_{k+1}\, dv_{k+1}\right|^q \leq \frac{C_{q,\sigma,d}}{\lambda^{qd/(2q^*)}(t)} \|K\|_{L^p}^q \int |f_{k+1,N}|^q e^{\lambda(t)e_{k+1}}\, dx_1\, dv_1 \cdots dx_{k+1}\, dv_{k+1}.$$

This finally lets us conclude, as claimed, that

$$\frac{d}{dt} \int |f_{k,N}|^q e^{\lambda(t)e_{k,\gamma}} \leq k \frac{N - \gamma k}{N} \frac{C_{q,\sigma}, d}{\lambda^{\theta_{q,d}}(t)} \|K\|_{L^p}^q \int |f_{k+1,N}|^q e^{\lambda(t)e_{k+1,\gamma}} + q \int |F_{k,N}|^{q-1} \operatorname{sign}(F_{k,N}) R_{k,N} e^{\lambda(t)e_{k,\gamma}}. \qquad \square$$

3.3. *Proof of technical results.* We start this subsection with the proof of Proposition 5.

Proof of Proposition 5. From the analysis in Section 3.1 and the assumptions (6) and (12) of Proposition 5, we have that $F_{k,N} = f_{k,N}$ is a renormalized solution to (17) and thus (22) with $\gamma = 1$. Moreover, $f_{k,N}$ satisfies the other assumptions in Lemma 9 with $R_{k,N} = 0$. Writing

$$X_k(t) = \int |f_{k,N}|^q e^{\lambda(t)e_k},$$

we hence observe that, by Lemma 9, we have the coupled dynamical inequality system

$$X_k(t) \leq X_k(0) + kL \int_0^t X_{k+1}(s)\, ds$$

for any $t \in [0, 1]$, where

$$L = \frac{C}{\lambda^\theta(1)} \|K\|^q_{L^p}.$$

From the assumptions of Proposition 5, we immediately have that

$$X_k(t) \le F_0^k + kL \int_0^t X_{k+1}(s)\, ds. \tag{23}$$

We now invoke the following simple lemma.

Lemma 10. *Consider any sequence $X_k(t)$ satisfying* (23). *Then one has*

$$X_k(t) \le \sum_{l=k}^{m} F_0^l L^{l-k} t^{l-k} \frac{(l-1)!}{(k-1)!\,(l-k)!} + L^{m+1-k} \int_0^t X_{m+1}(s)(t-s)^{m-k} \frac{m!}{(k-1)!\,(m-k)!}\, ds. \tag{24}$$

Assuming Lemma 10 holds, we use (24) up to $m+1 = N$ to derive through the assumptions on f_N that

$$X_k(t) \le \sum_{l=k}^{N-1} F_0^l L^{l-k} t^{l-k} \frac{(l-1)!}{(k-1)!\,(l-k)!} + L^{N-k} \int_0^t F^N (t-s)^{N-1-k} \frac{(N-1)!}{(k-1)!\,(N-1-k)!}\, ds,$$

that is

$$X_k(t) \le \sum_{l=k}^{N-1} F_0^l L^{l-k} t^{l-k} \frac{(l-1)!}{(k-1)!\,(l-k)!} + F^N L^{N-k} t^{N-k} \frac{(N-1)!}{(k-1)!\,(N-k)!}. \tag{25}$$

Note that

$$\frac{(l-1)!}{(k-1)!\,(l-k)!} = \binom{l-1}{k-1} \le 2^{l-1}.$$

Hence (25) implies

$$\begin{aligned} X_k(t) &\le \sum_{l=k}^{N} F_0^l L^{l-k} t^{l-k} 2^{l-1} + F^N L^{N-k} t^{N-k} 2^{N-1} \\ &= 2^{k-1} F_0^k \sum_{l=k}^{N-1} F_0^{l-k} 2^{l-k} L^{l-k} t^{l-k} + F^k 2^{k-1} F^{N-k} L^{N-k} t^{N-k} 2^{N-k} \\ &\le 2^{k-1} F_0^k (2 - 2^{k-N+1}) + F^k 2^{k-1} 2^{k-N} \\ &\le F_0^k 2^k + F^k 2^{2k-N-1}, \end{aligned}$$

provided that $4Lt \max(F_0, F) < 1$, which concludes the proof of the proposition. □

We finish with the quick proof of Lemma 10.

