ANALYSIS & PDE

Volume 18 No. 4 2025

msp

Analysis & PDE

msp.org/apde

Cover image: Eric J. Heller: “Linear Ramp”

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Analysis & PDE (ISSN 1948-206X electronic, 2157-5045 printed) at Mathematical Sciences Publishers, 798 Evans Hall #3840, c/o University of California, Berkeley, CA 94720-3840, is published continuously online.

APDE peer review and production are managed by EditFlow® from MSP.

PUBLISHED BY

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http://msp.org/

ANALYSIS AND PDE
Vol. 18 (2025), No. 4, pp. 805–856
DOI: 10.2140/apde.2025.18.805

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STOCHASTIC HOMOGENIZATION FOR VARIATIONAL SOLUTIONS OF HAMILTON–JACOBI EQUATIONS

CLAUDE VITERBO

Let (Ω, μ) be a probability space endowed with an ergodic action τ of $(\mathbb{R}^n, +)$. Let $H(x, p; \omega) = H_\omega(x, p)$ be a smooth Hamiltonian on $T^*\mathbb{R}^n$ parametrized by $\omega \in \Omega$ and such that $H(a + x, p; \tau_a\omega) = H(x, p; \omega)$. We consider for an initial condition $f \in C^0(\mathbb{R}^n, \mathbb{R})$ the family of variational solutions of the stochastic Hamilton–Jacobi equations

$$\begin{cases} \dfrac{\partial u^\varepsilon}{\partial t}(t, x; \omega) + H\left(\dfrac{x}{\varepsilon}, \dfrac{\partial u^\varepsilon}{\partial x}(t, x; \omega)\right) = 0, \\ u^\varepsilon(0, x; \omega) = f(x). \end{cases}$$

Under some coercivity assumptions on p — but without any convexity assumption — we prove that for a.e. $\omega \in \Omega$ we have $C^0\text{–}\lim u^\varepsilon(t, x; \omega) = v(t, x)$, where v is the variational solution of the homogenized equation

$$\begin{cases} \dfrac{\partial v}{\partial t}(t, x) + \overline{H}\left(x, \dfrac{\partial v}{\partial x}(t, x)\right) = 0, \\ v(0, x) = f(x). \end{cases}$$

This paper was written as the author was on leave at DMA, École Normale Supérieure, 45 Rue d'Ulm, 75230 Cedex 05, FRANCE. We also acknowledge support from ANR MICROLOCAL (ANR-15-CE40-0007) and NSF grant DMS- 1440140.
MSC2020: 35F21, 35R60, 37J39, 57R17.
Keywords: stochastic PDE, Hamilton–Jacobi equations, variational solutions.

1. Introduction

Let (Ω, μ) be a probability space endowed with an ergodic action τ of $(\mathbb{R}^n, +)$. This means that if $X \subset \Omega$ satisfies $\tau_a X \subset X$ for all $a \in \mathbb{R}^n$, then $\mu(X) = 0$ or 1.

Let $H(x, p; \omega) = H_\omega(x, p)$ be a smooth Hamiltonian on $T^*\mathbb{R}^n$ parametrized by $\omega \in \Omega$ and such that

$$H(a + x, p; \tau_a \omega) = H(x, p; \omega). \tag{Inv}$$

We shall specify later the assumptions satisfied by H. We now consider for an initial condition $f \in C^0(\mathbb{R}^n)$ the family of stochastic Hamilton–Jacobi equations

$$\begin{cases} \dfrac{\partial u^\varepsilon}{\partial t}(t, x; \omega) + H\Big(\dfrac{x}{\varepsilon}, \dfrac{\partial u^\varepsilon}{\partial x}(t, x; \omega); \omega\Big) = 0, \\ u^\varepsilon(0, x; \omega) = f(x). \end{cases} \tag{HJS$_\varepsilon$}$$

Fixing ω, we can consider different types of generalized solutions (there is generally no smooth solution) for this equation. The most interesting ones are either the viscosity solution of Crandall and Lions [1983] (see also [Barles 1994; Bardi and Capuzzo-Dolcetta 1997]), or the variational solutions defined in [Chaperon 1991; Viterbo 1996; 2006] (we also credit J. C. Sikorav [1989]), both requiring some assumptions on f and H that will be specified later. The problem of stochastic homogenization for the above equation is to determine whether, for μ-a.e. in ω, the sequence $u^\varepsilon(t, x; \omega)$ C^0-converges on compact sets to $\bar{u}(t, x)$, the solution of

$$\begin{cases} \dfrac{\partial v}{\partial t}(t, x) + \overline{H}\Big(\dfrac{\partial v}{\partial x}(t, x)\Big) = 0, \\ v(0, x) = f(x), \end{cases} \tag{HJH}$$

where $\overline{H}$ is to be determined (and in general cannot be defined explicitly). Note that $\overline{H}$ does *not* depend on ω by the ergodicity hypothesis. A classical case is the so-called (nonstochastic) periodic case, corresponding to the case where $\Omega = \mathbb{T}^n$ and τ_a is the translation on the torus. Then condition (Inv) means that there is a smooth function K on T^*T^n such that $H(x, p; \omega) = K(x - \omega, p)$. Then solving (HJS$_\varepsilon$) is equivalent to solving the (nonstochastic) equation

$$\frac{\partial u}{\partial t}(t, x) + K\Big(\frac{y}{\varepsilon}, \frac{\partial v^\varepsilon}{\partial y}(t, y)\Big) = 0$$

and in this case stochastic homogenization boils down[1] to deterministic homogenization for K. For viscosity solutions, homogenization in the periodic nonstochastic case has been settled in [Lions et al. 1988] in 1987, and for variational solutions in [Viterbo 2023] in 2014.

For the general stochastic case, this problem has been solved for viscosity solutions by Rezakhanlou and Tarver [2000] and Souganidis [1999], assuming H is convex in p. Beyond the quasiconvex case (i.e., functions having all their sublevels convex) and some very special cases (see for instance [Armstrong et al.

[1]Indeed, if $u^\varepsilon(t, x)$ is the solution (either viscosity or variational) of $\frac{\partial u^\varepsilon}{\partial t}(t, x) + K\big(\frac{x}{\varepsilon} - \omega, \frac{\partial u^\varepsilon}{\partial x}(t, x)\big) = 0$ then $v^\varepsilon(t, y) = u^\varepsilon(t, y + \varepsilon\omega)$ satisfies $\frac{\partial v^\varepsilon}{\partial t}(t, x) + K\big(\frac{y}{\varepsilon}, \frac{\partial v^\varepsilon}{\partial y}(t, y)\big) = 0$. Thus $u^\varepsilon(t, y) = v^\varepsilon(t, y - \varepsilon\omega)$, and convergence of v^ε to $\bar{v}$ as ε goes to 0 is equivalent to convergence of u^ε to $\bar{u} = \bar{v}$. See the proof of Corollary 1.7 for another method of reducing to the periodic case.

2015; Gao 2016]), nothing is known for viscosity solutions in the general (i.e., for H nonconvex in p) case, and counterexamples have been found, first by Ziliotto [2017] and then by Feldman and Souganidis [2017].

We settle here the case of variational solutions without any convexity assumption. Note that the construction of a variational solution relies on the choice of a field of coefficients for the homology theory we use, but once the field is chosen, the variational solution is uniquely defined.[2] We shall here fix once and for all a coefficient field (the reader can think of $\mathbb{Z}/2\mathbb{Z}$ or $\mathbb{R}$ for example). As in [Viterbo 2023], our results hold when H is either compactly supported or coercive in the p-direction. Note that fixing ω, if $V_t(H)f = u(t,x)$ is the variational solution operator[3] of the Hamilton–Jacobi equation, and $S_t(H)f$ is the viscosity semigroup, we know that for H convex in p we have $S_t(H) = V_t(H)$ [Zhukovskaya 1993; 1996]. Our result thus implies the stochastic homogenization for viscosity solutions in the convex case[4] as in [Rezakhanlou and Tarver 2000; Souganidis 1999]. In the general case it has been proved in [Wei 2013; 2014] (see also [Roos 2017, Theorem 1.19]) that

$$S_t(H) = \lim_{n\to+\infty} (V_{t/n}(H))^n.$$

Since there are counterexamples in the nonconvex case, stochastic homogenization of the viscosity solutions cannot hold in general.[5]

Of course, as in [Viterbo 2023], the equation (HJS$_\varepsilon$) is related to the Hamiltonian flow of $H\left(\frac{x}{\varepsilon}, p; \omega\right)$ given by

$$\varphi^t_{\varepsilon,\omega} = \rho_\varepsilon^{-1} \varphi_\omega^{t/\varepsilon} \rho_\varepsilon,$$

where φ^t_ω is the flow of $H(x,p;\omega)$ and $\rho_\varepsilon(x,p) = \left(\frac{x}{\varepsilon}, p\right)$.

We shall prove analogously to [Viterbo 2023] that, for almost all ω, we have

$$\varphi^t_{\varepsilon,\omega} \xrightarrow{\gamma_c} \bar{\varphi}^t_\omega,$$

but since we are on a noncompact base we have to redefine the γ-distance, which we shall denote by γ_c.

1.1. *Statement of the main results.* Our main result is:

Theorem 1.1 (Main Theorem). *Let $H(x,p;\omega)$ be a stochastic Hamiltonian on $T^*\mathbb{R}^n \times \Omega$, where (Ω,μ) is a probability space endowed with an action τ of $\mathbb{R}^n$. We assume the following conditions are satisfied*:

(1) *For all $a \in \mathbb{R}^n$, the map τ_a is measure-preserving and the action τ is ergodic for the measure μ (i.e., invariant sets have measure* 0 *or* 1).

---

[2] See for example [Cardin and Viterbo 2008] and more explicitly [Wei 2014] and Appendix B in [Roos 2019].

[3] This means that it sends f to the variational solution of

$$\begin{cases} \frac{\partial u}{\partial t}(t,x) + H\left(x, \frac{\partial u}{\partial x}(t,x)\right) = 0, \\ u(0,x) = f(x). \end{cases} \tag{HJS}$$

Note that the operator is not a semigroup (since variational solutions do not have the Markov property).

[4] However in that case our method is much more complicated.

[5] Of course if in some cases we knew that $V_t(\varepsilon) = V_t(H_\varepsilon) = \overline{V}_t + R_t(\varepsilon)$, where $\|R_t(\varepsilon)\| \le Ct\varepsilon$, and $\overline{V}_t = V_t(\overline{H})$ is the homogenized operator, we would get that $\|(V_{t/n}(\varepsilon))^n - (\overline{V}_{t/n})^n\| \le Ct\varepsilon$. Hence, setting $\overline{S}_t = \lim_n (\overline{V}_{t/n})^n$, we would have $\|S^t(\varepsilon) - \overline{S}_t\| \le Ct\varepsilon$ and then $\lim_{\varepsilon\to 0} S^t_0(\varepsilon) = \overline{S}^t_0$.

(2) *We have, for all* $a \in \mathbb{R}^n$, $(x,p) \in T^*\mathbb{R}^n$ *and almost all* $\omega \in \Omega$, *the identity* $H(x+a,p,\tau_a\omega) = H(x,p,\omega)$.

(3) *The map* $(x,p) \mapsto H(x,p,\omega)$ *is* $C^{1,1}$ *for* μ*-almost all* ω.

(4) *For almost all* ω, H *is compactly supported in the* p*-direction, i.e., the set*

$$\{p \mid \exists x \in \mathbb{R}^n,\ H(x,p;\omega) \neq 0\}$$

is bounded.

(5) *There exists* C *such that for almost all* ω *and for all* $(x,p) \in T^*\mathbb{R}^n$ *we have* $\left|\frac{\partial H}{\partial p}(x,p;\omega)\right| \le C$.

(6) *There exists* C *such that for almost all* ω *we have* $\sup_{(x,p)\in T^*\mathbb{R}^n} |H(x,p;\omega)| \le C$.

Then if $\varphi_{\varepsilon,\omega}$ *is the flow of* $H_{\varepsilon,\omega}(x,p) = H\left(\frac{x}{\varepsilon},p;\omega\right)$ *there is a function* $\overline{H}$ *in* $C^0(\mathbb{R}^n,\mathbb{R})$ *such that*

$$\varphi^t_{\varepsilon,\omega} \xrightarrow{\gamma_c} \bar{\varphi}^t_\omega$$

for the topology γ_c *that will be defined in Section 4. Here* $\varphi^t_{\overline{H}}$ *denotes the flow of* $\overline{H}$ *in* $\widehat{\mathfrak{DHam}}(T^*\mathbb{R}^n)$, *the* γ_c*-completion of* $\mathfrak{DHam}(T^*\mathbb{R}^n)$. *As a consequence if* f *is uniformly continuous on* $\mathbb{R}^n$, *then a.s. in* $\omega \in \Omega$ *the variational solution* $u^\varepsilon(t,x;\omega)$ *of* (HJS$_\varepsilon$) *converges to the variational solution* $\bar{u}(t,x)$ *of* (HJH).

Let us try to give some intuition for the γ_c metric. The γ_c metric is a version, in the noncompact case, of the γ-metric first defined in [Viterbo 1992]. For a compact base, it is easier to describe it on Lagrangians. For example if L_k is the graph of df_k and f_k C^0-converges to a smooth function f_∞, then L_k converges to L_∞, the graph of df_∞. For this reason, the γ-metric is often called a C^{-1}-metric. However, as is quite natural, the γ-completion of the set of smooth Lagrangians contains more objects and in particular contains the graphs of continuous functions. For Hamiltonians maps, if φ_k is the time-one flow of the Hamiltonian H_k and $(H_k)_{k\ge 1}$ C^0-converges to H_∞, then φ_k γ-converges to φ_∞, the time-one flow of H_∞. Here again the time-one flow of a C^0 Hamiltonian is well-defined in the completion (see [Viterbo 1992; 2006; Humilière 2008]).

Remarks 1.2. (1) Existence and uniqueness of the variational solution for (HJS$_\varepsilon$) follows from [Cardin and Viterbo 2008, pp. 266–276] (since we are in the case of a noncompact base). The *bounded propagation speed* condition in [loc. cit.] is more general than the one in the present paper and is obviously satisfied in the fiberwise compactly supported case.

(2) By ergodicity, each of the conditions (4), (5), (6) either holds a.s. or fails a.s. Indeed, set

$$\Omega_c = \left\{\omega \in \Omega \mid \sup_{(x,p)\in T^*\mathbb{R}^n} |H(x,p;\omega)| \ge c\right\}.$$

This set is τ-invariant. If for some c this set has measure 0, then (6) holds; otherwise

$$\sup_{(x,p)\in T^*\mathbb{R}^n} |H(x,p;\omega)| = +\infty$$

for a.e. ω. Similarly, the set $\Omega'_R = \{\omega \in \Omega \mid \operatorname{supp}(H) \subset \mathbb{R}^n \times B(R)\}$ is also invariant by τ. It thus either has measure 1 for some R, and then the bound in (4) is independent from ω in a set of full measure, or it

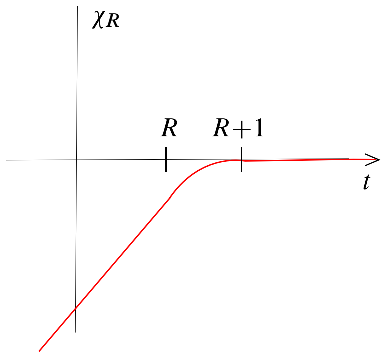

Figure 1. Graph of χ_R.

has measure 0 for all R and then, for a.e. ω, condition (4) is violated. In the first case, we shall say that the H_ω have *uniform fiber compact support*. This is assumption (4) of the Main Theorem.

(3) Let us compare our results to those of [Rezakhanlou and Tarver 2000; Souganidis 1999]. Note that if H is convex in p, then viscosity and variational solutions coincide. So consider a Hamiltonian H convex in p and uniformly coercive. In the ergodic case this implies that there exist functions $h_\pm(p)$ going to infinity such that $h_-(p) \le H(x, p; \omega) \le h_+(p)$ (this also follows from the assumptions in both [Rezakhanlou and Tarver 2000, (2.5)(ii) and (2.8), p. 280] and [Souganidis 1999, Condition 0.2]). Note that both authors assume $\lim_{|p|\to+\infty} h_\pm(p)/|p| = +\infty$, an assumption we do not require here.

Then we claim that the truncation $H_{\chi_R} = \chi_R(H)$, where χ_R is the function represented in Figure 1, satisfies assumption (5) of the Main Theorem (condition (4) is obvious) or equivalently, (2a) of the corollary. This is because

$$\frac{\partial H_{\chi_R}}{\partial p} = \chi_R'(H)\frac{\partial H}{\partial p},$$

so it is enough to prove that $\frac{\partial H}{\partial p}$ is bounded on a set $|p| \le C$. But if $\left|\frac{\partial H}{\partial p}(x_0, p_0)\right| \ge A$, we can find p_1 with $|p_1| \le 2C$ such that $p_0 - p_1$ is colinear with $\frac{\partial H}{\partial p}(x_0, p_0)$ and $|p_0 - p_1| = C$, so that

$$\sup_{|p|\le 2C} h_+(p) - \inf_{|p|\le 2C} h_-(p) \ge H(x, p_1) - H(x, p_0) \ge \left\langle \frac{\partial H}{\partial p}, p_0 - p_1 \right\rangle \ge C\left|\frac{\partial H}{\partial p}\right| = CA;$$

hence A is bounded.

The compactly supported case is usually not the most interesting in applications. However the above theorem implies

Corollary 1.3 (Main Corollary). *Let $H(x, p; \omega)$ be a stochastic Hamiltonian on $T^*\mathbb{R}^n \times \Omega$, where (Ω, μ) is a probability space endowed with an action τ of $\mathbb{R}^n$. We assume the following conditions are satisfied*:

(1a) *Conditions (1)–(3) as in the Main Theorem.*

(2a) *For all $(x, p; \omega)$ we have $\left|\frac{\partial H}{\partial p}(x, p; \omega)\right| \le h^1(|p|)$ for almost all ω for some continuous function $h^1 : \mathbb{R} \to \mathbb{R}$.*

(3a) *For almost all ω, H is coercive, that is $\lim_{|p|\to+\infty} |H(x, p; \omega)| = +\infty$ uniformly in x.*

If H satisfies the above assumptions and f is Lipschitz on $\mathbb{R}^n$, there is a coercive function $\overline{H}$ in $C^0(\mathbb{R}^n,\mathbb{R})$ such that a.e. in ω the variational solution $u^\varepsilon(t,x;\omega)$ of

$$\begin{cases} \dfrac{\partial u^\varepsilon}{\partial t}(t,x;\omega)+H\Big(\dfrac{x}{\varepsilon},\dfrac{\partial u^\varepsilon}{\partial x}(t,x;\omega);\omega\Big)=0,\\ u^\varepsilon(0,x;\omega)=f(x)\end{cases} \tag{HJS$_\varepsilon$}$$

converges to the variational solution $\bar{u}(t,x)$ of

$$\begin{cases} \dfrac{\partial v}{\partial t}(t,x)+\overline{H}\Big(\dfrac{\partial v}{\partial x}(t,x)\Big)=0,\\ v(0,x)=f(x).\end{cases} \tag{HJH}$$

Remark 1.4. We shall reduce the case (3a) where H is coercive to the uniformly fiberwise compactly supported case by replacing H by $\chi_R(H)$, which is compactly supported where $\chi_R:\mathbb{R}\to\mathbb{R}$ is a function supported in $]-\infty,R+1]$ such that $\chi'(t)=1$ for $t\le R$ (see [Cardin and Viterbo 2008, Appendix B]). Then $H_{\chi_R}=\chi_R(H)$ also satisfies $H_{\chi_R}(x+a,p;\tau_a\omega)=H_{\chi_R}(x,p;\omega)$.

Examples 1.5. (1) Let Ω be the space of C^1 functions on $\mathbb{R}^n$, $(\tau_a f)(x)=f(x+a)$ and μ be some measure on Ω invariant by τ_a and ergodic. Let V be a bounded function. Set $H(x,p;\omega)=\frac{1}{2}h(p)-V(\omega(x))$, where h is coercive. This satisfies the assumptions of the corollary and corresponds to a random potential, with probability μ.

(2) [Pelayo and Rezakhanlou 2018, Example 2.4(ii)] Let $H_0(q,p)$ be a Hamiltonian and $H(q,p;\omega)=\sum_{j\in\mathbb{Z}}H_0(q-q_j,p)$, where $\omega=(q_j)_{j\in\mathbb{Z}}$ is a stationary point process, that is, a probability on $\mathbb{R}^{\mathbb{Z}}$ invariant by translation. This makes sense provided H_0 decreases fast enough as q goes to infinity. Then H satisfies the assumption of the above corollary.

Remark 1.6. Here are a few comments:

(1) We could of course also state a convergence result in the coercive case for the sequence $\varphi_{\varepsilon,\omega}$; it is just that the statement of convergence would be a little more complicated to state.

(2) By ergodicity there exist $h_+(p)\in\mathbb{R}\cup\{+\infty\}$ and $h_-(p)\in\mathbb{R}\cup\{-\infty\}$ such that $\sup_{x\in\mathbb{R}^n}H(x,p;\omega)=h_+(p)$ a.e. in Ω and similarly $\inf_{x\in\mathbb{R}^n}H(x,p;\omega)=h_-(p)$ a.e. in Ω. Notice that (3a) implies that $h_\pm(p)$ is finite, and that $\lim_{|p|\to+\infty}h_\pm(p)=+\infty$. This condition is more or less explicit in both [Rezakhanlou and Tarver 2000, conditions (Aii)–(Aiii)] and [Souganidis 1999, Condition 0.2]. Similarly

$$h_+^1(p,\omega)=\sup_{x\in\mathbb{R}^n}\Big|\frac{\partial H}{\partial p}(x,p;\omega)\Big|$$

is invariant by τ, and hence independent from ω a.e. in Ω, and equal to $h_+^1(p)$, so (2a) and (5) either hold a.s. or do not hold a.s in Ω.

(3) Again by ergodicity, the coercivity is necessarily uniform: one has a function $f(r)$ such that $\lim_{r\to+\infty}f(r)=+\infty$ and for all $(x,p;\omega)$ we have $|H(x,p;\omega)|\ge f(|p|)$.

(4) Let us consider a Hamiltonian H convex in p and uniformly coercive. In the ergodic case this implies that there exist functions $h_\pm(p)$ going to infinity such that $h_-(p)\le H(x,p;\omega)\le h_+(p)$ (this also follows

from the assumptions in both [Rezakhanlou and Tarver 2000, (2.5)(ii) and (2.8), p. 280] and [Souganidis 1999, Condition 0.2]). Then we claim that its truncation $H_{\chi_R} = \chi_R(H)$ satisfies assumption (5) of the Main Theorem (condition (4) is obvious) or equivalently, (2a) of the corollary. This is because

$$\frac{\partial H_{\chi_R}}{\partial p} = \chi_R'(H)\frac{\partial H}{\partial p},$$

so it is enough to prove that $\frac{\partial H}{\partial p}$ is bounded on a set $|p| \le C$. But if $\left|\frac{\partial H}{\partial p}(x_0, p_0)\right| \ge A$, we can find p_1 with $|p_1| \le 2C$ such that $p_0 - p_1$ is colinear with $\frac{\partial H}{\partial p}(x_0, p_0)$ and $|p_0 - p_1| = C$, so that

$$\sup_{|p|\le 2C} h_+(p) - \inf_{|p|\le 2C} h_-(p) \ge H(x, p_1) - H(x, p_0) \ge \left\langle \frac{\partial H}{\partial p}, p_0 - p_1 \right\rangle \ge C\left|\frac{\partial H}{\partial p}\right| = CA;$$

hence A is bounded.

Our result can be easily extended, since we do not need the full action of $\mathbb{R}^n$. For example if we have an action of $\mathbb{Z}^n$ we get the following:

Corollary 1.7. *Take the same assumptions as in the Main Theorem except that we have an action of $\mathbb{Z}^n$ (instead of $\mathbb{R}^n$) on Ω, still denoted by τ, and the first two assumptions are replaced by:*

(1b) *For all $z \in \mathbb{Z}^n$, the map τ_z is measure-preserving and ergodic.*

(2b) *We have, for all $z \in \mathbb{Z}^n$, $(x, p) \in T^*\mathbb{R}^n$ and almost all $\omega \in \Omega$, the identity*

$$H(x + z, p, \tau_z\omega) = H(x, p, \omega),$$

while conditions (3)–(6) are unchanged. We then have the same conclusion as in the Main Theorem.

Finally, note that ergodicity of τ on Ω is not required, since we can use the ergodic decomposition theorem (see [Greschonig and Schmidt 2000]), which holds for Borel spaces[6] and obtain:

Corollary 1.8. *With the same assumptions as in the Main Theorem (resp. Corollary 1.7) except that the action τ is not supposed to be ergodic but we assume (Ω, μ) is a Borel space, we have the same conclusion, except that $\overline{H}(p; \omega)$ now depends on $\omega \in \Omega$ and is constant on each ergodic component of τ.*

1.2. ***Sketch of the proof of the Main Theorem.*** Our proof will require the following steps, starting from the uniformly fiber compactly supported case:

(1) On $\mathfrak{Ham}_{\mathrm{fc}}(T^*\mathbb{R}^n)$, the set of uniformly fiberwise compactly supported Hamiltonians on $T^*\mathbb{R}^n$, we define a metric γ_c (see Sections 3 and 4).

(2) We identify Ω to $\mathfrak{H}_\Omega$ the set of H_ω for $\omega \in \Omega$, and $\widehat{\mathfrak{H}}_\Omega$ its completion for γ_c. We then prove that ergodicity implies compactness of the metric space $(\widehat{\mathfrak{H}}_\Omega, \gamma_c)$ (see Sections 5 and 6). The action of $\mathbb{R}^n$ on $\mathfrak{H}_\Omega$ given by $(\tau_a H)(x, p; \omega) = H(x - a, p; \omega) = H(x, p; \tau_a\omega)$ extends to an action of a compact connected metric abelian group $\mathbb{A}_\Omega$ on $(\widehat{\mathfrak{H}}_\Omega, \gamma_c)$, and $\mathbb{R}^n$, through the action τ, is identified to a dense subgroup of $\mathbb{A}_\Omega$. Moreover we prove that for μ-almost all H in $\mathfrak{H}_\Omega$, the $\mathbb{A}_\Omega$ orbit of H is equal to $\widehat{\mathfrak{H}}_\Omega$.

[6]That is, isomorphic (as a measured space) to a complete separable metric space with a measure defined on its Borel algebra.

(3) In Section 7 we prove a regularization theorem showing that the action of $\mathbb{A}_\Omega$ on $\widehat{\mathfrak{H}}_\Omega$ can be approximated by an action of a finite-dimensional torus (note that $\mathbb{A}_\Omega$ is not in general a finite-dimensional torus, but is a projective limit of finite-dimensional tori).

(4) We prove in Section 8 that homogenization holds when $\mathbb{A}_\Omega$ is a finite-dimensional torus (the quasiperiodic case) and $\omega \mapsto H_\omega$ is continuous for the C^0-topology instead of the γ_c-topology.

(5) In Section 10 we conclude the proof in the fiberwise compact case, and in Section 11 for the coercive case and in Section 12 for the discrete case.

2. Notation and abbreviations

- Ω is a probability space with measure μ.
- a.s. or a.e. mean almost surely or almost everywhere in (Ω, μ).
- GFQI means "generating function quadratic at infinity".
- H^*, H_* are, respectively, cohomology and homology (either Čech or singular) with coefficients in some field $\mathbb{K}$.
- μ_N is the fundamental class in $H^d(N)$ (for a closed manifold) or $H^d(N, \partial N)$ (for a manifold with boundary) or $H^d_c(N)$ (for a noncompact manifold), where $d = \dim(N)$. When N is nonorientable, it is assumed that $\mathbb{K} = \mathbb{F}_2 = \mathbb{Z}/2\mathbb{Z}$.
- 1_N is the generator of $H^0(N)$.
- T^*N is the cotangent bundle of N with the standard symplectic form $\omega = d\lambda$, where $\lambda = p\,dq$.
- $\overline{T^*N}$ is the cotangent bundle of N with the opposite of the standard symplectic form $\omega = -d\lambda$, where $\lambda = p\,dq$.
- 0_N is the zero section of T^*N.
- $\mathfrak{Ham}_{\mathrm{fc}}(T^*N)$ is the set of smooth uniformly fiberwise compactly supported[7] autonomous Hamiltonians.
- $\mathfrak{Ham}_{\mathrm{fc}}([0,1] \times T^*N)$ is the set of smooth uniformly fiberwise compactly supported time-dependent Hamiltonians.
- $C^0_{\mathrm{fc}}([0,1] \times T^*N)$ is set of continuous functions on $[0,1] \times T^*N$ (viewed as "continuous Hamiltonians") which are fiberwise compact.
- For a Hamiltonian H on T^*N, $X_H(t,z)$ is the Hamiltonian vector field associated to H, defined by $\omega(X_H(t,z)) = -d_z H(t,z)$.
- For a Hamiltonian H on T^*N, φ^t_H is the solution of $\frac{d}{dt}\varphi^t_H(z) = X_H(t, \varphi^t_H(z))$ such that $\varphi^0_H(z) = z$. We set $\varphi_H = \varphi^1_H$.
- $\mathfrak{DHam}_{\mathrm{fc}}(T^*N)$ is the image by $H \mapsto \varphi_H$ of $\mathfrak{Ham}_{\mathrm{fc}}([0,1] \times T^*N)$.
- FPS means "finite propagation speed" (see Definition 3.1).

[7] That is, the support is contained in $\mathbb{R}^n \times B(R)$ for some R.

- BPS means "bounded propagation speed" (see Definition 3.8).
- $\mathfrak{DHam}_{\mathrm{FP}}(T^*N)$ (resp. $\mathfrak{Ham}_{\mathrm{FP}}(T^*N)$ or $\mathfrak{Ham}_{\mathrm{FP}}([0,1]\times T^*N)$) is the set of elements in $\mathfrak{DHam}(T^*N)$ (resp. $\mathfrak{Ham}(T^*N)$ or $\mathfrak{Ham}([0,1]\times T^*N)$) having FPS.
- $\mathfrak{DHam}_{\mathrm{BP}}(T^*N)$ (resp. $\mathfrak{Ham}_{\mathrm{BP}}(T^*N)$ or $\mathfrak{Ham}_{\mathrm{BP}}([0,1]\times T^*N)$) is the set of elements in $\mathfrak{DHam}(T^*N)$ (resp. $\mathfrak{Ham}(T^*N)$ or $\mathfrak{Ham}([0,1]\times T^*N)$) having BPS.
- $\mathfrak{L}(T^*N)$ is the set of pairs (L, f_L), where L is the image of 0_N by some element $\varphi \in \mathfrak{DHam}_{\mathrm{FP}}(T^*N)$ and f_L is a primitive of $\lambda_{|L}$. We often just write L if f_L is implicit.
- γ_c is the uniform topology on $\mathfrak{L}(T^*N)$ (see Definition 4.17).
- $\widehat{\mathfrak{L}}(T^*N)$ is the completion for γ_c of $\mathfrak{L}(T^*N)$ (see Definition 4.17).
- $\widehat{\mathfrak{DHam}}_{\mathrm{FP}}(T^*N)$ (resp. $\widehat{\mathfrak{DHam}}_{\mathrm{BP}}(T^*N)$ or $\widehat{\mathfrak{DHam}}_{\mathrm{fc}}(T^*N)$) is the completion for γ_c of $\mathfrak{DHam}_{\mathrm{FP}}(T^*N)$ (resp. $\mathfrak{DHam}_{\mathrm{BP}}(T^*N)$ or $\mathfrak{DHam}_{\mathrm{fc}}(T^*N)$) (see Definition 4.24)
- G_f is the graph of df in T^*N.
- $\bar{L}$: For $L \in \mathfrak{L}(T^*N)$ we define $\bar{L} = \{(x,-p) \mid (x,p) \in L\}$, where $f_{\bar{L}} = -f_L$.

3. Noncompactly supported Hamiltonians

Let N be a noncompact manifold. We shall assume that N is homeomorphic to the interior of a compact manifold with smooth boundary.[8]

Definition 3.1. Let $\varphi \in \mathfrak{DHam}(T^*N)$. We say that φ has *finite propagation speed* (FPS for short) if, for each bounded set U, there is a bounded set V such that $\varphi(T^*U) \subset T^*V$. A subset in $\mathfrak{DHam}(T^*N)$ has *uniformly finite propagation speed* if each element has finite propagation speed, and moreover, given U, the set V can be chosen to be the same for all the elements in the subset. We write $\mathfrak{DHam}_{\mathrm{FP}}(T^*N)$ for the set of Hamiltonian maps with finite propagation speed. By abuse of language, we use the same terminology in $\mathfrak{Ham}(T^*N)$: H has *finite propagation speed* if φ_H has finite propagation speed, etc. We use the notation $\mathfrak{Ham}_{\mathrm{FP}}(T^*N)$ for this set.

Note that for instance if $\left|\frac{\partial H}{\partial p}(t,q,p)\right| \le C_U$ for all $(q,p) \in T^*U$ then H has FPS.

The following lemma will prove useful.

Lemma 3.2. *Let $U \subset V$ be relatively compact open sets in N such that for any compact set K in N there exists an isotopy of N sending K in V. Let $\varphi \in \mathfrak{DHam}(T^*N)$ be such that $\varphi(T^*U) \subset T^*V$. Then we can find a Hamiltonian isotopy $(\varphi^t)_{t\in[0,1]}$ from the identity to φ such that for all $t \in [0,1]$ we have $\varphi^t(T^*U) \subset T^*V$.*

Proof. Let ψ^t be an isotopy from id to $\psi^1 = \varphi$. Let X be a vector field corresponding to the isotopy for a compact set containing the projection of $\bigcup_{t\in[0,1]} \psi^t(U) = K$ and pointing inwards on ∂V. Let ρ^t be the Hamiltonian vector field of $H(t,x,p) = \langle p, X(t,x)\rangle$ which projects on the flow of X. Possibly replacing ρ^t by a $\rho^{\alpha(t)}$, we may assume that for all $t \in [0,1]$ we have $\rho^t \circ \psi^t(T^*U) \subset T^*V$. Then

[8] We eventually only use the case $N = \mathbb{R}^n$. For this section we actually only need that there is an exhausting sequence of open bounded sets $(U_j)_{j\in\mathbb{N}}$ such that $U_j \subset U_{j+1}$ and, for j large enough, U_j is ambient isotopic to U_{j+1}.

$\rho^1\psi^1(T^*U) \subset T^*V$ and, since $\psi^1(T^*U) \subset T^*V$, the set of t such that $\rho^t\psi^1(T^*U) \subset T^*V$ is an interval — because X points inward on ∂V — it must contain [0, 1]; hence concatenating the Hamiltonian isotopy $t \mapsto \rho^t\psi^t$ with $t \mapsto \rho^{1-t}\psi^1$, we get a new Hamiltonian isotopy that we denote by φ^t such that $\varphi^t(T^*U) \subset T^*V$ for all $t \in [0, 1]$. □

Note that our hypothesis on N implies that we can find an exhausting sequence $(U_j)_{j\geq 1}$ of N satisfying the assumptions of Lemma 3.2.

We shall now prove that $\mathfrak{DHam}_{\mathrm{fc}}$, the set of Hamiltonians which are uniformly fiberwise compactly supported, is contained in $\mathfrak{Ham}_{\mathrm{FP}}$.

Proposition 3.3. *If $H \in \mathfrak{Ham}_{\mathrm{fc}}(T^*N)$ is uniformly fiberwise compactly supported, then H has FPS.*

Proof. Indeed, if for some C, φ is the identity outside of $DT^*_C(N) = \{(q, p) \mid |p| \leq C\}$, then $\varphi(T^*U) \subset T^*U \cup \varphi(T^*U \cap T^*_C N)$, but since $T^*U \cap T^*_C N$ is compact, its image is contained in some T^*V for V bounded, and we get $\varphi(T^*U) \subset T^*(U \cup V)$. □

The usefulness of this notion will be clear on several occasions. Remember that a generating function quadratic at infinity for (L, f_L), where L is a smooth Lagrangian and f_L a function such that of $df_L = \lambda_{|L}$, is a smooth function $S : E = N \times F \to \mathbb{R}$, where F is a finite-dimensional vector space,[9] such that

(1) $S(x, \xi)$ coincides with a nondegenerate quadratic form Q on the vector space F for ξ large enough,
(2) $(x, \xi) \mapsto \frac{\partial S}{\partial \xi}(x, \xi)$ is transverse to 0,
(3) setting $\Sigma_S = \left\{(x, \xi) \mid \frac{\partial S}{\partial \xi}(x, \xi)\right\}$ the image of this submanifold by $i_S : (x, \xi) \mapsto \frac{\partial S}{\partial x}(x, \xi)$ has image L,
(4) $f_L \circ i_S = S$.

Let S_1, S_2 be two GFQI. They are said to be equivalent if they are fiberwise diffeomorphic after stabilization, that is, there are two nondegenerate quadratic forms q_1, q_2 such that if

$$\widetilde{S}_j(x, \xi_j, \eta_j) = S_j(x, \xi_j) + q_j(\eta_j),$$

there is a fiber-preserving diffeomorphism

$$(x, \xi_1, \eta_1) \to (x, \xi_2(x, \xi_1, \eta_1), \eta_2(x, \xi_1, \eta_1))$$

such that

$$S_2(x, \xi_2(x, \xi_1, \eta_1), \eta_2(x, \xi_1, \eta_1)) = S_1(x, \xi_1, \eta_1).$$

We shall say that S_1, S_2 are equivalent over U if the fiber-preserving diffeomorphism is defined for $x \in U$. Note that the customary "addition of a constant" for the equivalence of generating functions is not needed here, since generating functions are normalized so that $S_{|\Sigma_S} = f_L \circ i_S$.

We cannot expect a noncompact Lagrangian to have a GFQI in this sense, since the number of variables required could go to infinity. We can either assume F is a Hilbert space, but then positive and

[9] All this discussion also works if we replace $N \times F$ by a general finite-dimensional vector bundle. Then we must replace in the sequel the Künneth isomorphism by the Thom isomorphism.

negative eigenspaces will generally be infinite-dimensional so that $H^*(S^b, S^a) = 0$, which is a notorious drawback.[10] Here we have:

Definition 3.4. We say that a Lagrangian $L \subset T^*N$ has a GFQI if, for each bounded set U, there is a GFQI defined over $U \times F$ (where F depends on U), S_U, and a set $V \supset U$ such that the S_W are all equivalent over U for $W \supset V$. Two GFQI are equivalent if they are equivalent over each bounded set.

Theorem 3.5. *Let φ be an element in $\mathfrak{DHam}_{\mathrm{FP}}(T^*N)$. Then $\varphi(0_N)$ has a GFQI. Moreover such a GFQI is unique up to equivalence.*

Proof. See Appendix A. □

Remarks 3.6. Notice that

(1) If φ does not have FPS, $\varphi(0_N)$ does not even need to have surjective projection on N: For example take on $T^*\mathbb{R}$ the Hamiltonian $\frac{\pi}{4}(x^2 + p^2)$. Then $\varphi(0_{\mathbb{R}}) = \{0\} \times \mathbb{R}$!

(2) Using Lemma 3.2 we may assume we have a sequence U_ν of domains such that for all $t \in [0, 1]$ we have $\varphi^t(T^*U_\nu) \subset T^*U_{\nu+1}$. We let $S_\nu = S_{U_\nu}$ and notice that we may assume that the restriction of S_μ over U_ν is exactly $S_\nu \oplus q_{\nu,\mu}$ by composing S_μ with an extension of the fiber-preserving diffeomorphism realizing the equivalence.[11] We shall always make this assumption in the sequel.

(3) We will use the expression "S is a GFQI for L" meaning "there is a sequence $(S_\nu)_{\nu \geq 1}$ of GFQI for L over U_ν" to avoid cumbersome indexes. Most of the time this means we consider S_ν for ν large enough.

Definition 3.7. We denote by $\mathfrak{L}(T^*N)$ the set of Lagrangians of the type $\varphi(0_N)$, where $\varphi \in \mathfrak{DHam}_{\mathrm{FP}}(T^*N)$.

On a Riemannian manifold, there is a more precise notion than FPS.

Definition 3.8. Let N be a manifold with a distance d and $\varphi \in \mathfrak{DHam}(T^*N)$. We say that φ has *bounded propagation speed* (BPS for short) if there is a constant r_0 such that for any ball $B(x_0, r)$ we have $\varphi(T^*B(x_0, r)) \subset T^*B(x_0, r + r_0)$. A subset in $\mathfrak{DHam}(T^*N)$ has *uniformly bounded propagation speed* if each element has bounded propagation speed, and moreover the constant r_0 can be chosen to be the same for all the elements in the subset. We write $\mathfrak{DHam}_{\mathrm{BP}}(T^*N)$ for the set of Hamiltonians maps with bounded propagation speed. By abuse of language, we use the same terminology in $\mathfrak{Ham}(T^*N)$: H has *bounded propagation speed* if φ_H has bounded propagation speed.

Example 3.9. If $\left|\frac{\partial H}{\partial p}(t, q, p)\right| \leq C$ for all $(q, p) \in T^*\mathbb{R}^n$ then H has BPS. In particular assumption (5) implies BPS.

Remark 3.10. (1) Of course bounded propagation speed implies finite propagation speed.

(2) Our definition of finite propagation speed does not exactly coincide with the terminology of [Cardin and Viterbo 2008, Definition B.5, p. 271]. Our definition is more involved and the notion of finite propagation speed defined there is weaker than the present one, but would still be sufficient to prove our theorems. However this would have made an already long paper even longer.

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[10] That we could avoid by using Floer homology everywhere, but would make reading this paper even harder for the Hamilton–Jacobi community!

[11] The existence of the extension follows from the fact that we may assume that, for μ, ν large enough, the inclusion $U_\nu \subset U_\mu$ is a homotopy equivalence.

4. Spectral invariants in cotangent bundles of noncompact manifolds

The goal of this section is to define and state the main properties of the metric γ that occurs in the statement of the Main Theorem. This has been done in [Viterbo 1992] in the case of a compact base; the present situation, for a noncompact base, is unfortunately slightly more involved. Even though we work on a general noncompact manifold, the reader can assume that $N = \mathbb{R}^n$. The general case will turn out to be useful for future applications, and the only extra difficulty is visual.

4.1. ***The case of Lagrangians.*** Let L be an exact Lagrangian in T^*N with N not necessarily compact (but assumed, for simplicity, to be connected). We assume a primitive of $\lambda_{|L}$, f_L, is given.[12]

We shall assume that L has a unique GFQI, S, such[13] that $f_L = S$ on L $\big($through the identification $i_S(x,\xi) = \big(x, \frac{\partial S}{\partial \xi}(x,\xi)\big)\big)$. For example according to Theorem 3.5, this is the case if $L = \varphi_H(0_N)$ with $\varphi \in \mathfrak{DHam}_{\mathrm{FP}}(T^*N)$. Note that in general, S_U, Q, F depend on U.

We denote by T_F the generator of $H^i(D(F^-), S(F^-))$, where F^- is the negative eigenspace of Q, $i = \dim(F^-)$ and $D(F^-), S(F^-)$ are respectively the disc and sphere in F^-, so that $\alpha \mapsto \alpha \otimes T_F$ is an isomorphism (the Künneth isomorphism) from $H^*(U)$ to

$$H^{*+i}(U \times D(F^-), U \times S(F^-)) = H^*(U) \otimes H^*(D(F^-), S(F^-))$$

for $U \subset N$. By abuse of language we again denote by T_F its homological counterpart in $H_i(D(F^-), S(F^-))$. We shall later write T instead of T_F.

We denote by $S_U^t = \{(x,\xi) \in U \times F \mid S(x,\xi) \leq t\}$ (we omit the subscript for $U = N$) and $S_U^{-\infty}$ (resp. $S_U^{+\infty}$) any of the S_U^{-c} (resp. S^c) for c large enough (by Morse's lemma they are all isotopic).

Classically we have a homotopy equivalence between $(S_U^{+\infty}, S_U^{-\infty})$ and $U \times (D(F^-), S(F^-))$. In the following definitions, we set $\mu_U \in H^n(U, \partial U)$, $1_U \in H^0(U)$ to be the generators of these cohomology groups.

Definitions 4.1. Let S be a GFQI for $L \in \mathfrak{L}(T^*N)$ and U a bounded open set with smooth boundary. We define:

(1) For $\alpha \in H^*(U)$,

$$c(\alpha, S) = \inf\{t \mid T \otimes \alpha \neq 0 \text{ in } H^*(S_{|U}^t, S_{|U}^{-\infty})\}.$$

(2) For $a \in H_*(U, \partial U)$,

$$c(a, S) = \inf\{t \mid T \otimes a \text{ is in the image of } H_*(S_{|U}^t, S_{|U}^{-\infty} \cup S_{|\partial U}^t)\}.$$

(3) For $\alpha \in H_c^*(U) = H^*(U, \partial U)$,

$$c(\alpha, S) = \inf\{t \mid T \otimes \alpha \neq 0 \text{ in } H^*(S_{|U}^t, S_{|U}^{-\infty} \cup S_{|\partial U}^t)\}.$$

[12]Even though we write L, we always mean the pair (L, f_L).

[13]Remember by Remarks 3.6(3) that this means there is a sequence S_ν of GFQI over U_ν such that, for $\nu \leq \mu$, the function S_μ restricts to the stabilization of S_ν over U_ν.

(4) For $a \in H_*(U)$,

$$c(a,S)=\inf\{t \mid T\otimes a \text{ is in the image of } H_*(S^t_{|U},S^{-\infty}_{|U})\}.$$

(5) For L_1, L_2 in $\mathfrak{L}(T^*N)$, having unique GFQI, S_1, S_2, we set $(S_1\ominus S_2)(x;\xi,\eta)=S_1(x;\xi)-S_2(x;\eta)$ and, for $\alpha\in H^*(U)$ or $H^*(U,\partial U)$,

$$c(\alpha,L_1,L_2)=c(\alpha,(S_1\ominus S_2))$$

and $c(\alpha,L)=c(\alpha,0_N,L)$.

(6) We set $\gamma_U(L_1,L_2)=c(\mu_U,L_1,L_2)-c(1_U,L_1,L_2)$ and $\gamma_U(L)=\gamma_U(0_N,L)$.

(7) We write $L_2\preceq_U L_1$ if $c(1_U,L_1,L_2)=0$. If this holds for all bounded sets U, we write $L_2\preceq L_1$.

(8) We set $GH^*(L_1,L_2;a,b)=H^{*-i}((S_1\ominus S_2)^b,(S_1\ominus S_2)^a)$.

Remark 4.2. We notice that

(1) As we said, S is shorthand for S_ν defined on U_ν. As long as $U\subset U_\nu$, it is easy to see that for $\alpha\in H^*(U)$ (resp. $H^*(U,\partial U)$) the $c(\alpha,S_\nu)$ do not depend on ν.

(2) The function $(S_1\ominus S_2)$ is not quadratic at infinity, but a standard trick allows us to deform it to a function quadratic at infinity (see [Viterbo 2006, Proposition 1.6]). The GH^* functor is called generating function homology (see [Traynor 1994]) and coincides with Floer homology[14] that we shall not introduce here.

(3) Note that if S has no fiber variables, $c(1_U,S)=\inf_{x\in U}S(x)$ and $c(\mu_U,S)=\sup_{x\in U}S(x)$.

It is often convenient to express the cohomological critical values in terms of their homology counterparts. Note that $H^*(U)$ is dual to $H_{n-*}(U,\partial U)$ and $H^*(U,\partial U)$ is dual to $H_{n-*}(U)$ by Lefschetz duality (see [Hatcher 2002, p. 254]). We have a fundamental class $\mu_U\in H^n(U,\partial U)$ dual to $[pt_U]\in H_0(U)$ and $1_U\in H^0(U)$ dual to $[U]\in H_n(U,\partial U)$. The following lemma will be useful.

Lemma 4.3. *We have for S a GFQI*:

(1) $c(1_U,S)=c([pt_U],S)$.

(2) $c(\mu_U,S)=c([U],S)$.

We also have the duality identity

$$c(1_U,\bar{L})=-c(\mu_U,L).$$

Proof. The first two properties follow from Proposition B.3 in [Viterbo 2023]. The duality identity is a consequence of the identity $c(1_U,-S)=-c(\mu_U,S)$. Both are easily adapted from the case $U=N$ closed to the present situation. This follows from the following argument (see [Viterbo 1992, Proposition 2.7, p. 692]). First notice that $(-S)^t=E\setminus S^{-t}$, so we look for the smallest t such that $1_U\neq 0$ in $H^*(E_{|U}\setminus S^{-t}_{|U},E_{|U}\setminus S^{-\infty}_{|U})$. We then apply Alexander duality (see [Spanier 1966, Theorem 10, p. 342]), which claims that for any closed pair (A,B) contained in an orientable manifold X we have an isomorphism

$$H_k(X-B,X-A)\simeq H_c^{d-k}(B,A).$$

[14] See [Viterbo 2003] (or [Milinković and Oh 1997]) for the equivalence of the two homologies.

Note that $H_c^{d-k}(B, A)$ is invariant by proper homotopy equivalence, so if there is a proper retraction of the pair (A, B) to a compact pair (A', B'), then

$$H_c^{d-k}(B, A) \simeq H_c^{d-k}(B', A') \simeq H^{d-k}(B', A') \simeq H^{d-k}(B, A).$$

In particular this is always the case for pairs (S^b, S^a), where S is a GFQI. We then get the following diagram, where vertical maps correspond to long exact sequences of triples, and horizontal to Alexander isomorphisms (omitting the subscript U):

$$\begin{array}{ccccc}
H_d(S^{-t}, S^{-\infty}) & \xrightarrow{\simeq} & H^{n+k-d}(E \setminus S^{-\infty}, E \setminus S^{-t}) & = & H^{n+k-d}((-S)^t, (-S)^{-\infty}) \\
\downarrow & & \downarrow & & \downarrow \\
H_d(S^{+\infty}, S^{-\infty}) & \xrightarrow{\simeq} & H^{n+k-d}(E \setminus S^{-\infty}, E \setminus S^{+\infty}) & = & H^{n+k-d}((-S)^{+\infty}, (-S)^{-\infty}) \\
\downarrow & & \downarrow & & \downarrow \\
H_d(S^{+\infty}, S^{-t}) & \xrightarrow{\simeq} & H^{n+k-d}(E \setminus S^{-t}, E \setminus S^{\infty}) & = & H^{n+k-d}((-S)^t, (-S)^{-\infty})
\end{array}$$

Using the universal coefficient theorem (recall, our coefficient ring is a field) we see that $H_*(S^{+\infty}_{|U}, S^{-\infty}_{|U})$ is a vector space dual to $H^*(S^{+\infty}_{|U}, S^{-\infty}_{|U})$. By abuse of language, we denote by 1_U the element $pt_U \in H_*(S^{+\infty}_{|U}, S^{-\infty}_{|U})$ sent to $1_U \in H^*(S^{+\infty}_{|U}, S^{-\infty}_{|U})$, and we see that $c(1_U, S)$ is the same whether we consider 1_U in homology or cohomology. On the other hand the second line of the diagram sends $T \otimes 1_U$ to $T \otimes \mu_U$, since in this case Alexander duality corresponds to Poincaré duality. Now saying that 1_U is in the image of $H_*(S^{-t}, S^{+\infty})$ is equivalent to saying that μ_U is in the image of $H^*((-S)^t, (-S)^{-\infty})$. In other words, $-t \geq c(1_U, S)$ is equivalent to $t \geq c(\mu_U, -S)$ and this means $c(1_U, S) = -c(\mu_U, -S)$. □

Definition 4.4. Let U be a bounded domain with smooth boundary, ∂U. We say that the sequence of smooth functions $(f_k)_{k\geq 1}$ in $C^\infty(N)$ *defines* U if

(1) there is a decreasing family F_k of closed subset of N such that $\bigcap_k F_k = \overline{U}$,

(2) $f_k = 0$ on F_k,

(3) f_k is a decreasing sequence converging to $-\infty$ on $N \setminus U$.

We say that $(f_k)_{k\geq 1}$ is a *standard defining sequence* if there is a function $r \in C^\infty(\mathbb{R})$ such that

(1) $r(t) = 0$ for $t \leq 0$,

(2) $r'(t) < 0$ for $0 < t < 1$,

(3) $r(t) = -1$ for $t \geq 1$,

and for some increasing sequence a_k converging to $+\infty$ we have

$$f_k(x) = a_k r_k(a_k \cdot d(x, U)).$$

Notice that given a sequence $(f_k)_{k\geq 1}$ defining U, we can find standard sequences $(g_k)_{k\geq 1}, (h_k)_{k\geq 1}$ such that $g_k \leq f_k \leq h_k$.

We define for a smooth function f the graph of its differential, $G_f = \{(x, df(x)) \mid x \in N\}$. This is an exact Lagrangian, with primitive f. If L is a Lagrangian with GFQI, S, we define $L + G_f$ to be the Lagrangian generated by $S + f$, where $S + f(x, \xi) = S(x, \xi) + f(x)$.

We notice that:

Lemma 4.5. *Let $(f_k)_{k\geq 1}$ be a sequence defining U, and V be any bounded open set such that $V \supset \bar{U}$. Then for $L_1, L_2 \in \mathfrak{L}(T^*N)$ we have*

$$c(1_U, L_1, L_2) = \lim_k c(1_V, L_1 - G_{f_k}, L_2) = \lim_k c(1_V, L_1, L_2 + G_{f_k}).$$

Proof. Let S_j be GFQI for L_j and $S = S_1 \ominus S_2$. We have $S^c_{|U} = \lim_k (S - f_k)^c_{|V}$; therefore for Čech cohomology, according to Theorem 5 in [Lee and Raymond 1968] we have

$$\lim_k H^*((S - f_k)^c_{|V}, (S - f_k)^b_{|V}) = H^*(S^c_{|U}, S^b_{|U})$$

and from the definition of $c(1_U, S)$ the proposition follows. □

Remark 4.6. One should be careful. We will often have to estimate $c(\mu_U, L_1, L_2)$ but it is not true that $c(\mu_U, L_1, L_2) = \lim_k c(\mu_N, L_1 - G_{f_k}, L_2)$. Indeed, if $L_1 = G_g$, $L_2 = 0_N$, then $c(\mu_U, L_1, L_2) = \sup_{x\in U} g(x) \neq \sup_{x\in N} g(x) - f_k(x)$. However it follows from Lemma 4.3 that

$$c(\mu_U, L_1, L_2) = -\lim_k c(1_N, L_2 + G_{f_k}, L_1).$$

Let U be an open set with smooth boundary and set $\nu(x) \in T^*_x U$ to be the exterior conormal to ∂U at $x \in \partial U$, i.e., $\nu(x) = 0$ on $T\partial U$ and $\langle \nu(x), n(x)\rangle = 1$, where $n(x)$ is the exterior unit normal to U at x. The conormal of U is then defined as

$$\nu^* U = \{(x, p) \in T^*N \mid x \in U, p = 0, \text{ or } x \in \partial U, p = c\nu(x), c \leq 0\}.$$

We now prove that the values of $c(\alpha, L)$ correspond to intersection points of L and ν^*U (or L and $\overline{\nu^*U}$).

Proposition 4.7 (representation theorem). *Let U be a bounded open set with smooth boundary and $(L_1, f_1), (L_2, f_2)$ be exact Lagrangians in T^*N. Then we have:*

(1) *For $\alpha \in H^*(U) \setminus \{0\}$, $c(\alpha; L_1, L_2)$ is given by $f_1(x_\alpha, p_{1,\alpha}) - f_2(x_\alpha, p_{2,\alpha})$, where $(x_\alpha, p_{1,\alpha}) \in L_\alpha$ and $(x_\alpha, p_{2,\alpha}) \in L_2$ and $(x_\alpha, p_{1,\alpha} - p_{2,\alpha}) \in \nu^*U$.*

(2) *The same holds for $\alpha \in H^*(U, \partial U) \setminus \{0\}$ but with ν^*U replaced by $\overline{\nu^*U}$.*

Proof. This is the representation theorem [Viterbo 1992, Proposition 2.4], using a standard defining sequence for U and the fact that $c(1_U; L_1, L_2) = \lim_k c(1_V; L_1 - G_{f_k}, L_2)$. Indeed, a converging sequence of points in G_{f_k} will converge to a point in ν^*U (remember f_k must also be bounded in the sequence!). Then the compactness of $L_1 \cap T^*U$ and $L_2 \cap T^*U$ implies the result. □

For $(f_k)_{k\geq 1}$ a defining sequence of U, we say ν^*U is the "limit" of the G_{f_k} for $k \geq 1$. We will formally write $c(\alpha, L, \nu^*U)$ for $c(\alpha_U, L)$.

Remarks 4.8. (1) The same will hold for $U \subset V$ and any $\alpha_V \in H^*(V)$ having restriction $\alpha_U \in H^*(U)$:

$$c(\alpha_U, L_1, L_2) = \lim_k c(\alpha_V, L_1 - G_{f_k}, L_2) = \lim_k c(\alpha_V, L_1, L_2 + G_{f_k}).$$

In particular, if M is a closed manifold containing N, we have

$$c(1_U, L_1, L_2) = \lim_k c(1_M, L_1 - G_{f_k}, L_2) = \lim_k c(1_M, L_1, L_2 + G_{f_k}) = c(1_M, L_1, L_2 + \nu^* U).$$

(2) Let $\overline{U} \subset V$. Then with obvious abuse of notation $c(1_V, \nu^* U) = -\infty$, $c(\mu_V, \nu^* U) = 0$ and of course $c(1_U, \nu^* V) = 0$, $c(\mu_U, \nu^* V) = +\infty$. This means that, for $(f_k)_{k \geq 1}$ and $(g_k)_{k \geq 1}$ defining U and V, we have $\lim_k c(1_M, G_{f_k}, G_{g_k}) = -\infty$ and $\lim_k c(\mu_M, G_{f_k}, G_{g_k}) = 0$.

We will now prove some of the properties of these invariants:

Proposition 4.9. *Let $\varphi \in \mathfrak{DHam}_{\mathrm{FP}}(T^*N)$ and $L = \varphi^1(0_N)$ be a Lagrangian submanifold. We have*

$$\gamma_U(L) := c(\mu_U, L) - c(1_U, L) \geq 0$$

and equality implies that $L \cap T^*U \supset 0_U$.

Proof. The proof follows from the triangle inequality (see [Viterbo 1992, Proposition 3.3, p. 693]) applied to the product

$$H^*(U) \otimes H_c^*(U) \to H_c^*(U).$$

Remember that the triangle inequality in [Viterbo 1992, Proposition 3.3] states that for two GFQI S_1, S_2 and two cohomology classes α, β, we have

$$c(\alpha \cup \beta, S_1 \oplus S_2) \geq c(\alpha, S_1) + c(\beta, S_2),$$

where $(S_1 \oplus S_2)(x; \xi, \eta) = S_1(x; \xi) + S_2(x; \eta)$. Here we apply it to S_1 a GFQI for L, and S_2 a nondegenerate quadratic form, that is, a GFQI for 0_N, $\alpha \in H^*(U), \beta \in H_c^*(U)$. We then have, since $c(\beta, 0_N) = 0$,

$$c(\alpha \cup \beta, L) \geq c(\alpha, L).$$

Thus we have $c(\mu_U, L) = c(1_U \cup \mu_U, L) \geq c(1_U, L)$ and equality implies that μ_U is nonzero in $K_c \simeq L \cap \overline{\nu^* U}$. But this implies $\pi(L \cap \nu^* U) \supset U$; hence L contains 0_U. Note that in general, contrary to the case where $N = U$ is compact, $L \cap T^*U$ may contain other connected components than 0_U. ☐

Proposition 4.10. *The following hold for $L_i \in \mathfrak{L}(T^*N)$:*

(1) *We have $c(\mu_U, L_1, L_2) = -c(1_U, L_2, L_1) = -c(1_U, \bar{L}_1, \bar{L}_2)$.*

(2) *For $U \subset V$ and L_1, L_2 Lagrangian submanifolds we have*

- $c(\mu_U, L_1, L_2) \leq c(\mu_V, L_1, L_2)$,
- $c(1_U, L_1, L_2) \geq c(1_V, L_1, L_2)$,
- $\gamma_U(L_1, L_2) \leq \gamma_V(L_1, L_2)$.

(3) *We have $\gamma_U(L_1, L_3) \leq \gamma_U(L_1, L_2) + \gamma_U(L_2, L_3)$.*

(4) *If* $\gamma_U(L_1, L_2) = 0$ *then* $L_1 \cap L_2$ *has a connected component with projection on* N *containing* U.

(5) *If* $L_1 \preceq L_2$ *then* $c(\alpha, L_1) \leq c(\alpha, L_2)$ *for all* $\alpha \neq 0$.

Proof. (1) The proof is the same as in Lemma 4.3, since $S_{L_1} \ominus S_{L_2} = -(S_{L_2} \ominus S_{L_1})$.

(2) If $U \subset N$ note that

$$c(1_U; L_1, L_2) = \lim_k c(1_N, L_1 - G_{f_k}, L_2).$$

Since we may choose defining sequences $(f_k)_{k\geq 1}, (g_k)_{k\geq 1}$ for U, V such that $f_k \leq g_k$, we have for S_1 a GFQI of L_1 that $S_1 - f_k \geq S_1 - g_k$, hence $c(1_N, L_1 - G_{f_k}) \geq c(1_N, L_1 - G_{g_k})$, and going to the limit, $c(1_U, L_1, L_2) \geq c(1_V, L_1, L_2)$. By the duality formula (1), we get $c(\mu_U; L_1, L_2) \leq c(\mu_V; L_1, L_2)$; hence $\gamma_U(L_1, L_2) \leq \gamma_V(L_1, L_2)$.

(3) We have

$$S_1 \ominus 2 \cdot f \ominus S_3 = (S_1 \ominus f) \ominus (S_3 \oplus f)$$

and $(S_1 \ominus f) \ominus S_2 = S_1 \ominus (f \oplus S_2)$. Now noting that if $(f_k)_{k\geq 1}$ defines U, then so does $(2 \cdot f_k)_{k\geq 1}$, we have

$$\begin{aligned}
\gamma_U(L_1, L_3) &= \lim_k \gamma_V(S_1 \ominus 2 \cdot f_k \ominus S_3) \\
&= \lim_k \gamma_V((S_1 \ominus f_k) \ominus (S_3 \oplus f_k)) \\
&\leq \lim_k \gamma_V(S_1 \ominus f_k \ominus S_2) + \lim_k \gamma_V(S_2 \ominus (f_k \oplus S_3)) \\
&= \gamma_U(L_1, L_2) + \gamma_U(L_2, L_3).
\end{aligned}$$

(4) This follows from Lusternik–Schnirelmann theory as in the proof of [Viterbo 1992, Proposition 2.2, p. 691] (see also Proposition 4.9).

(5) $L_1 \preceq L_2$ implies $c(\mu_U, L_1, L_2) = 0$ for all U. By the triangle inequality applied to $S_1 \oplus (-S_2)$ (where S_i is a GFQI for L_i) if $\beta \cup \alpha = \mu_U$, we have

$$0 = c(\mu_U, L_1, L_2) \geq c(\alpha, L_1, 0_N) + c(\beta, 0_N, L_2) \geq c(\alpha, L_1) - c(\alpha, L_2)$$

since $c(\beta, 0_N, L) = -c(\alpha, L, 0_N)$ according to the proof of Proposition B.3 in [Viterbo 2023]. □

We must now see what happens when we make a coordinates change in T^*N. We start with three lemmas.

Lemma 4.11. *Let* S *be a GFQI defined on* $E = Y \times F$ *and for* $f : X \to Y$ *a smooth map a map* $\tilde{f} : X \times F \to Y \times F$ *living over* f, *i.e., the diagram*

$$\begin{array}{ccc}
X \times F & \xrightarrow{\tilde{f}} & Y \times F \\
\downarrow & & \downarrow \\
X & \xrightarrow{f} & Y
\end{array}$$

is commutative. We then have, for $\alpha \in H^*(Y)$ *and* $(f)^*(\alpha) \in H^*(X)$,

$$c(\alpha, S) \leq c(f^*(\alpha), S \circ \tilde{f}).$$

Proof. Indeed, if $T \in H^*(D(F^-), S(F^-))$ is the Thom class for F^-, then $(\tilde{f})^*(T) = \widetilde{T}$ is the Thom class for $\tilde{f}^*(F^-)$ and we have, denoting by $\pi, \tilde{\pi}$ the projections on Y and X,

$$(\tilde{f})^*(T \cup \pi^*(\alpha)) = \pi^*(f^*(\alpha)) \cup \widetilde{T}.$$

Now if $c < c(\alpha, S)$ then $\pi^*(\alpha) \cup T$ vanishes in $H^*(S^c, S^{-\infty})$ and this implies that $(\tilde{f})^*(T \cup \pi^*(\alpha)) = \pi^*(f^*(\alpha)) \cup \widetilde{T}$ vanishes in $H^*((S \circ \tilde{f})^c, (S \circ \tilde{f})^{-\infty})$, i.e., $c \leq c(f^*(\alpha), S \circ \tilde{f})$. This implies the lemma. □

For the next lemma we use the notation $S_1 \boxtimes S_2$ to denote $S_1(x, y, \xi, \eta) = S_1(x; \xi) + S_2(y; \eta)$ (not to be confused with $S_1 \oplus S_2$) and $\alpha \otimes \beta$ to denote the class in $H^*(X \times Y)$ image of $\alpha \otimes \beta$ by Künneth's isomorphism.

Lemma 4.12. *We have*

$$c(1_U \otimes 1_U; L_1 \times L_2, \nu^* \Delta_N) = c(1_U; L_1, L_2).$$

Proof. Let $d^\varepsilon : N \times N \to \mathbb{R}$ be a smooth function vanishing on Δ_N and converging as ε goes to 0 to $-\infty \cdot (1 - \chi_{\Delta_N})$, where χ_{Δ_N} is the characteristic function of Δ_N. For example we can choose

$$d^\varepsilon(x, y) = -\frac{1}{\varepsilon} d(x, y).$$

Similarly define $d_U^\varepsilon(x, y) = d^\varepsilon(x, y) + f_U^\varepsilon(x) + f_U^\varepsilon(y)$, where f_U^ε converges to $-\infty(1 - \chi_U)$ as ε goes to 0.

Setting $[S_1 \boxtimes (-S_2)](x_1, x_2, \xi_1, \xi_2) = S_1(x_1, \xi_1) + S_2(x_2, \xi_2)$, and

$$[S_1 \oplus (-S_2)](x, \xi_1, \xi_2) = S_1(x, \xi_1) + S_2(x, \xi_2)$$

we may write

$$\begin{aligned} c(1_{U \times U}; L_1 \times L_2, \Delta_N) &= \lim_{\varepsilon \to 0} c(1_{N \times N}; (L_1 - G_{f_U^\varepsilon}) \times (L_2 - G_{f_U^\varepsilon}), \nu^* \Delta_N) \\ &= \lim_{\varepsilon \to 0} c(1_{N \times N}; (S_1 - f_U^\varepsilon) \boxtimes (-S_2 - f_U^\varepsilon), d^\varepsilon) \\ &= c(1_{N \times N}, [(S_1 - f_U^\varepsilon) \boxtimes (-S_2 - f_U^\varepsilon)] - d^\varepsilon). \end{aligned}$$

Now $\lim_{\varepsilon \to 0}(S_1 \boxtimes (-S_2) - d^\varepsilon)^c = (S_1 \oplus (-S_2))^c$ and if $\delta : \Delta_N \to N \times N$ is the diagonal map, $\delta^*(1_N \otimes 1_N) = 1_{\Delta_N}$, so from Lemma 4.11, we get

$$c(1_U \otimes 1_U; L_1 \times L_2, \nu^* \Delta_N) \leq c(1_N, (S_1 - f_U^\varepsilon) \oplus (S_2 - f_U^\varepsilon)) \leq c(1_U; L_1, L_2).$$

Conversely we notice that given c, for ε small enough, $(S_1 \boxtimes (-S_2) - d^\varepsilon)^c$ is contained in a neighborhood of Δ_N. Thus if $1_U \otimes 1_U$ does not vanish in

$$H_*([((S_1 - f_U^\varepsilon) \boxtimes (-S_2 - f_U^\varepsilon)) - d^\varepsilon]^c, [((S_1 - f_U^\varepsilon) \boxtimes (-S_2 - f_U^\varepsilon)) - d^\varepsilon]^{-\infty}),$$

i.e., $c \geq c(1_U \otimes 1_U; L_1 \times L_2, \nu^* \Delta_N)$, then its restriction to Δ_N, that is, 1_U does not vanish either, and $c \geq c(1_U; L_1, L_2)$, so

$$c(1_U \otimes 1_U; L_1 \times L_2, \nu^* \Delta_N) \geq c(1_U; L_1, L_2)$$

and we have equality. □

Lemma 4.13. *Let us consider a bounded open set with boundary $U \subset N$ and $\nu^*\Delta_U \subset T^*N \times \overline{T^*N}$, where Δ_U is the diagonal in U. Let φ^t be a Hamiltonian flow on T^*U such that $\varphi^1(T^*U) \subset T^*V$. We have*

$$(\varphi^1 \times \varphi^1)(\nu^*\Delta_U) \preceq \nu^*\Delta_V.$$

Proof. Let $(q, p, q, p') \in \nu^*\Delta_U$ and notice that unless $q \in \partial U$, we have $p = p'$. Then according to Lemma 3.2 we may assume $\varphi^t(T^*U) \subset T^*V$ for all $t \in [0, 1]$, so setting $(\varphi^t \times \varphi^t)(q, p, q, p') = (Q_t, P_t, Q'_t, P'_t)$ we know that when $(q, p, q, p') \in \nu^*\Delta_U \subset T^*\overline{U}$, we have $Q_t, Q'_t \notin \partial V$. So if $(Q_t, P_t, Q'_t, P'_t) \in \nu^*V$, we must have $Q_t = Q'_t$, $P_t = P'_t$, but then $p = p'$. In other words

$$(\varphi^t \times \varphi^t)(\nu^*\Delta_U) \cap \nu^*\Delta_V = (\varphi^t \times \varphi^t)(\Delta_{T^*U}) \cap \Delta_{T^*V} = (\varphi^t \times \varphi^t)(\Delta_{T^*U}).$$

So the intersection $(\varphi^t \times \varphi^t)(\nu^*\Delta_U) \cap \nu^*\Delta_V$ is constant and by a classical argument, this implies that as a function of t, $c(\alpha, (\varphi^t \times \varphi^t)(\nu^*\Delta_U), \nu^*\Delta_V)$ is constant. Since $\nu^*\Delta_U \preceq \nu^*\Delta_V$, we have for all t we have $(\varphi^t \times \varphi^t)(\nu^*\Delta_U) \preceq \nu^*\Delta_V$. □

Using Proposition 4.10(2), we may conclude that the limits in the following proposition are well-defined in $\mathbb{R} \cup \{\pm\infty\}$.

Definition 4.14. When U is an unbounded set we define $\mathscr{B}(U)$ to be the set of bounded subsets in U and

$$c(\mu_U, L_1, L_2) = \lim_{V \in \mathscr{B}(U)} c(\mu_V, L_1, L_2),$$
$$c(1_U, L_1, L_2) = \lim_{V \in \mathscr{B}(U)} c(1_V, L_1, L_2).$$

Remark 4.15. Symbolically we have for $\overline{U} \subset V$ that $\nu^*U + \nu^*V = \nu^*U$, meaning that if $(f_k)_{k\geq 1}$ defines U and $(g_k)_{k\geq 1}$ defines V then $(f_k + g_k)_{k\geq 1}$ defines U. More generally if $U \cap V \subset W$, we have $\nu^*U + \nu^*V \preceq \nu^*W$ where this means that if $(f_k)_{k\geq 1}$ defines U and $(g_k)_{k\geq 1}$ defines V, there is a sequence $(h_k)_{k\geq 1}$ defining W such that $f_k + g_k \leq h_k$.

Proposition 4.16. *We have for $\varphi \in \mathfrak{DHam}(T^*N)$ such that $\varphi(T^*U) \subset T^*V$ and $L_1, L_2 \in \mathfrak{L}(T^*N)$*

$$\gamma_U(\varphi(L_1), \varphi(L_2)) \leq \gamma_V(L_1, L_2).$$

Proof. We use Lemma 4.11 so we replace $c(1_U, \varphi(L_1), \varphi(L_2))$ by

$$c(1_U \otimes 1_U, \varphi(L_1) \times \varphi(L_2), \nu^*\Delta_N)$$

and this in turn equals

$$c(1_N \otimes 1_N, (\varphi \times \varphi)(L_1 \times L_2), \nu^*\Delta_N + \nu^*(U \times U)).$$

Using Remark 4.15 we have

$$\nu^*\Delta_N + \nu^*(U \times U) \preceq \nu^*(\Delta_N \cap (U \times U)) = \nu^*\Delta_U$$

and we get

$$\begin{aligned} c(1_N \otimes 1_N, (\varphi \times \varphi)(L_1 \times L_2), \nu^*\Delta_N + \nu^*(U \times U)) &\geq c(1_N \otimes 1_N, (\varphi \times \varphi)(L_1 \times L_2), \nu^*\Delta_U) \\ &= c(1_N \otimes 1_N, (L_1 \times L_2), (\varphi \times \varphi)^{-1}(\nu^*\Delta_U)) \end{aligned}$$

and using Lemma 4.13 we get that the last term is greater than

$$c(1_{N\times N}, L_1\times L_2, \nu^*\Delta_V)=c(1_V, L_1, L_2).$$

We may thus conclude that

$$c(1_V, L_1, L_2)\le c(1_U, \varphi(L_1), \varphi(L_2)).$$

By duality, we get

$$c(\mu_V, L_1, L_2)\ge c(\mu_U, \varphi(L_1), \varphi(L_2))$$

and our result follows. □

Definition 4.17. A sequence $(L_k)_{k\ge 1}\in\mathfrak{L}(T^*N)$ γ_c*-converges* to $L\in\mathfrak{L}(T^*N)$ if for all bounded domains U the sequence $\gamma_U(L_k, L)$ converges to 0. We shall write $L_k\xrightarrow{\gamma_c} L$. The γ_c*-completion* of $\mathfrak{L}(T^*N)$ for γ_c is the set of equivalence classes of γ_c-Cauchy sequences $(L_k)_{k\ge 1}$ for the following relation: $(L_k)_{k\ge 1}\simeq(L'_k)_{k\ge 1}$ if for all bounded domains U the sequence $\gamma_U(L_k, L'_k)$ converges to 0. We denote this completion by $\widehat{\mathfrak{L}}(T^*N)$.

Remark 4.18. Of course we may take a cofinal sequence U_k of bounded open sets in N and define

$$d(L_1, L_2)=\sum_{j=1}^{+\infty}2^{-j}\max\{1, \gamma_{U_j}(L_1, L_2)\}$$

and then take the completion with respect to this metric. It is easy to see that the completion coincides with the above, and hence does not depend on the choice of the sequence U_k (this is just rephrasing the fact that the γ_U define a uniform structure; see [Weil 1938] or [Bourbaki 2007, Chapter II]).

Example 4.19. Let f_k be a sequence of smooth functions. Then γ-convergence of the $L_k=\mathrm{gr}(df_k)$ is equivalent to uniform convergence on compact sets of the f_k.

We shall need the following proposition.

Proposition 4.20. *We have for* $L=\varphi_H^1(0_N)\in\mathfrak{L}(T^*N)$ *the inequalities*

$$c(\mu_U, L)\le\sup_{(q,p)\in T^*U}H(q,p),$$

$$c(1_U, L)\ge\inf_{(q,p)\in T^*U}H(q,p),$$

$$\gamma_U(L)\le\sup_{(q,p)\in T^*U}H(q,p)-\inf_{(q,p)\in T^*U}H(q,p)=\mathrm{osc}_{T^*U}(H)\le 2\|H\|_{C^0(T^*U)}.$$

Proof. Let $H(q,p)=h(q)$ and $L_h=\varphi_H(0_N)$. Then according to Remark 4.2(3) we have $c(\mu_U, L_h)\le\sup_{q\in U}h(q)$ and $c(1_U, L_h)\ge\inf_{q\in U}h(q)$ because $L_h=\{(q, dh(q))\mid q\in N\}$.

Now for general H, since for $H\le h(q)=\sup_{p\in T_q^*N}H(q,p)$ we have $H\le h$, we get $L\preceq L_h$, so $c(\mu_U, L)\le c(\mu, L_h)\le\sup_{q\in U}h(q)=\sup_{(q,p)\in T^*U}H(q,p)$ and we get the first inequality. The other two inequalities follow immediately from this one. □

4.2. ***The case of Hamiltonians in $T^*\mathbb{R}^n$.*** Let $H \in \mathfrak{Ham}_{\rm fc}([0,1]\times T^*\mathbb{R}^n)$ and φ_H^t be its flow. Let s_1, s_2 the symplectomorphisms

$$T^*\mathbb{R}^n \times \overline{T^*\mathbb{R}^n} \to T^*\Delta_{T^*\mathbb{R}^n}$$

defined respectively by

$$s_1(q,p,Q,P) = (q,P,p-P,Q-q),$$
$$s_2(q,p,Q,P) = (Q,p,p-P,Q-q).$$

Denoting by (x,y,X,Y) the coordinates in $T^*\Delta_{T^*\mathbb{R}^n}$, we have

$$s_i^*(dY\wedge dy + dX\wedge dx) = dp\wedge dq - dP\wedge dQ,$$

so the s_i are symplectic.

The graph of φ_H is $(\mathrm{id}\times\varphi_H)(\Delta_{T^*\mathbb{R}^n})$, and its image by s_1 is denoted by $\Gamma(\varphi_H)$, while its image by s_2 will be $\Gamma(\varphi_H^{-1})$. Let S_H be a GFQI for $\Gamma(\varphi_H)$ which exists and is unique if $H \in \mathfrak{Ham}_{\rm BP}(T^*\mathbb{R}^n)$ by Theorem 3.5.

Definition 4.21. We set for W a domain contained in $\Delta_{T^*\mathbb{R}^n}$. Then

(1) $c_W^-(\varphi_H,\varphi_K) = c(1_W;\Gamma(\varphi_H),\Gamma(\varphi_H))$.

(2) $c_W^+(\varphi_H,\varphi_K) = c(\mu_W;\Gamma(\varphi_H),\Gamma(\varphi_H))$.

(3) $\gamma_W(\varphi_H,\varphi_K) = c_W^+(\varphi_H,\varphi_K) - c_W^-(\varphi_H,\varphi_K)$.

(4) $c_W^-(\varphi_K)$, $c_W^+(\varphi_K)$ and $\gamma_W(\varphi_K)$ are abbreviations for $c_W^-(\mathrm{id},\varphi_K)$, $c_W^+(\mathrm{id},\varphi_K)$ and $\gamma_W(\mathrm{id},\varphi_K)$ respectively.

Remark 4.22. In T^*N we may define for $U\subset N$ the number

$$\hat\gamma_U(\varphi_H) = \sup_{L\in\mathfrak{L}(T^*N)} \gamma_U(L,\varphi_H(L)),$$

which corresponds to (even though we do not claim it is equal to) $\gamma_{(U\times\mathbb{R}^n)}(\varphi_H)$.

Analogously to Proposition 4.16 we prove:

Proposition 4.23. *For $\varphi_1,\varphi_2 \in \mathfrak{DHam}_{\rm BP}(T^*\mathbb{R}^n)$ such that $\varphi_j(T^*U)\subset T^*V$ and $L\in\mathfrak{L}(T^*\mathbb{R}^n)$ we have*

$$\gamma_U(\varphi_1(L),\varphi_2(L)) \le \gamma_{V\times\mathbb{R}^n}(\varphi_1,\varphi_2).$$

Proof. We have

$$\begin{aligned} c(1_U,\varphi(L),L) &= c(1_U\otimes 1_U;\varphi(L)\times L,\nu^*\Delta_N)\\ &\ge c(1_U\otimes 1_U;(\varphi\times\mathrm{id})(L\times L),(\varphi\times\mathrm{id})(\nu^*\Delta_N)) + c(1_U\otimes 1_U;(\varphi\times\mathrm{id})(\nu^*\Delta_N),\nu^*\Delta_N).\end{aligned}$$

Equality follows from Lemma 4.12 and the inequality is the triangle inequality.

Now if $(\varphi\times\mathrm{id})T^*(U\times U)\subset T^*(V\times V)$, we have

$$c(1_U\otimes 1_U;(\varphi\times\mathrm{id})(L\times L),(\varphi\times\mathrm{id})(\nu^*\Delta_N)) \ge c(1_V\otimes 1_V;L\times L,\Delta_{T^*N}) = c(1_V;L,L) = 0.$$

As a result we have

$$c(1_U,\varphi(L),L)\geq c(1_U\otimes 1_U;(\varphi\times\mathrm{id})(\nu^*\Delta_N),\nu^*\Delta_N)=c(1_U\otimes 1_U;\Gamma(\varphi),\Gamma(\mathrm{id})).$$

We must now compare this last invariant with $c(1_W;\Gamma(\varphi))$. The map $s_1 : T^*\mathbb{R}^n\times\overline{T^*\mathbb{R}^n}\to T^*\Delta_{\mathbb{R}^{2n}}$ given by $s_1(q,p,Q,P)=(q,P,p-P,Q-q)$ sends $T^*(V\times V)$ into $T^*(V\times\mathbb{R}^n)$, so we have

$$c(1_V\otimes 1_V;\Gamma(\varphi),\Gamma(\mathrm{id}))\geq c(1_{V\times\mathbb{R}^n},\Gamma(\varphi)).$$

We may then conclude that

$$c(1_U,\varphi(L),L)\geq c(1_{V\times\mathbb{R}^n},\Gamma(\varphi))$$

and using the dual inequality we get our result. □

Let then $(H_\nu)_{\nu\geq 1}$ be a sequence of Hamiltonians in $\mathfrak{Ham}_{\mathrm{FP}}(T^*\mathbb{R}^n)$ and $\varphi_\nu=\varphi_{H_\nu}$.

Definition 4.24. The sequence $(\varphi_\nu)_{\nu\geq 1}$ γ_c*-converges* to φ if for all bounded domains W we have $\lim_\nu \gamma_W(\varphi_\nu,\varphi)=0$.

The γ_c*-completion* $\widehat{\mathfrak{DHam}}_{\mathrm{FP}}(T^*\mathbb{R}^n)$ is defined as the set of Cauchy sequences in $\mathfrak{DHam}_{\mathrm{FP}}(T^*\mathbb{R}^n)$ for the uniform structure defined by the γ_W, in other words the set of sequences which are Cauchy for each γ_W, modulo the equivalence relation $(\varphi_\nu)_{\nu\geq 1}\simeq(\psi_\nu)_{\nu\geq 1}$ if for all W we have $\lim_\nu\gamma_W(\varphi_\nu,\psi_\nu)=0$.

Similarly we define for $H\in\mathfrak{Ham}_{\mathrm{FP}}(T^*\mathbb{R}^n)$ the pseudometric

$$\gamma_W(H,K)=\sup_{t\in[0,1]}\gamma_W(\varphi_H^t,\varphi_K^t).$$

We then define analogously the γ_c-convergence of a sequence in $\mathfrak{Ham}_{\mathrm{FP}}(T^*\mathbb{R}^n)$ and its completion $\widehat{\mathfrak{Ham}}_{\mathrm{FP}}(T^*\mathbb{R}^n)$.

Note that the property of having FPS or being in $\mathfrak{H}am_{\mathrm{fc}}$ can be checked in the γ_c-completion.

Proposition 4.25. *There exist closed sets in* $\widehat{\mathfrak{DHam}}_{\mathrm{FP}}(T^*\mathbb{R}^n)$ *that intersect* $\mathfrak{DHam}(T^*\mathbb{R}^n)$ *on* $\mathfrak{DHam}_{\mathrm{FP}}(T^*\mathbb{R}^n)$, $\mathfrak{DHam}_{\mathrm{BP}}(T^*\mathbb{R}^n)$ *and* $\{\varphi\in\mathfrak{DHam}(T^*T^n)\mid \mathrm{supp}(\varphi)\subset\{|p|\leq r\}$ *respectively.*

Proof. Indeed $\varphi(T^*U)\subset T^*V$ is equivalent to

$$\Gamma(\varphi)\cap\{(x,p_x,y,p_y)\mid x\in U, y\notin V\}=\varnothing$$

and being supported in $|p|\leq r$ is equivalent to

$$\Gamma(\varphi)\cap\{(x,p_x,y,p_y)\mid |p_x|\geq r\}\subset\Gamma(\mathrm{id})$$

and both are closed conditions, which makes sense in the completion (see [Humilière 2008]). □

When dealing with fiberwise compactly supported Hamiltonians, we have:

Definition 4.26. We set for $\varphi\in\mathfrak{Ham}_{\mathrm{fc}}(T^*\mathbb{R}^n)$

$$\gamma_r(\varphi)=\gamma_{\mathbb{R}^n\times B^n(r)}(\varphi)=\lim_{R\to+\infty}\gamma_{B^n(R)\times B^n(r)}(\varphi),$$

$$\gamma_\infty(\varphi)=\lim_{r\to\infty}\gamma_r(\varphi)\in\mathbb{R}\cup\{+\infty\}.$$

Notice that convergence for γ_c and γ_∞ coincides on sequences supported in a fixed bounded set in the p-direction.

Proposition 4.27. *If $h_-(p) \le H(t,q,p) \le h_+(p)$, we have the inequality*

$$\gamma_r(\varphi_H) \le \sup_{|p|\le r} h_+(p) - \inf_{|p|\le r} h_-(p).$$

In particular if $a \le H(q,p) \le b$, we have $\gamma_\infty(\varphi_H) \le b-a$.

Proof. Indeed, $c_W^+(H) \le c_W^+(h_+)$, but $c^+_{\mathbb{R}^n\times B^n(r)}(h_+) = \sup_{|p|\le r} h_+(p)$. Indeed, the flow of $h(p)$ is $(q,p)\mapsto(q+t\,dh(p),p)$ and its graph is given by $(q,p,0,t\,dh(p))$, so a GFQI is $S(q,P)=h(P)$, and

$$c^+_{\mathbb{R}^n\times B^n(r)}(H) \le c^+_{\mathbb{R}^n\times B^n(r)}(h_+) = \sup_{|p|\le r} h_+(p).$$

Similarly $c^-_{\mathbb{R}^n\times B^n(r)}(h_-) = \inf_{|p|\le r} h_-(p)$ and $c^-_{\mathbb{R}^n\times B^n(r)}(H) \ge c^+_{\mathbb{R}^n\times B^n(r)}(h_-) = \inf_{|p|\le r} h_-(p)$.

By taking the difference of the above inequalities, we prove the proposition. □

Remark 4.28. The quantity $\gamma_\infty(\varphi)$ is finite for $\varphi \in \mathfrak{Ham}(T^*\mathbb{R}^n)$ such that $\|H\|_{C^0(T^*\mathbb{R}^n)} < +\infty$.

Our last results in this section will be:

Proposition 4.29. *We have the following, remembering that $\rho(x,p) = \left(\frac{x}{\varepsilon}, p\right)$:*

(1) *Assume ψ, ψ^{-1} send $W = U\times V$ into $W' = U'\times V'$, where $U, U' \subset \mathbb{R}^n$, $V, V' \subset (\mathbb{R}^n)^*$. Then we have*

$$\gamma_W(\psi^{-1}\circ\varphi\circ\psi) \le \gamma_{W'}(\varphi).$$

(2) $\gamma_r(\tau_{-a}\circ\varphi\circ\tau_a) = \gamma_r(\varphi)$.

(3) $\gamma_r(\rho_\varepsilon^{-1}\circ\varphi\circ\rho_\varepsilon) = \varepsilon\gamma_r(\varphi)$.

Proof. In $T^*(\mathbb{R}^n\times\mathbb{R}^n)$ we have that $\Gamma(\varphi)$ is the set of $(q,P,P-p,q-Q)$, where $(Q,P)=\varphi(q,p)$, while $\Gamma(\psi\circ\varphi\circ\psi^{-1})$ is obtained by applying $\psi\times\psi$ to (q,p,Q,P). In other words writing $(q',p') = \psi(q,p), (Q',P') = \psi(Q,P)$, $\Gamma(\psi\circ\varphi\circ\psi^{-1})$ is obtained as

$$\{(q',P',P'-p',q'-Q') \mid \varphi(q,p)=(Q,P)\}.$$

Now if $q\in U$ and $P\in V$, we have $q'\in U'$ and $P'\in V'$; hence $(\psi\times\psi)(T^*(U\times V)) \subset T^*(U'\times V')$, where $U\times V, U'\times V'$ are considered subsets of $\Delta_{T^*\mathbb{R}^n}$.

As a result, since $\psi\times\psi$ preserves the diagonal (that is the zero section in the new coordinates) we have, using Proposition 4.16,

$$\gamma_{U\times V}(\psi^{-1}\varphi\psi) = \gamma_{U\times V}((\psi\times\psi)\Gamma(\varphi),(\psi\times\psi)(\Delta)) \le \gamma_{U'\times V'}(\Gamma(\varphi),\Delta) = \gamma_{U'\times V'}(\varphi).$$

Statement (2) follows from first applying (1) to $\psi=\tau_a$ so that, setting $U_a = \bigcup_{t\in[-a,a]}\tau_t(U)$,

$$\gamma_{U\times B(r)}(\tau_{-a}\varphi\tau_a) \le \gamma_{U_a\times B(r)}(\varphi).$$

Hence taking the limit for $U\subset\mathbb{R}^n$ we get

$$\gamma_r(\tau_{-a}\varphi\tau_a) \le \gamma_r(\varphi)$$

and changing a to $-a$ we get equality. The last equality is rather obvious since ρ_ε is $\frac{1}{\varepsilon}$-conformal and $\rho_\varepsilon(U \times B_r) = \left(\frac{1}{\varepsilon} \cdot U\right) \times B_r$. □

Remark 4.30. One should be careful, in particular $\gamma_U(\varphi_1, \varphi_2)$ is *not* in general equal to $\gamma_U(\varphi_2^{-1} \circ \varphi_1) = \gamma_U(\varphi_2^{-1} \circ \varphi_1, \mathrm{id})$. We thus have a priori two types of convergence. We could say that φ_ν converges to φ if for all bounded sets U either the sequence $\gamma_U(\varphi_\nu, \varphi)$ goes to 0 or if $\gamma_U(\varphi_\nu \varphi^{-1})$ goes to 0. However if the φ_ν have uniformly bounded propagation speed, that is, $\varphi_\nu(T^*B_r) \subset T^*B_{r+r_0}$ for all ν and all r, then the two conditions are equivalent.

5. Compactness and ergodicity

Let $H : T^*\mathbb{R}^n \times \Omega \to \mathbb{R}$ be Hamiltonian satisfying properties (1)–(6). Then each $H_\omega = H(\cdot, \cdot, \omega)$ is in $\mathfrak{Ham}_{\mathrm{fc}}(T^*\mathbb{R}^n)$ and we identify Ω with its image in $\mathfrak{Ham}_{\mathrm{fc}}(T^*\mathbb{R}^n)$, denoted by $\mathfrak{H}_\Omega$. Its closure for the γ_c-topology in the completion $\widehat{\mathfrak{Ham}}_{\mathrm{FP}}(T^*\mathbb{R}^n)$ is denoted by $\widehat{\mathfrak{H}}_\Omega$. The action τ of $\mathbb{R}^n$ on Ω induces an action on $\mathfrak{H}_\Omega$ by

$$(\tau_a H)(x, p; \omega) = H(x + a, p; \omega) = H(x, p; \tau_{-a}\omega).$$

This action translates into $\varphi \mapsto \tau_{-a}\varphi\tau_a$ on $\mathfrak{DHam}_{\mathrm{fc}}(T^*\mathbb{R}^n)$.

We first want to prove:

Proposition 5.1. *The abelian group $\mathbb{R}^n$ acts continuously by isometries on $(\mathfrak{Ham}_{\mathrm{fc}}(T^*\mathbb{R}^n), \gamma_c)$ and $(\mathfrak{DHam}_{\mathrm{fc}}(T^*\mathbb{R}^n), \gamma_c)$ and hence on $(\widehat{\mathfrak{Ham}}_{\mathrm{fc}}(T^*\mathbb{R}^n), \gamma_c)$ and $(\widehat{\mathfrak{DHam}}_{\mathrm{fc}}(T^*\mathbb{R}^n), \gamma_c)$. Therefore the action τ of $\mathbb{R}^n$ on $\mathfrak{H}_\Omega$ is a continuous action by isometries for γ_c which extends to a continuous action by isometries on $\widehat{\mathfrak{H}}_\Omega$.*

Proof. That $\mathbb{R}^n$ acts by isometries follows from Proposition 4.29(2). It is enough according to a theorem by Chernoff and Marsden[15] to prove the separate continuity of the map $\mathbb{R}^n \times \mathfrak{Ham}_{\mathrm{fc}}(T^*\mathbb{R}^n) \to \mathfrak{Ham}_{\mathrm{fc}}(T^*\mathbb{R}^n)$ in each variable. In other words — since τ_a is an isometry, it is obviously continuous in the second variable — we must prove that, for all $H \in \mathfrak{Ham}_{\mathrm{fc}}(T^*\mathbb{R}^n)$, we have

$$\lim_{a \to 0} \gamma_c(H, \tau_a H) = 0,$$

i.e., we want to prove that for all $r > 0$, $\lim_{a\to 0} \gamma_r(\tau_a^{-1}\varphi^{-1}\tau_a, \varphi) = 0$. But

$$\Gamma(\varphi) = \{(q, P, p - P, Q - q) \mid \varphi(q, p) = (Q, P)\},$$

while

$$\Gamma(\tau_a^{-1}\varphi\tau_a) = \{(q - a, P, p - P, Q - q) \mid \varphi(q, p) = (Q, P)\},$$

so that $S(q, P; \xi)$ is a GFQI for $\Gamma(\varphi)$ and $(\tau_a S)(q, P, \xi) = S(q - a, P; \xi)$ is a GFQI for $\Gamma(\tau_a^{-1}\varphi\tau_a)$. Since critical points of $S(q, P, \xi)$ are contained in $|P| \le R$ and $a \mapsto S(q-a, P; \xi)$ is uniformly continuous on $|P| \le R$, we get that $c_W(\alpha, S \ominus \tau_a S)$ depends continuously on a, and for $a = 0$ is equal to 0 (since it is equal to $c_W(\varphi, \varphi) = 0$). □

[15] Which claims that, under our assumptions, a separately continuous action is jointly continuous. See [Chernoff and Marsden 1970, Theorem 1], extending a theorem of Ellis [1957].

Proposition 5.1 extends the action τ to a continuous action by isometries of $\widehat{\mathfrak{H}}_\Omega$. Since $\mathrm{Isom}(\mathfrak{H}_\Omega, \gamma) \subset \mathrm{Isom}(\widehat{\mathfrak{H}}_\Omega, \gamma)$, the map $\tau : \mathbb{R}^n \to \mathrm{Isom}(\mathfrak{H}_\Omega, \gamma)$ extends to a map, still denoted by τ, from $\mathbb{R}^n$ to $\mathrm{Isom}(\widehat{\mathfrak{H}}_\Omega, \gamma)$. Since this is obviously a group morphism, its closure in $\mathrm{Isom}(\widehat{\mathfrak{H}}_\Omega, \gamma)$ is an abelian connected and complete metric group.

Proposition 5.2. *Let us denote the closure of $\tau(\mathbb{R}^n)$ in $\mathrm{Isom}(\widehat{\mathfrak{H}}_\Omega, \gamma)$ by $\mathbb{A}_\Omega$. Then $\mathbb{A}_\Omega$ is an abelian, connected and complete metric group.*

The goal of this section is to prove that our assumptions on H imply that $\mathbb{A}_\Omega$ is compact. For this it is enough to prove that $\mathrm{Isom}(\widehat{\mathfrak{H}}_\Omega, \gamma_c)$ is compact, but this follows immediately by the Arzelà–Ascoli theorem if we prove that $(\widehat{\mathfrak{H}}_\Omega, \gamma_c)$ is compact. Because by assumption $(\widehat{\mathfrak{H}}_\Omega, \gamma_c)$ is complete, it is enough to show that it is totally bounded, that is, for any $\varepsilon > 0$, $(\widehat{\mathfrak{H}}_\Omega, \gamma_c)$ can be covered by finitely many γ_c-balls of radius ε. Since $(\mathfrak{H}_\Omega, \gamma_c)$ is dense in $(\widehat{\mathfrak{H}}_\Omega, \gamma_c)$, it is enough to prove that $(\mathfrak{H}_\Omega, \gamma_c)$ is totally bounded. We shall prove slightly less but it will be good enough for our purposes:

Proposition 5.3. *Let $\hat{\mu}_\Omega$ be the push forward to $\widehat{\mathfrak{H}}_\Omega$ of the measure μ on Ω. Then the support of $\hat{\mu}_\Omega$ is totally bounded hence compact.*

This will follow from the following general result.

Proposition 5.4. *Let (X, μ) be a probability space endowed with a distance d such that (X, d) is separable.*[16] *Let G be a group acting ergodically on X by (measure-preserving) isometries. Then* $\mathrm{supp}(\mu)$ *is totally bounded.*

We shall first prove:

Lemma 5.5. *Let τ be a continuous ergodic action of a group G on a probability, separable metric space (X, μ, d). Then for μ-almost all points $x \in X$, the orbit $G \cdot x$ is dense in* $\mathrm{supp}(\mu)$.

Proof. This is an immediate consequence of Birkhoff's ergodic theorem, but we shall give a simpler (or at least easier) proof. Let Y be countable and dense in X and set

$$W = \bigcup_{\substack{y \in Y, r \in \mathbb{Q}_+^* \\ \mu(B(y,r))=0}} B(y, r).$$

If $\mu(B(x, r)) = 0$ for some $x \in X$, $r > 0$ then $x \in W$. Indeed, we may assume r is rational, and choose $y \in Y$ such that $d(y, x) < \frac{r}{2}$. Then $x \in B\left(y, \frac{r}{2}\right)$ so $B\left(y, \frac{r}{2}\right) \subset B(x, r)$ and we get $\mu\left(B\left(y, \frac{r}{2}\right)\right) = 0$. This argument implies that

$$W = \{x \in X \mid \exists U \text{ open } x \in U, \mu(U) = 0\}$$

and W is τ invariant since τ preserves μ and the open sets. Now because W is a countable union of open sets of measure 0, it is open and has measure 0. We may then replace X by $X \setminus W$, so we are reduced to the situation where all balls have > 0 measure, i.e., all open sets have positive measure.

[16] A separable topological space is a space having a countable dense subset.

Now let $(U_j)_{j\in\mathbb{N}}$ be a countable basis of open sets (since a separable metric space is second countable). Set $\tau_G A = \{\tau_g x \mid x \in A\}$; then the orbit of x misses U_j if and only if $\tau_G x \cap U_j = \varnothing$, i.e., $x \notin \tau_G(U_j)$. The points with nondense orbit must miss at least one $\tau_G(U_j)$ so they belong to

$$\bigcup_j (X \setminus \tau_G(U_j) = X \setminus \bigcap_j \tau_G(U_j),$$

but by ergodicity $\tau_G(U_j)$ being τ invariant has measure 1 (since it cannot be zero, as its measure is at least the measure of U_j that is positive by assumption). Therefore $\bigcap_j \tau_G(U_j)$ as a countable intersection of measure-1 sets has measure 1, and its complement has measure zero. □

We are now in a position to prove Proposition 5.4.

Proof of Proposition 5.4. By the lemma we may choose x such that $\tau_G x$ is a dense orbit in $\operatorname{supp}(\mu)$. We shall prove that $\tau_G(x)$ is totally bounded, arguing by contradiction.

Let $a_1, \dots, a_k, \dots \in G$ be a sequence in G such that:

- $\bigcup_{j=1}^k \bar{B}(\tau_{a_j} x, \varepsilon)$ does not cover $\tau_G x$, where $\bar{B}(x, r)$ is the closed ball of radius r.
- For all $i \neq j$ we have $B\bigl(\tau_{a_i} x, \frac{\varepsilon}{2}\bigr) \cap B\bigl(\tau_{a_j} x, \frac{\varepsilon}{2}\bigr) = \varnothing$.

We claim that if $\tau_G x$ cannot be covered by finitely many balls of size ε then we may construct such a sequence by induction. Indeed, assume $a_1, \dots, a_k$ have been constructed satisfying the above properties. Then by the first property we may find a_{k+1} such that $\tau_{a_{k+1}} x \notin \bigcup_{j=1}^k B(\tau_{a_j} x, \varepsilon)$ and this implies $B\bigl(\tau_{a_j} x, \frac{\varepsilon}{2}\bigr) \cap B\bigl(\tau_{a_{k+1}} x, \frac{\varepsilon}{2}\bigr) = \varnothing$. Hence $a_1, \dots, a_{k+1}$ satisfy both properties. But now we found infinitely many disjoint balls of radius $\frac{\varepsilon}{2}$ in $\tau_G x$. Since $\tau_{a_j} x \in \operatorname{supp}(\mu)$, we have $\mu(B\bigl(\tau_{a_j} x, \frac{\varepsilon}{2}\bigr)) > 0$ and since all the balls $B\bigl(\tau_a x, \frac{\varepsilon}{2}\bigr)$ are isometric, they have the same measure. But we cannot have infinitely many disjoint balls with the same positive measure, since the total measure of our space is 1. □

We may now conclude with:

Proof of Proposition 5.3. Here $G = \mathbb{R}^n$ and τ induces a measure-preserving ergodic action on $(\mathfrak{H}_\Omega, \gamma, \hat{\mu}_\Omega)$. This action is by isometries according to Proposition 5.1, so according to Proposition 5.3 the support of $\hat{\mu}_\Omega$ is totally bounded. □

Remark 5.6. As we pointed out already in [Viterbo 2023], there are not so many nontrivial examples of compact subset in $(\widehat{\mathfrak{Ham}}_{\mathrm{fc}}(T^*\mathbb{R}^n), \gamma)$ or $(\widehat{\mathfrak{DHam}}_{\mathrm{fc}}(T^*\mathbb{R}^n), \gamma)$, that is, sets that are not already compact for the C^0-topology (since γ is continuous for the C^0 topology on $\mathfrak{Ham}(T^*N)$ according to [Viterbo 1992]) and in $\mathfrak{DHam}(T^*N)$ according to [Seyfaddini 2012]). In [Viterbo 2023] we proved that in T^*T^n the sequence $(H_k)_{k\geq 1}$, where $H_k(q, p) = H(k \cdot q, p)$, is converging. Here we extend this to certain families of Hamiltonians on $T^*\mathbb{R}^n$.

We thus proved that $\mathbb{A}_\Omega$, the closure of $\mathbb{R}^n$ in $\operatorname{Isom}(\widehat{\mathfrak{H}}_\Omega, \gamma)$, is a compact, connected, metric abelian group.

We are thus in the following situation: we have an action — again denoted by τ — of the group $\mathbb{A}_\Omega$ acting by γ-isometries on the space $\widehat{\mathfrak{H}}_\Omega$ and preserving $\hat{\mu}_\Omega$. By compactness of $\mathbb{A}_\Omega$, we have that $\mathbb{A}_\Omega \cdot H$

is closed for all $H \in \widehat{\mathfrak{H}}_\Omega$. But since by Lemma 5.5 for almost all H, $\tau_{\mathbb{R}^n} H$ is dense, we conclude that for almost all H we have $\mathbb{A}_\Omega \cdot H = \widehat{\mathfrak{H}}_\Omega$.

Thus $\widehat{\mathfrak{H}}_\Omega \simeq \mathbb{A}_\Omega/\mathbb{K}_\Omega$, but $\mathbb{A}_\Omega/\mathbb{K}_\Omega$ is again a compact metric abelian group. Moreover the measure $\hat{\mu}_\Omega$ on $\widehat{\mathfrak{H}}_\Omega$ induces a measure on $\mathbb{A}_\Omega/\mathbb{K}_\Omega$, invariant by the action. It is therefore the Haar measure. To conclude, and writing from now on $\mathbb{A}_\Omega$ for $\mathbb{A}_\Omega/\mathbb{K}_\Omega$, we are reduced to the situation where:

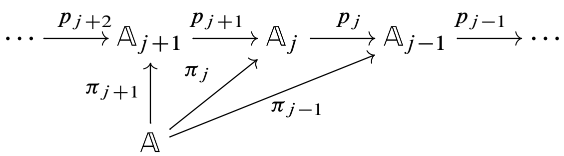
(1) $\Omega = \mathbb{A}_\Omega$.

(2) $\omega \to H_\omega \in \widehat{\mathfrak{Ham}}_{\mathrm{BP}}(T^*T^n)$ is continuous for the γ-topology.

(3) On the subgroup $\mathbb{R}^n$ in $\mathbb{A}_\Omega$ the action of $\mathbb{R}^n$ on Ω can be identified with the action by translation of $\mathbb{R}^n$ as a dense subgroup of $\mathbb{A}_\Omega$. The invariant measure on $\mathbb{A}_\Omega$ is the Haar measure and the action of $\mathbb{R}^n$ on $\mathbb{A}_\Omega$ is ergodic.

6. Some results on compact abelian metric groups

Let $\mathbb{A}$ be a compact metric abelian group having $\mathbb{R}^n$ as a dense subgroup (in particular $\mathbb{A}$ is connected). According to A. Weil [1965, p. 110] (see also [Hofmann and Morris 2013, Theorem 8.45]) $\mathbb{A}$ is the projective limit of finite-dimensional tori. In other words there are tori T^{n_j} and group morphisms $f_{j,i} : T^{n_j} \to T^{n_i}$ for $i < j$ integers such that $f_{k,j} \circ f_{j,i} = f_{k,i}$ and a map $f_{\infty,i} : \mathbb{A} \to T^{n_i}$ such that $\mathbb{A} = \varprojlim_j T^{n_j}$. We denote by $\mathbb{A}_j$ the image of $\mathbb{A}$ in T^{n_j}, which is clearly a connected compact subgroup of T^{n_j} and hence a subtorus, and we may replace T^{n_j} by $\mathbb{A}_j$. Setting by $p_j = f_{j+1,j}$ and $\pi_j = f_{\infty,j}$, we have the following sequence:



We set $\mathbb{K}_j = \mathrm{Ker}(\pi_j)$. We then have:

Lemma 6.1. *We have*

$$\lim_j \mathrm{diam}(\mathbb{K}_j) = 0.$$

Proof. The $\mathbb{K}_j$ are a decreasing sequence of closed — hence compact — subgroups such that $\bigcap_j \mathbb{K}_j = \{0\}$ by the definition of the projective limit. But this implies the lemma by an easy exercise (or [Rudin 1976, Theorem 3.10]). □

Now we need:

Lemma 6.2. *Let us consider the embeddings*

$$\pi_j^* : C^0(\mathbb{A}_j, \mathbb{R}) \to C^0(\mathbb{A}, \mathbb{R}), \quad f \mapsto f \circ \pi_j.$$

Then the union of the images of the π_j^* is dense in $C^0(\mathbb{A}, \mathbb{R})$.

Proof. Let $f \in C^0(\mathbb{A}, \mathbb{R})$. Then f is uniformly continuous by the Heine–Cantor theorem (see [Rudin 1976, Theorem 4.19]):

$$\forall \varepsilon > 0,\ \exists \eta > 0,\ \forall x, y \in \mathbb{A}, \quad d(x, y) < \eta \implies \delta(f(x), f(y)) < \varepsilon.$$

For j large enough we have $\operatorname{diam}(\mathbb{K}_j) < \eta$, so setting $f_j(x) = \min\{f(x+u) \mid u \in \mathbb{K}_j\}$, we see that by the compactness of $\mathbb{K}_j$ the function f_j is well-defined and continuous. Moreover $d(f(x), f_j(x)) < \varepsilon$ provided $\operatorname{diam}(\mathbb{K}_j) < \eta$. ☐

Now remember that we have a group morphism $\tau : \mathbb{R}^n \to \mathbb{A}$ with dense image. By the definition of a projective limit, the map τ is defined by a sequence of maps $\tau_j : \mathbb{R}^n \to \mathbb{A}_j$ such that $p_j \circ \tau_j = \tau_{j-1}$. Of course the density of $\tau(\mathbb{R}^n)$ implies the density of $\tau_j(\mathbb{R}^n)$ because the preimage by π_j of a proper closed subset is a proper closed subset (remember π_j is onto by assumption). Since the density of the image of τ is equivalent to the ergodicity of the action, we may conclude that τ is ergodic on $\mathbb{A}_j$.

We are now in the following situation: we have a subgroup $\mathbb{A}_\Omega$ in $\operatorname{Isom}(\widehat{\mathfrak{H}}_\omega, \gamma)$ and for almost every H (for the measure $\hat{\mu}_\Omega$) we have $\mathbb{A}_\Omega \cdot H = \widehat{\mathfrak{H}}_\omega$. Now $\mathbb{A}_\Omega \cdot H$ is approximated by $\mathbb{A}_j \cdot H$ for a finite-dimensional torus $\mathbb{A}_j$, and the action of $\mathbb{R}^n$ by τ yields a dense subgroup of $\mathbb{A}_j$. At the cost of an approximation, we have thus replaced H_ω for $\omega \in \mathbb{A}_\omega$ by the H_ω for $\omega \in \mathbb{A}_j$, that is, we have a continuous map $\mathbb{A}_j \to (\widehat{\mathfrak{Ham}}_{\mathrm{fc}}, \gamma)$ and $\mathbb{A}_j$ is a finite-dimensional torus.

7. Regularization of the Hamiltonians in $\widehat{\mathfrak{Ham}}_{\mathrm{fc}}$

Let $H \in \widehat{\mathfrak{Ham}}_{\mathrm{FP}}(T^*\mathbb{R}^n)$ and φ_H^t be its flow in $\widehat{\mathfrak{DHam}}_{\mathrm{FP}}(T^*\mathbb{R}^n)$. Let $S(q, p; \xi)$ be a GFQI for $\Gamma(\varphi_H)$, set $S_{(q,p)}(\xi) = S(q, p; \xi)$, and let $c(1_{(q,p)}, S) := c(1_{(q,p)}, S_{(q,p)})$ be the critical value corresponding to the unique cohomology class $1_{(q,p)} \in H^0(\{(q, p)\})$. The map $\varphi \mapsto c(1_{(q,p)}, \varphi)$ obviously extends to $\widehat{\mathfrak{DHam}}_{\mathrm{FP}}(T^*\mathbb{R}^n)$. We now set:

Definition 7.1. For $\eta > 0$ we set

$$H^\eta(q, p) = \frac{1}{\eta} c(1_{(q,p)}, \varphi_H^\eta) = \frac{1}{\eta} c(1_{(q,p)}, \Gamma(\varphi_H^\eta)).$$

This defines a map

$$\sigma_\eta : \widehat{\mathfrak{Ham}}_{\mathrm{fc}}(T^*\mathbb{R}^n) \to C^{0,1}_{\mathrm{fc}}(T^*\mathbb{R}^n),$$

where $C^{0,1}_{\mathrm{fc}}(T^*\mathbb{R}^n)$ is the set of Lipschitz functions with fiberwise compact support.

Our goal is to prove that σ_η is a regularizing operator. This is the content of:

Proposition 7.2. *We have for $H \in \widehat{\mathfrak{Ham}}_{\mathrm{fc}}(T^*\mathbb{R}^n)$:*

(1) $\gamma_c - \lim_{\eta \to 0} \sigma_\eta(H) = H$.

(2) *For each R there exists a constant C such that for H supported in $\mathbb{R}^n \times B(R)$ and such that $\varphi_H(T^*B(\rho)) \subset T^*B(\rho + r)$ we have $\sigma_\eta(H)$ is $\frac{C(R+r)}{\eta}$-Lipschitz.*

(3) $\sigma_\eta : \widehat{\mathfrak{Ham}}_{\mathrm{fc}}(T^*\mathbb{R}^n) \to C^0_{\mathrm{fc}}(T^*\mathbb{R}^n, \mathbb{R})$ *is continuous for the γ-topology.*

(4) $\sigma_\eta \circ \tau_a = \tau_a \circ \sigma_\eta$.

Remark 7.3. One should be careful: the γ_c-limit in (1) is of course not a C^0 limit, since H is not continuous in general — it is not even a function! But even if H is continuous, we do not claim this.

We need the following lemma, which we shall prove in Appendix C.

Lemma 7.4. *For η small enough we can find a GFQI for φ_K^η, $S_{K,\eta}$, such that*

$$\|S_{K,\eta}(q,p) - \eta K(q,p)\| \leq C\eta^2 \|\nabla K\|^2_{C^0}.$$

Proof of Proposition 7.2.

(1) By density we can find $K \in C^\infty_{\text{fc}}(T^*\mathbb{R}^n, \mathbb{R})$ such that $\gamma(H,K) \leq \varepsilon$. Now for $K \in C^\infty_{\text{fc}}(T^*\mathbb{R}^n, \mathbb{R})$ we may find a GFQI, $S_{K,\eta}$ of φ_K^η such that

$$S_{K,\eta}(q,p) = \eta \cdot K(q,p) + o(\eta)$$

as η goes to zero so that $K^\eta(q,p) = \frac{1}{\eta} c(1_{(q,p)}, S_{K,\eta}) = K(q,p) + o(1)$.

Now the formula $c(1_{(q,p)}, S_{K,\eta}) = \eta K(q,p) + o(\eta)$ follows immediately from the lemma by applying on one hand the triangle inequality (see [Viterbo 1992, Proposition 3.3, p. 693])

$$|c(1_x, L) - c(1_x, L')| \leq \gamma(L, L')$$

and on the other hand Proposition 4.20,

$$\|K^\eta(q,p) - K(q,p)\| \leq \eta \cdot \|\nabla K\|^2_{C^0}.$$

Now for η small enough we have $\gamma(K^\eta, K) \leq \varepsilon$. Remember from Definitions 4.1 that for $H, K \in \widehat{\mathfrak{Ham}}(T^*\mathbb{R}^n)$, $H \preceq K$ means $c(1_W, \varphi_K, \varphi_H) = 0$ for all W. The reduction inequality [Viterbo 1992, Proposition 5.1, p. 705] implies that $H^\eta(q,p) \leq K^\eta(q,p)$ for all $(q,p) \in T^*\mathbb{R}^n$.

Let $\zeta_R(p)$ be a function such that $0 \leq \eta_R(p) \leq 1$, vanishing for $|p| \leq R$ and equal to 1 for $|p|2 \geq R$. Now $\gamma(H,K) \leq \varepsilon$ implies that $K - \varepsilon\zeta_R \preceq H \preceq K + \varepsilon\zeta_R$ for R large enough: this follows from the formula $c(1_W, \varphi_{K+\varepsilon\zeta_R}, \varphi_H) = c(1_W, \varphi_K, \varphi_H) + \varepsilon$ for W large enough because if S is a GFQI for φ_K then $S_\varepsilon(q,p;\xi) = S_0(q,p;\xi) + \varepsilon\zeta_R(p)$ is a GFQI for $\varphi_{K+\varepsilon\zeta_R} = \varphi_K \circ \varphi_{\eta_R}$ and $c(1_W, S_\varepsilon) = c(1_W, S_0) + \varepsilon$ for R and W large enough.

Now we have $K^\eta - \varepsilon\zeta_R \preceq H^\eta \preceq K^\eta + \varepsilon\zeta_R$ and for η small enough we get $\|K - K^\eta\| \leq \varepsilon$ so

$$K - 2\varepsilon \preceq H^\eta \preceq K + 2\varepsilon.$$

Thus

$$H - 3\varepsilon \preceq K - 2\varepsilon \preceq H^\eta \preceq K + 2\varepsilon \preceq H + 3\varepsilon;$$

hence $\gamma(H^\eta, H) \leq 3\varepsilon$.

(2) We have for $|q_1 - q_2| + |p_1 - p_2| \leq r$

$$c(1_{(q_1,p_1)}\varphi_H^\eta) - c(1_{(q_2,p_2)}\varphi_H^\eta) \leq C(r)$$

because for $L_{(q,p)}$ Hamiltonianly isotopic to the vertical and coinciding with $T^*_{(q,p)}\Delta_{\mathbb{R}^{2n}}$ in $\Delta_{\mathbb{R}^{2n}} \times B_r^{2n}$ we have

$$c(1_{(q,p)}, \Gamma(\varphi_H^\eta)) = c(\Gamma(\varphi_H^\eta), L_{(q,p)})$$

and

$$|c(\Gamma(\varphi_H^\eta), L) - c(\Gamma(\varphi_H^\eta), \psi(L))| \le \gamma(L, \psi(L)) \le \gamma(\psi).$$

As a result, there is a Hamiltonian map ψ with $\gamma(\psi) \le C(r)$ such that

$$\psi(T^*_{(q_1,p_1)}\Delta_{\mathbb{R}^{2n}}) \cap (\Delta_{\mathbb{R}^{2n}} \times B_\rho^{2n}) = T^*_{(q_2,p_2)}\Delta_{\mathbb{R}^{2n}} \cap (\Delta_{\mathbb{R}^{2n}} \times B_\rho^{2n}),$$

where ρ is such that $\Gamma(\varphi_H^\eta) \subset \mathbb{R}^{2n} \times B_\rho^{2n}$. Since we assumed that H is supported in B_R we may assume $\rho = 2R$ and we have $C(r) = CR \cdot r$. Indeed if ψ_t is an isotopy such that ψ_1 sends (q_1, p_1) to (q_2, p_2), and Ψ_t its natural extension to a Hamiltonian isotopy $T^*(\Delta_{T^*\mathbb{R}})$, we truncate the Hamiltonian generating Ψ_t to $\mathbb{R}^{2n} \times B_\rho^{2n}$, where ρ is an upper bound for $|Q_H(q, p) - q| + |P_H(q, p) - p|$. Such an upper bound is given by $r + 2R$ (r for $|Q - q|$ and $2R$ for $|P - p|$). This proves the inequality.[17]

(3) We have

$$\|\sigma_\eta(H) - \sigma_\eta(K)\|_{C^0} \le \frac{1}{\eta} \sup_{(q,p)} c(1_{(q,p)}, \varphi^\eta(\psi^\eta)^{-1}) \le \frac{1}{\eta}\gamma(\varphi_H^\eta, \varphi_K^\eta) \le \frac{1}{\eta}\gamma(H, K),$$

where the first inequality is just the triangle inequality (see [Viterbo 1992, Proposition 3.3, p. 693]) and the second inequality follows by the reduction inequality in [loc. cit., Proposition 5.1, p. 705].

(4) We have $\sigma_\eta(H_\omega)(x + a, p) = \frac{1}{\eta} c(1_{x+a,p}, \varphi^\eta_{H_\omega}) = c(1, S^\omega(x + a, P; \xi))$ but $S^\omega(x + a, P; \xi)$ is the generating function corresponding to $\tau_a H_\omega$, i.e., $\Gamma(\tau_{-a}\varphi^\eta_{H_\omega}\tau_a)$ is the set of $(q + a, P, P - p, Q - q)$, where $\varphi^\eta_{H_\omega}(q, p) = (Q, P)$. So we have $\Gamma(\tau_{-a}\varphi^\eta_{H_\omega}\tau_a) = \tau_a(\Gamma(\varphi^\eta_{H_\omega}))$ and

$$S_{H_{\tau_{-a}\omega}}(x, P, \tau_{-a}\xi) = S_{\tau_a H_\omega}(x, P; \xi) = S_{H_\omega}(x + a, P; \xi).$$

We thus proved that

$$\tau_a\sigma_\eta(H_\omega)(x, p) = \sigma_\eta(H_\omega)(x + a, p) = \sigma_\eta(\tau_a H_\omega)(x, p) = \sigma_\eta(H_{\tau_{-a}\omega})(x, p) = \sigma_\eta(\tau_a H_\omega)(x, p). \quad \square$$

We are now in the following situation: we started from a continuous map

$$H : \mathbb{A}_j \to (\widehat{\mathfrak{Ham}}(T^*\mathbb{R}^n), \gamma)$$

and have constructed a map

$$H^\eta : \mathbb{A}_j \to (C^0_{\text{fc}}(T^*\mathbb{R}^n), d_{C_0})$$

which is continuous and satisfies $\tau_a H^\eta = H^\eta$. Note that we may replace if needed C^0_{fc} by C^k_{fc} by applying convolution since $\tau_a(H \star \chi) = (\tau_a H) \star \chi = H \star \chi$ (and of course, since $\|H \star \chi - H\| \to 0$ as $\chi \to \delta_0$, we also have γ_c-convergence).

Let us summarize our findings combining the results of Proposition 7.2 and the conclusions of Sections 5 and 6:

Corollary 7.5. *Let $H : T^*\mathbb{R}^n \times \Omega \to \mathbb{R}$ satisfy assumptions (1)–(6) of the Main Theorem. Define $\pi_d : \Omega \to \mathbb{A}_d = T^d$ be the projection defined in Section 6. Then, given $\varepsilon > 0$, there exist $d \in \mathbb{N}$ and $H^\varepsilon : T^*\mathbb{R}^n \times T^d \to \mathbb{R}$ such that:*

[17] We also can take $R \simeq \eta\|H\|_{C^{0,1}}$, and then $C(r) \simeq Cr\eta\|H\|_{C^{0,1}}$ but this requires H to be Lipschitz. But this proves that the map σ_η does increase the Lipschitz norm by a bounded multiplicative constant only.

(1) *$\omega \mapsto H^\varepsilon_\omega$ is continuous from T^d to $C^\infty_{\mathrm{fc}}(T^*\mathbb{R}^n, \mathbb{R})$.*

(2) *$\gamma(H_\omega, H^\varepsilon_{\pi_d(\omega)}) \le \varepsilon$ for all $\omega \in \Omega$.*

(3) *The Hamiltonians $H^\varepsilon_\omega, H^\varepsilon_{\pi_d(\omega)}$ satisfy assumptions (1)–(6).*

Proof. From Section 5 we get H from $\mathbb{A}_\Omega$ to $\widehat{\mathfrak{Ham}_c}(T^*\mathbb{R}^n)$. From Section 6 we can approximate H by a map from T^d to $\widehat{\mathfrak{Ham}_c}(T^*\mathbb{R}^n)$ and from the present section, we have an approximating map to $C^\infty_{\mathrm{fc}}(T^*\mathbb{R}^n, \mathbb{R})$. □

8. Homogenization in the almost periodic case

We assume in this section that we have a map $(q, p; \omega) \mapsto H(q, p; \omega) = H_\omega(q, p)$ such that:

(1) $\omega \in \Omega = T^d$.

(2) The map $\omega \mapsto H_\omega$ is continuous for the C^∞_{fc} topology. In particular the H_ω have uniformly fiberwise compact support and the H_ω are uniformly BPS by Proposition 3.3.

We set φ^t_ω to be the time t flow for H_ω and $\varphi_{\varepsilon,\omega} = \rho_\varepsilon^{-1} \varphi_\omega^{1/\varepsilon} \rho_\varepsilon$. By the compactness of Ω we also have a map $\omega \mapsto S_\omega(q, p; \xi)$ of GFQI for $\varphi_\omega = \varphi^1_\omega$, with ξ living in a vector space independent from ω: indeed its dimension is bounded by $2nN$ such that $\varphi_\omega^{1/N}$ is in a given neighborhood of id for all $\omega \in \Omega$ (see Appendix A for the number of fiber variables needed for a GFQI).

As we are going to use a number of results from [Viterbo 2023]. We will assume in the sequel that $\varepsilon = \frac{1}{k}$ and write ρ_k for $\rho_{1/k}$, h_k for $h_{1/k}$ and so on.

Definition 8.1. We set

$$h^\omega_{k,U}(p) = \lim_{V \ni p} c(\mu_{U \times V}, \varphi_{k,\omega})$$

and

$$h^\omega_k = \lim_{U \in \mathbb{R}^n} h^\omega_{k,U}.$$

Proposition 8.2. *The sequence h^ω_k is equicontinuous and equibounded. All its converging subsequences have the same limit $h_\omega(p)$, which is in fact independent from ω and denoted by $\overline{H}(p)$. We denote by $\varphi^t_{\overline{H}}$ the flow of $\overline{H}$ in $\widehat{\mathfrak{DHam}_{\mathrm{fc}}}(T^*\mathbb{R}^n)$ which belongs to $\widehat{\mathfrak{DHam}_{\mathrm{FP}}}(T^*\mathbb{R}^n)$.*

Proof. Let us start to examine what happens for fixed ω. For typographical reasons, the ω parameter will be omitted in the notation, but of course, everything depends on $\omega \in \Omega$, and the ω subscript will be reinstated when we prove that h_ω does not depend on ω.

Set $\varphi_k(q, p) = (Q_k(q, p), P_k(q, p))$ and $Q = Q_1$, $P = P_1$. By the definition of S_k we have

$$\frac{\partial S_k}{\partial \xi}(q, P_k(q, p); \xi) = 0 \quad \text{and} \quad \frac{\partial S_k}{\partial p}(q, P_k(q, p); \xi) = Q_k(q, p) - q.$$

By assumption we have

$$h_{k,U}(p) = S_k(q(p), p; \xi(p)),$$

where $(q(p), p; \xi(p))$ satisfies
$$\frac{\partial S_k}{\partial \xi}(q(p), p; \xi) = 0$$
and
$$\frac{\partial S_k}{\partial q}(q(p), p; \xi) = \begin{cases} 0 & \text{if } q \in U, \\ \lambda \cdot \nu_U(q) & \text{if } q \in \partial U \text{ and } \nu_U(q) \text{ is the exterior normal.} \end{cases}$$
Now as p varies, we can choose $p \mapsto (q(p), \xi(p))$ to be piecewise smooth, so that for p in the smooth locus
$$dh_{k,U}(p) = \frac{\partial S_k}{\partial p}(q(p), p; \xi(p)) + \frac{\partial S_k}{\partial q}(q(p), p; \xi(p)) \cdot \frac{\partial q}{\partial p} + \frac{\partial S_k}{\partial \xi}(q(p), p; \xi(p)) \cdot \frac{\partial \xi}{\partial p}.$$
Then we have
$$\frac{\partial S_k}{\partial \xi}(q, P_k(q, p); \xi) = 0 \quad \text{and} \quad \frac{\partial S_k}{\partial p}(q, P_k(q, p); \xi) = Q_k(q, p) - q.$$
But
$$h_{k,U}(p) = S_k(q(p), p; \xi(p)),$$
where
$$\frac{\partial S_k}{\partial \xi}(q(p), p; \xi) = 0$$
and
$$\frac{\partial S_k}{\partial q}(q(p), p; \xi) = \begin{cases} 0 & \text{if } q \in U, \\ \lambda \cdot \nu_U(q) & \text{if } q \in \partial U \text{ and } \nu_U(q) \text{ is the exterior normal.} \end{cases}$$
But if $q \in \partial U$, then $\frac{\partial q}{\partial p} \in T(\partial U)$, so that the term $\frac{\partial S_k}{\partial q}(q(p), p; \xi(p)) \cdot \frac{\partial q}{\partial p}$ also vanishes. We thus proved that where $h_{k,U}$ is smooth, we have
$$dh_k(p) = \frac{\partial S_k}{\partial p}(q(p), p; \xi(p)) = Q_k(q(p), p) - q(p) = \frac{1}{k}(Q(kq, p) - kq).$$
The assumption of finite propagation speed implies that this last quantity is uniformly bounded, so $|dh_{k,U}(p)|$ is uniformly bounded (independently from k, U).

From this we conclude that the sequence h_k is equicontinuous. Equiboundedness follows from Definition 4.8 in [Viterbo 2023] (or Proposition 9.1 of the current paper), which states that a GFQI S_k of φ_k is given by
$$S_k(q, p; \zeta) = \frac{1}{k}\left[S(kq, p_1) + \sum_{j=2}^{k-1} S(kq_j, p_j) + S((kq_k, p)\right] + B_k(q, p; \zeta),$$
where $S(q, p; \zeta) = S_1(q, p; \zeta)$ is a GFQI for $\varphi = \varphi_1$, $\zeta = (p_1, q_2, \ldots, p_{k-1}, q_k)$ and B_k is a nondegenerate quadratic form. As a result $|S_k - B_k| \le C$, where C is a bound for $|S(q, p; \zeta) - B_1(q, p; \zeta)|$.