Proof of Lemma 10. Taking $m = k$ in (24), we get

$$X_k(t) \le F_0^k + L \int_0^t X_{k+1}(s) \frac{k!}{(k-1)!\,(k-k)!}\, ds,$$

which is our starting point. Moreover, assuming that (24) holds for m, we may use (23) to find

$$X_k(t) \le \sum_{l=k}^{m} F_0^l L^{l-k} t^{l-k} \frac{(l-1)!}{(k-1)!\,(l-k)!} + L^{m+1-k} \int_0^t \left(F_0^{m+1} + L(m+1) \int_0^s X_{m+2}(s)\,ds \right) (t-s)^{m-k} \frac{m!}{(k-1)!\,(m-k)!}\,ds.$$

This yields

$$X_k(t) \le \sum_{l=k}^{m} F_0^l L^{l-k} t^{l-k} \frac{(l-1)!}{(k-1)!\,(l-k)!} + L^{m+1-k} F_0^{m+1} \frac{m!}{(k-1)!\,(m-k)!} \int_0^t (t-s)^{m-k}\,ds + L^{m+2-k} \int_0^t X_{m+2}(r) \int_r^t (t-s)^{m-k}\,ds\,dr \frac{(m+1)!}{(k-1)!\,(m-k)!},$$

or

$$X_k(t) \le \sum_{l=k}^{m} F_0^l L^{l-k} t^{l-k} \frac{(l-1)!}{(k-1)!\,(l-k)!} + L^{m+1-k} F_0^{m+1} \frac{m!}{(k-1)!\,(m+1-k)!} t^{m+1-k} + L^{m+2-k} \int_0^t X_{m+2}(r)(t-r)^{m+1-k}\,dr \frac{(m+1)!}{(k-1)!\,(m+1-k)!},$$

as claimed. □

3.4. *Proof of Theorem 2.* The proof of Theorem 2 follows closely the steps in the proof of Proposition 5, once appropriate bounds have been derived.

(1) *Uniform bounds on f_N in $L^q_{e_N}$.* First of all, note that from the assumptions of Theorem 2, we can easily obtain a bound on f_N^0 in $L^q_{\lambda^0 e_N}$ for Λ large enough. Indeed

$$\int_{\Pi^{dN}\times\mathbb{R}^{dN}} |f_N^0|^q e^{\lambda^0 e_N} = e^N \int_{\Pi^{dN}\times\mathbb{R}^{dN}} |f_N^0|^q e^{2\lambda^0 \sum_{i\le N} |v_i|^2} e^{(\lambda^0/N)\sum_{i,j\le N} \phi(x_i-x_j) - \lambda^0 \sum_{i\le N} |v_i|^2}.$$

We have straightforward L^r estimates on $e^{(\lambda^0/N)\sum_{i,j\le N} \phi(x_i-x_j) - \lambda^0 \sum_{i\le N} |v_i|^2}$ as, by the Hölder inequality,

$$\int_{\Pi^{dN}\times\mathbb{R}^{dN}} e^{(r\lambda^0/N)\sum_{i,j\le N} \phi(x_i-x_j) - r\lambda^0 \sum_{i\le N} |v_i|^2} = \frac{C^N}{\lambda_0^{N/2}} \int_{\Pi^{dN}} e^{(r\lambda^0/N)\sum_{i,j\le N} \phi(x_i-x_j)} \le \frac{C^N}{\lambda_0^{N/2}} \left(\Pi_{i\le N} \int_{\Pi^{dN}} e^{r\lambda^0 \sum_{j\le N} \phi(x_i-x_j)} \right)^{1/N} \le \frac{C^N}{\lambda_0^{N/2}}$$

for some constant C and by assumption (7) in Theorem 2, provided that $r\lambda^0 \le 1/\theta$. This implies, again by Hölder's inequality,

$$\int_{\Pi^{dN}\times\mathbb{R}^{dN}} |f_N^0|^q e^{\lambda^0 e_N} \le \frac{C^N}{\lambda_0^{N/2}} \int_{\Pi^{dN}\times\mathbb{R}^{dN}} |f_N^0|^{r^* q} e^{2r^*\lambda^0 \sum_{i\le N} |v_i|^2} \le \frac{C^N}{\lambda_0^{N/2}} \|f_N^0\|_{L^\infty}^{qr^*-1} \int_{\Pi^{dN}\times\mathbb{R}^{dN}} |f_N^0|^{r^* q} e^{2r^*\lambda^0 \sum_{i\le N} |v_i|^2}.$$