This implies that $|h_k(p)| \le C$ and since all these estimates are uniform in ω, this implies (uniform) equiboundedness.

We may thus apply the Arzelà–Ascoli theorem, and conclude that h_k^ω has a converging subsequence. Proving that the limit is unique follows as in [Viterbo 2023, Lemma 4.11 and Proposition 4.12].

Finally we prove that $h_\omega(p)$ is independent from ω, using the commutation of τ_a and ρ_k. We have

$$\begin{aligned} h_{k,\tau_a\omega}(p) &= \lim_{U\subset\mathbb{R}^n} c(\mu_U\otimes 1_p, \Gamma(\varphi_{k,\tau_a\omega})) \\ &= \lim_{U\subset\mathbb{R}^n} c(\mu_U\otimes 1_p, \Gamma(\tau_a^{-1}\varphi_{k,\omega}\tau_a)) \\ &= \lim_{U\subset\mathbb{R}^n} c(\mu_{\tau_a U}\otimes 1_p, \Gamma(\varphi_{k,\omega})) = h_{k,\omega}(p). \end{aligned}$$

Since $\omega\mapsto\varphi_{k,\omega}$ is γ-continuous, we infer that $\omega\mapsto h_{k,\omega}(p)$ is continuous and we just proved that it is τ-invariant. Ergodicity then implies that it is constant in ω. □

We define

$$\widehat{\mathfrak{Ham}}_{fc,BP}(T^*\mathbb{R}^n) = \widehat{\mathfrak{Ham}}_{\mathrm{BP}}(T^*\mathbb{R}^n)\cap\widehat{\mathfrak{Ham}}_{\mathrm{fc}}(T^*\mathbb{R}^n).$$

From now on we write $\bar{\varphi}^t$ instead of $\varphi^t_{\overline{H}}$ for typographical reasons.

The next proposition is the analog of Proposition 4.15 in [Viterbo 2023].

Proposition 8.3. *Let $\alpha\in\widehat{\mathfrak{DHam}}_{fc,BP}(T^*\mathbb{R}^n)$. There exists a sequence k_ν such that*

$$\lim_{\nu\to+\infty}\lim_{U\subset\mathbb{R}^n} c(\mu_U, \varphi_{k_\nu,\omega}\alpha) \le \lim_{U\subset\mathbb{R}^n} c(\mu_U, \bar{\varphi}\alpha).$$

Proof. The proof is identical to the proof of Proposition 4.15 in Section 4 of [Viterbo 2023] and can be found in Appendix D. □

The next proposition is the analog of Proposition 6.2 in [Viterbo 2023], but requires an adaptation. It will be proved in Section 9.

Proposition 8.4. *For each $\varepsilon>0$ there exists K such that, for all $k\ge K$ and U large enough, we have*

$$c(\mu_U\otimes 1_p, \varphi_{k,\omega}) \le c(1_U\otimes 1_p, \varphi_{k,\omega}) + \varepsilon.$$

This implies:

Corollary 8.5. *We have $\overline{\varphi^{-1}} = (\bar{\varphi})^{-1}$, or equivalently $\overline{H}_{\varphi^{-1}} = -\overline{H}_\varphi$.*

Now putting together Proposition 8.3 and Corollary 8.5 we get:

Proposition 8.6. *For almost all $\omega\in\Omega$, the sequence $\varphi_{k,\omega}$ γ_∞-converges to $\bar{\varphi}$.*

Proof assuming Corollary 8.5 and Proposition 8.3. Let us prove the above proposition as a consequence of Corollary 8.5 and Proposition 8.3. Indeed Proposition 8.3 implies

$$\lim_{k\to+\infty}\lim_U c(\mu_U, \varphi_{k,\omega}\bar{\varphi}^{-1}) \le \lim_U c(\mu_U, \mathrm{id}) = 0.$$

Applying the same inequality for φ^{-1} instead of φ and using the corollary, we get

$$\lim_{k\to+\infty}\lim_U c(\mu_U, \varphi_{k,\omega}^{-1}\bar{\varphi}) \le \lim_U c(\mu_U, \mathrm{id}) = 0$$

and this implies

$$\lim_{k\to+\infty}\lim_U \gamma(\mu_U, \varphi_{k,\omega}^{-1}\bar{\varphi}) = 0,$$

which proves our claim. □

Proof of Corollary 8.5 assuming Proposition 8.4. Set

$$h^+_{k,\omega}(\varphi;p)=\lim_{U\subset\mathbb{R}^n} c(\mu_U\otimes 1(p),\varphi_{k,\omega}),$$
$$h^-_{k,\omega}(\varphi;p)=\lim_{U\subset\mathbb{R}^n} c(1_U\otimes 1(p),\varphi_{k,\omega})$$

so that $h^-_{k,\omega}(\varphi;p)\le h^+_{k,\omega}(\varphi;p)$. Set $\sigma_{p_0}(q,p)=(q,p+p_0)$. If $S(q,p;\xi)$ is a GFQI for φ, then $S_p(x;\xi)=S(x,p;\xi)$ is a GFQI for $\sigma_p(0_{\mathbb{R}^n})-\varphi(\sigma_p(0_{\mathbb{R}^n}))$. If we assume φ has FPS we have from Proposition 4.16

$$c(\mu_U,\sigma_p(0_{\mathbb{R}^n})-\varphi(\sigma_p(0_{\mathbb{R}^n})))\le c(\mu_V,\sigma_{-p}\varphi^{-1}(\sigma_p(0_{\mathbb{R}^n})))$$

for V such that $\varphi(T^*U)\subset T^*V$. Taking the limit for $U\subset\mathbb{R}^n$ we get

$$\lim_{U\subset\mathbb{R}^n} c(\mu_U,S_p)=\lim_{U\subset\mathbb{R}^n} c(\mu_U,\sigma_{-p}\varphi^{-1}\sigma_p(0_{\mathbb{R}^n}))$$

and the same holds for 1_U instead of μ_U. Now we may write (again omitting the ω) using first Proposition 4.10(1) and then FPS of φ

$$\begin{aligned}h^+_k(\varphi^{-1};p)&=\lim_{U\subset\mathbb{R}^n} c(\mu_U\otimes 1(p),\sigma_{-p}\varphi_k\sigma_p(0_{\mathbb{R}^n}))\\&=-\lim_{U\subset\mathbb{R}^n} c(1_U\otimes 1(p),0_{\mathbb{R}^n}-\sigma_{-p}\varphi_k\sigma_p(0_{\mathbb{R}^n}))\\&\le-\lim_{V\subset\mathbb{R}^n} c(1_V\otimes 1(p),\sigma_{-p}\varphi_k^{-1}\sigma_p(0_{\mathbb{R}^n}))=-h^-_k(\varphi;p).\end{aligned}$$

As a result

$$h^+_k(\varphi^{-1};p)+h^-_k(\varphi;p)\le 0 \tag{a}$$

and as k goes to $+\infty$, Proposition 8.4 implies

$$h^+_k(\varphi^{-1};p)-h^-_k(\varphi;p)\le\varepsilon$$

and we get

$$h^+_k(\varphi^{-1};p)+h^+_k(\varphi;p)\le\varepsilon. \tag{b}$$

On the other hand, we have using again Proposition 4.10(1)

$$-c(1_U,\sigma_{-p}\varphi_k\sigma_p(0_{\mathbb{R}^n}))\le -c(1_V,0_{\mathbb{R}^n},\sigma_{-p}\varphi_k^{-1}\sigma_p(0_{\mathbb{R}^n}))=c(\mu_V,\sigma_{-p}\varphi_k^{-1}\sigma_p(0_{\mathbb{R}^n})),$$

so

$$-h^-_k(\varphi;p)\le h^+_k(\varphi;p),$$

and using (a) we get

$$h^+_k(\varphi;p)+h^-_k(\varphi;p)=0. \tag{c}$$

Using again Proposition 8.4 we get for k large enough

$$h^-_k(\varphi^{-1};p)+h^-_k(\varphi;p)\ge-\varepsilon. \tag{d}$$

Adding (b) and (d) we get

$$[h^+_k(\varphi^{-1};p)-h^-_k(\varphi^{-1};p)]+[h^+_k(\varphi;p)-h^-_k(\varphi;p)]\le 2\varepsilon. \tag{e}$$

Since $\overline{H}_{\varphi^{-1}} = \lim_k h_k^+(\varphi^{-1}; p)$, inequality (b) implies

$$\overline{H}_{\varphi^{-1}} + \overline{H}_{\varphi} \leq 0.$$

Using (d) and (e) we get

$$\overline{H}_{\varphi^{-1}} + \overline{H}_{\varphi} \geq 0$$

so we may conclude

$$\overline{H}_{\varphi^{-1}} + \overline{H}_{\varphi} = 0. \qquad \square$$

9. Proof of Proposition 8.4

We shall interchangeably use the notation $S_\omega(q, p; \xi)$ and $S(q, p; \xi; \omega)$ for the GFQI of φ_ω. We shall make repeated use of the iteration formula (see [Viterbo 2023, Lemma 4.5]), defining the GFQI $S_{k,\omega}$ for $\varphi_{k,\omega}$ in terms of the GFQI S_ω of φ_ω.

Proposition 9.1 (iteration formula). *Let S_ω be a GFQI for φ_ω. Then the following formula defines a GFQI for $\varphi_{k,\omega}$:*

$$S_{k,\omega}(x, y; \zeta, \xi) = \frac{1}{k}\left[S_\omega(kx, p_1; \xi_1) + \sum_{j=2}^{k-1} S_\omega(kq_j, p_j; \xi_j) + S_\omega(kq_k, y; \xi_k)\right] + B_k(x, y; \zeta),$$

where $\zeta = (p_1, q_2, \ldots, p_{k-1}, q_k)$, $q_1 = x$, $p_k = y$, $\xi = (\xi_1, \ldots, \xi_k)$ and

$$B_k(x, y; \zeta) = \langle p_1, q_2 - x\rangle + \sum_{j=2}^{k-1}\langle p_j, q_{j+1} - q_j\rangle + \langle y, x - q_k\rangle.$$

We shall set $F_{k,\omega} = S_{k,\omega} - B_k$.

The action of $\mathbb{R}^n$ is given by

$$\tau_a^{(k)}(x, y; \xi, \zeta; \omega) = \Big(x + \frac{a}{k}, y; \xi; \tau_{a/k}\zeta; \tau_a\omega\Big).$$

Remark 9.2. We will mostly use this formula when $S(q, p, \xi) = S(q, p)$, i.e., we have no fiber variables for S.

Lemma 9.3. *Assume $\omega \mapsto \varphi_\omega$ for $\omega \in \Omega = T^d$ to be continuous. Then we may choose $\omega \mapsto S_\omega(q, p; \xi)$ to be continuous and such that*

$$S(q + a, p; \tau_a\xi; \tau_a\omega) = S(q, p; \xi; \omega).$$

Proof. It is enough to prove this assuming φ_ω is C^1 small, that is, for $\varphi_\omega^{1/N}$ with N large enough, and then use iteration formula. But then the graph of φ_ω is the graph of a generating function with no fiber variable, which obviously depends continuously on ω and satisfies the above formula. $\square$

Now remember that τ_a is given on $\Omega = T^d$ by $\tau_a(\omega) = \omega + A\cdot a$, where $A : \mathbb{R}^n \to \mathbb{R}^d$ is a linear injective map with dense image in T^d. Consider triples α, β, γ, with $\alpha \in H^*(T^d)$, $\beta \in H^*(U)$ or $H^*(U, \partial U)$, $\gamma \in H^*(V)$ or $H^*(V, \partial V)$. We may then define[18] $c(\alpha \otimes \beta \otimes \gamma, S)$, and we have:

[18] Caveat: the cohomology class α corresponds to the last variable, ω!

Lemma 9.4. *We have the inequalities*

$$c(\mu_U \otimes 1(p); S_\omega) \le c(\mu_{T^d} \otimes \mu_U \otimes 1(p); S),$$
$$c(1_{T^d} \otimes \mu_U \otimes 1(p); S) \le c(1_U \otimes 1(p); S_\omega).$$

Proof. This is the reduction inequality (see [Viterbo 1992, Proposition 5.1, p. 705]). □

We now compare spectral invariants of S with those of S^0, where we define $S^0(p;\xi;\omega) = S(0, p;\xi;\omega)$.

Lemma 9.5. *We have*

$$\lim_{U\subset\mathbb{R}^n} c(\mu_{T^d} \otimes \mu_U \otimes 1(p); S) = c(\mu_{T^d} \otimes 1(0) \otimes 1(p); S) = c(\mu_{T^d} \otimes 1(p); S^0),$$
$$\lim_{U\subset\mathbb{R}^n} c(1_{T^d} \otimes \mu_U \otimes 1(p); S) = c(1_{T^d} \otimes 1(0) \otimes 1(p); S) = c(1_{T^d} \otimes 1(p); S^0).$$

Remarks 9.6. (1) The point of replacing S by S^0 is to avoid the complications related to the noncompactness of $x \in \mathbb{R}^n$. Our proofs could be adapted to work directly with S, but proving that the cycles we construct are in the right homology class is slightly more involved.

(2) This is an extension to GFQI of the following obvious identity for continuous functions $f : \mathbb{R}^n \times T^d \to \mathbb{R}$ such that $f(x + a, \tau_a\omega) = f(x, \omega)$: for any $x_0 \in \mathbb{R}^n$ we have

$$\sup_{(x,\omega)\in\mathbb{R}^n\times T^d} f(x,\omega) = \sup_{\omega\in T^d} f(x_0,\omega).$$

Moreover if the action of τ has dense orbits, this is also equal to $\sup_{x\in\mathbb{R}^n} f(x, \omega_0)$ for any $\omega_0 \in \Omega$. The analog of this last statement will be our main result.

Proof. Clearly if $0 \in U$, we have

$$c(\mu_{T^d} \otimes \mu_U \otimes 1(p); S) \ge c(\mu_{T^d} \otimes 1(0) \otimes 1(p); S)$$

and we need to prove the reverse inequality. Let C be a cycle representing $\mu_{T^d} \otimes 1(p) \in H_*((S_p^0)^c, S_p^0)^{-\infty})$ with $c \le c(\alpha \otimes 1(0) \otimes 1(p), S) + \varepsilon$ and set

$$\tilde{C}_U = \{(x, p, \tau_x\xi;\tau_x\omega) \mid (0, p;\xi;\omega) \in C, x \in U\}.$$

Then $\tilde{C}_U \subset S_p^c$ and clearly $[\tilde{C}_U] = \mu_{T^d} \otimes \mu_U \otimes 1(p)$. The above is in fact an abuse of language for $f_*(\mu_U \otimes [C])$, where

$$f : U \times ((S_p^0)^c, (S_p^0)^{-\infty}) \to ((S_p)^c, (S_p)^{-\infty})$$

is defined by $f(x; (0, p, \xi, \omega)) = (x, p, \tau_x\xi, \tau_x\omega)$.

Thus

$$c(\mu_{T^d} \otimes \mu_U \otimes 1(p), S) \le S(\tilde{C}_U) = S^0(C)$$

because $S(x, p, \tau_x\xi, \tau_x\omega) = S(0, p;\xi;\omega)$ and $S^0(C) \le c$.

This implies

$$c(\mu_{T^D} \otimes \mu_U \otimes 1(p); S) \le c(\mu_{T^D} \otimes 1(0) \otimes 1(p); S)$$

and proves the first equality. The second one is the dual of the first one, since $\mu_{T^d} \otimes \mu_U$ is dual to $1_{T^d} \otimes 1(U)$. □

Our Proposition 8.4 then follows from:

Proposition 9.7. *For each $\varepsilon > 0$ there exists K such that for $k \geq K$*

$$c(\mu_{T^d} \otimes 1(p), S_k^0) \leq c(1_{T^d} \otimes 1(p), S_k^0) + \varepsilon.$$

Remark 9.8. The idea behind the proof is that as we homogenize, the difference between the largest and smallest spectral invariants goes to zero. The proof is a Hamiltonian version of the following ancient result [Acerbi and Buttazzo 1983] that states that if we replace a metric g by a rescaled version g_k, so that the distance $d(x, y)$ becomes $d_k(x, y) = \frac{1}{k} d(k \cdot x, k \cdot y)$, then $\lim_{k\to\infty} d_k(x, y) = \bar{d}(x, y)$ is the distance associated to a flat Finsler metric, g_∞. In particular on a 2-torus for each homotopy class α of loops, α, there are two "spectral values" associated to the geodesic problem $l_1(g, \alpha) \leq l_2(g, \alpha)$, where $l_1(g, \alpha)$ is the shortest geodesic in the homotopy class α, while $l_2(g, \alpha)$ is the "second shortest", i.e., given by the Birkhoff minmax procedure:

$$l_2(g, \alpha) = \inf\Big\{c \;\Big|\; \exists \gamma_s \in C_\alpha^\infty(S^1, T^2),\ s \in S^1,\ \int_{S^1} |\dot{\gamma}_s(t)|\, dt \leq c,\ [s \mapsto \gamma_s(0)] \in \beta \neq \alpha\Big\}.$$

One then checks that $\lim_{k\to+\infty} l_1(g_k, \alpha) = \lim_{k\to\infty} l_2(g_k, \alpha) = l_1(g_\infty, \alpha) = l_2(g_\infty, \alpha)$. Our proof is the analog of the proof of the inequality $l_2(g_k, \alpha) \leq l_1(g_\infty, \alpha) + \varepsilon$ for k large enough, which obviously implies $\lim_{k\to+\infty} l_1(g_k, \alpha) = \lim_{k\to\infty} l_2(g_k, \alpha)$.

Proof. The proof will take up the rest of the section. We rewrite the iteration formula

$$S_{k,\omega}(x, y; \zeta; \omega) = \frac{1}{k}\Big[S_\omega(kx, p_1) + \sum_{j=2}^{k-1} S_\omega(kq_j, p_j) + S_\omega(kq_k, y)\Big] + B_k(x, y; \zeta),$$

where $\zeta = (p_1, q_2, \ldots, p_{k-1}, q_k)$, $q_1 = x$, $p_k = y$ and

$$B_k(x, y; \zeta) = \langle p_1, q_2 - x\rangle + \sum_{j=2}^{k-1} \langle p_j, q_{j+1} - q_j\rangle + \langle y, x - q_k\rangle$$

and $F_{k,\omega} = S_{k,\omega} - B_k$. The action of $\mathbb{R}^n$ is given by

$$\tau_a^{(k)}(x, y; \zeta; \omega) = \Big(x + \frac{a}{k}, y; \tau_{a/k}\zeta; \tau_a\omega\Big)$$

and now $S_{k,\omega}$ is $\tau_a^{(k)}$-invariant, i.e.,

$$S_k\Big(x + \frac{a}{k}, y; \tau_{a/k}\zeta; \tau_a\omega\Big) = S(x, y; \zeta; \omega).$$

Let $a \in \mathbb{R}^n$ such that for some $\nu \in \mathbb{Z}^d$ we have $|A \cdot a - \nu| \leq \delta$ (that is, $d_{T^d}(\tau_a(0), 0) \leq \delta$, where d_{T^d} is the distance on the torus). Then for some constant depending on H and provided δ is small enough

$$\forall t \in [0, 1],\ \forall (q, p; \xi; \omega) \in \mathbb{R}^n \times \mathbb{R}^n \times E \times \Omega, \quad |S(kq + ta, p; \xi; \omega) - S(kq, p; \xi; \omega)| \leq C \qquad (\star)$$

and

$$\forall (q, p; \xi; \omega) \in \mathbb{R}^n \times \mathbb{R}^n \times E \times \Omega,$$
$$|S(kq + a, p; \xi; \omega) - S(kq, p; \xi; \omega)| = |S(kq, p; \xi; \tau_{-a}\omega) - S(kq, p; \xi; \omega)| \leq \varepsilon. \qquad (\star\star)$$

Indeed the first inequality holds because

$$|S(q+a,p;\xi;\omega)-S(q,p;\xi;\omega)|=|S(q,p;\xi;\tau_{-a}\omega)-S(q,p;\xi;\omega)|\le\sup_{\omega,\omega'}|S(q,p;\xi;\omega)-S(q,p;\xi;\omega')|.$$

This follows by using the iteration formula. In this case we may assume $|S(q,p;\omega)-S(q,p;\omega')|\le \gamma(\varphi_\omega,\varphi_{\omega'})$. The second inequality follows from the fact that $d_{T^d}(\tau_a\omega,\omega)\le\delta$ and by the continuity of S.

Now let γ be the path in $\mathbb{R}^n$ defined by $\gamma(t)=t\cdot a$ for $0\le t\le 1$. Set $\tilde{\gamma}^{(k)}$ to be the path in $(\mathbb{R}^n)^k$ defined as the concatenation of the k paths

$$\begin{aligned}
&t\mapsto(\gamma(t),0,\dots,0) && \text{for } t\in\Big[0,\frac{1}{k}\Big],\\
&t\mapsto\Big(\gamma\Big(\frac{1}{k}\Big),\gamma\Big(t-\frac{1}{k}\Big),\dots,0\Big) && \text{for } t\in\Big[\frac{1}{k},\frac{2}{k}\Big],\\
&\quad\vdots && \quad\vdots\\
&t\mapsto\Big(\gamma\Big(\frac{1}{k}\Big),\gamma\Big(\frac{1}{k}\Big),\dots,\gamma\Big(\frac{1}{k}\Big),\gamma\Big(t-\frac{k-1}{k}\Big)\Big) && \text{for } t\in\Big[\frac{k-1}{k},1\Big].
\end{aligned}\tag{9-1}$$

The path $\tilde{\gamma}^{(k)}$ connects $\tilde{\gamma}^{(k)}(0)=(0,\dots,0)$ to $\tilde{\gamma}^{(k)}(1)=\big(\frac{a}{k},\frac{a}{k},\dots,\frac{a}{k}\big)$ through the points

$$\tilde{\gamma}^{(k)}\Big(\frac{1}{k}\Big)=\Big(\frac{a}{k},0,\dots,0\Big),\quad \tilde{\gamma}^{(k)}\Big(\frac{2}{k}\Big)=\Big(\frac{a}{k},\frac{a}{k},0,\dots,0\Big),\quad\dots,\quad \tilde{\gamma}^{(k)}\Big(\frac{k-1}{k}\Big)=\Big(\frac{a}{k},\frac{a}{k},\dots,\frac{a}{k},0\Big).$$

We shall omit the superscript k and set $\tilde{\gamma}^{(k)}(t)=\tilde{\gamma}(t)=(\gamma_1(t),\dots,\gamma_k(t))=(\gamma_1(t),\bar{\gamma}(t))$. We then set $\tau_{\bar{\gamma}(t)}\zeta=\tau_{\bar{\gamma}(t)}(p_1,q_2,\dots,p_{k-1},q_k)=(p_1,q_2+\gamma_2(t),\dots,p_{k-1},q_k+\gamma_k(t))$ and $\tau_{\tilde{\gamma}(t)}(x,y;\zeta)=(x+\gamma_1(t),y;\tau_{\bar{\gamma}(t)}\zeta)$. Now from $(\star)$ and $(\star\star)$ and the formula

$$F_k(x,y;\xi;\zeta;\omega)=\frac{1}{k}\Big[S_\omega(kx,p_1)+\sum_{j=2}^{k-1}S_\omega(kq_j,p_j)+S_\omega(kq_k,y)\Big],$$

we infer that on $\big[\frac{l}{k},\frac{l+1}{k}\big]$ for $1\le l\le k$

$$\begin{aligned}
F_k(\tau_{\tilde{\gamma}(t)}(x,y;\zeta;\omega))=F_{k,\omega}(x,y;\zeta;\omega)+\frac{1}{k}\Big[&S(kx+a,p_1;\zeta;\omega)-S(kx,p_1;\omega)\\
&+\sum_{k=2}^{l}(S(kq_j+a,p_j;\omega)-S(kq_j,p_j;\omega))\\
&\quad+S\Big(kq_{l+1}+\Big(t-\frac{l}{k}\Big)a,p_{l+1};\omega\Big)-S(kq_{l+1},p_{l+1};\omega)\Big],
\end{aligned}$$

so we get

$$|F_k(\tau_{\tilde{\gamma}(t)}(x,y;\zeta);\omega)-F_k(x,y;\zeta;\omega)|\le\frac{\varepsilon l}{k}+\frac{C}{k}\le\frac{C}{k}+\varepsilon.\tag{9-2}$$

We now want to estimate the variation of B_k on $\tau_{\tilde{\gamma}(t)}(x,y;\zeta)$ as t varies from 0 to 1. Note that the choice of this path is crucial to our argument: by changing coordinates one at the time, we achieve an increase of S by $O\big(\frac{1}{k}\big)$ instead of $O(1)$.

Lemma 9.9. *We have*

$$|B_k(\tau_{\tilde{\gamma}(t)}(x,y;\zeta))-B_k(x,y;\zeta)|\le(|p_{l+2}-p_{l+1}|+|p_{l+1}-p_l|)\frac{|a|}{k}.$$

Proof. Indeed,

$$B_k(x,y;\zeta)=\langle p_1,q_2-x\rangle+\sum_{j=2}^{k-1}\langle p_j,q_{j+1}-q_j\rangle+\langle y,x-q_k\rangle$$

and replacing $(x,q_2,\dots,q_l)$ by $\left(x+\frac{a}{k},q_2+\frac{a}{k},\dots,q_l+\frac{a}{k}\right)$ then q_{l+1} by $q_{l+1}+\left(t-\frac{l}{k}\right)a$, with $t\in\left[\frac{l}{k},\frac{l+1}{k}\right]$, and leaving $q_{l+2},\dots,q_k$ unchanged, we get for $t\in\left[\frac{l}{k},\frac{l+1}{k}\right]$

$$B_k(\tau_{\tilde{\gamma}(t)}(x,y;\zeta))=B_k(x,y;\zeta)+\left\langle p_{l+1}-p_l,\left(t-\frac{l}{k}\right)a-\frac{a}{k}\right\rangle-\left\langle p_{l+2}-p_{l+1},\left(t-\frac{l}{k}\right)a\right\rangle$$

and this proves the lemma since for t in $\left[\frac{l}{k},\frac{l+1}{k}\right]$, $\left|\left(t-\frac{l}{k}\right)a-\frac{a}{k}\right|$ and $\left|\left(t-\frac{l}{k}\right)a-\frac{a}{k}\right|$ are bounded by $\frac{|a|}{k}$. □

We must then bound the quantity $(|p_{l+2}-p_{l+1}|+|p_{l+1}-p_l|)\frac{|a|}{k}$ and we shall modify the cycle C representing the class in $H_k(S_k^c,S_k^{-\infty})$ so that the $|p_l|$ remain bounded. This follows from the lemma below.

Lemma 9.10 [Viterbo 2023, Lemma 6.5]. *There exist constants K, M such that, given a cycle $C\subset S_k^c$ representing a class $[C]\in H_*(S_k^c,S_k^{-\infty})$, we have a cycle $\widetilde{C}\subset S_k^c$ such that $[\widetilde{C}]=[C]$ in $H_*(S_k^c,S_k^{-\infty})$ and*

(1) $\widetilde{C}\subset S_k^{-4K}\cup(\{(x,y;\zeta;\omega)\mid \max_j|p_j|\le M\}\cap S_k^c)$,

(2) $\widetilde{C}\cap S_k^{-3K}\subset\{(x,y;\zeta;\omega)\mid\zeta\in E_k^-\}$, *where E_k^- is the negative eigenspace of B_k.*

The lemma means that we can deform C so that below a certain level of S_k it coincides with the negative bundle of B_k.

Proof. This is as in [Viterbo 2023, Lemma 6.5]. Let Z be the vector field defined by

$$\dot{q}_j=\chi(|p_j|)(p_{j+1}-p_j)=Z_{q_j}(q,p),\quad \dot{p}_j=0=Z_{p_j}(q,p),$$

where $\chi(r)$ vanishes for $r\le 1$. Denoting by ψ^s its flow, we have

$$\begin{aligned}\frac{d}{ds}S_k(\psi^s(q,p))&=dS_k(q,p)\cdot Z(q,p)=\left\langle\frac{\partial}{\partial q}S_k(q,p),Z_q(q,p)\right\rangle\\&=-\sum_{j=1}^{k}\chi(|p_j|)\left\langle\frac{d}{dq_j}S_k(q,p),p_{j+1}-2p_j+p_{j-1}\right\rangle\\&=-\sum_{j=1}^{k}\chi(|p_j|)|p_{j+1}-p_j|^2+\left\langle\frac{d}{dq}S_k(k\cdot q_j,p_j),p_{j+1}-p_j\right\rangle\\&=-\sum_{j=1}^{k}\chi(|p_j|)|p_{j+1}-p_j|^2;\end{aligned}$$

the last equality holds because S vanishes on the support of $\chi(|p_j|)$.

Now given $y=p_k$, if $\sup_j|p_j|\ge M$, we have that $\sum_{j=1}^{k}\chi(|p_j|)|p_{j+1}-p_j|^2$ is bounded from below by some positive quantity c_k (which is $O\left(\frac{1}{k}\right)$ but it does not matter). Thus, outside the region $\{(q,p)\mid |p_j|\le M\}$, the vector field Z is a pseudogradient vector field for F_k. Since Z is complete, its flow ψ^s has the following properties:

(1) It preserves the p_j.

(2) Outside $\{(q,p)\mid |p_j|\le M\}$, we have $\frac{d}{ds}S_k(\psi^s(q,p))\le -c_k$.

As a result if $F_k(q,p) \le c$, we have

$$\psi^{(c+4K)/c_k}(q,p) \in (\{(q,p) \mid |p_j| \le M\} \cup F_k^{-4K}) \cap F_k^c.$$

Thus $\widetilde{C}_1 = \psi^{(c+4K)/c_k}(C)$ satisfies (1).

Now to satisfy (2), we use a "cut and paste" as in [Viterbo 2023, Lemma 6.5]. □

Using Lemmas 9.9 and 9.10 and the inequality (9-2) we obtain the following:

Proposition 9.11. *Given a class a in $H_*(S_k^c, S_k^{-\infty})$, we can find a cycle C representing a and constants M_1, M_2 such that*

$$S_k(\tau_{\bar{\gamma}(t)}(C)) \le S_k(C) + \varepsilon + \frac{M_1}{k} + \frac{2M_2|a|}{k}.$$

Now let $a \in H_*(T^d)$ be represented by a map $u : C \to T^d$ and $b \in H_1(T^d)$ be represented by a map $v : S^1 \to T^d$. Then the Pontryagin product $a \cdot b$ is represented by $u \cdot v : S^1 \times C \to T^d$ given by $u \cdot v(z, \theta) = u(z) + v(\theta)$.

To conclude the proof of Proposition 9.7 (and as a consequence of Proposition 8.4) we need:

Lemma 9.12. *Let $\nu \in \mathbb{Z}^d$ be such that $|A \cdot a - \nu| \le \delta$, and let β_ν be the class in $H_1(T^d)$ of the loop $t \mapsto t \cdot \nu$ (for $t \in [0,1]$). Then given $\varepsilon > 0$, we have, for k large enough,*

$$c(\alpha \cdot \beta_\nu \otimes 1(p), S_k^0) \le c(\alpha \otimes 1(p), S_k^0) + \varepsilon.$$

Proof. Let C be a cycle representing a class in $H_*((S_k^0)^c, (S_k^0)^{-\infty})$. We may assume C satisfies properties (1) and (2). We are going to construct a cycle in the class of $\alpha \cdot \beta$ made of three pieces. First set

$$C_1 = \bigcup_{t \in [0,1]} C_1(t),$$

where

$$C_1(t) = \{(0, p; \tau_{-\gamma_1(t)}\tau_{\bar{\gamma}(t)}\zeta; \tau_{k\gamma_1(t)}\omega) \mid (0, p; \zeta; \omega) \in C\}.$$

According to Proposition 9.11 since

$$\begin{aligned} S_k(0, p; \tau_{-\gamma_1(t)}\tau_{\bar{\gamma}(t)}\zeta; \tau_{k\gamma_1(t)}\omega) &= S_k(\gamma_1(t), p; \tau_{\bar{\gamma}(t)}\zeta; \omega) \\ &= S_k(\tau_{\gamma(t)}(0, p; \zeta); \omega) \le S_k(C) + \varepsilon + \frac{M_1 + 2M_2|a|}{k}, \end{aligned}$$

as a result we have for each $t \in [0,1]$

$$S_k^0(C_1(t)) \le S_k^0(C) + \varepsilon + \frac{M_1 + 2M_2|a|}{k};$$

hence

$$S_k^0(C_1) \le S_k^0(C) + \varepsilon + \frac{M_1 + 2M_2|a|}{k}.$$

Note that

$$C_1(0) = C,$$
$$C_1(1) = \{(0, p; \zeta; \tau_a\omega) \mid (0, p; \zeta; \omega) \in C\}.$$

Now for $u \in [0,1]$ define the path $\eta(u) = (1-u)A \cdot a + u\nu$ so that $\eta(0) + \omega = \tau_a \omega$. Set

$$C_1(1+u) = \{(0, p; \zeta; \omega + \eta(u)) \mid (0, p; \zeta; \omega) \in C\}.$$

Now $C_1(2) = C$ and since $|\eta(u) - A \cdot a| \le \delta$, we have

$$S_k^0(C_1(1+u) \le S_k^0(C_1(1)) + \varepsilon$$

so that the cycle[19]

$$\widehat{C} = \bigcup_{s \in [0,2]} C_1(s)$$

satisfies for k large enough

$$S_k^0(\widehat{C}) \le S_k^0(C) + 2\varepsilon.$$

Moreover we claim that the cycle $\widehat{C}$ defines a cycle in the homology class of $\alpha \cdot \beta_\nu$. Indeed the lift of the variable ω starting from ω_0 is given

(1) for $s \in [0,1]$ by the path $s \mapsto \omega_0 + sA \cdot a$,

(2) for $s \in [1,2]$ by $s \mapsto \omega_0 + (2-s)A \cdot a + (s-1)\nu$,

and since it joins ω_0 to $\omega + \nu$, it belongs to the class β_ν. As a result $[\widehat{C}] = \alpha \cdot \beta \in H_*((S_k^0)^{+\infty}, (S_k^0)^{-\infty})$ and this proves the lemma. ☐

We shall also need:

Lemma 9.13. *Let $\varepsilon > 0$ and $A : \mathbb{R}^n \to \mathbb{R}^d$ be a linear map such that $A(\mathbb{R}^n)$ has dense projection on T^d. Then there are integral vectors $\nu_1, \dots, \nu_d$ in $\mathbb{Z}^d$ forming a basis of $\mathbb{R}^d$ such that there exist vectors $a_1, \dots, a_d$ in $\mathbb{R}^n$ such that*

$$|A \cdot a_j - \nu_j| \le \varepsilon.$$

Proof. See Appendix B. ☐

Proof of Propositions 9.7 and 8.4. Let $\alpha_j \in H_1(T^d)$ be the homotopy class of the path $t \mapsto t \cdot \nu_j$, where ν_j is a basis of $\mathbb{R}^d$ given by Lemma 9.13. Then $\alpha_1 \cdot \alpha_2 \cdot \dots \cdot \alpha_d = c_d \mu_{T^d}$ for some $c_d \neq 0$. since $c(c_d \mu_{T^d}, f) = c(\mu_{T^d}, f)$ we obtain by repeated applications of Lemma 9.12 that $c(\mu_{T^d} \otimes 1(p), S_k^0) \le c(1_{T^d} \otimes 1(p), S_k^0) + \varepsilon$ and this proves Proposition 9.7 and hence Proposition 8.4. ☐

10. Proof of the Main Theorem

We first prove that under assumptions (1)–(6) we have $\lim_{\varepsilon \to 0} \varphi^t_{\varepsilon,\omega} = \varphi^t_{\overline{H}}$ for almost all $\omega \in \Omega$. We start from H satisfying (1)–(6), then, using the results of Section 5, we get a map $H : \mathbb{A}_\Omega \to \widehat{\mathfrak{Ham}}(T^*T^n)$ such that $\mathbb{A}_\Omega$ is a compact connected metric abelian group. According to Section 6, $\mathbb{A}_\Omega$ is the projective limit of finite-dimensional tori, $\mathbb{A}_j$, on which τ_a is given by $\tau_a \omega = \omega + A_j \cdot a$, where the projection of $A_j(\mathbb{R}^n)$ is dense in $\mathbb{A}_j$ and $\omega \mapsto H(\dots, \cdot; \omega)$ is continuous from $\mathbb{A}_j$ to $C^\infty_{\mathrm{fc}}(T^*\mathbb{R}^n, \mathbb{R})$ and satisfies (1)–(6).

[19] Similarly to the proof of Lemma 9.5, this is an abuse of language for $f_*(\mathbb{R}/2\mathbb{Z} \times C)$, where

$$f(t, (0, p; \zeta; \omega)) = \begin{cases} (0, p; \tau_{-\gamma_1(t)} \tau_{\bar{\gamma}(t)} \zeta; \tau_{k\gamma_1(t)} \omega) & \text{for } 0 \le t \le 1, \\ (0, p; \zeta; \omega + \eta(t-1)) & \text{for } 1 \le t \le 2. \end{cases}$$

By Corollary 7.5 we find H^η in $C^\infty(T^*\mathbb{R}^n \times \mathbb{A}_j, \mathbb{R})$ such that

$$\gamma_\infty(H^\eta_{\pi_j(\omega)}, H_\omega) \leq \eta,$$

where $\pi_j : \mathbb{A}_\Omega \to \mathbb{A}_j$ is the projection map. According to Proposition 8.6, we know that

$$\gamma_c - \lim_k H^\eta_{k,\pi_j(\omega)} = \overline{H}^\eta_{\pi_j(\omega)}$$

and since, for all k, ω, we have $\gamma_\infty(H^\eta_{k,\pi_j(\omega)}, H_{k,\omega}) \leq \eta$, we infer for k large enough

$$\gamma_\infty(\overline{H}^\eta_{\pi(\omega)}, H_{k,\omega}) \leq 2\eta.$$

Now consider a sequence η_ν converging to 0 so that $H^{\eta_\nu}_{\pi_\nu(\omega)}$ is a γ_c-Cauchy sequence, γ_c-converging to H_ω uniformly in ω. Then $\overline{H}^{\eta_\nu}_{\pi_\nu(\omega)}$ is also a Cauchy sequence, so it converges to some $\overline{H} \in \widehat{\mathfrak{Ham}}(T^*T^n)$. But then $(H_{k,\omega})_{k\geq 1}$ converges a.s. in ω to $\overline{H}$.

For the second part of the Main Theorem, we must go from γ-convergence of the flow to γ-convergence of the solution of the corresponding Hamilton–Jacobi equation. In the case of a compact base this is achieved in [Viterbo 2006], and the extension to a noncompact base was spelled out in [Cardin and Viterbo 2008, pp. 266–276].

For $L \in \mathfrak{L}(T^*N)$ we define $u_L(x) = c(1_x, L)$. Our first claim is that γ-convergence for L implies C^0-convergence of the u_L uniformly on compact sets.

Lemma 10.1. *Let U be bounded domain in N. If $(L_\nu)_{\nu\geq 1}$ is a Cauchy sequence for γ_U, then the sequence u_{L_ν} is a Cauchy sequence for the topology of uniform convergence on U. As a result if $(L_\nu)_{\nu\geq 1}$ γ-converges to $L \in \widehat{\mathfrak{L}}(T^*N)$ then the sequence u_{L_ν} converges uniformly on compact sets to u_L.*

Proof. This is an immediate consequence of the reduction inequality [Viterbo 1992, Proposition 5.1, p. 705], which implies that, for any $x \in U$,

$$|c(1_x, L) - c(1_x, L')| \leq \gamma_U(L, L'). \qquad \square$$

Proposition 10.2. *Let $(\varphi_\nu)_{\nu\geq 1}$ be a sequence in $\widehat{\mathfrak{DHam}}_{c,\mathrm{FP}}$ γ_c-converging to $\varphi_\infty \in \widehat{\mathfrak{DHam}}_{c,\mathrm{FP}}$. Then for any $L \in \mathfrak{L}(T^*\mathbb{R}^n)$ (or in $\widehat{\mathfrak{L}}(T^*\mathbb{R}^n)$) the sequence $\varphi_\nu(L)$ γ_c-converges to $\varphi_\infty(L)$.*

Proof. Indeed, we proved in Proposition 4.23 that $\gamma_U(\psi_1(L), \psi_2(L)) \leq \gamma_{V\times B^n(r)}(\psi_1, \psi_2)$ provided $\psi_j^{\pm 1}$ sends T^*U to T^*V and $L \subset \mathbb{R}^n \times B(r)$. In our case, we get that for $L \subset \mathbb{R}^n \times B^n(r)$

$$\gamma_U(\varphi_\nu(L), \varphi_\infty(L)) \leq \gamma_{V\times B^n(r)}(\varphi_\nu, \varphi_\infty)$$

and since the right-hand side converges to 0, so does the left-hand side. $\square$

We may now conclude our proof. Since a.s. in ω, $\varphi^t_{k,\omega}$ γ_∞-converges to $\bar{\varphi}^t$ and is uniformly FPS for bounded t, we have by Proposition 10.2

$$\varphi^t_{k,\omega}(L_f) \underset{\gamma_\infty}{\to} \bar{\varphi}^t(L_f))$$

a.s. in ω. Then applying Lemma 10.1 to the sequence $(\varphi^t_{k,\omega})_{k\geq 1}$, this implies uniform convergence on compact sets of the sequence $(u_{k,\omega})_{k\geq 1}$ to its limit $\bar{u}$. This concludes the proof of our Main Theorem.

11. The coercive case

We now assume H satisfies assumptions (1a)–(3a) of Corollary 1.3. Let χ_A be a truncation function, that is, an increasing function such that $0 \le \chi'_A(t) \le 1$ and $\chi_A(t) = t - \frac{3}{2}A$ for $t \le A$ and $\chi_A(t) = 0$ for $t \ge 2A$. We set $H_A(x, p; \omega) = \chi_A(H(x, p, \omega))$. Then coercivity implies[20] that H_A has a.s. in $\omega \in \Omega$ the same flow as H in $U_R = \{(x, p) \mid |p| \le r(A)\}$ where $\lim_{A\to+\infty} r(A) = +\infty$. We apply the Main Theorem to H_A and obtain a homogenized Hamiltonian $\overline{H}_A$. We claim now that for $B \ge A$ we have $\overline{H}_A = \overline{H}_B$ on U_R. This follows the same proof as Section 11 in [Viterbo 2023]. Because f is Lipschitz, it can be approximated by functions f_k which have a bounded differential, so the image of the graph of df_k remains in some domain bounded in the p-direction. Therefore for A large enough, $\varphi^t_{H_A}(G_{f_k}) = \varphi_A(G_{f_k})$ for all k and all t. Therefore homogenization for H_A yields homogenization for H.

12. The discrete case (Proof of Corollary 1.7)

If we have a $\mathbb{Z}^n$ action on Ω, and its standard action on $\mathbb{R}^n$ we construct an $\mathbb{R}^n$ action on $\widetilde{\Omega} = \Omega \times \mathbb{R}^n / \simeq$, where

$$(\omega, t_1, \dots, t_n) \simeq (T_{-z}\omega, z_1 + t_1, \dots, z_n + t_n),$$

where $z = (z_1, \dots, z_n) \in \mathbb{Z}^n$. Then $\mathbb{R}^n$ acts on $\widetilde{\Omega}$ by translation, i.e.,

$$\widetilde{T}_a(\omega, t_1, \dots, t_n) = (\omega, t_1 + a_1, \dots, t_n + a_n).$$

Notice that if $z \in \mathbb{Z}^n$, we have $\widetilde{T}_z(\omega, t_1, \dots, t_n) = (T_z\omega, t_1, \dots, t_n)$.

Now it is easy to see that T is ergodic for the measure μ on Ω if and only if $\widetilde{T}$ is ergodic for the measure $\mu \times \lambda$ (where λ is the Lebesgue measure on $[0, 1[^n$), since any $\widetilde{T}$-invariant set will be of the form $U \times [0, 1[^n$ with U a T-invariant set. Then if H satisfies $H(x + z, p, T_z\omega) = H(x, p; \omega)$, we can consider $K(x, p, [\omega, t]) = H(x - t, p; \omega)$ and this satisfies

$$K(x + a, p, \widetilde{T}_a[\omega, t]) = K(x, p, [\omega, t])$$

for all $a \in \mathbb{R}^n$, and we can apply the stochastic homogenization from the Main Theorem.

13. Extending the Main Theorem

Note that one should be able to extend our methods to the case where we have a Hamiltonian satisfying the assumptions of the Main Theorem, but:

(1) We have a time-dependent Hamiltonian, $H(t, x, p; \omega)$, and an action of $\mathbb{R} \times \mathbb{R}^n$ such that $H(t + s, x + a, p; \tau(s, a)\omega)$ and consider the sequence $H\left(\frac{t}{\varepsilon}, \frac{x}{\varepsilon}, \omega\right)$ This has been reduced to our case in the nonstochastic situation in [Viterbo 2023, Section 11.2. The nonautonomous case].

(2) We consider partial homogenization. For example if $X = N \times \mathbb{R}^k$, then we should be able to apply the above propositions as in [loc. cit.].

[20] See Remark 1.6, since $h_-(p) \le H(x, p; \omega) \le h_+(p)$ a.s. in ω, where $\lim_{|p|\to+\infty} h_\pm(p) = +\infty$.

(3) We consider the homogenization $H_\varepsilon\big(x, \frac{p}{\varepsilon}; \omega\big)$ as ε goes to 0. This has been reduced to our case in the nonstochastic situation in [Viterbo 2023, Section 12. Homogenization in the p variable].

(4) We have a $\mathbb{Z}^n$ action on a manifold X such that the quotient $X/\mathbb{Z}^n$ is compact and the Hamiltonian satisfies $H(T_z x, T_z^* p, T_z \omega) = H(x, p; \omega)$, where T is the action of $\mathbb{Z}^n$ on X and we consider again the sequence $H_\varepsilon\big(x, \frac{p}{\varepsilon}, \omega\big)$ as ε goes to 0.

The proof in this last case should be the same as the Main Theorem. We just need to replace $\gamma(\varphi)$ (which is not defined on T^*X) with $\hat{\gamma}(\varphi)$ and we shall get an embedding of $\mathbb{Z}^n$ into $\mathrm{Isom}(\widehat{\mathfrak{H}}_\Omega, \gamma)$. According to [Weil 1965], the closure of the image of $\mathbb{Z}^n$ is the product of an abelian compact connected metric group, A^0_Ω, and a totally disconnected compact metric abelian group D_Ω. Since we have a morphism $c : \mathbb{Z}^n \to D_\Omega$ and the kernel L must be a cocompact free abelian group, hence a lattice, so L is isomorphic to $\mathbb{Z}^n$ and in suitable integral coordinates, we see that $L = a_1\mathbb{Z} \oplus a_2\mathbb{Z} \oplus \cdots \oplus a_n\mathbb{Z}$, so $D_\Omega = \mathbb{Z}^n/L \simeq \mathbb{Z}/a_1\mathbb{Z} \oplus \mathbb{Z}/a_1\mathbb{Z} \oplus \cdots \oplus \mathbb{Z}/a_n\mathbb{Z}$. Replacing $\mathbb{Z}^n$ by L, we can reduce ourselves to the case of a compact connected abelian group so we get $\overline{K}(p, \omega)$, where $\overline{K}(p, \cdot)$ is constant on the ergodic components of the action of L and the ergodic components are interchanged by an element of D_Ω; thus we get that $\overline{K}(p, \cdot)$ is indeed constant a.e.

It would be also interesting to see what can be done in the framework of more general groups, as explained in [Sorrentino 2019] (see also [Contreras et al. 2015]). In this setting a discrete group G is a quotient of the $\pi_1(M)$, where M is a compact manifold, and we see a Hamiltonian on M as a G-invariant one on $\widetilde{M}$ a cover of M. Then Sorrentino considers the Hamiltonian $H\big(x, \frac{1}{\varepsilon} p\big)$ as ε goes to zero, and proves that it converges in some weak sense (we would say in the γ topology) to a Hamiltonian defined on G_∞ a graded Lie group associated to G (at least if G is nilpotent).

Appendix A: Generating functions for noncompact Lagrangians: Proof of Theorem 3.5

The goal of this section is to prove Theorem 3.5 that is:

Theorem 3.5. *Let φ be an element in $\mathfrak{DHam}_{\mathrm{FP}}(T^*N)$. Then $\varphi(0_N)$ has a GFQI. Moreover such a GFQI is unique.*

First we claim that the fibration theorem of Théret [1999, Theorem 4.2] goes through. Here $\mathscr{F}$ is the set of sequences of GFQI $(S_\nu)_{\nu \geq 1}$ satisfying the above property and $\mathscr{L} = \mathfrak{L}(T^*\mathbb{R}^n)$ and we have:

Proposition A.1. *The projection $\pi : \mathscr{F} \to \mathscr{L}$ is a Serre fibration up to equivalence.*

The proof is the same as Theorem 4.2 in [Théret 1999]. We may reduce ourselves to the case of a single parameter (as in [loc. cit.]). The proof is then based on Sikorav's existence theorem, which uses only the fact that, for t small enough, if L has a GFQI over U_ν then so does $\varphi^t(L)$. Note that we may always assume that $\varphi^t(T^*U_\nu) \subset T^*U_{\nu+1}$ and by truncating φ^t beyond $T^*U_{\nu+1}$, we are reduced to the compact situation.

Proof of Theorem 3.5. Using Lemma 3.2 we may assume we have a sequence U_ν of domains such that $\varphi^t(T^*U_\nu) \subset T^*U_{\nu+1}$. Applying a sequence of cut-offs to the Hamiltonian defining φ we can then find a sequence L_ν of Lagrangians of the type $\varphi^1_\nu(0_N)$, where

(1) $\varphi_\nu^t(T^*U_\nu) \subset T^*U_{\nu+1}$ for all $t \in [0, 1]$,

(2) φ_ν^t has compact support in $T^*U_{\nu+1}$,

(3) setting $\varphi_\nu^t(0_N) = L_\nu(t)$, we have for $\mu \geq \nu$

$$L_\nu(t) \cap T^*U_\nu = L_\mu(t) \cap T^*U_\nu = \varphi^t(L) \cap T^*U_\nu.$$

Then each $L_\nu(t)$ has a GFQI, $S_\nu(t) : N \times E_\nu \to \mathbb{R}$ and we claim that, for $\mu \geq \nu$, $S_\nu(t)$ and $S_\mu(t)$ are equivalent over U_ν. Indeed, we have a deformation from L_ν to L_μ that is the identity on T^*U_ν. If we denote by S_s a GFQI covering this deformation (the existence of which follows from [Théret 1999], since we are again in the compactly supported case), then S_s generates a Lagrangian L_s that is constant over T^*U_ν. Then using [loc. cit., Lemma 5.3] we can assume, after applying a fiber-preserving diffeomorphism, that $\Sigma_s \cap (U \times F) = \Sigma_0 \cap (U \times F)$, where

$$\Sigma_s = \left\{ (x, \xi) \;\middle|\; \frac{\partial S_s}{\partial \xi}(x, \xi) = 0 \right\}.$$

But then as in [loc. cit., p. 259], using Hadamard's lemma we prove that there is a fiber-preserving diffeomorphism such that $S_1(x, \xi(x, \eta)) = S_0(x, \eta)$.

So may now assume that the restriction of S_μ over U_ν is exactly $S_\nu \oplus q_{\nu,\mu}$ by composing S_μ with an extension of the fiber-preserving diffeomorphism realizing the equivalence.[21] □

Appendix B: Proof of Lemma 9.13

Lemma 9.13. *Let $\varepsilon > 0$ and $A : \mathbb{R}^n \to \mathbb{R}^d$ be a linear map such that $A(\mathbb{R}^n)$ has dense projection on T^d. Then there are integral vectors $v_1, \dots, v_d$ in $\mathbb{Z}^d$ forming a basis of $\mathbb{R}^d$ such that there exist vectors $a_1, \dots, a_d$ in $\mathbb{R}^n$ such that*

$$|A \cdot a_j - v_j| \leq \varepsilon.$$

Remark B.1. We do not claim the basis is an integral basis, i.e., it does not necessarily have determinant 1.

Proof suggested by the referee. We know that $A(\mathbb{R}^n) + \mathbb{Z}^d$ is dense in $\mathbb{R}^d$, so we may find $b_1, \dots, b_d \in \mathbb{R}^n$, $w_1, \dots, w_n \in \mathbb{Z}^d$ such that

$$\left| Ab_i - w_i - \frac{e_i}{2} \right| \leq \frac{\varepsilon}{2}.$$

Then $a_i = 2b_i$, $v_i = e_i + 2w_i$ satisfy $|Aa_i - v_i| \leq \varepsilon$, and since $\det(v_i)$ is odd, $(v_1, \dots, v_d)$ is a basis of $\mathbb{R}^d$. □

Appendix C: Approximation of generating functions and symplectic integrators

Our goal is to prove Lemma 7.4. It is a consequence of the more precise result:

Lemma C.1. *Let φ_H^t have $S_t(q, p)$ as generating function. We have*

$$\|S_t(q, p) - tH(q, p)\|_{C^0} \leq \frac{t^2}{2} \left\| \frac{\partial H}{\partial q} \right\|_{C^0} \cdot \left\| \frac{\partial H}{\partial p} \right\|_{C^0}.$$

[21] The existence of the extension follows from the fact that we may assume that for μ, ν large enough, the inclusion $U_\nu \subset U_\mu$ is a homotopy equivalence.

Proof. In the sequel, $\|\cdot\|$ denotes the C^0 norm. Note that S_t has no fiber variable. It is a classical fact [Hamilton 1834; 1835; Jacobi 2009] (see also [Arnold 1978]) that S_t satisfies the Hamilton–Jacobi equation

$$\begin{cases} \dfrac{\partial}{\partial t} S_t(q,p) = H\Big(q + \dfrac{\partial S_t}{\partial p}(q,p), p\Big), \\ S_0(q,p) = 0. \end{cases}$$

Indeed, setting $\varphi_H^t(q,p) = (Q_t(q,p), P_t(q,p))$, the Lagrangian submanifold

$$\Lambda(\varphi) = \{(t, -H(t, Q_t(q,p), P_t(q,p)), q, p, Q_t(q,p), P_t(q,p)) \mid t \in \mathbb{R}, (q,p) \in T^*N\}$$

in $T^*\mathbb{R} \times T^*N \times \overline{T^*N}$ is contained in

$$\{(t, \tau, q, p, Q, P) \mid \tau + H(Q,P) = 0\}$$

since $\Lambda(\varphi)$ is easily seen to be invariant by the flow of the Hamiltonian $K(t,\tau,q,p,Q,P) = \tau + H(Q,P)$, which is given by

$$(t,\tau,q,p,Q,P) \to (t+s, \tau, q, p, Q_s(Q,P), P_s(Q,P)).$$

Since $Q_t = q + \frac{\partial S_t}{\partial q}$, the equation follows.

Now set $S_t(q,p) = t \cdot H(q,p) + R_t(q,p)$ and replace in the equation, using

$$|H(q+\xi, p) - H(q,p)| \le |\xi| \cdot \Big\| \frac{\partial H}{\partial q} \Big\|_{C^0},$$

$$\frac{\partial R_t}{\partial t}(q,p) \le \Big\| \frac{\partial H}{\partial q} \Big\|_{C^0} \Big| \frac{\partial S_t}{\partial p} \Big| \le t \cdot \Big\| \frac{\partial H}{\partial q} \Big\|_{C^0} \cdot \Big\| \frac{\partial H}{\partial p} \Big\|_{C^0} + \Big\| \frac{\partial H}{\partial q} \Big\|_{C^0} \cdot \Big| \frac{\partial R_t}{\partial p}(q,p) \Big|$$

and $R_0(q,p) = 0$. Now the relation

$$\partial_t R_t(q,p) \le tA + B \Big| \frac{\partial R_t}{\partial p} \Big|$$

implies by monotonicity of the solutions of the Hamilton–Jacobi equations[22] that R_t is bounded by the solution u_t of $\partial_t u = tA + B|\nabla_x u|$, that is, $u(t,x) = \frac{1}{2} t^2 A$, so

$$R_t(q,p) \le \frac{t^2}{2} \Big\| \frac{\partial H}{\partial q} \Big\|_{C^0} \cdot \Big\| \frac{\partial H}{\partial p} \Big\|_{C^0}.$$

The same argument gives an estimate from below. □

Appendix D: Proof of Proposition 8.3

Proposition 8.3. *Given any α, there exists a sequence $(\ell_\nu)_{\nu \ge 1}$ such that for almost all $\omega \in \Omega$*

$$\lim_{\nu \to \infty} \lim_{U \subset \mathbb{R}^n} c(\mu_U, \varphi_{\ell_\nu, \omega}\alpha) \le \lim_{U \subset \mathbb{R}^n} c(\mu_U, \bar{\varphi}\alpha).$$

The proof is essentially the same as in Section 5 of [Viterbo 2023]. We reproduce it here adapted to our situation and notation but notice that ω just appears as a parameter and so does not change the proof of Proposition 8.3. In particular the cycles we construct in the proof do not need to depend continuously on ω. We first need the next lemma. We define a cycle with closed support in X to be a cycle for the

[22]That is, $H \le K$ implies that the solutions v, w of $\partial_t u = H(x, D_x u)$ corresponding to the same initial condition satisfy $u \le v$.

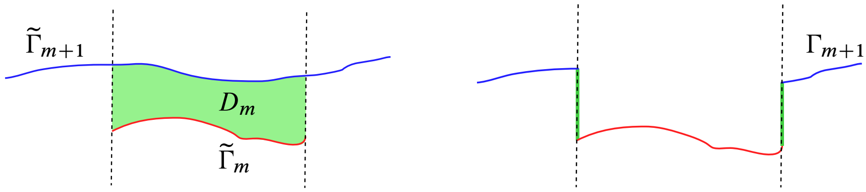


Figure 2. $\widetilde{\Gamma}_m$ in red, $\widetilde{\Gamma}_{m+1}$ in blue and D_m in green on the left and Γ_{m+1} on the right.

singular homology with locally finite support. These are the cycles of Borel–Moore homology (i.e., homology with closed supports) of X. Chains are infinite sums $\sum_\sigma a_\sigma \sigma$ of singular simplices such that there are only finitely many simplices with $a_\sigma \neq 0$ touching any compact set. As a result it is clear what it means for such a chain to be a cycle. For such homology, admissible maps are the proper maps, i.e., only a proper map $f : X \to Y$ will induce a map f_* between the corresponding Borel–Moore homology groups. Any proper submanifold without boundary represents a cycle in Borel–Moore homology, while in ordinary homology, this is the case only for compact submanifolds.

Lemma D.1. *Let S be a GFQI defined on E and $c = \lim_{U \subset N} c(\mu_U, S)$. There exists a closed cycle Γ such that $\Gamma_U = \Gamma \cap \pi^{-1}(U)$ satisfies $[\Gamma_U] = \mu_U$ in $H_*(S_U^{+\infty}, S^{-\infty})$ and $S(\Gamma_U) \leq c(\mu_U, S) + \varepsilon$ for U belonging to a sequence of exhausting open sets with smooth boundary.*

Remark D.2. Note that $[\Gamma_U]$ is assumed to be an ordinary cycle, so that its class in $H_*(S_U^{+\infty}, S_U^{-\infty})$ is well-defined.

Proof. Consider an increasing sequence U_n of open sets with smooth boundary such that $N = \bigcup_n U_n$. Notice that there is a restriction map for $U \subset V$ sending $H_*(V, \partial V) \to H_*(U, \partial U)$. It induces a map that we denote by $\rho_{U,V}$,

$$H_*(S_V^t, S_V^{-\infty} \cup E_{|\partial V}) \to H_*(S_U^t, S_U^{-\infty} \cup E_{|\partial U}),$$

and a diagram

$$\begin{array}{ccc} H_*(S_V^{+\infty}, S_V^{-\infty} \cup E_{|\partial V}) & \xrightarrow{\rho_{U,V}} & H_*(S_U^{+\infty}, S_U^{-\infty} \cup E_{|\partial U}) \\ \uparrow & & \uparrow \\ H_*(S_V^{c+\varepsilon}, S_V^{-\infty} \cup E_{|\partial V}) & \xrightarrow{\rho_{U,V}} & H_*(S_U^{c+\varepsilon}, S_U^{-\infty} \cup E_{|\partial U}) \end{array}$$

Now the upper horizontal map sends μ_V to μ_U, so applying this to the sequence U_n, we get a sequence $\widetilde{\Gamma}_n \in H_*(S_{U_n}^{c+\varepsilon}, S_{U_n}^{-\infty} \cup E_{|\partial U_n})$ with image μ_{U_n}, and we have a sequence such that $\rho_{U_n,U_m}[\widetilde{\Gamma}_n] = [\widetilde{\Gamma}_n \cap \pi^{-1}(U_m)]$ is constant for $n \geq m$. Then we may glue the $\widetilde{\Gamma}_n$ as follows: since $[\widetilde{\Gamma}_m] = [\widetilde{\Gamma}_{m+1} \cap \pi^{-1}(U_m)]$ in $H_*(S_{U_n}^{c+\varepsilon}, S_{U_n}^{-\infty} \cup E_{|\partial U_n})$, we have D_m such that $\partial D_m \cap \pi^{-1}(U_m) = \widetilde{\Gamma}_m - \widetilde{\Gamma}_{m+1} \cap \pi^{-1}(U_m)$ and we can assume $D_m \subset \pi^{-1}(\overline{U}_m)$. This is illustrated in Figure 2. Now we may consider the cycle

$$\Gamma_m = \widetilde{\Gamma}_m \cup (\partial D_m \cap \pi^{-1}(\overline{U}_m)) \cup \widetilde{\Gamma}_{m+1} \cap \pi^{-1}(U_{m+1} \setminus U_m).$$

We easily check that

(1) $\Gamma_m \cap \pi^{-1}(U_m) = \widetilde{\Gamma}_m$,

(2) $\Gamma_m \cap \pi^{-1}(U_{m+1} \setminus U_m) = (\widetilde{\Gamma}_{m+1} \cap \pi^{-1}(U_{m+1}) \cup (\partial D_m \cap \pi^{-1}(\partial U_m))),$

(3) $\partial\Gamma_m \subset E_{\partial U_{m+1}}$,

(4) $\Gamma_m \subset S^{c+\varepsilon}$.

By induction we can build a sequence Γ_m and we have $\Gamma_n \cap \pi^{-1}(U_m) = \Gamma_m \cap \pi^{-1}(U_m)$ for $n > m$. Therefore $\bigcup_n \Gamma_n$ is stationary over any compact set and defines a closed cycle Γ such that $S(\Gamma) \le c+\varepsilon$. □

Now the generating function for $\varphi_{k,\omega}$ is given by Proposition 9.1:

$$S_{k,\omega}(x,y;\xi,\zeta) = \frac{1}{k}\left[S_\omega(kx,p_1,\zeta_1) + \sum_{j=2}^{k-1} S_\omega(kq_j,p_j,\zeta_j) + S_\omega(kq_k,y,\zeta_k)\right] + B_k(x,y,\zeta),$$

where $\zeta = (\zeta_1,\dots,\zeta_k)$, $\xi = (p_1,q_2,\dots,q_{k-1},p_{k-1},q_k)$,

$$\tau_a\zeta = (p_1,q_2+a,\dots,q_{k-1}+a,p_{k-1},q_k+a)$$

and

$$B_k(x,y,\zeta) = \langle p_1,q_2-x\rangle + \sum_{j=2}^{k-1}\langle p_j,q_{j+1}-q_j\rangle + \langle y,x-q_k\rangle.$$

Now let $F(q,P;\eta)$ be a GFQI for the graph of α. Then

$$G_k^\omega(u,v;x,y,\eta;\xi,\zeta) = S_k^\omega(u,y;\xi) + F(x,v;\eta) + \langle y-v,u-x\rangle$$

is a GFQI of $\varphi_k\alpha$. We set

$$\bar{G}_k^\omega(u,v;x,y,\eta) = h_k^\omega(y) + F(x,v;\eta) + \langle y-v,u-x\rangle.$$

We shall omit the subscripts a,χ for the moment, so in the sequel, $\bar{G}_{k,a,\chi}^\omega$ means $\bar{G}_{k,a,\chi}^\omega$ Here the variables u,v,x,y are in $\mathbb{R}^n$ and we denote by E_k the space of the $\theta = (\zeta,\xi)$, where $\xi \in E^k$, $\zeta \in (\mathbb{R}^{2n})^k$ and $\eta \in V$. By definition we have a cycle Γ_U^ω in $U \times \mathbb{R}_v^n \times \mathbb{R}_x^n \times \mathbb{R}_y^n \times E \times V$ relative to $(\bar{G}_k^\omega)^{-\infty} \cup \partial U \times \mathbb{R}_v^n \times \mathbb{R}_x^n \times \mathbb{R}_y^n \times E \times V$ and homologous (as a closed cycle) to $U \times \mathbb{R}_v^n \times \Delta_{x,y} \times E^- \times V^-$ (where Δ is the diagonal) such that

$$\bar{G}_k^\omega(\Gamma_U^\omega) \le c(\mu_U,\bar{G}_k^\omega) + \varepsilon = c(\mu_U,\bar{\varphi}_{k,U}^\omega\alpha) + \varepsilon,$$

where $\bar{\varphi}_{k,U}^\omega$ is the flow of $h_{k,U}^\omega(y)$.

Moreover according to Lemma D.1, we can assume there is a closed (i.e., Borel–Moore) cycle Γ^ω such that $\Gamma_U^\omega = \Gamma^\omega \cap \pi^{-1}(U)$ (at least for a cofinal sequence of U's).

Now let $C_U^\omega(y)$ be a cycle in the class of $U \times E_k^-$ in $H_*((S_{k,y}^\omega)^{+\infty},(S_{k,y}^\omega)^{-\infty})$, depending continuously on y, such that[23]

$$S_k^\omega(y,C_U^\omega(y)) \le h_{k,U}^\omega(y) + a\chi(y) + \varepsilon.$$

As in [Viterbo 2023, Section 5, Lemma 5.1], this is possible provided χ is the characteristic function of Λ_δ, the complement of a disjoint union of sets of diameter less than δ. For example, we can take Λ_δ

[23] The notation is unfortunate since it does not respect the order of our variables. By $S_k(y,C_U(y))$ we mean the maximum of $S_k(x,y;\xi,\zeta)$, where $(x;\xi,\zeta) \in C(y)$

to be a neighborhood of $\Lambda(\delta)=\{(x_1,\dots,x_n)\mid \exists j,\ x_j\in\delta\mathbb{Z}\}$. Thus we set for $a\in\mathbb{R}_+$, $\chi\in C^\infty(\mathbb{R}^n)$

$$\bar{G}^\omega_{k,a,\chi}(u,v;x,y,\eta)=h^\omega_k(y)+F(x,v;\eta)+\langle y-v,u-x\rangle+a\chi(y).$$

We shall omit the subscripts a,χ for the moment, so in the sequel, $\bar{G}^\omega_{k,a,\chi}$ means $\bar{G}^\omega_{k,a,\chi}$.

We again invoke Lemma D.1 in order to obtain a (closed) cycle $C^\omega(y)$ such that for a cofinal sequence of U's we have $C^\omega_U(y)=C^\omega(y)\cap\pi^{-1}(U)$ and, like $C^\omega_U(y)$, the cycle $C(y)$ depends continuously on y.

We now construct a new (Borel–Moore) cycle, symbolically denoted by $\Gamma\times_Y C$ and defined as follows (everything depends on ω but for notational convenience we omit it):

$$\Gamma\times_Y C=\{(u,v;x,y,\theta,\eta)\mid (u,v,x,y,\eta)\in\Gamma,(u,\theta)\in C(y)\}.$$

We have

(1) $(\Gamma\times_Y C)_U$ is a Borel–Moore cycle homologous to $U\times\mathbb{R}^n_v\times\Delta_{x,y}\times E^-_k\times V^-$.

(2) $G^\omega_k((\Gamma\times C)_U)\le\bar{G}^\omega_{k,a,\chi}(\Gamma_U)+\varepsilon$.

Indeed for (1), it is a cycle by the continuity of $C(y)$ in y. That its homology class is the stated one follows from the fact that the homology class of $A\times_Y B$ only depends on the homology class of A,B and so $\Gamma_U\times_Y C^-_U$ is homologous to

$$\begin{aligned}(U\times\mathbb{R}^n_v\times\Delta_{x,y}\times V^-)\times_Y(U\times E^-_k)&=\{(u,v;x,y,\eta,\theta)\mid u\in U,\ x=y,\ \eta\in V^-,\ \theta\in E^-_k\}\\&=U\times\mathbb{R}^n_v\times\Delta_{x,y}\times V^-\times E^-_k.\end{aligned}$$

As for (2), we have $(\Gamma\times_Y C^-)_U=\Gamma_U\times_Y C^-_U$ and

$$G^\omega_k(\Gamma_U\times_Y C^-_U)\overset{\text{def}}{=}\sup\{S^\omega_k(u,y;\theta)+F(x,v;\eta)+\langle y-v,u-x\rangle\mid(u,v;x,y,\eta)\in\Gamma,\ (u;\theta)\in C(y)\}$$

but since $S_k(u,y;\theta)\le h^+_{k,U}(y)+a\chi(y)+\varepsilon$ for $(u;\theta)\in C(y)$, we have

$$\begin{aligned}&G^\omega_k(\Gamma_U\times_Y C^-_U)\\&\quad\le\sup\{F(x,v;\eta)+h^+_{k,U}(y)+a\chi(y)+\varepsilon+\langle y-v,u-x\rangle\mid(u,v;x,y,\eta)\in\Gamma^\omega_U,(u,\theta)\in C^\omega_U(y)\}\\&\quad\le\bar{G}^\omega_{k,a,\chi}(\Gamma_U)+\varepsilon.\end{aligned}$$

Now as in [Viterbo 2023, Section 5, p. 95], let us consider a collection of ℓ open sets Λ^j_δ for $1\le j\le\ell$ such that each of them is a translate of Λ_δ and any $n+1$ of them have empty intersection. We denote by χ_j $(1\le j\le\ell)$ the corresponding functions. We set $\bar{x}=(x_1,\dots,x_\ell)$, $\bar{y}=(y_1,\dots,y_\ell)$, $\bar{\theta}=(\theta_1,\dots,\theta_\ell)$ and define[24]

$$G_{k,\ell}(u,v,\bar{x},\bar{y},\bar{\theta},\eta)=F(x_1,v;\eta)+\frac{1}{\ell}\sum_{j=1}^{\ell}S_k(\ell x_j,y_j,\theta_j)+B_\ell(\bar{x},\bar{y})+\langle y_\ell-v,u-x_1\rangle$$

This is a GFQI for $\rho_\ell^{-1}\varphi_k^\ell\rho_\ell^{-1}\alpha=\rho_\ell^{-1}\rho_k^{-1}\varphi^{k\ell}\rho_k\rho_\ell\alpha=\varphi_{k\ell}\alpha$.

Let

$$\bar{G}_{k,\ell}(u,v;\bar{x},\bar{y},\eta)=F(x_1,v,\eta)+\frac{1}{\ell}\sum_{j=1}^{\ell}(h^+_k(y_j)+a\chi_j(y_j))+B_\ell(\bar{x},\bar{y})+\langle y_\ell-v,u-x_1\rangle.$$

[24] Here we omit ω from the notation, which would otherwise become unwieldy.

By definition there is a Borel–Moore cycle, $\Gamma_{k,\ell}$, such that

$$\overline{G}_{k,\ell}((\Gamma_{k,\ell})_U) \leq c(\mu_U, \overline{G}_{k,\ell}) + \varepsilon,$$

and using $C_j(y_j)$ as before for $1 \leq j \leq \ell$ and setting

$$\Gamma_{k,\ell} \times_Y C^-[\ell] = \{(u, v; \bar{x}, \bar{y}, \bar{\theta}, \eta) \mid (u, v, x, y, \eta) \in \overline{\Gamma}, (\ell x_j, \xi_j) \in C_j^-(y_j)\},$$

we have

$$c(\mu_U, G_{k,\ell}) \leq G_{k,\ell}((\Gamma_{k,\ell})_U \times_Y (C^-)_U[\ell]) \leq \overline{G}_{k,\ell}((\Gamma_{k,\ell})_U) \leq c(\mu_U, \overline{G}_{k,\ell}) + 2\varepsilon.$$

Finally we claim that

$$c(\mu_U, \overline{G}_{k,\ell}) \leq c(\mu_U, \overline{G}_k) + \frac{(n+1)a}{\ell}.$$

Indeed, $\overline{G}_{k,\ell}$ is the generating function of $\psi_{k,\ell} = \rho_\ell^{-1} \psi_k^1 \circ \cdots \circ \psi_k^\ell \rho_\ell$, where ψ_k^j is the time-one flow of $h_k(y) + a\chi_j(y)$. But these flows commute, so $\psi_{k,\ell}$ is the time-one flow of

$$K_{k,\ell}(y) = \frac{1}{\ell} \sum_{j=1}^{\ell} (h_{k,U}(y) + a\chi_j(y))$$

and we have $|K_{k,\ell}(y) - h_k(y)| \leq \frac{(1+n)a}{\ell} + \varepsilon$. Therefore

$$c(\mu_U, \overline{G}_{k,\ell}) \leq c(\mu_U, \psi_{k,\ell}\alpha) \leq c(\mu_U, \psi_k^1\alpha) + \frac{(1+n)a}{\ell} + \varepsilon \leq c(\mu_U, \bar{\varphi}_k\alpha) + \frac{(1+n)a}{\ell} + \varepsilon.$$

Thus for ℓ large enough $c(\mu_U, \overline{G}_{k,\ell}) \leq c(\mu_U, \psi_k^1\alpha) + 2\varepsilon$. Taking the limit as k goes to infinity, we get

$$c(\mu_U, \varphi_{k\ell}\alpha) = c(\mu_U, G_{k,\ell}) \leq c(\mu_U, \bar{\varphi}_k\alpha) + 2\varepsilon \leq c(\mu_U, \bar{\varphi}\alpha) + 3\varepsilon.$$

This concludes the proof of Proposition 8.3.

Acknowledgments and general remarks

I would like to acknowledge the hospitality of MSRI during the semester "Hamiltonian systems, from topology to applications through analysis" in the fall of 2018. The idea of this work came through attending talks and several discussions with Fraydoun Rezakhanlou, whom I want to thank here for generously sharing his ideas.

As the reader will check, and analogously to [Viterbo 2023], the methods used here are drawn from symplectic topology. This paper can be considered as part of a program to study symplectic topology in a random framework (or random phenomena having a symplectic structure) of which a foundational example is the random version of Poincaré–Birkhoff theorem from [Pelayo and Rezakhanlou 2018; 2025]. Last but not least, I would like to very warmly thank the referees for their in-depth reading of the manuscript and their many very thoughtful suggestions.

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Received 28 Jul 2021. Revised 27 Dec 2023. Accepted 18 Feb 2024.

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ANALYSIS AND PDE
Vol. 18 (2025), No. 4, pp. 857–894
DOI: 10.2140/apde.2025.18.857

EPSILON-REGULARITY FOR THE BRAKKE FLOW WITH BOUNDARY

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We prove that, if a Brakke flow with boundary is close enough to a stationary half-plane with density 1, then it is $C^{1,\alpha}$. Our approach is based on viscosity techniques introduced by Savin in the context of elliptic equations. The same techniques can be used to give an alternative proof of Brakke's (interior) regularity theorem.

1. Introduction

We state and prove a Brakke-type theorem for the mean curvature flow with boundary, that is, a flow of m-dimensional surfaces in $\mathbb{R}^d$ so that at every point the normal component of the velocity is equal to the mean curvature and the boundary is fixed. A weak notion of such a flow has been recently introduced in [White 2021] by using integral varifolds, as devised by Brakke [1978]. The objects in question are called *integral Brakke flows with boundary*.

In short, given an $(m-1)$-dimensional submanifold Γ, an integral Brakke flow with boundary Γ is a collection $\{V_t\}_{t\in I}$ of m-dimensional integral varifolds with the constraint that the first variation of V_t is a measure whose singular part with respect to $\|V_t\|$ behaves like $\mathcal{H}^{m-1} \llcorner \Gamma$ and the varifolds satisfy an evolution equation that encodes the information on the velocity. A precise definition will be given in Section 2.

The main result of this paper (Theorem 7.1) is that, if a Brakke flow in a ball of radius 1 is close enough (in some appropriate topology) to a unit-density half-plane (which is a stationary solution to the mean curvature flow with a prescribed straight boundary), then the Brakke flow becomes smooth up to the boundary in a smaller ball and after some fixed waiting time. Roughly stated, the main result is the following.

Theorem 1.1 (ε-regularity). *Let Γ be a $C^{1,\alpha}$, $(m-1)$-dimensional submanifold of B_1 and let $\{V_t\}_{t\in[-\Lambda,0]}$ be an integral Brakke flow with boundary Γ in $B_1\times[-\Lambda,0]$. Assume the following*:

(1) $0\in \operatorname{supp}\|V_0\|$.

(2) *At time $t=-\Lambda$, the mass measure $\|V_t\|$ is close to that of an m-dimensional half-disk.*

(3) *There exists a half-plane S^+ such that, for every $t\in[-\Lambda,0]$,*

$$\operatorname{supp}\|V_t\|\subset\{x\in\mathbb{R}^d : \operatorname{dist}(x,S^+)\le\varepsilon\}.$$

MSC2020: primary 53E10; secondary 35B65, 35D40.
Keywords: mean curvature flows, Brakke flows, Brakke's theorem, ε-regularity, varifolds, Allard's theorem, boundary regularity, viscosity, small perturbation solutions.

If ε and Λ are small enough, then there exist small constants η, β and a family $\{N_t\}_{t\in(-\eta^2,0]}$ of $C^{1,\beta}$ surfaces with boundary Γ such that

$$\operatorname{supp}\|V_t\| \cap B_\eta = N_t$$

for every $t \in (-\eta^2, 0]$.

We briefly comment on the assumptions. The key assumptions are (2) and (3), which describe how the Brakke flow is close to being a half-plane (with a straight boundary). Assumption (1), on the other hand, prevents a "pathological" behavior of Brakke flows, which is the possibility of a sudden loss of mass (see, for example, [Tonegawa 2019, Section 2.3]). The statement and the assumptions will be made more rigorous in Sections 4 and 7.

A central point in our work is that, under appropriate assumptions, the support of an integral Brakke flow with boundary satisfies a maximum principle. In order to fix ideas, assume that the support of the flow is the graph of some function $u : \mathbb{R}^m \to \mathbb{R}^{d-m}$. Then it can be proved that $|u|$ is a viscosity subsolution (in a suitable sense which we will describe at a later stage) to

$$\partial_t \varphi - \mathcal{M}^+(D^2\varphi) \le 0,$$

where $\mathcal{M}^+$ is a Pucci maximal operator. We may therefore exploit this property to adopt a technique developed by Savin [2007] in the framework of elliptic equations and later adapted by Wang [2013] to parabolic equations, which we now summarize in our case. The key step in proving Theorem 1.1 is proving the following *improvement of flatness*:

Proposition 1.2 (improvement of flatness). *Under the assumption of Theorem 1.1, there exist $\eta > 0$ and a half-plane T^+ close to S^+ such that, for every $t \in (-\eta^2, 0]$,*

$$\operatorname{supp}\|V_t\| \cap B_\eta \subset \left\{ x \in \mathbb{R}^d : \operatorname{dist}(x, T^+) \le \frac{\varepsilon}{2}\eta \right\}.$$

In summary, if the Brakke flow is "ε-flat" at scale 1, then it becomes "$\eta\varepsilon/2$-flat" at scale η for some η small and universal; from this, proving $C^{1,\alpha}$-regularity is classical.

The proof of Proposition 1.2 is based on a contradiction and compactness argument. Assume one can find a sequence of flatter and flatter Brakke flows for which the conclusion of Proposition 1.2 does not hold. Then appropriate rescalings of the supports of such flows converge in a suitable sense to the graph of a solution to the heat equation. The desired improvement of flatness is a straightforward consequence of classical Schauder estimates. The above convergence is achieved via a Harnack-type inequality, in the spirit of [Wang 2013], and a barrier argument that describes the behavior of the Brakke flow near the boundary.

Theorem 1.1 answers a question left open in [White 2021, Remark 11.2], that is, whether an integral Brakke flow with boundary that has a tangent flow which is a unit-density half-plane is smooth in a backward neighborhood. The reader should also compare our results with the regularity theorems proved in [loc. cit.]. The latter are proved under the additional assumption that the flow is *standard*: namely the flow has to be smooth at every point where a tangent flow is a unit-density half-plane (see [loc. cit., Definition 11.1]). Since we only prove backward regularity, our result does not guarantee (as it should not be expected) that an integral Brakke flow with boundary satisfying the assumptions of Theorem 1.1 is actually standard.

The first ε-regularity theorem for the mean curvature flow (without boundary) was proved in [Brakke 1978] and then refined in [Kasai and Tonegawa 2014], where the authors extended the result to mean curvature flow in general ambient manifolds. Another relevant reference is the recent work [Stuvard and Tonegawa 2024]. The above-mentioned proofs are variational and rely on L^2 energy estimates, somehow in the spirit of [Allard 1972]. We think that a variational proof of Theorem 1.1 may be performed by adapting the arguments in [Allard 1975; Bourni 2016] to account for the presence of the boundary. As mentioned, our proof is based on an argument first developed in [Savin 2007] for elliptic equations and then adapted to parabolic equations in [Wang 2013]. This method was used in [Savin 2018] to prove an Allard-type theorem for minimal surfaces. Although an adaptation of the same techniques to the mean curvature flow seems quite natural, to the best of the author's knowledge this paper is the first instance in which these techniques are used for the mean curvature flow.

The regularity of mean curvature flow with boundary has been briefly investigated also in [White 1995; 2005]. One should also see [Edelen 2020], where the author defines a Brakke flow with a free boundary condition. Another definition of Brakke flow with fixed boundary has been investigated in [Stuvard and Tonegawa 2021].

1.1. ***Structure of the paper.*** In Section 2, we collect some notation that will be used throughout the paper and some well-known facts about rectifiable measures. We then recall the definition of integral Brakke flow with boundary, as stated in [White 2021].

Section 3 is dedicated to collecting some known results about integral Brakke flows and to adapting them to the case of an integral Brakke flow with fixed boundary.

The core of the paper is Section 4, where we state and prove the improvement of flatness described in Proposition 1.2, which will later yield the desired $C^{1,\beta}$ regularity. The proof of this result is described in Section 4.2. The aforementioned barrier argument and Harnack-type inequality, which are crucial for obtaining the desired compactness, are discussed in Sections 5 and 6, respectively.

Finally, the proof of Theorem 1.1 is given in Section 7, where we iterate the improvement of flatness to obtain the desired regularity.

2. Preliminaries, notation and definitions

Throughout the paper, we consider fixed two positive integers m and d such that $m \le d$. All the constants taken in consideration in the present work depend, in general, on these two parameters, although we will mostly avoid stating such dependency.

For the present section, we introduce two generic positive integers $k \le n$ to define some objects in full generality.

2.1. ***Space-time.*** By $\mathbb{R}^{n,1}$ we denote the space $\{(x,t) : x \in \mathbb{R}^n \text{ and } t \in \mathbb{R}\}$. We use uppercase letters to denote points in $\mathbb{R}^{n,1}$, for example $X = (x,t)$.

For any pair $X = (x,t)$ and $Y = (y,s)$ of points in $\mathbb{R}^{n,1}$, we let

$$\rho(X,Y) = |x-y| + |t-s|^{1/2};$$

ρ is a metric on $\mathbb{R}^{n,1}$ (see, for example [Krylov 1996, Exercise 8.5.1]) and the topology that ρ induces on $\mathbb{R}^{n,1}$ coincides with the euclidean topology of $\mathbb{R}^{n+1}$. In particular, if $d_H(E, F)$ is the Hausdorff distance between E and F with respect to ρ and K is a compact subset of $\mathbb{R}^{n,1}$, then the space of nonempty closed subsets of K is a compact metric space, when endowed with the metric d_H.

If $x \in \mathbb{R}^n$ and $r > 0$, we set $B_r^n(x) = \{y \in \mathbb{R}^n : |y - x| < r\}$. When the dimension of the space is clear, we omit its indication and simply write $B_r(x)$. We also omit the indication of the center of the ball, whenever it coincides with 0, so that $B_r = B_r(0)$. If $(x, t) \in \mathbb{R}^{n,1}$, we define the parabolic cylinder

$$Q_r^n(x, t) = B_r^n(x) \times (t - r^2, t],$$

where the apex n indicates the dimension of the space component; as above, its indication will be omitted when no confusion shall arise. Lastly, we let $Q_r = Q_r(0, 0)$.

We denote by $\partial_p(U \times (a, b))$ the *parabolic boundary* of the cylinder $U \times (a, b)$, where $U \subset \mathbb{R}^n$:

$$\partial_p(U \times (a, b)) := (\bar{U} \times \{a\}) \cup (\partial U \times (a, b)).$$

We define the measures $\mathcal{L}^{n,1}$ and $\mathcal{H}^{s,1}$ (for any $0 \le s \le n$) on $\mathbb{R}^{n,1}$ by

$$\mathcal{L}^{n,1}(E \times F) = \mathcal{L}^n(E) \times \mathcal{L}^1(F), \quad \mathcal{H}^{s,1}(E \times F) = \mathcal{H}^s(E) \times \mathcal{L}^1(F)$$

for $E \subset \mathbb{R}^n$ and $F \subset \mathbb{R}$, where $\mathcal{L}^n$ is the Lebesgue measure in $\mathbb{R}^n$ and $\mathcal{H}^s$ is the s-dimensional Hausdorff measure in $\mathbb{R}^n$.

For any function $f : \mathbb{R}^{n,1} \to \mathbb{R}^k$, we denote by $\nabla f(x, t)$ the gradient of the function $f(\,\cdot\,, t)$ computed at x and by $\partial_t f(x, t)$ the derivative of $f(x, \cdot\,)$ computed at t, whenever they are defined.

Lastly, for a set $E \subset \mathbb{R}^n$ and $x \in \mathbb{R}^n$, we let $\chi_E(x) = 0$ if $x \notin E$ and $\chi_E(x) = 1$ if $x \in E$.

2.2. ***Linear functions and subspaces of the euclidean space.*** We let $\{\boldsymbol{e}_1, \dots, \boldsymbol{e}_n\}$ be the canonical orthonormal basis of $\mathbb{R}^n$.

We define the Grassmannian $\mathrm{Gr}(k, n)$ as the space of (unoriented) k-dimensional linear subspaces of $\mathbb{R}^n$; we identify $S \in \mathrm{Gr}(k, n)$ with the endomorphism $S : \mathbb{R}^n \to \mathbb{R}^n$ representing the orthogonal projection onto S. When no confusion shall arise and an orthonormal basis $\{\zeta_1, \dots, \zeta_k\}$ of S is fixed, we identify S with $\mathbb{R}^k$ via the canonical bijection

$$\iota : S \to \mathbb{R}^k, \quad x \mapsto (x \cdot \zeta_1, \dots, x \cdot \zeta_k);$$

therefore by Sx we denote both the point $Sx \in S \subset \mathbb{R}^n$ and its image via ι. In particular, when $S = \mathrm{span}\{\boldsymbol{e}_1, \dots, \boldsymbol{e}_k\}$ and $x \in \mathbb{R}^n$, we will often use the notation $x' = Sx = (x \cdot \boldsymbol{e}_1, \dots, x \cdot \boldsymbol{e}_k) \in \mathbb{R}^k$.

We also let $S : \mathbb{R}^{n,1} \to \mathbb{R}^{k,1}$ be the map $S(x, t) = (Sx, t)$ and, in the case $S = \mathrm{span}\{\boldsymbol{e}_1, \dots, \boldsymbol{e}_k\}$, for $X = (x, t) \in \mathbb{R}^{n,1}$ we let $X' = (x', t)$.

Lastly, if S and T are two endomorphisms of $\mathbb{R}^n$, we define the scalar product between S and T by

$$S : T = \sum_{i,j=1}^{n} S_{ij} T_{ij},$$

where (S_{ij}) is the representation of S as a $n \times n$ matrix such that

$$S_{ij} = \boldsymbol{e}_i \cdot (S\boldsymbol{e}_j).$$

We also let $|S| = \sqrt{S : S}$.

2.3. *Hölder regularity.* We point out some facts and definitions on Hölder regularity for several objects. In what follows, $\kappa \in (0, 1)$ is a fixed parameter.

(1) *Functions on* $\mathbb{R}^n$. Given a function $u : \mathbb{R}^n \supset U \to \mathbb{R}^k$, we say that $u \in C^{1,\kappa}(U; \mathbb{R}^k)$ if there exists $C > 0$ such that $\sup_{x\in U} |u(x)| \le C$ and for all $x \in U$ there is an affine function $L_x : \mathbb{R}^n \to \mathbb{R}^k$ such that, for every $y \in U$, it holds

$$|u(y) - L_x(y)| \le C|x - y|^{1+\kappa}.$$

(2) *Functions on* $\mathbb{R}^{n,1}$. Let $\Omega \subset \mathbb{R}^{n,1}$. We say that $u : \Omega \to \mathbb{R}^k$ is in $C^{1,\kappa}(\Omega; \mathbb{R}^k)$ if there exists $C > 0$ such that $\sup_{X\in\Omega} |u(X)| \le C$ and for all $X \in \Omega$ there is an affine function $L_X : \mathbb{R}^n \to \mathbb{R}^k$ such that, for every $Y = (y, s) \in \Omega$, it holds

$$|u(Y) - L_X(y)| \le C\rho(X, Y)^{1+\kappa}.$$

(3) *Submanifolds*. We say that a k-dimensional, properly embedded submanifold Γ of some open set $U \subset \mathbb{R}^n$ is $C^{1,\kappa}$ if there exists some $\kappa > 0$ such that, for every $x, y \in \Gamma$, it holds

$$[\Gamma]_{C^{1,\kappa}(U)} := \sup_{\substack{x,y\in\Gamma \\ x\neq y}} \frac{|T_x\Gamma - T_y\Gamma|}{|x - y|^\kappa} < \infty,$$

where $T_\cdot\Gamma \in \mathrm{Gr}(k, n)$ is the tangent space to Γ and $|T_x\Gamma - T_y\Gamma|$ should be intended as in Section 2.2.

Remark 2.1. We do not require the sets U and Ω in items (1) and (2) above to have any regularity. However, one can easily see that, if $U \subset \mathbb{R}^n$ has C^1 boundary and $u \in C^{1,\kappa}(U; \mathbb{R}^k)$, then u is actually bounded in U, it is differentiable at every point of $\operatorname{Int} E$ and the usual definition of $C^{1,\kappa}$ holds:

$$\|u\|_{C^{1,\kappa}(U)} := \sup_U |u| + \sup_{\substack{x,y\in U \\ x\neq y}} \frac{|\nabla u(x) - \nabla u(y)|}{|x - y|^\kappa} < \infty.$$

In fact, $\|u\|_{C^{1,\kappa}(U)}$ is bounded (up to some multiplicative constant depending only on U, n, k, κ) by the same constant C as in item (1) above.

Similarly, if $\Omega = U \times I \subset \mathbb{R}^{n,1}$ for some $U \subset \mathbb{R}^n$ with C^1 boundary and $I \subset \mathbb{R}$ some interval, then $u \in C^{1,\kappa}(\Omega; \mathbb{R}^k)$ yields that u is differentiable with respect to the space variable everywhere in $\operatorname{Int} U \times I$ and that

$$\|u\|_{C^{1,\kappa}(\Omega)} := \sup_\Omega |u| + \sup_{\substack{X,Y\in\Omega \\ X\neq Y}} \frac{|\nabla u(X) - \nabla u(Y)|}{\rho(X, Y)^\kappa} + \sup_{\substack{(x,t),(x,s)\in\Omega \\ s\neq t}} \frac{|u(x, t) - u(x, s)|}{|t - s|^{(1+\kappa)/2}} < \infty.$$

2.4. *Integral varifolds.* We adopt most of the terminology from [White 2021]. Let $U \subset \mathbb{R}^d$ be an open set and let $\mathcal{M}(U)$ be the set of nonnegative Radon measures on U; if φ is continuous and compactly supported on U, we let $M(\varphi) = \int \varphi(x)\, dM(x)$ for $M \in \mathcal{M}(U)$. Let $\mathcal{M}_m(U)$ be the set of m-dimensional rectifiable nonnegative Radon measures on U. Namely, $M \in \mathcal{M}_m(U)$ if and only if there exist an m-dimensional rectifiable set E and a nonnegative function $\theta \in L^1_{\mathrm{loc}}(\mathcal{H}^m \llcorner E)$ such that

$$M(\varphi) = \int_E \theta(x)\varphi(x)\, d\mathcal{H}^m(x) \quad \text{for all } \varphi \in C_c(U).$$

We also let $\mathcal{IM}_m(U)$ be the set of those $M \in \mathcal{M}_m(U)$ such that their density $\theta(x)$ is a nonnegative integer at M-a.e. x. If $M \in \mathcal{M}_m(U)$, then for M-a.e. x the approximate tangent space $T_x M \in \mathrm{Gr}(m, d)$ is well-defined (see, for instance, [Simon 1983, Chapter 3]). An m-dimensional varifold on U is a Radon measure on $U \times \mathrm{Gr}(m, d)$ (see [loc. cit., Chapter 8]). In particular, to each $M \in \mathcal{M}_m(U)$ we may associate an m-dimensional varifold $\mathrm{Var}(M)$ by

$$\mathrm{Var}(M)(\varphi) = \int \varphi(x, T_x M)\, dM(x) \quad \text{for all } \varphi \in C_c(U \times \mathrm{Gr}(m, d)).$$

Such an object is called a rectifiable varifold (see [loc. cit., Chapter 4]); $\mathrm{Var}(M)$ is said to be integral if and only if $M \in \mathcal{IM}_m(U)$. If $M \in \mathcal{M}_m(U)$, we say that $\mathrm{Var}(M)$ has bounded first variation if there exists $C > 0$ such that, for every smooth vector field $F : U \to \mathbb{R}^d$ with compact support in U, it holds

$$\int T_x M : \nabla F(x)\, dM(x) \le C\|F\|_\infty.$$

If $\mathrm{Var}(M)$ has bounded first variation, then there exist an M-locally integrable vector field H_M, a Radon measure β_M that is singular with respect to M and a β_M-locally integrable unit vector field ζ_M such that, for every $F \in C^1_c(U; \mathbb{R}^d)$, it holds

$$\int T_x M : \nabla F(x)\, dM(x) = -\int H_M \cdot F\, dM + \int F \cdot \zeta_M\, d\beta_M. \tag{2-1}$$

In the following, we will often write

$$\mathrm{div}_S F(x) = S : \nabla F(x).$$

When $M \in \mathcal{M}_m(U)$, we also let

$$\mathrm{div}_M F(x) := \mathrm{div}_{T_x M} F(x) = T_x M : \nabla F(x),$$

whenever it is well-defined.

Definition 2.2. Let Γ be a properly embedded $(m-1)$-dimensional submanifold of $U \subset \mathbb{R}^d$. We let $\mathcal{V}_m(U, \Gamma)$ be the space of those $M \in \mathcal{IM}_m(U)$ such that $\mathrm{Var}(M)$ has bounded first variation and the following hold true:

(1) $\beta_M(E) \le \mathcal{H}^{m-1}(E \cap \Gamma)$ for every $E \subset U$.

(2) $H_M(x)$ and $T_x M$ are perpendicular at M-a.e. x.

As mentioned in the remark following [White 2021, Definition 6], (2) is actually redundant, as it can be derived from [Brakke 1978, Section 5].

As in [White 2021], for $M \in \mathcal{V}_m(U, \Gamma)$ we let

$$\nu_M(x) = \lim_{r \searrow 0} \frac{1}{\omega_{m-1} r^{m-1}} \int_{B_r(x)} \zeta_M \, d\beta_M, \tag{2-2}$$

where the limit exists, and $\nu_M(x) = 0$ otherwise. Notice that the requirement $\beta_M \leq \mathcal{H}^{m-1} \llcorner \Gamma$ in Definition 2.2 yields $|\nu_M| \leq 1$ $\mathcal{H}^{m-1} \llcorner \Gamma$-a.e. Moreover, by [Allard 1975, Section 3.1], $\nu_M(y) \perp \Gamma$ for $\mathcal{H}^{m-1}$-a.e. $y \in \Gamma$.

In the following, whenever Γ is an $(m-1)$-dimensional submanifold of $\mathbb{R}^d$, by a small abuse of notation we denote by Γ the Hausdorff measure $\mathcal{H}^{m-1} \llcorner \Gamma$, if no confusion shall arise.

2.5. ***Integral Brakke flows with boundary.*** Let $U \subset \mathbb{R}^d$ be an open set, $I \subset \mathbb{R}$ be a nonempty interval and let Γ be a properly embedded $(m-1)$-dimensional submanifold of U.

Definition 2.3 (integral Brakke flow). An m-dimensional integral Brakke flow with boundary Γ in $U \times I$ is a collection $\boldsymbol{M} = \{M_t : t \in I\} \subset \mathcal{M}(U)$ such that the following hold true:

(1) For almost every t, $M_t \in \mathcal{V}_m(U, \Gamma)$.

(2) If $I' \Subset I$ and $U' \Subset U$, then $\int_{I'} \int_{U'} (1 + |H_{M_t}|^2) \, dM_t \, dt < +\infty$.

(3) If $[a, b] \subset I$ and u is a nonnegative, compactly supported, C^1 function on $U \times I$, then

$$\int u(\,\cdot\,, a) \, dM_a - \int u(\,\cdot\,, b) \, dM_b \geq \int_a^b \int (u|H_{M_t}|^2 - H_{M_t} \cdot \nabla u - \partial_t u) \, dM_t \, dt. \tag{2-3}$$

We denote by $\mathcal{BF}_m(U \times I, \Gamma)$ the set of all m-dimensional integral Brakke flows in $U \times I$ with boundary Γ.

When $\Gamma = \varnothing$, we drop its indication and simply write $\mathcal{BF}_m(U \times I)$; notice that in this case $\beta_{M_t} = 0$ for a.e. t, and the definition agrees with the one of integral Brakke flow (without boundary) given, for instance, in [Tonegawa 2019].

Given $\boldsymbol{M} \in \mathcal{BF}_m(U \times I, \Gamma)$, we define its *space-time mass measure* M by

$$\int \varphi(x, t) \, dM(x, t) = \iint \varphi(x, t) \, dM_t \, dt$$

for every $\varphi \in C_c(U \times I)$. We define the space-time track of $\boldsymbol{M}$ to be the closed set

$$\Sigma_{\boldsymbol{M}} = \operatorname{Clos}\Bigl(\bigcup_{t \in I} \operatorname{supp} M_t \times \{t\}\Bigr),$$

where the closure is taken in the euclidean topology of $\mathbb{R}^{d,1}$, and we let $\Sigma_{\boldsymbol{M}}(t)$ be the slice at time t of $\Sigma_{\boldsymbol{M}}$, namely $\Sigma_{\boldsymbol{M}}(t) = \{x \in \mathbb{R}^d : (x, t) \in \Sigma_{\boldsymbol{M}}\}$. It is straightforward to check that $\operatorname{supp} M \subset \Sigma_{\boldsymbol{M}}$. Under reasonable assumptions, the opposite inclusion holds true as well; we further discuss this point in Lemma 3.5. Whenever no confusion may arise, we write Σ and Σ_t in place of $\Sigma_{\boldsymbol{M}}$ and $\Sigma_{\boldsymbol{M}}(t)$, respectively.

Remark 2.4 (scaling properties). A Brakke flow $\boldsymbol{M} \in \mathcal{BF}_m(U \times I, \Gamma)$ may be translated and parabolically dilated while preserving the requirements in Definition 2.3. For $x_0 \in \mathbb{R}^d$ and $r > 0$, let $T_{x_0,r}(y) = (y - x_0)/r$. By $(T_{x_0,r})_\sharp \mu$ we denote the push-forward of $\mu \in \mathcal{M}(\mathbb{R}^d)$ through $T_{x_0,r}$. Then $\boldsymbol{M}' = \{M'_s\}$ given by

$$M'_s = r^{-m} (T_{x_0,r})_\sharp M_{t_0 + r^2 s}$$

is a Brakke flow in $(U - x_0)/r \times (I - t_0)/r^2$ with boundary $(\Gamma - x_0)/r$. In this case, we will write

$$\boldsymbol{M}' = \mathcal{D}_r(\boldsymbol{M} - X_0),$$

where, as usual, $X_0 = (x_0, t_0)$.

3. Properties of integral Brakke flows with boundary

We collect some known results about integral Brakke flows, which we will use throughout the rest of the paper.

3.1. ***Monotonicity properties.*** We denote by $\Psi : \mathbb{R}^d \times (-\infty, 0) \to \mathbb{R}$ the m-dimensional backward heat kernel

$$\Psi(x, t) = \frac{1}{(4\pi(-t))^{m/2}} \exp\left(-\frac{|x|^2}{4(-t)}\right).$$

We also pick a smooth cut-off function $\phi \in C_c^\infty([0, 2))$ such that $\phi \equiv 1$ in $[0, 1]$, $|\phi'| \le 2$ and $0 \le \phi \le 1$ everywhere, which from now on we consider fixed. For the chosen ϕ, for $R > 0$ we set

$$\Psi_R(x, t) = \Psi(x, t)\phi\left(\frac{|x|}{R}\right).$$

Proposition 3.1 (Huisken monotonicity formula). *There exists a universal constant $C > 0$ such that, if $\boldsymbol{M} \in \mathcal{BF}_m(U \times (-T, 0), \Gamma)$ and $B_{2R} \subset U$, then for every $-T < s \le t < 0$ it holds*

$$\int \Psi_R(x, t)\, dM_t - \int \Psi_R(x, s)\, dM_s \le \underbrace{\int_s^t \int \nu_{M_\tau} \cdot \nabla\Psi_R(\,\cdot\,, \tau)\, d\Gamma\, d\tau}_{\mathrm{I}} + \underbrace{C\frac{t-s}{R^2} \sup_{\tau\in[s,t]} \frac{M_\tau(B_{2R})}{R^m}}_{\mathrm{II}}, \tag{3-1}$$

where ν_{M_τ} is defined in (2-2).

Proof. See [White 2021, Theorem 6.1]. □

In several points of the present work, we are going to need some precise bounds on I and II in (3-1). While in most cases we will assume a uniform bound of the form

$$\sup_t \sup_{B_r(x)} \frac{M_t(B_r(x))}{r^m} \le E_1 < \infty,$$

which takes care of II, estimating I requires some more attention. What we prove in the following lemma is that, at a small enough scale, I is close to $\frac{1}{2}$ if $0 \notin \Gamma$; otherwise it is very small.

Lemma 3.2. *For every $\delta > 0$, there exist small positive constants Λ and c with the following property. Let $U \subset \mathbb{R}^d$ be open and let Γ be a $C^{1,\alpha}$ submanifold of U. Then, for every $R \le c/[\Gamma]_{C^{1,\alpha}(U)}$ and for every $(x, t) \in U \times \mathbb{R}$ such that $B_{2R}(x) \subset U$, it holds*

$$\int_{t-\Lambda R^2}^t \int |T_y\Gamma^\perp \nabla\Psi_R(y - x, s - t)|\, d\Gamma(y)\, ds \le \frac{\chi_{\Gamma^c}(x)}{2} + \delta. \tag{3-2}$$

The proof of Lemma 3.2 is somehow cumbersome and is therefore postponed to Appendix A.

Exploiting the above result, we may prove a sort of *clearing-out lemma*, in the spirit of [Tonegawa 2019, Proposition 3.6]. Namely, we prove that, provided we have some control on I and II in (3-1), if a point (x,t) is in the space-time track of $\boldsymbol{M}$, then M_s cannot be too small in a backward neighborhood of (x,t).

Before proceeding with this result, we introduce the following terminology:

Definition 3.3 (maximal density ratio). A Brakke flow $\boldsymbol{M}$ (possibly with boundary) in $U \times I$ is said to have *bounded maximal density ratio* in $U' \times I'$, where $U' \subset U$ and $I' \subset I$, if

$$\operatorname{mdr}(\boldsymbol{M}, U' \times I') := \sup_{B_r(x) \subset U'} \sup_{t \in I'} \frac{M_t(B_r(x))}{r^m} < \infty.$$

Proposition 3.4 (clearing-out lemma). *For every $K < \infty$ there exist positive constants c_1, c_2 with the following property. Let Γ be a $C^{1,\alpha}$ submanifold of U and let $\boldsymbol{M} \in \mathcal{BF}_m(U \times (a,b), \Gamma)$ be such that*

$$\operatorname{mdr}(\boldsymbol{M}, U \times (a,b)) \le K.$$

If $(x,t) \in \Sigma_{\boldsymbol{M}}$, and R is small enough depending on Γ, then

$$M_{t-c_1R^2}(B_{4R}(x)) \ge c_2 R^m.$$

Proof. The proof of the case without a boundary can be found, for example, in [Tonegawa 2019, Proposition 3.6]. For the sake of completeness, we sketch the proof along the same lines as in the case of an integral Brakke flow with boundary.

Corresponding to $\delta = \frac{1}{4}$, choose Λ and c as in Lemma 3.2. Let $(x,t) \in \Sigma$ and let $R \le c/[\Gamma]_{C^{1,\alpha}(U)}$.

We first assume that $x \in \operatorname{supp} M_t$ and that $M_t \in \mathcal{V}_m(U,\Gamma)$, so that in particular $M_t = \theta(\,\cdot\,)\mathcal{H}^m \llcorner E$ for some m-rectifiable set E. Then there exists $y \in B_R(x)$ such that

$$1 \le \theta(y) = \lim_{\tau \nearrow 0} M_t(\Psi_R(\,\cdot - y, \tau)). \tag{3-3}$$

Therefore, by centering Proposition 3.1 at a point $(y, t-\tau)$ and then letting $\tau \nearrow 0$, for any $t_1 < t$, it holds

$$M_{t_1}(\Psi_R(\,\cdot - y, t_1 - t)) \ge \theta(y) - CK\frac{t-t_1}{R^2} - \int_{t_1}^{t}\int \nu_{M_s} \cdot \nabla\Psi_R(\,\cdot - y, s-t)\,d\Gamma\,ds.$$

We now choose c_1 so small that both $CKc_1 \le \frac{1}{8}$ and $c_1 \le \Lambda$ and we set $t_1 = t - c_1R^2$. Then, using Lemma 3.2, we obtain

$$M_{t_1}(\Psi_R(\,\cdot - y, -c_1R^2)) \ge \theta(y) - \tfrac{1}{8} - \left(\tfrac{1}{2} + \tfrac{1}{4}\right) \ge \tfrac{1}{8},$$

where the second inequality is given by (3-3). Notice that, for every $z \in \mathbb{R}^d$, simple computations yield

$$\Psi_R(z - y, -c_1R^2) \le CR^{-m}\chi_{B_{2R}(y)}(z) \le CR^{-m}\chi_{B_{3R}(x)}(z)$$

for some $C > 0$ universal. Hence, by integrating the above inequality in $M_{t-c_1R^2}$, we obtain

$$M_{t-c_1R^2}(B_{3R}(x)) \ge \frac{R^m}{C} M_{t-c_1R^2}(\Psi_R(\,\cdot - y, -c_1R^2)) \ge \frac{R^m}{8C},$$

as desired.

If $x \notin \operatorname{supp} M_t$ or $M_t \notin \mathcal{V}_m(U, \Gamma)$, then one can find a sequence of points (x_i, t_i) such that $M_{t_i} \in \mathcal{V}_m(U, \Gamma)$, $x_i \in \operatorname{supp} M_{t_i}$ and such that $(x_i, t_i) \to (x, t)$. It is then sufficient to choose R_i so that $t_i - c_1 R_i^2 = t - c_1 R^2$ to obtain, for i large enough,

$$M_{t-c_1 R^2}(B_{4R}(x)) \geq M_{t-c_1 R^2}(B_{3R}(x_i)) \geq c_2 R^m. \qquad \square$$

We now state two important consequences of Proposition 3.4.

Lemma 3.5. *Let $\boldsymbol{M} \in \mathcal{BF}_m(U \times I, \Gamma)$ have bounded maximal density ratio in $U \times I$ and let $\Gamma \in C^{1,\alpha}(U)$. Then*

$$\Sigma_{\boldsymbol{M}} = \operatorname{supp} M.$$

Proof. The inclusion $\operatorname{supp} M \subset \Sigma$ is trivial. For the opposite inclusion, notice that, for a point $(x, t) \in \Sigma$ and for every $r > 0$ small enough, Proposition 3.4 gives

$$M_{t-cr^2}(B_r(x)) \geq cr^m.$$

It is now sufficient to integrate this inequality in r to obtain that for every $r > 0$ small enough, there is a set of the form

$$A_r = \left\{(y, s) : |y - x| \leq \theta\sqrt{t-s} \leq r\right\}$$

for some positive θ, c depending only on $\operatorname{mdr}(\boldsymbol{M})$ such that $M(A_r) > 0$; hence $(x, t) \in \operatorname{supp} M$, as claimed. $\square$

Lemma 3.6. *Let $\boldsymbol{M} \in \mathcal{BF}_m(U \times I, \Gamma)$ have bounded maximal density ratio and let $\Gamma \in C^{1,\alpha}(U)$. Then*

$$M \geq \mathcal{H}^{m,1} \llcorner \Sigma_{\boldsymbol{M}}.$$

For the proof of the above lemma, we refer the reader to Appendix B.

3.2. ***Maximum principle.*** In the present subsection, we assume that $U \subset \mathbb{R}^d$ is open and $I \subset \mathbb{R}$ is an interval of the form $(a, b]$. We also let Γ be an $(m-1)$-dimensional, $C^{1,\alpha}$ submanifold of U.

The main result of the present section is the following maximum principle.

Proposition 3.7 (maximum principle). *Let $\boldsymbol{M} \in \mathcal{BF}_m(U \times I, \Gamma)$ have bounded maximal density ratio.*

If there exist $u \in C^2(U \times I)$ and a point $(x_0, t_0) \in \Sigma \setminus \partial_p(U \times I)$ with $x_0 \notin \Gamma$ such that $u|_{\Sigma \cap \{t \leq t_0\}}$ has a local maximum at (x_0, t_0) and $\nabla u(x_0, t_0) \neq 0$, then

$$\partial_t u(x_0, t_0) - \inf_{\substack{T \in \mathrm{Gr}(m,d) \\ T \perp \nabla u(x_0,t_0)}} T : D^2 u(x_0, t_0) \geq 0.$$

Proof. This proposition is a corollary of the results in [Hershkovits and White 2023, Section 13]; see also [Ambrosio and Soner 1997]. For the reader's convenience, we give a self-contained proof, in the spirit of, for example, [White 2016].

We may assume, without loss of generality, that $(x_0, t_0) = (0, 0)$ and that $u|_{\Sigma \cap \{t \leq 0\}}$ has a strict local maximum at $(0, 0)$ (if not, replace u by $u - |x|^4 - |t|^2$).

Step 1: We first prove that

$$\partial_t u(0,0) - \operatorname{trace}_m D^2 u(0,0) \geq 0,$$

where $\operatorname{trace}_m D^2 u$ is the sum of the m smallest eigenvalues of $D^2 u$. Assume the result does not hold. Arguing as in [White 2016, Lemma 2.4], we may assume that, for some $\rho > 0$ and $\varepsilon > 0$ small, u satisfies

(i) $\partial_t u - \operatorname{trace}_m D^2 u < -\varepsilon < 0$ in Q_ρ,

(ii) $B_\rho \cap \Gamma = \varnothing$,

(iii) $u > \varepsilon > 0$ in $\Sigma \cap Q_{\rho/2}$ and $u < 0$ in $\Sigma \cap \{t \leq 0\} \setminus Q_\rho$.

We now let $\varphi(x,t) = (u^+(x,t))^4$, where $u^+ = \max\{u, 0\}$, and we use φ as a test function for (2-3). Since $\varphi(\,\cdot\,, -\rho^2) = 0$ by assumption, we have

$$\begin{aligned} 0 \leq \int \varphi(\,\cdot\,,0)\, dM_0 &= \int \varphi(\,\cdot\,,0)\, dM_0 - \int \varphi(\,\cdot\,,-\rho^2)\, dM_{-\rho^2} \\ &\leq \int_{-\rho^2}^0 \int (-|H|^2 \varphi + H \cdot \nabla\varphi + \partial_t \varphi)\, dM_t\, dt, \end{aligned}$$

where the last inequality is given by (2-3) and we have set $H(\,\cdot\,,t) = H_{M_t}(\,\cdot\,)$ for a.e. t. We now use the fact that $\operatorname{supp}\varphi \subset \Gamma^c$; thus

$$\int H \cdot \nabla\varphi\, dM_t = -\int \operatorname{div}_{M_t} \nabla\varphi\, dM_t$$

for a.e. t. Since the term $|H|^2\varphi$ is nonnegative, we obtain from the above chain of inequalities

$$0 \leq \int_{-\rho^2}^0 \int (-\operatorname{div}_{M_t} \nabla\varphi + \partial_t\varphi)\, dM_t\, dt.$$

Some straightforward computations show that

$$\operatorname{div}_{M_t} \nabla\varphi = 4(u^+)^3 \operatorname{div}_{M_t} \nabla u \geq 4(u^+)^3 \operatorname{trace}_m D^2 u$$

and $\partial_t \varphi = 4(u^+)^3 \partial_t u$. Therefore

$$0 \leq \int_{-\rho^2}^0 \int 4(u^+)^3 (\partial_t u - \operatorname{trace}_m D^2 u)\, dM_t\, dt \leq -4\varepsilon^4 M(Q_{\rho/2}),$$

where the last inequality is given by (i) and (iii) above. In particular, it must be

$$M(Q_{\rho/2}) = 0;$$

however, by Lemma 3.5, $(0,0) \in \Sigma = \operatorname{supp} M$; thus we reach a contradiction.

Step 2: We now prove the general result. It is sufficient to show that one can find an m-dimensional subspace $\overline{T}$ of $\mathbb{R}^d$ such that

$$\partial_t u(0,0) - \overline{T} : D^2 u(0,0) \geq 0$$

and $\overline{T} \nabla u(0,0) = 0$. Without loss of generality, assume that $u(0,0) = 0$. Let $\psi_j(z) = z + (j/2) z^2$ and let

$$u_j(X) = \psi_j(u(X)).$$

Then, for every j, $u_j|_{\Sigma\cap\{t\le 0\}}$ has a local maximum at $(0, 0)$. Thus, by Step 1, there is an m-dimensional subspace T_j of $\mathbb{R}^d$ such that, at $(0, 0)$,

$$0 \le \partial_t u_j - T_j : D^2 u_j = \partial_t u - T_j : D^2 u - jT_j : (\nabla u \otimes \nabla u).$$

Up to a subsequence, which we do not relabel, we have that $T_j \to \overline{T}$ for some m-dimensional subspace $\overline{T}$, and

$$\overline{T} : (\nabla u \otimes \nabla u) \le \liminf_j \frac{1}{j}(\partial_t u - T_j : D^2 u) = 0;$$

thus $\overline{T} \perp \nabla u$. On the other hand, since $jT_j : (\nabla u \otimes \nabla u) \ge 0$, we have

$$\overline{T} : D^2 u \le \liminf_j T_j : D^2 u \le \liminf_j (\partial_t u - jT_j : (\nabla u \otimes \nabla u)) \le \partial_t u,$$

as desired. □

Given an upper-semicontinuous function $u : \mathbb{R}^{m,1} \to [0, 1] \cup \{-\infty\}$ and a smooth function $\varphi : \mathbb{R}^{m,1} \to \mathbb{R}$, we say that φ *touches* u *from above at* $(x_0', t_0) \in \mathbb{R}^{m,1}$ if there exists $r > 0$ such that

$$\begin{cases} \varphi(x', t) \ge u(x', t) & \text{for every } (x', t) \in Q_r^m(x_0', t_0), \\ \varphi(x_0', t_0) = u(x_0', t_0). \end{cases}$$

We recall the definition of Pucci's maximal operator (see, for instance, [Caffarelli and Cabré 1995, Section 2.2]). For a symmetric matrix $N \in \mathbb{R}^{d\times d}$, we let

$$\mathcal{M}^+(N) := \mathcal{M}^+\Big(N, \frac{1}{2}, 2\Big) = \frac{1}{2}\sum_{\lambda_i<0}\lambda_i + 2\sum_{\lambda_i>0}\lambda_i, \tag{3-4}$$

where $\lambda_i = \lambda_i(N)$ are the eigenvalues of N. The following result is a consequence of Proposition 3.7.

Corollary 3.8. *Let* $M \in \mathcal{BF}_m(\mathbb{R}^{d,1})$ *have bounded maximal density ratio. For every* $(x', t) \in \mathbb{R}^{m,1}$, *let*

$$u(x', t) = \sup\{|z| : z \in \mathbb{R}^{d-m} \text{ and } (x', z) \in \Sigma_M(t)\},$$

with the convention $\sup\varnothing = -\infty$ *and assume that* $u \le 1$ *everywhere. There is* $\delta > 0$ *universal such that, whenever a smooth function* $\varphi : \mathbb{R}^{m,1} \to \mathbb{R}$ *touches* u *from above at* $X_0' = (x_0', t_0)$ *and* $\max\{|D^2\varphi(X_0')|, |\nabla\varphi(X_0')|\} \le \delta$, *it holds*

$$\partial_t \varphi(X_0') - \mathcal{M}^+(D^2\varphi(X_0')) \le 0.$$

Proof. We assume $x_0' = 0$ and $t_0 = 0$. Notice that, since Σ is closed and $u(0, 0) = \varphi(0, 0)$, the supremum in the definition of u is attained and, without loss of generality, we may assume that the contact point is $x_0 = \varphi(0, 0)\boldsymbol{e}_d \in \Sigma_0$. We let $S = \operatorname{span}\{\boldsymbol{e}_1, \dots, \boldsymbol{e}_m\}$ and $S' = \operatorname{span}\{\boldsymbol{e}_{m+1}, \dots, \boldsymbol{e}_{d-1}\}$, so that $\mathbb{R}^d = S + S' + \operatorname{span}\{\boldsymbol{e}_d\}$. Consider the function

$$H(x, t) = \tfrac{1}{4}|S'x|^2 + x \cdot \boldsymbol{e}_d - \varphi(Sx, t).$$

By assumption, in a neighborhood of $(x_0, 0)$ it holds

$$|S^\perp x| \le \varphi(Sx, t) \le 2$$

for every $(x,t)\in\Sigma$; therefore it can be checked that $H|_{\Sigma\cap\{t\le 0\}}\le 0$ in the same neighborhood. Since $H(x_0,0)=0$, we know $H|_{\Sigma\cap\{t\le 0\}}$ has a local maximum at $(x_0,0)$. Hence, by Proposition 3.7, it holds

$$\partial_t H(x_0,0)-\inf_{T\perp\nabla H(x_0,0)} T:D^2H(x_0,0)\ge 0. \tag{3-5}$$

We now estimate the two summands in the above inequality. In order to do so, we first remark that

$$\nabla H(x_0,0)=\begin{pmatrix}-\nabla\varphi(0,0)\\ 0\\ 1\end{pmatrix},\qquad D^2H(x_0,0)=\begin{pmatrix}-D^2\varphi(0,0) & 0 & 0\\ 0 & I_{S'}/2 & 0\\ 0 & 0 & 0\end{pmatrix}.$$

Consider $T\in\mathrm{Gr}(m,d)$ and an orthonormal basis $\zeta_1,\dots,\zeta_m$ of T. Then

$$\begin{aligned}T:D^2H&=\sum_{i=1}^m\langle D^2H\zeta_i;\zeta_i\rangle=\sum_{i=1}^m\Big(-\langle D^2\varphi(Sx,t)S\zeta_i;S\zeta_i\rangle+\frac{|S'\zeta_i|^2}{2}\Big)\\ &\ge-\sum_{i=1}^m\langle D^2\varphi(Sx,t)S\zeta_i;S\zeta_i\rangle.\end{aligned}$$

In particular, $S:D^2H(x,t)=-\Delta\varphi(Sx,t)$ and

$$\begin{aligned}T:D^2H&=S:D^2H+(T-S):D^2H\\ &\ge-\Delta\varphi-|T-S|\,|D^2\varphi|.\end{aligned}$$

Now, if $|T-S|\le c_1$, then the above inequality yields that, for some small c_1 universal,

$$T:D^2H(x_0,0)\ge-\mathcal{M}^+(D^2\varphi(0,0)).$$

On the other hand, if $|T-S|\ge c_1$, then we may choose an orthonormal basis $\zeta_1,\dots,\zeta_m$ of T such that $|S^\perp\zeta_1|\ge c_2$ for some c_2 universal. Since we are also assuming $T\perp\nabla H(x_0,0)$, we have

$$0=\zeta_1\cdot\nabla H(x_0,0)=-S\zeta_1\cdot\nabla\varphi(0,0)+\zeta_1\cdot\boldsymbol{e}_d.$$

Thus, in particular, $|\zeta_1\cdot\boldsymbol{e}_d|\le|\nabla\varphi|\le\delta$ and

$$|S'\zeta_1|\ge|S^\perp\zeta_1|-|\zeta_1\cdot\boldsymbol{e}_d|\ge c_2-\delta\ge\frac{c_2}{2},$$

provided $\delta\le c_2/2$. Therefore

$$T:D^2H(x_0,0)\ge-\sum_{i=1}^m\langle D^2\varphi(0,0)\zeta_i;\zeta_i\rangle+\frac{|S'\zeta_1|^2}{2}\ge-\Delta\varphi(0,0)+\frac{c_2^2}{8}\ge-C\delta+\frac{c_2^2}{8}$$

for some C universal, since $|D^2\varphi(0,0)|\le\delta$ by assumption. We may choose δ smaller, if needed, so that

$$-\mathcal{M}^+(D^2\varphi(0,0))\le C\delta\le-C\delta+\frac{c_2^2}{8}.$$

Therefore, whether $|T-S|\le c_1$ or not, it holds

$$T:D^2H(x_0,0)\ge-\mathcal{M}^+(D^2\varphi(0,0)).$$

We conclude the proof by remarking that

$$\partial_t H(x_0,0)=-\partial_t\varphi(0,0).$$

Thus (3-5) gives the desired result. □

Remark 3.9. With some more accurate computations, one may show that, actually, at the contact point φ satisfies the inequality

$$\partial_t\varphi - \sqrt{1+|\nabla\varphi|^2}\,\mathrm{div}\left(\frac{\nabla\varphi}{\sqrt{1+|\nabla\varphi|^2}}\right) \le 0.$$

However, the weaker result proved in Corollary 3.8 will be sufficient for the rest of the paper.

4. Improvement of flatness

This section is the core of the present work. We prove that if a Brakke flow with boundary is sufficiently flat in Q_1, then its flatness can be improved at a smaller universal scale. This is going to allow us to prove the desired $C^{1,\beta}$ regularity; see Section 7.

We introduce the following notation. We fix an m-dimensional subspace of $\mathbb{R}^d$, which we denote by S, and an $(m-1)$-dimensional subspace of $\mathbb{R}^d$, which we denote by Γ_0, such that $\Gamma_0 \subset S$. Up to changing coordinates in $\mathbb{R}^d$, we shall assume for the rest of the present section that $S = \mathrm{span}\{\boldsymbol{e}_1, \dots, \boldsymbol{e}_m\}$ and that $\Gamma_0 = \mathrm{span}\{\boldsymbol{e}_1, \dots, \boldsymbol{e}_{m-1}\}$. We also let $S^+ = S \cap \{x_m > 0\}$.

Given an $(m-1)$-dimensional submanifold Γ of B_R, we write $\Gamma \in \mathcal{F}_\alpha(\delta, B_R)$ if the following hold:

- Γ is a $C^{1,\alpha}$ submanifold of B_R and $[\Gamma]_{C^{1,\alpha}(B_R)} \le \delta R^{-\alpha}$.
- $0 \in \Gamma$ and $T_0\Gamma = \Gamma_0$.

In passing, we remark that if $\Gamma \in \mathcal{F}_\alpha(\delta, B_R)$ and $\theta > 0$, then $\theta\Gamma \in \mathcal{F}_\alpha(\delta, B_{\theta R})$.

Moreover, if $\Gamma \in \mathcal{F}_\alpha(\delta, B_R)$, and δ is smaller than some constant depending only on α, then there exists $\gamma : \Gamma_0 \cap B_R \to \Gamma_0^\perp$ such that $|\gamma(0)| = |\nabla\gamma(0)| = 0$, $\|\gamma\|_{C^{1,\alpha}(B_R)} \le \delta R^{-\alpha}$ and

$$\Gamma = \{x + \gamma(x) : x \in \Gamma_0 \cap B_R\} \cap B_R;$$

given $\Gamma \in \mathcal{F}_\alpha(\delta, B_R)$, we will always implicitly define γ as above.

The following is the main result of the present section.

Theorem 4.1 (improvement of flatness). *For every E_0 and α, there exist constants Λ, ε_0, η, β (small) and C (large) with the following property. Let $\varepsilon \le \varepsilon_0$, $\Gamma \in \mathcal{F}_\alpha(\varepsilon, B_1)$ and $\boldsymbol{M} \in \mathcal{BF}_m(B_1 \times [-\Lambda, 0], \Gamma)$ be such that $(0,0) \in \Sigma_{\boldsymbol{M}}$,*

$$\Sigma_{\boldsymbol{M}} \subset \{(z,\tau) : \mathrm{dist}(z, S^+) \le \varepsilon\},$$

$$\sup_{t\in[-\Lambda,0]} M_t(B_1) \le E_0$$

and

$$\int_{B_1} \Psi(\,\cdot\,, -\Lambda)\, dM_{-\Lambda} \le \frac{3}{4}. \tag{4-1}$$

Then there exists a half-plane T^+ of the form

$$T^+ = \{x + w\zeta : x \in \Gamma_0,\ w > 0\} \tag{4-2}$$

for some $\zeta \in \Gamma_0^\perp$ with $|\zeta - \boldsymbol{e}_m| \le C\varepsilon$ such that

$$\Sigma_{\boldsymbol{M}} \cap Q_\eta \subset \{(x,t) : \mathrm{dist}(x, T^+) \le \eta^{1+\beta}\varepsilon\}. \tag{4-3}$$

The proof of Theorem 4.1 is based on a contradiction and compactness argument. If one assumes the conclusion does not hold, then it is possible to find a sequence of Brakke flows which are flatter and flatter and satisfy the other assumptions of Theorem 4.1, for which, however, no half-plane of the form (4-2) can be found so that the flatness improves at any smaller scale. However, for such flows, one shows that, after an appropriate rescaling, the space-time tracks must converge in the Hausdorff distance to the graph of a solution to the heat equation. It is then sufficient to use Schauder estimates for the heat equation with Dirichlet boundary condition to recover the conclusion.

The central point of the proof is to obtain the desired compactness. This is achieved via the two following results. The first one provides a control over the oscillations near Γ of the space-time support of a Brakke flow satisfying the assumptions of Theorem 4.1.

Proposition 4.2 (boundary behavior). *For every E_0 and α, there exist small constants c_1 and r_1 with the following property. Let $\boldsymbol{M}$ and Γ satisfy the assumptions of Theorem 4.1. Then*

$$\Sigma \cap Q_{r_1} \subset \left\{ (x - \gamma(x'')) \cdot \boldsymbol{e}_m \geq -\varepsilon^2 + c_1 \frac{|S^\perp(x - \gamma(x''))|^2}{2\varepsilon^2} \right\}.$$

Here, x'' denotes the point $(x_1, \dots, x_{m-1}, 0, \dots, 0) \in \Gamma_0$.

With the above results at hand, we may prove that, if the Brakke flow is flat enough, then assumption (4-1) gives a Holder-type modulus of continuity in parabolic cylinders whose radii are controlled from below by some power of the flatness ε.

Proposition 4.3 (decay of oscillations). *For every E_0 and α, there exist constants ς, C_2 and r_2 with the following property. Let $\boldsymbol{M}$ and Γ satisfy the assumptions of Theorem 4.1 and let $(x, t), (y, s) \in \Sigma \cap Q_{r_2}$. If $\min\{x_m, y_m\} \geq 2\varepsilon$ and*

$$\rho := \rho((x', t), (y', s)) \geq C_2 \varepsilon^\varsigma,$$

then

$$|S^\perp(x - y)| \leq C_2 \varepsilon \rho^\varsigma.$$

The two above results are sufficient to prove, via an Arzelà–Ascoli-type argument, the convergence in the Hausdorff distance which we have described.

Before proceeding, it is worth spending a few words on how the constants in Theorem 4.1 will be chosen.

- We fix Λ once and for all in Proposition 4.4; it will be needed to prove that $\boldsymbol{M}$ has bounded maximal density ratio in a smaller parabolic cylinder, Q_{r_3}.
- Propositions 4.2, 4.3 and 4.4 hold true provided ε_0 is small enough (depending on E_0). We will therefore always assume that this is the case. The final value of ε_0 will not be determined explicitly, as Theorem 4.1 is proved by compactness.
- The constants r_1 and r_2 chosen in Propositions 4.2 and 4.3 are chosen smaller than r_3 (determined in Proposition 4.4) and they depend on E_0 and α. These two constants will give upper bounds for η. We will then give a further upper bound for η coming from the regularity properties of the heat equation.
- Lastly, the constants C and β depend only on α and on regularity properties for the heat equation.

We now briefly describe the rest of the present section. The proof of Theorem 4.1 is given in Section 4.2. In Section 4.1, we state and prove some lemmas which will be useful in the following. The proofs of Propositions 4.2 and 4.3 are postponed to Sections 5 and 6, respectively.

4.1. *Preliminaries to the proof of Theorem 4.1.* Some remarks on the assumptions of Theorem 4.1 will be needed for the proofs of Propositions 4.2 and 4.3 and, ultimately, of Theorem 4.1 itself. We begin by showing that (4-1) propagates in the interior of the domain.

Proposition 4.4 (propagation of small density). *For every E_0 and α, there is r_3 small with the following property. Let $\boldsymbol{M}$ and Γ satisfy the assumptions of Theorem 4.1. Then, for every $(x,t) \in Q_{r_3}$ and for every $\tau \in (-r_3^2, 0)$, it holds*

$$\int_{B_{r_3}(x)} \Psi(\,\cdot - x, \tau)\, dM_t \le \frac{7}{8} + \frac{\chi_{\Gamma^c}(x)}{2}.$$

Proof. We fix positive constants $r_3 \le \frac{1}{8}$, ε, Λ and δ, all of which we will determine later; we always assume that r_3 is much smaller than Λ. For simplicity of notation, in this proof we set $r = r_3$. For $(x,t) \in Q_r$ and $\tau \in (-r^2, 0)$, we let $t_0 = t - \tau$. Then, by Proposition 3.1, it holds

$$\begin{aligned}\int_{B_r(x)} \Psi(\,\cdot - x, t - t_0)\, dM_t &\le \int \Psi_{1/8}(\,\cdot - x, t - t_0)\, dM_t \\ &\le \int \Psi_{1/8}(\,\cdot - x, -\Lambda - t_0)\, dM_{-\Lambda} \\ &\qquad + \int_{-\Lambda}^{t} \int \nu_M \cdot \nabla \Psi_{1/8}(\,\cdot - x, s - t_0)\, d\Gamma\, d\tau + C E_0 (t + \Lambda). \end{aligned} \tag{4-4}$$

By Lemma 3.2, if ε and Λ are small enough and r is much smaller than Λ, then

$$\int_{-\Lambda}^{t} \int \nu_M \cdot \nabla \Psi_{1/8}(\,\cdot - x, s - t_0)\, d\Gamma\, d\tau \le \int_{t_0 - 2\Lambda}^{t_0} \int |T_y \Gamma^{\perp} \nabla \Psi_{1/8}(y - x, s - t_0)|\, d\Gamma(y)\, d\tau \le \frac{\chi_{\Gamma^c}(x)}{2} + \delta.$$

We then take Λ even smaller so that $C E_0 (t + \Lambda) \le C E_0 \Lambda \le \delta$.

So far, we have fixed ε and Λ depending only on E_0 and δ, and we have assumed that r is much smaller than Λ. The last step is to choose r even smaller in order to bound (4-4) from above. To this end, we let L be the Lipschitz constant of Ψ restricted to $\mathbb{R}^d \times (-\infty, -\Lambda/2]$. Since r is much smaller than Λ, $-\Lambda - t_0 \le -\Lambda/2$ and we can estimate, for every $y \in B_{1/4}(x)$,

$$\begin{aligned}\Psi_{1/8}(y - x, -\Lambda - t_0) &\le \Psi(y - x, -\Lambda - t_0) \\ &\le \Psi(y, -\Lambda) + L(|x| + |t_0|) \\ &\le \Psi(y, -\Lambda) + 2Lr.\end{aligned}$$

Let now $b = b(\Lambda) > 0$ be so small that $\Psi(y, -\Lambda) \ge b$ if $|y| \le \frac{1}{2}$. In particular, assuming that $r \le \frac{1}{4}$, for every $y \in B_{1/4}(x)$, it holds

$$\Psi_{1/8}(y - x, -\Lambda - t_0) \le \left(1 + \frac{2Lr}{b}\right) \Psi(y, -\Lambda).$$

The same bound holds, trivially, for any y such that $|y-x| \geq \frac{1}{4}$. We now choose r even smaller, if needed, so that $2Lr/b \leq \delta$. Therefore we may bound

$$\int \Psi_{1/8}(\cdot - x, t_0 + \Lambda)\, dM_{-\Lambda} \leq (1+\delta) \int_{B_1} \Psi(\cdot, -\Lambda)\, dM_{-\Lambda} \leq \frac{3(1+\delta)}{4},$$

which yields the desired conclusion, up to choosing δ small universal. □

Corollary 4.5 (bound on $\mathrm{mdr}(\boldsymbol{M})$). *Under the assumptions of Proposition 4.4, there exist E_1 universal such that, for every $t \in [-r_3^2, 0]$ and every $B_r(x) \subset B_{r_3}$, it holds*

$$M_t(B_r(x)) \leq E_1 r^m. \tag{4-5}$$

In particular, for every $(x,t) \in \Sigma_{\boldsymbol{M}} \cap Q_{r_3}$ and for every $r > 0$ small enough, it holds

$$M_{t-c_1 r^2}(B_r(x)) \geq c_2 r^m \tag{4-6}$$

for some c_1, c_2 small universal.

Proof. Let x, t and r be as in the statement. Then

$$M_t(B_r(x)) \leq C r^m \int_{B_r(x)} \Psi(\cdot - x, -r^2)\, dM_t \leq 2C r^m,$$

and (4-6) follows from (4-5) and Proposition 3.4. □

4.2. ***Proof of Theorem 4.1.*** As stated earlier, we are going to argue by contradiction and compactness. Namely, we fix E_0 and α, we let Λ be as specified in Proposition 4.4 and we assume there exist $\varepsilon_j \searrow 0$ and two sequences $\{\Gamma^j\}$, $\{\boldsymbol{M}^j\}$ such that, for every j, $\boldsymbol{M}^j$ and Γ^j satisfy the assumptions of Theorem 4.1 with ε_0 replaced by ε_j.

In particular, we assume that $\Gamma^j \in \mathcal{F}_\alpha(\varepsilon_j, B_1)$ and

$$\Sigma_{\boldsymbol{M}^j} \subset \{(z,\tau) : \mathrm{dist}(z, S^+) \leq \varepsilon_j\}. \tag{4-7}$$

We also assume, for the sake of contradiction, that for no j (4-3) is satisfied for any choice of T^+, η and β.

In the following, we let $\gamma^j : \Gamma_0 \cap B_1 \to \Gamma_0^\perp$ be such that $\Gamma^j \cap B_1 \subset \operatorname{graph} \gamma^j$, as in the definition of $\mathcal{F}_\alpha(\varepsilon_j, B_1)$, and we let $\Sigma^j := \Sigma_{\boldsymbol{M}^j}$. We also fix $r_0 = \min\{r_1, r_2, r_3\}$, so that the conclusions of Propositions 4.2, 4.3, 4.4 and of Corollary 4.5 hold true in Q_{r_0}.

Lemma 4.6 (compactness and convergence to hyperplane). *There exists a subsequence (not relabeled) such that, for almost every $t \in (-r_0^2, 0]$,*

$$M_t^j \rightharpoonup \mathcal{H}^m \llcorner S^+$$

as Radon measures in B_{r_0}.

Proof. By the Arzelà–Ascoli theorem, $\gamma^j \to 0$ in C^1 up to subsequences. By Corollary 4.5, we may apply the compactness theorems proven in [White 2021, Theorems 10.1 and 10.2] and find a further subsequence (not relabeled) and $\boldsymbol{M}^\infty \in \mathcal{BF}_m(Q_{r_0}, \Gamma_0)$ such that, for every $t \in (-r_0^2, 0]$,

$$M_t^j \rightharpoonup M_t^\infty.$$

In particular, the weak convergence stated above and (4-7) yield

$$M_t^\infty((S^+)^c) = 0$$

for every $t \in (-r_0^2, 0]$. Therefore, by Definition 2.3, for almost every t, there is an integer-valued function $\theta_t \in L^1_{\mathrm{loc}}(S^+)$ so that

$$M_t^\infty = \theta_t(\,\cdot\,)\mathcal{H}^m \llcorner (S^+ \cap B_{r_0}).$$

By testing (2-1) with vector fields $X \in C^1_c(B_{r_0} \setminus \Gamma_0; \mathbb{R}^d)$ such that $S^\perp X = 0$ everywhere, one deduces that, for almost every t, $\theta_t(\,\cdot\,)$ is an integer-valued $W^{1,1}_{\mathrm{loc}}$ function on S^+. Since $S^+ \cap B_{r_0}$ is connected, $\theta_t(\,\cdot\,)$ must be constant for almost every t. Moreover, by (4-6), $(0, 0) \in \Sigma_{\boldsymbol{M}^\infty}$; thus by Proposition 3.4 it must be $\theta_t > 0$ for every $t < 0$. We conclude by noting that, with the above remarks, for almost every t, $\beta_{M_t^\infty} = \theta_t \mathcal{H}^{m-1} \llcorner \Gamma_0$; then the assumption $M_t^\infty \in \mathcal{V}(B_{r_0}, \Gamma_0)$ yields $\theta_t = 1$. □

Before stating the next result, we define some objects that we will use in the rest of the subsection. First of all, let $F_\varepsilon : \mathbb{R}^d \to \mathbb{R}^d$ be the map

$$F_\varepsilon(x) = \Big(Sx, \frac{1}{\varepsilon} S^\perp x\Big);$$

with a small abuse of notation, we use the same notation for the map $F_\varepsilon : \mathbb{R}^{d,1} \to \mathbb{R}^{d,1}$ such that $F_\varepsilon(x, t) = (F_\varepsilon(x), t)$. We now define

$$\widetilde{\Sigma}^j = F_{\varepsilon_j}(\Sigma^j).$$

Notice that, by (4-7), $\widetilde{\Sigma}^j \subset \{(x, t) : |S^\perp x| \le 1\}$ for every j. For $j \in \mathbb{N}$ and $(x', t) \in Q_{r_0}$, we define

$$u^j(x', t) = \{z \in \overline{B_1^{d-m}} : ((x', z), t) \in \widetilde{\Sigma}^j\};$$

notice that such a set may well be empty or have more than one element. We also define $\tilde{\gamma}^j = F_{\varepsilon_j} \circ \gamma^j$; it is clear that

$$\tilde{\gamma}^j \cdot \boldsymbol{e}_m \to 0 \quad \text{in } C^{1,\alpha}.$$

Furthermore, since $\|\tilde{\gamma}^j\|_{C^{1,\alpha}(B_{r_0})} \le 1$, by the Arzelà–Ascoli theorem and up to passing to a subsequence (which we do not relabel) we may find $g : B_{r_0}^{m-1} \to \overline{B_1^{d-m}}$ such that, for every $0 < \varsigma < \alpha$,

$$S^\perp \tilde{\gamma}^j \to g \quad \text{in } C^{1,\varsigma}$$

and $\|g\|_{C^{1,\varsigma}} \le 1$.

In order to keep the notation light, in the following we denote by $E = \overline{B_{r_0}^m} \times \overline{B_1^{d-m}} \times [-r_0^2, 0] \subset \mathbb{R}^{d,1}$ and $E' = S(E) = \overline{Q_{r_0}^m} \subset \mathbb{R}^{m,1}$. We also let $E'_+ = E' \cap \{x_m \ge 0\}$.

Lemma 4.7 (uniform convergence). *There exist a subsequence* (*not relabeled*) *and* $u : E'_+ \to \overline{B_1^{d-m}}$ *with the following properties*:

(i) *It holds*

$$d_H(\widetilde{\Sigma}^j \cap E; \operatorname{graph} u) \to 0 \tag{4-8}$$

as $j \to \infty$.

(ii) *For every* $(x'', t) \in \overline{Q_{r_0}^{m-1}}$ *it holds* $u((x'', 0), t) = g(x'')$.

(iii) *For every* $X', Y' \in E'_+$,

$$|u(X') - u(Y')| \le 2C_2 \rho(X', Y')^{\varsigma},$$

where C_2 *and* ς *are as in Proposition 4.3.*

In (4-8), by graph u we mean the set $\{(x', u(x', t), t) : (x', t) \in E'_+\} \subset E$.

Proof. Step 1: Hausdorff convergence. By Lemma 4.6, $\widetilde{\Sigma}^j \cap E \neq \varnothing$ eventually. Thus one may extract a subsequence (not relabeled) so that $\widetilde{\Sigma}^j \cap E$ converges in the Hausdorff distance to some closed set $\widetilde{\Sigma} \subset E$. Since, by assumption, $\widetilde{\Sigma}^j \subset \{x_m \ge -\varepsilon_j\}$, it must also be $\widetilde{\Sigma} \subset \{x_m \ge 0\}$. We define the set-valued function

$$u(x', t) = \{y \in \overline{B_1^{d-m}} : ((x', y), t) \in \widetilde{\Sigma}\} \tag{4-9}$$

for $(x', t) \in E'_+$.

Step 2: $u(x', t) \neq \varnothing$ for every $(x', t) \in E'_+$. Assume, by contradiction, that there exists $(x', t) \in E'_+ \setminus S(\widetilde{\Sigma})$ (recall the notation $S(x, t) = (Sx, t) = (x', t)$). Then, since $S(\widetilde{\Sigma})$ is closed, there exists an open neighborhood U' of (x', t) such that $U' \subset (S(\widetilde{\Sigma}))^c$. If we let $U = S^{-1}(U') \subset \mathbb{R}^{d,1}$, then by Lemma 4.6 and Fatou's lemma

$$0 < \mathcal{H}^{m,1}(U \cap (S^+ \times \mathbb{R})) \le \liminf_j M^j(U).$$

Thus $M^j(U) > 0$ eventually. In particular, by taking smaller and smaller neighborhoods, one can pick a subsequence $j_\ell \to \infty$ and a sequence $X_\ell \in \Sigma^{j_\ell}$ so that $S(X_\ell) \to (x', t)$. By using the maps F_{ε_j} defined above, we rescale in the directions of $S^\perp$ and find that, up to subsequences, there exists $z \in \overline{B_1^{d-m}}$ such that

$$\widetilde{\Sigma}^{j_\ell} \ni F_{\varepsilon_{j_\ell}}(X_\ell) \to ((x', z), t).$$

By Step 1, $((x', z), t) \in \widetilde{\Sigma}$, which contradicts the fact that $u(x', t) = \varnothing$.

Step 3: $u((x'', 0), t) = \{g(x'')\}$. Let $(x'', t) \in \overline{Q_{r_0}^{m-1}}$. If $y \in u((x'', 0), t)$, then by Step 1 there exists a sequence $(x_j, t_j) \in \widetilde{\Sigma}^j$ such that $x_j \to ((x'', 0), y)$ and $t_j \to t$. In particular, by Proposition 4.2, it holds

$$\begin{aligned} |S^\perp(x_j - \tilde{\gamma}^j(x''_j))| &= \frac{1}{\varepsilon_j} |S^\perp(F_{\varepsilon_j}^{-1}(x_j) - \gamma^j(x''_j))| \\ &\le C|x_j \cdot \boldsymbol{e}_m + \varepsilon_j + \varepsilon_j^2|^{1/2} \to 0 \end{aligned}$$

as $j \to \infty$. Since $S^\perp \tilde{\gamma}^j$ converges uniformly to g and $S^\perp x_j \to y$, it must be

$$u((x'', 0), t) = \{g(x'')\}.$$

Step 4: $u(x', t)$ is a singleton and (iii) holds true. For $i = 1, 2$, let $X_i = (x_i, t_i) \in \widetilde{\Sigma}$. Let also $\rho := \rho(S(X_1), S(X_2))$ and, without loss of generality, assume $(x_2)_m \ge (x_1)_m$.

Case 1: $(x_1)_m = 0$. By Step 1 and Proposition 4.2, we have

$$|S^\perp x_2 - g(x''_2)| \le C(x_2)_m^{1/2} \le C\rho^{1/2}.$$

Moreover, $|S^\perp x_1 - g(x''_2)| = |g(x''_1) - g(x''_2)| \le C\rho$. Thus

$$|S^\perp x_2 - S^\perp x_1| \le C\rho^{1/2} + C\rho \le C\rho^{\varsigma}.$$

Case 2: $(x_1)_m > 0$ and $\rho = 0$. In this case, we prove that $S^\perp(x_1) = S^\perp(x_2)$. Fix ω much smaller than $(x_1)_m$. By Steps 1 and 2, we may pick j large enough and three points Y_1, Y_2, $W = (w, \tau) \in \widetilde{\Sigma}^j$ such that $\rho(X_i, Y_i) \le \omega$ and $2\omega \le \rho(S(W), S(X_i)) \le 4\omega$. Up to choosing j larger, we may assume that $\omega \ge C\varepsilon_j^\varsigma$ and $(y_i)_m \ge (x_i)_m - \omega \ge 2\varepsilon_j$. Therefore, by Proposition 4.3, since $\rho(S(W), S(Y_i)) \ge \omega$, we estimate

$$\begin{aligned}|S^\perp(x_1 - x_2)| &\le |S^\perp(x_1 - y_1)| + |S^\perp(x_2 - y_2)| + |S^\perp(y_1 - w)| + |S^\perp(y_2 - w)| \\ &\le 2\omega + C\omega^\varsigma.\end{aligned}$$

Since $\omega > 0$ is arbitrary, it holds $S^\perp(x_1) = S^\perp(x_2)$. In particular, $u(x', t)$ is a singleton for every $(x', t) \in E'_+$. With a small abuse of notation, from here onwards, we will denote by $u(x', t) \in \mathbb{R}^{d-m}$ the only element of the set defined in (4-9).

Case 3: $(x_1)_m > 0$ and $\rho > 0$. By Steps 1 and 2, we may choose j large enough and two points Y_1, Y_2 such that the following hold true:

(1) $Y_1, Y_2 \in \widetilde{\Sigma}^j$.

(2) For $i = 1, 2$, $\rho(X_i, Y_i) < \rho/8$.

(3) $C\varepsilon_j^\varsigma \le \rho/2$.

(4) For $i = 1, 2$, $(y_i)_m \ge 2\varepsilon_j$.

Then, by Proposition 4.3, it holds

$$\begin{aligned}|u(SX_1) - u(SX_2)| &\le |u(SX_1) - S^\perp(y_1)| + |u(SX_2) - S^\perp(y_2)| + |S^\perp(y_1 - y_2)| \\ &\le 2\frac{\rho}{8} + C\rho^\varsigma \le 2C\rho^\varsigma,\end{aligned}$$

as desired. □

The rest of the proof consists in proving that u defined in Lemma 4.7 solves the heat equation in the interior of E'_+. To this end, we recall some facts about the heat equation. First, recall that $E'_+ = \overline{Q^m_{r_0}} \cap \{x_m \ge 0\}$ and let us introduce the sets

$$\begin{gathered}\operatorname{Int}_p E'_+ = E'_+ \setminus \partial_p E'_+, \\ (E'_+)_r = \{x' \in \mathbb{R}^m : |x'| \le r_0 - r \text{ and } x'_m \ge r\} \times [-r_0^2 + r^2, 0].\end{gathered}$$

Notice that $\operatorname{Int}_p E'_+ = \bigcup_{r>0} (E'_+)_r$.

Lemma 4.8 (interior regularity for the heat equation). *Let $g \in C(\partial_p E'_+)$. Then there exists $h \in C^\infty(\operatorname{Int}_p E'_+) \cap C(E'_+)$ such that*

$$\begin{cases}\partial_t h - \Delta h = 0 & \text{in } \operatorname{Int}_p E'_+, \\ h = g & \text{on } \partial_p E'_+.\end{cases}$$

Moreover, for every $r > 0$ there exists $C > 0$ such that, for every $(x', t) \in (E'_+)_r$, it holds

$$\max\{|h(x', t)|, |\nabla h(x', t)|, |D^2 h(x', t)|, |\partial_t h(x', t)|\} \le C\|g\|_{L^\infty(\partial_p E'_+)}.$$

We now proceed with the proof of Theorem 4.1.

Lemma 4.9. *Let u be as in Lemma 4.7. Then $u \in C^\infty(\mathrm{Int}_p\, E'_+; \mathbb{R}^{d-m}) \cap C(E'_+; \mathbb{R}^{d-m})$ and*

$$\partial_t u - \Delta u = 0$$

in $\mathrm{Int}_p\, E'_+$.

Proof. We take as a model the proof of [Savin 2018, Lemma 2.4]. We show that u is equal to the solution $h : E'_+ \to \mathbb{R}^{d-m}$ to the boundary value problem

$$\begin{cases} \partial_t h - \Delta h = 0 & \text{in } \mathrm{Int}_p\, E'_+, \\ h = u & \text{on } \partial_p E'_+, \end{cases}$$

whose existence is guaranteed by Lemma 4.8. If not, there exist r, ω small and positive so that the function

$$E'_+ \ni (x', t) \mapsto |u(x', t) - h(x', t)|^2 + \omega |x'|^2$$

achieves its maximum at $(x'_0, t_0) \in (E'_+)_{2r}$. Since $\widetilde{\Sigma}^j$ converges in the Hausdorff distance to graph u, for some large j we may find $X_1 = (x_1, t_1) \in \Sigma^j$ such that $(x'_1, t_1) \in (E'_+)_r$ and the restriction to Σ^j of

$$H(x, t) := \left| \frac{S^\perp x}{\varepsilon_j} - h(Sx, t) \right|^2 + \omega |Sx|^2$$

achieves its maximum at X_1.

We claim that, if ε_j is small enough, depending on r and ω, then, for every m-dimensional subspace T, it holds $T : D^2 H(X_1) > \partial_t H(X_1)$. This would contradict Proposition 3.7, thus concluding the proof. To prove the claim, we define $f(x, t) = (1/\varepsilon_j) S^\perp x - h(Sx, t)$ and, with some straightforward computations, we write

$$H(x, t) = G_1(x, t) + G_2(x, t),$$

where

$$\begin{aligned} G_1(x, t) &= |f(X_1)|^2 + 2 f(X_1) \cdot (f(x, t) - f(X_1)) + \omega |Sx|^2, \\ G_2(x, t) &= |f(x, t) - f(X_1)|^2. \end{aligned}$$

Notice that, by Lemma 4.8, there exists C depending on r such that

$$|D^2 G_1(X_1)| \le C(\omega + |f(X_1)| |D^2 h(SX_1)|) \le C.$$

Then, just as in [Savin 2018], it is easy to show that, if $|T - S| \le c\omega$, then

$$T : D^2 G_1(X_1) > \partial_t H(X_1)$$

and $D^2 G_2(X_1) \ge 0$; thus in this case $T : D^2 H(X_1) > \partial_t H(X_1)$. On the other hand, if $|T - S| \ge c\omega$, then there exists a unit-vector $\nu \in T$ such that $S^\perp \nu \ge c\omega$. In particular, since $D^2 G_2(X_1) = 2 \nabla f(X_1) \nabla f(X_1)^T$, it holds

$$T : D^2 G_2(X_1) \ge -|S\nu|^2 |\nabla h(SX_1)|^2 + \frac{1}{\varepsilon_j^2} |S^\perp \nu|^2 \ge \frac{c\omega^2}{\varepsilon_j^2}.$$

We now conclude by remarking that $\partial_t H(X_1) = 2 f(X_1) \cdot \partial_t h(SX_1) \le C$ and

$$T : D^2 G_1(X_1) \ge -|D^2 G_1(X_1)| \ge -C.$$

Thus

$$T : D^2 H(X_1) \geq -C + \frac{c\omega^2}{\varepsilon_j^2} > \partial_t H(X_1),$$

provided ε_j is chosen small enough depending on ω and C (which, in turn, is a large constant depending on r). □

Once proven that u is a solution to the heat equation, it is sufficient to apply the following classical estimate:

Lemma 4.10 (boundary regularity for the heat equation). *For every $\alpha \in (0, 1)$, there exist positive constants C and β with the following property. Let $u \in C^2(\mathrm{Int}_p\, E'_+) \cap C(E'_+)$ be such that*

$$\partial_t u - \Delta u = 0 \quad \text{in } \mathrm{Int}_p\, E'_+.$$

Assume, moreover, that for all t, $u(\,\cdot\,, t)|_{\{x_m=0\}} = g \in C^{1,\alpha}(B_{r_0} \cap \{x_m = 0\})$ we have $|g(0)| = |Dg(0)| = 0$ and $|u| \leq 1$ everywhere. Then there exists a linear operator $L : \mathbb{R}^m \to \mathbb{R}^{d-m}$ with $|L| \leq C$ such that, for every $\eta \in \left(0, \frac{1}{4}\right)$,

$$|u(x', t) - L(x')| \leq C\eta^{1+\beta}$$

in $(B_\eta^m \cap \{x_m \geq 0\}) \times (-\eta^2, 0]$.

Proof. See [Wang 1992, Theorem 2.1]. □

Remark 4.11. From the fact that $g \in C^{1,\alpha}$ and that $Dg(0) = 0$, it follows that $L(x') = 0$ if $x_m = 0$.

Conclusion of the proof of Theorem 4.1. By Lemmas 4.9 and 4.10, there exists $L : \mathbb{R}^m \to \mathbb{R}^{d-m}$ linear such that $L(x') = 0$ if $x'_m = 0$, $|L| \leq C$ and, for every η small, it holds

$$\widetilde{\Sigma} \cap (B_{2\eta}^m \times B_1^{d-m} \times (-4\eta^2, 0]) \subset \{(x, t) : |S^\perp x - L(Sx)| \leq C\eta^{1+\beta}\}.$$

We fix η small, to be specified later, and we choose j sufficiently large so that the Hausdorff distance between $\widetilde{\Sigma}$ and $\widetilde{\Sigma}^j$ is smaller than $\eta^{1+\beta}$. We now let $T = \{x \in \mathbb{R}^d : S^\perp x = \varepsilon_j L(Sx)\}$. Then it holds

$$\Sigma^j \cap Q_\eta \subset \{|T^\perp x| \leq C'\varepsilon_j \eta^{1+\beta}\}.$$

Moreover, by Proposition 4.2 and the fact that $|\gamma_m^j(x'')| \leq \varepsilon_j |x''|^{1+\alpha}$, it holds

$$\Sigma^j \cap Q_\eta \subset \{x_m \geq -\varepsilon_j \eta^{1+\alpha} - \varepsilon_j^2\},$$

provided $\eta \leq r_2$. We choose j large enough so that $\varepsilon_j^2 \leq \eta^{1+\beta}$. Since β can be chosen smaller than α,

$$\Sigma^j \cap Q_\eta \subset \{x_m \geq -2\varepsilon_j \eta^{1+\beta}\} \cap \{|T^\perp x| \leq C\varepsilon_j \eta^{1+\beta}\}.$$

Up to choosing j larger, the above inclusion yields

$$\Sigma^j \cap Q_\eta \subset \{\mathrm{dist}(\,\cdot\,, T^+) \leq 2C\varepsilon_j \eta^{1+\beta}\}.$$

We conclude the proof by choosing $\beta' > \beta$ and η so small that $2C\eta^{1+\beta} \leq \eta^{1+\beta'}$ and we recover (4-3) (with β' instead of β). This contradicts the assumption made at the beginning of the present subsection, thus concluding the proof. □

5. Boundary behavior

We now prove Proposition 4.2. The setting is the following. Let E_0 and α be given and let r_3 be the constant given in Proposition 4.4. Assume $\boldsymbol{M}$ and Γ satisfy the assumptions of Theorem 4.1. Then $\mathrm{mdr}(\boldsymbol{M}, Q_{r_3}) < \infty$; therefore Proposition 4.2 follows from the following, more general, statement:

Proposition 5.1 (boundary behavior at scale R). *There exist c and ε_1 depending only on α with the following property. Let $0 < \delta < \varepsilon \le \varepsilon_1$, $\Gamma \in \mathcal{F}_\alpha(\delta, B_R)$ and $\boldsymbol{M} \in \mathcal{BF}_m(Q_R, \Gamma)$ be such that*

$$\Sigma \cap Q_R \subset \{(x,t) : \mathrm{dist}(x, S^+) \le \varepsilon R\}$$

and

$$\mathrm{mdr}(\boldsymbol{M}, Q_R) < \infty. \tag{5-1}$$

Then

$$\Sigma \cap Q_{R/2} \subset \left\{(x,t) : x_m \ge \gamma_m(x'') - R\delta^2 + cR\frac{|S^\perp(x-\gamma(x''))|^2}{2(\varepsilon R)^2}\right\}.$$

Remark 5.2. The role of (5-1) is to guarantee that the maximum principle (Proposition 3.7) holds true.

Proof. By a simple rescaling argument, it is sufficient to prove the result in the case $R = 1$. We fix c small and $\varepsilon_1 \le c$, to be specified later. By contradiction, assume there exist $0 < \delta \le \varepsilon \le \varepsilon_1$, Γ and $\boldsymbol{M}$ as above, and a point $(\bar{x}, \bar{t}) \in \Sigma \cap Q_{1/2}$ such that

$$0 < \omega := \frac{c}{2\varepsilon^2}|S^\perp(\bar{x} - \gamma(\bar{x}''))|^2 - \delta^2 + \gamma_m(\bar{x}'') - \bar{x}_m.$$

We show that, if this is the case, then we may build a family of surfaces sliding in the direction of $\boldsymbol{e}_m$ that touch Σ at some point where the conclusion of Proposition 3.7 fails.

In order to do so, we first define the functions $g : \mathbb{R}^{d-m} \to \mathbb{R}$ and $h : \mathbb{R}^m \to \mathbb{R}$ as

$$g(z) = c\frac{|z - S^\perp\gamma(\bar{x}'')|^2}{2\varepsilon^2},$$
$$h(y) = P(y'') - |y'' - \bar{x}''|^2 - y_m,$$

where

$$P(y'') = \gamma_m(\bar{x}'') + \nabla\gamma_m(\bar{x}'') \cdot (y'' - \bar{x}'') - \delta^2 - C|y'' - \bar{x}''|^2,$$

and C is a constant depending only on α chosen so that

$$P(x'') \le \gamma_m(x'') \tag{5-2}$$

(to show that such C depending only on α exists, use the fact that $\gamma \in C^{1,\alpha}(B_R)$ and Young's inequality). Then, choose a smooth function $f : \mathbb{R} \to \mathbb{R}$ such that $f(-1) = -4c$, $f|_{t \ge -1/4} \ge -\omega/2$, $f < 0$ everywhere and $f'(t) \le 8c$ everywhere. We now set

$$H(x,t) = g(S^\perp x) + h(Sx) + f(t).$$

This way, the zero-level set of H is a surface sliding in the $\boldsymbol{e}_m$-direction. Notice that

$$H(\bar{x}, \bar{t}) = \omega + f(\bar{t}) > 0. \tag{5-3}$$

We now show that, if $(x,t) \in \Sigma \cap ((\Gamma \times \mathbb{R}) \cup \partial_p Q_1)$, then $H(x,t) \le 0$.

(1) If $x \in \Sigma_{-1}$, then

- $g(S^\perp x) \le c(\varepsilon+\delta)^2/(2\varepsilon^2) = 2c$, since $|S^\perp x| \le \varepsilon$ and $|\gamma(x'')| \le \delta \le \varepsilon$,
- by (5-2), $h(Sx) \le \gamma_m(x'') - x_m \le 2\varepsilon$.

The two above facts, along with the assumption $f(-1) = -4c$, yield

$$H(x,-1) \le 2c + 2\varepsilon - 4c \le 0$$

provided $\varepsilon \le c$.

(2) If $x \in \partial B_1 \cap \Sigma_t$, then $|S^\perp x| \le \varepsilon$ and $x_m \ge -\varepsilon$. Thus

$$|x''| \ge \sqrt{1-\varepsilon^2 - x_m^2} \ge \tfrac{3}{4} - x_m,$$

provided ε is small enough. In particular, $|x'' - \bar{x}''| \ge \frac{1}{4} - x_m$. Hence:

- Since $\|\gamma\|_{C^{1,\alpha}(B_1)} \le \delta$, we have
$$h(Sx) \le 2\delta - (C+1)|x''-\bar{x}''|^2 - x_m \le 2\delta - (C+1)\big(\tfrac{1}{4} - x_m\big)^2 - x_m.$$
- As in (1), $g(S^\perp x) \le 2c$,
- $f(t) \le 0$.

Therefore

$$H(x,t) \le 2c + 2\delta - (C+1)\big(\tfrac{1}{4} - x_m\big)^2 - x_m \le 2c + 2\delta - \frac{C}{4(1+C)} \le 0$$

provided $C \ge 1$ and c, δ are small enough.

(3) Lastly, for every $x \in \Gamma$ and $t \in (-1,0)$, under the assumptions $\delta \le \varepsilon$ and $c \le 1$, it holds

$$g(S^\perp x) = \frac{c}{2\varepsilon^2}|S^\perp(\gamma(x'') - \gamma(\bar{x}''))|^2 \le \frac{c}{2\varepsilon^2}\|\nabla\gamma\|_\infty^2 |x''-\bar{x}''|^2 \le |x''-\bar{x}''|^2.$$

Since $f \le 0$ and $h(Sx) \le \gamma_m(x'') - |x''-\bar{x}''|^2 - x_m$, we have

$$H(x,t) \le \gamma_m(x'') - x_m = 0.$$

Points (1)–(3) above and (5-3) show that there must exist $Y = (y,s) \in Q_1 \cap \Sigma$ with $y \notin \Gamma$ such that $H|_{\{t \le s\}}$ has a local maximum at (y,s).

We now show that one can choose c even smaller, if needed, so that the existence of such a point would contradict the maximum principle. Indeed, since $|S^\perp y| \le \varepsilon$, if c is small enough then

$$\frac{|\nabla h(Sy)|^2}{|\nabla g(S^\perp y)|^2} \ge \varepsilon.$$

Thus

$$|S^\perp \nabla H(Y)| = |\nabla g(S^\perp y)| \le (1-\varepsilon)|\nabla H(Y)|.$$

Therefore, if T is an m-dimensional subspace of $\mathbb{R}^d$ such that $T \perp \nabla H(Y)$, then

$$T : S^\perp \ge \varepsilon$$

and

$$T : D^2H(Y) = T : \begin{pmatrix} D^2h & 0 \\ 0 & D^2g \end{pmatrix} \geq -|D^2h(Sy)| + \varepsilon|D^2g(S^\perp y)|.$$

Now, simple computations show that, up to multiplication by constants depending only on m and α, $|D^2h(Sy)| \leq 1$ and $|D^2g(S^\perp y)| \geq c/\varepsilon^2$. Therefore, if ε is much smaller than c, then $T : D^2H(y) \geq c/(2\varepsilon)$. However, by Proposition 3.7, it holds

$$\inf_{T \perp \nabla H(Y)} T : D^2H(Y) \leq \partial_t H(Y) = f'(s) \leq 8c,$$

which is a contradiction. □

6. Decay of oscillations: proof of Proposition 4.3

We begin by giving the following definition:

Definition 6.1. Let $u : \mathbb{R}^{m,1} \to [-\infty, 1]$ be an upper-semicontinuous function. Assume that, whenever a smooth function $\varphi : \mathbb{R}^{m,1} \to \mathbb{R}$ touches u from above at some $(x'_0, t_0) \in U \times I$ (according to the terminology set in Section 3.2) and $|\nabla\varphi(x'_0, t_0)|$, $|D^2\varphi(x'_0, t_0)|$ are smaller than some fixed universal constant δ_0, then

$$\partial_t\varphi - \mathcal{M}^+(D^2\varphi) \leq 0 \tag{6-1}$$

at (x'_0, t_0) (see Section 3.2 for the definition of $\mathcal{M}^+$). Then u is said to be a viscosity subsolution to (6-1) in $U \times I$.

The reader should notice that the classical definition of viscosity solution is slightly different than ours, in that the test function φ usually has no restrictions on the magnitude of $|\nabla\varphi|$ and $|D^2\varphi|$ at the touching point.

The proof of Proposition 4.3 is achieved in three steps:

(1) First of all, one sees that the support of a $\boldsymbol{M}$ behaves, in some sense, like the graph of a viscosity subsolution to (6-1), as in the definition above; this was proved in Corollary 3.8.

(2) By exploiting the results in [Wang 2013], one shows that, if a Σ has a point far enough from S, then the mass of $\boldsymbol{M}$ near that point cannot be too small.

(3) If Σ does not have the decay of oscillations stated in Proposition 4.3, then by the previous step the mass of $\boldsymbol{M}$ in some parabolic cylinder must be large; this contradicts the small density assumption (4-1).

Before proceeding, we introduce some notation that we are going to use in the present subsection. Given $\theta \in (0, 1)$, we define the set

$$\mathcal{P}_1^\theta = \left\{(x', t) \in \mathbb{R}^{m,1} : |x'|^2 < -\frac{t}{\theta^2} < 1\right\}.$$

One should compare these sets with those which, in [Wang 2013], are called "parabolic balls". Our definition slightly differs from theirs; notice that with our choice $\mathcal{P}_1^\theta \subset B_1^m \times (-\theta^2, 0)$.

Lemma 6.2 (measure estimate [Wang 2013]). *For every $\theta > 0$ and $\mu \in (0,1)$, there exist small constants η', r with the following property. Let $u : \mathbb{R}^{m,1} \to [-\infty, 1]$ be a viscosity subsolution to (6-1) in $B_1^m \times (-\theta^2, 0)$ and assume that*

$$u(Y_0) \geq 1 - \eta'$$

for some $Y_0 \in B_r^m \times (-\theta^2 r^2, 0)$. Then

$$\mathcal{L}^{m,1}(\{u \geq 1-\mu\} \cap \mathcal{P}_1^\theta) \geq (1-\mu)\mathcal{L}^{m,1}(\mathcal{P}_1^\theta). \tag{6-2}$$

Proof. This result corresponds, essentially, to [Wang 2013, Lemma 4.3]. Apart from some trivial adjustment of constants, there are two caveats:

- The results in [loc. cit.] are stated with the classical definition of viscosity solutions, where no bound on the test function at the touching point is required. However, it is easy to see that the results are valid for our definition of viscosity solution, as well.

- In our setting, we allow u to be merely upper-semicontinuous and, possibly, take infinite values, while in [loc. cit.] u is required to be continuous. This minor point can be easily overcome by looking at the sup-convolution of u,

$$u_\delta(x,t) = \sup\Big\{u(y,s) - \frac{1}{\delta}(|x-y|^2 + (t-s)^2)\Big\},$$

which conserves the property of being a viscosity subsolution to (6-1) and for which (6-2) holds true, by [loc. cit., Lemma 4.3]. Letting $\delta \searrow 0$ gives the desired conclusion. □

Before stating the next result, we fix some further notation. For any closed set $\Sigma \subset \mathbb{R}^{d,1}$ and any $\Omega \subset \mathbb{R}^{d,1}$, we let

$$\operatorname{osc}(\Sigma, \Omega) = \inf\big\{h > 0 : \text{there is } y \in \mathbb{R}^d \text{ such that } \Sigma \cap \Omega \subset \{x : |S^\perp(x-y)| \leq h\}\big\}.$$

We also let

$$C_r = \{x \in \mathbb{R}^d : |Sx| < r\}.$$

Lemma 6.3 (Harnack inequality). *For every $\delta \in (0,1)$, there exist small constants $\varepsilon_2, \theta, r, \eta$ with the following property. Let $\varepsilon \leq \varepsilon_2$ and $\boldsymbol{M} \in \mathcal{BF}_m(C_1 \times (-\theta^2, 0])$ be such that*

$$\Sigma \subset \{|S^\perp x| \leq \varepsilon\}, \tag{6-3}$$

$$\int_{C_1} \Psi(\,\cdot\,, t)\, dM_t \leq 2 - \delta \quad \textit{for all } t \in (-\theta^2, 0), \tag{6-4}$$

and

$$\operatorname{mdr}(\boldsymbol{M}, C_1 \times (-\theta^2, 0]) < \infty. \tag{6-5}$$

Then

$$\operatorname{osc}(\Sigma, C_r \times (-\theta^2 r^2, 0]) \leq (1-\eta)\varepsilon. \tag{6-6}$$

The proof of the above result involves some technical estimates. It is therefore convenient to give an overview of the strategy. If (6-6) does not hold, then one finds two points Y_1 and Y_2 in Σ that are far enough in $S^\perp$. By applying Lemma 6.2 twice, we find that in $C_1 \times (-\theta^2, 0)$ the mass of $\boldsymbol{M}$ must be almost that of two m-dimensional disks. This contradicts (6-4), which encodes the fact that the mass of $\boldsymbol{M}$ must not exceed by too much that of a single disk.

Proof of Lemma 6.3. Let $\delta \in (0, 1)$ be given. Fix θ and μ, which we will specify later, and let r and η' be chosen accordingly as in Lemma 6.2. Moreover, fix ε much smaller than μ and $\eta \le \eta'$, to be specified later. Assume, by contradiction, that there exist $\boldsymbol{M} \in \mathcal{BF}_m(C_1 \times (-\theta^2, 0])$ that satisfies the assumptions of the present result with the choices made above, and two points $Y_1 = (y_1, s_1)$, $Y_2 = (y_2, s_2) \in \Sigma \cap (C_r \times (-\theta^2 r^2, 0])$ with $|S^\perp y_1 - S^\perp y_2| \ge 2(1-\eta)\varepsilon$. For every $(x', t) \in B_1^m \times (-\theta^2, 0]$ and for $i = 1, 2$, let

$$u_i(x', t) = \frac{1}{2\varepsilon} \sup\{|z - S^\perp y_i| : z \in S^\perp \text{ and } (x', z) \in \Sigma_t\}.$$

Notice that u_1 and u_2 are upper-semicontinuous and, for every (x', t), either $u_1(x', t), u_2(x', t) \in [0, 1]$ or $u_1 = u_2 = -\infty$. By Corollary 3.8 and (6-5), both u_1 and u_2 are viscosity subsolutions to (6-1). Moreover,

$$u_1(Sy_2, s_2) \ge \frac{1}{2\varepsilon} |S^\perp y_2 - S^\perp y_1| \ge 1 - \eta \ge 1 - \eta';$$

hence, by Lemma 6.2,

$$\mathcal{L}^{m,1}(\{u_1 \ge 1 - \mu\} \cap \mathcal{P}_1^\theta) \ge (1 - \mu)\mathcal{L}^{m,1}(\mathcal{P}_1^\theta).$$

With the same argument, one also obtains

$$\mathcal{L}^{m,1}(\{u_2 \ge 1 - \mu\} \cap \mathcal{P}_1^\theta) \ge (1 - \mu)\mathcal{L}^{m,1}(\mathcal{P}_1^\theta). \tag{6-7}$$

We now want to estimate

$$\int_{C_1 \times (-\theta^2, 0)} \Psi \, dM.$$

We first define, for $i = 1, 2$, the sets

$$A_i = \left\{(x, t) \in \mathbb{R}^{d,1} : (Sx, t) \in \mathcal{P}_1^\theta, |S^\perp(x - y_i)| \le \frac{\varepsilon}{2} \text{ and } t \le -\frac{2\varepsilon^2}{\delta}\right\}.$$

Notice that $A_1 \cap A_2 = \varnothing$ and, by (6-3), for M-a.e. $(x, t) \in A_i$, it holds

$$\Psi(x, t) = \exp\left(\frac{|S^\perp x|^2}{4t}\right)\Psi'(Sx, t) \ge \exp\left(-\frac{\varepsilon^2}{8\varepsilon^2/\delta}\right)\Psi'(Sx, t) \ge \left(1 - \frac{\delta}{8}\right)\Psi'(Sx, t),$$

where $\Psi'(x', t) := \Psi((x', 0), t)$.

Therefore we have

$$\begin{aligned}\int_{C_1 \times (-\theta^2, 0)} \Psi \, dM &\ge \int_{A_1} \Psi \, dM + \int_{A_2} \Psi \, dM \\ &\ge \left(1 - \frac{\delta}{8}\right)\left(\int_{A_1} \Psi'(Sx, t) \, dM(x, t) + \int_{A_2} \Psi'(Sx, t) \, dM(x, t)\right).\end{aligned} \tag{6-8}$$

Moreover, by Lemma 3.6 and by the coarea formula,

$$\int_{A_i} \Psi'(Sx, t) \, dM(x, t) \ge \int_{S(A_i \cap \Sigma)} \Psi'(x', t) \, d\mathcal{L}^{m,1}(x', t). \tag{6-9}$$

We may assume that μ and η are smaller that some universal constant so that, if $z \in \mathbb{R}^{d-m}$ with $|z| \le \varepsilon$ is such that $|z - S^{\perp} y_2|/(2\varepsilon) \ge 1 - \mu$, then

$$|z - S^{\perp} y_1| \le \frac{\varepsilon}{2}.$$

In particular, we have

$$S(A_1 \cap \Sigma) \supset \{u_2 \ge 1 - \mu\} \cap \mathcal{P}_1^{\theta} \cap \Big\{t \le -\frac{2\varepsilon^2}{\delta}\Big\},$$

which, together with (6-7), yields that $S(A_1 \cap \Sigma)$ covers a large portion of $\mathcal{P}_1^{\theta}$; namely

$$\begin{aligned}\mathcal{L}^{m,1}(S(A_1 \cap \Sigma)) &\ge \mathcal{L}^{m,1}\Big(\mathcal{P}_1^{\theta} \cap \{u_2 \ge 1 - \mu\} \cap \Big\{t \le -\frac{2\varepsilon^2}{\delta}\Big\}\Big) \\ &\ge \mathcal{L}^{m,1}(\mathcal{P}_1^{\theta} \cap \{u_2 \ge 1 - \mu\}) - \frac{2\varepsilon^2}{\delta} \ge (1 - 2\mu)\mathcal{L}^{m,1}(\mathcal{P}_1^{\theta}),\end{aligned}$$

provided $\varepsilon^2 \le c\delta\mu$ for some c small universal.

We are now ready to choose μ, depending on δ, so that the above inequality and the fact that $\Psi \in L^1(\mathcal{L}^{m,1} \llcorner \mathcal{P}_1^{\theta})$ yield

$$\int_{S(A_1 \cap \Sigma)} \Psi' \, d\mathcal{L}^{m,1} \ge \int_{\mathcal{P}_1^{\theta}} \Psi' \, d\mathcal{L}^{m,1} - \frac{\delta\theta^2}{8} = \theta^2 \int_{B_1^m} \Psi'(\,\cdot\,, -\theta^2) \, d\mathcal{L}^m - \frac{\delta\theta^2}{8}. \tag{6-10}$$

Finally, we also choose θ small such that

$$\int_{B_1^m} \Psi'(\,\cdot\,, -\theta^2) \, d\mathcal{L}^m \ge \int_{\mathbb{R}^m} \Psi'(\,\cdot\,, -\theta^2) \, d\mathcal{L}^m - \frac{\delta}{8} = 1 - \frac{\delta}{8}. \tag{6-11}$$

By (6-10) and (6-11), it holds

$$\int_{S(A_1 \cap \Sigma)} \Psi' \, d\mathcal{L}^{m,1} \ge \theta^2\Big(1 - \frac{\delta}{4}\Big). \tag{6-12}$$

The same argument can be repeated for A_2, thus giving

$$\int_{S(A_2 \cap \Sigma)} \Psi' \, d\mathcal{L}^{m,1} \ge \theta^2\Big(1 - \frac{\delta}{4}\Big). \tag{6-13}$$

We conclude the proof by combining (6-8), (6-9), and (6-12), (6-13), obtaining

$$\int_{C_1 \times (-\theta^2, 0)} \Psi \, dM \ge 2\theta^2\Big(1 - \frac{\delta}{8}\Big)\Big(1 - \frac{\delta}{4}\Big) \ge \theta^2\Big(2 - \frac{3\delta}{4}\Big),$$

which contradicts (6-4). This concludes the proof. □

A simple rescaling argument allows one to iterate Lemma 6.3 and obtain the following

Proposition 6.4. *For every $\delta \in (0, 1)$ there exist C (large) and ς (small) with the following property. Let $\boldsymbol{M} \in \mathcal{BF}_m(C_R \times (-R^2, R^2))$ be such that*

$$\operatorname{mdr}(\boldsymbol{M}, C_R \times (-R^2, R^2)) < \infty \tag{6-14}$$

and assume that, for every $(x,t)\in C_{R/2}\times(-R^2/4,R^2/4))$ *and every* $s\in(t-R^2/4,t)$*, it holds*

$$\int_{C_{R/2}(x)}\Psi(\cdot-x,s-t)\,dM_s\le 2-\delta. \tag{6-15}$$

If $\varepsilon=\operatorname{osc}(\Sigma,C_R\times(-R^2,R^2))$*, then, for any couple* $(x,t),(y,s)\in C_{R/2}\times(-R^2/4,R^2/4)\cap\Sigma$ *such that* $\rho=\rho(X',Y')\ge CR^{1-\varsigma}\varepsilon^{\varsigma}$*, it holds*

$$|S^\perp(x-y)|\le C\varepsilon\left(\frac{\rho}{R}\right)^{\varsigma}.$$

Proof. We prove the result for $R=1$, as the general case follows by replacing $\boldsymbol{M}$ with $\mathcal{D}_R\boldsymbol{M}$.

Let $\varepsilon_2,\theta,r,\eta$ be the constants given in Lemma 6.3 in correspondence to δ. Without loss of generality, we may assume that $\varepsilon\le\varepsilon_2$; otherwise the result follows by choosing C large enough. Consider the rescaled flows $\boldsymbol{M}^k=\mathcal{D}_{r^k}(\boldsymbol{M}-X)$. By induction, the assumptions of Lemma 6.3 are in place for every integer k such that

$$\left(\frac{1-\eta}{r}\right)^k\varepsilon\le\varepsilon_2. \tag{6-16}$$

Therefore, scaling back to the original flow, we see that for those k

$$\operatorname{osc}(\Sigma_{\boldsymbol{M}},C_{r^k}(x)\times(t-\theta^2r^{2k},t])\le(1-\eta)^k\varepsilon.$$

Let now $X=(x,t)$ and $Y=(y,s)$ be two points in $C_{1/2}\times\left(-\frac14,\frac14\right)\cap\Sigma$ and let $\rho=\rho((x',t),(y',s))$. Without loss of generality, we may assume that $t\ge s$. Furthermore, by taking $C\ge 2/\theta$, we may clearly reduce ourselves to the case $\rho\le\theta/2$. By choosing ς small enough and C larger than the choice made above, if necessary, we infer from $\rho\ge C\varepsilon^{\varsigma}$ that there exists $k\in\mathbb{N}$ satisfying (6-16) such that $r^{k+1}\le 2\rho/\theta\le r^k$. Thus

$$Y\in C_{2\rho}(x)\times(t-4\rho^2,t]\subset C_{r^k}(x)\times(t-\theta^2r^{2k},t];$$

hence it must be $|S^\perp(x-y)|\le 2(1-\eta)^k\varepsilon$. We conclude the proof by taking C larger and ς smaller, if needed, so that $2(1-\eta)^k\le C\rho^{\varsigma}$. □

We finally prove Proposition 4.3.

Proof of Proposition 4.3. Let $r_2=\frac12\min\{r_1,r_3\}$, where r_1 and r_3 are given in Propositions 4.2 and 4.4, respectively. Let also $X=(x,t)$, $Y=(y,s)$ be two points in $\Sigma\cap Q_{r_2}$. Without loss of generality, we assume that $R:=x_m\ge y_m\ge 2\varepsilon$. Let $\rho=\rho((x',t),(y',s))$; finally, let ς be the constant determined in Proposition 6.4 corresponding to $\delta=\frac12$. We shall distinguish two cases.

If $\rho\le R/8$, then we may find $t'\in(-r_2^2,0]$ such that $X,Y\in C_{R/4}(x)\times(t'-R^2/16,t'+R^2/16)$ and $C_{R/4}(x)\subset\Gamma^c$. Since $R\le 1$, the assumption $\rho\ge C\varepsilon^{\varsigma}$ yields $\rho\ge CR^{1-\varsigma}\varepsilon^{\varsigma}$. By Proposition 4.4 and Corollary 4.5, (6-14) and (6-15) hold true; thus Proposition 6.4 applies and we obtain

$$|S^\perp(x-y)|\le C\left(\frac{\rho}{R}\right)^{\varsigma}\operatorname{osc}(\Sigma,\mathcal{U}(X)),$$

where $\mathcal{U}(X) := C_{R/4}(x) \times (t' - R^2/16, t' + R^2/16)$. By Proposition 4.2, we may estimate

$$\operatorname{osc}(\Sigma, \mathcal{U}(X)) \leq 2C\varepsilon\sqrt{2R + \varepsilon + \varepsilon^2} + CR\|\nabla\gamma\|_\infty \leq C\varepsilon R^{1/2},$$

since $\varepsilon \leq R/2$ and $\|\nabla\gamma\|_\infty \leq C\varepsilon$. Thus

$$|S^\perp(x - y)| \leq C\varepsilon R^{1/2-\varsigma}\rho^\varsigma \leq C\varepsilon\rho^\varsigma,$$

since ς can be chosen smaller than $\frac{1}{2}$.

On the other hand, if $\rho \geq R/8$, then it is sufficient to use Proposition 4.2 twice and the fact that $\|\nabla\gamma\|_\infty \leq C\varepsilon$ to estimate

$$|S^\perp(x - \gamma(x''))| \leq C\varepsilon(R + \varepsilon + \varepsilon^2)^{1/2} \leq C\varepsilon\rho^{1/2},$$
$$|S^\perp(y - \gamma(y''))| \leq C\varepsilon(2R + \varepsilon + \varepsilon^2)^{1/2} \leq C\varepsilon\rho^{1/2},$$
$$|S^\perp(\gamma(y'') - \gamma(x''))| \leq C\varepsilon\rho.$$

We therefore conclude

$$|S^\perp(x - y)| \leq C\varepsilon\rho^{1/2} \leq C\varepsilon\rho^\varsigma,$$

which is the desired result. □

7. $C^{1,\beta}$ regularity

In the present section, we prove the following ε-regularity theorem:

Theorem 7.1 ($C^{1,\beta}$ regularity). *For every E_0 and α, there are small constants ε_3, Λ, η and β with the following property. Let $\varepsilon \leq \varepsilon_3$, $\Gamma \in \mathcal{F}_\alpha(\varepsilon, B_1)$, $\boldsymbol{M} \in \mathcal{BF}_m(B_1 \times [-\Lambda, 0], \Gamma)$ be such that $(0, 0) \in \Sigma_{\boldsymbol{M}}$,*

$$\Sigma_{\boldsymbol{M}} \subset \{(z, \tau) : \operatorname{dist}(z, S^+) \leq \varepsilon\},$$

$$\sup_{t\in[-\Lambda,0]} M_t(B_1) \leq E_0$$

and

$$\int_{B_1} \Psi(\,\cdot\,, -\Lambda)\, dM_{-\Lambda} \leq \frac{3}{4}.$$

Then there is $u \in C^{1,\beta}(Q^m_\eta, \mathbb{R}^{d-m})$ with $\|u\|_{C^{1,\beta}} \leq C\varepsilon$ such that $\Sigma_{\boldsymbol{M}} \cap Q^d_\eta = \operatorname{graph} u$ and, for all $t \in (-\eta^2/4, 0]$, it holds $\partial\Sigma_t \cap B_\eta \subset \Gamma$.

Before proving the above result, we record the following consequence of Theorem 4.1

Proposition 7.2 (iteration of the improvement of flatness). *Under the assumptions of Theorem 4.1, for every $X = (x, t) \in \Sigma \cap Q_\eta$:*

- *If $x \in \Gamma$, then there exists an m-dimensional half-plane T^+_X such that $\partial T^+_X = T_x\Gamma$ and*

$$\Sigma \cap Q_{\eta^k}(X) \subset \{\operatorname{dist}(\,\cdot\,, x + T^+_X) \leq 2\varepsilon\eta^{k(1+\beta)}\}$$

for every $k \in \mathbb{N}$.

- *If $x \notin \Gamma$, then there exists an m-dimensional plane T_X such that*

$$\Sigma \cap Q_{\eta^k}(X) \subset \{\operatorname{dist}(\,\cdot\,, x + T_X) \leq 2\varepsilon\eta^{k(1+\beta)}\}$$

for every $k \in \mathbb{N}$.

Proof. This result is a straightforward consequence of an iteration of Theorem 4.1. Namely, given $X \in \Sigma \cap Q_\eta$ with $x \in \Gamma$, we may find a sequence of half-planes T_k^+ such that

$$\Sigma \cap Q_{\eta^k} \subset \{\operatorname{dist}(\cdot, T_k^+) \le \eta^{k(1+\beta)}\varepsilon\}.$$

Moreover, $|T_k^+ - T_{k-1}^+| \le C\varepsilon\eta^{k(1+\beta)}/\eta^k$ for some C depending only on E_0 and α. Therefore, $\{T_k^+\}$ converges to some half-plane T_X^+ for which the conclusion of the proposition holds true.

For the case $x \notin \Gamma$, one may see [Tonegawa 2019] or replicate the techniques of the previous sections. □

Remark 7.3. Given $x \in \Gamma$ and $T_{(x,t)}^+$ as in Proposition 7.2, throughout the rest of the present section, we let $T_{(x,t)}$ be the m-dimensional plane obtained by reflecting $T_{(x,t)}^+$ across $T_x\Gamma$. We note the following conclusion of Theorem 4.1: there is C depending only on E_0 and α such that, for every $X \in \Sigma \cap Q_\eta$, it holds

$$|T_X - S| \le C\varepsilon.$$

We are now ready to prove Theorem 7.1.

Proof of Theorem 7.1. Up to a rotation, we may assume, without loss of generality, that $T_{(0,0)}$ defined in Proposition 7.2 coincides with the plane S that satisfies the assumptions of the present result.

Step 1: Σ is the graph of a $C^{1,\beta}$ function over $S(\Sigma)$. Let $X \in \Sigma \cap Q_\eta^d$ and, for simplicity of notation, let $T = T_X$ as defined in Remark 7.3; recall that $|T - S| \le C\varepsilon$. For any other point $Y \in \Sigma \cap Q_\eta^d$, we may write

$$\begin{aligned}|S^\perp(x-y)| &\le |T^\perp(x-y)| + |S^\perp - T^\perp|\,|x-y| \\ &\le C\varepsilon\rho(X,Y)^{1+\beta} + C\varepsilon\rho(X,Y) \\ &\le 2C\varepsilon\rho(X,Y).\end{aligned}$$

If ε_3 is smaller than some universal constant, we conclude

$$|S^\perp(x-y)| \le 3C\varepsilon\rho(SX, SY).$$

The above inequality, together with Proposition 7.2, yields

$$|T^\perp(x-y)| \le C\varepsilon\rho(SX, SY)^{1+\beta}. \tag{7-1}$$

Now, by using the identities $S + S^\perp = T + T^\perp = I$, it may be checked by direct computations that

$$(I - S^\perp T)S^\perp(x-y) - S^\perp T S(x-y) = S^\perp T^\perp(x-y). \tag{7-2}$$

Since $|S - T| \le C\varepsilon$, $(I - S^\perp T)$ is invertible and $|(I - S^\perp T)^{-1}| \le 2$ provided ε is small enough. In particular, by letting $L = (I - S^\perp T)^{-1}S^\perp$, we have $|L| \le 2$. Then (7-1) and (7-2) above give

$$|S^\perp x - S^\perp y - LT(Sx - Sy)| = |LT^\perp(x-y)| \le 2|T^\perp(x-y)| \le C\varepsilon\rho(SX, SY)^{1+\beta}. \tag{7-3}$$

This proves that, indeed, there is $u \in C^{1,\beta}(S(\Sigma \cap Q_\eta); \mathbb{R}^{d-m})$ with $\|u\|_{C^{1,\beta}} \le C\varepsilon$ such that $\Sigma \cap Q_\eta =$ graph u.

Step 2: Absence of holes. We now "split" the parabolic cylinder $Q_{\eta/2}$ into two components, on two opposite sides of Γ. To this end, we define the sets

$$E'_+ := \{x' \in B^m_{\eta/2} : x'_m > \gamma_m(x'')\}, \qquad E'_- := \{x' \in B^m_{\eta/2} : x'_m < \gamma_m(x'')\},$$

their parabolic counterparts

$$\mathcal{E}'_\pm := E'_\pm \times (-\eta^2/4, 0] \subset \mathbb{R}^{m,1}$$

and, lastly,

$$\mathcal{E}_\pm = \{(x,t) \in \mathbb{R}^{d,1} : (x',t) \in E_\pm\}.$$

Arguing for the positive side (as the argument applies for the other case) we claim that, if $X_1 = (x_1, t_1) \in \Sigma \cap \mathcal{E}_+$, then

$$\mathcal{E}'_+ \cap \{t \le t_1\} \subset S(\Sigma \cap Q_\eta). \tag{7-4}$$

To prove this, assume by contradiction that there is $(x'_0, t_0) \in \mathcal{E}'_+ \setminus S(\Sigma)$ with $t_0 < t_1$. Since $\gamma \in C^{1,\alpha}$, it is easy to see that there exist a smooth curve $p : [t_0, t_1] \to E'_+$ and $\rho > 0$ with the following properties:

$$\begin{cases} p(t_0) = x'_0 \ \text{ and } \ p(t_1) = x'_1, \\ Q^m_\rho(p(t), t) \subset \mathcal{E}'_+, \\ Q^m_\rho(p(t_0), t_0) \subset S(\Sigma)^c. \end{cases}$$

The fact that Σ is closed and Proposition 3.4 yield the existence of a time $\bar{t} \in (t_0, t_1)$ such that $Q^m_\rho(p(t), t) \subset S(\Sigma)^c$ for every $t < \bar{t}$ and a point $(y_0, s_0) \in \Sigma$ such that $(y'_0, s_0) \in \partial B^m_\rho(p(\bar{t})) \times [\bar{t} - \rho^2, \bar{t}]$.

Let us now consider a sequence $r_j \searrow 0$ and define the dilations

$$\boldsymbol{M}^j = \mathcal{D}_{r_j}(\boldsymbol{M} - (y_0, s_0)).$$

Since $\boldsymbol{M}$ has bounded maximal density ratio, the compactness theorems in [White 2021, Section 10] yield that, up to passing to a subsequence, $\boldsymbol{M}^j$ converges to a limit Brakke flow $\boldsymbol{M}^\infty$.

Then (7-3) implies that there exists an m-dimensional half-plane T^+ such that

$$\Sigma_{\boldsymbol{M}^\infty} \subset T^+ \times (-\infty, 0].$$

Moreover, since $(y_0, s_0) \in \Sigma_{\boldsymbol{M}}$, we have $(0,0) \in \Sigma_{\boldsymbol{M}^\infty}$.

We finally show that this violates the maximum principle. Up to a change of coordinates, say $T^+ = \{x_{m+1} = \cdots = x_d = 0 \text{ and } x_m > 0\}$ and let

$$f(x,t) = \frac{|T^\perp x|^2}{2} - \frac{|x''|^2}{2m} + \frac{|x_m|^2}{2} - x_m + \frac{1}{2m} t,$$

where $x'' = (x_1, \dots, x_{m-1})$. Then $f|_{\Sigma_{\boldsymbol{M}^\infty} \cap \{t \le 0\}}$ has a local maximum at $(0,0)$. However, it holds $\partial_t f(0,0) = 1/(2m)$ and

$$\inf_{\substack{T \in \mathrm{Gr}(m,d) \\ T \perp \nabla f(0,0)}} T : D^2 f(0,0) = 1 - (m-1)\frac{1}{m} = \frac{1}{m},$$

which contradicts Proposition 3.7, thus proving (7-4).

Step 3: Conclusion. Since, by assumption, $(0,0) \in \Sigma_M$, by Proposition 3.4 it must be $M_t(B_r) > 0$ for every $t < 0$ and $r > 0$. However, if there were $t_1 < 0$ and $r > 0$ such that

$$M_{t_1}(B_r \cap \{x_m < \gamma_m(x'')\}) > 0,$$

by the previous step it should be $\mathcal{E}'_- \cap \{t \le t_1\} \subset S(\Sigma \cap Q_\eta)$. However, this would contradict Proposition 5.1 with $R = \eta$, provided ε_3 is chosen small enough. Therefore we have that, for every $t < 0$ sufficiently close to 0, it must be

$$M_t(B_r \cap \{x_m > \gamma_m(x'')\}) > 0$$

and, by the previous step, $\mathcal{E}'_+ \subset S(\Sigma \cap Q_\eta)$ which, together with (7-3) amounts to saying that there exists $u : \mathcal{E}'_+ \to \mathbb{R}^{d-m}$ such that

$$\Sigma \cap (\mathcal{E}_+ \cup \mathcal{E}_-) = \operatorname{graph} u = \{(x,t) : Sx \in \mathcal{E}'_+ \text{ and } S^\perp x = u(Sx,t)\}$$

and $\|u\|_{C^{1,\beta}} \le C\varepsilon$.

We only have to prove that $\partial\Sigma_t \cap B_{\eta/2} \subset \Gamma$ for every $t \in (-\eta^2/4, 0]$. If there were $x \in \partial\Sigma_t \cap B_{\eta/2} \setminus \Gamma$, then by the fact that $u \in C^{1,\beta}$ there would be some blow-up of Σ around (x,t) that is contained in an m-dimensional half-plane for all times. Arguing as in the previous step, one finds a contradiction to Proposition 3.7. □

Appendix A: Proof of Lemma 3.2

Up to rescaling and translating, it is sufficient to prove that there exist A and Λ small and positive, depending only on δ and α, such that, if Γ is an $(m-1)$-dimensional properly embedded submanifold of B_2 with $[\Gamma]_{C^{1,\alpha}(B_2)} \le A$, then

$$\int_{-\Lambda}^{0} \int |T_y\Gamma^\perp \nabla\Psi_1(y,t)|\, d\Gamma(y)\, dt \le \frac{\chi_{\Gamma^c}(0)}{2} + \delta. \tag{A-1}$$

For brevity, we denote by Γ_y the space $T_y\Gamma$. Throughout the proof, C will denote constants (possibly changing from one expression to another) depending only on m, d, α.

Case 1: $0 \in \Gamma$. We start by remarking that

$$\int_{-\Lambda}^{0} \int |\Gamma_y^\perp \nabla\Psi_1(\cdot,t)|\, d\Gamma\, dt \le \int_{-\Lambda}^{0} \int_{B_2} |\Gamma_y^\perp \nabla\Psi(\cdot,t)|\, d\Gamma(y)\, dt + C\Gamma(B_2) \int_{-\Lambda}^{0} \frac{e^{1/(4t)}}{(-t)^{m/2}}\, dt.$$

If A is smaller than some universal constant, then $\Gamma(B_2) \le C$; thus we may take Λ small depending on δ so that the last term in the above inequality is smaller than $\delta/2$. Therefore we reduce ourselves to proving that, if A is small, then

$$I_1 := \int_{-\Lambda}^{0} \int_{B_2} |\Gamma_y^\perp \nabla\Psi(\cdot,t)|\, d\Gamma\, dt \le \frac{\delta}{2}.$$

Since $[\Gamma]_{C^{1,\alpha}(B_2)} \le A$ is small, for every $(y,t) \in \Gamma \times (-\infty, 0)$,

$$|\Gamma_y^\perp \nabla\Psi(y,t)| \le C\frac{e^{|y|^2/t}}{(-t)^{1+m/2}} |\Gamma_y^\perp y| \le CA\frac{e^{|y|^2/t}}{(-t)^{1+m/2}} |y|^{1+\alpha}.$$

We then use the fact that, if A is smaller than some universal constant, then $\Gamma \cap B_2$ is the graph over Γ_0 of some function $\gamma : \mathbb{R}^{m-1} \to \mathbb{R}^{d-m+1}$ such that $\|\gamma\|_{C^{1,\alpha}} \le CA$. In particular, by using the area formula and the fact that $\|\nabla\gamma\|_{L^\infty(B_2)} \le 1$ for A small enough, we obtain

$$\int_{B_2} |\Gamma_y^\perp \nabla\Psi(y,t)|\, d\Gamma(y) \le CA\frac{1}{(-t)^{1+m/2}} \int_{\mathbb{R}^{m-1}} |y|^{1+\alpha} e^{|y|^2/t}\, d\mathcal{L}^{m-1}(y) = CA(-t)^{\alpha/2-1}$$

for some C depending only on m and α. Therefore, assuming $\Lambda \le 1$,

$$I_1 \le CA \int_{-1}^{0} (-t)^{\alpha/2-1}\, dt \le CA.$$

We conclude the proof in the case $0 \in \Gamma$ by choosing $A \le \delta/(2C)$.

Case 2: $0 \notin \Gamma$. Let E_Γ be the m-dimensional Hausdorff measure restricted to the exterior cone

$$C_\Gamma := \{\lambda y : \lambda \ge 1 \text{ and } y \in \Gamma\}$$

with multiplicity, as defined in [White 2021, Section 7]. With similar computations to those in the proof of [loc. cit., Theorem 7.1], we may show that

$$\begin{aligned}\int_{-\Lambda}^{0}\int |\Gamma_y^\perp \nabla\Psi_1(y,t)|\, d\Gamma(y)\, dt &\le -\lim_{\tau\nearrow 0}\int \Psi_1(\,\cdot\,,\tau)\, dE_\Gamma + \int \Psi_1(\,\cdot\,,-\Lambda)\, dE_\Gamma + C\Lambda E_\Gamma(B_2)\\ &= \int \Psi_1(\,\cdot\,,-\Lambda)\, dE_\Gamma + C\Lambda E_\Gamma(B_2),\end{aligned}$$

where the last equality comes from the fact that $0 \notin \Gamma$.

In order to prove (A-1), we argue by contradiction: assume there is a sequence $\{\Gamma^j\}$ with $0 \notin \Gamma^j$ such that $\|\Gamma^j\|_{C^{1,\alpha}(B_2)} \le 1/j$ for which the left-hand side of (A-2) is greater than $\frac{1}{2} + \delta$. One may show that, up to extracting a subsequence, E_{Γ^j} converges weakly to $\mathcal{H}^m \llcorner S^+$, where S^+ is some m-dimensional half-plane such that $0 \notin \operatorname{Int}(S^+)$. Therefore

$$\limsup_{j\to\infty}\left\{\int \Psi_1(\,\cdot\,,-\Lambda) E_{\Gamma^j} + C\Lambda E_{\Gamma^j}(B_2)\right\} \le \int_{S^+} \Psi_1(\,\cdot\,,-\Lambda)\, d\mathcal{H}^m + C\Lambda\mathcal{H}^m(\bar{B}_2 \cap S^+).$$

Since $0 \notin \operatorname{Int}(S^+)$, the integral in the right-hand side of the above inequality is smaller than $\frac{1}{2}$ for every choice of Λ. On the other hand, Λ may be chosen so small that

$$C\Lambda\mathcal{H}^m(\bar{B}_2 \cap S^+) \le \frac{\delta}{2},$$

which contradicts the assumption made above, thus concluding the proof.

Appendix B: Proof of Lemma 3.6

We refer the reader to [Kasai and Tonegawa 2014, Lemma 9.4] for a detailed proof of Lemma 3.6. Since some minor modifications are needed, in this section we sketch the outline of the proof.

Let $U \subset \mathbb{R}^d$ be open, $I \subset \mathbb{R}$ be a nonempty interval, Γ be an $(m-1)$-dimensional $C^{1,\alpha}$ submanifold of U and let $\boldsymbol{M} \in \mathcal{BF}_m(U \times I, \Gamma)$. We assume that $\boldsymbol{M}$ satisfies a bound of the form

$$\operatorname{mdr}(\boldsymbol{M}, U \times I) \le E_1 < \infty \tag{B-1}$$

and we let $\Sigma = \Sigma_{\boldsymbol{M}}$ be its space-time support.

Before proceeding, by virtue of Proposition 3.4, we fix small constants c_1, c_2 and R_0, depending on E_1 and Γ, such that, for every $(x,t) \in \Sigma$ and every $R \le R_0$ such that $B_R(x) \times (t - c_1 R^2, t) \Subset U \times I$, it holds

$$M_{t-c_1R^2}(B_{R/2}(x)) \ge c_2 R^m.$$

By Definition 2.3, for almost every $t \in I$ there exist an m-dimensional rectifiable set $E \subset U$ and a positive, integer-valued function $\theta_t : E_t \to \mathbb{N}$ such that $M_t = \theta_t(\,\cdot\,)\mathcal{H}^m \llcorner E_t$. We choose a time t as above, with the additional condition that $s \mapsto M_s(\varphi)$ is continuous at t for every $\varphi \in C_c(U)$. By [Tonegawa 2019, Proposition 3.3], almost every $t \in I$ satisfies the latter condition.

We claim that, for every such t and for every $B_{3r}(x_0) \Subset U$, it holds

$$\mathcal{H}^m((\Sigma_t \setminus E_t) \cap B_r(x_0)) = 0; \tag{B-2}$$

this clearly implies (3-4).

In order to prove (B-2), we argue by contradiction. Assume that there is $(x_0, t_0) \in U \times I$ and $r > 0$ such that $B_{3r} \Subset U$ and $\mathcal{H}^m(A \cap B_r(x_0)) > 0$. Without loss of generality, we may take $x_0 = 0$, $t = 0$ and set $A := \Sigma_0 \setminus E_0$.

Let

$$A_k := \{x \in A \cap B_r : M_0(B_R(x)) \le c_2 R^m/2 \text{ for all } R \in (0, r/k)\}.$$

Since, for $\mathcal{H}^m$-a.e. $x \in E_0^c$, it holds

$$\lim_{R \searrow 0} \frac{M_0(B_R(x))}{R^m} = 0,$$

we have

$$0 < \mathcal{H}^m(A \cap B_r) = \mathcal{H}^m\Bigl(\bigcup_{k \in \mathbb{N}} A_k\Bigr).$$

Therefore we may find $k \in \mathbb{N}$ such that $b_0 := \mathcal{H}^m(A_k) > 0$.

By standard measure-theoretic arguments, it is not hard to show that there exists c small universal such that, for every R small enough, we may find $N \in \mathbb{N}$ and a finite collection of points $\{x_j\}_{j=1}^N \subset A_k$ such that $\{B_R(x_j)\}$ are mutually disjoint and

$$N R^m \ge c b_0. \tag{B-3}$$

By the definition of A, since $x_j \in A_k$, we have

$$M_0\Bigl(\bigcup_{j=1}^N B_R(x_j)\Bigr) \le N c_2 \frac{R^m}{2}. \tag{B-4}$$

On the other hand, by Proposition 3.4 and the fact that $x_j \in \Sigma_0$, we have

$$M_{-c_1R^2}\Bigl(\bigcup_{j=1}^N B_{R/2}(x_j)\Bigr) \ge N c_2 R^m. \tag{B-5}$$

We now fix a cut-off function $\varphi \in C_c^\infty(B_1)$ such that $\varphi \in [0,1]$ everywhere, $\varphi|_{B_{1/2}} \equiv 1$ and $|\nabla\varphi| \le 4$. Then, given R small, we let $\varphi_0(x) = \varphi(x/(2r))$, $\varphi_j(x) = \varphi((x-x_j)/R)$ and

$$\tilde\varphi = \varphi_0 - \sum_{j=1}^N \varphi_j.$$

Then clearly $\tilde\varphi \in [0,1]$ everywhere and $|\nabla\tilde\varphi| \le C/R$. Notice, moreover, that

$$\sum_{j=1}^N \chi_{B_{R/2}(x_j)} \le \sum_{j=1}^N \varphi_j \le \sum_{j=1}^N \chi_{B_R(x_j)}.$$

For brevity, set $s = -c_1 R^2$. By (B-4) and (B-5), we have

$$\begin{aligned} M_0(\varphi_0) - M_s(\varphi_0) &= (M_0(\tilde\varphi) - M_s(\tilde\varphi)) + \Big(M_0\Big(\sum \varphi_j\Big) - M_s\Big(\sum\varphi_j\Big)\Big) \\ &\le (M_0(\tilde\varphi) - M_s(\tilde\varphi)) + (Nc_2R^m/2 - Nc_2R^m) \\ &\le (M_0(\tilde\varphi) - M_s(\tilde\varphi)) - c_3 b_0 \end{aligned} \tag{B-6}$$

for some c_3 small, where (B-3) was used in the last inequality.

We now estimate, by using Definition 2.3,

$$M_0(\tilde\varphi) - M_s(\tilde\varphi) \le \int_s^0\int H\cdot\nabla\tilde\varphi\, dM_t\, dt \le \Big(\int_s^0\int_{B_{2r}} |H|^2\Big)^{1/2}\Big(\int_s^0\int |\nabla\tilde\varphi|^2\Big)^{1/2}. \tag{B-7}$$

By (B-1) and the fact that $s = -c_1R^2$, we have, for some C large,

$$\int_s^0\int |\nabla\tilde\varphi|^2\, dM_t\, dt \le (-s)\|\nabla\tilde\varphi\|_\infty^2 M_t(B_{2r}) \le CE_1 r^m;$$

therefore (B-6) and (B-7) yield

$$\Big(\int_{-c_2R^2}^0\int_{B_{2r}} |H|^2\, dM_t\, dt\Big)^{1/2} \ge \Big(\frac{1}{CE_1r^m}\Big)^{1/2}(M_0(\varphi_0) - M_{-c_1R^2}(\varphi_0) + c_3b_0).$$

By assumption, $t \mapsto M_t(\varphi_0)$ is continuous at 0. Thus we may choose R so small that the right-hand side of the above inequality is larger than $cb_0/(E_1r^m)^{1/2}$ for some c small enough.

Finally, we consider the function $\hat\varphi = \varphi(x/(3r))$. By Definition 2.3 and the Cauchy–Schwarz inequality, we have

$$\begin{aligned} M_0(\hat\varphi) - M_{-c_1R^2}(\hat\varphi) &\le \int_{-c_1R^2}^0\int_{B_{3r}} (-\hat\varphi|H|^2 + \nabla\hat\varphi\cdot H)\, dM_t\, dt \\ &\le -\int_{-c_1R^2}^0\int_{B_{3r}} \frac{1}{2}\hat\varphi|H|^2\, dM_t\, dt + \int_{-c_1R^2}^0\int_{B_{3r}} \frac{|\nabla\hat\varphi|^2}{2\hat\varphi}\, dM_t\, dt \\ &\le -\frac{1}{2}\int_{-c_1R^2}^0\int_{B_{2r}} |H|^2\, dM_t\, dt + CE_1r^mR^2 \le -c\frac{b_0^2}{E_1r^m}, \end{aligned}$$

provided R is chosen small enough. This contradicts the continuity of $t \mapsto M_t(\hat\varphi)$ at 0, thus concluding the proof.

Acknowledgements

The author is extremely grateful to Guido De Philippis for his crucial help in writing this paper and to Luigi De Masi for countless discussions and comments. This material is based upon work partially supported by the European Research Council (ERC), under the European Union's Horizon 2020 research and innovation program, through the project ERC VAREG-Variational approach to the regularity of the free boundaries (grant agreement no. 853404) and by INDAM-GNAMPA. This work was completed when the author was a Ph.D. student at SISSA, Trieste.

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Received 22 Jun 2022. Revised 4 Sep 2023. Accepted 3 Jan 2024.

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ANALYSIS AND PDE
Vol. 18 (2025), No. 4, pp. 895–962
DOI: 10.2140/apde.2025.18.895

OPTIMAL BLOWUP STABILITY FOR THREE-DIMENSIONAL WAVE MAPS

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We study corotational wave maps from (1+3)-dimensional Minkowski space into the three-sphere. We establish the asymptotic stability of an explicitly known self-similar wave map under perturbations that are small in the critical Sobolev space. This is accomplished by proving Strichartz estimates for a radial wave equation with a potential in similarity coordinates. Compared to earlier work, the main novelty lies with the fact that the critical Sobolev space is of fractional order.

1. Introduction

The present work is concerned with the wave maps equation, the prototypical example of a geometric wave equation. The wave maps equation is a natural generalization of the wave equation when the unknown takes values in a Riemannian manifold. Here, we are only interested in the case where the manifold is the round sphere; i.e., we consider maps $U:\mathbb{R}^{1,d}\to\mathbb{S}^d\subset\mathbb{R}^{d+1}$, where $\mathbb{R}^{1,d}$ is the $(1+d)$-dimensional Minkowski space. In this special case, the wave maps equation takes the form

$$\partial^\mu\partial_\mu U+(\partial^\mu U\cdot\partial_\mu U)U=0, \tag{1-1}$$

where $\cdot$ denotes the Euclidean inner product on $\mathbb{R}^{d+1}$ and Einstein's summation convention[1] is in force. Equation (1-1) is a hyperbolic partial differential equation and it is natural to study the Cauchy problem. To this end, one prescribes initial data $U(0,\cdot):\mathbb{R}^d\to\mathbb{S}^d$, $\partial_0 U(0,\cdot):\mathbb{R}^d\to\mathbb{R}^{d+1}$, with

This work was supported by the Austrian Science Fund FWF, Projects P 30076 "Self-similar blowup in dispersive wave equations" and P 34560 "Stable blowup in supercritical wave equations".
MSC2020: primary 35L71; secondary 35B44, 35L15.
Keywords: nonlinear PDE, wave equations, blowup.

[1]As is common in relativity, we number the slots of a function on Minkowski space from 0 to d and $\partial^0=-\partial_0$, whereas $\partial^j=\partial_j$ for $j\in\{1,2,\dots,d\}$. Two indices, where one occurs upstairs and the other one downstairs, are automatically summed over and Greek indices take on the values $0,1,\dots,d$.

$U(0,\cdot)\cdot\partial_0 U(0,\cdot)=0$ and aims to construct a unique solution to (1-1) satisfying these initial conditions. Intriguingly, for $d\geq 2$, it is in general impossible to construct global-in-time solutions to the Cauchy problem for the wave maps equation, even if the initial data $(U(0,\cdot),\partial_0 U(0,\cdot))$ are smooth and nicely behaved towards infinity. For $d\geq 3$, this is evidenced by an explicit one-parameter family of self-similar solutions. Indeed, for $T>0$, let $U_*^T(t,x)=F_*(x/(T-t))$, where $F_*:\mathbb{R}^d\to\mathbb{S}^d\subset\mathbb{R}^{d+1}$ is given by

$$F_*(\xi):=\frac{1}{d-2+|\xi|^2}\begin{pmatrix}2\sqrt{d-2}\,\xi\\ d-2-|\xi|^2\end{pmatrix}=\begin{pmatrix}\sin(f_*(\xi))\frac{\xi}{|\xi|}\\ \cos(f_*(\xi))\end{pmatrix},$$

with

$$f_*(\xi):=2\arctan\left(\frac{|\xi|}{\sqrt{d-2}}\right).$$

Then U_*^T is a wave map, as one may convince oneself by a straightforward computation. The solution U_*^T, which was discovered in [Turok and Spergel 1990; Shatah 1988; Bizoń and Biernat 2015], starts from smooth initial data but develops a singularity in finite time in the sense that the gradient blows up. Moreover, by the finite speed of propagation property inherent to the wave maps equation, the behavior of the data at spatial infinity is completely irrelevant. A natural question that arises immediately is as to whether this explicit blowup solution has any bearing on the generic behavior of the Cauchy evolution. Perhaps U_*^T actually belongs to a larger family of solutions which exhibit similar kinds of singular behavior? And if so, how large is this family? To answer these questions it is necessary to study the stability of U_*^T under perturbations of the initial data. For the case $d=3$, we show that all solutions that start out close to U_*^T develop a singularity with the same asymptotic profile as U_*^T. Furthermore, the smallness of the perturbation is measured in the weakest possible (L^2-based) Sobolev norm.

In order to state our main theorem precisely, we set

$$u_*^T(t,x)=\frac{2}{|x|}\arctan\left(\frac{|x|}{T-t}\right)$$

and define $\Omega_T\subset\mathbb{R}\times\mathbb{R}^3$ for $T>0$ by

$$\Omega_T:=([0,\infty)\times\mathbb{R}^3)\setminus\{(t,x)\in[T,\infty)\times\mathbb{R}^3:|x|\leq t-T\}.$$

In words, Ω_T is all of the future of the initial surface $t=0$ minus the forward lightcone emanating from the blowup point $(T,0)$. Furthermore, for $R>0$ and $x_0\in\mathbb{R}^d$, we set $\mathbb{B}_R^d(x_0):=\{x\in\mathbb{R}^d:|x-x_0|<R\}$ and abbreviate $\mathbb{B}_R^d:=\mathbb{B}_R^d(0)$.

Theorem 1.1. *There exist constants $\delta_0, M>0$ such that the following holds. Let $F:\mathbb{R}^3\to\mathbb{S}^3\subset\mathbb{R}^4$ and $G:\mathbb{R}^3\to\mathbb{R}^4$ be given by*

$$F(x)=\begin{pmatrix}\sin(|x|f(x))\frac{x}{|x|}\\ \cos(|x|f(x))\end{pmatrix},\quad G(x)=\begin{pmatrix}\cos(|x|f(x))g(x)x\\ -\sin(|x|f(x))|x|g(x)\end{pmatrix}$$

for smooth, radial functions $f,g:\mathbb{R}^3\to\mathbb{R}$. Assume further that $\delta\in[0,\delta_0]$ and

$$\||\cdot|[(f,g)-(u_*^1(0,\cdot),\partial_0 u_*^1(0,\cdot))]\|_{H^{3/2}\times H^{1/2}(\mathbb{B}^3_{1+\delta})}\leq\frac{\delta}{M}.$$

Then there exists a $T \in [1-\delta, 1+\delta]$ *and a unique smooth wave map* $U : \Omega_T \to \mathbb{S}^3 \subset \mathbb{R}^4$ *that satisfies* $U(0,x) = F(x)$ *and* $\partial_0 U(0,x) = G(x)$ *for all* $x \in \mathbb{R}^3$*. Furthermore, in the backward lightcone of the point* $(T, 0)$*, we have the weighted Strichartz estimates*

$$\int_0^T \||\cdot|^{-\frac{4}{5}}(U(t,\cdot) - U_*^T(t,\cdot))\|^2_{L^{10}(\mathbb{B}^3_{T-t})}\,dt \le \delta^2,$$

$$\int_0^T \||\cdot|^{-\frac{4}{15}}(\partial_j U(t,\cdot) - \partial_j U_*^T(t,\cdot))\|^6_{L^{30/11}(\mathbb{B}^3_{T-t})}\,dt \le \delta^6$$

for $j \in \{1,2,3\}$.

1.1. *Discussion.* We would like to make a couple of remarks.

1.1.1. *Stability of blowup.* Note that

$$U_*^T(t,0) = \binom{0}{1}$$

for all $t \in [0,T)$. Hence, a scaling argument shows that

$$\||\cdot|^{-\frac{4}{5}}(U_*^T(t,\cdot) - U_*^T(t,0))\|_{L^{10}(\mathbb{B}^3_{T-t})} \simeq (T-t)^{-\frac{1}{2}},$$

from which we infer that

$$\int_0^T \||\cdot|^{-\frac{4}{5}}(U_*^T(t,\cdot) - U_*^T(t,0))\|^2_{L^{10}(\mathbb{B}^3_{T-t})}\,dt \simeq \int_0^T (T-t)^{-1}\,dt = \infty.$$

Similarly,

$$\int_0^T \||\cdot|^{-\frac{4}{15}}\partial_j U_*^T(t,\cdot)\|^6_{L^{30/11}(\mathbb{B}^3_{T-t})}\,dt \simeq \int_0^T (T-t)^{-1}\,dt = \infty.$$

Consequently, these Strichartz norms detect self-similar blowup and Theorem 1.1 shows that U_*^T is asymptotically stable in the backward lightcone of the singularity. Put differently, our solution U can be trivially written as

$$U(t,x) = U_*^T(t,x) + \underbrace{U(t,x) - U_*^T(t,x)}_{\text{small}}$$

and this shows that U exhibits the same blowup as U_*^T modulo an error which is small in suitable Strichartz spaces.

1.1.2. *Optimality.* Equation (1-1) is invariant under the scaling $U(t,x) \mapsto U(t/\lambda, x/\lambda)$ for $\lambda > 0$ and the corresponding scaling-invariant Sobolev space is $\dot{H}^{d/2} \times \dot{H}^{d/2-1}(\mathbb{R}^d)$. Moreover, from the ill-posedness of the wave maps equation below scaling [Shatah and Tahvildar-Zadeh 1994] it follows that the smallness condition imposed on the initial data is measured in the optimal topology in terms of regularity.

1.1.3. *Symmetry.* The prescribed initial data belong to the class of corotational maps, a symmetry preserved by the wave maps flow. Further, our Strichartz estimates are not translation-invariant and so are inherently corotational.

1.1.4. *Maximal domain of existence.* The domain on which we construct solutions is all of $[0,\infty)\times\mathbb{R}^3$ except for the part of spacetime that is causally influenced by the singularity. Whether one can extend the solution even further in a meaningful way is an intriguing open question.

1.1.5. *Supercriticality.* Lastly, we want to emphasize the fact that Theorem 1.1 is a large-data result for an energy-supercritical geometric wave equation.

1.2. ***Related results.*** Due to the sheer volume of intriguing works on the wave maps equation, we can only mention a handful of results which are directly linked to the present paper. For the local theory of corotational wave maps at low regularity we refer to [Shatah and Tahvildar-Zadeh 1994]. The general case is the focus of the works [Klainerman and Machedon 1995; Klainerman and Selberg 1997; Tao 2000; Masmoudi and Planchon 2012]. Establishing results concerning the small data global Cauchy problem is of course most delicate when one measures smallness in a scaling-invariant space. This challenging problem was intensely studied in the 1990s and the beginning of the 2000s and was resolved in [Tataru 1998; 2001; 2005; Tao 2001a; 2001b; Klainerman and Rodnianski 2001; Shatah and Struwe 2002; Krieger 2003; 2004; Nahmod et al. 2003; Candy and Herr 2018].

Turning to the large-data problem, we start with the case $d = 2$, where the strongest results are available, given that this is the energy-critical case where energy conservation yields invaluable global information. However, despite the conservation of energy, finite-time blowup is possible, albeit via a different, more complicated mechanism than in our case. Singularity formation takes place via a dynamical rescaling of a soliton (a harmonic map). Consequently, already the construction of finite-time blowup is highly nontrivial and was first achieved in [Krieger et al. 2008; Rodnianski and Sterbenz 2010; Raphaël and Rodnianski 2012], inspired by numerical evidence [Bizoń et al. 2001]; see also [Gao and Krieger 2015]. Stability results for blowup are proven in [Raphaël and Rodnianski 2012; Krieger and Miao 2020]. Subsequently, the question of large-data global existence has to be addressed in view of the fact that finite-time blowup is possible. Since the blowup takes place via the shrinking of a harmonic map, the "first" harmonic map provides a natural threshold for global existence. This is expressed in the *threshold conjecture* [Sterbenz and Tataru 2010a; 2010b; Krieger and Schlag 2012; Lawrie and Oh 2016; Chiodaroli et al. 2018]; see also the series of unpublished preprints [Tao 2008a; 2008b; 2008c; 2009a; 2009b] and the earlier [Struwe 2003; Côte et al. 2008] for the corotational setting. Recent works on energy-critical wave maps focus on the precise asymptotic behavior and the *soliton resolution conjecture* [Côte et al. 2015a; 2015b; Côte 2015; Grinis 2017; Jia and Kenig 2017; Jendrej and Lawrie 2018; Duyckaerts et al. 2018].

The present paper is concerned with the energy-supercritical case $d \geq 3$, where the conservation of energy is of no use for the study of the Cauchy problem. Therefore, the understanding of large-data evolutions is still comparatively poor. The existence of self-similar blowup for $d \geq 3$ is established in [Shatah 1988; Turok and Spergel 1990; Cazenave et al. 1998; Bizoń 2000; Bizoń and Biernat 2015]. Motivated by numerical evidence [Bizoń et al. 2000], the stability of self-similar blowup under perturbations that are small in Sobolev spaces of sufficiently high order is proved in [Donninger et al. 2012; Donninger 2011; Costin et al. 2016; 2017; Chatzikaleas et al. 2017; Donninger and Glogić 2019; Biernat et al. 2021]. We also remark that starting from dimension 7, another blowup mechanism occurs which is more reminiscent of the energy-critical case [Ghoul et al. 2018]; see [Dodson and Lawrie 2015; Chiodaroli and Krieger 2017] for other large-data results. Blowup stability in critical Sobolev spaces has so far been established for the four-dimensional wave maps equation [Donninger and Wallauch 2023] and the simpler energy-critical wave equation in dimensions $3 \le d \le 6$ [Donninger 2017; Donninger and Rao 2020; Wallauch 2023]; see also [Bringmann 2020] for an extension to randomized perturbations.

We would like to emphasize that in contrast to previous work [Donninger and Wallauch 2023; Donninger 2017; Donninger and Rao 2020; Wallauch 2023], which relied crucially on the fact that the critical Sobolev space is of integer order, the present paper is the first instance of an optimal blowup stability result in a Sobolev space of fractional order. This seemingly technical feature turns out to be much more substantial than one might expect. In fact, it adds several layers of new difficulties to the analysis, most notably the necessity of developing a suitable interpolation technique that is compatible with the nonself-adjoint spectral structure of the problem.

1.3. ***Outline of the proof.*** To prove Theorem 1.1 we follow the strategy laid out in [Donninger and Wallauch 2023] and which itself built on the previous works [Donninger 2017; Donninger and Rao 2020]. However, in contrast to the four-dimensional case studied in [Donninger and Wallauch 2023], the optimal Sobolev spaces here are of fractional order. This causes major additional problems throughout our analysis of (1-1) which were not present in previous works.

The first step in analyzing (1-1) is the symmetry reduction. From the special corotational form of the prescribed data and the preservation of that symmetry class by the wave maps flow it follows that the associated solution U is of the form

$$U(t,x) = \begin{pmatrix} \sin(|x|u(t,x))\frac{x}{|x|} \\ \cos(|x|u(t,x)) \end{pmatrix} \tag{1-2}$$

for a smooth function $u : [0,T) \times \mathbb{R}^3 \to \mathbb{R}$ such that $u(t,\cdot)$ is radial for each $t \in [0,T)$. Further, the corotational ansatz simplifies (1-1) to the semilinear equation

$$\Big(\partial_t^2 - \partial_r^2 - \frac{4}{r}\partial_r\Big)\tilde{u}(t,r) + \frac{\sin(2r\tilde{u}(t,r)) - 2r\tilde{u}(t,r)}{r^3} = 0 \tag{1-3}$$

for $r > 0$, where $u(t,x) = \tilde{u}(t,|x|)$. Note that (1-3) is a five-dimensional equation rather than a three-dimensional one, as one would perhaps expect. Therefore, it is natural to view u as a radial function on $[0,T) \times \mathbb{R}^5$ instead of $[0,T) \times \mathbb{R}^3$. Moreover, a Taylor expansion shows that the apparent singularity in (1-3) is in fact removable and the nonlinearity is perfectly smooth. Theorem 1.1 is then essentially a consequence of the following result.

Theorem 1.2. *There exist $\delta_0, M > 0$ such that the following holds. Let $f, g \in C^\infty(\overline{\mathbb{B}^5_{1+\delta}})$ be radial and let $\delta \in [0,\delta_0]$ be such that*

$$\|(f,g) - (u_*^1(0,\cdot), \partial_0 u_*^1(0,\cdot))\|_{H^{3/2} \times H^{1/2}(\mathbb{B}^5_{1+\delta})} \le \frac{\delta}{M}.$$

Then there exists a blowup time $T \in [1-\delta, 1+\delta]$ and a unique smooth solution

$$u : \{(t,x) \in [0,T) \times \mathbb{R}^5 : |x| \le T - t\} \to \mathbb{R}$$

of (1-3) satisfying $u(0,\cdot)=f$ *and* $\partial_0 u(0,\cdot)=g$ *on* $\overline{\mathbb{B}^5_T}$. *Furthermore, we have the Strichartz estimates*

$$\int_0^T \|u(t,\cdot)-u_*^T(t,\cdot)\|^2_{L^{10}(\mathbb{B}^5_{T-t})}\,dt \le \delta^2, \tag{1-4}$$

$$\int_0^T \|u(t,\cdot)-u_*^T(t,\cdot)\|^6_{\dot W^{1,30/11}(\mathbb{B}^5_{T-t})}\,dt \le \delta^6. \tag{1-5}$$

We now give a nontechnical outline of the proof of Theorem 1.2.

- First, we perform preliminary coordinate transformations and choose the right functional setup. Given the self-similar nature of the blowup, we recast (1-3) in the similarity coordinates

$$\tau=-\log(T-t)+\log(T), \quad \rho=\frac{r}{T-t}.$$

Then, we proceed to show that the operator corresponding to the free wave equation in these coordinates is densely defined and closable in different topologies and that each of these closures generates a semigroup $\boldsymbol{S}_0$. More precisely, we show that

$$\|\boldsymbol{S}_0(\tau)\|_{H^2\times H^1(\mathbb{B}^5_1)} \lesssim e^{-\frac{\tau}{2}}, \quad \|\boldsymbol{S}_0(\tau)\|_{H^1\times L^2(\mathbb{B}^5_1)} \lesssim e^{\frac{\tau}{2}},$$

which we interpolate to obtain

$$\|\boldsymbol{S}_0(\tau)\|_{H^{3/2}\times H^{1/2}(\mathbb{B}^5_1)} \lesssim 1.$$

We then linearize the nonlinearity around u_*^T and study the resulting linear operator $\boldsymbol{L}$. Utilizing [Costin et al. 2017] enables us to infer that $\boldsymbol{L}$, viewed as a densely defined operator on $H^{3/2}\times H^{1/2}(\mathbb{B}^5_1)$, has precisely one eigenvalue $\lambda=1$ in the (closed) complex right half-plane with a corresponding rank-1 spectral projection $\boldsymbol{P}$.

- To control the evolution, we next derive Strichartz estimates for $\boldsymbol{S}(\tau)(\boldsymbol{I}-\boldsymbol{P})$, where $\boldsymbol{S}$ is the semigroup generated by $\boldsymbol{L}$. We accomplish this by asymptotically constructing the resolvent of $\boldsymbol{L}$ and representing the semigroup as the Laplace inversion of $(\lambda-\boldsymbol{L})^{-1} =: \boldsymbol{R}_{\boldsymbol{L}}(\lambda)$. For the resolvent construction it is crucial that the spectral equation $(\lambda-\boldsymbol{L})\boldsymbol{u}=\boldsymbol{f}$, with $\boldsymbol{u}=(u_1,u_2)$ and $\boldsymbol{f}=(f_1,f_2)$, reduces to the second-order ODE

$$(\rho^2-1)u_1''(\rho)+\Big(2(\lambda+2)\rho-\frac{4}{\rho}\Big)u_1'(\rho)+(\lambda+2)(\lambda+1)u_1(\rho)-\frac{16}{(1+\rho^2)^2}u_1(\rho)=F_\lambda(\rho), \tag{1-6}$$

with $F_\lambda(\rho)=f_2(\rho)+(\lambda+2)f_1(\rho)+\rho f_1'(\rho)$ and $\rho\in(0,1)$. The construction of $\boldsymbol{R}_{\boldsymbol{L}}(\lambda)$ is carried out by an intricate asymptotic ODE analysis of (1-6) based on a Liouville–Green transform, Bessel asymptotics, and Volterra iterations.

- Having done this, we turn to the somewhat lengthy task of obtaining Strichartz estimates by estimating the oscillatory integrals occurring in the Laplace inversion of $\boldsymbol{R}_{\boldsymbol{L}}$. A first idea would be to obtain estimates of the form

$$\begin{aligned}\|[\boldsymbol{S}(\tau)(\boldsymbol{I}-\boldsymbol{P})\boldsymbol{f}]_1\|_{L^{p_1}_\tau(\mathbb{R}_+)L^{q_1}(\mathbb{B}^5_1)} &\lesssim \|\boldsymbol{f}\|_{H^1\times L^2(\mathbb{B}^5_1)},\\ \|[\boldsymbol{S}(\tau)(\boldsymbol{I}-\boldsymbol{P})\boldsymbol{f}]_1\|_{L^{p_2}_\tau(\mathbb{R}_+)L^{q_2}(\mathbb{B}^5_1)} &\lesssim \|\boldsymbol{f}\|_{H^2\times H^1(\mathbb{B}^5_1)},\end{aligned} \tag{1-7}$$

and interpolate between them, where $[\boldsymbol{S}(\tau)(\boldsymbol{I}-\boldsymbol{P})\boldsymbol{f}]_1$ denotes the first component of $\boldsymbol{S}(\tau)(\boldsymbol{I}-\boldsymbol{P})\boldsymbol{f}$. There is, however, a problem with this naive ansatz. In the $H^1 \times L^2$ universe the spectrum of $\boldsymbol{L}$, $\sigma(\boldsymbol{L})$, satisfies

$$\left\{z \in \mathbb{C} : \operatorname{Re} z < \tfrac{1}{2}\right\} \cup \{1\} \subset \sigma(\boldsymbol{L}).$$

Consequently, the best estimate one can hope for is of the form

$$\|[e^{-\frac{\tau}{2}}\boldsymbol{S}(\tau)(\boldsymbol{I}-\boldsymbol{P})\boldsymbol{f}]_1\|_{L^{p_1}_\tau(\mathbb{R}_+)L^{q_1}(\mathbb{B}^5_1)} \lesssim \|\boldsymbol{f}\|_{H^1\times L^2(\mathbb{B}^5_1)}. \tag{1-8}$$

Thus, for the interpolation argument to work, the corresponding $H^2 \times H^1$ estimate needs to compensate for the added decay in τ. In other words, we have to derive estimates of the type

$$\|[e^{\frac{\tau}{2}}\boldsymbol{S}(\tau)(\boldsymbol{I}-\boldsymbol{P})\boldsymbol{f}]_1\|_{L^{p_2}_\tau(\mathbb{R}_+)L^{q_2}(\mathbb{B}^5_1)} \lesssim \|\boldsymbol{f}\|_{H^2\times H^1(\mathbb{B}^5_1)}. \tag{1-9}$$

However, we cannot rigorously exclude the existence of finitely many eigenvalues with real parts bigger than $-\frac{1}{2}$. But what we do know is that all of these possible eigenvalues have finite algebraic multiplicities. Hence, the semigroup $\boldsymbol{S}(\tau)$ generated by $\boldsymbol{L}$ satisfies

$$\|\boldsymbol{S}(\tau)(\boldsymbol{I}-\boldsymbol{Q})(\boldsymbol{I}-\boldsymbol{P})\|_{H^2\times H^1(\mathbb{B}^5_1)} \lesssim_\eta e^{\eta\tau}$$

for any $\eta > -\frac{1}{2}$, where $\boldsymbol{Q}$ is the spectral projection associated to all eigenvalues λ_i with $-\frac{1}{2} < \operatorname{Re}\lambda_i < 0$. Furthermore, there might also be eigenvalues sitting on the boundary of the essential spectrum in the $H^2 \times H^1$ universe (i.e., the line $\operatorname{Re} z = -\frac{1}{2}$). Thus, we can only derive an estimate of the form

$$\|[e^{(\frac{1}{2}-\delta)\tau}\boldsymbol{S}(\tau)(\boldsymbol{I}-\boldsymbol{Q})(\boldsymbol{I}-\boldsymbol{P})\boldsymbol{f}]_1\|_{L^{p_2}_\tau(\mathbb{R}_+)L^{q_2}(\mathbb{B}^5_1)} \lesssim \|\boldsymbol{f}\|_{W^{2,2/(1+\delta)}\times W^{1,2/(1+\delta)}(\mathbb{B}^5_1)}, \tag{1-10}$$

with δ very close to 0. Hence, instead of proving (1-8), we show that

$$\|[e^{-(\frac{1}{2}-\delta)\tau}\boldsymbol{S}(\tau)(\boldsymbol{I}-\boldsymbol{P})\boldsymbol{f}]_1\|_{L^{p_1}_\tau(\mathbb{R}_+)L^{q_1}(\mathbb{B}^5_1)} \lesssim \|\boldsymbol{f}\|_{W^{1,2/(1-\delta)}\times L^{2/(1-\delta)}(\mathbb{B}^5_1)} \tag{1-11}$$

so that interpolation puts us in the correct spaces. As a consequence, we still have to control the evolution on the image of $\boldsymbol{Q}$. For this, we will make use of the following lemma.

Lemma 1.3. *Let H be a Hilbert space. Then, for any densely defined operator $T : D(T) \subset H \to H$ with finite rank, there exists a dense subset $X \subset H$ with $X \subset \mathcal{D}(T)$ and a bounded linear operator $\widehat{T} : H \to H$ such that*

$$T|_X = \widehat{T}|_X.$$

By applying this result to $\boldsymbol{Q}$, viewed as a densely defined unbounded operator on $H^{3/2} \times H^{1/2}(\mathbb{B}^5_1)$, we manage to arrive at the desired estimates

$$\|[\boldsymbol{S}(\tau)(\boldsymbol{I}-\boldsymbol{P})\boldsymbol{f}]_1\|_{L^{p}_\tau(\mathbb{R}_+)L^{q}(\mathbb{B}^5_1)} \lesssim \|\boldsymbol{f}\|_{H^{3/2}\times H^{1/2}(\mathbb{B}^5_1)}. \tag{1-12}$$

Analogously, we derive other spacetime estimates involving (fractional) derivatives on the left-hand side.

- Finally, the full nonlinear problem is treated by fixed-point arguments in an appropriate Strichartz space.

2. Transformations and semigroup theory

In all that follows we identify radial functions with their radial representatives. Moreover, any vector space, for instance $H^k(\mathbb{B}^5_1)$ or $C^k(\overline{\mathbb{B}^5_1})$, always denotes the corresponding radial subspace within that space. Before we can properly analyze (1-3) in the lightcone $\Gamma^T := \{(t,r) \in [0,\infty)^2 : r \le T-t\}$, we first need the right choice of coordinates. For our purposes, suitable coordinates are given by the similarity coordinates

$$\tau = -\log(T-t) + \log(T), \qquad \rho = \frac{r}{T-t}. \tag{2-1}$$

Thus, we set $\psi(\tau,\rho) = Te^{-\tau}u(T - Te^{-\tau}, Te^{-\tau}\rho)$ and switch to the similarity coordinates, which turns (1-3) into

$$\Big(2 + 3\partial_\tau + \partial_\tau^2 + 2\rho\partial_\tau\partial_\rho + 4\rho\partial_\rho - \frac{4}{\rho}\partial_\rho + (\rho^2-1)\partial_\rho^2\Big)\psi + \frac{\sin(2\rho\psi) - 2\rho\psi}{\rho^3} = 0, \tag{2-2}$$

where we omit the arguments of ψ for brevity. Next, we define

$$\psi_1(\tau,\rho) := \psi(\tau,\rho),$$
$$\psi_2(\tau,\rho) := (1 + \partial_\tau + \rho\partial_\rho)\psi_1(\tau,\rho),$$

which yields the system

$$\begin{aligned} \partial_\tau\psi_1 &= \psi_2 - \psi_1 - \rho\partial_\rho\psi_1, \\ \partial_\tau\psi_2 &= \partial_\rho^2\psi_1 + \frac{4}{\rho}\partial_\rho\psi_1 - \rho\partial_\rho\psi_2 - 2\psi_2 - \frac{3\sin(2\rho\psi_1) - 6\rho\psi_1}{2\rho^3}, \end{aligned} \tag{2-3}$$

with initial data

$$\psi_1(0,\rho) = Tf(T\rho), \qquad \psi_2(0,\rho) = T^2 g(T\rho).$$

We also remark that in these coordinates the blowup function u_*^T is of the form

$$\Psi_*(\rho) = \begin{pmatrix} \frac{2}{\rho}\arctan(\rho) \\ \frac{2}{1+\rho^2} \end{pmatrix}.$$

2.1. ***Semigroup theory.*** Motivated by the above evolution equation, we define the differential operator $\tilde{\boldsymbol{L}}_0$ as

$$\tilde{\boldsymbol{L}}_0\boldsymbol{u}(\rho) := \begin{pmatrix} -\rho u_1'(\rho) - u_1(\rho) + u_2(\rho) \\ u_1''(\rho) + \frac{4}{\rho}u_1'(\rho) - \rho u_2'(\rho) - 2u_2(\rho) \end{pmatrix},$$

where $\boldsymbol{u} = (u_1, u_2)$, with domain

$$D(\tilde{\boldsymbol{L}}_0) := \{\boldsymbol{u} \in C^3 \times C^2(\overline{\mathbb{B}^5_1}) : \boldsymbol{u} \text{ radial}\}.$$

We also define two inner products $(\cdot\,,\cdot)_{\mathcal{E}_1}$ and $(\cdot\,,\cdot)_{\mathcal{E}_2}$ on $D(\tilde{\boldsymbol{L}}_0)$ as

$$(\boldsymbol{u},\boldsymbol{v})_{\mathcal{E}_1} := \int_0^1 u_1'(\rho)\overline{v_1'(\rho)}\rho^4\,d\rho + \int_0^1 u_2(\rho)\overline{v_2(\rho)}\rho^4\,d\rho + u_1(1)\overline{v_1(1)}$$

and

$$(\boldsymbol{u},\boldsymbol{v})_{\mathcal{E}_2} := 8\int_0^1 u_1''(\rho)\overline{v_1''(\rho)}\rho^4\,d\rho + 32\int_0^1 u_1'(\rho)\overline{v_1'(\rho)}\rho^2\,d\rho + 2\int_0^1 u_2'(\rho)\overline{v_2'(\rho)}\rho^4\,d\rho + u_1(1)\overline{v_1(1)} + u_2(1)\overline{v_2(1)}.$$

Further, we denote the associated norms by $\|\cdot\|_{\mathcal{E}_j}$. Then the following estimate holds.

Lemma 2.1. *The operator* $\widetilde{\boldsymbol{L}}_0$ *satisfies*

$$\operatorname{Re}(\widetilde{\boldsymbol{L}}_0\boldsymbol{u},\boldsymbol{u})_{\mathcal{E}_1} \leq \tfrac{1}{2}\|\boldsymbol{u}\|_{\mathcal{E}_1}^2$$

for all $\boldsymbol{u} \in D(\widetilde{\boldsymbol{L}}_0)$.

Proof. Integrating by parts shows

$$-\int_0^1 u_1''(\rho)\overline{u_1'(\rho)}\rho^5\,d\rho = -|u_1'(1)|^2 + 5\int_0^1 |u_1'(\rho)|^2\rho^4\,d\rho + \int_0^1 u_1'(\rho)\overline{u_1''(\rho)}\rho^5\,d\rho,$$

and so

$$-\operatorname{Re}\int_0^1 u_1''(\rho)\overline{u_1'(\rho)}\rho^5\,d\rho = -\frac{|u_1''(1)|^2}{2} + \frac{5}{2}\int_0^1 |u_1'(\rho)|^2\rho^4\,d\rho.$$

Consequently,

$$\int_0^1 [\widetilde{\boldsymbol{L}}_0\boldsymbol{u}]_1'(\rho)\overline{u_1'(\rho)}\rho^4\,d\rho = \frac{1}{2}\int_0^1 |u_1'(\rho)|^2\rho^4\,d\rho + \int_0^1 u_2'(\rho)\overline{u_1'(\rho)}\rho^4\,d\rho - \frac{1}{2}|u_1'(1)|^2.$$

Similarly,

$$\begin{aligned}\int_0^1 [\widetilde{\boldsymbol{L}}_0\boldsymbol{u}]_2(\rho)\overline{u_2(\rho)}\rho^4\,d\rho \\ = \frac{1}{2}\int_0^1 |u_2(\rho)|^2\rho^4\,d\rho + 4\int_0^1 u_1'(\rho)\overline{u_2(\rho)}\rho^3\,d\rho + \int_0^1 u_1''(\rho)\overline{u_2(\rho)}\rho^4\,d\rho - \frac{1}{2}|u_1'(1)|^2.\end{aligned}$$

Further, given that

$$\int_0^1 u_1''(\rho)\overline{u_2(\rho)}\rho^4\,d\rho = u_1'(1)\overline{u_2(1)} - \int_0^1 u_1'(\rho)\overline{u_2'(\rho)}\rho^4 d\rho - 4\int_0^1 u_1'(\rho)\overline{u_2(\rho)}\rho^3\,d\rho,$$

we obtain

$$\begin{aligned}\operatorname{Re}(\widetilde{\boldsymbol{L}}_0\boldsymbol{u},\boldsymbol{u})_{\mathcal{E}_1} = \frac{1}{2}\int_0^1 (|u_1'(\rho)|^2\rho^4 + |u_2(\rho)|^2\rho^4)\,d\rho - \frac{1}{2}(|u_1'(1)|^2 + |u_2(1)|^2) \\ + \operatorname{Re}\big(u_2(1)\overline{u_1'(1)} - u_1'(1)\overline{u_1(1)} - |u_1(1)|^2 + u_2(1)\overline{u_1(1)}\big).\end{aligned}$$

By employing the elementary inequality

$$\operatorname{Re}(a\bar{b} + a\bar{c} - b\bar{c}) \leq \tfrac{1}{2}(|a|^2 + |b|^2 + |c|^2),$$

with $a = u_2(1),\ b = u_2(1),\ c = u_1(1)$ we deduce that

$$\operatorname{Re}(\widetilde{\boldsymbol{L}}_0\boldsymbol{u},\boldsymbol{u})_{\mathcal{E}_1} \leq \frac{1}{2}\int_0^1 (|u_1'(\rho)|^2\rho^4 + |u_2(\rho)|^2\rho^4)\,d\rho \leq \frac{1}{2}\|\boldsymbol{u}\|_{\mathcal{E}_1}^2.$$ □

For the inner product $(\cdot\,,\cdot)_{\mathcal{E}_2}$ we can derive a similar but better estimate.

Lemma 2.2. *The operator* $\widetilde{\boldsymbol{L}}_0$ *satisfies*

$$\operatorname{Re}(\widetilde{\boldsymbol{L}}_0\boldsymbol{u},\boldsymbol{u})_{\mathcal{E}_2} \leq -\tfrac{1}{2}\|\boldsymbol{u}\|_{\mathcal{E}_2}^2$$

for all $\boldsymbol{u} \in D(\widetilde{\boldsymbol{L}}_0)$.

Proof. Let $\boldsymbol{u} \in D(\widetilde{\boldsymbol{L}}_0)$. Integrating by parts as above shows that

$$\begin{aligned}\operatorname{Re}\int_0^1 [\widetilde{\boldsymbol{L}}_0\boldsymbol{u}]_1''(\rho)\overline{u_1''(\rho)}\rho^4\,d\rho &= \operatorname{Re}\Big(-\int_0^1 u_1^{(3)}(\rho)\overline{u_1''(\rho)}\rho^5\,d\rho+\int_0^1 u_2''(\rho)\overline{u_1''(\rho)}\rho^4\,d\rho\Big)-3\int_0^1 |u_1''(\rho)|^2\rho^4\,d\rho\\ &= -\frac{1}{2}\int_0^1 |u_1''(\rho)|^2\rho^4\,d\rho-\frac{|u_1''(1)|^2}{2}+\operatorname{Re}\int_0^1 u_2''(\rho)\overline{u_1''(\rho)}\rho^4\,d\rho.\end{aligned}$$

Similarly, we see that

$$\begin{aligned}\operatorname{Re}\int_0^1 [\widetilde{\boldsymbol{L}}_0\boldsymbol{u}]_2'(\rho)\overline{u_2'(\rho)}\rho^4\,d\rho &= \operatorname{Re}\int_0^1 u_1^{(3)}(\rho)\overline{u_2'(\rho)}\rho^4\,d\rho+\operatorname{Re}\Big(4\int_0^1 [u_1''(\rho)\overline{u_2'(\rho)}\rho^3-u_1'(\rho)\overline{u_2'(\rho)}\rho^2]\,d\rho\Big)\\ &\qquad -\frac{1}{2}\int_0^1 |u_2'(\rho)|^2\rho^4\,d\rho-\frac{|u_2'(1)|^2}{2}\\ &= \operatorname{Re}\Big(-\int_0^1 u_1''(\rho)\overline{u_2''(\rho)}\rho^5\,d\rho-4\int_0^1 u_1'(\rho)\overline{u_2'(\rho)}\rho^3\,d\rho\Big)\\ &\qquad +\operatorname{Re}(u_1''(1)\overline{u_2'(1)})-\frac{|u_2'(1)|^2}{2}-\frac{1}{2}\int_0^1 |u_2'(\rho)|^2\rho^4\,d\rho.\end{aligned}$$

It follows that

$$\begin{aligned}&\operatorname{Re}\int_0^1 \big([\widetilde{\boldsymbol{L}}_0\boldsymbol{u}]_1''(\rho)\overline{u_1''(\rho)} + [\widetilde{\boldsymbol{L}}_0\boldsymbol{u}]_2'(\rho)\overline{u_2'(\rho)}\big)\rho^4\,d\rho\\ &\qquad = -\frac{1}{2}(|u_1''(1)|^2+|u_2'(1)|^2)+\operatorname{Re}(u_1''(1)\overline{u_2'(1)})-\frac{1}{2}\int_0^1 |u_1''(\rho)|^2\rho^4\,d\rho\\ &\qquad\qquad -\frac{1}{2}\int_0^1 |u_2'(\rho)|^2\rho^4\,d\rho-4\operatorname{Re}\int_0^1 u_1'(\rho)\overline{u_2'(\rho)}\rho^3\,d\rho =: I_1.\end{aligned}$$

A short calculation then shows

$$\begin{aligned}8I_1+32\operatorname{Re}\int_0^1 [\widetilde{\boldsymbol{L}}_0\boldsymbol{u}]_1'(\rho)\overline{u_1'(\rho)}\rho^2\,d\rho &\le -4\int_0^1 |u_1''(\rho)|^2\rho^4\,d\rho-4\int_0^1 |u_2'(\rho)|^2\rho^4\,d\rho\\ &\quad -16\int_0^1 |u_1'(\rho)|^2\rho^2\,d\rho-\frac{1}{2}(|u_1''(1)|^2+|u_2'(1)|^2)\\ &\quad -16|u_1'(1)|^2+\operatorname{Re}(u_1''(1)\overline{u_2'(1)}).\end{aligned}$$

Consequently, adding up all the boundary terms yields

$$\begin{aligned}&-\tfrac{1}{2}(|u_1''(1)|^2+|u_2'(1)|^2)-16|u_1'(1)|^2+\operatorname{Re}(u_1''(1)\overline{u_2'(1)})+[\widetilde{\boldsymbol{L}}_0\boldsymbol{u}]_1(1)\overline{u_1(1)}+[\widetilde{\boldsymbol{L}}_0\boldsymbol{u}]_2(1)\overline{u_2(1)}\\ &= -\tfrac{1}{2}(|u_1''(1)|^2+|u_2'(1)|^2)-16|u_1'(1)|^2+\operatorname{Re}(u_1''(1)\overline{u_2'(1)})+\operatorname{Re}(u_1(1)\overline{u_2(1)}-u_1'(1)\overline{u_1(1)})-|u_1(1)|^2\\ &\qquad +\operatorname{Re}\big(u_1''(1)\overline{u_2(1)}+4u_1'(1)\overline{u_2(1)}-u_2'(1)\overline{u_2(1)}\big)-2|u_2(1)|^2.\end{aligned}$$

By again employing the inequality

$$\operatorname{Re}(a\bar{b}+a\bar{c}-b\bar{c}) \le \tfrac{1}{2}(|a|^2+|b|^2+|c|^2),$$

once with $a=u_2(1),\ b=u_1'(1),\ c=u_1(1)$ and once with $a=u_1''(1),\ b=u_2'(1),\ c=u_2(1)$, we obtain

$$\begin{aligned}\operatorname{Re}(\widetilde{\boldsymbol{L}}_0\boldsymbol{u},\boldsymbol{u})_{\mathcal{E}_2} &\le -4\int_0^1 |u_1''(\rho)|^2\rho^4\,d\rho-4\int_0^1 |u_2'(\rho)|^2\rho^4\,d\rho-16\int_0^1 |u_1'(\rho)|^2\rho^2\,d\rho\\ &\qquad +\operatorname{Re}(3u_1'(1)\overline{u_2(1)})-15|u_1'(1)|^2-\tfrac{1}{2}|u_1(1)|^2-|u_2(1)|^2\\ &\le -\tfrac{1}{2}\|\boldsymbol{u}\|_{\mathcal{E}_2}^2.\end{aligned}$$

□

To be able to invoke the Lumer–Phillips theorem we carry on by showing the density of the range of $(1-\widetilde{\boldsymbol{L}}_0)$.

Lemma 2.3. *Let $\boldsymbol{f} \in C^\infty \times C^\infty(\overline{\mathbb{B}^5_1})$. Then there exists a $\boldsymbol{u}$ in $D(\widetilde{\boldsymbol{L}}_0)$ such that*

$$(1-\widetilde{\boldsymbol{L}}_0)\boldsymbol{u} = \boldsymbol{f}.$$

Proof. The equation $(\lambda-\widetilde{\boldsymbol{L}}_0)\boldsymbol{u} = \boldsymbol{f}$ written out explicitly reads

$$(1+\lambda)u_1(\rho)+\rho u_1'(\rho)-u_2(\rho) = f_1(\rho),$$
$$(2+\lambda)u_2(\rho)+\rho u_2'(\rho)-u_1''(\rho)-\frac{4}{\rho}u_1'(\rho) = f_2(\rho)$$

and the first of the above equations implies that

$$u_2(\rho) = (1+\lambda)u_1(\rho)+\rho u_1'(\rho)-f_1(\rho). \tag{2-4}$$

Setting $\lambda = 1$ and plugging this into the second one yields

$$(\rho^2-1)u_1''(\rho)+\Big(6\rho-\frac{4}{\rho}\Big)u_1'(\rho)+6u_1(\rho) = F_1(\rho), \tag{2-5}$$

with $F_1(\rho) = f_2(\rho)+3f_1(\rho)+\rho f_1'(\rho)$. A fundamental system for the homogeneous equation

$$(\rho^2-1)u_1''(\rho)+\left(6\rho-\frac{4}{\rho}\right)u_1'(\rho)+6u_1(\rho) = 0$$

is given by

$$\psi_0(\rho) := \frac{\tanh^{-1}(\rho)-\rho}{\rho^3}, \quad \psi_1(\rho) := \rho^{-3},$$

and the Wronskian of these two is given by

$$W(\psi_0,\psi_1)(\rho) = -\frac{1}{\rho^4(1-\rho^2)}.$$

By the variation of constants formula, a solution u_1 of (2-5) is then given by

$$\begin{aligned} u_1(\rho) &= \psi_0(\rho)\int_\rho^1 \frac{\psi_1(s)F_1(s)}{W(\psi_0,\psi_1)(s)(s^2-1)}\,ds+\psi_1(\rho)\int_0^\rho \frac{\psi_0(s)F_1(s)}{W(\psi_0,\psi_1)(s)(s^2-1)}\,ds \\ &= \frac{\tanh^{-1}(\rho)-\rho}{\rho^3}\int_\rho^1 sF_1(s)\,ds+\rho^{-3}\int_0^\rho (s\tanh^{-1}(s)-s^2)F_1(s)\,ds. \end{aligned}$$

From standard ODE theory it follows that $u_1 \in C^\infty((0,1))$. Moreover, a Taylor expansion shows that ψ_0 is a smooth even function on $[0,1)$ and so

$$\rho \mapsto \psi_0(\rho)\int_\rho^1 sF_1(s)\,ds \in C^\infty([0,1)).$$

Next, we rescale according to $\rho t = s$ to obtain that

$$i_1(\rho) := \rho^{-3}\int_0^\rho (s\tanh^{-1}(s)-s^2)F_1(s)\,ds = \int_0^1 t\left(\frac{\tanh^{-1}(\rho t)}{\rho}-t\right)F_1(\rho t)\,dt.$$

For ρ close to 0 a Taylor expansion shows that

$$\frac{\tanh^{-1}(\rho t)}{\rho} - t = \frac{\rho^2 t}{3} - \frac{\rho^4 t^5}{5} + O(\rho^6 t^7),$$

where the O-term is a smooth function. Consequently, we infer that $i_1 \in C^\infty([0,1))$, with

$$i_1'(0) = i_1^{(3)}(0) = 0. \tag{2-6}$$

Thus, $u_1 \in C^\infty([0,1))$ and by combining (2-6) with the fact that ψ_0 is even, one easily establishes that $u_1(0) = u_1^{(3)}(0) = 0$. Therefore, we are left with checking the behavior of u_1 at $\rho = 1$. For this we remark that we can recast u_1 as

$$u_1(\rho) = \frac{\tanh^{-1}(\rho)}{\rho^3} \int_\rho^1 s F_1(s)\, ds + \rho^{-3} \int_0^\rho s \tanh^{-1}(s) F_1(s)\, ds + r_1(\rho),$$

where r_1 is a smooth function at $\rho = 1$. So, we only have to show that

$$v_1(\rho) := \tanh^{-1}(\rho) \int_\rho^1 s F_1(s)\, ds + \int_0^\rho s \tanh^{-1}(s) F_1(s)\, ds$$

is regular enough at 1. Clearly, v_1 is continuous at 1 and

$$v_1'(\rho) = \frac{1}{1-\rho^2} \int_\rho^1 s F_1(s)\, ds.$$

Further,

$$\int_\rho^1 s F_1(s)\, ds = \int_{\rho-1}^0 (s+1) F_1(s+1)\, ds = \int_{\frac{\rho-1}{1-\rho^2}}^0 (y(1-\rho^2)+1) F_1(y(1-\rho^2)+1)(1-\rho^2)\, ds.$$

Hence,

$$v_1'(\rho) = \int_{-(1+\rho)^{-1}}^0 (y(1-\rho^2)+1) F_1(y(1-\rho^2)+1)\, ds$$

and this is visibly smooth at $\rho = 1$. Summarizing, we see that

$$u_1 \in C^3([0,1]), \quad u_1'(0) = u_1^{(3)}(0) = 0$$

and from (2-4) it follows that $\boldsymbol{u} \in D(\widetilde{\boldsymbol{L}}_0)$. □

The last few lemmas allow us to invoke the Lumer–Phillips theorem. However, since we would rather like to work in standard H^k spaces, we first prove the equivalences of the norms $\mathcal{E}_j$ with standard radial Sobolev norms. For this we will require the following version of Hardy's inequality

Lemma 2.4. *The estimates*

$$\| |\cdot|^{-1} f \|_{L^2(\mathbb{B}_1^5)} \lesssim \|f\|_{H^1(\mathbb{B}_1^5)},$$
$$\| |\cdot|^{-2} f \|_{L^2(\mathbb{B}_1^5)} \lesssim \|f\|_{H^2(\mathbb{B}_1^5)}$$

hold for all $f \in C^2(\overline{\mathbb{B}_1^5})$.