Using now assumption (6), provided that $2r^*\lambda_0 \le \beta$, we conclude

$$\int_{\Pi^{dN}\times\mathbb{R}^{dN}} |f_N^0|^q e^{\lambda^0 e_N} \le \left(\frac{CVM}{\lambda_0}\right)^N \tag{26}$$

for any $q < \infty$. We now choose any fixed $2 < q < \infty$ such that $1/p + 1/q < 1$, and we remark that the Liouville equation (4) is included in (22) for $\gamma = 1$, $R_{k,N} = 0$, and $k = N$. Thus, we next invoke Lemma 9 for f_N with $k = N$ and $\gamma = 1$ to find that f_N solves

$$\frac{d}{dt}\int_{\Pi^{dN}\times\mathbb{R}^{dN}} |f_N(t,\cdot,\cdot)|^q e^{\lambda(t)e_N} \le 0,$$

so that, from (26), we obtain

$$\sup_{t\le 1}\int_{\Pi^{dN}\times\mathbb{R}^{dN}} |f_N(t,\cdot,\cdot)|^q e^{\lambda(t)e_N} \le \left(\frac{CVM}{\lambda_0}\right)^N.$$

This finally implies that there exists some constant $F > 0$ such that

$$\sup_{t\le 1}\int_{\Pi^{dN}\times\mathbb{R}^{dN}} |f_N(t,\cdot,\cdot)|^q e^{\lambda(t)e_N} \le F^N. \tag{27}$$

(2) *Uniform estimates on the marginals and passing the limit in the hierarchy* (17). First of all we can perform the same bounds on each $f_{k,N}^0$ to find similarly to (26) that

$$\int_{\Pi^{kd}\times\mathbb{R}^{kd}} |f_{k,N}^0|^q e^{\lambda^0 e_k} \le \left(\frac{CVM}{\lambda_0}\right)^k.$$

As a consequence, every assumption of Proposition 5 holds and, in particular, assumption (12) holds. This implies that, for some time $T^* > 0$ depending only on V, M, $\|K\|_{L^p}$, and the choice of q, we have

$$\sup_N \sup_{t\le T^*}\int_{\Pi^{kd}\times\mathbb{R}^{kd}} |f_{k,N}|^q e^{\lambda(t)e_k} \le \bar{M}^k$$

for some constant $\bar{M}$. At this point, we will no longer need the potential in the reduced energy e_k, which was required to handle the L_k operator that vanishes at the limit. For this reason and since $\phi \ge 0$, we deduce from the previous inequality that

$$\sup_N \sup_{t\le T^*}\int_{\Pi^{kd}\times\mathbb{R}^{kd}} |f_{k,N}|^q e^{\lambda(T^*)\sum_{i\le k}|v_i|^2} \le \bar{M}^k. \tag{28}$$

These uniform bounds let us extract a converging subsequence such that all $f_{k,N}$ converge weak-$\star$ to some $\bar{f}_k$ in $L^\infty([0,T^*], L^q_{x,v})$ which also satisfies

$$\sup_{t\le T^*}\int_{\Pi^{kd}\times\mathbb{R}^{kd}} |\bar{f}_k|^q e^{\lambda(T^*)\sum_{i\le k}|v_i|^2} \le \bar{M}^k, \tag{29}$$

where we have used classical convex estimates. We emphasize that for the moment we only have convergence of a subsequence, though we still denote it by N for simplicity. We eventually obtain the convergence of the whole sequence only after the uniqueness of the limit is proved in the next step.