Proof. The first estimate is just Lemma 4.7 in [Donninger and Rao 2020]. For the second one, we let

$$E : H^2(\mathbb{B}^5_1) \to H^2(\mathbb{R}^5)$$

be a bounded extension operator. Then, by Hardy's inequality,

$$\||\cdot|^{-2} f\|_{L^2(\mathbb{B}^5_1)} \le \||\cdot|^{-2} Ef\|_{L^2(\mathbb{R}^5)} \lesssim \|Ef\|_{\dot{H}^2(\mathbb{R}^5)} \lesssim \|f\|_{H^2(\mathbb{B}^5_1)}. \qquad \square$$

Lemma 2.5. *The estimate*

$$\||\cdot|^{-1} f'\|_{L^2(\mathbb{B}^5_1)} \lesssim \|f\|_{H^2(\mathbb{B}^5_1)}$$

holds for all $f \in C^2(\overline{\mathbb{B}^5_1})$.

Proof. This is an immediate consequence of Lemma 4.1 in [Donninger and Schörkhuber 2016]. $\square$

Lemma 2.6. *The norms* $\|\cdot\|_{\mathcal{E}_j}$ *and* $\|\cdot\|_{H^j \times H^{j-1}(\mathbb{B}^5_1)}$ *are equivalent on* $D(\widetilde{\boldsymbol{L}}_0)$. *Consequently, they are also equivalent on* $H^2 \times H^1(\mathbb{B}^5_1)$.

Proof. For $j = 1$ this is Lemma 2.2 in [Donninger and Rao 2020]. For $j = 2$, the inequality

$$\|\cdot\|_{H^2 \times H^1(\mathbb{B}^5_1)} \lesssim \|\cdot\|_{\mathcal{E}_2}$$

is an immediate consequence of the estimate

$$\int_0^1 |u(\rho)|^2 \rho^4 \, d\rho \lesssim \int_0^1 |u'(\rho)|^2 \rho^4 \, d\rho + |u(1)|^2$$

and the triangle inequality. For the other inequality, we first note that

$$|u(1)| \lesssim \left| \int_0^1 \partial_\rho (u(\rho)\rho^4) \, d\rho \right| \lesssim \|u\|_{H^1(\mathbb{B}^5_1)} + \||\cdot|^{-1} u\|_{L^2(\mathbb{B}^5_1)} \lesssim \|u\|_{H^1(\mathbb{B}^5_1)}$$

for all $u \in C^1(\overline{\mathbb{B}^5_1})$. Therefore,

$$|u_1(1)|^2 + |u_2(1)|^2 \lesssim \|(u_1, u_2)\|^2_{H^2 \times H^1(\mathbb{B}^5_1)}.$$

Further,

$$\int_0^1 |u_1'(\rho)|^2 \rho^2 \, d\rho \lesssim \|(u_1, u_2)\|^2_{H^2 \times H^1(\mathbb{B}^5_1)}$$

thanks to Lemma 2.5. Finally,

$$\int_0^1 |u_1''(\rho)|^2 \rho^4 \, d\rho \lesssim \|(u_1, u_2)\|^2_{H^2 \times H^1(\mathbb{B}^5_1)} + \int_0^1 |u_1'(\rho)|^2 \rho^2 \, d\rho \lesssim \|(u_1, u_2)\|^2_{H^2 \times H^1(\mathbb{B}^5_1)}. \qquad \square$$

Thus, the Lumer–Phillips theorem immediately yields the following lemma.

Lemma 2.7. *The operator* $\widetilde{\boldsymbol{L}}_0$ *is closable and its closure, denoted by* $\boldsymbol{L}_0$, *generates a semigroup* $\boldsymbol{S}_0$ *on* $H^1 \times L^2(\mathbb{B}^5_1)$ *such that*

$$\|\boldsymbol{S}_0(\tau) \boldsymbol{f}\|_{H^1 \times L^2(\mathbb{B}^5_1)} \lesssim e^{\frac{\tau}{2}} \|\boldsymbol{f}\|_{H^1 \times L^2(\mathbb{B}^5_1)}$$

for all $\boldsymbol{f} \in H^1 \times L^2(\mathbb{B}^5_1)$ *and all* $\tau \geq 0$. *Furthermore, the restriction of* $\boldsymbol{S}_0$ *to* $H^2 \times H^1(\mathbb{B}^5_1)$ *satisfies*

$$\|\boldsymbol{S}_0(\tau)\boldsymbol{f}\|_{H^2\times H^1(\mathbb{B}^5_1)} \lesssim e^{-\frac{\tau}{2}}\|\boldsymbol{f}\|_{H^2\times H^1(\mathbb{B}^5_1)}$$

for all $\boldsymbol{f} \in H^2 \times H^1(\mathbb{B}^5_1)$ *and all* $\tau \geq 0$.

To proceed, we use that $H^{3/2} \times H^{1/2}(\mathbb{B}^5_1)$ is an exact interpolation space of $H^2 \times H^1(\mathbb{B}^5_1)$ and $H^1 \times L^2(\mathbb{B}^5_1)$ of order $\frac{1}{2}$, see [Triebel 1995, p. 317, Section 4.3.1.1, Theorem 1], to conclude the next result.

Lemma 2.8. *The semigroup* $\boldsymbol{S}_0$ *satisfies*

$$\|\boldsymbol{S}_0(\tau)\boldsymbol{f}\|_{H^{3/2}\times H^{1/2}(\mathbb{B}^5_1)} \lesssim \|\boldsymbol{f}\|_{H^{3/2}\times H^{1/2}(\mathbb{B}^5_1)}$$

for all $\boldsymbol{f} \in H^{3/2} \times H^{1/2}(\mathbb{B}^5_1)$ *and all* $\tau \geq 0$.

It is also vital for us that $\boldsymbol{S}_0$ satisfies appropriate Strichartz estimates, provided we restrict T to the interval $\left[\frac{1}{2}, \frac{3}{2}\right]$. This restriction leads to no loss of generality for us, as we are only interested in values of T which lie close to 1 anyway. Henceforth, we assume that $T \in \left[\frac{1}{2}, \frac{3}{2}\right]$.

Lemma 2.9. *Let* $p \in [2, \infty]$ *and* $q \in \left[\frac{10}{3}, \infty\right]$ *be such that* $\frac{1}{p} + \frac{5}{q} = 1$. *Then we have the estimate*

$$\|[\boldsymbol{S}_0(\tau)\boldsymbol{f}]_1\|_{L^p_\tau(\mathbb{R}_+)L^q(\mathbb{B}^5_1)} \lesssim \|\boldsymbol{f}\|_{H^{3/2}\times H^{1/2}(\mathbb{B}^5_1)}$$

for all $\boldsymbol{f} \in H^{3/2} \times H^{1/2}(\mathbb{B}^5_1)$. *Furthermore, also the inhomogeneous estimate*

$$\left\|\int_0^\tau [\boldsymbol{S}_0(\tau-\sigma)\boldsymbol{h}(\sigma)]_1\, d\sigma\right\|_{L^p_\tau(I)L^q(\mathbb{B}^5_1)} \lesssim \|\boldsymbol{h}\|_{L^1(I)H^{3/2}\times H^{1/2}(\mathbb{B}^5_1)}$$

holds for all $\boldsymbol{h} \in L^1(\mathbb{R}_+, H^{3/2} \times H^{1/2}(\mathbb{B}^5_1))$ *and all intervals* $I \subset [0, \infty)$ *containing* 0.

Proof. This follows by restricting the standard Strichartz estimates for the free wave equation to the lightcone; see [Donninger and Wallauch 2023]. □

Lemma 2.10. *The estimates*

$$\|[\boldsymbol{S}_0(\tau)\boldsymbol{f}]_1\|_{L^2_\tau(\mathbb{R}_+)W^{1/2,5}(\mathbb{B}^5_1)} \lesssim \|\boldsymbol{f}\|_{H^{3/2}\times H^{1/2}(\mathbb{B}^5_1)},$$
$$\|[\boldsymbol{S}_0(\tau)\boldsymbol{f}]_1\|_{L^6_\tau(\mathbb{R}_+)W^{1,30/11}(\mathbb{B}^5_1)} \lesssim \|\boldsymbol{f}\|_{H^{3/2}\times H^{1/2}(\mathbb{B}^5_1)}$$

hold for all $\boldsymbol{f} \in H^{3/2} \times H^{1/2}(\mathbb{B}^5_1)$. *Furthermore, also the inhomogeneous estimates*

$$\left\|\int_0^\tau [\boldsymbol{S}_0(\tau-\sigma)\boldsymbol{h}(\sigma)]_1\, d\sigma\right\|_{L^2_\tau(\mathbb{R}_+)W^{1/2,5}(\mathbb{B}^5_1)} \lesssim \|\boldsymbol{h}\|_{L^1(I)H^{3/2}\times H^{1/2}(\mathbb{B}^5_1)},$$
$$\left\|\int_0^\tau [\boldsymbol{S}_0(\tau-\sigma)\boldsymbol{h}(\sigma)]_1\, d\sigma\right\|_{L^6_\tau(\mathbb{R}_+)W^{1,30/11}(\mathbb{B}^5_1)} \lesssim \|\boldsymbol{h}\|_{L^1(I)H^{3/2}\times H^{1/2}(\mathbb{B}^5_1)}$$

hold for all $\boldsymbol{h} \in L^1(\mathbb{R}_+, H^{3/2} \times H^{1/2}(\mathbb{B}^5_1))$ *and all intervals* $I \subset [0, \infty)$ *containing* 0.

To get a better understanding of the dynamics of solutions which are close to u_*^T, we linearize the nonlinearity around this solution. For this, we set $\Psi = \Phi + \Psi^*$, where Ψ^* is the transformed blow up solution u_*^T, and formally linearize the nonlinearity around Ψ^*. This results in a linear operator $\boldsymbol{L}'$ given by

$$\boldsymbol{L}'\boldsymbol{u}(\rho) = \begin{pmatrix} 0 \\ \frac{16}{(1+\rho^2)^2} u_1(\rho) \end{pmatrix}$$

and a formal nonlinear operator $\boldsymbol{N}$ given by

$$\boldsymbol{N}(\boldsymbol{u})(\rho) := \begin{pmatrix} 0 \\ N(\psi_{*_1} + u_1)(\rho) - N(\psi_{*_1})(\rho) - \frac{16}{(1+\rho^2)^2} u_1(\rho) \end{pmatrix}.$$

Lastly, we define $\boldsymbol{L} := \boldsymbol{L}_0 + \boldsymbol{L}'$ and note that we have the following result.

Lemma 2.11. *The operator $\boldsymbol{L}'$ is a compact operator on $H^s \times H^{s-1}(\mathbb{B}_1^5)$ for any $s \geq 1$.*

Proof. This is an immediate consequence of the compactness of the embedding $H^s(\mathbb{B}_1^5) \hookrightarrow H^{s'}(\mathbb{B}_1^5)$ for $s > s' \geq 0$. □

Consequently, the bounded perturbation theorem implies that $\boldsymbol{L}$ will also generate a semigroup on each of the previously employed Sobolev spaces $H^s \times H^{s-1}(\mathbb{B}_1^5)$, which we denote by $\boldsymbol{S}$. With this, we can at least formally rewrite our equation in Duhamel form as

$$\Phi(\tau) = \boldsymbol{S}(\tau)\boldsymbol{u} + \int_0^\tau \boldsymbol{S}(\tau - \sigma)\boldsymbol{N}(\Phi(\sigma))\, d\sigma. \tag{2-7}$$

To make sense of this equation, we will show in the following that $\boldsymbol{S}$ satisfies Strichartz estimates as in Lemma 2.9, provided we project away the unstable direction. This will naturally give meaning to (2-7) in an appropriate Strichartz space.

2.2. ***Spectral analysis of L.*** From now on $\boldsymbol{L}$ will always denote the version of $\boldsymbol{L}$ that is a densely defined closed operator with

$$\boldsymbol{L} : D(\boldsymbol{L}) \subset H^2 \times H^1(\mathbb{B}_1^5) \to H^2 \times H^1(\mathbb{B}_1^5),$$

unless specifically stated otherwise. Then, for any $\lambda \in \mathbb{C}$ with $\operatorname{Re}\lambda > -\frac{1}{2}$, we have $\lambda \in \rho(\boldsymbol{L}_0)$ since

$$\|\boldsymbol{S}_0(\tau)\boldsymbol{f}\|_{H^2\times H^1(\mathbb{B}_1^5)} \lesssim e^{-\frac{1}{2}\tau}\|\boldsymbol{f}\|_{H^2\times H^1(\mathbb{B}_1^5)}$$

for all $\tau \geq 0$ and all $\boldsymbol{f} \in H^2 \times H^1(\mathbb{B}_1^5)$. As a consequence, the identity

$$\lambda - \boldsymbol{L} = (1 - \boldsymbol{L}'\boldsymbol{R}_{\boldsymbol{L}_0}(\lambda))(\lambda - \boldsymbol{L}_0),$$

with $\boldsymbol{R}_{\boldsymbol{L}_0}(\lambda) := (\lambda - \boldsymbol{L}_0)^{-1}$ implies that any spectral point λ with $\operatorname{Re}\lambda > -1$ has to be an eigenvalue of finite algebraic multiplicity by the spectral theorem for compact operators.

Lemma 2.12. *The point spectrum $\sigma_p(\boldsymbol{L})$ of $\boldsymbol{L}$ is contained in the set $\{z \in \mathbb{C} : \operatorname{Re} z < 0\} \cup \{1\}$. Furthermore, the eigenvalue* 1 *has geometric and algebraic multiplicity* 1 *and an associated eigenfunction is given by*

$$\boldsymbol{g}(\rho) = \begin{pmatrix} \frac{1}{1+\rho^2} \\ \frac{2}{(1+\rho^2)^2} \end{pmatrix}.$$

Proof. That the point spectrum really is a subset of $\{z \in \mathbb{C} : \operatorname{Re} z < 0\} \cup \{1\}$ follows as in [Donninger and Wallauch 2023]. To discern the properties of the eigenvalue 1, we start by noting that obviously $\boldsymbol{g} \in D(\boldsymbol{L})$ and a straightforward computation shows that $(1-\boldsymbol{L})\boldsymbol{g} = \boldsymbol{0}$. Moreover, the calculations in the proof of Lemma 2.3 show that the equation $(\lambda - \boldsymbol{L})\boldsymbol{u} = 0$ is equivalent to

$$u_2(\rho) = (1+\lambda)u_1(\rho) + \rho u_1'(\rho) \tag{2-8}$$

and the second-order linear differential equation

$$(\rho^2-1)u_1''(\rho) + \Big(2(\lambda+2)\rho - \frac{4}{\rho}\Big)u_1'(\rho) + (\lambda+2)(\lambda+1)u_1(\rho) - \frac{16}{(1+\rho^2)^2}u_1(\rho) = 0. \tag{2-9}$$

For $\lambda = 1$ we use reduction of order to obtain a second solution to (2-9),

$$\tilde{g}_1(\rho) = \frac{12\rho^3 \tanh^{-1}(\rho) - 9\rho^2 - 1}{\rho^3(\rho^2+1)}.$$

Hence, any solution of (2-9) has to be a linear combination of g_1 and $\tilde{g}_1$. As $\tilde{g}_1 \notin H^1(\mathbb{B}^5_1)$, we conclude that an eigenfunction has to be a multiple of $\boldsymbol{g}$ since the second component of any eigenfunction is uniquely determined by its first through (2-8). Therefore, the geometric multiplicity of the eigenvalue 1 is 1. Moving on, we define $\boldsymbol{P}$ to be the spectral projection associated to this eigenvalue, i.e.,

$$\boldsymbol{P} := \int_\gamma \boldsymbol{R}_{\boldsymbol{L}}(\lambda)\, d\lambda,$$

where $\gamma : [0,1] \to \mathbb{C}$, $\gamma(t) = 1 + \frac{1}{2}e^{2\pi i t}$. Moreover, as the essential spectrum of $\boldsymbol{L}_0$ is invariant under compact perturbations, we see that $\dim \boldsymbol{P} < \infty$. Now, given that $\boldsymbol{P}$ is a projection, we can decompose $H^2 \times H^1(\mathbb{B}^5_1)$ into the closed subspaces $\operatorname{rg} \boldsymbol{P}$ and $\ker \boldsymbol{P}$. This also yields a decomposition of $\boldsymbol{L}$ into the operators $\boldsymbol{L}_{\operatorname{rg} \boldsymbol{P}}$ and $\boldsymbol{L}_{\ker \boldsymbol{P}}$, which act as operators on $\operatorname{rg} \boldsymbol{P}$ and $\ker \boldsymbol{P}$, respectively. The inclusion $\langle \boldsymbol{g} \rangle \subset \operatorname{rg} \boldsymbol{P}$ is immediate and we claim that in fact $\operatorname{rg} \boldsymbol{P} = \langle \boldsymbol{g} \rangle$. To show this, we first remark that the finite-dimensional operator $(\boldsymbol{I}_{\operatorname{rg} \boldsymbol{P}} - \boldsymbol{L}_{\operatorname{rg} \boldsymbol{P}}) : \operatorname{rg} \boldsymbol{P} \to \operatorname{rg} \boldsymbol{P}$ is nilpotent as its only eigenvalue is 0. Thus, there exists a minimal $n \in \mathbb{N}$ such that $(\boldsymbol{I}_{\operatorname{rg} \boldsymbol{P}} - \boldsymbol{L}_{\operatorname{rg} \boldsymbol{P}})^n \boldsymbol{u} = 0$ for all $\boldsymbol{u} \in \operatorname{rg} \boldsymbol{P}$. If $n = 1$, we are done. If not, then there exists a $\boldsymbol{v} \in \operatorname{rg} \boldsymbol{P}$ such that $(\boldsymbol{I}_{\operatorname{rg} \boldsymbol{P}} - \boldsymbol{L}_{\operatorname{rg} \boldsymbol{P}})\boldsymbol{v} = \boldsymbol{g}$. This implies that v_1 satisfies the inhomogeneous ODE

$$(\rho^2-1)v_1''(\rho) + \Big(6\rho - \frac{4}{\rho}\Big)v_1'(\rho) + \Big(6 - \frac{16}{(1+\rho^2)^2}\Big)v_1(\rho) = G(\rho),$$

with

$$G(\rho) = g_2(\rho) + 3g_1(\rho) + \rho g_1'(\rho) = \frac{7+\rho^2}{(1+\rho^2)^2}.$$

By the variation of constants formula, v_1 has to be of the form

$$v_1(\rho) = c_1 g_1(\rho) + c_2 \tilde{g}_1(\rho) - g_1(\rho)\int_\rho^1 \frac{\tilde{g}_1(s)G(s)}{(1-s^2)W(g_1,\tilde{g}_1)(s)}\,ds - \tilde{g}_1(\rho)\int_0^\rho \frac{g_1(s)G(s)}{(1-s^2)W(g_1,\tilde{g}_1)(s)}\,ds,$$

with $c_1, c_2 \in \mathbb{C}$. Note that

$$W(g_1,\tilde{g}_1)(\rho) = \frac{3}{\rho^4(1-\rho^2)}$$

is strictly positive on $(0, 1)$ and therefore nonvanishing on that interval. Evidently, both

$$\frac{g_1(\rho)G(\rho)}{(1-\rho^2)W(g_1,\tilde{g}_1)(\rho)} \quad \text{and} \quad \frac{\tilde{g}_1(\rho)G(\rho)}{(1-\rho^2)W(g_1,\tilde{g}_1)(\rho)}$$

are continuous on $[0, 1]$. Consequently, since $\tilde{g}_1 \notin L^2(\mathbb{B}^5_1)$, we must have $c_2 = 0$. Furthermore, $|\tilde{g}_1'(\rho)| \simeq (1-\rho)^{-1}$ near $\rho = 1$ and thus, for v to be in $H^1(\mathbb{B}^5_1)$, we must necessarily have

$$\int_0^1 \frac{g_1(s)G(s)}{(1-s^2)W(g_1,\tilde{g}_1)(s)}\, ds = 0.$$

This is however impossible due to the strict positivity of the integrand on $(0, 1)$. □

Lemma 2.13. *The essential spectrum of $\boldsymbol{L}$, denoted by $\sigma_e(\boldsymbol{L})$ satisfies*

$$\sigma_e(\boldsymbol{L}) \subset \left\{z \in \mathbb{C} : \operatorname{Re}(z) \leq -\tfrac{1}{2}\right\}.$$

In addition, any spectral point λ with $\operatorname{Re}\lambda > -\frac{1}{2}$ is an eigenvalue of finite algebraic multiplicity and there exist only finitely many such spectral points.

Proof. The first claim is an immediate consequence of the growth bound

$$\|\boldsymbol{S}_0(\tau)\boldsymbol{f}\|_{H^2\times H^1(\mathbb{B}^5_1)} \lesssim e^{-\frac{1}{2}\tau}\|\boldsymbol{f}\|_{H^2\times H^1(\mathbb{B}^5_1)}.$$

The second follows from invoking Theorem B.1 in [Glogić 2022]. □

A calculation which is very similar to the one done in the proof of Lemma 2.6 in [Donninger and Rao 2020] yields our next result.

Lemma 2.14. *Let $\eta > -\frac{1}{2}$. Then there exist constants $C_\eta, K_\eta > 0$ such that*

$$\|\boldsymbol{R}_{\boldsymbol{L}}(\lambda)\boldsymbol{f}\|_{H^2\times H^1(\mathbb{B}^5_1)} \leq C_\eta\|\boldsymbol{f}\|_{H^2\times H^1(\mathbb{B}^5_1)}$$

for all $\lambda \in \mathbb{C}$ satisfying $|\lambda| \geq K_\eta$ and $\operatorname{Re}\lambda \geq \eta$ and all $\boldsymbol{f} \in H^2 \times H^1(\mathbb{B}^5_1)$.

Let now $\boldsymbol{Q}$ be the spectral projection associated to the finite set of eigenvalues

$$\left\{\lambda \in \sigma(\boldsymbol{L}) : -\tfrac{1}{2} < \operatorname{Re}\lambda < 0\right\}.$$

Moreover, we remark that when viewed as a densely defined, closed operator on $H^{3/2} \times H^{1/2}(\mathbb{B}^5_1)$, the calculations in the proof of Lemma 2.12 show that in this case we have that

$$\sigma(\boldsymbol{L}) \subset \{z \in \mathbb{C} : \operatorname{Re}(z) \leq 0\} \cup \{1\} \quad \text{and} \quad \sigma_p(\boldsymbol{L}) \subset \{z \in \mathbb{C} : \operatorname{Re}(z) < 0\} \cup \{1\}$$

and 1 remains a simple eigenvalue. We denote by $\boldsymbol{P}$ the corresponding bounded projection $\boldsymbol{P} : H^{3/2} \times H^{1/2}(\mathbb{B}^5_1) \to \langle \boldsymbol{g} \rangle$.

Lemma 2.15. *Let $\eta > -\frac{1}{2}$. Then there exists a constant $C_\eta > 0$ such that*

$$\|\boldsymbol{S}(\tau)(\boldsymbol{I}-\boldsymbol{Q})(\boldsymbol{I}-\boldsymbol{P})\boldsymbol{f}\|_{H^2\times H^1(\mathbb{B}^5_1)} \leq C_\eta e^{\eta\tau}\|\boldsymbol{f}\|_{H^2\times H^1(\mathbb{B}^5_1)}$$

for all $\boldsymbol{f} \in H^2 \times H^1(\mathbb{B}^5_1)$ and all $\tau \geq 0$.

Proof. This lemma follows immediately from Lemma 2.14 and the Gearhart–Prüss–Greiner theorem, see, e.g., [Engel and Nagel 2000, p. 302, Theorem 1.11], since

$$\sigma(\boldsymbol{L}_{\ker \boldsymbol{P} \cap \ker \boldsymbol{Q}}) \subset \{\lambda \in \mathbb{C} : \operatorname{Re}(\lambda) \leq -\tfrac{1}{2}\}. \qquad \square$$

As the growth estimate from Lemma 2.15 does not help us at the critical regularity, at which analogous considerations would yield an exponentially growing bound for the semigroup, a more sophisticated analysis is needed. So, let $\boldsymbol{f} \in C^\infty \times C^\infty(\overline{\mathbb{B}^5_1})$ and set

$$\tilde{\boldsymbol{f}} := (\boldsymbol{I}-\boldsymbol{Q})(\boldsymbol{I}-\boldsymbol{P})\boldsymbol{f} \in D(\boldsymbol{L}).$$

Then, for any $\eta > -\frac{1}{2}$, Laplace inversion yields

$$\boldsymbol{S}(\tau)\tilde{\boldsymbol{f}} = \lim_{N\to\infty} \frac{1}{2\pi i} \int_{\eta-iN}^{\eta+iN} e^{\lambda\tau} \boldsymbol{R}_{\boldsymbol{L}}(\lambda)\tilde{\boldsymbol{f}}\, d\lambda; \tag{2-10}$$

see [Engel and Nagel 2000, p. 234, Corollary 5.15]. Hence, to obtain enough qualitative information on the semigroup $\boldsymbol{S}(\boldsymbol{I}-\boldsymbol{Q})(\boldsymbol{I}-\boldsymbol{P})$, we need to investigate $\boldsymbol{R}_{\boldsymbol{L}}(\lambda)$. To that end we remark that $\boldsymbol{u} = \boldsymbol{R}_{\boldsymbol{L}}(\lambda)\tilde{\boldsymbol{f}}$ implies $(\lambda - \boldsymbol{L})\boldsymbol{u} = \tilde{\boldsymbol{f}}$, which in turn implies

$$(\rho^2-1)u_1''(\rho) + \Big(2(\lambda+2)\rho - \frac{4}{\rho}\Big)u_1'(\rho) + (\lambda+2)(\lambda+1)u_1 - \frac{16}{(1+\rho^2)^2}u_1(\rho) = F_\lambda(\rho), \tag{2-11}$$

where $F_\lambda(\rho) = f_2(\rho) + (\lambda+2) f_1(\rho) + \rho f_1'(\rho)$. Accordingly, our next step will be a detailed analysis of (2-11).

3. ODE analysis

3.1. *Preliminary transformations.* To put many of the tediously involved functions into a manageable fashion, we introduce function of symbol type as follows. Let $I \subset \mathbb{R}$, $\rho_0 \in \mathbb{R}\setminus I$, and $\alpha \in \mathbb{R}$. We say that a smooth function $f : I \to \mathbb{C}$ is of symbol type and write $f(\rho) = \mathcal{O}((\rho_0-\rho)^\alpha)$ if

$$|\partial_\rho^n f(\rho)| \lesssim_n |\rho_0 - \rho|^{\alpha-n}$$

for all $\rho \in I$ and all $n \in \mathbb{N}_0$. Similarly, for $g : \mathbb{C} \to \mathbb{R}$ we write $g(\lambda) = \mathcal{O}(\langle\omega\rangle^\alpha)$ if

$$|\partial_\omega^n f(\omega)| \lesssim_n \langle\omega\rangle^{\alpha-n},$$

where $\langle\omega\rangle$ denotes the Japanese bracket $\sqrt{1+|\cdot|^2}$. Analogously,

$$h(\rho,\lambda) = \mathcal{O}((\rho-\rho_0)^\alpha\langle\omega\rangle^\beta) \quad \text{if } |\partial_\rho^n\partial_\omega^k h(\rho,\lambda)| \lesssim_{n,k} |\rho_0-\rho|^{\alpha-n}\langle\omega\rangle^{\beta-k}$$

for all $\ell, k \in \mathbb{N}$ and $\alpha, \beta \in \mathbb{R}$. Motivated by the spectral equation (2-11), we study the ODE

$$(1-\rho^2)u''(\rho) + \Big(\frac{4}{\rho} - 2(\lambda+2)\rho\Big)u'(\rho) - (\lambda+1)(\lambda+2)u(\rho) - V(\rho)u(\rho) = -F_\lambda(\rho) \tag{3-1}$$

for $\operatorname{Re}\lambda \in [-\frac{3}{4}, \frac{3}{4}]$, $\lambda \neq 0$, and an arbitrary even potential $V \in C^\infty([0,1])$. To get rid of the first-order term we set

$$v(\rho) = \rho^2(1-\rho^2)^{\frac{\lambda}{2}}u(\rho),$$

which, for $F_\lambda = 0$, turns (3-1) into

$$v''(\rho) + \frac{-2+\rho^2(2+2\lambda-\lambda^2)}{\rho^2(1-\rho^2)^2} v(\rho) = \frac{V(\rho)}{1-\rho^2} v(\rho). \tag{3-2}$$

One of the main tools to study (3-2) is the diffeomorphism $\varphi : (0,1) \to (0,\infty)$, given by

$$\varphi(\rho) := \tfrac{1}{2}(\log(1+\rho) - \log(1-\rho)).$$

Observe that

$$\varphi'(\rho) = \frac{1}{1-\rho^2}$$

and that the associated Liouville–Green potential Q_φ, defined by

$$Q_\varphi(\rho) := -\frac{3}{4}\frac{\varphi''(\rho)^2}{\varphi'(\rho)^2} + \frac{1}{2}\frac{\varphi'''(\rho)}{\varphi'(\rho)},$$

is given by

$$Q_\varphi(\rho) = \frac{1}{(1-\rho^2)^2}.$$

Hence, we rewrite (3-2) as

$$\begin{aligned} v''(\rho) + \frac{-1+2\lambda-\lambda^2}{(1-\rho^2)^2} v(\rho) - \frac{2}{\varphi(\rho)^2(1-\rho^2)^2} v(\rho) + Q_\varphi(\rho) v(\rho) \\ = \frac{V(\rho)}{1-\rho^2} + \left(\frac{2}{\rho^2(1-\rho^2)^2} - \frac{2}{(1-\rho^2)^2} - \frac{2}{\varphi(\rho)^2(1-\rho^2)^2} \right) v(\rho). \end{aligned} \tag{3-3}$$

Next, we perform a Liouville–Green transformation, that is, we set

$$w(\varphi(\rho)) := \varphi'(\rho)^{\frac{1}{2}} v(\rho),$$

which transforms

$$v''(\rho) + \frac{-1+2\lambda-\lambda^2}{(1-\rho^2)^2} v(\rho) - \frac{2}{\varphi(\rho)^2(1-\rho^2)^2} v(\rho) + Q_\varphi(\rho) v(\rho) = 0 \tag{3-4}$$

into

$$w''(\varphi(\rho)) - (1-\lambda)^2 w(\varphi(\rho)) - \frac{2}{\varphi(\rho)^2} w(\varphi(\rho)) = 0. \tag{3-5}$$

This is now a Bessel equation with a fundamental system given by

$$\cos(a(\lambda)\varphi(\rho)) - \frac{\sin(a(\lambda)\varphi(\rho))}{a(\lambda)\varphi(\rho)} \quad \text{and} \quad \sin(a(\lambda)\varphi(\rho)) + \frac{\cos(a(\lambda)\varphi(\rho))}{a(\lambda)\varphi(\rho)},$$

with $a(\lambda) = i(1-\lambda)$. From this we infer that

$$\begin{aligned} b_1(\rho,\lambda) &= \sqrt{1-\rho^2}\left(\frac{\sin(a(\lambda)\varphi(\rho))}{a(\lambda)\varphi(\rho)} - \cos(a(\lambda)\varphi(\rho)) \right), \\ b_2(\rho,\lambda) &= \sqrt{1-\rho^2}\left(\sin(a(\lambda)\varphi(\rho)) + \frac{\cos(a(\lambda)\varphi(\rho))}{a(\lambda)\varphi(\rho)} \right) \end{aligned}$$

is a fundamental system of (3-4).

3.2. *Construction of fundamental systems.*

Lemma 3.1. *There exist $r > 0$ and $\rho_0 \in [0, 1)$ such that for $\rho \in [\rho_\lambda, 1)$, where $\rho_\lambda := \min\{r/|1-\lambda|, \rho_0\}$, and $\lambda \neq 0$ with $-\frac{3}{4} \leq \operatorname{Re}\lambda \leq \frac{3}{4}$ the equation*

$$v''(\rho) + \frac{-2+\rho^2(2+2\lambda-\lambda^2)}{\rho^2(1-\rho^2)^2} v(\rho) = 0 \tag{3-6}$$

has a fundamental system of the form

$$h_1(\rho,\lambda) = \sqrt{1-\rho^2}\left(\frac{1-\rho}{1+\rho}\right)^{\frac{1-\lambda}{2}} [1 + (1-\rho)\mathcal{O}(\langle\omega\rangle^{-1}) + \mathcal{O}(\rho^{-1}(1-\rho)^2\langle\omega\rangle^{-1})],$$
$$h_2(\rho,\lambda) = \sqrt{1-\rho^2}\left(\frac{1+\rho}{1-\rho}\right)^{\frac{1-\lambda}{2}} [1 + (1-\rho)\mathcal{O}(\langle\omega\rangle^{-1}) + \mathcal{O}(\rho^{-1}(1-\rho)^2\langle\omega\rangle^{-1})],$$

where $\omega = \operatorname{Im}\lambda$.

Proof. We rewrite (3-6) as

$$v''(\rho) + \frac{2\lambda-\lambda^2}{(1-\rho^2)^2} v(\rho) = \frac{2}{\rho^2(1-\rho^2)} v(\rho)$$

and note that the equation

$$w''(\rho) + \frac{2\lambda-\lambda^2}{(1-\rho^2)^2} w(\rho) = 0$$

has a fundamental system of solutions given by

$$w_1(\rho,\lambda) = \sqrt{1-\rho^2}\left(\frac{1-\rho}{1+\rho}\right)^{\frac{1-\lambda}{2}},$$
$$w_2(\rho,\lambda) = \sqrt{1-\rho^2}\left(\frac{1+\rho}{1-\rho}\right)^{\frac{1-\lambda}{2}}.$$

The Wronskian of these solutions is given by

$$W(w_1(\cdot,\lambda), w_2(\cdot,\lambda)) = 2(1-\lambda).$$

Therefore, Duhamel's formula suggests the Volterra equation

$$w(\rho,\lambda) = w_1(\rho,\lambda) + \int_\rho^{\rho_1} \frac{w_1(\rho,\lambda)w_2(s,\lambda)}{(1-\lambda)s^2(1-s^2)} w(s,\lambda)\,ds - \int_\rho^{\rho_1} \frac{w_2(\rho,\lambda)w_1(s,\lambda)}{(1-\lambda)s^2(1-s^2)} w(s,\lambda)\,ds \tag{3-7}$$

for $\rho > 1/|1-\lambda|$. As $w_1(\cdot,\lambda)$ does not vanish on $(0, 1)$, we can divide (3-7) by w_1. For the new variable $\tilde{w} = w/w_1$, we then obtain the equation

$$\begin{aligned}\tilde{w}(\rho,\lambda) &= 1 + \int_\rho^{\rho_1} \frac{w_1(s,\lambda)w_2(s,\lambda)}{(1-\lambda)s^2(1-s^2)} \tilde{w}(s,\lambda)\,ds - \int_\rho^{\rho_1} \frac{w_2(\rho,\lambda)w_1^2(s,\lambda)}{w_1(\rho,\lambda)(1-\lambda)s^2(1-s^2)} \tilde{w}(s,\lambda)\,ds \\ &= 1 + \int_\rho^{\rho_1} \frac{1-\left(\frac{1+\rho}{1-\rho}\frac{1-s}{1+s}\right)^{1-\lambda}}{(1-\lambda)s^2} \tilde{w}(s,\lambda)\,ds \\ &=: 1 + \int_\rho^{\rho_1} K(\rho,s,\lambda)\tilde{w}(s,\lambda)\,ds.\end{aligned}$$

From $1/|1-\lambda| \le \rho \le s$, we conclude that

$$\int_{\frac{1}{|1-\lambda|}}^{\rho_1} \sup_{\rho\in[\frac{1}{|1-\lambda|},s]} \left| \frac{1-\left(\frac{1+\rho}{1-\rho}\frac{1-s}{1+s}\right)^{1-\lambda}}{(1-\lambda)s^2} \right| ds \lesssim \int_{\frac{1}{|1-\lambda|}}^{\rho_1} \frac{1}{s^2|1-\lambda|}\, ds \lesssim 1$$

independent of $\rho_1 \in [1/|1-\lambda|, 1]$. Consequently, we are able to set $\rho_1 = 1$ and use Lemma B.1 in [Donninger et al. 2011] to infer the existence of a unique solution $\tilde{w}$ to (3-7) of the form

$$\tilde{w}(\rho,\lambda) = 1 + O(\rho^{-1}\langle\omega\rangle^{-1}).$$

Strictly speaking, the O-term also depends on $\operatorname{Re}\lambda$ but as this dependence is of no relevance to us, we suppress it in our notation. Having established the existence of $h_1 = w\tilde{w}_1$, one proceeds in the same manner as in [Donninger and Wallauch 2023, Lemma 4.1] to conclude that h_1 is indeed of the desired form and that a second solution h_2 to (3-6) of the claimed form can be constructed. □

Without loss of generality we can assume that neither $h_1(\cdot,\lambda)$ nor $h_2(\cdot,\lambda)$ vanishes anywhere on $[\rho_\lambda, 1)$, as we can enlarge r and ρ_0 if necessary. We now set $\hat{\rho}_\lambda := \min\{\frac{1}{2}(\rho_0+1), 2r/|a(\lambda)|\} \in (\rho_\lambda, 1)$ and with this we turn to the full equation (3-2).

Lemma 3.2. *Equation (3-2) has a fundamental system of the form*

$$\begin{aligned}\psi_1(\rho,\lambda) &= b_1(\rho,\lambda)[1+\mathcal{O}(\rho^2\langle\omega\rangle^0)],\\ \psi_2(\rho,\lambda) &= b_2(\rho,\lambda)[1+\mathcal{O}(\rho^2\langle\omega\rangle^0)] + \mathcal{O}(\rho\langle\omega\rangle^{-2})\end{aligned}$$

for all $\rho \in (0, \hat{\rho}_\lambda]$ and all $\lambda \neq 0$ with $-\frac{3}{4} \le \operatorname{Re}\lambda \le \frac{3}{4}$.

Proof. We start by noting that for $\rho \in (0, \hat{\rho}_\lambda]$, the functions b_1 and b_2 satisfy

$$\begin{aligned}b_1(\rho,\lambda) &= \mathcal{O}(\rho^2\langle\omega\rangle^2),\\ b_2(\rho,\lambda) &= \mathcal{O}(\rho^{-1}\langle\omega\rangle^{-1})\end{aligned} \tag{3-8}$$

and that their Wronskian is given by

$$W(b_1(\cdot,\lambda), b_2(\cdot,\lambda)) = i(1-\lambda).$$

Therefore, we have to solve the fixed-point problem

$$b(\rho,\lambda) = b_1(\rho,\lambda) - \int_0^\rho \frac{b_1(\rho,\lambda)b_2(s,\lambda)\widetilde{V}(s)}{i(1-\lambda)(1-s^2)} b(s,\lambda)\, ds + \int_0^\rho \frac{b_2(\rho,\lambda)b_1(s,\lambda)\widetilde{V}(s)}{i(1-\lambda)(1-s^2)} b(s,\lambda)\, ds, \tag{3-9}$$

with

$$\widetilde{V}(\rho) = V(\rho) + \frac{2}{\rho^2(1-\rho^2)} - \frac{2}{1-\rho^2} - \frac{2}{\varphi(\rho)^2(1-\rho^2)}.$$

Observe that $\widetilde{V} \in C^\infty([0,1))$. We claim that $b_1(\cdot,\lambda)$ does not vanish on $(0,1)$. This follows from the fact that the zeros of $J_{3/2}$ are all real (see [Olver 1974, p. 244, Theorem 6.2]) and any zero of $b_1(\cdot,\lambda)$ is a zero of $J_{3/2}$. Since $a(\lambda)$ always has nonzero imaginary part for $\operatorname{Re}\lambda \in \left[-\frac{3}{4}, \frac{3}{4}\right]$, we see that the argument of the Bessel function is always nonreal. Hence, we can divide (3-9) by b_1. Upon setting $\tilde{b} = b/b_1$ we obtain the Volterra equation

$$\begin{aligned}\tilde{b}(\rho,\lambda) &= 1 - \int_0^\rho \frac{b_1(s,\lambda)b_2(s,\lambda)\tilde{V}(s)}{i(1-\lambda)(1-s^2)}\tilde{b}(s,\lambda)\,ds + \int_0^\rho \frac{b_2(\rho,\lambda)b_1^2(s,\lambda)\tilde{V}(s)}{i(1-\lambda)(1-s^2)b_1(\rho,\lambda)}\tilde{b}(s,\lambda)\,ds \\ &=: 1 + \int_0^\rho K(\rho,s,\lambda)\tilde{b}(s,\lambda)\,ds.\end{aligned}$$

Using the estimates (3-8), we see that

$$|b_2(\rho,\lambda)b_1(\rho,\lambda)| \lesssim \rho\langle\omega\rangle, \qquad \left|\frac{b_2(\rho,\lambda)}{b_1(\rho,\lambda)}b_1(s,\lambda)^2\right| \lesssim s\langle\omega\rangle$$

for all $0 \le s \le \rho \le \hat{\rho}_\lambda$ and so

$$\int_0^{\hat{\rho}_\lambda} \sup_{\rho\in[s,\hat{\rho}_\lambda]} |K(\rho,s,\lambda)|\,ds \lesssim \langle\omega\rangle^{-2}.$$

Hence, a Volterra iteration yields the existence of a unique solution $\tilde{b}(\rho,\lambda)$ to (3-9) that satisfies $\tilde{b}(\rho,\lambda) = 1 + O(\rho^2\langle\omega\rangle^0)$. Furthermore, since all the involved functions behave like symbols,[2] Appendix B of [Donninger et al. 2011] shows that $\tilde{b}(\rho,\lambda) = 1 + \mathcal{O}(\rho^2\langle\omega\rangle^0)$ and thus, we obtain the existence of a solution to (3-2) of the form

$$\psi_1(\rho,\lambda) = b_1(\rho,\lambda)[1 + \mathcal{O}(\rho^2\langle\omega\rangle^0)].$$

To construct the second solution stated in the lemma, we pick a $\rho_1 \in (0,1]$ such that ψ_1 does not vanish for $\rho \le \min\{\rho_1, \hat{\rho}_\lambda\} =: \tilde{\rho}_\lambda$ for any $\lambda \in \mathbb{C}\setminus\{0\}$ with $-\frac{3}{4} \le \operatorname{Re}\lambda \le \frac{3}{4}$. Moreover, as $\tilde{b}_1(\rho,\lambda) := b_1(\rho,\lambda)\int_\rho^{\tilde{\rho}_\lambda} b_1(s,\lambda)^{-2}\,ds$ is also a solution of (3-4), there exist constants $c_1(\lambda), c_2(\lambda)$ such that

$$b_2(\rho,\lambda) = c_1(\lambda)b_1(\rho,\lambda) + c_2(\lambda)\tilde{b}_1(\rho,\lambda).$$

Explicitly, these constants are given by

$$c_1(\lambda) = \frac{W(b_2(\cdot,\lambda),\tilde{b}_1(\cdot,\lambda))}{W(b_1(\cdot,\lambda),\tilde{b}_1(\cdot,\lambda))}, \qquad c_2(\lambda) = -\frac{W(b_2(\cdot,\lambda),b_1(\cdot,\lambda))}{W(b_1(\cdot,\lambda),\tilde{b}_1(\cdot,\lambda))}.$$

Using that $W(b_2(\cdot,\lambda),b_1(\cdot,\lambda)) = -\frac{2}{\pi}$ and $W(b_1(\cdot,\lambda),\tilde{b}_1(\cdot,\lambda)) = -1$, we infer that $c_2 = -\frac{2}{\pi}$ and $c_1(\lambda) = -W(b_2(\cdot,\lambda),\tilde{b}_1(\cdot,\lambda))$. Next, evaluating $W(b_2(\cdot,\lambda),\tilde{b}_1(\cdot,\lambda))$ at $\tilde{\rho}_\lambda$ yields

$$W(b_2(\cdot,\lambda),\tilde{b}_1(\cdot,\lambda)) = -b_2(\tilde{\rho}_\lambda,\lambda)b_1(\tilde{\rho}_\lambda,\lambda)^{-1} = \mathcal{O}(\langle\omega\rangle^0).$$

Keeping these facts in mind, we now turn our attention to constructing ψ_2. For this, we remark that a second solution of (3-2) is given by

$$\tilde{\psi}_1(\rho,\lambda) = \psi_1(\rho,\lambda)\int_\rho^{\tilde{\rho}_\lambda} \psi_1(s,\lambda)^{-2}\,ds.$$

[2] Strictly speaking, $\hat{\rho}_\lambda$ not differentiable at $\frac{1}{2}(\rho_0+1)$. However, this is inessential and can easily be remedied by using a smoothed out version of $\hat{\rho}_\lambda$.

Considering this, we calculate

$$\begin{aligned}\psi_2(\rho,\lambda) &:= c_1(\lambda)\psi_1(\rho,\lambda)+c_2\psi_1(\rho,\lambda)\int_\rho^{\tilde\rho_\lambda}\psi_1(s,\lambda)^{-2}\,ds\\ &= c_1(\lambda)\psi_1(\rho,\lambda)+c_2\psi_1(\rho,\lambda)\int_\rho^{\tilde\rho_\lambda}b_1(s,\lambda)^{-2}\,ds+c_2\psi_1(\rho,\lambda)\int_\rho^{\tilde\rho_\lambda}[\psi_1(s,\lambda)^{-2}-b_1(s,\lambda)^{-2}]\,ds\\ &= b_2(\rho,\lambda)[1+\mathcal{O}(\rho^2\langle\omega\rangle^0)]+c_2\psi_1(\rho,\lambda)\int_\rho^{\tilde\rho_\lambda}\frac{\mathcal{O}(s^2\langle\omega\rangle^0)}{b_1(s,\lambda)^2[1+\mathcal{O}(s^2\langle\omega\rangle^0)]^2}\,ds.\end{aligned}$$

Since $b_1(\rho,\lambda)^{-2}=\mathcal{O}(\rho^{-4}\langle\omega\rangle^{-4})$, we obtain

$$\int_\rho^{\tilde\rho_\lambda}\frac{\mathcal{O}(s^2\langle\omega\rangle^0)}{b_1(s,\lambda)^2[1+\mathcal{O}(s^2\langle\omega\rangle^0)]^2}\,ds=\mathcal{O}(\rho^0\langle\omega\rangle^{-3})+\mathcal{O}(\rho^{-1}\langle\omega\rangle^{-4})=\mathcal{O}(\rho^{-1}\langle\omega\rangle^{-4}),$$

where the last inequality follows as we only consider values of ρ that are smaller than $\tilde\rho_\lambda$. Finally, for $|\lambda|$ large enough, we see that $\tilde\rho_\lambda=\hat\rho_\lambda$ and so we can safely assume that $\tilde\rho_\lambda=\hat\rho_\lambda$. ☐

One final Volterra iteration based on h_1 and h_2 yields the following result.

Lemma 3.3. *There exists a fundamental system for (3-2) of the form*

$$\begin{aligned}\psi_3(\rho,\lambda)&=h_1(\rho,\lambda)[1+(1-\rho)\mathcal{O}(\langle\omega\rangle^{-1})+\mathcal{O}(\rho^0(1-\rho)^2\langle\omega\rangle^{-1})],\\ \psi_4(\rho,\lambda)&=h_2(\rho,\lambda)[1+(1-\rho)\mathcal{O}(\langle\omega\rangle^{-1})+\mathcal{O}(\rho^0(1-\rho)^2\langle\omega\rangle^{-1})]\end{aligned}$$

for all $\rho\in[\rho_\lambda,1)$ *and all* $\lambda\neq 0$ *with* $-\frac{3}{4}\le\operatorname{Re}\lambda\le\frac{3}{4}$.

Lemma 3.4. *For* $\rho\in[\rho_\lambda,\hat\rho_\lambda]$ *the solutions* ψ_3 *and* ψ_4 *have the representations*

$$\begin{aligned}\psi_3(\rho,\lambda)&=c_{1,3}(\lambda)\psi_1(\rho,\lambda)+c_{2,3}(\lambda)\psi_2(\rho,\lambda),\\ \psi_4(\rho,\lambda)&=c_{1,4}(\lambda)\psi_1(\rho,\lambda)+c_{2,4}(\lambda)\psi_2(\rho,\lambda),\end{aligned}$$

with

$$\begin{aligned}c_{1,3}(\lambda)&=\frac{W(h_1(\cdot,\lambda),b_2(\cdot,\lambda))(\rho_\lambda)}{i(1-\lambda)}+\mathcal{O}(\langle\omega\rangle^{-1}),\\ c_{2,3}(\lambda)&=-\frac{W(h_1(\cdot,\lambda),b_1(\cdot,\lambda))(\rho_\lambda)}{i(1-\lambda)}+\mathcal{O}(\langle\omega\rangle^{-1}),\\ c_{1,4}(\lambda)&=\frac{W(h_2(\cdot,\lambda),b_2(\cdot,\lambda))(\rho_\lambda)}{i(1-\lambda)}+\mathcal{O}(\langle\omega\rangle^{-1}),\\ c_{2,4}(\lambda)&=-\frac{W(h_2(\cdot,\lambda),b_1(\cdot,\lambda))(\rho_\lambda)}{i(1-\lambda)}+\mathcal{O}(\langle\omega\rangle^{-1}).\end{aligned}$$

Proof. We know the explicit representations

$$c_{1,3}(\lambda)=\frac{W(\psi_3(\cdot,\lambda),\psi_2(\cdot,\lambda))}{W(\psi_1(\cdot,\lambda),\psi_2(\cdot,\lambda))},\quad c_{2,3}(\lambda)=-\frac{W(\psi_3(\cdot,\lambda),\psi_1(\cdot,\lambda))}{W(\psi_1(\cdot,\lambda),\psi_2(\cdot,\lambda))}$$

and computing the connection coefficients reduces to calculating these Wronskians. Evaluating the Wronskian $W(\psi_1(\cdot,\lambda),\psi_2(\cdot,\lambda))$ at $\rho=0$ yields

$$W(\psi_1(\cdot,\lambda),\psi_2(\cdot,\lambda)) = i(1-\lambda),$$

while an evaluation at ρ_λ yields

$$\begin{aligned} &W(\psi_3(\cdot,\lambda),\psi_2(\cdot,\lambda)) \\ &\qquad = W(h_1(\cdot,\lambda),b_2(\cdot,\lambda))(\rho_\lambda)[1+\mathcal{O}(\langle\omega\rangle^{-1})] + h_1(\rho_\lambda,\lambda)b_2(\rho_\lambda,\lambda)[\mathcal{O}(\langle\omega\rangle^{0})+\mathcal{O}(\langle\omega\rangle^{-1})] \\ &\qquad = W(h_1(\cdot,\lambda),b_2(\cdot,\lambda))(\rho_\lambda) + \mathcal{O}(\langle\omega\rangle^{0}). \end{aligned}$$

Consequently,

$$c_{1,3}(\lambda) = \frac{W(h_1(\cdot,\lambda),b_2(\cdot,\lambda))(\rho_\lambda)}{i(1-\lambda)} + \mathcal{O}(\langle\omega\rangle^{-1})$$

and the remaining coefficients are computed analogously. □

We can patch together the solutions of the “free equation” in the same fashion. To this end, let ψ_{f_1} and ψ_{f_2} be the solutions obtained from Lemma 3.2 in the case $V=0$ and, for notational convenience, we set $\psi_{\mathrm{f}_3} := h_1$ and $\psi_{\mathrm{f}_4} := h_2$.

Lemma 3.5. *For $\rho\in[\rho_\lambda,\hat{\rho}_\lambda]$, the solutions ψ_{f_3} and ψ_{f_4} have the representations*

$$\begin{aligned} \psi_{\mathrm{f}_3}(\rho,\lambda) &= c_{\mathrm{f}_{1,3}}(\lambda)\psi_{\mathrm{f}_1}(\rho,\lambda) + c_{\mathrm{f}_{2,3}}(\lambda)\psi_{\mathrm{f}_2}(\rho,\lambda), \\ \psi_{\mathrm{f}_4}(\rho,\lambda) &= c_{\mathrm{f}_{1,4}}(\lambda)\psi_{\mathrm{f}_1}(\rho,\lambda) + c_{\mathrm{f}_{2,4}}(\lambda)\psi_{\mathrm{f}_2}(\rho,\lambda), \end{aligned}$$

with

$$\begin{aligned} c_{\mathrm{f}_{1,3}}(\lambda) &= = \frac{W(h_1(\cdot,\lambda),b_2(\cdot,\lambda))(\rho_\lambda)}{i(1-\lambda)} + \mathcal{O}(\langle\omega\rangle^{-1}), \\ c_{\mathrm{f}_{2,3}}(\lambda) &= -\frac{W(h_1(\cdot,\lambda),b_1(\cdot,\lambda))(\rho_\lambda)}{i(1-\lambda)} + \mathcal{O}(\langle\omega\rangle^{-1}), \\ c_{\mathrm{f}_{1,4}}(\lambda) &= \frac{W(h_2(\cdot,\lambda),b_2(\cdot,\lambda))(\rho_\lambda)}{i(1-\lambda)} + \mathcal{O}(\langle\omega\rangle^{-1}), \\ c_{\mathrm{f}_{2,4}}(\lambda) &= -\frac{W(h_2(\cdot,\lambda),b_1(\cdot,\lambda))(\rho_\lambda)}{i(1-\lambda)} + \mathcal{O}(\langle\omega\rangle^{-1}). \end{aligned}$$

Next, let $\chi:[0,1]\times\{z\in\mathbb{C}:-\frac{1}{2}\le \operatorname{Re} z\le \frac{3}{4}\}\to[0,1]$, $\chi_\lambda(\rho):=\chi(\rho,\lambda)$, be a smooth cut-off function that satisfies $\chi_\lambda(\rho)=1$ for $\rho\in[0,\rho_\lambda]$, $\chi_\lambda(\rho)=0$ for $\rho\in[\hat{\rho}_\lambda,1]$, and $|\partial_\rho^k\partial_\omega^\ell\chi_\lambda(\rho)|\le C_{k,\ell}\langle\omega\rangle^{k-\ell}$ for $k,\ell\in\mathbb{N}_0$. We then define two solutions of (3-2) as

$$\begin{aligned} v_1(\rho,\lambda) &:= \chi_\lambda(\rho)[c_{1,4}(\lambda)\psi_1(\rho,\lambda)+c_{2,4}(\lambda)\psi_2(\rho,\lambda)] + (1-\chi_\lambda(\rho))\psi_4(\rho,\lambda), \\ v_2(\rho,\lambda) &:= \chi_\lambda(\rho)[c_{1,3}(\lambda)\psi_1(\rho,\lambda)+c_{2,3}(\lambda)\psi_2(\rho,\lambda)] + (1-\chi_\lambda(\rho))\psi_3(\rho,\lambda), \end{aligned}$$

and note that an evaluation at $\rho=1$ yields

$$W(v_1(\cdot,\lambda),v_2(\cdot,\lambda)) = W(\psi_4(\cdot,\lambda),\psi_3(\cdot,\lambda)) = 2(1-\lambda).$$

With this remark we return to the full equation (3-1).

4. Resolvent construction

We now return to our specific potential $V(\rho)=-16/(1+\rho^2)^2$. Setting $u_j(\rho,\lambda)=\rho^{-2}(1-\rho^2)^{-\lambda/2}v_j(\rho,\lambda)$ for $j\in\{1,2\}$ yields two solutions to (3-1) with $F_\lambda=0$.

Lemma 4.1. *The solutions u_1 and u_2 are of the form*

$$\begin{aligned} u_1(\rho,\lambda) &= \rho^{-2}(1-\rho^2)^{-\frac{\lambda}{2}}h_2(\rho,\lambda)[1+(1-\rho)\mathcal{O}(\langle\omega\rangle^{-1})+\mathcal{O}(\rho^0(1-\rho)^2\langle\omega\rangle^{-1})] \\ &= \rho^{-2}(1+\rho)^{1-\lambda}[1+(1-\rho)\mathcal{O}(\langle\omega\rangle^{-1})+\mathcal{O}(\rho^{-1}(1-\rho)^2\langle\omega\rangle^{-1})], \\ u_2(\rho,\lambda) &= \rho^{-2}(1-\rho^2)^{-\frac{\lambda}{2}}h_1(\rho,\lambda)[1+(1-\rho)\mathcal{O}(\langle\omega\rangle^{-1})+\mathcal{O}(\rho^0(1-\rho)^2\langle\omega\rangle^{-1})] \\ &= \rho^{-2}(1-\rho)^{1-\lambda}[1+(1-\rho)\mathcal{O}(\langle\omega\rangle^{-1})+\mathcal{O}(\rho^{-1}(1-\rho)^2\langle\omega\rangle^{-1})] \end{aligned}$$

for all $\rho\geq\hat\rho_\lambda=\min\{\frac{1}{2}(\rho_0+1),2r/|a(\lambda)|\}$ and all $\lambda\in\mathbb{C}\setminus\{0\}$ with $-\frac{3}{4}\leq\operatorname{Re}\lambda\leq\frac{3}{4}$. Moreover, we have $u_1(\cdot,\lambda)\in C^\infty((0,1])$ for all such values of λ.

Proof. The explicit forms of u_1 and u_2 follow immediately from our ODE construction. To see that u_1 is smooth away from $\rho=0$, we first remark that clearly $u_1(\cdot,\lambda)\in C^\infty(0,1)$. Furthermore, the Frobenius indices of (2-9) at $\rho=1$ are $\{0,1-\lambda\}$. Hence, there exist coefficients $c_1(\lambda)$ and $c_2(\lambda)$ such that $c_1(\lambda)u_1(\cdot,\lambda)+c_2(\lambda)u_2(\cdot,\lambda)$ is nontrivial and smooth on $(0,1]$. However, since $\operatorname{Re}\lambda\geq-\frac{1}{2}$, we clearly have that $u_1(\cdot,\lambda)\in C^2((0,1])$, while $u_2(\cdot,\lambda)\notin C^2((0,1])$ and so $c_2(\lambda)=0$. □

4.1. *Considerations on the point spectrum.*

We aim to establish $H^{3/2}\times H^{1/2}$-type Strichartz estimates by proving bounds of the form

$$\begin{aligned} \|[e^{\frac{\tau}{2}}\boldsymbol{S}(\tau)(\boldsymbol{I}-\boldsymbol{P})\boldsymbol{f}]_1\|_{L^p_\tau L^q(\mathbb{B}^5_1)} &\lesssim \|\boldsymbol{f}\|_{H^2\times H^1(\mathbb{B}^5_1)}, \\ \|[e^{-\frac{\tau}{2}}\boldsymbol{S}(\tau)(\boldsymbol{I}-\boldsymbol{P})\boldsymbol{f}]_1\|_{L^p_\tau L^q(\mathbb{B}^5_1)} &\lesssim \|\boldsymbol{f}\|_{H^1\times L^2(\mathbb{B}^5_1)} \end{aligned}$$

and interpolating between these two. An obvious obstruction to the first estimate is the existence of eigenvalues λ of $\boldsymbol{L}$ with $-\frac{1}{2}\leq\operatorname{Re}\lambda<0$. Unfortunately, we cannot rigorously rule out such eigenvalues, even though they are not expected to exist (see [Bizoń 2005; Donninger and Aichelburg 2010] for numerical evidence). To circumvent this, we recall Lemmas 2.12 and 2.13, which tell us that

$$\sigma_u(\boldsymbol{L}):=\sigma(\boldsymbol{L})\cap\{\lambda\in\mathbb{C}:\operatorname{Re}\lambda>-\tfrac{1}{2}\}=\{\lambda_1,\dots,\lambda_n,1\},$$

with $n\in\mathbb{N}_0$ and where $\operatorname{Re}\lambda_i<0$ for $i=1,\dots,n$. Moreover, each element of $\sigma_u(\boldsymbol{L})$ is an eigenvalue of finite algebraic multiplicity. This alone of course still does not settle our spectral problem, but the following general property of finite-rank operators will enable us to deal with these eigenvalues.

Lemma 4.2. *Let H be a Hilbert space. Then, for any densely defined operator $T:D(T)\subset H\to H$ with finite rank, there exists a dense subset $X\subset H$ with $X\subset D(T)$ and a bounded linear operator $\widehat{T}:H\to H$ such that*

$$T|_X=\widehat{T}|_X.$$

Proof. If T is bounded, we choose $X=H$ and $\widetilde{T}$ the unique extension of T to all of H. Consequently, we may assume that T is not bounded. We prove the result by induction on the rank of T. Let $\dim\operatorname{rg}T=1$.

Then we have $Tx = \varphi(x)x_0$ for a suitable $x_0 \in H$ and a linear functional $\varphi : D(T) \subset H \to \mathbb{C}$ that is not bounded. Thus, we find a sequence $(\tilde{y}_n)_{n\in\mathbb{N}} \subset D(T)$ with $\|\tilde{y}_n\|_H \le 1$ and $|\varphi(\tilde{y}_n)| \ge n$ for all $n \in \mathbb{N}$. We set $y_n := \tilde{y}_n/\varphi(\tilde{y}_n)$. Then we have $\|y_n\|_H \le 1/n$ and $\varphi(y_n) = 1$. Now let $x \in D(T)$ be arbitrary and set $x_n := x - \varphi(x)y_n$. Then we have $\varphi(x_n) = \varphi(x) - \varphi(x)\varphi(y_n) = 0$ and thus, $x_n \in \ker T$. Furthermore, $\|x - x_n\|_H = |\varphi(x)|\,\|y_n\| \le |\varphi(x)|/n$ and thus, $x_n \to x$ as $n \to \infty$. Consequently, $\ker T$ is dense in H.

Assume now that the claim has been established for operators with rank n. Let T be a densely defined operator with rank $n+1$ that is not bounded and let $\{e_1, \dots, e_{n+1}\}$ be an orthonormal basis of $\operatorname{rg} T$. Further, denote by $P_j : \operatorname{rg} T \to \operatorname{span}\{e_j\}$ the associated orthonormal projections. Then $T = \sum_{j=1}^{n+1} P_j T$ and thus, at least one of the operators $T_j := P_j T$ cannot be bounded. After relabeling we can assume that T_1 is not bounded. Since T_1 has rank 1, we see by the above that $\ker T_1$ is dense in H. Consider now the restriction of T to the kernel of T_1. Then this is either a bounded operator and we are done or it is not bounded and has rank n, and we can use the induction hypothesis to conclude as well. □

Recall now that $\boldsymbol{Q} : H^2 \times H^1(\mathbb{B}_1^5) \to H^2 \times H^1(\mathbb{B}_1^5)$ denotes the Riesz projection associated to the set $\sigma_u(\boldsymbol{L}) \setminus \{1\}$. We can of course also view $\boldsymbol{Q}$ as a potentially unbounded operator on $H^{3/2} \times H^{1/2}(\mathbb{B}_1^5)$ with domain $D(\boldsymbol{Q}) = H^2 \times H^1(\mathbb{B}_1^5)$. Then Lemma 4.2 applies and we obtain a dense subset $X \subset H^{3/2} \times H^{1/2}(\mathbb{B}_1^5)$ together with a bounded linear operator $\hat{\boldsymbol{Q}}$ on $H^{3/2} \times H^{1/2}(\mathbb{B}_1^5)$ which agrees with $\boldsymbol{Q}$ on X. Concretely, we have the following lemma.

Lemma 4.3. *There exists a dense subset X in $H^{3/2} \times H^{1/2}(\mathbb{B}_1^5)$ with $X \subset H^2 \times H^1(\mathbb{B}_1^5)$ and a bounded linear operator $\hat{\boldsymbol{Q}}$: $H^{3/2} \times H^{1/2}(\mathbb{B}_1^5) \to H^{3/2} \times H^{1/2}(\mathbb{B}_1^5)$ such that*

$$\hat{\boldsymbol{Q}}|_X = \boldsymbol{Q}|_X.$$

Unfortunately, this does not help us with eigenvalues that lie on the line $\operatorname{Re} z = -\frac{1}{2}$. We can however pick a $1 \gg \delta > 0$ such that $\sigma_u \cap \{z \in C : -\frac{1}{2}(1-\delta) \le \operatorname{Re} z \le -\frac{1}{2}(1-3\delta)\} = \varnothing$ and aim to prove estimates of the form

$$\|[e^{(\frac{1}{2}-\delta)\tau}\boldsymbol{S}(\tau)(\boldsymbol{I}-\boldsymbol{Q})(\boldsymbol{I}-\boldsymbol{P})\boldsymbol{f}]_1\|_{L^p_\tau L^q(\mathbb{B}_1^5)} \lesssim \|(\boldsymbol{I}-\boldsymbol{Q})\boldsymbol{f}\|_{W^{2,2/(1+2\delta)}\times W^{1,2/(1+2\delta)}(\mathbb{B}_1^5)},$$
$$\|[e^{-(\frac{1}{2}-\delta)\tau}\boldsymbol{S}(\tau)(\boldsymbol{I}-\boldsymbol{Q})(\boldsymbol{I}-\boldsymbol{P})\boldsymbol{f}]_1\|_{L^p_\tau L^q(\mathbb{B}_1^5)} \lesssim \|(\boldsymbol{I}-\boldsymbol{Q})\boldsymbol{f}\|_{W^{1,2/(1-2\delta)}\times L^{2/(1-2\delta)}(\mathbb{B}_1^5)}.$$

For this the following classification will be vital for us.

Lemma 4.4. *Any point $\lambda \in \mathbb{C} \setminus \{0\}$ with $-\frac{1}{2} \le \operatorname{Re}\lambda \le \frac{3}{4}$ is an eigenvalue of $\boldsymbol{L}$ if and only if $c_{2,4}(\lambda) = 0$.*

Proof. Assume that $c_{2,4}(\lambda) = 0$. Then, as the Frobenius indices of (2-9) are $\{0, -3\}$ at $\rho = 0$ and $\{0, 1-\lambda\}$ at $\rho = 1$, one readily checks $u_1(\cdot, \lambda) \in H^2(\mathbb{B}_1^5)$ and from (2-4) we see that the associated vector-valued function $\boldsymbol{u}_1$ satisfies $\boldsymbol{u}_1(\cdot, \lambda) \in H^2 \times H^1(\mathbb{B}_1^5)$. Thus, $\lambda \in \sigma_p(\boldsymbol{L})$. Conversely, let $\lambda \in \mathbb{C} \setminus \{0\}$ be an eigenvalue of $\boldsymbol{L}$ with $\operatorname{Re}\lambda \in [-\frac{1}{2}, \frac{3}{4}]$ and let $\boldsymbol{f}$ be an eigenfunction. Now, the first component of any eigenfunction has to be a linear combination of u_1 and u_2. Since $u_2 \notin H^2((\frac{1}{2}, 1))$ and $u_1 \in H^2((\frac{1}{2}, 1))$, f_1 has to be a multiple of u_1. However, given that $|\cdot|^{-2}\psi_1(\cdot, \lambda) \in L^2(\mathbb{B}_{\rho_\lambda}^5)$ while $\rho^{-2}\psi_2(\rho, \lambda) \simeq \rho^{-3}$ as $\rho \to 0$, we see that $u_1(\cdot, \lambda) \in H^2(\mathbb{B}_1^5)$ is only possible if $c_{2,4}(\lambda) = 0$. □

4.2. ***The reduced resolvent.*** Lemma 4.4 enables us to construct a third solution to (3-1) with $F_\lambda = 0$, whenever $\operatorname{Re}(\lambda) \in \left[-\frac{1}{2}, \frac{3}{4}\right]$ and $\lambda \notin \sigma(\boldsymbol{L}) \cup \{0\}$, by setting

$$u_0(\rho, \lambda) := u_2(\rho, \lambda) - \frac{c_{2,3}(\lambda)}{c_{2,4}(\lambda)} u_1(\rho, \lambda).$$

Note that

$$W(u_1(\cdot\,, \lambda), u_0(\cdot\,, \lambda))(\rho) = 2(1-\lambda)\rho^{-4}(1-\rho^2)^{-\lambda}.$$

So, to solve (3-1) when $\operatorname{Re}\lambda \in \left(0, \frac{3}{4}\right]$, we make the ansatz

$$\begin{aligned} u(\rho, \lambda) &= -u_0(\rho, \lambda) \int_\rho^1 \frac{u_1(s, \lambda)}{W(u_1(\cdot\,, \lambda), u_0(\cdot\,, \lambda))(s)} \frac{F_\lambda(s)}{1-s^2}\, ds \\ &\qquad - u_1(\rho, \lambda) \int_0^\rho \frac{u_0(s, \lambda)}{W(u_1(\cdot\,, \lambda), u_0(\cdot\,, \lambda))(s)} \frac{F_\lambda(s)}{1-s^2}\, ds \\ &= -\frac{u_0(\rho, \lambda)}{2(1-\lambda)} \int_\rho^1 \frac{s^4 u_1(s, \lambda) F_\lambda(s)}{(1-s^2)^{1-\lambda}}\, ds - \frac{u_1(\rho, \lambda)}{2(1-\lambda)} \int_0^\rho \frac{s^4 u_0(s, \lambda) F_\lambda(s)}{(1-s^2)^{1-\lambda}}\, ds \end{aligned} \tag{4-1}$$

and one can check that $u(\cdot\,, \lambda) \in H^2(\mathbb{B}_1^5)$. However, for $\operatorname{Re}\lambda \le 0$ we need some more considerations stemming from the simple fact that for any F_λ with $F_\lambda(1) \neq 0$, one has

$$\int_\rho^1 \frac{s^4 u_1(s, \lambda) F_\lambda(s)}{(1-s^2)^{1-\lambda}}\, ds = \infty$$

for all $\rho \in (0, 1)$. To remedy this, we slightly modify our ansatz. Thus, for $F_\lambda \in C^\infty(\overline{\mathbb{B}_1^5})$, let $\rho, \rho_1 \in (0, 1)$, $c \in \mathbb{C}$, and set

$$u(\rho, \lambda) = c u_0(\rho, \lambda) - \frac{u_0(\rho, \lambda)}{2(1-\lambda)} \int_\rho^{\rho_1} \frac{s^4 u_1(s, \lambda) F_\lambda(s)}{(1-s^2)^{1-\lambda}}\, ds - \frac{u_1(\rho, \lambda)}{2(1-\lambda)} \int_0^\rho \frac{s^4 u_0(s, \lambda) F_\lambda(s)}{(1-s^2)^{1-\lambda}}\, ds,$$

as well as

$$U_j(\rho, \lambda) = \frac{1}{2(1-\lambda)} \int_0^\rho \frac{s^4 u_j(s, \lambda)}{(1-s^2)^{1-\lambda}}\, ds$$

for $j = 0, 1, 2$. Integrating by parts yields

$$\begin{aligned} u(\rho, \lambda) &= u_0(\rho, \lambda) \left[c + U_1(\rho, \lambda) F_\lambda(\rho) - U_1(\rho_1, \lambda) F_\lambda(\rho_1) + \int_\rho^{\rho_1} U_1(s, \lambda) F_\lambda'(s)\, ds \right] \\ &\qquad - u_1(\rho, \lambda) U_0(\rho, \lambda) F_\lambda(\rho) + u_1(\rho, \lambda) \int_0^\rho U_0(s, \lambda) F_\lambda'(s)\, ds, \end{aligned}$$

which, upon setting $c = \tilde{c} + U_1(\rho_1, \lambda) F_\lambda(\rho_1)$ with $\tilde{c} \in \mathbb{C}$, reduces to

$$\begin{aligned} &u_0(\rho, \lambda) \left[\tilde{c} + U_1(\rho, \lambda) F_\lambda(\rho) + \int_\rho^{\rho_1} U_1(s, \lambda) F_\lambda'(s)\, ds \right] \\ &\qquad - u_1(\rho, \lambda) U_0(\rho, \lambda) F_\lambda(\rho) + u_1(\rho, \lambda) \int_0^\rho U_0(s, \lambda) F_\lambda'(s)\, ds \end{aligned}$$

and also allows us to safely take the limit $\rho_1 \to 1$.

Lemma 4.5. *Let* $f \in C^\infty(\overline{\mathbb{B}^5_1})$ *and* $\operatorname{Re}\lambda \in \left[-\frac{1}{2}, \frac{3}{4}\right]$ *with* $\lambda \notin \sigma_p(\boldsymbol{L}) \cup \{0\}$. *Then the unique solution* $u(\,\cdot\,,\lambda) \in H^2(\mathbb{B}^5_1)$ *of the equation*

$$(1-\rho^2)u''(\rho) + \Big(\frac{4}{\rho} - 2(\lambda+2)\rho\Big)u'(\rho) - (\lambda+1)(\lambda+2)u(\rho) + \frac{16}{(1+\rho^2)^2}u(\rho) = -f(\rho),$$

with $\rho \in (0,1)$ *is given by*

$$\mathcal{R}(f)(\rho,\lambda) := u_0(\rho,\lambda)[b_\lambda(f) + U_1(\rho,\lambda)f(\rho)] + u_0(\rho,\lambda)\int_\rho^1 U_1(s,\lambda)f'(s)\,ds \\ - u_1(\rho,\lambda)U_0(\rho,\lambda)f(\rho) + u_1(\rho,\lambda)\int_0^\rho U_0(s,\lambda)f'(s)\,ds,$$

with

$$b_\lambda(f) := -\frac{f(1)}{2\lambda(1-\lambda)}\int_0^1 \partial_s[s^4 u_1(s,\lambda)(1+s)^{-1+\lambda}](1-s)^\lambda\,ds.$$

Proof. As the Frobenius indices of (2-9) are given by $\{0,-3\}$ at $\rho=0$ and $\{0,1-\lambda\}$ at $\rho=1$, we see that $u_0(\,\cdot\,,\lambda) \in H^2(\mathbb{B}^5_{1/2}) \cap C^2(\mathbb{B}^5_1)$, while $u_1(\,\cdot\,,\lambda)$ is smooth for $0<\rho\le 1$. Therefore, one easily verifies that $\mathcal{R}(f)(\,\cdot\,,\lambda) \in H^2(\mathbb{B}^5_{1/2})$. To study the behavior of $\mathcal{R}(f)(\rho,\lambda)$ at $\rho=1$, we rewrite the boundary terms as

$$u_2(\rho,\lambda)[b_\lambda(f) + U_1(\rho,\lambda)f(\rho)] - u_1(\rho,\lambda)U_2(\rho,\lambda)f(\rho) - b_\lambda(f)\frac{c_{2,3}(\lambda)}{c_{2,4}(\lambda)}u_1(\rho,\lambda).$$

Observe now that $U_2(\,\cdot\,,\lambda)$ and hence $u_1(\,\cdot\,,\lambda)U_2(\,\cdot\,,\lambda)f$ is twice continuously differentiable at $\rho=1$. Further, since $u_1(\,\cdot\,,\lambda) \in C^2((0,1])$, the only remaining boundary term we have to check is

$$u_2(\rho,\lambda)[b_\lambda(f) + U_1(\rho,\lambda)f(\rho)].$$

For this we integrate by parts once more to infer that

$$U_1(\rho,\lambda) = -\frac{u_1(\rho,\lambda)\rho^4(1-\rho)^\lambda(1+\rho)^{-1+\lambda}}{2\lambda(1-\lambda)} + \frac{1}{2\lambda(1-\lambda)}\int_0^\rho \partial_s[s^4 u_1(s,\lambda)(1+s)^{-1+\lambda}](1-s)^\lambda\,ds. \tag{4-2}$$

Then,

$$b_\lambda(f) + U_1(\rho,\lambda)f(\rho) \\ = \frac{1}{2\lambda(1-\lambda)}\bigg[[f(\rho)-f(1)]\int_0^1 \partial_s[s^4 u_1(s,\lambda)(1+s)^{-1+\lambda}](1-s)^\lambda\,ds \\ - u_1(\rho,\lambda)\rho^4(1-\rho)^\lambda(1+\rho)^{-1+\lambda}f(\rho) - f(\rho)\int_\rho^1 \partial_s[s^4 u_1(s,\lambda)(1+s)^{-1+\lambda}](1-s)^\lambda\,ds\bigg] \tag{4-3}$$

and by using this form, one readily checks that $u_2(\rho,\lambda)[b_\lambda(f) + U_1(\rho,\lambda)f(\rho)]$ belongs to $C^2((0,1])$. We turn to the integral terms, which we rewrite as

$$u_2(\rho,\lambda)\int_\rho^1 U_1(s,\lambda)f'(s)\,ds + u_1(\rho,\lambda)\int_0^\rho U_2(s,\lambda)f'(s)\,ds - \frac{c_{2,3}(\lambda)}{c_{2,4}(\lambda)}u_1(\rho,\lambda)\int_0^1 U_1(s,\lambda)f'(s)\,ds.$$

In this form one can promptly verify by scaling that all these terms are elements of $C^2((0,1])$ as well. $\square$

For $\lambda > 0$ we can use the ansatz (4-1) to recast $\mathcal{R}(f)$ in a simpler form.

Lemma 4.6. *Let $f \in C^\infty(\overline{\mathbb{B}^5_1})$ and $\operatorname{Re}\lambda \in \left(0, \frac{3}{4}\right]$. Then $\mathcal{R}(f)$ satisfies*

$$\mathcal{R}(f)(\rho,\lambda) = -\frac{u_0(\rho,\lambda)}{2(1-\lambda)}\int_\rho^1 \frac{s^4 u_1(s,\lambda) f(s)}{(1-s^2)^{1-\lambda}}\,ds - \frac{u_1(\rho,\lambda)}{2(1-\lambda)}\int_0^\rho \frac{s^4 u_0(s,\lambda) f(s)}{(1-s^2)^{1-\lambda}}\,ds \tag{4-4}$$

for all $\rho \in (0,1)$.

Proof. This can either be seen by directly undoing the integrations by parts in the construction of $\mathcal{R}(f)$ or by noting that both $\mathcal{R}(f)$ and

$$\tilde{\mathcal{R}}(f)(\rho,\lambda) := -\frac{u_0(\rho,\lambda)}{2(1-\lambda)}\int_\rho^1 \frac{s^4 u_1(s,\lambda) f(s)}{(1-s^2)^{1-\lambda}}\,ds - \frac{u_1(\rho,\lambda)}{2(1-\lambda)}\int_0^\rho \frac{s^4 u_0(s,\lambda) f(s)}{(1-s^2)^{1-\lambda}}\,ds$$

solve (3-1) and are elements of $H^2(\mathbb{B}^5_1)$ (that $\tilde{\mathcal{R}}(f) \in H^2(\mathbb{B}^5_1)$ for $\operatorname{Re}\lambda > 0$ follows in the same manner as $\mathcal{R}(f) \in H^2(\mathbb{B}^5_1)$). Given that both of these functions solve (3-1) their difference has to be a linear combination of u_1 and u_0. However, as $u_j \notin H^2(\mathbb{B}^5_1)$ for $j = 1, 2$, we see that $\mathcal{R}(f)$ and $\tilde{\mathcal{R}}(f)$ have to coincide. ☐

Having constructed a suitable solution to (3-1), we remark that we can copy the same construction in the "free" case $V = 0$. This follows from the fact that $\boldsymbol{L}_0$ generates a semigroup on $H^2 \times H^1(\mathbb{B}^5_1)$ which satisfies the growth bound $\|\boldsymbol{S}_0(\tau)\|_{H^2\times H^1(\mathbb{B}^5_1)} \lesssim e^{-\tau/2}$. We denote the corresponding free solutions by $\mathcal{R}_{\mathrm{f}}(f)$. For $\boldsymbol{f} \in C^\infty \times C^\infty(\overline{\mathbb{B}^5_1})$ we set

$$\tilde{\boldsymbol{f}} = (\boldsymbol{I} - \boldsymbol{Q})(\boldsymbol{I} - \boldsymbol{P})\boldsymbol{f}$$

and use Laplace inversion to explicitly write down $[\boldsymbol{S}(\tau)(\boldsymbol{I} - \boldsymbol{Q})(\boldsymbol{I} - \boldsymbol{P})\boldsymbol{f}]_1$ for any such $\boldsymbol{f}$ and $\kappa \in \left(-\frac{1}{2}, \frac{3}{4}\right]$ as

$$[\boldsymbol{S}(\tau)\tilde{\boldsymbol{f}}]_1(\rho) = [\boldsymbol{S}_0(\tau)\tilde{\boldsymbol{f}}]_1(\rho) + \frac{1}{2\pi i}\lim_{N\to\infty}\int_{\kappa-iN}^{\kappa+iN} e^{\lambda\tau}[\boldsymbol{R}_{\boldsymbol{L}}(\lambda)\tilde{\boldsymbol{f}} - \boldsymbol{R}_{\boldsymbol{L}_0}(\lambda)\tilde{\boldsymbol{f}}]_1\,d\lambda. \tag{4-5}$$

5. A first set of Strichartz estimates

Using (4-5) we can obtain establish the desired Strichartz estimates on $\boldsymbol{S}$ by bounding the integral term

$$\lim_{N\to\infty}\int_{\kappa-iN}^{\kappa+iN} e^{\lambda\tau}[\mathcal{R}(F_\lambda)(\rho,\lambda) - \mathcal{R}_{\mathrm{f}}(F_\lambda)(\rho,\lambda)]\,d\lambda. \tag{5-1}$$

To accomplish this we will need some preliminary lemmas.

5.1. ***Preliminary and technical lemmas.*** The first set of lemmas will be concerned with oscillatory integrals.

Lemma 5.1. *Let $\alpha > 0$. Then*

$$\left|\int_{\mathbb{R}} e^{i\omega a}\mathcal{O}(\langle\omega\rangle^{-(1+\alpha)})\,d\omega\right| \lesssim \langle a\rangle^{-2}$$

for any $a \in \mathbb{R}$.

Proof. Since the integral is absolutely convergent the claim follows by two integrations by parts. ☐

Lemma 5.2. *Let* $\alpha \in (0,1)$. *Then*

$$\left| \int_{\mathbb{R}} e^{i\omega a} \mathcal{O}(\langle\omega\rangle^{-\alpha})\, d\omega \right| \lesssim |a|^{\alpha-1} \langle a\rangle^{-2}$$

holds for $a \in \mathbb{R} \setminus \{0\}$.

Proof. See Lemma 4.2 in [Donninger and Rao 2020]. □

Lemma 5.3. *We have*

$$\left| \int_{\mathbb{R}} e^{i\omega a} (1-\chi_\lambda(\rho)) \mathcal{O}(\rho^{-n} \langle\omega\rangle^{-(n+1)})\, d\omega \right| \lesssim \langle a\rangle^{-2}$$

for all $n \geq 1$, $\rho \in (0,1)$, *and* $a \in \mathbb{R}$.

Proof. This can be proven in the same manner as Lemma 4.3 in [Donninger and Rao 2020]. □

Lemma 5.4. *We have*

$$\left| \int_{\mathbb{R}} e^{i\omega a} (1-\chi_\lambda(\rho)) \mathcal{O}(\rho^{-n} \langle\omega\rangle^{-n})\, d\omega \right| \lesssim |a|^{-1} \langle a\rangle^{-2}$$

for any $n \geq 2$, $\rho \in (0,1)$, *and* $a \in \mathbb{R} \setminus \{0\}$.

Proof. This can be proven as Lemma 4.4 in [Donninger and Rao 2020]. □

Finally, by interpolating between Lemmas 5.3 and 5.4 one obtains the following result.

Lemma 5.5. *We have*

$$\left| \int_{\mathbb{R}} e^{i\omega a} (1-\chi_\lambda(\rho)) \mathcal{O}(\rho^{-n} \langle\omega\rangle^{-n})\, d\omega \right| \lesssim \rho^{-\theta} |a|^{-(1-\theta)} \langle a\rangle^{-2}$$

for any $n \geq 2$, $\rho \in (0,1)$, $\theta \in [0,1]$, *and* $a \in \mathbb{R} \setminus \{0\}$.

We will also rely on the following estimate.

Lemma 5.6. *Let* $\alpha \in (0,1)$ *and* $\beta \in [0,1)$. *Then we have the estimate*

$$\int_0^1 s^{-\beta} |a + \log(1 \pm s)|^{-\alpha}\, ds \lesssim |a|^{-\alpha}$$

for all $a \in \mathbb{R} \setminus \{0\}$.

Proof. We only prove the "−" case as the "+" case can be shown analogously. For $a < 0$ the estimate

$$|a + \log(1-s)|^{-\alpha} \leq |a|^{-\alpha}$$

holds for all $s \in [0,1]$ and so the claim follows. For $a > 0$ we change variables according to $s = 1 - e^{ax}$ and compute

$$\begin{aligned}
&\int_0^1 s^{-\beta} |a + \log(1-s)|^{-\alpha}\, ds \\
&\quad = \int_{-\infty}^0 (1-e^{ax})^{-\beta} |a + ax|^{-\alpha} a e^{ax}\, dx \\
&\quad \lesssim |a|^{1-\alpha} \int_{-\frac{1}{2}}^0 (1-e^{ax})^{-\beta} e^{ax}\, dx \\
&\qquad\qquad + |a|^{1-\alpha} (1-e^{-\frac{a}{2}})^{-\beta} e^{-\frac{a}{2}} \int_{-2}^{-\frac{1}{2}} |1+x|^{-\alpha}\, dx + |a|^{1-\alpha} \int_{-\infty}^{-2} (1-e^{ax})^{-\beta} e^{ax}\, dx.
\end{aligned}$$

The claimed estimate is now an immediate consequence of the two identities

$$\partial_x \frac{(1-e^{ax})^{1-\beta}}{a(1-\beta)} = -(1-e^{ax})^{-\beta} e^{ax}$$

and

$$(1-e^{-\frac{a}{2}})^{-\beta} e^{-\frac{a}{2}} \lesssim a^{-\beta}. \qquad \square$$

Similarly, one can show the next technical lemma.

Lemma 5.7. *Let* $\alpha \in (0,1)$ *and* $\beta \in [0,1)$. *Then the estimate*

$$\int_0^1 s^{-\beta} \big| a \pm \tfrac{1}{2}\log(1-s^2)\big|^{-\alpha}\, ds \lesssim |a|^{-\alpha}$$

holds for all $a \in \mathbb{R} \setminus \{0\}$.

Lastly, we will also require the following result on weighted norms.

Lemma 5.8. *The estimate*

$$\| |\cdot| f \|_{L^6(\mathbb{B}^5_1)} \lesssim \| f \|_{H^1(\mathbb{B}^5_1)}$$

holds for all $f \in C^\infty(\overline{\mathbb{B}^5_1})$.

Proof. This follows by a minor adaptation of the argument given in the proof of Lemma 4.8 in [Donninger and Rao 2020]. $\square$

5.2. *Kernel estimates.* We will now begin bounding the integral term (5-1). We start with the case $\kappa = \frac{1}{2} - \delta$. Therefore, we suppose $f \in C^\infty(\overline{\mathbb{B}^5_1})$ and take a look at the difference $\mathcal{R}(f) - \mathcal{R}_{\mathrm{f}}(f)$.

Lemma 5.9. *Let* $\operatorname{Re}\lambda = \frac{1}{2} - \delta$ *and* $f \in C^\infty(\overline{\mathbb{B}^5_1})$. *Then, we can decompose* $\mathcal{R}(f) - \mathcal{R}_{\mathrm{f}}(f)$ *as*

$$\mathcal{R}(f)(\rho,\lambda) - \mathcal{R}_{\mathrm{f}}(f)(\rho,\lambda) = \sum_{j=1}^{9} G_j(f)(\rho,\lambda),$$

where

$$G_1(f)(\rho,\lambda) = \rho^{-2}(1-\rho^2)^{-\frac{\lambda}{2}} b_1(\rho,\lambda) \int_\rho^1 \frac{s^2 \chi_\lambda(s)[b_1(s,\lambda)\alpha_1(\rho,s,\lambda) + b_2(s,\lambda)\alpha_2(\rho,s,\lambda)]}{2(1-\lambda)(1-s^2)^{1-\frac{\lambda}{2}}} f(s)\, ds$$
$$+ \rho^{-2}(1-\rho^2)^{-\frac{\lambda}{2}} b_1(\rho,\lambda)[1+\mathcal{O}(\rho^2)] \int_\rho^1 \frac{s^2 \chi_\lambda(s)\mathcal{O}(s\langle\omega\rangle^{-2})}{2(1-\lambda)(1-s^2)^{1-\frac{\lambda}{2}}} f(s)\, ds,$$

$$G_2(f)(\rho,\lambda) = \chi_\lambda(\rho)\rho^{-2}(1-\rho^2)^{-\frac{\lambda}{2}} b_1(\rho,\lambda) \int_\rho^1 \frac{s^2(1-\chi_\lambda(s))h_2(s,\lambda)\beta_1(\rho,s,\lambda)}{2(1-\lambda)(1-s^2)^{1-\frac{\lambda}{2}}} f(s)\, ds,$$

$$G_3(f)(\rho,\lambda) = (1-\chi_\lambda(\rho))\rho^{-2}(1-\rho^2)^{-\frac{\lambda}{2}} h_2(\rho,\lambda) \int_\rho^1 \frac{s^2 h_2(s,\lambda)\gamma_1(\rho,s,\lambda)}{2(1-\lambda)(1-s^2)^{1-\frac{\lambda}{2}}} f(s)\, ds,$$

$$G_4(f)(\rho,\lambda) = (1-\chi_\lambda(\rho))\rho^{-2}(1-\rho^2)^{-\frac{\lambda}{2}} h_1(\rho,\lambda) \int_\rho^1 \frac{s^2 h_2(s,\lambda)\gamma_2(\rho,s,\lambda)}{2(1-\lambda)(1-s^2)^{1-\frac{\lambda}{2}}} f(s)\, ds,$$

$$G_5(f)(\rho,\lambda)=\chi_\lambda(\rho)\rho^{-2}(1-\rho^2)^{-\frac{\lambda}{2}}b_1(\rho,\lambda)\int_0^\rho \frac{s^2b_1(s,\lambda)\alpha_1(s,\rho,\lambda)}{2(1-\lambda)(1-s^2)^{1-\frac{\lambda}{2}}}f(s)\,ds,$$

$$G_6(f)(\rho,\lambda)=\chi_\lambda(\rho)\rho^{-2}(1-\rho^2)^{-\frac{\lambda}{2}}b_2(\rho,\lambda)\int_0^\rho \frac{s^2b_1(s,\lambda)\alpha_2(s,\rho,\lambda)}{2(1-\lambda)(1-s^2)^{1-\frac{\lambda}{2}}}f(s)\,ds + \chi_\lambda(\rho)(1-\rho^2)^{-\frac{\lambda}{2}}\mathcal{O}(\rho^{-1}\langle\omega\rangle^{-2})\int_0^\rho \frac{s^2b_1(s,\lambda)[1+\mathcal{O}(s^2)]}{2(1-\lambda)(1-s^2)^{1-\frac{\lambda}{2}}}f(s)\,ds,$$

$$G_7(f)(\rho,\lambda)=(1-\chi_\lambda(\rho))\rho^{-2}(1-\rho^2)^{-\frac{\lambda}{2}}h_2(\rho,\lambda)\int_0^\rho \frac{s^2\chi_\lambda(s)b_1(s,\lambda)\beta_1(s,\rho,\lambda)}{2(1-\lambda)(1-s^2)^{1-\frac{\lambda}{2}}}f(s)\,ds,$$

$$G_8(f)(\rho,\lambda)=(1-\chi_\lambda(\rho))\rho^{-2}(1-\rho^2)^{-\frac{\lambda}{2}}h_2(\rho,\lambda)\int_0^\rho \frac{s^2(1-\chi_\lambda(s))h_2(s,\lambda)\gamma_1(s,\rho,\lambda)}{2(1-\lambda)(1-s^2)^{1-\frac{\lambda}{2}}}f(s)\,ds,$$

$$G_9(f)(\rho,\lambda)=(1-\chi_\lambda(\rho))\rho^{-2}(1-\rho^2)^{-\frac{\lambda}{2}}h_2(\rho,\lambda)\int_0^\rho \frac{s^2(1-\chi_\lambda(s))h_1(s,\lambda)\gamma_2(s,\rho,\lambda)}{2(1-\lambda)(1-s^2)^{1-\frac{\lambda}{2}}}f(s)\,ds,$$

where

$$\alpha_j(\rho,s,\lambda)=\mathcal{O}(\langle\omega\rangle^{-1})+\mathcal{O}(\rho^2\langle\omega\rangle^0)+\mathcal{O}(s^2\langle\omega\rangle^0)+\mathcal{O}(\rho^2s^2\langle\omega\rangle^0),$$
$$\beta_1(\rho,s,\lambda)=\mathcal{O}(\langle\omega\rangle^{-1})+\mathcal{O}(\rho^2\langle\omega\rangle^0)+\mathcal{O}(s^0(1-s)\langle\omega\rangle^{-1})+\mathcal{O}(\rho^2s^0(1-s)\langle\omega\rangle^{-1}),$$
$$\gamma_j(\rho,s,\lambda)=\mathcal{O}(\langle\omega\rangle^{-1})+\mathcal{O}(\rho^0(1-\rho)\langle\omega\rangle^{-1})+\mathcal{O}(s^0(1-s)\langle\omega\rangle^{-1})+\mathcal{O}(\rho^0(1-\rho)s^0(1-s)\langle\omega\rangle^{-2}).$$

Proof. This follows by plugging the definitions of the u_j into (4-4) and a straightforward calculation using estimates like

$$\psi_1(\rho,\lambda)-\psi_{f_1}(\rho,\lambda)=b_1(\rho,\lambda)\mathcal{O}(\rho^2\langle\omega\rangle^0)$$

and

$$c_{2,3}(\lambda)-c_{f_{2,3}}(\lambda)=\mathcal{O}(\langle\omega\rangle^{-1}).$$

□

Next, we will recast the G_j into a more controllable form.

Lemma 5.10. *The functions $G_j(f)$ satisfy*

$$G_1(f)(\rho,\lambda)=(1-\rho^2)^{-\frac{\lambda}{2}}\int_\rho^1 \frac{\chi_\lambda(s)\mathcal{O}(\rho^0s\langle\omega\rangle^{-1})}{(1-s^2)^{1-\frac{\lambda}{2}}}f(s)\,ds,$$

$$G_2(f)(\rho,\lambda)=\chi_\lambda(\rho)(1-\rho^2)^{-\frac{\lambda}{2}}\mathcal{O}(\rho^0\langle\omega\rangle^2)\times\int_\rho^1 \frac{s^2(1-\chi_\lambda(s))[1+\mathcal{O}(s^{-1}(1-s)\langle\omega\rangle^{-1})]\beta_1(\rho,s,\lambda)}{2(1-\lambda)(1-s)^{1-\lambda}}f(s)\,ds,$$

$$G_3(f)(\rho,\lambda)=(1-\chi_\lambda(\rho))\rho^{-2}(1+\rho)^{1-\lambda}[1+\mathcal{O}(\rho^{-1}(1-\rho)\langle\omega\rangle^{-1})]\times\int_\rho^1 \frac{s^2(1-\chi_\lambda(s))[1+\mathcal{O}(s^{-1}(1-s)\langle\omega\rangle^{-1})]\gamma_1(\rho,s,\lambda)}{2(1-\lambda)(1-s)^{1-\lambda}}f(s)\,ds,$$

$$G_4(f)(\rho,\lambda)=(1-\chi_\lambda(\rho))\rho^{-2}(1-\rho)^{1-\lambda}[1+\mathcal{O}(\rho^{-1}(1-\rho)\langle\omega\rangle^{-1})]\times\int_\rho^1 \frac{s^2(1-\chi_\lambda(s))[1+\mathcal{O}(s^{-1}(1-s)\langle\omega\rangle^{-1})]\gamma_2(\rho,s,\lambda)}{2(1-\lambda)(1-s)^{1-\lambda}}f(s)\,ds,$$

$$G_5(f)(\rho,\lambda)=\chi_\lambda(\rho)(1-\rho^2)^{-\frac{\lambda}{2}}\int_0^\rho \frac{\mathcal{O}(\rho^{-1}s^2\langle\omega\rangle^{-1})}{(1-s^2)^{1-\frac{\lambda}{2}}}f(s)\,ds,$$

$$G_6(f)(\rho,\lambda)=\chi_\lambda(\rho)(1-\rho^2)^{-\frac{\lambda}{2}}\int_0^\rho \frac{\mathcal{O}(\rho^{-1}s^2\langle\omega\rangle^{-1})}{(1-s^2)^{1-\frac{\lambda}{2}}}f(s)\,ds,$$

$$G_7(f)(\rho,\lambda)=(1-\chi_\lambda(\rho))\rho^{-2}(1+\rho)^{1-\lambda}[1+\mathcal{O}(\rho^{-1}(1-\rho)\langle\omega\rangle^{-1})]\times\int_0^\rho \frac{\chi_\lambda(s)\mathcal{O}(s^4\langle\omega\rangle)\beta_1(s,\rho,\lambda)}{(1-s^2)^{1-\frac{\lambda}{2}}}f(s)\,ds,$$

$$G_8(f)(\rho,\lambda)=(1-\chi_\lambda(\rho))\rho^{-2}(1+\rho)^{1-\lambda}[1+\mathcal{O}(\rho^{-1}(1-\rho)\langle\omega\rangle^{-1})]\times\int_0^\rho \frac{s^2(1-\chi_\lambda(s))[1+\mathcal{O}(s^{-1}(1-s)\langle\omega\rangle^{-1})]\gamma_1(s,\rho,\lambda)}{2(1-\lambda)(1-s)^{1-\lambda}}f(s)\,ds,$$

$$G_9(f)(\rho,\lambda)=(1-\chi_\lambda(\rho))\rho^{-2}(1+\rho)^{1-\lambda}[1+\mathcal{O}(\rho^{-1}(1-\rho)\langle\omega\rangle^{-1})]\times\int_0^\rho \frac{s^2(1-\chi_\lambda(s))[1+\mathcal{O}(s^{-1}(1-s)\langle\omega\rangle^{-1})]\gamma_2(s,\rho,\lambda)}{2(1-\lambda)(1+s)^{1-\lambda}}f(s)\,ds.$$

Motivated by this decomposition we define operators $T_j(\tau)f(\rho)$ for $j=1,\dots,9$ and $f\in C^\infty(\overline{\mathbb{B}^5_1})$ as

$$T_j(\tau)f(\rho):=\lim_{N\to\infty}\int_{-N}^N e^{i\omega\tau}G_j(f)\big(\rho,\tfrac12-\delta+i\omega\big)\,d\omega.$$

Given that the integrals above are absolutely convergent, which follows from Lemma 5.10, one concludes that $T_j(\tau)f(\rho)$ is meaningful for all $j=1,\dots,9$, $\tau\in\mathbb{R}$, $\rho\in(0,1)$, and $f\in C^\infty(\overline{\mathbb{B}^5_1})$. In addition, we have the following estimates.

Lemma 5.11. *The operators T_j satisfy the estimates*

$$\|T_j(\tau)f\|_{L^{2/(1-\delta)}_\tau(\mathbb{R}_+)L^{45/8}(\mathbb{B}^5_1)}\lesssim\|f\|_{L^{2/(1-2\delta)}(\mathbb{B}^5_1)},$$
$$\|T_j(\tau)f\|_{L^\infty_\tau(\mathbb{R}_+)L^{10/3}(\mathbb{B}^5_1)}\lesssim\|f\|_{L^{2/(1-2\delta)}(\mathbb{B}^5_1)}$$

for all $f\in C^\infty(\overline{\mathbb{B}^5_1})$ and $j=1,\dots,9$.

Proof. We start with T_1 and use

$$\chi_\lambda(s)\mathcal{O}(\rho^0 s\langle\omega\rangle^{-1})=\chi_\lambda(s)\mathcal{O}(\rho^{-\frac45}s^{\frac74}\langle\omega\rangle^{-1-\frac{1}{20}}),\tag{5-2}$$

which holds for $0<\rho\le s$. This enables us to use dominated convergence and Fubini's theorem to conclude that

$$\begin{aligned}T_1(\tau)f(\rho)&=\int_\rho^1\int_{\mathbb{R}} e^{i\omega\tau}(1-\rho^2)^{-\frac14+\frac{\delta}{2}-\frac{i\omega}{2}}\frac{\chi_{\frac12-\delta+i\omega}(s)\mathcal{O}(\rho^{-\frac45}s^{\frac74}\langle\omega\rangle^{-1-\frac{1}{20}})}{(1-s^2)^{\frac34+\frac{\delta}{2}-\frac{i\omega}{2}}}f(s)\,d\omega\,ds\\&=\int_\rho^1\int_{\mathbb{R}} e^{i\omega\tau}(1-\rho^2)^{-\frac14+\frac{\delta}{2}-\frac{i\omega}{2}}1_{(0,\rho_1)}(s)\frac{\chi_{\frac12-\delta+i\omega}(s)\mathcal{O}(\rho^{-\frac45}s^{\frac74}\langle\omega\rangle^{-1-\frac{1}{20}})}{(1-s^2)^{\frac34+\frac{\delta}{2}-\frac{i\omega}{2}}}f(s)\,d\omega\,ds\end{aligned}$$

for some $\rho_1 < 1$ and where $1_{(0,\rho_1)}$ is the characteristic function of the interval $(0,\rho_1)$. Consequently, Lemma 5.1 yields

$$|T_1(\tau)f(\rho)| \lesssim \rho^{-\frac{4}{5}} \int_0^{\rho_1} \langle \tau - \tfrac{1}{2}\log(1-\rho^2) + \tfrac{1}{2}\log(1-s^2)\rangle^{-2} s^{\frac{7}{4}} |f(s)|\, ds$$
$$\lesssim \rho^{-\frac{4}{5}} \langle\tau\rangle^{-2} \int_0^1 s^{\frac{7}{4}} |f(s)|\, ds \lesssim \langle\tau\rangle^{-2} \rho^{-\frac{4}{5}} \|f\|_{L^2(\mathbb{B}^5_1)} \| |\cdot|^{-\frac{1}{4}} \|_{L^2((0,1))}$$
$$\lesssim \langle\tau\rangle^{-2} \rho^{-\frac{4}{5}} \|f\|_{L^{2/(1-2\delta)}(\mathbb{B}^5_1)}.$$

Thus, given that

$$\| |\cdot|^{-\frac{4}{5}} \|_{L^{45/8}(\mathbb{B}^5_1)} = \left(\int_0^1 \rho^{-\frac{1}{2}}\, d\rho \right)^{\frac{8}{45}} \lesssim 1,$$

we see that

$$\|T_1(\tau) f\|_{L^{45/8}(\mathbb{B}^5_1)} \lesssim \langle\tau\rangle^{-2} \|f\|_{L^{2/(1-2\delta)}(\mathbb{B}^5_1)}$$

and so the estimates on T_1 follow.

We move on to T_2, which, after interchanging the order of integration and using an estimate similar to (5-2), takes the form

$$T_2(\tau)f(\rho) = \int_\rho^1 \int_{\mathbb{R}} e^{i\omega\tau} \chi_{\frac{1}{2}-\delta+i\omega}(\rho)(1-\rho^2)^{-\frac{1}{4}-\frac{i\omega}{2}} \mathcal{O}(\rho^{-\frac{5}{6}}\langle\omega\rangle^{\frac{1}{6}})$$
$$\times \frac{s^2(1-\chi_{\frac{1}{2}-\delta+i\omega}(s))[1+\mathcal{O}(s^{-1}(1-s)\langle\omega\rangle^{-1})]\beta_1(\rho,s,\frac{1}{2}-\delta+i\omega)}{(1-s)^{\frac{1}{2}+\delta-i\omega}} f(s)\, d\omega\, ds$$

and we can apply Lemma 5.2 to infer that

$$|T_2(\tau)f(\rho)| \lesssim \rho^{-\frac{5}{6}} \int_\rho^1 \langle\tau + \log(1-s)\rangle^{-2} \left|\tau - \tfrac{1}{2}\log(1-\rho^2) + \log(1-s)\right|^{-\frac{1}{6}} s^2 |f(s)| (1-s)^{-\frac{1}{2}-\delta}\, ds.$$

Hence, Minkowski's inequality implies that

$$\|T_2(\tau)f\|_{L^{\frac{45}{8}}(\mathbb{B}^5_1)} \lesssim \int_0^1 s^2|f(s)|(1-s)^{-\frac{1}{2}-\delta}\langle\tau+\log(1-s)\rangle^{-2}$$
$$\times \left(\int_0^1 \rho^{-\frac{11}{16}} |\tau - \tfrac{1}{2}\log(1-\rho^2) + \log(1-s)|^{-\frac{15}{16}}\, d\rho \right)^{\frac{8}{45}} ds$$

and by employing Lemma 5.7 we obtain

$$\|T_2(\tau)f\|_{L^{\frac{45}{8}}(\mathbb{B}^5_1)} \lesssim \int_0^1 s^2|f(s)|(1-s)^{-\frac{1}{2}-\delta}\langle\tau+\log(1-s)\rangle^{-2}|\tau+\log(1-s)|^{-\frac{1}{6}}\, ds.$$

By now changing variables according to $s = 1-e^{-y}$ and using Young's inequality, we compute

$$\|T_2(\tau)f\|_{L^{2/(1-\delta)}_\tau(\mathbb{R}_+)L^{45/8}(\mathbb{B}^5_1)}$$
$$\lesssim \left\| \int_0^1 s^2|f(s)|(1-s)^{-\frac{1}{2}-\delta}\langle\tau+\log(1-s)\rangle^{-2}|\tau+\log(1-s)|^{-\frac{1}{6}}\, ds \right\|_{L^{2/(1-\delta)}_\tau(\mathbb{R}_+)}$$

$$\lesssim \left\| \int_0^\infty (1-e^{-y})^2 |f(1-e^{-y})| e^{-(\frac{1}{2}-\delta)y} \langle \tau - y\rangle^{-2} |\tau - y|^{-\frac{1}{6}}\, dy \right\|_{L^{2/(1-\delta)}_\tau(\mathbb{R})}$$
$$\lesssim \|(1-e^{-y})^2 |f(1-e^{-y})| e^{-(\frac{1}{2}-\delta)y}\|_{L^{2/(1-2\delta)}_y(\mathbb{R}_+)} \|\langle\cdot\rangle^{-2}|\cdot|^{-\frac{1}{6}}\|_{L^1(\mathbb{R}_+)}$$
$$\lesssim \|f\|_{L^{2/(1-2\delta)}(\mathbb{B}^5_1)}.$$

As a consequence, the first of the desired estimates on T_2 follows. Since the second can be obtained likewise we turn to T_3. To bound T_3, we employ Lemma 5.5 to deduce that

$$|T_3(\tau) f(\rho)| \lesssim \rho^{-\frac{5}{6}} \int_\rho^1 \langle \tau + \log(1-s)\rangle^{-2} |\tau - \log(1+\rho) + \log(1-s)|^{-\frac{1}{6}} s^2 |f(s)| (1-s)^{-\frac{1}{2}-\delta}\, ds.$$

So, Minkowski's inequality combined with an application of Lemma 5.6 yields

$$\|T_3(\tau) f\|_{L^{45/8}(\mathbb{B}^5_1)} \lesssim \int_0^1 \langle \tau + \log(1-s)\rangle^{-2} |\tau + \log(1-s)|^{-\frac{1}{6}} s^2 |f(s)| (1-s)^{-\frac{1}{2}-\delta}\, ds$$

and one can bound T_3 in the same manner as T_2. Further, since the estimates on the remaining operators can be established by analogous means, we conclude this proof. □

Unfortunately, the operators T_j alone do not suffice to establish the necessary estimates on the semigroup $\boldsymbol{S}$ since one of the terms in the definition of F_λ consists of $(\lambda+2) f_1(\rho)$. To remedy this we define another set of operators $\dot{T}_j$ for $j = 1, \dots, 9$ and $f \in C^\infty(\overline{\mathbb{B}^5_1})$ by

$$\dot{T}_j(\tau) f(\rho) := \lim_{N\to\infty} \int_{-N}^{N} i\omega e^{i\omega\tau} G_j(f)\big(\rho, \tfrac{1}{2} - \delta + i\omega\big)\, d\omega.$$

This additional power of ω spoils the absolute convergence of the integral and so, to see that $\dot{T}_j(\tau) f$ is a meaningful expression, one cannot argue as simply as for the operators T_j. However, the following lemma shows that the above-defined operators $\dot{T}_j(\tau)$ exist as bounded linear operators from a dense subset of $W^{1,2/(1-2\delta)}(\mathbb{B}^5_1)$ into certain Strichartz spaces.

Lemma 5.12. *The operators $\dot{T}_j$ satisfy the estimates*

$$\|\dot{T}_j(\tau) f\|_{L^{2/(1-\delta)}_\tau(\mathbb{R}_+) L^{45/8}(\mathbb{B}^5_1)} \lesssim \|f\|_{W^{1,2/(1-2\delta)}(\mathbb{B}^5_1)},$$
$$\|\dot{T}_j(\tau) f\|_{L^\infty_\tau(\mathbb{R}_+) L^{10/3}(\mathbb{B}^5_1)} \lesssim \|f\|_{W^{1,2/(1-2\delta)}(\mathbb{B}^5_1)}$$

for all $f \in C^\infty(\overline{\mathbb{B}^5_1})$ and $j = 1, \dots, 9$.

Proof. For $\dot{T}_1$ we argue similarly as we did for T_1 and use the identity

$$\chi_\lambda(s) \mathcal{O}(\rho^0 s \langle\omega\rangle^0) = \chi_\lambda(s) \mathcal{O}(\rho^{-\frac{4}{5}} s^{\frac{3}{4}} \langle\omega\rangle^{-1-\frac{1}{20}}), \tag{5-3}$$

which holds for $0 < \rho \le s$ to infer that

$$\dot{T}_1(\tau) f(\rho) = \int_\rho^1 \int_{\mathbb{R}} e^{i\omega\tau} (1-\rho^2)^{-\frac{1}{4}+\frac{\delta}{2}-\frac{i\omega}{2}} 1_{(0,\rho_1)}(s) \frac{\chi_{\frac{1}{2}-\delta+i\omega}(s) \mathcal{O}(\rho^{-\frac{4}{5}} s^{\frac{3}{4}} \langle\omega\rangle^{-1-\frac{1}{20}})}{(1-s^2)^{\frac{3}{4}+\frac{\delta}{2}-\frac{i\omega}{2}}} f(s)\, d\omega\, ds,$$

with $\rho_1 < 1$. Thus,

$$
\begin{aligned}
|\dot{T}_1(\tau) f(\rho)| &\lesssim \rho^{-\frac{4}{5}} \int_0^{\rho_1} \big\langle \tau - \tfrac{1}{2}\log(1-\rho^2) + \tfrac{1}{2}\log(1-s^2)\big\rangle^{-2} s^{\frac{3}{4}} |f(s)|\, ds \\
&\lesssim \langle \tau\rangle^{-2} \rho^{-\frac{4}{5}} \| |\cdot|^{-1} f\|_{L^2(\mathbb{B}^5_1)} \| |\cdot|^{-\frac{1}{4}} \|_{L^2((0,1))} \\
&\lesssim \langle \tau\rangle^{-2} \rho^{-\frac{4}{5}} \| |\cdot|^{-1} f\|_{L^2(\mathbb{B}^5_1)} \lesssim \langle \tau\rangle^{-2} \rho^{-\frac{4}{5}} \|f\|_{H^1(\mathbb{B}^5_1)} \lesssim \langle \tau\rangle^{-2} \rho^{-\frac{4}{5}} \|f\|_{W^{1,2/(1-2\delta)}(\mathbb{B}^5_1)}
\end{aligned}
$$

by Lemma 2.4. Consequently, the claimed estimates on $\dot{T}_1$ follow. For $\dot{T}_2$, we perform one integration by parts and exchange powers of ρ for decay in ω to derive that

$$
\begin{aligned}
\dot{T}_2(\tau) f(\rho) = &\int_{\mathbb{R}} e^{i\omega\tau} \chi_{\frac{1}{2}-\delta+i\omega}(\rho)(1-\rho^2)^{-\frac{1}{4}+\frac{\delta}{2}-\frac{i\omega}{2}} \mathcal{O}(\rho^0 \langle\omega\rangle^2) \\
&\times \int_\rho^1 \frac{s^2(1-\chi_{\frac{1}{2}-\delta+i\omega}(s))[1+\mathcal{O}(s^{-1}(1-s)\langle\omega\rangle^{-1})]\beta_1\big(\rho, s, \frac{1}{2}-\delta+i\omega\big)}{(1-s)^{\frac{1}{2}+\delta-i\omega}} f(s)\, ds\, d\omega \\
= &\int_{\mathbb{R}} e^{i\omega\tau} \chi_{\frac{1}{2}-\delta+i\omega}(\rho)(1-\rho^2)^{-\frac{1}{4}+\frac{\delta}{2}-\frac{i\omega}{2}} \rho \mathcal{O}(\rho^{-1} \langle\omega\rangle^{-1}) \\
&\times \frac{(1-\chi_{\frac{1}{2}-\delta+i\omega}(\rho))[1+\mathcal{O}(\rho^{-1}(1-\rho)\langle\omega\rangle^{-1})]\beta_1\big(\rho, \rho, \frac{1}{2}-\delta+i\omega\big)}{(1-\rho)^{-\frac{1}{2}+\delta-i\omega}} f(\rho)\, d\omega \\
&+ \int_\rho^1 \int_{\mathbb{R}} e^{i\omega\tau} \chi_{\frac{1}{2}-\delta+i\omega}(\rho)(1-\rho^2)^{-\frac{1}{4}-\frac{i\omega}{2}} \mathcal{O}(\rho^{-\frac{9}{10}} \langle\omega\rangle^{\frac{1}{10}}) \\
&\times \frac{\partial_s\big(s^2(1-\chi_{\frac{1}{2}-\delta+i\omega}(s))[1+\mathcal{O}(s^{-1}(1-s)\langle\omega\rangle^{-1})]\beta_1\big(\rho, s, \frac{1}{2}-\delta+i\omega\big) f(s)\big)}{(1-s)^{-\frac{1}{2}+\delta-i\omega}}\, d\omega\, ds \\
=: &\, B_2(f)(\tau,\rho) + I_2(f)(\tau,\rho).
\end{aligned}
$$

An application of Lemma 5.3 then yields

$$
|B_2(f)(\tau,\rho)| \lesssim \langle\tau\rangle^{-2} \rho |f(\rho)|
$$

and the estimates for $B_2(f)$ follow from Lemma 5.8. To bound $I_2(f)$ we first remark that if the derivative hits f, one can argue as for T_2. Similarly, if the cut-off function gets differentiated, one can argue as for $\dot{T}_1$. Making use of Lemmas 5.3 and 5.5 one sees that the remaining terms $\widehat{I}_2(f)(\tau,\rho)$ satisfy

$$
\begin{aligned}
|\widehat{I}_2(f)(\tau,\rho)| &\lesssim \rho^{-\frac{11}{10}} \int_\rho^1 \langle \tau + \log(1-s)\rangle^{-2} s |f(s)| (1-s)^{\frac{1}{2}-\delta}\, ds \lesssim \langle\tau\rangle^{-2} \rho^{-\frac{11}{10}} \int_0^1 s|f(s)|\, ds, \\
|\widehat{I}_2(f)(\tau,\rho)| &\lesssim \rho^{-\frac{5}{6}} \int_\rho^1 \langle \tau + \log(1-s)\rangle^{-2} \big|\tau - \tfrac{1}{2}\log(1-\rho^2) + \log(1-s)\big|^{-\frac{1}{6}} s|f(s)|(1-s)^{\frac{1}{2}-\delta}\, ds.
\end{aligned}
$$

As a consequence, we can argue as we did for T_2 to derive the desired estimates on $\dot{T}_2$. For $\dot{T}_3$ we cannot straight away take the limit $N \to \infty$ as the integral is not absolutely convergent. However, by proceeding as before and performing a similar integration by parts, this can be remedied. More precisely, we compute that

$$
\begin{aligned}
&\dot{T}_3(\tau) f(\rho) \\
&\quad := \lim_{N\to\infty} \int_{-iN}^{iN} i\omega e^{i\omega\tau}(1-\chi_{\frac{1}{2}-\delta+i\omega}(\rho))\rho^{-2}(1+\rho)^{\frac{1}{2}+\delta-i\omega}[1+\mathcal{O}(\rho^{-1}(1-\rho)\langle\omega\rangle^{-1})] \\
&\qquad\qquad \times \int_\rho^1 \frac{s^2(1-\chi_{\frac{1}{2}-\delta+i\omega}(s))[1+\mathcal{O}(s^{-1}(1-s)\langle\omega\rangle^{-1})]}{(1-2i\omega)(1-s)^{\frac{1}{2}+\delta-i\omega}} \gamma_1\big(\rho, s, \tfrac{1}{2}-\delta+i\omega\big) f(s)\, ds\, d\omega \\
&\quad = \int_{\mathbb{R}} e^{i\omega\tau}(1-\chi_{\frac{1}{2}-\delta+i\omega}(\rho))^2(1+\rho)^{\frac{1}{2}+\delta-i\omega}[1+\mathcal{O}(\rho^{-1}(1-\rho)\langle\omega\rangle^{-1})] \\
&\qquad\qquad \times \frac{[1+\mathcal{O}(\rho^{-1}(1-\rho)\langle\omega\rangle^{-1})]\mathcal{O}(\langle\omega\rangle^{-1})\gamma_1\big(\rho,\rho,\frac{1}{2}-\delta+i\omega\big)}{(1-\rho)^{-\frac{1}{2}+\delta-i\omega}} f(\rho)\, d\omega \\
&\qquad + \int_\rho^1 \int_{\mathbb{R}} e^{i\omega\tau}\big(1-\chi_{\frac{1}{2}-\delta+i\omega}(\rho)\big)\rho^{-2}(1+\rho)^{\frac{1}{2}+\delta-i\omega}[1+\mathcal{O}(\rho^{-1}(1-\rho)\langle\omega\rangle^{-1})] \\
&\qquad \times \frac{\partial_s\big(s^2(1-\chi_{\frac{1}{2}-\delta+i\omega}(s))[1+\mathcal{O}(s^{-1}(1-s)\langle\omega\rangle^{-1})]\gamma_1\big(\rho,s,\frac{1}{2}-\delta+i\omega\big) f(s)\big)}{(1-s)^{-\frac{1}{2}+\delta-i\omega}} \mathcal{O}(\langle\omega\rangle^{-1})\, d\omega\, ds,
\end{aligned}
$$

and one can readily check that $\dot{T}_3$ can be bounded in a similar fashion as $\dot{T}_2$. Furthermore, as the remaining $\dot{T}_j$ can be bounded by analogous means, we conclude this proof. □

As a result of the last two lemmas one readily establishes the following proposition.

Proposition 5.13. *The difference of the semigroups $\boldsymbol{S}$ and $\boldsymbol{S}_0$ satisfies the Strichartz estimates*

$$
\begin{aligned}
&\|e^{-(\frac{1}{2}-\delta)\tau}[(\boldsymbol{S}(\tau)-\boldsymbol{S}_0(\tau))(\boldsymbol{I}-\boldsymbol{Q})(\boldsymbol{I}-\boldsymbol{P})\boldsymbol{f}]_1\|_{L^{2/(1-2\delta)}_\tau(\mathbb{R}_+)L^{45/8}(\mathbb{B}^5_1)} \\
&\qquad\qquad\qquad\qquad \lesssim \|(\boldsymbol{I}-\boldsymbol{Q})\boldsymbol{f}\|_{W^{1,2/(1-2\delta)}\times L^{2/(1-2\delta)}(\mathbb{B}^5_1)}, \\
&\|e^{-(\frac{1}{2}-\delta)\tau}[(\boldsymbol{S}(\tau)-\boldsymbol{S}_0(\tau))(\boldsymbol{I}-\boldsymbol{Q})(\boldsymbol{I}-\boldsymbol{P})\boldsymbol{f}]_1\|_{L^\infty_\tau(\mathbb{R}_+)L^{10/3}(\mathbb{B}^5_1)} \\
&\qquad\qquad\qquad\qquad \lesssim \|(\boldsymbol{I}-\boldsymbol{Q})\boldsymbol{f}\|_{W^{1,2/(1-2\delta)}\times L^{2/(1-2\delta)}(\mathbb{B}^5_1)}
\end{aligned}
$$

for all $\boldsymbol{f} \in C^\infty \times C^\infty(\overline{\mathbb{B}^5_1})$.

Proof. By construction the first component of $(\boldsymbol{S}(\tau)-\boldsymbol{S}_0(\tau))(\boldsymbol{I}-\boldsymbol{Q})(\boldsymbol{I}-\boldsymbol{P})\boldsymbol{f}$ with $\boldsymbol{f} \in C^\infty \times C^\infty(\overline{\mathbb{B}^5_1})$ is up to multiplicative constants given by

$$
e^{(\frac{1}{2}-\delta)\tau}\bigg(\sum_{j=1}^9 T_j(\tau)(\tilde{f}_1+\tilde{f}_2) + \sum_{j=1}^9 \dot{T}_j(\tau)\tilde{f}_1\bigg),
$$

with $\tilde{f}_j = [(\boldsymbol{I}-\boldsymbol{Q})(\boldsymbol{I}-\boldsymbol{P})\boldsymbol{f}]_j$ for $j = 1, 2$. Consequently, the claim follows immediately from Lemmas 5.11 and 5.12. □

5.3. ***Further estimates.*** To be able to control the nonlinearity, we will also need estimates on derivatives. For this we have to exchange derivatives with integrals which are not absolutely convergent. We achieve this by performing enough integrations by parts to render the oscillatory integral absolutely convergent.

This allows us to invoke Lemma 5.14 (see below) and variations thereof, which enables us to carry out said interchanging. After this we simply undo the integrations by parts.

Lemma 5.14 [Donninger and Wallauch 2023, Lemma 6.1]. *Let $f(\omega)=\mathcal{O}(\langle\omega\rangle^{-1-\alpha})$ with $\alpha>0$. Then*

$$\partial_a \int_{\mathbb{R}} e^{i\omega a} f(\omega)\,d\omega = i\int_{\mathbb{R}} \omega e^{i\omega a} f(\omega)\,d\omega$$

for $a\in\mathbb{R}\setminus\{0\}$.

Suppose now, as before, that $\boldsymbol{f}\in C^\infty\times C^\infty(\overline{\mathbb{B}^5_1})$, $\tilde{\boldsymbol{f}}=(\boldsymbol{I}-\boldsymbol{Q})(\boldsymbol{I}-\boldsymbol{P})\boldsymbol{f}$, and $\lambda=\frac{1}{2}-\delta+i\omega$. Then, by using variations of Lemma 5.14, we obtain

$$\partial_\rho[\boldsymbol{S}(\tau)\tilde{\boldsymbol{f}}]_1(\rho)=\partial_\rho[\boldsymbol{S}_0(\tau)\tilde{\boldsymbol{f}}]_1(\rho)+\frac{1}{2\pi i}\lim_{N\to\infty}\int_{\frac{1}{2}-\delta-iN}^{\frac{1}{2}-\delta+iN} e^{\lambda\tau}\partial_\rho[\mathcal{R}(F_\lambda)(\rho,\lambda)-\mathcal{R}_{\mathrm{f}}(F_\lambda)(\rho,\lambda)]\,d\lambda,$$

with

$$\partial_\rho\mathcal{R}(f)(\rho,\lambda)=-\frac{\partial_\rho u_0(\rho,\lambda)}{2(1-\lambda)}\int_\rho^1\frac{s^4u_1(s,\lambda)f(s)}{(1-s^2)^{1-\lambda}}\,ds-\frac{\partial_\rho u_1(\rho,\lambda)}{2(1-\lambda)}\int_0^\rho\frac{s^4u_0(s,\lambda)f(s)}{(1-s^2)^{1-\lambda}}\,ds.$$

Hence, our next step is to investigate the oscillatory integral above.

Lemma 5.15. *Let $\operatorname{Re}\lambda=\frac{1}{2}-\delta$ and $f\in C^\infty(\overline{\mathbb{B}^5_1})$. Then we can decompose*

$$\partial_\rho[\mathcal{R}(f)(\rho,\lambda)-\mathcal{R}_{\mathrm{f}}(f)(\rho,\lambda)]$$

as

$$\partial_\rho[\mathcal{R}(f)(\rho,\lambda)-\mathcal{R}_{\mathrm{f}}(f)(\rho,\lambda)]=\sum_{j=1}^{9}G_j'(f)(\rho,\lambda),$$

with

$$G_1'(f)(\rho,\lambda)=(1-\rho^2)^{-\frac{\lambda}{2}}\int_\rho^1\frac{\chi_\lambda(s)\mathcal{O}(\rho^{-1}s\langle\omega\rangle^{-1})}{(1-s^2)^{1-\frac{\lambda}{2}}}f(s)\,ds,$$

$$\begin{aligned}G_2'(f)(\rho,\lambda)&=[\lambda\rho(1-\rho)^{-1}+\rho^{-1}]\chi_\lambda(\rho)(1-\rho^2)^{-\frac{\lambda}{2}}\mathcal{O}(\rho^0\langle\omega\rangle^2)\\&\quad\times\int_\rho^1\frac{s^2(1-\chi_\lambda(s))[1+\mathcal{O}(s^{-1}(1-s)\langle\omega\rangle^{-1})]\beta_1(\rho,s,\lambda)}{2(1-\lambda)(1-s)^{1-\lambda}}f(s)\,ds+\widetilde{G}_2(f)(\rho,\lambda),\end{aligned}$$

$$\begin{aligned}G_3'(f)(\rho,\lambda)&=\left[\frac{1-\lambda}{1+\rho}-2\rho^{-1}\right]G_3(f)(\rho,\lambda)\\&\quad+\mathcal{O}(\rho^{-2}(1-\rho)^0\langle\omega\rangle^{-1})(1-\chi_\lambda(\rho))\rho^{-2}(1+\rho)^{1-\lambda}\\&\quad\times\int_\rho^1\frac{s^2(1-\chi_\lambda(s))[1+\mathcal{O}(s^{-1}(1-s)\langle\omega\rangle^{-1})]\gamma_1(\rho,s,\lambda)}{2(1-\lambda)(1-s)^{1-\lambda}}f(s)\,ds+\widetilde{G}_3(f)(\rho,\lambda),\end{aligned}$$

$$\begin{aligned}G_4'(f)(\rho,\lambda)&=\left[-\frac{1-\lambda}{1-\rho}-2\rho^{-1}\right]G_4(f)(\rho,\lambda)\\&\quad+\mathcal{O}(\rho^{-2}(1-\rho)^0\langle\omega\rangle^{-1})(1-\chi_\lambda(\rho))\rho^{-2}(1-\rho)^{1-\lambda}\\&\quad\times\int_\rho^1\frac{s^2(1-\chi_\lambda(s))[1+\mathcal{O}(s^{-1}(1-s)\langle\omega\rangle^{-1})]\gamma_2(\rho,s,\lambda)}{2(1-\lambda)(1-s)^{1-\lambda}}f(s)\,ds+\widetilde{G}_4(f)(\rho,\lambda),\end{aligned}$$

$$G_5'(f)(\rho,\lambda) = \chi_\lambda(\rho)(1-\rho^2)^{-\frac{\lambda}{2}} \int_0^\rho \frac{\mathcal{O}(\rho^{-2}s^2\langle\omega\rangle^{-1})}{(1-s^2)^{1-\frac{\lambda}{2}}} f(s)\,ds,$$

$$G_6'(f)(\rho,\lambda) = \chi_\lambda(\rho)(1-\rho^2)^{-\frac{\lambda}{2}} \int_0^\rho \frac{\mathcal{O}(\rho^{-2}s^2\langle\omega\rangle^{-1})}{(1-s^2)^{1-\frac{\lambda}{2}}} f(s)\,ds,$$

$$\begin{aligned} G_7'(f)(\rho,\lambda) = {}& \left[\frac{1-\lambda}{1+\rho} - 2\rho^{-1}\right] G_7(f)(\rho,\lambda) \\ &+ \mathcal{O}(\rho^{-2}(1-\rho)^0\langle\omega\rangle^{-1})(1-\chi_\lambda(\rho))\rho^{-2}(1+\rho)^{1-\lambda} \\ &\qquad \times \int_0^\rho \frac{\chi_\lambda(s)\mathcal{O}(s^4\langle\omega\rangle)\beta_1(s,\rho,\lambda)}{(1-s^2)^{1-\frac{\lambda}{2}}} f(s)\,ds + \widetilde{G}_7(f), \end{aligned}$$

$$\begin{aligned} G_8'(f)(\rho,\lambda) = {}& \left[\frac{1-\lambda}{1+\rho} - 2\rho^{-1}\right] G_8(f)(\rho,\lambda) \\ &+ \mathcal{O}(\rho^{-2}(1-\rho)^0\langle\omega\rangle^{-1})(1-\chi_\lambda(\rho))\rho^{-2}(1+\rho)^{1-\lambda} \\ &\qquad \times \int_0^\rho \frac{s^2(1-\chi_\lambda(s))[1+\mathcal{O}(s^{-1}(1-s)\langle\omega\rangle^{-1})]\gamma_1(s,\rho,\lambda)}{2(1-\lambda)(1-s)^{1-\lambda}} f(s)\,ds + \widetilde{G}_8(f), \end{aligned}$$

$$\begin{aligned} G_9'(f)(\rho,\lambda) = {}& \left[\frac{1-\lambda}{1+\rho} - 2\rho^{-1}\right] G_9(f)(\rho,\lambda) \\ &+ \mathcal{O}(\rho^{-2}(1-\rho)^0\langle\omega\rangle^{-1})(1-\chi_\lambda(\rho))\rho^{-2}(1+\rho)^{1-\lambda} \\ &\qquad \times \int_0^\rho \frac{s^2(1-\chi_\lambda(s))[1+\mathcal{O}(s^{-1}(1-s)\langle\omega\rangle^{-1})]\gamma_2(s,\rho,\lambda)}{2(1-\lambda)(1+s)^{1-\lambda}} f(s)\,ds + \widetilde{G}_9(f), \end{aligned}$$

where $\widetilde{G}_j(f)(\rho,\lambda)$ are the terms obtained from differentiating either β_1 or γ_j with respect to ρ.

Proof. This is just a straightforward computation. ☐

Proceeding as above, we define operators T_j' and $\dot{T}_j'$ for $j = 1,\dots,9$ and $f \in C^\infty(\overline{\mathbb{B}_1^5})$ as

$$T_j'(\tau)f(\rho) := \lim_{N\to\infty}\int_{-N}^{N} e^{i\omega\tau} G_j'(f)\big(\rho, \tfrac{1}{2}-\delta+i\omega\big)\,d\omega,$$
$$\dot{T}_j'(\tau)f(\rho) := \lim_{N\to\infty}\int_{-N}^{N} i\omega e^{i\omega\tau} G_j'(f)\big(\rho, \tfrac{1}{2}-\delta+i\omega\big)\,d\omega.$$

Again, these integrals are not necessarily absolutely convergent. Nevertheless, the operators can be made sense of, as is visible from the following lemma.

Lemma 5.16. *The estimates*

$$\|T_j'(\tau)f\|_{L^6_\tau(\mathbb{R}_+)L^{45/23}(\mathbb{B}_1^5)} \lesssim \|f\|_{L^{2/(1-2\delta)}(\mathbb{B}_1^5)},$$
$$\|\dot{T}_j'(\tau)f\|_{L^6_\tau(\mathbb{R}_+)L^{45/23}(\mathbb{B}_1^5)} \lesssim \|f\|_{W^{1,2/(1-2\delta)}(\mathbb{B}_1^5)}$$

hold for $j = 1,\dots,9$ and all $f \in C^\infty(\overline{\mathbb{B}_1^5})$.

Proof. We start with $j = 1$, in which case we can take the limit $N\to\infty$ for $\dot{T}_1'$ and T_1'. Combining this with

$$\chi_{\frac{1}{2}-\delta+i\omega}(s)\mathcal{O}(\rho^{-1}s\langle\omega\rangle^{-1}) = \chi_{\frac{1}{2}-\delta+i\omega}(s)\mathcal{O}(\rho^{-\frac{13}{8}}s^{\frac{25}{16}}\langle\omega\rangle^{-\frac{17}{16}}),$$

which is valid for $0<\rho\leq s$, yields

$$T_1'(\tau)f(\rho)=\int_\rho^1\int_{\mathbb{R}} e^{i\omega\tau}(1-\rho^2)^{-\frac14+\frac{\delta}{2}-\frac{i\omega}{2}}1_{(0,\rho_1)}(s)\frac{\chi_{\frac12-\delta+i\omega}(s)\mathcal{O}(\rho^{-\frac{13}{8}}s^{\frac{25}{16}}\langle\omega\rangle^{-\frac{17}{16}})}{(1-s^2)^{\frac14+\frac{\delta}{2}-\frac{i\omega}{2}}}f(s)\,d\omega\,ds.$$

Thus, by employing Lemma 5.2 we obtain

$$\begin{aligned}|T_1'(\tau)f(\rho)|&\lesssim\langle\tau\rangle^{-2}\rho^{-\frac{13}{8}}\int_0^1 s^{\frac{25}{16}}|f(s)|\,ds\lesssim\langle\tau\rangle^{-2}\rho^{-\frac{13}{8}}\left(\int_0^1 s^4|f(s)|^2\,ds\int_0^1 s^{-\frac78}\,ds\right)^{\frac12}\\&\lesssim\langle\tau\rangle^{-2}\rho^{-\frac{13}{8}}\|f\|_{L^2(\mathbb{B}^5_1)}.\end{aligned}$$

Consequently,

$$\|T_1'(\tau)f\|_{L^6_\tau(\mathbb{R}_+)L^{45/23}(\mathbb{B}^5_1)}\lesssim\|f\|_{L^{2/(1-2\delta)}(\mathbb{B}^5_1)}.$$

Moreover, to bound $\|\dot{T}_1'(\tau)f\|_{L^6_\tau(\mathbb{R}_+)L^{45/23}(\mathbb{B}^5_1)}$ one argues similarly to deduce that

$$\|\dot{T}_1'(\tau)f\|_{L^6_\tau(\mathbb{R}_+)L^{45/23}(\mathbb{B}^5_1)}\lesssim\||\cdot|^{-1}f\|_{L^2(\mathbb{B}^5_1)}\lesssim\|f\|_{W^{1,2/(1-2\delta)}(\mathbb{B}^5_1)}$$

by Lemma 2.4. For $j=2$ we can again interchange the order of integration and take the limit $N\to\infty$ in both T_2' and $\dot{T}_2'$. Note that the hardest term over which we have to obtain control in order to bound T_2' is given by

$$\begin{aligned}&\int_\rho^1\int_{\mathbb{R}} e^{i\omega\tau}\chi_{\frac12-\delta+i\omega}(\rho)(1-\rho^2)^{-\frac14+\frac{\delta}{2}-\frac{i\omega}{2}}\mathcal{O}(\rho^{-1}\langle\omega\rangle^2)\\&\qquad\times\frac{s^2(1-\chi_{\frac12-\delta+i\omega}(s))[1+\mathcal{O}(s^{-1}(1-s)\langle\omega\rangle^{-1})]\beta_1\big(\rho,s,\frac12-\delta+i\omega\big)}{(1-2i\omega)(1-s)^{\frac12+\delta-i\omega}}f(s)\,d\omega\,ds.\end{aligned}$$

By using that

$$\chi_{\frac12-\delta+i\omega}(\rho)\mathcal{O}(\rho^{-1}\langle\omega\rangle^0)=\chi_{\frac12-\delta+i\omega}(\rho)\mathcal{O}(\rho^{-\frac52}\langle\omega\rangle^{-\frac32})$$

we deduce that

$$\|T_2'(\tau)f\|_{L^{45/23}(\mathbb{B}^5_1)}\lesssim\||\cdot|^{-\frac52}\|_{L^{45/23}(\mathbb{B}^5_1)}\int_0^1\langle\tau+\log(1-s)\rangle^{-2}|f(s)|s^2(1-s)^{-\frac12-\delta}\,ds.$$

Consequently, by employing previously used arguments, one readily establishes the desired estimate on T_2'. Next, when estimating $\dot{T}_2'$ the hardest term is given by

$$\begin{aligned}&\int_{\mathbb{R}}\int_\rho^1 e^{i\omega\tau}\chi_{\frac12-\delta+i\omega}(\rho)(1-\rho^2)^{-\frac14+\frac{\delta}{2}-\frac{i\omega}{2}}\mathcal{O}(\rho^{-1}\langle\omega\rangle^3)\\&\qquad\times\frac{s^2(1-\chi_{\frac12-\delta+i\omega}(s))[1+\mathcal{O}(s^{-1}(1-s)\langle\omega\rangle^{-1})]\beta_1\big(\rho,s,\frac12-\delta+i\omega\big)}{(1-2i\omega)(1-s)^{\frac12+\delta-i\omega}}f(s)\,ds\,d\omega\\&=\int_{\mathbb{R}} e^{i\omega\tau}\chi_{\frac12-\delta+i\omega}(\rho)(1-\chi_{\frac12-\delta+i\omega}(\rho))(1-\rho^2)^{-\frac14+\frac{\delta}{2}-\frac{i\omega}{2}}\mathcal{O}(\rho\langle\omega\rangle^1)(1-\rho)^{\frac12-\delta+i\omega}\\&\qquad\times[1+\mathcal{O}(\rho^{-1}(1-\rho)\langle\omega\rangle^{-1})]\beta_1\big(\rho,\rho,\tfrac12-\delta+i\omega\big)f(\rho)\,d\omega\\&\quad+\int_{\mathbb{R}}\int_\rho^1 e^{i\omega\tau}\chi_{\frac12-\delta+i\omega}(\rho)(1-\rho^2)^{-\frac14+\frac{\delta}{2}-\frac{i\omega}{2}}\mathcal{O}(\rho^{-1}\langle\omega\rangle)(1-s)^{\frac12-\delta+i\omega}\\&\qquad\times\partial_s\big[s^2(1-\chi_{\frac12-\delta+i\omega}(s))[1+\mathcal{O}(s^{-1}(1-s)\langle\omega\rangle^{-1})]\beta_1\big(\rho,s,\tfrac12-\delta+i\omega\big)f(s)\big]\,d\omega\,ds\\&=:\dot{B}_2'(\tau)f(\rho)+\dot{I}_2'(\tau)f(\rho).\end{aligned}$$

For $\dot{B}_2'(\tau)f(\rho)$ we use Lemma 5.1 to compute that

$$|\dot{B}_2'(\tau)f(\rho)| \lesssim \langle\tau\rangle^{-2}\rho^{-\frac{1}{6}}|f(\rho)|.$$

Hence,

$$\|\dot{B}_2'(\tau)f\|_{L^6_\tau(\mathbb{R}_+)L^{45/23}(\mathbb{B}^5_1)} \lesssim \|f\|_{W^{1,2/(1-2\delta)}(\mathbb{B}^5_1)},$$

thanks to Lemma 2.4. Similarly,

$$|\dot{I}_2'(\tau)f(\rho)| \lesssim \langle\tau\rangle^{-2}\rho^{-2-\frac{1}{6}}\int_0^1 s|f(s)|+s^2|f'(s)|\,ds$$

and so,

$$\|\dot{I}_2'(\tau)f\|_{L^6_\tau(\mathbb{R}_+)L^{45/23}(\mathbb{B}^5_1)} \lesssim \|f\|_{W^{1,2/(1-2\delta)}(\mathbb{B}^5_1)}.$$

We proceed with T_3', which we estimate according to

$$|T_3'(\tau)f(\rho)| \lesssim \rho^{-2}\int_0^1 s^2|f(s)|(1-s)^{-\frac{1}{2}-\delta}\langle\tau+\log(1-s)\rangle^{-2}[1+|\tau-\log(1+\rho)+\log(1-s)|^{-\frac{1}{8}}]\,ds.$$

Further,

$$\|\rho^{-2}[1+|\tau-\log(1+\rho)+\log(1-s)|^{-\frac{1}{8}}]\|_{L^{45/23}_\rho(\mathbb{B}^5_1)} \lesssim [1+|\tau+\log(1-s)|^{-\frac{1}{8}}]$$

and the claimed estimate on T_3' follows. The bound on $\dot{T}_3'$ follows by integrating parts once and then arguing in similar fashion. Moving on, Lemma 5.2 shows that

$$\begin{aligned}|T_4'(\tau)f(\rho)| \lesssim \rho^{-2}\int_0^1 & s^2|f(s)|(1-s)^{-\frac{1}{2}-\delta}(1-\rho)^{-\frac{1}{2}-\delta}\\ &\times\langle\tau-\log(1-\rho)+\log(1-s)\rangle^{-2}[1+|\tau-\log(1-\rho)+\log(1-s)|^{-\frac{1}{10^4}}]\,ds.\end{aligned}$$

Observe now, that the estimate

$$(1-\rho)^{\frac{1}{100}}\langle\tau-\log(1-\rho)+\log(1-s)\rangle^{-2} \lesssim \langle\tau+\log(1-s)\rangle^{-2}$$

holds. Hence,

$$\begin{aligned}&\|\rho^{-2}(1-\rho)^{-\frac{1}{2}-\delta}\langle\tau-\log(1-\rho)+\log(1-s)\rangle^{-2}[1+|\tau-\log(1-\rho)+\log(1-s)|^{-\frac{1}{10^4}}]\|_{L^{45/23}_\rho(\mathbb{B}^5_1)}\\
&\quad\lesssim \langle\tau+\log(1-s)\rangle^{-2}\|(1-\rho)^{-\frac{1}{2}-\frac{1}{100}-\delta}[1+|\tau-\log(1-\rho)+\log(1-s)|^{-\frac{1}{10^4}}]\|_{L^{45/23}_\rho((0,1))}\\
&\quad\lesssim \langle\tau+\log(1-s)\rangle^{-2}\|(1-\rho)^{-\frac{51}{100}-\delta}\|_{L^{49/25}_\rho((0,1))}\|1+|\tau-\log(1-\rho)+\log(1-s)|^{-\frac{1}{10^4}}\|_{L^{2205/2}_\rho((0,1))}\\
&\quad\lesssim \langle\tau+\log(1-s)\rangle^{-2}[1+|\tau+\log(1-s)|^{-\frac{1}{10^4}}].\end{aligned}$$

Therefore,

$$\|T_4'(\tau)f\|_{L^{45/23}(\mathbb{B}^5_1)} \lesssim \int_0^1 s^2|f(s)|(1-s)^{-\frac{1}{2}-\delta}\langle\tau+\log(1-s)\rangle^{-2}[1+|\tau+\log(1-s)|^{-\frac{1}{10^4}}]\,ds$$

and the claimed estimate follows. Furthermore, estimating $\dot{T}_4'$ is achieved by first integrating by parts once in the s integral to recover decay in ω and a similar calculation. Analogously, one can bound the remaining operators, so we conclude this proof. □

Proposition 5.17. *The difference of $\boldsymbol{S}$ and $\boldsymbol{S}_0$ satisfies*

$$\|e^{-(\frac{1}{2}-\delta)\tau}[(\boldsymbol{S}(\tau)-\boldsymbol{S}_0(\tau))(\boldsymbol{I}-\boldsymbol{Q})(\boldsymbol{I}-\boldsymbol{P})\boldsymbol{f}]_1\|_{L^6_\tau(\mathbb{R}_+)W^{1,45/23}(\mathbb{B}^5_1)} \lesssim \|(\boldsymbol{I}-\boldsymbol{Q})\boldsymbol{f}\|_{W^{1,2/(1-\delta)}\times L^{2/(1-2\delta)}(\mathbb{B}^5_1)}$$

for all $\boldsymbol{f} \in C^\infty \times C^\infty(\overline{\mathbb{B}^5_1})$.

With this result, our task of establishing Strichartz estimates on the $W^{1,2/(1-\delta)} \times L^{2/(1-2\delta)}$ level has come to an end and we move on to the next set of estimates.

6. Strichartz estimates in $W^{2,2/(1+2\delta)}$

We now move on to $W^{2,2/(1+2\delta)} \times W^{1,2/(1+2\delta)}$-type Strichartz estimates, i.e., estimates of the form

$$\|[e^{(\frac{1}{2}-\delta)\tau}\boldsymbol{S}(\tau)\tilde{\boldsymbol{f}}]_1\|_{L^p_\tau(\mathbb{R}_+)L^q(\mathbb{B}^5_1)} \lesssim \|\tilde{\boldsymbol{f}}\|_{W^{2,2/(1+2\delta)}\times W^{1,2/(1+2\delta)}(\mathbb{B}^5_1)}.$$

For this we break the difference $\mathcal{R}(f)-\mathcal{R}_{\mathrm{f}}(f)$ into smaller pieces. The first part we look at is given by

$$W_1(f)(\rho,\lambda) := b_\lambda(f)u_0(\rho,\lambda) - b_{\mathrm{f}_\lambda}(f)u_{\mathrm{f}_0}(\rho,\lambda).$$

Lemma 6.1. *Let* $\operatorname{Re}\lambda = -\frac{1}{2}+\delta$. *Then we can decompose $W_1(f)(\rho,\lambda)$ as*

$$W_1(f)(\rho,\lambda) = f(1)\sum_{j=1}^{3} H_j(\rho,\lambda),$$

where

$$\begin{aligned}
H_1(\rho,\lambda) &:= \chi_\lambda(\rho)(1-\rho^2)^{-\lambda}\mathcal{O}(\rho^0\langle\omega\rangle^{-2}),\\
H_2(\rho,\lambda) &:= (1-\chi_\lambda(\rho))\rho^{-2}(1+\rho)^{1-\lambda}[1+\mathcal{O}(\rho^{-1}(1-\rho)\langle\omega\rangle^{-1})]\\
&\qquad \times[\mathcal{O}(\langle\omega\rangle^{-4}) + (1-\rho)\mathcal{O}(\langle\omega\rangle^{-5}) + \mathcal{O}(\rho^{-1}(1-\rho)^2\langle\omega\rangle^{-5})],\\
H_3(\rho,\lambda) &:= (1-\chi_\lambda(\rho))\rho^{-2}(1-\rho)^{1-\lambda}[1+\mathcal{O}(\rho^{-1}(1-\rho)\langle\omega\rangle^{-1})]\\
&\qquad \times[\mathcal{O}(\langle\omega\rangle^{-4}) + (1-\rho)\mathcal{O}(\langle\omega\rangle^{-5}) + \mathcal{O}(\rho^{-1}(1-\rho)^2\langle\omega\rangle^{-5})].
\end{aligned}$$

Proof. We start by looking at

$$\begin{aligned}
b_\lambda(f) &= \frac{f(1)}{2\lambda(1-\lambda)}\int_0^1 \partial_s[s^4u_1(s,\lambda)(1+s)^{-1+\lambda}](1-s)^\lambda\,ds\\
&= \frac{f(1)}{2\lambda(1-\lambda)}\bigg(\int_0^1 \chi_\lambda(s)\partial_s[s^4u_1(s,\lambda)(1+s)^{-1+\lambda}](1-s)^\lambda\,ds\\
&\qquad + \int_0^1 (1-\chi_\lambda(s))\partial_s[s^4u_1(s,\lambda)(1+s)^{-1+\lambda}](1-s)^\lambda\,ds\bigg)
\end{aligned}$$

and claim that $b_\lambda(f) = f(1)\mathcal{O}(\langle\omega\rangle^{-3})$. For the first of the above terms on the right side one readily computes that

$$\int_0^1 \chi_\lambda(s)\partial_s[s^4 u_1(s,\lambda)(1+s)^{-1+\lambda}](1-s)^\lambda\, ds = \int_0^1 \chi_\lambda(s)\partial_s\mathcal{O}(s\langle\omega\rangle^{-1})\, ds = \mathcal{O}(\langle\omega\rangle^{-1}).$$

The second term we split according to

$$\begin{aligned}\int_0^1 &(1-\chi_\lambda(s))\partial_s[s^2[1+(1-s)\mathcal{O}(\langle\omega\rangle^{-1}) + \mathcal{O}(s^{-1}(1-s)^2\langle\omega\rangle^{-1})]](1-s)^\lambda\, ds\\ &= \int_0^1 (1-\chi_\lambda(s))\partial_s[s^2[1+(1-s)\mathcal{O}(\langle\omega\rangle^{-1})]](1-s)^\lambda\, ds\\ &\qquad + \int_0^1 (1-\chi_\lambda(s))\partial_s[s^2[\mathcal{O}(s^{-1}(1-s)^2\langle\omega\rangle^{-1})]](1-s)^\lambda\, ds\\ &=: I_1(\lambda) + I_2(\lambda).\end{aligned}$$

Observe that $I_2(\lambda) = \mathcal{O}(\langle\omega\rangle^{-1})$, while an integration by parts yields

$$\begin{aligned}I_1(\lambda) &= \mathcal{O}(\langle\omega\rangle^{-1})\int_0^1 \partial_s\big((1-\chi_\lambda(s))\partial_s[s^2[1+(1-s)\mathcal{O}(\langle\omega\rangle^{-1})]]\big)(1-s)^{1+\lambda}\, ds\\ &= \mathcal{O}(\langle\omega\rangle^{-1}).\end{aligned}$$

Consequently, the claim follows. Similarly, one computes that

$$b_\lambda(f) - b_{\mathfrak{f}_\lambda}(f) = f(1)\mathcal{O}(\langle\omega\rangle^{-4}).$$

Therefore, one establishes the desired decomposition by plugging in the explicit forms of the solutions and a straightforward computation. □

Motivated by this decomposition we define the operators

$$\begin{aligned}S_j(\tau)f(\rho) &= \int_{\mathbb{R}} e^{i\omega\tau} f(1) H_j\big(\rho, -\tfrac{1}{2}+\delta+i\omega\big)\, d\omega,\\ \dot{S}_j(\tau)f(\rho) &= \int_{\mathbb{R}} \omega e^{i\omega\tau} f(1) H_j\big(\rho, -\tfrac{1}{2}+\delta+i\omega\big)\, d\omega\end{aligned}$$

for $j = 1,2,3$ and $f \in C^\infty(\overline{\mathbb{B}^5_1})$.

Lemma 6.2. *The estimates*

$$\begin{aligned}\|S_j(\tau)f\|_{L^{2/(1+2\delta)}_\tau(\mathbb{R}_+)L^{45}(\mathbb{B}^5_1)} &\lesssim \|f\|_{W^{1,2/(1+2\delta)}(\mathbb{B}^5_1)},\\ \|S_j(\tau)f\|_{L^\infty_\tau(\mathbb{R}_+)L^{10}(\mathbb{B}^5_1)} &\lesssim \|f\|_{W^{1,2/(1+2\delta)}(\mathbb{B}^5_1)}\end{aligned}$$

and

$$\begin{aligned}\|\dot{S}_j(\tau)f\|_{L^{2/(1+2\delta)}_\tau(\mathbb{R}_+)L^{45}(\mathbb{B}^5_1)} &\lesssim \|f\|_{W^{2,2/(1+2\delta)}(\mathbb{B}^5_1)},\\ \|\dot{S}_j(\tau)f\|_{L^\infty_\tau(\mathbb{R}_+)L^{10}(\mathbb{B}^5_1)} &\lesssim \|f\|_{W^{2,2/(1+2\delta)}(\mathbb{B}^5_1)}\end{aligned}$$

hold for $j = 1,2,3$ *and* $f \in C^\infty(\overline{\mathbb{B}^5_1})$.

Proof. From Lemma 5.1 we see that

$$|S_1(\tau)f(\rho)| \lesssim |f(1)|\langle\tau\rangle^{-2};$$

hence, the bounds on S_1 follow from the Sobolev embedding

$$W^{1,\frac{2}{1+2\delta}}((0,1)) \hookrightarrow L^\infty([0,1])$$

and Lemma 2.4. To establish the estimates on $\dot{S}_1$ we use that

$$\chi_{-\frac{1}{2}+\delta+i\omega}(\rho)\mathcal{O}(\rho^0\langle\omega\rangle^{-1}) = \chi_{-\frac{1}{2}+\delta+i\omega}(\rho)\mathcal{O}(\rho^{-\frac{1}{100}}\langle\omega\rangle^{-1-\frac{1}{100}})$$

and employ Lemma 5.1 to establish that

$$|\dot{S}_1(\tau)f(\rho)| \lesssim \rho^{-\frac{1}{100}}|f(1)|\langle\tau\rangle^{-2}.$$

Consequently, the estimates on $\dot{S}_1$ follow. The remaining bounds can be obtained in a similar fashion by making use of Lemma 5.3. □

Next, we take a closer look at

$$W_2(f)(\rho,\lambda) := u_0(\rho,\lambda)U_1(\rho,\lambda)f(\rho) - u_{f_0}(\rho,\lambda)U_{f_1}(\rho,\lambda)f(\rho) - u_1(\rho,\lambda)U_0(\rho,\lambda)f(\rho) + u_{f_1}(\rho,\lambda)U_{f_0}(\rho,\lambda)f(\rho).$$

Lemma 6.3. *Let* $\operatorname{Re}\lambda = -\frac{1}{2}+\delta$. *Then we can decompose* $W_2(f)$ *as*

$$W_2(f)(\rho,\lambda) = f(\rho)\sum_{j=4}^{8} H_j(\rho,\lambda),$$

where

$$H_4(\rho,\lambda) := (1-\rho^2)^{-\frac{\lambda}{2}}\chi_\lambda(\rho)\int_0^\rho \frac{\mathcal{O}(\rho^0 s\langle\omega\rangle^{-1})}{(1-s^2)^{1-\frac{\lambda}{2}}}\,ds,$$

$$H_5(\rho,\lambda) := (1-\chi_\lambda(\rho))\rho^{-2}(1+\rho)^{1-\lambda}[1+\mathcal{O}(\rho^{-1}(1-\rho)\langle\omega\rangle^{-1})]\int_0^\rho \frac{\chi_\lambda(s)\mathcal{O}(s\langle\omega\rangle^{-2})}{(1-s^2)^{1-\frac{\lambda}{2}}}\beta_2(\rho,s,\lambda)\,ds,$$

$$H_6(\rho,\lambda) := (1-\chi_\lambda(\rho))\rho^{-2}(1-\rho)^{1-\lambda}[1+\mathcal{O}(\rho^{-1}(1-\rho)\langle\omega\rangle^{-1})]\int_0^\rho \frac{\chi_\lambda(s)\mathcal{O}(s\langle\omega\rangle^{-2})}{(1-s^2)^{1-\frac{\lambda}{2}}}\beta_3(\rho,s,\lambda)\,ds,$$

$$H_7(\rho,\lambda) := (1-\chi_\lambda(\rho))\rho^{-2}(1-\rho)^{1-\lambda}[1+\mathcal{O}(\rho^{-1}(1-\rho)\langle\omega\rangle^{-1})] \times\int_0^\rho \frac{s^2(1-\chi_\lambda(s))[1+\mathcal{O}(s^{-1}(1-s)\langle\omega\rangle^{-1})]\gamma_3(\rho,s,\lambda)}{2(1-\lambda)(1-s)^{1-\lambda}}\,ds,$$

$$H_8(\rho,\lambda) := (1-\chi_\lambda(\rho))\rho^{-2}(1+\rho)^{1-\lambda}[1+\mathcal{O}(\rho^{-1}(1-\rho)\langle\omega\rangle^{-1})] \times\int_0^\rho \frac{s^2(1-\chi_\lambda(s))[1+\mathcal{O}(s^{-1}(1-s)\langle\omega\rangle^{-1})]\gamma_4(\rho,s,\lambda)}{2(1-\lambda)(1+s)^{1-\lambda}}\,ds,$$

with β_j *and* γ_j *as in Lemma 5.9.*

As before we define operators corresponding to the kernels H_j as

$$S_j(\tau)f(\rho) := \lim_{N\to\infty}\int_{-N}^{N} e^{i\omega\tau} f(\rho)H_j\big(\rho,-\tfrac{1}{2}+\delta+i\omega\big)\,d\omega,$$
$$\dot S_j(\tau)f(\rho) := \lim_{N\to\infty}\int_{-N}^{N} \omega e^{i\omega\tau} f(\rho)H_j\big(\rho,-\tfrac{1}{2}+\delta+i\omega\big)\,d\omega$$

for $j=4,\dots,8$ and $f\in C^\infty(\overline{\mathbb{B}^5_1})$.

Lemma 6.4. *The estimates*

$$\|S_j(\tau)f\|_{L^{2/(1+2\delta)}_\tau(\mathbb{R}_+)L^{45}(\mathbb{B}^5_1)} \lesssim \|f\|_{W^{1,2/(1+2\delta)}(\mathbb{B}^5_1)},$$
$$\|S_j(\tau)f\|_{L^{\infty}_\tau(\mathbb{R}_+)L^{10}(\mathbb{B}^5_1)} \lesssim \|f\|_{W^{1,2/(1+2\delta)}(\mathbb{B}^5_1)}$$

and

$$\|\dot S_j(\tau)f\|_{L^{2/(1+2\delta)}_\tau(\mathbb{R}_+)L^{45}(\mathbb{B}^5_1)} \lesssim \|f\|_{W^{2,2/(1+2\delta)}(\mathbb{B}^5_1)},$$
$$\|\dot S_j(\tau)f\|_{L^{\infty}_\tau(\mathbb{R}_+)L^{10}(\mathbb{B}^5_1)} \lesssim \|f\|_{W^{2,2/(1+2\delta)}(\mathbb{B}^5_1)}$$

hold for $j=4,\dots,8$ and $f\in C^\infty(\overline{\mathbb{B}^5_1})$.

Proof. For $j=4$ exchanging a small power of ρ for decay in ω and applying Lemma 5.1 yields the estimate

$$|S_4(\tau)f(\rho)| \lesssim \langle\tau\rangle^{-2}\rho^{2-\delta}|f(\rho)|.$$

So,

$$\|S_4(\tau)f(\rho)\|_{L^{45}_\rho(\mathbb{B}^5_1)} \lesssim \langle\tau\rangle^{-2}\|\rho^{2+\frac{1}{90}}f(\rho)\|_{L^{45}_\rho((0,1))}$$

provided that δ is sufficiently small. Hence, from the embedding $W^{1,2/(1+2\delta)}((0,1))\hookrightarrow L^{45}([0,1])$ we conclude that

$$\begin{aligned}\|\rho^{2+\frac{1}{90}}f(\rho)\|_{L^{45}_\rho((0,1))} &\lesssim \|\rho^{2+\frac{1}{90}}f(\rho)\|_{W^{1,2/(1+2\delta)}_\rho((0,1))}\\ &\lesssim \|f(\rho)\|_{W^{1,2/(1+2\delta)}(\mathbb{B}^5_1)} + \|\rho^{1+\frac{1}{90}}f(\rho)\|_{L^1_\rho((0,1))}\\ &\lesssim \|f(\rho)\|_{W^{1,2/(1+2\delta)}(\mathbb{B}^5_1)},\end{aligned}$$

where the last inequality follows from Theorem 1 in [Ostermann 2025]. Hence the desired estimates on S_4 follow. To estimate $\dot S^4$ one computes that

$$|\dot S_4(\tau)f(\rho)| \lesssim \langle\tau\rangle^{-2}\rho^{1-\delta}|f(\rho)|$$

and so the estimates on $\dot S_4$ follow from similar considerations. For $j=5$ we apply Lemma 5.3 to see that

$$|S_5(\tau)f(\rho)| \lesssim \langle\tau\rangle^{-2}\rho^{2}|f(\rho)|,$$
$$|\dot S_5(\tau)f(\rho)| \lesssim \langle\tau\rangle^{-1}\rho^{2}|f(\rho)|.$$

Thus, the bounds on S_5 and $\dot S_5$ follow. Moreover, as the estimates for $j=6$ can be obtained likewise, we move on to S_7. Here, an application of Lemma 5.3 shows that

$$|S_7(\tau)f(\rho)| \lesssim \rho^{-1}|f(\rho)|(1-\rho)^{\frac{3}{2}-\delta}\int_0^\rho \frac{\langle\tau-\log(1-\rho)+\log(1-s)\rangle^{-2}s^2}{(1-s)^{\frac{3}{2}-\delta}}\,ds \lesssim \langle\tau\rangle^{-2}\rho^2|f(\rho)|$$

and again the desired bounds follow. To bound $\dot{S}_7$ we integrate by parts once to see that

$$\begin{aligned}H_7(\rho,\lambda) &= (1-\chi_\lambda(\rho))\rho^{-2}(1-\rho)^{1-\lambda}[1+\mathcal{O}(\rho^{-1}(1-\rho)\langle\omega\rangle^{-1})] \\ &\qquad\times\int_0^\rho \frac{s^2(1-\chi_\lambda(s))[1+\mathcal{O}(s^{-1}(1-s)\langle\omega\rangle^{-1})]\gamma_3(\rho,s,\lambda)}{2(1-\lambda)(1-s)^{1-\lambda}}\,ds \\ &= (1-\chi_\lambda(\rho))^2(1-\rho)[1+\mathcal{O}(\rho^{-1}(1-\rho)\langle\omega\rangle^{-1})][1+\mathcal{O}(\rho^{-1}(1-\rho)\langle\omega\rangle^{-1})]\frac{\gamma_3(\rho,\rho,\lambda)}{2\lambda(1-\lambda)} \\ &\quad + (1-\chi_\lambda(\rho))\rho^{-2}(1-\rho)^{1-\lambda}[1+\mathcal{O}(\rho^{-1}(1-\rho)\langle\omega\rangle^{-1})] \\ &\qquad\times\int_0^\rho \frac{\partial_s[s^2(1-\chi_\lambda(s))[1+\mathcal{O}(s^{-1}(1-s)\langle\omega\rangle^{-1})]\gamma_3(\rho,s,\lambda)]}{2\lambda(1-\lambda)(1-s)^{-\lambda}}\,ds.\end{aligned}$$

By recasting H_7 as such and employing Lemma 5.3, the claimed bounds follow. Finally, as S_8 and $\dot{S}_8$ can be bounded likewise, we conclude this proof. □

To proceed, we take a closer look at

$$\begin{aligned}W_3(f)(\rho,\lambda) &:= u_0(\rho,\lambda)\int_\rho^1 U_1(s,\lambda)f'(s)\,ds - u_{\mathrm{f}_0}(\rho,\lambda)\int_\rho^1 U_{\mathrm{f}_1}(s,\lambda)f'(s)\,ds \\ &\quad + u_1(\rho,\lambda)\int_0^\rho U_0(s,\lambda)f'(s)\,ds - u_{\mathrm{f}_1}(\rho,\lambda)\int_0^\rho U_{\mathrm{f}_0}(s,\lambda)f'(s)\,ds.\end{aligned}$$

Lemma 6.5. *Let* $\operatorname{Re}\lambda = -\frac{1}{2}+\delta$. *Then we can decompose* $W_3(\rho,\lambda)$ *as*

$$W_3(f)(\rho,\lambda) = \sum_{j=9}^{18} H_j(f)(\rho,\lambda),$$

where

$$H_9(f)(\rho,\lambda) := (1-\rho^2)^{-\frac{\lambda}{2}}\chi_\lambda(\rho)\int_\rho^1 f'(s)\int_0^s \frac{\chi_\lambda(t)\mathcal{O}(\rho^0 t\langle\omega\rangle^{-1})}{(1-t^2)^{1-\frac{\lambda}{2}}}\,dt\,ds,$$

$$\begin{aligned}H_{10}(f)(\rho,\lambda) &:= (1-\rho^2)^{-\frac{\lambda}{2}}\chi_\lambda(\rho) \\ &\qquad\times\int_\rho^1 f'(s)\int_0^s \frac{(1-\chi_\lambda(t))\mathcal{O}(\rho^0 t^2\langle\omega\rangle)[1+\mathcal{O}(t^{-1}(1-t)\langle\omega\rangle^{-1})]}{(1-t)^{1-\lambda}}\beta_4(\rho,t,\lambda)\,dt\,ds,\end{aligned}$$

$$\begin{aligned}H_{11}(f)(\rho,\lambda) &:= (1-\chi_\lambda(\rho))\rho^{-2}(1-\rho)^{1-\lambda}[1+\mathcal{O}(\rho^{-1}(1-\rho)\langle\omega\rangle^{-1})] \\ &\qquad\times\int_\rho^1 f'(s)\int_0^s \frac{\chi_\lambda(t)\mathcal{O}(t\langle\omega\rangle^{-2})}{(1-t^2)^{1-\frac{\lambda}{2}}}\beta_5(t,\rho,\lambda)\,dt\,ds,\end{aligned}$$

$$\begin{aligned}H_{12}(f)(\rho,\lambda) &:= (1-\chi_\lambda(\rho))\rho^{-2}(1+\rho)^{1-\lambda}[1+\mathcal{O}(\rho^{-1}(1-\rho)\langle\omega\rangle^{-1})] \\ &\qquad\times\int_\rho^1 f'(s)\int_0^s \frac{\chi_\lambda(t)\mathcal{O}(t\langle\omega\rangle^{-2})}{(1-t^2)^{1-\frac{\lambda}{2}}}\beta_6(t,\rho,\lambda)\,dt\,ds,\end{aligned}$$

$$H_{13}(f)(\rho,\lambda) := (1-\chi_\lambda(\rho))\rho^{-2}(1-\rho)^{1-\lambda}[1+\mathcal{O}(\rho^{-1}(1-\rho)\langle\omega\rangle^{-1})] \times \int_\rho^1 f'(s)\int_0^s \frac{t^2(1-\chi_\lambda(t))[1+\mathcal{O}(t^{-1}(1-t)\langle\omega\rangle^{-1})]\gamma_3(\rho,t,\lambda)}{2(1-\lambda)(1-t)^{1-\lambda}}\,dt\,ds,$$

$$H_{14}(f)(\rho,\lambda) := (1-\chi_\lambda(\rho))\rho^{-2}(1+\rho)^{1-\lambda}[1+\mathcal{O}(\rho^{-1}(1-\rho)\langle\omega\rangle^{-1})] \times \int_\rho^1 f'(s)\int_0^s \frac{t^2(1-\chi_\lambda(t))[1+\mathcal{O}(t^{-1}(1-t)\langle\omega\rangle^{-1})]\gamma_4(\rho,t,\lambda)}{2(1-\lambda)(1-t)^{1-\lambda}}\,dt\,ds,$$

$$H_{15}(f)(\rho,\lambda) := (1-\rho^2)^{-\frac{\lambda}{2}}\chi_\lambda(\rho)\int_0^\rho f'(s)\int_0^s \frac{\mathcal{O}(\rho^0 t\langle\omega\rangle^{-1})}{(1-t^2)^{1-\frac{\lambda}{2}}}\,dt\,ds,$$

$$H_{16}(f)(\rho,\lambda) := (1-\chi_\lambda(\rho))\rho^{-2}(1+\rho)^{1-\lambda}[1+\mathcal{O}(\rho^{-1}(1-\rho)\langle\omega\rangle^{-1})] \times \int_0^\rho f'(s)\int_0^s \frac{\chi_\lambda(t)\mathcal{O}(t\langle\omega\rangle^{-1})}{(1-t^2)^{1-\frac{\lambda}{2}}}\beta_7(t,\rho,\lambda)\,dt\,ds,$$

$$H_{17}(f)(\rho,\lambda) := (1-\chi_\lambda(\rho))\rho^{-2}(1+\rho)^{1-\lambda}[1+\mathcal{O}(\rho^{-1}(1-\rho)\langle\omega\rangle^{-1})] \times \int_0^\rho f'(s)\int_0^s \frac{t^2(1-\chi_\lambda(t))[1+\mathcal{O}(t^{-1}(1-t)\langle\omega\rangle^{-1})]\gamma_5(\rho,t,\lambda)}{2(1-\lambda)(1-t)^{1-\lambda}}\,dt\,ds,$$

$$H_{18}(f)(\rho,\lambda) := (1-\chi_\lambda(\rho))\rho^{-2}(1+\rho)^{1-\lambda}[1+\mathcal{O}(\rho^{-1}(1-\rho)\langle\omega\rangle^{-1})] \times \int_0^\rho f'(s)\int_0^s \frac{t^2(1-\chi_\lambda(t))[1+\mathcal{O}(t^{-1}(1-t)\langle\omega\rangle^{-1})]\gamma_6(\rho,t,\lambda)}{2(1-\lambda)(1+t)^{1-\lambda}}\,dt\,ds,$$

with β_j and γ_j as in Lemma 5.9.

Continuing, we set

$$S_j(\tau)f(\rho) = \lim_{N\to\infty}\int_N^{-N} e^{i\omega\tau}H_j(f)\big(\rho,-\tfrac{1}{2}+\delta+i\omega\big)\,d\omega,$$
$$\dot S_j(\tau)f(\rho) = \lim_{N\to\infty}\int_N^{-N} \omega e^{i\omega\tau}H_j(f)\big(\rho,-\tfrac{1}{2}+\delta+i\omega\big)\,d\omega$$

for $j = 9,\dots,18$ and $f \in C^\infty(\overline{\mathbb{B}^5_1})$.

Lemma 6.6. *The estimates*

$$\|S_j(\tau)f\|_{L^{2/(1+2\delta)}_\tau(\mathbb{R}_+)L^{45}(\mathbb{B}^5_1)} \lesssim \|f\|_{W^{1,2/(1+2\delta)}(\mathbb{B}^5_1)},$$
$$\|S_j(\tau)f\|_{L^{\infty}_\tau(\mathbb{R}_+)L^{10}(\mathbb{B}^5_1)} \lesssim \|f\|_{W^{1,2/(1+2\delta)}(\mathbb{B}^5_1)}$$

and

$$\|\dot S_j(\tau)f\|_{L^{2/(1+2\delta)}_\tau(\mathbb{R}_+)L^{45}(\mathbb{B}^5_1)} \lesssim \|f\|_{W^{2,2/(1+2\delta)}(\mathbb{B}^5_1)},$$
$$\|\dot S_j(\tau)f\|_{L^{\infty}_\tau(\mathbb{R}_+)L^{10}(\mathbb{B}^5_1)} \lesssim \|f\|_{W^{2,2/(1+2\delta)}(\mathbb{B}^5_1)}$$

hold for $j = 9,\dots,18$ and $f \in C^\infty(\overline{\mathbb{B}^5_1})$.

Proof. Lemma 5.1 yields the estimate

$$|S_9(\tau)f(\rho)| \lesssim \langle\tau\rangle^{-2}\int_0^1 |f'(s)|s^{\frac{7}{4}}\,ds$$

and so, from the Cauchy–Schwarz inequality we can immediately infer the desired estimates on S_9. Analogously, one derives that

$$\|\dot{S}_9(\tau)f\|_{L^p_\tau(\mathbb{R}_+)L^q(\mathbb{B}^5_1)} \lesssim \||\cdot|^{-1}f\|_{H^1(\mathbb{B}^5_1)} \lesssim \|f\|_{H^2(\mathbb{B}^5_1)}.$$

For $j=10$ we perform an integration by parts to conclude that

$$\begin{aligned}
&H_{10}(f)(\rho,\lambda)\\
&=(1-\rho^2)^{-\frac{\lambda}{2}}\chi_\lambda(\rho)\int_\rho^1 f'(s)\frac{(1-\chi_\lambda(s))\mathcal{O}(\rho^0 s^2\langle\omega\rangle^0)[1+\mathcal{O}(s^{-1}(1-s)\langle\omega\rangle^{-1})]}{(1-s)^{-\lambda}}\beta_1(\rho,s,\lambda)\,ds\\
&\quad+(1-\rho^2)^{-\frac{\lambda}{2}}\chi_\lambda(\rho)\int_\rho^1 f'(s)\int_0^s\frac{\partial_t\big((1-\chi_\lambda(t))\mathcal{O}(\rho^0t^2\langle\omega\rangle^0)[1+\mathcal{O}(t^{-1}(1-t)\langle\omega\rangle^{-1})]\beta_1(\rho,t,\lambda)\big)}{(1-t)^{-\lambda}}\,dt\,ds\\
&=:B_{10}(f)(\rho,\lambda)+I_{10}(f)(\rho,\lambda).
\end{aligned}$$

By employing Lemma 5.2 and substituting $s=1-e^{-y}$ we estimate

$$\begin{aligned}
&\left|\int_{\mathbb{R}} e^{i\omega\tau}B_{10}(f)(\rho,-\tfrac{1}{2}+\delta+i\omega)\,d\omega\right|\\
&\quad\lesssim \rho^{-\frac{1}{90}}\int_\rho^1 \left|\tau-\tfrac{1}{2}\log(1-\rho^2)+\log(1-s)\right|^{-\frac{1}{8}}\langle\tau+\log(1-s)\rangle^{-2}|f'(s)|s^{2+\frac{1}{90}}(1-s)^{-\frac{1}{2}+\delta}\,ds\\
&\quad\lesssim \rho^{-\frac{1}{90}}\int_0^\infty \left|\tau-\tfrac{1}{2}\log(1-\rho^2)-y\right|^{-\frac{1}{8}}\langle\tau-y\rangle^{-2}|f'(1-e^{-y})|(1-e^{-y})^{2+\frac{1}{90}}e^{-(\frac{1}{2}+\delta)y}\,ds\\
&\quad\lesssim \rho^{-\frac{1}{90}}\left(\int_0^\infty \left|\tau-\tfrac{1}{2}\log(1-\rho^2)-y\right|^{-\frac{1}{3}}\langle\tau-y\rangle^{-2}\,dy\right)^{\frac{1-2\delta}{2}}\\
&\qquad\qquad\times\left(\int_0^\infty\langle\tau-y\rangle^{-2}|f'(1-e^{-y})|^{\frac{2}{1+2\delta}}(1-e^{-y})^4e^{-y}\,dy\right)^{\frac{1+2\delta}{2}}.
\end{aligned}$$

Observe that

$$\begin{aligned}
\int_0^\infty \left|\tau-\tfrac{1}{2}\log(1-\rho^2)-y\right|^{-\frac{1}{3}}\langle\tau-y\rangle^{-2}\,dy &\lesssim \int_{\mathbb{R}}\left|\tfrac{1}{2}\log(1-\rho^2)+y\right|^{-\frac{1}{3}}\langle y\rangle^{-2}\,dy\\
&\lesssim \int_{-\frac{1}{2}\log(1-\rho^2)-1}^{-\frac{1}{2}\log(1-\rho^2)+1}\left|\tfrac{1}{2}\log(1-\rho^2)+y\right|^{-\frac{1}{3}}dy+\int_{\mathbb{R}}\langle y\rangle^{-2}\,dy.
\end{aligned}$$

Thus, as

$$\int_{-\frac{1}{2}\log(1-\rho^2)-1}^{-\frac{1}{2}\log(1-\rho^2)+1}\left|\tfrac{1}{2}\log(1-\rho^2)+y\right|^{-\frac{1}{3}}dy=\int_{-1}^1|y|^{-\frac{1}{3}}\,dy\lesssim 1,$$

we deduce that

$$\begin{aligned}
&\left\|\int_{\mathbb{R}} e^{i\omega\tau}B_{10}(f)\big(\rho,-\tfrac{1}{2}+\delta+i\omega\big)\,d\omega\right\|^2_{L^2_\tau(\mathbb{R}_+)L^\infty_\rho(\mathbb{B}^5_1)}\\
&\qquad\lesssim \int_0^\infty\int_0^\infty\langle\tau-y\rangle^{-2}|f'(1-e^{-y})|^{\frac{2}{1+2\delta}}(1-e^{-y})^4e^{-y}\,dy\,d\tau.
\end{aligned}$$

Hence, the bounds on $\int_{\mathbb{R}} e^{i\omega\tau} B_{10}(f)\big(\rho, -\frac{1}{2}+\delta+i\omega\big)\,d\omega$ follow from Young's inequality. To proceed, we illustrate the general procedure on how to bound

$$\int_{\mathbb{R}} e^{i\omega\tau} I_{10}(f)\big(\rho, -\tfrac{1}{2}+\delta+i\omega\big)\,d\omega,$$

with

$$\begin{aligned}
&\widetilde{I}_{10}(f)(\rho,\lambda)\\
&\quad = (1-\rho^2)^{-\frac{\lambda}{2}}\chi_\lambda(\rho)\int_\rho^1 f'(s)\int_0^s \frac{(1-\chi_\lambda(t))\mathcal{O}(\rho^0 t^2\langle\omega\rangle^0)\mathcal{O}(t^{-2}(1-t)^0\langle\omega\rangle^{-1})]\beta_1(\rho,t,\lambda)}{(1-t)^{-\lambda}}\,dt\,ds,
\end{aligned}$$

i.e., the term we obtain when the t-derivative hits $[1+\mathcal{O}(t^{-1}(1-t)\langle\omega\rangle^{-1})]$. Here, we use Lemma 5.3 to derive that

$$\left|\int_{\mathbb{R}} e^{i\omega\tau}\widetilde{I}_{10}(f)\big(\rho,-\tfrac{1}{2}+\delta+i\omega\big)\,d\omega\right| \lesssim \int_0^1 |f'(s)|\int_0^s \langle\tau+\log(1-t)\rangle^{-2}\frac{t}{(1-t)^{\frac{1}{2}-\delta}}\,dt\,ds.$$

So,

$$\begin{aligned}
&\left\|\int_{\mathbb{R}} e^{i\omega\tau}\widetilde{I}_{10}(f)\big(\rho,-\tfrac{1}{2}+\delta+i\omega\big)\,d\omega\right\|_{L_\tau^{2/(1+2\delta)}(\mathbb{R}_+)L_\rho^{45}(\mathbb{B}_1^5)}\\
&\qquad\qquad \lesssim \int_0^1 |f'(s)|\left\|\int_0^s\langle\tau+\log(1-t)\rangle^{-2}\frac{t}{(1-t)^{\frac{1}{2}-\delta}}\,dt\right\|_{L_\tau^{2/(1+2\delta)}(\mathbb{R}_+)} ds.
\end{aligned}$$

Furthermore,

$$\begin{aligned}
&\left\|\int_0^s\langle\tau+\log(1-t)\rangle^{-2}\frac{t}{(1-t)^{\frac{1}{2}-\delta}}\,dt\right\|_{L_\tau^{2/(1+2\delta)}(\mathbb{R}_+)} ds\\
&\qquad\qquad = \left\|\int_0^{-\log(1-s)}\langle\tau-y\rangle^{-2}(1-e^{-y})e^{-(\frac{1}{2}+\delta)}\,dy\right\|_{L_\tau^{2/(1+2\delta)}(\mathbb{R}_+)}
\end{aligned}$$

and using Young's inequality yields

$$\begin{aligned}
\left\|\int_0^{-\log(1-s)}\langle\tau-y\rangle^{-2}(1-e^{-y})e^{-\frac{y}{2}}\,dy\right\|_{L_\tau^{2/(1+2\delta)}(\mathbb{R}_+)} &\lesssim \|1_{(0,-\log(1-s))}(y)(1-e^{-y})e^{-\frac{y}{2}}\|_{L_y^{3/2}(\mathbb{R})}\\
&\lesssim \left(\int_0^s (1-t)^{-\frac{1}{4}}t^{\frac{3}{2}}\,dt\right)^{\frac{2}{3}} \lesssim (1-s)^{-\frac{1}{6}}s^{\frac{5}{3}}.
\end{aligned}$$

Thus,

$$\begin{aligned}
\left\|\int_{\mathbb{R}} e^{i\omega\tau}\widetilde{I}_{10}(f)\big(\rho,-\tfrac{1}{2}+\delta+i\omega\big)\,d\omega\right\|^2_{L_\tau^{2/(1+2\delta)}(\mathbb{R}_+)L_\rho^{45}(\mathbb{B}_1^5)} &\lesssim \left(\int_0^1 |f'(s)|(1-s)^{-\frac{1}{6}}s^{\frac{5}{3}}\,ds\right)^2\\
&\lesssim \|f\|^2_{W^{1,2/(1+2\delta)}(\mathbb{B}_1^5)}\int_0^1 (1-s)^{-\frac{2}{3}}s^{-\frac{3}{4}}\,ds\\
&\lesssim \|f\|^2_{W^{1,2/(1+2\delta)}(\mathbb{B}_1^5)}.
\end{aligned}$$

To bound the remaining terms, one integrates by parts once more and uses similar reasoning. Hence,

$$\|S_{10}(\tau)f\|_{L_\tau^{2/(1+2\delta)}(\mathbb{R}_+)L^{45}(\mathbb{B}_1^5)} \lesssim \|f\|_{W^{1,2/(1+2\delta)}(\mathbb{B}_1^5)}.$$

The second estimate on S_{10} then follows from similar reasoning and we move to $\dot{S}_{10}$. To derive the stated estimates on $\dot{S}_{10}$, we again first take a look at $B_{10}(f)$. Integrating by parts once again yields

$$\begin{aligned}
&B_{10}(f)(\rho,\lambda)\\
&=(1-\rho^2)^{-\frac{\lambda}{2}}\chi_\lambda(\rho)\int_\rho^1 f'(s)\frac{(1-\chi_\lambda(s))\mathcal{O}(\rho^0 s^2\langle\omega\rangle^0)[1+\mathcal{O}(s^{-1}(1-s)\langle\omega\rangle^{-1})]}{(1-s)^{-\lambda}}\beta_4(\rho,s,\lambda)\,ds\\
&=(1-\rho^2)^{-\frac{\lambda}{2}}\chi_\lambda(\rho)f'(\rho)(1-\chi_\lambda(\rho))\mathcal{O}(\rho^2\langle\omega\rangle^{-1})[1+\mathcal{O}(\rho^{-1}(1-\rho)\langle\omega\rangle^{-1})](1-\rho)^{1+\lambda}\beta_1(\rho,\rho,\lambda)\\
&\quad+(1-\rho^2)^{-\frac{\lambda}{2}}\chi_\lambda(\rho)\int_\rho^1(1-s)^{1+\lambda}\partial_s\big[f'(s)(1-\chi_\lambda(s))\mathcal{O}(\rho^0 s^2\langle\omega\rangle^0)[1+\mathcal{O}(s^{-1}(1-s)\langle\omega\rangle^{-1})]\beta_4(\rho,s,\lambda)\big]\,ds\\
&:=B_{10}^1(f)(\rho,\lambda)+B_{10}^2(f)(\rho)(\lambda).
\end{aligned}$$

Now, by using that $\chi_\lambda(\rho)\mathcal{O}(\rho^2\langle\omega\rangle^{-1})=\chi_\lambda(\rho)\mathcal{O}(\rho\langle\omega\rangle^{-2})$ and Lemma 5.3, one establishes the estimate

$$\left|\int_{\mathbb{R}}\omega e^{i\omega\tau}B_{10}^1(f)\big(\rho,-\tfrac{1}{2}+\delta+i\omega\big)\,d\omega\right|\lesssim\langle\tau\rangle^{-2}\rho^2|f'(\rho)|$$

from which one concludes the desired bounds by already-exhibited means. Moreover, the remaining kernels can be bounded by implementing essentially the same strategies that we used for $j=9,10$ in Lemma 5.12 and we conclude this proof. □

These last couple of estimates now add together to our next set of Strichartz estimates.

Proposition 6.7. *The difference of the semigroups* $\boldsymbol{S}$ *and* $\boldsymbol{S}_0$ *satisfies the Strichartz estimates*

$$\begin{aligned}
\|e^{(\frac{1}{2}-\delta)\tau}[(\boldsymbol{S}(\tau)-\boldsymbol{S}_0(\tau))(\boldsymbol{I}-\boldsymbol{Q})(\boldsymbol{I}-\boldsymbol{P})\boldsymbol{f}]_1\|_{L^2_\tau(\mathbb{R}_+)L^{45}(\mathbb{B}^5_1)}&\lesssim\|(\boldsymbol{I}-\boldsymbol{Q})\boldsymbol{f}\|_{W^{2,2/(1+2\delta)}\times W^{1,2/(1+2\delta)}(\mathbb{B}^5_1)},\\
\|e^{(\frac{1}{2}-\delta)\tau}[(\boldsymbol{S}(\tau)-\boldsymbol{S}_0(\tau))(\boldsymbol{I}-\boldsymbol{Q})(\boldsymbol{I}-\boldsymbol{P})\boldsymbol{f}]_1\|_{L^\infty_\tau(\mathbb{R}_+)L^{10}(\mathbb{B}^5_1)}&\lesssim\|(\boldsymbol{I}-\boldsymbol{Q})\boldsymbol{f}\|_{W^{2,2/(1+2\delta)}\times W^{1,2/(1+2\delta)}(\mathbb{B}^5_1)}
\end{aligned}$$

for all $\boldsymbol{f}\in C^\infty\times C^\infty(\overline{\mathbb{B}^5_1})$.

6.1. ***Even more estimates.*** Unfortunately, we still need one more estimate at the $W^{2,2/(1+2\delta)}\times W^{1,2/(1+2\delta)}$ level, which is of the form

$$\|e^{(\frac{1}{2}-\delta)\tau}[(\boldsymbol{S}(\tau)-\boldsymbol{S}_0(\tau))(\boldsymbol{I}-\boldsymbol{Q})(\boldsymbol{I}-\boldsymbol{P})\boldsymbol{f}]_1\|_{L^6_\tau(\mathbb{R}_+)W^{1,9/2}(\mathbb{B}^5_1)}\lesssim\|\boldsymbol{f}\|_{W^{2,2/(1+2\delta)}\times W^{1,2/(1+2\delta)}(\mathbb{B}^5_1)}.$$

As done above we use a variant of Lemma 5.14 to reduce the problem to estimating

$$\begin{aligned}
&\int_{\mathbb{R}}e^{i\omega\tau}\partial_\rho[\mathcal{R}(F_\lambda)(\rho,\lambda)-\mathcal{R}_{\mathrm{f}}(F_\lambda)(\rho,\lambda)]\,d\omega,\\
&\int_{\mathbb{R}}e^{i\omega\tau}\omega\partial_\rho[\mathcal{R}(F_\lambda)(\rho,\lambda)-\mathcal{R}_{\mathrm{f}}(F_\lambda)(\rho,\lambda)]\,d\omega,
\end{aligned}$$

with $\lambda=-\frac{1}{2}+\delta+i\omega$. For this we remark that

$$\begin{aligned}
\partial_\rho\mathcal{R}(f)(\rho,\lambda):=\partial_\rho u_0(\rho,\lambda)[b_\lambda(f)+U_1(\rho,\lambda)f(\rho)]+\partial_\rho u_0(\rho,\lambda)\int_\rho^1 U_1(s,\lambda)f'(s)\,ds&\\
-\partial_\rho u_1(\rho,\lambda)U_0(\rho,\lambda)f(\rho)+\partial_\rho u_1(\rho,\lambda)\int_0^\rho U_0(s,\lambda)f'(s)\,ds.&
\end{aligned}$$

We kick off this round of estimates by first looking at

$$W_1'(f)(\rho,\lambda) := b_\lambda(f)\,\partial_\rho u_0(\rho,\lambda) - b_{\mathrm{f}_\lambda}(f)\,\partial_\rho u_{\mathrm{f}_0}(\rho,\lambda).$$

Lemma 6.8. *Let* $\operatorname{Re}\lambda = -\frac{1}{2}+\delta$. *Then we can decompose* $W_1'(f)(\rho,\lambda)$ *as*

$$W_1'(f)(\rho,\lambda) = f(1)\sum_{j=1}^{3} H_j'(\rho,\lambda),$$

where

$$\begin{aligned}
H_1'(\rho,\lambda) &:= \chi_\lambda(\rho)(1-\rho^2)^{-\lambda}\mathcal{O}(\rho^{-1}\langle\omega\rangle^{-2}),\\
H_2'(\rho,\lambda) &:= [(1+\rho)^{-1}\langle\omega\rangle - 2\rho^{-1}]H_2(\rho,\lambda)\\
&\quad + (1-\chi_\lambda(\rho))\rho^{-2}(1+\rho)^{1-\lambda}\mathcal{O}(\rho^{-2}(1-\rho)^{-1}\langle\omega\rangle^{-1})\\
&\quad \times[\mathcal{O}(\langle\omega\rangle^{-4}) + (1-\rho)\mathcal{O}(\langle\omega\rangle^{-5}) + \mathcal{O}(\rho^{-1}(1-\rho)^2\langle\omega\rangle^{-5})] + \widetilde{H}_2(\rho,\lambda),\\
H_3'(\rho,\lambda) &:= [(1-\rho)^{-1}\langle\omega\rangle - 2\rho^{-1}]H_3(\rho,\lambda)\\
&\quad + (1-\chi_\lambda(\rho))\rho^{-2}(1-\rho)^{1-\lambda}\mathcal{O}(\rho^{-2}(1-\rho)^{-1}\langle\omega\rangle^{-1})\\
&\quad \times[\mathcal{O}(\langle\omega\rangle^{-4}) + (1-\rho)\mathcal{O}(\langle\omega\rangle^{-5}) + \mathcal{O}(\rho^{-1}(1-\rho)^2\langle\omega\rangle^{-5})] + \widetilde{H}_3(\rho,\lambda),
\end{aligned}$$

where $\widetilde{H}_j(\rho,\lambda)$ *are the terms we obtain when a* ρ*-derivative hits the perturbative terms*

$$(1-\rho)\mathcal{O}(\langle\omega\rangle^{-5}) + \mathcal{O}(\rho^{-1}(1-\rho)^2\langle\omega\rangle^{-5}).$$

Proof. This follows immediately by differentiating W_1 and noting that the derivatives which hit cut-offs cancel each other. ☐

Proceeding, we set

$$\begin{aligned}
S_j'(\tau)f(\rho) &= \int_{\mathbb{R}} e^{i\omega\tau} f(1) H_j\big(\rho, -\tfrac{1}{2}+\delta+i\omega\big)\,d\omega,\\
\dot{S}_j'(\tau)f(\rho) &= \int_{\mathbb{R}} \omega e^{i\omega\tau} f(1) H_j\big(\rho, -\tfrac{1}{2}+\delta+i\omega\big)\,d\omega
\end{aligned}$$

for $j = 1,2,3$ and $f \in C^\infty(\overline{\mathbb{B}_1^5})$.

Lemma 6.9. *The estimates*

$$\begin{aligned}
\|S_j'(\tau)f\|_{L^6_\tau(\mathbb{R}_+)L^{9/2}(\mathbb{B}_1^5)} &\lesssim \|f\|_{W^{1,2/(1+2\delta)}(\mathbb{B}_1^5)},\\
\|\dot{S}_j'(\tau)f\|_{L^6_\tau(\mathbb{R}_+)L^{9/2}(\mathbb{B}_1^5)} &\lesssim \|f\|_{W^{2,2/(1+2\delta)}(\mathbb{B}_1^5)}
\end{aligned}$$

hold for $j = 1,2,3$ *and* $f \in C^\infty(\overline{\mathbb{B}_1^5})$.

Proof. In essence H_j' differs from H_j by a loss of either one power in ρ or one power of decay in ω. But since $|\cdot|^{-1} \in L^{9/2}(\mathbb{B}_1^5)$, this loss can be compensated for and the claimed estimates follow just as the ones established in Lemma 6.2. ☐

Lemma 6.10. *Let* $\operatorname{Re}\lambda = -\frac{1}{2}+\delta$. *Then we can decompose*

$$\begin{aligned}
W_2'(f)(\rho,\lambda) := \partial_\rho u_0(\rho,\lambda)U_1(\rho,\lambda)f(\rho) - \partial_\rho u_{\mathrm{f}_0}(\rho,\lambda)U_{\mathrm{f}_1}(\rho,\lambda)f(\rho)\\
- \partial_\rho u_1(\rho,\lambda)U_0(\rho,\lambda)f(\rho) + \partial_\rho u_{\mathrm{f}_1}(\rho,\lambda)U_{\mathrm{f}_0}(\rho,\lambda)f(\rho)
\end{aligned}$$

as

$$W_2'(f)(\rho,\lambda)=f(\rho)\sum_{j=4}^{8}H_j(\rho,\lambda),$$

where

$$H_4'(\rho,\lambda)=(1-\rho^2)^{-\frac{\lambda}{2}}\chi_\lambda(\rho)\int_0^\rho\frac{\mathcal{O}(\rho^{-1}s\langle\omega\rangle^{-1})}{(1-s^2)^{1-\frac{\lambda}{2}}}\,ds,$$

$$H_5'(\rho,\lambda)=\left[\frac{1-\lambda}{1+\rho}-2\rho^{-1}\right]H_5(\rho,\lambda)$$
$$+(1-\chi_\lambda(\rho))\rho^{-2}(1+\rho)^{1-\lambda}\mathcal{O}(\rho^{-2}(1-\rho)^0\langle\omega\rangle^{-1})\int_0^\rho\frac{\chi_\lambda(s)\mathcal{O}(s\langle\omega\rangle^{-2})}{(1-s^2)^{1-\frac{\lambda}{2}}}\beta_2(\rho,s,\lambda)\,ds+\widetilde{H}_5(\rho,\lambda),$$

$$H_6'(\rho,\lambda)=\left[-\frac{1-\lambda}{1-\rho}-2\rho^{-1}\right]H_6(\rho,\lambda)$$
$$+(1-\chi_\lambda(\rho))\rho^{-2}(1-\rho)^{1-\lambda}\mathcal{O}(\rho^{-2}(1-\rho)^0\langle\omega\rangle^{-1})\int_0^\rho\frac{\chi_\lambda(s)\mathcal{O}(s\langle\omega\rangle^{-2})}{(1-s^2)^{1-\frac{\lambda}{2}}}\beta_3(\rho,s,\lambda)\,ds+\widetilde{H}_6(\rho,\lambda),$$

$$H_7'(\rho,\lambda)=\left[-\frac{1-\lambda}{1-\rho}-2\rho^{-1}\right]H_7(\rho,\lambda)$$
$$+(1-\chi_\lambda(\rho))\rho^{-2}(1-\rho)^{1-\lambda}\mathcal{O}(\rho^{-2}(1-\rho)^0\langle\omega\rangle^{-1})$$
$$\times\int_0^\rho\frac{s^2(1-\chi_\lambda(s))[1+\mathcal{O}(s^{-1}(1-s)\langle\omega\rangle^{-1})]\gamma_3(\rho,s,\lambda)}{2(1-\lambda)(1-s)^{1-\lambda}}\,ds+\widetilde{H}_7(\rho,\lambda),$$

$$H_8'(\rho,\lambda)=\left[\frac{1-\lambda}{1+\rho}-2\rho^{-1}\right]H_8(\rho,\lambda)$$
$$+(1-\chi_\lambda(\rho))\rho^{-2}(1+\rho)^{1-\lambda}\mathcal{O}(\rho^{-1}(1-\rho)^0\langle\omega\rangle^{-1})$$
$$\times\int_0^\rho\frac{s^2(1-\chi_\lambda(s))[1+\mathcal{O}(s^{-1}(1-s)\langle\omega\rangle^{-1})]\gamma_4(\rho,s,\lambda)}{2(1-\lambda)(1+s)^{1-\lambda}}\,ds+\widetilde{H}_8(\rho,\lambda),$$

with β_j and γ_j as in Lemma 5.9 and where $\widetilde{H}_j(\rho,\lambda)$ are the terms we obtain when ∂_ρ hits either β_j or γ_j.

Again, we define operators corresponding to the kernels H_j' as

$$S_j'(\tau)f(\rho):=\lim_{N\to\infty}\int_{-N}^{N}e^{i\omega\tau}f(\rho)H_j\left(\rho,-\tfrac{1}{2}+\delta+i\omega\right)d\omega,$$
$$\dot{S}_j'(\tau)f(\rho):=\lim_{N\to\infty}\int_{-N}^{N}\omega e^{i\omega\tau}f(\rho)H_j\left(\rho,-\tfrac{1}{2}+\delta+i\omega\right)d\omega$$

for $j=4,\dots,8$ and $f\in C^\infty(\overline{\mathbb{B}_1^5})$.

Lemma 6.11. *The estimates*

$$\|S_j'(\tau)f\|_{L^6_\tau(\mathbb{R}_+)L^{9/2}(\mathbb{B}^5_1)}\lesssim\|f\|_{W^{1,2/(1+2\delta)}(\mathbb{B}^5_1)},$$
$$\|\dot{S}_j'(\tau)f\|_{L^6_\tau(\mathbb{R}_+)L^{9/2}(\mathbb{B}^5_1)}\lesssim\|f\|_{W^{2,2/(1+2\delta)}(\mathbb{B}^5_1)}$$

hold for $j=4,\dots,8$ and $f\in C^\infty(\overline{\mathbb{B}_1^5})$.

Proof. The lemma follows by adapting the proof of Lemma 6.4 slightly. ☐

Lastly, we come to

$$W_3'(f)(\rho,\lambda) := \partial_\rho u_0(\rho,\lambda)\int_\rho^1 U_1(s,\lambda)f'(s)\,ds - \partial_\rho u_{\mathrm{f}_0}(\rho,\lambda)\int_\rho^1 U_{\mathrm{f}_1}(s,\lambda)f'(s)\,ds + \partial_\rho u_1(\rho,\lambda)\int_0^\rho U_0(s,\lambda)f'(s)\,ds - \partial_\rho u_{\mathrm{f}_1}(\rho,\lambda)\int_0^\rho U_{\mathrm{f}_0}(s,\lambda)f'(s)\,ds.$$

Lemma 6.12. *Let* $\operatorname{Re}\lambda = -\frac{1}{2}+\delta$. *Then we can decompose* $W_3'(\rho,\lambda)$ *as*

$$W_3'(f)(\rho,\lambda) = \sum_{j=9}^{18} H_j'(f)(\rho,\lambda),$$

where

$$H_9'(f)(\rho,\lambda) := (1-\rho^2)^{-\frac{\lambda}{2}}\chi_\lambda(\rho)\int_\rho^1 f'(s)\int_0^s \frac{\chi_\lambda(t)\mathcal{O}(\rho^{-1}t\langle\omega\rangle^{-1})}{(1-t^2)^{1-\frac{\lambda}{2}}}\,dt\,ds,$$

$$H_{10}'(f)(\rho,\lambda) := (1-\rho^2)^{-\frac{\lambda}{2}}\chi_\lambda(\rho) \times\int_\rho^1 f'(s)\int_0^s (1-\chi_\lambda(t))\frac{\mathcal{O}(\rho^{-1}t^2\langle\omega\rangle)[1+\mathcal{O}(t^{-1}(1-t)\langle\omega\rangle^{-1})]}{(1-t)^{1-\lambda}}\beta_4(\rho,t,\lambda)\,dt\,ds + \widetilde{H}_{10}(f)(\rho,\lambda),$$

$$H_{11}'(f)(\rho,\lambda) := \left[-\frac{1-\lambda}{1-\rho}-2\rho^{-1}\right]H_{11}(f)(\rho,\lambda) +(1-\chi_\lambda(\rho))\rho^{-2}(1-\rho)^{1-\lambda}\mathcal{O}(\rho^{-2}(1-\rho)^0\langle\omega\rangle^{-1}) \times\int_\rho^1 f'(s)\int_0^s \frac{\chi_\lambda(t)\mathcal{O}(t\langle\omega\rangle^{-2})}{(1-t^2)^{1-\frac{\lambda}{2}}}\beta_5(t,\rho,\lambda)\,dt\,ds+\widetilde{H}_{11}(f)(\rho,\lambda),$$

$$H_{12}'(f)(\rho,\lambda) := \left[\frac{1-\lambda}{1+\rho}-2\rho^{-1}\right]H_{12}(f)(\rho,\lambda) +(1-\chi_\lambda(\rho))\rho^{-2}(1+\rho)^{1-\lambda}\mathcal{O}(\rho^{-2}(1-\rho)^0\langle\omega\rangle^{-1}) \times\int_\rho^1 f'(s)\int_0^s \frac{\chi_\lambda(t)\mathcal{O}(t\langle\omega\rangle^{-2})}{(1-t^2)^{1-\frac{\lambda}{2}}}\beta_6(t,\rho,\lambda)\,dt\,ds+\widetilde{H}_{12}(f)(\rho,\lambda),$$

$$H_{13}'(f)(\rho,\lambda) := \left[-\frac{1-\lambda}{1-\rho}-2\rho^{-1}\right]H_{13}(f)(\rho,\lambda) +(1-\chi_\lambda(\rho))\rho^{-2}(1-\rho)^{1-\lambda}\mathcal{O}(\rho^{-2}(1-\rho)^0\langle\omega\rangle^{-1}) \times\int_\rho^1 f'(s)\int_0^s \frac{t^2(1-\chi_\lambda(t))[1+\mathcal{O}(t^{-1}(1-t)\langle\omega\rangle^{-1})]\gamma_3(\rho,t,\lambda)}{2(1-\lambda)(1-t)^{1-\lambda}}\,dt\,ds+\widetilde{H}_{13}(f)(\rho,\lambda),$$

$$H_{14}'(f)(\rho,\lambda) := \left[\frac{1-\lambda}{1+\rho}-2\rho^{-1}\right]H_{14}(f)(\rho,\lambda) +(1-\chi_\lambda(\rho))\rho^{-2}(1+\rho)^{1-\lambda}\mathcal{O}(\rho^{-2}(1-\rho)^0\langle\omega\rangle^{-1}) \times\int_\rho^1 f'(s)\int_0^s \frac{t^2(1-\chi_\lambda(t))[1+\mathcal{O}(t^{-1}(1-t)\langle\omega\rangle^{-1})]\gamma_4(t,\rho,\lambda)}{2(1-\lambda)(1-t)^{1-\lambda}}\,dt\,ds+\widetilde{H}_{14}(f)(\rho,\lambda),$$

$$H'_{15}(f)(\rho,\lambda) := (1-\rho^2)^{-\frac{\lambda}{2}}\chi_\lambda(\rho)\int_0^\rho f'(s)\int_0^s \frac{\mathcal{O}(\rho^{-1}t\langle\omega\rangle^{-1})}{(1-t^2)^{1-\frac{\lambda}{2}}}\,dt\,ds,$$

$$\begin{aligned}H'_{16}(f)(\rho,\lambda) := &\left[\frac{1-\lambda}{1+\rho}-2\rho^{-1}\right]H_{16}(f)(\rho,\lambda)\\&+(1-\chi_\lambda(\rho))\rho^{-2}(1+\rho)^{1-\lambda}\mathcal{O}(\rho^{-2}(1-\rho)^0\langle\omega\rangle^{-1})\\&\times\int_0^\rho f'(s)\int_0^s \frac{\chi_\lambda(t)\mathcal{O}(t\langle\omega\rangle^{-1})}{(1-t^2)^{1-\frac{\lambda}{2}}}\beta_7(t,\rho,\lambda)\,dt\,ds+\widetilde{H}_{16}(f)(\rho,\lambda),\end{aligned}$$

$$\begin{aligned}H'_{17}(f)(\rho,\lambda) := &\left[\frac{1-\lambda}{1+\rho}-2\rho^{-1}\right]H_{17}(f)(\rho,\lambda)\\&+(1-\chi_\lambda(\rho))\rho^{-2}(1+\rho)^{1-\lambda}\mathcal{O}(\rho^{-2}(1-\rho)^0\langle\omega\rangle^{-1})\\&\times\int_0^\rho f'(s)\int_0^s \frac{t^2(1-\chi_\lambda(t))[1+\mathcal{O}(t^{-1}(1-t)\langle\omega\rangle^{-1})]\gamma_5(\rho,t,\lambda)}{2(1-\lambda)(1-t)^{1-\lambda}}\,dt\,ds+\widetilde{H}_{17}(f)(\rho,\lambda),\end{aligned}$$

$$\begin{aligned}H'_{18}(f)(\rho,\lambda) := &\left[\frac{1-\lambda}{1+\rho}-2\rho^{-1}\right]H_{18}(f)(\rho,\lambda)\\&+(1-\chi_\lambda(\rho))\rho^{-2}(1+\rho)^{1-\lambda}\mathcal{O}(\rho^{-2}(1-\rho)^0\langle\omega\rangle^{-1})\\&\times\int_0^\rho f'(s)\int_0^s \frac{t^2(1-\chi_\lambda(t))[1+\mathcal{O}(t^{-1}(1-t)\langle\omega\rangle^{-1})]\gamma_6(\rho,t,\lambda)}{2(1-\lambda)(1+t)^{1-\lambda}}\,dt\,ds+\widetilde{H}_{18}(f)(\rho,\lambda),\end{aligned}$$

with β_j and γ_j as in Lemma 5.9 and where $\widetilde{H}_j(f)(\rho,\lambda)$ are terms obtained when a ρ-derivative hits β_j or γ_j.

One last time we define operators S'_j and $\dot{S}'_j$

$$S'_j(\tau)f(\rho) := \lim_{N\to\infty}\int_{-N}^{N} e^{i\omega\tau}H'_j(f)\big(\rho,-\tfrac{1}{2}+\delta+i\omega\big)\,d\omega,$$
$$\dot{S}'_j(\tau)f(\rho) := \lim_{N\to\infty}\int_{-N}^{N} \omega e^{i\omega\tau}H'_j(f)\big(\rho,-\tfrac{1}{2}+\delta+i\omega\big)\,d\omega$$

for $j = 9,\dots,18$ and $f \in C^\infty(\overline{\mathbb{B}^5_1})$.

Lemma 6.13. *The estimates*

$$\|S'_j(\tau)f\|_{L^6_\tau(\mathbb{R}_+)L^{9/2}(\mathbb{B}^5_1)} \lesssim \|f\|_{W^{1,2/(1+2\delta)}(\mathbb{B}^5_1)},$$
$$\|\dot{S}'_j(\tau)f\|_{L^6_\tau(\mathbb{R}_+)L^{9/2}(\mathbb{B}^5_1)} \lesssim \|f\|_{W^{2,2/(1+2\delta)}(\mathbb{B}^5_1)}$$

hold for $j = 9,\dots,18$ and $f \in C^\infty(\overline{\mathbb{B}^5_1})$.

Proof. The estimates can be established by adapting the procedures used in the proof of Lemma 6.6 in a straightforward way. ☐

Proposition 6.14. *The difference of $\boldsymbol{S}$ and $\boldsymbol{S}_0$ satisfies*

$$\|e^{(\frac{1}{2}-\delta)\tau}[(\boldsymbol{S}(\tau)-\boldsymbol{S}_0(\tau))(\boldsymbol{I}-\boldsymbol{Q})(\boldsymbol{I}-\boldsymbol{P})\boldsymbol{f}]_1\|_{L^6_\tau(\mathbb{R}_+)W^{1,9/2}(\mathbb{B}^5_1)} \lesssim \|(\boldsymbol{I}-\boldsymbol{Q})\boldsymbol{f}\|_{W^{2,2/(1+2\delta)}\times W^{1,2/(1+2\delta)}(\mathbb{B}^5_1)}$$

for all $\boldsymbol{f} \in C^\infty\times C^\infty(\overline{\mathbb{B}^5_1})$.

We now turn to interpolating the previously derived Strichartz estimates to obtain estimates on the $H^{3/2} \times H^{1/2}$ level. For the notation and conventions appearing in the context of interpolation throughout the following proof we refer the reader to the Appendix and [Bergh and Löfström 1976]. We also recall that we constructed a subset $X \subset H^3 \times H^2(\mathbb{B}^5_1)$ which lies dense in $\mathcal{H} := H^{3/2} \times H^{1/2}(\mathbb{B}^5_1)$ such that the spectral projection $\boldsymbol{Q}$ agrees on X with a bounded linear operator $\widehat{\boldsymbol{Q}} : \mathcal{H} \to \mathcal{H}$.

Proposition 6.15. *Let $p \in [2, \infty]$ and $q \in [5, 10]$ be such that $\frac{1}{p} + \frac{5}{q} = 1$. Then, the semigroup $\boldsymbol{S}$ satisfies the Strichartz estimates*

$$\|[\boldsymbol{S}(\tau)(\boldsymbol{I} - \boldsymbol{P})\boldsymbol{f}]_1\|_{L^p_\tau(\mathbb{R}_+)L^q(\mathbb{B}^5_1)} \lesssim \|\boldsymbol{f}\|_{\mathcal{H}}$$

for all $\boldsymbol{f} \in \mathcal{H}$. Furthermore, also the inhomogeneous estimate

$$\left\| \int_0^\tau [\boldsymbol{S}(\tau - \sigma)(\boldsymbol{I} - \boldsymbol{P})\boldsymbol{h}(\sigma)]_1 \, d\sigma \right\|_{L^p_\tau(I)L^q(\mathbb{B}^5_1)} \lesssim \|\boldsymbol{h}\|_{L^1(I)\mathcal{H}}$$

holds for all $\boldsymbol{h} \in L^1(\mathbb{R}_+, \mathcal{H})$ and all intervals $I \subset [0, \infty)$ containing 0.

Proof. We start by setting

$$\|u\|^p_{L^p(\mathbb{R}_+, e^{a\tau}\, d\tau)L^q(\mathbb{B}^5_1)} := \int_{\mathbb{R}_+} \|u(\tau)\|^p_{L^q(\mathbb{B}^5_1)} e^{a\tau} \, d\tau$$

for $a \in \mathbb{R}$ and $\widetilde{\boldsymbol{S}}(\tau) = \boldsymbol{S}(\tau) - \boldsymbol{S}_0(\tau)$. Then, by a density argument we have that

$$\begin{aligned}
&\|[\widetilde{\boldsymbol{S}}(\tau)(\boldsymbol{I} - \boldsymbol{Q})(\boldsymbol{I} - \boldsymbol{P})\boldsymbol{f}]_1\|_{L^{2/(1-2\delta)}(\mathbb{R}_+, e^{-\tau(1+2\delta)/(1-2\delta)}\, d\tau)L^{45/8}(\mathbb{B}^5_1)} \\
&\qquad = \|e^{-(\frac{1}{2}+\delta)\tau}[\widetilde{\boldsymbol{S}}(\tau)(\boldsymbol{I} - \boldsymbol{Q})(\boldsymbol{I} - \boldsymbol{P})\boldsymbol{f}]_1\|_{L^{2/(1-2\delta)}_\tau(\mathbb{R}_+)L^{45/8}(\mathbb{B}^5_1)} \\
&\qquad \lesssim \|(\boldsymbol{I} - \boldsymbol{Q})\boldsymbol{f}\|_{W^{1,2/(1-2\delta)} \times L^{2/(1-2\delta)}(\mathbb{B}^5_1)}
\end{aligned}$$

for all $\boldsymbol{f} \in X$ thanks to Proposition 5.13. Similarly, from Proposition 6.7 we know that

$$\begin{aligned}
&\|[\widetilde{\boldsymbol{S}}(\tau)(\boldsymbol{I} - \boldsymbol{Q})(\boldsymbol{I} - \boldsymbol{P})\boldsymbol{f}]_1\|_{L^{2/(1+2\delta)}(\mathbb{R}_+, e^{\tau 1/(1+2\delta)}\, d\tau)L^{45}(\mathbb{B}^5_1)} \\
&\qquad = \|e^{(\frac{1}{2}-\delta)\tau}[\widetilde{\boldsymbol{S}}(\tau)(\boldsymbol{I} - \boldsymbol{Q})(\boldsymbol{I} - \boldsymbol{P})\boldsymbol{f}]_1\|_{L^{2/(1+2\delta)}_\tau(\mathbb{R}_+)L^{45}(\mathbb{B}^5_1)} \\
&\qquad \lesssim \|(\boldsymbol{I} - \boldsymbol{Q})\boldsymbol{f}\|_{W^{2,2/(1+2\delta)} \times W^{1,2/(1+2\delta)}(\mathbb{B}^5_1)},
\end{aligned}$$

Hence, by invoking Proposition A.1 and using that

$$\mathcal{H} = H^{\frac{3}{2}} \times H^{\frac{1}{2}}(\mathbb{B}^5_1) = (W^{2,\frac{2}{1+2\delta}} \times W^{1,\frac{2}{1+2\delta}}(\mathbb{B}^5_1), W^{1,\frac{2}{1-2\delta}} \times L^{\frac{2}{1-2\delta}}(\mathbb{B}^5_1))_{[\frac{1}{2}]},$$

see [Triebel 1995, p. 317, Section 4.3.1.1, Theorem 1], we conclude that

$$\|[\widetilde{\boldsymbol{S}}(\tau)(\boldsymbol{I} - \boldsymbol{Q})(\boldsymbol{I} - \boldsymbol{P})\boldsymbol{f}]_1\|_{L^2(\mathbb{R}_+)L^{10}(\mathbb{B}^5_1)} \lesssim \|(\boldsymbol{I} - \boldsymbol{Q})\boldsymbol{f}\|_{\mathcal{H}}. \tag{6-1}$$

In addition, since

$$\begin{aligned}
\|e^{-(\frac{1}{2}-\delta)\tau}[\widetilde{\boldsymbol{S}}(\tau)(\boldsymbol{I} - \boldsymbol{Q})(\boldsymbol{I} - \boldsymbol{P})\boldsymbol{f}]_1\|_{L^{10/3}(\mathbb{B}^5_1)} &\lesssim \|(\boldsymbol{I} - \boldsymbol{Q})\boldsymbol{f}\|_{W^{1,2/(1+2\delta)} \times L^{2/(1+2\delta)}(\mathbb{B}^5_1)}, \\
\|e^{(\frac{1}{2}-\delta)\tau}[\widetilde{\boldsymbol{S}}(\tau)(\boldsymbol{I} - \boldsymbol{Q})(\boldsymbol{I} - \boldsymbol{P})\boldsymbol{f}]_1\|_{L^{10}(\mathbb{B}^5_1)} &\lesssim \|(\boldsymbol{I} - \boldsymbol{Q})\boldsymbol{f}\|_{W^{2,2/(1+2\delta)} \times W^{1,2/(1+2\delta)}(\mathbb{B}^5_1)}
\end{aligned} \tag{6-2}$$

for all $\tau \geq 0$ interpolating yields

$$\|[\widetilde{\boldsymbol{S}}(\tau)(\boldsymbol{I}-\boldsymbol{Q})(\boldsymbol{I}-\boldsymbol{P})\boldsymbol{f}]_1\|_{L^\infty(\mathbb{R}_+)L^5(\mathbb{B}^5_1)} \lesssim \|(\boldsymbol{I}-\boldsymbol{Q})\boldsymbol{f}\|_{\mathcal{H}}$$

for all $\boldsymbol{f} \in X$. Hence, elementary interpolation between (6-1) and (6-2) combined with the estimates on $\boldsymbol{S}_0$ in Lemma 2.9 yields

$$\|[\boldsymbol{S}(\tau)(\boldsymbol{I}-\boldsymbol{Q})(\boldsymbol{I}-\boldsymbol{P})\boldsymbol{f}]_1\|_{L^p(\mathbb{R}_+)L^q(\mathbb{B}^5_1)} \lesssim \|(\boldsymbol{I}-\boldsymbol{Q})\boldsymbol{f}\|_{\mathcal{H}},$$

where $p \in [2,\infty]$ and $q \in [5,10]$ are such that $\frac{1}{p}+\frac{5}{q}=1$. Furthermore, by construction $\boldsymbol{Q}$ agrees with a bounded linear operator $\widehat{\boldsymbol{Q}} : \mathcal{H} \to \mathcal{H}$ on X and so

$$\|(\boldsymbol{I}-\boldsymbol{Q})\boldsymbol{f}\|_{\mathcal{H}} = \|(\boldsymbol{I}-\widehat{\boldsymbol{Q}})\boldsymbol{f}\|_{\mathcal{H}} \lesssim \|\boldsymbol{f}\|_{\mathcal{H}}$$

for all $\boldsymbol{f} \in X$. Next, we turn to $\boldsymbol{S}(\tau)\boldsymbol{Q}$. From the Sobolev embedding $H^2(\mathbb{B}^5_1) \hookrightarrow L^{10}(\mathbb{B}^5_1)$, we deduce that

$$\|[\boldsymbol{S}(\tau)\boldsymbol{Q}(\boldsymbol{I}-\boldsymbol{P})\boldsymbol{f}]_1\|_{L^p_\tau(\mathbb{R}_+)L^q(\mathbb{B}^5_1)} \lesssim \|[\boldsymbol{S}(\tau)\boldsymbol{Q}(\boldsymbol{I}-\boldsymbol{P})\boldsymbol{f}]_1\|_{L^p_\tau(\mathbb{R}_+)H^2(\mathbb{B}^5_1)}$$

for all admissible pairs (p,q). Given that the range of $\boldsymbol{Q}$ is contained in the union of finitely many generalized eigenspaces corresponding to eigenvalues which all have negative real part, we infer the existence of an $\varepsilon > 0$ such that

$$\|[\boldsymbol{S}(\tau)\boldsymbol{Q}(\boldsymbol{I}-\boldsymbol{P})\boldsymbol{f}]_1\|_{H^2(\mathbb{B}^5_1)} \lesssim e^{-\varepsilon\tau}\|\boldsymbol{Q}\boldsymbol{f}\|_{H^2\times H^1(\mathbb{B}^5_1)}$$

on X. Moreover, since the range of $\boldsymbol{Q}$ is finite-dimensional, we see that

$$\|\boldsymbol{Q}\boldsymbol{f}\|_{H^2\times H^1(\mathbb{B}^5_1)} \lesssim \|\boldsymbol{Q}\boldsymbol{f}\|_{\mathcal{H}} = \|\widehat{\boldsymbol{Q}}\boldsymbol{f}\|_{\mathcal{H}} \lesssim \|\boldsymbol{f}\|_{\mathcal{H}}$$

for all $\boldsymbol{f} \in X$. Thus, the estimate

$$\|\boldsymbol{S}(\tau)[(\boldsymbol{I}-\boldsymbol{P})\boldsymbol{f}]_1\|_{L^p(\mathbb{R}_+)L^q(\mathbb{B}^5_1)} \lesssim \|\boldsymbol{f}\|_{\mathcal{H}}$$

holds for all claimed pairs (p,q) and all $\boldsymbol{f} \in X$ and by density for all $\boldsymbol{f} \in \mathcal{H}$. For the inhomogeneous estimates one uses Minkowski's inequality as in the proof of Lemma 3.7 in [Donninger and Wallauch 2023]. □

Analogously, one proves Strichartz estimates involving (fractional) derivatives.

Proposition 6.16. *The estimates*

$$\|[\boldsymbol{S}(\tau)(\boldsymbol{I}-\boldsymbol{P})\boldsymbol{f}]_1\|_{L^2_\tau(\mathbb{R}_+)W^{1/2,5}(\mathbb{B}^5_1)} \lesssim \|\boldsymbol{f}\|_{\mathcal{H}},$$
$$\|[\boldsymbol{S}(\tau)(\boldsymbol{I}-\boldsymbol{P})\boldsymbol{f}]_1\|_{L^6_\tau(\mathbb{R}_+)W^{1,30/11}(\mathbb{B}^5_1)} \lesssim \|\boldsymbol{f}\|_{\mathcal{H}}$$

hold for all $\boldsymbol{f} \in \mathcal{H}$. Furthermore, also the inhomogeneous estimates

$$\left\|\int_0^\tau [\boldsymbol{S}(\tau-\sigma)(\boldsymbol{I}-\boldsymbol{P})\boldsymbol{h}(\sigma)]_1\,d\sigma\right\|_{L^2_\tau(I)W^{1/2,5}(\mathbb{B}^5_1)} \lesssim \|\boldsymbol{h}\|_{L^1(I)\mathcal{H}},$$
$$\left\|\int_0^\tau [\boldsymbol{S}(\tau-\sigma)(\boldsymbol{I}-\boldsymbol{P})\boldsymbol{h}(\sigma)]_1\,d\sigma\right\|_{L^6_\tau(I)L^{30/11}(\mathbb{B}^5_1)} \lesssim \|\boldsymbol{h}\|_{L^1(I)\mathcal{H}}$$

hold for all $\boldsymbol{h} \in L^1(\mathbb{R}_+,\mathcal{H})$ and all intervals $I \subset [0,\infty)$ containing 0.

Proof. Note that

$$(W^{1,\frac{9}{2}}(\mathbb{B}_1^5), L^{\frac{45}{8}}(\mathbb{B}_1^5))_{[\frac{1}{2}]} = W^{\frac{1}{2},5}(\mathbb{B}_1^5),$$
$$(W^{1,\frac{9}{2}}(\mathbb{B}_1^5), W^{1,\frac{45}{23}}(\mathbb{B}_1^5))_{[\frac{1}{2}]} = W^{1,\frac{30}{11}}(\mathbb{B}_1^5),$$

thanks to [Triebel 1995, p. 317, Section 4.3.1.1, Theorem 1]. Consequently, the desired estimates follow from Propositions 5.17, 6.14, A.1 and the arguments employed in the proof of Proposition 6.15. ☐

7. Nonlinear theory

We now take a closer look at our nonlinearity

$$N(u)(\rho) = \frac{\sin(4\arctan(\rho) + 2\rho u(\rho)) - 2\rho u(\rho)}{\rho^3} - \frac{\sin(4\arctan(\rho))}{\rho^3} + \frac{16}{(1+\rho^2)^2}u(\rho),$$

which we recast as

$$N(u)(\rho) = \underbrace{-\frac{16(1-\rho^2)}{(1+\rho^2)^2}}_{=:V_N(\rho)} u_1(\rho)^2 - 4\int_0^{u(\rho)} \cos(4\arctan(\rho) + 2\rho t)(u(\rho)-t)^2\, dt$$

by performing a Taylor expansion.

Lemma 7.1. *The estimates*

$$\|N(u)\|_{H^{1/2}(\mathbb{B}_1^5)} \lesssim \|u\|^2_{L^{10}(\mathbb{B}_1^5)} + \|u\|^3_{L^5(\mathbb{B}_1^5)} + \|u\|^4_{L^{20/3}(\mathbb{B}_1^5)}$$
$$+ \|u\|_{W^{1/2,5}(\mathbb{B}_1^5)}\|u\|_{L^{10}(\mathbb{B}_1^5)} + \|u\|_{W^{1,30/11}(\mathbb{B}_1^5)}\|u\|^2_{L^{60/7}(\mathbb{B}_1^5)}$$

and

$$\begin{aligned}\|N(u)-N(v)\|_{H^{1/2}(\mathbb{B}_1^5)} \lesssim{}& \|u-v\|_{L^{10}(\mathbb{B}_1^5)}(\|u\|_{L^{10}(\mathbb{B}_1^5)} + \|v\|_{L^{10}(\mathbb{B}_1^5)} + \|u\|^2_{L^{20/3}(\mathbb{B}_1^5)} + \|v\|^2_{L^{20/3}(\mathbb{B}_1^5)})\\ &+ \|u-v\|_{L^{10}(\mathbb{B}_1^5)}(\|u\|^3_{L^6(\mathbb{B}_1^5)} + \|v\|^3_{L^6(\mathbb{B}_1^5)})\\ &+ \|u-v\|_{L^{10}(\mathbb{B}_1^5)}(\|u\|_{W^{1/2,5}(\mathbb{B}_1^5)} + \|v\|_{W^{1/2,5}(\mathbb{B}_1^5)})\\ &+ \|u-v\|_{W^{1/2,5}(\mathbb{B}_1^5)}(\|u\|_{L^{10}(\mathbb{B}_1^5)} + \|v\|_{L^{10}(\mathbb{B}_1^5)})\\ &+ \|u-v\|_{W^{1,30/11}(\mathbb{B}_1^5)}\|u\|^2_{L^{60/7}(\mathbb{B}_1^5)}\\ &+ \|u-v\|_{L^{60/7}(\mathbb{B}_1^5)}\|v\|_{W^{1,30/11}(\mathbb{B}_1^5)}(\|u\|_{L^{60/7}(\mathbb{B}_1^5)} + \|u\|_{L^{60/7}(\mathbb{B}_1^5)})\end{aligned}$$

hold for all $u, v \in C^\infty(\overline{\mathbb{B}_1^5})$.

Proof. We start off with the easier quadratic term and use the product rule for fractional derivatives twice to compute that

$$\begin{aligned}\|V_N u^2\|_{H^{1/2}(\mathbb{B}_1^5)} &\lesssim \|V_N\|_{W^{1/2,10/3}(\mathbb{B}_1^5)}\|u^2\|_{L^5(\mathbb{B}_1^5)} + \|V_N\|_{L^5(\mathbb{B}_1^5)}\|u^2\|_{W^{1/2,10/3}(\mathbb{B}_1^5)}\\ &\lesssim \|u\|^2_{L^{10}(\mathbb{B}_1^5)} + \|u\|_{W^{1/2,5}(\mathbb{B}_1^5)}\|u\|_{L^{10}(\mathbb{B}_1^5)}.\end{aligned}$$

For the cubic term, we use the Sobolev inequality $\|\cdot\|_{H^{1/2}(\mathbb{B}^5_1)} \lesssim \|\cdot\|_{W^{1,5/3}(\mathbb{B}^5_1)}$ to estimate that

$$\begin{aligned}
\left\| \int_0^{u(\rho)} \cos(4\arctan(\rho)+2\rho t)(u(\rho)-t)^2\,dt \right\|_{H^{1/2}_\rho(\mathbb{B}^5_1)} & \\
&\lesssim \left\| \int_0^{u(\rho)} \cos(4\arctan(\rho)+2\rho t)(u(\rho)-t)^2\,dt \right\|_{W^{1,5/3}_\rho(\mathbb{B}^5_1)} \\
&\lesssim \|u^2u'\|_{L^{5/3}(\mathbb{B}^5_1)} + \|u^3\|_{L^{5/3}(\mathbb{B}^5_1)} + \|u^4\|_{L^{5/3}(\mathbb{B}^5_1)} \\
&\lesssim \|u'\|_{L^{30/11}(\mathbb{B}^5_1)}\|u\|^2_{L^{60/7}(\mathbb{B}^5_1)} + \|u\|^3_{L^5(\mathbb{B}^5_1)} + \|u\|^4_{L^{20/3}(\mathbb{B}^5_1)}.
\end{aligned}$$

To establish local Lipschitz estimates we let $u, v \in C^\infty(\overline{\mathbb{B}^5_1})$ and again first take a look at the easier quadratic term

$$\begin{aligned}
\|V_N(u^2-v^2)\|_{H^{1/2}(\mathbb{B}^5_1)} &\lesssim \|V_N\|_{W^{1/2,10/3}(\mathbb{B}^5_1)}\|u^2-v^2\|_{L^5(\mathbb{B}^5_1)} + \|V_N\|_{L^5(\mathbb{B}^5_1)}\|u^2-v^2\|_{W^{1/2,10/3}(\mathbb{B}^5_1)} \\
&\lesssim \|(u-v)(u+v)\|_{L^5(\mathbb{B}^5_1)} + \|(u-v)(u+v)\|_{W^{1/2,10/3}(\mathbb{B}^5_1)} \\
&\lesssim \|u-v\|_{L^{10}(\mathbb{B}^5_1)}(\|u\|_{L^{10}(\mathbb{B}^5_1)} + \|v\|_{L^{10}(\mathbb{B}^5_1)}) \\
&\quad + \|u-v\|_{W^{1/2,5}(\mathbb{B}^5_1)}(\|u\|_{L^{10}(\mathbb{B}^5_1)} + \|v\|_{L^{10}(\mathbb{B}^5_1)}) \\
&\quad\quad + \|u-v\|_{L^{10}(\mathbb{B}^5_1)}(\|u\|_{W^{1/2,5}(\mathbb{B}^5_1)} + \|v\|_{W^{1/2,5}(\mathbb{B}^5_1)}).
\end{aligned}$$

Next, consider the function $n : \mathbb{R}\times[0,1] \to \mathbb{R}$,

$$n(x,\rho) := 4\int_0^x \cos(4\arctan(\rho)+2\rho t)(x-t)^2\,dt$$

and note that

$$|\partial_1 n(x,\rho)| \lesssim |x|^2, \quad |\partial_2 n(x,\rho)| \lesssim |x|^4, \quad |\partial_1^2 n(x,\rho)| \lesssim |x|, \quad |\partial_1\partial_2 n(x,\rho)| \lesssim |x|^3.$$

Consequently,

$$\begin{aligned}
\|n(u,\cdot)-n(v,\cdot)\|_{L^{5/3}(\mathbb{B}^5_1)} &\lesssim \|(|u|^2+|v|^2)(u-v)\|_{L^{5/3}(\mathbb{B}^5_1)} \\
&\lesssim \|u-v\|_{L^{10}(\mathbb{B}^5_1)}(\|u\|^2_{L^4(\mathbb{B}^5_1)} + \|v\|^2_{L^4(\mathbb{B}^5_1)}),
\end{aligned}$$

as well as

$$\begin{aligned}
\|n(u,\cdot)-n(v,\cdot)\|_{\dot W^{1,5/3}(\mathbb{B}^5_1)} &\lesssim \|\partial_2(n(u,\cdot)-n(v,\cdot))\|_{L^{5/3}(\mathbb{B}^5_1)} + \|u'\partial_1 n(u,\cdot)-v'\partial_1 n(v,\cdot)\|_{L^{5/3}(\mathbb{B}^5_1)} \\
&:= N_1 + N_2.
\end{aligned}$$

For N_1 we obtain

$$N_1 \lesssim \|(|u|^3+|v|^3)|u-v|\|_{L^{5/3}(\mathbb{B}^5_1)} \lesssim \|u-v\|_{L^{10}(\mathbb{B}^5_1)}(\|u\|^3_{L^6(\mathbb{B}^5_1)} + \|v\|^3_{L^6(\mathbb{B}^5_1)}).$$

Further,

$$\begin{aligned}
N_2 &\lesssim \|(u'-v')\partial_1 n(u,\cdot)\|_{L^{5/3}(\mathbb{B}^5_1)} + \|v'(\partial_1 n(u,\cdot)-\partial_1 n(v,\cdot))\|_{L^{5/3}(\mathbb{B}^5_1)} \\
&\lesssim \||u'-v'|u^2\|_{L^{5/3}(\mathbb{B}^5_1)} + \||v'|(|u|+|v|)|u-v|\|_{L^{5/3}(\mathbb{B}^5_1)} \\
&\lesssim \|u'-v'\|_{L^{30/11}(\mathbb{B}^5_1)}\|u\|^2_{L^{60/7}(\mathbb{B}^5_1)} + \|u-v\|_{L^{60/7}(\mathbb{B}^5_1)}\|v'\|_{L^{30/11}(\mathbb{B}^5_1)}(\|u\|_{L^{60/7}(\mathbb{B}^5_1)} + \|u\|_{L^{60/7}(\mathbb{B}^5_1)}). \quad \square
\end{aligned}$$

Motivated by these estimates on the nonlinearity, we define the space $\mathcal{X}$ to be the completion of $C_c^\infty(\mathbb{R}_+ \times \overline{\mathbb{B}_1^5})$ with respect to the norm

$$\|\phi\|_{\mathcal{X}} = \|\phi\|_{L^2(\mathbb{R}_+)L^{10}(\mathbb{B}_1^5)} + \|\phi\|_{L^{12/5}(\mathbb{R}_+)L^{60/7}(\mathbb{B}_1^5)} + \|\phi\|_{L^3(\mathbb{R}_+)L^{15/2}(\mathbb{B}_1^5)} + \|\phi\|_{L^4(\mathbb{R}_+)L^{20/3}(\mathbb{B}_1^5)} + \|\phi\|_{L^2(\mathbb{R}_+)W^{1/2,5}(\mathbb{B}_1^5)} + \|\phi\|_{L^6(\mathbb{R}_+)W^{1,30/11}(\mathbb{B}_1^5)}.$$

Moreover, we set

$$\mathcal{X}_\delta := \{\phi \in \mathcal{X} : \|\phi\|_{\mathcal{X}} \le \delta\}$$

and for $\boldsymbol{u} \in \mathcal{H}$ and $\phi \in C_c^\infty(\mathbb{R}_+ \times \overline{\mathbb{B}_1^5})$ we define

$$\boldsymbol{K_u}(\phi)(\tau) := \left[\boldsymbol{S}(\tau)\boldsymbol{u} + \int_0^\tau \boldsymbol{S}(\tau-\sigma)\boldsymbol{N}((\phi(\sigma),0))\,d\sigma - \boldsymbol{C}(\boldsymbol{u},\phi)(\tau)\right]_1,$$

where the correction term $\boldsymbol{C}$, which we add to suppress the unstable direction induced by the eigenvalue 1, is given by

$$\boldsymbol{C}(\boldsymbol{u},\phi)(\tau) := \boldsymbol{P}\left(e^\tau \boldsymbol{u} + \int_0^\infty e^{\tau-\sigma}\boldsymbol{N}((\phi(\sigma),0))\,d\sigma\right).$$

Lemma 7.2. *We have that $\boldsymbol{K_u}(\phi) \in \mathcal{X}$ for all $\boldsymbol{u} \in \mathcal{H}$ and all $\phi \in C_c^\infty(\mathbb{R}_+ \times \overline{\mathbb{B}_1^5})$. Moreover,*

$$\|\boldsymbol{K_u}(\phi)\|_{\mathcal{X}} \lesssim \|\boldsymbol{u}\|_{\mathcal{H}} + \|\phi\|_{\mathcal{X}}^2 + \|\phi\|_{\mathcal{X}}^4$$

for all $\boldsymbol{u} \in \mathcal{H}$ and all $\phi \in C_c^\infty(\mathbb{R}_+ \times \overline{\mathbb{B}_1^5})$.

Proof. We split $\boldsymbol{K_u}(\phi)$ into

$$\begin{aligned}\boldsymbol{K_u}(\phi) &= \left[(\boldsymbol{I}-\boldsymbol{P})\boldsymbol{S}(\tau)\boldsymbol{u} + (\boldsymbol{I}-\boldsymbol{P})\int_0^\tau \boldsymbol{S}(\tau-\sigma)\boldsymbol{N}((\phi(\sigma),0))\,d\sigma\right]_1 \\ &\quad + \left[\boldsymbol{PS}(\tau)\boldsymbol{u} + \boldsymbol{P}\int_0^\tau \boldsymbol{S}(\tau-\sigma)\boldsymbol{N}((\phi(\sigma),0))\,d\sigma - \boldsymbol{C}(\boldsymbol{u},\phi)(\tau)\right]_1 \\ &=: (\boldsymbol{I}-\boldsymbol{P})\boldsymbol{K_u}(\phi) + \boldsymbol{PK_u}(\phi)\end{aligned}$$

and investigate $(\boldsymbol{I}-\boldsymbol{P})\boldsymbol{K_u}(\phi)$ and $\boldsymbol{PK_u}(\phi)$ separately. For the first one we observe that

$$(\boldsymbol{I}-\boldsymbol{P})\boldsymbol{K_u}(\phi)(\tau) = \left[\boldsymbol{S}(\tau)(\boldsymbol{I}-\boldsymbol{P})\boldsymbol{u} + \int_0^\tau \boldsymbol{S}(\tau-\sigma)(\boldsymbol{I}-\boldsymbol{P})\boldsymbol{N}((\phi(\sigma),0))\,d\sigma\right]_1.$$

Hence, we use Propositions 6.15 and 6.16 to deduce that

$$\begin{aligned}\|(\boldsymbol{I}-\boldsymbol{P})\boldsymbol{K_u}(\phi)\|_{\mathcal{X}} &\lesssim \|\boldsymbol{u}\|_{\mathcal{H}} + \left\|\int_0^\tau [\boldsymbol{S}(\tau-\sigma)(\boldsymbol{I}-\boldsymbol{P})\boldsymbol{N}(\phi(\sigma),0))]_1\,d\sigma\right\|_{\mathcal{X}} \\ &\lesssim \|\boldsymbol{u}\|_{\mathcal{H}} + \int_0^\infty \|N(\phi(\sigma))\|_{H^{1/2}(\mathbb{B}_1^5)}\,d\sigma \\ &\lesssim \|\boldsymbol{u}\|_{\mathcal{H}} + \int_0^\infty \|\phi(\sigma)\|_{W^{1,30/11}(\mathbb{B}_1^5)}\|\phi(\sigma)\|_{L^{60/7}(\mathbb{B}_1^5)}^2 + \|\phi(\sigma)\|_{L^5(\mathbb{B}_1^5)}^3\,d\sigma \\ &\qquad + \int_0^\infty \|\phi(\sigma)\|_{L^{20/3}(\mathbb{B}_1^5)}^4 + \|\phi(\sigma)\|_{L^{10}(\mathbb{B}_1^5)}^2 + \|\phi(\sigma)\|_{W^{1/2,5}(\mathbb{B}_1^5)}\|\phi(\sigma)\|_{L^{10}(\mathbb{B}_1^5)}\,d\sigma \\ &\lesssim \|\boldsymbol{u}\|_{\mathcal{H}} + \|\phi\|_{\mathcal{X}}^2 + \|\phi\|_{\mathcal{X}}^4.\end{aligned}$$

We move on to $\boldsymbol{P}\boldsymbol{K}_{\boldsymbol{u}}(\phi)$, where we first discern that

$$\boldsymbol{P}\boldsymbol{K}_{\boldsymbol{u}}(\phi)(\tau) = \left[\int_\tau^\infty e^{\tau-\sigma}\boldsymbol{P}\boldsymbol{N}((\phi(\sigma),0))\,d\sigma\right]_1.$$

We also remark that as $\boldsymbol{P}$ has rank 1, there exists a unique $\tilde{\boldsymbol{g}} \in \mathcal{H}$ such that $\boldsymbol{P}\boldsymbol{f} = (\boldsymbol{f}, \tilde{\boldsymbol{g}})_{\mathcal{H}}\boldsymbol{g}$ for all $\boldsymbol{f} \in \mathcal{H}$. Hence,

$$\|\boldsymbol{P}\boldsymbol{K}_{\boldsymbol{u}}(\phi)(\tau)\|_{L^p(\mathbb{B}^5_1)} + \|\boldsymbol{P}\boldsymbol{K}_{\boldsymbol{u}}(\phi)(\tau)\|_{W^{1/2,5}(\mathbb{B}^5_1)} + \|\boldsymbol{P}\boldsymbol{K}_{\boldsymbol{u}}(\phi)(\tau)\|_{W^{1,30/11}(\mathbb{B}^5_1)} \lesssim \|N(\phi(\tau))\|_{H^{1/2}(\mathbb{B}^5_1)}$$

for any $2 \le p \le \infty$. So,

$$\|\boldsymbol{P}\boldsymbol{K}_{\boldsymbol{u}}(\phi)(\tau)\|_{L^p_\tau(\mathbb{R}_+)L^q(\mathbb{B}^5_1)} \lesssim \left\|\int_\tau^\infty e^{\tau-\sigma}\|N(\phi(\sigma))\|_{H^{1/2}(\mathbb{B}^5_1)}\,d\sigma\right\|_{L^p_\tau(\mathbb{R}_+)}$$

and Young's inequality implies that

$$\begin{aligned}\|\boldsymbol{P}\boldsymbol{K}_{\boldsymbol{u}}(\phi)(\tau)\|_{L^p_\tau(\mathbb{R}_+)L^q(\mathbb{B}^5_1)} &\lesssim \|N(\phi)\|_{L^1(\mathbb{R}_+)H^1/2(\mathbb{B}^5_1)}\|1_{(-\infty,0]}(\tau)e^\tau\|_{L^p_\tau(\mathbb{R}_+)}\\ &\lesssim \|\phi\|^2_{\mathcal{X}} + \|\phi\|^4_{\mathcal{X}}.\end{aligned}$$

As the remaining spacetime norms can be bounded likewise, one obtains the desired estimate

$$\|\boldsymbol{P}\boldsymbol{K}_{\boldsymbol{u}}(\phi)\|_{\mathcal{X}} \lesssim \|\phi\|^2_{\mathcal{X}} + \|\phi\|^4_{\mathcal{X}}. \qquad \square$$

Lemma 7.3. *The estimate*

$$\|\boldsymbol{K}_{\boldsymbol{u}}(\phi) - \boldsymbol{K}_{\boldsymbol{u}}(\psi)\|_{\mathcal{X}} \lesssim (\|\phi\|_{\mathcal{X}} + \|\phi\|^3_{\mathcal{X}} + \|\psi\|_{\mathcal{X}} + \|\psi\|^3_{\mathcal{X}})\|\phi-\psi\|_{\mathcal{X}}$$

holds for all $\boldsymbol{u} \in \mathcal{H}$ *and all* $\phi, \psi \in C^\infty_c(\mathbb{R}_+ \times \overline{\mathbb{B}^5_1})$.

Proof. Invoking Propositions 6.15 and 6.16 yields

$$\begin{aligned}&\|(\boldsymbol{I}-\boldsymbol{P})(\boldsymbol{K}_{\boldsymbol{u}}(\phi) - \boldsymbol{K}_{\boldsymbol{u}}(\psi))\|_{\mathcal{X}}\\
&\lesssim \int_0^\infty \|N(\phi(\sigma)) - N(\psi(\sigma))\|_{H^{1/2}(\mathbb{B}^5_1)}\,d\sigma\\
&\lesssim \int_0^\infty \|\phi(\sigma)-\psi(\sigma)\|_{L^{10}(\mathbb{B}^5_1)}(\|\phi(\sigma)\|_{L^{10}(\mathbb{B}^5_1)} + \|\psi(\sigma)\|_{L^{10}(\mathbb{B}^5_1)})\\
&\quad + \|\phi(\sigma)-\psi(\sigma)\|_{L^{10}(\mathbb{B}^5_1)}(\|\phi(\sigma)\|^2_{L^{20/3}(\mathbb{B}^5_1)} + \|\psi(\sigma)\|^2_{L^{20/3}(\mathbb{B}^5_1)})\\
&\quad + \|\phi(\sigma)-\psi(\sigma)\|_{W^{1/2,5}(\mathbb{B}^5_1)}(\|\phi(\sigma)\|_{L^{10}(\mathbb{B}^5_1)} + \|\psi(\sigma)\|_{L^{10}(\mathbb{B}^5_1)})\\
&\quad + \|\phi(\sigma)-\psi(\sigma)\|_{L^{10}(\mathbb{B}^5_1)}(\|\phi(\sigma)\|_{W^{1/2,5}(\mathbb{B}^5_1)} + \|\psi(\sigma)\|_{W^{1/2,5}(\mathbb{B}^5_1)})\\
&\quad + \|\phi(\sigma)-\psi(\sigma)\|_{L^{10}(\mathbb{B}^5_1)}(\|\phi(\sigma)\|^3_{L^6(\mathbb{B}^5_1)} + \|\psi(\sigma)\|^3_{L^6(\mathbb{B}^5_1)})\\
&\quad + \|\phi(\sigma)-\psi(\sigma)\|_{W^{1,30/11}(\mathbb{B}^5_1)}\|\phi(\sigma)\|^2_{L^{60/7}(\mathbb{B}^5_1)}\\
&\quad + \|\phi(\sigma)-\psi(\sigma)\|_{L^{60/7}(\mathbb{B}^5_1)}\|\psi(\sigma)\|_{W^{1,30/11}(\mathbb{B}^5_1)}(\|\phi(\sigma)\|_{L^{60/7}(\mathbb{B}^5_1)} + \|\psi(\sigma)\|_{L^{60/7}(\mathbb{B}^5_1)})\,d\sigma\\
&\lesssim \|\phi-\psi\|_{\mathcal{X}}(\|\phi\|_{\mathcal{X}} + \|\psi\|_{\mathcal{X}} + \|\phi\|^3_{\mathcal{X}} + \|\psi\|^3_{\mathcal{X}}).\end{aligned}$$

Estimating $\boldsymbol{P}(\boldsymbol{K}_{\boldsymbol{u}}(\phi) - \boldsymbol{K}_{\boldsymbol{u}}(\psi))$ can be done by employing the same strategy as in the proof of Lemma 7.2. $\square$

The last two lemmas combined with an application of the contraction mapping principle yield the next result.

Lemma 7.4. *For any $\boldsymbol{u} \in \mathcal{H}$ fixed, the operator $\boldsymbol{K_u}$ extends to an operator on all of $\mathcal{X}$. Moreover, there exist $\delta > 0$ and $C > 1$ such that there exists a unique $\phi \in \mathcal{X}_\delta$ with*

$$\boldsymbol{K_u}(\phi) = \phi$$

whenever $\|\boldsymbol{u}\|_{\mathcal{H}} \leq \delta/C$.

7.1. ***Proof of Theorem 1.2.*** To prove Theorem 1.2 we still have to get rid of the correction term $\boldsymbol{C}$. We achieve this by picking the right blowup time T close to 1. For this, we recall that the prescribed initial data

$$\Phi(0) = (\phi_1(0, \cdot), \phi_2(0, \cdot))$$

are given by

$$\phi_1(0,\rho) = \psi_1(0,\rho) - \frac{2\arctan(\rho)}{\rho} = Tf(T\rho) - \frac{2\arctan(\rho)}{\rho},$$
$$\phi_2(0,\rho) = \psi_2(0,\rho) - \frac{2}{1+\rho^2} = T^2 g(T\rho) - \frac{2}{1+\rho^2}.$$

Furthermore, $u_*^1[0]$ transformed to similarity coordinates is given by

$$\psi^1_{1_*}(0,\rho) = \frac{2\arctan(T\rho)}{\rho}, \quad \psi^1_{2_*}(0,\rho) = \frac{2T^2}{1+T^2\rho^2}.$$

This explicit dependence of T of the initial data motivates the definition of the operator

$$\boldsymbol{U} : [1-\delta, 1+\delta] \times (H^{\frac{3}{2}} \times H^{\frac{1}{2}})(\mathbb{B}^5_{1+\delta}) \to \mathcal{H}$$

by

$$\boldsymbol{U}(T,\boldsymbol{v})(\rho) = (Tv_1(T\rho), T^2 v_2(T\rho)) + (\psi^1_{1_*}(0,\rho), \psi^1_{2_*}(0,\rho)) - \left(\frac{2\arctan(\rho)}{\rho}, \frac{2}{1+\rho^2}\right).$$

Note that for $\delta \in \left(0, \frac{1}{2}\right)$ and any $\boldsymbol{v}$ fixed, this defines a continuous map

$$U(\cdot, \boldsymbol{v}) : [1-\delta, 1+\delta] \to \mathcal{H}$$

(this follows as the first part of Lemma 8.2 in [Glogić 2025]). Also, the two identities

$$\boldsymbol{U}(1, \boldsymbol{0}) = \boldsymbol{0}$$

and

$$\Phi(0,\rho) = \boldsymbol{U}\left(T, \left(f(\rho) - \frac{2\arctan(\rho)}{\rho}, g(\rho) - \frac{2}{1+\rho^2}\right)\right)$$

hold. Furthermore, by arguing as in the proof of Lemma 8.2 in [Glogić 2025], one shows that the estimate

$$\|\boldsymbol{U}(T,\boldsymbol{v})\|_{\mathcal{H}} \lesssim \|\boldsymbol{v}\|_{H^{3/2}\times H^{1/2}(\mathbb{B}^5_{1+\delta})} + |1-T|$$

is true for all $T \in [1-\delta, 1+\delta]$.

Lemma 7.5. *There exist constants $M \geq 1$ and $\delta > 0$ such that if $\boldsymbol{v} \in H^{3/2} \times H^{1/2}(\mathbb{B}^5_{1+\delta})$ satisfies $\|\boldsymbol{v}\|_{H^{3/2}\times H^{1/2}(\mathbb{B}^5_{1+\delta})} \leq \delta/M$, then there exists a unique $T^* \in [1-\delta, 1+\delta]$ and a unique $\phi \in \mathcal{X}_\delta$ with $\phi = \boldsymbol{K}_{\boldsymbol{U}(T^*,\boldsymbol{v})}(\phi)$ and $\boldsymbol{C}(\phi, \boldsymbol{U}(T^*, \boldsymbol{v})) = 0$.*

Proof. Since

$$\partial_T\left(\frac{2\arctan(T\rho)}{\rho}, \frac{2T^2}{1+T^2\rho^2}\right)\bigg|_{T=1} = 2\boldsymbol{g}(\rho),$$

the claim follows by an application of Brouwer's fixed-point theorem; see the proof of Lemma 6.5 in [Donninger 2017] for the details. □

This allows us to give rigorous meaning to the notion of solutions in our topology.

Definition 7.6. Let

$$\Gamma^T := \{(t,r) \in [0,T) \times [0,\infty) : r \le T-t\}.$$

We say that $u : \Gamma^T \to \mathbb{R}$ is a Strichartz solution of

$$\Big(\partial_t^2 - \partial_r^2 - \frac{4}{r}\partial_r\Big)u(t,r) + \frac{\sin(2ru(t,r)) - 2ru(t,r)}{r^3} = 0$$

if $\phi = \Phi_1 := [\Psi - \Psi_*]_1$, with

$$\Psi(\tau,\rho) := \begin{pmatrix} \psi(\tau,\rho) \\ (1+\partial_\tau+\rho\partial_\rho)\psi(\tau,\rho)\end{pmatrix}, \quad \psi(\tau,\rho) := Te^{-\tau}u(T-Te^{-\tau}, Te^{-\tau}\rho),$$

belongs to $\mathcal{X}$ and satisfies

$$\phi = \boldsymbol{K}_{\Phi(0)}(\phi)$$

and $\boldsymbol{C}(\phi, \Phi(0)) = \boldsymbol{0}$.

Proof of Theorem 1.2. Let $\delta > 0$ be small enough, choose $M \ge 0$ sufficiently large, and, let $\boldsymbol{v} = (f,g) - u_*^1[0] \in C^\infty \times C^\infty(\overline{\mathbb{B}^5_{1+\delta}})$ be such that

$$\|(f,g) - u_*^1[0]\|_{H^{3/2}\times H^{1/2}(\mathbb{B}^5_{1+\delta})} \le \frac{\delta}{M}.$$

Then, by Lemmas 7.4 and 7.5 there exists a Strichartz solution u with that initial data. Therefore, the associated ϕ is the unique fixed point of $\boldsymbol{K}$ in $\mathcal{X}_\delta$ with vanishing correction term. Moreover, by standard partition arguments one shows that this ϕ is in fact the unique fixed point in all of $\mathcal{X}$; see for instance [Donninger and Wallauch 2023, Lemma 7.6]. Furthermore, by classical Gronwall-type arguments one shows that u is in fact a smooth function on Γ^T, where T denotes the blowup time. We calculate

$$\begin{aligned}
\delta^2 \ge \|\phi\|^2_{L^2(\mathbb{R}_+)L^{10}(\mathbb{B}^5_1)} &= \int_0^\infty \|\psi(\tau,\cdot) - 2|\cdot|^{-1}\arctan(|\cdot|)\|^2_{L^{10}(\mathbb{B}^5_1)}\,d\tau \\
&= \int_0^T \|\psi(-\log(T-t)+\log T,\cdot) - 2|\cdot|^{-1}\arctan(|\cdot|)\|^2_{L^{10}(\mathbb{B}^5_1)}\frac{dt}{T-t} \\
&= \int_0^T \left\|(T-t)^{-1}\psi\Big(-\log(T-t)+\log T, \frac{\cdot}{T-t}\Big) - 2|\cdot|^{-1}\arctan\Big(\frac{|\cdot|}{T-t}\Big)\right\|^2_{L^{10}(\mathbb{B}^5_{T-t})} dt \\
&= \int_0^T \|u(t,\cdot) - u_*^T(t,r)\|^2_{L^{10}(\mathbb{B}^5_{T-t})}\,dt
\end{aligned}$$

and similarly one computes

$$\delta^6 \geq \int_0^T \|u(t,\cdot)-u_*^T(t,\cdot)\|^6_{\dot W^{1,30/11}(\mathbb{B}^5_{T-t})}\,dt, \tag{7-1}$$

completing the proof. □

Proof of Theorem 1.1. Establishing Theorem 1.1 reduces to two tasks. First one needs to prove that u can be extended to all of

$$\Omega^5_T := ([0,\infty)\times\mathbb{R}^5)\setminus\{(t,x)\in[T,\infty)\times\mathbb{R}^5 : |x|\leq t-T\}.$$

This is a consequence of N being a smooth bounded function away from $r=0$ and we refer the reader to Section 2 and Lemma 8.3 of [Donninger and Wallauch 2023], where this was done for one dimension higher. Secondly, one has to show that all estimates on u ascend to estimates on

$$U_u(t,\cdot) = \begin{pmatrix} \sin(|\cdot|(u(t,\cdot))\frac{\cdot}{|\cdot|} \\ \cos(|\cdot|u(t,\cdot)) \end{pmatrix}.$$

This procedure was also carried out for $d=4$ in Section 8 of [Donninger and Wallauch 2023] and can be adapted in a straightforward way to the three-dimensional case. □

Appendix: Interpolation theory

This appendix is concerned with our required interpolation result for weighted Strichartz spaces. The presentation given here is based on the book by J. Bergh and J. Löfström [1976]. Following this reference, we let (X_0, X_1) be a tuple of Banach spaces out of which we form the Banach space $(X_0+X_1, \|\cdot\|_{X_0+X_1})$, where

$$\|x\|_{X_0+X_1} := \inf_{x=x_0+x_1,\,x_j\in X_j,\,j=1,2}(\|x_0\|_{X_0}+\|x_1\|_{X_1})$$

for $x\in X_0+X_1$. We now set $S:=\{z\in\mathbb{C}: 0\leq z\leq 1\}$ and consider the set $F(X_0,X_1)$ consisting of all continuous functions $f:S\to X_0+X_1$ that are analytic on the interior of S. Moreover, for f to be an element of $F(X_0,X_1)$, we require the function $t\mapsto f(j+it)$, for $j=0,1$, to be a continuous function from $\mathbb{R}$ to X_j which tends to 0 as $|t|\to\infty$. Then, $F(X_0,X_1)$ is a vector space and by equipping it with the norm

$$\|f\|_{F(X_0,X_1)} := \max\Bigl\{\sup_{t\in\mathbb{R}}\|f(it)\|_{X_0}, \sup_{t\in\mathbb{R}}\|f(1+it)\|_{X_1}\Bigr\}$$

it becomes a Banach space; see [Bergh and Löfström 1976, p. 88, Lemma 4.1.1]. Next, for $\theta\in(0,1)$, we define the interpolation functor C_θ as follows. Let $(X_0,X_1)_{[\theta]} = C_\theta(X_0,X_1)$ be the set of all $x\in X_0+X_1$ for which there exists an $f\in F(X_0,X_1)$ with $f(\theta)=x$. Furthermore, for any such x we set

$$\|x\|_{(X_0,X_1)_{[\theta]}} := \inf\{\|f\|_{F(X_0,X_1)} : f\in F(X_0,X_1),\ f(\theta)=x\}.$$

Then, $((X_0,X_1)_{[\theta]}, \|\cdot\|_{(X_0,X_1)_{[\theta]}})$ is Banach space and C_θ is an exact interpolation functor of order θ (see [Bergh and Löfström 1976, p. 88, Theorem 4.1.2.]). Moreover, for a given Sobolev norm $\|\cdot\|_{W^{s,q}(\mathbb{B}^5_1)}$, with $s \geq 0$ and $1 \leq q \leq \infty$, as well as $a \in \mathbb{R}$, we let $L^p(\mathbb{R}_+, e^{a\tau}\,d\tau)W^{s,q}(\mathbb{B}^5_1)$ with $1 \leq p < \infty$ be the completion of $C^\infty_c(\mathbb{R}_+ \times \overline{\mathbb{B}^5_1})$ with respect to the norm

$$\|f\|^p_{L^p(\mathbb{R}_+, e^{a\tau}\,d\tau)W^{s,q}(\mathbb{B}^5_1)} := \int_{\mathbb{R}_+} \|f(\tau,\cdot)\|^p_{W^{s,q}(\mathbb{B}^5_1)} e^{a\tau}\,d\tau.$$

Finally, we once more employ [Triebel 1995, p. 317, Section 4.3.1.1, Theorem 1] to infer that for any $1 \leq p, q_0, q_1 \leq \infty$ and $0 \leq s_0, s_1 < \infty$ one has that

$$(W^{s_0,q_0}(\mathbb{B}^5_1), W^{s_1,q_1}(\mathbb{B}^5_1))_{[\frac{1}{2}]} = W^{s_{1/2},q_{1/2}}(\mathbb{B}^5_1),$$

where $s_{1/2} = \frac{1}{2}(s_0 + s_2)$ and $1/q_{1/2} = \frac{1}{2}(1/q_0 + 1/q_1)$. Having concluded these preliminaries, we come to the desired interpolation result.