From estimate (28) and since $1/q + 1/p \le 1$, we may simply bound

$$\Big\| \sum_{i\le k} \frac{1}{N} \sum_{j\le k} K(x_i - x_j)\cdot \nabla_{v_i} f_{k,N} \Big\|_{L^\infty_t L^1_{x,v,\mathrm{loc}}} \lesssim \frac{k^2}{N} \|K\|_{L^p} \|f_{k,N}\|_{L^\infty_t L^q_{x,v}}.$$

For any fixed k, the corresponding term vanishes as $N\to\infty$. Similarly estimate (28) allows us to pass to the limit

$$\int_{\Pi^d\times\mathbb{R}^d} K(x_i - x_{k+1}) f_{k+1,N}\, dx_{k+1}\, dv_{k+1} \to \int_{\Pi^d\times\mathbb{R}^d} K(x_i - x_{k+1}) \bar f_{k+1}\, dx_{k+1}\, dv_{k+1}$$

for the weak-$\star$ topology of $L^\infty([0, T^*], L^q_{x,v})$. It is straightforward to pass to the limit in the sense of distributions in all other terms of the hierarchy (17), so we deduce that $\bar f_k$ is a solution to the limiting hierarchy (21) in the sense of distributions.

We can also easily identify the initial value of $\bar f_k$. From (17) and the bounds derived from (28), we immediately obtain a uniform bound on $\partial_t f_{k,N}$ in $L^\infty_t W^{-1,q}_{x,v,\mathrm{loc}}$. By the assumption of Theorem 2, $f^0_{k,N}$ converges weakly to $(f^0)^{\otimes k}$, so we have

$$\bar f_k(t=0) = (f^0)^{\otimes k}.$$

(3) *Uniqueness on the limiting hierarchy and conclusion.* We first argue that $\bar f_k$ is automatically a renormalized solution to (21). Indeed, (21) can be seen as a linear advection-diffusion equation with a locally Lipschitz velocity field $(v_1, \dots, v_k)$ and a remainder

$$\nabla_{v_i}\cdot \int_{\Pi^d\times\mathbb{R}^d} K(x_i - x_j)\bar f_{k+1}\, dx_{k+1}\, dv_{k+1}$$

that belongs to $L^\infty_t L^q_{x,v}$ with $q > 2$ per our prior estimates.

Next we note that, since f is a classical solution to the Vlasov equation (3), the $f^{\otimes k}$ also yield renormalized solutions to the Vlasov hierarchy (21) for every $k \ge 1$. Due to the linearity in terms of the sequence $\{f_k\}_{k\in\mathbb{N}^\star}$ of the Vlasov hierarchy, we get that each $F_k = \bar f_k - f^{\otimes k}$ is also a renormalized solution to the Vlasov Hierarchy (21) for every k. Moreover, since $\bar f_k$ and $f^{\otimes k}$ are identical at the initial time $t = 0$, we have that $F_k(t = 0) = 0$.

Furthermore, by (29) and the assumption of Gaussian decay on f^0, we have

$$\sup_{t\le T^*} \int_{\Pi^{kd}\times\mathbb{R}^{kd}} |F_k|^q e^{\tilde\beta \sum_{i\le k}(1+|v_i|)^2} \le \widetilde M^k \tag{30}$$

for some $\tilde\beta$ and some $\widetilde M$. Equation (21) corresponds to (22) in the case $\gamma = 0$, where $e_{k,\gamma}$ reduces to $e_{k,0} = \sum_{i\le k}(1+|v_i|)^2$. Hence, provided we choose some $\widetilde\Lambda$ possibly lower than Λ, we satisfy all assumptions from Lemma 9.

Defining $Y_k = \int |F_k|^q e^{\tilde\lambda(t) e_{k,0}}$, we get for all $k\in\mathbb{N}^\star$

$$Y_k(t) \le k\widetilde L \int_0^t Y_{k+1}\, ds. \tag{31}$$

We can then use Lemma 10 with $F_0 = 0$ up to any arbitrary m to show, together with (30), that

$$Y_k(t) \le \widetilde{L}^{m+1-k}\widetilde{M}^{m+1}\int_0^t (t-s)^{m-k}\frac{m!}{(k-1)!\,(m-k)!}\,ds \le \widetilde{L}^{m+1-k}\widetilde{M}^{m+1}t^{m+1-k}\binom{m}{k-1} \le 2^k\widetilde{M}^k(2\widetilde{L}\widetilde{M}t)^{m+1-k}. \tag{32}$$

By taking $t < T_0$ with T_0 small enough and sending m to ∞, we obtain that $Y_k(t) = 0$, and hence $\bar{f}_k = f^{\otimes k}$ on $[0, T_0]$. This allows us to repeat the argument starting from $t = T_0$ instead of $t = 0$ until we reach the maximum time T^*. This finally allows us to conclude as claimed that $\bar{f}_k = f^{\otimes k}$ over the whole interval $[0, T^*]$.