Proposition A.1. *Let* $1 \leq q_0, q_1 \leq \infty$, $0 \leq s_0, s_1 < \infty$, $1 \leq p_0, p_1 < \infty$, *and* $a \in \mathbb{R}$. *Then*

$$\big(L^{p_0}(\mathbb{R}_+, e^{-ap_0\tau}\,d\tau)W^{s_0,q_0}(\mathbb{B}^5_1), L^{p_1}(\mathbb{R}_+, e^{ap_1\tau}\,d\tau)W^{s_1,q_1}(\mathbb{B}^5_1)\big)_{[\frac{1}{2}]} = L^{p_{1/2}}(\mathbb{R}_+)W^{s_{1/2},q_{1/2}}(\mathbb{B}^5_1).$$

Proof. The proposition follows by slightly modifying the ideas of [Bergh and Löfström 1976, p. 107, Theorem 5.1.2], which we illustrate here for the convenience of the reader. To simplify notation, we set $W_0 = W^{s_0,q_0}(\mathbb{B}^5_1)$, $W_1 = W^{s_1,q_1}(\mathbb{B}^5_1)$ and $p = p_{1/2}$. By construction, $C^\infty_c(\mathbb{R}_+ \times \overline{\mathbb{B}^5_1})$ lies dense in $L^{p_0}(\mathbb{R}_+, e^{-ap_0\tau}\,d\tau)W_0 \cap L^{p_1}(\mathbb{R}_+, e^{ap_1\tau}\,d\tau)W_1$ and so by [Bergh and Löfström 1976, p. 91, Theorem 4.2.2] also in

$$\big(L^{p_0}(\mathbb{R}_+, e^{-ap_0\tau}\,d\tau)W_0, L^{p_1}(\mathbb{R}_+, e^{ap_1\tau}\,d\tau)W_1\big)_{[\frac{1}{2}]} \quad \text{and} \quad L^p(\mathbb{R}_+)(W_0, W_1)_{[\frac{1}{2}]}.$$

Consequently, it suffices to consider $C^\infty_c(\mathbb{R}_+ \times \overline{\mathbb{B}^5_1})$. We start with the inequality

$$\|u\|_{(L^{p_0}(\mathbb{R}_+, e^{-ap_0\tau}\,d\tau)W_0, L^{p_1}(\mathbb{R}_+, e^{ap_1\tau}\,d\tau)W_1)_{[1/2]}} \leq \|u\|_{L^p(\mathbb{R}_+)(W_0,W_1)_{[1/2]}}.$$

Let $u \in C^\infty_c(\mathbb{R}_+ \times \overline{\mathbb{B}^5_1})$ with $u \neq 0$. Then, for every $\varepsilon > 0$ and every $\tau \geq 0$, there exists an $f(\tau) \in F(W_0, W_1)$ with $f(\tau)\big(\frac{1}{2}\big) = u(\tau,\cdot)$ and

$$\|f(\tau)\|_{F(W_0,W_1)} \leq (1+\varepsilon)\|u(\tau,\cdot)\|_{(W_0,W_1)_{[1/2]}}.$$

Set

$$g(\tau)(z) = f(\tau)(z)e^{2a(\frac{1}{2}-z)\tau}\left(\frac{\|u(\tau)\|_{(W_0,W_1)_{[1/2]}}}{\|u\|_{L^p(\mathbb{R}_+)(W_0,W_1)_{[1/2]}}}\right)^{p(\frac{1}{p_0}-\frac{1}{p_1})(\frac{1}{2}-z)}.$$

Then, clearly $g(\tau)\big(\frac{1}{2}\big) = u(\tau,\cdot)$ and since

$$p_0 + p_0\frac{p}{2}\left(\frac{1}{p_0} - \frac{1}{p_1}\right) = p$$

one readily computes that

$$\begin{aligned}
\|g(\tau)(it)\|^{p_0}_{L^{p_0}(\mathbb{R}_+, e^{-ap_0\tau}\,d\tau)W_0} &= \int_{\mathbb{R}_+} \|f(\tau)(it)\|^{p_0}_{W_0}\left(\frac{\|u(\tau)\|_{(W_0,W_1)_{[1/2]}}}{\|u\|_{L^p(\mathbb{R}_+)(W_0,W_1)_{[1/2]}}}\right)^{p_0\frac{p}{2}(\frac{1}{p_0}-\frac{1}{p_1})} d\tau \\
&\leq (1+\varepsilon)^{p_0}\|u\|^{-p_0\frac{p}{2}(\frac{1}{p_0}-\frac{1}{p_1})}_{L^p(\mathbb{R}_+)(W_0,W_1)_{[1/2]}} \int_{\mathbb{R}_+} \|u(\tau,\cdot)\|^p_{(W_0,W_1)_{[1/2]}}\,d\tau \\
&= (1+\varepsilon)^{p_0}\|u\|^{p_0}_{L^p(\mathbb{R}_+)(W_0,W_1)_{[1/2]}}
\end{aligned}$$

and similarly

$$\|g(\tau)(1+it)\|^{p_1}_{L^{p_1}(\mathbb{R}_+,e^{ap_1\tau}\,d\tau)W_1} \le (1+\varepsilon)^p \|u\|^{p_1}_{L^p(\mathbb{R}_+)(W_0,W_1)_{[1/2]}}.$$

Hence, as $\varepsilon > 0$ was chosen arbitrarily, the claim follows.

For the other inequality, we invoke [Bergh and Löfström 1976, p. 93, Lemma 4.3.2], which states that any $f \in F(W_0, W_1)$ satisfies

$$\|f\|_{(W_0,W_1)_{[\theta]}} \le \left(\frac{1}{1-\theta}\int_{\mathbb{R}} \|f(it)\|_{W_0} P_0(\theta,t)\,dt\right)^{1-\theta} \left(\frac{1}{\theta}\int_{\mathbb{R}} \|f(1+it)\|_{W_1} P_1(\theta,t)\,dt\right)^{\theta}, \tag{A-1}$$

where

$$P_j(x+iy,t) := \frac{e^{-\pi(t-y)}\sin(\pi x)}{\sin(\pi x)^2 + (\cos(\pi x) - e^{ij\pi-\pi(t-y)})^2}$$

are the Poisson kernels of the strip S. Further, for $u \in C^\infty_c(\mathbb{R}_+ \times \overline{\mathbb{B}^5_1})$ let $f(\tau) \in F(W_0, W_1)$ be such that $f(\tau)\big(\frac{1}{2}\big) = u(\tau,\cdot)$. Then, (A-1), Hölder's inequality, and the identity $1/p = 1/(2p_0) + 1/(2p_1)$ imply that

$$\begin{aligned}
&\|u\|_{L^p(\mathbb{R}_+)(W_0,W_1)_{[1/2]}} \\
&\quad\le 4\left\|\int_{\mathbb{R}_+}\left(\int_{\mathbb{R}} \|f(\tau)(it)\|_{W_0} P_0\big(\tfrac{1}{2},t\big)\,dt\right)^{\frac{1}{2}}\left(\int_{\mathbb{R}} \|f(\tau)(1+it)\|_{W_1} P_1\big(\tfrac{1}{2},t\big)\,dt\right)^{\frac{1}{2}}\right\|_{L^p_\tau(\mathbb{R}_+)} \\
&\quad\le 4\left\|e^{-a\tau}\int_{\mathbb{R}} \|f(\tau)(it)\|_{W_0} P_0\big(\tfrac{1}{2},t\big)\,dt\right\|^{\frac{1}{2}}_{L^{p_0}_\tau(\mathbb{R}_+)} \left\|e^{a\tau}\int_{\mathbb{R}} \|f(\tau)(1+it)\|_{W_1} P_1\big(\tfrac{1}{2},t\big)\,dt\right\|^{\frac{1}{2}}_{L^{p_1}_\tau(\mathbb{R}_+)}.
\end{aligned}$$

Next, by Minkowski's inequality

$$\begin{aligned}
\left\|\int_{\mathbb{R}} \|f(\tau)(it)\|_{W_0} P_0\big(\tfrac{1}{2},t\big)\,dt\,e^{-a\tau}\right\|_{L^{p_0}_\tau(\mathbb{R}_+)} &\le \int_{\mathbb{R}} \big\|\cdot\|f(\tau)(it)\|_{W_0} e^{-a\tau}\big\|_{L^{p_0}_\tau(\mathbb{R}_+)} P_0\big(\tfrac{1}{2},t\big)\,dt \\
&\le \sup_{t\in\mathbb{R}} \|f(\tau)(it)\|_{L^{p_0}(\mathbb{R}_+,e^{-ap_0\tau}\,d\tau)W_0} \int_{\mathbb{R}} P_0\big(\tfrac{1}{2},t\big)\,dt
\end{aligned}$$

and analogously one estimates the second factor. Observe now that for $j = 0, 1$

$$\int_{\mathbb{R}} P_j\big(\tfrac{1}{2},t\big)\,dt = \int_{\mathbb{R}} \frac{e^{-\pi t}}{1+e^{-2\pi t}}\,dt = \frac{1}{2}.$$

Therefore,

$$\begin{aligned}
\|u\|^p_{L^p(\mathbb{R}_+)(W_0,W_1)_{[1/2]}} &\le \sup_{t\in\mathbb{R}} \|f(\tau)(it)\|^{\frac{1}{2}}_{L^{p_0}(\mathbb{R}_+,e^{-ap_0\tau}\,d\tau)W_0} \sup_{t\in\mathbb{R}} \|f(\tau)(1+it)\|^{\frac{1}{2}}_{L^{p_1}(\mathbb{R}_+,e^{ap_1\tau}\,d\tau)W_1} \\
&\le \|f\|_{F(L^p(\mathbb{R}_+,e^{-ap_0\tau}\,d\tau)W_0,\,L^p(\mathbb{R}_+,e^{ap_1\tau}\,d\tau)W_1)}.
\end{aligned}$$

□

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Received 21 Dec 2022. Revised 3 Oct 2023. Accepted 21 Dec 2023.

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ANALYSIS AND PDE
Vol. 18 (2025), No. 4, pp. 963–1035
DOI: 10.2140/apde.2025.18.963

QUANTITATIVE STABILITY OF GEL'FAND'S INVERSE BOUNDARY PROBLEM

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Dedicated to the memory of Yaroslav Kurylev

In Gel'fand's inverse problem, one aims to determine the topology, differential structure and Riemannian metric of a compact manifold M with boundary from the knowledge of the boundary ∂M, the Neumann eigenvalues λ_j and the boundary values of the eigenfunctions $\varphi_j|_{\partial M}$. We show that this problem has a stable solution with quantitative stability estimates in a class of manifolds with bounded geometry. More precisely, we show that finitely many eigenvalues and the boundary values of corresponding eigenfunctions, known up to small errors, determine a metric space that is close to the manifold in the Gromov–Hausdorff sense. We provide an algorithm to construct this metric space. This result is based on an explicit estimate on the stability of the unique continuation for the wave operator.

1. Introduction

Gel'fand's inverse problem, formulated by I. Gel'fand [1957], concerns finding the topology, differential structure and Riemannian metric of a compact manifold with boundary from the spectral data for the Neumann Laplacian on the boundary, that is, the Neumann eigenvalues and the boundary values of the corresponding eigenfunctions. The problem is closely related to an inverse problem for the wave equation that can be solved using the boundary control method developed by Belishev [1987] on domains of $\mathbb{R}^n$. The uniqueness of Gel'fand's inverse problem on manifolds was proved in 1992 by Belishev and Kurylev [1992], see also [Anderson et al. 2004; Belishev 2007; 2017; Caday et al. 2019; Krupchyk et al. 2008; Kurylev et al. 2018], in the form of an inverse spectral problem: the geometry of a compact Riemannian manifold with boundary is uniquely determined by the boundary spectral data for the Neumann Laplacian.

MSC2020: 35R30, 58J50, 53C21, 58J45.
Keywords: Gel'fand's inverse problem, stability, quantitative unique continuation, wave operator, boundary control method.

On a given domain of the Euclidean space, Gel'fand's problem was reduced in [Nachman et al. 1988] to inverse coefficient problems for elliptic equations which were solved in [Astala and Päivärinta 2006; Nachman 1988; 1996; Sylvester and Uhlmann 1987], see also [Dos Santos Ferreira et al. 2009; Guillarmou and Tzou 2011; Isozaki 2004; Kenig and Salo 2013; Kenig et al. 2007; Uhlmann 1998], and the stability of the solutions of these problems has been studied in [Alessandrini 1988; Alessandrini and Sylvester 1990; Sylvester and Uhlmann 1988]. Gel'fand's inverse problem is ill-posed in the sense of Hadamard, as one can make large changes to the geometry of the interior without affecting the boundary spectral data much. One approach of stabilizing the inverse problem is to study the conditional stability by assuming a priori knowledge of the desired quantities, for instance higher regularity of coefficients [Alessandrini 1988], and higher regularity of Riemannian metrics if they are close to Euclidean [Stefanov and Uhlmann 1998]. For a general Riemannian manifold, it is natural to impose a priori bounds on geometric parameters such as the diameter, injectivity radius and sectional curvature. An abstract continuity result for the stability of the problem was proved in [Anderson et al. 2004], however with no stability estimates, and the related determination of the smooth structure was shown in [Fefferman et al. 2020]. With additional geometric assumptions, strong stability estimates for this problem can be obtained, e.g., [Bellassoued and Dos Santos Ferreira 2011; Stefanov and Uhlmann 2005], when the metric is close to simple (i.e., with strictly convex boundary and no conjugate points). One could also consider the inverse interior problem, that is, an inverse problem on closed manifolds analogous to Gel'fand's problem. For the inverse interior problem where the eigenfunctions are measured in a ball of a closed manifold, the unique solvability of the problem was proved in [Krupchyk et al. 2008] and a quantitative stability estimate for general metric has recently been obtained in [Bosi et al. 2022]. A quantitative stability of Gel'fand's inverse problem for manifolds with boundary in the general case was yet unknown. The main purpose of the present paper is to provide an answer to this question.

The key result for establishing the uniqueness of Gel'fand's inverse problem was Tataru's unique continuation theorem [1995] for the wave operator. Its stability, i.e., quantitative unique continuation, is essential to the stability of the inverse problem. The quantitative unique continuation for the wave operator on Riemannian manifolds, from sets of the form $\Gamma \times [-T, T]$, where Γ is the observation region, has been investigated independently in [Bosi et al. 2016; 2018] for closed manifolds, and in [Laurent and Léautaud 2019] when T is larger than the diameter of the manifold. Using [Bosi et al. 2016; 2018], the authors established a log-log type of stability estimate [Bosi et al. 2022] for the analogous inverse problem on a closed manifold where spectral data are measured in a ball. However, for manifolds with boundary, the quantitative unique continuation for arbitrary time T is yet unclear, partly due to the lack of smoothness caused by geodesics touching the boundary. This brought substantial difficulty into propagating the local unique continuation to a global one without losing any domain of dependence. It turns out that it is beneficial for us to treat these geodesics as distance-minimizing paths in Alexandrov spaces with curvature bounded above, instead of handling them in boundary normal coordinates. As our main technical task occupying most of Sections 3 and 6, we focus on geometric issues brought by geodesics near the boundary, and give a fully explicit stability estimate for the unique continuation in the optimal domain of dependence. Our result also makes it possible to obtain quantitative stability of other inverse problems that are solved using the boundary control method.

We hope our results may have applications in medicine, especially to cancer treatment, more concretely, to imaging necessary for radiation therapy (e.g., the navigation of cyber knives) and for ultrasound surgery, see, e.g., [Western et al. 2015]. In these treatments, many thin beams of X-rays or high-amplitude ultrasound waves are concentrated in the cancerous tissue and the planning of the treatment requires stable imaging methods. A significant potential instance is the focused ultrasound surgery [Tempany et al. 2011], where a cancerous tissue is destroyed by an excessive heat dose generated by focused ultrasound waves. The location where the ultrasound waves are focused is determined by the intrinsic Riemannian metric corresponding to the wave speed of acoustic waves; see [Dahl et al. 2009; Lassas 2018]. In particular, in an anisotropic medium where the inverse problem is not uniquely solvable in Euclidean coordinates, see [Sylvester 1990], it is beneficial to do imaging in the same Riemannian structure that determines the wave propagation. The imaging of the Riemannian metric associated with the wave propagation is an inverse problem for the wave equation, which is equivalent, see [Katchalov et al. 2004], to Gel'fand's inverse problem studied in this paper. Numerical methods to solve these problems have been studied in [de Hoop et al. 2016; 2018]. The quantitative stability of reconstruction from other types of data, e.g., the Dirichlet-to-Neumann map or the source-to-solution map for the wave equation, has not yet been studied; however, in the light of [Bosi et al. 2022; Katchalov et al. 2004], we think a similar stability estimate might be possible.

Let (M, g) be a compact, connected, orientable Riemannian manifold of dimension $n \geqslant 2$ with smooth boundary ∂M. We consider the manifold M in the class $\mathcal{M}_n(D, K_1, K_2, i_0, r_0)$ of bounded geometry defined by the bounds on the diameter $\operatorname{diam}(M)$, the injectivity radius $\operatorname{inj}(M)$, the Riemannian curvature tensor R_M of M, and the second fundamental form S of the boundary ∂M embedded in M:

$$\begin{aligned} \operatorname{diam}(M) \leqslant D, &\quad \operatorname{inj}(M) \geqslant i_0, \\ \|R_M\|_{C^0} \leqslant K_1^2, &\quad \|S\|_{C^0} \leqslant K_1, \\ \sum_{i=1}^{5} \|\nabla^i R_M\|_{C^0} \leqslant K_2, &\quad \sum_{i=1}^{4} \|\nabla^i S\|_{C^0} \leqslant K_2, \end{aligned} \tag{1-1}$$

where ∇^i denotes the i-th covariant derivative on M. The injectivity radius for a manifold with boundary is defined in Section 2.1. In addition, we impose the lower bound on the following quantity $r_{\mathrm{CAT}}(M)$ (Definition 2.1):

$$r_{\mathrm{CAT}}(M) \geqslant r_0, \tag{1-2}$$

where $r_{\mathrm{CAT}}(M)$ is defined as the largest number r such that any pair of points with distance less than r is connected by a unique distance-minimizing geodesic (possibly touching the boundary) of M. This quantity is known to be positive for a compact Riemannian manifold with smooth boundary. For Riemannian manifolds without boundary, the condition (1-2) is already incorporated in the lower bound for the injectivity radius.

Denote by λ_j $(j \geqslant 1)$ the j-th eigenvalue of the (nonnegative) Laplace–Beltrami operator $-\Delta_g$ on (M, g) with the Neumann boundary condition at ∂M, and by φ_j a (smooth) eigenfunction with respect to λ_j. We know that $0 = \lambda_1 < \lambda_2 \leqslant \cdots \leqslant \lambda_j \leqslant \lambda_{j+1} \leqslant \cdots$, and $\lambda_j \to +\infty$ as $j \to +\infty$. Assume the eigenfunctions are orthonormalized with respect to the L^2-norm of M. In particular $\varphi_1 = \operatorname{vol}_n(M)^{-1/2}$.

The *Neumann boundary spectral data* of M refers to the collection of data

$$(\partial M, g_{\partial M}, \{\lambda_j, \varphi_j|_{\partial M}\}_{j=1}^{\infty}),$$

which consists of the boundary ∂M and its intrinsic metric $g_{\partial M}$, the Neumann eigenvalues and the boundary values of a choice of orthonormalized Neumann eigenfunctions.

Definition 1.1. We say a collection of data $(\partial M, g_{\partial M}, \{\lambda_j^a, \varphi_j^a|_{\partial M}\}_{j=1}^{J})$ is a δ-approximation of the Neumann boundary spectral data of (M, g) (in C^2) for some $\delta \geqslant J^{-1}$ if there exists a choice of Neumann boundary spectral data $\{\lambda_j, \varphi_j|_{\partial M}\}_{j=1}^{\infty}$ such that the following three conditions are satisfied for all $j \leqslant \delta^{-1}$:

(1) $\lambda_j^a \in [0, \infty)$, $\varphi_j^a|_{\partial M} \in C^2(\partial M)$.

(2) $|\sqrt{\lambda_j} - \sqrt{\lambda_j^a}| < \delta$.

(3) $\|\varphi_j - \varphi_j^a\|_{C^{0,1}(\partial M)} + \|\nabla^2_{\partial M}(\varphi_j - \varphi_j^a)|_{\partial M}\|_{C^0} < \delta$, where $\nabla^2_{\partial M}$ denotes the second covariant derivative with respect to the induced metric $g_{\partial M}$ on ∂M.

Let M_1, M_2 be two Riemannian manifolds with isometric boundaries, and let $\Phi : \partial M_1 \to \partial M_2$ be the Riemannian isometry (diffeomorphism) between boundaries. We say the Neumann boundary spectral data of M_1, M_2 are δ-close if the pull-back via Φ of the Neumann boundary spectral data of M_2 (or M_1) is a δ-approximation of the Neumann boundary spectral data of M_1 (or M_2).

Note that the definition above is coordinate-free. The second covariant derivative of a function is called the Hessian of the function, which is a symmetric $(0, 2)$-tensor. In a local coordinate on ∂M, Definition 1.1(3) translates to $(\varphi_j - \varphi_j^a)|_{\partial M}$ having small C^2-norm. A similar definition in L^2-norm was seen in [Bosi et al. 2022].

If finite boundary spectral data $\{\lambda_j, \varphi_j|_{\partial M}\}_{j=1}^{J}$ are known without error, then this set of finite data is a δ-approximation of the Neumann boundary spectral data with $\delta = J^{-1}$ by definition. If we are given a certain choice of Neumann boundary spectral data, then Definition 1.1(3) is equivalent to the existence of orthogonal matrices acting on eigenfunctions in eigenspaces, such that the condition is satisfied by the given spectral data after applying these matrices.

The main purpose of this paper is to prove the following stability estimate for the reconstruction of a manifold from the Neumann boundary spectral data.

Theorem 1. *There exists $\delta_0 = \delta_0(n, D, K_1, K_2, i_0, r_0) > 0$ such that the following holds. If we are given a δ-approximation of the Neumann boundary spectral data of a Riemannian manifold with boundary $M \in \mathcal{M}_n(D, K_1, K_2, i_0, r_0)$ for $\delta < \delta_0$, then one can construct a finite metric space X directly from the given boundary data such that*

$$d_{\mathrm{GH}}(M, X) < C_1(\log(|\log \delta|))^{-C_2},$$

where d_{GH} denotes the Gromov–Hausdorff distance between metric spaces. The constant C_1 depends on n, D, K_1, K_2, i_0, r_0, and the constant C_2 explicitly depends only on n.

Theorem 1 implies the stability of Gel'fand's inverse problem.

Theorem 2. *There exists* $\delta_0 = \delta_0(n, D, K_1, K_2, i_0, r_0) > 0$ *such that the following holds. Suppose two Riemannian manifolds* $M_1, M_2 \in \mathcal{M}_n(D, K_1, K_2, i_0, r_0)$ *have isometric boundaries and their Neumann boundary spectral data are* δ*-close for* $\delta < \delta_0$*. Then* M_1 *is diffeomorphic to* M_2*, and*

$$d_{\mathrm{GH}}(M_1, M_2) < C_1(\log(|\log\delta|))^{-C_2}.$$

Remark 1.1. The dependency of C_1, δ_0 is not explicit. An explicit estimate with dependence additionally on $\mathrm{vol}_n(M)$, $\mathrm{vol}_{n-1}(\partial M)$ can be obtained, but this process results in a third logarithm. More details can be found in the Appendix.

If any explicitness for the results is not of interest, the bounds (1-1) we assumed on the Riemannian curvature tensor and the second fundamental form can be relaxed to bounds on Ricci curvatures of M, ∂M and the mean curvature of ∂M, due to Corollary 2 in [Katsuda et al. 2007].

We do not know if the log-log type of estimates above is optimal. While strong (Hölder-type) stability results [Bellassoued and Dos Santos Ferreira 2011; Stefanov and Uhlmann 1998; 2005] were known near simple metrics, the stability of the problem is likely weak in the general case; see [Koch et al. 2021; Mandache 2001].

The key result in proving Theorem 1 is a uniform stability estimate for the unique continuation in the class of Riemannian manifolds with bounded geometry, and without loss of domain in the domain of dependence. Let Γ be an open subset of the boundary ∂M and $T > 0$. The *domain of influence* of the set Γ at a time $t \in [0, T]$ is defined as

$$M(\Gamma, t) = \{x \in M : d(x, \Gamma) < t\}, \tag{1-3}$$

where d is the intrinsic distance function of M. The *double cone of influence* of $\Gamma \times [-T, T]$ is defined as

$$K(\Gamma, T) = \{(x, t) \in M \times [-T, T] : d(x, \Gamma) < T - |t|\}. \tag{1-4}$$

Recall Tataru's unique continuation theorem [1995]: if the Cauchy boundary data of a wave u vanish on $\Gamma \times [-T, T]$, i.e.,

$$u|_{\Gamma\times[-T,T]} = 0, \quad \partial_{\boldsymbol{n}} u|_{\Gamma\times[-T,T]} = 0,$$

then the wave u vanishes in the double cone of influence $K(\Gamma, T)$, and in particular, the initial value $u(\cdot, 0)$ vanishes in the domain of influence $M(\Gamma, T)$. Note that the domain $K(\Gamma, T)$ (and $M(\Gamma, T)$ for the initial value) in this result is optimal due to finite speed of propagation of waves. The stability of the unique continuation, i.e., quantitative unique continuation, asks if u is small when the Cauchy boundary data are small.

Theorem 3. *Let* M *be a compact, orientable Riemannian manifold with smooth boundary* ∂M*, and let* Γ *(possibly* $\Gamma = \partial M$*) be a connected open subset of* ∂M *with smooth boundary. Suppose* $u \in H^2(M \times [-T, T])$ *is a solution of the wave equation* $(\partial_t^2 - \Delta_g)u(x, t) = 0$ *with the Neumann boundary condition* $\partial_{\boldsymbol{n}} u|_{\partial M\times[-T,T]} = 0$ *and the initial condition* $\partial_t u(\cdot, 0) = 0$*. Assume the Dirichlet boundary value of* u *satisfies*

$$u|_{\partial M\times[-T,T]} \in H^2(\partial M \times [-T, T]).$$

If

$$\|u(\,\cdot\,,0)\|_{H^1(M)} \leqslant \Lambda, \quad \|u\|_{H^2(\Gamma\times[-T,T])} \leqslant \varepsilon_0,$$

then, for $0 < h < h_0$, *the following estimate holds*:

$$\|u(\,\cdot\,,0)\|_{L^2(M(\Gamma,T))} \leqslant C_3^{1/3} h^{-2/9} \exp(h^{-C_4 n}) \frac{\Lambda + h^{-1/2}\varepsilon_0}{(\log(1+h+h^{3/2}\Lambda/\varepsilon_0))^{1/6}} + C_5 \Lambda h^{1/(3\max\{n,3\})}.$$

The constants h_0, C_3, C_4, C_5 *explicitly depend only on intrinsic geometric parameters of* M *and* Γ (*in particular*, *independent of* ε_0).

Quantitative unique continuation for the wave operator has been investigated independently in [Bosi et al. 2016; 2018] for closed manifolds and in [Laurent and Léautaud 2019], both inspired by the ideas in [Tataru 1998]. In particular, the case for manifolds with boundary for large T was studied in [Laurent and Léautaud 2019], however without addressing how the geometry of the manifold affects the estimate. Our result explicitly shows how the constants depend on the geometry and how close the domain of quantitative unique continuation can approach the optimal domain. These are crucial questions frequently showing up in the stability of inverse problems. In Theorem 3, the stability estimate is obtained up to the optimal domain for arbitrary T, and can be made fully explicit only in terms of intrinsic geometric parameters. The estimate comprises two parts. One is by propagating local unique continuation up to the $\sqrt{h}$-neighborhood of the boundary of the optimal domain. This is the most technical part of the paper and gives a global estimate (Theorem 3.1) on a domain arbitrarily close to the optimal domain, see Figure 1, since h is a small parameter one can freely choose in advance. The second part is to estimate the L^2-norm on the region which the first part does not reach. Once we prove that this region has uniformly controlled volume (Proposition 3.14), the second part of the estimate immediately follows from the a priori H^1-norm. We remark that one can also balance the parameters ε_0, h in Theorem 3 and arrive at a log-log type of estimate with a single parameter ε_0.

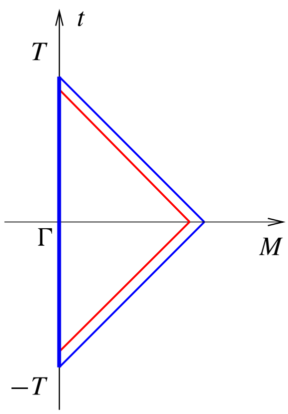

Figure 1. Domains of unique continuation. The blue vertical line is $\Gamma\times[-T,T]$. The domain enclosed by the blue lines is the optimal domain $K(\Gamma,T)$. The domain enclosed by the red lines is $\Omega(h)$ defined in (2-4), obtained by propagating local unique continuation. The distance between the blue and red lines is $\sqrt{h}$.

Equipped with Theorem 3, we can adopt the approach introduced in [Katsuda et al. 2004] to obtain a stability estimate for Gel'fand's inverse problem. Namely, we apply a quantitative version of the boundary control method to evaluate an approximate volume for the domain of dependence. The error of the approximate volume can be made arbitrarily small as long as sufficient boundary spectral data are known. Then we define approximations to the boundary distance functions through slicing procedures, from which the manifold can be reconstructed [Katsuda et al. 2007].

The method we use to obtain the quantitative unique continuation may be of independent interest. Essentially it is proved by propagating local stability estimates to obtain a global estimate. However, the presence of general manifold boundaries brings significant trouble in defining the process, especially when the path of propagation touches the boundary. One straightforward approach would be to avoid the boundary. Namely, one can approximate a geodesic touching the boundary with a curve in the interior of the manifold, and propagate local estimates through balls along this curve. This approach works well if the time domain is larger than the diameter of the manifold, in which case the domain of dependence is smooth, i.e., the whole manifold. However, difficulties arise for an arbitrary time domain, where the domain of dependence in the manifold has corners. An estimate obtained with this approach may not be uniform in a class of manifolds.

Our method directly defines a series of noncharacteristic domains through which local estimates are propagated, using the intrinsic distance of the manifold and the distance to the boundary. This is made possible by directly handling geodesics near the boundary. These domains are globally defined in a coordinate-free way. The boundaries of these domains normally have the shape of a hyperboloid and warp quickly near the boundary (and the injectivity radius). In this way, the local estimates propagate (almost) along distance-minimizing geodesics, and naturally produce a uniform global estimate depending only on intrinsic geometric parameters.

This paper is organized as follows. We review relevant concepts and the unique continuation in Section 2. Section 3 is devoted to proving Theorem 3, an explicit stability estimate for the unique continuation from a subset of the boundary. Section 3 uses several technical lemmas, and their proofs can be found in Section 6. In Section 4, we apply Theorem 3.1 to introduce the essential step of our reconstruction method where we compute, in a stable way, how the Fourier coefficients of a function (with respect to the basis of eigenfunctions) change, when the function is multiplied by an indicator function of a union of balls with center points on the boundary. The new feature of this method is that it is directly based on the unique continuation theorem. The main results, Theorems 1 and 2, are proved in Section 5, with the dependency of constants on geometric parameters derived in the Appendix.

2. Preliminaries

2.1. ***Bounded geometry.*** Let $(M, g) \in \mathcal{M}_n(D, K_1, K_2, i_0, r_0)$ be a compact, connected, orientable Riemannian manifold of dimension $n \geqslant 2$ with smooth boundary ∂M. The C^0-norm of the Riemannian curvature tensor R_M appearing in (1-1) is defined as

$$\|R_M\|_{C^0} = \sup_{x \in M} |R_M|_x|,$$

where $|R_M|_x|$ denotes the operator norm of R_M at $x \in M$ as a multilinear operator to $\mathbb{R}$. The C^0-norms of S and the covariant derivatives are defined in the same way. In this paper, we usually omit the subscript C^0 for brevity.

Since the Riemannian curvature tensor is completely determined by the sectional curvatures, assuming a bound on the curvature tensor is equivalent to assuming a bound on sectional curvatures. By the Gauss equation, the bounds on the curvature tensor of M and the second fundamental form of ∂M yield a bound on the curvature tensor $R_{\partial M}$ of ∂M (when ∂M is at least two-dimensional), also denoted by K_1^2. Without loss of generality, assume $K_1, K_2 > 0$.

From now on, we write $\|A\| = \|A\|_{C^0}$ for a tensor field A on M. For convenience, we define

$$\|R_M\|_{C^k} = \|R_M\| + \sum_{i=1}^{k} \|\nabla^i R_M\|, \quad \|S\|_{C^k} = \|S\| + \sum_{i=1}^{k} \|\nabla^i S\|.$$

Then the curvature bound assumptions in (1-1) are written as

$$\|R_M\| \leqslant K_1^2, \quad \|S\| \leqslant K_1, \quad \|R_{\partial M}\| \leqslant K_1^2,$$
$$\|R_M\|_{C^5} \leqslant K_1^2 + K_2, \quad \|S\|_{C^4} \leqslant K_1 + K_2, \quad \|R_{\partial M}\|_{C^4} \leqslant C(K_1, K_2).$$

The boundary ∂M is said to admit a boundary normal neighborhood of width r if the exponential map $(z, s) \mapsto \exp_z(s\boldsymbol{n}_z)$ defines a homeomorphism from $\partial M \times [0, r]$ to the r-neighborhood of ∂M, where $\boldsymbol{n}_z$ denotes the inward-pointing unit normal vector at $z \in \partial M$ (see, e.g., Section 2.1.16 in [Katchalov et al. 2001]). The *boundary injectivity radius* $i_b(M)$ of M is defined as the largest number with the following property that ∂M admits a boundary normal neighborhood of width r for any $r < i_b(M)$. The injectivity radius $\operatorname{inj}(M)$ of M is usually defined as the largest number $r \leqslant \min\{\operatorname{inj}(\partial M), i_b(M)\}$ satisfying the following condition: the open ball $B_r(x)$ of radius r is a domain of Riemannian normal coordinates on M centered at any $x \in M$ with $d(x, \partial M) \geqslant r$.

This definition of the injectivity radius for a manifold with boundary gives little information on the geometry near the boundary. We find it convenient to consider the following quantity.

Definition 2.1. For $x \in M$, $r_{\mathrm{CAT}}(x)$ is defined to be the largest number r such that the (distance-)minimizing geodesic of M connecting x and any $y \in B_r(x)$ is unique. Define

$$r_{\mathrm{CAT}}(M) = \inf_{x \in M} r_{\mathrm{CAT}}(x).$$

We call this quantity the radius of radial uniqueness (or CAT radius).

The radius of radial uniqueness is positive for a compact Riemannian manifold with smooth boundary (Lemma 6.2(1)). This definition is a natural extension of the injectivity radius for manifolds without boundary. More precisely, for a Riemannian manifold without boundary, $\min\{\pi/\sqrt{K}, r_{\mathrm{CAT}}\}$ gives a lower bound for the injectivity radius, where K is the upper bound for the sectional curvatures.

The radius of radial uniqueness has an immediate connection with metric spaces of curvature bounded above in the sense of Alexandrov. A metric space has curvature bounded above (globally) by $K > 0$ if every minimizing geodesic triangle in the space has perimeter less than $2\pi/\sqrt{K}$, and has each of its angles at most equal to the corresponding angle in a triangle with the same side-lengths in the surface of constant curvature K. This space is denoted by CAT(K). A CAT(K) space has the property that any pair of points with distance less than $\pi/\sqrt{K}$ is connected by a unique (within the space) minimizing geodesic, and the geodesic continuously depends on its endpoints. It is well known that a Riemannian manifold M with smooth boundary is locally CAT(K), where K is the upper bound for the sectional curvatures of M and the second fundamental form of ∂M [Alexander et al. 1993, characterization theorem]. In fact, more is known: the open ball around any point in M of the radius $\min\{\pi/2\sqrt{K}, r_{\mathrm{CAT}}(M)\}$ is CAT(K) [Alexander and Bishop 1996, Theorem 4.3]. This is where the notation r_{CAT} comes from. The CAT space provides useful nondifferential tools to work with manifold boundaries where the standard differential machinery is often problematic.

2.2. *Wave operator and the unique continuation.* The Laplace–Beltrami operator Δ_g with respect to the metric g has the following form in local coordinates $(x^1, \dots, x^n)$:

$$\Delta_g = \frac{1}{\sqrt{\det(g_{ij})}} \sum_{i,j=1}^{n} \frac{\partial}{\partial x^i}\Big(\sqrt{\det(g_{ij})}\, g^{ij} \frac{\partial}{\partial x^j}\Big). \tag{2-1}$$

Then the wave operator $P = \partial_t^2 - \Delta_g$ has the following form in local coordinates:

$$\begin{aligned} P &= \frac{\partial^2}{\partial t^2} - \frac{1}{\sqrt{\det(g_{ij})}} \sum_{i,j=1}^{n} \frac{\partial}{\partial x^i}\Big(\sqrt{\det(g_{ij})}\, g^{ij} \frac{\partial}{\partial x^j}\Big) \\ &= \frac{\partial^2}{\partial t^2} - \sum_{i,j=1}^{n} g^{ij} \frac{\partial^2}{\partial x^i \partial x^j} + \text{lower-order terms}. \end{aligned} \tag{2-2}$$

The Riemannian metric g approximates the standard Euclidean metric in small scale. In sufficiently small coordinate charts, the Laplace–Beltrami operator is a strongly elliptic operator given by the formula (2-1). However, the wave operator of the form above is only locally defined on manifolds, different from the wave operator on Euclidean spaces with global coefficients.

In the boundary normal neighborhood of ∂M, it is convenient to use the boundary normal coordinate $(x^1, \dots, x^{n-1}, x^n)$, where $(x^1, \dots, x^{n-1})$ is a choice of coordinate at the nearest point on ∂M and $x^n = d(x, \partial M)$. In other words, the coordinate $(x^1, \dots, x^{n-1}, d(x, \partial M))$ is defined by pushing forward the local coordinate $(x^1, \dots, x^{n-1})$ on ∂M via the family of exponential maps $z \mapsto \exp_z(s\boldsymbol{n}_z)$ from the boundary in the normal direction. Note that the choice of coordinate on ∂M is fixed. Hence by the Gauss lemma, the metric g has the form of a product metric in such a coordinate:

$$g = (dx^n)^2 + \sum_{\alpha,\beta=1}^{n-1} g_{\alpha\beta}\, dx^\alpha\, dx^\beta.$$

On the boundary ∂M, two frequent choices of coordinate are the geodesic normal coordinate and the harmonic coordinate. In this paper, we use the geodesic normal coordinate of ∂M. Namely, at any point on ∂M, we have a geodesic normal coordinate $(x^\alpha)_{\alpha=1}^{n-1}$ in the ball (of ∂M) of a sufficiently small radius such that

$$\tfrac{1}{2}|\xi|^2 \leqslant \sum_{\alpha,\beta=1}^{n-1} g^{\alpha\beta}\xi_\alpha\xi_\beta \leqslant 2|\xi|^2 \quad (\xi \in \mathbb{R}^{n-1}), \tag{2-3}$$
$$\|g_{\alpha\beta}\|_{C^1} \leqslant 2, \quad \|g_{\alpha\beta}\|_{C^4} \leqslant C(n, K_1, K_2, i_0).$$

It is known that the radius of the ball in which the conditions above are satisfied is uniformly bounded below by a positive number explicitly depending on n, $\|R_{\partial M}\|_{C^1}$, i_0 [Hebey and Vaugon 1995, Lemma 8; Eichhorn 1991, Theorem A]. We denote this uniform radius by $r_g(\partial M)$.

Recall that the wave operator P enjoys the unique continuation property from the boundary; namely if the Cauchy boundary data of a wave u (a solution of the wave equation $Pu=0$) vanish on $\Gamma\times[-T,T]$, i.e.,

$$u|_{\Gamma\times[-T,T]}=0, \qquad \frac{\partial u}{\partial \boldsymbol{n}}\Big|_{\Gamma\times[-T,T]}=0,$$

then the wave vanishes in the double cone of influence $K(\Gamma,T)$ defined in (1-4); see [Tataru 1995] or, e.g., Theorem 3.16 in [Katchalov et al. 2001]. Here $\boldsymbol{n}$ denotes the unit normal vector field on ∂M pointing inwards. We are interested in its stability: when the Cauchy boundary data are small on $\Gamma\times[-T,T]$, we consider if the wave is small in the double cone. The following global stability result on Tataru's unique continuation principle [1995] was proved in [Bosi et al. 2016], from which the stability of the unique continuation from a ball on a closed Riemannian manifold can be obtained [Bosi et al. 2016, Theorem 3.3].

Theorem 2.2 [Bosi et al. 2016, Theorem 1.2]. *Let Ω_{bd} be a bounded connected open subset of $\mathbb{R}^n\times\mathbb{R}$ and P be the wave operator (2-2). Assume $u\in H^1(\Omega_{\mathrm{bd}})$ and $Pu\in L^2(\Omega_{\mathrm{bd}})$. In Ω_{bd}, we assume the existence of a finite number of connected open subsets Ω_j^0 and Ω_j, $j=1,2,\dots,J$, a connected set Υ and functions ψ_j satisfying the following assumptions*:

1. $\psi_j\in C^{2,1}(\Omega_{\mathrm{bd}})$; *$p(\,\cdot\,,\nabla\psi_j)\neq 0$ and $\nabla\psi_j\neq 0$ in Ω_j^0, where p denotes the principle symbol of the wave operator P.*
2. $\operatorname{supp}(u)\cap\Upsilon=\varnothing$; *there exists $\psi_{\max,j}\in\mathbb{R}$ such that $\varnothing\neq\{y\in\Omega_j^0:\psi_j(y)>\psi_{\max,j}\}\subset\overline{\Upsilon}_j$, where $\Upsilon_j=\Omega_j^0\cap\big(\bigcup_{l=1}^{j-1}\Omega_l\cup\Upsilon\big)$.*
3. $\Omega_j=\{y\in\Omega_j^0-\overline{\Upsilon}_j:\psi_j(y)>\psi_{\min,j}\}$ *for some $\psi_{\min,j}\in\mathbb{R}$, and* $\operatorname{dist}(\partial\Omega_j^0,\Omega_j)>0$.
4. $\overline{\Omega}$ *is connected, where $\Omega=\bigcup_{j=1}^J\Omega_j$.*

Then the following estimate holds for Ω and $\Omega^0=\bigcup_{j=1}^J\Omega_j^0$:

$$\|u\|_{L^2(\overline{\Omega})}\leqslant C\frac{\|u\|_{H^1(\Omega^0)}}{(\log(1+\|u\|_{H^1(\Omega^0)}/\|Pu\|_{L^2(\Omega^0)}))^\theta},$$

where $\theta\in(0,1)$ is arbitrary, and the constant C explicitly depends on θ, ψ_j, $\operatorname{dist}(\partial\Omega_j^0,\Omega_j)$, $\|g^{ij}\|_{C^1}$, $\mathrm{vol}_{n+1}(\Omega_{\mathrm{bd}})$.

The intuition behind this result is propagating the unique continuation step by step to cover a large domain, as long as the error introduced in each step is small. The set Υ is the initial domain where the function u vanishes, and Ω_j is the domain propagated by the unique continuation at the j-th step. The estimate is obtained by propagating local stability estimates, and the assumptions make sure that certain support conditions [Bosi et al. 2018, Assumption A1] required by the local stability estimates are satisfied at every step. For some simple cases, one choice of the domains and functions is enough, for example if the function u initially vanishes over a ball in $\mathbb{R}^n$. However, these assumptions are rather restrictive for general cases, and multiple iterations of the domains and functions need to be carefully constructed to handle the difficulties brought by the geometry of the boundary and the injectivity radius. Note that the constant in the estimate depends on higher derivatives of ψ_j in Ω_j^0. It is crucial to construct the required domains where ψ_j has uniformly bounded higher derivatives. Although Theorem 2.2 is formulated in Euclidean spaces, it applies to manifolds since it is obtained by propagating local stability estimates, which can be done in local coordinate charts.

2.3. ***Notation.*** We introduce notation that we will frequently use in this paper. Denote by vol_k the k-dimensional Hausdorff measure on M. When the Hausdorff dimension of a set in question is clear, we omit the subscript k. In particular, we denote by $\mathrm{vol}(M)$ the Riemannian volume of M, and by $\mathrm{vol}(\partial M)$ the Riemannian volume of ∂M with respect to the induced metric on ∂M.

Given an open subset $\Gamma \subset \partial M$, we define the following domain with a positive parameter $h < 1$ by

$$\Omega_{\Gamma,T}(h) = \{(x,t) \in M \times [-T,T] : T - |t| - d(x,\Gamma) > \sqrt{h},\ d(x, \partial M - \Gamma) > h\}, \tag{2-4}$$

and we write $\Omega(h)$ for short. Note that $\Omega(h)$ is a subset of the double cone of influence $K(\Gamma,T)$, and $\Omega(h)$ approximates $K(\Gamma,T)$ as $h \to 0$. If $\Gamma = \partial M$, the set above is defined with the last condition dropped. In this paper, our consideration always includes the possibility that $\Gamma = \partial M$. For the sole purpose of incorporating this special case notationwise in later proofs, we set any distance from the empty set to be infinity.

Given a function $u : \partial M \times [-T,T] \to \mathbb{R}$ and an open subset $\Gamma \subset \partial M$, we define the norm

$$\|u\|^2_{H^{2,2}(\Gamma\times[-T,T])} = \int_{-T}^{T} \bigl(\|u(\cdot,t)\|^2_{H^2(\Gamma)} + \|\partial_t u(\cdot,t)\|^2_{L^2(\Gamma)} + \|\partial_t^2 u(\cdot,t)\|^2_{L^2(\Gamma)} \bigr)\, dt \tag{2-5}$$

if $u(\cdot,t) \in H^2(\Gamma)$ and $\partial_t u(\cdot,t),\ \partial_t^2 u(\cdot,t) \in L^2(\Gamma)$ for all $|t| \leqslant T$. We say $u \in H^{2,2}(\Gamma \times [-T,T])$ if the norm above is finite, and we call it the $H^{2,2}$-norm.

3. Stability of the unique continuation

In this section, we obtain an explicit estimate on the stability of the unique continuation for the wave operator, provided small Cauchy data on a connected open subset of the manifold boundary. First we state this result as follows.

Theorem 3.1. *Let $M \in \mathcal{M}_n(D, K_1, K_2, i_0, r_0)$ be a compact, orientable Riemannian manifold with smooth boundary ∂M, and let Γ (possibly $\Gamma = \partial M$) be a connected open subset of ∂M with smooth boundary. Denote by $i_b(\overline{\Gamma})$ the boundary injectivity radius of $\overline{\Gamma}$. Then there exist a constant $C_3 > 0$ that explicitly depends on n, T, D, K_1, $\|\nabla R_M\|_{C^0}$, $\|\nabla S\|_{C^0}$, i_0, r_0, $\mathrm{vol}_n(M)$, $\mathrm{vol}_{n-1}(\Gamma)$, an absolute constant $C_4 > 0$, and a sufficiently small constant $h_0 > 0$, that explicitly depends on n, T, K_1, K_2, i_0, r_0, $i_b(\overline{\Gamma})$, $\mathrm{vol}_{n-1}(\partial M)$, such that the following holds.*

Suppose $u \in H^2(M \times [-T,T])$ *is a solution of the nonhomogeneous wave equation* $Pu = f$ *with* $f \in L^2(M \times [-T,T])$. *Assume the Cauchy data satisfy*

$$u|_{\partial M \times [-T,T]} \in H^{2,2}(\partial M \times [-T,T]), \qquad \frac{\partial u}{\partial \boldsymbol{n}} \in H^{2,2}(\partial M \times [-T,T]). \tag{3-1}$$

If

$$\|u\|_{H^1(M \times [-T,T])} \leqslant \Lambda_0, \qquad \|u\|_{H^{2,2}(\Gamma \times [-T,T])} + \left\| \frac{\partial u}{\partial \boldsymbol{n}} \right\|_{H^{2,2}(\Gamma \times [-T,T])} \leqslant \varepsilon_0, \tag{3-2}$$

then, for $0 < h < h_0$, *we have*

$$\|u\|_{L^2(\Omega(h))} \leqslant C_3 \exp(h^{-C_4 n}) \frac{\Lambda_0 + h^{-1/2}\varepsilon_0}{\left(\log(1 + (\Lambda_0 + h^{-1/2}\varepsilon_0)/(\|Pu\|_{L^2(M \times [-T,T])} + h^{-3/2}\varepsilon_0))\right)^{1/2}}.$$

The domain $\Omega(h)$ *and the* $H^{2,2}$*-norm are defined in Section 2.3.*

As a consequence, the following estimate holds for any $\theta \in (0,1)$ *by interpolation*:

$$\|u\|_{H^{1-\theta}(\Omega(h))} \leqslant C_3^{\theta} \exp(h^{-C_4 n}) \frac{\Lambda_0 + h^{-1/2}\varepsilon_0}{\left(\log(1 + (\Lambda_0 + h^{-1/2}\varepsilon_0)/(\|Pu\|_{L^2(M \times [-T,T])} + h^{-3/2}\varepsilon_0))\right)^{\theta/2}}.$$

Remark 3.2. In Theorem 3.1, the different smoothness indexes of the Sobolev spaces in the qualitative smoothness assumption $u \in H^2(M \times [-T,T])$ and in the quantitative bounds for the Sobolev norms (3-2) are related to the smooth extension of the weak solution of the wave equation to a boundary layer. We note that the nonuniform smoothness assumptions are typical, and sometimes also optimal, for the weak solutions of the wave equation with the Neumann boundary condition; see [Lasiecka and Triggiani 1991]. We also note that in Theorem 3.1 the assumption $u \in H^2(M \times [-T,T])$ can be relaxed to the assumption that u is a weak solution of the wave equation $Pu = f$ with the Neumann boundary condition, where $f \in L^2(M \times [-T,T])$, and u and its Neumann boundary value $\partial_{\boldsymbol{n}} u|_{\partial M \times [-T,T]}$ satisfy

$$u \in C([-T,T]; H^1(M)) \cap C^1([-T,T]; L^2(M)),$$
$$\partial_{\boldsymbol{n}} u|_{\partial M \times [-T,T]} \in L^2(\partial M \times [-T,T]).$$

Then, by [Lasiecka and Triggiani 1991, Theorem A], the Dirichlet boundary value is a well-defined function $u|_{\partial M \times [-T,T]} \in L^2(\partial M \times [-T,T])$. In this case, (3-1) can be viewed as an additional smoothness requirement for the Dirichlet and the Neumann boundary values of u. This relaxation of the smoothness assumptions only affects the last part of the proof of Lemma 3.5, and this lemma can be proved via the weak version of Green's formula.

Our method can also be used to derive a stability estimate for the unique continuation from any open domain in the interior of M, as long as the boundary of the domain is smoothly embedded in M. In this way, a stability estimate can be obtained on domains arbitrarily close to the double cone of influence from the interior domain in question, which provides a generalization of Theorem 3.3 in [Bosi et al. 2016]. We remark that as the domain approaches the double cone of influence, the estimate above grows exponentially. This exp-dependence and the log-type of the estimate itself eventually lead to the two logarithms in Theorem 1. We also mention that Proposition 3.14 may be of independent interest, which provides an explicit uniform bound for the Hausdorff measure of the boundary of the domain of influence.

Most of this section is occupied by the proof of Theorem 3.1. First we properly extend the manifold, the wave operator P and the wave u, so that Pu stays small on the manifold extension over Γ, given sufficiently small Cauchy data on Γ. The extension of u is cut off near the boundary in the manifold extension, from which we start propagating the unique continuation. Then we carefully construct a series of domains satisfying the assumptions in Theorem 2.2 such that the union of these domains approximates the double cone of influence. Thus Theorem 2.2 gives a stability estimate on domains arbitrarily close to the double cone of influence.

The main difficulty lies in actually finding that series of domains satisfying the properties stated above, as the assumptions in Theorem 2.2 (essentially assumptions for local estimates) are rather restrictive for a general manifold with boundary. This requires us to directly deal with the intrinsic distance and (distance-minimizing) geodesics of the manifold. In this section, we use several technical lemmas and their proofs can be found in Section 6.

Theorem 3.1 yields the following stable continuation result on the whole domain of influence $M(\Gamma, T)$.

Proposition 3.3. *Let $M \in \mathcal{M}_n(D, K_1, K_2, i_0, r_0)$ be a compact Riemannian manifold with smooth boundary ∂M, and let Γ (possibly $\Gamma = \partial M$) be a connected open subset of ∂M with smooth boundary. Suppose $u \in H^2(M \times [-T, T])$ is a solution of the wave equation $Pu(x,t) = 0$ with the Neumann boundary condition $\partial_{\boldsymbol{n}} u|_{\partial M \times [-T,T]} = 0$ and the initial condition $\partial_t u(\,\cdot\,, 0) = 0$. Assume the Dirichlet boundary value of u satisfies*

$$u|_{\partial M \times [-T,T]} \in H^{2,2}(\partial M \times [-T, T]).$$

If

$$\|u(\,\cdot\,, 0)\|_{H^1(M)} \leqslant \Lambda, \quad \|u\|_{H^{2,2}(\Gamma \times [-T,T])} \leqslant \varepsilon_0,$$

then, for $0 < h < h_0$, the following estimate holds:

$$\|u(\,\cdot\,, 0)\|_{L^2(M(\Gamma,T))} \leqslant C_3^{1/3} h^{-2/9} \exp(h^{-C_4 n}) \frac{\Lambda + h^{-1/2}\varepsilon_0}{(\log(1 + h + h^{3/2}\Lambda/\varepsilon_0))^{1/6}} + C_5 \Lambda h^{1/(3\max\{n,3\})}.$$

Here C_3 explicitly depends on n, T, D, $\|R_M\|_{C^1}$, $\|S\|_{C^1}$, i_0, r_0, $\mathrm{vol}(M)$, $\mathrm{vol}_{n-1}(\Gamma)$; C_4 is an absolute constant; C_5 explicitly depends on n, $\|R_M\|_{C^1}$, $\|S\|_{C^1}$, i_0, $\mathrm{vol}(M)$, $\mathrm{vol}(\partial M)$; $h_0 > 0$ is a sufficiently small constant explicitly depending on n, T, K_1, K_2, i_0, r_0, $i_b(\overline{\Gamma})$, $\mathrm{vol}(\partial M)$.

We postpone the proof of Proposition 3.3 after the proof of Theorem 3.1.

3.1. *Extension of manifolds.* Let $(M, g) \in \mathcal{M}_n(D, K_1, K_2, i_0, r_0)$ be a compact, orientable Riemannian manifold with bounded geometry defined in the Introduction.

Lemma 3.4. *For sufficiently small δ_{ex} explicitly depending on n, K_1, K_2, i_0, $\mathrm{vol}(\partial M)$, we can extend (M, g) to a Riemannian manifold $(\widetilde{M}, \tilde{g})$ with smooth boundary such that the following properties are satisfied:*

(1) *$\widetilde{M} - M$ lies in a normal neighborhood of ∂M in $\widetilde{M}$, and $\tilde{d}(x, \partial M) = \delta_{\mathrm{ex}}$ for any $x \in \partial\widetilde{M}$, where $\tilde{d}$ denotes the distance function of $\widetilde{M}$.*

(2) $\tilde{g}$ *is of* $C^{3,1}$ *in some atlas on* $\widetilde{M}$, *in which*

$$\|\tilde{g}_{ij}|_{\widetilde{M}-M}\|_{C^1} \leqslant C(K_1), \quad \|\tilde{g}_{ij}|_{\widetilde{M}-M}\|_{C^4} \leqslant C(n, K_1, K_2, i_0).$$

(3) $\|R_{\widetilde{M}}\| \leqslant 2K_1^2$, $\|S_{\partial\widetilde{M}}\| \leqslant 2K_1$ *and* $\|\nabla R_{\widetilde{M}}\| \leqslant 2K_2$, *where* $S_{\partial\widetilde{M}}$ *denotes the second fundamental form of* $\partial\widetilde{M}$ *in* $\widetilde{M}$.

As a consequence, we have:

(4) $r_{\mathrm{CAT}}(\widetilde{M}) \geqslant \min\{C(K_1), i_0/4, r_0/2\}$.

Proof. We glue a collar $\partial M \times [-\delta_{\mathrm{ex}}, 0]$ for $0 < \delta_{\mathrm{ex}} < \min\{1, i_0/2\}$ onto M by identifying $\partial M \times \{0\}$ of the collar with ∂M. Denote the topological space after the gluing procedure by $\widetilde{M}$. Any $(y, \rho) \in \partial M \times [-\delta_{\mathrm{ex}}, 0]$ admits coordinate charts by extending boundary normal coordinate charts at $(y, -\rho) \in M$. The transition maps are clearly smooth and therefore $\widetilde{M}$ is a smooth manifold.

Let $\{y_i\}$ be a maximal $r_g(\partial M)/2$-separated set (and hence an $r_g(\partial M)/2$-net) in ∂M. Let U_i be the ball of radius $r_g(\partial M)$ in ∂M around y_i, and therefore $\{U_i\}$ is an open cover of ∂M. We take a partition of unity $\{\phi_i\}$ subordinate to $\{U_i\}$ satisfying

$$\|\phi_i\|_{C^s} \leqslant C\, r_g(\partial M)^{-s} \quad \text{for } s \in [1, 4].$$

Then $\{\widetilde{U}_i := U_i \times [-\delta_{\mathrm{ex}}, 0]\}$ is an open cover of the collar $\partial M \times [-\delta_{\mathrm{ex}}, 0]$, and $\{\tilde{\phi}_i\}$ is a partition of unity subordinate to this cover satisfying the same bound on C^s-norm, where $\tilde{\phi}_i$ is defined by $\tilde{\phi}_i(y, \rho) = \phi_i(y)$ for $(y, \rho) \in \partial M \times [-\delta_{\mathrm{ex}}, 0]$.

We choose the geodesic normal coordinate $(y^\alpha)_{\alpha=1}^{n-1}$ on each U_i such that (2-3) holds. Within each coordinate chart $\widetilde{U}_i$, we define the metric components at $(y, \rho) \in \widetilde{U}_i$ as follows: $\tilde{g}^{(i)}_{\rho\rho} = 1$, and $\tilde{g}^{(i)}_{\alpha\rho} = 0$ for $\alpha, \beta = 1, \ldots, n-1$, and

$$\tilde{g}^{(i)}_{\alpha\beta}(y, \rho) = g^{(i)}_{\alpha\beta}(y, 0) + \rho \frac{\partial g^{(i)}_{\alpha\beta}}{\partial \rho}(y, 0) + \frac{\rho^2}{2} \frac{\partial^2 g^{(i)}_{\alpha\beta}}{\partial \rho^2}(y, 0) + \frac{\rho^3}{6} \frac{\partial^3 g^{(i)}_{\alpha\beta}}{\partial \rho^3}(y, 0) \quad \text{for } \rho \leqslant 0.$$

Then one can define a Riemannian metric $\tilde{g}$ on $\partial M \times [-\delta_{\mathrm{ex}}, 0]$ through partition of unity:

$$\tilde{g}|_{(y,\rho)} = \sum_i \tilde{\phi}_i(y, \rho) g^{(i)}|_{(y,\rho)} = \sum_i \phi_i(y) g^{(i)}|_{(y,\rho)} \quad \text{for } \rho \leqslant 0. \tag{3-3}$$

At $(y, \rho \in \mathbb{R}_+) \in M$ with respect to the boundary normal coordinate of ∂M in M, define $\tilde{g} = g$. Due to the Riccati equation (e.g., [Petersen 2006, Theorem 2, p. 44]), the derivatives of $g^{(i)}_{\alpha\beta}$ with respect to ρ at $\rho = 0$ up to the third order can be expressed in terms of the components of S, R_M and ∇R_M. Then the curvature bound assumptions (1-1) implies that $\tilde{g}^{(i)}_{\alpha\beta}$ is of C^4 within each coordinate chart $\widetilde{U}_i$.

Now let us consider the coordinate charts $U_i \times [-\delta_{\mathrm{ex}}, i_0)$. In this coordinate, the components $\tilde{g}^{(i)}_{\alpha\beta}$ are of $C^{3,1}$ in the normal direction, and C^4 in other directions. Therefore $\tilde{g}$ is of $C^{3,1}$ in the local coordinate charts $\{U_i \times [-\delta_{\mathrm{ex}}, i_0)\}$.

Furthermore, it follows from a straightforward calculation that, for $\rho \leqslant 0$,

$$\left| \frac{\partial^{k+l} \tilde{g}^{(i)}_{\alpha\beta}}{\partial x_T^k \partial \rho^l}(y, \rho) - \frac{\partial^{k+l} g^{(i)}_{\alpha\beta}}{\partial x_T^k \partial \rho^l}(y, 0) \right| \leqslant C(\|R_M\|_{C^5}, \|S\|_{C^4})|\rho| \quad \text{for } k + l \leqslant 4,\ l \leqslant 3.$$

Note that $\partial^4 \tilde{g}^{(i)}_{\alpha\beta}/\partial\rho^4 = 0$ by definition. Recall that the C^4-norm of ϕ_i is uniformly bounded by $C\, r_g(\partial M)^{-4}$, and $r_g(\partial M)$ explicitly depends on n, $\|R_{\partial M}\|_{C^1}$, i_0. Furthermore, the total number of coordinate charts U_i is bounded by $C(n, K_1)\,\mathrm{vol}(\partial M)\, r_g(\partial M)^{-n+1}$. Hence by (3-3), the estimates above hold for $\tilde{g}_{\alpha\beta}$ and $g_{\alpha\beta}$ with another constant $C(n, \|R_M\|_{C^5}, \|S\|_{C^4}, i_0, \mathrm{vol}(\partial M))$.

Therefore we can restrict the extension width δ_{ex} to be sufficiently small explicitly depending only on n, K_1, K_2, i_0, $\mathrm{vol}(\partial M)$ such that the matrix $(\tilde{g}_{\alpha\beta})$ is nondegenerate and hence a metric, and

$$\begin{gathered}\|\tilde{g}_{\alpha\beta}|_{\widetilde{M}-M}\|_{C^1} \leqslant 4K_1+4, \qquad \|\tilde{g}_{\alpha\beta}|_{\widetilde{M}-M}\|_{C^4} \leqslant C(n, K_1, K_2, i_0), \\ \|R_{\widetilde{M}}\| \leqslant 2K_1^2, \qquad \|S_{\partial\widetilde{M}}\| \leqslant 2K_1, \qquad \|\nabla R_{\widetilde{M}}\| \leqslant 2K_2.\end{gathered} \tag{3-4}$$

Here the first inequality is due to (2-3) and the definition that $\partial_\rho g_{\alpha\beta}|_{\partial M} = 2S_{\alpha\beta}$, where $S_{\alpha\beta}$ denotes the components of the second fundamental form S of ∂M. The bound on $S_{\partial\widetilde{M}}$ follows from the bound on $\partial_\rho \tilde{g}_{\alpha\beta}|_{\widetilde{M}-M}$.

With this type of extension, $\tilde{g}$ is also a product metric in the collar, which implies that the integral curve of $\partial/\partial\rho$ minimizes length and is hence a minimizing geodesic. This shows that, for any $x=(y,\rho) \in \partial M \times [-\delta_{\mathrm{ex}}, 0]$, we have $\tilde{d}(x, \partial M) = -\rho$, which yields property (1). The property (4) is due to properties (1)–(3) and Lemma 6.2(2). □

Coordinate system. From now on, we extend the manifold (M, g) to $(\widetilde{M}, \tilde{g})$ such that Lemma 3.4 holds. We say $(\widetilde{M}, \tilde{g})$ is an extension of (M, g) with the extension width δ_{ex}. We choose a coordinate system on $\widetilde{M}$ as follows.

In the boundary normal (tubular) neighborhood of ∂M, we choose the boundary normal coordinate of ∂M. Let $\{y_i\}$ be a maximal $r_g(\partial M)/2$-separated set in ∂M, and U_i be the ball of radius $r_g(\partial M)$ in ∂M around y_i. The proof of Lemma 3.4 shows that $\tilde{g}$ is of $C^{3,1}$ in the coordinate charts $U_i \times [-\delta_{\mathrm{ex}}, i_0)$. In each coordinate chart, we choose the boundary normal coordinate $(x^1, \ldots, x^{n-1}, \rho(x))$ of ∂M, where $(x^1, \ldots, x^{n-1})$ is the geodesic normal coordinate of ∂M such that (2-3) holds. The coordinate function $\rho(x)$ in the normal direction is defined as

$$\rho(x) = \begin{cases} d(x, \partial M) & \text{if } x \in M, \\ -\tilde{d}(x, \partial M) & \text{if } x \in \widetilde{M} - M. \end{cases} \tag{3-5}$$

Note that $\tilde{d}(x, \partial M) = d(x, \partial M)$ for $x \in M$. Lemma 3.4(2) shows that the metric components on $\widetilde{M} - M$ have uniformly bounded C^4-norm. On the other side, due to Lemma 6.1, we can find a uniform width $r_b = r_b(K_1, i_0)$ such that the C^4-norm of metric components is uniformly bounded by $C(n, K_1, K_2, i_0)$ in the boundary normal coordinate of width r_b in M. Consequently, we have a uniform bound for the $C^{3,1}$-norm of metric components in the coordinate charts $U_i \times [-\delta_{\mathrm{ex}}, i_0)$.

For any point $x \in M$ with $d(x, \partial M) > r_b/2$, we choose the geodesic normal coordinate of M around x of the radius $\min\{r_b/2, r_g(x)\}$ such that the C^4-norm of metric components is uniformly bounded. By [Hebey and Vaugon 1995, Lemma 8] and [Eichhorn 1991, Theorem A], this radius is uniformly bounded below by n, $\|R_M\|_{C^1}$, i_0, r_b. Denote by r_g the minimum of this radius and $r_g(\partial M)$, and therefore r_g explicitly depends only on n, $\|R_M\|_{C^1}$, $\|S\|_{C^1}$, i_0.

Combining these two types of coordinates, we have a coordinate system on $\widetilde{M}$ in which the metric components satisfy the properties

$$\tfrac{1}{4}|\xi|^2 \leqslant \sum_{i,j=1}^{n} \tilde{g}^{ij}\xi_i\xi_j \leqslant 4|\xi|^2 \quad (\xi \in \mathbb{R}^n),$$
$$\|\tilde{g}_{ij}\|_{C^1} \leqslant C(n, \|R_M\|_{C^1}, \|S\|_{C^1}), \qquad \|\tilde{g}_{ij}\|_{C^{3,1}} \leqslant C(n, K_1, K_2, i_0). \tag{3-6}$$

Observe that, for any $x \in \widetilde{M}$, the ball $\widetilde{B}_{r_g/2}(x)$ of $\widetilde{M}$ or the cylinder $B_{\partial M}(y, r_g/2)\times(\rho-r_g/2, \rho+r_g/2)$ is contained in at least one of the coordinate charts defined above, where $x = (y, \rho)$ if x is in the boundary normal coordinate of ∂M. To see this, it suffices to show that, for any $y \in \partial M$, the ball $B_{\partial M}(y, r_g/2)$ of ∂M is contained in at least one of U_i. The latter statement is a direct consequence of the fact that $\{y_i\}$ is an $(r_g(\partial M)/2)$-net in ∂M.

3.2. *Extension of functions.* Let $(\widetilde{M}, \tilde{g})$ be an extension of (M, g) satisfying Lemma 3.4 with the extension width δ_{ex}. Points in the boundary normal neighborhood of ∂M have the coordinate $(x^1, \dots, x^{n-1}, \rho(x))$, where $\rho(x)$ is defined in (3-5). We write the coordinate as $(x_T, \rho(x))$ for short, where $x_T = (x^1, \dots, x^{n-1})$ denotes the tangential coordinate.

We define an extension of functions on M to $\widetilde{M}$ as follows. Given a function u on M and its Cauchy data $u,\ \partial u/\partial \boldsymbol{n}$ on ∂M, we extend u to a function $\tilde{u}_{\text{ex}}$ on $\widetilde{M}$ by

$$\tilde{u}_{\text{ex}}(x_T, \rho, t) = \begin{cases} u(x_T, \rho, t) & \text{if } \rho \geqslant 0,\\ u(x_T, 0, t) + \rho \dfrac{\partial u}{\partial \boldsymbol{n}}(x_T, 0, t) & \text{if } \rho < 0.\end{cases}$$

For $0 < h < \delta_{\text{ex}}$, we define another function $\tilde{u} : \widetilde{M} \times [-T, T] \to \mathbb{R}$ by $\tilde{u} = u$ on $M \times [-T, T]$, and

$$\tilde{u}(x_T, \rho, t) = \phi\Big(\frac{\rho}{h}\Big)\tilde{u}_{\text{ex}}(x_T, \rho, t) \quad \text{for } \rho < 0, \tag{3-7}$$

where ϕ is a monotone increasing smooth function vanishing on $(-\infty, -1]$ and equal to 1 on $[0, \infty)$ with $\|\phi\|_{C^2} \leqslant 8$. Then $\tilde{u} = 0$ when $\rho \leqslant -h$.

Lemma 3.5. *Let $(\widetilde{M}, \tilde{g})$ be an extension of (M, g) satisfying Lemma 3.4 with the extension width δ_{ex}. Let Γ be a connected open subset of ∂M. Assume*

$$u|_{\partial M\times[-T,T]} \in H^{2,2}(\partial M \times [-T, T]), \qquad \frac{\partial u}{\partial \boldsymbol{n}} \in H^{2,2}(\partial M \times [-T, T]).$$

Then we have

$$\|\tilde{u}\|^2_{H^1(\Omega_\Gamma\times[-T,T])} \leqslant Ch^{-1}\|u\|^2_{H^1(\Gamma\times[-T,T])} + Ch\Big\|\frac{\partial u}{\partial \boldsymbol{n}}\Big\|^2_{H^1(\Gamma\times[-T,T])},$$
$$\|(\partial_t^2 - \Delta_{\tilde{g}})\tilde{u}\|^2_{L^2(\Omega_\Gamma\times[-T,T])} \leqslant Ch^{-3}\|u\|^2_{H^{2,2}(\Gamma\times[-T,T])} + Ch^{-1}\Big\|\frac{\partial u}{\partial \boldsymbol{n}}\Big\|^2_{H^{2,2}(\Gamma\times[-T,T])},$$

where $\Omega_\Gamma = \Gamma \times [-\delta_{\text{ex}}, 0]$ denotes the part of the manifold extension over Γ, and the constants explicitly depend on n, K_1.

Furthermore, suppose $u \in H^2(M \times [-T, T])$ is a solution of the nonhomogeneous wave equation $Pu = f$ with $f \in L^2(M \times [-T, T])$. Then $\tilde{u} \in H^1(\widetilde{M} \times [-T, T])$ and $(\partial_t^2 - \Delta_{\tilde{g}})\tilde{u} \in L^2(\widetilde{M} \times [-T, T])$.

Proof. First we estimate the H^1-norm of $\tilde{u}$ over Ω_Γ. Here we only estimate the dominating term in h; the other terms can be done in the same way. Denote by ∂_α, ∂_n, ∂_t the derivatives with respect to x^α-, x^n-coordinates and time t, respectively. We denote $\partial_\alpha u$, $\partial_n u$, $\partial_t u$ evaluated at $(x_T, 0, t)$ by u_α, u_n, u_t and $\phi'(s) = (d/ds)\phi(s)$, evaluated at $s = \rho/h$. In addition, whenever we write the function u without specifying where it is evaluated, the evaluation is also done at $(x_T, 0, t)$. By the definition of $\tilde{u}$,

$$(\partial_n \tilde{u})(x_T, \rho, t) = h^{-1}(u + \rho u_n)\phi' + u_n \phi. \tag{3-8}$$

Since $\tilde{u}$ vanishes unless $\rho \in [-h, 0]$, we have

$$\begin{aligned}\|\partial_n \tilde{u}\|^2_{L^2(\Omega_\Gamma \times [-T,T])} &= \int_{-T}^{T}\int_\Gamma \int_{-\delta_{\mathrm{ex}}}^{0} |h^{-1}(u+\rho u_n)\phi' + u_n\phi|^2 \, dx_T \, d\rho \, dt \\ &\leqslant C \int_{-T}^{T}\int_\Gamma \int_{-h}^{0} (h^{-2}u^2 + h^{-2}\rho^2 u_n^2 + u_n^2) \, dx_T \, d\rho \, dt \\ &\leqslant Ch^{-1}\|u\|^2_{L^2(\Gamma\times[-T,T])} + Ch \left\| \frac{\partial u}{\partial \boldsymbol{n}} \right\|^2_{L^2(\Gamma\times[-T,T])}.\end{aligned}$$

Next we estimate the Laplacian of $\tilde{u}$ over Ω_Γ for $\rho \in [-h, 0]$. In the boundary normal coordinate of our choice, by definition (2-1) we have

$$\begin{aligned}\Delta_{\tilde{g}}\tilde{u} &= \sum_{i,j=1}^{n} \frac{1}{\sqrt{|\tilde{g}|}} \partial_i\big(\sqrt{|\tilde{g}|}\tilde{g}^{ij}\partial_j \tilde{u}\big) \\ &= \frac{1}{\sqrt{|\tilde{g}|}} \partial_n\big(\sqrt{|\tilde{g}|}\tilde{g}^{nn}\partial_n \tilde{u}\big) + \sum_{\alpha,\beta=1}^{n-1} \frac{1}{\sqrt{|\tilde{g}|}} \partial_\alpha\big(\sqrt{|\tilde{g}|}\tilde{g}^{\alpha\beta}\partial_\beta \tilde{u}\big) = A_1 + A_2,\end{aligned}$$

where $|\tilde{g}|$ denotes the determinant of the matrix $(\tilde{g}_{ij})$. We estimate A_2 as

$$\begin{aligned}A_2(x_T, \rho, t) &= \sum_{\alpha,\beta=1}^{n-1} \frac{1}{\sqrt{|\tilde{g}|}} \partial_\alpha\big(\sqrt{|\tilde{g}|}\tilde{g}^{\alpha\beta}\partial_\beta \tilde{u}\big) \\ &= \sum_{\alpha,\beta} \frac{\partial_\alpha |\tilde{g}|}{2|\tilde{g}|} \tilde{g}^{\alpha\beta}\partial_\beta \tilde{u} + (\partial_\alpha \tilde{g}^{\alpha\beta})(\partial_\beta \tilde{u}) + \tilde{g}^{\alpha\beta}\partial_\alpha\partial_\beta \tilde{u}.\end{aligned}$$

Hence we have

$$\begin{aligned}|A_2(x_T, \rho, t)| &\leqslant C\sum_{\alpha,\beta}(|u_\beta| + h|u_{n\beta}|) + C\sum_{\alpha,\beta}|\partial_\alpha\partial_\beta(u+\rho u_n)|(x_T, 0, t) \\ &\leqslant C\sum_{\alpha,\beta}(|u_{\alpha\beta}| + h|u_{n\alpha\beta}|) + C\sum_{\beta}(|u_\beta| + h|u_{n\beta}|),\end{aligned}$$

where the constants explicitly depend on n, K_1 due to the C^1 metric bound (3-4).

Finally we estimate A_1 and the time derivatives. Since $\tilde{g}^{nn} = 1$, we know that

$$A_1(x_T, \rho, t) = \frac{\partial_n |\tilde{g}|}{2|\tilde{g}|}\partial_n \tilde{u} + \partial_n^2 \tilde{u}.$$

We differentiate (3-8) again:

$$(\partial_n^2 \tilde{u})(x_T, \rho, t) = h^{-2}(u+\rho u_n)\phi'' + 2h^{-1}u_n\phi'.$$

Hence we have

$$\begin{aligned}|((\partial_t^2-\partial_n^2)\tilde{u})(x_T,\rho,t)| &= |(u_{tt}+\rho u_{ntt})\phi-(\partial_n^2\tilde{u})(x_T,\rho,t)| \\ &\leqslant Ch^{-2}|u|+Ch^{-1}|u_n|+C|u_{tt}|+Ch|u_{ntt}|,\end{aligned}$$

which leads to a similar estimate for $(\partial_t^2\tilde{u}-A_1)(x_T,\rho,t)$ by (3-8). Thus,

$$|((\partial_t^2-\Delta_{\tilde{g}})\tilde{u})(x_T,\rho,t)|\leqslant Ch^{-2}|u|+Ch^{-1}|u_n|+C(|u_{tt}|+h|u_{ntt}|)+C\sum_{\alpha,\beta}(|u_\alpha|+|u_{\alpha\beta}|+h|u_{n\alpha}|+h|u_{n\alpha\beta}|),$$

where all terms on the right-hand side are boundary data evaluated at $(x_T,0,t)$. Then the second estimate of the lemma immediately follows from integrating the last inequality.

Now we additionally assume that $u\in H^2(M\times[-T,T])$ is a (strong) solution of the nonhomogeneous wave equation $Pu=f$ with $f\in L^2(M\times[-T,T])$. By the regularity result for the wave equation (e.g., Theorem 2.30 in [Katchalov et al. 2001]), the solution u is in the energy class

$$u\in C([-T,T];H^1(M))\cap C^1([-T,T];L^2(M)).$$

From the definition (3-7), the weak derivatives of $\tilde{u}(\,\cdot\,,t)$ exist on $\widetilde{M}$ for any fixed $t\in[-T,T]$. Since the Cauchy data are in $H^{2,2}$, we have $\tilde{u}(\,\cdot\,,t)\in H^1(\widetilde{M})$ for all t directly by definition (3-7), and therefore $\tilde{u}\in H^1(\widetilde{M}\times[-T,T])$.

Since the Cauchy data are in $H^{2,2}$, the definition (3-7) also indicates that $\tilde{u}\in H^{2,2}((\widetilde{M}-M)\times[-T,T])$. Hence over $\widetilde{M}-M$,

$$\tilde{f}_{\mathrm{ex}}:=(\partial_t^2-\Delta_{\tilde{g}})\tilde{u}\in L^2((\widetilde{M}-M)\times[-T,T]).$$

Define a function $\tilde{f}:\widetilde{M}\times[-T,T]\to\mathbb{R}$ by $\tilde{f}=f$ over M and $\tilde{f}=\tilde{f}_{\mathrm{ex}}$ over $\widetilde{M}-M$. Clearly $\tilde{f}\in L^2(\widetilde{M}\times[-T,T])$. Thus the only part left is to show that $(\partial_t^2-\Delta_{\tilde{g}})\tilde{u}=\tilde{f}$ on $\widetilde{M}\times[-T,T]$ in the weak form. Observe that the wave equation on either M or $\widetilde{M}-M$ is well-defined pointwise. Then for any test function $\varphi\in H_0^1(\widetilde{M}\times[-T,T])$, by applying the wave equation separately on M, $\widetilde{M}-M$ and Green's formula, we have

$$\begin{aligned}\int_{-T}^{T}\int_{\widetilde{M}}(-\partial_t\tilde{u}\,\partial_t\varphi+\langle\nabla\tilde{u},\nabla\varphi\rangle_{\tilde{g}}) &= \int_{-T}^{T}\int_{M\cup(\widetilde{M}-M)}(-\partial_t\tilde{u}\,\partial_t\varphi+\langle\nabla\tilde{u},\nabla\varphi\rangle_{\tilde{g}}) \\ &= \int_{-T}^{T}\int_M f\varphi-\int_{-T}^{T}\int_{\partial M}\frac{\partial u}{\partial\boldsymbol{n}}\varphi+\int_{-T}^{T}\int_{\widetilde{M}-M}\tilde{f}_{\mathrm{ex}}\varphi+\int_{-T}^{T}\int_{\partial M}\frac{\partial\tilde{u}}{\partial\boldsymbol{n}}\varphi.\end{aligned}$$

Due to the definition (3-7), the normal derivative of $\tilde{u}$ from either side of ∂M coincides and hence the boundary terms cancel out. This shows that the wave equation is satisfied on $\widetilde{M}\times[-T,T]$ in the weak form, with the source term in $L^2(\widetilde{M}\times[-T,T])$. □

3.3. ***Distance functions.*** Later in the proof of Theorem 3.1, we will need to switch back and forth to different distance functions. The following lemma shows relations between distance functions.

Lemma 3.6. *Let $(\widetilde{M},\tilde{g})$ be an extension of (M,g) satisfying Lemma 3.4 with the extension width δ_{ex}. Denote the distance functions of M and $\widetilde{M}$ by d and $\tilde{d}$, respectively. Then there exists a uniform constant r_b*

explicitly depending only on K_1, i_0 *such that the following inequality holds for any* $x, y \in M$ *as long as* $\delta_{\text{ex}} \leqslant r_b$:

$$\tilde{d}(x, y) \leqslant d(x, y) \leqslant (1 + 3K_1\delta_{\text{ex}})\tilde{d}(x, y).$$

If $x, y \in \widetilde{M} - M$, *then the second inequality holds after replacing* $d(x, y)$ *with* $d(x^\perp, y^\perp)$, *where* $x^\perp$ *denotes the normal projection of* x *onto* ∂M. *If* $x \in \widetilde{M} - M$, $y \in M$, *then the second inequality holds for* $d(x^\perp, y)$.

Furthermore, if a minimizing geodesic of $\widetilde{M}$ *between* $x, y \in \widetilde{M}$ *lies in the boundary normal* (*tubular*) *neighborhood of* ∂M *of width* δ_{ex}, *then we have*

$$d_{\partial M}(x^\perp, y^\perp) \leqslant (1 + 3K_1\delta_{\text{ex}})\tilde{d}(x, y),$$

where $d_{\partial M}$ *denotes the intrinsic distance function of* ∂M.

Proof. The first inequality is trivial and we prove the second inequality. Consider any (distance) minimizing geodesic $\tilde{\gamma}$ of $\widetilde{M}$ from x to y; its length $L(\tilde{\gamma})$ satisfies $L(\tilde{\gamma}) = \tilde{d}(x, y)$ by definition. It is known that $\tilde{\gamma}$ is a C^1 curve with arclength parametrization (e.g., Section 2 in [Alexander et al. 1987]). Observe that the second inequality follows trivially if $\tilde{\gamma}$ lies entirely in M. Since the statement of the lemma is independent of the choice of coordinate, we work in the boundary normal coordinate $(x^1, \dots, x^{n-1}, \rho(x))$ of ∂M.

Suppose $\tilde{\gamma}$ lies entirely in $\widetilde{M} - \text{int}(M)$ with both endpoints x, y on ∂M. Consider the normal projection, denoted by γ, of $\tilde{\gamma}$ onto the boundary ∂M with respect to the boundary normal coordinate. More precisely, if $\tilde{\gamma}(s) = (x_1(s), \dots, x_{n-1}(s), x_n(s))$ in a boundary normal coordinate near a point on $\tilde{\gamma}$, then its normal projection has the form $\gamma(s) = (x_1(s), \dots, x_{n-1}(s), 0)$. The fact that $\tilde{\gamma}$ is of C^1 implies that $x_i(s)$ is a C^1 function for any i. Hence γ is a C^1 (possibly not regular or simple) curve in ∂M from x to y with the induced parametrization from $\tilde{\gamma}$. Note that γ may not be differentiable with respect to its own arclength parameter.

As a consequence, the length $L(\gamma)$ of γ can be written as

$$L(\gamma) = \int_0^{L(\tilde{\gamma})} \sqrt{g(\gamma'(s), \gamma'(s))}\, ds = \int_0^{L(\tilde{\gamma})} \sqrt{g(\tilde{\gamma}_T'(s)|_{\gamma(s)}, \tilde{\gamma}_T'(s)|_{\gamma(s)})}\, ds,$$

where $\tilde{\gamma}_T'(s)$ denotes the vector field with constant coefficients in the frame $(\partial/\partial x^1, \dots, \partial/\partial x^{n-1})$, with the coefficients being the tangential components of the tangent vector $\tilde{\gamma}'(s)$ of $\tilde{\gamma}$. Note that $\tilde{\gamma}_T'(s)$ is a Jacobi field for the normal coordinate function $\rho(x)$. For every fixed s, by the definition of the second fundamental form (more precisely the shape operator),

$$\frac{\partial}{\partial \rho}\tilde{g}_\rho(\tilde{\gamma}_T', \tilde{\gamma}_T') = 2\tilde{g}_\rho(S_\rho(\tilde{\gamma}_T'), \tilde{\gamma}_T'),$$

where $\tilde{g}_\rho$ and S_ρ denote the metric and the shape operator of the equidistant hypersurface from ∂M (in $\widetilde{M} - M$) with distance $|\rho|$ (i.e., the level set $\tilde{d}(\,\cdot\,, \partial M) = |\rho|$). Observe that Lemma 6.1 holds in the boundary normal neighborhood of ∂M regardless of which side the neighborhood extends to, thanks to Lemma 3.4(3). Then the first part of Lemma 6.1 indicates that for sufficiently small $|\rho|$ depending only on K_1, i_0,

$$\left|\frac{\partial}{\partial\rho}\tilde{g}_\rho(\tilde{\gamma}'_T,\tilde{\gamma}'_T)\right| \leqslant 4K_1\tilde{g}_\rho(\tilde{\gamma}'_T,\tilde{\gamma}'_T).$$

Thus by Gronwall's inequality, we have

$$g(\tilde{\gamma}'_T|_\gamma,\tilde{\gamma}'_T|_\gamma) \leqslant \tilde{g}_\rho(\tilde{\gamma}'_T,\tilde{\gamma}'_T)e^{4K_1|\rho|}.$$

Since the extended metric $\tilde{g}$ is a product metric in the boundary normal coordinate, then $\tilde{g}(\tilde{\gamma}'_T|_{\tilde{\gamma}},\tilde{\gamma}'_T|_{\tilde{\gamma}}) \leqslant \tilde{g}(\tilde{\gamma}',\tilde{\gamma}')$. Hence for sufficiently small δ_{ex} depending only on K_1 and $|\rho| \leqslant \delta_{\text{ex}}$, we obtain

$$\begin{aligned} L(\gamma) &\leqslant e^{2K_1|\rho|}\int_0^{L(\tilde{\gamma})}\sqrt{\tilde{g}_\rho(\tilde{\gamma}'_T(s),\tilde{\gamma}'_T(s))}\,ds \\ &\leqslant e^{2K_1\delta_{\text{ex}}}\int_0^{L(\tilde{\gamma})}\sqrt{\tilde{g}(\tilde{\gamma}'(s),\tilde{\gamma}'(s))}\,ds \leqslant (1+3K_1\delta_{\text{ex}})\tilde{d}(x,y), \end{aligned}$$

which yields the second inequality by definition.

In general, if $\tilde{\gamma}$ crosses ∂M with both endpoints in M, we can divide $\tilde{\gamma}$ into segments in M and segments in $\widetilde{M}-M$. The lemma is trivially satisfied for the endpoints of any segment in M. Any (continuous) segment in $\widetilde{M}-M$ has endpoints on ∂M and lies entirely in $\widetilde{M}-\text{int}(M)$. Thus we apply the argument above for every segment in $\widetilde{M}-M$ and the estimate follows. Finally, if the endpoints of $\tilde{\gamma}$ are not both in M, then its projection γ is a curve between the projections of the endpoints of $\tilde{\gamma}$ onto M. This concludes the proof for the first part of the lemma.

Now we prove the second part of the lemma. Let $\tilde{\gamma}$ be the minimizing geodesic of $\widetilde{M}$ from x to y lying in the boundary normal tubular neighborhood of ∂M. If $\tilde{\gamma}$ lies entirely in M or $\widetilde{M}-\text{int}(M)$, one can use the previous argument to project $\tilde{\gamma}$ to a curve on ∂M and show the same estimate as the first part. The only difference is that when x, y are not in ∂M, the projection γ is a curve on ∂M from $x^\perp$ to $y^\perp$. In general, the estimate follows from dividing $\tilde{\gamma}$ into segments in M and in $\widetilde{M}-M$, and projecting both types of segments onto ∂M. □

Definition 3.7. For $h < i_0/2$, we consider the submanifold

$$M_h = \{x \in M : d(x,\partial M) \geqslant h\}.$$

Denote by $d_h : M_h \times M_h \to \mathbb{R}$ the intrinsic distance function of the submanifold M_h, and we extend it to any point $x \in \widetilde{M}-M_h$ by

$$d_h(x,z) = d_h(x^{\perp_h},z) + h^{-1}\tilde{d}(x,x^{\perp_h}) \quad \text{for } z \in M_h,\ x \in \widetilde{M}-M_h, \tag{3-9}$$

where $x^{\perp_h} \in \partial M_h$ is the unique normal projection of $x \in \widetilde{M}-M_h$ onto ∂M_h within the boundary normal neighborhood of ∂M such that $\tilde{d}(x,x^{\perp_h}) = \tilde{d}(x,\partial M_h)$. In this definition we require at least one of the points to belong to M_h. Note that a similar notation $x^\perp$ denotes the normal projection of x onto ∂M.

Thus the path between $z \in M_h$ and a point $x \in \widetilde{M}-M_h$ realizing $d_h(x,z)$ is a broken curve consisting of a geodesic of M_h and a vertical line of the boundary neighborhood (see Figure 2).

In general, the intrinsic distance function of a manifold with boundary is at most of $C^{1,1}$: the function $d_h(\cdot\,,z)$ is at most of $C^{1,1}$ even on $M_h-\{z\}$. We need to smoothen it in order to match the $C^{2,1}$ regularity required by Theorem 2.2.

Definition 3.8. For a fixed $z\in M_h$ and any $x\in M$, we denote by $d_h^s(x,z)$ the smoothening of $d_h(x,z)$ via convolution in a ball of radius $r<\delta_{\mathrm{ex}}/2$ around the center x with respect to the distance $\tilde d$ of $\widetilde M$. More precisely,

$$d_h^s(x,z)=c_n r^{-n}\int_{\widetilde M} k_1\biggl(\frac{\tilde d(y,x)}{r}\biggr)\,d_h(y,z)\,dy, \tag{3-10}$$

where $k_1:\mathbb R\to\mathbb R$ is a nonnegative smooth mollifier supported on $\bigl[\frac12,1\bigr]$, and dy denotes the Riemannian volume form on $\widetilde M$. The constant c_n is the normalization constant such that

$$c_n r^{-n}\int_{\mathbb R^n} k_1\biggl(\frac{|v|}{r}\biggr)\,dv=1, \tag{3-11}$$

where dv denotes the Euclidean volume form on $\mathbb R^n$.

Lemma 3.9. *Let δ_{ex} be sufficiently small determined in Lemma 3.4. For sufficiently small r depending on $n, K_1, K_2, i_0, r_0, r_g$, the function $d_h^s(\cdot\,,z)$ is of $C^{2,1}$ on M for any fixed $z\in M_h$. Furthermore, in the coordinate of our choice, the $C^{2,1}$-norm of $d_h^s(\cdot\,,z)$ is uniformly bounded explicitly depending on r, n, $\|R_M\|_{C^1}$.*

Proof. By Lemma 3.4(4), for sufficiently small δ_{ex}, we know $r_{\mathrm{CAT}}(\widetilde M)$ is bounded below by $C(K_1,i_0,r_0)$. We restrict the smoothening radius to be less than this lower bound: $r<C(K_1,i_0,r_0)$. Then for any $y\in\widetilde B_r(x)$, there is a unique minimizing geodesic between x and y. Furthermore, no conjugate points occur along geodesics of length less than $\pi/(2K_1)$ [Alexander et al. 1993, Corollary 3]. Since $\widetilde B_r(x)\cap\partial\widetilde M=\varnothing$ for any $x\in M$ as $r<\delta_{\mathrm{ex}}/2$, we know $\tilde d(\cdot\,,x)$ is simply a geodesic distance function in the ball of the smoothening radius around any $x\in M$. As a consequence, $\tilde d(\cdot\,,x)$ is differentiable on $\widetilde B_r(x)$ and $|\nabla\tilde d(\cdot\,,x)|=1$.

By our choice of coordinate charts in Section 3.1, for any $x'\in\widetilde M$, the ball $\widetilde B_{r_g/2}(x')$ or the cylinder $B_{\partial M}(y,r_g/2)\times(\rho-r_g/2,\rho+r_g/2)$ is contained in at least one of the coordinate charts defined in Lemma 3.4, where $x'=(y,\rho)$ if x' is in the boundary normal coordinate of ∂M. Then by Lemma 3.6, the ball $\widetilde B_{r_g/4}(x')$ of $\widetilde M$ is contained in one of the coordinate charts if we choose a smaller r_b depending on K_1. Hence, for $r<r_g/4$, $\widetilde B_r(x)$ is contained in one of these coordinate charts for any $x\in M$, and therefore $\tilde d(\cdot\,,x)$ is of $C^{2,1}$ on $\widetilde B_r(x)-\{x\}$ by Lemma 3.4(2) and Theorem 2.1 in [DeTurck and Kazdan 1981]. Observe that $\tilde d(\cdot\,,x)$ is bounded below by $r/2$ in the support of k_1, which yields a bound on higher derivatives of $\tilde d(\cdot\,,x)$. This shows that the function $d_h^s(\cdot\,,z)$ is of $C^{2,1}$.

To estimate the $C^{2,1}$-norm of $d_h^s(\cdot\,,z)$, it suffices to estimate the $C^{2,1}$-norm of $\tilde d(\cdot\,,y)$ on the annulus $\widetilde B_r(y)-\widetilde B_{r/2}(y)$. Due to the Hessian comparison theorem (e.g., [Petersen 2006, Theorem 27, p. 175]), for sufficiently small r depending on K_1, we have $\|\widetilde\nabla^2\tilde d(\cdot\,,y)\|\leqslant 4r^{-1}$ on the annulus, where $\widetilde\nabla^2$ denotes the second covariant derivative on $\widetilde M$. In a local coordinate $(x^1,\dots,x^n)$ on $\widetilde M$, the covariant derivative has the form (e.g., [Petersen 2006, Chapter 2, p. 32])

$$(\widetilde\nabla^2\tilde d(\cdot\,,y))\Bigl(\frac{\partial}{\partial x^k},\frac{\partial}{\partial x^l}\Bigr)=\frac{\partial^2}{\partial x^k\partial x^l}\tilde d(\cdot\,,y)-\sum_{i=1}^n\widetilde\Gamma^i_{kl}\frac{\partial}{\partial x^i}\tilde d(\cdot\,,y),\quad k,l=1,\dots,n. \tag{3-12}$$

Hence in the coordinate charts of our choice, for sufficiently small r, (3-6) yields

$$\|\tilde{d}(\cdot, y)\|_{C^2} \leqslant Cr^{-1} \quad \text{on } \widetilde{B}_r(y) - \widetilde{B}_{r/2}(y). \tag{3-13}$$

An estimate on the $C^{2,1}$-norm can be obtained by differentiating the Riccati equation in polar coordinates $\tilde{g} = dr^2 + \tilde{g}_r$ around y, where ∂_r is the radial direction in the geodesic normal coordinate. Then on the annulus, examining the proof of Lemma 8 in [Hebey and Vaugon 1995] gives a bound

$$\|\widetilde{\nabla}^3 \tilde{d}(\cdot, y)\| \leqslant C(n, \|R_{\widetilde{M}}\|_{C^1}) r^{-2}.$$

Hence by differentiating the formula (3-12), for sufficiently small r depending on n, K_1, K_2, i_0, we obtain

$$\|\tilde{d}(\cdot, y)\|_{C^{2,1}} \leqslant C(n, \|R_{\widetilde{M}}\|_{C^1}) r^{-2} \quad \text{on } \widetilde{B}_r(y) - \widetilde{B}_{r/2}(y). \tag{3-14}$$

Then a straightforward differentiation yields an estimate on the $C^{2,1}$-norm of $d_h^s(\cdot, z)$. ☐

3.4. *Proof of Theorem 3.1.* Now we prove the main technical result Theorem 3.1, by constructing the functions and domains assumed in Theorem 2.2. The proof consists of several parts.

To begin with, let h be a positive number satisfying $h < \min\{1/5, i_0/10, r_b/10\}$, where $r_b = r_b(K_1, i_0)$ is the width of the boundary normal neighborhood determined in Lemma 6.1. For sufficiently small h only depending on n, K_1, K_2, i_0, $\mathrm{vol}(\partial M)$, we extend (M, g) to $(\widetilde{M}, \tilde{g})$ with the extension width $\delta_{\mathrm{ex}} = 5h$ such that Lemma 3.4 holds. Then we extend u to $\tilde{u}$ by (3-7) with the cut-off width h. Let r_g be the uniform radii of C^1 geodesic normal coordinates of M and ∂M such that metric bounds (3-6) hold. We have shown that r_g explicitly depends on n, $\|R_M\|_{C^1}$, $\|S\|_{C^1}$, i_0. Now we collect all these relevant parameters and impose the following requirements on the choice of h due to technical reasons:

$$0 < h < \min\Big\{\frac{1}{10}, \frac{T}{8}, \frac{i_0}{10}, \frac{r_0}{10}, \frac{r_g}{10}, \frac{r_b}{10}, \frac{i_b(\overline{\Gamma})}{10}, \frac{\pi}{12K_1}\Big\}. \tag{3-15}$$

The part of the manifold extension over Γ is denoted by $\Omega_\Gamma = \Gamma \times [-5h, 0]$. The number $\min\{1, T^{-1}\}$ will be frequently used in this proof and we denote it by

$$a_T = \min\{1, T^{-1}\}. \tag{3-16}$$

We restrict the choice of h once again such that, for sufficiently small h,

$$r_{\mathrm{CAT}}(M_h) \geqslant \min\Big\{\frac{2r_0}{3}, \frac{\pi}{2K_1}\Big\}, \quad r_{\mathrm{CAT}}(\widetilde{M}) \geqslant \min\Big\{\frac{2r_0}{3}, \frac{\pi}{2K_1}\Big\}. \tag{3-17}$$

This is possible due to Lemma 6.2. We remark that the dependency of h is not explicit in Lemma 6.2(3), and one can instead use the explicit lower bound in Lemma 6.2(2).

With the choice of $\delta_{\mathrm{ex}} = 5h$ and h as above, the function $d_h(\cdot, z)$ defined in (3-9) is Lipschitz with a Lipschitz constant $2h^{-1}$ (Lemma 6.3(3)). In Definition 3.8, we set the smoothening radius to be $r = a_T h^3$. Then it follows that $|d_h^s(x, z) - d_h(x, z)| < 2a_T h^2$ for any $x \in M$ (Lemma 6.3(4)).

Assume h is sufficiently small so that Lemma 3.9 holds. For any $z \in M_h$ and $x \in M$ satisfying $h/4 \leqslant d_h(x, z) \leqslant \min\{i_0/2, r_0/2, \pi/(6K_1)\}$, we have $|\nabla_x d_h^s(x, z)| > 1 - 2h$ (Lemma 6.5). Outside the injectivity radius this gradient can be 0 if cut points are involved. This lower bound being close to 1 is crucial for our method to ensure no loss of domain, and we define d_h (3-9) with the h^{-1} scaling in the boundary neighborhood specifically to guarantee it. While this lower bound is almost trivial when z is far from ∂M_h, careful treatment is required when the manifold boundary is involved.

For $|b| \leqslant 5h$, we define the set

$$\Gamma_b(h) = \{x \in \widetilde{M} : \rho(x) = b,\ x^\perp \in \Gamma,\ d_{\partial M}(x^\perp, \partial\Gamma) \geqslant h\}, \tag{3-18}$$

where $\partial\Gamma$ denotes the boundary of Γ in ∂M. The function $\rho(x)$ is the coordinate function in the normal direction defined in (3-5). Note that if $\Gamma = \partial M$, the last two conditions above are automatically satisfied, and then the set above is simply the level set for the normal coordinate function.

Recall that $\tilde{u}$, the extension of u to $\widetilde{M}$ defined by (3-7), vanishes on $\Gamma_b(0)$ for all $b \leqslant -h$. The set $\Gamma_{-2h}(0)$ is the set from which we intend to propagate the unique continuation. More precisely, we start the propagation from an h-net in $\Gamma_{-2h}(8h)$. The reason of this specific choice is the following.

Sublemma 1. *For sufficiently small h only depending on K_1, we have*

$$\tilde{d}(z, \partial(M \cup \Omega_\Gamma) - \partial\widetilde{M}) \geqslant 7h \quad \textit{for any } z \in \Gamma_{-2h}(8h),$$

where $\Omega_\Gamma = \Gamma \times [-5h, 0]$ is the part of the manifold extension over Γ.

Proof. Let y be a point in $\partial(M \cup \Omega_\Gamma) - \partial\widetilde{M}$ realizing the distance to z. Suppose $\tilde{d}(z, y) < 7h$. Then the minimizing geodesic of $\widetilde{M}$ from z to y lies in the boundary normal (tubular) neighborhood of ∂M of width $5h$. Hence Lemma 3.6 implies that

$$d_{\partial M}(z^\perp, y^\perp) \leqslant (1 + 15K_1 h)\tilde{d}(z, y) < 7h(1 + 15K_1 h).$$

However, we know $d_{\partial M}(z^\perp, y^\perp) \geqslant 8h$ by the definition (3-18). Hence we get a contradiction for sufficiently small h only depending on K_1. □

Initial step. As the initial step, we propagate the unique continuation from outside the manifold M to a region close to Γ in M.

Consider the function $\xi : [0, +\infty) \to \mathbb{R}$ defined by

$$\xi(x) = \frac{(h - x)^3}{h^3} \quad \text{for } x \in [0, h], \tag{3-19}$$

and $\xi(x) = 0$ for $x > h$. The function $\xi(x)$ on negative numbers can be defined in any way so that $\xi(x) \geq 1$ for $x < 0$, and $\xi(x)$ is smooth on $(-\infty, h)$. The function $\xi(x)$ is of $C^{2,1}$ on $\mathbb{R}$ and monotone decreasing on $[0, +\infty)$. Let $\{z_{0,j}\}_{j=1}^{J(0)}$ be an h-net in $\Gamma_{-2h}(8h)$: that is, for any $z \in \Gamma_{-2h}(8h)$, there exists some $z_{0,j}$ such that $\tilde{d}(z, z_{0,j}) < h$. We define

$$\psi_{0,j}(x, t) = \big((1 - \xi(6h - \tilde{d}(x, z_{0,j})))T - \tilde{d}(x, z_{0,j})\big)^2 - t^2, \tag{3-20}$$

and consider the following domains (see Figure 2):

$$\Omega^0_{0,j} = \{(x, t) \in \widetilde{M} \times [-T, T] : \psi_{0,j}(x, t) > h^2,\ \rho(x) > -3h/2\}. \tag{3-21}$$

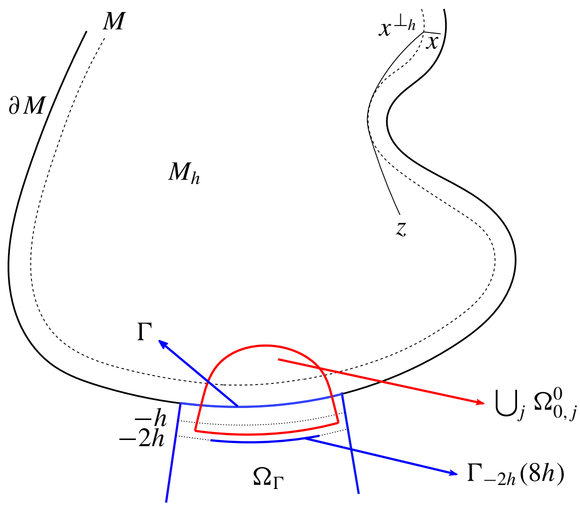

Figure 2. Domains for the initial step. Enclosed by the red solid line is the domain we work in, and it is close to Γ.

Note that in general, the domain characterized by $\psi_{0,j}(x,t) > h^2$ has two connected components. Here we define $\Omega^0_{0,j}$ to be the connected component characterized by $(1-\xi(6h-\tilde{d}(x,z_{0,j})))T-\tilde{d}(x,z_{0,j})>0$.[1] Observe that in this connected component, it holds that $\tilde{d}(x,z_{0,j}) < 6h$ due to the definition of the function ξ.

Then we define

$$\Upsilon = \{x \in \Omega_\Gamma : -2h \leqslant \rho(x) \leqslant -h\} \times [-T, T], \tag{3-22}$$

$$\Omega_{0,j} = \{(x,t) \in \Omega^0_{0,j} - \Upsilon : \psi_{0,j}(x,t) > 4h^2\}. \tag{3-23}$$

Now we prove that the conditions assumed in Theorem 2.2 are satisfied for $\psi_{0,j}$, $\Omega^0_{0,j}$, $\Omega_{0,j}$, Υ, $\psi_{\max,0} = (T-h)^2$, and therefore Theorem 2.2 applies. A stability estimate will be derived at the end of the proof.

(1) We show that $\psi_{0,j}$ is of $C^{2,1}$ and noncharacteristic in $\Omega^0_{0,j}$. Indeed, for any $(x,t) \in \Omega^0_{0,j}$, we have $\tilde{d}(x,z_{0,j}) < 6h$ by the definition of $\psi_{0,j}$. Hence any minimizing geodesic of $\widetilde{M}$ from $z_{0,j}$ to x must not intersect $\partial\widetilde{M}$; otherwise the length of such geodesic would exceed $6h$ due to the condition that $\rho(x) > -3h/2$. Furthermore, by our choice $h < \min\{r_0/10, \pi/(12K_1)\}$ and (3-17), the minimizing geodesic from $z_{0,j}$ to any $x \in \widetilde{B}_{6h}(z_{0,j})$ is unique and no conjugate points can occur. Therefore $\tilde{d}(\,\cdot\,, z_{0,j})$ is a $C^{2,1}$ geodesic distance function in $\Omega^0_{0,j}$, which shows that $\psi_{0,j}$ is of $C^{2,1}$ in $\Omega^0_{0,j}$. Moreover, since $\tilde{d}(x,z_{0,j}) > h/2$ for any $(x,t) \in \Omega^0_{0,j}$ by definition, the $C^{2,1}$-norms of $\tilde{d}(\,\cdot\,, z_{0,j})$ and $\psi_{0,j}$ are uniformly bounded in $\Omega^0_{0,j}$ due to (3-14).

Next we prove that $\psi_{0,j}$ is noncharacteristic in $\Omega^0_{0,j}$. For any $(x,t) \in \Omega^0_{0,j}$,

$$\nabla_x \psi_{0,j} = 2\big((1-\xi(6h-\tilde{d}(x,z_{0,j})))T - \tilde{d}(x,z_{0,j})\big)(\xi' T \nabla_x \tilde{d}(x,z_{0,j}) - \nabla_x \tilde{d}(x,z_{0,j})).$$

Note that ξ' is evaluated at $6h - \tilde{d}(x,z_{0,j})$ in the formula above. Since $\xi' \leqslant 0$, we have

$$|\xi' T \nabla_x \tilde{d}(x,z_{0,j}) - \nabla_x \tilde{d}(x,z_{0,j})| \geqslant |\nabla_x \tilde{d}(x,z_{0,j})| = 1.$$

[1] Throughout the proof, whenever we define a domain using level sets of a similar function, we exactly mean this one type of connected component.

Hence,

$$p((x,t),\nabla\psi_{0,j}) = \sum_{k,l=1}^{n} \tilde{g}^{kl}(\partial_{x_k}\psi_{0,j})(\partial_{x_l}\psi_{0,j}) - |\partial_t\psi_j|^2 = |\nabla_x\psi_{0,j}|^2 - |\partial_t\psi_{0,j}|^2$$
$$\geqslant 4\big((1-\xi(6h-\tilde{d}(x,z_{0,j})))T-\tilde{d}(x,z_{0,j})\big)^2 - 4t^2 = 4\psi_{0,j}^2(x,t) > 4h^2.$$

(2) The extended function $\tilde{u}$ defined by (3-7) vanishes on Υ. We claim that

$$\varnothing \neq \{(x,t)\in\Omega_{0,j}^0 : \psi_{0,j}(x,t) > (T-h)^2\} \subset \Upsilon.$$

Indeed, for any (x,t) in the set, it satisfies that $\tilde{d}(x,z_{0,j}) < h$, which indicates $\rho(x) < -h$. On the other hand, Sublemma 1 implies that $x \in \Omega_\Gamma$, and therefore $(x,t)\in\Upsilon$. For the nonemptyness, consider the point $x_j \in \Gamma_{-5h/4}(0)$ such that $\tilde{d}(x_j,z_{0,j}) = 3h/4$ (i.e., x_j is the projection of $z_{0,j}$ onto $\Gamma_{-5h/4}(0)$). By definition, we have $\psi_{0,j}(x_j,0) = (T-3h/4)^2 > (T-h)^2$. This also shows that $(x_j,0)\in\Omega_{0,j}^0$ by definition when $T>2h$, which yields the nonemptyness.

(3) We show that $\mathrm{dist}_{\widetilde{M}\times\mathbb{R}}(\partial\Omega_{0,j}^0,\Omega_{0,j}) > 0$. It suffices to prove $\overline{\Omega}_{0,j}\subset\Omega_{0,j}^0$. For any $(x,t)\in\Omega_{0,j}^0$, we have $\tilde{d}(x,z_{0,j}) < 6h$ by the definition of $\psi_{0,j}$, which implies that $\Omega_{0,j}^0 \subset M\cup\Omega_\Gamma$ due to Sublemma 1. This indicates that the boundaries of $\Omega_{0,j}^0$, $\Omega_{0,j}$ are determined only by $\psi_{0,j}$ and $\rho(x)$. Since we know $\rho(x) > -h$ for any $(x,t)\in\Omega_{0,j}$ by definition, clearly $\overline{\Omega}_{0,j}\subset\Omega_{0,j}^0$.

(4) We claim that $\bigcup_{j=1}^{J(0)}\Omega_{0,j}$ is connected and therefore its closure is connected. Take two reference points z_{0,j_1}, z_{0,j_2} satisfying $\tilde{d}(z_{0,j_1},z_{0,j_2}) < 3h$. Consider $(z_{0,j_1}^\perp,0)\in\partial M\times[-T,T]$. Directly checked by the definition of $\Omega_{0,j}$, this point $(z_{0,j_1}^\perp,0)$ is in both Ω_{0,j_1} and Ω_{0,j_2}. In particular, this shows $\Omega_{0,j_1}\cap\Omega_{0,j_2}\neq\varnothing$ if $\tilde{d}(z_{0,j_1},z_{0,j_2}) < 3h$. Since each $\Omega_{0,j}$ is path connected, so is $\Omega_{0,j_1}\cup\Omega_{0,j_2}$. The claim follows from the fact that for any two points in the h-net $\{z_{0,j}\}$ we can find a chain of $\{z_{0,j}\}$ such that every pair of adjacent points in this chain has distance less than $3h$.

In order to propagate further in subsequent steps, we need to estimate how much $\bigcup_j\Omega_{0,j}$ covers in the original manifold M.

Sublemma 2. $\big(\bigcup_{b\in[0,2h]}\Gamma_b(8h)\big)\times[-T+6h,T-6h]\subset\bigcup_{j=1}^{J(0)}\Omega_{0,j}$.

Proof. For any (x,t) in the left-hand set, there exists j_0 such that $\tilde{d}(x,z_{0,j_0}) < 5h$ due to the definition of h-net, which indicates that the ξ-term in ψ_{0,j_0} (3-20) vanishes. Thus

$$\psi_{0,j_0}(x,t) = (T-\tilde{d}(x,z_{0,j_0}))^2 - t^2 > (T-5h)^2-(T-6h)^2 > 5h^2,$$

where we used $T>8h$. This shows that (x,t) is in both Ω_{0,j_0}^0 and Ω_{0,j_0}. □

Subsequent steps. After the initial step, the reference set is moved to $\Gamma_h(8h)$ and unique continuation is propagated up to $\Gamma_{2h}(8h)$. Let $\{z_{1,j}\}$ be an h-net in $\Gamma_h(10h)\subset M_h$ with respect to d_h. Note that here the range of the j index is different from that of the j index in the initial step, and a precise notation would be $\{z_{1,j}\}_{j=1}^{J(1)}$. We omit this dependence on the step number to keep the notations short. Set $T_1 = T-6h$ and $\rho_0 = \min\{i_0/2, r_0/2, r_g/4, \pi/(6K_1)\}$. We divide into two cases depending on if T is larger than ρ_0.

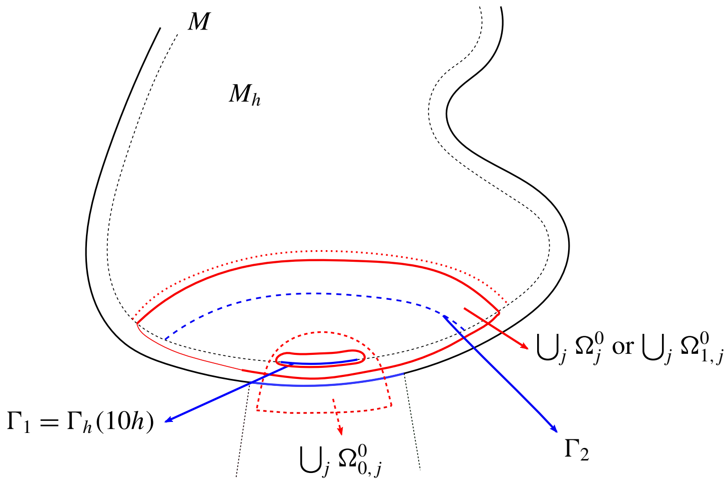

Figure 3. Domains for Case 1 or the first step in Case 2. Enclosed by the red solid lines is the domain we work in, and its boundary consists of two disjoint parts. This domain never reaches outside distance ρ_0, which is marked by the upper red dotted line. The blue dashed line Γ_2 is the reference set for the second step in Case 2.

Case 1: $T \leqslant \rho_0 = \min\{i_0/2, r_0/2, r_g/4, \pi/(6K_1)\}$. For any $(x,t) \in M \times [-T_1, T_1]$, we define the $C^{2,1}$ functions

$$\psi_j(x,t) = \big((1-\xi(d(x,\partial M)))T_1 - d_h^s(x, z_{1,j})\big)^2 - t^2, \tag{3-24}$$

and consider the domains[2]

$$\Omega_j^0 = \{(x,t) \in M \times [-T_1, T_1] : \psi_j(x,t) > 8T^2 h\} - \{x : d_h^s(x, z_{1,j}) \leqslant h/2\} \times [-T_1, T_1]. \tag{3-25}$$

Observe that $\xi(d(x,\partial M)) < 1$ in Ω_j^0 and hence Ω_j^0 never intersect with ∂M at any time. For any $(x,t) \in \Omega_j^0$, we have $h/2 < d_h^s(x, z_{1,j}) < T_1 \leqslant \rho_0 - 6h$ by definition. Then Lemma 6.3(4) indicates that $h/4 < d_h(x, z_{1,j}) < \min\{i_0/2, r_0/2, \pi/(6K_1)\}$, and hence Lemma 6.5 applies.

Then we define

$$\Omega_j = \Big\{(x,t) \in \Omega_j^0 - \bigcup_j \bar{\Omega}_{0,j} : \psi_j(x,t) > 9T^2 h\Big\}. \tag{3-26}$$

Now we prove that the conditions assumed in Theorem 2.2 are satisfied for ψ_j, Ω_j^0, Ω_j, $\psi_{\max} = (T_1 - 3h/4)^2$, together with relevant functions and domains in the initial step. The relevant domains are illustrated in Figure 3.

First we show that ψ_j is noncharacteristic at any $(x,t) \in \Omega_j^0$. For $x \in M - M_h$,

$$\nabla_x \psi_j = 2\big((1-\xi(d(x,\partial M)))T_1 - d_h^s(x, z_{1,j})\big)\big(-\xi' T_1 \nabla_x d(x,\partial M) - \nabla_x d_h^s(x, z_{1,j})\big).$$

Note that ξ' is evaluated at $d(x,\partial M)$ in the formula above.

For $x \in M - M_h$ with $d(x, \partial M_h) \geqslant a_T h^3$, the vectors $\nabla_x d_h(x, z_{1,j})$ and $\nabla_x d_h^s(x, z_{1,j})$ only differ by a small component $C(n, K_1, K_2)h^2$ due to (6-9). In particular, $\langle \nabla_x d_h(x, z_{1,j}), \nabla_x d_h^s(x, z_{1,j})\rangle > 0$ for

[2] The connected component characterized by $(1-\xi(d(x,\partial M)))T_1 - d_h^s(x, z_{1,j}) > 0$.

sufficiently small h depending on n, K_1, K_2. Hence by the definition of d_h (3-9),

$$\langle \nabla_x d(x,\partial M), \nabla_x d_h^s(x,z_{1,j})\rangle = -h\langle \nabla_x d_h(x,z_{1,j}), \nabla_x d_h^s(x,z_{1,j})\rangle < 0.$$

Then by Lemma 6.5 and $\xi' \leqslant 0$, we have

$$|-\xi' T_1 \nabla_x d(x,\partial M) - \nabla_x d_h^s(x,z_{1,j})| \geqslant |\nabla_x d_h^s(x,z_{1,j})| > 1-2h.$$

For $x \in M - M_h$ with $d(x,\partial M_h) < a_T h^3$, we have $|\xi'(d(x,\partial M))| < 3a_T^2 h^3 \leqslant 3T^{-1}h^3$ at such points by definitions (3-19) and (3-16). Therefore, for any $x \in M - M_h$ and sufficiently small h, we have

$$\begin{aligned} |\nabla_x \psi_j| &> 2|(1-\xi)T_1 - d_h^s|(1-2h-3h^3) \\ &> 2|(1-\xi)T_1 - d_h^s|(1-3h). \end{aligned} \tag{3-27}$$

On the other hand, if $x \in M_h$, then the ξ-term vanishes and the estimate above holds. Hence, for any $(x,t) \in \Omega_j^0$,

$$\begin{aligned} p((x,t),\nabla\psi_j) &= |\nabla_x \psi_j|^2 - |\partial_t \psi_j|^2 \\ &> 4((1-\xi)T_1 - d_h^s)^2(1-3h)^2 - 4t^2 \\ &> 4\psi_j(x,t) - 24T^2 h > 8T^2 h. \end{aligned} \tag{3-28}$$

This shows that ψ_j is noncharacteristic at any $(x,t) \in \Omega_j^0$.

It is straightforward to show the connectedness of $\big(\bigcup_j \bar{\Omega}_j\big) \cup \big(\bigcup_j \bar{\Omega}_{0,j}\big)$ in the same way as we did for $\bigcup_j \Omega_{0,j}$ in the initial step. The other conditions assumed in Theorem 2.2 follow from Sublemma 3 below and Sublemma 2.

Sublemma 3. *For sufficiently small $h < \frac{1}{8}$ depending on K_1, we have*

$$\varnothing \neq \{(x,t) \in \Omega_j^0 : \psi_j(x,t) > (T_1 - 3h/4)^2\} \subset \Big(\bigcup_{b\in[0,2h]} \Gamma_b(8h)\Big) \times [-T_1, T_1],$$

and $\operatorname{dist}_{\widetilde{M}\times\mathbb{R}}(\partial\Omega_j^0, \Omega_j) > 0$.

Proof. The nonemptyness follows from definition. For any (x,t) in the left-hand set, we know $d_h^s(x,z_{1,j}) < 3h/4$ by definition. Hence it suffices to show that

$$\big\{x : d_h^s(x,z_{1,j}) \leqslant 3h/4\big\} \subset \bigcup_{b\in[0,2h]} \Gamma_b(8h). \tag{3-29}$$

For any x in the left-side set in (3-29), Lemma 6.3(4) indicates that $d_h(x,z_{1,j}) < h$ and hence $\rho(x) < 2h$. This checks the condition on $\rho(x)$ in (3-18). We proceed to check the rest of the conditions in (3-18).

If $x \in M_h$, then by Lemma 3.6,

$$d_{\partial M}(x^\perp, z_{1,j}^\perp) \leqslant (1+15K_1 h)\tilde{d}(x,z_{1,j}) \leqslant (1+15K_1 h)d_h(x,z_{1,j}) < h(1+15K_1 h).$$

If $x \in M - M_h$, then $d_h(x^{\perp_h}, z_{1,j}) < d_h(x,z_{1,j}) < h$ by definition (3-9). Hence,

$$d_{\partial M}(x^\perp, z_{1,j}^\perp) = d_{\partial M}((x^{\perp_h})^\perp, z_{1,j}^\perp) \leqslant (1+15K_1 h)d_h(x^{\perp_h}, z_{1,j}) < h(1+15K_1 h),$$

where we used the fact that $(x^{\perp_h})^\perp = x^\perp$.

Therefore in either case, for sufficiently small h depending only on K_1, we have $d_{\partial M}(x^\perp, z^\perp_{1,j}) < 2h$. Then the fact that $d_{\partial M}(z^\perp_{1,j}, \partial\Gamma) \geqslant 10h$ yields $x^\perp \in \Gamma$ and $d_{\partial M}(x^\perp, \partial\Gamma) > 8h$. This completes the proof of (3-29) and consequently the first statement of the sublemma.

For the second statement, it suffices to prove $\overline{\Omega}_j \subset \Omega^0_j$. For any $(x,t) \in \overline{\Omega}_j$, clearly we have $\psi_j(x,t) \geqslant 9T^2h > 8T^2h$ and $(x,t) \notin \bigcup_j \Omega_{0,j}$ by definition (3-26). To show $(x,t) \in \Omega^0_j$, we only need to show $(x,t) \notin \{x : d^s_h(x, z_{1,j}) \leqslant h/2\} \times [-T_1, T_1]$. This is a direct consequence of the fact that a larger cylinder $\{x : d^s_h(x, z_{1,j}) \leqslant 3h/4\} \times [-T_1, T_1]$ is strictly contained in the open set $\bigcup_j \Omega_{0,j}$, due to (3-29) and Sublemma 2. An explicit lower bound for the distance between their boundaries is estimated in Lemma 6.6. □

Error estimate for Case 1. We prove that $\overline{\Omega} = \left(\bigcup_j \overline{\Omega}_j\right) \cup \left(\bigcup_j \overline{\Omega}_{0,j}\right)$ almost covers the domain of influence in the original manifold M. More precisely, we prove that there exists $C' = C'(T, K_1)$ such that $\Omega(C'h) \subset \overline{\Omega}$. Since $\Omega(C'h) \subset M \times [-T,T]$, it suffices to show that $M \times [-T,T] - \overline{\Omega} \subset M \times [-T,T] - \Omega(C'h)$.

For any $(x,t) \in M \times [-T,T] - \overline{\Omega}$, by the definitions (3-24), (3-25), (3-26), we know that one of the following two situations must happen:

(1) $d(x, \partial M) < h$.

(2) $x \in M_h$ and $d^s_h(x, z_{1,j}) > T_1 - \sqrt{t^2 + 9T^2h}$ for any $z_{1,j}$.

We analyze these two situations separately as follows.

(1) By virtue of Sublemma 2 and the definition (3-18), the situation (1) implies that $x^\perp \notin \Gamma$, or $x^\perp \in \Gamma$ and $d_{\partial M}(x^\perp, \partial\Gamma) < 8h$, or $|t| > T - 6h$. The condition $x^\perp \notin \Gamma$ indicates that $d(x, \partial M - \Gamma) < h$. If $x^\perp \in \Gamma$ and $d_{\partial M}(x^\perp, \partial\Gamma) < 8h$, then, by the triangle inequality,

$$d(x, \partial\Gamma) \leqslant d(x, x^\perp) + d(x^\perp, \partial\Gamma) \leqslant h + d_{\partial M}(x^\perp, \partial\Gamma) < 9h,$$

which yields $d(x, \partial M - \Gamma) < 9h$ due to $\partial\Gamma \subset \partial M - \Gamma$. If $|t| > T - 6h$, then the following inequality is trivially satisfied:

$$T - |t| - \sqrt{6h} < 6h - \sqrt{6h} < 0 \leqslant d(x, \Gamma).$$

Note that if $\Gamma = \partial M$, the first two possibilities automatically do not occur and hence only the last inequality above is valid under the first situation.

(2) By Lemma 6.3(4), the situation (2) implies that $d_h(x, z_{1,j}) > T_1 - |t| - 3T\sqrt{h} - 2h^2$ for $x \in M_h$ and any $z_{1,j}$. Since $\{z_{1,j}\}$ is an h-net in $\Gamma_h(10h)$ with respect to d_h, we have

$$d_h(x, \Gamma_h(10h)) > T_1 - |t| - 3T\sqrt{h} - h - 2h^2.$$

Then we apply Lemma 3.6 after replacing $M, \widetilde{M}$ with M_h, M:

$$d(x, \Gamma_h(10h))(1 + 6K_1h) \geqslant d_h(x, \Gamma_h(10h)) > T_1 - |t| - 3T\sqrt{h} - h - 2h^2,$$

where we used the fact that the second fundamental form of ∂M_h is bounded by $2K_1$ due to Lemma 6.1. Hence by the triangle inequality,

$$d(x, \Gamma_0(10h)) > (T_1 - |t| - 3T\sqrt{h} - h - 2h^2)(1 + 6K_1h)^{-1} - h.$$

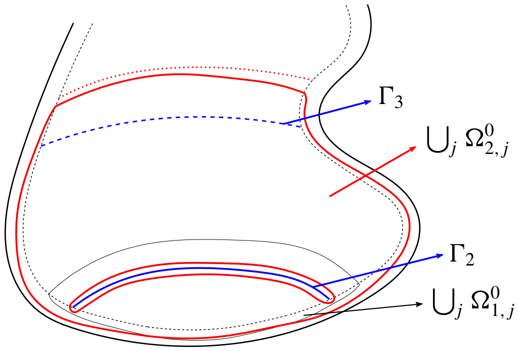

Figure 4. Domains for the second step in Case 2. Enclosed by the red solid lines is the domain we work in. The blue dashed line Γ_3 is the reference set for the third step. From here, the procedure is entirely done in M.

For any $y \in \Gamma - \Gamma_0(10h)$, y lies in the boundary normal neighborhood of $\partial\Gamma$ in Γ due to $10h < i_b(\overline{\Gamma})$. Hence $d(y, \Gamma_0(10h)) \leqslant d_{\partial M}(y, \Gamma_0(10h)) \leqslant 10h$. Then,

$$d(x, y) \geqslant d(x, \Gamma_0(10h)) - d(y, \Gamma_0(10h)) > (T - |t| - 3T\sqrt{h} - 7h - 2h^2)(1 + 6K_1 h)^{-1} - 11h,$$

where we used $T_1 = T - 6h$. Hence we arrive at

$$d(x, \Gamma) > T - |t| - C(T, K_1)\sqrt{h}.$$

Finally we combine these two situations together, and we have proved that $(x, t) \in M \times [-T, T] - \Omega(Ch)$ for $C = \max\{C(T, K_1)^2, 9\}$ by definition (2-4). Therefore, there exists $C' = C'(T, K_1)$ such that $\Omega(C'h) \subset \overline{\Omega}$, and a stability estimate can be obtained on $\Omega(C'h)$ from Theorem 2.2. The stability estimate will be derived at the end of the proof.

Case 2: $T > \rho_0 = \min\{i_0/2, r_0/2, r_g/4, \pi/(6K_1)\}$. As Lemma 6.5 is only valid within the injectivity radius, we define the procedure step by step and each step is done within the injectivity radius. Recall that $\{z_{1,j}\}$ is an h-net in $\Gamma_h(10h) \subset M_h$ with respect to d_h, and $T_1 = T - 6h$. For the first step, we define functions $\psi_{1,j}$ by adding to (3-24) another term associated with T_1,

$$\psi_{1,j}(x, t) = \big((1 - \xi(d(x, \partial M)) - \xi(\rho_0 - d_h^s(x, z_{1,j})))T_1 - d_h^s(x, z_{1,j})\big)^2 - t^2, \tag{3-30}$$

and consider the domains

$$\Omega^0_{1,j} = \{(x, t) \in M \times [-T_1, T_1] : \psi_{1,j}(x, t) > 8T^2 h\} - \{x : d_h^s(x, z_{1,j}) \leqslant h/2\} \times [-T_1, T_1]. \tag{3-31}$$

One can compare these definitions here with those in Case 1. Note that the regions $\Omega^0_{1,j}$ stay within half the injectivity radius due to the definition of the function ξ. The gradient of $\psi_{1,j}$ has the form

$$\nabla_x \psi_{1,j} = 2\big((1 - \xi(d(x, \partial M)) - \xi(\rho_0 - d_h^s(x, z_{1,j})))T_1 - d_h^s(x, z_{1,j})\big)$$
$$\cdot \big(-\xi' T_1 \nabla_x d(x, \partial M) + \xi' T_1 \nabla_x d_h^s(x, z_{1,j}) - \nabla_x d_h^s(x, z_{1,j})\big).$$

The vector part of $\nabla_x \psi_{1,j}$ consists of $\nabla_x d(x, \partial M)$ and $\nabla_x d_h^s(x, z_j)$, the same as in Case 1. Furthermore, the form for the vector part is the same as that in Case 1 up to multiplication by a positive function, since

$\xi' \leqslant 0$. Hence one obtains the same lower bounds for the length of the gradient and the principle symbol as (3-27) and (3-28). It follows that $\psi_{1,j}$ is noncharacteristic in $\Omega^0_{1,j}$. And we define $\psi_{\max,1}$ and $\Omega_{1,j}$ the same as in Case 1 (see Figure 3). More precisely, define $\psi_{\max,1} = (T_1 - 3h/4)^2$ and

$$\Omega_{1,j} = \Big\{(x,t) \in \Omega^0_{1,j} - \bigcup_j \bar{\Omega}_{0,j} : \psi_{1,j}(x,t) > 9T^2 h\Big\}. \tag{3-32}$$

Since (3-29) is still valid, Sublemma 3 holds for $\psi_{1,j}$, $\Omega^0_{1,j}$, $\Omega_{1,j}$. Hence Theorem 2.2 applies to the first step. We stop the procedure right after the first step if $T_1 - \rho_0 - 3T\sqrt{h} \leqslant 2h$.

For the second step, we need to choose a new set of reference points. Observe that the first step propagates past the level set $\Gamma_2 := \{x \in M_h : d_h(x, \Gamma_h(10h)) = \rho_0 - 4h\}$ due to Lemma 6.3(4) and the procedure-stopping criterion $T_1 - \rho_0 - 3T\sqrt{h} > 2h$. We choose the new reference points $\{z_{2,j}\}$ as an h-net in Γ_2 with respect to d_h. At Γ_2, the square of the maximal time allowed is $(T_1 - \rho_0 + 4h)^2 - 9T^2 h$, and we set the time range T_2 for the second step as $T_2 = T_1 - \rho_0 - 3T\sqrt{h}$. The procedure-stopping criterion indicates that $T_2 > 2h$. Then we define the functions

$$\psi_{2,j}(x,t) = \big((1 - \xi(d(x,\partial M)) - \xi(\rho_0 - d^s_h(x, z_{2,j})))T_2 - d^s_h(x, z_{2,j})\big)^2 - t^2.$$

To apply Theorem 2.2, we need to ensure that small neighborhoods around the new reference points are contained in the regions already propagated by the unique continuation in the first step. To that end, we define $\psi_{\max,2} = (T_2 - a_T h)^2$, where $a_T = \min\{1, T^{-1}\}$, and

$$\Omega^0_{2,j} = \{(x,t) \in M \times [-T_2, T_2] : \psi_{2,j}(x,t) > 8T^2 h\} - \{x : d^s_h(x, z_{2,j}) \leqslant a_T h/2\} \times [-T_2, T_2],$$
$$\Omega_{2,j} = \Big\{(x,t) \in \Omega^0_{2,j} - \Big(\Big(\bigcup_j \bar{\Omega}_{1,j}\Big) \cup \Big(\bigcup_j \bar{\Omega}_{0,j}\Big)\Big) : \psi_{2,j}(x,t) > 9T^2 h\Big\}.$$

These domains are illustrated in Figure 4. The specific choice of $\psi_{\max,2}$ is justified in Sublemma 5 a bit later, to ensure that $\varnothing \neq \{(x,t) \in \Omega^0_{2,j} : \psi_{2,j}(x,t) > \psi_{\max,2}\} \subset \big(\bigcup_j \bar{\Omega}_{1,j}\big) \cup \big(\bigcup_j \bar{\Omega}_{0,j}\big)$.

Now we define the remaining steps iteratively. We define the reference sets as

$$\Gamma_i = \{x \in M_h : d_h(x, \Gamma_1) = (i-1)(\rho_0 - 4h)\}, \quad i \geqslant 2,$$

where $\Gamma_1 = \Gamma_h(10h) \subset M_h$. The reference points $\{z_{i,j}\}$ are defined as an h-net in Γ_i with respect to d_h. Note that the range of the j index for each step i is different, and the notation $\{z_{i,j}\}$ here is short for $\{z_{i,j}\}_{j=1}^{J(i)}$. We define the $C^{2,1}$ functions $\psi_{i,j}$ as

$$\psi_{i,j}(x,t) = \big((1 - \xi(d(x,\partial M)) - \xi(\rho_0 - d^s_h(x, z_{i,j})))T_i - d^s_h(x, z_{i,j})\big)^2 - t^2,$$

where $T_i = T_{i-1} - \rho_0 - 3T\sqrt{h}$ with $T_1 = T - 6h$. We stop the procedure at the i-th step if $T_{i+1} \leqslant 2h$ or $\Gamma_{i+1} = \varnothing$. The regions $\Omega^0_{i,j}$ and $\Omega_{i,j}$ for $i \geqslant 2$ are defined as[3]

$$\Omega^0_{i,j} = \{(x,t) \in M \times [-T_i, T_i] : \psi_{i,j}(x,t) > 8T^2 h\} - \{x : d^s_h(x, z_{i,j}) \leqslant a_T h/2\} \times [-T_i, T_i],$$
$$\Omega_{i,j} = \Big\{(x,t) \in \Omega^0_{i,j} - \bigcup_{l=0}^{i-1} \bigcup_j \bar{\Omega}_{l,j} : \psi_{i,j}(x,t) > 9T^2 h\Big\},$$

[3]The connected component characterized by $\big(1 - \xi(d(x,\partial M)) - \xi(\rho_0 - d^s_h(x, z_{i,j}))\big)T_i - d^s_h(x, z_{i,j}) > 0$.

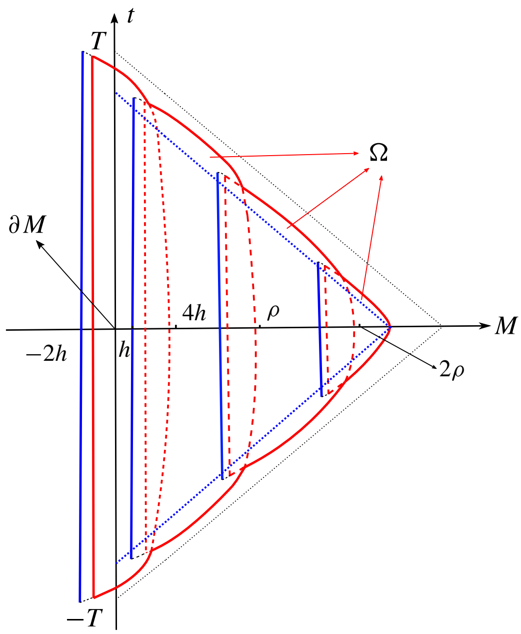

Figure 5. The procedure of a three-step propagation besides the initial step. The red solid lines enclose the whole region $\Omega = \bigcup_{i,j} \Omega_{i,j}$ propagated by the unique continuation. The black dotted line represents the optimal region, while the blue dotted line represents the actual region we can estimate.

where $a_T = \min\{1, T^{-1}\}$ in (3-16). It follows that $\psi_{i,j}$ is noncharacteristic in $\Omega^0_{i,j}$ in the same way as for $\psi_{1,j}$. Due to Sublemma 5 below, Theorem 2.2 applies with $\psi_{\max,i} = (T_i - a_T h)^2$.

Sublemma 4. *For $i \geqslant 2$ and any $z \in \Gamma_i$, we have $d_h(z, \Gamma_{i-1}) = \rho_0 - 4h$.*

Proof. Let $z_1 \in \Gamma_1$ be a point in Γ_1 such that $d_h(z, z_1) = d_h(z, \Gamma_1)$. Take a minimizing geodesic of M_h from z to z_1 and the geodesic intersects with Γ_{i-1} at $z_{i-1} \in \Gamma_{i-1}$. This geodesic has length $(i-1)(\rho_0 - 4h)$, and its segment from z_{i-1} to z_1 has length at least $(i-2)(\rho_0 - 4h)$ by definition. Hence $d_h(z, \Gamma_{i-1}) \leqslant d_h(z, z_{i-1}) \leqslant \rho_0 - 4h$. On the other hand, for any $z' \in \Gamma_{i-1}$, we have $d_h(z, z') \geqslant d_h(z, \Gamma_1) - d_h(z', \Gamma_1) = \rho_0 - 4h$, which shows $d_h(z, \Gamma_{i-1}) \geqslant \rho_0 - 4h$. □

Sublemma 5. *For $i \geqslant 2$ and sufficiently small $h < \min\{1/2, T/4\}$ depending on n, K_1, i_0, we have* $\mathrm{dist}_{\widetilde{M}\times\mathbb{R}}(\partial\Omega^0_{i,j}, \Omega_{i,j}) > 0$, *and*

$$\varnothing \neq \{(x,t) \in \Omega^0_{i,j} : \psi_{i,j}(x,t) > (T_i - a_T h)^2\} \subset \bigcup_{l=0}^{i-1} \bigcup_j \overline{\Omega}_{l,j}.$$

Proof. We prove the following stronger statement:

$$\{x : d^s_h(x, z_{i,j}) \leqslant a_T h\} \times [-T_i - h, T_i + h] \subset \bigcup_{l=0}^{i-1} \bigcup_j \overline{\Omega}_{l,j}. \tag{3-33}$$

More precisely, for any (x,t) in the left-hand set, we prove that if $(x,t) \notin \bigcup_{l=0}^{i-2} \bigcup_j \overline{\Omega}_{l,j}$, then $(x,t) \in \bigcup_j \Omega_{i-1,j}$.

By Sublemma 4 and the fact that $\{z_{i-1,j}\}$ is an h-net in Γ_{i-1}, we can find some z_{i-1,j_0} such that $d_h(z_{i,j}, z_{i-1,j_0}) < \rho_0 - 3h$. Then for any (x,t) in the left-hand set in (3-33), Lemma 6.3(2) implies that for sufficiently small h depending on n, K_1, i_0,

$$d_h^s(x, z_{i-1,j_0}) < d_h^s(x, z_{i,j}) + (\rho_0 - 3h)(1 + CnK_1^2 h^6) < \rho_0 - 3h/2, \tag{3-34}$$

which indicates that $\xi(\rho_0 - d_h^s(x, z_{i-1,j_0}))$ vanishes.

We claim that $(x,t) \in \Omega_{i-1,j_0}$. To prove this, by the definition of $\psi_{i-1,j}$, $\Omega_{i-1,j}$ and the condition that $(x,t) \notin \bigcup_{l=0}^{i-2} \bigcup_j \bar{\Omega}_{l,j}$, we only need to show that

$$\psi_{i-1,j_0}(x,t) = \big((1 - \xi(d(x, \partial M)))T_{i-1} - d_h^s(x, z_{i-1,j_0})\big)^2 - t^2 > 9T^2 h.$$

Since $|t| \leqslant T_i + h$, it is enough to show

$$(1 - \xi(d(x, \partial M)))T_{i-1} - d_h^s(x, z_{i-1,j_0}) > T_i + h + 3T\sqrt{h}.$$

Now since $d_h^s(x, z_{i,j}) \leqslant h/T$, by the definition of d_h and Lemma 6.3(4) we have

$$d(x, \partial M) \geqslant h - d_h(x, z_{i,j})h > h - \left(\frac{h}{T} + \frac{2h^2}{T}\right)h > h - \frac{2h^2}{T},$$

which implies by the definition of ξ (3-19)

$$\xi(d(x, \partial M)) < \xi\left(h - \frac{2h^2}{T}\right) = \frac{8h^3}{T^3}.$$

Since $T_i = T_{i-1} - \rho_0 - 3T\sqrt{h}$ by definition, we have by (3-34)

$$\begin{aligned}(1 - \xi(d(x, \partial M)))T_{i-1} - d_h^s(x, z_{i-1,j_0}) &> T_{i-1} - \xi(d(x, \partial M))T_{i-1} - \rho_0 + \frac{3h}{2} \\ &> T_i + 3T\sqrt{h} + \frac{3h}{2} - \frac{8h^3}{T^3}T \\ &> T_i + 3T\sqrt{h} + h.\end{aligned}$$

This proves $(x,t) \in \Omega_{i-1,j_0}$ and hence (3-33).

The inclusion (3-33) shows $\{x : d_h^s(x, z_{i,j}) \leqslant a_T h/2\} \times [-T_i, T_i]$ is strictly contained in $\bigcup_{l=0}^{i-1} \bigcup_j \bar{\Omega}_{l,j}$, which implies that $\bar{\Omega}_{i,j} \subset \Omega_{i,j}^0$. An explicit lower bound for the distance between their boundaries is estimated in Lemma 6.6.

For the second statement of the sublemma, by (3-33),

$$\{\psi_{i,j}(x,t) > (T_i - a_T h)^2\} \subset \{d_h^s(x, z_{i,j}) < a_T h\} \times (-T_i, T_i) \subset \bigcup_{l=0}^{i-1} \bigcup_j \bar{\Omega}_{l,j}.$$

The nonemptyness directly follows from the definition of $\Omega_{i,j}^0$. □

Error estimate for Case 2. Finally we show that $\bar{\Omega} = \bigcup_{i \geqslant 0} \bigcup_j \bar{\Omega}_{i,j}$ almost covers the domain of influence in the original manifold M (see Figure 5). More precisely, we prove that there exists $C' = C'(T, D, K_1, i_0, r_0, r_g)$ such that $\Omega(C'h) \subset \bar{\Omega}$. The idea of the proof is similar to that for Case 1, and we omit the parts of the proof identical to Case 1.

For any $(x,t)\in M\times[-T,T]-\overline{\Omega}$, one of the following two situations must happen:

(1) $d(x,\partial M)<h$.

(2) $x\in M_h$ and $d_h^s(x,z_{i,j})>(1-\xi(\rho_0-d_h^s(x,z_{i,j})))T_i-\sqrt{t^2+9T^2h}$ for any $z_{i,j}$ $(i\geqslant 1)$.

The situation (1) implies that $d(x,\partial M-\Gamma)<9h$ or $d(x,\Gamma)>T-|t|-\sqrt{6h}$ by the same argument as for Case 1.

Now we focus on the situation (2) when $x\in M_h$. Lemma 6.3(4) yields that, for any $z_{i,j}$ $(i\geqslant 1)$,

$$d_h(x,z_{i,j})>(1-\xi(\rho_0-d_h^s(x,z_{i,j})))T_i-|t|-3T\sqrt{h}-2h^2. \tag{3-35}$$

Let $z_1\in\Gamma_1$ be a point in Γ_1 such that $d_h(x,z_1)=d_h(x,\Gamma_1)$, and take a minimizing geodesic of M_h from x to z_1. Observe that this minimizing geodesic intersects with each Γ_i at most once; otherwise it would fail to minimize the distance $d_h(x,\Gamma_1)$. Furthermore, due to the continuity of the distance function $d_h(\cdot,\Gamma_1)$, if the minimizing geodesic intersects with Γ_i, then it intersects with Γ_l for all $1\leqslant l<i$. Suppose the minimizing geodesic intersects with Γ_i at $z_i\in\Gamma_i$ for $1\leqslant i\leqslant m$, and the intersection does not occur at any nonempty Γ_i for $i>m$. Then by Sublemma 4, we have

$$d_h(x,\Gamma_1)=d(x,z_1)=d_h(x,z_m)+\sum_{i=1}^{m-1}d_h(z_i,z_{i+1})\geqslant d_h(x,z_m)+(m-1)(\rho_0-4h). \tag{3-36}$$

We claim that $d_h(x,z_m)\leqslant\rho_0-3h$. Suppose not, and by the inequality above, we have $d_h(x,\Gamma_1)>m(\rho_0-4h)$. This implies that $\Gamma_{m+1}\neq\varnothing$ and any minimizing geodesic from x to Γ_1 must intersect with Γ_{m+1}, which is a contradiction.

Since $\Gamma_m\neq\varnothing$ by assumption, the step m of our procedure takes place as long as $T_m>2h$ by our stopping criterion. However if $T_m\leqslant 2h$, the procedure stops at some previous step.

(i) $T_m>2h$. On Γ_m, we can find some $z_{m,j}$ such that $d_h(z_m,z_{m,j})<h$ since $\{z_{m,j}\}$ is an h-net. Then it follows that $d_h(x,z_{m,j})<\rho_0-2h$. Lemma 6.3(4) indicates that $d_h^s(x,z_{m,j})<\rho_0-h$. Hence $\xi(\rho_0-d_h^s(x,z_{m,j}))$ in (3-35) vanishes. Then by (3-36),

$$\begin{aligned}d_h(x,\Gamma_1)&>d_h(x,z_{m,j})-h+(m-1)(\rho_0-4h)\\&>T_m-|t|-3T\sqrt{h}-h-2h^2+(m-1)(\rho_0-4h)\\&=T_1-|t|-3mT\sqrt{h}-h-4(m-1)h-2h^2,\end{aligned}$$

where we used $T_m=T_1-(m-1)(\rho_0+3T\sqrt{h})$ by the definition of T_i.

(ii) $T_m\leqslant 2h$. From $T_m=T_1-(m-1)(\rho_0+3T\sqrt{h})$, we have

$$T_1\leqslant(m-1)(\rho_0+3T\sqrt{h})+2h.$$

Hence by (3-36), we still get a similar estimate as the previous situation:

$$\begin{aligned}d_h(x,\Gamma_1)\geqslant(m-1)(\rho_0-4h)&\geqslant T_1-3(m-1)T\sqrt{h}-2h-4(m-1)h\\&\geqslant T_1-|t|-3(m-1)T\sqrt{h}-2h-4(m-1)h.\end{aligned}$$

From here, one can follow the rest of the estimates for Case 1 and obtain

$$d(x,\Gamma) > T - |t| - C(m, T, K_1)\sqrt{h}.$$

Combining these situations together, we have proved that $(x,t) \in M \times [-T, T] - \Omega(Ch)$ for $C = \max\{C(m, T, K_1)^2, 9\}$ by definition (2-4). Therefore, there exists $C' = C'(m, T, K_1)$ such that $\Omega(C'h) \subset \overline{\Omega}$.

The only part left is to estimate the upper bound for m. By assumption, $\Gamma_m \neq \varnothing$ and hence Γ_m must be taken before $d_h(\,\cdot\,, \Gamma_1)$ reaches outside the diameter of M_h. Due to Lemma 3.6 for M_h, M, the diameter of M_h is bounded by $6D/5$ for sufficiently small h depending only on K_1. Thus by the definition of Γ_i, we have

$$m \leqslant \left[\frac{6D}{\rho_0}\right] + 1,$$

where $\rho_0 = \min\{i_0/2, r_0/2, r_g/4, \pi/(6K_1)\}$ depends only on n, $\|R_M\|_{C^1}$, $\|S\|_{C^1}$, i_0, r_0.

Stability estimate. With all the functions and domains we have constructed, the only part left is to apply Theorem 2.2. From the error estimate above, we have proved that there exists $C' = C'(T, D, K_1, i_0, r_0, r_g)$ such that $\Omega(C'h) \subset \overline{\Omega} = \bigcup_{i\geqslant 0}\bigcup_j \overline{\Omega}_{i,j}$, where r_g is a constant depending only on n, $\|R_M\|_{C^1}$, $\|S\|_{C^1}$, i_0. Recall that $\tilde{u}$ is an extension of u to $\widetilde{M}$ defined by (3-7). Theorem 2.2 yields the following stability estimate on $\overline{\Omega}$ and hence on $\Omega(C'h)$:

$$\|u\|_{L^2(\Omega(C'h))} \leqslant \|\tilde{u}\|_{L^2(\overline{\Omega})} \leqslant C \frac{\|\tilde{u}\|_{H^1(\Omega^0)}}{(\log(1 + \|\tilde{u}\|_{H^1(\Omega^0)}/\|P\tilde{u}\|_{L^2(\Omega^0)}))^{1/2}},$$

where $\Omega^0 = \bigcup_{i\geqslant 0}\bigcup_j \Omega^0_{i,j}$. During the initial step, we have shown $\Omega^0_{0,j} \subset M \cup \Omega_\Gamma$, and $\Omega^0_{i,j}$ is defined in $M \times [-T, T]$ for all $i \geqslant 1$. Hence $\Omega^0 \subset (M \cup \Omega_\Gamma) \times [-T, T]$. Since the function $x \mapsto x(\log(1+x))^{-1/2}$ is nondecreasing on $[0, +\infty)$, we have

$$\|u\|_{L^2(\Omega(C'h))} \leqslant C \frac{\|\tilde{u}\|_{H^1((M\cup\Omega_\Gamma)\times[-T,T])}}{(\log(1 + \|\tilde{u}\|_{H^1((M\cup\Omega_\Gamma)\times[-T,T])}/\|P\tilde{u}\|_{L^2((M\cup\Omega_\Gamma)\times[-T,T])}))^{1/2}}.$$

Therefore, the desired stability estimate follows from Lemma 3.5 after replacing h by h/C'. The number of domains in each step is not consequential to the estimate as long as relevant quantities of $\psi_{i,j}$ are uniformly bounded. The dependency of the constant is calculated in the Appendix.

The second statement of the theorem is due to the following interpolation formula for bounded domains with locally Lipschitz boundary:

$$\|u\|_{H^{1-\theta}} \leqslant \|u\|^{\theta}_{L^2}\|u\|^{1-\theta}_{H^1}, \qquad \theta \in (0, 1).$$

This concludes the proof of Theorem 3.1.

Remark 3.10. If we define d_h (3-9) with h^{-2} scaling in the boundary neighborhood and require $h < T^{-1}$, then the level sets of ψ_j (3-24) automatically do not intersect with ∂M even without the $\xi(d(x, \partial M))$-term. However, the extra condition $h < T^{-1}$ is not ideal and we want to choose the parameter h as large as possible for a large T, considering the stability estimate grows exponentially in h. In addition, we frequently used the number $a_T = \min\{1, T^{-1}\}$ exactly for the same purpose.

Remark 3.11. In the definition of $\Omega^0_{i,j}$ for Case 2, we removed the region where points are $a_T h/2$-close to the reference points, and this region is contained in the set propagated by the unique continuation from previous steps by Sublemma 5. The h^{-1} scaling in the definition of d_h (3-9) directly affects the order of this number $a_T h/2$. Without the scaling, the order of this number would be of h^2.

3.5. *Applications of the quantitative unique continuation.*

Due to the trace theorem, Theorem 3.1 yields the following estimate on the initial value.

Corollary 3.12. *Let $M \in \mathcal{M}_n(D, K_1, K_2, i_0, r_0)$ be a compact Riemannian manifold with smooth boundary ∂M, and let Γ (possibly $\Gamma = \partial M$) be a connected open subset of ∂M with smooth boundary. Suppose $u \in H^2(M \times [-T, T])$ is a solution of the wave equation $Pu = 0$. Assume the Cauchy data satisfy*

$$u|_{\partial M\times[-T,T]} \in H^{2,2}(\partial M \times [-T, T]), \qquad \frac{\partial u}{\partial \boldsymbol{n}} \in H^{2,2}(\partial M \times [-T, T]).$$

If

$$\|u\|_{H^1(M\times[-T,T])} \leqslant \Lambda_0, \quad \|u\|_{H^{2,2}(\Gamma\times[-T,T])} + \left\| \frac{\partial u}{\partial \boldsymbol{n}} \right\|_{H^{2,2}(\Gamma\times[-T,T])} \leqslant \varepsilon_0,$$

then for sufficiently small h, we have

$$\|u(x, 0)\|_{L^2(\Omega(2h,0,3))} \leqslant C_3^{1/3} h^{-2/9} \exp(h^{-C_4 n}) \frac{\Lambda_0 + h^{-1/2}\varepsilon_0}{(\log(1 + h + h^{3/2}\Lambda_0/\varepsilon_0))^{1/6}},$$

where C_3, C_4 are constants independent of h, and their dependency on geometric parameters is stated in Theorem 3.1. For a fixed $t \in [-T, T]$, the domain $\Omega(h, t, m)$ is defined as

$$\Omega(h, t, m) = \{x \in M : T - |t| - d(x, \Gamma) > h^{1/m},\ d(x, \partial M - \Gamma) > h^{1/m}\}. \tag{3-37}$$

Proof. Observe that $\Omega(2h, 0, 3) \times (-t_0, t_0) \subset \Omega(h)$ with $t_0 = (\sqrt[3]{2} - 1)\sqrt[3]{h}$ by definition. Then we take $\theta = \frac{1}{3}$ in Theorem 3.1 and apply the trace theorem [Bergh and Löfström 1976, Theorem 6.6.1]: there exists a constant C such that

$$\begin{aligned}\|u(x, 0)\|_{L^2(\Omega(2h,0,3))} &\leqslant C t_0^{-2/3} \|u(x, t)\|_{H^{2/3}(\Omega(2h,0,3)\times(-t_0,t_0))}\\ &\leqslant 4Ch^{-2/9} \|u(x, t)\|_{H^{2/3}(\Omega(h))}.\end{aligned}$$

□

Remark 3.13. Note that the constant C_3 in Corollary 3.12 is not exactly the same as the constant C_3 in Theorem 3.1. However, they depend on the same set of geometric parameters. In this paper, we keep the same notation for constants if operations do not introduce any new parameter.

The following independent result gives an explicit estimate on the Hausdorff measure of the boundary of the domain of influence, which shows that the region Corollary 3.12 does not cover has a uniformly controlled small volume.

Proposition 3.14. *Let M be a compact Riemannian manifold with smooth boundary. For any measurable subset $\Gamma \subset \partial M$ and any $t \geqslant 0$, the following explicit estimate applies*:

$$\text{vol}_{n-1}(\partial M(\Gamma, t)) < C_5(n, \|R_M\|_{C^1}, \|S\|_{C^1}, i_0, \text{vol}(M), \text{vol}(\partial M)),$$

where $M(\Gamma, t)$ is defined in (1-3). As a consequence, the estimate above implies the following volume estimate due to the coarea formula. Namely, for any $t, \gamma \geqslant 0$, we have

$$\mathrm{vol}_n(M(\Gamma, t+\gamma) - M(\Gamma, t)) < C_5(n, \|R_M\|_{C^1}, \|S\|_{C^1}, i_0, \mathrm{vol}(M), \mathrm{vol}(\partial M))\gamma.$$

Proof. Denote the level set of the distance function by $\Sigma_t = \{x \in \mathrm{int}(M) : d(x, \Gamma) = t\}$. For any point in Σ_t, there exists a minimizing geodesic from the point to the subset Γ. These minimizing geodesics do not intersect with Σ_t except at the initial points by definition. Moreover, they do not intersect each other in the interior of M, as geodesics would fail to minimize distance past a common point in the interior of M. Define $l(x)$ to be the infimum of the distances between a point $x \in \Sigma_t$ and the first intersection points with the boundary along all minimizing geodesics from x to Γ, and to be infinity if any minimizing geodesic from x to Γ does not intersect $\partial M - \overline{\Gamma}$.

For sufficiently small $\varepsilon > 0$ chosen later, define

$$\Sigma_t(\varepsilon) = \{x \in \Sigma_t : \varepsilon/2 < l(x) \leqslant \varepsilon\}.$$

Denote by $U(\Sigma_t(\varepsilon))$ the set of all points on all minimizing geodesics from $\Sigma_t(\varepsilon)$ to Γ and consider the set $U(\Sigma_t(\varepsilon)) \cap \Sigma_{t'}$ for $t' \in [t - \varepsilon/4, t)$. Clearly the set $U(\Sigma_t(\varepsilon)) \cap \Sigma_{t'}$ does not intersect with ∂M by definition. Furthermore, it is contained in the $C(n, \|R_M\|_{C^1}, \|S\|_{C^1})\varepsilon^2$-neighborhood of the boundary ∂M if ε is not greater than $\varepsilon_0(n, \|R_M\|_{C^1}, \|S\|_{C^1}, i_0)$, due to Lemma 6.7.

Since the distance function $d(\,\cdot\,, \Gamma)$ is Lipschitz with the Lipschitz constant 1, it is differentiable almost everywhere by Rademacher's theorem and its gradient has length at most 1. The existence of minimizing geodesics from Γ yields that the gradient of $d(\,\cdot\,, \Gamma)$ has length at least 1 wherever it exists. Hence the gradient of $d(\,\cdot\,, \Gamma)$ has unit length almost everywhere. We apply the coarea formula (e.g., Theorem 3.1 in [Federer 1959]) to the sets $U(\Sigma_t(\varepsilon)) \cap \Sigma_{t'}$ with the distance function $d(\,\cdot\,, \Gamma)$. Then by Lemma 6.8 and Lemma 6.7, we have

$$\begin{aligned} \frac{\varepsilon}{4}\mathrm{vol}_{n-1}(\Sigma_t(\varepsilon)) &< 5^{n-1}\int_{t-\varepsilon/4}^{t} \mathrm{vol}_{n-1}(U(\Sigma_t(\varepsilon)) \cap \Sigma_{t'})\,dt' \\ &= 5^{n-1}\mathrm{vol}_n\Bigg(\bigcup_{t'\in[t-\varepsilon/4,t)} (U(\Sigma_t(\varepsilon)) \cap \Sigma_{t'})\Bigg) \\ &< 5^{n-1}C(n, \|R_M\|_{C^1}, \|S\|_{C^1})\varepsilon^2\,\mathrm{vol}(\partial M). \end{aligned}$$

Then for $\varepsilon \leqslant \varepsilon_0$ we get

$$\mathrm{vol}_{n-1}(\Sigma_t(\varepsilon)) < C(n, \|R_M\|_{C^1}, \|S\|_{C^1}, \mathrm{vol}(\partial M))\varepsilon.$$

Hence we have an estimate on the measure of $U_t(\varepsilon_0) := \{x \in \Sigma_t : l(x) \leqslant \varepsilon_0\}$:

$$\begin{aligned} \mathrm{vol}_{n-1}(U_t(\varepsilon_0)) = \mathrm{vol}_{n-1}\Bigg(\bigcup_{k=0}^{\infty}\Sigma_t(\varepsilon_0 2^{-k})\Bigg) &= \sum_{k=0}^{\infty}\mathrm{vol}_{n-1}(\Sigma_t(\varepsilon_0 2^{-k})) \\ &< C(n, \|R_M\|_{C^1}, \|S\|_{C^1}, \mathrm{vol}(\partial M))\varepsilon_0\sum_{k=0}^{\infty}2^{-k} \\ &< C(n, \|R_M\|_{C^1}, \|S\|_{C^1}, i_0, \mathrm{vol}(\partial M)). \end{aligned}$$

As for the other part $\Sigma_t - U_t(\varepsilon_0)$, if $t > \varepsilon_0$, the minimizing geodesics from the points of $\Sigma_t - U_t(\varepsilon_0)$ to Γ do not intersect the boundary within distance ε_0. By the same argument as above, we can control the measure in question in terms of the volume of the manifold,

$$\frac{\varepsilon_0}{2}\,\mathrm{vol}_{n-1}(\Sigma_t - U_t(\varepsilon_0)) < 5^{n-1}\,\mathrm{vol}(M),$$

which implies that

$$\mathrm{vol}_{n-1}(\Sigma_t - U_t(\varepsilon_0)) < C(n, \|R_M\|_{C^1}, \|S\|_{C^1}, i_0, \mathrm{vol}(M)).$$

Since the part of $\partial M(\Gamma, t)$ on the boundary is bounded by $\mathrm{vol}(\partial M)$, the measure estimate for $\partial M(\Gamma, t)$ follows.

If $t \leqslant \varepsilon_0$, the domain of influence is contained in the boundary normal neighborhood of width t. The minimizing geodesics from points of $\Sigma_t - U_t(\varepsilon_0)$ to Γ do not intersect the boundary within distance $t/2$. Then by the same argument as before, we have

$$\frac{t}{2}\,\mathrm{vol}_{n-1}(\Sigma_t - U_t(\varepsilon_0)) < 5^{n-1}\,\mathrm{vol}(\partial M)t,$$

which completes the measure estimate for $\partial M(\Gamma, t)$.

The n-dimensional volume estimate directly follows from the measure estimate for $\partial M(\Gamma, t)$ and the coarea formula. □

Due to the Sobolev embedding theorem and Corollary 3.12, we next prove Proposition 3.3.

Proof of Proposition 3.3. Due to Corollary 3.12, we only need an estimate in $M(\Gamma, T) - \Omega(2h, 0, 3)$. By the definition (3-37) and Proposition 3.14, we have

$$\mathrm{vol}(M(\Gamma, T) - \Omega(2h, 0, 3)) < \mathrm{vol}(M(\Gamma, T) - M(\Gamma, T - (2h)^{1/3})) + \mathrm{vol}(\partial M)(2h)^{1/3} < Ch^{1/3}.$$

Since $u(x, 0) \in H^1(M)$, by the Sobolev embedding theorem we have, for $n \geqslant 3$,

$$\|u(x, 0)\|_{L^{2n/(n-2)}(M)} \leqslant C\|u(x, 0)\|_{H^1(M)} \leqslant C\Lambda,$$

and, for $n = 2$,

$$\|u(x, 0)\|_{L^6(M)} \leqslant C\|u(x, 0)\|_{W^{1,3/2}(M)} \leqslant C\Lambda.$$

Hence Hölder's inequality gives an estimate on the L^2-norm of $u(x, 0)$ over $M(\Gamma, T) - \Omega(2h, 0, 3)$. Then the proposition follows from Corollary 3.12, and the regularity result for the wave equation (e.g., Theorem 2.30 in [Katchalov et al. 2001]): namely,

$$\max_{t\in[-T,T]} \|u(x, t)\|_{H^1(M)} \leqslant C(T)\|u(x, 0)\|_{H^1(M)}.$$

This proves Proposition 3.3. □

4. Fourier coefficients and the multiplication by an indicator function

In this section, we present the essential step of our reconstruction method where we compute how the Fourier coefficients of a function (with respect to the basis of eigenfunctions) change when the function is multiplied by an indicator function of a union of balls with center points on the boundary. This step is based on the stability estimate for the unique continuation we have obtained in Section 3. The results in this section will be applied to study the stability of the manifold reconstruction from boundary spectral data in the next section.

Let M be a compact Riemannian manifold with smooth boundary ∂M. Given a small number $\eta > 0$, we choose subsets of ∂M in the following way. Suppose $\{\Gamma_i\}_{i=1}^N$ are disjoint open connected subsets of ∂M satisfying

$$\partial M = \bigcup_{i=1}^{N} \overline{\Gamma}_i, \quad \operatorname{diam}(\Gamma_i) \leqslant \eta,$$

where the diameter is measured with respect to the distance of M. Assume that every Γ_i contains a ball (of ∂M) of radius $\eta/6$. Without loss of generality, we assume every $\partial \Gamma_i$ is smooth embedded and admits a boundary normal neighborhood of width $\eta/10$. This is because one always has the choice to propagate the unique continuation from the smaller ball of radius $\eta/6$. An error of order η does not affect our final result.

Let $\alpha = (\alpha_0, \alpha_1, \ldots, \alpha_N)$, with $\alpha_k \in [\eta, D] \cup \{0\}$ $(k = 0, \ldots, N)$, be a multi-index, where D is the upper bound for the diameter of M. Set $\Gamma_0 = \partial M$. We define the domain of influence associated with α by

$$M_\alpha := \bigcup_{k=0}^{N} M(\Gamma_k, \alpha_k) = \bigcup_{k=0}^{N} \{x \in M : d(x, \Gamma_k) < \alpha_k\}. \tag{4-1}$$

We will only be concerned with (nonempty) domains of influence with the initial time range $\alpha_k \geqslant \eta$. Hence for sufficiently small η explicitly depending on geometric parameters, Proposition 3.3 applies with $h < \eta/100$, since $i_b(\overline{\Gamma}_k) \geqslant \eta/10$ for all $k \geqslant 1$ by assumption.

We are given a function $u \in H^3(M)$ with

$$\|u\|_{L^2(M)} = 1, \quad \|u\|_{H^3(M)} \leqslant \Lambda.$$

Lemma 4.1. *For a small parameter $\gamma \in (0, N^{-2})$, we can construct a function $u_0 \in H^3(M)$ such that*

$$u_0|_{M_\alpha} = 0, \quad u_0|_{M^c_{\alpha+\gamma}} = u, \quad \|u_0\|_{L^2(M)} \leqslant 1,$$
$$\|u_0\|_{H^s(M)} \leqslant C_0 \Lambda \gamma^{-s} \quad \textit{for } s \in [1, 3], \tag{4-2}$$

where $\alpha + \gamma = (\alpha_0 + \gamma, \alpha_1 + \gamma, \ldots, \alpha_N + \gamma)$, and C_0 is a constant explicitly depending on geometric parameters.

Proof. Let $\{x_l\}$ be a maximal $\gamma/2$-separated set in M, and $\{\phi_l\}$ be a partition of unity subordinate to the open cover $\{B_{\gamma/2}(x_l)\}$ of M such that $\|\phi_l\|_{C^s} \leqslant C\gamma^{-s}$. Then the desired function u_0 can be defined as

$$u_0(x) = \sum_{\operatorname{supp}(\phi_l) \cap M_\alpha = \varnothing} \phi_l(x) u(x), \quad x \in M. \tag{4-3}$$

The first three conditions are clearly satisfied.

To prove the H^s-norm condition, we only need to show that the number of nonzero terms in the sum (4-3) is uniformly bounded. Given an arbitrary point $x \in M$, any $B_{\gamma/2}(x_l)$ with $\phi_l(x) \neq 0$ is contained in $B_\gamma(x)$. By the definition of a $\gamma/2$-separated set, $\{B_{\gamma/4}(x_l)\}$ do not intersect with each other. Hence it suffices to estimate the number of disjoint balls of radius $\gamma/4$ in a ball of radius γ. For sufficiently small γ, the volume of a ball of radius γ is bounded from both sides by $C\gamma^n$, which yields that the maximal number of balls is bounded by a constant independent of γ. To obtain an explicit estimate, it is convenient to work in a Riemannian extension of M, for instance in $\widetilde{M}$ defined in Lemma 3.4. Then an explicit estimate for the maximal number follows from Lemma 3.6 and (6-4). □

Note that due to Proposition 3.14, we have

$$\mathrm{vol}(M_{\alpha+\gamma} - M_\alpha) < (N+1)C_5\gamma < 2C_5\gamma^{1/2}. \tag{4-4}$$

4.1. ***Approximation results with spectral data without error.*** Suppose the first J Neumann boundary spectral data $\{\lambda_j, \varphi_j|_{\partial M}\}_{j=1}^J$ are known without error. Let $u \in H^3(M)$ be a given function with $\|u\|_{L^2(M)} = 1$ and $\|u\|_{H^3(M)} \leqslant \Lambda$. Let u_0 be defined in Lemma 4.1. We define u_J to be the projection of u_0 onto the first J eigenspaces $\mathcal{V}_J = \mathrm{span}\{\varphi_1, \dots, \varphi_J\} \subset C^\infty(M)$ with respect to the $L^2(M)$-norm:

$$u_J = \sum_{j=1}^J \langle u_0, \varphi_j\rangle \varphi_j \in \mathcal{V}_J. \tag{4-5}$$

We consider the following initial value problem for the wave equation with the Neumann boundary condition:

$$\begin{aligned} \partial_t^2 W - \Delta_g W = 0 \quad &\text{on } \mathrm{int}(M)\times\mathbb{R}, \\ \frac{\partial W}{\partial \boldsymbol{n}}\Big|_{\partial M\times\mathbb{R}} = 0, \quad &\partial_t W|_{t=0} = 0, \\ W|_{t=0} &= v. \end{aligned}$$

Denote by $W(v)$ the solution of the wave equation above with the initial value v. Then we define $\mathcal{U}$ to be the set of initial values $v \in \mathcal{V}_J$ for which the corresponding waves $W(v)$ are small at all $\Gamma_k \times [-\alpha_k, \alpha_k]$; namely,

$$\mathcal{U}(J,\Lambda,\gamma,\varepsilon_1) = \bigcap_{k=0}^N \{v \in \mathcal{V}_J : \|v\|_{H^1(M)} \leqslant 3C_0\Lambda\gamma^{-3},\ \|W(v)\|_{H^{2,2}(\Gamma_k\times[-\alpha_k,\alpha_k])} \leqslant \varepsilon_1\}. \tag{4-6}$$

When the parameters $J, \Lambda, \gamma, \varepsilon_1$ are clearly specified in a certain context, we denote this set by simply $\mathcal{U}$ for short.

Note that since functions in $\mathcal{V}_J$ are smooth on M, the wave $W(v)$ for $v \in \mathcal{V}_J$ is also smooth and hence its $H^{2,2}$-norm is well-defined. Given the Fourier coefficients of $v \in \mathcal{V}_J$, the conditions of $\mathcal{U}$ can be checked only using the boundary spectral data. In fact, if a function v has the form $v = \sum_{j=1}^J v_j\varphi_j$, then $\|v\|_{H^1(M)} = \sum_{j=1}^J (1+\lambda_j)v_j^2$, and the wave $W(v)$ over ∂M is given by

$$W(v)(x,t)|_{\partial M\times\mathbb{R}} = \sum_{j=1}^J v_j \cos(\sqrt{\lambda_j}t)\varphi_j(x)|_{\partial M}. \tag{4-7}$$

For convenience, we use the following equivalent Sobolev norm (e.g., Theorem 2.22 in [Katchalov et al. 2001]) for a function $v \in H^s(M)$ with the Fourier expansion $v = \sum_{j=1}^\infty v_j\varphi_j$:

$$\|v\|^2_{H^s(M)} = \sum_{j=1}^\infty (1+\lambda_j^s)v_j^2 \quad \text{for } s \in [1,3]. \tag{4-8}$$

Lemma 4.2. *Let $u \in H^3(M)$ be a given function with $\|u\|_{H^3(M)} \leqslant \Lambda$, and u_0, u_J be defined in Lemma 4.1 and (4-5). Then, for any $\varepsilon_1 > 0$, there exists $J_0 = J_0(D, \Lambda, \gamma, \varepsilon_1)$ such that $u_J \in \mathcal{U}(J, \Lambda, \gamma, \varepsilon_1)$ for any $J \geqslant J_0$.*

Proof. Assume J is sufficiently large such that $\lambda_J > 1$. Suppose u_0, u_J have expansions:

$$u_0 = \sum_{j=1}^{\infty} d_j \varphi_j, \quad u_J = \sum_{j=1}^{J} d_j \varphi_j \in \mathcal{V}_J.$$

By (4-8) we know

$$\|u_0\|^2_{H^3(M)} \geqslant \sum_{j=J+1}^{\infty} d_j^2 \lambda_j^3 \geqslant \lambda_J \sum_{j=J+1}^{\infty} d_j^2 \lambda_j^2, \tag{4-9}$$

and hence by (4-2),

$$\|u_0 - u_J\|^2_{H^2(M)} = \sum_{j=J+1}^{\infty} (1+\lambda_j^2) d_j^2 \leqslant 2 \sum_{j=J+1}^{\infty} \lambda_j^2 d_j^2 \leqslant 2C_0^2 \Lambda^2 \lambda_J^{-1} \gamma^{-6}. \tag{4-10}$$

As a consequence, u_J satisfies the H^1-norm condition of $\mathcal{U}$ (4-6):

$$\begin{aligned}\|u_J\|_{H^1(M)} &\leqslant \|u_0\|_{H^1(M)} + \|u_0 - u_J\|_{H^1(M)} \\ &\leqslant C_0 \Lambda \gamma^{-1} + \sqrt{2} C_0 \Lambda \lambda_J^{-1/2} \gamma^{-3} < 3 C_0 \Lambda \gamma^{-3}.\end{aligned}$$

Next we show that u_J also satisfies the $H^{2,2}$-norm condition of $\mathcal{U}$ (4-6) for sufficiently large J. This condition is trivially satisfied when $\alpha_k = 0$. Due to the finite speed propagation of waves, the condition $u_0|_{M_\alpha} = 0$ implies that $W(u_0)|_{\Gamma_k \times (-\alpha_k, \alpha_k)} = 0$ for all k with $\alpha_k \neq 0$. Thus it suffices to show that $W(u_0) - W(u_J)$ has small $H^{2,2}$-norm on $\partial M \times [-D, D]$.

Since $u_0 \in H^3(M)$, the regularity result for the wave equation (e.g., Theorem 2.45 in [Katchalov et al. 2001]) shows that

$$W(u_0)|_{M \times [-D,D]} \in C([-D, D]; H^3(M)) \cap C^3([-D, D]; L^2(M)).$$

Hence from (4-7), we have

$$(W(u_0) - W(u_J))(x,t)|_{\partial M \times [-D,D]} = \sum_{j=J+1}^{\infty} d_j \cos(\sqrt{\lambda_j} t) \varphi_j(x)|_{\partial M}.$$

Then the trace theorem and (4-8) imply that

$$\begin{aligned}\|W(u_0) - W(u_J)\|^2_{H^2(\partial M)} &\leqslant C \|W(u_0) - W(u_J)\|^2_{H^{11/4}(M)} \\ &= C \sum_{j=J+1}^{\infty} (1+\lambda_j^{11/4}) d_j^2 \cos^2(\sqrt{\lambda_j} t) \leqslant 2C \sum_{j=J+1}^{\infty} d_j^2 \lambda_j^{11/4} \leqslant C(\Lambda) \lambda_J^{-1/4} \gamma^{-6},\end{aligned}$$

where the last inequality is due to a similar estimate to (4-9). For the time derivatives, the trace theorem and (4-8) imply

$$\begin{aligned}\|\partial_t^2 W(u_0) - \partial_t^2 W(u_J)\|^2_{L^2(\partial M)} &\leqslant C \|\partial_t^2 W(u_0) - \partial_t^2 W(u_J)\|^2_{H^{3/4}(M)} \\ &= C \sum_{j=J+1}^{\infty} (1+\lambda_j^{3/4}) d_j^2 \lambda_j^2 \cos^2(\sqrt{\lambda_j} t) \leqslant 2C \sum_{j=J+1}^{\infty} d_j^2 \lambda_j^{11/4} \leqslant C(\Lambda) \lambda_J^{-1/4} \gamma^{-6}.\end{aligned}$$

Similarly by using (4-9),

$$\|\partial_t W(u_0) - \partial_t W(u_J)\|^2_{L^2(\partial M)} \leqslant C(\Lambda)\lambda_J^{-1}\gamma^{-6}.$$

Hence by the definition of $H^{2,2}$-norm (2-5),

$$\begin{aligned}\|W(u_0) - W(u_J)\|^2_{H^{2,2}(\partial M\times[-D,D])} &\leqslant 2D\, C(\Lambda)(2\lambda_J^{-1/4}\gamma^{-6} + \lambda_J^{-1}\gamma^{-6}) \\ &\leqslant C(D,\Lambda)\lambda_J^{-1/4}\gamma^{-6}.\end{aligned}$$

Therefore, for all $k = 0, \dots, N$ with $\alpha_k \neq 0$, we have

$$\|W(u_J)\|^2_{H^{2,2}(\Gamma_k\times[-\alpha_k,\alpha_k])} = \|W(u_0) - W(u_J)\|^2_{H^{2,2}(\Gamma_k\times[-\alpha_k,\alpha_k])} \leqslant C(D,\Lambda)\lambda_J^{-1/4}\gamma^{-6}.$$

For any $\varepsilon_1 > 0$, choose sufficiently large J such that $\lambda_J \geqslant C(D,\Lambda)\gamma^{-24}\varepsilon_1^{-8}$ and the lemma follows. □

Remark 4.1. The choice of J_0 in Lemma 4.2 also depends on geometric parameters, which is brought in when applying the trace theorem. Those relevant parameters are part of the parameters we considered in Section 3, so we omit them in this section for brevity. The same goes for the next two propositions, where the dependency on geometric parameters is brought in when applying Proposition 3.3.

We prove the following approximation result for finite spectral data.

Proposition 4.3. *Let $u \in H^3(M)$ be a given function with $\|u\|_{L^2(M)} = 1$ and $\|u\|_{H^3(M)} \leqslant \Lambda$. Let $\alpha = (\alpha_0, \dots, \alpha_N)$, $\alpha_k \in [\eta, D] \cup \{0\}$, be given, and M_α be defined in (4-1). Then, for any $\varepsilon > 0$, there exists sufficiently large $J = J(D, N, \Lambda, \eta, \varepsilon)$ such that by only knowing the first J Neumann boundary spectral data $\{\lambda_j, \varphi_j|_{\partial M}\}_{j=1}^J$ and the first J Fourier coefficients $\{a_j\}_{j=1}^J$ of u, we can find $\{b_j\}_{j=1}^J$ and $u^a = \sum_{j=1}^J b_j\varphi_j$ such that*

$$\|u^a - \chi_{M_\alpha}u\|_{L^2(M)} < \varepsilon,$$

where χ denotes the characteristic function.

Proof. We consider the following minimization problem in $\mathcal{U}(J, \Lambda, \gamma, \varepsilon_1)$ (denoted by $\mathcal{U}$ from now on) defined in (4-6), where the parameters J, γ, ε_1 will be determined later. Let $u_{\min} \in \mathcal{U}$ be the solution of the minimization problem

$$\|u_{\min} - u\|_{L^2(M)} = \min_{w\in\mathcal{U}} \|w - u\|_{L^2(M)}. \tag{4-11}$$

Observe that given the first J Fourier coefficients of u, finding the minimum of the norm $\|w-u\|_{L^2(M)}$ is equivalent to finding the minimum of a polynomial in terms of the (J number of) Fourier coefficients of w. Since the conditions of $\mathcal{U}$ (4-6) can be checked with finite boundary spectral data by (4-7) and (4-8), the minimization problem transforms into a polynomial minimization problem in a bounded domain in $\mathbb{R}^J$ (the space of Fourier coefficients). Hence the Fourier coefficients of the minimizer $u_{\min}$ are solvable by only using the finite spectral data.

Next, we investigate what properties this minimizer $u_{\min}$ satisfies. By Proposition 3.3 and the fact that the Neumann boundary condition is imposed, $w \in \mathcal{U}$ implies that $\|w\|_{L^2(M(\Gamma_k,\alpha_k))} < \varepsilon_2(h, \Lambda, \eta, \gamma, \varepsilon_1)$ for all $k = 0, 1, \dots, N$ with $\alpha_k \neq 0$, where

$$\varepsilon_2 = C_3^{1/3} h^{-2/9} \exp(h^{-C_4 n}) \frac{\Lambda\gamma^{-3} + h^{-1/2}\varepsilon_1}{(\log(1 + h^{3/2}\gamma^{-3}\Lambda/\varepsilon_1))^{1/6}} + C_5 \Lambda \gamma^{-3} h^{1/(3n+3)}. \tag{4-12}$$

Hence,

$$\|w\|_{L^2(M_\alpha)} < (N+1)\varepsilon_2.$$

Then, for any $w \in \mathcal{U}$ and in particular for $w = u_{\min}$,

$$\begin{aligned} \|w-u\|^2_{L^2(M)} &= \|w-u\|^2_{L^2(M_\alpha)} + \|w-u\|^2_{L^2(M_\alpha^c)} \\ &> \|u\|^2_{L^2(M_\alpha)} - 4N\varepsilon_2 + \|w-u\|^2_{L^2(M_\alpha^c)}. \end{aligned} \tag{4-13}$$

On the other hand the following estimate holds for u_J:

$$\begin{aligned} \|u_J - u\|^2_{L^2(M)} &\leqslant (\|u_J - u_0\|_{L^2(M)} + \|u_0 - u\|_{L^2(M)})^2 \\ &\leqslant \|u_J - u_0\|^2_{L^2(M)} + 4\|u_J - u_0\|_{L^2(M)} + \|u_0 - u\|^2_{L^2(M)} \\ &\leqslant C(\Lambda)\lambda_J^{-1/2}\gamma^{-2} + \|u\|^2_{L^2(M_\alpha)} + \|u_0 - u\|^2_{L^2(M_{\alpha+\gamma} - M_\alpha)}, \end{aligned}$$

where the last inequality is due to an estimate for $\|u_J - u_0\|_{L^2}$ similar to (4-10), and the definition of u_0. The definition of partition of unity in (4-3), the Sobolev embedding theorem (see the proof of Proposition 3.3) and (4-4) yield that

$$\|u_0 - u\|_{L^2(M_{\alpha+\gamma} - M_\alpha)} \leqslant \|u\|_{L^2(M_{\alpha+\gamma} - M_\alpha)} < 2C_5 \Lambda \gamma^{1/(2\max\{n,3\})}.$$

Hence,

$$\|u_J - u\|^2_{L^2(M)} < C(\Lambda)\lambda_J^{-1/2}\gamma^{-2} + \|u\|^2_{L^2(M_\alpha)} + 4C_5^2\Lambda^2\gamma^{1/(n+1)}.$$

For sufficiently large $J = J(D, \Lambda, \gamma, \varepsilon_1)$, we have $u_J \in \mathcal{U}$ by Lemma 4.2. This indicates that the minimizer $u_{\min}$ also satisfies

$$\|u_{\min} - u\|^2_{L^2(M)} < C(\Lambda)\lambda_J^{-1/2}\gamma^{-2} + \|u\|^2_{L^2(M_\alpha)} + 4C_5^2\Lambda^2\gamma^{1/(n+1)}. \tag{4-14}$$

Combining the two inequalities (4-13) and (4-14), we have

$$\|u_{\min} - u\|^2_{L^2(M_\alpha^c)} < 4N\varepsilon_2 + C(\Lambda)\lambda_J^{-1/2}\gamma^{-2} + 4C_5^2\Lambda^2\gamma^{1/(n+1)}.$$

The fact that $\|u_{\min}\|_{L^2(M_\alpha)} < N\varepsilon_2$ implies that

$$\begin{aligned} \|\chi_{M_\alpha}u - (u - u_{\min})\|^2_{L^2(M)} &= \|u_{\min} - \chi_{M_\alpha^c}u\|^2_{L^2(M)} \\ &= \|u_{\min} - \chi_{M_\alpha^c}u\|^2_{L^2(M_\alpha^c)} + \|u_{\min}\|^2_{L^2(M_\alpha)} \\ &< 4N\varepsilon_2 + C(\Lambda)\lambda_J^{-1/2}\gamma^{-2} + 4C_5^2\Lambda^2\gamma^{1/(n+1)} + 4N^2\varepsilon_2^2. \end{aligned}$$

From our discussion at the beginning of this proof, we know the Fourier coefficients of $u_{\min}$ are solvable. Suppose we have found a minimizer $u_{\min} = \sum_{j=1}^J c_j\varphi_j$. Since the first J Fourier coefficients of u are given as a_j, we can replace the function $u - u_{\min}$ in the last inequality by $\sum_{j=1}^J a_j\varphi_j - u_{\min}$ and the error in L^2-norm is controlled by $\Lambda\lambda_J^{-1/2}$. Hence by the Cauchy–Schwarz inequality, we obtain

$$\Big\|\chi_{M_\alpha}u-\sum_{j=1}^{J}(a_j-c_j)\varphi_j\Big\|^2_{L^2(M)}<8N\varepsilon_2+8N^2\varepsilon_2^2+C(\Lambda)\lambda_J^{-1/2}\gamma^{-2}+8C_5^2\Lambda^2\gamma^{1/(n+1)}, \tag{4-15}$$

which makes $u^a:=\sum_{j=1}^{J}b_j\varphi_j$ with $b_j=a_j-c_j$ our desired function.

Finally, we determine the relevant parameters. For any $\varepsilon>0$, we first choose and fix γ such that the last (4-15) term $8C_5^2\Lambda^2\gamma^{1/(n+1)}$ is equal to $\varepsilon^2/4$, and choose sufficiently large J such that the third term is smaller than $\varepsilon^2/4$. Then we choose ε_2 so that the first two terms satisfy $8N\varepsilon_2+8N^2\varepsilon_2^2=\varepsilon^2/4$. Next we determine ε_1. We choose and fix $h<\eta/100$ such that the second term in (4-12) is equal to $\varepsilon_2/2$, and choose ε_1 such that the first term in (4-12) is equal to $\varepsilon_2/2$. By Lemma 4.2, there exists sufficiently large J such that $u_J\in\mathcal{U}$, which validates all the estimates. □

4.2. *Approximation results with spectral data with error.* Now suppose that not only do we not know all the spectral data, we also only know them up to an error. More precisely, suppose we are given a set of data $\{\lambda_j^a,\varphi_j^a|_{\partial M}\}$ which is a δ-approximation of the Neumann boundary spectral data, where $\lambda_j^a\in\mathbb{R}_{\geqslant 0}$ and $\varphi_j^a|_{\partial M}\in C^2(\partial M)$. By Definition 1.1, there exists a choice of Neumann boundary spectral data $\{\lambda_j,\varphi_j|_{\partial M}\}_{j=1}^{\infty}$ such that, for all $j\leqslant\delta^{-1}$,

$$|\sqrt{\lambda_j}-\sqrt{\lambda_j^a}|<\delta,\qquad \|\varphi_j-\varphi_j^a\|_{C^{0,1}(\partial M)}+\|\nabla^2_{\partial M}(\varphi_j-\varphi_j^a)|_{\partial M}\|<\delta. \tag{4-16}$$

Since $\varphi_j^a\in C^2(\partial M)$ by assumption, the bound on the $C^{0,1}$-norm above yields

$$\|\varphi_j-\varphi_j^a\|_{C^0(\partial M)}+|\nabla(\varphi_j-\varphi_j^a)|_{\partial M}|<\delta\quad\text{for } j\leqslant\delta^{-1}. \tag{4-17}$$

In a local coordinate $(x^1,\dots,x^{n-1})$ on ∂M, for any $f\in C^2(\partial M)$, we have the formula

$$(\nabla^2_{\partial M}f)\Big(\frac{\partial}{\partial x^k},\frac{\partial}{\partial x^l}\Big)=\frac{\partial^2 f}{\partial x^k\partial x^l}-\sum_{i=1}^{n-1}\Gamma^i_{kl}\frac{\partial f}{\partial x^i},\qquad k,l=1,\dots,n-1.$$

Furthermore, we can choose to work in the geodesic normal coordinate. Then the norm of the second covariant derivative (the Hessian), the formula above and (2-3) yield a bound $C\delta$ on the second derivative of $(\varphi_j-\varphi_j^a)|_{\partial M}$:

$$\Big|\frac{\partial^2}{\partial x^k\partial x^l}(\varphi_j-\varphi_j^a)|_{\partial M}\Big|<C\delta\quad\text{for } j\leqslant\delta^{-1},\ k,l=1,\dots,n-1. \tag{4-18}$$

We prove the following approximation result analogous to Proposition 4.3.

Proposition 4.4. *Let $u\in H^3(M)$ be a given function with $\|u\|_{L^2(M)}=1$ and $\|u\|_{H^3(M)}\leqslant\Lambda$. Let $\alpha=(\alpha_0,\dots,\alpha_N)$, $\alpha_k\in[\eta,D]\cup\{0\}$ be given, and M_α be defined in (4-1). Then, for any $\varepsilon>0$, there exists sufficiently large $J=J(D,N,\Lambda,\eta,\varepsilon)$ such that the following holds.*

There exists $\delta=\delta(D,\operatorname{vol}(\partial M),N,\Lambda,J,\eta,\varepsilon)\leqslant J^{-1}$ such that by knowing a δ-approximation $\{\lambda_j^a,\varphi_j^a|_{\partial M}\}$ of the Neumann boundary spectral data, and knowing the first J Fourier coefficients $\{a_j\}_{j=1}^{J}$ of u, we can find $\{b_j\}_{j=1}^{J}$ and $u^a=\sum_{j=1}^{J}b_j\varphi_j$ such that

$$\|u^a-\chi_{M_\alpha}u\|_{L^2(M)}<\varepsilon.$$

Here the known Fourier coefficients of u are with respect to $\{\varphi_j\}$, *which is a choice of orthonormalized eigenfunctions satisfying* (4-16) *for* $\{\lambda_j^a, \varphi_j^a|_{\partial M}\}$.

Proof. Since we only know an approximation of the boundary spectral data, an error appears when we determine if a function belongs to the space $\mathcal{U}$ (4-6) in the minimization problem (4-11). The norms appearing in the conditions of $\mathcal{U}$ can be written in terms of the Fourier coefficients and boundary spectral data. However in this case, the actual spectral data are unknown and we can only check these norm conditions with a given approximation of the spectral data. First we need to estimate how these conditions change when the spectral data are perturbed.

For a function $v(x) = \sum_{j=1}^J v_j\varphi_j(x)$ with $\sum_{j=1}^J v_j^2 \leqslant 1$, the error for the H^1-norm condition of $\mathcal{U}$ is

$$\left| \|v\|_{H^1(M)}^2 - \sum_{j=1}^J (1+\lambda_j^a)v_j^2 \right| = \sum_{j=1}^J |\lambda_j - \lambda_j^a| v_j^2 < (2\sqrt{\lambda_J}+\delta)\delta. \tag{4-19}$$

For the $H^{2,2}$-norm condition of $\mathcal{U}$, from (4-7) we know

$$W(v)(x,t)|_{\partial M\times\mathbb{R}} = \sum_{j=1}^J v_j \cos(\sqrt{\lambda_j}t)\varphi_j(x)|_{\partial M}.$$

To check if this condition is satisfied, we can only use the approximate spectral data:

$$W^a(v)(x,t)|_{\partial M\times\mathbb{R}} = \sum_{j=1}^J v_j \cos(\sqrt{\lambda_j^a}t)\varphi_j^a(x)|_{\partial M}.$$

In fact, we are only concerned with a finite time range $t\in[-D,D]$. Since

$$|\cos(\sqrt{\lambda_j}t) - \cos(\sqrt{\lambda_j^a}t)| \leqslant |\sqrt{\lambda_j}t - \sqrt{\lambda_j^a}t| < D\delta,$$

we have the following estimate on the error:

$$\begin{aligned}
\|W(v)-W^a(v)\|_{H^2(\partial M)} &\leqslant \left\| \sum_{j=1}^J v_j\cos(\sqrt{\lambda_j}t)\varphi_j - \sum_{j=1}^J v_j \cos\Big(\sqrt{\lambda_j^a}t\Big)\varphi_j \right\|_{H^2(\partial M)} \\
&\qquad + \left\| \sum_{j=1}^J v_j \cos(\sqrt{\lambda_j^a}t)\varphi_j - \sum_{j=1}^J v_j\cos(\sqrt{\lambda_j^a}t)\varphi_j^a \right\|_{H^2(\partial M)} \\
&\leqslant D\delta\sum_{j=1}^J |v_j| \|\varphi_j\|_{H^2(\partial M)} + \sum_{j=1}^J |v_j| \|\varphi_j - \varphi_j^a\|_{H^2(\partial M)} \\
&< D\delta \sum_{j=1}^J \|\varphi_j\|_{H^2(\partial M)} + CJ\delta\sqrt{\mathrm{vol}(\partial M)},
\end{aligned}$$

where the last inequality is due to (4-17) and (4-18). By the trace theorem and (4-8), we know

$$\|\varphi_j\|_{H^2(\partial M)}^2 \leqslant C\|\varphi_j\|_{H^3(M)}^2 = C(1+\lambda_j^3),$$

and hence we obtain

$$\|W(v) - W^a(v)\|_{H^2(\partial M)} < C(D, \mathrm{vol}(\partial M)) J\lambda_J^{3/2}\delta.$$

Similarly for the time derivatives, we have

$$\|\partial_t W(v)-\partial_t W^a(v)\|_{L^2(\partial M)} < C(D,\mathrm{vol}(\partial M))J\lambda_J\delta,$$
$$\|\partial_t^2 W(v)-\partial_t^2 W^a(v)\|_{L^2(\partial M)} < C(D,\mathrm{vol}(\partial M))J\lambda_J^{3/2}\delta.$$

Therefore by definition (2-5), for some $C_0'=C_0'(D,\mathrm{vol}(\partial M))$, we have

$$\|W(v)-W^a(v)\|_{H^{2,2}(\partial M\times[-D,D])} < C_0' J\lambda_J^{3/2}\delta. \tag{4-20}$$

Now following the proof of Proposition 4.3, we still consider the minimization problem (4-11), however in a perturbed space of $\mathcal{U}$. We define an approximate space $\mathcal{U}^a$ of $\mathcal{U}$ as follows:

$$\mathcal{U}^a=\bigcap_{k=0}^{N}\Bigl\{v=\sum_{j=1}^{J}v_j\varphi_j:\sum_{j=1}^{J}v_j^2\leqslant 1,\ \sum_{j=1}^{J}(1+\lambda_j^a)v_j^2\leqslant 9C_0^2\Lambda^2\gamma^{-6}+3\lambda_J^{1/2}\delta,\\ \|W^a(v)\|_{H^{2,2}(\Gamma_k\times[-\alpha_k,\alpha_k])}\leqslant \varepsilon_1+C_0'J\lambda_J^{3/2}\delta\Bigr\}.$$

Clearly this space $\mathcal{U}^a$ can be determined with only Fourier coefficients and the given approximation $\{\lambda_j^a,\varphi_j^a|_{\partial M}\}$ of the boundary spectral data. Then we consider the minimization problem (4-11) with the space $\mathcal{U}$ replaced by $\mathcal{U}^a$. Hence this perturbed minimization problem is solvable by only using the given approximation of the spectral data.

By Lemma 4.2, there exists sufficiently large J such that $u_J\in\mathcal{U}$, and it follows from (4-19) and (4-20) that $u_J\in\mathcal{U}^a$. Then one can follow the rest of the proof for Proposition 4.3. The only part changed is ε_2, since the actual H^1- and $H^{2,2}$-norms of $v\in\mathcal{U}^a$ differ from the original conditions of $\mathcal{U}$. More precisely, for any $v\in\mathcal{U}^a$, again by (4-19) and (4-20), we have

$$\|v\|_{H^1(M)} < \sqrt{9C_0^2\Lambda^2\gamma^{-6}+6\lambda_J^{1/2}\delta} < 3C_0\Lambda\gamma^{-3}+3\lambda_J^{1/4}\sqrt{\delta},$$
$$\|W(v)\|_{H^{2,2}(\Gamma_k\times[-\alpha_k,\alpha_k])} < \varepsilon_1+2C_0'J\lambda_J^{3/2}\delta.$$

Therefore following the proof of Proposition 4.3, for $\delta<\lambda_J^{-1}$, one obtains an estimate almost the same as (4-15) with $\varepsilon_2(\delta)$:

$$\Bigl\|\chi_{M_\alpha}u-\sum_{j=1}^{J}(a_j-c_j)\varphi_j\Bigr\|_{L^2(M)}^2 < 8N\varepsilon_2(\delta)+8N^2\varepsilon_2^2(\delta)+C(\Lambda)\lambda_J^{-1/2}\gamma^{-2}+8C_5^2\Lambda^2\gamma^{1/(n+1)}, \tag{4-21}$$

where c_j is the j-th Fourier coefficient of a minimizer, and

$$\varepsilon_2(\delta)=C_3^{1/3}h^{-2/9}\exp(h^{-C_4 n})\frac{\Lambda\gamma^{-3}+h^{-1/2}(\varepsilon_1+2C_0'J\lambda_J^{3/2}\delta)}{(\log(1+h^{3/2}\gamma^{-3}\Lambda/(\varepsilon_1+2C_0'J\lambda_J^{3/2}\delta)))^{1/6}}+C_5\Lambda\gamma^{-3}h^{1/(3n+3)}.$$

Finally we determine the relevant parameters. For any $\varepsilon>0$, we first choose and fix γ, $\varepsilon_2(0)$, ε_1 such that the right-hand side of (4-21) with $\delta=0$ is equal to $3\varepsilon^2/4$ in the same way as in Proposition 4.3. By Lemma 4.2 we choose and fix sufficiently large J such that $u_J\in\mathcal{U}$, which validates all the estimates if we restrict $\delta \leqslant J^{-1}$. At last we choose sufficiently small $\delta < \lambda_J^{-1}$ such that

$$N\varepsilon_2(\delta) + N^2\varepsilon_2^2(\delta) - N\varepsilon_2(0) - N^2\varepsilon_2^2(0) < \varepsilon^2/32,$$

and then the proposition follows. □

Remark 4.2. We point out that in Propositions 4.3 and 4.4, it suffices to know the boundary data on $\bigcup_{\alpha_i>0}\Gamma_i$ to obtain the estimate for M_α with $\alpha_0 = 0$. This may be useful when only partial boundary spectral data (measured only on a part of the boundary) are known.

5. Approximations to boundary distance functions

Let M be a compact Riemannian manifold with smooth boundary ∂M. For $x \in M$, the *boundary distance function* $r_x : \partial M \to \mathbb{R}$ is defined by

$$r_x(z) = d(x, z), \quad z \in \partial M.$$

Then the boundary distance functions define a map $\mathcal{R} : M \to L^\infty(\partial M)$ by $\mathcal{R}(x) = r_x$. It is known that the map $\mathcal{R}$ is a homeomorphism and the metric of the manifold can be reconstructed from its image $\mathcal{R}(M)$ (e.g., Section 3.8 in [Katchalov et al. 2001]). Furthermore, the reconstruction is stable (Theorem 5.7). Therefore, to construct a stable approximation of the manifold from boundary spectral data, we only need to construct a stable approximation to the boundary distance functions $\mathcal{R}(M)$. In this section, we construct an approximation to the boundary distance functions through slicing procedures.

Given $\eta > 0$, let $\{\Gamma_i\}_{i=1}^N$ be a partition of the boundary ∂M into disjoint open connected subsets satisfying the assumptions at the beginning of Section 4: $\operatorname{diam}(\Gamma_i) \leqslant \eta$ and every Γ_i contains a ball (of ∂M) of radius $\eta/6$, where the diameter is measured with respect to the distance of M. We can also choose Γ_i to be the closure of these open sets. For example, one can choose Γ_i to be the Voronoi regions corresponding to a maximal $\eta/2$-separated set on ∂M with respect to the intrinsic distance $d_{\partial M}$ of ∂M. It is straightforward to check that these Voronoi regions satisfy our assumptions with

$$N \leqslant C(n, \operatorname{vol}(\partial M))\eta^{-n+1}. \tag{5-1}$$

The approximation results in Section 4 enable us to approximate the volume on M by only knowing an approximation of the Neumann boundary spectral data.

Lemma 5.1. *Let $\alpha = (\alpha_0, \dots, \alpha_N)$, $\alpha_k \in [\eta, D] \cup \{0\}$, be given, and M_α be defined in (4-1). Then, for any $\varepsilon > 0$, there exists sufficiently small $\delta = \delta(\eta, \varepsilon)$ such that by only knowing a δ-approximation $\{\lambda_j^a, \varphi_j^a|_{\partial M}\}$ of the Neumann boundary spectral data, we can compute a number $\operatorname{vol}^a(M_\alpha)$ satisfying*

$$|\operatorname{vol}^a(M_\alpha) - \operatorname{vol}(M_\alpha)| < \varepsilon.$$

Proof. Recall that $\varphi_1 = \operatorname{vol}(M)^{-1/2}$ on M and it follows that

$$\|\chi_{M_\alpha}\varphi_1\|^2_{L^2(M)} = \frac{\operatorname{vol}(M_\alpha)}{\operatorname{vol}(M)}.$$

Since the eigenspace with respect to $\lambda_1 = 0$ is 1-dimensional, the Fourier coefficients of φ_1 with respect to any choice of orthonormalized Neumann eigenfunctions are $(1, 0, \dots, 0, \dots)$. Apply Proposition 4.4

to $u=\varphi_1$, and we obtain the Fourier coefficients of $u^a=\sum_{j=1}^J b_j\varphi_j$ for sufficiently large J, and that the L^2-norm of u^a approximates $\|\chi_{M_\alpha}\varphi_1\|_{L^2(M)}$. Therefore $\sum_{j=1}^J b_j^2$ approximates $\mathrm{vol}(M_\alpha)/\mathrm{vol}(M)$, and equivalently $\mathrm{vol}(M)\sum_{j=1}^J b_j^2$ approximates $\mathrm{vol}(M_\alpha)$. If $\mathrm{vol}(M)$ is known, then $\mathrm{vol}(M)\sum_{j=1}^J b_j^2$ is the number we are looking for.

However, we do not exactly know $\mathrm{vol}(M)$ since we do not exactly know the first eigenfunction; we only know an approximation of $\mathrm{vol}(M)$ in terms of the first approximate eigenfunction φ_1^a. More precisely,

$$\delta > \|\varphi_1-\varphi_1^a\|_{C^0(\partial M)} \geqslant |\mathrm{vol}(M)^{-1/2}-\|\varphi_1^a\|_{C^0(\partial M)}|.$$

Hence an approximate volume can be defined as

$$\mathrm{vol}^a(M_\alpha):=\|\varphi_1^a\|_{C^0(\partial M)}^{-2}\sum_{j=1}^J b_j^2,$$

and then it satisfies the statement of the lemma. □

Besides the conditions we discussed earlier for the partition $\{\Gamma_i\}$, we need to further restrict the choice of the partition. We start with the following independent lemma regarding the *boundary distance coordinate*. One may refer to Section 2.1.21 in [Katchalov et al. 2001] for a brief introduction on this subject. This type of coordinate will be used to reconstruct the inner part (bounded away from the boundary) of the manifold.

Lemma 5.2. *Let $M\in\mathcal{M}_n(D,K_1,K_2,i_0)$. Then there exist a constant L and boundary points $\{z_i\}_{i=1}^L$, $z_i\in\partial M$, such that the following two properties hold*:

(1) *For any $x\in M$ with $d(x,\partial M)\geqslant i_0/2$, there exist n boundary points $\{z_{i_1(x)},\dots,z_{i_n(x)}\}\subset\{z_i\}_{i=1}^L$ such that the distance functions $(d(\cdot,z_{i_1(x)}),\dots,d(\cdot,z_{i_n(x)}))$ define a bi-Lipschitz local coordinate in a neighborhood of x.*

(2) *The map $\Phi_L:M\to\mathbb{R}^L$ defined by*

$$\Phi_L(x)=(d(x,z_1),\dots,d(x,z_L))$$

is bi-Lipschitz on $\{x\in M: d(x,\partial M)\geqslant i_0/2\}$, where the Lipschitz constant and L depend only on $n,D,K_1,K_2,i_0,\mathrm{vol}(\partial M)$.

Furthermore, the boundary points $\{z_i\}_{i=1}^L$ can be chosen as any r_L-maximal separated set on ∂M, where $r_L<i_0/8$ is a constant depending only on n,D,K_1,K_2,i_0.

Proof. Given $x\in M$ with $d(x,\partial M)\geqslant i_0/2$, let $z\in\partial M$ be a nearest boundary point; i.e., $d(x,z)=d(x,\partial M)$. Then it follows that z is not conjugate to x along the minimizing geodesic from x to z. That is to say, the differential $d\exp_x|_v$ is nondegenerate, where $\exp_x$ denotes the exponential map of M and $v=\exp_x^{-1}(z)$. Hence by the inverse function theorem, there exists a neighborhood of $(x,v)\in TM$ (with respect to the Sasaki metric on the tangent bundle) such that the exponential map is a diffeomorphism to a neighborhood of z. Furthermore, one can find a uniform radius r_1 depending on n,D,K_1,K_2,i_0 for the size of these neighborhoods [Katsuda et al. 2007, Lemma 4].

We take $\{z_i\}$ to be an r_2-net on ∂M (with respect to the intrinsic distance $d_{\partial M}$ of ∂M), where the parameter $r_2 < r_1/8$ is determined later. By definition, there exists $z_1 \in \{z_i\}$ such that $d_{\partial M}(z, z_1) < r_2$. Then we search for $n-1$ points $z_2, \dots, z_n$ such that ${}_{\partial M}\exp_{z_1}^{-1}(z_j)$ (for $j = 2, \dots, n$) form a basis in $T_{z_1}(\partial M)$, where ${}_{\partial M}\exp$ denotes the exponential map of ∂M. We claim that this is possible for sufficiently small r_2 explicitly depending on r_1, n, K_1. This claim can be proved as follows. Take $v_2, \dots, v_n$ to be an orthonormal basis of $T_{z_1}(\partial M)$, and consider the points $z_j' = {}_{\partial M}\exp_{z_1}(s v_j) \in \partial M$ for a fixed $s \in (r_1/4, r_1/2)$. By the definition of r_2-net, there exist points $z_2, \dots, z_n \in \{z_i\}$ such that $d_{\partial M}(z_j', z_j) < r_2$ (for $j = 2, \dots, n$). We consider the triangle with the vertices z_1, z_j', z_j. Since the lengths of the sides $z_1 z_j'$ and $z_1 z_j$ are at least $r_1/8$, for sufficiently small r_2 explicitly depending on K_1, the angle of the triangle at z_1 is small (by Toponogov's theorem) and therefore ${}_{\partial M}\exp_{z_1}^{-1}(z_j)$ (for $j = 2, \dots, n$) also form a basis. Then by the same argument as Lemma 2.14 in [Katchalov et al. 2001], one can show $z_1, z_2, \dots, z_n$ are the desired boundary points, from which a boundary distance coordinate is admitted in a neighborhood of x.

From now on, we choose $\{z_i\}_{i=1}^{L}$ to be a maximal r_2-separated set on ∂M, which is indeed an r_2-net by maximality. The cardinality L of this net is bounded by $C(n, \mathrm{vol}(\partial M)) r_2^{-n+1}$. The bi-Lipschitzness of the boundary distance coordinate follows from the fact that the differential of the exponential map is uniformly bounded in the relevant domain by a constant depending on n, D, K_1, K_2, i_0 [Katsuda et al. 2007, Lemma 3 and Proposition 1]. This concludes the proof for the first part of the lemma.

Next we prove the second part of the lemma. We claim that there exists $r_3 > 0$ such that Φ_L with respect to any maximal r_3-separated set on ∂M is bi-Lipschitz on $\{x \in M : d(x, \partial M) \geqslant i_0/2\}$. Note that Φ_L is automatically Lipschitz with the Lipschitz constant $\sqrt{L}$ by the triangle inequality. Suppose there exist a sequence of manifolds $M_k \in \mathcal{M}_n(D, K_1, i_0)$ and points $x_k, y_k \in \{x \in M_k : d(x, \partial M_k) \geqslant i_0/2\}$ such that

$$\frac{|\Phi_{L,k}(x_k) - \Phi_{L,k}(y_k)|}{d_{M_k}(x_k, y_k)} \to 0 \quad \text{as } k \to \infty,$$

where $\Phi_{L,k}$ is defined with respect to some maximal $1/k$-separated set on ∂M_k. The precompactness of $\mathcal{M}_n(D, K_1, i_0)$ [Anderson et al. 2004, Theorem 3.1] yields that there exists a subsequence of M_k that converges to a limit M in the C^1-topology. We choose subsequences of x_k, y_k that converge to limit points $x, y \in M$. The assumption implies that $\Phi_L(x) = \Phi_L(y)$ with respect to a dense subset of ∂M. Due to the fact that the boundary distance map $\mathcal{R}$ is a homeomorphism [Katchalov et al. 2001, Lemma 3.30], it follows that $x = y$. Moreover, we have $d(x, \partial M) \geqslant i_0/2$. However, for sufficiently large k such that $x_k, y_k \in B_{r_1}(x)$, the points x_k, y_k lie in the same boundary distance coordinate neighborhood by the first part of the lemma, on which $\Phi_{L,k}$ is locally bi-Lipschitz with a uniformly bounded Lipschitz constant. This is a contradiction to the assumption. Therefore there exists some $r_3 > 0$ depending on n, D, K_1, i_0 such that Φ_L with respect to any maximal r_3-separated set on ∂M is bi-Lipschitz.

Finally, we further restrict $\{z_i\}_{i=1}^{L}$ to be a maximal $\min\{r_1, r_2, r_3\}$-separated set on ∂M. Hence the cardinality L satisfies

$$L \leqslant C(n, \mathrm{vol}(\partial M)) \min\{r_1, r_2, r_3\}^{-n+1},$$

which depends only on $n, D, K_1, K_2, i_0, \mathrm{vol}(\partial M)$. We define $r_L = \min\{r_1, r_2, r_3\}$, which depends on n, D, K_1, K_2, i_0. □

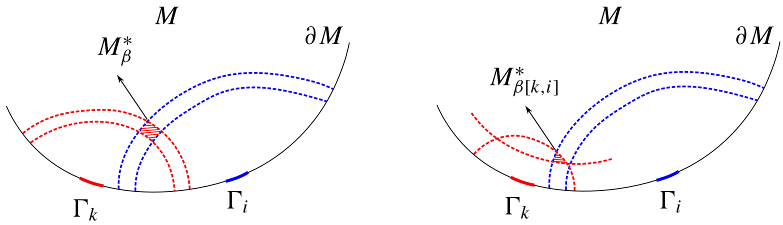

Figure 6. Subdomains from two subsets of the boundary. The former type is used to reconstruct the inner part of the manifold, while the latter type is used to reconstruct the boundary normal neighborhood.

Choice of partition. Let $\eta > 0$ be given. We choose boundary points $\{z_i\}_{i=1}^N$ and a partition $\{\Gamma_i\}_{i=1}^N$ of ∂M as follows. Let $\{z_1, \dots, z_L\}$ be the boundary points determined in Lemma 5.2, and then we add $N-L$ number of boundary points such that $\{z_1, \dots, z_N\}$ is a maximal $\eta/2$-separated set on ∂M. This is possible because $\{z_1, \dots, z_L\}$ can be chosen as any r_L-maximal separated set on ∂M, with r_L being a uniform constant independent of η. We take $\{\Gamma_i\}_{i=1}^N$ to be a partition of ∂M (e.g., Voronoi regions corresponding to $\{z_i\}_{i=1}^N$) satisfying the assumptions at the beginning of this section: $\operatorname{diam}(\Gamma_i) \leqslant \eta$, $z_i \in \Gamma_i$, and every Γ_i contains a ball (of ∂M) of radius $\eta/6$. The cardinality N of the partition is bounded above by (5-1).

Definition 5.3. Let $\eta > 0$ be given. For multi-indices β of the form $\beta = (\beta_0, \beta_1, \dots, \beta_N)$, with $\beta_0 \in \{0, 1\}$, $\beta_1, \dots, \beta_N \in \mathbb{N}$, we consider the following two types of subdomains (see Figure 6):

(1) Given a multi-index $\beta = (0, \beta_1, \dots, \beta_N)$, we define a slicing of the manifold by

$$M^*_\beta = \bigcap_{i:\,\beta_i>0} \{x \in M : d(x, \Gamma_i) \in [\beta_i\eta - 2\eta, \beta_i\eta)\}. \tag{5-2}$$

We also consider the following modified multi-index by setting specific components equal to zero:

$$\beta\langle l\rangle := (0, \beta_1, \dots, \beta_L, 0, \dots, 0, \beta_l, 0, \dots, 0), \quad l \in \{L+1, \dots, N\}.$$

(2) Given a multi-index $\beta = (1, \beta_1, \dots, \beta_N)$, we define a modified multi-index by

$$\beta[k, i] := (1, 0, \dots, 0, \beta_k, 0, \dots, 0, \beta_i, 0, \dots, 0), \quad k \neq i.$$

In other words, $\beta[k, i]$ can only have nonzero k-th and i-th components besides the 0-th component. Then we define the subdomain

$$M^*_{\beta[k,i]} = \bigl\{x \in M : d(x, \partial M) \geqslant \beta_k\eta - 2\eta,\ d(x, \Gamma_k) < \beta_k\eta,\ d(x, \Gamma_i) \in [\beta_i\eta - 2\eta, \beta_i\eta)\bigr\}. \tag{5-3}$$

By definition (5-2), we only slice the manifold from Γ_i if $\beta_i > 0$. Hence $M^*_\beta \subset M^*_{\beta\langle l\rangle}$ for any $l \in \{L+1, \dots, N\}$. Since the diameter of the manifold is bounded above by D, it suffices to consider a finite number of choices $\beta_i \leqslant 2 + D/\eta$ for each β_i. Notice that we always use a fixed number (independent of η) of Γ_i to slice the manifold. This keeps the total number of slicings from growing too large as η gets small.

Similar to Lemma 5.1, we can also evaluate approximate volumes for $\operatorname{vol}(M^*_{\beta\langle l\rangle})$, $\operatorname{vol}(M^*_{\beta[k,i]})$, and the error can be made as small as needed given sufficient boundary spectral data.

Lemma 5.4. *Let $\eta > 0$ be given, and $M^*_{\beta\langle l\rangle}$, $M^*_{\beta[k,l]}$ be defined in Definition 5.3. Then, for any $\varepsilon > 0$, there exists sufficiently small $\delta = \delta(\eta, \varepsilon)$ such that by only knowing a δ-approximation $\{\lambda^a_j, \varphi^a_j|_{\partial M}\}$ of the Neumann boundary spectral data, we can compute numbers $\mathrm{vol}^a(M^*_{\beta\langle l\rangle})$, $\mathrm{vol}^a(M^*_{\beta[k,i]})$ satisfying*

$$|\mathrm{vol}^a(M^*_{\beta\langle l\rangle}) - \mathrm{vol}(M^*_{\beta\langle l\rangle})| < 2^{L+1}\varepsilon \quad \text{for any } l \in \{L+1, \dots, N\},$$
$$|\mathrm{vol}^a(M^*_{\beta[k,i]}) - \mathrm{vol}(M^*_{\beta[k,i]})| < 4\varepsilon \qquad \text{for any } i \neq k,$$

where L is a uniform constant independent of η determined in Lemma 5.2.

Proof. Observe that for any $\beta = (0, \beta_1, \dots, \beta_N)$ with $\beta_1, \dots, \beta_N > 0$, the subdomain M^*_β can be obtained as a finite number of unions, intersections and complements of the subdomains M_α of the form (4-1) with $\alpha_0 = 0$. More precisely,

$$M^*_\beta = \bigcap_{i=1}^N (M(\Gamma_i, \beta_i\eta) - M(\Gamma_i, \beta_i\eta - 2\eta))$$
$$= \bigcap_{i=1}^N M(\Gamma_i, \beta_i\eta) - \bigcup_{i=1}^N M(\Gamma_i, \beta_i\eta - 2\eta).$$

Then the volume of M^*_β can be written in terms of the volumes of M_α with $\alpha_0 = 0$ through the following operations. For any n-dimensional Hausdorff measurable subset $\Omega_1, \Omega_2 \subset M$,

$$\mathrm{vol}(\Omega_1 - \Omega_2) = \mathrm{vol}(\Omega_1 \cup \Omega_2) - \mathrm{vol}(\Omega_2),$$
$$\mathrm{vol}(\Omega_1 \cap \Omega_2) = \mathrm{vol}(\Omega_1) + \mathrm{vol}(\Omega_2) - \mathrm{vol}(\Omega_1 \cup \Omega_2).$$

Moreover, for any multi-indices α, α',

$$\mathrm{vol}(M_\alpha \cup M_{\alpha'}) = \mathrm{vol}(M_{\alpha_{\max}}), \quad \text{where } (\alpha_{\max})_i = \max\{\alpha_i, \alpha'_i\}.$$

Therefore the approximate volume $\mathrm{vol}^a(M^*_\beta)$ for M^*_β can be defined by replacing the volumes of M_α in the expansion with the approximate volume $\mathrm{vol}^a(M_\alpha)$.

On the other hand, for a multi-index of the form $\beta[k, i]$, we have

$$M^*_{\beta[k,i]} = M(\Gamma_k, \beta_k\eta) \cap M(\Gamma_i, \beta_i\eta) - M(\partial M, \beta_k\eta - 2\eta) \cup M(\Gamma_i, \beta_i\eta - 2\eta).$$

Recall that the volume information from the whole boundary ∂M is incorporated in the α_0-component of the multi-index α. Thus the volume of $M^*_{\beta[k,l]}$ can be written in terms of the volumes of M_α with $\alpha_0 \geqslant 0$.

For a multi-index of the form $\beta\langle l\rangle$, the total number of volume terms of M_α in $\mathrm{vol}(M^*_{\beta\langle l\rangle})$ is at most 2^{L+1}. For a multi-index of the form $\beta[k, i]$, the total number of volume terms of M_α in $\mathrm{vol}(M^*_{\beta[k,i]})$ is at most 4. Then the error estimates directly follow from Lemma 5.1. □

Now we are in place to define an approximation to the boundary distance functions $\mathcal{R}(M)$. We consider the following candidate.

Definition 5.5. Let $\eta, \varepsilon > 0$ be given. For a multi-index $\beta = (\beta_0, \beta_1, \dots, \beta_N)$ with $\beta_0 \in \{0, 1\}$, $\beta_1, \dots, \beta_N \in \mathbb{N}_+$, if either of the following two situations happens, we associate with this β a piecewise constant function $r_\beta \in L^\infty(\partial M)$ defined by

$$r_\beta(z) = \beta_i\eta \quad \text{if } z \in \Gamma_i :$$

(1) $\beta_0 = 0$; $\beta_i \eta > i_0/2$ for all $i = 1, \dots, N$, and $\mathrm{vol}^a(M^*_{\beta\langle l\rangle}) \geqslant \varepsilon$ for all $l = L+1, \dots, N$.

(2) $\beta_0 = 1$; there exists $k \in \{1, \dots, N\}$ such that $\beta_k \eta \leqslant i_0/2$ and $\mathrm{vol}^a(M^*_{\beta[k,i]}) \geqslant \varepsilon$ for all $i = 1, \dots, N$ with $i \neq k$.

We test all multi-indices β up to $\beta_i \leqslant 2 + D/\eta$ for each β_i, and denote the set of all functions r_β chosen this way by $\mathcal{R}^*_\varepsilon$.

Intuitively, the first situation in Definition 5.5 describes a small neighborhood in the interior of the manifold away from the boundary. The second situation describes a small neighborhood near the boundary with the help of the boundary normal neighborhood. We prove that $\mathcal{R}^*_\varepsilon$ is an approximation to the boundary distance functions $\mathcal{R}(M)$ for sufficiently small ε.

Proposition 5.6. *Let $M \in \mathcal{M}_n(D, K_1, K_2, i_0, r_0)$. For any $\eta > 0$, there exist $\varepsilon = \varepsilon(\eta)$ and sufficiently small $\delta = \delta(\eta)$ such that by only knowing a δ-approximation $\{\lambda^a_j, \varphi^a_j|_{\partial M}\}$ of the Neumann boundary spectral data we can construct a set $\mathcal{R}^*_\varepsilon \subset L^\infty(\partial M)$ such that*

$$d_H(\mathcal{R}^*_\varepsilon, \mathcal{R}(M)) \leqslant C_6\sqrt{\eta},$$

where d_H denotes the Hausdorff distance between subsets of the metric space $L^\infty(\partial M)$ and the constant C_6 depends only on $n, D, K_1, K_2, i_0, \mathrm{vol}(\partial M)$.

Proof. Let $\eta < \min\{1, i_0/8\}$. Given any $x \in M$, take a point $x' \in M$ such that $d(x, x') \leqslant \eta$ and $d(x', \partial M) \geqslant \eta$. Clearly there exist positive integers $\beta_i > 0$ such that $d(x', \Gamma_i) \in [\beta_i\eta - 2\eta, \beta_i\eta)$ for all $i = 1, \dots, N$. In fact, there are two choices for each β_i, and we choose the one satisfying $d(x', \Gamma_i) \in [\beta_i\eta - 3\eta/2, \beta_i\eta - \eta/2)$ for all i. In particular, we see that each β_i satisfies $\beta_i\eta - 2\eta \leqslant D$.

If $\beta_i\eta > i_0/2$ for all $i = 1, \dots, N$, then we consider the multi-index $\beta = (0, \beta_1, \dots, \beta_N)$. It follows from the triangle inequality that $B_{\eta/2}(x') \subset M^*_\beta$. Since $B_{\eta/2}(x')$ does not intersect ∂M, we have $\mathrm{vol}(M^*_\beta) > \mathrm{vol}(B_{\eta/2}(x')) \geqslant c_n\eta^n$ for sufficiently small η, which implies that $\mathrm{vol}(M^*_{\beta\langle l\rangle}) > c_n\eta^n$ for all $l = L+1, \dots, N$. We define

$$\varepsilon_* = c_n\eta^n/2, \tag{5-4}$$

and set $\varepsilon = 2^{-L-1}\varepsilon_*$ in Lemma 5.4. Then we consider the set of functions $\mathcal{R}^*_{\varepsilon_*}$. Since $\mathrm{vol}^a(M^*_{\beta\langle l\rangle}) > c_n\eta^n - \varepsilon_* = \varepsilon_*$ by Lemma 5.4, we have $r_\beta \in \mathcal{R}^*_{\varepsilon_*}$ by the first situation in Definition 5.5. Then by the condition $\mathrm{diam}(\Gamma_i) \leqslant \eta$ and the triangle inequality, we have

$$\|r_x - r_\beta\|_{L^\infty(\partial M)} \leqslant \|r_x - r_{x'}\|_{L^\infty(\partial M)} + \|r_{x'} - r_\beta\|_{L^\infty(\partial M)} \leqslant \eta + 2\eta = 3\eta. \tag{5-5}$$

If there exists $k \in \{1, \dots, N\}$ such that $\beta_k\eta \leqslant i_0/2$, then we consider the multi-index $\beta = (1, \beta_1, \dots, \beta_N)$. Without loss of generality, assume k is the index such that $\beta_k = \min_{i>0}\beta_i$. Hence

$$d(x', \partial M) = \min\{d(x', \Gamma_1), \dots, d(x', \Gamma_N)\} \geqslant \beta_k\eta - 3\eta/2,$$

which shows $x' \in M^*_{\beta[k,i]}$ for all $i = 1, \dots, N$ with $i \neq k$ by definition (5-3). Moreover, we also have $B_{\eta/2}(x') \subset M^*_{\beta[k,i]}$ for all i. Thus by choosing the same ε_* and ε as the previous case, we have $r_\beta \in \mathcal{R}^*_{\varepsilon_*}$ by the second situation in Definition 5.5, and (5-5) still holds. This concludes the proof for one direction.

On the other hand, given any $r_\beta \in \mathcal{R}^*_{\varepsilon_*}$, Definition 5.5 and Lemma 5.4 indicates that either $\mathrm{vol}(M^*_{\beta\langle l\rangle}) > 0$ for all $l = L+1, \dots, N$, or there exists k such that $\mathrm{vol}(M^*_{\beta[k,i]}) > 0$ for all i. Recall that $\beta_1, \dots, \beta_N > 0$ by definition.

(i) The first situation allows us to pick an arbitrary point x_l in every $M^*_{\beta\langle l\rangle}$. Then by $\mathrm{diam}(\Gamma_i) \leqslant \eta$ and the triangle inequality, we have

$$\|r_\beta - r_{x_l}\|_{L^\infty(\Gamma_1\cup\cdots\cup\Gamma_L\cup\Gamma_l)} \leqslant 3\eta \quad \text{for any } l \in \{L+1, \dots, N\}. \tag{5-6}$$

Notice that all x_l are in fact bounded away from the boundary. More precisely, for any x_l, we know from Definition 5.5 that

$$d(x_l, \Gamma_i) \geqslant \beta_i\eta - 2\eta > i_0/2 - 2\eta > i_0/4 \quad \text{for all } i = 1, \dots, L.$$

Since the boundary points $\{z_i\}_{i=1}^L$ can be chosen as an r_L-maximal separated set on ∂M, where $r_L < i_0/8$ is a uniform constant independent of η (Lemma 5.2), we have, for any x_l,

$$d(x_l, \partial M) > i_0/8.$$

Hence for any other $j \in \{L+1, \dots, N\}$ with $j \neq l$, Lemma 5.2 yields that

$$d(x_l, x_j) \leqslant C(n, D, K_1, i_0)|\Phi_L(x_l) - \Phi_L(x_j)| \leqslant C\sqrt{L}\,\eta,$$

where $\Phi_L(\cdot) = (d(\cdot, z_1), \dots, d(\cdot, z_L))$. Then it follows from the triangle inequality and (5-6) that

$$\|r_\beta - r_{x_l}\|_{L^\infty(\Gamma_j)} \leqslant \|r_\beta - r_{x_j}\|_{L^\infty(\Gamma_j)} + \|r_{x_j} - r_{x_l}\|_{L^\infty(\Gamma_j)} \leqslant (C\sqrt{L}+3)\eta.$$

Thus by ranging $j \neq l$ over $\{L+1, \dots, N\}$, we obtain

$$\|r_\beta - r_{x_l}\|_{L^\infty(\partial M)} \leqslant (C\sqrt{L}+3)\eta.$$

(ii) The second situation allows us to pick an arbitrary point x_i in every $M^*_{\beta[k,i]}$. Observe from Definition 5.5 that, for any x_i, we have

$$d(x_i, \partial M) \leqslant d(x_i, \Gamma_k) < \beta_k\eta \leqslant i_0/2.$$

The fact that $d(x, \Gamma_k) \geqslant d(x, \partial M)$ implies that

$$\|r_\beta - r_{x_i}\|_{L^\infty(\Gamma_k\cup\Gamma_i)} \leqslant 2\eta.$$

For any other $j \in \{1, \dots, N\}$ with $j \neq k, i$, we have

$$d(x_i, x_j) \leqslant C\sqrt{\eta}.$$

This is due to the fact that the diameter of the subdomain $\{x \in M : d(x, \partial M) \geqslant \beta_k\eta - 2\eta,\ d(x, \Gamma_k) < \beta_k\eta\}$ for $\beta_k\eta \leqslant i_0/2$ is bounded above by $C\sqrt{\eta}$. Hence by ranging $j \neq k, i$ over $\{1, \dots, N\}$, we obtain

$$\|r_\beta - r_{x_i}\|_{L^\infty(\partial M)} \leqslant C\sqrt{\eta} + 2\eta. \qquad \square$$

Remark 5.1. We only used a fixed number (independent of η) of subsets of the boundary to slice the manifold, so that the total number of slicings does not grow too large as η gets small. To reconstruct the inner part of the manifold, we used $L+1$ subsets with L being a uniform constant (however not explicit).

Near the boundary, we took advantage of the boundary normal neighborhood and essentially only used two subsets. Instead if we use all N subsets to slice the manifold, it would result in a third logarithm in Theorem 1.

Remark 5.2. By virtue of Remark 4.2, the approximate volume for M_α with $\alpha_0 = 0$ in Lemma 5.1 can be found by only knowing the boundary data on $\bigcup_{\alpha_i>0} \Gamma_i$. This implies that the approximate volume for M_β^* (with $\beta_0 = 0$) in Lemma 5.4 can be found by only knowing the boundary data on $\bigcup_{\beta_i>0} \Gamma_i$. Thus in a way similar to but simpler than Definition 5.5 and Proposition 5.6, one can define an approximation to $\mathcal{R}(M)$ restricted on a part of the boundary using partial boundary spectral data. Furthermore in the case of partial data, a similar calculation to that in the Appendix yields a log-log-log estimate on the stability of the reconstruction of $\mathcal{R}(M)$.

The following result shows that the reconstruction of a manifold from $\mathcal{R}(M)$ is stable.

Theorem 5.7 [Katsuda et al. 2007, Theorem 1]. *Let M be a compact Riemannian manifold with smooth boundary. Suppose $\mathcal{R}^*$ is an η-approximation to the boundary distance functions $\mathcal{R}(M)$ for sufficiently small η. Then one can construct a finite metric space X directly from $\mathcal{R}^*$ such that*

$$d_{\mathrm{GH}}(M, X) < C_7(n, D, K_1, K_2, i_0)\, \eta^{1/36},$$

where d_{GH} denotes the Gromov–Hausdorff distance between metric spaces.

Finally we prove the main results Theorems 1 and 2.

Proof of Theorem 1. The estimate directly follows from Proposition 5.6 and Theorem 5.7. The dependency of constants is derived in the Appendix.

The only part left is to find an upper bound for $\mathrm{vol}(\partial M)$, $\mathrm{vol}(M)$ in terms of other geometric parameters. Due to Corollary 2(b) in [Katsuda et al. 2007], the (intrinsic) diameter of ∂M is uniformly bounded by a constant depending on $n, D, \|R_M\|_{C^1}, \|S\|_{C^2}, i_0$, however not explicitly. Then by the volume comparison theorem for ∂M, $\mathrm{vol}(\partial M)$ is uniformly bounded by the same set of parameters. As for $\mathrm{vol}(M)$, the manifold M is covered by harmonic coordinate charts with the total number of charts bounded (not explicitly) by a constant depending on $n, D, \|R_M\|_{C^1}, \|S\|_{C^2}, i_0$ [Katsuda et al. 2007, Theorem 3]. Away from the boundary, the volumes of balls of a small radius are uniformly bounded. Near the boundary, we can use the boundary normal neighborhood of ∂M since $\mathrm{vol}(\partial M)$ is already shown to be bounded. Hence $\mathrm{vol}(M)$ is uniformly bounded by the same set of parameters. □

Proof of Theorem 2. We take the first δ^{-1} Neumann boundary spectral data of M_2, and by Definition 1.1, this set of finite data (without error) is a δ-approximation of the Neumann boundary spectral data of M_2. By Proposition 5.6, we can construct an approximation to $\mathcal{R}(M_2)$. On the other hand, the finite spectral data of M_2 is δ-close to the Neumann boundary spectral data of M_1 by Definition 1.1, since the Neumann boundary spectral data of M_1 and M_2 are δ-close by assumption. Then from the pull-back of the finite spectral data of M_2 via the boundary isometry, we can construct an approximation to $\mathcal{R}(M_1)$. Since the boundary isometry (diffeomorphism) preserves Riemannian metrics on the boundaries, the pull-back of the finite spectral data via the boundary isometry produces an isometric approximation to the boundary distance functions. Hence Theorem 2 follows from Corollary 1 in [Katsuda et al. 2007]. □

6. Technical lemmas

This section contains the proofs of several lemmas used in Section 3. Some of the lemmas in this section, especially Lemma 6.5, are important technical results, and we prove them here without interrupting the structure of the main proof. Some other lemmas are known facts. We did not find precise references for them, so we present short proofs here.

Lemma 6.1. *Let $(M, g) \in \mathcal{M}_n(D, K_1, K_2, i_0)$. Denote by S_ρ the second fundamental form of the equidistant hypersurface in M defined by the level set $d(\cdot\,, \partial M) = \rho$ for $\rho < i_0$. Then there exists a uniform constant r_b explicitly depending only on K_1, i_0 such that, for any $\rho \leqslant r_b$, we have $\|S_\rho\| \leqslant 2K_1$.*

Moreover, if the metric components satisfy (2-3) with respect to a coordinate chart in a ball U of ∂M, then the metric components with respect to the boundary normal coordinate in $U \times [0, r_b]$ satisfy

$$\|g_{ij}\|_{C^1} \leqslant C(n, \|R_M\|_{C^1}, \|S\|_{C^1}), \quad \|g_{ij}\|_{C^4} \leqslant C(n, K_1, K_2, i_0) \quad \textit{for all } 1 \leqslant i, j \leqslant n.$$

Proof. At an arbitrary point $z \in \partial M$, take an arbitrary unit vector V in $T_z(\partial M)$ and extend it to $V(\rho) \in T_{\gamma_{z,\boldsymbol{n}}(\rho)} M$ ($\rho < i_0$) via the parallel translation along $\gamma_{z,\boldsymbol{n}}$, where $\gamma_{z,\boldsymbol{n}}$ denotes the geodesic of M from z with the initial normal vector $\boldsymbol{n}$ at z. We still use the notation S_ρ to denote the shape operator of the equidistant hypersurface with distance ρ from ∂M. Consider the function

$$\kappa_V(\rho) = \langle S_\rho(V(\rho)), V(\rho)\rangle_g.$$

The bound on the second fundamental form of ∂M indicates $|\kappa_V(0)| \leqslant K_1$. For convenience, we omit the evaluation at ρ and use V to denote the vector field $V(\rho)$.

Since V is a parallel vector field with respect to the normal vector field $\partial/\partial\rho$ (or simply ∂_ρ), we have

$$\frac{d}{d\rho}\kappa_V = \langle \nabla_{\partial_\rho}(S_\rho V), V\rangle + \langle S_\rho V, \nabla_{\partial_\rho} V\rangle = \langle (\nabla_{\partial_\rho} S_\rho)V, V\rangle.$$

Then the Riccati equation (e.g., [Petersen 2006, Theorem 2, p. 44]) leads to the formula

$$\frac{d}{d\rho}\kappa_V = -\langle S_\rho^2 V, V\rangle + R_M(V, \partial_\rho, V, \partial_\rho). \tag{6-1}$$

Due to the fact that S_ρ is symmetric and $|V| = 1$, we have

$$\langle S_\rho^2 V, V\rangle = |S_\rho V|^2 \geqslant |\langle S_\rho V, V\rangle|^2.$$

Hence,

$$\frac{d}{d\rho}\kappa_V(\rho) \leqslant -\kappa_V^2(\rho) + K_1^2. \tag{6-2}$$

On the other hand, we need a lower bound for $d\kappa_V/d\rho$. This is possible because we a priori know the solution of the Riccati equation exists up to i_0, and the equidistant hypersurfaces vary smoothly in a neighborhood of ∂M. This implies that there exists a positive number $\rho_{\max} \leqslant i_0/2$ satisfying

$$\rho_{\max} = \sup\{\rho \in [0, i_0/2] : \|S_\tau\| \leqslant 2K_1 \text{ for all } \tau \in [0, \rho]\}.$$

Hence, for any $\rho \in [0, \rho_{\max}]$, we have $|S_\rho V| \leqslant 2K_1$ as the condition above is a closed condition. Then from (6-1),

$$\frac{d}{d\rho}\kappa_V(\rho) \geqslant -4K_1^2 - K_1^2 = -5K_1^2. \tag{6-3}$$

Combining (6-2) and (6-3), we have

$$\left|\frac{d}{d\rho}\kappa_V(\rho)\right| \leqslant 5K_1^2, \quad \rho \in [0, \rho_{\max}].$$

Thus for any $\rho \leqslant \min\{\rho_{\max}, (10K_1)^{-1}\}$, we have $|\kappa_V(\rho)| \leqslant 3K_1/2$. Since z and V are arbitrary, this shows $\|S_\rho\| \leqslant 3K_1/2$.

We claim that the uniform constant r_b can be chosen as $r_b = \min\{i_0/2, (10K_1)^{-1}\}$. This choice is obviously justified if $\rho_{\max} = i_0/2$. Now if $\rho_{\max} < i_0/2$, we prove that $\rho_{\max} > (10K_1)^{-1}$. Suppose otherwise, and it implies that $\|S_\rho\| \leqslant 3K_1/2$ satisfies for any $\rho \leqslant \rho_{\max}$. We know the solution of the Riccati equation exists in a neighborhood of $\rho_{\max}$, and therefore there exists a larger $\rho > \rho_{\max}$ satisfying the condition for $\rho_{\max}$ since $\rho_{\max} < i_0/2$ by assumption. This contradicts the maximality of $\rho_{\max}$. As a consequence, our estimate holds up to $\rho \leqslant (10K_1)^{-1}$ in this case. On the other hand, the fact that $(10K_1)^{-1} < \rho_{\max} < i_0/2$ justifies our choice of r_b in this case. This completes the proof for the first part of the lemma.

For the second part, we consider the matrix Riccati equation in the boundary normal coordinate. This time we use the Lie derivative version of the Riccati equation (e.g., [Petersen 2006, Proposition 7(3), p. 47]). The components of the shape operator are denoted by $S_\alpha^l = \sum_{\beta=1}^{n-1} g^{\beta l} S_{\alpha\beta}$, where $S_{\alpha\beta}$ denotes the components of the second fundamental form of the equidistant hypersurfaces. Here the evaluation at ρ is omitted. Then the Riccati equation has the form

$$\frac{d}{d\rho}S_{\alpha\beta} = \sum_{\gamma,l=1}^{n-1} g_{\gamma l} S_\alpha^\gamma S_\beta^l + R_M\Big(\frac{\partial}{\partial x^\alpha}, \frac{\partial}{\partial \rho}, \frac{\partial}{\partial x^\beta}, \frac{\partial}{\partial \rho}\Big).$$

By definition we have the equation on the distortion of metric:

$$\frac{d}{d\rho}g_{\alpha\beta} = 2S_{\alpha\beta}.$$

Due to the first part of the lemma, $dg_{\alpha\beta}/d\rho$ is uniformly bounded. As a consequence, $g_{\alpha\beta}$ is uniformly bounded since it is bounded in the coordinate chart on ∂M. The tangential derivatives of $g_{\alpha\beta}$ are estimated as follows.

The Riccati equation can be written in terms of $(S_{\alpha\beta})$ and $(g_{\alpha\beta})$ using the formula for the matrix inverse. We differentiate these two equations with respect to all tangential directions $x^1, \dots, x^{n-1}$, and we get a system of first-order ODEs with the variable $\boldsymbol{v}$:

$$\boldsymbol{v}(\rho) = \Big(\dots, \frac{\partial g_{\alpha\beta}}{\partial x_T}(\rho), \dots, \frac{\partial S_{\gamma l}}{\partial x_T}(\rho), \dots\Big), \quad \alpha, \beta, \gamma, l = 1 \dots, n-1,$$

where x_T ranges over all tangential directions $x^1, \dots, x^{n-1}$. This system of equations can be written in the form

$$\frac{d}{d\rho}\boldsymbol{v} = B_1\boldsymbol{v} + B_2\boldsymbol{v} + \nabla R_M^*.$$

The matrix B_1 is obtained by differentiating the term of the S^2 form in the Riccati equation, and only consists of components of the second fundamental form $(S_{\alpha\beta})$ and the metric $(g_{\alpha\beta})$. The matrix B_2 is obtained by differentiating the curvature term, and only consists of components of the curvature tensor and $(g_{\alpha\beta})$. The vector ∇R_M^* absorbs all the remaining terms and is considered as a constant vector. More precisely, the vector ∇R_M^* is made up of components of the covariant derivative ∇R_M, and components of R_M, $(S_{\alpha\beta})$, $(g_{\alpha\beta})$.

Due to the first part of the lemma, the components $(S_{\alpha\beta})$ and $(g_{\alpha\beta})$ are uniformly bounded in the boundary normal neighborhood of width r_b. Then it follows that the components $(g^{\alpha\beta})$ are also uniformly bounded. This implies that the matrices B_1, B_2 have norms bounded above by $C(n, K_1)$, and the vector ∇R_M^* has length bounded above by $C(n, K_1, \|\nabla R_M\|)$. The initial condition $|\boldsymbol{v}(0)|$ is bounded above by n, $\|\nabla S\|$. Then the standard theory of ODEs yields a bound for $|\boldsymbol{v}|$ and hence for all components of $\boldsymbol{v}$. In particular, $\partial g_{\alpha\beta}/\partial x_T$ are uniformly bounded, which implies that $\|g_{ij}\|_{C^1} \leqslant C(n, \|R_M\|_{C^1}, \|S\|_{C^1})$ for all $1 \leqslant i, j \leqslant n$.

We keep differentiating the matrix Riccati equation with respect to x_T and ρ up to the fourth order. By the same argument, all relevant coefficients of that system of ODEs are uniformly bounded by $\|R_M\|_{C^4}$, $(S_{\alpha\beta})$, $(g_{\alpha\beta})$ and previous lower-order estimates. Since the initial condition at $\rho = 0$ is bounded by n, $\|g_{\alpha\beta}(0)\|_{C^4}$, $\|S\|_{C^4}$ and $\|R_M\|_{C^3}$, the C^4 estimate for the metric components directly follows from (2-3). □

Lemma 6.2. (1) *For any $M \in \mathcal{M}_n(K_1)$, we have $r_{\mathrm{CAT}}(M) > 0$.*

Assume further $M \in \mathcal{M}_n(D, K_1, K_2, i_0)$. The submanifold M_h is defined in Definition 3.7. Suppose $\widetilde{M}$ is an extension of M satisfying Lemma 3.4(1)–(3) with the extension width δ_{ex}. Then:

(2) *For sufficiently small h, δ_{ex} explicitly depending on K_1, K_2, i_0, we have*

$$r_{\mathrm{CAT}}(M_h) \geqslant \min\{C(n, \|R_M\|_{C^1}, \|S\|_{C^1}), r_{\mathrm{CAT}}(M)\},$$
$$r_{\mathrm{CAT}}(\widetilde{M}) \geqslant \min\left\{C(n, \|R_M\|_{C^1}, \|S\|_{C^1}), \frac{i_0}{4}, \frac{r_{\mathrm{CAT}}(M)}{2}\right\}.$$

(3) *For sufficiently small h, δ_{ex}, we have*

$$r_{\mathrm{CAT}}(M_h) \geqslant \min\left\{\frac{2r_{\mathrm{CAT}}}{3}(M), \frac{\pi}{2K_1}\right\}, \quad r_{\mathrm{CAT}}(\widetilde{M}) \geqslant \min\left\{\frac{2r_{\mathrm{CAT}}}{3}(M), \frac{\pi}{2K_1}\right\}.$$

Proof. Due to the characterization theorem in [Alexander et al. 1993], any point $x \in M$ has an open ball U_x such that U_x has curvature bounded above by K_1^2 in the sense of Alexandrov. In particular, for any point $p, q \in U_x$ satisfying $d_{U_x}(p, q) < \pi/K_1$, there is a unique minimizing geodesic in U_x (not necessarily a minimizer of M) connecting p and q (e.g., Theorem 9.8 in [Alexander et al. 2024]).

(1) Suppose $r_{\mathrm{CAT}}(M) = 0$, and there exist sequences of points p_i, q_i, such that there are two minimizing geodesics of M joining each pair of points p_i, q_i with $d(p_i, q_i) \to 0$. By the compactness of M, we can find converging subsequences of points, still denoted by p_i and q_i. Let x be their limit point. For sufficiently large i, there are two minimizing geodesics of M connecting p_i, q_i and they both lie in U_x, which is a contradiction to the property of U_x.

(2) Given an arbitrary point $p \in M_h$, suppose $q \in M_h$ is a point such that there are two minimizing geodesics of M_h connecting p, q. Without loss of generality, assume $d_h(p,q) < \min\{\pi/(2K_1), r_{\mathrm{CAT}}(M)\}$. We choose h sufficiently small such that $\|S_{\partial M_h}\| \leqslant 2\|S\|$ and $\|S_{\partial M_h}\|_{C^1} \leqslant 2\|S\|_{C^1}$. Recall that no conjugate points occur along geodesics (of M_h) of length less than $\pi/(2K_1)$ [Alexander et al. 1993, Corollary 3]. Furthermore, we consider p, q to be the closest pair: $d_h(p,q) = r_{\mathrm{CAT}}(M_h)$. Then by the first variation formula (e.g., Proposition 3 in [Alexander et al. 1993]), the two geodesics connecting p, q form a closed geodesic of M_h. It is known that geodesics on manifolds with smooth boundary are of $C^{1,1}$. Hence their geodesic curvature exists almost everywhere and is bounded by $C(n, \|R_M\|_{C^1}, \|S\|_{C^1})$ due to (6-14). Now consider these two geodesics of M_h connecting p, q as a closed $C^{1,1}$-curve of M, and it lies in the ball of M centered at p of the radius $\min\{\pi/(2K_1), r_{\mathrm{CAT}}(M)\}$, which is $\mathrm{CAT}(K_1)$ due to Theorem 4.3 in [Alexander and Bishop 1996]. Hence by Corollary 1.2(c) in [Alexander and Bishop 1996], the length of this closed curve is bounded below by $C(n, \|R_M\|_{C^1}, \|S\|_{C^1})$, and therefore $d_h(p,q)$ is bounded below by $C(n, \|R_M\|_{C^1}, \|S\|_{C^1})$.

Next we derive a lower bound for $r_{\mathrm{CAT}}(\widetilde{M})$. Suppose $p, q \in \widetilde{M}$ is the closest pair of points such that there are two minimizing geodesics of $\widetilde{M}$ joining p, q. Assume $\tilde{d}(p,q) < \min\{\pi/(4K_1), i_0/4, r_{\mathrm{CAT}}(M)/2\}$. Then we immediately see that at least one of these two geodesics intersects $\widetilde{M} - M$. This implies that both geodesics lie in the boundary normal (tubular) neighborhood of ∂M by assumption. Furthermore, the two geodesics connecting p, q form a closed geodesic of $\widetilde{M}$ by the first variation formula. We move inwards on this closed geodesic along the family of geodesics normal to ∂M by distance $\delta_{\mathrm{ex}} < i_0/2$. This process results in a closed $C^{1,1}$-curve of M contained in the boundary normal neighborhood. For sufficiently small δ_{ex} depending on K_1, K_2, this closed $C^{1,1}$-curve of M has length at most $3\tilde{d}(p,q)$ and its geodesic curvature is bounded by $C(n, \|R_M\|_{C^1}, \|S\|_{C^1})$ almost everywhere. Hence this closed curve of M lies in a ball of M of the radius $\min\{\pi/(2K_1), r_{\mathrm{CAT}}(M)\}$ (which is $\mathrm{CAT}(K_1)$), and therefore its length is bounded below by $C(n, \|R_M\|_{C^1}, \|S\|_{C^1})$ by Corollary 1.2(c) in [Alexander and Bishop 1996]. This shows that the length of the original closed geodesic of $\widetilde{M}$ is bounded below by $C(n, \|R_M\|_{C^1}, \|S\|_{C^1})$, which gives the lower bound for $\tilde{d}(p,q)$.

(3) Here we only prove the statement for M_h; the proof for $\widetilde{M}$ is the same. Suppose not, and we can find $p_i, q_i \in M_{h_i}$ ($h_i \to 0$) such that there are two minimizing geodesics of M_{h_i} connecting each pair p_i, q_i with $d_{h_i}(p_i, q_i) < \min\{2r_{\mathrm{CAT}}(M)/3, \pi/(2K_1)\}$. Moreover, we can assume q_i is the closest point from p_i such that this happens, and therefore the two geodesics connecting p_i, q_i form a closed geodesic of M_{h_i}. Thus we have a sequence of closed C^1-curves with lengths less than $4r_{\mathrm{CAT}}(M)/3$. This sequence of closed curves also has lengths uniformly bounded away from 0 due to (2). Hence by the Arzelà–Ascoli theorem, we can find a subsequence converging to a limit closed curve in M of nonzero length (not necessarily of C^1). Let $p, q \in M$ be the limit points of p_i, q_i. Since d_{h_i} converges to d (Lemma 3.6), the lower semicontinuity of length yields that the limit closed curve has length at most $2d(p,q)$.

Consider the segment of the limit closed curve from p to q and the other segment from q to p. Both segments must have lengths at least the distance $d(p,q)$. Since the limit closed curve has length at most $2d(p,q)$, each segment is a minimizing geodesic of M. If these two segments do not coincide, then we get two minimizing geodesics of M from p to q of lengths at most $2r_{\mathrm{CAT}}(M)/3$, which contradicts the condition for $r_{\mathrm{CAT}}(M)$. If the two segments coincide, we pick a point $y \in M$ on the limit curve close to p, and consider points y_1, y_2 on the closed geodesic of M_{h_i} near (fixed) y at opposite sides from p_i. For sufficiently large i, the points y_1, y_2 can be arbitrarily close in M_{h_i} and meanwhile bounded away from p_i. However, the angle between the geodesic segment of M_{h_i} from p_i to y_1 and the segment from p_i to y_2 is always π, since the curve in question is a closed C^1-curve. This is a contradiction to the local CAT condition for M_{h_i} combined with (2). □

Lemma 6.3. *Let h be sufficiently small determined at the beginning of Section 3.4. Let $d_h^s(\cdot, z)$ (Definition 3.8) be the smoothening of the function $d_h(\cdot, z)$ (Definition 3.7) with the smoothening radius $r = a_T h^3$, where $a_T = \min\{1, T^{-1}\}$. Then the following properties are satisfied for $z, z_1, z_2 \in M_h$, $x \in M$ and $x_1, x_2 \in \widetilde{M}$:*

(1) $|d_h(x_1, z_1) - d_h(x_1, z_2)| \leqslant d_h(z_1, z_2)$.

(2) $|d_h^s(x, z_1) - d_h^s(x, z_2)| \leqslant (1 + CnK_1^2 h^6) d_h(z_1, z_2)$.

(3) *For sufficiently small h only depending on K_1, we have*

$$|d_h(x_1, z) - d_h(x_2, z)| < \frac{3h^{-1}}{2} \tilde{d}(x_1, x_2).$$

(4) *For sufficiently small h depending on n, K_1, i_0, if $d_h(x, z) < i_0$, then*

$$|d_h^s(x, z) - d_h(x, z)| < 2a_T h^2.$$

Proof. (1) This directly follows from the definition of d_h.

(2) Let $r = a_T h^3$. Observe that the ball of radius h^3 centered at any $x \in M$ does not intersect $\partial \widetilde{M}$, and hence the distance function $\tilde{d}(\cdot, x)$ for $x \in M$ is simply a geodesic distance function. Due to (3-4), the Jacobian $J_x(v)$ of the exponential map $\exp_x(v)$ of $\widetilde{M}$ at $v \in \mathcal{B}_r(0) \subset T_x \widetilde{M}$ satisfies

$$|J_x(v) - 1| \leqslant CnK_1^2 |v|^2 \leqslant CnK_1^2 h^6. \tag{6-4}$$

Then it follows from (3-11) that

$$\int_{\widetilde{M}} k_1\left(\frac{\tilde{d}(y, x)}{r}\right) dy = \int_{\mathcal{B}_r(0) \subset T_x \widetilde{M}} k_1\left(\frac{|v|}{r}\right) J_x(v)\, dv \leqslant (1 + CnK_1^2 h^6) \int_{\mathbb{R}^n} k_1\left(\frac{|v|}{r}\right) dv. \tag{6-5}$$

This inequality (6-5), (3-11) and (1) yield (2).

(3) Recall that the second fundamental form of ∂M_h is bounded by $2K_1$ due to Lemma 6.1, and $\widetilde{M}$ can be considered as an extension of M_h by gluing a collar of width $6h$. If $x_1, x_2 \in M_h$, then Lemma 3.6 applies by replacing M with M_h and we have

$$|d_h(x_1, z) - d_h(x_2, z)| \leqslant d_h(x_1, x_2) \leqslant (1 + 36K_1 h) \tilde{d}(x_1, x_2). \tag{6-6}$$

If $x_1, x_2 \in \widetilde{M} - M_h$, then Lemma 3.6 yields

$$d_h(x_1^{\perp_h}, x_2^{\perp_h}) \leqslant (1 + 36K_1 h) \tilde{d}(x_1, x_2).$$

Then by the definition of d_h (3-9) and (6-6), we have

$$\begin{aligned}|d_h(x_1,z)-d_h(x_2,z)| &\leqslant |d_h(x_1^{\perp_h},z)-d_h(x_2^{\perp_h},z)|+h^{-1}|\tilde{d}(x_1,x_1^{\perp_h})-\tilde{d}(x_2,x_2^{\perp_h})| \\ &\leqslant d_h(x_1^{\perp_h},x_2^{\perp_h})+h^{-1}|\tilde{d}(x_1,\partial M_h)-\tilde{d}(x_2,\partial M_h)| \\ &\leqslant (1+36K_1h)\tilde{d}(x_1,x_2)+h^{-1}\tilde{d}(x_1,x_2).\end{aligned}$$

Thus the desired estimate follows for sufficiently small h only depending on K_1.

If $x_1\in\widetilde{M}-M_h,\ x_2\in M_h$, then similarly we have

$$\begin{aligned}|d_h(x_1,z)-d_h(x_2,z)| &\leqslant |d_h(x_1^{\perp_h},z)-d_h(x_2,z)|+h^{-1}\tilde{d}(x_1,x_1^{\perp_h}) \\ &\leqslant d_h(x_1^{\perp_h},x_2)+h^{-1}\tilde{d}(x_1,\partial M_h) \\ &\leqslant (1+36K_1h)\tilde{d}(x_1,x_2)+h^{-1}\tilde{d}(x_1,x_2),\end{aligned}$$

and the same estimate follows.

(4) In view of (6-4) and (6-5), the Jacobian only generates error terms of order at least h^6. Hence we only need to prove that for any point y in the ball (of $\widetilde{M}$) of the smoothening radius $a_T h^3$ around the center $x\in M$, it satisfies that $|d_h(y,z)-d_h(x,z)|<3a_T h^2/2$, which is guaranteed by (3). □

Lemma 6.4. *Let $\gamma_1,\gamma_2:[0,l]\to\mathbb{R}^n$ be two $C^{1,1}$ curves. If $\|\gamma_1-\gamma_2\|_{C^0}\leqslant\varepsilon<l^2/4$ and $\|\gamma_i''\|_{L^\infty}\leqslant\kappa$ for $i=1,2$, then $\|\gamma_1'-\gamma_2'\|_{C^0}\leqslant C(\kappa)\sqrt{\varepsilon}$.*

Proof. Since γ_i is of $C^{1,1}$, we know γ_i' is absolutely continuous. Hence Taylor's theorem with the integral form of the remainder applies:

$$\gamma_i(s_2)=\gamma_i(s_1)+\gamma_i'(s_1)(s_2-s_1)+\int_{s_1}^{s_2}\gamma_i''(\tau)(s_2-\tau)\,d\tau \quad \text{for all } 0\leqslant s_1<s_2\leqslant l.$$

From $\|\gamma_i''\|_{L^\infty}\leqslant\kappa$, we have

$$|\gamma_i(s_2)-\gamma_i(s_1)-\gamma_i'(s_1)(s_2-s_1)|\leqslant\frac{\kappa}{2}(s_2-s_1)^2.$$

Taking the inequality above for γ_1 and for γ_2, adding them together and using the triangle inequality, we obtain

$$\big|(\gamma_1(s_2)-\gamma_2(s_2))-(\gamma_1(s_1)-\gamma_2(s_1))-(\gamma_1'(s_1)-\gamma_2'(s_1))(s_2-s_1)\big|\leqslant\kappa(s_2-s_1)^2.$$

Then by $\|\gamma_1-\gamma_2\|_{C^0}\leqslant\varepsilon$,

$$|\gamma_1'(s_1)-\gamma_2'(s_1)|\leqslant\frac{2\varepsilon}{s_2-s_1}+\kappa(s_2-s_1).$$

Take $s_2-s_1=\sqrt{\varepsilon}$ if exists, and we have

$$|\gamma_1'(s_1)-\gamma_2'(s_1)|\leqslant(\kappa+2)\sqrt{\varepsilon}.$$

Since $\sqrt{\varepsilon}<l/2$, we can find $s_2=s_1+\sqrt{\varepsilon}$ for any $s_1\in[0,l/2]$. For $s_1\in(l/2,l]$, one can repeat the whole process backwards. Hence the estimate above holds for all $s_1\in[0,l]$, which proves the lemma. □

Lemma 6.5. *Let h be sufficiently small determined at the beginning of Section 3.4. Let $d_h^s(\cdot, z)$ (Definition 3.8) be the smoothening of the function $d_h(\cdot, z)$ (Definition 3.7) with the smoothening radius $r = a_T h^3$, where $a_T = \min\{1, T^{-1}\}$. Then, for sufficiently small h depending on n, K_1, K_2, given any $x \in M$ and $z \in M_h$ satisfying $h/4 \leqslant d_h(x, z) \leqslant \min\{i_0/2, r_0/2, \pi/(6K_1)\}$, we have*

$$|\nabla_x d_h^s(x, z)| > 1 - 2h.$$

Proof. Let $r = a_T h^3$. By the definition (3-10), we have

$$\begin{aligned}\nabla_x d_h^s(x, z) &= c_n r^{-n} \int_{\widetilde{M}} \nabla_x k_1\left(\frac{\tilde{d}(y, x)}{r}\right) d_h(y, z)\, dy \\ &= c_n r^{-n} \int_{\widetilde{B}_r(x)\subset \widetilde{M}} k_1' \frac{1}{r}\left(\frac{-\exp_x^{-1}(y)}{\tilde{d}(y, x)}\right) d_h(y, z)\, dy,\end{aligned}$$

where $\exp_x$ denotes the exponential map of $\widetilde{M}$ at $x \in M$. Now we change to the geodesic normal coordinate of $\widetilde{M}$ around x, and identify vectors in the tangent space $T_x\widetilde{M}$ with points in $\mathbb{R}^n$:

$$\begin{aligned}\nabla_x d_h^s(x, z) &= c_n r^{-n} \int_{\mathcal{B}_r(0)\subset T_x\widetilde{M}} k_1' \frac{1}{r}\frac{(-v)}{|v|}\, d_h(\exp_x(v), z) J_x(v)\, dv \\ &= c_n r^{-n} \int_{\mathcal{B}_r(0)\subset T_x\widetilde{M}} -\nabla_v\left(k_1\left(\frac{|v|}{r}\right)\right) d_h(\exp_x(v), z) J_x(v)\, dv \\ &= c_n r^{-n} \int_{\mathcal{B}_r(0)\subset T_x\widetilde{M}} k_1\left(\frac{|v|}{r}\right) \nabla_v(d_h(\exp_x(v), z) J_x(v))\, dv,\end{aligned}$$

where $J_x(v)$ denotes the Jacobian of $\exp_x$ at v. Here we have used integration by parts in the last equality.