Coming back to our extracted subsequence on $f_{k,N}$, since all such subsequences have the same limit, we have convergence of the whole sequence to the $f^{\otimes k}$, concluding the proof.

3.5. ***Proof of Theorem 3.*** The aim of this result is to provide a quantitative estimate between $f_{k,N}$ and f_k that satisfies (17) and (21), respectively, for the tensorized limits $f_k = f^{\otimes k}$. First let us note that $F_k^N = f_{k,N} - f_k$ satisfies

$$\partial_t F_k^N + L_k F_k^N + \frac{N-k}{N}\sum_{i=1}^k \nabla_{v_i}\cdot\int_{\Pi^d\times\mathbb{R}^d} F_{k+1}^N K(x_i - x_{k+1})\,dx_{k+1}\,dv_{k+1} = \frac{\sigma^2}{2}\sum_{i=1}^k \Delta_{v_i}F_{k,N} + R_{k,N},$$

where L_k is defined in (18) and

$$R_{k,N} = \sum_{i=1}^k\left[\left(K\star\int_{\mathbb{R}^d} f\right)(t,x_i) - \frac{1}{N}\sum_{j=1}^k K(x_i - x_j)\right]\cdot\nabla_{v_i} f_k. \\ -\frac{N-k}{N}\sum_{i=1}^k \nabla_{v_i}\cdot\int_{\Pi^d\times\mathbb{R}^d} f_{k+1}K(x_i - x_{k+1})\,dx_{k+1}\,dv_{k+1}. \tag{33}$$

We again use Lemma 9 with $q = 2$ to deduce

$$\frac{d}{dt}\int_{\Pi^{kd}\times\mathbb{R}^{kd}} |F_{k,N}|^2 e^{\lambda(t)e_{k,\gamma}} + \frac{\sigma^2}{4}\sum_{i\le k}\int_{\Pi^{kd}\times\mathbb{R}^{kd}} |\nabla_{v_i}F_{k,N}|^2 e^{\lambda(t)e_k} \\ \le k\frac{N-k}{N}\frac{C_{2,\sigma,d}}{\lambda^{\theta_{2,d}}(t)}\|K\|_{L^2}^2\int_{\Pi^{kd}\times\mathbb{R}^{kd}} |F_{k+1,N}|^2 e^{\lambda(t)e_{k+1}} \\ + \lambda'(t)\int_{\Pi^{kd}\times\mathbb{R}^{kd}} e_k|F_{k,N}|^2 e^{\lambda(t)e_k} + \int_{\Pi^{kd}\times\mathbb{R}^{kd}} R_{k,N}F_{k,N}e^{\lambda(t)e_k}. \tag{34}$$

Note that $R_{k,N}$ may be written as

$$R_{k,N} = \sum_{i=1}^k\frac{1}{N}\sum_{j=1}^k\left[\left(K\star\int_{\mathbb{R}^d} f\right)(t,x_i) - K(x_i - x_j)\right]\cdot\nabla_{v_i} f_k \\ -\frac{N-k}{N}\sum_{i=1}^k\left[\nabla_{v_i}\cdot\int_{\Pi^d\times\mathbb{R}^d} f_{k+1}K(x_i - x_{k+1})\,dx_{k+1}\,dv_{k+1} - \left(K\star\int_{\mathbb{R}^d}\bar{f}\right)(t,x_i)\cdot\nabla_{v_i} f_k\right]. \tag{35}$$