It is known that $|\nabla_v J_x(v)| \leqslant C(n, K_1, K_2)|v| \leqslant C(n, K_1, K_2)h^3$ due to the C^1-estimate for the metric components [Hebey and Vaugon 1995, Lemma 8] and Lemma 3.4(3). Then by (3-11), we have

$$\begin{aligned}&\left| c_n r^{-n} \int_{\mathcal{B}_r(0)} k_1\left(\frac{|v|}{r}\right) d_h(\exp_x(v), z)(\nabla_v J_x(v))\, dv \right| \\ &\qquad\qquad \leqslant c_n r^{-n} \int_{\mathcal{B}_r(0)} k_1\left(\frac{|v|}{r}\right) \frac{\pi}{4K_1} C(n, K_1, K_2)h^3\, dv \leqslant C(n, K_1, K_2)h^3.\end{aligned}$$

Hence we only need to estimate the lower bound for the length of the dominating term

$$A_0 = c_n r^{-n} \int_{\mathcal{B}_r(0)\subset T_x\widetilde{M}} k_1\left(\frac{|v|}{r}\right)(\nabla_v\, d_h(\exp_x(v), z)) J_x(v)\, dv. \tag{6-7}$$

We start by considering the following two simple cases.

Case 1: $d_h(z, \partial M_h) > \min\{i_0/2, r_0/2, \pi/(6K_1)\}$. In this case, we know $x \in M_h$ and no geodesic from z to x intersects with ∂M_h in this case. Then the distance function $d_h(\cdot, z)$ in the relevant domain is simply a geodesic distance function with the second derivative bounded by $5/h$ for sufficiently small h depending on K_1 (e.g., [Petersen 2006, Theorem 27, p. 175]. Since the exponential map and its inverse are uniformly bounded up to C^2 in the relevant domain for sufficiently small h depending on K_1, K_2, we have

$$|\nabla_v d_h(\exp_x(v), z) - \nabla_v d_h(\exp_x(v), z)|_{v=0}| \leqslant Ch^{-1}|v| \leqslant Ch^2.$$

Note that vectors in $T_v(T_x\widetilde{M})$ are identified with vectors in $T_x\widetilde{M}$. Observe that at $v=0$ we know

$$\nabla_v d_h(\exp_x(v), z)|_{v=0} = (d\exp_x|_{v=0})^{-1}\nabla_x d_h(x, z) = \nabla_x d_h(x, z).$$

Hence by the Jacobian estimate (6-4) and the normalization (3-11), we obtain

$$\begin{aligned}|\nabla_x d_h^s(x, z) - \nabla_x d_h(x, z)| &\leqslant |A_0 - \nabla_x d_h(x, z)| + C(n, K_1, K_2)h^3\\ &\leqslant Ch^2 + C(n, K_1, K_2)h^3,\end{aligned}$$

which gives the desired lower bound for $|\nabla_x d_h^s(x, z)|$ for sufficiently small h, due to $|\nabla_x d_h(x, z)| = 1$.

Case 2: $x \in M - M_h$ and $\tilde{d}(x, \partial M_h) > r$. In this case, the gradient $\nabla_x d_h(x, z)$ is equal to $h^{-1}\nabla_x\tilde{d}(x, \partial M_h)$ by the definition of d_h (3-9). The second derivative of $\tilde{d}(\cdot, \partial M_h)$ is bounded by $2K_1$ on the second fundamental forms of the equidistant hypersurfaces from ∂M in the boundary normal neighborhood of ∂M (Lemma 6.1). Hence we have

$$|\nabla_v d_h(\exp_x(v), z) - \nabla_v d_h(\exp_x(v), z)|_{v=0}| \leqslant C(K_1)h^{-1}|v| \leqslant C(K_1)h^2. \tag{6-8}$$

Then the same argument as in Case 1 shows that

$$|\nabla_x d_h^s(x, z) - \nabla_x d_h(x, z)| \leqslant C(K_1)h^2 + C(n, K_1, K_2)h^3, \tag{6-9}$$

which yields a lower bound considering $|\nabla_x d_h(x, z)| = h^{-1}$.

The general case when x is close to ∂M_h requires more careful treatment. We spend the rest of the proof addressing it.

Case 3: $x \in M - M_h$ with $\tilde{d}(x, \partial M_h) \leqslant r$ or $x \in M_h$. Since $d_h(x, z) \leqslant \min\{r_0/2, \pi/(6K_1)\}$ is bounded by the radius of radial uniqueness (3-17), the gradient $|\nabla_x d_h(x, z)|$ equals 1 or h^{-1} depending on whether x is in M_h. It is known that geodesics of M_h are of $C^{1,1}$ and the second derivative of a geodesic exists except at countably many switch points (switching between interior segments and boundary segments) where both one-sided second derivatives exist (e.g., Section 2 in [Alexander et al. 1987]). Furthermore, the second derivative exists and vanishes at intermittent points which are the accumulation points of switch points. It was also proved that if the endpoints of a family of geodesics converge, then the geodesics converge uniformly in C^1 (see the first lemma in Section 4 of [Alexander et al. 1987]). However, the estimates in that work were done in terms of an extrinsic parameter (depending on how a manifold is embedded in the ambient space), and we show the following modification in terms of intrinsic parameters.

The manifold M_h has curvature bounded above by $4K_1^2$ locally in the sense of Alexandrov due to the characterization theorem in [Alexander et al. 1993]. Furthermore by [Alexander and Bishop 1996, Theorem 4.3] and (3-17), for any $z \in M_h$, the ball of M_h around z of the radius $\min\{2r_0/3, \pi/(4K_1)\}$ is a metric space of curvature bounded above by $4K_1^2$. Denote by γ_x, γ_y the minimizing geodesics of M_h from $x, y \in M_h$ to z. Denote the length of γ_x by L_x (i.e., $L_x = d_h(x, z)$). The geodesics γ_x, γ_y are parametrized in the arclength parameter on $[0, L_x]$, $[0, L_y]$ respectively. Without loss of generality, assume $L_x \leqslant L_y$. Hence

$$d_h(\gamma_y(L_x), \gamma_x(L_x)) = d_h(\gamma_y(L_x), \gamma_y(L_y)) = L_y - L_x \leqslant d_h(x, y),$$

where we used $\gamma_x(L_x) = \gamma_y(L_y) = z$. Then Corollary 9.13 in [Alexander et al. 2024] shows that if $d_h(x, z) \leqslant \pi/(6K_1)$ and $d_h(x, y)$ is sufficiently small depending on K_1, we have

$$\|\gamma_x - \gamma_y\|_{C^0([0,L_x])} < 2d_h(x, y),$$

where the C^0-norm is the uniform norm with respect to d_h. This leads to $\|\gamma_x - \gamma_y\|_{C^0([0,L_x])} < C\tilde{d}(x, y)$ if $\tilde{d}(x, y)$ is sufficiently small by (6-6). On the other hand, due to Lemma 6.1 and (6-14), the second derivatives of γ_x, γ_y are bounded by $C(n, K_1, K_2)$ whenever they exist in the boundary normal coordinate of ∂M_h, and both one-sided second derivatives respect the same bound at switch points.

We lift the part of the curves γ_x, γ_y near x, y onto the tangent space $T_x\widetilde{M}$. Without loss of generality, assume all of γ_x, γ_y lie in the image of $\exp_x$. Since the exponential map and its inverse are uniformly bounded up to C^2, the properties stated above satisfied by γ_x, γ_y are also satisfied by their lifts: namely, if $\tilde{d}(x, y)$ is sufficiently small depending on K_1,

$$\| \exp_x^{-1} \circ\gamma_x - \exp_x^{-1} \circ\gamma_y\|_{C^0([0,L_x])} < C\tilde{d}(x, y),$$

and the second derivatives of $\exp_x^{-1} \circ\gamma_x$, $\exp_x^{-1} \circ\gamma_y$ are uniformly bounded by $C(n, K_1, K_2)$ in L^∞-norm. Here the C^0-norm is the uniform norm with respect to the Euclidean distance in $T_x\widetilde{M}$. Hence Lemma 6.4 applies:

$$\|(\exp_x^{-1} \circ\gamma_x)' - (\exp_x^{-1} \circ\gamma_y)'\|_{C^0([0,L_x])} < C(n, K_1, K_2)\sqrt{\tilde{d}(x, y)}. \tag{6-10}$$

At the starting point $y = \gamma_y(0)$ of γ_y, we know $\gamma_y'(0) = -\nabla_y d_h(y, z)$ and hence

$$(\exp_x^{-1} \circ\gamma_y)'(0) = (d\exp_x|_v)^{-1}\gamma_y'(0) = -\nabla_v d_h(\exp_x(v), z),$$

where $v = \exp_x^{-1}(y)$. At the starting point $x = \gamma_x(0)$ of γ_x, we simply have $(\exp_x^{-1} \circ\gamma_x)'(0) = -\nabla_x d_h(x, z)$ by definition. Thus for sufficiently small h depending on K_1, if $y \in M_h$ and $\tilde{d}(x, y) \leqslant h^3$, the estimate (6-10) at starting points gives

$$|\nabla_v d_h(\exp_x(v), z) - \nabla_x d_h(x, z)| < C\sqrt{\tilde{d}(x, y)} \leqslant C(n, K_1, K_2)h^{3/2}. \tag{6-11}$$

The difference between this case and Case 1 is that the formula for $\nabla_x d_h^s(x, z)$ (at the beginning of the proof) may split into two parts: the integral over points in M_h and over points in $M - M_h$. The key observation is that in a small neighborhood intersecting ∂M_h, the gradient $\nabla_x d_h(x, z)$ for $x \in M - M_h$ is essentially normal to ∂M_h, which has almost the same direction as the normal component (with respect to ∂M_h) of $\nabla_x d_h(x, z)$ for $x \in M_h$. A precise version of this observation will be shown later. The h^{-1} scaling in the definition of d_h (3-9) plays a crucial role in obtaining the desired lower bound.

Denote the part of the integral A_0 (6-7) over points in M_h by A_1, and the part of A_0 over points in $M - M_h$ by A_2. We divide Case 3 into the following three situations depending on where x lies.

Case 3(i): $x \in M_h$ and $\tilde{d}(x, \partial M_h) > r$. In this case, the integral A_0 only involves points in M_h and $A_0 = A_1$. Then the same argument as in Case 1 and (6-11) imply that

$$|\nabla_x d_h^s(x,z) - \nabla_x d_h(x,z)| < C(n, K_1, K_2) h^{3/2}.$$

Case 3(ii): $x \in \partial M_h$. Denote by $\boldsymbol{n}_x \in T_x(\widetilde{M})$ the outward-pointing unit vector normal to ∂M_h. The estimate (6-11) yields the closeness between normal components:

$$|\langle \nabla_v d_h(\exp_x(v), z), \boldsymbol{n}_x \rangle - \langle \nabla_x d_h(x,z), \boldsymbol{n}_x \rangle| < Ch^{3/2} \quad \text{if } \exp_x(v) \in M_h.$$

Since clearly $\langle \nabla_x d_h(x,z), \boldsymbol{n}_x \rangle \geqslant 0$ for $x \in \partial M_h$, we have

$$\langle \nabla_v d_h(\exp_x(v), z), \boldsymbol{n}_x \rangle > -Ch^{3/2} \quad \text{if } \exp_x(v) \in M_h, \tag{6-12}$$

which implies that $\langle A_1, \boldsymbol{n}_x \rangle > -Ch^{3/2}$.

On the other hand, we replace the evaluation at $v = 0$ in the estimate (6-8) with $v = \exp_x^{-1}(x')$ for an arbitrary point $x' \in M - M_h$ close to x. Then consider their normal components similarly. Since $\nabla_x d_h(x', z)$ can be arbitrarily close to $h^{-1}\boldsymbol{n}_x$ and the exponential map only changes the inner product by a higher-order $C(K_1)r^2$-term, we have

$$\langle \nabla_v d_h(\exp_x(v), z), \boldsymbol{n}_x \rangle \geqslant h^{-1} - Ch^2 \quad \text{if } \exp_x(v) \in M - M_h. \tag{6-13}$$

Furthermore by (6-8), the tangential component of $\nabla_v d_h(\exp_x(v), z)$ can only have length at most Ch^2 if $\exp_x(v) \in M - M_h$. This implies that $|A_2 - \langle A_2, \boldsymbol{n}_x \rangle \boldsymbol{n}_x| < Ch^2$.

(1) If $c_n r^{-n} \int_{\{v \in \mathcal{B}_r(0) : \exp_x(v) \in M - M_h\}} k_1(|v|/r)\, dv \geqslant h$, then (6-13) yields that $\langle A_2, \boldsymbol{n}_x \rangle \geqslant 1 - Ch^3$. Thus by (6-12),

$$|A_0| \geqslant |\langle A_0, \boldsymbol{n}_x \rangle| = |\langle A_1 + A_2, \boldsymbol{n}_x \rangle| > 1 - Ch^{3/2} - Ch^3.$$

(2) If $c_n r^{-n} \int_{\{v \in \mathcal{B}_r(0) : \exp_x(v) \in M - M_h\}} k_1(|v|/r)\, dv < h$, then by (6-11) and (3-11), we have

$$|A_1| > \left| c_n r^{-n} \int_{\{v \in \mathcal{B}_r(0) : \exp_x(v) \in M_h\}} k_1\left(\frac{|v|}{r}\right) \left(\nabla_x d_h(x,z)\right) J_x(v)\, dv \right| - Ch^{3/2} > 1 - h - Ch^{3/2}.$$

Observe that (6-13) implies that $\langle A_2, \boldsymbol{n}_x \rangle > 0$ for sufficiently small h. If $\langle A_1, \boldsymbol{n}_x \rangle \geqslant 0$, then

$$\begin{aligned} |A_0| = |A_1 + A_2| &\geqslant |A_1 + \langle A_2, \boldsymbol{n}_x \rangle \boldsymbol{n}_x| - |A_2 - \langle A_2, \boldsymbol{n}_x \rangle \boldsymbol{n}_x| \\ &> |A_1| - Ch^{3/2} > 1 - h - Ch^{3/2} - Ch^2. \end{aligned}$$

If $\langle A_1, \boldsymbol{n}_x \rangle < 0$, then $|\langle A_1, \boldsymbol{n}_x \rangle| < Ch^{3/2}$ by (6-12). This shows that $|A_1 - \langle A_1, \boldsymbol{n}_x \rangle \boldsymbol{n}_x| > 1 - h - Ch^{3/2}$. Hence we have

$$\begin{aligned} |A_0| &\geqslant |A_1 + A_2 - \langle A_1 + A_2, \boldsymbol{n}_x \rangle \boldsymbol{n}_x| \\ &\geqslant |A_1 - \langle A_1, \boldsymbol{n}_x \rangle \boldsymbol{n}_x| - |A_2 - \langle A_2, \boldsymbol{n}_x \rangle \boldsymbol{n}_x| \\ &> 1 - h - Ch^{3/2} - Ch^2. \end{aligned}$$

Case 3(iii): $x \notin \partial M_h$ and $\tilde{d}(x, \partial M_h) \leqslant r$. In this case, we choose an arbitrary point $x_0 \in \partial M_h$ such that $\tilde{d}(x_0, x) \leqslant r$. By the triangle inequality, (6-11) yields that

$$|\nabla_v d_h(\exp_x(v), z) - \nabla_v d_h(\exp_x(v), z)|_{v=v_0}| < C(n, K_1, K_2)h^{3/2} \quad \text{if } \exp_x(v) \in M_h,$$

where $v_0 = \exp_x^{-1}(x_0)$. Then we consider the normal component with respect to $(d\exp_x|_{v_0})^{-1}\boldsymbol{n}_{x_0} \in T_x(\widetilde{M})$ and replace the vector $\boldsymbol{n}_x$ in Case 3(ii) with $(d\exp_x|_{v_0})^{-1}\boldsymbol{n}_{x_0}$. Since $\langle \nabla_x d_h(x, z)|_{x=x_0}, \boldsymbol{n}_{x_0}\rangle_{x_0} \geqslant 0$ with respect the inner product of $T_{x_0}\widetilde{M}$, after lifting the vectors onto $T_x\widetilde{M}$ via the exponential map, we have

$$\big\langle (d\exp_x|_{v_0})^{-1}(\nabla_x d_h(x, z)|_{x=x_0}), (d\exp_x|_{v_0})^{-1}(\boldsymbol{n}_{x_0})\big\rangle_x \geqslant -C(K_1)r^2.$$

Then the rest of the argument in Case 3(ii) applies up to a higher-order term as $d\exp_x|_{v_0}$ only changes the inner product by a higher-order $C(K_1)r^2$-term.

Finally, combining all the cases together, we obtain

$$|\nabla_x d_h^s(x, z)| \geqslant |A_0| - C(n, K_1, K_2)h^3 > 1 - h - C(n, K_1, K_2)h^{3/2},$$

and therefore the lemma follows. □

Lemma 6.6. *For $i \geqslant 1$ and sufficiently small h depending on n, T, K_1, i_0, we have*

$$\operatorname{dist}_{\widetilde{M}\times\mathbb{R}}(\partial\Omega^0_{i,j}, \Omega_{i,j}) > \min\Big\{\frac{h^3}{100}, \frac{h^2}{20T}\Big\}.$$

For $i = 0$, we have

$$\operatorname{dist}_{\widetilde{M}\times\mathbb{R}}(\partial\Omega^0_{0,j}, \Omega_{0,j}) > \frac{h^3}{6T^2}.$$

Proof. There are two types of boundaries involved. The first type is from the level sets of $d_h^s(\cdot, z_{i,j})$. For $i \geqslant 2$, the distance of the first type is from the boundary of the cylinder

$$\{x : d_h^s(x, z_{i,j}) \leqslant \min\{1, T^{-1}\}h/2\} \times [-T_i, T_i]$$

and the boundary of $\bigcup_{l=0}^{i-1}\bigcup_j \bar{\Omega}_{l,j}$. Since a larger cylinder

$$\{x : d_h^s(x, z_{i,j}) \leqslant \min\{1, T^{-1}\}h\} \times [-T_i - h, T_i + h]$$

is also contained in $\bigcup_{l=0}^{i-1}\bigcup_j \bar{\Omega}_{l,j}$ due to (3-33), the distance of this type is bounded below by the distance between these two cylinders, which is bounded below by $\min\{1, T^{-1}\}h^2/20$ by Lemma 6.3(4,3) if $h < 1/10$.

For $i = 1$, the distance of the first type is from the boundary of the cylinder

$$\{x : d_h^s(x, z_{1,j}) \leqslant h/2\} \times [-T_1, T_1]$$

and the boundary of $\bigcup_j \Omega_{0,j}$. By (3-29) and Sublemma 2, the cylinder

$$\{x : d_h^s(x, z_{1,j}) \leqslant 3h/4\} \times [-T_1, T_1]$$

is contained in the open set $\bigcup_j \Omega_{0,j}$, and hence the distance between the boundary of the cylinder and that of $\bigcup_j \Omega_{0,j}$ is bounded away from 0. To obtain an explicit estimate, one can prove a slightly tighter estimate than Sublemma 2 if $T > 10h$:

$$\Big(\bigcup_{b\in[0,2h]} \Gamma_b(8h)\Big) \times [-T+11h/2, T-11h/2] \subset \bigcup_j \Omega_{0,j}.$$

With (3-29), this shows that a larger cylinder

$$\{x : d_h^s(x, z_{1,j}) \leqslant 3h/4\} \times [-T_1 - h/2, T_1 + h/2]$$

is contained in $\bigcup_j \Omega_{0,j}$. Then Lemma 6.3(4,3) yields a lower bound $h^2/40$ if $h < 1/20$.

For $i \geqslant 1$, the other type of boundary is generated by the level sets of $\psi_{i,j}$. Suppose boundary points (x_1, t_1) and (x_2, t_2) belong to $\{\psi_{i,j} = 9T^2h\}$ and $\{\psi_{i,j} = 8T^2h\}$ respectively, and hence by the definition of $\psi_{i,j}$ we have

$$\big((1-\xi(d(x_1,\partial M)) - \xi(\rho_0 - d_h^s(x_1)))T_i - d_h^s(x_1)\big)^2 \\ - \big((1-\xi(d(x_2,\partial M)) - \xi(\rho_0 - d_h^s(x_2)))T_i - d_h^s(x_2)\big)^2 - t_1^2 + t_2^2 = T^2h.$$

Then,

$$2T^2|\xi(\rho_0 - d_h^s(x_1)) - \xi(\rho_0 - d_h^s(x_2))| + 2T^2|\xi(d(x_1,\partial M)) - \xi(d(x_2,\partial M))| \\ + 2T|d_h^s(x_1) - d_h^s(x_2)| + 2T|t_1 - t_2| > T^2h.$$

By the definition of ξ,

$$\frac{6T^2}{h}|d_h^s(x_1, z_{i,j}) - d_h^s(x_2, z_{i,j})| + \frac{6T^2}{h}|d(x_1,\partial M) - d(x_2,\partial M)| \\ + 2T|d_h^s(x_1, z_{i,j}) - d_h^s(x_2, z_{i,j})| + 2T|t_1 - t_2| > T^2h.$$

Then it follows that at least one of the four absolute values must be larger than $h^2/24$ if $h < 3T$, which implies that at least one of $|d_h(x_1, z_{i,j}) - d_h(x_2, z_{i,j})|$, $|d(x_1,\partial M) - d(x_2,\partial M)|$ or $|t_1 - t_2|$ is larger than $h^2/50$ by Lemma 6.3(4). Here we divided the smoothening radius by a constant to keep the error brought by the convolution relatively small. Since $d(x,\partial M) = \tilde{d}(x,\partial M)$ for $x \in M$, Lemma 6.3(3) yields that at least one of $\tilde{d}(x_1, x_2)$ or $|t_1 - t_2|$ is larger than $h^3/100$ and hence the lemma follows.

Finally for the initial step $i = 0$, the first type of boundary distance is from $\{\rho(x) = -3h/2\}$ and the boundary of Υ, which is clearly bounded below by $h/2$. The second type of boundary distance is between level sets of $\psi_{0,j}$. One can follow the same argument as for $i \geqslant 1$ for this type of boundary distance, and obtain a lower bound $h^3/6T^2$. □

Lemma 6.7. *Suppose $\gamma(s)$ is a geodesic of M satisfying $\gamma(0) \in \partial M$ and the initial vector $\gamma'(0) \in T_{\gamma(0)}\partial M$. Then there exists a constant ε_0 explicitly depending on n, $\|R_M\|_{C^1}$, $\|S\|_{C^1}$, i_0 such that, for any $s \leqslant \varepsilon_0$, we have $d(\gamma(s), \partial M) \leqslant C(n, \|R_M\|_{C^1}, \|S\|_{C^1})s^2$.*

Proof. Without loss of generality, assume the geodesic $\gamma(s)$ lies entirely in the interior of M except for the initial point. Consider another geodesic of ∂M with the same initial point $\gamma(0)$ and the same initial vector $\gamma'(0)$. We claim that the distance between this geodesic of ∂M and $\gamma(s)$ is bounded above by Cs^2 for sufficiently small s. Clearly this claim yields the lemma.

Denote the geodesics of $M, \partial M$ in question with the arclength parametrization by γ_1, γ_2. Take $\varepsilon_0 < i_0$ and we consider the geodesics $\gamma_i(s)$ $(i = 1, 2)$ in a C^1 boundary normal coordinate $(x^1, \dots, x^n)$. Due to Lemma 8 in [Hebey and Vaugon 1995] and Lemma 6.1, within a uniform radius explicitly depending on $n, \|R_M\|_{C^1}, \|S\|_{C^1}, i_0$, the C^1-norm of metric components is uniformly bounded by a constant explicitly depending on $n, \|R_M\|_{C^1}, \|S\|_{C^1}$. Since γ_1, γ_2 have the same initial point and the same initial vector, we know $\gamma_1^j(0) = \gamma_2^j(0)$ and $\partial_s\gamma_1^j(0) = \partial_s\gamma_2^j(0)$ for all $j = 1, \dots, n$, where γ_i^j denotes the j-th component of γ_i with respect to the coordinate x^j. The fact that $|\partial_s\gamma_1(s)|_M = |\partial_s\gamma_2(s)|_{\partial M} = 1$ yields $|\partial_s\gamma_i^j(s)| \leqslant C$ for any j due to the C^0 metric bound in bilinear form. Moreover, the geodesic equation in local coordinates has the form

$$\partial_s^2\gamma^j + \sum_{k,l}\Gamma_{kl}^j(\partial_s\gamma^k)(\partial_s\gamma^l) = 0,$$

and γ_1, γ_2 satisfy this equation with Γ_{kl}^j of $M, \partial M$ respectively. Hence by applying the C^1 bound for metric components, we have an estimate for the second derivative:

$$|\partial_s^2\gamma_i^j(s)| \leqslant C(n, \|R_M\|_{C^1}, \|S\|_{C^1}) \quad \text{for all } j = 1, \dots, n. \tag{6-14}$$

Since γ_1, γ_2 lie entirely in $\mathrm{int}(M), \partial M$ by assumption, they are at least of C^2 and hence

$$|\gamma_1^j(s) - \gamma_2^j(s)| \leqslant \frac{s^2}{2}\sup_{s' \in (0,s)}|\partial_s^2\gamma_1^j(s') - \partial_s^2\gamma_2^j(s')| \leqslant C(n, \|R_M\|_{C^1}, \|S\|_{C^1})s^2.$$

This implies $d(\gamma_1(s), \gamma_2(s)) \leqslant C(n, \|R_M\|_{C^1}, \|S\|_{C^1})s^2$ due to the C^0 metric bound. □

Lemma 6.8. *Let $A_t(\varepsilon) = \{x \in \Sigma_t : l(x) > \varepsilon\}$ and denote by $U(A_t(\varepsilon))$ the set of all points on all minimizing geodesics from $A_t(\varepsilon)$ to Γ. Then for sufficiently small ε explicitly depending on K_1 and any $t' \in [t - \varepsilon/2, t)$, we have*

$$\mathrm{vol}_{n-1}(A_t(\varepsilon)) < 5^{n-1}\,\mathrm{vol}_{n-1}(U(A_t(\varepsilon)) \cap \Sigma_{t'}).$$

Proof. We define a function $F : U(A_t(\varepsilon)) \cap \Sigma_{t'} \to A_t(\varepsilon)$ by mapping a point $x \in U(A_t(\varepsilon)) \cap \Sigma_{t'}$ to the initial point of the particular minimizing geodesic containing x from $A_t(\varepsilon)$ to Γ. This function is well-defined since minimizing geodesics cannot intersect at $\Sigma_{t'}$; otherwise they would fail to minimize length past an intersection point. To show the measure estimate in question, it suffices to show that F is locally Lipschitz with a Lipschitz constant 5 for sufficiently small ε depending on K_1. Since the measure in question is an $(n-1)$-dimensional Hausdorff measure, the Lipschitz continuity of F implies the measure estimate with the constant 5^{n-1} [Burago et al. 2001, Section 5.5.2].

Here we show that the function F is locally Lipschitz. For any point

$$y_0 \in U(A_t(\varepsilon)) \cap \{x : t - \varepsilon/2 \leqslant d(x, \Gamma) \leqslant t\},$$

there exists $x_0 \in U(A_t(\varepsilon)) \cap \Sigma_{t-\varepsilon}$ such that x_0 lies on a minimizing geodesic from y_0 to Γ, which indicates $d(y_0, \Gamma) = d(y_0, x_0) + d(x_0, \Gamma)$. Observe that the geodesic segment from y_0 to x_0 does not intersect the boundary. Then there exists a small neighborhood of y_0 such that, for any y in this neighborhood, the minimizing geodesic from x_0 to y does not intersect the boundary. Thus the distance function $d(\cdot, x_0)$ in the small neighborhood of y_0 is just a geodesic distance function with the second derivative bounded by $3/\varepsilon$ for sufficiently small ε depending on K_1 (e.g., [Petersen 2006, Theorem 27, p. 175]). Hence we have

$$\begin{aligned} d(y,\Gamma) &\leqslant d(x_0,\Gamma)+d(y,x_0)=d(y_0,\Gamma)-d(y_0,x_0)+d(y,x_0) \\ &\leqslant d(y_0,\Gamma)+\nabla_y d(y_0,x_0)\cdot \exp_{y_0}^{-1}(y)+\frac{3}{2\varepsilon}d(y,y_0)^2+o(d(y,y_0)^2). \end{aligned}$$

This shows that the distance function $d(\cdot\,,\Gamma)$ is a semiconcave function in

$$U(A_t(\varepsilon))\cap\{x : t-\varepsilon/2\leqslant d(x,\Gamma)\leqslant t\}$$

for sufficiently small ε with the semiconcavity constant $3/\varepsilon$. Now consider the gradient flow by the distance function $d(\cdot\,,\Gamma)$, and the function F is simply the gradient flow restricted to this region $U(A_t(\varepsilon))\cap\{x : t'\leqslant d(x,\Gamma)\leqslant t\}$ for $t'\in[t-\varepsilon/2,t)$. By Lemma 2.1.4(i) in [Petrunin 2007], the restricted gradient flow (or F) is locally Lipschitz with a Lipschitz constant $e^{3/2}<5$. □

Appendix: Dependency of constants

In this section, we show explicitly how the constant in Theorem 1 depends on geometric parameters. We first show the dependency of constants in Theorem 3.1, and then trace the dependency through the proofs in Sections 4 and 5.

For $i\geqslant 1$, the lower bounds (3-27) and (3-28) hold:

$$\min_{(x,t)\in\Omega^0_{i,j}}|\nabla_x\psi_{i,j}|>2T\sqrt{h},\qquad \min_{(x,t)\in\Omega^0_{i,j}} p((x,t),\nabla\psi_{i,j})>8T^2h.$$

From the definition (3-10), Lemma 3.4(3), (6-4), (3-13) and (3-14), for sufficiently small h depending on n, K_1, K_2, i_0, we have

$$\begin{gathered} \|d_h^s(x,z_{i,j})\|_{C^0(\Omega^0_{i,j})}<\rho_0<i_0,\qquad \|\nabla_x d_h^s(x,z_{i,j})\|_{C^0(\Omega^0_{i,j})}<2h^{-1}, \\ \|\nabla_x^2 d_h^s(x,z_{i,j})\|_{C^0(\Omega^0_{i,j})}<C(n,i_0)h^{-6}, \\ \|\nabla_x^2 d_h^s(x,z_{i,j})\|_{\mathrm{Lip}(\Omega^0_{i,j})}<C(n,\|R_M\|_{C^1},i_0)h^{-9}. \end{gathered}$$

On the other hand, the $C^{2,1}$-norm of $d(\cdot\,,\partial M)$ is bounded by $2\|S\|_{C^1}$ for sufficiently small h. Therefore by the definition of $\psi_{i,j}$, for sufficiently small h depending on n, K_1, K_2, i_0,

$$\begin{gathered} \|\psi_{i,j}\|_{C^0(\Omega^0_{i,j})}<T^2,\qquad \|\psi_{i,j}\|_{C^1(\Omega^0_{i,j})}<C(T)h^{-2}, \\ \|\psi_{i,j}\|_{C^2(\Omega^0_{i,j})}<C(n,T,i_0)h^{-7},\qquad \|\psi_{i,j}\|_{C^{2,1}(\Omega^0_{i,j})}<C(n,T,\|R_M\|_{C^1},i_0)h^{-10}. \end{gathered}$$

For $i=0$, we have

$$\min_{(x,t)\in\Omega^0_{0,j}}|\nabla_x\psi_{0,j}|>2h,\qquad \min_{(x,t)\in\Omega^0_{0,j}} p((x,t),\nabla\psi_{0,j})>4h^2,$$

and the bounds above for $\psi_{i,j}$ also hold for $\psi_{0,j}$.

Now we calculate the parameters in the table (4.3) in [Bosi et al. 2016] for our case. The following notation (until (A-1)) was used in that paper and is not used in our present paper; we write it here only for the convenience of the reader:

$$M_1 \sim M_2 \sim h^{-15}, \quad \lambda \sim h^{-15}, \quad R_1 \sim h^{17}, \quad \varepsilon_0 \sim h^{34},$$
$$R_2 \sim R \sim h^{51}, \quad r \sim h^{135}, \quad \delta \sim h^{138},$$
$$N = C(n, T, \|R_M\|_{C^1}, i_0, \|g^{ij}\|_{C^1}, \mathrm{vol}(M), \mathrm{vol}_{n-1}(\Gamma))h^{-135(n+1)}.$$
$$c_{161} \sim c_{158} \sim c_{155,N} + c_{156} \sim c_{155,N} + c_{156}^{-1/(1-\alpha)}, \quad \alpha^N = \tfrac{1}{2}.$$

The quotient $c_{155,j}/c_{155,j-1}$ is polynomial large in h, and $c_{156} \sim c_{106}/c_{131}$ is also polynomial large, where the exponents are explicit multiples of n. Therefore c_{161} has at most exponential growth with an explicit exponent $C_4 n$ for some absolute constant C_4. Then we turn to the constant in our result:

$$\begin{aligned} C(h) &\sim C\big(n, T, \|R_M\|_{C^1}, \|S\|_{C^1}, i_0, \mathrm{vol}(M), \mathrm{vol}_{n-1}(\Gamma)\big)h^{-C(n)h^{-C(n)}} \\ &< C_3\big(n, T, \|R_M\|_{C^1}, \|S\|_{C^1}, i_0, \mathrm{vol}(M), \mathrm{vol}_{n-1}(\Gamma)\big)\exp(h^{-C_4 n}), \end{aligned} \tag{A-1}$$

where we have used the fact that the C^1-norm of metric components is bounded by a constant depending on n, $\|R_M\|_{C^1}$, $\|S\|_{C^1}$. The dependency on the diameter D, r_0 is introduced after replacing h by h/C' during the last part of the proof of Theorem 3.1.

From here, we come back to the notation of our present paper. Next we show the dependency of C_1 and C_2 in Theorem 1. The final parameter is η in Proposition 5.6 and we start from η to work out the parameters J, δ. The criteria for determining parameters are already described during the proofs of relevant lemmas and propositions in Sections 4 and 5. Let $\eta \in (0, 1)$ be the parameter in Proposition 5.6. Then,

$$\Lambda = 1, \quad N = C(n, \mathrm{vol}(\partial M))\eta^{-n+1},$$
$$\varepsilon(\text{volume}, M_\beta^*) = \varepsilon_* = C\eta^n \quad (\text{Proposition 5.6}),$$

and by Lemmas 5.1 and 5.4

$$\varepsilon = \varepsilon(\text{projection}) = \frac{\varepsilon(\text{volume}, M_\alpha)}{2\,\mathrm{vol}(M)} = \frac{\varepsilon(\text{volume}, M_\beta^*)}{2^{L+1} 2\,\mathrm{vol}(M)} = C(\mathrm{vol}(M), L)\eta^n.$$

The following three parameters are determined by (4-21) in Proposition 4.4:

$$\begin{aligned} 8C_5^2\Lambda^2\gamma^{1/(n+1)} = \frac{\varepsilon^2}{4} \quad &\Longrightarrow \quad \gamma = \left(\frac{\varepsilon^2}{32C_5^2\Lambda^2}\right)^{n+1} = \left(\frac{\varepsilon^2}{32C_5^2}\right)^{n+1}, \\ C(\Lambda)\lambda_J^{-1/2}\gamma^{-2} \leqslant \frac{\varepsilon^2}{4} \quad &\Longrightarrow \quad \lambda_J \geqslant 16C^2\gamma^{-4}\varepsilon^{-4}, \\ 8N\varepsilon_2(0) + 8N^2\varepsilon_2^2(0) = \frac{\varepsilon^2}{4} < 1 \quad &\Longrightarrow \quad \varepsilon_2(0) = \frac{\varepsilon^2}{64N}. \end{aligned} \tag{A-2}$$

By the formula for $\varepsilon_2(0)$ in (4-12),

$$\begin{aligned} \varepsilon_2(0) &= C_3^{1/3}h^{-2/9}\exp(h^{-C_4 n})\frac{\Lambda\gamma^{-3} + h^{-1/2}\varepsilon_1}{(\log(1 + h^{3/2}\gamma^{-3}\Lambda/\varepsilon_1))^{1/6}} + C_5\Lambda\gamma^{-3}h^{1/(3n+3)} \\ &< C_3^{1/3}\exp(h^{-C_4 n})\frac{\gamma^{-3}h^{-1}}{(\log(1 + h^{3/2}\gamma^{-3}(1/\varepsilon_1)))^{1/6}} + C_5\gamma^{-3}h^{1/(3n+3)}. \end{aligned} \tag{A-3}$$

We choose h such that the second term in (A-3) equals $\varepsilon_2(0)/2 = \varepsilon^2/(128N)$:

$$C_5\gamma^{-3}h^{1/(3n+3)} = \frac{\varepsilon_2(0)}{2} \quad\Longrightarrow\quad h = \left(\frac{\varepsilon_2(0)\gamma^3}{2C_5}\right)^{3n+3} = \left(\frac{\varepsilon^{6n+8}}{8^{3n+3}128NC_5^{6n+7}}\right)^{3n+3}. \tag{A-4}$$

Then the first term of (A-3) being $\varepsilon_2(0)/2 = \varepsilon^2/(128N)$ yields that

$$\varepsilon_1 = h^{3/2}\gamma^{-3}\exp\left(-\frac{\gamma^{-18}h^{-6}128^6N^6C_3^2\exp(6h^{-C_4 n})}{\varepsilon^{12}}\right), \tag{A-5}$$

which indicates the choice of J by Lemma 4.2:

$$\lambda_J \geqslant C(D,\Lambda)\gamma^{-24}\varepsilon_1^{-8} = C(D)h^{-12}\exp\left(\frac{8\gamma^{-18}h^{-6}128^6N^6C_3^2\exp(6h^{-C_4 n})}{\varepsilon^{12}}\right). \tag{A-6}$$

For the choice of δ, choose $N\varepsilon_2(\delta)+N^2\varepsilon_2^2(\delta)-N\varepsilon_2(0)-N^2\varepsilon_2^2(0) < \varepsilon^2/32$, or simply $N\varepsilon_2(\delta)-N\varepsilon_2(0) < \varepsilon^2/64$. By differentiating (A-3) with respect to ε_1,

$$\frac{C_3^{1/3}\exp(h^{-C_4 n})\gamma^{-6}\sqrt{h}}{(\log(1+h^{3/2}\gamma^{-3}(1/\varepsilon_1)))^2(\varepsilon_1^2+h^{3/2}\gamma^{-3}\varepsilon_1)}2C_0'J\lambda_J^{3/2}\delta < \frac{\varepsilon^2}{64N}.$$

Hence it suffices to choose δ satisfying

$$\frac{C_3^{1/3}\exp(h^{-C_4 n})\gamma^{-3}}{h\varepsilon_1}C_0'J\lambda_J^{3/2}\delta < \frac{\varepsilon^2}{128N}. \tag{A-7}$$

From now on, we absorb polynomial terms into exponential terms and denote by $\sim$ if two quantities differ by a factor of some constant in the exponent. Inserting the choice of γ (A-2) to ε_1 (A-5) and λ_J (A-6), we get

$$\begin{aligned}\varepsilon_1 &\sim \exp(-C_5^{36n+36}C_3^2\exp(h^{-C_4 n})\varepsilon^{-36n-48}N^6),\\ \lambda_J &\sim C(D)\exp(C_5^{36n+36}C_3^2\exp(h^{-C_4 n})\varepsilon^{-36n-48}N^6).\end{aligned}$$

By Weyl's asymptotic formula for eigenvalues: $\lambda_j \sim C(n,\mathrm{vol}(M))j^{2/n}$, we know

$$J \sim C(n,\mathrm{vol}(M))\lambda_J^{n/2},$$

and hence by (A-7), we have

$$\begin{aligned}\delta &\sim \frac{1}{C_0'C_3^{1/3}}\frac{\varepsilon^{6n+8}\exp(-h^{-C_4 n})\varepsilon_1}{C_5^{6n+6}NJ\lambda_J^{3/2}}\\ &\sim C(D,\mathrm{vol}(\partial M))C_3^{-1/3}\frac{\exp(-C_5^{36n+36}C_3^2\exp(h^{-C_4 n})\varepsilon^{-36n-48}N^6)}{J}\\ &\sim C(n,D,\mathrm{vol}(M),\mathrm{vol}(\partial M))\exp(-nC_5^{36n+36}C_3^2\exp(h^{-C_4 n})\varepsilon^{-36n-48}N^6).\end{aligned}$$

The terms we need to estimate are $\exp(h^{-C_4 n})\varepsilon^{-36n-48}N^6$. By the choice of h (A-4), we get

$$\exp(h^{-C_4 n}) \sim \exp\left(\left(\frac{\varepsilon^{6n+8}}{NC_5^{6n+7}}\right)^{-C_4 n(3n+3)}\right),$$

which absorbs $\varepsilon^{-36n-48}N^6$. Then from

$$\varepsilon = C(\mathrm{vol}(M), L)\eta^n, \quad N \sim \eta^{-n+1},$$

it follows that

$$\begin{aligned}\delta &\sim C(n, D, \mathrm{vol}(M), \mathrm{vol}(\partial M)) \exp\bigl(-C(C_3, C_4, C_5) \exp(C\varepsilon^{-C_4 n(3n+3)(6n+9)})\bigr)\\ &\sim C(n, D, \mathrm{vol}(M), \mathrm{vol}(\partial M)) \exp\bigl(-C(C_3, C_4, C_5) \exp(C(L)\eta^{-C_2'(n)})\bigr)\\ &\sim \exp\bigl(-\exp(C_1'\eta^{-C_2'})\bigr),\end{aligned}$$

where $C_1' = C_1'(n, D, \mathrm{vol}(M), \mathrm{vol}(\partial M), C_3, C_4, C_5, L)$ and $C_2' = C_2'(n) > 1$. The dependency of C_3, C_4, C_5 is stated in Proposition 3.3, and the dependency of L is stated in Lemma 5.2. Therefore we obtain

$$\eta \sim (C_1')^{1/C_2'}(\log(|\log \delta|))^{-1/C_2'},$$

and the dependency of constants in Theorem 1 follows from Proposition 5.6 and Theorem 5.7. More precisely, the constant $C_1 = C_1(C_1', C_2', C_6, C_7)$ explicitly depends only on n, D, $\|R_M\|_{C^1}$, $\|S\|_{C^1}$, i_0, r_0, $\mathrm{vol}(M)$, $\mathrm{vol}(\partial M)$, L, C_6, C_7, and the constant $C_2 = C_2(C_2')$ explicitly depends only on n. Note that the dependency of L, C_6, C_7 is not explicit. The choice of the parameter δ depends on all present parameters including all curvature bounds assumed for K_2, and the choice of small η in Theorem 5.7.

We remark that one can obtain an explicit estimate without using the parameter L. To do this, one can use all N of Γ_i to slice the manifold, and evaluate an approximate volume for M_β^* similar to Lemma 5.4. The error of the approximate volume would be $2^N\varepsilon$, and the parameter ε would be $\varepsilon = C2^{-N}\eta^n$. In addition, the constant C_6 can be replaced by an absolute constant. However, the number 2^N grows exponentially in η. This process results in an explicit estimate with three logarithms, and the constants explicitly depend only on n, D, $\|R_M\|_{C^1}$, $\|S\|_{C^1}$, i_0, r_0, $\mathrm{vol}(M)$, $\mathrm{vol}(\partial M)$.

Acknowledgements

Yaroslav Kurylev worked with us until the middle of this project and quit when he could not work anymore, just a couple of months before he passed away. We cannot formally include him as a coauthor, but we still consider him to be one of the authors of this paper.

We would like to thank A. Petrunin for helpful discussions. Burago was partially supported by NSF grant DMS-1205597. Ivanov was partially supported by RFBR grant 20-01-00070. Lassas and Lu were partially supported by Academy of Finland, grants 273979, 284715, 312110, and Finnish Centre of Excellence in Inverse Modelling and Imaging.

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Received 17 Feb 2023. Revised 5 Dec 2023. Accepted 20 Mar 2024.

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ANALYSIS AND PDE
Vol. 18 (2025), No. 4, pp. 1037–1064
DOI: 10.2140/apde.2025.18.1037

A NEW APPROACH TO THE MEAN-FIELD LIMIT OF VLASOV–FOKKER–PLANCK EQUATIONS

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We introduce a novel approach to the mean-field limit of stochastic systems of interacting particles, leading to the first ever derivation of the mean-field limit to the Vlasov–Poisson–Fokker–Planck system for plasmas in dimension 2 together with a partial result in dimension 3. The method is broadly compatible with second-order systems that lead to kinetic equations and it relies on novel estimates on the BBGKY hierarchy. By taking advantage of the diffusion in velocity, those estimates bound weighted L^p norms of the marginals or observables of the system, uniformly in the number of particles. This allows us to qualitatively derive the mean-field limit for very singular interaction kernels between the particles, including repulsive Poisson interactions, together with quantitative estimates for a general kernel in L^2.

1. Introduction

The rigorous derivation of kinetic models such as the Vlasov–Poisson system from many-particle systems has been a long standing open question, ever since the introduction of the Vlasov–Poisson system in [Vlasov 1938; 1967]. While our understanding of the mean-field limit for singular interactions has made significant progress for first-order dynamics, the mean-field limit for second-order systems has remained frustratingly less understood. This article proposes a new approach that is broadly applicable to second-order systems with repulsive interactions and diffusion in velocity. In particular, this allows us to derive for the first time the Vlasov–Poisson–Fokker–Planck system in dimensions higher than 1 without any truncation or regularizing.

We more precisely consider the classical second-order Newton dynamics

$$\begin{aligned} \frac{d}{dt} X_i(t) &= V_i(t), \quad X_i(t=0) = X_i^0, \\ dV_i(t) &= \frac{1}{N} \sum_{j \neq i} K(X_i - X_j)\, dt + \sigma\, dW_i, \quad V_i(t=0) = V_i^0, \end{aligned} \tag{1}$$

where the W_i are N independent Wiener processes. For simplicity we take the positions X_i on the torus Π^d, while the velocities lie in $\mathbb{R}^d$. The kernel K models the pairwise interaction between particles and is taken to be *repulsive* throughout this paper, in the basic sense that it derives from a potential $K = -\nabla\phi$ that is even and positive, $\phi \geq 0$.

Remark 1. For simplicity, we write $\phi(0) = 0$ and $K(0) = 0$ even if ϕ and K are not continuous at 0. This simplifies the notation by allowing us to sum over all j in (1) since the term $j = i$ trivially vanishes.

MSC2020: 35Q70, 35Q83, 60H10, 70F45, 82D10.
Keywords: kinetic equations, mean-field limit, Vlasov–Poisson–Fokker–Planck, marginals.

We naturally focus on singular kernels K with, as a main guiding example, the case of Coulombian interactions

$$K = \alpha \frac{x}{|x|^d} + K_0(x), \tag{2}$$

with $\alpha > 0$ and K_0 a smooth correction to periodize K. This corresponds, if $d \geq 3$, to the choice $\phi = \alpha(d-2)^{-1}|x|^{2-d} + \text{correction}$, or, if $d = 2$, the choice $\phi = -\alpha \ln|x| + \text{correction}$.

The Coulombian kernel (2) typically models electrostatic interactions between point charges, such as ions or electrons in a plasma, when the velocities are small enough with respect to the speed of light. In that setting, diffusion in (1) may for example represent collisions against a random background, such as the collision of the faster electrons against the background of ions. Such random collisions may also involve some friction in velocity, which we did not include in (1) but could be added to our method without difficulty. This makes (1) with (2) one of the most classical and important starting points for the modeling of plasmas; we refer in particular to the classical [Bogoliubov 1946].

Coulombian interactions are also a natural scaling in many models. The obvious counterpart to plasmas concerns the Newtonian dynamics of point masses through gravitational interactions. This consists in taking $\alpha < 0$ in (2) and leads to attractive interactions with a negative potential and for this reason cannot be handled with the method presented here.

The system (1) usually involves a very large number of particles, typically up to 10^{20}–10^{25} in plasmas for example. This makes the mean-field limit especially attractive. This is a kinetic, Vlasov–Fokker–Planck equation posed on the limiting one-particle density $f(t, x, v)$:

$$\partial_t f + v \cdot \nabla_x f + (K \star_x \rho) \cdot \nabla_v f = \frac{\sigma^2}{2} \Delta_v f, \quad \text{with } \rho = \int_{\mathbb{R}^d} f \, dv. \tag{3}$$

Well posedness for mean-field kinetic equations such as (3) is now reasonably well understood, including for singular Coulombian interactions such as (2) in dimension $d \leq 3$. For the nondiffusive case $\sigma = 0$, weak solutions were established in [Arsenev 1975], while classical solutions were obtained in dimension 2 in [Ukai and Okabe 1978]. The dimension 3 case is harder and obtaining classical solutions requires more difficult dispersive arguments and were only obtained later in [Lions and Perthame 1991; Pfaffelmoser 1992; Schaeffer 1991], see also the more recent [Gasser et al. 2000; Holding and Miot 2018; Loeper 2006; Pallard 2014]. In the case with diffusion $\sigma > 0$, we refer to [Victory 1991] for weak solutions, and to [Bouchut 1993; Degond 1986; Ono and Strauss 2000; Rein and Weckler 1992; Victory and O'Dwyer 1990] for classical solutions.

Of course the mean-field scaling is not the only possible scaling on systems such as (1). We mention in particular the likely even more critical Boltzmann–Grad limit, such as obtained in the classical [Lanford 1975] and the major results in [Bodineau et al. 2018; 2020; Gallagher et al. 2014; Pulvirenti and Simonella 2017; Pulvirenti et al. 2014]. We note as well that the derivation of macroscopic equations from mesoscopic systems such as (3) is another important and challenging question. For example the passage to the fluid macroscopic system from Vlasov–Poisson–Fokker–Planck has been approached in different low-field (parabolic) or high-field (hyperbolic) regimes depending on the space dimension; see for example [Carrillo et al. 2022; Goudon et al. 2005; Nieto et al. 2001; Poupaud and Soler 2000].

Mean-field limits have been rigorously derived for general systems, including second-order dynamics such as (1), in the case of Lipschitz interaction kernels K. We refer the reader to the classical works [McKean 1967; Sznitman 1991] in the stochastic case and [Braun and Hepp 1977; Dobrushsin 1979] for the deterministic case. Uniform-in-time propagation of chaos has also been obtained in the locally Lipschitz case, notably in a close to convex case in [Bolley et al. 2010] and more recently in a nonconvex setting in [Guillin et al. 2022].

There now exists a large literature on the question of the mean-field limits; see for example the survey in [Golse 2016; Jabin 2014; Jabin and Wang 2017]. However in the specific case of second-order systems such as (1) very little is known. In dimension $d = 1$, the Vlasov–Poisson–Fokker–Planck system was derived in [Guillin et al. 2023; Hauray and Salem 2019]. In dimensions $d \geq 2$, the only results for unbounded interaction kernels were obtained in [Hauray and Jabin 2007; 2015]. But those are valid only in the deterministic case $\sigma = 0$ and for only mildly singular kernels with

$$|K(x)| \lesssim |x|^{-\alpha} \quad \text{and} \quad |\nabla K| \lesssim |x|^{-\alpha-1} \qquad \text{for } \alpha < 1.$$

Jabin and Wang [2016] derived the mean-field limit with $K \in L^\infty$ and without extra derivative. Those cannot cover Coulombian interactions, even in dimension 2.

More is known for singular interaction kernels K that are smoothed or truncated at some N-dependent scale ε_N. In that truncated case, one can mention in particular [Ganguly and Victory 1989; Ganguly et al. 1991; Victory and Allen 1991; Wollman 2000] for the convergence of so-called particle methods. The recent works [Boers and Pickl 2016; Lazarovici 2016; Lazarovici and Pickl 2017] in the deterministic case and [Huang et al. 2020] in the stochastic case considerably extended the results for such truncated kernels and allowed for almost reaching the critical physical scale $\varepsilon_N \sim N^{-1/d}$. One can also mention [Carrillo et al. 2019] with polynomial cut-off. It is also possible to derive the Vlasov–Poisson system directly from many-particle quantum dynamics such as the Hartree equation, for which we briefly refer to [Golse and Paul 2019; Lafleche 2021; Saffirio 2020].

The mean-field limits for first-order systems with singular interactions appear to be more tractable. A classical example concerns the dynamics of point vortices or stochastic point vortices where the mean-field limit corresponds to the vorticity formulation of two-dimensional incompressible Euler or Navier–Stokes equations. The interaction between vortices obey the Biot–Savart law, which has the same singularity as the Coulombian kernel in dimension 2. In the deterministic case, the mean-field limit was classically obtained for example in [Goodman and Hou 1991; Goodman et al. 1990] or [Schochet 1995; 1996] for the two-dimensional Euler equation and extended remarkably to essentially any Riesz kernels in [Serfaty 2020]. In the stochastic case, we refer in particular to [Fournier et al. 2014; Jabin and Wang 2018; Osada 1987] for the limit to two-dimensional Navier–Stokes equations, to [Bresch et al. 2020; 2023] for singular attractive kernels, or to [Nguyen et al. 2022] for multiplicative noise. Uniform-in-time propagation of chaos was even recently obtained in [Guillin et al. 2024; Rosenzweig and Serfaty 2023].

One of the reasons second-order systems appear more difficult to handle stems from how the structure of the singularity interacts with the distribution of velocities. Because of the term $K(X_i - X_j)$, the singularity in pairwise interactions is typically localized on collisions $X_i = X_j$. For first-order systems this corresponds to a point singularity, while for second-order systems the presence of the additional velocity variables makes it into a plane. In that regard, we also note that the derivation of macroscopic systems directly from second-order dynamics is in fact better understood than the derivation of kinetic equations like (3). We refer to the derivation of incompressible Euler equations in [Han-Kwan and Iacobelli 2021], or to the derivation of monokinetic solutions to (3) (which are essentially equivalent to a macroscopic system) in [Serfaty 2020].

The main argument in our proof is a new quantitative estimate on the so-called marginals of the system through the BBGKY hierarchy. This leads to the propagation of some weighted L^p estimates on the marginals. It implies a weak propagation of chaos in the sense of [Sznitman 1991] but it applies more broadly to initial data that are not chaotic or not close to being chaotic.

Recently, new approaches have been introduced to bound marginals on systems with appropriate nondegenerate diffusion. Using relative entropy, Lacker [2023] was the first to derive quantitative estimates comparing the marginals to the limiting tensorized solution, thus deriving optimal rates for the propagation of chaos in $O(1/N)$, instead of $O(1/\sqrt{N})$ on the convergence of the marginals (as observed for smoother interactions in [Duerinckx 2021]). While formulated for first-order systems, the method also applies to second-order systems with diffusion in velocity, as observed by Lacker. The method takes advantage of the regularizing provided by the diffusion to avoid "losing" a derivative in the hierarchy estimates. The use of the relative entropy however imposes that the interaction kernel belongs to an exponential Orlicz space. In a different context of nonexchangeable systems, [Jabin et al. 2025] later used the propagation of L^2 norms on some equivalent of the marginals, again taking advantage of the diffusion but requiring that the interaction kernel K be in L^∞.

The present article focuses mostly on second-order singular systems, where our method combines this general idea with a specific choice of weights for the L^p norms that are propagated. Those weights are based on a total energy reduced to k particles when dealing with the marginal of order k. They allow us to take advantage of a further regularizing effect in the hierarchy to only require kernels K to be in some L^p with $p > 1$. The same idea to propagate L^p norms on the marginals also applies to first-order systems in confined domains, without then requiring weights.

A direct consequence of our approach is the first ever derivation of the mean-field limit for the repulsive Vlasov–Poisson–Fokker–Planck over a finite time interval. This applies to any chaotic initial data in dimension $d = 2$ and for initial data with more restrictive energy bound in any dimension $d \geq 3$. We are expecting to extend this derivation in a future work to any chaotic initial data in any dimension $d \geq 2$ by decomposing appropriately the initial data.

The paper is structured as follows: We start in Section 2 with the notation and main results. We first state our main result, Theorem 2, that proves the convergence to the Vlasov–Fokker–Planck equation as N tends to infinity followed with Theorem 3 proving quantitative estimates for singular kernels in L^2. We next introduce Proposition 5, which states the explicit propagation of weighted L^p bounds on the marginals. We in particular discuss more thoroughly the limitations and possible extensions of our approach after stating Proposition 5. Section 3 is devoted to the proof of Proposition 5 and Theorem 2 from the key technical contribution of the article around Lemma 9 and ends with the proof of Theorem 3.

2. Main results

2.1. *The new result.* We introduce the full N-particle joint law of the system f_N which satisfies the Liouville or forward Kolmogorov equation

$$\partial_t f_N + \sum_{i=1}^{N} v_i \cdot \nabla_{x_i} f_N + \sum_{i=1}^{N} \frac{1}{N} \sum_{j=1}^{N} K(x_i - x_j) \cdot \nabla_{v_i} f_N = \frac{\sigma^2}{2} \sum_i \Delta_{v_i} f_N, \tag{4}$$

which is a linear advection-diffusion equation. However the marginals $f_{k,N}$ of f_N will also play a critical role in the analysis. They correspond to the law of k among N particles and are represented through

$$f_{k,N}(t, x_1, v_1, \ldots, x_k, v_k) = \int_{\Pi^{d(N-k)} \times \mathbb{R}^{d(N-k)}} f_N(t, x_1, v_1, \ldots, x_N, v_N)\, dx_{k+1}\, dv_{k+1} \cdots dx_N\, dv_N. \tag{5}$$

The question of well-posedness for (4) can be delicate and is separate from the issue of the mean-field limit considered here. For this reason, we consider the notion of an entropy solution $f_N \in L^\infty(\mathbb{R}_+ \times \Pi^{dN} \times \mathbb{R}^{dN})$ to (4), fully described later in Section 2.4, to which we impose some Gaussian decay in velocity:

$$\sup_{t \le 1} \int_{\Pi^{dN} \times \mathbb{R}^{dN}} e^{\beta \sum_{i \le N} |v_i|^2} f_N\, dx_1\, dv_1 \cdots dx_N\, dv_N \le V^N \quad \text{for some } \beta > 0, \quad V > 0, \tag{6}$$

for which we refer to the short discussion in Section 2.4.

Our main result is the derivation of the mean-field limit for a broad class of singular kernels.

Theorem 2. *Assume that there exists some constant $\theta > 0$ such that the potential ϕ satisfies*

$$\int_{\Pi} e^{\theta \phi(x)}\, dx < +\infty \tag{7}$$

and that

$$K = -\nabla \phi \in L^p(\Pi^d) \quad \textit{for some } p > 1.$$

Let f be the unique smooth solution to the Vlasov equation (3) with initial data $f^0 \in C^\infty(\Pi^d \times \mathbb{R}^d)$ such that $\int_{\Pi^d \times \mathbb{R}^d} f^0 e^{\beta |v|^2} < \infty$. Consider moreover an entropy solution f_N to (4) (in the sense of Section 2.4) *satisfying (6) with initial data $f_N^0 \in L^\infty(\Pi^{dN} \times \mathbb{R}^{dN})$. Assume that $f_{k,N}^0$ converges weakly in L^1 to $(f^0)^{\otimes k}$ for each fixed k and that*

$$\| f_{k,N}^0 \|_{L^\infty(\Pi^{dN} \times \mathbb{R}^{dN})} \le M^k$$

for some $M > 0$ and for all $k \le N$. Then there exists T^* depending only on M, V, and $\|K\|_{L^p}$ such that the $f_{k,N}$, given by (5), weakly converge to

$$f_k = f^{\otimes k} \quad \textit{in } L^q_{\text{loc}}([0, T^\star] \times \Pi^{kd} \times \mathbb{R}^{kd})$$

for any k and any $2 < q < \infty$, with $1/q + 1/p \le 1$.

Our estimates can also provide quantitative rates of convergence though we need to use a stronger assumption, namely $K \in L^2$.

Theorem 3. *Assume the same conditions and hypotheses of Theorem 2, with moreover $p = 2$. We also assume that there exists a constant C independent of N and $\varepsilon_N \to 0$ such that*

$$\int_{\Pi^{kd}\times\mathbb{R}^{kd}} |f^0_{k,N} - (f^0)^{\otimes k}|^2 e^{\lambda(0)e_k} \le C^k \varepsilon_N$$

for all k, with

$$e_k(x_1, v_1, \dots, x_k, v_k) = \sum_{i\le k}(1 + |v_i|^2) + \frac{1}{N}\sum_{i,j\le k}\phi(x_i - x_j) \tag{8}$$

and

$$\lambda(t) = \frac{1}{\Lambda(1+t)} \quad \text{for a positive constant } \Lambda.$$

Then, there exists T^* such that $f_{k,N}$ converges strongly to f_k in $L^2_{\mathrm{loc}}([0, T^\star] \times \Pi^{kd} \times \mathbb{R}^{kd})$ for any k, and we have the quantitative estimate

$$\sup_{t\le T^\star}\int_{\Pi^{kd}\times\mathbb{R}^{kd}} |f_{N,k} - f^{\otimes k}|^2 e^{\lambda(t)e_k} \le \widetilde{C}^k \varepsilon_N$$

for some $\widetilde{C}$ independent of N.

In addition to the mean-field limit, Theorem 2 implies the weak propagation of chaos in the sense of the famous [Sznitman 1991], although with strong conditions on f^0_N. Theorem 2 also justifies for the first time the convergence to the Vlasov–Poisson–Fokker–Planck in two space dimensions. More precisely, we highlight the following result.

Corollary 4. *Let $d = 2$, and consider the Poisson kernel $K = -\nabla\phi$ with its associated potential $\phi(x) \simeq -\ln|x|$. Then, the convergence properties given by Theorem 2 hold true, leading to the Vlasov–Poisson–Fokker–Planck system.*

2.2. ***New stability estimates.*** Theorem 2 relies on a new approach to derive estimates on the BBGKY hierarchy solved by the marginals $f_{k,N}$, which is of significant interest in itself. In general, deriving bounds on either the BBGKY or limiting Vlasov hierarchy is complex. We refer for example to [Golse et al. 2013] for the Vlasov hierarchy, and to [Duerinckx and Saint-Raymond 2021] for the study of long-time corrections to mean-field limits. Bounds on the hierarchy are critical for the derivation of collisional models such as the Boltzmann equation, ever since [Lanford 1975]. Even a partial discussion of the challenges in the collisional setting would go well beyond the scope of this paper, and we simply refer again to [Bodineau et al. 2017; 2018; 2020; Gallagher et al. 2014; Kac 1956; Lanford 1975; Pulvirenti et al. 2014; Pulvirenti and Simonella 2017].

The main difficulty in handling the hierarchy consists in the term

$$\nabla_{v_i}\int_{\Pi^d\times\mathbb{R}^d} K(x_i - x_{k+1})\, f_{k+1,N}\, dx_{k+1}\, dv_{k+1}, \tag{9}$$

as seen in (17), because this introduces the next-order marginal $f_{k+1,N}$ into the equation for $f_{k,N}$. When treated naively as a source term, it leads to a loss of one derivative on each equation of the hierarchy.

However, it was noticed first in [Lacker 2023] and then in [Jabin et al. 2025] that one may avoid this loss of derivative in the stochastic case for nondegenerate diffusion: any L^2 estimate then gains an additional H^1 dissipation which can be used to control the loss of one derivative. This idea still appears applicable in the present kinetic context: even though we only have diffusion in velocity, the derivative in (9) is also only on the velocity variable.

Both [Jabin et al. 2025] and [Lacker 2023] require high integrability on the kernel: $K \in L^\infty$ for [Jabin et al. 2025] and some sort of exponential Orlicz space of the type $\int e^{\lambda |K(x)|}\,dx < C$ for [Lacker 2023]. Lacker [2023] used quantitative relative entropy estimates to prove uniqueness on the BBGKY hierarchy, while [Jabin et al. 2025] proved uniqueness on a tree-indexed limiting hierarchy through L^2 bounds. Hence, in both cases, the corresponding bounds on the marginals was already known uniformly in N, and the challenge was to prove that the norm of the difference with the limit is small.

This leads to a first key difference with respect to the present approach and to the first critical new idea introduced in this paper. In essence, we note that the integral in (9) leads to a regularizing effect that has the same scaling as the convolution at the limit: one has by Hölder estimates that

$$\left\| \int_{\Pi^d} K(x_i - x_{k+1}) f(x_1, \dots, x_{k+1})\, dx_{k+1} \right\|_{L^q(\Pi^{dk})} \le \|K\|_{L^p(\Pi^d)} \|f\|_{L^q(\Pi^{d(k+1)})}, \tag{10}$$

provided that $1/p + 1/q \le 1$.

Taking advantage of (10) for singular $K \in L^p$ with p small naturally leads us to try to propagate L^q norms of the marginals $f_{k,N}$ for large exponents q; in opposition to [Jabin et al. 2025; Lacker 2023]. But it also leads to an additional major difficulty, due to the velocity variable in the unbounded space $\mathbb{R}^d$ in (9). In fact, trying to use (10) in (9) as is would force the use of a mixed norm $L^q_x L^1_v$ on the marginals. Unfortunately such mixed norms are notoriously ill-behaved on kinetic equations.

Instead, a more natural idea, from the point of view of kinetic equations, consists in using some moments or fast decay in velocity. Even if they are less usual for kinetic equations, the use of Gaussian moments is especially attractive in the current case because they are naturally tensorized. For example, one has the extension of (10)

$$\int_{\Pi^{dk}\times\mathbb{R}^{dk}} e^{|v_1|^2+\dots+|v_k|^2} \left| \int_{\Pi^d\times\mathbb{R}^d} K(x_i - x_{k+1}) f_{k+1,N}\, dx_{k+1}\, dv_{k+1} \right|^q \\ \le C_d \|K\|^q_{L^p(\Pi^d)} \int_{\Pi^{d(k+1)}\times\mathbb{R}^{d(k+1)}} e^{|v_1|^2+\dots+|v_{k+1}|^2} |f_{k+1,N}|^q, \tag{11}$$

still provided $1/p + 1/q \le 1$.

However, pure Gaussian moments in velocity do not seem to be naturally propagated at the discrete level of the hierarchy, even though they would trivially be propagated on the limiting mean-field equation at least for short time. This leads to the final critical idea of the paper, which is to incorporate the potential energy in the Gaussian: namely to consider $e^{\lambda(t)e_k}$ instead of a pure Gaussian with e_k defined by (8).

We observe that our definition of e_k uses $1 + |v_i|^2$ but could just as well be reduced to $|v_i|^2$ instead as (11) suggests. The extra constant in e_k allows us to normalize the weight of each marginal by a factor $e^{\lambda(0)k}$, which saves some extra numerical constants in the proof.

We also remark that the use of a dynamical weights argument has been recently developed in [Bresch et al. 2023] for first-order particle systems with singular kernels. We also note that Proposition 5, stated below, shows the propagation of weighted L^q bounds on the marginals, without requiring the initial data to be chaotic or close to chaotic as introduced in [Kac 1956]. It hence applies to a broader framework than just the mean-field limit.

Proposition 5. *Let us assume $K \in L^p(\Pi^d)$ for some $p > 1$ and define*

$$\lambda(t) = \frac{1}{\Lambda(1+t)} \quad \text{and} \quad L = \frac{C}{\lambda(1)^\theta} \|K\|_{L^p}^q$$

for positive constants Λ and C, θ depending only on q, d, and σ, and provided that $1/q + 1/p \leq 1$. Consider a renormalized solution f_N to (4) satisfying (6) with initial data $f_N^0 \in L^\infty(\Pi^{dN} \times \mathbb{R}^{dN})$ and satisfying

$$\begin{aligned} \int_{\Pi^{kd}\times\mathbb{R}^{kd}} |f_{k,N}^0|^q e^{\lambda(0)e_k} &\leq F_0^k, \\ \sup_{t\leq 1} \int_{\Pi^{Nd}\times\mathbb{R}^{Nd}} |f_N|^q e^{\lambda(t)e_N} &\leq F^N \end{aligned} \tag{12}$$

for some $F > 0$, $F_0 > 0$, and q such that $2 \leq q < \infty$, with $1/q + 1/p \leq 1$. Then, one has that

$$\sup_{t\leq T} \int_{\Pi^{kd}\times\mathbb{R}^{kd}} |f_{k,N}|^q e^{\lambda(t)e_k} \leq 2^k F_0^k + F^k 2^{2k-N-1}, \tag{13}$$

where T is given by

$$T = \min\left(1, \frac{1}{4L\max(F_0, F)}\right).$$

Proposition 5 shows that the corresponding L^q norm of a marginal at order k behaves like C^k for some constant C. This is the expected scaling for propagation of chaos and tensorized marginals $f_k = f^{\otimes k}$.

However, Proposition 5 also presents several intriguing features that we want to highlight.

- *Vlasov–Poisson–Fokker–Planck in higher dimensions*. Proposition 5 handles just as easily Coulombian interactions in any dimension d, and not only dimension $d = 2$ as Theorem 2. Therefore, Proposition 5 would imply some form of propagation of chaos for the Vlasov–Poisson–Fokker–Planck system in any dimension if we are able to consider initial N-particle laws f_N^0 which are f^0-chaotic as $N \to +\infty$ and whose marginals f^0 and associated solution f_n to the forward Kolmogorov equation satisfy (12). While there are examples of such initial data, take $f_N^0 = Z\exp(-e_N)$ for instance, they demand some sort of truncation or decay of the configurations with high energy. This is not satisfying because we cannot even take $f_N^0 = (f^0)^{\otimes N}$: Assumption (12) cannot hold in such a case as $e^{\lambda(0)e_k}$ is not integrable if K is the Poisson kernel in dimension $d > 2$. The issue is that by taking $f_N^0 = (f^0)^{\otimes N}$, we allow some configurations with high potential energy. And roughly speaking the existence time T in the proposition vanishes as the starting potential energy increases in that case.

- *Repulsive potentials*. Proposition 5 does require repulsive potentials $\phi \geq 0$ as this assumption is critical in the proof. The repulsive assumption on the potential only appears to be needed to handle the discrete many-particle system. The extension to nonrepulsive settings remains an open problem.

• *Extension to the stochastic case of mildly singular kernels.* A special case concerns mildly singular kernels K with $K \in L^p$ for some $p > 1$ such that $\phi \in L^\infty$. In that situation, by considering $\phi + \|\phi\|_{L^\infty}$ instead of ϕ, yielding the same interaction kernel K, we can always ensure that $\phi \geq 0$. For example this easily extends for the first time to the stochastic settings the results of [Hauray and Jabin 2007; 2015], which had been obtained only for deterministic second-order systems with

$$|K| \lesssim |x|^{-\alpha} \quad \text{for } \alpha < 1.$$

• *Convergence for finite times.* We finally emphasize that, like Theorem 2, Proposition 5 holds over a finite time interval, independent of N. This may initially appear puzzling since we are dealing with linear equations for any fixed N. However, because those estimates are essentially independent of N, they also extend to the nonlinear limiting Vlasov equation. Moreover Proposition 5 includes a propagation of Gaussian moments in velocity over the marginals from the term $e^{\lambda(t)e_k}$ and the definition (8) of e_k. The propagation for all times of such moments for Vlasov–Poisson is only known in dimension $d = 2$, see [Degond 1986; Ukai and Okabe 1978], and dimension $d = 3$, see [Bouchut 1993; Gasser et al. 2000; Holding and Miot 2018; Lions and Perthame 1991; Ono and Strauss 2000; Pallard 2014; Pfaffelmoser 1992; Rein and Weckler 1992; Schaeffer 1991; Victory and O'Dwyer 1990] as cited in the introduction; it also requires in dimension 3 the use of dispersion estimates that are not present in our proof. As we already noted, Proposition 5 is in fact valid in any dimension which naturally limits it to some given finite time interval.

2.3. ***The case of first-order systems.*** While we focus on second-order systems, we also emphasize that our method directly applies to first-order systems on bounded domains (in a much simpler manner in fact) and provides the mean-field limit under very weak assumptions on the kernel K again. Consider in that case

$$\begin{gathered} \frac{d}{dt} X_i(t) = \frac{1}{N} \sum_{j \neq i} K(X_i - X_j)\, dt + \sigma\, dW_i, \\ X_i(t=0) = X_i^0, \end{gathered} \tag{14}$$

fully on the torus Π^d. The mean-field limit is similar to (3):

$$\partial_t f + (K \star_x f) \cdot \nabla_x f = \frac{\sigma^2}{2} \Delta_x f. \tag{15}$$

Similarly, the joint law $f_N(t, x_1, \ldots, x_N)$ solves an appropriately modified Liouville equation

$$\partial_t f_N + \sum_{i=1}^N \frac{1}{N} \sum_{j=1}^N K(x_i - x_j) \cdot \nabla_{x_i} f_N = \frac{\sigma^2}{2} \sum_i \Delta_{x_i} f_N. \tag{16}$$

Because system (14) does not involve velocities, many technical difficulties in our proofs actually vanish. For example, we no longer need to add assumptions such as (6). We also do not need to require that K derives from a potential, and hence do not require assumptions like (7). We then have the following equivalent of Theorem 2.

Theorem 6. *Assume that*

$$K \in L^p(\Pi^d) \text{ for some } p > 1, \quad (\text{div } K)_- \in L^\infty(\Pi^d),$$

where x_- denotes the negative part of x. Let f be the unique smooth solution to the Vlasov equation (15) with initial data $f^0 \in C^\infty(\Pi^d)$. Consider moreover an entropy solution f_N to (16) (still in the sense of Section 2.4) *with initial data $f_N^0 \in L^\infty(\Pi^{dN})$. Assume that $f_{k,N}^0$ converges weakly in L^1 to $(f^0)^{\otimes k}$ for each fixed k and that*

$$\|f_{k,N}^0\|_{L^\infty(\Pi^{dN})} \le M^k$$

for some $M > 0$ and for all $k \le N$. Then there exists T^* depending only on M, $\|K\|_{L^p}$, and $\|(\text{div } K)_-\|_{L^\infty}$ such that the $f_{k,N}$, given by (5), weakly converge to $f_k = f^{\otimes k}$ in $L^q_{\text{loc}}([0, T^\star] \times \Pi^{kd})$ for any k and any $2 < q < \infty$, with $1/q + 1/p \le 1$.

Because it is not our main focus, we do not give a distinct proof of Theorem 6.

As mentioned above, there exists now a large literature for the mean-field limit of first-order systems in the stochastic case, with much recent progress for singular kernels. We refer for example to the derivation of two-dimensional Navier–Stokes equations from a system of many vortices in [Fournier et al. 2014; Jabin and Wang 2018; Osada 1987]. The derivation of the two-dimensional Keller–Segel system, corresponding to attractive Coulombian potentials, was recently obtained in [Bresch et al. 2020; Tardy 2024]; see also [Fournier and Tardy 2024] for a precise description of the collisions leading to the blow-up. We also cite [Lacker 2023] which only requires the kernel to be in an Orlicz space similar to Exp, together with [Lacker and Le Flem 2023] which obtains global-in-time regularity for Lipschitz kernels with a smallness assumption on div K.

All those results require stronger assumptions on the kernel K than just $K \in L^p$ with $p > 1$ as here. A similar scaling was however obtained in [Serfaty 2020] on first-order systems with no diffusion. The breakthrough method in that seminal paper is based on a modulated energy between the empirical measure and the limit and it applies to Riesz kernels where $K \sim 1/|x|^\alpha$ with $\alpha < d$ (corresponding to $K \in L^p$ with $p > 1$), with either a repulsive gradient flow or Hamiltonian interactions, or alternatively where $K * f \in W^{1,\infty}$. Uniform-in-time propagation of chaos was later obtained in [Rosenzweig and Serfaty 2023] including diffusion with the restriction $\alpha < d - 1$ using the modulated energy method and some relaxation rates properties. This was recently improved in [Chodron de Courcel et al. 2023] to again $\alpha < d$ combining precise relaxation rates with the new modulated free energy introduced in [Bresch et al. 2020]. One obvious advantage of our method here is that it allows for a much more general form of interaction, with singularities far away from the origin. On the other hand, Theorem 6 does require a nonvanishing diffusion and is again only valid for a finite time, instead of the much stronger uniform-in-time estimates above.

Contrary to the case of second-order systems, this short-time limitation appears less fundamental as many limiting systems do not blow up, with the obvious exception of attractive interactions such as Keller–Segel. We conjecture that the present method could lead to large-time results by taking advantage of the full nondegenerate diffusion for first-order systems.

2.4. ***Our notion of entropy solution for the hierarchy*: *the well-posedness of* (4).**

The definition. Being nonlinear, our estimates cannot be performed on any weak solutions. Moreover, the concept of a solution for f_N is carried over the marginals $f_{k,N}$ and not just the joint law f_N, so we also need an appropriate notion of entropy solutions on those marginals.

The hierarchy for the marginals from the Liouville equation. From (4), the $f_{k,N}$ solve the so-called BBGKY hierarchy

$$\partial_t f_{k,N} + \sum_{i=1}^{k} v_i \cdot \nabla_{x_i} f_{k,N} + \sum_{i\le k} \frac{1}{N} \sum_{j\le k} K(x_i - x_j)\cdot \nabla_{v_i} f_{k,N} \\ + \frac{N-k}{N} \sum_{i\le k} \nabla_{v_i} \cdot \int_{\Pi^d\times\mathbb{R}^d} f_{k+1,N} K(x_i - x_{k+1})\, dx_{k+1}\, dv_{k+1} = \frac{\sigma^2}{2} \sum_{i\le k} \Delta_{v_i} f_{k,N}. \tag{17}$$

If f_N belongs to L^∞ and satisfies (6), then all marginals $f_{k,N}$ belong to $L^\infty_t L^q_{x,v}$ for every $q < \infty$ with similar Gaussian decay. For simplicity, we denote here abstractly by $L^q_{x,v}$ any space $L^q(\Pi^{kd} \times \mathbb{R}^{kd})$ when there is no confusion about the dimension k, as in our case. We also denote by $L^q_{\lambda e_k}$ the weighted L^q space

$$\|f\|^q_{L^q_{\lambda e_k}} = \int_{\Pi^{kd}\times\mathbb{R}^{kd}} |f|^q e^{\lambda e_k}.$$

Since $K \in L^p$ for some $p > 1$, by using a direct Hölder inequality, those bounds on the $f_{k,N}$ imply that

$$\int_{\Pi^d\times\mathbb{R}^d} f_{k+1,N} K(x_i - x_{k+1})\, dx_{k+1}\, dv_{k+1} \in L^\infty_t L^q_{x,v}$$

for all $q < \infty$. This allows us to immediately and rigorously derive (17) from (4).

Definition of entropy solutions. We write the advection component of (17) as

$$L_k = \sum_{i\le k} v_i \cdot \nabla_{x_i} + \frac{1}{N} \sum_{i,j\le k} K(x_i - x_j)\cdot \nabla_{v_i}. \tag{18}$$

The argument above implies that the only difficulties to propagate our estimates in (17) stem from L_k. Consequently we define our entropy solution as follows: a function $f_N \in L^\infty([0,1] \times \Pi^{dN} \times \mathbb{R}^{dN})$ satisfying (6) is an entropy solution if and only if all marginals $f_{k,N}$ for $1 \le k \le N$, as defined by (5), satisfy

$$\int_0^T \int_{\Pi^{dk}\times\mathbb{R}^{dk}} e^{\lambda e_k} |f_{k,N}|^{q-1} \operatorname{sign}(f_{k,N}) L_k f_{k,N}\, dx_1\, dv_1 \cdots dx_k\, dv_k\, dt \ge 0 \tag{19}$$

for any $T \in [0, 1]$, any $1 < q < \infty$, and any $\lambda < \lambda_0$. Inequality (19) is still somewhat formal and should be understood in the following rigorous sense: for some smooth convolution kernel K_ε, one has that

$$\liminf_{\varepsilon\to 0} \int_0^T \int_{\Pi^{dk}\times\mathbb{R}^{dk}} e^{\lambda e_k} |K_\varepsilon^{\otimes k} \star f_{k,N}|^{q-1} \operatorname{sign}(K_\varepsilon^{\otimes k} \star f_{k,N}) K_\varepsilon^{\otimes k} \star (L_k f_{k,N})\, dx_1\, dv_1 \cdots dx_k\, dv_k\, dt \ge 0, \tag{20}$$

where we define

$$K_\varepsilon^{\otimes k} \star g = \int_{\Pi^{dk}\times\mathbb{R}^{dk}} K_\varepsilon(x_1-y_1, v_1-w_1)\cdots K_\varepsilon(x_k-y_k, v_k-w_k) g(y_1, w_1, \dots, y_k, w_k)\, dy_1\, dw_1 \cdots dy_k\, dw_k,$$

with $K_\varepsilon \to \delta$ when $\varepsilon \to 0$. However, it is usually more delicate to determine whether any weak solution f_N in L^∞ and with the bound (6) is an entropy solution according to our definition. For linear advection-diffusion equations such as (4), this is usually approached through the notion of renormalized solutions as introduced in [DiPerna and Lions 1989]. In that context, (20) is obviously similar to the classical commutator estimate at the basis of many methods for renormalized solutions.

Remark 7. (1) We first remark that (19) is automatically satisfied if we have classical solutions. Indeed, L_k is an antisymmetric operator, so we expect it to propagate L^q norms such that, if all terms are smooth, we have

$$|f_{k,N}|^{q-1}\mathrm{sign}(f_{k,N}) L_k f_{k,N} = L_k |f_{k,N}|^q.$$

(2) We immediately observe that the reduced energy e_k is formally invariant under the advection component of (17):

$$L_k e_k = \frac{2}{N}\sum_{i,j\le k} v_i \cdot \nabla_{x_i}\phi(x_i-x_j) + \frac{2}{N}\sum_{i,j\le k} K(x_i-x_j)\cdot v_i = 0$$

since $K = -\nabla_x \phi$. In the same way, we have $L_k \Phi(e_k) = 0$ for any locally Lipschitz function Φ.

(3) If K is smooth and f_N is a classical solution to (4), we would hence immediately have equality in (19). With K only in L^p, it would be straightforward to obtain one entropy solution in the sense defined above, through passing to the limit in a sequence of solutions for a smoother kernel K.

Remark 8. There exists an extensive literature on renormalized solutions with a comparably large variety of potential assumptions that one may consider. While we cannot do justice to this question in this short discussion, we briefly mention for instance [Hauray 2004] that studies the specific case of the Liouville equation (4) for second-order systems without diffusion. In the present setting of a constant nonvanishing diffusion, we also refer to [Bogachev et al. 2015; Le Bris and Lions 2008; 2019] that provide broad results of well-posedness for velocity fields in L^p.

We in particular note that renormalized solutions apply to the case $K \in L^p$ with $p > 2$ and f_N in L^∞ with $\nabla_{v_i} f_N^{q/2} \in L^2$ for any $q < \infty$ and satisfying the extension of (6)

$$\sup_{t\le 1}\int_{\Pi^{dN}\times\mathbb{R}^{dN}} e^{\lambda_0 e_k} f_N\, dx_1\, dv_1 \cdots dx_N\, dv_N < \infty.$$

The latter estimates are natural for the Liouville equation (4), as demonstrated by Lemma 9 for the case $k = N$ in Section 3. In that situation, all marginals $f_{k,N}$ belong to $L_t^\infty L_{x,v}^q$ for every $q < \infty$ with similar exponential decay in e_k and with as well $\nabla_{v_i} f_{k,N}^{q/2} \in L_{t,x,v}^r$ for any $r < 2$. This regularity easily allows us to prove that (20) holds for $\lambda < \lambda_0$.

We also mention that so-called mild solutions can also offer a natural way to prove (20). We simply refer to [Bouchut 1993; Carrillo and Soler 1997] for such formulations through the Fokker–Planck kernel in the whole space, or to [Clark 1993] or [Degond 1986; Victory and O'Dwyer 1990] for periodic conditions.

Strong solutions up to the first collision. We also emphasize that, in the case of repulsive kernels smooth out of the origin but with singular potentials $\lim_{x\to 0}\phi(x)=+\infty$, a straightforward bound on the energy of the system can easily lead to strong solutions on the many-particle system (1), bypassing the need for entropy or renormalized solutions.

Very roughly, if $K\in C^\infty(\Pi^d\setminus\{0\})$, then up to the conditional time of first collision in (1), we may write

$$d\Big(\sum_{i=1}^N |V_i|^2+\frac{1}{N}\sum_{i\neq j}\phi(X_i-X_j)\Big)=\sigma^2\,dt+\sum_{i=1}^N 2\sigma V_i\cdot dW_i.$$

This implies that, with probability 1, the total energy remains finite if it was so initially. Because $\lim_{x\to 0}\phi(x)=+\infty$, it also implies that collisions almost surely never happen. This argument would in particular apply to the Coulombian case in any dimension $d\geq 2$.

To conclude this discussion of the well-posedness of (4) or (1) for a fixed N, we emphasize the estimates that we described here cannot easily be made uniform in N. The previous discussion of the energy bound on the system (1) for the Coulombian interaction in dimension $d=2$ is an excellent illustration: if we have the bound

$$\sum_{i=1}^N |V_i|^2+\frac{1}{N}\sum_{i\neq j}\phi(X_i-X_j)\leq E$$

with some large probability on some time interval and for $\phi(x)=-\log|x|$, then this only proves that, for any $i\neq j$,

$$|X_i-X_j|\geq e^{-NE},$$

which is indeed finite for any fixed N but is completely unhelpful when considering the limit $N\to\infty$.

Hence the present discussion remains focused on renormalized solutions for a fixed N. Quantitative approaches to renormalized solutions have for example been introduced in [Crippa and De Lellis 2008], which are based on the propagation of a sort of log-derivative on the characteristics; see also for example the discussion on Eulerian variants in [Bresch and Jabin 2018]. This leads to an interesting and so far mostly fully open question as to whether it would be possible to obtain quantitative bounds that would combine the limit $N\to\infty$ with some regularity estimates on the solution for a fixed N.

3. Proof of the main results

3.1. ***The BBGKY and Vlasov hierarchies.*** Using (3), the tensorized limits $f_k=\bar f^{\otimes k}$ satisfy the Vlasov hierarchy

$$\partial_t f_k+\sum_{i=1}^k v_i\cdot\nabla_{x_i} f_k+\sum_{i=1}^k\Big(K\star\int_{\mathbb{R}^d} f\,dv\Big)\cdot\nabla_{v_i} f_k=\frac{\sigma^2}{2}\sum_{i=1}^k \Delta_{v_i} f_k. \tag{21}$$

To avoid repeating the analysis working on (17) or (21), we introduce the generalized hierarchy equation

$$\partial_t F_{k,N}+\sum_{i=1}^{k} v_i\cdot\nabla_{x_i}F_{k,N}+\sum_{i\le k}\frac{\gamma}{N}\sum_{j\le k}K(x_i-x_j)\cdot\nabla_{v_i}F_{k,N}$$
$$+\frac{N-\gamma k}{N}\sum_{i\le k}\nabla_{v_i}\cdot\int_{\Pi^d\times\mathbb{R}^d}F_{k+1,N}K(x_i-x_{k+1})\,dx_{k+1}\,dv_{k+1}=\frac{\sigma^2}{2}\sum_{i\le k}\Delta_{v_i}F_{k,N}+R_{k,N}. \quad (22)$$

Note that (22) is exactly (21) for $\gamma=0,\ R_{k,N}=0$ and exactly (17) for $\gamma=1,\ R_{k,N}=0$. In the same spirit we define

$$e_{k,\gamma}=\sum_{i\le k}(1+|v_i|^2)+\frac{\gamma}{N}\sum_{i,j\le k}\phi(x_i-x_j),$$
$$L_{k,\gamma}=\sum_{i\le k}v_i\cdot\nabla_{x_i}+\frac{\gamma}{N}\sum_{i,j\le k}K(x_i-x_j)\cdot\nabla_{v_i}$$

and observe that we of course still have $L_{k,\gamma}e_{k,\gamma}=0$.

The main technical contribution of this section and of the paper is Lemma 9 stated in Section 3.2, which provides estimates for the solutions to (17). We will then use the uniform bound on the k-marginals $f_{k,N}$ for the proof of Proposition 5. Proposition 5 allows passing to the limit in the hierarchy (17), and a final use of Lemma 9 leads to proving uniqueness of the limiting hierarchy (21) to conclude the result of Theorem 2.

3.2. *The key technical lemma.* We first present the key technical lemma which links the k-marginal L^q_w control to the $(k+1)$-marginal L^q_w estimate control.

Lemma 9. *Assume that $K\in L^p(\Pi^d)$ for some $p>1$. There exist some constants Λ, C, and θ depending only on q, d, and σ such that*

$$\|F_{k,N}\|^q_{L^q_{\lambda(t)e_k}}\le\|F_{k,N}(t=0)\|^q_{L^q_{\lambda(0)e_k}}+q\int_0^t\int|F_{k,N}|^{q-1}\mathrm{sign}(F_{k,N})R_{k,N}e^{\lambda(s)e_{k,\gamma}}\,ds$$
$$+k\frac{N-\gamma k}{N}\frac{C}{\lambda^\theta(t)}\|K\|^q_{L^p}\int_0^t\|F_{k+1,N}(s)\|^q_{L^q_{\lambda(s)e_{k+1}}}\,ds$$

for any entropy solution $F_{k,N}$ to (22) (in the sense of Section 2.4) *and satisfying* (6) *with $F_{k,N}\in L^q_{\lambda(t)e_{k,\gamma}}$ and for any $2\le q<\infty$ such that $1/q+1/p\le 1$, with $\lambda(t)$ defined by $\lambda(t)=(\Lambda(1+t))^{-1}$.*

Proof. To be made fully rigorous, many calculations in this proof should involve a convolution kernel K_ε, estimating

$$\frac{d}{dt}\int|K_\varepsilon^{\otimes k}\star F_{k,N}|^q e^{\lambda(t)e_{k,\gamma}},$$

and passing to the limit in $\varepsilon\to 0$ while using appropriately the entropy condition (20). For simplicity, however, we will only present the corresponding formal calculations.

We hence calculate in a straightforward manner

$$\frac{d}{dt}\int|F_{k,N}|^q e^{\lambda(t)e_{k,\gamma}}=q\int|F_{k,N}|^{q-1}\mathrm{sign}(F_{k,N})\partial_t F_{k,N}e^{\lambda(t)e_{k,\gamma}}+\lambda'(t)\int e_{k,\gamma}|F_{k,N}|^q e^{\lambda(t)e_{k,\gamma}}.$$

Inserting now in this identity the definition of $\lambda(t)$ and the (17), we find

$$\begin{aligned}
&\frac{d}{dt}\int |F_{k,N}|^q e^{\lambda(t)e_{k,\gamma}}\\
&= -q\int |F_{k,N}|^{q-1}\mathrm{sign}(F_{k,N})(L_{k,\gamma}F_{k,N})e^{\lambda(t)e_{k,\gamma}} + q\frac{\sigma^2}{2}\int |F_{k,N}|^{q-1}\mathrm{sign}(F_{k,N})\Big(\sum_{i\le k}\Delta_{v_i}F_{k,N}\Big)e^{\lambda(t)e_{k,\gamma}}\\
&\qquad - q\frac{N-\gamma k}{N}\sum_{i\le k}\int |F_{k,N}|^{q-1}\mathrm{sign}(F_{k,N})\nabla_{v_i}\cdot\int K(x_i-x_{k+1})F_{k+1,N}\,dx_{k+1}\,dv_{k+1}e^{\lambda(t)e_{k,\gamma}}\\
&\qquad - \Lambda\lambda^2(t)\int e_{k,\gamma}|F_{k,N}|^q e^{\lambda(t)e_{k,\gamma}} + q\int |F_{k,N}|^{q-1}\mathrm{sign}(F_{k,N})R_{k,N}e^{\lambda(t)e_{k,\gamma}}.
\end{aligned}$$

Note that

$$q|F_{k,N}|^{q-1}\mathrm{sign}(F_{k,N})(L_{k,\gamma}F_{k,N}) = L_{k,\gamma}|F_{k,N}|^q,$$

so that by integration by parts, we formally have

$$q\int |F_{k,N}|^{q-1}\mathrm{sign}(F_{k,N})(L_{k,\gamma}F_{k,N})e^{\lambda(t)e_{k,\gamma}} = -\int |F_{k,N}|^q L_{k,\gamma}e^{\lambda(t)e_{k,\gamma}} = 0.$$

On the other hand, again by integration by parts,

$$\begin{aligned}
&q\frac{\sigma^2}{2}\int |F_{k,N}|^{q-1}\mathrm{sign}(F_{k,N})\Big(\sum_{i\le k}\Delta_{v_i}F_{k,N}\Big)e^{\lambda(t)e_{k,\gamma}}\\
&= -q(q-1)\sum_{i\le k}\frac{\sigma^2}{2}\int |F_{k,N}|^{q-2}|\nabla_{v_i}F_{k,N}|^2 e^{\lambda(t)e_{k,\gamma}}\\
&\qquad - 2q\lambda(t)\sum_{i\le k}\frac{\sigma^2}{2}\int |F_{k,N}|^{q-1}\mathrm{sign}(F_{k,N})v_i\cdot\nabla_{v_i}F_{k,N}e^{\lambda(t)e_{k,\gamma}}.
\end{aligned}$$

By the Cauchy–Schwartz inequality, since $q\ge 2$, we obtain

$$\begin{aligned}
&q\frac{\sigma^2}{2}\int |F_{k,N}|^{q-1}\mathrm{sign}(F_{k,N})\Big(\sum_{i\le k}\Delta_{v_i}F_{k,N}\Big)e^{\lambda(t)e_{k,\gamma}}\\
&\qquad \le -q(q-1)\sum_{i\le k}\frac{\sigma^2}{4}\int |F_{k,N}|^{q-2}|\nabla_{v_i}F_{k,N}|^2 e^{\lambda(t)e_{k,\gamma}} + \frac{q}{q-1}\lambda^2\frac{\sigma^2}{2}\int |F_{k,N}|^q\sum_{i\le k}|v_i|^2 e^{\lambda(t)e_{k,\gamma}}.
\end{aligned}$$

Note that, since $\phi\ge 0$, we have $\sum_{i\le k}|v_i|^2\le e_k$ and, therefore, combining all our estimates so far, we deduce that

$$\begin{aligned}
&\frac{d}{dt}\int |F_{k,N}|^q e^{\lambda(t)e_{k,\gamma}}\\
&\le -q(q-1)\sum_{i\le k}\frac{\sigma^2}{4}\int |F_{k,N}|^{q-2}|\nabla_{v_i}F_{k,N}|^2 e^{\lambda(t)e_{k,\gamma}}\\
&\qquad - q\frac{N-\gamma k}{N}\sum_{i\le k}\int |F_{k,N}|^{q-1}\mathrm{sign}(F_{k,N})\nabla_{v_i}\cdot\int K(x_i-x_{k+1})F_{k+1,N}\,dx_{k+1}\,dv_{k+1}e^{\lambda(t)e_{k,\gamma}}\\
&\qquad - \frac{\Lambda}{2}\lambda^2(t)\int e_{k,\gamma}|F_{k,N}|^q e^{\lambda(t)e_{k,\gamma}} + q\int |F_{k,N}|^{q-1}\mathrm{sign}(F_{k,N})R_{k,N}e^{\lambda(t)e_{k,\gamma}},
\end{aligned}$$

provided that $\Lambda\ge q\sigma^2/(q-1)$.

We integrate by parts the second term in the right-hand side to obtain

$$\sum_{i\le k}\int |F_{k,N}|^{q-1}\mathrm{sign}(F_{k,N})\nabla_{v_i}\cdot\int K(x_i-x_{k+1})F_{k+1,N}\,dx_{k+1}\,dv_{k+1}e^{\lambda(t)e_{k,\gamma}}=\mathrm{RH}_1+\mathrm{RH}_2,$$

with

$$\mathrm{RH}_1=-(q-1)\sum_{i\le k}\int |F_{k,N}|^{q-2}\nabla_{v_i}F_{k,N}\int K(x_i-x_{k+1})F_{k+1,N}\,dx_{k+1}\,dv_{k+1}e^{\lambda(t)e_{k,\gamma}}$$

and

$$\mathrm{RH}_2=-2\lambda(t)\sum_{i\le k}\int |F_{k,N}|^{q-1}\mathrm{sign}(F_{k,N})v_i\int K(x_i-x_{k+1})F_{k+1,N}\,dx_{k+1}\,dv_{k+1}e^{\lambda(t)e_{k,\gamma}}.$$

We perform a straightforward Cauchy–Schwartz inequality on both terms to find that

$$\mathrm{RH}_2\le\lambda^2(t)\sum_{i\le k}\int |F_{k,N}|^q|v_i|^2e^{\lambda(t)e_{k,\gamma}}+\sum_{i\le k}\int |F_{k,N}|^{q-2}\left|\int K(x_i-x_{k+1})F_{k+1,N}\,dx_{k+1}\,dv_{k+1}\right|^2e^{\lambda(t)e_{k,\gamma}},$$

and similarly

$$\mathrm{RH}_1\le\frac{\sigma^2}{4}\sum_{i\le k}\int |F_{k,N}|^{q-2}|\nabla_{v_i}F_{k,N}|^2e^{\lambda(t)e_{k,\gamma}}\\+\frac{(q-1)^2}{\sigma^2}\sum_{i\le k}\int |F_{k,N}|^{q-2}\left|\int K(x_i-x_{k+1})F_{k+1,N}\,dx_{k+1}\,dv_{k+1}\right|^2e^{\lambda(t)e_{k,\gamma}}.$$

Note that by Young estimates

$$\int |F_{k,N}|^{q-2}\left|\int K(x_i-x_{k+1})F_{k+1,N}\,dx_{k+1}\,dv_{k+1}\right|^2e^{\lambda(t)e_{k,\gamma}}\\\le\frac{q-2}{q}\lambda^2\int |F_{k,N}|^qe^{\lambda(t)e_{k,\gamma}}+\frac{2}{q\lambda^{q-2}}\int e^{\lambda(t)e_{k,\gamma}}\left|\int K(x_i-x_{k+1})F_{k+1,N}\,dx_{k+1}\,dv_{k+1}\right|^q.$$

Therefore, combining together all those terms, we obtain the further estimate

$$\sum_{i\le k}\int |F_{k,N}|^{q-1}\mathrm{sign}(F_{k,N})\nabla_{v_i}\cdot\int K(x_i-x_{k+1})F_{k+1,N}\,dx_{k+1}\,dv_{k+1}e^{\lambda(t)e_{k,\gamma}}\\\le\frac{\sigma^2}{4}\sum_{i\le k}\int |F_{k,N}|^{q-2}|\nabla_{v_i}F_{k,N}|^2e^{\lambda(t)e_{k,\gamma}}+\lambda^2(t)\left(1+\frac{(q-2)(q-1)^2}{q\sigma^2}\right)\sum_{i\le k}\int |F_{k,N}|^q(1+|v_i|^2)e^{\lambda(t)e_{k,\gamma}}\\+\frac{2}{q\lambda^{q-2}}\left(1+\frac{(q-1)^2}{\sigma^2}\right)\sum_{i\le k}\int e^{\lambda(t)e_{k,\gamma}}\left|\int K(x_i-x_{k+1})F_{k+1,N}\,dx_{k+1}\,dv_{k+1}\right|^q.$$

Hence, provided that

$$\Lambda\ge 2q\left(1+\frac{(q-2)(q-1)^2}{q\sigma^2}\right),$$

we obtain

$$\frac{d}{dt}\int |F_{k,N}|^qe^{\lambda(t)e_{k,\gamma}}\le C_{q,\sigma,d}k\frac{N-\gamma k}{\lambda^{q-2}N}\int e^{\lambda(t)e_{k,\gamma}}\left|\int K(x_1-x_{k+1})F_{k+1,N}dx_{k+1}dv_{k+1}\right|^q.$$

At this point is where we take advantage of the specific structure of the hierarchy. Denoting by q^* the conjugate of q, namely such that $1/q^* + 1/q = 1$, we bound

$$\left|\int K(x_1 - x_{k+1}) F_{k+1,N}\, dx_{k+1}\, dv_{k+1}\right|^q \\ \leq \left(\int |K(x_1 - x_{k+1})|^{q^*} e^{-(q^*/q)\lambda(t)|v_{k+1}|^2}\, dx_{k+1}\, dv_{k+1}\right)^{q/q^*} \int |F_{k+1,N}|^q e^{\lambda(t)|v_{k+1}|^2}\, dx_{k+1}\, dv_{k+1},$$

which implies

$$\left|\int K(x_1 - x_{k+1}) F_{k+1,N}\, dx_{k+1}\, dv_{k+1}\right|^q \leq \frac{C_{q,\sigma,d}}{\lambda^{qd/(2q^*)}(t)} \|K\|_{L^p}^q \int |F_{k+1,N}|^q e^{\lambda(t)|v_{k+1}|^2}\, dx_{k+1}\, dv_{k+1}$$

since $q \geq p^*$. Consequently

$$\int e^{\lambda(t) e_{k,\gamma}} \left|\int K(x_1 - x_{k+1}) F_{k+1,N}\, dx_{k+1}\, dv_{k+1}\right|^q \\ \leq \frac{C_{q,\sigma,d}}{\lambda^{qd/(2q^*)}(t)} \|K\|_{L^p}^q \int |F_{k+1,N}|^q e^{\lambda(t)|v_{k+1}|^2 + \lambda(t) e_{k,\gamma}}\, dx_1\, dv_1 \cdots dx_{k+1}\, dv_{k+1}.$$

Note that

$$e_{k+1,\gamma} = e_{k,\gamma} + 1 + |v_{k+1}|^2 + \frac{2\gamma}{N} \sum_{i \leq k} \phi(x_i - x_{k+1}) \geq e_{k,\gamma} + 1 + |v_{k+1}|^2,$$

so that

$$\int e^{\lambda(t) e_k} \left|\int K(x_i - x_{k+1}) f_{k+1,N}\, dx_{k+1}\, dv_{k+1}\right|^q \\ \leq \frac{C_{q,\sigma,d}}{\lambda^{qd/(2q^*)}(t)} \|K\|_{L^p}^q \int |f_{k+1,N}|^q e^{\lambda(t) e_{k+1}}\, dx_1\, dv_1 \cdots dx_{k+1}\, dv_{k+1}.$$

This finally lets us conclude, as claimed, that

$$\frac{d}{dt} \int |f_{k,N}|^q e^{\lambda(t) e_{k,\gamma}} \\ \leq k \frac{N - \gamma k}{N} \frac{C_{q,\sigma}, d}{\lambda^{\theta_{q,d}}(t)} \|K\|_{L^p}^q \int |f_{k+1,N}|^q e^{\lambda(t) e_{k+1,\gamma}} + q \int |F_{k,N}|^{q-1} \mathrm{sign}(F_{k,N}) R_{k,N} e^{\lambda(t) e_{k,\gamma}}. \qquad \square$$

3.3. ***Proof of technical results.*** We start this subsection with the proof of Proposition 5.

Proof of Proposition 5. From the analysis in Section 3.1 and the assumptions (6) and (12) of Proposition 5, we have that $F_{k,N} = f_{k,N}$ is a renormalized solution to (17) and thus (22) with $\gamma = 1$. Moreover, $f_{k,N}$ satisfies the other assumptions in Lemma 9 with $R_{k,N} = 0$. Writing

$$X_k(t) = \int |f_{k,N}|^q e^{\lambda(t) e_k},$$

we hence observe that, by Lemma 9, we have the coupled dynamical inequality system

$$X_k(t) \leq X_k(0) + kL \int_0^t X_{k+1}(s)\, ds$$

for any $t \in [0, 1]$, where

$$L = \frac{C}{\lambda^\theta(1)} \|K\|^q_{L^p}.$$

From the assumptions of Proposition 5, we immediately have that

$$X_k(t) \le F_0^k + kL \int_0^t X_{k+1}(s)\, ds. \tag{23}$$

We now invoke the following simple lemma.

Lemma 10. *Consider any sequence $X_k(t)$ satisfying (23). Then one has*

$$X_k(t) \le \sum_{l=k}^m F_0^l L^{l-k} t^{l-k} \frac{(l-1)!}{(k-1)!\,(l-k)!} + L^{m+1-k} \int_0^t X_{m+1}(s)(t-s)^{m-k} \frac{m!}{(k-1)!\,(m-k)!}\, ds. \tag{24}$$

Assuming Lemma 10 holds, we use (24) up to $m+1 = N$ to derive through the assumptions on f_N that

$$X_k(t) \le \sum_{l=k}^{N-1} F_0^l L^{l-k} t^{l-k} \frac{(l-1)!}{(k-1)!\,(l-k)!} + L^{N-k} \int_0^t F^N (t-s)^{N-1-k} \frac{(N-1)!}{(k-1)!\,(N-1-k)!}\, ds,$$

that is

$$X_k(t) \le \sum_{l=k}^{N-1} F_0^l L^{l-k} t^{l-k} \frac{(l-1)!}{(k-1)!\,(l-k)!} + F^N L^{N-k} t^{N-k} \frac{(N-1)!}{(k-1)!\,(N-k)!}. \tag{25}$$

Note that

$$\frac{(l-1)!}{(k-1)!\,(l-k)!} = \binom{l-1}{k-1} \le 2^{l-1}.$$

Hence (25) implies

$$\begin{aligned} X_k(t) &\le \sum_{l=k}^{N} F_0^l L^{l-k} t^{l-k} 2^{l-1} + F^N L^{N-k} t^{N-k} 2^{N-1} \\ &= 2^{k-1} F_0^k \sum_{l=k}^{N-1} F_0^{l-k} 2^{l-k} L^{l-k} t^{l-k} + F^k 2^{k-1} F^{N-k} L^{N-k} t^{N-k} 2^{N-k} \\ &\le 2^{k-1} F_0^k (2 - 2^{k-N+1}) + F^k 2^{k-1} 2^{k-N} \\ &\le F_0^k 2^k + F^k 2^{2k-N-1}, \end{aligned}$$

provided that $4Lt \max(F_0, F) < 1$, which concludes the proof of the proposition. □

We finish with the quick proof of Lemma 10.

Proof of Lemma 10. Taking $m = k$ in (24), we get

$$X_k(t) \le F_0^k + L \int_0^t X_{k+1}(s) \frac{k!}{(k-1)!\,(k-k)!}\, ds,$$

which is our starting point. Moreover, assuming that (24) holds for m, we may use (23) to find

$$X_k(t) \le \sum_{l=k}^{m} F_0^l L^{l-k} t^{l-k} \frac{(l-1)!}{(k-1)!\,(l-k)!} + L^{m+1-k} \int_0^t \left(F_0^{m+1} + L(m+1) \int_0^s X_{m+2}(s)\,ds \right) (t-s)^{m-k} \frac{m!}{(k-1)!\,(m-k)!}\,ds.$$

This yields

$$X_k(t) \le \sum_{l=k}^{m} F_0^l L^{l-k} t^{l-k} \frac{(l-1)!}{(k-1)!\,(l-k)!} + L^{m+1-k} F_0^{m+1} \frac{m!}{(k-1)!\,(m-k)!} \int_0^t (t-s)^{m-k}\,ds + L^{m+2-k} \int_0^t X_{m+2}(r) \int_r^t (t-s)^{m-k}\,ds\,dr \frac{(m+1)!}{(k-1)!\,(m-k)!},$$

or

$$X_k(t) \le \sum_{l=k}^{m} F_0^l L^{l-k} t^{l-k} \frac{(l-1)!}{(k-1)!\,(l-k)!} + L^{m+1-k} F_0^{m+1} \frac{m!}{(k-1)!\,(m+1-k)!} t^{m+1-k} + L^{m+2-k} \int_0^t X_{m+2}(r)(t-r)^{m+1-k}\,dr \frac{(m+1)!}{(k-1)!\,(m+1-k)!},$$

as claimed. □

3.4. ***Proof of Theorem 2.*** The proof of Theorem 2 follows closely the steps in the proof of Proposition 5, once appropriate bounds have been derived.

(1) *Uniform bounds on f_N in $L^q_{e_N}$.* First of all, note that from the assumptions of Theorem 2, we can easily obtain a bound on f_N^0 in $L^q_{\lambda^0 e_N}$ for Λ large enough. Indeed

$$\int_{\Pi^{dN}\times\mathbb{R}^{dN}} |f_N^0|^q e^{\lambda^0 e_N} = e^N \int_{\Pi^{dN}\times\mathbb{R}^{dN}} |f_N^0|^q e^{2\lambda^0 \sum_{i\le N} |v_i|^2} e^{(\lambda^0/N)\sum_{i,j\le N} \phi(x_i-x_j) - \lambda^0 \sum_{i\le N} |v_i|^2}.$$

We have straightforward L^r estimates on $e^{(\lambda^0/N)\sum_{i,j\le N} \phi(x_i-x_j)-\lambda^0\sum_{i\le N}|v_i|^2}$ as, by the Hölder inequality,

$$\int_{\Pi^{dN}\times\mathbb{R}^{dN}} e^{(r\lambda^0/N)\sum_{i,j\le N}\phi(x_i-x_j) - r\lambda^0 \sum_{i\le N}|v_i|^2} = \frac{C^N}{\lambda_0^{N/2}} \int_{\Pi^{dN}} e^{(r\lambda^0/N)\sum_{i,j\le N}\phi(x_i-x_j)} \le \frac{C^N}{\lambda_0^{N/2}} \left(\Pi_{i\le N} \int_{\Pi^{dN}} e^{r\lambda^0 \sum_{j\le N}\phi(x_i-x_j)} \right)^{1/N} \le \frac{C^N}{\lambda_0^{N/2}}$$

for some constant C and by assumption (7) in Theorem 2, provided that $r\lambda^0 \le 1/\theta$. This implies, again by Hölder's inequality,

$$\int_{\Pi^{dN}\times\mathbb{R}^{dN}} |f_N^0|^q e^{\lambda^0 e_N} \le \frac{C^N}{\lambda_0^{N/2}} \int_{\Pi^{dN}\times\mathbb{R}^{dN}} |f_N^0|^{r^* q} e^{2r^*\lambda^0\sum_{i\le N}|v_i|^2} \le \frac{C^N}{\lambda_0^{N/2}} \|f_N^0\|_{L^\infty}^{qr^*-1} \int_{\Pi^{dN}\times\mathbb{R}^{dN}} |f_N^0|^{r^* q} e^{2r^*\lambda^0\sum_{i\le N}|v_i|^2}.$$

Using now assumption (6), provided that $2r^*\lambda_0 \le \beta$, we conclude

$$\int_{\Pi^{dN}\times\mathbb{R}^{dN}} |f_N^0|^q e^{\lambda^0 e_N} \le \left(\frac{CVM}{\lambda_0}\right)^N \tag{26}$$

for any $q < \infty$. We now choose any fixed $2 < q < \infty$ such that $1/p + 1/q < 1$, and we remark that the Liouville equation (4) is included in (22) for $\gamma = 1$, $R_{k,N} = 0$, and $k = N$. Thus, we next invoke Lemma 9 for f_N with $k = N$ and $\gamma = 1$ to find that f_N solves

$$\frac{d}{dt}\int_{\Pi^{dN}\times\mathbb{R}^{dN}} |f_N(t,\cdot,\cdot)|^q e^{\lambda(t)e_N} \le 0,$$

so that, from (26), we obtain

$$\sup_{t\le 1}\int_{\Pi^{dN}\times\mathbb{R}^{dN}} |f_N(t,\cdot,\cdot)|^q e^{\lambda(t)e_N} \le \left(\frac{CVM}{\lambda_0}\right)^N.$$

This finally implies that there exists some constant $F > 0$ such that

$$\sup_{t\le 1}\int_{\Pi^{dN}\times\mathbb{R}^{dN}} |f_N(t,\cdot,\cdot)|^q e^{\lambda(t)e_N} \le F^N. \tag{27}$$

(2) *Uniform estimates on the marginals and passing the limit in the hierarchy* (17). First of all we can perform the same bounds on each $f_{k,N}^0$ to find similarly to (26) that

$$\int_{\Pi^{kd}\times\mathbb{R}^{kd}} |f_{k,N}^0|^q e^{\lambda^0 e_k} \le \left(\frac{CVM}{\lambda_0}\right)^k.$$

As a consequence, every assumption of Proposition 5 holds and, in particular, assumption (12) holds. This implies that, for some time $T^* > 0$ depending only on V, M, $\|K\|_{L^p}$, and the choice of q, we have

$$\sup_N \sup_{t\le T^*}\int_{\Pi^{kd}\times\mathbb{R}^{kd}} |f_{k,N}|^q e^{\lambda(t)e_k} \le \overline{M}^k$$

for some constant $\overline{M}$. At this point, we will no longer need the potential in the reduced energy e_k, which was required to handle the L_k operator that vanishes at the limit. For this reason and since $\phi \ge 0$, we deduce from the previous inequality that

$$\sup_N \sup_{t\le T^*}\int_{\Pi^{kd}\times\mathbb{R}^{kd}} |f_{k,N}|^q e^{\lambda(T^*)\sum_{i\le k}|v_i|^2} \le \overline{M}^k. \tag{28}$$

These uniform bounds let us extract a converging subsequence such that all $f_{k,N}$ converge weak-$\star$ to some $\bar{f}_k$ in $L^\infty([0,T^*], L^q_{x,v})$ which also satisfies

$$\sup_{t\le T^*}\int_{\Pi^{kd}\times\mathbb{R}^{kd}} |\bar{f}_k|^q e^{\lambda(T^*)\sum_{i\le k}|v_i|^2} \le \overline{M}^k, \tag{29}$$

where we have used classical convex estimates. We emphasize that for the moment we only have convergence of a subsequence, though we still denote it by N for simplicity. We eventually obtain the convergence of the whole sequence only after the uniqueness of the limit is proved in the next step.

From estimate (28) and since $1/q + 1/p \leq 1$, we may simply bound

$$\left\| \sum_{i \leq k} \frac{1}{N} \sum_{j \leq k} K(x_i - x_j) \cdot \nabla_{v_i} f_{k,N} \right\|_{L_t^\infty L^1_{x,v,\mathrm{loc}}} \lesssim \frac{k^2}{N} \|K\|_{L^p} \|f_{k,N}\|_{L_t^\infty L^q_{x,v}}.$$

For any fixed k, the corresponding term vanishes as $N \to \infty$. Similarly estimate (28) allows us to pass to the limit

$$\int_{\Pi^d \times \mathbb{R}^d} K(x_i - x_{k+1}) f_{k+1,N} \, dx_{k+1} \, dv_{k+1} \to \int_{\Pi^d \times \mathbb{R}^d} K(x_i - x_{k+1}) \bar{f}_{k+1} \, dx_{k+1} \, dv_{k+1}$$

for the weak-$\star$ topology of $L^\infty([0, T^*], L^q_{x,v})$. It is straightforward to pass to the limit in the sense of distributions in all other terms of the hierarchy (17), so we deduce that $\bar{f}_k$ is a solution to the limiting hierarchy (21) in the sense of distributions.

We can also easily identify the initial value of $\bar{f}_k$. From (17) and the bounds derived from (28), we immediately obtain a uniform bound on $\partial_t f_{k,N}$ in $L_t^\infty W^{-1,q}_{x,v,\mathrm{loc}}$. By the assumption of Theorem 2, $f^0_{k,N}$ converges weakly to $(f^0)^{\otimes k}$, so we have

$$\bar{f}_k(t = 0) = (f^0)^{\otimes k}.$$

(3) *Uniqueness on the limiting hierarchy and conclusion.* We first argue that $\bar{f}_k$ is automatically a renormalized solution to (21). Indeed, (21) can be seen as a linear advection-diffusion equation with a locally Lipschitz velocity field $(v_1, \dots, v_k)$ and a remainder

$$\nabla_{v_i} \cdot \int_{\Pi^d \times \mathbb{R}^d} K(x_i - x_j) \bar{f}_{k+1} \, dx_{k+1} \, dv_{k+1}$$

that belongs to $L_t^\infty L^q_{x,v}$ with $q > 2$ per our prior estimates.

Next we note that, since f is a classical solution to the Vlasov equation (3), the $f^{\otimes k}$ also yield renormalized solutions to the Vlasov hierarchy (21) for every $k \geq 1$. Due to the linearity in terms of the sequence $\{f_k\}_{k \in \mathbb{N}^\star}$ of the Vlasov hierarchy, we get that each $F_k = \bar{f}_k - f^{\otimes k}$ is also a renormalized solution to the Vlasov Hierarchy (21) for every k. Moreover, since $\bar{f}_k$ and $f^{\otimes k}$ are identical at the initial time $t = 0$, we have that $F_k(t = 0) = 0$.

Furthermore, by (29) and the assumption of Gaussian decay on f^0, we have

$$\sup_{t \leq T^*} \int_{\Pi^{kd} \times \mathbb{R}^{kd}} |F_k|^q e^{\tilde{\beta} \sum_{i \leq k} (1+|v_i|)^2} \leq \widetilde{M}^k \tag{30}$$

for some $\tilde{\beta}$ and some $\widetilde{M}$. Equation (21) corresponds to (22) in the case $\gamma = 0$, where $e_{k,\gamma}$ reduces to $e_{k,0} = \sum_{i \leq k} (1 + |v_i|)^2$. Hence, provided we choose some $\widetilde{\Lambda}$ possibly lower than Λ, we satisfy all assumptions from Lemma 9.

Defining $Y_k = \int |F_k|^q e^{\tilde{\lambda}(t) e_{k,0}}$, we get for all $k \in \mathbb{N}^\star$

$$Y_k(t) \leq k \widetilde{L} \int_0^t Y_{k+1} \, ds. \tag{31}$$

We can then use Lemma 10 with $F_0 = 0$ up to any arbitrary m to show, together with (30), that

$$Y_k(t) \le \widetilde{L}^{m+1-k}\widetilde{M}^{m+1}\int_0^t (t-s)^{m-k}\frac{m!}{(k-1)!\,(m-k)!}\,ds$$
$$\le \widetilde{L}^{m+1-k}\widetilde{M}^{m+1}t^{m+1-k}\binom{m}{k-1} \le 2^k\widetilde{M}^k(2\widetilde{L}\widetilde{M}t)^{m+1-k}. \tag{32}$$

By taking $t < T_0$ with T_0 small enough and sending m to ∞, we obtain that $Y_k(t) = 0$, and hence $\bar{f}_k = f^{\otimes k}$ on $[0, T_0]$. This allows us to repeat the argument starting from $t = T_0$ instead of $t = 0$ until we reach the maximum time T^*. This finally allows us to conclude as claimed that $\bar{f}_k = f^{\otimes k}$ over the whole interval $[0, T^*]$.

Coming back to our extracted subsequence on $f_{k,N}$, since all such subsequences have the same limit, we have convergence of the whole sequence to the $f^{\otimes k}$, concluding the proof.

3.5. *Proof of Theorem 3.* The aim of this result is to provide a quantitative estimate between $f_{k,N}$ and f_k that satisfies (17) and (21), respectively, for the tensorized limits $f_k = f^{\otimes k}$. First let us note that $F_k^N = f_{k,N} - f_k$ satisfies

$$\partial_t F_k^N + L_k F_k^N + \frac{N-k}{N}\sum_{i=1}^k \nabla_{v_i}\cdot\int_{\Pi^d\times\mathbb{R}^d} F_{k+1}^N K(x_i - x_{k+1})\,dx_{k+1}\,dv_{k+1} = \frac{\sigma^2}{2}\sum_{i=1}^k \Delta_{v_i}F_{k,N} + R_{k,N},$$

where L_k is defined in (18) and

$$R_{k,N} = \sum_{i=1}^k\bigg[\Big(K\star\int_{\mathbb{R}^d} f\Big)(t,x_i) - \frac{1}{N}\sum_{j=1}^k K(x_i - x_j)\bigg]\cdot\nabla_{v_i} f_k.$$
$$-\frac{N-k}{N}\sum_{i=1}^k \nabla_{v_i}\cdot\int_{\Pi^d\times\mathbb{R}^d} f_{k+1}K(x_i - x_{k+1})\,dx_{k+1}\,dv_{k+1}. \tag{33}$$

We again use Lemma 9 with $q = 2$ to deduce

$$\frac{d}{dt}\int_{\Pi^{kd}\times\mathbb{R}^{kd}} |F_{k,N}|^2 e^{\lambda(t)e_{k,\gamma}} + \frac{\sigma^2}{4}\sum_{i\le k}\int_{\Pi^{kd}\times\mathbb{R}^{kd}} |\nabla_{v_i}F_{k,N}|^2 e^{\lambda(t)e_k}$$
$$\le k\frac{N-k}{N}\frac{C_{2,\sigma,d}}{\lambda^{\theta_{2,d}}(t)}\|K\|_{L^2}^2\int_{\Pi^{kd}\times\mathbb{R}^{kd}} |F_{k+1,N}|^2 e^{\lambda(t)e_{k+1}}$$
$$+\lambda'(t)\int_{\Pi^{kd}\times\mathbb{R}^{kd}} e_k|F_{k,N}|^2 e^{\lambda(t)e_k} + \int_{\Pi^{kd}\times\mathbb{R}^{kd}} R_{k,N}F_{k,N}e^{\lambda(t)e_k}. \tag{34}$$

Note that $R_{k,N}$ may be written as

$$R_{k,N} = \sum_{i=1}^k\frac{1}{N}\sum_{j=1}^k\bigg[\Big(K\star\int_{\mathbb{R}^d} f\Big)(t,x_i) - K(x_i - x_j)\bigg]\cdot\nabla_{v_i} f_k$$
$$-\frac{N-k}{N}\sum_{i=1}^k\bigg[\nabla_{v_i}\cdot\int_{\Pi^d\times\mathbb{R}^d} f_{k+1}K(x_i - x_{k+1})\,dx_{k+1}\,dv_{k+1} - \Big(K\star\int_{\mathbb{R}^d}\bar{f}\Big)(t,x_i)\cdot\nabla_{v_i} f_k\bigg]. \tag{35}$$

Then, using that $f_k = f^{\otimes k}$, we have

$$\int_{\Pi^{kd}\times\mathbb{R}^{kd}} R_{k,N} F_{k,N} e^{\lambda(t)e_k} = \int_{\Pi^{kd}\times\mathbb{R}^{kd}} \frac{k}{N}\sum_{i=1}^{k}\Big[\Big(K\star\int_{\mathbb{R}^d} f\Big)(t,x_i) - K(x_i - x_1)\Big]\cdot\nabla_{v_i} f_k F_{k,N} e^{\lambda(t)e_k},$$

where we have used the fact that the particles are interchangeable. Integrating by parts with respect to v_i and using Young's inequality, we obtain

$$\int_{\Pi^{kd}\times\mathbb{R}^{kd}} R_{k,N} F_{k,N} e^{\lambda(t)e_k} \le \frac{\sigma^2}{4}\frac{k}{N}\sum_{i=1}^{k}\int_{\Pi^{kd}\times\mathbb{R}^{kd}} |\nabla_{v_i} F_{k,N}|^2 e^{\lambda(t)e_k} + \frac{1}{\sigma^2}\frac{k}{N}\sum_{i=1}^{k}\int_{\Pi^{kd}\times\mathbb{R}^{kd}} |\widetilde{R}^1_{k,N}|^2 e^{\lambda(t)e_k}$$
$$+\lambda(t)\int_{\Pi^{kd}\times\mathbb{R}^{kd}} e_k |F_{k,N}|^2 e^{\lambda(t)e_k} + \frac{1}{2}\int_{\Pi^{kd}\times\mathbb{R}^{kd}} |\widetilde{R}^2_{k,N}|^2 e^{\lambda(t)e_k}, \quad (36)$$

where

$$\widetilde{R}^1_{k,N} = \Big[\Big(K\star\int_{\mathbb{R}^d} f\,dx\Big)(t,x_i) - K(x_i - x_1)\Big] f_k,$$
$$\widetilde{R}^2_{k,N} = \sum_{i=1}^{k}\Big[\Big(K\star\int_{\mathbb{R}^d} f\,dx\Big)(t,x_i) - K(x_i - x_1)\Big] f_k.$$

We observe that

$$\|\widetilde{R}^i_{k,N}\|^2_{L^2_{\lambda(t)e_k}} \le Ck\int_{\Pi^{kd}\times\mathbb{R}^{kd}} |f_k|^p e^{\lambda(t)e_k},$$

with a constant C that does not depend on k. We have also used the fact that, in particular, $K \in L^2(\Pi^d)$ and $f \in L^\infty(\Pi^d\times\mathbb{R}^d)$.

Then, using (13) and letting $N \to +\infty$, we get

$$\sup_{t\le T^*}\int_{\Pi^{kd}\times\mathbb{R}^{kd}} |f_k|^p e^{\lambda(t)e_{k,\gamma}} \le 2^k F_0^k.$$

We can insert this estimate into (36) for $p = 2$ to derive

$$\int_{\Pi^{kd}\times\mathbb{R}^{kd}} R_{k,N} F_{k,N} e^{\lambda(t)e_k}$$
$$\le \frac{\sigma^2}{4}\frac{k}{N}\sum_{i=1}^{k}\int_{\Pi^{kd}\times\mathbb{R}^{kd}} |\nabla_{v_i} F_{k,N}|^2 e^{\lambda(t)e_k} + \lambda(t)\int_{\Pi^{kd}\times\mathbb{R}^{kd}} e_k|F_{k,N}|^2 e^{\lambda(t)e_k} + Ck2^kF_0^k.$$

Once this estimate is incorporated into (34) and using that $\lambda'(t) = -\lambda(t)/(1+t)$, we can, following the same lines of the proof of Proposition 5, repeat the estimate on the ODE inequality with the extra term coming from the interaction of $F_{k,N}$ with rest term $R_{k,N}$. This provides the conclusion that there exists T^* such that

$$\sup_{t\le T^\star}\int_{\Pi^{kd}\times\mathbb{R}^{kd}} |f_{N,k} - f_k|^2 e^{\lambda(t)e_{k,\gamma}} \le \widetilde{C}^k\varepsilon_N + \widetilde{C}^k\int_{\Pi^{kd}\times\mathbb{R}^{kd}} |f^0_{N,k} - f^0_k|^2 e^{\lambda(0)e_{k,\gamma}},$$

where $\widetilde{C}$ is a positive constant that does not depend on N and $\varepsilon_N = O(\varepsilon^N)$, where $\varepsilon < 1$ depends on a small enough T^*. This expression can be deduced in a similar way as (32) in the proof of Theorem 2. We finally emphasize that the quantitative bounds of Theorem 3 would allow us to recover the optimal convergence rate in $O(1/N)$ recently obtained in [Lacker 2023].

Acknowledgements

D. Bresch is partially supported by SingFlows project, grant ANR-18-CE40-0027. P. E. Jabin is partially supported by NSF DMS grants 161453, 1908739, 2049020. J. Soler is partially supported by the State Research Agency (SRA) of the Spanish Ministry of Science and Innovation and European Regional Development Fund (ERDF), project PID2022-137228OB-I00, and by Modeling Nature Research Unit, project QUAL21-011.

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Received 7 Mar 2023. Revised 21 Sep 2023. Accepted 21 Dec 2023.

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ANALYSIS & PDE

Volume 18 No. 4 2025