Then, using that $f_k = f^{\otimes k}$, we have

$$\int_{\Pi^{kd}\times\mathbb{R}^{kd}} R_{k,N}F_{k,N}e^{\lambda(t)e_k} = \int_{\Pi^{kd}\times\mathbb{R}^{kd}} \frac{k}{N}\sum_{i=1}^{k}\biggl[\biggl(K\star\int_{\mathbb{R}^d} f\biggr)(t,x_i) - K(x_i - x_1)\biggr]\cdot\nabla_{v_i} f_k F_{k,N}e^{\lambda(t)e_k},$$

where we have used the fact that the particles are interchangeable. Integrating by parts with respect to v_i and using Young's inequality, we obtain

$$\int_{\Pi^{kd}\times\mathbb{R}^{kd}} R_{k,N}F_{k,N}e^{\lambda(t)e_k} \le \frac{\sigma^2}{4}\frac{k}{N}\sum_{i=1}^{k}\int_{\Pi^{kd}\times\mathbb{R}^{kd}} |\nabla_{v_i}F_{k,N}|^2 e^{\lambda(t)e_k} + \frac{1}{\sigma^2}\frac{k}{N}\sum_{i=1}^{k}\int_{\Pi^{kd}\times\mathbb{R}^{kd}} |\widetilde{R}^1_{k,N}|^2 e^{\lambda(t)e_k} + \lambda(t)\int_{\Pi^{kd}\times\mathbb{R}^{kd}} e_k|F_{k,N}|^2 e^{\lambda(t)e_k} + \frac{1}{2}\int_{\Pi^{kd}\times\mathbb{R}^{kd}} |\widetilde{R}^2_{k,N}|^2 e^{\lambda(t)e_k}, \tag{36}$$

where

$$\widetilde{R}^1_{k,N} = \biggl[\biggl(K\star\int_{\mathbb{R}^d} f\,dx\biggr)(t,x_i) - K(x_i - x_1)\biggr] f_k,$$

$$\widetilde{R}^2_{k,N} = \sum_{i=1}^{k}\biggl[\biggl(K\star\int_{\mathbb{R}^d} f\,dx\biggr)(t,x_i) - K(x_i - x_1)\biggr] f_k.$$

We observe that

$$\|\widetilde{R}^i_{k,N}\|^2_{L^2_{\lambda(t)e_k}} \le Ck\int_{\Pi^{kd}\times\mathbb{R}^{kd}} |f_k|^p e^{\lambda(t)e_k},$$

with a constant C that does not depend on k. We have also used the fact that, in particular, $K \in L^2(\Pi^d)$ and $f \in L^\infty(\Pi^d\times\mathbb{R}^d)$.

Then, using (13) and letting $N \to +\infty$, we get

$$\sup_{t\le T^*}\int_{\Pi^{kd}\times\mathbb{R}^{kd}} |f_k|^p e^{\lambda(t)e_{k,\gamma}} \le 2^k F_0^k.$$

We can insert this estimate into (36) for $p = 2$ to derive

$$\int_{\Pi^{kd}\times\mathbb{R}^{kd}} R_{k,N}F_{k,N}e^{\lambda(t)e_k} \le \frac{\sigma^2}{4}\frac{k}{N}\sum_{i=1}^{k}\int_{\Pi^{kd}\times\mathbb{R}^{kd}} |\nabla_{v_i}F_{k,N}|^2 e^{\lambda(t)e_k} + \lambda(t)\int_{\Pi^{kd}\times\mathbb{R}^{kd}} e_k|F_{k,N}|^2 e^{\lambda(t)e_k} + Ck2^k F_0^k.$$

Once this estimate is incorporated into (34) and using that $\lambda'(t) = -\lambda(t)/(1+t)$, we can, following the same lines of the proof of Proposition 5, repeat the estimate on the ODE inequality with the extra term coming from the interaction of $F_{k,N}$ with rest term $R_{k,N}$. This provides the conclusion that there exists T^* such that

$$\sup_{t\le T^\star}\int_{\Pi^{kd}\times\mathbb{R}^{kd}} |f_{N,k} - f_k|^2 e^{\lambda(t)e_{k,\gamma}} \le \widetilde{C}^k\varepsilon_N + \widetilde{C}^k\int_{\Pi^{kd}\times\mathbb{R}^{kd}} |f^0_{N,k} - f^0_k|^2 e^{\lambda(0)e_{k,\gamma}},$$

where $\widetilde{C}$ is a positive constant that does not depend on N and $\varepsilon_N = O(\varepsilon^N)$, where $\varepsilon < 1$ depends on a small enough T^*. This expression can be deduced in a similar way as (32) in the proof of Theorem 2. We finally emphasize that the quantitative bounds of Theorem 3 would allow us to recover the optimal convergence rate in $O(1/N)$ recently obtained in [Lacker 2023].

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